C*-algebras of commuting endomorphisms

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dedicated to Șerban Strătilă on his 60th birthday

Abstract

Given a compact space $X$ and two commuting continuous open surjective maps $\sigma_1, \sigma_2 : X \to X$, we construct certain C*-algebras that reflect the dynamics of the $\mathbb{N}^2$-action. When the maps $\sigma_1, \sigma_2$ are local homeomorphisms, these are groupoid algebras, but in general, we will use a Cuntz-Pimsner algebra associated to a product system of Hilbert bimodules in the sense of Fowler.

The motivating example for our construction is the dynamical system associated with a rank two graph by Kumjian and Pask. We consider also a two-dimensional subshift of Ledrappier, the case of two covering maps of the circle, and the two-dimensional Bernoulli shift.

§1. Introduction

There are different approaches to define the crossed product of a C*-algebra by a semigroup of endomorphisms in the works of G. Murphy, I. Raeburn, M. Laca and others (see [M1], [M2], [LR], [LiR], [F]). They are using covariant isometric or partial isometric representations of the positive cone in a totally ordered abelian group, are realizing the crossed product by a cancellative abelian semigroup of injective endomorphisms as a corner of the covariance algebra of a classical C*-dynamical system, or are using a family of Hilbert bimodules indexed by the quasi-lattice ordered semigroup and generalized Cuntz-Pimsner algebras.

Our examples are mainly related to an abelian $\mathbb{N}^2$-dynamical system, and we will consider in section 2 a groupoid approach, similar to [D], and a Hilbert bimodule approach in the next section. We will see how the definition of the inner product, right and left actions will affect the outcome, and how the dynamics of the $\mathbb{N}^2$-action is reflected in the corresponding C*-algebra. We define the inner products by using the transfer operators associated to some conditional expectations onto the ranges of the endomorphisms as in [E1], [E2].

In the case of local homeomorphisms, the resulting Hilbert bimodules are finitely generated and projective, therefore the left action consists entirely of compact operators. In general, the transfer operators are defined using a family of measures as in [Br], and it may happen that the left action has trivial intersection with the compacts. For two commuting endomorphisms, we will construct a product system of Hilbert bimodules over the semigroup $\mathbb{N}^2$(see [F]), and the corresponding Cuntz-Pimsner algebra will play the role of the groupoid algebra. Sometimes, this algebra could be understood as an iterated Pimsner construction, by extending the scalars and by using the universal properties.
§2. The groupoid approach

Consider a compact space $X$ with two commuting continuous open surjections $\sigma_1, \sigma_2 : X \to X$. In analogy to the groupoids associated to a covering map (see [D]), we define the equivalence relations

$$R_n = \{(x, y) \in X \times X \mid \sigma^n x = \sigma^n y\} \quad \text{and} \quad R = \bigcup_{n \in \mathbb{N}^2} R_n.$$  

Here $n = (n_1, n_2)$ and $\sigma^n = \sigma_1^{n_1} \sigma_2^{n_2}$. We consider $\mathbb{N}^2$ directed by the partial order

$$(n_1, n_2) \leq (m_1, m_2) \text{ if } n_i \leq m_i, i = 1, 2.$$  

We put the induced topology on $R_n$, and the inductive limit topology on $R$.

**Proposition 1.** If $\sigma_i$ are local homeomorphisms, then all $R_n$ are r-discrete, $C^*(R)$ is well defined, and

$$C^*(R) = \lim_{\rightarrow} C^*(R_n).$$

**Proof.** See pages 122-123 in [Re].

Let $\Gamma = \Gamma(\sigma_1, \sigma_2) = \{(x, p - q, y) \in X \times \mathbb{Z}^2 \times X \mid \sigma^p x = \sigma^q y\}$. With the usual operations,

$$(x, k, y) \cdot (y, l, z) = (x, k + l, z), \quad (x, k, y)^{-1} = (y, -k, x),$$

$\Gamma$ is a groupoid with the unit space identified with $X$. The isotropy group bundle is

$$I = \{(x, p - q, x) \in X \times \mathbb{Z}^2 \times X \mid \sigma^p x = \sigma^q x\}.$$

**Proposition 2.** If all the equivalence relations $R_n$ are r-discrete, then $\Gamma$ is a locally compact Hausdorff groupoid with a Haar system.

**Proof.** The topology of $\Gamma$ is defined by the cylinder sets

$$Z(U, V, p, q) = \{(x, p - q, (\sigma \mid V)^{-q} \circ \sigma^p(x)), x \in U\},$$

where $p, q \in \mathbb{N}^2$, and $U$ and $V$ are open sets of $X$ such that $\sigma^p \mid U$ and $\sigma^q \mid V$ are homeomorphisms. A Haar system on $R$ could be extended to $\Gamma$.

**Remark 1.** Assuming that $C^*(\Gamma)$ is well defined (which certainly happens in the case $\sigma_i$ are local homeomorphisms), the cocycle

$$c : \Gamma \to \mathbb{Z}^2, c(x, n, y) = n,$$
induces a $T^2$-action on $C^*(\Gamma)$, via

$$(z \cdot f)(x, n, y) = z^n \cdot f(x, n, y), f \in C_c(\Gamma), z \in T^2.$$ 

**Remark 2.** If $\sigma_i$ are homeomorphisms, then $C^*(\Gamma)$ is isomorphic to the usual crossed product $C(X) \times \mathbb{Z}^2$.

In general, we will see that $C^*(\Gamma)$ is a certain crossed product of $C(X)$ by the semigroup $\mathbb{N}^2$.

**Definition.** The orbit of $x \in X$ is defined to be

$$O(x) = \bigcup_{k \in \mathbb{N}^2} \sigma^{-k}(\sigma^k x),$$

where $\sigma^{-k}y = \{z \in X | \sigma^k z = y\}$.

We say that $\sigma$ is minimal if each orbit is dense and that $\sigma$ is essentially free if

$$\{x \in X | \forall k, l, \sigma^k x = \sigma^l x \Rightarrow k = l\}$$

is dense in $X$.

**Proposition 3.** If $\sigma$ is minimal, then $C^*(R)$ is simple. Moreover, if $\sigma$ is essentially free, then $C^*(\Gamma)$ is also simple.

**Proof.** Since $\sigma$ is minimal and

$$O(x) = \{y \in X | (x, y) \in R\},$$

it follows that there are no nontrivial open invariant subsets. If $\sigma$ is also essentially free, then the groupoid $\Gamma$ is essentially principal in the sense of Renault (see definition II.4.3 of [Re]). We can apply proposition II.4.6 of [Re], where the ideals of an essentially principal groupoid are characterized.

**Example 1.** Consider $X$ the infinite path space of a rank two graph in the sense of Kumjian and Pask, and $\sigma_1, \sigma_2$ the horizontal and vertical shifts, which are local homeomorphisms (see [KP]). For simplicity, we describe $X$ in a particular situation. Start with two finite graphs $(G_1, V)$ and $(G_2, V)$ with the same set of vertices $V$, such that there is a bijection $\rho : G_1 * G_2 \to G_2 * G_1$. Here

$$G_1 * G_2 = \{(e, f) \in G_1 \times G_2 | s(e) = r(f)\},$$

where $s$ and $r$ are the source and range maps. For example, this happens if the vertex matrices commute. Then $X$ could be thought as an infinite grid in the first quadrant, where the vertices at the lattice points are joined by horizontal edges (from $G_1$) and vertical edges (from $G_2$). The unique factorization property translates in the fact that the bijection $\rho$ determines the other two edges of each unit square, once we fix
a horizontal edge, followed by a vertical edge (see section 6 in [KP]). It follows that any finite rectangular grid is determined by one horizontal side $e_1 e_2 \ldots e_m$ and one vertical side $f_1 f_2 \ldots f_n$. This ensures that $\sigma_i$ are local homeomorphisms of the Cantor set $X$.

The C*-algebra of the corresponding groupoid $\Gamma$ is strongly Morita equivalent to a crossed product of an AF-algebra by the group $\mathbb{Z}^2$. Under some mild conditions, the groupoid is essentially free, and the C*-algebra is simple and purely infinite (see section 4 in [KP]).

**Example 2.** Let $X$ be the unit circle $\mathbb{T}$, and $\sigma_1, \sigma_2$ two covering maps, of indices $p_1$ and $p_2$, respectively, with $|p_i| \geq 2$. In this case, each $C^*(R_n)$ is of the form $C(\mathbb{T}) \otimes M_k$ for some positive integer $k = k(n)$, $C^*(R)$ is a circle algebra, and $C^*(\Gamma)$ is simple and purely infinite, since the orbits are dense, and the groupoid is essentially free and locally contracting (see [A-D]).

**Example 3** (Ledrappier). Let $X \subset \{0, 1\}^{\mathbb{N}^2}$ be the subgroup defined by

$$x_{i+1,j} + x_{i,j} + x_{i,j+1} = 0 \mod 2.$$

The admissible patterns are

$$P = \left\{ \begin{array}{ccc} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right\}.$$

The maps $\sigma_1$ and $\sigma_2$ are the horizontal and the vertical shifts, which are local homeomorphisms in this case. Indeed, let $x \in X$ be fixed, and consider the open neighborhood of $x$, defined by

$$U = \{ y \in X \mid y(0,0) = x(0,0) \}.$$

Then $\sigma_1(U) = X$, and if $y, z \in U$ with $\sigma_1(y) = \sigma_1(z)$, then $y(i,j) = z(i,j)$ for all $i \geq 1$, $j \geq 0$. But also $y(0,0) = z(0,0) = x(0,0)$, and since the only admissible patterns are the ones in $P$, we get $y(0,1) = z(0,1)$. By induction, we get $y(0,j) = z(0,j)$ for all $j \geq 0$, therefore $y = z$ and $\sigma_1 |_U$ is one to one. Similarly, $\sigma_2$ is a local homeomorphism.

The C*-algebra $C^*(R)$ is isomorphic to the CAR algebra $UHF(2^\infty)$, and $C^*(\Gamma)$ is again strongly Morita equivalent to a crossed product of an AF-algebra by $\mathbb{Z}^2$, like in the case of a rank 2 graph.

More general, we may consider $V$ to be a finite alphabet and a closed subset $X \subset V^{\mathbb{N}^2}$ (in the product topology) that is $\sigma$-invariant, where

$$\sigma^l(x)(k) = x(k+l), \ x \in X, \ k, l \in \mathbb{N}^2.$$

Such a dynamical system $(X, \sigma)$ is a (two-dimensional) *Markov shift* or a *subshift of finite type* if there exists a finite set $F \subset \mathbb{N}^2$ and a set of *admissible patterns* $P \subset V^F$ such that

$$X = X(F, P) = \{ x \in V^{\mathbb{N}^2} \mid \pi_F(\sigma^m x) \in P \text{ for every } m \in \mathbb{N}^2 \}.$$
where $\pi_F$ is the projection onto $V^F$. If the shift maps are local homeomorphisms, then the C*-algebra $C^*(\Gamma)$ will be strongly Morita equivalent to a crossed product of an AF-algebra by $\mathbb{Z}^2$. But, in general, the shift maps are not local homeomorphisms, as it can be seen in the case of the full shift $X = \{0, 1\}^\mathbb{N}$. The equivalence classes in the equivalence relations $R_n$ are not discrete anymore, in fact they are Cantor sets. For other interesting examples of higher dimensional subshifts of finite type, see S.

§3. The Hilbert bimodule approach

For a C*-algebra $A$ and an injective unital endomorphism $\alpha \in \text{End}(A)$ such that there is a conditional expectation $P$ onto the range $\alpha(A)$, one can define a Hilbert bimodule $E = A(\alpha, P)$, using the transfer operator $L = \alpha^{-1} \circ P$ (see [E1], [E2]). We start with the vector space $A$, and define the inner product, right and left multiplications by the equations

$$<\xi, \eta > = L(\xi^* \eta), \quad \xi \cdot a = \xi \alpha(a), \quad a \cdot \xi = a \xi.$$

Then we have

$$<\xi, \eta \cdot a > = L(\xi^* \eta \alpha(a)) = \alpha^{-1}(P(\xi^* \eta \alpha(a))) = \alpha^{-1}(P(\xi^* \eta)\alpha(a)) = \alpha^{-1}(P(\xi^* \eta)a) = <\xi, \eta > a.$$

Using the Pimsner construction (see [P]), we get a Cuntz-Pimsner algebra $O_{\alpha, P}$. Recall

**Definition.** A Toeplitz representation of a Hilbert $A$-bimodule $E$ in a C*-algebra $B$ is a pair $(\psi, \pi)$ with $\psi : E \to B$ a linear map and $\pi : A \to B$ a homomorphism, such that

$$\psi(\xi \cdot a) = \psi(\xi) \pi(a)$$

$$\psi(\xi)^* \psi(\eta) = \pi(<\xi, \eta >)$$

$$\psi(a \cdot \xi) = \pi(a) \psi(\xi).$$

The corresponding universal C*-algebra is called the Toeplitz algebra of $E$, denoted by $T_E$. There is a homomorphism $\pi^{(1)} : K(E) \to B$ which satisfies

$$\pi^{(1)}(\Theta_{\xi, \eta}) = \psi(\xi) \psi(\eta)^*.$$

We say that $(\psi, \pi)$ is Cuntz-Pimsner covariant if

$$\pi^{(1)}(\phi(a)) = \pi(a) \quad \forall \ a \in \phi^{-1}(K(E)),$$

where $\phi : A \to L(E)$ is the left action. The Cuntz-Pimsner algebra $O_E$ is universal for Toeplitz representations which are Cuntz-Pimsner covariant. There is a gauge action of the circle group $\mathbb{T}$ on $O_E$, and the fixed point algebra is denoted by $F_E$. 

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Example 1. For $A = C(X)$ and $\alpha$ induced by a local homeomorphism $\sigma : X \to X$, we can take

$$(Pf)(x) = \frac{1}{\nu(x)} \sum_{\sigma(y) = \sigma(x)} f(y),$$

where $\nu(x)$ is the number of elements in the fiber $\sigma^{-1}(x)$. The corresponding algebra $O_{A(\alpha, p)}$ is isomorphic to $C^*(\Gamma(\sigma))$, where $\Gamma(\sigma)$ is the Renault groupoid as in [D],

$$\Gamma(\sigma) = \{(x, p - q, y) \in X \times \mathbb{Z} \times X \mid \sigma^p x = \sigma^q y\}.$$

Indeed, it is known that $\mathcal{K}(E) = E \otimes_A E^*$, where $E^*$ is the adjoint of $E$. That means $E^* = C(X)$ with the conjugate multiplication of scalars, and with the left and right actions of $A$ interchanged. Since $\xi \cdot f \otimes \eta^* = \xi \otimes \phi(f)\eta^*$, it follows that $\xi(x)f(\sigma(x))\eta^*(y) = \xi(x)f(\sigma(y))\eta^*(y)$ for all $(x, y) \in X \times X$ and $f \in C(X)$. Hence $\sigma(x) = \sigma(y)$, and $\mathcal{K}(E) = C(R_1)$ as sets. We have

$$\Theta_{\xi,\eta} \Theta_{\xi',\eta'}(\xi) = \Theta_{\xi,\eta}(\xi' < \eta', \xi) = \xi < \eta, \xi' < \eta', \xi' > = \xi < \eta, \xi' > < \eta', \xi' > = \Theta_{\xi,\eta,\xi',\eta'}(\xi),$$

therefore

$$(\xi \otimes \eta^*)(\xi' \otimes \eta^*)(x, y) = \xi(x) < \eta, \xi' > (\sigma(x))\eta^*(y) = \sum_{\sigma(z) = \sigma(x)} \xi(x)\eta^*(z)\xi'(z)\eta^*(y),$$

and the multiplication of compact operators is exactly the convolution product on $C(R_1)$. Hence, $\mathcal{K}(E) = C^*(R_1)$ as $C^*$-algebras. In the same way, using the fact that $\mathcal{K}(E^\otimes n) = (E^\otimes n) \otimes_A (E^\otimes n)^*$, we get $\mathcal{K}(E^\otimes n) = C^*(R_n)$. Taking inductive limits, it follows that $\mathcal{F}_E = C^*(R)$, since in this case $\mathcal{F}_E$ is generated by all $\mathcal{K}(E^\otimes n), \ n \geq 0$. The isometry $\nu$ which induces the corner endomorphism of $C^*(R)$ is induced by the function $\gamma(x) = 1$, regarded as an element of $E$. Indeed, $< \gamma, \gamma > = 1$, therefore $\psi(\gamma)$ is an isometry in any Cuntz-Pimsner representation. The endomorphism of $\mathcal{F}_E$ is given by the formula

$$(\xi_1 \otimes \ldots \otimes \xi_n) \otimes (\eta_1 \otimes \ldots \otimes \eta_n)^* \mapsto (\gamma \otimes \xi_1 \otimes \ldots \otimes \xi_n) \otimes (\gamma \otimes \eta_1 \otimes \ldots \otimes \eta_n)^*.$$

Remark. The above Cuntz-Pimsner algebra $O_{A(\alpha, p)}$ could also be described as the universal $C^*$-algebra generated by a copy of $C(X)$ and an isometry $S$ subject to the relations

(i) $Sf = \alpha(f)S$,  (ii) $S^*fS = L(f)$,  (iii) $1 = \sum_{i=1}^m u_i SS^* u_i^*$,

for all $f \in C(X)$ where

$$L(f) = \frac{1}{\nu(x)} \sum_{\sigma(y) = x} f(y),$$

and $\{u_1, u_2, \ldots, u_m\} \subset C(X)$ such that $f = \sum_{i=1}^m u_i P(u_i^* f)$ for all $f \in C(X)$ (Theorem 9.2. in [EV]). The corresponding Toeplitz algebra $\mathcal{T}_{A(\alpha, p)}$ satisfies only the first two relations.
Note. If \( \sigma : X \to X \) is continuous, open and surjective, but not a local homeomorphism, the groupoid construction from [D] may fail. The equivalence relation \( R \) does not have an obvious Haar system, since the Haar systems on \( R_n \) may not be compatible. Since \( R_n \) is not necessarily an open subset of \( R_m \) for \( n \leq m \), a continuous function on \( R_n \) has no natural extension to a function on \( R_m \).

Example 2. Let \( X = \prod_{1}^{\infty} [0, 1] \) and \( \sigma(x_1, x_2, ...) = (x_2, x_3, ...) \). Then \( \sigma \) is not a local homeomorphism, in fact the fibers are homeomorphic to the interval \([0, 1]\). Also, \( R_n \subset R_{n+1} \) is not open, therefore we can not conclude that a continuous function on \( R_n \) is naturally extended to a continuous function on \( R_{n+1} \). We try to overcome the lack of an obvious groupoid by using the existence of certain families of measures on the fibers of \( \sigma \) which will define a transfer operator and a Hilbert bimodule over \( C(X) \).

Definition. Suppose that \( \pi : X \to Y \) is a continuous open surjection between locally compact Hausdorff spaces. A family \( \lambda = \{ \lambda_y \}_{y \in Y} \) of positive Radon measures on \( X \) is called a \( \pi \)-system if the support of \( \lambda_y \) is contained in \( \pi^{-1}(y) \) for each \( y \in Y \), and for each \( f \in C_c(X) \), the function

\[
\lambda(f)(y) := \int f(x) d\lambda_y(x)
\]

lies in \( C_c(Y) \). We say that a \( \pi \)-system is full if the support of each \( \lambda_y \) is all of \( \pi^{-1}(y) \). The existence of a full \( \pi \)-system is proved in (B, Théorème 3.3) for \( X \) separable and \( Y \) second countable.

For the above example, one can take the usual Lebesgue measure on \([0, 1]\) for each fiber. We get a conditional expectation, a transfer operator, and a Hilbert bimodule \( E \). The corresponding Cuntz-Pimsner algebra is isomorphic to the Toeplitz algebra \( T_E \) since \( \phi^{-1}(K(E)) = 0 \) in this case (see 3.3 in Sc and K).

It was observed by Raeburn and Sims (RaS) that the \( C^* \)-algebras associated to higher rank graphs could be obtained from a product system of Hilbert bimodules in the sense of Fowler (F). We will do something similar for two commuting endomorphisms of a \( C^* \)-algebra \( A \). Recall

Definition (Fowler). Given \( Q \) a countable semigroup with identity \( e \) and \( p : E \to Q \) a family of Hilbert bimodules over \( A \), we say that \( E \) is a product system over \( Q \) if \( E \) is a semigroup, \( p \) is a morphism, and for each \( s, t \in Q \setminus \{ e \} \), the map \( (\xi, \eta) \in E_s \times E_t \mapsto \xi \eta \in E_{st} \) extends to an isomorphism between \( E_s \otimes_A E_t \) and \( E_{st} \). We require that \( E_e = A \) (with \( \phi_e(a)b = ab \) ), that the multiplications \( E_e \times E_s \to E_s \) and \( E_s \times E_e \to E_s \) satisfy \( a\xi = \phi_s(a)\xi \), \( \xi a = \xi \cdot a \), and that each \( E_s \) is essential as left \( A \)-modules, in the sense that \( E_s \) is the closed span of the elements \( \phi_s(a)\xi \) with \( a \in A \), \( \xi \in E_s \), and \( \phi_s : A \to L(E_s) \) defining the left multiplication.

Definition. A Toeplitz representation of the product system \( E \) in a \( C^* \)-algebra \( B \) is a map \( \psi : E \to B \) such that

1. For each \( s \in Q, (\psi_s, \psi_e) \) is a Toeplitz representation of \( E_s \), where \( \psi_s = \psi|_{E_s} \), and
2. \( \psi(\xi \eta) = \psi(\xi) \psi(\eta) \) for \( \xi, \eta \in E \).
If in addition each \((\psi_s, \psi_e)\) is Cuntz-Pimsner covariant, then \(\psi\) is a Cuntz-Pimsner representation.

It was proved by Fowler (Propositions 2.8, 2.9 [F]) that the Toeplitz algebra \(T_E\) and Cuntz-Pimsner algebra \(O_E\) exist and are unique up to isomorphism.

Given \(\alpha_i, i = 1, 2\) two commuting, injective unital endomorphisms of a \(C^*\)-algebra \(A\) such that there exist commuting conditional expectations \(P_i\) onto the ranges \(\alpha_i(A)\), we will construct a product system of Hilbert bimodules over the semigroup \(\mathbb{N}^2\). We take \(E_{(0,0)} = A, E_{(1,0)} = A(\alpha_1, P_1), E_{(0,1)} = A(\alpha_2, P_2), \) and \(E_{(m,n)} = E_{(1,0)} \otimes_A E_{(0,1)}^{\otimes n} \) for all \((m, n) \in \mathbb{N}^2\). The semigroup structure is given by the tensor product.

**Lemma.** We have \(A(\alpha_1, P_1) \otimes_A A(\alpha_2, P_2) \simeq A(\alpha_1 \circ \alpha_2, P_1 \circ P_2)\).

**Proof.** The map \(\Phi : A(\alpha_1, P_1) \otimes_A A(\alpha_2, P_2) \to A(\alpha_1 \circ \alpha_2, P_1 \circ P_2)\), \(\Phi(\xi_1 \otimes \xi_2) = \xi_1 \alpha_1(\xi_2)\) induces the Hilbert bimodules isomorphism. The inverse is \(\Phi^{-1}(\xi) = \xi \otimes 1\).

**Proposition.** If \(\sigma_i : X \to X, i = 1, 2\) are two commuting local homeomorphisms of a compact space \(X\), and \(P_i\) are given by

\[(P_i f)(x) = \frac{1}{\nu_i(x)} \sum_{\sigma_i(y) = \sigma_i(x)} f(y),\]

where \(\nu_i(x)\) is the number of elements in the fiber \(\sigma_i^{-1}(x)\), then \(P_1 \circ P_2 = P_2 \circ P_1\), and the Cuntz-Pimsner algebra \(O_E\) associated to the above product system is isomorphic to the groupoid algebra \(C^*(\Gamma(\sigma_1, \sigma_2))\) considered in §2.

**Remark.** Such a product system over \(\mathbb{N}^2\) could be constructed from two more general Hilbert bimodules \(E_1, E_2\) over a \(C^*\)-algebra \(A\), such that \(E_1 \otimes_A E_2 \simeq E_2 \otimes_A E_1\), after we made the identifications which will ensure the associativity of the multiplication. For example, let \(A = \mathbb{C}\) and let \(E_i = \mathbb{C}^{n_i}\) for \(i = 1, 2\) be the Hilbert spaces with the usual inner products and right and left multiplications. If the isomorphism \(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \to \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_1}\) is given by \(e_i \otimes f_j \mapsto f_j \otimes e_i\), then \(O_E \simeq O_{n_1} \otimes O_{n_2}\), where \(O_n\) is the Cuntz algebra, and \(\{e_1, e_2, \ldots, e_{n_1}\}, \{f_1, f_2, \ldots, f_{n_2}\}\) are the canonical bases in \(\mathbb{C}^{n_i}, i = 1, 2\).

**Example 3.** Let \(A\) be a \(C^*\)-algebra, and \(\alpha_1, \alpha_2\) two commuting automorphisms. The Cuntz-Pimsner algebra of the corresponding product system is isomorphic to the crossed product \(A \rtimes_{\alpha_1, \alpha_2} \mathbb{Z}^2\). This \(C^*\)-algebra could be obtained also from an iterated Pimsner construction. If we take \(E_i = A(\alpha_i, id)\), then \(O_{E_i} \simeq A \rtimes_{\alpha_1} \mathbb{Z}\). By extending the scalars, \(E_2 \otimes O_{E_1}\) becomes a Hilbert bimodule over \(O_{E_1}\), and \(O_{E_2} \otimes O_{E_1} \simeq A \rtimes_{\alpha_1, \alpha_2} \mathbb{Z}^2\).

**Example 4.** For the full shift \(X = \{0, 1\}^{\mathbb{N}^2}\), let \(\sigma_i, i = 1, 2\), be the horizontal and vertical shifts. In this case, the equivalence classes in \(R_n\) are homeomorphic to the Cantor set. The existence of \(\sigma_i\)-systems \(\mu^i\) for
i = 1, 2 will allow us to construct a product system over \( \mathbb{N}^2 \), and its Cuntz-Pimsner algebra plays the role of \( C^*(\Gamma) \) defined in §2. More precisely, take \( \alpha_i \) the endomorphisms of \( A = C(X) \) induced by \( \sigma_i \), and \( P_i \) the conditional expectations

\[
(P_i f)(x) = \int_{\sigma_i(y) = \sigma_i(x)} f(y) d\mu^i_x(y), \quad i = 1, 2,
\]

which commute. The Hilbert bimodules \( A(\alpha_i, P_i) \) are isomorphic to \( C(X) \) as vector spaces, with inner products

\[
< \xi, \eta >_i (x) = \int_{\sigma_i(y) = x} \overline{\xi(y)} \eta(y) d\mu^i_x(y),
\]

the left actions \( (f \cdot \xi)(x) = f(x) \xi(x) \), and the right actions given by

\[
(\xi \cdot f)(x) = \xi(x) f(\sigma_i(x)), \quad i = 1, 2, f \in C(X).
\]

The fibers \( E(m, n) \) are defined as above, by taking tensor products, and using the isomorphisms given by the Lemma. Since none of the measures is supported on a finite set, the resulting C*-algebra \( O_E \) is isomorphic to the Toeplitz algebra \( T_E \).

**References**

[A-D] C. Anantharaman-Delaroche, Purely infinite C*-algebras arising from dynamical systems, Bull. Soc. Math. France 125 (1997), no. 2, pp. 199–225.

[Bl] E. Blanchard, Déformations de C*-algèbres de Hopf, Bull. Soc. Math. France 124 (1996) pp. 141–215.

[Br] B. Brenken, C*-algebras associated with topological relations, Preprint.

[D] V. Deaconu, Groupoids associated with endomorphisms, Trans. Amer. Math. Soc. 347 (1995), no. 5, 1779–1786.

[E1] R. Exel, Crossed-products by finite index endomorphisms and KMS states, J. Funct. Anal. 199 (2003), no. 1, 153–188.

[E2] R. Exel, A new look at the crossed-product of a C*-algebra by an endomorphism, to appear in Ergodic Theory Dynam. Systems.

[EV] R. Exel, A. Vershik, C*-algebras of irreversible dynamical systems, Preprint, arXiv:math.OA/0203185 v1.

[F] N. Fowler, Discrete product systems of Hilbert bimodules, Pacific J. Math. 204 (2002), no. 2, pp. 335–375.
[Ka] T. Katsura, On C*-algebras associated with C*-correspondences, Preprint, arXiv:math.OA/0309088 v2.

[K] A. Kumjian, On certain Cuntz-Pimsner algebras, to appear in Pacific J. Math.

[KP] A. Kumjian and D. Pask, Higher rank graph C*-algebras, New York J. of Math. 6(2000) pp. 1–20.

[KS] M. Khoshkam and G. Skandalis, Toeplitz algebras associated with endomorphisms and Pimsner-Voiculescu exact sequences, Pacific J. Math. vol 181(1997) no. 2, pp. 315–331.

[LR] M. Laca, I. Raeburn, Semigroup crossed products and the Toeplitz algebras of nonabelian groups, J. of Funct. Anal. 139(1996) pp. 415–440.

[LiR] J. Lindiarni and I. Raeburn, Partial-isometric crossed products by semigroup of endomorphisms, Preprint, arXiv:math.OA/0210364 v1.

[M1] G.J. Murphy, Ordered groups and crossed products of C*-algebras, Pacific J. Math. 148 (1991) no. 2, pp. 319–349.

[M2] G.J. Murphy, Crossed products of C*-algebras by endomorphisms, Integral Equations Operator Theory 24 (1996) no. 3, pp. 298–319.

[Pa] W.L. Paschke, The crossed product of a C*-algebra by an endomorphism, Proc.Amer.Math.Soc. 80(1980) pp. 113–118.

[P] M. Pimsner, A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z, Free probability theory, Amer. Math. Soc., Providence, RI, 1997, pp. 189–212.

[RaS] I. Raeburn and A. Sims, Product systems of graphs and the Toeplitz algebras of higher-rank graphs, Preprint, arXiv:math.OA/0305371 v1.

[Re] J. Renault, A Groupoid Approach to C*-algebras, Lecture Notes in Mathematics no. 793, Springer-Verlag 1980.

[S] K. Schmidt, Algebraic ideas in ergodic theory, CBMS Regional Conference Series in Mathematics 76, Providence, RI, 1990.

[Sc] J. Schweizer, Crossed products by C*-correspondences and Cuntz-Pimsner algebras, C*-algebras (Münster, 1999), Springer, Berlin, 2000, pp. 203–226.