One-loop Noncommutative $U(1)$ Gauge Theory from Bosonic Worldline Approach

Youngjai Kiem$^{a,c}$, Yeonjung Kim$^b$, Cheol Ryou$^a$, AND Haru-Tada Sato$^a$

$^a$ BK21 Physics Research Division and Institute of Basic Science, Sungkyunkwan University, Suwon 440-746, Korea

$^b$ Department of Physics, Korea Advanced Institute of Science and Technology, Taejon 305-701, Korea

$^c$ Physics Department, Princeton University, Princeton, NJ08544, USA

ykiem, cheol, haru@newton.skku.ac.kr, geni@muon.kaist.ac.kr

Abstract

We develop a method to compute the one-loop effective action of noncommutative $U(1)$ gauge theory based on the bosonic worldline formalism, and derive compact expressions for $N$-point 1PI amplitudes. The method, resembling perturbative string computations, shows that open Wilson lines emerge as a gauge invariant completion of certain terms in the effective action. The terms involving open Wilson lines are of the form reminiscent of closed string exchanges between the states living on the two boundaries of a cylinder. They are also consistent with recent matrix theory analysis and the results from noncommutative scalar field theories with cubic interactions.

*Current address: Department of Physics, Korea Advanced Institute of Science and Technology, Taejon 305-701, Korea; E-mail: ykiem@muon.kaist.ac.kr
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1 Introduction

Noncommutative field theories are a miniaturized version of string theory, through which we can discuss issues such as nonlocality and off-shell physics in a controlled fashion (for reviews, see [1]). A notable aspect in this regard is the appearance of open Wilson lines [2]; they allow us to form off-shell gauge invariant observables [3], and capture the dipole nature of noncommutative field theories representing their inherent nonlocality [4]. Typically, the low energy effective description of noncommutative D-branes corresponds to noncommutative gauge theory, a prime example of noncommutative field theories. Even in its simplest setup of noncommutative $U(1)$ gauge theory, the complete computation of the one-loop effective action is non-trivial. In particular, to directly test if open Wilson lines emerge as expected, one has to sum over an infinite number of gauge field insertions. An efficient method that produces the manageable form of the $N$-point 1PI amplitudes and further allows the summation over $N$ is thus desirable. The development of such a method for noncommutative $U(1)$ gauge theory at one-loop is the task achieved in this paper following the bosonic worldline approach [5]-[9]. We derive a scheme that resembles perturbative string computations of amplitudes involving gauge boson vertex operators. The main difference is that being a field theory construction, our method is valid off-shell, as well as on-shell. The worldline formalism, that has been known to produce string-theory-like schemes in the commutative context [7], turns out to be a useful device that allows us to keep track of all possible terms in the effective action. It is especially helpful when taking care of various contact terms that appear in the case of noncommutative $U(1)$ gauge theory, whose appearance is rather similar to the case of nonabelian gauge theories in commutative space-time.

The $N$-point terms in the one-loop effective action computed by our method are summarized by Eqs. (3.16)-(3.19) for the ghost loop contributions, and by Eqs. (3.38), (3.42) and (3.43) for the gauge loop contributions. They can be compactly rewritten as (4.2) and (4.3). After summing over certain class of (an infinite number of) terms in the effective action, we find that it contains the sequence of terms:

$$\Gamma = \frac{D-2}{2} \int \frac{d^Dk}{(2\pi)^D} W_k[A] K_0(k) W_{-k}[A]$$
\[ +\frac{1}{2} \sum_{n_1, n_2} \frac{(-1)^{n_1}}{n_1! n_2!} \int \frac{d^D k}{(2\pi)^D} g^{(n_1)}_k [A] K_{n_1+n_2}(k) g^{(n_2)}_{-k} [A], \]  

where the \( n \)-th descendent \( g^{(n)}_k \) \(( W_k[A] = g^{(0)}_k \) of open Wilson lines involving \( n \)-copies of field strength tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig(A_\mu \star A_\nu - A_\nu \star A_\mu) \) is defined as

\[ g^{(n)}_k = 2^n (ig)^n \int d^D x P_* \left[ W(x, C) \prod_{i=1}^n F(x) \right] \star e^{ikx} \]  

in terms of the ‘straight’ open Wilson line \([2]\)

\[ W(x, C) = P_* \exp \left( ig \int_0^1 dt \frac{\partial y^\mu(t)}{\partial t} A_\mu(x + y(t)) \right), \quad y^\mu(t) = \theta^{\mu\nu} k_\nu t. \]  

Basic notations are given at the end of this section, the space-time indices are explained in detail later in this paper, and \( P_* \) is the path-ordering with respect to the \( \star \)-product. The explicit expression for the kernel \( K_n \) is given in (1.3). The summation \( \sum_{n_1, n_2} \) represents \( \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \) excluding the case \( n_1 = n_2 = 0 \); the contribution coming from \( n_1 = n_2 = 0 \) is separated out into the first term of (1.1), and it also contains the ghost loop contribution. Each term in (1.1) involves an infinite number of higher-point terms originating from the expansion of the exponential function \( W(x, C) \) and extra higher-point terms from the noncommutative commutator term in the field strength tensor. Partially due to the factor \((-1)^n_1\), all the terms in (1.1) vanish in the commutative limit, which is not the case in the noncommutative setup. When there are more fields transforming adjointly under the noncommutative gauge group, it was argued in [14, 15] that the coefficient \((D-2)\) in (1.1) is in general replaced with \((N_B - N_F)\) where \( N_B \) \((N_F)\) is the bosonic \((\text{fermionic})\) degrees of freedom (see also [16, 17]). In the maximally supersymmetric \( \mathcal{N} = 4 \), the terms up to \( n_1 + n_2 < 4 \) vanish, as the consequence of boson-fermion cancellations [18].

Our results (1.1) are consistent with the gauge invariant completion suggested in Ref. [3]. In [19], the leading nonvanishing four-point terms in the \( D = 4, \mathcal{N} = 4 \) noncommutative

\footnote{In noncommutative real scalar field theory with a cubic interaction [10, 11, 12, 13], the whole one-loop effective action in the large noncommutativity limit can be summed up to the form of \( n_1 = n_2 = 0 \) term with coefficient 1/2. In this context, the open Wilson line is replaced with a scalar analog of the vector Wilson line [13]. The kernel \( K_0 \) in this case is the leading term in the asymptotic expansion of \( K_0 \) in (1.1) for the large value of the argument.}
supersymmetric Yang-Mills theory were computed by adopting the perturbative string theory technique and taking the Seiberg-Witten limit \([20]\). They turned out to be the type of the \(n_1 + n_2 = 4\) terms of \((1.1)\) when taking \(W(x, C) = 1\) and neglecting the commutator terms, modulo the detailed space-time index structure on which we will comment later. Based on the requirement of (off-shell) gauge invariance, it was conjectured in \([3]\) that the higher point terms that make up the \(\mathcal{F}_k^{(n)}\) should be present in the one-loop effective action. Further perturbative evidence supporting this conjecture was provided by Refs. \([21]\). More recently, the matrix theory side considerations of open Wilson lines in noncommutative gauge theory were reported in \([14]\) for one-loop case and in \([22]\) for two-loop case. In addition, the authors of \([15]\) obtained \(n_1 = n_2 = 0\) result of \((1.1)\) by explicitly summing up certain Feynman diagrams. It should be noted however that there are extra terms in the effective action that can not be easily represented in terms of open Wilson lines, as our analysis will show. We will make further comments on them later. It is an interesting outstanding problem to see if they can also be written in terms of (variants of) open Wilson lines, and to see if they are absent in the supersymmetric gauge theories.

The analysis of \(U(1)\) gauge theory closely parallels the analysis of scalar field theory with a cubic interaction reported in \([10, 11, 12, 13]\), where it was pushed to two-loops. As is clear from the comparison of the final answers, our results indicate that the appearance of open Wilson lines is a universal feature of noncommutative field theories regardless of detailed spin contents \([23]\). Main physical features found in \([10, 11, 12, 13]\) are still present without any essential modifications; for example, the terms \((1.1)\) are of the form reminiscent of ‘closed string’ exchanges between the states living on the two boundaries of a cylinder. There is, however, an important difference. In the case of the scalar noncommutative field theories, to obtain the scalar version of open Wilson lines, one needs to take a large noncommutativity limit to reduce the ‘closed string kernel’ \(\mathcal{K}_{n_1+n_2}\) to its leading term in the asymptotic expansion. In the gauge theory case, it is not necessary to take a similar limit; the summation over \(N\)-point functions goes through for finite value of the noncommutativity parameter. It suggests that the appearance of open Wilson lines is closely related to the existence of \(U(\infty)\) symmetry of a theory in consideration.
This paper is organized as follows. In section 2, we review the action of noncommutative $U(1)$ gauge theory in the context of the background field method \[24\]. The choice of the background field gauge allows us to formulate a scheme with manifest gauge invariance that closely resembles the perturbative string theory computations. In section 3, we evaluate the $N$-point functions resulting from the ghost loop and the gauge loop, adopting the bosonic worldline formalism. Particular attentions are paid to the treatment of various contact terms that are essential in maintaining the gauge invariance. Some of the details along this line are presented in Appendix B and C. A detailed implementation of noncommutative worldline formalism is relegated to Appendix A. Eventually, we find an expression for the ghost loop that involves gauge boson vertex operators similar to the vertex operators of bosonic string theory. For the gauge loop, appropriate vertex operators turn out to be similar to the 0-picture gauge boson vertex operators in superstring theory. In section 4, we sum certain class of $N$-point terms to obtain (1.1), showing the emergence of open Wilson lines and their descendents in the one-loop effective action.

The notations adopted throughout this paper are as follows. We consider the noncommutative $U(1)$ gauge theory on a noncommutative plane $R^D$, where the coordinates satisfy

$$[x^\mu, x^{\nu}] = i\theta^{\mu\nu} .$$

(1.4)

Via the Weyl-Moyal correspondence, the product between fields is given by the $\star$-product

$$\phi \star \phi(x) = e^{i\theta^{\mu\nu}\partial_\mu y \partial_\nu z} \phi(y) \phi(z) \bigg|_{y=z=x} .$$

(1.5)

In addition, the following notations are used:

$$p \wedge k = p_\mu \theta^{\mu\nu} k_\nu , \quad p ~ k = p_\mu (-\theta^2)^{\mu\nu} k_\nu \geq 0 .$$

(1.6)

We only consider the space-space noncommutativity in this paper. The Lorentz indices, therefore, will be frequently put upside down for convenience.
2 Review: Background field action of noncommutative $U(1)$ gauge theory

Our main interest is the computation of one-loop effective action maintaining the explicit gauge invariance. For this purpose, we employ the background field method, splitting the gauge field $A$ into $A + Q$, where $A$ and $Q$ are the background fields and quantum fluctuations, respectively. The noncommutative $U(1)$ gauge theory is then described by the following action:

$$S = \int d^D x \left\{ -\frac{1}{4} F_{\mu\nu}(A + Q) \star F^{\mu\nu}(A + Q) - \frac{1}{2\alpha}(\bar{D}^\mu Q_\mu)^2 + \bar{C} \bar{D}_\mu \star D^{A+Q}_\mu C \right\} ,$$

(2.1)

where we include ghost field $C$ and anti-ghost field $\bar{C}$. Eventually, we will choose the Feynman gauge setting $\alpha = 1$ (keeping the gauge invariance for $A$). The covariant derivatives and field strength for noncommutative gauge fields $X_\mu$ (either $A_\mu$ or $Q_\mu$) are given by

$$D_\mu X = \partial_\mu X - ig[X_\mu, Y]_* , \quad \bar{D}_\mu \equiv D^A_\mu ,$$

(2.2)

$$F_{\mu\nu}(X) = \partial_\mu X_\nu - \partial_\nu X_\mu - ig[X_\mu, X_\nu]_* , \quad \bar{F}_{\mu\nu} \equiv F_{\mu\nu}(A) ,$$

(2.3)

where the $\star$-commutator represents $[X, Y]_* = X \star Y - Y \star X$. It is useful to use the following relation:

$$F_{\mu\nu}(A + Q) = \bar{F}_{\mu\nu} + \bar{D}_\mu Q_\nu - \bar{D}_\nu Q_\mu - ig[Q_\mu, Q_\nu]_* .$$

(2.4)

After some algebra, we organize the action up to surface terms

$$S = \int d^D x \left[ -\frac{1}{4} \bar{F}^{\mu\nu} \star \bar{F}_{\mu\nu} - \bar{F}^{\mu\nu} \star \bar{D}_\mu Q_\nu \\
- \frac{1}{2} (\bar{D}^\mu Q_\nu) \star (\bar{D}_\mu Q_\nu) + \frac{1}{2} (1 - \frac{1}{\alpha})(\bar{D}^\mu Q_\mu) \star (\bar{D}_\nu Q_\nu) + ig\bar{F}^{\mu\nu} \star [Q_\mu, Q_\nu]_* \\
+ \frac{1}{4} g^2 [Q^\mu, Q^\nu]_* \star [Q_\mu, Q_\nu]_* + ig[Q^\mu, Q^\nu]_* \star D_\mu Q_\nu \right]$$

$$+ \int d^D x \left\{ \bar{C} \bar{D}_\mu \bar{D}^\mu C - ig\bar{C} \bar{D}_\mu [Q_\mu, C]_* \right\}_* .$$

(2.5)

Discarding one-particle-reducible terms and setting $\alpha = 1$, the one-loop relevant parts of $S$, needed when computing the one-loop effective action, read $S \sim S^{\text{gauge}} + S^{\text{ghost}}$ where

$$S^{\text{gauge}} = \int d^D x \frac{1}{2} Q^\mu \star [g_{\mu\nu} \bar{D}^\rho \star \bar{D}_\rho - 4ig\bar{F}_{\mu\nu}] \star Q^\nu ,$$

(2.6)

$$S^{\text{ghost}} = \int d^D x \bar{C} \star \bar{D}_\mu \star \bar{D}^\mu \star C .$$

(2.7)
To effectively deal with the space-time nonlocality, it is convenient to adopt the momentum basis via Fourier transformation. Utilizing the following relations for the terms in $S^{\text{gauge}}$:

\[
\int d^D x \bar{D}^\rho Q^\mu \ast \bar{D}_\rho Q_\mu = \int d^D x \left[ -Q^\mu \partial^2 Q_\mu + 2igA_\rho \ast [\partial^\rho Q^\mu, Q_\mu] \right] + 2g^2(A^\rho B_\rho Q^\mu Q_\mu - A^\rho A_\rho A_\mu Q_\mu) \ast \right], \quad (2.8)
\]

\[
-2ig \int d^D x Q^\mu \ast \bar{F}_{\mu \nu} \ast Q^\nu = -ig \int d^D x Q^\mu \ast [\bar{F}_{\mu \nu}, Q^\nu] \ast = \int d^D x Q^\mu \ast [\bar{D}_\mu, \bar{D}_\nu] \ast Q^\nu, \quad (2.9)
\]

valid up to surface terms, we realize that (2.8) produces $S_2$ part, and (2.9) yields $S_3$ and $S_4$ parts given by:

\[
S^{\text{gauge}} = S_2 + S_3 + S_4, \quad (2.10)
\]

where

\[
S_2(Q) = -\frac{1}{2} \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \left\{ ik^\mu_1 - ig \int \frac{d^D p}{(2\pi)^D} (e^{\frac{i}{2} k_1 \wedge p} - e^{-\frac{i}{2} k_1 \wedge p}) \hat{A}^\mu(p) \right\} \\
\times \left\{ ik^\mu_2 - ig \int \frac{d^D p}{(2\pi)^D} (e^{\frac{i}{2} k_2 \wedge p} - e^{-\frac{i}{2} k_2 \wedge p}) \hat{A}_\mu(p) \right\} \hat{Q}_\nu(k_1) \hat{Q}^\nu(k_2), \quad (2.11)
\]

\[
S_3(Q) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} g \int \frac{d^D p}{(2\pi)^D} (e^{\frac{i}{2} k_1 \wedge p} - e^{-\frac{i}{2} k_1 \wedge p}) \\
\times (p_\mu \hat{A}_\nu(p) - p_\nu \hat{A}_\mu(p)) \hat{Q}^\mu(k_1) \hat{Q}^\nu(k_2), \quad (2.12)
\]

\[
S_4(Q) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} g^2 \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} (e^{\frac{i}{2} k_1 \wedge p_1} - e^{-\frac{i}{2} k_1 \wedge p_1}) (e^{\frac{i}{2} k_2 \wedge p_2} - e^{-\frac{i}{2} k_2 \wedge p_2}) \\
\times (\hat{A}_\mu(p_1) \hat{A}_\nu(p_2) - \hat{A}_\nu(p_1) \hat{A}_\mu(p_2)) \hat{Q}^\mu(k_1) \hat{Q}^\nu(k_2). \quad (2.13)
\]

In deriving these expressions, we have assumed the momentum conservation as usual, and $\hat{X}$ stands for the Fourier transforms of $X = Q_\nu, A_\nu$. The ghost action $S^{\text{ghost}}$ is simply given by replacing the $Q$ field dependence in $S_2$ with the (anti-) ghost field dependence, i.e., symbolically, $S^{\text{ghost}} = S_2(C)$.

When we set $\theta^{\mu \nu} = 0$, the interaction terms between $A$ and $Q$ disappear and the theory becomes trivial, as the commutative $U(1)$ gauge theory should be. In the presence of nonvanishing $\theta^{\mu \nu}$, however, the $U(1)$ gauge theory in consideration closely mimics the behavior of nonabelian gauge theories in commutative space-time. To summarize, the ghost and gauge
loop contributions to the one-loop effective action $\Gamma[A]$ are given by the logarithm of formal one-loop determinants
\[
\Gamma^\text{ghost}[A] = \ln \text{Det} \left[ \frac{-\delta^2 S_2(C)}{\delta \hat{C}(k_1) \delta \hat{C}(k_2)} \right], \tag{2.14}
\]
\[
\Gamma^\text{gauge}[A] = \ln \text{Det}^{-1/2} \left[ \frac{-\delta^2}{\delta \hat{Q}(k_1) \delta \hat{Q}(k_2)} (S_2 + S_3 + S_4) \right], \tag{2.15}
\]
which we will evaluate adopting the bosonic worldline formalism.

3 Computation of $N$-point functions in the one-loop effective action

In this section, we show how we can extract the worldline path integral expressions for $N$-point proper amplitudes. The starting point for the computation is to consider the ghost contribution in detail, where the action consisting of only $S_2$ is simpler. Particular attention will be paid to the treatment of contact terms (quartic terms) in $S_2$ and the necessary point-splitting regularization. Once we understand the ghost loop contribution, the treatment of the gauge loop is straightforward. We will eventually show that $S_3$ and $S_4$ parts combine to form field strength tensors, where the (quartic) $S_4$ part provides the $\star$-commutator terms.

3.1 The ghost loop

We first consider the ghost loop (2.14) following the ‘stripping method’ developed in the treatment of noncommutative scalar field theories [12]. The details not presented in that reference are available in Appendix A\textsuperscript{2}. The basic idea is to separate out the overall Filk phase factor that is responsible for the $\star$-products (or $\star^
u$-products in the one-loop context) between background fields (stripping). We then attach an extra phase factor depending on the loop momentum when a ‘nonplanar crossing’ happens. It shows up for an external

\textsuperscript{2}One should note that the computations here closely parallel those using ‘nonstripping method’ of [11] as well. It turns out that both methods produce exactly the same result at the one-loop level. For the treatment of higher loops, however, the stripping method of [12] is more convenient.
insertion along the ‘inner’ boundary of a one-loop diagram in the double-line notation. See Figure 1, for example.

Taking into account of stripping factors from (A.3) and (A.4), we extract the effective vertex \( G_1 + G_2 \) out of the second derivatives of \( S_2 \):

\[
G_1(k) = k^2 + 2gk^\mu \int \frac{d^Dp}{(2\pi)^D} (1 - e^{ik\wedge p}) \hat{A}_\mu(p),
\]

\[
G_2(k) = g^2 \int \frac{d^Dp_1}{(2\pi)^D} \int \frac{d^Dp_2}{(2\pi)^D} (1 - e^{ik\wedge p_1})(1 - e^{i(k+p_1)\wedge p_2}) \hat{A}_\mu(p_1) \hat{A}_\nu(p_2) g^{\mu\nu}.
\]

This leads to a path integral expression for (2.14) given by

\[
\Gamma_{\text{ghost}} = -\int \frac{dT}{T} \int Dk D\hat{P} \exp \left[ -\int_0^T \left( k^2 - i\dot{k} \cdot \dot{x} + G_1(k) + G_2(k) \right) \right].
\]

The contribution from \( G_1 \) comes from the three-point vertex with a single external line (and two internal lines) familiar from \( \phi^3 \) scalar theory in the background field method [12]: two terms, 1 and \(-e^{ik\wedge p}\), represent the outer (without a crossing) and inner (with a crossing) insertions in the double-line notation, respectively. The factor \(-e^{ik\wedge p}\) is the aforementioned extra phase factor depending on the internal momentum \( k \), which was also used in [16]. The extra minus sign for the inner insertion is a property of gauge theories. The interaction vertex in \( G_2 \) involves two external fields inserted at the same point. See Figure 1 and, as such, they are what we call contact terms. These terms will play a crucial role in simplifying the final result and enforcing the gauge invariance, as will be shown shortly.

It is convenient to set \( G_2 = 0 \) first, and the resulting effective action will be called \( \Gamma^{(0)\text{ghost}} \).

Expanding the \( \hat{A} \)-dependent piece in (3.1) in the exponential part of (3.3), we have

\[
\Gamma^{(0)\text{ghost}} = -\int \frac{dT}{T} \int Dk D\hat{P} \exp \left[ -\int_0^T \left( k^2 - i\dot{k} \cdot \dot{x} \right) \right]
\]

\[
\times \sum_{N=0}^{\infty} \sum_{n=0}^N (-2g)^N \prod_{l=1}^n \int_0^{\tau_{l+1}} d\tau_l \int \frac{d^Dp_l}{(2\pi)^D} \left[ k \cdot \hat{A}_\star(p_l) \right]
\]

\[
\times \prod_{j=1}^{N-n} \int_0^{\tau_j' + 1} d\tau_j' \int \frac{d^Dp_j'}{(2\pi)^D} \left[ -k \cdot \hat{A}_\star(p_j') \right] \exp \left[ -i \sum_{j=1}^{N-n} p_j' \wedge k(\tau_j') \right],
\]

where \( n \) is the number of outer insertions and \( N-n \) is the number of inner insertions.

In order to generate all possible Feynman diagrams automatically, we replace the external
background fields \( \hat{A} \) with sums of all plane wave modes (after then, inserting the factor \( \frac{1}{N!} \) is necessary):

\[
\prod_{l=1}^{n} T_{l=0}^{\tau_{l+1}} \int d\tau l \int \frac{d^D p l}{(2\pi)^D} [k \cdot \hat{A}_\ast(p l)] \prod_{j=1}^{N-n} T_{j=0}^{\tau_{j+1}} \int d\tau j \int \frac{d^D p j}{(2\pi)^D} [-k \cdot \hat{A}_\ast(p j)] \\
\rightarrow \frac{1}{N!} \prod_{l=1}^{n} T_{l=0}^{\tau_{l+1}} \prod_{j=1}^{N-n} T_{j=0}^{\tau_{j+1}} \int d\tau \int d\tau \left(e^{ip_1 x(\tau_1)} \ast \nu e^{ip_2 x(\tau_2)} \ast \nu \cdots e^{ip_N x(\tau_N)} + (\text{all } p_i \text{ permutations}) \right),
\]

where \( \nu \) is the polarization vector. The stripped overall Filk phase shows itself through \( \ast \nu \) defined in (A.12) as

\[
\phi \ast \nu \varphi(x) = e^{\frac{i}{2} \nu \partial_\mu \phi \partial^\mu \varphi} \phi(y) \varphi(z) \bigg|_{y=z=x} \quad \text{for} \quad \nu = 0, \pm 1,
\]

where \( \nu = 1 \) when both \( \phi \) and \( \varphi \) are outer insertions, \( \nu = -1 \) when both of them are inner insertions, and \( \nu = 0 \) otherwise. The \( \ast \nu \)-products at tree level are precisely the \( \nu = 1 \) (outer boundary) \( \ast \nu \)-products. The quantities with primes represent inner insertions. When compared to scalar field theories with a cubic interaction, there is an extra \((-1)^{N-n}\) factor (Cf. (A.11)).

To summarize the result so far, the \( N \)-point contributions are given by

\[
\Gamma_{N}^{(0)\text{ghost}} = -(-2g)^N \sum_{\{\nu_i\}} \int dT \left(\prod_{i=1}^{N} T_{i=0}^{\tau_i}\right) e^{\frac{i}{4} \sum_{\nu_i \gamma_j p_i p_j (\nu_i + \nu_j) \varepsilon(\tau_{ij})} \left(\prod_{j=1}^{N} (-)^{\alpha_j}\right)} \\
\times \int D x e^{i \sum_j p_j x(\tau_j)} \int D k (k(\tau_1) \cdot \epsilon_1) \cdots (k(\tau_N) \cdot \epsilon_N) \\
\times \exp \left(- \int_0^T (k^2(\tau) - ik \hat{x}(\tau)) d\tau \right) \prod_{j=1}^{N} e^{-i \alpha_j p_j k(\tau_j)},
\]

which is to be summed with the weight \( 1/N! \) to become \( \Gamma_{N}^{(0)\text{ghost}} \). For inner insertions, we have \( \alpha_i = 1, \nu_i = -1 \), and for outer insertions, \( \alpha_j = 0, \nu_j = 1 \) (defined in (A.18)). The summation \( \sum_{\nu_i} \) is the summation over all possible inner/outer insertions consisting of \( 2^N \) terms. The summation over all possible permutations of external momenta is encoded in the integration range of \( \tau \) variables, and the \( \ast \nu \) phase factor effect gives the phase factor in the first line where \( \varepsilon(\tau_{ij}) = \text{sign}(\tau_i - \tau_j) \).
The next task is the evaluation of the path integral over $k(\tau)$. If we replace the products of $k(\tau_j) \cdot \epsilon_j$’s with the functional derivatives $\epsilon_j^\mu \delta / \delta (i \dot{x}_\mu(\tau_j))$, the resulting path integral is a Gaussian type that can be straightforwardly evaluated:

$$
\Gamma^{(0)\text{ghost}}_N = - (-2g)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^N \int_0^T d\tau_i \right) e^{\frac{i}{2} \sum_{i<j} p_i \wedge p_j (\nu_i + \nu_j) \varepsilon(\tau_i)} \left( \prod_{j=1}^N (-\alpha_j) \right) \times \int Dx \ e^{i \sum_i p_i x(\tau_i)} N \prod_{j=1}^N \epsilon_j^\mu \frac{\delta}{\delta (i \dot{x}_\mu(\tau_j))} K ,
$$

where $K$ is given by

$$
K = \mathcal{N}(T) e^{-\frac{i}{4} \int_0^T \dot{x}^2 d\tau} \prod_{j=1}^N e^{-\frac{i}{4} \dot{x}^j (\tau_j) \wedge \alpha_j}
$$

with the normalization factor $\mathcal{N}(T)$:

$$
\mathcal{N}(T) = \int Dk \ e^{-\int_0^T k^2 d\tau} \mathcal{N}(T) \int Dx \ e^{-\frac{i}{4} \int_0^T \dot{x}^2 d\tau} = \left( \frac{1}{4\pi T} \right)^{\frac{N}{2}}.
$$

The functional derivatives in (3.8) acting on the exponential part of $K$ generate large number of terms when $N > 1$. In particular, the derivative operator can hit the $\dot{x}^\nu(\tau_j)$ already taken out from the exponential part

$$
\frac{\partial}{\partial \dot{x}_\nu(\tau_i)} \dot{x}_\mu(\tau_j) = g^\nu\mu \delta(\tau_i - \tau_j)
$$

producing extra ‘contact terms’. There are progressively many terms of this kind as $N$ increases, and we write down the result after taking the functional derivatives as

$$
\Gamma^{(0)\text{ghost}}_N = - (-ig)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^N \int_0^T d\tau_i \right) e^{\frac{i}{2} \sum_{i<j} p_i \wedge p_j (\nu_i + \nu_j) \varepsilon(\tau_i)} \left( \prod_{j=1}^N (-\alpha_j) \right) \times \mathcal{N}(T) \int Dx \ e^{-\frac{i}{4} \int_0^T \dot{x}^2(\tau) d\tau} \prod_{j=1}^N \mathcal{O}_j(\tau_j) + \cdots ,
$$

where the gauge particle insertion in a ghost loop is given by $\mathcal{O}_j(\tau)$ defined as

$$
\mathcal{O}_j(\tau) = \epsilon_j^\rho \left( \dot{x}^\rho(\tau) + \Theta^\rho(\tau) \right) \exp \left[ ip_j^\mu \left( x^\mu(\tau) - \frac{i}{2} \alpha_j \theta^{\mu\nu} \dot{x}^\nu(\tau) \right) \right] ,
$$

and

$$
\Theta^\mu(\tau) = \sum_{i=1}^N \theta^{\mu\nu} p_i^\nu \alpha_i \delta(\tau - \tau_i) .
$$
The \( \cdots \) parts that we do not explicitly write down in (3.12) are the contributions involving the aforementioned extra ‘contact terms’.

Our first main combinatorial result shown in Appendix B is that: these extra ‘contact terms’ are completely cancelled when we include the contributions from \( G_2 \). This immediately implies that

\[
\Gamma^\text{ghost}_N = -(-ig)^N \sum_{\{\nu_i\}} \frac{dT}{T} \left( \prod_{i=1}^{N} \int_0^T d\tau_i \right) \frac{1}{\alpha^4} \sum_{i<j} p_i \wedge p_j (\nu_i + \nu_j) \varepsilon(\tau_{ij}) \left( \prod_{j=1}^{N} (-) \alpha_j \right) \times N(T) \int D x e^{-\frac{1}{2}g^2(\tau) d\tau} \prod_{j=1}^{N} \phi_j(\tau_j). \tag{3.15}
\]

![Figure 1: Four types of \( G_2 \) vertices. Whenever nonplanar crossings happen, extra phase factors are attached.](image)

**Figure 1:** Four types of \( G_2 \) vertices. Whenever nonplanar crossings happen, extra phase factors are attached.

### 3.2 Derivation of Wick contraction rule for ghost loop

We will now derive the following formula:

\[
\Gamma^\text{ghost}_N = -(-ig)^N \sum_{\{\nu_i\}} \frac{dT}{T} \left( \prod_{i=1}^{N} \int_0^T d\tau_i \right) \frac{1}{\alpha^4} \sum_{i<j} p_i \wedge p_j (\nu_i + \nu_j) \varepsilon(\tau_{ij}) \left( \prod_{j=1}^{N} (-) \alpha_j \right) \times \left( \frac{1}{4\pi T} \right)^{\frac{D}{2}} \langle \prod_{j=1}^{N} V_j(\tau_j) \rangle_x, \tag{3.16}
\]

where \( V_j \)'s are the “gluon”-ghost vertex operators

\[
V_j(\tau) = e_j^\rho \delta_\rho(\tau) \exp\left[ ip_j \cdot x(\tau) \right]. \tag{3.17}
\]
A rule is that $\langle \prod_{j=1}^{N} V_j(\tau_j) \rangle_\theta$ should be computed with the use of Wick contraction in terms of

$$\langle x^\mu(\tau_i)x^\nu(\tau_j) \rangle_\theta = -G_{\theta}^{\mu\nu}(\tau_i, \tau_j; \alpha_i, \alpha_j),$$

where the ‘Green function’ is given by (A.25):

$$G_{\theta}^{\mu\nu}(\tau_i, \tau_j; \alpha_i, \alpha_j) = g^{\mu\nu}G_B(\tau_i, \tau_j) - \frac{i}{T} \theta^{\mu\nu} \alpha_{ij}(\tau_i + \tau_j) - \frac{1}{4T} \alpha_{ij}^2 (\theta^2)^{\mu\nu},$$

(3.19)

$\alpha_{ij} = \alpha_i - \alpha_j$, and the ordinary worldline Green function $G_B(\tau_i, \tau_j) = |\tau_{ij}| - \tau_{ij}^2/T$. This Wick contraction rule for the ghost loop is remarkably similar to the rules for computing perturbative string amplitudes in the presence of a constant NS two-form background field.

We observe that the propagator (3.19) fits well with the Seiberg-Witten limit of the bosonic string worldsheet propagator [18, 25, 26]. A previous example in commutative space-time that is close to our analysis can be found in Ref. [8] for the scalar and spinor QED cases, where the worldline formalism is adopted and string-theory-like rules for perturbative computations are given. In the noncommutative setup, the $U(1)$ gauge theory behaves much like nonabelian gauge theories; purely gauge degrees of freedom are enough to produce nontrivial answers without adding extra matter fields. At technical level, furthermore, to properly take care of the Filk phases, interaction vertices should be expanded first before performing the path integral over $k(\tau)$. This forces us to introduce the functional derivatives in (3.8) causing extra complications here, compared to the commutative case analysis.

As an application of the rule, we compute the two-point contribution to the amplitude to obtain:

$$\Gamma_2^{\text{ghost}} = g^2(\epsilon_1^\mu \epsilon_2^\nu p_1^\rho p_2^\sigma - \epsilon_1^\nu \epsilon_2^\mu p_1^\sigma p_2^\rho) \int \frac{dT}{T} \left( \frac{1}{4\pi T} \right) \int_0^T d\tau_1 \int_0^T d\tau_2 \times \left\{ \partial_1 G_B^{\mu\nu}(\tau_1, \tau_2; \alpha_1, \alpha_2) \partial_2 G_B^{\rho\sigma}(\tau_2, \tau_1; \alpha_2, \alpha_1) \right\} \times \exp\left[ \frac{1}{2} \sum_{i,j=1}^{2} p_i^{\rho} G_{B\theta}^{\mu\nu}(\tau_i, \tau_j; \alpha_i, \alpha_j) p_j^{\sigma} \right],$$

(3.20)

where we have used an integration by parts. Throughout the rest of this subsection, we will derive the Wick contraction rule.
For a direct proof of (3.16), we consider the evaluation of $x(\tau)$ path integral in (3.15):
\[
I \equiv N(T) \int Dxe^{\frac{1}{2} \int_0^T \dot{x}^2 d\tau} \prod_{j=1}^N \mathcal{O}_j(\tau_j)
\]
with the decomposition
\[
x^\mu(\tau) = x_0^\mu + \sum_{n=1}^\infty x_n^\mu \sin \left( \frac{n\pi \tau}{T} \right),
\]
satisfying the periodicity condition. The path integral under this decomposition becomes
\[
\int Dx \rightarrow \int_{-\infty}^{\infty} \prod_{n=0}^\infty dx_n,
\]
and we have to evaluate a Gaussian integral for each $n > 0$. The zero-mode ($n = 0$) integral gives the momentum conservation condition. These computations proceed parallel to the evaluation of (A.19), however the main difference now is the polynomial (polarization) part in $\mathcal{O}_j(\tau_j)$ involving $\dot{x}$. We notice that the Gaussian integrals for modes involve a variable shift:
\[
\dot{x}^\mu(\tau) \rightarrow \dot{x}^\mu(\tau) + \frac{Ai}{\pi} \sum_{k=1}^N p_k^\mu \sum_{n=1}^\infty \frac{\sin(\frac{n\pi \tau}{T})}{n} \cos(\frac{n\pi \tau}{T}) - \frac{2}{T} \sum_{k=1}^N \theta^{\mu \nu} p_k^\nu \sum_{n=1}^\infty \cos(\frac{n\pi \tau_k}{T}) \cos(\frac{n\pi \tau_k}{T}) - \Theta^\mu(\tau) \cos(\frac{n\pi \tau}{T}),
\]
to ‘complete the square’ in the presence of linear terms in $x$ in the exponential coming from $\mathcal{O}_j(\tau_j)$. One can show that (3.24) can be rewritten as
\[
\dot{x}^\mu(\tau) \rightarrow \dot{x}^\mu(\tau) - i \sum_{k=1}^N p_k^\nu \partial_\tau G^{\mu\nu}_{B\theta}(\tau, \tau_k; \alpha, \alpha_k) - \Theta^\mu(\tau).
\]
This shift transforms the polarization part of (3.13) into
\[
\dot{x}^\mu(\tau_j) + \Theta^\mu(\tau_j) \rightarrow \dot{x}^\mu(\tau_j) - i \sum_{k=1}^N p_k^\nu \partial_\tau G^{\mu\nu}_{B\theta}(\tau_j, \tau_k; \alpha_j, \alpha_k) \equiv \dot{\mathcal{O}}^\mu_j(\tau_j).
\]
\(^3\)When one evaluates the summation over $n$ in the second term of (3.24), there is a subtlety related to the existence of winding numbers along a circle, which should be carefully analyzed. The issue shows up when one tries to use the formula (A.21) in the derivation. In that case, one should cut open the loop so that $\tau$ becomes larger than all of $\tau_j$ to satisfy the condition in (A.21). At the end, in order to join both ends of the line to form a loop, one should reinstall the contribution $\sum_{j=1}^N p_j \varepsilon(\tau - \tau_j)$, which is zero when $\tau > \tau_j$ for all $j$ due to the momentum conservation. At this stage, one can relax the condition $\tau_j \leq \tau$. A safer way is not to use the formula. Either way, we can obtain (3.25).
The second term in $\tilde{O}_j^\nu(\tau_j)$ represents the contractions between $\dot{x}^\mu$ and the exponentials of (3.17). We note that, even if we deleted $k = j$ (self-contraction) from the summation, that contribution actually vanishes. We therefore arrive at the expression

$$\Gamma^\text{ghost}_N = -(-i g)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^N \int_0^T d\tau_i \right) e^{\frac{i}{2} \sum_{i<j} p_i \wedge p_j (\nu_i + \nu_j) \varepsilon(\tau_{ij})} \left( \prod_{j=1}^N (-)^{\alpha_j} \right) \times \left( \frac{1}{4\pi T} \right)^N \left\langle \prod_{i=1}^N \epsilon_i \cdot \tilde{O}_i(\tau_i) \right\rangle \exp \left[ \frac{1}{2} \sum_{i,j=1}^N p_i^\mu G^\mu_{\alpha\beta}(\tau_i, \tau_j; \alpha_i, \alpha_j) p_j^\beta \right],$$

(3.27)

where the contractions between $\dot{x}$'s in $\langle \cdots \rangle$ should be evaluated with the ordinary bosonic worldline Green function $G_B$. Using the relation

$$\langle \dot{x}^\mu(\tau_i) \dot{x}^\nu(\tau_j) \rangle = \langle \dot{x}^\mu(\tau_i) \dot{x}^\nu(\tau_j) \rangle_{\theta}$$

(3.28)

changes this prescription, and we immediately verify that Eq.(3.27) is exactly the same as (3.16).

### 3.3 The gauge loop

We analyze the gauge loop in this subsection. Since the details are similar to those of the ghost loop case, we will mainly highlight the genuine features of the gauge loop. In addition to $G_1$ and $G_2$, we also have the following contributions in the gauge loop case:

$$G_3(k) = 2g \int \frac{d^Dp}{(2\pi)^D} (1 - e^{ik \wedge p}) p_\alpha \hat{A}_\beta(p) \delta^\mu_{\alpha\beta},$$

$$G_4(k) = 2g^2 \int \frac{d^Dp_1}{(2\pi)^D} \int \frac{d^Dp_2}{(2\pi)^D} (1 - e^{ik \wedge p_1})(1 - e^{i(k+p_1) \wedge p_2}) \hat{A}_\alpha(p_1) \hat{A}_\beta(p_2) \delta^\mu_{\alpha\beta},$$

(3.29) (3.30)

which lead to the following path integral expression for (2.15),

$$\Gamma^\text{gauge} = \frac{1}{2} \int \frac{dT}{T} \int Dx Dk \exp \left[ -\int_0^T d\tau \left\{ k^2 - ik \dot{x} + G_1(k) + G_2(k) + G_3(k) + G_4(k) \right\} \right].$$

(3.31)

Here, we introduce $\delta^\mu_{\alpha\beta} = g^\mu\alpha g^{\nu\beta} - g^{\mu\beta} g^\nu\alpha$. To include both (3.29) and the second term of (3.1), the functional derivatives in (3.8) should be modified to

$$\epsilon^\mu_j \frac{\delta}{\delta (i\dot{x}_\mu(\tau_j))} \rightarrow \epsilon^\mu_j \left( \prod \frac{\delta}{\delta (i\dot{x}_\alpha(\tau_j))} + i \sum_{\alpha \beta} \delta^\mu_{\alpha\beta} P_j^\beta \right),$$

(3.32)
where $I$ and $J$ are matrices in the Lorentz space:

$$
(I)^{\mu\nu} = g^{\mu\nu}, \quad (J_{\alpha\beta})^{\mu\nu} = i \delta_{\alpha\beta}^{\mu\nu}.
$$

After the integration of $k(\tau)$, contributions purely from $G_1$ and $G_3$ are expressed as

$$
\Gamma^{(0)\text{gauge}}_N = \frac{1}{2} (-2g)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^N \int d\tau_i \right) e^{i \sum_{i<j} p_i \cdot p_j (\nu_i + \nu_j) \varepsilon(\tau_{ij})} \left( \prod_{j=1}^N (-)^{\alpha_j} \right) \times \text{Tr}_L \int Dx e^{i \sum_i p_i x(\tau_i)} \prod_{j=1}^N \epsilon_j^\alpha \left( \frac{\delta}{\delta (i \dot{\alpha}_\alpha(\tau_j))} + i J_{\alpha\beta} p_j^\beta \right) K, 
$$

where $\text{Tr}_L$ stands for the Lorentz index trace. Similarly, the Lorentz index structure of the contact interactions $G_2$ plus $G_4$ (see (3.2) and (3.30)) should also be generalized from $\hat{A}_\mu \hat{A}_\nu g^{\mu\nu}$ to $\hat{A}_\alpha \hat{A}_\beta (I - 2i J_{\alpha\beta})^{\mu\nu}$. Taking the same procedure as the ghost loop case, these contact interactions can be encapsulated in the full $N$-point expression (Appendix C):

$$
\Gamma^{\text{gauge}}_N = \frac{1}{2} (-ig)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^N \int d\tau_i \right) e^{i \sum_{i<j} p_i \cdot p_j (\nu_i + \nu_j) \varepsilon(\tau_{ij})} \left( \prod_{j=1}^N (-)^{\alpha_j} \right) \times N(T) \int Dx e^{-\frac{T}{2} \dot{x}^2(\tau) d\tau} \text{Tr}_L \left( \prod_{j=1}^N \Phi_j(\tau_j) \right) + \cdots,
$$

where the Lorentz matrix $\Phi_j(\tau)$ is

$$
\Phi_j(\tau) = \epsilon_j^\alpha \left( \{ \dot{x}^\alpha(\tau) + \Theta^\alpha(\tau) \} I + 2 J_{\alpha\beta} p_j^\beta \right) \exp \left[ i p_j^\rho (x_\rho(\tau) - \frac{i}{2} \alpha_j \theta^{\rho\sigma} \dot{x}_\sigma(\tau)) \right].
$$

The $\cdots$ parts of (3.36), here and from now on, are just the pinching contributions from (3.30) given by the replacement rule

$$
(\epsilon_i^\alpha J_{\alpha\beta} p_i^\beta)(\epsilon_j^\gamma J_{\gamma\delta} p_j^\delta) \rightarrow \epsilon_i^\alpha J_{\alpha\beta} \epsilon_j^\delta.
$$

The proof of this rule is based on the following combinatorial observation. Let us consider the two-point function. Take two vertices from the second order term of the exponential series (3.29) and compare with (3.30). The numerical coefficient precisely describes the $G_4$
coupling coefficient: $\frac{1}{2}(2g)^2 = 2g$. This matching behavior can be generalized to arbitrary $N$. We also note the following identity, which will be useful in the next section as well:

$$\int_0^1 \int_0^1 d\tau_i d\tau_j e^{\frac{i}{\hbar}(\nu_i+\nu_j)p_i \wedge p_j \varepsilon(\tau_i) \varepsilon(\tau_j)} \varepsilon(\tau_{ij})$$

$$\int_0^1 \int_0^1 d\tau_i \int_0^{\tau_i} d\tau_j e^{\frac{i}{\hbar}(\nu_i+\nu_j)p_i \wedge p_j \varepsilon(\tau_i) \varepsilon(\tau_j)} - \int_0^1 \int_0^1 d\tau_i \int_0^{\tau_i} d\tau_j e^{\frac{i}{\hbar}(\nu_i+\nu_j)p_i \wedge p_j \varepsilon(\tau_i) \varepsilon(\tau_j)}$$

$$= \frac{1}{2} \int_0^1 d\tau e^{\frac{i}{\hbar}(\nu_i+\nu_j)p_i \wedge p_j} - \frac{1}{2} \int_0^1 d\tau e^{-\frac{i}{\hbar}(\nu_i+\nu_j)p_i \wedge p_j}.$$  \hspace{1cm} (3.39)

We understand the first formal expression in the sense of the point-splitting regularization \[20\]. In (3.39), $\varepsilon(\tau_{ij})$ in the first expression originates from the antisymmetry under the exchange of $i$ and $j$ due to the antisymmetric nature of the matrix $\mathbb{J}_{\alpha\beta}$ (see (3.30) and (3.38)). Due to the relative $-$ sign in the last line of (3.39), a single insertion of contact vertex on the inner boundary ($\nu = -1$) comes with a relative $(-)$ sign compared to the outer insertion ($\nu = +1$) case, precisely the same as an insertion from $G_3$ (3.29).

The rest of the analysis is straightforward. After performing the $x$ integration, we formally have the $N$-point expression

$$\Gamma_{N}^{\text{gauge}} = \frac{1}{2} (-ig)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^{N} \int_0^T d\tau_i \right) e^{\frac{i}{\hbar} \sum_{i<j} p_i \wedge p_j (\nu_i+\nu_j) \varepsilon(\tau_i) \varepsilon(\tau_j)} \left( \prod_{j=1}^{N} (-)^{\alpha_j} \right)$$

$$\times \left[ \frac{1}{4\pi T} \Tr \left( \sum_{i=1}^{N} \hat{\phi}_i(\tau_i) \sum_{\nu \neq \tau} \varepsilon(\tau_{ij}) \prod_{j=1}^{N} p_i^\mu G_{B0}^{\mu\nu}(\tau_i, \tau_j; \alpha_i, \alpha_j, \alpha_k) p_j^\nu + \cdots \right) \right].$$  \hspace{1cm} (3.40)

with the matrix

$$\hat{\phi}_j(\tau_j) = e_j^\alpha \left( \{ \hat{x}^\alpha(\tau_j) - i \sum_{k=1}^{N} \hat{p}_k^\beta \partial_j G_{B0}^{\alpha\beta}(\tau_j, \tau_k; \alpha_j, \alpha_k) \} \mathbb{I} + 2\mathbb{J}_{\alpha\beta} p_j^\beta \right).$$  \hspace{1cm} (3.41)

Following the same procedures as those in the ghost loop case, this is summarized as the Wick contraction formula

$$\Gamma_{N}^{\text{gauge}} = \frac{1}{2} (-ig)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^{N} \int_0^T d\tau_i \right) e^{\frac{i}{\hbar} \sum_{i<j} p_i \wedge p_j (\nu_i+\nu_j) \varepsilon(\tau_i) \varepsilon(\tau_j)} \left( \prod_{j=1}^{N} (-)^{\alpha_j} \right)$$

$$\times \left[ \frac{1}{4\pi T} \Tr \left( \prod_{j=1}^{N} \mathbb{V}_j(\tau_j) \right) \sum_{\nu \neq \tau} \varepsilon(\tau_{ij}) + \cdots \right],$$  \hspace{1cm} (3.42)

where $\mathbb{V}_j$ is the usual bosonic “gluon” vertex operator

$$\mathbb{V}_j(\tau) = e_j^\alpha \left( \hat{x}^\alpha(\tau) \mathbb{I} + 2\mathbb{J}_{\alpha\beta} p_j^\beta \right) \exp \left[ ip_j \cdot x(\tau) \right].$$  \hspace{1cm} (3.43)
As an illustration, we compute the two-point contribution from the gauge loop using (3.42):

\[
\Gamma^{\text{gauge}}_2 = -\frac{1}{2} g^2 (\epsilon_1^\mu \epsilon_2^\nu p_2^\rho p_1^\sigma - \epsilon_1^\mu p_2^\nu \epsilon_2^\rho p_1^\sigma) \int \frac{dT}{T} \left( \frac{1}{4\pi T} \right) \frac{\partial T}{T} \int_0^T d\tau_1 \int_0^T d\tau_2 \times \left\{ D \partial_1 G^{\mu\nu}_{B\theta}(\tau_1, \tau_2; \alpha_1, \alpha_2) \partial_2 G^{\rho\sigma}_{B\theta}(\tau_2, \tau_1; \alpha_2, \alpha_1) + 8g^{\mu\nu} g^{\rho\sigma} \right\}
\times \exp \left[ \frac{1}{2} \sum_{i,j=1}^2 p_i^\mu G^{\mu\nu}_{B\theta}(\tau_i, \tau_j; \alpha_i, \alpha_j) p_j^\nu \right].
\] (3.44)

Adding this contribution to the ghost contribution (3.20), we have the following expression for the self-energy part

\[
\Pi_{\mu\nu} = -\frac{1}{2} g^2 (\epsilon_1^\mu \epsilon_2^\nu p_2^\rho p_1^\sigma - \epsilon_1^\mu p_2^\nu \epsilon_2^\rho p_1^\sigma) \int \frac{dT}{T} \left( \frac{1}{4\pi T} \right) \frac{\partial T}{T} \int_0^T d\tau_1 \int_0^T d\tau_2 \times \left\{ (D-2) \partial_1 G^{\mu\nu}_{B\theta}(\tau_1, \tau_2; \alpha_1, \alpha_2) \partial_2 G^{\rho\sigma}_{B\theta}(\tau_2, \tau_1; \alpha_2, \alpha_1) + 8g^{\mu\nu} g^{\rho\sigma} \right\}
\times \exp \left[ \frac{1}{2} \sum_{i,j=1}^2 p_i^\mu G^{\mu\nu}_{B\theta}(\tau_i, \tau_j; \alpha_i, \alpha_j) p_j^\nu \right].
\] (3.45)

This is identical to the results obtained from the conventional Feynman diagrammatics [17, 25] and from the Seiberg-Witten limit of the perturbative string computations [18, 25, 26].

4 Emergence of open Wilson lines

We have developed an efficient method of computing the multi-point 1PI amplitudes resembling the perturbative string theory computations. We now investigate how our method helps us obtain the gauge invariant completions of various terms. Let us briefly illustrate the idea with the two-point function example (\( p_1^\mu = -p_2^\mu = p^\mu \)) [25]:

\[
\Gamma^{NP}_2 = g^2 (\epsilon_1^\mu \epsilon_2^\nu p_2^\rho p_1^\sigma - \epsilon_1^\mu p_2^\nu \epsilon_2^\rho p_1^\sigma) \int \frac{dT}{T} \left( \frac{1}{4\pi T} \right) \frac{\partial T}{T} \int_0^1 dx \times \left\{ -(D-2)g^{\mu\nu} g^{\rho\sigma}(1-2x)^2 + 8g^{\mu\nu} g^{\rho\sigma} + (D-2)\frac{1}{T^2} \theta^{\mu\nu} \theta^{\rho\sigma} \right\}
\times \exp \left( -p^2Tx(1-x) - \frac{1}{4T} p \circ p \right),
\] (4.1)
written explicitly from (3.45) for the nonplanar case with one inner insertion and one outer insertion. We will find that our results immediately produce the gauge invariant completions
of the second term (and their generalizations) and the third term in the curly bracket of (4.1) in the low momentum limit; open Wilson lines emerge for these types of terms. It is not yet clear how to find the gauge invariant completion of the first term even in the low momentum limit. We will make further comments on this point later.

The correlation functions from the ghost loop (3.16) and the gauge loop (3.42) can be computed to be:

\[
\langle \prod_{j=1}^N V_j(\tau_j) \rangle_{\theta} = \exp \left[ \sum_{i<j}^N \epsilon_i^\mu \epsilon_j^\nu \hat{G}_{B\theta ij} - i \sum_{i,j=1}^N \epsilon_i^\mu p_j^\nu \hat{G}_{B\theta ij} \right]_{m.l.} e^{\frac{1}{2} \sum_{i,j} p_i G_{B\theta i j} p_j} \quad (4.2)
\]

\[
\langle \prod_{j=1}^N \mathbb{V}_j(\tau_j) \rangle_{\theta} = \exp \left[ \left( \sum_{i<j}^N \epsilon_i^\mu \epsilon_j^\nu \hat{G}_{B\theta ij} - i \sum_{i,j=1}^N \epsilon_i^\mu p_j^\nu \hat{G}_{B\theta ij} \right) I + 2 \sum_{i=1}^N \epsilon_i^\mu \mathbb{J}_{\mu \nu} p_i^\nu \right]_{m.l.} \times e^{\frac{1}{2} \sum_{i,j} p_i G_{B\theta i j} p_j} \quad (4.3)
\]

Here the subscript \textit{m.l.}, i.e., “multi-linear”, means that we expand the exponential and retain only the terms which are linear in all \(N\)-polarization vectors \(\mathbb{J}\). The expression \(\hat{G}\) denotes a derivative with respect to the first argument \(\tau\) of \(G\), and \(\ddot{G}\), double derivatives with respect to the same first argument. In fact, using more combinatorics, we can simplify (4.3) further. Exponentiating \(\mathbb{J}\) term in (3.43) and taking Wick contractions, we can re-express the \(\mathbb{J}\) term on the right hand side of (4.3) as

\[
\exp \left[ 4 \sum_{i<j}^N (\epsilon_i^\mu p_i^\nu)(\epsilon_j^\nu p_j^\nu) \right]_{m.l.} \quad (4.4)
\]

and we apply (3.38) to this expression to reproduce the pinching contribution parts. In this way, we can absorb the pinching part \(\cdots\) of (3.42) in a compact form:

\[
\langle \prod_{j=1}^N \mathbb{V}_j(\tau_j) \rangle_{\theta} + \cdots = \exp \left[ \left( \sum_{i<j}^N \epsilon_i^\mu \epsilon_j^\nu \hat{G}_{B\theta ij} - i \sum_{i,j=1}^N \epsilon_i^\mu p_j^\nu \hat{G}_{B\theta ij} \right) I + 2 \sum_{i=1}^N \epsilon_i^\mu \mathbb{J}_{\mu \nu} p_i^\nu + 4 \sum_{i<j}^N \epsilon_i^\mu \mathbb{J}_{\mu \nu} \epsilon_j^\nu (\tau_i - \tau_j) \right]_{m.l.} \times e^{\frac{1}{2} \sum_{i,j} p_i G_{B\theta i j} p_j} \quad (4.5)
\]

The expressions (4.2) and (4.5) will be used for further discussions.

Let us first consider the polarization dependent part of (4.2) and (4.5). The \(\mathbb{J}\)-dependent part of (4.3), that we will call the \((a)\) part, generates the second term of (4.1) at the two-point level. When it comes to \(\hat{G}\) and \(\ddot{G}\) parts of (4.2) and (4.5), there are \(\theta^{\mu \nu}\)-independent

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terms, which will be called the \((b)\) part. The \((b)\) part is responsible for the first term of (4.1), upon using the integration by parts for \(\dot{G}\) (see also [9]). These two parts \((a)\) and \((b)\) are the sources for generating the field strength tensor \(\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\). In addition, there is an extra contribution from \(\dot{G} - 1\).

\[
\sum_{i,j=1}^{N} \epsilon_i^\mu p_j^\nu \theta^{\mu\nu} \alpha_{ij} = -\frac{1}{T} \sum_{i,j=1}^{N} \epsilon_i^\mu \wedge p_j^\nu (\alpha_i - \alpha_j) = \frac{1}{T} \sum_{i,j=1}^{N} \epsilon_i^\mu \wedge p_j^\nu \alpha_{ij},
\]  \hspace{1cm} (4.6)

which depends linearly on \(\theta^{\mu\nu}\) (\(\sum_j p_j = 0\)). One should note that, when compared to the \((a)\) and \((b)\) parts, (4.6) comes with a prefactor \(1/T\), and it is responsible for the third term of (4.1). Similarly, after the scaling \(\tau \to T\), the term \(\epsilon_i^\mu \mathbb{J}_{\mu\nu} \epsilon_j^\nu \delta(\tau_i - \tau_j)\) of (4.3) also has the prefactor \(1/T\). As we increase the number of insertions, each position moduli integral supplies a factor \(T\) after \(\tau \to T\) scaling. For the insertions with the prefactor \(1/T\), increasing the number of insertions does not generate extra powers of \(T\), and this is an important fact that allows the straightforward summation over these terms.

We now show that for the terms (4.6) (\(\theta\)-dependent part of \(\dot{G}\)) the higher-point functions precisely combine to form open Wilson lines in the low momentum limit. We will concentrate on the gauge loop expression for the moment, for the ghost loop contribution (4.2) is a simplified version of (4.3). Since the external insertions are classified into the inner and outer boundary insertions, we introduce a momentum flow between two boundaries

\[
k = \sum_{r=1}^{N_1} p_r = -\sum_{a=1}^{N_2} p_a = -\sum_{i=1}^{N} \alpha_i p_i
\]  \hspace{1cm} (4.7)

where \(N_1\) and \(N_2\); \(N = N_1 + N_2\) are the number of insertions on outer \((N_1, \alpha = 0, \nu = 1; r, s, \cdots)\) and inner \((N_2, \alpha = 1, \nu = -1; a, b, \cdots)\) boundaries, respectively. Scaling the moduli parameters \(\tau_i \to T\tau_i\) and using an identity following from the momentum conservation,

\[
- \imath \sum_{i,j=1}^{N} p_i \wedge p_j \alpha_{ij} (\tau_i + \tau_j) = - \imath \sum_{i,j=1}^{N} p_i \wedge p_j (\nu_i + \nu_j) \tau_{ij}.
\]  \hspace{1cm} (4.8)

we derive the following formula for the \(N\)-point amplitudes:

\[
\Gamma_{N, \{\nu_i\}}^{\text{gauge}} = \frac{1}{2} \int dT \frac{1}{T} \left( \frac{1}{4\pi T} \right)^D \exp \left[ -m^2 T - \frac{k \circ k}{4T} \right]
\]

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\[ \times (-igT)^{N_1} \prod_{r=1}^{N_1} 1 \int_0^1 d\tau_r \exp \left( \frac{i}{2} \sum_{r<s} p_r \wedge p_s \varepsilon(\tau_{rs}) - i p_r \wedge p_s \tau_{rs} \right) \]

\[ \times (igT)^{N_2} \prod_{a=1}^{N_2} 1 \int_0^1 d\tau_a \exp \left( -\frac{i}{2} \sum_{a<b} p_a \wedge p_b \varepsilon(\tau_{ab}) + i p_a \wedge p_b \tau_{ab} \right) \]

\[ \times \text{Tr}_L \exp \left[ \left( \sum_{i<j} \epsilon_i^\mu \epsilon_j^\nu \hat{G}^{\mu\nu}_{B^i_{B^j}} - i \sum_{i,j=1}^N \epsilon_i^\mu p_j^\nu \hat{G}^{\mu\nu}_{B^i_{B^j}} \right) I \right] \]

\[ + 2 \sum_{i=1}^N \epsilon_i^\mu J_{\mu\nu} p_i^\nu + \frac{4}{T} \sum_{i<j}^N \epsilon_i^\mu J_{\mu\nu} \epsilon_j^\nu \delta(\tau_i - \tau_j) \left| \frac{\exp(\sum p_i \cdot p_j G_B)}{m.l.} \right. \]

(4.9)

In this expression, we have introduced the IR cutoff mass \( m^2 \). If one wants to use our \( U(1) \) gauge theory to simulate the broken \( U(1) \) by separating two (bosonic) D-branes, the mass \( m \) could be interpreted as being proportional to the separation distance. As assumed at the outset, we neglect the \( \theta^{\mu\nu} \)-independent \( I \) parts in (4.9), since they are the \( (b) \) part terms. Furthermore, by taking the low momentum limit, the \( G_B = T(|\tau_{ij}| - \tau_{ij}^2) \) part will also be neglected.

We first sum up the terms which have zero number of \( J \)-insertions. From the \( \theta^{\mu\nu} \)-dependent \( I \) part, we derive

\[ \exp \left[ -\frac{1}{T} \left( \sum_{r=1}^{N_1} \epsilon_r \wedge k + \sum_{a=1}^{N_2} \epsilon_a \wedge k \right) I \right] \bigg|_{m.l.} = (-1)^{N_1+N_2} T^{N_1-N_2} \left( \prod_{r=1}^{N_1} \epsilon_r \wedge k \right) \left( \prod_{a=1}^{N_2} \epsilon_a \wedge k \right) I . \]

(4.10)

It is important to note that it comes with negative powers of \( T \), which cancels the positive powers of \( T \) coming from the position moduli integrals. We also note that the corresponding ghost contribution has the same expression as this except for the Lorentz unit matrix \( I \).

Since the two sets of position moduli integrations, the inner and the outer, describe the phase parts of the generalized \( \ast \)-products, \( \ast_{N_1} \) and \( \ast_{N_2} \) \[3, 23, 27\],

\[ J_N(p_1, \ldots, p_N; k) = \left( \prod_{i=1}^N \int_0^1 d\tau_i \right) \exp \left( \frac{i}{2} \sum_{a<b} p_a \wedge p_b \{ \varepsilon(\tau_{ab}) - 2\tau_{ab} \} \right) , \]

(4.11)

and the effective action sums up with the following combinatorics

\[ \Gamma[A] = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\{\nu_i\}} \Gamma_{N,\{\nu_i\}} = \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \frac{1}{N_1!N_2!} \Gamma_{N,\{\nu_i\}} , \]

(4.12)
the effective action can be expressed in terms of the straight open Wilson line \[3\]:

$$W_k[A] = \sum_{N=0}^{\infty} \frac{(ig)^N}{N!} \int \frac{d^Dp_1}{(2\pi)^D} \cdots \int \frac{d^Dp_N}{(2\pi)^D} (2\pi)^D \delta^D(k - \sum_{i=1}^{N} p_i) \times \left[ J_N(p_1, \cdots, p_N; k) (l \cdot \hat{A})(p_1) \cdots (l \cdot \hat{A})(p_N) \right],$$  \hspace{0.5cm} (4.13)

where

$$l^\mu = \theta^{\mu\nu} k_\nu,$$ \hspace{0.5cm} (4.14)

living on each boundary. The planar contributions \(N_1 = 0\) or \(N\) vanish because \(l = 0\) (no momentum flow between two boundaries). The completely factorized \(T\)-integral provides the ‘closed string propagator’ between two boundaries \((n = 0\) for all \(N\)):

$$\mathcal{K}_n(k) := \int_0^{\infty} \frac{dT}{T} \left( \frac{1}{4\pi T} \right)^{\frac{D}{2}} T^n \exp \left[ -m^2 T - \frac{k \circ k}{4T} \right],$$ \hspace{0.5cm} (4.15)

which can be straightforwardly evaluated to yield an expression involving modified Bessel functions \(K_n(z)\)

$$\mathcal{K}_n(k) = 2 \left( \frac{1}{4\pi} \right)^{\frac{D}{2}} \left( \frac{1}{2} \sqrt{\frac{k \circ k}{m^2}} \right)^{n-\frac{D}{2}} K_{n-\frac{D}{2}}(\sqrt{m^2 k \circ k}).$$ \hspace{0.5cm} (4.16)

In the above, we have also defined its derivatives (arbitrary positive \(n\)) for later convenience.

Combining the calculations so far, we obtain the effective action in the following form:

$$\Gamma[A] = \frac{D-2}{2} \int \frac{d^Dk}{(2\pi)^D} W_k[A] \mathcal{K}_0(k) W_{-k}[A],$$ \hspace{0.5cm} (4.17)

where we have included the ghost contribution. The outer boundary \((N_1\text{-summation})\) produces the factor \(W_k[A]\), while the inner boundary gives the factor \(W_{-k}[A]\) \((N_2\text{-summation})\), which is Hermitian conjugate to \(W_k[A]\).

Next, we turn our attention to the terms containing nonzero number of insertions from the \(J\) parts in \([1.9]\), which are also of our interest. The ghost part does not contribute to this case. As noticed in Appendix \([3]\) the \(\delta(\tau_{ij})\)-contact term exists only for pairs inserted on the same boundary (figures (a) and (c) in Figure 1); for other types of insertions, they cancel out due to \(\nu_i + \nu_j = 0\) in \([3.39]\). Hence these parts can also be factorized as

$$\exp \left( 2 \sum_{r=1}^{N_1} \epsilon_r^\mu \epsilon_r^\nu p_r^\mu p_r^\nu + \frac{4}{T} \sum_{r<s}^{N_1} \epsilon_r^\mu \epsilon_r^\nu \epsilon_s^\rho \epsilon_s^\sigma \delta(\tau_{rs}) \right),$$ \hspace{0.5cm} (4.18)
and a similar form for the other boundary. The first summation in (4.18) is identical to 
∂μAμ − ∂νAν with a plane wave substitution Aμ → \( \sum_{r=1}^{N_1} \epsilon^\mu_r \exp(ip_r \cdot x) \). On the same ground, 
the second summation corresponds to a commutator form accompanied by the Filk phase 
factor \( \exp[\frac{i}{2} \sum_{r<s} p_r \wedge p_s \varepsilon(\tau_{rs})] \), which yields the \( \ast \)-commutator between two \( \dot{A}' \)s in a contact 
term (see (3.39)). When there are \( N = 2n_{\text{contact}} + n_{\text{cubic}} \) insertions along one boundary, 
except for the Filk phase factor for the \( \ast \)-commutator, the would-be \( \ast_{N} \)-kernel \( J_N \) of (4.11) 
rearranges itself to \( \ast_{N'} \)-kernel \( J_{N'} \), where \( N' = n_{\text{contact}} + n_{\text{cubic}} \), because of the \( \delta(\tau_{rs}) \) part of a 
contact term insertion. In other words, even if the \( \ast \)-commutator part involves two insertions 
of \( \dot{A}' \)s, it counts as a single insertion when it comes to the \( \ast_{N} \)-kernel. The same is true for 
the counting of the power of \( T \) due to the extra \( 1/T \) factor for the contact term. After all we 
notice that (4.18) is nothing but the Fourier transform of \( 2 \hat{\Phi}_{\mu \nu} \) as naturally expected from 
(2.6). The descendents of an open Wilson line are thus defined as follows 
\[
\mathcal{G}_k^{(n)}[A] = 2^n (ig)^n \sum_{N=0}^{\infty} \frac{(ig)^N}{N!} \left( \prod_{i=1}^{n+N} \int \frac{d^D p_i}{(2\pi)^D} \right) \hat{\Phi}(p_1) \cdots \hat{\Phi}(p_n)(l \cdot \hat{A})(p_{n+1}) \cdots (l \cdot \hat{A})(p_{n+N}) 
\times J_{n+N}(p_1, \cdots, p_{n+N}; k) (2\pi)^D \delta^D(k - \sum_{i=1}^{n+N} p_i), 
\]
where \( (\hat{\Phi})_{\mu \nu} = \hat{F}_{\mu \nu} \) containing the \( \ast \)-commutator term, and obviously \( \mathcal{G}_k^{(0)}[A] = W_k[A] \). 
Dividing \( N \) into \( (N_1 + n_1) + (N_2 + n_2) \), where \( n_i; (i = 1, 2) \) represent the numbers of 
outer/inner \( \Phi \) insertions, the combinatorics for the effective action reads 
\[
\Gamma[A] = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\{\nu_i\}} \Gamma_{N,\{\nu_i\}} = \sum_{N_1} \sum_{N_2} \sum_{n_1} \sum_{n_2} \frac{1}{N_1! N_2! n_1! n_2!} \Gamma_{N,\{\nu_i\}}. 
\]
These considerations immediately yield 
\[
\Gamma[A] = \frac{1}{2} \text{Tr}_k \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_1}}{n_1! n_2!} \int \frac{d^D k}{(2\pi)^D} \mathcal{G}_k^{(n_1)}[A] \mathcal{K}_{n_1+n_2}(k) \mathcal{G}_k^{(n_2)}[A]. 
\]
The result is precisely the gauge invariant completion of field strength tensors in terms of 
the insertion of an open Wilson line for each boundary. 

An outstanding issue is whether one can find the simple gauge-invariant completion of 
the terms involving the field strength coming from the \( (b) \) part. There are two sources of 
complications for the computations; first, the terms \( \hat{G} \cdots \hat{G} \) appear to perturb the expressions
for the $\star_N$ kernel. Secondly, the integration by parts involved in turning $\tilde{G}$'s into $\hat{G}$'s, in general, can generate extra terms. Under any circumstances, it remains to be seen if the analog of the (b) part will be present in the supersymmetric setup. In fact, the terms computed in [19] (and further considered in [3]) appear to be related to the terms from the (a) part involving the four $J$'s. It is amusing to note that the “gluon” vertex operator of \textcolor{red}{(3.43)} is formally similar to the 0-picture gauge boson vertex operator

$$V^0 = g_o(2\alpha')^{-1/2}t^a(i\dot{X}^\mu + 2\alpha' k_\nu \psi^\nu \psi^\mu)e^{ik \cdot X}$$

\textcolor{red}{(4.22)}

of superstring theory, where $t^a$ is the Chan-Paton matrix and $g_o$ is the open string coupling, once we replace the Fermion bilinear $\psi^\nu \psi^\mu$ with $J^{\mu\nu}$. The terms considered in [19] actually originate from the $\psi^\nu \psi^\mu$ part of the 0-picture vertex operators. Regarding the $J$ matrix as the bilinear of worldline fermion fields in our formalism produces the precisely the same answer as that of [19]. It will also be interesting to understand this connection closely, for example, by constructing the supersymmetric version of our formulation.

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Appendix

A The stripping method

In this appendix, we present the details of the stripping method. For simplicity, we choose to consider the noncommutative real scalar field theory with a cubic interaction as a concrete example:

\[ S = \int d^D x \left( \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{3!} \phi \times \phi \times \phi \right)(x) . \] (A.1)

Decomposing \( \hat{\phi} \) (Fourier transform of \( \phi \)) into classical \( \hat{\phi}_0 \) and quantum \( \hat{\varphi} \) fields and adopting the procedure of [28], we have the following one-loop relevant part:

\[ S_{1-loop} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} (2\pi)^D \left\{ \frac{1}{2} (k_1^2 + m^2) \delta^D(k_1 + k_2) \right\} \]
\[ + \frac{g}{4} \int \frac{d^D p}{(2\pi)^D} \delta^D(k_1 + k_2 + p) (e^{\frac{i}{2} k_1 \wedge p} + e^{-\frac{i}{2} k_1 \wedge p}) \hat{\phi}_0(p) \hat{\varphi}(k_1) \hat{\varphi}(k_2) . \] (A.2)

If one regards \( \exp[\pm \frac{i}{2} k_1 \wedge p] \hat{\phi}_0 \) terms as planar and nonplanar interactions, one has to precisely go through the computation of [11]. In this case, there is no phase factor stripping process (the nonstripping method).

The stripping method is based on the following phase space observation. From the formulae for general bosonic functions

\[ \int \varphi_1 \times \phi_0 \times \varphi_2(x) d^D x = \frac{1}{(2\pi)^{2D}} \int d^D k_1 d^D k_2 d^D p \delta^D(k_1 + k_2 + p) \]
\[ \times e^{-\frac{i}{2} k_1 \wedge p} \hat{\varphi}_1(k_1) \hat{\phi}_0(p) \hat{\varphi}_2(k_2) , \] (A.3)
\[ \int \varphi_1 \times \phi_1 \times \phi_2 \times \varphi_2(x) d^D x = \frac{1}{(2\pi)^{4D}} \int d^D k_1 d^D k_2 d^D p_1 d^D p_2 \delta^D(k_1 + k_2 + p_1 + p_2) \]
\[ \times e^{-\frac{i}{2} (k_1 \wedge p_1 + p_2 \wedge k_2)} \hat{\varphi}_1(k_1) \hat{\phi}_1(p_1) \hat{\phi}_2(p_2) \hat{\varphi}_2(k_2) , \] (A.4)

appropriate Fourier bases for the noncommutative determinant appear to be \( e^{-\frac{i}{2} k_1 \wedge p} \) for a cubic vertex (this might be interpreted as a phase factor for functional derivatives of second order) and to be \( e^{-\frac{i}{2} (k_1 \wedge p_1 + p_2 \wedge k_2)} \) for a contact vertex if exists. We hence remove this factor from (A.2) in the three-body interaction. This should be understood as an inclusion of
\textasteriskcentered\textasteriskcentered-operation into the background field (denoted as \(\tilde{\phi}_0(p)\)); we thus have

\[
\Gamma = \ln \det \frac{-1}{2} \left[ (k_i^2 + m^2)\delta^D(k_i + k_j) + \frac{g}{2} \int \frac{d^Dp}{(2\pi)^D} \delta^D(k_i + k_j + p) (1 + e^{ik\wedge p})\tilde{\phi}_0(p) \right]. \quad (A.5)
\]

Alternatively we have an option of not including \(e^{-\frac{i}{2}k_i\wedge p}\) into the background field (nonstripping method):

\[
\Gamma = \ln \det \frac{-1}{2} \left[ (k_i^2 + m^2)\delta^D(k_i + k_j) + \frac{g}{2} \int \frac{d^Dp}{(2\pi)^D} \delta^D(k_i + k_j + p) (e^{\frac{i}{2}k\wedge p} + e^{-\frac{i}{2}k\wedge p})\tilde{\phi}_0(p) \right]. \quad (A.6)
\]

This time, the products between the background fields should be understood as conventional commuting products. This approach was examined in [11], and we will not repeat it here.

Further computations following the approach based on (A.5) should be in order. Even if this process is almost parallel to the nonstripping method, a subtlety should be taken care of; namely, one should first define the notion of \(\textasteriskcentered\textasteriskcentered\text{-products for background fields in the presence of two boundaries in the double-line notation. At tree level, the number of boundary is one. We expand the action assuming a path ordered exponential:

\[
\Gamma = \frac{1}{2} \int \frac{dT}{T} \int DxDk \exp \left[ -\int \frac{T}{0} (k^2 + m^2 - ik\dot{x})d\tau \right] \times \sum_{N=0}^{\infty} \sum_{n=0}^{N} (-\frac{g}{2})^N \prod_{l=1}^{n} \int d\tau_l \int \frac{d^Dp_l}{(2\pi)^D} \hat{\phi}_0(p_l) \prod_{j=1}^{N-n} \int d\tau_j' \int \frac{d^Dp_j'}{(2\pi)^D} \hat{\phi}_0(p_j') \times \exp \left[ -i \sum_{j=1}^{N-n} p'_j \wedge k(\tau_j') \right], \quad (A.7)
\]

where \(\tau_{n+1}\) and \(\tau_{N+1-n}\) are equal to \(T\) for given \(n\) and \(N\). The \(p_l\) and \(p'_j\) are the external momenta corresponding to vertex insertions in either outer or inner boundaries. We shall assign the sign factor \(\nu_l = 1\) to the outer insertion case and \(\nu_j = -1\) to the inner insertion case. The Feynman amplitudes are defined as functions of the set of external momenta

\[
\{ p_i \} = \{ p_i \text{ for } i = 1, 2, \cdots, n; \quad p'_j \text{ for } i = n + 1, \cdots, N \}, \quad (A.8)
\]

\[
\{ \tau_i \} = \{ \tau_l \text{ for } i = 1, 2, \cdots, n; \quad \tau_j \text{ for } i = n + 1, \cdots, N \}. \quad (A.9)
\]

Corresponding to the product \(\prod_{i=1}^{N} \hat{\phi}_0(p_i)\) in (A.7), it is necessary to replace the whole product with

\[
\phi_0(x(\tau_1)) \star' \phi_0(x(\tau_2)) \star' \cdots \star' \phi_0(x(\tau_N)) \quad (A.10)
\]
in the configuration space when performing the plane wave substitution \( \phi_0 \rightarrow \sum_{n=1}^{N} \exp[ip_n x] \), where \( \nu = 1 \) is applied to the products from \( \phi_0(x(\tau_1)) \) to \( \phi_0(x(\tau_n)) \), \( \nu = -1 \) to those from \( \phi_0(x(\tau_{n+1})) \) to \( \phi_0(x(\tau_N)) \), and \( \nu = 0 \) to the products between those two sets. According to the Chan-Paton charge assignment in string theory, for example, when a charge is attached to one boundary, an anti-charge should be attached to the other boundary. Remembering that \( N! \) overcountings occur by definition in the plane wave substitution, we obtain for \( N \) point function part as

\[
\prod_{l=1}^{n} \int_0^{\tau_{l+1}} d\tau_l \hat{\phi}_{0*}(p_l) \prod_{j=1}^{\tau'_{j+1}} \int_0^{\tau_j} d\tau'_{j} \hat{\phi}_{0*}(p'_j) \\
\rightarrow \frac{1}{N!} \prod_{l=1}^{n} \int_0^{\tau_{l+1}} d\tau_l \prod_{j=1}^{N-n} \int_0^{\tau'_{j+1}} d\tau'_{j} \left( e^{ip_1 x(\tau_1)} \ast \nu e^{ip_2 x(\tau_2)} \ast \cdots \ast \nu e^{ip_N x(\tau_N)} \right) \\
+ \text{(all } p_i \text{ permutations)} \right). \tag{A.11}
\]

Furthermore, since we want to take account of all orderings of \( \tau \) as \( \ast \)-product effects, it is very natural to incorporate \( \tau \) dependence into \( \ast \nu \)-product by defining

\[
e^{ip_i x(\tau_i)} \ast \nu e^{ip_j x(\tau_j)} \overset{\text{def.}}{=} \exp\left[-\frac{i}{2} \nu \varepsilon(\tau_{ij}) \theta^{\mu\nu} \partial_{y}^{\mu} \partial_{z}^{\nu} \right] e^{ip_i y} e^{ip_j z} \bigg|_{y=x(\tau_i), z=x(\tau_j)} \\
= \exp\left[\frac{i}{2} p_i \wedge p_j \nu \varepsilon(\tau_{ij}) \right] e^{ip_i x(\tau_i)+ip_j x(\tau_j)} , \tag{A.12}
\]

where \( \nu \) is related to the average

\[
\nu = \frac{\nu_i + \nu_j}{2} , \tag{A.13}
\]

and

\[
\tau_{ij} = \tau_i - \tau_j \tag{A.14}
\]

(see [20] as well for a related string theory discussion). The symbol \( \varepsilon(x) \) picks up the sign of its argument \( x \) and it will be typically understood via the point-splitting regularization. Interchanging integration variables \( \tau_i \), all permutation terms in \( \text{(A.11)} \) can be arranged as all possible ordered integrals having the same integrand. We thus conclude that the right hand side of \( \text{(A.11)} \) is

\[
\frac{1}{N!} \sum_{\{\nu_i\}} \int_0^{T} \cdots \int_0^{T} \int_0^{T} \prod_{j=1}^{N} e^{ip_j x(\tau_j)} , \tag{A.15}
\]

27
where the summation on \( \{ \nu_i \} \) denotes the sum over all possible outer/inner insertions. Here, the ‘stripped’ Filk phase is given by

\[
\Xi = \exp \left[ \frac{i}{4} \sum_{i<j} p_i \wedge p_j (\nu_i + \nu_j) \varepsilon(\tau_{ij}) \right]. \tag{A.16}
\]

The \( N \)-point amplitudes (\( \Gamma_N \equiv \sum \frac{1}{N!} \Gamma_N \)) are therefore obtained as

\[
\Gamma_N = \frac{1}{2} \left( \frac{-g}{2} \right)^N \sum_{\{ \nu_i \}} \int \frac{dT}{T} \left( \prod_{i=1}^{N} \int_0^T d\tau_i \right) \Xi \times \int \mathcal{D} x e^{i \sum_i p_i x(\tau_i)} \int \mathcal{D} k e^{-\int (k^2(\tau)+m^2-i k \dot{x}(\tau)) d\tau} \prod_{j=1}^{N} e^{-i \alpha_j p_j \wedge k(\tau_j)}, \tag{A.17}
\]
where \( \alpha_j \) takes either 0 or 1 for outer and inner boundary insertions, respectively;

\[
\alpha_j = \frac{1 - \nu_j}{2}. \tag{A.18}
\]

Note that in this expression we have new quantities \( \Xi \) and \( \alpha_j \) instead of \( \nu_j \), which do not appear in the nonstripping method \cite{11}. After performing the \( k \) integration, we compute the remaining \( x \) integration (the counterpart in the nonstripping method is written as \( X \)):

\[
\tilde{X} \equiv \int_{x(0)=x(T)} \mathcal{D} x e^{\frac{i}{4} \int_0^T \dot{x}^2 d\tau} \prod_{j=1}^{N} \exp \left[ i p_j \mu \left( x_\mu(\tau_j) - \frac{i}{2} \alpha_j \theta_{\mu\nu} \dot{x}_\nu(\tau_j) \right) \right]. \tag{A.19}
\]

In the present case, we have to use three of the following formulae (assuming \( 0 < \tau_i \pm \tau_j < 2T \)):

\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{1}{4} (|x| - \pi)^2 - \frac{\pi^2}{12}, \quad 0 \leq x \leq 2\pi \tag{A.20}
\]
\[
\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{1}{2} (\pi - x), \quad 0 \leq x < 2\pi \tag{A.21}
\]
\[
\sum_{n=1}^{\infty} \cos n(x-a) = \pi \delta(x-a) - \frac{1}{2}, \quad a - \pi < x < a + \pi \tag{A.22}
\]

In contrast, we use only two of them in the nonstripping method; we use the derivative of (A.20) instead of (A.21) (See Eq.(2.11) in Ref. \cite{11}). Note that the second formula is the only source of \( \wedge \)-product phase factors. Direct computation yields

\[
\tilde{X} = \left( \frac{1}{4\pi T} \right)^\frac{N}{2} \exp \left[ -\frac{T}{4} \sum_{i,j=1}^{N} p_i \cdot p_j \left( \left( 1 - \frac{|\tau_i - \tau_j|}{T} \right)^2 - \left( 1 - \frac{\tau_i + \tau_j}{T} \right)^2 \right) \right]
\]

28
\[ - \frac{i}{2} \sum_{i,j=1}^{N} p_i \wedge p_j \{ \alpha_j \left( 1 - \frac{\tau_i - \tau_j}{T} \right) + \alpha_j \left( 1 - \frac{\tau_i + \tau_j}{T} \right) \} \]
\[ - \frac{1}{4T} \sum_{i,j=1}^{N} p_i \circ p_j \alpha_i \alpha_j \]  
(A.23)

or more compactly

\[ \tilde{X} = \left( \frac{1}{4\pi T} \right)^{\frac{D}{2}} \exp \left[ \frac{i}{2} \sum_{i,j=1}^{N} G^\mu_\nu_{B\theta}(\tau_i, \tau_j; \alpha_i, \alpha_j) p_i^\mu p_j^\nu \right], \]  
(A.24)

where \( G^\mu_\nu_{B\theta} \) is the noncommutative version of \( G^\mu_\nu_B \):

\[ G^\mu_\nu_{B\theta}(\tau_i, \tau_j; \alpha_i, \alpha_j) = g^\mu_\nu G^\mu_\nu_B(\tau_i, \tau_j) - \frac{i}{T} \theta^\mu_\nu \alpha_{ij}(\tau_i + \tau_j) - \frac{1}{4T} \alpha_{ij}^2 (\theta^2)^\mu_\nu, \]  
(A.25)

and \( \alpha_{ij} = \alpha_i - \alpha_j \). By comparing the result with \([11]\), we establish the relation between two methods

\[ X = \Xi \tilde{X}, \]  
(A.26)

and we verify that \((A.17)\) leads to the same results as those presented in \([11]\). The equivalence between these two methods was verified only at one-loop level. To tackle multi-loops using nonstripping method, more subtle ‘branch choice’ similar to the one given by the Appendix of \([19]\) appears necessary. The stripping method is safer in this sense, since the overall Filk phase \((\star^\nu \text{ with multiple boundaries})\) can be unambiguously determined within the purely field theory considerations \([12, 28]\).

### B Cancellation of \( G_2 \) in ghost loop

As seen in \((3.7)\), the nonplanar phase contributions from \( G_1 \) are summarized by the phase factor \( e^{-i\alpha_j p_j \wedge k(\tau_j)} \) at each vertex position \( \tau_j \). When two vertices converge into one position

\[ e^{-i\alpha_1 p_1 \wedge k(\tau_1)} e^{-i\alpha_2 p_2 \wedge k(\tau_2)} \rightarrow e^{-i\alpha_1 p_1 \wedge k(\tau_1)} e^{-i\alpha_2 p_2 \wedge (k(\tau_1) + p_1)}, \]  
(B.1)

where the momentum conservation should be taken into account along with the point-splitting regularization (note that \( k(\tau_1) \neq k(\tau_2) \) as \( \tau_1 \rightarrow \tau_2 \) when \( \tau_1 > \tau_2 \)). This identification \((B.1)\) holds independently of whether two vertices are converging to a point or remain separated, as long as there are not any extra insertions between them. In view of this relation,
the phase factors in $G_2$ are in fact the same as those in $G_1^2$ form. This is clearly seen in the original form of $S_2$ in (2.11), and is rather trivial in the nonstripping approach. Hence we simply attach the overall phase factor $\Xi$, where a pair of converging positions $\tau_i$ and $\tau_j$ are to be understood as having an infinitesimal separation so that $\varepsilon(\tau_{ij})$ can reproduce the phase factors for intertwining pairs. In this way, the phase part of $(G_1)^2$ and $G_2$ perfectly match when two points converge into one vertex. We only have to discuss the rest; namely, we can concentrate on the cancellation problem of $\delta$-functions produced by functional derivatives, simply omitting phase factors by hand.

The above argument leads us to examine the (commutative) scalar QED case, which is known as a fact that all four point contributions are contained in the $\delta$-function of a worldline two-point function: $\dot{x}^\mu(\tau_1)\dot{x}^\nu(\tau_2) = 2g^{\mu\nu}\delta(\tau_{12}) + \langle \dot{x}^\mu(\tau_1)\dot{x}^\nu(\tau_2) \rangle_{\text{regular}}$. Our purpose is to show how the $\delta$-functions (extra ‘contact terms’) generated by functional derivative operation in (3.8) are canceled. The full action that we consider in view of the previous argument is given by

$$
\Gamma^{\text{scalar}} = + \int \frac{dT}{T} \int DkDp \exp\left[ - \int_0^T d\tau \left\{ k^2 - ik\dot{x} + G_1(k) + G_2(k) \right\} \right],
$$

with

$$
G_1(k) = k^2 + 2gk^\mu \int \frac{dp}{(2\pi)^D} \hat{A}_\mu(p),
$$

$$
G_2(k) = g^2 \int \frac{dp_1}{(2\pi)^D} \int \frac{dp_2}{(2\pi)^D} \hat{A}_\mu(p_1)\hat{A}_\nu(p_2)g^{\mu\nu}.
$$

Instead of (3.8) and (3.9), let us consider the following expressions:

$$
\Gamma^{(0)}_N = (-2g)^N \int \frac{dT}{T} \left( \prod_{i=1}^N \int d\tau_i \right) \int Dx \ e^{\int \sum_i \dot{x}^\mu(\tau_i) \int_0^\tau d\tau_j \delta(\dot{x}^\mu(\tau_j))} \mathcal{K},
$$

and

$$
\mathcal{K} = \mathcal{N}(T) e^{-\frac{1}{4} \int_0^T \dot{x}^2 d\tau}.
$$

We can also write down a formula for the $N$-point contributions originated from purely $G_2$ parts (i.e. no inclusion of three-point vertex contributions):

$$
\Gamma^{(1)} = \sum_{n=0}^\infty \frac{(-g^2)^n}{(2n)!} \int \frac{dT}{T} \int DkDx e^{-\int (k^2 - ik\dot{x})}
$$

30
\begin{equation}
\times \prod_{i,j=1,i\neq j}^{n} \int_{0}^{T} d\tau_{i} d\tau_{j} \int \frac{d^{D}p_{i}d^{D}q_{j}}{(2\pi)^{2D}} \hat{A}_{\mu}(p_{i})\hat{A}_{\nu}(q_{j})g^{\mu\nu}\delta(\tau_{i} - \tau_{j}) , \tag{B.7}
\end{equation}

where we have used the fact that the number of ways of inserting \( \delta \)-functions \((2n - 1)!! \) times the number of shuffling external legs \((2n)!! \) is equal to \( N! \) \((N = 2n)\). We thus find:

\[ \Gamma_{N}^{(1)} = (ig)^{N} \int \frac{dT}{T} \int D_{\mathcal{C}} e^{\sum_{n=1}^{\infty} p_{n}(\tau_{n})} \left( \prod_{i=1}^{N} \int_{0}^{T} d\tau_{i} \right) \frac{2n}{n!} \left( \sum_{i<j}^{N} \epsilon_{i} \cdot \epsilon_{j} \delta(\tau_{i} - \tau_{j}) \right) \bigg|_{m.l.} \mathcal{K} , \tag{B.8}\]

where the subscript \( m.l. \) means that only the contributions “multi-linear” in all \( N \) polarization vectors should be retained. In the \( N = 2 \) case, the \( \delta \)-function contribution from (B.9)

\[ (-2g)^{N} \frac{\delta^{2} \mathcal{K}}{\delta(i\dot{x})\delta(i\dot{x})}\bigg|_{\delta(i\dot{x})}\bigg|_{\delta(i\dot{x})} = g^{2} \left( 2g^{\mu\nu}\delta(\tau_{1} - \tau_{2}) - \dot{x}^{\mu}(\tau_{1})\dot{x}^{\nu}(\tau_{2}) \right) \mathcal{K} \tag{B.9} \]

cancels the (B.8). Thus, \( \Gamma_{2} = \Gamma_{2}^{(0)} + \Gamma_{2}^{(1)} \) is given by \( \langle \dot{x}^{\mu}(\tau_{1})\dot{x}^{\nu}(\tau_{2}) \rangle \).

It is useful to see \( N = 3 \) case, where we can observe the cross term cancellation between \( G_{1} \) and \( G_{2} \). Though there is not any \( \Gamma_{3}^{(1)} \) contribution in this case, we have \( \Gamma_{2}^{(1)} \) and \( \Gamma_{1}^{(0)} \) combination (denote \( \Gamma_{N}^{(0+1)} \)), whose integrand is given by multiplying those two integrands. From \( \Gamma_{3}^{(0)} \), we have

\[ \frac{(-2g)^{N} \delta^{3} \mathcal{K}}{\delta(i\dot{x})^{3}\delta(i\dot{x})^{3}} = -ig^{3} \left( 2\delta_{12}\dot{x}_{3}g^{\mu\nu} + 2\delta_{13}\dot{x}_{2}g^{\mu\nu} + 2\delta_{23}\dot{x}_{1}g^{\mu\nu} + \dot{x}_{1}\dot{x}_{2}\dot{x}_{3} \right) \mathcal{K} , \tag{B.10}\]

where \( \dot{x}_{i}^{\mu} \equiv \dot{x}^{\mu}(\tau_{i}) \) and \( \delta_{ij} \equiv \delta(\tau_{ij}) \). From the cross terms, we have the integrand for \( \Gamma_{3}^{(0+1)} \):

\[ \left( (-2g)\frac{i}{2}\dot{x}_{1}^{\mu} \right) \left( (ig)^{2}2\epsilon_{2} \cdot \epsilon_{3} \delta_{23} \right) + \text{cyclic permutations} . \tag{B.11}\]

Again the \( \delta \)-function cancellation happens in \( \Gamma_{3} = \Gamma_{3}^{(0)} + \Gamma_{3}^{(0+1)} \), which is thus given by \( \langle \dot{x}_{1}\dot{x}_{2}\dot{x}_{3} \rangle \). Note also that (B.9) and (B.10) are regular parts themselves.

In the \( N = 4 \) case, we have three kinds of contributions \( \Gamma_{4}^{(0)} \), \( \Gamma_{4}^{(1)} \) and \( \Gamma_{4}^{(0+1)} \), whose integrand is given by those of \( \Gamma_{2}^{(0)} \) and \( \Gamma_{2}^{(1)} \):

\[ \Gamma_{4}^{(0)} \sim (-2g)^{4} \epsilon_{1}^{\mu} \epsilon_{2}^{\nu} \epsilon_{3}^{\sigma} \frac{\delta^{4} \mathcal{K}}{\delta(i\dot{x})^{4}\delta(i\dot{x})^{4}} = 2g^{4} \left( \sum_{i<j}^{N} \epsilon_{i} \cdot \epsilon_{j} \delta(\tau_{i} - \tau_{j}) \right) \bigg|_{m.l.} - 2g^{4} \left( \sum_{i<j}^{N} (\epsilon_{i} \cdot \epsilon_{j}) \delta_{ij} \right) \left( \sum_{k<l}^{N} \epsilon_{k} \cdot \dot{x}_{k} \epsilon_{l} \cdot \dot{x}_{l} \right) \bigg|_{m.l.} \]

\[ + g^{4} \epsilon_{1} \cdot \dot{x}_{1} \epsilon_{2} \cdot \dot{x}_{2} \epsilon_{3} \cdot \dot{x}_{3} \epsilon_{4} \cdot \dot{x}_{4} . \tag{B.12}\]
\[
\Gamma_4^{(0+1)} \sim \left( g^2 \sum_{i<j}^4 (\epsilon_i^\mu \epsilon_j^\nu) (2g^{\mu\nu} \delta_{ij} - \dot{x}_i^\mu \dot{x}_j^\nu) \right) \left( (ig)^2 \sum_{j<k}^4 2\epsilon_j \cdot \epsilon_k \delta_{jk} \right) \bigg|_{m.l.} \\
= -4g^4 \left( \sum_{i<j}^4 \epsilon_i \cdot \epsilon_j \delta_{ij} \right)^2 \bigg|_{m.l.} + 2g^4 \left( \sum_{i<j}^4 \epsilon_i \cdot \epsilon_j \delta_{ij} \right) \left( \sum_{k<l}^4 \epsilon_k \cdot \dot{x}_k \epsilon_l \cdot \dot{x}_l \right) \bigg|_{m.l.} , \quad (B.13)
\]

\[
\Gamma_4^{(1)} \sim (ig)^4 2 \left( \sum_{i<j}^4 \epsilon_i \cdot \epsilon_j \delta_{ij} \right)^2 \bigg|_{m.l.} \\
= 4g^4 \left( (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)\delta_{12}\delta_{34} + (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4)\delta_{13}\delta_{24} + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot \epsilon_4)\delta_{23}\delta_{14} \right) .
\]

Gathering these up, all \(\delta\)-functions cancel except for the part \(\langle \dot{x}_1^\mu \dot{x}_2^\nu \dot{x}_3^\rho \dot{x}_4^\sigma \rangle\), which gives \(\Gamma_4\). It is not difficult to generalize the above expressions for higher values of \(N\); one can check the cancellations in the cases of higher values of \(N\) straightforwardly.

Finally, in order to confirm the \(N\)-point function \((3.15)\), we have to replace \((B.6)\) with \((3.9)\) in the above argument. The derivative of \(K\) then generates an additional \(\Theta^\mu(\tau)\) term (see \((B.14)\) for definition):

\[
\frac{\delta K}{\delta (\dot{x}_\mu(\tau_1))} = \frac{i}{2} \left( \dot{x}^\mu(\tau_1) + \Theta^\mu(\tau_1) \right) K \overset{\text{def.}}{=} \frac{i}{2} \nu^\mu(\tau_1) K . \quad (B.15)
\]

One may wonder if the derivative singularities proportional to \(\Theta^\mu\) survive. However \(\Theta^\mu\) appears symmetric in the exchange \(\dot{x}^\mu(\tau) \leftrightarrow \Theta^\mu(\tau)\) as understood from \((B.17)\). Since all derivative singularities proportional to \(\dot{x}\) vanish in the above argument, this kind of new terms are also canceled out in a parallel way. We therefore conclude that the total \(N\)-point contribution of a ghost loop is given by \((3.15)\).

### C Cancellation of \(G_2\) in gauge loop

Denoting \(G_2 + G_4\) contribution as \(\Gamma^{(1)}\) similar to \((B.7)\), let us define

\[
\Gamma^{(1)}_{\text{gauge}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-g^2)^n}{(2n)!} \int \frac{dT}{T} \int DkDx e^{-\int (k^2 - ik\dot{x})} \prod_{i,j=1,i\neq j}^{n} \int_0^T d\tau_i d\tau_j \times \text{Tr}_L \int \frac{d^Dp_i d^Dq_j}{(2\pi)^{2D}} \hat{A}_\alpha(p_i) \hat{A}_\beta(q_j) \left( \frac{1}{2} \int_{\mu\nu} \alpha^\mu \delta(\tau_i - \tau_j) \right) . \quad (C.1)
\]
The changes from the ghost loop case (Appendix B) are the overall factor \(-\frac{1}{2}\) and the Lorentz structure \((3.35)\). Here the local phase factors of four Feynman diagram combinations in Figure. 1 are ignored for the same reason as before, and the overall phase factor \(\Xi\) is also omitted for simplicity: one should notice that this omission is irrelevant if one follows the nonstripping approach of Appendix A. The \(N (= 2n)\) point functions for \((C.1)\) are then defined as

\[
\Gamma^{(1)\text{gauge}}_N = \frac{1}{2} (ig)^N \int \frac{dT}{T} \int \mathcal{D}x e^i \sum_i p_i x(\tau_i) \left( \prod_{i=1}^{N} \int_{0}^{T} d\tau_i \right) \times \frac{1}{n!} \text{Tr}_L \left( \sum_{i,j} \epsilon_i^\alpha \epsilon_j^\beta (\mathbb{J} - 2iJ_{\mu
u})^{\alpha\beta} \delta(\tau_i - \tau_j) \right) \bigg|_{m.l.} K .
\]  

(C.2)

Suppose that a pair of converging points are separated by point splitting, and the overall phase factor \(\Xi\) is then attached in the above expression. In the case when a pair of external lines are inserted on different boundaries, the pair does not contribute to \(\Xi\) (see \((3.39)\) with \(\nu_i + \nu_j = 0\)); such kinds of four-point contribution from \((C.2)\) vanish due to the antisymmetric nature of the \(J\) matrix. Other four-point contributions from the \(J\) term survive. These nonvanishing contributions have nothing to do with the following argument on cancellations of the derivative-induced \(\delta\)-functions. We hence consider the following expression for this purpose:

\[
\Gamma^{(0)\text{gauge}}_N = 1/2 (-2g)^{N/2} \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^{N} \int_{0}^{T} d\tau_i \right) \frac{D2^n}{n!} \left( \sum_{i<j} \epsilon_i^\alpha \epsilon_j^\beta (\tau_i - \tau_j) \right) \bigg|_{m.l.} K + \cdots .
\]  

(C.3)

Let us consider the \(\Gamma^{(0)}\) part without phase factors,

\[
\Gamma^{(0)\text{gauge}}_N = 1/2 (-2g)^N \sum_{\{\nu_i\}} \int \frac{dT}{T} \left( \prod_{i=1}^{N} \int_{0}^{T} d\tau_i \right) \times \text{Tr}_L \int \mathcal{D}x e^i \sum_i p_i x(\tau_i) \prod_{j=1}^{N} \epsilon_j^\beta \left( \frac{\delta}{\delta (\tau_j)} \right) + iJ_{\alpha\beta} \delta_{\mu\nu}^\alpha \delta_{\mu\nu}^\beta \bigg|_{m.l.} K ,
\]  

(C.4)

and define the cross term contribution \(\Gamma^{(0+1)\text{gauge}}_n\) whose integrand is given by multiplying those of \(\Gamma^{(0)\text{gauge}}_{n-n}\) and \(\Gamma^{(1)\text{gauge}}_n\):

\[
\Gamma^{(0+1)\text{gauge}}_n = 1/2 \sum_{n=1}^{N} \sum_{\{\nu_i\}} \int \frac{dT}{T} \int \mathcal{D}x e^i \sum_i p_i x(\tau_i) \prod_{i=1}^{N} \int_{0}^{T} d\tau_i \bigg|_{m.l.} K .
\]
where the orderings of contact term insertions should be understood properly, although it is not explicit here. In the same way as in Appendix B, the $\Gamma_{N}^{(1)}_{\text{gauge}}$ and $\Gamma_{N}^{(0+1)}_{\text{gauge}}$ contributions cancel the $\delta$-functions produced by $\frac{\delta}{\delta x}$ in (C.4). It is straightforward to explicitly verify this up to $N = 4$, and further generalizations are also possible. Gathering (C.3), (C.4) and (C.5), we therefore conclude that the $N$-point function is given by (3.36).

References

[1] J. A. Harvey, hep-th/0102076;
M. R. Douglas and N. A. Nekrasov, hep-th/0106048;
R. J. Szabo, hep-th/0109162

[2] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, Nucl. Phys. B 573, 573 (2000), hep-th/9910004;
J. Ambjorn, Y. M. Makeenko, J. Nishimura and R.J. Szabo, JHEP 0005, 023 (2000), hep-th/0004147;
S.-J. Rey and R. von Unge, Phys. Lett. B 499, 215 (2001), hep-th/0007089;
S. R. Das, S.-J. Rey, Nucl. Phys. B 590, 453 (2000), hep-th/0008042;
D. J. Gross, A. Hashimoto, N. Itzhaki, Adv. Theor. Math. Phys. 4, 893 (2000), hep-th/0008075;
Y. Okawa and H. Ooguri, Nucl. Phys. B 599, 55 (2001), hep-th/0012218

[3] H. Liu, Nucl. Phys. B 614, 305 (2001), hep-th/00111213

[4] D. Bigatti and L. Susskind, Phys. Rev. D 62, 066004 (2000), hep-th/9908056;
Z. Yin, Phys. Lett. B 466, 234 (1999), hep-th/9908152;
H. Liu and J. Michelson, Phys. Rev. D 62, 066003 (2000), hep-th/0004013.

[5] Z. Bern and D.A. Kosower, Nucl. Phys. B 379, 451 (1992).
[6] Z. Bern, Phys. Lett. B 296, 85 (1992).

[7] M. J. Strassler, Nucl. Phys. B 385, 145 (1992).

[8] M. G. Schmidt and C. Schubert, Phys. Rev. D 53, 2150 (1996), hep-th/9410100.

[9] H.-T. Sato, M. G. Schmidt, C. Zahlten, Nucl. Phys. B 579, 492 (2000), hep-th/0003070.

[10] Y. Kiem, S.-J. Rey, H.-T. Sato and J.-T. Yee, Phys. Rev. D 65, 026002 (2002), hep-th/0106121.

[11] Y. Kiem, S.-J. Rey, H.-T. Sato and J.-T. Yee, hep-th/0107106, to appear in Euro. Phys. Journ. C.

[12] Y. Kiem, S.S. Kim, S.-J. Rey and H.-T. Sato, hep-th/0110066.

[13] Y. Kiem, S. Lee, S.-J. Rey and H.-T. Sato, hep-th/0110213, to appear in Phys. Rev. D.

[14] M. van Raamsdonk, JHEP 0111, 006 (2001), hep-th/0110093.

[15] A. Armoni and E. Lopez, hep-th/0110113.

[16] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP 0002, 020 (2000), hep-th/9912072.

[17] M. Hayakwa, hep-th/9912167.
   A. Matusis, L. Susskind and N. Toumbas, JHEP 0012, 002 (2000), hep-th/0002075.
   A. Armoni, Nucl. Phys. B 593, 229 (2001), hep-th/0005208.
   C. P. Martin and F. Ruiz Ruiz, Nucl. Phys. B 597, 197 (2001), hep-th/0007131.
   M. Pernici, A. Santambrogio and D. Zanon, Phys. Lett. B 504, 131 (2001), hep-th/0011140.
   D. Zanon, Phys. Lett. B 502, 265 (2001), hep-th/0012009.
   F. R. Ruiz, Phys. Lett. B 502, 274 (2001), hep-th/0012171.
   K. Landsteiner, E. Lopez and M. H. Tytgat, JHEP 0106, 055 (2001), hep-th/0104133.

[18] Y. Kiem and S. Lee, Nucl. Phys. B 586, 303 (2001), hep-th/0003145.
[19] H. Liu and J. Michelson, Nucl. Phys. B 614, 279 (2001), hep-th/0008205.

[20] N. Seiberg and W. Witten, JHEP 9909, 032 (1999), hep-th/9908142.

[21] H. Liu and J. Michelson, Phys. Lett. B 518, 143 (2001), hep-th/0104139;
H. Liu and J. Michelson, Nucl. Phys. B 614, 330 (2001), hep-th/0107172.

[22] L. Jiang and E. Nicholson, hep-th/0111143.

[23] Y. Kiem, S. Lee and D.-H. Park, Phys. Rev. D 63, 126006 (2001), hep-th/0011233.

[24] L. F. Abbot, Nucl. Phys. B 185, 189 (1981);
Z. Bern and D. C. Dunbar, Nucl. Phys. B 379, 567 (1992).

[25] J. Gomis, M. Kleban, T. Mehen, M. Rangamani and S. Shenker, JHEP 0008, 011 (2000), hep-th/0003213.

[26] A. Bilal, C-S. Chu and R. Russo, Nucl. Phys. B 582, 65 (2000), hep-th/0003180;
C.-S. Chu, R. Russo, S. Sciuto, Nucl. Phys. B 585, 193 (2000), hep-th/0004183;
S. Chaudhuri and E. G. Novak, JHEP 0008, 027 (2000), hep-th/0006014.

[27] T. Mehen and M. B. Wise, JHEP 0012, 008 (2000), hep-th/0010204.

[28] I. Chepelev and R. Roiban, JHEP 0005, 037 (2000), hep-th/9911098.