Primary and secondary invariants of Dirac operators on $G$-proper manifolds

Paolo Piazza and Xiang Tang

For the 40th birthday of cyclic cohomology

Abstract. In this article, we survey the recent constructions of cyclic cocycles on the Harish-Chandra Schwartz algebra of a connected real reductive Lie group $G$ and their applications to higher index theory for proper cocompact $G$-actions.

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1. Introduction

Let $G$ be a connected reductive linear Lie group. The study of index theory on proper cocompact $G$-manifolds goes back to the 70s, in work of Atiyah [2], Atiyah-Schmid [3], and Connes-Moscovici [13]. It is deeply connected to geometric analysis, noncommutative geometry, and representation theory. In September 2021, the first author was supported in part by Ministero Università e Ricerca, through the PRIN Spazi di moduli e teoria di Lie. The second author was supported in part by NSF Grants DMS-1800666 and DMS-1952551.

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in the Conference “Cyclic Cohomology at 40”, organized by the Fields Institute, the authors reported on the recent studies of cyclic cocycles on the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$ and their applications to higher index theory. In this paper, we expand our two talks and provide a more complete account of related topics. The article is organized as follows.

In Section 2, we introduce the geometric setup for proper cocompact $G$-manifolds and define the $C^*$-algebraic index of an invariant elliptic operator on a proper cocompact $G$-manifold. We also explain the connection to the Connes-Kasparov isomorphism conjecture/theorem.

In Section 3, after briefly introducing the definition of cyclic cohomology, we present two approaches to construct cyclic cocycles on the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$. One source of cyclic cocycles is the differentiable group cohomology of $G$; the other source is the orbital integral and its generalization to higher cyclic cocycles.

In Section 4, we study higher index theory for proper cocompact $G$-manifolds that have no boundaries. More precisely, we discuss two higher index theorems corresponding to the above two types of cyclic cocycles on $\mathcal{C}(G)$. We also explain these higher index theorems in the special case of $X = G/K$, with their connections to representation theory.

In Section 5, we extend the study of higher index theory to proper cocompact $G$-manifolds with boundary. We introduce the geometric setup for proper cocompact $G$-manifold with boundary and adapt the Melrose $b$-calculus to investigate the higher index on these manifolds.

In Section 6, we introduce the framework of relative cyclic cohomology and apply it to present two higher Atiyah-Patodi-Singer index theorems associated to the cyclic cocycles introduced in Section 3. Higher rho invariants are introduced as spectral invariants for proper cocompact $G$-manifolds without boundary.

In Section 7, we discuss applications of these results to interesting problems in topology and geometry; in particular we introduce higher genera for $G$-proper manifolds without boundary and explain their stability properties and their cut-and-paste behaviour; we also discuss bordism invariance of the rho numbers we have introduced.

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2. Invariant elliptic operators on proper cocompact manifolds

Let $G$ be a connected reductive linear Lie group with a maximal compact subgroup $K$. We introduce in this section the index of a $G$-equivariant elliptic operator on a proper cocompact $G$-manifold.

2.1. Geometry of Proper Cocompact Manifolds.
2.1. Definition. A smooth manifold $X$ is called a $G$-proper manifold if $X$ is equipped with a proper $G$-action, that is, the associated map

$$G \times X \to X \times X, \quad (g, x) \mapsto (x, gx), \quad g \in G, x \in X,$$

is a proper map. This implies that the stabilizer groups $G_x$ of all points $x \in X$ are compact and that the quotient space $X/G$ is Hausdorff. The action is said to be cocompact if the quotient $X/G$ is compact. Finally, a cocompact $G$-proper manifold can always be endowed with a cut-off function $c_X$, a smooth compactly supported function on $X$ satisfying

$$\int_G c_X(g^{-1}x)dg = 1, \quad \text{for all } x \in X.$$

The following theorem was proved by Abels [1] for proper cocompact $G$-manifolds, and will be used crucially in our study of index theory.

2.2. Theorem. Let $X$ be a proper cocompact $G$-manifold (with or without boundary). There is a compact submanifold $Z$ (with or without boundary) which is equipped with a $K$-action such that

$$X \cong G \times_K Z.$$

2.3. Assumption. We assume in this paper that the manifolds $G/K$, $X$ and $Z$ are even dimensional.

Choose a $K$-invariant inner product on the Lie algebra $\mathfrak{g}$ of $G$. We then have an orthogonal decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ where $\mathfrak{t}$ is the Lie algebra of $K$ and $\mathfrak{p}$ its orthogonal complement. Accordingly, we have an isomorphism

$$TY \cong G \times_K (\mathfrak{p} \oplus TZ),$$

where in the above decomposition we have abused notation and employed $\mathfrak{p}$ for the trivial vector bundle $\mathfrak{p} \times Z \to Z$.

2.5. Definition. We say that the $G$-invariant metric $h$ on $X$ is slice-compatible, if it is obtained by a $K$-invariant metric on $Z$ and a $K$-invariant metric on $\mathfrak{p}$ via Equation (2.4).

We assume that that adjoint representation $\text{Ad} : K \to SO(\mathfrak{p})$ admits a lift $\tilde{\text{Ad}} : K \to \text{Spin}(\mathfrak{p})$. By the following exact sequence of vector bundles and the two out of three lemma for Spin$^c$-structures,

$$0 \to G \times_K \mathfrak{p} \to TX \to G \times_K TZ \to 0,$$

we see that a $K$-invariant Spin$^c$-structure on $Z$ induces a $G$-invariant Spin$^c$-structure on $X$.

2.6. Definition. We shall say that a $G$-invariant Spin$^c$-structure on $X$ is slice-compatible if it is associated to a $K$-invariant metric and a $K$-invariant Spin$^c$-structure on the slice in the above way.

We consider a $G$-equivariant twisted spinor bundle $E$ on $X$, i.e. $E = S \otimes W$ (for the spinor bundle $S$ on $X$ and a $G$-equivariant Hermitian vector bundle $W$). We shall consider $E$ as arising from a $K$-invariant twisted spinor bundle $E_Z$ on $Z$, defined by the slice-compatible Spin$^c$-structure on $Z$ and by an auxiliary $K$-equivariant vector bundle on $Z$. Then $E$ is equipped with a $G$-equivariant
Cliff(TX)-module structure. A Clifford connection $\nabla^E$ on $E$ is a connection on $E$ satisfying
\[ [\nabla^E_V, c(V')] = c(\nabla^E_{[V,V']}), \quad V,V' \in C^\infty(X,TX) \]
where $c$ is the Clifford action and $\nabla^T_X$ is the Levi-Civita connection on $X$. The Dirac operator associated to the Clifford connection is given by the following composition
\[ D : C^\infty(X,E) \xrightarrow{\nabla^E} C^\infty(X,T^*X \otimes E) \cong C^\infty(X,TX \otimes E) \xrightarrow{c} C^\infty(X,E). \]

The vector bundle $E$ on $X$ is defined by a $K$-equivariant vector bundle $E_Z$ on $Z$, i.e.
\[ E \cong G \times_K (S_p \otimes E_Z), \]
and $E_Z$ admits a $K$-equivariant Clifford(TZ)-module structure. Accordingly, we can decompose
\[ L^2(Y,E) \cong [L^2(G) \otimes S_p \otimes L^2(Z,E_Z)]^K. \]

By the assumption that the metric is slice-compatible, we introduce a new split Dirac operator $D_{\text{split}}$ by the following formula
\[ D_{\text{split}} = D_{G,K} \otimes 1 + 1 \otimes D_Z, \]
where $D_{G,K}$ is the Spin$^c$-Dirac operator on $(L^2(G) \otimes S_p)^K$, $D_Z$ is a $K$-equivariant Dirac operator on $E_Z$, and $\otimes$ means the graded tensor product. We make the following observation,
\[ [D_{G,K} \otimes 1, 1 \otimes D_Z] = 0, \quad D_{\text{split}}^2 = D_{G,K}^2 + D_Z^2. \]

There was a confusion in literature that the split Dirac operator $D_{\text{split}}$ is identical to $D$, e.g. [24, Section 3]. It is not hard to see that the two Dirac operators $D$ and $D_{\text{split}}$ have identical principal symbols when the metric is slice compatible. Accordingly, the two operators define the same $G$-equivariant $K$-homology class on $X$ and therefore the same index element $\text{Ind}(D)$ and $\text{Ind}(D_{\text{split}})$ in $K_0(C^*(G))$. However, in general, the two operators are not identical. And therefore, the spectral invariants associated to $D$ and $D_{\text{split}}$ are a priori not same. We refer to [49] for a more detailed discussion.

2.2. Roe’s $C^*$-algebra. Roe’s $C^*$-algebra [54] for a complete proper metric space $X$ is a powerful tool to study higher index theory. More precisely, for $r > 0$, let $C_{fp}(X \times X)$ be the space of bounded continuous functions on $X \times X$ satisfying $f(x,y) = 0$ for $d(x,y) > r$ for some $r > 0$. Elements of $C_{fp}(X \times X)$ naturally define kernels of bounded linear operators on $L^2(X)$. Define $C^*(X)$ to be the completion of $C_{fp}(X \times X)$ with respect to the operator norm on $L^2(X)$. Composition of kernels makes $C^*(X)$ into a $C^*$-algebra. For a proper cocompact $G$-manifold $X$ equipped with a $G$-invariant proper complete metric, we consider $C_{fp}^G(X \times X)$ which is a subspace of $C_{fp}(X \times X)$ consisting of $G$-invariant bounded continuous functions. And we denote the completion of $C_{fp}^G(X \times X)$ with respect to the operator norm by $C^*_G(X)$, which is a $C^*$-algebra with respect to operator composition.

In this paper, we will work with smooth subalgebra of $C^*_G(X)$ using the slice theorem (Theorem 2.2), i.e. $X \cong G \times_K Z$. Under the above diffeomorphism, $C_{fp}^G(X \times X)$ can be identified with $A_G(X)$ which is defined as
\[ A_G(X) := (C_\text{c}(G) \otimes C(S \times S))^K \times K. \]
Let $\Psi^{-\infty}(S)$ be the space of smoothing operators on $X$. Then we consider the subspace $\mathcal{A}_G^c(X) \subset \mathcal{A}_G(X)$ with

$$\mathcal{A}_G^c(X) := (C_c^\infty(G) \otimes \Psi^{-\infty}(S))^{K \times K} \subset \mathcal{A}_G(X).$$

This is an algebra and it corresponds, under Abels’ identification, to the algebra of smoothing $G$-equivariant operators on $X$ of $G$-compact support. Notice that the latter or, equivalently, $\mathcal{A}_G^c(X)$ is a subalgebra of the Roe algebra $C^*_G(X)$ in a natural way. More generally, if $E$ is a $G$-equivariant vector bundle on $X$, define

$$\mathcal{A}_G^c(X, E) := (C_c^\infty(G) \otimes \Psi^{-\infty}(S, E|_S))^{K \times K},$$

where $\Psi^{-\infty}(S, E|_S)$ is the space of smoothing operators on $E|_S$. We remark that $C_c^\infty(G)$ and $\Psi^{-\infty}(S)$ are Fréchet algebras, and the tensor products in the above definitions of $\mathcal{A}_G^c(X)$ and $\mathcal{A}_G^c(X, E)$ are projective tensor products. The following description of $\mathcal{A}_G^c(X, E)$ is proved in [48 Prop. 1.7]

\begin{equation}
\mathcal{A}_G^c(X, E) \\
\cong \{ \Phi : G \to \Psi^{-\infty}(S, E|_S), \text{ smooth, compactly supported and } K \times K \text{ invariant} \}.
\end{equation}

2.3. The Harish-Chandra Schwartz algebra $\mathcal{C}(G)$. Let $\hat{G}$ be the tempered dual of isomorphism classes of unitary irreducible representations of $C^*_r(G)$. In noncommutative geometry, $C^*_r(G)$ can be viewed as the algebra of “continuous functions” on $\hat{G}$.

In the following, we introduce the noncommutative version of smooth functions on $\hat{G}$ which is the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$.

Let $v_0$ be a unit vector in the spherical representation $\pi$ of $G$, and $\Theta$ be the Harish-Chandra spherical function on $G$ defined by

$$\Theta(g) := \langle v_0, \pi(g)v_0 \rangle.$$

Let $g$ be the Lie algebra of $G$ and $U(g)$ be the universal enveloping algebra associated to $g$. For $V, W \in U(g)$, let $L_V$ be the differential operator on $G$ associated to the left $G$-action on $G$, and $R_V$ be the differential operator on $G$ associated to the right $G$-action on $G$.

Consider the Cartan decomposition $G = K \exp p$ where $p$ is the Lie algebra of the associated parabolic subgroup $P$. For $X \in p$, let $||X||$ be a $K$-invariant Euclidean norm on $p$. Let $L : G \to \mathbb{R}^+$ be the function on $G$ defined by,

$$L(g) := ||X||$$

for $g = k \exp(X)$ with $k \in K$ and $X \in p$.

2.9. Definition. Define the seminorm $\nu_{V, W, m}(-)$ by

$$\nu_{V, W, m}(f) := \sup_{g \in G} ||1 + L(g)||^m \Theta(g)^{-1} L_V R_W (f) (g)||, \text{ for } f \in C_c^\infty(G).$$

The Harish-Chandra Schwartz algebra $\mathcal{C}(G)$ for $G$ is the space of smooth functions $f$ on $G$ such that

$$\nu_{V, W, m}(f) < \infty, \text{ for } V, W \in U(g), m \geq 0.\footnote{In the latest version of our paper [49], inspired by [31], we found it more convenient to work with a Banach algebra version of the Harish-Chandra Schwartz algebra, which we call the Lafforgue algebra. We refer the reader to [49] for its precise definition.}$$
By [31], the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$ is a subalgebra of $C^*_r(G)$ stable under holomorphic functional calculus. Therefore, we have

$$K_i(\mathcal{C}(G)) \cong K_i(C^*_r(G)), \ i = 0, 1.$$ 

Starting from the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$ we can also define an algebra of smoothing $G$-equivariant operators on $X$ by considering

$$\mathcal{A}_{G}^\infty(X, E) := (\mathcal{C}(G) \otimes \Psi^{-\infty}(S, E|_S))^{K \times K}.$$ 

One can extend (2.8) and prove the following description of $\mathcal{A}_G$:

(2.10) $$\mathcal{A}_G^\infty(X, E) \cong \left\{ \Phi : G \to \Psi^{-\infty}(S), K \times K \text{ invariant and } g \to \nu_{V,W,m}(\|\Phi(g)\|_\alpha) \text{ bounded } \forall \alpha, m, V, W \right\}$$

The algebra $\mathcal{A}_G^\infty(X, E)$ is a subalgebra of the Roe $C^*$-algebra $C^*_r(G)(X, E)$ and it is not difficult to prove, see [47] Proposition 3.9, that it is dense and holomorphically closed. Thus we have

$$K_i(\mathcal{A}_G^\infty(X, E)) \cong K_i(C^*_r(G)(X, E)), \ i = 0, 1.$$ 

We end this subsection by pointing out that there exists a Morita isomorphism $\mathcal{M}$ between $C^*_c(M, E)$ and $C^*_r(G)$; this can be justified by general principles, since $C^*_c(M, E)$ is isomorphic to the $C^*$-algebra of compact operators $\mathcal{K}(E)$, with $E$ denoting the $C^*_r(G)$-Hilbert module obtained by closing the space of compactly supported sections of $E$ on $X$, $C^\infty_c(X, E)$, endowed with the $C^*_c$-valued inner product $(e, e')_{C^*_c}(g) := (e, g \cdot e')_{L^2(X, E)}$, $e, e' \in C^\infty_c(X, E)$, $g \in G$ (and it is well known that then $K_i(\mathcal{K}(E)) = K_i(C^*_c(G))$.) Following Hochs-Wang [29], we prefer to implement explicitly this isomorphism as follows: we consider $\mathcal{A}^\infty_{G}(M, E)$ and $\mathcal{C}(G)$ and consider a partial trace map $\text{Tr}_S : \mathcal{A}_G^\infty(X, E) \to \mathcal{C}(G)$ associated to the slice $S$: if $f \otimes k \in (\mathcal{C}(G) \otimes \Psi^{-\infty}(S, E|_S))^{K \times K}$ then

$$\text{Tr}_S(f \otimes k) := f \text{ Tr}(T_k) = f \int_S \text{ Tr}(k(s, s))ds,$$

with $T_k$ denoting the smoothing operator on $S$ defined by $k$ and $\text{Tr}(T_k)$ its functional analytic trace on $L^2(S, E|_S)$. It is proved in [29] that this map induces the isomorphism $\mathcal{M}$ between $K_*(C^*_c(X, E))$ and $K_*(C^*_r(G))$.

### 2.4. The index class of a $G$-invariant elliptic operator.

Recall that $X$ is even-dimensional. Let $D$ be an odd $\mathbb{Z}_2$-graded Dirac operator, equivariant with respect to the $G$-action. Recall, first of all, the classical Connes-Skandalis idempotent. Let $Q$ be a $G$-equivariant parametrix of $G$-compact support with remainders $S_\pm \in \mathcal{A}^\infty(X, E)$; consider the $2 \times 2$ idempotent

$$P_Q := \begin{pmatrix} S_+^2 & S_+ (I + S_+) Q \\ S_+ D^+ & I - S_+^2 \end{pmatrix}. $$

This produces a well-defined class

(2.12) $$\text{Ind}_e(D) := [P_Q] - [e_1] \in K_0(\mathcal{A}_G^\infty(X, E)) \text{ with } e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. $$

2.13. **Definition.** The $C^*$-index associated to $D$ is the class $\text{Ind}_{C^*_r(M, E)}(D) \in K_0(C^*_r(G)(X, E))$ obtained by considering $[P_Q] - [e_1]$ as a formal difference of idempotents with entries in $C^*_r(G)(X, E)$, under the continuous inclusion $\iota : \mathcal{A}_G^\infty(X, E) \hookrightarrow C^*_r(G)(X, E)$. 


One can also give a definition of $\operatorname{Ind}_{C^*_G(X,E)}(D) \in K_0(C^*(X,E))^G$ using Coarse Index Theory, as in the book of Higson and Roe, see [21]; the compatibility of the two definitions is proved in [51] Proposition 2.1.

We denote the image through the Morita isomorphism $\mathcal{M}$ of the index class $\operatorname{Ind}_{C^*_G(X,E)}(D) \in K_0(C^*_G(X,E))$ in the group $K_0(C^*_r(G))$ by $\operatorname{Ind}_{C^*_r(G)}(D)$. There are other, well-known descriptions of the latter index class: one, following Kasparov, see [30], describes the $C^*_r(G)$-index class as the difference of two finitely generated projective $C^*_r(G)$-modules, using the invertibility modulo $C^*_r(G)$-compact operators of (the bounded-transform of) $D$; the other description is via assembly and $KK$-theory, as in the classic article by Baum, Connes and Higson [4]. All these descriptions of the class $\operatorname{Ind}_{C^*_r(G)}(D) \in K_0(C^*_r(G))$ are equivalent. See [55] and [51] Proposition 2.1.

There is another way of expressing the index class $\operatorname{Ind}_{C^*_G(M,E)}(D) \in K_0(C^*_G(X,E))$; this is due to Connes–Moscovici [12] and employs the parametrix

$$Q := \frac{I - \exp(-\frac{1}{2}D^-D^+)}{D^-D^+}D^+$$

with $I - QD^+ = \exp(-\frac{1}{2}D^-D^+)$, $I - D^+Q = \exp(-\frac{1}{2}D^+D^-)$. This particular choice of parametrix produces the idempotent

$$V_{CM}(D) = \begin{pmatrix} e^{-\frac{1}{2}D^-D^+} & e^{-\frac{1}{2}D^+D^-} \\ e^{-\frac{1}{2}D^+D^-} & I - e^{-\frac{1}{2}D^-D^+} \end{pmatrix} D^-$$

where CM stands for Connes and Moscovici. We certainly have $\operatorname{Ind}_{C^*_G(M,E)}(D) = [V_{CM}(D)] - [e_1]$.

The following result is proved in Piazza-Posthuma [48]:

2.16. PROPOSITION. The idempotent $V_{CM}(D)$ is an element in $M_{2 \times 2}(A^\infty_G(X,E))$ (with the identity adjoined).

As $A^\infty(M,E)$ is holomorphically closed in $C^*_r(X,E)$ we have

$$\operatorname{Ind}_{C^*_G(X,E)}(D) = \operatorname{Ind}_{A^\infty(M,E)}(D) = [V_{CM}(D)] - [e_1] \in K_0(A^\infty(M,E)) = K_0(C^*(M,E)^G).$$

We call $[V_{CM}(D)] - [e_1] \in K_0(A^\infty(M,E))$ the smooth index class associated to $D$.

As in [40], [16] we shall also consider the adjoint $V_{CM}^*(D)$, which is again an element with entries in $A^\infty_G(X,E)$. It is easy to prove that $[V_{CM}(D)] - [e_1] = [V_{CM}^*(D)] - [e_1]$ in $K_0(A^\infty_G(X,E)) = K_0(C^*_G(X,E))$ through an explicit homotopy.

2.18. DEFINITION. We set $V(D) := V_{CM}(D) \oplus V_{CM}^*(D)$.

Let $f_1 = e_1 \oplus e_1$. Then

$$[V(D)] - [f_1] = 2([V_{CM}(D)] - [e_1]) \equiv 2 \operatorname{Ind}_{C^*_r(M,E)^G}(D) \quad \text{in} \ K_0(C^*_G(X,E)).$$

\footnote{In [29] and [25] the authors established, through a different argument with respect to the one provided in [48], that $V_{CM}(D_{\text{split}})$ is an element in the smooth algebra $M_{2 \times 2}(A^\infty_G(X,E))$ (with the identity adjoined). However, $D$ and $D_{\text{split}}$ are not identical. One might complete this proof of Proposition 2.16 by applying an argument involving the Volterra series connecting $V_{CM}(D)$ to $V_{CM}(D_{\text{split}})$.}
We call \( V(D) \) the symmetrized Connes-Moscovici projector; as we shall see, it plays an important role in the proof of explicit higher index formulae.

Let \( X = G/K \) with the proper left \( G \) action. For a highest weight \( \mu \) of \( K \), let \( V_\mu \) be the corresponding unitary irreducible \( K \)-representation associated to \( \mu \), and \( \tilde{V}_\mu \) be the associated vector bundle on \( G/K \) defined by \( E_\mu := G \times V_\mu / K \). We consider the Dirac operator \( D_\mu \) associated to the vector bundle \( E_\mu \) on \( X \). Applying the construction in Definition 2.18, we obtain an element \( \text{Ind}_{C^*_r(X,E_\mu)}(D_\mu) \in K_0(C^*_r(X,E_\mu)^G) = K_0(C^*_r(G)) \). Varying \( \mu \), we obtain the following morphism of abelian groups

\[
\text{Ind} : \text{Rep}(K) \to K_0(C^*_r(G)),
\]

where \( \text{Rep}(K) \) is the representation ring of the compact group \( K \).

Connes and Kasparov conjectured in the early 80s that the above index map \( \text{Ind} \) is an isomorphism, providing a geometric approach to compute the \( K \)-theory groups of \( C^*_r(G) \). The study of the index map has been one of the driving forces in the study of higher index theory on proper cocompact \( G \)-manifolds. The isomorphism theorem is now established in great generality: it is satisfied by all almost connected topological groups \( K \).

3. Cyclic cocycles on \( \mathcal{C}(G) \)

In this section, we present two methods to construct cyclic cocycles on \( \mathcal{C}(G) \).

3.1. Cyclic cohomology. Cyclic cohomology was introduced by Alain Connes as the noncommutative version of de Rham cohomology. We briefly recall it below.

3.1. Definition. Let \( A \) be a Fréchet algebra over \( \mathbb{C} \). The space of Hochschild cochains of degree \( k \) of \( A \) is defined to be the space

\[
C^k(A) := \text{Hom}_\mathbb{C}(A^{\otimes (k+1)}, \mathbb{C})
\]

of all bounded \((k+1)\)-linear functionals on \( A \). The Hochschild codifferential \( b : C^k(A) \to C^{k+1}(A) \) is defined by

\[
b \Phi(a_0 \otimes \cdots \otimes a_{k+1}) = \sum_{i=0}^{k} (-1)^i \Phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) + (-1)^{k+1} \Phi(a_{k+1} a_0 \otimes a_1 \otimes \cdots \otimes a_k).
\]

The Hochschild cohomology of \( A \) is the cohomology of the complex \((C^\bullet(A), b)\).

3.2. Definition. A Hochschild \( k \)-cochain \( \Phi \in C^k(A) \) is called cyclic if

\[
\Phi(a_k, a_0, \ldots, a_{k-1}) = (-1)^k \Phi(a_0, a_1, \ldots, a_k), \quad \forall a_0, \ldots, a_k \in A.
\]

The subspace \( C^k_c(A) \) of cyclic cochains is closed under the Hochschild codifferential. The cyclic cohomology \( HC^k_c(A) \) is defined to be the cohomology of the subcomplex of cyclic cochains.

There exists a natural operator \( S : HC^k_c(A) \to HC^{k+2}_C(A) \), the periodicity operator, and the periodic cyclic cohomology is defined as follows

\[
HP^k(A) := \lim_{n \to \infty} HC^{k+2n}_C(A), \quad k = 0, 1.
\]
To understand the definition of Hochschild and cyclic cohomology, we look at the definition for $k = 0$. A degree 0 Hochschild cochain $\Phi$ on $A$ is a linear functional on $A$. The coboundary of $\Phi$ is a 1-cochain defined as follows

$$b\Phi(a_0, a_1) = \Phi(a_0 a_1) - \Phi(a_1 a_0).$$

The above formula for $b\Phi$ suggests that $\Phi$ is a Hochschild cocycle if $\Phi(a_0 a_1) = \Phi(a_1 a_0)$, $\forall a_0, a_1 \in A$, i.e. $\Phi$ is a trace on $A$. For $k = 0$, the cyclic property trivially holds. Hence, the degree 0 cyclic cohomology of $A$ consists of traces on $A$. In general, cyclic cohomology is a natural generalization of traces.

The cyclic cohomology can also be computed by the following $b$-$B$ bicomplex on $C^\bullet(A)$. Consider the operator $B : C^k(A) \to C^{k-1}(A)$ by the formula

$$B\Phi(a_0 \otimes \ldots \otimes a_{k-1}) := \sum_{i=0}^{k-1} (-1)^{(k-1)i} \Phi(1, a_i, \ldots, a_{k-1}, a_0, \ldots, a_{i-1}) - (-1)^{(k-1)i} \Phi(a_i, 1, a_{i+1}, \ldots, a_{k-1}, a_0, \ldots, a_{i-1}).$$

This defines a differential, i.e., $B^2 = 0$, and we have $[B, b] = 0$, so we can form the $(b, B)$-bicomplex.

$$\begin{array}{c}
\cdots \\
C^2(A) \xrightarrow{B} C^1(A) \xrightarrow{B} C^0(A) \\
\uparrow b \\
C^1(A) \xrightarrow{B} C^0(A) \\
\uparrow b \\
C^0(A)
\end{array}$$

The cyclic cohomology $HC_k^k(A)$ is isomorphic to the degree $k$ total cohomology of the above $b$-$B$ bicomplex, c.f. [37] Sec. 2.4.

For the application to index theory, we are interested in the pairing between cyclic cohomology and $K$-theory of $A$. Let $P = (p_{ij})$, $i, j = 1, \ldots, m$ be an idempotent in $M_m(A)$, the space of $m \times m$ matrices with entries in $A$. The following formula

$$\langle [\Phi], [P] \rangle := \frac{1}{k!} \sum_{i_0, \ldots, i_{2k}=1}^m \Phi(p_{i_0 i_1}, p_{i_1 i_2}, \ldots, p_{i_{2k} i_0})$$

defines a natural pairing between $[\Phi] \in H^{\text{even}}(A)$ and $K_0(A)$, i.e.

$$\langle \cdot, \cdot \rangle : H^{\text{even}}(A) \otimes K_0(A) \to \mathbb{C}.$$ 

In this article, we are interested in constructing cyclic cocycles of the Harish-Chandra Schwartz algebra $C(G)$ and applying them on the one hand to study the structure of $K_\bullet(C^*(G)) \equiv K_\bullet(C^r_c(G))$ via the above pairing and, on the other hand, to obtain and study numeric invariants of a Dirac operator through the pairing of the associated $K$-theory index class with such cyclic cocycles.

Our study is inspired by the computation of the periodic cyclic cohomology of the group algebra of a discrete group, [7][11]. Let $\Gamma$ be a discrete group and $\mathbb{C}\Gamma$ be
the group algebra, i.e. \( C\Gamma := \bigoplus_{\gamma \in \Gamma} C\gamma \) such that \( \gamma_1 \ast \gamma_2 := \gamma_1 \gamma_2 \).

\[
(3.3) \quad HP^\bullet(C\Gamma) = \prod_{\hat{\gamma} \in (\Gamma)'} (H^\bullet(N\gamma, C) \otimes HP^\bullet(C)) \times \prod_{\hat{\gamma} \in (\Gamma)''} H^\bullet(N\gamma, C),
\]

where \( (\Gamma) \) is the set of conjugacy classes of \( \Gamma \), \( (\Gamma)' \subset (\Gamma) \) is the subset of classes of elements \( \gamma \in \Gamma \) of finite order, \( (\Gamma)'' \) its complement, and \( N\gamma \) is the normalizer of \( \gamma \) defined by \( C\gamma/\langle \gamma \rangle \), the quotient of the centralizer \( C\gamma \) of \( \gamma \) by the subgroup \( \langle \gamma \rangle \) generated by \( \gamma \). Let \( O_\gamma \) be the conjugacy class associated to \( \gamma \). On \( C\Gamma \), we have the following trace \( tr_\gamma \) associated to \( O_\gamma \):

\[
(3.4) \quad tr_\gamma(f) = \sum_{\alpha \in O_\gamma} f(\alpha) = \sum_{\beta \in G/C\gamma} f(\beta\gamma\beta^{-1}),
\]

where we recall that \( C\gamma \) is the centralizer of \( \gamma \). The trace \( tr_\gamma \) is a degree 0 cyclic cocycle on \( C\Gamma \) associated to the conjugacy class \( O_\gamma \in (\Gamma) \). The computation (3.3) shows that there are many interesting higher degree cyclic cocycles on the group algebra \( C\Gamma \).

A complete generalization of the computation (3.3) for a general Lie group is still under search. In (3.3), the cyclic cohomology of \( C\Gamma \) is decomposed into conjugacy classes of the group \( \Gamma \). As a step toward understanding the cyclic cohomology of \( C(G) \), we will discuss below recent developments in the following special cases.

(1) We shall study cyclic cocycles associated to the identity conjugacy class of \( G \). In [45,47] a chain map was introduced from differentiable group cohomology of \( G \) to the cyclic cohomology of \( C(G) \) generalizing the trace \( tr_e \) associated to the identity element \( e \).

(2) We shall study the orbital integral associated to a conjugacy class of \( G \), a direct generalization of (3.4). Next, following [56], we shall define higher orbital integrals on \( C(G) \) and study their properties.

3.2. Case of \( \mathbb{R}^n \). In this subsection, we look at the abelian group \( \mathbb{R}^n \). The Harish-Chandra Schwartz algebra \( C(\mathbb{R}^n) \) consists of Schwartz functions on \( \mathbb{R}^n \) with the convolution product

\[
f \ast g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.
\]

Under the Fourier transform,

\[
\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) \exp^{-2\pi i \xi \cdot x} dx,
\]

the image of \( \mathcal{F}(f) \) is a Schwartz function on \( \mathbb{R}^n \) for \( f \in C(\mathbb{R}^n) \), and the convolution product is transformed to the pointwise multiplication

\[
f \cdot g(\xi) = f(\xi)g(\xi), \quad \mathcal{F}(f \ast g) = \mathcal{F}(f) \cdot \mathcal{F}(g).
\]

Using the Fourier transform, we conclude that the cyclic cohomology of \( C(\mathbb{R}^n) \) is isomorphic to the one of \( S(\mathbb{R}^n) \), which by [10] Theorem 46 is computed as follows

\[
HP^i(C(\mathbb{R}^n)) = \begin{cases} 
0, & i \not\equiv n \pmod{2}, \\
\mathbb{C}, & i \equiv n \pmod{2}.
\end{cases}
\]
On $\mathcal{S}(\mathbb{R}^n)$, the $n$-th cyclic cohomology group is generated by the following cocycle

$$
\Psi(f_0, \cdots, f_n) = \int_{\mathbb{R}^n} f_0 df_1 \cdots df_n.
$$

On $C(\mathbb{R}^n)$, the Fourier transform of $\Psi$ has the following form. Define a function $C : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ by

$$
C(x_1, \cdots, x_n) := \begin{vmatrix}
x_1 & \cdots & x_1 \\
\vdots & \ddots & \vdots \\
x_n & \cdots & x_n
\end{vmatrix},
$$

where we have written $x \in \mathbb{R}^n$ as $x = (x^1, \cdots, x^n)$.

Define $\Phi$ to be a cocycle on $C(\mathbb{R}^n)$ by

$$
\Phi(f_0, \cdots, f_n) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} dx_1 \cdots dx_n
$$

$$
C(x_1, \cdots, x_n)f_0(-x_1 - \cdots - x_n)f_1(x_1) \cdots f_n(x_n).
$$

It is by direct computation that one can show that up to a constant $\Phi$ is the Fourier transform of the cyclic cocycle $\Psi$ on $\mathcal{S}(\mathbb{R}^n)$.

3.6. Proposition. The cyclic cocycle $\Phi$ is the generator of the cyclic cohomology of $C(\mathbb{R}^n)$.

The generalization of $\Phi$ to general groups plays the central role in the following constructions.

3.3. Differentiable group cohomology. We observe that the function $C$ defined in Eq. (3.5) is a group cocycle on $\mathbb{R}^n$. And the correspondence, $C \mapsto \Phi$, can be viewed as the analog of $H^*(N, \mathbb{R})$ in the cyclic cohomology of $C(G)$ for $\gamma = e$ and $G = \mathbb{R}^n$ with $N_e = G$. In [45][47], this construction was generalized to general Lie group(oid)s. In literature, two different but equivalent models have been used in the definition of differentiable group cohomology. We recall them below.

3.7. Definition. Let $C^\infty(G^{\times k})$ be the space of smooth functions on $G \times \cdots \times G$.

Define the differential $\delta : C^\infty(G^{\times k}) \to C^\infty(G^{\times k})$ by

$$
\delta(\varphi)(g_1, \cdots, g_{k+1})
= \varphi(g_2, \cdots, g_{k+1}, g_{k+1}) - \varphi(g_1, g_2, \cdots, g_{k+1})
+ \cdots + (-1)^k\varphi(g_1, \cdots, g_{k+1}, g_{k+1}) + (-1)^{k+1}\varphi(g_1, \cdots, g_k).
$$

The differentiable group cohomology $H^*_\text{diff}(G)$ is defined to be the cohomology of $(C^\infty(G^{\times \bullet}), \delta)$, which is called the normalized differentiable group cohomology complex.

3.8. Definition.

$$
C^k_{\text{diff}}(G) := \left\{ c : G^{\times (k+1)} \to \mathbb{C} \text{ smooth}, c(g_0, \cdots, g_k) = c(g_0, \cdots, g_k), \forall g_0, \ldots, g_k \in G \right\}.
$$

Define the differential $\partial : C^k_{\text{diff}}(G) \to C^{k+1}_{\text{diff}}(G)$ as

$$
(\partial c)(g_0, \ldots, g_{k+1}) := \sum_{i=0}^{k+1} (-1)^i c(g_1, \ldots, \hat{g}_i, \ldots, g_{k+1}).
$$
The differentiable group cohomology \( H_{\text{diff}}^\bullet(G) \) can also be computed by the cohomology of the chain complex \( (C^\bullet_{\text{diff}}(G), \partial) \), which is called the homogenous differentiable group cohomology complex.

In this paper, following the literature, we will use both chain complexes introduced in Definition 3.7 and 3.8. And we will explicitly point out the models used in the respective formulas below.

Fix a Haar measure \( dg \) on \( G \).

3.9. Definition. Define a pairing between \( C^\infty(G \times k) \) and \( C^\infty_c(G)^{\otimes (k+1)} \) by
\[
\langle \varphi, f_0 \otimes \cdots \otimes f_k \rangle := \int f_0(g_k^{-1} \cdots g_1^{-1})f_1(g_1) \cdots f_k(g_k) \varphi(g_1, \cdots, g_k) dg_1 \cdots dg_k.
\]
In the above formula, we have used the normalized differentiable cohomology complex.

3.10. Theorem. Assume that \( G \) is unimodular, i.e. the Haar measure is bi-invariant. The above pairing descends to a character morphism \( \chi \),
\[
\chi : H_{\text{diff}}^\bullet(G) \to H^\bullet(\mathcal{C}(G)).
\]

3.11. Remark. In [45], the character morphism was introduced as a map from \( H_{\text{diff}}^\bullet(G) \) to \( H^\bullet(\mathcal{C}_c(G)) \), where \( C^\infty_c(G) \) is the convolution algebra of compactly supported functions. When \( G \) has property RD and the homogeneous space \( G/K \) has nonpositive sectional curvature, Piazza and Posthuma [47] improved this character morphism as a map from \( H_{\text{diff}}^\bullet(G) \) to the cyclic cohomology of \( H^\bullet_L(G) \), which for a length function \( L \) on \( G \) is defined as follows,
\[
H_{\text{cyclic}}^\bullet(G) = \left\{ f \in L^2(G) \mid \int_G (1 + L(g))^k |f(g)|^2 dg < \infty, \forall k \in \mathbb{N} \right\}.
\]
For a connected reductive linear Lie group \( G \), \( G/K \) is a Hermitian symmetric space of noncompact type [20], i.e. \( G/K \) has nonpositive sectional curvature. We observe that the same argument for \( H_{\text{cyclic}}^\bullet(G) \) can be applied to the Harish-Chandra Schwarz algebra \( \mathcal{C}(G) \) via the techniques in [56] Appendix A]. This is how we reached the final statement for Theorem 3.10.

3.12. Example. For \( G = SL_2(\mathbb{R}) \), by the van Est isomorphism, \( H_{\text{diff}}^\bullet(SL_2(\mathbb{R})) \) is generated by the area cocycle, which has the following geometric description. Let \( \mathbb{H} \) be the upper half space identified with the homogenous space \( SL_2(\mathbb{R})/SO(2) \). Let \( [e] \) be the point in \( \mathbb{H} \) corresponding to the coset \( eSO(2) \) in \( SL_2(\mathbb{R})/SO(2) \). As \( \mathbb{H} \) is equipped with a metric of constant negative curvature, any two points in \( \mathbb{H} \) are connected by a unique geodesic. Given \( g_1, g_2, g_3 \in SL_2(\mathbb{R}) \), we consider \( g_i[e] \), \( i = 1, 2, 3 \), in \( \mathbb{H} \), and the corresponding geodesic triangle \( \Delta(g_1[e], g_2[e], g_3[e]) \) with vertices \( g_1[e], g_2[e], g_3[e] \). Define a smooth function \( A \) on \( SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \) as follows,
\[
A(g_0, g_1, g_2) := \text{Area}_{\mathbb{H}}(\Delta(g_0[e], g_1[e], g_2[e])),
\]
where \( g_i \in SL(2, \mathbb{R}) \) acts on \( \mathbb{H} \) as usual by Möbius transformations. It is straightforward to check that \( A \) is a smooth 2-cocycle on \( SL_2(\mathbb{R}) \) (in the homogeneous differentiable group cohomology chain complex introduced in Definition 3.8, c.f. [47] Remark 2.20, (ii]), generalizing the volume cocycle \( \text{Vol} \) for \( \mathbb{R}^n \) in Sec. 3.2.
Furthermore, its image \( \chi(A) \) under the character morphism can be identified with...
the Connes-Chern character of the fundamental $K$-cycle of $SL_2(\mathbb{R})$ in \[10\], Lemma 5]. For $f_0, f_1, f_2 \in \mathcal{C}(SL_2(\mathbb{R}))$, 
\[
\chi(A)(f_0, f_1, f_2) = \int_{SL_2(\mathbb{R})} \int_{SL_2(\mathbb{R})} f_0((g_1 g_2)^{-1}) f_1(g_1) f_2(g_2) \text{Area}_G(\Delta([e], g_1 [e], g_1 g_2 [e])) dg_1 dg_2.
\]

3.4. Higher orbital integrals. The trace $\text{tr}_\gamma$ introduced in Equation (3.4) on $G$ has a natural generalization $\text{tr}_g$ for $G$, i.e.
\[
(3.13) \quad \text{tr}_g(f) := \int_{SL_2(\mathbb{R})} f(h g h^{-1}) d(h Z_g).
\]

For a discrete group $\Gamma$, it is not clear whether the trace $\text{tr}_\gamma$ on $G$, for a general element $\gamma$, pairs with $K_0(C_r^*(\Gamma))$. Still, although a general statement is missing, the pairing is well-defined for groups of polynomial growths and, more importantly, for Gromov hyperbolic groups. The latter example has been a challenge for some time and has been settled in the affirmative way by Puschnigg, who defined a dense holomorphically closed subalgebra of $C_r^*(\Gamma)$ to which the trace $\text{tr}_\gamma$ extends. See [53]. Higher cyclic cocycles associated to the other elements in the Burghelea’s decomposition were studied in detail by Chen-Wang-Xie-Yu in [9] and by Piazza-Schick-Zenobi in [52]; in the above articles (and references therein) many purely geometric applications were also given. Notice that there is an analogy between the Puschnigg algebra and the extendability of delocalized (higher) cyclic cocycles for discrete Gromov hyperbolic groups and the Harish-Chandra Schwartz algebra and the extendability of delocalized (higher) cyclic cocycles for connected real reductive Lie groups.

We go back to the orbital integral (3.13). Hochs and Wang [29], building on results of Harish-Chandra [19], proved that for a semisimple element $g$ the trace functional $\text{tr}_g$ is well defined on the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$ and therefore pairs with $K_0(\mathcal{C}(G)) \cong K_0(C_r^*(G))$. Hochs and Wang [29] computed the pairing between $\text{tr}_g$ and the index element $\text{Ind}_{C_r^*(G)}(D)$ for a $G$-equivariant Dirac operator on a proper cocompact $G$-manifold $X$; this explicit formula will be recalled in Theorem 4.11 below. As we shall see in Theorem 4.11 when $G$ does not have a compact Cartan subgroup the trace $\text{tr}_g$ is a trivial cyclic cohomology class. This raises a natural question to find a generalization of $\text{tr}_g$ that pairs nontrivially with $K_0(C_r^*(G))$. The problem was raised and solved in [56], as we shall now explain.

Let $K < G$ be a maximal compact subgroup and let $P < G$, $P = MAN$, be a cuspidal parabolic subgroup of $G$. Using the Iwasawa decomposition $G = MAN$ we write an element $g \in G$ as
\[
g = \kappa(g) \mu(g) e^{H(g)} n \in KMAN = G.
\]
We observe that the function $H(g)$ is well defined on $G$ though the components $\kappa(g)$ and $\mu(g)$ in the Iwasawa decomposition may not be unique. Let $\dim(A) = m$.
Choosing coordinates of the Lie algebra $\mathfrak{a}$ of $A$, we define the function
\[
H = (H_1, \ldots, H_m) : G \to \mathfrak{a}.
\]
Song and Tang defined in [56] the following cyclic cocycle on $\mathcal{C}(G)$, the Harish-Chandra Schwartz algebra of $G$.

\[\text{tr}_g(f) := \int_{SL_2(\mathbb{R})} f(h g h^{-1}) d(h Z_g).\]
3.14. Definition. For \( f_0, \ldots, f_m \in C(G) \) and a semi-simple element \( g \in M \), define \( \Phi^0_g \) by the following integral,

\[
\Phi^0_g(f_0, f_1, \ldots, f_m) := \int_{h \in M/Z} Z_{MN} Z_{G} \int_{G^m} C(H(g_1 \ldots g_m k), H(g_2 \ldots g_m k), \ldots, H(g_m k)) f_0(kgh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}) f_1(g_1) \cdots f_m(g_m) dg_1 \cdots dg_m dk dndh,
\]

where \( Z_M(x) \) is the centralizer of \( x \) in \( M \).

It is easy to check that the integral in Equation (3.15) is convergent for \( f_0, \ldots, f_m \in C^\infty_c(G) \). Song and Tang proved \[56\], Theorem 3.5 the following property for \( \Phi^0_g \).

3.16. Theorem. For a cuspidal parabolic subgroup \( P = MAN \) of \( G \), and a semisimple element \( g \in M \), the cochain \( \Phi^0_g \) is a cyclic cocycle on \( C(G) \).

3.5. From cocycles on \( C(G) \) to cocycles on \( A^\infty_\infty(G, E) \). In Section 2.3, we introduced \( A^\infty_\infty(G, E) \), a dense subalgebra of the Roe algebra \( C^*_G(X, E) \) that has the same \( K \)-theory groups as \( C(G) \). In this subsection, we explain how to lift the cocycles on \( C(G) \) to \( A^\infty_\infty(G, E) \).

Recall, see (2.10), that an element in \( A^\infty_\infty(G, E) \) is a \( K \) bi-invariant smooth function on \( G \) with values in \( \Psi^{-\infty}(\mathcal{S}) \). The partial trace map \( \text{Tr}_S : A^\infty_\infty(G, E) \to C(G), \text{Tr}_S(A)(g) := \text{Tr}(A(g)) = \int_S \text{Tr}(A(g)(s, s)) ds. \)

Generalizing this partial trace map, we define the map

\[
\tau : C^k(C(G)) \to C^k(A^\infty_\infty(G, E))
\]

as follows,

\[
\tau_{\varphi}(A_0, \ldots, A_k) := \varphi \left( \text{Tr} \left( A_0 \circ A_1 \circ \cdots \circ A_k(g_k) \right) \right).
\]

It is easy to check that \( \tau \) induces a chain map on the Hochschild and cyclic complexes. Thus, see \[47\], we have:

3.17. Proposition. The chain map \( \tau \) induces a morphism \( \tau : HP^\bullet(C(G)) \to HP^\bullet(A^\infty_\infty(G, E)) \).

We shall work with the following cyclic cocycles on \( A^\infty_\infty(G, E) \):

- for a differentiable group cocycle \( \varphi \) of \( G \), we shall consider \( \tau(\chi(\varphi)) \), denoted in the sequel by \( \tau^X_{\varphi} \);
- for \( g \in G \), we shall consider \( \tau(\text{tr}_g) \), denoted by \( \tau^X_{\varphi} \) in the sequel;
- for \( g \in G \), we shall consider \( \tau(\Phi^0_g) \), denoted by \( \Phi^0_{X,g} \) in the sequel.

4. Higher indices for proper cocompact manifolds without boundaries

In this section, we present higher index theorems on proper cocompact \( G \)-manifolds associated to the cyclic cocycles on \( C(G) \) introduced in Section 3.
4.1. *L2*-index theorem. We start by reviewing Atiyah’s *L2*-index theorem on Galois coverings. Let $M$ be a compact smooth manifold without boundary, and $E$ a twisted spinor bundle on $M$, and $D$ an odd $\ZZ_2$-graded Dirac operator acting on the sections of $E$.

Let $\Gamma$ be the fundamental group of $M$ and $X$ be the universal covering space of $M$. $X$ is equipped with a proper, free, and cocompact $\Gamma$ action such that the quotient is $M$. As $X$ is a covering space of $M$, the operator $D$, as a differential operator on $M$, lifts to an odd $\ZZ_2$-graded $\Gamma$-equivariant operator $\tilde{D}$ on $E$, the pullback of $E$ to $X$.

Let $C^*_r(\Gamma)$ be the reduced group $C^*$-algebra of $\Gamma$. Following Equation (2.17), we consider the index $\text{Ind}_{C^*_r(\Gamma)}(\tilde{D})$ of the operator $\tilde{D}$, which is an element of $K_0(C^*_r(\Gamma))$.

The trace $\text{tr}_c$ on $C\Gamma$ naturally extends naturally to a trace on $C^*_r(\Gamma)$. The pairing between $\text{tr}_c$ and the index element $\text{Ind}_{C^*_r(\Gamma)}(D)$ was computed by Atiyah [2] in the following theorem.

**4.1. Theorem.**

$$\langle \text{tr}_c, \text{Ind}_{C^*_r(\Gamma)}(\tilde{D}) \rangle = \text{ind}(D).$$

where on the right hand side we have the Fredholm index of $D$.

Generalizing Atiyah’s Theorem [4,1] to proper Lie group actions, Connes and Moscovici [13] proved the following index formula on homogeneous spaces. Let $K$ be a maximal compact subgroup of $G$, and $X := G/K$ be the associated homogeneous space, which is equipped with a proper left $G$ action with the quotient being a point. We assume that $X$ is equipped with a $G$-equivariant Spin$^c$-structure and consider the $G$-equivariant Dirac operator $D$ on $X$ obtained by twisting the Spin$^c$-Dirac operator with the bundle $E_\mu := G \times V_\mu/K$, with $V_\mu$ an irreducible unitary $K$ representation associated with highest weight $\mu$. We obtain a well defined index class for $D$, an element $\text{Ind}_{C^*_r(G)}(D) \in K_0(C^*_r(G))$.

**4.2. Theorem.** Assume that $G$ is unimodular. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$.

$$\langle \text{tr}_c, \text{Ind}(D_\mu) \rangle = \langle \hat{A}(\mathfrak{g}, K) \wedge \text{ch}(V_\mu)p^*, [V] \rangle,$$

where $p^* \subset \mathfrak{g}^*$ is the conormal space of $\mathfrak{k}$ in $\mathfrak{g}$, and $[V]$ is the fundamental class of $p^*$.

Hang Wang [58] generalized the Connes-Moscovici theorem, Theorem 4.2 to general proper cocompact $G$-manifolds as follows.

**4.3. Theorem.** Let $X$ be a proper cocompact $G$-manifold which is equipped with a $G$-equivariant Spin$^c$-structure. Suppose that $D$ is a $G$-equivariant Dirac operator on $X$ associated to a twisted $G$-equivariant twisted spinor bundle $E = S \otimes F$. The pairing between $\text{tr}_c$ and $\text{Ind}_{C^*_r(G)}(D)$ is computed as follows.

$$\langle \text{tr}_c, \text{Ind}_{C^*_r(G)}(D) \rangle = \int_X c_X \text{AS}(D),$$

with $c_X$ a cut-off function for the proper action of $G$ on $X$ and

$$\text{AS}(D) := \hat{A} \left( \frac{R_X}{2\pi i} \right) \text{tr} \left( \exp \left( \frac{R_F}{2\pi i} \right) \right).$$

Here $R_X$ is the curvature form for the Levi-Civita connection on $X$ and $R_F$ is the curvature form associated to a $G$-invariant Hermitian metric on $G$. 

4.2. Higher indices associated to differentiable group cohomology.

In [12], Connes and Moscovici established a higher version of Atiyah’s $L^2$-index Theorem [4] and computed a geometric formula for the pairing between the group cohomology of the fundamental group $\Gamma$ and the index class $\text{Ind}_G(A) \in K_*(A_G(X,E))$. Inspired by this result, Pflaum, Posthuma, and Tang [45, 46] proved a generalization of Theorem 4.3, computing the pairing between the cyclic cohomology of the fundamental group $\Gamma$ and the index class $\text{Ind}_A$. Theorem 4.1 and computing the pairing between the cyclic cocycles in the image of the character morphism $\chi : H^*_\text{diff}(G) \to H^*_\text{inv}(\mathcal{C}(G))$ in Theorem 3.10 and the smooth index class

$\text{Ind}_{\mathcal{C}(G)}(D) \in K_0(\mathcal{C}(G)) \equiv \text{Ind}_{\mathcal{C}_G(G)}(D) \in K_0(C^*(G))$.

Let $X$ be a manifold equipped with proper $G$ action, and $\Omega^k_{\text{inv}}(X)$ be the space of $G$-invariant differential forms on $X$. The de Rham differential restricts to $\Omega^k_{\text{inv}}(X)$ and the associated cohomology is denoted by $H^k_{\text{inv}}(X)$.

4.4. Definition. The van Est morphism $\Phi : C^k_{\text{diff}}(G) \to \Omega^k_{\text{inv}}(X)$ is defined as follows.

$\Phi(\varphi) := (d_{x_1} f_{\varphi}(x), \ldots, d_{x_k} f_{\varphi}(x)) .

In the above formulas, we have used the homogeneous differentiable group cohomology chain complex in Definition 3.8, and $f_{\varphi}$ is a smooth function on $X^{\times (k+1)}$, and $d_{x_i} f_{\varphi}$ is the differential of $f_{\varphi}$ along the $x_i$ component. As the cut-off function $c_X$ is compactly supported along each $G$ orbit, the above integral is convergent.

The following property is proved in [47] Prop. 2.5.

4.5. Proposition. The map $\Phi$ induces a morphism $\Phi : H^k_{\text{diff}}(G) \to H^k_{\text{inv}}(X)$.

The following theorem computes the explicit formula about the pairing between $\chi(\varphi)$ and $\text{Ind}_{\mathcal{C}_G(G)}(D)$.

4.6. Theorem. Suppose that $G$ is unimodular. Let $G$ be a connected reductive linear Lie group acting properly and cocompactly on a manifold $X$. For any $\varphi \in H^k_{\text{diff}}(G)$, the index pairing is given by

$$\langle \chi(\varphi), \text{Ind}_{\mathcal{C}_G(G)}(D) \rangle = \frac{1}{(2\pi i)^k(2k)!} \int_{T^*X} c_X \Phi(\varphi) \wedge \hat{A}(T^*X) \wedge \text{ch}(F),$$

where $c_X \in C^\infty_c(X)$ is a cut-off function, $\hat{A}(T^*X) := \frac{\text{tr} \left( \exp \left( \frac{R}{2\pi i} \right) \right)}{2\pi i}$, as in Theorem 4.3.

4.7. Remark. Pflaum, Posthuma, and Tang [45] proved Theorem 4.6 through the algebraic index theorem method developed by Fedosov [14] and [43] for general $G$-invariant elliptic operators on $X$ on a proper cocompact $G$-manifold for a Lie groupoid $G$. Recently, Piazza and Posthuma [48] presented a new proof of this theorem for the case of Dirac operators using the heat kernel and Getzler’s rescaling techniques.

4.8. Remark. The assumption of unimodularity in Theorem 4.6 can be dropped by working with smooth group cohomology of $G$ with coefficients. This is developed in [45].
4.9. Example. We consider the case in which \( G = \mathbb{R}^2 \) and \( K \) is trivial. In this case, \( X = G/K \) is identified with \( \mathbb{R}^2 \). Via the isomorphism with \( \mathbb{C} \), \( \mathbb{R}^2 \) is equipped with the \( \mathbb{R}^2 \) invariant Dolbeault operator \( D \). As there is no \( L^2 \)-harmonic form on \( \mathbb{R}^2 \), it was observed by Connes and Moscovici \( 13 \) that the \( L^2 \)-index \( \langle \text{tr}_g, \text{Ind}_{C^*_r(\mathbb{R}^2)}(D) \rangle \) vanishes. However, the determinant \( C \) function in Section 3.2 is a 2-cocycle on \( \mathbb{R}^2 \), and its image \( \chi(C) \) in \( H^{even}(\mathcal{C}(\mathbb{R}^2)) \) coincides with the cyclic cocycle \( \Phi \) introduced in Section 3.2 We can apply Theorem 4.6 to compute
\[
\langle \Phi, \text{Ind}_{C^*_r(\mathbb{R}^2)}(D) \rangle = \langle \chi(C), \text{Ind}_{C^*_r(\mathbb{R}^2)}(D) \rangle = \frac{1}{2}.
\]
This example shows that higher indices contain interesting information of the operator \( D \) beyond the \( L^2 \)-index. The computation also extends naturally to \( \mathbb{R}^{2n} = \mathbb{C}^n \).

4.3. Delocalized indices. Hochs and Wang \( 29 \) computed the pairing between \( \text{tr}_g \) and \( \text{Ind}_{C^*_r(G)}(D) \) for a twisted \( \text{Spin}^c \) Dirac operator on a proper cocompact \( G \)-manifold on \( X \). To introduce their result, we fix the following set up.

- \( X^g \) is the \( g \)-fixed point submanifold;
- \( N_{X^g} \) is the normal bundle of \( X^g \) in \( X \) and \( R^N \) is the curvature form associated to the Hermitian connection on \( N_{X^g} \otimes \mathbb{C} \);
- \( L_{\text{det}} \) is the determinant line bundle of the \( \text{Spin}^c \)-structure on \( X \) and \( L_{\text{det}|_{X^g}} \) is its restriction to \( X^g \) and \( R^L \) is the curvature form associated to the Hermitian connection on \( L_{\text{det}|_{X^g}} \);
- \( R_{X^g} \) is the Riemannian curvature form associated to the Levi-Civita connection on the tangent bundle of \( X^g \);
- The \( \text{AS}_g(D) \) has the following expression for a twisted Dirac operator on \( E = S \otimes F \):

\[
\text{AS}_g(D) := \frac{\text{A}(\frac{R_{X^g}}{2\pi i}) \text{tr}(g \exp(\frac{R^N}{2\pi i})) \exp(\text{tr}(\frac{R^L}{2\pi i}))}{\text{det}(1 - g \exp(-\frac{R^N}{2\pi i}))^{\frac{1}{2}}}.
\]

(4.10)

This will also be denoted by \( \text{AS}_g(X, E) \) or, if there is no confusion on the vector bundle \( E \), simply by \( \text{AS}_g(X) \).

Hochs and Wang proved the following result using heat kernel and Getzler’s rescaling techniques\( 4 \).

4.11. Theorem. (1) If \( G \) has a compact Cartan subgroup, then

\[
\langle \text{tr}_g, \text{Ind}_{C^*_r(G)}(D) \rangle = \int_{X^g} c_{X^g} \text{AS}_g(D),
\]

for a cutoff function \( c_{X^g} \in C^\infty_c(X^g) \) for the \( Z_g \)-action on \( X^g \), with \( Z_g \) equal to the centralizer subgroup of \( g \) in \( G \);

(2) If \( G \) does not have a compact Cartan subgroup,

\[
\langle \text{tr}_g, \text{Ind}_{C^*_r(G)}(D) \rangle = 0.
\]

4.4. Delocalized higher indices. To improve Theorem 4.11 to allow \( G \) to be nonequal rank, in \( 25 \), Hochs, Song, and Tang computed the pairing between the cocycle \( \Phi^g \) and the index element \( \text{Ind}_{C^*_r(G)}(D) \). Let \( X/AN \) be the quotient of \( X \) with respect to the \( AN < G \) action. The group \( M \) acts properly and cocompactly on the quotient \( X/AN \); for \( g \) a semisimple element of \( G \), \( (X/AN)^g \) is the fixed

\[\text{We refer also to the recent article} \ 49 \ \text{for a related, detailed discussion of this result.}\]
submanifold of the action on the quotient $X/AN$: $c^g_{X/AN}$ is a smooth compactly supported cut-off function on $(X/AN)^g$.

Let $\mathfrak{m}$ be the Lie algebra of the group $M$, and $\mathfrak{a}$ the Lie algebra of the group $A$. Using the $K$-invariant metric on $\mathfrak{p}$, we obtain a $K \cap M$-invariant decomposition

$$\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a} \oplus (\mathfrak{t}/(\mathfrak{t} \cap \mathfrak{m})),$$

which induces the $K \cap M$ decomposition of spinors,

$$S_\mathfrak{p} \cong S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_\mathfrak{a} \otimes S_{\mathfrak{t}/(\mathfrak{t} \cap \mathfrak{m})}.$$

Using the slice theorem, Theorem 2.2, $X/AN$ can be identified as $X/AN = M \times_{K \cap M} S$. On $X/AN$, we consider the bundle $E_{X/AN}$,

$$E_{X/AN} = M \times_{M \cap K} (S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_{\mathfrak{t}/(\mathfrak{t} \cap \mathfrak{m})} \otimes \mathcal{W}|_S) \cong M \times_{M \cap K} (S_{\mathfrak{p} \cap \mathfrak{m}} \otimes \hat{\mathcal{W}}),$$

where we denote $S_{\mathfrak{t}/(\mathfrak{t} \cap \mathfrak{m})} \otimes \mathcal{W}|_S$ by $\hat{\mathcal{W}}$ and $M \times_{M \cap K} \hat{\mathcal{W}}$ by $W_{X/AN}$. We observe that $W_{X/AN}$ is an $M$-equivariant Hermitian vector bundle on $X/AN$. On $E_{X/AN}$, we consider the associated Dirac operator $D_{X/AN}$. The index of $D_{X/AN}$ is an element of $\text{Ind}_{C^\bullet(M)}(D_{X/AN})$ in $K^*(C(M))$.

Hochs, Song, and Tang proved a reduction theorem relating the index pairing for $D$ and the index pairing for $D_{X/AN}$.

4.12. Theorem.

$$\langle \Phi_g^D, \text{Ind}_{C^\bullet(G)}(D) \rangle = \langle \text{tr}_g^M, \text{Ind}_{C^\bullet(M)}(D_{X/AN}) \rangle.$$

The following theorem follows directly from Theorem 4.11 and 4.12.

4.13. Theorem.

$$\langle \Phi_g^P, \text{Ind}_{C^\bullet(G)}(D) \rangle = \int_{(X/AN)^g} c_{(X/AN)^g} AS(X/AN)_g.$$ 

4.5. The example of $G/K$. We look at the example of $X = G/K$ with the left $G$ action. Song and Tang prove the following theorem using Harish-Chandra's theory of orbital integrals; it can also be derived from Theorem 4.13 as a corollary.

4.14. Theorem. Let $H$ be a Cartan subgroup of $G$, and $T := K \cap H$ with $t \in T$. Assume that $T < M$ is a Cartan subgroup of $M$, and $P$ is a maximal cuspidal parabolic subgroup, i.e., $T$ is maximal among all possible $P$. Let $\Delta_M^H$ be the corresponding Weyl denominator.

(1)

$$\langle \Phi_e^P, \text{Ind}_{C^\bullet(G)}(D) \rangle = \frac{1}{|W_{M \cap K}|} \sum_{w \in W_K} m\left(\sigma^M(w \cdot \mu)\right),$$

where $\sigma^M(w \cdot \mu)$ is the discrete series representation of $M$ with Harish-Chandra parameter $w \cdot \mu$, and $m\left(\sigma^M(w \cdot \mu)\right)$ is its Plancherel measure, and $W_{M \cap K}$ is the Weyl group of $M \cap K$;

(2)

$$\langle \Phi_e^P, \text{Ind}(D) \rangle = \frac{\sum_{w \in W_K} (-1)^w e^{w \cdot \mu}(t)}{\Delta_M^H(t)}.$$
4.17. **Remark.** When \( G \) is equal rank, Eq. (4.15) was proved by Connes-Moscovici [13], i.e. the \( L^2 \)-trace of the operator \( \overline{D} \) is the formal degree of the associated discrete series representation of \( G \).

4.18. **Remark.** When the kernel of \( D \) gives a discrete series representation, i.e. \( G \) has equal rank, Equation (4.16) in Theorem 4.14 may be derived from Theorem 4.13 and the computation in Hochs and Wang [28]. Here, we do not assume \( G \) to have equal rank in Theorem 4.14 and allow the kernel of \( D \) to be a limit of discrete series representation. Equation (4.16) suggests that the index pairing can be used to detect some of the limits of discrete series representations of \( G \).

4.19. **Remark.** In Theorem 4.15, we have assumed \( P \) to be maximal. When \( P \) is not maximal, it can be shown that the pairing in Theorem 4.14 vanishes. As a special case of Theorem 4.6, we have the following theorem generalizing the Connes-Moscovici \( L^2 \)-index theorem, Theorem 4.2, for homogeneous spaces.

4.20. **Theorem.** Suppose that \( G \) is unimodular. For a \( G \)-invariant Dirac operator on \( X = G/K \) and \( \varphi \in H^*_\text{diff}^k(\mathbb{G}) \), we have

\[
\chi(\varphi)(\text{Ind}_{\mathcal{C}^*_r(G)}(D)) = \frac{1}{(2\pi\sqrt{-1})^k} \left< A(g, K) \wedge \text{ch}(V_\mu) \wedge \Phi([\varphi]), [V] \right>,
\]

where \( \Phi([\varphi]) \) is a class in \( H^{2k}(g, K) \) defined by a \( G \)-invariant closed differential form on \( G/K \) via the van Est isomorphism in Proposition 4.5.

4.6. **Summary of results for manifolds without boundary.** We have introduced 0-cyclic cocycles \( \text{tr}_r \) and \( \text{tr}_g \) on the Harish-Chandra algebra \( \mathcal{C}(G) \) and we have stated index theorems computing the pairing of these 0-cyclic cocycles with the index class \( \text{Ind}_{\mathcal{C}^*_r(G)}(D) \in K_0(\mathcal{C}(G)) \equiv \text{Ind}_{\mathcal{C}^*_r(G)}(D) \in K_0(\mathcal{C}^*(G)) \):

\[
\text{tr}_r(\text{Ind}_{\mathcal{C}^*_r(G)}(D)), \quad \text{tr}_g(\text{Ind}_{\mathcal{C}^*_r(G)}(D))
\]

These results are due to Wang, c.f. Theorem 4.3 and Hochs-Wang, c.f. Theorem 4.11. We have then stated generalizations of these two theorems: the first theorem, c.f. Theorem 4.6, by Pflaum-Posthuma-Tang, computes the pairing of the index class with the cyclic cocycles associated to cocycles in the differentiable group cohomology introduced in Theorem 3.10; the second theorem, c.f. Theorem 4.13 by Hochs-Song-Tang, computes the pairing between the index class and the higher orbital integrals introduced in Theorem 3.16.

5. **Proper cocompact \( G \)-manifolds with boundaries**

In this section, we introduce the index class for a Dirac operator on a proper cocompact \( G \)-manifold.

5.1. **Geometry on \( G \) proper manifolds with boundary.** We start with some generalities. Let \( Y_0 \) be a manifold with boundary, \( G \) a finitely connected Lie group acting properly and cocompactly on \( Y_0 \). We denote by \( X \) the boundary of \( Y_0 \). There exists a collar neighbourhood \( U \) of the boundary \( \partial Y_0 \), \( U \cong [0, 2] \times \partial Y_0 \), which is \( G \)-invariant and such that the action of \( G \) on \( U \) is of product type. We assume that \( Y_0 \) is endowed with a \( G \)-invariant metric \( h_0 \) which is of product type near the boundary. We let \( (Y_0, h_0) \) be the resulting Riemannian manifold with boundary; in the collar neighborhood \( U \cong [0, 2] \times \partial Y_0 \) the metric \( h_0 \) can be written, through...
the above isomorphism, as \( dt^2 + h_X \), with \( h_X \) a \( G \)-invariant Riemannian metric on \( X = \partial Y_0 \). We denote by \( c_{Y_0} \) a cut-off function for the action of \( G \) on \( Y_0 \); since the action is cocompact, this is a compactly supported smooth function. We consider the associated manifold with cylindrical ends \( \tilde{Y} := Y_0 \cup \partial Y_0 \times (-\infty, 0] \times \partial Y_0 \), endowed with the extended metric \( \tilde{h} \) and the extended \( G \)-action. We denote by \((Y, h)\) the \( b \)-manifold associated to \((\tilde{Y}, \tilde{h})\). We shall often treat \((\tilde{Y}, \tilde{h})\) and \((Y, h)\) as the same object. We denote by \( c_Y \) the obvious extension of the cut-off function \( c_{Y_0} \) for the action of \( G \) on \( Y_0 \) (constant along the cylindrical end); this is a cut-off function of the extended action of \( G \) on \( Y \). If \( x \) is a boundary defining function for the cocompact \( G \)-manifold \( Y_0 \), then the \( b \)-metric \( h \) has the following product-structure near the boundary \( X \):

\[
\frac{dx^2}{x^2} + h_X.
\]

Let us fix a slice \( Z_0 \) for the \( G \) action on \( Y_0 \); thus

\[ Y_0 \cong G \times_K Z_0 \]

with \( K \) a maximal compact subgroup of \( G \) and \( Z_0 \) a smooth compact manifold with boundary endowed with a \( K \)-action. We denote by \( S \) the boundary \( \partial Z_0 \). Consequently, \( Y \cong G \times_K Z \) with \( Z \) the \( b \)-manifold associated to \( Z_0 \) and the boundary

\[ X = \partial Y_0 \cong G \times_K S. \]

We shall concentrate on the case where the symmetric space \( G/K \), the \( G \)-manifold \( Y_0 \) and the \( K \)-slice \( Z_0 \) are all even dimensional.

5.1. Example. Start with an inclusion \( K \subset G \) of Lie groups with \( K \) compact, and let \( Z_0 \) be a compact \( K \)-manifold with boundary \( \partial Z_0 =: S_0 \). Then \( Y_0 := G \times_K Z_0 \) is an example of manifold with boundary \( \partial Y_0 = G \times_K \partial S_0 \), equipped with a proper, cocompact action of \( G \). We also have the associated \( b \)-manifolds \( Z \) and \( Y = G \times_K Z \).

If we choose a \( K \)-invariant inner product on the Lie algebra \( \mathfrak{g} \) of \( G \) and a \( K \)-invariant \( b \)-metric \( g_Z \) on \( Z \) of product-type near the boundary then, as in the closed case, we obtain a \( G \)-invariant metric on \( Y \); we call such a metric slice compatible.

As an explicit example, consider \( G = SL(2, \mathbb{R}) \) and \( K = SO(2) \) acting on the unit disk \( \mathbb{D}^2 \) in the complex plane by rotations around the origin. The resulting manifold \( Y := SL(2, \mathbb{R}) \times_{SO(2)} \mathbb{D}^2 \) is a 4-dimensional fiber bundle over hyperbolic 2-space \( SL(2, \mathbb{R})/SO(2) \cong \mathbb{H}^2 \) with fiber \( \mathbb{D}^2 \). The boundary \( \partial Y \) of this manifold is isomorphic to \( SL(2, \mathbb{R}) \).

5.2. Dirac operators.

We assume the existence of a \( G \)-equivariant bundle of Clifford modules \( E_0 \) on \( Y_0 \), endowed with a Hermitian metric, product-type near the boundary, for which the Clifford action is unitary, and equipped with a Clifford connection also of product type near the boundary. Associated to these structures there is a generalized \( G \)-invariant Dirac operator \( D \) on \( Y_0 \) with product structure near the boundary acting on the sections of \( E_0 \). We denote by \( D_0 \) the operator induced on the boundary.

We employ the same symbol, \( D \), for the associated \( b \)-Dirac operator on \( M \), acting on the extended Clifford module \( E \). We also have \( D_{\text{cyl}} \) on \( \mathbb{R} \times \partial Y_0 \equiv \text{cyl}(\partial Y_0) \). We shall make the following fundamental assumption.

5.2. Assumption. There exists \( \alpha > 0 \) such that

\[
\text{spec}_{L^2}(D_0) \cap [-\alpha, \alpha] = \emptyset.
\]
We also have $D_{\text{cyl}}$ on $\mathbb{R} \times \partial Y_0 \equiv \text{cyl}(\partial Y_0)$. It should be noticed that because of the self-adjointness of $D_{\partial}$, assumption (5.3) implies the $L^2$-invertibility of $D_{\text{cyl}}$.

5.4. Example. As an example where this condition is satisfied we can consider a $G$-proper manifold with boundary with a $G$-invariant riemannian metric and a $G$-invariant Spin structure with the property that the metric on the boundary is of positive scalar curvature. We would then consider the Spin-Dirac operator $D$; because of the psc assumption on the boundary we do have that $D_{\partial}$ is $L^2$-invertible. These manifolds arise as in 17 from the slice theorem and a $K$-Spin-manifold with boundary $S$ endowed with a $K$-invariant metric which is of psc on $\partial S$. Such compact manifolds $S$ arise, for example, as follows. Consider a compact $K$-manifold without boundary $N$ endowed with a $K$-invariant metric of positive scalar curvature. For the existence of such manifolds see for example 18 32 60. We can now perform on this manifold $K$-equivariant surgeries and produce along the process a $K$-manifold with boundary $W$. Under suitable conditions (for example, equivariant surgeries only of codimension at least equal to 3) this manifold with boundary $W$ will have a $K$-invariant metric of positive scalar curvature. We can now take the connected sum of $W$ with a closed $K$-manifold not admitting a $K$-invariant metric of psc. See 18 60. The result will be a manifold $S$ with a $K$-invariant metric which is of psc (only) on $\partial S$.

5.3. The index class and $b$-calculus. Let $Y_0$ be a $G$-proper manifold with boundary, with compact quotient, and let $Y$ be the associated manifold with cylindrical ends, or, equivalently, the associated $b$-manifold. In the $b$-picture we consider $\mathcal{E}_b$, the $C^*_b\text{-}G$-Hilbert module obtained by (double) completion of $\mathcal{C}^\infty_c(Y, E)$ endowed with the $C^\infty_c(G)$-valued inner product

$$(e, e')_{C^\infty_c(G)}(x) := (e, x \cdot e')_{L^2_b(Y, E)}, \quad e, e' \in \mathcal{C}^\infty_c(Y, E), \quad x \in G.$$  

Here the dot means vanishing of infinite order at the boundary; notice that on the right hand side we employ $L^2_b(Y, E)$, the $L^2$ space associated to the volume form associated to the $b$ metric $h$. One can prove that $\mathcal{K}(\mathcal{E}_b)$ is isomorphic to the relative Roe algebra $C^*_0(Y_0 \subset Y, E)^G$, the latter defined by completion of operators with propagation at a finite distance from $Y_0$. See [51] for precise definitions and proofs.

We then have the following canonical isomorphisms

$$(5.5) \quad K_\ast(\mathcal{K}(\mathcal{E}_b)) \cong K_\ast(C^*_0(Y_0 \subset Y, E)^G) \cong K_\ast(C^*_0(Y_0, E_0)^G) \cong K_\ast(C^*_0(G))$$

with the second isomorphism also explained in [51] and the third one the Morita isomorphism already explained in these notes.

We now want to prove the existence of an index class $\text{Ind}(D) \in K_0(C^*_0(Y \subset Y, E)^G)$ under Assumption 5.2 next, we shall want to find a smooth, i.e. dense and holomorphically closed, subalgebra $A^\infty(Y, E)$ of $C^*_0(Y_0 \subset Y, E)^G$ and a smooth representative $\text{Ind}_\infty(D) \in K_0(A^\infty(Y, E))$ corresponding to $\text{Ind}(D) \in K_0(C^*_0(Y \subset Y, E)^G)$ under the natural isomorphism between $K_0(A^\infty(Y, E))$ and $K_0(C^*_0(Y \subset Y, E)^G)$.

We use the $b$-pseudodifferential calculus as in 48, even though this is not strictly necessary for the first task. For a detailed account of the $b$-calculus we refer the reader to Melrose book, 38.

Let us recall very briefly the basics of the $b$-calculus. For simplicity we assume that the boundary of our cocompact $G$-proper manifold $Y_0$ is connected. As above, we denote by $Y$ the associated $b$-manifold. We sometimes denote the boundary of $Y_0$...
by $X$; for notational convenience we set $\partial Y = X$. Finally, we often expunge
the vector bundle $E$ from the notation.

We can define the $b$-stretched product $Y \times_b Y$ which inherits in a natural
way an action of $G \times G$ and a diagonal action of $G$, see [33]. Proceeding as in these
references, we can define the algebra of $G$-equivariant $b$-pseudodifferential operators
on $Y$ with $G$-compact support, denoted $b\Psi^{G,c}(Y)$, which is a $\mathbb{Z}$-graded algebra.

Let us fix $\epsilon > 0$. Then, as in the compact case, we can extend the algebra
$b\Psi^{G,c}(Y)$ and consider the $G$-equivariant $b$-calculus of $G$-compact support with
$\epsilon$-bounds, denoted $b\Psi^{G,c}(Y,\epsilon)$. Thus, by definition,

$$b\Psi^{G,c}(Y,\epsilon) = b\Psi^{G,c}(Y) + b\Psi^{-\infty,\epsilon}(Y) + \Psi^{-\infty,\epsilon}(Y).$$

The second term on the right hand side corresponds to the Schwartz kernels on
$Y \times_b Y$, smooth in the interior, conormal of order $\epsilon$ on the left and right boundaries
of $Y \times_b Y$ and smooth up to the front face (and of course $G$-equivariant). The third
term on the right hand side corresponds instead to Schwartz kernels on $Y \times Y$, smooth in the interior and conormal of order $\epsilon$ on the boundary hypersurfaces
of $Y \times Y$. See [38] and for a quick treatment the Appendix in [39]. Elements
in $\Psi^{G,c}(Y)$ are called residual. Composition formulae for the elements in the
algebra with bounds are as in [39] Theorem 4).

Restriction to the front face of $Y \times_b Y$ defines the indicial operator $I : b\Psi^{\infty,\epsilon}(Y) \to b\Psi^{\infty,\epsilon}_G(\partial Y)$ and, more generally,

$$I : b\Psi^{-\infty,\epsilon}_G(Y) \to b\Psi^{-\infty,\epsilon}_G(\partial Y)$$

with $\partial Y$ denoting the compactified inward-pointing normal bundle to the boundary
and the subscript $\mathbb{R}^+$ denoting equivariance with respect to the natural $\mathbb{R}^+$-
action. See [38] Section 4.15], where $\partial Y$ is denoted $Y$. Recall that there is a
diffeomorphism of $b$-manifolds: $\partial Y \simeq \partial Y \times [-1,1]$. Metrically we can think of
$\partial Y$ as the cylinder $\partial Y := \mathbb{R} \times \partial Y$ and with a small abuse of notation we
shall adopt this notation henceforth, thus writing (5.6) as

$$I : b\Psi^{-\infty,\epsilon}_G(Y) \to b\Psi^{-\infty,\epsilon}_G(\partial Y)$$

By fixing a cut-off function $\chi$ on the collar neighborhood of the boundary, equal
to 1 on the boundary, we can define a section $s$ to the indicial homomorphism $I$:

$$s : b\Psi^{-\infty,\epsilon}_G(\partial Y) \to b\Psi^{-\infty,\epsilon}_G(Y)$$

the linear map $s$ associates to an $\mathbb{R}^+$-invariant operator $G$ on $\partial Y$ an operator
on the $b$-manifold $Y$; the latter is obtained by pre-multiplying and post-multiplying
by the cut-off function $\chi$.

The Mellin transform in the $s$-variable for an $\mathbb{R}^+$-invariant kernel $\kappa(s, y, y') \in b\Psi^{-\infty,\epsilon}_G(\partial Y)$ defines an isomorphism $\mathcal{M}$ between $b\Psi^{-\infty,\epsilon}_G(\partial Y)$ and holomorphic families of operators

$$\{ \mathbb{R} \times i(-\epsilon, \epsilon) \ni \lambda \to \Psi^{G,c}_G(\partial Y) \}$$

which rapidly decrease as $|\text{Re}\lambda| \to \infty$ as functions with values in the Fréchet algebras $\Psi^{G,c}_G(\partial Y)$. We denote by $I(R, \lambda)$ or sometimes by $\tilde{R}(\lambda)$, the Mellin transform
of $R \in b\Psi^{-\infty,\epsilon}_G(\partial Y)$; if $P \in b\Psi^{-\infty,\epsilon}_G(Y)$ then its indicial family is by definition
the indicial family associated to its indicial operator \( I(P) \). The inverse \( \mathcal{M}^{-1} \) is obtained by associating to a holomorphic family \( \{ S(\lambda), |\text{Im}\lambda| < \epsilon \} \), rapidly decreasing in \( \text{Re}\lambda \), the \( \mathbb{R}^+ \)-invariant Schwartz kernel \( \kappa_S \) that in projective coordinates is given by

\[
(5.9) \quad \kappa_G(s, y, y') = \int_{\text{Im}\lambda=r} s^\lambda G(\lambda)(y, y') d\lambda
\]

with \( r \in (-\epsilon, \epsilon) \).

Let \( D \) be an equivariant \( \mathbb{Z}_2 \)-graded odd Dirac operator, of product type near the boundary and let us make Assumption 5.2. We sketch the proof of the existence of a \( C^* \)-index class associated to \( D^+ \). One begins by finding a symbolic parametrix \( Q_\sigma \) to \( D^+ \), with remainders \( R_\sigma^\pm \):

\[
(5.10) \quad Q_\sigma D^+ = \text{Id} - R_\sigma^+, \quad D^+ Q_\sigma = \text{Id} - R_\sigma^-.
\]

We can choose \( Q_\sigma \in b\Psi_{G,c}^{-1}(Y) \), i.e. of \( G \)-compact support and then get \( R_\sigma^\pm \in b\Psi_{G,\infty}^{-1}(Y) \).

Consider now the indicial operator \( I(D^+) \equiv D^+_\text{cyl} \equiv \partial/\partial t + D_\partial; \) by Assumption 5.2 we know that this operator is \( L^2 \)-invertible on the \( b \)-cylinder \( \text{cyl}(\partial Y) \). Consider the smooth kernel \( K(Q') \) on \( Y \times_Y Y \) which is zero outside a neighbourhood \( U_H \) of the front face and such that in \( U_H \) with (projective) coordinates \( (x, s, y, y') \) is equal to

\[
K(Q')(s, x, y, y') := \chi(x) \int_{-\infty}^{\infty} s^\lambda K((D_\partial + i\lambda)^{-1} \circ I(R_\sigma^-, \lambda))(y, y') d\lambda.
\]

Put it differently, \( Q' = s(I(D)^{-1}I(R_\sigma)) \), with \( s \) as in (5.8) and where we have used the inverse Mellin transform for the expression of \( I(D)^{-1}I(R_\sigma) \). Notice that because of the presence of \((D_\partial + i\lambda)^{-1}\) this is not compactly supported. Still, by proceeding exactly as in [35 Lemma 4 and 5] we can establish the following fundamental result:

5.11. THEOREM. The kernel \( K(Q') \) defines a bounded operator on the \( C^* \)-\( G \)-module \( \mathcal{E}_b \): \( Q' \in \mathbb{B}(\mathcal{E}_b) \). If \( Q^b := Q_\sigma - Q' \), then \( Q^b \) provides an inverse of \( D^+ \) modulo elements in \( \mathbb{K}(\mathcal{E}_b) \). Thus, for a Dirac operator \( D \) satisfying the Assumption 5.2 there is a well defined index class

\[
\text{Ind}_{c^*}(D) \in K_0(\mathbb{K}((\mathcal{E}_b)) \equiv K_0(C^*(Y_0 \subset Y, E)^G)\] .

We sometime refer to the passage from \( Q_\sigma \) to \( Q^b = Q_\sigma - Q' \) as the passage from a symbolic parametrix to a true parametrix or an improved parametrix.

The \( b \)-calculus approach to the \( C^* \) index class will prove to be useful in establishing the existence of a smooth index class, see below, and for proving a higher APS index formula. However, there are other approaches to the \( C^* \)-index class. Indeed, in the recent article [27] a coarse approach is explained. It should not be difficult to show that the two index classes, the one defined here and the one defined in [27] are in fact the same. (The proof should proceed as for Galois coverings, where the analogous result is established in [51 Proposition 2.4].)

As in the case of Galois coverings, and always under the invertibility Assumption 5.2 it should also be possible to establish the existence of an APS \( C^* \)-index class through a boundary value problem defined through the spectral projection.
χ(0,∞)(D₀) and show that it is equal to the index class as defined above. See [34] for the case of Galois coverings.

Assume now that G is connected, or more generally that |π₀(G)| < ∞; we can apply to Y the slice theorem and obtain a diffeomorphism

$$G \times_K Z \cong Y$$

with Z a compact manifold with boundary. One can prove, as in the closed case, that

$$bΨ_{G,c}^{-∞,ε}(Y) = (C^∞_c(G) \otimes bΨ_{∞,ε}(Z))^{K \times K}.$$ A similar description can be given for Ψ_{G,c}^{-∞,ε}(Y).

We set

$$−∞ \rightarrow A \rightarrow \cdots$$

and observe that the indicial operator induces a surjective algebra homomorphism

$$bA_{G,c}^{-∞,ε}(Y) \rightarrow bA_{G,∞}(cyl(∂Y))$$

where I is equal to the indicial operator on $$bΨ_{G,c}^{-∞,ε}(Y) = (C^∞_c(G) \otimes bΨ_{∞,ε}(Z))^{K \times K}$$ and it is defined as zero on $$Ψ_{G,c}^{-∞,ε}(Y) = (C^∞_c(G) \otimes Ψ_{∞,ε}(Z))^{K \times K}$$.

**Notation:** from now on we shall omit the ε from the notations for $$bA_{G,c}^{-∞,ε}(Y)$$ and other related algebras.

We obtain in this way algebras of operators on Y fitting into a short exact sequence

$$0 → A_{G}(Y) → bA_{G}(Y) → bA_{G,R}(cyl(∂Y)) → 0.$$ Here

$$0 → A_{G}(Y) := \text{Ker}(bA_{G,c}(Y) \rightarrow bA_{G,R}(cyl(∂Y))).$$

5.4. Harish-Chandra smoothing operators. Recall that

$$bA_{G,c}^{-∞,ε}(Y) = (C^∞_c(G) \otimes bΨ_{∞,ε}(Z))^{K \times K} + (C^∞_c(G) \otimes Ψ_{∞,ε}(Z))^{K \times K},$$

$$bA_{G,R}(cyl(∂Y)) = (C^∞_c(G) \otimes bΨ_{∞,ε}(cyl(∂Z)))^{K \times K}.$$ Employing the Harish-Chandra algebra C(G) instead of $$C^∞_c(G)$$ on the right hand side we can define the algebras $$bA_{G}^{-∞}(Y)$$ and $$bA_{G,R}(cyl(∂Y))$$ fitting into the short exact sequence

$$0 → A_{G}^{-∞}(Y) → bA_{G}^{-∞}(Y) → bA_{G,R}(cyl(∂Y)) → 0$$

with $$A_{G}^{-∞}(Y) = \text{Ker}(bA_{G,c}^{-∞}(Y) \rightarrow bA_{G,R}(cyl(∂Y))).$$ This extends [5.13] and will play a crucial role in our analysis. Similar short exact sequences can be defined when the operators act on sections of a vector bundle E. One can prove, see [48], the following result:

5.16. Proposition. The algebra $$A_{G}^{-∞}(Y,E)$$ is a dense and holomorphically closed subalgebra of the Roe algebra $$C^*(Y_0 \subset Y,E)^G$$. Consequently, $$K_*(A_{G}^{-∞}(Y,E)) \simeq K_*(C^*(Y_0 \subset Y,E)^G).$$
5.5. The smooth index class. Bearing in mind the last proposition and analyzing in great detail the Schwartz kernels of the operator involved, one can prove the following crucial results, see [48].

5.17. Theorem. Let $D$ be as above and let $Q^*$ be a symbolic $b$-parametrix for $D$. The Connes-Skandalis projector

$$P^b_Q := \begin{pmatrix} bS_2^+ & bS_+(I + bS_\mp)Q^b \\ bS_-D^+ & I - bS_-^2 \end{pmatrix}$$

associated to a true parametrix $Q^b = Q^* - Q'$ with remainders $bS_\pm$ in $\mathcal{A}_G^{\infty}(Y_0)$ is a $2 \times 2$ matrix with entries in $\mathcal{A}_G^{\infty}(Y)$. We thus have a well-defined smooth index class

$$(5.19) \quad \text{Ind}_\infty(D) := [P^b_Q] - [e_1] \in K_0(\mathcal{A}_G^{\infty}(Y)) \equiv K_0(\mathcal{C}^*(Y_0 \subset Y)^G) \quad \text{with} \quad e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

We can consider in particular the parametrix

$$Q_{\exp} := \frac{I - \exp(-\frac{1}{2}D - D^+)}{D - D^+} D^+$$

and its associated true parametrix $Q^b_{\exp}$. Using the first, we can define the Connes-Moscovici projector $V_{CM}(D)$; using the latter we define the improved Connes-Moscovici projector $V^b_{CM}(D)$. We then have:

5.20. Theorem. The projector $V^b_{CM}(D)$ obtained from the true parametrix $Q^b_{\exp}$ associated to $Q_{\exp}$ is a $2 \times 2$ matrix with entries in $\mathcal{A}_G^{\infty}(Y)$ and defines the same smooth index class in $K_0(\mathcal{A}_G^{\infty}(Y))$ as the Connes-Skandalis projector of Theorem 5.17.

We can of course extend these results to the associated symmetrized projectors.

5.6. Higher APS-indices. We now want to extract numbers out of our index class $\text{Ind}_\infty(D) \in K_0(\mathcal{A}_G^{\infty}(Y)) = K_0(\mathcal{C}^*(Y_0 \subset Y)^G)$ and we shall do so by pairing the index class with the cyclic cocycles already encountered in the closed case, that is

- the 0-cocycle $\text{tr}_g$, $g \in G$, with $g = e$ or $g \neq e$ a semisimple element,
- the higher cocycles $\tau^\varphi$, with $\varphi \in H^*_\text{diff}(G)$,
- the higher delocalized cocycles $\Phi^P_g$, with $P$ a cuspidal parabolic subgroup and $g$ a semisimple element.

These cocycles do define elements in $H P^*(\mathcal{A}_G^c(Y, E))$, because the Schwartz kernels of the operators in $\mathcal{A}_G^c(Y, E)$ vanish on all of the boundary hypersurfaces of $Y \times \mathcal{A}_G^c(Y)$. The following proposition is proved as in the closed case, using the information that the corresponding cyclic cocycles on the group $G$, that is

$$\text{tr}_g, \quad \chi(\varphi), \quad \Phi^P_g,$$

do extend from $\mathcal{C}^c(G)$ to $\mathcal{C}(G)$.

As usual, the entries really live in a slightly extended algebra because of the identity appearing in the right lower corner of the matrix.
5.21. Proposition. The cyclic cocycles

\[ \text{tr}^Y, \; \tau^Y, \; \Phi^P_{Y,g} \]

extend continuously from \( \mathcal{A}_G^\infty(Y,E) \) to \( \mathcal{A}_G^{\text{even}}(Y,E) \), thus defining elements in the periodic cyclic cohomology \( H^{\text{even}}(\mathcal{A}_G^{\text{even}}(Y,E)) \).

Using the pairing between \( H^{\text{even}}(\mathcal{A}_G^{\text{even}}(Y,E)) \) and \( K_0(\mathcal{A}_G^{\text{even}}(Y,E)) \) we obtain the numerical indices

\[ \langle \text{tr}^Y, \text{Ind}_\infty(D) \rangle, \; \langle \tau^Y, \text{Ind}_\infty(D) \rangle, \; \langle \Phi^P_{Y,g}, \text{Ind}_\infty(D) \rangle. \]

In the next section we shall present formulae à la Atiyah-Patodi-Singer for these (higher) indices.

6. Higher APS index theorems

In this section we shall finally present index formulae of Atiyah-Patodi-Singer type for the numerical indices appearing in (5.23). We shall employ in a crucial way relative K-theory and relative cyclic cohomology, as in previous work of Moriyoshi-Piazza, Lesch-Moscovici-Pflaum and Gorokhovsky-Moriyoshi-Piazza, see [16, 36, 41]. To be more precise, and forgetting the bundle \( E \) in the notation, we shall proceed as follows. Consider the surjective homomorphism

\[ b \mathcal{A}_G^\infty(Y) \xrightarrow{\text{cyl}(\partial Y)} b \mathcal{A}_{G,R+}^\infty(\text{cyl}(\partial Y)). \]

Then, as a first step, associated to (6.1), we shall define a relative index class

\[ \text{Ind}_\infty(D, D_0) \in K_0(b \mathcal{A}_G^\infty(Y), b \mathcal{A}_{G,R+}^\infty(\text{cyl}(\partial Y))) \]

where, with a small abuse of notation, we do not write the indicial homomorphism.

Next, let \( \omega^Y \) be one of the cyclic cocycles appearing in (5.22); then

- to the cyclic cocycle \( \omega^Y \) on \( \mathcal{A}_G^\infty(Y) \) we associate a regularised cyclic cochain on \( b \mathcal{A}_{G,R+}^\infty(\text{cyl}(\partial Y)) \) and a cyclic cocycle \( \xi^{\partial Y} \) on \( b \mathcal{A}_{G,R+}^\infty(\text{cyl}(\partial Y)) \) so that \( \langle \omega^Y, \xi^{\partial Y} \rangle \) is a relative cyclic cocycle for the homomorphism (6.1);
- we call \( \xi^{\partial Y} \) the eta cocycle associated to \( \omega^Y \) and we call \( \langle \omega^Y, \xi^{\partial Y} \rangle \) the relative cyclic cocycle associated to \( \omega^Y \);
- we prove that

\[ \langle \omega^Y, \text{Ind}_\infty(D) \rangle = \langle (\omega^Y, \xi^{\partial Y}), \text{Ind}_\infty(D, D_0) \rangle. \]

Let us see some of details involved in this program.

6.1. Relative index classes. Recall that if \( 0 \to J \to A \xrightarrow{\pi} B \to 0 \) is a short exact sequence of Fréchet algebras then we set \( K_0(J) := K_0(J^+, J) \cong \text{Ker}(K_0(J^+) \to \mathbb{Z}) \) and \( K_0(A^+, B^+) = K_0(A, B) \) with \( (\cdot)^+ \) denoting unitalization. See [6, 21]. Recall that a relative \( K_0 \)-element for \( A \xrightarrow{\pi} B \) with unital algebras \( A, B \) is represented by a triple \( (P, Q, p_t) \) with \( P \) and \( Q \) idempotents in \( M_{n \times n}(A) \) and \( p_t \in M_{n \times n}(B) \) a path of idempotents connecting \( \pi(P) \) to \( \pi(Q) \). The excision isomorphism

\[ \alpha_{\pi} : K_0(J) \to K_0(A, B) \]

is given by \( \alpha_{\pi}([P, Q]) = [(P, Q, c)] \) with \( c \) denoting the constant path.

Let us go back to the parametrix

\[ Q_{\exp} := \frac{I - \exp(-\frac{1}{2}D^-D^+)}{D^-D^+} D^+, \]

and its associated true parametrix $Q^b_{\exp}$. Using the latter we have defined the improved Connes-Moscovici projector $V^b_{CM}(D)$, with entries in $\mathcal{A}^\infty(Y, E)$. Using the former we can now define the usual Connes-Moscovici projector $V_{CM}(D)$. We have the following important result:

6.4. Theorem.

1) The Connes-Moscovici projector $V_{CM}(D)$,

$$V_{CM}(D) := \begin{pmatrix} e^{-D^+D^-} & e^{-\frac{1}{2}D^+D^-} \left( \frac{I - e^{-D^+D^-}}{D^+D^-} \right) D^- \\ e^{-\frac{1}{2}D^+D^-} & I - e^{-D^+D^-} \end{pmatrix},$$

is a $2 \times 2$ matrix with entries in $b \mathcal{A}^\infty_G(Y)$;

2) the Connes-Moscovici projector $V_{CM}(D_{cyl})$ is a $2 \times 2$ matrix with entries in $b \mathcal{A}^\infty_{G, \mathbb{R}}(cyl(\partial Y))$.

Consider the Connes-Moscovici projections $V_{CM}(D)$ and $V_{CM}(D_{cyl})$ associated to $D$ and $D_{cyl}$. With $e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have the triple,

$$(V_{CM}(D), e_1, q_t), \quad t \in [1, +\infty], \quad \text{with } q_t := \begin{cases} V_{CM}(tD_{cyl}), & \text{if } t \in [1, +\infty), \\ e_1, & \text{if } t = \infty. \end{cases}$$

(6.5)

6.6. Proposition. Under the invertibility Assumption \[5.3\], the Connes-Moscovici idempotents $V_{CM}(D)$ and $V_{CM}(D_{cyl})$ define through formula (6.5) a relative class in $K_0(b \mathcal{A}^\infty_G(Y), b \mathcal{A}^\infty_{G, \mathbb{R}}(cyl(\partial Y)))$, the relative $K$-theory group associated to the surjective homomorphism

$$b \mathcal{A}^\infty_G(Y) \xrightarrow{\mathcal{L}} b \mathcal{A}^\infty_{G, \mathbb{R}}(cyl(\partial Y)).$$

With a small abuse of notation we denote this class by $[V_{CM}(D), e_1, V_{CM}(tD_{cyl})]$.

6.7. Definition. We define the relative (smooth) index class as

$$\text{Ind}_\infty(D, D_0) := [V_{CM}(D), e_1, V_{CM}(D_{cyl})] \in K_0(b \mathcal{A}^\infty_G(Y), b \mathcal{A}^\infty_{G, \mathbb{R}}(cyl(\partial Y))).$$

Recall now the smooth index class $\text{Ind}_\infty(D) \in K_0(\mathcal{A}^\infty_G(Y))$ defined through Theorem 5.20. The following result plays a crucial role:

6.8. Theorem. Let $\alpha_{ex}$ be the excision isomorphism for the short exact sequence $0 \to \mathcal{A}^\infty_G(Y) \to b \mathcal{A}^\infty_G(Y) \overset{\mathcal{L}}{\to} b \mathcal{A}^\infty_{G, \mathbb{R}}(cyl(\partial Y)) \to 0$. Then

$$\alpha_{ex}(\text{Ind}_\infty(D)) = \text{Ind}_\infty(D, D_0).$$

(6.9)

We summarize the content of subsection 5.5 and the present subsection 6.1 as follows:

using the Connes-Moscovici projector(s) we have proved the existence of smooth index classes

$$\text{Ind}_\infty(D) \in K_0(\mathcal{A}^\infty_G(Y)) \text{ and } \text{Ind}_\infty(D, D_0) \in K_0(b \mathcal{A}^\infty_G(Y), b \mathcal{A}^\infty_{G, \mathbb{R}}(cyl(\partial Y))),$$

with the first one sent to the second one by the excision isomorphism $\alpha_{ex}$. 
6.2. Relative cyclic cocycles. Recall that the relative cyclic complex associated to a short exact sequence $0 \to J \to A \xrightarrow{\pi} B \to 0$ of algebras is given by

$$CC^k(A, B) := CC^k(A) \oplus CC^{k+1}(B),$$

equipped with the differential

$$
\begin{pmatrix}
  b + B & -\pi^* \\
  0 & -(b + B)
\end{pmatrix},
$$

where $b, B$ are the usual Hochschild and cyclic differential and $\pi^*$ denotes the pullback of functionals through the surjective morphism $\pi : A \to B$. In our case, the relevant extension is given, first of all, by

$$0 \to \mathcal{A}_G^b(Y) \to b\mathcal{A}_G^b(Y) \xrightarrow{I} b\mathcal{A}_G^{b, R+}(\text{cyl}(Y)) \to 0, \quad \mathcal{A}_G^b(Y) := \text{Ker } I,$$

where this is now, for simplicity, the short exact sequence for the small $b$-calculus.

As in the closed case, given a global slice $Z \subset Y$, we can view $A \in b\mathcal{A}_G^b(Y)$ as a map $\Phi_A : G \to b\mathcal{A}_G^b(Y)$ by setting $\Phi_A(g, s_1, s_2) := A(s_1, gs_2)$. Likewise, an element $B \in b\mathcal{A}_G^{b, R+}(\text{cyl}(\partial Z))$ gives rise, by Mellin transform, to a map $\Phi_B : G \times \mathbb{R} \to \Psi^{-\infty}(\partial Z)$, which is compactly supported on $G$, and rapidly decreasing on $\mathbb{R}$. In the following, we shall denote by

$$(6.10) \quad \Phi_A \mapsto \hat{\Phi}_A \text{ the morphism } I : b\mathcal{A}_G^b(Y) \to b\mathcal{A}_G^{b, R+}(\text{cyl}(\partial Z))$$

followed by the Mellin transform.

6.3. The $b$-Trace on $G$-proper $b$-manifolds. Let us construct the correct analogue of the $b$-trace of Melrose in this setting where we have a proper group action. In our geometric setting, a choice of cut-off function $c_{Y_0}$ for the action of $G$ on $Y_0$ restricts to give a cut-off function $c_{\partial Y_0} := c_{Y_0}|_{\partial Y_0}$ for the $G$-action on $\partial Y_0$. We shall also write briefly $c_b$. We consider as usual the associated $b$-manifold $Y$, endowed with a product-type $b$-metric $h$, so that, metrically, $Y$ is a manifold with cylindrical ends, and we shall, by a small abuse of notation, write $c_Y$ for the extension of $c_{Y_0}$ on $Y_0$ which is constant in the cylindrical coordinate.

Using the $b$-integral of Melrose, see [38], we now define

6.11. Definition. For $A \in b\mathcal{A}_G^b(Y)$ its $b$ $G$-trace is defined as

$$b\text{Tr}_{c_Y}(A) := \int_Y K_A(x, x)c_Y(x)\text{dvol}(x).$$

Remark that the cut-off function on $Y_0$ has compact support, so the $b$-regularized integral is indeed well defined. The argument in [46, Prop. 2.3] shows that $b\text{Tr}_{c_Y}$ is independent of the choice of cut-off function $c_Y$. Next, using a simple trick with a family $\{c_{Y, \epsilon}\}_{\epsilon > 0}$ of cut-off functions converging to the characteristic function on $Z$, we can rewrite

$$(6.12) \quad b\text{Tr}_{c_Y}(A) = b\text{Tr}_Y(\hat{\Phi}_A(\epsilon)).$$

As in the usual $b$-calculus, $b\text{Tr}_{c_Y}$ is not a trace, but we have a precise formula for its defect on commutators, directly inspired by Melrose’ $b$-trace formula :

6.13. Lemma. For $A_1, A_2 \in b\mathcal{A}_G^b(Y)$, we have

$$b\text{Tr}_{c_Y}([\Phi_{A_1}, \Phi_{A_2}]) = \frac{i}{2\pi} \int_R \int_G \text{Tr}_{\partial Z} \left( \frac{\partial I(\hat{\Phi}_{A_1}, h^{-1}, \lambda)}{\partial \lambda} \circ I(\Phi_{A_2}, h, \lambda) \right) dhd\lambda,$$
with $\text{Tr}_Z$ denoting the usual functional analytic trace on smoothing operators on closed compact manifolds.

6.4. From absolute to relative cyclic cocycles. Let $\omega^Y$ be any one of the cyclic cocycles for $\mathcal{A}^\infty_G(Y)$ appearing in (5.22). We can write

$$\omega^Y = \omega \circ \text{Tr}_Z$$

where $\omega$ is the corresponding cyclic cocycle on the group $G$ and where $\text{Tr}_Z : \mathcal{A}^\infty_G(Y, E) \rightarrow C(G)$ is the homomorphism of integration along the slice. The integrals along the slice are well defined because the operators in $\mathcal{A}^\infty_G(Y)$ vanish to order $\epsilon$ at all the boundary hypersurfaces of $Y \times_b Y$. If we pass to $b\mathcal{A}^\infty_G(Y)$ we must replace ordinary integration with $b$-integration, as operators in $b\mathcal{A}^\infty_G(Y)$ do not vanish on the front face of $Y \times_b Y$; we obtain in this way a regularized multilinear functional $\omega^Y, \epsilon$; this will not give us a cyclic cocycle anymore, precisely because of Lemma 6.13 however, using the exact form of the Lemma 6.13 we will be able to produce a cyclic cocycle on $b\mathcal{A}^\infty_{G, \mathbb{R}^+}(\text{cyl}(\partial Y))$, call it $\xi^\partial Y$, in such a way that the pair $(\omega^Y, \epsilon^\partial Y)$ is in fact a relative cyclic cocycle for the surjective homomorphism $b\mathcal{A}^\infty_G(Y) \xrightarrow{I} b\mathcal{A}^c_{G, \mathbb{R}^+}(\text{cyl}(\partial Y))$.

6.5. Relative cyclic cocycles associated to orbital integrals. Let us see how the general principle put forward in the previous subsection works in the case $\omega^Y = \text{tr}_g^Y$. If $Y_0$ is a cocompact $G$-proper manifold with boundary and $Y$ is the associated $b$-manifold, then associated to the orbital integral $\text{tr}_g$ we have the trace-homomorphism

$$\text{tr}_g^Y : \mathcal{A}^\infty_G(Y) \rightarrow \mathbb{C},$$

given explicitly by

$$\text{tr}_g^Y(\kappa) := \int_{G/Z_g} \int_Y c_Y(hgh^{-1}y)\text{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}y, y))dx dy.$$  

Here $dy$ denotes the $b$-density associated to the $b$-metric $h$ and the cut-off function $c_Y$ on $Y_0$ is extended constantly along the cylinder to define $c_Y$. As already explained, the $Y$ integral converges, given that $\kappa$ vanishes of order $\epsilon > 0$ on all the boundary hypersurfaces of $Y \times_b Y$.

Now, let $X$ be a closed manifold equipped with a proper, cocompact action of $G$. We consider $\text{cyl}(X) = X \times \mathbb{R}$ the cylinder over $X$, equipped with the action of $G \times \mathbb{R}$. Using the Mellin transforms we have an algebra homomorphism

$$b\mathcal{A}^c_{G, \mathbb{R}}(\text{cyl}(Y)) \rightarrow \mathcal{S}(\mathbb{R}, \mathcal{A}^c_G(Y)), \ A \mapsto \hat{A}.
$$

6.17. Proposition. Let $X$ be a cocompact $G$-proper manifold without boundary; for example $X = \partial Y_0$. Define the following 1-cochain on $b\mathcal{A}^c_{G, \mathbb{R}}(\text{cyl}(X))$

$$\sigma_g^X(A_0, A_1) = \frac{i}{2\pi} \int_{\mathbb{R}} \text{tr}_g^X(\partial_1 \hat{A}_0(\lambda) \circ \hat{A}_1(\lambda))d\lambda.
$$

Then $\sigma_g^X(\cdot, \cdot)$ is well-defined and is a cyclic 1-cocycle.

Let $Y$ be now a $b$-manifold and let $\partial Y$ be its boundary. Let $\text{tr}_g^{Y, r}$ be the functional on $b\mathcal{A}^c_G(Y)$:

$$\text{tr}_g^{Y, r}(T) := \int_{G/Z_g} \int_Y c_Y(hgh^{-1}y)\text{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}y, y))dy d(hZ).$$
Here $\kappa$ is the kernel of the operator $G$, and Melrose’s $b$-integral has been used. This is the regularization of $\text{tr}^g$ that one needs to consider when one passes from $A^c_G(Y)$ to $bA^c_G(Y)$ (for the time being on kernels of $G$-compact support). Observe that

$$\text{tr}^Y = \text{tr}_g \circ b \text{Tr}_Z$$

with $b \text{Tr}_Z : bA^c_G(Y) \to C^\infty_c(G)$ denoting $b$-integration along the slice $Z$. More precisely, as in the closed case, we have an isomorphism

$$bA^c_G(Y) \cong \{ \Phi : G \to b\psi^{-\infty, c}(Z), \text{ smooth, compactly supported and } K \times K \text{ invariant} \}$$

and $b \text{Tr}_Z$ associates to $\Phi$ the function $G \ni \gamma \to b \text{Tr}(\Phi(\gamma))$.

One can prove the following

6.19. Proposition. The pair $(\text{tr}^Y, \sigma^\partial_Y)$ defines a relative 0-cocycle for

$$bA^c_G(Y) \xrightarrow{f} bA^c_G(\text{cyl}(\partial Y)).$$

Moreover, the 0-degree cyclic cocycle $(\text{tr}^Y, \sigma^\partial_Y)$ extends continuously to a relative 0-cocycle for

$$bA^{\infty}_G(Y) \xrightarrow{f} bA^{\infty}_G(\text{cyl}(\partial Y)).$$

Finally, the following formula holds:

$$(6.20) \quad \langle \text{tr}^Y, \text{Ind}_\infty(D) \rangle = \langle (\text{tr}^Y, \sigma^\partial_Y), \text{Ind}_\infty(D, D_0) \rangle.$$  

6.6. The delocalized APS index formula on $G$-proper manifolds. We shall prove the delocalized APS index formula using crucially $[6.20]$. On the right hand-side we have the pairing of the relative cocycle $(\tau^Y, \sigma^\partial_Y)$ with the relative index class defined by

$$(6.21) \quad (V_{CM}(D), e_1, q_t), \quad t \in [1, +\infty], \quad \text{with } q_t := \begin{cases} V_{CM}(tD_{cyl}) & \text{if } t \in [1, +\infty) \\ e_1 & \text{if } t = \infty \end{cases}$$

and with $e_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

By definition of relative pairing we have:

$$(6.22) \quad \langle (\text{tr}^Y, \sigma^\partial_Y), (V_{CM}(D), e_1, q_t) \rangle = \text{tr}^Y_g(e^{-D^-D^+}) - \text{tr}^Y_g(e^{-D^+D^-}) + \int_1^\infty \sigma^\partial_Y(\dot{q}_t, q_t) dt.$$

A complicated but totally elementary computation shows that the following Proposition holds:

6.23. Proposition. The term $\int_1^\infty \sigma^\partial_Y(\dot{q}_t, q_t) dt$, with $q_t := V_{CM}(tD_{cyl})$ is equal to

$$-\frac{1}{2} \left( \frac{1}{\sqrt{\pi}} \int_1^\infty \text{tr}^\partial_Y(D_0 \exp(-tD_{\partial}^2)) \frac{dt}{\sqrt{t}} \right).$$

Thanks to this Proposition we have that

$$(6.24) \quad \langle (\text{tr}^Y, \sigma^\partial_Y), (V(D), e_1, q_t) \rangle = \text{tr}^Y_g(e^{-D^-D^+}) - \text{tr}^Y_g(e^{-D^+D^-}) - \frac{1}{2} \int_1^\infty \frac{1}{\sqrt{\pi}} \sigma^\partial_Y(D_\partial \exp(-tD_{\partial}^2)) \frac{dt}{\sqrt{t}}.$$
As a last step we replace $D$ by $sD$; in the equality
\[
\langle \tr_g^Y, \Ind_\infty(D) \rangle = \langle (\tr_g^Y, c^Y_g), \Ind_\infty(D, D_0) \rangle
\]
the left hand side $\langle \tr_g^Y, \Ind_\infty(D) \rangle$ remains unchanged whereas the right hand side becomes
\[
\tr_g^Y r(e^{-s^2D^-D^+}) - \tr_g^Y r(e^{-s^2D^+D^-}) - \frac{1}{2} \int_s^\infty \frac{1}{\sqrt{\pi}} \tr_g^Y(D\partial Y) \exp(-tD^2\partial Y) \frac{dt}{\sqrt{t}}.
\]
Summarizing, for each $s > 0$ we have
\[
\tr_g^Y r(e^{-s^2D^-D^+}) - \tr_g^Y r(e^{-s^2D^+D^-}) = \langle \tr_g^Y, \Ind_\infty(D) \rangle + \frac{1}{2} \int_s^\infty \frac{1}{\sqrt{\pi}} \tr_g^Y(D\partial Y) \exp(-tD^2\partial Y) \frac{dt}{\sqrt{t}}.
\]
Now we take the limit as $s \downarrow 0$; using Getzler’ rescaling in the $b$-context one can prove that
\[
\lim_{s \downarrow 0} \frac{1}{2} \int_s^\infty \frac{1}{\sqrt{\pi}} \tr_g^Y(D\partial Y) \exp(-tD^2\partial Y) \frac{dt}{\sqrt{t}}
\]
exists and equals
\[
\int_{Y_0^g} c^Y_{g_0} \AS_g(D_0) - \langle \tr_g^Y, \Ind_\infty(D) \rangle.
\]
We conclude that the following Theorem holds:

6.25. THEOREM. (0-degree delocalized APS)
Let $G$ be a connected, linear real reductive group. Let $g$ be a semisimple element. Let $Y_0$, $Y$, $D$, $D_0\partial Y$ as above. Assume that $D_0\partial Y$ is $L^2$-invertible. Then
\[
\eta_g(D\partial Y) := \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{\pi}} \tr_g^Y(D\partial Y) \exp(-tD^2\partial Y) \frac{dt}{\sqrt{t}}
\]
exists and for the pairing of the index class $\Ind(D_\infty) \in K_0(A^*_C(Y)) \equiv K_0(C^*(Y_0 \subset Y)^G)$ with the 0-cocycle $\tr_g^Y \in HC^0((A^*_C(Y)))$ the following delocalized 0-degree APS index formula holds:
\[
\langle \tr_g^Y, \Ind_\infty(D) \rangle = \int_{Y_0^g} c^Y_{g_0} \AS_g(D_0) - \frac{1}{2} \eta_g(D\partial Y),
\]
where the integrand $c^Y_{g_0} \AS_g(D_0)$ is defined in the same way as the one in Equation (4.10).

6.27. REMARK. This result was first discussed in the work of Hochs-Wang-Wang [27]. Our treatment, centred around the interplay between absolute and relative cyclic cohomology and the $b$-calculus, is completely different; moreover our treatment allows us to get sharper results compared to [27] in the case of a connected linear real reductive group $G$. More precisely, in Theorem [6.25] we only assume that $g$ is a semisimple element of $G$ to obtain the index formula (6.26), while in [27] Theorem 2.1], the authors require that $G/Z_g$ is compact.
6.28. Remark. If we take \( g = e \) then we get
\[
\langle \text{tr}_e^Y, \text{Ind}_{C^*(G)}(D) \rangle = \int_Y c_Y \text{AS}(Y) - \frac{1}{2} \eta_G(D_0) = \int_{Y_0} c_{Y_0} \text{AS}(Y_0) - \frac{1}{2} \eta_G(D_0)
\]
with
\[
\eta_G(D_0) = \frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr}_e^{\partial Y} D_0 e^{-(tD_0)^2} dt.
\]
Notice however that this particular result holds under much more general assumptions on \( G \) than the ones we are currently imposing (\( G \) connected reductive linear Lie group). Indeed, the pairing of the index class with \( \text{tr}_e \) obtained the well-definedness of \( G \)-equivariant metrics on \( C^*G \).

6.7. More on delocalized eta invariants. In the previous section we have obtained the well-definedness of \( \eta_g(D_0) \), with \( D_0 \) being \( L^2 \)-invertible, as a byproduct of the proof of the delocalized APS index theorem for 0-degree cocycles. In fact, one can show that \( \eta_g(D) \) is well defined on a cocompact \( G \)-proper manifold even if \( D \) does not arise as a boundary operator. This is the content of the next theorem, partially discussed also in [26] [27].

6.29. Theorem. Let \((X, h)\) be a cocompact \( G \)-proper manifold without boundary endowed with a \( G \)-equivariant metric and let \( D \) be an \( L^2 \)-invertible Dirac-type operator of the form \((2.7)\). Let \( g \) be a semisimple element. The integral
\[
\frac{1}{\sqrt{\pi}} \int_0^\infty \text{tr}_g^X(D \exp(-tD^2)) \frac{dt}{\sqrt{t}}
\]
converges and defines the delocalized eta invariant associated to \( D \), \( \eta_g(D) \).

Notice that the proof of this result is rather delicate, both at \( t = 0 \), where results of Zhang [62] are used, and at \( t = +\infty \), where a delicate analysis of the large time behaviour of the heat kernel is needed.

We also point out that in a previous version of this survey, based on an earlier version of our work with Posthuma and Song [49], this result was wrongly stated without the invertibility assumption; our proof of the convergence at \( t = +\infty \) does apply to \( D_{\text{split}} \) but not to \( D \), which is why we do need the invertibility assumption.
6.8. Relative cyclic cocycles associated to smooth group cocycles. In this subsection we want to see how, given a smooth group cocycle \( \phi \in Z^k_{\text{diff}}(G) \), we can pass from the cyclic cocycle \( \tau_Y^\phi \) on \( A_{c,\mathcal{G}}(Y) \) to a relative cyclic cocycle for the surjective homomorphism \( b_{A_{c,\mathcal{G}}(Y)} \overset{I}{\to} b_{A_{c,\mathcal{G},\mathbb{R}}(\text{cyl}(\partial Y))} \). As we have already explained, the first step is to pass from \( \tau_Y^\phi \) to \( \tau_Y^{\phi,r} \), and this is achieved by replacing integrals by \( b \)-integrals or, equivalently, traces by \( b \)-traces. This is what we do in the next definition.

6.31. Definition. Let \( Y \) be a proper \( G \)-manifold with boundary, and \( \phi \in Z^k_{\text{diff}}(G) \) be a smooth group cocycle. For \( A_0, \ldots, A_k \in b_{A_{c,\mathcal{G}}(Y)} \), define

\[
\tau_Y^{\phi,r}(A_0, \ldots, A_k) := \int_{G^{k+1}} b_{\text{Tr}} \left( \Phi_{A_0}((g_1 \cdots g_k)^{-1}) \circ \Phi_{A_1}(g_1) \circ \cdots \circ \Phi_{A_k}(g_k) \right)
\varphi(e, g_1, g_1 g_2, \ldots, g_1 \cdots g_k) dg_1 \cdots dg_k.
\]

In the above equation we have used the homogeneous differentiable group cohomology complex introduced in Definition 3.8.

Next, following the general strategy explained at the beginning of this section, we want to define the eta cocycle associated to \( \phi \).

6.32. Definition. Let \( X \) be a closed manifold equipped with a proper, cocompact action of \( G \), and let \( \phi \in C^k_{\text{diff}}(G) \) be a smooth group cochain. The eta cochain on \( b_{A_{c,\mathcal{G},\mathbb{R}}(\text{cyl}(X))} \) associated to \( \phi \) is defined as

\[
\sigma_X^\phi(B_0, \ldots, B_{k+1}) := \frac{(-1)^{k+1}}{2\pi} \int_{G^{k+1}} \int_{\mathbb{R}} \text{Tr} \left( \hat{B}_0((g_1 \cdots g_{k+1})^{-1}, \lambda) \circ \hat{B}_1(g_1, \lambda) \circ \cdots \circ \hat{B}_k(g_k, \lambda) \circ \frac{\partial \hat{B}_{k+1}(g_{k+1}, \lambda)}{\partial \lambda} \right)
\varphi(e, g_1, g_1 g_2, \ldots, g_1 \cdots g_k, \lambda) d\lambda dg_1 \cdots dg_{k+1},
\]

where the notation in (6.10) has been used, and where we have used the homogeneous differentiable group cohomology complex introduced in Definition 3.8.

Using Lemma 6.13 one can prove the following result [48]:

6.33. Proposition. If \( \phi \in Z^k_{\text{diff}}(G) \) is a smooth group cocycle then \( (\tau_Y^{\phi,r}, \sigma_\partial^\phi) \) is a relative cyclic cocycle for \( b_{A_{c,\mathcal{G}}(Y)} \overset{I}{\to} b_{A_{c,\mathcal{G},\mathbb{R}}(\text{cyl}(\partial Y))} \).

In addition, one can prove the following:

6.34. Proposition. The pair \( (\tau_Y^{\phi,r}, \sigma_\partial^\phi) \) extends continuously to a relative \( k \)-cocycle for

\[
b_{A_{c,\mathcal{G}}}(Y) \overset{I}{\to} b_{A_{c,\mathcal{G},\mathbb{R}}(\text{cyl}(\partial Y))}.
\]

Moreover the following formula holds:

\[
(\tau_Y^\phi, \text{Ind}_\infty(D)) = ((\tau_Y^{\phi,r}, \sigma_\partial^\phi), \text{Ind}_\infty(D, D_\partial))
\]
6.36. **Example.** As explained in Example [3.12](#), besides the trivial group cocycle, there is an interesting degree 2 cocycle $A$ given by the area of a hyperbolic triangle in $\mathbb{H}$. The corresponding eta 3-cocycle on $\mathcal{S}(\mathbb{R}, C^\infty_c(G))$ is given by

$$\sigma_A(B_0, B_1, B_2, B_3) := -\frac{1}{2\pi} \int_{G^3} \int_{\mathbb{R}} \hat{B}_0((g_1 g_2 g_3)^{-1}, \lambda) \star \hat{B}_1(g_1, \lambda) \star \hat{B}_2(g_2, \lambda) \star \frac{\partial \hat{B}(g_3, \lambda)}{\partial \lambda}$$

$$\text{Area}(\Delta_{[e, g_1 e, g_1 g_2 e])} \lambda dg_1 dg_2 dg_3.$$

6.9. **Higher APS index theorem associated to a group cocycle $\varphi \in Z^k_{\text{diff}}(G).** Using (6.35), proceeding as we did for the delocalized trace $\text{tr}^Y_g$ (and for the corresponding relative cocycle $(\text{tr}^Y_g, \sigma_\varphi^{\partial Y})$), using the heat kernel approach to the Pflaum-Posthuma-Tang index formula developed in [48](#), one can establish the following Atiyah-Patodi-Singer index theorem.

6.37. **Theorem.** Let $Y_0, Y$ and $D$ as above. Assume that the boundary operator $D_\partial$ is $L^2$-invertible. We consider $[\varphi] \in H^2_{\text{diff}}(G)$ and

$$\text{Ind}_\varphi(D) := (-1)^p \frac{2p!}{p!} \langle \tau^Y_\varphi, \text{Ind}_\infty(D) \rangle.$$

(6.38)

$$\text{Ind}_\varphi(D) = \int_\partial c_Y \text{AS}(Y) \wedge \Phi([\varphi]) - \frac{1}{2} \eta_\varphi(D_\partial) \equiv \int_{Y_0} c_Y \text{AS}(Y_0) \wedge \Phi([\varphi]) - \frac{1}{2} \eta_\varphi(D_\partial),$$

where

(6.39)

$$\eta_\varphi(D_\partial) := c_p \left[ \sum_{l=0}^{2p} \int_{0}^\infty \sigma_\varphi^{\partial Y}(p_t, \ldots, [p_t, p_t], \ldots, p_t)dt \right]$$

with $p_t = V(tD_{\text{cyl}})$ and $c_p = (-1)^p \frac{2p!}{p!}.$

For more details we refer to [48](#).

6.10. **Higher delocalized APS index theorem.** We finally come to the higher delocalized cyclic cocycles $\Phi_{Y, g}^P$, with $P$ a cuspidal parabolic subgroup with Langlands decomposition $MAN$ and $g$ a semisimple element in $M$. The cyclic cocycle $\Phi_{Y, g}^P$ on $\mathcal{A}^G_c(Y)$ can be written explicitly as

$$\Phi_{Y, g}^P(A_0, A_1, \ldots, A_m) := \int_{b \in M/Z_M(g)} \int_K N \int_{G^m} C(H(g_1 \ldots g_m k), H(g_2 \ldots g_m k), \ldots, H(g_m k))$$

$$\text{Tr} \left(A_0(khgh^{-1}(g_1 \ldots g_m)^{-1}) \circ A_1(g_1) \cdots \circ A_m(g_m) \right) dg_1 \cdots dg_m dk dh.$$
where the notation in (6.10) has been used. One proves that the pair \((\Phi^r, P)\) defines a relative cyclic cocycle for the homomorphism \(b_{G}^{\ast}: \mathcal{A}_{G}^{c}(\text{cyl}(\partial Y)) \to \mathcal{A}_{G}^{c}(\text{cyl}(\partial Y))\). Moreover:

- the relative cyclic cocycle \((\Phi^r, P, \sigma_{\partial Y,g})\) extends continuously from the pair \(\mathcal{A}_{G}^{c}(\text{cyl}(\partial Y))\) to the pair \(\mathcal{A}_{G}^{\infty}(\partial Y) \to \mathcal{A}_{G}^{\infty}(\text{cyl}(\partial Y))\).
- the crucial formula \(\langle \Phi^r, \text{Ind}_{\infty}(D) \rangle = \langle (\Phi^r, \sigma_{\partial Y,g}), \text{Ind}_{\infty}(D, D_{0}) \rangle\) holds.

Using the last formula and proceeding as in the previous cases we arrive at the following result:

For each \(s \in (0, 1]\)

\[
c_m(\text{Ind}_{\infty}(D), [\Phi^r_{Y,g}]) = \Phi^r_{Y,g}(V(sD), \ldots , V(sD)) \mp \frac{1}{2} \int_{s}^{\infty} \eta_{g}^{P}(t) dt
\]

with \(c_m = (-1)^{m-m!2}\) and \(\eta_{g}^{P}(t) := \frac{2c}{E} \sum_{i=0}^{m} \sigma_{\partial Y,g}^{P}(p_{t}, \ldots , [p_{t}, p_{t}], \ldots , p_{t})\). Part of the statement is of course that the \(t\)-integral converges at \(+\infty\).

We would like to take the limit as \(s \downarrow 0\); unfortunately we do not know how to compute the limit of the first term on the right hand side (this should produce the local term in the index formula); in fact we cannot compute this limit even in the closed case.

**Open problem.** Let \(P < G\) a cuspidal parabolic subgroup with Langland’s decomposition \(P = MAN\) and let \(g \in M\) be a semisimple element. Let \(D\) be a \(G\)-equivariant Dirac operator on a closed cocompact \(G\)-proper manifold \(X\); let \(V(D)\) be the symmetrized Connes-Moscovici projector. Can one prove that

\[
\lim_{s \downarrow 0} \Phi^r_{Y,g}(V(sD), \ldots , V(sD))
\]

exists? If so, can one give an explicit formula for it?

Because of these difficulties we go back to the proof of the higher delocalized index formula in the closed case by reduction. Thus we consider \(Y_{M} := Y/AN\), an \(M\)-proper manifold, which has a slice decomposition given by \(Y_{M} := M \times K \cap M Z\). The arguments of Hochs, Song and Tang in Theorem [4.12] can be extended (with some efforts) to the case of manifolds with boundary yielding the following theorem (which is one of the main results in [49]):

**6.40. Theorem.** Suppose that the metric on \(Y\) is slice-compatible. Assume that \(D_{\partial Y}^{\ast}\) is \(L^{2}\)-invertible and consider the higher index \(\langle \Phi^{r}_{Y,g}, \text{Ind}_{\infty}(D) \rangle\). The following formula holds:

\[
\langle \Phi^{r}_{Y,g}, \text{Ind}_{\infty}(D) \rangle = \int_{(Y_{0}/AN)^{g}} c_{(Y_{0}/AN)^{g}}^{q} AS(Y_{0}/AN)_{g} - \frac{1}{2} \eta_{g}(D_{\partial Y_{M}})
\]
with
\[ \eta_g(D\partial Y_M) = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{tr} g(D\partial Y_M \exp(-tD^2\partial Y_M)) \frac{dt}{\sqrt{t}}. \]

Here \( c_{(Y_0/AN)^g} \) is a compactly supported smooth cutoff function on \((Y_0/AN)^g\) associated to the \( Z_{M,g} \) action on \((Y_0/AN)^g\).

7. Geometric applications

Now that we have explained various index theorems associated to a \( G \)-equivariant Dirac operator on a \( G \)-proper manifold, we discuss some geometric applications.

7.1. Higher genera. We begin by observing that the homogeneous space \( G/K \) is a smooth model for \( EG \), the classifying space for proper actions of \( G \), see [4]: for any smooth proper action of \( G \) on a manifold \( X \), there exists a smooth \( G \)-equivariant classifying map \( \psi_X : X \to G/K \), unique up to \( G \)-equivariant homotopy. For any proper action of \( G \) on manifold \( X \) we consider \( \Omega_{\text{inv}}^\bullet(X) \), the complex of \( G \)-invariant differential forms on \( X \) and its cohomology denoted by \( H_{\text{inv}}^\bullet(X) \).

For a connected real reductive linear group \( G \), we have the Van Est isomorphism:
\[ H^\bullet_{\text{diff}}(G) \cong H^\bullet_{\text{inv}}(G/K). \]

Consider now \([\alpha] \in H^\bullet_{\text{inv}}(G/K)\) and let \( \alpha \in \Omega^\bullet_{\text{inv}}(G/K) \) a representative; consider its pull-back \( \psi_X^* \alpha \in \Omega^\bullet_{\text{inv}}(X) \) such that \([\psi_X^* \alpha] = \Phi([\alpha])\).

The higher signature associated to \([\alpha]\) is the real number
\[
\sigma(X, [\alpha]) := \int_X c_X L(X) \wedge \psi_X^* (\alpha),
\]

where \( L(X) \) is the invariant de Rham form representing the \( L \)-class of \( X \). The insertion of the cut-off function \( c_X \), which as we know has compact support, ensures that the integral is well-defined (and it can be shown that it only depends on the class \([L(X) \wedge \psi_X^* \alpha] \in H^\bullet_{\text{inv}}(X)\)). The collection
\[
\{ \sigma(X, [\alpha]), [\alpha] \in H^\bullet_{\text{inv}}(G/K) \}
\]
are called the higher signatures of \( X \); by the Van Est isomorphism they are labelled by the elements in \( H^\bullet_{\text{diff}}(G) \). Similarly, the higher \( \hat{A} \) genus associated to \( X \) and to \([\alpha] \in H^\bullet_{\text{inv}}(G/K)\) is the real number
\[
\hat{A}(X, [\alpha]) := \int_X c_X \hat{A}(X) \wedge \psi_X^* (\alpha)
\]

with \( \hat{A}(X) \) the de Rham class associated to the \( \hat{A} \)-differential form for a \( G \)-invariant metric. The collection
\[
\{ \hat{A}(X, [\alpha]), [\alpha] \in H^\bullet_{\text{inv}}(G/K) \}
\]
are called the higher \( \hat{A} \)-genera of \( Y \).

We have:

7.5. Theorem. Let \( G \) be a connected real reductive Lie group. Let \( X \) be an orientable manifold with a proper, cocompact action of \( G \). Then the following holds true:

(i) each higher signatures \( \sigma(X, [\alpha]), [\alpha] \in H^\bullet_{\text{inv}}(G/K) \), is a \( G \)-homotopy invariant of \( X \).
(ii) if $X$ admits a $G$-invariant Spin structure and a $G$-invariant metric of positive scalar curvature, then each higher $\tilde{A}$-genus $\tilde{A}(X, [\alpha])$, $[\alpha] \in H^*_{\text{inv}}(G/K)$, vanishes.

The theorem follows easily from the higher index formula for differentiable group cocycles explained in the first part of this article and the usual stability properties of the $C^*$ index class of the signature operator and of the Spin-Dirac operator, established in this context by Fukumoto \cite{15} and Guo-Mathai-Wang \cite{17}. As we have already explained, the theorem in fact holds more generally for $G$ a Lie group with finitely many connected components satisfying property RD, and such that $G/K$ is of non-positive sectional curvature for a maximal compact subgroup $K$. See \cite{47} for more details.

The corresponding APS index theorems can be used to introduce relative higher genera $\sigma(X, \partial X, [\alpha])$ and $\tilde{A}(X, \partial X, [\alpha])$ and prove, for example, additivity results for closed manifolds that are obtained by gluing manifolds with boundary along $G$-diffeomorphic boundaries. In addition, the relative higher $\tilde{A}$-genera can be used to produce obstructions to the existence of an isotopy from a $G$-invariant PSC metric on $\partial X$ to a $G$-invariant metric on $\partial X$ which extends to a $G$-invariant metric which is PSC on all of $X$.

### 7.2. Rho invariants

In this subsection we shall briefly introduce (higher) rho numbers associated to positive scalar curvature metrics and $G$-equivariant homotopy equivalences. All our Dirac operators will be $L^2$-invertible; indeed, if we want to consider bordism properties of these numbers we do need $L^2$-invertibility so as to be able to define an APS index class on the manifold with boundary realising the bordism.

**Rho numbers associated to delocalized 0-cocycles.** We consider a closed $G$-proper manifold $X$ without boundary, $G$ connected, linear real reductive, $g \in G$ a semisimple element, $D_X$ a $G$-equivariant $L^2$-invertible Dirac operator of the form \cite{2.7}. We consider

$$
\eta_g(D_X) := \frac{1}{\sqrt{\pi}} \int_0^\infty \text{tr}_g(D_X \exp(-tD_X^2) \frac{dt}{\sqrt{t}}).
$$

Let $X$ be $G$-equivariantly Spin and $D_X \equiv D_h$, the Spin Dirac operator associated to a $G$-equivariant PSC metric $h$. Then we know that $D_h$ is $L^2$-invertible. We define

$$
\rho_g(h) := \eta_g(D_h).
$$

If, on the other hand, $f : X_1 \to X_2$ is a $G$-homotopy equivalence, then using \cite{15} we know that there exists a bounded perturbation $B_f$ of the signature operator on $X := X_1 \sqcup (-X_2)$, where $-X_2$ is the same manifold as $X_2$ with the opposite orientation, that makes it invertible. Moreover, one can prove that this perturbation $B_f$ is in $C^*(X, \Lambda^*)^G$. Hence, by density, we conclude that there exists a perturbation $B_f^\infty \in \mathcal{A}_C^\infty(X, \Lambda^*)$ such that $D_X^{\text{sign}} + B_f^\infty$ is $L^2$-invertible. It is possible to extend Theorem 6.29\footnote{from now on we shall briefly write PSC} to this perturbed situation and define the rho-number of the homotopy equivalence $f$ as

$$
\rho_g(f) := \eta_g(D_X^{\text{sign}} + B_f^\infty).
$$
We refer to \[50\] for a detailed discussion of the index theory associated to perturbed operators such as the one appearing above.

**Rho numbers associated to higher delocalized cocycles** We can generalize the above definitions and define rho numbers associated the higher cocycles $\Phi_{P_g}$. More precisely, let $P = MAN$, $g \in M$ as above and consider $\Phi_{P_g}$ and $\Phi_{P_{X,g}}$ (we recall that $X$ is without boundary). Assume that $h$ is a slice-compatible $G$-invariant PSC metric on $X$. Then

\[
\rho_{P_g}(h) := \eta_g(D_{X,M})
\]  
(with $X_M$ the reduced manifold associated to $X$) is well defined. Notice that it is proved \[49\] that if $D_X$ invertible, then $D_{X,M}$ is also invertible.

**Bordism properties.** The APS index theorems introduced in this article can be used to study the bordism properties of these rho invariants. We concentrate on the case of psc metrics. We assume, unless otherwise stated, that we are on a $G$-proper manifold which is endowed with a slice-compatible $G$-invariant metric and a slice-compatible $G$-invariant Spin structure.

Let $(Y, h_0)$ and $(Y, h_1)$ be two slice-compatible psc metrics. We say that they are $G$-concordant if there exists a $G$-invariant metric $h$ on $Y \times [0, 1]$ which is of PSC, product-type near the boundary and restricts to $h_0$ at $Y \times \{0\}$ and to $h_1$ at $Y \times \{1\}$.

The following Proposition is an example of the applications one can envisage for these secondary invariants. Before stating it, we remark that if $g$ is non-elliptic, that is, does not conjugate to a compact element, then every element of the conjugacy class $C(g) := \{hgh^{-1} | h \in G\}$ in $G$ does not have any fixed point on a $G$-proper manifold.

**7.7. Proposition.**
1] Assume that the $G$-invariant psc metrics $h_0$ and $h_1$ on $Y$ are $G$-concordant. Assume that $g$ is non-elliptic on $Y$. Then $\rho_{P_g}(h_0) = \rho_{P_g}(h_1)$.
2] Let $P = MAN < G$ be a cuspidal parabolic subgroup and let $x \in M$ be a semisimple element. Assume that the $G$-invariant slice-compatible psc metrics $h_0$ and $h_1$ on $Y$ are $G$-concordant. Assume that $g$ is non-elliptic on $Y$. Then $\rho_{P_{x,g}}(h_0) = \rho_{P_{x,g}}(h_1)$.

Put it differently, for such $g$ our (higher) rho invariants are in fact concordance invariants.

**Proof.** In both cases the proof is an immediate consequence of the relevant delocalized APS index theorems. □

**7.8. Remark.** In the case of free proper actions of discrete groups, higher rho invariants have been employed very successfully in studying the moduli space of concordant metrics of positive scalar curvature. See \[61\] and \[52\]. It is a challenge to understand whether the results we have just explained in the context of proper actions of Lie groups can be employed in studying the space $R_{G,\text{slice}}(Y)$ of slice-compatible metrics of PSC (if non-empty) or the space $R_{G,\text{eq}}(Y)$ (arbitrary $G$-equivariant metrics of PSC). For these questions it would be interesting to develop a $G$-equivariant Stolz’ sequence and investigate its basic properties. We leave this task to future research.
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DIPARTIMENTO DI MATEMATICA, SAPIENZA UNIVERSITÀ DI ROMA, I-00185 ROMA, ITALY
Email address: piazza@mat.uniroma1.it

DEPARTMENT OF MATHEMATICS AND STATISTICS, WASHINGTON UNIVERSITY, ST. LOUIS, MO, 63130, U.S.A.
Email address: xtang@math.wustl.edu