Semitransparent one-dimensional potential: a Green’s function approach

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Abstract
We study the unstable harmonic oscillator and the unstable linear potential in the presence of the point potential, which is the superposition of the Dirac $\delta(x)$ and its derivative $\delta'(x)$. Using the physical boundary conditions for the Green’s function we derive for both systems the resonance poles and the resonance wave functions. The matching conditions for the resonance wave functions coincide with those obtained by the self-adjoint extensions of the point potentials and also by the modelling of the $\delta'(x)$ function. We find that, with our definitions, the pure $b\delta'(x)$ barrier is semi-transparent independent of the value of $b$.

Keywords: point potentials, unstable systems, Green’s functions

(Some figures may appear in colour only in the online journal)

1. Introduction

The present paper is a contribution to the discussion about the physical properties of one-dimensional point potentials as well as the study of quantum models with resonances. As is well known, point potentials are quantum perturbations of the Hamiltonian which are supported in one point or a countable number of them, either finite or denumerable. A typical example is the Dirac delta or its derivative. In the present work, our point potentials will be supported at the origin and will appear in combination with another potential term.

An important property of point potentials is that they are often solvable or quasi-solvable. In the first case, we may find exact solutions for the eigenvalue problem associated to systems with this kind of potential. In the second, energy values and other data, like resonance poles, can be obtained numerically as solutions of transcendental equations. This feature has its origin in the fact that point potentials are often semitransparent, i.e., they have non-vanishing transmission coefficient. Therefore, point potentials are useful tools to construct one-dimensional models with resonances, which will be of great help in the study of physical and mathematical properties of resonances.

The use of point potentials in quantum mechanics has a long tradition. It began with Kronning-Penney [1], who used them as an approximation for a periodic medium. Next, Bethe and Peierls used a similar approximation for a nucleon model [2]. Berezin and Faddev [3] made a complete mathematical study of a three-dimensional problem with such potentials. In addition, the study of point potentials has been useful in other areas of physics for the solution of various types of problems, see for example [4].

The construction of one-dimensional models with resonances making use of point potentials is not new. Thus far, there are several studies on this field dealing with specific models [5–7]. One of the most interesting is the unstable quantum oscillator proposed by Espinosa and Kielanowski [8]. It is interesting not only because it shows all kinds of features like bound and antibound states and resonances, but because of the (sometimes unexpected) behavior under the variations of certain parameters [9]. Our one-dimensional unstable quantum oscillator has a potential, which is a harmonic oscillator for $x < 0$, zero for $x > 0$ and at the origin a point potential of the form $a\delta(x) + b\delta'(x)$. Here, $a$ and $b$ are real parameters. Eventually, we can put a discontinuity of mass at the origin [18], which adds one more parameter (the ratio of masses) in the model description.

Although the characterization of a point potential of the Dirac delta type, $\delta(x)$ is clear, this is usually not the case with its derivative $\delta'(x)$. The latter is a perturbation which has been defined in several non-equivalent ways [10–14]. In general, terms in the potential of the type $a\delta(x) + b\delta'(x)$ are defined through the theory of self-adjoint extensions of symmetric operators with equal deficiency indices [11, 15]. This is also...
the case when dealing with self-adjoint Hamiltonians in models with a discontinuity in the mass [16].

Another equivalent method to study the properties of these models is through Green’s functions. As the Green’s function can be viewed as a kernel of the resolvent of the Hamiltonian, it is a possible tool to study the resonances in the models under consideration. Resolvent of self-adjoint extensions of symmetric Hamiltonians can be obtained through the Krein formula, and by Green’s function method which is much more familiar to physicists. There are some precedents in the use of Green’s functions with point potentials: for instance in the study of the variation of the energy levels of the infinite square well or the harmonic oscillator in the presence of a point potential [9]. A detailed study of the Green’s function of the harmonic oscillator perturbed by point potentials is given in [17].

In this context, one of the obvious applications of Green’s functions is the location of resonances (in terms of its energy and width) and other features like antibound states in one-dimensional models with point potentials. In the present paper, we shall focus our attention on two models: the unstable harmonic oscillator and a similar linear potential with a perturbation of the type $a\delta(x) + b\delta'(x)$.

It is important to remark that an unambiguous definition of the perturbation $a\delta(x) + b\delta'(x)$ is needed i.e. how the point potential acts on the wavefunctions. To this end, the definition proposed in [11] is widely used, for instance in [18, 19]. This definition is based on the use of matching conditions at the origin.

However, here we shall adopt a different point of view. We first assume that both distributions the Dirac delta $\delta(x)$ and its derivative $\delta'(x)$ may be multiplied by functions having, along with their derivatives, a finite discontinuity at the origin. We have to define these types of products that are distributions, and instead of using the usual product proposed by Kurasov [11], we prefer a more general form given by Zolotaryuk [5]. The Schrödinger equation with the perturbation $a\delta(x) + b\delta'(x)$ becomes an equation for distributions.

Then, the use of the Green’s function, that in our case can be exactly evaluated, and of its properties, like their boundary conditions, will finally determine the matching conditions required for the self-adjoint determination of the Hamiltonian with potential $a\delta(x) + b\delta'(x)$. Although this has been done for two particular cases, we think that the procedure is quite general. This is important, since there has been some confusion on the proper definition of the perturbation $a\delta(x) + b\delta'(x)$, due to the presence of the term $\delta'(x)$ that in our approach is defined unambiguously.

The plan of the paper is the following. In section 2 we fix our notation and include a general discussion of the Green’s function of the Schrödinger equation, including the definition of the resonance. In section 3 we discuss the Green’s functions of two types of potential: harmonic oscillator and linear potential for $x < 0$ and a free motion for $x > 0$. These potentials do not have resonances and bound states, but the Green’s functions for these potentials are used for the determination of the Green’s functions of the point potentials in section 4. In this section we also find conditions for resonances that follow from the Green’s function. In section 5 we determine from the Green’s functions the matching conditions for the wave functions of the resonances at the singularity (support) of the point potentials. Section 6 contains a discussion of our results and conclusions.

2. The Green’s function for the Schrödinger equation

The time independent Schrödinger equation (SE) is

$$H\psi(x) = E\psi(x),$$

(1)

where $H$ is the Hamiltonian of the system, $E$ is an energy eigenvalue of $H$ and $\psi(x)$ is the associated eigenfunction. A way to solve equation (1) is by using the solution of the equation

$$(H - z)G(x, x'; z) = \delta(x - x').$$

(2)

$G(x, x'; z)$ is called the Green function (GF) for the operator $H$. Although the GF is initially defined for real values of $z$, the self adjointness of the Hamiltonian guarantees that the GF, as a function of $z$, admits analytic continuations from below to above and from above to below through the spectrum (usually the positive real line). Thus, the GF is a meromorphic function of $z$ with possible isolated singularities. In general, these singularities are pairs of complex conjugate poles with a positive real part, which can be associated to resonances. The analytic continuation process is detailed in [20], some of the physical content of the Green’s function is described in [21]. Thus, in general $G(x, x'; z)$ is an analytic function in the whole plane $z$, except for a set of points, which can be continuous or discrete. Such a set contains the bound states and the resonances values of the system and it is called the spectrum of $H$. It is denoted by $\text{Sp}(H)$.

For $z \notin \text{Sp}(H)$ the solution of the equation

$$(H - z)\psi(x) = \phi(x),$$

(3)

can be written as

$$\psi(x) = \int G(x, x'; z)\phi(x')dx',$n

(4)

whereas for $z \in \text{Sp}(H)$ the solution of equation (3) is

$$\psi(x) = \int G(x, x'; z^+)\phi(x')dx' + \psi_0(x), \quad \text{with} \quad z^+ = \lim_{b \to 0^+} (a + ib), \quad a, b \in \mathbb{R}.$$  

(5)

The term $\psi_0(x)$ is the solution to the homogeneous equation

$$(H - z)\psi_0(x) = 0.$$  

(6)

The function $G(x, x'; z)$ usually has a branch cut that can be identified with the positive real semi-axis. In addition, it may have poles on the real axis, which correspond to bound states and poles in the analytic continuation of the variable $z$ of $G(x, x'; z)$ (which in the language of Riemann surfaces are...
located on the second Riemann sheet) of the form
\[ z = E_0 - i \frac{\Gamma}{2}. \]  
(7)

For \( \Gamma > 0 \) there is a state with a finite mean lifetime, given by:
\[ \tau = \frac{1}{\Gamma}. \]  
(8)

Let \( G_0(x, x'; z) \) be the GF for the Hamiltonian \( H_0 \) and let \( H = H_0 + \hat{V} \) be the new Hamiltonian under our scrutiny. The GF \( G(x, x'); z) \) of \( H \) can be formally calculated using the \( G_0(x, x'; z) \) of \( H_0 \). To do this one can write equation (2) as
\[ (H_0 - z)G(x, x'; z) = \delta(x - x') - \hat{V}(x)G(x, x'; z). \]  
(9)

As \( G_0(x, x'; z) \) satisfies
\[ (H_0 - z)G_0(x, x'; z) = \delta(x - x'). \]  
(10)

Equation (9) can be multiplied from the left by \( G_0(y, x; z) \) and one obtains
\[ \delta(y - x)G(x, x'; z) = G_0(y, x; z)\delta(x - x') \]
\[ - G_0(y, x; z)\hat{V}(x)G(x, x'; z). \]  
(11)

Formally, equation (11) can be integrated and \( G(x, x'; z) \) can be written as
\[ G(y, x'; z) = G_0(y, x'; z) \]
\[ - \int G_0(y, x; z)\hat{V}(x)G(x, x'; z)d\tau. \]  
(12)

This derivation is well known and is presented here purely formally for the sake of completeness.

3. The Green’s function for the unstable quadratic and linear potentials

Along this section we shall derive the Green’s functions for two potentials. The first one will be the so-called unstable harmonic oscillator, which is a semi-oscillator with a point potential supported in the origin. The second one we call the linear unstable potential, as the semi-oscillator is replaced by the straight line potential.

3.1. The free GF for the unstable quadratic potential

Let us start with the unstable harmonic oscillator. It was introduced by Espinosa and Kielanowski in their study of resonances, see \[8\]. The form of the potential is a harmonic oscillator for \( x < 0 \) (and hence called semi-oscillator) plus a point potential supported at the origin and of the form \( ao(x) + b\delta(x) \), \( a \) and \( b \) being real numbers. The original idea in \[8\] was to introduce a Dirac delta in order to produce resonances in the model, although it was shown in \[18\] that resonances appear even in the absence of any kind of point potential. In the mentioned paper in \[18\], we also have made a thorough analysis of the behavior of resonances for this model.

Let us begin our analysis by defining the unperturbed Hamiltonian \( H_0 \) as:
\[ H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{2} \theta(-x), \]  
(13)

where \( \theta(x) \) is the Heaviside step function. Later, we shall define the total Hamiltonian as the sum of \( H_0 \) plus the point potential. To obtain the GF \( G_0^\pm(x, x'; z) \) for \( H_0 \) in equation (13), it is necessary to take equation (10) and observe that there exist two equations
\[ (H_0 - z)G_0^+(x, x'; z) = 0, \]  
for \( x > x' \);
\[ (H_0 - z)G_0^-(x, x'; z) = 0, \]  
for \( x < x' \).
(14)

We need to solve (14) for \( G_0^+(x, x'; z) \) and \( G_0^-(x, x'; z) \) to construct \( G_0(x, x'; z) \). Using the set of equations (14) along with the following conditions:

(i) continuity of \( G_0^\pm(x, x'; z) \)
\[ \lim_{x \to 0} [G_0^+(x, x + \epsilon; z) - G_0^+(x, x - \epsilon; z)] = 0, \]  
(15)

(ii) jump discontinuity for the derivative with respect to the \( x \) (denoted by the superscript (1, 0))
\[ \lim_{x \to 0} [G_0^+(x, x + \epsilon; z) - G_0^+(x, x - \epsilon; z)] = -2, \]  
(16)

(iii) purely outgoing boundary condition for \( G_0^+(x, x'; z) \) as \( x \to \infty \)
\[ G_0^+(x, x'; z) \sim e^{ikx} \]  
for \( x \to \infty \),
(17)

(iv) localization of the \( G_0^-(x, x'; z) \)
\[ G_0^-(x, x'; z) \to 0 \]  
for \( x \to -\infty \),
(18)

the GF \( G_0(x, x'; z) \) can be calculated explicitly. The conditions (i) and (ii) can be deduced from equation (14) and (iii) and (iv) are well established physical requirements [22].

Then, let us use equation (14) with the Hamiltonian \( H_0 \) given in equation (13). We obtain that the GF satisfying the conditions given by equations (15), (16), (17) and (18) is given by:
\[ G_0^+(x, x'; z) \]
\[ = A(x') \begin{cases} y_1(2z, x) + ik y_2(2z, x) & \text{for } x < 0 \\ e^{ikx} & \text{for } x > 0 \end{cases} \]  
(19)

\[ G_0^-(x, x'; z) \]
\[ = C(x') \begin{cases} y_1(2z, x) + 2g(z)y_2(2z, x) & \text{for } x < 0 \\ \frac{1}{2} \left( 1 - \frac{2ig(z)}{k} \right) e^{ikx} & \text{for } x > 0 \end{cases} \]  
(20)

1 The constants in this work are chosen as: \( h = m = 1 \), \( h \) is the Planck’s constant, \( m \) the mass of the particle involved. Moreover, for the particular potentials studied here, we will use all the coupling constants equal to one.
with

\[ A(x') = \frac{1}{2} \left( 1 - \frac{2i g(z)}{k} \right) e^{i k x'} + \frac{1}{2} \left( 1 + \frac{2i g(z)}{k} \right) e^{-i k x'}. \]

\[ C(x') = -\frac{e^{i k x'}}{ik - 2g(z)}. \]

\[ g(z) = \Gamma \left( \frac{3 - 2z}{4} \right) \Gamma \left( \frac{1 - 2z}{4} \right)^{-1}. \]

(21)

The functions \( y_1(a, x) \) and \( y_2(a, x) \) are the parabolic cylinder functions and \( \Gamma(z) \) is the Gamma function (see for example [23]). It is a simple exercise to check that (19) and (20) satisfy all the properties required for the Green’s functions.

3.2. The free GF for the unstable linear potential

The case of the linear potential is also interesting, since it is one of the few potentials for which the Schrödinger equation admits a known analytic solution. In addition, it appears in the first order term in the WKB method for SE, see [24]. In this case, the free Hamiltonian \( H_0 \) is given by

\[ H_0(x) = -\frac{1}{2} \frac{d^2}{dx^2} - x \delta(-x), \]

where the intensity of interaction has been set to one. Such an interaction can be interpreted like a presence of a constant electric field for a charged particle [25].

Following the same procedure as in the previous case and taking into account the mentioned properties that the GF must have, one finds the GF, \( G_0(x, x'; z) \), as:

\[ G_0^+(x, x'; z) = D(x') \begin{cases} i^{2\delta h} e^{-\sqrt{2} \zeta(x-x')} & x > 0 \\ F(x')(\text{Bi}(-\sqrt{2}(z + x))\cos(\sqrt{2}\sqrt{z}x) - \text{Ai}(\sqrt{2}\sqrt{z}x)\sin(\sqrt{2}\sqrt{z}x)) \end{cases} \]

(23)

\[ G_0^-(x, x'; z) = C(x') \begin{cases} \sqrt{2} (\frac{3}{4})^{-1/6} e^{\sqrt{2}\sqrt{z}x} \text{Ai}(\sqrt{2}\sqrt{z}x) & x > 0 \\ 2^{3/2} \text{Ai}(\sqrt{2}\sqrt{z}x) & x < 0 \end{cases}. \]

(24)

where

\[ C(x') = \frac{i}{\sqrt{2} \zeta(z + x')} - i \sqrt{\frac{2}{\pi}} \text{Ai}(\sqrt{2}\sqrt{z}x) \]

\[ D(x') = \frac{\text{Ai}(\sqrt{2}(z + y))}{\sqrt{2}} \cos(\sqrt{2}\sqrt{z}x) \]

\[ F(x') = \frac{\text{Bi}(\sqrt{2}\sqrt{z}x)}{\sqrt{2} \zeta(z + x')} - \frac{\text{Ai}(\sqrt{2}\sqrt{z}x)}{\sqrt{2} \zeta(z + x')} \]

The functions \( \text{Ai}(x), \text{Bi}(x) \) are the Airy functions. One can find their properties in [23].

Once we have obtained the explicit form for the GF of the free Hamiltonian for both cases under our study, it is time to add the perturbation in the form of a point potential. This will be done in the next section.

4. Point potentials

Now, we shall construct the Green’s functions corresponding to the Hamiltonians studied in the previous section, perturbed with a point potential. In the present case, we have a total Hamiltonian of the form

\[ H = H_0 + V(x), \]

(25)

where \( H_0 \) is either (13) or (22), for which the Green’s function \( G_0(x, x'; z) \) has been already derived and the perturbation \( V(x) \) is the contribution of the point potential given by

\[ V(x) = a \delta(x) + b \delta'(x). \]

(26)

In order to give a proper self-adjoint definition for (25), we need the theory of self-adjoint extensions of symmetric operators with equal deficiency indices. This theory was originally proposed by von Neumann, a complete treatment of the details can be found in [15]. In order to choose the domain for the self-adjoint determination of (25), we need to give matching conditions at the origin [11]. This allows all the self-adjoint extensions of our Hamiltonians to be constructed [19]. Note that the functions in the domain of \( H \) show in general a discontinuity at the origin.
To obtain the GF for $H$ as in (26), we insert (26) into (12), so that:

\[
G(x, x'; z) = G_0(x, x', z)
- a \int G_0(x, x', z) \delta(x') G(x', x', z) \, dx'
- b \int G_0(x, x', z) \delta(x') G(x', x', z) \, dx',
\]

(27)

where $G(x, x'; z)$ is the GF of $H$ as in (25). This is an integral equation for $G(x, x'; z)$, which is solvable in our case due to the specific form of the potential (27). However, one should take into account that $G(x, x'; z)$ and its partial derivatives may be discontinuous in its arguments $x$ and $x'$. Then, it is necessary to define the action of the Dirac delta $\delta(x)$ and its derivative $\delta'(x)$ on functions showing a finite jump at the origin. This includes the form of the products $f(x)\delta(x)$ and $f(x)\delta'(x)$, where $f(x)$ is an arbitrary function with discontinuity at the origin. A first definition has been proposed by Kurasov [11] and applied with similar purposes in other papers [18, 19]. A more general version has been proposed by Zolotaryuk [5–7, 26] and this will be used here. This allows the problem to be viewed with greater generality. Then, the formulas for the products $f(x)\delta(x)$ and $f(x)\delta'(x)$ we have chosen are defined by:

\[
\begin{align*}
\int f(x) \delta(x) \, dx &= \zeta f(0+) + \eta f(0-), \\
\int f(x) \delta'(x) \, dx &= -(\zeta' f(0+) + \eta' f(0-)),
\end{align*}
\]

(28)

where $\zeta$ and $\eta$ are non-negative real numbers such that $\zeta + \eta = 1$. For any function $f(x)$, we denote by $f(0+)$ and $f(0-)$ the right and left limits at the origin, respectively.

From now on, we shall use the following abridged notation: $G(x, x') = G(x, x'; z)$. Next, using (28) in (27), we obtain the following equation:

\[
G(x, x') = G_0(x, x') - a(\eta G(0-, x') + \zeta G(0+, x')) G_0(x, 0)
+ bG_0(x, 0)(\zeta G^{(1,0)}(0+, x') + \eta G^{(1,0)}(0-, x'))
+ \eta G^{(0,1)}(x, 0-)(0, x').
\]

(29)

The superscripts $(1, 0)$ and $(0, 1)$ in $G^{(1,0)}(x, y)$, $G^{(0,1)}(x, y)$ refer to the partial derivatives of $G(x, y)$ with respect to the first and second variable, respectively and $G^{(1,1)}(x, y)$ is the mixed, second derivative. In (29), we have used the following notation:

\[
G(0\pm, y) = \lim_{x \to 0\pm} G(x, y).
\]

(30)

Let us go back to equation (29). This is a relation between the GF of $H_0$ and the GF for $H$ and their partial derivatives. Fortunately, it can be solved so that $G(x, x')$ can be written in terms of $G_0(x, x')$ including the partial derivatives of both GF. Needless to say that, in order to solve (29), the functions $G(0, x')$, $G^{(1,0)}(0\pm, x')$, $G^{(0,1)}(0\pm, x')$ and the value $G^{(12)}(0, 0)$ have to be known.

If we take the derivative in (29) with respect to $x$, we obtain:

\[
\begin{align*}
G^{(1,0)}(0, x') &= G_0^{(1,0)}(0, x') - a(\zeta G(0+, x') + \eta G(0-, x'))
+ \eta G(0-, x') G_0^{(1,0)}(0+, 0)
+ bG_0(0+, 0)(\zeta G^{(1,0)}(0+, x') + \eta G^{(1,0)}(0-, x'))
+ \eta G^{(0,1)}(0-, 0)(0+, x')
+ \eta G_0^{(1,2)}(0+, 0-)(0+, x').
\end{align*}
\]

(31)

Now, first take the limits $x \to 0^+$ and $x \to 0^-$ in (31), i.e., the limits when $x$ goes to zero from the right and from the left, respectively. One finds for $x \to 0^+$

\[
\begin{align*}
G^{(1,0)}(0+, x') &= G_0^{(1,0)}(0+, x') - a(\zeta G(0+, x') + \eta G(0-, x') G_0^{(1,0)}(0+, 0)
+ bG_0(0+, 0)(\zeta G^{(1,0)}(0+, x') + \eta G^{(1,0)}(0-, x'))
+ \eta G^{(0,1)}(0-, 0)(0+, x')
+ \eta G_0^{(1,2)}(0+, 0-)(0+, x').
\end{align*}
\]

(32)

and for $x \to 0^-$

\[
\begin{align*}
G^{(1,0)}(0-, x') &= G_0^{(1,0)}(0-, x') - a(\zeta G(0-, x') + \eta G(0+, x') G_0^{(1,0)}(0-, 0)
+ bG_0(0-, 0)(\zeta G^{(1,0)}(0-, x') + \eta G^{(1,0)}(0+, x'))
+ \eta G^{(0,1)}(0+, 0)(0-, x')
+ \eta G_0^{(1,2)}(0-, 0-)(0+, x').
\end{align*}
\]

(33)

We also need

\[
G(0+, x') = G_0(0+, x') - a(\eta G(0-, x') + \zeta G(0+, x')) G_0(0+, 0)
+ bG_0(0+, 0)(\eta G^{(1,0)}(0+, x') + \zeta G^{(1,0)}(0-, x'))
+ \eta G^{(0,1)}(0+, 0)(0-, x')
+ \eta G_0^{(1,2)}(0+, 0-)(0-, x').
\]

(34)

and

\[
G(0-, x') = G_0(0-, x') - a(\eta G(0-, x') + \zeta G(0+, x')) G_0(0-, 0)
+ bG_0(0-, 0)(\eta G^{(1,0)}(0-, x') + \zeta G^{(1,0)}(0+, x'))
+ \eta G^{(0,1)}(0-, 0)(0+, x')
+ \eta G_0^{(1,2)}(0-, 0-)(0+, x').
\]

(35)
Equations (32), (33), (34) and (35) look rather complicated. Nevertheless, they can be solved for \( G(0+, x') \), \( G(0-, x') \) and its derivatives.

After solving, \( G(x, x'; z) \) can be written as:

\[
G(x, x'; z) = \frac{N(x, x'; z)}{M(z)}.
\]

(36)

Here the functions \( N(x, x'; z) \) and \( M(z) \) are given by:

\[
N(x, x'; z) = -G(0, y)\left(G(x, 0)\left(a - b^2G^{(1,0)}(0, 0)\right) + b\left(b_2G^{(1,0)}(0+, 0) + b_3G^{(1,0)}(0-, 0) - 1\right)\right) + G(x, y)\left(aG(0, 0) + b\left(b_2G^{(1,0)}(0, 0) + 1\right)\right) \times \left(\zeta^{(1,0)}(0+, 0) + \eta^{(1,0)}(0-, 0)\right) + \eta^{(0,1)}(0, 0) - bG(0, 0)\left(\zeta^{(0,1)}(x, 0+) + \eta^{(0,1)}(x, 0-)\right).
\]

(37)

\[
M(z) = 1 + aG(0, 0) - b^2G(0, 0)G^{(1,1)}(0, 0) + b^2\left(\eta^{(1,0)}(0, 0)\right) \times \left(\zeta^{(1,0)}(0+, 0) + \eta^{(1,0)}(0-, 0)\right) - b\left(\zeta^{(0,1)}(0, 0) + \eta^{(0,1)}(0, 0)\right) + b\left(b_2G^{(0,1)}(0, 0) + 1\right) \times \left(\zeta^{(0,1)}(0+, 0) + \eta^{(0,1)}(0-, 0)\right).
\]

(38)

As we have already mentioned, the GF contains a complete information about the system under consideration. In particular, isolated singularities of the GF \( G(x, x'; z) \) in the variable \( z \) give the energy eigenvalues and complex singularities provide energy and mean life of resonances. Thus, in order to find the location of resonance poles, we choose \( M(z) = 0 \) in (38). That the poles of \( G(x, x'; z) \) are given only by the roots of \( M(z) = 0 \) is justified by the following argument: it is true that \( G_0(x, x'; z) \) has singularities, that appear in \( N(x, x'; z) \) and \( M(z) \), but to construct \( G(x, x'; z) \) we need to divide (37) by (38). This operation makes the singularities of \( G_0(x, x'; z) \) not appear in those of \( G(x, x'; z) \) because they cancel out in the division. This means that resonance poles are located at

\[
M(z) = 1 + aG(0, 0) - b^2G(0, 0)G^{(1,1)}(0, 0) + b^2\left(\eta^{(1,0)}(0, 0)\right) \times \left(\zeta^{(1,0)}(0+, 0) + \eta^{(1,0)}(0-, 0)\right) - b\left(\zeta^{(0,1)}(0, 0) + \eta^{(0,1)}(0, 0)\right) + b\left(b_2G^{(0,1)}(0, 0) + 1\right) \times \left(\zeta^{(0,1)}(0+, 0) + \eta^{(0,1)}(0-, 0)\right).
\]

(39)

Formulas (37), (38) and (39) can be applied to any problem no matter which is the chosen Hamiltonian \( H_0 \), provided that the point potential is given by (26). This obviously includes both situations under our consideration. For the case of the unstable harmonic oscillator, one obtains from (39), along with (19) and (20) the following equation for resonance poles position:

\[
ik = \frac{2a}{(2b\zeta + 1)(2b\eta + 1)} + \frac{1 - 2b\eta}{(2b\zeta + 1)(2b\eta + 1)}\delta(z).
\]

(40)

from which one can obtain the resonance energy and width. Note that the resonance condition depends on the parameters \( \zeta \) and \( \eta \). One can find an asymptotic expression for the resonances energies equation (40). Such a formula is an expansion around the eigenvalues of the energy for the special case \( a \to \infty \), these eigenvalues are given by

\[
E_n = 2n + \frac{3}{2},
\]

where \( n \) is a positive integer. So, if in (40) we put \( z = E_n + \delta z_n \), with \( |\delta z_n| < 1 \) we obtain:

\[
\delta z_n = \frac{2}{\pi} \arctan \left(\frac{1 - 2b\eta}{1 - 2b\zeta}\right) + \frac{2a}{\sqrt{4\eta + 1} + i(1 + 2b\zeta)(1 + 2b\eta)}.
\]

(41)

For the case of the linear potential, using again formula (39) and replacing the values for equations (23) and (24), one obtains

\[
ik = \frac{2a}{(2b\zeta + 1)(2b\eta + 1)} - \frac{2}{\sqrt{2}}\frac{1 - 2b\zeta}{\sqrt{2}}\left(\frac{2b\zeta + 1}{2b\eta + 1}\right)Ai(-\sqrt{2}z).
\]

(42)

In this case an asymptotic formula for equation (42) is obtained, using a procedure similar to the one that was used to derive (41). In this case the energy eigenvalues for the infinite well are located at

\[
E_n = \frac{(-1 + 4n)^{2/3}(3n)^{2/3}}{2^{7/3}},
\]

which is actually an approximation. This value of \( E_n \) corresponds to the vanishing of the Airy function \( Ai(x) \) in
an approximate formula for \( \text{Ai}(x) \) [23], such approximation is
\[
\text{Ai}(-z) \sim \sin \left( \frac{2\zeta^{3/2} + \pi}{4} \right). 
\]

If again we put \( z = E_n + \delta\zeta_n \) we obtain
\[
\delta_n = \left( -1 + 4n \right)^{2/3} \left( 3\pi \right)^{2/3} \frac{2\sqrt{3}}{\pi}\left[ \left( \frac{2\sqrt{3}}{\pi} - \left( -1 + 4n \right)^{1/3} (3\pi)^{1/3} \right)^{2/3} \left( 1 - 2\zeta \right) \right] \left( 1 + 2\eta \right) \left( 1 + 2\eta \right) 
\]
\[
+ \frac{2\sqrt{3}}{(1 + 2\zeta)(1 + 2\eta)} \frac{\left( -1 + 4n \right)^{1/3} (3\pi)^{1/3}}{2\sqrt{3}}.
\]

(43)

In both cases, the harmonic and the linear potential, the complex resonance energies depend on the parameters \( \zeta, \eta, a \) and \( b \). Figures 1, 2 and 3 show the variation of complex resonance energies for different values of the parameters. Figure 1 deserves attention: it shows a dependence on the parameters \( \zeta \) and \( \eta \), so one can say the transparency is determined by the parameters \( \eta \) and \( \zeta \). Figure 4(a) shows the difference between the analytic asymptotic formulae (41) and the numerical solutions of (40). Comparison between the numerical solution of (42) and the asymptotic formula (43) is shown in figure 4(b).

Equations (40) and (42) have the same structure, this suggests the form of the resonance condition for a problem with a general regular potential \( V_0(x) \) for which the exact solution is known. Let \( V_0(x) \) be a continuous and monotone increasing potential with the following behavior:

\[
V_0(0) = 0 \quad \text{and} \quad V_0(x) \to \infty \quad \text{as} \quad x \to -\infty.
\]

(44)

Then, the resonance condition should have the structure implied by equations (40) and (42). This indeed is the case, because if \( G_0(x, x'; z) \) is the Green’s function for \( H_0 \), then from equation (39) the general resonance condition can be
calculated. Let $G_0(x, x'; E)$ be denoted by $u(x, x'; E)$, the resonance condition is then,
\[
\zeta + \eta = \frac{1}{b} \left( 1 + 2a \zeta^2 (1 + 2b\eta) \right) \frac{u(0, 0; \zeta)}{u(0, 0; \zeta)}.
\] (45)

Equation (45), involves the function $u(0, 0; \zeta)$, which is position independent and depends only on $\zeta$, by solving it we obtain the complex values of $E$ which are the resonance energies and widths for the Hamiltonian pair \{$H_0, H = H_0 + V$\}.

5. The matching conditions

5.1. The matching conditions using the Green’s function

As we have mentioned earlier, Hamiltonians with point potentials are self-adjoint extensions of the symmetric Hamiltonian with equal deficiency indices. These extensions are determined by choosing their domains, i.e., the space of (square integrable) functions on which the extension acts. The functions on these domains are, in general discontinuous at the points supporting the potentials and they are determined by defining some matching conditions at these points, which, in our case is the only one: the origin. The forms of domains and, therefore, the form of matching conditions give the conditions for the self-adjointness of the extension.

Usually the matching conditions that the functions in the domain of $H = H_0 + a\delta(x) + b\delta'(x)$ have to fulfill at the origin are given a priori. In our case, we have not chosen these matching conditions prior to our discussion and moreover they can be derived from the previously discussed Green’s function.

In order to see it, let us consider the Schrödinger equation for the harmonic semi-oscillator with the point potential $a\delta(x) + b\delta'(x)$, which reads
\[
-\frac{1}{2} \frac{d^2}{dx^2} \psi(x) + \frac{x^2}{2} \psi(x) \theta(x) = E \psi(x) = -a\delta(x) \psi(x) - b\delta'(x) \psi(x).
\] (46)

The objective is to obtain the form of the discontinuity of the solutions $\psi(x)$ of (46) and of their first derivatives, $\psi'(x)$,
at the origin. Now, using equation (5) and the GF $G_0(x, x')$, from equation (10), one has

$$
\psi(x) = \psi_0(x) - a \int G_0(x, x') \delta(x') \psi(x')dx'
- b \int G_0(x, x') \delta'(x') \psi(x')dx',
$$

(47)

where $\psi_0(x)$ is the solution of the equation without the $a\delta(x)$ + $b\delta(x')$ perturbation.

Next, integrating equation (47) and using relations (28), we obtain the following equation:

$$
\psi(x) = \psi_0(x) - aG_0(x, 0)(\zeta\psi(0+) + \eta\psi(0-))
+ b\left(G_0(x, 0)(\zeta\psi'(0+) + \eta\psi'(0-)) + \zeta G_0^{(0, 1)}(x, 0+)\psi(0+)
+ \eta G_0^{(0, 1)}(x, 0-\psi(0-))\right).
$$

(48)

Since the potential $V_0(x)$ in $H_0$ is continuous at $x = 0$, we conclude that both $\psi_0(x)$ and $\psi_0'(x)$ are continuous at origin. From equation (10) one can verify that $G_0(x, 0)$ and $G_0^{(1, 1)}(x, 0)$ are continuous at $x = 0$, while

$$
\begin{align*}
G_0^{(1, 0)}(0+, 0) = G_0^{(1, 0)}(0-, 0) &= -2, \\
G_0^{(0, 1)}(0, 0+) = G_0^{(0, 1)}(0, 0-) &= 2.
\end{align*}
$$

(49)

Using equation (49) in equation (48) one gets the following expression for the discontinuity of $\psi(x)$ at the origin:

$$
\begin{align*}
\psi(0+) - \psi(0-) &= 2b(\zeta\psi(0+) + \eta\psi(0-))
\Rightarrow \psi(0+)
= (1 - 2b\zeta)\psi(0-)
\end{align*}
$$

(50)

Then, take the derivative of (48) with respect to $x$. We obtain:

$$
\begin{align*}
\psi'(x) &= -aG_0^{(1, 0)}(x, 0)(\zeta\psi(0+) + \eta\psi(0-))
+ b\left(G_0^{(1, 0)}(x, 0)(\zeta\psi'(0+) + \eta\psi'(0-)) + \zeta G_0^{(1, 1)}(x, 0+)\psi(0+)
+ \eta G_0^{(1, 1)}(x, 0-)\psi(0-)\right).
\end{align*}
$$

(51)

Using equation (49) in equation (51), we obtain the discontinuity of the derivative at the origin:

$$
\begin{align*}
\psi'(0+) - \psi'(0-) &= 2a(\zeta\psi(0+) + \eta\psi(0-))
- 2b(\zeta\psi'(0+) + \eta\psi'(0-)),
\end{align*}
$$

(52)

which can be transformed into

$$
\begin{align*}
\psi'(0+) + (1 - 2b\zeta)\psi(0+) &= 2a\eta\psi(0-) + (1 - 2b\eta)\psi'(0-).
\end{align*}
$$

(53)

Equations (50) and (53) can be written together in a matrix form as,

$$
\begin{pmatrix}
(1 - 2b\zeta) & 0 \\
-2a\zeta & (1 + 2b\zeta)
\end{pmatrix}
\begin{pmatrix}
\psi(0+) \\
\psi'(0+)
\end{pmatrix}
= 
\begin{pmatrix}
(1 + 2b\eta) & 0 \\
2a\eta & (1 - 2b\eta)
\end{pmatrix}
\begin{pmatrix}
\psi(0-) \\
\psi'(0-)
\end{pmatrix}
$$

(54)

or equivalently,

$$
\begin{align*}
\psi(0+) &= \begin{pmatrix}
\frac{1 + 2b\eta}{1 - 2b\zeta} & 0 \\
2a & (1 - 2b\eta)
\end{pmatrix}
\begin{pmatrix}
\psi(0-) \\
\psi'(0-)
\end{pmatrix}
\end{align*}
$$

(55)

Equation (55) is the final form of the matching conditions for $\psi(x)$ and $\psi'(x)$ at the support of the point potentials, for $b \neq \pm 1/(2\zeta)$. Equations (50), (51) and (52) for the cases $\zeta = \eta = 1/2$ and $b = \pm 1$ give the relations:

$$
\begin{align*}
\psi(0+) &= 0, \\
\psi'(0+) &= \mp \frac{a}{2}\psi(0+) - \psi(0-).
\end{align*}
$$

(56)

The same method can be used to derive the matching conditions for the unstable linear potential and the result coincides with the one in equation (55).

We know equation (26) can be obtained by using the self-adjoint extension method for the free Hamiltonian $H_0$ [11]. In our paper the Hamiltonian without the point potential contains an additional self-adjoint term $V_0(x)\delta(-x)$. Such a term does not alter the procedure used in [11], so that the Hamiltonian (25) can be considered as the self-adjoint extension of the Hamiltonian $H = H_0 + V_0(x)$. From this follows that the wave function $\psi(x)$ and its derivative should obey the same matching conditions at $x = 0$ as in [11], which are

$$
\begin{pmatrix}
\psi(0+) \\
\psi'(0+)
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1 + b}{1 - b} & 0 \\
\frac{1}{1 - b/2} & 1 + b
\end{pmatrix}
\begin{pmatrix}
\psi(0-) \\
\psi'(0-)
\end{pmatrix}
$$

(57)

One can see that equation (57) is not valid for the cases $b = \pm 1$, the matching conditions for these values of $b$ are given in [11]. It is easy to see that equation (57) is a special case of equation (55), by choosing the values $\zeta = \eta = 1/2$. For the cases $b = \pm 1$ the results of [11] coincide with equation (56).

6. Conclusions

We have considered the unstable harmonic oscillator and unstable linear potential with point potentials. In both cases, we found that the resonances are formed and the point potential is semi-transparent. Our method of derivation of the resonance condition and the matching condition for the wave function at the support of the point potential is based on the analysis of the Green’s function of the system. From the poles positions of the Green’s function we obtain the resonance

2 Only the symmetric case $\zeta = \eta = 1/2$ is considered in [11].
conditions and from the residues of the poles we get the wave functions of the resonances. Our assumptions are minimal: we only assume the standard boundary conditions imposed by the physical requirements. We make no assumptions about the model of the $\delta'(x)$ as in [26] or about the self-adjoint extensions for point interactions [11]. It is remarkable, that our results for the matching condition of the wave function coincide with those of [11, 26].

It is very interesting to note that within our approach the barrier formed by the pure $b\delta'(x)$ point potential is semi-transparent for all values of the parameter $b \neq \pm 1/(2\zeta)$. This is in contrast to the results from [26], where it was found that such a potential is semi-transparent only for certain discrete values of $b$. There is a similar work to this [27], in which the Green’s function approach to the problem with highly singular potentials is used, however, the result obtained there is that the transmission through the barrier is zero.

This work shows the equivalence between the method of self-adjoint extensions for the Schrödinger operator with the method of Green’s functions. Whereas the continuity conditions in the former are used only in one dimension the latter has the advantage, that it can be extended to several dimensions in a relatively easy way.

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