Visibility moments and power spectrum of turbulence velocity

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ABSTRACT
Here we introduce moments of visibility function and discuss how those can be used to estimate the power spectrum of the turbulent velocity of external spiral galaxies. We perform numerical simulation to confirm the credibility of this method and found that for galaxies with lower inclination angles it works fine. The estimator outlined here is unbiased and has the potential to recover the turbulent velocity spectrum completely from radio interferometric observations.

Key words: physical data and process: turbulence-galaxy:disc-galaxies:ISM

1 INTRODUCTION
It is found that the power spectra of the H\textsc{i} column density fluctuations in our Galaxy (Crovisier & Dickey 1983, Green 1993) follow power law at length scales ranging a few parsec to few hundred parsecs. Different probes to the statistics of the turbulent velocity fluctuations of our Galaxy: statistics of centroid of velocities (Esquivel & Lazarian 2009), velocity coordinate spectrum or VCS (Pogosyan & Lazarian 2009), velocity channel analysis or VCA (Lazarian & Pogosyan 2004), spectral correlation function (Padoan et al. 2001) and nearby dwarf galaxies (Stanimirovi\textsuperscript{c} & Lazarian 2001; Dutta et al. 2008) and spiral galaxies (Dutta et al. 2008, 2013) are also found to follow power law at length scales ranging from a few parsec to ∼10 kpc. Generating mechanism of these large scale structures are yet to be understood (Walker et al. 2014). Estimating the velocity fluctuation power spectrum would give more clue here. For this purpose, VCA is the most efficient technique and has been used by several authors successfully for our galaxy (Elmegreen & Scalo 2004) driven by supernovae shocks, self gravity etc. Recently, H\textsc{i} column density power spectrum from external dwarf (Begum et al. 2006; Dutta et al. 2008) and spiral galaxies (Dutta et al. 2008, 2013) are also found to follow power law at length scales ranging from a few parsec to ∼10 kpc. Generating mechanism of these large scale structures are yet to be understood (Walker et al. 2014). Estimating the velocity fluctuation power spectrum would give more clue here.

In this letter we propose to use moments of the observed visibilities to perform a model dependent estimate of the power spectrum of the turbulent velocity fluctuations and access it’s efficacy for a noise free simulated data. Section (2) defines visibility moments and the estimator. Simulation parameters are given in section (3) and in section (4) we explain the procedure to use the estimator on the simulated data. We discuss the results and conclude in the last section (5).

2 POWER SPECTRUM OF VISIBILITY MOMENTS
Specific intensity of radiation with rest frequency ν\textsubscript{0} from H\textsc{i} gas at \textit{r} = (x, y, z) = (R, z) at a distance D and having temperature \textit{T} can be written for small optical depth limit as (Draine 2011)

\[ I(\vec{\theta}, \nu) = I_0 \int dz n_{\text{HI}}(\vec{r}) \phi(v). \]  

Here \textit{v} = c(\nu\textsubscript{0} - \nu)/\nu\textsubscript{0}, \textit{v} is the frequency of observation, \textit{I}_0 = \frac{h\nu_0^2 c}{4\pi k_B T_0} and \textit{z} is the line of sight direction. \textit{R} is in the plane of the sky, angular separation of a direction in the field of view with respect to the field centre is \textit{\vec{\theta}} = \frac{\vec{b}}{\text{\textit{D}}}. The line shape function \phi(\nu) can be written as

\[ \phi(v) = \phi_0 \exp \left[ \frac{-{(v - v_s(\vec{r}))}^2}{2\sigma^2(\vec{r})} \right], \]  

were \sigma(\vec{r}) = \sqrt{\frac{h \nu_c}{m_{\text{HI}} c}} is the thermal velocity dispersion and \textit{v}_s(\vec{r}) is the line of sight velocity component of the H\textsc{i} gas.

We use radio interferometers to observe H\textsc{i} emissions from the spiral galaxies. Every antenna pair of an interferometric array measures the quantity termed as visibility \textit{V}(\vec{U}, \nu) at a baseline \textit{U} given by the projected antenna separation at the plane of the sky in units of observing wavelength. Visibilities are direct Fourier transform of the specific
intensity distribution, i.e.,

\[ V(\vec{U}, v) = \int d\vec{\theta} \, e^{2\pi i \vec{\theta} \cdot \vec{U}} I(\vec{\theta}, v). \]  

(3)

Depending on the antenna positions of the particular telescope array, integration time of observation and declination of the source, visibilities can only be measured at a finite set of points in the \( \vec{U} \) plane. While reconstructing the image by performing an inverse Fourier transform of the observed visibilities, above sampling results in a complicated telescope beam pattern, usually known as dirty beam (Taylor et al. 1994). Hence, image obtained by an inverse Fourier transform of the visibilities is a convolution of the dirty beam with the sky specific intensity. Various deconvolution techniques are used to estimate the specific intensity from the image. However, all these algorithms effectively interpolate the visibilities in the unsampled part of the \( \vec{U} \) plane. This gives rise to an unknown but correlated noise in the reconstructed image (Dutta 2011). Hence, any estimate of power spectrum of column density or line of sight velocity, evaluated using the reconstructed specific intensity is expected to have an unknown bias.

We define the zeroth moment of the visibility function as follows

\[ \nu_0(\vec{U}) = \int dv \, V(\vec{U}, v). \]  

(4)

The velocity integral above has to be performed over the entire spectral range of \( \text{HI} \) emission. Clearly the moment \( \nu_0(\vec{U}) \) is proportional to the Fourier transform of columns density \( n_{\text{HI}}(\vec{\theta}) = \int dz \, n_{\text{HI}}(\vec{R}, z) \). In fact, the quantity \( P_0(\vec{U}) = | \nu_0(\vec{U}) \nu_0(\vec{U})^* | \) has been used in literature (Greaves 1993; Dutta & Bharadwaj 2013) to estimate the power spectrum of the column density fluctuations of our Galaxy and external galaxies. We define the first moment of visibility function as

\[ \nu_1(\vec{U}) = \int dv \, v \, V(\vec{U}, v) \]  

(5)

\[ = \int d\vec{\theta} \, e^{2\pi i \vec{\theta} \cdot \vec{U}} I_0 \int dz \, n_{\text{HI}}(\vec{r}) \, v_z(\vec{r}). \]

Here \( v_z(\vec{r}) \), the line of sight velocity, has two components: \( v_z^{\Omega}(\vec{\theta}) \), systematic rotation of the galaxy, and \( v_z^T(\vec{\theta}) \), random motion because of turbulence. Hence, \( v_z(\vec{r}) = v_z^{\Omega}(\vec{r}) + v_z^T(\vec{r}) \). In case of the spiral galaxies with relatively small inclination angle, we may assume that the line of site component of the rotational velocity is independent of the distance along the line of sight, then

\[ \nu_1(\vec{U}) = \nu_0(\vec{U}) \otimes v_z^{\Omega}(\vec{U}) + \chi(\vec{U}), \]  

(6)

where \( v_z^{\Omega}(\vec{U}) \) is the Fourier transform of \( v_z^{\Omega}(\vec{\theta}) \) and \( \chi(\vec{U}) \) is the Fourier transform of

\[ \chi(\vec{\theta}) = I_0 \int dz \, n_{\text{HI}}(\vec{r}) v_z^T(\vec{r}). \]  

(7)

2.1 An estimator of the turbulent velocity fluctuations

It is safe to assume that the systematic rotational velocity of the galaxy and the random turbulent velocity are not correlated. We define

\[ P_\chi(\vec{U}) = | \chi(\vec{U}) |^2 = | \nu_1(\vec{U}) |^2 - P_0(\vec{U}) \otimes | v_z^T(\vec{U}) |^2. \]  

(8)

The angular brackets \(< ... >\) denote ensemble averaging, which has to be taken over many statistical realisations of the observed sky. As we are provided with only one realisation in reality, this can not be performed. Turbulence generated fluctuations in density and velocity, particularly for magnetohydrodynamic (MHD) turbulence, can have preferred orientations (Enqvist & Lazarian 2005). However, it is not clear, if at the length scales of 1 kpc to 10 kpc or higher one would expect to see anisotropic fluctuations in neutral hydrogen resulting because of MHD turbulence. Here, we assume that we are probing homogenous and isotropic fluctuations only. The method presented here can be generalized for the anisotropic case, however, observations with higher signal to noise than presently available may be required. With this assumptions, the ensemble average in eqn. (8) can be replaced by averages over annular bins in \( \vec{U} \). Note that the quantities \( | \nu_1(\vec{U}) |^2 \) and \( P_0(\vec{U}) \) can be estimated directly from the observed visibilities. The line of sight component of the rotational velocity \( v_z^{\Omega}(\vec{\theta}) \), inclination or position angle etc depends only on the galacto-centric radius. These quantities, and hence also \( | v_z^T(\vec{U}) | \), can be estimated from the reconstructed image without any bias. It is clear from eqn. (7) that \( P_\chi(\vec{U}) \) contain in it informations about \( \text{HI} \) density as well as \( \text{HI} \) velocity fluctuation power spectra, whereas the \( \text{HI} \) density fluctuation power spectra can be estimated independently using \( P_0(\vec{U}) \).

Next we assume that the turbulent velocity and the density fluctuations are not correlated and refer to the following two cases.

- If we are interested in estimating the power spectra at length scales comparable to the scale of the galaxy’s disk, i.e. scales much larger than the scale height of the galaxy in consideration, we may approximate the galaxy as a thin disk. In this case, we can write

\[ P_\chi(\vec{U}) = P_0(\vec{U}) \otimes P_v^T(\vec{U}), \]  

(9)

where \( P_\theta(\vec{U}) = | v_z^T(\vec{U}) |^2 \) is the power spectrum of the turbulence velocity. It is to be noted that the quantities \( P_0(\vec{U}) \) and \( P_\theta(\vec{U}) \) are two dimensional power spectra of the column density and the velocity fluctuations respectively.

We have already discussed that \( P_0(\vec{U}) \) can be measured from the zeroth moment of the observed visibilities. Since we expect all three functions in the above expression to be smooth and continuous, we can estimate \( P_v^T(\vec{U}) \) either by deconvolving \( P_0(\vec{U}) \) from \( P_\chi(\vec{U}) \) or by regression analysis using a parametric model for \( P_v^T(\vec{U}) \). Since we have an independent and exact measure of \( P_0(\vec{U}) \) from the zeroth moment of visibilities, we can have a complete estimate of \( P_v^T(\vec{U}) \) in this case.

- On the other hand, if we are to estimate the power spectra at scales comparable to the scale height of the galaxy’s disk, we may write

\[ P_\chi(\vec{U}) = P_{n_{\text{HI}}}^3(\vec{U},0) \otimes P_v^T(\vec{U},0), \]  

(10)

where \( P_{n_{\text{HI}}}^3(\vec{U},0) \) and \( P_v^T(\vec{U},0) \) are the three dimensional power spectra of \( n_{\text{HI}}(\vec{r}) \) and \( v_z^T(\vec{r}) \) evaluated at the zero line of sight wavenumber. It can be shown that \( P_{n_{\text{HI}}}^3(\vec{U},0) \)
is proportional to $P_{0}(U)$ for a thick disk. Hence, just like the case for the thin disk, we may use $P_{0}(U)$ to find the turbulent velocity fluctuation power spectrum from above relation either by deconvolution or by regression analysis. However, our ability to estimate the amplitude of $F_{v_{T}^{2}}$ would depend on the details of the morphology of the galaxy in the line of sight direction.

### 3 SIMULATING H I OBSERVATIONS FROM A SPIRAL GALAXY

A major assumption we took in deriving the expression for $P_{\chi}(U)$ in the previous section is that $v_{T}^{2}$ is independent of the line of sight direction. In case of galaxies with lower inclination angle, this assumption is justified. However, for galaxies with higher inclination angles, line of sight pass through different galacto-centric radius and this assumption may not be justified. We simulate H I observation from an external spiral galaxy and use it to study the effect of galaxy’s inclination on the turbulent velocity power spectrum estimator described in the previous section. Simulation of the H I signal is based on eqn. (1) and (2) and discussed in Dutta (2013) in detail, we note the salient features here. We consider an exponential radial profile and a gaussian vertical profile to model the H I distribution in the galaxy, i.e.,

$$n_{HI}(r_{g}) = n_{0} \exp\left(-\frac{\left|R_{ag}\right|}{R_{0}}\right) \exp\left(-\frac{1}{2}(z/\sigma_{z})^{2}\right),$$

where $r_{g} = (R_{g}, z_{g})$ is a vector centred at the galactic centre with $R_{g}$ at the plane of the galaxy. We assume a constant inclination angle $i$ between the line of sight direction and $z_{g}$ for simplicity. This gives a rotation mapping between the coordinates $\hat{r}$ and $\hat{r}_{g}$. Turbulent H I number density fluctuation $\delta n_{HI}(r_{g})$ is expected to have a power law spectrum and we quantify it with a slope $\alpha$. We assume that the tangential rotation velocity $v_{T}(R_{g})$ follow Brandt profile (Brandt 1960) of the form

$$v_{T}(R_{g}) = \frac{R_{g}}{R_{0} v_{0}^{3/2}} \left(1 + 2 f_{HI} R_{g}^{3/2} \right)^{-2} + f_{v_{T}} (\hat{r}).$$

We shall discuss the other parameters in above expressions shortly.

We consider 512^3 cubic grids in our simulation volume with each cube representing an individual H I cloud. To accommodate the simulated galaxy’s disk inside the simulation volume and adopt a generic rotation curve, we considered both $R_{0} = 200$ grid units. Assuming H I extent of a typical spiral galaxy is $\sim 50$ kpc these choice gives us a spatial resolution of $\sim 200$ pc. Simulating a thin disk requires relatively higher computational resources. On the other hand inclination angle affects both the thick and thin disks in similar way, we opted for simulating a thick disk here and adopt a value of $z_{0} = 100$ grid units. We show these radial and vertical profiles normalised to their central value in Figure (1) along with the tangential velocity profile normalised to $v_{0}$.

Dutta & Bharadwaj (2013) have reported that the amplitude of the H I column density fluctuation of the spiral galaxies in the THINGS sample are approximately $1/10^{th}$ of the mean column density. Following this, we choose $f_{HI} = 0.1$. Line of sight velocity dispersion of H I for the galaxies in the THINGS sample varies in the range $\sim 5$ to 20 km sec^{-1} (Tamburro et al. 2009), while the flattening value of the rotation curve are found to be in the range $\sim 100$ to 200 km sec^{-1} (de Blok et al. 2008). Since this velocity dispersion is mostly because of turbulence, we adopt $f_{v_{T}} = v_{0}/10$ in the simulation, the actual value of $v_{0}$ is unimportant here.
Figure 3. Best fit values of $\beta$, i.e., $\hat{\beta}$ found using the estimator to the simulated data is plotted against the input values of $\beta$ with $3\sigma$ errors for four different sets of simulations. The solid line represents the ideal case of $\hat{\beta} = \beta$. Clearly at high inclination angles the estimated values of $\beta$ has relatively higher uncertainty.

$\textbf{Kanekar et al. (2011)}$ found that in our Galaxy, line of sights with column density $> 2 \times 10^{20}$ atoms cm$^{-2}$ have temperature of 240 K or lower, whereas the mean column density of the spiral galaxies in the THINGS sample is $> 3 \times 10^{20}$ atoms cm$^{-2}$ ($\textbf{Walter et al. 2008, Dutta & Bharadwa} 2013$). This suggest existence of large amount of cold gas in the spiral galaxies. In view of this, thermal velocity dispersion $\sigma$ is considered to be $v_0/100$. For a $v_0 = 200$ km sec$^{-1}$, this corresponds to a gas temperature 500 K, i.e from cold neutral medium.

$\textbf{H i}$ column density power spectra of 18 spiral galaxies in the THINGS sample are found to follow power laws ($\textbf{Dutta et al. 2013}$) for length scales more than the scale height of the disk and the slope of the power law for most of the galaxies were found to be in the range $-1.5$ to $-1.8$. To see the effect of $\alpha$ on the efficacy of our estimator we perform the simulation for two different $\alpha = (-2.5, -1.5)$.

We consider a nearly face on disk with $i = 10^\circ$ and a more edge on disk with $i = 45^\circ$. With these choices, we run four sets of simulations as (a) ($i = 10^\circ, \alpha = -1.5$), (b) ($i = 10^\circ, \alpha = -2.5$), (c) ($i = 45^\circ, \alpha = -1.5$) and (d) ($i = 45^\circ, \alpha = -2.5$). Turbulent velocity component $v_T(\vec{r})$ is also taken to follow power law power spectrum ($\textbf{Federrath et al. 2009}$), with a slope $\beta$. $\textbf{Goldman (2000)}$ pointed out that it is likely that the ISM density fluctuations passively follow the velocity fluctuations and hence they would have similar power spectra. Observations show that the column density fluctuation power spectra has a slope of $\sim -2.5$ ($\textbf{Green 1993}$) in our Galaxy and $\sim -1.5$ ($\textbf{Dutta & Bharadwa} 2013$) in case of the external galaxies. We simulate the $\text{H i}$ emission for $\beta$ varying in the range $-0.5$ to $-3.5$ in steps of 0.1 and generate $\text{H i}$ specific intensity data cubes following eqn. (1) with a velocity resolution of $\sigma/2$. This amounts in calculating the line shape function for at least six points within it’s support, necessary and sufficient to avoid the shot noise bias discussed in $\textbf{Chepurnov & Lazarian 2009}$. We convert these specific intensity maps to corresponding visibilities and use them as the observed visibilities $V(\vec{U}, \nu)$ for further analysis. In this letter we do not consider any measurement noise. Further, visibilities are estimated at grids in $\vec{U}$ space and hence the baseline coverage is assumed to be complete.
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4 ESTIMATING TURBULENT VELOCITY FLUCTUATION FROM THE SIMULATED DATA

We proceed to estimate the power spectrum of turbulent velocity fluctuations through the following steps.

- We estimate the visibility moments \( \mathcal{V}_0(\bar{U}) \) and \( \mathcal{V}_1(\bar{U}) \) from the visibilities \( V(\bar{U}, v) \).
- We reconstruct the image data cube from visibilities and estimate the rotation curve using a standard tilted ring fit to the data. Since, in the simulation we have kept the inclination angle constant, we also keep it to be constant with radius while performing the tilted ring fit. For a real data, we would need to consider the uncertainties in the estimate of the rotation curve (and also inclination and position angles). At the present case these uncertainties are not considered. Using the rotation curve estimate we construct \( \bar{v}_0(\bar{U}) \).
- We bin the data in annular bins and estimate the azimuthally averaged \( P_0(\bar{U}) \) from the visibility moment \( \mathcal{V}_0(\bar{U}) \) and \( P_2(\bar{U}) \) from \( \mathcal{V}_1(\bar{U}) \) and \( \bar{v}_0(\bar{U}) \) in the same bins.
- Assuming the power spectrum of the turbulent velocity fluctuations to follow power law, we consider \( P_{\chi,\text{mod}}^T(\bar{U}) \sim \bar{U}^{-\gamma} \) to model its power spectrum. We convolve \( P_{\chi,\text{mod}}^T(\bar{U}) \) with \( P_0(\bar{U}) \) to make the azimuthally averaged model \( P_{\chi,\text{mod}}^T(\bar{U}) \). It is important to realise here that as discussed in Section (3) our model of \( P_{\chi,\text{mod}}^T(\bar{U}) \) derived this way does not have the proper amplitude. We shall restrict ourself in estimating the slope of the turbulent velocity power spectra here. The solid line in Figure (2) shows the column density power spectra for the case (b). The dotted line corresponds to \( P_T^V(\bar{U}) \) for \( \beta = -2.0 \) while the dot-dash line shows the model \( P_{\chi,\text{mod}}^T(\bar{U}) \) with \( \gamma = -2.5 \). As expected, the model and the estimated value are in mismatch here.
- We vary \( \gamma \) and compare \( P_{\chi,\text{mod}}^T(\bar{U}) \) with \( P_T^V(\bar{U}) \) by a least square estimator with likelihood function of the form \( \exp(-L) \), where \( L \) is defined as

\[
L(\gamma) = \frac{L_0}{L_{\min}} \quad L(\gamma) = \sum_{i=1}^{N} \left[ P_T^V \left( \frac{U_i}{U_0} \right) - P_{\chi,\text{mod}}^T \left( \frac{U_i}{U_0} \right) \right]^2.
\]

Here the index ‘i’ refer to a particular annular bin and \( N \) is the total number of bins. Note that both the power spectra are normalised at a given baseline \( U_0 \) and hence this estimator is not sensitive to the amplitude of the velocity fluctuation spectra. We find the best fit value of \( \gamma \), i.e., \( \beta \) and also error in estimating this value.

Figure (3) summarises the efficacy of the model dependent maximum likelihood estimation of the slope of the turbulent velocity fluctuation power spectrum from the four sets of simulations. Let us concentrate on the panel (a) first. Here \( \beta \) is plotted against the different values of input \( \beta \) for \( \alpha = -1.5 \) and \( i = 10^5 \) with \( 3\sigma \) error bars. The solid line corresponds to an exact match between the input and estimated values of \( \beta \).

Clearly, for a lower inclination angle, i.e \( i = 10^5 \), the estimator does a better job in accessing the real value of \( \beta \). For the higher inclination angle, i.e \( i = 45^5 \), for all the values of \( \beta \) the best estimate has systematically higher uncertainty. This is expected as for high inclination angle our assumption of independency of \( v_\alpha \) on line of sight coor-
suring the statistics of the turbulent velocity fluctuations from external galaxies at large scales. We plan to use this estimator to the spiral galaxies with lower inclination angle from the THINGS sample in near future.

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