um-TOPOLOGY IN MULTI-NORMED VECTOR LATTICES

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Abstract. Let $\mathcal{M} = \{m_\lambda\}_{\lambda \in \Lambda}$ be a separating family of lattice semi-norms on a vector lattice $X$, then $(X, \mathcal{M})$ is called a multi-normed vector lattice (or MNVL). We write $x_\alpha \overset{m}{\to} x$ if $m_\lambda(x_\alpha - x) \to 0$ for all $\lambda \in \Lambda$. A net $x_\alpha$ in an MNVL $X = (X, \mathcal{M})$ is said to be unbounded $m$-convergent (or $um$-convergent) to $x$ if $|x_\alpha - x| \wedge u \overset{m}{\to} 0$ for all $u \in X_+$. $um$-Convergence generalizes $un$-convergence \cite{7, 15} and $uaw$-convergence \cite{25}, and specializes $up$-convergence \cite{3} and $ur$-convergence \cite{6}. $um$-Convergence is always topological, whose corresponding topology is called unbounded $m$-topology (or $um$-topology). We show that, for an $m$-complete metrizable MNVL $(X, \mathcal{M})$, the $um$-topology is metrizable iff $X$ has a countable topological orthogonal system. In terms of $um$-completeness, we present a characterization of MNVLs possessing both Lebesgue’s and Levi’s properties. Then, we characterize MNVLs possessing simultaneously the $\sigma$-Lebesgue and $\sigma$-Levi properties in terms of sequential $um$-completeness. Finally, we prove that any $m$-bounded and $um$-closed set is $um$-compact iff the space is atomic and has Lebesgue’s and Levi’s properties.

1. Introduction and preliminaries

Unbounded convergences have attracted many researchers (see for instance \cite{13, 9, 10, 8, 7, 25, 15, 3, 19, 17, 16, 11, 12, 21, 6}). Unbounded convergences are well-investigated in vector and normed lattices (cf. \cite{7, 10, 15, 22, 24}). In the present paper, we also extend several previous results from \cite{7, 10, 15, 22, 24, 25} to multi-normed setting. This work is a continuation of \cite{6}, in which unbounded topological convergence was studied in locally solid vector lattices.

For a net $x_\alpha$ in a vector lattice $X$, we write $x_\alpha \overset{\alpha}{\to} x$ if $x_\alpha$ converges to $x$ in order. That is, there is a net $y_\beta$, possibly over a different index set, such that $y_\beta \downarrow 0$ and, for every $\beta$, there exists $\alpha_\beta$ satisfying $|x_\alpha - x| \leq y_\beta$ whenever $\alpha \geq \alpha_\beta$. A net $x_\alpha$ in a vector lattice $X$ is unbounded order convergent (uo-convergent) to $x \in X$ if $|x_\alpha - x| \wedge u \overset{\alpha}{\to} 0$ for every $u \in X_+$.

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We write $x_\alpha \xrightarrow{u_o} x$ in this case. Clearly, order convergence implies $u_o$-convergence and they coincide for order bounded nets. For a measure space $(\Omega, \Sigma, \mu)$ and a sequence $f_n$ in $L_p(\mu)$ ($0 \leq p \leq \infty$), $f_n \xrightarrow{u_o} 0$ iff $f_n \rightarrow 0$ almost everywhere [10, Rem. 3.4]. It is known that almost everywhere convergence is not topological. Therefore, $u_o$-convergence might not be topological in general. It was also shown recently that order convergence is never topological in infinite dimensional vector lattices [5].

Let $(X, \|\cdot\|)$ be a normed lattice. For a net $x_\alpha$ in $X$, we write $x_\alpha \stackrel{\|\cdot\|}{\longrightarrow} x$ if $x_\alpha$ converges to $x$ in norm. We say that $x_\alpha$ unbounded norm converges to $x$ ($x_\alpha$ un-converges to $x$ or $x_\alpha \xrightarrow{u_n} x$) if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in X_+$. Clearly, norm convergence implies un-convergence. The un-convergence is topological, and the corresponding topology (which is known as un-topology) was investigated in [15]. A net $x_\alpha$ uaw-converges to $x$ if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for all $u \in X_+$, where “w” stands for the weak convergence. Absolute weak convergence implies uaw-convergence. uaw-Convergence and uaw-topology were introduced and investigated in [25].

All topologies considered throughout this article are assumed to be Hausdorff. If a linear topology $\tau$ on a vector lattice $X$ has a base at zero consisting of solid sets, then the pair $(X, \tau)$ is called a locally solid vector lattice. Furthermore, if $\tau$ has base at zero consisting of convex-solid sets, then $(X, \tau)$ is called a locally convex-solid vector lattice. It is known that a linear topology $\tau$ on $X$ is locally convex-solid iff there exists a family $\mathcal{M} = \{m_\lambda\}_{\lambda \in \Lambda}$ of lattice seminorms that generates $\tau$ (cf. [11 Thm. 2.25]). Moreover, for such $\mathcal{M}$, $x_\alpha \tau \rightarrow x$ iff $m_\lambda(x_\alpha - x) \xrightarrow{\alpha} 0$ in $\mathbb{R}$ for each $m_\lambda \in \mathcal{M}$. Since $\tau$ is Hausdorff then the family $\mathcal{M}$ is separating.

A subset $A$ in a topological vector space $(X, \tau)$ is called $\tau$-bounded if, for every $\tau$-neighborhood $V$ of zero, there exists $\lambda > 0$ such that $A \subseteq \lambda V$. In the case when the topology $\tau$ is generated by a family $\{m_\lambda\}_{\lambda \in \Lambda}$ of seminorms, a subset $A$ of $X$ is $\tau$-bounded iff $\sup_{a \in A} m_\lambda(a) < \infty$ for all $\lambda \in \Lambda$.

Recall that a locally solid vector lattice $(X, \tau)$ is said to have the Lebesgue property if $x_\alpha \downarrow 0$ in $X$ implies $x_\alpha \tau \rightarrow 0$ or, equivalently, if $x_\alpha \xrightarrow{\tau} 0$ implies $x_\alpha \xrightarrow{\tau} 0$; $(X, \tau)$ is said to have the $\sigma$-Lebesgue property if $x_n \downarrow 0$ in $X$ implies $x_n \tau \rightarrow 0$; and $(X, \tau)$ is said to have the pre-Lebesgue property if $0 \leq x_n \uparrow \leq x$ implies only that $x_n$ is $\tau$-Cauchy. Finally, $(X, \tau)$ is said to have the Levi property if, when $0 \leq x_\alpha \uparrow$ and $x_\alpha$ is $\tau$-bounded, then $x_\alpha \tau \rightarrow x$ for some $x \in X$; $(X, \tau)$ is said to have the $\sigma$-Levi property if $x_n$ has supremum in $X$ provided by $0 \leq x_n \uparrow$ and by the $\tau$-boundedness of $x_n$, see [11 Def. 3.16].

2. Multi-Normed Vector Lattices

Let $(X, \tau)$ be a locally convex-solid vector lattice with an upward directed family $\mathcal{M} = \{m_\lambda\}_{\lambda \in \Lambda}$ of lattice seminorms generating $\tau$. Throughout this
article, the pair \((X, M)\) will be referred to as a multi-normed vector lattice (MNVL). Also, \(\tau\)-convergence, \(\tau\)-Cauchy, \(\tau\)-complete, etc. will be denoted by \(m\)-convergence, \(m\)-Cauchy, \(m\)-complete, etc.

Let \(X\) be a vector space, \(E\) be a vector lattice, and \(p : X \to E_+\) be a vector norm (i.e. \(p(x) = 0 \iff x = 0\), \(p(\lambda x) = |\lambda|p(x)\) for all \(\lambda \in \mathbb{R}\), \(x \in X\), and \(p(x + y) \leq p(x) + p(y)\) for all \(x, y \in X\)), then \((X, p, E)\) is called a lattice-normed space, abbreviated as LNS, see \([18]\). If \(X\) is a vector lattice, and the vector norm \(p\) is monotone (i.e. \(|x| \leq |y| \Rightarrow p(x) \leq p(y)\)), then the triple \((X, p, E)\) is called a lattice-normed vector lattice, abbreviated as LNVL (cf. \([3, 4]\)).

Given an LNS \((X, p, E)\). Recall that a net \(x_\alpha\) in \(X\) is said to be \(p\)-convergent to \(x\) (see \([3]\)) if \(p(x_\alpha - x) \xrightarrow{m} 0\) in \(E\). In this case, we write \(x_\alpha \xrightarrow{p} x\). A subset \(A\) of \(X\) is called \(p\)-bounded if there exists \(e \in E\) such that \(p(a) \leq e\) for all \(a \in A\).

**Proposition 1.** Every MNVL induces an LNVL. Moreover, for arbitrary nets, \(p\)-convergence in the induced LNVL implies \(m\)-convergence, and they coincide in the case of \(p\)-bounded nets.

**Proof.** Let \((X, M)\) be an MNVL, then there is a separating family \(\{m_\lambda\}_{\lambda \in \Lambda}\) of lattice seminorms on \(X\). Let \(E = \mathbb{R}^\Lambda\) be the vector lattice of all real-valued functions on \(\Lambda\), and define \(p : x \mapsto p_x\) from \(X\) into \(E_+\) such that \(p_x[\lambda] := m_\lambda(x)\).

It is clear that \(p\) is a monotone vector norm on \(X\). Therefore \((X, p, E)\) is an LNVL. Let \(x_\alpha\) be a net in \(X\). If \(x_\alpha \xrightarrow{p} 0\), then \(\{x_\alpha\} \xrightarrow{m} 0\) in \(\mathbb{R}^\Lambda\), and so \(p_{x_\alpha}[\lambda] \xrightarrow{m} 0\) or \(m_\lambda(x_\alpha) \xrightarrow{m} 0\) for all \(\lambda \in \Lambda\). Hence \(x_\alpha \xrightarrow{m} 0\).

Finally, assume a net \(x_\alpha\) to be \(p\)-bounded. If \(x_\alpha \xrightarrow{m} 0\), then \(m_\lambda(x_\alpha) \xrightarrow{m} 0\) or \(p_{x_\alpha}[\lambda] \xrightarrow{m} 0\) for each \(\lambda \in \Lambda\). Since \(x_\alpha\) is \(p\)-bounded, then \(\{x_\alpha\} \xrightarrow{m} 0\) in \(\mathbb{R}^\Lambda\). That is \(x_\alpha \xrightarrow{p} 0\). \(\square\)

Let \(X\) be a vector lattice. An element \(0 \neq e \in X_+\) is called a strong unit if the ideal \(I_e\) generated by \(e\) is \(X\) or, equivalently, for every \(x \geq 0\), there exists \(n \in \mathbb{N}\) such that \(x \leq ne\); a weak unit if the band \(B_e\) generated by \(e\) is \(X\) or, equivalently, \(x \land ne \uparrow x\) for every \(x \in X_+\). If \((X, \tau)\) is a topological vector lattice, then \(0 \neq e \in X_+\) is called a quasi-interior point if the principal ideal \(I_e\) is \(\tau\)-dense in \(X\) (see Definition 6.1 in \([20]\)). It is known that

\[
\text{strong unit} \Rightarrow \text{quasi-interior point} \Rightarrow \text{weak unit}.
\]

The following proposition characterizes quasi-interior points, and should be compared with \([2\text{ Thm.4.85}].\)

**Proposition 2.** Let \((X, M)\) be an MNVL, then the following statements are equivalent:

1. \(e \in X_+\) is a quasi-interior point;
Proof. (1)⇒(2) Suppose that \( e \) is a quasi-interior point of \( X \), then \( \overline{I}_e^m = X \). Let \( x \in X_+ \). Then \( x \in \overline{I}_e^m \), so there exists a net \( x_\alpha \) in \( I_e \) that \( m \)-converges to \( x \). But \( x_\alpha \xrightarrow{m} x \) implies \( |x_\alpha| \xrightarrow{m} |x| = x \). Moreover, \( x_\alpha \wedge x \xrightarrow{m} x \wedge x = x \), and \( x_\alpha \wedge x \leq x_\alpha \) implies that \( x_\alpha \wedge x \in I_e \), because \( I_e \) is an ideal. So we can assume also that \( x_\alpha \leq x \). Hence, for any \( x \in X_+ \), there is a net \( 0 \leq x_\alpha \in I_e \) and \( x_\alpha \leq x \). Then \( 0 \leq x_\alpha \wedge ne \leq x \wedge ne \leq x \) for all \( n \in \mathbb{N} \). Now, take \( \lambda \in \Lambda \), and let \( \varepsilon > 0 \), then there is \( \alpha_{\varepsilon} \geq \varepsilon \) such that \( m_{\lambda}(x-x_{\alpha_{\varepsilon}}) < \varepsilon \). But \( 0 \leq x_{\alpha_{\varepsilon}} \in I_e \), so \( 0 \leq x_{\alpha_{\varepsilon}} \leq k_{\varepsilon}e \) for some \( k_{\varepsilon} \in \mathbb{N} \). Since \( 0 \leq x_{\alpha_{\varepsilon}} = x_{\alpha_{\varepsilon}} \wedge k_{\varepsilon}e \leq x \wedge k_{\varepsilon}e \leq x \), then \( m_{\lambda}(x-x \wedge ne) \leq m_{\lambda}(x-x \wedge k_{\varepsilon}e) \leq m_{\lambda}(x-x_{\alpha_{\varepsilon}} \wedge k_{\varepsilon}e) = m_{\lambda}(x-x_{\alpha_{\varepsilon}}) < \varepsilon \) for all \( n \geq k_{\varepsilon} \). Hence \( m_{\lambda}(x-x \wedge ne) \xrightarrow{n} 0 \) as \( n \to \infty \). Since \( \lambda \in \Lambda \) was chosen arbitrary, we get \( x-x \wedge ne \xrightarrow{m} 0 \).

The proofs of the implications (2)⇒(3), and (3)⇒(1) are similar to the proofs of the corresponding implications of Theorem 4.85 in [2].

\[ \square \]

3. \( um \)-Topology

In this section we introduce the \( um \)-topology in an analogous manner to the \( un \)-topology [15] and \( awu \)-topology [25]. First we define the \( um \)-convergence.

Definition 1. Let \((X, \mathcal{M})\) be an MNVL, then a net \( x_\alpha \) is said to be unbounded \( m \)-convergent to \( x \), if \( |x_\alpha - x| \wedge u \xrightarrow{m} 0 \) for all \( u \in X_+ \). In this case, we say \( x_\alpha \) \( um \)-converges to \( x \) and write \( x_\alpha \xrightarrow{um} x \).

Clearly, that \( um \)-convergence is a generalization of \( un \)-convergence. The following result generalizes [15] Cor.4.5).

Proposition 3. If \((X, \mathcal{M})\) is an MNVL possessing the Lebesgue and Levi properties, and \( x_\alpha \xrightarrow{um} 0 \) in \( X \), then \( x_\alpha \xrightarrow{um} 0 \) in \( X^{**} \).

Proof. It follows from Theorem 6.63 of [1] that \((X, \mathcal{M})\) is \( m \)-complete and \( X \) is a band in \( X^{**} \). Now, [1] Thm.2.22 shows that \( X^{**} \) is Dedekind complete, and so \( X \) is a projection band in \( X^{**} \). The conclusion follows now from [6] Thm.3(3)).

In a similar way as in [7] Section 7, one can show that \( N_0 \), the collection of all sets of the form

\[ V_{e,u,\lambda} = \{ x \in X : m_{\lambda}(|x| \wedge u) < \varepsilon \} \quad (\varepsilon > 0, 0 \neq u \in X_+, \lambda \in \Lambda) \]

forms a neighborhood base at zero for some Hausdorff locally solid topology \( \tau \) such that, for any net \( x_\alpha \) in \( X \): \( x_\alpha \xrightarrow{um} 0 \) iff \( x_\alpha \xrightarrow{\tau} 0 \). Thus, the \( um \)-convergence is topological, and we will refer to its topology as the \( um \)-topology.
Clearly, if \( x_\alpha \overset{m}{\to} 0 \), then \( x_\alpha \overset{\text{um}}{\to} 0 \), and so the \( m \)-topology, in general, is finer than \( \text{um} \)-topology. On the contrary to Theorem 2.3 in \cite{15}, the following example provides an MNVL which has a strong unit, yet the \( m \)-topology and \( \text{um} \)-topology do not agree.

**Example 1.** Let \( X = C[0,1] \). Let \( \mathcal{A} := \{ [a, b] \subseteq [0, 1] : a < b \} \). For \( [a, b] \in \mathcal{A} \) and \( f \in X \), let \( m_{[a, b]}(f) := \frac{1}{b-a} \int_a^b |f(t)| dt \). Then \( \mathcal{M} = \{ m_{[a, b]} : [a, b] \in \mathcal{A} \} \) is a separating family of lattice seminorms on \( X \). Thus, \((X, \mathcal{M})\) is an MNVL. For each \( 2 \leq n \in \mathbb{N} \), let

\[
f_n = \begin{cases} \frac{n^2(1-n)x + n^2}{n^2}, & \text{if } x \in \left[0, \frac{1}{n}\right], \\ \frac{1}{n^2}, & \text{if } x \in \left[\frac{1}{n}, \frac{1}{n-1}\right], \\ 0, & \text{if } x \in \left[\frac{1}{n-1}, 1\right]. \end{cases}
\]

So we have

\[
f_n \wedge 1 = \begin{cases} 1, & \text{if } x \in \left[0, \frac{n+1}{n^2}\right], \\ \frac{n^2(1-n)x + n^2}{n^2}, & \text{if } x \in \left[\frac{n+1}{n^2}, \frac{1}{n-1}\right], \\ 0, & \text{if } x \in \left[\frac{1}{n-1}, 1\right]. \end{cases}
\]

Now, let \( 0 < b \leq 1 \), then there is \( n_0 \in \mathbb{N} \) such that \( \frac{1}{n_0-1} < b \). So, for \( n \geq n_0 \), we have \( \frac{1}{n_0-1} < b \), and so we get \( m_{[0, b]}(f_n) = \frac{1}{b}(1 + \frac{1}{n-1}) \to \frac{1}{b} \neq 0 \) as \( n \to \infty \). Thus, \( f_n \overset{m}{\not\to} 0 \). On the other hand, if \( [a, b] \in \mathcal{A} \) then there is \( n_0 \in \mathbb{N} \) such that \( \frac{1}{n_0-1} < b \) so, for \( n \geq (n_0 - 1) \), we have \( m_{[a, b]}(f_n \wedge 1) = \frac{1}{b-a} \left( \frac{n+1}{n^2} + \frac{1}{2n^2(n-1)} \right) \to 0 \) as \( n \to \infty \). Since \( 1 \) is a strong unit in \( X \) then, by \cite[Cor.5]{6}, \( f_n \overset{\text{um}}{\not\to} 0 \).

4. **Metrizability of \( \text{um} \)-topology**

The main result in this section is Proposition which shows that the \( \text{um} \)-topology is metrizable iff the space has a countable topological orthogonal system.

It is well known (cf. \cite[Thm.2.1]{11}) that a topological vector space is metrizable iff it has a countable neighborhood base at zero. Furthermore, an MNVL \((X, \mathcal{M})\) is metrizable iff the \( m \)-topology is generated by a countable family of lattice seminorms, see \cite[Theorem VII.8.2]{23}.

Notice that, in an MNVL \((X, \mathcal{M})\) with countable \( \mathcal{M} = \{ m_k \}_{k \in \mathbb{N}} \), an equivalent translation-invariant metric \( \rho_\mathcal{M} \) can be constructed by the formula

\[
\rho_\mathcal{M}(x, y) = \sum_{k=1}^{\infty} \frac{m_k(x - y)}{2^k(m_k(x - y) + 1)} \quad (x, y \in X).
\]
Lemma 1. 

Recall that a collection \( \{ e_\gamma \}_{\gamma \in \Gamma} \) of positive vectors in a vector lattice \( X \) is called an orthogonal system if \( e_\gamma \wedge e_{\gamma'} = 0 \) for all \( \gamma \neq \gamma' \). If, moreover, \( x \wedge e_\gamma = 0 \) for all \( \gamma \in \Gamma \) implies \( x = 0 \), then \( \{ e_\gamma \}_{\gamma \in \Gamma} \) is called a maximal orthogonal system. It follows from the Zorn’s lemma that every vector lattice containing at least one non-zero element has a maximal orthogonal system. Next, we recall the following notion.

**Definition 2.** [6 Def.1] Let \((X, \tau)\) be a topological vector lattice. An orthogonal system \( Q = \{ e_\gamma \}_{\gamma \in \Gamma} \) of non-zero elements in \( X_+ \) is said to be a topological orthogonal system, if the ideal \( I_Q \) generated by \( Q \) is \( \tau \)-dense in \( X \).

A series \( \sum_{i=1}^{\infty} x_i \) in a multi-normed space \((X, \mathcal{M})\) is called absolutely \( m \)-convergent if \( \sum_{i=1}^{\infty} m_\lambda(x_i) < \infty \) for all \( \lambda \in \Lambda \); and the series is \( m \)-convergent, if the sequence \( s_n := \sum_{i=1}^{n} x_i \) of partial sums is \( m \)-convergent. The following lemma can be proven by combining a diagonal argument with the proof of [14 Prop. 3 in Section 3.3] and therefore we omit its proof.

**Lemma 1.** A metrizable multi-normed space \((X, \mathcal{M})\) is \( m \)-complete iff every absolutely \( m \)-convergent series in \( X \) is \( m \)-convergent.

The following result extends [15 Thm.3.2].

**Proposition 4.** Let \((X, \mathcal{M})\) be a metrizable \( m \)-complete MNVL. Then the following conditions are equivalent:

(i) \( X \) has a countable topological orthogonal system;

(ii) the \( um \)-topology is metrizable;

(iii) \( X \) has a quasi interior point.

**Proof.** Since \((X, \mathcal{M})\) is metrizable, we may suppose that \( \mathcal{M} = \{ m_k \}_{k \in \mathbb{N}} \) is countable and directed.

(i) \( \Rightarrow \) (ii) It follows directly from [6 Prop.5]. Notice also that a metric \( d_{um} \) of the \( um \)-topology can be constructed by the following formula:

\[
d(x, y) = \sum_{k,n=1}^{\infty} \frac{1}{2^{k+n}} \cdot \frac{m_k(|x - y| \wedge e_n)}{1 + m_k(|x - y| \wedge e_n)},
\]

where \( \{ e_n \}_{n \in \mathbb{N}} \) is a countable topological orthogonal system for \( X \).

(ii) \( \Rightarrow \) (iii) Assume that the \( um \)-topology is generated by a metric \( d_{um} \) on \( X \). For each \( n \in \mathbb{N} \), let \( B_{um}(0, \frac{1}{n}) = \{ x \in X : d_{um}(x, 0) < \frac{1}{n} \} \). Since the \( um \)-topology is metrizable, then, for each \( n \in \mathbb{N} \), there are \( k_n \in \mathbb{N}, 0 < u_n \in X_+ \), and \( \varepsilon_n > 0 \) such that \( V_{\varepsilon_n, u_n, k_n} \subseteq B_{um}(0, \frac{1}{n}) \), where

\[
V_{\varepsilon, u_n, k_n} = \{ x \in X : m_k(|x| \wedge u_n) < \varepsilon \}.
\]
Notice that \( \{V_{\varepsilon, m, n}\}_{\varepsilon, m, n \in \mathbb{N}} \) is a base at zero of the um-topology on \( X \).

Let \( B_m(0, 1) = \{ x \in X : d_m(x, 0) < 1 \} \), where \( d_m \) is the metric generating the \( m \)-topology. There is a zero neighborhood \( V \) in the \( m \)-topology such that \( V \subseteq B_m(0, 1) \). Since \( V \) is absorbing, then, for every \( n \in \mathbb{N} \), there is \( c_n \geq 1 \) such that \( \frac{1}{c_n} u_n \in V \). Thus \( \frac{1}{c_n} u_n \in V \subseteq B_m(0, 1) \) for each \( n \in \mathbb{N} \). Hence, the sequence \( \frac{1}{c_n} u_n \) is \( d_m \)-bounded and so it is bounded with respect to the multi-norm \( \mathcal{M} = \{m_k\}_{k \in \mathbb{N}} \). Let

\[
(4.3) \quad e := \sum_{n=1}^{\infty} \frac{u_n}{2^n c_n}.
\]

Fix \( k \in \mathbb{N} \). Since the sequence \( \frac{u_n}{c_n} \) is bounded with respect to \( \mathcal{M} \), there exists \( r_k \in \mathbb{R}^+ \) such that \( m_k(\frac{u_n}{c_n}) \leq r_k < \infty \) for all \( n \in \mathbb{N} \). Hence,

\[
\sum_{n=1}^{\infty} m_k\left( \frac{u_n}{2^n c_n} \right) = \sum_{n=1}^{\infty} \frac{1}{2^n} m_k\left( \frac{u_n}{c_n} \right) \leq r_k \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.
\]

Thus, the series \( \sum_{n=1}^{\infty} \frac{u_n}{2^n c_n} \) is absolutely \( m \)-convergent. Since \( X \) is \( m \)-complete, Lemma 4 assures that the series \( \sum_{n=1}^{\infty} \frac{u_n}{2^n c_n} \) is \( m \)-convergent to some \( e \in X \).

Now, we use Theorem 2 in [6] to show that \( e \) is a quasi-interior point in \( X \). Let \( x_\alpha \) be a net in \( X \) such that \( x_\alpha \wedge e \xrightarrow{m} 0 \). Our aim is to show that \( x_\alpha \xrightarrow{\text{um}} 0 \). Since

\[
x_\alpha \wedge u_n \leq 2^n c_n x_\alpha \wedge 2^n c_n e = 2^n c_n (x_\alpha \wedge e) \xrightarrow{m} 0 \quad (\alpha \to \infty),
\]

then \( x_\alpha \wedge u_n \xrightarrow{\text{um}} 0 \) for all \( n \in \mathbb{N} \). In particular, \( m_{k_n}(x_\alpha \wedge u_n) \to 0 \). Thus, there exists \( \alpha_n \) such that \( m_{k_n}(x_\alpha \wedge u_n) < \varepsilon_n \) for all \( \alpha \geq \alpha_n \). That is \( x_\alpha \in V_{\varepsilon_n, u_n, k_n} \) for all \( \alpha \geq \alpha_n \), which implies \( x_\alpha \in B_{u_n}(0, \frac{1}{n}) \). Therefore, \( x_\alpha \xrightarrow{\text{um}} 0 \) and so \( x_\alpha \xrightarrow{\text{um}} 0 \). Hence, \( e \) is a quasi interior point.

\((iii) \Rightarrow (i)\) It is trivial. \( \square \)

Similar to [15] Prop.3.3], we have the following result.

**Proposition 5.** Let \( (X, \mathcal{M}) \) be an \( m \)-complete metrizable MNVNL. The um-topology is stronger than a metric topology iff \( X \) has a weak unit.

**Proof.** The sufficiency follows from [6] Prop.6.

For the necessity, suppose that the um-topology is stronger than the topology generated by a metric \( d \). Let \( e \) be as in (4.3) above. Assume \( x \wedge e = 0 \). Since \( e \geq \frac{u_n}{2^n c_n} \) for all \( n \in \mathbb{N} \), we get \( x \wedge e \xrightarrow{m} 0 \), and hence \( x \wedge u_n = 0 \) for all \( n \). Then \( x \in V_{\varepsilon_n, u_n, k_n} \) for all \( n \), and \( x \in B(0, \frac{1}{n}) = \{ x \in X : d(x, 0) < \frac{1}{n} \} \) for each \( n \in \mathbb{N} \). So \( x = 0 \), which means that \( e \) is a weak unit. \( \square \)
5. um-Completeness

A subset $A$ of an MNVL $(X, M)$ is said to be (sequentially) um-complete if, it is (sequentially) complete in the um-topology. In this section, we characterize um-complete subsets of $X$ in terms of the Lebesgue and Levi properties. We begin with the following technical lemma.

**Lemma 2.** Let $(X, M)$ be an MNVL, and $A \subseteq X$ be m-bounded, then $\overline{A}^{um}$ is m-bounded.

**Proof.** Given $\lambda \in \Lambda$, then $M_\lambda = \sup_{a \in A} m_\lambda(a) < \infty$. Let $x \in \overline{A}^{um}$, then there is a net $a_\alpha$ in $A$ such that $a_\alpha \xrightarrow{um} x$. So $m_\lambda(|a_\alpha - x| \wedge u) \to 0$ for any $u \in X_\times$. In particular,

$$m_\lambda(|x|) = m_\lambda(|x| \wedge |x|) = m_\lambda(|x - a_\alpha + a_\alpha| \wedge |x|) \leq m_\lambda(|x - a_\alpha| \wedge |x|) + \sup_{a \in A} m_\lambda(a) = m_\lambda(|x - a_\alpha| \wedge |x|) + M_\lambda.$$

Letting $\alpha \to \infty$, we get $m_\lambda(x) = m_\lambda(|x|) \leq M_\lambda < \infty$ for all $x \in \overline{A}^{um}$. □

**Theorem 1.** Let $(X, M)$ be an MNVL and let $A$ be an m-bounded and um-closed subset in $X$. If $X$ has the Lebesgue and Levi properties, then $A$ is um-complete.

**Proof.** Suppose that $x_\alpha$ is um-Cauchy in $A$, then, without lost of generality, we may assume that $x_\alpha$ consists of positive elements.

Case (1): If $X$ has a weak unit $e$, then $e$ is a quasi-interior point, by the Lebesgue property of $X$ and Proposition[2] Note that, for each $k \in \mathbb{N}$,

$$|x_\alpha \wedge ke - x_\beta \wedge ke| \leq |x_\alpha - x_\beta| \wedge ke,$$

hence the net $(x_\alpha \wedge ke)_\alpha$ is m-Cauchy in $X$. Now, [1] Thm.6.63] assures that $X$ is m-complete, and so the net $(x_\alpha \wedge ke)_\alpha$ is m-convergent to some $y_k \in X$. Given $\lambda \in \Lambda$. Then

$$m_\lambda(y_k) = m_\lambda(y_k - x_\alpha \wedge ke + x_\alpha \wedge ke) \leq m_\lambda(y_k - x_\alpha \wedge ke) + m_\lambda(x_\alpha) \leq m_\lambda(y_k - x_\alpha \wedge ke) + \sup_{\alpha} m_\lambda(x_\alpha).$$

Taking limit over $\alpha$, we get $m_\lambda(y_k) \leq \sup_{\alpha} m_\lambda(x_\alpha) < \infty$. Hence the sequence $y_k$ is m-bounded in $X$. Note also that $y_k$ is increasing in $X$, but $X$ has the Lebesgue and Levi properties, so, by [1] Thm.6.63], $y_k$ m-converges to some $y \in X$.

It remains to show that $y$ is the um-limit of $x_\alpha$. Given $\lambda \in \Lambda$. Note that, by Birkhoff’s inequality,

$$|x_\alpha \wedge ke - x_\beta \wedge ke| \wedge e \leq |x_\alpha - x_\beta| \wedge e.$$
Thus

\[ m_\lambda(\|x_\alpha \wedge ke - x_\beta \wedge ke\|) \leq m_\lambda(\|x_\alpha - x_\beta\|). \]

Taking limit over \( \beta \), we get

\[ m_\lambda(\|x_\alpha \wedge ke - y_{k}\|) \leq \lim_{\beta} m_\lambda(\|x_\alpha - x_\beta\|). \]

Now taking limit over \( k \), we have

\[ m_\lambda(\|x_\alpha - y\|) \leq \lim_{\beta} m_\lambda(\|x_\alpha - x_\beta\|). \]

Finally, as \( x_\alpha \) is \textit{um}-Cauchy, taking limit over \( \alpha \), yields

\[ \lim_{\alpha} m_\lambda(\|x_\alpha - y\|) \leq \lim_{\alpha,\beta} m_\lambda(\|x_\alpha - x_\beta\|) = 0. \]

Thus, \( x_\alpha \overset{\text{um}}{\longrightarrow} y \) and, since \( A \) is \textit{um}-closed, \( y \in A \).

Case (2): If \( X \) has no weak unit. Let \( \{e_\gamma\}_{\gamma \in \Gamma} \) be a maximal orthogonal system in \( X \). Let \( \Delta \) be the collection of all finite subsets of \( \Gamma \). For each \( \delta \in \Delta \), \( \delta = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \), consider the band \( B_\delta \) generated by \( \{e_{\gamma_1}, e_{\gamma_2}, \ldots, e_{\gamma_n}\} \). It follows from [1, Thm.3.24] that \( B_\delta \) is a projection band. Then \( B_\delta \) is an \( m \)-complete MNVL in its own right. Moreover, the \( m \)-topology restricted to \( B_\delta \) possesses the Lebesgue and Levi properties. Note that \( B_\delta \) has a weak unit, namely \( e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_n} \). Let \( P_\delta \) be the band projection corresponding to \( B_\delta \).

For \( \delta \in \Delta \), since \( x_\alpha \) is \textit{um}-Cauchy in \( X \) and \( P_\delta \) is a band projection, then \( P_\delta x_\alpha \) is \textit{um}-Cauchy in \( B_\delta \). Lemma 2 assures that \( P_\delta(A)^{\textit{um}} \) is \( m \)-bounded in \( B_\delta \). Thus, by Case (1), there is \( z_\delta \in B_\delta \) such that

\[ P_\delta x_\alpha \overset{\textit{um}}{\longrightarrow} z_\delta \geq 0 \text{ in } B_\delta \quad (\alpha \to \infty). \]

Since \( B_\delta \) is a projection band, then \( P_\delta x_\alpha \overset{\textit{um}}{\longrightarrow} z_\delta \geq 0 \text{ in } X \) (over \( \alpha \)). It is easy to see that \( 0 \leq z_\delta \uparrow, \) and \( z_\delta \) is \( m \)-bounded. Since \( X \) has the Lebesgue and Levi properties, it follows from [1, Thm.6.63], that there is \( z \in X_+ \) such that \( z_\delta \overset{\text{m}}{\longrightarrow} z \), and so \( z_\delta \uparrow z \). It remains to show that \( x_\alpha \overset{\textit{um}}{\longrightarrow} z \). The argument is similar to the proof of [13, Thm.4.7], and we leave it as an exercise. Since \( A \) is \textit{um}-closed, then \( z \in A \) and so \( A \) is \textit{um}-complete. \( \square \)

The following lemma and its proof are analogous to Lemma 1.2 in [15].

\textbf{Lemma 3.} Let \((X, \mathcal{M})\) be an MNVL. If \( x_\alpha \) is an increasing net in \( X \) and \( x_\alpha \overset{\textit{um}}{\longrightarrow} x \), then \( x_\alpha \uparrow x \) and \( x_\alpha \overset{\text{m}}{\longrightarrow} x \).

\textbf{Lemma 4.} Let \((X, \mathcal{M})\) be an MNVL possessing the pre-Lebesgue property. Let \( x_n \) be a positive disjoint sequence which is not \( m \)-null. Put \( s_n := \sum_{k=1}^{n} x_k \). Then the sequence \( s_n \) is \textit{um}-Cauchy, which is not \textit{um}-convergent.
Proof. The sequence $s_n$ is monotone increasing and, since $x_n$ is not $m$-null, $s_n$ is not $m$-convergent. Hence, by Lemma 3 the sequence $s_n$ is not $um$-convergent. To show that $s_n$ is $um$-Cauchy, fix any $\varepsilon > 0$ and take $0 \neq w \in X_+$. Since $x_n$ is a positive disjoint sequence, we have $s_n \wedge w = \sum_{k=1}^{\infty} w \wedge x_k$. The sequence $s_n \wedge w$ is increasing and order bounded by $w$, hence it is $m$-Cauchy, by [1, Thm.3.22]. Let $\lambda \in \Lambda$. We can find $n_{\varepsilon, \lambda}$ such that $m_\lambda(s_m \wedge w - s_n \wedge w) < \varepsilon$ for all $m \geq n \geq n_{\varepsilon, \lambda}$. Observe that

$$s_m \wedge w - s_n \wedge w = \sum_{k=1}^{m} w \wedge x_k - \sum_{k=1}^{n} w \wedge x_k = \sum_{k=n+1}^{m} w \wedge x_k = w \wedge |s_m - s_n|.$$ 

It follows $m_\lambda(|s_m - s_n| \wedge w) < \varepsilon$ for all $m \geq n \geq n_{\varepsilon, \lambda}$. But $\lambda \in \Lambda$ was chosen arbitrary. Hence $s_n$ is $um$-Cauchy. \[\square\]

Next theorem generalizes Theorem 6.4 in [15].

**Theorem 2.** Let $(X, M)$ be an $m$-complete MNVL with the pre-Lebesgue property. Then $X$ has the Lebesgue and Levi properties iff every $m$-bounded $um$-closed subset of $X$ is $um$-complete.

**Proof.** The necessity follows directly from Theorem 1.

For the sufficiency, first notice that, in an $m$-complete MNVL, the pre-Lebesgue and Lebesgue properties coincide [1, Thm.3.24].

If $X$ does not have the Levi property then, by [1, Thm.6.63], there is a disjoint sequence $x_n \in X_+$, which is not $m$-null, such that its sequence of partial sums $s_n = \sum_{j=1}^{n} x_j$ is $m$-bounded. Let $A = \{s_n : n \in \mathbb{N}\}^{um}$. By Lemma 2, we have that $A$ is $m$-bounded. By Lemma 3, the sequence $s_n$ is $um$-Cauchy in $X$ and so in $A$, in contrary with that the sequence $s_{n+1} - s_n = x_{n+1}$ is not $m$-null. \[\square\]

**Theorem 3.** Let $(X, M)$ be an $m$-complete metrizable MNVL, and let $A$ be an $m$-bounded sequentially $um$-closed subset of $X$. If $X$ has the $\sigma$-Lebesgue and $\sigma$-Levi properties then $A$ is sequentially $um$-complete. Moreover, the converse holds if, in addition, $X$ is Dedekind complete.

**Proof.** Suppose $M = \{m_k\}_{k \in \mathbb{N}}$. Let $0 \leq x_n$ be a $um$-Cauchy sequence in $A$. Let $e := \sum_{n=1}^{\infty} \frac{x_n}{2^n}$. For $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} m_k\left(\frac{x_n}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} m_k(x_n) \leq c_k \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty,$$

where $m_k(a) \leq c_k < \infty$ for all $a \in A$. Since $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ is absolutely $m$-convergent, then, by Lemma 1, $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ is $m$-convergent in $X$. Note that, $x_n \leq 2^ne$, so $x_n \in B_e$ for all $n \in \mathbb{N}$. Since $X$ has the Levi property, then $X$
is $\sigma$-order complete (see [1, Definition 3.16]). Thus $B_e$ is a projection band. Also $e$ is a weak unit in $B_e$. Then, by the same argument as in Theorem [1] we get that there is $x \in B_e$ such that $x_n \xrightarrow{um} x$ in $B_e$ and so $x_n \xrightarrow{um} x$ in $X$. Since $A$ is sequentially $um$-closed, we get $x \in A$. Thus $A$ is sequentially $um$-complete.

The converse follows from Proposition 8 in [6]. □

6. $um$-Compact sets

A subset $A$ of an MNVL $(X, M)$ is said to be (sequentially) $um$-compact if it is (sequentially) compact in the $um$-topology. In this section, we characterize $um$-compact subsets of $X$ in terms of the Lebesgue and Levi properties. We begin with the following result which shows that $um$-compactness can be “localized” under certain conditions.

**Theorem 4.** Let $(X, M)$ be an MNVL possessing the Lebesgue property. Let $\{e_\gamma\}_{\gamma \in \Gamma}$ be a maximal orthogonal system. For each $\gamma \in \Gamma$, let $B_\gamma$ be the band generated by $e_\gamma$, and $P_\gamma$ be the corresponding band projection onto $B_\gamma$. Then $x_\alpha \xrightarrow{um} 0$ in $X$ iff $P_\gamma x_\alpha \xrightarrow{um} 0$ in $B_\gamma$ for all $\gamma \in \Gamma$.

**Proof.** For the forward implication, we assume that $x_\alpha \xrightarrow{um} 0$ in $X$. Let $b \in (B_\gamma)_+$. Then

$$|P_\gamma x_\alpha| \wedge b = P_\gamma |x_\alpha| \wedge b \leq |x_\alpha| \wedge b \xrightarrow{m} 0,$$

that implies $P_\gamma x_\alpha \xrightarrow{um} 0$ in $B_\gamma$.

For the backward implication, without lost of generality, we may assume that $x_\alpha \geq 0$ for all $\alpha$. Let $u \in X_+$. Our aim is to show that $x_\alpha \wedge u \xrightarrow{m} 0$. It is known that $x_\alpha \wedge u = \sum_{\gamma \in \Gamma} P_\gamma (x_\alpha \wedge u)$. Let $F$ be a finite subset of $\Gamma$. Then

\begin{equation}
(6.1) \quad x_\alpha \wedge u = \sum_{\gamma \in F} P_\gamma (x_\alpha \wedge u) + \sum_{\gamma \in \Gamma \setminus F} P_\gamma (x_\alpha \wedge u).
\end{equation}

Note

\begin{equation}
(6.2) \quad \sum_{\gamma \in F} P_\gamma (x_\alpha \wedge u) = \sum_{\gamma \in F} P_\gamma x_\alpha \wedge P_\gamma u \xrightarrow{m} 0.
\end{equation}

We have to control the second term in (6.1).

\begin{equation}
(6.3) \quad \sum_{\gamma \in \Gamma \setminus F} P_\gamma (x_\alpha \wedge u) \leq \frac{1}{n} \sum_{\gamma \in F} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F} P_\gamma u,
\end{equation}

where $n \in \mathbb{N}$. Let $\mathcal{F}(\Gamma)$ be the collection of all finite subsets of $\Gamma$. Let $\Delta = \mathcal{F}(\Gamma) \times \mathbb{N}$. For each $\delta = (F, n)$, put

$$y_{\delta} = \frac{1}{n} \sum_{\gamma \in F} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F} P_\gamma u.$$
We show that $y_\delta$ is decreasing. Let $\delta_1 \leq \delta_2$ then $\delta_1 = (F_1, n_1), \delta_2 = (F_2, n_2)$. Then $\delta_1 \leq \delta_2$ iff $F_1 \subseteq F_2$ and $n_1 \leq n_2$. But $n_1 \leq n_2$ iff $\frac{1}{n_1} \geq \frac{1}{n_2}$. So,

\begin{equation}
\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u.
\end{equation}

Note also

\begin{equation}
\frac{1}{n_2} \sum_{\gamma \in F_2} P_\gamma u = \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u.
\end{equation}

Since $F_1 \subseteq F_2$, then $\Gamma \setminus F_1 \supseteq \Gamma \setminus F_2$ and hence, $\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u$. Note, that

\begin{equation}
\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u = \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u.
\end{equation}

Now,

\begin{equation}
\sum_{\gamma \in F_2 \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u.
\end{equation}

Combining (6.6) and (6.7), we get

\begin{equation}
\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u.
\end{equation}

Adding (6.4) and (6.8), we get

\begin{equation}
\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u.
\end{equation}

It follows from (6.5), that

\begin{equation}
\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u,
\end{equation}

that is $y_{\delta_1} \geq y_{\delta_2}$. Next, we show $y_\delta \downarrow 0$. Assume $0 \leq x \leq y_\delta$ for all $\delta \in \Delta$. Let $\gamma_0 \in \Gamma$ be arbitrary and fix it. Let

\[ F = \{ \gamma_0 \}, \quad n \in \mathbb{N}, \quad 0 \leq x \leq \frac{1}{n} P_{\gamma_0} u + \sum_{\gamma \in \Gamma \setminus \{ \gamma_0 \}} P_\gamma u. \]

We apply $P_{\gamma_0}$ for the expression above, so $0 \leq P_{\gamma_0} x \leq \frac{1}{n} P_{\gamma_0} u$ for all $n \in \mathbb{N}$, and so $P_{\gamma_0} x = 0$. Since $\gamma_0 \in \Gamma$ was chosen arbitrary, we get $P_{\gamma_0} x = 0$ for all $\gamma \in \Gamma$. Hence, $x = 0$ and so $y_\delta \downarrow 0$. Since $(X, \mathcal{M})$ has the Lebesgue property, we get $y_\delta \xrightarrow{m} 0$. Therefore, by (6.3),

\begin{equation}
\sum_{\gamma \in \Gamma \setminus F} P_\gamma (x \land u) \leq y_\delta \xrightarrow{m} 0.
\end{equation}

Hence (6.1), (6.2), and (6.9) imply $x \land u \xrightarrow{m} 0$. \qed
The following result and its proof are similar to Theorem 7.1 in \[15\]. Therefore we omit its proof.

**Theorem 5.** Let \((X, \mathcal{M})\) be an MNVL possessing the Lebesgue and Levi properties. Let \(\{e_\gamma\}_{\gamma \in \Gamma}\) be a maximal orthogonal system. Let \(A\) be a \(um\)-closed \(m\)-bounded subset of \(X\). Then \(A\) is \(um\)-compact iff \(P_\gamma(A)\) is \(um\)-compact in \(B_\gamma\) for each \(\gamma \in \Gamma\), where \(B_\gamma\) is the band generated by \(e_\gamma\) and \(P_\gamma\) is the band projection corresponding to \(B_\gamma\).

**Theorem 6.** Let \((X, \mathcal{M})\) be an MNVL. The following are equivalent:

1. Any \(m\)-bounded and \(um\)-closed subset \(A\) of \(X\) is \(um\)-compact.
2. \(X\) is an atomic vector lattice and \((X, \mathcal{M})\) has the Lebesgue and Levi properties.

**Proof.** (1) \(\Rightarrow\) (2). Let \([a, b]\) be an order interval in \(X\). For \(x \in [a, b]\), we have \(a \leq x \leq b\) and so \(0 \leq x - a \leq b - a\). Consider the order interval \([0, b - a]\) \(\subseteq X_+\). Clearly, \([0, b - a]\) is \(m\)-bounded and \(um\)-closed in \(X\). By (1), the order interval \([0, b - a]\) is \(um\)-compact. Let \(x_\alpha\) be a net in \([0, b - a]\). Then there is a subset \(x_{\alpha} \rightarrow um\) \(x\) in \([0, b - a]\). That is \(|x_{\alpha} - x| \wedge u \rightarrow m 0\) for all \(u \in [0, b - a]\). Hence, \(|x_{\alpha} - x| = |x_{\alpha} - x| \wedge (b - a) \rightarrow m 0\). So, \(x_{\alpha} \rightarrow m x\) in \([0, b - a]\). Thus, \([0, b - a]\) is \(m\)-compact. Consider the following shift operator \(T_a : X \rightarrow X\) given by \(T_a(x) := x + a\). Clearly, \(T_a\) is continuous, and so \(T_a([0, b - a]) = [a, b]\) is \(m\)-compact.

Since any order interval in \(X\) is \(m\)-compact, then it follows from [1] Cor.6.57 that \(X\) is atomic and has the Lebesgue property. It remains to show that \(X\) has the Levi property. Suppose \(0 \leq x_\alpha \rightarrow\) and is \(m\)-bounded. Let \(A = \{x_\alpha\}^{um}\). Then \(A\) is \(um\)-closed and, by Lemma 2, \(A\) is an \(m\)-bounded subset of \(X\). Thus, \(A\) is \(um\)-compact and so, there are a subnet \(x_{\alpha} \rightarrow um\) and \(x \in A\) such that \(x_{\alpha} \rightarrow um\) \(x\). Hence, by Lemma 3, \(x_{\alpha} \rightarrow x\), and so \(x_\alpha \rightarrow x\). Hence, \(X\) has the Levi property.

(2) \(\Rightarrow\) (1). Let \(A\) be an \(m\)-bounded and \(um\)-closed subset of \(X\). We show that \(A\) is \(um\)-compact. Since \(X\) is atomic, there is a maximal orthogonal system \(\{e_\gamma\}_{\gamma \in \Gamma}\) of atoms. For each \(\gamma \in \Gamma\), let \(P_\gamma\) be the band projection corresponding to \(e_\gamma\). Clearly, \(P_\gamma(A)\) is \(m\)-bounded. Now, by the same argument as in the proof of Theorem 7.1 in [15], we get that \(P_\gamma(A)\) is \(um\)-closed in \(\prod_{\gamma \in \Gamma} B_\gamma\), and so it is \(um\)-closed in \(B_\gamma\). But \(um\)-closedness implies \(m\)-closedness. So \(P_\gamma(A)\) is \(m\)-bounded and \(m\)-closed in \(B_\gamma\) for all \(\gamma \in \Gamma\). Since each \(e_\gamma\) is an atom in \(X\), then \(B_\gamma = \text{span}\{e_\gamma\}\) is a one-dimensional subspace. It follows from the Heine-Borel theorem that \(P_\gamma(A)\) is \(m\)-compact in \(B_\gamma\), and so it is \(um\)-compact in \(B_\gamma\) for all \(\gamma \in \Gamma\). Therefore, Theorem 5 implies that \(A\) is \(um\)-compact in \(X\).

**Proposition 6.** Let \(A\) be a subset of an \(m\)-complete metrizable MNVL \((X, \mathcal{M})\).
(1) If $X$ has a countable topological orthogonal system, then $A$ is sequentially um-compact iff $A$ is um-compact.

(2) Suppose that $A$ is $m$-bounded, and $X$ has the Lebesgue property. If $A$ is um-compact, then $A$ is sequentially um-compact.

Proof. (1). It follows immediately from Proposition 4.

(2). Let $x_n$ be a sequence in $A$. Find $e \in X_+$ such that $x_n$ is contained in $B_e$ (e.g., take $e = \sum_{n=1}^{\infty} \frac{|x_n|}{n^2}$). Since $A$ is um-compact, then $A \cap B_e$ is um-compact in $B_e$. Now, since $X$ is $m$-complete and has the Lebesgue property, then $B_e$ is also $m$-complete and has the Lebesgue property. Moreover, $e$ is a quasi-interior point of $B_e$. Hence, by Proposition 4, the um-topology on $B_e$ is metrizable, consequently, $A \cap B_e$ is sequentially um-compact in $B_e$. It follows that there is a subsequence $x_{n_k}$ that um-converges in $B_e$ to some $x \in A \cap B_e$. Since $B_e$ is a projection band, then [6, Thm.3(3)] implies $x_{n_k} \xrightarrow{um} x$ in $X$. Thus, $A$ is sequentially um-compact. □

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