REALIZABILITY IN OCA\(\text{s}\) AND AKS\(\text{s}\)

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Abstract. In the context of the OCA\(\text{s}\) associated to an AKS\(\text{s}\) we introduce a closure operator and two associated maps that replace the closure and the maps defined in [5]. We were motivated by the search of a full adjunction to the original implication map. We show that all the constructions from OCA\(\text{s}\) to triposes developed in [5] can be also implemented in the new situation.

1. Introduction

I. In this report we revisit some aspects of the important construction presented in the paper: Krivine’s Classical Realizability from a Categorical Perspective by Thomas Streicher –see [13]–. We put special emphasis in the algebrization of the structure of an AKS and the associated OCA –see [4] or [13] and the definitions and notations therein–. Streicher’s paper points towards interpreting the classical realizability of Krivine, as an instance of the categorical approach started by Hyland–see [9]; [10]; [11]; [12] for Krivine’s approach and [8] and [14] for the categorical approach. The present paper concentrates upon the basic algebraic set up of these important categorical aspects of the theory. The categorical aspects will be considered in future work by the authors.

II. In what follows we describe briefly the contents of each of the sections of the current paper.

In Section 2 and in order to fix notations, we recall general concepts on ordered structures and indexed ordered structures. In particular, we write the definitions of preorder, meet semilattice and Heyting preorder and its indexed versions i.e., contravariant functors from the category Set –of sets– with codomain the adequate subcategory of the category Cat –of all categories –(e.g. the category of preorders, of meet semilatices, of Heyting preorders). As a particular case of indexed Heyting preorders we recall the crucial definition of tripos (see [8]). At the end, and as a tool for later usage, we briefly recall how to program directly in an OCA\(\text{a}\) using the standard codification in combinatory algebras (combinatory completeness).

In Section 3 we present the classes of ordered combinatory algebras that we need in the paper, quasi implicative ordered combinatory: \(\text{qOCA}\(\text{s}\)\), implicative ordered combinatory algebras: \(\text{iOCA}\(\text{s}\)\) and full adjunction implicative ordered combinatory algebras: \(\text{fOCA}\(\text{s}\)\). We use the completeness properties of \(\mathcal{A}\) –the OCA\(\text{a}\) we are considering– to define a new product \(\#: A \times A \rightarrow A\) that is bounded below by the application. We also show how to define realizability in high order arithmetics for \(\text{fOCA}\(\text{s}\)\) (almost in the same manner that we did in [5], Section 6 for \(\text{iOCA}\(\text{s}\)\)).

In Section 4 in preparation for the definition of Abstract Krivine Structure we introduce realizability lattices and some operations between the sets of terms and stacks of the lattice.

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which are: the double perpendicular introduced by Streicher in [13] and another smaller one that
we call \((-\)\). Then, if there is an additional push map we say that we have a realizability lattice
—similarly to the situation treated in [5] Section 2. Streicher’s Abstract Krivine Structures. We
define the basic operations of conductor and push—which are the predecessors of the application
and the implication of the OCA and that are defined between subsets of the set of stacks. All in
all we define three conductors and three push operations—including the ones used by Streicher—,
and the introduction of these operations will be profitable for the understanding of the adjunction
relations for Streicher’s operations, because the new operations are better behaved with respect to
adjunction than the original one.

In Section 5 we add the remaining pieces of the structure: the additional application operation in
the set of terms, the subset of terms called the set of quasi–proofs with its distinguished elements,
and the axioms relating the application with the push through perpendicularity.

We define the last operation, the application–like operation —called the square product and denoted
as \(\square\)– and a distinguished term called the “adjunctor”. The adjunctor should be viewed as a func-
tor that will allow –partially, or in a “lax” sense– the recuperation of the full adjunction property
between the conductor –application– and the push –implication–, connecting thus with Streicher’s
perspective in [13]. Later, we use the three pairs of application/implication–like operations to con-
struct different OCA's starting from the same AKS. Only in Streicher’s construction one needs
the adjunctor, and for this reason, the new OCA—that we call \(\mathcal{A}_{KS}^r\) and that has stronger adjunc-
tion properties, might be a good candidate to construct the tripos we are searching for, in a simpler
manner than the original appearing in [13].

The content of Section 6 and Section 7 can be described using the diagram below that summarizes
the constructions appearing in [5]:

\[
\begin{array}{ccc}
\mathcal{A}_{KS} & \rightarrow & \mathcal{HPO} \\
\text{IOCA} & \nearrow & \searrow \\
\end{array}
\]

In this diagram the concept of \(\text{IOCA}\) and the construction of the right diagonal map appears in
[5], the horizontal arrow represents the construction of a \(\text{IOCA}\) using the closure operator associated
to \(\perp\) and the left diagonal arrow is Streicher’s construction as it appears in [13].

Along these two sections, we reproduce the triangle of constructions described above with the
modifications we mention below.

The \(\text{IOCA}\)s are substituted by the \(\text{FOCA}\)s where no adjunctor is needed and the new right di-
agonal arrow is formally the same construction that the one in [5], that can be performed in the
same manner thanks to presence of the filter. Moreover, in this paper the horizontal arrow is the
construction of a \(\text{FOCA}\) using the closure operator \((-\)\) as described before.

Concerning the passage from ordered structures to triposes, that is the main content of Section
7 we need to describe one more construction that is a crucial part of the theory developed in [5].
The upper arrow in the diagram below consists of the building of an \(\mathcal{AKS}\) from a \(\text{IOCA}\) that was
performed in [5] Section 5].
The content of [5, Theorem 5.16], is the following: if we start from \( \mathcal{A} \in I\text{OCA} \), take \( \mathcal{K}_{\mathcal{A}} \) and perform on it Streicher’s construction, the associated tripos is equivalent to the one obtained directly from \( \mathcal{A} \).

In Theorem 7.5 we prove a version of the above mentioned result starting now from a \( I\text{OCA} \) and performing similar steps to the point where we prove an analogous result concerning the equivalence of the associated triposes.

III. In future work we intend to explore the use of the above methods in order to explore the basic foundational pillars of the categorical approach to realizability.

2. ORDERED STRUCTURES, INDEXED ORDERED STRUCTURES, TRIPoses

In this section—in order to fix notations—we recall besides the basic notion of tripos some general definitions related to ordered structures and indexed ordered structures.

IV. We list some definitions and basic results.

**Definition 2.1.** A preorder \((D, \leq)\) is a pair of a set \(D\) with a reflexive and transitive relation \(\leq\). A morphism of preorders is a monotonic map between the sets.

1. If \(d \leq d'\), and \(d' \leq d\), with \(d, d' \in D\) we say that \(d\) and \(d'\) are isomorphic, and write \(d \equiv d'\).
2. If \((C, \leq)\) and \((D, \leq)\) are two preorders and \(f, g : (C, \leq) \to (D, \leq)\) are morphisms of preorders, we say that \(f \leq g : \iff \forall d \in D, f(d) \leq g(d)\) and that \(f\) and \(g\) are isomorphic \((f \equiv g)\) if \(f \leq g\) and \(g \leq f\). A monotonic map \(f : (C, \leq) \to (D, \leq)\) is an equivalence if and only if:
   - \(g \circ f \equiv \text{id}_C\), and \(f \circ g \equiv \text{id}_D\). The map \(g\) is called a weak inverse of \(f\). In this case we say that \((C, \leq)\) and \((D, \leq)\) are equivalent (denoted as \((C, \leq) \equiv (D, \leq)\)).
3. Given monotonic maps \(f : (C, \leq) \to (D, \leq)\), \(g : (D, \leq) \to (C, \leq)\), we say that “\(f\) is left adjoint to \(g\)”, or “\(g\) is right adjoint to \(f\)”, and write \(f + g\), if \(\text{id}_C \leq g \circ f\) and \(f \circ g \leq \text{id}_D\).
4. We call \(\text{Ord}\) the category of preorders and, order preserving maps, i.e. monotonic maps.
5. (a) A principal filter in \((D, \leq)\) is a subset of \(D\) of the form \(\uparrow d_0 := \{d \in D : d \leq d_0\}\), for some \(d_0 \in D\).
   (b) Dually, a principal ideal in \(D\) is a subset of the form \(\downarrow d_0 := \{d \in D : d \leq d_0\}\), for \(d_0 \in D\).

**Observation 2.2.**

1. Given monotonic maps \(f : (C, \leq) \to (D, \leq)\), \(g : (D, \leq) \to (C, \leq)\), \(f\) is left adjoint to \(g\) if and only if \(\forall c \in C, d \in D, f(c) \leq d \iff c \leq g(d)\).
2. We will use the following standard result in order theory:
   A monotonic map \(f : (C, \leq) \to (D, \leq)\) is an equivalence if and only if:
   - \(f\) is order reflecting: \(\forall c, c' \in C, f(c) \leq f(c') \Rightarrow c \leq c'\)
   - \(f\) is essentially surjective: \(\forall d \in D \exists c \in C, f(c) \equiv d\).
3. Concerning principal filters and principal ideals we have that: \(\inf(\uparrow d_0) = d_0 = \sup(\downarrow d_0)\).
4. The above standard notion of filter, is different from the one we use for OCA’s later.
Definition 2.3. A meet semi-lattice is a preorder \((A, \leq)\) equipped with a binary operation \(\land\) and a distinguished element \(\top\) such that for all \(a, b, c \in C\):

1. \(a \land b \leq a\);
2. \(a \land b \leq b\);
3. \(c \leq a\) and \(c \leq b\) \(\Rightarrow c \leq a \land b\);
4. \(a \leq \top\).

We call \(\text{SLat}\) the category of meet semi-lattices, and meet preserving monotonic maps, i.e. monotonic maps \(f : (C, \leq) \to (D, \leq)\) such that \(f(c) \land f(c') \equiv f(c \land c')\) and \(f(\top) \equiv \top\).

Definition 2.4.

1. A Heyting preorder is a meet semi-lattice \((A, \leq)\) with a binary operation \(\to: A \times A \to A\) (called Heyting implication) satisfying:

   \[
   \text{for all } a, b, c \in A \quad a \land b \leq c \text{ if and only if } a \leq b \to c \quad (2.4.1)
   \]

2. A morphism of Heyting preorders is a monotonic map \(f : (A, \leq) \to (B, \leq)\) such that \(f(\top) \equiv \top\),

   \[
   f(a \land b) \equiv f(a) \land f(b), \quad f(a \to b) \equiv f(a) \to f(b) \quad \text{for all } a, b \in A.
   \]

We call \(\text{HPO}\), the category of Heyting preorders and its morphisms.

Observation 2.5. It follows from the definition above that in a Heyting preorder the meet is monotonic in both arguments and the implication is antitone in the first argument and monotonic in the second one.

V. Next we generalize the definitions above, to the situation of indexed structures. We write down the definitions for the category of indexed preorders–for the other structures the definitions are similar (see [5]).

Definition 2.6. An indexed category is a functor \(F : \text{Set}^{\text{op}} \to \text{Cat}\), where \(\text{Cat}\) is the category of all categories.

Example 2.7. Some examples of indexed categories that we use in this work are the following—with special codomains inside \(\text{Cat}\):

1. An indexed preorder is a functor \(F : \text{Set}^{\text{op}} \to \text{Ord}\).
2. An indexed meet-semi-lattice is a functor \(L : \text{Set}^{\text{op}} \to \text{SLat}\).
3. An indexed Heyting preorder is a functor \(P : \text{Set}^{\text{op}} \to \text{HPO}\).

Definition 2.8. Given indexed preorders \(C, D : \text{Set}^{\text{op}} \to \text{Ord}\) an indexed monotonic map \(\sigma : C \to D\) is a family \(\sigma_I : C(I) \to D(I)\) \((I \in \text{Set})\) of monotonic functions, such that for all functions \(f : J \to I\) and predicates \(\varphi \in C(I)\):

   \[
   \sigma_J(f^*(\varphi)) \equiv f^*(\sigma_I(\varphi)) \quad (2.8.2)
   \]

where the symbol \(f^*\) denotes \(C(f)\) when on the left side of the equation and \(D(f)\) when on the right side.

Definition 2.9. Let \(C, D : \text{Set}^{\text{op}} \to \text{Ord}\) be indexed preorders.

1. For indexed monotonic maps \(\sigma, \tau : C \to D\), we define

   \[
   \sigma \leq \tau \iff \forall I \in \text{Set} \quad \sigma_I \leq \tau_I \iff \forall I \in \text{Set} \text{ and } \forall d \in C(I), \sigma_I(d) \leq \tau_I(d).
   \]

   We say that \(\sigma\) and \(\tau\) are isomorphic, and write \(\sigma \equiv \tau\), if \(\sigma \leq \tau\) and \(\tau \leq \sigma\).

2. An indexed monotonic map \(\sigma : C \to D\) is called an equivalence, if there exists a indexed monotonic map \(\tau : D \to C\) such that \(\tau \circ \sigma \equiv \text{id}_C\), and \(\sigma \circ \tau \equiv \text{id}_D\). In this case, \(\tau\) is called an (indexed) weak inverse of \(\sigma\).

3. We say that \(C\) and \(D\) are equivalent, and write \(C \simeq D\), if there exists an equivalence \(\sigma : C \to D\).
A proof of the next Lemma follows from the considerations appearing at beginning of Paragraph TV.

**Lemma 2.10.** An indexed monotonic map \( \sigma : C \to D \) is an equivalence, if and only if for every set \( I \), the monotonic map \( \sigma_I : C(I) \to D(I) \) is order reflecting and essentially surjective.

VI. Next we consider a special kind of indexed Heyting preorders, called triposes, see [8].

**Definition 2.11.** A tripos is a functor \( P : \text{Set}^{op} \to \text{HPO} \) such that:
1. For every function \( f : J \to I \), the reindexing map \( f^* : P(I) \to P(J) \) has a right adjoint \( \forall_f : P(J) \to P(I) \).
2. If \( \mathcal{A} \), \( \mathcal{B} \), \( \mathcal{C} \) are polynomials, then:
   \[
   \mathcal{A} \cdot \mathcal{B} \leq \mathcal{C}
   \]
   \[
   f_{\mathcal{A}} \Rightarrow f_{\mathcal{B}} 
   \]
3. \( P \) has a generic predicate, i.e. there exists a set \( \text{Prop} \), and a \( \text{tr} \in P(\text{Prop}) \) such that for every set \( I \) and \( \varphi \in P(I) \) there exists a (not necessarily unique) function \( \chi_\varphi : I \to \text{Prop} \) with \( \varphi \equiv \chi_\varphi(\text{tr}) \).

**Remark 2.12.** Assume that \( C \) and \( P \) are equivalent indexed preorders, and that \( P \) is a tripos, then so is \( C \).

VII. We briefly recall how to program directly in an \( OCA \mathcal{A} \), using the standard codification in combinatorial algebras—see for example [4] or [5, Theorem 3.4] for details. For any finite set of variables \( \{x_1, \ldots, x_k, y\} \), there is a function \( \lambda^* y : A[x_1, \ldots, x_k, y] \to A[x_1, \ldots, x_k] \) satisfying the following property: If \( t \in A[x_1, \ldots, x_k, y] \) and \( u \in A[x_1, \ldots, x_k] \), then \( \lambda^* y(t) \circ u \leq t[y := u] \).

The function \( \lambda^* y \) is defined recursively: i) If \( y \neq x \), then \( \lambda^* y(x) := k x \); ii) \( \lambda^* y(y) := \mathsf{s} k k \); iii) if \( p, q \) are polynomials, then: \( \lambda^* y(pq) := \mathsf{s} (\lambda^* y(p))(\lambda^* y(q)) \).

Sometimes we write: \( \lambda^* y(t) = \lambda^* y.t \).

3. **Ordered combinatory algebras**

VIII. We start by recalling the following basic structure, compare with [5, Section 3.2].

**Definition 3.1.** Let \((A, \leq)\) be an inf–complete partially ordered set equipped with:
1. **(OP) binary operations**
   - \( \text{app} : A \times A \to A, \quad (a, b) \mapsto ab \)
   - called *application*, monotone in both arguments, and
   \[
   \text{imp} : A \times A \to A, \quad (a, b) \mapsto a \Rightarrow b
   \]
   - called *implication*, antimonotone in the first argument and monotone in the second;
2. **(CO) distinguished elements** \( \mathsf{s}, k \in A \) such that for all \( a, b, c \in A \) the following holds:
   - \( \mathsf{PK} \) \( k ab \leq a \)
   - \( \mathsf{PS} \) \( s abc \leq ac(bc) \)
   - \( \mathsf{PA} \) If \( a \leq b \Rightarrow c \), then \( ab \leq c \).
3. **(FL) a subset \( \Phi \subseteq A \) (called *filter*) which is closed under application and such that \( \mathsf{s}, k \in \Phi \).

In above context we establish the following definitions:

1. The structure \( \mathcal{A} = (A, \text{app}, \text{imp}, \mathsf{s}, k, \Phi) \) as above is called a *quasi implicative ordered combinatory algebra* – a \( qOCA \).
(2) If there exists a distinguished element $e \in \Phi$ such that:

$$(PE) \quad \text{If } ab \leq c, \text{ then } e \ a \leq b \to c$$

we say that $\mathcal{A} = (A, \text{app}, \text{imp}, s, k, e, \Phi)$ is an implicative ordered combinatory algebra $-$ an $I\text{OCA}$.

(3) If for all $a, b, c \in A$

$$(PE)' \quad \text{If } ab \leq c, \text{ then } a \leq b \to c,$$

we say that $\mathcal{A} = (A, \text{app}, \text{imp}, s, k, e, \Phi)$ is a full adjunction implicative ordered combinatory algebra $-$ a $F\text{OCA}$.

We refer the reader to [5, Section 3] for motivation and more details concerning the definition of $I\text{OCA}$, in particular we mention that the element $e$ is called an adjunctor. When there is no danger of confusion we denote $A = (A, \text{app}, \text{imp}, s, k, e, \Phi)$ as $A$.

**Observation 3.2.** Notice that the element $i = s \ k \ k \in \Phi$, and that given a $F\text{OCA}$ called $A$, for all $a, b, c \in A$ if $ab \leq c$ implies that $i \ a \leq a \ b \to c$ and thus $A$ is also a $F\text{OCA}$ with adjunctor $i$.

**Observation 3.3.** Notice that the application $\text{app}(a, b)$ is denoted as $ab$ (c.f.: 3.1(OP)). Eventually $ab$ will be denoted $a \circ b$, when there is danger of confusion.

When operating with the function $\text{app}$ we associate to the left and when operating with the function $\text{imp}$ we associate to the right, so that $abc$ means $(ab)c$ and $a \to b \to c$ means $a \to (b \to c)$.

**IX.** We use the completeness property to establish the following definition.

**Definition 3.4.** Let $A$ be a $Q\text{OCA}$, for $a, b \in A$ define

$$a \# b = \inf\{c : a \leq (b \to c)\}.$$

**Theorem 3.5.**

(1) If $A$ is a $Q\text{OCA}$, then for all $a, b \in A$, $ab \leq a \# b$.

(2) If $A$ is a $I\text{OCA}$ with adjunctor $e$, we have that for all $a, b \in A$: $(e \ a) \# b \leq ab$.

(3) If $A$ is a $F\text{OCA}$ we have that for all $a, b \in A$: $a \# b = ab$.

**Proof.**

(1) Follows from condition (PA): if $a \leq (b \to c)$ we have that $ab \leq c$, and then $ab \leq a \# b$.

(2) From $ab \leq ab$, we obtain that $e \ a \leq b \to (ab)$, and this implies that $(e \ a) \# b \leq ab$.

(3) Is proved as (2), but without $e$.

$\square$

**X.** In [5], Section 6, we illustrated that we can define realizability for $I\text{OCA}$s and thus, to define realizability in higher-order arithmetics. In what follows we mention briefly how to adapt the constructions therein, to the context of a $F\text{OCA}$, and hence we show that the ordered combinatory algebras considered above, can be taken as an adequate platform to do realizability.

The adaptations are minor: the kinds, the language and the type system are the same. We have to slightly modify the typing rule of the implication that becomes:

$$\Gamma, x : A^\circ + p : B^\circ \quad \Gamma + A^\circ x.p : (A^\circ \Rightarrow B^\circ)^\circ \quad (\to,)$$
The interpretation of $\mathcal{L}^\circ$ is defined in the same manner than in [5], and the adequacy follows also in the same manner, the only new problem to deal with is the adequacy for the implication rules. This follows directly using that the $\text{IFOC}_t\mathcal{A}$ satisfies the full adjunction property. Indeed, for the implication rule: $(\rightarrow)$, we assume $A \models \Gamma, x : A^\circ \vdash p : B^\circ$ where $\Gamma = x_1 : A_1^\circ, \ldots, x_k : A_k^\circ$. Consider an assignment $a$ and $b_1, \ldots, b_k \in A$ such that $b_i \leq [A_i^\circ]$. We get:

$$(\lambda^*x.p)(x_1 := b_1, \ldots, x_k := b_k) \circ [A^\circ] = \lambda^*x.(p[x_1 := b_1, \ldots, x_k := b_k]) \circ [A^\circ] \leq p[x_1 := b_1, \ldots, x_k := b_k, x := [A^\circ]] \leq [B^\circ]$$

for the first inequality see Paragraph VII, the last inequality follows directly from the assumption that $A \models \Gamma, x : A^\circ \vdash p : B^\circ$. Applying the full adjunction relation between $\circ$ and $\rightarrow$ we deduce that $\lambda^*x.p(x_1 := b_1, \ldots, x_k := b_k) \leq [(A^\circ \Rightarrow B^\circ)^\circ]$. As this is proved for all assignments $a$, we conclude that $A \models \Gamma \vdash \lambda^*x.p : (A^\circ \Rightarrow B^\circ)^\circ$.

4. Operations in sets of terms and stacks

XI. We briefly recapitulate the definitions and the basic operations in an Abstract Krivine Structure a.k.a. AKS and introduce new operations. Compare with [5] where some aspects of the work of J.L. Krivine and T. Streicher are reformulated (see [11] and [13] respectively).

XII. Polarities and closure operations. A triple of sets $(\Lambda, \Pi, \perp)$ with $\perp \subseteq \Lambda \times \Pi$ being a subset or a relation (see [11] Chapter V, Section 7), induces a polarity as illustrated below. A generic triple as above will be denoted as $\mathcal{R} = (\Lambda, \Pi, \perp)$.

The elements of $\Lambda$ and of $\Pi$ are called respectively terms and stacks and the elements of $\Lambda \times \Pi$ are called processes. The processes are pairs $(t, \pi)$, usually denoted by $t \star \pi$ (c.f.: [10]). If $t \star \pi \in \perp$, we write $t \perp \pi$, and say that “$t$ is orthogonal to $\pi$” or that “$t$ realizes $\pi$”. If $P \subseteq \Pi$ and $t \perp \pi$ for all $\pi \in P$ we say that “$t$ realizes $P$” and write $t \perp P$. Symetrically if $t \star \pi \in \perp$, we say that “$\pi$ is orthogonal to $t$”. If $L \subseteq \Lambda$ and $t \perp \pi$ for all $t \in L$ we say that $\pi$ is perpendicular to $L$.

Definition 4.1. Given a triple $\mathcal{R}$ as above, we define the following order reversing maps and families of sets:

$$(\ )^\perp : \mathcal{P}(\Lambda) \longrightarrow \mathcal{P}(\Pi)$$

$L \mapsto L^\perp = \{\pi \in \Pi | \forall t \in L, t \perp \pi\}$;

$$(\ )^\perp : \mathcal{P}(\Pi) \longrightarrow \mathcal{P}(\Lambda)$$

$P \mapsto ^\perp P = \{t \in \Lambda | \forall \pi \in P, t \perp \pi\}$.

The maps are called the polar maps and the subsets $L^\perp$ and $^\perp P$ are called the polars –or perpendiculars– of $L$ and $P$ respectively.

Observation 4.2. The maps $(\ )^\perp : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Pi)$ and $(\ )^\perp : \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Lambda)$ are order reversing.

For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that $(^\perp L)^\perp \supseteq L$ and $(^\perp P)^\perp \supseteq P$.

For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that $(^\perp (L^\perp))^\perp = L^\perp$ and $(^\perp (^\perp P)^\perp) = ^\perp P$.

The roles of $\Lambda$ and $\Pi$ up to now, are completely symmetric. For reasons that will be clear later, for the next considerations we work with $\Pi$, i.e. with subsets of stacks, noting that the corresponding definitions for $\Lambda$ are identical.

Definition 4.3. The set $((^\perp P)^\perp)$ is abreviated as $\overline{P}$ (when $P$ is a wide expression, we write $(P)^\perp$ instead of $\overline{P}$).
The map \( P \mapsto (\mathcal{I} P) = \overline{P} \) is called the closure operation associated to the polarity. The family of closed subsets associated to the closure operation (see [1, Chapter V, Section 7, Theorem 19]) is denoted as:

\[ \mathcal{P}_\perp(\Pi) = \{ P \subseteq \Pi : \overline{P} = P \} \]

**Observation 4.4.** The basic properties of the inclusion \( \mathcal{P}_\perp(\Pi) \subseteq \mathcal{P}(\Pi) \) can be expressed by saying that the first is a reflective subcategory of the second. The map \( (P \mapsto \overline{P} : \mathcal{P}(\Pi) \to \mathcal{P}_\perp(\Pi)) \) is the reflector (see [2]). The reflective property can be expressed as: for \( P \in \mathcal{P}(\Pi) \) and \( Q \in \mathcal{P}_\perp(\Pi) \), then:

\[ P \subseteq Q \iff \overline{P} \subseteq \overline{Q} \iff \perp Q \subseteq \perp P \]

Notice that the polar maps are antitone bijections between \( \mathcal{P}_\perp(\Lambda) \) and \( \mathcal{P}_\perp(\Pi) \), each inverse of the other.

We define another closure operator (not coming in the standard fashion from the polarity) as follows:

**Definition 4.5.**

\[ P \mapsto \hat{P} := \bigcup \{ \overline{\pi} : \pi \in P \} : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi) \]

When \( P \) is a too wide expression, we write \((P)\hat{\hat{}}\) instead of \( \hat{P} \).

The family of closed subsets associated to this closure operation is denoted as:

\[ \mathcal{P}_*(\Pi) = \{ P \subseteq \Pi : \hat{P} = P \} \]

**Observation 4.6.** A reflective property also holds for \((P)\hat{\hat{}} : P \in \mathcal{P}(\Pi) \) and \( Q \in \mathcal{P}_*(\Pi) \), then:

\[ P \subseteq Q \iff \hat{P} \subseteq Q \]

We define now an interior operator as follows:

**Definition 4.7.**

\[ P \mapsto \tilde{P} := \{ \pi \in P : \pi \subseteq \overline{P} \} : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi) \]

When \( P \) is a too wide expression, we write \((P)\tilde{\tilde{}}\) instead of \( \tilde{P} \).

Notice –see the proof below– that we also have:

\[ \mathcal{P}_*(\Pi) = \{ P \subseteq \Pi : \tilde{P} = P \} \]

**Observation 4.8.** The corresponding reflection property in this case reads as: \( P \in \mathcal{P}_*(\Pi) \) and \( Q \subseteq \mathcal{P}(\Pi) \), then:

\[ P \subseteq Q \iff P \subseteq \tilde{Q} \]

**Observation 4.9.** Let us recall that a closure operator in \( \Pi \) is a map \( P \mapsto c(P) : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi) \) such that:

- For all \( P \subseteq \Pi \), \( P \subseteq c(P) \).
- For all \( P \subseteq \Pi \), \( c(c(P)) = c(P) \).
- For all \( P, Q \subseteq \Pi \), \( P \subseteq Q \) implies that \( c(P) \subseteq c(Q) \).

The operator is said to be a topological closure operator if:

- For all \( P, Q \subseteq \Pi \), then \( c(P \cup Q) = c(P) \cup c(Q) \).
- \( c(\emptyset) = \emptyset \)

It is said to satisfy the Alexandroff condition if:

- For every \( \{ P_i : i \in I \} \subseteq \mathcal{P}(\Pi) \) we have \( c(\bigcup\{ P_i : i \in I \}) = \bigcup\{ c(P_i) : i \in I \} \).
In the above situation it is customary to define the set of $c$–closed subsets of $\Pi$ as: $\mathcal{P}_c(\Pi) = \{ P \subseteq \Pi : c(P) = P \}$. Observe that if $\{ P_i : i \in I \} \subseteq \mathcal{P}(\Pi)$ is a family of $c$–closed sets, the set $\bigcap P_i$ is also closed.

The notion of interior operator is dual to the one of topological closure operator. The open sets defined by an interior operator $i$ are the sets $P$ such that $i(P) = P$.

The operators defined above satisfy the following:

**Observation 4.10.** The operator $P \mapsto \bar{P} : \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Pi)$ is a closure operator $P \mapsto \bar{P} : \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Pi)$ is a topological closure operator and $P \mapsto \bar{P} : \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Pi)$ is an interior operator. Moreover, the topology in $\Pi$ associated to the closure operator $P \mapsto \bar{P}$ is an Alexandroff topology (c.f.: 4.12).

The proof of this fact follows directly from the adjunction properties, of the operators: $P \mapsto \bar{P}, P \mapsto \bar{P}$.

**Theorem 4.11.** For an arbitrary subset $P \subseteq \Pi$:

(1) $\bar{P} \subseteq P \subseteq \bar{P} \subseteq \bar{P}$,

(2) $\hat{P} = \bar{P}$ and $\bar{P} = \bar{P}$,

(3) $\mathcal{P}_c(\Pi) \subseteq \mathcal{P}_c(\Pi) \subseteq \mathcal{P}(\Pi)$,

(4) The following are equivalent:
   
   (a) $P \in \mathcal{P}_c(\Pi)$
   
   (b) $\pi \in P \Leftrightarrow \pi \subseteq P$.
   
   (c) $P = \bar{P}$

(5) For $P \subseteq \Pi$, then $P \in \mathcal{P}_c(\Pi) \Leftrightarrow \bar{P} = P = \bar{P} = \bar{P}$.

**Proof.**

(1) Firstly observe that $\bar{P} = \bigcap \{ \llcorner \pi \rrbracket : \pi \in P \}$. Thus $\bar{P} = (\bar{P})^\perp = (\bigcap \{ \llcorner \pi \rrbracket : \pi \in P \})^\perp \supseteq \bigcup \{ \llcorner \pi \rrbracket : \pi \in P \} = \bar{P}$. The rest of the inclusions are straightforward.

(2) It follows directly from (1) that $\bar{P} \supseteq \bar{P}$. For the reverse inclusion, observe that $t \perp \pi$ if and only if $t \perp \pi$ (c.f.: 4.2). The other equality is obtained applying the corresponding polar map to $\bar{P} = \bar{P}$.

(3) Let us consider $P \in \mathcal{P}_c(\Pi)$, i.e.: $\bar{P} = P$. By (1) we get $P \subseteq \bar{P} \subseteq \bar{P} = P$, which implies that $P \in \mathcal{P}_c(\Pi)$, thus proving the left inclusion. The other one is evident.

(4) The equivalence of (4a) and (4b) follows directly from the definition of $\bar{P}$. The equivalence of (4b) and (4c) follows from the definition of $\bar{P}$.

(5) This equivalence follows directly from (3) and (4).

□

**Observation 4.12.** By the equivalence of (4.11)(4a) and (4.11)(4c), we have that the open sets induced by $\hat{P}$ are just the closed sets induced by $\hat{P}$, thus concluding that both topologies are Alexandroff.

**XIII. Order and Completeness.** If $R$ is as above, all the three sets: $\mathcal{P}_c(\Pi) \subseteq \mathcal{P}_c(\Pi) \subseteq \mathcal{P}(\Pi)$ are complete with the order $\subseteq$. Indeed, the functors $(-)^\wedge : \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Pi)$ and $(-)^\wedge : \mathcal{P}(\Pi) \rightarrow \mathcal{P}_c(\Pi)$ are retractions of $\mathcal{P}(\Pi)$ into $\mathcal{P}_c(\Pi)$ and $\mathcal{P}_c(\Pi)$ respectively and the considerations that follow about the form of the product and coproduct (or inf and sup) in $\mathcal{P}_c(\Pi)$ and $\mathcal{P}_c(\Pi)$ are particular cases of general results about the construction of limits and colimits in reflexive subcategories—see [2] and [3] pg. 118 and pg. 196 respectively. (1) It is clear that for $X \subseteq \mathcal{P}(\Pi), \sup(X) = \bigcup X$, $\inf(X) = \bigcap X$. 
We give one more step heading towards the definition due to Streicher in [13] of an:

XIV.

Example 4.13. Consider the triple \((\Lambda, \Pi, \perp)\) with \(\Lambda = \Pi = \mathbb{R}^3\), and \(\perp\) being the usual perpendicularity relation. In this context for \(P \subseteq \mathbb{R}^3\), \(\overline{P}\) is the linear subspace generated by \(P\) and \(\overline{P}\) is the cone with vertex in the origin generated by \(P\). Then \(\mathcal{P}_\perp(\mathbb{R}^3)\) is the set of linear subspaces and \(\mathcal{P}_\Lambda(\mathbb{R}^3)\) is the set of cones.

XV. We give one more step heading towards the definition due to Streicher in [13] of an \(\mathcal{AKS}\).

Definition 4.14. Realizability lattice.

A quadruple \((\Lambda, \Pi, \perp, \text{push})\) consisting of a pair of sets \(\Lambda\) –set of terms– \(\Pi\) –set of stacks–, a subset \(\perp \subseteq \Lambda \times \Pi\) and a function \(\text{push}: \Lambda \times \Pi \rightarrow \Pi\) is called a realizability lattice.

Notation: The family of the realizability lattices is denoted as \(\mathcal{RL}\) and one of its generic elements will be called \(\mathcal{R}\).

We use also the notation \(\text{push}(t, \pi) = t \sim \pi = t \cdot \pi\) for \(t \in \Lambda\) and \(\pi \in \Pi\).

XVI. The operations of application and implication: first approximation. Below we define three pairs of operations. As the role of the first and third operations in the formalization by Streicher of realizability theory (see [13]) has been considered in [5, Section 2], we concentrate our attention in the new set \(\mathcal{P}_\Lambda(\Pi)\) and the new operations \(*\) and \(\sim\). The first and third are mentioned only for comparison reasons.

Definition 4.15. For \(L \in \mathcal{P}(\Lambda)\), \(P \in \mathcal{P}(\Pi)\) we define the following subsets, given by the push operations:

\[
\begin{align*}
\sim & : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Pi) \quad L \sim P := \{t \cdot \pi : t \in L, \pi \in P\} \\
\sim_* & : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \rightarrow \mathcal{P}_*(\Pi) \quad L \sim_* P := (L \sim P)^\wedge \\
\sim_\perp & : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \rightarrow \mathcal{P}_\perp(\Pi) \quad L \sim_\perp P := (L \sim P)^\perp
\end{align*}
\]

For \(L \in \mathcal{P}(\Lambda)\), \(P \in \mathcal{P}(\Pi)\) we define the following conductors of \(L\) into \(P\):

\[
\begin{align*}
* & : \mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Pi) \quad P * L := \{\pi \in \Pi : L \sim \pi \subseteq P\} = \{\pi \in \Pi : L \cdot \pi \subseteq P\} \\
_* & : \mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \rightarrow \mathcal{P}_*(\Pi) \quad P_* L := \{\pi \in \Pi : L \sim \overline{\pi} \subseteq P\} = \{\pi \in \Pi : L \cdot \overline{\pi} \subseteq P\} \\
*_\perp & : \mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \rightarrow \mathcal{P}_\perp(\Pi) \quad P_* \perp L := \{\pi \in \Pi : L \sim \pi \subseteq P\} = (P * L)^\perp
\end{align*}
\]

Observation 4.16. By definition of \(*\), \(L \overline{\pi} \subseteq P\) if and only if \(\overline{\pi} \subseteq P * L\). We conclude that \(P *_* L = (P * L)^\perp\).

Notice a basic difference in the definition of the pair of operations \((*_\perp, \sim_\perp)\) and the pair \((*_*, \sim_*)\). In the first row the operations are given in terms of the closure \((-)\) while in the second row by the interior operator \((-)^\wedge\) and the closure \((-)^\perp\).

| Conductor | Push |
|-----------|------|
| \(P * \perp L\) | \((P * L)^\perp\) |
| \(L \sim \perp P\) | \((L \sim P)^\perp\) |
| \(P *_* L\) | \((P * L)^\wedge\) |
| \(L \sim_* P\) | \((L \sim P)^\wedge\) |

Observation 4.17.
(1) Applying (4.16) and (4.11) [11], we get the following table:

| Conductor | Push |
|-----------|------|
| $P \star L \subset P * L \subset P \perp L$ | $L \leadsto P \subset L \leadsto \perp P \subset L \leadsto \perp P$ |

(2) The reflection properties of the closures and of the interior operator –see Paragraph [11][XII] observations (4.4), (4.6) and (4.8)– leads to the following tables:

(a) If $R \in \mathcal{P}_\perp(\Pi)$, $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$:

| Conductor | Push |
|-----------|------|
| $P * L \subseteq R$ | $R \supseteq P * L$ |

(b) If $R \in \mathcal{P}_\perp(\Pi)$, $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$:

| Conductor | Push |
|-----------|------|
| $R \subseteq P * L$ | $R \supseteq P * L$ |

XVII. The adjunction of the new operators: first approximation. We have the following adjunction result for $\mathcal{R}$, a realizability lattice.

**Theorem 4.18.** The maps $\star : \mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Pi), \leadsto : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Pi)$ satisfy the adjunction property: If $P, R \in \mathcal{P}_\perp(\Pi)$ and $L \in \mathcal{P}(\Lambda)$, then $L \leadsto R \subseteq P$ if and only if $R \subseteq P * L$.

**Proof.** In the above situation $L \leadsto R \subseteq P$, if and only if $L \leadsto R \subseteq P$ if and only if $R \subseteq P * L$ if and only if $R \subseteq P * L$ –see Observation [4.17] [2].

XVIII. Concerning the adjunction property for the other products the following holds as it is proved in [5] Section 2).

(1) The maps $\star : \mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Pi)$ and $\leadsto : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \rightarrow \mathcal{P}(\Pi)$, satisfy the full adjunction property: $L \leadsto R \subseteq P$ if and only if $R \subseteq P * L$.

(2) The other half of the adjunction property can be partially recuperated using the combinator $\varepsilon$ as follows –see [5] Section 2, §2.3, Theorem 2.13] (and for another proof see Theorem [5.9] below):

\[
\text{If } R \subseteq P * L \text{ then } L \leadsto R \subseteq \{\varepsilon\}^* \star \perp P.
\]

XIX. We can complete the Example presented in Paragraph [3.13] adding: push$(v, w) = \langle v, w \rangle v \wedge w$ –if $v, w \in \mathbb{R}^3$ where we are denoting as $\langle v, w \rangle$, $v \wedge w \in \mathbb{R}^3$ the usual inner and vector product respectively. Let $\{e_1, e_2, e_3\}$ be a positive orthonormal basis.

(1) We construct an example where $P \in \mathcal{P}_\perp(\Pi), L \in \mathcal{P}_\perp(\Lambda)$ and $P * L = P * L \neq P \perp L$. Consider $L = \mathbb{R} e_2$ and $P = \mathbb{R} e_1$. Then: $P * L = \{w \in \mathbb{R}^3 : \langle e_2, w \rangle e_2 \wedge w \in \mathbb{R} e_1\}$. If $\langle e_2, w \rangle = 0$, then $w \in \mathbb{R} e_1 + \mathbb{R} e_3$; otherwise $e_2 \wedge w \in \mathbb{R} e_1$ if and only if $w \in \mathbb{R} e_2 + \mathbb{R} e_3$. Then, $P * L = P * L = (\mathbb{R} e_1 + \mathbb{R} e_3) \cup (\mathbb{R} e_2 + \mathbb{R} e_3) \neq P \perp L = \mathbb{R}^3$.

(2) Moreover, the full adjunction property does not hold in this situation for the operators $\leadsto$ and $\star$. Indeed, we have that if we take $P, L$ as above and $R = \mathbb{R} (e_1 + e_2)$ a direct computation shows that: $R = \mathbb{R} (e_1 + e_2) \subseteq P * L = \mathbb{R}^3$, but $L \leadsto P (e_1 + e_2) = \mathbb{R} e_3 \not\subseteq \mathbb{R} e_1 = P$.  

(3) Next we give an example where $L \in \mathcal{P}_\perp(\Lambda)$ and $P \in \mathcal{P}_\perp(\Pi)$ but $P \ast_\perp L$ and $P \ast L$ are different.

In the same context than above we take $\text{push}(v, w) = (v, w_0 - w_0 w) \vee w$ where $w_0 \notin \mathbb{R}e_1 + \mathbb{R}e_3$, also consider $L = \mathbb{R}e_2$ and $P = \mathbb{R}e_1$. Then, $P \ast_\perp L = \mathbb{R}e_2 + \mathbb{R}e_3$, $P \ast L = (\mathbb{R}e_1 + \mathbb{R}e_3 + w_0) \cup (\mathbb{R}e_2 + \mathbb{R}e_3)$.

5. Abstract Krivine structures: Operations and combiners

XX. Recall the definition of an Abstract Krivine Structure a.k.a. \textsc{AKS} see\[^5\] where parts of the work of J.L. Krivine and T. Streicher in \[^{11}\] and \[^{13}\] are reformulated.

**Definition 5.1.** Let $\mathcal{R} = (\Lambda, \Pi, \sqcup, \text{push})$ be an $\mathcal{R}_L$, and assume that we have the following additional elements:

(1) A function $\text{app} : \Lambda \times \Lambda \to \Lambda$ (denoted also as $\ast$ when viewed as a map of type $\mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$). We also write $\text{app}(t, s) = ts$ for $t, s \in \Lambda$. We use the convention that associates to the left when operating with more than three elements,

(2) A set $\mathcal{R} \subseteq \Lambda$ of “quasi-Proofs”, which is closed under application,

(3) A fixed pair of elements $\kappa, s \in \mathcal{R} \subseteq \Lambda$, satisfying the following:
   
   (a) For all $t, s \in \Lambda$ if $t \perp s \cdot \pi$, then $ts \perp \pi$;
   
   (b) For all $t \perp \pi$ and for all $s \in \Lambda$ we have that $\kappa \perp t \cdot s \cdot \pi$;
   
   (c) If $tu(su) \perp \pi$ for $t, s, u \in \Lambda$, then $s \perp t \cdot s \cdot u \cdot \pi$.

The structure thus obtained is called an Abstract Krivine Structure a.k.a. \textsc{AKS} usually $\mathcal{K}$ will denote a generic $\mathcal{AKS}[^1]$.

XXI. We introduce a new operation $\mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$ that allows to express in terms of this operation the basic reduction rule of an \textsc{AKS}: “If $t \perp s \cdot \pi$, then $ts \perp \pi$” as well as the axioms for $\kappa$ and $s$.

**Definition 5.2.** We define the operation $\Box : \mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$ as: $P \Box L := (\perp \mathcal{P}L)^\perp$.

Observe that $P \Box L$ is in $\mathcal{P}_\perp(\Pi)$ because it is the perpendicular of a subset of $\Lambda$.

This operation is really the “transfer” of the operation app from $\mathcal{P}(\Lambda)$ to $\mathcal{P}(\Pi)$ through the perpendicularity map, as the diagram below illustrates:

\[
\begin{array}{ccc}
\mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) & \xrightarrow{\Box} & \mathcal{P}(\Pi) \\
\downarrow^{\perp(-) \circ \text{id}} & & \downarrow^{(-) \perp} \\
\mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda) & \xrightarrow{\text{app}} & \mathcal{P}(\Lambda)
\end{array}
\]

**Lemma 5.3.** The following statements are equivalent:

(1) For all $t, s \in \Lambda, \Pi \in \Pi$: if $t \perp s \cdot \pi$ then $ts \perp \pi$.

(2) $P \ast_\perp L \subseteq P \Box L$ for all $L \in \mathcal{P}(\Lambda), P \in \mathcal{P}(\Pi)$.

(3) $P \ast_\perp L \subseteq P \Box L$ for all $L \in \mathcal{P}(\Lambda), P \in \mathcal{P}(\Pi)$.

(4) For $P \subseteq \Pi$ and $L \subseteq \Lambda$: if $t \perp P$ and $\ell \in L$ then $t\ell \perp P \ast_\perp L \subseteq P \ast L$.

**Proof.**

- To prove that (1) implies (2) take $\pi \in P \ast_\perp L, t \perp P$ and $\ell \in L$. Since $\pi \in P \ast L$ then $\ell \cdot \pi \in P$, then $t \perp \ell \cdot \pi$. Applying (1) we get $t\ell \perp \pi$ and hence $\pi \in P \Box L$. To prove that (2) implies (1), just take $L := \{s\}$ and $P := \{t\}$.

- Statements (2) and (3) are equivalent because $P \Box L \in \mathcal{P}_\perp(\Pi)$ and $(-) \perp$ is a reflection.

- Condition (4) can be formulated as the inclusion: $\perp \mathcal{P}L \subseteq (\perp \mathcal{P}L)^\perp$ that is clearly equivalent to condition (3).

---

\[^1\] Strictu sensu, the standard definition of \textsc{AKS} includes additional structure—see \[^4\] or \[^3\]. We only take the basic elements we need for our present considerations.
Observation 5.4. In particular, in an $\mathcal{AKS}$ we have the following chain of inequalities (see Observation 4.17): For all $P \subseteq \Pi$ and $L \subseteq \Lambda$,
\[ P \star_* L \subseteq P \star L \subseteq P \star_\perp L \subseteq P \square L. \quad (5.4.4) \]

Notice that the operation $\square$, can be defined only when we furnish the original $\mathcal{RL}$ with the necessary additional structure necessary to produce an $\mathcal{AKS}$—i.e. with: the application map; the set of quasi proofs with its distinguished elements, and the axioms relating the application with the push through perpendicularity, that also characterize the distinguished elements—. This $\square$—operation is an upper bound for the operations $\star, \star_*$ and $\star_\perp$, which are more elementary and defined in the realm of the underlying $\mathcal{RL}$.

XXII. Next we recall some constructions and results based upon the properties of the basic combinators $\kappa, s$. Most of this appears in [5 Section 2, §2.3, Lemma 12.2]. The introduction of the new operations allows some additional precision in the inequalities obtained.

1. Define for an $\mathcal{AKS}$ called $\mathcal{K}$ the following elements of $QP \subseteq \Lambda$:
   (a) $\bot := \mathcal{S}(\mathcal{K})$,
   (b) $\in := \mathcal{S}(\mathcal{K}(\mathcal{K})) = \mathcal{S}(\mathcal{K}(1))$.
   (c) $\bowtie := \mathcal{S}(\mathcal{K}(\mathcal{S}))$.

2. The elements given above satisfy the following conditions for all $t, s, \ell \in \Lambda, \pi \in \Pi$:
   (a) $t \perp \pi$ implies that: $1 \perp t \cdot \pi$.
   (b) $t' \perp \pi$ implies that: $\in \perp t \cdot \ell \cdot \pi$.
   (c) $t \perp (s\ell) \cdot \pi$ implies that: $\bowtie \perp t \cdot s \cdot \ell \cdot \pi$.

3. The following results hold:
   (a) For all $P \in \mathcal{P}(\Pi), L \in \mathcal{P}(\Lambda)$, then $\kappa \perp P \leadsto (L \leadsto P)$.
   (b) For all $P, Q, R \in \mathcal{P}(\Pi)$, then $\subseteq \perp P \leadsto \left(1 \perp Q \leadsto (1 \perp R \leadsto ((P \star R) \star (Q \star R))) \right)$.
   (c) For all $P \in \mathcal{P}(\Pi), L \in \mathcal{P}(\Lambda)$; $\in \perp P \leadsto (L \leadsto (P \square L))$.

As mentioned above the validity of (3): (a), (b) and (c) is proved in [5 Section 2, §2.3, Lemma 12.2].

Lemma 5.5. For all $P \in \mathcal{P}(\Pi), L \in \mathcal{P}(\Lambda)$:

1. $\in \perp P \leadsto (L \leadsto (P \square L))$.
2. $P \square L \subseteq (\{\in\} \perp \star_* P) \star_* L \subseteq (\in \perp P) \perp \star_* L$.

Proof.

1. Take $t \perp P, \ell \in L$ and $\pi \in (\perp PL) \perp = P \square L$, hence $t\ell \perp \pi$ and it follows from Paragraph [XXII (2)] that $\in \perp t \cdot \ell \cdot \pi$ that implies that $\in \perp t \cdot \ell \cdot \pi$ or in other words that $\in \perp t \cdot \ell \cdot \pi$.

Hence $\in \perp (L \cdot (P \square L)) = \perp (L \leadsto (P \square L)) = \perp (L \leadsto (P \square L)) = \perp (L \leadsto (P \square L))$ where the second equality comes from the statement in Paragraph [XXII (4.1.1)] and the others are simply the definitions.

Stating this assertion as $\in \perp t \perp \rho$ for $t \perp P$ and $\rho \in L \leadsto (P \square L)$, and reasoning as before, we obtain that $\in \perp t \perp \rho$ and $\in \perp t \perp \rho$.

This means that $\in \perp (\perp (t \cdot (L \leadsto (P \square L)) \perp = (P \leadsto (L \leadsto (P \square L)))) \perp = (L \leadsto (P \square L)) \perp = (L \leadsto (P \square L)))$ where the last equality is obtained in the same manner than above.

2. If we write the perpendicularity relation just proved as: $[\in \perp \supseteq (P \leadsto (L \leadsto (P \square L)))) \perp = (P \leadsto (L \leadsto (P \square L)))) \perp = (L \leadsto (P \square L)))$, and then apply twice the full adjunction between $\star,*$ and $\leadsto$, we obtain $P \square L \subseteq ([\in \perp \star_* P) \star_* L$.

The second inclusion of (2) is a consequence of the inclusion of the first into the last term in (5.4.4) applied to $[\in \perp \star_* P$.
XXIII. The adjunctor. In this paragraph we show that with the help of the element $\varepsilon$ and the new operations $\cdot$ and $\sim$, we can obtain, similarly than in [5] control over the “other half” of the adjunction between the functors $\cdot \perp$ and $\sim \perp$ –compare with [5, Section 2, §2.3, Theorem 2.13].

**Definition 5.6.** Let $\mathcal{K}$ be an $\mathcal{AKS}$, define a map $P \mapsto \eta P : \mathcal{P}(\Pi) \to \mathcal{P}_*(\Pi)$ by the formula:

$$\eta P = \{\varepsilon\varepsilon\} \perp P \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp P \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp \cdot \perp P \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp \cdot \perp \cdot \perp P.$$  

The element $\varepsilon$ that is a particular $\eta$–expansor, will be called an adjunctor.

**Observation 5.7.**

1. For future reference we write down the basic inequality involving the definition of $\eta P$ –and that follows from equation (5.4.4).

$$\eta P \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp P \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp \cdot \perp \cdot \perp P \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp \cdot \perp \cdot \perp \cdot \perp P.$$  

2. Taking into account that some of the sets in the above inequality are elements of $\mathcal{P}_\perp(\Pi)$, taking closures we obtain that:

$$\overline{\eta P} \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp P \subseteq \{\varepsilon\varepsilon\} \perp \cdot \perp \cdot \perp \cdot \perp P = \overline{(\{\varepsilon\varepsilon\} \perp P)} \perp \subseteq (\{\varepsilon\varepsilon\} \perp P) \perp.$$  

The theorem that follows summarizes the inclusion relations obtained before.

**Theorem 5.8.** Assume that $\mathcal{K}$ is an $\mathcal{AKS}$ and the notations are as above. Then the following inclusions hold for $P \in \mathcal{P}(\Pi)$ and $L \in \mathcal{P}(\Lambda)$:

$$P \cdot \perp L \subseteq P \cdot L \subseteq P \cdot \perp \cdot \perp L \subseteq P \cdot \cdot \perp \perp L \subseteq \cdot \perp \cdot \perp \cdot \perp L \subseteq \cdot \perp \cdot \perp \cdot \perp \cdot \perp L.$$  

**Proof.** The first two as well as the last two inclusions follow from the general properties of the closure operators, the third is the content of Lemma [5, 5.3] that is supported mainly on the basic reduction rule of an $\mathcal{AKS}$ (Definition [5, 5.1] (3)). The fourth follows from Lemma [5, 5.5] (2) that is based on the properties of the combinator $\varepsilon$. See also [5, Section 2, §2.3]. □

XXIV. Next we show that in the case that we work on $\mathcal{P}_\perp$, we recuperate partially the adjunction property –compare with Paragraph XVII. The name adjunctor for the combinator $\varepsilon$ is justified by its role in the inclusions below. See [5, Section 2, §2.3, Theorem 2.13] for a proof that does not use the new products introduced in this work, but that is more precise in the sense that uses the combinator $\varepsilon$ instead of $\varepsilon \varepsilon$ as it is used below.

**Theorem 5.9.** Assume that $L \in \mathcal{P}_\perp(\Lambda)$, $P, R \in \mathcal{P}_\perp(\Pi)$. Then

1. If $L \sim \perp R \subseteq P$ then $R \subseteq P \cdot \perp L$.
2. If $R \subseteq P \cdot \perp L$ then $P \sim \perp R \subseteq \eta P = \{\varepsilon\varepsilon\} \perp \cdot \perp P \subseteq \{\varepsilon\varepsilon\} \cdot \perp \cdot \perp \cdot \perp P.$

**Proof.** We concentrate in the second assertion: it follows from Theorem 5.8 that $R \subseteq P \cdot \perp L \subseteq P \cdot \perp \cdot \perp L \subseteq \eta P \cdot \perp \cdot \perp L$. From the adjunction property for $\cdot$ and $\sim$, see Paragraph XVII, we deduce that $L \sim \cdot R \subseteq \eta \cdot P$. Taking double perpendicularity we deduce that $L \sim \cdot R \subseteq \eta \cdot P$ and the rest of the inequalities follows from Observation 5.7. □
XXV. The definitions of the basic operations given before: the push–type operations in $\mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ and the conductor–type operations in $\mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$, can be transferred to $\mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ by taking perpendiculars in the first variable or in the second—in other words composing with the map $(-)^\perp : \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$ in the first or in the second variable. Next we look at the basic properties of this redefined operations.

**Definition 5.10.** Given the push–type functions $\leadsto, \leadsto_\ast, \leadsto_\perp : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ we define the new operations: $\to, \to_\ast, \to_\perp : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ as shown in the table below.

$$
\begin{align*}
P \to Q &:= \perp P \leadsto Q \\
P \to_\ast Q &:= \perp P \leadsto_\ast Q \\
P \to_\perp Q &:= \perp P \leadsto_\perp Q
\end{align*}
$$

Given the conductor–type operations $*, *_\ast, *_\perp$ and the $\perp$–operation, all of type $\mathcal{P}(\Pi) \times \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$; we define the new operations: $\circ, \circ_\ast, \circ_\perp : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ as shown in the table below.

$$
\begin{align*}
P \circ Q &:= P \ast \perp Q \\
P \circ_\ast Q &:= P \ast_\ast \perp Q \\
P \circ_\perp Q &:= P \ast_\perp \perp Q \\
P \circ \perp Q &:= P \perp \perp \perp Q
\end{align*}
$$

Thus $\eta P = \{EE\}^\perp \circ_\ast P$. We reformulate the properties of the new operations:

1. **Monotony.** The operations $\circ, \circ_\ast, \circ_\perp, \circ$ are monotone in both variables and $\to, \to_\ast, \to_\perp$ are antitone in the first variable and monotone in the second.

2. **Adjunctions.**
   - $Q \to R \subseteq P$ if and only if $R \subseteq P \circ Q$;
   - $Q \to_\ast R \subseteq P$ if and only if $R \subseteq P \circ_\ast Q$.

3. **Two half adjunctions.**
   - If $Q \to_\perp R \subseteq P$ then $R \subseteq P \circ_\perp Q$;
   - If $R \subseteq P \circ_\perp Q$ then $Q \to_\perp R \subseteq \eta P = \{EE\}^\perp \circ_\ast P \subseteq \{EE\}^\perp \circ_\perp P$.

4. **Inclusion relations.**
   - $P \circ_\ast Q \subseteq P \circ Q \subseteq P \circ_\perp Q \subseteq P \circ Q \subseteq \eta P \circ_\ast Q \subseteq \eta P \circ Q \subseteq \eta P \circ_\perp Q$;
   - $P \to Q \subseteq P \to_\ast Q \subseteq P \to_\perp Q$.

5. **Properties of the combinators**
   - $K \perp P \to (Q \to P)$ (5.10.5)
   - $s \perp P \to (Q \to (R \to ((P \circ R) \circ (Q \circ R)))$ (5.10.6)
   - $E \perp P \to (Q \to (P \circ Q))$ (5.10.7)

**Definition 5.11.** Assume that $\mathcal{K}$ is an $\mathcal{A\{\hfill\}}$KS

1. The operations $\circ, \to : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ are called the **application** and the **implication** in $\mathcal{P}(\Pi)$.

2. The operations $\circ_\ast, \to_\ast : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ when restricted to $\mathcal{P}_*(\Pi) \times \mathcal{P}_*(\Pi)$ yield maps from $\mathcal{P}_*(\Pi) \times \mathcal{P}_*(\Pi) \to \mathcal{P}_*(\Pi)$ and are called the **application** and the **implication** in $\mathcal{P}_*(\Pi)$. 

(3) The operations $\odot_{\bot}, \rightarrow_{\bot}: \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ when restricted to $\mathcal{P}_{\bot}(\Pi) \times \mathcal{P}_{\bot}(\Pi)$ yield maps from $\mathcal{P}_{\bot}(\Pi) \times \mathcal{P}_{\bot}(\Pi) \to \mathcal{P}_{\bot}(\Pi)$ and are called the application and the implication in $\mathcal{P}_{\bot}(\Pi)$.

XXVI. In accordance with Definition 3.4 the implication of a complete $OCA$, induces an application like operation that we called $\sharp$. Given $K \in AKS$, as defined in [5], there is a $OCA A_K$ which is associated to $K$. The theorem that follows shows that there is a close connection between the $\sharp$ defined in $A_K$ and the operation $\odot$ of $K$. The corresponding considerations for the case of the other operations are not too interesting because in those cases, as there is a full adjunction, the results of Theorem 3.5 guarantee that the $\chi$–product and the $\odot$–product, coincide.

**Theorem 5.12.** For a $OCA$ of the form $A_K$ where $K$ is an $AKS$ and for all $a, b \in \mathcal{P}_{\bot}(\Pi)$ then $a \odot b = a \#$ $b$. See Paragraph XVII for the definition of $a \odot b$ and Definition 3.4 for the meaning of $a \# b$.

**Proof.** Consider:

$$a \# b = \inf\{c \in \mathcal{P}_{\bot}(\Pi) : a \leq b \rightarrow_{\bot} c\} = \sup\{c \in \mathcal{P}_{\bot}(\Pi) : b \rightarrow_{\bot} c \subseteq a\} =$$

$$\bigcup\{c \in \mathcal{P}_{\bot}(\Pi) : b \rightarrow_{\bot} c \subseteq a\} = \bigcup\{c \in \mathcal{P}_{\bot}(\Pi) : b \rightarrow c \subseteq a\} =$$

$$\bigcup\{c \in \mathcal{P}_{\bot}(\Pi) : c \subseteq a \odot b\},$$

where the fourth equality is justified by the considerations in paragraph XIII item (3), the fifth is consequence of 4.17, item (b), the sixth is by XVIII item (1) and the last one is consequence of 4.17, item (1).

Now, if $\pi \in a \odot b$ then $\pi \subseteq a \odot b$ and it follows by the above equality that $\pi \subseteq a \# b$, and hence $a \odot b \subseteq a \# b$ and also $a \odot b \subseteq a \# b$.

Conversely, it is clear that $\bigcup\{c \in \mathcal{P}_{\bot}(\Pi) : c \subseteq a \odot b\} \subseteq a \odot b$, and taking closures we obtain that $a \# b \subseteq a \# b$. □

**Observation 5.13.**

(1) Notice that in accordance with Paragraph 4.13 the product $a \odot b$ need not be closed with respect to the bar operator. Hence in general the operations $\odot$ and $\#$ are different. The above proof guarantees that in general for all $a, b \in \mathcal{P}_{\bot}(\Pi)$, we have that $a \odot b \subseteq a \# b$.

(2) In general for all $a, b \in \mathcal{P}_{\bot}(\Pi)$, we have that:

$$a \odot b \subseteq a \# b = a \odot b \subseteq a \odot b.$$  

6. **From AKSs and IOCA as Heyting preorders**

XXVII. In [5] we interpreted the main results appearing in [13], as a triangle of constructions between the main objects under consideration:
The left diagonal arrow in the diagram is basically Streicher’s construction in [13], i.e. a map between abstract Krivine structures and preorders, and the triangle can be interpreted as a factorization of this map.

The horizontal arrow (compare with [5]) represents two possible constructions of an \( {\cal OCA} \), one using the operations \((\circ, \to)\) and the other \((\circ_\perp, \to_\perp)\). The first are defined in \( \cal P(\Pi) \) and the others in \( \cal P_\perp(\Pi) \).

In this section we intend to perform a similar construction of an horizontal map, using the operations \((\circ_\ast, \to_\ast)\) on \( \cal P_\ast(\Pi) \), and thereby obtaining a diagram similar to the one above (with small changes due to the new operations):

\[
\begin{array}{c}
\mathcal{AKS} \\
\downarrow \\
\mathcal{HPO}
\end{array}
\quad\quad
\begin{array}{c}
{\cal OCA} \\
\downarrow \\
{\cal I OCA}
\end{array}
\]

XXVIII. In this paragraph we start by recalling briefly the construction of the arrows of the first diagram, i.e. the construction of a \( {\cal I OCA} \) from a \( \mathcal{AKS} \) and of a Heyting preorder from a \( {\cal I OCA} \), see [5] Section 5, Definition 5.10, Theorem 5.11; Section 4, Definition 4.13, Theorem 4.15] and the construction of a Heyting preorder from an \( \mathcal{AKS} \).

1. From \( \mathcal{AKS} \) to \( {\cal I OCA} \). For \( \mathcal{K} \) an \( \mathcal{AKS} \), the following elements produce a \( {\cal I OCA} \) denoted as

\( \mathcal{A}_{\perp}: A_\perp = \mathcal{P}_\perp(\Pi), a \leq b \in A_\perp \text{ iff } a \supseteq b; \text{ and } a \circ_\perp b, a \to_\perp b \in A_\perp \) are as in Paragraph XXV Definition [5,11] \( k = \{ k \}^\downarrow, s = \{ s \}^\downarrow, e = \{ e \}^\downarrow \). Moreover, \( \Phi = \{ P \in \mathcal{P}_\perp(\Pi) : \exists t \in \mathcal{Q} P t \sqsubseteq P \} \).

2. From \( {\cal I OCA} \) to \( \mathcal{HPO} \). If \( \mathcal{A} \) is a \( {\cal I OCA} \) with filter \( \Phi \) and maximum element \( \top \in \Phi \), define:

(a) The relation \( \sqsubseteq \in A: a \sqsubseteq b, \text{ if and only if } \exists f \in \Phi \text{ such that } fa \leq b, \)

(b) A map \( \wedge: A \times A \to A \) as \( a \wedge b := p \cdot ab \).

If \( \mathcal{A} \) is a \( {\cal I OCA} \) then defining \( H_\mathcal{A} := (A, \sqsubseteq, \wedge, \to) \), it is clear that it is a Heyting preorder where \( \sqsubseteq, \wedge \) are as above and \( \to \) is the original implication of \( \mathcal{A} \). For the definition of \( p \) see [5] Section 3, Definition 3.5] where it is defined as the combinator \( p := \lambda x.\lambda y.\lambda z.(zxy) \) using the combinatory completeness to guarantee that it belongs to \( \Phi \). An element \( f \) as above is said to be “a realizer of the relation \( a \sqsubseteq b \)”, and write this assertion as \( f \not\vdash a \sqsubseteq b \). See [5] Section 4, Lemma 4.14] for the proof that it is a meet semilattice and [5 Section 4, Theorem 4.15] for the rest of the proof.

3. From \( \mathcal{AKS} \) to \( \mathcal{HPO} \). Let \( \mathcal{K} = (A, \Pi, \sqsubseteq_\perp, \text{push, app, } k, s, \mathcal{Q} P) \) be an abstract Krivine structure. We define the relation \( \sqsubseteq_\perp \in \mathcal{P}(\Pi) \) as follows: \( P, Q \in \mathcal{P}(\Pi), P \sqsubseteq_\perp Q \Leftrightarrow \exists t \in \mathcal{Q} P t \perp P \to_\perp Q \). An element \( t \in \mathcal{Q} P \) as above is said to be “a realizer of the relation” \( P \sqsubseteq_\perp Q \). It follows that for \( P, Q \in \mathcal{P}(\Pi) \) we have that \( t \in A \) is a realizer of the relation \( P \sqsubseteq Q \) if and only if it is a realizer of \( P \sqsubseteq_\perp Q \). Prima facie we have only considered a preorder \( \sqsubseteq_\perp \in \mathcal{P}_\perp(\Pi) \) and thus produced an object in \( \text{Ord} \), called \( \mathcal{H}_{\mathcal{K}_\perp} \). In [5 Theorem 5.11] it is proved that \( \mathcal{H}_{\mathcal{K}_\perp} \) and \( \mathcal{H}_{\mathcal{A}_{\mathcal{K}_\perp}} \) are isomorphic, and that implies that the first one is in fact an element in \( \mathcal{HPO} \).

XXIX. Next we want to construct a triangle of the second kind mentioned above. Regarding the horizontal arrow, for \( \mathcal{K} \in \mathcal{AKS} \), we construct an ordered combinatory algebra that we call \( \mathcal{A}_{\mathcal{K}_\perp} \in {\cal I OCA} \). It has as basic set \( \mathcal{P}_\perp(\Pi) \), operations \( \circ_\ast, \to_\ast \) and combinators \( k_\ast, s_\ast \) where \( k_\ast = \{ k \}^\downarrow, s_\ast = \{ e ((b \in s))^\downarrow \) (see Section [5]). This \( {\cal I OCA} \) has a filter that is the natural extension of the filter of \( \mathcal{A}_{\mathcal{K}_\perp} \).

Compared with the above construction of \( \mathcal{A}_{\mathcal{K}_\perp} \), if we choose to work with \( \mathcal{A}_{\mathcal{K}_\perp} \) we will enjoy the following benefits:
(1) The construction of the associated tripos is more direct than in the case of \( \mathcal{A}_{\mathbb{K}_{\perp}} \). This is due to the fact that in this context we do not have to recur to an \( \eta \)-expansor or adjurator \( \mathbf{e} \) –compare with the construction appearing in [13] or [4].

(2) The use \( \mathcal{P}_*(\Pi) \), that is a priori “larger” than \( \mathcal{P}_+(\Pi) \) as the set of falsity values, might have some conceptual advantages. Besides producing more programs expressible in the semantics, it is closer to Krivine’s original definition that took \( \mathcal{P}(\Pi) \) as the set of falsity values.

Then we revisit the construction of the left diagonal arrow in our context, adapting the case of \( \mathcal{P}_+ \) to the context of the new closure operator i.e. the case of \( \mathcal{P}_* \).

Finally, with respect to the remaining arrow in the triangle, the right diagonal arrow, there is not much to be done as the construction performed for a \( \mathcal{F} \mathcal{O} \mathcal{C} \mathcal{A} \) rather than a \( \mathcal{A} \mathcal{C} \mathcal{A} \) is identical.

XXX. Next we present the precise statements and the proofs of the above assertions.

**Theorem 6.1.** Let \( \mathcal{K} = (\Lambda, \Pi, \sqcup, \text{push}, k, s, QP) \in \mathcal{A}K\mathcal{S} \), \( k_* = \{e_\mathcal{E} \}_\mathcal{E} \), \( s_* = \{e((\mathcal{E}, s)) \}_\mathcal{E} \in \mathcal{P}_* (\Pi) \) and \( \Phi = \{P \in \mathcal{P}_* (\Pi) \} t \in \mathcal{Q}P \), \( t \perp P \} \subseteq \mathcal{P}_* (\Pi) \). Then \( \mathcal{A}_{\mathcal{K}*} = (\mathcal{P}_* (\Pi), \circ_*, \rightarrow_*, \supseteq, k_* , s_* , \Phi) \) is a \( \mathcal{F} \mathcal{O} \mathcal{C} \mathcal{A} \).

**Proof.** Clearly \( k_* , s_* \in \mathcal{P}_* (\Pi) \subseteq \mathcal{P}_+ (\Pi) \) because both are perpendiculars of subsets of \( \Lambda \). The operation \( \circ_* \) is monotonic in both arguments and \( \rightarrow_* \) is antimonotonic on the first argument and monotonic in the second –see Paragraph [XXV]. To prove that \( k_* \) and \( s_* \) satisfy the required properties we proceed as follows.

1. The inequality \( k_* \circ_* P \circ_* Q \supseteq P \) holds for generic subsets \( P, Q \in \mathcal{P}_+ (\Pi) \). Indeed, if we take \( p \in \perp P, p \in P \) implies: see Definition [5.1], (b), that \( k \perp q \cdot p \cdot \pi \) for all \( q \in \Lambda \) in particular for all \( q \perp Q \) and consequently we have that in the same sets: \( k p \perp q \cdot \pi \). In other words we have that \( k p \in (\perp Q \cdot P) = (Q \rightarrow P) = ((Q \rightarrow P)) = (Q \rightarrow P) \). The equalities being guaranteed by the considerations in Paragraph [XII.11],(2) and by Definition [5.10] Hence, it follows from Paragraph [XXVII],(2) that for all \( p \perp P \), \( p \in Q \rightarrow \mathcal{P} \), \( p \leq k \perp q \cdot p \cdot \rho \) and also that \( E \perp p \cdot \rho \). In the same manner than before, this implies that \( E \perp k \in (\perp P \rightarrow_\mathcal{F} (Q \rightarrow_\mathcal{F} P) \). This inequality is equivalent with \( k_* \circ_\mathcal{F} P \circ_\mathcal{F} (Q \rightarrow_\mathcal{F} P) \) and the adjunction property for \( (\rightarrow_\mathcal{F} , \circ_\mathcal{F}) \) implies that \( k_* \circ_\mathcal{F} P \circ_\mathcal{F} Q \supseteq P \).

2. Now we have to prove that \( s_* \circ_\mathcal{F} P \circ_\mathcal{F} Q \circ_\mathcal{F} R \supseteq (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R) \) for generic \( P, Q, R \in \mathcal{P}_+ (\Pi) \). It follows from the adjunction property that the above is equivalent with \( s_* \supseteq \mathcal{P}_+ (\Pi) \supseteq \mathcal{P}_+ (\Pi) \) appearing in Paragraph [XXVII],(2) we deduce first that \( \langle e \rangle \perp q \cdot \rho \cdot p \cdot \pi \) and then that \( E \perp e \perp (\mathcal{E}, s) \cdot p \cdot q \cdot \rho \cdot p \cdot \pi \) and thus then in the same manner than before that \( s_p \perp R \rightarrow_\mathcal{F} (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R) \). It follows then that \( s_p \perp q \cdot p \cdot \pi \) as above and for all \( p \in (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R) \). Using the properties for the combinators \( \mathcal{E} \), \( \mathcal{B} \) appearing in Paragraph [XXVII],(2) we deduce first that \( \perp (s_p) \cdot q \cdot p \cdot \pi \) as above and then that \( E \perp e \perp (\mathcal{E}, s) \cdot p \cdot q \cdot \rho \cdot p \cdot \pi \). Then, \( (\mathcal{E}, s) \perp q \cdot p \cdot \pi \) and as before write down this property as: \( (\mathcal{E}, s) \perp q \cdot p \perp R \rightarrow_\mathcal{F} (R \rightarrow_\mathcal{F} (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R)) \). If call \( \gamma \) a generic element in \( Q \rightarrow_\mathcal{F} (R \rightarrow_\mathcal{F} (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R)) \), from \( (\mathcal{E}, s) \perp q \cdot p \cdot \pi \) we deduce first that \( E \perp \perp (\mathcal{E}, s) \cdot p \cdot q \cdot \rho \cdot p \cdot \pi \) as above and then that \( (\mathcal{E}, s) \perp q \cdot p \cdot q \cdot \rho \cdot p \cdot \pi \). As before, a perpendicularity relation like the above means that \( e ((\mathcal{E}, s)) \perp P \rightarrow_\mathcal{F} (Q \rightarrow_\mathcal{F} (R \rightarrow_\mathcal{F} (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R))) \) and then that \( e ((\mathcal{E}, s)) \perp (P \rightarrow_\mathcal{F} (Q \rightarrow_\mathcal{F} (R \rightarrow_\mathcal{F} (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R)))) \) that \( e ((\mathcal{E}, s)) \perp (P \rightarrow_\mathcal{F} (Q \rightarrow_\mathcal{F} (R \rightarrow_\mathcal{F} (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R)))) \). Then, \( e ((\mathcal{E}, s)) \perp (P \rightarrow_\mathcal{F} (Q \rightarrow_\mathcal{F} (R \rightarrow_\mathcal{F} (P \circ_\mathcal{F} R) \circ_\mathcal{F} (Q \circ_\mathcal{F} R)))) \).

(3) To prove that \( \Phi \subseteq A \) is a filter that contains \( k_* , s_* \) we proceed as follows. The subset \( \Phi \) is closed under application. Indeed, if \( f, g \in \Phi \), we have that \( t f \in \perp f \cap \mathcal{Q}P \) and \( t g \in \perp g \cap \mathcal{Q}P \), then \( t f t g \in \perp f \cdot g \cap \mathcal{Q}P \). It follows directly from Lemma [5.3],(4) that \( f \cdot g \perp f \cdot g \cdot f \cdot g \), and hence \( \Phi \) is closed under application. Moreover, \( k_* \) is the set of elements perpendicular to a fixed element.
of \(QP\), and the same for \(s_\ast\). It is clear – by the definition of \(\Phi\) that this set of this kind are elements of \(\Phi\). 

\[\Box\]

XXXI. In this paragraph, given \(K\) an \(\mathcal{AKS}\) and taking into account the realizability relation, we define a preorder \(\mathcal{H}_K\), based on the set \(\mathcal{P}_\ast(\Pi)\), that is similar to the one defined by Streicher in \([13]\) and summarized in Paragraph [XXVIII][3] and used in \([5]\) to define realizability in terms of \(\mathcal{OC}\mathcal{A}\)s.

**Definition 6.2.** Let \(K = (\Lambda, \Pi, \perp, \text{push, app, } \kappa, s, \mathcal{P}Q)\) be an abstract Krivine structure. We define the relation \(\sqsubseteq\) in \(\mathcal{P}(\Pi)\) as follows:

\[P, Q \in \mathcal{P}(\Pi), \quad P \sqsubseteq Q \iff \exists t \in \mathcal{P}Q \exists tP \rightarrow P \ast Q = (\perp P).Q,\]

for \(P, Q \in \mathcal{P}_\ast(\Pi)\). An element \(t \in \mathcal{Q}P\) as above is said to be “a realizer of the relation” \(P \sqsubseteq Q\).

**Observation 6.3.** From Definition [4.1] and Theorem [4.11] it follows that for \(P, Q \in \mathcal{P}(\Pi)\) we have that \(\perp (P \rightarrow Q) = (\perp P \rightarrow \ast Q) = (\perp P \rightarrow \perp Q)\). Hence, \(t \in \Lambda\) is a realizer of the relation \(P \sqsubseteq Q\) if and only if \(t \in \mathcal{P}Q\) if and only if \(t\) is a realizer of the relation \(P \sqsubseteq Q\).

**Lemma 6.4.** Let \(K\) be an abstract Krivine structure, then the relation \(\sqsubseteq\) is a preorder on \(\mathcal{P}(\Pi)\) and the relation \(\sqsubseteq\) is a preorder on \(\mathcal{P}_\ast(\Pi)\)

**Proof.** The first assertion appears proved in \([5\), Section 4, Lemma 4.12] and the second follows from the above Observation 6.3. 

**Lemma 6.5.** The canonical inclusion \((\mathcal{P}_\ast(\Pi), \sqsubseteq) \hookrightarrow (\mathcal{P}(\Pi), \sqsubseteq)\) is an equivalence of preorders.

**Proof.** By the comments at the beginning of Paragraph [IV] it suffices to show that the inclusion is order reflecting and essentially surjective. Since the order on \(\mathcal{P}_\ast(\Pi)\) is defined as restriction of the order on \(\mathcal{P}(\Pi)\), the first assertion is clear.

As \((\perp P)^\perp \in \mathcal{P}_\ast(\Pi)\) for all \(P \in \mathcal{P}_\ast(\Pi)\), once we prove that \(P \sqsubseteq (\perp P)^\perp\) and \((\perp P)^\perp \sqsubseteq P\), the fact that the inclusion is essentially surjective follows directly.

To prove the above inclusions, first recall from \([5\), Section 2, Lemma 2.12, (4)] that for all \(Q \subseteq \Pi\) \(\perp Q \cdot Q\), and then apply this perpendicularity relation to the situation where \(Q := (\perp P)^\perp\) and \(Q := P\) respectively.

**Definition 6.6.** For \(K\) an \(\mathcal{AKS}\) we define the preorder \(\mathcal{H}_K := (\mathcal{P}_\ast(\Pi), \sqsubseteq) \in \mathcal{Ord}\).

**Observation 6.7.** We may state the above result, in terms of Streicher’s construction of the map \(\mathcal{AKS} \rightarrow \mathcal{OC}A \rightarrow \mathcal{Ord}\), as guaranteeing that the three constructions: (1) \(\mathcal{K} \rightarrow \mathcal{A}_K\), (2) \(\mathcal{K} \rightarrow \mathcal{A}_K\), (3) \(\mathcal{K} \rightarrow \mathcal{A}_K\); composed with the construction from \(\mathcal{OC}A \rightarrow \mathcal{HPO}\) produce equivalent preorders.

XXXII. In this paragraph, we show that if \(K\) is in \(\mathcal{AKS}\) the preorders \(\mathcal{H}_K\) and \(\mathcal{H}_\kappa\) are isomorphic. (see Paragraphs [XXII][XXIII] and the definition below).

**Definition 6.8.** If \(\mathcal{A}\) is a \(\mathcal{OC}A\) with with filter \(\Phi\) and maximum element \(\top \in \Phi\), define the relation \(\sqsubseteq\) in \(A\): \(a \sqsubseteq b\), if and only if \(\exists f \in \Phi\) such that \(fa \leq b\) and a map \(\land : A \times A \rightarrow A\) as \(a \land b := pab\). We call \(\mathcal{H}_\mathcal{A} = (A, \sqsubseteq, \land, \rightarrow)\) the Heyting preorder associated to \(\mathcal{A}\).

It is clear (compare with the definitions in Paragraph [XXVIII]), that if \(\mathcal{A}\) is a \(\mathcal{OC}A\) one has that \(\mathcal{H}_\mathcal{A}\) is a Heyting preorder.

**Theorem 6.9.** Let \(K\) be an \(\mathcal{AKS}\), then \(\mathcal{H}_K\) and \(\mathcal{H}_\kappa\) are isomorphic preorders. In particular, we conclude that \(\mathcal{H}_K\) is a Heyting preorder.
Proof. In both cases the basic sets of the preorders is $P_*\(\Pi\)$. We only have to check that the two definitions of the preorder coincide. The order relation in $\mathcal{H}_{\mathcal{K}_*}$ is given by: $\exists P \in \Phi, P \circ Q \supseteq R$, which using the full adjunction can be formulated equivalently as: $\exists P \in \Phi, P \supseteq Q \rightarrow \ast R$.

As to the equivalence we have that:
$$\exists P \in \Phi, P \supseteq Q \rightarrow \ast R \iff \exists t \in QP, t \perp (\perp Q R) \ast \iff \exists t \in QP, t \perp (\perp Q R) \iff \exists t \in QP, t \perp Q \rightarrow R,$$

and the last line is the definition of the preorder in $\mathcal{H}_{\mathcal{K}_*}$. □

XXXIII. Next we consider and adapt to the current context, the construction of an inverse of the horizontal map that appears in [5, Definition 5.12].

In this weaker context the map going in the opposite direction than $\mathcal{K} \rightarrow \mathcal{A}_{\mathcal{K}_*}$, that we denote as $\mathcal{A} \rightarrow \mathcal{K}_{\mathcal{A}_*}$, will not be a full inverse but only a Galois injection (see Observation 6.15):

$$\begin{array}{c}
\mathcal{A}_{\mathcal{K}_*} \\
\mathcal{K}_{\mathcal{A}_*} \\
\mathcal{A} \\
\mathcal{OCA} \\
\text{HPO}
\end{array}$$

Definition 6.10. Given a $\mathcal{OCA}$ called $\mathcal{A} = (A, \leq, \text{app}, \text{imp}, k, s, \Phi)$, we define the structure: $\mathcal{K}_{\mathcal{A}_*} = (\Lambda, \Pi, \perp, \text{app}, \text{push}, k, s, \Phi)$ as follows.

(1) $\Lambda = \Pi := A$;
(2) $\perp : \leq$, i.e. $s \perp \pi :\iff s \leq \pi$;
(3) $\text{app}(s, t) := st$, $\text{push}(s, \pi) := \text{imp}(s, \pi) = s \rightarrow \pi$;
(4) $k := k$, $s := s$;
(5) $QP := \Phi$.

Theorem 6.11. In the notations of Definition 6.10, $\mathcal{K}_{\mathcal{A}_*}$ is an $\mathcal{A}_{\mathcal{K}_*}$.

Proof. It is clear that QP is closed under application and contains $k, s$. Next we check the axioms concerning the orthogonality relation (see Definition 5.1). Substituting the above definitions, these axioms become:

(S1) $t \leq u \rightarrow \pi \Rightarrow tu \leq \pi$;
(S2) $t \leq \pi \Rightarrow k \leq t \rightarrow u \rightarrow \pi$;
(S3) $tv(uv) \leq \pi \Rightarrow s \leq t \rightarrow u \rightarrow v \rightarrow \pi$.

(S1) follows from Definition 5.1 (PA).

(S2) is shown by the following derivation based on the definition of $k$ (5.1 (PK)) and the full adjunction, i.e. [3.1] (PE)’:
$$t \leq \pi \Rightarrow kt \leq u \rightarrow \pi \Rightarrow k \leq t \rightarrow u \rightarrow \pi.$$

(S3) is proved using repeatedly the full adjunction property as before as well as [3.1] (PS):
$$tv(uv) \leq \pi \Rightarrow stuv \leq \pi \Rightarrow stu \leq v \rightarrow \pi \Rightarrow st \leq u \rightarrow v \rightarrow \pi \Rightarrow s \leq t \rightarrow u \rightarrow v \rightarrow \pi.$$

□
Observation 6.12. To avoid confusion, when we view \( A \) as the set \( \Pi \) of the corresponding AKS \( \mathcal{K}_{\mathcal{A}} \) we write it as \( A_\Pi \) and when we view it as \( \Lambda \) we write \( A_\Lambda \).

XXXIV. In what follows we compare both constructions, considering the relationship between \( \mathcal{A} \), the original \( \mathcal{T} \mathcal{OCA} \), and the iterated construction of \( \mathcal{A}_{\mathcal{K}_{\mathcal{A}}} \).

Lemma 6.13. Let \( \mathcal{A} \) be a \( \mathcal{T} \mathcal{OCA} \) and \( \mathcal{K}_{\mathcal{A}} \) the AKS of Definition 6.10

(1) For \( C \subseteq A_\Pi \) we have \( \uparrow C = \downarrow (\inf C) \), and for \( C \subseteq A_\Lambda \) we have \( C^\bot = \uparrow (\sup C) \).

(2) In particular:

\[
\downarrow (\uparrow a) = \downarrow a; \quad \uparrow (\downarrow a) = \uparrow a; \quad \uparrow (\downarrow (\uparrow a)) = \uparrow a; \quad \downarrow (\uparrow (\downarrow a)) = \downarrow a \quad \text{for} \quad a \in A_\Pi;
\]

\[
\downarrow (\uparrow a); \quad \uparrow (\downarrow a) = \top; \quad a^\bot = \uparrow a; \quad \uparrow (\downarrow a) = \downarrow a; \quad \downarrow (\uparrow a) = \uparrow a \quad \text{for} \quad a \in A_\Lambda.
\]

(3) If and only if \( C \subseteq A_\Pi \), then \( \widehat{D} = \uparrow (\inf D) \) and \( \widehat{D} = \bigcup \{ \uparrow c : c \in D \} \). In other words: \( P_\bot(A_\Pi) \) consists of the set of all principal filters of \( A \) and \( \mathcal{P}_\bot(A_\Pi) \) is the set of all subsets that are union of principal filters of \( A \). In particular \( \inf D = \inf(\widehat{D}) = \inf(\overline{D}) \).

(4) For \( C, D \in \mathcal{P}(A_\Pi) \) we have \( (\inf C \rightarrow \inf D) \leq \inf(\sup C \rightarrow \sup D) = \inf(C \rightarrow \uparrow \sup D) \).

Proof. The verification of the first three items is clear, the remaining can be proved as follows:

\[ C \rightarrow D = \downarrow C \cdot \uparrow D \subseteq (\downarrow C \cdot \uparrow D) = C \rightarrow \uparrow D \cdot \downarrow \sup C \leq \uparrow \sup C \rightarrow \sup D, \]

then \( \inf(C \rightarrow \uparrow \sup D) = \inf(C \rightarrow \sup D) = \inf(C \rightarrow D) \) – see part (3) of this Lemma. The inequality that remains to be proved, is a consequence of the antimomony of the arrow with respect to the first variable: \( \inf(\inf C \rightarrow d : d \in D) \leq \inf(c \rightarrow d : c < \inf C, \; d \in D) \leq \inf(C \rightarrow D) \). Clearly \( \inf(C \rightarrow d : d \in D) = \inf C \rightarrow \inf D \) by the preservation of limits by right adjoints and hence the conclusion follows.

\[ \Box \]

Theorem 6.14. Let \( \mathcal{A} \) be a \( \mathcal{T} \mathcal{OCA} \), \( \mathcal{K}_{\mathcal{A}} \) the AKS of Definition 6.10 and \( \mathcal{A}_{\mathcal{K}_{\mathcal{A}}} = (\mathcal{P}_\bot(A_\Pi), \circ, \rightarrow, \geq, k, s, \Phi) \) the associated \( \mathcal{T} \mathcal{OCA} \) – Theorem 6.1. Then, the functors:

\[
(A, \leq) \xrightarrow{\iota} \mathcal{P}_\bot(A_\Pi), \geq,
\]

\[
iota : A \rightarrow \mathcal{P}_\bot(A_\Pi), \; a \mapsto \uparrow a \quad \text{and} \quad \rho : \mathcal{P}_\bot(A_\Pi) \rightarrow A, \; C \mapsto \inf C, \; \text{form and adjoint pair (a Galois connection), i.e. if} \quad a \in A, \; C \subseteq A_\Pi, \; \widehat{C} = C \text{ then}:
\]

\[
a \leq \rho(C) \iff \iota(a) \leq C : \iff \iota(a) \supseteq C.
\]

and the unit of the adjunction \( \iota \dashv \rho \) is an isomorphism: \( \id_A \cong \rho \iota \), while the counit is the natural inclusion \( \iota \rho \leq \id_{\mathcal{P}_\bot(A_\Pi)} \).

Proof. The codomain of \( \iota \) lies in \( \mathcal{P}_\bot(A_\Pi) \), see Lemma 6.13 (2) where we proved that \( \uparrow a \in \mathcal{P}_\bot(A_\Pi) \), and it is clear that the maps \( \iota \) and \( \rho \) are monotonic. As \( \inf(\uparrow a) = a \) it follows that \( \rho \iota = \id_A \). Also for \( C \subseteq A_\Pi \), \( \widehat{C} = C \), \( \iota(\rho(C)) = \uparrow(\inf C) = \widehat{C} \supseteq C = \id_{\mathcal{P}_\bot(A_\Pi)}(C) \).

Once we have the unit and counit, the proof of the assertion (6.14.9) follows directly.

\[ \Box \]

Observation 6.15. (1) A pair of preorders as above, equipped with a Galois connection

\[
(A, \leq) \xrightarrow{\iota} (B, \leq), \quad \text{i.e.} \; \iota \dashv \rho,
\]

with the property that the unit is an isomorphism, is sometimes called a Galois injection. It is easy to prove that to give such a connection is equivalent to give an operator on \( B \) satisfying the following axioms:

\[
(a) \; b \leq b' \Rightarrow \zeta(b) \leq \zeta(b'),
\]
To prove the above assertion just take $\zeta := \iota \rho : B \to B$. Conversely, given $\zeta : B \to B$ as above, take $A = \zeta(B)$, $\rho = \zeta$ and $\iota = \text{inc}$.

(2) In our case the operator $\zeta : \mathcal{P}_*(A_{M}) \to \mathcal{P}_*(A_{M})$ is $\zeta(C) = \overline{C}$.

XXXV. From $\mathcal{KOCAs}$ to $\mathcal{AKS}$s.

Let us look again at the above triangle of constructions.

We want to show that the family of $\mathcal{HPOs}$ produced by $\mathcal{FOCAs}$, coincides with the family of $\mathcal{HPOs}$ produced by Streicher’s construction –named $S$ in the diagram. This is a direct consequence of the following theorem:

**Theorem 6.16.** In the notations above, the diagram

![Diagram](attachment:image.png)

is commutative, up to equivalence of preorders.

**Proof.** If $\mathcal{K}$ is an $\mathcal{AKS}$ the map $S$ is defined as $S(\mathcal{K}) = \mathcal{H}_\zeta$. Hence, we need to prove that the Heyting preorders $(A, \sqsubseteq)$ and $(\mathcal{P}_*(A_{M}), \sqsubseteq)$, are equivalent. Recall that the order relations are the following:

- $C, C' \in \mathcal{P}_*(A_{M})$, $C \sqsubseteq C' \iff \exists t \in \text{QP} : t \perp C \implies t \subseteq C'$,
- $a, a' \in A$, $a \sqsubseteq a' \iff \exists r \in \Phi : ra \leq a'$.

Consider the map $\rho : \mathcal{P}_*(A_{M}) \to A$, $\rho(C) = \inf C$.

In order to guarantee that $\rho$ is an equivalence, we prove that $\rho$ is order reflecting and essentially surjective (see Observation 2.2).

The surjectivity condition for $\rho$ comes from the fact that the adjunction $\iota \dashv \rho$ is a Galois injection (see Theorem 6.14). The fact that $\rho$ is order reflecting is guaranteed by the following reasoning.

The assertion $C \sqsubseteq C'$ in our situation means that: $\exists t \in \text{QP} : s \leq b \forall b \{ s \to \pi, s \leq C, \pi \in C' \}$. Using the full adjunction we deduce that the condition $C \sqsubseteq C'$ is:

$$\exists t \in \text{QP} = \Phi : ts \leq \pi \forall s \leq C, \pi \in C'. \quad (6.16.10)$$

The assertion

$$\rho(C) \sqsubseteq \rho(C'), \quad (6.16.11)$$

can be written as: $\exists r \in \Phi : r \inf C \leq \inf C'$. It is clear that if $s \leq C$, then $s \leq \inf C$, hence for such an $s \leq C$, $rs \leq r \inf C \leq \inf C'$. We can take then $t := r$ to obtain a proof that from the condition (6.16.11), we deduce (6.16.10), i.e. that $\rho$ is order reflecting. \qed
XXXVI. In [5] Section 5] we illustrated how to construct triposes from ordered combinatory algebras and from abstract Krivine structures and the relations between them. As the situation is almost the same, we very briefly revisit that construction in our situation where we are dealing with $^F\text{OCA}$s.

Given an $\mathsf{AKS}$ called $\mathcal{K}$, define:

1. The entailment relation $\vdash$ in $\mathcal{P}_*(\Pi)^I$ is: $\varphi, \psi \in \mathcal{P}_*(\Pi)^I$, $\varphi \vdash \psi$ $\iff \exists t \in Q \forall i \in I \ t \perp \varphi(i) \rightarrow \psi(i)$, for $\varphi, \psi : I \rightarrow \mathcal{P}_*(\Pi)$. An element $t \in \Phi$ as above is said to be “a realization of the entailment $\varphi \vdash \psi$”.

2. For any function $f : J \rightarrow I$, $f^*$ is the monotonic map $f^* : (\mathcal{P}_*(\Pi)^I, \vdash) \rightarrow (\mathcal{P}_*(\Pi)^J, \vdash)$, $\varphi \mapsto \varphi \circ f$.

3. The preceding constructions yield an indexed preorder:

$$\mathcal{P}_*(\mathcal{K}) : \text{Set}^{\text{op}} \rightarrow \text{Ord}, \quad I \mapsto (\mathcal{P}(\Pi)^I, \vdash), \quad f \mapsto f^*.$$ 

Remark 7.1. (1) Notice that the entailment relation above, could have been defined using the arrow $\rightarrow$, because $t \perp \varphi(i) \rightarrow \psi(i)$ if and only if $t \perp \psi(i) \rightarrow \varphi(i)$, compare with Observation 6.3.

(2) In particular from this follows that if we change in the above definition $\mathcal{P}_*(\Pi)$ by $\mathcal{P}(\Pi)$, we may define the indexed preorder $\mathcal{P}(\mathcal{K})$ and in this situation we have a natural inclusion $\mathcal{P}_*(\mathcal{K}) \hookrightarrow \mathcal{P}(\mathcal{K})$.

Lemma 7.2. The canonical inclusion $\mathcal{P}_*(\mathcal{K}) \hookrightarrow \mathcal{P}(\mathcal{K})$ is an equivalence of indexed preorders.

Proof. By Lemma 2.10 it suffices to show that the inclusion $(\mathcal{P}_*(\Pi)^I, \vdash) \hookrightarrow (\mathcal{P}(\Pi)^I, \vdash)$ is an equivalence for all sets $I$ and this is the content of Lemma 6.5. \qed

XXXVII. In this Paragraph we prove the equivalence of all the relevant tripos constructed previously in this paper.

Remark 7.3. Let $\mathcal{A}$ be a $^F\text{OCA}$:

1. The entailment relation is defined by $\varphi \vdash \psi$ $\iff \exists r \in \Phi \forall i \in I \ r(\varphi(i)) \leq \psi(i)$, for $\varphi, \psi : I \rightarrow \mathcal{A}$. An element $r \in \Phi$ as above is said to be “a realization of the entailment $\varphi \vdash \psi$”. The entailment relation is a preorder on $A^I$, and $(A^I, \vdash)$ is a meet-semi-lattice with: $\top : I \rightarrow \mathcal{A}$; $\top(i) = \top = k$ and $(\varphi \wedge \psi)(i) = \varphi(i) \wedge \psi(i)$.

2. For any function $f : J \rightarrow I$, we define map $f^* : (A^I, \vdash) \rightarrow (A^J, \vdash)$, $\varphi \mapsto \varphi \circ f$ that preserves meets and is monotonic.

3. Define the indexed meet-semi-lattice: $\mathcal{P}(\mathcal{A}) : \text{Set}^{\text{op}} \rightarrow \text{Slat}$, $I \mapsto (A^I, \vdash)$, $f \mapsto f^*$.

4. If $\mathcal{A} = (A, \leq, \text{app}, \text{imp}, \Phi, k, s)$ is a $^F\text{OCA}$, then $\mathcal{P}(\mathcal{A})$ is a tripos.

The last assertion can be proved in an identical manner than the corresponding result for a $^F\text{OCA}$, compare with [5] Section 5, Theorem 5.8.

Theorem 7.4. Let $\mathcal{K}$ and $\mathcal{A}_\mathcal{K}_*$ be as in Paragraph XXX Then, the associated indexed preorders $\mathcal{P}_*(\mathcal{K})$ and $\mathcal{P}(\mathcal{A}_\mathcal{K}_*)$ are isomorphic.

Proof. The proof is a direct consequence of the fact that the Heyting preorders $\mathcal{H}_{\mathcal{K}_*}$ and $\mathcal{H}_{\mathcal{A}_\mathcal{K}_*}$ are isomorphic, see Theorem 6.9. \qed

Theorem 7.5. Let $\mathcal{A}$ and $\mathcal{K}_{\mathcal{A}_*}$ be as in Paragraph XXXIII Then, the associated indexed triposes $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{K}_{\mathcal{A}_*})$ are equivalent.

Proof. The equivalence of the triposes follow immediately from the commutativity –up to equivalence– of the diagram:
as guaranteed by Theorem 6.16.

**References**

[1] Birkhoff, G. *Lattice theory* Colloquium Publications, Vol 25, American Mathematical Society, (1995), Third edition.

[2] Borceux, F. *Handbook of Categorical Algebra Volume 1. Basic Category Theory*, Encyclopedia of Mathematics and its Applications, 50. Cambridge Univ. Press. Cambridge.(2008)

[3] Borceux, F. *Handbook of Categorical Algebra Volume 2. Categories and Structures*, Encyclopedia of Mathematics and its Applications, 51. Cambridge Univ. Press. Cambridge.(2008)

[4] Ferrer Santos, W., Guillermo, M. and Malherbe, O. *A Report on realizability*, arXiv:1309.0706v2 [math.LO], 2013, pp. 1–25.

[5] Ferrer Santos, W., Frey, J., Guillermo, M. and Malherbe, O., Miquel, A. *Ordered combinatory algebras and realizability*. To appear in: Mathematical Structures in Computer Science.

[6] Hofstra, P and van Oosten, J. *Ordered partial combinatory algebras*, Math. Proc. Cambridge Philos. Soc. 134 (2004), no. 3, pp. 445–463.

[7] Hofstra, P. *All realizability is relative*, Math. Proc. Cambridge Philos. Soc. 141 (2006), no. 2, pp. 239–264.

[8] Hyland, J.M.E. *The effective topos*, Proc. of The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout 1981) pp. 165-216, North Holland 1982.

[9] Krivine, J.-L. *Types lambda-calculus in classical Zermelo-Fraenkel set theory*, Arch. Math. Log. 40 (2001), no. 3, pp. 189–205.

[10] Krivine, J.-L. *Dependent choice, quote and the clock*, Th. Comp. Sc. 308 (2003), pp. 259–276.

[11] Krivine, J.-L. *Structures de réalisabilité, RAM et ultrafiltre sur N*, (2008). http://www.pps.jussieu.fr/~krivine/Ultrafiltre.pdf.

[12] Krivine, J.-L. *Realizability in classical logic in Interactive models of computation and program behaviour*, Panoramas et synthèses 27 (2009), SMF.

[13] Streicher, T. *Krivine’s Classical Realizability from a Categorical Perspective*, Math. Struct. in Comp. Science

[14] van Oosten, J. *Realizability, an Introduction to its Categorical Side*, (2008), Elsevier.

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