The strong metric dimension of the power graph of a finite group

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Abstract

We characterize the strong metric dimension of the power graph of a finite group. As applications, we compute the strong metric dimension of the power graph of a cyclic group, an abelian group, a dihedral group or a generalized quaternion group.

Keywords: Power graph, finite group, strong resolving set, strong metric dimension.

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1 Introduction

Given a graph $\Gamma$, denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of $\Gamma$, respectively. For $x, y, z \in V(\Gamma)$, we say that $z$ \textit{strongly resolves} $x$ and $y$ if there exists a shortest path from $z$ to $x$ containing $y$, or a shortest path from $z$ to $y$ containing $x$. A subset $S$ of $V(\Gamma)$ is a \textit{strong resolving set} of $\Gamma$ if every pair of vertices of $\Gamma$ is strongly resolved by some vertex of $S$. The smallest cardinality of a strong resolving set of $\Gamma$ is called the \textit{strong metric dimension} of $\Gamma$ and is denoted by $\text{sdim}(\Gamma)$.

In the 1970s, metric dimension was first introduced, independently by Harary and Melter \cite{10} and by Slater \cite{25}. This parameter has appeared in various applications...
(see [3] and [4] for more information). In 2004, Sebő and Tannier [24] introduced the strong metric dimension of a graph and presented some applications of strong resolving sets to combinatorial searching. The strong metric dimension of corona product graphs, rooted product graphs and strong products of graphs were studied in [18–20], respectively. The problem of computing strong metric dimension is NP-hard [20]. Some theoretical results, computational approaches and recent results on strong metric dimension can be found in [17].

The power graph $\Gamma_G$ of a finite group $G$ has the vertex set $G$ and two distinct elements are adjacent if one is a power of the other. In 2000, Kelarev and Quinn [12] introduced the concept of a power graph. Recently, many interesting results on power graphs have been obtained, see [1, 5–9, 13–15, 21, 22]. A detailed list of results and open questions on power graphs can be found in [2].

This paper is organized as follows. In Section 2, we express the strong metric dimension of a graph with diameter two in terms of the clique number of its reduced graph. Sections 3 and 4 study the clique number of the reduced graph of the power graph of a finite group $G$. Therefore, the strong metric dimension of $\Gamma_G$ is characterized. In Section 5, we compute the strong metric dimension of the power graph of a cyclic group, an abelian group, a dihedral group or a generalized quaternion group.

## 2 Reduced graphs

Let $\Gamma$ be a connected graph. The distance $d_{\Gamma}(x, y)$ between vertices $x$ and $y$ is the length of a shortest path from $x$ to $y$ in $\Gamma$. The closed neighborhood of $x$ in $\Gamma$, denoted by $N_{\Gamma}[x]$, is the set of vertices which have distance at most one from $x$. The greatest distance between any two vertices in $\Gamma$ is called the diameter of $\Gamma$. A subset of $V(\Gamma)$ is a clique if any two distinct vertices in this subset are adjacent in $\Gamma$. The clique number $\omega(\Gamma)$ is the maximum cardinality of a clique in $\Gamma$.

### Proposition 2.1.

Let $\Gamma$ be a connected graph with diameter two. Then a subset $S$ of $V(\Gamma)$ is a strong resolving set of $\Gamma$ if and only if the following conditions hold:

1. $V(\Gamma) \setminus S$ is a clique in $\Gamma$;
2. $N_{\Gamma}[u] \neq N_{\Gamma}[v]$ for any two distinct vertices $u$ and $v$ of $V(\Gamma) \setminus S$.

### Proof.

Assume that $S$ is a strong resolving set of $\Gamma$. Let $u$ and $v$ be two distinct vertices of $V(\Gamma) \setminus S$. Since $\Gamma$ has diameter two, we have $d_{\Gamma}(u, v) = 1$ by [16, Property 2]. Then (i) holds. If $N_{\Gamma}[u] = N_{\Gamma}[v]$, then $d_{\Gamma}(u, w) = d_{\Gamma}(v, w)$ for any $w \in S$, and so $u$ and $v$ can not be strongly resolved by any vertex in $S$, a contradiction. Hence (ii) holds.
For the converse, it follows from (ii) that there exists a vertex \( w \) in \( N_{\Gamma}[u] \setminus N_{\Gamma}[v] \) or \( N_{\Gamma}[v] \setminus N_{\Gamma}[u] \). Without loss of generality, let \( w \in N_{\Gamma}[u] \setminus N_{\Gamma}[v] \). By (i), we have \( w \in S \). Note that \((w,u,v)\) is a shortest path. Therefore, \( w \) strongly resolves \( u \) and \( v \), as desired.

For vertices \( x \) and \( y \) in a graph \( \Gamma \), define \( x \equiv y \) if \( N_{\Gamma}[x] = N_{\Gamma}[y] \). Observe that \( \equiv \) is an equivalence relation. Let \( U(\Gamma) \) be a complete set of distinct representative elements for this equivalence relation. The reduced graph \( R_{\Gamma} \) of \( \Gamma \) has the vertex set \( U(\Gamma) \) and two vertices are adjacent if they are adjacent in \( \Gamma \). We get the following result immediately from Proposition \( 2.1 \).

**Theorem 2.2.** Let \( \Gamma \) be a connected graph with diameter two. Then

\[
sdim(\Gamma) = |V(\Gamma)| - \omega(R_{\Gamma}).
\]

### 3 The clique number of \( R_G \)

In the remaining of this paper, we always use \( G \) to denote a finite group. For simplify, denote by \( R_G \) the reduced graph \( R_{\Gamma_G} \). Note that the diameter of \( \Gamma_G \) is at most two. In order to compute \( sdim(\Gamma_G) \), we only need to study \( \omega(R_G) \) from Theorem 2.2.

For a positive integer \( n \), write

\[
n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m},
\]

where \( p_1, p_2, \ldots, p_m \) are distinct prime numbers and \( r_i \geq 1 \) for \( 1 \leq i \leq m \). Write

\[
\sigma_n = \begin{cases} 1, & \text{if } m = 1; \\ \sum_{i=1}^{m} r_i, & \text{if } m \geq 2. \end{cases}
\]

Let \( Z_n \) be the cyclic group of order \( n \).

**Theorem 3.1.** \( \omega(R_{Z_n}) = \sigma_n. \)

In the rest of this section, assume that \( G \) is a noncyclic group. Denote by \( \mathcal{M} \) the set of all maximal cyclic subgroups of \( G \). Given a prime \( p \), let \( \mathcal{M}_p \) be the set of all \( p \)-subgroups in \( \mathcal{M} \). Suppose \( \mathcal{M}_p \neq \emptyset \). Let

\[
\mathcal{M}_p = \{M_1, M_2, \ldots, M_t\}.
\]

(2)

For \( i \in \{1, \ldots, t\} \), write

\[
\{M_i \cap M_j : j \in \{1, \ldots, t\}\} = \{C_{i_1}, \ldots, C_{i_{s_i}}\}.
\]
Note that $C_{i_1}, \ldots, C_{i_s}$ are subgroups of $M_i$ which is a cyclic group of prime power order. Without loss of generality, we may assume that

$$C_{i_1} \subsetneq C_{i_2} \subsetneq \cdots \subsetneq C_{i_s} = M_i. \quad (3)$$

Let

$$p_{\lambda_i}^M = \begin{cases} \max\{|M_i \cap M| : M \in \mathcal{M} \setminus \mathcal{M}_p\}, & \text{if } \mathcal{M}_p \subsetneq \mathcal{M}; \\ p^{-1}, & \text{if } \mathcal{M}_p = \mathcal{M}. \end{cases}$$

Define

$$\alpha_p = \max\{s_i - s'_i + \lambda_i + 2 : 1 \leq i \leq t\}, \quad (4)$$

where $s'_i = \min\{u : 1 \leq u \leq s_i, p_{\lambda_i}^M < |C_{iu}|\}$. If $\mathcal{M}_p = \emptyset$, we define $\alpha_p = 0$.

A finite group is called a CP-group [11] if every element of the group has prime power order. The set of all prime divisors of a positive integer $n$ is denoted by $\pi_n$.

**Theorem 3.2.** Let $G$ be a noncyclic group of order $n$.

(i) If $G$ is a CP-group, then $\omega(\mathcal{R}_G) = \max\{\alpha_p : p \in \pi_n\}$.

(ii) If $G$ is not a CP-group, then

$$\omega(\mathcal{R}_G) = \max\{\alpha_p : p \in \pi_n\} \cup \{|\sigma_M| + 1 : M \in \mathcal{M} \setminus (\bigcup_{p \in \pi_n} \mathcal{M}_p)\}.$$  

Combining Theorems 2.2, 3.1 and 3.2, we get a characterization of the strong metric dimension of $\Gamma_G$.

4 Proofs of Theorems 3.1 and 3.2

**Proof of Theorem 3.1:** Let $n$ be as in (1). Note that the power graph of a cyclic group of prime power order is complete. Then $\omega(\mathcal{R}_{\mathbb{Z}_n}) = 1$ if $m = 1$. In the following proof, we assume that $m \geq 2$.

Let $S$ be a clique of $\mathcal{R}_{\mathbb{Z}_n}$, and write $S = \{u_1, u_2, \ldots, u_s\}$. We may assume that $m_1 \leq m_2 \leq \cdots \leq m_s$, where $m_i = |u_i|$. Since $\{u_i, u_j\} \in E(\Gamma_{\mathbb{Z}_n})$ for $1 \leq i < j \leq s$, one has $m_i | m_j$. Note that $N_{\Gamma_{\mathbb{Z}_n}}[u_i] \neq N_{\Gamma_{\mathbb{Z}_n}}[u_j]$. It follows from [8, Proposition 3.6] that the elements with the same order have the same closed neighborhoods, and the closed neighborhoods of the identity and any generator are equal. Hence, one gets $m_i \neq m_j$ and

$$|S \cap \{x \in \mathbb{Z}_n : |x| = 1 \text{ or } n\}| \leq 1.$$  

Therefore, we get $s = |S| \leq \sigma_n$, and so

$$\omega(\mathcal{R}_{\mathbb{Z}_n}) \leq \sigma_n.$$ 

Let $T = \{a_1, a_2, \ldots, a_{\sigma_n}\}$, where

$$\begin{align*}
|a_1| &= p_m, |a_2| = p_m^2, \ldots, |a_{r_m}| = p_m^{r_m}, \\
|a_{r_m+1}| &= p_{m-1}p_m, \ldots, |a_{r_m+r_{m-1}}| = p_{m-1}^{r_{m-1}}, \\
|a_{r_m+r_{m-1}+1}| &= p_{m-2}p_m, \ldots, |a_{\sigma_n-1}| = p_1^{r_1}p_2^{r_2} \cdots p_m^{r_m}, |a_{\sigma_n}| = p_1^{r_1}p_2^{r_2} \cdots p_m^{r_m}.
\end{align*}$$

Then any pair of elements in $T$ has distinct closed neighborhoods in $\Gamma_{\mathbb{Z}_n}$. Therefore, $T$ is a clique in $R_{\mathbb{Z}_n}$ with size $\sigma_n$, as desired. \qed

In the following, suppose that $G$ is a noncyclic group. With reference to (2) and (3), for $1 \leq i \leq t$, let $c_{iu}$ be a generator of $C_{iu}$ and $C_{i0} = \emptyset$, where $1 \leq u \leq s_i$.

**Lemma 4.1.** Let $1 \leq u, v \leq s_i$ and $c \in G$.

(i) If $u \neq v$, then $N_{\Gamma_G}[c_{iu}] \neq N_{\Gamma_G}[c_{iv}]$;

(ii) Suppose that the maximal cyclic subgroups of $G$ that contains $\langle c \rangle$ are $p$-groups. If $C_{i(u-1)} \subsetneq \langle c \rangle \subseteq C_{iu}$, then $N_{\Gamma_G}[c] = N_{\Gamma_G}[c_{iu}]$.

**Proof.** (i) Without loss of generality, assume that $u < v$. Clearly, there exists a maximal cyclic subgroup $M_j \in \mathcal{M}_p \setminus \{M_i\}$ such that $C_{iu} = M_i \cap M_j$. Since $C_{iu} \subsetneq C_{iv}$, one has $C_{iv} \subsetneq M_j$. Hence, any generator of $M_j$ belongs to $N_{\Gamma_G}[c_{iu}] \setminus N_{\Gamma_G}[c_{iv}]$, which implies that (1) holds.

(ii) We only need to show that $x \in N_{\Gamma_G}[c]$ is equivalent to $x \in N_{\Gamma_G}[c_{iu}]$. Note that $\langle c \rangle \subseteq C_{iu}$ and $C_{iu}$ is a cyclic $p$-group. Hence, it suffices to prove that if $\langle c \rangle \subseteq \langle x \rangle$, then $x \in N_{\Gamma_G}[c_{iu}]$. Let $M$ be a maximal cyclic subgroup of $G$ that contains $\langle x \rangle$. Then $\langle c \rangle \subseteq M$, and so $M \in \mathcal{M}_p$. Write $M_j = M$. Then $\langle c \rangle \subseteq M_i \cap M_j$. By (3), one has $C_{iu} \subseteq M_i \cap M_j$. Hence, both $\langle x \rangle$ and $C_{iu}$ are subgroup of the cyclic $p$-group $M_j$, and so $x \in N_{\Gamma_G}[c_{iu}]$, as wanted. \qed

**Lemma 4.2.** Let $p$ be a prime number and $\{a_1, a_2, \ldots, a_k\}$ be a subset of $G$. Then $k \leq \alpha_p$ if the following conditions hold:

(i) $\langle a_1 \rangle \subsetneq \langle a_2 \rangle \subsetneq \cdots \subsetneq \langle a_k \rangle$;

(ii) $|a_k|$ is a power of $p$;

(iii) $N_{\Gamma_G}[a_u] \neq N_{\Gamma_G}[a_v]$ for $1 \leq u < v \leq k$.

**Proof.** By condition (ii), there exists $M_i \in \mathcal{M}_p$ such that $\langle a_k \rangle \subseteq M_i$. Set $|a_k| = p^m$. If $m \leq \lambda_i$, then $k \leq m + 1 \leq \lambda_i + 1 < \alpha_p$ by condition (i) and equation (4). In the following proof, suppose $m > \lambda_i$. Let $l = \min\{j : p^\lambda_j < |a_j|, 1 \leq j \leq k\}$. Then $|a_{l-1}| \leq p^\lambda_l$, and so

\begin{equation}
 l - 1 \leq \lambda_i + 1.
\end{equation}

For every $a_j \in \{a_i, a_{i+1}, \ldots, a_k\}$, the maximal cyclic subgroups of $G$ containing $\langle a_j \rangle$ are $p$-groups. As refer to (3), we get $C_{i(u-1)} \subsetneq \langle a_j \rangle \subseteq C_{iu}$, where $u \in \{s_i, s_i' + 1, \ldots, s_i\}$.\
If \( k - l > s_i - s_i' \), then there exists indices \( j \in \{ l, l + 1, \ldots, k \} \) and \( u \in \{ s_i', s_i' + 1, \ldots, s_i \} \) such that \( C_i(u-1) \not\subseteq \langle a_j \rangle \subseteq \langle a_{j+1} \rangle \subseteq C_iu \), and it follows from Lemma 4.1 (ii) that \( N_{\Gamma_G}[a_j] = N_{\Gamma_G}[a_{j+1}] \), contradicting condition (iii). Hence, one has

\[
k - l \leq s_i - s_i'.
\]

Combining (5) and (6) we get \( k \leq s_i - s_i' + \lambda_i + 2 \), and so \( k \leq \alpha_p \).

**Lemma 4.3.** If \( \mathcal{M}_p \neq \emptyset \), then \( \mathcal{R}_G \) has a clique with size \( \alpha_p \).

**Proof.** With reference to (2) and (4), take \( M_i \in \mathcal{M}_p \) such that \( \alpha_p = s_i - s_i' + \lambda_i + 2 \). As refer to (3), we have

\[
\langle c_{i1} \rangle \subseteq \cdots \subseteq \langle c_i(s_i'-1) \rangle \subseteq \langle c_i(s_i'+1) \rangle \subseteq \cdots \subseteq \langle c_{i\lambda_i} \rangle.
\]

For \( j \in \{ 0, 1, \ldots, \lambda_i \} \), let \( x_j \) be an element of order \( p^j \) in \( \langle c_i(s_i') \rangle \). Write

\[
S = \{ x_0, x_1, x_2, \ldots, x_{\lambda_i}, c_{i(s_i')}, c_{i(s_i'+1)}, \ldots, c_{i\lambda_i} \}.
\]

Then \( |S| = \alpha_p \) and \( S \) is a clique of \( \Gamma_G \). Now we only need to show that for any two distinct vertices \( a \) and \( b \) in \( S \),

\[
N_{\Gamma_G}[a] \neq N_{\Gamma}[b]. \tag{7}
\]

If \( \{ a, b \} \subseteq \{ c_{i(s_i')}, c_{i(s_i'+1)}, \ldots, c_{i\lambda_i} \} \), then (7) holds by Lemma 4.1 (i). In the following, suppose that \( \{ a, b \} \not\subseteq \{ c_{i(s_i')}, c_{i(s_i'+1)}, \ldots, c_{i\lambda_i} \} \). Without loss of generality, assume that \( a = x_u \) for \( 0 \leq u \leq \lambda_i \). Let \( \langle x \rangle \) be a maximal cyclic subgroup containing \( x_{\lambda_i} \) in \( \mathcal{M} \setminus \mathcal{M}_p \). If \( b = x_v \) for \( 0 \leq v \leq \lambda_i \), then \( |x| \) is not a prime power, which implies that (7) holds as \( \{ a, b \} \subseteq \langle x \rangle \). If \( b = c_{iv} \) for \( s_i' \leq v \leq s_i \), then \( x \in N_{\Gamma_G}[a] \setminus N_{\Gamma_G}[b] \), and so (7) holds.

Now we prove Theorem 3.2.

**Proof of Theorem 3.2:** Let \( S \) be a clique \( \mathcal{R}_G \) with size \( \omega(\mathcal{R}_G) \), and write

\[
S = \{ x_1, x_2, \ldots, x_s \}.
\]

In view of [23, Lemma 1], we may assume that \( \langle x_1 \rangle \not\subseteq \langle x_2 \rangle \subseteq \cdots \subseteq \langle x_s \rangle \subseteq K \), where \( K \in \mathcal{M} \).

Suppose that \( K \in \mathcal{M}_q \) for some \( q \in \pi_n \). Then \( |x_s| \) is a power of \( q \). Furthermore, we have that \( \langle x_u \rangle \not\subseteq \langle x_{u+1} \rangle \) and \( N_{\Gamma_G}[x_u] \neq N_{\Gamma_G}[x_v] \) for any \( 1 \leq u < v \leq s \). According to Lemma 4.2 we get

\[
s \leq \alpha_q \leq \max\{ \alpha_p : p \in \pi_n \}, \tag{8}
\]

which implies that \( \omega(\mathcal{R}_G) \leq \max\{ \alpha_p : p \in \pi_n \} \), and so (i) holds by Lemma 4.3.
In the following, we prove (ii). If \( K \in \mathcal{M} \setminus \left( \bigcup_{p \in \pi_n} \mathcal{M}_p \right) \), then
\[
s \leq \sigma_{|K|} + 1 \leq \max\{\sigma_{|M|} + 1 : M \in \mathcal{M} \setminus \left( \bigcup_{p \in \pi_n} \mathcal{M}_p \right)\}. \tag{9}
\]
Combining (8) and (9) we have
\[
\omega(\mathcal{R}_G) \leq l, \quad \text{where } l = \max\{\{\alpha_p : p \in \pi_n\} \cup \{\sigma_{|M|} + 1 : M \in \mathcal{M} \setminus \left( \bigcup_{p \in \pi_n} \mathcal{M}_p \right)\}\}.
\]

Now it suffices to show that \( \mathcal{R}_G \) has a clique with size \( l \). If \( l = \alpha_{q'} \) for some \( q' \in \pi_n \), it follows from Lemma 4.3 that \( \mathcal{R}_G \) has a clique with size \( l \), as wanted. Now suppose that \( l = \sigma_{|M|} + 1 \) for some \( M \in \mathcal{M} \setminus \left( \bigcup_{p \in \pi_n} \mathcal{M}_p \right) \). By the proof of Theorem 3.1, there exists a subset \( A \) of \( \pi_n \) and \( |A| = l - 1 \) such that \( K \setminus A \) is a clique of \( \mathcal{R}_M \), where \( e \) is the identity of \( G \). Let \( T = A \cup \{e\} \). It follows from [5, Proposition 4] that \( T \) is a clique of \( \mathcal{R}_G \) with size \( l \). The proof is now complete.

## 5 Examples

The following result is immediate from Theorems 2.2 and 3.2.

**Corollary 5.1.** Let \( G \) be a noncyclic \( p \)-group of order \( n \). With reference to (3),
\[
\text{sdim}(\Gamma_G) = n - \max\{s_i : 1 \leq i \leq t\}.
\]

**Remark 5.2.** Take \( p \in \pi_{|G|} \). With reference to (2), (3) and (4), we have
\[
s_i \leq f_i + 1 \quad \text{and} \quad \alpha_p \leq \max\{f_i : 1 \leq i \leq t\} + 1,
\]
where \( p^{f_i} = |M_i| \).

Let \( \mathbb{Z}_p^n \) be the elementary abelian \( p \)-group of order \( p^n \). Then any maximal cyclic subgroup of \( \mathbb{Z}_p^n \) is of order \( p \). Thus, by Corollary 5.1 we have the following result.

**Example 5.3.** \( \text{sdim}(\Gamma_{\mathbb{Z}_p^n}) = p^n - 2 \).

Let \( D_{2n} \) denote the dihedral group of order \( 2n \).

**Example 5.4.** For \( n \geq 3 \), we have \( \text{sdim}(\Gamma_{D_{2n}}) = 2n - (\sigma_n + 1) \).

*Proof.* Let \( D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle \). Then every maximal cyclic subgroup of \( D_{2n} \) is either \( \langle a \rangle \) or isomorphic to \( \mathbb{Z}_2 \). If \( n \) is a power of 2, then \( s_i = 2 \) for each \( i \), which implies that \( \text{sdim}(\Gamma_{D_{2n}}) = 2n - 2 = 2n - (\sigma_n + 1) \) from Corollary 5.1. If \( n \) is a power of an odd prime \( q \), then \( \alpha_q = \alpha_2 = 2 \), the required result follows...
form Theorems 2.2 and 3.2 (i). If \( n \) is not a power of any prime, it follows from Theorems 2.2 and 3.2 (ii) and Remark 5.2 that \( \text{sdim}(\Gamma_{D_{2n}}) = 2n - (\sigma_n + 1) \), as wanted.

The generalized quaternion group is defined by

\[
Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle, \quad n \geq 2.
\]

**Example 5.5.** \( \text{sdim}(\Gamma_{Q_{4n}}) = 4n - (\sigma_{2n} + 1) \).

**Proof.** Note that every maximal cyclic subgroup of \( Q_{4n} \) is either \( \langle x \rangle \) or isomorphic to \( \mathbb{Z}_4 \), and the intersection of any two distinct maximal cyclic subgroups is of order 2. If \( n \) is a power of 2, then \( s_i = 2 \) for all \( i \), which implies that \( \text{sdim}(\Gamma_{Q_{4n}}) = 2n - (\sigma_{2n} + 1) \) by Corollary 5.1. If \( n \) is not a power of 2, then by Theorems 2.2 and 3.2 (ii) and Remark 5.2, we get the desired result.

Let \( A \) be an abelian group of non-prime-power order. By fundamental theorem of finite abelian groups, we may assume that

\[
A \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_k},
\]

where \( d_i | d_{i+1} \) for \( 1 \leq i \leq k - 1 \). Note that \( d_k \) is not a power of a prime, and the order of any maximal cyclic subgroup of \( A \) is a divisor of \( d_k \). The following result follows from Remark 5.2 and Theorems 2.2 and 3.2 (ii).

**Example 5.6.** \( \text{sdim}(\Gamma_A) = \prod_{i=1}^k d_i - (\sigma_{d_k} + 1) \).

It is clear that \( \text{sdim}(\Gamma_G) = |G| - 1 \) if and only if \( G \) is a cyclic group of prime power order. Finally, we classify the group \( G \) with \( \text{sdim}(\Gamma_G) = |G| - 2 \).

**Corollary 5.7.** Let \( G \) be a group of order \( n \). Then \( \text{sdim}(\Gamma_G) = n - 2 \) if and only if \( G \) is one of the following groups:

- (i) \( \mathbb{Z}_{pq} \), where \( p \) and \( q \) are two distinct prime numbers;
- (ii) \( Q_{2^m} \), where \( m \geq 3 \);
- (iii) A noncyclic CP-group such that each two maximal cyclic subgroups have trivial intersection.

**Proof.** Suppose that \( \text{sdim}(\Gamma_G) = n - 2 \). If \( G \) is a cyclic group, by Theorems 2.2 and 3.1, one has that \( G \cong \mathbb{Z}_{pq} \), where \( p \) and \( q \) are two distinct prime numbers. Now suppose that \( G \) is not cyclic. If \( G \) has a nonidentity element \( x \) with \( N_{\Gamma_G}[x] = G \), then \( G \cong Q_{2^m} \) for some \( m \geq 3 \) by [5, Proposition 4]. Assume that any nonidentity element \( x \) of \( G \) satisfies \( N_{\Gamma_G}[x] \neq G \). It follows from Theorems 2.2 and 3.2 (ii) that \( G \) is a CP-group. If there exist two maximal cyclic subgroups \( \langle x \rangle \) and \( \langle y \rangle \) in \( G \) such that
they has nontrivial intersection \( \langle z \rangle \), then take \( T = \{ e, z, x \} \) and so \( \text{sdim}(\Gamma_G) \leq n - 3 \) by Peoposition 2.1, a contradiction. Thus, the necessity is valid.

By Theorems 2.2, 3.1 and 3.2, it is routine to check that every group \( G \) in (i), (ii) and (iii) satisfies \( \text{sdim}(\Gamma_G) = n - 2 \).

\[ \Box \]

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