Phase-delay induced variation of synchronization bandwidth and frequency stability in a micromechanical oscillator

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Abstract Phase feedback is commonly utilized to set up a synchronized MEMS oscillator for high performance sensor application. It is a consensus that the synchronization region varies with phase delay with an ‘Anti-U’ mode. In this paper, phase-delay induced variation of synchronization bandwidth and frequency stability in a micromechanical oscillator is investigated analytically and experimentally. The analytical expression for predicting the synchronization bandwidth with phase delay is derived based on the mathematic model. An additional dynamic extreme point of synchronization bandwidth actuated by nonlinearity is observed, which leads to three different types (‘U’, ‘Anti-U’ and ‘M’) of variation pattern of synchronization bandwidth as feedback tuning. The variation of frequency stability along phase delay is also studied. The synchronization bandwidth and the frequency stability have exactly opposite variation pattern with phase delay in linear oscillators while they are totally consistent in nonlinear oscillators. Experimental results validate the analytical observations. Our work provides a precise way for achieving best performance of a synchronized MEMS oscillator in the sensor application.

Keywords Micromechanical oscillator · Nonlinear dynamics · Synchronization · Feedback tuning · Frequency stability

1 Introduction

Microelectromechanical systems (MEMS) have increasingly attracted considerable interest due to their small size and low power consumption since the integrated circuit (IC) technology rise in 1990. Related investigations not only focus on fundamental questions, such as frequency locking [29–31], internal resonance [7,21,44,45], mode coupling [4,10,32], pull-in [15,50], exit time [46], bilayer effect [18], damping effect [24,28,48,49], etc., but also in tremendous potential applications including gas/mass sensing [13,36,37], filters [1], memory devices [42] and time keeping [2]. Due to the size effect, MEMS system is easily driven into nonlinear regime. In MEMS, nonlinearity can typically arise from external potentials [9], or geometric deformation [42]. A typical phenomenon is that nonlinearity can enhance the synchronization behavior. Synchronization usually refers to the harmonization between the phases of at least two vibration systems.
The earliest research on synchronization phenomenon can be traced back to the observation of the synchronization phenomenon of coupled pendulum by C. Huygens in 1673. When one of the vibration systems is a perturbation source which is generated by external signal generator, it shows a behavior that the oscillator’s frequency will synchronously changing with the external perturbation frequency, which makes it a great choice to utilize this property as frequency sensing. Great quantity of works have been done in synchronization, such as synchronization in chaotic systems [19,20], superharmonic synchronization [6,26], subharmonic synchronization [27,41,43], etc. Several works have shown the advantages of synchronization for sensing such as reduction of the phase noise [16,39], enhancement of the frequency stability [12,33,47], etc. Meanwhile, the effect of time delay in system are also considered in energy harvesting [5,35], traffic flow [14] as well as in synchronization of Micromechanical systems [12]. Most importantly, enhancing the synchronization bandwidth has become an active area of research. Some researchers find that nonlinearity in the oscillator can be exploited to enhance the synchronization bandwidth [3]. Their experimental and analytical results show that nonlinearities are the key determinants in the nonlinear regime in a self-sustained micromechanical oscillator. Phase feedback is commonly utilized to keep a self-sustained micro-oscillator. Previous studies show that the synchronization bandwidth varies with the phase delay in an ‘Anti-U’ pattern. In theoretical and experimental studies the phase delay is always fixed at π/2 to obtain a maximum amplitude [22,23,38,40], or to get a maximum synchronization bandwidth [3,12]. Nevertheless, after in-depth analysis and experimental verification, we present some new findings.

In this paper, the variation of synchronization region and frequency stability with phase delay in a micromechanical oscillator is investigated, and three different variation patterns (‘U’, ‘Anti-U’ and ‘M’) of synchronization bandwidth are observed analytically and experimentally. The novelty and contributions of our work lie in the following: (1) the analytical expression for predicting the synchronization bandwidth with phase delay in micromechanical oscillators is derived, from which three variation patterns (‘U’, ‘Anti-U’ and ‘M’) of synchronization bandwidth with phase delay are observed as feedback tuning; (2) the synchronization bandwidth and the frequency stability are found have exactly opposite variation pattern along phase delay in linear oscillators while they are consistent in nonlinear oscillators; (3) our work presented enriches the synchronization phenomenon and provides a precise way for achieving best performance of a synchronized MEMS oscillator in the sensor application.

2 Theoretical analysis

2.1 Basic features of the micro-resonator

A clamped–clamped (C–C) micromechanical beam resonator with dimensions 482 µm long, 12 µm wide, and 25 µm thick, respectively, as shown in Fig. 1a, is studied in this paper. This micromechanical resonator is electrostatically actuated to vibrate at its primary mode by applying a combination of constant DC bias voltage $V_{DC}$ and AC dynamic voltage $V_{ac}$ on the mid electrostatic comb. The primary modal shape simulation obtained by Comsol of the micro-resonator is shown in Fig. 1b. Figure 1c shows the open-loop amplitude-frequency responses of the resonator for a fixed $V_{DC}$ ($V_{DC} = 20$ V) and various $V_{ac}$ from 10 to 550 mV. Inset displays the resonator vibrating under small driving strength $V_{ac} = 10$ mV. It is seen that for small driving strength, the resonator vibrates with small amplitude in its linear regime, while exhibiting strong nonlinearity when driving strength increases.

Based on the Euler–Bernoulli beam carrying a concentrated mass in the mid-span, the governing equation of the transverse deflection $\hat{w}(\hat{u}, \hat{t})$ for the microbeam, is written as [11],

$$EI \frac{\partial^4 \hat{w}}{\partial \hat{u}^4} + (\rho S + m_c \delta \left(\hat{u} - L/2\right)) \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} + \hat{c} \frac{\partial \hat{w}}{\partial \hat{t}}$$

$$= \frac{ES}{2L} \int_0^L \left(\frac{\partial \hat{w}}{\partial \hat{u}}\right)^2 d\hat{u} \frac{\partial^2 \hat{w}}{\partial \hat{t}^2}$$

$$+ \hat{F}(\hat{\delta}, \hat{\omega}, \hat{\Omega}_{syn}) \delta \left(\hat{u} - L/2\right)$$

(1)

where $E$ is the Young’s modulus, $I$ and $S$ are the moment of inertia and area of the cross section. $\hat{u}$ is the location along the beam, $\hat{t}$ is time, $L$ and $\rho$ are the total length and material density, respectively. $\hat{c}$ is the viscous damping per unit length and $m_c$ is the mass of the middle comb structure. $\hat{F}$ is the combination of the applied electrostatic feedback sustaining force $\hat{F}_0$.
and the synchronizing perturbation force $\hat{F}_s$, $\hat{\Omega}$ is the instantaneous frequency of the oscillator, $\hat{\Omega}_{\text{syn}}$ is the synchronization perturbation frequency.

Introducing the following nondimensional variables

$$w = \frac{\bar{w}}{b}, \quad u = \frac{\bar{u}}{L}, \quad t = \frac{\bar{t}}{L},$$

Eq. (1) can be simplified as,

$$\frac{\partial^4 w}{\partial u^4} + \left(1 + \frac{\bar{m}}{\rho L^2} \right) \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} = a \int_0^1 \left( \frac{\partial w}{\partial u} \right)^2 \frac{\partial^2 w}{\partial u^2} + F(\Omega, \Omega, t, \Omega, t) \delta \left( u - \frac{1}{2} \right) \quad (2)$$

The C–C beam subjected to the following boundary conditions,

$$w(0, t) = 0, \quad w(1, t) = 0,$$

$$\frac{\partial w(0, t)}{\partial u} = 0, \quad \frac{\partial w(1, t)}{\partial u} = 0. \quad (3)$$

Note that in our experiments the feedback AC driving voltage is set much lower than the DC bias voltage ($V_{\text{ac}} \ll V_{\text{DC}}$) in order to minimize the effect from harmonic excitations, and the perturbation strength $V_{\text{acs}}$ is much lower than the feedback voltage ($V_{\text{acs}} \ll V_{\text{ac}}$). When synchronization arises, retaining the first three orders after Taylor expansion, $F$ is simplified as [17, 34],

$$F(w, \omega t, \omega_{\text{t}}) = \frac{L^3 \gamma}{EId} (4w + 8w^3)$$

$$+ 2L^3 \gamma \frac{V_{\text{ac}}}{EId} \frac{\partial^2 w}{\partial t^2} + 1 + 2w + 3w^2 + 4w^3\cos(\Omega t + \phi_0)$$

$$+ 2L^3 \gamma V_{\text{acs}} \frac{\partial^2 w}{\partial t^2} + 1 \cos(\Omega_{\text{syn}} t) \quad (4)$$

The first term in the right hand of Eq. (4) will tune the linear and nonlinear stiffness of the system. The second term is the feedback sustaining force, and the third one is the weak synchronization perturbation force. The first mode is dominant in our experiments when the resonator works near its natural frequency. By using the Galerkin decomposition method, the solution of Eq. (2) is approximated as,

$$w(u, t) = x(t) q(u), \quad (5)$$

where $x(t)$ is the time varying first mode coordinate, $q(u)$ is the first modal shape function. For a C–C beam, $q(u)$ usually has the form,

$$q(u) = -\frac{\bar{m}}{4} \left( \sinh \sqrt{\omega_n u} - \frac{\sin \sqrt{\omega_n u}}{\cosh \sqrt{\omega_n L}} \right)$$

$$- 2U \left( u - \frac{1}{2} \right) \left[ \sin \sqrt{\omega_n L} \left( u - \frac{1}{2} \right) \right]$$

$$- \sin \sqrt{\omega_n L} \left( u - \frac{1}{2} \right) \right] \quad (6)$$

where $\omega_n$ is the natural frequency. Substituting Eq. (5) into Eq. (6), integrating the resulting equation from 0 to 1, we obtain

$$M \frac{\partial^2 x}{\partial t^2} + C \frac{dx}{dt} + K_1 x + K_3 x^3$$

$$= F_{\text{fb}} \cos(\Omega t + \phi_0) + F_{\text{syn}} \cos(\Omega_{\text{syn}} t) \quad (7)$$
where,

\[ M = \int_0^1 q(u)^2 \, du, \quad K_1 \]

\[ = \int_0^1 q(u)^2 \, du - \frac{4\gamma L^3}{EId} q^2 \left( \frac{1}{2} \right) , \]

\[ C = c \int_0^1 q(u)^2 \, du, \quad K_3 \]

\[ = \alpha \left( \int_0^1 q(u)^2 \, du \right)^2 - \frac{8\gamma L^3}{EId} q^2 \left( \frac{1}{2} \right) , \]

\[ F_{fb} = \frac{2L^3\gamma V_{acf}}{EIdV_{DC}} q \left( \frac{1}{2} \right) \left( \frac{d^2}{ga} + 1 \right) , \]

\[ F_{syn} = \frac{2L^3\gamma V_{acs}}{EIdV_{DC}} q \left( \frac{1}{2} \right) \left( \frac{d^2}{ga} + 1 \right) . \]  

(8)

where \( M, C, K_1, K_3, F_{fb} \) and \( F_{syn} \) are the effective mass, damping coefficient, linear mechanical stiffness, cubic mechanical stiffness, feedback force and external synchronization force, respectively. The feedback control force \( F_{fb} \) is generated by a close-loop feedback circuit to compensate the energy dissipation, and make the resonator a self-sustained oscillator. \( F_{syn} \) is the perturbation force injected into the self-oscillation oscillator. It is known that if the injected signal frequency \( \Omega_{syn} \) approaches the resonator’s oscillation frequency \( \Omega \) in a small range, synchronization occurs. After redefining time units \((t \sqrt{K/M} \to t)\), the normalized motion equation becomes

\[ \ddot{x} + Q^{-1}x + x + \beta x^3 = f_0 \cos(\phi + \phi_0) + f_s \cos(\Omega_s t) \]  

(9)

where \( \phi = \Omega t \) is the oscillator’s instantaneous phase, \( Q = \sqrt{K/M} \) denotes the quality factor, which represents the rate of energy loss, \( \beta = \frac{K_3}{K_1} \), \( f_0 = \frac{F_{fb}}{K_1} \), \( f_s = \frac{F_{syn}}{K_1} \) and \( \Omega_s = \frac{\Omega_{syn}}{\sqrt{K_1/M}} \) are the normalized Duffing non-linearity, the amplitude of feedback force, weak periodic perturbation force and normalized perturbation frequency, respectively.

2.2 Synchronization bandwidth with phase delay

The analytical solution of Eq. (9) can be obtained using the harmonic approximation,

\[ x = A \cos(\phi) \]

\[ \dot{x} = -A\Omega \sin(\phi) \]  

(10)

Substituting Eq. (10) into Eq. (9), we have

\[ -A\Omega^2 \cos(\phi) - Q^{-1}A\Omega \sin(\phi) + A \cos(\phi) + \beta A^3 \cos^3(\phi) = f_0 \cos(\phi + \phi_0) + f_s \cos(\Omega_s t) \]  

(11)

In the synchronized case, due to the frequency locking effect, the oscillation frequency \( \Omega \) of the resonator is exactly equal to the perturbation frequency \( \Omega_s \), i.e., \( \Omega = \Omega_s \). Then, we have \( \Omega_s t = \phi + \phi_s \). Equation (11) becomes

\[ -A\Omega^2 \cos(\phi) - Q^{-1}A\Omega \sin(\phi) + A \cos(\phi) + \beta A^3 \cos^3(\phi) = f_0 \cos(\phi + \phi_0) - f_0 \sin(\phi) \cos(\phi_0) + f_s \cos(\phi_s) - f_s \sin(\phi) \cos(\phi_s) \]  

(12)

Neglecting higher-order harmonic contributions, we have the approximate expression [25]

\[ \cos^3(\phi) \approx \frac{3}{4} \cos(\phi) \]  

(13)

Substituting Eq. (13) into Eq. (12) and separating terms in proportional to \( \cos(\phi) \) and \( \sin(\phi) \), respectively, we get two equations which can be merged into a single equation in the complex domain

\[ \left\{ (1 - \Omega^2) A + \frac{3}{4} \beta A^3 - f_0 \cos(\phi_0) \right\} e^{-i\phi} + \left\{ A\Omega Q^{-1} + f_0 \sin(\phi_0) \right\} i = f_s e^{-i\phi} \]  

(14)

Firstly, we consider the dynamic characteristics of the self-sustained resonator without the external perturbation, i.e., \( f_0 \neq 0 \), \( f_s = 0 \). In this case, two separate equations can be obtained from Eq. (14) as

\[ (1 - \Omega^2) A + \frac{3}{4} \beta A^3 - f_0 \cos(\phi_0) = 0 \]  

(15)

\[ A\Omega Q^{-1} + f_0 \sin(\phi_0) = 0 \]  

(16)

\( \Omega_0 \) and \( A_0 \) are obtained by solving Eq. (15) and (15)

\[ A_0 = \frac{Q f_0 \sin(\phi_0)}{\Omega_0} \]  

(17)

\[ \Omega_0 = \frac{1}{\sqrt{2}} \left( 1 + \left( 1 + 3 Q^2 f_0^2 \sin^2(\phi_0) \right)^{\frac{1}{2}} \right)^2 \]  

(18)
Here, $\Omega_0$ and $A_0$ are vibration frequency and amplitude of the self-sustained oscillator.

When the external perturbation is applied, i.e. $f_s \neq 0$, $f \neq 0$, the frequency and amplitude of the oscillator will deviate from self-oscillation frequency. For weak perturbation, (i.e. $f_s < f_0$), the variation of frequency $\delta \Omega$ and amplitude $\delta A$ of the oscillator are expected to be small compared to $\Omega_0$ and $A_0$. Introducing the perturbative parameter $\epsilon = f_s / f_0 \ll 1$, and rewriting $\delta \Omega$ and $\delta A$ as $\epsilon \delta \Omega$ and $\epsilon \delta A$, respectively. The instantaneous frequency and amplitude are

$$\Omega = \Omega_0 + \epsilon \delta \Omega, \quad A = A_0 + \epsilon \delta A \tag{19}$$

Substituting Eq. (19) into Eq. (14), by neglecting higher-order terms, we get

$$\left\{ (1 - \Omega_0)^2 + \frac{9}{4} \beta A_0^2 - i \Omega_0 Q^{-1} \right\} \delta A$$

$$- \left\{ 2A_0 \Omega_0 + i A_0 Q^{-1} \right\} \delta \Omega = f_0 e^{-(i\phi_s)} \tag{20}$$

The modulus of the complex equation Eq. (20) satisfies

$$N_r^2 + N_i^2 = f_0^2 \tag{21}$$

where $N_r = \left\{ (1 - \Omega_0)^2 + \frac{9}{4} \beta A_0^2 \right\} \delta A - 2A_0 \Omega_0 \delta \Omega$ and $N_i = Q^{-1} (A_0 \delta \Omega + \Omega_0 \delta A)$ are real and imaginary terms of Eq. (20). Equation (21) can be written as a quadratic equation in terms of $\delta A$,

$$\left\{ (1 - \Omega_0)^2 + \frac{9}{4} \beta A_0^2 \right\} \delta A^2$$

$$+ \left\{ \frac{2A_0 \Omega_0 \delta \Omega}{Q^2} - 4A_0 \Omega_0 \left( (1 - \Omega_0)^2 + \frac{9}{4} \beta A_0^2 \right) \right\} \delta A$$

$$+ \left\{ \frac{4\Omega_0^2 + \frac{1}{Q^2}}{A_0^2} \right\} \delta \Omega^2 - f_0^2 = 0 \tag{22}$$

To ensure the real solution of Eq. (22), the following expression needs to be satisfied

$$\left\{ \frac{2A_0 \Omega_0 \delta \Omega}{Q^2} - 4A_0 \Omega_0 \left( (1 - \Omega_0)^2 + \frac{9}{4} \beta A_0^2 \right) \right\}^2$$

$$- 4 \left\{ \left( (1 - \Omega_0)^2 + \frac{9}{4} \beta A_0^2 \right)^2 + \frac{\Omega_0^2}{Q^2} \right\}$$

$$\left\{ \frac{4\Omega_0^2 + \frac{1}{Q^2}}{A_0^2} \right\} \delta \Omega^2 - f_0^2 \geq 0 \tag{23}$$

Equation (23) holds under the synchronization condition. Solving Eq. (23) the limit frequency deviation $\delta \Omega_c$ is determined, which leads to a synchronization range $(-\delta \Omega_c, \delta \Omega_c)$.

$$|\delta \Omega_c| \leq \epsilon \frac{Q}{A_0} \left( 1 - \frac{144 \beta A_0^2 + 64 - \frac{16}{Q^2} \Omega_0^2}{(4 + 4\Omega_0^2 + 9 \beta A_0^2)^2} \right)^{\frac{1}{2}} \tag{24}$$

Thus, the analytical expression of full synchronization bandwidth is obtained as $B = 2\delta \Omega_c$, i.e.,

$$B = 2 \frac{Q f_s}{f_0 A_0} \left( 1 - \frac{144 \beta A_0^2 + 64 - \frac{16}{Q^2} \Omega_0^2}{(4 + 4\Omega_0^2 + 9 \beta A_0^2)^2} \right)^{\frac{1}{2}} \tag{25}$$

where $\Omega_0$ and $A_0$ are determined by Eqs. (17) and (18). From Eq. (25) we can see that the synchronization bandwidth is nonlinear function of nonlinearity $\beta$ and quality factor $Q$, which would be distinct in different oscillators. We plot the variation form of synchronization bandwidth along with phase delay $\phi_0$ under different feedback forces in Fig. 2. It can be seen that the quality factor $Q$ and nonlinearity $\beta$ has similar effects on the variation forms of synchronization bandwidth along with phase delay under different strength of feedback force $f_0$. For an oscillator with lower quality factor $Q$, the variation of the synchronization bandwidth exhibits a ‘U’ form, the minimum synchronization bandwidth appears at phase delay equals $\pi / 2$, and feedback force $f_0$ almost has no effect on the size of bandwidth, which is similar with a linear case shown in Fig. 2c2 ($\beta = 0$). While for larger quality factor $Q = 50$ (Fig. 2b1) and relatively larger nonlinearity $\beta = 0.01$ (Fig. 2b2), the synchronization bandwidth also presents a ‘U’ form under small strength of feedback force. However, as $f_0$ increases to 0.05 (black line), the variation form begins to bulge from the bottom and evolves toward a ‘Anti-U’ form. Most interestingly, for a large quality factor or a strong nonlinearity under strong feedback force (Black lines in Fig. 2a1, a2), the variation form of synchronization bandwidth exhibits ‘M’ form, with two maximum values and a minimum value. From Eqs. (17) and (18) we can see that the oscillator’s self-oscillation amplitude is governed by the quality factor $Q$, nonlinearity $\beta$, feedback force $f_0$ and phase delay $\phi_0$ (the perturbation strength $f_s$ is small enough, i.e. $|f_s| \approx 0$). For a given nonlinearity $\beta$, small quality $Q$ and small feedback force $f_0$ will lead to small amplitude $A_L$, thus the nonlinearity term $\beta A_L^2$ can be neglected compared to linearity term.
A_L, the oscillator will exhibit as linear. For a large quality factor Q_H and nonlinearity β_H, the self oscillation amplitude A_H will become considerable, so the nonlinearity term β A^3_H can be comparable with linear term A_H. The nonlinearity become active, even plays a leading role. Above results remind us that for different oscillators, we can obtain the maximum synchronization bandwidth by adjusting the phase delay and the feedback force simultaneously, which can be achieved in our experimental tests.

2.3 Variation forms of synchronization region

For a given oscillator driven in a stable test environment, we just consider the effect of the feedback force and phase delay. The synchronization bandwidth B is a nonlinear function of phase delay φ_0 as shown in Eq. (25). After ignoring small terms, \( \frac{dB}{d\phi_0} = 0 \) leads to the following two solutions

\[
\phi_{0,1} = \frac{\pi}{2}, \quad \phi_{0,2} = \arcsin \left( \frac{1}{Q \sqrt{\beta} f_0} \right) \tag{26}
\]

where \( \phi_{0,1} \) is a fixed external point at \( \pi/2 \), which is a traditional result in previous studies [3,12], \( \phi_{0,2} \) is an additional dynamic one, which is activated by the nonlinearity β of the oscillator. Figure 3 shows the \( \phi_{0,1} \) and \( \phi_{0,2} \) varying with feedback force \( f_0 \) in different oscillators. The black dots (\( Q \cdot \sqrt{\beta} = 0 \)) represent a linear oscillator, where the peak value of synchronization bandwidth always exists at stable phase delay \( \pi/2 \), which is not affected by the strength of feedback force \( f_0 \). The red dots (\( Q \cdot \sqrt{\beta} = 0.1 \)) represent a oscillator with weakly nonlinearity. When the strength of feedback force is weak, the phase delay of peak value fixed at \( \pi/2 \). However, as the strength of feedback force increasing, the peak phase delay split into two symmetrical points about 90°. And when the nonlinearity becomes strong (blue dots, where \( Q \cdot \sqrt{\beta} = 1 \)), the bifurcation point emerges at smaller feedback force \( f_0 \).

For convenience of experimental verification in a specific oscillator with approximately changeless nonlinearity and a stable quality factor Q, we fix Q \( \cdot \sqrt{\beta} = 1 \) and plot the synchronization bandwidth against phase delay under different feedback forces. Figure 4 displays the synchronization region B varying with phase delay φ_0 for different feedback strengths \( f_0 \), calculated from Eq. (25) with parameters Q = 10, \( \beta = 0.1 \), \( \epsilon = f_s/f_0 = 0.1 \). Interestingly, three different types of variation pattern of the synchronization bandwidth with the phase delay is observed.
Phase-delay induced variation of synchronization

2.3.1 ‘U’ form

For small feedback strength, the synchronization bandwidth varies with the phase delay with a ‘U’ form as shown in Fig. 4b(I). The minimum synchronization bandwidth is achieved when \( \phi_0 = \pi/2 \). This is because, for small feedback strength, the oscillator vibrates in the linear regime with a very small amplitude (i.e., \( \beta = 0 \)), which leads to the reduced expression of synchronization bandwidth from Eq. (25)

\[
B = \frac{f_s}{Q f_0 \sin (\phi_0)} = \epsilon \cdot \left( \frac{1}{Q \sin (\phi_0)} \right)
\]

(27)

Only fixed extremal point \( \phi_{0,1} \) is found in Eq. (27) and dynamic one \( \phi_{0,2} \) disappears. Equation (27) gives a ‘U’ mode curve of synchronization bandwidth \( B \) with phase delay \( \phi_0 \) within \((0, \pi)\). Synchronization bandwidth \( B \) decreases first and increases again as \( \phi_0 \) increases from 0 to \( \pi \). There exists a minimum value at \( \phi_0 = \pi/2 \).

2.3.2 ‘Anti-U’ form

As feedback strength increases, an ‘Anti-U’ pattern variation of the synchronization bandwidth of the resonator is observed as shown in Fig. 4b(II). In this case, the synchronization bandwidth has a maximum value at phase delay \( \phi_0 = \pi/2 \), which is consistent with the previous studies [3,12]. Since for a relatively large feedback strength, the nonlinear term is not negligible and there exists two extreme points \( \phi_{0,1} \) and \( \phi_{0,2} \) as seen in Eq. (26). However, for small feedback strength, the second extreme point \( \phi_{0,2} \) is very close to the first one \( \phi_{0,1} \), which leads to the ‘Anti-U’ form curve as shown in Fig. 4b(II).

2.3.3 ‘M’ form

As the feedback strength increases to a much larger value, \( \phi_{0,2} \) will separate from \( \phi_{0,1} \) due to the decrease of \( \frac{1}{Q \sqrt{\beta f_0}} \) in Eq. (26). Two peaks will exist in the synchronization bandwidth curves, which leads to an ‘M’ shape curve as shown in Fig. 4b(III). As the feedback strength further increases, the values of these two peaks of the synchronization bandwidth curves get larger and two peaks grow apart from each other.

The colored regions in Fig. 4b display the difference between the maximum synchronization bandwidth obtained from expression (25) (black line) and the synchronization bandwidth at a fixed phase delay \( \phi_0 = \pi/2 \) (red line) for varying feedback strength. It can be seen that only when the strength of feedback \( f_0 \) is in a certain region (regime II), the maximum synchronization bandwidth is achieved at \( \phi_0 = \pi/2 \). This means that the previous method of synchronization bandwidth calculation is only suitable for a small range of \( f_0 \).
3 Experimental results

In this section, experimental tests are carried out to validate the analytical results above. The experimental setup is shown in Fig. 5. The micro single beam resonator is clamped to clamped (C–C) and works near its natural frequency actuated by a combination of dynamic voltage $V_{\text{ac}}$ and bias voltage $V_{\text{DC}}$. The motional signal is converted to electrical signal and detected by the HF2LI lock-in amplifier after diminish feedthrough [8] and pre-amplified by the differential circuit. The vacuum chamber can create a vacuum environment to reduce damping. The lock-in amplifier HF2LI with a built-in Phase Locked Loop (PLL) can output and detect signals simultaneously. Meanwhile, it can accurately tune the phase of the feedback signal. The signal generator can output synchronous disturbance signal to generate synchronization. All the measurements are tested in the vacuum chamber at a pressure below 0.1 Pa at room temperature. The measured oscillation frequency of the free running self-sustained oscillator is 214,500 Hz and the measured quality factor $Q \sim 20,000$ under the condition of $V_{\text{ac}}$ equals to 10 mV.

3.1 Synchronization region

The synchronization regions under different excitation voltages are measured in this section. We set $V_{\text{DC}} = 20$ V and $V_{\text{ac}}$ fixed at a specific voltage while tuning the phase delay $\phi_0$ from 30° to 90° in steps of 5°. A periodic perturbation signal is injected into the oscillator by a signal generator. Set $V_{\text{DC}} = 20$ V and feedback voltage strength $V_{\text{ac}} = 400$ mV. The oscillation frequencies of the resonator varying with the perturbation frequency when sweeping up (red solid line) and down (black solid line) are shown in Fig. 6a. It is obvious that there is a region (orange region) where the oscillation frequency is captured by the perturbation frequency. We extract the values of upper limit, central frequency and lower limit, then calculate the synchronization bandwidth for each test. Figure 6b–e shows the synchronization region varying with phase delay for different feedback strength $V_{\text{ac}}$. For small feedback (10 mV in Fig. 6b), the bandwidth of synchronization region decreases monotonously with phase delay. It is expected that the bandwidth has a minimum value at $\phi_0 = \pi / 2$, which is consistent with the analytically predicted ‘U’ case. As the feedback increases to 200 mV...
Phase-delay induced variation of synchronization

Fig. 6 Measured synchronization region. a Measured synchronization region under sweep up and sweep down perturbation at phase delay equals 60° and feedback force exciting voltage $V_{\text{act}} = 400$ mV. The red line and the black line are the oscillation frequency varying as the perturbation frequency sweep and down, respectively, and the orange region is the synchronization region at this condition. b, c, d, e are measured synchronization region under a fixed $V_{\text{DC}} = 20$ V and vary $V_{\text{act}}$, where b ($V_{\text{act}} = 10$ mV, $V_{\text{acs}}/V_{\text{act}} = 0.1$), c ($V_{\text{act}} = 100$ mV, $V_{\text{acs}}/V_{\text{act}} = 0.1$), d ($V_{\text{act}} = 500$ mV, $V_{\text{acs}}/V_{\text{act}} = 0.02$), e ($V_{\text{act}} = 550$ mV, $V_{\text{acs}}/V_{\text{act}} = 0.02$) ($V_{\text{acs}}$ is the perturbation strength), where the blue line, red line and black line in b, c, d, e are measured upper limit, central frequency and lower limit, respectively; f The measured synchronization bandwidth at $V_{\text{act}}$ equals 10 mV, 100 mV, 500 mV and 550 mV, respectively. The purple arrow between two orange circles which are the peak values of synchronization bandwidth at $V_{\text{act}} = 500$ mV and $V_{\text{act}} = 550$ mV represents the movement of the peak value from 60° to 55° as $V_{\text{act}}$ varies from 500 mV to 550 mV. (Color figure online)

(Fig. 6c), the bandwidth is observed to be a monotonically increasing function of phase delay. The calculated synchronization bandwidth agrees with the analytically predicted ‘Anti-U’ case. Further increasing the feedback strength to 500 mV and 550 mV, causes a second extreme point to appear in addition to $\phi_0 = \pi/2$ as shown in Fig. 6d, e, which validates the analytical ‘M’ case. The peak value gets larger and corresponding phase delay reduces from 60° to 55° as feedback strength increases from 500 to 550 mV as shown in Fig. 6f, which is consistent with the analytical results in Fig. 4a(III). The above experimental results verify our theoretical analysis and suggest that we can achieve the maximum synchronization bandwidth by adjusting the phase delay in MEMS sensor applications.

3.2 Frequency stability

Experiments are performed to investigate the effect of phase delay on the frequency stability of the oscillator. Allan deviation is an effective indicator for frequency stability evaluation, which is defined by the frequency fluctuations averaged over an integration time $\tau$. Figure 7 plots the Allan deviation at integration time $\tau = 1$ s as functions of phase delay for different feedback strength. The whole Allan deviation curves dur-
**Fig. 7** Measured Allan deviation at integration time $\tau = 1$ s from experimental self-oscillation frequency at a fixed $V_{DC} = 20$ V and $V_{acf}$ equals 10 mV, 200 mV, 400 mV and 550 mV, respectively, where scatters are measured results and real line are the fit line of scatters. Red scatter ($V_{acf} = 10$ mV) shows Allan deviation decrease as phase delay varies from 30° to 90° in linear regime, which has same variation pattern with synchronization region (green line in Fig. 3). Better frequency stability is achieved for the price of the bandwidth of synchronization region. Blue scatter ($V_{acf} = 200$ mV), purple scatter ($V_{acf} = 400$ mV) and green scatter ($V_{acf} = 550$ mV) show the Allan deviation varies with phase delay for nonlinear oscillator, which have totally opposite variation pattern with synchronization region. At the same time, as the feedback force strength increases, the most stable point of the self-oscillation frequency also decreases from 90° to 0°. There exists a good balance between the synchronization bandwidth and frequency stability. (Color figure online)

We can see that for longer integration time shown in Fig. 8, the Allan deviation gradually gets smaller, which means that the frequency stability of the oscillation frequency works in its synchronization regime gradually tends to the stability of the signal generator as the phase changes from 0° to 90°. For a long time, the frequency stability of synchronization tends to approach the generator’s stability, this reminds us that for application, a stable synchronization source will bring better detection results. For MEMS sensors, the synchronization bandwidth reflects the detection range of the sensor, while the frequency stability is used to evaluate the accuracy of the sensor, the higher the frequency stability, the closer the detected frequency to the precise frequency. Experimental results show that if the oscillator operated with proper phase delay in deep nonlinear regime, it can reach a maximum detection range and excellent accuracy simultaneously.
Fig. 8  Measured Allan deviation. Measured Allan deviation of oscillation frequency before and after synchronization at different phase delays and exciting voltages. The dashed and solid lines are measured Allan deviation before and after synchronization respectively. The thick black solid line is the Allan deviation of signal generator. Where a $V_{acf} = 10$ mV; b $V_{acf} = 200$ mV; c $V_{acf} = 400$ mV; d $V_{acf} = 550$ mV. (Color figure online)

4 Conclusion

In this paper, phase delay induced variation of synchronization bandwidth and frequency stability in a micromechanical oscillator, subject to feedback tuning was investigated. The expression for predicting the synchronization bandwidth with phase delay is derived, from which an additional dynamic extreme point $\phi_{0,2}$ actuated by nonlinearity in addition to the traditional one $\phi_{0,1} = \pi/2$ is observed, which leads to three different variation patterns (‘U’, ‘Anti-U’ and ‘M’) of synchronization bandwidth. The evolution of the synchronization bandwidth alters the distribution of frequency stability with phase delay. For linear oscillator, the synchronization bandwidth and the frequency stability have exactly opposite variation pattern along phase delay, which implies better frequency stability will be achieved at the cost of the bandwidth of synchronization region. However, in nonlinear oscillators, the synchronization bandwidth and the frequency stability are totally consistent, and there exists a good balance between the synchronization bandwidth and frequency stability. When the oscillator is working in a strong nonlinear regime, such as $V_{acf}$ equals 550 mV,
the maximum synchronization bandwidth obtained by adjusting the phase delay can be increased by 60 Hz compared with the traditional strategy that fix the phase delay at 90°, the bandwidth is increased by about 25%, and the frequency fluctuation reduced from 1.2e−7 to 8e−8, approximately reduced by 33%. The synchronization bandwidth evolution presented here provides a precise way for achieving best performance of a synchronized MEMS oscillator in the sensor application. However, there still exist some limitations in our strategy. First, in experimental sets, the nonlinear damping caused by ohmic losses in electrostatic actuation will increase and have effects on the measurement when the AC voltage increase to above 1 V, so the tests are limited with the region 0 < 550 mV. Second, the experiment was not able to verify all phase delays from 0 < 180° because of hysteresis, but the existing measured experimental results are consistent with the theoretical analysis. Third, although the synchronization bandwidth can be significantly enhanced in nonlinear regime, the frequency stability still become worse compared within linear regime. Therefore, when using this strategy as an application of synchronous sensing, it is a trade-off between frequency stability and synchronous bandwidth to make a suitable choice. Our contribution lies in the realization of a closed-loop feedback system that utilizes the adjustment of the feedback phase and the magnitude of the feedback force to achieve the maximum synchronization bandwidth and the improvement of frequency stability. The multiple patterns of synchronization bandwidth vary with phase delay under different feedback forces is first observed and also interesting. We highlight that these results are applicable not only to MEMS resonator but also to any type of synchronized oscillator utilizing feedback.

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Data availability statement The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

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