THE HALL ALGEBRA OF A CYCLIC QUIVER AT $q = 0$

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Abstract. We show that the generic Hall algebra of nilpotent representations of an oriented cycle specialised at $q = 0$ is isomorphic to the generic extension monoid in the sense of Reineke. This continues the work of Reineke in [3].

1. The generic extension monoid and the generic Hall algebra

For the following let $Q$ be a finite quiver with vertex set $Q_0$ and arrow set $Q_1$. We consider only finite dimensional and nilpotent representations and modules. For any field $K$ we work in $KQ\text{mod}^0$, the category of nilpotent, finite dimensional $KQ$-representations.

The generic extension monoid. Fix an algebraically closed field $K$. For each dimension vector $d \in \mathbb{N}^{Q_0}$ define

$$\text{Rep}_d := \bigoplus_{\alpha:i \to j \in Q_1} \text{Hom}_K(K^{d_i}, K^{d_j}).$$

Each point of $\text{Rep}_d$ corresponds to a representation of $Q$ over $K$ and, by choosing bases, for each representation $M$ of $Q$ over $K$ there is point $m$ in $\text{Rep}_{\dim(M)}$ such that $M \cong m$ as a representation of $Q$.

$$\text{GL}_d := \prod_{i \in Q_0} \text{GL}_{d_i}(K)$$

acts on $\text{Rep}_d$ by base change. The $\text{GL}_d$ orbits are in 1:1 correspondence to isomorphism classes of $Q$ representations of dimension vector $d$. Denote such an orbit for a finite dimensional module $M$ by $O_M$. We say that a module $M$ degenerates to $N$, $M \leq_{\text{deg}} N$, if $O_N \subseteq O_M$, where we take the closure in the Zariski topology.

For two arbitrary sets $U \subseteq \text{Rep}_d, V \subseteq \text{Rep}_e$ we define

$$\mathcal{E}(U, V) := \{ M \in \text{Rep}_{d+e} \mid \exists A \in U, B \in V \text{ and a short exact sequence } 0 \to B \to M \to A \to 0 \}.$$

The multiplication on closed irreducible $\text{GL}_d$-stable respectively $\text{GL}_e$-stable subvarieties $A \subseteq \text{Rep}_d, B \subseteq \text{Rep}_e$ is defined as:

$$A * B := \mathcal{E}(A, B)$$

This multiplication is well defined, associative and has a unit: $\text{Rep}_0$. The set of closed irreducible subvarieties of nilpotent representations with this multiplication is called the generic extension monoid $\mathcal{M}(Q)$. The composition monoid $\mathcal{CM}(Q)$ is the submonoid generated by the orbits of simple modules without self extensions. For all this see Reineke [4]. For any word $w = (N_1, \ldots, N_r)$ in semisimples we define $A_w := O_{N_1} * \cdots * O_{N_r}$. This is an element of $\mathcal{M}(Q)$.

If $A$ is a closed irreducible subvariety of some $\text{Rep}_d$, define $[A] := A / \text{GL}_d$ as a topological space with the quotient topology. Hence points of $[A]$ correspond to isomorphism classes in $A$.

2000 Mathematics Subject Classification. 16G20.
The generic Hall algebra. Now let $K$ be a finite field. Let $M, N, X \in KQ \mod 0$ be three nilpotent representations of $Q$ over $K$. Then define

$$F_{MN}^{X} := \# \{U \leq X \mid U \text{ submodule}, U \cong N, X/U \cong M\}$$

This is a finite number.

Now let $Q$ be Dynkin or an oriented cycle. Then the isomorphism classes of nilpotent indecomposable modules are given by some combinatorial set $\Phi$, finitely many for each dimension vector, independent of the field. Hence an isomorphism class is given by a function $\alpha : \Phi \to \mathbb{N}$ with finite support. For each such $\alpha$ and any field $K$ choose a module $M(\alpha, K)$ in this isomorphism class. Then there are polynomials $f_{\mu, \nu}(q) \in \mathbb{Z}[q]$ such that for each finite field $K$ we have

$$F_{M(\mu, K)M(\nu, K)}^{M(\xi, K)} = f_{\mu, \nu}(|K|).$$

Then we define the generic Hall algebra $\mathcal{H}_{q}(Q)$ to be the free $\mathbb{Z}[q]$ module with basis $\{u_\alpha \mid \alpha \in \Phi\}$ with finite support and multiplication given by:

$$u_\mu \odot u_\nu = \sum_\xi f_{\mu, \nu}(q)u_\xi.$$  

The generic composition algebra $C_{q}(Q)$ is then the subalgebra of $\mathcal{H}_{q}(Q)$ generated by the simple modules without self extensions. If the quiver is fixed then we often write $\mathcal{H}_{q}$ and $C_{q}$ instead of $\mathcal{H}_{q}(Q)$ and $C_{q}(Q)$. Moreover, for convenience we identify for any module $M \cong M(\alpha, K)$, $u_M$ with $u_\alpha$ and $F_{MN}^{X}$ with $f_{\mu, \nu}(q)$. For this see Ringel [6] and [7] but also Hubery [2].

2. THE COMPOSITION ALGEBRA OF A CYCLIC QUIVER

Consider the cyclic quiver $\tilde{A}_{n}$ (all arrows in one direction), $R := \mathbb{F}_{q}\tilde{A}_{n}$ and $R \mod 0$, the category of nilpotent representations over $R$, where $\mathbb{F}_{q}$ is a finite field with $q$ elements. Let $S_{0}, S_{1}, \ldots, S_{n-1}$ denote the simple modules in $R \mod 0$ numbered in such a way that $\text{Ext}(S_{i}, S_{i+1}) \neq 0$ (now and in the remainder of this section always count modulo $n$). For generalities on the cyclic quiver see [1].

Up to isomorphism there is exactly one indecomposable representation $S_{i}[l]$ of length $l$ with socle $S_{i}$. Hence the isomorphism classes of modules in $R \mod 0$ can be described as elements of $\Pi := \{(\pi^{(0)}, \ldots, \pi^{(n-1)}) \mid \pi^{(i)} \text{ is a partition } \forall i\}$, where each partition $\pi^{(i)}$ describes the indecomposable summands with socle $S_{i}$. In the notation of the previous section $\Phi = \{0, \ldots, n-1\} \times \mathbb{N}$ and the elements of $\Pi$ are in obvious bijection to elements of $\Phi^{*}$ with finite support. So identify these functions with $\Pi$.

Now let $M$ be an arbitrary module isomorphic to a $\pi \in \Pi$ and we denote with $u_{M}$ or $u_{\pi}$ its symbol in the generic Hall algebra of $R \mod 0$. If $p = (p_{1}, p_{2}, \ldots, p_{m})$ ($p_{1} \geq \cdots \geq p_{m} > 0$) is a partition we denote by $p(k) := (p_{1}, \ldots, p_{k}, p_{k+1}, \ldots, p_{m})$ the partition obtained by deleting the $k$th component and by $p-1 := (p_{1}-1, \ldots, p_{m}-1)$. If $T \subseteq \{1, \ldots, m\}$ we denote by $p_{T}$ the subpartition of $p$ by taking the components at the positions in $T$. For $\pi \in \Pi$ let $\pi_{a, k}$ be equal to the set of partitions created by deleting the $k$th component of $\pi^{(a-1)}$ and setting $\pi_{a, k}^{(a)} = \pi^{(a)} \cup (\pi_{k}^{(a-1)} + 1)$ (take the $k$th component of the $(a-1)$st partition, add one to it and then push it to the right).

Lemma 2.1. Let $S_{i}$ be a simple module and $p$ a partition with $n$ components. Let $A \subseteq \{1, \ldots, m\}$, $|A| = k$ and $B = \{1, \ldots, m\} \setminus A$. Let $M := S_{i-1}[pA - 1] \oplus S_{i}[pB]$ and $X = S_{i}[p]$.

Then $f_{MS_{i}}^{X} = 0 \mod q$ if $p_{A} \neq p_{(1, \ldots, k)}$ and $f_{MS_{i}}^{X} = 1 \mod q$ if $p_{A} = p_{(1, \ldots, k)}$. 
In such a situation define

\[ Q(X, k) := S_{i-1}[p_{\{1, \ldots, k\}} - 1] \oplus S_{i}[p_{\{k+1, \ldots, m\}}]. \]

We will see later that \( Q(X, k) \) is the quotient of \( X \) by \( S_i^k \) with the smallest orbit dimension.

**Proof.** Each submodule isomorphic to \( S_i^k \) is given by an inclusion of \( S_i^k \) into the socle of \( X \). Since \( S_i^k \) is semisimple and \( \text{End}(S_i) = \mathbb{F}_q \) these inclusions correspond to elements of \( \text{Mat}(m, k) \) with rank \( k \). Two inclusions \( f_1, f_2 \in \text{Mat}(m, k) \) describe the same submodule if \( \text{Im}(f_1) = \text{Im}(f_2) \). Column transforms on a matrix don’t change the image. Using the Gauss algorithm we can bring the matrix to column echelon form (this corresponds to applying an automorphism on \( S_i^k \), \( \text{Aut}(S_i^k) = \text{GL}_k(\mathbb{F}_q) \)). Hence a submodule is given by one \( f \in \text{Mat}(m, k) \) of the form:

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ast & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

Here the \( j \)th row corresponds to the summand of \( X \) isomorphic to \( S_i[p_j] \). This is actually a certain cell in the cell decomposition of the Grassmannian \( Gr(m) \). Now we want to show that the quotient of this submodule just depends on the cell.

Let \( J \) be the set of indices of the pivot rows, i.e. the rows with a leading 1. Now fix a \( J \) (i.e. fix a cell). Then replacing each \( \ast \) with an arbitrary element of \( \mathbb{F}_q \) gives us a different submodule of \( X \).

For all \( j < l \) there are embeddings \( \phi_{l,j} : S_i[p_l] \rightarrow S_i[p_j] \), unique up to scalar multiply. Without loss of generality we can choose them in such a way that \( \phi_{j,p} \circ \phi_{l,j} = \phi_{l,p} \). These map the socle of one module to the socle of the other one.

Hence one can do row transforms which add \( \lambda \) times a row to a row more to the top by applying an automorphism to \( X \), so without changing the factor. Hence our matrix becomes one with just ones and zeros, at most one 1 in each row and column. Hence the quotient is

\[ S_{i-1}[p_J - 1] \oplus S_{i}[p_{\{1, \ldots, m\}\setminus J}] \]

Therefore the quotient is \( M \) iff \( \pi_J = \pi_A \). Hence \( J \) determines the quotient (i.e. the quotient is independent of the element of the cell). If \( J \) is not completely embedded in the top part of the matrix (i.e. \( J = \{1, \ldots, k\} \)), then the number of submodules in this cell is \( q^{r} \), where \( r \) is the number of \( \ast \) respectively the dimension of the cell. Moreover, if \( J = \{1, \ldots, k\} \) then the number of submodules is 1. Hence the claim follows.

**Corollary 2.2.** Let \( X \) be an extension of \( M \) by a simple \( S_a \), \( X \) corresponding to \( \rho \in \Pi \) and \( M \) to \( \pi \in \Pi \), where \( \rho = \pi_{a,k} \). Let \( l = \pi_k^{(a-1)} + 1 \), \( N_j := \left\{ i \mid \rho_{i}^{(a)} = j \right\} \), \( n_j := |N_j| \) and \( M_j := \left\{ i \mid \rho_{i}^{(a)} > j \right\} \), \( m_j := |M_j| \).

Then

\[ \frac{X_{\text{MS}}(q)}{q^{ni} - 1} = q^{ni}. \]

**Proof.** Using the proof of the previous lemma we just have to count the elements of the cells which give us the right quotient. Since we look at one dimensional
subspaces there is just one column and \( J = \{ j \} \) consists just of one element. The factor is \( M \) if \( j \in N_t \). The number of elements in this cell is \( q^j - 1 \). So we obtain

\[
 f^X_{MS}(q) = \sum_{j \in N_t} q^j - 1 = q^{n_t} \sum_{i=0}^{n_t-1} q^i = \frac{q^{n_t} - 1}{q - 1} q^{n_t}
\]

\[\square\]

Now we are able to describe the coefficients modulo \( q \) for an extension with a semisimple.

**Lemma 2.3.** Let \( N = \bigoplus_{i=0}^{n-1} S_i^{a_i}, a_i \in \mathbb{N} \) be a semisimple module. Let \( M \in R \mod 0 \) and \( X = \bigoplus_{i=0}^{n-1} S_i[\pi^{(i)}], \pi \in \Pi \) be arbitrary modules. Then \( f^X_M = 1 \mod q \) iff

\[
 M \cong \bigoplus Q(X_i, a_i) =: Q(X, N)
\]

with \( X_i := S_i[\pi^{(i)}] \) and \( f^X_M = 0 \mod q \) otherwise.

**Proof.** Since \( \text{Hom}(S_1^{a_1}, S_2[\pi^{(2)}]) = 0 \) for \( i \neq j \) every short exact sequence \( 0 \to N \to X \to M \to 0 \) is the direct sum of short exact sequences of the form:

\[
 0 \to S_i \to X_i \to M_i \to 0
\]

for some modules \( M_i \) such that \( \bigoplus_{i=0}^{n-1} M_i \cong M \). So we have

\[
 f^X_{MN} = \sum_{(M_0, \ldots, M_{n-1}) : \bigoplus M_i \cong M} \prod_{i=0}^{n-1} f^X_{M_iS_i^{a_i}}
\]

where \( X_i := S_i[\pi^{(i)}] \). But now \( f^X_{M_iS_i^{a_i}} \) is non-zero modulo \( q \) iff

\[
 M_i \cong S_i - [\pi^{(i)}] - 1 \oplus S_i[\pi^{(i)}] = Q(X_i, a_i)
\]

for all \( i \) by lemma 2.3. Moreover, the same lemma yields \( f^X_{Q(X_i, a_i)} S_i^{a_i} = 1 \mod q \).

Hence we are done. \[\square\]

Now we show that the quotients with coefficients 1 mod \( q \) are maximal with respect to the degeneration order.

**Lemma 2.4.** Let \( S \) be a simple module, \( X_1, X_2 \) be two arbitrary modules, \( f_1 : S \to X_1 \) and \( f_2 : S \to X_2 \) be two injections. Moreover, let \( g : X_1 \to X_2 \) be a morphism such that \( g(f_1(S)) = f_2(S) \).

Then \( X_2 \oplus (X_1/f_1(S)) \leq_{\text{deg}} X_1 \oplus (X_2/f_2(S)) \) and this is even an extension degeneration.

**Proof.** We construct an extension degeneration. Let \( \pi_1 : X_1 \to X_1/f_1(S) \) and \( \pi_2 : X_2 \to X_2/f_2(S) \) be the canonical projections and let \( \overline{g} : X_1/f_1(S) \to X_2/f_2(S) \) be the map induced by \( \pi_2 g \) (exists since \( f_1(S) \subseteq \ker(\pi_2 g) \)). By construction we have the following commutative diagram:

\[
\begin{array}{cccccc}
 0 & \longrightarrow & S & \xrightarrow{f_1} & X_1 & \xrightarrow{\pi_1} & X_1/f_1(S) & \longrightarrow & 0 \\
| & & \downarrow g & & \downarrow \overline{g} & & \\
0 & \longrightarrow & S & \xrightarrow{f_2} & X_2 & \xrightarrow{\pi_2} & X_2/f_2(S) & \longrightarrow & 0
\end{array}
\]

This is a pushout, therefore

\[
 0 \longrightarrow X_1 \xrightarrow{(\pi_1)} X_2 \oplus X_1/f_1(S) \xrightarrow{(\pi_2 - \overline{g})} X_2/f_2(S) \longrightarrow 0
\]

is a short exact sequence, the desired extension degeneration. \[\square\]
Corollary 2.5. Let $M$ be any modules and $h: S \to M$ be any injection. Let $g \in \text{End}(M)$ be any endomorphism such that $gh \neq 0$. Then
\[ M/h(S) \leq_{\text{deg}} M/gh(S). \]

Proof. From lemma (2.4) we have an extension of the form
\[ 0 \to M \to M \oplus M/h(S) \to M/gh(S) \to 0. \]
By a result of Riedtmann [5] prop. 4.3] this yields that
\[ M/h(S) \leq_{\text{deg}} M/gh(S). \]

Corollary 2.6. Let $X, M \in R \ mod \ p^\beta$, $N$ a semisimple module such that there is a short exact sequence $0 \to N \to X \to M \to 0$. Then $M$ degenerates to $Q(X, N)$.

Proof. It is enough to show that this is true for every $X_i$ and $M_i$. So let $X = S_i[p]$, $M = S_{i-1}[pA-1] \oplus S_i[pB]$ and $N = S_i^p$. We want to apply the previous corollary to $X$, so we just have to choose an endomorphism $f$ which embeds the module $S_i[pA]$ into $S_i[p_{1, \ldots, k}]$. But this is no problem at all since the lengths are bigger to the left. So the corollary follows.

Now we are able to describe the monomial elements of $\mathcal{H}_{\overline{A}_n}(q)$, the generic Hall algebra of the cyclic quiver.

Theorem 2.7. Let $w = (N_1, \ldots, N_r)$ be a sequence of semisimple modules. Then
\[ u_w := u_{N_1} \circ u_{N_2} \circ \cdots \circ u_{N_r} = \sum_{M \in [A_w]} u_M \mod q \]

Proof. We proof the theorem by induction on $r$. For $r = 1$ the statement is trivial. Now let $w' = wN$ for a semisimple module $N$ and show that we have
\[ \sum_{M \in [A_{w'}]} u_M = u_{w'} = u_w \circ u_N \mod q. \]
First, it is obvious that every isomorphism class appearing in $u_w \circ u_N$ with non-zero coefficient is in $[A_{w'}]$.

Now let $X$ be in $[A_{w'}]$. We have to show that the coefficient of $u_X$ equals 1. This coefficient is $\sum_{M \in [A_w]} u^X_M$. By lemma (2.3) this sum is 1 mod $q$ iff $Q(X, N)$ is in $A_w$, and 0 otherwise. So it remains to show that $Q(X, N) \in A_w$. Now, since $X \in A_{w'}$, we know that there is at least one $M \in A_w$ such that there is a short exact sequence $0 \to N \to X \to M \to 0$. Moreover, $A_w$ is closed and via lemma (2.1) we know that $M$ degenerates to $Q(X, N)$, hence $Q(X, N) \in A_w$. □

By using this we obtain the following result.

Theorem 2.8. The map
\[ \Psi: \mathbb{Z}M(\overline{A}_n) \to \mathcal{H}_0(\overline{A}_n) \]
\[ \mathcal{A} \mapsto \sum_{M \in [A]} u_M \]
is an isomorphism of graded rings.

Proof. Since the isomorphism classes of representations of $\overline{A}_n$ are independent of the field, the map is well defined. Since $M(\overline{A}_n)$ is generated by the sets $O_N$ for $N$ semisimple we obtain by theorem (2.7) that $\Psi$ is a homomorphism of rings. The images of the basis elements $O_N$ are homogeneous of degree $\dim(N)$ in $\mathcal{H}_0$. Now we show that $\Psi$ is an isomorphism of free $\mathbb{Z}$-modules by showing that it is an isomorphism for every degree. The degeneration order is a partial order on the
isomorphism classes of modules and we have that $\Psi(\mathcal{O}_X) = u_X + \sum_{X < \deg Y} u_Y$. But all elements of $\mathcal{M}(\tilde{A}_n)$ are orbit closures and we have that $\Psi$ is injective. Moreover the dimension of the graded components of $\mathcal{M}$ and $\mathcal{H}_0$ are both equal to the number of isomorphism classes of this degree. So $\Psi$ is an isomorphism. \hfill \Box

**Corollary 2.9.** $\mathcal{ZCM}(\tilde{A}_n)$ is isomorphic to the generic composition algebra of the cyclic quiver at $q = 0$.

**Proof.** Everything follows from the theorem since $\Psi$ maps $\mathcal{O}_{S_i}$ to $u_{S_i}$ and these are the generators for $\mathcal{ZCM}(\tilde{A}_n)$ respectively the composition algebra. \hfill \Box

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