Kazantsev dynamo in turbulent compressible flows

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We consider the kinematic fluctuation dynamo problem in a flow that is random, white-in-time, with both solenoidal and potential components. This model is a generalization of the well-studied Kazantsev model. If both the solenoidal and potential parts have the same scaling exponent, then, as the compressibility of the flow increases, the growth rate decreases but remains positive. If the scaling exponents for the solenoidal and potential parts differ, in particular if they correspond to typical Kolmogorov and Burgers values, we again find that an increase in compressibility slows down the growth rate but does not turn it off. The slow down is, however, weaker and the critical magnetic Reynolds number is lower than when both the solenoidal and potential components display the Kolmogorov scaling. Intriguingly, we find that there exist cases, when the potential part is smoother than the solenoidal part, for which an increase in compressibility increases the growth rate. We also find that the critical value of the scaling exponent above which a dynamo is seen is unity irrespective of the compressibility. Finally, we realize that the dimension $d = 3$ is special, since for all other values of $d$ the critical exponent is higher and depends on the compressibility.

I. INTRODUCTION

Astrophysical objects typically have magnetic fields over many ranges of scales. The growth and saturation of these magnetic fields are the subject of dynamo theory. Most astrophysical flows are turbulent, hence the astrophysically relevant magnetic fields are generated by turbulent dynamos. Broadly speaking, we understand two different kinds of dynamo mechanism in turbulent fluids: (a) one that generates magnetic fields whose characteristic length scales are significantly larger than the energy-containing length scales of the flow — large-scale dynamos; and (b) one that generates small-scale magnetic fields — the fluctuation dynamo. The small-scale tangled magnetic fields in the interstellar medium (ISM) or the Sun are supposed to be generated by the fluctuation dynamo. If we are interested in only the initial growth of the magnetic field but not its saturation, then we can ignore the Lorentz force by which the magnetic field acts back on the flow and reduce a non-linear problem to a linear one — the kinematic dynamo problem. A pioneering work on the kinematic fluctuation dynamo was by Kazantsev (1968), who approximated the velocity field by a random function of space and time. In particular, the velocity field is assumed to be statistically stationary, homogeneous and isotropic with zero divergence (under the latter assumption the velocity is called solenoidal or incompressible). It is also assumed to be short-correlated in time, more specifically white-in-time, and its correlation function in space is assumed to have a power-law behavior with an exponent $0 \leq \xi \leq 2$. By virtue of the white-in-time nature of the velocity field, it is possible to write down a closed equation for the equal-time two-point correlation function of the magnetic field. Consequently, it is possible to show that a dynamo exists — the magnetic energy grows exponentially in time — iff the exponent $\xi > 1$ (Vergassola, 1996). This result is an example of an anti-dynamo theorem. The Kazantsev model has played an important role in our understanding of the kinematic fluctuation dynamo excited by turbulent flows à la Kolmogorov, see e.g. the review by Brandenburg & Subramanian (2005) and references therein. The same model for the velocity field has been extensively used to study the scaling behavior and intermittency of advected passive scalar fields — the Kraichnan model (Kraichnan, 1968) of passive-scalar turbulence, see e.g. Falkovich et al. (2001) for a review.

In this paper we are interested in a generalization of the Kazantsev model to compressible turbulence, which is relevant for the fluctuation dynamo in the ISM. To the best of our knowledge, the earliest attempt to generalize the Kazantsev model to compressible flows was by Kazantsev, Ruzmaikin & Sokolov (1985), who modelled the flow not by a white-in-time process but by random weakly-damped sound waves, with energy concentrated on a single wave number, to find that the growth rate of the dynamo is proportional to the fourth power...
of the Mach number. More recently, Rogachevskii & Kleeromin (1997) generalized the Kazantsev model to compressible flows, preserving its white-in-time nature, but adding a potential (irrotational) component to the velocity field. Both the solenoidal and the potential components are assumed to have the same scaling exponent, \(0 \leq \xi \leq 2\). In this model, Rogachevskii & Kleeromin (1997) found that the inclusion of compressible modes always makes it more difficult to excite the dynamo. Later Schekochihin & Kulsrud (2001) and Schekochihin et al. (2002) studied the same problem for the case of smooth velocity fields, \(\xi = 2\), which corresponds to the limit of large magnetic Prandtl numbers. Schober et al. (2012) also tried to extend the Kazantsev model with the purpose of understanding the magnetic-field growth in ISM. However, for a tensor function to be the correlation of an isotropic vector field, its Fourier transform must be positive semi-definite (Monin & Yaglom, 1975), and the correlation tensor that Schober et al. (2012) prescribed for the velocity field violates such condition.

Recent direct numerical simulations of compressible fluid turbulence (Wang et al., 2013, 2018) have shown that the spectra of the flow can be separated into a solenoidal and a potential part, where the solenoidal part scales with an exponent quite close to the classical Kolmogorov result for incompressible turbulence — the energy spectrum of the solenoidal part scales with exponent \(-5/3\) in the inertial range — and the scaling exponent for the potential part is close to that of the turbulent Burgers equation — the energy spectrum of the potential part scales with an exponent of \(-2\) in the inertial range. How will the Kazantsev dynamo problem change if we use two independent scaling exponents for the solenoidal and the potential parts of the energy spectrum of the velocity field? This is the main task we set for ourselves in this paper.

We end this introduction by a warning to our reader: because of the popularity of the Kazantsev model as an analytically tractable model for the turbulent dynamo, the literature on it is large and diverse. We have so far cited only those papers that are directly relevant to our work. For a detailed introduction we suggest several reviews (Brandenburg & Subramanian, 2005; Falkovich et al., 2001; Tobias, Cattaneo & Boldyrev, 2012; Zeldovich, Ruzmaikin & Sokolov, 1990) and references therein.

II. MODEL

We model the plasma as a conducting fluid at scales where the magnetohydrodynamics (MHD) equations are valid. Consequently the magnetic field, \(B(x, t)\), evolves according to the induction equation (Chandrasekhar, 1961)

\[
\partial_t B = \nabla \times (u \times B - \kappa J),
\]

where \(J = \nabla \times B\) is the current, \(\kappa\) the magnetic diffusivity, and \(u(x, t)\) the velocity. Furthermore, Maxwell’s equations imply \(\nabla \cdot B = 0\). Two dimensionless numbers characterize the MHD equations: the Reynolds number \(Re \equiv V L / \nu\) and the magnetic Reynolds number \(Re_m \equiv V L / \kappa\), where \(V\) is a typical large-scale velocity, \(L\) is the correlation length of the velocity field, and \(\nu\) the kinematic viscosity of the fluid. The ratio of \(Re_m\) and \(Re\) is the magnetic Prandtl number \(Pr \equiv \nu / \kappa\).

Instead of solving the momentum equation for the velocity, as is usual in MHD, the Kazantsev model assumes that the velocity \(u\) is a statistically stationary, homogeneous, isotropic and parity invariant, Gaussian random field with zero mean and correlation

\[
\langle u_i(x+r, t)u_j(x, t') \rangle = \mathcal{D}_{ij}(r)\delta(t-t'), \quad i, j = 1, \ldots, d,
\]

where \(d\) is the spatial dimension of the flow. In this paper, we use \(d = 3\) except in Section IV where results for general \(d\) are presented. The velocity field is also assumed to be Galilean invariant, hence it is useful to introduce the second-order structure function \(S_{ij}(r) = \mathcal{D}_{ij}(0) - \mathcal{D}_{ij}(r)\), which is a Galilean invariant quantity that describes the statistics of the velocity increments. In view of statistical isotropy and parity invariance, \(S_{ij}(r)\) takes the form (Monin & Yaglom, 1975; Robertson, 1940)

\[
S_{ij}(r) = S_N(r)\delta_{ij} + [S_L(r) - S_N(r)]\tilde{r}_i\tilde{r}_j,
\]

where \(\tilde{r}_i = r_i / r\) and \(S_N(r)\) and \(S_L(r)\) are called the normal (also known as transverse or lateral) and longitudinal second-order structure functions, respectively.

A. Solenoidal random flows

Kazantsev (1968) considered the three-dimensional solenoidal case (\(\nabla \cdot u = 0\)) and the limits of vanishing \(Pr\) and infinite \(Re\) and \(Re_m\), in which the velocity structure functions are scale invariant:

\[
S_L(r) = 2Dr^\xi, \quad S_N(r) = (\xi + 2)Dr^\xi.
\]

The positive constant \(D\) determines the magnitude of the fluctuations of the velocity increments, while the scaling exponent \(\xi\) varies between 0 and 2, the latter value describing a spatially smooth velocity field. The above choice for the structure functions correspond to a kinetic-energy spectrum of the form \(E(k) \propto k^{-1-\xi}\). We remind the reader that, owing to the \(\delta\)-correlation in time, some care should be taken in comparing the flow \(u\) defined in equation (2) with a time-correlated flow. Consider indeed a flow with the same statistical properties as \(u\) but with nonzero correlation time and structure functions that scale as \(r^\alpha\). The two flows yield the same laws of Lagrangian dispersion provided \(\xi = 1 + \alpha / 2\) (Falkovich et al., 2001). In particular, the Kolmogorov phenomenology of fully developed turbulence is reproduced by the \(\delta\)-correlated flow if \(\xi = 4 / 3\).

\[\text{Note that our definitions of the structure functions differ from those of Monin & Yaglom (1975) by a factor of 2.}\]
B. Compressible random flows

One way to include the effect of compressibility into the Kazantsev model is to modify, in d-dimensions, $S_L(r)$ and $S_N(r)$ in the following manner (Celani, Lanotte & Mazzino, 1999; Epenal, Kleerorin & Rogachevskii, 1995; Gawędzki & Vergassola, 2000):

$$S_L(r) = Dr^d(d-1)(\rho \xi + 1), \quad S_N(r) = Dr^d(-\rho \xi + \xi + d-1),$$  \hspace{1cm} (5)

by introducing a new parameter $\rho$ which is the degree of compressibility of the velocity field and varies between 0 (solenoidal, or incompressible, flow) and 1 (potential, or irrotational, flow). In this model both the solenoidal and the potential parts of the flow have the same scaling exponent.

III. RESULTS

A. Closed equation for the correlation function of the magnetic field

Given the velocity field the task is to calculate the equations obeyed by the correlation functions of the magnetic field, where the averages are calculated over the statistics of the velocity field. The magnetic field is assumed to have the same spatial statistical symmetries as the velocity field. Its correlations are thus written as

$$C_{ij}(r,t) = C_L(r,t)\delta_{ij} + \frac{r}{d-1}\partial_r C_L(r,t)(\delta_{ij} - \hat{r}_i \hat{r}_j),$$  \hspace{1cm} (6)

where a single scalar function $C_L(r,t)$ is sufficient to define $C_{ij}(r,t)$ thanks to the solenoidality of the magnetic field (Monin & Yaglom, 1975; Robertson, 1940).

By virtue of the Gaussian and white-in-time nature of the velocity field, it is possible to write a closed equation for the evolution of $C_{ij}(r,t)$ in a straightforward manner, by using the induction equation (1) and then averaging over the statistics of the velocity field. To obtain a closed equation, however, we need to calculate averages of a triple product: that of the velocity field, of the magnetic field and of its spatial derivative. This has been obtained by many different methods. A well established approach employs a result known as “Gaussian integration by parts”: for a Gaussian, not-necessarily white-in-time, vector-valued noise $z(t)$, with components $z_j(t)$, and its arbitrary functional $F(z)$,

$$\langle F(z)z_j(t) \rangle = \int ds \langle z_j(t)z_k(s) \rangle \frac{\delta F}{\delta z_k(s)},$$  \hspace{1cm} (7)

see e.g. Zinn-Justin (1996) Section 4.2) for a proof. At the next step we need to integrate equation (1) formally to obtain the magnetic field as a function of the velocity field and then to calculate the necessary functional derivatives. Then we have to take the limit $s \to t$. As the noise correlation becomes singular in this limit, we need to replace the Dirac delta function by a regularised even function and then take the limits. This regularisation is equivalent to using the Stratonovich prescription for the noise. This method has been used extensively in dynamo theory (see, e.g., Mitra & Brandenburg, 2012; Schekochihin et al., 2002; Schekochihin & Kulsrud, 2001; Seshasayanan & Alexakis, 2016; Vergassola, 1996), to calculate similar or higher-order correlation functions. Hence we skip the details of the derivation and directly write down the result (see also Schekochihin et al., 2002 and references therein):

$$\partial_t C_L = (2\kappa + S_L)\partial_r^2 C_L + \left( \frac{2(d+1)}{r} \right) \kappa 
+ \partial_r S_L + \frac{2(d-1)}{r} S_L + \frac{3-d}{r} S_N \partial_r C_L 
+ \frac{d-1}{r} \left[ \partial_r S_L + \partial_r S_L + \frac{d-2}{r} (S_L - S_N) \right] C_L,$$

\hspace{1cm} (8)

B. Schrödinger formulation

One of Kazantsev’s main results is the formulation of the turbulent dynamo effect as the trapping of a quantum particle in a one-dimensional potential whose shape is determined by the functional form of the velocity structure functions. This formulation leads to an appealing interpretation of the dynamo effect.

Equation (8) is a partial differential equation with one space and one time dimension. By the substitution

$$C_L(r,t) = \psi(r,t)r^{-(d-1)}[2\kappa + S_L(r)]^{-1/2} \times \exp \left[ \frac{d-3}{2} \int r^{-1}2\kappa + S_N(r) \frac{2\kappa + S_L(r)}{2\kappa + S_L(r)} dr \right],$$  \hspace{1cm} (9)

equation (8) is turned into an imaginary-time Schrödinger equation with space-dependent mass for the function $\psi(r,t)$ (Schekochihin et al., 2002):

$$m(r)\partial_t \psi = \partial_r^2 \psi - m(r)U(r)\psi,$$

\hspace{1cm} (10)

with

$$m(r) = \frac{1}{2\kappa + S_L(r)}$$  \hspace{1cm} (11)

and

$$U(r) = \frac{\partial_r^2 S_L(r)}{2} - \frac{[\partial_r S_L(r)]^2}{4[2\kappa + S_L(r)]} + \frac{(d-3)^2[2\kappa + S_N(r)]^2}{4r^2[2\kappa + S_L(r)]} 
+ \frac{(3d-5)[2\kappa + S_N(r) - r\partial_r S_N(r)]}{2r^2}.$$  \hspace{1cm} (12)

The function $\psi(r,t)$ has the same time dependence as $C_L(r,t)$. We therefore seek solutions of the form $\psi(r,t) = \psi_s(r)e^{-\gamma t}$, where $\psi_s(r)$ are the eigenstates of the Schrödinger operator in equation (10) and $\gamma$ the associate energies. The magnetic correlation grows in time if there exist negative-energy ($\gamma < 0$) eigenstates. If so, the energy of
the ground state, $\gamma_0$, yields the asymptotic growth rate $|\gamma_0|^2$. From equation (10), $\gamma$ can be written as:

$$\gamma = \frac{\int m(r)U(r)\psi_0^2(r)dr + \int [\psi'_0(r)]^2 dr}{\int m(r)\psi_0^2(r)dr}.$$  

(13)

Since $m(r) > 0$ for all $r$ (Schekochihin et al., 2002), $\gamma$ is negative if and only if the effective potential $V_{\text{eff}}(r)$ admits negative energies. Thus the problem of the growth of the magnetic correlations is mapped into that of the existence of negative-energy states for the one-dimensional potential $V_{\text{eff}}(r)$ — the energy for the ground state of the potential $V_{\text{eff}}(r)$ is equal to the growth rate of the fastest growing mode in the kinematic dynamo problem.

C. Solenoidal random flows

In the original Kazantsev model, the flow is assumed to be solenoidal and the longitudinal and normal structure functions have the same scaling exponent as given in equation (4). The effective potential is obtained by substituting from equation (4) to equation (8) and simplifying (e.g., Vergassola et al., 1996):

$$U_{\text{eff}}(r) = \frac{8\kappa^2 - 2(3\xi - 8)\kappa Dr^2 - (3\xi^2 + 6\xi - 8)D^2 r^2}{2\psi(0,0)} = \frac{8\kappa^2 - 2(3\xi^2 + 6\xi - 8)(\kappa + Dr)^2}{4\kappa^2(\kappa + Dr)^2}.$$  

(14)

where in the second step we have introduced the diffusive scale, $r_\kappa \equiv (\kappa/D)^{1/\xi}$, at which the diffusive and the advective terms in the induction equation (1) balance each other. For $r \ll r_\kappa$, $U_{\text{eff}}(r) \sim 2/r^2$ and is therefore repulsive. For $r \gg r_\kappa$, $U_{\text{eff}}(r) \sim -(3\xi^2/4 + 3\xi/2 - 2)/r^2$; hence the effective potential is negative at large $r$. By examining the shape of $U_{\text{eff}}(r)$, it is possible to conclude that the critical value of $\xi$ for the kinematic dynamo effect is $\xi_{\text{crit}} = 1$ (Kazantsev, 1968, see also Vergassola, 1996). For $\xi < 1$, $U_{\text{eff}}(r)$ is indeed everywhere greater than a potential of the form $-c/r^2$ with $c < 1/4$, which does not have any negative-energy eigenstates (Landau & Lifshitz, 1958). Hence the same holds for $U_{\text{eff}}(r)$, and consequently no dynamo exists. For $\xi > 1$, the effective potential behaves at large distances as $U_{\text{eff}}(r) \sim -c/r^2$ with $c > 1/4$. A potential with such a large-$r$ behavior has a discrete spectrum containing an infinite number of negative-energy eigenvalues (Landau & Lifshitz, 1958), so the magnetic correlation can grow exponentially.

For a smooth flow ($\xi = 2$), the value of the growth rate is known analytically: $|\gamma_0|^2 = 15D/2$ (Chertkov et al., 1999).

Guzinov et al. (1996), Kazantsev (1968), and the magnetic correlation decays as $C_L(r) \sim r^{-5/2}$ (Chertkov et al., 1999). For a spatially rough flow, $1 < \xi < 2$, the analytical calculation of $|\gamma_0|^2$ is a difficult task. It was obtained numerically by Kazantsev (1968) and Vincenzi (1996), and an analytical estimate was given by Arponen & Horvai (2007). The growth rate increases as a function of $\xi$, i.e., a greater degree of spatial regularity of the velocity field favours the magnetic growth. In addition, the magnetic correlation decays as a stretched-exponential of the spatial separation with stretching exponent $1 - \xi/2$ (Vincenzi, 2002).

The incompressible case was generalized to $d$ dimensions by Guzinov et al. (1996) and Vergassola (1996). In particular, for $d = 2$, $U_{\text{eff}}(r)$ is repulsive everywhere and hence there cannot be dynamo effect, in accordance with Zel’dovich’s anti-dynamo theorem (Zel’dovich, 1957).

D. Compressible random flows

Next we consider the model of compressible flow given by equation (5), where both the solenoidal and the potential components of the structure functions scale with the same exponent. Upon substitution of equation (5) into equations (11) and (12), the mass and potential functions are found to have the form

$$m(r) = \frac{1}{2\kappa + Dr}(\psi_0^2 + 1)$$  

(15)

and

$$U(r) = \frac{a_0 + a_1\psi^2 + a_2\psi^4}{4r^2(2\kappa + Dr)(\psi_0^2 + 1)}.$$  

(16)

where the coefficients $a_0$, $a_1$, $a_2$ are given in appendix A.

Let us examine the $d = 3$ case. At small distances ($r \ll r_\kappa$), the asymptotic behavior of $U_{\text{eff}}(r)$ is unchanged compared to the incompressible case. For $r \gg r_\kappa$,

$$U_{\text{eff}}(r) \sim \frac{\psi_0^2 + 2\psi^2 - 3\psi^2 - 4\psi_0^2 - 6\psi + 8}{4(\psi_0^2 + 1)^2}.$$  

(17)

By studying the form of $U_{\text{eff}}(r)$ as in Section III.B it can be shown that the critical value of $\xi$ for the dynamo effect is not affected by the degree of compressibility, i.e., $\xi_{\text{crit}} = 1$ for all values of $\varphi$ (Rogachevskii & Klecker, 1997). Indeed, for $\xi < 1$, $U_{\text{eff}}(r)$ behaves asymptotically as $-c/r^2$ with $c < 1/4$, and both $a_0$ and $a_1$ are positive for all values of $\varphi$. Hence $U_{\text{eff}}(r) > -c/r^2$ with $c < 1/4$ for all $r$, and it does not admit negative-energy eigenstates. For $\xi > 1$, $U_{\text{eff}}(r) \sim -c/r^2$ with $c > 1/4$, and an infinite number of states with negative energy can exist. Even though $\xi_{\text{crit}}$ does not vary with $\varphi$, for $\xi > 1$ the magnetic growth is affected by the degree of compressibility of the flow. For $\xi = 2$, the growth rate is $|\gamma_0| = (15/2 - 5\psi_0)D$, while at large separations and long times the correlation $C_L(r,t)$ behaves, up to logarithmic corrections, as $r^{-5/2} \exp(|\gamma_0|t)$ (Schekochihin et al., 2002).
For $1 < \xi < 2$, to our knowledge no analytical expression for the growth rate $|\gamma_0|$ is available. We calculate it numerically by applying to equation (10) a variation-iteration method that is similar to techniques used to find the largest eigenvalue of very large matrices. This method is described in detail in appendix B. The result is shown in Figure 1 where we plot $|\gamma_0| t_\kappa$ versus $\xi$ for several values of $\wp$; here $t_\kappa \equiv r_\kappa^2 / \kappa$ is the time scale associated with magnetic diffusion. We find that, irrespective of $\wp$, the growth rate is positive for all values of $\xi > 1$ and increases with $\xi$, i.e., as the spatial regularity of the flow improves. For a fixed $\xi$, the effect of compressibility is always to decrease the growth rate, i.e., to make the dynamo weaker. Rogachevskii & Kleeorin (1997) tried to estimate the growth rate by asymptotic matching; they also found the same qualitative effect of compressibility.

We find out the behavior of $C_L(r, t)$ for large $r$ by replacing $n(r)$ and $U(r)$ in equation (10) with their asymptotic expressions. The resulting equation for $\psi_\gamma(r)$ is a Bessel differential equation, whose non-diverging solution is

$$\psi_\gamma(r) \propto \sqrt{r} K_\beta \left[ \beta \sqrt{|\gamma|} r \left( \frac{r}{r_\kappa} \right)^{1-\xi/2} \right],$$

where $\beta = b\sqrt{2/(\wp \xi + 1)}$ and $K_\beta$ is the modified Bessel function of the second kind of order $b = 1/(2 - \xi)$. At large $r$ this solution behaves as $\exp[-\beta \sqrt{|\gamma|} t_\kappa (r/r_\kappa)^{1-\xi/2}]$. Furthermore, the long-time behavior of the magnetic correlation is determined by the ground state $\psi_{\gamma_0}(r)$. From equation (9) we therefore conclude that, for $r \gg r_\kappa$ and $t \gg t_\kappa$,

$$C_L(r, t) \propto r^{-2-\xi/2} \exp \left[ -\beta \sqrt{|\gamma|} t_\kappa \left( \frac{r}{r_\kappa} \right)^{1-\xi/2} \right] e^{\gamma_0 t}.$$  

(19)

As far as the spatial dependence of the magnetic correlation is concerned, we thus recover the stretched-exponential behavior of the incompressible case; the degree of compressibility of the flow only affects $|\gamma_0|$, and hence the rate at which the stretched exponential function decays in space, but not the stretching exponent.

E. Solenoidal–potential decomposition

The results of the previous section indicate that the spatial regularity of the flow favours the magnetic growth, whereas compressibility hinders it. It has now been realized that for turbulence at moderate Mach numbers the solenoidal and potential parts of the flow have different scaling exponents; more precisely, the solenoidal part of the kinetic-energy spectrum displays the Kolmogorov scaling $k^{-5/3}$, while the potential one displays the Burgers scaling $k^{-2}$ (Wang et al., 2013, 2018). It is therefore interesting to investigate the interplay between spatial regularity and compressibility in the case where the potential part of the flow is more regular than the solenoidal one. We thus consider the following velocity structure functions (see Monin & Yaglom (1975) p. 57) for the solenoidal–potential decomposition of an isotropic random field:

$$S_L(r) = 2(1 - \wp) D_S r^{\xi_S} + 2\wp D_P (1 + \xi_P) r^{\xi_P},$$

(20)

$$S_N(r) = (1 - \wp) D_S (2 + \xi_S) r^{\xi_S} + 2\wp D_P r^{\xi_P},$$

(21)

where $\xi_S$ and $\xi_P$ are the scaling exponents of the solenoidal ($\wp = 0$) and potential ($\wp = 1$) components, respectively, and $D_P$ and $D_S$ are positive coefficients. Varying the degree of compressibility thus interpolates linearly between a solenoidal flow with scaling exponent $\xi_S$ and a potential flow with exponent $\xi_P$.

The mass and the potential functions can be calculated by inserting $S_L(r)$ and $S_N(r)$ from equations (20) and (21) into...
The product of $m(r)$ and $U(r)$ yields the effective potential. In the diffusive range, $r \ll r_\kappa, U_{\text{eff}}(r) \sim 2/r^2$ independently of the values of $\xi_S$ and $\xi_P$; the dynamics is indeed dominated by magnetic diffusion in this range. If $\xi_S > \xi_P (\xi_S < \xi_P)$ the asymptotic behavior of $U_{\text{eff}}(r)$ is the same as that of a solenoidal flow with exponent $\xi_S$ (a potential flow with an exponent $\xi_P$). As noted in Section II.D the critical value of the scaling exponent equals unity, $\xi_{\text{crit}} = 1$, in both cases. Therefore, the critical scaling exponent for the dynamo effect does not depend on whether or not the solenoidal and potential components scale differently, and if at least one of the two exponents is greater than unity, a dynamo is obtained.

The magnetic growth rate, by contrast, is expected to vary with the spatial regularity of the potential component. Since for a generic $\varphi$ the solenoidal and potential components of the velocity field now scale differently, two scales, $r_\kappa^{(S)} = (\kappa/D_S)^{1/\xi_S}$ and $r_\kappa^{(P)} = (\kappa/D_P)^{1/\xi_P}$, can be formed by balancing the diffusion term in equation (1) with either the solenoidal or the potential contribution to the advection term. In the calculation of the magnetic growth rate, we set $r_\kappa^{(S)} = r_\kappa^{(P)}$. The relative weight of the solenoidal and the potential components is thus determined solely by $\varphi$. Furthermore, such choice allows us to define a single diffusive time scale $\tau_\kappa$, which is used to rescale the growth rate. We do not know any analytical method allowing us to exactly calculate the growth rate as a function of $\xi$ and $\varphi$. Hence we use the variation–iteration method to calculate it numerically.

In Fig. 2 we plot the non-dimensionalized growth rate as a function of $\varphi$ for $\xi_S = 4/3$ and for several different values of $\xi_P$. According to the discussion at the end of Section II.A the scaling behavior of the kinetic-energy spectrum observed in [Wang et al., 2013 [2018]], i.e., $k^{-5/3}$ for the solenoidal component and $k^{-2}$ for the potential one, corresponds to a white-in-time flow with $\xi_S = 4/3$ and $\xi_P = 3/2$ (green square symbols in Fig. 2). The case $\xi_S = \xi_P = 4/3$ corresponds to the case where both the solenoidal and the potential components scale with Kolmogorov exponent (red * symbols in Fig. 2). In both of these cases the growth rate monotonically decreases as a function of $\varphi$. But the growth rate for the case $\xi_S = 4/3$ and $\xi_P = 3/2$ is higher for all values of $\varphi$. Remarkably, for greater values of $\xi_P$, we find that $|\gamma_0|$ varies non-monotonically or even increases with $\varphi$ (inset of Fig. 2). Hence compressibility can increase the magnetic growth if the potential component of the flow is sufficiently regular. This is a new qualitative result.

F. Critical magnetic Reynolds number

The magnetic Reynolds number $Re_m$ measures the relative intensity of the stretching of magnetic lines by the flow and Ohmic dissipation. Therefore, for the dynamo effect to take place, $Re_m$ must be sufficiently high and exceed a critical value $Re_{m\text{crit}}$.

We consider first the case in which both the solenoidal and the potential components of the velocity field display the same scaling exponent ($\xi_S = \xi_P = \xi$). In the original version of the Kazantsev model, $Re_m$ is infinite, since the structure functions of the velocity field are assumed to increase indefinitely with the scale separation. It is possible, however, to modify the form of $S_L(r)$ and $S_N(r)$ in such a way as to introduce a finite correlation length $L$ beyond which the structure functions saturate to a constant value [Artamonova & Sokolov, 1986 [Novikov, Ruzmaikin & Sokolov, 1983 [Ruzmaikin & Sokolov, 1981 [Ruzmaikin, Sokoloff & Shukurov, 1989 [Schekochihin et al., 2002]]. This allows the definition of the magnetic Reynolds number as $Re_m = L/r_\kappa = L/(D/\kappa)^{1/\xi}$. Here we follow Monin & Yaglom (1975, p. 55) (see also Arponen & Horvai, 2007 [Falkovich et al., 2001 [Vincenzi, 2002]) and introduce $L$ in the model by modifying the kinetic-energy spectrum as follows:

$$E(k) \propto \frac{k^4}{(k^2 + L^{-2})^{(5+\xi)/2}}$$

(24)

The reader is referred to appendix C for more details. The specific form of $E(k)$ is immaterial and only affects numerical details. What is important is that the power-law behavior $E(k) \sim k^{-1-\xi}$ is recovered for $k \gg L^{-1}$, while $E(k)$ is regular for $k \ll L^{-1}$. The structure functions for finite $Re_m$
can then be calculated from $E(k)$ (see appendix C) and can be inserted into equation (11) and (12) to obtain the effective potential $U_{\text{eff}}(r)$, and hence the magnetic growth rate via the variation–iteration method.

Within the Schrödinger formulation of the Kazantsev model, the existence of a critical magnetic Reynolds number can be interpreted thus: a finite value of $L$ generates a potential barrier at spatial scales of the order of $L$ and greater than it (Novikov, Ruzmakin & Sokolov 1983; Schekochihin et al., 2002). As $L$ (and hence $Re_m$) is decreased, the potential well raises, until negative-energy states cease to exist and the dynamo effect disappears.

Figure 3 shows the rescaled magnetic growth rate as a function of $Re_m$ for (a) for fixed $\xi = 4/3$ and different values of $\varphi$ and (b) for fixed $\varphi = 0$ and different values of $\xi$. We find yet another instance in which the degree of compressibility and the spatial regularity of the flow have an opposite effect on the dynamo effect: compressibility increases $Re_m^{\text{crit}}$, whereas the smoothness of the flow makes it decrease.

It is thus interesting to examine the interplay between these two effects by considering, as in Sect. III.E, a kinetic-energy spectrum such that the solenoidal component of the velocity field displays the Kolmogorov scaling, while the potential one displays the Burgers scaling:

$$E(k) = (1 - \varphi) \frac{A_S}{L^{\xi_S}} \frac{k^4}{(k^2 + L^{-2})^{(5 + \xi_S)/2}} + \varphi \frac{A_P}{L^{\xi_P}} \frac{k^4}{(k^2 + L^{-2})^{(5 + \xi_P)/2}}; \quad \text{(25)}$$

Here $A_S$ and $A_P$ are positive coefficients. The structure functions corresponding to the above spectrum can be found in appendix C.

Once more we calculate the magnetic growth rate numerically by using the variation–iteration method. The scales $r_\kappa^{(S)}$ and $r_\kappa^{(P)}$ are defined as in Sect. III.E with $D_S$ and $D_P$ given in equations (C7) and (C8). We again set the ratio $r_\kappa^{(S)}/r_\kappa^{(P)}$ to unity; the values of $Re_m$ and $t_\kappa$ are thus unaffected by whether $r_\kappa^{(S)}$ or $r_\kappa^{(P)}$ is used in the definitions. The magnetic growth rate as a function of $Re_m$ is shown in Fig. 4 for the case $\xi_S = 4/3$, $\xi_P = 3/2$ and for representative values of the degree of compressibility. Compared to the case $\xi_S = \xi_P = 4/3$, the critical magnetic Reynolds number is always lower when the potential component of the flow displays the Burgers scaling. In this latter case, $Re_m^{\text{crit}}$ depends weakly on $\varphi$, which suggests that the higher spatial regularity compensates the effect of the increase in compressibility.
IV. KINEMATIC DYNAMO IN $d$ DIMENSIONS

In this section, we examine the dependence of the Kazantsev model (with single scaling exponent) on the space dimension. In equation (4), the dimension $d$ may indeed be regarded as a parameter that can also take values different from 2 or 3. So far in this paper we have limited ourselves to the case $d = 3$. Gruzinov et al. (1996) showed that if the flow is smooth ($\xi = 2$) and incompressible ($\varphi = 0$), there is dynamo effect only when the spatial dimension is in the interval $2.103 \leq d \leq 8.765$. Arponen & Horvai (2007) extended this result to a rough flow ($0 < \xi < 2$) and showed that the curve $\xi_{\text{crit}}$ vs $d$ is convex and has its minimum in $d = 3$. Therefore, the range of spatial dimensions over which the dynamo effect can take place shrinks as the flow becomes rougher or, in other words, a higher degree of spatial regularity of the flow is necessary if $d \neq 3$.

To study the effect of compressibility on $\xi_{\text{crit}}$ for a general dimension $d$, we once again examine the large-$r$ form of $U_{\text{eq}}(r)$. By repeating the argument used in Sections III.B and III.D, we obtain that $\xi_{\text{crit}}$ is the root of the following equation,

$$
\frac{a_2}{4[D(d-1)(\varphi \xi + 1)]^2} = -\frac{1}{4},
$$

such that $0 \leq \xi_{\text{crit}} \leq 2$ (the coefficient $a_2$ is defined in equations (16) and (A.2)). Figure 5 shows $\xi_{\text{crit}}$ as a function of $d$ for different values of $\varphi$. Clearly, the three-dimensional case is peculiar inasmuch as it is the only one for which $\xi_{\text{crit}}$ does not depend upon $\varphi$, and the critical exponent is the lowest for $d = 3$. For all other values of $d$, increasing $\varphi$ broadens the range of spatial dimensions over which the magnetic correlation grows, and lowers the critical exponent for the appearance of the dynamo effect. No exponential growth of the magnetic field is found for $d = 2$ irrespective of the value of $\varphi$.

V. CONCLUSIONS

In this paper we have used a model that is a generalization of the Kazantsev model to flows that have both solenoidal and potential components. We find, in agreement with earlier results, that the critical value of the exponent above which a dynamo is seen is unity in all cases. If both the solenoidal and the potential parts have the same scaling exponent, then, as the compressibility of the flow increases, the growth rate of the dynamo decreases but does not go to zero. In other words, even when the flow has only a potential component, the dynamo is not turned off but merely slowed down. This qualitative result was already known from the approximate analytical work of Rogachevskii & Kleeroin (1997); we have now provided numerical results that support the same conclusion. We also show that the flow compressibility increases the critical magnetic Reynolds number for the dynamo effect, whereas spatial regularity lowers it. More importantly, we have considered the more realistic case where the solenoidal and the potential parts have different scaling exponents. If we consider the case where these scaling exponents for the solenoidal and potential parts correspond to typical Kolmogorov and Burgers values, then we again reach the same qualitative conclusion — an increase in compressibility slows down the growth rate of the dynamo but does not turn it off. The slow down due to compressibility is, however, weaker than in the case in which both the solenoidal and the potential components of the flow display a Kolmogorov scaling. Intriguingly, we find that there exist cases, if the potential part is smoother than the solenoidal part, where an increase in compressibility can actually increase the dynamo growth rate. Unfortunately, this result remains a curiosity because we are not aware of any realistic flow that corresponds to such a case. If the potential component of the flow displays the Burgers scaling, we also show that the critical Reynolds number is lower than when both the solenoidal and the potential components display the Kolmogorov scaling. Finally, by considering the same model in general $d$ dimensions, we realize that the dimension $d = 3$ is special; for all other values of $d$ the critical value of the exponent depends on the degree of compressibility of the flow, in particular, increasing the compressibility lowers the critical value of the exponent.

The Kazantsev model is a rare example of flow allowing an analytical study of the turbulent dynamo effect. For this reason, it has attracted a lot of attention in the literature, and several extensions of the model have been proposed to include certain properties of a turbulent velocity field that were not present in the original version. One should of course be careful in translating lessons learned from Kazantsev-type models to real astrophysical flows or to direct numerical simulations. These models are indeed limited by their white-in-time nature and their Gaussian statistics. Direct quantitative comparison between such models and simulations may therefore not be straightforward. Our results are furthermore limited by the
fact that we consider a vanishing magnetic Prandtl number. Care should also be taken in extending the results for the temporal growth of the second moment of the magnetic field to moments of a different order (Seshasayanan & Pétrélis 2018). Nevertheless, the large literature on Kazantsev-type models shows that the qualitative insight gained from the study of such models is robust.

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Appendix A: Potential $U$ for general $d$ and $\varphi$

For general $d$ and $\varphi$, the coefficients in the expression of the potential $U(r)$, see equation (16), are:

$$a_0 = 4(d + 1)(d - 1)n^2,$$

$$a_1 = 4D[2(\varphi^3 + \varphi \varphi^2 - \xi^2 + 2\varphi \xi - 3\xi + 8) + (\varphi^3 + 2\varphi^2 \xi^2 - 2\xi^2 + 7\varphi \xi - 8\xi + 20)(d - 3) + 2(\varphi^3 - \xi + 4)(d - 3)^2 + (d - 3)^3],$$

$$a_2 = D^2[4(\varphi^2 \xi^4 + 2\varphi^2 \xi^3 - 2\varphi^2 \xi^2 - 4\varphi^2 \xi^2 - 4\varphi^2 \xi^2 - 3\xi^2 + 4\varphi \xi - 6\xi + 8) + 4(\varphi^2 \xi^4 + 3\varphi^2 \xi^3 - 3\varphi^3 \xi^2 - 5\varphi^2 \xi^2 - 5\varphi \xi^2 - 4\xi^2 + 9\varphi \xi - 11\xi + 14)(d - 3) + (\varphi^2 \xi^4 + 4\varphi^2 \xi^3 - 4\varphi^3 \xi^2 - 5\varphi^2 \xi^2 - 26\varphi \xi^2 - 26\xi^2 + 22\varphi \xi - 24\xi + 66)(d - 3)^2 - 2(3\varphi^2 \xi^2 - 2\varphi \xi + 2\xi - 5)(d - 3)^3 + (d - 3)^4],$$

with $\mu > 0$, which transforms equation (B1) as follows:

$$\mathcal{L}\psi_\gamma = \lambda \mathcal{M}\psi_\gamma,$$

with

$$\mathcal{L} = -[\ln(1 + \mu)]^2 \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} \right) + \frac{m(\rho)}{\rho} \left[ U(\rho) - U_{\text{min}} \right],$$

$$\mathcal{M} = m(\rho)/\rho,$$

and $\lambda = \gamma - U_{\text{min}}$. Here $U_{\text{min}}$ denotes the minimum of the potential $U(\rho)$. The operators $\mathcal{L}$ and $\mathcal{M}$ are positive-definite and self-adjoint with respect to the usual inner product on $L^2([0, 1])$. In addition, for $\xi > 1$, the spectrum of eigenvalues $\lambda$ is bounded from below and extends to infinity.

Let $\phi^{(0)}(\rho)$ be an initial trial function that is not orthogonal to $\psi_{\gamma_0}(\rho)$ and satisfies the boundary conditions $\phi^{(0)}(0) = \phi^{(0)}(1) = 0$. No other conditions are required for $\phi^{(0)}(\rho)$. The technique for calculating $\psi_{\gamma_0}(r)$ and $[\gamma_0]$ consists in iteratively applying the operator $\mathcal{L}^{-1}\mathcal{M}$, in such a way that the $n$th iterate $\phi^{(n)}(\rho)$ is given by

$$\phi^{(n)}(\rho) = \mathcal{L}^{-1}\mathcal{M}\phi^{(n-1)}(\rho) = (\mathcal{L}^{-1}\mathcal{M})^n\phi^{(0)}(\rho)$$

and the corresponding eigenvalue estimate is:

$$\lambda^{(n)}_0 = \frac{\int_0^1 \phi^{(n)}(\rho)\mathcal{L}\phi^{(n)}(\rho) d\rho}{\int_0^1 \phi^{(n)}(\rho)\mathcal{M}\phi^{(n)}(\rho) d\rho}.$$

As $n$ is increased, $\phi^{(n)}(\rho)$ tends to $\psi_{\gamma_0}(\rho)$ and $\lambda^{(n)}_0$ to $\gamma_0 - U_{\text{min}}$ (from above). The iteration is stopped once the desired convergence is obtained.

In numerical calculations, the interval $[0, 1]$ is partitioned into $N$ subintervals of size $\Delta$ and, for a generic function $f(\rho)$, the differential part of the operator $\mathcal{L}$ is discretized as follows:

$$\frac{d}{d\rho} \left( \rho \frac{df}{d\rho} \right)_{\rho_k} = \frac{1}{2\Delta} \left( \rho_{k+1}f_{k+1} - \rho_k f_k + \rho_k f_k - \rho_{k-1} f_{k-1} \right)$$

$$= \frac{1}{2\Delta} \left( \rho_k f_{k+1} - f_k - \rho_k f_k - f_{k-1} \right)$$

$$+ \rho_k \left( \rho_{k+1} - \rho_k \right) f_{k+1} - \rho_{k-1} f_{k-1} - \rho_k f_k - f_{k-1}$$

$$= \frac{1}{2\Delta} \left[ \left( \rho_{k+1} + \rho_k \right) f_{k+1} + \left( \rho_k + \rho_{k-1} \right) f_{k-1} \right]$$

$$- \left( \rho_{k+1} + \rho_k - 1 + 2\rho_k \right) f_k,$$

where $\rho_k = k\Delta$, $f_k = f(\rho_k)$ and $f'_k = f'(\rho_k)$. The above discretization allows us to preserve the vanishing boundary conditions on $\phi^{(n)}(\rho)$ at each step of the iteration. The functions $\phi^{(0)}(\rho)$ and $\phi^{(n)}(\rho)$ in equation (B4) are replaced with the $N$-dimensional arrays $\phi^{(0)}(\rho_k)$ and $\phi^{(n)}(\rho_k)$, $k = 1, \ldots, N$, while the operators $\mathcal{M}$ and $\mathcal{L}$ with the $N \times N$ ma-
traces $M_{ij} = \delta_{ij} m(\rho_i)/\rho_i$ and
\[
\Lambda_{ij} = \delta_{ij} \frac{m(\rho_i)}{\rho_i} [U(\rho_i) - U_{\text{Min}}] + \frac{\ln(1+\mu)^2}{2\Delta^2} \times \begin{cases} 
(-\rho_{i-1} - \rho_i) & j = i - 1 \\
(-\rho_{i-1} + 2\rho_i + \rho_{i+1}) & j = i \\
(-\rho_i - \rho_{i+1}) & j = i + 1 \\
0 & \text{otherwise,}
\end{cases} 
\]
where $i,j = 1, \ldots, N$ (no implicit sum on $i$). Likewise, the $n$th approximation to the lowest eigenvalue is calculated as:
\[
\lambda^{(n)}_0 = \frac{\sum_{i,j=1}^N \phi^{(n)}(\rho_i) A_{ij} \phi^{(n)}(\rho_j)}{\sum_{i,j=1}^N \phi^{(n)}(\rho_i) M_{ij} \phi^{(n)}(\rho_j)}, \tag{B8}
\]
To summarize, one starts from a trial array $\phi^{(0)}(\rho_k)$ satisfying the appropriate boundary conditions, repeatedly applies the matrix $\Lambda^{-1} M$ and calculates $\lambda^{(n)}_0$ from equation (B8), until $|(\lambda^{(n)}_0 - \lambda^{(n-1)}_0)/\lambda^{(n)}_0| < \epsilon$ for a given $\epsilon$. The array $\phi^{(n)}(\rho_k)$ is renormalized after every iteration.

The above procedure involves the parameters $N$, $\mu$, and $\epsilon$. $N$ and $\mu$ must be chosen in such a way as to accurately resolve $U_{\text{eff}}(\rho)$, namely the repulsive barrier at $\rho = 1$, the potential well near to the diffusive scale, and the decay of the potential to zero at $\rho = 0$. In particular, the choice of the values of $N$ and $\mu$ requires some care for $\xi$ near to 2, since the effective potential decays to zero slowly in that case. In our simulations we varied $N$ between $5 \times 10^4$ and $10^5$ and $\mu$ between $10^{-5}$ and $5 \times 10^{-2}$ according to the values of $\xi$ and $\varphi$. The threshold for the convergence, $\epsilon$, needs to be sufficiently small, especially for $\xi$ or $\varphi$ close to 1, because the convergence to the theoretical eigenvalue may be rather slow for these values of $\xi$ and $\varphi$. In our simulations $\epsilon$ was varied between $10^{-14}$ and $10^{-8}$.

Appendix C: Structure functions for a finite $R_{\text{em}}$

Following [Monin & Yaglom (1975)], we introduce a finite $R_{\text{em}}$ in the Kazantsev model by assuming that the kinetic-energy spectrum has the form given in equation (25). Such a spectrum can be obtained by choosing the longitudinal and normal three-dimensional kinetic-energy spectra as follows:
\[
F_L(k) = \varphi \frac{A_P}{2\pi L^{5/2}} \frac{k^2}{(k^2 + L^2)^{5/2}}, \tag{C1}
\]
\[
F_N(k) = (1 - \varphi) \frac{A_S}{4\pi L^{5/2}} \frac{k^2}{(k^2 + L^2)^{5/2}}. \tag{C2}
\]
The structure functions corresponding to the above spectra are (Monin & Yaglom [1975], p. 108):
\[
S_L(r) = (1 - \varphi) S_{L}^{(S)}(r) + \varphi S_{L}^{(P)}(r), \tag{C3}
\]
\[
S_N(r) = (1 - \varphi) S_{N}^{(S)}(r) + \varphi S_{N}^{(P)}(r). \tag{C4}
\]
with
\[
S_L^{(S)}(r) = \frac{A_S \sqrt{\pi}}{4(2 + \xi_S)} \left[ \Gamma \left( \frac{\xi_S}{2} \right) - 2^{1-\xi_S} \left( \frac{r}{L} \right)^{\xi_S} K_{\xi_S} \left( \frac{r}{L} \right) \right], \tag{C5}
\]
\[
S_N^{(P)}(r) = \frac{A_P \sqrt{\pi}}{4(2 + \xi_P)} \left[ \Gamma \left( \frac{\xi_P}{2} \right) - 2^{1-\xi_P} \left( \frac{r}{L} \right)^{\xi_P} K_{\xi_P} \left( \frac{r}{L} \right) \right], \tag{C6}
\]
\[
S_N^{(S)}(r) = S_L^{(S)}(r) + \frac{1}{2} r \, dS_L^{(S)}/dr \quad \text{and} \quad S_N^{(P)}(r) = S_L^{(P)}(r) + r \, dS_L^{(P)}/dr \quad \text{[we remind the reader that our definition of the structure functions differ from that of Monin & Yaglom (1975) by a factor of 2].}
\]
For $r \ll L$, $S_L(r)$ and $S_N(r)$ reduce to the functions given in equations (20) and (21) with
\[
D_S = \frac{A_S \Gamma(-1 - \xi_S)}{(3 + \xi_S)L^{5/2}} \cos \frac{\pi \xi_S}{2}, \tag{C7}
\]
\[
D_P = \frac{A_P \Gamma(-1 - \xi_P)}{(3 + \xi_P)L^{5/2}} \cos \frac{\pi \xi_P}{2}. \tag{C8}
\]
The structure functions for the case in which the solenoidal and potential parts of the velocity field have same scaling exponent can be derived from the above expressions by setting $\xi_S = \xi_P = \xi$.

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