FREQUENCY DOMAIN APPROACH TO DECAY RATES FOR A COUPLED HYPERBOLIC-PARABOLIC SYSTEM

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Abstract. We consider the asymptotic behavior of a linear model arising in fluid-structure interactions. The system is formed by a heat equation and a wave equation in two distinct domains, which are coupled by a transmission condition along the interface of the domains. By means of the frequency domain approach, we establish some decay rates for the whole system. Our results also show that the decay of the fluid-structure interaction depends not only on the transmission of the damping from the heat equation to the wave equation, but also on the location of the damping for the wave equation.

1. Introduction and main results. Let $G$ be a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$) with boundary $\Gamma \in C^2$ and let $\omega \subset G$ be a sub-domain of $G$ with boundary $\gamma \in C^2$ such that $\omega \subset G$. Let $\Omega = G \setminus \omega$. The boundary $\partial \Omega$ of $\Omega$ is equal to $\gamma \cup \Gamma$. We denote by $\nu = \nu(x) = (\nu^1(x), \cdots, \nu^n(x))$ the unit outward normal vector of $\Omega$ at $x \in \partial \Omega$, and by $\mu = \mu(x) = (\mu^1(x), \cdots, \mu^n(x))$ the unit outward normal vector of $\omega$ at $x \in \gamma$. Clearly, we have $\nu = -\mu$ on the interface $\gamma$.

Let us consider the following coupled heat-wave system

\[
\begin{align*}
&y_{tt} - \Delta y = 0 \quad \text{in } \omega, \\
&u_t - \Delta u = 0 \quad \text{in } \Omega
\end{align*}
\]

associated with the homogeneous Dirichlet boundary condition on the outer boundary of the heat domain $\Omega$:

\[
u\big|_\Gamma = 0 \quad \text{on } \Gamma
\]

and the transmission condition on the interface $\gamma$:

\[
y = u \quad \text{and} \quad \partial_\nu y = -\partial_\nu u \quad \text{on } \gamma.
\]

The above system is a rough linear model for fluid-structure interactions ([18, 20, 21, 22]). Approximately, the heat part models the fluid structure and the wave one

2020 Mathematics Subject Classification. Primary: 35B35, 37L15; Secondary: 74F10, 93D20.
Key words and phrases. Parabolic-hyperbolic system, fluid-structure interaction, decay rate, interface, frequency domain approach.

This work is partially supported by the NSF of China under grants 11931011, 11821001 and 11831011, and by the Science Development Project of Sichuan University under grant 2020SCUNL201.

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models the elastic structure. To get more physical models, one could replace the heat equation by the Navier-Stokes system and the wave one by an elastic equation.

The coupled system (1.1) describes a configuration where the elastic body is enveloped by the fluid. We will also consider the configuration where the fluid is inside the elastic body:

\[
\begin{aligned}
    u_t - \Delta u &= 0 \quad \text{in } \omega, \\
y_{tt} - \Delta y &= 0 \quad \text{in } \Omega
\end{aligned}
\]  

(1.4)

associated with the viscous damping on the outer boundary of wave domain \( \omega \):

\[
\partial_\nu y + by + y_t = 0 \quad \text{on } \Gamma
\]

(1.5)

(for a given positive constant \( b \)) and the condition of transmission on the interface \( \gamma \):

\[
u_u = y \quad \text{and } \quad \partial_\gamma u = -\partial_\nu y \quad \text{on } \gamma.
\]

(1.6)

Let \( H_a \) be the complex Hilbert space defined by

\[
H_a = \{ U = (y, z, u) \in H^1(\omega) \times L^2(\omega) \times H^1(\Omega) \mid u|\gamma = 0, \quad u|\gamma = y|\gamma \},
\]

equipped with the inner product:

\[
\langle (y_1, z_1, u_1), (y_2, z_2, u_2) \rangle = \int_\omega (\nabla y_1 \cdot \nabla y_2 + z_1 \bar{z}_2) dx + \int_\Omega (\nabla u_1 \cdot \nabla u_2) dx,
\]

\[
\forall (y_1, z_1, u_1), (y_2, z_2, u_2) \in H_a.
\]

Define a linear unbounded operator \( A_a \) on \( H_a \) by

\[
\begin{aligned}
D(A_a) &= \{ (y, z, u) \in H^2(\omega) \times H^1(\omega) \times H^1(\Omega) \mid \Delta u \in H^1(\Omega), \\
        & \quad u = \Delta u = 0 \text{ on } \Gamma, \quad y = u, \quad \partial_\nu y = -\partial_\nu u, \quad \Delta u = z \text{ on } \gamma \}, \\
A_a(y, z, u) &= (z, \Delta y, \Delta u), \quad \forall (y, z, u) \in D(A_a).
\end{aligned}
\]

For any \( U = (y, z, u) \in D(A_a) \), a straightforward computation gives the following dissipation (arising from the heat equation):

\[
\text{Re} \langle A_a U, U \rangle = -\int_\omega |\Delta u|^2 dx \leq 0.
\]

(1.7)

The asymptotic behavior of the whole system depends on how the dissipation affects the wave equation. It was shown in [18, Theorem 2.1] that the operator \( A_a \) (associated with the system (1.1)–(1.3)) generates a contractive \( C_0 \)-semigroup \( S_a(\cdot) \), which is strongly (but not uniformly exponentially) stable. Moreover, based on a weak observability inequality via an elementary perturbation argument, the decay rate

\[
\|S_a(t)U_0\|_{H_a} \leq \frac{M}{t^s} \|U_0\|_{D(A_a)}, \quad \forall \ t \geq 1,
\]

(1.8)

for \( s = 1/2 \) was also established in [18, Theorem 5.2]. Later in [9, Theorem 2], under some further conditions on \( \Omega \) and \( \omega \), using some tools from micro-local analysis (e.g., [8]), the decay rate (1.8) has been improved up to any given \( s < 1 \). Further in [1], using a delicate micro-local argument, the decay rate (1.8) (for a closely related model in 2–3 dimensions) has been obtained for \( s = 1 \), which turns out to be sharp (for multi-dimensional spaces), as conjectured in [18, Remark 5.2].

Motivated by the fundamental works [10, 17] and the subsequent ones [3, 6, 7, 11, 19], the frequency domain approach has become a very useful tool to analyze the long-time behaviors (including exponential/polynomial/logarithmic stability) of many autonomous systems in infinite dimensions. This approach leads to not only a relatively concise and elementary argument, but also an optimal decay rate.
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estimate for the problem under consideration at least for some relatively simple geometrical configurations, such as one-dimensional or rectangular domains (e.g., [4, 5, 13, 15]). Meanwhile, compared to the arguments based on micro-local analysis, this approach requires considerably weaker smoothness conditions on the data (including particularly the boundaries of the space domains in which the systems evolve).

As for the asymptotic behavior of the coupled heat-wave systems arising in fluid-structure interactions, by means of the frequency domain approach, the following results have been obtained:

1. In [2], the decay rate (1.8) (for a closely related model with \( n = 2 \) or 3) was proved for \( s = \frac{1}{2} \);
2. When \( n = 1 \), [4] revisited the models analyzed in [20, 21] and proved the same sharp decay rate (1.8) for \( s = 2 \) by means of a relatively simpler technique;
3. Recently, when \( n = 2 \), [5] considered the special case of a rectangular domain (for which the usual geometric control condition fails) and derived a very interesting sharp decay rate (1.8) (for the system introduced in [22, p. 52] rather than that of (1.1)–(1.3) or (1.4)–(1.6)) with \( s = \frac{1}{3} \).

The main purpose of this work is to focus further on the applications of the above-mentioned frequency domain approach to the systems (1.1)–(1.3) and (1.4)–(1.6). For the first system, we have the following result:

**Theorem 1.1.** There exists a positive constant \( M > 0 \), independent of the initial data, such that solutions to the system (1.1)–(1.3) satisfy

\[
\|S_0(t)U_0\|_{H_a} \leq \frac{M}{t^{3/4}} \|U_0\|_{D(A_a)}, \quad \forall \ t \geq 1
\]

for all initial data \( U_0 \in D(A_a) \).

Clearly, the decay rate 3/4 in (1.9) is better than that in [18, Theorem 5.2]. On the other hand, though the decay rate in (1.9) is not as sharp as that in [9, Theorem 2], our assumptions on the domains \( \Omega \) and \( \omega \) (in the present work) are much weaker than those in [9] because we do not need to use the tools from micro-local analysis. Also, motivated by the main result in [1], one may expect that, under our assumptions (instead of the micro-local analysis type conditions introduced in [1]) on \( \Omega \) and \( \omega \), the decay rate 3/4 in (1.9) should be improved to be 1 (which turns out to be sharp for multi-dimensional spaces, as indicated in [18, Remark 5.2]) but this is an unsolved problem.

Now, we consider the second system (1.4)–(1.6), for which the heat domain is inside the wave one. Let

\[
H_b = \left\{ (u, y, z) \in H^1(\omega) \times H^1(\Omega) \times L^2(\Omega) \mid u = y \text{ on } \gamma \right\}.
\]

Equipped with the following inner product

\[
\langle (u_1, y_1, z_1), (u_2, y_2, z_2) \rangle = \int_{\omega} \nabla u_1 \cdot \nabla u_2 dx + \int_{\Omega} (\nabla y_1 \cdot \nabla y_2 + z_1 z_2) dx + b \int_{\Gamma} y_1 y_2 ds,
\]

\[ \forall (u_1, y_1, z_1), (u_2, y_2, z_2) \in H_b, \]
the space $H_b$ is clearly a Hilbert space. Moreover we define a linear unbounded operator $A_b$ on $H_b$ by
\[
D(A_b) = \{(u, y, z) \in H^2(\omega) \times H^2(\Omega) \times H^1(\Omega) \mid \Delta u \in H^1(\Omega), \partial_\nu y + z = 0 \text{ on } \Gamma, \text{ and } u = y, \partial_\nu u = -\partial_\nu y, \Delta u = z \text{ on } \gamma \},
\]
\[
A_b(u, y, z) = (\Delta u, z, \Delta y), \quad \forall (u, y, z) \in D(A_b).
\]
For any $U = (u, y, z) \in D(A_b)$, a direct computation gives
\[
\text{Re} \langle A_b U, U \rangle = -\int_\omega |\Delta u|^2\,dx - \int_\Gamma |z|^2\,d\sigma \leq 0. \tag{1.10}
\]
Similarly to [18, Theorem 2.1], it can be easily proved that $A_b$ generates a $C_0$-semigroup $S_b(\cdot)$ of contractions on the energy space $H_b$. By (1.10), it is clear that the dissipation arises from both the heat equation on $\omega$ and the viscous damping of the wave equation on the boundary $\Gamma$.

We have the following result:

**Theorem 1.2.** Assume that there exists a point $x_0 \in \mathbb{R}^n$ and $\delta > 0$ so that, for $m = x - x_0$,
\[
(m \cdot \nu) > \delta, \quad \forall x \in \Gamma. \tag{1.11}
\]
Then there exists a positive constant $M > 0$, independent of the initial data, such that solutions to the system (1.4)–(1.6) satisfy
\[
\|S_b(t)U_0\|_{H_b} \leq \frac{M}{t^{1/2}}\|U_0\|_{D(A_b)}, \quad \forall t \geq 1, \tag{1.12}
\]
for all initial data $U_0 \in D(A_b)$.

Because of the double dissipations in (1.10), the solutions to (1.1)–(1.3) should decay more quickly than those to the system (1.4)–(1.6). However, the decay (1.12) is weaker than (1.9). Comparing Theorems 1.1 and 1.2, it seems that when the heat domain is located inside the wave one, the heat dissipation is poorly transmitted to the wave equation. However, under some further geometrical condition on the interface $\gamma$, the nature of stability will radically change. In fact, we have the following result:

**Theorem 1.3.** Assume that the condition (1.11) in Theorem 1.2 holds, and that
\[
(m \cdot \mu) \geq 0, \quad \forall x \in \gamma. \tag{1.13}
\]
Then the semigroup $S_b(\cdot)$ associated to the system (1.4)–(1.6) is uniformly (exponentially) stable in the space $H_b$.

To the best of our knowledge, Theorem 1.3 is the first exponential decay result for hyperbolic-parabolic coupled systems arising in fluid-structure interactions. Note that, when $n > 1$ the additional dissipation from the viscous damping of the wave equation on the boundary $\Gamma$ (in the form of (1.5)) is indispensable for the polynomial/exponential decay results in Theorems 1.2 and 1.3. In fact, as pointed out in [9, Remark 1.5], for the case of multi-dimensional spaces, only the logarithmic decay result could be expected for similar systems without the boundary damping from the wave side, say for the system (1.4)–(1.6) with (1.5) replaced by $\partial_\nu y = 0$ or $y = 0$ on $\Gamma$.

The rest of this paper is organized as follows. In Section 2, we shall collect first some preliminary results which will be used later, and then, in order to highlight the main idea in the proofs of our main results (i.e., Theorems 1.1–1.3), we consider two simplified one-dimensional models, for which, because of the very simple geometry,
it is easy to establish the polynomial or exponential decay rates. Then, in Section 3, we give a proof of Theorem 1.1. Finally, in Section 4 we prove Theorems 1.2 and 1.3.

2. Some preliminaries. First of all, let us recall the following characterization of decay rates for \(C_0\)-semigroups of contractions on Hilbert spaces in terms of the growth of the resolvent of the infinitesimal generators on the imaginary axis (See [3, 6, 11, 19]):

**Lemma 2.1.** Let \(A\) be the infinitesimal operator of a \(C_0\)-semigroup \(S(\cdot)\) of contraction on a Hilbert space \(H\), and let \(\ell > 0\) be given. Then there exists a constant \(M = M(A, \ell) > 0\) such that

\[
\|S(t)U_0\|_H \leq \frac{M}{t^{\ell}}\|U_0\|_{D(A)}, \quad \forall \ t \geq 1, \tag{2.1}
\]

for all initial data \(U_0 \in D(A)\) if and only if

\[
i\mathbb{R} \subseteq \rho(A) \quad \text{and} \quad \sup_{|\beta| \geq 1} \beta^{-\ell}\|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty. \tag{2.2}
\]

For the case \(\ell = 0\), we have the following well-known result (See [10, 17]).

**Lemma 2.2.** Let \(A\) be the infinitesimal operator of a \(C_0\)-semigroup \(S(\cdot)\) of contraction on a Hilbert space \(H\). Then there exist two constants \(M > 0\) and \(r > 0\) such that

\[
\|S(t)U_0\|_H \leq Me^{-rt}\|U_0\|_H, \quad \forall \ t \geq 1, \tag{2.3}
\]

for all initial data \(U_0 \in H\) if and only if

\[
i\mathbb{R} \subseteq \rho(A), \quad \sup_{|\beta| \geq 1} \|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty. \tag{2.4}
\]

In practice, one can verify the growth of the resolvent (of the \(C_0\)-semigroup under consideration) by means of the multiplier method, which may produce sharp results for one-dimensional problems (e.g., Proposition 2.1 in the next subsection). When employing this method, we need the following result.

**Lemma 2.3.** Let \(A\) be the infinitesimal operator of a \(C_0\)-semigroup \(S(\cdot)\) of contraction on a Hilbert space \(H\), and let \(\ell \geq 0\) be given. Assume that

\[
i\mathbb{R} \subseteq \rho(A) \quad \text{and} \quad \sup_{|\beta| \geq 1} \beta^{-\ell}\|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} = +\infty. \tag{2.5}
\]

Then there exist a sequence \((\beta_k)\) of reals and a sequence \((U_k)\) in \(D(A)\), such that

\[
\beta_k \to +\infty, \quad \|U_k\|^2_H = 1, \quad \beta_k^\ell(i\beta_k I - A)U_k \to 0 \quad \text{in} \ H. \tag{2.6}
\]

**Proof.** This result is actually known and well developed in [12]. However, for the sake of readers’ convenience, we shall give here a detailed proof. First, if

\[
\sup_{|\beta| \geq 1} \beta^{-\ell}\|(i\beta I - A)^{-1}\|_H < +\infty
\]

for any given \(U \in H\), then by the Uniform Boundedness Theorem, there exists a constant \(C > 0\) such that

\[
\sup_{|\beta| \geq 1} \beta^{-\ell}\|(i\beta I - A)^{-1}\|_H < C,
\]

which contradicts the second condition in (2.5). Hence, there do exist a sequence of reals \((\beta_k)\) and an element \(U_0 \in H\) with \(\|U_0\| = 1\) such that

\[
\beta_k^\ell\|(i\beta_k I - A)^{-1}U_0\|_H \to +\infty \text{ as } k \to +\infty.
\]
Because of the continuity of the resolvent \((i\beta_k I - A)^{-1}\), we can extract a subsequence, still denoted by \((\beta_k)\), such that \(\beta_k \to +\infty\). Then, let

\[
U_k = \frac{(i\beta_k I - A)^{-1}U_0}{\| (i\beta_k I - A)^{-1}U_0 \|},
\]

we easily check that \(U_k\) satisfies all the requirements in (2.6).

The following interpolation inequality is also known (See [12, Theorem 1.4.4]) and will be used in this paper. For the sake of completeness, we shall give a simple proof.

**Lemma 2.4.** Let \(\omega\) be a bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\gamma\) of class \(C^2\). There exists a positive constant \(c\) depending only on \(\omega\) such that

\[
\int_\gamma |y|^2 d\sigma \leq c \| y \|_{H^1(\omega)} \| y \|_{L^2(\omega)}, \quad \forall y \in H^1(\omega). \tag{2.7}
\]

**Proof.** Let \(h \in C^1(\bar{\omega}; \mathbb{R}^n)\) be such that \(h = \mu\) on \(\gamma\). By the divergence theorem, we have

\[
\int_\omega h \cdot \nabla |y|^2 dx = \int_\gamma (h \cdot \mu)|y|^2 d\sigma - \int_\omega \text{div}(h)|y|^2 dx.
\]

Noting that \((h \cdot \mu) = (\mu \cdot \mu) = 1\) on \(\gamma\), we obtain that

\[
\int_\gamma |y|^2 d\sigma = \int_\omega \text{div}(h)|y|^2 dx + \int_\omega h \cdot \nabla |y|^2 dx
\]

\[
\leq \| \text{div}(h) \|_\infty \int_\omega |y|^2 dx + 2\| h \|_\infty \int_\omega |y| |\nabla y| dx
\]

\[
\leq \| \text{div}(h) \|_\infty \| y \|_{L^2(\omega)}^2 + 2\| h \|_\infty \| y \|_{L^2(\omega)} \| \nabla y \|_{L^2(\omega)}
\]

\[
\leq (\| \text{div}(h) \|_\infty \| y \|_{L^2(\omega)} + 2\| h \|_\infty \| \nabla y \|_{L^2(\omega)}) \| y \|_{L^2(\omega)}
\]

The proof is thus complete.

Next, for the readers’ convenience, before giving the proofs of Theorems 1.1–1.3 (in Sections 3–4), in the next two subsections we shall consider the decay rates for two simplified models in one-dimensional space. Because of the very simple geometry and equations, the calculations are quite explicit and easy, and this allows us to exhibit the main idea of the approach.

### 2.1. Case of the coupled wave equation with homogeneous boundary condition.

Consider the following one-dimensional model:

\[
\begin{align*}
y_{tt} - y'' &= 0, \quad t > 0, \quad -1 < x < 0, \\
u_t - u'' &= 0, \quad t > 0, \quad 0 < x < 1
\end{align*}
\tag{2.8}
\]

associated with the homogeneous Dirichlet boundary conditions

\[
y(-1, t) = 0 \quad \text{and} \quad u(1, t) = 0 \tag{2.9}
\]

and the transmission conditions

\[
y(0, t) = u(0, t) \quad \text{and} \quad y'(0, t) = u'(0, t). \tag{2.10}
\]

The system (2.8)–(2.10) can be regarded as a simplified one-dimensional version of the system (1.1)–(1.3) (We refer to [15] for a more natural one-dimensional version of (1.1)–(1.3), which is then a wave-heat-wave system).
Let $H$ be the complex Hilbert space
\[ H = \{ (y, z, u) \in H^1(-1,0) \times L^2(-1,0) \times H^1(0,1) \mid y(-1) = u(1) = 0, \ u(0) = y(0) \}, \]
(2.11)
equipped with the following inner product:
\[ \langle (y_1, z_1, u_1), (y_2, z_2, u_2) \rangle = \int_{-1}^{0} \left( V_1 y_1' z_2' + z_1 z_2 + u_1 u_2' \right) dx + \int_{0}^{1} u_1 u_2' dx, \] \forall (y_1, z_1, u_1), (y_2, z_2, u_2) \in H. \]
(2.12)

Define the linear unbounded operator $A$ on the above $H$ by
\[
\begin{cases}
D(A) = \{ (y, z, u) \in H^2(-1,0) \times H^1(-1,0) \times H^3(0,1) \mid \\
y(-1) = 0, \ u(1) = u''(1) = 0, \ y(0) = u(0), \ y'(0) = u'(0) \}, \\
A(y, z, u) = (z, y'', u''), \quad \forall (y, z, u) \in D(A).
\end{cases}
\]
(2.13)

For any $U = (y, z, u) \in D(A)$, a straightforward computation gives
\[ \text{Re} \langle AU, U \rangle = - \int_{0}^{1} |u''|^2 dx \leq 0. \] \(2.14\)

It was shown in [21, Section 2] and [18, Theorem 2.1] that $A$ generates a $C_0$-semigroup $S(\cdot)$ of contractions on $H$. Moreover, using a Riesz basis approach, the following result was also established in [21, Theorem 3.1]. In this section, we shall give a short proof of this result by the frequency domain approach (such an approach was also sketched in [4, Section 5] without details).

**Proposition 2.1.** There exists a positive constant $M > 0$, independent of initial data, such that solutions to the system (2.8)–(2.10) satisfy the sharp decay
\[ \| S(t)U_0 \| \leq \frac{M}{t^2} \| U_0 \|_{D(A)}, \quad \forall \ t \geq 1, \] \(2.15\)
for all $U_0 \in D(A)$.

**Proof.** First of all, let us check the conditions of Lemma 2.1. Since $A^{-1}$ is compact and $A$ has no pure imaginary eigenvalues (see [21, Section 2] and [18, Theorem 2.1]), the first condition in (2.2) is clearly satisfied.

Then, we will use the contradiction argument to show that the second condition in (2.2) holds for $\ell = 1/2$. If the condition (2.2) failed, by Lemma 2.3, there would exist a sequence $\beta_k \to +\infty$ and a sequence $(y_k, z_k, u_k) \in D(A)$ such that
\[ \|(y_k, z_k, u_k)\|_H^2 = \|y_k\|_{H^1(-1,0)}^2 + \|z_k\|_{L^2(-1,0)}^2 + \|u_k\|_{H^1(0,1)}^2 = 1 \] \(2.16\)
and
\[ \beta_k^{1/2}(i\beta_k I - A)(y_k, z_k, u_k) \to 0 \quad \text{in } H. \] \(2.17\)
Taking an inner product of (2.17) with $(y_k, z_k, u_k)$, which is a bounded sequence in $H$, we get
\[ i\beta_k^{3/2}\|(y_k, z_k, u_k)\|_H^2 - \beta_k^{1/2}\langle A(y_k, z_k, u_k), (y_k, z_k, u_k) \rangle = o(1). \]
Taking the real part, we get
\[ \beta_k^{1/2} \text{Re} \left( \langle A(y_k, z_k, u_k), (y_k, z_k, u_k) \rangle \right) = \text{Re} \left( i\beta_k^{3/2}\|(y_k, z_k, u_k)\|_H^2 \right) + o(1) = o(1). \]
Using (2.14), it follows that
\[ \beta_k^{1/2}\|u_k''\|_{L^2(0,1)}^2 = o(1). \] \(2.18\)
Now, we express the condition (2.17) as follows

\[
\begin{align*}
\begin{cases}
i\beta_k y_k - z_k &= \frac{f_k}{\beta_k^{1/2}} \quad \text{in } H^1(-1, 0), \\
i\beta_k z_k - y_k'' &= \frac{g_k}{\beta_k^{1/2}} \quad \text{in } L^2(-1, 0), \\
i\beta_k u_k - u_k'' &= \frac{h_k}{\beta_k^{1/2}} \quad \text{in } H^1(0, 1),
\end{cases}
\end{align*}
\]  

(2.19)

where

\[
\|f_k\|_{H^1(-1, 0)} = o(1), \quad \|g_k\|_{L^2(-1, 0)} = o(1), \quad \|h_k\|_{H^1(0, 1)} = o(1).
\]

Then using (2.16) and (2.18) in the expressions,

\[
i\beta_k y_k = z_k + \frac{f_k}{\beta_k^{1/2}}, \quad u_k = \frac{u_k''}{i\beta_k} + \frac{h_k}{i\beta_k^{1/2}},
\]

(2.21)

we deduce that

\[
\|y_k\|_{H^1(-1, 0)} = O(1), \quad \|\beta_k y_k\|_{L^2(-1, 0)} = O(1), \quad \|u_k\|_{L^2(0, 1)} = o(1). \quad \beta_k
\]

(2.22)

Using (2.18), (2.22), we have

\[
\|u_k\|_{H^2(0, 1)}^2 = \|u_k''\|_{L^2(0, 1)}^2 + \|u_k\|_{L^2(0, 1)}^2 = \frac{o(1)}{\beta_k^{1/2}}.
\]

(2.23)

Then by the Nirenberg-Gagliardo inequality, (2.22) and (2.23), it follows that

\[
\|u_k\|_{H^1(0, 1)}^2 \leq C\|u_k\|_{H^2(0, 1)}\|u_k\|_{L^2(0, 1)} = \frac{o(1)}{\beta_k}. \quad \beta_k
\]

(2.24)

Using the interpolation inequality (2.7) with (2.22) and (2.24), it follows that

\[
|u_k(0)|^2 \leq C\|u_k\|_{H^1(0, 1)}\|u_k\|_{L^2(0, 1)} = \frac{o(1)}{\beta_k^2}.
\]

(2.25)

Next, by the first two equations of (2.19), we get

\[
\beta_k^2 y_k + y_k'' = -i\beta_k^{1/2} f_k - \frac{g_k}{\beta_k^{1/2}}.
\]

(2.26)

Since the sequence \(y_k\) is bounded in \(H^1(-1, 0)\), multiplying (2.26) by \(2(x+1)\bar{y}_k\) we get

\[
2\beta_k^2 \int_{-1}^0 (x+1)y_k\bar{y}_k' dx + 2\int_{-1}^0 (x+1)y_k''\bar{y}_k' dx = -2i\beta_k^{1/2} \int_{-1}^0 (x+1)f_k\bar{y}_k' dx + \frac{o(1)}{\beta_k^{1/2}}.
\]

Using integration by parts, we obtain that

\[
\int_{-1}^0 |\beta_k y_k|^2 dx + \int_{-1}^0 |y_k'|^2 dx
\]

\[
= |\beta_k y_k(0)|^2 + |y_k'(0)|^2 - \text{Re} \left(2i\beta_k^{1/2} \int_{-1}^0 (x+1)f_k\bar{y}_k' dx + \frac{o(1)}{\beta_k^{1/2}} \right),
\]

(2.27)

and

\[
\beta_k^{1/2} \int_{-1}^0 (x+1)f_k\bar{y}_k' dx = \beta_k^{1/2} f_k(0)\bar{y}_k(0) - \frac{1}{\beta_k^{1/2}} \int_{-1}^0 ((x+1)f_k)'\beta_k\bar{y}_k dx.
\]

(2.28)
By (2.20) and (2.22), \(\|f_k\|_{H^1(-1,0)} = o(1)\) and \(\|\beta_k y_k\|_{L^2(-1,0)} = O(1)\), it follows that
\[-\frac{1}{\beta_k^{1/2}} \int_{-1}^{0} ((x+1)f_k' + f_k)\beta_k \bar{y}_k dx = \frac{o(1)}{\beta_k^{1/2}}. \quad (2.29)\]

Inserting (2.28) and (2.29) into (2.27), it follows that
\[\int_{-1}^{0} |\beta_k y_k|^2 dx + \int_{-1}^{0} |y_k'|^2 dx = |\beta_k y_k(0)|^2 + |y_k'(0)|^2 - \text{Re} \left(2i\beta_k^{1/2} f_k(0)\bar{y}_k(0)\right) + \frac{o(1)}{\beta_k^{1/2}}. \]

Using (2.10), (2.18), (2.20), (2.23) and (2.25), we get
\[
\begin{cases}
  f_k(0) = o(1), \\
  y_k(0) = u_k(0) = \frac{o(1)}{\beta_k}, \\
  |y_k'(0)| = |u_k'(0)| \leq C\|u_k\|_{H^2(0,1)} = \frac{o(1)}{\beta_k^{1/4}}.
\end{cases}
\]

It follows that
\[\int_{-1}^{0} |\beta_k y_k|^2 dx + \int_{-1}^{0} |y_k'|^2 dx = o(1),\]
which together with the first equation in (2.21) implies that
\[\int_{-1}^{0} |z_k|^2 dx + \int_{-1}^{0} |y_k'|^2 dx = o(1).\]

Then noting (2.24), we deduce
\[\int_{0}^{1} |u_k'|^2 dx + \int_{-1}^{0} |z_k|^2 dx + \int_{-1}^{0} |y_k'|^2 dx = o(1),\]
which contradicts (2.16).

Finally, thanks to the asymptotic expansion (2.30) in [21, p. 392], the hyperbolic branch of eigenvalues of \(A\) satisfies
\[\lambda^h_m = -\frac{1}{\sqrt{2m+1}\pi} + \left(\frac{1}{2} + m\right)\pi i + \frac{\text{sgn}(m)}{\sqrt{2m+1}\pi} i + O\left(\frac{1}{m}\right).\]

Since
\[\text{Re}(\lambda^h_m) \sim \frac{-1}{\sqrt{4m(\lambda^h_m)}},\]
the decay rate (2.15) is indeed optimal (See [13, Theorem 2.1]). The proof is thus complete.

**Remark 1.** The trace estimate (2.25) is directly obtained from the heat dissipation \(\|u_k''\|_{L^2(0,1)}\), which is equivalent to \(\|u_k\|_{H^2(0,1)}\) by (2.23). This yields the sharp decay rate (2.15). However, the situation in high-dimensional spaces is much more complicated (see Remark 3).
2.2. Case of the coupled wave equation with boundary damping. Consider the following one-dimensional model

\[
\begin{align*}
  y_{tt} - y'' &= 0, \quad t > 0, \quad -1 < x < 0, \\
  u_t - u'' &= 0, \quad t > 0, \quad 0 < x < 1
\end{align*}
\]  

(2.30)

subject to a viscous damping on the left-side of the wave equation and a homogeneous Dirichlet boundary condition on the right-side of the heat equation

\[
y'(-1, t) + y(-1, t) = 0 \quad \text{and} \quad u(1, t) = 0
\]  

(2.31)

and the transmission conditions

\[
y(0, t) = u(0, t) \quad \text{and} \quad y'(0, t) = u'(0, t).
\]  

(2.32)

Similarly to (2.8)–(2.10), the system (2.30)–(2.32) can be regarded as a simplified one-dimensional version of the system (1.4)–(1.6) (Actually, a more natural one-dimensional version of (1.4)–(1.6) should be a heat-wave-heat system, corresponding to the wave-heat-wave one in [15]).

Following the same procedure as in [21, Section 2] and the proof of [18, Theorem 2.1], it is easy to show that the system (2.30)–(2.32) generates a \( C_0 \)-semigroup of contractions in the Hilbert space:

\[
H = \{(y, z, u) \in H^1(-1, 0) \times L^2(-1, 0) \times H^1(0, 1) \mid u(0) = y(0), u(1) = 0\},
\]

with the inner product given by that in (2.12). Moreover, we have the following result.

**Proposition 2.2.** The one-dimensional system (2.30)–(2.32) is uniformly exponentially stable.

**Proof.** Define the energy by

\[
E(t) = \frac{1}{2} \int_{-1}^0 (|y'|^2 + |y_t|^2)dx + \frac{1}{2} \int_0^1 |u'|^2dx.
\]

A straightforward computation gives that

\[
E_t(t) = \int_{-1}^0 (y'y_t + yty_{tt})dx + \int_0^1 u'u_tdx
\]

\[
= \int_{-1}^0 (-y''y_t + yty_{tt})dx - \int_0^1 u''u_tdx + \left[y'y_{t-1} |_{x=0} + [u'u_t]_{x=1}\right]
\]

\[
= -\int_0^1 |u''|^2dx - y'(-1)y'(-1) + y'(0)y'(0) - u'(0)u_t(0) + u'(1)u_t(1).
\]

Using (2.31)–(2.32), we obtain

\[
E_t(t) = -|y_t(-1, t)|^2 - \int_0^1 |u''|^2dx.
\]

Integrating this expression over the time interval \([0, T]\) with \(T > 0\), we have

\[
\int_0^T |y_t(-1, t)|^2dt + \int_0^T \int_0^1 |u''|^2dxdt = E(0) - E(T) \leq E(0).
\]  

(2.33)

Multiplying the wave equation in (2.30) by \(xy'\) and integrating over \((-1, 0) \times (0, T)\), we get

\[
\frac{1}{2} \int_0^T \int_{-1}^0 (|y|^2 + |y'|^2)dxdt = -\left[\int_{-1}^0 xyy'dx\right]_0^T + \int_0^T |y_t(-1, t)|^2dt.
\]
By the Cauchy-Schwartz inequality, it follows that
\[
\left| \int_{-1}^{0} xy'y' \, dx \right| \leq \frac{1}{2} \int_{-1}^{0} (|y|^2 + |y'|^2) \, dx \leq E(t) \leq E(0), \quad \forall \ t \geq 0.
\]
Then,
\[
\left| \int_{-1}^{0} xy'y' \, dx \right|_0^T \leq E(T) + E(0) \leq 2E(0).
\] (2.34)
By (2.33) and (2.34), we get
\[
\frac{1}{2} \int_{0}^{T} \int_{-1}^{0} (|y|^2 + |y'|^2) \, dx \, dt \leq E(T) + E(0) \leq 2E(0).
\] (2.35)
By the Nirenberg-Gagliardo and Poincaré inequalities, we have
\[
\|u''\|_{L^2(0,1)} \leq c \|u''\|_{L^2(0,1)} \|u\|_{L^2(0,1)} \leq C \|u''\|_{L^2(0,1)} \|u''\|_{L^2(0,1)}.
\]
Hence,
\[
\|u''\|_{L^2(0,1)} \leq C \|u''\|_{L^2(0,1)},
\]
which together with (2.33) gives
\[
\int_{0}^{T} \int_{0}^{1} |u''|^2 \, dx \, dt \leq C \int_{0}^{T} \int_{0}^{1} |u''|^2 \, dx \, dt \leq CE(0).
\] (2.36)
Then, combining (2.35) and (2.36), we get
\[
\int_{0}^{+\infty} E(t) \, dt \leq C' E(0).
\]
By the Datko Theorem (See [16, Theorem 4.4.1]), we obtain the desired uniform exponential stability. 

**Remark 2.** Comparing Propositions 2.1 and 2.2, we see that the stability of the coupled heat/wave system depends not only on the transmission on the interface, but also on the boundary condition on the other side of the wave equation. We will examine the general situation in Section 4.

3. **Proof of Theorem 1.1.** The main purpose of this section is addressed to proving Theorem 1.1.

**Proof of Theorem 1.1.** We will check the conditions of Lemma 2.1. Unlike the one-dimensional case in Subsection 2.2, where \( \|u_k''\|_{L^2(0,1)}^2 \) is a norm in \( H^2(0,1) \), because of the lack of boundary condition on \( \gamma \), the heat dissipation \( \|\Delta u_k\|_{L^2(\Omega)}^2 \) in (1.7) is no longer a norm on \( H^2(\Omega) \). We have to use some interpolation inequalities to establish the decay rate, which is weaker than that in the one-dimensional problem.

Since \( A_{n-1}^{-1} \) is compact and \( A_n \) has no purely imaginary eigenvalues (see [18, Theorem 2.1]), the first condition of (2.2) is satisfied. We will show that the second condition of (2.2) holds for \( \ell = 4/3 \). Otherwise, by Lemma 2.3, we could find a sequence \( \beta_k \to +\infty \) and a sequence \( (y_k, z_k, u_k) \in D(A_n) \) such that
\[
\|y_k\|_{H^1(\omega)}^2 + \|z_k\|_{L^2(\omega)}^2 + \|u_k\|_{H^1(\Omega)}^2 = 1
\] (3.1)
and
\[
\beta_k^{4/3} (i\beta_k I - A_n)(y_k, z_k, u_k) \to 0 \quad \text{in} \ H_n.
\] (3.2)
Taking an inner product of (3.2) with \((y_k, z_k, u_k)\), which is a bounded sequence in \(H_\alpha\), we get
\[
    i\beta_k^{7/3} \|y_k, z_k, u_k\|_{H_\alpha}^2 - \beta_k^{4/3} \langle A_\alpha(y_k, z_k, u_k), (y_k, z_k, u_k) \rangle = o(1).
\]
Taking the real part, we get
\[
    \beta_k^{4/3} Re\langle A_\alpha(y_k, z_k, u_k), (y_k, z_k, u_k) \rangle = Re(i\beta_k^{7/3} \|y_k, z_k, u_k\|_{H_\alpha}^2) + o(1) = o(1).
\]
By the dissipation (1.7), it follows that
\[
    \beta_k^{4/3} \|\Delta u_k\|_{L^2(\Omega)}^2 = o(1).
\]
We next express the condition (3.2) as follows
\[
    i\beta_k y_k - z_k = \frac{f_k}{\beta_k^{4/3}} \quad \text{in } H^1(\omega),
\]
\[
    i\beta_k z_k - \Delta y_k = \frac{g_k}{\beta_k^{4/3}} \quad \text{in } L^2(\omega),
\]
\[
    i\beta_k u_k - \Delta u_k = \frac{h_k}{\beta_k^{4/3}} \quad \text{in } H^1(\Omega),
\]
where
\[
    \|f_k\|_{H^1(\omega)} = o(1), \quad \|g_k\|_{L^2(\omega)} = o(1), \quad \|h_k\|_{H^1(\Omega)} = o(1).
\]
It follows that
\[
    \|\beta_k y_k\|_{L^2(\omega)} = O(1), \quad \|\Delta u_k\|_{L^2(\Omega)} = \frac{o(1)}{\beta_k^{2/3}}, \quad \|u_k\|_{L^2(\Omega)} = \frac{o(1)}{\beta_k^{5/3}}.
\]
For clarity, we divide the remainder of the proof into three steps.

(i) Estimating the normal derivative \(\partial_n u_k\) on \(\Gamma\). Let \(\epsilon > 0\) be a small real number and define the \(\epsilon\)-neighborhood of \(\gamma\) as follows
\[
    \gamma^\epsilon = \{ x \in \Omega \mid \inf_{\tilde{x} \in \gamma} |\tilde{x} - x| \leq \epsilon \}.
\]
Let \(\theta \in C^2(\overline{\Omega})\) be such that
\[
    \theta \equiv 0 \quad \text{in } \gamma^\epsilon; \quad \theta \equiv 1 \quad \text{in } \Omega \setminus \gamma^{2\epsilon}.
\]
Introducing a new variable \(w_k = \theta u_k\), we have
\[
    \Delta w_k = \theta \Delta u_k + 2\nabla \theta \cdot \nabla u_k + u_k \Delta \theta.
\]
Using the third equation in (3.3), we get
\[
    \left\{ \begin{array}{l}
    \Delta w_k = i\beta_k \theta u_k - \frac{\theta h_k}{\beta_k^{4/3}} + 2\nabla \theta \cdot \nabla u_k + u_k \Delta \theta \quad \text{in } \Omega, \\
    w_k = 0 \quad \text{on } \gamma \cup \Gamma.
    \end{array} \right.
\]
Noting (3.4) and (3.5), we obtain that
\[
    -\Delta w_k = \frac{o(1)}{\beta_k^{2/3}} \quad \text{in } H^{-1}(\Omega).
\]
Since \(-\Delta\) is an isomorphism from \(H_0^1(\Omega)\) into \(H^{-1}(\Omega)\), we get \(w_k = \frac{o(1)}{\beta_k^{2/3}} \text{ in } H_0^1(\Omega)\).
In particular, since \(\theta \equiv 1\) in \(\Omega \setminus \gamma^{2\epsilon}\), we get
\[
    w_k |_{\Omega \setminus \gamma^{2\epsilon}} = u_k |_{\Omega \setminus \gamma^{2\epsilon}} = \frac{o(1)}{\beta_k^{2/3}} \quad \text{in } H^1(\Omega \setminus \gamma^{2\epsilon}).
\]
We next estimate $\partial_\nu u_k$ on $\Gamma$. Let $\hat{\theta} \in C^2(\overline{\Omega})$ be such that
\[ \hat{\theta} \equiv 0 \quad \text{in } \gamma^{2\epsilon}; \quad \hat{\theta} \equiv 1 \quad \text{in } \Omega \setminus \gamma^{4\epsilon}. \]
Introducing a new variable $\hat{w}_k = \hat{\theta} u_k$, we have
\[ \Delta \hat{w}_k = i\beta \hat{\theta} u_k - \frac{\hat{\theta} h_k}{\beta_k^{4/3}} + 2\nabla \hat{\theta} \cdot \nabla u_k + u \Delta \hat{\theta} \quad \text{in } \Omega, \]
\[ \hat{w}_k = 0 \quad \text{on } \gamma \cup \Gamma. \]
Noting (3.6), we have
\[ \nabla \hat{\theta} \cdot \nabla u_k = \frac{o(1)}{\beta_k^{2/3}} \quad \text{in } L^2(\Omega). \]
which together with (3.5) implies that
\[ \Delta \hat{w}_k = \frac{o(1)}{\beta_k^{2/3}} \quad \text{in } L^2(\Omega). \]
Since $-\Delta$ is an isomorphism from $H^1_k(\Omega) \cap H^2(\Omega)$ into $L^2(\Omega)$, we get $\hat{w}_k = \frac{o(1)}{\beta_k^{2/3}}$ in $H^1_k(\Omega) \cap H^2(\Omega)$. In particular, noting that $\hat{w}_k = u_k$ in $\Omega \setminus \gamma^{4\epsilon}$, we have
\[ \int_{\Gamma} |\partial_\nu u_k|^2 d\sigma = \int_{\Gamma} |\partial_\nu \hat{w}_k|^2 d\sigma \leq C \|\hat{w}_k\|_{H^1(\Omega)}^2 = \frac{o(1)}{\beta_k^{4/3}}. \tag{3.7} \]
(ii) Estimating the trace of $u_k$ on $\gamma$. Let
\[ F_k = \begin{cases} \beta_k^2 y_k + i \frac{f_k}{\beta_k^{1/3}} + \frac{g_k}{\beta_k^{4/3}} & \text{in } \omega, \\ -i \beta u_k + \frac{h_k}{\beta_k^{4/3}} & \text{in } \Omega, \end{cases} \quad \phi_k = \begin{cases} y_k & \text{in } \omega, \\ u_k & \text{in } \Omega. \end{cases} \]
By the transmission condition (1.3) on $\gamma$, we have
\[ \begin{cases} -\Delta \phi_k = F_k & \text{in } G, \\ \phi_k = 0 & \text{on } \Gamma. \end{cases} \tag{3.8} \]
Since $-\Delta$ is an isomorphism from $H^1_k(G) \cap H^2(G)$ into $L^2(G)$, there exists a positive constant $C > 0$ independent of $\beta_k$ such that
\[ \|\phi_k\|_{H^1(G)} \leq C \|F_k\|_{L^2(G)}, \]
namely,
\[ \|y_k\|_{H^2(\omega)} + \|u_k\|_{H^2(\Omega)} \leq C \beta_k \left(\|y_k\|_{H^2(\omega)} + \|u_k\|_{L^2(\Omega)}\right) + \frac{o(1)}{\beta_k^{4/3}}. \]
Noting (3.1), we get
\[ \|y_k\|_{H^2(\omega)} + \|u_k\|_{H^2(\Omega)} \leq C \beta_k. \tag{3.9} \]
Using the Nirenberg-Gagliardo inequality, we get
\[ \|u_k\|_{H^1(\Omega)}^2 \leq C \|u_k\|_{H^2(\Omega)} \|u_k\|_{L^2(\Omega)} \leq C \beta_k \|u_k\|_{L^2(\Omega)}^3. \]
Then by the interpolation inequality (2.7), we obtain
\[ \int_{\gamma} |u_k|^2 d\sigma \leq C \|u_k\|_{H^1(\Omega)} \|u_k\|_{L^2(\Omega)} \leq C \frac{\beta_k^{1/2}}{\beta_k^{3/2}} \|u_k\|_{L^2(\Omega)}^3. \]
which, together with (3.5), gives
\[ \int_\gamma |u_k|^2 \, d\sigma = \frac{o(1)}{\beta_k^{4/3}}. \]  
(3.10)

(iii) Estimating the total energy. Let \( \mathcal{O} \) be an open set in \( \mathbb{R}^n \) and \( x_0 \in \mathbb{R}^n \). Recalling that \( m = x - x_0 \), for all \( w \in H^2(\mathcal{O}) \), we have the following Green formula
\[ \Re \int_\mathcal{O} w(2m \cdot \nabla w + (n - 1)\overline{w}) \, dx = - \int_\mathcal{O} |\nabla w|^2 \, dx + \Re \int_{\partial \mathcal{O}} (m \cdot \nu)|\nabla w|^2 \, d\sigma \]  
(3.11)
and Rellich identity (see [14, Ch. III]).
\[ \Re \int_\mathcal{O} \Delta w(2m \cdot \nabla w + (n - 1)\overline{w}) \, dx = - \int_\mathcal{O} |\nabla w|^2 \, dx + 2\Re \int_{\partial \mathcal{O}} \partial_\nu w(m \cdot \nabla \overline{w}) \, d\sigma - \int_{\partial \mathcal{O}} (m \cdot \nu)|\nabla w|^2 \, d\sigma \]  
(3.12)

Now, eliminating the unknown \( z \) in the first two equations of (3.3), we get
\[ \beta_k^2 y + \Delta y_k = - \frac{if_k}{\beta_k^{4/3}} - \frac{g_k}{\beta_k^{4/3}} \]  
in \( \omega \).  
(3.13)

Multiplying (3.13) by \( 2m \cdot \nabla y_k + (n - 1)\overline{y}_k \), which is a bounded sequence in \( L^2(\omega) \) by (3.1), we get
\[ \beta_k^2 \int_\omega y_k(2m \cdot \nabla \overline{y}_k + (n - 1)\overline{y}_k) \, dx + \int_\omega \Delta y_k(2m \cdot \nabla \overline{y}_k + (n - 1)\overline{y}_k) \, dx = \frac{o(1)}{\beta_k^{4/3}}. \]  

Using the Green formula (3.11) for the first term and the Rellich identity (3.12) for the second term with \( \mathcal{O} = \omega, \ w = y_k \), we get
\[ \int_\omega |\beta_k y_k|^2 \, dx + \int_\omega |\nabla y_k|^2 \, dx = 2\Re \int_\gamma \partial_\nu y_k(m \cdot \nabla \overline{y}_k) \, d\sigma - \int_\gamma (m \cdot \mu)|\nabla y_k|^2 \, d\sigma \]  
\[ + (n - 1)\Re \int_\gamma \partial_\nu y_k \overline{y}_k \, d\sigma + \int_\gamma (m \cdot \mu)|\beta_k y_k|^2 \, d\sigma + \frac{o(1)}{\beta_k^{4/3}}. \]  
(3.14)

By the transmission condition (1.3) and the trace estimate (3.10), it follows that
\[ \int_\omega |\beta_k y_k|^2 \, dx + \int_\omega |\nabla y_k|^2 \, dx = 2\Re \int_\gamma \partial_\nu y_k(m \cdot \nabla \overline{y}_k) \, d\sigma - \int_\gamma (m \cdot \mu)|\nabla y_k|^2 \, d\sigma \]  
\[ + (n - 1)\Re \int_\gamma \partial_\nu y_k \overline{y}_k \, d\sigma + o(1). \]  
(3.14)

On the other hand, using (3.5), the last equation in (3.3) can be written as
\[ \Delta u_k = i\beta_k u_k - \frac{h_k}{\beta_k^{4/3}} = \frac{o(1)}{\beta_k^{4/3}} \]  
in \( \Omega \).  
(3.15)
Multiplying (3.15) by $2m \cdot \nabla \bar{u}_k + (n-1)\bar{u}_k$ which is bounded in $L^2(\Omega)$ by (3.1), we obtain that
\[
\int_{\Omega} \Delta u_k (2m \cdot \nabla \bar{u}_k + (n-1)\bar{u}_k) \, dx = \frac{o(1)}{\beta_k^{2/3}}.
\]
Using again the Rellich identity (3.12) with $O = \Omega, w = u_k$, we get
\[
\int_{\Omega} |\nabla u_k|^2 \, dx = 2 \text{Re} \int_{\Gamma \cup \gamma} \partial_\nu u_k (m \cdot \nabla \bar{u}_k) \, d\sigma + \int_{\Gamma \cup \gamma} (m \cdot \nu) |\nabla u_k|^2 \, d\sigma + (n-1) \int_{\Gamma \cup \gamma} \bar{u}_k \partial_\nu u_k \, d\sigma = o(1).
\]
Using (3.1), (3.7) and the boundary conditions
\[
u = -\mu, \quad u_k = 0, \quad \nabla u_k = (\partial_\nu u_k)\nu \quad \text{on} \Gamma,
\]
we get
\[
\begin{align*}
\int_{\Gamma} \partial_\nu u_k (m \cdot \nabla \bar{u}_k) \, d\sigma &= \int_{\Gamma} (m \cdot \nu) |\partial_\nu u_k|^2 \, d\sigma = \frac{o(1)}{\beta_k^{3/2}}, \\
\int_{\Gamma} (m \cdot \nu) |\nabla u_k|^2 \, d\sigma &= \int_{\Gamma} (m \cdot \nu) |\partial_\nu u_k|^2 \, d\sigma = \frac{o(1)}{\beta_k^{1/2}}, \\
\int_{\Gamma} \bar{u}_k \partial_\nu u_k \, d\sigma &\leq C \|u_k\|_{H^1(\Omega)} \|\partial_\nu u_k\|_{L^2(\Gamma)} = \frac{o(1)}{\beta_k^{3/2}}.
\end{align*}
\]
Inserting the estimates in (3.17) into (3.16), we get
\[
\int_{\Omega} |\nabla u_k|^2 \, dx = 2 \text{Re} \int_{\Gamma \cup \gamma} \partial_\nu u_k (m \cdot \nabla \bar{u}_k) \, d\sigma
\]
\[
- \int_{\gamma} (m \cdot \nu) |\nabla u_k|^2 \, d\sigma + (n-1) \int_{\gamma} \bar{u}_k \partial_\nu u_k \, d\sigma + o(1).
\]
Recall that the transmission condition (1.3) implies
\[
\nu = -\mu, \quad u_k = y_k, \quad \partial_\nu u_k = -\partial_\mu y_k, \quad \nabla u_k = \nabla y_k \quad \text{on} \gamma.
\]
It follows from (3.18) that
\[
\int_{\Omega} |\nabla u_k|^2 \, dx = -2 \text{Re} \int_{\gamma} \partial_\mu y_k (m \cdot \nabla \bar{y}_k) \, d\sigma
\]
\[
+ \int_{\gamma} (m \cdot \mu) |\nabla y_k|^2 \, d\sigma - (n-1) \text{Re} \int_{\gamma} \bar{y}_k \partial_\mu y_k \, d\sigma + o(1).
\]
When we add (3.14) and (3.20), the boundary integrals on the right-hand side of (3.14) and (3.20) cancel each other out, we thus get
\[
\int_{\omega} |\beta_k y_k|^2 \, dx + \int_{\omega} |\nabla y_k|^2 \, dx + \int_{\Omega} |\nabla u_k|^2 \, dx = o(1).
\]
Using the first equation in (3.3), we deduce that
\[
\int_{\omega} |z_k|^2 \, dx + \int_{\omega} |\nabla y_k|^2 \, dx + \int_{\Omega} |\nabla u_k|^2 \, dx = o(1),
\]
which contradicts (3.1). The proof of Theorem 1.1 is complete. 
\qed
Remark 3. The key points in the proof of Theorem 1.1 are the trace estimates (3.7) and (3.10). By lack of boundary condition on $\gamma$, the heat dissipation $\|\Delta u_k\|_{L^2(\Omega)}$ is not a norm in $H^2(\Omega)$. In order to get the estimate of $u_k$, we construct an auxiliary $\phi_k$ using the equation (3.8) in the global domain $\omega \cup \Omega$ with the homogeneous boundary condition on $\Gamma$. However, the estimate (3.9) is not optimal. This is why the decay rate in Theorem 1.1 is limited by $3/4$ instead of $2$ in Proposition 2.1.

4. Proofs of Theorems 1.2 and 1.3. This section is mainly addressed to prove Theorem 1.2 and 1.3.

Proof of Theorems 1.2 and 1.3. As in the proof of Theorem 1.1, we will apply Lemmas 2.1 and 2.2 (To simplify the presentation, in what follows we shall view Lemma 2.2 as a “limit” case of Lemma 2.1 with $\ell = 0$). It can be easily proved that $A_b$ generates a $C_0$-semigroup $S_b(t)$ of contractions on the energy space $H_b$. Moreover $A_b^{-1}$ is compact and $A_b$ has no purely imaginary eigenvalues, therefore, the first condition of (2.2) is satisfied. The proof is standard and similar to [18, Theorem 2.1].

It suffices to check the second condition of (2.2) for $\ell = 2$, or $\ell = 0$ respectively. Otherwise, by Lemma 2.3, we would find a sequence $\beta_k \to +\infty$ and a sequence $(u_k, y_k, z_k) \in D(A_k)$ such that

$$
\|u_k\|_{H^1(\omega)}^2 + \|y_k\|_{H^1(\Omega)}^2 + \|z_k\|_{L^2(\Omega)}^2 = 1
$$

and

$$
\beta_k^\ell (i\beta_k I - A_k)(u_k, y_k, z_k) \to 0 \text{ in } H_b.
$$

First, taking an inner product of (4.2) with $(y_k, z_k, u_k)$, which is a bounded sequence in $H_b$, we get

$$
i\beta_k^{\ell+1} \|(u_k, y_k, z_k)\|_{H_b}^2 - \beta_k^\ell \langle A_b(u_k, y_k, z_k), (u_k, y_k, z_k) \rangle = o(1).
$$

Taking the real part, we get

$$
\beta_k^\ell Re(A_b(u_k, y_k, z_k), (u_k, y_k, z_k)) = Re(i\beta_k^{\ell+1} \|(u_k, y_k, z_k)\|_{H_b}^2) + o(1) = o(1).
$$

Using the dissipation (1.10), we get

$$
\beta_k^\ell \int_\omega |\Delta u_k|^2 dx + \beta_k^\ell \int_\Gamma |z_k|^2 dx = o(1).
$$

It follows that

$$
\|\Delta u_k\|_{L^2(\omega)} = o(1) \beta_k^{\ell/2}, \quad \|z_k\|_{L^2(\Gamma)} = o(1) \beta_k^{\ell/2}.
$$

(4.3)

Now we express the condition (4.2) as follows

$$
\begin{cases}
  i\beta_k u_k - \Delta u_k = \frac{h_k}{\beta_k} & \text{in } H^1(\omega), \\
  i\beta_k y_k - z_k = \frac{f_k}{\beta_k} & \text{in } H^1(\Omega), \\
  i\beta_k z_k - \Delta y_k = \frac{g_k}{\beta_k} & \text{in } L^2(\Omega),
\end{cases}
$$

(4.4)

where

$$
\|h_k\|_{H^1(\omega)} = o(1), \quad \|f_k\|_{H^1(\Omega)} = o(1), \quad \|g_k\|_{L^2(\Omega)} = o(1).
$$

(4.5)
Using (4.1), (4.3), the boundary condition \( \partial_\nu y_k + by_k + z_k = 0 \) on \( \Gamma \) and the second equation in (4.4), we get

\[
\|\beta_k y_k\|_{L^2(\Omega)} = O(1), \quad \|y_k\|_{L^2(\Gamma)} = \frac{o(1)}{\beta_k^{1+\epsilon/2}}, \quad \|\partial_\nu y_k\|_{L^2(\Gamma)} = \frac{o(1)}{\beta_k^{1+\epsilon/2}}
\]

and

\[
\|u_k\|_{L^2(\omega)} = \frac{o(1)}{\beta_k^{1+\epsilon/2}}.
\]

Noting (4.1), \( \|u_k\|_{H^1(\omega)} \) is bounded. Then using the interpolation inequality (2.7), we have

\[
\int_\gamma |u_k|^2 d\sigma \leq C\|u_k\|_{H^1(\omega)}\|u_k\|_{L^2(\omega)} = \frac{o(1)}{\beta_k^{1+\epsilon/2}}.
\]

Next, using (4.3) and the first equation in (4.4), we get \( \|\beta_k u_k\|_{L^2(\omega)} = \frac{o(1)}{\beta_k^{1+\epsilon/2}} \). Then multiplying the first equation in (4.4) by \( 2m \cdot \nabla \pi_k + (n - 1)\pi_k \), which is bounded in \( L^2(\omega) \) by (4.1), we arrive at

\[
\int_\omega \Delta u_k (2m \cdot \nabla \pi_k + (n - 1)\pi_k) dx = \frac{o(1)}{\beta_k^{1+\epsilon/2}}.
\]

Using the Rellich identity (3.12) with \( \mathcal{O} = \omega \) and \( w = u_k \), we get

\[
\int_\omega \|
abla u_k \|^2 dx = 2Re \int_\gamma \partial_\nu u_k (m \cdot \nabla \pi_k) d\sigma - \int_\gamma (m \cdot \mu) |\nabla u_k|^2 d\sigma + (n - 1)Re \int_\gamma \pi_k \partial_\nu u_k d\sigma + \frac{o(1)}{\beta_k^{1+\epsilon/2}}.
\]

Now, we write the last two equations of (4.4) as

\[
\beta_k^2 y_k + \Delta y_k = -\frac{if_k}{\beta_k} - \frac{g_k}{\beta_k} \quad \text{in} \quad \Omega.
\]

Then multiplying (4.10) by \( 2m \cdot \nabla \bar{y}_k + (n - 1)\bar{y}_k \), which is bounded in \( L^2(\Omega) \) by (4.1), we get

\[
\beta_k^2 \int_\Omega y_k (2m \cdot \nabla \bar{y}_k + (n - 1)\bar{y}_k) dx + \int_\Omega \Delta y_k (2m \cdot \nabla \bar{y}_k + (n - 1)\bar{y}_k) dx
\]

\[
+ \frac{2i}{\beta_k^{1+\epsilon/2}} \int_\Omega f_k m \cdot \nabla \bar{y}_k dx = \frac{o(1)}{\beta_k^{1+\epsilon/2}}.
\]

Using the Green formula (3.11) for the first term and the Rellich identity (3.12) for the second term with \( \mathcal{O} = \Omega \) and \( w = y_k \), we get

\[
\int_\Omega |\beta_k y_k|^2 dx + \int_\Omega |\nabla y_k|^2 dx
\]

\[
= 2Re \int_{\gamma \cup \Gamma} \partial_\nu y_k (m \cdot \nabla \bar{y}_k) d\sigma - \int_{\gamma \cup \Gamma} (m \cdot \nu) |\nabla y_k|^2 d\sigma
\]

\[
+ (n - 1)Re \int_{\gamma \cup \Gamma} \partial_\nu y_k \bar{y}_k d\sigma + \int_{\gamma \cup \Gamma} (m \cdot \nu) |\beta_k y_k|^2 d\sigma
\]

\[
+ Re \left( \frac{2i}{\beta_k^{1+\epsilon/2}} \int_\Omega f_k m \cdot \nabla \bar{y}_k dx \right) + \frac{o(1)}{\beta_k^{1+\epsilon/2}}.
\]

(4.11)
Using the geometrical control condition (1.11) on $\Gamma$ and the boundary dissipation (4.6), we easily check that

$$
2Re \int_{\Gamma} \partial_v y_k (m \cdot \nabla y_k) d\sigma - \int_{\Gamma} (m \cdot \nu) |\nabla y_k|^2 d\sigma
\leq 2R \|\partial_v y_k\|_{L^2(\Gamma)} \|\nabla y_k\|_{L^2(\Gamma)} - \delta \|\nabla y_k\|^2_{L^2(\Gamma)}
$$

\begin{equation}
(4.12)
\end{equation}

where $R = \max_{x \in \Gamma} |x - x_0|$. Using the boundary dissipation (4.6), we have

$$
\left| (n - 1)Re \int_{\Gamma} \partial_v y_k \bar{\gamma}_k d\sigma \right| \leq (n - 1) \|\partial_v y_k\|_{L^2(\Gamma)} \|\bar{\gamma}_k\|_{L^2(\Gamma)} = \frac{o(1)}{\beta_k}\end{equation}

(4.13) and

\begin{equation}
\left| \int_{\Gamma} (m \cdot \nu) |\beta_k y_k|^2 d\sigma \right| \leq R \|\beta_k y_k\|^2_{L^2(\Gamma)} = \frac{o(1)}{\beta_k}.
\end{equation}

(4.14)

Inserting the estimates (4.12)-(4.14) into (4.11), we find that

\begin{align*}
\int_{\Omega} |\beta_k y_k|^2 dx + \int_{\Omega} |\nabla y_k|^2 dx & \leq 2Re \int_{\gamma} \partial_v y_k (m \cdot \nabla y_k) d\sigma - \int_{\gamma} (m \cdot \nu) |\nabla y_k|^2 d\sigma \\
& + (n - 1)Re \int_{\gamma} \partial_v y_k \bar{\gamma}_k d\sigma + \int_{\gamma} (m \cdot \nu) |\beta_k y_k|^2 d\sigma \\
& + Re \left( \frac{2i}{\beta_k^{1+\ell/2}} \int_{\Omega} f_k m \cdot \nabla \bar{\gamma}_k dx \right) + \frac{o(1)}{\beta_k}.
\end{align*}

(4.15)

Integrating by parts gives

$$
\int_{\Omega} f_k m \cdot \nabla \bar{\gamma}_k dx = - \int_{\Omega} div(f_k m) \bar{\gamma}_k dx + \int_{\gamma \cup \Gamma} (m \cdot \nu) f_k \bar{\gamma}_k d\sigma.
$$

Noting (4.5) and (4.6), we obtain

$$
\int_{\Omega} div(f_k m) \bar{\gamma}_k dx = \frac{o(1)}{\beta_k}, \quad \int_{\Gamma} (m \cdot \nu) f_k \bar{\gamma}_k d\sigma = \frac{o(1)}{\beta_k^{1+\ell/2}}.
$$

It follows that

$$
\int_{\Omega} f_k m \cdot \nabla \bar{\gamma}_k dx = \int_{\gamma} (m \cdot \nu) f_k \bar{\gamma}_k d\sigma + \frac{o(1)}{\beta_k}.
$$

Inserting the above estimate into (4.15), we get

\begin{align*}
\int_{\Omega} |\beta_k y_k|^2 dx + \int_{\Omega} |\nabla y_k|^2 dx & \leq 2Re \int_{\gamma} \partial_v y_k (m \cdot \nabla y_k) d\sigma - \int_{\gamma} (m \cdot \nu) |\nabla y_k|^2 d\sigma \\
& + (n - 1)Re \int_{\gamma} \partial_v y_k \bar{\gamma}_k d\sigma + \int_{\gamma} (m \cdot \nu) |\beta_k y_k|^2 d\sigma \\
& + Re \left( \frac{2i}{\beta_k^{1+\ell/2}} \int_{\gamma} (m \cdot \nu) f_k \bar{\gamma}_k d\sigma \right) + \frac{o(1)}{\beta_k}.
\end{align*}

(4.16)
Under the transmission condition (1.6) on $\gamma$, we have
\[ \nu = -\mu, \quad u_k = y_k, \quad \partial_n u_k = -\partial_n y_k, \quad \nabla u_k = \nabla y_k \text{ on } \gamma. \] (4.17)

Inserting (4.17) into (4.16), we get
\[ \int_{\Omega} |\beta_k y_k|^2 \, dx + \int_{\Omega} |\nabla y_k|^2 \, dx \]
\[ \leq -2Re \int_{\gamma} \partial_\nu u_k (m \cdot \nabla \pi_k) \, d\sigma + \int_{\gamma} (m \cdot \mu)|\nabla u_k|^2 \, d\sigma \]
\[ - (n - 1)Re \int_{\gamma} \partial_\nu u_k \pi_k \, d\sigma - \int_{\gamma} (m \cdot \mu)|\beta_k u_k|^2 \, d\sigma \]
\[ - Re \left( \frac{2i}{\beta_{k}^{(t-1)}} \int_{\gamma} (m \cdot \mu) f_k \pi_k \, d\sigma \right) + o(1) \frac{1}{\beta_{k}^{t/2}}. \] (4.18)

Adding (4.9) and (4.18), the first three terms on the right-hand side of (4.9) and (4.18) cancel each other out, and we have
\[ \int_{\Omega} |\nabla u_k|^2 \, dx + \int_{\Omega} |\beta_k y_k|^2 \, dx + \int_{\Omega} |\nabla y_k|^2 \, dx \]
\[ \leq - \int_{\gamma} (m \cdot \mu)|\beta_k u_k|^2 \, d\sigma - Re \left( \frac{2i}{\beta_{k}^{(t-1)}} \int_{\gamma} (m \cdot \mu) f_k \pi_k \, d\sigma \right) + o(1) \frac{1}{\beta_{k}^{t/2}}. \] (4.19)

In the case of Theorem 1.2, there is no geometrical condition on $\gamma$. It follows from the transmission condition (1.6) on $\gamma$ and the trace estimate (4.8) that
\[ \int_{\gamma} (m \cdot \mu)|\beta_k u_k|^2 \, d\sigma = \frac{o(1)}{\beta_{k}^{(t-2)/2}}, \quad \frac{2i}{\beta_{k}^{(t-1)}} \int_{\gamma} (m \cdot \mu) f_k \pi_k \, d\sigma = \frac{o(1)}{\beta_{k}^{(5t-2)/4}}. \]

Then, it follows from (4.19) that
\[ \int_{\Omega} |\nabla u_k|^2 \, dx + \int_{\Omega} |\beta_k y_k|^2 \, dx + \int_{\Omega} |\nabla y_k|^2 \, dx \leq \frac{o(1)}{\beta_{k}^{(t-2)/2}}, \]
which together with the second equation in (4.4) implies that
\[ \int_{\Omega} |\nabla u_k|^2 \, dx + \int_{\Omega} |z_k|^2 \, dx + \int_{\Omega} |\nabla y_k|^2 \, dx \leq \frac{o(1)}{\beta_{k}^{(t-2)/2}}. \]

This contradicts (4.1) with $\ell = 2$.

In the case of Theorem 1.3, we have the geometrical control condition (1.13). Using the Cauchy-Schwarz inequality, we get
\[ - Re \left( \frac{2i}{\beta_{k}^{(t-1)}} \int_{\gamma} (m \cdot \mu) f_k \pi_k \, d\sigma \right) \leq \int_{\gamma} (m \cdot \mu)|\beta_k u_k|^2 \, d\sigma + \frac{1}{\beta_{k}^{2t}} \int_{\gamma} (m \cdot \mu)|f_k|^2 \, d\sigma. \] (4.20)

Noting (4.5), it follows from (4.20) that
\[ - Re \left( \frac{2i}{\beta_{k}^{t}} \int_{\gamma} (m \cdot \mu) f_k \pi_k \, d\sigma \right) \leq \int_{\gamma} (m \cdot \mu)|\beta_k u_k|^2 \, d\sigma + \frac{o(1)}{\beta_{k}^{2t}}. \] (4.21)

Inserting (4.21) into (4.19), it follows that
\[ \int_{\Omega} |\nabla u_k|^2 \, dx + \int_{\Omega} |\beta_k y_k|^2 \, dx + \int_{\Omega} |\nabla y_k|^2 \, dx \leq \frac{o(1)}{\beta_{k}^{t/2}}, \]
which together with the second equation in (4.4) implies that
\[
\int_{\omega} |\nabla u_k|^2 \, dx + \int_{\Omega} |z_k|^2 \, dx + \int_{\Omega} |\nabla y_k|^2 \, dx \leq \frac{\alpha(1)}{\beta_k^{\ell/2}}.
\]
This contradicts (4.1) with \(\ell = 0\). We have thus shown that
\[
\sup_{|\beta| \geq 1} \|(i\beta I - A)^{-1}\|_{L(H_b)} < +\infty. \tag{4.22}
\]
Then, by Lemma 2.2, the semigroup \(S_b(\cdot)\) is uniformly exponentially stable in the space \(H_b\). The proof is thus complete.

**Remark 4.** In Theorem 1.3, because of the geometrical control condition (1.13), the term “\(-\int_{\gamma} (m \cdot \mu) |\beta_k u_k|^2 \, d\sigma\)” in (4.19) is negative and is used as an additional boundary damping to produce the uniform exponential stability. This is the novelty in this situation.

**Acknowledgments.** The authors would like to thank the three anonymous referees for their very careful and helpful suggestions and comments which led to this improved version.

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Received January 2021; revised June 2021; early access July 2021.

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