The rôle of city geometry in determining the utility of a small urban light rail/tram system

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Abstract

In this work, we show the importance of considering a city’s shape, as much as its population density figures, in urban transport planning. We consider in particular cities that are “circular” (the most common shape) compared to those that are “rectangular”. For the latter case we show greater utility for a single line light rail/tram system. A particular case study is presented for Galway City.

1 Introduction

There are many factors to consider when constructing a light rail / tram system in a city. Some of these factors can be scientifically analyzed (as is the case in this paper), but others perhaps not (aspects that are political, sociological, financial,...). These latter aspects are of course important, but are not analyzed here.

Further, amongst considerations that lend themselves to scientific analysis, we restrict further to small tram systems, indeed while we look a little at a two line system, most of this paper considers a single line system. To make the model tractable, in this current work we restrict ourselves to circular or rectangular “geometries” (which should none the less approximate many real world city shapes).

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2 City Geometries

2.1 Rectangular vs. circular

We analyze here idealized models where the city shape is either rectangular (of dimensions $p$ (kilometres) times $l$, where $p$ is the short side of the rectangle and $l \geq 1$) or circular (of radius $r$ kilometres). For the rectangle, $p$ defines its “size”, while $l$ defines its “shape” (varying from $l = 1$ (square) to large values of $l$ for a “long”, “skinny” rectangle). As a (very) simplifying assumption, we assume uniform population density inside the city shape - the results of the analysis in this paper only apply in cities that approximate this distribution.

Assuming a uniform population density inside the city, let us suppose a person wants to go from a random location $i$ to a random location $j$ inside the city. One question we may ask is, what is the average length $d$ of such trips? It should be clear that this average distance will be an increasing function of all our parameters $r, p, l$. It should also be intuitively clear that, given a “rectangular” city and a “circular” one of equal area (and the aforementioned uniform population density), if $l \gg 1$ then $d$ will be greater in the rectangular case. These mathematical questions are studied in the discipline of Geometric Probability, see [1, 7].

The analysis of [6] gives for the circular case

$$d_{\text{circ}} = \frac{128r}{45\pi},$$

while for the rectangular case [8] gives

$$d_{\text{rect}} = \frac{p}{15} \left[ l^3 + \frac{1}{l^2} + \sqrt{l^2 + 1} \right] \left[ 3 - l^2 - \frac{1}{l^2} + \frac{5}{2} \ln \left( l + \sqrt{l^2 + 1} \right) + \frac{5l^2}{2} \ln \left( \frac{1 + \sqrt{l^2 + 1}}{l} \right) \right].$$

A natural question to ask is, how does $d_{\text{rect}}$ vary, for a city of fixed area, as the shape varies from square to a long/thin rectangle? For fixed $A = lp^2$, the $l$ dependence is given by

$$d_{\text{rect}} = \frac{\sqrt{A}}{15\sqrt{l}} \left[ l^3 + \frac{1}{l^2} + \sqrt{l^2 + 1} \right] \left[ 3 - l^2 - \frac{1}{l^2} + \frac{5}{2} \ln \left( l + \sqrt{l^2 + 1} \right) + \frac{5l^2}{2} \ln \left( \frac{1 + \sqrt{l^2 + 1}}{l} \right) \right].$$

which presents as a curve with a reasonably uniform slope close to $\sqrt{A}/15$ (see figure 1). Note further that for the case of a square ($l = 1$), equation (3) gives

$$d_{\text{square}} = \left( \frac{2 + \sqrt{2} + 5\ln(1 + \sqrt{2})}{15} \right) p \approx 0.52p$$

which is slightly larger than [1] (setting $p = 2r$), as expected.

**Observation 1.** For rectangular cities, the average distance travelled by inhabitants is larger the more rectangular the city is. Therefore, the need for a rail/tram system is larger.

2.2 Where to put the rail/tram line?

In what follows, we will just use the word “tram”, instead of mentioning rail/tram each time.

2.2.1 Rectangular city

Suppose we put a single line tram in our “rectangular” city. Assume that residents will consider using the tram if they live within $d_t$ kilometres of it (later we will set $d_t$ to 0.5km). In our idealized model, we put the tram along the centre of the rectangle, running along the longer side, stopping at each end $d_t$ from the end. To explain it using the cartesian plane, if our rectangular city is placed with the shorter side going from $(0, 0)$ to $(0, p)$, and the longer side from $(0, 0)$ to $(lp, 0)$, then the tram line will run from $(d_t, p/2)$ to $(lp - d_t, p/2)$. We assume the tram has $n$ stops along the line (including the ends), and an average of $\tau$ minutes to go between stops (so one full run along the line takes $\tau(n - 1)$ minutes).

The average speed of the journey between any two tram stations for the rectangular city is simply

$$\bar{v}_{\text{rect}} = \frac{lp - 2d_t}{(n - 1)\tau}$$

1So, the results here will not apply in “large” cities, where there are skyscrapers / tower blocks / large apartment blocks in city centres. They will also not apply in small cities in countries where there is a tradition of people living in apartment blocks in city centres (much of continental Europe, for example). But our results will apply in smaller cities in USA, UK, Ireland, for example, which generally do not have large apartment blocks in their centres.
Figure 1: The average distance ($\bar{d}_{\text{rect}}$) between two randomly chosen points in a rectangle of dimensions $(p) \times (lp)$, and fixed area $A = lp^2$. The curve in blue (triangles) is a plot of equation (3). The intercept on the y-axis ($l = 1$) corresponds to a square (see equation (4)).

2.2.2 Circular city

In what follows, we assume a circular city of equal area to the rectangular one (and of equal population, hence of equal population density). We have that $lp^2 = A = \pi r^2$, where $r$ is the radius (in kilometres) of the city. Since we assumed $l \geq 1$, $p$ must be less than the diameter of the circle ($p < 2r$).

1 line

Starting at $l = 1$, our rectangular city is a square. Since $lp^2 = A = \pi r^2 = p^2$, we have that $p = r \sqrt{\pi} \approx 1.77r$, so the sides of the square are less than the circle diameter. As we increase $l$, keeping areas constant, our line of length $lp - 2d_t$ becomes longer until its length equals that of the longest straight line (tram) we put in the circle, of length $2r - 2d_t$, so we get

$$lp = 2r = 2p \sqrt{\frac{T}{\pi}} \Rightarrow \sqrt{l} = \frac{2}{\sqrt{\pi}} \Rightarrow l = \frac{4}{\pi} \approx 1.27$$

(6)

2 lines

Increasing $l$ further above $4/\pi$, we match this (one) line in the rectangle with two intersecting lines in the circle, whose lengths sum to $lp - 2d_t$. In our idealized model, we run the two lines at right angles to one another, intersecting at the centre of the circle. As $l$ increases, eventually the sum of the maximum lengths of the two lines in the circle ($4r - 4d_t$) must match $lp - 2d_t$, so we get

$$4r - 4d_t = lp = lr \sqrt{\frac{\pi}{T}} = r \sqrt{\pi l} \Rightarrow \sqrt{l} = \frac{2}{\sqrt{\pi}} \left(2 - \frac{d_t}{r}\right) \Rightarrow l \approx \frac{16}{\pi} \left(1 - \frac{d_t}{r}\right)$$

(7)

where we ignore the terms quadratic in $d_t/r$. We anticipate (later) that $d_t/r$ will be approximately 0.1, and we may ignore this in later approximations.

Observation 2. Obviously the circular city with 2 lines also demands other infrastructure - an intersection tram station at the center, where the two lines meet, to allow passengers to swap lines. This junction may imply further cost and/or restrictions:

1. Cost: Build a full overpass/underpass system (bridge) so trams pass freely.

2. Restriction: In the absence of a bridge, there may be a restriction on schedules (2 trams can’t pass the junction at the same time), perhaps causing inferior service.

3 “Quality of Service”

While we cannot “guesstimate” scientifically how many people may use a tram service, we can say that the better the quality of service, the more will use it. Two main factors (amongst others) that affect quality of service are
1. Frequency of service
2. Time to get “from A to B”

These considerations may not be independent: In the rectangular model, the travel time does not depend on the service frequency, but in the circular model, it may, in trips where one has to transfer lines. In this section we analyze the average time required to get “from A to B” in our “circular” and “rectangular” cities. Let $\lambda$ be the average distance between neighboring stops on the line, so, the average speed is

$$\bar{s} = \frac{\lambda}{\tau}$$

(8)

If there is a tram every $t_f$ minutes (frequency of service), the average waiting time for a tram is $t_f/2$: It will be convenient in what follows to write this time in terms of $\tau$,

$$t_f/2 = q\tau$$

(9)

and we will consider situations where $q = 1, 2, 3, \ldots$

We will call the region of service of the tram the set of all points $(x,y)$ that are within distance $d_t$ of the tram line. Within this region of service, we now compare the average speed of a journey between two tram stations, for the case defined by equation (7), for our circular and rectangular cities. (Note that for the case of equation (6), this average speed will be identical in both cases, and equal to $\lambda/\tau$.)

Rectangular city: The average speed is $\lambda/\tau$.

Circular city: For journeys using only one single line (i.e. from stations on the blue line to other stations on the blue line, or from stations on the red line to other stations on the red line), the average speed is $\lambda/\tau$ as before. However, if we make a journey between two randomly chosen stations, half of these journeys involve changing line, and we will show this results in considerably reduced speed.

Letting the origin be at the usual position of the intersection of the $x$ (red) and $y$ (blue) lines, we label the stations with integers $i$ (along $x$) and $j$ (along $y$), so that the average distance between station $i$ and $j$ is $\lambda\sqrt{i^2 + j^2}$. There is a certain amount of waiting time at the junction at $(0,0)$, which we write in terms of $\tau$ as described in equation (9). Then the total transit time from $i$ to $j$ is $(i+j+q)\tau$. This results in an average speed from $i$ to $j$ of

$$s_{ij} = \frac{\lambda\sqrt{i^2 + j^2}}{(i+j+q)\tau}$$

(10)

Since the numerator is less than $(i+j)\lambda$, and the denominator is greater than $(i+j)\tau$, this speed can be considerably less than $\lambda/\tau$. Averaging over all $i,j \leq m$ gives an average speed of

$$\bar{s} = \frac{1}{m^2} \left( \frac{\lambda}{\tau} \right) \sum_{i,j=1}^{m} \frac{\sqrt{i^2 + j^2}}{(i+j+q)}$$

(11)

We plot $\bar{s}$ as a function of $m$ and $q$ in figure 3.

**Observation 3.** Note that for a substantial range of the parameters $m$ and $q$, the average speed falls to 50%.

**Observation 4.** For fixed $m$, the larger the value of $q$ (i.e. the larger the wait at the junction), the smaller the overall point-to-point speed.

**Observation 5.** For fixed values of $q$, as we increase $m$ we reduce the impact of $q$ on the overall speed.
Figure 3: The average speed between randomly chosen stations on different tram lines (see figure 2). All parameters are dimensionless. The average speed is written as a percentage of \( \bar{s} = \lambda / \tau \) (see equation 5). \( m \) is the total number of stations (counting from the origin/junction point of the two lines) while \( q \) is the average waiting time at the junction as a multiple of \( \tau \) (see equation 6).

4 Infeasible Regions

For all choices of departure point \((x_1, y_1)\) and arrival point \((x_2, y_2)\), we now compare travelling with or without the tram. If we fix a particular departure point \((x_1, y_1)\) and consider all possible arrival points \((x_2, y_2)\), it is intuitively clear that for some arrival points, using the tram would serve no purpose (we say the journey is infeasible via the tram). In this section we define this notion precisely, and calculate what proportion of trips are infeasible. We show that the infeasible region for a circular city is larger than that for a rectangular city.

4.1 Metrics

We denote by \(d_{\text{euc}}\) the standard Euclidean metric on the plane \( \mathbb{R}^2 \), with the distance between two points \( p_1, p_2 \in \mathbb{R}^2 \) given by

\[
d_{\text{euc}}(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

(12)

where \( p_1 = (x_1, y_1) \) and \( p_2 = (x_2, y_2) \).

We now define a metric for measuring the (effective) distance for the journey between \( p_1 \) and \( p_2 \) using the tram (see [2] for general information on metrics). This is comprised of three parts:

1. The journey (not using tram) from \( p_1 \) to the nearest tram station
2. The journey on the tram to the station nearest the destination
3. The journey (not using tram) from that station to the destination \( p_2 \)

To calculate this we need to determine the nearest point on the tram line to an arbitrary point \( p = (x, y) \). Let our tram line run between points \((-a,0)\) and \((a,0)\), so it is represented by a horizontal line segment centered at the origin \((0,0)\). Let us define

\[
x_t = \begin{cases} 
-a, & x < -a \\
x, & -a \leq x \leq a \\
a, & x > a
\end{cases}
\]

(13)

or equivalently

\[
x_t = \begin{cases} 
\max(x,-a), & x < 0 \\
\min(x,a), & x > 0
\end{cases}
\]

(14)

The nearest point on the tram line to any point \( p = (x, y) \) is \( \text{Proj}(p) = (x_t, 0) \). Our tram metric \( d_{\text{tr}} \) is

\[
d_{\text{tr}}(p_1, p_2) = d_{\text{euc}}(p_1, \text{Proj}(p_1)) + |x_1^t - x_2^t| / (\bar{s}_{\text{nt}} / \bar{s}_t) + d_{\text{euc}}(\text{Proj}(p_2), p_2)
\]

(15)

where \( \bar{s}_t \) is the average speed along the tram line, while \( \bar{s}_{\text{nt}} \) is the average speed on the other two (non-tram) segments of the journey. Note here that the physical distance \( |x_1^t - x_2^t| \) along the tram line is reduced by a factor of \( \bar{s}_{\text{nt}} / \bar{s}_t \) because of the superior speed of the tram \( \bar{s}_t \) (compared to \( \bar{s}_{\text{nt}} \)).

\[\text{For example: By foot, } \bar{s}_{\text{nt}} \approx 0.1 \text{ km/minute, by bicycle } \bar{s}_{\text{nt}} \approx 0.2 \text{ km/minute. On the Paris metro, } \bar{s}_t \approx 0.5 \text{ (measured by the author on line 4, between Jussieu and Mairie d’Ivry, November 2018), while the Dublin Luas has } \bar{s}_t \approx 0.28.\]
4.2 Infeasible Journeys

To compare $d_{\text{euc}}(p_1, p_2)$ with $d_{\text{tr}}(p_1, p_2)$ we have to consider one further term: Travelling via the tram, when one arrives at the point $\text{Proj}(p_1)$ one must wait on average time $t_f/2$ for a tram. In this time, if the traveller had not taken the tram, they could have travelled a distance of $\bar{s}_{nt} t_f/2$ kilometers. Thus, we should compare $d_{\text{euc}}(p_1, p_2)$ with $d_{\text{tr}}(p_1, p_2) + \bar{s}_{nt} t_f/2$.

**Definition 1.** We say a journey between points $p_1$ and $p_2$ is $(\alpha, \beta)$–infeasible if and only if

$$d_{\text{euc}}(p_1, p_2) - d_{\text{tr}}(p_1, p_2) < \bar{s}_{nt} t_f/2$$

where $\alpha = \bar{s}_t/\bar{s}_{nt}$ is the tram speed relative to the non-tram speed, and $\beta = t_f/2$ is the average waiting time.

**Definition 2.** We say a journey is infeasible if and only if it is $(\infty, 0)$–infeasible.

Definition 2 corresponds to an ideal world of zero waiting time for the tram, and infinite tram speed!

**Definition 3.** The infeasible region corresponding to a point $p$ is the set of all points $q$ such that the journey from $p$ to $q$ is an infeasible journey. We denote this using the function

$$I : \mathbb{R}^2 \to 2^{\mathbb{R}^2}, I(p) = \{ q \in \mathbb{R}^2 | d_{\text{euc}}(p, q) < d_{\text{tr}}(p, q) \}$$

For a city whose area is $A$ (we will soon denote by $R$ the area of a rectangular city, and by $C$ the area of a circular city), we further define

**Definition 4.** The Infeasibility Factor (IF($p$)), corresponding to a particular departure point $p$ within the area of the city, is the area of $I(p)$ relative to the overall city area, expressed as a percentage, i.e.

$$\text{IF}(p) = (100) \frac{\iint_{I(p)} dx \, dy}{\iint_{\mathbb{R}^2} dx \, dy}$$

**Definition 5.** The Infeasibility Factor (IF) for the city as a whole is the average over all points of the Infeasibility Factors for each point,

$$IF = \frac{\iint_A \text{IF}((x, y)) \, dx \, dy}{\iint_A dx \, dy},$$

where we write $p$ as $(x, y)$.

In Figure 4 we show examples of infeasible regions for a number points, superimposed on a circular city and two different rectangular cities. In Appendix A we present further plots showing infeasible regions for various values of $\alpha$ and $\beta$. 


Figure 4: Infeasible regions for four different (departure) points. In each image, the point is a bold black dot and the tram line is a solid (red) line centered at (0, 0). Three different city shapes, of equal area (about 300 square kilometres), are indicated: (i) circular (blue), (ii) rectangular with $l = 3$ (grey) and (iii) rectangular with $l = 4$ (yellow). The infeasible region is shaded in (red) x signs. Because of symmetries, our examples are all of points in the upper right quadrant.

4.3 Infeasibility in circular and rectangular cases

We carried out the calculations for Figure 4 by discretizing a grid of $40 \times 40$ nodes, and writing code in PYTHON, using equation (16), to determine infeasible journeys.

For each city shape, we calculate the Infeasibility Factor for each departure point $p = (x, y)$. From Figure 4, this corresponds to the area marked by (red) x signs within the city, divided by the total city area. In Figure 5 we show the dependence of $IF(p)$ on the position $p = (x, y)$ for a rectangular city with $l = 3$. Figure 6 presents the corresponding results for a circular city.
Figure 5: Infeasibility Factors for a rectangular city with $a = 5, l = 3$.

Figure 6: Infeasibility Factors for a circular city with $a = 5$.

The infeasibility Factors, for a range of realistic values of $\alpha$ and $\beta$, in the circular and rectangular cases, are presented in Appendix B. From Figures 9 and 10 we note
Observation 6. The infeasibility factors for the circular case are higher in all cases (except for Figures 9 (f) and 10 (f), where they are identical) than for the rectangular case. The difference is particularly noticeable comparing Figure 9 (b), (c), (d), (e) with 10 (b), (c), (d), (e), where it is over 20%.

We discretize equation 16 to calculate a single number (percentage) for each city shape, representing an average over \( p \) of all \( IF(p) \). This number still depends on

- \( \alpha \)
- \( \beta \)
- \( a \) (fixed by the length of the tram line)
- the city shape

We fix \( a = r/2 \) and present in Figure 7 results for the Infeasibility Factor (IF) as it depends on \( \alpha \) and \( \beta \) for the rectangular and circular cases.

We present in Appendix C a case study for Galway City (Ireland), a rectangular city whose “length” is about 3 times its “width” \( (l = 3) \).

5 Case Study: Galway City

We present in Appendix C a case study for Galway City (Ireland), a rectangular city whose “length” is about 3 times its “width” \( (l = 3) \).

6 Conclusions

We have presented a model here that enables mathematical calculation of the utility of a light rail / tram line in a city of a certain shape. We have presented in detail the calculations for a single line tram in rectangular and circular cities, showing the lower infeasibility factors for such a tram in the rectangular case: We therefore argue that building such a tram in a rectangular city is more feasible than in a circular city. This is illustrated in the case study for Galway City.

For further work we envisage the following:
• We will investigate the dependence of the infeasibility factors, not just on the tram speeds and tram frequencies, but also on the length of the tram line (relative to the city size).

• We will construct a more elaborate model, using the metrics presented here, for more elaborate tram networks with multiple (intersecting) lines. This should model larger real-world city transit systems.

• In our calculations, we assume all journeys are equally probable. This assumption could be relaxed, building a model which calculates infeasibility factors that are weighted averages of different journeys. In real-world scenarios, the probability of going from point $i$ to $j$ depends not just on population densities (which in any case will not be uniform), but on other features. For example, one or other of $i$, $j$ may correspond to a place of work, a hospital, a University, a transport hub, a shopping centre: Journeys to/from these locations may have higher weightings, even though the population densities at these locations may be lower. (Thanks to Ulf Strohmayer for pointing this out.)

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Appendices

A Examples of \((\alpha,\beta)\)–infeasible regions

To help the reader appreciate the role of the tram speed, and the frequency of service, on the infeasible regions, we show in Figure 8 some further plots of infeasible regions for some realistic values of \(\alpha\) and \(\beta\) and departure point \((8,3)\). We remind the reader that \(\alpha = \bar{s}_t / \bar{s}_nt\) measures the (relative) tram speed while \(\beta = t_f / 2\) measures the frequency. Note

- Figure 8(a) corresponds to the extreme case scenario where the tram does not move (its speed is zero) and/or the time interval between trams is infinite. In this (ridiculous!) situation, obviously the infeasibility factor is 100% (there is no reason to take such a tram!)

- Plots (c) and (d) of Figure 8 are very similar: (d) has higher speed trams, while (c) has lower speed trams but more frequent ones.

B Infeasibility Factor Examples

We present in Figure 9 plots of \(IF(p)\) for various values of \(\alpha\) and \(\beta\). Our city here has area 300 square kilometres, and is a rectangle of size 10 \(\times\) 30. Figures 9(a) and 9(f) present the extreme case scenarios with fewest/most infeasible journeys, respectively. As we would expect, points at right angles to the tram line show highest Infeasibility Factors for example points \((5,0)\) or \((-5,0)\) in Figures 9(b), (c), (d), (e). (Figure 9(a) is identical to Figure 5, where a different coloring scheme is used for the contours.)

In Figure 10 we present the corresponding contour plots for a circular city of approximately equal area (radius 10km).
(a) \((\alpha, \beta) = (0, \infty)\). \(IF((8, 3)) = 100\%\) (see Definition 4).

(b) \((\alpha, \beta) = (3, 10)\). \(IF((8, 3)) \approx 66/49/48\%\) for the circle/\(l = 3/l = 4\) cases respectively.

(c) \((\alpha, \beta) = (3, 5)\). \(IF((8, 3)) \approx 61/46/45\%\) for the circle/\(l = 3/l = 4\) cases respectively.

(d) \((\alpha, \beta) = (5, 10)\). \(IF((8, 3)) \approx 58/46/45\%\) for the circle/\(l = 3/l = 4\) cases respectively.

(e) \((\alpha, \beta) = (5, 5)\). \(IF((8, 3)) \approx 54/43/43\%\) for the circle/\(l = 3/l = 4\) cases respectively.

(f) \((\alpha, \beta) = (\infty, 0)\). \(IF((8, 3)) \approx 42/38/40\%\) for the circle/\(l = 3/l = 4\) cases respectively.

Figure 8: Infeasible regions for the (departure) point \((8, 3)\) with six different pairs of \((\alpha, \beta)\) values. The departure point is marked with a bold black dot and the tram line is a solid (red) line centered at \((0, 0)\). Three different city shapes, of equal area (about 300 square kilometres), are indicated: (i) circular (blue), (ii) rectangular with \(l = 3\) (grey) and (iii) rectangular with \(l = 4\) (yellow). The infeasible region is shaded in (red) \(x\) signs.
(a) $(\alpha, \beta) = (\infty, 0), IF = 37\%$
(b) $(\alpha, \beta) = (5, 5), IF = 47\%$
(c) $(\alpha, \beta) = (5, 10), IF = 51\%$
(d) $(\alpha, \beta) = (3, 5), IF = 52\%$
(e) $(\alpha, \beta) = (3, 10), IF = 56\%$
(f) $(\alpha, \beta) = (0, \infty), IF = 100\%$

Figure 9: Infeasibility Factors (IF) for a rectangular city with $l = 3$ and $a = 5$, for some (plausible) values of $(\alpha, \beta)$. The same color scale (on the right) is used for all plots.
Figure 10: Infeasibility Factors (IF) for a circular city with radius $r = 10$. The tram line runs between points ($-5, 0$) and $(5, 0)$. The same color scale (on the right of (b)) is used for all plots.
Figure 11: Schematic (drawn to scale) of population densities in Galway City. The figures are taken from [5]. Grid squares are square kilometres.

Figure 12: Table / grid of population densities in Galway City. Each square represents a square kilometre, and is drawn to scale. Multiply each number by 10 to get the population in that square. (Note that the blank squares at the bottom/left correspond to Galway Bay / Atlantic Ocean - no population!)

C Case study of Galway City

We present here a few generic\(^3\) schematic\(^3\) suggestions for the layout of a single line light rail/tram for Galway City. Each suggestion differs by taking in to account in greater detail the variations of population density, and building a successively longer line. So in Figure 13 our (short) line just links the highest population densities (from Figure 12), leading eventually to Figure 17 which is the longest line, taking in to account all the data.

Figure 11 shows the overall population densities, drawn to scale, with some important points of interest indicated. Figure 19 shows the single tram line, connecting highest density areas, superimposed on a GOOGLE map of Galway City. Figure 18 presents data from the CSO (Central Statistics Office, see [3]), via AIRO (All-Island Research Observatory, see [4]), showing the linear/rectangular nature of (the population distribution of) Galway City.

\(^3\)We do not consider any details of city topography, road layout, physical geography (river, etc.) here, and leave it to others to take these in to account. Because our layout models link square kilometer areas, they do not have fine grain detail, and so leave room for the actual line to be placed within hundreds of metres of the centre of each square in our grid, without affecting substantially the details of our calculations.
Figure 13: Schematic (drawn to scale) of a light rail/tram line connecting only the highest density areas in Galway City (see also Figure 12). We consider here only areas with more than 3500 people per square kilometre. Grid squares are square kilometres. This line services about 19 thousand people (the number of people living within $d_t = 500$ metres of the line).

Figure 14: Schematic (drawn to scale) of a light rail/tram line in Galway City connecting areas with more than 3000 people per square kilometre. (see also Figure 12). Grid squares are square kilometres. This line services about 39 thousand people (the number of people living within $d_t = 500$ metres of the line).

Figure 15: Schematic (drawn to scale) of a light rail/tram line in Galway City connecting areas with more than 2000 people per square kilometre. (see also Figure 12). Grid squares are square kilometres. This line services about 42 thousand people (the number of people living within $d_t = 500$ metres of the line).
Figure 16: Schematic (drawn to scale) of a light rail/tram line in Galway City connecting areas with more than 1000 people per square kilometre. (see also Figure 12). Grid squares are square kilometres. This line services about 47 thousand people (the number of people living within \( d_t = 500 \) metres of the line).

Figure 17: Schematic (drawn to scale) of a light rail/tram line in Galway City connecting areas with more than 500 people per square kilometre. (see also Figure 12). Grid squares are square kilometres. This line services about 48 thousand people (the number of people living within \( d_t = 500 \) metres of the line).

Figure 18: Fine-grained population density in Galway City and surroundings. This is taken from [4, 3], with a superimposed rectangle hand-drawn in black by the author to illustrate the city shape.
Figure 19: Population density grid squares and tram line superimposed on GOOGLE map of Galway City. The legend for the colored kilometre squares (corresponding to population densities) is as used previously (e.g. Figure [17]).