Mixed-symmetry massive fields in AdS(5)

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Abstract

Free mixed-symmetry arbitrary spin massive bosonic and fermionic fields propagating in AdS(5) are investigated. Using the light-cone formulation of relativistic dynamics, we study bosonic and fermionic fields on an equal footing. Light-cone gauge actions for such fields are constructed. Various limits of the actions are discussed.

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1 Introduction

The study of higher spin massive fields theories in $AdS$ space-time is motivated by the conjectured duality of the conformal $\mathcal{N} = 4$ SYM theory and the theory of type $IIB$ superstring in $AdS_5 \times S^5$ background \[1\]. Recent interesting discussions of various aspects of this theme may be found, e.g., in \[2, 3, 4, 5\]. As is well known, quantization of GS superstring propagating in flat space is straightforward only in the light-cone gauge. It is the light-cone gauge that removes unphysical degrees of freedom explicitly and reduces the action to the quadratic form in string coordinates. The light-cone gauge in string theory implies the corresponding light-cone formulation for target space fields. In the case of strings in $AdS$ background, this suggests that we should study a light-cone form dynamics of target space fields propagating in $AdS$ space-time. It is expected that $AdS$ massive fields form the spectrum of states of $AdS$ strings. Therefore, understanding the light-cone description of $AdS$ massive target space fields might be helpful in discussion of various aspects of $AdS$ string dynamics. This is what we do in this paper.

Let us first formulate the main problem we solve in this paper. Fields propagating in $AdS_5$ space are associated with positive-energy unitary lowest weight representations of the $SO(4,2)$ group. A positive-energy lowest weight irreducible representation of the $SO(4,2)$ group, denoted as $D(E_0, \mathbf{h})$, is defined by $E_0$, the lowest eigenvalue of the energy operator, and by $\mathbf{h} = (h_1, h_2)$, which is the highest weight of the unitary representation of the $SO(4)$ group. The highest weights $h_i$ are integers and half-integers for bosonic and fermionic fields respectively and satisfy the standard restriction

$$h_1 \geq |h_2|.$$  \hspace{1cm} (1.1)

The fields with $\mathbf{h} = (0, 0)$ and $\mathbf{h} = (1/2, \pm 1/2)$ correspond to the respective scalar bosonic and spin one-half fermionic fields. The actions for such fields are well known. $E_0$ and $\mathbf{h}$ associated with the remaining fields in $AdS_5$ are given by

$$E_0 > h_1 + 1, \quad h_1 = |h_2| > 1/2,$$  \hspace{1cm} (1.2)

$$E_0 = h_1 + 2, \quad h_1 > |h_2|,$$  \hspace{1cm} (1.3)

$$E_0 > h_1 + 2, \quad h_1 > |h_2|.$$  \hspace{1cm} (1.4)

The fields with $E_0, \mathbf{h}$ in (1.3) and (1.2) are referred to as massless and self-dual massive fields respectively. The fields with $E_0$, $\mathbf{h}$ in (1.4) are referred to as massive fields and these fields can be divided into two groups

$$E_0 > h_1 + 2, \quad h_1 > |h_2|, \quad h_2 = 0, \pm 1/2,$$  \hspace{1cm} (1.5)

$$E_0 > h_1 + 2, \quad h_1 > |h_2|, \quad |h_2| > 1/2.$$  \hspace{1cm} (1.6)

The massive fields in (1.5) are referred to as totally symmetric bosonic ($h_2 = 0$) and fermionic ($h_2 = \pm 1/2$) fields respectively, while the massive fields in (1.6) are referred to as mixed-symmetry fields\(^2\). In manifestly Lorentz covariant formulation the bosonic(fermionic)

\(^1\)We recall that the representations with $E_0 = h_1 + 1, h_1 = |h_2|$ do not admit field theoretical realization in $AdS_5$.

\(^2\)We note that the case $\mathbf{h} = (1, 0)$ corresponds to spin one massive field, the case $\mathbf{h} = (2, 0)$ is the massive spin two field. The labels $h_i$ are the standard Gelfand–Zeitlin labels. They are related to Dynkin labels $h_i^D$ by the formula $(h_i^D, h_2^D) = (h_1 - h_2, h_1 + h_2)$.
totally symmetric and mixed-symmetry massive representation are described by a set of the tensor(tensor-spinor) fields whose $SO(4,1)$ space-time tensor indices have the structure of the Young tableaux with one and two rows respectively. Lorentz covariant actions for the bosonic totally symmetric massive fields in $AdS_5$ space were found in [6].

Light-cone actions for both the bosonic and fermionic totally symmetric massive fields in $AdS_d$ with arbitrary $E_0$ and $h$ were obtained in [13]. Light-cone actions for $AdS_5$ mixed-symmetry massless [1,3], $|h_2| > 1/2$, and self-dual massive fields [1,2] were found in [14]. Mixed-symmetry massive $AdS_5$ fields [1,6] (bosonic and fermionic ones) have not been described at the field theoretical level so far\(^3\). In this paper, we develop a light-cone gauge formulation for such fields at the action level. In manifestly Lorentz covariant formulation these mixed-symmetry massive fields correspond to Young tableaux with two rows.

\[
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\]

Using a new version [13] of the old light-cone gauge formalism in $AdS$ space [23], we describe both the bosonic and fermionic fields on an equal footing. Since, by analogy with flat space, we expect that a quantization of the Green-Schwarz $AdS$ superstring with a Ramond-Ramond charge will be straightforward only in the light-cone gauge [21], it seems that from the stringy perspective of $AdS/CFT$ correspondence the light-cone approach is the fruitful direction to go\(^7\). An interesting recent discussion of the relationship between the light-cone gauge and BRST quantizations of superstrings may be found in [27].

The content of this paper is as follows. In section 2, we summarize some relevant facts about light-cone gauge actions and discuss realization of relativistic symmetries of such actions in the light-cone approach. Light-cone actions for bosonic and fermionic fields are constructed in sections 3 and 4 respectively. In section 5, we discuss various limits of the actions: limits of massive self-dual fields and massless fields in $AdS_5$ and the flat space limit. Section 6 suggests directions for future research. Appendices contain some mathematical details and useful formulas.

\(^3\)Massive self-dual spin fields in $AdS_3$ were investigated in [7]. Spin two massive fields were studied in [8] (see also [9, 10]). A discussion of massive totally symmetric fields in $(A)dS_d$, $d \geq 4$, may be found in [11]. Description of massive fields on constant curvature spaces corresponding to two-column Young tableaux irreducible representations of the general linear group may be found in [12].

\(^4\)Lorentz covariant actions for the mixed-symmetry massless fields in $AdS_5$ were obtained recently in [15] (see also [16, 17]).

\(^5\)Study of unitary representations of the $so(4,2)$ algebra may be found, e.g., in [18]. Group theoretical description of various massive representation of the $so(4,2)$ algebra via the oscillator method [19] may be found, e.g., in [20]. Lorentz covariant equations of motion for such fields with special values of $E_0$ were discussed in [21]. Lorentz covariant actions for mixed-symmetry massive $AdS_5$ fields corresponding to particular values of $h$ were constructed in [22].

\(^6\)Bosonic mixed-symmetry fields correspond to Young tableaux with row lengths $h_1$ and $|h_2|$, while fermionic fields correspond to Young tableaux with row lengths $h_1 - 1/2$ and $|h_2| - 1/2$.

\(^7\)Note that sometimes a light-cone gauge formulation turns out to be a good starting point for deriving a Lorentz covariant formulation. This is to say that for fields in flat space, interesting methods were developed [23, 24] which admit manifest covariantization of the results obtained in the light-cone gauge. It is highly likely that these methods can be generalized to the case of fields in $AdS$ space.
2 Light-cone gauge action and its global symmetries

In this section we present a new version [13] of the old light-cone formalism [23]. Let \( \phi(x) \) and \( \psi(x) \) be respective bosonic and fermionic arbitrary spin fields propagating in \( AdS_5 \) space. If we collect spin degrees of freedom in ket-vectors \( |\phi\rangle \) and \( |\psi\rangle \), then the respective light-cone gauge actions for the fields \( \phi \) and \( \psi \) can be cast into the following ‘covariant form’

\[
S_{l.c.} = \frac{1}{2} \int d^5x \langle \phi | (\Box - \frac{1}{z^2} A) |\phi\rangle, \quad \text{for bosonic fields, (2.1)}
\]

\[
S_{l.c.} = \int d^5x \langle \psi | i \frac{\partial}{\partial^+} (\Box - \frac{1}{z^2} A) |\psi\rangle, \quad \text{for fermionic fields, (2.2)}
\]

where a light-cone representation of the flat Lorentz covariant D’Alembertian is given by (details of the notation may be found in Appendix A)

\[
\Box = 2 \partial^+ \partial^- + \partial_i \partial_i + \partial_z^2. \quad (2.3)
\]

The operator \( A \) does not depend on space-time coordinates and their derivatives. This operator acts only on spin indices collected in the ket-vectors \( |\phi\rangle \) and \( |\psi\rangle \). We call the operator \( A \) the \( AdS \) mass operator.

It is instructive to present the \( AdS \) mass operator for simplest cases of the scalar field, \( h = (0, 0) \), and spin one-half field, \( h = (1/2, \pm 1/2) \). For a massive scalar field, the operator \( A \) takes the form

\[
A = (mR)^2 + \frac{d(d-2)}{4}, \quad d = 5, \quad (2.4)
\]

where we show the dependence on the dimension \( d \) of \( AdS \) space explicitly, while for the case of a spin one-half Dirac field corresponding to \( h = (1/2, \pm 1/2) \), the operator \( A \) takes the form

\[
A = (Rm)^2 + Rm \gamma^z \quad (2.5)
\]

that is valid for an arbitrary dimension of \( AdS \) space (see Appendix B).

We now turn to discussion of the \( AdS \) mass operator \( A \) for arbitrary spin massive fields. It turns out that the operator \( A \) admits the representation [13]

\[
A = 2B^2 + M^{zi}M^{zi} + \frac{1}{2} M^{IJ}M^{IJ} - \langle Q_{AdS} \rangle + \frac{15}{4}, \quad (2.6)
\]

where \( M^{IJ} \) is a spin operator of the \( so(3) \) algebra

\[
[M^{IJ}, M^{KL}] = \delta^{JK} M^{IL} + 3 \text{ terms} \quad (2.7)
\]

The operator \( M^{IJ} \) is acting only on spin degrees of freedom of wave function \( |\phi\rangle \). In formula (2.6), \( \langle Q_{AdS} \rangle \) is an eigenvalue of the second order Casimir operator of the \( so(4, 2) \) algebra for the representation labelled by \( D(E_0, h) \):

\[
- \langle Q_{AdS} \rangle = E_0(E_0 - 4) + h_1(h_1 + 2) + h_2^2, \quad (2.8)
\]

\[^8\text{In the \( so(2) \) algebra basis, the operator \( M^{IJ} \) splits into } M^{ij}, M^{zi}, i, j = 1, 2.\]
while $B^z$ is the $z$-component of the $so(3)$ algebra vector $B^I$ that satisfies the equation

$$[B^I, B^J] + (M^3)^{I[I,J]} + (\langle Q_{AdS} \rangle - \frac{1}{2} M^2 - 4) M^{IJ} = 0. \quad (2.9)$$

As noted, the operator $B^I$ transforms in the vector representation of the $so(3)$ algebra,

$$[B^I, M^{JK}] = \delta^{IJ} B^K - \delta^{IK} B^J. \quad (2.10)$$

This operator should be hermitian with respect to an appropriate scalar product that will be discussed below,

$$B^I = B^I. \quad (2.11)$$

We now turn to discussion of global $so(4,2)$ symmetries of the light-cone gauge actions of arbitrary spin fields. The choice of the light-cone gauge spoils the manifest global symmetries, and in order to demonstrate that these global invariances are still present, one needs to find the Noether charges which generate them. Noether charges (or generators) can be split into kinematical and dynamical generators. For $x^+ = 0$, the kinematical generators are quadratic in the physical field $|\phi\rangle$, while the dynamical generators receive corrections in the interacting theory. In this paper, we deal with free fields. Let us first consider bosonic fields. At the quadratic level both kinematical and dynamical generators have the following standard representation in terms of the bosonic physical field $|23|$

$$\hat{G} = \int d^4 x (\partial^+ \phi |G|\phi), \quad d^4 x \equiv dx^- dz d^2 x. \quad (2.12)$$

The representation for the kinematical generators in terms of differential operators $G$ acting on the bosonic physical field $|\phi\rangle$ is given by

$$P^i = \partial^i, \quad P^+ = \partial^+, \quad (2.13)$$

$$D = x^+ P^- + x^- \partial^+ + x^i \partial^i + \frac{3}{2}, \quad (2.14)$$

$$J^{+-} = x^+ P^- - x^- \partial^+, \quad (2.15)$$

$$J^{ij} = x^j \partial^i - x^i \partial^j + M^{ij}, \quad (2.16)$$

$$K^+ = -\frac{1}{2} (2x^+ x^- + x^j x^j) \partial^+ + x^+ D, \quad (2.17)$$

$$K^i = -\frac{1}{2} (2x^+ x^- + x^j x^j) \partial^i + x^i D + M^{ij} x^j + M^{i-} x^+, \quad (2.18)$$

while the representation for the dynamical generators takes the form

$$P^- = -\frac{\partial^j \partial^j}{2\partial^+} + \frac{1}{2z^2 \partial^+} A, \quad (2.19)$$

$$J^{-i} = x^- \partial^i - x^i P^- + M^{-i}, \quad (2.20)$$

$$K^- = -\frac{1}{2} (2x^+ x^- + x^j x^j) P^- + x^- D + \frac{1}{\partial^+} x^i \partial^j M^{ij} - \frac{x^i}{2z \partial^+} [M^{zi}, A] + \frac{1}{\partial^+} B, \quad (2.21)$$

We use the notation $(M^3)^{I[I,J]} = \frac{1}{2} M^{IK} M^{LJ} - (I \leftrightarrow J)$, $M^2 \equiv M^{IJ} M^{IJ}$. We note that for the spin operator of the $so(3)$ algebra, one has the relation $(M^3)^{I[I,J]} = -\frac{1}{2} M^2 M^{IJ}$.

These charges play a crucial role in formulating interaction vertices in field theory. Application of Noether charges in formulating superstring field theories may be found in [28].
where
\[ M^{-i} \equiv M^{ij} \frac{\partial^j}{\partial^+} - \frac{1}{2z\partial^+}[M^{zi}, A], \quad M^{-i} = -M^{i-} \]  
and the new operator $B$ that enters $K^-$ in (2.22) admits the representation
\[ B = B^z + M^{zi}M^{zi}. \]  
Making use of the above formulas, one can check that the light-cone gauge action (2.1) is invariant under the global symmetries generated by the $so(4, 2)$ algebra taken in the form
\[ \delta \hat{G} |\phi\rangle = G|\phi\rangle. \]  
Generalization of above formulas to the case of fermionic fields is straightforward. This is to say that at the quadratic level, both kinematical and dynamical generators have the following standard representation in terms of the fermionic physical light-cone field
\[ \hat{G}^{ferm} = \int d^4x \langle \psi | G^{ferm} | \psi \rangle, \]  
where differential operators $G^{ferm}$ are obtainable from those of bosonic fields (2.13)-(2.22) by making there the following substitution
\[ x^- \rightarrow x^- + \frac{1}{2\partial^+}. \]  
In addition to this, in expressions for generators in (2.13)-(2.22), we should use the spin operator $M^{IJ}$ suitable for fermionic fields. Defining equation (2.9) for the operator $B^I$ and the representations for the operators $A, B$ given in (2.6),(2.24) do not change.

The action for fermionic fields (2.2) is invariant under the global relativistic symmetries generated by the $so(4, 2)$ algebra:
\[ \delta \hat{G}^{ferm} |\psi\rangle = G^{ferm} |\psi\rangle. \]  
To summarize the procedure for finding light-cone description consists of the following steps:

i) choose the form of a realization of spin degrees of freedom for the field $|\phi\rangle$ (or $|\psi\rangle$);
ii) fix an appropriate representation for the spin operator $M^{IJ}$;
iii) find a solution to the defining equations for the operator $B^I$ in (2.9).

Following this procedure, we now discuss bosonic and fermionic fields in turn.

3 Bosonic fields

To discuss field theoretical description of a massive $AdS_5$ field, we could use a complex-valued tensor field $\phi$ that is associated with the weight-$h$ representation of the $so(4)$ algebra. We prefer to decompose such field into traceless totally symmetric complex-valued tensors of the $so(3)$ algebra $\phi^{I_1...I_{s'}}$, $I = 1, 2, 3$; $s' = |h_2|, |h_2| + 1, \ldots, h_1$:
\[ \phi = \sum_{s' = |h_2|}^{h_1} \oplus \phi^{I_1...I_{s'}}. \]
The number of complex-valued degrees of freedom (DoF) of mixed-symmetry massive field $\phi$ associated with the weight-$h$ representation of the $so(4)$ algebra is given by

$$N_{DoF}^{c}(\phi) = (h_1 + 1)^2 - h_2^2,$$

and this number of complex-valued DoF can be presented as

$$N_{DoF}^{c}(\phi) = \sum_{s' = |h_2|}^{h_1} (2s' + 1).$$

(3.2) (3.3)

Formula (3.3) tells us that the tensor field $\phi$ can indeed be decomposed into totally symmetric tensor fields as shown in (3.1).

As usual, to avoid cumbersome tensor expressions, we introduce creation and annihilation oscillators $\alpha^I$ and $\bar{\alpha}^I$

$$[\bar{\alpha}^I, \alpha^J] = \delta^{IJ}, \quad \bar{\alpha}^I|0\rangle = 0,$$

and make use of ket-vectors $|\phi_{s'}\rangle$ defined by

$$|\phi_{s'}\rangle \equiv \alpha^{I_1} \ldots \alpha^{I_{s'}} \phi^{I_1 \ldots I_{s'}}(x)|0\rangle.$$ (3.4) (3.5)

The ket-vector $|\phi_{s'}\rangle$ satisfies the algebraic constraints

$$\left(\alpha\bar{\alpha} - s'\right)|\phi_{s'}\rangle = 0, \quad \alpha\bar{\alpha} \equiv \alpha^I\bar{\alpha}^I,$$ (3.6)

$$\bar{\alpha}^I\alpha^J|\phi_{s'}\rangle = 0, \quad \text{tracelessness}.$$ (3.7)

Equation (3.6) tells us that $|\phi_{s'}\rangle$ is a monomial of degree $s'$ in the oscillator $\alpha^I$. Tracelessness of the tensor field $\phi^{I_1 \ldots I_{s'}}$ is reflected in (3.7). Thus, in the language of ket-vectors, the massive field $\phi$ is representable as

$$|\phi\rangle = \sum_{s' = |h_2|}^{h_1} \oplus |\phi_{s'}\rangle$$ (3.8)

and the scalar product $\langle \varphi||\phi\rangle$ that enters light-cone action (2.1) is defined to be\footnote{Because we use complex-valued fields, the bra-vectors $\langle \phi_{s'}|$ are built in terms of complex conjugate tensors fields $\phi^{I_1 \ldots I_{s'}}^*$, where the asterisk is used to denote complex conjugation.}

$$\langle \varphi||\phi\rangle \equiv \sum_{s' = |h_2|}^{h_1} \langle \varphi_{s'}||\phi_{s'}\rangle.$$ (3.9)

The spin operator $M^{IJ}$ of the $so(3)$ algebra for the above defined fields $|\phi_{s'}\rangle$ then takes the form

$$M^{IJ} = \alpha^I\bar{\alpha}^J - \alpha^J\bar{\alpha}^I.$$ (3.10)

The action of the operator $B^I$ on the physical fields $|\phi_{s'}\rangle$ is found to be (details of derivation may be found in Appendix C)

$$B^I|\phi_{s'}\rangle = b^0\tilde{M}^I|\phi_{s'}\rangle + b^0\tilde{A}_{s'-1}^I|\phi_{s'-1}\rangle + b_s^+\tilde{A}_{s'}^I|\phi_{s'+1}\rangle.$$ (3.11)
where the coefficients $b_{s'}^\pm$, $b_0^{s'}$ are given by

\[ b_{s'}^- = f(E_0, h, s'), \]
\[ b_{s'}^+ = f^*(E_0, h, s' + 1), \]
\[ b_0^{s'} = i\frac{(h_1 + 1)h_2(E_0 - 2)}{s'(s' + 1)} \]

and a complex-valued function $f(E_0, h, s')$ is found to be

\[ |f(E_0, h, s')|^2 \equiv \frac{((h_1 + 1)^2 - s'^2)(s'^2 - h_2^2)((E_0 - 2)^2 - s'^2)}{(2s' + 1)s'^2}. \]

The operators $\tilde{M}^I$, $A^I_s$ that enter the operator $B^I$ in (3.11) are defined by the relations

\[ \tilde{M}^I \equiv \epsilon^{IJK} \alpha^J \bar{\alpha}^K, \]
\[ A^I_s \equiv \alpha^I - \alpha^2 \bar{\alpha}^I \frac{2}{2s + 1}; \quad \alpha^2 \equiv \alpha^I \alpha^I, \]

where $\epsilon^{IJK}$ ($\epsilon^{123} = 1$) is the Levi-Civita tensor of so(3). Note that the operators $M^{IJ}$ (3.10) and $\tilde{M}^I$ (3.16) are related by the formulas

\[ \tilde{M}^I = \frac{1}{2} \epsilon^{IJK} M^{JK}, \quad M^{IJ} = \epsilon^{IJK} \tilde{M}^K. \]

Below, we present various relations for the operators $\tilde{M}^I$ and $A^I_s$ which are helpful in solving the defining equation for the spin operator $B^I$ given in (2.9):

\[ A^I_s A^{J}_{s-1} - (I \leftrightarrow J) = 0, \]
\[ \alpha^I \tilde{M}^J - \alpha^J \tilde{M}^I \approx -\epsilon^{IJK} (\alpha \bar{\alpha} + 2) \bar{\alpha}^K, \]
\[ \tilde{M}^I \alpha^J - \tilde{M}^J \alpha^I \approx \epsilon^{IJK} \alpha \bar{\alpha} \bar{\alpha}^K, \]
\[ A^I_s \tilde{M}^J - A^J_s \tilde{M}^I \approx \epsilon^{IJK} s A^K_s, \]
\[ \tilde{M}^I A^J_s - \tilde{M}^J A^I_s \approx -\epsilon^{IJK} (s + 2) A^K_s, \]
\[ A^I_s \bar{\alpha}^J - (I \leftrightarrow J) = M^{IJ}, \]
\[ \bar{\alpha}^I A^J_s - (I \leftrightarrow J) = -\frac{2s + 3}{2s + 1} M^{IJ}, \]

where we use the sign $\approx$ to indicate those relations that are valid on the space of ket-vectors subject to the constraints (3.6), (3.7).

A few remarks are in order.

i) From formula (3.15), it is easy to see that if $h_1 > |h_2|$, then $E_0$ should satisfy the restriction $E_0 \geq 2 + \max s'$. Taking into account that $\max s' = h_1$, we find a restriction on $E_0$,

\[ E_0 \geq h_1 + 2. \]
The boundary values $E_0 = h_1 + 2$ correspond to the massless fields \(1.3\), while values $E_0 > h_1 + 2$ correspond to the massive fields \(1.4\). Thus, our analysis involves massless fields \(1.3\) as a particular case and reproduces the restriction on $E_0$ corresponding to the massive fields \(1.4\). Moreover, we will demonstrate below that our results involve the self-dual massive fields \(1.2\) as a particular case.

ii) In formulas \(3.12\)-(\(3.15\)), phase factors of the coefficients $b^\pm_s \(3.12\), \(3.13\)$, which are determined by phase factors of the functions $f(E_0, h, s')$, have not been fixed. It is easy to demonstrate that making use of field redefinitions, the phase factors of $b^\pm_s$ could be normalized to be equal to 1, i.e., all coefficients $b^\pm_s$ could be chosen real-valued and positive. However, for flexibility we do not fix phase factors of the coefficients $b^\pm_s$.

iii) We found a realization of the spin operator $B^I$ on the vectors $|\phi_{s'}\rangle$ which depends on $s' \(3.11\). In contrast to this, the realization of the spin operator $M^IJ$ given in \(3.10\) does depend on $s'$, i.e., the operator $M^IJ$ takes the same form for all vectors $|\phi_{s'}\rangle$. Now we would like to describe a realization of the operator $B^I$, which also does not depend on $s'$. To this end, we introduce new creation and annihilation oscillators $\zeta, \bar{\zeta}$:

$$[\bar{\zeta}, \zeta] = 1, \quad \zeta|0\rangle = 0 \quad (3.27)$$

and build the new ket-vector

$$|\phi\rangle = \sum_{s'=|s_0|}^{h_1} \frac{\zeta^{h_1-s'}\bar{\zeta}^{s-s'}|\phi_{s'}\rangle}{\sqrt{(h_1-s')!}}, \quad (3.28)$$

where the normalization factors in expansion \(3.28\) are chosen so as to keep the normalization used in the scalar product \(3.39\). For this realization of the spin degrees of freedom we obtain the following representation for the operator $B^I$ on the vector $|\phi\rangle \(3.28\)$:

$$B^I = \hat{f}^0(E_0, h, \alpha \bar{\alpha}) \tilde{M}^I + \frac{\hat{f}(E_0, h, \alpha \bar{\alpha})}{\alpha \bar{\alpha} - \frac{1}{2}} k^I \bar{\zeta} + \hat{f}^\dagger(E_0, h, \alpha \bar{\alpha} + 1) \zeta \bar{\alpha}^I, \quad (3.29)$$

where the operator-valued functions $\hat{f}, \hat{f}^0$ are defined by\(^{12}\)

$$|\hat{f}(E_0, h, \alpha \bar{\alpha})|^2 = \frac{(h_1 + 1 + \alpha \bar{\alpha})((\alpha \bar{\alpha})^2 - h_0^2)((E_0 - 2)^2 - (\alpha \bar{\alpha})^2)}{(2\alpha \bar{\alpha} + 1)(\alpha \bar{\alpha})^2}, \quad (3.30)$$

$$\hat{f}^0(E_0, h, \alpha \bar{\alpha}) = i\frac{(h_1 + 1)h_2(E_0 - 2)}{\alpha \bar{\alpha}(\alpha \bar{\alpha} + 1)} \quad (3.31)$$

and we use the convention $|\hat{f}|^2 = \hat{f} \hat{f}^\dagger$. The operator $k^I$ in \(3.29\) is defined by

$$k^I = -\frac{1}{2} \alpha^2 \bar{\alpha}^I + \alpha' (\alpha \bar{\alpha} + \frac{1}{2}) \quad (3.32)$$

and this operator is a counterpart of the operator $A^I_s$ in \(3.14\) used in the representation for $B^I$ given in \(3.11\). The operator $k^I$ is similar to the conformal boost operator and one readily verifies the commutator $[k^I, k^J] = 0$.

\(^{12}\) The operator $\alpha \bar{\alpha}$ appearing in denominators of expressions \(3.29\), \(3.31\) is well defined because this operator acts on the vector $|\phi\rangle \(3.8\)$ that does not involve terms of degree zero in $\alpha'$ (see \(1.6\)).
The representations for the operator $B^I$ given in (3.11) and (3.29) are related as

$$B^I|\phi\rangle = \sum_{s'=[h_2]}^{h_1} \frac{\zeta^{h_1-s'}}{\sqrt{(h_1-s')!}} B^I|\phi_{s'}\rangle,$$  \hspace{1cm} (3.33)

where in the l.h.s. of (3.33) we should use the representation given in (3.29), while in the r.h.s. of (3.33) we should use the representation given in (3.11).

### 4 Fermionic fields

Before studying the concrete form of the spin operators $M^{IJ}$ and $B^I$ we should fix a field theoretical realization of the spin degrees of freedom collected in $|\psi\rangle$. To discuss field theoretical description of fermionic massive field, we could use complex-valued tensor-spinor field $\psi$ that is associated with the $so(4)$ algebra representation of weight $h$. However, we prefer to decompose such a field into traceless totally symmetric tensor-spinor fields of the $so(3)$ algebra $\psi^{I_1...I_{s'}\alpha}$, $I = 1, 2, 3; s' = |h_2| - \frac{1}{2}, |h_2| + \frac{1}{2}, ..., h_1 - \frac{1}{2}$:

$$\psi = \sum_{s'=[h_2]-\frac{1}{2}}^{h_1-\frac{1}{2}} \psi^{I_1...I_{s'}\alpha}.$$  \hspace{1cm} (4.1)

As before, to avoid cumbersome expressions we use the creation and annihilation oscillators $\alpha^I$ and $\bar{\alpha}^I$ (3.4) and make use of the ket-vectors $|\psi_{s'}\rangle$ defined by

$$|\psi_{s'}\rangle \equiv \alpha^{I_1}...\alpha^{I_{s'}}\psi^{I_1...I_{s'}\alpha}(x)|0\rangle.$$  \hspace{1cm} (4.2)

The spinor index $\alpha = 1, 2, 3, 4$ is to remind that we deal with Dirac tensor-spinor fields $|\psi_{s'}\rangle$. Below, all spinor indices are implicit. The ket-vector $|\psi_{s'}\rangle$ satisfies the algebraic constraints

$$\Pi^{\oplus}|\psi_{s'}\rangle = |\psi_{s'}\rangle,$$  \hspace{1cm} (4.3)

$$(\alpha\bar{\alpha} - s')|\psi_{s'}\rangle = 0,$$  \hspace{1cm} (4.4)

$$\gamma\bar{\alpha}|\psi_{s'}\rangle = 0, \hspace{1cm} \gamma\bar{\alpha} \equiv \gamma^{I}\bar{\alpha}^I,$$  \hspace{1cm} (4.5)

$$\bar{\alpha}^I\bar{\alpha}^I|\psi_{s'}\rangle = 0.$$  \hspace{1cm} (4.6)

Constraint (4.3) is the standard constraint of the light-cone formalism, which allows us to deal with physical degrees of freedom of the Dirac tensor-spinor field (some details may be found in Appendices A and B). Equation (4.4) tells us that $|\psi_{s'}\rangle$ is a monomial of degree $s'$ in the oscillator $\alpha^I$. Tracelessness of the tensor-spinor field $\psi^{I_1...I_{s'}\alpha}$ is reflected in (4.6). Thus, in the language of ket-vectors the massive field $\psi$ is representable as

$$|\psi\rangle = \sum_{s'=[h_2]-\frac{1}{2}}^{h_1-\frac{1}{2}} \ominus |\psi_{s'}\rangle$$  \hspace{1cm} (4.7)

and the scalar product $\langle \chi||\psi\rangle$ that enters light-cone action (2.2) is defined to be

$$\langle \chi||\psi\rangle \equiv \sum_{s'=[h_2]-\frac{1}{2}}^{h_1-\frac{1}{2}} \langle \chi_{s'}||\psi_{s'}\rangle.$$  \hspace{1cm} (4.8)
Realization of the spin operator $M^{I,J}$ on the space of ket-vectors $|\psi_{s'}\rangle$ is given by

$$M^{I,J} = \alpha^I \bar{\alpha}^J - \alpha^J \bar{\alpha}^I + \frac{1}{2} \gamma^{I,J}, \quad \gamma^{I,J} \equiv \frac{1}{2} (\gamma^I \gamma^J - \gamma^J \gamma^I).$$  \hspace{1cm} (4.9)

The action of the operator $B^I$ on the physical fermionic fields $|\psi_{s'}\rangle$ is then found to be

$$B^I |\psi_{s'}\rangle = b^0_s \tilde{M}^I |\psi_{s'}\rangle + b^+_s A^I_{s' - 1} |\psi_{s' - 1}\rangle + b^-_s \bar{\alpha}^I |\psi_{s' + 1}\rangle.$$ \hspace{1cm} (4.10)

As before, the coefficients $b^+_s, b^0_s$ depend on $E_0, h_1, h_2, s'$ and are given by

$$b^-_s = f(E_0, h, s'), \quad b^+_s = f^*(E_0, h, s' + 1), \quad b^0_s = \frac{(h_1 + 1)h_2(E_0 - 2)}{(s' + \frac{1}{2})(s' + \frac{3}{2})},$$ \hspace{1cm} (4.11)-(4.13)

where a complex-valued function $f(E_0, h, s')$ is found to be

$$|f(E_0, h, s')|^2 = \frac{((h_1 + 1)^2 - (s' + \frac{1}{2})^2)((s' + \frac{1}{2})^2 - h_2^2)((E_0 - 2)^2 - (s' + \frac{1}{2})^2)}{2s'(s' + 1)}. \hspace{1cm} (4.14)$$

The operators $\tilde{M}^I$ and $A^I_s$ that enter the spin operator $B^I$ in (4.10) are given by

$$\tilde{M}^I \equiv \epsilon^{IJK} \alpha^J \bar{\alpha}^K + \frac{i}{2} \gamma^I, \hspace{1cm} (4.15)$$

$$A^I_s \equiv \alpha^I - \frac{\alpha^2 \bar{\alpha}^J + \gamma \alpha \gamma^J}{2s + 3}. \hspace{1cm} (4.16)$$

Here, we present various relations for the operators $\tilde{M}^I$ and $A^I_s$, which are helpful in solving the defining equation for the spin operator $B^I$ given in (4.10):

$$A^J_s A^I_{s - 1} - (I \leftrightarrow J) = 0,$$ \hspace{1cm} (4.17)

$$\bar{\alpha}^I \tilde{M}^J - (I \leftrightarrow J) \approx -(\alpha \bar{\alpha} + \frac{5}{2})\epsilon^{IJK} \bar{\alpha}^K,$$ \hspace{1cm} (4.18)

$$\tilde{M}^I \bar{\alpha}^J - (I \leftrightarrow J) \approx (\alpha \bar{\alpha} + \frac{1}{2})\epsilon^{IJK} \bar{\alpha}^K,$$ \hspace{1cm} (4.19)

$$A^J_s \tilde{M}^I - (I \leftrightarrow J) \approx (s + \frac{1}{2})\epsilon^{IJK} A^K_s,$$ \hspace{1cm} (4.20)

$$\tilde{M}^I A^J_s - (I \leftrightarrow J) \approx -(s + \frac{5}{2})\epsilon^{IJK} A^K_s,$$ \hspace{1cm} (4.21)

$$A^J_s \bar{\alpha}^J - (I \leftrightarrow J) \approx \frac{2(s + 1)}{2s + 3} M^{I,J},$$ \hspace{1cm} (4.22)

$$\bar{\alpha}^I A^J_s - (I \leftrightarrow J) \approx \frac{2(s + 2)}{2s + 3} M^{I,J},$$ \hspace{1cm} (4.23)
where we use the sign \(\approx\) to indicate those relations that are valid on the space of ket-vectors subject to the constraints (4.3)-(4.6).

The above realization of the spin operator \(B^I (4.10)\) on the vectors \(|\psi_{s'}\rangle\) depends on \(s'\). By analogy with bosonic fields this realization can be rewritten in the form independent of \(s'\). To this end, we use the oscillators \(\zeta, \bar{\zeta} (3.27)\) and build the ket-vector

\[
|\psi\rangle = \sum_{s'=|h_2|^{-\frac{1}{2}}} \zeta^{h_1 - \frac{1}{2} - s'} \sqrt{(h_1 - \frac{1}{2} - s')!} |\psi_{s'}\rangle ,
\]

where normalization factors in expansion (4.24) are chosen so as to keep normalization of the scalar product used in (4.8). The realization of spin degrees of freedom given in (4.24) leads to the following representation for the operator \(B^I\) on the vector \(|\psi\rangle (4.24)\):

\[
B^I = \hat{f}^0(E_0, h, \alpha \bar{\alpha}) \tilde{M}^I + \frac{\hat{f}(E_0, h, \alpha \bar{\alpha})}{\alpha \bar{\alpha} - \frac{1}{2}} k^I \zeta + \hat{f}^\dagger(E_0, h, \alpha \bar{\alpha} + 1) \zeta \alpha^I ,
\]

where operator-valued functions \(\hat{f}^0, \hat{f}\) are defined by

\[
|\hat{f}(E_0, h, \alpha \bar{\alpha})|^2 = \frac{(h_1 + \frac{3}{2} + \alpha \bar{\alpha}) ((\alpha \bar{\alpha} + \frac{1}{2})^2 - h_2^2) ((E_0 - 2)^2 - (\alpha \bar{\alpha} + \frac{1}{2})^2)}{(2\alpha \bar{\alpha} + 1)\alpha \bar{\alpha}(\alpha \bar{\alpha} + 1)} ,
\]

\[
\hat{f}^0(E_0, h, \alpha \bar{\alpha}) = i \frac{(h_1 + 1)h_2(E_0 - 2)}{(\alpha \bar{\alpha} + 1)\alpha \bar{\alpha} + \frac{1}{2}} .
\]

As before we use the convention \(|\hat{f}|^2 \equiv \hat{f} \hat{f}^\dagger\), while the operator \(k^I\) is modified compared to that in (3.32) as follows

\[
k^I \equiv -\frac{1}{2} \alpha^2 \alpha^I + \alpha^I (\alpha \bar{\alpha} + 1) + \frac{1}{2} \gamma^I \alpha^J .
\]

This operator satisfies the commutator \([k^I, k^J] = 0\).

5 Various limits of mixed-symmetry massive fields

In previous sections, we found the light-cone gauge actions (2.1),(2.2) for the mixed-symmetry massive fields (1.6). Note that these actions can also be used for description of totally symmetric massive fields (1.6). All that is required to obtain the actions for totally symmetric fields is to set \(h_2 = 0\) (for bosonic fields) and \(h_2 = \pm 1/2\) (for fermionic fields) in the above actions for mixed-symmetry fields. In various limits the actions (2.1),(2.2) lead to actions for self-dual massive fields (1.2) and massless fields (1.3) which were discussed in detail in [14]. Another interesting case is the flat space limit. In this case, our actions become the actions for massive fields propagating in flat space. We discuss these limits in turn.

5.1 Limit of massless fields in \(AdS_5\)

To realize limit of massless fields (1.3), we take

\[
E_0 \to h_1 + 2 , \quad h_1 > |h_2| .
\]

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It can be verified that this limit leads to appearance of the respective bosonic and fermionic invariant subspaces in $|\phi\rangle$ \textsuperscript{(3.8)} and $|\psi\rangle$ \textsuperscript{(4.7)} and these invariant subspaces, denoted as $\phi^\text{subsp}_{E_0=h_1+2}$ and $\psi^\text{subsp}_{E_0=h_1+2}$, are given by highest terms in the respective expansions \textsuperscript{(3.8)} and \textsuperscript{(4.7)}:

$$
\phi^\text{subsp}_{E_0=h_1+2} = |\phi_{h_1}\rangle, \quad \text{for bosonic fields,} \quad (5.2)
$$

$$
\psi^\text{subsp}_{E_0=h_1+2} = |\psi_{h_1-1/2}\rangle, \quad \text{for fermionic fields,} \quad (5.3)
$$

In other words, in the limit \textsuperscript{(5.1)}, the vectors $|\phi_{h_1}\rangle$ and $|\psi_{h_1-1/2}\rangle$ transform into themselves under the action of the respective spin operators $M^{IJ}$ \textsuperscript{(3.10)-(4.9)} and $B^I$ \textsuperscript{(3.11)-(4.10)} and these vectors describe spin degrees of freedom of the respective bosonic and fermionic massless fields \textsuperscript{(1.3)}. The realization of the spin operator $M^{IJ}$ on $|\phi_{h_1}\rangle$ and $|\psi_{h_1-1/2}\rangle$ is given by \textsuperscript{(3.10)} and \textsuperscript{(4.9)} respectively. To find the realization of the spin operator $B^I$ on the vectors $|\phi_{h_1}\rangle$ and $|\psi_{h_1-1/2}\rangle$, we note that in limit \textsuperscript{(5.1)}, the action of the operator $B^I$ on $|\phi_{h_1}\rangle$ \textsuperscript{(3.11)} and $|\psi_{h_1-1/2}\rangle$ \textsuperscript{(4.10)} is governed by terms proportional to $b^0_{h_1}$, $b^0_{h_1-1/2}$ respectively. By taking limit \textsuperscript{(5.1)} in \textsuperscript{(3.14)-(4.13)} we obtain

$$
b^0_{h_1} = i h_2, \quad b^0_{h_1-1/2} = i h_2 \quad (5.4)
$$

and this leads to the following realization for the operator $B^I$:

$$
B^I = i h_2 \tilde{M}^I, \quad (5.5)
$$

which is valid for both bosonic $|\phi_{h_1}\rangle$ and fermionic $|\psi_{h_1-1/2}\rangle$ fields. The realization of the AdS mass operator $A$ on the massless fields $|\phi_{h_1}\rangle$ and $|\psi_{h_1-1/2}\rangle$ can then be obtained using formula \textsuperscript{(2.6)}. To this end we should evaluate $B^z$. Making use of the convention $\epsilon^{ij} = \epsilon^{ij}$, $\epsilon^{12} = -\epsilon^{21} = 1$ and the respective formulas for the spin operator $\tilde{M}^I$ given in \textsuperscript{(3.10)-(4.13)} we obtain the following representation for AdS mass operator \textsuperscript{(14)}:

$$
A = -\frac{1}{2} m^{ij} m^{ij} - \frac{1}{4}, \quad m^{ij} \equiv M^{ij} - i \epsilon^{ij} h_2, \quad (5.6)
$$

where the $so(2)$ spin operator $M^{ij}$ takes the form given in \textsuperscript{(3.10)} and \textsuperscript{(4.9)} for bosonic and fermionic fields respectively.

We finish with comparison of the number of physical degrees of freedom in AdS and flat spaces. On the one hand the number of real-valued physical degrees of freedom (DoF) for mixed-symmetry massless field \textsuperscript{(1.3)}\textsuperscript{13} in AdS space is equal to $2(2h_1 + 1)$ (see Refs. \textsuperscript{(14)}), while the numbers of the DoF for totally symmetric bosonic and fermionic massless fields are equal to $2h_1 + 1$ and $2(2h_1 + 1)$ respectively. On the other hand, it is well known that the numbers of the DoF for massless spin $h_1$ bosonic and fermionic fields\textsuperscript{14} in flat space are equal to $2h_1 + 1$ and $2(2h_1 + 1)$ respectively. This implies that:

i) the number of the DoF for mixed-symmetry massless bosonic field \textsuperscript{(1.3)} in AdS space is twice that for the massless spin $h_1$ bosonic field in flat space-time, while the

\textsuperscript{13}Massless fields \textsuperscript{(1.3)} with $|h_2| > 1/2$ are referred to as mixed-symmetry fields, while those with $h_2 = 0, \pm 1/2$ are referred to as totally symmetric fields.

\textsuperscript{14}In 5d flat space physical DoF of the massless spin $h_1$ field transform in irreps of the $so(3)$ algebra. Such a field has the only label $h_1$ that is a weight of the $so(3)$ algebra irreps, and there is no label similar to $h_2$. Therefore, all massless fields in 5d flat space can be considered totally symmetric fields.
number of the DoF for mixed-symmetry massless fermionic field in AdS space is equal to that for massless fermionic fields in flat space-time\textsuperscript{15};

ii) the number of the DoF for totally symmetric massless field is equal to that for massless field in flat space-time (this was well known previously).

5.2 Limit of self-dual massive fields in AdS\textsubscript{5}

The limit of self-dual massive fields (1.2) is realized by taking

\[ h_1 \to |h_2|. \]

(5.7)

In view of \( h_1 = |h_2| \) and formulas (3.8), (4.7), we see that physical degrees of freedom are described by

\[ |\phi\rangle_{h_1=|h_2|} = |\phi_{h_1}\rangle, \quad \text{for bosonic fields,} \]

(5.8)

\[ |\psi\rangle_{h_1=|h_2|} = |\psi_{h_1 - \frac{1}{2}}\rangle, \quad \text{for fermionic fields.} \]

(5.9)

The spin operators \( M^{IJ} \) preserve their form given in (3.10), (4.9), while the operator \( B^I \) is reduced to the operator \( \tilde{M}^I \) and is governed by terms proportional to \( b^0_{h_1}, b^0_{h_1 - 1/2} \). By taking limit (5.7) in (3.14), (4.13), we obtain the respective coefficients for the bosonic and fermionic fields

\[ b^0_{h_1} = i(E_0 - 2) \text{sign} h_2, \quad b^0_{h_1 - 1/2} = i(E_0 - 2) \text{sign} h_2, \]

(5.10)

where \( \text{sign} h = +1(-1) \) for \( h > 0(h < 0) \). These expressions lead to the following representation for the spin operator \( B^I \):

\[ B^I = i(E_0 - 2) \text{sign} h_2 \tilde{M}^I. \]

(5.11)

Evaluating \( B^z \) and making use of formula (2.6), we obtain the AdS mass operator \textsuperscript{14}:

\[ A = -\frac{1}{2} m^{ij} m^{ij} - \frac{1}{4}, \quad m^{ij} \equiv M^{ij} - i\varepsilon^{ij}(E_0 - 2) \text{sign} h_2, \]

(5.12)

where the operator \( M^{ij} \) is given in (3.10) for bosonic fields and in (4.9) for fermionic fields.

5.3 Flat space limit

To realize the flat space limit we introduce a new coordinate \( \phi \) defined by

\[ z = R e^{\phi/R}, \]

(5.13)

and take

\[ R \to \infty, \quad \phi, x^\pm, x^i - \text{fixed}, \]

(5.14)

\textsuperscript{15}We thank K. Alkalaev for raising the question concerning the comparison of DoF for fermionic fields in AdS space and those in flat space.
where $x^\pm$, $x^i$ are four isometric coordinates along the boundary directions (see (A.1)). In this limit the light-cone actions for AdS massive fields $(2.1), (2.2)$ become the actions for massive fields propagating in flat space-time:

$$S_{l.c.} = \frac{1}{2} \int d^5x \, \langle \phi | (\Box - m^2) | \phi \rangle , \quad \text{for bosonic fields,} \quad (5.15)$$

$$S_{l.c.} = \int d^5x \, \langle \psi | \frac{i}{2\Theta^+} (\Box - m^2) | \psi \rangle , \quad \text{for fermionic fields.} \quad (5.16)$$

Note that in this limit the number of physical degrees of freedom does not change, i.e., we can use the fields $|\phi\rangle$ in (3.9) and $|\psi\rangle$ in (4.7) for description of the respective bosonic and fermionic massive fields in flat space-time. In other words arbitrary spin massive mixed-symmetry fields in AdS space and those in flat space have the same number of physical degrees of freedom. Let us comment on how the actions in flat space $(5.15), (5.16)$ can be obtained from those in AdS space $(2.1), (2.2)$.

Relations between various quantities that we are using for the description of AdS and flat space massive fields can be obtained by exploiting the two relations

$$E_0 \to Rm ,$$

$$M^{IJ} = M^{IJ} ,$$

where $M^{IJ}$ stands for the $so(d-2)$ algebra spin operator of the massive field in flat space. Here and below, we use boldface letters for the generators of the Poincaré algebra and spin operators of massive fields in flat space. Relation $(5.17)$ is well known from the study of various massive fields with particular values of spin and it can also be easily obtained from the general results of Refs. [13]. Relation $(5.18)$ reflects the fact that the spin operator of the $so(d-2)$ algebra in AdS and flat space does not depend on $R$ at all. Making use of $(5.17), (5.18)$ and of the equation for the AdS mass operator $(2.6)$ and the spin operator $B^I (2.9)$, it is easy to establish the following relations in limit as $R \to \infty$:

$$A \to R^2 m^2 ,$$

$$B^I \to Rm M^I ,$$

where the flat space spin operator $M^I$ satisfies the commutators

$$[M^I, M^K] = \delta^{IJ} M^K - \delta^{IK} M^J , \quad [M^I, M^J] = M^{IJ} . \quad (5.21)$$

From $(5.20)$, we see that the spin operator $B^I$ is an AdS counterpart of the flat space spin operator $M^I$. We note that $M^I$ and $M^{IJ}$ form the commutators of the $so(d-1)$ algebra and satisfy the following hermitian conjugation rules $M^{IJ\dagger} = -M^{JI}$, $M^{I\dagger} = M^I$. Plugging formulas $(5.13), (5.19)$ in $(2.1), (2.2)$, we see that actions $(2.1), (2.2)$ for massive fields in AdS space are indeed reduced to those in flat space $(5.15), (5.16)$. We finish with the description of interrelations between the relativistic symmetries in AdS space and those in flat space.

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16Formulas $(5.62), (5.63)$ of Ref. [13] which relate $E_0$ and the mass parameter $m$ for massive fields of arbitrary spin are given for the normalization of AdS radius $R = 1$. In order to restore the dependence on the radius $R$ space, one needs to make the rescaling $m \to Rm$ in those formulas.
In the limit (5.14) the generators of the AdS algebra given in (2.13)-(2.22) tend to those of the Poincaré algebra as

$$P^- \to P^-, \quad J^{-i} \to J^{-i}, \quad J^{+-} \to J^{+-}, \quad (5.22)$$

$$D \to R P^z, \quad (5.23)$$

$$K^i \to -\frac{1}{2} R^2 P^i + R J^{iz}, \quad (5.24)$$

$$K^+ \to -\frac{1}{2} R^2 P^+ + R J^{+z}, \quad (5.25)$$

$$K^- \to -\frac{1}{2} R^2 P^- + R J^{-z}. \quad (5.26)$$

The remaining generators of the AdS algebra (2.13),(2.16),(2.17) coincide with those of the Poincaré algebra:

$$P^i = P^i, \quad P^+ = P^+, \quad J^{+i} = J^{+i}, \quad J^{ij} = J^{ij}. \quad (5.27)$$

To make our presentation self-contained, we write the representation for generators of the Poincaré algebra which we used in establishing the relations in (5.22)-(5.27):

$$P^I = \partial^I, \quad P^+ = \partial^+, \quad P^- = \frac{-\partial^I \partial^I + m^2}{2 \partial^z}, \quad (5.28)$$

$$J^{+I} = x^+ \partial^I - x^I \partial^+, \quad (5.29)$$

$$J^{+-} = x^+ P^- - x^- \partial^+, \quad (5.30)$$

$$J^{IJ} = x^I \partial^J - x^J \partial^I + M^{IJ}, \quad (5.31)$$

$$J^{-I} = x^- \partial^I - x^I P^- + \frac{1}{\partial^+} (M^{IJ} \partial^J + m M^I ). \quad (5.32)$$

In (5.28)-(5.32), the coordinates $x^I$ stand for $x^i, \phi$. Accordingly, the derivatives $\partial_I$ stand for $\partial_i = \partial/\partial x^i; \partial_\phi = \partial/\partial \phi$. In (5.23)-(5.26), we use the identifications $P^z = P^\phi, J^{\pm z} = J^{\pm \phi}$.

6 Conclusions

The results presented here should have a number of interesting applications and generalizations, some of which are: i) In this paper, we develop a light-cone formulation for massive fields propagating in AdS$_5$. It would be interesting to extend this formulation to the study of massive arbitrary spin fields propagating in AdS$_5 \times S^5$; ii) AdS/CFT correspondence for various AdS$_d$ massive fields and the corresponding boundary operators was studied in [29, 30, 31, 32, 33] (see [34] for a review). AdS/CFT correspondence for massless and self-dual massive fields in AdS$_5$ and the corresponding boundary conformal operators has been demonstrated at the level of two point functions in [14] (see also [35] for some related interesting discussion). It would be interesting to extend the results of
this paper to the study of AdS/CFT correspondence for mixed-symmetry massive AdS fields along the lines of Refs. 36, 37, 38.

In this paper, we studied the classical fields propagating in AdS space-time. It would be interesting to extend the results of this paper to the case of quantized fields, to study the space-time correlation functions of such fields, and to clarify the relation between the positivity of the energy and the analyticity properties of the correlation functions. We hope to return to these problems in future publications.

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Appendix A Notation and conventions

We use the Poincaré parametrization of AdS space in which
\[ ds^2 = \frac{R^2}{z^2} (-dx^0 dx^0 + dx^i dx^i + dz dz + dx^4 dx^4), \]  
(A.1)
where \( R \) is the radius of AdS space, and we use indices
\[ i, j, k = 1, 2; \quad I, J, K = 1, 2, 3 \]
(A.2)
and often identify the radial coordinate \( z \) as
\[ z = x^3. \]
(A.3)
The light-cone coordinates in the ± directions are defined as
\[ x^\pm = \frac{1}{\sqrt{2}}(x^4 \pm x^0) \]
(A.4)
and \( x^+ \) is taken to be the light-cone time. We adopt the conventions
\[ \partial^I = \partial_I \equiv \partial/\partial x^I, \quad \partial^\pm = \partial_{\mp} \equiv \partial/\partial x^{\pm}. \]
(A.5)
The scalar product of the \( so(3) \) algebra vectors \( X^I = (X^i, X^z), Y^I = (Y^i, Y^z) \) is given by
\[ X^I Y^I = X^i Y^i + X^z Y^z, \quad X^3 \equiv X^z, \quad Y^3 \equiv Y^z. \]
(A.6)
We use \( 4 \times 4 \) Dirac \( so(4, 1) \) \( \gamma \)-matrices:
\[ \{ \gamma^a, \gamma^b \} = 2\eta^{ab}, \quad a, b = 0, 1, \ldots, 4, \]
(A.7)
where \( \eta^{ab} \) is the mostly positive flat metric tensor. The \( \gamma \)-matrices are normalized as
\[ \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4 = i. \]
(A.8)
In the light-cone frame, the \( \gamma \)-matrices are decomposed in the standard way
\[ \gamma^a = \gamma^+, \gamma^-, \gamma^I, \quad \gamma^\pm \equiv \frac{1}{\sqrt{2}}(\gamma^4 \pm \gamma^0). \]
(A.9)
The four component Dirac spinor field of the $so(4,1)$ algebra $\psi_{Dirac}$ can be decomposed as

$$\psi_{Dirac} = \psi^\oplus + \psi^\ominus, \quad \psi^\oplus \equiv \Pi^\oplus \psi_{Dirac}, \quad \psi^\ominus \equiv \Pi^\ominus \psi_{Dirac},$$

(A.10)

where the standard projectors $\Pi^\oplus, \Pi^\ominus$ defined by

$$\Pi^\oplus \equiv \frac{1}{2} \gamma^- \gamma^+, \quad \Pi^\ominus \equiv \frac{1}{2} \gamma^+ \gamma^-,$$

(A.11)

satisfy the relations

$$\Pi^\oplus \Pi^\oplus = \Pi^\oplus, \quad \Pi^\ominus \Pi^\ominus = \Pi^\ominus, \quad \Pi^\ominus \Pi^\oplus = 0,$$

(A.12)

$$\Pi^\oplus + \Pi^\ominus = 1.$$  \hspace{1cm} (A.13)

In the light-cone formalism, the field $\psi^\oplus$ turns out to be physical field, while the field $\psi^\ominus$ is a non-physical field (for more details concerning the spin one-half Dirac field, see Appendix B). Therefore, the light-cone action is formulated entirely in terms of the field $\psi^\oplus$.

It turns out that the relations for $\gamma$-matrices are simplified on the space of physical fields $\psi^\oplus$. This is to say that normalization (A.8) leads to the relation

$$\gamma^{IJ} = i\epsilon^{IJK}\gamma^K (\Pi^\oplus - \Pi^\ominus),$$

(A.14)

where $\epsilon^{IJK}$ is Levi-Civita symbol subject to the normalization $\epsilon^{123} = 1$. Applying relation (A.14) to $\psi^\oplus$ and using the first formula in (A.12) we obtain the simplified formula

$$\gamma^{IJ}\psi^\oplus = i\epsilon^{IJK}\gamma^K \psi^\oplus.$$  \hspace{1cm} (A.15)

This formula implies that the $4 \times 4$ $\gamma^I$-matrices being restricted to the space of physical fields $\psi^\oplus$ satisfy the same relations as the standard $2 \times 2$ Pauli matrices. This property of the $4 \times 4$ $\gamma^I$-matrices is used throughout this paper. Note that in main body of the paper we use the simplified notation $\psi$ for the physical field $\psi^\oplus$ (i.e., we drop the superscript $\oplus$).

### Appendix B Derivation of light-cone action for spin one-half Dirac field

Here we discuss the transformation of the Lorentz covariant action for the spin one-half Dirac field to the light-cone action given in (2.2),(2.2). The standard Lorentz covariant action for the spin one-half Dirac field in $AdS_d$

$$S = - \int d^d x e \bar{\Psi} (\gamma^\mu D_\mu + m) \Psi,$$

(B.1)

considered in Poincaré coordinates (A.1), takes the form

$$S = - \int d^d x \left( \frac{z}{R} \right)^{-d} \bar{\Psi} \left( \frac{z}{R} \gamma^\alpha \partial_\alpha + \frac{1-d}{2R} \gamma^z + m \right) \Psi.$$  \hspace{1cm} (B.2)
Introducing the canonically normalized field $\psi$,
\[
\Psi = \left(\frac{z}{R}\right)^{\frac{d-4}{2}} \psi
\]  
we obtain
\[
S = -\int d^d x \; \bar{\psi} \left( \gamma^a \partial_a + \frac{Rm}{z} \right) \psi. \tag{B.4}
\]

Making use of the light-cone splitting for the Dirac spinor
\[
\psi^\oplus = q \Pi^\oplus \psi, \quad \psi^\ominus = q \Pi^\ominus \psi, \quad q^2 = \sqrt{2}, \tag{B.5}
\]
where compared to (A.10) we use an extra normalization factor $q$, we obtain the following light-cone representation for the action (B.4):
\[
S = \frac{i}{2} \int d^d x \; \left( 2 \psi^{\oplus\dagger} \partial^- \psi^\oplus - 2 \psi^{\ominus\dagger} \partial^+ \psi^\ominus \right.
\left. - \psi^{\oplus\dagger} (\gamma^I \partial^I - \frac{Rm}{z}) \gamma^- \psi^\ominus + \psi^{\ominus\dagger} (\gamma^I \partial^I - \frac{Rm}{z}) \gamma^+ \psi^\oplus \right). \tag{B.6}
\]

This action leads to the equation for the field $\psi^\ominus$,
\[
\partial^+ \psi^\ominus = \frac{1}{2} (\gamma^I \partial^I - \frac{Rm}{z}) \gamma^+ \psi^\oplus, \tag{B.7}
\]
which tells us that $\psi^\ominus$ is not a dynamical field and can be expressed in terms of the physical field $\psi^\oplus$. Thus, by inserting the solution for $\psi^\ominus$ into action (B.6), we obtain the light-cone action for the physical field $\psi^\oplus$:
\[
S_{l.c.} = \int d^d x \; \psi^{\oplus\dagger} \left( \Box - \frac{1}{z^2} \left( (Rm)^2 + Rm \gamma^2 \right) \right) \psi^\oplus. \tag{B.8}
\]

Comparison of (B.8) and (2.2) leads to the AdS mass operator given in (2.5).

**Appendix C  Derivation of expression for spin operator $B^I$.**

Here, we describe the method of solving equations for the operator $B^I$ given in (2.9)- (2.11). Unitarity equation (2.11) and so(3) covariance equation (2.10) are easiest to treat. The main problem is to solve Eq. (2.9), which is nonlinear in the spin operator $M^{IJ}$. Fortunately, it turns out that Eq. (2.9) can be reduced to the analysis of commutation relations for spin operators of the so(4) algebra. We demonstrate our method for the bosonic fields. Extension to the fermionic fields is straightforward. Our method consists of the following steps:

i) First, we start with the analysis of Eq. (2.9). Our basic observation is that a solution of Eq. (2.9) can be presented as
\[
B^I = y M^I - M^{IJ} M^J, \quad y \equiv E_0 - 1, \tag{C.1}
\]
provided a new spin operator $M^I$ that transforms in the vector representation with respect to $so(3)$ transformations generated by $M^{IJ}$ satisfies the commutation relation

$$[M^I, M^J] = M^{IJ},$$

and the constraint

$$M^I M^I - \frac{1}{2} M^{IJ} M^{IJ} + \langle Q_{so(4)} \rangle = 0,$$

$$-\langle Q_{so(4)} \rangle \equiv h_1(h_1 + 2) + h_2^2,$$

where $\langle Q_{so(4)} \rangle$ is an eigenvalue of the second order Casimir operator for irreps of the $so(4)$ algebra labelled by $h = (h_1, h_2)$. Because the operator $M^I$ transforms in the vector representation of the $so(3)$ algebra transformation generated by the spin operator $M^{IJ}$, the operators $M^I$ and $M^{IJ}$ form commutators of the $so(4)$ algebra. Thus, we reduced Eq. (2.9) to the problem of solving Eqs. (C.2), (C.3). Compared to non-linear Eq. (2.9), equations (C.2), (C.3) are easier to solve because they are the standard equations for the spin operator of the $so(4)$ algebra. We begin with the analysis of Eq. (C.2). To this end, we write the most general $so(3)$ covariant expression for the action of the operator $M^I$ on the ket-vectors $|\phi_{s'}\rangle$:

$$M^I |\phi_{s'}\rangle = m^0_{s'} \tilde{M}^I |\phi_{s'}\rangle + m^+_s A^I_{s'} + m^-_{s'} \alpha^I |\phi_{s'+1}\rangle,$$

where the coefficients $m^0_{s'}$, $m^\pm_s$ are still to be fixed. The representation for $M^I$ given in (C.5) is fixed simply by the requirement that action of $M^I$ on $|\phi_{s'}\rangle$ should respect the constraints given in (3.6), (3.7). Equation (C.5) should be supplemented by the 'initial' conditions

$$m^-_{|h_2|} = 0, \quad m^+_h = 0$$

which express the fact that the action of the operator $M^I$ is defined on those vectors $|\phi_{s'}\rangle$ whose spin $s'$ takes values in the domain $s' = |h_2|, \ldots, h_1$. Now we demonstrate that the coefficients $m^0_{s'}$, $m^\pm_s$ can be found by analyzing Eq. (C.2) and 'initial' conditions (C.6). To this end, we write the action of two spin operators $M^I$ on $|\phi_{s'}\rangle$:

$$M^I M^J |\phi_{s'}\rangle = (m^0_{s'} \tilde{M}^I \tilde{M}^J + m^-_{s'} A^I_{s'-1} A^J_{s'-1} \alpha^J + m^+_s \alpha^I \alpha^J A^J_{s'}) |\phi_{s'}\rangle$$

$$+ m^-_{s'-1} A^I_{s'-1} A^J_{s'-2} |\phi_{s'-2}\rangle$$

$$+ (m^0_{s'} m^-_{s'} \tilde{M}^I A^J_{s'-1} + m^-_{s'} m^0_{s'-1} A^I_{s'-1} \tilde{M}^J) |\phi_{s'-1}\rangle$$

$$+ (m^0_{s'} m^+_s \tilde{M}^I \alpha^J + m^+_s m^0_s \alpha^I \tilde{M}^J) |\phi_{s'+1}\rangle$$

$$+ m^+_s m^+_s \alpha^I \alpha^J |\phi_{s'+2}\rangle.$$

From this formula and (3.19)-(3.25), we find that commutator (C.2) gives the equations

$$s' m^0_{s'} - (s' + 2) m^0_{s'+1} = 0,$$

$$-m^0_{s'} + m^-_s m^+_{s'-1} - \frac{2s' + 3}{2s'} m^-_{s'+1} m^+_s = 1.$$

20
Equations (C.8), (C.9) and (C.6) lead to the desired relations
\[ m_s^0 m^+_s = \frac{((h_1 + 1)^2 - s'^2)(s'^2 - h_2^2)}{(2s' + 1)s'^2}, \]  
(C.10)

\[ m^0_s = \frac{(h_1 + 1)h_2}{s'(s' + 1)}. \]  
(C.11)

Making use of (C.10), (C.11) and the formulas
\[ M^{I}M^{I} | \phi_{s'} \rangle = \left( h_1(h_1 + 2) + h_2^2 - s'(s' + 1) \right) | \phi_{s'} \rangle, \]  
(C.12)

\[ -\frac{1}{2}M^{IJ}M^{IJ} | \phi_{s'} \rangle = s'(s' + 1) | \phi_{s'} \rangle \]  
(C.13)

we check that Eq. (C.3) is satisfied automatically, i.e., Eq. (C.3) does not impose additional constraints on the coefficients \( m^0_s, m^+_s \). Helpful relations in deriving (C.12) are
\[ A^I_{s+1}A^I_s | \phi_s \rangle = 0, \]  
(C.14)

\[ A^I_{s-1} \tilde{\alpha}^I | \phi_s \rangle = s | \phi_s \rangle, \]  
(C.15)

\[ \tilde{\alpha}^I A^I_s | \phi_s \rangle = \frac{(2s + 3)(s + 1)}{2s + 1} | \phi_s \rangle. \]  
(C.16)

ii) Second, we analyze the requirement that the spin operator \( B^I \) be hermitian with respect to the scalar product defined in (3.9). Making use of formulas (C.1), (C.5), we find the following representation for the operator \( B^I \):
\[ B^I | \phi_{s'} \rangle = b^0_{s'} \tilde{M}^I | \phi_{s'} \rangle + b^-_{s'} A^I_{s'-1} | \phi_{s'-1} \rangle + b^+_{s'} \tilde{\alpha}^I | \phi_{s'+1} \rangle, \]  
(C.17)

where coefficients \( b^0_{s'}, b^\pm_{s'} \) are expressible in terms of \( m^0_s, m^+_s \) as
\[ b^-_{s'} = (y - s' - 1)m^-_{s'}, \]  
(C.18)

\[ b^+_{s'} = (y + s')m^+_{s'}, \]  
(C.19)

\[ b^0_{s'} = (y - 1)m^0_{s'}. \]  
(C.20)

These relations can be obtained from (C.1), (C.5) by using the formulas
\[ M^{IJ}A^I_s | \phi_s \rangle = (s + 2)A^I_s | \phi_s \rangle, \]  
(C.21)

\[ M^{IJ} \tilde{\alpha}^I | \phi_s \rangle = (1 - s)\tilde{\alpha}^I | \phi_s \rangle, \]  
(C.22)

\[ M^{IJ} \tilde{M}^J = \tilde{M}^I. \]  
(C.23)

The expression for \( y \) in (C.1) and formulas (C.11), (C.20) lead to \( b^0_{s'} \) given in (3.14).

Making use of the representation for the operator \( B^I \) given in (C.17), we find that the requirement for the operator \( B^I \) to be hermitian with respect to the scalar product in (3.9)
\[ \langle \varphi | B^I \varphi \rangle = \langle B^I \varphi | \varphi \rangle \]  
(C.24)
leads to the relations

\[ b_{s'}^+ = b_{s'+1}^- , \quad b_{s'}^0 = -b_{s'}^{0*} . \]  \hspace{1cm} (C.25)

Note that the coefficient \( b_{s'}^0 \) (see (C.11), (C.21)) satisfies unitarity requirement (C.25) automatically. Now we are ready to demonstrate that equations (C.10), relations (C.18), (C.19), and unitarity requirement (C.25) allow us to fix the coefficients \( b_{s'}^\pm \). Indeed, multiplying \( b_{s'}^- \) by \( b_{s'-1}^+ \) and using formulas (C.10), (C.18), (C.19) we obtain

\[ b_{s'}^- b_{s'-1}^+ = \frac{((h_1 + 1)^2 - s'^2)(s'^2 - h_2^2)}{(2s' + 1)s'^2} \left((E_0 - 2)^2 - s'^2\right) . \]  \hspace{1cm} (C.26)

From this formula and the first relation in (C.25), we obtain the solution for the coefficients \( b_{s'}^\pm \) given in (3.12), (3.13).
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