Fredholm solvability of time-periodic boundary value hyperbolic problems

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Abstract

We investigate a large class of linear boundary value problems for the general first-order one-dimensional hyperbolic systems in the strip \([0,1] \times \mathbb{R}\). We state rather broad natural conditions on the data under which the operators of the problems satisfy the Fredholm alternative in the spaces of continuous and time-periodic functions. A crucial ingredient of our analysis is a non-resonance condition, which is formulated in terms of the data responsible for the bijective part of the Fredholm operator. In the case of \(2 \times 2\) systems with reflection boundary conditions, we provide a criterium for the non-resonant behavior of the system.

Keywords: first-order hyperbolic systems, periodic conditions in time, boundary conditions in space, non-resonance conditions, Fredholm alternative

1 Introduction

1.1 Motivation

We investigate the general linear first-order hyperbolic system in a single space variable

\[
\partial_t u_j + a_j(x,t)\partial_x u_j + \sum_{k=1}^n b_{jk}(x,t)u_k = f_j(x,t), \quad (x,t) \in (0,1) \times \mathbb{R}, \quad j \leq n, \quad (1.1)
\]
subjected to periodic conditions in time

\[ u_j(x, t) = u_j(x, t + 2\pi), \quad j \leq n, \quad t \in \mathbb{R} \]  

and boundary conditions in space

\[ u_j(0, t) = (Ru)_j(t), \quad 1 \leq j \leq m, \quad t \in \mathbb{R}, \]
\[ u_j(1, t) = (Ru)_j(t), \quad m < j \leq n, \quad t \in \mathbb{R}, \]  

\[(1.3)\]

where \(0 \leq m \leq n\) are positive integers and \(R = (R_1, \ldots, R_n)\) is a bounded linear operator.

From the physical point of view (see Examples 1.3–1.5 in Section 1.3.1), systems of the type (1.1)–(1.3) describe models of laser dynamics [14, 20, 21, 22], chemical kinetics [1, 15, 24], and population dynamics [2, 4]. These systems also have applications in the area of optimal boundary control problems [3, 19].

From the mathematical point of view, there is a need for developing a theory of local smooth continuation [12] and bifurcation [10] for Fredholm hyperbolic operators, in particular, such tools as Lyapunov-Schmidt reduction. Another source of our motivation is developing a stability theory of time-periodic solutions to hyperbolic PDEs, in particular, such tools as exponential dichotomies. Note that the known theorems about exponential dichotomies for ODEs and abstract evolution equations (see, e.g., [13, 17, 18]) are stated in terms of Fredholm solvability. For hyperbolic operators, even proving a Fredholm property is a nontrivial issue, and this is the subject that we consider in the present paper.

A particular case of (1.1)–(1.3) is studied in [6], where an existence result is obtained for solutions in the space of continuous and periodic in \(t\) functions. Specifically, the authors consider the system (1.1), (1.2) with the boundary conditions

\[ u_j(0, t) = \mu_j(t), \quad 1 \leq j \leq m, \]
\[ u_j(1, t) = \mu_j(t), \quad m < j \leq n, \]  

\[(1.4)\]

where \(\mu_j(t)\) are time-periodic. An essential assumption made in [6] is the smallness of all \(b_{jk}\). It comes from the Banach fixed point argument used in the proof of the main result. In the present paper we do not need this assumption and allow \(b_{jk}\) to be arbitrary elements of the space of continuous and time-periodic functions. Our main assumption, which is the non-resonance condition (1.14) stated in Section 1.2, is fulfilled in the setting of [6] (this is easy to see after the changing of variables \(u_j \rightarrow v_j = u_j - \mu_j(t)\)).

Time-periodic solutions to the system (1.1) with some reflection boundary conditions are investigated in [9, 11]. These papers suggest a rather general approach to proving the Fredholm alternative in the scale of Sobolev-type spaces of time-periodic functions (in the autonomous case [9]) and in the space of continuous and time-periodic functions (in the non-autonomous case [11]). In the present paper, we extend the approach from [11] to a quite general boundary operator \(R\) which covers periodic boundary conditions as well as boundary conditions with delays.
1.2 Our contribution

By \(C_{n,2\pi}\) we denote the vector space of all \(2\pi\)-periodic in \(t\) and continuous maps \(u : [0, 1] \times \mathbb{R} \to \mathbb{R}^n\), with the norm

\[
\|u\|_\infty = \max_{j \leq n} \max_{x \in [0,1]} \max_{t \in \mathbb{R}} |u_j|.
\]

Similarly, \(C^1_{n,2\pi}\) denotes the Banach space of all \(u \in C_{n,2\pi}\) such that \(\partial_x u, \partial_t u \in C_{n,2\pi}\), with the norm

\[
\|u\|_1 = \|u\|_\infty + \|\partial_x u\|_\infty + \|\partial_t u\|_\infty.
\]

Also, we use the notation \(C_{n,2\pi}(\mathbb{R})\) for the space of all continuous and \(2\pi\)-time-periodic maps \(v : \mathbb{R} \to \mathbb{R}^n\) and the notation \(C^1_{n,2\pi}(\mathbb{R})\) for the space of all \(v \in C_{n,2\pi}(\mathbb{R})\) with \(v' \in C_{n,2\pi}(\mathbb{R})\). For simplicity, we will skip the subscript \(n\) if \(n = 1\) and write simply \(C_{2\pi}\) for \(C_{1,2\pi}\) (similarly, we will write \(C^1_{2\pi}, C_{2\pi}(\mathbb{R}), C^1_{2\pi}(\mathbb{R})\) for \(C^1_{1,2\pi}, C_{1,2\pi}(\mathbb{R}), C^1_{1,2\pi}(\mathbb{R})\), respectively).

We make the following assumptions on the coefficients of (1.1):

\[
a_j, b_{jk} \in C^1_{2\pi} \text{ for all } j \leq n \text{ and } k \leq n, \tag{1.5}
\]

\[
a_j(x,t) \neq 0 \text{ for all } (x,t) \in [0, 1] \times \mathbb{R} \text{ and } j \leq n, \tag{1.6}
\]

and

\[
\text{for all } 1 \leq j \neq k \leq n \text{ there exists } \tilde{b}_{jk} \in C^1_{2\pi} \text{ such that } b_{jk} = \tilde{b}_{jk}(a_k - a_j). \tag{1.7}
\]

The operator \(R\) is supposed to be a bounded linear operator from \(C_{n,2\pi}\) to \(C_{n,2\pi}(\mathbb{R})\) satisfying the following condition:

- the restriction of the operator \(R\) to \(C^1_{n,2\pi}\) is a bounded linear operator from \(C^1_{n,2\pi}\) to \(C^1_{n,2\pi}(\mathbb{R})\). \tag{1.8}

Our goal is to prove the Fredholm alternative for (1.1)–(1.3). More specifically, we intend to show that, under a certain non-resonance condition on the coefficients \(a_j, b_{jj}\), and the boundary operator \(R\), either the space of nontrivial solutions to (1.1)–(1.3) with \(f = (f_1, ..., f_n) = 0\) is not empty and has finite dimension or the system (1.1)–(1.3) has a unique solution for any \(f\).

Let us introduce the characteristics of the hyperbolic system (1.1). Given \(j \leq n\), \(x \in [0, 1]\), and \(t \in \mathbb{R}\), the \(j\)-th characteristic is defined as the solution \(\xi \in [0, 1] \mapsto \omega_j(\xi, x, t) \in \mathbb{R}\) of the initial value problem

\[
\partial_\xi \omega_j(\xi, x, t) = \frac{1}{a_j(\xi, \omega_j(\xi, x, t))}, \quad \omega_j(x, x, t) = t. \tag{1.9}
\]

To shorten notation, we will simply write \(\omega_j(\xi) = \omega_j(\xi, x, t)\). Set

\[
c_j(\xi, x, t) = \exp \int_x^\xi \left( \frac{b_{jj}}{a_j} \right)(\eta, \omega_j(\eta)) d\eta, \quad d_j(\xi, x, t) = \frac{c_j(\xi, x, t)}{a_j(\xi, \omega_j(\xi))}. \tag{1.10}
\]
Integration along the characteristic curves brings the system (1.1)–(1.3) to the integral form

\[ u^j(x, t) = c^j(0, x, t)(Ru)^j(\omega_j(0)) + \int_0^x d_j(\xi, x, t) \sum_{k \neq j} b^j_k(\xi, \omega_j(\xi))u_k(\xi, \omega_j(\xi))d\xi + \int_0^x d_j(\xi, x, t)f^j(\xi, \omega_j(\xi))d\xi, \]

for \( 1 \leq j \leq m, \) (1.11)

\[ u^j(x, t) = c^j(1, x, t)(Ru)^j(\omega_j(1)) + \int_1^x d_j(\xi, x, t) \sum_{k \neq j} b^j_k(\xi, \omega_j(\xi))u_k(\xi, \omega_j(\xi))d\xi + \int_1^x d_j(\xi, x, t)f^j(\xi, \omega_j(\xi))d\xi, \]

for \( m < j \leq n. \) (1.12)

By straightforward calculation, one can easily show that a \( C^1 \)-map \( u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n \) is a solution to the PDE problem (1.1)–(1.3) if and only if it satisfies the system (1.11)–(1.12). This motivates the following definition.

**Definition 1.1** A function \( u \in C_{n,2\pi} \) is called a continuous solution to (1.1)–(1.3) if it satisfies (1.11) and (1.12).

Introduce an operator \( C \in \mathcal{L}(C_{n,2\pi}) \) by

\[ (Cv)^j(x, t) = \begin{cases} 
   c^j(0, x, t)(Ru)^j(\omega_j(0)) & \text{for } 1 \leq j \leq m, \\
   c^j(1, x, t)(Ru)^j(\omega_j(1)) & \text{for } m < j \leq n. 
\end{cases} \] (1.13)

**Theorem 1.2** Suppose that the conditions (1.5)–(1.8) are fulfilled. Assume that there exists \( \ell \in \mathbb{N} \) such that

\[ \|C^\ell\|_{\mathcal{L}(C_{n,2\pi})} < 1, \] (1.14)

for the operator \( C \) defined by (1.13). Let \( \mathcal{K} \) denote the vector space of all continuous solutions to (1.1)–(1.3) with \( f = 0. \) Then

(i) \( \dim \mathcal{K} < \infty \) and the vector space of all \( f \in C_{n,2\pi} \) such that there exists a continuous solution to (1.1)–(1.3) is a closed subspace of codimension \( \dim \mathcal{K} \) in \( C_{n,2\pi}. \)

(ii) If \( \dim \mathcal{K} = 0, \) then for any \( f \in C_{n,2\pi} \) there exists a unique continuous solution \( u \) to (1.1)–(1.3).

In Section 1.3 we comment about our crucial conditions (1.7) and (1.14) and give examples of the practical cases of the problem (1.1), (1.3) related to real life applications. Theorem 1.2 is proved in Section 2. Moreover, in Section 3 we consider the case of reflection boundary conditions and provide non-resonance conditions that are broader than (1.14). In the particular case of only two equations in the hyperbolic system (1.1), we derive a necessary and sufficient non-resonance condition, which is stable with respect to data perturbations.
1.3 Further comments

1.3.1 Examples related to applications

Example 1.3 Chemical kinetics. The paper [24] discusses catalytic processes in a chemical reactor. A reaction has first order if the reaction rate linearly depends on the amount of reactants. In the presence of a catalyst and the internal heat exchange, such reactions are described by the following boundary value problem for a $3 \times 3$-semilinear hyperbolic system:

$$\begin{align*}
\beta u_t + u_x &= KQe^u(1 - x) - \gamma (u - v), \\
v_t - v_x &= \gamma (u - v), \\
w_t - w_x &= K(1 - x), \\
u(0, t) &= v(0, t), \\
v(1, t) &= h(t), \\
w(0, t) &= 0,
\end{align*}$$

(1.15)

where $u$ denotes the temperature in the reactor, $v$ is the temperature in the refrigerator and $w$ is the concentration of the reactant. The positive constants $\gamma$, $K$, $\beta$, and $Q$ characterize a catalyst and a reactant.

It is easy to see that linearizations of (1.15) are particular cases of (1.1), (1.3).

Example 1.4 Chemotaxis. The following correlated random walk model for chemotaxis (chemosensitive movement, see [4]) consists of the hyperbolic system

$$\begin{align*}
\partial_t u^+ + \partial_x(a_1(x)u^+) &= -\mu_1(x)u^+ + \mu_2(x)u^-, \\
\partial_t u^- - \partial_x(a_2(x)u^-) &= -\mu_2(x)u^- + \mu_1(x)u^+
\end{align*}$$

(1.16)

and the boundary conditions

$$a^+(x)u^+(x, t) = a^-(x)u^-(x, t), \quad x = 0, 1,$$

of the type (1.3). Here $u^+$ and $u^-$ are the densities for right and left moving particles. Furthermore, $\mu_1$, $\mu_2$ are the turning rates and $a_1$, $a_2$ are the particle speeds that depend on the external signal $x$.

Example 1.5 Laser dynamics. The dynamic behavior of distributed feedback multisec- tion semiconductor lasers is represented by means of traveling wave models, describing the forward and backward propagating complex amplitudes of the light $u = (u_1, u_2)$. The model consists of a hyperbolic system coupled to an equation for the carrier density $v$, namely

$$\begin{align*}
\partial_t u(x, t) &= (-\partial_x u_1(x, t), \partial_x u_2(x, t)) + G(x, u(x, t), v(x, t)), \\
\partial_t v(x, t) &= I(x, t) + H(x, u(x, t), v(x, t)) \\
&\quad + \sum_{k=1}^m b_k \chi_{S_k}(x) \left( \frac{1}{x_k - x_{k-1}} \int_{S_k} v(y, t)dy - v(x, t) \right),
\end{align*}$$
which is supplemented with the reflection boundary conditions

\[ u_1(0, t) = r_0 u_2(0, t) + \alpha(t), \]
\[ u_2(1, t) = r_1 u_1(1, t). \]

Here \( 0 < r_0 < 1 \) and \( 0 < r_1 < 1 \) are reflection coefficients. This model describes the longitudinal dynamics of edge emitting lasers [14]. A linearization of the main, hyperbolic part of the model is covered by our system (1.1), (1.3).

1.3.2 About the non-resonance condition (1.14)

Suppose that there is \( \ell \in \mathbb{N} \) such that \( C^\ell = 0 \) in \( C_{n,2\pi} \). Such boundary conditions appear, for example, in optimal boundary control problems [19] and chemical kinetics [24]; they are smoothing in the sense of [7, 8, 16]. The condition (1.14) is satisfied by trivial reasons in this case, and the system (1.1)–(1.3) is non-resonant. Even this case shows that the assumption of Theorem 1.2, involving the existence of a suitable \( \ell \), is broader than the condition \( \|C\|_{\mathcal{L}(C_{n,2\pi})} < 1 \) (corresponding to \( \ell = 1 \)). Indeed, it is easy to see that, for each \( \ell > 1 \), there is an operator \( C \) such that \( C^\ell = 0 \) while (1.14) is not true for any smaller value of \( \ell \). One can easily check that this is exactly the case for the problem from chemical kinetics (1.15) with \( l = 2 \). Specifically, for the linearization of (1.15) at a stationary solution \( (u, v, w) = (u_0(x), v_0(x), w_0(x)) \) we have

\[
(C(u, v, w))_1(x, t) = \exp \left\{ \beta \int_0^x b_{11}(\eta) \, d\eta \right\} v(0, -\beta x + t), \\
(C(u, v, w))_2(x, t) = 0, \\
(C(u, v, w))_3(x, t) = 0,
\]

where \( b_{11}(x) = -K Q e^{u_0(x)}(1 - x) \). Evidently, \( C^2 = 0 \).

Consider now practical sufficient conditions making the assumption (1.14) true for small \( \ell \). For \( \ell = 1 \) such a condition is

\[
\|R\|_{\mathcal{L}(C_{n,2\pi})} \max \exp \int_x^{x_j} \left( \frac{b_{ij}}{a_j} \right) (\eta, \omega_j(\eta)) \, d\eta < 1. \quad (1.17)
\]

This easily follows from (1.13).

Now consider (1.14) for \( \ell = 2 \). Using the notation

\[
x_j = \begin{cases} 
0 & \text{if } 1 \leq j \leq m, \\
1 & \text{if } m < j \leq n
\end{cases} \quad (1.18)
\]

and the definition (1.13) of the operator \( C \), we have

\[
(C^2 u)_j(x, t) = c_j(x_j, x, t)(Ru)_j(\omega_j(x_j)), 
\]

where

\[
(Ru)_j(\omega_j(x_j)) = (R[c_1(x_1, x, t)(Ru)_1(\omega_1(x_1)), ..., c_n(x_n, x, t)(Ru)_n(\omega_n(x_n))])_j(\omega_j(x_j)).
\]
Therefore, the condition \( \| C^2 \|_{L(C_n,2\pi)} < 1 \) follows from
\[
\| R C \|_{L(C_n,2\pi)} \max_{j,x,t} \exp \int_x^{x_j} \left( \frac{b_{ij}}{a_j} \right) (\eta, \omega_j(\eta)) d\eta < 1.
\] (1.20)

There are simple examples when (1.20) is true while (1.17) is not.

1.3.3 About the conditions (1.7)

The following two examples show that the condition (1.7) plays a crucial role for our result.

**Example 1.6** Consider the \( 2 \times 2 \)-system
\[
\begin{align*}
\partial_t u_1 + \frac{1}{2\pi} \partial_x u_1 - u_2 &= 0, \\
\partial_t u_2 + \frac{1}{2\pi} \partial_x u_2 + u_1 &= 0,
\end{align*}
\] (1.21)
with periodic conditions in both \( t \) and \( x \), namely
\[
\begin{align*}
u_1(x,t) &= u_1(x,t + 2\pi), & u_2(x,t) &= u_2(x,t + 2\pi), \\
u_1(x,t) &= u_1(x+1,t), & u_2(x,t) &= u_2(x+1,t).
\end{align*}
\] (1.22)
(1.23)

This problem is a particular case of (1.1), (1.3) and satisfies all assumptions of Theorem 1.2 with the exception of (1.7). It is straightforward to check that
\[
\begin{align*}
u_1 &= \sin(2\pi x) \sin l(t - 2\pi x), & l &\in \mathbb{N}, \\
u_2 &= \cos(2\pi x) \sin l(t - 2\pi x), & l &\in \mathbb{N},
\end{align*}
\]
are infinitely many linearly independent solutions to the problem (1.21)–(1.23) and, therefore, the kernel of the operator of (1.21)–(1.23) is infinite dimensional. Thus, the conclusion of Theorem 1.2 is not true without (1.7).

**Example 1.7** Consider the \( 2 \times 2 \)-system
\[
\begin{align*}
\partial_t u_1 + \partial_x u_1 &= 0, \\
\partial_t u_2 + \partial_x u_2 + bu_1 &= 0,
\end{align*}
\] (1.24)
with the periodic conditions in time
\[
\begin{align*}u_1(x,t + 2\pi) &= u_1(x,t), & u_2(x,t + 2\pi) &= u_2(x,t),
\end{align*}
\]
and the reflection conditions in space
\[
\begin{align*}u_1(0,t) &= r_0u_2(0,t), & u_2(1,t) &= r_1u_1(1,t).
\end{align*}
\]
Here \( r_0 \) and \( r_1 \) are real numbers and \( b \) is a non-zero constant. If \( r_0 r_1 < 1 \), then all but (1.7) assumptions of Theorem 1.2 are fulfilled. If, moreover,

\[
b = \frac{r_0 r_1 - 1}{r_0},
\]

then

\[
\begin{align*}
  u_1(x, t) &= \sin t \sin l(t - x), \\
  u_2(x, t) &= b \left( \frac{1}{1 - r_0 r_1} - x \right) \sin l(t - x), \\
  l &\in \mathbb{N},
\end{align*}
\]

are infinitely many linearly independent solutions. Again, the conclusion of Theorem 1.2 is not true.

1.3.4 About the boundary conditions (1.3)

The boundary operator \( R \) covers different kinds of reflections, in particular, periodic boundary conditions in \( x \) and reflection boundary conditions with delays (see, e.g., [15] and references therein), for example, if

\[
(Ru)_j(t) = \sum_{k=1}^{n} \sum_{s=1}^{p} \left[ r_{jk}^0(t) u_k(0, t - \theta_s) + r_{jk}^1(t) u_k(1, t - \theta_s) \right], \quad j \leq n,
\]

where \( r_{jk}^0 \) and \( r_{jk}^1 \) are \( t \)-periodic and continuous functions and \( \theta_s \) are fixed real numbers.

2 Fredholm alternative (proof of Theorem 1.2)

Define bounded linear operators \( B, F : C_{n,2\pi} \to C_{n,2\pi} \) by

\[
(Bu)_j(x, t) = - \int_{x_j}^{x} d_j(\xi, x, t) \sum_{\xi \neq \zeta} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) d\xi, \quad j \leq n, \tag{2.1}
\]

and

\[
(Ff)_j(x, t) = \int_{x_j}^{x} d_j(\xi, x, t) f_j(\xi, \omega_j(\xi)) d\xi, \quad j \leq n, \tag{2.2}
\]

where \( x_j \) is given by (1.18). On the account of (1.13), (2.1), and (2.2), the system (1.11)–(1.12) can be written as the operator equation

\[
u = Cu + Bu + Ff.\]

Note that Theorem 1.2 says exactly that the operator \( I - C - B : C_{n,2\pi} \to C_{n,2\pi} \) is Fredholm of index zero.

**Lemma 2.1** The operator \( I - C : C_{n,2\pi} \to C_{n,2\pi} \) is bijective.
The proof is a straightforward consequence of the condition (1.14) and the Banach fixed-point theorem.

By Lemma 2.1, the operator \( I - C - B : C_{n,2\pi} \to C_{n,2\pi} \) is Fredholm of index zero if and only if
\[
I - (I - C)^{-1}B : C_{n,2\pi} \to C_{n,2\pi} \text{ is a Fredholm operator of index zero.} \quad (2.3)
\]

To prove (2.3), we will use Nikolsky’s criterion of Fredholmness in Banach spaces [5, Theorem XIII.5.2]. This criterion says that an operator \( I + K \) on a Banach space is Fredholm of index zero whenever \( K^2 \) is compact. It is interesting to note that the compactness of \( K^2 \) and the identity \( I - K^2 = (I + K)(I - K) \) imply that the operator \( I - K \) is a parametrix of the operator \( I + K \); see [23].

We, therefore, have to show that the operator \([(I - C)^{-1}B]^2 = (I - C)^{-1}B(I - C)^{-1}B : C_{n,2\pi} \to C_{n,2\pi}\) is compact. As the composition of a compact and a bounded operator is a compact operator, it is enough to show that
\[
B(I - C)^{-1}B : C_{n,2\pi} \to C_{n,2\pi} \text{ is compact.}
\]

Since \( B(I - C)^{-1}B = B(I + C + C^2 + ...)B = B^2 + BC(I + C + C^2 + ...)B = B^2 + BC(I - C)^{-1}B \) and \((I - C)^{-1}B\) is bounded, it suffices to prove that
\[
B^2 \text{ and } BC \text{ are compact operators from } C_{n,2\pi} \text{ to } C_{n,2\pi}. \quad (2.4)
\]

By the Arzela-Ascoli theorem, \( C^1_{n,2\pi} \) is compactly embedded into \( C_{n,2\pi} \). The desired compactness property (2.4) will follow if we show that
\[
B^2 \text{ and } BC \text{ map continuously } C_{n,2\pi} \text{ into } C^1_{n,2\pi}. \quad (2.5)
\]

Using (1.13), (2.1) and the equalities
\[
\partial_x\omega_j(\xi) = -\frac{1}{a_j(x,t)} \exp \int_{\xi}^x \left( \frac{\partial_2 a_j}{a_j^2} \right)(\eta,\omega_j(\eta))d\eta, \quad (2.6)
\]
\[
\partial_t\omega_j(\xi) = \exp \int_{\xi}^x \left( \frac{\partial_2 a_j}{a_j^2} \right)(\eta,\omega_j(\eta))d\eta, \quad (2.7)
\]
being true for all \( j \leq n, \xi, x \in [0, 1], \) and \( t \in \mathbb{R} \), we see that the partial derivatives \( \partial_x B^2 u, \partial_t B^2 u, \partial_x BC u, \partial_t BC u \) exist and are continuous for each \( u \in C^1_{n,2\pi} \). Here and below by \( \partial_i \) we denote the partial derivative with respect to the \( i \)-th argument. Since \( C^1_{n,2\pi} \) is dense in \( C_{n,2\pi} \), the desired condition (2.5) will follow from the next lemma, whose proof will therefore complete proving Theorem 1.2.

**Lemma 2.2** For all \( u \in C^1_{n,2\pi} \) we have
\[
\|B^2 u\|_1 + \|BC u\|_1 = O(\|u\|_\infty). \quad (2.8)
\]
Proof. Claim 1. The following estimate is true:

$$\|B^2 u\|_1 = O(\|u\|_\infty) \text{ for all } u \in C^1_{n,2\pi}. \quad (2.9)$$

Given \( j \leq n \) and \( u \in C^1_{n,2\pi} \), let us consider the following representation for \((B^2 u)_j(x,t)\) obtained by application of the Fubini theorem:

$$(B^2 u)_j(x,t) = \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^\pi d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta, \quad (2.10)$$

where

$$d_{jkl}(\xi, \eta, x, t) = d_j(\xi, x, t) d_k(\eta, \xi, \omega_j(\xi)) b_{kl}(\eta, \omega_k(\eta, \xi, \omega_j(\xi))). \quad (2.11)$$

From (2.10) we immediately get the bound

$$\|B^2 u\|_\infty = O(\|u\|_\infty).$$

We now claim that

$$\|[(\partial_t + a_j(x,t)\partial_x)(B^2 u)_j]\|_\infty = O(\|u\|_\infty) \text{ for all } j \leq n \text{ and } u \in C^1_{n,2\pi}. \quad (2.12)$$

To prove this, we use the identity (which follows from (2.6) and (2.7))

$$(\partial_t + a_j(x,t)\partial_x)\varphi(\omega_j(\xi, x, t)) \equiv 0,$$

being true for all \( j \leq n, \varphi \in C^1(\mathbb{R}), x, \xi \in [0,1], \) and \( t \in \mathbb{R} \). On the account of (1.10) and (2.11), this entails that for all \( j \leq n, k \leq n, \) and \( l \leq n \) we have

$$(\partial_t + a_j(x,t)\partial_x) d_{jkl}(\xi, \eta, x, t) \equiv 0,$$

$$(\partial_t + a_j(x,t)\partial_x) b_{jk}(\xi, \omega_j(\xi)) \equiv 0,$$

$$(\partial_t + a_j(x,t)\partial_x) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \equiv 0.$$ 

Using (2.10), we conclude that

$$(\partial_t + a_j(x,t)\partial_x)(B^2 u)_j$$

$$= (\partial_t + a_j(x,t)\partial_x) \left( \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^\pi d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta \right)$$

$$= a_j(x,t) \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x d_{jkl}(x, \eta, x, t) b_{jk}(x, \omega_j(x)) u_l(\eta, \omega_k(\eta, x, \omega_j(x))) d\eta$$

$$= a_j(x,t) \sum_{k \neq j} b_{jk}(x,t) \sum_{l \neq k} \int_{x_j}^x d_{jkl}(x, \eta, x, t) u_l(\eta, \omega_k(\eta)) d\eta.$$ 

The estimate (2.12) now easily follows.

In order to prove (2.9), we have to prove two estimates

$$\|\partial_x B^2 u\|_\infty = O(\|u\|_\infty). \quad (2.13)$$
and
\[ \| \partial_t B^2 u \|_\infty = O(\| u \|_\infty). \] (2.14)

Since (2.13) follows from (2.14) by (2.12) and (1.6), it is enough to prove (2.14).

To this end, we start with the following consequence of (2.10):
\[
\begin{align*}
\partial_t [(B^2 u)_j(x, t)] & = \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x \frac{d}{dt} \left[ d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) \right] u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta \\
& + \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) \\
& \times \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_t \omega_j(\xi) \partial_t u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta.
\end{align*}
\]

Let us transform the second summand. Using (1.9), (2.6), and (2.7), we get
\[
\begin{align*}
\frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) & = \left[ \partial_\xi \omega_k(\eta, \xi, \omega_j(\xi)) + \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_\xi \omega_j(\xi) \right] \partial_t u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \\
& = \left( \frac{1}{a_j(\xi, \omega_j(\xi))} - \frac{1}{a_k(\xi, \omega_j(\xi))} \right) \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_t u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))).
\end{align*}
\]

Therefore,
\[
\begin{align*}
b_{jk}(\xi, \omega_j(\xi)) & \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_t u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \\
& = a_j(\xi, \omega_j(\xi)) a_k(\xi, \omega_j(\xi)) b_{jk}(\xi, \omega_j(\xi)) \frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))),
\end{align*}
\]
where the functions $b_{jk} \in C_{2\pi}$ are fixed so that they satisfy (1.7). Note that $b_{jk}$ are not uniquely defined by (1.7) for $(x, t)$ with $a_j(\xi, x, t) = a_k(\xi, x, t)$. Nevertheless, as it follows from (2.15), the right-hand side (and, hence, the left-hand side of (2.16)) do not depend on the choice of $b_{jk}$, since $\frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) = 0$ if $a_j(\xi, x, t) = a_k(\xi, x, t)$.

Write
\[
\tilde{d}_{jkl}(\xi, \eta, x, t) = d_{jkl}(\xi, \eta, x, t) \partial_t \omega_j(\xi) a_k(\xi, \omega_j(\xi)) a_j(\xi, \omega_j(\xi)) b_{jk}(\xi, \omega_j(\xi)),
\]
where $d_{jkl}$ is introduced by (2.11) and (1.10). Using (1.9) and (2.6), we see that the function $\tilde{d}_{jkl}(\xi, \eta, x, t)$ is $C^1$-regular in $\xi$ due to the regularity assumptions (1.5) and (1.7). Similarly, using (2.7), we see that the functions $d_{jkl}(\xi, \eta, x, t)$ and $b_{jk}(\xi, \omega_j(\xi))$ are $C^1$-smooth in $t$.

By (2.16) we have
\[
\begin{align*}
\partial_t [(B^2 u)_j(x, t)] & = \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x \frac{d}{dt} \left[ d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) \right] u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta.
\end{align*}
\]
The desired estimate (2.14) now easily follows from the assumptions (1.5), (1.6) and (1.7).

Claim 2. The following estimate is true:

$$\|BCu\|_1 = O(\|u\|_\infty) \text{ for all } u \in C^1_{n,2\pi}.$$  

We are done if we show that

$$\|BCu\|_\infty + \|\partial_t BCu\|_\infty = O(\|u\|_\infty) \text{ for all } u \in C^1_{n,2\pi},$$  \hspace{1cm} (2.17)

as the estimate for $\partial_x BCu$ follows similarly to the case of $\partial_x B^2u$. In order to prove (2.17), we consider an arbitrary integral contributing into $BCu$, namely

$$\int_{x}^{x_j} e_{jk}(\xi, x, t)b_{jk}(\xi, \omega_j(\xi))(Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi)))d\xi,$$  \hspace{1cm} (2.18)

where

$$e_{jk}(\xi, x, t) = d_j(\xi, x, t)c_k(x_k, \xi, \omega_j(\xi))$$

and $j \leq n$ and $k \leq n$ are arbitrary fixed. From (2.18) it follows the bound

$$\|BCu\|_\infty = O(\|u\|_\infty).$$

Differentiating (2.18) in $t$, we get

$$\int_{x}^{x_j} \frac{d}{dt}\left[e_{jk}(\xi, x, t)b_{jk}(\xi, \omega_j(\xi))(Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi)))\right]d\xi \hspace{1cm} (2.19)$$

$$+ \int_{x}^{x_j} e_{jk}(\xi, x, t)b_{jk}(\xi, \omega_j(\xi))\partial_t \omega_k(x_k, \xi, \omega_j(\xi))\partial_t \omega_j(\xi)(Ru)_k'(\omega_k(x_k, \xi, \omega_j(\xi)))d\xi.$$ 

Our task is to estimate the second integral; for the first one the desired estimate is obvious. Similarly to the above, we use (1.9), (2.6), and (2.7) to obtain

$$\frac{d}{d\xi}(Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi)))$$

$$= \left[\partial_x \omega_k(x_k, \xi, \omega_j(\xi)) + \partial_t \omega_k(x_k, \xi, \omega_j(\xi))\partial_x \omega_j(\xi)\right](Ru)_k'(\omega_k(x_k, \xi, \omega_j(\xi)))$$

$$= \left(-\frac{1}{a_j(\xi, \omega_j(\xi))} - \frac{1}{a_k(\xi, \omega_j(\xi))}\right)\partial_t \omega_k(x_k, \xi, \omega_j(\xi))(Ru)_k'(\omega_k(x_k, \xi, \omega_j(\xi))).$$
Taking into account (1.7), the last expression reads
\[
\begin{align*}
b_{jk}(\xi, \omega_j(\xi)) &= a_j(\xi, \omega_j(\xi))a_k(\xi, \omega_j(\xi))\tilde{b}_{jk}(\xi, \omega_j(\xi)) \frac{d}{d\xi} (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))).
\end{align*}
\] (2.20)

Set
\[
\tilde{e}_{jk}(\xi, x, t) = e_{jk}(\xi, x, t)\partial_t \omega_j(\xi)a_k(\xi, \omega_j(\xi))a_j(\xi, \omega_j(\xi))\tilde{b}_{jk}(\xi, \omega_j(\xi)).
\]

Using (2.20), let us transform the second summand in (2.19) as
\[
\int_x^{x_j} e_{jk}(\xi, x, t)b_{jk}(\xi, \omega_j(\xi))\partial_t \omega_k(x_k, \xi, \omega_j(\xi))\partial_t \omega_j(\xi)(Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi)))d\xi
\]
\[
= \int_x^{x_j} \tilde{e}_{jk}(\xi, x, t)\frac{d}{d\xi} (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi)))d\xi
\]
\[
= \left[ \tilde{e}_{jk}(\xi, x, t)(Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \right]_{\xi=x_j}^{\xi=x} - \int_x^{x_j} \partial_\xi \tilde{e}_{jk}(\xi, x, t)(Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi)))d\xi.
\] (2.21)

The bound (2.17) now easily follows from (2.19) and (2.21). The lemma is therewith proved.

\[\square\]

3 Reflection boundary conditions and non-resonant behavior

As we have seen in Section 1.3.1, in many mathematical models the system (1.1) is controlled by the so-called reflection boundary conditions. We intend to show that for such problems the basic assumption (1.14) of Theorem 1.2 can be extended.

3.1 The case of 2 \times 2 systems

Let (1.1) be a system of two equations, namely
\[
\begin{align*}
\partial_t u_1 + a_1(x, t)\partial_x u_1 + b_{11}(x, t)u_1 + b_{12}(x, t)u_2 &= f_1(x, t), \\
\partial_t u_2 + a_2(x, t)\partial_x u_2 + b_{21}(x, t)u_1 + b_{22}(x, t)u_2 &= f_2(x, t),
\end{align*}
\] (3.1)

endowed with the periodic conditions in time
\[
u_j(x, t) = u_j(x, t + 2\pi), \quad j = 1, 2,
\] (3.2)

and the boundary conditions
\[
\begin{align*}
u_1(0, t) &= (Ru)_1(t) = p_0(t)u_2(0, t), \\
u_2(1, t) &= (Ru)_2(t) = p_1(t)u_1(1, t),
\end{align*}
\] (3.3)
where $p_0, p_1 \in C_{2\pi}(\mathbb{R})$. We are able to derive a sharp non-resonance condition (ensuring the bijectivity of the operator $I - C$, where $C$ is introduced by (1.13)), which is stable with respect to data perturbations. Accordingly to (3.1)–(3.3), the operator $C : C_{2,2\pi} \rightarrow C_{2,2\pi}$ reads
\[
(Cv)_j(t) = \begin{cases} 
 c_1(0, x, t)p_0(\omega_1(0))v_2(0, \omega_1(0)) & \text{for } j = 1, \\
 c_2(1, x, t)p_1(\omega_2(1))v_1(1, \omega_2(1)) & \text{for } j = 2.
\end{cases}
\]
Then the bijectivity of $I - C : C_{2,2\pi} \rightarrow C_{2,2\pi}$ means that the system
\[
\begin{align*}
  u_1(x, t) &= c_1(0, x, t)p_0(\omega_1(0))u_2(0, \omega_1(0)) \\
  u_2(x, t) &= c_2(1, x, t)p_1(\omega_2(1))u_1(1, \omega_2(1))
\end{align*}
\]
has a unique (trivial) solution in $C_{2,2\pi}$ or, the same, the system
\[
\begin{align*}
  u_1(x, t) &= c_1(0, x, t)p_0(\omega_1(0))c_2(1, 0, \omega_1(0))p_1(\omega_2(1, 0, \omega_1(0)))u_1(1, \omega_2(1, 0, \omega_1(0))) \\
  u_2(x, t) &= c_2(1, x, t)p_1(\omega_2(1))u_1(1, \omega_2(1))
\end{align*}
\]
has a unique solution in $C_{2,2\pi}$. The first equation at $x = 1$ reads
\[
u_1(1, t) = c_1(0, 1, t)p_0(\omega_1(0, 1, t))c_2(1, 0, \omega_1(0, 1, t))
\times p_1(\omega_2(1, 0, \omega_1(0, 1, t)))u_1(1, \omega_2(1, 0, \omega_1(0, 1, t))).
\tag{3.4}
\]
Consider two maps $z(t) = \omega_2(1, 0, t)$ and $z(t) = \omega_1(0, 1, t)$. Due to (1.6), both of them are monotonically increasing from $\mathbb{R}$ to $\mathbb{R}$. Hence, the map $z(t) = \omega_2(1, 0, \omega_1(0, 1, t))$ is bijective. Moreover, the equation (3.4) is uniquely solvable in $C_{2,2\pi}$ if and only if
\[
|c_1(0, 1, t)p_0(\omega_1(0, 1, t))c_2(1, 0, \omega_1(0, 1, t))p_1(\omega_2(1, 0, \omega_1(0, 1, t)))| \neq 1 \text{ for all } t \in \mathbb{R},
\]
or, the same, if and only if
\[
\exp \int_0^1 \left[ \left( \frac{b_{22}}{a_2} \right) (\eta, \omega_2(\eta, 0, \omega_1(0, 1, t))) - \left( \frac{b_{11}}{a_1} \right) (\eta, \omega_1(\eta, 1, t)) \right] d\eta
\times |p_0(\omega_1(0, 1, t))p_1(\omega_2(1, 0, \omega_1(0, 1, t)))| \neq 1 \text{ for all } t \in \mathbb{R}.
\tag{3.5}
\]
This is the desired non-resonance condition, which is obviously sharp. Moreover, it is stable with respect to data perturbation. Note that, if (3.5) is not fulfilled, then (3.1)–(3.3) demonstrates the so-called completely resonance behavior.

We also see that the non-resonant behavior of the system (3.1)–(3.3) is controlled by the coefficients $a_1, a_2$ of the differential part and by the coefficients $b_{11}, b_{22}$ of the diagonal lower order part of the hyperbolic system, as well as by the reflection coefficients $p_0, p_1$.

### 3.2 The case of $n \times n$ systems

Let us consider the system (1.1) with the reflection boundary conditions
\[
\begin{align*}
  u_j(0, t) &= \sum_{k=m+1}^n p_{jk}(t)u_k(0, t) + \sum_{k=1}^m p_{jk}(t)u_k(1, t), \quad 1 \leq j \leq m, \\
  u_j(1, t) &= \sum_{k=m+1}^n p_{jk}(t)u_k(0, t) + \sum_{k=1}^m p_{jk}(t)u_k(1, t), \quad m < j \leq n,
\end{align*}
\tag{3.6}
\]
where \( p_{jk} \in C_{2\pi}(\mathbb{R}) \). Then the operator \( C : C_{n,2\pi} \to C_{n,2\pi} \) reads

\[
(Cu)_j(x, t) = c_j(x_j, x, t) \left[ (1 - x_j) \sum_{k=1}^{n} p_{jk}(\omega_j(0))v_k(1 - x_k, \omega_j(0)) + x_j \sum_{k=1}^{n} p_{jk}(\omega_j(1))v_k(1 - x_k, \omega_j(1)) \right], \quad j \leq n.
\]

Introduce the functions

\[
S_j(t) = \begin{cases} 
  c_j(0, 1, t) \sum_{k=1}^{n} |p_{jk}(\omega_j(0, 1, t))| & \text{for } 1 \leq j \leq m, \\
  c_j(1, 0, t) \sum_{k=1}^{n} |p_{jk}(\omega_j(1, 0, t))| & \text{for } m < j \leq n.
\end{cases}
\]

A non-resonance condition analogous to (1.14) can be stated as

\[
\max_{j \leq n} \max_{t \in \mathbb{R}} S_j(t) < 1. \tag{3.7}
\]

Using the strategy of the proof of Theorem 1.2, let us show that under the condition (3.7) the system

\[
u_j(x, t) = c_j(x_j, x, t) \left[ (1 - x_j) \sum_{k=1}^{n} p_{jk}(\omega_j(0))u_k(1 - x_k, \omega_j(0)) + x_j \sum_{k=1}^{n} p_{jk}(\omega_j(1))u_k(1 - x_k, \omega_j(1)) \right], \quad j \leq n
\]

is uniquely solvable in \( C_{n,2\pi} \) with respect to \( u_j, j \leq n \). Putting \( x = 0 \) for \( m < j \leq n \) and \( x = 1 \) for \( j \leq m \) in (3.8), we get the following system of \( n \) equations with respect to \( n \) unknowns \( u_j(0, t), m < j \leq n \) and \( u_j(1, t), j \leq m \):

\[
u_j(0, t) = c_j(1, 0, t) \sum_{k=1}^{n} p_{jk}(\omega_j(0, 1, 0))u_k(1 - x_k, \omega_j(1, 0, t)), \quad m < j \leq n,
\]

\[
u_j(1, t) = c_j(0, 1, t) \sum_{k=1}^{n} p_{jk}(\omega_j(0, 1, 0))u_k(1 - x_k, \omega_j(0, 1, t)), \quad 1 \leq j \leq m.
\tag{3.9}
\]

Notice that the unique solvability of (3.9) in \( C_{n,2\pi} \) entails the unique solvability of (3.8) in \( C_{n,2\pi} \). From (3.9) we have

\[
\max_{m < j \leq n} \max_{j \leq m} \max_{t, \tau \in \mathbb{R}} \{ |u_t(0, t)|, |u_j(1, \tau)| \} \\
\leq \max_{t, \tau \in \mathbb{R}} \left\{ \max_{j \leq m} c_j(0, 1, \tau) \sum_{k=1}^{n} |p_{jk}(\omega_j(0, 1, \tau))|u_k(1 - x_k, \omega_j(0, 1, \tau)) \right\},
\]

\[
\max_{m < j \leq n} c_j(1, 0, t) \sum_{k=1}^{n} |p_{jk}(\omega_j(1, 0, t))|u_k(1 - x_k, \omega_j(1, 0, t)) \}
\leq \max_{j \leq n} \max_{t \in \mathbb{R}} S_j(t) \max_{k \leq m} \max_{m < i \leq n} \max_{t, \tau \in \mathbb{R}} \{ |u_t(0, t)|, |u_k(1, \tau)| \}.
\tag{3.10}
\]
Using (3.10) and applying the Banach fixed-point argument to (3.9), we conclude that (3.7) ensures the unique solvability of (3.8), as desired.

We now show, in addition to (3.7), another sufficient non-resonance condition. To this end, we change the variable $t$ to $\tau = \omega_j(1, 0, t)$ for $m < j \leq n$ and $t$ to $\tau = \omega_j(0, 1, t)$ for $j \leq m$ in (3.9). This allows us to rewrite the system (3.9) as follows:

\begin{align*}
    u_j(0, \omega_j(1, 0, \tau)) &= c_j(1, 0, \omega_j(1, 0, \tau)) \sum_{k=1}^{n} p_{jk}(\tau) u_k(1 - x_k, \tau), \quad m < j \leq n, \\
    u_j(1, \omega_j(0, 1, \tau)) &= c_j(0, 1, \omega_j(0, 1, \tau)) \sum_{k=1}^{j} p_{jk}(\tau) u_k(1 - x_k, \tau), \quad j \leq m.
\end{align*}

Set $v(t) = (u_1(1, t), \ldots, u_m(1, t), u_{m+1}(0, t), \ldots, u_n(0, t))$ and rewrite (3.11) in the operator-matrix form

\[(Gv)(t) = Q(t)v(t),\]

where the operator $G \in \mathcal{L}(C_{n, 2\pi}(\mathbb{R}))$ is given by

\[(Gv)(t) = (u_1(1, \omega_j(0, 1, \tau)), \ldots, u_m(1, \omega_j(0, 1, \tau)), u_{m+1}(0, \omega_j(1, 0, \tau)), \ldots, u_n(0, \omega_j(1, 0, \tau))),\]

and the matrix $Q(t)$ is defined by the right-hand side of (3.11). Assume that the matrix $Q(t)$ is invertible for all $t \in \mathbb{R}$, and, moreover,

\[\|Q^{-1}(t)\|_{\infty} < 1.\]

(3.12)

Then the system (3.11) and, hence, the system (3.8) is uniquely solvable. This means that (3.12) is, additionally to (3.7), a non-resonance condition for the problem (1.1), (3.6).

To illustrate applicability of these two non-resonance conditions, suppose that the coefficients $a_j$, $b_{jj}$, and $p_{jk}$ are constant. In this case the condition (3.7) is simplified to

\[\exp\left\{\left(-1\right)^{1-x_j}b_{jj}\frac{a_j}{a_j}\right\} \sum_{k=1}^{n} |p_{jk}| < 1 \quad \text{for all } j \leq n.\]

The matrix $Q$ in this case does not depend on $t$ and reads

\[Q = \begin{pmatrix}
    p_{11} \exp\left\{\frac{-b_{11}}{a_j}\right\} & \cdots & p_{1n} \exp\left\{\frac{-b_{1n}}{a_j}\right\} \\
    \vdots & \ddots & \vdots \\
    p_{m1} \exp\left\{\frac{-b_{m1}}{a_m}\right\} & \cdots & p_{mn} \exp\left\{\frac{-b_{mn}}{a_m}\right\} \\
    p_{m+1,1} \exp\left\{\frac{b_{m+1,1}}{a_{m+1}}\right\} & \cdots & p_{m+1,n} \exp\left\{\frac{b_{m+1,n}}{a_{m+1}}\right\} \\
    \vdots & \ddots & \vdots \\
    p_{n1} \exp\left\{\frac{b_{n1}}{a_n}\right\} & \cdots & p_{nn} \exp\left\{\frac{b_{nn}}{a_n}\right\}
\end{pmatrix}.\]

If $Q$ is invertible and the norm of $Q^{-1}$ is less than one, then we meet our second non-resonance condition (3.12).
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