Q′-curvature flow on Pseudo-Einstein CR manifolds

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Abstract In this paper we consider the problem of prescribing the Q′-curvature on three dimensional Pseudo-Einstein CR manifolds. We study the gradient flow generated by the related functional and we will prove its convergence to a limit function under suitable assumptions.

Keywords: Pseudo-Einstein CR manifolds, P′-operator

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1 Introduction and statement of the results

Let (M, T1,0M, θ) be a CR three manifold, which we will always assume smooth and closed. It is known that one can construct a pair (Q, Pθ) such that under a conformal change of the contact form ˆθ = e2uθ, one has

Pθu + Qθ = Qθ e4u

where the Paneitz operator Pθ = (∆θ)2 + T2 + l.o.t.; in particular the operator Pθ contains the space of CR pluriharmonic functions P in its kernel, moreover the total Q-curvature is always zero [16], hence it does not provide any extra geometric information. Therefore, one considers another pair (P′, Q′), see [3], where P′ is a Paneitz type operator satisfying P′ = 4(∆θ)2 + l.o.t. and is defined on the space of pluriharmonic functions and the Q′-curvature is defined implicitly so that

P′θu + Q′θ = P′θ(θ)2 = Q′θ e2u,

which is equivalent to

P′θu + Q′θ = Q′θ e2u mod P⊥.

(1)

In the case of pseudo-Einstein three dimensional CR manifolds (we refer the reader to the next section for further details), in [10] the authors showed that the total Q′-curvature is not always zero and it is invariant under the conformal change of the contact structure; in

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particular it is proportional to the Burns-Epstein invariant $\mu(M)$ (see [5], [13]) and if $(M, J)$ is the boundary of a strictly pseudo-convex domain $X$, then

$$\int_M Q' \theta \wedge d\theta = 16\pi^2 \left( \chi(X) - \int_X (c_2 - \frac{1}{3}c_1^2) \right),$$

where $c_1$ and $c_2$ are the first and second Chern forms of the Kähler-Einstein metric on $X$ obtained by solving Fefferman’s equation.

At this point, in order to avoid the problem of solving orthogonally to the infinite dimensional space $P^\perp$, in [11] it is introduced on pseudo-Einstein three dimensional CR manifolds a new couple $(\tilde{P}, \tilde{Q})$ which comes from the projection of equation (1) on to the space of $L^2$ CR pluriharmonic functions $\tilde{P}$, the completion of $P$ under the $L^2$-norm. Since the $P'$-operator is only defined after projection on $P$, if $\Gamma : L^2(M) \rightarrow \tilde{P}$ denotes the orthogonal projection and we let $\tilde{P}' = \Gamma \circ P'$ and $\tilde{Q}' = \Gamma \circ Q'$, then one can consider the problem of prescribing the $\tilde{Q}'$-curvature, under conformal change of the contact structure on pseudo-Einstein CR manifolds. In particular, for a given a function $f \in \tilde{P}$, one wants to solve the following equation

$$P'u + Q' = fe^{2u} \mod P^\perp,$$ (2)

that is equivalent to

$$\tilde{P}'u + \tilde{Q}' = \Gamma(fe^{2u}).$$

Therefore, if $u$ solves (2), then for $\tilde{\theta} = e^{u}\theta$, one has $\tilde{Q}'_{\tilde{\theta}} = f$. Let us explicitly notice the differences between the two projections, since the space of $L^2$ CR pluriharmonic functions $\tilde{P}$ does not depend on the contact form. Thus, $\tilde{Q}'$ is the orthogonal projection of $Q'$ on $\tilde{P}$ with respect to the $L^2$-inner product induced by $\theta$, while $\tilde{Q}'_{\tilde{\theta}}$ is the orthogonal projection of $Q'_{\tilde{\theta}}$ with respect to the $L^2$-inner product induced by $\tilde{\theta}$; in particular $\phi \in \tilde{P}_\theta$ if and only if $\phi \in \tilde{P}_{\tilde{\theta}}$ and $\psi \in P^\perp$ if and only if $e^{-2u}\psi \in P^\perp_{\tilde{\theta}}$. Therefore, by denoting $\Gamma_u$ the orthogonal projection induced by $\tilde{\theta}$, one has $\Gamma_u(Q'_{\tilde{\theta}}) = f$. Let us also recall that in [10], the authors show that the non-negativity of the Paneitz operator $P_\theta$ and the positivity of the CR-Yamabe invariant imply that $\tilde{P}'$ is non-negative and $\ker \tilde{P}' = \mathbb{R}$. Moreover, $\int_M Q' = \int_M \tilde{Q}' \leq 16\pi^2$ with equality if an only if $(M, T^{1,0}M, \theta)$ is the standard sphere; in particular the previously assumptions imply that the $(M, T^{1,0}M, \theta)$ is embeddable (see [12]). Notice that unlike the Riemannian case, it remains unclear if the non-negativity of $\tilde{P}'$ and $\ker \tilde{P}' = \mathbb{R}$ is a sufficient condition for $\int_M \tilde{Q}' \leq 16\pi^2$. In particular, the results presented in this paper do not fully cover the case $\tilde{P}' \geq 0$ and $\ker \tilde{P}' = \mathbb{R}$.

Thus, from now on we will always assume that $(M, T^{1,0}M, \theta)$ is a pseudo-Einstein CR three dimensional manifold such that $\tilde{P}'$ is non-negative and $\ker \tilde{P}' = \mathbb{R}$. The problem in (2) was first studied in [10] for $f$ constant and in the subcritical case, namely $\int_M \tilde{Q}' < 16\pi^2$. Then in [21] the problem was solved for $f > 0$ via a probabilistic approach again in the subcritical case; also, a solution of the problem was provided in [17] for $f > 0$ and $0 < \int_M \tilde{Q}' < 16\pi^2$ via direct minimization.

In this paper we will study the equation (2) allowing $f$ to change sign: our approach follows closely the methods in [11], where the authors study the analogous problem in the Riemannian setting. In particular, we will use a variational approach by defining a suitable functional on an appropriate space and then we will study the evolution problem along the negative gradient flow lines: the convergence at infinity will provide a solution to the initial problem.
Indeed, with respect to \([1]\), new technical issues will appear, which are essentially due to our sub-Riemannian setting: in particular all the computations and the estimates regarding the convergence along the flow lines have to be done accordingly to the projection on the space of \(L^2\) CR pluriharmonic functions, that we defined earlier. Moreover, some technical estimates on the sphere will be adapted to the CR setting as we will see in Section 5 and the Appendix. Therefore, let us define the following functional \(E : H \to \mathbb{R}\), by

\[
E(u) = \int_M u\mathcal{P}'u + 2\int_M Q'u
\]

where \(H = \hat{\mathcal{P}} \cap S^2(M)\) and \(S^2(M)\) is the Folland-Stein Sobolev space equipped with the equivalent norm (see section 2), defined by

\[
\|u\|^2 = \int_M u\mathcal{P}'u + \int_M u^2. \tag{3}
\]

We consider the following space, which will serve as a constraint

\[
X = \left\{ u \in H; N(u) := \int_M \Gamma(f e^{2u}) = \int_M \overline{Q}' \right\};
\]

we notice that the space is well defined since \(e^u \in L^2\), see [9], Theorem 3.1.

As in the classical case, we will need the following hypotheses, depending on the sign of \(\int_M \overline{Q}'\), namely:

\[
\begin{align*}
(i) & \quad \inf_{x \in M} f(x) < 0, & \text{if } \int_M \overline{Q}' < 0 \\
(ii) & \quad \sup_{x \in M} f(x) > 0, \quad \inf_{x \in M} f(x) < 0 & \text{if } \int_M \overline{Q}' = 0 \\
(iii) & \quad \sup_{x \in M} f(x) > 0, & \text{if } 0 < \int_M \overline{Q}' \leq 16\pi^2. \tag{4}
\end{align*}
\]

In the case when \(\int_M \overline{Q}' = 0\), we let \(\ell\) the unique CR pluriharmonic function satisfying \(\mathcal{P}'\ell + Q' = 0\) and \(\int_M \ell = 0\), see [9], Theorem 1.1. Notice that \(\overline{Q}_{e^{2\theta}} = 0\). We also recall that in the critical case \(M = S^3\), there are some extra compatibility condition of Kazdan-Warner type that \(f\) needs to satisfy in order to be the \(\overline{Q}'\)-curvature of a contact structure conformal to the standard one on the sphere (see Theorem 1.3. in [17]).

Now, in order to define the flow equation, we compute the first variation of \(E, N\), and their \((S^2)\) gradient, respectively:

\[
\langle \nabla E(u), \phi \rangle = 2 \int_M \left( \mathcal{P}'u + \overline{Q}' \right) \phi, \forall \phi \in H,
\]

\[
\langle \nabla N(u), \phi \rangle = 2 \int_M \Gamma(f e^{2u}) \phi, \forall \phi \in H,
\]

\[
\nabla E(u) = 2 \left( \mathcal{P} + I \right)^{-1} \left( \mathcal{P}'u + \overline{Q}' \right),
\]

\[
\nabla N(u) = 2 \left( \mathcal{P} + I \right)^{-1} \Gamma(f e^{2u}).
\]

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In addition, since by hypotheses (4), \( \nabla N \neq 0 \) on \( X \), then \( X \) is a regular hypersurface in \( H \) and a unit normal vector field on \( X \) is given by \( \nabla N/\|\nabla N\| \). Indeed, \( \nabla N(u) \neq 0 \) if and only if \( \Gamma(e^{2u}f) \neq 0 \). This last identity is clear for the hypothesis (i) and (iii). But for (ii), recall that \( f \in \hat{P} \), so if \( \Gamma(e^{2u}f) = 0 \), then \( \int_M e^{2u}f^2 = 0 \), leading to a contradiction. The gradient of \( E \) restricted to \( X \) is then

\[
\nabla^X E = \nabla E - \left\langle \nabla E, \frac{\nabla N}{\|\nabla N\|} \right\rangle \frac{\nabla N}{\|\nabla N\|}.
\]

Finally, the (negative) gradient flow equation is given by

\[
\begin{cases}
  \partial_t u = -\nabla^X E(u) \\
  u(0) = u_0 \in X
\end{cases}
\] (5)

Now we can state our main results.

**Theorem 1.1.** Let \((M, T^{1,0} M, \theta)\) be a pseudo-Einstein CR three dimensional manifold such that \( \hat{P} \) is non-negative and \( \ker \hat{P} = \mathbb{R} \). Let us assume that \( \int_M \hat{Q} < 0 \) and let \( f \in C(M) \cap \hat{P} \) as in (4). Then there exists a positive constant \( C_0 \) depending on \( f^- = \max\{-f, 0\} \), \( M \) and \( \theta \), such that if

\[
e^{\tau\|u_0\|^2} \sup_{x \in M} f(x) \leq C_0
\]

for a constant \( \tau > 1 \) depending on \( M \) and \( \theta \), then as \( t \to \infty \), the flow converges in \( H \) to a solution \( u_\infty \) of (1). Moreover, there exist constants \( B, \beta > 0 \) such that

\[
\|u(t) - u_\infty\| \leq B(1 + t)^{-\beta},
\]

for all \( t \geq 0 \).

**Theorem 1.2.** Let \((M, T^{1,0} M, \theta)\) be a pseudo-Einstein CR three dimensional manifold such that \( \hat{P} \) is non-negative and \( \ker \hat{P} = \mathbb{R} \). Let us assume that \( \int_M \hat{Q} = 0 \) and let \( f \in C(M) \cap \hat{P} \) as in (4). Then as \( t \to \infty \), the flow converges in \( H \) to a function \( u_\infty \) and there exists a constant \( \lambda \) such that \( v = u_\infty + \lambda \) satisfies

\[
\hat{P} v + \hat{Q} = \delta \Gamma(f e^{2w}),
\]

where \( \delta \in \{+1, 0, -1\} \). Moreover, there exist constants \( B, \beta > 0 \) such that

\[
\|u(t) - u_\infty\| \leq B(1 + t)^{-\beta},
\]

for all \( t \geq 0 \). If in addition, we assume that \( \int_M f e^{2t} \neq 0 \), then \( \delta \neq 0 \).

**Theorem 1.3.** Let \((M, T^{1,0} M, \theta)\) be a pseudo-Einstein CR three dimensional manifold such that \( \hat{P} \) is non-negative and \( \ker \hat{P} = \mathbb{R} \). Let us assume that \( 0 < \int_M \hat{Q} < 16\pi^2 \) and let \( f \in C(M) \cap \hat{P} \) as in (4). Then as \( t \to \infty \), the flow converges in \( H \) to a solution \( u_\infty \) of (1). Moreover, there exist constants \( B, \beta > 0 \) such that

\[
\|u(t) - u_\infty\| \leq B(1 + t)^{-\beta},
\]

for all \( t \geq 0 \).
Finally, the critical case of the sphere, which is a bit different. We will consider a group $G$ acting on $S^3$ preserving the CR structure. We denote by $\Sigma$ the set of points fixed by $G$, that is

$$\Sigma = \{ x \in S^3; \ g \cdot x = x, \ \forall g \in G \}$$

and we will assume $f$ being invariant under $G$, namely $f(g \cdot x) = f(x), \forall g \in G$. Then we have the following

**Theorem 1.4.** Let us consider the sphere $M = S^3$ equipped with its standard contact structure and let $f \in C(M) \cap \mathcal{P}$ as in (4) and invariant under $G$. Let us assume that also $u_0 \in X$ is invariant under $G$. If $\Sigma = \emptyset$ or

$$\sup_{x \in \Sigma} f(x) \leq e^{-\frac{E(u_0)}{16\pi^2}},$$

then as $t \to \infty$, the flow converges in $H$ to a solution (invariant under $G$) $u_\infty$ of (1). Moreover, there exist constants $B, \beta > 0$ such that

$$\|u(t) - u_\infty\| \leq B(1 + t)^{-\beta},$$

for all $t \geq 0$.

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## 2 Some definition in pseudo-Hermitian geometry

We will follow the notations in [10]. Let $M^3$ be a smooth, oriented three-dimensional manifold. A CR structure on $M$ is a one-dimensional complex sub-bundle $T^{1,0}M \subset T_C M := TM \otimes \mathbb{C}$ such that $T^{1,0}M \cap T^{0,1}M = \{0\}$ for $T^{0,1}M := \overline{T^{1,0}M}$. Let $\mathcal{H} = ReT^{1,0}M$ and let $J : \mathcal{H} \to \mathcal{H}$ be the almost complex structure defined by $J(Z + \bar{Z}) = i(Z - \bar{Z})$, for all $Z \in T^{1,0}M$. The condition that $T^{1,0}M \cap T^{0,1}M = \{0\}$ is equivalent to the existence of a contact form $\theta$ such that $ker \theta = \mathcal{H}$. We recall that a 1-form $\theta$ is said to be a contact form if $\theta \wedge d\theta$ is a volume form on $M^3$. Since $M$ is oriented, a contact form always exists, and is determined up to multiplication by a positive real-valued smooth function. We say that $(M^3, T^{1,0}M)$ is strictly pseudo-convex if the Levi form $d\theta(\cdot, J \cdot)$ on $\mathcal{H} \otimes \mathcal{H}$ is positive definite for some, and hence any, choice of contact form $\theta$. We shall always assume that our CR manifolds are strictly pseudo-convex.

Notice that in a CR-manifold, there is no canonical choice of the contact form $\theta$. A pseudo-Hermitian manifold is a triple $(M^3, T^{1,0}M, \theta)$ consisting of a CR manifold and a contact form. The Reeb vector field $T$ is the vector field such that $\theta(T) = 1$ and $d\theta(T, \cdot) = 0$. The choice of $\theta$ induces a natural $L^2$-dot product $\langle \cdot, \cdot \rangle$, defined by

$$\langle f, g \rangle = \int_M f(x)g(x) \ \theta \wedge d\theta.$$ 

A $(1,0)$-form is a section of $T^*_c M$ which annihilates $T^{0,1}M$. An admissible coframe is a non-vanishing $(1,0)$-form $\theta^1$ in an open set $U \subset M$ such that $\theta^1(T) = 0$. Let $\theta^1 := \theta^1$ be
its conjugate. Then \( d\theta = ih_{11}\theta^1 \wedge \theta^\dagger \) for some positive function \( h_{11} \). The function \( h_{11} \) is equivalent to the Levi form. We set \( \{Z_1, Z_\dagger, T\} \) to the dual of \( (\theta^1, \theta^\dagger, \theta) \). The geometric structure of a CR manifold is determined by the connection form \( \omega^1_1 \) and the torsion form \( \tau_1 = A_{11}\theta^1 \) defined in an admissible coframe \( \theta^1 \) and is uniquely determined by

\[
\begin{align*}
  d\theta^1 &= \theta^1 \wedge \omega^1_1 + \theta \wedge \tau^1, \\
  \omega^1_1 + \omega^\dagger_1 &= dh_{11},
\end{align*}
\]

where we use \( h_{11} \) to raise and lower indices. The connection forms determine the pseudo-Hermitian connection \( \nabla \), also called the Tanaka-Webster connection, by

\[
\nabla Z_1 := \omega^1_1 \otimes Z_1.
\]

The scalar curvature \( R \) of \( \theta \), also called the Webster curvature, is given by the expression

\[
d\omega^1_1 = R\theta^1 \wedge \theta^1 \mod \theta.
\]

**Definition 2.1.** A real-valued function \( w \in C^\infty(M) \) is CR pluriharmonic if locally \( w = \Re f \) for some complex-valued function \( f \in C^\infty(M, \mathbb{C}) \) satisfying \( Z_1 f = 0 \).

Equivalently, \( [20] \), \( w \) is a CR pluriharmonic function if

\[
P_3 w := \nabla_1\nabla_1\nabla_1^\dagger w + iA_{11}\nabla_1^\dagger w = 0
\]

for \( \nabla_1 := \nabla_{Z_1} \). We denote by \( \mathcal{P} \) the space of all CR pluriharmonic functions and \( \hat{\mathcal{P}} \) the completion of \( \mathcal{P} \) in \( L^2(M) \), also called the space of \( L^2 \) CR pluriharmonic functions. Let \( \Gamma : L^2(M) \to \hat{\mathcal{P}} \) be the orthogonal projection on the space of \( L^2 \) pluriharmonic functions. If \( S : L^2(M) \to \ker \bar{\partial}_b \) denotes the Szego kernel, then

\[
\Gamma = S + \bar{S} + F,
\]

where \( F \) is a smoothing kernel as shown in \([19]\). The Paneitz operator \( P_\theta \) is the differential operator

\[
P_\theta (w) := 4\text{div}(P_3 w) = \Delta_b^2 w + T^2 - 4\text{Im}(A_{11}\nabla_1^\dagger f)
\]

for \( \Delta_b := \nabla_1\nabla_1 + \nabla_\dagger\nabla_\dagger \) the sub-Laplacian. In particular, \( \mathcal{P} \subset \ker P_\theta \). Hence, \( \ker P_\theta \) is infinite dimensional. For a thorough study of the analytical properties of \( P_\theta \) and its kernel, we refer the reader to \([19, 6, 8]\). The main property of the Paneitz operator \( P_\theta \) is that it is CR covariant \([16]\). That is, if \( \theta = e^w\theta \), then \( e^{2w}P_\theta = P_\theta \).

**Definition 2.2.** Let \((M^3, T^{1,0}M, \theta)\) be a pseudo-Hermitian manifold. The Paneitz type operator \( P'_\theta : \mathcal{P} \to C^\infty(M) \) is defined by

\[
P'_\theta f = 4\Delta_b^2 f - 8\text{Im}(\nabla^\alpha(A_{\alpha\beta}\nabla^\beta f)) - 4\text{Re}(\nabla^\alpha(R\nabla_\alpha f)) + \frac{8}{3}\text{Re}(\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta})\nabla^\alpha f - \frac{4}{3}f\nabla^\alpha(\nabla_\alpha R - i\nabla^\beta A_{\alpha\beta})
\]

for \( f \in \mathcal{P} \).
The main property of the operator $P'_\theta$ is its “almost” conformal covariance as shown in [2,10]. That is if $(M^3, T^{1,0} M, \theta)$ is a pseudo-Hermitian manifold, $w \in C^\infty(M)$, and we set $\hat{\theta} = e^w \theta$, then
\[
e^{2w} P'_{\hat{\theta}}(u) = P'_\theta(u) + P_{\theta}(uw)
\]
for all $u \in \mathcal{P}$. In particular, since $P_{\theta}$ is self-adjoint and $\mathcal{P} \subset \ker P_{\theta}$, we have that the operator $P'$ is conformally covariant, mod $\mathcal{P}^\perp$.

**Definition 2.3.** A pseudo-Hermitian manifold $(M^3, T^{1,0} M, \theta)$ is pseudo-Einstein if
\[
\nabla_\alpha R - i \nabla^\beta A_{\alpha\beta} = 0.
\]
Moreover, if $\theta$ induces a pseudo-Einstein structure then $e^w \theta$ is pseudo-Einstein if and only if $u \in \mathcal{P}$. The definition above was stated in [10], but it was implicitly mentioned in [16]. In particular, if $(M^3, T^{1,0} M, \theta)$ is pseudo-Einstein, then $P'_\theta$ takes a simpler form:
\[
P'_{\theta} f = 4 \Delta_\theta^2 f - 8 \text{Im} (\nabla^1 (A_{11} \nabla^1 f)) - 4 \text{Re} (\nabla^1 (R \nabla^1 f)).
\]
In particular, one has
\[
\int_M u P'_{\theta} u \geq 4 \int_M |\Delta_\theta u|^2 - C \int_M |\nabla_\theta u|^2.
\]
Using the interpolation inequality
\[
\int_M |\nabla_\theta u|^2 \leq C ||u||_{L^2} ||\Delta_\theta u||_{L^2},
\]
and $2ab \leq a^2 + \frac{1}{e} b^2$, we have the existence of $C_1 > 0$ and $C_2 > 0$, such that
\[
\int_M u P'_{\theta} u \geq C_1 \int_M |\Delta_\theta u|^2 - C_2 \int_M u^2.
\]
Hence, if $P'_{\theta}$ is non-negative, with trivial kernel, one has the equivalence of the Folland-Stein Sobolev norm and (3).

**Definition 2.4.** Let $(M^3, T^{1,0} M, \theta)$ be a pseudo-Einstein manifold. The $Q'$-curvature is the scalar quantity defined by
\[
Q'_{\theta} = 2 \Delta_\theta R - 4|A|^2 + R^2.
\]
The main equation that we will be dealing with is the change of the $Q'$-curvature under conformal change. Let $(M^3, T^{1,0} M, \theta)$ be a pseudo-Einstein manifold, let $w \in \mathcal{P}$, and set $\hat{\theta} = e^w \theta$. Hence $\hat{\theta}$ is pseudo-Einstein. Then [2,10]
\[
e^{2w} Q'_{\hat{\theta}} = Q'_{\theta} + P'_{\theta}(w) + \frac{1}{2} P_{\theta}(w^2).
\]
In particular, $Q'_{\theta}$ behaves as the $Q$-curvature for $P'_{\theta}$, mod $\mathcal{P}^\perp$. Since we are working modulo $\mathcal{P}^\perp$ it is convenient to project the previously defined quantities on $\bar{\mathcal{P}}$. So we define the operator $\mathcal{P}'_{\theta} = \Gamma \circ P'_{\theta}$ and the $Q'$-curvature by $\overline{Q'_{\theta}} = \Gamma(Q'_{\hat{\theta}})$. Notice that
\[
\int_M Q'_{\theta} \theta \wedge d\theta = \int_M \overline{Q'_{\theta}} \theta \wedge d\theta.
\]
Moreover, the operator $\mathcal{T}_\theta'$ has many interesting analytical properties. Indeed, $\mathcal{T}_\theta': \mathcal{P} \to \hat{\mathcal{P}}$ is an elliptic pseudo-differential operator (see [9]) and if we assume that $\ker \mathcal{T}_\theta' = \mathbb{R}$, then its Green’s function $G$ satisfies

$$\mathcal{T}_\theta' G(\cdot, y) = \Gamma(\cdot, y) - \frac{1}{V},$$

where $V = \int_M \theta \wedge d\theta$ is the volume of $M$ and $\Gamma(\cdot, \cdot)$ is the kernel of the projection operator $\Gamma$. Moreover,

$$G(x, y) = -\frac{1}{4\pi^2} \ln(|xy^{-1}|) + \mathcal{K}(x, y),$$

where $\mathcal{K}$ is a bounded kernel as proved in [7].

3 Preliminary results on the flow

First we recall one fundamental inequality that we will be using all along this paper, namely the CR version of the Beckner-Onofri inequality. This inequality was first proved in the odd dimensional spheres in [3] and then naturally extended to pseudo-Einstein 3-manifolds in [9, Theorem 3.1].

**Theorem 3.1.** Assume that $\mathcal{T}'$ is non-negative and $\ker \mathcal{T}' = \mathbb{R}$. Then, there exists $C > 0$ such that for all $u \in \hat{\mathcal{P}} \cap S^2(M)$ with $\int_M u = 0$, we have

$$\frac{1}{16\pi^2} \int_M u' u + C \geq \ln \left( \int_M e^{2u} \right).$$

In the case of the sphere, $C$ can be taken to be 0 and equality holds if and only if $u = J(h)$ with $h \in Aut(S^3)$ and $J(h) = det(Jac(h))$ is the determinant of the Jacobian determinant of $h$. The dual version of the above inequality was also investigated in [22], where the existence of extremals was investigated.

Now, we prove the global existence of solutions of (5):

**Lemma 3.1.** Let $(M, T^{1,0} M, \theta)$ be a pseudo-Einstein CR three dimensional manifold such that $\mathcal{T}'$ is non-negative and $\ker \mathcal{T}' = \mathbb{R}$. Let $f \in C(M) \cap \hat{\mathcal{P}}$ as in (4). Then for any $u_0 \in X$ there exists a solution $u \in C^\infty([0, \infty), H)$ of problem (5) such that $u(t) \in X$, for all $t \geq 0$. Moreover it holds

$$\int_0^t \| \partial_t u(s) \|^2 ds = E(u_0) - E(u(t)),$$

for all $t \geq 0$.

**Proof.** Since all the functionals involved are regular, the short time existence of a solution $u$ for (5) is ensured by the Cauchy-Lipschitz Theorem. In order to extend it to all $t \geq 0$, we notice that

$$\| \partial_t u \| = \| \nabla^X E(u) \| \leq 2 \| \nabla E(u) \| \leq C_1 \| u \| + C_2.$$

Thus, since

$$\partial_t \| u \|^2 = 2 \langle u, \partial_t u \rangle \leq C_3 \| u \|^2 + C_4,$$

by Gronwall’s lemma, the solution $u$ exists for all $t \geq 0$. In addition

$$\partial_t N(u) = \langle \nabla N(u), \partial_t u \rangle = -\langle \nabla N(u), \nabla^X E(u) \rangle = 0,$$
therefore \( u(t) \in X \), for all \( t \geq 0 \). Finally, we have
\[
\partial_t E(u) = \langle \nabla E(u), \partial_t u \rangle = -\|\partial_t u\|^2.
\]
Hence, \( E \) is decreasing along the flow and the following energy identity holds
\[
\int_0^t \|\partial_s u\|^2 ds = E(u_0) - E(u(t)). \tag{11}
\]

Next we prove the following lemma about the convergence

**Lemma 3.2.** Let \((M, T^{1,0}M, \Theta)\) be a pseudo-Einstein CR three dimensional manifold such that \( \overline{P}' \) is non-negative and \( \ker \overline{P}' = \mathbb{R} \). Let \( f \in C(M) \cap \mathcal{P} \) and let \( u \) be the solution of problem \((5)\) obtained in the previous Lemma \((3.1)\). If there exists a constant \( C > 0 \) such that \( \|u(t)\| \leq C \), for all \( t \geq 0 \), then when \( t \to \infty \), \( u(t) \to u_\infty \) in \( H \) and \( u_\infty \) solves the equation
\[
\overline{P}' u + \overline{Q} u = \lambda \Gamma(f e^{2u}),
\]
for a certain \( \lambda \in \mathbb{R} \). Moreover, there exist constants \( B, \beta > 0 \) such that
\[
\|u(t) - u_\infty\| \leq B(1 + t)^{-\beta},
\]
for all \( t \geq 0 \).

**Proof.** Since \( \|u\| \leq C \), we have that
\[
|E(u)| \leq 2\|u\|^2 + C_2.
\]
Therefore, by the previous energy estimate
\[
\int_0^\infty \|\partial_t u\|^2 dt < \infty.
\]
So there exists a sequence \( t_k \to \infty \) such that
\[
\|\partial_t u(t_k)\| = \|\nabla^X E(u(t_k))\| \to 0.
\]

Now, from the boundedness of \( \|u\| \), we also have the convergence \( u(t_k) \to u_\infty \) strongly in \( L^2(M) \) and weakly in \( \mathcal{S}^2(M) \). From Theorem \((3.1)\) we have that \( e^{2u(t_k)} \in L^p(M) \), with \( p \geq 1 \), and \( \|e^{2u(t_k)}\|_{L^p} \) is uniformly bounded. Thus by Egorov’s lemma, we can deduce that
\[
\left\| f e^{2u(t_k)} - f e^{2u_\infty} \right\|_{L^p} \to 0, \ 1 \leq p < \infty.
\]
Indeed, we fix \( \varepsilon > 0 \). Then, there exists a set \( A \) with \( Vol(A) < \varepsilon \) such that \( f e^{2u(t_k)} \) converges uniformly to \( f e^{2u_\infty} \) on \( M \setminus A \). Therefore,
\[
\left\| f e^{2u(t_k)} - f e^{2u_\infty} \right\|_{L^p} \leq C \left( \left\| f e^{2u(t_k)} - f e^{2u_\infty} \right\|_{L^\infty(M \setminus A)} + \left( \left\| f e^{2u(t_k)} \right\|_{L^p} + \left\| f e^{2u_\infty} \right\|_{L^p} \right) Vol(A)^{\frac{1}{p} - \frac{1}{p'}},
\]
for $p > 0 < \infty$. So the conclusion follows from the uniform boundedness of $\|e^{2u(t_k)}\|_{L^p}$, for all $1 \leq p < \infty$.

Thus $u_\infty \in X$. Now we have that

$$\nabla N(u(t_k)) = 2(\overline{P} + I)^{-1}\Gamma(f e^{2u(t_k)})$$

and since $f e^{2u(t_k)}$ converges strongly in $L^p(M)$ and $\Gamma$ maps continuously $L^p(M)$ to $L^p(M)$ (this follows from [23] and [24]), we have by the compactness of $\overline{P} + I$ that $\nabla N(u(t_k))$ converges strongly to $\nabla N(u_\infty)$. Also,

$$\nabla E(u(t)) = 2(\overline{P} + I)^{-1}(\overline{P} u + \overline{Q} ) = 2u(t) + 2(\overline{P} + I)^{-1}(\overline{Q} - u).$$

Thus, since $\partial_t u(t_k) \to 0$ in $S^2(M)$, we have that $u(t_k)$ converges strongly to $u_\infty$ and $\nabla X E(u_\infty) = 0$. Moreover, we have

$$(\overline{P} + I)^{-1}(\overline{P} u_\infty + \overline{Q} ) = \lambda(u_\infty)(\overline{P} + I)^{-1}\Gamma(f e^{2u_\infty})$$

where

$$\lambda(u_\infty) = \frac{\langle \nabla E(u_\infty), \nabla N(u_\infty) \rangle}{\|\nabla N(u_\infty)\|^2}.$$

Now by integration we have that

$$\int_M \overline{Q} = \lambda(u_\infty) \int_M \Gamma(f e^{2u_\infty}),$$

and since $u_\infty \in X$, we have if $\int_M \overline{Q} \neq 0$, that $\lambda(u_\infty) = 1$ and hence $u_\infty$ solves the desired equation. On the other hand, if $\int_M \overline{Q} = 0$, either $\lambda(u_\infty) = 0$ and thus

$$\overline{P} u_\infty + \overline{Q} = 0,$$

or $\lambda(u_\infty) > 0$, thus setting $v = u_\infty + \frac{1}{2} \ln(\lambda(u_\infty))$ we have

$$\overline{P} v + \overline{Q} = \Gamma(f e^{2v}),$$

and similarly if $\lambda(u_\infty) < 0$, we have a function $v = u_\infty + \frac{1}{2} \ln(-\lambda(u_\infty))$ such that

$$\overline{P} v + \overline{Q} = -\Gamma(f e^{2v}).$$

In particular, if we assume that $\int_M f e^{2\ell} \neq 0$ in the case $\lambda(u_\infty) = 0$, we have $u_\infty - \ell$ is constant. Hence,

$$0 = \int_M f e^{2u_\infty} = e^{2(u_\infty - \ell)} \int_M f e^{2\ell} \neq 0,$$

which is a contradiction.

The polynomial convergence of the flow can be deduced from the Lojasiewicz-Simon inequality following Theorem 3 in [25] and Lemma 3.2 in [1]. Let $\eta : H \to T_{u_\infty}X$ be the natural
projection, where $T_{u_\infty}X$ denotes the tangent space of the manifold $X$ at the point $u_\infty$. We have, for $v \in T_{u_\infty}X$

$$(\nabla X)^2 E(u_\infty)v = \eta \left( \nabla^2 E(u_\infty)v - \frac{\langle \nabla E(u_\infty), \nabla N_1(u_\infty) \rangle}{\|\nabla N(u)\|^2} \nabla^2 N_1(u_\infty)v + R^\perp v \right)$$

where $R^\perp v$ is the component along $\nabla N(u_\infty)$. Thus, since $\eta \left( R^\perp v \right) = 0$, we have that

$$((\nabla X)^2 E(u_\infty)v = 2 \left( I - \eta(T' + 1) \right) v - 4 \frac{\langle \nabla E(u_\infty), \nabla N_1(u_\infty) \rangle}{\|\nabla N(u)\|^2} (T' + 1)^{-1} \Gamma(f e^{2u_\infty} v).$$

It can be checked that $(\nabla X)^2 E(u_\infty) : T_{u_\infty}X \to T_{u_\infty}X$ is a Fredholm operator, then there exists a constant $\delta > 0$ and $0 < \kappa < \frac{1}{2}$ such that if $\|u(t) - u_\infty\| < \delta$, it holds

$$\|\nabla X E(u)\| \geq (E(u(t)) - E(u_\infty))^{1-\kappa}.$$  

We note that if $E(u(t_0)) = E(u_\infty)$ for some $t_0 \geq 0$, then the flow is stationary and the estimate is trivially satisfied. So we can assume that $E(u(t)) - E(u_\infty) > 0$, for all $t \geq 0$. Since $\lim_{n \to \infty} \|u(t_n) - u_\infty\| = 0$, for a given $\varepsilon > 0$, there exists $n_0 > 0$ such that for $n \geq n_0$ we have,

$$\|u(t_n) - u_\infty\| < \frac{\varepsilon}{2}$$

and

$$\frac{1}{\kappa} (E(u(t_n)) - E(u_\infty))^{\kappa} < \frac{\varepsilon}{2}.$$  

We set $\varepsilon = \frac{\delta}{2}$ and

$$T := \sup \{ t \geq t_{n_0}; \|u(s) - u_\infty\| < \delta; s \in [t_{n_0}, t] \},$$

and we assume for the sake of contradiction that $T < \infty$. Now we have

$$-\partial_t [E(u(t)) - E(u_\infty)]^{\kappa} = -\kappa \partial_t E(u(t)) [E(u(t)) - E(u_\infty)]^{\kappa-1},$$

but

$$-\partial_t E(u(t)) = - (E(u), \partial_t u) = \|\nabla X E(u)\| \|\partial_t u\|.$$  

Thus, for $t \in [t_{n_0}, T]$ we have

$$-\partial_t [E(u(t)) - E(u_\infty)]^{\kappa} \geq \kappa \|\partial_t u\|,$$

and since $E$ is non-increasing along the flow, we have after integration in the interval $[t_{n_0}, T]$

$$\|u(T) - u(t_{n_0})\| \leq \int_{t_{n_0}}^{T} \|\partial_t u\| ds \leq \frac{1}{\kappa} [E(u(t_{n_0})) - E(u_\infty)]^{\kappa} < \frac{\varepsilon}{2}.$$  

Hence,

$$\|u(T) - u_\infty\| \leq \|u(T) - u(t_{n_0})\| + \|u(t_{n_0}) - u_\infty\| < \varepsilon = \frac{\delta}{2}$$

which is a contradiction and so $T = +\infty$. We set now $g(t) = E(u(t)) - E(u_\infty)$, for $t \in [t_{n_0}, +\infty)$. Then we have

$$g'(t) = -\|\nabla X E(u)\|^2 \geq g^{2\kappa-1}(t).$$
By integration we have

\[ g^{2\kappa - 1}(t) \geq g^{2\kappa - 1}(t_{n_0}) + (1 - 2\kappa)(t - t_{n_0}). \]

Since \(2\kappa - 1 < 0\), we have

\[ g(t) \leq \left[ g^{2\kappa - 1}(t_{n_0}) + (1 - 2\kappa)(t - t_{n_0}) \right]^{\frac{1}{2\kappa - 1}} \leq Ct^{\frac{\kappa}{2\kappa - 1}}. \]

Now, by taking \(t' > t\), we have

\[ \|u(t) - u(t')\| \leq \int_t^{t'} \|\partial_s u\| ds \leq \frac{1}{\theta}[E(u(t)) - E(u_\infty)]^\kappa \leq \frac{1}{\kappa}g^{\kappa}(t) \leq Ct^{\frac{\kappa}{2\kappa - 1}}. \]

For \(t' = t_n\), letting \(n \to \infty\) and setting \(\beta = \frac{\kappa}{1 - 2\kappa}\), we get that for \(t > t_{n_0}\)

\[ \|u(t) - u_\infty\| \leq Ct^{-\beta} \]

Therefore, since \(\|u(t) - u_\infty\|\) is bounded for \(t > t_{n_0}\), we have the existence of \(B > 0\) such that for all \(t \geq 0\)

\[ \|u(t) - u_\infty\| \leq B(1 + t)^{-\beta}. \]

\[ \square \]

**Corollary 3.1.** Let \((M, T^{1,0} M, \theta)\) be a pseudo-Einstein CR three dimensional manifold such that \(P^f\) is non-negative and \(\ker P^f = \mathbb{R}\). Let \(f \in C(M) \cap \hat{P}\) as in (4) and let \(u\) be the solution of problem (3) obtained in the Lemma (3.1). If \(\bar{u} = \frac{1}{V} \int_M u\) is uniformly bounded then the flow converges. Here \(V = \int_M \theta \wedge d\theta\) is the volume of \(M\).

**Proof.** From the energy identity (11) we have that

\[ \int_M u^{P^f} u + 2 \int_M Q u \leq E(u_0), \]

but we also have from the Poincaré-type inequality (or the non-negativity of the operator \(P^f\)), that

\[ \int_M u^{P^f} u \geq \lambda_1 \int_M (u - \bar{u})^2. \]

Here \(\lambda_1\) is the first non-zero eigenvalue of the operator \(P^f\). In particular, from Young’s inequality, we have that

\[ \int_M u^{P^f} u \leq E(u_0) + \varepsilon \int_M (u - \bar{u})^2 + C(\varepsilon)\|\bar{Q}\|^2_{L^2} - 2\bar{u} \int_M \bar{Q} \]

Hence, for \(\varepsilon\) small enough, we have that

\[ \int_M u^{P^f} u \leq C, \]

since \(\bar{u}\) is uniformly bounded, then the uniform boundedness of \(\|u\|\) and the conclusion follows from Lemma 3.2 \(\square\)

Therefore, in the rest of the paper, we will show the uniform boundedness of \(\bar{u}\) along the flow, in order to have convergence at infinity.
4 The sub-critical case

Along all this section we will assume that $P'$ is non-negative and $\ker P' = \mathbb{R}$. Next we consider the three separate cases in which $\int_M Q < 16\pi^2$. Also we let $V = \int_M \theta \wedge d\theta$ be the volume of $M$.

4.1 Case $\int_M Q < 0$ and proof of Theorem 1.1

Lemma 4.1. There exists a positive constant $C > 0$ depending on $M$ and $\theta$ such that for any measurable subset $K \subset M$ with $\text{Vol}(K) > 0$, we have

$$\int_M u \leq |E(u_0)| + \frac{C}{\text{Vol}(K)} + \frac{4V}{\text{Vol}(K)} \max \left( \int_K u, 0 \right)$$

Proof. Without loss of generality we can assume that $\int_M u > 0$ otherwise the inequality is trivially satisfied. First we have that

$$\int_M uP' u \leq E(u_0) - 2\int_M Q' u$$

and

$$\|u - \bar{u}\|_{L^2}^2 \leq \frac{1}{\lambda_1} \int_M uP' u.$$

Hence,

$$\int_M u^2 \leq \frac{1}{\lambda_1} E(u_0) - \frac{2}{\lambda_1} \int_M Q' u + \frac{1}{V} \left( \int_M u \right)^2$$

Now if $\int_K u \leq 0$, then we have

$$\left( \int_M u \right)^2 \leq \left( \int_{K^c} u \right)^2 \leq \text{Vol}(K^c) \int_M u^2,$$

hence

$$\frac{\text{Vol}(K)}{V} \int_M u^2 \leq \frac{1}{\lambda_1} E(u_0) - \frac{2}{\lambda_1} \int_M Q' u.$$

Again using Young’s inequality we have

$$\int_M u^2 \leq \frac{2V}{\lambda_1 \text{Vol}(K)} E(u_0) + \frac{4\|Q\|_{L^2}^2 V^2}{\lambda_1^2 \text{Vol}(K)^2},$$

but

$$\left( \int_M u \right)^2 \leq V \int_M u^2 \leq \frac{2V^2}{\lambda_1 \text{Vol}(K)} E(u_0) + \frac{4\|Q\|_{L^2}^2 V^3}{\lambda_1^2 \text{Vol}(K)^2}$$

$$\leq |E(u_0)|^2 + \frac{V^4}{\lambda_1^2 \text{Vol}(K)^2} + \frac{4\|Q\|_{L^2}^2 V^3}{\lambda_1^2 \text{Vol}(K)^2},$$
which yields
\[ \int_M u \leq |E(u_0)| + \frac{C}{\text{Vol} K}. \]

We assume now that \( \int_K u > 0 \). Then one has
\[ \int_M u^2 \leq \frac{1}{\lambda_1} E(u_0) - \frac{2}{\lambda_1} \int_M \overline{Q}' u + \frac{1}{V} \left( \left( \int_K u \right)^2 + \left( \int_{K^c} u \right)^2 + 2 \int_{K^c} u \int_K u \right), \]
and
\[ \frac{2}{V} \int_{K^c} u \int_K u \leq \frac{2 \text{Vol}(K^c)}{\text{Vol}(K) V} \left( \int_K u \right)^2 + \frac{\text{Vol}(K)}{2V} \int_M u^2. \]

Hence,
\[ \frac{\text{Vol}(K)}{2V} \int_M u^2 \leq \frac{1}{\lambda_1} E(u_0) - \frac{2}{\lambda_1} \int_M \overline{Q}' u + \frac{3}{\text{Vol}(K)} \left( \int_K u \right)^2. \]

By using that
\[ \left| \frac{2}{\lambda_1} \int_M \overline{Q}' u \right| \leq \frac{\text{Vol}(K)}{4V} \int_M u^2 + \frac{4V \| \overline{Q}' \|^2_{L^2}}{\lambda_1^2 \text{Vol}(K)}, \]
we have,
\[ \int_M u^2 \leq \frac{4V}{\lambda_1 \text{Vol}(K)} |E(u_0)| + \frac{16V^2 \| \overline{Q}' \|^2_{L^2}}{\lambda_1^2 \text{Vol}(K)^2} + \frac{12V}{\text{Vol}(K)^2} \left( \int_M u \right)^2. \]

Hence,
\[ \left( \int_M u \right)^2 \leq \frac{4V^2}{\lambda_1 \text{Vol}(K)} |E(u_0)| + \frac{16V^3 \| \overline{Q}' \|^2_{L^2}}{\lambda_1^2 \text{Vol}(K)^2} + \frac{12V^2}{\text{Vol}(K)^2} \left( \int_M u \right)^2, \]
and therefore
\[ \int_M u \leq |E(u_0)| + \frac{C}{\text{Vol}(K)} + \frac{4V}{\text{Vol}(K)} \int_K u. \]

\[ \square \]

**Lemma 4.2.** Let \( K \) be a measurable subset of \( M \) such that \( \text{Vol}(K) > 0 \). Then there exists a constant \( \alpha > 1 \) depending on \( M \) and \( \theta \) and a constant \( C_K > 1 \) depending on \( \text{Vol}(K) \) such that
\[ \int_M e^{2u} \leq C_K e^{\alpha \|u\|_2} \max \left( \left( \int_K e^{2u} \right)^\alpha, 1 \right). \]

**Proof.** Recall that from Theorem 3.1 one has the existence of \( C > 0 \) such that
\[ \int_M e^{2u} \leq C \exp \left( \frac{1}{16\pi^2} \int_M u \overline{P}' u + \frac{2}{V} \int_M u \right). \]

Again, by the energy identity (11) and Young’s inequality, we have
\[ \int_M u \overline{P}' u \leq E(u_0) - 2 \int_M \overline{Q}' (u - \bar{u}) - 2\bar{u} \int_M \overline{Q}' \]
\[ \leq E(u_0) - 2\bar{u} \int_M \overline{Q}' + \frac{1}{\varepsilon \| \overline{Q}' \|^2_{L^2}} + \frac{\varepsilon}{\lambda_1} \int_M u \overline{P}' u. \]
Thus, for $\varepsilon = \frac{\lambda}{2}$,
\[
\frac{1}{2} \int_M u' u \leq E(u_0) - 2\bar{u} \int_M Q' + \frac{2}{\lambda_1} \|Q'\|_{L^2}^2.
\]
Therefore
\[
\int_M e^{2u} \leq C \exp \left( \frac{1}{8\pi^2} E(u_0) + \frac{\|Q'\|_{L^2}^2}{4\lambda_1} + \left( 2 - \frac{1}{4\pi^2} \int_M Q' \right) \bar{u} \right).
\]
Now we notice that $E(u_0) \leq \|u_0\|^2 + \|Q\|^2_{L^2}$, hence there exist constants $C_1$ and $C_2$ such that
\[
\int_M e^{2u} \leq C_1 \exp \left( \frac{1}{8\pi^2} \|u_0\|^2 + C_2 \int_M u \right).
\]
By using Lemma 4.1 we get
\[
\int_M e^{2u} \leq \tilde{C}_K \exp \left( A_1 \|u_0\|^2 + \frac{A_2}{Vol(K)} \max \left( \int_K u, 0 \right) \right),
\]
where $\tilde{C}_K$ depends on $Vol(K)$. Now, we set $\alpha = \max \left( A_1, \frac{A_2}{2}, 2 \right) > 1$, and we get
\[
\int_M e^{2u} \leq \tilde{C}_K \exp \left( \alpha \|u_0\|^2 + \frac{\alpha}{Vol(K)} \max \left( \int_K 2u, 0 \right) \right).
\]
But Jensen’s inequality yields
\[
\exp \left( \frac{1}{Vol(K)} \int_K 2u \right) \leq \frac{1}{Vol(K)} \int_K e^{2u},
\]
in particular
\[
\exp \left( \frac{\alpha}{Vol(K)} \max \left( \int_K u, 0 \right) \right) \leq \max \left( \left( \frac{1}{Vol(K)} \int_K e^{2u} \right)^\alpha, 1 \right).
\]
Therefore, by adjusting the constant eventually
\[
\int_M e^{2u} \leq C_K e^{\alpha \|u_0\|^2} \max \left( \left( \int_K e^{2u} \right)^\alpha, 1 \right)
\]
which completes the proof.

Next we move to the proof of Theorem 1.1. We set
\[
K = \left\{ x \in M; f(x) \leq \frac{1}{2} \inf_{x \in M} f(x) \right\}.
\]
From the compatibility condition (i) in (4), we have that $Vol(K) > 0$, and since
\[
\int_M Q = \int_M f e^{2u_0},
\]
we have
\[
\inf_{x \in M} f(x) \leq \int_M e^{2u_0}.
\]
Thus, there exists $C > 0$ (we will assume $C > 1$ actually) such that
\[
\int_M e^{2u_0} \leq C \exp \left[ C \left( \int_M u_0 P' u_0 + \int_M u_0^2 \right) \right] = C e^{C \|u_0\|^2}.
\]
Hence,
\[
\int_M \overline{Q} \inf_{x \in M} f(x) \leq C e^{C \|u_0\|^2}.
\] (12)

Next we will prove the following

**Lemma 4.3.** Let $C_K$ and $\alpha$ be the constants found in Lemma 4.2. Let
\[
r = C_K(8C)\alpha e^{(C+1)\alpha\|u_0\|^2},
\]
and let us assume that
\[
e^{\tau\|u_0\|^2} \sup_{x \in M} f(x) \leq C_0,
\]
where $\tau = \alpha(C + 1) - C$ and
\[
C_0 = -\inf_{x \in M} f(x) \quad \frac{1}{8^\alpha C_K C^{\alpha-1}}.
\]
Then for all $t \geq 0$, it holds
\[
\int_M e^{2u} \leq 2r.
\]

**Proof.** Let
\[
T = \sup \left\{ s \geq 0; \int_M e^{2u} \leq 2r \text{ in } [0,s] \right\}
\]
and let us assume for the sake of contradiction that $T < \infty$. We notice that by continuity, we have that
\[
\int_M e^{2u(T)} = 2r.
\]
We assume first that
\[
\int_M f^+ e^{2u(T)} \leq \frac{1}{2} \int_M f^- e^{2u(T)},
\]
where $f^+ := \max\{f, 0\}$ and $f^- = f^+ - f$ denote the positive and negative part of $f$ respectively. Then we have
\[
\int_M f^- e^{2u(T)} \leq -2 \int_M f e^{2u(T)} = -2 \int_M \overline{Q} \leq -4 \int_M \overline{Q}.
\]
Since in $K$ we have $f^-(x) \geq -\frac{1}{\tau} \inf_{x \in M} f(x)$, we have
\[
\int_K e^{2u(T)} \leq \frac{8 \int_M \overline{Q}}{\inf_{x \in M} f(x)}
\]
which combined with (12) gives
\[
\int_K e^{2u(T)} \leq 8C e^{C \|u_0\|^2}.
\]
But from Lemma 4.2 we have
\[
\int_M e^{2u(T)} \leq C_K e^{a\|u_0\|^2} \max \left( \left( \int_K e^{2u} \right)^\alpha , 1 \right).
\]
Thus
\[
\int_M e^{2u(T)} \leq C_K e^{\alpha\|u_0\|^2} \left( 8C e^C\|u_0\|^2 \right)^\alpha = r,
\]
which is a contradiction.
So we move to the next case, where
\[
\int_M f + e^{2u(T)} > \frac{1}{2} \int_M f - e^{2u(T)}.
\]
Then we have
\[
- \frac{1}{2} \inf_{x \in M} f(x) \int_K e^{2u(T)} \leq \int_M f - e^{2u(T)} < 2 \int_M f + e^{2u(T)} \leq 4 \sup_{x \in M} f(x).
\]
Hence,
\[
\int_K e^{2u(T)} \leq - \frac{8r \sup_{x \in M} f(x)}{\inf_{x \in M} f(x)}.
\]
By using our assumption, we have that
\[
\int_K e^{2u(T)} \leq C_K e^{\alpha\|u_0\|^2} \left( 8re^{-\tau\|u_0\|^2}C_0 / \frac{\inf_{x \in M} f(x)}{-\inf_{x \in M} f(x)} \right)^\alpha \leq r,
\]
leading again to a contradiction. Hence \( T = +\infty \) and \( \int_M e^{2u} \) is uniformly bounded.

Now, by Jensen’s inequality we have
\[
\exp \left( \frac{1}{V} \int_M 2u \right) \leq \frac{1}{V} \int_M e^{2u} \leq \frac{2r}{V},
\]
thus \( \bar{u} \) is bounded from above. Now again using the energy identity (11), we have
\[
\int_M u\overrightarrow{P} u + 2 \int_M \overrightarrow{Q}(u - \bar{u}) + 2\bar{u} \int_M \overrightarrow{Q} \leq E(u_0),
\]
and
\[
\int_M u\overrightarrow{P} u + 2 \int_M \overrightarrow{Q}(u - \bar{u}) \geq \frac{1}{2} \int_M u\overrightarrow{P} u - \frac{2\|\overrightarrow{Q}\|_{L^2}^2}{\lambda_1} \geq -C_3.
\]
Therefore
\[
2\bar{u} \int_M \overrightarrow{Q} \leq E(u_0) + C_3,
\]
and since \( \int_M \overrightarrow{Q} < 0 \) we have that \( \bar{u} \) is uniformly bounded from below which finishes the proof of Theorem 1.1. \( \square \)
4.2 Case $\int_M \overline{Q'} = 0$ and proof of Theorem 1.2

Since $\int_M \overline{Q'} = 0$, we have that

$$\langle \nabla E(u), 1 \rangle = 2 \int_M \overline{P'} u = 0$$

and

$$\langle \nabla N(u), 1 \rangle = 2 \int_M \Gamma (fe^{2u}) = 2 \int_M \overline{Q'} = 0$$

Hence,

$$0 = \int_M \partial_t u = \partial_t \int_M u,$$

which means that the average value of $u$ is preserved. Therefore $\bar{u} = \bar{u}_0$ and by Corollary 3.1, we have the convergence of the flow. This ends the proof of Theorem 1.2.

4.3 Case $0 < \int_M \overline{Q'} < 16\pi^2$ and proof of Theorem 1.3

First, we have again from the energy identity (11)

$$\int_M u\overline{P' u} + 2 \int_M \overline{Q'} (u - \bar{u}) + 2\bar{u} \int_M \overline{Q'} \leq E(u_0).$$

Hence

$$2\bar{u} \int_M \overline{Q'} \leq E(u_0) - \frac{1}{2} \int_M u\overline{P' u} + \frac{2}{\lambda_1} \|\overline{Q'}\|_{L^2}^2$$

and then $\bar{u}$ is bounded from above; we will need a bound from below. Since $u \in X$, we get

$$\int_M \overline{Q'} = \int_M fe^{2u} \leq \|f\|_{\infty} \int_M e^{2u},$$

and therefore

$$\ln \left( \frac{\int_M \overline{Q'}}{\|f\|_{\infty}} \right) \leq \ln \left( \int_M e^{2u} \right).$$

Now again from Theorem 3.1 we have

$$\ln \left( \frac{\int_M \overline{Q'}}{\|f\|_{\infty}} \right) \leq C + \frac{1}{16\pi^2} \int_M u\overline{P' u} + \frac{2}{V} \int_M u.$$  

Let $\delta > 0$ to be determined later, we sum equation (13) and $-\delta$ times equation (14), obtaining

$$\ln \left( \frac{\int_M \overline{Q'}}{\|f\|_{\infty}} \right) - \delta E(u_0) \leq C + \left( \frac{1}{16\pi^2} - \delta \right) \int_M u\overline{P' u} + 2 \left( 1 - \delta \int_M \overline{Q'} \right) \bar{u} - 2\delta \int_M \overline{Q'} (u - \bar{u}).$$
Since \( \int_M \overline{Q} < 16 \pi^2 \), we choose \( \delta \) such that \( \int_M \overline{Q} < \frac{1}{\delta} < 16 \pi^2 \), and we set
\[
c_1 = 2 \left( 1 - \delta \int_M \overline{Q} \right), \quad c_2 = \delta - \frac{1}{16 \pi^2}.
\]
We have
\[
\ln \left( \frac{\int_M \overline{Q}}{\|f\|_\infty} \right) - \delta E(u_0) - C + c_2 \int_M u \overline{P}' u + 2 \delta \int_M \overline{Q} (u - \bar{u}) \leq c_1 \bar{u}.
\]
Now we notice that
\[
c_2 \int_M u \overline{P}' u + 2 \delta \int_M \overline{Q} (u - \bar{u}) \geq (c_2 \lambda_1 - \delta \varepsilon) \|u - \bar{u}\|^2_{L^2} - \frac{\delta}{\varepsilon} \|\overline{Q}\|^2_{L^2},
\]
therefore for \( \varepsilon \) small enough we have that
\[
c_2 \int_M u \overline{P}' u + 2 \delta \int_M \overline{Q} (u - \bar{u}) \geq -c_3.
\]
It follows that \( \bar{u} \) is bounded from below and therefore from Corollary 3.1 this finishes the proof.

5 The critical case and proof of Theorem 1.4

Here we will study the case \( \int_M \overline{Q} = 16 \pi^2 \), where \( M = S^3 \) is the sphere equipped with its standard contact structure. We will see \( S^3 \) as a subset of \( \mathbb{C}^2 \) with coordinates \( (\zeta_1, \zeta_2) \) such that
\[
S^3 = \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 : |\zeta_1|^2 + |\zeta_2|^2 = 1 \right\}.
\]
Recall that the set \( Aut(\mathbb{H}^1) \) of conformal transformation of the Heisenberg group \( \mathbb{H}^1 \) is generated by left translations, dilations, rotations and inversion. Using the Cayley transform \( C : \mathbb{H}^1 \to S^3 \{ (0, -i) \} \) one has a clear description of the set \( Aut(S^3) \):
\[
Aut(S^3) = \{ C \circ h \circ C^{-1} : h \in Aut(\mathbb{H}^1) \}.
\]
For \( p \in S^3 \) and \( r \geq 1 \), we will write \( h_{p,r} \) the element of \( Aut(S^3) \) corresponding to a Cayley transform centered at \( p \) and a dilation of size \( r \). Now, for \( u \in X \) we set
\[
v_{p,r} = u \circ h_{p,r} + \frac{1}{2} \ln(J(h_{p,r})),
\]
where we denoted \( J(h) = det(Jac(h)) \), the Jacobian determinant of \( h \). We have
\[
E(v_{p,r}) = E(u) \leq E(u_0),
\]
and since \( u \in X \)
\[
\int_{S^3} f \circ h_{p,r} e^{2v_{p,r}} = \int_{S^3} f e^{2u},
\]
hence
\[ \int_{S^3} e^{2v_{p,r}} \geq \frac{16 \pi^2}{\sup_{x \in S^3} f(x)}. \]

From [3, page 38], we know that for all \( t \geq 1 \) there exists \( r(t) \geq 1 \) and \( p(t) \in S^3 \) such that
\[ \int_{S^3} \xi_i e^{2v_{p(t),r(t)}} = 0, \ i = 1, 2. \]

So we let \( v(t) = v_{p(t),r(t)} \) and \( h(t) = h_{p(t),r(t)} \). Then using Corollary A.2 in the Appendix, one has the existence of \( a < \frac{1}{16 \pi^2} \) and a constant \( C_1 \) such that
\[ a \int_{S^3} v(t) \bar{\nabla} v(t) + 2 \int_{S^3} v(t) - \ln \left( \int_{S^3} e^{2v(t)} \right) + C_1 \geq 0. \]

Since \( E(v(t)) \leq E(u_0) \), we find that
\[ \int_{S^3} v(t) \bar{\nabla} v(t) \leq C, \]
and
\[ \left| \int_{S^3} v(t) \right| \leq C. \]

In particular we have that for all \( p \geq 1 \)
\[ \int_{S^3} e^{2|v(t)|} \leq C_p, \]
and hence
\[ \int_{S^3} v^2(t) \leq C \]
leading to the boundedness of \( v(t) \) in \( H \). We need the following concentration-compactness lemma in order to prove uniform boundedness.

**Lemma 5.1.** Either

(i) \( \|u(t)\| \leq C \), for some constant \( C \);

or

(ii) there exists a sequence \( t_n \to \infty \) and a point \( p_0 \in S^3 \) such that for all \( r > 0 \)
\[ \lim_{n \to \infty} \int_{B_r(p_0)} f e^{2u(t_n)} = 16 \pi^2. \]

Moreover, for any \( \bar{x} \in S^3 \setminus \{p_0\} \), and any \( r < d(\bar{x}, p_0) \), we have
\[ \lim_{n \to \infty} \int_{B_r(\bar{x})} f e^{2u(t_n)} = 0. \]
Proof. We assume first that $r(t)$ is bounded. Then we have that

$$0 < C_1 \leq J(h_p(t), r(t)) \leq C_2.$$ 

Thus, from the uniform boundedness of $v(t)$ we have

$$\int_{S^3} |u(t)| \leq C.$$

Therefore, from Lemma 3.1 it follows that $\|u(t)\|$ is uniformly bounded.

So now we assume that $r(t)$ is not bounded, then there exists a sequence $t_n \to \infty$ such that $r(t_n) \to \infty$ and without loss of generality, by compactness of $S^3$ we can assume that $p(t_n) \to p_0$. From the uniform boundedness of $v(t)$, we can also assume that $v(t_n) \to v_\infty$ strongly in $L^2(S^3)$ and weakly in $H$. We let then $r > 0$ and set $K_n = h(t_n)^{-1}(B_r(p_0))$. Then we have

$$\left| \int_{S^3} f \circ h(t_n)e^{2v(t_n)} - \int_{K_n} f \circ h(t_n)e^{2v(t_n)} \right| \leq \left( \sup_{x \in S^3} f(x) \right) \left( Vol(K_n) \int_{S^3} e^{-4v(t_n)} \right)^{\frac{1}{2}}.$$

Since $h(t_n)(x) \to p_0$ a.e. then $\lim_{n \to \infty} Vol(K_n) = V$, and thus

$$\int_{B_r(p_0)} f e^{2u(t_n)} = \int_{K_n} f \circ h(t_n)e^{2v(t_n)} = \int_{S^3} f \circ h(t_n)e^{2v(t_n)} + o(1).$$

We have also

$$\int_{S^3} f \circ h(t_n)e^{2v(t_n)} = 16\pi^2,$$

and then

$$\lim_{n \to \infty} \int_{B_r(p_0)} f e^{2u(t_n)} = 16\pi^2.$$ 

Now if we consider $\tilde{x} \in S^3 \setminus \{p_0\}$ and $r < d(p_0, \tilde{x})$ we have that $h(t_n)(x) \not\in B_r(\tilde{x})$ for $n$ big enough, since $\lim_{n \to \infty} h(t_n)(x) = p_0$ a.e.; in particular

$$\lim_{n \to \infty} \chi h(t_n)^{-1}(B_r(\tilde{x})) = 0,$$

where $\chi$ is the characteristic function. Therefore

$$\lim_{n \to \infty} \int_{B_r(\tilde{x})} f e^{2u(t_n)} = \lim_{n \to \infty} \int_{h(t_n)^{-1}(B_r(\tilde{x}))} f \circ h(t_n)e^{2v(t_n)} = 0.$$

Let us assume now that $\Sigma = \emptyset$. By using the previous lemma, if $\|u(t)\|$ is not uniformly bounded, then there exists $p_0 \in S^3$ such that

$$\lim_{n \to \infty} \int_{B_r(p_0)} f e^{2u(t_n)} = 16\pi^2,$$
and if $p_1 \neq p_0$ and $r < d(p_0, p_1)$, then
\[
\lim_{n \to \infty} \int_{B_r(p_1)} f e^{2u(t)} = 0.
\]
Since $\Sigma = \emptyset$, then there exists $g \in G$ such that $p_1 = g \cdot p_0 \neq p_0$. But
\[
16\pi^2 = \lim_{n \to \infty} \int_{B_r(p_0)} f e^{2u(t)} = \lim_{n \to \infty} \int_{B_r(p_0)} f e^{2u(t)} = \lim_{n \to \infty} \int_{B_r(p_1)} f e^{2u(t)} = 0
\]
which is a contradiction. Hence $\|u(t)\|$ is uniformly bounded.

Now we assume that $\Sigma \neq \emptyset$ and that $\|u(t)\|$ is not uniformly bounded. We have that the concentration point $p_0 \in \Sigma$, otherwise we reach a contradiction arguing as in the previous case. So we have
\[
\int_{B_r(p_0)} f e^{2u(t_n)} \leq \sup_{x \in B_r(p_0)} f(x) \int_{B_r(p_0)} e^{2u(t_n)} \leq \max \left( \sup_{x \in B_r(p_0)} f(x), 0 \right) \int_{S^3} e^{2u(t_n)}.
\]
By using the the sphere version of Theorem 3.1 proved in [3], we have that
\[
\frac{1}{V} \int_{S^3} e^{2u(t_n)} \leq e^{-\frac{E(u(t_n))}{V}}.
\]
Thus
\[
\int_{B_r(p_0)} f e^{2u(t_n)} < \max \left( \sup_{x \in B_r(p_0)} f(x), 0 \right) V e^{-\frac{E(u(t_n))}{V}}.
\]
Now we first let $n \to \infty$, then $r \to \infty$ and we get
\[
16\pi^2 < V \max(f(p_0), 0)e^{-\frac{E(u(t_n))}{V}}.
\]
Therefore $f(p_0) > 0$ and
\[
1 < f(p_0) e^{\frac{E(u(t_n))}{16\pi^2}},
\]
which leads to a contradiction of the assumption in Theorem 1.4. Therefore we have the uniform boundedness of $\|u\|$ also in this case, which yields the convergence of the flow and it ends the proof.

A Appendix: Improved Moser-Trudinger Inequality

In what follows we will consider $S^3$ as a subset of $\mathbb{C}^2$ with coordinates $(\zeta_1, \zeta_2)$ such that $|\zeta_1|^2 + |\zeta_2|^2 = 1$. We recall here the Improved Moser-Trudinger inequality introduced in [3] in order to prove the existence of a minimizer:

**Proposition A.1.** ([3], Proposition 3.4) Given $\frac{1}{2} < a < 1$, there exist constants $C_1(a)$, $C_2(a)$ such that for $u \in H$ with $\int_{S^3} \zeta_i e^{2u} = 0$, $i = 1, 2$, it holds:
\[
\frac{a}{16\pi^2} \int_{S^3} u P' u + 2 \int_{S^3} u - \ln \left( \int_{S^3} e^{2u} \right) + C_1(a) \|(-\Delta_b)^{\frac{a}{2}} u\|_2^2 + C_2(a) \geq 0.
\]
This improved estimate will not be useful to us in our setting since it contains the term 
\[ C_1(a)\|(-\Delta_b)^{\frac{a}{2}}u\|_2^2 \] 
that we cannot bound along the flow. Notice that in [3], the authors exploit Ekeland’s principle to exhibit a good minimizing Palais-Smale sequence that allows the control of this extra term. In our setting, we will prove a result that can be seen as intermediate between Proposition A.1 and the usual Moser-Trudinger inequality in Theorem 3.1. In fact in [3] the authors gave hints on how to prove this result, knowing that this method only works in dimension 3 and 5. We will follow a technique used in [11], since it is simpler and it allows even more improved estimates.

We set 
\[ P_k := \left\{ \text{polynomials of } \mathbb{C}^2 \text{ with degree at most } k \right\} \]
and 
\[ P_{k,0} := \left\{ f \in P_k; \int_{S^3} f = 0 \right\}. \]

For a given \( m \in \mathbb{N} \) we let 
\[ N_m := \left\{ N \in \mathbb{N}; \exists x_1, \ldots, x_N \in S^3, \nu_1, \ldots, \nu_N \in \mathbb{R}^+ \text{ with } \sum_{k=1}^{N} \nu_k = 1 \text{ and for any } f \in P_{m,0}; \sum_{k=1}^{N} \nu_k f(x_k) = 0 \right\}. \]

We let then \( N_m = \min N_m \). As it was shown in [11], one has \( N_1 = 2 \) and \( N_2 = 4 \). We recall from [3] that one has the following inequality on the standard sphere:

There exists a constant \( A_2 > 0 \) such that 
\[ \int_{S^3} \exp \left[ A_2 \frac{u - \bar{u}}{\|\Delta_b u\|_{L^2}}^2 \right] \leq C_0. \]

In fact the sharp constant \( A_2 \) was explicitly computed in [3] and it has the value \( A_2 = 32 \). With this result we can easily deduce that if \( u \in S^2(S^3) \) then \( e^{2u} \in L^p(S^3) \) for all \( 1 \leq p < \infty \).

**Lemma A.1.** Consider a sequence of functions \( u_k \in S^2(S^3) \) such that 
\[ \bar{u}_k = 0, \quad \|\Delta_b u_k\|_{L^2} \leq 1 \]
and suppose that \( u_k \rightharpoonup u \) weakly in \( S^2(S^3) \) and 
\[ |\Delta_b u_k|^2 - |\Delta_b u|^2 + \sigma \text{ in measure }, \]
where \( \sigma \) is a measure on \( S^3 \). Let \( K \subset S^3 \) be a compact set with \( \sigma(K) < 1 \), then for all \( 1 \leq p < \frac{1}{\sigma(K)} \) we have 
\[ \sup_{k} \int_{K} \exp \left[ pA_2 u_k^2 \right] < \infty. \]

**Proof.** Let \( \varphi \) be a fixed smooth compactly supported function on \( S^3 \). We set \( v_k = u_k - u \). Then \( v_k \to 0 \) strongly in \( L^2 \) and weakly in \( S^2(S^3) \). Now we compute 
\[ \int_{S^3} |\Delta_b (\varphi v_k)|^2 = \int_{S^3} (\varphi \Delta_b v_k + v_k \Delta_b \varphi + 2\nabla H \varphi \nabla H v_k)^2 \]
\[ = \int_{S^3} \varphi^2 (\Delta_b v_k)^2 + v_k^2 (\Delta_b \varphi)^2 + 4|\nabla H v_k \nabla H \varphi|^2 + 2\varphi v_k \Delta_b \varphi \Delta_b v_k + \]
\[ + 4\varphi (\nabla H \varphi \nabla H v_k) \Delta_b v_k + 4v_k (\nabla H v_k \nabla H \varphi) \Delta_b \varphi. \]

\[ \text{(15)} \]
Hence,
\[ \int_{S^3} |\Delta_b(\varphi v_k)|^2 \to \int_{S^3} \varphi^2 d\sigma. \]

Assume that \( 1 \leq p_1 < \frac{1}{\sigma(K)} \) and take \( \varphi \) so that \( \varphi|_K = 1 \), and \( \int_{S^3} \varphi^2 d\sigma < \frac{1}{p_1} \). Then we have for \( k \) large,
\[ \|\Delta_b(\varphi v_k)\|_{L^2}^2 < \frac{1}{p_1}. \]

Therefore,
\[ \int_K \exp \left( p_1 A_2(v_k - \overline{v}v_k) \right)^2 \leq \int_{S^3} \exp \left( p_1 A_2(\varphi v_k - \overline{\varphi}v_k)^2 \right) \leq \int_{S^3} \exp \left( A_2 \frac{(\varphi v_k - \overline{\varphi}v_k)^2}{\|\Delta_b \varphi v_k\|_{L^2}^2} \right) \leq C_0. \]

Thus, if we fix \( \varepsilon > 0 \), we can write
\[
\begin{align*}
    u_k^2 &= (v_k - \overline{\varphi}v_k + u + \overline{\varphi}v_k)^2 \\
         &= (v_k - \overline{\varphi}v_k)^2 + 2(v_k - \overline{\varphi}v_k)(u + \overline{\varphi}v_k) + (u + \overline{\varphi}v_k)^2 \\
         &\leq (1 + \varepsilon)(v_k - \overline{\varphi}v_k)^2 + 2(1 + \frac{1}{\varepsilon})u^2 + 2(1 + \frac{1}{\varepsilon})^2 \varphi v_k^2.
\end{align*}
\]

Hence, given \( p < \frac{1}{\sigma(K)} \) we can take \( p_1 \in (p, \frac{1}{\sigma(K)}) \) such that
\[ \int_K e^{A_2 u_k^2} < C_0, \]
which finishes the proof.

**Corollary A.1.** We consider the same assumptions as in Lemma A.1 and we let \( \ell = \max_{x \in S^3} \sigma(\{x\}) \leq 1 \). Then the following hold

- If \( \ell < 1 \), then for any \( 1 \leq p < \frac{1}{\ell} \), \( e^{A_2 u_k^2} \) is bounded in \( L^p(S^3) \). In particular \( e^{A_2 u_k^2} \to e^{A_2 u^2} \) in \( L^1 \).
- If \( \ell = 1 \), then there exists \( x_0 \in S^3 \) such that \( \sigma = \delta_{x_0} \), \( u = 0 \) and after passing to a subsequence if necessary, we have
\[ e^{A_2 u_k^2} \to 1 + c_0 \delta_{x_0}, \]
for some \( c_0 \geq 0 \).

**Proof.** Assume that \( \ell < 1 \) and let \( 1 \leq p < \frac{1}{\ell} \). Then for all \( x \in S^3 \), \( \sigma(\{x\}) < \frac{1}{p} \). By continuity, there exists \( r_x > 0 \) such that \( \sigma(B_{r_x}(x)) < \frac{1}{p} \). Since \( S^3 \) is compact we can find a finite collection of balls of the form \( B_{r_i}(x_i) \) such that
\[ S^3 = \bigcup_{i=1}^{N} B_{r_i}(x_i). \]

So using Lemma A.1, we have
\[ \sup_k \int_{B_{r_i}(x_i)} \exp \left[ pA_2 u_k^2 \right] < \infty. \]
Thus,

$$\sup_k \int_{S^3} \exp \left[ p A_2 u_k^2 \right] < \infty.$$  

We assume now that $\ell = 1$. Since $\|\Delta_b u\|^2 \leq 1$ we have that $\|\Delta_b u\|^2 + \sigma(S^3) \leq 1$. Therefore, we have $u = 0$ and there exists $x_0 \in S^3$ such that $\sigma = \delta_{x_0}$. Hence, for $r$ small, we have that

$$\sup_k \int_{S^3 \setminus B_r(x_0)} \exp \left[ q A_2 u_k^2 \right] < \infty,$$

for all $q \geq 1$. Therefore, $e^{A_2 u_k^2} \to 1$ in $L^1(S^3 \setminus B_r(x_0))$ for every $r > 0$ and small. Hence, after passing to a subsequence if necessary we have that $e^{A_2 u_k^2} \rightharpoonup 1 + c_0 \delta_{x_0}$ in measure. □

**Proposition A.2.** Let $\alpha > 0$ and consider a sequence $m_k \to \infty$ and $u_k \in S^2(S^3)$ such that $\overline{\mu_k} = 0$ and $\|\Delta_b u_k\|_{L^2} = 1$ such that $u_k \rightharpoonup u$ weakly in $S^2(S^3)$ and $(\Delta_b u_k)^2 \rightharpoonup (\Delta_b u)^2 + \sigma$ in measure. We assume moreover that

$$\ln \left( \int_{S^3} e^{2m_k u_k} \right) \geq \alpha m_k,$$

and

$$\frac{e^{2m_k u_k}}{\int_{S^3} e^{2m_k u_k}} \to \nu$$

in measure.

We set $R = \left\{ x \in S^3; \sigma(\{x\}) \geq 2 A_2 \alpha \right\} = \{x_1, \cdots, x_N\}$. Then $\nu = \sum_{i=1}^N \nu_i \delta_{x_i}$ with $\nu_i \geq 0$ and $\sum_i \nu_i = 1$.

**Proof.** Let $K \subset S^3$ such that $\sigma(K) < 2 A_2 \alpha$. By continuity, we can find a compact set $K_1$ such that $K \subset \text{int}(K_1)$ and $\sigma(K_1) < 2 A_2 \alpha$. Now given $\frac{1}{2 A_2} < p < \frac{1}{\sigma(K_1)}$, we have

$$\sup_k \int_{K_1} e^{p A_2 u_k^2} \leq C_0.$$

Since $2m_k u_k \leq p A_2 u_k^2 + \frac{m_k^2}{p A_2}$, we have

$$\int_{K_1} e^{2m_k u_k} \leq C e^{m_k^2 \alpha \frac{1}{2 A_2}}.$$

Therefore,

$$\frac{\int_{K_1} e^{2m_k u_k}}{\int_{S^3} e^{2m_k u_k}} \leq C e^{\left( \frac{1}{2 A_2} - \alpha \right) m_k^2}.$$

So $\nu(K) \leq \nu(K_1) = 0$ and $\nu(K) = 0$. Thus, if $\sigma(\{x\}) < 2 A_2 \alpha$, then there exists $r_x > 0$ small enough so that $\sigma(B_{r_x}(x)) < 2 A_2 \alpha$. Hence, $\nu(B_{r_x}(x)) = 0$. We deduce then that $\nu(S^3 \setminus R) = 0$. Therefore

$$\nu = \sum_{k=1}^N \nu_k \delta_{x_k},$$

with $\nu_k \geq 0$ and $\sum_{k=1}^N \nu_k = 1$. □
Let \( f_1, \ldots, f_\ell \in C(S^3) \). We define
\[
S_f = \left\{ u \in S^2(S^3); \overline{u} = 0; \int_{S^3} f_k e^{2u} = 0; k = 1, \ldots, \ell \right\}.
\]
We assume that the inequality
\[
\ln \left( \int_{S^3} e^{2u} \right) \leq \alpha \| \Delta_b u \|_{L^2}^2 + C
\]
does not hold for \( u \in S_f \). Then there exists a sequence \( u_k \in S_f \) such that
\[
\ln \left( \int_{S^3} e^{2u_k} \right) - \alpha \| \Delta_b u_k \|_{L^2}^2 \to \infty.
\]
Therefore, it follows that \( \int_{S^3} e^{2u_k} \to \infty \) and \( \| \Delta_b u_k \|_{L^2} \to \infty \). So we let \( m_k = \| \Delta_b u_k \|_{L^2} \) and \( v_k = \frac{u_k}{m_k} \). Then \( m_k \to \infty \), \( \| \Delta_b v_k \|_{L^2} = 1 \). Hence, after passing to a subsequence, we have
\[
\left\{ \begin{array}{l}
v_k \rightharpoonup v \text{ weakly in } S^2(S^3), \\
|\Delta_b v_k|^2 \rightarrow |\Delta_b v|^2 + \sigma \text{ in measure}, \\
\int_{S^3} e^{2m_k v_k} \rightarrow \nu \text{ in measure.}
\end{array} \right.
\]
So we let \( R = \{ x \in S^3; \sigma(\{ x \}) \geq A_2 \alpha \} = \{ x_1, \ldots, x_N \} \). It follows from Proposition A.2 that \( \nu = \sum_{j=1}^N \nu_j \delta_{x_j} \), with \( \sum_{j=1}^N \nu_j = 1 \) and \( \nu_j \geq 0 \).
But since \( u_k \in S_f \), we have
\[
\int_{S^3} f_j d\nu = 0.
\]
Therefore,
\[
\sum_{i=1}^N \nu_i f_j(x_i) = 0, \text{ for all } 1 \leq j \leq \ell.
\]
On the other hand, \( A_2 \alpha N \leq 1 \). In particular, if \( f_j \in P_{m,0} \), we have that \( N \in \mathbb{N} \). Therefore,
\[
\alpha \leq \frac{1}{A_2 N} \leq \frac{1}{A_2 N_m}.
\]
Hence, if \( \alpha = \frac{1}{A_2 N_m} + \varepsilon \) we get a contradiction and the result holds. Therefore, if we define
\[
S_0 = \left\{ u \in S^2(S^3); \overline{u} = 0; \int_{S^3} fe^{2u} = 0 \text{ for all } f \in P_{1,0} \right\},
\]
the following corollary holds

**Corollary A.2.** There exist \( a < \frac{1}{16\pi} \) and \( C > 0 \) such that for all \( u \in \mathcal{P} \cap S_0 \), we have
\[
a \int_{S^3} u \overline{u} + 2 \int_{M} u - \ln \left( \int_{M} e^{2u} \right) \geq -C.
\]
Indeed, this corollary follows from the fact that
\[
\int_{S^3} u \overline{u} \geq \int_{S^3} |2\Delta_b u|^2
\]
for all \( u \in \mathcal{P} \) and \( 8A_2 > 16\pi^2 \).
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