EXTENSION OF WEAKLY AND STRONGLY $F$-REGULAR RINGS BY FLAT MAPS

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§1. Introduction

Throughout this paper all rings will be Noetherian of positive characteristic $p$. Hence tight closure theory [HH1–4] takes a prominent place (see §2 for tight closure definitions and terminology). The purpose of this note is to help answer the following question: if $R$ is weakly (resp. strongly) $F$-regular and $\phi : R \rightarrow S$ is a flat map then under what conditions on the fibers is $S$ weakly (resp. strongly) $F$-regular. This question (among many others) is raised in [HH4] in section 7. It is shown there that if $\phi$ is a flat map of local rings, $S$ is excellent and the generic and closed fibers are regular then weak $F$-regularity of $R$ implies that of $S$ (Theorem 7.24). One of our main results weakens the hypotheses considerably.

Theorem 3.4. Let $\phi : (R, m) \rightarrow (S, n)$ be a flat map. Assume that $S/mS$ is Gorenstein and $R$ is weakly $F$-regular and Cohen-Macaulay. Suppose that either

1. $c \in R^\circ$ is a common test element for $R$ and $S$, and $S/mS$ is $F$-injective, or
2. $c \in S - mS$ is a test element for $S$ and $S/mS$ is $F$-rational, or
3. $R$ is excellent and $S/mS$ is $F$-rational.

Then $S$ is weakly $F$-regular.

We note that the Gorenstein assumption on the fiber is essential, even if $R$ is regular. Even weakening the assumption on the fiber to $\mathbb{Q}$-Gorenstein is not strong enough to give a good theorem, as Singh [Si] gives an example of $R \rightarrow S$ flat, where $R$ is a discrete valuation domain, $S/mS$ is $\mathbb{Q}$-Gorenstein and strongly $F$-regular, yet $S$ is not weakly $F$-regular!

We also prove a corresponding result for strong $F$-regularity.

Theorem 3.6. Let $(R, m, K) \rightarrow (S, n, L)$ be a flat map of $F$-finite reduced rings with Gorenstein closed fiber. Assume that $R$ is strongly $F$-regular. If $S/mS$ is $F$-rational then $S$ is strongly $F$-regular.

In order to prove the first of these theorems we investigate how flat maps $\phi : (R, m) \rightarrow (S, n)$ with Gorenstein closed fibers affect tight closure for $I \subseteq R$ such that $l(R/I) < \infty$.

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and \( I \) is irreducible in \( R \). In general these results do not depend on the relationship of \( R/\mathfrak{m} \to S/\mathfrak{n} \) (e.g., separability or finiteness).

While not directly relevant to this paper, we note that other authors have recently investigated tight closure properties under good flat maps. For instance Enescu [En] and Hashimoto [Ha] have recently shown that for a flat map with \( F \)-rational base and \( F \)-rational closed fiber, the target is \( F \)-rational (in the presence of a common test element).

\section{Background for tight closure}

Let \( R \) be a Noetherian ring of characteristic \( p > 0 \). We use \( q = p^e \) for a varying power of \( p \) and for an ideal \( I \subseteq R \) we let \( I^{[q]} = \langle i^q : i \in I \rangle \). Also let \( R^c \) be the complement in \( R \) of the union of the minimal primes of \( R \). Then \( x \) is in the tight closure of \( I \) if and only if there exists \( c \in R^c \) such that \( cx^q \in I^{[q]} \) for all \( q \gg 0 \). If \( I^* = I \) then \( I \) is said to be tightly closed. We will say that \( I \) is Frobenius closed if \( x^q \in I^{[q]} \) for some \( q \) always implies that \( x \in I \).

There is a tight closure operation for a submodule \( N \subseteq M \), but we will not discuss this case in general. It is however useful to discuss tight closure in the case of a particular type of direct limit. Suppose that \( M = \varinjlim_t R/I_t \) for a sequence of ideals \( \{I_t\} \). Let \( u \in M \) be an element which is given by \( \{u_t\} \) where in the direct limit system \( u_t \mapsto u_{t+1} \). We will say that \( u \in 0^*_M \) if there exists \( c \in R^c \) and a sequence \( t_q \) such that for all \( q \gg 0 \), \( cu^q_{t_q} \in I_t^{[q]} \). We will say that \( u \) is in the finitistic tight closure of \( 0 \) in \( M \), \( 0^*_{fg} \), if there exists \( c \in R^c \) and \( t > 0 \) such that \( cu^q_t \in I_t^{[q]} \) for all \( q \). This definition of finitistic tight closure agrees with that in [HH2] for this case. Clearly \( 0^*_{fg} \subseteq 0^*_M \).

A ring \( R \) in which every ideal is tightly closed is called weakly \( F \)-regular. If every localization of \( R \) is weakly \( F \)-regular then \( R \) is \( F \)-regular. When \( R \) is reduced then \( R^{1/p} \) denotes the ring of \( p \)-th roots of elements of \( R \). More generally, \( R^{1/q} \) is the ring of \( q \)-th roots. Clearly \( R \subseteq R^{1/q} \). If \( R \) is \( F \)-finite and reduced \( (R^{1/p} \) is a finite \( R \)-module) then \( R \) is called strongly \( F \)-regular if for all \( c \in R^c \), there exists a \( q \) such that the inclusion \( Rc^{1/q} \subseteq R^{1/q} \) splits over \( R \). If \( R \) is \( F \)-finite and \( R_c \) is strongly \( F \)-regular for some \( c \in R^c \), then \( R \) is strongly \( F \)-regular if and only if there exists \( q \) such that \( Rc^{1/q} \subseteq R^{1/q} \) splits over \( R \) [HH1, Theorem 3.3]. Strongly \( F \)-regular rings are \( F \)-regular, and weakly \( F \)-regular rings are normal and under mild conditions (e.g., excellent) are Cohen-Macaulay.

The equivalence of the three conditions is an important open question. Let \((R, \mathfrak{m})\) be an excellent reduced local ring and let \( E \) be an injective hull of the residue field of \( R \). Then \( E \) can be written as a direct limit of the form above since \( R \) is approximately Gorenstein. Weak \( F \)-regularity of \( R \) is equivalent to \( 0^*_{fg} = 0 \) [HH2, Theorem 8.23], while strong \( F \)-regularity is equivalent to \((F\text{-finiteness and}) \ 0^*_E = 0 \) [LS, Proposition 2.9].

By a parameter ideal in \((R, \mathfrak{m})\) we mean an ideal generated by part of a system of parameters. We say that \((R, \mathfrak{m})\) is \( F \)-rational if every parameter ideal is tightly closed, and \( F \)-injective if every parameter ideal is Frobenius closed (this is a slightly different notion of \( F \)-injectivity from that in [FW], but is equivalent for CM rings). \( F \)-rational rings are normal and under mild conditions are Cohen-Macaulay. In a Gorenstein ring, \( F \)-rationality is equivalent to all forms of \( F \)-regularity.

A priori, the multiplier element \( c \) in the definition of tight closure depends on both \( I \) and \( x \). If \( c \) works for every tight closure test then we say that \( c \) is a test element for \( R \).
If $c$ works for every tight closure test for every completion of every localization of $R$ then we say that $c$ is a completely stable test element. It is shown in [HH4] that if $(R, \mathfrak{m})$ is a reduced excellent domain, $c \in R^\circ$, and $R_c$ is Gorenstein and weakly $F$-regular then $c$ has a power which is a completely stable test element for $R$.

In [HH2, HH3] it is shown that the multiplier $c$ in the definition of tight closure need not remain constant. Let $R$ be a domain. One may have a sequence of elements $c_q$ such that $c_q x^g \in I^{[q]}$ where $c_q$ must have “small order.” We can obtain a notion of order, denoted ord, by taking a $\mathbb{Z}$-valued valuation on $R$ which is non-negative on $R$ and positive on $\mathfrak{m}$. Let $R^+$ be the integral closure of $R$ in an algebraic closure of the fraction field of $R$ ($R^+$ has many wonderful properties, such as being a big Cohen-Macaulay algebra for $R$ when $R$ is excellent [HH5]). The valuation then extends to a function on $R^+$ which takes values in $\mathbb{Q}$. In particular, $\text{ord}(c^{1/q}) = \text{ord}(c)/q$. We will need to use the following theorem [HH3, Theorem 3.1]:

**Theorem 2.1.** Let $(R, \mathfrak{m})$ be a complete local domain of characteristic $p$, let $x \in R$ and let $I \subseteq R$. Then $x \in I^*$ if and only if there exists a sequence of elements $\epsilon_n \in (R^+)^\circ$ such that $\text{ord}(\epsilon_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\epsilon_n x \in IR^+$.

In fact we would like to strengthen this theorem in order to apply it to tight closure calculations for non finitely generated modules which are defined by a direct limit system of ideals. The proof we give is just an altered version of the proof of Theorem 3.1 given in [HH3]. The key component is [HH3, Theorem 3.3]:

**Theorem 2.2.** Let $(R, \mathfrak{m}, k)$ be a complete local domain. Let ord be a $\mathbb{Q}$-valued valuation on $R^+$ nonnegative on $R$ (and hence on $R^+$) and positive on $\mathfrak{m}$ (and, hence, on $\mathfrak{m}^+$. Then there exists a fixed real number $\nu > 0$ and a fixed positive integer $r$ such that for every element $u$ of $R^+$ of order $< \nu$ there is an $R$-linear map $\phi : R^+ \rightarrow R$ such that $\phi(u) \notin \mathfrak{m}^r$.

The generalization of Theorem 2.1 is given below.

**Theorem 2.3.** Let $(R, m)$ be a complete local domain of characteristic $p$. Let $M = \varinjlim R/I_t$ be an $R$-module and let $x \in M$. Suppose that $x$ comes from the sequence $\{x_t\}$ where $x_t \mapsto x_{t+1}$. Then $x \in 0^*_M$ if and only if there exists a sequence of elements $\epsilon_n \in (R^+)^\circ$ such that $\text{ord}(\epsilon_n) \rightarrow 0$ as $n \rightarrow \infty$ and for each $n$ there exists $t$ such that $\epsilon_n x_t \in I_t R^+$.

**Proof.** The “only if” part is trivial, as if $c x^g = 0$ for all $q \gg 0$ then we can take $\epsilon_q = c^{1/q}$.

To see the “if” direction, choose $\nu > 0$ and $r$ as in Theorem 2.2. Fix $q = p^r > 0$. Choose $n$ large enough that $\text{ord}(\epsilon_n) < \nu/q$. Let $\epsilon = \epsilon_n^r$. Then there exists $t$ such that $\epsilon x_t^r \in I_t^{[q]} R^+$ and $\text{ord}(\epsilon) < \nu$. Applying an $R$ linear map $\phi$ as in Theorem 2.2 we find that $c_q x_t^r \in I^{[q]} \subseteq (I_t^{[q]})^\circ$ with $c_q = \phi(\epsilon) \in R - \mathfrak{m}^r$. Thus, setting $J_q = \cup_t (I_t^{[q]})^\circ : R x_t^q$ we have $c_q \in J_q$ for all $q$.

The sequence $J_q$ is nonincreasing. If for some $t$, $yx_t^q \in (I_t^{[pq]})^\circ$ then $c'(yx_t^{pq})q' \in (I_t^{[pq]})^{[q']} = (I_t^{[pq]})^{[q']} = (I_t^{[pq]})$ for all $q' \gg 0$ where $c' \neq 0$. But then $c'(yx_t^{pq})q' \in (I_t^{[q']})^{[pq]}$ for all $q' \gg 0$ and hence $yx_t^q \in (I_t^{[q]})^\circ$, as required.
Since the sequence \( \{J_q\}_q \) is nonincreasing, it cannot have intersection 0, or Chevalley’s theorem would give \( J_q \subseteq \mathfrak{m}^r \) for \( q \gg 0 \). As \( e_q \in J_q - \mathfrak{m}^r \) for all \( q \), we can choose a nonzero element \( d \in \cap_q J_q \). Then for each \( q \) there exists \( t \) such that \( dx_i^q \in \langle I_t^{[q]} \rangle^* \). If \( c \) is a test element for \( R \) then \( c dx_i^q \in I_t^{[q]} \). Thus \( x \in 0^*_M \). □

**Proposition 2.4.** Let \((R, \mathfrak{m})\) be an excellent local domain such that its completion is a domain. Let \( M = \lim_{t \to \infty} R/I_t \) be a direct limit system. Fix \( u \notin 0^*_M \). Then there exists \( q_0 \) such that \( J_q = \cap_q (I_t^{[q]} : u_t^q) \subseteq \mathfrak{m}^{[q/q_0]} \) for all \( q \gg 0 \) (where \( \{u_t\} \) represents \( u \in M \) and \( u_t \mapsto u_{t+1} \)). In particular if \( I \subseteq R \) we may take \( M = R/I \) where the limit system consists of equalities. Then \( u \notin I^* \) implies that \( \langle I^{[q]} : u^q \rangle \subseteq \mathfrak{m}^{[q/q_0]} \).

**Proof.** Suppose that we can show that the proposition holds in \( \hat{R} \). Then \((I_t^{[q]} : R \ u_t^q) \subseteq (I_t^{[q]} : R \ u_t^q) \cap R \subseteq \mathfrak{m}^{[q/q_0]} \hat{R} \cap R \subseteq \mathfrak{m}^{[q/q_0]} R \). Thus we may assume that \( R \) is complete.

For \( x \in R \) let \( f(x) \) be the largest power of \( \mathfrak{m} \) that \( x \) is in, and set \( f(x) = \lim_{n \to \infty} f(x^n/n) \). By the valuation theorem [Re, Theorem 4.16], there exist a finite number of \( \mathbb{Z} \)-valued valuations \( v_1, \ldots, v_k \) on \( R \) which are non-negative on \( \mathfrak{m} \) and positive on \( \mathfrak{m} \) and positive rational numbers \( e_1, \ldots, e_k \) such that \( f(x) = \min\{v_i(x)/e_i\} \). Furthermore, since \( R \) is analytically unramified, there exists a constant \( L \) such that for all \( x \in R \), \( f(x) \leq \lceil f(x) \rceil \leq f(x) + L \) ([Re, Theorem 5.32 and 4.16]).

Now, by Theorem 2.3, for each \( v_i \) there exists a positive real number \( \alpha_i \) such that if \( c \in (I_t^{[q]} : u_t^q) \) then \( v_i(c) \geq \alpha_i q \). Combined with the valuation theorem we see that \( f(c) \geq \min\{q v_i/e_i\} \). Let \( \alpha = \min\{\alpha_i/e_i\} \). Then \( f(c) \geq \alpha q - L - 1 \). Let \( s = \mu(\mathfrak{m}) \). Choose \( q_1 > 1/\alpha \), \( q_2 \geq L + 1 \), and \( q_3 \geq s \) (all powers of \( p \)). Set \( q_0 = q_1 q_2 q_3 \). Then \( f(c) \geq \alpha q - (L + 1) \geq q/q_1 - (L + 1) \geq q/q_1 q_2 - 1 \geq (q/q_0)s - 1 \). A simple combinatorial argument shows that \( \mathfrak{m}^{(q/q_0)s-1} \subseteq \mathfrak{m}^{[q/q_0]} \). Hence \( c \in \mathfrak{m}^{[q/q_0]} \). □

**§3. Tight closure in flat extension maps**

We show in this section that extending a weakly (respectively, strongly) \( F \)-regular ring by a flat map with sufficiently nice Gorenstein closed fiber yields another weakly (resp., strongly) \( F \)-regular ring. These results are Theorems 3.4 and 3.6 (see also Corollary 3.5 for the \( F \)-regular case).

By saying that \( \phi : (R, \mathfrak{m}) \to (S, \mathfrak{n}) \) is flat we mean that \( \phi \) is flat and that \( \phi(\mathfrak{m}) \subseteq \mathfrak{n} \). Since the map is flat we then know that given ideals \( A, B \subseteq R \) we have \( AS :_S BS = (A :_RB)S \) (\( B \) finitely generated). The next lemma merely asserts that modding out by elements which are regular in the closed fiber preserves flatness.

**Lemma 3.1.** Let \( \phi : (R, \mathfrak{m}) \to (S, \mathfrak{n}) \) be a flat map. Let \( z_1, \ldots, z_d \in S \) be elements whose images in \( S/\mathfrak{m}S \) are a regular sequence. Then for any ideal \( I \) generated by monomials in the \( z \)’s, the ring \( S/IS \) is flat over \( R \).

**Proof.** See, for example [HH4, Theorem 7.10a,b]. □

The next proposition shows that tight closure behaves well for irreducible \( \mathfrak{m} \)-primary ideals when extending to \( S \). Given a sequence of elements \( z = z_1, \ldots, z_d \) we will use \( z^{[2]} \) to denote \( z_1^2, \ldots, z_d^2 \).
Proposition 3.2. Let \( \phi : (R, m, K) \to (S, n, L) \) be a flat map with Gorenstein closed fiber. Let \( z = z_1, \ldots, z_d \in S \) be elements whose images form a s.o.p. in \( S/mS \). Let \( I \subseteq R \) be such that \( l(R/I) < \infty \) and \( \dim_K(0 :_{R/I} m) = 1 \). Suppose that either

1. \( R \) and \( S \) have a common test element and \( S/mS \) is F-injective, or
2. \( c \in S - mS \) is a test element for \( S \), and \( S/mS \) is F-rational, or
3. \( R \) is excellent, \( \hat{R} \) is a domain, and \( S/mS \) is F-rational.

Then \( I \) is tightly closed in \( R \) \( \iff \) for all \( t > 0 \), \( IS + (z)^{[t]}S \) is tightly closed in \( S \) \( \iff \) there exists \( t > 0 \) such that \( IS + (z)^{[t]}S \) is tightly closed in \( S \).

Proof. Let \( b \in S \) have as its image the socle element in \( S/mS + (z)S \). Let \( u \in R \) be the socle element mod \( I \). Then the socle element of \( S/(IS + (z)S) \) is \( ub \) since the map \( R/I \to R/I \otimes S = S/IS \) is flat with Gorenstein fibers (there is only one fiber).

Suppose that \( I \) is tightly closed. There is no loss of generality in taking \( t = 1 \). If \( IS + (z)S \) is not tightly closed in \( S \) then we have \( c(ub)^q \in (I^{[q]} + (z)^{[q]}S) \) for all \( q \). In case (1) we may take \( c \in R^c \), so that

\[
b^q \in (I^{[q]} + (z)^{[q]}S) :S \quad \text{for all } q > 0.
\]

The first equality is a consequence of flatness, while the inclusion follows since \( u \notin I^* \). By our assumption that \( S/mS \) is F-injective we reach the contradictory conclusion that \( b \in ((z) + m)S \). In case (2) we have

\[
cb^q \in (I^{[q]} + (z)^{[q]}S) :S \quad \text{for all } q > 0.
\]

As \( S/mS \) is F-rational, it is a domain, so \( c \neq 0 \) in \( S/mS \). This contradicts our hypothesis that \( S/mS \) is F-rational (in fact it is enough to assume that \( I \) is Frobenius closed to reach this conclusion). In case (3) we can choose \( q_0 \) as in Proposition 2.4, and then

\[
c(b^{q_0})^{q/q_0} \in (I^{[q]} + (z)^{[q]}S) :S \quad \text{for all } q/q_0 > 0.
\]

But then \( b^{q_0} \in (mS + (z)^{[q_0]})^* \). By persistence, the image of \( b^{q_0} \) is in \( ((z)^{[q_0]}S/mS)^* \), which contradicts the F-rationality of \( S/mS \).

Suppose now that \( IS + (z)^{[t]}S \) is tightly closed in \( S \) for all \( t \), but \( I \) is not tightly closed in \( R \). Then \( u \in (IR)^* \subseteq (I + (z)^{[t]})^* \) (since \( R^c \subseteq S^0 \)). But then \( u \in \cap_t (IS + (z)^{[t]}S) \cap R \subseteq IS \cap R = IR \).

Finally, suppose that \( (IS + (z)^{[t_0]}S \) is tightly closed for some \( t_0 \). Given any \( t \), the socle element of \( (IS + (z)^{[t]}S \) is \( (z_1 \cdots z_d)^{t-1}ub \). If \( c((z_1 \cdots z_d)^{t-1}ub)^q \in (IS + (z)^{[t]}S) \) then by flatness, \( c((z_1 \cdots z_d)^{t_0-1}ub)^q \in (IS + (z)^{[t_0]}S) \). Therefore, one such ideal tightly closed shows that all such ideals are tightly closed. \( \square \)

To deal with strong F-regularity we need to give a similar proposition with \( R/I \) replaced by the injective hull \( E_R(R/m) \). Suppose that we can write \( E = E_R(R/m) = \lim_n R/J_n \), the set \( \{u_t\} \subseteq R \) is a collection of elements such that \( u_t \to u_{t+1} \) in the map \( R/J_t \to R/J_{t+1} \) and the image of each \( u_t \) in \( E \) is the socle element of \( E \). It suffices that \( R \) be approximately Gorenstein \([Ho2]\) (e.g., excellent and normal, or even reduced) to obtain \( E \) in this manner. In particular an F-finite ring is excellent \([Ku]\), so a reduced F-finite ring is approximately Gorenstein.
Proposition 3.3. Let \((R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)\) be a flat map of \(F\)-finite reduced rings with Gorenstein closed fiber.

(1) If \(Rc^{1/q} \subseteq R^{1/q}\) splits for some \(q\) (over \(R\)) and \(S/\mathfrak{m}S\) is \(F\)-injective then \(Sc^{1/q} \subseteq S^{1/q}\) splits for some \(q\) (over \(S\)).

(2) If \(0\) is Frobenius closed in \(E_{R}(K)\), \(S/\mathfrak{m}S\) is \(F\)-rational and \(c \in S - \mathfrak{m}S\) then there exists \(q\) such that \(Sc^{1/q} \subseteq S^{1/q}\) splits (over \(S\)).

Proof. Choose \(z = z_{1}, \ldots, z_{d} \in S\) elements which generate a s.o.p. in \(S/\mathfrak{m}S\). By [HH4, Lemma 7.10] we have \(E_{S}(L) = \lim_{v \to} S/(z^{[v]}) \otimes_{R} E_{R}(K) = \lim_{t, \mathfrak{m} \to} S/(z^{[v]}) \otimes_{R} R/J_{t} = \lim_{t} S/(z^{[t]}, J_{t})S\). If \(b \in S\) generates the socle element in \(S/(\mathfrak{m} + (z))S\) then the image of \((z_{1} \cdots z_{d})^{t-1}bu_{t}\) in \(S/((z^{[t]}) + J_{t})S\) maps to the socle element of \(E_{S}\) (where \(u_{t}\) is as given above).

In case (1), if for all \(q\) the inclusion \(Sc^{1/q} \to S^{1/q}\) fails to split, by [Ho1, Theorem 1 and Remark 2] for all \(q\) there exists \(t_{q}\) such that

\[c(z_{1} \cdots z_{d})^{(t_{q}-1)q}b^{q}u_{t_{q}}^{q} \in ((z^{[t_{q}]}, J_{t_{q}})[q])S.\]

Hence \((z_{1} \cdots z_{d})^{(t_{q}-1)q}b^{q} \in ((z^{[t_{q}]}, J_{t_{q}})[q])S \cup cu_{t_{q}}^{q} \subseteq (J_{t_{q}}^{[q]} : R cu_{t_{q}}^{q})S + (z^{[t_{q}]})^{[q]}S \subseteq \mathfrak{m}S + (z^{[t_{q}]})^{[q]}S\) for \(q \gg 0\) (we are using here that if \(Rc^{1/q} \subseteq R^{1/q}\) splits for some \(q\) then \(Rc^{1/q} \subseteq R^{1/q}\) splits for all \(q' \geq q\)). Thus \(b^{q} \in \mathfrak{m}S + (z^{[q]})\) since \(S/\mathfrak{m}S\) is CM. This contradicts the \(F\)-injectivity of \(S/\mathfrak{m}S\).

To see (2), if there is no splitting we obtain

\[c(z_{1} \cdots z_{d})^{(t_{q}-1)q}b^{q} \in (z^{[t_{q}]}, J_{t_{q}})[q] : S cu_{t_{q}}^{q} \subseteq (J_{t_{q}}^{[q]} : R cu_{t_{q}}^{q})S + (z^{[t_{q}]})^{[q]}S \subseteq \mathfrak{m}S + (z^{[t_{q}]})^{[q]}S\]

and hence \(cub \in \mathfrak{m}S + (z)^{[q]}\). This contradicts the \(F\)-rationality of \(S/\mathfrak{m}S\). □

We can now give our main theorems on the extension of weakly and strongly \(F\)-regular rings by flat maps with Gorenstein closed fiber.

Theorem 3.4. Let \(\phi : (R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a flat map. Assume that \(S/\mathfrak{m}S\) is Gorenstein and \(R\) is weakly \(F\)-regular and \(CM\). Suppose that either

(1) \(c \in R^{c}\) is a common test element for \(R\) and \(S\), and \(S/\mathfrak{m}S\) is \(F\)-injective, or

(2) \(c \in S - \mathfrak{m}S\) is a test element for \(S\) and \(S/\mathfrak{m}S\) is \(F\)-rational, or

(3) \(R\) is excellent and \(S/\mathfrak{m}S\) is \(F\)-rational.

Then \(S\) is weakly \(F\)-regular.

Proof. To see that \(S\) is weakly \(F\)-regular it suffices to show that there exists a sequence of irreducible tightly closed ideals of \(S\) cofinite with the powers of \(\mathfrak{n}\). As \(R\) is weakly \(F\)-regular (so normal) and \(CM\) it is approximately Gorenstein. Say that \(\{J_{t}\}\) is a sequence of irreducible ideals cofinite with the powers of \(\mathfrak{m}\). Let \(z = z_{1}, \ldots, z_{d} \in S\) be elements which form a s.o.p. in \(S/\mathfrak{m}S\). Then \((J_{t} + z^{[t]})S\) is a sequence of irreducible ideals in \(S\) cofinite with the powers of \(\mathfrak{n}\). By Proposition 3.2, in cases (1), (2), and (3), the ideals \((J_{t} + z^{[t]})S\) are tightly closed in \(S\) (in case (3), \(\tilde{R}\) is still weakly \(F\)-regular, so is a domain).
Therefore $S$ is weakly $F$-regular. We note that in case (2) we may weaken the assumption that $R$ is weakly $F$-regular to the assumption that $R$ is $F$-pure (see the comment in the proof of Proposition 3.2, part (2)).

The next corollary should be compared with [HH4, Theorem 7.25(c)].

**Corollary 3.5.** Let $(R, m) \rightarrow (S, n)$ be a flat map of excellent rings with Gorenstein fibers. Suppose that the generic fiber is $F$-rational and all other fibers are $F$-injective. If $R$ is $F$-regular then $S$ is $F$-regular.

**Proof.** By hypothesis the generic fiber is Gorenstein and $F$-rational, therefore there is a $c \in R^e$ which is a common completely stable test element. $F$-regularity is local on the prime ideals of $S$ and the fiber of such a localization is the localization of a fiber, hence Gorenstein and $F$-injective (the property of $F$-injectivity is easily seen to localize). Therefore Theorem 3.4(1) always applies. □

**Theorem 3.6.** Let $(R, m, K) \rightarrow (S, n, L)$ be a flat map of $F$-finite reduced rings with Gorenstein closed fiber. Assume that $R$ is strongly $F$-regular. If $S/mS$ is $F$-rational then $S$ is strongly $F$-regular.

**Proof.** We must show that there exists an element $c \in S^0$ such that $S_c$ is strongly $F$-regular and $S_{c^{1/q}} \subseteq S^{1/q}$ splits for some $q$.

If there exists $c \in R^e$ such that $S_c$ is strongly $F$-regular (i.e., a power of $c$ is a common test element for $R$ and $S$) then we are done by Proposition 3.3(1). Even if $R$ and $S$ have no (apparent) common test element, however, we claim that there exists $c \in S - mS$ such that $S_c$ is strongly $F$-regular. Once we have shown this, the theorem follows by Proposition 3.3(2).

Since the non-strongly $F$-regular locus is closed [HH1, Theorem 3.3] it suffices to show that $S_{mS}$ is strongly $F$-regular, for then there exists an element $c \in S - mS$ such that $S_c$ is strongly $F$-regular. Let $B = S_{mS}$. Then $R \rightarrow B$ is flat and the closed fiber is a field. In particular $E_B(B/mB) = E_R(K) \otimes_R B$. As $R$ is strongly $F$-regular (so normal) it is approximately Gorenstein. Say $E_R = \lim_{\rightarrow q} R/J_t$ with socle element mapped to by $u_t$ (as before). Then $u_t \in B/J_t B$ will still map to the socle element $u$ in $E_B$. Suppose that $u \in 0_{E_B}$. This means there exists $b \in B_0$ such that for all $q$ there exists $t_q$ such that $bu_{t,q}^q \in J_{t,q}^q B$. Hence $b \in J_{t,q}^q :_B u_{t,q}^q = (J_{t,q}^q :_R u_{t,q}^q)B$. Note that $R$ is an excellent normal domain, so its completion remains a domain. Thus by Proposition 2.4 we see that as $q \rightarrow \infty$, $(J_{t,q}^q :_R u_{t,q}^q)$ gets into larger and larger powers of the maximal ideal, since 0 is tightly closed in $E_R$. Thus $b \in \cap_N m^N B = 0$, a contradiction. □

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