Quantum state reconstruction from dynamical systems theory

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When an informationally incomplete set of observables is considered there are several solutions to the quantum state reconstruction problem using von Neumann measurements. The set of solutions are known as Pauli partners, which are not easy to find even numerically. We present, in a self-contained paper, a new way to find this solutions using the physical imposition operator. We show that every Pauli partner is an attractive fixed point of this operator, which means that we can find complete sets of Pauli partners very efficiently. As a particular case, we found numerically 24 mutually unbiased bases in dimension $N = 23$ in less than 30 seconds in a standard PC. We hope that the algorithm presented can be adapted to construct MU Constellations, SIC-POVMs, Equiangular Tight Frames and Quantum t-Designs, which could open new possibilities to find numerical solutions to these open problems related with quantum information theory.

Keywords: Quantum state reconstruction, Mutually unbiased bases, Pauli partners.
PACS: 03.67.-a;03.67.Ac

I. INTRODUCTION

Quantum state reconstruction from complete sets of physical distributions is an important open problem in the foundations of quantum mechanics. It has been proved that it belongs to the complex category NP-(Hard), where NP means non polynomial. That is, this is a problem harder than NP-problems. An algorithm is NP when the time required to solve it increases faster than any polynomial function depending on the relevant parameters of the system, for example, the space dimension. Sometimes, there exists polynomial algorithms that can solve NP problems efficiently. In this case, we say the problem is NP-EASY. For example, the sort operation is a NP problem, but there is a polynomial operation (quick sort), that is able to solve it in polynomial time. A review of NP problems that can be reduced to polynomial ones can be found in [4]. In our work, we could not find a polynomial algorithm for quantum state reconstruction. However, we have an exponential algorithm that has a very small exponential factor, what means that we can

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reconstruct efficiently quantum states, and complete sets of Pauli partners, in high dimensional Hilbert spaces. For example, we found numerically 24 mutually unbiased bases in a 23-dimensional Hilbert space in 27 seconds with a standard PC. Here, we present a new way for studying quantum state reconstruction using dynamical system theory. We show that different states containing the same information about given observables, that is Pauli partners, are attractive fixed points of the *physical imposition operator*. We will present an algorithm able to find complete sets of Pauli partners when informationally incomplete sets of observables are considered and a unique solution when the observables are informationally complete. As a particular case, our algorithm is able to recognize when a set of observables is informationally complete.

This paper is organized in the following way: in Section II, we define the quantum state reconstruction problem and we mention some related open problems. In Section III, we present an algorithm for state reconstruction and we prove several properties using dynamical systems theory. Additionally, we give a sufficient condition for defining informationally complete set of observables. In Section IV, we discuss numerical results and we mention some possible applications for the algorithm, and in Section V we resume our work. In the appendices there are some proofs an we define a generalization of Hellinger metric.

II. PAULI PROBLEM

The pure state of a quantum system is a ray in the Hilbert space, and it contains the complete information about every relevant physical property of the system. In this work, we consider the quantum state reconstruction from von Neumann measurements, that is, from complete sets of eigenvalues distributions corresponding to several observables.

In the laboratory, we are able to make measurements that contain information about the quantum state. However, these measurements are not the projections of the state in a basis of the Hilbert space, like happens in classical mechanics. What we are able to measure are only the amplitudes of these projections, and this means that we should solve a set of non linear equations in order to find the state of the system. When we consider distributions from an incomplete set of observables, these equations do not have a unique solution, and this situation is precisely the Pauli problem: there are more than one state with the same partial information about the physical system, and the problem is to find the complete set of them. This situation was suggested originally by W. Pauli in a footnote, and the problem is also known as the quantum state reconstruction problem. Different states containing the same set of eigenvalues distributions of the quantum system are named Pauli partners. The original version of Pauli considers position and momentum observables, and there are several examples of different quantum states with the same distributions for these observables. We particularly remark Theorem 2.5 in where it is proved that there exists wave functions with an infinite number of Pauli partners. Today, we know that the Pauli problem appears when we do not consider an informationally complete set of observables. In the case of position and momentum, we know that their eigenvectors basis are not
the maximal set of mutually unbiased bases (MUBs). For example, in a $N$-dimensional Hilbert space we need $N - 1$ observables beside position and momentum \cite{10} to reconstruct every quantum state efficiently \cite{11}. In quantum information theory, position and momentum bases are known as the eigenvectors bases of the \textit{shift} ($X$) and \textit{phase} ($Z$) operators. Shift and phase operators are physically relevant, because they are the generators of the translation and impulsion of states in a unity of magnitude. Also, these operators are very close to two important open problems in quantum information theory: existence of MUBs and SIC-POVMs in arbitrary dimensional Hilbert space.

Let us give an introduction to the quantum state reconstruction problem. Let $A, B$ be two observables and $\{\phi_k\}_{k=1..N}, \{\phi_p\}_{p=1..N}$ be their eigenvectors basis. The information about $A$ and $B$ contained in the pure state $\Phi$ is given by the eigenvalues probability distributions

$$\rho^A_k = |\langle \phi_k, \Phi \rangle|^2, k = 1..N$$

and

$$\rho^B_p = |\langle \phi_p, \Phi \rangle|^2, p = 1..N$$

respectively. The state reconstruction problem for the observables $A$ and $B$ consists in finding $\Phi$ from the knowledge of the distributions $\{\rho^A_k\}$ and $\{\rho^B_p\}$.

We remark that Pauli partners exist in every finite dimensional Hilbert spaces, even in the smallest dimension $N = 2$. For example, a 2-dimensional state $\Phi$ have the same position and momentum distributions than $\Phi^* \cite{14}$. The spin $s = 1/2$ version of this problem do not have Pauli partners because the observables $S_x, S_y, S_z$ are a maximal set of mutually unbiased operators, that is, observables with mutually unbiased eigenvector basis. Pauli partners disappear because these eigenvectors basis determine a complete set of Mutually Unbiased Bases (MUBs) in dimension $N = 2$, and it has been proved that complete set of MUBs are optimal for quantum state reconstruction \cite{11}. However, for $s \neq 1/2$ the spin eigenvectors basis are not a set of MUBs and there exists a null measure set of states having Pauli partners \cite{15,16}. In a previous work \cite{17}, we found analytically the complete set of states with Pauli partners corresponding to a particle with spin $s = 1$ predicted by Amiet-Weigert in \cite{16} using our physical imposition operator.

Another interesting problem related to quantum state reconstruction is to find the maximal set of MUBs. Two orthonormal bases defined in a $N$ dimensional Hilbert space $\{\phi_k\}_{k=1..N}$ and $\{\phi_p\}_{p=1..N}$ are MUBs if

$$|\langle \phi_k, \phi_p \rangle|^2 = \frac{1}{N}, \forall k, p = 1..N.$$  

It is well known that there exists a maximal set of $N + 1$ MUBs when $N$ is a power of a prime number and at most $N + 1$ MUBs in every dimension \cite{10}, but the maximal set of MUBs in non power of prime dimensions is still unknown, even in the lowest dimensional case $N = 6 \cite{18,20}$. The MUBs problem is a particular case of quantum state reconstruction problem, that appears when we consider MUBs eigenvectors basis for the observables and when every probability distribution
considered is flat. A distribution is named flat when every probability take the same value $1/N$. For this reason, our algorithm may be applied to try solve the MUBs problem.

III. ALGORITHM FOR STATE RECONSTRUCTION

In finite dimensions, the quantum state reconstruction problem is defined as follows: Suppose that we have a quantum system with $m$ incompatible observables. Each one has an eigenvectors basis $\{\psi_k^r\}_{r=1..m}^k=1..N$, where $N$ is the dimension of the Hilbert space. Suppose also that we have realized measurements of the $m$ observables on an ensemble prepared in the same unknown state $\Phi$, obtaining the eigenvalues distributions $\{\rho_k^r\}, r = 1..m, k = 1..N$. Then, the Pauli problem implies to find the complete set of solutions for $\Phi$ satisfying the set of non linear equations

$$\rho_k^r = |\langle \psi_k^r, \Phi \rangle|^2, \forall k = 1..N, r = 1..m. \quad (4)$$

The maximal set of different states satisfying Eq. (4) determines the complete set of Pauli partners of the system. Its general solution is still unknown and only a few particular cases were solved or partially solved, for example [13, 16, 17, 21]. Next, we will define a non linear operator which will be an useful tool for finding complete sets of Pauli partners.

A. Physical imposition operator

The physical imposition operator is a natural way for transmitting to any state the complete eigenvalues distribution obtained from measurements over a system. To understand how it works let us analyse the increase of knowledge that has an observer about an unknown ensemble when he realize measurements. We should do not confuse this situation with the knowledge that has the observer about the collapsed state, but with the partial knowledge acquired about the original state, unperturbed by measurements. A way to quantify the partial knowledge about an ensemble is to consider the volume of all possible states that are compatible with the information collected about the state. Before any measure, the observer has no knowledge about the system. So, he is not able to rule out any Hilbert space element. In this situation, his best option is to choose a state at random, namely $\Psi_0$. Obviously, this state has any property about the system. In order to obtain a better description of the quantum system we must impose some information on $\Psi_0$ that should be obtained from measurements. Then, let us suppose that the observer considers an observable $A$ with eigenvectors basis $\{\varphi_k\}_{k=1..N}$ and he realizes a standard tomography process over an ensemble prepared in the unknown state $\Phi$. After measurements, the information obtained is the distribution

$$\rho_k = |\langle \varphi_k, \Phi \rangle|^2, \forall k = 1..N. \quad (5)$$

In the measurement process, we obtain information about the system and, after that, all states in the Hilbert space are not equally probable for describing the original state. Suppose that we want
to establish a connection between the state \( \Psi_0 \) chosen random, that contains null information about the system and the state \( \Psi_1 \), having the complete information about the probability distribution \( \{ \rho_k \} \) and no more information than this about the system. Of course, there are infinite options for \( \Psi_1 \) but, a simple one involves the following operator

**DEFINITION III.1** Let \( A \) be an observable with eigenvectors basis \( \{ \varphi_k \}_{k=1..N} \) and \( \Phi, \Psi_0 \in \mathcal{H} \). Then, we define the Physical Imposition Operator as

\[
T_A \Phi \Psi_0 = \sum_{k=1}^{N} \frac{|\langle \varphi_k, \Phi \rangle|}{|\langle \varphi_k, \Psi_0 \rangle|} \varphi_k.
\]  

(6)

Then, our state \( \Psi_1 \) is given by \( T_A \Phi \Psi_0 \). This operator is well defined for every quantum states except when \( \Psi_0 = \varphi_k \), for any \( k = 1..N \). When this happen, we replace \( \frac{|\langle \varphi_k, \Psi_0 \rangle|}{|\langle \varphi_k, \Psi_0 \rangle|} \varphi_k \) by the unity. Let us analyze in three steps how \( T_A \Phi \) acts on a state \( \Psi_0 \):

1- Expands \( \Psi_0 \) in the eigenvectors basis \( \{ \varphi_k \} \):

\[
\sum_{k=1}^{N} \frac{|\langle \varphi_k, \Psi_0 \rangle|}{|\langle \varphi_k, \Psi_0 \rangle|} \varphi_k.
\]

2- Removes the information that \( \Psi_0 \) contains about \( A \):

\[
\sum_{k=1}^{N} \frac{|\langle \varphi_k, \Psi_0 \rangle|}{|\langle \varphi_k, \Psi_0 \rangle|} \varphi_k.
\]

3- Imposes the information about \( A \) obtained in the laboratory:

\[
\sum_{k=1}^{N} \frac{|\langle \varphi_k, \Phi \rangle|}{|\langle \varphi_k, \Psi_0 \rangle|} \varphi_k.
\]

Notice that in the step 2 we have not a quantum state, because the vector is not normalized. However, in the step 3 normalization is restored because the distribution imposed is normalized. The physical imposition operator is non linear, idempotent and preserve norms. These properties can be proved very easily from the Definition III.1. However, it is neither a projector nor an unitary operator due to the nonlinearity. The physical imposition operator gives us the first approach to the state \( \Phi \) of the system. Notice that the complex unitary phases in the state \( \Psi_0 \) remind unchanged when \( T_A \Phi \) is applied. The reconstruction problem would be solved if we could define an observable \( B \) that let us find the unknown phases in the laboratory but, as we know, it is not possible in general. This is the main reason why the quantum state reconstruction problem exists. The knowledge about only one observable is not enough for the reconstruction of the state, except in the trivial case when the system is prepared in an eigenvector state of the observable.

Then, we needed to take into account additional information of, at least, a second observable \( B \) that does not commute with \( A \) with the aim to fix the state. In order to do this we consider a second physical imposition operator \( T_B \Phi \), related to the observable \( B \), given by

\[
T_B \Phi \Psi_1 = \sum_{p=1}^{N} \frac{|\langle \phi_p, \Phi \rangle|}{|\langle \phi_p, \Psi_1 \rangle|} \phi_p,
\]

(7)

where \( \pi_p = |\langle \phi_p, \Phi \rangle|^2 \) is the eigenvalues distribution of \( B \) in the state \( \Phi \) and \( \phi_p \) are the eigenvectors of \( B \). The state \( \Psi_1 = T_A \Phi \Psi_0 \) has more information than \( \Psi_0 \) about the system and the next step is natural, we should consider the state \( \Psi_2 = T_B \Phi \Psi_1 = T_B \Phi T_A \Phi \Psi_0 \). Notice that \( \Psi_2 \) contains the complete information about the distribution \( \{ \pi_p \} \) but a partial information about the distribution \( \{ \rho_k \} \), because the imposition operator \( T_B \Phi \) destroys the information about \( A \) when the modulus of
the coefficients of \( \Psi_1 \) in the basis \( \{ \phi_p \} \) are replaced. However, part of the information about \( \{ \rho_k \} \) remains in the unchanged phases when \( T_{B\Phi} \) is applied. It is important to remark that the single imposition operators are idempotent, because they exhaust the information about an observable, but the multiple imposition operator \( T_{AB\Phi} = T_{B\Phi} T_{A\Phi} \) does not exhaust the information about \( A \) and \( B \), and then \( T_{AB\Phi} \) is not idempotent. So, we can define the sequence \( \Psi_n = (T_{AB\Phi})^n \Psi_0 \) and hope that the successive impositions could give a state with the complete information about \( A \) and \( B \). If we have more than two observables we should consider the sequence \( \Psi_n = (T_{ABC..})^n \Psi_0 \). In every step of the sequence the unknown phases gain information about the set of distributions and, intuitively, we hope that after an infinite number of steps this insistent process converges to a solution. In order to study the convergence of this sequence we must define a metric. Basically, we need two metrics because 1) We need to know when a sequence converges, and 2) We need to know when two states are partners. Therefore we need a metric in the space of Hilbert space rays and another metric in the space of eigenvalues distributions. In order to analyze the convergence of a sequence of quantum states we cannot take into account the usual metric in Hilbert space, because quantum states are defined up to a global complex phase. The usual distance between the states \( \Phi \) and \( e^{i\alpha} \Psi \) is given by \( \delta(\Phi, e^{i\alpha} \Psi) \) and it depends on \( \alpha \). A good choice is the Bures metric \( d(\cdot, \cdot) \) that, for pure states, is given by
\[
d(\Phi, \Psi) = \min_{\alpha \in [0, 2\pi]} \delta(\Phi, e^{i\alpha} \Psi) = \sqrt{2 - 2|\langle \Phi, \Psi \rangle|}.
\] (8)
Also, we need to compare distributions, and we consider Hellinger metric \( d(\cdot, \cdot) \) and a natural generalization to several distributions. In Appendix B, explicit expressions of these metrics and their properties can be found.

The physical imposition operator needs, as input, sets of compatible distributions (in order to agree with the indeterminacy principle), and in computer simulations we need a way to obtain them. This is why it is important to define the notion of generator state: Let \( \Phi \in \mathcal{H} \) be a quantum state and \( \{ A_j \} \) be a set of observables with eigenvectors bases \( \{ \phi_{jk} \} \) respectively. Then, we are able to generate the set of distributions
\[
\rho_{jk} = |\langle \phi_{jk}, \Phi \rangle|^2, k = 1..N, j = 1..m,
\] (9)
from the state \( \Phi \). After this process, we remember the distributions \( \rho_{jk} \) but we forget the generator state \( \Phi \), and we try to reconstruct it using the physical imposition operator, the eigenvectors bases, and the information contained in the distributions \( \{ \rho_{jk} \} \). Generator state appears as an useful subscript in the physical imposition operator \( T_{AB\Phi} \), but we remark that the state \( \Phi \) is not a known state in the reconstruction process. We only have knowledge about the distributions generated from it.

B. Quantum state reconstruction from dynamical system

In this section, we show that the physical imposition operator has a beautiful description from dynamical system theory, and we prove that each Pauli partner is an attractive fixed point of the
physical imposition operator. First, let us introduce some basic notions from dynamical systems theory.

**Definition III.2 (Fixed point)** Let $\phi \in \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ be an operator. We say that $\phi$ is a fixed point of $T$ iff $\phi$ is invariant when $T$ is applied. That is, $T\phi = \phi$.

**Definition III.3 (Attractive fixed point)** Let $T : \mathcal{H} \to \mathcal{H}$, $d(\cdot, \cdot)$ a metric defined on the rays of $\mathcal{H}$ and $\phi$ a fixed point of $T$. We say that $\phi$ is an attractive fixed point of $T$ if $d(T\psi, \phi) \leq d(\psi, \phi)$ for all $\psi$ contained in a neighborhood of $\phi$.

It is easy to notice that every fixed point of a single physical imposition operator has the same distribution than the generator state. Then, a state is a fixed point of the single physical imposition operator iff it is a Pauli partner for the single observable considered. However, if we consider more than one observable in the physical imposition operator there are more fixed points than partners. That is, there could exist a state $\eta$ satisfying $T_{AB}\Phi \eta = T_{B}\Phi T_{A}\Phi \eta = \eta$, but $T_{A}\Phi \eta \neq \eta$ (notice that $T_{AB}\Phi \eta = \eta$ necessarily imply that $T_{B}\Phi \eta = \eta$, because $T_{B}\Phi$ is the last single operator applied and it impose the eigenvalues distribution about $B$ contained in $\Phi$). Let us define the set of fixed points that are interesting for the quantum state reconstruction problem.

**Definition III.4** Let $A$ be an observable and $\Phi$ a generator state. We define $\Gamma_{A\Phi}$ as the set of fixed points of the single imposition physical operator $T_{A\Phi}$. That is,

$$\Gamma_{A\Phi} = \{ \Psi \in \mathcal{H} / T_{A\Phi} \Psi = \Psi \}. \tag{10}$$

Here, we are considering only one representant $\Psi$ from the ray $\Psi_\alpha = e^{i\alpha} \Psi$. Now, we are going to define a particular set of fixed point of the multiple physical imposition operator.

**Definition III.5** Let $\{A^j\}_{j=1..m}$ be a set of incompatible observables and $\Phi \in \mathcal{H}$ a generator state. We define $\Gamma_{A^1..A^m,\Phi}$ as

$$\Gamma_{A^1..A^m,\Phi} = \Gamma_{A^1,\Phi} \cap \cdots \cap \Gamma_{A^m,\Phi}. \tag{11}$$

Notice that since $\Phi$ is a generator of distributions we always have $|\Gamma_{A^1..A^m,\Phi}| \neq 0$ where the symbol $|\cdot|$ is the cardinality of the set. The next two propositions are easy to understand and their purpose is to clarify the recent definitions. The proofs are trivial.

**Proposition III.1** Let $\{A^j\}_{j=1..m}$ be an informationally incomplete set of observables and $\Phi \in \mathcal{H}$ a generator state. Then, the number of Pauli partners $N$ is given by

$$N = |\Gamma_{A^1..A^m,\Phi}|. \tag{12}$$

In case of $N = 1$ we say that $\Phi$ is a Pauli unique.

**Proposition III.2** A set of observables $\{A^j\}_{j=1..m}$ is informationally complete iff

$$|\Gamma_{A^1..A^m,\Phi}| = 1, \forall \Phi \in \mathcal{H}. \tag{13}$$
PROPOSITION III.3 Let \( \{A^j\}_{j=1}^m \) be a set of \( m \)-observables and \( \Phi \in \mathcal{H} \) a generator state. Then, \( \Psi \in \Gamma_{A^1 \ldots A^m} \) iff \( \Psi \) is a Pauli partner of \( \Phi \).

Now we are going to present the most important fact on this paper:

PROPOSITION III.4 Let \( \{A^j\}_{j=1}^m \) be a set of \( m \)-observables, \( \Phi \in \mathcal{H} \) a generator state and \( \Psi \in \Gamma_{A^1 \ldots A^m} \Phi \). Let \( d(\cdot, \cdot) \) be the Bures metric for quantum states and \( T_{A^1 \ldots A^m} \Phi \) the physical imposition operator related to the observables \( \{A^j\}_{j=1}^m \). Then, \( \Psi \) is an attractive fixed point of \( T_{A^1 \ldots A^m} \Phi \).

The proof of this proposition is easy but not short, and it can be found in Appendix A. This proposition means that all fixed points in \( \Gamma_{A^1 \ldots A^m} \), that is, the complete set of Pauli partners of the system, are attractive fixed points of \( T_{A^1 \ldots A^m} \Phi \), considering Bures Metric. The most important result given by the above proposition is that the multiple physical imposition operator allows us to find the complete set of Pauli partners of a system for every set of observables and every generator chosen. The above proposition together with the fact that probability distributions come from a generator state are necessary and sufficient conditions to have a convergent sequence \( \Psi_n = T_{A^1 \ldots A^m}^n \Phi_0 \). In Section IV we present some results obtained from numerical simulations.

We can extend the definition of the physical imposition operator to mixed states and composite systems, considering tensor product of singles ones. Another interesting case is to consider entangled physical imposition operators, entangled generators or both of them. A further study of our algorithm in these cases will be presented in another work.

C. Bifurcations

Sometimes, for some particular values of the parameters of an operator, the stability of their fixed points change, fixed points are added or removed. When one of these situations happen we say that the operator has a bifurcation. In our work, bifurcations are very important, because they tell us about where Pauli partners appear or disappear. The parameters in the physical imposition operator are given by the physical distributions. Let us give an example of bifurcations in the physical imposition operator: let \( (A, B) \) be two observables with MUBs eigenvector bases, defined in a \( N \) dimensional Hilbert space where \( N \) is a power of a prime number. Let us consider the following generators: \( \Phi_1 \) be an eigenvector of \( A \) or \( B \), and \( \Phi_2 \) a mutually unbiased vector to \( A \) and \( B \) eigenvectors. This means that \( \Phi_1 \) generates (sharp,flat) or (flat,sharp) distributions on \( (A, B) \) whereas \( \Phi_2 \) generates (flat,flat) distributions on \( (A, B) \). We know that \( \Phi_1 \) has no partners for the considered observables, because it is an eigenvector of \( A \) or \( B \), whereas \( \Phi_2 \) has at least \( N(N-1) - 1 \) partners corresponding, together with \( \Phi_2 \), to the set of eigenvectors of the \( N-1 \) complementary MUBs to the eigenvectors of \( A \) and \( B \). Then, every curve in the state space connecting the generators \( \Phi_1 \) with \( \Phi_2 \) contains at least one bifurcation.

The basin of attraction of Pauli partners has an interesting property. Let us notice that the
sequences

\[
\Psi_n = T^n_{ABC} \cdot \Phi \Psi_0, \quad (14)
\]

and

\[
\tilde{\Psi}_n = T^n_{ABC} \cdot \tilde{\Phi} \tilde{\Psi}_0, \quad (15)
\]

are identical when \( T_{A\Phi} \Psi_0 = T_{A\Phi} \tilde{\Psi}_0 \), that is, when \( \Psi_0 \) have the same set of unitary phases than \( \tilde{\Psi}_0 \) in the basis of eigenvectors of \( A \), namely \( \{ \varphi_k \}_{k=1..N} \). This means that the information about the basin of attraction of Pauli partners is only contained in the \( N - 1 \) relevant phases of the initial state \( \Psi_0 \). Then, if we consider the decomposition of \( \Phi_0 \) in the basis \( \{ \varphi_k \} \)

\[
\Psi_0 = (\sqrt{\rho_0}, \sqrt{\rho_1} e^{i \alpha_1}, \ldots, \sqrt{\rho_{N-1}} e^{i \alpha_{N-1}}), \quad (16)
\]

the \( N - 1 \) dimensional real vector

\[
\vec{r} = (\alpha_1, \ldots, \alpha_{N-1}), \quad (17)
\]

contains the complete information about where the sequence \( \Psi_n = T^n_{ABC} \cdot \Phi \Psi_0 \) converges. This fact is an advantage in numerical simulations, because we should consider a \( N - 1 \) dimensional seed \( \vec{r} \) instead of the \( 2(N - 1) \) dimensional seed \( \Psi_0 \).

Until now, we have been studying Pauli partners, that is, considering informationally incomplete sets of observables. Let us present a proposition that connects bifurcations with informationally complete set of observables in a univocal way.

**PROPOSITION III.5** Let \( A^1..A^m \) be a set of incompatible observables. Then, \( T_{A^1..A^m} \Phi \) has no bifurcations for all \( \Phi \in H \) iff \( \{ A^1..A^m \} \) is an informationally complete set of observables.

Proof: Suppose that \( T_{A^1..A^m} \Phi \) has no bifurcations. We know that every eigenvector of \( A^1..A^m \) has no Pauli partner, because there is a unique state that has a sharp distribution. Since \( T_{A^1..A^m} \Phi \) has no bifurcations, Pauli partners cannot appear for any generator state and the quantum state can be reconstructed efficiently for all distributions. Then, \( A^1..A^m \) is an informationally complete set of observables. The reciprocal argument can be immediately proved by definition of informationally complete set of observables.

Notice that this proposition imply a high cost in numerical simulations because we should consider a huge set of generators. However, some theorems related to necessary conditions for the existence of bifurcations could be very useful [20]. Interesting theorems related with informationally complete sets can be found in [9, 27–29].

### IV. NUMERICAL RESULTS

**A. Mutually unbiased bases (MUBs)**

We know that it is possible to construct a maximal set of \( N + 1 \) MUBs in dimension \( N = p^r \), where \( p \) is a prime number and \( r \) is a positive integer. For non power of prime dimensions the
maximal set of MUBs is still unknown. For example, in dimension $N = 6$ three MUBs have been found, but not 7, and in general we know that there exists at least 3 MUBs for every dimension. This last case is particularly interesting in physics because, the 3 MUBs are conformed by the eigenvector basis of the shift generator $Z_x = e^{-ixP}$, the boost $B_p = e^{ipX}$ and the operator $Z_x B_p = e^{-ix_p P} e^{ip_0 X} \neq e^{i(p_0 X - x_0 P)}$ defined in a finite dimensional Hilbert space of dimension $N$ $\Rightarrow$ [30]. Notice that the last equality is not allowed in finite dimensions, but it is valid when $N \to \infty$.

The problem of determining the maximal number of MUBs is contained in the quantum state reconstruction problem. For example, given two observables with mutually unbiased eigenvectors basis, like position and momentum, we can ask to our algorithm about the maximal set of states having flat distributions for these observables. If $N$ is prime, the algorithm presented in this work is an useful tool to reconstruct the complete set of $N + 1$ MUBs. Starting with position and momentum eigenvectors bases we could found $N + 1$ MUBs for $N = 2, \ldots, 37$ ($N$ a prime number).

The algorithm is very efficient. For example, we could reconstruct 24 MUBs in $N = 23$ in 27 seconds and 38 MUBs in $N = 37$ in 75 minutes in a standard Pentium IV. When $N$ is not prime, starting with position and momentum basis we could only obtain 3 MUBs. This means that the second basis considered (momentum basis) is not adequate for constructing $N + 1$ MUBs for all dimensions (if they exist) $\Rightarrow$ [31]. We know that if $N = p^r$ (power of prime) a suitable second basis is $\tilde{\phi}_p = \bigotimes_{j=1}^{r} \phi_p^j$, where $\{ \phi_p \}_{p=1..N}$ is the momentum basis in dimension $p$ and $\bigotimes$ is the tensorial productory. This second basis is a very particular option, and it cannot be found from searches realized over random orthogonal bases. Of course, when we put the correct second basis, the algorithm is able to reconstruct $N + 1$ MUBs when $N$ is a power of a prime number, but this process is not possible without the additional information about the second basis. Several simulations had been realized for $N = 6$ and we could not find more than 3 MUBs, but these numerical results are not enough to refuse the existence of 7 MUBs in $N = 6$. The problem of determining maximal sets of MUBs in non power of prime dimensions starts in the very huge set of options for the second basis. Moreover, the choice of the first vector of the second MUB is determinant to construct a maximal set of MUBs. Considering the first MUB as the canonical (computational) basis, every vector of the second basis has $N - 1$ complex phases to be fixed, because all the amplitudes are equal to $1/\sqrt{N}$. The additional condition needed is the orthogonality of the bases ($N^2/2 - N/2$ conditions). Then, given that we have $N(N - 1)$ phases, there exist a $N^2/2 - N/2$-manifold for the free parameters, namely $\{ \chi_k \}_{k=1..N^2/2-N/2}$. In numerical simulations, we must take a finite partition of each parameter $\chi_k$ in order to explore all possible second basis. Considering that every partition is regular and it has $m$ intervals, then we should analyze $m^{N^2/2-N/2}$ possibilities for the second MUB. When $N = 6$, and considering $m = 5$, which is not a good partition, we should analyze $5^{15} \approx 3 \times 10^{10}$ possibilities. Supposing that we have an algorithm that decides if 2 MUBs can be extended to $N + 1$ MUBs in only 1 second we need wait 968 thousand years to analyze every case. Because of this problem, numerical existence of maximal sets of MUBs should be explored in a different way, considering MU constellations $\Rightarrow$ [32–35].
B. Basin of attraction

Finally, we present some numerical results for the basin of attraction of the physical imposition operator. In simulations we could see that basin of attractions are very complex sets in general, and due to this complexity we do not have any hope of obtaining in a simple way analytical expressions neither for the basin of attraction of Pauli partner nor for the undesirables initial states. Undesirable initial states are initial states $\Psi_0$ within the basin of attraction of an attractive fixed point that is not a Pauli partner. In our several simulations realized in dimension $N = 3$ we could not detect basins of attraction with fractal behavior, and we have no knowledge about the existence of fractal basin of attraction in higher dimensions.

As we have seen in Section III C, we can show in a plane the complete basins of attraction for every set of observables and every generator when $N = 3$. Let us consider, then, two observables with MUBs eigenvectors basis defined on $\mathcal{H} = \mathbb{C}^3$, specifically position and momentum. In Fig. 1 we show the basin of attraction of Pauli partners in case of the MUBs problem, that is, considering a generator of flat distributions. We found 6 partners, corresponding to the two complementary mutually unbiased bases to the eigenvectors bases of the observables. This result tells us that there is only one way to complete a maximally set of MUBs from position and momentum when $N = 3$. Another interesting result is that there is no isolated MU partner, though they exist in other lower dimensions. Fig. 2 and Fig. 3 correspond to two random generators and we find 4 and 5 Pauli partners respectively. In Fig. 4 and Fig. 5 we consider two random generators without partners. Each color in the figures is identifying a different basin of attraction. Violet regions shows basin of attraction of the undesirable fixed points. Sometimes, an undesirable point may be a saddle point instead of a fixed point, having an attractive manifold of null measure, named stable manifold. Applying our algorithm to the maximally MUBs problem ($N = 3$) we could see that every basin of attraction of a partner is a triangle of identical area. In Fig. 1, undesirable states are contained in a null measure stable manifold living in the border of the triangles. This null measure attractive manifold can be numerically detected due to the nature of the convergence process in numerical simulations, that is, a null measure manifold becomes a little set with the same dimension than the Hilbert space, and the size of these sets is a growing function of the numerical convergence bound condition. Unfortunately, undesirables states in the MUBs problem, in case of $N > 3$, are fixed points and consequently they have a basin of attraction that has not null measure. Moreover, the number of undesirable fixed points increases faster than the number of Pauli partners with the dimension $N$. For example, in case of $N = 31$ most of time the algorithm is rejecting undesirable states. This is the reason why we do not have a polynomial algorithm for quantum state reconstruction. If undesirable states could be avoided the algorithm would be polynomial (linear in case of MUBs problem), and the quantum state reconstruction problem would be reduced to a NP-EASY problem.
V. CONCLUSIONS

We have presented an algorithm for state reconstruction in finite dimensional Hilbert spaces. We showed that every Pauli partner is an attractive fixed point of the physical imposition operator, independently of the set of observables considered. This fact allows us to obtain the complete set of Pauli partners for a given system. The algorithm was applied to the following particular cases: 1) the MUBs problem, where we could reconstruct the maximal set of MUBs from $N = 2$ to $N = 37$ ($N$ prime), 2) the study of basin of attraction of Pauli partners in case of $N = 3$ considering position and momentum observables. An interesting property found for the physical imposition operator is when it has no bifurcations, because this is a necessary and sufficient condition to have an informationally complete set of observables. Unfortunately, our algorithm can not be applied, at least in the way considered, to contribute to numerical evidences of the non existence of a maximal set of MUBs in non power of prime dimensions, and this is due to the high dimensional manifold that characterizes the different options for the second MUB. The physical imposition operator for mixed states and composited systems will be considered in another work. Extensions of our algorithm will be studied in order to try to find numerical solutions to the problems of MU Constellations, SIC-POVMs, Equiangular Tight Frames and Quantum t-Designs.

We thank to Dr. T. Santhanam for sending us his works about quantum mechanics in finite dimensions. This work contains the last results of the PhD Thesis of DG, supported by a CONICET scholarship. This work was supported by CONICET and CONICyT PFB-0824.
Appendix A

Our intention here is to give a proof of Proposition III.4. First, we need to proof some previous geometrical properties for the single physical imposition operator:

**PROPOSITION A.1** Let \( A \) be an observable, \( \Phi \in \mathcal{H} \) a generator state, \( T_A \Phi \) the physical imposition operator and \( d(\cdot, \cdot) \) the Bures metric. Then,

1. \( d(T_A \Phi, \Psi) \leq d(\Psi, \Phi), \forall \Phi, \Psi \in \mathcal{H}. \)

2. \( d(T_A \Phi, \varphi_k) = d(\Phi, \varphi_k), \forall k = 1...N. \)

3. \( d(T_A \Phi, \Phi) \leq 2 \min_k d(\Phi, \varphi_k) = 2\sqrt{2} \sqrt{1 - \max_k \sqrt{\rho_k}}, \forall \Psi \in \mathcal{H}. \)

4. \( d(T_A \Phi, \Phi) \leq 2d(\Psi, \Phi), \forall \Phi, \Psi \in \mathcal{H}. \)

5. \( T_A \Phi = \xi \iff d(T_A \Phi, \xi) \leq d(\Psi, \xi), \forall \Psi \in N_A(\xi), \)

where \( N_A(\xi) \) is a neighborhood of \( \xi \).

**proof:**

1. 

\[
|\langle T_A \Phi, \Psi \rangle| = \sum_{k=1}^{N} |\langle \varphi_k, \Phi \rangle \langle \varphi_k, \Psi \rangle| = \sum_{k=1}^{N} |\langle \varphi_k, \Psi \rangle \langle \varphi_k, \Phi \rangle| \geq \left| \sum_{k=0}^{N-1} \langle \varphi_k, \Psi \rangle \varphi_k, \Phi \right| = |\langle \Psi, \Phi \rangle|. \tag{A1}
\]

Then,

\[
d(T_A \Phi, \Psi) = \sqrt{2} \sqrt{1 - |\langle T_A \Phi, \Psi \rangle|} \leq \sqrt{2} \sqrt{1 - |\langle \Psi, \Phi \rangle|} = d(\Psi, \Phi). \tag{A2}
\]

2. Remembering the definition of the physical imposition operator we can find that

\[
|\langle T_A \Phi, \varphi_k \rangle| = \left| \langle \varphi_k, \Phi \rangle \langle \varphi_k, \Psi \rangle \right| = |\langle \Phi, \varphi_k \rangle|. \tag{A3}
\]

Then,

\[
d(T_A \Phi, \varphi_k) = \sqrt{2} \sqrt{1 - |\langle T_A \Phi, \varphi_k \rangle|} = \sqrt{2} \sqrt{1 - |\langle \Phi, \varphi_k \rangle|} = d(\Phi, \varphi_k) \forall k = 1...N. \tag{A4}
\]
3. Using triangle inequality and Eq. (A12)

\[ d(T_{A\Phi}\Psi,\Phi) \leq d(T_{A\Phi}\Psi,\varphi_k) + d(\varphi_k,\Phi) = 2d(\varphi_k,\Phi), \ \forall k = 1...N. \] (A13)

The more restrictive condition is given by

\[ d(T_{A\Phi}\Psi,\Phi) \leq 2 \min_k d(\varphi_k,\Phi). \] (A14)

Equation

\[ \min_k d(\varphi_k,\Phi) = \sqrt{2} \sqrt{1 - \max_k \sqrt{\rho_k}}, \]

is proven immediately from Bures metric definition.

4. From triangle inequality and property 1) from this proposition we have

\[ d(T_{A\Phi}\Psi,\Phi) \leq d(T_{A\Phi}\Psi,\Psi) + d(\Psi,\Phi). \] (A15)

\[ \leq 2d(\Psi,\Phi). \] (A16)

5. Taking into account Eq. (A7) and doing the following parameter change \( \Psi \to \xi + \delta \xi, \Phi \to \xi \) we have

\[ d(T_{A\Phi}(\xi + \delta \xi),\xi + \delta \xi) \leq d(\xi + \delta \xi,\xi). \] (A17)

Notice that in the last equation we have written \( T_{A\Phi} \) instead of \( T_{A\xi} \). This is in order to consider the general situation, when \( \Phi \) is a partner of \( \xi \) and not necessarily \( \xi \). Taking \( \xi + \delta \xi \in N_{A}(\xi) \),

\[
|\langle T_{A\Phi}(\xi + \delta \xi),\xi + \delta \xi | = |\langle T_{A\Phi}(\xi + \delta \xi),\xi \rangle + \langle T_{A\Phi}(\xi + \delta \xi),\delta \xi \rangle |
\approx |\langle T_{A\Phi}(\xi + \delta \xi),\xi \rangle|,
\]

or, equivalently

\[ d(T_{A\Phi}(\xi + \delta \xi),\xi + \delta \xi) \approx d(T_{A\Phi}(\xi + \delta \xi),\xi). \] (A18)

Taking into account Eqs. (A17) and (A18) we have

\[ d(T_{A\Phi}(\xi + \delta \xi),\xi) \approx d(T_{A\Phi}(\xi + \delta \xi),\xi + \delta \xi) \] (A19)

\[ \leq d(\xi + \delta \xi,\xi). \] (A20)

Item 4) in previous proposition defines a relationship between the distance between two elements before and after physical imposition operator is applied. The factor 2 on this equation is related with the existence of Pauli partners. A factor less than one would mean a contradiction to existence of Pauli partner, because, in this case, the physical imposition operator would be a contraction.
and, by Banach’s fixed point theorem, it would have a unique fixed point, rejecting the idea of Pauli partners.

Now, we are able to give a proof of Proposition III.4. Considering the last proposition for several observables \( A, B, C, \ldots \) we have

\[
\begin{align*}
\| \Psi - \xi \| & \geq \| T_A \Phi \Psi - \xi \| \quad (A21) \\
& \geq \| T_B T_A \Phi \Psi - \xi \| \quad (A22) \\
& \geq \| T_C T_B T_A \Phi \Psi - \xi \| \quad (A23) \\
& \vdots \quad (A24) \\
& \geq \| \cdots T_C T_B T_A \Phi \Psi - \xi \| \quad (A25) \\
& \geq \| T_{ABC} \cdots \Phi \Psi - \xi \|, \quad (A26)
\end{align*}
\]

where we consider that \( \Psi \in N_{ABC} \cdots (\xi) = N_A(\xi) \cap N_B(\xi) \cap N_C(\xi) \cdots \). Then, \( \xi \) is an attractive fixed point of \( T_{ABC} \cdots \Phi \). Notice that the neighborhood \( N_{ABC} \cdots (\xi) \) cannot be an empty set or a set of null measure in state space, because each set \( N_\xi \) contain an open set around \( \xi \). Given that an intersection of a finite number of open sets is an open set, then \( N_{ABC} \cdots (\xi) \) contain, at least, an open set. Notice that the basin of attraction of the multiple physical imposition operator contain the set \( N_{ABC} \cdots (\xi) \).

Appendix B

In this appendix we will define two useful metrics in quantum mechanics. Suppose that we have two different states \( \Phi, \Psi \in \mathcal{H} \) and we want to know if they have the same distribution for the eigenvalues of some observable \( A \). Several metrics can be defined for doing that, and a nice option is Hellinger metric \( D(\cdot, \cdot) \).

**DEFINITION B.1** Let \( A \) be an observable defined on a \( N \)-dimensional Hilbert space \( \mathcal{H} \), \( \{ \varphi_k \}_{k=1}^N \) its eigenvectors basis and \( \Phi, \Psi \in \mathcal{H} \). Then, Hellinger metric is given by

\[
D_A(\Phi, \Psi) = \left( \sum_{k=1}^N (\sqrt{\rho_k} - \sqrt{\sigma_k})^2 \right)^{1/2}, \quad (B1)
\]

where \( \rho_k = |\langle \varphi_k, \Phi \rangle|^2 \) and \( \sigma_k = |\langle \varphi_k, \Psi \rangle|^2 \).

Hellinger metric is able to compare distributions for only one observable. However, Pauli partners has the same distributions for several observables and we need to define a metric that considers all of them. In order to do this, let us define an intuitive generalization of Hellinger metric, that we named *distributional metric*.

**DEFINITION B.2** Let \( A_1^j \ldots A_m^j \) be a set of \( m \)-observables defined on a \( N \)-dimensional Hilbert space \( \mathcal{H} \), \( \{ \varphi_k^j \}_{k=1}^N \) their eigenvectors basis and \( \Phi, \Psi \in \mathcal{H} \). Then, the distributional metric is
given by the expression

$$D_{A^1 \ldots A^m}(\Phi, \Psi) = \left( \frac{1}{m} \sum_{j=1}^{m} (D_{A^j}(\Phi, \Psi))^2 \right)^{1/2}. \quad (B2)$$

It is easy to prove that the distributional metric satisfies all metric conditions. Moreover, the distributional metric is proportional to the usual metric in $\mathbb{R}^N$. Let us establish an upper bound for distributional metric considering Bures metric.

**PROPOSITION B.1** Let $A^1, \ldots, A^m$ be a set of m-observables and $\Phi, \Psi$ two different states. Then, Bures metric is an upper bound for every Hellinger metric and for the distributional metric. That is,

$$d(\Phi, \Psi) \geq D_{A^j}(\Phi, \Psi), \ \forall \Phi, \Psi \in \mathcal{H}, \forall j = 1 \ldots m \quad (B3)$$

and

$$d(\Phi, \Psi) \geq D_{A^1 \ldots A^m}(\Phi, \Psi), \ \forall \Phi, \Psi \in \mathcal{H}, \forall A^1 \ldots A^m. \quad (B4)$$

**Proof:**

Let $A$ be an observable with eigenvector basis $\{\varphi_k\}_{k=1..N}$. The decomposition of the states $\Phi$ and $\Psi$ in the eigenvector basis $\varphi_k$ is

$$\Phi = \sum_{k=1}^{N} \sqrt{\rho_k} e^{i\alpha_k} \varphi_k \quad (B5)$$

$$\Psi = \sum_{k=1}^{N} \sqrt{\sigma_k} e^{i\beta_k} \varphi_k. \quad (B6)$$

Using the triangular inequality we can see that

$$|\langle \Phi, \Psi \rangle| = \left| \sum_{k=1}^{N} \sqrt{\rho_k} \sqrt{\sigma_k} e^{i(\beta_k - \alpha_k)} \right| \quad (B7)$$

$$\leq \sum_{k=1}^{N} \sqrt{\rho_k} \sqrt{\sigma_k} \quad (B8)$$

Remembering the expression of the Bures metric

$$d(\Phi, \Psi) = \sqrt{2} \sqrt{1 - |\langle \Phi, \Psi \rangle|}, \quad (B10)$$

and Hellinger metric

$$D_A(\Phi, \Psi) = \sqrt{2} \sqrt{1 - \sum_{k=1}^{N} \sqrt{\rho_k} \sqrt{\sigma_k}}, \quad (B11)$$
we obtain
\[ d(\Phi, \Psi) \geq D_A(\Phi, \Psi), \quad \forall \Phi, \Psi \in \mathcal{H}. \]  

(B12)

Then, Eq. (B13) is proven.

Now we are going to proof Eq. (B13). The summatory of the m-inequalities Eq. (B13) considering every m-observable is given by
\[ md^2(\Phi, \Psi) \geq \sum_{j=1}^{m} (D_A_j(\Phi, \Psi))^2, \]  

(B13)

or, equivalently
\[ d(\Phi, \Psi) \geq \sqrt{\frac{1}{m} \sum_{j=1}^{m} (D_A_j(\Phi, \Psi))^2}, \]  

(B14)

Then,
\[ d(\Phi, \Psi) \geq D_{A^1...A^m}(\Phi, \Psi), \quad \forall \Phi, \Psi \in \mathcal{H}, \forall A^1...A^m. \]  

(B15)

This proposition is another manifestation of the existence of Pauli partners, because
\[ d(\Phi, \Psi) = 0 \implies D_{A^1...A^m}(\Phi, \Psi) = 0, \]  

(B16)

but
\[ D_{A^1...A^m}(\Phi, \Psi) = 0 \not\implies d(\Phi, \Psi) = 0. \]  

(B17)

However, Eq. (B17) becomes an implication when \( A^1...A^m \) is an informationally complete set of observables. For example, it is valid when we consider a complete set of \( N + 1 \) mutually unbiased observables in an \( N \) dimensional Hilbert space (\( N \)-power of prime).

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A set of observables $\{O_k\}$ is informationally complete when the information obtained from their measurements is enough to reconstruct every state $\Psi \in H$. For example: Maximal set of MUBs and SIC-POVM.

Consider periodical conditions.
FIGURE CAPTIONS (Figures attached)

FIG 1: Basin of attraction of Pauli partners in case of position and momentum observables with a flat generator \((N = 3)\). The six partners found form two orthogonal MU bases.

FIG 2: Basin of attraction of Pauli partners in case of position and momentum observables with a random generator state (4 partners found). Violet regions correspond to basin of attraction of undesirable fixed points.

FIG 3: Basin of attraction of Pauli partners in case of position and momentum observables with a random generator state (5 partners found). Violet regions correspond to basin of attraction of undesirable fixed points.

FIG 4: Basin of attraction of Pauli partners in case of position and momentum observables with a random generator state. In this case there is a Pauli unique state. Violet regions correspond to basin of attraction of undesirable fixed points.

FIG 5: Basin of attraction of Pauli partners in case of position and momentum observables with a random generator state. In this case there is a Pauli unique state. Violet regions correspond to basin of attraction of undesirable fixed points.
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