ANALYSIS OF THE $P_{cs}(4338)$ AND RELATED PENTAQUARK MOLECULAR STATES VIA THE QCD SUM RULES

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Abstract

In this work, we tentatively identify the $P_{cs}(4338)$ as the $D\Xi_c$ molecular state, and distinguish the isospins of the current operators to explore the $D\Xi_c, D\Lambda_c, D_s\Xi_c, D_s\Lambda_c, D^*\Xi_c, D^*\Lambda_c, \bar{D}\Xi_c$ and $\bar{D}\Lambda_c$ molecular states without strange, with strange and with double strange in the framework of the QCD sum rules in details. The present explorations favor identifying the $P_{cs}(4338)$ ($P_{cs}(4459)$) as the $D\Xi_c$ ($D^*\Xi_c$) molecular state with the spin-parity $J^P = \frac{3}{2}^-$ (rather than the reversed parity $\frac{5}{2}^+$ and $\frac{7}{2}^-$) respectively in the $\Lambda_c^0 \to J/\psi K^- p$ decays [11]. If they are really resonant states (not re-scattering effects, threshold effects, cusp effects), they must have minimal valence quark content of $uudc\bar{c}$ and are excellent pentaquark candidates. In 2019, the LHCb collaboration confirmed the structure $P_{cs}(4459)$, which is resolved with two narrow overlapping peaks $P_{cs}(4440)$ and $P_{cs}(4457)$ with the statistical significance of $5.4\sigma$. In addition, they observed a narrow structure $P_{cs}(4312)$ in the $J/\psi p$ invariant mass distribution with the statistical significance of $7.3\sigma$ in a much larger data sample, the $P_{cs}(4312)$ is also an excellent pentaquark candidate with the minimal valence quark content $uudc\bar{c}$ [2].

In 2020, the LHCb collaboration reported an evidence of new structure $P_{cs}(4459)$ in the $J/\psi \Lambda$ invariant mass distribution with a significance of $3.1\sigma$ in the $\Xi_c^- \to J/\psi K^- \Lambda$ decays [3]. If the $P_{cs}(4459)$ is confirmed to be a real resonance, it is an excellent pentaquark candidate with the minimal valence quark content $udsc\bar{c}$. In fact, the $P_{cs}(4459)$ is also consistent with being due to two resonances, just like in the case of the $P_{cs}(4450)$.

In 2021, the LHCb collaboration observed an evidence for a new structure $P_{cs}(4337)$ in the $J/\psi p$ and $J/\psi\bar{p}$ systems in the $B^0_c \to J/\psi p\bar{p}$ decays with a significance in the range of $3.1 - 3.7\sigma$ depending on the assigned $J^P$ hypothesis [4]. The existence of the $P_{cs}(4337)$ is still need confirmation and its spin-parity are still need measurement.

Recently, the LHCb collaboration observed an evidence for a new structure $P_{cs}(4338)$ in the $J/\psi \Lambda$ mass distribution in the $B^- \to J/\psi \Lambda \bar{p}$ decays [5]. The measured Breit-Wigner mass and width are $4338.2 \pm 0.7 \pm 0.4$ MeV and $7.0 \pm 1.2 \pm 1.3$ MeV respectively and the favored spin-parity is $J^P = \frac{1}{2}^-$. The $P_{cs}(4338)$ and $P_{cs}(4459)$ are observed in the $J/\psi \Lambda$ invariant mass distribution, they have the isospin $I = 0$, as the strong decays conserve the isospin, the observation of their isospin cousins are of crucial importance.

The $P_{cs}(4312)$, $P_{cs}(4380)$, $P_{cs}(4440)$, $P_{cs}(4457)$, $P_{cs}(4459)$ and $P_{cs}(4338)$ lie slightly below or above the thresholds of the charmed meson-baryon pairs $D\Sigma_c, D\Sigma^*_c, D^*\Sigma_c, D^*\Sigma^*_c, \bar{D}\Xi_c, \bar{D}\Xi^*_c, \bar{D}\Lambda_c$, and $\bar{D}\Xi_c$, respectively. Just as what we expected, the $P_{cs}(4312)$, $P_{cs}(4380)$, $P_{cs}(4440)$, $P_{cs}(4457)$, $P_{cs}(4459)$ and $P_{cs}(4338)$ have been tentatively assigned to be the charmed meson-baryon molecular states according to several phenomenological analysis [7] [8] [9] [10] [11] [12] [13]. While it is difficult to identify the $P_{cs}(4337)$ as the molecular state without resorting to the help of

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1 Introduction

In 2015, the LHCb collaboration observed the $P_{cs}(4380)$ and $P_{cs}(4450)$ in the $J/\psi p$ invariant mass distribution with the favored spin-parity $J^P = \frac{3}{2}^-$ and $\frac{5}{2}^+$ (rather than the reversed parity $\frac{3}{2}^+$ and $\frac{7}{2}^-$) respectively in the $\Lambda_c^0 \to J/\psi K^- p$ decays [1]. If they are really resonant states (not re-scattering effects, threshold effects, cusp effects), they must have minimal valence quark content of $uudc\bar{c}$ and are excellent pentaquark candidates. In 2019, the LHCb collaboration confirmed the structure $P_{cs}(4450)$, which is resolved with two narrow overlapping peaks $P_{cs}(4440)$ and $P_{cs}(4457)$ with the statistical significance of $5.4\sigma$. In addition, they observed a narrow structure $P_{cs}(4312)$ in the $J/\psi p$ invariant mass distribution with the statistical significance of $7.3\sigma$ in a much larger data sample, the $P_{cs}(4312)$ is also an excellent pentaquark candidate with the minimal valence quark content $uudc\bar{c}$ [2].

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The $P_{cs}(4312)$, $P_{cs}(4380)$, $P_{cs}(4440)$, $P_{cs}(4457)$, $P_{cs}(4459)$ and $P_{cs}(4338)$ lie slightly below or above the thresholds of the charmed meson-baryon pairs $D\Sigma_c, D\Sigma^*_c, D^*\Sigma_c, D^*\Sigma^*_c, \bar{D}\Xi_c, \bar{D}\Xi^*_c, \bar{D}\Lambda_c$, and $\bar{D}\Xi_c$, respectively. Just as what we expected, the $P_{cs}(4312)$, $P_{cs}(4380)$, $P_{cs}(4440)$, $P_{cs}(4457)$, $P_{cs}(4459)$ and $P_{cs}(4338)$ have been tentatively assigned to be the charmed meson-baryon molecular states according to several phenomenological analysis [7] [8] [9] [10] [11] [12] [13]. While it is difficult to identify the $P_{cs}(4337)$ as the molecular state without resorting to the help of

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large coupled-channel effects due to lacking nearby meson-baryon thresholds, it is more natural to identify the $P_c(4337)$ as the $A−A−\bar{c}$-type hidden-charm pentaquark state with the spin-parity $J^P = \frac{1}{2}^-$, where the $A$ denotes the axialvector diquark states [14].

In fact, we can reproduce the masses of the $P_c(4312), P_c(4337), P_c(4380), P_c(4440), P_c(4557)$ and $P_{cs}(4459)$ in the picture of diquark-diquark-antiquark type (or diquark-triquark type) pentaquark states via the theoretical method of QCD sum rules [14, 15].

In Ref. [16], we adopt the hadronic dressing mechanism to compromise the pentaquark and molecule interpretations, which are two quite different schemes but both can give satisfactory experimental masses. The hadronic dressing mechanism was introduced previously to interpret the exotic hierarchy of the masses of the scalar mesons below 1 GeV [17, 18, 19], and we expect the same mechanism exists here. The pentaquark states maybe have a diquark-diquark-antiquark type intrinsic pentaquark kernel $qqqc\bar{c}$ with the typical size of the conventional $qqq$ baryons, say about $0.5 \sim 0.7$ fm, the strong couplings of the intrinsic kernels $qqqc\bar{c}$ to the nearby charmed meson-baryon pairs result in some molecule components, and the pentaquark states maybe spend a rather large time as the molecular states, thus they maybe display properties of the molecules.

According to our previous calculations in the framework of the QCD sum rules, the lowest diquark-diquark-antiquark type hidden-charm pentaquark state without strange has the mass about 4.31 GeV, it is difficult to assign the $P_{cs}(4338)$ as the diquark-diquark-antiquark type pentaquark state with strange, or there exists contradiction in identifying the $P_c(4312)$ and $P_{cs}(4338)$ in the same picture as pentaquark states. The lowest diquark-diquark-antiquark type hidden-charm pentaquark state with strange has a mass about 4.45 GeV according to the direct calculations in the framework of the QCD sum rules and qualitative analysis based on the $SU(3)$ mass-breaking effects [14, 15], which is much larger than the mass of the $P_{cs}(4338)$.

In the QCD sum rules, we usually choose the local currents to interpolate the tetraquark or pentaquark molecular states which have two color-neutral clusters [20, 21, 22, 23, 24, 25, 26], the color-neutral clusters are not necessary to be the physical mesons and baryons, they just have the same quantum numbers as the physical mesons and baryons. The local currents require that the molecular states have the average spatial sizes $\sqrt{\langle r^2 \rangle}$ of the same magnitudes as the conventional mesons and baryons, and they are also compact objects, just like the diquark-antidiquark type tetraquark states or diquark-diquark-antiquark type pentaquark states, and those molecular states are not necessary to be loosely bound, as the conventional mesons and baryons are compact objects, in the local limit, the conventional mesons and baryons lose themselves and merge into color-singlet-color-singlet type tetraquark or pentaquark states [27].

In the present work, we extend our previous works on the pentaquark molecular states to investigate the $\bar{D}\Xi_{c}, \bar{D}\Lambda_{c}, \bar{D}\Xi_{c}, \bar{D}\Lambda_{c}, \bar{D}^*\Xi_{c}, \bar{D}^*\Lambda_{c}$ and $\bar{D}^*\Lambda_{c}$ molecular states with distinguished isospins in the framework of the QCD sum rules [25, 28, 29, 30]. We carry out the operator product expansion up to the vacuum condensates of dimension 13 consistently, just like what we did previously, and determine the best energy scales of the spectral densities using the modified energy-scale formula by considering the light-flavor $SU(3)$ mass-breaking effects, and try to obtain the lowest color-singlet-color-singlet type pentaquark states as one of the color-neutral clusters has the same quantum numbers as the lowest charmed baryons in the flavor anti-triplet, and make possible assignments of the $P_{cs}(4338)$ and $P_{cs}(4459)$.

The article is arranged as follows: we acquire the QCD sum rules for the pentaquark molecular states in Sect. 2; in Sect. 3, we present the numerical results and discussions; and Sect. 4 is reserved for our conclusion.
2 QCD sum rules for the pentaquark molecular states

Firstly, let us write down the two-point correlation functions $\Pi(p)$ and $\Pi_{\mu\nu}(p)$,

\[
\Pi(p) = i \int d^4xe^{ipx} \langle 0| J(x)\bar{J}(0) | 0 \rangle ,
\]
\[
\Pi_{\mu\nu}(p) = i \int d^4 xe^{ipx} \langle 0| J_\mu(x)\bar{J}_\nu(0) | 0 \rangle ,
\]

where the interpolating currents,

\[
J(x) = J^D_\Xi(x), J^{D_0}_\Xi(x), J^{D_0}_\Xi(x), J^{D_0\Lambda_0^+}(x), J^{D_0\Lambda^-}(x), J_{\mu}(x) = J^{D_0\Sigma^-}(x), J^{D_0\Sigma^0}(x), J^{D_0\Sigma^+}(x), J^{D_0\Lambda_0^+}(x), \]

\[
J^{D_0\Sigma^-}(x) = \frac{1}{\sqrt{2}} J^{D_0\Sigma^0}(x) J^{D_0\Sigma^0}(x) - \frac{1}{\sqrt{2}} J^{D_0\Sigma^0}(x) J^{D_0\Sigma^0}(x),
\]
\[
J^{D_0\Sigma^0}(x) = \frac{1}{\sqrt{2}} J^{D_0\Sigma^0}(x) J^{D_0\Sigma^0}(x) + \frac{1}{\sqrt{2}} J^{D_0\Sigma^0}(x) J^{D_0\Sigma^0}(x),
\]

\[
J^{D_0\Sigma^+}(x) = J^{D_0\Sigma^0}(x) J^{D_0\Sigma^0}(x),
\]
\[
J^{D_0\Sigma^-}(x) = J^{D_0\Sigma^0}(x) J^{D_0\Sigma^0}(x),
\]

and

\[
J^{D_0\Sigma^0}(x) = \bar{c}(x)i\gamma_5 u(x),
\]
\[
J^{D_0\Sigma^0}(x) = \bar{c}(x)i\gamma_5 d(x),
\]
\[
J^{D_0\Sigma^0}(x) = \bar{c}(x)i\gamma_5 s(x),
\]
\[
J^{D_0\Sigma^0}(x) = \bar{c}(x)\gamma_\mu u(x),
\]
\[
J^{D_0\Sigma^0}(x) = \bar{c}(x)\gamma_\mu d(x),
\]
\[
J^{D_0\Sigma^0}(x) = \bar{c}(x)\gamma_\mu s(x),
\]

the super(sub)scripts $i, j, k$ are color indices, and the $C$ represents the charge conjugation matrix, in fact, the $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$, $J^{D_0\Sigma^0}(x)$ and $J^{D_0\Sigma^0}(x)$ are
the commonly used meson and baryon currents, respectively, the subscripts \((1, 0), (0, 0)\) and \((\frac{1}{2}, \frac{1}{2})\) represent the isospins \((I, I_z)\).

According to quark-hadron duality, the currents \(J(0)\) couple potentially to the hidden-charm molecular states with the spin-parity \(J^P = \frac{1}{2}^\pm\), as for the currents \(J_\mu(0)\), they couple potentially to the hidden-charm molecular states with the spin-parity \(J^P = \frac{1}{2}^\pm\) and \(\frac{3}{2}^\pm\),

\[
\begin{align*}
(0|J(0)|P^-_\frac{1}{2}(p)) &= \lambda^-_\frac{1}{2} U^-(p, s), \\
(0|J(0)|P^+_\frac{1}{2}(p)) &= \lambda^+_\frac{1}{2} i\gamma_5 U^+(p, s),
\end{align*}
\]

where the \(\lambda^\pm_{\frac{1}{2}}, \lambda^\pm_{\frac{3}{2}}\) and \(f^\pm_{\frac{1}{2}}\) are the current-molecule coupling constants (or pole residues), the \(U^\pm(p, s)\) and \(U^\pm_\mu(p, s)\) are the Dirac spinors and Rarita-Schwinger spinors, respectively \[14, 25, 31, 32\].

At the hadron side of the correlation functions \(\Pi(p)\) and \(\Pi_{\mu\nu}(p)\), we isolate the ground state contributions from the hidden-charm molecular states with the spin-parity \(J^P = \frac{1}{2}^\pm\) and \(\frac{3}{2}^\pm\), respectively, and acquire the hadronic representation \[14, 25, 31, 32\].

\[
\begin{align*}
\Pi(p) &= \left(\lambda^\pm_{\frac{1}{2}}\right)^2 \frac{\not{p} + M_{\frac{1}{2}}}{M_{\frac{1}{2}}^2 - p^2} + \left(\lambda^\pm_{\frac{3}{2}}\right)^2 \frac{\not{p} - M_{\frac{3}{2}}}{M_{\frac{3}{2}}^2 - p^2} + \cdots, \\
\Pi_{\mu\nu}(p) &= \left(\lambda^\pm_{\frac{1}{2}}\right)^2 \frac{\not{p} + M_{\frac{1}{2}}}{M_{\frac{1}{2}}^2 - p^2} (-g_{\mu\nu}) + \left(\lambda^\pm_{\frac{3}{2}}\right)^2 \frac{\not{p} - M_{\frac{3}{2}}}{M_{\frac{3}{2}}^2 - p^2} (-g_{\mu\nu}) + \cdots,
\end{align*}
\]

we choose the components \(\Pi^{1/0}(p^2)\) and \(\Pi^{1/0}(p^2)\) to explore the molecular states with the spin-parity \(J^P = \frac{1}{2}^\pm\) and \(\frac{3}{2}^\pm\), respectively.

In the following, we omit the subscripts of the pole residues and correlation functions in above equations, see Eqs.\[(10), (10)\], and mark them as the \(\lambda_{\pm}\) and \(\Pi^{1/0}(s)\), respectively. It is direct to get the hadronic spectral densities through dispersion relation,

\[
\begin{align*}
\text{Im}\Pi^1(s) &= \lambda^2_{\pm} \delta\left( s - M^2_{\frac{1}{2}} \right) + \lambda^2_{\pm} \delta\left( s - M^2_{\frac{3}{2}} \right) = \rho^1_H(s), \\
\text{Im}\Pi^0(s) &= M_{-\lambda^2_{\pm}} \delta\left( s - M^2_{\frac{1}{2}} \right) - M_{+\lambda^2_{\pm}} \delta\left( s - M^2_{\frac{3}{2}} \right) = \rho^0_H(s),
\end{align*}
\]

where we introduce the index \(H\) to stand for the hadron side, then we get the QCD sum rules at the hadron side with the help of the weight functions \(\sqrt{s} \exp\left(-\frac{s}{T^2}\right)\) and \(\exp\left(-\frac{s}{T^2}\right),\)

\[
\int_{4m^2}^{s_0} ds \left[ \sqrt{s} \rho^1_H(s) + \rho^0_H(s) \right] \exp\left(-\frac{s}{T^2}\right) = 2M_{-\lambda^2_{\pm}} \exp\left(-\frac{M^2_{\frac{1}{2}}}{T^2}\right),
\]

where the \(s_0\) are the continuum threshold parameters and the \(T^2\) are the Borel parameters.

It is also direct to carry out the operator product expansion routinely \[14, 25, 31, 32\]. We contract the \(u, d, s\) and \(c\) quark fields in the correlation functions \(\Pi(p)\) and \(\Pi_{\mu\nu}(p)\) with the Wick's
then observe that there are three full light-quark propagators \((U_{ij}(x), D_{ij}(x), S_{ij}(x))\) in the coordinate space, and two full charm-quark propagators \((C_{ij}(x))\) in the momentum space,

\[
\begin{align*}
U/D_{ij}(x) &= \frac{i\delta_{ij}}{2\pi^2 x^2} - \frac{\delta_{ij}(\bar{q}q)}{12} - \frac{\delta_{ij}x^2(\bar{q}q)Gq}{192} - \frac{ig_\alpha\delta_{ij}x^2(\bar{q}q)Gq}{32\pi^2 x^2} - \frac{\delta_{ij}x^4(\bar{q}q)Gq}{27648} - \frac{1}{8}(\bar{q}j\sigma_{\mu
u}q_i)\sigma_{\mu
u} + \cdots, \\
S_{ij}(x) &= \frac{i\delta_{ij}}{2\pi^2 x^2} - \frac{\delta_{ij}m_s}{32\pi^2 x^2} - \frac{\delta_{ij}(\bar{s}s)}{48} + \frac{\delta_{ij}x^2(\bar{s}s)Gq}{192} + \frac{\delta_{ij}x^2(\bar{s}s)Gq}{27648} - \frac{1}{8}(\bar{s}j\sigma_{\mu
u}s_i)\sigma_{\mu
u} + \cdots, \\
C_{ij}(x) &= \frac{i}{(2\pi)^4} \int d^4k e^{-ikx} \left\{ \frac{\delta_{ij}}{k^2 - m_c^2} - \frac{g_\alpha\delta_{ij}x^2(\bar{s}s)Gq}{4(k^2 - m_c^2)^2} - \frac{g_\alpha^2x^4(\bar{s}s)Gq}{4(k^2 - m_c^2)^5} + \cdots \right\}, \\
f^{\alpha\beta\gamma} &= (k + m_c)^\gamma(\bar{q}q)\gamma^\alpha(k + m_c)\gamma^\beta(k + m_c)\gamma^\mu(k + m_c),
\end{align*}
\]

and \(t^\alpha = \frac{1}{2}\lambda^\alpha\), the \(\lambda^\alpha\) is the Gell-Mann matrix. If each charm-quark line emits a gluon and each light-quark line contributes a quark-antiquark pair, we acquire a quark-gluon operator \(g_\alpha^2G_{\alpha\beta}^\gamma\bar{q}q\bar{q}q\) (with \(q = u, d\) or \(s\)) of dimension 13, therefore, we have to deal with the condensates at least up to dimension 13 to judge the convergent behavior of the operator product expansion, as the condensates are vacuum expectations of the quark-gluon operators in the QCD vacuum.

We retain the possible operators \((\bar{q}j\sigma_{\mu
u}q_i)\) and \((\bar{s}j\sigma_{\mu
u}s_i)\) from the Fierz transformations of the quark operators \((q_i\bar{q}_j)\) and \((s_i\bar{s}_j)\) (before the Wick's contractions) to absorb the gluons emitted from other quark lines to extract the mixed condensates \((\bar{q}q)\sigma Gq)\) and \((\bar{s}q)\sigma Gs)\), respectively. Then we compute all the integrals in the coordinate space and momentum space sequentially to obtain the representations at the quark-gluon level.

We count the vacuum condensates by the strong fine structure constant \(\alpha_s = \frac{g^2}{4\pi}\) with the orders \(O(\alpha_s^k)\), where \(k = 0, 1, \frac{1}{2}, 1, \frac{3}{2}, \cdots\). In this work, we prefer the truncations \(k \leq 1\) consistently, and deal with the quark-gluon operators of the orders \(O(\alpha_s^k)\) with \(k \leq 1\). To be more precise and concrete, we take account of the vacuum condensates \((\bar{q}q), (\bar{q}q)G, (\bar{q}q)^2, \cdots\), \((\bar{q}q)\alpha G, (\bar{q}q)\alpha G, (\bar{q}q)\alpha G^2, \cdots\), \((\bar{q}q)\alpha G, (\bar{q}q)\alpha G, (\bar{q}q)\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), \((\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G, (\bar{q}q)^3\alpha G^2, \cdots\), and with the assumption of vacuum saturation consistently to assess the convergent behaviors. Where we set the masses of the \(u, d\) and \(s\) quarks to be zero and consider the contributions of the order \(O(m_s)\) consistently for the \(s\) quark so as to take account of the light-flavor \(SU(3)\) mass-breaking effects.

At last, we acquire the QCD spectral densities \(\rho_{QCD}(s)^{1/2}\) and \(\rho_{QCD}(s)^{1/2}\) through dispersion relation, their explicit expressions are available through contacting the corresponding author via E-mail. Then we assume (and implement) the quark-hadron duality below the continuum thresholds \(s_0\), and again we acquire the QCD sum rules with the help of the weight functions \(\sqrt{s}\exp \left(\frac{-s}{M^2}\right)\) and \(\exp \left(\frac{-s}{M^2}\right)\):

\[
2M_\lambda^2 \exp \left(\frac{-M^2}{T^2}\right) = \int_{4m_s^2}^{s_0} ds \left[ \sqrt{s}\rho_{QCD}(s) + \rho_{QCD}(s) \right] \exp \left(\frac{-s}{M^2}\right).
\]

We differentiate Eq.\([15]\) in regard to \(\tau = \frac{1}{T^2}\), then delete the pole residues \(\lambda_+\) by adopting a
fraction to get the QCD sum rules for the molecule masses,

\[ M^2 = -\frac{\frac{4}{\pi} \int_{4m_c^2}^{s} ds \left[ \sqrt{s} \rho_{QCD}(s) + \rho_{QCD}(s) \right] \exp(-\tau s)}{\int_{4m_c^2}^{s} ds \left[ \sqrt{s} \rho_{QCD}(s) + \rho_{QCD}(s) \right] \exp(-\tau s)}. \quad (16) \]

3 Numerical results and discussions

At the beginning points, we take the conventional (or commonly used) values of the vacuum condensates \((\bar{q}q) = -(0.24 \pm 0.01 \text{ GeV})^3\), \((\bar{s}s) = (0.8 \pm 0.1)(\bar{q}q)\), \((\bar{q}g, \sigma Gq) = m_Q^0(\bar{q}q)\), \((\bar{s}g, \sigma Gs) = m_S^0(\bar{s}s)\), \(m_0^2 = 0.8 \pm 0.1 \text{ GeV}^2\), \((\frac{\alpha_{QCD}}{\pi}) = 0.012 \pm 0.004 \text{ GeV}^4\) at the energy scale \(\mu = 1 \text{ GeV}\) \[33, 37, 38\], and take the \(M_S\) masses \(m_c(m_c) = (1.275 \pm 0.025) \text{ GeV}\) and \(m_s(\mu = 2 \text{ GeV}) = (0.095 \pm 0.005) \text{ GeV}\) from the Particle Data Group \[39\]. Then, we take account of the energy-scale dependence of all the input parameters \[39\],

\[
\langle \bar{q}q \rangle (\mu) = \langle \bar{q}q \rangle (1 \text{ GeV}) \left[ \frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^{12/5 - 2\eta}, \\
\langle \bar{s}s \rangle (\mu) = \langle \bar{s}s \rangle (1 \text{ GeV}) \left[ \frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^{2 - 2\eta}, \\
\langle \bar{q}g, \sigma Gq \rangle (\mu) = \langle \bar{q}g, \sigma Gq \rangle (1 \text{ GeV}) \left[ \frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^{2 - 2\eta}, \\
\langle \bar{s}g, \sigma Gs \rangle (\mu) = \langle \bar{s}g, \sigma Gs \rangle (1 \text{ GeV}) \left[ \frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^{2 - 2\eta}, \\
m_c(\mu) = m_c(m_c) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{12/5 - 2\eta}, \\
m_s(\mu) = m_s(2 \text{ GeV}) \left[ \frac{\alpha_s(\mu)}{\alpha_s(2 \text{ GeV})} \right]^{12/5 - 2\eta}, \\
\alpha_s(\mu) = \frac{1}{b_0} \left[ 1 - \frac{b_1 \log t}{b_0^2} + \frac{b_2^2(\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right], \quad (17)
\]

where \(t = \log \frac{\mu^2}{\Lambda^2}\), \(b_0 = \frac{33 - 2n_f}{12\pi^2}\), \(b_1 = \frac{153 - 19n_f}{24\pi^2}\), \(b_2 = \frac{2857 - 583n_f + 37n_f^2}{128\pi^2}\), \(\Lambda = 213 \text{ MeV}\), 296 MeV and 339 MeV for the quark flavors \(n_f = 3, 4\) and 5, respectively \[6, 39\], and evolve them from the energy scales \(\mu = 1 \text{ GeV}\), \(m_c\) and \(2 \text{ GeV}\) to a particular uniform energy scale \(\mu\) in the QCD sum rules for a molecular state to extract the hadron mass.

In this work, we explore the lowest hidden-charm molecular states without strange, with strange, and with double strange, it is better to choose the quark flavor numbers \(n_f = 4\), and evolve all the input parameters to the particular energy scales \(\mu\), which satisfy the modified energy scale formula \(\mu = \sqrt{M^2_{X/Y/Z/P} - (2M_c)^2 - kM_s}\) with the effective \(c\)-quark mass \(M_c = 1.85 \pm 0.01 \text{ GeV}\) and effective \(s\)-quark mass \(M_s = 0.2 \text{ GeV}\), the subscripts \(X, Y, Z\) and \(P\) denote the exotic states with hidden-charm, we take account of the light-flavor \(SU(3)\) mass-breaking effects via counting the \(s\)-quark numbers \(k = 0, 1\) and 2 to assess the impact on choosing the energy scales \[21, 30\].

In the hidden-charm (or hidden-bottom) four- and five-quark systems \(QQq\) and \(QQqq\), we discriminate the heavy and light degree’s of freedoms explicitly, and describe them as \(2M_Q\) and \(\mu + kM_s\), respectively. We assume that the hadron masses satisfy a Regge-trajectory-like relation,

\[ M^2_{X/Y/Z/P} = (\mu + kM_s)^2 + C, \quad (18) \]

where the constants \(C = 4M_Q^2\), and fit the effective masses \(M_Q\) and \(M_s\) by the QCD sum rules themselves. Direct and explicit calculations indicate that the \(M_Q\) and \(M_s\) have universal values and
work well for all the exotic \( X, Y, Z \) and \( P \) states. We only use the universal parameters \( M_Q \) and \( M_s \) to determine the appropriate energy scales \( \mu \) of the QCD spectral densities in a self-consistent way. While in the QCD spectral densities, we take the \( \overline{MS} \) (modified minimal subtraction scheme) quark masses. The modified energy scale formula serves as a powerful and useful constraint to obey. On the other hand, if we set

\[
M^2_{X/Y/Z/P} = (\mu + k M_s + 2 M_Q)^2,
\]

and take the best energy scales \( \mu = 1.3 \) GeV and 2.2 GeV for the \( Z_c(3900) \) and \( P_c(4312) \) respectively as the input parameters \([21, 29]\), we obtain the effective \( c \)-quark mass \( M_c = 1.30 \) GeV and 1.06 GeV, respectively, no uniform/self-consistent parameter can be reached.

We search for the suitable Borel parameters and continuum threshold parameters to obey the two elementary criteria of the QCD sum rules (pole dominance and convergence of the operator product expansion play an essential role to warrant reliability) via trial and error, in fact, it is not easy to achieve such requirements for multiquark states. As the spectrum of the exotic states are unclear, we have no robust guides to choose the continuum thresholds, the two criteria manifest themselves in this aspect. Then we acquire the Borel windows and continuum threshold parameters, therefore the optimal energy scales of the QCD spectral densities and pole contributions of the ground states, which are all presented plainly in Table 1.

From the table, we can see clearly that the contributions from the ground states are about or slightly larger than \((40 - 60)\%\), the pole dominance criterion is satisfied very well, we choose the uniform pole contributions in all the channels to assess the reliability, if the predictions are reliable in one channel, then they are reliable in another channel, vice versa. The normalized contributions of the condensates of dimension \( n \) are defined by,

\[
D(n) = \frac{\int_{4m^2}^{s_0} ds \rho_n(s) \exp \left(-\frac{s}{T^2} \right)}{\int_{4m^2}^{s_0} ds \rho(s) \exp \left(-\frac{s}{T^2} \right)},
\]

as we choose the spectral densities \( \rho(s) \Theta(s - s_0) \) to approximate the continuum states, where the \( \rho_n(s) \) represents the terms involving the condensates of dimension \( n \) in the total QCD spectral densities \( \rho(s) = \sqrt{s} \rho_{QCD}(s) + \rho_{QCD}^0(s) \). In calculations, we observe that the normalized contributions \( D(6) \) serve as a milestone, in all the channels, if we choose the same Borel parameter \( T^2 \), the absolute values \( |D(n)| \) with \( n \geq 6 \) decrease monotonically and quickly with the increase of the \( n \) (except that the values \( |D(7)| \) are very small), and the values \( |D(13)| \ll 1\% \), the convergent behavior of the operator product expansion is very good. In Fig 1 we plot the absolute values of the \( D(n) \) with central values of all the parameters for the \( D\Xi_c \) molecular state having the isospin \((I, I_3) = (0, 0)\) as an example. For reader’s convenience, we present the lengthy QCD spectral densities in the Appendix.

At the last step, we take account of all uncertainties of the input parameters including the quark masses, vacuum condensates, Borel parameters, continuum threshold parameters, and acquire the masses and pole residues of the hidden-charm molecular states without strange, with strange and with double strange, and present them explicitly in Table 2 and Figs 2-3. From Tables 1-2 we can see clearly that the modified energy scale formula \( \mu = \sqrt{M^2_{X/Y/Z/P} - (2 M_c)^2 - k M_s} \) with the \( s \)-quark numbers \( k = 0, 1 \) and 2 is satisfied very well \([21]\). In Figs 2-3 we plot the masses of the \( D\Xi_c \) and \( D^*\Xi_c \) molecular states with the isospins \((I, I_3) = (0, 0) \) and \((1, 0) \) with variations of the Borel parameters at much larger ranges than the Borel windows, which site between the two short perpendicular lines. There appear very flat platforms in the Borel windows indeed, the uncertainties originate from the Borel parameters can be ignored safely, which is congruous with the supplementary nature of the \( T^2 \), it is reliable to extract the molecule masses.

The present investigations with the same constraints indicate that there may exist the \( D\Xi_c \) and \( D^*\Xi_c \) molecular states with the isospin \((I, I_3) = (0, 0) \), which lie near (irrespective slightly above or below) the corresponding charmed meson-baryon thresholds, respectively. While the
Figure 1: The absolute values of the $D(n)$ with central values of all the parameters for the $\bar{D}\Xi_c$ molecular state having the isospin $(I, I_3) = (0,0)$.

Figure 2: The masses of the $\bar{D}\Xi_c$ molecular states with variations of the Borel parameters $T^2$, where the (a) and (b) denote the isospins $(0, 0)$ and $(1, 0)$, respectively.

Figure 3: The masses of the $\bar{D}^*\Xi_c$ molecular states with variations of the Borel parameters $T^2$, where the (c) and (d) denote the isospins $(0, 0)$ and $(1, 0)$, respectively.
Table 1: The best energy scales $\mu$, Borel windows $T^2$, continuum threshold parameters $s_0$ and pole contributions (pole) for the hidden-charm pentaquark molecular states.

| $\bar{D} \Xi_c^+$ | $(I, I_3)$ | $\mu$(GeV) | $T^2$(GeV$^2$) | $\sqrt{s_0}$(GeV) | Pole |
|-------------------|------------|-------------|----------------|-----------------|------|
| $\bar{D} \Xi_c^+$ | (0, 0)     | 2.1         | 3.2 - 3.8      | 5.00 ± 0.10     | (41 - 60)% |
| $\bar{D} \Xi_c^+$ | (1, 0)     | 2.3         | 3.1 - 3.7      | 5.09 ± 0.10     | (42 - 61)% |
| $\bar{D}_s \Lambda_c$ | $(\frac{1}{2}, \frac{1}{2})$ | 2.5         | 3.2 - 3.8      | 5.11 ± 0.10     | (42 - 60)% |
| $\bar{D}_s \Lambda_c$ | $(\frac{1}{2}, -\frac{1}{2})$ | 2.2         | 3.2 - 3.8      | 5.15 ± 0.10     | (41 - 59)% |
| $\bar{D}_s \Lambda_c$ | (0, 0)     | 2.3         | 3.2 - 3.8      | 5.13 ± 0.10     | (43 - 61)% |
| $\bar{D}^* \Xi_c^+$ | (0, 0)     | 2.3         | 3.2 - 3.8      | 5.10 ± 0.10     | (43 - 61)% |
| $\bar{D}^* \Xi_c^+$ | (1, 0)     | 2.6         | 3.3 - 3.9      | 5.27 ± 0.10     | (43 - 61)% |
| $\bar{D}^* \Lambda_c$ | $(\frac{1}{2}, \frac{1}{2})$ | 2.7         | 3.3 - 3.9      | 5.23 ± 0.10     | (41 - 61)% |
| $\bar{D}^* \Lambda_c$ | $(\frac{1}{2}, -\frac{1}{2})$ | 2.4         | 3.3 - 3.9      | 5.28 ± 0.10     | (42 - 59)% |
| $\bar{D}^* \Lambda_c$ | (0, 0)     | 2.4         | 3.2 - 3.8      | 5.14 ± 0.10     | (42 - 60)% |

The $\bar{D} \Xi_c^+$ and $\bar{D}^* \Xi_c^+$ molecular states with the isospin $(I, I_3) = (1, 0)$, the $\bar{D} \Lambda_c$, $\bar{D}_s \Xi_c^+$, $\bar{D}_s \Lambda_c$, and $\bar{D}^* \Xi_c^+$ molecular states with the isospin $(I, I_3) = (\frac{1}{2}, \frac{1}{2})$, and the $\bar{D}_s \Lambda_c$ and $\bar{D}^*_c \Lambda_c$ molecular states with the isospin $(I, I_3) = (0, 0)$, lie above the corresponding charmed meson-baryon thresholds, they might be the charmed meson-baryon resonances and would have much larger widths than the $P_{cs}(4338)$ and $P_{cs}(4459)$.

The present investigations favor identifying the $P_{cs}(4338)$ as the $\bar{D} \Xi_c^+$ molecular state with the spin-parity $J^P = \frac{1}{2}^-$ and isospin $I = 0$, the observation of its cousin with the isospin $I = 1$ in the $J/\psi \Sigma^0 / \eta_c \Sigma^0$ invariant mass distributions would decipher the inner structure of the $P_{cs}(4338)$ and lead to more robust assignment. The present investigations also favor identifying the $P_{cs}(4459)$ as the $\bar{D}^* \Xi_c^+$ molecular state with the spin-parity $J^P = \frac{3}{2}^-$ and isospin $I = 0$. In Ref. [28], we obtain the masses $M = 4.45 \pm 0.12$ GeV for the $\bar{D} \Xi_c^+$ molecular state with the $J^P = \frac{1}{2}^-$ and $M = 4.51 \pm 0.11$ GeV for the $\bar{D}^* \Xi_c^+$ molecular state with the $J^P = \frac{3}{2}^-$, which favor identifying the $P_{cs}(4459)$ as the $\bar{D} \Xi_c^+$ molecular state with the spin-parity $J^P = \frac{1}{2}^-$ and isospin $I = 0$. At the present time, we cannot exclude the possibility of identifying the $P_{cs}(4459)$ as the $\bar{D}^* \Xi_c^+$ molecular state with the spin-parity $J^P = \frac{3}{2}^-$ and isospin $I = 0$ considering uncertainty of the mass. Precise measurement of the mass and more experimental data on the quantum numbers, such as the spin and parity, are still needed. Furthermore, the observation of the $P_{cs}(4459)$’s cousin with the isospin $I = 1$ in the $J/\psi \Sigma^0 / \eta_c \Sigma^0$ invariant mass distributions is of crucial importance, and would decipher the inner structure of the $P_{cs}(4459)$, and lead to more robust assignment.

4 Conclusion

In the present work, we extend our previous works on the pentaquark (molecular) states and distinguish the isospins of the interpolating currents to investigate the $\bar{D} \Xi_c^+$, $\bar{D} \Lambda_c$, $\bar{D}_s \Xi_c^+$, $\bar{D}_s \Lambda_c$, $\bar{D}^* \Xi_c^+$, $\bar{D}^* \Lambda_c$, $\bar{D}^*_c \Xi_c^+$ and $\bar{D}^*_c \Lambda_c$ molecular states without strange, with strange and with double strange in the framework of the QCD sum rules in details. Here the $D$, $D_s$, $D^*$ and $D^*_c$ stand for the color-singlet clusters having the same quantum numbers as the physical mesons, the $\Lambda_c$ and $\Xi_c$ stand for the color-singlet clusters having the same quantum numbers as the physical ground state flavor-antitriplet charmed baryons. As the charmed baryons in flavor antitriplet have smaller masses than that in flavor sextet having the same valence quarks, we expect to acquire the lowest molecular states (to be more precise, the color-singlet-color-singlet type pentaquark states).

We accomplish the operator product expansion up to the vacuum condensates of dimension 13 consistently, and choose the best energy scales of the QCD spectral densities with the help
Table 2: The masses and pole residues of the pentaquark molecular states with the possible assignments.

|        | $I, I_3$ | $M$ (GeV) | $\lambda(10^{-3}\text{GeV}^6)$ | Thresholds (MeV) | Assignments |
|--------|----------|-----------|-------------------------------|------------------|-------------|
| $D \Xi_c$ | (0, 0)   | 4.34$^{+0.07}_{-0.06}$ | 1.43$^{+0.12}_{-0.18}$     | 4337             | $P_{cs}(4338)$ |
| $D_1 \Xi_c$ | (1, 0)   | 4.46$^{+0.07}_{-0.06}$ | 1.37$^{+0.12}_{-0.18}$     | 4337             |             |
| $D_2 \Lambda_c$ | $(\frac{1}{2}, \frac{1}{2})$ | 4.46$^{+0.07}_{-0.06}$ | 1.47$^{+0.12}_{-0.18}$     | 4151             |             |
| $D_3 \Xi_c$ | $(\frac{3}{2}, \frac{1}{2})$ | 4.54$^{+0.07}_{-0.06}$ | 1.58$^{+0.12}_{-0.18}$     | 4337             |             |
| $D_3^* \Lambda_c$ | (0, 0)   | 4.48$^{+0.07}_{-0.06}$ | 1.57$^{+0.12}_{-0.18}$     | 4255             |             |
| $D_4 \Xi_c$ | (0, 0)   | 4.46$^{+0.07}_{-0.06}$ | 1.55$^{+0.12}_{-0.18}$     | 4151             |             |
| $D_4^* \Xi_c$ | (0, 0)   | 4.46$^{+0.07}_{-0.06}$ | 1.55$^{+0.12}_{-0.18}$     | 4151             |             |
| $D_1^* \Lambda_c$ | $(\frac{1}{2}, \frac{1}{2})$ | 4.63$^{+0.07}_{-0.06}$ | 1.69$^{+0.12}_{-0.18}$     | 4479             |             |
| $D_2^* \Lambda_c$ | $(\frac{1}{2}, \frac{1}{2})$ | 4.59$^{+0.07}_{-0.06}$ | 1.57$^{+0.12}_{-0.18}$     | 4255             |             |
| $D_3^* \Xi_c$ | $(\frac{5}{2}, \frac{1}{2})$ | 4.65$^{+0.07}_{-0.06}$ | 1.66$^{+0.12}_{-0.18}$     | 4580             |             |
| $D_4^* \Lambda_c$ | (0, 0)   | 4.50$^{+0.07}_{-0.06}$ | 1.52$^{+0.12}_{-0.18}$     | 4398             |             |

of the modified energy scale formula, which plays a crucial important role in matching the two fundamental criteria of the QCD sum rules, and we acquire the masses and pole residues of those molecular states. The present investigations favor identifying the $P_{cs}(4338)$ ($P_{cs}(4459)$) as the $D \Xi_c$ ($D^* \Xi_c$) molecular state with the spin-parity $J^P = \frac{1}{2}^-$ ($\frac{3}{2}^-$) and the isospin $(I, I_3) = (0, 0)$, which are in congruence with the decays to the final states $J/\psi \Lambda$, the observation of their cousins with the isospin $(I, I_3) = (1, 0)$ in the $J/\psi \Sigma^0/\eta_c \Sigma^0$ invariant mass distributions would decipher the inner structures of the $P_{cs}(4338)$ and $P_{cs}(4459)$, and lead to more robust assignments. While in the picture of diquark-diquark-antiquark type pentaquark states, the $P_{cs}(4338)$ cannot find its position, the $P_{cs}(4459)$ can be identified as the strange partner of the $P_e(4312)$ tentatively.

More experimental data are still needed to reach final assignments. Furthermore, we make predictions for other pentaquark molecular states with the isospins $(I, I_3) = (\frac{1}{2}, 0)$ and $(0, 0)$, which lie above the corresponding charmed meson-baryon thresholds, and it is better to call them resonances, and they would have much larger widths than the $P_{cs}(4338)$ and $P_{cs}(4459)$. All the predictions can be confronted to the experimental data in the future.

Appendix

The detailed QCD spectral densities for the current $J^P_{(0,0)}(x)$,

$$
\rho_{QCD}^1(s) = \sum_n [\rho_{a}^1(n) + \rho_{b}^1(n) + \rho_{c}^2(n)\delta(s - m_c^2) + \rho_{d}^1(n)\delta(s - m_d^2)],
$$

$$
\rho_{QCD}^2(s) = \sum_n [\rho_{a}^2(n) + \rho_{b}^2(n) + \rho_{c}^1(n)\delta(s - m_c^2) + \rho_{d}^2(n)\delta(s - m_d^2)],
$$

where the $a, b, c$ and $d$ refer to four types of integrals, $n$ are the dimension of the condensates. In those integrals, we introduce the notations $m_c^2 = \frac{m_c^2}{y(y-1)}, m_d^2 = \frac{(y+z)m_d^2}{y^2}, \xi = y + z - 1, \zeta = 1 - y$ and $\omega = s - m_c^2$. For the types $a$ and $b$, $y_i = \frac{1}{2} \left( 1 - \sqrt{1 - 4m_i^2/s} \right), y_f = \frac{1}{2} \left( 1 + \sqrt{1 - 4m_i^2/s} \right)$ and

$z_i = \frac{ym_i^2}{ys_i - m_i^2}$. For the types $c$ and $d$, $y_i = 0, y_f = 1$ and $z_i = 0$.

The $a$ type integrals:

$$
\rho_{a}^1(8) = \frac{m_cm_c[9(g_s\sigma Gq)\langle ss\rangle - 39(q\bar{q})(q\bar{q}g_s\sigma Gq) + 7(q\bar{q})(q\bar{g}_s\sigma Gs)]}{9216\pi^4} \int_{y_i}^{y_f} dy_i \zeta,
$$
\[
\rho_a(9) = \frac{13m_c\langle ss\rangle\langle qq\rangle^2}{1152\pi^2} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta \frac{m_c g_s^2\langle qq\rangle}{262208\pi^4} \left[ 7\langle qq\rangle^2 + \langle qq\rangle\langle ss\rangle + 7\langle ss\rangle^2 \right] \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta

+ \frac{m_s\langle ss\rangle\langle qq\rangle^2}{768\pi^2} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta \frac{m_s g_s^2\langle qq\rangle^2[14\langle qq\rangle - 13\langle ss\rangle]}{41472\pi^4} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta,
\]

\[
\rho_a(10) = \frac{11\langle qq,\sigma Gq\rangle\langle \bar{s}g_s\sigma Gs\rangle}{8192\pi^4} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta \frac{\langle g_s^2 GG\rangle\langle qq\rangle[\langle qq\rangle + 14\langle ss\rangle]}{36864\pi^4} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta,
\]

\[
\rho_a(8) = -\frac{m_s m_c^2[39\langle qq\rangle\langle \bar{s}g_s\sigma Gq\rangle - 9\langle \bar{q}g_s\sigma Gq\rangle\langle ss\rangle - 7\langle \bar{q}g_s\sigma Gs\rangle]}{9216\pi^4} \int_{y_i}^{y_f} dy,
\]

\[
\rho_a(9) = -\frac{13m_c^2\langle qq\rangle^2\langle ss\rangle}{1152\pi^2} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta \frac{m_c g_s^2\langle qq\rangle^2}{62208\pi^4} \left[ 7\langle qq\rangle^2 + \langle qq\rangle\langle ss\rangle + 7\langle ss\rangle^2 \right] \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta

+ \frac{m_s m_c\langle qq\rangle^2\langle ss\rangle}{1152\pi^2} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta \frac{m_s g_s^2\langle qq\rangle^2[14\langle qq\rangle - 13\langle ss\rangle]}{62208\pi^4} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta,
\]

\[
\rho_a(10) = \frac{11m_c\langle \bar{q}q,\sigma Gq\rangle\langle \bar{s}g_s\sigma Gs\rangle}{12288\pi^4} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta \frac{m_c\langle g_s^2 GG\rangle\langle qq\rangle[\langle qq\rangle + 14\langle ss\rangle]}{55926\pi^4} \int_{y_i}^{y_f} dy \int_{\zeta_i}^{\zeta_f} d\zeta.
\]

The \( b \) type integrals:

\[
\rho_b(0) = \frac{13}{1572864\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \frac{3\omega^5}{5} + s\omega^4 - \frac{m_s m_c}{393216\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \omega^3 \omega^4,
\]

\[
\rho_b(3) = -\frac{m_c[14\langle \bar{q}q\rangle + \langle ss\rangle]}{24576\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \omega^3 + \frac{m_s m_c[13\langle \bar{s}g_s\sigma Gs\rangle - 28\langle \bar{q}g_s\sigma Gq\rangle]}{16384\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \omega^2 \omega^2,
\]

\[
\rho_b(4) = -\frac{13m_c^2\langle g_s^2 GG\rangle}{786432\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \frac{\xi^3}{y^2} \left( \omega^2 + \frac{2s\omega}{3} \right) - \frac{29\langle g_s^2 GG\rangle}{1572864\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \omega^3 \omega^2

+ \frac{m_s m_c\langle g_s^2 GG\rangle}{32768\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \omega^2 \omega^2 + \frac{m_s m_c\langle g_s^2 GG\rangle}{1179648\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \xi^2 \omega

+ \frac{m_s m_c\langle g_s^2 GG\rangle}{131072\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \left( \xi^2 \omega^2 - z\xi^2 \right),
\]

\[
\rho_b(5) = -\frac{m_c\langle \bar{q}g_s\sigma Gq\rangle}{16384\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \left( 11z\xi^2 + \frac{14z\xi^2 \omega^2}{y} \right)

+ \frac{m_s}{24576\pi^6} \left[ -36\langle \bar{q}g_s\sigma Gq\rangle + 13\langle \bar{s}g_s\sigma Gs\rangle \right] \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \omega^3 \omega^3

- \frac{m_s \langle \bar{q}g_s\sigma Gq\rangle}{8192\pi^8} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \int_{\zeta_i}^{\zeta_f} d\zeta \frac{3\omega^2}{2} + s\omega,
\]
\[
\rho_{0}^1(6) = \frac{\langle qq \rangle \left[ \langle qq \rangle - 14 \langle \bar{s}s \rangle \right]}{1024\pi^4} \int_0^y f \mathrm{d}y \int_0^z z f \left( \frac{2\omega}{3} + \omega^2 \right)
- \frac{13}{110592\pi^6} \left[ 2g_s^2 \langle \bar{q}q \rangle^2 + g_s^2 \langle \bar{s}s \rangle^2 \right] \int_0^y f \mathrm{d}y \int_0^z z f \left( \frac{2\omega}{3} + \omega^2 \right)
+ \frac{m_s m_c \langle \bar{q}q \rangle [13 \langle \bar{q}q \rangle - 7 \langle \bar{s}s \rangle]}{1536\pi^4} \int_0^y f \mathrm{d}y \int_0^z z f + \frac{m_s m_c g_s^2 \langle \bar{q}q \rangle^2}{82944\pi^6} \int_0^y f \mathrm{d}y \int_0^z z f ,
\]

\[
\rho_{0}^1(7) = \frac{m_s^3 g_s^2 G G \left[ 14 \langle \bar{q}q \rangle + \langle \bar{s}s \rangle \right]}{294912\pi^6} \int_0^y f \mathrm{d}y \int_0^z z \frac{\xi^2}{y^2} f
+ \frac{m_c g_s^2 G G \left[ 14 \langle \bar{q}q \rangle + \langle \bar{s}s \rangle \right]}{98304\pi^6} \int_0^y f \mathrm{d}y \int_0^z z \frac{\xi^2}{y^2} \left( \frac{\omega}{3} - \frac{sy}{3} \right)
+ \frac{m_c^2 g_s^2 G G \left[ 3 \langle \bar{q}q \rangle + 2 \langle \bar{s}s \rangle \right]}{49152\pi^6} \int_0^y f \mathrm{d}y \int_0^z z \frac{\xi^2}{y^2}
- \frac{m_c g_s^2 G G \left[ 56 \langle \bar{q}q \rangle + 7 \langle \bar{s}s \rangle \right]}{294912\pi^6} \int_0^y f \mathrm{d}y \int_0^z z \frac{\xi^2}{y^2} f
+ \frac{7 m_s m_c g_s^2 G G \left[ 2 \langle \bar{q}q \rangle - \langle \bar{s}s \rangle \right]}{49152\pi^6} \int_0^y f \mathrm{d}y \int_0^z z \frac{\xi^2}{y^2}
+ \frac{m_s g_s^2 G G \left[ 3 \langle \bar{s}s \rangle - 14 \langle \bar{q}q \rangle \right]}{294912\pi^6} \int_0^y f \mathrm{d}y \int_0^z z \frac{\xi^2}{y^2} f ,
\]

\[
\rho_{0}^1(8) = \frac{-12 \langle \bar{q}q, \sigma G \rangle \langle \bar{s}s \rangle + \langle \bar{q}q \rangle \left[ \langle \bar{q}q, \sigma G \rangle + 13 \langle \bar{s}s, \sigma G \rangle \right]}{6144\pi^4} \int_0^y f \mathrm{d}y \int_0^z z f \left( \omega + s \omega \right)
- \frac{2 \langle \bar{q}q, \sigma G \rangle \langle \bar{s}s \rangle + \langle \bar{q}q \rangle \left[ 2 \langle \bar{q}q, \sigma G \rangle + \langle \bar{s}s, \sigma G \rangle \right]}{6144\pi^4} \int_0^y f \mathrm{d}y \int_0^z z f \left( \omega + s \omega \right)
+ \frac{7 m_s m_c \langle \bar{q}q, \sigma G \rangle \langle \bar{s}s \rangle - \langle \bar{s}s \rangle}{3072\pi^4} \int_0^y f \mathrm{d}y \int_0^z z f \left( \omega + s \omega \right) ,
\]

\[
\rho_{0}^1(10) = \frac{\langle g_s^2 G G \rangle \left[ \langle \bar{q}q \rangle + 20 \langle \bar{s}s \rangle \right]}{24576\pi^4} \int_0^y f \mathrm{d}y \int_0^z z \xi f
- \frac{\langle \bar{q}q, \sigma G \rangle \left[ 24 \langle \bar{q}q, \sigma G \rangle + 35 \langle \bar{s}s, \sigma G \rangle \right]}{196608\pi^4} \int_0^y f \mathrm{d}y \int_0^z z f ,
\]

\[
\rho_{0}^0(0) = \frac{13 m_s^2}{1572864\pi^5} \int_0^y f \mathrm{d}y \int_0^z z y f \left( \frac{2\omega^5}{5} + s\omega^4 \right)
- \frac{m_s m_c^2}{393216\pi^5} \int_0^y f \mathrm{d}y \int_0^z z f \left( \frac{2\omega^5}{3} + s\omega^2 \right),
\]

\[
\rho_{0}^0(3) = \frac{-m_s^2 \left[ 14 \langle \bar{q}q \rangle + \langle \bar{s}s \rangle \right]}{24576\pi^6} \int_0^y f \mathrm{d}y \int_0^z z f \xi^3
- \frac{m_c m_s \left[ 28 \langle \bar{q}q \rangle - 13 \langle \bar{s}s \rangle \right]}{16384\pi^6} \int_0^y f \mathrm{d}y \int_0^z z f \xi^2 \left( \frac{2\omega^3}{3} + s\omega^2 \right) ,
\]
$$\rho_0^0(4) = -\frac{13m_c^2(g^2 GG)}{4718592\pi^8} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi^4}{y^2} (\omega^2 + s\omega)$$

$$-\frac{13m_c(g^2 GG)}{4718592\pi^8} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi^4}{y^2} (s^2 y \omega + y \omega^3 - s\omega^2)$$

$$-\frac{13m_c(g^2 GG)}{1572864\pi^8} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi^4}{y^2} (s \omega^2 - \frac{2\omega^3}{9})$$

$$-\frac{29m_c(g^2 GG)}{1572864\pi^8} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi^3}{y^2} (\frac{2\omega^3}{3} + s\omega^2)$$

$$+ \frac{m_c(g^2 GG)}{32768\pi^8} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz y \xi^2 \left(\frac{2\omega^3}{3} + s\omega^2\right)$$

$$- \frac{m_c^2(g^2 GG)}{589824\pi^8} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi^3}{y^2} (\omega^2 - s\omega)$$

$$- \frac{m_c^2(m^2 GG)}{131072\pi^8} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \left(\frac{\xi^2\omega^2}{y} - \xi\omega^2\right),$$

$$\rho_b^0(5) = -\frac{m^2 [14(\bar{q}g_s Gq) + (\bar{g}s Gs)]}{16384\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \xi \omega^2$$

$$-\frac{7m_c^2(\bar{q}g_s Gq)}{8192\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi^2\omega^2}{y}$$

$$+ \frac{m_c^2 [3(\bar{q}g_s Gq) + (\bar{g}s Gs)]}{16384\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \xi \omega^2$$

$$- \frac{m_c [42(\bar{q}g_s Gq) - 13(\bar{g}s Gs)]}{24576\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz y \xi (\omega^2 + s\omega)$$

$$- \frac{m_c^2(\bar{q}g_s Gq)}{8192\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \xi^2 (\omega^2 + s\omega)$$

$$+ \frac{m_c^2(\bar{q}g_s Gq)}{4096\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz y \xi (\omega^2 + s\omega),$$

$$\rho_b^0(6) = -\frac{m_c(\bar{q}q) [14(\bar{q}q) + 14(\bar{s}s)]}{1536\pi^5} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz y \xi (\omega^2 + s\omega)$$

$$+ \frac{m_c^2(\bar{q}q) [13(\bar{q}q) - 7(\bar{s}s)]}{1536\pi^4} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \omega + \frac{m_c^2 m_c^2(\bar{q}q)\omega^2}{82944\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz \omega$$

$$- \frac{13m_c [2g^2(\bar{q}q)^2 + g^2(\bar{s}s)^2]}{16588\pi^6} \int_{y_1}^{y_f} dy \int_{z_i}^{z_f} dz y \xi (\omega^2 + s\omega),$$

13
\[
\rho_0'(7) = \frac{m_c^2 (g_G^2 \sigma \bar{G}q) [14\langle \bar{q}q \rangle + \langle \bar{s}s \rangle]}{73728\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi^2}{y^2} \left( -\xi\omega + \frac{sy}{2} \right) \\
+ \frac{m_c^2 (g_G^2 \sigma \bar{G}q) [3\langle \bar{q}q \rangle + 2\langle \bar{s}s \rangle]}{49152\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi}{y} \\
- \frac{7m_c^2 (g_G^2 \sigma \bar{G}q) [8\langle \bar{q}q \rangle + \langle \bar{s}s \rangle]}{294912\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \omega \\
+ \frac{m_s m_c (g_G^2 \sigma \bar{G}q) [28\langle \bar{q}q \rangle - 13\langle \bar{s}s \rangle]}{294912\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \left( \frac{\xi^2}{y^2} + \frac{z\xi^2}{y^3} \right) \\
- \frac{m_s m_c (g_G^2 \sigma \bar{G}q) [28\langle \bar{q}q \rangle - 13\langle \bar{s}s \rangle]}{196608\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \left( s + 2\omega \right) \\
+ \frac{7m_s m_c (g_G^2 \sigma \bar{G}q) [2\langle \bar{q}q \rangle - \langle \bar{s}s \rangle]}{49152\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \xi \left( s + 2\omega \right) \\
+ \frac{m_s m_c (g_G^2 \sigma \bar{G}q) [\bar{q}q]}{98304\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \left( s + 2\omega \right) \\
- \frac{7m_s m_c (g_G^2 \sigma \bar{G}q) [\bar{q}q]}{147456\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \left( s + 2\omega \right). 
\]

\[
\rho_0'(8) = -\frac{m_c \{7\langle \bar{q}q, \sigma \bar{G}q \rangle + \langle \bar{q}q \rangle [\langle \bar{q}q, \sigma \bar{G}q \rangle + 7\langle \bar{s}s, \sigma \bar{G}s \rangle] \}}{3072\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz (s y + 2y\omega) \\
- \frac{m_c \{2\langle \bar{q}q, \sigma \bar{G}q \rangle + \langle \bar{q}q \rangle [2\langle \bar{q}q, \sigma \bar{G}q \rangle + \langle \bar{s}s, \sigma \bar{G}s \rangle] \}}{6144\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz (s y + 2s\omega) \\
+ \frac{m_c \{2\langle \bar{q}q, \sigma \bar{G}q \rangle + \langle \bar{q}q \rangle [\langle \bar{q}q, \sigma \bar{G}q \rangle + \langle \bar{s}s, \sigma \bar{G}s \rangle] \}}{6144\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz (s y + 2y\omega) \\
- \frac{7m_s m_c^2 \langle \bar{q}q, \sigma \bar{G}q \rangle [\langle \bar{q}q \rangle + \langle \bar{s}s \rangle]}{3072\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{1}{y}, 
\]

\[
\rho_0'(10) = \frac{m_c (g_G^2 \sigma \bar{G}q) [\langle \bar{q}q \rangle + 14\langle \bar{s}s \rangle]}{18432\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{z\xi}{y^2} \left( y - \frac{2}{3} \right) \\
+ \frac{m_c (g_G^2 \sigma \bar{G}q) [\langle \bar{q}q \rangle + 28\langle \bar{s}s \rangle]}{36864\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \\
- \frac{m_c (\bar{q}q, \sigma \bar{G}q) [24\langle \bar{q}q, \sigma \bar{G}q \rangle + 35\langle \bar{s}s, \sigma \bar{G}s \rangle]}{294912\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz. 
\]

The c type integrals:

\[
\rho_c'(9) = \frac{-m_s g_c^2 (\bar{q}q)^2 [14\langle \bar{q}q \rangle - 13\langle \bar{s}s \rangle]}{124416\pi^4} \int_{y_i}^{y_f} dy \zeta y \bar{m}_c^2 + \frac{m_s (\bar{q}q)^2 \langle \bar{s}s \rangle}{2304\pi^2} \int_{y_i}^{y_f} dy \zeta y \bar{m}_c^2, 
\]

The c type integrals:
\[
\rho_c^{(10)} = \frac{11\langle \bar{q}q, \sigma Gq \rangle \langle \bar{s}g, \sigma Gs \rangle}{24576\pi^4} \int_{y_1}^{y_f} dy \zeta y \bar{m}_c^2 + \frac{\langle g^2 G G \rangle \langle \bar{q}q \rangle [\langle \bar{q}q \rangle + 14\langle ss \rangle]}{110592\pi^4} \int_{y_1}^{y_f} dy \zeta y \bar{m}_c^2 \\
- m_s m_c \langle \bar{q}g, \sigma Gq \rangle [-39 \langle \bar{q}g, \sigma Gq \rangle + 14 \langle \bar{s}g, \sigma Gs \rangle] \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right) \\
- \frac{m_s m_c \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle}{36864\pi^4} \int_{y_1}^{y_f} dy \zeta \frac{\zeta}{y} - m_s m_c \langle g^2 G G \rangle \langle \bar{q}q \rangle [-26 \langle \bar{q}q \rangle + 7 \langle ss \rangle] \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right) \\
- \frac{7m_s m_c \langle \bar{q}g, \sigma Gq \rangle [3 \langle \bar{q}g, \sigma Gq \rangle - \langle \bar{s}g, \sigma Gs \rangle]}{18432\pi^4} \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right) \\
+ \frac{m_s m_c \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}g, \sigma Gs \rangle}{18432\pi^4} \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right),
\]

\[
\rho_c^{(11)} = \frac{13m_c \langle \bar{q}q \rangle [2\langle \bar{q}g, \sigma Gq \rangle \langle ss \rangle + \langle \bar{q}q \rangle \langle \bar{s}g, \sigma Gs \rangle]}{2304\pi^2} \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right) \\
+ \frac{\frac{7m_c \langle \bar{q}g, \sigma Gq \rangle G^2(s) \langle \bar{s}s \rangle^2}{124416\pi^4} \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right) \\
- m_c \langle \bar{q}q \rangle [28 \langle \bar{q}g, \sigma Gq \rangle \langle ss \rangle + \langle \bar{q}q \rangle \langle \bar{s}g, \sigma Gs \rangle] \int_{y_1}^{y_f} dy \zeta \frac{\zeta}{y} \\
+ \frac{m_c G^2 \langle \bar{q}q \rangle [7 \langle \bar{q}g, \sigma Gq \rangle + \langle \bar{s}g, \sigma Gs \rangle]}{124416\pi^4} \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right) \\
- m_s \langle \bar{q}q \rangle [3 \langle \bar{q}g, \sigma Gq \rangle \langle ss \rangle + \langle \bar{q}q \rangle \langle \bar{s}g, \sigma Gs \rangle] \int_{y_1}^{y_f} dy \zeta y \zeta \left(3 + \frac{\bar{m}_c^4}{2T^4} + \frac{2\bar{m}_c^2}{T^2}\right) \\
+ \frac{m_s G^2 \langle \bar{q}q \rangle [21 \langle \bar{q}g, \sigma Gq \rangle - 13 \langle \bar{s}g, \sigma Gs \rangle]}{373248\pi^4} \int_{y_1}^{y_f} dy \zeta \left(3 + \frac{\bar{m}_c^4}{2T^4} + \frac{2\bar{m}_c^2}{T^2}\right) \\
+ \frac{m_s \langle \bar{q}q \rangle \langle \bar{q}g, \sigma Gq \rangle \langle ss \rangle}{2304\pi^2} \int_{y_1}^{y_f} dy \zeta \left(1 + \frac{\bar{m}_c^2}{2T^2}\right),
\]

\[
\rho_c^{(12)} = \frac{g^2 \langle \bar{q}q \rangle^2 \langle ss \rangle [14 \langle \bar{q}q \rangle + \langle ss \rangle]}{46656\pi^2} \int_{y_1}^{y_f} dy \zeta y \zeta \left(3 + \frac{\bar{m}_c^4}{2T^4} + \frac{2\bar{m}_c^2}{T^2}\right) \\
- \frac{7m_s m_c G^2 \langle \bar{q}q \rangle^3}{93312\pi^2} \int_{y_1}^{y_f} dy \zeta \left(\frac{\bar{m}_c^4}{T^6} + \frac{2\bar{m}_c^2}{T^4} + \frac{2}{T^2}\right),
\]
\begin{align*}
\rho_c^1(13) &= - \frac{13 m_c \langle ggGq \rangle \langle ggGq \rangle \langle ss \rangle}{18432\pi^2} + 2 \langle qq \rangle \langle sgGq \rangle + \frac{13 m_c \langle g^2Gq \rangle \langle qq \rangle^2 \langle ss \rangle}{41472\pi^2} \int_{y_i}^{y_f} dy \frac{1}{y} + \frac{13 m_c \langle g^2Gq \rangle \langle qq \rangle^2 \langle ss \rangle}{55296\pi^2} \int_{y_i}^{y_f} dy \frac{\zeta}{y^2} \left( \frac{m_c^2}{T^2} + \frac{2}{T^2} \right) - \frac{13 m_c \langle g^2Gq \rangle \langle qq \rangle^2 \langle ss \rangle}{82944\pi^2} \int_{y_i}^{y_f} dy \frac{\zeta}{y^2} \left( \frac{m_c^4}{T^6} + \frac{2m_c^2}{T^4} + \frac{1}{T^2} \right) + \frac{13 m_c \langle g^2Gq \rangle \langle qq \rangle^2 \langle ss \rangle}{55296\pi^2} \int_{y_i}^{y_f} dy \frac{\zeta}{y^2} \left( \frac{m_c^4}{T^6} + \frac{2m_c^2}{T^4} + \frac{2}{T^2} \right) + m_c \langle ggGq \rangle \frac{14 \langle ggGq \rangle \langle ss \rangle + 15 \langle qq \rangle \langle sgGq \rangle}{9216\pi^2} \int_{y_i}^{y_f} dy \frac{\zeta}{y^2} \left( \frac{m_c^2}{T^2} + \frac{1}{T^2} \right) + m_c \langle ggGq \rangle \frac{3 \langle ggGq \rangle \langle ss \rangle + 4 \langle qq \rangle \langle sgGq \rangle}{110592\pi^2} \int_{y_i}^{y_f} dy \frac{\zeta}{y^2} \left( \frac{6m_c^2}{T^2} + \frac{3m_c^4}{T^6} + \frac{1}{T^2} \right) - m_c \langle ggGq \rangle \frac{2 \langle ggGq \rangle \langle ss \rangle + 14 \langle qq \rangle \langle sgGq \rangle}{27648\pi^2} \int_{y_i}^{y_f} dy \frac{\zeta}{y^2} \left( \frac{m_c^2}{T^2} + \frac{m_c^4}{T^6} + \frac{1}{T^2} \right), \\
\rho_c^0(9) &= m_c \langle g^2Gq \rangle \langle qq \rangle^2 \int_{y_i}^{y_f} dy \frac{y m_c^2}{2304\pi^2} - \frac{124416\pi^4}{m_c \langle g^2Gq \rangle \langle qq \rangle^2 \int_{y_i}^{y_f} dy \frac{y m_c^2}{2304\pi^2} \int_{y_i}^{y_f} dy m_c^2, \\
\rho_c^0(10) &= - \frac{11 m_c \langle ggGq \rangle \langle ggGq \rangle \langle ss \rangle}{24576\pi^4} + m_c \langle g^2Gq \rangle \langle qq \rangle \frac{14 \langle ss \rangle}{110592\pi^4} \int_{y_i}^{y_f} dy \frac{y m_c^2}{2304\pi^2} + m_c \langle g^2Gq \rangle \langle qq \rangle \frac{39 \langle ggGq \rangle - 14 \langle sgGq \rangle}{73728\pi^4} \int_{y_i}^{y_f} dy \left( \frac{1}{y^2} \right) - \frac{m_c \langle g^2Gq \rangle \langle qq \rangle \langle ss \rangle}{36864\pi^4} \int_{y_i}^{y_f} dy \left( \frac{1}{y} \right) + m_c \langle g^2Gq \rangle \langle qq \rangle \frac{26 \langle qq \rangle - 7 \langle ss \rangle}{221184\pi^4} \int_{y_i}^{y_f} dy \left( \frac{1}{y^2} \right) - \frac{m_c m_c \langle ggGq \rangle \langle qq \rangle \langle ss \rangle}{18432\pi^4} \int_{y_i}^{y_f} dy \left( \frac{1}{y^2} \right) + m_c \langle g^2Gq \rangle \langle ggGq \rangle \frac{3 \langle ggGq \rangle - \langle sgGq \rangle}{36864\pi^4} \int_{y_i}^{y_f} dy \left( \frac{1}{y^2} \right),
\end{align*}
\[ \rho^0_c(11) = \frac{13m^2_c}{4608\pi^2} \langle \bar{q}q \rangle \{ 2 \langle \bar{q}q, \sigma Gq \rangle \langle \bar{s}s \rangle + \langle \bar{q}q \rangle \langle \bar{s}g, GsG \rangle \} \int_{y_i}^{y_f} dy \left( 1 + \frac{\tilde{m}^2_c}{T^2} \right) \\
+ \frac{m^2_c}{248832\pi^4} \langle \bar{q}q \rangle \{ 7 \langle \bar{q}g, \sigma Gq \rangle g^*_c \{ \langle \bar{s}s \rangle \}^2 + g^*_c \{ \langle \bar{s}s \rangle \}^2 \{ 7 \langle \bar{q}g, \sigma Gq \rangle + \langle \bar{s}g, GsG \rangle \} \} \int_{y_i}^{y_f} dy \left( 1 + \frac{\tilde{m}^2_c}{T^2} \right) \\
- \frac{m^2_c}{4608\pi^2} \langle \bar{q}q \rangle \{ 28 \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle + \langle \bar{q}q \rangle \langle \bar{s}g, GsG \rangle \} \int_{y_i}^{y_f} dy \left( 1 + \frac{\tilde{m}^2_c}{2T^4} + \frac{\tilde{m}^2_c}{T^2} \right) \\
+ \frac{m^2_c g^*_c \{ \langle \bar{q}q \rangle \}^2}{373248\pi^4} \{ 21 \langle \bar{q}g, \sigma Gq \rangle - 13 \langle \bar{s}g, GsG \rangle \} \int_{y_i}^{y_f} dy \left( 1 + \frac{\tilde{m}^2_c}{2T^4} + \frac{\tilde{m}^2_c}{T^2} \right) \\
+ \frac{m^2_c m^*_c \langle \bar{q}q \rangle \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle}{4608\pi^2} \int_{y_i}^{y_f} dy \left( 1 + \frac{\tilde{m}^2_c}{T^2} \right), \]

\[ \rho^0_c(12) = \frac{m_c g^*_c \{ \langle \bar{q}q \rangle \}^2 \{ \langle \bar{s}s \rangle \}^2}{46656\pi^2} \int_{y_i}^{y_f} dy \left( 1 + \frac{\tilde{m}^2_c}{T^2} + \frac{\tilde{m}^2_c}{2T^4} \right) \\
- \frac{7m_c m^*_c g^*_c \{ \langle \bar{q}q \rangle \}^3 \{ \langle \bar{s}s \rangle \}^2}{93312\pi^2} \int_{y_i}^{y_f} dy \frac{\tilde{m}^2_c}{T^6}, \]

\[ \rho^0_c(13) = \frac{13m^2_c \langle \bar{q}g, \sigma Gq \rangle \{ \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle + 2 \langle \bar{q}q \rangle \langle \bar{s}g, GsG \rangle \} \int_{y_i}^{y_f} dy \frac{\tilde{m}^2_c}{T^6} \\
+ \frac{13m^2_c g^2 \langle \bar{q}g, \sigma Gq \rangle \{ \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle \}^2}{20736\pi^2} \int_{y_i}^{y_f} dy \frac{1}{y^2} \left( \frac{1}{T^2} - \frac{\tilde{m}^2_c}{2T^4} \right) \\
- \frac{13m^2_c \langle \bar{q}g, \sigma Gq \rangle \{ \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle \}^2}{55296\pi^2} \int_{y_i}^{y_f} dy \frac{\tilde{m}^2_c}{T^6} \\
+ \frac{m^2_c \langle \bar{q}g, \sigma Gq \rangle \{ 14 \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle + 15 \langle \bar{q}q \rangle \langle \bar{s}g, GsG \rangle \} \int_{y_i}^{y_f} dy \frac{\tilde{m}^2_c}{yT^3} \\
+ \frac{m_c m_c \langle \bar{q}g, \sigma Gq \rangle \{ 3 \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle + 4 \langle \bar{q}q \rangle \langle \bar{s}g, GsG \rangle \} \int_{y_i}^{y_f} dy \frac{\tilde{m}^2_c}{yT^6} \\
- \frac{m_c m_c \langle \bar{q}g, \sigma Gq \rangle \{ 3 \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle + 2 \langle \bar{q}q \rangle \langle \bar{s}g, GsG \rangle \} \int_{y_i}^{y_f} dy \frac{\tilde{m}^2_c}{yT^6} \\
+ \frac{m_c m^*_c \langle \bar{q}g, \sigma Gq \rangle \{ \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle \}^2}{165888\pi^2} \int_{y_i}^{y_f} dy \left( \frac{3\tilde{m}^2_c}{y^2T^4} + \frac{\tilde{m}^2_c}{T^8} \right) \\
- \frac{m_c m^*_c \langle \bar{q}g, \sigma Gq \rangle \{ \langle \bar{q}g, \sigma Gq \rangle \langle \bar{s}s \rangle \}^2}{165888\pi^2} \int_{y_i}^{y_f} dy \left( \frac{\zeta^2 \tilde{m}^2_c}{y^2} + \frac{1}{y^2} \right) \left( \frac{\tilde{m}^2_c}{T^6} - \frac{1}{T^4} \right). \]

The \( d \) type integrals:

\[ \rho^0_d(7) = -\frac{m_c m^*_c \langle \bar{q}g, \sigma Gq \rangle \{ -28 \langle \bar{q}q \rangle + 13 \langle \bar{s}s \rangle \} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{z^2 \tilde{m}^2_c}{y^2}, \]
\[
\rho_d^{10}(10) = \frac{m_c^2(g_2^G Gq)(\bar{q}q)[(\bar{q}q) + 14(\bar{s}s)]}{27648\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{z \xi}{y^2} \left(1 + \frac{m_c^2}{2T^2}\right) \\
+ \frac{(g_2^G Gq)(\bar{q}q)[(\bar{q}q) + 20(\bar{s}s)]}{73728\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{z m_c^2}{y^2} \\
- \frac{(q_g q_G q)([24(q_g q_G q) + 35(s_g G q_s)]}{589824\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{z m_c^2}{y^2} \\
+ \frac{m_c^2 m_c^2 g_2^G Gq(\bar{q}q)[-13(\bar{q}q) + 7(\bar{s}s)]}{110592\pi^2 T^2} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{1}{y^2}, \\
\rho_d^0(7) = \frac{m_c^2 m_c^2 g_2^G Gq(28(\bar{q}q) - 13(\bar{s}s))}{589824\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \left(\frac{\xi^2}{y^2} + \frac{z \xi^2}{y^2}\right) \frac{m_c^2}{y^2}, \\
\rho_d^0(10) = \frac{m_c^2 g_2^G Gq(\bar{q}q)[(\bar{q}q) + 14(\bar{s}s)]}{110592\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{\xi}{y^2} \left(1 + \frac{m_c^2}{T^2}\right) \\
- \frac{m_c g_2^G Gq(\bar{q}q)[(\bar{q}q) + 14(\bar{s}s)]}{27648\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{z \xi}{y^2} \left(\frac{m_c^2}{2} - \frac{m_c^2 y}{4T^2} - m_c^2 y\right) \\
+ \frac{m_c^2 g_2^G Gq(\bar{q}q)[(\bar{q}q) + 28(\bar{s}s)]}{73728\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{1}{y^2} \\
- \frac{m_c(\bar{q}q q_G q)([24(q_g q_G q) + 35(s_g G q_s)]}{589824\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \frac{1}{y^2} \left(1 - \frac{m_c^2 y}{2T^2} - \frac{y}{2}\right). 
\]

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