EXISTENCE AND UNIQUENESS OF SOLUTIONS TO NON-ABELIAN MULTIPLE VORTEX EQUATIONS ON GRAPHS

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ABSTRACT. Let \( G = (V, E) \) be a connected finite graph. We study a system of non-Abelian multiple vortex equations on \( G \). We establish a necessary and sufficient condition for the existence and uniqueness of solutions to the non-Abelian multiple vortex equations.

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1. INTRODUCTION

Vortices play important roles in many areas of theoretical physics including condensed-matter physics, cosmology, superconductivity theory, optics, electroweak theory, and quantum Hall effect. In the past two decades, the topological, non-topological and doubly periodic multivortices to self-dual Chern-Simons model, Chern-Simons Higgs model, the generalized self-dual Chern-Simons model, Abelian Higgs model, the generalized Abelian Higgs model and non-Abelian Chern–Simons model were established; see, for example, [5, 10, 18, 22, 23, 24, 27] and the references therein. Wang and Yang [25] studied Bogomol'nyi system arising in the abelian Higgs theory defined on a rectangular domain and subject to a 't Hooft type periodic boundary condition and established a sufficient and necessary condition for the existence of multivortex solutions of the Bogomol’nyi system. Caffarelli and Yang [6] established the existence of periodic multivortices in the Chern–Simons Higgs Model. In particular, Lin and Yang [20] investigated a system of non-Ablian multiple vortex equations governing coupled \( SU(N) \) and \( U(1) \) gauge and Higgs fields which may be embedded in a supersymmetric field theory framework.

In recent years, equations on graphs have attracted extensive attention; see, for example, [3, 4, 7, 8, 11, 14, 15, 16, 17, 26] and the references therein. Ge, Hua and Jiang [9] proved that there exists a uniform lower bound for the energy, \( \sum_G e^u \) of any solution \( u \) to the equation \( \Delta u + e^u = 0 \) on graphs. Huang, Wang and Yang [14] studied the Mean field equation and the relativistic Abelian Chern-Simons equations (involving two Higgs particles and any two gauge fields) on any finite connected graphs and established some existence results. Huang, Lin and Yau [15] proved the existence of solutions to the following mean field equations

\[
\Delta u + e^u = \rho_0
\]

and

\[
\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j}
\]

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on graphs.

Let $G = (V, E)$ be a connected finite graph, $V$ denote the vertex set and $E$ denote the edge set.

Inspired by the work of Huang-Lin-Yau [15], we investigate a system of non-Abelian multiple vortex equations

$$
\Delta u_1 = -N m_e^2 + m_e^2 \left( \frac{u_1}{N} + \frac{(N-1)}{N} u_2 + (N - 1) \frac{u_1}{N} \frac{u_2}{N} \right) + 4\pi \sum_{j=1}^{n} \delta_{p_j}(x),
$$

$$
\Delta u_2 = m_g^2 \left( \frac{u_1}{N} + \frac{(N-1)}{N} u_2 - \frac{u_1}{N} \frac{u_2}{N} \right) + 4\pi \sum_{j=1}^{n} \delta_{p_j}(x)
$$

on $G$, where $n, N$ are positive integers, $m_e, m_g$ are constants and $\delta_{p_j}$ is the dirac mass at vertex $p_j$.

Let $\mu : V \to (0, +\infty)$ be a finite measure, and $|V| = \text{Vol}(V) = \sum_{x \in V} \mu(x)$ be the volume of $V$.

We state our main result as follows.

**Theorem 1.1.** Equations (1.1) admits a unique solution if and only if

$$
|V| > \frac{4\pi n}{Nm_e^2} + \frac{4\pi n (N - 1)}{Nm_g^2}.
$$

The paper is organized as follows. In Section 2, we introduce preliminaries. Section 3 is devoted to the proof of Theorem 1.1.

## 2. Preliminary results

For each edge $xy \in E$, we suppose that its weight $w_{xy} > 0$ and that $w_{xy} = w_{yx}$. For any function $u : V \to \mathbb{R}$, the Laplacian of $u$ is defined by

$$
\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{yx} (u(y) - u(x)),
$$

where $y \sim x$ means $xy \in E$. The gradient form of $u$ is defined by

$$
\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy} (u(y) - u(x))(v(y) - v(x)).
$$

Denote the length of the gradient of $u$ by

$$
|\nabla u(x)| = \sqrt{\Gamma(u, u)(x)} = \left( \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy} (u(y) - u(x))^2 \right)^{1/2}.
$$

We denote, for any function $u : V \to \mathbb{R}$, an integral of $u$ on $V$ by $\int_{V} u d\mu = \sum_{x \in V} \mu(x)u(x)$. For $p \geq 1$, denote $\|u\|_p := \left( \int_{V} |u|^p d\mu \right)^{1/p}$. As in [3], we define a sobolev space and a norm by

$$
W^{1,2}(V) = \left\{ u : V \to \mathbb{R} : \int_{V} \left( |\nabla u|^2 + u^2 \right) d\mu < +\infty \right\},
$$
and
\[ \|u\|_{H^1(V)} = \|u\|_{W^{1,2}(V)} = \left( \int_V (|\nabla u|^2 + u^2) \, d\mu \right)^{1/2}. \]

The following Sobolev embedding and Poincaré inequality will be used later in the paper.

**Lemma 2.1.** (3 Lemma 5) Let \( G = (V, E) \) be a finite graph. The Sobolev space \( W^{1,2}(V) \) is precompact. Namely, if \( u_j \) is bounded in \( W^{1,2}(V) \), then there exists some \( u \in W^{1,2}(V) \) such that up to a subsequence, \( u_j \to u \) in \( W^{1,2}(V) \).

**Lemma 2.2.** (3 Lemma 6) Let \( G = (V, E) \) be a finite graph. For all functions \( u : V \to \mathbb{R} \) with \( \int_V u \, d\mu = 0 \), there exists some constant \( C \) depending only on \( G \) such that \( \int_V u^2 \, d\mu \leq C \int_V |\nabla u|^2 \, d\mu \).

### 3. The Proof of Theorem 1.1

Since \( \int_V -\frac{4\pi n}{|V|} + 4\pi \sum_{j} \delta_{x_j}(x) \, d\mu = 0 \), the equation
\[ \Delta u_0 = -\frac{4\pi n}{|V|} + 4\pi \sum_{j} \delta_{x_j}(x), \quad x \in V; \quad u_0 \leq 0 \tag{3.1} \]
admits a solution \( u_0 \). Let \( v_1 = u_1 - u_0, v_2 = u_2 - u_0 \). Then we know \( (v_1, v_2) \) satisfies
\[ \Delta v_1 = -Nm_e^2 + \frac{4\pi n}{|V|} + m_e^2 \left( e^{u_0 + \frac{v_1}{m} + \frac{(N-1)}{m}v_2} + [N-1]e^{\frac{v_1}{m} - \frac{v_2}{m}} \right), \]
\[ \Delta v_2 = \frac{4\pi n}{|V|} + m_g^2 \left( e^{u_0 + \frac{v_1}{m} + \frac{(N-1)}{m}v_2} - e^{\frac{v_1}{m} - \frac{v_2}{m}} \right). \tag{3.2} \]

Define the energy functional
\[ J(v_1, v_2) = \int_V \left\{ \frac{1}{2m_e^2} \Gamma(v_1, v_1) + \frac{(N-1)}{2m_g^2} \Gamma(v_2, v_2) + Ne^{u_0 + \frac{v_1}{m} + \frac{(N-1)}{m}v_2} + N(N-1)e^{\frac{v_1}{m} - \frac{v_2}{m}} \right\} \, d\mu. \tag{3.3} \]

We give a necessary condition for the existence of solutions to (1.1) by the following lemma.

**Lemma 3.1.** If (1.1) admits a solution, then
\[ N|V| > \frac{4\pi n}{m_e^2} + \frac{4\pi n(N-1)}{m_g^2}. \tag{3.4} \]

**Proof.** Integrating (3.2), we deduce that
\[ \int_V \left( e^{u_0 + \frac{v_1}{m} + \frac{(N-1)}{m}v_2} + [N-1]e^{\frac{v_1}{m} - \frac{v_2}{m}} \right) \, d\mu = N|V| - \frac{4\pi n}{m_e^2}, \]
\[ \int_V \left( e^{u_0 + \frac{v_1}{m} + \frac{(N-1)}{m}v_2} - e^{\frac{v_1}{m} - \frac{v_2}{m}} \right) \, d\mu = -\frac{4\pi n}{m_g^2}, \tag{3.5} \]
which is equivalent to
\[
N \int_V e^{\frac{v_0 + N}{N} + \frac{(N-1)}{N} v_2} \, d\mu = N |V| - \frac{4\pi n}{m_e^2} - \frac{4\pi n (N-1)}{m_h^2},
\]
\[
N \int_V e^{\frac{\frac{1}{N}}{N} + \frac{w}{N}} \, d\mu = \left( |V| - \frac{4\pi n}{m_e^2} \right) + \frac{4\pi n}{m_h^2}.
\]

Then the desired conclusion follows.

We now complete the proof. \(\square\)

Next, we give a priori bounds for a solution to (1.1).

**Lemma 3.2.** Suppose that \((v, w)\) is a solution of (1.1). Then we have \(v < 0, w < 0\) and \(v - w < \frac{N}{N-1}\).

**Proof.** Let \(M := \max_{x \in V} w = w(x_0)\). We claim that \(M < 0\). Otherwise, \(w(x_0) \geq 0\). Thus, we have
\[
\Delta w(x_0) = m_g^2 \left( e^{\frac{N}{N} w} - e^{\frac{N}{N} w} \right) + 4\pi \sum_{j=1}^{n} \delta_{p_j}(x) \mid_{x=x_0} > 0.
\]

On the other hand, by (2.1), we obtain
\[
\Delta w(x_0) \leq 0.
\]
This is impossible. Thus, we have
\[
w(x) < 0
\]
for all \(x \in V\).

Next, we show that \(M_1 := \max_{x \in V} v = v(x_1) < 0\). Suppose by way of contradiction that \(M_1 \geq 0\). Let
\[
F(t) := e^{\frac{N-1}{N} t} + (N-1) e^{-\frac{t}{N}}.
\]
Then it is easy to check that
\[
F'(t) := \frac{N-1}{N} e^{-\frac{t}{N}} (e^t - 1).
\]
Thus we have
\[
F(t) > F(0) = N, \ t < 0.
\]
It follows that
\[
e^{\frac{N-1}{N} t} + (N-1) e^{-\frac{t}{N}} > N, \ t < 0.
\]
Thus, we have
\[
\Delta v(x_1) = -Nm_e^2 + m_e^2 (e^{\frac{N}{N} w} + (N-1) e^{\frac{N}{N} w}) + 4\pi \sum_{j=1}^{n} \delta_{p_j}(x) > 0.
\]
By (2.1), we see that \(0 \geq \Delta v(x_1)\), this a contradiction. Thus we obtain \(v < 0\) for all \(x \in V\).
Now, we show that $M_3 := \max_{x \in V} (v - w) = (v - w)(y_0) < N \ln \frac{N}{N - 1}$. Assume the assertion is false, then we deduce that

$$\Delta \left( \frac{v}{N} - \frac{w}{N} \right)(y_0) = \left( \frac{m_e^2 - m_g^2}{N} \right) e^{N-1} w + \left( \frac{N - 1}{N} m_e^2 + \frac{m_g^2}{N} \right) e^{N-1} - m_e^2 \bigg|_{y=y_0}$$

$$> \frac{N - 1}{N} m_e^2 e^{N-1} w - m_e^2 \bigg|_{y=y_0}$$

$$\geq 0. \quad (3.10)$$

By (2.1), we have

$$0 \geq \Delta \left( \frac{v}{N} - \frac{w}{N} \right)(y_0). \quad (3.11)$$

This is impossible. Thus we have

$$v - w < N \ln \frac{N}{N - 1} \leq \frac{N}{N - 1} \tag{3.12}$$

for all $x \in V$. □

Let $\lambda_1 = m_e^2$, $\lambda_2 = m_g^2$, $v = v_1$ and $w = v_2$ in (3.2). Then we have

$$\Delta v = \lambda_1 \left( e^{u_0} e^{w} \frac{N-1}{N} w + (N - 1) e^{w} - N \right) + \frac{4\pi n}{|V|}, \quad (3.13)$$

$$\Delta w = \lambda_2 \left( e^{u_0} e^{w} \frac{N-1}{N} w - e^{w} \right) + \frac{4\pi n}{|V|}. \quad (3.14)$$

In order to prove Lemma 3.4, we need the following lemma.

**Lemma 3.3.** Suppose that $u$ satisfies $\Delta u = f$ and $\int_V u d\mu = 0$. Then we there exists $\hat{C} > 0$ such that

$$\max_{x \in V} |u(x)| \leq \hat{C} ||f||_{L^2(V)}. \tag{3.19}$$

**Proof.** From $\Delta u = f$, we deduce that

$$\int_V \Gamma(u, u) d\mu = -\int_{x \in V} f u d\mu. \quad (3.15)$$

By Cauchy inequality with $\epsilon (\epsilon > 0)$ and Lemma 2.2, there exists $C > 0$ such that

$$\int_V \Gamma(u, u) d\mu \leq \frac{1}{4\epsilon} \int_V f^2 d\mu + \epsilon C \int_V \Gamma(u, u) d\mu. \tag{3.16}$$

Taking $\epsilon = \frac{1}{2\hat{C}}$ in (3.16), we have

$$\int_V \Gamma(u, u) d\mu \leq C \int_V f^2 d\mu. \quad (3.17)$$

Applying Lemma 2.2, we know that

$$||u||_{L^2(V)} \leq C ||f||_{L^2(V)}. \tag{3.18}$$

Then we deduce that there exists constant $\bar{C} > 0$ such that

$$|u(x)| \leq \bar{C} ||f||_{L^2(V)} \tag{3.19}$$
for all \( x \in V \).

We now complete the proof. \( \square \)

To show that Theorem 1.1, we need the following Lemma.

**Lemma 3.4.** Let \( \lambda_1 = m_2^+ \) and \( \lambda_2 = m_2^- \). Set \( \{(v_k, w_k)\} \) be a sequence of solutions to equations (3.13) - (3.14) with \( \lambda_1 = \lambda_{1,k} \) and \( \lambda_2 = \lambda_{2,k} \). Assume that \( \lambda_{1,k} \to \lambda_1, \lambda_{2,k} \to \lambda_2 \) and

\[
\sup \{|v_k(x)| + |w_k(x)| \mid x \in V\} \to \infty \quad (3.20)
\]

as \( k \to +\infty \). Then \( \lambda_1 \) and \( \lambda_2 \) satisfy

\[
|V| = \frac{4\pi n}{N\lambda_1} + \frac{4\pi n(N - 1)}{N\lambda_2}. \quad (3.21)
\]

**Proof.** Denote

\[
\Delta v_k = \lambda_{1,k} \left( e^{u_0} e^{\frac{N-1}{N} w_k(x)} + (N-1)e^{\frac{w_k-w_k}{N}} - N \right) + \frac{4\pi n}{|V|} := f_k, 
\]

\[
\Delta w_k = \lambda_{2,k} \left( e^{u_0} e^{\frac{w_k-w_k}{N}} - e^{\frac{w_k-w_k}{N}} \right) + \frac{4\pi n}{|V|} := g_k. 
\]

Denote \( \bar{v}_k := \int_V v_kd\mu \) and \( \bar{w}_k := \int_V w_kd\mu \). Since \( \int_V v_k - \bar{v}_k = 0 \), by Lemma 3.3 and Lemma 3.2 we deduce that there exists \( C_N > 0 \) so that

\[
\max_V (|v_k - \bar{v}_k|) \leq C_1 ||f_k||_{L^2(V)} \leq C_N \quad (3.24)
\]

and

\[
\max_V (|w_k - \bar{w}_k|) \leq C_2 ||g_k||_{L^2(V)} \leq C_N. \quad (3.25)
\]

Suppose \( \sup_V \{|v_k(x)| \mid x \in V\} \to \infty \). Since \( v_k + u_0 < 0 \), we deduce that

\[
\bar{v}_k \leq -\int_V u_0d\mu.
\]

From (3.24), we deduce that \( v_k(x) \to -\infty \) and \( \bar{v}_k \to -\infty \) uniformly on \( V \) as \( k \to +\infty \). From Lemma 3.2, we see that

\[
\bar{v}_k - \bar{w}_k \leq \frac{N}{N - 1} |V|.
\]

Suppose that

\[
\liminf_{k \to \infty} (\bar{v}_k - \bar{w}_k) = -\infty.
\]

Subject to passing a subsequence, we have

\[
\lim_{k \to \infty} (\bar{v}_k - \bar{w}_k) = -\infty.
\]

From (3.24) and (3.25), we deduce that

\[
v_k(x) - w_k(x) \to -\infty \text{ uniformly on } V \text{ as } k \to +\infty.
\]

It follows that \( f_k \to -N\lambda_1 + \frac{4\pi n}{|V|} \). It follows from (3.24) that, by passing to a subsequence, \( v_k - \bar{v}_k \to v \) (say). Letting \( k \to +\infty \) in \( \Delta(v_k - \bar{v}_k) = f_k \). Then we have \( \Delta v = -N\lambda_1 + \frac{4\pi n}{|V|} \) on \( V \). This implies that

\[
N\lambda_1 |V| = 4\pi n.
\]
By Lemma 3.1 we deduce that
\[ N|V| > \frac{4\pi n}{\lambda_{1,k}} + \frac{4\pi (N-1)n}{\lambda_{2,k}}, \]  
and hence that \( |V| > \frac{4\pi n}{\lambda_{1,N}} \). This is impossible. Thus \( \{\bar{v}_k - \bar{w}_k\} \) is bounded. Therefore, \( \bar{w}_k \to -\infty \) as \( k \to \infty \). By (3.25), we see that
\[ v_k \to -\infty \quad \text{as} \quad k \to \infty. \]

By passing to a subsequence, we have
\[ v_k - \bar{v}_k \to v, \quad w_k - \bar{w}_k \to W \]  
uniformly for \( x \in V \) as \( k \to \infty \). Thus, we deduce that
\[ \Delta v = \lambda_1 \left( (N-1)e^{\frac{v-w+\sigma}{n}} - N \right) + \frac{4\pi n}{|V|}, \]
\[ \Delta W = \frac{4\pi n}{|V|} - \frac{\lambda_2 e^{\frac{v-w+\sigma}{n}}}{N}, \]  
and hence that
\[ \int_V e^{\frac{v-w+\sigma}{n}} d\mu = \frac{N|V|}{N-1} - \frac{4\pi n}{\lambda_1 (N-1)}, \]
\[ \int_V e^{\frac{v-w+\sigma}{n}} d\mu = \frac{4\pi n}{\lambda_2}. \]  
Therefore, we conclude that
\[ |V| = \frac{4\pi n}{N\lambda_1} + \frac{4\pi (N-1)n}{N\lambda_2}. \]  

We now complete the proof.

We will give the proof of Theorem 1.1 by applying Lemma 3.4 and the following Lemma.

**Lemma 3.5.** Assume that \( \lambda_1 = \lambda_2 \). Then equations (3.13) - (3.14) admits a unique solution if and only if \( |V| > \frac{4\pi n}{\lambda_1} \).

**Proof.** Suppose \((v, w)\) is a solution to equations (3.13) - (3.14). Due to \( \lambda_1 = \lambda_2 > 0 \), by mean value Theorem, we deduce that there exists \( \xi \) such that
\[ \Delta (v - w) = \lambda_1 e^\xi (v - w). \]
Let \( M := \max_V (v - w) = (v - w)(x_0) \). We claim that \( M \leq 0 \). Otherwise, \( M > 0 \). Then
\[ \Delta (v - w)(x_0) = \lambda_1 e^\xi (v - w) \bigg|_{x=x_0} > 0. \]  
By (2.1), we see that
\[ 0 \geq \Delta (v - w)(x_0). \]  
This is a contradiction. Thus we have \( v \leq w \) on \( V \). By a similar argument as above, we deduce that \( v \geq w \) on \( V \). Therefore, we conclude that \( v \equiv w \) on \( V \). Thus, \( v \) satisfies
\[ \Delta v = \lambda_1 (e^{w_0 + v} - 1) + \frac{4\pi n}{|V|}. \]  
It follows from [12] that (3.32) admits a unique solution if and only if \( |V| > \frac{4\pi n}{\lambda_1} \).
Proof of Theorem 1.1. Define
\[ \bar{H}^1(V) := \{ u \in H^1(V) | \bar{u} := \int_V u d\mu = 0 \} \]
and \( X := \bar{H}^1(V) \times \bar{H}^1(V) \). Let
\[
\begin{align*}
\int_V f(x, v(x) + a, w(x) + b)dx &= 0, \\
\int_V g(x, v(x) + a, w(x) + b)dx &= 0,
\end{align*}
\]
where
\[
\begin{align*}
f(x, v, w) &= \lambda_1 \left( e^{u_0(x)} e^{\frac{N-1}{N}w} + (N-1) e^{\frac{v-w}{N}} - N \right) + \frac{4\pi n}{|V|}, \\
g(x, v, w) &= \lambda_2 \left( e^{u_0(x)} e^{\frac{N-1}{N}w} - e^{\frac{v-w}{N}} \right) + \frac{4\pi n}{|V|}.
\end{align*}
\]
Denote \( A = \int_V e^{u_0 + \frac{N-1}{N}w} d\mu \), \( B = \int_V e^{\frac{v-w}{N}} d\mu \) and \( C = -\frac{N|V|}{4\pi n} \lambda_2 + \frac{\lambda_2}{\lambda_1} \). Then there exists a unique pair
\[
\begin{align*}
b &= b(v, w) = \ln \frac{BC + (N-1)B}{A(C-1)}, \\
a &= a(v, w) = \frac{1}{N} \ln \frac{BC + (N-1)B}{A(C-1)} + \ln \left( \frac{\lambda_1 N|V| - 4\pi n}{\frac{BC+(N-1)B}{A(C-1)}A + (N-1)B} \right) \lambda_1
\end{align*}
\]
such that
\[
\begin{align*}
\int_\Omega f(x, v(x) + a, w(x) + b)dx &= 0, \\
\int_\Omega g(x, v(x) + a, w(x) + b)dx &= 0.
\end{align*}
\]
For any \( (v, w) \in X \), define
\[ (Q, W) := T(v, w) \in X, \]
where \( (Q, W) \in X \) is the unique solution to the equations
\[
\begin{align*}
\Delta Q &= f(x, v + a, w + b), \\
\Delta W &= g(x, v + a, w + b).
\end{align*}
\]
By a similar argument as Lemma 3.3, we know that \( T \) is completely continuous. Furthermore, by Lemma 3.4 there exists \( M > 0 \) such that
\[
\|Q\|_{H^1(V)} + \|W\|_{H^1(V)} \leq M. \tag{3.35}
\]
Thus, we may define the Leray-Schauder degree \( d(\lambda_1, \lambda_2) \) for \( T \). From Lemma 3.5 there exists a sufficiently large \( \lambda_0 > 0 \) so that \( d(\lambda_0, \lambda_0) = 1 \). In view of
\[
\left\{ (\lambda_1, \lambda_2) \middle| \frac{4\pi n}{N\lambda_1} + \frac{4\pi n(N-1)}{N\lambda_2} \right\}
\]
is path-connected. We see that \( d(\lambda_1, \lambda_2) = d(\lambda_0, \lambda_0) = 1 \). Therefore, \((3.13)-(3.14)\) admits at least one solution. It is easy to check that \( J \) defined by \((3.3)\) is convex in \( H^1(V) \). Thus the solution of \((1.1)\) is unique.

We now complete the proof. □

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