ON WEAKLY $\delta$-SEMIPRIMARY IDEALS OF COMMUTATIVE RINGS

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Abstract. Let $R$ be a commutative ring with $1 \neq 0$. We recall that a proper ideal $I$ of $R$ is called a semiprimary ideal of $R$ if whenever $a,b \in R$ and $ab \in I$, then $a \in \sqrt{I}$ or $b \in \sqrt{I}$. We say $I$ is a weakly semiprimary ideal of $R$ if whenever $a,b \in R$ and $0 \neq ab \in I$, then $a \in \sqrt{I}$ or $b \in \sqrt{I}$. In this paper, we introduce a new class of ideals that is closely related to the class of weakly semiprimary ideals. Let $I(R)$ be the set of all ideals of $R$ and let $\delta : I(R) \to I(R)$ be a function. Then $\delta$ is called an expansion function of ideals of $R$ if whenever $L, I, J$ are ideals of $R$ with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Let $\delta$ be an expansion function of ideals of $R$. Then a proper ideal $I$ of $R$ (i.e., $I \neq R$) is called a ($\delta$-semiprimary) weakly $\delta$-semiprimary ideal of $R$ if $(ab \in I) \neq 0 \neq ab \in I$ implies $a \in \delta(I)$ or $b \in \delta(I)$. For example, let $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$. Then $\delta$ is an expansion function of ideals of $R$ and hence a proper ideal $I$ of $R$ is a ($\delta$-semiprimary) weakly $\delta$-semiprimary ideal of $R$ if and only if $I$ is a (semiprimary) weakly semiprimary ideal of $R$. A number of results concerning weakly $\delta$-semiprimary ideals and examples of weakly $\delta$-semiprimary ideals are given.

1. Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let $R$ be a commutative ring. An ideal $I$ of $R$ is said to be proper if $I \neq R$. Let $I$ be a proper ideal of $R$. Then $\sqrt{I}$ denotes the radical ideal of $I$ (i.e., $\sqrt{I} = \{ x \in R \mid x^n \in I$ for some positive integer $n \geq 1 \}$). Note that $\sqrt{\{0\}}$ is the set (ideal) of all nilpotent elements of $R$.

Let $I$ be a proper ideal of $R$. We recall from [1] and [6] that $I$ is said to be weakly semiprime if $0 \neq x^2 \in I$ implies $x \in I$. We recall from [11] (11) that a proper ideal $I$ of $R$ is said to be weakly prime (weakly primary) if $0 \neq ab \in I$, then $a \in I$ or $b \in I$ (i.e., $I \neq R$). Let $L$ be a proper ideal of $R$. Then $L$ is called a prime ideal of $R$. We recall from [1] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of $R$. We recall from [14] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of $R$. We recall from [14] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of $R$. We recall from [14] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of $R$.
if whenever $L, I, J$ are ideals of $R$ with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Recall from [14] that a proper ideal $I$ of $R$ is said to be a $\delta$-primary ideal of $R$ if whenever $a, b \in R$ with $ab \in I$ implies $a \in I$ or $b \in \delta(I)$, where $\delta$ is an expansion function of ideals of $R$. Let $\delta$ be an expansion function of ideals of $R$. In this paper, a proper ideal $I$ of $R$ (i.e., $I \neq R$) is called a ($\delta$-semiprimary) weakly $\delta$-semiprimary ideal of $R$ if $ab \in I$ implies $a \in \delta(I)$ or $b \in \delta(I)$. For example, let $\delta: I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$. Then $\delta$ is an expansion function of ideals of $R$ and hence a proper ideal $I$ of $R$ is a ($\delta$-semiprimary) weakly $\delta$-semiprimary ideal of $R$ if and only if $I$ is a (semiprimary) weakly semiprimary ideal of $R$. A number of results concerning weakly $\delta$-semiprimary ideals and examples of weakly $\delta$-semiprimary ideals are given.

Let $\delta$ be an expansion function of ideals of a ring $R$. Among many results in this paper, it is shown (Theorem 2.7) that if $I$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary, then $I^2 = \{0\}$ and hence $I \subseteq \sqrt{\{0\}}$. If $I$ is a proper ideal of $R$ and $I^2 = \{0\}$, then $I$ need not be a weakly $\delta$-semiprimary ideal of $R$ (Example [1]). It is shown (Example [7]) that if $I, J$ are weakly $\delta$-semiprimary ideals of $R$ such that $\delta(I) = \delta(J)$ and $I + J \neq R$, then $I + J$ need not be a weakly $\delta$-semiprimary ideal of $R$. It is shown (Theorem 2.11) that if $R$ is a Boolean ring, then every weakly semiprimary ideal of $R$ is weakly prime. It is shown (Theorem 3.1) that if $S$ is a multiplicatively closed subset of $R$ such that $S \cap Z(R) = \emptyset$ (where $Z(R)$ is the set of all zerodivisor elements of $R$) and $I$ is a weakly semiprimary ideal of $R$ such that $S \cap \sqrt{I} = \emptyset$, then $I_S$ is a weakly semiprimary ideal of $R_S$. It is shown (Corollary 3.3) that if $I$ is a weakly semiprimary ideal of $R$ and $\frac{R}{I}$ is a weakly semiprimary ideal of $\frac{R}{I}$, then $J$ is a weakly semiprimary ideal of $R$. It is shown (Corollary 4.2) that if $R = R_1 \times R_2$, where $R_1, R_2$ are some rings with $1 \neq 0$, and $I$ is a proper ideal of $R$, then $I$ is a weakly semiprimary ideal of $R$ if and only if $I = \{(0,0)\}$ or $I$ is a semiprimary ideal of $R$. It is shown (Theorem 5.4) that if $I$ is a weakly $\delta$-semiprimary ideal of $R$ and $\{0\} \neq AB \subseteq I$ for some ideals $A, B$ of $R$, then $A \subseteq \delta(I)$ or $B \subseteq \delta(I)$.

2. WEAKLY $\delta$-SEMIPRIMARY IDEALS

Definition 2.1. Let $I(R)$ be the set of all ideals of $R$. We recall from [14] that a function $\delta: I(R) \to I(R)$ is called an expansion function of ideals of $R$ if whenever $L, I, J$ are ideals of $R$ with $J \subseteq I$, then $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$.

In the following example, we give some expansion functions of ideals of a ring $R$.

Example 1. [S] Let $\delta: I(R) \to I(R)$ be a function. Then

1. If $\delta(I) = I$ for every ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$.
2. If $\delta(I) = \sqrt{I}$ (note that $\sqrt{R} = R$) for every ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$.
3. Suppose that $R$ is a quasi-local ring (i.e., $R$ has exactly one maximal ideal) with maximal ideal $M$. If $\delta(I) = M$ for every proper ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$.
4. Let $I$ be a proper ideal of $R$. Recall from [13] that an element $r \in R$ is called integral over $I$ if there is an integer $n \geq 1$ and $a_i \in I^i$, $i = 1, \ldots, n$, $r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_{n-1}r + a_n = 0$. Let $I = \{r \in R \mid r$ is integral over $I\}$. Let $I \in I(R)$. It is known (see [13]) that $I$ is an ideal of $R$ and
Let $\delta$ be an expansion function of ideals of a ring $R$.

1. A proper ideal $I$ of $R$ is called a ($\delta$-semiprimary) weakly $\delta$-semiprimary ideal of $R$ if whenever $a, b \in R$ and $(ab \in I) \neq 0 \neq ab \in I$, then $a \in \delta(I)$ or $b \in \delta(I)$.

2. Recall that if $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$ for every proper ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$. In this case, a proper ideal $I$ of $R$ is called a (semiprimary) weakly semiprimary ideal of $R$ if whenever $a, b \in R$ and $(ab \in I) \neq 0 \neq ab \in I$, then $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

3. A proper ideal $I$ of $R$ is called a ($\delta$-primary) weakly $\delta$-primary ideal of $R$ if whenever $a, b \in R$ and $(ab \in I) \neq 0 \neq ab \in I$, then $a \in I$ or $b \in \delta(I)$.

4. A proper ideal $I$ of $R$ is called a weakly prime ideal of $R$ if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in I$.

5. A proper ideal $I$ of $R$ is called a weakly primary ideal of $R$ if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in \sqrt{I}$.

We have the following trivial result, and hence we omit its proof.

**Theorem 2.3.** Let $I$ be a proper ideal of $R$ and let $\delta$ be an expansion function of ideals of $R$. Then

1. If $I$ is a $\delta$-primary ideal of $R$, then $I$ is a weakly $\delta$-semiprimary ideal of $R$. In particular, if $I$ is a primary ideal of $R$, then $I$ is a weakly semiprimary ideal of $R$.

2. If $I$ is a weakly $\delta$-primary ideal of $R$, then $I$ is a weakly $\delta$-semiprimary ideal of $R$. In particular, if $I$ is a weakly primary ideal of $R$, then $I$ is a weakly semiprimary ideal of $R$.

3. If $I$ is a $\delta$-semiprimary ideal of $R$, then $I$ is a weakly $\delta$-semiprimary ideal of $R$.

4. $\sqrt{\{0\}}$ is a weakly prime ideal of $R$ if and only if $\sqrt{\{0\}}$ is a weakly semiprimary ideal of $R$.

5. If $I$ is a weakly prime ideal of $R$, then $I$ is a weakly semiprimary ideal of $R$.

The following is an example of a proper ideal of a ring $R$ that is a weakly semiprimary ideal of $R$ but it is neither weakly primary nor weakly prime.
Example 2. Let $A = \mathbb{Z}_2[X, Y]$ where $X, Y$ are indeterminates. Then $I = (Y^2, XY)A$ and $J = (Y^2, X^2Y^2)A$ are ideals of $A$. Set $R = A/J$. Then $L = I/J$ is an ideal of $R$ and $\sqrt{L} = (Y, XY)A/J$. Since $0 \neq XY + J \in L$ and neither $X + J \in \sqrt{L}$ nor $Y + J \in L$, we conclude that $L$ is not a weakly primary ideal of $R$. Since $0 + J \neq XY + J \in L$ but neither $X + J \in L$ nor $Y + J \in L$, $L$ is not a weakly prime ideal of $R$. It is easy to check that $L$ is a weakly semiprimary ideal of $R$.

The following is an example of an ideal that is weakly semiprimary but not semiprimary.

Example 3. Let $R = \mathbb{Z}_{36}$. Then $I = \{0\}$ is a weakly semiprimary ideal of $R$ by definition. Note that $\sqrt{I} = 6R$. Since $0 = 4 \cdot 9 \in I$ but neither $4 \in \sqrt{I}$ nor $9 \in \sqrt{I}$, we conclude that $I$ is not a semiprimary ideal of $R$.

Definition 2.4. Let $\delta$ be an expansion function of ideals of a ring $R$. Suppose that $I$ is a weakly $\delta$-semiprimary ideal of $R$ and $x \in R$. Then $x$ is called a dual-zero element of $I$ if $xy = 0$ for some $y \in R$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$ (note that $y$ is also a dual-zero element of $I$).

Remark 2.5. Let $\delta$ be an expansion function of ideals of a ring $R$. Note that if $I$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary, then $I$ must have a dual-zero element of $R$.

Theorem 2.6. Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a weakly $\delta$-semiprimary ideal of $R$. If $x \in R$ is a dual-zero element of $I$, then $xI = \{0\}$.

Proof. Assume that $x \in R$ is a dual-zero element of $I$. Then $xy = 0$ for some $y \in R$ such that neither $x \in \delta(I)$ nor $y \in \delta(I)$. Let $i \in I$. Then $x(y + i) = 0 + xi = xi \in I$. Suppose that $xi \neq 0$. Since $0 \neq x(y + i) = xi \in I$ and $I$ is a weakly $\delta$-semiprimary ideal of $R$, we conclude that $x \in \delta(I)$ or $(y + i) \in \delta(I)$ and hence $x \in \delta(I)$ or $y \in \delta(I)$, a contradiction. Thus $xi = 0$.

Theorem 2.7. Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary. Then $I^2 = \{0\}$ and hence $I \subseteq \sqrt{\{0\}}$.

Proof. Since $I$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary, we conclude that $I$ has a dual-zero element $x \in R$. Since $xy = 0$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$, we conclude that $y$ is a dual-zero element of $I$. Let $i, j \in I$. Then by Theorem 2.6 we have $(x + i)(y + j) = ij \in I$. Suppose that $ij \neq 0$. Since $0 \neq (x + i)(y + j) = ij \in I$ and $I$ is a weakly $\delta$-semiprimary ideal of $R$, we conclude that $x + i \in \delta(I)$ or $y + j \in \delta(I)$ and hence $x \in \delta(I)$ or $y \in \delta(I)$, a contradiction. Thus $ij = 0$ and hence $I^2 = \{0\}$.

In view of Theorem 2.7 we have the following result.

Corollary 2.8. Let $I$ be a weakly semiprimary ideal of $R$ that is not semiprimary. Then $I^2 = \{0\}$ and hence $I \subseteq \sqrt{\{0\}}$.

The following example shows that a proper ideal $I$ of $R$ with the property $I^2 = \{0\}$ need not be a weakly semiprimary ideal of $R$.

Example 4. Let $R = \mathbb{Z}_{12}$. Then $I = \{0, 6\}$ is an ideal of $R$ and $I^2 = \{0\}$. Note that $\sqrt{I} = I$. Since $0 \neq 2 \cdot 3 \in I$ and neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I}$, we conclude that $I$ is not a weakly semiprimary ideal of $R$. 
Theorem 2.9. Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a proper ideal of $R$. If $\delta(I)$ is a weakly prime of $R$, then $I$ is a weakly $\delta$-semiprimary ideal of $R$. In particular, if $\sqrt{I}$ is a weakly prime of $R$, then $I$ is a weakly semiprimary ideal of $R$.

Proof. Suppose that $0 \neq xy \in I$ for some $x, y \in R$. Hence $0 \neq xy \in \delta(I)$. Since $\delta(I)$ is weakly prime, we conclude that $x \in \delta(I)$ or $y \in \delta(I)$. Thus $I$ is a weakly $\delta$-semiprimary ideal of $R$. $\square$

Note that if $I$ is a weakly semiprimary ideal of a ring $R$ then $\sqrt{I}$ need not be a weakly prime ideal of $R$. We have the following example.

Example 5. $I = \{0\}$ is a weakly semiprimary ideal of $Z_{12}$. However, $\sqrt{I} = \{0, 6\}$ is not a weakly prime ideal of $Z_{12}$. For $0 \neq 2 \cdot 3 \in \sqrt{I}$, but neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I}$.

Remark 2.10. Note that a weakly prime ideal of a ring $R$ is weakly semiprimary but the converse is not true. Let $R = \mathbb{Z}[X]/(X^2)$. Then $\frac{X^2}{X+X^2}$ is an ideal of $R$. Since $0 \neq (X + (X^2))(X + (X^2)) = X^2 + (X^3) \in I$ but $X + (X^3) \notin I$, we conclude that $I$ is not a weakly prime ideal of $R$. Since $\sqrt{I} = \{\frac{X}{X+X^2}\}$ is a prime ideal of $R$, $I$ is a (weakly) semiprimary ideal of $R$.

Let $R$ be a Boolean ring (i.e., $x^2 = x$ for every $x \in R$). Since $\sqrt{I} = I$ for every proper ideal $I$ of $R$, we have the following result.

Theorem 2.11. Let $R$ be a Boolean ring and $I$ be a proper ideal of $R$. The following statements are equivalent.

(1) $I$ is a weakly semiprimary ideal of $R$.
(2) $I$ is a weakly prime ideal of $R$.

Theorem 2.12. Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a weakly $\delta$-semiprimary ideal of $R$. Suppose that $\delta(I) = \delta(\{0\})$. The following statements are equivalent.

(1) $I$ is not $\delta$-semiprimary.
(2) $\{0\}$ has a dual-zero element of $R$.

Proof. $(1) \Rightarrow (2)$. Since $I$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary, there are $x, y \in R$ such that $xy = 0$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$. Since $\delta(I) = \delta(\{0\})$, we conclude that $x$ is a dual-zero element of $\{0\}$.

$(2) \Rightarrow (1)$. Suppose that $x$ is a dual-zero element of $\{0\}$. Since $\delta(I) = \delta(\{0\})$, it is clear that $x$ is a dual-zero element of $I$. $\square$

In view of Theorem 2.12 we have the following result.

Corollary 2.13. Let $I \subseteq \sqrt{\{0\}}$ be a proper ideal of $R$ such that $I$ is a weakly semiprimary ideal of $R$. The following statements are equivalent.

(1) $I$ is not semiprimary.
(2) $\{0\}$ has a dual-zero element of $R$.

Proof. Since $\delta: I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$ for every proper ideal $I$ of $R$ is an expansion function of ideals of $R$, we have $\delta(I) = \delta(\{0\})$. Thus the claim is clear by Theorem 2.12. $\square$
We show that the hypothesis ”δ(I) = δ({0})” in Theorem 2.12 is crucial, i.e. the following is an example of an ideal I of a ring R such that I ⊆ √{0} and {0} has a dual-zero element of R but I is a δ-semiprimary ideal of R for some expansion function δ of ideals of R.

**Example 6.** Let R = Z₈, δ : I(R) → I(R) such that δ(I) = √I for every nonzero proper ideal I of R, and δ({0}) = {0}. Let I = 4R. Then δ(I) = √I = 2R. It is clear that I is a δ-semiprimary ideal of R and 2 is a dual-zero element of {0}.

**Theorem 2.14.** Let δ be an expansion function of ideals of a ring R and I be a weakly δ-semiprimary ideal of R. If J ⊆ I and δ(J) = δ(I), then J is a weakly δ-semiprimary ideal of R.

**Proof.** Suppose that 0 ≠ xy ∈ J for some x, y ∈ R. Since J ⊆ I, we have 0 ≠ xy ∈ I. Since I is a weakly δ-semiprimary ideal of R, we conclude that x ∈ δ(I) or y ∈ δ(I). Since δ(I) = δ(J), we conclude that x ∈ δ(J) or y ∈ δ(J). Thus J is a weakly δ-semiprimary ideal of R. □

In view of Theorem 2.14 we have the following result.

**Corollary 2.15.** Let I be a weakly semiprimary ideal of R such that I ⊆ √{0}. If J ⊆ I, then J is a weakly semiprimary ideal of R. In particular, if L is an ideal of R, then LI and L ∩ I are weakly semiprimary ideals of R. Furthermore, if n ≥ 1 is a positive integer, then Iⁿ is a weakly semiprimary ideal of R.

**Theorem 2.16.** Let Iₙ, i ∈ J be a collection of weakly semiprimary ideals of a ring R that are not semiprimary. Then I = ∩ₙ Iₙ is a weakly semiprimary ideal of R.

**Proof.** Note that √I = ∩ₙ √Iₙ = √{0} by Theorem 2.14. Hence the result follows. □

If I, J are weakly semiprimary ideals of a ring R such that √I = √J and I + J ≠ R, then I + J need not be a weakly semiprimary ideal of R. We have the following example.

**Example 7.** Let A = ℤ₂[T, U, X, Y], H = (T², U², XY + T + U, TU, TX, TY, UX, UY)A be an ideal of A, and R = A/H. Then by construction of R, I = (TA + H)/H = {0, T + H} and J = (UA + H)/H = {0, U + H} are weakly semiprimary ideals of R such that |I| = |J| = 2 and √I = √J = √{0} (in R) = (T, U, XY)A/H. Let L = I + J = (H + (T, U)A)/H. Then √L = √{0} (in R) and L is not a weakly semiprimary ideal of R. For 0 ≠ X + H · Y + H = XY + H, X + H ∉ L, and Y + H ∉ √L.

**Theorem 2.17.** Let δ be an expansion function of ideals of R such that δ({0}) is a δ-semiprimary ideal of R and δ(δ({0})) = δ({0}). Then

1. δ({0}) is a prime ideal of R.
2. Suppose that I be a weakly δ-semiprimary ideal of R. Then I is a δ-semiprimary ideal of R.

**Proof.** (1) Suppose that ab ∈ δ({0}) for some a, b ∈ R. Suppose that a ∉ δ(δ({0})) = δ({0}). Since δ({0}) is a δ-semiprimary ideal of R and a ∉ δ(δ({0})), we have b ∈ δ(δ({0})) = δ({0}). Thus δ({0}) is a prime ideal of R.
(2) Suppose that $I$ is not $\delta$-semiprimary. Clearly, $\delta(\{0\}) \subseteq \delta(I)$. Since $I^2 = \{0\}$ by Theorem 2.7 and $\delta(\{0\})$ is a prime ideal of $R$, we have $I \subseteq \delta(\{0\})$. Since $\delta(\{0\}) = \delta(\{0\})$, we have $I \subseteq \delta(I) \subseteq \delta(\{0\}) = \delta(\{0\})$. Since $\delta(\{0\}) \subseteq \delta(I)$ and $\delta(I) \subseteq \delta(\{0\})$, we have $\delta(I) = \delta(\{0\})$ is a prime ideal of $R$. Since $\delta(I)$ is prime, $I$ is a $\delta$-semiprimary ideal of $R$, which is a contradiction.

\[ \square \]

Theorem 2.18. Let $\delta$ be an expansion function of ideals of $R$ such that $\delta(\{0\})$ is a weakly $\delta$-semiprimary ideal of $R$, $\sqrt{\{0\}} \subseteq \delta(\{0\})$, and $\delta(\{0\}) = \delta(\{0\})$. Then

1. $\delta(\{0\})$ is a weakly prime ideal of $R$.
2. Suppose that $I$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary. Then $\delta(I) = \delta(\{0\}) = \delta(\sqrt{\{0\}})$ is a weakly prime ideal of $R$ that is not prime. Furthermore, if $J \subseteq \sqrt{\{0\}}$, then $J$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary and $\delta(J) = \delta(\{0\})$.

Proof. (1) Suppose that $0 \neq ab \in \delta(\{0\})$ for some $a, b \in R$. Suppose that $a \notin \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\})$ is a weakly $\delta$-semiprimary ideal of $R$ and $a \notin \delta(\delta(\{0\}))$, we have $b \in \delta(\delta(\{0\})) = \delta(\{0\})$. Thus $\delta(\{0\})$ is a weakly prime ideal of $R$.

(2) Suppose that $I$ is not $\delta$-semiprimary. Then $I^2 = \{0\}$ by Theorem 2.7 and hence $I \subseteq \sqrt{\{0\}}$. Since $\sqrt{\{0\}} \subseteq \delta(\{0\})$, we have $I \subseteq \delta(\{0\})$. Hence $\delta(J) \subseteq \delta(\{0\}) = \delta(\{0\})$. Since $\delta(\{0\}) \subseteq \delta(J)$ and $\delta(J) \subseteq \delta(\{0\})$, we conclude that $\delta(J) = \delta(\{0\})$. In particular, $\delta(I) = \delta(\{0\}) = \delta(\sqrt{\{0\}})$ is a weakly prime ideal of $R$. Since $\delta(I)$ is a weakly $\delta$-semiprimary ideal of $R$ and $\delta(J) = \delta(I)$, we conclude that $J$ is a weakly $\delta$-semiprimary ideal of $R$. Since $I$ is not $\delta$-semiprimary, we conclude that $\delta(I) = \delta(\{0\})$ is not a prime ideal of $R$. Since $\delta(J) = \delta(\{0\})$ is a weakly prime ideal of $R$ that is not prime, we conclude that $J$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary.

\[ \square \]

3. Weakly $\delta$-semiprimary ideals under localization and ring-homomorphism

For a ring $R$, let $Z(R)$ be the set of all zerodivisors of $R$.

Theorem 3.1. Let $S$ be a multiplicatively closed subset of $R$ such that $S \cap Z(R) = \emptyset$. If $I$ is a weakly semiprimary ideal of $R$ and $S \cap \sqrt{I} = \emptyset$, then $IS$ is a weakly semiprimary ideal of $RS$.

Proof. Since $S \cap \sqrt{I} = \emptyset$, we conclude that $\sqrt{IS} = (\sqrt{I})S$. Let $a, b \in R$, $s, t \in S$ such that $0 \neq \frac{a}{s}, \frac{b}{t} \in IS$. Then there exists $u \in S$ such that $0 \neq uab \in I$. Since $u \in S$ and $S \cap \sqrt{I} = \emptyset$, we conclude that $0 \neq ab \in \sqrt{I}$. Since $I$ is a weakly semiprimary ideal of $R$, we conclude that $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Thus $\frac{a}{s} \in IS$ or $\frac{b}{t} \in IS$. Thus $IS$ is a weakly semiprimary ideal of $RS$.

\[ \square \]

Theorem 3.2. Let $\gamma$ be an expansion function of ideals of $R$ and let $I, J$ be proper ideals of $R$ with $I \subseteq J$. Let $\delta : I(\frac{I}{J}) \to I(\frac{I}{J})$ be an expansion function of ideals of $S = \frac{R}{I}$ such that $\delta(L+I) = \frac{\gamma(L+I)}{1}$ for every $L \in I(R)$. Then the followings statements hold.
(1) If \( J \) is a weakly \( \gamma \)-semiprimary ideal of \( R \), then \( J/J \) is a weakly \( \delta \)-semiprimary ideal of \( S \).

(2) If \( I \) is a weakly \( \gamma \)-semiprimary ideal of \( R \) and \( I/J \) is a weakly \( \delta \)-semiprimary ideal of \( S \), then \( J \) is a a weakly \( \gamma \)-semiprimary ideal of \( R \).

Proof. First observe that since \( I \subseteq J \), we have \( I \subseteq J \subseteq \gamma(J) \) and \( \delta(J) = \gamma(J) \).

(1) Assume that \( ab \in J \setminus I \) for some \( a, b \in R \). Then \( 0 \neq ab \in J \). Hence \( a \in \gamma(J) \) or \( b \in \gamma(J) \). Thus \( a + I \in \gamma(J) \) or \( b + I \in \gamma(J) \). Thus \( J/J \) is a weakly \( \delta \)-semiprimary ideal of \( S = R/J \).

(2) Since \( I \subseteq J \), we have \( \gamma(I) \subseteq \gamma(J) \). Assume that \( 0 \neq ab \in J \) for some \( a, b \in R \). Assume \( ab \in I \). Since \( I \) is a weakly \( \gamma \)-semiprimary ideal of \( R \), we have \( a \in \gamma(I) \) or \( b \in \gamma(I) \). Assume that \( ab \in J \setminus I \). Hence \( I \neq ab + I \in J \). Since \( J/J \) is a weakly \( \delta \)-semiprimary ideal of \( S \), we have \( a + I \in \gamma(J) \) or \( b + I \in \gamma(J) \). Thus \( a \in \gamma(J) \) or \( b \in \gamma(J) \). Thus \( J \) is a a weakly \( \gamma \)-semiprimary ideal of \( R \).

\[ \square \]

In view of Theorem 3.2, we have the following result.

**Corollary 3.3.** Let \( I, J \) be proper ideals of \( R \) with \( I \subseteq J \). Then the followings statements hold.

(1) If \( J \) is a weakly semiprimary ideal of \( R \), then \( J/J \) is a weakly semiprimary ideal of \( R/J \).

(2) If \( I \) is a weakly semiprimary ideal of \( R \) and \( I/J \) is a weakly semiprimary ideal of \( R/J \), then \( J/J \) is a a weakly semiprimary ideal of \( R/J \).

**Theorem 3.4.** Let \( R, S \) be rings and \( f : R \to S \) be a surjective ring-homomorphism. Then

(1) If \( I \) is a weakly semiprimary ideal of \( R \) and \( \ker(f) \subseteq I \), then \( f(I) \) is a weakly semiprimary ideal of \( S \).

(2) If \( I \) is a weakly semiprimary ideal of \( R \) and \( \ker(f) \) is a weakly semiprimary ideal of \( R \), then \( f^{-1}(I) \) is a weakly semiprimary ideal of \( R \).

**Proof.**

(1) Since \( I \) is a weakly semiprimary ideal of \( R \), we conclude that \( L/\ker(f) \) is a weakly semiprimary ideal of \( R/\ker(f) \) by Corollary 3.1. Since \( R/\ker(f) \) is ring-isomorphic to \( S \), the result follows.

(2) Let \( L = f^{-1}(I) \). Then \( \ker(f) \subseteq L \). Since \( R/\ker(f) \) is ring-isomorphic to \( S \), we conclude that \( L/\ker(f) \) is a weakly semiprimary ideal of \( R/\ker(f) \). Since \( \ker(f) \) is a weakly semiprimary ideal of \( R \) and \( L/\ker(f) \) is a weakly semiprimary ideal of \( R/\ker(f) \), we conclude that \( L = f^{-1}(I) \) is a weakly semiprimary ideal of \( R \) by Corollary 3.2. 

\[ \square \]

4. Weakly \( \delta \)-semiprimary ideals in product of rings

Let \( R_1, \ldots, R_n \), where \( n \geq 2 \), be commutative rings with \( 1 \neq 0 \). Assume that \( \delta_1, \ldots, \delta_n \) are expansion functions of ideals of \( R_1, \ldots, R_n \), respectively. Let \( R = R_1 \times \cdots \times R_n \). We define a function \( \delta : I(R) \to I(R) \) such that \( \delta_x(I_1 \times \cdots \times I_n) = \delta_1(I_1) \times \cdots \times \delta_n(I_n) \) for every \( I_i \in I(R_i) \), where \( 1 \leq i \leq n \). Then it is clear that \( \delta_x \)
is an expansion function of ideals of $R$. Note that every ideal of $R$ is of the form $I_1 \times \cdots \times I_n$, where each $I_i$ is an ideal of $R_i$, $1 \leq i \leq n$.

**Theorem 4.1.** Let $R_1$ and $R_2$ be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, $\delta_1, \delta_2$ be expansion functions of ideals of $R_1, R_2$, respectively. Let $I$ be a proper ideal of $R$. Then the following statements are equivalent.

1. $I \times R_2$ is a weakly $\delta_x$-semiprimary ideal of $R$.
2. $I \times R_2$ is a $\delta_x$-semiprimary ideal of $R$.
3. $I$ is a $\delta_1$-semiprimary ideal of $R_1$.

**Proof.** (1)$\Rightarrow$(2). Let $J = I \times R_2$. Then $J^2 \neq \{(0,0)\}$. Hence $J$ is a $\delta_x$-semiprimary ideal of $R$ by Theorem 2.14.

(2)$\Rightarrow$(3). Suppose that $I$ is not a $\delta_1$-semiprimary ideal of $R_1$. Then there exist $a, b \in R_1$ such that $ab \in I$, but neither $a \in \delta_1(I)$ nor $b \in \delta_1(I)$. Since $(a,1)(b,1) = (ab,1) \in I \times R_2$, we have $(a,1) \in \delta_x(I \times R_2)$ or $(b,1) \in \delta_x(I \times R_2)$. It follows that $a \in \delta_1(I)$ or $b \in \delta_1(I)$, a contradiction. Thus $I$ is a $\delta_1$-semiprimary ideal of $R_1$.

(3)$\Rightarrow$(1). Let $I$ be a $\delta_1$-semiprimary ideal of $R_1$. Then it is clear that $I \times R_2$ is a (weakly) $\delta_x$-semiprimary ideal of $R$.

**Theorem 4.2.** Let $R_1$ and $R_2$ be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, and $\delta_1, \delta_2$ be expansion functions of ideals of $R_1, R_2$, respectively such that $\delta_2(K) = R_2$ for some ideal $K$ of $R_2$ if and only if $K = R_2$. Let $I = I_1 \times I_2$ be a proper ideal of $R$, where $I_1, I_2$ are some ideals of $R_1$ and $R_2$, respectively. Suppose that $\delta_1(I_1) \neq R_1$. The following statements are equivalent.

1. $I$ is a weakly $\delta_x$-semiprimary ideal of $R$.
2. $I = \{(0,0)\}$ or $I = I_1 \times R_2$ is a $\delta_x$-semiprimary ideal of $R$ (and hence $I_1$ is a $\delta_1$-semiprimary ideal of $R_1$).

**Proof.** (1)$\Rightarrow$(2). Assume that $\{(0,0)\} \neq I = I_1 \times I_2$ is a weakly $\delta_x$-semiprimary ideal of $R$. Then there exists $(0,0) \neq (x,y) \in I$ such that $x \in I_1$ and $y \in I_2$. Since $I$ is a weakly $\delta_x$-semiprimary ideal of $R$ and $(0,0) \neq (x,1)(1,y) = (x,y) \in I$, we conclude $(x,1) \in \delta_x(I)$ or $(1,y) \in \delta_x(I)$. Since $\delta_1(I_1) \neq R_1$, we conclude that $(1,y) \notin \delta_x(I)$. Thus $(x,1) \in \delta_x(I)$ and hence $1 \notin \delta_2(I_2)$. Since $1 \in \delta_2(I_2)$, we conclude that $\delta_2(I_2) = R_2$ and hence $I_2 = R_2$ by hypothesis. Thus $I = I_1 \times R_2$ is a $\delta_x$-semiprimary ideal of $R$ by Theorem 4.1.

(2)$\Rightarrow$(1). No comments. 

**Corollary 4.3.** Let $R_1$ and $R_2$ be commutative rings with $1 \neq 0$ and $R = R_1 \times R_2$. Let $I$ be a proper ideal of $R$. The following statements are equivalent.

1. $I$ is a weakly semiprimary ideal of $R$.
2. $I = \{(0,0)\}$ or $I$ is a semiprimary ideal of $R$.
3. $I = \{(0,0)\}$ or $I = I_1 \times R_2$ for some semiprimary ideal $I_1$ of $R_1$ or $I = R_1 \times I_2$ for some semiprimary ideal $I_2$ of $R_2$.

5. **Strongly weakly $\delta$-semiprimary ideals**

**Definition 5.1.** Let $\delta$ be an expansion function of ideals of a ring $R$. A proper ideal $I$ of $R$ is called a strongly weakly $\delta$-semiprimary ideal of $R$ if whenever $\{0\} \neq AB \subseteq I$ for some ideals $A, B$ of $R$, then $A \subseteq \delta(I)$ or $B \subseteq \delta(I)$. Hence, a proper ideal $I$ of $R$ is called a strongly weakly semiprimary ideal of $R$ if whenever $\{0\} \neq AB \subseteq I$ for some ideals $A, B$ of $R$, then $A \subseteq \sqrt{\{0\}}$ or $B \subseteq \sqrt{\{0\}}$. 
Remark 5.2. Let \( \delta \) be an expansion function of ideals of a ring \( R \). It is clear that a strongly weakly \( \delta \)-semiprimary ideal of \( R \) is a weakly \( \delta \)-semiprimary ideal of \( R \). In this section, we show that a proper ideal \( I \) of \( R \) is a strongly weakly \( \delta \)-semiprimary ideal of \( R \) if and only if \( I \) is a weakly \( \delta \)-semiprimary ideal of \( R \).

Theorem 5.3. Let \( \delta \) be an expansion function of ideals of a ring \( R \) and \( I \) be a weakly \( \delta \)-semiprimary ideal of \( R \). Suppose that \( AB \subseteq I \) for some ideals \( A, B \) of \( R \) and suppose that \( ab = 0 \) for some \( a \in A \) and \( b \in B \) such that neither \( a \in \delta(I) \) nor \( b \in \delta(I) \). Then \( AB = \{0\} \).

Proof. First we show that \( aB = bA = \{0\} \). Suppose that \( aB \neq \{0\} \). Then \( 0 \neq ac \in I \) for some \( c \in B \). Since \( I \) is a weakly \( \delta \)-semiprimary ideal of \( R \) and \( a \notin \delta(I) \), we conclude that \( c \in \delta(I) \). Hence \( 0 \neq a(b + c) = ac \in I \). Thus \( a \in \delta(I) \) or \( b + c \in \delta(I) \). Since \( c \in \delta(I) \), we conclude that \( a \in \delta(I) \) or \( b \in \delta(I) \), a contradiction. Thus \( aB = \{0\} \). Similarly, \( bA = \{0\} \). Now suppose that \( AB \neq \{0\} \). Then there is an element \( d \in A \) and there is an element \( e \in B \) such that \( 0 \neq de \in I \). Since \( I \) is a weakly \( \delta \)-semiprimary ideal of \( R \), we conclude that \( d \in \delta(I) \) or \( e \in \delta(I) \). We consider three cases:  

**Case I.** Suppose that \( d \notin \delta(I) \) and \( e \notin \delta(I) \). Since \( AB = \{0\} \), we have \( 0 \neq e(d + a) = de \in I \), we conclude that \( e \in \delta(I) \) or \( d + a \in \delta(I) \). Since \( d \in \delta(I) \), we have \( e \in \delta(I) \) or \( a \in \delta(I) \), a contradiction.  

**Case II.** Suppose that \( d \notin \delta(I) \) and \( e \in \delta(I) \). Since \( bA = \{0\} \), we have \( 0 \neq d(e + b) = de \in I \), we conclude that \( d \in \delta(I) \) or \( e + b \in \delta(I) \). Since \( e \in \delta(I) \), we have \( d \in \delta(I) \) or \( b \in \delta(I) \), a contradiction.  

**Case III.** Suppose that \( d, e \in \delta(I) \). Since \( AB = bA = \{0\} \), we have \( 0 \neq (b + e)(d + a) = de \in I \), we conclude that \( b + e \in \delta(I) \) or \( d + a \in \delta(I) \). Since \( d, e \in \delta(I) \), we have \( b \in \delta(I) \) or \( a \in \delta(I) \), a contradiction. Thus \( AB = \{0\} \). \( \square \)

Theorem 5.4. Let \( \delta \) be an expansion function of ideals of a ring \( R \) and \( I \) be a weakly \( \delta \)-semiprimary ideal of \( R \). Suppose that \( \{0\} \neq AB \subseteq I \) for some ideals \( A, B \) of \( R \). Then \( A \subseteq \delta(I) \) or \( B \subseteq \delta(I) \) (i.e., \( I \) is a strongly weakly \( \delta \)-semiprimary ideal of \( R \)).

Proof. Since \( AB \neq \{0\} \), by Theorem 5.3 we conclude that whenever \( ab \in I \) for some \( a \in A \) and \( b \in B \), then \( a \in \delta(I) \) or \( b \in \delta(I) \). Assume that \( \{0\} \neq AB \subseteq I \) and \( A \nsubseteq \delta(I) \). Then there is an \( x \in A \setminus \delta(I) \). Let \( y \in B \). Since \( xy \in AB \subseteq I \) and \( \{0\} \neq AB \) and \( x \notin \delta(I) \), we conclude that \( y \in \delta(I) \) by Theorem 5.3. Hence \( B \subseteq \delta(I) \). \( \square \)

In view of Theorem 5.4, we have the following result.

Corollary 5.5. Let \( I \) be a weakly semiprimary ideal of \( R \). Suppose that \( \{0\} \neq AB \subseteq I \) for some ideals \( A, B \) of \( R \). Then \( A \subseteq \sqrt{I} \) or \( B \subseteq \sqrt{I} \) (i.e., \( I \) is a strongly weakly semiprimary ideal of \( R \)).

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