Symmetry constraints for real dispersionless Veselov-Novikov equation

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Abstract

Symmetry constraints for dispersionless integrable equations are discussed. It is shown that under symmetry constraints the dispersionless Veselov-Novikov equation is reduced to the 1 + 1-dimensional hydrodynamic type systems.

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1 Introduction.

Dispersionless integrable equations have attracted recently a considerable interest (see e.g. [1]-[5]). They arise in various fields of physics, mathematical physics and applied mathematics. Several methods and approaches have been used to study dispersionless systems, from the quasi-classical Lax pair representation with its close relationship with the Whitham universal hierarchy [2, 3] to the quasi-classical version of the inverse scattering method. In particular, the quasi-classical $\partial$-dressing method [6, 7, 8], recently formulated, is a general and systematic approach to construct dispersionless
integrable systems and to find their solutions. On the other hand a reduction method (see e.g. [9, 10, 11]) provides us also with the effective way to solve 2 + 1-dimensional dispersionless equations. It was shown [12] that symmetry constraints for dispersionless equations provide us with an efficient way to construct reductions. In [12] certain hydrodynamic type reductions of dispersionless Kadomtsev-Petviashvili (dKP) and dispersionless two-dimensional Toda Lattice (2DdTl) equations have been constructed using symmetry constraints.

In this paper we study symmetry constraints for dispersionless Veselov-Novikov equation (dVN) and analyze the corresponding hydrodynamic type equations.

In section 2 we remind the definition of symmetry constraint. In sections 3 and 4 symmetry constraints for soliton (dispersive) and dispersionless equations respectively are considered, where the Kadomtsev-Petviashvili (KP) and dKP equation are considered as examples. Symmetry constraints for real dVN equation are discussed in section 5. In section 6 we demonstrate how symmetry constraints for dVN equation allow us to reduce this 2+1-dimensional equation to 1+1-dimensional hydrodynamic type systems.

2 Symmetry constraints.

Let us consider a partial differential equation for the scalar function \( u = u(t) = u(t_1, t_2, \ldots) \)

\[
F(u, u, u_{t_1}, u_{t_2}, \ldots) = 0, \tag{1}
\]

where \( u_{t_i} = \partial u/\partial t_i \). By definition, a symmetry of equation (1) is a transformation \( u(t) \rightarrow u'(t') \), such that \( u'(t') \) is again a solution of (1) (for more details see e.g. [13]). Infinitesimal continuous symmetry transformations

\[
\delta t_i = t_i + \delta t_i; \quad \delta u = u + \epsilon \delta u = u + \epsilon u_\epsilon. \tag{2}
\]

are defined by the linearized equation (1) \( L \delta u = 0 \), (3)

where \( L \) is the Gateaux derivative of \( F \)

\[
L \delta u := \frac{dF}{d\epsilon} \left. \left( u + \epsilon u_\epsilon, \frac{\partial}{\partial t_i} (u + \epsilon u_\epsilon), \ldots \right) \right|_{\epsilon=0}. \tag{4}
\]
Any linear superposition \( \delta u = \sum_k c_k \delta_k u \) of infinitesimal symmetries \( \delta_k u \) is, obviously, an infinitesimal symmetry. By definition a \textit{symmetry constraint} is a requirement that certain superposition of infinitesimal symmetries vanishes, i.e.

\[
\sum_k c_k \delta_k u = 0. \tag{5}
\]

Since null function is a symmetry of equation (1), the constraint (5) is compatible with equation (1). Symmetry constraints allow us to select a class of solutions which possess some invariance properties. For instance, well-known symmetry constraint \( \delta u = \epsilon u_{t_k} = 0 \), selects solutions which are stationary with respect to the “time” \( t_k \).

3 Soliton equations.

Symmetry constraints for 2 + 1-dimensional soliton equations have been discussed the first time in the papers [14, 15]. Here, we discuss the KP equation equation (see e.g. [16])

\[
\begin{align*}
    u_t &= \frac{3}{2} uu_x + u_{xxx} + \frac{3}{4} \omega_y, \\
    \omega_x &= u_y, \tag{6}
\end{align*}
\]

where \( x := t_1, \ y := t_2 \) and \( t := t_3 \). KP equation (6) is equivalent to the compatibility of the following linear problems [16]

\[
\begin{align*}
    \psi_y &= \psi_{xx} + u\psi, \\
    \psi_t &= \psi_{xxx} + \frac{3}{2} u\psi_x + \left( \frac{3}{2} u_x + \frac{3}{4} \omega \right) \psi. \tag{7}
\end{align*}
\]

The symmetries equation (3) for KP assumes the form

\[
\begin{align*}
    (\delta u)_t &= \frac{3}{2} (u_x \delta u + u(\delta u)_x) + (\delta u)_{xxx} + \frac{3}{4} (\delta \omega)_y, \tag{8} \\
    (\delta \omega)_x &= (\delta u)_y. \tag{9}
\end{align*}
\]

Now, introducing the adjoint linear problems of (7) defined as

\[
\begin{align*}
    -\psi_y^* &= \psi_{xx}^* + u\psi^*, \\
    \psi_t^* &= \psi_{xxx}^* + \frac{3}{2} u\psi_x^* + \left( \frac{3}{2} u_x - \frac{3}{4} \omega \right) \psi^*, \tag{10}
\end{align*}
\]
one can verify directly that the function \( \phi = (\psi \psi^*)_x \) obeys the linearized KP equation (8), i.e. \( (\psi \psi^*)_x \) is an infinitesimal symmetry of the KP equation. A class of symmetry constraint can be taken as

\[
u_n = (\psi \psi^*)_x, \quad n = 1, 2, 3.
\]

(11)

The simplest of them is

\[
u_x = (\psi \psi^*)_x, \quad (12)
\]

which can be integrated to

\[u = \psi \psi^*. \quad (13)\]

Substituting (13) in the first equation of (7) and its adjoint (10), one obtains

\[
\psi_y + \psi_{xx} + \psi^2 \psi^* = 0 \quad (14)
\]

\[
-\psi_y^* + \psi_{xx}^* + (\psi^*)^2 \psi = 0, \quad (15)
\]

that is the AKNS system (16), which is reduced to the nonlinear Schrödinger equation if \( \psi^* = \bar{\psi} \), where the “bar” means complex conjugation.

Substituting (13) in the second equations of linear problems and its adjoint and observing that \( \omega = \psi_x^* \psi - \psi_x \psi^* \), one gets the higher AKNS system

\[
\psi_t = 3\psi \psi^* \psi_x + \psi_{xxx} \quad (16)
\]

\[
\psi_t^* = 3\psi \psi^* \psi_x^* + \psi_{xxx}^*. \quad (17)
\]

It is a straightforward check that if \( \psi \) and \( \psi^* \) obey equations (14)-(17), then \( u = \psi \psi^* \) solves KP equation. Thus, symmetry constraints can be used to find solutions of 2+1-dimensional system using solutions of the 1+1-dimensional integrable systems.

4 Nonlinear dispersionless equations.

The dispersionless limit of soliton equations can be performed introducing slow variables, formally substituting \( t_n \rightarrow \epsilon^{-1} t_n \), and looking for solutions having a certain behavior when \( \epsilon \rightarrow 0 \), for instance

\[
u \left( \frac{t_n}{\epsilon} \right) \rightarrow u(t_n) + O(\epsilon), \quad \epsilon \rightarrow 0. \quad (18)
\]
For example, the dispersionless limit of KP equation is
\begin{equation}
\begin{aligned}
    u_t &= \frac{3}{2} uu_x + \frac{3}{4} \omega_y \\
    \omega_x &= u_y.
\end{aligned}
\end{equation}
(19)

The dispersionless limit of an integrable equation corresponds to the quasi-classical limit of the corresponding linear problems. In fact, representing the solution \( \psi \) of (7) as
\begin{equation}
\psi = \psi_0 \exp \left( \frac{S}{\epsilon} \right),
\end{equation}
(20)
where \( S(\lambda; x, y, t) \to S(\lambda; x, y, t) + O(\epsilon) \), and \( \lambda \) is the so-called spectral parameter, in the limit \( \epsilon \to 0 \) one gets from (7) the following pair of Hamilton-Jacobi type equations
\begin{equation}
\begin{aligned}
    S_y &= S_x^2 + u \\
    S_t &= S_x^3 + \frac{3}{2} u S_x + \frac{3}{2} u_x + \frac{3}{4} \omega,
\end{aligned}
\end{equation}
(21)
where \( \omega_x = u_y \). The compatibility condition for (21) is just the dKP equation (19). Similarly to the dispersionful case, we have the linearized dKP
\begin{equation}
\begin{aligned}
    (\delta u)_t &= \frac{3}{2} (u_x \delta u + u(\delta u)_x) + \frac{3}{4} (\delta \omega)_y, \\
    (\delta \omega)_x &= (\delta u)_y,
\end{aligned}
\end{equation}
(22)
whose solutions are infinitesimal symmetries of dKP.

**Theorem 1** Given any solutions \( S_i \) and \( \tilde{S}_i \) of the Hamilton-Jacobi equations (21), the quantity \( \delta u = \sum_{i=1}^{N} c_i \left( S_i - \tilde{S}_i \right)_{xx} \), where \( c_i \) are arbitrary constants, is a symmetry of dKP equation.

*Proof.* It is straightforward to check that \( \left( S_i - \tilde{S}_i \right)_{xx} \) satisfies equation (22). \( \square \)

This type of symmetries has been introduced for the first time in [12], within the quasiclassical \( \bar{\partial} \)-dressing approach. A simple example of symmetry constraint for dKP, parallel to [12], is
\begin{equation}
    u_x = S_{xx},
\end{equation}
(23)
Under this constraint the Hamilton-Jacobi system \((21)\) gives rise to the following hydrodynamic type system (the dispersionless nonlinear Schrödinger equation) \([1]\)

\[
\check{u}_y = (\check{u}^2 + u)_x, \\
u_y = 2 (\check{u}u)_x,
\]

(24)

where \(\check{u} = \partial S_x/\partial \lambda\).

5 Real dVN equation.

The Veselov-Novikov (VN) equation has been introduced as the two-dimensional integrable extension of KdV in 1984 \([18]\). It looks like

\[
u_t = (uV)_z + (u\bar{V})_\bar{z} + u_{zzz} + u_{\bar{z}\bar{z}\bar{z}} \tag{25}
\]

\[V_\bar{z} = -3u_z, \tag{26}\]

where \(z = x + iy\). It is equivalent to the compatibility condition for equations

\[
\psi_{zz} = u\psi, \tag{27}
\]

\[
\psi_t = \psi_{zzz} + \psi_{\bar{z}\bar{z}\bar{z}} + (V\psi_z) + (\bar{V}\psi_{\bar{z}}), \tag{28}
\]

The VN equation has applications in differential geometry \([19, 20]\). Recently, it was shown that the dVN equation governs the propagation of light in a special class of nonlinear media in the limit of geometrical optics \([21]\).

The dVN equation can be obtained as slow variables expansion of the VN equation \((25)\). Setting \(\psi = \psi_0(\lambda, \epsilon^{-1}z, \epsilon^{-1}\bar{z}, \epsilon^{-1}t) \exp \epsilon^{-1}S(\lambda, z, \bar{z}, t)\) just like in the previous section, one has the following pair of Hamilton-Jacobi equations \([2, 8]\)

\[
S_zS_{\bar{z}} = u, \tag{29}
\]

\[
S_t = S_z^3 + S_{\bar{z}}^3 + VS_z + \bar{V}S_{\bar{z}}, \tag{30}
\]

and the equation

\[
u_t = (uV)_z + (u\bar{V})_\bar{z} \tag{31}
\]

\[V_\bar{z} = -3u_z.\]
In his paper we consider the case of real-valued $u$.

Linearized version of (31) is of the form

\[
(\delta u)_t = (V \delta u + u \delta V)_z + (\bar{V} \delta u + \delta V u)_{\bar{z}}
\]

\[V_{\bar{z}} = -3u_z; \quad (\delta V)_{\bar{z}} = -3(\delta u)_z.\]  

(32)

**Theorem 2** Given any solutions $S_i$ and $\tilde{S}_i$ of the Hamilton-Jacobi equations (29)-(30), the quantity

\[
\delta u = \sum_{i=1}^{N} c_i \left(S_i - \tilde{S}_i\right)_{z\bar{z}},
\]

where $c_i$ are arbitrary constants, is a symmetry of dVN equation.

**Proof.** It is straightforward to check that $\left(S_i - \tilde{S}_i\right)_{z\bar{z}}$ satisfies equation (32). \hfill \Box

In particular, one can choose $S_i = S(\lambda = \lambda_i)$ and $\tilde{S}_i = S(\lambda = \lambda_i + \mu_i)$. In the case $\mu_i \to 0$ and $c_i = \bar{c}_i / \mu_i$, one has the class of symmetries given by

\[
\delta u = \sum_{i=1}^{N} \bar{c}_i \phi_{i z\bar{z}}
\]

\[
\phi_i = \frac{\partial S}{\partial \lambda} (\lambda = \lambda_i).
\]

(34)

(35)

In what follows we will discuss three particular cases of real reductions, providing real solutions of dVN.

If $S$ is a solution of Hamilton-Jacobi equations (29), then $-\bar{S}$ is a solution as well. Thus, for real-valued $S$ ($S = \bar{S}$), specializing constraint (33) for $N = 1$, we have a simple constraint

**Case I**

\[u_x = (S)_{z\bar{z}}.\]  

(36)

For complex valued $S$ one has the constraint

**Case II**

\[u_x = \frac{1}{2} (S + \bar{S})_{z\bar{z}}.\]  

(37)

The last example of constraint is nothing but a particular case of (34), i.e.

**Case III**

\[u_x = \phi_{z\bar{z}}.\]  

(38)
6 Hydrodynamic type reductions of the dVN equation.

6.1 Case I

Let us introduce the functions $\rho_1 := S_x$ and $\rho_2 := S_y$. Thus, the symmetry constraint (36) can be written as follows

$$u_x = \frac{1}{4} (S_{xx} + S_{yy}) = \frac{1}{4} (\rho_{1x} + \rho_{2y}).$$

(39)

In order to analyze constraint (39) it is more convenient to consider equations (29) in Cartesian coordinates $(x, y)$, i.e.

$$S_x^2 + S_y^2 = 4u$$

(40)

$$S_t = \frac{1}{4} S_x^3 - \frac{3}{4} S_x S_y^2 + V_1 S_x + V_2 S_y,$$

(41)

where $V = V_1 + iV_2$, while dVN equation acquires the form

$$u_t = (uV_1)_x + (uV_2)_y$$

(42)

$$V_{1x} - V_{2y} = -3u_x$$

(43)

$$V_{2x} + V_{1y} = 3u_y,$$

(44)

Substituting (40) into (39), one gets the following hydrodynamic type system

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}_y = \begin{pmatrix} 0 & 1 \\ 2\rho_1 - 1 & 2\rho_2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}_x.$$  

(45)

Now, let us focus on definition $V_z := -3u_z$. Differentiating it with respect to $x$, using constraint (36) and equations (45), one obtains the equations

$$V_{1z} = -\frac{3}{2} \rho_{1z} + \frac{3}{4} (\rho_1^2 + \rho_2^2)_x$$

(46)

$$V_{2z} = \frac{3}{2} \rho_{2z},$$

(47)

which can be trivially integrated providing the following explicit formulas for $V_1$ and $V_2$ in terms of $\rho_1$ and $\rho_2$:

$$V_1 = -\frac{3}{2} \rho_1 + \frac{3}{4} (\rho_1^2 + \rho_2^2)$$

$$V_2 = \frac{3}{2} \rho_2.$$  

(48)
At this point we can derive $t$-dependent equations for $\rho_1$ and $\rho_2$. Differentiating equation (41) and using (45) and (48), one obtains the system

$$
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\varphi_1 \\
\varphi_2
\end{pmatrix}_t =

\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\varphi_1 \\
\varphi_2
\end{pmatrix}_x,
$$

where

$$
A_{11} = 3\rho_1 (\rho_1 - 1), \quad A_{12} = 3\rho_2, \\
A_{21} = 3\rho_2 (2\rho_1 - 1), \quad A_{22} = 3\rho_1 (\rho_1 - 1) + 6\rho_2^2.
$$

### 6.2 Case II

In this case (presenting the complex-valued function $S$ in terms of its real and imaginary parts, $S = \rho + i\varphi$) the symmetry constraint (37) acquires the form

$$
u_x = \frac{1}{4} (\rho_{xx} + \rho_{yy}).
$$

Equation (40) is equivalent to the system

$$
\begin{align*}
(\nabla \rho)^2 - (\nabla \varphi)^2 &= 4u \\
\nabla \rho \cdot \nabla \varphi &= 0,
\end{align*}
$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ and notation $\nabla \rho := (\rho_1, \rho_2)$ and $\nabla \varphi := (\varphi_1, \varphi_2)$ is introduced. Let us note that equation (52) allows to express, for instance, the component $\varphi_2$ in terms of the other ones ($\varphi_2 = -\rho_1 \varphi_1/\rho_2$), so that only the functions $\rho_2$, $\varphi_2$, and $\varphi_1$ are independent. By using the constraint (50), similar to the previous case, one shows that $\rho_1$, $\rho_2$, and $\varphi_1$, satisfy the hydrodynamic system

$$
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\varphi_1 \\
\varphi_2
\end{pmatrix}_y =

\begin{pmatrix}
0 & 1 & 0 \\
0 & a_1 & a_2 \\
b_1 & b_2 & b_3
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\varphi_1 \\
\varphi_2
\end{pmatrix}_x,
$$

where

$$
a_1 = 2\rho_1 \left( 1 - \frac{\varphi_1^2}{\rho_2^2} \right) - 1, \quad a_2 = 2 \left( \rho_2 + \frac{\rho_1^2 \varphi_1^2}{\rho_2^3} \right), \\
a_3 = -2\varphi_1 \left( 1 + \frac{\rho_1^2}{\rho_2^2} \right), \quad b_1 = -\frac{\rho_1}{\rho_2}, \quad b_2 = \frac{\rho_1 \varphi_1}{\rho_2}, \quad b_3 = -\frac{\rho_1}{\rho_2}.
$$
Just like in the previous section, starting with the definition of $V$ and differentiating it with respect to $x$, it is possible to express its real and imaginary parts in terms of $\rho_1, \rho_2, \varphi_1$ and $\varphi_2$

$$V_1 = -\frac{3}{2}\rho_1 + \frac{3}{4}(\rho_1^2 + \rho_2^2 - \varphi_1^2 - \varphi_2^2)$$

$$V_2 = \frac{3}{2}\rho_2. \quad (54)$$

or

$$V_1 = -\frac{3}{2}\rho_1 + \frac{3}{4}\left(\rho_1^2 + \rho_2^2 - \varphi_1^2 - \frac{\rho_1^2 \varphi_1^2}{\rho_2^2}\right)$$

$$V_2 = \frac{3}{2}\rho_2. \quad (54)$$

Separating real and imaginary parts in equation (41), one gets the system

$$\rho_t = \frac{1}{4}\left(\rho_x^3 - 3\rho_x \varphi_x^2\right) - \frac{3}{4}\left(\rho_x \rho_y^2 - \rho_x \varphi_y^2 - 2\rho_y \varphi_x \varphi_y\right) + V_1 \rho_x + V_2 \rho_y, \quad (55)$$

$$\varphi_t = \frac{1}{4}\left(-\varphi_x^3 + 3 \rho_x^2 \varphi_x\right) - \frac{3}{4}\left(2 \rho_x \rho_y \varphi_y + \varphi_x \rho_y^2 - \varphi_x \varphi_y^2\right) + V_1 \varphi_x + V_2 \varphi_y. \quad (56)$$

Substituting expressions (54) into (55) and (56) and differentiating with respect to $x$ and $y$, one obtains the hydrodynamic type system for $\rho_1, \rho_2$ and $\varphi_1$

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \varphi_1 \end{pmatrix}_t = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \varphi_1 \end{pmatrix}_x, \quad (57)$$

where

$$B_{11} = 3\left(\rho_1^2 - \varphi_1^2\right) - \frac{3}{2}\rho_1, \quad B_{12} = 0, \quad B_{13} = -3\rho_1 \varphi_1,$$

$$B_{21} = \rho_2 \left(6\rho_1 - 3\right) - \frac{9\rho_1}{\rho_2} \varphi_1^2, \quad B_{22} = 3 \left(\rho_1 \left(\rho_1 - 1\right) + 2\rho_2^2 - \varphi_1^2\right),$$

$$B_{23} = -6\rho_2 \varphi_1, \quad B_{31} = \frac{3}{2} \varphi_1 \left(4\rho_1 - 1\right), \quad B_{32} = \frac{3\rho_1^2}{2\rho_2} \varphi_1 \left(\rho_2 + 1\right),$$

$$B_{33} = 3 \left(\rho_1^2 - \varphi_1^2\right) - \frac{3}{2}\rho_1.$$
6.3 Case III

Let us note that symmetry constraint \(38\) implies that function \(\phi\) must be real-valued, and we denote \((\sigma_1, \sigma_2) := \nabla \phi\). Hence, the symmetry constraint \(38\) looks like

\[
u_x = \frac{1}{4} (\sigma_{1x} + \sigma_{2y}). \quad (58)
\]

Moreover, for sake of simplicity, we assume function \(S\) to be real-valued as well, and denote \((\rho_1, \rho_2) := \nabla S\). Differentiating equation \(40\) with respect to \(\lambda\), we obtain the algebraic relation

\[
\rho_1 \sigma_1 + \rho_2 \sigma_2 = 0, \quad (59)
\]

which allows us to eliminate, for instance, \(\rho_2\). Using these assumptions, we obtain the following hydrodynamic type system in the variables \(x\) and \(y\), for the functions \(\sigma_1, \sigma_2\) and \(\rho_1\)

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\rho_1
\end{pmatrix}_y =
\begin{pmatrix}
0 & 1 & 0 \\
\sigma_1 & \sigma_2 & \rho_1
\end{pmatrix}_x
\quad (60)
\]

where

\[
c_1 = 2 \frac{\sigma_1 \rho_1^2}{\sigma_2^2} - 1, \quad c_2 = -2 \frac{\sigma_1^2 \rho_1^2}{\sigma_2^4}, \quad c_3 = 2 \rho_1 \left(1 + \frac{\sigma_1^2}{\sigma_2^2}\right),
\]

\[
d_1 = -\frac{\rho_1}{\sigma_2}, \quad d_2 = \frac{\sigma_1^2}{\sigma_2^2} \rho_1, \quad d_3 = -\frac{\sigma_1}{\sigma_2}. \quad (61)
\]

Using equation \(41\), the corresponding equation for \(\phi\), obtained by differentiation of \(41\) with respect to \(\lambda\) and the system \(60\), one gets the following expressions of \(V_1\) and \(V_2\)

\[
V_1 = -\frac{3}{2} \sigma_1 + \frac{3}{4} (\rho_1^2 + \rho_2^2),
\]

\[
V_2 = \frac{3}{2} \sigma_2. \quad (62)
\]

Expressing \(\rho_2\) in terms of \(\sigma_1, \sigma_2\) and \(\rho_1\), one gets

\[
V_1 = -\frac{3}{2} \sigma_1 + \frac{3}{4} \rho_1^2 + \frac{3 \rho_1^2 \sigma_1^2}{4 \sigma_2^2} \quad (63)
\]

\[
V_2 = \frac{3}{2} \sigma_2. \quad (64)
\]
Using the formula (63), one obtains
\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\rho_1
\end{pmatrix}_t = \begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\rho_1
\end{pmatrix}
\]
where
\[
C_{11} = 3 \left( \frac{3}{2} \rho_1^2 - \sigma_1 \right), \quad C_{12} = 3 \sigma_2, \quad C_{13} = 9 \rho_1 \sigma_1,
\]
\[
C_{21} = \frac{3 \sigma_1^2}{\sigma_2^2} \rho_1^4 \left( \frac{1}{\sigma_2^2} - 1 \right) + \frac{3}{2} \sigma_1 \rho_1^2 \left( 1 - \frac{3}{\sigma_2^2} \right) - 3 \sigma_2,
\]
\[
C_{22} = \frac{3}{2} \rho_1^2 \left( 3 + 2 \frac{\sigma_1^2}{\sigma_2^2} \right) - 3 \sigma_1,
\]
\[
C_{23} = 3 \rho_1 \sigma_2 \left( 2 + \frac{\sigma_1^2}{\sigma_2^2} \right),
\]
\[
C_{31} = -3 \rho_1, \quad C_{32} = 0, \quad C_{33} = 3 \left( \rho_1^2 - \sigma_1 \right).
\]

Physical and geometrical meanings of the hydrodynamic type systems obtained in this paper will be discussed elsewhere.

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