Some Inverse Problems in Periodic Homogenization of Hamilton-Jacobi Equations

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Abstract

We look at the effective Hamiltonian $\overline{H}$ associated with the Hamiltonian $H(p, x) = H(p) + V(x)$ in the periodic homogenization theory. Our central goal is to understand the relation between $V$ and $\overline{H}$. We formulate some inverse problems concerning this relation. Such types of inverse problems are, in general, very challenging. In this paper, we discuss several special cases in both convex and nonconvex settings.

1. Introduction

1.1. Setting of the Inverse Problem

For each $\varepsilon > 0$, let $u^{\varepsilon} \in C(\mathbb{R}^n \times [0, \infty))$ be the viscosity solution to the following Hamilton-Jacobi equation:

\[
\begin{align*}
    u_t + H(Du^{\varepsilon}, \frac{x}{\varepsilon}) &= 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
    u^{\varepsilon}(x, 0) &= g(x) & \text{on } \mathbb{R}^n.
\end{align*}
\]

The Hamiltonian $H = H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies that

(H1) $x \mapsto H(p, x)$ is $\mathbb{Z}^n$-periodic,

(H2) $p \mapsto H(p, x)$ is coercive uniformly in $x$, that is,

$$\lim_{|p| \to +\infty} H(p, x) = +\infty \text{ uniformly for } x \in \mathbb{R}^n.$$
and the initial data \( g \in \text{BUC}(\mathbb{R}^n) \) for the set of bounded, uniformly continuous functions on \( \mathbb{R}^n \).

It was proved by Lions, Papanicolaou and Varadhan [16] that \( u^\varepsilon \), as \( \varepsilon \to 0 \), converges locally uniformly to \( u \), the solution of the effective equation being

\[
\begin{aligned}
  u_t + \nabla H(Du) &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
  u(x, 0) &= g(x) \quad \text{on } \mathbb{R}^n.
\end{aligned}
\]

The effective Hamiltonian \( \overline{H} : \mathbb{R}^n \to \mathbb{R} \) is determined by the cell problems as follows. For any \( p \in \mathbb{R}^n \), we consider the cell problem

\[
H(p + Dv, x) = c \quad \text{in } \mathbb{T}^n,
\]

where \( \mathbb{T}^n \) is the \( n \)-dimensional torus \( \mathbb{R}^n/\mathbb{Z}^n \). We here look for a pair of unknowns \((v, c) \in C(\mathbb{T}^n) \times \mathbb{R}\) in the viscosity sense. It was established in [16] that there exists a unique constant \( c \in \mathbb{R} \) such that (1.3) has a solution \( v \in C(\mathbb{T}^n) \). We then denote by \( \overline{H}(p) = c \).

In this paper, we always consider the Hamiltonian \( H \) of the form \( H(p, x) = H(p) + V(x) \). Our main goal is to study the relation between the potential energy \( V \) and the effective Hamiltonian \( \overline{H} \). In the case where \( H \) is uniformly convex, Concordel [6,7] provided some first general results on the properties of \( \overline{H} \), which is convex in this case. In particular, she achieved some representation formulas of \( \overline{H} \) by using the optimal control theory, and showed that \( \overline{H} \) has a flat part under some appropriate conditions on \( V \). The connection between the properties of \( \overline{H} \) and the weak KAM theory can be found in E [10], Evans and Gomes [11], Fathi [13] and the references therein. We refer the reader to Evans [12, Section 5] for a list of interesting viewpoints and open questions. To date, deep properties of \( \overline{H} \) are still not well understood.

In the case where \( H \) is not convex, there have been not so many results on the qualitative and quantitative properties of \( \overline{H} \). Very recently, Armstrong, Tran and Yu [1,2] studied nonconvex stochastic homogenization and derived qualitative properties of \( \overline{H} \) in the general one dimensional case and in some special cases in multi-dimensional spaces. The general case in multi-dimensional spaces is still out of reach.

We present here a different question concerning the relation between \( V \) and \( \overline{H} \). In its simplest way, the question can be thought of as: how much can we recover the potential energy \( V \) provided that we know \( H \) and \( \overline{H} \)? More precisely, we are interested in the following inverse type problem:

**Question 1.1.** Let \( H \in C(\mathbb{R}^n) \) be a given coercive function, and \( V_1, V_2 \in C(\mathbb{R}^n) \) be two given potential energy functions which are \( \mathbb{Z}^n \)-periodic. Set \( H_1(p, x) = H(p) + V_1(x) \) and \( H_2(p, x) = H(p) + V_2(x) \) for \((p, x) \in \mathbb{R}^n \times \mathbb{R}^n\). Suppose that \( \overline{H}_1 \) and \( \overline{H}_2 \) are two effective Hamiltonians corresponding to the two Hamiltonians \( H_1 \) and \( H_2 \) respectively. If

\[
\overline{H}_1 \equiv \overline{H}_2,
\]

then what can we conclude about the relations between \( V_1 \) and \( V_2 \)? In particular, can we identify some common “computable” properties shared by \( V_1 \) and \( V_2 \)?