Complex Path Integrals and the Space of Theories

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The Feynman Path Integral is extended in order to capture all solutions of a quantum field theory. This is done via a choice of appropriate integration cycles, parametrized by \( M \in \text{SL}(2, \mathbb{C}) \), i.e., the space of allowed integration cycles is related to certain \( Dp \)-branes and their properties, which can be further understood in terms of the “physical states” of another theory. We also look into representations of the Feynman Path Integral in terms of a Mellin–Barnes transform, bringing the singularity structure of the theory to the foreground. This implies that, as a sum over paths, we should consider more generic paths than just Brownian ones. Finally, we are able to study the Space of Theories through our examples in terms of their Quantum Phases and associated Stokes’ Phenomena (wall-crossing).

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1. Introduction

This paper has the goal to extend our previous work, [1], bringing to the foreground some issues of Quantum Field Theory that have not quite found their place yet. In particular, it aims to introduce quantum phases as the analogous object to chamber walls (wall-crossing and Stokes' phenomena [8]), in order to study the “Space of Theories” via suitable generalizations of the [Feynman] Path Integral.

We will start by considering the quantization procedure as defined by the path integral and its respective algebra of observables,

$$\mathcal{Z}_\tau^\Sigma[J] = \int e^{iS_\tau^\Sigma[\varphi]} e^{iJ \varphi} D\varphi;$$

$$= \mathcal{F}[e^{iS_\tau^\Sigma[\varphi]}][J];$$

$$\langle O_1 \cdots O_n \rangle^\Sigma = \int O_1 \cdots O_n e^{iS_\tau^\Sigma[\varphi]} e^{iJ \varphi} D\varphi;$$

$$= \mathcal{F}[O_1 \cdots O_n e^{iS_\tau^\Sigma[\varphi]}][J];$$

where $S_\tau^\Sigma$ is the Action of the theory, with $\Sigma$ representing a possible surface term needed in order to establish the self-adjointness of the Laplace–Beltrami operator in the kinetic term of the Action, interpolating among the possible boundary conditions suitable for the problem at hand, and $\tau$ represents the possible coupling constants of the theory; $O_i = O_i[\varphi], i = 1, \ldots , n$ are operators forming an algebra in an appropriate Hilbert space; and $\mathcal{F}$ represents the Fourier Transform (with respect to the source $J$), and the path integral and the trace of the operators are cast as suitable Fourier transforms of the Feynman weight, $e^{iS_\tau^\Sigma[\varphi]}$.

In order for the Fourier transforms above to be well defined, the integrand and the kernel have to be single-valued. To that end, we can consider their Complexification and look for suitable integration cycles labeled by $\tau$ and $\Sigma$, $\mathcal{C}_\tau^\Sigma$, i.e., Riemann sheets rendering our problem well defined. If $\mathcal{C}_\tau^\Sigma$ is ultimately $\mathbb{R}$, we recover the above definitions in the sense of Fourier transforms. However, for non-trivial cycles we end up with a more general integral transform, one that preserves the symplectic structure of the problem at hand. In fact, we can ask ourselves what is the most general integral transform that preserves the symplectic structure of a problem. And, as we will see in sections 2 and 3, this is accomplished by the Linear Canonical Transform (a linear map of the phase space preserving the symplectic form).

In this sense, it is clear that $\tau$ and $\Sigma$ will determine the analytic structure of our quantization procedure: in fact, the integration cycle $\mathcal{C}_\tau^\Sigma$ ultimately determines the fall-off properties of the fields in our theory, thus relating the decaying properties of the fields with the analyticity of the integral transform at hand, [16, in particular, the Paley–Wiener theorem (and its various generalizations), describing functions through their analytic extensions in the complex domain]. Actually, making this relation explicit will motivate us, in sections 4 and 5, to cast the Path Integral as a Mellin–Barnes transform, bringing the role of the analytic structure of our theory to the foreground.

Thus, in a sense that will become clearer later, the central concept will be the analytical structure of our quantization procedure via the Feynman Path Integral, including its source term, $e^{iJ \varphi}$: both, the integrand and the source term, have to live in the same Riemann sheet, otherwise we get an ill-defined theory. Therefore, realizing that the source term reflects the
analytic structure of the Feynman weight (after all, \(J\) and \(\varphi\) are related by a Legendre transform), we can already infer that \(\varphi \rightarrow -i \partial J\) is not going to express the most general case: this will only be true in the situation where \(\mathcal{O}^\Sigma_\tau\) defines a Fourier transform; otherwise, the Legendre-dual relation between \(J\) and \(\varphi\) will typically be more elaborate.

As a side remark, we can note that in order for the path integral quantization to be consistent with its respective Schwinger–Dyson equations, the field, \(\varphi\), has to be determined by its respective quantum equations of motion. Therefore, given that \(\tau\) and \(\Sigma\) determine the properties (monodromy, Stokes’ phenomena, etc) of the quantum equations of motion, i.e., of the Schwinger–Dyson equations, the integrand and \(e^{iJ \varphi}\) have to live in the same Riemann sheet. This is a trivial observation if the quantum equations of motion are of 1st degree. However, they are typically of higher degree, meaning that there are multiple solutions, which we can label through \(\tau\) and \(\Sigma\), thinking that each solution is associated to a quantum equation of motion with suitable values of \(\tau\) and \(\Sigma\).

Thus, in order to bring these issues to the foreground, we will generalize the path integral (thought as an integral transform in a general sense, or as a Fourier transform in a more specific one) in two different ways: as a Linear Canonical Transform (sections 2 and 3), and as a Mellin–Barnes Transform (sections 4 and 5). These are particularly useful, since the former aims to be invariant by canonical transformations, while the latter is relevant to make explicit the analytic structure of the theory, e.g., the modern subject of scattering amplitudes in \(\mathcal{N} = 4\) Super Yang–Mills. In this way, we will be able to associate certain theories formulated as Linear Canonical Transforms with their respective Mellin–Barnes transforms, making clear how one framework is related to the other, i.e., we will be able to relate certain orbits of the \(\text{SL}(2, \mathbb{C})\) parametrizing the Linear Canonical Transform (as seen below) with appropriate Mellin–Barnes transforms, connecting the symmetries described by these orbits to the singularity structure made explicit in the Mellin–Barnes transforms.

This brings us to the question of what kinds of paths are allowed within the context of the path integral. After all, taking the analytic structure of the quantization process, of the path integral, seriously, how do we know that more exotic paths (woven across several allowed and distinct Riemann sheets) are not contributing, despite cancellations due to the highly oscillatory nature of the integrand?

In quantum mechanics, the usual story says that all paths are allowed, although the equivalence between the integral formulation (path integral) and the differential one (Schrödinger equation) is only established at the level of Brownian paths, where we think of Schrödinger’s equation as a diffusion equation, analytically continued. We are interested in this connection because of its possible implications in Quantum Field Theory, where we deal, instead, with Feynman’s Path Integral and Schwinger–Dyson’s equations as the two sides of the quantization procedure. We want to study the connection between the dimension of the solution space of the Schwinger–Dyson equations and the number of vacuum states present in a physical theory: in quantum mechanics, the dimension of the solution space of Schrödinger’s equation determines the number of ground states a certain model has — as is the case with all differential equations and the dimension of their respective solution spaces: the dimension of the solution space (vector space of analytic functions) of a \(n\)th order linear homogeneous differential equation with analytic coefficients is \(n\). How does the equivalent problem work in Quantum Field Theory? Does Feynman’s Path Integral include only Brownian paths or does it include more generic paths? It is to make sense of questions such as these that we want...
to study the connections between the integral and differential formulations of Quantum Field Theories.

The first extension of this result was done in [2] and can be achieved if one consider slightly more generic paths, called Lévy flights. In this case, the path integral becomes a Fractional Fourier Transform. The gist of this approach is that Schrödinger’s equation becomes fractional (i.e., rather than differential operators one uses fractional differential operators), while the path integral now includes paths of fractal dimension. More to the point, while Brownian motions have a scaling relation of the form $x^2 \propto t$ and fractal dimension $d_{\text{Brownian}} = 2$, Lévy flights scale as $x^\alpha \propto t$ and have fractal dimension $d_{\text{Lévy fractal}} = \alpha$, with $0 < \alpha \leq 2$. This means that, for example, in the case of a nonrelativistic particle, its dispersion relation is given by $E = D_\alpha p^\alpha$, where $D_\alpha$ is the generalized fractional quantum diffusion coefficient. If $\alpha = 2$, we have that $D_2 = 1/2m$ and $E = p^2/2m$, which is the case for Brownian paths.

The connection between this analysis in terms of scaling and dispersion relations, and our previous argument, is through the fall-off properties of the field content of our theory. Roughly speaking, the integration cycle $C^\Sigma_\tau$ ultimately determines these decaying properties (Stokes’ phenomena, monodromy of the quantum equations of motion, etc). And, as such, $C^\Sigma_\tau$ is related to the scaling and dispersion relations above, determining whether they are isotropic or anisotropic (and the respective scaling factors). Finally, the subject of anisotropic scaling relations has been studied before in Hořava–Lifshitz gravity, where the scaling goes as $x \propto t^z$, meaning that $\alpha \sim 1/z$, and considering the Lorentz symmetry as emergent when the theory flows to some value of $z \sim 1/\alpha$ (e.g., $z = 3 = 1/\alpha$), [6]. Unfortunately, the relation between anisotropic scaling, fractality, and integration cycles of the path integral does not seem to have been analyzed before.

However, this is not the most general case one can consider. In fact, this is just one among several possibilities. The bottom line is that the path integral can be parametrized by $\text{SL}(2, \mathbb{C})$, where different orbits of the group correspond to distinct possible transformations: from a Fourier and Fractional Fourier, to a Linear Canonical Transform.

Furthermore, we will use the expression of the path integral as a Linear Canonical Transform in order to introduce a representation of the path integral as a Mellin–Barnes transform, given by Fox’s $H$-function. This is possible, as we will see later, because of a result stating that all stable probability densities can be represented in terms of Fox’s $H$-function, [9]. The advantage of such a construction is that we make explicit the analytic structure of our theory: the monodromy of the integration cycles defining the $H$-function representing the path integral will determine the ramification of the theory at hand. The study of ramifications is generally described by Picard–Lefschetz theory.

Thus, we tie in with our previous construction in different ways. Firstly, this approach has been used before, namely in [2], in order to express the fundamental solutions for a fractional diffusion process in terms of an appropriate $H$-function. Secondly, the properties of the $\text{SL}(2, \mathbb{C})$ parametrization of the Linear Canonical Transform are reflected in the properties of the $H$-function: given that we explicitly have the analytic structure of the theory, it is not difficult to see the results of certain analytic continuations. Finally, different $H$-functions, representing different path integrals, will be associated to different stable probability distributions and, as such, with different allowed paths in the path integral, giving further evidence that we need to include more exotic paths than just Brownian ones.
2. Linear Canonical Transform

At this point, a note is in order: the above results are not dependent on the particulars of our example(s). For instance, $\varphi$ could be $\mathbb{R}$- or $\mathbb{C}$-valued, it could also be Matrix-valued (or, more generally, Tensor-like in character), or even Lie algebra-valued (including graded Lie algebras, i.e., SUSY): the exact same results would be true (with minor modifications, e.g., tracing out the matrix degrees-of-freedom, or defining the Fourier Transform over the appropriate group, etc).

The outline of this paper is as follows: in section 2 we will introduce the Linear Canonical Transform, and in section 3 we will describe the path integral as an LCT; in section 4 we will introduce Fox’s $H$-function and some of its generalizations, while in section 5 we will describe the path integral as a Mellin–Barnes transform via the use of the $H$-function; in section 6 we will develop several examples supporting our previous constructions, and in section 7 we will suggest how this approach can be used to deal with higher-dimensional systems. Finally, in section 8 we will make a synopsis and hint at possible further developments.

2. Linear Canonical Transform

Canonical Transforms are those transformations which map the operators $Q$ and $P$ into linear combinations of themselves. The Linear Canonical Transformation is the linear map which preserves the symplectic form: it is understood as a family of integral transforms, parametrized by $\text{SL}(2, \mathbb{R})$, generalizing several classical transforms, such as the Fourier, the fractional Fourier, the Laplace, the Gauss–Weierstrass, the Bargmann, and the Fresnel transforms.

One of the Fourier Transform’s main properties is that the operator $Q$ (multiplication by the argument: $(Qf)(q) = qf(q)$) is mapped into the operator $P$ (differentiation: $(Pf)(q) = -i \frac{df(q)}{dq}$), and vice-versa with a minus sign. Such behavior is analogous of a rotation by $\pi/2$ in the $Q$–$P$ plane, the Phase Space. However, we will look for linear operators $C$ which map $Q$ and $P$ into linear combinations of each other,

$$Q' = CQC^{-1} = dQ - bP;$$
$$P' = CP^{-1} = -cQ + aP;$$

where $a, b, c, d \in \mathbb{R}$ for now, with the caveat that $[Q', P'] = C[Q, P]C^{-1} = i \mathbb{1}$, implying that

$$ad - bc = 1.$$ 

We readily see that the Fourier Transform corresponds to $a = 0 = d$, $b = 1 = -c$, and that the identity transformation corresponds to $a = 1 = d$, $b = 0 = c$. So, we will label the transform operator as $C_M$ with the unimodular matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{det } M = 1.$$
The operator $C_M$ is linear and has the following properties:

\[
(C_M f)(q) = f^M(q) ;
\]

\[
(C_M Q f)(q) = (C_M Q C_M^{-1} C_M f)(q) ;
\]

\[
= d (q f^M(q)) + i b \left( \frac{df^M(q)}{dq} \right) ;
\]

\[
(C_M P f)(q) = (C_M P C_M^{-1} C_M f)(q) ;
\]

\[
= -c (q f^M(q)) - i a \left( \frac{df^M(q)}{dq} \right) .
\]

**Integral Transform and Kernel**

To concretely realize the operator $C_M$ let us use an integral transform with the kernel $C_M(q', q)$,

\[
f^M(q') = (C_M f)(q') = \int f(q) C_M(q', q) dq ; (1)
\]

Combined with the properties previously derived, we have the following expression for the kernel $C_M$,

\[
C_M(q', q) = \frac{e^{-i \pi/4}}{\sqrt{2 \pi b}} e^{i(a q^2 - 2 q q' + d q'^2)/2 b} . (2)
\]

For Real parameters, (2) oscillates strongly in the limits where $|q| \to \infty$ and $|q'| \to \infty$ but with a fixed modulus at $1/\sqrt{2 \pi |b|}$.

The inverse of the $C_M$ transform (1) can be shown to be given by,

\[
f(q) = (C_M^{-1} f^M)(q) = \int f^M(q') C_M^*(q, q') dq' . (3)
\]

The composition of two such transforms, $C_{M_1}$ and $C_{M_2}$, is given by the transform whose matrix parameter is the product of the constituent transforms, i.e., $C_{M_1 M_2}$. In Group Theory, the $C_M$ are a ray representation of $SL(2, \mathbb{R})$ ($2 \times 2$ unimodular matrices), also known as the metaplectic representation (Weil).

The identity $C_M$ transform corresponds to $M = 1$, as expected: $\lim_{M \to 1} C_M(q', q) = \delta(q - q')$. From this result, it is possible to show that all lower-triangular transformations $M$, where $b = 0$, have the form,

\[
(C_M(b=0) f)(q) = \frac{e^{i c q^2/2a}}{\sqrt{a}} f(q/a) .
\]

These are called geometric transformations, once they consist of dilations by $a$ and/or multiplication by an oscillating Gaussian.

As for the inverse canonical transform, $C_{M^{-1}}(q', q) = C_M^*(q, q')$, we just have to note that,

\[
M^{-1} = \begin{pmatrix} d & e^{-i \pi} b \\ e^{i \pi} c & a \end{pmatrix} ;
\]
where we have to watch out for the appropriate sheet in the complex $b$-plane.

Thus, we have showed that the set of LCTs $C_M$ forms a group of unitary transformations, $\text{SL}(2, \mathbb{R})$; i.e., LCTs are a family of integral transforms parametrized by a 3-manifold that can be understood as the action of $\text{SL}(2, \mathbb{R})$ on Phase Space, the field-source plane. The name comes from the map preserving the symplectic (or multiplectic, for the case of multi-dimensional transforms) structure, canonical transformations, once $\text{SL}(2, \mathbb{R})$ can be understood as the symplectic group $\text{Sp}_2$. Thus, the LCTs are linear maps in the coordinate–momentum domain $\mathbb{Q}$–$\mathbb{P}$ (i.e., phase space), preserving the symplectic form.

The LCT, parameterized by $M \in \text{SL}(2, \mathbb{R})$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$, can also be defined more straightforwardly as follows,

$$
\mathcal{F}_M[J] = \sqrt{-i} \int_{-\infty}^{\infty} e^{i \pi J^2d/b} F[\varphi] e^{i \pi \varphi^2a/b} e^{-2\pi i \varphi J/b} d\varphi ; \quad b \neq 0 ;
$$

$$
\mathcal{F}_{M'}[J] = \sqrt{d} \int_{-\infty}^{\infty} e^{i \pi J^2c d} F[J d] ; \quad b = 0 ;
$$

where the prime denotes those elements where $b = 0$: $M' = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$.

Let us describe some of the special cases generalized by the LCT, i.e., physically relevant cases characterized by symmetries described by orbits of $\text{SL}(2, \mathbb{C})$:

**Parabolic Subgroup or Dilation Transform** Corresponds to a dilation by a factor $\Delta$ (and are a family of parabolic elements, represented by $a = e^{-\beta/2}$, $\beta \in \mathbb{R}$),

$$
M_{\text{DT}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta^{-1} \end{pmatrix} .
$$

**Fresnel or Gauss–Weierstrass Transform** The Fresnel Transform corresponds to shearing by $\Lambda$ (cutoff), while the Gauss–Weierstrass “diffusive” transform is its analytical continuation, $\Lambda \mapsto \Lambda = -i b$:

$$
M_{\text{GWT}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \Lambda \\ 0 & 1 \end{pmatrix} .
$$

**Gaussian Transform** Multiplication by a Gaussian of width $-i w$,

$$
M_{\text{GT}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w^{-1} & 1 \end{pmatrix} .
$$

**Elliptic Subgroup or Fractional Fourier Transform** Corresponds to a rotation by an arbitrary angle $\omega$; they are the elliptic elements of $\text{SL}(2, \mathbb{R})$, represented by:

$$
M_{\text{FrFT}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} ;
$$

**Fourier Transform** Corresponds to a rotation by $90^\circ$, which is a special case of the Fractional Fourier Transform for $M_{\text{FrFT}}(\omega = \pi/2)$, with the corresponding integral kernel given by
\( C_{M_{FT}} = e^{-i \pi/4} M_{FT} \), where

\[
M_{FT} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};
\]

**Hyperbolic Subgroup**

\[
M_{HS} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cosh \omega & -\sinh \omega \\ -\sinh \omega & \cosh \omega \end{pmatrix};
\]

**Laplace Transform** The bilateral Laplace Transform can be obtained once we extend into the Complex domain, i.e., \( SL(2, \mathbb{C}) \),

\[
M_{LT} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};
\]

**Bargmann Transform** This is a particular case of the Complex LCTs, analogous to the Fourier Transform for Real LCTs, where we use \(-i = e^{-i \pi/2}\)

\[
M_{BT} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.
\]

For transformations with \( a \neq 0 \) (i.e., excluding FTs and special cases of the FrFTs), we can show that every \( C_M \in SL(2, \mathbb{R}) \) can be written as the product of a GT, a DT and a GWT, as follows:

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} = M_{GT}(c/a) M_{DT}(a) M_{GWT}(b/a).
\]

When involving the Fourier Transform, we have that,

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 1/b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a/b & 1 \end{pmatrix} = M_{DT}(b) M_{FT} M_{GT}(a/b).
\]

As expected, the generalized uncertainty relation we get from \( \Delta_f \) and \( \Delta_{f^M} \), respectively the dispersion of \( f \) and \( f^M \), is given by:

\[
\Delta_f \Delta_{f^M} \geq \frac{b^2}{4}.
\]

Finally, we notice that the transform kernel, \( C_M \), plays the role of the Dirac \( \delta \), in the sense of closing the relation of completeness. This is called the Coherent-State basis.
Hyperdifferential Operator Realization  We want to find a differential operator in the following context: given a 1-parameter family of integral transforms,

\[(C_{M(\tau)} f)(q') = \int C_{M(\tau)}(q', q) f(q) \, dq = f(q', \tau) ; \tag{4}\]

including the identity for \( \tau = 0 \), i.e., \( M(0) = 1 \) and \( f(q', 0) = f(q') \), we want a differential operator \( H \) that can be written as,

\[(C_{M(\tau)} f)(q) = \exp(i \tau H) f(q) = \sum_{n=0}^{\infty} \frac{(i \tau H)^n}{n!} f(q) .\]

Operator of this kind appear mainly in connection with the time evolution of the wave and diffusion equations, translations and dilatations.

Formally, we can write

\[
H f(q') = -i \frac{\partial}{\partial \tau} \int \left. C_{M(\tau)}(q', q) f(q) \, dq \right|_{\tau=0} .
\]

The operator \( H \) generates the 1-parameter integral transform family (4).

Of particular interest are the following 1-parameter subgroups of \( SL(2, \mathbb{R}) \):

Parabolic Subgroup or Dilatation Transforms  \( H^D = \frac{1}{2} \left( Q P + P Q \right) \),

\[
M^{DT}(e^{-\tau}) = \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix} ;
\]

\[
\left. \frac{\partial}{\partial \tau} C_{M^{DT}}(q', q) \right|_{\tau=0} = \frac{1}{2} \delta(q - q') - q' \frac{\partial}{\partial q} \delta(q - q') ;
\]

\[
= \left( \frac{1}{2} - q \frac{\partial}{\partial q} \right) C_{M^{DT}}(q', q) \right|_{\tau=0} ;
\]

Fresnel or Gauss–Weierstrass Transforms  \( H^{GW} = \frac{1}{2} \pi^2 \),

\[
M^{GWT}(\tau) = \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} ;
\]

\[
\left. \frac{\partial}{\partial \tau} C_{M^{GWT}}(q', q) \right|_{\tau=0} = -\frac{i}{2} \frac{\partial^2}{\partial q^2} C_{M^{GWT}}(q', q) \right|_{\tau=0} ;
\]

Gaussian Transform  \( H^G = \frac{1}{2} \pi^2 \),

\[
M^{GT}(\tau) = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} ;
\]

\[
\left. \frac{\partial}{\partial \tau} C_{M^{GT}}(q', q) \right|_{\tau=0} = \frac{i}{2} q^2 C_{M^{GT}}(q', q) \right|_{\tau=0} ;
\]
### Hyperbolic Subgroup $\mathcal{H}^{HS} = \frac{1}{2} (P^2 - Q^2)$,

$$
\mathbf{M}^{HS}(\tau) = \begin{pmatrix}
\cosh \tau & -\sinh \tau \\
-\sinh \tau & \cosh \tau
\end{pmatrix};$

$$
\frac{\partial}{\partial \tau} \mathcal{C}_{\mathbf{M}^{HS}}(q', q) \bigg|_{\tau=0} = \frac{i}{2} \left( -\frac{\partial^2}{\partial q^2} - q^2 \right) \mathcal{C}_{\mathbf{M}^{HS}}(q', q) \bigg|_{\tau=0};$

### Elliptic Subgroup or Fractional Fourier Transform $\mathcal{H}^{FrF} = \frac{1}{2} (P^2 + Q^2)$,

$$
\mathbf{M}^{FrFT}(\tau) = \begin{pmatrix}
\cos \tau & -\sin \tau \\
\sin \tau & \cos \tau
\end{pmatrix};$

$$
\frac{\partial}{\partial \tau} \mathcal{C}_{\mathbf{M}^{FrFT}}(q', q) \bigg|_{\tau=0} = \frac{i}{2} \left( -\frac{\partial^2}{\partial q^2} + q^2 \right) \mathcal{C}_{\mathbf{M}^{FrFT}}(q', q) \bigg|_{\tau=0};$

Again, if we allow for complex parameters in the expressions above, we obtain the group $\text{SL}(2, \mathbb{C})$ of complex 2-dimensional special (unimodular) linear transformations.

We now see that linear canonical transformations are generated by all operators constructed out of quadratic expressions in $Q$ and $P$. Further, we have the following objects,

1. Integral Transforms $\mathcal{C}_\mathbf{M}$;
2. Hyperdifferential Operators $\exp(i \tau \mathcal{H})$; &
3. $2 \times 2$ matrices $\mathbf{M}$.

For every element in one there are corresponding elements in the other two, and this correspondence is preserved under composition, sum and multiplication.

### 3. Path Integral as a Linear Canonical Transform

One of the more common quantization techniques is via the Feynman Path Integral formulation, generalizing the Action Principle. Furthermore, the Path Integral allows us to easily change coordinates between different canonically conjugated descriptions of the same quantum system. However, it is rather unfortunate that most treatments of the Path Integral do not do it full justice, e.g., a priori assuming that its measure is $\mathbb{R}$-valued, and thus missing on possible extra solutions to the quantum equations of motion and extensions of the theory (c.f., $[7, 8, 10, 16]$, roughly summarized in appendix A).

Historically, defining the Path Integral’s measure, in any way (ranging from more mathematically rigorous approaches all the way to the more formal and heuristic ones), has always been the Achilles’ heel of the Path Integral quantization scheme. What we will show below is a generalization that brings to the foreground the Legendre-dual relation between the field, $\varphi$, and its source, $J$. Once again, note that $\varphi$ can be quite general: $\mathbb{R}$- or $\mathbb{C}$-valued, Vector- or Matrix- or Tensor-valued, and even Lie- or graded Lie Algebra-valued (SUSY): the necessary modifications, e.g., tracing matrix degrees-of-freedom, do not affect our discussion: there
is no loss of generality in the results below dealing with scalar fields (spin 0 bosonic fields) or D0-branes.

The idea of generalizing the Path Integral not only as an integral transform, but as a general Linear Canonical transform, explicitly bringing forth the symplectic structure of the theory, is motivated by several reasons. In what follows, we can consider a “finite” (discrete) version of the Path Integral in several different but analogous ways: we can think of our theory as having an IR- and an UV-cutoff (including defining the theory via an appropriate Vertex Operator Algebra that implements the desired IR- and UV-cutoffs), or we can consider different discretization schemes (discrete differential forms, finite exterior calculus, Matrix models — BFSS/IKKT, infinite momentum frame: \( N \times N \) matrices describing \( N \) Dirichlet particles —, naïve lattice discretizations, etc). In these cases the Path Integral as well as the questions and motivations below are well-defined and tractable.

**Nonlinear Stokes’ Phenomena & Path Integral Convergence:** We want the Path Integral to be well-defined in a certain sense, even if it is mathematically loose (and only satisfies the physicist’s level of rigour). One of the ways to achieve this is through studying the asymptotics of the oscillatory Riemann–Hilbert problem associated to the nonlinear Schwinger–Dyson equation describing the system at hand: this means that the fall-off behavior of the fields of our theory will ultimately determine their algebra and quantization procedure. One of the methods used to attack such problems is the *steepest descents* (also called ‘saddle-points’ or ‘stationary phase’), which identifies the contribution from the dominant saddle-point: once the integration contour is appropriately deformed, the original integral is extended (by a change of variables) to the entire \( \mathbb{R} \)-axis. In this sense, we will consider the Path Integral over a \( \mathbb{C} \)-dimensional integration cycle (a multi-dimensional integration contour), that we deform in order to obtain a Path Integral over the \( \mathbb{R} \)-axis (which is the usual way they are presented to us). Moreover, we want to associate to each integration cycle a *Matrix label* whose job is to ‘lift’ the Feynman Path Integral to an integral transform called *Linear Canonical Transform*, [3].

A previous generalization of the Path Integral along these lines has already been done, albeit only using the Elliptic subgroup of \( \text{SL}(2, \mathbb{R}) \): it re-interprets the Path Integral as a Fractional Fourier Transform, [2]. Unfortunately, this is but a subset of all that can be done, although we can already see some of the features of the general case: the Elliptic subgroup of \( \text{SL}(2, \mathbb{R}) \) consists of rotations parametrized by an arbitrary angle,

\[
\mathbf{M}_{\text{FrFT}} = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix};
\]

which means that we are performing a rotation of \( \omega \) in Phase Space (\( \varphi-J \) plane). This means that the Fractional Fourier Transform, i.e., the Path Integral, has real and imaginary parts, and its imaginary part only vanishes for special values of \( \omega \).

However, this rotation is limited by the integration cycle defining the Fractional Fourier Transform (viz. Path Integral): we cannot cross the poles of the integrand being transformed, given that the integration cycle delimits the allowed regions. The boundary between two allowed regions is subjected to Stokes’ phenomena, also known as wall-crossing phenomena in this setting, [12].
These Stokes’ lines are curves in the $\mathbb{C}$omplex $\varphi$-plane, where the integrand of the Path Integral decays without oscillations; analogously, anti-Stokes’ lines are curves where the integrand oscillates without change in amplitude. Upon crossing the Stokes’ lines, the integrand jumps discontinuously; equivalently, the integrand becomes more and more wildly oscillatory as it approaches an anti-Stokes’ line. The steepest descent path is relevant for topological reasons (more about this on the next item below), i.e., we need to know which saddle-point lies on the path. Lastly, we should remind ourselves between the connection of Stokes’ phenomena and the study of hyperasymptotics: the analysis of divergent asymptotic series, where key parts of the answer lie ‘beyond all orders’, because the desired features are exponentially small in the perturbation parameter, [8]. This should gives us a clear connection between the perturbative series for our model, Stokes’ phenomena, and wall-crossing.

Along these lines, we can define each “branch” (wall chamber) of the theory determined by $S_\tau[\varphi]$, as bounded by appropriate Stokes’ lines (codimension one manifold). Thus, we can translate the quantities computed in one “branch” (wall chamber) into another using the discontinuous jump determined by crossing Stokes’ lines: this enables the computation of the discontinuous change of quantities such as integer invariants, space of BPS states, etc, across walls of marginal stability. In this sense, maybe we can borrow the definition of quantum phases, which originally means to indicate different quantum states of matter at zero temperature: the idea is that varying parameters of the theory, quantum fluctuations can drive phase transitions. In our current case, the integration cycle and associated Stokes’ phenomena essentially determine the allowed range of continuous values of the coupling constants. It is at the point where the coupling constants suffer a discontinuous leap (along a Stokes’ line) that a quantum phase transition happens. Thus, a given theory $S_\tau[\varphi]$ may have more than one quantum phase associated to it, provided we find more than one integration cycle rendering the Path Integral well-defined.

Here we depict a “single theory”, $S_\tau[\varphi]$, that has three quantum phases for different values of the couplings: $\tau = \tau_1, \tau_2, \tau_3$. Separating each set of allowed couplings are Stokes’ lines (codimension one manifolds). In this sense, we can say that this theory has three quantum phases (wall chambers), such that crossing a Stokes’ line (wall-crossing) is analogous to a quantum phase transition.

Different Quantum Phases are associated with distinct integration cycles that are bounded by the appropriate Stokes’ lines. This yields distinct values for the matrix parameter of the Linear Canonical Transform, $M \in \text{SL}(2, \mathbb{C})$, as shown in the examples below. Further, because each quantum phase is delimited by Stokes’ lines, crossing from one quantum
phase to another implies the picking up of a Stokes phase factor, i.e., a finite contribution that accounts for the discontinuity between the two quantum phases, and is related to the monodromy of the Schwinger–Dyson equations.

Finally, given the nature of the Linear Canonical Transform as generalizing several different kinds of integral transforms, it also represents an extension in the allowed classes of paths present in the Path Integral (viewed as a sum over paths): on top of Brownian paths, one should also consider more generic cases, such as Lévy flights, and so on, [2].

**Symplectic Structure & Complex Path Integral:** We can consider the above as the problem of quantizing a symplectic manifold, namely the Phase Space of the particular theory we have at hands: we choose a Lagrangian splitting (a maximal isotropic subspace, a choice of coordinates and momenta, i.e., Phase Space $\mathcal{P} \simeq \mathcal{L} \otimes \mathcal{L}'$ where $\mathcal{L}$ and $\mathcal{L}'$ are isotropic subspaces and $\dim\mathbb{R}\mathcal{L} = \dim\mathbb{R}\mathcal{L}' = \frac{1}{2} \dim\mathbb{R}\mathcal{P}$: this is tantamount to choosing a polarization for the phase space at hand), and the integration cycle for the Path Integral is the extra information provided towards integrability.

In this sense, this is very reminiscent of the Bargmann–Segal representation (cf. Bargmann–Fock or holomorphic representation) equipped with a Kähler polarization and compatible affine and Complex structures, [17]. This is reasonable given that the Linear Canonical Transform also generalizes the Bargmann transform, as we saw in the previous section, with the matrix parameter being $M_{BT} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix}$ and $-i = e^{-i\pi/2}$.

The relation between Phase Space quantization (also known as Weyl or deformation quantization: focus on the multiplicative structure of quantum observables), geometric quantization (focus on the representation of observables), and the $A$-model in string theory is discussed in [17, 18]. Essentially, the basic idea is to start with a symplectic manifold and choose a suitable complexification for it. There are three points of view to this subject:

1. Complexify the phase space (i.e., the original symplectic manifold) to a complex symplectic manifold that has a good $A$-model with respect to the symplectic structure given by the imaginary part of the complexified one;

2. Start with a complex symplectic manifold and pick a suitable coisotropic brane, assuming it to be an $A$-brane with respect to the symplectic structure given by the imaginary part of the original complex one; &

3. This is the more natural one from the viewpoint of 2-dimensional TFTs, bringing to the foreground the $A$-model of the complexified phase space: there may be many inequivalent choices of $A$-branes, corresponding to different choices of a complex structure. Thus, the same $A$-model can lead to quantization of the phase space in different symplectic structures.

The point is to choose a suitable behavior at infinity for the functions that we can quantize. Generally, the condition chosen is that only functions of polynomial growth are allowed. However, the integration cycle that we choose for our Path Integral will ultimately determine this fall-off behavior, [16]: understanding the Path Integral as a Fourier Transform, the idea is to relate the decay properties at infinity of its integrand
with the analiticity properties of the Path Integral (in a loose analogy with the Paley–Wiener theorem, [16]). This very same observation can be made through the study of the [global] boundary conditions of the Schwinger–Dyson operator (viz. Bethe–Salpeter equation, Ward–Takahashi identities, and anomalies). Thinking in terms of differential equations (local systems and Higgs bundles, [11]) makes the discussion about boundary conditions more straightforward, where different boundary conditions are related to distinct extensions of the respective differential operators, [10] (c.f., appendix A).

Historically, this study can be traced back to some early ideas by V. I. Arnold, regarding Hermitian operators and its ground states, as well as the role played by solitons/instantons/monopoles (loosely referred to as “quasimodes”), [25]. Later, these were extended to symmetric operators with multiple ground states (c.f., appendix A, and [10]), and also to Complex structures on moduli spaces, [25]. Finally, the above can also be understood in terms of the Symplectic Field Theory developed by Y. Eliashberg, [26], including its connection to Floer Homology. (Seiberg–Witten–Floer homology yields knot invariants formally similar to combinatorially defined Khovanov homology: these are closely related to Donaldson and Seiberg invariants of 4-manifolds, as well as the Gromov invariant of symplectic 4-manifolds — solutions to the relevant differential equations, that we take to be the Schwinger–Dyson equations.)

**Effective Action & Schwinger's Action Principle:** Expanding the motivations for this proposed generalization of the Path Integral, we now focus on the construction of the Legendre conjugation between \( \varphi \) and \( J \), and explore its implications:

\[
\begin{align*}
\mathcal{W}[J] &= -i \log \mathcal{Z}[J] ; \\
\varphi \mapsto -i \frac{\delta}{\delta J} \mathcal{W}[J] ; \\
\text{and} \quad \Gamma[\varphi] &= \mathcal{W}[J] - J \varphi ; \\
\text{and} \quad J \mapsto -i \frac{\delta}{\delta \varphi} \Gamma[\varphi] ; \\
\text{yielding} \quad \Gamma[\varphi] &= S[\varphi] + \frac{i}{2} \log \det \left( \frac{\delta^2}{\delta \varphi \delta \varphi} S[\varphi] \right).
\end{align*}
\]
As long as the relation established by the Legendre transform between $J$ and $\varphi$ is at most quadratic (in the expression defining the Effective Action, $\Gamma$, above), the Linear Canonical Transform is the most general integral transform preserving the symplectic (resp. metaplectic) form at hand, i.e., preserving the structure defined by the Legendre conjugation of $\varphi$ and $J$. This implies that, in fact, we can even extend this construction from 1PI to 2PI, [15], and the Canonical Linear Transform can again be used to construct the Path Integral of the system, given that it still preserves the symplectic structure of the theory (once the 2PI construction saturates the quadratic relation between $\varphi$ and $J$, [15]). Speaking in terms of Deformation Quantization, the Linear Canonical Transform preserves the structure of the theory up to $O(\hbar^3)$ deformations.

Thus, considering the symplectic structure of the plane $\varphi-J$, i.e., the theory’s Phase Space, we can think of the case where $\varphi, J \in \mathbb{C} \cup \{\infty\}$, i.e., the field and source are valued in the Riemann Sphere: in such a scenario the symplectic structure is then preserved by the Möbius group, $\text{SL}(2, \mathbb{C})$, which is the automorphism group of the Riemann Sphere.

In the special case where the coefficients of a Möbius transformation are integers, we obtain the Modular Group, $\text{SL}(2, \mathbb{Z})$, which is relevant for $S$- and $T$-dualities.

Finally, with a bit of hindsight from (5) and (6), we can discuss the Legendre Transform construction of the Effective Action based on the Linear Canonical Transform expression for the Path Integral, in analogy to what is usually done when we treat the Path Integral as a Fourier Transform. Let us assume that $b = -2 \pi$ in the Linear Canonical Transform (cf. (5)), for if $b = 0$ we only have trivial [free] dynamics (cf. (6)). Now, we have to discern between two cases: $d = 0$ and $d \neq 0$. If $d$ vanishes, we have that $\varphi \mapsto -i \delta_J$, meaning that $\langle \varphi^n \rangle = \delta_J^n Z_M / (-2 \pi i)^n$, implying $\langle \varphi \rangle = 0$. However, if $d$ does not vanish, we get $\varphi \mapsto b^2 / 4 \pi^2 d J \delta_J$, and $\langle \varphi \rangle \neq 0$. This gives us a direct way to account for symmetry breaking straight from the parametrization of the Path Integral given by $M = (a \ b \ c \ d) \in \text{SL}(2, \mathbb{C})$.

In this sense, thinking of the Path Integral as a “sum over paths”, we see once again that we are required to include more exotic paths than just Brownian ones, such as Lévy flights, and so on: a generic Linear Canonical Transform, parametrized by a generic $M \in \text{SL}(2, \mathbb{C})$, will be characterized by different moments, $\langle \varphi^n \rangle$, depending on the physical system at hand, ultimately describing distinct [stable] probability densities, [9], that are associated to their respective paths and diffusion processes. Further, in anticipation of Section 5, it is worth noting that all stable probability distributions can be associated with a certain Fox $H$-function, [9].

Lastly, we can use the connection between stochastic processes and orthogonal polynomials in order to try and establish a Wiener–Askey polynomial chaos expansion of the Path Integral, [14], loosely along the following lines,

$$\mathcal{Z}[J] = \sum_{i=0}^{\infty} c_i Z_i[\zeta];$$

where $\{Z_i\}$ is the complete orthogonal polynomial basis of the Wiener–Askey chaos ex-
pansion, and \( \zeta \) is a random variable chosen according to the type of the random distribution at hand, e.g., free fields, i.e., Gaussian random variables, yield Hermite-chaos, [14, cf. table 4.1].

The idea is to use appropriate Fox \( H \)-functions for the \( Z_i \)’s, thus developing an “\( H \)-chaos expansion” for the Path Integral. This should become more clear in Section 5, when we venture into writing the Path Integral in terms of Fox’s \( H \)-functions.

**Penrose Transform:** Finishing the list of motivations for our generalization of the Path Integral, we will borrow an analogy with the Penrose and the Penrose–Ward transform. The insight is that we are also trying to define some sort of Complex analogue of the Radon transform, where the straight lines we are looking for are not the light-cone as done in the Penrose Transform (massless fields, as in twistor theory), but the lines delimiting the Stokes wedge where a certain integration cycle is defined, i.e., the Stokes’ lines of the integrand. In this sense, we would like to establish a parametrization of the space of integration cycles by the Matrix index, \( M \in SL(2, \mathbb{C}) \), labeling the Linear Canonical Transform, i.e., we would like to parametrize a certain space of compact Complex manifolds (the integration cycles) by a particular 3-dimensional submanifold of \( SL(2, \mathbb{C}) \) determined by the allowed values of \( M \).

For massless fields, we already know that the Penrose Transform establishes a correspondence between two spaces, where one parametrizes certain compact complex submanifolds (cycles) of the other. Further, its non-linear extension, the Penrose–Ward transform, relates certain holomorphic vector bundles (on \( \mathbb{C}P^3 \)) with solutions of the self-dual Yang–Mills equations (on \( S^4 \)).

The idea is to construct a map between integration cycles (compact Complex manifolds) and solutions to the Schwinger–Dyson equations, given by the Linear Canonical Transform evaluated at a particular \( M \in SL(2, \mathbb{C}) \).

Therefore, given all of the above, we will define our extended Feynman Path Integral to be given by the expressions below:

\[
\mathcal{Z}_M[J] = \sqrt{-1} e^{i \pi J^2 d/b} \int \mathcal{O}[\varphi] e^{i \pi \varphi^2 a/b} e^{-2 \pi i \varphi J/b} \mathcal{D}\varphi ; \quad b \neq 0 ; \quad (5)
\]

\[
\mathcal{Z}_{M'}[J] = \sqrt{d} e^{i \pi J^2 c d} \mathcal{O}[J d ] ; \quad b = 0 ; \quad (6)
\]

where \( M = (a \ b \ c \ d) \in SL(2, \mathbb{C}) \) parametrizes our Path Integral, the prime denotes those elements where \( b = 0 \), i.e., \( M' = (a \ b) \in SL(2, \mathbb{C}) \), and \( \mathcal{O}[\varphi] \) is a certain functional of the fields.

**4. Fox’s \( H \)-, and Generalized \( H \)-function Transforms**

The history of Mellin–Barnes transforms is intimately related to that of differential and difference equations, as well as the development of hypergeometric functions: the robust understanding of the asymptotic properties of functions, hyperasymptotics, Stokes’ phenomena, monodromy, and so on, [8].
Fox’s $H$-function was introduced to better the asymptotic control and to be the most generalized symmetrical Fourier kernel (a generalized Fredholm operator, [24]) involving Mellin–Barnes integrals. The $H$-function includes and generalizes several elementary, higher, and special functions, ranging from the exponential, Airy, Bessel, special polynomials, hypergeometric, Riemann Zeta function, up to Meijer’s $G$-function, multi-index Mittag–Leffler function, polylogarithms, etc. This means that we can give a Mellin–Barnes representation to a wide range of functions.

In particular, the $H$-function has recently found applications in a large variety of situations, ranging from diffusion all the way up to fractional differential and integral equations, and so on. It is worth noting that the $H$-function can be extended to have Matrix arguments ($\mathbb{R}$- or $\mathbb{C}$-valued), to include polylogarithms, and so on, [4, 5].

The notation we will use for the $H$-function is as follows,

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}[(\bar{a}_p, \bar{A}_p \; ; \; \bar{b}_q, \bar{B}_q \; ; \; z)] = H^{m,n}_{p,q}[(a_1, A_1), \ldots, (a_p, A_p) \; ; \; (b_1, B_1), \ldots, (b_q, B_q) \; ; \; z];$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s) \prod_{j=m+1}^{p} \Gamma(a_j + A_j s) z^{-s} \, ds; \tag{7}$$

where $i = \sqrt{-1}$, $z \neq 0$, $z^{-s} = \exp(-s(\log |z| + i \arg z))$, and $\arg z$ is not necessarily the principal value, and an empty product is always taken as unity. There are three different integration cycles $\mathcal{L}$, schematically depicted below, where red ticks are singularities of $\Gamma(1 - a_j - A_j s)$ and green ones are singularities of $\Gamma(b_j + B_j s)$:

- $\mathcal{L}_1$: goes from $r - i \infty$ to $r + i \infty$, $r \in \mathbb{R}$, so that all singularities of $\Gamma(b_j + B_j s)$ lie to the right of $\mathcal{L}_1$, while all singularities of $\Gamma(1 - a_j - A_j s)$ lie to the left of it.

- $\mathcal{L}_2$: loop beginning and ending at $+\infty$ and encircling all the singularities of $\Gamma(b_j + B_j s)$ once in the clockwise direction, but none of the singularities of $\Gamma(1 - a_j - A_j s)$.

- $\mathcal{L}_3$: loop beginning and ending at $-\infty$ and encircling all the singularities of $\Gamma(1 - a_j - A_j s)$ once in the anti-clockwise direction, but none of the singularities of $\Gamma(b_j + B_j s)$.

The $H$-function is, in general, multi-valued, due to the presence of the $z^{-s}$ factor in the integrand. However, it is one-valued on the Riemann surface of $\log(z)$. 
For example, here are some functions expressed in terms of the \( H \)-function, i.e., the Mellin–Barnes representations of some functions:

\[
H_{0,1}^{1,0}(b, B) \left| \begin{array}{c} z \\ \end{array} \right. = B^{-1} z^{b/B} \exp(-z^{1/B});
\]

\[
H_{1,1}^{1,1}(1 - \nu, 1, (0, 1)) \left| \begin{array}{c} z \\ \end{array} \right. = \Gamma(\nu)(1 + z)^{-\nu} = \Gamma(\nu)F_0(\nu; -z);
\]

\[
H_{1,2}^{1,1}(1 - \gamma, 1, (0, 1), (1 - \beta, \alpha)) \left| \begin{array}{c} z \\ \end{array} \right. = \Gamma(\gamma)E_\alpha^\beta(z);
\]

\[
H_{p,q}^{m,n}(a_p, A_p = 1) \left| \begin{array}{c} z \\ \end{array} \right. = G_{p,q}^{m,n}(\vec{a}_p; \vec{b}_q; z);
\]

\[
G_{s+1, s+1}^{1, 1}(0, 1 - a, \ldots, 1 - a, 0, -a, \ldots, -a) \left| \begin{array}{c} 1 \\ \end{array} \right. = \zeta(s, a);
\]

where \( \Gamma(z) \) is the Gamma function, \( \pFq{p}{q}{a_p}{A_p; \vec{b}_q; z} \) is the generalized hypergeometric function, \( E_\alpha^\beta(z) \) is the generalized Mittag–Leffler function, \( G_{p,q}^{m,n}(\vec{a}_p; \vec{b}_q; z) \) is Meijer’s \( G \)-function, and \( \zeta(s, a) \) is Hurwitz’s zeta function (that yields Riemann’s zeta function as \( \zeta(s, 1) \)).

The \( H \)-function can be further generalized along similar lines as the construction done above, in order to include some functions which are not particular cases of Fox’s \( H \)-function, e.g., Polylogarithms, including Polylogs of complex order, [5, Appendix A.5], and some further examples from Physics, [4].

For Polylogarithms and generalizations of the Riemann-zeta function, among others, we need to use one certain generalization of the \( H \)-function, called the \( \tilde{H} \)-function, \( \tilde{H}_{p,q}^{m,n}(z) \). The \( \tilde{H} \)-function is defined as,

\[
\tilde{H}_{p,q}^{m,n}(a_j, \alpha_j, A_j)_{1,n}, (a_j, \alpha_j)_{n+1,p} \left| \begin{array}{c} z \\ \end{array} \right. = \frac{1}{2\pi i} \int_{\mathcal{L}_{i\infty}} \chi(s) z^{-s} ds ;
\]

where \( \chi(s) \) is given by,

\[
\chi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma^{A_j}(1 - a_j - \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma^{B_j}(1 - b_j - \beta_j s) \prod_{j=n+1}^{p} \Gamma^{A_j}(a_j + \alpha_j s)} ;
\]

where the contour \( \mathcal{L}_{i\infty} \) starts at \( r - i\infty \) and goes to \( r + i\infty \), with \( r \in \mathbb{R} \), separating all the singularities of \( \Gamma(b_j - \beta_j s) \) from those of \( \Gamma(1 - a_j + \alpha_j s) \). For non-integer \( A_j \) or \( B_j \) the poles of the Gamma functions in (9) become branch points, whose branch cuts can be chosen so that
the integration cycle can be distorted for each of the three $\mathcal{L}$'s shown above as long as there is no coincidence of poles from any pair(s) of $\Gamma$'s.

Then, the polylog of Complex order $\nu$, $L^{\nu}(z)$, is represented as follows:

$$L^{\nu}(z) = H^{1,2}_{2,2} \left[ \begin{array}{c} (0, 1, 1), (1, 1, \nu) \\ (0, 1), (0, 1, \nu - 1) \end{array} \right] - z$$

(10)

and a generalization of Hurtwitz's zeta function is given by,

$$\phi(z, q, \eta) = H^{1,2}_{2,2} \left[ \begin{array}{c} (0, 1, 1), (1 - \eta, 1, q) \\ (0, 1), (-\eta, 1, q) \end{array} \right] - z = \sum_{k=0}^{\infty} \frac{z^k}{(\eta + k)^q}.$$  

(11)

The reason for our studying of the $H$-function is because it shows up when one generalizes from derivative operators to Fractional Differentiation, $[2,5,9]$. By extending the Path Integral to be represented by a Linear Canonical Transform, we have brought into play not only the elliptic subgroup of $SL(2, \mathbb{C})$ — related to fractional differential operators —, but also the parabolic, the Fresnel and Gauss–Weierstrass, the hyperbolic, the Bargmann, etc, subgroups of $SL(2, \mathbb{C})$, as already explained in previous sections. Moreover, as mentioned in the beginning of this section, the $H$-function is intimately related to the asymptotic behavior of the system it is describing, as well as its Stokes' phenomena and monodromy properties. In this fashion, if we can express the Path Integral as an $H$-function, we can condense all this information in the analytic structure of the $H$-function. This is the subject of the next section.

5. Path Integral as a Fox $H$-, or Generalized $H$-function Transform

Throughout the paper, we will also try and express the calculated Path Integral in terms of the $H$-function and its generalizations, $[4]$. The reason for this stems from the fact that the $H$-function is a Mellin–Barnes transform, which shows up in generalizations from derivative operators to Fractional Differentiation, etc, $[2,5,9]$.

As such, we look for a representation of our theory in terms of a Mellin–Barnes transform, because of its relation to asymptotic expansions (i.e., Stokes Phenomena and wall-crossing), and its closeness to Fourier Transforms: in this sense we would like to capture the properties of the theory straight from its singularities. We will consider the Path Integral as a hypergeometric integral transform, or simply an $H$-transform, $[2,19]$. This view in terms of an $H$-transform is of particular significance given Fox's $H$-function's definition in terms of a Mellin–Barnes transform, that clearly brings the singularity structure of the particular $H$-function being used to the foreground. This shows that the Path Integral considered as an $H$-transform encodes all of the interesting and relevant singularity structure, which can be as simple as that of Polylogarithms, or far more intricate as we will show in the examples below.

In the particular case of Polylogs, represented by the $\tilde{H}$-function, their recent importance for the computation of scattering amplitudes (maximally helicity violating amplitudes, MHV, and Wilson loops in $\mathcal{N} = 4$ SYM), their relation with Grassmannians and Cluster Varieties (viz. Stokes Phenomena), their relation to the Bloch–Wigner–Ramakrishnan–Zagier functions and
volumes of polytopes in AdS and mixed Tate motives, makes them a very relevant case to be studied. In this way, by expressing the Path Integral in terms of a generic $H$-function, we can more easily see what is induced by this structure.

As an example of how the singularity structure of the $H$-function determines a theory, we can look at $\left[2\right]$, where the fundamental solution (Green's function) for a fractional diffusion process is given by a generalization of the Airy function, expressed in terms of Fox's $H$-function: $H_{1,0}^{1,0}[(1 - \beta/2, \beta/2); (0, 1); \varphi]$, where the limit $\beta \rightarrow 1$ recovers the Gaussian density $H_{1,0}^{1,0}[(1/2, 1/2); (0, 1); \varphi] = \exp(-\varphi^2/4)/2\sqrt{\pi}$, and the limit $\beta \rightarrow 2/3$ recovers the Airy function, $H_{1,1}^{1,1}[(1/3, 2/3); (0, 1); \varphi] = 3^{2/3}\text{Ai}(\varphi/3^{1/3})$. In fact, a stronger result holds: all stable probability densities can be represented in terms of Fox $H$-functions $\left[9\right]$. (A random variable is called stable (and has a stable distribution) if a linear combination of two independent copies of itself has the same distribution, up to translations and scalings. Examples of stable distributions include the Gaussian, Cauchy and Lévy distributions.)

6. D0-brane Examples

We perform the analysis suggested above in three different examples, progressively showing more and more relevant details and properties of our extension for the Path Integral.

The models we study include the free field, the cubic, and quartic interactions, revealing their different structure dependent on $\mathbb{M} \in \text{SL}(2, \mathbb{C})$, i.e., using a 3-dimensional manifold as a parameter space.

We would like to re-interpret our construction along the lines of a BFSS/IKKT matrix-model and consider its large-$N$ limit, which is conjectured to be equivalent to $M$-theory. As such, rather than considering the problem from a bottom-up, constructionist, viewpoint, we would like to think about it using a top-down strategy: rather than solving a $D0$-brane model and considering infinitely many copies of it in order to build an extended model, we can think in “reversed” terms, where certain extended structures of the theory determine its properties. This has been named Brane Quantization, $\left[18\right]$. Our plan is to use our $D0$-brane examples in this section, considering the implications of the large-$N$ limit of these models to certain structures in each one of them, e.g., the convergence of the Path Integral defining the model or, analogously, the boundary conditions of the Schwinger–Dyson equation each model obeys. For example, let us consider a $\text{Matrix-valued D0}$-brane model with a generic polynomial for an Action. Its Feynman Path Integral (FPI) and Schwinger–Dyson Equation (SDE) are given by,

$$\mathcal{Z}_\tau[J] = \int e^{i \text{tr} \sum_{k=0}^n A_k \varphi^k} e^{i J \varphi} \mathcal{D} \varphi < \infty; \quad (12)$$

$$\text{tr} \left( \sum_{k=1}^n k A_k (-i \partial J)^{k-1} - J \right) \mathcal{Z}_\tau[J] = 0; \quad (13)$$

where $\varphi \in \mathbb{C}^{n \times n}$ is our $\text{Matrix-valued field}$ and $J \in \mathbb{C}^{n \times n}$ is its source, $\mathcal{D} \varphi$ is the canonical integration measure in $\mathbb{C}^{n \times n}$, $\mathcal{C}$ is an integration cycle rendering (12) finite, and $\tau$ is a label to
remind us of the connection between $\mathcal{C}$ and $\{A_k\}_{k=0}^n$ (this will be further explored later), i.e., different sets of coupling constants will yield Actions that will be convergent under distinct cycles $\mathcal{C}$, and $\tau$’s role is to perform this bookkeeping.

We could as well have written the above for a graded Lie Algebra-valued field, i.e., for a superfield $\Phi$ with superpotential $W[\Phi],$

$$\mathcal{L}_\tau^{[\tilde{J}, J]} = \int_{\mathcal{C}} e^{i(L\phi + \frac{1}{2} M^{ab} \phi_a \phi_b + \frac{1}{6} \phi_a \phi_b \phi_c + \text{c.c.)}} e^{i \tilde{J}^a \phi_a} e^{i \tilde{\phi}_a \tilde{J}^a} \text{D}\Phi \text{D}\bar{\Phi} < \infty ; \quad (14)$$

$$\left( M^{ab} \partial_{\tilde{J}^b} - i y^{abc} \partial_{\tilde{J}^b} \partial_{\tilde{J}^c} - \tilde{J}^a \right) \mathcal{L}_\tau^{[\tilde{J}]} = 0 ; \quad (15)$$

$$\left( \tilde{M}^{ab} \partial_{\tilde{J}^b} - i \tilde{y}^{abc} \partial_{\tilde{J}^b} \partial_{\tilde{J}^c} - \tilde{J}^a \right) \mathcal{L}_\tau^{[\tilde{J}]} = 0 ; \quad (16)$$

where $\tilde{J}^a = J^a + L^a$, i.e., the source is shifted by the linear term (ditto for its complex conjugate term). This motivates our treatment of the cubic Action on Section 6.2, i.e., cubic potentials are relevant to SUSY models, as well as to [non-Abelian] Chern-Simons theory.

At this point we have some observations to make regarding the coupling constants of the model at hand, that analogously to the fields themselves, can be valued in $\mathbb{R}$ or $\mathbb{C}$, or be Vector or Matrix valued, or Lie algebra or graded Lie algebra valued.

The couplings can be understood as background fields, meaning that the renormalized, effective Action is constrained by symmetries (superselection rules), by its holomorphic properties (Lee–Yang and Fisher zeros), and by its different properties in different limits (weak and strong coupling limits). Moreover, a holomorphic function (i.e., a section in an appropriate bundle) is determined by its asymptotic behavior and singularities, connecting us to he formalism developed earlier, where the Path Integral is represented in terms of the appropriate $H$-function encoding its asymptotic behavior and singularities.

Furthermore, we can implement desired boundary conditions for the Schwinger–Dyson equations in terms of a VEV(s) for the field(s) (at the boundary), $\langle \phi \rangle |_{\partial M} = \text{const}$ (flux quantization), or via explicit terms in the action. See Appendix A and [10, 18].

So, considering the couplings as parametrizing a family of functions between two Morse functions (namely, the Hamiltonians described by the initial and final values of the couplings in a certain range), its degeneracies involve a birth (branching out) or death (merging together) transition of critical points. This study can be done via Cerf theory, which tackles stratified spaces labelled by the co-dimension of the strata, [25]. As such, in the examples below we will show plots to illustrate these transitions at particular values of the couplings.

### 6.1. Free Field

In order to get our feet wet and highlight some of the properties that will be relevant later on, let us start with the simplest possible example: the free field.

This $D0$-brane model has an Action given by $S[\varphi] = \mu \varphi^2 / 2$, yielding the following FPI and SDE:
\[ \mathcal{F}[J] = \int_{\mathcal{C}} e^{i \mu \varphi^2/2} e^{i \varphi J} \, D\varphi < \infty; \quad (17) \]

\[ (-i \mu \partial_j - J) \mathcal{F}[J] = 0 \Rightarrow \mathcal{F}[J] = a e^{i J^2/2\mu}; \quad (18) \]

where \( a \) is an arbitrary constant, and \( \mathcal{C} \) is an integration cycle.

In order to cast it as an LCT, we will follow (5) and choose \( \Theta[\varphi] = 1 \) and \( M = \left( \frac{-\mu - 2\pi}{2\pi}, 0 \right) \).

This choice of \( M \in \text{SL}(2, \mathbb{C}) \) gives us the rational function determining the analytic structure of this transformation,

\[ M(\varphi) = f_\mu(\varphi) = \frac{-2\pi (\mu \varphi + 2\pi)}{\varphi}. \quad (19) \]

These are plots of \( f_\mu(\varphi) \) for different values of \( \mu \):
6. D0-brane Examples

Plots of \( f_\mu(\varphi) = -2\pi \mu \varphi - 4\pi^2 \varphi \), for \( \mu = 1, -1, i, -i \), respectively. The plots on the right have \( \Re(f_{\mu=\pm1,\pm i}) \) as the x-axis and \( \Im(f_{\mu=\pm1,\pm i}) \) as the y-axis, with variations of color (phase \( \theta \)) and brightness (magnitude \( \rho \)) to indicate \( z = \rho e^{i\theta} = f_{\mu=\pm1,\pm i} \); plots on the left are simply their wrapping over the Riemann Sphere.

It is not difficult to see that these graphs are related to each other by a Möbius transform of the Complex \( \varphi \)-plane.

For the special case where \( \mu = i \) we recover a well known identity about Fourier Transforms: the Hermite polynomials are its eigenfunctions. This can be seen in (17) simply by noting that the 0th order Hermite polynomial is given by \( \text{He}_0(\varphi) = 1 \), as defined in (31) and (32), and \( \mathcal{F}[e^{-\varphi^2/2} \text{He}_n(\varphi)] = (i^n e^{-J^2/2} \text{He}_n(J) \), taking \( n = 0 \) for this special case of ours. This result can be extended to other values of \( \mu \) with a change of variables given by \( \varphi \rightarrow \sqrt{i \mu} \varphi \) and \( J \rightarrow J / \sqrt{i \mu} \), provided we stay within the same Stokes wedge defining the original integral transform. This is in complete analogy with the following extension of Hermite polynomials:

\[
\text{He}_n^{[\alpha]}(\varphi) = \alpha^{-n/2} \text{He}_n^{[1]}(\varphi / \sqrt{\alpha}) = e^{-\alpha \varphi^2/2} \varphi^n ;
\]

meaning that we can choose \( \alpha = i \mu \) and talk in terms of \( \text{He}_n^{[i \mu]}(\varphi) \).

As mentioned before, we want to represent these results in terms of Fox’s \( H \)-function. In order to do so, we will use the following conversion table:

\[
\text{He}_{2n}(\varphi) = (-1)^n \frac{(2n)!}{n!} \text{F}_1(-n; \frac{1}{2}; \varphi^2) ;
\]

\[
= (-1)^n \frac{(2n)!}{n!} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(-n)} \text{H}_{1,2}^{1,1}\left[(1 + n, 1); (0, \frac{1}{2}, 1); \varphi^2\right] ;
\]

\[
\text{He}_{2n+1}(\varphi) = (-1)^n \frac{(2n+1)!}{n!} \text{F}_1(-n; \frac{3}{2}; \varphi^2) ;
\]

\[
= (-1)^n \frac{(2n+1)!}{n!} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(-n)} \text{H}_{1,2}^{1,1}\left[(1 + n, 1); (0, -\frac{1}{2}, 1); \varphi^2\right] ;
\]

where \( \text{F}_1 \) is Kummer’s confluent hypergeometric function, \( \Gamma \) is the Gamma function, and \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) and \( \Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}/2 \).
With this in mind, it is not difficult to write the FPI as an $H$-transform,

$$
\mathcal{F}_M[J] = \left\{ e^{i\mu \varphi^2/2} e^{i\varphi J} D\varphi \right\}^{\text{FPI}} = a e^{iJ^2/2\mu} \quad \text{where } M = \left( \begin{array}{cc} -\mu & -2\pi \\ 0 & 1 \end{array} \right); \quad \text{(20)}
$$

$$
= a \sqrt{\pi} H_{1,1}^0(\frac{1}{2}, \frac{1}{2}; (0, 1); \sqrt{-\frac{2\pi}{\mu}} J). \quad \text{(21)}
$$

In this particular case, it is possible to reinterpret (20) in terms of an LCT with $b = 0$, following (6). All we need to do is choose $\Theta[J d] = 1$ and,

$$
M' = \left( \begin{array}{cc} 1 & 0 \\ \omega^{-1} & 1 \end{array} \right);
$$

where $\omega = i 2 \pi \mu$. This means that $\mathcal{F}_M'[J]$ is a Gaussian Transform of width $-i \omega$.

Now, we can use (17) parametrized by (19) in order to construct Coherent States $\Upsilon^\mu_\varphi[J]$, parametrized by $\mu$ and $\varphi_0 = \langle \varphi \rangle$.

The general expression for $\Upsilon^\mu_\varphi[J]$ is given in terms of the matrix $A = (a b; c d)$ diagonalizing $M$, $A^{-1} M A = \text{diag}(\lambda_1, \lambda_2)$, in the following fashion:

$$
\Upsilon^\mu_\varphi(J) = (C_A \delta_\varphi)(J) = \sqrt{-i} e^{i\pi J^2 d/b} \int_{-\infty}^{\infty} \delta(\varphi - \varphi_0) e^{i J \varphi^2 a/b} e^{-2\pi i \varphi J/b} d\varphi; \quad \text{(22)}
$$

where the notation is the same as in (1), (2), (3): that is, the coherent states are the Linear Canonical Transforms of the Dirac-delta function parametrized by the diagonalization transform, i.e., localized states defining each quantum phase (representing each integration cycle and Stokes’ wedge).

For this case of ours, the coherent state is given by,

$$
\Upsilon^\mu_\varphi[J] = \sqrt{-i} e^{iJ^2/2\lambda_\mu} e^{i\varphi_0^2/\lambda_\mu} e^{-2\pi i \varphi_0 J}; \quad \text{(23)}
$$

where $\lambda_\mu = \frac{-\mu - \sqrt{\mu^2 - 4}}{2}$, and $\lambda_{\mu = \pm 2} = \mp 1$, $\lambda_{\mu = \pm 2i} = \mp i - i \sqrt{2}$.

The plots below depict $\Upsilon^\mu_\varphi[J]$ for $\mu = \pm 2$ and $\mu = \pm 2i$: the graphs on the left are the wrapping over the Riemann Sphere of the ones on the right.
Coherent State plots for different combinations of the parameters \( \mu \) and \( \varphi_0 \). The plots on the right have \( \Re(\Upsilon_{\mu=\pm2,\pm2i}[J]) \) as the \( x \)-axis and \( \Im(\Upsilon_{\varphi_0=0}[J]) \) as the \( y \)-axis, with variations of color (phase \( \theta \)) and brightness (magnitude \( \rho \)) to indicate \( z = \rho e^{i\theta} = \Upsilon_{\varphi_0=0}[J] \); plots on the left are simply their wrapping over the Riemann Sphere.

### 6.2. Cubic Action

At this point we can try and study a slightly non-trivial model, given by a D0-brane with \( S[\varphi] = \varphi^3/3 \). It's Feynman Path Integral and Schwinger–Dyson equation are shown below:
\[ \mathcal{Z}[J] = \int_\mathcal{C} e^{i\varphi^3/3} e^{iJ\varphi} D\varphi < \infty \Rightarrow \mathcal{C} : \text{Arg}(\varphi) = \{0, \pm 2\pi/3\} ; \quad (24) \]

\[ (\partial^2_J - J) \mathcal{Z}[J] = 0 \Rightarrow \mathcal{Z}[J] = a\text{Ai}[J] + b\text{Bi}[J] ; \quad (25) \]

where the contour \(\mathcal{C}\) over which the Path Integral is well defined is along the lines of \(\varphi\) such that its arguments are either 0 or \(\pm 2\pi/3\), and \(\text{Ai}\) and \(\text{Bi}\) are the Airy functions, that can also be expressed in terms of Fox's \(H\)-function as \(\frac{3}{2} / 3 \text{Ai}(\varphi/3) = H^{1,0}_{1,1}(\frac{2}{3}, \frac{1}{3}; (0, 1); \varphi/3^1)\).

The \(\text{Bi}\) function can be expressed in terms of the \(\text{Ai}\) function via a combination of different integration cycles: \(\text{Bi}[J] = e^{-i\pi/6} \text{Ai}[Je^{-2\pi i/3}] + e^{i\pi/6} \text{Ai}[Je^{2\pi i/3}]\). That is, we scale and rotate the argument. The combination of the two integration cycles can be understood as follows: the pre-factor \(e^{\pm i\pi/6}\) is a critical value for the scaling, and \(g = e^{\pm 2\pi i/3}\) is the appropriate Stokes' factor associated to crossing each cycle (upon which we cross a cycle and pick up another Stokes' factor).

Now, we want to choose an appropriate \(M\) and \(O[\varphi]\) in order to parametrize our Path Integral along the lines of (5). Let us choose \(M_{\text{Ai}} = (0, -2\pi/0)\) and \(O[\varphi] = e^{i\varphi^3/3}\).

The shaded regions are Stokes' wedges and represent zones of convergence: cycles beginning and ending within these regions will yield meaningful integrals. \(\mathcal{C}_1\) and \(\mathcal{C}_2\) are particular cycles giving rise to the two independent solutions indicated with their respective Stokes' factor.

As we mentioned before, \(M_{\text{Ai}}\) can be decomposed in the following fashion: \(M_{\text{Ai}} = M_{\text{DT}}(-2\pi) M_{\text{FT}} M_{\text{GT}}(0)\), i.e., we use a composition of a Gaussian Transform with a Fourier Transform and with a Dilation. However, because we now have an \(O[\varphi] = e^{i\varphi^3/3}\) that is non-trivial, it is easier to read the analytic structure of our Path Integral straight from its representation in terms of its associated \(H\)-function.

As a passing comment, let us just note that (25) can be easily generalized in two different ways:

\[ (\partial^2_J \pm g^2 J) \mathcal{Z}_{M_{\text{Ai}}} [J] = 0 ; \]
\[ (\partial^3_J - 4 J \partial_J - 2) \mathcal{Z}_{M_{\text{Ai}}} [J] = 0 ; \]
that are finite at the origin, and where the first generalization can be understood in terms of a scaling by a factor of $g^{2/3}$, while the second generalization can be interpreted in terms of completing the squares — as expected, these solutions can easily be expressed in terms of $H$-functions (via dilations of its variable or by taking powers of it), and are also parametrizable by distinct $M$’s. That is, completing the polynomial potential such that we have an Action along the lines of $S[\varphi] = a \varphi^2/2 + b \varphi^3/3$, the coefficient of the highest power (“top coupling”) determines the analytical structure. As $b \to 0$, $a$ plays a more and more relevant role: only a cycle contained in both regions would yield convergent solutions for both powers — a cycle in non-matching convergence regions yields a divergent $Z$, while an integration cycle that can be deformed from one convergence region to another yields a vanishing $Z$ (analogous to the accumulation of Lee–Yang zeros).

Therefore, the quantum phases of this system are given by the $Ai$ and $Bi$ functions, and their integral representation corresponds to the appropriate Path Integral associated to each quantum phase. Moreover, their respective coherent states are given by

$$\Upsilon_{\varphi_0}[J] = \sqrt{-i} \ e^{-J^2/2} e^{-i \pi \varphi_0^2} e^{\varphi_0 J} ;$$

where $\varphi_0$ is a constant (something like $\varphi_0 = \langle Ai | \varphi | Ai \rangle$ or $\varphi_0 = \langle Bi | \varphi | Bi \rangle$, for each quantum phase).

Below you see plots of (26) for $\varphi_0 = \pm 1$: the plots on the right side are such that the complex argument (phase) is shown as color (hue) and the magnitude is show as brightness; the plots on the left side are nothing but the wrapping of the former over the Riemann Sphere (keeping in mind that its group of automorphisms is given by $\text{SL}(2, \mathbb{C})/\{\pm 1\}$).
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Coherent State plots for each quantum phase. The plots on the right have \( \Re(T_{\varphi_0=\pm 1}[J]) \) as the x-axis and \( \Im(T_{\varphi_0=\pm 1}[J]) \) as the y-axis, with variations of color (phase \( \theta \)) and brightness (magnitude \( \rho \)) to indicate \( z = \rho e^{i\theta} = T_{\varphi_0=\pm 1}[J] \); plots on the left are simply their wrapping over the Riemann Sphere.

6.3. Quartic Action

The quartic model is that of a D0-brane with \( S[\varphi] = g \varphi^2/2 + \varphi^4/4 \), where \( g = \mu^2/\lambda \) just for convenience’s sake, i.e., \( g \to +\infty \) corresponds to the weak coupling regime, while \( g \to 0 \) represents the strong coupling one, and \( g \to -\infty \) represents the “broken symmetric” one. Accordingly, the Path Integral and Schwinger–Dyson equation for this model are given by,

\[
\mathcal{Z}[J] = \int_{\mathcal{C}} e^{i \left( \frac{g}{2} \varphi^2 + \frac{\varphi^4}{4} \right) - J \varphi} \, D\varphi < \infty \Rightarrow \mathcal{C} : \text{Arg}(\varphi) = \{0, \pm \pi/2\}; \quad (27)
\]

\[
\left( \partial_j^3 - g \partial_j - J \right) \mathcal{Z}[J] = 0 \Rightarrow \mathcal{Z}[J] = a U[g,J] + b V[g,J] + c W[g,J]; \quad (28)
\]

where the cycle \( \mathcal{C} \) over which (27) is well defined is such that the arguments of \( \varphi \) are either 0 or \( \pm \pi/2 \), and \( U, V \) and \( W \) are the Parabolic Cylinder Functions, that can also be cast in terms of Fox’s \( H \)-function as follows:
\[ y_1(g; \varphi) = PCyl_{\text{even}}(g; \varphi) = e^{-\varphi^2/4} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{2g+1}{4})} H_{1,2}^{1,1}\left(\frac{3-2g}{4}, 1; (0, \frac{1}{2}, 1); \varphi^2/2\right) ; \]
\[ y_2(g; \varphi) = PCyl_{\text{odd}}(g; \varphi) = \varphi e^{-\varphi^2/4} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{2g+3}{4})} H_{1,2}^{1,1}\left(\frac{1-2g}{4}, 1; (0, -\frac{1}{2}, 1); \varphi^2/2\right) ; \]

where \( U, V, W \) are particular combinations of \( PCyl_{\text{even}} \) and \( PCyl_{\text{odd}} \) satisfying the boundary conditions of (28) or, equivalently, compatible with the integration cycle chosen in (27) (in the sense of yielding a well-defined measure assuring the convergence of the integral). Thus, the Path Integral of this model is given by the integral representation of Fox’s \( H \)-function, (7), making it clear how the coupling constant \( g \) determines the singularity structure and integration cycle of the particular Path Integral in question.

We can now appropriately parametrize (27) using \( M_{PCyl} = \begin{pmatrix} -g & -2\pi \\ -2\pi & 0 \end{pmatrix} \) and \( \vartheta[\varphi] = e^{i\varphi^4/4} \); in this way, the quantum phases of our model are given by the Parabolic Cylinder Functions with respect to the physical parameter \( g \) — varying \( g \) will change the ground state across Stokes’ lines due to quantum fluctuations. As before, \( M_{PCyl} \) can be understood in terms of a composition of a Gaussian Transform of width \( \omega = g/2i\pi \) with a Fourier Transform and with a Dilation of \( \Delta = -2\pi \): \( M_{PCyl} = M_{DT}(\Delta = -2\pi) M_{FT} M_{GT}(\omega = g/2\pi) \).

The shaded regions represent convergence contours for the following terms, respectively: red \( \Leftrightarrow \) quartic, blue \( \Leftrightarrow \) cubic, and yellow \( \Leftrightarrow \) quadratic.

The coherent states for this model can also be computed, yielding,
\[ \Upsilon_{\varphi_0}^\mathbb{F}[J] = \sqrt{-i} \exp\left\{ i \frac{J^2}{(\sqrt{g^2-4}-g)} \right\} \exp\left\{ i \pi \varphi_0^2 g + \sqrt{g^2-4} \right\} \exp\left\{ -2i \varphi_0 \frac{J}{\sqrt{g^2-4}} \right\} ; \quad (29) \]

where \( \varphi_0 \) is a constant (à la \( \varphi_0 = \langle U|\varphi|U \rangle \), for each quantum phase: \( U, V, W \)), and \( g \) is the model’s coupling constant.
Below you find plots of (29) for $g = +2, 0, -2$ and $\varphi_0 = +2i, 0, +2$, respectively. The plots on the right side show the phase as color (hue) and the magnitude as brightness. The plots on the left side are the wrapping of the former over the Riemann Sphere (keeping in mind that its group of automorphisms is given by $SL(2, \mathbb{C})/\{\pm 1\}$).

Coherent state plots for each quantum phase. The plots on the right have $\Re(Y^{g=+2,0,-2}_{\varphi_0=+2i,0,+2})$ as $x$-axis and $\Im(Y^{g=+2,0,-2}_{\varphi_0=+2i,0,+2})$ as $y$-axis, with variations of color (phase $\theta$) and brightness (magnitude $\rho$) to indicate $z = \rho e^{i\theta} = Y^{g=+2,0,-2}_{\varphi_0=+2i,0,+2}$. Plots on the left are simply their wrapping over the Riemann Sphere. Note that the values for $g$ and $\varphi_0$ are obtained from their relation established via the Schwinger–Dyson equation: $\varphi = 0$ or $\varphi = \pm \sqrt{-2g}$, where we have chosen the positive branch of the square root.
It is very interesting to note that $\Upsilon_{g=+2}^{\phi_{0}=+2} \rightarrow \Upsilon_{g=-2}^{\phi_{0}=+2}$ through a rotation of $\pi/2$ ($J \rightarrow iJ$ and we also pick up a Stokes' factor, the multiplier $e^{-i(8\pi)}$), which can be interpreted in terms of the rolling of the Riemann Sphere around the base manifold (which is just a point in this case). Moreover, the fixed-point locus of our Action is simply given by the set $\{0, \pm \sqrt{-2g}\}$. Both of these together imply that the Gromov–Witten invariant for this model is given by the sum of the stationary phase approximation (viz. Atiyah–Bott fixed-point theorem) of the Path Integral computed at each point of the fixed-point set, for different values of the coupling constant $g$: this brings the asymptotic behavior of $U, V, W$ to the forefront, making the Stokes' phenomena (wall-crossing) associated to this model quite relevant in regards to the asymptotic behavior of each quantum phase.

The Stokes' Phenomena mentioned above is associated to understanding the differential operator defining the Schwinger–Dyson equation (28), $\left(\partial_{j}^{3} - g \partial_{j} - J\right)$, as a connection on the Riemann Sphere, forming the basis of an interpretation in terms of the Riemann–Hilbert correspondence, which can be thought of as describing the Schwinger–Dyson equation (28) in terms of the monodromies of its solutions. The underlying geometry is sometimes called “wild geometry” (in analogy with “wild ramification” in arithmetic): the matrix of the connection cannot be reduced to [a matrix] having logarithmic poles, which is in turn related to the fact that (28) is represented via Fox's $H$-function, and not some form of polylogarithm (which is very common in models dealing with MHV amplitudes).

Both of the observations made in the two paragraphs above are intimately related to the topic of nonlinear Fredholm theory, [24], in fact, with the nonlinear Fredholm theory of the differential operator defining the Schwinger–Dyson equation (28), describing the deformations of the given pseudoholomorphic curves (determined by the different solutions to (28)). As such, as long as the Path Integral representation of each solution, (27), can be understood as a Fredholm map, these observations are also related to the construction of Seiberg–Witten invariants.

### 6.4. Non-Polynomial Actions

For the case of non-polynomial actions, we can perform the same construction as for polynomial ones, just the integration cycle may be a bit more intricate. For example, if $S = \beta \cos \theta$ and $\mathcal{Z}[J, \tilde{J}] = \int_{-\pi}^{\pi} e^{\beta \cos \theta} e^{iJ} e^{i\tilde{J}} e^{-i\theta} D\theta$, the two relevant integration cycles are $C_{1}$ and $C_{2}$, where $\int_{-\pi}^{\pi} = \int_{C_{1}} - \int_{C_{2}}$, as depicted below,
7. Dimensional Extensions

Firstly, it is important to realize and clarify a few different facts:

**Airy Property:** the so-called “Airy property”, as defined by [31], needs to be satisfied if we want to dimensionally extend the models above;

**Topological Field Theory:** our D0-dimensional examples above can be understood as TFTs in the sense of Atiyah (the spacetime metric does not appear anywhere in these theories, and there is no real dynamics or propagation);

**Σ-model analogy:** when we extend our field $\varphi$ from being scalar-valued to being, e.g., Matrix-valued, we enter the realm of Σ-models, in the sense that the space of Matrices is considered the target space of our model — the same argument holds true if rather than Matrix-valued we have Tensor-valued, Lie or graded (SUSY) Lie Algebra-valued fields, or even more generic target spaces;

**BFSS/IKKT analogy:** the D0-branes of our examples can be understood as being in the infinite momentum frame (light-cone frame) and, as such, as a random matrix model (extending the fields from scalar- to matrix-valued) where the $N \to \infty$ is conjectured to be equivalent to M-theory — in this sense, matrices of finite $N$ can be thought of as a discretization scheme;

**Causal Dynamical Triangulation:** Tensor-valued fields are a generalization of Matrix models and correspond to dynamical triangulations, [23], and, as such, the results previously obtained are relevant to the field of CDTs;

**Group Field Theory:** GFTs were also developed as a generalization of Matrix models, where the fields are valued in an appropriate Lie Algebra (possibly graded, SUSY) — once again, our results are thus relevant to this field as well;
Boundary Conditions: the boundary conditions of the Schwinger–Dyson equations (resp. integration cycles defining the Feynman Path Integrals) are required to not be affected in the process of taking the \( N \to \infty \) limit, meaning that the quantum phases of the system should remain well-defined within their respective Stokes' wedges (topological \( D_p \)-brane). However, as \( N \to \infty \), we have more and more points satisfying the boundary conditions, eventually yielding themselves a \( D_p \)-brane in their own right. Thus, we can revert our reasoning and say that such \( D_p \)-branes built out of the suitable boundary conditions are responsible for the quantization of our system, in analogy with [18].

Therefore, we have the following general picture. We start with a \( D_0 \)-brane model of fields that can be either scalar-valued (e.g., \( \mathbb{R}, \mathbb{C} \)), yielding a topological field theory, or valued in more general target spaces (Vector, Matrix, Tensor, \( \mathcal{L} \)ie or graded \( \mathcal{L} \)ie algebra, etc), yielding a topological \( \Sigma \)-model.

In any case, the quantization is done via a generalization of the Feynman Path Integral given in terms of a Linear Canonical Transform, which is parametrized generically by a matrix \( M \in \text{SL}(2, \mathbb{C}) \) (keeping in mind that \( \text{SL}(2, \mathbb{C}) \) is a 3-dim manifold parametrizing our model). This parameter \( M \) essentially labels the allowed integration cycles that render the Linear Canonical Transform finite and well-defined. Furthermore, it is possible to combine different cycles using the rule that \( M_3 = M_2 \cdot M_1 \): if \( M_1 \) and \( M_2 \) are both labels of allowed integration cycles, so is \( M_3 \), i.e.,

\[
\mathcal{Z}_{M_3 = M_2 \cdot M_1} = \mathcal{Z}_{M_2} \circ \mathcal{Z}_{M_1} ;
\]

\[
= -i e^{i \pi J^2 (d_2 \cdot b_2^{-1} + d_1 \cdot b_1^{-1})} \times \int_{C_{M_3} = C_{M_2} \circ C_{M_1}} e^{-2i \pi J \varphi (b_2^{-1} + b_1^{-1})} e^{i \pi \varphi^2 (a_2 \cdot b_2^{-1} + a_1 \cdot b_1^{-1})} e^{i S[\varphi]} \mathcal{D} \varphi .
\]

This composition rule is analogous to the gluing of bordisms together, following Atiyah’s definition of a TFT, in the sense of [32, Definition 1.1.1 and 1.1.5, Example 1.1.9]. This is relevant for dimensional extensions for the reason that if we want to combine several \( D_0 \)-branes into a \( D_1 \)-brane, certain relations among the boundary conditions of their respective Schwinger–Dyson equations must be satisfied in order for this dimensional extension to be “stable” in some suitable sense. These relations are more easily encoded in terms of the algebra of the parameters \( M \).

Thus, if we want to dimensionally extend our 0-dimensional systems into bona fide \( n \)-dimensional ones, we need to first pick an integration cycle, i.e., we need to pick an \( M \) labeling the particular boundary conditions of the Schwinger–Dyson equations, determining which solution we are about to extend. Then, we can proceed in two ways,

**BFSS/IKKT inspired:** Interpreting the model as a \( \Sigma \)-model (where scalar-valued systems are understood and extended to Vector-valued linear \( \Sigma \)-models), taking the large-\( N \) limit and Lorentz boosting it to a desired frame (i.e., away from the infinite momentum frame);

**Many-Body System inspired:** Constructing a discretized version of the Feynman Path Integral out of several copies of our system, extending the class of allowed paths to include...
not only Brownian paths, but also Lévy flights, and even more generic ones, using the full power of the Linear Canonical Transformation.

These two processes, in principle, should give us the same resulting dimensionally extended theory. Analogously, we can revert the above arguments saying that given a $D$-dimensional system, we can define its quantization through the implementation of the appropriate boundary conditions of its respective Schwinger–Dyson equation (resp. integration cycles of its Feynman Path Integral) in terms of suitable $D_p$-branes: the solutions to the quantum equations of motion are only fully determined when we implement its compatible boundary conditions, which is achieved through certain $D_p$-branes that implement them.

8. Summary and Outlook

We have shown how to extend the notion of the Feynman Path Integral in a way to include more general paths than the original formulation allows (e.g., Brownian paths, Lévy flights, etc). This extension comes labelled by a matrix parameter that effectively determines the kind of path being used.

Furthermore, we have established the integro-differential nature of the quantization problem, associating the boundary conditions of the Schwinger–Dyson equations to the integration cycle of the Feynman Path Integral. In this fashion, the matrix label previously mentioned has the job to perform the bookkeeping of all possible allowed solutions, a fact that is reflected by the existence of more than one integration cycle guaranteeing the convergence of the Feynman Path Integral.

A relevant property of Linear Canonical Transforms that will be important to us is their eigenfunctions, forming a complete orthonormal set — in analogy, for instance, with the Fourier Transform’s eigenfunctions,

$$\Psi_n[\varphi] = \frac{2^{1/4}}{\sqrt{n!}} e^{-\pi \varphi^2} \text{He}_n[2 \sqrt{\pi} \varphi]; \quad (31)$$

where the Hermite polynomials are given by,

$$\text{He}_n[\varphi] = (-1)^n e^{\varphi^2/2} \frac{d^n}{d\varphi^n} e^{-\varphi^2/2}. \quad (32)$$

These will give us information about the coherent states associated to the Linear Canonical Transform and each of its components (Fractional Fourier, Hyperbolic, Bargmann, etc).

A stratified space (or filtration) is one that has been decomposed into subspaces called strata, that are required to fit together in a certain way. Stratified spaces provide a setting for the study of singularities via Morse Theory on manifolds with boundary and manifolds with corners (e.g., orbifolds).
allowing for a more general transform to be introduced and used in field-source analysis (i.e., quantization) via Path Integrals or Schwinger–Dyson equations, namely the Linear Canonical Transform.

For example, thinking of the Path Integral as a Sum over Paths (considering the space of paths suitably stratified according to the problem at hand), the Path Integral as a Fourier Transform implies these paths are Brownian; the Path Integral as a Fractional Fourier Transform implies these paths are Lévy flights; and so on. In this sense, we are trying to formalize the notion of a “transform” between two [fundamental] groupoids\(^2\), the space of paths, \(\varphi\), and the space of sources, \(J\), where the Path Integral plays the role of this transform.

On top of the explicit analysis of three different D0-brane examples, we also showed how to dimensionally extend such models in two different ways. This enabled us to reverse the line of thought and think of the quantization of a system as being determined by the \(Dp\)-branes that act as boundaries to it.

As for the connection with wall-crossing and crystal melting, we clearly stated the role that Stokes’ Phenomena plays in our models, showing how the asymptotic structure of each solution is relevant in determining the allowed integration cycles of the appropriate Feynman Path Integral representing the system. Along these lines, we used an intrinsic property of the Linear Canonical Transformations, namely its symmetry with respect to \(\text{SL}(2, \mathbb{C})\), plotting the relevant objects over the Riemann Sphere, to make a connection with some dualities (S-duality, T-duality) and to establish the modular character of the Feynman Path Integral representation we used.

Finally, we have determined the appropriate representation of the Feynman Path Integral of each of our examples in terms of Fox’s \(H\)-function, which is a Mellin–Barnes transform. Thus, we brought to the forefront discussions regarding the singularity structure of our models, showing that not all systems can be described in terms of Polylogarithms (as happens in the case of MHV amplitudes): more complicated systems require a more elaborate description, where the use of the \(H\)-function can be particularly handy. This fact is associated with the nonlinear Fredholm theory of the Schwinger–Dyson operator in question, which in turn has something to say about the construction of Seiberg–Witten invariants in each of our systems.

To illustrate this point further, we will use some Cerf theory, which is a way to study families of smooth functions on smooth manifolds, their generic singularities, and the topology of the subspaces these singularities define, as subspaces of the function space. Essentially, two Morse functions can be approximated by one that is Morse at all but finitely many degenerate times. The degeneracies involve a birth/death transition of critical points, i.e., the attachment of a 1-handle connecting the manifold \(f_{t>0}\) to \(f_{t<0}\). Something similar happens with \(f_1(x) = x^4 + tx^2\), which is Morse for \(t > 0\) (1 critical points) and for \(t < 0\) (3 critical points).

\(^2\)A groupoid generalises the notion of a group in several equivalent ways, e.g., an oriented graph, or a category in which every morphism is invertible, etc. The fundamental group measures the 1-dimensional singularity structure of a space; for studying “higher-dimensional singularities”, the homotopy groups are used. The fundamental groupoid, rather than singling out one point and considering the loops based at that point up to homotopy, considers all paths in the space (up to homotopy).
8. Summary and Outlook

points), while \( t = 0 \) indicates the birth/death transition of critical points, i.e., the attachment of the handlebody connecting the manifolds \( f_{t>0} \) and \( f_{t<0} \).

The plots below highlight this construction for two relevant Potentials, the cubic and quartic ones: \( V_J(\phi) = \phi^3/3 - J \phi \), \( V_g(\phi) = \phi^4 + g \phi^2 \). The birth/death transition, at \( J = 0 \) and \( g = 0 \) (marked as “critical points” in the plots), represents the addition of extra handles. We can think of this in terms of constructing the Potential manifold via the attachment of handlebodies. In physics parlance, we can give the fluxes that determine the handlebody decomposition of a manifold (in particular, the vacuum manifold, the moduli space): once the handlebodies determined by the appropriate fluxes are known, the desired manifold is determined — this is known as the Heegaard diagram of a manifold. Using the help of the contour (equipotential) lines on the plots, we can determine the set of flow lines outgoing or incoming a certain critical point (birth/death transition), thus defining ascending and descending manifolds (analogous to the ones computed in [36]). Finally, the intersection of these ascending and descending manifolds marks the Stokes’ lines determining the boundaries between adjacent Stokes’ wedges. These intersections, called “handle slides”, are the \( Dp \)-branes establishing the desired boundary conditions for our model.
Plots of $V_J(\phi) = \phi^3/3 - J \phi$ and $V_g(\phi) = \phi^4 + g \phi^2$ for ranges of the parameters $J$ and $g$, showing the “critical point” where the birth/death transition of critical points between two Morse functions happens, i.e., the point where the transition handlebody should be attached. The points where $J, g = 0$ represent the transition between $J, g > 0$ and $J, g < 0$, where an extra monomial gets added to the top-coupling term.

At this point we can make some comments about similar approaches and constructions available in the literature.

1. Multi-cut matrix models (which can be extended to Tensor models and thus Group Field Theory) have been a research topic for a couple of decades now, [33]. However, their associated Stokes’ phenomena, and their non-perturbative completion, are much more modern features. Their connection with our approach has only become clear in the past two years or so. Here is a rough summary of the approach, highlighting the common points.

- Riemann–Hilbert and Deligne–Simpson problem: given a system of differential equations on the Riemann Sphere, we want to reconstruct the system from its monodromy data, [11], i.e., we want to solve the inverse monodromy problem in order to obtain the suitable Stokes’ multipliers and factors. In our case, the system of differential equations is given by the Schwinger–Dyson equations, while its monodromy is established by the boundary conditions we choose, i.e., by the fall-off properties of the fields in our system. In the case of multi-cut boundary conditions, these turn out to be the quantum integrable $T$-systems; and in the case of non-critical string theory, this is described by the position of the cuts.

- Non-perturbative amplitude and corrections: the $D$-instanton chemical potentials are the missing information in [perturbative] string theory, [33]. These are equivalent to the $\theta$-vacua in [1] which, in turn, are the $Dp$-branes establishing the appropriate boundary conditions rendering our Path Integral well defined. Further, these $\theta$-vacua correspond to the appropriate Stokes’ multipliers and factors which, as seen above, are related to the Riemann–Hilbert and Deligne–Simpson problems via inverse monodromy, thus establishing the Stokes’ phenomena of our system. This enables us to relate different perturbative vacua and study the string theory Landscape from first principles, where the relation between different string theory vacua in the Landscape happens via gluing the spectral curves non-perturbatively.

- Spectral Curves and perturbative string theory: the resolvent of the multi-cut matrix model defines the spectral curve, and thus the perturbative correlators, in the following way,

$$R(x) = \left\langle \frac{1}{N} \text{tr} \frac{1}{x - M} \right\rangle;$$
where,

\[ \mathcal{Z} = \int e^{-N \text{tr} V(M)} \, dM; \]
\[ = \int e^{-N \sum_i V(\lambda_i)} \prod_{i>j} (\lambda_i - \lambda_j)^2 \, d^N \lambda; \]

and the perturbative correlators are given by,

\[ R_n(x_1, \ldots, x_n) = \left\langle \prod_{i=1}^n \frac{1}{N} \text{tr} \frac{1}{x_i - M} \right\rangle; \]

i.e., a \( N \)-body problem with potential \( V \) (and eigenvalues \( \lambda_i \)). The resolvent operator allows us to determine the position of the eigenvalues and the eigenvalue density, [33]. The non-perturbative corrections are given by the \( D \)-instantons, i.e., by the \( \theta \)-vacua parameters that we compute via the Stokes’ factors of the model. The geometric meaning of these \( \theta \)-vacua is to determine the position of the “eigenvalue cuts”, i.e., the determination of the Stokes’ wedges.

\( \Rightarrow \) Quantum Integrability: Stokes’ phenomena of the Schwinger–Dyson Equations satisfy the \( T \)-systems of quantum integrable models, linking the spectral analysis of differential equations and integrable models, establishing a Differential Equation/Integrable Model correspondence, [33, 34].

2. \( PT \)-symmetric Quantum Mechanics started as a proposal to broaden the usual formulation of Quantum Mechanics by relaxing the requirement of Hermiticity of the Hamiltonian, [35]. Rather than using Hermiticity in order to guarantee that the Hamiltonian has a \( \mathbb{R} \)eal spectrum, it was required that the Hamiltonian satisfied the weaker condition of \( PT \) symmetry, i.e., that the Hamiltonian be invariant by parity reflection and time reversal. Finally, this generalization is usually studied via a \( \mathbb{C} \)omplex deformation of the harmonic oscillator, \( H = p^2 + x^2 (i x)^{\epsilon} \), where \( \epsilon \in \mathbb{R} \), exhibiting two phases: when \( \epsilon \geq 0 \) and when \( -1 < \epsilon < 0 \); the phase transition occurring at \( \epsilon = 0 \).

This scenario can be understood in terms of appropriate integration cycles for the Path Integral: effectively, the integration cycle is related to \( \epsilon \) in the following way. Let us choose \( \epsilon = 1 \) to represent the \( PT \)-symmetric phase, and \( \epsilon = -1 \) for the broken-\( PT \) one. When \( \epsilon = 0 \) we have a simple harmonic oscillator (or, in QFT terms, a free theory). Now, we can associate a Matrix model to each one of these phases, the \( PT \)-symmetric one being given by \( V_{\text{symm}}(X) = i X^3 \), and the broken-\( PT \) one being \( V_{\text{bsymm}}(X) = -i X^2 X^{-1} \). Moreover, as long as the broken-\( PT \) model also has \( \det X \neq 0 \), we have \( V_{\text{bsymm}}(X) = -i X \).

Thus, from their polynomial form alone we can already see that the same integration cycle will not guarantee the convergence of both models. In this sense, \( PT \)-symmetry can be associate with the suitable cycle guaranteeing the convergence of each respective Path Integral.

Now, following the guidance of our examples, let us define a model without a definite
PT-symmetric behavior, given by $V(X) = V_{\text{symm}}(X) + V_{\text{bsymm}}(X) = i(uX^3 - vX)$, where we have introduced the modular parameters $u, v \in \mathbb{C}$, and where the moduli space is given by $\mathcal{M} = \mathbb{C}P^1$, once only the ratio $u/v$ is relevant. This is in complete analogy to what was done in subsection 6.2. As such, $u$ (i.e., the top coupling) controls the analytic structure of the theory, and as $u \to 0$, $v$ plays an ever more relevant role, and only an integration cycle contained in both Stokes’ wedges would yield a convergent solution for both powers. As before, an integration cycle in a non-matching region of both Stokes’ wedges yields a divergent $\mathcal{Z}$, and an integration cycle that can be deformed from one wedge to another yields a vanishing $\mathcal{Z}$, analogous to Lee–Yang zeros (see the plots in subsection 6.2). In fact, we can recast this model as $V(X) = igX^3$, where $g = -u/v \in \mathbb{C}P^1$, and the linear term, $iX$, is absorbed in the source term, $\mathcal{J}X \to \tilde{\mathcal{J}}X$, where $\tilde{\mathcal{J}} = \mathcal{J} + 1$. Thus, we end up with a complete parallel to the example of subsection 6.2, scaled by a [power of the] factor $g$. Therefore, we can choose a particular linear combination of the cycles we obtained in subsection 6.2 in order to endow one of the solutions of this theory to be PT-symmetric, while the other solution will have broken-PT symmetry. This will be reflected in the allowed ranges of the coupling $g$, just like the $A_i$ and $B_i$ solutions in subsection 6.2 depend on the integration cycles themselves. As such, we have a phase transition at $g = 0$, and symmetric and broken-symmetric phases for $\Re(g) > 0$ and $\Re(g) < 0$.

3. The Complexification of the Path Integral of Quantum Mechanics, the analytic continuation of Chern–Simons theory, as well as that of Liouville theory, have recently been shown to relate branes in a two-dimensional $A$-model with possibly new integration cycles for their respective Path Integrals, [36]. The novelty introduced in these works is that a new integration cycle for the Path Integral is possible considering Complex-valued paths that are boundary values of pseudoholomorphic maps, giving a middle-dimensional cycle in the loop space of a symplectic manifold (classical phase space), which is one of the main ideas in Floer cohomology, [36].

A pseudoholomorphic curve (or $J$-holomorphic curve, where $J$ is a suitable almost-Complex structure, $J^2 = 1$) is a smooth map from a Riemann surface, $\Sigma$, into an almost-Complex manifold, satisfying the Cauchy–Riemann equations, [40]. Moreover, if $\Sigma$ is compact (e.g., Riemann Sphere, $\mathbb{C}P^1$), there is a nonlinear Fredholm theory describing the deformations of the given pseudoholomorphic curve, i.e., these deformations are parametrised by a finite-dimensional moduli space which, in turn, will be smoothly deformed by variations in $J$, [40]. This can be useful in a strategy where $J$-holomorphic curves are used as sigma-models, $\sigma : \Sigma \to (M,J)$, since the properties above can be used to determine the moduli space of the theory. Further, if $J$ is compatible with a symplectic structure on $M$ (think Phase Space), this provides us with an integrability condition extending this framework to a well-defined global theory (symplectic topology), [40]. In such a case, $I = \int_{\Sigma} \sigma$ is a topological invariant of $\sigma$, and controlled by topological data. In physics parlance, if $J$ is compatible with the Poisson bracket of our sigma-model’s Phase Space (a fact that will depend on the boundary conditions of the Schwinger–Dyson equations of the model), the Action of the theory is a [homotopy] invariant. As such, $\sigma$ yields numerical invariants: the Gromov–Witten invariants, opening the door to the use of Floer homology, Fukaya category (mirror symmetry, Khovanov...
homology), and, in four dimensions, to Seiberg–Witten invariants, [40].

In [36], the \( \sigma \)-model mentioned above is a topologically twisted \( A \)-model, where the target space is the Complexification of the original classical Phase Space. The integration cycle is described by an \( A \)-brane called a coisotropic brane, establishing a relationship between the \( A \)-model and quantization. In this sense, the Path Integral localizes in the new integration cycle(s) of the \( A \)-model.

This is not different from what we presented in this paper (particularly in the case of fields valued in a graded Lie algebra, so we can incorporate SUSY to our model and cancel eventual bosonic determinants appearing through the implementation of the boundary conditions via delta-functionals, on top of being able to use the superpotential in order to make things a bit simpler), provided we never cross the Stokes’ lines delimiting a certain Stokes’ wedge. Otherwise, we have to account for the Stokes’ factor (hyperasymptotics) associated to this crossing: the bottom-line is the same as in [36], the single-valuedness of the integrand, i.e., the choice of the \( n \)th root of the canonical bundle is an important part of the framework guaranteeing its global existence. Furthermore, the coherent states, \( \Upsilon \), that we computed in the examples are Poincaré dual to their respective defining integration cycles.

Finally, this is a good point to bring forward the similarities between the present paper, [19], and [36]. For example, integral discriminants are an intrinsic property of the Action \( S \), as shown below,

\[
J_{n|\tau} = \int e^{-S_{\tau_1,\ldots,\tau_r} \varphi_{\tau_1} \cdots \varphi_{\tau_r}} d^n\varphi ;
\]

where the simplest cases are just powers of the algebraic discriminant, which determines only the singularities of \( J_{n|\tau} \). The theory of integral discriminants is related to invariant theory, and can be viewed as one of the branches of \textit{nonlinear} algebra, [19, and references therein].

The connections are not difficult to be seen: equation (33) can be understood in terms of a Vector field, or we can think of the \( \varphi_i \)'s as \( n \) matrices in a multi-Matrix model, or we can interpret the fields in terms of a Tensor model (Group Field Theory). It is very interesting to note that [19] expresses the integral discriminants in terms of generalized hypergeometric functions, which is in complete analogy with our use of Fox’s \( H \)-function to describe the Path Integral, given that the \( H \)-function is a much broader generalization of hypergeometric functions, bringing the singularities of the theory to the foreground (in parallel to the relation between integral and algebraic discriminants).

Moreover, [19] uses Ward identities in order to specify appropriate integration cycles for the integral discriminants at hand. This is the same role played by the Schwinger–Dyson equation(s) in our framework. It is interesting to see the use of a diagrammatic technique representing the tensor fields of our model, [19]: this gives us a graphical way to compute integral discriminants (Path Integrals), which we have not explored in this paper — although it is possible to reconstruct these diagrams from the particular form of the \( H \)-function representing our theory.

Further, by \( \beta \)-deforming our Matrix model representation (i.e., assuming \( \beta \neq 1 \) in the
equation below), we can make connections with the Seiberg–Witten prepotentials,

\[ \mathcal{Z} = \int e^{-N \sum \lambda_i V(\lambda_i)} \prod_{i>j} (\lambda_i - \lambda_j)^{2\beta} \text{d}^N \lambda ; \]

where the Seiberg–Witten prepotential is given by \( \log(\mathcal{Z}) \), [19].

It is definitely a very positive reinforcement to see so many equivalences among such varied topics and approaches, and the framework we have constructed in this paper.

4. Complex Action Theory extends the position and momentum operators to be non-Hermitian, as well as their eigenvalues to be complex, allowing a description of a theory with a complex action or of a theory with a real action and complex saddle-points, [37]. The basic idea is twofold:

a) Redefine the inner product of the quantum states in a physically reasonable way, such that the Hamiltonian’s eigenstates are orthogonal with respect to it, i.e., the Hamiltonian has a well-defined Hermiticity under this new inner product — let us call this inner product \( I_Q \) and say the Hamiltonian is \( Q \)-Hermitian; &

b) Suppress the effect of the anti-\( Q \)-Hermitian part of the Hamiltonian after long times, thus the effect of the imaginary part of the \( Q \)-Hamiltonian (the anti-\( Q \)-Hermitian part) is smoothed out except for an irrelevant constant.

Compared to our approach, this is the same as realizing that the integration cycle (i.e., the particular Stokes’ wedge where the Path Integral is defined) determines the allowed values of the parameters of our theory (the coupling constants), and that gives us control over the asymptotic behavior of our theory: that is, ultimately the selected Stokes’ wedge determines the parameters of the theory which, in turn, control the fall-off behavior of the fields in the theory and thus its asymptotic properties.

In some sense, this is not very different from the \( PT \)-symmetric Quantum Mechanical case mentioned above, where the inner product is modified via a charge conjugation symmetry, \( C \), such that \( CPT \) is ultimately conserved in the theory. In this fashion, as soon as we choose a suitable integration cycle retaining \( PT \) symmetry, it suffices to choose a compatible \( I_C \) inner product. Thus, this charge conjugation symmetry, \( C \), plays the role of the deformation \( Q \) mentioned above, [37], defining a \( C \)-Hermitian property. This can be understood in terms of the extensions of operators, as done in Appendix A, where the extension is given by \( C \). In short, the properties of the particular chosen integration cycle select the parameters of the model which determine not only its asymptotic behavior but also the \( C \)-extension, clearly implying that different Stokes’ wedges (i.e., integration cycles) yield distinct sets of coherent states labelled \( 1_C = \int_c |\Phi \rangle \langle \Phi | \text{d} \Phi \), where \( C \) is the integration cycle associated to the charge conjugation symmetry, \( C \).

Complex Action Theory is then an extension of \( PT \)-symmetric Quantum Mechanics in the sense that one is free to pick any of the allowed integration cycles and define the charge conjugation symmetry, \( C \), through it. Then, it suffices to define a compatible \( PT \) symmetry, and build the theory from there. This enables the use of all Stokes’ wedges, as opposed to the situation in \( PT \)-symmetric Quantum Mechanics.
5. A new framework for Complex-valued classical fields in the case of quantum field theories describing neutral particles has recently been proposed, generalizing Osterwalder and Schrader’s construction of Euclidean fields via reflection positivity, studying scenarios that ensure and preserve it, [38].

The relevant part of this framework is that the neutral fields can be either Real or Complex — the usual distinction between neutral and charged fields does not correspond with the distinction between Real- and Complex-valued fields: Complex neutral fields allow for some novelty. As made quite clear in [38], the two-point function of the fields is the integral kernel of an operator, $D$, which need not be Hermitian. However, two requirements are made:

a) The Hermitian part of $D$ should have a strictly positive spectrum; &

b) The transformation $D$ should be reflection positive.

Charged fields are also treated in an innovative fashion: distinct charged fields are introduced, not related by Complex conjugation, where charge conjugation acts unitarily, $U_C \Phi_\pm U_C^{-1} = \Phi_{\mp}$.

Reflection Positivity not only gives meaning to inverse Wick rotations, but is also used in the analytic continuation of group representations, in understanding the properties of the spectrum of the Transfer matrix, and in the theory of phase transitions, [38].

The connection to our approach can be established when we are able to use the functional integral description of the above, which describes the decaying properties of the fields of the theory, i.e., their fall-off properties at the boundary. This is equivalent to setting boundary conditions to the Schwinger–Dyson equations appropriate to our model. In this sense, the problem of quantization is reduced to that of establishing suitable boundary conditions determining the desired decaying properties for the field content of our theory. This procedure can be accomplished in different ways, one of them being the construction of an appropriate Fourier Transform respecting the constraints of the problem, that is then understood as the partition function for the model (in an analogy with the Paley–Wiener theorem, [16]).

Essentially, this can be understood as an extension of $PT$-symmetric Euclidean QFT, where we can choose different combinations of $C$, $P$, and $T$ to be held by the Hamiltonian, and the remaining symmetry “corrects” the inner product. For example, we can choose a $CP$-symmetric Euclidean QFT whose inner product is corrected by $T$, etc. As such, this can be understood in terms of $J$-extensions of $J$-symmetric operators, [7,8,10].

In this sense, this discussion can be reduced to that of $PT$-symmetric Quantum Mechanics and of Complex Action Theory above.

As a final remark, it is worth saying that we tried to understand the process of quantization via the singularity structure and poles of the path integral, $Z^{\Sigma}_\tau[J]$, appropriately decorated by the parameters of the theory, $\tau$, and suitably compatible boundary conditions, $\Sigma$. In the language of [41], we can loosely say that we tried to study the functoriality of quantization via the singularity structure of the path integral: facts on a topological setting (e.g., fall-off properties of the fields, asymptotic structure of the path integral) can be translated to
an algebraic setting (algebra of observables, classification of the vacua of the theory). The representation of the path integral as a Fox $H$-function encodes the analytic structure of the theory in a Mellin–Barnes transform, which can be understood as an appropriate $L$-function (viz. functoriality via poles of $L$-functions), [41].

In particular, as suggested in [41], one can think of Quantization $\subset$ Deformation: study the symmetries of a certain equation of motion to be quantized and deform the Lie algebra associated to each symmetry. These should be able to be expressed in terms of their own path integral. Deformation and Functoriality then can be seen as a reflection of the wave-particle duality, respectively: functoriality $\sim$ particle $\sim$ Heisenberg picture, deformation $\sim$ wave $\sim$ Schrödinger picture.

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After this work was completed, the following references were kindly brought to our attention, [42,43]. Also, [38] has recently been augmented by [39].

### A. Extensions of Operators

Another approach to the material developed above would be in terms of extensions of symmetric operators, [7]. Denoting the Schwinger–Dyson operator by $\Theta_{SD}$, it is said to be symmetric if

$$\langle \Theta_{SD} \phi | \varphi \rangle = \langle \phi | \Theta_{SD} \varphi \rangle .$$

Its self-adjoint extensions can be understood in terms of its boundary values (resp. $Dp$-branes), i.e., the monodromy of the boundary conditions (resp. brane monodromy) parameterize the possible extensions. Analogously, we can study $\Theta_{SD}$ via its Cayley transform, $U_{\Theta_{SD}} = (\Theta_{SD} - i \mathbb{1})(\Theta_{SD} + i \mathbb{1})^{-1}$, where $\mathbb{1}$ is the unit operator: if $U_{\Theta_{SD}}$ is unitary then $\Theta_{SD}$ is self-adjoint. This can be measured through the deficiency indices of $\Theta_{SD}$, which are defined as the dimension of the orthogonal complements of its domain and range: if both deficiency indices are identical then $U_{\Theta_{SD}}$ is unitary.

The interesting thing to notice is that different self-adjoint extensions may have different spectrum. That is, by varying the monodromy of the boundary conditions (i.e., varying the brane monodromy) we obtain different operators $\Theta'_{SD}$, which may define different theories in terms of their particle spectrum.

This can be further understood in terms of an extension of the resolution of the identity, i.e., coherent state quantization à la [11],

$$\mathbb{1} = \left( \int_{c_1} + \cdots + \int_{c_n} \right) \ket{\Phi} \bra{\Phi} = \frac{1}{\sum_{j=1}^{n} \alpha_j} \sum_{j=1}^{n} \alpha_j e^{i c_j} ;$$
where $C_{m=1,...,n}$ labels the integration cycle associated to each monodromy generator, and $e^{iC_{m=1,...,n}}$ represents the contribution from each cycle.

This can be further generalized by relaxing the condition that the extension be self-adjoint, [10]. Instead, we can talk in terms of $J$-extensions, where $J$ is a conjugation: $\langle J \phi, J \varphi \rangle = \langle \phi, \varphi \rangle$. That is, a $J$-extension of $\Theta_{SD}$ is such that $J \Theta_{SD} J = \Theta_{SD}^\dagger \Leftrightarrow \langle \Theta_{SD} \phi, J \varphi \rangle = \langle \phi, J \Theta_{SD} \varphi \rangle$. For example, $\mathcal{P} \mathcal{T}$-symmetric quantum mechanics falls in this case if we choose $J = \mathcal{P} \mathcal{T}$, where we have to modify the inner product slightly in order for $J$ to remain a conjugation (so we can still retain $\mathcal{C} \mathcal{P} \mathcal{T}$ invariance, [13]).

This should not necessarily come as a surprise because of the following observation. We can understand the boundary conditions for $\Theta_{SD}$ via explicit boundary terms in the Action. These explicit boundary terms are implemented via new fields that are subjected to the actual boundary conditions we desire. The new fields are related to the original ones: they are merely convenient objects used to express the constraints necessary for the boundary conditions of $\Theta_{SD}$ to be [explicitly] reflected in the Action. Finally, these boundary terms in the Action are equivalent to $Dp$-branes described by these new fields: this is the rationale behind brane quantization, [18].

Following this construction, the notion of $J$-extensions described above is translated in terms of deformations of the integration cycle for the path integral, done in such a way as to implement the particulars of the conjugation at hand.

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