Million-Body Star Cluster Simulations: Comparisons between Monte Carlo and Direct N-body

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ABSTRACT

We present the first detailed comparison between million-body globular cluster simulations computed with a Hénon-type Monte Carlo code, CMC, and a direct N-body code, NBODY6++GPU. Both simulations start from an identical cluster model with 10^6 particles, and include all of the relevant physics needed to treat the system in a highly realistic way. With the two codes “frozen” (no fine-tuning of any free parameters or internal algorithms of the codes) we find excellent agreement in the overall evolution of the two models. Furthermore, we find that in both models, large numbers of stellar-mass black holes (> 1000) are retained for 12 Gyr. Thus, the very accurate direct N-body approach confirms recent predictions that black holes can be retained in present-day, old globular clusters. We find only minor disagreements between the two models and attribute these to the small-N dynamics driving the evolution of the cluster core for which the Monte Carlo assumptions are less ideal. Based on the overwhelming general agreement between the two models computed using these vastly different techniques, we conclude that our Monte Carlo approach, which is more approximate, but dramatically faster compared to the direct N-body, is capable of producing a very accurate description of the long-term evolution of massive globular clusters even when the clusters contain large populations of stellar-mass black holes.

Key words: binaries: close — globular clusters: general — Gravitational waves — Methods: numerical — Stars: black holes — Stars: kinematics and dynamics

1 INTRODUCTION

In the last few years, our understanding of the dynamical evolution of globular clusters (GC), especially, as a result of the detailed dynamical evolution and fate of the large numbers of black holes (BHs) that are bound to form in these large-N clusters, has shifted significantly. While it was once thought that present-day old GCs should have at most a couple of BHs remaining, more recent studies have shown that if BH-formation kicks are not sufficiently high to eject all BHs from the GCs, a significant fraction of the formed BHs are retained up to the typical old ages (∼12 Gyr) of the GCs (Breen & Heggie 2013; Heggie & Giersz 2014; Mackey et al. 2008; Sippel & Hurley 2013; Morscher et al. 2015). This new perspective has been driven by major advances in parallel computing, which enables the modeling of GC systems with realistic number of stars and all the relevant physics.

Since the early theoretical work predicting the rapid ejection of BHs from clusters, several groups have performed ever more realistic evolutionary simulations. While several of the first attempts confirmed the theoretical prediction for complete evaporation, the most recent models have predicted that at least some BHs, and likely many, may remain in old GCs today. These results are coming at an exciting time when stellar BHs are being discovered at a rapid pace in GCs in the Milky Way and in other Galaxies through X-ray and radio surveys (Maccarone et al. 2007; Barnard et al. 2011; Maccarone et al. 2011; Shih et al. 2010; Strader et al. 2012; Chomiuk et al. 2013).

We are also beginning to understand why these new results are at odds with the early theoretical prediction of rapid BH evaporation. The original argument assumed that the BHs would be-
come Spitzer unstable due to their rapid mass-segregation, dynamically decoupling from the cluster. Once decoupled, the BHs would evolve as an isolated cluster, with an evaporation timescale of $\sim 1$ Gyr. While there were a few theoretical hints from simulations that at least some BHs could be retained for 12 Gyr, there was no good explanation for how this could be possible, and where the old prediction breaks down. Using two-component direct $N$-body models, Breen & Heggie (2013) showed that after the BHs began to segregate, their rate of energy generation was controlled by the rate of energy flow through the cluster as a whole, and so it was set by the relaxation timescale of the entire cluster, not simply by that of the small subpopulation of BHs. Recently, Morscher et al. (2015) (hereafter MOR15) presented a grid of 42 detailed Monte Carlo models for realistic star clusters containing substantial initial populations of stellar BHs. This study was the first to present a large number of realistic simulations with a range of initial cluster masses (from $\sim 10^5 - 10^6 M_\odot$) as well as variation in other important parameters, such as the virial radius, that also includes all of the relevant physics required to describe these systems accurately. Starting from standard assumptions for the initial conditions of MW-like clusters and BH formation processes, MOR15 found that nearly all of the models did indeed retain significant fractions of their BHs up to the end of the simulation at 12 Gyr. The retained BHs heated the full cluster, leading to large final core sizes; however, the most compact initial cluster models were able to eject the majority of their BHs by 12 Gyr, causing their cores to contract (the so-called second core collapse) to sizes similar to those observed in MWGCs. Work is in progress to study the effect of varying the initial mass function and the BH birth kick distribution, on the initial BH populations in clusters, and ultimately their impact on cluster dynamics and long-term BH retention.

Perhaps most interestingly, MOR15 has revealed a new theoretical understanding of the complex dynamical interplay between BHs and clusters, and how it may be possible for clusters to retain large numbers of BHs for many Gyr. These MC models showed that, in realistic systems, the Spitzer instability does not involve all of the BHs in the cluster. Rather, the BHs power core oscillations during which only a small subset of the BHs segregate from the cluster to form a deep cusp, which then promptly re-expands upon formation of three-body binaries. Thus, as a result of the energy generated by the dynamics of the BHs, the BH interaction rate, and therefore also the evaporation rate, is kept much lower than previously thought, making it possible for star clusters to retain significant numbers of BHs for 12 Gyr. It seems that we should not expect the BHs to succomb to the Spitzer Instability; most of the BHs, in fact, always remain spread throughout the cluster, far from the central cusp. This new understanding of BH dynamics in star clusters has provided a theoretical basis for explaining the recent discoveries of several BH candidates in old GCs.

As always, we would like to be able to compare our results to simulations done with other codes, especially those that use different modeling techniques. The direct $N$-body technique is the most accurate and assumption-free method for modeling the dynamics of star clusters. It resolves physics on the dynamical timescale, which means it can in principle be used to model clusters of any size, because it is valid even for clusters with relaxation timescales not much longer than their dynamical timescales. This is in contrast to the MC technique, whose assumptions rely on the relaxation timescale being long compared to the dynamical timescale. For large enough $N$, however, the MC technique used in our study is an extremely powerful tool that is capable of computing many large-$N$ dynamical cluster models in a short amount of time, which means it can be used to explore the parameter space of initial conditions and test the robustness of our results. This feature sets it apart from direct $N$-body simulations, which are significantly more computationally expensive, and therefore are usually restricted to $N \lesssim 10^5$. The speed of the MC technique comes at a price, however, in that two-body relaxation, as well as other dynamical processes, are treated in an approximate way via a single interaction per pair of stars per time step, where the timestep is chosen to be a small fraction of the relaxation timescale. Physics occurring on much shorter timescales, such as the dynamical timescale, cannot be resolved accurately with the MC method, but is handled naturally with the direct $N$-body technique.

Our MC code has been compared extensively to direct $N$-body simulations whenever possible, and has shown excellent agreement (Joshi et al. 2000, 2001; Fregeau et al. 2003; Fregeau & Rasio 2007; Chatterjee et al. 2010; Umbreit et al. 2012). However, the types of clusters we are now capable of simulating are quite different than those we have simulated (and thus tested) in the past. In particular, the inclusion of large populations of BHs with a broad spectrum of masses has a significant effect on cluster evolution, producing clusters at 12 Gyr with very different properties than models without such objects (cf. Chatterjee et al. 2010, in which the BH mass spectrum was truncated at just above $2 M_\odot$). For example, the deep BH-driven core oscillations that occur ubiquitously in our new CMC models are a new phenomenon that we have not seen before in previous models. During these collapses, the physics may be governed by a very small number of objects ($\sim 10$) that are partially decoupled from the rest of the cluster. If the relaxation timescale of the subset of stars is much smaller than the relaxation timescale of the cluster as a whole, then its evolution is perhaps not being computed accurately by our approximate MC scheme. Furthermore, it is possible that our crude prescription for forming three-body binaries is not capable of properly capturing the physics of this dynamically important process.

These limitations of our MC calculations could potentially have an impact on our results regarding the dynamical evolution of BHs in dense clusters, including long-term BH retention and the resulting heating that produces puffy cluster cores. For these reasons, it would be desirable to re-test the results produced by the latest version of our MC code with the best currently-available direct $N$-body simulations. For our specific interests, there have previously been no suitable large-$N$ direct $N$-body simulations available to which we could compare our large-$N$ MC simulations. Recently, however, Wang et al. 2015 developed a new direct $N$-body code that employs hybrid parallelization methods to speed up the NBODY6++ code (Spurzem 1999; Spurzem et al. 2008), which is considered to be the gold standard in direct $N$-body modeling. This new code, NBODY6++GPU, is capable of modeling clusters with $10^5$ stars within a year (Wang et al. 2015; Wang et al. 2016). This provides an excellent opportunity for us to test our predictions regarding the evolution of clusters with BHs through a direct comparison.

Here we present a comparison between models generated with our MC code (the $\mathcal{MC}$ model) and with NBODY6++GPU (the $\mathcal{BB}$ model). We present this as an honest comparison using what are believed to be the best versions of the two codes, and we do not do any fine-tuning of free parameters. This rest of this paper is organized as follows. In Section 2 we describe the two codes in more detail and give the initial conditions for the model that is the focus of this comparison. In Section 3 we present and compare the results of the two simulations, including the global structural evolution as well as the evolution of the BH populations. We discuss some of
the differences between the models and uncertainties and compare to other studies in Section 4. Finally in Section 5 we summarize our results and state our conclusions.

2 NUMERICAL SETUP

We compare the results of a small set of MC models to a single direct N-body simulation, all starting with identical initial conditions. The comparison model is a million-particle direct N-body simulation that has been computed with NBODY6++GPU (Wang et al. 2013; Wang et al. 2016), a newly developed optimized version of NBODY6++ with improved hybrid parallelization methods (MPI, GPU, OpenMP and AVX/SSE).

This new code combines the MPI parallelized NBODY6++ (Spurzem 1999; Hensendorf, Khalisi, Omarov & Spurzem 2003) with the GPU and AVX/SSE libraries of Nitadori & Aarseth 2012 to significantly accelerate the direct N-body integration. They have also made improvements to speed up the time-step scheduling and stellar evolution, which had become bottlenecks after the hybrid parallelization scheme was implemented. With these modifications, for cluster models consisting of single objects only, distributed over 320 CPU cores (across 16 nodes), with 86016 GPU cores (on 32 GPUs), they achieve a speed-up factor of 400-2000, depending on the number of stars. NBODY6++GPU is the first direct N-body code capable of simulating the evolution of a million-body collisional star cluster over many Gyr. For a detailed description of this hybrid code see Wang et al. (2015).

Wang et al. have shared the results for the million-body simulation presented in Wang et al. (2016). Provided with the exact initial cluster model used in their N-body simulation, we have performed a simulation with our own MC code, CMC, which has been described in great detail in several earlier papers (Joshi et al. 2000, 2001; Fregeau et al. 2003; Fregeau & Rasio 2007; Chatterjee et al. 2010; Umbreit et al. 2012). Briefly, we use a variation of the so-called “orbit-averaged Monte Carlo method” developed by Hénon (1971) for solving the Fokker-Planck equation. With our recently parallelized MC code (Pattabiraman et al. 2013), we can calculate million-body simulations such as the one presented here in less than 2 days1 when distributed over just 48 CPU cores.

While these two codes employ very different techniques for modeling stellar dynamics, they include nearly all of the same physical processes, including two-body relaxation (treated in an approximate way by the MC code), direct physical collisions, higher-order strong binary encounters (integrated directly in both codes), and single and interacting binary stellar evolution (using SSE and BSE, Hurley et al. 2000, 2002). These stellar evolution prescriptions have been modified from the original publicly available versions. In the original SSE/BSE codes, only very low-mass BHs were formed. This was pointed out by Belczynski et al. (2002), who also provided a new prescription for forming more realistic BH masses including fallback based on Fryer & Kalogera (2001).

In both CMC and NBODY6++GPU, the remnant masses are selected following the metallicity-dependent prescription of Belczynski et al. (2002) which forms BHs in the range of about 3–30 M\(_{\odot}\) for typical GC metallicities. Also, the magnitudes of the natal kicks received by stellar remnants that form via a supernova should depend on the amount of fallback material, and both codes have also updated their birth kick prescriptions to depend on fallback according to Belczynski et al. (2002). The kicks are drawn from a Maxwellian velocity distribution with \(\sigma = 265\ \text{km}\ \text{s}^{-1}\), then reduced proportionally to the mass of the material that falls back onto the newly-formed compact object.

While three-body binary formation occurs naturally in direct N-body calculations, the MC technique relies on pair-wise interactions between two neighboring particles, precluding the dynamical formation of a binary from encounters involving three particles. Therefore we opt to use a simple analytic prescription that relies on an estimate of the local probability of binary formation, as described in detail in MOR15. In the MC code, we use the exact physical assumptions and parameters as presented in MOR15 (e.g., those associated with three-body binary formation; see Section 2.2, Equations 1 and 2 from MOR15). One difference is that the MC code is not able to treat stable triple systems that form through four-body (binary-binary) encounters. These systems are thus broken upon formation. However, since they form relatively rarely, we would not expect these to have a significant impact on the evolution of the cluster.

The initial conditions used in the model of Wang et al. (2016) were selected to be similar to large GCs in the Milky Way. They use a King model with concentration \(W_0 = 6, N = 10^6\), half-mass radius \(R_h = 7.5\ \text{pc}\), binary fraction of 5\%, Galactocentric distance \(R_G = 7.1\ \text{kpc}\), and metallicity Z = 0.00016. Stellar masses were chosen in the range 0.1 – 100 M\(_{\odot}\) according to the initial mass function (IMF) of Kroupa (2001), which is a broken power-law of the form \(dN/dm \propto m^{-\alpha}\), with \(\alpha = 1.3\) for 0.08 \(\leq m/M_{\odot} \leq 0.5\), and \(\alpha = 2.3\) for \(m/M_{\odot} \geq 0.5\). Binaries are created by choosing an existing single star randomly to become a binary, and drawing a companion mass from the distribution 0.6(m\(_1)/m_j\)^{−0.4} (with \(m_1 \leq m_j\)) (Kouwenhoven et al. 2007). The semi-major axis is then drawn from a distribution that is uniform in \(\log a\) in the range 0.005-50 AU, and eccentricity is chosen from the thermal distribution. The main difference between the initial model used here and the types of models presented in MOR15 is that the model presented here is much more extended, with a half-mass radius of 7.56 pc, compared to the typical half-mass radii of around 2 pc for the models from our main study.

We have converted the initial conditions used by Wang et al. into a format that can be used in CMC, which means our model is an identical star-by-star representation of the N-body model. Of course, to retain spherical symmetry we needed to convert the star coordinates \(x, y, z\) to a radial coordinate \(r\) and the velocity vectors \(v_r, v_t, v_z\) to their corresponding radial and transverse vectors, \(v_r\) and \(v_t\). All other initial properties, including stellar masses, positions, and binary properties, are identical.

3 COMPARISON TO DIRECT N-BODY

In what follows, we make direct comparisons between the two simulations up to a time of 12 Gyr. In Section 3.1, we explore the evolution and dynamics of the BHs in both simulations, while in Section 3.2 we highlight the agreement (and disagreement) between the structural properties of both cluster models.

3.1 Black Hole Retention and Ejection

We start with our most exciting result: the direct N-body simulation confirms the findings of MOR15 that large numbers of BHs

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1 This is true for models with relatively large initial half-mass radii, as is the case here. However, MC simulations starting from compact models with half mass radii of about 1 pc can take more than a month.
Figure 1. The BH mass (top) and radial distributions (bottom) for BHs in both cluster models at 100 Myr (left) and 12 Gyr (right). On the top, the BH masses for model MC are shown by the solid black line, and those for model NB are indicated by the dashed red line. Overall the number of retained BHs and the mass distributions agree very well. There are a small number of BHs with masses between 30 $M_\odot$ and 55 $M_\odot$, highlighted in the insert, which are only formed in the NB model. The bottom panels show the radial distributions for the BHs (solid curves) and the non-BHs (dotted curves) separately, with model MC in black and model NB in red. Each star is counted individually, regardless of whether it is a part of a binary (or an even higher-order system, which is possible in the direct N-body simulation).

Figure 2. Comparison of the total number of BHs retained in each cluster model as a function of time. Model MC is shown in black and model NB in red. As already seen in Figure 3, both models forms and retains initially (at 20 Myr, after BH formation) a roughly identical number of BHs. Over time, BHs are slowly ejected in both models at a similar rate, with model MC ejecting 336 BHs and model NB ejecting 400 BHs by 12 Gyr.

On the bottom panels in the same figure we show the cumulative radial distributions of the BHs (solid curves) and the non-BHs (dotted curves) after BH formation and at the end of the simulation (solid curves). The two models show excellent agreement in the radial distributions of both star types, with only slight disagreement growing in the outskirts of the cluster models for the non-BHs. The crude tidal treatment in the MC code is not expected to reproduce the more accurate three-dimensional treatment in the direct N-body code. It is clear that the BHs, while more centrally concentrated can be retained for many Gyr in old GCs. At the end of the simulation, there are 1085 BHs remaining in model MC and 1036 in model NB. Figure 1 shows the initial and final distributions of BHs masses for the two models, which are in nearly perfect agreement. Dynamically, the two models produce extremely similar results, with model MC ejecting 336 BHs and model NB ejecting 400 BHs from the population that is retained initially, and, in agreement with MOR15, ejecting the most massive BHs first. We do note that model NB contains a handful of BHs with masses above 35 $M_\odot$, whereas model MC has none. These abnormally massive BHs are formed as a result of a minor bug in the N-body treatment of merged stellar binaries, which was discovered after the simulation had begun.
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Figure 3. The properties of the binary BHs ejected from each simulation. The top plot shows the number of ejected binaries over time for the MC model (solid black) and the NB model (dashed red). The middle plot shows the component masses of the binaries (MC in black circles, NB in red crosses), while the bottom plot shows the eccentricity and semi-major axis distributions of the binaries (with same color scheme).

Figure 4. Top: Number of bound stars remaining in the cluster as a function of time for model MC (solid black) and model NB (dashed red). Each star is counted individually, regardless of whether it is a part of a binary (or an even higher-order system, which is possible in the direct N-body simulation). Bottom: binary fraction as a function of time, with the same color scheme as above.

than the non-BHs, are still quite spread out over the cluster. About half of the BHs lie outside of the inner 4 pc, and about 10% lie beyond 10 pc.

In Figure 2 we show the total number of BHs retained in each model as a function of time. The agreement is excellent throughout the formation of nearly all of the BHs, with the MC model forming and retaining an initial population of 1421 BHs and the NB model forming 1436 BHs. By about 20 Myr, after all BHs have formed and the ones with large natal kicks have been ejected from the cluster, the number of BHs in each model levels out to the initially retained numbers given above. Over time the number of BHs decreases slowly in each model at a similar rate, with the NB model ejecting 64 more BHs than the MC model by 12 Gyr. Part of this increase can be attributed to the more-massive BHs (35-55 $M_\odot$) found in the N-body simulation which will be ejected faster than the $\sim 25M_\odot$ BHs. However, the 18 erroneously large BHs cannot account for the full discrepancy in the number of ejected BHs between the two models.
Finally, we compare the properties of the ejected BH binaries. In Figure 3, we show the number of BH binaries ejected by each model over time. As the NB model is ejecting BHs more quickly than the MC model, the rate of binary BH ejection is also larger in the N-body model (54 versus 41 binaries). Here, about half of these ejected binaries can be attributed to binaries whose components are abnormally large. Even then, this suggests that the Monte Carlo method may underestimate the production rate of binary BHs. We also compare the component masses, eccentricities, and semi-major axes of the ejected binaries (the middle and bottom panels of the same figure). In this instance, we find very good agreement between the NB and MC models. Such good agreement is to be expected, as both codes model binary hardening and partner exchanges (via three and four-body encounters) using a direct N-body integration.

3.2 Structural Cluster Properties

Next we compare the overall structural evolution of the two models. In Figure 4 we show the number of bound stars remaining in the cluster (top) and the binary fraction (bottom) as a function of time, with model MC shown in black (solid line) and model NB in red (dashed line). Here the number of bound stars is determined by counting every star individually (i.e., a binary counts as two stars). By the final time of 12 Gyr, both models have lost roughly 30% of their initial mass (4.5×10^5 stars). The two models agree to within 5% (with models MC and NB having 6.9×10^5 and 7.29×10^5 stars, respectively). The N-body code uses a radial tidal criterion such that any star that goes beyond twice the cluster tidal radius is removed. In contrast, the Monte Carlo code relies on an energy criterion, which has shown to produce better agreement with direct N-body than a radial criterion, but has still been observed to strip stars on a slightly shorter timescale (Chatterjee et al. 2010). The binary fraction in the models shows even better agreement to within about 1%, decreasing rather slowly with time from f_b=0.05 initially down to f_b=0.0465 in MC and f_b=0.0462 in model NB.

In Figure 5 we show the time evolution of the core radius, r_c, and the half-mass radius, r_h, again with model MC in black and model NB in red. Here we use the standard N-body definition of the core radius from Cassertano & Hut (1985), which is a density-weighted estimate of the average of the star positions. This theoretical core radius is of course unrelated to that which an observer would calculate, since it depends on mass rather than luminosity. The half-mass radius is the radius that encloses half of the total cluster mass. We find nearly perfect agreement between the two models in the evolution of r_c for the entire span of the simulations. The agreement in the core radius is also very good, although at early times (within a few hundred Myr) the core radius in model MC expands slightly more than that of model NB. This is likely caused by subtle differences in our stellar evolution routines (e.g., wind mass loss), which have both been modified from the original publicly-available SSE and BSE codes (Hurley et al. 2000, 2002). These differences can be difficult to track down, and therefore we leave this topic for future investigations. In any case, since the core radii in both models have contracted down to a very similar value within a Gyr, and then remain in good agreement up to the end, this slight disagreement early on appears to have very little influence on long-term evolution of the core radius.

In Figure 6 we show a comparison of the Lagrange radii for the two models, with model MC in black and model NB in red. From bottom to top, the pairs of black and red lines correspond to the radii enclosing 1%, 10%, 50%, 90% and 99% of the total mass in the cluster. Overall we find strong agreement between the two models. In the outer parts of the cluster (beyond a few tens of pc), model NB expands slightly more than in model MC early on, and the memory of this seems to last through the end of the simulations. Again, we do not necessarily expect the two codes to produce identical results because of the very different tidal stripping prescriptions employed. More importantly, the 10% and 50% Lagrange radii agree nearly perfectly. The 1% radius in model MC is noisier and displays slightly deeper collapses than that of model NB, although the outer envelope of the two curves matches very well. The behavior of the 1% radius is similar to that of the core radius shown in Figure 5. Since the MC approach is not designed to handle the physics of small numbers of objects, it is actually quite impressive that the behavior on these small scales agrees as well as it does with the direct N-body simulation.

In Figure 7 we again show the Lagrange radii, only this time we separate the BHs from the other types of objects in order to better understand the behavior of the BHs. With model MC in black and model NB in red, we show (from bottom to top) the 1%, 10%, 50% and 90% radii for the BHs, and the same Lagrange radii for the non-BHs in blue for model MC and cyan for model NB. We can see that the innermost 1% of the BH mass in both models participates in oscillations similar to those seen in the core radii in Figure 5 (i.e., deeper oscillations in the MC model, more shallow ones in the N-body), but these oscillations are not seen in the non-BH population. This tells us that these collapses are primarily driven by the BHs. This innermost 1% mass bin for the BHs consists of only ~10 objects, so it is unreasonable to expect the MC code to treat this subsystem perfectly. In all but the innermost 1% BH mass, we find excellent agreement between the two models in the radial distributions of both species (BHs and non-BHs) as a function of time. The outermost 90% Lagrange radii for the BHs falls at about 10 pc for both models at the end of the simulations, and the 90% radius for the non-BHs lies just a bit further out at around 30 pc. Thus the BHs remain quite spread out in the cluster, well-mixed with the other objects, all the way to the end of the simulation. Despite the deeper collapses of the central BH subsystem in model MC, the fact that the BH ejection rate agrees so well provides encouraging evidence that BH dynamics and retention is not necessarily governed by the depth of the BH-driven core oscillations. The only noticeable change in the BH distributions between 100 Myr and 12 Gyr (Figure 1) is at the high mass end, with the heavy BHs being among the first to be ejected. The two models agree very well in this regard. Thus, not only do the numbers of ejected BHs agree, but so too do the masses of the ejected BHs.

4 DISCUSSION

We have found that our MC approach and the new code NBODY6++GPU produce similar evolution for a million-star cluster over 12 Gyr, both in terms of global structural properties and the dynamical evolution of the BH populations. Despite the significant agreement in the dynamics, we are still interested in exploring why the BHs in model MC experience deeper collapses than in model NB, as this may help us better understand the nature of the interactions between BHs and the rest of the cluster. One potential cause for the deeper collapses in the MC model would be the presence of more massive BHs than in the N-body model. MOR15 found that it is always the most massive BHs that are found near the center and driving the deep core oscillations. As a result, they are also the first to be ejected. However, we actually find the opposite to be true: for the first Gyr, the N-body model contains about ten BHs in the mass
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Figure 5. Comparison of the time evolution of the core radius ($r_c$) and half-mass radius ($r_h$) between MC (solid black line) and NB (dashed red line) up to the current time of the N-body simulation. The two models show good agreement in the evolution of the half-mass radius (to within $\sim 1$ pc), and fairly good agreement in the core radius evolution. We note that the core-collapses in model MC go deeper than those seen in NB (see Figure 7 for a more detailed look at what is happening in the central region).

Figure 6. Comparison of the Lagrange radii for models MC (black) and NB (red). The pairs of black and red curves show the radii enclosing 1%, 10%, 50%, 90% and 99% (bottom to top) of the total cluster mass for the two respective models over the entire simulation.

range $35 - 55 M_\odot$, whereas the MC model contains none in this range. By 4 Gyr, these heavy BHs have been ejected from model NB. In any case, it is clear that the deep collapses in the MC model cannot be explained by the presence of more massive BHs. Furthermore, we saw in Figure 1 that, besides the handful of BHs with masses between $35 - 55 M_\odot$, the overall distribution of BH masses retained in the two models agrees fairly well throughout the entire simulation.

Another possible explanation for the difference in the collapse depths could be differences in the population of BH binaries that arise from the simplified three-body binary formation prescription used in CMC. In our standard prescription, we allow only BHs to form binaries through three-body encounters, a choice motivated
Figure 7. Lagrange radii for the BHs and the non-BHs separately. The black (MC) and red (NB) curves show the radii enclosing 1%, 10%, 50% and 90% of the BH mass. Similarly, the blue (MC) and cyan (NB) curves show the radii enclosing the same fractions of the non-BH mass. Here we can see that it is primarily the BHs that are driving the core oscillations shown in Figure 5. The 10%, 50% and 90% BH Lagrange radii for both models agree extremely well, although in model MC the 10% radius is a bit noisier. The 1% BH Lagrange radius in model MC collapses significantly deeper than in model NB, however this bin typically contains only about 10 BHs. The BHs remain very well mixed with other stars (even at 12 Gyr, 90% of the BH mass is spread throughout the $\sim 10^5$ stars comprising the inner 10% of non-BH mass).

by previous studies suggesting that for non-compact stars, the high densities required to form three-body binaries would instead lead to physical collisions of these stars, rather than forming binaries (Chernoff & Huang 1996), and is thus never important. This means that in model MC we are suppressing the creation of BH-non-BH binaries by three-body interactions, and our population of BH-non-BH binaries is limited to the number of BHs that can exchange into existing binaries through binary interactions. The smaller number of BH binaries (of any type) in the MC model may be responsible for the deep core collapses which would be overturned much earlier in the collapse if we had a larger number of BH binaries. Alternatively, the collapse depths could be sensitive to the hardness of the three-body binaries that are formed, or the actual rate of binary formation, both of which are determined in part by the parameter $\eta_{\text{min}}$, the minimum hardness of the binaries that are allowed to form in the simple prescription (see MOR15 for the full details of the procedure). This parameter affects binary formation in two different ways: first, $\eta_{\text{min}}$ enters the rate equation, so the rate of binary formation in a given timestep with this minimum hardness $\Gamma(\eta > \eta_{\text{min}}) \sim \eta^{-3.5}_{\text{min}}$, meaning that it is easier to form softer binaries (smaller $\eta$). Secondly, if our procedure determines that a binary should form, then we choose the actual hardness for the binary from a distribution according to the differential rate, $d\Gamma/d\eta$, with lower limit $\eta_{\text{min}}$. This function falls off rapidly for increasing values of $\eta$, so a lower limit will tend to result in significantly softer binaries being formed. In model MC, as well as in the models presented in MOR15, we have strictly used a hardness cutoff $\eta_{\text{min}} = 5$.

For full details of the three-body binary formation prescription employed in CMC, see MOR15.

To test the effect of modifications to our binary formation procedure we performed two additional simulations identical to MC, except for a slight modification to our three-body binary treatment. We have performed one simulation in which all stars, including non-BHs, are allowed to participate in three-body binary formation. We find, however, that this change has essentially no effect on the BH-driven collapses nor on the BH evaporation rate. Next we reduced $\eta_{\text{min}}$ to 0.5 (a factor of ten smaller than used in model MC). In this case we saw a slight reduction in the collapse depths, but the number of retained BHs remained unaffected. It seems that when softer binaries are allowed to form, which also increases the overall rate of production, the BH-driven collapses can be overturned slightly sooner, and the very deepest collapses are avoided. While these modifications to the simple treatment of binary formation do not seem to have much of an effect on overall cluster evolution or on the retention of BHs, it may very well affect the details of the BH-binary populations, such as the production of BH-X-ray binaries and tight BH-BH binaries. It would be possible to study these binary populations in greater detail, and perhaps even use the direct N-body model to calibrate the three-body binary procedure used in CMC, but we leave this for a future study.

In contrast to the MC models presented in MOR15, the initial model used in this study is much more extended. Other than slight differences in the initial binary populations, the rest of the initial model details and the physics implementation remains the same. The behavior of the core radius and the inner BH Lagrange radii
in model MC is nonetheless very similar to that seen in the much more compact MC models presented in MOR15. Also, the order in which BHs are ejected, from heavy to light, agrees with the findings of MOR15. The main difference in the evolution of model MC presented here is that, being much more extended, the cluster has a significantly longer relaxation timescale, and therefore it processes (and thus ejects) BHs much more slowly than more compact models with similar N from MOR15, which have typically ejected ~1000s of BHs on the same physical timescale.

5 SUMMARY AND CONCLUSIONS

In this study we have compared the results of realistic large-N star cluster simulations created using two very different techniques: an orbit-averaged MC approach (CMC) and new direct N-body code (NBODY6++GPU). In terms of overall dynamical structure and evolution, as well as the dynamics of the BHs and long-term BH retention, we find quite remarkable agreement between the two techniques. Most notably, the direct N-body model confirms the finding of MOR15 that very large numbers of BHs (~1000) can be retained for ~10 Gyr timescales in old GCs.

We see slight differences in the evolution of the innermost 1% BH Lagrange radius: model MC displays core oscillations driven by the heaviest BHs, which do not occur in model NB. However, this disagreement does not seem to have any significant impact on the overall evolution of the cluster models. For example, the half-mass radius evolution of model MC models agrees with that of the N-body model. Additionally, both models eject BHs at roughly the same rate, losing just over 300 of the initially retained BHs over the entire simulation and thus still retaining more than a thousand BHs at the end of the simulation. This verifies that, while our approximate MC approach may not be capable of treating the physics of the small number of BHs in the core perfectly, our technique is nonetheless capable of reproducing the bulk evolution of the cluster as seen in the N-body model rather well, making it a viable technique for modeling the dynamical evolution of realistic clusters containing large numbers of stellar-mass BHs. With our parallel CMC code, a cluster model like the one studied here can be completed (evolved up to 12 Gyr) within about a day. The same model computed with the Spitzer N-Body code ++GPU requires more than 6 months to complete.

MOR15 argued that the long-term survivability of BHs in clusters relies on the fact that the BHs mostly avoid the Spitzer instability, in contrast to what has often been assumed. The lack of BH-driven collapses in the N-body model presented here provides evidence that the BHs are even less susceptible to the Spitzer instability than predicted previously by MOR15. This would provide strong support for the main conclusion from MOR15, namely that if large numbers of BHs are retained initially (as they are under standard assumptions regarding star cluster initial conditions and BH formation processes), then many will be retained at 12 Gyr, especially for a model with a large initial virial radius (as studied in this work).

6 ACKNOWLEDGMENTS

This work was supported by NSF Grant AST-1312945 and NASA Grant NNX14AP99G. All computations using CMC were performed on Northwestern University’s HPC cluster Quest. CR was supported by an NSF GRFP Fellowship, award DGE-0824162. MM was supported by an NSF GK-12 Fellowship, award DGE-0948017. W. L. acknowledge the support by the Silk Road Project at the National Astronomical Observatories of China (NAOC, http://silkroad.bao.ac.cn), the Max-Planck-Institute for Astrophysics and the Max Planck Computing and Data Facility (MPCDF, http://www.mpcdf.mpg.de/) in Garching, Germany. All direct N-body simulations were run on their Hydra GPU cluster.

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