ON ISOMETRIC DILATIONS OF PRODUCT SYSTEMS OF
$C^*$-CORRESPONDENCES AND APPLICATIONS TO FAMILIES
OF CONTRACTIONS ASSOCIATED TO HIGHER-RANK
GRAPHS

ADAM SKALSKI

Abstract. Let $E$ be a product system of $C^*$-correspondences over $N_0^r$. Some
sufficient conditions for the existence of a not necessarily regular isometric
dilation of a completely contractive representation of $E$ are established and
difference between regular and $^*$-regular dilations discussed. It is in particular
shown that a minimal isometric dilation is $^*$-regular if and only if it is doubly
commuting. The case of product systems associated with higher-rank graphs
is analysed in detail.

Classical multi-dimensional dilation theory ([SzF]) for Hilbert space operators is
concerned with dilating tuples of contractions to tuples of isometries or unitaries,
preserving as many properties of the original family as possible. In particular if the
tuple with which we start consists of mutually commuting operators, it is desirable
to obtain a commuting dilation. Celebrated examples of S. Parrott, N. Varopoulos
and others show that a joint dilation of three or more commuting contractions to
commuting isometries need not exist. In general it is difficult to decide whether
a given commuting tuple has a commuting isometric dilation. On the other hand
the existence of so-called regular or $^*$-regular dilations (i.e. dilations satisfying ad-
ditional conditions with respect to products of the original contractions and their
adjoints, see for example [Tim]) can be detected via simple conditions correspond-
ing to positive-definiteness of certain operator-valued functions associated with the
initial tuple.

In a recent paper [SZ2] together with J. Zacharias we considered dilations of $\Lambda$-
contractions, that is tuples of operators satisfying commutation relations encoded
by a (higher rank) graph $\Lambda$. It has now become clear that using the constructions
provided by I. Raeburn and A. Sims in [RaS] some of the results of [SZ2] can be
viewed as statements on completely contractive representations of the canonical
product system of $C^*$-correspondences associated to $\Lambda$. Product systems of $C^*$-
correspondences were first defined in [Fow] as generalisations of product systems of
Hilbert spaces and quickly proved to provide a natural framework for extensions
of the classical multi-dimensional dilation theory to more complicated objects (see
[So2] and references therein). The questions about the existence of a joint dilation of
a family of contractions satisfying certain commutativity relations to an analogous
family of isometries translates here into a question on the existence of an isometric
dilation of a (completely) contractive representation of a given product system.

Permanent address of the author: Department of Mathematics, University of Łódź, ul. Ba-
nacha 22, 90-238 Łódź, Poland.

2000 Mathematics Subject Classification. Primary 47A20, Secondary 05C20, 46L08, 47A13.

Key words and phrases. Multi-dimensional dilations, product systems of $C^*$-correspondences,
higher-rank graphs.
Motivated by the observations above we show in this paper that the generalised Poisson transform constructed in [SZ2] (see also [Pop], [MS2]) can be associated to a completely contractive representation of a product system of $C^*$-correspondences over $\mathbb{N}_0^r$, if only the system enjoys what we call a normal ordering property and the representation satisfies a so-called ‘Popescu condition’. This implies in turn that any such representation admits an isometric dilation. These sufficient conditions for the existence of an isometric dilation should be compared with recent results of [So2], where sufficient and necessary conditions for the existence of a regular dilation were established. The dilations constructed via the Poisson transform in the case of product systems related to graphs are of a $^*$-regular type. It is shown that in general a minimal (not necessarily regular) isometric dilation of a contractive representation is doubly commuting if and only if it satisfies the $^*$-regularity property. Contrary to the classical case of commuting Hilbert space contractions, here the difference between the regular and $^*$-regular dilations is fundamental, as there is no natural adjoint operation on a class of representations of a given product system (moreover we cannot always assume that the dilations have natural ‘unitary’ extensions, see [SZ1] and references therein).

In the second part of the paper we consider the case of certain families of contractions associated with a higher-rank graph $\Lambda$ and formalise heuristic observations listed in the second paragraph of this introduction. It is shown that the dilations of [SZ2] can indeed be viewed as dilations of representations of the canonical product system $E(\Lambda)$. General results of the first part of the paper specialised to this context can be interpreted as giving sufficient conditions on existence of regular or $^*$-regular dilations of certain tuples of contractions satisfying the commutation relations encoded by a higher-rank graph. In particular one can deduce immediately from [So1] that any $\Lambda$-contraction associated with a rank-2 graph has a dilation to a Toeplitz-type family.

The detailed plan of the paper is as follows: after listing some general notations we proceed to introduce in Section 1 basic notions of $C^*$-correspondences, their product systems over $\mathbb{N}_0^r$ and (covariant completely) contractive representations of such objects. In Section 2 we proceed to define isometric dilations of contractive representations and to quote fundamental results of B. Solel ([So1], [So2]) on the existence of dilations in the two-dimensional case and on sufficient and necessary conditions for the existence of regular dilations. A notion of a $^*$-regular isometric dilation is also introduced and a fact that a minimal isometric dilation is $^*$-regular if and only if it is doubly commuting established. Section 3 is devoted to the construction of a generalised Poisson transform associated to a representation of a product system with the normal ordering property satisfying the Popescu condition and to applications of the transform to isometric dilations. In Section 4 we recall the canonical product system of $C^*$-correspondences associated to a higher-rank graph $\Lambda$ ([RaS]) and describe its representations in terms of the $\Lambda$-families of operators on a Hilbert space. Finally Section 5 presents the general results of Sections 2 and 3 specified and adapted to the case of dilations of $\Lambda$-families described in Section 4.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Fix now and for the rest of the paper $r \in \mathbb{N}$. The canonical ‘basis’ in $\mathbb{N}_0^r$ will be denoted by $(e_1, \ldots, e_r)$, with $e := \sum_{i=1}^r e_i$. The componentwise maximum (respectively, minimum) of $n, m \in \mathbb{Z}^r$ is denoted by $n \vee m$ (respectively, $n \wedge m$) and we write $|n| = n_1 + \cdots + n_r$, $n_+ = n \vee 0$, $n_- = -(n \wedge 0)$. 

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Fix now and for the rest of the paper $r \in \mathbb{N}$. The canonical ‘basis’ in $\mathbb{N}_0^r$ will be denoted by $(e_1, \ldots, e_r)$, with $e := \sum_{i=1}^r e_i$. The componentwise maximum (respectively, minimum) of $n, m \in \mathbb{Z}^r$ is denoted by $n \vee m$ (respectively, $n \wedge m$) and we write $|n| = n_1 + \cdots + n_r$, $n_+ = n \vee 0$, $n_- = -(n \wedge 0)$. 

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Fix now and for the rest of the paper $r \in \mathbb{N}$. The canonical ‘basis’ in $\mathbb{N}_0^r$ will be denoted by $(e_1, \ldots, e_r)$, with $e := \sum_{i=1}^r e_i$. The componentwise maximum (respectively, minimum) of $n, m \in \mathbb{Z}^r$ is denoted by $n \vee m$ (respectively, $n \wedge m$) and we write $|n| = n_1 + \cdots + n_r$, $n_+ = n \vee 0$, $n_- = -(n \wedge 0)$.
1. Product systems of $C^*$-correspondences and their representations

Let $A$ be a $C^*$-algebra. By a $C^*$-correspondence $E$ over $A$ is meant a Hilbert $C^*$-module over $A$, equipped with the structure of a left $A$-module (via a nonzero *-homomorphism $\phi$ mapping $A$ into the $C^*$-algebra of adjointable operators on $E$). $E$ is essential as a left $A$-module if the closed linear span of $\phi(A)E$ is equal to $E$. Each $C^*$-correspondence is considered with the usual operator space structure (i.e. the one coming from viewing it as a corner in the appropriate linking algebra). The $C^*$-algebra of adjointable operators on $E$ is denoted by $\mathcal{L}(E)$. Further details can be found in [Lan] or [RaW]; note that we will often use the concept of internal tensor products in the category of Hilbert $C^*$-modules equipped with left actions. In particular any representation $\sigma$ of $A$ on a Hilbert space $H$ allows us to consider a new Hilbert space $E \otimes_{\sigma} H$ equipped with the representation of $A$ arising from the left action of $A$ on $E$.

Fix now and for the rest of the paper $r \in \mathbb{N}$. As explained in [So1–2] a product system $E$ of $C^*$-correspondences over $\mathbb{N}_0^r$ (formally introduced in [Fow] for a general countable semigroup with a neutral element) can be thought of as a family of $r$ $C^*$-correspondences $\{E_1, \ldots, E_r\}$ over the same $C^*$-algebra together with the unitary isomorphisms $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ ($i > j$) satisfying the natural associativity relations:

$$(\text{id}_{E_i} \otimes t_{i,j})(t_{i,l} \otimes \text{id}_{E_i}) = (t_{i,l} \otimes \text{id}_{E_i})(\text{id}_{E_i} \otimes t_{i,j})$$

for all $1 \leq i < j < l \leq k$. This point of view entails identifying for all $n = (n_1, \ldots, n_r) \in \mathbb{N}_0^r$ the correspondence $\mathbb{E}(n)$ with $E_1^{\otimes n_1} \otimes \cdots \otimes E_r^{\otimes n_r}$. We write $t_{i,i} = \text{id}_{E_i} \otimes E_i$, $t_{i,j} = t_{j,i}^{-1}$ for $i < j$ and also define unitary isomorphisms $t_{m,n} : \mathbb{E}(m) \otimes \mathbb{E}(n) \to \mathbb{E}(n) \otimes \mathbb{E}(m)$ ($m, n \in \mathbb{N}_0^r$) by obvious compositions of tensor extensions of appropriate $t_{i,j}$’s.

Let $\mathcal{F}_E := \bigoplus_{n \in \mathbb{N}_0^r} \mathbb{E}(n)$ denote the Fock module of $E$ (see [Fow] for the details of the construction). It is a $C^*$-correspondence over $A$. For each $n \in \mathbb{N}_0^r$ and $e \in \mathbb{E}(n)$ define the creation operator $L_e : \mathcal{F}_E \to \mathcal{F}_E$ by the formula

$$L_e(f) = e \otimes f, \quad f \in \mathcal{F}_E.$$ 

The Toeplitz algebra associated with $E$ is a concrete $C^*$-algebra in $\mathcal{L}(\mathcal{F}_E)$ generated by all creation operators as above. It will be denoted by $\mathcal{T}_E$.

**Definition 1.1.** A product system $E$ of $C^*$-correspondences over $\mathbb{N}_0^r$ is called compactly aligned if given $n, m \in \mathbb{N}_0^r$ and two operators $S \in \mathcal{K}(\mathbb{E}(n)), T \in \mathcal{K}(\mathbb{E}(m))$ the operator $S_n^{\oplus m} T_m^{\oplus n} \in \mathcal{K}(\mathbb{E}(n \oplus m))$, where $S_n^{\oplus m} := S \otimes I_{\mathbb{E}(n \oplus m - n)}$ and $T_m^{\oplus n} := T \otimes I_{\mathbb{E}(n \oplus m - m)}$.

The notion of compact alignment may seem rather technical, but it proved to be very useful ([Fow]). For the product system associated with a higher-rank graph it is equivalent to the graph in question being finitely aligned (see Sections 4 and 5). Examples coming from graphs suggest also that compact alignment of a product system is closely related to a form of ‘normal ordering’ in the Toeplitz algebra. As we have not been able to determine whether these two properties coincide in general, we introduce the following definition:

**Definition 1.2.** A product system $E$ of $C^*$-correspondences over $\mathbb{N}_0^r$ is said to have a normal ordering property if $\mathcal{T}_E = \overline{\text{span}} \{L_e L_f^* : e, f \in \bigcup_{n \in \mathbb{N}_0^r} \mathbb{E}(n)\}$. 

3
The normal ordering property may be thought of as a strong form of ‘double commutativity’ of the creation operators in the Toeplitz algebra. This is naturally seen when we work with product systems associated with higher-rank graphs in Sections 4 and 5. Note that if \( A = \mathbb{C} \) then each \( E_j \) is a Hilbert space and the structure of a product system is determined by the Hilbert space unitaries \( t_{i,j} : E_i \to E_j \) (precise description can be found in [So2] or in [SZ1]). If each \( E_j \) is additionally assumed to be finite-dimensional we are in the situation analysed in [PoS] and it is easy to see that the corresponding product system has a normal ordering property (and is compactly aligned).

**Representations of \( C^* \)-correspondences.** The notions presented in this subsection have been introduced and developed in the series of papers by P. Muhly and B. Solel (see [MS1] and references therein).

**Definition 1.3.** Let \( H \) be a Hilbert space. By a (completely contractive covariant) representation of a \( C^* \)-correspondence \( E \) over a \( C^* \)-algebra \( A \) on \( H \) is meant a pair \((\sigma,T)\), where \((\sigma,H)\) is a representation of \( A \) on \( H \), and \( T : E \to B(H) \) is a linear completely contractive map such that

\[
T(ab) = \sigma(a)T(\xi)\sigma(b), \quad a,b \in A, \xi \in E.
\]

It is called isometric if for each \( \xi, \eta \in E \)

\[
T(\xi)\*T(\eta) = \sigma((\xi,\eta)).
\]

A representation \((T,\sigma)\) determines a contraction \( \tilde{T} : E \otimes_H H \to H \) given by

\[
\tilde{T}(\xi \otimes h) = T(\xi)h \quad (\xi \in E, h \in H).
\]

This satisfies:

\[
(1.1) \quad \tilde{T}(\phi(a) \otimes I_H) = \sigma(a)\tilde{T}, \quad a \in A
\]

(\( \phi \) denoting the left action of \( A \) on \( E \)), and one can in fact show that, given a representation \( \sigma \), there is a 1-1 correspondence between contractions satisfying (1.1) and representations of \( E \) ([MS1] Lemma 2.1). The isometric representations are exactly those for which \( \tilde{T} \) is an isometry. The representation \((\sigma,T)\) is said to be (fully) coisometric if \( \tilde{T}T^* = I_H \).

It is easy to see how the notion of a representation of a \( C^* \)-correspondence extends to a product system.

**Definition 1.4.** Let \( E \) be a product system of \( C^* \)-correspondences over \( N_0^r \). By a (completely contractive covariant) representation of \( E \) on a Hilbert space \( H \) is meant a tuple \((\sigma,T(1),\ldots,T(r))\), where \((\sigma,T(i))\) is a representation of \( E_i \) on \( H \) and

\[
(1.2) \quad \tilde{T}^{(i)}(I_{E_i} \otimes \tilde{T}^{(j)}) = \tilde{T}^{(j)}(I_{E_j} \otimes \tilde{T}^{(i)})(t_{i,j} \otimes I_H)
\]

for \( i,j \in \{1,\ldots,r\} \). Such a representation is called isometric if each \((\sigma,T(i))\) is isometric, and coisometric if each \((\sigma,T(i))\) is coisometric.

To lighten the notation we will occasionally write \( \tilde{T} \) for \((\sigma,T(1),\ldots,T(r))\). We will also exploit the inductively defined maps \( T(n)(e) \in B(H) \) \( (n \in N_0^r, e \in E(n)) \) (see [MS1] or [SZ1]) and their natural partners \( \tilde{T}(n) : E(n) \otimes_H H \to H \). It is important to note that because of the condition (1.1) operators \( \tilde{T}(n)\tilde{T}(n)^* \) belong to \( \sigma(A) \).

If we represent the Toeplitz algebra faithfully on a Hilbert space, then the map \( \bigcup_{n \in N_0^r} E(n) \ni e \mapsto T(e) \in T_E \) yields in a natural way a representation of \( E \), called further the Fock-Toeplitz representation. It is easily seen to be isometric.
Definition 2.1. Let $\sigma, T$ both sides of the equation (1.3) hold on $\text{Ran}(\Gamma)$.

Definition 1.5. A representation $(\sigma, T^{(1)}, \ldots, T^{(r)})$ of $E$ on a Hilbert space $H$ is called doubly commuting if for each $i, j \in \{1, \ldots, r\}, i \neq j$ implies
\[
\widehat{T}^{(j)}(z)^* \widehat{T}^{(i)} = (I_{E_j} \otimes \widehat{T}^{(i)})(t_{i,j} \otimes I_H)(I_{E_i} \otimes \widehat{T}^{(j)^*}).
\]

For isometric representations of product systems over $\mathbb{N}_0^r$ double commutativity is exactly the same as Nica-covariance considered in [RS] ([So2], Remark 3.12). It also has the following equivalent characterisation:

Lemma 1.6. An isometric representation $(\sigma, T^{(1)}, \ldots, T^{(r)})$ of $E$ on a Hilbert space $H$ is doubly commuting if and only if for each $i, j \in \{1, \ldots, r\}, i \neq j$
\[
\widehat{T}^{(j)}(\text{Ker}(I_{E_i} \otimes \widehat{T}^{(i)^*})) \subset \text{Ker}(\widehat{T}^{(j)^*}).
\]

Proof. Let $i, j$ be as above and denote the operator $I_{E_i} \otimes \widehat{T}^{(j)} : E_i \otimes E_j \otimes \sigma H \to E_i \otimes \sigma H$ by $\Gamma_{ij}$. Note that the Ker$(\Gamma_{ij}^*) = E_i \otimes \sigma \text{Ker}(\widehat{T}^{(j)^*})$. It can be proved exactly in the same way as the well known statement for kernel of the operator $K_1 \otimes S$, where $K_1, K_2$ are Hilbert spaces and $S \in B(K_2)$ (at least if you know how to show the latter without using an orthonormal basis in $K_1$).

If $\widehat{T}$ is doubly commuting, then (1.4) is easily seen to be satisfied. Suppose then that (1.4) holds. Any vector in $E_i \otimes E_j \otimes \sigma H$ can be decomposed as a sum of an element in Ker$(\Gamma_{ij}^*)$ and in Ker$(\Gamma_{ij}^*)^\perp = \text{Ran}(\Gamma_{ij})$. It is therefore enough to show that the both sides of the equation (1.3) hold on $\text{Ran}(\Gamma_{ij})$. Let then $z \in E_i \otimes E_j \otimes \sigma H$. Then
\[
\widehat{T}^{(j)}(\Gamma_{ij})(z) = \widehat{T}^{(j)}(I_{E_i} \otimes \widehat{T}^{(j)})(z) = \widehat{T}^{(j)^*}(I_{E_i} \otimes \widehat{T}^{(j)^*})(z) = (I_{E_i} \otimes \widehat{T}^{(j)^*})(t_{i,j} \otimes I_H)(I_{E_i} \otimes \widehat{T}^{(j)^*})(z)
\]
This ends the proof. \hfill \square

2. General properties of isometric dilations of representations of product systems of $C^*$-correspondences

In this section we discuss several classes of isometric dilations of a representation of product system of $C^*$-correspondences.

Definition 2.1. Let $E$ be a product system of $C^*$-correspondences over $\mathbb{N}_0^r$ and let $(\sigma, T^{(1)}, \ldots, T^{(r)})$ be a representation of $E$ on a Hilbert space $K$. We say that $(\pi, V^{(1)}, \ldots, V^{(r)})$, an isometric representation of $E$ on a Hilbert space $K \supset H$, is an isometric dilation of $(\sigma, T^{(1)}, \ldots, T^{(r)})$ if
\[
\begin{align*}
(i) & \quad \forall a \in A \quad \pi(a)|_H = \sigma(a); \\
(ii) & \quad \forall i \in \{1, \ldots, r\} \forall \xi \in H \quad (V^{(i)^*}(\xi)) = (\widehat{T}^{(i)^*})(\xi).
\end{align*}
\]
The dilation is called minimal if $K = \text{lin}\{V(n)(e)\xi : n \in \mathbb{N}_0^r, e \in E(n), \xi \in H\}$.

Note that the condition (ii) above in particular exploits the identification of $E_i \otimes \sigma H$ with a subspace of $E_i \otimes \sigma K$. Moreover $P_H \in \pi(A)'$, so that the operators of the form $T \otimes P_H$ ($T \in L(E(n))$) are well defined as operators in $L(E(n) \otimes \pi K)$. This is used in Definition 2.4 below.
It is known that two commuting contractions can be always jointly dilated to commuting isometries. The following result established by B. Solel shows that this phenomenon persists in the category of representations of $C^*$-correspondences.

**Theorem 2.2** ([So1], Theorem 4.4). Let $E$ be a product system of $C^*$-correspondences over $\mathbb{N}_0^2$. Every (completely contractive covariant) representation $(\sigma, T^{(1)}, T^{(2)})$ of $E$ on a Hilbert space has a minimal isometric dilation $(\pi, V^{(1)}, V^{(2)})$. If $\sigma$ is non-degenerate and $E_1, E_2$ are essential then $\rho$ is non-degenerate.

**Regular isometric dilations (after B. Solel).** To formulate the next result we need a few more definitions. For $u = \{u_1, \ldots, u_k\} \subset \{1, \ldots, r\}$ write $e(u) = e_{u_1} + \cdots + e_{u_k}$.

**Definition 2.3.** Let $E$ be a product system of $C^*$-correspondences over $\mathbb{N}_0^2$. A representation $(\sigma, T^{(1)}, \ldots, T^{(r)})$ of $E$ on a Hilbert space is said to satisfy the Brehmer-Solel condition if for each $v \subset \{1, \ldots, r\}$

$$\sum_{u \subset v} (-1)^{|u|} (I_{E(e(v) - e(u))} \otimes \tilde{T}(e(u))^* \tilde{T}(e(u))) \geq 0.$$  

The condition above first appeared in the context of commuting families of contractions in [Bre]; recently it was exploited in the context of product systems of $C^*$-correspondences in [So2].

**Definition 2.4.** Let $(\sigma, T^{(1)}, \ldots, T^{(r)})$ be a representation of $E$ on a Hilbert space $H$. An isometric dilation $\tilde{V}$ of $T$ is said to be regular if for all $n \in \mathbb{Z}^r$

$$(2.1) \quad (I_{E(n-)} \otimes P_H) \tilde{V}(n-) V^*(n+)|_{E(n+) \otimes H} = \tilde{T}(n-) V^*(n+).$$

B. Solel showed that the condition described in Definition 2.3 characterises these representations which allow regular isometric dilations.

**Theorem 2.5** ([So2], Theorem 3.5). Let $E$ be a product system of $C^*$-correspondences over $\mathbb{N}_0^2$. A (completely contractive covariant) representation $(\sigma, T^{(1)}, \ldots, T^{(r)})$ of $E$ on a Hilbert space has a regular isometric dilation if and only if it satisfies the Brehmer-Solel condition.

Note that minimal regular dilations are necessarily unique, in the sense that any two such dilations respectively on Hilbert spaces $K$ and $K'$ are intertwined by a unitary $U : K \to K'$.

**$^*$-regular dilations.** Let us begin with a simple equivalent characterisation of regularity of an isometric dilation.

**Lemma 2.6.** An isometric dilation $(\pi, V^{(1)}, \ldots, V^{(r)})$ of a representation $(\sigma, T^{(1)}, \ldots, T^{(r)})$ of $E$ is regular if and only if for all $n, m \in \mathbb{N}_0^r$ such that $n_j \neq 0$ implies $m_j = 0$ ($j \in \{1, \ldots, r\}$) and all $e \in E(n)$, $f \in E(m)$

$$(2.2) \quad P_H(V(n)(e))^* V(m)(f)|_H = (T(n)(e))^* T(m)(f).$$

**Proof.** Note that condition (2.1) is satisfied for all $n \in \{1, \ldots, r\}$ if and only if for all $n, m \in \mathbb{N}_0^r$ such that $n_j \neq 0$ implies $m_j = 0$ ($j \in \{1, \ldots, r\}$) there is

$$(I_{E(n)} \otimes P_H) \tilde{V}(n)^* \tilde{V}(m)|_{E(m) \otimes H} = \tilde{T}(n)^* \tilde{T}(m).$$

The last condition is equivalent to the fact that for all $\xi \in H$, $f \in E(m)$

$$(I_{E(n)} \otimes P_H) \tilde{V}(n)^* V(m)(f)\xi = \tilde{T}(n)^* T(m)(f)\xi,$$
and further to the fact that for all \( e \in \mathbb{E}(n), \eta \in \mathcal{H} \)
\[
\langle e \otimes \eta, (I_{\mathbb{E}(n)} \otimes P_{\mathcal{H}})(V)(n)f \rangle = \langle e \otimes \eta, T(n)f \rangle.
\]
This in turn holds if and only if
\[
\langle V(n)(e)\eta, V(m)f \rangle = \langle T(n)e\eta, T(m)f \rangle,
\]
if and only if
\[
\langle \eta, P_{\mathcal{H}}(V(n)(e))^*V(m)f \rangle = \langle \eta, (T(n)(e))^*T(m)f \rangle.
\]
This ends the proof. \( \square \)

Recall ([Tim]) that if \((S, T)\) is a commuting pair of contractions on a Hilbert space \( \mathcal{H} \) and \((U, V)\) is a commuting isometric dilation of \((S, T)\) then it is said to be \( \ast \)-regular if for all \( k, l \in \mathbb{N} \)
\[
P_{\mathcal{H}}(U)^lV^k|_{\mathcal{H}} = T^k(S)^l.
\]

How should a corresponding definition look here? Note that if \( \overrightarrow{V} = (\pi, V^{(1)}, \ldots, V^{(r)}) \)
is an isometric dilation of a representation \( \overrightarrow{T} = (\sigma, T^{(1)}, \ldots, T^{(r)}) \) of \( \mathbb{E} \) then for all \( n, m \in \mathbb{N}_0^r \) and all \( e \in \mathbb{E}(n), f \in \mathbb{E}(m) \)
\[
(2.3) \quad P_{\mathcal{H}}V(n)(e)(V(m)(f))^*|_{\mathcal{H}} = T(n)(e)(T(m)(f))^*.
\]
Moreover one can see that similarly for all \( n \in \mathbb{Z}^r \)
\[
(2.4) \quad (I_{\mathbb{E}(n_-)} \otimes P_{\mathcal{H}})(I_{\mathbb{E}(n_-)} \otimes \overrightarrow{V}(n_+))(t_{n_+,n_-} \otimes I_{\mathcal{H}})(I_{\mathbb{E}(n_+)} \otimes \overrightarrow{V}(n_-))^*|_{\mathbb{E}(n_+) \otimes \mathcal{H}}
= (I_{\mathbb{E}(n_-)} \otimes \overrightarrow{T}(n_+))(t_{n_+,n_-} \otimes I_{\mathcal{H}})(I_{\mathbb{E}(n_+)} \otimes \overrightarrow{T}(n_-))^*;
\]
it is enough to observe that from the definition of the dilation it follows that for all \( n, m \in \mathbb{N}_0^r \)
\[
(I_{\mathbb{E}(n)} \otimes \overrightarrow{V}(m))^*|_{\mathbb{E}(n) \otimes \mathcal{H}} = (I_{\mathbb{E}(n)} \otimes \overrightarrow{T}(m))^*|_{\mathbb{E}(n) \otimes \mathcal{H}}.
\]

If the dilation \( \overrightarrow{V} \) is doubly commuting then the condition (2.4) reduces to
\[
(I_{\mathbb{E}(n_-)} \otimes P_{\mathcal{H}})(V(n_-))^*\overrightarrow{V}(n_+)|_{\mathbb{E}(n_+) \otimes \mathcal{H}} = (I_{\mathbb{E}(n_-)} \otimes \overrightarrow{T}(n_+))(t_{n_+,n_-} \otimes I_{\mathcal{H}})(I_{\mathbb{E}(n_+)} \otimes \overrightarrow{T}(n_-))^*.
\]
The latter can be seen as a natural generalisation of the notion of \( \ast \)-regularity.

In the classical context of commuting contractions Theorem 2 of [Tim] (see also [GaS]) shows that a minimal isometric dilation is \( \ast \)-regular if and only if it is doubly commuting. The same remains true in our context, as the next theorem shows. The proof is a natural generalisation of that in [Tim]. The basic idea is the following: as the last equation implies ‘double commutativity on \( \mathcal{H} \), we need to exploit minimality to deduce ‘double commutativity on \( \mathcal{K} \).

**Theorem 2.7.** A minimal isometric dilation \((\pi, V^{(1)}, \ldots, V^{(r)})\) of a representation \((\sigma, T^{(1)}, \ldots, T^{(r)})\) of \( \mathbb{E} \) is doubly commuting if and only if it is \( \ast \)-regular, that is if for all \( n \in \mathbb{Z}^r \)
\[
(2.5) \quad (I_{\mathbb{E}(n_-)} \otimes P_{\mathcal{H}})(\overrightarrow{V}(n_-))^*\overrightarrow{V}(n_+)|_{\mathbb{E}(n_+) \otimes \mathcal{H}} =
(I_{\mathbb{E}(n_-)} \otimes \overrightarrow{T}(n_+))(t_{n_+,n_-} \otimes I_{\mathcal{H}})(I_{\mathbb{E}(n_+)} \otimes \overrightarrow{T}(n_-))^*.
\]
Therefore, let now \( l \in \mathbb{N}_0^r \). Note that as \( v \in \operatorname{Ran}(\tilde{V}^{(j)}) \), it suffices if we can show that \( z - v \downarrow \operatorname{Ran}(\tilde{V}^{(j)}) \). We are going to exploit minimality once more. Let \( f \in E_j, n \in \mathbb{N}_0^r, h \in \mathbb{E}(n), \eta \in \mathbb{H} \) and compute

\[
A := \langle \tilde{V}^{(j)}(f \otimes \tilde{V}(n)(\eta \otimes \eta)), z \rangle = \langle f \otimes h \otimes \eta, \tilde{V}(n + e_j)^* \tilde{V}(m)(g \otimes \eta) \rangle
\]

Let now \( l = (n + e_j) \land m, p = n + e_j - l, q = m - l \). Note that \( l = 0, p, q \neq 0 \). Then

\[
A = \langle f \otimes h \otimes \eta, (t_{j,l} \otimes I_{E(p-e_j)} \otimes P_H)(I_{E(l)} \otimes \tilde{V}(p)^* \tilde{V}(l)(I_{E(l)} \otimes \tilde{V}(q))(g \otimes \eta) \rangle
\]

\[
= \langle (t_{j,l} \otimes I_{E(p-e_j)} \otimes H_0)(f \otimes h \otimes \eta), (I_{E(l)} \otimes I_{E(p)} \otimes P_H)(I_{E(l)} \otimes \tilde{V}(p)^*)(I_{E(l)} \otimes \tilde{V}(q))(g \otimes \eta) \rangle
\]

The \(*\)-regularity condition (2.5) implies that for all \( l \in \mathbb{N}_0^r \) and all \( p, q \in \mathbb{N}_0^r \) such that \( p \land q = 0 \) there is

\[
(I_{E(l)} \otimes I_{E(p)} \otimes P_H)(I_{E(l)} \otimes \tilde{V}(p)^*)(I_{E(l)} \otimes \tilde{V}(q))|_{E(l+q)\otimes H} = (I_{E(l)} \otimes I_{E(p)} \otimes \tilde{T}(q))(I_{E(l)} \otimes t_{q,p} \otimes P_H)(I_{E(l)} \otimes \tilde{V}(q))(g \otimes \eta)\]

Therefore

\[
A = \langle (t_{j,l} \otimes I_{E(p-e_j)} \otimes H_0)(f \otimes h \otimes \eta), (I_{E(l)} \otimes I_{E(p)} \otimes \tilde{T}(q))(I_{E(l)} \otimes t_{q,p} \otimes P_H)(I_{E(l)} \otimes I_{E(q)} \otimes \tilde{T}(p)^*)(g \otimes \eta) \rangle.
\]

Similarly

\[
B := \langle \tilde{V}^{(j)}(f \otimes \tilde{V}(n)(\eta \otimes \eta)), v \rangle = \langle f \otimes h \otimes \eta, \tilde{V}(n + e_j)^* \tilde{V}(m)(g \otimes \tilde{V}^{(j)}(\tilde{T}^{(j)^*}) \eta) \rangle
\]

\[
= \langle f \otimes h \otimes \eta, \tilde{V}(n + e_j)^* \tilde{V}(m + e_j)(g \otimes \tilde{T}^{(j)^*}) \eta) \rangle.
\]
Put now \( l' = (n + e_j) \land (m + e_j), \) \( p' = n + e_j - l', \) \( q' = m + e_j - l'. \) Note that \( l' = l + e_j, \) \( p' = p - e_j, \) \( q' = q. \) Continuing as before we obtain (note that \( t_{j,l} \) no longer features, as \( l'_j \neq 0)\)

\[
B = \langle f \otimes h \otimes \eta, (I_{E'_{l'}} \otimes I_{E'(p')} \otimes P_H)(I_{E'(l')} \otimes \tilde{V}(p'^*)\tilde{V}(l'^*)\tilde{V}(q')(g \otimes \tilde{T}^{(j)^*}\xi) \rangle
\]

\[
= \langle f \otimes h \otimes \eta, (I_{E'(l')} \otimes I_{E'(p')} \otimes \tilde{T}(q'))(I_{E'(l')} \otimes t_{q',p} \otimes P_H)(I_{E'(l')} \otimes I_{E'(q')} \otimes \tilde{T}(p'^*)\tilde{T}(l'^*))\xi \rangle
\]

\[
= \langle f \otimes h \otimes \eta, (I_{E(l+e_j)} \otimes I_{E(p-e_j)} \otimes \tilde{T}(q))
\]

\[
(I_{E(l+e_j)} \otimes t_{q,p-e_j} \otimes P_H)(I_{E(l)} \otimes t_{q,j} \otimes I_{E(p-e_j)})(I_{E(l)} \otimes I_{E(q)} \otimes \tilde{T}(p'^*)\xi \rangle
\]

The comparison of the formulas above shows that \( A = B \) if only

\[
(t_{i,j} \otimes I_{E(p-e_j)} \otimes I_{E(q)})(I_{E(l)} \otimes I_{E(p)} \otimes I_{E(q)})(I_{E(l)} \otimes t_{q,p})(I_{E(l)} \otimes I_{E(q)} \otimes I_{E(p)})
\]

\[
= (I_{E(l+e_j)} \otimes I_{E(p-e_j)} \otimes I_{E(q)})(I_{E(l+e_j)} \otimes t_{q,p-e_j})(I_{E(l)} \otimes t_{q,j} \otimes I_{E(p-e_j)})(I_{E(l)} \otimes I_{E(q)} \otimes I_{E(p)})
\]

This can be further reduced to checking two equalities

\[
t_{i,j} \otimes I_{E(p-e_j)} = I_{E(l+e_j)} \otimes I_{E(p-e_j)}
\]

and

\[
I_{E(l)} \otimes t_{q,p} = (I_{E(l+e_j)} \otimes t_{q,p-e_j})(I_{E(l)} \otimes t_{q,j} \otimes I_{E(p-e_j)});
\]

these finally are simple consequences of the definition of \( t_{m,n} \) in the beginning of Section 1.

The equality \( A = B \) implies that

\[
(\tilde{V}^{(j)}(f \otimes \tilde{V}(n)(h \otimes \eta)), z - v) = 0.
\]

As \( f \in E_j, n \in \mathbb{N}_0^*, h \in E(n), \eta \in H \) are arbitrary and \( \tilde{V} \) is minimal, \( z - v \perp \text{Ran}(\tilde{V}^{(j)}) \) and (2.6) is proved. Note now that

\[
\tilde{V}^{(i)}(P_{K_0}(e \otimes z)) = \tilde{V}^{(i)}(e \otimes (z - v))
\]

\[
= \tilde{V}^{(i)}(e \otimes \tilde{V}(m)(g \otimes \xi - g \otimes \tilde{V}^{(j)^*}\xi))
\]

\[= \tilde{V}^{(m+e_j)}(e \otimes g \otimes \xi - e \otimes g \otimes \tilde{V}^{(j)^*}\xi)
\]

\[= P_{K_0}\tilde{V}^{(m+e_j)}(e \otimes g \otimes \xi) \in K_0.
\]

This ends the proof. \( \square \)

It follows from the theorem above that a minimal isometric doubly commuting dilation of a representation of a product system is unique up to a unitary equivalence, as condition (2.5) together with minimality determines scalar products between all vectors in \( K. \) In general a minimal isometric dilation need not be unique. Concrete examples of this phenomenon can be found in [DPY] (Examples 4.3 and 4.4).

The following result can be shown in a similar way to Proposition 2.6 in [SZ].

**Proposition 2.8.** If \( \tilde{V} \) is a minimal isometric doubly commuting dilation of a coisometric representation \( \overline{T}, \) then \( \tilde{V} \) is coisometric.
3. Generalised Poisson transform and isometric dilations

In this section we describe how to construct isometric dilations via the generalised Poisson transform associated with a given representation of a product system. In a similar multi-dimensional context it has been first introduced in [Pop]; the one-dimensional counterpart for representations of $W^*$-correspondences has been recently investigated in [MS].

The next definition describes a natural variation on the type of conditions considered when one wants to construct isometric dilations of higher-rank objects (see [SzF] and references therein) and should be compared to the introduced earlier Brehmer-Solel condition.

**Definition 3.1.** Let $E$ be a product system of $C^*$-correspondences over $N_0$. For a representation $(\sigma, T^{(1)}, \ldots, T^{(r)})$ of $E$ on a Hilbert space $H$ define the defect operator $(s \in (0, 1))$

\[
\Delta_s(\overline{T}) = \sum_{n \in N_0, n \leq e} (-s^2)^{|n|} \overline{T}(n)\overline{T}(n)^\ast.
\]

The representation $\overline{T}$ is said to satisfy the Popescu condition (or condition ‘P’) if there exists $\rho \in (0, 1)$ such that for all $s \in (\rho, 1)$ the operator $\Delta_s(\overline{T})$ is positive.

The condition above in a similar form first appeared in [Pop]. Its variant for families of contractions associated with higher-rank graphs was extensively studied in [SZ]. Because of the condition (1.1) the defect operator $\Delta_s(\overline{T})$ is in $\sigma(A)'$. It is easy to see that if $\overline{T}$ is doubly commuting or coisometric then it satisfies the Popescu condition.

**Theorem 3.2.** Let $E$ be a product system of $C^*$-correspondences over $N_0$ having a normal ordered property and let $\overline{T} = (\sigma, T^{(1)}, \ldots, T^{(r)})$ be a representation of $E$ on a Hilbert space $H$ satisfying the Popescu condition. Then there exists a unique continuous linear map $R_{\overline{T}} : E \rightarrow B(H)$ satisfying

\[R_{\overline{T}}(L_e L_f^\ast) = T(n)(e)(T(m)(f))^\ast, \quad n, m \in N_0, e \in E(n), f \in E(m).\]

The map $R_{\overline{T}}$ will be called the generalised $E$-Poisson transform (associated with $\overline{T}$). It is completely positive and contractive, unital if $T_E$ is unital.

**Proof.** The proof is almost identical to the one given for the case of $\Lambda$-Poisson transforms associated with higher-rank graphs in [SZ]. We will therefore only indicate the main points and extra difficulties arising here. Let $s \in (0, 1)$ and consider the operator $\Gamma_s(\overline{T}) \in B(H)$ given by

\[\Gamma_s(\overline{T})(\xi) = \sum_{n \in N_0^+} s^{|n|} \overline{T}(n)(I_E(n) \otimes \Delta_s(\overline{T}))\overline{T}(n)^\ast \xi\]

($\xi \in H$). It can be checked that $\Gamma_s(\overline{T}) = I_H$ (see Lemma 2.1 of [SZ]). CHECK!!!

Let $\rho \in (0, 1)$ be such that for all $s \in (\rho, 1)$ the operator $\Delta_s(\overline{T})$ is positive. As $\Delta_s(\overline{T}) \in \sigma(A)'$, hence $\Delta_s(\overline{T})^{1/2} \in \sigma(A)'$ and moreover for each $n \in N_0$ the operator $I_E(n) \otimes \Delta_s(\overline{T})$ on $E(n) \otimes \sigma H$ is positive,

\[(I_E(n) \otimes \Delta_s(\overline{T}))^{1/2} = I_E(n) \otimes \Delta_s(\overline{T})^{1/2}.\]
Similarly, if \( n \in \mathbb{N}^r, T \in \mathcal{L}(E(n)), S \in \sigma(A)' \), then
\[
(T \otimes S)^* = T^* \otimes S^*.
\]
These properties will be further used without any comments.

Define the isometry \( W_s(T) : H \rightarrow F_E \otimes_\sigma H \) by
\[
W_s(T) \xi = \bigoplus_{n \in \mathbb{N}_0^r} s^{|n|} (I_{E(n)} \otimes \Delta_s(T^*)) \tilde{T}(n)^* \xi.
\]
Let the map \( R_{s,T} : \mathcal{L}(F_E) \rightarrow B(H) \) be given by the formula
\[
R_{s,T}(x) = W_s(T)^*(x \otimes I_H) W_s(T),
\]
It is clear that \( R_{s,T} \) is completely positive and contractive. Moreover for any \( e \in E(n), f \in E(m) \) \((n, m \in \mathbb{N}_0^r)\)
\[
R_{s,T}(L_e T_f) = s^{|n| + |m|} T(n)(e) T(m)(f)^*.
\]
Indeed, let \( e, f \) be as above. Note first that for all \( n' \in \mathbb{N}_0^r \)
\[
\tilde{T}(n')(L_e \otimes I_{E(n'-n) \otimes \sigma_h}) = T(n)(e) \tilde{T}(n'-n),
\]
so also
\[
(L_e^* \otimes I_{E(n'-n) \otimes \sigma_h}) \tilde{T}(n')^* = (\tilde{T}(n'-n)^* T(n)(e))^*.
\]
Compute further \((\xi, \eta) \in H\):
\[
\langle \eta, R_{s,T}(L_e T_f) \xi \rangle = \langle W_s(T) \eta, (L_e T_f \otimes I_H) W_s(T) \xi \rangle
= \langle \sum_{n' \in \mathbb{N}_0^r} s^{|n'|} (I_{E(n')} \otimes \Delta_s(T^*)) \tilde{T}(n')^* \eta, (L_e T_f \otimes I_H) \sum_{m' \in \mathbb{N}_0^r} s^{|m'|} (I_{E(m')} \otimes \Delta_s(T^*)) \tilde{T}(m')^* \xi \rangle
= \langle \sum_{n' \in \mathbb{N}_0^r, n', \geq n} s^{|n'|} (I_{E(n'-n)} \otimes \Delta_s(T^*)) (L_{e}^* \otimes I_{E(n'-n) \otimes \sigma_h}) \tilde{T}(n')^* \eta, \sum_{m' \in \mathbb{N}_0^r, m', \geq m} s^{|m'|} (I_{E(m'-m)} \otimes \Delta_s(T^*)) (L_{f}^* \otimes I_{E(m'-m) \otimes \sigma_h}) \tilde{T}(m')^* \xi \rangle
= \sum_{p \in \mathbb{N}_0^r} \langle s^{|p| + |n|} (I_{E(p)} \otimes \Delta_s(T^*)) \tilde{T}(p)^* (T(n)(e))^* \xi, s^{|p| + |m|} (I_{E(p)} \otimes \Delta_s(T^*)) \tilde{T}(p)^* (T(m)(f))^* \eta \rangle
= s^{|n| + |m|} \langle (T(n)(e))^* \xi, \sum_{p \in \mathbb{N}_0^r} s^{|2p|} \tilde{T}(p)(I_{E(p)} \otimes \Delta_s(T^*)) \tilde{T}(p)^* (T(m)(f))^* \eta \rangle
= s^{|n| + |m|} \langle (T(n)(e))^* \xi, \Gamma_s(T)(T(m)(f))^* \eta \rangle = \langle \xi, s^{|n| + |m|} T(n)(e) T(m)(f)^* \eta \rangle.
\]
It is now easy to see that by the normal ordering property the limit \( \lim_{x \rightarrow 1^-} R_{s,T}(x) \) in the norm topology exists for each \( x \in T_E \), and moreover the map \( R_{s,T} : T_E \rightarrow \)
Further given that this shows that \((K, a)\) constructed in Theorem 3.2. This provides us with a Hilbert space \(H^*\) should be \(\rho\) on a Hilbert space \(H\). B∗ double commutativity, this will be the case (see Theorem 5.6). Recall that minimal necessary extending \(R\) normal ordering property and let \(\sigma, T^{(s)}\) be a representation of \(E\) on a Hilbert space \(H\) satisfying the Popescu condition. Then \(\overline{T}\) has an isometric dilation.

Proof. Consider the minimal Stinespring dilation of the Poisson transform \(R_{\overline{T}}\) constructed in Theorem 3.2. This provides us with a Hilbert space \(K\), a representation \(\rho : T_E \to B(K)\) and an operator \(V \in B(H, K)\) such that for all \(x \in T_E\)

\[R_V(x) = V^*\rho(x)V\]

and \(K = \overline{\text{Lin}} \{\rho(x)V\xi : x \in T_E, \xi \in H\}\). We may assume that \(V\) is an isometry, if necessary extending \(R_{\overline{T}}\) in the unital manner to the unitisation of \(T_E\) in \(L(F_E)\). This allows us to view \(H\) as a subspace of \(K\). Define for each \(i \in \mathbb{N}_0, e \in E_i\)

\[V^{(i)}(e) = \rho(L_e);\]

and for \(a \in A\) (note that in our framework \(A = E(0) \subset F_E\))

\[\pi(a) = \rho(L_a).\]

It is clear that the tuple \(\overline{V} = (\pi, V^{(1)}, \ldots, V^{(r)})\) is an isometric representation of \(E\), as it is a *-homomorphic image of the Fock-Toeplitz representation.

The fact that condition (2.3) is satisfied follows directly from the definition of \(\overline{V}\), so it remains to establish that each \((V(n)(e))^* (n \in \mathbb{N}_0, e \in E(n))\) leaves \(H\) invariant. By the minimality of the Stinespring dilation we know that

\[K = \overline{\text{Lin}} \{V(m)(f)(V(p)(g))^*\xi : m, p \in \mathbb{N}_0, f \in E(m), g \in E(p), \xi \in H\}.\]

Further given \(m, p \in \mathbb{N}_0, f \in E(m), g \in E(p)\) and \(\xi, \eta \in H\),

\[
\begin{align*}
&\langle V(m)(f)(V(p)(g))^*\xi, (V(n)(e))^*\eta \rangle = \langle V(n)(e)V(m)(f)(V(p)(g))^*\xi, \eta \rangle \\
&= \langle V(n+m)(e \otimes f)(V(p)(g))^*\xi, \eta \rangle = \langle P_H V(n+m)(e \otimes f)(V(p)(g))^* P_H \xi, \eta \rangle \\
&= \langle T(n)(f)(g)^* T(m)(f)(g))^* P_H \xi, (T(n)(e))^* \eta \rangle \\
&= \langle (V(n)(f)(V(p)(g))^*\xi, (T(m)(e))^* \eta \rangle.
\end{align*}
\]

This shows that \((V(n)(e))^*|_H = (T(n)(e))^*\). In particular

\[K = \overline{\text{Lin}} \{V(m)(f)\xi : m \in \mathbb{N}_0, f \in E(m), \xi \in H\}.\]

The approach via a Poisson transform suggests that the constructed dilation should be *-regular. If the creation operators in \(T_E\) satisfy some variant of the double commutativity, this will be the case (see Theorem 5.6). Recall that minimal *-regular dilations are unique, as explained in the comments after Theorem 2.7. Once again we see here potential analogies between the normal ordering property, compact alignment and double commutativity of the creation operators.
4. Product system of Hilbert bimodules associated to a higher rank graph

In this section we recall a construction of a product system of $C^*$-correspondences associated to a higher-rank graph $\Lambda$ introduced in [RaS] and describe its representations in terms of the $\Lambda$-families of operators on a Hilbert space.

A rank-$r$ graph $\Lambda$ is a small category with set of objects $\Lambda^0$ and shape functor $\sigma : \Lambda \to \mathbb{N}^r$ (where $\mathbb{N}^r$ is viewed as the category with one object and morphisms $\mathbb{N}^r_0$) satisfying the factorisation property defined in [KuPa]. If $n \in \mathbb{N}^r_0$ the set of morphisms in $\Lambda$ of shape $n$ is denoted by $\Lambda^n$. Further for each $a \in \Lambda^0$ and $n \in \mathbb{N}^r_0$ write $\Lambda^n_0 := \{ \lambda \in \Lambda : s(\lambda) = a, \sigma(\lambda) = n \}$ and $|\lambda| = |\sigma(\lambda)|$. The morphisms in $\Lambda$ may be thought of as paths in a ‘multi-coloured’ graph with vertices indexed by $\Lambda^0$. The range and source maps are respectively denoted by $r : \Lambda \to \Lambda^0$ and $s : \Lambda \to \Lambda^0$. The factorisation property says that if $m, n \in \mathbb{N}^r_0$ then every morphism $\lambda \in \Lambda^{m+n}$ is a unique product $\lambda = \mu \nu$ of a $\mu \in \Lambda^m$ and $\nu \in \Lambda^n$, where $s(\mu) = r(\nu)$.

A rank-$r$ graph $\Lambda$ is called finitely aligned if for each $\lambda, \mu \in \Lambda$ the set of minimal common extensions of $\lambda$ and $\mu$, that is $MCE(\lambda, \mu) := \{ \nu \in \Lambda : \exists a, b \in \Lambda \nu = \lambda a = \mu b, \sigma(\lambda a) = \sigma(\lambda) \land \sigma(\mu) \}$, is finite.

In [RaS] it was shown that every higher-rank graph can be viewed as a product system of rank-1 graphs and this point of view leads to associating to such a graph a product system of $C^*$-correspondences. We rephrase this construction below - note that our conventions on the rank and source follow [Rae] rather than [RaS] and we are solely interested in product systems over $\mathbb{N}^r_0$ (which leads to certain simplifications).

Let $A_0 = C_0(\Lambda^0)$ denote the $C^*$-algebra of all complex-valued functions on $\Lambda^0$ vanishing at infinity. Let $j \in \{1, \ldots, r\}$. Define the $C^*$-correspondence $E_j(\Lambda)$ over $A_0$ as follows: $E_j(\Lambda)$ consists of these functions $x : \Lambda^{e_j} \to \mathbb{C}$ which are ‘locally square integrable’, i.e. for each $a \in \Lambda^0$

$$x_a := \sum_{\lambda \in \Lambda^n_a} |x(\lambda)|^2 < \infty$$

and the function $a \to x_a$ vanishes at infinity. The actions of $A_0$ on $E_j(\Lambda)$ are defined via $(f \in A_0, x \in E_j(\Lambda), \lambda \in \Lambda^{e_j})$

$$(x \cdot f)(\lambda) = x(\lambda)f(s(\lambda)), \quad (f \cdot x)(\lambda) = f(r(\lambda)) x(\lambda),$$

and the $A_0$ valued scalar product by $(x, y \in E_j(\Lambda), a \in \Lambda^0)$

$$(x, y)(a) = \sum_{\lambda \in \Lambda^n_a} \overline{x(\lambda)} y(\lambda).$$

As finitely supported functions are dense in $E_j(\Lambda)$, it is easy to see that each of the $C^*$-correspondences $E_j(\Lambda)$ is essential: $\overline{\mathcal{A}_0 E_j(\Lambda)} = E_j(\Lambda)$.

It will also be important at a certain point to consider the natural operator space structure of $E_j(\Lambda)$. Intuitively one should think of $E_j$ as a bundle of Hilbert spaces over $\Lambda^0$ and observe that in the linking algebra picture the Hilbert spaces in question act as columns. Therefore the natural operator space structure on $E_j$ is the one coming from viewing it as a bundle of operator spaces $(H_a)_a$. The explicit
formula for the matricial norms is as follows:

\[
\|(x_{ij})_{i,j=1}^n\| = \sup_{a \in \Lambda^0} \left\| \left( \sum_{i=1}^n \sum_{\lambda \in \Lambda_{ij}^0} t_{ik}(\lambda)x_{ik}(\lambda) \right)_{i,k=1}^n \right\|_{M_n}.
\]

To introduce on \((E_1(\Lambda), \ldots, E_r(\Lambda))\) the structure of a product system we identify \(E_i(\Lambda) \otimes E_j(\Lambda)\) with the space of all functions \(z : \Lambda^{e_i+e_j} \to \mathbb{C}\) such that for each \(a \in \Lambda^0\)

\[
z_a := \sum_{\lambda \in \Lambda_{ij}^{e_i+e_j}} |z(\lambda)|^2 < \infty
\]

and the function \(a \to z_a\) vanishes at infinity. The identification is implemented via the factorisation property: given \(\nu \in \Lambda^{e_i+e_j}\) we can decompose it uniquely as \(\nu = \lambda \mu\), \(\nu \in \Lambda^{e_i}, \mu \in \Lambda^{e_j}\) and for \(x \in E_i(\Lambda), y \in E_j(\Lambda)\) define

\[
(x \otimes y)(\nu) := x(\lambda)y(\mu).
\]

In other words, if \(x \in E_i(\Lambda), y \in E_j(\Lambda)\) then for \(\nu = \lambda \mu\) \(\nu \in \Lambda^{e_i+e_j}\)

\[
t_{ij}(x \otimes y)(\nu) = x(\lambda)y(\mu), \quad \text{where} \quad \lambda \in \Lambda^{e_i}, \mu \in \Lambda^{e_j}, \nu = \lambda \mu.
\]

Note that this leads to natural identifications of \(E(n)\) with the spaces of 'locally square integrable' functions on \(\Lambda^n\). Precisely speaking, for each \(n \in \mathbb{N}_0\) we define \(E_n(\Lambda)\) to be the space of all functions \(x : \Lambda^n \to \mathbb{C}\) such that for each \(a \in \Lambda^0\)

\[
x_a := \sum_{\lambda \in \Lambda_n^a} |x(\lambda)|^2 < \infty
\]

and the function \(a \to x_a\) vanishes at infinity. The actions of \(A_0\) and the \(A_0\) valued scalar product on \(E_n(\Lambda)\) are defined again via formulas (4.1) and (4.2) (this time \(f \in A_0, x, y \in E_n, \lambda \in \Lambda^n\)). Define for all \(n, m \in \{1, \ldots, r\}\) the map \(U_{n,m} : E_n(\Lambda) \otimes E_m(\Lambda) \to E_{n+m}(\Lambda)\) via the continuous linear extension of the formula

\[
U_{n,m}(x \otimes y)(\lambda) = x(\mu)y(\nu),
\]

where \(x \in E_n(\Lambda), y \in E_m(\Lambda), \lambda \in \Lambda^{n+m}, \lambda = \mu \nu, \mu \in \Lambda^n, \nu \in \Lambda^m\). It can be checked that \(U_{n,m}\) is an isomorphism in the category of C*-correspondences. Because of that we will identify \(E(n)\) with \(E_n(\Lambda)\) without any further comments. The resulting product system of C*-correspondences will be called the product system of the graph \(\Lambda\) and denoted by \(E(\Lambda)\). In what follows we will often view the Dirac functions \(\delta_a\) \((a \in \Lambda^0)\) and \(\delta_\lambda\) \((\lambda \in \Lambda^{e_i})\) as elements respectively of \(A_0\) and of \(E_j(\Lambda)\).

**Representations of \(E(\Lambda)\).**

**Definition 4.1.** Suppose that \(\Lambda\) is a higher-rank graph. A family of partial isometries \(\{x_\lambda : \lambda \in \Lambda\}\) in a C*-algebra \(B\) is called a Toeplitz \(\Lambda\)-family if the following are satisfied:

(i) \(\{x_\lambda : a \in \Lambda^0\}\) is a family of mutually orthogonal projections;
(ii) \(x_\lambda x_\mu = x_{\lambda \mu}\) if \(\lambda, \mu \in \Lambda, s(\lambda) = r(\mu)\);
(iii) \(x_\lambda^* x_\lambda = x_{s(\lambda)}\) if \(\lambda \in \Lambda\);
(iv) if \(n \in \mathbb{N}_0 \setminus \{0\}, a \in \Lambda^0\) and \(F \subset \{\lambda \in \Lambda^n : r(\lambda) = a\}\) is finite then

\[
\sum_{\lambda \in F} x_\lambda x_\lambda^*.
\]
If Λ is finitely aligned and additionally the condition
\[(v) \quad x_\mu^* x_\nu = \sum_{\mu\alpha=\nu\beta\in MCE(\mu,\nu)} x_\alpha x_\beta^* \]
is satisfied for all \(\mu, \nu \in \Lambda\), the family \(\{x_\lambda : \lambda \in \Lambda\}\) is called a Toeplitz-Cuntz-Krieger family.

In [RaS] isometric representations of \(E(\Lambda)\) are called Toeplitz representations. They are given by Toeplitz families.

**Theorem 4.2** ([RaS], Theorem 4.2). Let \(\Lambda\) be a higher-rank graph. There is a 1-1 correspondence between isometric representations of \(E(\Lambda)\) on a Hilbert space \(H\) and Toeplitz \(\Lambda\) families in \(B(H)\). The correspondence is given by
\[
\sigma(\delta_a) = x_a, \quad a \in \Lambda^0, \\
T^{(j)}(\delta_\lambda) = x_\lambda, \quad j \in \{1, \ldots, r\}, \lambda \in \Lambda^e_j.
\]

Note that \(\sigma\) defined as above is nondegenerate if and only if \(\sum_{a \in \Lambda^0} x_a = I_H\), where the sum is understood in the strong operator topology.

It is also possible to give an easy characterisation of those isometric representations which are doubly commuting (equivalently, Nica-covariant).

**Lemma 4.3** ([RaS], Proposition 6.4). Let \(\Lambda\) be a finitely aligned higher-rank graph. An isometric representation of \(E(\Lambda)\) on a Hilbert space \(H\) is doubly commuting if and only if the corresponding Toeplitz family is a Toeplitz-Cuntz-Krieger family.

It is easy to see that if \(\Lambda\) is finitely aligned then \(E(\Lambda)\) satisfies the normal ordering condition. Note that by Theorem 5.4 of [RaS] \(\Lambda\) is finitely aligned if and only if \(E(\Lambda)\) is compactly aligned.

We are now ready to define objects which were the main subject of investigation in [SZ2].

**Definition 4.4.** Let \(H\) be a Hilbert space. A family \(V = \{V_\lambda : \lambda \in \Lambda\}\) of operators in \(B(H)\) is called a \(\Lambda\)-contraction if the following conditions are satisfied:
\[
(i) \quad \forall_{\lambda, \mu \in \Lambda, s(\lambda) \neq r(\mu)} V_\lambda V_\mu = 0; \\
(ii) \quad \forall_{\lambda, \mu \in \Lambda, s(\lambda) = r(\mu)} V_\lambda V_\mu = V_\lambda V_\mu; \\
(iii) \quad \forall_{n \in \mathbb{N}^0} \sum_{\lambda \in \Lambda^e_j} V_\lambda V_\lambda^* \leq I; \\
(iv) \quad \text{each } V_a (a \in \Lambda^0) \text{ is an orthogonal projection.}
\]

All infinite sums here and in what follows are understood in the strong operator topology.

The definition in [SZ2] was slightly different as we additionally requested that \(\sum_{a \in \Lambda^0} V_a = I\). As explained in that paper the distinction is not very important: then conditions (ii) and (iii) imply that each \(V_a\) for \(a \in \Lambda^0\) is a contractive idempotent, hence a projection (so that in particular (iv) is a consequence of (ii) and (iii)). Further (i) shows that \(V_a V_b = 0\) if \(b \in \Lambda^0\) and \(a \neq b\). Denoting by \(p\) the sum \(\sum_{a \in \Lambda^0} V_a\) we see that \(V_\lambda = pV_\lambda p\) (by (i) and (ii)). Therefore even if \(\sum_{a \in \Lambda^0} V_a = I\) is not satisfied at the outset, it will be fulfilled by the obvious \(\Lambda\)-contraction on \(pH\).

In condition (iii) above it is enough to assume that the inequalities hold only for \(n\) of the form \(e_j, j \in \{1, \ldots, r\}\).

We write \(V_\lambda \mu := 0\) if \(s(\lambda) \neq r(\mu)\). Sometimes we will also write \(V_\emptyset = I_H\).
The following observation is not very complicated but lies at the heart of this section; it was actually the motivating point for trying to extend the results of \[SZ\] to the framework of representations of product systems of $C^*$-correspondences.

**Lemma 4.5.** Let $\Lambda$ be a higher-rank graph. There is a 1-1 correspondence between completely contractive representations of $\mathcal{E}(\Lambda)$ on a Hilbert space $\mathcal{H}$ and $\Lambda$-contractions in $\mathcal{B}(\mathcal{H})$. The correspondence is given by

\begin{align}
\sigma(\delta_a) &= V_a, \quad a \in \Lambda_0, \\
T^{(j)}(\delta_{\lambda}) &= V_{\lambda}, \quad j \in \{1, \ldots, r\}, \lambda \in \Lambda^{(j)}.
\end{align}

**Proof.** Let $\mathcal{V}$ be a $\Lambda$-contraction. Fix for a moment $j \in \{1, \ldots, r\}$ and write $E$ instead of $E_j(\Lambda)$. To show that $T := T^{(j)}$ defined by the linear extension of the formula (4.4) is completely contractive consider a matrix $(x_{\alpha})_{\alpha \in \Lambda}$ of finitely supported functions in $E$ and let $\xi_1, \ldots, \xi_n$ be vectors in $\mathcal{H}$. Let $T_n$ denote the $n$-th matrix lifting of $T$ and write $\xi = [\xi_1, \ldots, \xi_n]^T \in \mathbb{H}^n$. Then

\[ ||T_n \left( (x_{i,k})_{i,k=1}^n \right) \xi ||^2 = \sum_{i,k=1}^n \langle T(x_{i,k}) \xi_k, T(x_{i,l}) \xi_l \rangle \]

\[ = \sum_{i,k=1}^n \left( \sum_{\lambda \in \Lambda^{(j)}} x_{i,k}(\lambda) V_{\lambda} \xi_k, \sum_{\lambda' \in \Lambda^{(j)}} x_{i,l}(\lambda') V_{\lambda'} \xi_l \right) \]

\[ = \sum_{i=1}^n \left( \sum_{\lambda \in \Lambda^{(j)}} V_{\lambda} \sum_{k=1}^n x_{i,k}(\lambda) V_{\lambda}(\cdot) \xi_k, \sum_{\lambda' \in \Lambda^{(j)}} V_{\lambda'} \sum_{l=1}^n x_{i,l}(\lambda') V_{\lambda'}(\cdot) \xi_l \right). \]

Define for each $i = 1, \ldots, n$ and $\lambda \in \Lambda^{(j)}$

\[ \zeta^{i}_{\lambda} = \sum_{k=1}^n x_{i,k}(\lambda) V_{\lambda}(\cdot) \xi_k. \]

Then

\[ ||T_n \left( (x_{i,k})_{i,k=1}^n \right) \xi ||^2 = \sum_{i=1}^n \left( \sum_{\lambda \in \Lambda^{(j)}} V_{\lambda} \zeta^{i}_{\lambda}, \sum_{\lambda' \in \Lambda^{(j)}} V_{\lambda'} \zeta^{i}_{\lambda'} \right) = \sum_{i=1}^n \left\| \sum_{\lambda \in \Lambda^{(j)}} V_{\lambda} \zeta^{i}_{\lambda} \right\|^2. \]

The condition (iii) in Definition 4.4 implies that $\| \sum_{\lambda \in \Lambda^{(j)}} V_{\lambda} \zeta^{i}_{\lambda} \|^2 \leq \sum_{\lambda \in \Lambda^{(j)}} || \zeta^{i}_{\lambda} ||^2$. Moreover for $\lambda \in \Lambda_{a}^{(j)}$

\[ || \zeta^{i}_{\lambda} ||^2 \leq \sum_{k=1}^n x_{i,k}(\lambda) x_{i,k}(\cdot) \xi_k = \sum_{k,l=1}^n \overline{x_{i,k}(\lambda)} x_{i,l}(\cdot) \langle V_{\lambda} \xi_k, V_{\lambda} \xi_l \rangle \]

so that

\begin{align}
T_n \left( (x_{i,k})_{i,k=1}^n \right) \xi ||^2 &\leq \sum_{i=1}^n \sum_{\lambda \in \Lambda^{(j)}} || \zeta^{i}_{\lambda} ||^2 \leq \sum_{i=1}^n \sum_{\alpha \in \Lambda} \sum_{\lambda \in \Lambda_{a}^{(j)}} \sum_{k,l=1}^n \overline{x_{i,k}(\lambda)} x_{i,l}(\cdot) \langle V_{\lambda} \xi_k, V_{\lambda} \xi_l \rangle.
\end{align}

Define for each $a \in \Lambda_0$ a matrix $A_a \in M_n$ by

\[ (A_a)_{k,l} = \sum_{i=1}^n \sum_{\lambda \in \Lambda_{a}^{(j)}} \overline{x_{i,k}(\lambda)} x_{i,l}(\cdot). \]
Note that the (4.3) implies that
\[ \|(x_{i,k})_{k=1}^n\|_{M_n(E)} = \sup_{a \in A^0} \|A_a\|. \]
On the other hand (4.6) implies that
\[ \|T_n((x_{i,k})_{k=1}^n)\| \leq \sum_{a \in A^0} (P_a^{(n)}\xi, (A_a \otimes I_H)P_a\xi), \]
where \( P_a^{(n)} = P_a \oplus \cdots \oplus P_a \in B(H^\otimes n) \). Thus finally
\[ \|T_n((x_{i,k})_{k=1}^n)\| \leq \sup_{a \in A^0} \{\|A_a\|\} \sum_{a \in A^0} \|P_a^{(n)}\xi\| = \sup_{a \in A^0} \{\|A_a\|\}\|\xi\|^2, \]
so that \( T \) extends to a complete contraction from \( E \) to \( B(H) \). The fact that the continuous linear extension of (4.4) yields a representation of \( A_0 \) is immediate and then the routine check shows that \( (\sigma, T^{(1)}, \ldots, T^{(r)}) \) is a representation of \( E(\Lambda) \).

Conversely, if \( (\sigma, T^{(1)}, \ldots, T^{(r)}) \) is a representation of \( E(\Lambda) \) we can use formulas (4.4) and (4.5) to define operators \( V_\lambda \) for \( \lambda \in \Lambda^0 \cup \bigcup_{j=1}^r \Lambda^{e_j} \). Given any \( \mu \in \Lambda \) due to the factorisation property we can always write it as a concatenation of elements in \( \Lambda^0 \cup \bigcup_{j=1}^r \Lambda^{e_j} \) and define \( V_\mu \) as a corresponding composition. The fact that this gives a unique prescription is a consequence of the fact that \( (\sigma, T^{(1)}, \ldots, T^{(r)}) \) is a representation of \( E(\Lambda) \); moreover it is easy to check that conditions (i), (ii) and (iv) of Definition 4.4 are satisfied (contractive idempotents in \( B(H) \)).

It remains to check (iii). By (i) and remarks after the definition of a \( \Lambda \)-contraction it is enough to do it for \( n = e_j \) (\( j \in \{1, \ldots, r\} \)). Let then \( n \in \mathbb{N} \) and let \( \lambda_1, \ldots, \lambda_n \) be distinct elements in \( \Lambda^{e_j} \). Then the row matrix \([V_{\lambda_1} \cdots V_{\lambda_n}]\) is equal to \( T^n((x_{1k})_{k=1}^n)\), where \( x_{1k} = \delta_{\lambda_k} \). It follows easily from (4.3) that \( \|(x_{1k})_{k=1}^n\|_{M_n(E_0)} = 1 \), and as \( T \) is assumed to be a complete contraction we obtain \( \|[V_{\lambda_1} \cdots V_{\lambda_n}]\| \leq 1 \) and the result follows. \( \square \)

Note that the representation \( \sigma \) of \( A_0 \) associated to a \( \Lambda \)-contraction \( \mathcal{V} \) is nondegenerate if and only if \( \sum_{a \in A^0} V_a = I_H \).

**Lemma 4.6.** Let \( \Lambda \) be a rank-r graph and let \( \mathcal{V} \) be a \( \Lambda \)-contraction on a Hilbert space \( H \). The representation \( \mathcal{T} \) of \( E(\Lambda) \) associated with \( \mathcal{V} \) is doubly commuting if and only if for all \( i, j \in \{1, \ldots, r\} \), \( \lambda \in \Lambda^{e_i} \), \( \mu \in \Lambda^{e_j} \), there is
\[ V_\lambda V_\mu = \sum_{\alpha \in \Lambda^{e_i}, \beta \in \Lambda^{e_j}} V_\alpha V_\beta. \]

**Proof.** Let \( i \in \{1, \ldots, r\} \), \( \lambda \in \Lambda^{e_i} \) and \( \xi \in H \). Then
\[ T^{(i)}(\delta_\lambda \otimes \xi) = V_\lambda \xi \]
and it follows that for \( \eta \in H \)
\[ (T^{(i)})^* \eta = \sum_{\lambda \in \Lambda^{e_i}} \delta_\lambda \otimes V_\lambda^* \eta. \]

The equivalence of the conditions in the lemma follows from straightforward computations. \( \square \)

It would be interesting and nontrivial to analyse how the results of this section extend to topological higher-rank graphs as discussed for example in [Yee].
5. Dilating graph-contractions via dilating representations of associated product systems of Hilbert $C^*$-correspondences

Here we apply the conclusions of the discussions of previous two sections to obtain the dilations of $\Lambda$-contractions to Toeplitz-type families.

**Theorem 5.1.** Let $\Lambda$ be a rank-2 graph and let $\mathcal{V}$ be a $\Lambda$-contraction on a Hilbert space $\mathcal{H}$. There exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a $\Lambda$-contraction $\mathcal{W}$ on $\mathcal{K}$ consisting of partial isometries forming a Toeplitz family such that for each $\lambda \in \Lambda$

$$W^*_\lambda|_{\mathcal{H}} = \mathcal{V}^*_\lambda.$$  

One may assume that $\mathcal{K} = \overline{\text{Lin}}\{W_\lambda \mathcal{H} : \lambda \in \Lambda\}$. Under this assumption $\sum_{a \in \Lambda^0} W_a = I_\mathcal{K}$ if $\sum_{a \in \Lambda^0} V_a = I_\mathcal{H}$.

**Proof.** Let $\overrightarrow{T}$ be the representation of $E(\Lambda)$ associated with $\mathcal{V}$ by Lemma 4.5. From Theorem 4.2 it follows that any isometric dilation of $\overrightarrow{T}$ has to be given by a Toeplitz family $\mathcal{W}$ such that (5.1) holds. The main statement therefore follows directly from Theorem 2.2.

Before we identify necessary conditions for the representation of $E(\lambda)$ associated to a given $\Lambda$-contraction to satisfy the Brehmer-Solel condition we need to understand the Hilbert spaces involved. Observe that if $\sigma$ is a representation of $\Lambda_0$ on a Hilbert space $\mathcal{H}$ then for each $n \in \mathbb{N}_0$ the Hilbert space $E(\Lambda)(n) \otimes_n \mathcal{H}$ is isometrically isomorphic to the Hilbert space $\bigoplus_{a \in \Lambda_0} l^2(\Lambda^a_0; P_a \mathcal{H})$, where $P_a = \sigma(\delta_a)$ for each $a \in \Lambda_0$.

**Lemma 5.2.** Let $r \in \mathbb{N}$, let $\Lambda$ be a rank-$r$ graph and let $\mathcal{V}$ be a $\Lambda$-contraction on a Hilbert space $\mathcal{H}$. Define for each $u \subset v \subset \{1, \ldots, r\}$ an operator $P_{u,v}$ on the Hilbert space $\bigoplus_{a \in \Lambda^0} l^2(\Lambda^a_0; V_a \mathcal{H})$ via the continuous linear extension of the formula

$$P_{u,v}(\delta_{\mu \nu} \otimes \xi) = \bigoplus_{a \in \Lambda^0} \sum_{\lambda \in \Lambda^a_0} \delta_{\mu \lambda} \otimes V^*_\lambda V_{\nu} \xi,$$

where $\mu \in \Lambda^{v(u)-v(\nu)}, \nu \in \Lambda^{v(\nu)}, r(\nu) = s(\mu), \xi \in V_{s(\nu)} \mathcal{H}$. Then the associated representation of $E(\Lambda)$ satisfies the Brehmer-Solel condition if and only if

$$\forall_{u \subset v \subset \{1, \ldots, r\}} \sum_{u \subset v} (-1)^{|u|} P_{u,v} \geq 0.$$  

**Proof.** Direct consequence of the remark before the lemma and the formula (4.7).

**Theorem 5.3.** Let $\Lambda$ be a higher rank graph and let $\mathcal{V}$ be a $\Lambda$-contraction on a Hilbert space $\mathcal{H}$. Suppose that $\mathcal{V}$ satisfies the condition (5.3), where the operators $P_{u,v}$ are defined by (5.2). Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a $\Lambda$-contraction $\mathcal{W}$ on $\mathcal{K}$ consisting of partial isometries forming a Toeplitz family such that each $W^*_\lambda$ leaves $\mathcal{H}$ invariant and for $\lambda \in \Lambda$, $\nu \in \Lambda$ such that $\sigma(\lambda)_j \neq 0$ implies $\sigma(\nu)_j = 0$ ($j \in \{1, \ldots, r\}$)

$$P_{H\lambda} W^*_\lambda W_\mu|_{\mathcal{H}} = V^*_\lambda V_\mu.$$  

One may assume that $\mathcal{K} = \overline{\text{Lin}}\{W_\lambda \mathcal{H} : \lambda \in \Lambda\}$; under this assumption the family $\mathcal{W}$ is unique up to unitary equivalence.
Proof. Let $\overrightarrow{T}$ be the representation of $E(\Lambda)$ associated with $V$ by Lemma 4.5. From Theorem 4.2 it follows that any isometric dilation of $\overrightarrow{T}$ has to be given by a Toeplitz family $W$ such that (5.1) holds. The existence of a regular dilation to a Toeplitz family is a consequence of Theorem 2.5; Lemma 2.6 implies that regularity of the dilation can be expressed by a simple formula above. □

The next corollary is a consequence of Theorem 3.15 of [So2] and Lemma 4.3 above.

**Corollary 5.4.** Let $\Lambda$ be a finitely-aligned higher rank graph and let $V$ be a doubly commuting $\Lambda$-contraction on a Hilbert space $H$. Then there exists a Hilbert space $K \supset H$ and a $\Lambda$-contraction $W$ on $K$ consisting of partial isometries forming a Toeplitz-Cuntz-Krieger family such that each $W_\lambda^*$ leaves $H$ invariant and for $\mu \in \Lambda$ such that $\sigma(\lambda)_j \neq 0$ implies $\sigma(\mu)_j = 0$ ($j \in \{1, \ldots, r\}$)

$$P_H W_\lambda^* W_\mu |_H = V_\lambda^* V_\mu.$$ 

One may assume that $K = \overline{\text{Lin}\{W_\lambda H : \lambda \in \Lambda\}}$; under this assumption the family $W$ is unique up to unitary equivalence.

The following definition was introduced in [SZ2] as a generalisation of the notion of condition ‘P’ suggested in [Pop].

**Definition 5.5.** Let $V$ be a $\Lambda$-contraction and define for $s \in (0, 1)$ the defect operator

$$\Delta_s(V) = \sum_{\mu \in \Lambda, \sigma(\mu) \leq e} (-s^2)^{|\mu|} V_\mu V_\mu^*.$$ 

The family $V$ is said to satisfy the Popescu condition (or condition ‘P’) if there exists $\rho \in (0, 1)$ such that for all $s \in (\rho, 1)$ the operator $\Delta_s(V)$ is positive.

In this context Theorem 3.3 can be used to establish the following:

**Theorem 5.6** (Theorem 3.1, [SZ2]). Let $\Lambda$ be a finitely-aligned higher rank graph and let $V$ be a $\Lambda$-contraction on a Hilbert space $H$ which satisfies the Popescu condition. Then there exists a Hilbert space $K \supset H$ and a $\Lambda$-contraction $W$ on $K$ consisting of partial isometries forming a Toeplitz-Cuntz-Krieger family such that $W_\lambda^* |_H = V_\lambda^*$ for each $\lambda \in \Lambda$. One may assume that $K = \overline{\text{Lin}\{W_\lambda H : \lambda \in \Lambda\}}$; under this assumption the family $W$ is unique up to unitary equivalence.

Proof. It follows from the remark stated after Lemma 4.3 that $E(\Lambda)$ has the normal ordering property. Most of the statements in the theorem follow therefore immediately from Theorem 3.3 by now standard applications of the identifications obtained in Section 4. The only extra element is double commutativity and uniqueness of the dilation. The first follows from the fact that if the graph is finitely aligned, then the Fock-Toeplitz representation is automatically doubly commuting in the natural sense and therefore so is its *-homomorphic image yielding the dilation in Theorem 3.3. The uniqueness follows from the remarks after Theorem 2.7. □

**References**

[Bre] S. Brehmer, Über vetauschbare Kontraktionen des Hilbertschen Raumes, *Acta Sci. Math. Szeged* 22 (1961), 106–111.

[DPY] K. Davidson, S.C. Power and D. Yang, Dilation theory for rank 2 graph algebras, *J. Operator Theory*, to appear.
[Fow] N. Fowler, Discrete product systems of Hilbert bimodules, *Pacific J. Math.* **204** (2002), 335–375.

[FoR] N. Fowler and I. Raeburn, The Toeplitz algebra of a Hilbert bimodule, *Indiana Univ. Math. J.* **48** (1999), 155–181.

[GaS] D. Gaspar and N. Suciu, On the intertwinings of regular dilations, *Ann. Pol. Math.* **LXVI** (1997), 105–121.

[KuPa] A. Kumjian and D. Pask, Higher rank graph $C^*$-algebras, *New York J. Math.* **6** (2000), 1–20.

[Lan] E.C. Lance, “Hilbert $C^*$-modules”, LMS Lecture Notes Series **210**, Cambridge University Press, Cambridge, 1995.

[MS1] P. Muhly and B. Solel, Tensor algebras over $C^*$-correspondences (Representations, dilations and $C^*$-envelopes), *J. Funct. Anal.* **158** (1998), 389–457.

[MS2] P. Muhly and B. Solel, The Poisson Kernel for Hardy Algebras, *Complex Analysis and Operator Theory*, to appear.

[Pop] G. Popescu, Poisson transforms on some $C^*$-algebras generated by isometries, *J. Funct. Anal.* **161** (1999), no.1, 27–61.

[PoS] S.C. Power and B. Solel, Operator algebras associated with unitary commutation relations, *preprint*, arXiv:0704.0079v1.

[Rae] I. Raeburn, “Graph $C^*$-algebras”, CBMS Regional Conference Series in Mathematics, 103, Providence, RI, 2005.

[RaS] I. Raeburn and A. Sims, Product systems of graphs and the Toeplitz algebras of higher-rank graphs, *J. Operator Theory* **53** (2005), no. 2, 399–429.

[RaW] I. Raeburn and D. Williams, “Morita Equivalence and Continuous-Trace-$C^*$-Algebras”, Mathematical Surveys and Monographs, 60, American Mathematical Society, Providence, RI, 1998.

[SZ1] A. Skalski and J. Zacharias, Wold decomposition for representations of product systems of $C^*$-correspondences, *International Journal of Mathematics* **19** (2008), no.4, 455-479.

[SZ2] A. Skalski and J. Zacharias, Poisson transform for higher-rank graph algebras and its applications, *J. Operator Theory*, to appear.

[Sol1] B. Solel, Representations of product systems over semigroups and dilations of commuting CP maps, *J. Funct. Anal.* **235** (2006), no. 2, 593–618.

[Sol2] B. Solel, Regular dilations of representations of product systems, *Math. Proc. Royal Irish Soc.*, to appear.

[SzF] B. Sz.-Nagy and C. Foias, “Harmonic analysis of operators on Hilbert space”, North Holland, Amsterdam, 1970.

[Tim] D. Timotin, Regular dilations and models for multicontractions, *Indiana Univ. Math. J.* **47** (1998), no. 2, 593–618.

[Yee] T. Yeend, Topological higher-rank graphs and the $C^*$-algebras of topological 1-graphs, in *Operator theory, operator algebras, and applications*, 231–244, *Contemp. Math.* **414**, Amer. Math. Soc., Providence, RI, 2006.

**Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, United Kingdom**

*E-mail address: a.skalski@lancaster.ac.uk*