Infinite Volume Limit for Correlation functions in the Dipole Gas

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Abstract

We study a classical lattice dipole gas with low activity in dimension $d \geq 3$. We investigate long distance properties by a renormalization group analysis. We prove that various correlation functions have an infinite volume limit. We also get estimates on the decay of correlation functions.

1 Introduction

1.1 Overview

In this paper we continue to study the classical dipole gas on a unit lattice $\mathbb{Z}^d$ with $d \geq 3$. Each dipole is described by its position coordinate $x \in \mathbb{Z}^d$ and a unit polarization vector (moment) $p \in S^{d-1}$.

Let $\{e_1, \ldots, e_d\}$ be the standard basis for $\mathbb{Z}^d$. For $\varphi : \mathbb{Z}^d \to \mathbb{R}$ and $\mu \in \{1, \ldots, d\}$ we define $\partial_\mu \varphi$ as $\partial_\mu \varphi(x) = \varphi(x+e_\mu) - \varphi(x)$. Let $e_{-\mu} = -e_\mu$ with $\mu \in \{1, \ldots, d\}$. Then the definition of $\partial_\mu \varphi$ can be used to define the forward or backward lattice derivative along the unit vector $e_\mu$ with $\mu \in \{\pm 1, \ldots, \pm d\}$.

We have that $\partial_\mu$ and $\partial_{-\mu}$ are adjoint to each other and $-\Delta = \frac{1}{2} \sum_{\pm \mu = 1}^d \partial^*_\mu \partial_\mu = \frac{1}{2} \sum_{\pm \mu = 1}^d \partial_{-\mu} \partial_\mu$.

As in [5], the potential energy between unit dipoles $(x, p_1)$ and $(y, p_2)$ is

$$ (p_1 \cdot \partial)(p_2 \cdot \partial)C(x - y) $$

where $x, y \in \mathbb{Z}^d$ are positions, $p_1, p_2 \in S^{d-1}$ are moments, $\partial = (\partial_1, \ldots, \partial_d)$ and $C(x - y)$ is the Coulomb potential on the unit lattice $\mathbb{Z}^d$, which is the kernel of the inverse Laplacian

$$ C(x, y) = (-\Delta)^{-1}(x, y) = (2\pi)^{-d} \int_{[\pi, \pi]^d} \frac{e^{ip \cdot (x-y)}}{2 \sum_{\mu=1}^d (1 - \cos p_\mu)} dp $$

And the potential energy of $n$ dipoles, including self energy, has the form

$$ \sum_{1 \leq k, j \leq n} (p_k \cdot \partial)(p_j \cdot \partial)C(x_k, x_j) $$

Let $\Lambda_N$ be a box in $\mathbb{R}^d$

$$ \Lambda_N = \left[ -\frac{L^N}{2}, \frac{L^N}{2} \right]^d $$

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1We distinguish forward and backward derivatives to facilitate a symmetric decomposition of $V(\Lambda_N)$ (defined in $\mathbb{R}$) into blocks.
where \( L \geq 2^{d+3} + 1 \) is a very large, odd integer. For \( \Lambda_N \cap \mathbb{Z}^d \), the classical statistical mechanics of a gas of such dipoles with inverse temperature (for convenience) \( \beta = 1 \) and activity (fugacity) \( z > 0 \) is given by the grand canonical partition function

\[
Z_N = \sum_{n \geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \sum_{x_i \in \mathbb{Z}^d \cap \Lambda_N} \int_{S^{d-1}} dp_i \exp \left( -\frac{1}{2} \sum_{1 \leq k, j \leq n} (p_k \cdot \partial)(p_j \cdot \partial)C(x_k, x_j) \right)
\]

All fundamental objects to study, such as the pressure, truncated correlation functions, etc. are derived from \( Z_N \) and similar objects.

The model can be equivalently expressed as a Euclidean field theory (due to Kac [13] and Siegert [16]) and is given by

\[
\theta Z_N \equiv Z_N = \int \exp \left( zW(\Lambda_N, \phi) \right) d\mu_C(\phi)
\]

where

\[
W(\Lambda_N, \phi) = \sum_{x \in \Lambda_N \cap \mathbb{Z}^d} \int_{S^{d-1}} dp \cos(p \cdot \partial \phi(x))
\]

with

- \( dp \) : the standard normalized rotation invariant measure on \( S^{d-1} \).
- The fields \( \phi(x) \) : a family of Gaussian random variables (on some abstract measure space) indexed by \( x \in \mathbb{Z}^d \) with mean zero and covariance \( C(x, y) \) which is a positive definite function as given above.
- The measure \( \mu_C \) : the underlying measure (see section 11.2 [11] and Appendix A [15] for more detail). We discuss about the equivalence of \([7]\) and \([8]\) in appendix [A] of this paper.

For investigating the truncated correlation functions, we consider a more general version of \([6]\):

\[
j Z_N = \int \exp \left( if(\phi) + zW(\Lambda_N, \phi) \right) d\mu_C(\phi)
\]

where \( f(\phi) \) can be:

1. \( f(\phi) = 0 \) as in [5].
2. \( f(\phi) = \sum_{k=1}^{m} t_k \partial_{\mu_k} \phi(x_k) \). We use this \( f(\phi) \) to study the truncated correlation functions

\[
\langle \prod_{k=1}^{m} \partial_{\mu_k} \phi(x_k) \rangle^t = \left. i^m \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \log j Z \right|_{t_1=0, \ldots, t_m=0}
\]

which is nontrivial and previously investigated by Dimock and Hurd [7].

3. \( f(\phi) = \sum_{k=1}^{m} t_k \exp(i\partial_{\mu_k} \phi(x_k)) \) (for studying the dipole correlation

\[
\left. \langle \prod_{k=1}^{m} \exp(i\partial_{\mu_k} \phi(x_k)) \rangle^t \right|_{t_1=0, \ldots, t_m=0}
\]

4. Other general form which will be discussed at the end of this paper. The general form can be applied for truncated correlations of density of the dipoles which also has been studied by Brydges and Keller [2]. We think that this general form has more applications.

Here \( x_k \in \mathbb{Z}^d \) are different points; \( \mu_k \in \{ \pm 1, \ldots, \pm d \} \) and \( t_k \) small and complex; \( m \geq 2 \). For the set \( \{x_1, \ldots, x_m\} \subset \mathbb{Z}^d \), let \( \text{diam}(x_1, \ldots, x_m) = \max_{1 \leq i,j \leq m} \text{dist}(x_i, x_j) \) where \( \text{dist}(x_i, x_j) \) is the distance between \( x_i \) and \( x_j \) on lattice \( \mathbb{Z}^d \).

To get rid of the boundary and study the long distance properties of the system, we would like to take the thermodynamic limit for these quantities, i.e. the limit as \( N \to \infty \) which is so called infinite...
volume limit. Actually $Z_N$ is not expected to have a limit as $N \to \infty$. In [5], Dimock has established an infinite volume limit for the pressure defined by

$$p_N = \left| \Lambda_N \right|^{-1} \log Z_N$$  

(with $|z|$ sufficiently small). Such infinite volume limits have also been obtained by Frohlich and Park [10] and by Frohlich and Spencer [11]. They used a method of correlation inequalities.

In this paper, we continue the study of the long distance properties of the dipole gas model. For long distance (i.e., when $|x - y|$ large), the potential $\partial_\mu \partial_\nu C(x - y)$ behaves like $O(|x - y|^{-d})$, that means it is not integrable and we could not use the theory of the Mayer expansion to establish such results. To overcome this problem, we use the method of the renormalization group.

We follow particularly a Renormalization Group approach recently developed by Brydges and Slade [1] and Dimock [5]. We generalize Dimock’s framework with an external field and obtain some estimates on the correlation functions as in Dimock and Hurd [7], and Brydges and Keller [2]. The main result is the existence of the infinite volume limit for correlations functions, which is new. Earlier work using RG approach to the dipole gas can be found in Gawedski and Kupiainen [12], Brydges and Yau [4].

Besides the dipole gas papers mentioned above, we would like to cite some other papers on the Coulomb gas in $d = 2$ which has a dipole phase. There are the works of Dimock and Hurd [8], Falco [9] and Zhao [17].

1.2 The main result

For our RG approach we follow the analysis of Brydges’ lecture [1]. Instead of (7), we use a different finite volume approximation. First, we add an extra term $(1 - \varepsilon) V(\Lambda_N, \phi)$ where $0 < \varepsilon$ is closed to 1 and

$$V(\Lambda_N, \phi) = \frac{1}{4} \sum_{x \in \Lambda_N \cap \mathbb{Z}^d} \sum_{\pm \mu = 1}^d (\partial_\mu \phi(x))^2$$  

By replacing the covariance $C$ by $\varepsilon^{-1} C$, this extra term will be partially compensated. Hence instead of (7) we will consider a new finite volume generating function

$$f Z_N = f Z_N' / Z_N''$$

where

$$f Z_N' = \int e^{if(\phi)} \exp \left( z W(\Lambda_N, \phi) - (1 - \varepsilon) V(\Lambda_N, \phi) \right) d\mu_{\varepsilon^{-1} C}(\phi)$$

and

$$Z_N'' = \int \exp \left( - (1 - \varepsilon) V(\Lambda_N, \phi) \right) d\mu_{\varepsilon^{-1} C}(\phi)$$

We have

$$f Z_N = f Z_N' / Z_N'' = \int e^{if(\phi)} \exp \left( z W(\Lambda_N, \phi) \right) \left[ (Z_N'')^{-1} \exp \left( - (1 - \varepsilon) V(\Lambda_N, \phi) \right) d\mu_{\varepsilon^{-1} C}(\phi) \right]$$

When $N \to \infty$, $\exp \left( - (1 - \varepsilon) V(\Lambda_N, \phi) \right)$ formally becomes $\exp \left( 1/2(1 - \varepsilon)(\phi, -\Delta \phi) \right)$, and $d\mu_{\varepsilon^{-1} C}(\phi) = (1/2(\varepsilon)(\phi, -\Delta \phi)) d\phi$. So the bracketed expression formally converges to (const.) $1/2(\phi, -\Delta \phi) d\phi = (\text{const.}) d\mu_{C}(\phi)$ when $N \to \infty$. Formally this new $f Z_N$ gives the same limit as (7). This result holds for any choice of $\varepsilon$. By definition (9), the extra term $(1 - \varepsilon)V(\Lambda_N, \phi) = (1 - \varepsilon)\frac{1}{4} \sum_{x \in \Lambda_N \cap \mathbb{Z}^d} \sum_{\pm \mu = 1}^d (\partial_\mu \phi(x))^2$. 

3
Therefore the choice of $\varepsilon$ is a choice of how much $(\partial \phi)^2$ one is putting in the interaction and how much in the measure.

Similarly to the Theorem 1 in [5], our main theorems are:

**Theorem 1** For $|z|$ and $\max_k |t_k|$ sufficiently small there is an $\varepsilon = \varepsilon(z)$ close to 1 so that

\[
\int f^{N} = |\Lambda_{N}|^{-1} \log (f^{N} Z_{N}) \text{ has a limit as } N \to \infty
\]

Using $f(\phi) = \sum_{k=1}^{m} t_k \partial_{\mu_k} \phi(x_k)$, we achieve some estimate for the correlation functions:

**Theorem 2** For any small $\varepsilon > 0$, with $L, A$ sufficiently large (depending on $\varepsilon$), $\eta = \min\{d/2, 2\}$, we have:

\[
\left| \langle \prod_{k=1}^{m} \partial_{\mu_k} \phi(x_k) \rangle \right| \leq \frac{m^t}{a_m} \text{diam}^{-\eta+\varepsilon}(x_1, \ldots, x_m)
\]  

(14)

where $a$ depends on $\varepsilon, L, A$.

And we also can obtain the existence of infinite volume limit for correlation functions.

**Theorem 3** With $L, A$ sufficiently large, the infinite volume limit of truncated correlation function

\[
\lim_{N \to \infty} \langle \prod_{k=1}^{m} \partial_{\mu_k} \phi(x_k) \rangle^t
\]

exists when $d = 3$ or 4, the result in Theorem 2 looks like the result in [7], but here it is obtained with the new method.

Using $f(\phi) = \sum_{k=1}^{m} t_k \exp (i \partial_{\mu_k} \phi(x_k))$, we can obtain Theorems [10] and [11] which are similar to Theorems [2] and [3] just with different $f$.

At the end of this paper, we investigate a general form of $f$ and obtain Theorems [12] and [13].

Applying theorem [12] with a special $f$ for density of dipoles, we have obtained some estimates for truncated correlation functions of density of dipoles with $(m \geq 2)$ points (Corollary [1]) instead of only 2 points as Theorem 1.1.2 in [2]. Then we apply theorem [13] to establish the infinite volume limit for truncated correlation functions of density of dipoles (Corollary [2]).

For the proof of Theorem [1] we will show that, with a suitable choice of $\varepsilon = \varepsilon(z)$, the density $\exp (zW - (1 - \varepsilon)V)$ likely goes to zero under the renormalization group flow and leaves a measure like $\mu_{z}(z)^{-1} C$ to describe the long distance behavior of the system. Accordingly $\varepsilon(z)$ can be interpreted as a dielectric constant.

Now we rewrite the generating function $f Z_{N}$. First we scale $\phi \to \phi / \sqrt{z}$ and then let $\sigma = \varepsilon^{-1} - 1$. Because $\varepsilon$ is closed to 1, we have $\sigma$ is near zero. We also have

\[
\begin{align*}
Z_{N}'(z, \sigma) & = \int e^{i f(\phi)} \exp \left( z W(\Lambda_{N}, \sqrt{1 + \sigma \phi}) - \sigma V(\Lambda_{N}, \phi) \right) d\mu_{C}(\phi) \\
Z_{N}''(\sigma) & = \int \exp \left( - \sigma V(\Lambda_{N}, \phi) \right) d\mu_{C}(\phi) \\
f Z_{N}(z, \sigma) & = f Z_{N}'(z, \sigma)/Z_{N}''(\sigma)
\end{align*}
\]

Then we need to show that with $|z|$ sufficiently small there is a (smooth) $\sigma = \sigma(z)$ near zero such that,

\[
|\Lambda_{N}|^{-1} \log f Z_{N}(z, \sigma(z)) = |\Lambda_{N}|^{-1} \log f Z_{N}'(z, \sigma(z)) - |\Lambda_{N}|^{-1} \log Z_{N}''(\sigma(z))
\]

(16)

has a limit when $N \to \infty$. And theorem [1] is proved just by putting $\varepsilon(z) = (1 + \sigma(z))^{-1}$ back. Dimock has proved that, for small real $\sigma$ with $|\sigma| < 1$, we have $|\Lambda_{N}|^{-1} \log Z_{N}''(\sigma)$ converges as $N \to \infty$ (Theorem 2, [3]). Hence we only need to investigate the first term in (15).

The paper is organized as follows:

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In Theorem [1] $f(\phi)$ can be $0$, $\sum_{k=1}^{m} t_k \partial_{\mu_k} \phi(x_k)$, or $\sum_{k=1}^{m} t_k \exp (i \partial_{\mu_k} \phi(x_k))$. 

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4
In section 2 we give some general definitions on the lattice and its properties. We also give definitions about the norms we use together with their crucial properties and estimates. Then we define the basic Renormalization Group transformation as in [5].

In section 3 we accomplish the detailed analysis of the Renormalization Group transformation to isolate the leading terms. Then we simplify them for the next scale.

In section 8 we study the RG flow and find the stable manifold \( \sigma = \sigma(z) \).

In section 5 we assemble the results and prove the infinite volume limit for \( |\Lambda_N|^{-1} \log f Z_N' \) exists.

Finally in section 6, by combining all the other estimates, we obtain some estimates for correlation functions and establish the infinite volume limit of correlation functions.

2 Preliminaries

In this section, we quote all notations and basic result from Dimock [5]. At the same time, we introduce some new notations which are useful for this paper.

2.1 Multiscale decomposition

RG methods are based upon a multiscale decomposition of the basic lattice covariance \( C \) into a sequence of more controllable integrals and analyze the effects separately at each stage. Especially we choose a decomposition into finite range covariances which is developed by Brydges, Guadagni, and Mitter [3]. The decomposition of the lattice covariance \( C \) has the form

\[
C(x - y) = \sum_{j=1}^{\infty} \Gamma_j(x - y)
\]

such that

- \( \Gamma_j(x) \) is defined on \( \mathbb{Z}^d \), is positive semi-definite, and satisfies the finite range property: \( \Gamma_j(x) = 0 \) if \( |x| \geq L_j/2 \).
- There is a constant \( c_0 \) independent of \( L \) such that, for all \( j, x \), we have
  \[
  |\Gamma_j(x)| \leq c_0 L^{-(j-1)(d-2)}
  \]
  This implies that the series converges uniformly.
- There are constants \( c_\alpha \) independent of \( L \) such that
  \[
  |\partial^\alpha \Gamma_j(x)| \leq c_\alpha L^{-(j-1)(d-2)+|\alpha|}
  \]
  where \( \partial^\alpha = \prod_{\mu=1}^d \partial_{\mu}^\alpha \) is a multi-derivative and \( |\alpha| = \sum_\mu |\alpha_\mu| \). Thus the differentiated series converges uniformly to \( \partial^\alpha C \).
- (Lemma 2, [5]) There are some constants \( C_{L,\alpha} \) such that
  \[
  |\partial^\alpha C(x)| \leq C_{L,\alpha} (1 + |x|)^{-d+2-|\alpha|}
  \]
  For our RG analysis we need to break off pieces of covariance \( C(x - y) \) one at a time. So we define
  \[
  C_k(x - y) = \sum_{j=k+1}^{\infty} \Gamma_j(x - y)
  \]
  Hence we have \( C = C_0 \) and
  \[
  C_k(x - y) = C_{k+1}(x - y) + \Gamma_{k+1}(x - y)
  \]
2.2 Renormalization Group Transformation

The generating function (15) can be rewritten as

\[ f Z'_N(z, \sigma) = \int f Z'_0(\phi)d\mu_{C_0}(\phi) \]  

(23)

with

\[ f Z'_0(\phi) = e^{if(\phi)} \exp \left( zW(\Lambda_N, \sqrt{1 + \sigma \phi}) - \sigma V(\Lambda_N, \phi) \right) \]  

(24)

We use the left subscript \( f \) as an extra notation for 3 cases at the same time:

- \( f(\phi) = 0 \) as in (Dimock, [5]);
- \( f(\phi) = \sum_{k=1}^{m} t_k \partial_{\mu_k} \phi(x_k) \);
- \( f(\phi) = \sum_{k=1}^{m} t_k \exp (i \partial_{\mu_k} \phi(x_k)) \).

Since \( C_0 = C_1 + \Gamma_1 \) we replace an integral over \( \mu_{C_0} \) by an integral over \( \mu_{\Gamma_1} \) and \( \mu_{C_1} \). So we have

\[ f Z'_N(z, \sigma) = \int f Z'_0(\phi + \zeta)d\mu_{\Gamma_1}(\zeta)d\mu_{C_1}(\phi) = \int f Z'_1(\phi)d\mu_{C_1}(\phi) \]  

(25)

We define a new density by the fluctuation integral

\[ f Z'_1(\phi) = (\mu_{\Gamma_1} * f Z'_0)(\phi) = \int f Z'_0(\phi + \zeta)d\mu_{\Gamma_1}(\zeta) \]  

(26)

Because \( \Gamma_1, C_1 \) are only positive semi-definite, these are degenerate Gaussian measures\(^3\). By continuing this way, we will have the representation for \( j = 0, 1, 2, \ldots \)

\[ f Z'_j(z, \sigma) = \int f Z'_j(\phi)d\mu_{C_j}(\phi) \]  

(27)

here the density \( f Z'_j(\phi) \) is defined by

\[ f Z'_j(\phi) = (\mu_{\Gamma_{j+1}} * f Z'_j)(\phi) = \int f Z'_j(\phi + \zeta)d\mu_{\Gamma_{j+1}}(\zeta) \]  

(28)

Our job is to investigate the growth of these densities when \( j \) go to \( \infty \).

2.3 Local expansion

We will rewrite each density \( f Z'_j(\phi) \) in a form which presents its locality properties known as a polymer representation. The localization becomes coarser when \( j \) gets bigger. First we will give some basic definitions on the lattice \( \mathbb{Z}^d \).

2.3.1 Basic definitions on the lattice \( \mathbb{Z}^d \)

For \( j = 0, 1, 2, \ldots \) we partition \( \mathbb{Z}^d \) into \( j \)-blocks \( B \). These blocks have side \( L^j \) and are translates of the center \( j \)-blocks

\[ B_j^0 = \{ x \in \mathbb{Z}^d : |x| < 1/2(L^j - 1) \} \]

(29)

by points in the lattice \( L^j \mathbb{Z}^d \). The set of all \( j \)-blocks in \( \Lambda = \Lambda_N \) is denoted \( B_j(\Lambda_N) \), \( B_j(\Lambda) \) or just \( B_j \). A union of \( j \)-blocks \( X \) is called a \( j \)-polymer. Note that \( \Lambda \) is also a \( j \)-polymer for \( 0 \leq j \leq N \). The set of

\(^3\)Dimock has discussed these in Appendix A, [5].
all \( j \)-polymers in \( \Lambda = \Lambda_N \) is denoted \( \mathcal{P}_j(\Lambda) \) or just \( \mathcal{P}_j \). The set of all connected \( j \)-polymers is denoted by \( \mathcal{P}_j^c \). Let \( X \in \mathcal{P}_j \), the closure \( \overline{X} \) is the smallest \( Y \in \mathcal{P}_{j+1} \) such that \( X \subseteq Y \).

For a \( j \)-polymer \( X \), let \( |X|_j \) be the number of \( j \)-blocks in \( X \). We call \( j \)-polymer \( X \) a small set if it is connected and contains no more than \( 2^d \) \( j \)-blocks. The set of all small set \( j \)-polymers in \( \Lambda \) is denoted by \( S_j(\Lambda) \) or just \( S_j \). A \( j \)-block \( B \) has a small set neighborhood \( B^* = \bigcup \{ Y \in S_j : Y \supset B \} \).

**Note:** If \( B_1, B_2 \) are \( j \)-blocks and \( B_2 \in B_1^* \) then, using above definition, we also have that \( B_1 \in B_2^* \). Similarly a \( j \)-polymer \( X \) has a small set neighborhood \( X^* \).

For \( l \geq 1 \) and integer \( d \), we define some constants \( n_1(d), n_2(d), n_3(d, l) \) which are bounded and, for every \( j \geq 0 \), we have:

\[
\begin{align*}
    n_1(d) &\equiv \sum_{x \in S_0 \cap X \supseteq 0} 1/|X|_0 = \sum_{x \in S_j \cap X \supset B_0^j} 1/|X|_j \\
    n_2(d) &\equiv \sum_{x \in S_0 \cap X \supseteq 0} 1 = \sum_{x \in S_j \cap X \supset B_0^j} 1 \\
    n_3(d, l) &\equiv \sum_{x \in S_0 \cap X \supseteq 0} l^{-|X|_0} = \sum_{x \in S_j \cap X \supset B_0^j} l^{-|X|_j} \\
    n_3(d, l) &\leq n_3(d, 1) = n_1(d) \leq n_2(d) \leq (2^d)!/(2d)^{2d}
\end{align*}
\]

Furthermore, with a fixed \( d \), we can get

\[
0 \leq \lim_{l \to \infty} n_3(d, l) \leq \lim_{l \to \infty} \frac{n_1(d)}{l} = 0
\]

### 2.3.2 Local expansion

Using the same approach as in \[5\], we rewrite the density \( fZ_j(\phi) \) for \( \phi : \mathbb{Z}^d \to \mathbb{R} \) in the the general form

\[
fZ = (fI \circ fK)(\Lambda) = \sum_{X \in \mathcal{P}_j(\Lambda)} fI(\Lambda - X)fK(X)
\]

Here \( fI(Y) \) is a background functional which is explicitly known and carries the main contribution to the density. The \( fK(X) \) is so called a polymer activity. It represents small corrections to the background.

In section \[5\] we will show that the initial density \( fI_0 \) has the factor property. We want to keep this factor property at all scales. Then we can use the analysis of Brydges’ lecture \[1\]. Therefore we assume \( fI(Y) \) always is in the form of

\[
fI(Y) = \prod_{B \in B_j : B \subseteq Y} fI(B)
\]

and \( fI(B, \phi) \) depends on \( \phi \) only \( B^* \), the small set neighborhood of \( B \). Moreover we assume \( fK(X) \) factors over the connected components \( C(X) \) of \( X \)

\[
fK(X) = \prod_{Y \subseteq C(X)} fK(Y)
\]

and that \( fK(X, \phi) \) only depends on \( \phi \) in \( X^* \).

As in \[5\], the background functional \( fI(B) \) has a special form: \( fI(fE, \sigma, B) = \exp(-V(fE, \sigma, B)) \) where

\[
V(fE, \sigma, B, \phi) = fE(B) + \frac{1}{4} \sum_{x \in B} \sum_{\mu \nu} \sigma_{\mu \nu}(B) \partial_{\mu} \phi(x) \partial_{\nu} \phi(x)
\]

\[\text{Sums over } \mu \text{ are understood to range over } \mu = \pm 1, \ldots, \pm d, \text{ unless otherwise specified.} \]
for some functions \( f, \sigma_{\mu\nu} : B_j \rightarrow \mathbb{R} \). Indeed we usually can take \( \sigma_{\mu\nu}(B) = \sigma\delta_{\mu\nu} \) for some constant \( \sigma \). Then \( V(f, \sigma, B, \phi) \) becomes
\[
V(f, \sigma, B, \phi) = fE(B) + \frac{\sigma}{4} \sum_{x \in B} \sum_{\mu}(\partial_\mu \phi(x))^2 \equiv fE(B) + \sigma V(B) \tag{36}
\]

Also in our model, when \( f = 0 \), we will have
\[
oK(X, \phi) = oK(X, -\phi) \quad oK(X, \phi) = oK(X, \phi + c) \tag{37}
\]
The later holds for any constant \( c \) which means that \( oK(X, \phi) \) only depends on derivatives \( \partial \phi \).

### 2.4 About norms and their properties

In this paper we use exactly the same norms and notations as in Dimock [5]. Now we consider potential \( V(s, B, \phi) \) of the form
\[
V(s, B, \phi) = \frac{1}{4} \sum_{x \in B} \sum_{\mu\nu} s_{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) \tag{38}
\]
here the norms of functions \( s_{\mu\nu}(x) \) are defined by
\[
\|s\|_{j, B} = \sup_{B \in B_j} |B|^{-1} \|s\|_{1, B} = \sup_{B \in B_j} L^{-dj} \sum_{\mu\nu} \sum_{x \in B} |s_{\mu\nu}(x)| \tag{39}
\]
If \( s_{\mu\nu}(x) = \sigma\delta_{\mu\nu} \) then \( V(s, B) = \sigma V(B) \) as defined in (36) and the norm \( \|s\|_j = 2d \sigma \).

The following lemmas are some results from Section 3 in [5]:

**Lemma 1** (Lemma 3, [5])

1. For any functional \( s_{\mu\nu}(x) \), we have
   \[
   \|V(s, B)\|_{s,j} \leq \hbar^2 \|s\|_j \tag{40}
   \]
   \[
   \|V(s, B)\|_{s,j} \leq \hbar^2 \|s\|_j \tag{40}
   \]

2. The function \( \sigma \rightarrow \exp(-\sigma V(B)) \) is complex analytic and if \( \hbar^2 \sigma \) is sufficiently small, we have
   \[
   \|e^{-\sigma V(B)}\|_{s,j} \leq 2 \tag{41}
   \]
   \[
   \|e^{-\sigma V(B)}\|_{s,j} \leq 2 \tag{41}
   \]

Let \( c \) be a constant such that the function \( \sigma \rightarrow \exp(-\sigma V(B)) \) is analytic in \( |\sigma| \leq ch^{-2} \) and satisfies \( \|\exp(-\sigma V(B))\|_{s,j} \leq 2 \) on that domain.

To start the RG transformation, we also need some estimate on the initial interaction. When \( j = 0, B \in B_0 \) is just a single site \( x \in \mathbb{Z}^d \), so we consider
\[
W(u, B, \phi) = \int_{\mathbb{R}^{d-1}} dp \cos(p \cdot \partial \phi(x) u) \tag{42}
\]

**Lemma 2** (Lemma 4, [5])

1. \( W(u, B) \) is bounded by
   \[
   \|W(u, B)\|_{s,0} \leq 2e^{\sqrt{d}\hbar u} \tag{43}
   \]
   We also have that \( W(u, B) \) is strongly continuously differentiable in \( u \).

2. \( e^{zW(u, B)} \) is complex analytic in \( z \) and satisfies, for \( |z| \) sufficiently small (depending on \( d, h, u \)), we have
   \[
   \|e^{zW(u, B)}\|_{s,0} \leq 2 \tag{44}
   \]
   And \( e^{zW(u, B)} \) is also strongly continuously differentiable in \( u \).
3 Analysis of the RG Transformation

Now we use the Brydges-Slade RG analysis and follow the framework of Dimock [5], but with an external field $f$.

3.1 Coordinates $(fI_j, fK'_j)$

Continuing to the subsection 2.3.2 (Local Expansion), we suppose that we have $fZ(\phi) = (fI \circ fK)(\Lambda, \phi)$ with polymers on scale $j$. We rewrite it as

$$fZ'(\phi') = (\mu_{j+1} \ast fZ)(\phi') \equiv \int fZ(\phi' + \zeta)d\mu_{j+1}(\zeta)$$

(45)

here we try to put it back to the form

$$fZ'(\phi') = (fI' \circ fK')(\Lambda, \phi')$$

(46)

where the polymers are now on scale $(j + 1)$. Furthermore, supposed that we have chosen $fI'$, we will find $fK'$ so the identity holds. As explained before, our choice of $fI'$ is to have the form

$$fI'(B', \phi') = \prod_{B \in B_j, B \subset B'} fI(B, \phi')$$

(47)

Now we define

$$\delta fI(B, \phi', \zeta) = fI(B, \phi' + \zeta) - \tilde{f}I(B, \phi')$$

$$fK \circ \delta fI \equiv \tilde{f}K(X, \phi', \zeta) = \sum_{Y \subset X} fK(Y, \phi' + \zeta)\delta fI^{X-Y}(\phi', \zeta)$$

(48)

For connected $X$ we write $\tilde{f}K(X, \phi', \zeta)$ in the form

$$\tilde{f}K(X, \phi', \zeta) = \sum_{B \subset X} fJ(B, X, \phi') + \tilde{f}K(X, \phi', \zeta)$$

(49)

Given $fK$ and $fJ$ the equation (49) would give us a definition of $\tilde{f}K(X)$ for $X$ connected. And for any $X \in P_j$, we define

$$\tilde{f}K(X, \phi', \zeta) = \prod_{Y \in C(X)} \tilde{f}K(Y, \phi', \zeta)$$

(50)

After using the finite range property and making some rearrangements as Proposition 5.1, Brydges [1], we have (51) holds with

$$fK'(U, \phi') = \sum_{X, \chi \to U} fJ^\chi(\phi') fJ^{U-(X \cup X)}(\phi') f\tilde{K}^\#(X, \phi') \quad U \in P_{j+1}$$

(51)

where $\chi = (B_1, X_1, \ldots B_n, X_n)$ and the condition $X, \chi \to U$ means that $X_1, \ldots X_n, X$ be strictly disjoint and satisfy $(B_1^1 \cup \cdots \cup B_n^1 \cup X) = U$. Moreover

$$fJ^\chi(\phi') = \prod_{i=1}^n fJ(B_i, X_i, \phi')$$

$$fJ^{U-(X \cup X)}(\phi') = \prod_{B \in U-(X \cup X)} fJ(B, \phi')$$

(52)

As in (Dimock, [3]), $fJ(B, X)$ will be chosen to depend on $fK$ and required $fJ(B, X) = 0$ unless $X \in S_j, B \subset X$ and that $fJ(B, X, \phi')$ depend on $\phi'$ only in $B^*$. 




3.2 Choosing $J$

For a smooth function $g$ with $X = \cup \mathcal{X}_i$. And $fK^\#(X, \phi')$ is $fK(X, \phi', \zeta)$ integrated over $\zeta$ as (78) in [5].

At this point $fK'$ is considered as a function of $fI, fI, fJ, fK$. It vanishes at the point $(fI, fI, fJ, fK) = (1, 1, 0, 0)$ since $\chi = 0$ and $X = \emptyset$ if $U = \emptyset$. We study its behavior in a neighborhood of this point. We have the norm on $fK$ as (75) in [5] and we define

\[
\|fI\|_{s,j} = \sup_{B \in B_j} \|fI(B)\|_{s,j}
\]

\[
\|fI\|_{s,j}' = \sup_{B \in B_j} \|fI(B)\|_{s,j}
\]

\[
\|fJ\|_j' = \sup_{X \in S_j, B \subset X} \|fJ(X, B)\|_j'
\]

We also set

\[
\delta_fK = fK - 0K
\]

Using the same argument as Theorem 3 in [5], we have the following result.

**Theorem 4** Let $A$ be sufficiently large.

1. For $R > 0$ there is a $r > 0$ such that the following holds for all $j$. If $\|fI - 1\|_{s,j} < r$, $\|fI - 1\|_{s,j}' < r$, $\max\{\|fJ\|_j', \|0J\|_j'\} < r$ and $\max\{\|fK\|_j, \|0K\|_j\} < r$ then $\max\{\|fK'\|_{j+1}, \|0K'\|_{j+1}\} < R$. Furthermore $fK'$ is a smooth function of $fI, fI, fJ, fK$ on this domain with derivatives bounded uniformly in $j$. The analyticity of $fK'$ in $t_1, \ldots, t_m$ still holds when we go from $j$-scale to $(j + 1)$-scale.

2. If also

\[
\sum_{X \in S_j, X \supset B} fJ(B, X) = 0
\]

then the linearization of $fK' = fK'(fI, fI, fJ, fK)$ at $(fI, fI, fJ, fK) = (1, 1, 0, 0)$ is

\[
\sum_{X \in \mathcal{P}_j, \mathcal{X} = U} \left( fK^\#(X) + (fI^\#(X) - 1)1_{X \in B_j} - (fI(X) - 1)1_{X \in B_j} - \sum_{B \subset X} fJ(B, X) \right)
\]

where

\[
fK^\#(X, \phi) = \int fK(X, \phi + \zeta)d\mu_{j+1}(\zeta)
\]

and $0J$ actually is $fJ$ at $f = 0$.

### 3.2 Choosing $J$ and Estimating $L_1, L_2$

#### 3.2.1 Choosing $J$

For a smooth function $g(\phi)$ on $\phi \in \mathbb{R}^A$ let $T_2g$ denote a second order Taylor expansion:

\[
(T_2g)(\phi) = g(0) + g_1(0; \phi) + \frac{1}{2}g_2(0; \phi, \phi)
\]

\[
(T_0g)(\phi) = g(0)
\]

With $fK^\#$ defined in (57), for $X \in S_j$, $X \supset B$, $X \neq B$, we pick:

\[
fJ(B, X) = \frac{1}{|X_j|} \left[ T_2(fK^\#(X)) + T_0(fK^\#(X)) - T_0(0K^\#(X)) \right]
\]

\[
= \frac{1}{|X_j|} \left[ T_2(fK^\#(X)) + T_0(\delta fK^\#(X)) \right]
\]
and choose \( f_J(B, B) \) so that (55) is satisfied. Otherwise, we let \( f_J(B, X) = 0 \).

As in (56) we have picked
\[
\begin{align*}
I(B) &= f_I(f_E, \sigma, B) = \exp(-V(f_E, \sigma, B)) \\
J(B) &= f_J(f_E, \sigma, B) = \exp(-V(f_E, \sigma, B))
\end{align*}
\]

So we require \( \tilde{I} \) to have the same form
\[
\begin{align*}
\tilde{I}(B) &= f_I(f_{\tilde{E}}, \tilde{\sigma}, B) = \exp(-V(f_{\tilde{E}}, \tilde{\sigma}, B)) \\
\text{with } f_{\tilde{E}}, \tilde{\sigma} \text{ which will be defined later.}
\end{align*}
\]

Because \( \sum_{B < B'} V(B) = V(B') \), we have
\[
\begin{align*}
\tilde{I}'(B') &= f_I(f_{\tilde{E}'}', \sigma', B') = \exp(-V(f_{\tilde{E}'}', \sigma', B'))
\end{align*}
\]

with
\[
\begin{align*}
f_{\tilde{E}'}(B') &= \sum_{B < B'} f_{\tilde{E}}(B) \\
\sigma' &= \tilde{\sigma}
\end{align*}
\]

The map \( f_K' \) becomes \( f_K' = f_K'(f_{\tilde{E}}, \tilde{\sigma}, f_E, \sigma, f_{\tilde{E}}, 0K) \). We use the standard norm on the energy
\[
\|f_E\| = \sup_{B \in B_j} |f_E(B)|
\]

And the theorem becomes:

**Theorem 5** Let \( A \) be sufficiently large.

1. For \( R > 0 \) there is a \( r > 0 \) such that the following holds for all \( J \). If \( \|f_{\tilde{E}}\|, |\tilde{\sigma}|, \|f_E\|, |\sigma|, \max\{\|f_K\|, \|f_{\tilde{K}}\|\} < r \) then \( \max\{\|f_K'\|, \|f_{\tilde{K}}'\|\} < R \). Furthermore \( f_K' \) is a smooth function of \( f_E, \tilde{\sigma}, f_E, \sigma, f_{\tilde{E}}, 0K \) on this domain with derivatives bounded uniformly in \( J \). The analyticity of \( f_K' \) in \( t_1, \ldots, t_m \) still holds when we go from \( J \)-scale to \((J + 1)\)-scale.

2. The linearization of \( f_K' \) at the origin has the form
\[
\begin{align*}
\mathcal{L}_1(f_K) + \mathcal{L}_2(f_K) + \mathcal{L}_3(f_E, \sigma, f_{\tilde{E}}, \tilde{\sigma}, f_{\tilde{K}}, 0K)
\end{align*}
\]

where
\[
\begin{align*}
\mathcal{L}_1(f_K)(U) &= \sum_{X \in S_j: \nabla X = U} f_K^\#(X) \\
&= \sum_{X \in S_j: \nabla X = U} oK^\#(X) + \sum_{X \in S_j: \nabla X = U} \delta f_K^\#(X) \\
&= \mathcal{L}_1(oK)(U) + \mathcal{L}_1(\delta f_K)(U)
\end{align*}
\]
\[
\begin{align*}
\mathcal{L}_2(f_K)(U) &= \sum_{X \in S_j: \nabla X = U} (f_K^\#(X) - [T_2(oK^\#(X)) + T_0(\delta f_K^\#(X))]) \\
&= \sum_{X \in S_j: \nabla X = U} (I - T_2)(oK^\#(X)) + \sum_{X \in S_j: \nabla X = U} (I - T_0)(\delta f_K^\#(X)) \\
&= \mathcal{L}_2(oK)(U) + \mathcal{L}_2(\delta f_K)(U)
\end{align*}
\]
\[
\begin{align*}
\mathcal{L}_3(f, \sigma, f_{\tilde{E}}, \tilde{\sigma}, f_{\tilde{K}}, 0K)(U) &= \sum_{B \in U} \left( V(f_{\tilde{E}}, \tilde{\sigma}, B) - V^\#(f_E, \sigma, B) \right) \\
&+ \sum_{B \in U} \sum_{X \in S_j: \nabla X = U} \frac{1}{|X|} [T_2(oK^\#(X)) + T_0(\delta f_K^\#(X))]
\end{align*}
\]
Proof. The new map actually is the composition of the map \( fK' = fK'(fI, fJ, fJ, fK) \) of theorem\(^{[3]}\) with the maps \( fI = fI(fE, \sigma), fI = fI(fE, \tilde{\sigma}), fJ = fJ(fJ, J, fK) \). Thus it suffices to establish uniform bounds and smoothness for the latter.

In the case \( f = 0 \), we’ve already have the proof in Theorem 4, Dimock\(^{[5]}\). So we only consider the case \( f \neq 0 \).

For \( fI = fI(fE, \sigma), fI = fI(fE, \tilde{\sigma}) \) the proof is the same as the proof for \( I, J' \) in (Theorem 4, \(^{[5]}\)). We have:

\[
fJ(B, X) = \frac{1}{|X_j|} [T_2(0K^#(X)) + T_0(fK^#(X)) - T_0(0K^#(X))] = \frac{1}{|X_j|} [(T_2 - T_0)(0K^#(X)) + T_0(fK^#(X))]
\]

With the same argument as in (Theorem 4, \(^{[5]}\)), we obtain:

\[
\|(T_2 - T_0)(0K^#(X))\|_j' \leq \mathcal{O}(1)\|0K^#(X)\|_j'
\]

\[
\|T_0(fK^#(X))\|_j' \leq \mathcal{O}(1)\|fK^#(X)\|_j'
\]

By (79) in \(^{[5]}\) these are bounded by \( \mathcal{O}(1)||fK||_j + ||0K||_j \). The same bound holds for \( ||fJ(B, B)||_j' \).

The linearization is just a computation. Indeed \( fJ(B, X) \) is designed so that

\[
\sum_{X \in J, X \subseteq X} \left( fK^#(X) - \sum_{B \subseteq X} fJ(B, X) \right)
\]

\[
= \sum_{B \subseteq X} \sum_{X \in J, X \subseteq B} \frac{1}{|X_j|} [T_2(0K^#(X)) + T_0(\delta fK^#(X))] + \sum_{X \in J, X \subseteq U} fK^#(X) - [T_2(0K^#(X)) + T_0(\delta fK^#(X))]
\]

which accounts for the presence of these terms. Also the linearization of \((fI^#(B) - 1)\) is \(-V^#(fE, \sigma, B)\), and so forth. This completes the proof.

3.2.2 Estimating \( L_1, L_2 \) - the first two linearization parts

Next we make some estimates on the linearization’s parts. First we estimate \( L_1 \) which is the linearization on the large \( j \)-polymers

Lemma 3 Let \( A \) be sufficiently large depending on \( L \). Then the operator \( L_1 \) is a contraction with a norm which goes to zero as \( A \to \infty \).

Proof. We use the same proof as in (Dimock, \(^{[5]}\), Lemma 5), but with updated notations. We estimate by (79) and (80) in \(^{[5]}\)

\[
\|L_1(0K)(U)||_{j+1} \leq \|L_1(0K)(U)||_j' \leq \sum_{X \in J, X \subseteq U} \|0K^#(X)\|_j'
\]

\[
\leq \sum_{X \in J, X \subseteq U} (A/2)^{-|X_j|} ||0K||_j
\]

12
and
\[
\|L_1(\delta_f K)(U)\|_{j+1} \leq \|L_1(\delta_f K)(U)\|_j' \leq \sum_{x \not\in S_j, X = U} \|\delta_f K^#(X)\|_j'
\leq \sum_{x \not\in S_j, X = U} (A/2)^{-|X|_j} \|\delta_f K\|_j
\]

(71)

Multiplying by \(A^{[U]_{j+1}}\) then taking the supremum over \(U\), these yield
\[
\|L_1(0K)\|_{j+1} \leq \left[ \sup_U A^{[U]_{j+1}} \sum_{x \not\in S_j, X = U} (A/2)^{-|X|_j} \right] \|0K\|_j
\]
\[
\|L_1(\delta_f K)\|_{j+1} \leq \left[ \sup_U A^{[U]_{j+1}} \sum_{x \not\in S_j, X = U} (A/2)^{-|X|_j} \right] \|\delta_f K\|_j
\]

(72)

Using lemma 6.18 [1], the bracketed expression goes to zero as \(A \to \infty\). And we also have
\[
\|L_1(fK)\|_{j+1} \leq \|L_1(0K)\|_{j+1} + \|L_1(\delta_f K)\|_{j+1}
\]
\[
\|\delta_f K\|_j \leq \|0K\|_j + \|fK\|_j
\]

Hence for \(A\) sufficiently large \(\|L_1(fK)\|_{j+1}\) is arbitrarily small. The idea of lemma 6.18 [1] is that, for large polymers \(X\) such that \(X = U\), the quantity \(|X|_j\) must be much larger than \(|U|_{j+1}\).

(Q.E.D)

Now we estimate and find an explicit upper bound for \(L_2\)

**Lemma 4** Let \(L\) be sufficiently large . Then the operator \(L_2\) is a contraction with a norm which goes to zero as \(L \to \infty\).

**Proof.** For \(f = 0\), we have the Lemma 6, in [5].

For \(f \neq 0\), we can write
\[
L_2(fK)(U) = \sum_{X \in S_j, X = U} \{ (I - T_2)0K^#(X) + (I - T_0)\delta_f K^#(X) \}
\]
\[
= L_2(0K)(U) + L_2(\delta_f K)(U)
\]

where
\[
L_2(\delta_f K)(U) = \sum_{X \in S_j, X = U} (I - T_0)\delta_f K^#(X)
\]

(75)

Using (60, 6.40) as well as (55, Lemma 6) we get:
\[
\| (I - T_2)_0K^#(X, \phi) \|_{j+1} \leq (1 + \| \phi \|_{\Phi_{j+1}(X^*)})^3 \|0K^#_3(X, \phi)\|_{j+1}
\leq 4 \left( 1 + \| \phi \|_{\Phi_{j+1}(X^*)}^3 \right) \|0K^#_3(X, \phi)\|_{j+1}
\]
\[
\| (I - T_0)\delta_f K^#(X, \phi) \|_{j+1} \leq (1 + \| \phi \|_{\Phi_{j+1}(X^*)}) \|\delta_f K^#_1(X, \phi)\|_{j+1}
\]

(76)

Notice that \(\delta_f K^#(X, 0) = fK^#(X, 0) - 0K^#(X, 0)\) only depend on \(\phi\) in \(X^*\). Moreover, \(0K\) and \(fK\) are different only on \(\text{supp}(f) = \{x_1, ..., x_m\}\). So, if \(X^* \cap \{x_1, x_2, ..., x_m\} = \emptyset\) then \(fK^#(X, 0) = 0K^#(X, 0)\) which means \(\delta_f K^#(X, 0) = 0\). Therefore \(\delta_f K^#(X, 0) = 0\) unless \(X^* \cap \{x_1, x_2, ..., x_m\} \neq \emptyset\)
Using property (64) in \([5]\), we have

\[
\|_0 K^\#(X, \phi)\|_{j+1} \leq L^{-3d/2}\|_0 K^\#(X, \phi)\|_j
\leq 6(L^{-3d/2})\|_0 K^\#(X, \phi)\|_j
\leq 6(L^{-3d/2})\|_0 K^\#(X)\|_j G_j(X, \phi, 0)
\]

\[
\|\delta f K^\#(X, \phi)\|_{j+1} \leq L^{-d/2}\|\delta f K^\#(X, \phi)\|_j
\leq L^{-d/2}\|\delta f K^\#(X, \phi)\|_j
\leq L^{-d/2}\left(\|\delta f K^\#(X)\|_j\right) G_j(X, \phi, 0)
\]

(77)

and for \(\phi = \phi' + \zeta\), using \((\Pi), (6.58)\) we get:

\[
(1 + \|\phi\|_{\Phi_j+1(X-1)}) G_j(X, \phi, 0) \leq (1 + \|\phi\|_{\Phi_j+1(X-1)})^3 G_j(X, \phi, 0)
\leq 4 \left(1 + \|\phi\|_{\Phi_j+1(X-1)}^3\right) G_j(X, \phi, 0)
\leq 4q G_{j+1}(\bar{X}, \phi', \zeta)
\]

(78)

with \(q\) as in \((\Pi), (6.127)\). Combining all of them yields

\[
\| (I - T_2)_{0} K^\#(X, \phi)\|_{j+1} \leq 24q\left(L^{-3d/2}\right)\|_0 K^\#(X)\|_{j} G_{j+1}(\bar{X}, \phi', \zeta)
\]

\[
\| (I - T_0)\delta f K^\#(X, \phi)\|_{j+1} \leq 4q L^{-d/2} \left(\|\delta f K^\#(X)\|_j\right) G_{j+1}(\bar{X}, \phi', \zeta)
\]

(79)

and also using (79) in \([5]\), we obtain:

\[
\| (I - T_2)_{0} K^\#(X)\|_{j+1} \leq 24q\left(L^{-3d/2}\right)\|_0 K^\#(X)\|_{j}
\leq 24q\left(L^{-3d/2}\right)(A/2)^{-|X_j|} \|_0 K\|_j
\]

\[
\| (I - T_0)\delta f K^\#(X)\|_{j+1} \leq 4q L^{-d/2} \left(\|\delta f K\|_j\right) A^{-|X_j|} 2^{|X_j|}
\]

(80)

Therefore

\[
\| L_2(f K)\|_{j+1} \leq \| L_2(\delta f K)\|_{j+1} + \| L_2(\alpha K)\|_{j+1}
\leq 24q\left(L^{-3d/2}\right)\sup_{X \in S_j, X = U} \sum_{X \in S_j, X = U} \left(A/2\right)^{-|X_j|} \|_0 K\|_j
\]

\[
+ 4q L^{-d/2} \left(\|\delta f K\|_j\right) \sup_{X \in S_j, X = U} \sum_{X \in S_j, X = U} \left(A/2\right)^{-|X_j|} 2^{|X_j|}
\]

(81)

The bracketed expression is less than \(2^d 2^{2d} n_2(d) L^d\) (using \((\Pi), (6.90)\)) so we have

\[
\| L_2(\alpha K)\|_{j+1} \leq 24q 2^d 2^{2d} n_2(d)(L^{-d/2}) \|_0 K\|_j
\]

(82)

Because \(|U|_{j+1} \leq |X|_j \leq 2^d\), we get:
\[ 4qL^{-d/2}(\|\delta f K\|_j) \sup_{U} \sum_{X \in S_j, X = U \atop X^* \cap \{x_1, x_2, \ldots, x_m\} \neq \emptyset} A^{[U\setminus j]} A^{-|X|} J^{|X|} \]
\[ \leq 4qL^{-d/2}(\|\delta f K\|_j) \sum_{X \in S_j, X^* \cap \{x_1, x_2, \ldots, x_m\} \neq \emptyset} 2^{d^2} \tag{83} \]
\[ \leq 4qL^{-d/2}|S_j| \sum_{i=1}^{m} \sum_{X \in S_j, X^* \cap \{x_i\} \neq \emptyset} 1 \]
\[ \leq 4qmL^{-d/2}|S_j| n_2(d) \]

Thus
\[ \|\mathcal{L}_2(fK)\|_{j+1} \leq 24q2^{d^2} n_2(d)(L^{-d/2})\|0K\|_j + L^{-d/2}(\|\delta f K\|_j)n_2(d)2^{d^2}4qm \] (84)

Moreover \(\|\delta f K\|_j \leq \|0K\|_j + ||fK||_j\). So we have the Lemma 4.

(Q.E.D)

3.3 Splitting \(\mathcal{L}_3\)

3.3.1 Splitting \(\mathcal{L}_3\)

Similarly in \([5]\), we have a special treatment for the term \(\mathcal{L}_3\). First we rewrite the final term in \(\mathcal{L}_3\) which is

\[ \sum_{B = U \atop X \supset B} \sum_{X \in S_j} \frac{1}{|X|} [T_2(0K^\#(X)) + T_0(\delta f K^\#(X))] \]
\[ = \sum_{B = U \atop X \supset B} \sum_{X \in S_j} \frac{1}{|X|} \left(0K^\#(X, 0) + \frac{1}{2}0K^\#_2(X, 0; \phi, \phi) + fK^\#(X, 0) - 0K^\#(X, 0)\right) \tag{85} \]
\[ = \sum_{B = U \atop X \supset B} \sum_{X \in S_j} \frac{1}{|X|} \left(fK^\#(X, 0) + \frac{1}{2}0K^\#_2(X, 0; \phi, \phi)\right) \]

In \(0K^\#_2(X, 0; \phi, \phi)\) we pick a point \(z \in B\), then use the same argument as section 4.3 \([5]\) by replacing \(\phi(x)\) with \(z\)

\[ \phi(z) + \frac{1}{2}(x - z) \cdot \partial \phi(z) \equiv \phi(z) + \frac{1}{2} \sum_{\mu} (x_\mu - z_\mu) \partial_\mu \phi(z) \] (86)

\[ \text{We need the factor } \frac{1}{2} \text{ since the sum is over } \pm \mu = 1, \ldots, d \text{ and } x_{-\mu} = -x_\mu \]
If we also average over \( z \in B \), we have \(86\) becomes

\[
\sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \left( fK^\#(X, 0) + \frac{1}{2 |B|} \sum_{z \in B} oK_2^\#(X, 0; \phi, \phi) \right)
\]

\[
= \sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \left( fK^\#(X, 0) \right)
\]

\[
+ \sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \left( \frac{1}{8 |B|} \sum_{z \in B} \sum_{\mu \nu} oK_2^\#(X, 0; x_\mu, x_\nu) \partial_\mu \phi(z) \partial_\nu \phi(z) \right)
\]

\[
+ \sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \sum_{z \in B} \frac{1}{|B|} \left( \frac{1}{2} oK_2^\#(X, 0; \phi, \phi) \right)
\]

\[
= \sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \left( fK^\#(X, 0) + \frac{1}{8 |B|} \sum_{z \in B} \sum_{\mu \nu} oK_2^\#(X, 0; x_\mu, x_\nu) \partial_\mu \phi(z) \partial_\nu \phi(z) \right)
\]

\[
+ L'_3(fK)(U)
\]

where \( L'_3(fK)(U) = L'_3(oK)(U) \) is so-called the error, namely

\[
L'_3(oK)(U) = \sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \sum_{z \in B} \frac{1}{|B|} \left( \frac{1}{2} oK_2^\#(X, 0; \phi, \phi) - \frac{1}{8} oK_2^\#(X, 0; x \cdot \partial \phi(z), x \cdot \partial \phi(z)) \right)
\]

and we can say \( L'_3(\delta fK)(U) = 0 \). Next we define

\[
f \beta(B) = f \beta(fK, B) = - \sum_{X \in S_j, X \supset B} \frac{1}{|X|} fK^\#(X, 0)
\]

\[
\alpha_{\mu \nu}(B) = \alpha_{\mu \nu}(fK, B) = \alpha_{\mu \nu}(oK, B) = - \frac{1}{2 |B|} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} oK_2^\#(X, 0; x_\mu, x_\nu)
\]

Note that \( \alpha_{\mu \nu} \) is symmetric and satisfies \( \alpha_{-\mu \nu} = -\alpha_{\mu \nu} \). We also let \( \alpha_{\mu \nu} \) stand for the function \( \alpha_{\mu \nu}(x) \) which takes the constant value \( \alpha_{\mu \nu}(B) \) for \( x \in B \).

Now we write \(87\) as

\[
\sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \left( fK^\#(X, 0) + \frac{1}{8 |B|} \sum_{z \in B} \sum_{\mu \nu} oK_2^\#(X, 0; x_\mu, x_\nu) \partial_\mu \phi(z) \partial_\nu \phi(z) \right) + L'_3(oK)(U)
\]

\[
= \sum_{B = U} \left( \frac{1}{4} \sum_{z \in B} \sum_{\mu \nu} \frac{1}{2 |B|} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} oK_2^\#(X, 0; x_\mu, x_\nu) \partial_\mu \phi(z) \partial_\nu \phi(z) \right)
\]

\[
+ \sum_{B = U} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} fK^\#(X, 0) + L'_3(oK)(U)
\]
\[
- \sum_{B=U} \left( f^\beta(B) + \frac{1}{4} \sum_{z \in B} \sum_{\mu \nu} \alpha_{\mu \nu}(B) \partial_\mu \phi(z) \partial_\nu \phi(z) \right) + \mathcal{L}'_3(\alpha K)(U)
\]
\[
= - \left( \sum_{B=U} \left( V(f^\beta, \alpha, B, \phi) \right) + \mathcal{L}'_3(\alpha K)(U) \right)
\]

where \( V(f^\beta, \alpha, B, \phi) \) defined in (95). Combining all of the above, we get:
\[
\mathcal{L}_3(f E, \sigma, f \tilde{E}, \tilde{\sigma}, f K, \alpha K)(U) = \sum_{B=U} \left( V(f \tilde{E}, \tilde{\sigma}, B) - V^\#(f E, \sigma, B) - V(f^\beta, \alpha, B) \right) + \mathcal{L}'_3(\alpha K)(U)
\]

### 3.3.2 Estimating \( \alpha, f^\beta \) and \( \mathcal{L}'_3 \)

First we find some explicit upper bounds for \( \alpha \) and \( f^\beta \)

**Lemma 5** *(Estimates \( f^\beta \) and \( \alpha \))*

\[
\|f^\beta\|_j \leq \sup_{B \in B_j} |f^\beta(B)| \leq 2n_2(d)A^{-1}\|f K\|_j
\]
\[
\|\alpha\|_j \leq \sup_{B \in B_j} \sum_{\mu \nu} |\alpha_{\mu \nu}(B)| \leq 4(2d)^2n_2(d)h^{-2}A^{-1}\|0 K\|_j
\]

**Remark.** The norm \( \|\alpha\|_j \) agrees with the norm \( \|s\|_j \) in (59) if \( s_{\mu \nu}(x) = \alpha_{\mu \nu}(B) \) for \( x \in B \).

**Proof.** By (70) and (79) in [5], with \( A \) very large, we have:
\[
|f K^\#(X, 0)| \leq \|f K^\#(X)\|_j \leq (A/2)^{-1}\|f K\|_j
\]
\[
\|0 K_2^\#(X, 0)\|_j \leq 2\|0 K^\#(X)\|_j \leq 4A^{-1}\|0 K\|_j
\]

From (30), the number of small sets containing a block \( B \) is \( n_2(d) \) which is bounded and depends only on \( d \), we have:
\[
|f^\beta(B)| \leq \sum_{X \in S_j, X \supset B} |f K^\#(X, 0)|
\]
\[
\leq \sum_{X \in S_j, X \supset B} 2A^{-1}\|f K\|_j
\]
\[
\leq 2n_2(d)A^{-1}\|f K\|_j
\]

We also have \( \|x_{\mu \nu}\|_{\phi_j(X^*)} = h^{-1}L^{d/2} \) and \( |B| = L^d \). By using (66) in [5] we get
\[
|B|^{-1}\|0 K_2^\#(X, 0; x_{\mu \nu})\| \leq (h^{-1}L^{d/2})^2L^{-d}\|0 K^\#_2(X, 0)\|_j
\]
\[
= h^{-2}\|0 K^\#_2(X, 0)\|_j
\]
\[
\leq 4h^{-2}A^{-1}\|0 K\|_j
\]

then
\[
\sum_{\mu \nu} |\alpha_{\mu \nu}(B)| \leq \sum_{\mu \nu} \sum_{X \in S_j, X \supset B} |B|^{-1}\|0 K_2^\#(X, 0; x_{\mu \nu}, x_{\nu \nu})\|_j
\]
\[
\leq \sum_{\mu \nu} \sum_{X \in S_j, X \supset B} 4h^{-2}A^{-1}\|0 K\|_j
\]
\[
\leq \sum_{\mu \nu} n_2(d)4h^{-2}A^{-1}\|0 K\|_j
\]
\[
\leq (2d)^2n_2(d)4h^{-2}A^{-1}\|0 K\|_j
\]
Now we give some estimate for \( L' \).

**Lemma 6** Let \( L \) be sufficiently large. Then the operator \( L' \) is a contraction with arbitrarily small norm.

\[
\|L'_3(0K)\|_{j+1} \leq 72d^22^d n_1(d)(L^{-2})\|0K\|_j
\]

**Proof.** Based on the proof of Lemma 8 in [5], we make some modifications and obtain a better upper bound with some explicit coefficient.

We have

\[
L'_3(U) = \sum_{B=U} \sum_{X \subseteq B} \frac{1}{|X|} \sum_{z \in B} \frac{1}{|B|} \frac{1}{2} \frac{1}{2} K^#(X, 0; \phi - \frac{1}{2} x \cdot \partial \phi(z), \phi + \frac{1}{2} x \cdot \partial \phi(z))
\]

Using (5, (152)- (154)) we get:

\[
\|\phi - \frac{1}{2} x \cdot \partial \phi(z)\| \leq 3d2^d(L^{-d/2-1})\|\phi\|_{j+1}(X^*)
\]

\[
\|\phi + \frac{1}{2} x \cdot \partial \phi(z)\| \leq 3d2^d(L^{-d/2-1})\|\phi\|_{j+1}(X^*)
\]

Now we estimate

\[
0H_X(U, \phi) = 0K^#(X, 0; \phi - \frac{1}{2} x \cdot \partial \phi(z), \phi + \frac{1}{2} x \cdot \partial \phi(z))
\]

Using the same argument as (156)-(157) in [5], we obtain:

\[
\|0H_X(U, \phi)\|_{j+1} \leq 18d^22^d(L^{-d-2})\|K^#(X, 0)\|_{j+1} + \|\phi\|^2_{j+1}(U^*)
\]

But for \( \phi = \phi' + \zeta \)

\[
(1 + \|\phi\|^2_{j+1}(U^*)) \leq G_{s,j+1}(U, \phi, 0) \leq G_{s,2,j+1}(U, \phi', \zeta) \leq G_{s,j+1}(U, \phi', \zeta)
\]

Also using (94) we can get:

\[
\|H_X(U)\|_{j+1} \leq 72d^22^d(L^{-d-2})A^{-1}\|K\|_j
\]

which yields to

\[
\|L'_3K(U)\|_{j+1} \leq n_1(d) \sum_{B=U} \|H_X(U)\|_{j+1}
\]

\[
\leq n_1(d) L^d 72d^22^d(L^{-d-2})A^{-1}\|K\|_j
\]

\[
\leq 72d^22^d n_1(d)(L^{-2})A^{-1}\|K\|_j
\]

Since \( L'_3K(U) \) is zero unless \( |U|_{j+1} = 1 \) this gives

\[
\|L'_3K\|_{j+1} \leq 72d^22^d n_1(d)(L^{-2})\|K\|_j
\]

(Q.E.D)
3.4 Identifying invariant parts and estimating the others

Now we investigate the 1st term of (92). We notice that \( \alpha_{\mu \nu}(B) = \alpha_{\mu \nu}(\hat{f}K, B) = \alpha_{\mu \nu}(\check{0}K, B) \) is independent from \( f(\phi) \) and \( qE(B), \check{0}K(X, \phi) \) actually is the same as \( E(B), K(X, \phi) \) in lemma 9 [5]. Therefore we have the same result as lemma 9 [5].

**Lemma 7** (Lemma 9, Dimock [5])

Suppose \( qE(B), \check{0}K(X, \phi) \) are invariant under lattice symmetries away from the boundary of \( \Delta N \) and \( \hat{q}E(B) \) is invariant for \( B^* \) away from the boundary. Then

1. \( \hat{q}E(B'), \check{0}K(U, \phi) \) are invariant for \( B', U \) away from the boundary

2. If \( B^* \) is away from the boundary then \( \hat{q}E(B), \alpha_{\mu \nu}(B) \) are independent of \( B \) and \( \alpha_{\mu \nu}(B) = \check{\alpha}_{\mu \nu}(B) \) defined for all \( B \) by

   \[
   \check{\alpha}_{\mu \nu}(B) = \frac{\alpha}{2} (\delta_{\mu \nu} - \delta_{\mu, -\nu})
   \]

   where \( \alpha \) is a constant.

For all \( B \in B_j \) we define

\[
\alpha'_{\mu \nu}(B) = \alpha \delta_{\mu \nu}
\]

and write, for any \( U \in B_{j+1} \)

\[
\sum_{B=U} V(f\beta, \alpha, B) = \sum_{B=U} V(f\beta, \alpha', B) - \mathcal{L}_4(fK)(U) - \Delta(fK)(U)
\]

where, for \( U \subset B_{j+1} \),

\[
\mathcal{L}_4(fK)(U) = \mathcal{L}_4(\check{0}K)(U) = \sum_{B=U} V(0, \alpha' - \hat{\alpha}, B) = V(0, \alpha' - \hat{\alpha}, U)
\]

\[
\Delta(fK)(U) = \Delta(\check{0}K)(U) = \sum_{B=U} V(0, \hat{\alpha} - \alpha, B) = V(0, \hat{\alpha}, U)
\]

where \( \hat{\alpha}_{\mu \nu}(x) = \hat{\alpha}_{\mu \nu}(B) - \alpha_{\mu \nu}(B) \) if \( x \in B \). Then we can write that \( \mathcal{L}_4(\delta fK)(U) = 0 \) and \( \Delta(\delta fK)(U) = 0 \). By the above definition \( \Delta(\check{0}K)(U) \) vanishes unless \( U \) touches the boundary. Now (92) becomes

\[
\mathcal{L}_3(fE, \sigma, \hat{f}E, \hat{\sigma}, fK, \check{0}K)(U)
\]

\[
= \sum_{B=U} \left( V(f\hat{E}, \hat{\sigma}, B) - V^\#(fE, \sigma, B) - V(f\beta, \alpha', B) \right)
\]

\[
+ \mathcal{L}_4(\check{0}K)(U) + \mathcal{L}_4(\check{0}K)(U) + \Delta(\check{0}K)(U)
\]

**Remark.** Because \( \mathcal{L}_4(fK) = \mathcal{L}_4(\check{0}K) \) and \( \Delta(fK) = \Delta(\check{0}K) \) are independent from \( f \), we will have the same results as Lemma 10 and Lemma 11 in (Dimock, [5]). Moreover, by using Lemma 5 above, we can obtain some explicit upper bounds for \( \mathcal{L}_4(\check{0}K) \) and \( \Delta(\check{0}K) \).

**Lemma 8** Let \( L \) be sufficiently large. Then the operator \( \mathcal{L}_4 \) is a contraction with

\[
\| \mathcal{L}_4(\check{0}K) \|_{j+1} \leq 4(2d)^5 n_2(d) L^{-1} \| \check{0}K \|_j
\]

**Lemma 9** Let \( L \) be sufficiently large. Then the operator \( \Delta \) is a contraction with

\[
\| \Delta(\check{0}K) \| \leq 4(2d)^5 2^d n_2(d) L^{-1} \| \check{0}K \|_j
\]
3.5 Simplifying for the next scale

We now pick \( jE(B), \tilde{\sigma} \) so that the V terms in (111) cancel. We have:

\[
\begin{align*}
V^\#(jE, \sigma, B, \phi) &= jE(B) + \int \frac{\sigma}{4} \sum_{x \in B} \sum_{\mu} (\partial_\mu \phi(x) + \partial_\mu \zeta(x))^2 d\mu_{\Gamma_{j+1}}(\zeta) \\
&= jE(B) + \frac{\sigma}{4} \sum_{x \in B} \sum_{\mu} \partial_\mu \phi(x)^2 \int d\mu_{\Gamma_{j+1}}(\zeta) \\
&\quad + \frac{\sigma}{2} \sum_{x \in B} \sum_{\mu} \partial_\mu \phi(x) \int \partial_\mu \zeta(x) d\mu_{\Gamma_{j+1}}(\zeta) \\
&\quad + \frac{\sigma}{4} \sum_{x \in B} \sum_{\mu} \int \partial_\mu \zeta(x)^2 d\mu_{\Gamma_{j+1}}(\zeta)
\end{align*}
\]

(114)

because

\[
\begin{align*}
\int \partial_\mu \zeta(x) d\mu_{\Gamma_{j+1}}(\zeta) &= 0 \\
\int \partial_\mu \zeta(x)^2 d\mu_{\Gamma_{j+1}}(\zeta) &= \int (\zeta, \partial_\mu^* \delta_x)(\zeta, \partial_\mu^* \delta_x) d\mu_{\Gamma_{j+1}}(\zeta) \\
&= (\partial_\mu^* \delta_x, \Gamma_{j+1} \partial_\mu^* \delta_x) \\
&= (\delta_x, \partial_\mu \Gamma_{j+1} \partial_\mu^* \delta_x) \\
&= (\partial_\mu \Gamma_{j+1} \partial_\mu^* \delta_x)(x, x)
\end{align*}
\]

(115)

If we choose \( \tilde{f}E = fE(jE, \sigma, fK) \)

\[
\tilde{f}E(B) = jE(B) + \frac{\sigma}{4} \sum_{\mu} \text{Tr}(1_B \partial_\mu \Gamma_{j+1} \partial_\mu^* ) + f\beta(jK, B)
\]

(116)

then the constant terms of (114) will be canceled. The second order terms of (114) would be vanish if we define \( \tilde{\sigma} = \tilde{\sigma}(\sigma, jK) = \sigma(\sigma, 0K) \) by

\[
\tilde{\sigma} = \sigma + \alpha(jK) = \sigma + \alpha(0K)
\]

(117)

Here we are canceling the constant term exactly for all \( B \), but for the quadratic term we only cancel the invariant version away from the boundary.

By composing \( jK' = jK'(fE, jE, jE, jE, K, 0K) \) in theorem 5 with newly defined \( fE = \tilde{f}E(jE, \sigma, jK) \) and \( \tilde{\sigma} = \tilde{\sigma}(\sigma, jK) = \sigma(\sigma, 0K) \) we obtain a new map \( jK' = jK'(fE, \sigma, jE, 0K) \). We also have new quantities \( jE'(fE, \sigma, jE) \) defined by \( jE'(B') = \sum_{B \subset B'} jE(B) \) and \( \sigma' = \sigma'(\sigma, jE) = \sigma'(\sigma, 0K) \) defined by \( \sigma' = \sigma + \alpha(jK) = \sigma + \alpha(0K) \) as normal. These quantities satisfy (15)

\[
\mu_{\Gamma_{j+1}} * (fI(fE, \sigma) \circ jK)(\Lambda) = (fI'(fE', \sigma') \circ jK')(\Lambda)
\]

(118)

Here we still assume that \( L \) is sufficiently large, and that \( A \) is sufficiently large depending on \( L \).

**Theorem 6** 1. For \( R > 0 \) there is a \( r > 0 \) such that the following holds for all \( j \). If \( \|fE\|_j, \|\sigma\|, \max\{\|jK\|_j, \|0K\|\} < r \) then \( \|jE'\|_{j+1}, \|\sigma'\|, \max\{\|jK'\|_{j+1}, \|0K'\|_{j+1}\} < R \). Furthermore \( jE', jK', \sigma' \) are smooth functions of \( fE, \sigma, jK, 0K \) on this domain with derivatives bounded uniformly in \( j \). The analyticity of \( jK' \) in \( t_1, \ldots, t_m \) still holds when we go from \( j \)-scale to \( (j + 1) \)-scale.
2. The linearization of $fK' = fK'(fE, \sigma, fK, 0K)$ at the origin is the contraction $L(fK)$ where

$$L = L_1 + L_2 + L_3 + L_4 + \Delta$$  \hspace{1cm} (119)

Proof. For the first part, by combining with theorem 5 it suffices to show that the linear maps $\tilde{f}E$ and $\tilde{\sigma}$ have norms bounded uniformly in $j$. Using the estimate $|\alpha(fK)| = |\alpha(0K)| = 4(2d)^2n_2(d)h^{-1}||0K||_j$ from lemma 5, we have $\tilde{\sigma}$ is bounded. From lemma 5 we also have the bound on $||\beta(fK)||_j \leq 2n_2(d)A^{-1}||fK||_j$. For $B \in B_j$, the estimate (119) gives us

$$\left| \frac{\sigma}{4} \sum_{\mu} Tr(1_B(\partial_{\mu} \Gamma_{j+1} \partial_{\mu}^*) \right| \leq dc_{1,1} |\sigma| \sum_{n \in B} L^{-n} \leq dc_{1,1} |\sigma|$$  \hspace{1cm} (120)

where $c_{1,1}$ as in (119). Combining with (116) we have that $\tilde{f}E = \tilde{f}E(fE, \sigma, fK)$ satisfies

$$||f\tilde{E}||_j \leq ||f\tilde{E}||_j + C(|\sigma| + A^{-1}||fK||_j)$$  \hspace{1cm} (121)

where $C = \max\{dc_{1,1}, 2n_2(d)\}$.

The second part follows since the linearization of the new function $fK'$ is the linearization of the old function $fK'$ in theorem 5 composed with $fE = fE(fE, \sigma, fK), \tilde{\sigma} = \tilde{\sigma}(fK) = \tilde{\sigma}(\sigma, 0K)$. (All of them vanish at zero.) The cancellation gives us only with $L(fK)$.

3.6 Forming RG Flow

It is easier for us if we can extract the energy from the other variables. Assume that we start with $E(B) = 0$ in (118)

$$\mu_{\gamma_{j+1}}(fI(0, \sigma) \circ fK)(\Lambda_N) = (fI'(fE', \sigma') \circ fK')(\Lambda)$$  \hspace{1cm} (122)

where $\sigma' = \sigma'(fK) = \sigma'(\sigma, 0K)$ and $fK' = fK'(0, \sigma, fK)$ and $fE' = fE'(0, \sigma, fK)$ as above. Then we remove the $fE'$ by making an adjustment in $fK'$.

$$\mu_{\gamma_{j+1}}(fI(0, \sigma) \circ fK)(\Lambda_N) = (fI'(fE', \sigma') \circ fK')(\Lambda)$$

where $fE'(\sigma, fK, B'), \sigma'(\sigma, fK)$, $fK'(\sigma, fK, U)$ are defined as following ($U \in \mathcal{P}_{j+1}, B' \in \mathcal{B}_{j+1}$)

$$fE'(\sigma, fK, B') \equiv fE(0, \sigma, fK, B') = \sum_{B \subseteq B'} fE(0, \sigma, fK, B)$$

$$\sigma'(\sigma, fK) \equiv \sigma'(\sigma, fK) = \sigma'(\sigma, 0K) = \sigma + \alpha(0K)$$  \hspace{1cm} (124)

$$fK'(\sigma, fK, U) \equiv \exp \left( \sum_{B' \in \mathcal{B}_{j+1}(U)} fE'+(B') \right) fK'(0, \sigma, fK, U)$$
The dynamical variables are now $\sigma^+(\sigma, fK)$ and $fK^+(\sigma, fK)$. The extracted energy $fE^+(\sigma, K)$ is controlled by the other variables. Because everything vanishes at the origin the linearization of $fK^+(\sigma, fK)$ is still $\mathcal{L}(fK)$. The bound (121) on $fE$ would give us an upper bound on $fE^+$ and our theorem (3) becomes:

**Theorem 7**  1. For $R > 0$ there is a $r > 0$ such that the following holds for all $j$. If $|\sigma|$, max$\{||fK||_j, ||0K||_j\} < r$ then $|\sigma^+|$, max$\{||fK^+||_{j+1}, ||0K^+||_{j+1}\} < R$. Furthermore $\sigma^+, fK^+$ are smooth functions of $\sigma, fK$ on this domain with derivatives bounded uniformly in $j$. The analyticity of $fK^+$ in $t_1, \ldots, t_n$ still holds when we go from $j$-scale to $(j+1)$-scale.

2. The extracted energies satisfy

$$||fE^+(\sigma, fK)||_{j+1} \leq C(L^4)(|\sigma| + A^{-1}||fK||_j)$$

(125)

3. The linearization of $K^+$ at the origin is the contraction $\mathcal{L}$.

**4 The stable manifold**

Up to now, we have not specialized to the dipole gas, but take a general initial point $\sigma_0, fK_0$ corresponding to an integral $\int (fI(0, \sigma_0) \circ fK_0)(\Lambda_N) d\mu_{C_0}$. We assume $0K_0(X, \phi)$ has the lattice symmetries and satisfies the conditions (87). We also assume $|\sigma_0|$, max$\{||fK||_0, ||0K||_0\} < r$ where $r$ is small enough so the theorem (7) holds, say with $R = 1$, then we can take the first step. We apply the transformation (123) for $j = 0, 1, 2, \ldots$ and continue as far as we can. Then we get a sequence $\sigma_j, fK^N_j(X)$ by $\sigma_j = \sigma^+(\sigma_j, fK^N_j)$ and $fK^N_{j+1} = fK^+(\sigma_j, fK^N_j)$ with extracted energies $fE^N_{j+1} = fE^+(\sigma_j, fK^N_j)$. Then we have, for any $l$, with $fI_j(\sigma_j) = fI_j(0, \sigma_j)$

$$\int (fI_0(\sigma_0) \circ fK_0)(\Lambda_N) d\mu_{C_0} = \exp \left( \sum_{j=1}^{l} \sum_{B \in B_j(\Lambda_N)} fE^N_j(B) \right) \int (fI_j(\sigma_j) \circ fK^N_j)(\Lambda_N) d\mu_{C_j}$$

(126)

The quantities $0K^N_j(X)$ and $0E^N_j(B)$ are independent of $N$ and have the lattice symmetries if $X, B$ are away from the boundary $\partial \Lambda_N$ in the sense that they have no boundary blocks. These properties are true initially and are preserved by the iteration. In these cases we denote these quantities by just $0K_j(X)$ and $0E_j(B)$.

With our construction $\alpha$ defined in (89), (107) only depends on $0K_j$. By splitting $K^+$ into a linear and a higher order piece the sequence $\sigma_j, fK^N_j(X)$ is generated by the RG transformation

$$\sigma_{j+1} = \sigma_j + \alpha(K_j)$$

$$0K^N_{j+1} = \mathcal{L}(0K^N_j) + 0g(\sigma_j, 0K^N_j)$$

$$\delta_jK^N_{j+1} = (\mathcal{L}_1 + \mathcal{L}_2) (\delta_jK^N_j) + f\delta_jK^N_j, 0K^N_j - 0g(\sigma_j, 0K^N_j)$$

(127)

This is regarded as a mapping from the Banach space $\mathbb{R} \times (K_j(\Lambda_N) \times K_j(\Lambda_N))$ to the Banach space $\mathbb{R} \times (K_j(\Lambda_N) \times K_j(\Lambda_N))$. The 2nd equation of (127) defines $0g$ which is smooth with derivatives bounded uniformly in $j$ and satisfies $0g(0, 0) = 0$, $D(0g)(0, 0) = 0$. The last equation of (127) defines $f\delta_j$ which is also smooth with derivatives bounded uniformly in $j$ and satisfies $f(0, 0) = 0$, $D(f\delta_j)(0, 0) = 0$.

Now we consider the first two equations in (127). Around the origin there are a neutral direction $\sigma_j$ and a contracting direction $K_j$ (since $\mathcal{L}$ is a contraction.). Hence we expect there is a stable manifold. We quote a version of the stable manifold theorem due to Brydges [1], as applied in Theorem 7 in Dimock [5].
\textbf{Theorem 8} (Theorem 7, Dimock [3])

Let $L$ be sufficiently large, $A$ sufficiently large (depending on $L$), and $r$ sufficiently small (depending on $L, A$). Then there is $0 < r < r$ and a smooth real-valued function $\sigma_0 = h(0K_0)$, $h(0) = 0$, mapping $\|K_0\|_0 < r$ into $|\sigma_0| < r$ such that with these start values the sequence $\sigma_j, 0K_j^N$ is defined for all $0 \leq j \leq N$ and

$$|\sigma_j| \leq r2^{-j} \quad \|0K_j^N\|_j \leq r2^{-j} \quad (128)$$

Furthermore the extracted energies satisfy

$$\|0E_j^N\|_{j+1} \leq 2C(L^d)2^{-j} \quad (129)$$

\textbf{Remark.} Using the Lemma [10] below, given $r > 0$, we can always choose $z, \sigma_0$ and $\max_k |t_k|$ sufficiently small then max\{\|fK\|_0, \|0K\|_0\} $\leq r$. Now we claim that $\|fK_j^{N}\|_j$ has the same bound as the $\|0K_j^N\|_j$ in the last theorem.

Supposed that at $j = k$, we have: $\|fK_j^N\|_j \leq r2^{-k}$. As in the proof of Theorem 7 in (Dimock, [4]), we can say that $L$ and $(L_1 + L_2)$ is a contraction with norm less than 1/8 and $fg(\sigma_j, fK_j^N, 0K_j)$ is second order. Hence there are some constant $H$ such that: $\|fg(\sigma_j, fK_j^N, 0K_j)\| \leq H(|\sigma_j|^2 + \|0K_j^N\|_j^2 + \|fK_j^N\|_j^2)$ with $|\sigma_j|, \|0K_j^N\|_j, \|fK_j^N\|_j$ small. Then we have:

$$\|fK_j^{N+1}\|_{j+1} \leq \frac{1}{8}\left(\|0K_j^{N+1}\|_j + \|fK_j^{N+1}\|_j\right) + H\left(|\sigma_j|^2 + \|0K_j^{N+1}\|_j^2 + \|fK_j^{N+1}\|_j^2\right)$$

$$\leq \frac{1}{8}\left(\|fK_j^{N+1}\|_j + \|fK_j^{N+1}\|_j\right) + 3H\left(2r^{-j}\right)^2$$

$$\leq \frac{1}{8}\left(32r^{-j}\right) + 3H\left(2r^{-j}\right)^2 \quad (130)$$

for $r$ sufficiently small.

The bound for $\|fE_j^{N}\|_{j+1}$ comes from the bound on $\|fK_j^{N}\|_j, 125$ and $A > 1$.

Combining with the last theorem, for all $0 \leq j \leq N$ we can have:

$$|\sigma_j| \leq r2^{-j} \quad \|fK_j^N\|_j \leq r2^{-j} \quad (131)$$

and the extracted energies satisfy

$$\|fE_j^{N}\|_{j+1} \leq 2C(L^d)r2^{-j}. \quad (132)$$

\section{5 The dipole gas}

\subsection{5.1 The initial density}

Now we consider the generating function:

$$fZ_N(z, \sigma) = \int e^{if(\phi)} \exp\left(zW(\Lambda_N, \sqrt{1 + \sigma}\phi) - \sigma V(\Lambda_N, \phi)\right) d\mu_{C_0}(\phi) \quad (133)$$

When $f = 0$, it becomes

$$0Z_N(z, \sigma) = \int \exp\left(zW(\Lambda_N, \sqrt{1 + \sigma}\phi) - \sigma V(\Lambda_N, \phi)\right) d\mu_{C_0}(\phi) \quad (134)$$
For $B \in B_0$, we define: $W_0(B) = zW(\sqrt{1 + \sigma_0}, B)$ as in (12) and $V_0(B) = \sigma_0 V(B)$ as in (36). Then we follow with a Mayer expansion to put the density in the form we want.

$$f Z_0^N = \prod_{B \subset \Lambda_N} e^{if(\phi)+W_0(B)-V_0(B)}$$

$$= \prod_{B \subset \Lambda_N} \left(e^{-V_0(B)} + (e^{if(\phi)+W_0(B)} - 1)e^{-V_0(B)}\right)$$

$$= \sum_{X \subset \Lambda_N} f I_0(\sigma_0, \Lambda_N - X)f K_0(X)$$

$$= (f I_0(\sigma_0) \circ f K_0)(\Lambda_N)$$

where $I_0(\sigma_0, B) = e^{-V_0(B)}$ and $f K_0(X) = f K_0(z, \sigma_0, X)$ is given by

$$f K_0(X) = \prod_{B \subset X} (e^{if(\phi)|_B + W_0(B)} - 1)e^{-V_0(B)}$$

when $f(\phi) = \sum_{k=1}^m t_k \exp (i\partial_{\mu_k} \phi(x_k))$, $if(\phi)|_B = t_k \exp (i\partial_{\mu_k} \phi(x_k))$ if $B = \{x_k\}$ for some $k$, otherwise $if(\phi)|_B = 0$

$$f K_0(X) = \prod_{B \subset X} (e^{if(\phi)|_B + W_0(B)} - 1)e^{-V_0(B)}$$

when $f(\phi) = f(\phi) = \sum_{k=1}^m t_k \partial_{\mu_k} \phi(x_k)$, $if(\phi)|_B = t_k \partial_{\mu_k} \phi(x_k)$ if $B = \{x_k\}$ for some $k$, otherwise $if(\phi)|_B = 0$

$$f K_0(X) = \prod_{B \subset X} (e^{W_0(B)} - 1)e^{-V_0(B)}$$

when $f(\phi) = 0$

Note that, when $f = 0$, $\partial K_0$ actually is the $K_0$ in lemma 12, [5]. We also can prove the same result for $f K_0$.

**Lemma 10** Given $1 > r > 0$, there are some sufficiently small $a(r), b(r)$ and $c(r)$ such that if $\max_k |t_k| \leq a(r)$, $|z| \leq b(r)$ and $|\sigma_0| \leq c(r)$ then $\|f K_0(z, \sigma_0)\|_0 \leq r$. Furthermore $f K_0$ is a smooth function of $(z, \sigma_0)$, and analytic in $t_k$ for all $k = 1, \ldots, m$.

**Proof.**

*When $f = 0$, using lemma 12 [5], we have some $b_0(r), c_0(r)$ such that $\|0 K_0(z, \sigma_0)\|_0 \leq r$ if $|z| \leq b_0(r)$ and $|\sigma_0| \leq c_0(r)$

*In the case $f(\phi) = \sum_{k=1}^m t_k \partial_{\mu_k} \phi(x_k)$, using (5), (95), for $\phi = \phi' + \zeta$, we have:

$$\|f(\phi)\|_0 = \|e^{if(\phi)|_B + W_0(B)} - 1\|_0 \leq \sum_{n=1}^{\infty} \frac{1}{n!} \|W(\sqrt{1 + \sigma_0}, B) + if(\phi)|_B\|_0^n$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left(\|W(\sqrt{1 + \sigma_0}, B)\|_0 + \|if(\phi)|_B\|_0\right)^n \tag{139}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left(2|z|e^{h(1+\sigma_0)} + \max_k |t_k| h^{-1}\|\phi|_{\Phi_0(B^*)}\right)^n$$

24
We can assume that \( \max_k |t_k|h^{-1} \leq 1 \). Applying lemma 3 in [5], we get \( \|e^{-V_0(B)}\|_{s,0} \leq 2 \).

\[
\|fK_0(B)\|_0 = \sup_{\phi', \zeta} \|fK_0(B, \phi' + \zeta)\|_0 \mathcal{G}_0(X, \phi', \zeta)^{-1}
\leq \sup_{\phi', \zeta} \|(e^{f(\phi)})_B + W_0(B) - 1\|_0 \|e^{-V_0(B)}\|_0 \mathcal{G}_{s,0}(X, \phi', \zeta)^{-2}
\leq \|e^{-V_0(B)}\|_{s,0} \sup_{\phi', \zeta} \|(e^{f(\phi)})_B + W_0(B) - 1\|_0 \mathcal{G}_{s,0}(X, \phi', \zeta)^{-1}
\leq 2 \sup_{\phi', \zeta} \left( \exp \left( 2|z|e^{\sqrt{(1 + \sigma_0)}} \right) + \max_k |t_k|h^{-1}||\phi' + \zeta||_{\Phi(B)} \right)^{-1} \mathcal{G}_{s,0}(X, \phi', \zeta)^{-1}
\leq 2 \sup_{\phi', \zeta} \left( \exp \left( 2|z|e^{\sqrt{(1 + \sigma_0)}} \right) - 1 \right) \exp \left( \max_k |t_k|h^{-1}||\phi' + \zeta||_{\Phi(B)} \right) \mathcal{G}_{s,0}(X, \phi', \zeta)^{-1}
\]

(140)

\[
+ 2 \sup_{\phi', \zeta} \left( \exp \left( \max_k |t_k|h^{-1}||\phi' + \zeta||_{\Phi(B)} \right) - 1 \right) \mathcal{G}_{s,0}(X, \phi', \zeta)^{-1}
\leq 2 \sup_{\phi', \zeta} \left( \exp \left( 2|z|e^{\sqrt{(1 + \sigma_0)}} \right) - 1 \right) \exp \left( ||\phi' + \zeta||_{\Phi(B)} \right) e^{-||\phi'||_{\Phi_0(B)}^2 - ||\zeta||_{\Phi_0(B)}^2}
\]

(142)

Because \( \exp \left( ||\phi' + \zeta||_{\Phi_0(B)} \right) e^{-||\phi'||_{\Phi_0(B)}^2 - ||\zeta||_{\Phi_0(B)}^2} \) is bounded and

\[
\lim_{z, \sigma_0 \to 0} \left( \exp \left( 2|z|e^{\sqrt{(1 + \sigma_0)}} \right) - 1 \right) = 0
\]

there exist some sufficiently small \( b_1(r), c_1(r) > 0 \) such that we have

\[
2 \sup_{\phi', \zeta} \left( \exp \left( 2|z|e^{\sqrt{(1 + \sigma_0)}} \right) - 1 \right) \exp \left( ||\phi' + \zeta||_{\Phi_0(B)} \right) e^{-||\phi'||_{\Phi_0(B)}^2 - ||\zeta||_{\Phi_0(B)}^2} \leq \frac{r}{4A}
\]

for all \( |z| \leq b_1(r) \) and \( |\sigma_0| \leq c_1(r) \).

For other part, we have:

\[
2 \sup_{\phi', \zeta} \left( \exp \left( \max_k |t_k|h^{-1}||\phi' + \zeta||_{\Phi_0(B)} \right) - 1 \right) \exp(-||\phi'||_{\Phi_0(B)}^2 - ||\zeta||_{\Phi_0(B)}^2)
\]

(143)

\[
\leq 2 \sup_{\phi', \zeta} \left( \exp \left( ||\phi'||_{\Phi_0(B)} + ||\zeta||_{\Phi_0(B)} \right) - 1 \right) \exp(-||\phi'||_{\Phi_0(B)}^2 - ||\zeta||_{\Phi_0(B)}^2)
\]

We also can find some sufficiently large \( H \) such that if \( ||\phi'||_{\Phi_0(B)} + ||\zeta||_{\Phi_0(B)} \geq H \) then

\[
2 \left( \exp \left( ||\phi'||_{\Phi_0(B)} + ||\zeta||_{\Phi_0(B)} \right) - 1 \right) \exp(-||\phi'||_{\Phi_0(B)}^2 - ||\zeta||_{\Phi_0(B)}^2) \leq \frac{r}{4A}
\]

(144)

For \( ||\phi'||_{\Phi_0(B)} + ||\zeta||_{\Phi_0(B)} \leq H \), we have \( ||\phi' + \zeta||_{\Phi_0(B)} \leq ||\phi'||_{\Phi_0(B)} + ||\zeta||_{\Phi_0(B)} \leq H \). So with \( \max_k |t_k| \leq a_1(r) \) sufficiently small and \( ||\phi'||_{\Phi_0(B)} + ||\zeta||_{\Phi_0(B)} \leq H \),

\[
2 \left( \exp \left( \max_k |t_k|h^{-1}||\phi' + \zeta||_{\Phi_0(B)} \right) - 1 \right) \exp(-||\phi'||_{\Phi_0(B)}^2 - ||\zeta||_{\Phi_0(B)}^2) \leq \frac{r}{4A}
\]

(145)

In summary we can always choose sufficiently small \( a(r), b(r), c(r) \) such that if \( \max_k |t_k| \leq a_1(r), |z| \leq b_1(r), \text{ and } |\sigma_0| \leq c_1(r) \) then

\[
\|fK_0(B)\|_0 \leq 2 \frac{r}{4A} = \frac{r}{2A} \quad \forall B \in B_0
\]

(146)
For those \(a_1(r), b_1(r), c_1(r), \max_k|t_k| \leq a_1(r), |z| \leq b_1(r), \) and \(|\sigma_0| \leq c_1(r)\) we have

\[
\|fK_0\|_0 = \sup_{X \in \mathcal{P}_{0,c}} \|fK_0(X)\|_0 A^{\|X\|_0} \\
\leq \sup_{X \in \mathcal{P}_{0,c}} \left( \prod_{B \subset X} \|fK_0(B)\|_0 \right) A^{\|X\|_0} \\
\leq \sup_{X \in \mathcal{P}_{0,c}} \left( \frac{r}{2A} \right)^{\|X\|_0} A^{\|X\|_0} \leq \frac{r}{2} < r 
\]

(147)

* In the last case, \(f(\phi) = \sum_{k=1}^m t_k \exp(i\partial_{\mu_k} \phi(x_k))\), we have:

\[
\|(e^{i\phi_0})_{A_t} W_0(B) - 1\|_0 \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left( 2|z| e^{\lambda A_2 + \sigma_0} + \max_k |t_k| \right)^n \\
= \exp \left( 2|z| e^{\lambda A_2 + \sigma_0} + \max_k |t_k| \right) - 1 
\]

(148)

Using the same argument as above, we can choose some sufficiently small \(a_2(r), b_2(r), c_2(r)\) such that \(\|fK_0(z, \sigma_0)\|_0 \leq r\) when \(\max_k |t_k| \leq a_2(r), |z| \leq b_2(r)\) and \(|\sigma_0| \leq c_2(r)\).

Now we just simply pick \(a(r) = \max\{a_1(r), a_2(r)\}, b(r) = \max\{b_0(r), b_1(r), b_2(r)\}\) and \(c(r) = \max\{c_0(r), c_1(r), c_2(r)\}\).

The smoothness follows similarly from Lemma 12, (Dimock, [5]).

**Remark.** We have \(fK_0\) is analytic. For each step when we jump from \(j\)-scale to \((j+1)\)-scale, the analyticity of \(fK\) still holds for the next scale.

Noticing that \(aK_0\) is just the \(K_0\) in Section 6, (Dimock, [5]), we need the following lemma to apply Theorem 8.

**Lemma 11** (Lemma 13, [5])

The equation \(\sigma = h(aK_0(z, \sigma))\) defines a smooth implicit function \(\sigma = \sigma(z)\) near the origin which satisfies \(\sigma(0) = 0\).

Taking \(|z|\) sufficiently small and choosing \(\sigma_0 = \sigma(z)\), we can apply theorem 8. For \(0 \leq l \leq N\), we have

\[
fZ_N = \exp \left( \sum_{j=1}^l \sum_{B \in B_j(\Lambda_N)} fE_j^N(B) \right) \int (fI_l(\sigma_l) \circ fK_l^N)(\Lambda_N) d\mu_{C_1} 
\]

(150)

where \(|\sigma_j| \leq r2^{-j}\) and \(\|fK_j\|_j \leq r2^{-j}\) and \(\|fE_j^N\|_{j+1} \leq O(L^4) r2^{-j}\).

**5.2 Completing the proof of Theorem 1**

**Theorem 9** For \(|z|\) and \(\max_k |t_k|\) sufficiently small the following limit exists:

\[
\lim_{N \to \infty} |\Lambda_N|^{-1} \log fZ_N'(z, \sigma(z)) 
\]

(151)

\(^7\)Instead of using the usual estimates, such as \(1 + \|\phi\|^2_{\Phi_j(B^*)} \leq \exp(\|\phi\|^2_{\Phi_j(B^*)}) = G_{s,j}(B, \phi, 0)\), we can use

\[
1 + \|\phi\|^2_{\Phi_j(B^*)} = k \left( \frac{1}{k} + \frac{1}{\lambda} \|\phi\|^2_{\Phi_j(B^*)} \right) \leq k \exp \left( \frac{1}{k} \|\phi\|^2_{\Phi_j(B^*)} \right) = kG_{s,j}(B, \phi, 0) 
\]

(149)

for any positive integer \(k\), and so forth.
Proof.
With updated index, the proof can go exactly the same as the proof of Theorem 8, [5]. We take \( l = N \) in (150). At this scale there is only one block \( \Lambda_N \in B_N(\Lambda_N) \) and so we have
\[
|\Lambda_N|^{-1} \log f Z'(z, \sigma(z)) = |\Lambda_N|^{-1} \sum_{j=1}^{N} \sum_{B \in B_j(\Lambda_N)} f E_j^N(B)
\]
\[
+ |\Lambda_N|^{-1} \log \left( \int [f I_N(\sigma_N, \Lambda_N) + f K_N^N(\Lambda_N)] \, d\mu_{CN} \right)
\]
(152)

The second term has the form
\[
|\Lambda_N|^{-1} \log \left( 1 + \int f T_N d\mu_{CN} \right)
\]
(153)

where
\[
f T_N = \left( 1 + \int f F_N d\mu_{CN} \right)
\]
(154)

By (75) and (126) in [5], we have
\[
\| f I_N(\sigma_N, \Lambda_N) - 1 \|_N \leq 4c^{-1} h^2 |\sigma_N| \leq 4c^{-1} h^2 r 2^{-N}
\]
\[
\| f K_N^N(\Lambda_N) \|_N \leq A^{-1} \| f K_N^N \|_N \leq A^{-1} r 2^{-N}
\]
(155)

so that \( \| f F_N(\Lambda_N) \|_N \leq (4c^{-1} h^2 + A^{-1}) r (2^{-N}) \) which is \( O(2^{-N}) \) as \( N \to \infty \).

In lemma 14 [5]) Dimock has proved that for \( h \) sufficiently large
\[
\int G_N(\Lambda_N, 0, \zeta) d\mu_{CN}(\zeta) \leq 2
\]
(156)

Then we estimate
\[
\left| \int f F_N(\Lambda_N) d\mu_{CN} \right| \leq \| f F_N(\Lambda_N) \|_N \int G_N(\Lambda_N, 0, \zeta) d\mu_{CN}(\zeta)
\]
\[
\leq 2 \| F(\Lambda_N) \|_N
\]
\[
\leq 2 (4c^{-1} h^2 + A^{-1}) r (2^{-N})
\]
(157)

Hence the expression \( 156 \) is \( O(2^{-N}) |\Lambda_N|^{-1} \) and goes to zero very quickly as \( N \to \infty \).

The rest of the proof came as in the proof of Theorem 8 in [5].

6 Correlation functions: estimates and infinite volume limit

Note: We always can assume that \( L \gg 2^{d+3} + 1 \)

6.1 In the case: \( f(\phi) = \sum_{k=1}^{m} t_k \partial_{\mu_k} \phi(x_k) \)

For \( x_k \in \mathbb{Z}^d \) are different points; \( \mu_k \in \{ \pm 1, \ldots, \pm d \} \) and \( t_k \) complex and \( |t_k| \leq a = a(r) \) for \( \forall k : 1, 2, \ldots, m. \)
6.1.1 Proof of Theorem 2

Using (158) with $l = N$, for the truncated correlation functions, we have:

$$G_l(x_1, x_2, \ldots, x_m) \equiv \left( \prod_{k=1}^{m} \partial_{\mu_k} \phi(x_k) \right)^l \equiv i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log f Z^l \bigg|_{t_1=0, \ldots, t_m=0}$$

$$= i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \left( \sum_{j=1}^{N} \sum_{B \in B_j(\Lambda_N)} f E_j^{N}(B) \right) \bigg|_{t_1=0, \ldots, t_m=0}$$

$$+ i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log \left( \int (f I_1(\sigma_N) \circ f K_N(\Lambda_N)) d\mu_{CN} \right) \bigg|_{t_1=0, \ldots, t_m=0}$$

(158)

Now we consider the quantity:

$$f F_N \equiv \sum_{j=1}^{N} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \ldots \partial t_m} f E_j^{N}(B) \bigg|_{t_1=0, \ldots, t_m=0}$$

$$= \sum_{j=0}^{N-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \ldots \partial t_m} f (f K_j^{N}, B) \bigg|_{t_1=0, \ldots, t_m=0}$$

(159)

$$= \sum_{j=0}^{N-1} \sum_{B \in B_j(\Lambda_N)} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \int f K_j^{N}(X, 0) \bigg|_{t_1=0, \ldots, t_m=0}$$

by the definition of $f \beta$ in (89).

We notice that $\frac{\partial^m}{\partial t_1 \ldots \partial t_m} f E_j^{N}(B) = 0$ unless $B^* \supset \{x_1, x_2, \ldots, x_m\}$. Therefore,

$$f F_N = \sum_{j=0}^{N-1} \sum_{B \in B_j(\Lambda_N)} \sum_{X \in S_j, X \supset B} \frac{1}{|X|} \int f K_j^{N}(X, 0) \bigg|_{t_1=0, \ldots, t_m=0}$$

(160)

**Note:** Let $\eta = \min\{d/2, 2\}$. For any small $\epsilon > 0$, we can always find $A, L$ sufficiently large such that:

$$\| (L_1 + L_2 + L_3 + L_4) (0K) \|_{j+1} \leq \frac{1}{4L^\eta - \epsilon} ||K||_j$$

$$\| (L_1 + L_2) (\delta f K) \|_{j+1} \leq \frac{1}{4L^\eta - \epsilon} (||\delta f K||_j) \leq \frac{1}{4L^\eta - \epsilon} (||0K||_j + ||f K||_j)$$

(161)

with $j \geq 1$ by using the explicit upper bounds in Lemmas 8, 9, 11, and 8.

Then we can replace $\mu = 1/2$ in Theorem 7 by $\mu = 1/M$ for $M = L^\eta - \epsilon \geq 2$. We still have $|\sigma_j| \leq r M^{-j}$ and $||f K_j^{N}||_j \leq r M^{-j}$ and $||f E_j^{N}||_{j+1} \leq \mathcal{O}(L^2) r M^{-j}$ with max $k \leq a$ sufficiently small and $0 \leq j \leq N - 1$. Because $f K_j^{N}(X, 0)$ is analytic, using Cauchy’s bound and (79) in [3], we have:

$$\left| \frac{\partial^m}{\partial t_1 \ldots \partial t_m} f K_j^{N}(X, 0) \bigg|_{t_1=0, \ldots, t_m=0} \right| \leq \frac{m!}{a^m} \left( \frac{A}{2} \right)^{-|X|} ||f K_j^{N}||_j$$

$$\leq \frac{m!}{a^m} \left( \frac{A}{2} \right)^{-|X|} r M^{-j}$$

(162)
Then
\[
\left| \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{x \in S_j, x \in B} \frac{1}{|X|} fK_j^N(x, 0) \right|_{t_1=0, \ldots, t_m=0}
\]
\[
\leq \sum_{x \in S_j, x \in B} \frac{1}{|X|} \left| \frac{\partial^m}{\partial t_1 \cdots \partial t_m} fK_j^N(x, 0) \right|_{t_1=0, \ldots, t_m=0}
\]
\[
\leq \sum_{x \in S_j, x \in B} \frac{m!}{2^m} \left( \frac{A}{2} \right)^{|X|} rM^{-j}
\]
\[
\leq n_3(d, \frac{A}{2}) \frac{m!rM^{-j}}{a^m}
\]

So
\[
|fF_N| \leq \sum_{j=0}^{N-1} \sum_{B \in B_j(\Lambda_N)} n_3(d, \frac{A}{2}) \frac{m!rM^{-j}}{a^m}
\]

By \((153) + (157)\), we have:
\[
\left| \log \left( 1 + \int (fI_N(\sigma_N) - 1 + fK_N^N)(\Lambda_N)d\mu_{C_N} \right) \right|
\]
\[
\leq \log \left( 1 + 2\|F(\Lambda_N)\|_N \right)
\]
\[
\leq \log \left( 1 + 2[4c^{-1}h^2 + A^{-1}]r2^{-N} \right)
\]

Using the Cauchy’s bound as above, we obtain:
\[
\left| \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \log \left( 1 + \int (fI_N(\sigma_N) - 1 + fK_N^N)(\Lambda_N)d\mu_{C_N} \right) \right|_{t_1=0, \ldots, t_m=0}
\]
\[
\leq \frac{m!}{a^m} \log \left( 1 + 2[4c^{-1}h^2 + A^{-1}]r2^{-N} \right)
\]

So
\[
\lim_{N \to \infty} \left| \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \log \left( 1 + \int (fI_N(\sigma_N) - 1 + fK_N^N)(\Lambda_N)d\mu_{C_N} \right) \right|_{t_1=0, \ldots, t_m=0} = 0
\]

Now let \(j_0\) be the smallest integer such that \(\exists B \in B_{j_0} : B^* \supset \{x_1, x_2, \ldots, x_m\} \).
Without loosing the generality, we can assume that \(|x_1 - x_2| = \text{diam}(x_1, \ldots, x_m)\).
For every \(j \geq j_0\), let \(B_j^1 \in B_j\) be the unique \(j\)-block that contains \(\{x_1\}\). For any \(B \in B_j, j \geq j_0\)
with \(B^* \supset \{x_1, x_2, \ldots, x_m\}\), \(B\) must be in \(B_j^1\).
We have
\[
|fF_N| \leq \sum_{j=0}^{N-1} \sum_{B \in B_j(\Lambda_N)} n_3(d, \frac{A}{2}) \frac{m!rM^{-j}}{a^m}
\]
\[
= \sum_{j=j_0}^{N-1} \sum_{B \in B_j(\Lambda_N)} n_3(d, \frac{A}{2}) \frac{m!rM^{-j}}{a^m}
\]
Since $M \geq 2$, the last part of (168) is bounded by

$$\sum_{j=0}^{N-1} \sum_{B \in B_j(\Lambda_N)} n_3(d, A, m) \frac{m! r M^{-j}}{a^m} \leq \sum_{j=0}^{N-1} \sum_{B \in B_j(\Lambda_N)} n_3(d, A, m) \frac{m! r M^{-j}}{a^m}$$

$$\leq \sum_{j=0}^{N-1} (2d)^d n_3(d, A, m) \frac{m! r M^{-j}}{a^m}$$

$$\leq 2^{d(d+1)} n_3(d, A, m) \frac{m! r M^{-j}}{a^m}$$

(169)

Therefore, we have:

$$|f F_N| \leq 2^{d(d+1)} n_3(d, A, m) \frac{m! r M^{-j}}{a^m}$$

(170)

By the definition of $j_0$, we have: $|x_1 - x_2| \leq 2^{d+1} L j_0$. Because $M = L^{\eta-\epsilon}$, we get

$$M^{-j_0} = L^{-j_0(\eta-\epsilon)} \leq (2^{d+1})^j |x_1 - x_2|^{-\eta+\epsilon}$$

$$= (2^{d+1})^j \text{diam}^{-\eta+\epsilon}(x_1, \ldots, x_m)$$

(171)

Hence, we have:

$$|f F_N| \leq 2^{d(d+1)} n_3(d, A, m) \frac{m! r M^{-j}}{a^m} \text{diam}^{-\eta+\epsilon}(x_1, \ldots, x_m) \left(2^{d+1}\eta(d+1)\right)$$

(172)

Using this with (167), we obtain:

$$\left| \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \log Z \right|_{t_1=0, \ldots, t_m=0} \leq 2^{d(d+1)} 4 n_3(d, A, m) \frac{m! r M^{-j}}{a^m} \text{diam}^{-\eta+\epsilon}(x_1, \ldots, x_m) \left(2^{d+1}\eta(d+1)\right)$$

(173)

Combining with (171), we get $n_3(d, A, m) 2^{d(d+1)} 4 (2^{d+1})^\eta \leq 1$ with sufficiently large $A$. Therefore, with sufficiently large $A$, we have:

$$\left| G^t(x_1, x_2, \ldots, x_m) \right| = \left| \prod_{k=1}^m \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \phi(x_k) \right|^t \leq \left| \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \log Z \right|_{t_1=0, \ldots, t_m=0}$$

$$\leq \frac{m!}{a^m} \text{diam}^{-\eta+\epsilon}(x_1, \ldots, x_m)$$

(174)

We complete the proof of Theorem 2.

**Remark.** Actually for any $N - 1 \geq q \geq j_0$, similarly to (169), we have

$$\left| \sum_{j=q}^{N-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in \Sigma, X \supset B} \frac{1}{|X|^j} K^N_j(X, 0) \right|_{t_1=0, \ldots, t_m=0} \leq \sum_{j=q}^{N-1} \sum_{B \in B_j(\Lambda_N)} n_3(d, A, m) \frac{m! r M^{-j}}{a^m}$$

(175)
6.1.2 Proof of Theorem 3

Now we fix the set \( \{x_1, x_2, \ldots, x_m\} \). Let \( j_1 \) be the smallest integer such that \( B^0_{j_1} \supset \{x_1, x_2, \ldots, x_m\} \). Then \( j_1 \) is the smallest integer which is greater than \( \log_q \max\|x_i\|_\infty \). We also have: \( j_0 \leq j_1 \).

Let \( q \) be any number such that \( q \geq j_1 + 1 \geq j_0 + 1 \). And let \( N_1, N_2 \) be any integers such that \( N_2 \geq N_1 > q \). Using the definition of \( j_0 \), we have

\[
jF_{N_1} = \sum_{j=j_0}^{q-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0} + \sum_{j=q}^{N_1-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0} \tag{176}
\]

\[
jF_{N_2} = \sum_{j=j_0}^{q-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0} + \sum_{j=q}^{N_1-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0}
\]

We also notice that:

\[
\sum_{j=j_0}^{q-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0}
\]

\[
= \sum_{j=j_0}^{q-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0}
\tag{177}
\]

because for \( 0 \leq j \leq q-1 \), \( f K_j^N (X, 0) \) only depend on \( \phi \) within \( X^* \) and \( X^* \subset \Lambda_q \) which is the center \( q \)-block of \( \Lambda_{N_1} \subset \Lambda_{N_2} \). Therefore,

\[
|jF_{N_2} - jF_{N_1}| \leq \sum_{j=q}^{N_1-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0} \tag{178}
\]

Then using (175) with \( \mu = 1/2 \) instead of \( \mu = 1/M = L^{-\eta+\epsilon} \), we obtain:

\[
\left| \sum_{j=q}^{N_1-1} \sum_{B \in B_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in S_{j-1}} \frac{1}{|X_j|} f K_{N_2}^j(X, 0) \bigg|_{t_1=0, \ldots, t_m=0} \right| \leq 2^{d(d+1)} n_3 (d, \frac{A}{2} j_2^2) m^2 \sum_{j=q}^{N_1-1} \frac{2^{-q}}{a^m}
\tag{179}
\]
and

\[ \left| \sum_{j=q}^{N_1-1} \sum_{B \in \mathcal{B}_j(A_N)} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \sum_{X \in \mathcal{S}_j, X \supset B} \frac{1}{|X|^j} f_{N_1}(X, 0) \right|_{t_1=0, \ldots, t_m=0} \leq 2^{d(d+1)} n_3(d, A) \frac{m!r^{2-q}}{a^m} \]  

(180)

That means we have:

\[ |f_{N_1} - f_{N_2}| \leq 2^{d(d+1)} n_3(d, A) \frac{m!r^{2-q}}{a^m} \rightarrow 0 \]  

(181)

when \( q \rightarrow \infty \).

Combining this with (158) and (167), we can conclude that \( \lim_{N \rightarrow \infty} \langle \prod_{k=1}^m \partial \mu_k \phi(x_k) \rangle^t \) exists.

**Remark.** We have \( N \)-uniformly boundedness on correlation functions and \( \lim_{N \rightarrow \infty} \mathcal{G}(x_1, x_2, \ldots, x_m) \) exists. Therefore the bounds are held for infinite volume limit.

### 6.2 When \( f(\phi) = \sum_{k=1}^m t_k \exp(i \partial \mu_k \phi(x_k)) \)

Using exactly the same argument as the above subsection, we obtain these following results:

**Theorem 10** For any small \( \epsilon > 0 \), with \( L, A \) sufficiently large (depending on \( \epsilon \)), let \( \eta = \min\{d/2, 2\} \) we have:

\[ \left| \langle \prod_{k=1}^m \exp(i \partial \mu_k \phi(x_k)) \rangle^t \right| \leq \frac{m!}{a^m} \text{diam}^{-\eta+\epsilon}(x_1, \ldots, x_2) \]  

(182)

where \( a \) depends on \( \epsilon, L, A \).

**Theorem 11** With \( L, A \) sufficiently large, the infinite volume limit of the truncated correlation function \( \lim_{N \rightarrow \infty} \langle \prod_{k=1}^m \exp(i \partial \mu_k \phi(x_k)) \rangle^t \) exists.

### 6.3 Other cases

We can consider \( f(\phi) = \sum_{k=1}^m t_k f_k(\phi)(x_k) \) with

* \( t_k \in \mathbb{C} \)
* \( x_k \in \mathbb{Z}^d \) are different points.
* \( f_k \) is bounded in the sense that there are some \( M_k, m_k \geq 0 \) such that

\[ \| f_k(\{x_k\}, \phi) \|_0 \leq M_k \| \phi \| \phi_0 + m_k \]  

(183)

With the same argument as above cases, we have:

**Theorem 12** For any small \( \epsilon > 0 \), with \( L, A \) sufficiently large (depending on \( \epsilon \)), let \( \eta = \min\{d/2, 2\} \) we have:

\[ \left| \langle \prod_{k=1}^m f_k(\phi)(x_k) \rangle^t \right| \leq \frac{m!}{a^m} \text{diam}^{-\eta+\epsilon}(x_1, \ldots, x_2) \]  

(184)

where \( a \) depends on \( \epsilon, L, A \).

**Theorem 13** With \( L, A \) sufficiently large, the infinite volume limit of the truncated correlation function \( \lim_{N \rightarrow \infty} \langle \prod_{k=1}^m f_k(\phi)(x_k) \rangle^t \) exists.
In the case \( f = \sum_{k=1}^{m} t_k W_0(\{x_k\}) \), with \( W_0(\{x_k\}) = z W(1, \{x_k\}) \) as in (12). Using the Lemma 2 (or the lemma 4 in [5]), these \( W_0(\{x_k\}) \) satisfy those above conditions. The \( W_0(\{x_k\}) \) are actually the density of the dipoles at \( x_k \) used in [2]. Applying theorems [12] and [13] we obtain these results:

**Corollary 1** For any small \( \epsilon > 0 \), with \( L, A \) sufficiently large (depending on \( \epsilon \)), let \( \eta = \min\{d/2, 2\} \) we have:

\[
\left| \left\langle \prod_{k=1}^{m} W_0(\{x_k\}) \right\rangle^t \right| \leq \frac{m!}{a^n} \text{diam}^{-\eta + \epsilon}(x_1, \ldots, x_2)
\]

This result somehow looks like the theorem (1.1.2) in [2]. However it gives estimates for truncated correlation functions of (\( p \geq 2 \)) points instead of some estimate for only 2 points.

**Corollary 2** With \( L, A \) sufficiently large, the infinite volume limit of the truncated correlation function \( \lim_{N \to \infty} \left\langle \prod_{k=1}^{m} W_0(\{x_k\}) \right\rangle^t \) exists

**Remark.** We can consider the more general form \( f(\phi) = \sum_{k=1}^{m} t_k f_k(\phi) \) with

* \( t_k \in \mathbb{C} \)
* \( A_k \equiv \text{supp} f_k \) are pairwise disjoint and \( |A_k| < \infty \)
* \( f_k \) is bounded in the sense that there are some \( M_k, m_k \geq 0 \) such that

\[
\|f_k(A_k, \phi)\|_0 \leq M_k \|\phi\|_{\Phi_0} + m_k
\]

Then we still get similar results as in Theorems 12 and 13.

**A Kac-Siegert Transformation**

By expanding the exponential in (6) and carrying out the Gaussian integrals, we can rewrite \( a Z_N \) as

\[
a Z_N = \int \left( \sum_{n \geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \sum_{x_i \in \Lambda_A \cap \mathbb{Z}^d} \int_{S^{d-1}} dp_i (e^{ip_i \cdot \partial \phi(x_i)} + e^{-ip_i \cdot \partial \phi(x_i)}) / 2 \right) \mu_C(\phi)
\]

\[
= \int \left( \sum_{n \geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \sum_{x_i \in \Lambda_A \cap \mathbb{Z}^d} \int_{S^{d-1}} dp_i e^{ip_i \cdot \partial \phi(x_i)} \right) \mu_C(\phi)
\]

\[
= \sum_{n \geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \sum_{x_i \in \Lambda_A \cap \mathbb{Z}^d} \int_{S^{d-1}} dp_i e^{\int_{1}^{n} \sum_{k=1}^{p_k} \partial \phi(x_k)} \mu_C(\phi)
\]

which is exactly the same as the grand canonical partition function [6].

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34