Weak Hardy Spaces $W H^p_L(\mathbb{R}^n)$ Associated to Operators Satisfying $k$-Davies-Gaffney Estimates

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Abstract Let $L$ be a one-to-one operator of type $\omega$ having a bounded $H_\infty$ functional calculus and satisfying the $k$-Davies-Gaffney estimates with $k \in \mathbb{N}$. In this paper, the authors introduce the weak Hardy space $W H^p_L(\mathbb{R}^n)$ associated to $L$ for $p \in (0, 1]$ via the non-tangential square function $S_L$ and establish a weak molecular characterization of $W H^p_L(\mathbb{R}^n)$. Typical examples of such operators include the $2k$-order divergence form homogeneous elliptic operator $L_1 := (-1)^k \sum_{|\alpha| = k} \partial^\alpha (a_{\alpha, \beta} \partial^\beta)$, where $\{a_{\alpha, \beta}\}_{|\alpha| = k}$ are complex bounded measurable functions, and the $2k$-order Schrödinger type operator $L_2 := (-\Delta)^k + V^k$, where $\Delta$ is the Laplacian operator and $0 \leq V \in L^{2k}_\text{loc}(\mathbb{R}^n)$. As applications, for $i \in \{1, 2\}$ and $p \in (n/k, 1]$, the authors prove that the associated Riesz transform $\nabla^k(L_i^{-1/2})$ is bounded from $W H^p_L(\mathbb{R}^n)$ to the classical weak Hardy space $W H^p(\mathbb{R}^n)$ and, for all $0 < p < r \leq 1$ and $\alpha = n(1/p - 1/r)$, the fractional power $L_i^{-\alpha/2}$ is bounded from $W H^p_L(\mathbb{R}^n)$ to $W H^r_L(\mathbb{R}^n)$. Furthermore, the authors find the dual space of $W H^p_L(\mathbb{R}^n)$ for $p \in (0, 1]$, which can be defined via mean oscillations based on some subtle coverings of bounded open sets and, even when $L := -\Delta$, are also previously unknown. In particular, if $L$ is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates, the authors further establish the weak atomic characterization of $W H^p_L(\mathbb{R}^n)$.

1 Introduction

It is well known that Stein and Weiss [55] originally inaugurated the study of real Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ on the Euclidean space $\mathbb{R}^n$. Later, a real-variable theory of $H^p(\mathbb{R}^n)$ for $p \in (0, 1]$ was systematically developed by Fefferman and Stein in [28]. Since then, the real-variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ has found many important applications in various fields of analysis and partial differential equations; see,
for example, [15, 16, 19, 32, 44, 50, 52, 53, 56]. It is now known that $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$ when studying the boundedness of operators; for example, when $p \in (0, 1]$, the Riesz transform $\nabla(-\Delta)^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$, but bounded on $H^p(\mathbb{R}^n)$, where $\Delta$ is the Laplacian operator $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $\nabla$ is the gradient operator $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ on $\mathbb{R}^n$. Moreover, when considering some weak type inequalities for some of the most important operators from harmonic analysis and partial differential equations, we are led to the more general weak Hardy space $WH^p(\mathbb{R}^n)$ (see, for example, [29, 46, 31, 2, 33, 51, 1, 49]). It is well known that the weak Hardy space $WH^p(\mathbb{R}^n)$ is a suitable substitute of both the weak Lebesgue space $WL^p(\mathbb{R}^n)$ and the Hardy space $H^p(\mathbb{R}^n)$ when studying the boundedness of operators in the critical case.

For example, let $\delta \in (0, 1]$, $T$ be a $\delta$-Calderón-Zygmund operator and $T^*(1) = 0$, where $T^*$ denotes the adjoint operator of $T$. It is known that $T$ is bounded on $H^p(\mathbb{R}^n)$ for all $p \in (\frac{n}{n+\delta}, 1]$, and bounded from $H^{\frac{np}{n+\delta}}(\mathbb{R}^n)$ to $WH^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ (see [46, 2]). Recall that Riesz transform $\nabla(-\Delta)^{-1/2}$ is a 1-Calderón-Zygmund operator with convolution kernel, which is smooth on $\mathbb{R}^n \times \mathbb{R}^n$ except on the diagonal points $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$. For more related history and properties about $WH^p(\mathbb{R}^n)$, we refer to [27, 29, 46, 2, 47, 51, 1] and the references cited therein. We should point out that Fefferman, Rivièrè and Sagher [27] proved that the weak Hardy space $WH^p(\mathbb{R}^n)$ occurs as the intermediate spaces in the real method of interpolation between the Hardy space $H^p(\mathbb{R}^n)$. It is easy to see that the classical Hardy spaces $H^p(\mathbb{R}^n)$ and the weak Hardy spaces $WH^p(\mathbb{R}^n)$ are essentially related to the operator $\Delta$.

In recent years, the study of Hardy spaces and their generalizations associated to differential operators attracts a lot of attentions; see, for example, [3, 4, 6, 9, 10, 12, 13, 14, 15, 16, 19, 32, 44, 50, 52, 53, 56] and their references. In particular, Auscher, Duong and McIntosh [4] first introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated to $L$, where the heat kernel generated by $L$ satisfies a pointwise Poisson type upper bound. Later, Duong and Yan [23, 24] introduced the dual space $\text{BMO}_L(\mathbb{R}^n)$ and showed that the dual space of $H^1_L(\mathbb{R}^n)$ is $\text{BMO}_L^*(\mathbb{R}^n)$, where $L^*$ denotes the adjoint operator of $L$ in $L^2(\mathbb{R}^n)$. Yan [57] further introduced the Hardy space $H^1_L(\mathbb{R}^n)$ for some $p \in (0, 1]$ near to 1 and generalized these results to $H^p_L(\mathbb{R}^n)$ and their dual spaces. A theory of Orlicz-Hardy spaces and their dual spaces associated to $L$ was also developed in [43, 41].

Recently, the (Orlicz-) Hardy space associated to a one-to-one operator of type $\omega$ satisfying the $k$-Davies-Gaffney estimate and having a bounded $H_\infty$ functional calculus was introduced in [10, 21, 8, 20]. Typical examples of such operators include the $2k$-order divergence form homogeneous elliptic operator

\begin{equation}
L_1 := (-1)^k \sum_{|\alpha|=k=|\beta|} \partial^\beta (a_{\alpha, \beta} \partial^\alpha)
\end{equation}

interpreted in the usual weak sense via a sesquilinear form, with complex bounded measurable coefficients $a_{\alpha, \beta}$ for all multi-indices $\alpha$ and $\beta$ satisfying the elliptic condition, namely, there exist constants $0 < \lambda \leq \Lambda < \infty$ such that, for all $\alpha$, $\beta$ with $|\alpha| = k = |\beta|$, \[\|a_{\alpha, \beta}\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda\] and, for all $f \in W^{k,2}(\mathbb{R}^n)$, \[\Re(L_1 f, f) \geq \lambda \|\nabla^k f\|_{L^2(\mathbb{R}^n)}^2.\] Here and in what follows, $\Re z$ denotes the real part of $z$ for all $z \in \mathbb{C}$. Another example of such
operators is the 2k-order Schrödinger-type operator

$$L_2 := (-\Delta)^k + V^k,$$

where $0 \leq V \in L^k_{\text{loc}}(\mathbb{R}^n)$. Notice that, when $k = 1$ and $L$ is nonnegative and self-adjoint, $H^p_L(\mathbb{R}^n)$ is the Hardy space associated to the nonnegative self-adjoint operators satisfying the Davies-Gaffney estimates, which was introduced by Hofmann et al. [36] and Jiang and Yang [40]. Notice also that, when $k = 1$, $H^p_{L_2}(\mathbb{R}^n)$ is the Hardy space associated to the Schrödinger operator $-\Delta + V$, which was introduced by Dziubański and Zienkiewicz [25, 26], Hofmann et al. [36], and Jiang and Yang [40] for $V$ satisfying different conditions. Moreover, $H^p_{L_1}(\mathbb{R}^n)$ is the Hardy space associated to the second order divergence form elliptic operator on $\mathbb{R}^n$ with complex bounded measurable coefficients, which was introduced by Hofmann and Mayboroda [37, 38], Hofmann et al. [39] and Jiang and Yang [42]. It is known that the associated Riesz transform $\nabla^k L_i^{-1/2}$, for $i \in \{1, 2\}$, is bounded from $H^p_{L_1}(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ for all $p \in \left(\frac{n}{n+k}, 1\right]$ (see [10]). Unlike the classical case, in this case, $\nabla^k L_i^{-1/2}$ may even not have a smooth convolution kernel. Thus, the boundedness of $\nabla^k L_i^{-1/2}$ can not be extended to the full range of $p \in (0, \infty)$ as before.

However, when considering the endpoint boundedness of the associated Riesz transforms, it is found that the weak Hardy space is useful. For example, it was proved in [45] that $\nabla^k L_i^{-1/2}$ is bounded from $H^{n/(n+k)}_{L_i}(\mathbb{R}^n)$ to the weak Hardy space $WH^{n/(n+k)}(\mathbb{R}^n)$.

Motivated by the above results, it is our goal to establish a theory of weak Hardy spaces associated to a class of differential operators and study their applications for all $p \in (0, 1]$ in this paper. More precisely, we always assume that $L$ is a one-to-one operator of type $\omega$ having a bounded $H^\infty$ functional calculus and satisfying the k-Davies-Gaffney estimates. For $p \in (0, 1]$, we introduce the weak Hardy space $WH^p_L(\mathbb{R}^n)$ associated to $L$ via the non-tangential square function $S_L$ and establish its weak molecular characterization. In particular, if $L$ is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates, we further establish the weak atomic decomposition of $WH^p_L(\mathbb{R}^n)$. By their atomic characterizations, we easily see that $WH^p_{-\Delta}(\mathbb{R}^n)$ and the closure of $WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ on the quasi-norm $\| \cdot \|_{WH^p(\mathbb{R}^n)}$ coincide with equivalent quasi-norms. As applications, for $i \in \{1, 2\}$ and $p \in \left(\frac{n}{n+k}, 1\right)$, we prove that the associated Riesz transform $\nabla^k(L_i^{-1/2})$ is bounded from $WH^p_{L_i}(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$. Furthermore, for all $0 < p < r \leq 1$ and $\alpha = n\left(\frac{1}{p} - \frac{1}{r}\right)$, the fractional power $L_i^{-\frac{\alpha}{k}}$ is bounded from $WH^p_{L_i}(\mathbb{R}^n)$ to $WH^r_{L_i}(\mathbb{R}^n)$. Moreover, for $p \in (0, 1]$, we give out the dual space of $WH^p_{L}(\mathbb{R}^n)$, which is defined via mean oscillations of distributions based on some subtle coverings of bounded open sets, and prove that the elements in $WAM^p_{L}(\mathbb{R}^n)$ can be viewed as a weak type Carleson measure of order $\alpha$. We point out that, even when $L := -\Delta$, the dual spaces of $WH^p_{L}(\mathbb{R}^n)$ are also previously unknown, since the seminal paper [29] of Fefferman and Soria on $WH^1(\mathbb{R}^n)$ was published in 1986. The results of this paper round out the picture on weak Hardy spaces associated to operators satisfying k-Davies-Gaffney estimates. As in the aforementioned papers on the theory of Hardy spaces associated with operators, the achievement of all results in this paper stems from subtle atomic decompositions of weak tent spaces introduced in this paper. To
the best of our knowledge, all results obtained in this paper are new even when $L$ is the Laplace operator.

This paper is organized as follows.

In Section 2, we first present some assumptions on the operator $L$ used throughout the whole paper (see Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$) and recall some basic facts concerning the $k$-Davies-Gaffney estimates (see Lemmas 2.3, 2.4 and 2.5) in Subsection 2.1. In Subsection 2.2, we introduce the weak tent space and establish its weak atomic decomposition (see Theorem 2.6). Later, via the Whitney-type covering lemma, we obtain a weak atomic block decomposition of $WT^p(\mathbb{R}^{n+1}_+)$ (see Theorem 2.12). Here, the atomic block can be viewed as an “atom” associated to the union of a family of balls (see Definition 2.11). Moreover, the weak atomic block decomposition possesses a good property that the infimum of the radiuses of all the balls appearing in the weak atomic block decomposition is strictly larger than 0 (see Theorem 2.12(ii)). This property is often used when we do the off-diagonal estimates on a general bounded open set in Section 3, especially, in the proof of Theorem 3.6. Observe that, from the Whitney-type covering lemma (see Lemma 2.7(iv)), it follows that the above property fails in the weak atomic decomposition of Theorem 2.6.

Finally, in Section 2.3, after recalling some necessary results on the Hardy space $H^p_{L_i}(\mathbb{R}^n)$ associated to $L$, we introduce the weak Hardy space $WH^p_{L_i}(\mathbb{R}^n)$ associated to $L$ (see Definition 2.13) and establish its weak molecular characterization (see Theorem 2.21). As applications of this weak molecular characterization of $WH^p_{L_i}(\mathbb{R}^n)$, we prove the boundedness of the associated Riesz transform $\nabla^k(L^{-1/2}_i)$ and fractional power $L^{-\alpha/(2k)}_i$ on $WH^p_{L_i}(\mathbb{R}^n)$ (see Theorems 2.23 and 2.24). Moreover, when $L$ is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates, we obtain its weak atomic characterization (see Theorem 2.16). Recall that in [29, 46], for all $p \in (0, 1]$, a weak atomic decomposition of the classical weak Hardy space $WH^p(\mathbb{R}^n)$ is obtained. However, the “atoms” appeared in the weak atomic characterization of $WH^p(\mathbb{R}^n)$ in [29, 46] are essentially more in the spirit of the classical “$L^\infty(\mathbb{R}^n)$-atoms”, while the “atoms” appeared in our weak atomic characterization of $WH^p_{L_i}(\mathbb{R}^n)$ are just $H^p_{L_i}(\mathbb{R}^n)$-atoms associated to $L$ from [36] when $p = 1$ and [40] when $p \in (0, 1]$ (see Theorem 2.16).

We also point out that to establish the weak atomic decomposition of weak tent spaces in Theorem 2.6, we borrow some ideas from the proof of [17, Proposition 2], with some necessary adjustments by changing the formation of the norm from the original strong version to the present weak version. We also remark that this weak atomic characterization still holds true under some small modifications of the level set of the $A$-functional (see (2.1) and Remark 2.10). An innovation of Theorem 2.6 is to establish an explicit relation between the supports of $T^p(\mathbb{R}^{n+1})$-atoms and the corresponding coefficients, which plays a key role in establishing the weak atomic/molecular characterizations of weak Hardy spaces associated to $L$ (see Theorems 2.6(ii), 2.16 and 2.21). Indeed, the proof of Theorem 2.16 strongly depends on Theorem 2.6 and a superposition principle on weak type estimate (see Lemma 2.18). We point out that, without Theorem 2.6(ii), Theorem 2.16 seems impossible (see (2.10) and (2.11)). The proof of Theorem 2.21 is similar to that of Theorem 2.16, but needs more careful off-diagonal estimates because of the lack of the support condition of molecules.
Section 3 is devoted to the dual theory of $WH_p^p(R^n)$. We first introduce the notion of the weak Lipschitz space $WA^α_{\Delta,L^*N}(R^n)$ via the mean oscillation over bounded open sets, then we prove that the elements in $WA^α_{\Delta,L^*N}(R^n)$ can be viewed as some weak Carleson measures of order $α$ (see Proposition 3.7) and prove that the dual space of $WH_p^p(R^n)$ is $WA_{L^*N}^{n(1/p-1)}(R^n)$ (see Theorem 3.6), where $L^*$ denotes the adjoint operator of $L$ in $L^2(R^n)$.

Recall that the dual space of the classical weak Hardy space $WH^1(R^n)$ was first considered by Fefferman and Soria [29]. More precisely, for any bounded open set $Ω ⊂ R^n$ and function $φ$ on $R^n$, the oscillation $O(φ, Ω)$ of $φ$ over $Ω$ was defined in [29] by

$$O(φ, Ω) := \sup_{Qk} \frac{1}{|Q|} \sum_k \int_{Qk} |φ(x) - φ_Qk| \, dx$$

where $φ_Q := \frac{1}{|Q|} \int_Q φ(x) \, dx$ and the supremum is taken over all collections $\{Qk\}$ of subcubes of $Ω$ with bounded $C(n)$-overlap (which means that there exists a positive constant $C(n)$ such that $\sum_k χ_{Qk} ≤ C(n)$). Let $ω(δ) := \sup_{Ω=δ} O(φ, Ω)$.

Let $L_0^1(R^n) := \{ f \in L^1(R^n) : \int_{R^n} f(x) \, dx = 0 \}$ and $L_0^2(R^n)$ be the closure of $L_0^1(R^n)$ in the norm of the weak Hardy space $WH^1(R^n)$. In [29], Fefferman and Soria proved that the dual of $L_0^2(R^n)$ is the set of all the function $φ$ satisfying

$$\|φ\|_* := \int_0^∞ \frac{ω(δ)}{δ} \, dδ.$$

In the present paper, we show in Theorem 3.6 that the dual space of $WH_p^p(R^n)$ for all $p ∈ (0, 1]$ is $WA_{L^*N}^{n(1/p-1)}(R^n)$, which is defined by means of a similar integral of the mean oscillation based on some smart coverings of bounded open sets (see Definitions 3.1 and 3.2). Here the integral mean $φ_Q$ is replaced by some approximation of identity, and the collections of subcubes of an open set with bounded $C(n)$-overlap by another new class of sets (see Definition 3.1). In particular, when $L = −Δ$, $WA_{−Δ,N}^{n(1/p-1)}(R^n)$ is the dual space of the space $WH^p(R^n) \cap L^2(R^n)$ on the quasi-norm $\| · \|_{WH^p(R^n)}$.

The proof of Theorem 3.6 strongly depends on a Cardéron reproducing formula obtained in [37], a weak atomic block decomposition of the weak tent space (see Theorem 2.12), and a resolvent characterization of $WH_p^p(R^n)$ (see Proposition 3.5) and Proposition 3.7.

Recall that a key ingredient to prove the duality between Hardy spaces and Lipschitz space is to represent the Lipschitz norm by means of a dual norm expression of some Hilbert spaces. It is known that, in the case of the classical “strong” Lipschitz space $Λ^α_L(R^n)$, this Hilbert space can be chosen to be $L^2(B)$, where $B$ is some ball (see the proof of [39, Theorem 3.51]). Observe also that the mean oscillation appearing in the norm of “strong” Lipschitz space $Λ^α_L(R^n)$ has the form

$$\left\{ \frac{1}{|B|} \int_B \left( I - e^{-r^2_B L} \right)^M f(x) \, dx \right\}^{1/2}.$$
2.1 Assumptions on \( L \) paper, we recall some useful technical lemmas on the calculus. Then, after stating our assumptions on the operator hypotheses on the operator \( L \) plane \( C \) and fractional power on molecular characterization, we obtain the boundedness of the associated Riesz transform \( \{ \)

2 The Weak Hardy space \( WH^p_\sigma(\mathbb{R}^n) \)

The main purpose of this section is to introduce the weak Hardy space \( WH^p_\sigma(\mathbb{R}^n) \) and establish its weak atomic and molecular characterizations. As applications of this weak molecular characterization, we obtain the boundedness of the associated Riesz transform and fractional power on \( WH^p_\sigma(\mathbb{R}^n) \). In order to achieve this goal, we need to describe our hypotheses on the operator \( L \) throughout the whole paper.

2.1 Assumptions on \( L \)

In this subsection, we first survey some known results on the bounded \( H_\infty \) functional calculus. Then, after stating our assumptions on the operator \( L \) throughout the whole paper, we recall some useful technical lemmas on the \( k \)-Davies-Gaffney estimates.

For \( \theta \in [0, \pi] \), the open and closed sectors, \( S^\theta_0 \) and \( S_\theta \), of angle \( \theta \) in the complex plane \( \mathbb{C} \) are defined, respectively, by setting \( S^\theta_0 := \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta \} \) and \( S_\theta := \{ z \in \mathbb{C} : |\arg z| \leq \theta \} \). Let \( \omega \in [0, \pi) \). A closed operator \( T \) on \( L^2(\mathbb{R}^n) \) is said to be of type \( \omega \), if

(i) the spectrum of \( T \), \( \sigma(T) \), is contained in \( S_\omega \);
(ii) for each $\theta \in (\omega, \pi)$, there exists a nonnegative constant $C$ such that, for all $z \in \mathbb{C}\setminus S_\theta$,

$$ \| (T - zI)^{-1} \|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C|z|^{-1}, $$

here and hereafter, for any normed linear space $\mathcal{H}$, $\| S \|_{\mathcal{L}(\mathcal{H})}$ denotes the operator norm of the linear operator $S : \mathcal{H} \to \mathcal{H}$.

For $\mu \in [0, \pi)$ and $\sigma, \tau \in (0, \infty)$, let $H(S^0_\mu) := \{ f : f$ is a holomorphic function on $S^0_\mu \}$,

$$ H_\infty(S^0_\mu) := \{ f \in H(S^0_\mu) : \| f \|_{L^\infty(S^0_\mu)} < \infty \} $$

and

$$ \Psi_{\sigma,\tau}(S^0_\mu) := \{ f \in H(S^0_\mu) : \text{there exists a positive constant } C \text{ such that,} $$

$$ \text{for all } \xi \in S^0_\mu, |f(\xi)| \leq C \inf\{ |\xi|^\sigma, |\xi|^{-\tau} \} \} . $$

It is known that every one-to-one operator $T$ of type $\omega$ in $L^2(\mathbb{R}^n)$ has a unique holomorphic functional calculus; see, for example, [48]. More precisely, let $T$ be a one-to-one operator of type $\omega$, with $\omega \in [0, \pi)$, $\mu \in (\omega, \pi)$, $\sigma, \tau \in (0, \infty)$ and $f \in \Psi_{\sigma,\tau}(S^0_\mu)$. The function of the operator $T$, $f(T)$, can be defined by the $H_\infty$ functional calculus in the following way,

$$ f(T) = \frac{1}{2\pi i} \int_{\Gamma} (\xi I - T)^{-1} f(\xi) \, d\xi, $$

where $\Gamma := \{ re^{i\nu} : \infty > r > 0 \} \cup \{ re^{-i\nu} : 0 < r < \infty \}, \nu \in (\omega, \mu)$, is a curve consisting of two rays parameterized anti-clockwise. It is known that $f(T)$ is independent of the choice of $\nu \in (\omega, \mu)$ and the integral is absolutely convergent in $\| \cdot \|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ (see [48, 34]).

In what follows, we always assume $\omega \in [0, \pi/2)$. Then, it follows, from [34, Proposition 7.1.1], that for every operator $T$ of type $\omega$ in $L^2(\mathbb{R}^n)$, $-T$ generates a holomorphic $C_0$-semigroup $\{ e^{-zT} \}_{z \in S^0_\pi/2-\omega}$ on the open sector $S^0_{\pi/2-\omega}$ such that, for all $z \in S^0_{\pi/2-\omega}$,

$$ \| e^{-zT} \|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq 1 $$

and, moreover, every nonnegative self-adjoint operator is of type 0.

Let $\Psi(S^0_\mu) := \cup_{\sigma,\tau>0} \Psi_{\sigma,\tau}(S^0_\mu)$. It is well known that the above holomorphic functional calculus defined on $\Psi(S^0_\mu)$ can be extended to $H_\infty(S^0_\mu)$ via a limit process (see [48]). Recall that, for $\mu \in (0, \pi)$, the operator $T$ is said to have a bounded $H_\infty(S^0_\mu)$ functional calculus in the Hilbert space $\mathcal{H}$, if there exists a positive constant $C$ such that, for all $\psi \in H_\infty(S^0_\mu)$,

$$ \| \psi(T) \|_{\mathcal{L}(\mathcal{H})} \leq C \| \psi \|_{L^\infty(S^0_\mu)} $$

and $T$ is said to have a bounded $H_\infty$ functional calculus in the Hilbert space $\mathcal{H}$ if there exists $\mu \in (0, \pi)$ such that $T$ has a bounded $H_\infty(S^0_\mu)$ functional calculus.

Throughout the whole paper, we always assume that $L$ satisfies the following three assumptions:

**Assumption (L)$_1$.** The operator $L$ is a one-to-one operator of type $\omega$ in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \pi/2)$.

**Assumption (L)$_2$.** The operator $L$ has a bounded $H_\infty$ functional calculus in $L^2(\mathbb{R}^n)$. 
**Assumption (L)3.** Let $k \in \mathbb{N}$. The operator $L$ generates a holomorphic semigroup \{e^{-tL}\}_{t > 0} which satisfies the $k$-Davies-Gaffney estimates, namely, there exist positive constants $C$ and $C_1$ such that, for all closed sets $E$ and $F$ in $\mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$ supported in $E$,

$$\|e^{-tL}f\|_{L^2(F)} \leq C \exp \left\{ -C_1 \frac{[\text{dist}(E, F)]^{2k/(2k-1)}}{t^{1/(2k-1)}} \right\} \|f\|_{L^2(E)},$$

here and hereafter, for any $p \in (0, \infty)$, $\|f\|_{L^p(E)} := \left\{ \int_E |f(x)|^p \, dx \right\}^{1/p}$ and $\text{dist}(E, F) := \inf_{x \in E, y \in F} |x - y|$ denotes the distance between $E$ and $F$.

**Remark 2.1.** The notion of the off-diagonal estimates (or the so called Davies-Gaffney estimates) of the semigroup \{e^{-tL}\}_{t > 0} are first introduced by Gaffney [30] and Davies [18], which serves as good substitutes of the Gaussian upper bound of the associated heat kernel; see also [7, 5] and related references. We point out that, when $k = 1$, the $k$-Davies-Gaffney estimates are the usual Davies-Gaffney estimates (or the $L^2$ off-diagonal estimates or just the Gaffney estimates) (see, for example, [36, 37, 38, 40, 39]).

**Proposition 2.2 ([10]).** The operators $L_1$ as in (1.1) and $L_2$ as in (1.2) satisfy Assumptions (L)1, (L)2 and (L)3.

In order to make this paper self-contained, we list the following three technical lemmas which are used in the proofs of our main results.

**Lemma 2.3 ([10]).** Assume that the operator $L$ defined on $L^2(\mathbb{R}^n)$ satisfies Assumptions (L)1, (L)2 and (L)3. Then, for all $m \in \mathbb{N}$, the family of operators, \{(tL)^m e^{-tL}\}_{t > 0}, also satisfy the $k$-Davies-Gaffney estimate.

**Lemma 2.4 ([10]).** Let $\{A_t\}_{t > 0}$ and $\{B_t\}_{t > 0}$ be two families of linear operators satisfy the $k$-Davies-Gaffney estimates. Then, the families of linear operators $\{A_t B_t\}_{t > 0}$ also satisfy the $k$-Davies-Gaffney estimates.

**Lemma 2.5 ([10]).** Let $M \in \mathbb{N}$, and $L_1$ and $L_2$ be respectively as in (1.1) and (1.2). Then, there exists a positive constant $C$ such that, for all $i \in \{1, 2\}$, closed sets $E$, $F$ in $\mathbb{R}^n$ with $\text{dist}(E, F) > 0$, $f \in L^2(\mathbb{R}^n)$ supported in $E$ and $t \in (0, \infty)$,

$$\left\| \nabla^k L_i^{-1/2} (I - e^{-tL_i})^M f \right\|_{L^2(F)} \leq C \left( \frac{t}{\text{dist}(E, F)^{2k}} \right)^M \|f\|_{L^2(E)},$$

and

$$\left\| \nabla^k L_i^{-1/2} (tL_i e^{-tL_i})^M f \right\|_{L^2(F)} \leq C \left( \frac{t}{\text{dist}(E, F)^{2k}} \right)^M \|f\|_{L^2(E)}.$$
2.2 The weak tent spaces $WT^p(\mathbb{R}^{n+1}_+)$

In this subsection, we introduce the weak tent space and establish its weak atomic characterization. This construction constitutes a crucial component to obtain the weak atomic or molecular characterizations of the weak Hardy space.

We first recall the definition of the tent space. Let $F$ be a function on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. For all $x \in \mathbb{R}^n$, the $A$-functional $A(F)(x)$ of $F$ is defined by setting

\[(2.1) \quad A(F)(x) := \left\{ \int \int_{\Gamma(x)} |F(y, t)|^2 \frac{dy \, dt}{t^{n+1}} \right\}^{\frac{1}{2}},\]

where $\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ : |y - x| < t\}$ is a cone with vertex $x$. For all $p \in (0, \infty)$, the tent space $T^p(\mathbb{R}^{n+1}_+)$ is defined by

\[T^p(\mathbb{R}^{n+1}_+) := \left\{ F : \mathbb{R}^{n+1}_+ \to C : \|F\|_{T^p(\mathbb{R}^{n+1}_+)} := \|A(F)\|_{L^p(\mathbb{R}^n)} < \infty \right\}.

For all open sets $\Omega$, let $\tilde{\Omega} := \mathbb{R}^{n+1}_+ \setminus \bigcup_{x \in \mathbb{R}^n} \Omega \Gamma(x)$ be the tent over $\Omega$. For all $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, let $B := B(x_B, r_B)$ be the ball in $\mathbb{R}^n$. It is easy to see that $\tilde{B} = \{(y, t) : |y - x_B| \leq r_B - t\}$. For all $p \in (0, \infty)$ and balls $B$, a function $A$ defined on $\mathbb{R}^{n+1}_+$ is called a $T^p(\mathbb{R}^{n+1}_+)$-atom associated to $B$, if $\text{supp } A \subset \tilde{B}$ and

\[\left\{ \int \int_{\tilde{B}} |A(x, t)|^2 \frac{dx \, dt}{t} \right\}^{\frac{1}{2}} \leq \|B\|^{\frac{1}{2} - \frac{1}{p}}.

For $p \in (0, \infty)$, let $WL^p(\mathbb{R}^n)$ be the weak Lebesgue space with the quasi-norm

\[\|f\|_{WL^p(\mathbb{R}^n)} := \left( \sup_{\alpha \in (0, \infty)} \alpha^p \left\{|x \in \mathbb{R}^n : |f(x)| > \alpha\right\} \right)^{1/p}.

The weak tent space $WT^p(\mathbb{R}^{n+1}_+)$ is defined to be the collection of all functions $F$ on $\mathbb{R}^{n+1}_+$ such that its $A$-functional $A(F) \in WL^p(\mathbb{R}^n)$. For any $F \in WT^p(\mathbb{R}^{n+1}_+)$, define its quasi-norm by $\|F\|_{WT^p(\mathbb{R}^{n+1}_+)} := \|A(F)\|_{WL^p(\mathbb{R}^n)}$.

For the weak tent space, we have the following weak atomic decomposition.

**Theorem 2.6.** Let $p \in (0, 1]$. For each $F \in WT^p(\mathbb{R}^{n+1}_+)$, there exists a sequence $\{A_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \in T^p(\mathbb{R}^{n+1}_+)$-atoms associated, respectively, to the balls $\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+}$ such that

(i) $F = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} A_{i,j}$ pointwisely almost everywhere on $\mathbb{R}^{n+1}_+$, where

\[\lambda_{i,j} := 2^i |B_{i,j}|^{1/p};\]

(ii) there exists a positive constant $C$, independent of $F$, such that

\[\sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right)^{\frac{1}{p}} \leq C \|F\|_{WT^p(\mathbb{R}^{n+1}_+)};\]

(iii) there exists an $M_0 \in \mathbb{N}$ such that, for all $i \in \mathbb{Z}$, $\sum_{j \in \mathbb{Z}_+} \lambda_{i,j} \chi_{\frac{1}{2}B_{i,j}} \leq M_0$. 
In order to prove this theorem, we need the following Whitney-type covering lemma.

**Lemma 2.7 (\cite{19}).** Let $\Omega \subset \mathbb{R}^n$ be an open set and $C_0 \in [1, \infty)$ be a positive constant. For all $x \in \mathbb{R}^n$, let $r(x) := d(x, \mathbb{R}^n \setminus \Omega)/(2C_0)$. Then there exist two sequences $\{x_i\}_{i \in \mathbb{N}}$ of points contained in $\Omega$ and $\{r_i\}_{i \in \mathbb{N}} := \{r(x_i)\}_{i \in \mathbb{N}}$ of positive numbers such that

(i) $\{B(x_i, r_i)\}_{i \in \mathbb{N}}$ are disjoint;

(ii) $\Omega = \bigcup_{i \in \mathbb{N}} B(x_i, r_i)$;

(iii) $B(x_i, C_0r_i) \subset \Omega$;

(iv) for all $x \in B(x_i, C_0r_i)$, $C_0r_i \leq d(x, \mathbb{R}^n \setminus \Omega) \leq 3C_0r_i$;

(v) for all $i \in \mathbb{N}$, there exists a point $x_i^* \in \mathbb{R}^n \setminus \Omega$ such that $d(x_i^*, x_i) < 3C_0r_i$;

(vi) there exists a positive constant $M_0$ such that, for all $x \in \Omega$, $\sum_{i \in \mathbb{N}} \chi_{B(x_i, C_0r_i)}(x) \leq M_0$.

For any fixed $\gamma \in (0, 1)$ and bounded open set $\Omega$ in $\mathbb{R}^n$ with the complementary set $F$, let $\Omega^*_\gamma := \{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega)(x) > 1 - \gamma\}$ and $F^*_\gamma := \mathbb{R}^n \setminus \Omega^*_\gamma$, where $\mathcal{M}$ is the usual Hardy-Littlewood maximal function, namely, for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls containing $x$. We also need the following auxiliary lemma.

**Lemma 2.8 (\cite{17}).** Let $\alpha \in (0, \infty)$. There exist constant $\gamma \in (0, 1)$, sufficiently close to 1, and positive constant $C$ such that, for any closed set $F$ whose complement (denoted by $\Omega$) has finite measure and non-negative function $\Phi$ on $\mathbb{R}^{n+1}_+$,

$$\int_{\mathcal{R}(F^*_\gamma)} \Phi(y, t) t^\alpha dy dt \leq C \int_F \left\{ \int_{\Gamma(x)} \Phi(y, t) dy dt \right\} dx,$$

where $\mathcal{R}(F^*_\gamma) := \bigcup_{x \in F^*_\gamma} \Gamma(x)$.

**Proof of Theorem 2.6.** We carry out the proof in the order of (ii), (i) and (iii). Let $F \in W^p_{\text{loc}}(\mathbb{R}^{n+1}_+)$ and $F^*_\gamma := \mathbb{R}^n \setminus \Omega^*_\gamma$. For all $i \in \mathbb{Z}$, let $O_i := \{x \in \mathbb{R}^n : \mathcal{A}(F)(x) > 2^i\}$. It is easy to see that $O_{i+1} \subset O_i$. Moreover, since $F \in W^p_{\text{loc}}(\mathbb{R}^{n+1}_+)$, we readily see that $|O_i| < \infty$. For fixed $\gamma \in (0, 1)$ satisfying the same restriction as in Lemma 2.8, let

$$\Omega^*_\gamma := \{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega)(x) > 1 - \gamma\}.$$

By abuse of notation, we simply write $O^*_i$ instead of $(O^*_i)_{i \in I}$. Since $O_i$ is open, we easily see that $O_i \subset O^*_i$ and, by the weak $(1, 1)$ boundedness of $\mathcal{M}$, that there exists a positive constant $C(\gamma)$, depending on $\gamma$, such that $|O^*_i| \leq C(\gamma)|O_i|$. For each $O^*_i$, using Lemma 2.7, we obtain a Whitney-type covering $\{B_{i,j}\}_{j \in \mathbb{Z}^+}$ of $O^*_i$.

For all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}^+$, let $\Delta_{i,j} := (B_{i,j} \times (0, \infty)) \cap (\hat{O}^*_i \setminus \hat{O}^*_{i+1})$, $\lambda_{i,j} := 2^i |B_{i,j}|^{\frac{1}{p}}$ and $A_{i,j} : = F(\chi_{\Delta_{i,j}} - 2^{-i}|B_{i,j}|^{\frac{1}{p}})$, where $\hat{O}^*_i$ is the tent over $O^*_i$. From the fact that $\text{supp} F \subset \bigcup_{i \in \mathbb{Z}} \bigcup_{j \in \mathbb{Z}^+} \Delta_{i,j}$, it follows that

$$F(x, t) = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}^+} F(x, t) \chi_{\Delta_{i,j}}(x, t) = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}^+} \lambda_{i,j} A_{i,j}.$$
pointwisely almost everywhere on $\mathbb{R}^{n+1}$.

Moreover, for all $i \in \mathbb{Z}$, by the definition of $\lambda_{i,j}$, (i) and (iii) of Lemma 2.7, the fact that $|O_i| \leq C(\gamma)|O_i|$ and the definition of $O_i$, we conclude that

$$
(2.3) \quad \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p = 2^{np} \sum_{j \in \mathbb{Z}_+} |B_{i,j}| \lesssim 2^{np}|O_i| \sim 2^{np}\{|x \in \mathbb{R}^n : \mathcal{A}(F)(x) > 2^j\}|
$$

which immediately implies (ii).

On the other hand, for any closed set $F$, let $\mathcal{R}(F) := \bigcup_{x \in F} \Gamma(x)$. For all $(y, t) \in \Delta_{i,j}$, it follows, from the fact that $\widehat{O}_i^* = \{y, t : \text{dist}(y, \mathbb{R}^n \setminus O_i^*) \geq t\}$ and Lemma 2.7(iv), that $t \leq \text{dist}(y, \mathbb{R}^n \setminus O_i^*) < 3C_0r_{B_{i,j}}$. Thus, for all $H \in T^2(\mathbb{R}^{n+1})$ satisfying $\|H\|_{T^2(\mathbb{R}^{n+1})} \leq 1$, by the fact that $\mathbb{R}^n \setminus \widehat{O}_{i+1}^* = \mathcal{R}(\mathbb{R}^n \setminus O_{i+1}^*)$, the support condition of $A_{i,j}$, Lemma 2.8, Lemma 2.7(iv), Hölder’s inequality and the definitions of $A_{i,j}$ and $O_{i+1}$, we see that

$$
(2.4) \quad \left|\langle A_{i,j}, H \rangle_{T^2(\mathbb{R}^{n+1})}\right| \leq \int_{\mathcal{R}(\mathbb{R}^n \setminus O_{i+1}^*)} |A_{i,j}(y, t)H(y, t)| \chi_{\Delta_{i,j}}(y, t) \frac{dy dt}{t}
$$

Moreover, from the definitions of $\Delta_{i,j}$ and $\widehat{O}_i^*$, and Lemma 2.7(iv), we deduce that $\Delta_{i,j} \subset 3C_0B_{i,j}$. Thus, $A_{i,j}$ is a $T^p(\mathbb{R}^{n+1})$-atom associated to the ball $3C_0B_{i,j}$, which, combined with (2.2), implies (i). Moreover, (iii) follows readily from Lemma 2.7(vi), which finishes the proof of Theorem 2.6.

If a function $F \in WT^p(\mathbb{R}^{n+1})$ also belongs to $T^2(\mathbb{R}^{n+1})$, then the weak atomic decomposition obtained in Theorem 2.6 also converges in $T^2(\mathbb{R}^{n+1})$.

**Corollary 2.9.** Let $p \in (0, 1]$. For all $F \in WT^p(\mathbb{R}^{n+1}) \cap T^2(\mathbb{R}^{n+1})$, the weak atomic decomposition $F = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j}A_{i,j}$ obtained in Theorem 2.6 also holds true in $T^2(\mathbb{R}^{n+1})$. 

\[ \square \]
Proof. Let $F \in WT^p(\mathbb{R}_+^{n+1}) \cap T^2(\mathbb{R}_+^{n+1})$. We use the same notation as in Theorem 2.6 and its proof. By Theorem 2.6, we see that the weak atomic decomposition $F = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} A_{i,j}$ holds true pointwisely almost everywhere on $\mathbb{R}_+^{n+1}$. Thus, for all positive numbers $N_1$ and $N_2 \in \mathbb{Z}_+$, from Fubini’s theorem, the definitions of $A_{i,j}$ and $\lambda_{i,j}$ and the bound overlap property of $\{\Delta_{i,j}\}_{i,j \in \mathbb{Z}_+}$, we deduce that

$$
\left\| \sum_{|i| \geq N_1 \text{ or } j \geq N_2} \lambda_{i,j} A_{i,j} \right\|^2_{T^2(\mathbb{R}_+^{n+1})} \lesssim \int_0^{\infty} \int_{\mathbb{R}^n} \left| \sum_{|i| \geq N_1 \text{ or } j \geq N_2} \lambda_{i,j} A_{i,j}(y, t) \right|^2 \frac{dy dt}{t}.
$$

By letting $N_1, N_2 \to \infty$ and using the condition that $F \in T^2(\mathbb{R}_+^{n+1})$, we know that $F = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} A_{i,j}$ holds true in $T^2(\mathbb{R}_+^{n+1})$, which completes the proof of Corollary 2.9.

Remark 2.10. Let $i_0 := 0 =: \tilde{i}_0$ and $O_{i_0} := \{ x \in \mathbb{R}^n : A(F)(x) > 2^{i_0} \}$. Let

$$i_1 := \min \{ i \in \mathbb{Z} : |\{ x \in \mathbb{R}^n : A(F)(x) > 2^i \}| < |O_{i_0}| \}$$

and $O_{i_1} := \{ x \in \mathbb{R}^n : A(F)(x) > 2^{i_1} \}$. Choose $\tilde{i}_1 \in \mathbb{Z}_+$ satisfying

$$|O_{i_1}| = [2^{-\tilde{i}_1-1}|O_{i_0}|, 2^{-\tilde{i}_1}|O_{i_0}|].$$

Notice that $|O_i| < \infty$ for all $i \in \mathbb{Z}$ since $F \in WT^p(\mathbb{R}_+^{n+1})$. Now define

$$i_2 := \min \{ i \in \mathbb{Z} : |\{ x \in \mathbb{R}^n : A(F)(x) > 2^i \}| < 2^{-(i+1)|O_{i_0}|} \},$$

$$O_{i_2} := \{ x \in \mathbb{R}^n : A(F)(x) > 2^{i_2} \} \text{ and } \tilde{i}_2 \in \mathbb{Z} \text{ satisfying } |O_{i_2}| \subset [2^{-\tilde{i}_2-1}|O_{i_0}|, 2^{-\tilde{i}_2}|O_{i_0}|].$$

Continuing this process, we obtain a sequence $\{O_{i_j}\}_{j \in \mathbb{Z}}$ of decreasing open sets, and sequences $\{i_j\}_{j \in \mathbb{Z}}$, $\{\tilde{i}_j\}_{j \in \mathbb{Z}}$ of increasing numbers (for $j < 0$, the sequences $\{O_{i_j}\}_{j \in \mathbb{Z}}$ and $\{i_j\}_{j \in \mathbb{Z}}$ can also be defined in a similar way). For simplicity, we denote the index sets $\{i_j\}_{j \in \mathbb{Z}}$ and $\{\tilde{i}_j\}_{j \in \mathbb{Z}}$ respectively by $I$ and $\tilde{I}$. By following the same line of the proof of Theorem 2.6, but replacing $\{O_{i_j}\}_{i \in I}$ by $\{O_{i_j}\}_{j \in \tilde{I}}$ defined here, we also obtain a weak atomic decomposition of $T^p(\mathbb{R}_+^{n+1})$ with the same properties. In this case, the achieved atomic decomposition is of the following form

$$F = \sum_{i \in I} \sum_{j \in \mathbb{N}} \lambda_{i,j} A_{i,j}.$$

Now, we establish a weak atomic block decomposition of $WT^p(\mathbb{R}_+^{n+1})$.

Definition 2.11. Let $C_0 \in [1, \infty)$. For any $p \in (0, \infty)$ and bounded open set $W$, a function $A$ defined on $\mathbb{R}_+^{n+1}$ is called a $C_0$-atomic block associated to $W$, if there exist an index set $\Lambda \subset \mathbb{N}$ and a sequence $\{B_j\}_{j \in \Lambda}$ of balls in $\mathbb{R}^n$ such that
(i) $W = \bigcup_{j \in \Lambda} B_j$ and $\text{supp } A \subset \bigcup_{j \in \Lambda} \tilde{B}_j$;
(ii) $\{C_0 B_j\}_{j \in \Lambda}$ have bounded overlap;
(iii) 
\[
\left\{ \int_{j \in \Lambda} \int \frac{|A(x,t)|^2 \, dx \, dt}{t} \right\}^{\frac{1}{2}} \leq |W|^{\frac{1}{2}} - \frac{1}{p}.
\]

**Theorem 2.12.** Let $C_0 \in [1, \infty)$ and $p \in (0, 1]$. For all $F \in W^{p,p}(\mathbb{R}^{n+1}_+)$, there exist sets $I$, $\Lambda_1$, $\Lambda_2 \subset \mathbb{N}$ of indices, $\{A_{i,1}\}_{i \in I}$ of $C_0 \cdot T^p(\mathbb{R}^{n+1}_+)$-atomic block associated to the open sets $\{\bigcup_{i \in I} 3B_{i,j_1,j_2}\}_{i \in I}$, and $\{A_{i,2,j_2}\}_{i \in I, j_2 \in \Lambda_2}$ of $T^p(\mathbb{R}^{n+1}_+)$-atoms associated to balls $\{B_{i,j_2}\}_{i \in I, j_2 \in \Lambda_2}$ such that

(i) $F = \sum_{i \in I} \left[ \lambda_{i,1} \tilde{A}_{i,1} + \sum_{j_2 \in \Lambda_2} \lambda_{i,2,j_2} A_{i,2,j_2} \right]$ pointwisely almost everywhere on $\mathbb{R}^{n+1}_+$, where, for all $i \in I$, $\tilde{\lambda}_{i,1} := 2^j \sum_{j_2 \in \Lambda_1} |B_{i,1,j_2}|^{\frac{1}{p}}$ and $\lambda_{i,2,j_2} := 2^{|B_{i,2,j_2}|^{\frac{1}{p}}}$;
(ii) for all $i \in I$ and $j_1 \in \{1, 2\}$, $r_{i,j_1} := \inf_{j_2 \in \Lambda_1} \{r_{B_{i,j_1,j_2}}\} > 0$ and $\{4C_0 B_{i,j_1,j_2}\}_{j_1 \in \{1,2\}, j_2 \in \Lambda_1}$ has uniformly bounded overlap on $i \in I$;
(iii) there exists a positive constant $C$, depending only on $n$, $p$, and $C_0$, such that, for all $i \in I$,
\[
\left\{ \left| \tilde{\lambda}_{i,1} \right|^p + \sum_{j_2 \in \Lambda_2} |\lambda_{i,2,j_2}|^p \right\}^{\frac{1}{p}} \leq C \|F\|_{W^{p,p}(\mathbb{R}^{n+1}_+)}.\]

**Proof.** We first prove (i) and (ii) of Theorem 2.12. Let $I$ be as in Remark 2.10, $i \in I$ and $O^*_i$ be as in the proof of Theorem 2.6. For all $i \in I$, let $\epsilon \in (0, \infty)$ such that the open set
\[
O^*_{i,\epsilon} := O^*_i \cup \left\{ x \in (O^*_i)^c : \text{dist}(x, \partial O^*_i) < \epsilon \right\}
\]
satisfies $|O^*_{i,\epsilon}| < 2|O^*_i|$. Let $\{B_{i,j}\}_{j \in \mathbb{Z}^+}$ be a Whitney-type covering of $O^*_i$ as in Lemma 2.7 with $C_0$ therein replaced by $4C_0$, where $C_0$ is as in the assumption of Theorem 2.12. Assume that $\{B_{i,j}\}_{j \in \Lambda_1}$ is the maximal subsequence such that, for all $j \in \Lambda_1$, $B_{i,j} \cap \partial O^*_i \neq \emptyset$ and that $\{B_{i,j}\}_{j \in \Lambda_2}$ is the maximal subsequence such that, for all $j \in \Lambda_2$, $B_{i,j} \cap \partial O^*_i \neq \emptyset$ and $B_{i,j} \cap \partial O^*_i = \emptyset$. From Lemma 2.7(iv), it follows that, for all $j \in \Lambda_1$, $\frac{1}{4C_0} \epsilon < r_{B_{i,j}} < \frac{1}{C_0} \epsilon$, which implies that, for all $j$, $\tilde{j} \in \Lambda_1$, $r_{B_{i,j}} \sim r_{B_{i,\tilde{j}}}$. Now, for all $i \in I$, $j_1 \in \{1, 2\}$ and $j_2 \in \Lambda_{j_1}$, let
\[
r_{i,j_1} := \inf_{j_2 \in \Lambda_{j_1}} \{r_{B_{i,j_1,j_2}}\} \quad \text{and} \quad W_{i,j_1} := \bigcup_{j_2 \in \Lambda_{j_1}} B_{i,j_1,j_2},
\]
where, $B_{i,j_1,j_2} := B_{i,j_2} \in \{B_{i,j_2}\}_{j_2} \in \Lambda_j$ and, from the definition of $\{B_{i,j}\}_{j \in \mathbb{Z}^+}$ and Lemma 2.7(iv), it follows that $r_{i,j_1} > 0$.

For all $i \in I$, it is easy to see that $\{W_{i,j_1}\}_{j_1=1}^2$ satisfies the following three properties:

(i) $O_i^* \subset W_{i,1} \cup W_{i,2}$;

(ii) There exist two sequences $\{B_{i_1, j_2}\}_{j_2} \in \Lambda_1$ and $\{B_{i_2, j_2}\}_{j_2} \in \Lambda_2$, of balls satisfying that

$$\inf_{j_2 \in \Lambda_1} \{r_{B_{i_1, j_1}, j_2}\} =: r_{i, j_1} > 0,$$

(iii) for all $j_1 \in \{1, 2\}$, $W_{i,j_1} = \bigcup_{j_2 \in \Lambda_j} B_{i,j_1,j_2}$.

For all $(i)$ and (ii), we conclude (ii) of Theorem 2.12.

Now, for all $i \in I$, $j_1 \in \{1, 2\}$ and $j_2 \in \Lambda_j$, let

$$\Delta_{i,j_1,j_2} := (B_{i,j_1,j_2} \times (0, \infty)) \bigcap \hat{O}_i^* \setminus \hat{O}_{i+1}^*.$$

For all $F \in W^{T^p}(\mathbb{R}^{n+1}_+)$, from the fact that

$$\text{supp } F \subset \bigcup_{i \in I} \left( (W_{i,1} \times (0, \infty)) \cap (\hat{O}_i^* \setminus \hat{O}_{i+1}^*) \right) \bigcup \left( (W_{i,2} \times (0, \infty)) \cap (\hat{O}_i^* \setminus \hat{O}_{i+1}^*) \right)$$

we deduce that, for almost every $(x, t) \in \mathbb{R}^{n+1}_+$,

$$F(x, t) = \sum_{i \in I} \left( F \chi_{\Delta_{i,1}}(x, t) + \sum_{j_2 \in \Lambda_2} \left( F \chi_{\Delta_{i,2,j_2}}(x, t) \right) \right)$$

$$= \sum_{i \in I} \left\{ \left( 2^i \left[ \sum_{j_2 \in \Lambda_1} |B_{i,j_1,j_2}| \right] \right)^{\frac{1}{p'}} \left( 2^i \left[ \sum_{j_2 \in \Lambda_1} |B_{i_1,j_2}| \right] \right)^{-\frac{1}{p'}} \left( F \chi_{\Delta_{i,1}}(x, t) \right) \right\}$$

$$+ \sum_{j_2 \in \Lambda_2} \left( 2^i |B_{i,j_2}| \right)^{\frac{1}{p'}} \left( 2^i |B_{i_1,j_2}| \right)^{-\frac{1}{p'}} \left( F \chi_{\Delta_{i,2,j_2}}(x, t) \right)$$

$$= \sum_{i \in I} \left[ \lambda_{i,1} \tilde{A}_{i,1}(x, t) + \sum_{j_2 \in \Lambda_2} \lambda_{i,2,j_2} A_{i,2,j_2}(x, t) \right].$$

From the proof of Theorem 2.6, it follows that $A_{i,2,j_2}$ is a $T^p(\mathbb{R}^{n+1}_+)$-atom. On the other hand, for all $j_2 \in \Lambda_1$, since $B_{i_1,j_2} \cap \partial O_i^* \neq \emptyset$, we know that $\Delta_{i,j_2} \subset 3B_{i_1,j_2}$, which immediately implies that $\text{supp } \tilde{A}_{i,1} \subset \bigcup_{j_2 \in \Lambda_1} 3B_{i_1,j_2} \subset \mathcal{R}(\mathbb{R}^n \setminus O_i^*).$ Moreover, for any
\( H \in T^2(\mathbb{R}^{n+1}_+) \) with \( \|H\|_{T^2(\mathbb{R}^{n+1}_+)} \leq 1 \), by the definitions of \( \tilde{A}_{i,1} \), \( \tilde{\chi}_{i,1} \) and \( \Delta_{i,2,j_2} \), Lemma 2.8, the estimates similar to (2.4), the fact that \( \{4C_0B_{i,1,j_2}\}_{j_2 \in \Lambda_1} \) has uniformly bounded overlap on \( i \in I \), Hölder’s inequality and the definition of \( O_i \), we conclude that

\[
\left| \left\langle \tilde{A}_{i,1}, H \right\rangle \right|_{T^2(\mathbb{R}^{n+1}_+)} = 2^{-i} \left[ \sum_{j_2 \in \Lambda_1} |B_{i,1,j_2}| \right]^{-\frac{1}{p}} \left[ \int_{\mathbb{R}(\mathbb{R}^n \setminus O^{n+1}_i)} \left| F\tilde{\chi}_{\tilde{\chi}_{i,1}}(y,t)H(y,t) \frac{dy dt}{t} \right| \right]^{\frac{1}{p}} 
\leq 2^{-i} \left[ \sum_{j_2 \in \Lambda_1} |B_{i,1,j_2}| \right]^{-\frac{1}{p}} \left[ \int_{\mathbb{R}^n \setminus O^{n+1}_i} \left[ \int_0^\infty \int_{|y-x|<t} \left| F\tilde{\chi}_{\tilde{\chi}_{i,1}}(y,t) \right| |H(y,t)| \frac{dy dt}{t^{n+1}} \right] dx \right]^{\frac{1}{p}} 
\leq 2^{-i} \left[ \sum_{j_2 \in \Lambda_1} |B_{i,1,j_2}| \right]^{-\frac{1}{p}} \left[ \sum_{j_2 \in \Lambda_1} \int_{4B_{i,1,j_2} \setminus O^{n+1}_i} \left[ \int_0^\infty \int_{|y-x|<t} \left| F\tilde{\chi}_{\tilde{\chi}_{i,1}}(y,t) \right| dx \right] dy dt \right]^{\frac{1}{p}} 
\leq 2^{-i} \left[ \sum_{j_2 \in \Lambda_1} |B_{i,1,j_2}| \right]^{-\frac{1}{p}} \left[ 2^{i} \left\| \bigcup_{j_2 \in \Lambda_1} B_{i,1,j_2} \right\| \right]^{\frac{1}{2}} \sim \left[ \sum_{j_2 \in \Lambda_1} |B_{i,1,j_2}| \right]^{-\frac{1}{p}} \sim |W_{i,1}|^{-\frac{1}{2} - \frac{1}{p}},
\]

which immediately implies that

\[
\left\| \tilde{A}_{i,1} \right\|_{T^2(\mathbb{R}^{n+1}_+)} \lesssim |W_{i,1}|^{-\frac{1}{2} - \frac{1}{p}}.
\]

Thus, \( \tilde{A}_{i,1} \) is a \( C^0-T^p(\mathbb{R}^{n+1}_+) \)-atomic block associated to \( \cup_{j_2 \in \Lambda_1} (3B_{i,1,j_2}) \). Therefore, (i) of Theorem 2.12 holds true.

To prove (iii) of Theorem 2.12, by the definitions of \( \tilde{\chi}_{i,1} \) and \( \tilde{\chi}_{i,2,j_2} \), together with (2.3), we see that, for all \( i \in I \),

\[
|\tilde{\chi}_{i,1}|^p + \sum_{j_2 \in \Lambda_2} |\tilde{\chi}_{i,2,j_2}|^p = 2^{ip} \sum_{j_2 \in \Lambda_1} |B_{i,1,j_2}| + 2^{ip} \sum_{j_2 \in \Lambda_2} |B_{i,2,j_2}| \sim 2^{ip}|O_i| \lesssim \|F\|_{W^{T^p(\mathbb{R}^{n+1})}}^p,
\]

which completes the proof of Theorem 2.12. \( \square \)

### 2.3 The weak Hardy spaces \( WH^p_L(\mathbb{R}^n) \)

In this subsection, we study the weak Hardy space \( WH^p_L(\mathbb{R}^n) \). First, we recall the definition of the classical weak Hardy space from [29, 46, 47]. Let \( p \in (0, 1] \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \)
supported in the unit ball $B(0, 1)$. The \textit{weak Hardy space} $WH^p(\mathbb{R}^n)$ is defined to be the space
\[
\left\{ f \in S'(\mathbb{R}^n) : \|f\|_{W^{H^p}(\mathbb{R}^n)} := \sup_{\alpha > 0} \left( \alpha^p \left\{ x \in \mathbb{R}^n : \sup_{t > 0} |\varphi_t * f(x)| > \alpha \right\} \right)^{1/p} < \infty \right\}.
\]

Now, let $L$ satisfy Assumptions ($\mathcal{L}$)$_1$, ($\mathcal{L}$)$_2$ and ($\mathcal{L}$)$_3$. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the \textit{L-adapted non-tangential square function} $S_L f$ is defined by
\[
S_L f(x) := \left\{ \int\int_{\Gamma(x)} \left| t^{2k} L e^{-t^{2k}L} f(y) \right|^2 \frac{dy \, dt}{t^{n+1}} \right\}^{1/2},
\]

Let $p \in (0, 1]$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{H}^p_L(\mathbb{R}^n)$ if $S_L f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{\mathbb{H}^p_L(\mathbb{R}^n)} := \|S_L f\|_{L^p(\mathbb{R}^n)}$. The \textit{Hardy space} $\mathbb{H}^p_L(\mathbb{R}^n)$ associated to $L$ is then defined to be the completion of $\mathbb{H}^p_L(\mathbb{R}^n)$ with respect to the quasi-norm $\| \cdot \|_{\mathbb{H}^p_L(\mathbb{R}^n)}$ (see [10]).

Now, we introduce the definition of the weak Hardy space $WH^p_L(\mathbb{R}^n)$.

**Definition 2.13.** Let $p \in (0, 1]$ and $L$ satisfy Assumptions ($\mathcal{L}$)$_1$, ($\mathcal{L}$)$_2$ and ($\mathcal{L}$)$_3$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{W}^p_L(\mathbb{R}^n)$ if $S_L f \in W^p_L(\mathbb{R}^n)$; moreover, define $\|f\|_{\mathbb{W}^p_L(\mathbb{R}^n)} := \|S_L f\|_{W^p_L(\mathbb{R}^n)}$. The \textit{weak Hardy space} $\mathbb{W}^p_L(\mathbb{R}^n)$ associated to $L$ is then defined to be the completion of $\mathbb{W}^p_L(\mathbb{R}^n)$ with respect to the quasi-norm $\| \cdot \|_{\mathbb{W}^p_L(\mathbb{R}^n)}$.

**Remark 2.14.** We point out that, unlike the Hardy space $H^p(\mathbb{R}^n)$, with $p \in (0, 1]$, in which the space $L^2(\mathbb{R}^n)$ is dense (see, for example, [47, Proposition 3.2]), the space $L^2(\mathbb{R}^n)$ is not dense in the weak Hardy space $WH^p(\mathbb{R}^n)$ in the sense of Fefferman and Soria [29] (see also a very recent work of He [35]). Thus, when $L = -\Delta$, the weak Hardy space $WH^p_{-\Delta}(\mathbb{R}^n)$ defined as in Definition 2.13 coincides with the space
\[
WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),
\]

namely, the closure of $WH^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ on the quasi-norm $\| \cdot \|_{WH^p(\mathbb{R}^n)}$, which is a proper subspace of $WH^p(\mathbb{R}^n)$.

Now, let $T$ be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. It is known that $T$ is a special case of operators $L$ satisfying Assumptions ($\mathcal{L}$)$_1$, ($\mathcal{L}$)$_2$ and ($\mathcal{L}$)$_3$. We first establish the weak atomic decomposition of the weak Hardy space $WH^p_T(\mathbb{R}^n)$.

**Definition 2.15** ([36, 40]). Let $p \in (0, 1]$, $M \in \mathbb{N}$ and $B := B(x_B, r_B)$ be a ball. A function $a \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M)_T$-\textit{atom associated to} $B$, if the following conditions are satisfied:

(i) there exists a function $b$ belonging to the domain of $T^M$, $D(T^M)$, such that $a = T^M b$;
(ii) for all $\ell \in \{0, \ldots, M\}$, supp $(T^\ell b) \subset B$;
(iii) for all $\ell \in \{0, \ldots, M\}$, $\|r_B^2 T^\ell b\|_{L^2(\mathbb{R}^n)} \leq \frac{2^M}{r_B} |B|^\frac{1}{2} - \frac{1}{p}$.
For all $p \in (0, 1]$ and $M \in \mathbb{N}$, let $f \in L^2(\mathbb{R}^n)$ and \{a_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+}$ be a sequence of $(p, 2, M)_T$-atoms associated to balls \{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+}. The equality $f = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} a_{i,j}$ in $L^2(\mathbb{R}^n)$ is called a weak atomic $(p, 2, M)_T$-representation of $f$, if

(i) $\lambda_{i,j} := 2^i |B_{i,j}|^\frac{1}{p}$;

(ii) there exists a positive constant $C_2$, depending only on $f, n, p$ and $M$, such that

$$\sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right)^{\frac{1}{p}} \leq C_2.$$

The weak atomic Hardy space $WH_{T, at, M}^p(\mathbb{R}^n)$ is defined to be the completion of the space

$$WH_{T, at, M}^p(\mathbb{R}^n) := \{ f \in L^2(\mathbb{R}^n) : f \text{ has a weak atomic } (p, 2, M)_T\text{-representation} \}$$

with respect to the quasi-norm

$$\| f \|_{WH_{T, at, M}^p(\mathbb{R}^n)} := \inf \left\{ \sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right)^{\frac{1}{p}} \right\},$$

where the infimum is taken over all the weak atomic $(p, 2, M)_T$-representations of $f$.

We have the following weak atomic characterization of $WH_{T}^p(\mathbb{R}^n)$.

**Theorem 2.16.** Let $p \in (0, 1]$ and $T$ be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. Assume that $M \in \mathbb{N}$ satisfies $M > \frac{n}{2} (\frac{1}{p} - \frac{1}{2})$. Then $WH_{T}^p(\mathbb{R}^n) = WH_{T, at, M}^p(\mathbb{R}^n)$ with equivalent quasi-norms.

To prove this theorem, we need to recall some notions and known results from [36].

Let $C_3 \in [1, \infty)$. Assume that $\varphi \in C_c^\infty(\mathbb{R})$ is even, supp $\varphi \subset (-\frac{1}{2C_3}, \frac{1}{2C_3})$, $\varphi \geq 0$ and there exists a positive constant $C_4$ such that, for all $t \in (-\frac{1}{2C_3}, \frac{1}{2C_3})$, $\varphi(t) \geq C_4$. Let $\Phi$ be the Fourier transform of $\varphi$ and $\Psi(t) := \varphi(2^{(M+1)}t)$ for all $t \in [0, \infty)$. For $T$ as in Theorem 2.16, all $F \in T^2(\mathbb{R}^{n+1}_+)$ and $x \in \mathbb{R}^n$, define the operator $\Pi_{\Psi, T}(F)(x)$ by setting

$$\Pi_{\Psi, T}(F)(x) := \int_0^\infty \Psi(t\sqrt{T})(F(\cdot, t))(x) \frac{dt}{t}. \tag{2.6}$$

From Fubini’s theorem and the quadratic estimates, it follows that $\Pi_{\Psi, T}$ is bounded from $T^2(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^n)$. By using the finite speed of the propagation of the wave equation and the Paley-Wiener theorem, Hofmann et al. proved the following conclusion.

**Lemma 2.17 ([36]).** Let $p \in (0, 1]$, $M \in \mathbb{N}$ and $T$ be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates. Assume that $A$ is $T^p(\mathbb{R}^{n+1}_+)$-atom associated to the ball $B$ and $\Pi_{\Psi, T}$ is as in (2.6). Then there exists a positive constant $C(M)$, independent of $A$, such that $[C(M)]^{-1} \Pi_{\Psi, T}(A)$ is a $(p, 2, M)_T$-atom associated to the ball $2B$.
We also need the following superposition principle on the weak type estimate.

**Lemma 2.18** ([54]). Let \( p \in (0, 1) \) and \( \{f_j\}_{j \in \mathbb{Z}_+} \) be a sequence of measurable functions. If \( \sum_{j \in \mathbb{Z}_+} |\lambda_j|^p < \infty \) and there exists a positive constant \( C \) such that, for all \( \alpha \in (0, \infty) \) and \( j \in \mathbb{Z}_+ \), \( \{|x \in \mathbb{R}^n : |f_j(x)| > \alpha\| \leq C \alpha^{-p} \). Then, there exists a positive constant \( \tilde{C} \) such that, for all \( \alpha \in (0, \infty) \),

\[
\left\{ x \in \mathbb{R}^n : \sum_{j \in \mathbb{Z}_+} \lambda_j f_j(x) > \alpha \right\} \leq \frac{\tilde{C}^2 - p}{1 - p} \alpha^{-p} \sum_{j \in \mathbb{Z}_+} |\lambda_j|^p.
\]

With the above preparations, we now prove Theorem 2.16.

**Proof of Theorem 2.16.** In order to prove Theorem 2.16, it suffices to show that

\[
WH^p_T(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = WH^p_{T,at,M}(\mathbb{R}^n)
\]

with equivalent quasi-norms. We first prove the inclusion that \( WH^p_T(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset WH^p_{T,at,M}(\mathbb{R}^n) \). Let \( f \in WH^p_T(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). From its definition and the quadratic estimate, it follows that \( t^2Te^{-t^2T}f \in WT^p(\mathbb{R}^{n+1}_+)^* \). By Theorem 2.6, there exist sequences \( \{\lambda_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \subset \mathbb{C} \) and \( \{A_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \) of \( T^p(\mathbb{R}^{n+1}_+) \)-atoms associated to the balls \( \{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \) such that \( t^2Te^{-t^2T}(f) = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} A_{i,j} \) pointwisely almost everywhere on \( \mathbb{R}^{n+1}_+ \), \( \lambda_{i,j} = 2^i |B_{i,j}|^{1/p} \) and

\[
\sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right)^{\frac{1}{p}} \lesssim \left\| t^2Te^{-t^2T}(f) \right\|_{WT^p(\mathbb{R}^{n+1}_+)} \sim \left\| f \right\|_{WH^p(\mathbb{R}^{n+1}_+)}.\tag{2.7}
\]

Moreover, by the bounded \( H_\infty \) functional calculus in \( L^2(\mathbb{R}^n) \), Corollary 2.9 and the fact that \( \Pi_{\Psi,T} \) is bounded from \( T^2(\mathbb{R}^{n+1}_+) \) to \( L^2(\mathbb{R}^n) \), we conclude that there exists a constant \( C(\Psi) \), depending on \( \Psi \), such that

\[
f = C(\Psi) \int_0^\infty \Psi(t\sqrt{T})(t^2Te^{-t^2T}f) \frac{dt}{t} = C(\Psi) \int_0^\infty \Psi(t\sqrt{T}) \left( \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} A_{i,j} \right) \frac{dt}{t},
\]

where the above equalities hold true in \( L^2(\mathbb{R}^n) \). For all \( i \in \mathbb{Z} \) and \( j \in \mathbb{Z}_+ \), let \( a_{i,j} := \int_0^\infty \Psi(t\sqrt{T})(A_{i,j}) \frac{dt}{t} \). By Lemma 2.17, we see that \( a_{i,j} \) is a \( (p, 2, M)_T \)-atom associated to \( \{2B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \) up to a harmless constant. Thus, we conclude that \( f \) has a weak atomic \( (p, 2, M)_T \)-representation \( \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} a_{i,j} \) and \( f \in WH^p_{T,at,M}(\mathbb{R}^n) \). Moreover, from (2.7), we deduce that

\[
\left\| f \right\|_{WH^p_{T,at,M}(\mathbb{R}^n)} \leq \left( \sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \lesssim \left\| f \right\|_{WH^p_T(\mathbb{R}^n)},
\]
which immediately implies that \( W^p_T(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset W^p_{T,at,M}(\mathbb{R}^n) \).

Now we prove the converse, namely, \( W^p_{T,at,M}(\mathbb{R}^n) \subset W^p_T(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). Let \( f \in W^p_{T,at,M}(\mathbb{R}^n) \). From its definition, it follows that there exists a sequence \( \{a_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \) of \((p, 2, M)\)-atoms associated to the balls \( \{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \) such that \( f = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \lambda_{i,j} a_{i,j} \) in \( L^2(\mathbb{R}^n) \) and, for all \( i \in \mathbb{Z} \),

\[
\sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \lesssim \|f\|_{W^p_{T,at,M}(\mathbb{R}^n)},
\]

where \( \{\lambda_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} := \{2^i |B_{i,j}|^{1/p}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+} \).

Given \( \alpha \in (0, \infty) \), let \( i_0 \in \mathbb{Z} \) satisfy that \( 2^{i_0} \leq \alpha < 2^{i_0+1} \). We then see that

\[
f = \sum_{i=-\infty}^{\infty} \sum_{j \in \mathbb{Z}_+} \lambda_{i,j} a_{i,j} + \sum_{i=i_0+1}^{\infty} \sum_{j \in \mathbb{Z}_+} \lambda_{i,j} a_{i,j} =: f_1 + f_2
\]

holds true in \( L^2(\mathbb{R}^n) \).

We first estimate \( f_2 \). Let \( \tilde{B}_{i_0} := \cup_{i=i_0+1}^{\infty} \cup_{j \in \mathbb{Z}_+} 8B_{i,j} \). From the definition of \( \lambda_{i,j} \) and (2.8), we deduce that

\[
|\tilde{B}_{i_0}| \lesssim \sum_{i=i_0+1}^{\infty} \sum_{j \in \mathbb{Z}_+} |B_{i,j}| \lesssim \sum_{i=i_0+1}^{\infty} 2^{-ip} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right) \lesssim \sum_{i=i_0+1}^{\infty} 2^{-ip} \|f\|_{W^p_{T,at,M}(\mathbb{R}^n)},
\]

\[
\lesssim \frac{1}{\alpha^p} \|f\|_{W^p_{T,at,M}(\mathbb{R}^n)}.
\]

Now, for \( q \in (0, p) \), we write

\[
f_2 = \sum_{i=i_0+1}^{\infty} \sum_{j \in \mathbb{Z}_+} \left( 2^i |B_{i,j}|^{1/p} \right) \left( |B_{i,j}|^{1/p} \right)^{q-1} a_{i,j} =: \sum_{i=i_0+1}^{\infty} \sum_{j \in \mathbb{Z}_+} \tilde{\lambda}_{i,j} \tilde{a}_{i,j}.
\]

By (2.8) and the fact that \( q \in (0, p) \), we know that

\[
\sum_{i=i_0+1}^{\infty} \sum_{j \in \mathbb{Z}_+} \tilde{\lambda}_{i,j}^q \lesssim \sum_{i=i_0+1}^{\infty} 2^q \sum_{j \in \mathbb{Z}_+} |B_{i,j}| \lesssim \sum_{i=i_0+1}^{\infty} 2^{i(p-q)} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right) \lesssim \|f\|_{W^p_{T,at,M}(\mathbb{R}^n)},
\]

which, combined with (2.9), (2.10) and Lemma 2.18, implies that, to show that \( S_T(f_2) \in WL^p(\mathbb{R}^n) \), it suffices to show that, for all \( \alpha \in (0, \infty), i \in \mathbb{Z} \) and \( j \in \mathbb{Z}_+ \),

\[
|\left\{ x \in (C_0B_{i,j})^C : S_T(\tilde{a}_{i,j})(x) > \alpha \right\}| \lesssim \frac{1}{\alpha^q}.
\]
Indeed, if (2.12) holds true, then, for $N_1 \in \mathbb{N} \cap (i_0 + 1, \infty)$ and $N_2 \in \mathbb{N}$, by Tchebychev’s inequality, and the sub-linearity and the $L^2(\mathbb{R}^n)$ boundedness of $S_T$, we conclude that

$$\left| \{ x \in \mathbb{R}^n : S_T(f_2)(x) > \alpha \} \right| \leq \left\{ x \in \mathbb{R}^n : \sum_{i=i_0+1}^{N_1} \sum_{j=0}^{N_2} \tilde{\lambda}_{i,j} S_T(\tilde{a}_{i,j})(x) > \frac{\alpha}{2} \right\} + \left\{ x \in \mathbb{R}^n : S_T \left( \sum_{N_1+1 \leq l < \infty \atop \text{or} j \geq N_2+1} \lambda_{i,j} a_{i,j} \right)(x) > \frac{\alpha}{2} \right\},$$

where

$$\sum_{l=N_1+1}^{\infty} \sum_{j=0}^{N_2} := \sum_{i=N_1+1}^{\infty} \sum_{j=\infty}^{\infty} + \sum_{i=0}^{\infty} \sum_{j=\infty}^{\infty} + \sum_{i=\infty}^{\infty} \sum_{j=\infty}^{\infty}.$$

By letting $N_1$ and $N_2 \to \infty$, the fact that $f_2 = \sum_{i=i_0+1}^{\infty} \sum_{j=\infty}^{\infty} \lambda_{i,j} a_{i,j}$ holds true in $L^2(\mathbb{R}^n)$ and (2.11), we see that

$$\left| \{ x \in \mathbb{R}^n : S_T(f_2)(x) > \alpha \} \right| \lesssim \alpha^{-q} \sum_{i=i_0+1}^{\infty} \sum_{j=\infty}^{\infty} |\tilde{\lambda}_{i,j}|^q + \alpha^{-2} \sum_{N_1+1 \leq l < \infty \atop \text{or} j \geq N_2+1} \lambda_{i,j} a_{i,j} \left\| \frac{1}{\alpha} \right\|^p W_{H^p_{\alpha}(\mathbb{R}^n)},$$

which is desired.

To prove (2.12), from Chebyshev’s inequality and Hölder’s inequality, we deduce that

$$\left| \{ x \in (8B_{i,j})^c : S_T(\tilde{a}_{i,j})(x) > \alpha \} \right| \lesssim \alpha^{-q} \int_{(8B_{i,j})^c} [S_T(\tilde{a}_{i,j})(x)]^q \, dx$$

$$\lesssim \alpha^{-q} \sum_{i=4}^{\infty} \left\{ \int_{S_i(B_{i,j})} |S_T(\tilde{a}_{i,j})(x)|^2 \, dx \right\}^{\frac{q}{2}} |S_i(B_{i,j})|^{1-\frac{q}{2}},$$

where $S_i(B_{i,j}) := 2^l B_{i,j} \setminus (2^{l-1} B_{i,j})$ for all $l \in \mathbb{N}$. For $l \geq 4$, let

$$I_{l,i,j} := \left\{ \int_{S_i(B_{i,j})} |S_T(\tilde{a}_{i,j})(x)|^2 \, dx \right\}^{\frac{q}{2}},$$

$b_{i,j} := T^{-M} a_{i,j}$ and $\tilde{b}_{i,j} := |B_{i,j}|^{\frac{1}{p} - \frac{1}{q}} b_{i,j}$.

By Minkowski’s inequality, we write $I_{l,i,j}$ into

$$I_{l,i,j} \lesssim \left\{ \int_{S_i(B_{i,j})} \int_0^{T^{B_{i,j}}} \int_{|y-x|<t} \left| t^{2} Te^{-t^2} \tilde{a}_{i,j}(y,t) \right|^2 \frac{dy \, dt}{t^{n+1}} \, dx \right\}^{\frac{q}{2}}.$$
To estimate $J_{l,i,j}$, notice that $\text{dist}(S_l(B_{i,j}), B_{i,j}) > 2^{l-2}r_{B_{i,j}}$, when $l \geq 4$. Let $E_{l,i,j} := \{ x \in \mathbb{R}^n : \text{dist}(x, S_l(B_{i,j})) < r_{B_{i,j}} \}$. We easily see that $\text{dist}(E_{l,i,j}, B_{i,j}) > 2^{l-3}r_{B_{i,j}}$, which, together with Fubini’s theorem, Lemma 2.3 and Definition 2.15, implies that, there exists a positive constant $\alpha_0 > \frac{n}{q}(1 - \frac{q}{2})$ such that

\[
J_{l,i,j} \lesssim \left\{ \int_0^{r_{B_{i,j}}} \int_{E_{l,i,j}} t^2Te^{-t^2T}(y,t) \, dy \, dt \right\}^{\frac{q}{2}} \\
\lesssim \left\{ \int_0^{r_{B_{i,j}}} \exp \left\{ -C_1 \frac{[\text{dist}(E_{l,i,j}, B_{i,j})]^2}{t^2} \right\} \, dt \right\}^{\frac{q}{2}} \|\hat{a}_{i,j}\|_{L^2(B_{i,j})} \\
\lesssim 2^{-lq\alpha_0} |B_{i,j}|^{\frac{q}{2}-1} \sim 2^{-l[q\alpha_0 - n(1 - \frac{q}{2})]} |S_l(B_{i,j})|^{\frac{q}{2}-1}.
\]

To estimate $K_{l,i,j}$, let $F_{l,i,j} := \{ x \in \mathbb{R}^n : \text{dist}(x, S_l(B_{i,j})) < \frac{\text{dist}(x, B_{i,j})}{4} \}$. It is easy to see that $\text{dist}(F_{l,i,j}, B_{i,j}) > 2^{l-3}r_{B_{i,j}}$. Moreover, by Fubini’s theorem, Lemma 2.3 and Definition 2.15, we know that there exists a positive constant $\alpha_1 \in (\frac{q}{2}(1 - \frac{q}{2}), 2M)$ such that

\[
K_{l,i,j} \lesssim \left\{ \int_{r_{B_{i,j}}}^{\infty} \int_{F_{l,i,j}} t^{2(M+1)}T^{M+1}e^{-t^2T}(y) \, dy \, dt \right\}^{\frac{q}{2}} \\
\lesssim \left\{ \int_{r_{B_{i,j}}}^{\infty} \exp \left\{ -C_1 \frac{[\text{dist}(F_{l,i,j}, B_{i,j})]^2}{t^2} \right\} \, dt \right\}^{\frac{q}{2}} \|\hat{b}_{i,j}\|_{L^2(B_{i,j})} \\
\lesssim \left\{ \int_{r_{B_{i,j}}}^{\infty} \left[ \frac{t^2}{2^M r_{B_{i,j}}} \right]^{\alpha_1} \, dt \right\}^{\frac{q}{2}} r_{B_{i,j}}^{2Mq} |B_{i,j}|^{\frac{q}{2}-1} \\
\lesssim 2^{-l[q\alpha_1 - n(1 - \frac{q}{2})]} |S_l(B_{i,j})|^{\frac{q}{2}-1}.
\]

Similar to the estimates of (2.14) and (2.15), we obtain

\[
Q_{l,i,j} \lesssim \left\{ \int_{2^{l-2}r_{B_{i,j}}}^{\infty} \int_{\mathbb{R}^n} |(t^2T)^{M+1}e^{-t^2T}(y)|^2 \, dy \, dt \right\}^{\frac{q}{2}} \\
\lesssim \left\{ \int_{2^{l-2}r_{B_{i,j}}}^{\infty} \, dt \right\}^{\frac{q}{2}} \|\hat{b}_{i,j}\|_{L^2(B_{i,j})}^2 \lesssim 2^{-l[2qM - n(1 - \frac{q}{2})]} |S_l(B_{i,j})|^{\frac{q}{2}-1},
\]
which, together with (2.13), (2.14) and (2.15), implies that, for all \( \alpha \in (0, \infty) \), \( i \in \mathbb{Z} \) and \( j \in \mathbb{Z}_+ \),

\[
\left\{ x \in (C_0 B_{i,j})^c : S_T(a_{i,j})(x) > \alpha \right\} \lesssim \alpha^{-q} \sum_{l=4}^{\infty} (J_{l,i,j} + K_{l,i,j} + Q_{l,i,j}) |S_l(B_{i,j})|^{1-\frac{q}{2}} \lesssim \frac{1}{\alpha^q}.
\]

Thus, (2.12) holds true.

The estimate for \( f_1 \) is similar to that for \( f_2 \), but we need to replace \( q \in (0, p) \) by another \( \tilde{q} \in (p, 2) \setminus \{1\} \), the details being omitted here; see the below proof of Theorem 2.21 for some details. Combining the estimates for \( f_1 \) and \( f_2 \), we then complete the proof of Theorem 2.16.

Now, we try to establish the molecular characterization of \( WH^p_L(\mathbb{R}^n) \) with \( L \) being an operator on \( L^2(\mathbb{R}^n) \) satisfying Assumptions \((\mathcal{L})_1\), \((\mathcal{L})_2\) and \((\mathcal{L})_3\). We first recall the notion of \((p, \epsilon, M)_L\)-molecules.

**Definition 2.19** ([10]). Let \( k \in \mathbb{N} \), \( p \in (0, 1] \), \( \epsilon \in (0, \infty) \), \( M \in \mathbb{N} \) and \( L \) satisfy Assumptions \((\mathcal{L})_1\), \((\mathcal{L})_2\) and \((\mathcal{L})_3\). A function \( m \in L^2(\mathbb{R}^n) \) is called a \((p, \epsilon, M)_L\)-molecule, if there exists a ball \( B := B(x_B, r_B) \) such that

(i) for each \( \ell \in \{1, \ldots, M\} \), \( m \) belongs to the range of \( L^\ell \) in \( L^2(\mathbb{R}^n) \);

(ii) for all \( i \in \mathbb{Z}_+ \) and \( \ell \in \{0, \ldots, M\} \),

\[
\left\| \left( r_B^{-2k} L^{-1} \right)^\ell m \right\|_{L^2(S_i(B))} \leq (2^i r_B)^{n(\frac{1}{2} - \frac{1}{p})} \cdot 2^{-i \epsilon}.
\]

**Definition 2.20.** Let \( \epsilon \in (0, \infty) \), \( M \in \mathbb{Z}_+ \), \( p \in (0, 1] \) and \( L \) satisfy Assumptions \((\mathcal{L})_1\), \((\mathcal{L})_2\) and \((\mathcal{L})_3\). Assume that \( \{m_{i,j}\}_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_+} \) is a sequence of \((p, \epsilon, M)_L\)-molecules associated to balls \( \{B_{i,j}\}_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_+} \) and \( \{\lambda_{i,j}\}_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_+} \subset \mathbb{C} \) satisfying the conditions that

(i) for all \( i \in \mathbb{Z} \) and \( j \in \mathbb{Z}_+ \), \( \lambda_{i,j} := 2^{l} |B_{i,j}|^{1/p} \);

(ii) there exists a positive constant \( C_5 \), depending only on \( f \), \( n \), \( p \), \( \epsilon \) and \( M \), such that

\[
\sup_{i \in \mathbb{Z}_+} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right)^{1/p} \leq C_5.
\]

Then, for any \( f \in L^2(\mathbb{R}^n) \), \( f = \sum_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_+} \lambda_{i,j} m_{i,j} \) is called a \textit{weak molecular} \((p, \epsilon, M)_L\)-representation of \( f \), if \( f = \sum_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_+} \lambda_{i,j} m_{i,j} \) holds true in \( L^2(\mathbb{R}^n) \). The \textit{weak molecular Hardy space} \( WH^p_{L,mol,\epsilon,M}(\mathbb{R}^n) \) is then defined to be the completion of the space

\[
WH^p_{L,mol,\epsilon,M}(\mathbb{R}^n) := \{ f : f \text{ has a weak molecular } (p, \epsilon, M)_L\text{-representation} \}
\]

with respect to the quasi-norm

\[
\|f\|_{WH^p_{L,mol,\epsilon,M}(\mathbb{R}^n)} := \inf \left\{ \sup_{i \in \mathbb{Z}_+} \left( \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p \right)^{1/p} : f = \sum_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_+} \lambda_{i,j} m_{i,j} \text{ is a weak } \right\}\]

We only remark that, in this case, the operator $\Pi \Psi$ follows from a similar argument to the corresponding part of the proof of Theorem 2.16.

Moreover, from Definition 2.20 and the fact $q > p$, we deduce that

$$\sum_{i=-\infty}^{i_0} \sum_{j \in \mathbb{Z}_+} |\tilde{\lambda}_{i,j}|^q \lesssim \sum_{i=-\infty}^{i_0} 2^{i q} \sum_{j \in \mathbb{Z}_+} |B_{i,j}| \lesssim \sum_{i=-\infty}^{i_0} 2^{i(q-p)} \sum_{j \in \mathbb{Z}_+} |\lambda_{i,j}|^p.$$
\[ \|f\|_{W^p_{H_x, mol, c, L}^\infty}^p \lesssim \sum_{i=0}^{i_0} 2^{i(q-p)} \lesssim 2^{i_0(q-p)} \|f\|_{W^p_{H_x, mol, c, L}^\infty}^p, \]

which, together with Lemma 2.18, implies that, to show that \( S_L(f_1) \in W^p_{L^p}(\mathbb{R}^n) \), it suffices to show that, for all \( \alpha \in (0, \infty) \), \( i \in \mathbb{Z} \) and \( j \in \mathbb{Z}_+ \),

\[ \{ x \in \mathbb{R}^n : S_L(\bar{m}_{i,j})(x) > \alpha \} \lesssim \frac{1}{\alpha^{\gamma_q}}. \]

(2.17)

To prove (2.17), by Chebyshev’s inequality and Hölder’s inequality, we write

\[ \{ x \in \mathbb{R}^n : S_L(\bar{m}_{i,j})(x) > \alpha \} \lesssim \sum_{i=0}^{\infty} 2^{-i\alpha q} \| S_L(\bar{m}_{i,j}) \|_{L^q(S(B_{i,j}))}^q |S(B_{i,j})|^{1 - \frac{q}{2}} \]

(2.18)

For \( l \in \{0, \ldots, 4\} \), by Fubini’s theorem, the \( L^2(\mathbb{R}^n) \) boundedness of \( S_L \), Definition 2.19 and (2.16), we conclude that

\[ \| S_L(\bar{m}_{i,j}) \|_{L^2(S(B_{i,j}))} \lesssim \| S_L(\bar{m}_{i,j}) \|_{L^2(\mathbb{R}^n)} \lesssim \| \bar{m}_{i,j} \|_{L^2(\mathbb{R}^n)} \lesssim |B_{i,j}|^{-\frac{
}{2}}. \]

(2.19)

For \( l \geq 5 \), let

\[ \tilde{J}_{l,i,j} := \left\{ J_{S(B_{i,j})} \left[ \int_{S(B_{i,j})} \int_{|y-x|<t} |2^k L e^{-t^2 L} \bar{m}_{i,j}(y)|^2 \frac{dy dt}{t^{n+1}} \right] dx \right\}^{\frac{1}{2}}, \]

\[ \tilde{K}_{l,i,j} := \left\{ K_{S(B_{i,j})} \left[ \int_{S(B_{i,j})} \int_{|y-x|<t} \left(2^k L \right)^{M+1} e^{-t^2 L} \left( \alpha(\bar{L}^{-M} \bar{m}_{i,j})(y) \right)^2 \frac{dy dt}{t^{4kM+n+1}} \right] dx \right\}^{\frac{1}{2}}, \]

and

\[ \tilde{Q}_{l,i,j} := \left\{ Q_{S(B_{i,j})} \left[ \int_{S(B_{i,j})} \int_{|y-x|<t} \left(2^k L \right)^{M+1} e^{-t^2 L} \left( \alpha L^{-M} \bar{m}_{i,j}(y) \right)^2 \frac{dy dt}{t^{4kM+n+1}} \right] dx \right\}^{\frac{1}{2}}. \]

To estimate \( \tilde{J}_{l,i,j} \), let \( \tilde{E}_{l,i,j} := \{ x \in \mathbb{R}^n : \text{dist}(x, S(B_{i,j})) < r_{B_{i,j}} \} \) and

\[ \tilde{G}_{l,i,j} := \{ x \in \mathbb{R}^n : \text{dist}(x, E_{l,i,j}) < 2l^{-3} r_{B_{i,j}} \}. \]

It is easy to see that \( \text{dist}(\mathbb{R}^n \setminus \tilde{G}_{l,i,j}, \tilde{E}_{l,i,j}) > 2l^{-4} r_{B_{i,j}} \). Moreover, by Fubini’s theorem, the \( L^2(\mathbb{R}^n) \) boundedness of \( S_L \), Assumption (L)3, Definition 2.19, (2.16) and the fact that \( n(\frac{1}{p} - \frac{1}{q}) < \epsilon \), we see that there exists a positive constant \( \alpha_2 \in (n(\frac{1}{q} - \frac{1}{2}), \infty) \) such that

\[ \tilde{J}_{l,i,j} \lesssim \left\{ \int_{\tilde{E}_{l,i,j}} \int_{\tilde{G}_{l,i,j}} \left[ \chi_{\tilde{G}_{l,i,j}} + \chi_{\mathbb{R}^n \setminus \tilde{G}_{l,i,j}} \right] \bar{m}_{i,j}(y)^2 \frac{dy dt}{t^l} \right\}^{\frac{1}{2}}. \]

(2.20)
that there exist a ball $B_{f}$ and molecule as notion of $(p, \epsilon, M)$, namely, there exist positive constants $C$ and $\omega_{0}$ such that, for all closed sets $E, F \subseteq \mathbb{R}^{n}$, respectively as in (1.1) and (1.2), satisfies the following assumption:

$$\|e^{-tL}f\|_{L^{q}(F)} \leq C t^{\frac{k}{2k}(\frac{1}{q} - \frac{1}{p})} \exp \left\{ -C_{1} \frac{\text{dist}(E, F)^{2k/(2k-1)}}{t^{1/(2k-1)}} \right\} \|f\|_{L^{p}(E)}.$$ 

Moreover, the corresponding weak molecular Hardy space $WH_{L,mol,q,\epsilon,M}^{p}(\mathbb{R}^{n})$ can be defined analogously to Definition 2.20.

Assume further that $L$ satisfies the following assumption:

**Assumption** (L)$_{4}$. Let $k \in \mathbb{Z}_{+}$ and $(p_{-}(L), p_{+}(L))$ be the range of exponents $p$ for which the holomorphic semigroup $\{e^{-tL}\}_{t>0}$ is bounded on $L^{p}(\mathbb{R}^{n})$. Assume that $q \in (p_{-}(L), p_{+}(L))$. Similar to the notion of $(p, \epsilon, M)$-molecules as in Definition 2.19, we can also define the $(p, q, \epsilon, M)$-molecule as $m \in L^{q}(\mathbb{R}^{n})$ belonging to the range of $L^{\ell}$ for all $\ell \in \{0, \ldots, M\}$ and satisfying that there exist a ball $B := (x_{B}, r_{B})$ and a positive constant $C$ such that, for all $i \in \mathbb{Z}_{+}$,

$$\left\| \left( \frac{r_{B}^{k}}{L} \right)^{-\ell} m \right\|_{L^{q}(S_{i}(B))} \leq C 2^{-i} |S_{i}(B)|^{\frac{1}{h} - \frac{1}{q}}.$$

Moreover, the corresponding weak molecular Hardy space $WH_{L,mol,q,\epsilon,M}^{p}(\mathbb{R}^{n})$ can be defined analogously to Definition 2.20.

Assume further that $L$ satisfies the following assumption:

**Assumption** (L)$_{4}$. Let $k \in \mathbb{Z}_{+}$ and $(p_{-}(L), p_{+}(L))$ be the range of exponents $p$ for which the holomorphic semigroup $\{e^{-tL}\}_{t>0}$ is bounded on $L^{p}(\mathbb{R}^{n})$. Assume that, for all $p_{-}(L) < p < q < p_{+}(L)$, $\{e^{-tL}\}_{t>0}$ satisfies the $L^{p} - L^{q}$ k-off-diagonal estimate, namely, there exist positive constants $C$ and $C_{1}$ such that, for all closed sets $E, F \subseteq \mathbb{R}^{n}$ and $f \in L^{p}(\mathbb{R}^{n}) \cap L^{2}(\mathbb{R}^{n})$ supported in $E$,

$$\left\| e^{-tL}f \right\|_{L^{q}(F)} \leq C t^{\frac{k}{2k}(\frac{1}{q} - \frac{1}{p})} \exp \left\{ -C_{1} \frac{\text{dist}(E, F)^{2k/(2k-1)}}{t^{1/(2k-1)}} \right\} \|f\|_{L^{p}(E)}.$$ 

By using the method in the proof of [42, Proposition 4.2] and [8, Theorem 2.23], we can also prove the equivalence between $WH_{L,mol,q,\epsilon,M}^{p}(\mathbb{R}^{n})$ and the molecular weak Hardy space $WH_{L,mol,q,\epsilon,M}^{p}(\mathbb{R}^{n})$. Recall that, in [8, Proposition 2.10], it was proved that both $L_{1}$ and $L_{2}$, respectively as in (1.1) and (1.2), satisfy Assumption (L)$_{4}$.
Let $L$ be as in (1.1) or (1.2). Applying the weak molecular characterization, we may now study the boundedness of the associated Riesz transform $\nabla^k L^{-1/2}$ and the fractional power $L^{-\alpha/(2k)}$ as follows.

**Theorem 2.23.** Let $k \in \mathbb{N}$ and $L$ be as in (1.1) or (1.2). For all $p \in (\frac{n}{n+k}, 1]$, the Riesz transform $\nabla^k L^{-1/2}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to $WH^p(\mathbb{R}^n)$.

**Proof.** Let $f \in WH^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. From Theorem 2.21, we deduce that there exist sequences $\{\lambda_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}^+} \subset \mathbb{C}$ and $\{m_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}^+}$ of $(p, \epsilon, M)_L$-molecules such that

$$f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}^+} \lambda_{i,j} m_{i,j}$$

in $L^2(\mathbb{R}^n)$ and

$$\sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}^+} |\lambda_{i,j}|^p \right)^{1/p} \sim \|f\|_{WH^p_L(\mathbb{R}^n)}.$$ 

By the proof of [10, Theorem 6.2], we know that, for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}^+$, $\nabla^k L^{-1/2}m_{i,j}$ is a classical $H^p(\mathbb{R}^n)$-molecule. From this and Remark 2.14, together with Theorem 2.21 in the case $L = -\Delta$, it follows that $\nabla^k L^{-1/2} f \in WH^p(\mathbb{R}^n)$ and

$$\left\| \nabla^k L^{-1/2} f \right\|_{WH^p(\mathbb{R}^n)} \lesssim \sup_{i \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}^+} |\lambda_{i,j}|^p \right)^{1/p} \sim \|f\|_{WH^p_L(\mathbb{R}^n)},$$

which, together with a density argument, then completes the proof of Theorem 2.23. □

**Theorem 2.24.** Let $k \in \mathbb{N}$ and $L$ be as in (1.1) or (1.2). For all $0 < p < r \leq 1$ and $\alpha = n\left(\frac{1}{p} - \frac{1}{r}\right)$, the fractional power $L^{-\alpha/(2k)}$ is bounded from $WH^p_L(\mathbb{R}^n)$ to $WH^r_L(\mathbb{R}^n)$.

**Proof.** Similar to the proof of [42, Theorems 7.2 and 7.3], we know that $L^{-\alpha/(2k)}$ maps each $(p, \epsilon, M)_L$-molecule to a $(r, q, \epsilon, M)_L$-molecule with $\alpha = n\left(\frac{1}{p} - \frac{1}{r}\right)$, up to a harmless constant. This, together with the fact that $L^{-\alpha/(2k)}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ (see [8, Lemma 3.10]) and Remark 2.22, then finishes the proof of Theorem 2.24. □

**Remark 2.25.** (i) The boundedness of the Riesz transform $\nabla^k(L^{-1/2})$, for $k \in \mathbb{N}$ and $L$ as in (1.1) or (1.2), on the Hardy space $H^p_L(\mathbb{R}^n)$ associated to $L$ has already been known. It was proved in [10, 11] that, for all $p \in (\frac{n}{n+k}, 1]$, $\nabla^k L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ and, at the endpoint case $p = \frac{n}{n+k}$, $\nabla^k L^{-1/2}$ is bounded from $H^{n/(n+k)}_L(\mathbb{R}^n)$ to the classical weak Hardy space $WH^{n/(n+k)}(\mathbb{R}^n)$.

(ii) The boundedness of the fractional power $L^{-\alpha/(2k)}$, for $k \in \mathbb{N}$ and $L$ as in (1.1) or (1.2), on the Hardy space $H^p_L(\mathbb{R}^n)$ associated to $L$ is also well known. It is proved that, for all $0 < p < r \leq 1$ and $\alpha = n\left(\frac{1}{p} - \frac{1}{r}\right)$, $L^{-\alpha/(2k)}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $H^r_L(\mathbb{R}^n)$ (see [8, 39, 42]).
3 The dual space of $WH_L^p(\mathbb{R}^n)$

In this section, we study the dual space of $WH_L^p(\mathbb{R}^n)$. It turns out that the dual of $WH_L^p(\mathbb{R}^n)$ is some weak Lipschitz space, which can be defined via the mean oscillation over some bounded open sets.

Before giving the definition of weak Lipschitz spaces, we first introduce a class of coverings of all bounded open sets, which is motivated by the subtle covering, appearing in the proof of Theorem 2.12, of the level sets of $A$-functionals, obtained via the Whitney-type covering lemma.

**Definition 3.1.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. A family $\{W_1, W_2\}$ of two open sets is said to be in the class $\mathcal{W}_\Omega$, if

(i) $\Omega \subset W_1 \cup W_2$;

(ii) for all $j_1 \in \{1, 2\}$, there exist an index set $\Lambda_{j_1} \subset \mathbb{N}$ and a sequence $\{B_{j_1,j_2}\}_{j_2 \in \Lambda_{j_1}}$ of balls such that

(ii)$_1$ for $j_1 \in \{1, 2\}$, $W_{j_1} = \bigcup_{j_2 \in \Lambda_{j_1}} B_{j_1,j_2}$,

(ii)$_2$ for $j_1 \in \{1, 2\}$, $\{B_{j_1,j_2}\}_{j_2 \in \Lambda_{j_1}}$ has bounded overlap,

(ii)$_3$ for all $j_1 \in \{1, 2\}$ and $\tilde{j}_2 \in \Lambda_{j_1}$, $r_{B_{j_1,\tilde{j}_2}} = \inf_{j_2 \in \Lambda_{j_1}} r_{B_{j_1,j_2}} =: r_1 > 0$, where the implicit positive constants are independent of $\Omega$ and $\{W_{j_1}\}_{j_1=1}$, and $r_2 := \inf_{j_2 \in \Lambda_{j_2}} r_{B_{j_2,\tilde{j}_2}} > 0$;

(iii) $|W_1| + |W_2| < 4|\Omega|$.

From the argument below (2.5), it follows that, for any bounded open set $\Omega$, $\mathcal{W}_\Omega \neq \emptyset$. Moreover, for a bounded open set $\Omega$, $\{W_{j_1}\}_{j_1=1}^2 = \{\cup_{j_2 \in \Lambda_{j_1}} B_{j_1,j_2}\}_{j_1=1}^2 \in \mathcal{W}_\Omega$ and $i \in \mathbb{N}$, we define the *annuli associated to $W_{j_1}$* by setting

\[
S_i(W_{j_1}) := \left( \bigcup_{j_2 \in \Lambda_{j_1}} 2^i B_{j_1,j_2} \right) \setminus \left( \bigcup_{j_2 \in \Lambda_{j_1}} 2^{i-1} B_{j_1,j_2} \right) \quad \text{and} \quad S_0(W_{j_1}) := W_{j_1}.
\]

For $j_1 \in \{1, 2\}$, since $\Omega$ is bounded, $\{B_{j_1,j_2}\}_{j_2 \in \Lambda_{j_1}}$ has bounded overlap and $r_{j_1} > 0$, we know that the cardinality of $\Lambda_{j_1}$ is finite. Thus, for any $j_1 \in \{1, 2\}$ and $i \in \mathbb{Z}_+$, there exists a positive constant $C_{j_1,i}$, depending on $j_1$ and $i$, such that

\[
\sum_{j_2 \in \Lambda_{j_1}} \lambda_{S_i(B_{j_1,j_2})} \leq C_{j_1,i} \lambda_{\cup_{j_2 \in \Lambda_{j_1}} S_i(B_{j_1,j_2})}.
\]

We define the *control exponent*, $N_{i,(W_{j_1})_{j_1=1}^2}$, associated to $\{W_{j_1}\}_{j_1=1}^2$ of order $i$ by setting that $N_{i,(W_{j_1})_{j_1=1}^2} \in \mathbb{N}$ satisfies that, for all $j_1 \in \{1, 2\}$,

\[
C_{j_1,i} 2^{-i N_{i,(W_{j_1})_{j_1=1}^2}} \leq \sum_{\ell=0}^{i+1} 2^{\ell n(\frac{i}{i+1} - \frac{1}{2})} \leq 1.
\]

Now, let $k \in \mathbb{N}$, $\alpha \in [0, \infty)$, $\epsilon \in (0, \infty)$, $M \in \mathbb{Z}_+$ satisfy $M > \frac{1}{2k}(\alpha + \frac{\epsilon}{2})$ and $L$ Assumptions $(L)_1$, $(L)_2$ and $(L)_3$. The space $\mathcal{M}_{\alpha,L}^n(\mathbb{R}^n)$ is defined to be the space of all functions $u$ in $L^2(\mathbb{R}^n)$ satisfying the following two conditions:
(i) for all \( \ell \in \{0, \ldots, M\} \), \( L^{-\ell}u \in L^2(\mathbb{R}^n) \);
(ii) let \( Q_0 \) be the unit cube with its center at the origin. Then

\[
\|u\|_{\mathcal{M}^{M,M}(\mathbb{R}^n)} := \sup_{j \in \mathbb{Z}_+} 2^{-j(\frac{n}{2} + \alpha)} \sum_{\ell=0}^{M} \|L^{-\ell}u\|_{L^2(S_j(Q_0))} < \infty.
\]

Let \( \mathcal{M}^{M,M}(\mathbb{R}^n) := \cap_{\varepsilon \in (0, \infty)} (\mathcal{M}^{M,M}(\mathbb{R}^n))^\varepsilon \). For all \( r \in (0, \infty) \), let

\[
\mathcal{A}_r := \left( I - e^{-tL^*} \right)^M,
\]

where \( L^* \) denotes the adjoint operator of \( L \) in \( L^2(\mathbb{R}^n) \). For any \( f \in \mathcal{M}^{M,M}(\mathbb{R}^n) \) and bounded open set \( \Omega \), let \( \mathcal{O}_{\mathcal{N}_\Omega} (f, \Omega) \) be the mean oscillation of \( f \) over \( \Omega \) defined by

\[
\mathcal{O}_{\mathcal{N}_\Omega} (f, \Omega) := \sup_{i \in \mathbb{Z}_+} \sup_{\{W_j\}_{j=1}^{2}} 2^{-iN_i(W_{j_1})_{j_1=1}^{2}} \left\{ \frac{1}{|\Omega|^{1+\frac{2\alpha}{n}}} \left[ \int_{S_i(W_1)} |A_{r_1}f(x)|^2 \, dx \right] ^{\frac{1}{2}} + \sum_{j_2 \in \Lambda_2} \int_{S_i(B_{2,j_2})} |A_{r_2}f(x)|^2 \, dx \right\} ^{\frac{1}{2}},
\]

where, for \( i \in \mathbb{Z}_+ \), \( j_1 \in \{1, 2\} \), \( r_{j_1} \), \( S_i(W_1) \) and \( A_{r_{j_1}} \) are, respectively, as in Definition 3.1, (3.1) and (3.4), and \( \mathcal{N}_\Omega := \{N_i(W_{j_1})_{j_1=1}^{2}, i \in \mathbb{Z}_+ \} \) with \( N_i(W_{j_1})_{j_1=1}^{2} \) as in (3.3).

By the \( k \)-Davies-Gaffney estimates, we know that the above integral is well defined.

For all \( \delta \in (0, \infty) \), define

\[
\omega(\delta\mathcal{N}_{\delta}) := \sup_{|\Omega| = \delta} \mathcal{O}_{\mathcal{N}_{\delta}} (f, \Omega),
\]

where \( \mathcal{N}_{\delta} := \{\mathcal{N}_{\Omega}\}_{|\Omega| = \delta} \). From its definition, it follows that \( \omega \) is a decreasing function on \( (0, \infty) \). Indeed, assume that \( f \in \mathcal{M}^{M,M}(\mathbb{R}^n) \), \( \delta \in (0, \infty) \) and \( \Omega \) is an open set satisfying \( |\Omega| = \delta \). For all \( \{W_{j_1}\}_{j_1=1}^{2} \in \mathcal{W}_\Omega \), we know, from Definition 3.1, that there exists a positive constant \( \bar{C} \in (1, \infty) \) such that \( |W_1| + |W_2| < \frac{4}{\bar{C}}|\Omega| \). This implies that, for all open sets \( \bar{\Omega} \subset \Omega \) satisfying that \( |\bar{\Omega}| =: \bar{\delta} \in [\frac{\delta}{\bar{C}}, \delta) \),

\[
|W_1| + |W_2| < 4|\bar{\Omega}|,
\]

which immediately shows that \( \{W_{j_1}\}_{j_1=1}^{2} \in \mathcal{W}_{\bar{\Omega}} \). Thus, from its definition, it follows that \( \mathcal{O}_{\mathcal{N}_{\bar{\Omega}}} (f, \Omega) \leq \mathcal{O}_{\mathcal{N}_{\delta}} (f, \bar{\Omega}) \) and hence \( \omega_{\mathcal{N}_{\bar{\Omega}}} (\bar{\delta}) \leq \omega_{\mathcal{N}_{\delta}} (\delta) \) for all \( \bar{\delta} \in [\frac{\delta}{\bar{C}}, \delta) \). This implies that \( \omega \) is decreasing.

Now, we introduce the notion of the weak Lipschitz space associated to \( L \).
Definition 3.2. Let $k \in \mathbb{N}$, $\alpha \in [0, \infty)$, $\epsilon \in (0, \infty)$, $M \in \mathbb{Z}_+$ satisfy $M > \frac{1}{2k}(\alpha + \frac{n}{2})$ and $L$ Assumptions (L)$_1$, (L)$_2$ and (L)$_3$. The weak Lipschitz space $W^\alpha_{L^\epsilon}(\mathbb{R}^n)$ is defined to be the space of all functions $f \in \mathcal{M}_{\alpha,L}(\mathbb{R}^n)$ such that

$$
\|f\|_{W^\alpha_{L^\epsilon}(\mathbb{R}^n)} := \int_0^\infty \frac{\omega_{\mathcal{D}}(\delta)}{\delta} d\delta < \infty,
$$

where $\omega_{\mathcal{D}}(\delta)$ for $\delta \in (0, \infty)$ is as in (3.6) and $\mathcal{D} := \{\mathcal{D}_\delta\}_{\delta \in (0, \infty)}$ is called the sequence of control exponents associated to $\{W_\Omega : \Omega$ is a bounded open set$\}$.

We also introduce the notion of the resolvent weak Lipschitz space. To this end, we need another class of open sets as follows, which is a slight variant of the class $W_\Omega$.

Definition 3.3. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. A family $\{W_1, W_2\}$ of open sets is said to be in the class $\tilde{W}_\Omega$, if

(i) $\Omega \subset W_1 \cup W_2$;

(ii) for all $j_1 \in \{1, 2\}$, there exist an index set $\Lambda_{j_1} \subset \mathbb{N}$ and a sequence $\{B_{j_1,j_2}\}_{j_2 \in \Lambda_{j_1}}$ of balls such that

(i) for $j_1 \in \{1, 2\}$, $W_{j_1} = \cup_{j_2 \in \Lambda_{j_1}} B_{j_1,j_2}$,

(ii) for $j_1 \in \{1, 2\}$, $\{2B_{j_1,j_2}\}_{j_2 \in \Lambda_{j_1}}$ have bounded overlap.

(iii) for all $j_1 \in \{1, 2\}$ and $j_2 \in \Lambda_{j_1}$, $r_{B_{j_1,j_2}} \sim \inf_{j_2 \in \Lambda_{j_1}} \{r_{B_{j_1,j_2}}\} := r_1 > 0$, where the implicit constants are independent of $\Omega$ and $\{W_{j_1}\}_{j_1=1}^2$, and $r_2 := \inf_{j_2 \in \Lambda_{j_1}} \{r_{B_{j_1,j_2}}\} > 0$;

(iii) $|W_1| + |W_2| < 2|\Omega|$.

It is easy to see that, for any bounded open set $\Omega$, $\tilde{W}_\Omega \subset W_\Omega$. For all $\alpha \in [0, \infty)$, $k \in \mathbb{N}$ and $M \in \mathbb{N}$ satisfying $M > \frac{1}{2k}(\alpha + \frac{n}{2})$ and $r \in (0, \infty)$, let

$$
B_r := \left[ I - \left( I + r^{2k}L^* \right)^{-1} \right]^M,
$$

with $L^*$ being the adjoint operator of $L$ in $L^2(\mathbb{R}^n)$, and assume that $f \in \mathcal{M}_{\alpha,L}(\mathbb{R}^n)$ and $\delta \in (0, \infty)$. Let

$$
\mathcal{O}_{\text{res},\tilde{W}_\Omega}(f, \Omega) := \sup_{i \in \mathbb{N}_+} \sup_{\{W_{j_1}\}_{j_1=1}} 2^{-i\mathcal{N}_i(\{W_{j_1}\}_{j_1=1})^2} \left\{ \frac{1}{|\Omega|^{1 + \frac{2k}{n}}} \int_{S_r(\Omega)} |B_{r_1} f(x)|^2 \, dx \right. \\
+ \sum_{j_2 \in \Lambda_{j_1}} \int_{S(\{B_{j_2,j_2}\})} |B_{r_2} f(x)|^2 \, dx \right\}^{\frac{1}{2}},
$$

where $\mathcal{N}_\Omega := \{\mathcal{N}_i(\{W_{j_1}\}_{j_1=1})\}_{i \in \mathbb{N}_+}$, $\{W_{j_1}\}_{j_1=1}$ in $\tilde{W}_\Omega$ with $\mathcal{N}_i(\{W_{j_1}\}_{j_1=1})$, as in (3.3) and, for $j_1 \in \{1, 2\}$, $r_{j_1}$ and $B_{r_{j_1}}$ are respectively as in Definition 3.3 and (3.7). Let also, for any $\delta \in (0, \infty)$,

$$
\omega_{\text{res},\tilde{W}_\Omega}(\delta) := \sup_{|\Omega| = \delta} \mathcal{O}_{\text{res},\tilde{W}_\Omega}(f, \Omega),
$$

where $\tilde{W}_\Omega := \{\mathcal{N}_\Omega\}_{|\Omega| = \delta}$. 

**Definition 3.4.** Let $k \in \mathbb{N}$, $\alpha \in [0, \infty)$, $\epsilon \in (0, \infty)$, $M \in \mathbb{Z}_+$ satisfy $M > \frac{1}{2k}(\alpha + \frac{\epsilon}{2})$ and $L$ Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. The resolvent weak Lipschitz space $WA_{L^r, \text{res}, \mathcal{N}}^\alpha(\mathbb{R}^n)$ is then defined to be the space of all functions $f \in \mathcal{M}_{\alpha,L}^M(\mathbb{R}^n)$ satisfying that

$$
\|f\|_{WA_{L^r, \text{res}, \mathcal{N}}^\alpha(\mathbb{R}^n)} := \int_0^\infty \frac{\omega_{\text{res}, \mathcal{N}}^\alpha(\delta)}{\delta} d\delta < \infty,
$$

where $\omega_{\text{res}, \mathcal{N}}^\alpha(\delta)$ for $\delta \in (0, \infty)$ is as in (3.8) and $\mathcal{N} := \{\mathcal{N}_\delta\}_{\delta \in (0, \infty)}$ is called the sequence of control exponents associated with $\{\mathcal{W}_\Omega : \Omega$ is a bounded open set of $\mathbb{R}^n\}$.

Observe that, for any bounded open set $\Omega$, $\mathcal{W}_\Omega \subset \mathcal{W}_\Omega$ and, for all $W_{j_1}$

$$
W_{j_1}^2 := \left\{ \begin{array}{l}
\sum_{j_2 \in \lambda_1} B_{j_1,j_2}^2 \in \mathcal{W}_\Omega
\end{array} \right. \quad \forall j_1 \in \{1, 2\},
$$

and $s_{j_1} \in [1, 2]$ for $j_1 \in \{1, 2\}$, $\{s_{j_1}W_{j_1}\}_{j_1=1}^2 := \left\{ \sum_{j_2 \in \lambda_1} s_{j_1} B_{j_1,j_2}^2 \right\}_{j_1=1}^2 \in \mathcal{W}_\Omega$. Let $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ be the sequences of control exponents as in Definitions 3.2 and 3.4, respectively, associated with the classes of sets $\{\mathcal{W}_\Omega : \Omega$ is a bounded open set of $\mathbb{R}^n\}$ and

$$
\mathcal{W}_\Omega : \Omega$ is a bounded open set of $\mathbb{R}^n$.
$$

We call $\mathcal{N}^{(1)} < \mathcal{N}^{(2)}$ if there exist $s_{j_1} \in [1, 2]$ for all $j_1 \in \{1, 2\}$ such that $N^{(1)}_{i, \{s_{j_1}W_{j_1}\}_{j_1=1}^2} \leq N^{(2)}_{i, \{s_{j_1}W_{j_1}\}_{j_1=1}^2}$ for all $i \in \mathbb{Z}_+$, open sets $\Omega$ and $\{W_{j_1}\}_{j_1=1}^2 \in \mathcal{W}_\Omega$, where $N^{(k)}_{i, \{W_{j_1}\}_{j_1=1}^2} \in \mathcal{N}^{(k)}$ for $k \in \{1, 2\}$.

We have the following relationship between the weak Lipschitz space and the resolvent weak Lipschitz space.

**Proposition 3.5.** Let $\alpha \in [0, \infty)$ and $L$ satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. Let $\mathcal{N}^{(1)}$ be the sequence of control exponents as in Definition 3.2. Then there exists a sequence $\mathcal{N}^{(2)}$ of control exponents as in Definition 3.4 such that $\mathcal{N}^{(1)} < \mathcal{N}^{(2)}$ and $WA_{L^r, \mathcal{N}^{(1)}}^\alpha(\mathbb{R}^n) \subset WA_{L^r, \mathcal{N}^{(2)}}^\alpha(\mathbb{R}^n)$.

Proof. We prove this proposition by showing that, for all $f \in WA_{L^r, \mathcal{N}^{(1)}}^\alpha(\mathbb{R}^n)$,

$$
\|f\|_{WA_{L^r, \mathcal{N}^{(2)}}^\alpha(\mathbb{R}^n)} \leq \|f\|_{WA_{L^r, \mathcal{N}^{(1)}}^\alpha(\mathbb{R}^n)}.
$$

By an argument similar to that used in the proof of [39, (3.42)], which corresponds to the case that $k = 1$ here, we see that

$$
f = (2k)^M \left[ r_{j_1}^{2k} \int_{r_{j_1}}^{r_{j_2}} s^{2k-1} (1 - e^{-s^{2k}L^*})^M ds + \sum_{m=1}^M \binom{M}{m} \right] f
$$

$$
\circ \left[ \sum_{l=0}^{M} e^{-i\ell_1^{2k}L^*} \right] f
$$
\[ = (2k)^M \left( \sum_{j_1}^{2^{1/(2k)} s^{2k-1}} \left( I - e^{-s^{2k} L^*} \right)^M ds \right) + (2k)^M \left( \sum_{j_1}^{M} \sum_{\ell_1, \ldots, \ell_M} \left( I - e^{-\ell_1 s^{2k} L^*} \right)^y \right) \]

\[ \times \left[ \sum_{j_1}^{2^{1/(2k)} s^{2k-1}} \left( I - e^{-\ell_1 s^{2k} L^*} \right)^M ds \right] \]

\[ \times \ldots \times \sum_{j_1}^{2^{1/(2k)} s^{2k-1}} \left( I - e^{-\ell_1 s^{2k} L^*} \right)^M ds \]

\[ =: A_{0,j_1} f + \sum_{\ell_1, \ldots, \ell_M} A_{\ell_1, \ldots, \ell_M, j_1} f + A_{M_1, j_1} f, \]

where \( {M \choose \ell} \) denotes the binomial coefficients.

For all \( i \in \mathbb{Z}_+ \) and \( \left\{ W_{j_1} \right\}_{j_1=1}^{2} \in \mathcal{W}_{11} \), let \( \mathcal{N}_{i_1, W_{j_1}} \) be as in (3.3), which is chosen later. We first estimate

\[ D_2 := 2^{-i_1 \mathcal{N}_{i_1, W_{j_1}}^{(2)}} \left\{ \int_{S_i(B_2, 2)} \left| B_{r_2} A_{\ell_0, \ldots, \ell_M, 2 f(x)} \right|^2 dx \right\}^{1/2}. \]

Let

\[ \tilde{A}_{\ell_0, \ldots, \ell_M, 2} := \left( r_2^{2k s^{2k-1}} \right)^{\ell_1 + \cdots + \ell_M} \left( r_2^{2k s^{2k-1}} \right)^{-1}. \]

From the functional calculus and the fact that \( \{ e^{-t L^*} \}_{t>0} \) satisfies the \( k \)-Davies-Gaffney estimates, we deduce that \( B_{r_2} \) satisfies the \( k \)-Davies-Gaffney estimates with \( t \sim r_2^{2k} \), namely, there exists a positive constant \( C_1 \) such that, for all closed sets \( E \) and \( F \) in \( \mathbb{R}^n \), \( t \in (0, \infty) \) and \( f \in L^2(\mathbb{R}^n) \) supported in \( E \),

\[ \| B_{r_2} f \|_{L^2(F)} \lesssim \exp \left\{ -C_1 \frac{[\text{dist}(E, F)]^{2k/(2k-1)}}{r_2^{2k/(2k-1)}} \| f \|_{L^2(E)} \right\}. \]

Moreover, from Lemmas 2.3 and 2.4, it follows that \( \tilde{A}_{\ell_0, \ldots, \ell_M, 2} \) also satisfies the \( k \)-Davies-Gaffney estimates with \( t \sim r_2^{2k} \). Similarly,

\[ \left( \sum_{j_1}^{2^{1/(2k)} s^{2k-1}} \left( I - e^{-\ell_1 s^{2k} L^*} \right)^M ds \right) \]
also satisfies the $k$-Davies-Gaffney estimates with $t \sim t^{2k}$. Moreover, let

$$F_{s,2} := \frac{1}{t^{2k}} \int_{r_2}^{2^{1/(2k)}r_2} s^{2k-1} \left( I - e^{-s^{2k}t^2} \right)^M f \, ds.$$ 

By Minkowski’s inequality, the $k$-Davies-Gaffney estimates and Hölder’s inequality, there exists a positive constant $\alpha_3 \in (0, \infty)$ such that

$$D_2 \lesssim 2^{-\frac{i\alpha_3}{2}}_{i,\{W_{j1}\}^2_{j1=1}} \left\{ \frac{1}{|\Omega|^{1+\frac{m}{n}}} \sum_{j_2 \in \Lambda_2} \left\| \left( \int_{S_i(B_{2,j_2})} |F_{s,2}(x)|^2 \, dx \right)^{1/2} \right\|^2 \right\}^{\frac{1}{2}}$$

$$\times \mathcal{A}_{\ell_0,\ldots,\ell_M,2} F_{s,2}(x) \, dx \right\}^{\frac{1}{2}}$$

$$\lesssim 2^{-\frac{i\alpha_3}{2}}_{i,\{W_{j1}\}^2_{j1=1}} \left\{ \frac{1}{|\Omega|^{1+\frac{m}{n}}} \sum_{j_2 \in \Lambda_2} \left\{ \sum_{l \in \mathbb{Z}_+} \left( \sum_{i=1}^{l+1} \left\| \int_{S_i(B_{2,j_2})} |F_{s,2}(x)|^2 \, dx \right\|^{1/2} \right) \right\}^{1/2}$$

$$+ \sum_{l \geq i+2 \text{ or } l \leq i-2} 2^{-\frac{i\alpha_3}{2}}_{i,\{W_{j1}\}^2_{j1=1}} \left\{ \sum_{l \geq i+2 \text{ or } l \leq i-2} 2^{-\frac{i\alpha_3}{2}}_{i,\{W_{j1}\}^2_{j1=1}} \left\| \int_{S_i(B_{2,j_2})} |F_{s,2}(x)|^2 \, dx \right\|^{1/2} \right\}^{1/2}$$

$$\times \left\{ \sum_{j_2 \in \Lambda_2} \left\| \int_{S_i(B_{2,j_2})} |F_{s,2}(x)|^2 \, dx \right\|^{1/2} \right\}^{1/2}$$

$$\lesssim \sup_{i \in \mathbb{Z}_+} 2^{-\frac{i\alpha_3}{2}}_{i,\{W_{j1}\}^2_{j1=1}} \left\{ \frac{1}{|\Omega|^{1+\frac{m}{n}}} \mathcal{C}_{2,i} \int_{\bigcup_{j_2 \in \Lambda_2} S_i(B_{2,j_2})} \left\| F_{s,2}(x) \, ds \right\|^{1/2} \right\}^{1/2},$$

where $\mathcal{C}_{2,i}$ is as in (3.2) with $j_1 = 2$.

From the fact that \( e^{-tL^*} \) is a strongly continuous semigroup on \( L^2(\mathbb{R}^n) \), we know that, for all \( f \in L^2(\mathbb{R}^n) \) and \( \Omega \subset \mathbb{R}^n \), \( e^{-tL^*}f \mid_{\Omega} \) is a continuous function on \( t \). Now, for \( j \in \{1, 2\} \), let \( s_j \in (r_{j1}, 2^\frac{1}{j_1}r_{j1}) \) and

$$\tilde{W}_{j1} := \bigcup_{j_2 \in \Lambda_1} s_j r_{j1}. $$
From the definitions of $\mathcal{W}_\Omega$ and $\tilde{\mathcal{W}}_\Omega$, it follows that \( \{\tilde{W}_{j_1}\}_{j_1=1}^2 \in \mathcal{W}_\Omega \). We now choose \( \mathcal{N}_{i,\{W_{j_1}\}_{j_1=1}^2} \) such that, for all \( i \in \mathbb{Z}_+ \) and \( j_1 \in \{1, 2\} \),

\[
C_{j_1,2}^{1/2} 2^{-i\mathcal{N}_{i,\{W_{j_1}\}_{j_1=1}^2}} \lesssim 2^{-i\mathcal{N}_{1,\{W_{j_1}\}_{j_1=1}^2}}.
\]

These, combined with Minkowski’s integral inequality, the mean value theorem for integrals and the choice of the control exponents, imply that there exists a positive constant \( r_{s_2} \in (r_2, 2\frac{r_1}{r_2}) \) such that

\[
D_2 \lesssim \sup_{i \in \mathbb{Z}_+} 2^{-i\mathcal{N}_{i,\{W_{j_1}\}_{j_1=1}^2}} \left\{ \frac{C_{2,i}}{\Omega^{1+\frac{2k}{n}}} \left[ \left( \int_{\bigcup_{j_2 \in \Lambda_2} S_i(B_{2,j_2})} \left| \frac{1}{r_2} \int_{r_2}^{r_2^{1/(2k)\frac{r_1}{r_2}}} s^{2k-1} \right| f(x) \ ds \right)^2 \ dx \right] \right\}^{\frac{1}{2}}
\]

\[
\times \left( I - e^{-s^{2k}L^*} \right)^M f(x) \ ds \right] \right\}^{\frac{1}{2}}
\]

\[
\lesssim \sup_{i \in \mathbb{Z}_+} 2^{-i\mathcal{N}_{i,\{W_{j_1}\}_{j_1=1}^2}} \left\{ \frac{C_{2,i}}{\Omega^{1+\frac{2k}{n}}} \int_{\bigcup_{j_2 \in \Lambda_2} S_i(B_{2,j_2})} \left| \left( I - e^{-r_2^{2k}L^*} \right)^M f(x) \right| \ dx \right\}^{\frac{1}{2}}
\]

\[
\sim \sup_{i \in \mathbb{Z}_+} \sup_{\{\tilde{W}_{j_1}\}_{j_1=1}^2 \in \mathcal{W}_\Omega} 2^{-i\mathcal{N}_{1,\{W_{j_1}\}_{j_1=1}^2}} \left\{ \frac{1}{\Omega^{1+\frac{2k}{n}}} \sum_{j_2 \in \Lambda_2} \int_{S_i(B_{2,j_2})} |A_{r_2}f(x)|^2 \ dx \right\}^{\frac{1}{2}}
\]

\[
\sim O_{\mathcal{N}_{\Omega}}(f, \Omega).
\]

To estimate

\[
D_1 := 2^{-i\mathcal{N}_{i,\{W_{j_1}\}_{j_1=1}^2}} \left\{ \frac{1}{\Omega^{1+\frac{2k}{n}}} \int_{S_1(W_1)} \left| B_{r_1} A_{t_0,\ldots,t_M,1} f(x) \right|^2 \ dx \right\}^{\frac{1}{2}}
\]

Similar to the estimates of \( D_2 \), we see that there exists a positive constant \( \alpha_4 \) such that

\[
D_1 \lesssim 2^{-i\mathcal{N}_{i,\{W_{j_1}\}_{j_1=1}^2}} \left\{ \frac{1}{\Omega^{1+\frac{2k}{n}}} \left[ \sum_{i \in \mathbb{Z}_+} \left( \int_{S_1(W_1)} \left[ (r_1^{-2k}(L^*)^{-1})^{\ell_1+\cdots+\ell_M} \right] B_{r_1} \right) \right] \right\}^{\frac{1}{2}}.
\]
Proposition 3.7. Let $\omega$ be as in Definition 3.2, and $L$ as in Definition 3.3. Then, for all bounded open sets $\Omega$, let $\mu$ be a positive measure on $\mathbb{R}^{n+1}$. For all bounded open sets $\Omega$, let

$$\mathcal{C}_\alpha(\mu, \Omega) := \sup_{\{W_{j_1}^2\}_{j_1=1}^{N_1} \in \mathcal{W}_\Omega} \left\{ \frac{1}{|\Omega|^{1+\frac{\alpha}{n}}} \sum_{j_1=1}^{2} \mu \left( \bigcup_{j_2 \in \Lambda_{j_1}} \tilde{B}_{j_1,j_2} \right) \right\}^{\frac{1}{2}}$$

where $\mathcal{W}_\Omega$ is as in Definition 3.3.

For all $\delta \in (0, \infty)$, let

$$\omega_{\mathcal{C}}(\delta) := \sup_{|\Omega|=\delta} \mathcal{C}_\alpha(\mu, \Omega).$$

Then $\mu$ is called a weak Carleson measure of order $\alpha$, denoted by $\mathcal{C}_\alpha$, if

$$\|\mu\|_{\mathcal{C}_\alpha} := \int_0^\infty \omega_{\mathcal{C}}(\delta) \frac{d\delta}{\delta} < \infty.$$

Proposition 3.7. Let $k \in \mathbb{N}$, $\alpha \in [0, \infty)$, $M \in \mathbb{N}$ satisfy that $M > \frac{\alpha k}{2k^2} (\frac{1}{p} - \frac{1}{2})$, $\tilde{N}$ be as in Definition 3.2, and $L$ satisfy Assumptions $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathcal{L})_3$. Then, for all $f \in W^{\alpha}_{L^r, \tilde{N}}(\mathbb{R}^n)$,

$$\mu_f(x, t) := \left( \int_{2k^2}^M e^{-t^{\alpha} L^*} f(x) \right)^2 \frac{dx dt}{t}, \quad \forall \ (x, t) \in \mathbb{R}^{n+1}.$$
is a weak Carleson measure of order $\alpha$. Moreover, there exists a positive constant $C$ such that, for all $f \in W_{L^p, N^q}^\alpha(\mathbb{R}^n)$,

$$
\|\mu f\|_{C_\alpha} \leq C\|f\|_{W_{L^p, N^q}^\alpha(\mathbb{R}^n)}.
$$

**Proof.** Let $f \in W_{L^p, N^q}^\alpha(\mathbb{R}^n)$. For all bounded open sets $\Omega$ and

$$
\{W_{j_1}\}_{j_1=1}^2 := \{\cup_{j_2 \in \Lambda_{j_1}} B_{j_1,j_2}\}_{j_1=1}^2 \in \overline{W}_\Omega,
$$

by the definition of weak Carleson measures, for all $j_1 \in \{1, 2\}$, we need to estimate

$$
\left\{ \frac{1}{|\Omega|^{1 + \frac{2\alpha}{n}}} \mu f \left( \bigcup_{j_2 \in \Lambda_{j_1}} \hat{B}_{j_1,j_2} \right) \right\}^{\frac{1}{2}}.
$$

Without loss of generality, we may only consider the case when $j_1 = 2$. By Minkowski’s inequality, we write

$$
\left\{ \frac{1}{|\Omega|^{1 + \frac{2\alpha}{n}}} \mu f \left( \bigcup_{j_2 \in \Lambda_{j_1}} \hat{B}_{j_1,j_2} \right) \right\}^{\frac{1}{2}} \lesssim \left\{ \frac{1}{|\Omega|^{1 + \frac{2\alpha}{n}}} \sum_{j_2 \in \Lambda_{2}} \int_{\hat{B}_{j_2}} \left| \left( t^{2k} L^* \right)^M e^{-t^{2k} L^*} B_{r_2} f(x) \right|^2 dx \frac{dt}{t} \right\}^{\frac{1}{2}}
$$

$$
+ \left\{ \frac{1}{|\Omega|^{1 + \frac{2\alpha}{n}}} \sum_{j_2 \in \Lambda_{2}} \int_{\hat{B}_{j_2}} \left| \left( t^{2k} L^* \right)^M e^{-t^{2k} L^*} (I - B_{r_2}) f(x) \right|^2 dx \frac{dt}{t} \right\}^{\frac{1}{2}} =: I_1 + I_2.
$$

To estimate $I_1$, by Minkowski’s inequality, Lemmas 2.3 and 2.4, and Hölder’s inequality, we conclude that there exists a positive constant $\alpha_5$ such that

$$
\left\{ \frac{1}{|\Omega|^{1 + \frac{2\alpha}{n}}} \sum_{j_2 \in \Lambda_{2}} \left[ \left( \int_{\hat{B}_{j_2}} \left| \left( t^{2k} L^* \right)^M e^{-t^{2k} L^*} \left( \sum_{i \in Z_n} \chi_{S_i(B_{2,j_2})} \right) B_{r_2} f(x) \right|^2 dx \frac{dt}{t} \right)^{\frac{1}{2}} \right]^{2} \right\}^{\frac{1}{2}}
$$

$$
\lesssim \left\{ \frac{1}{|\Omega|^{1 + \frac{2\alpha}{n}}} \sum_{j_2 \in \Lambda_{2}} \left[ \left( \sum_{i \in Z_n} \int_{\hat{B}_{j_2}} \left( t^{2k} L^* \right)^M e^{-t^{2k} L^*} \chi_{S_i(B_{2,j_2})} B_{r_2} f(x) \right)^2 dx \frac{dt}{t} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}
$$

$$
\lesssim \left\{ \frac{1}{|\Omega|^{1 + \frac{2\alpha}{n}}} \sum_{j_2 \in \Lambda_{2}} \left[ \sum_{i \in Z_n} \int_0^{r_{B_{2,j_2}}} \exp \left\{ -C_1 \frac{\text{dist}(S_i(B_{2,j_2}), B_{2,j_2})^{\frac{2k}{2^{k-1}}} \sqrt{t}}{t^{\frac{2k}{2^{k-1}}} \sqrt{t}} \right\} dt \right] \left( \int_{S_i(B_{2,j_2})} |B_{r_2} f(x)|^2 dx \right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}.
$$
\[ \left\{ \frac{1}{|\Omega|^{1 + \frac{2M}{n}}} \sum_{j_2 \in \Lambda_2} \left[ \sum_{i \in \mathbb{Z}^+} 2^{-iN_i(w_{j_1})^2_{j_1=1}} \|B_{r_2} f\|_{L^2(S_i(B_{r_2}, j_2))}^2 \right] \right\}^{\frac{1}{2}} \]

\[ \lesssim \sup_{i \in \mathbb{Z}^+} \sup_{w_{j_1} \in \mathcal{W}_0} 2^{-iN_i(w_{j_1})^2_{j_1=1}} \left\{ \frac{1}{|\Omega|^{1 + \frac{2M}{n}}} \sum_{j_2 \in \Lambda_2} \int S_i(B_{r_2}, j_2) |B_{r_2} f(x)|^2 \, dx \right\}^{\frac{1}{2}} \]

\[ \lesssim \mathcal{O}_{_{\text{res}N_{\Omega}}}(f, \Omega), \]

where \( N_i(w_{j_1})^2_{j_1=1} \) is as in (3.3). Thus, \( I_1 \lesssim \mathcal{O}_{_{\text{res}N_{\Omega}}}(f, \Omega) \).

The estimate of \( I_2 \) is similar to that for \( I_1 \). In this case, we need the following operator equality that, for all \( r \in (0, \infty) \),

\[ \left( I - \left[ I - (I + r^{2k} L^*)^{-1} \right]^M \right) \left[ I - \left( I + r^{2k} L^* \right)^{-1} \right]^{-M} = \sum_{\ell=1}^M \frac{M!}{(M-\ell)!\ell!} (r^{2k} L^*)^{-\ell}. \]

We omit the details.

By combining the estimates for \( I_1 \) and \( I_2 \), we conclude that \( \mathcal{C}_\alpha(\mu_f, \Omega) \lesssim \mathcal{O}_{_{\text{res}N_{\Omega}}}(f, \Omega) \), which, together with Proposition 3.5, shows that \( \mu_f \) is a weak Carleson measure of order \( \alpha \). This finishes the proof of Proposition 3.7. \( \square \)

We now turn to the proof of Theorem 3.6.

\textbf{Proof of Theorem 3.6.} We first prove that \( WA^{n(\frac{1}{p-1})}_{L^*N}(\mathbb{R}^n) \subset (WH^p_L(\mathbb{R}^n))^* \). For all \( g \in WA^{n(\frac{1}{p-1})}_{L^*N}(\mathbb{R}^n) \), since

\[ g \in \bigcap_{\epsilon \in (0, \infty)} (M_{\mathcal{M}^\epsilon M_n(\frac{1}{p-1}), L}^\infty(\mathbb{R}^n))^*, \]

from the fact that all \((p, \epsilon, M)_L\)-molecules \( m \) belong to \( \bigcup_{\epsilon \in (0, \infty)} \mathcal{M}_{\epsilon M_n(\frac{1}{p-1}), L}^\infty(\mathbb{R}^n) \), it follows that, for all \((p, \epsilon, M)_L\)-molecules \( m \), \( \langle g, m \rangle \) is well defined. Moreover, for all \( f \in WH^p_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), by Theorem 2.21 and Remark 2.10, we see that \( f \) has a weak molecular \((p, \epsilon, M)_L\)-representation \( \sum_{i \in \mathcal{I}, j \in \mathbb{Z}^+} \lambda_{i,j} m_{i,j} \), where \( \mathcal{I} \) is as in Remark 2.10. For \( N_1, N_2 \in \mathbb{N} \), let

\[ f_{N_1, N_2} := \sum_{|i| \leq N_1} \sum_{j \leq N_2} \lambda_{i,j} m_{i,j}. \]

It is easy to see that \( t^{2k} L e^{-t^{2k} L} f_{N_1, N_2} \in WT^p(\mathbb{R}^{n+1}_+) \), which, together with Remark 2.10 and Theorem 2.12, implies that there exist sequences \{\( \lambda_{i,1} \)\}_{i \in \mathcal{I}}, \{\lambda_{i,2,j_2} \}_{i \in \mathcal{I}, j_2 \in \Lambda_2} \subset \mathbb{C}, \{A_{i,1} \}_{i \in \mathcal{I}} \) of \( 2T^p(\mathbb{R}^{n+1}_+) \)-atomic blocks associated with \{\( W_{i,1} \)\}_{i \in \mathcal{I}} := \{\cup_{j_2 \in \Lambda_1} B_{i,1,j_2} \}_{i \in \mathcal{I}}, \) and \{\( A_{i,2,j_2} \)\}_{i \in \mathcal{I}, j_2 \in \Lambda_2} of \( T^p(\mathbb{R}^{n+1}_+) \)-atoms associated with \{\( B_{i,2,j_2} \)\}_{i \in \mathcal{I}, j_2 \in \Lambda_2} such that

\[ t^{2k} L e^{-t^{2k} L} f_{N_1, N_2} = \sum_{i \in \mathcal{I}} \bar{\lambda}_{i,1} A_{i,1} + \sum_{j_2 \in \Lambda_2} \lambda_{i,2,j_2} A_{i,2,j_2}. \]
pointwisely almost everywhere and in $T^2(\mathbb{R}^{n+1})$. For all $i \in I$, let $\Omega_i := \bigcup_{j_1=1}^{2} \bigcup_{j_2 \in A_{j_1}} B_{i,j_1,j_2}$, where $A_{j_1}$ and $B_{i,j_1,j_2}$ are as in Definition 2.11. It is easy to see that $\{W_{i,j_1}\}_{j_1=1}^{2} \in W_{\Omega_i}$. By comparing the quasi-norms between $W^*_{\Lambda_{L^p}^{p,n}}(\mathbb{R}^n)$ and $\Lambda^*_{L^p}^{n,\frac{1}{p}-1}(\mathbb{R}^n)$ and $\Lambda^*_{L^p}^{n,\frac{1}{p}-1}(\mathbb{R}^n)$, we see that $\Lambda^*_{L^p}^{n,\frac{1}{p}-1}(\mathbb{R}^n) \subset \Lambda^*_{L^p}^{n,\frac{1}{p}-1}(\mathbb{R}^n)$, where $\Lambda^*_{L^p}^{n,\frac{1}{p}-1}(\mathbb{R}^n)$ denotes the "strong" Lipschitz space associated to $L$ defined as in [39, (1.26)]. This, together with the Calderón reproducing formula [37, Lemma 8.4], Theorem 2.6, Remark 2.10, Theorem 2.12, Definition 2.11 and Hölder’s inequality, implies that

$$
\left| \langle g, f_{N_1,N_2} \rangle \right| \\
\lesssim \left| \int_{\mathbb{R}^{n+1}}^{} \left( t^{2k}L^* \right)^M e^{-t^{2k}L^*} g(x) t^{2k} L^* f_{N_1,N_2}(x) \frac{dx}{t} \right|
$$

$$
\lesssim \int_{\mathbb{R}^{n+1}} \left( t^{2k}L^* \right)^M e^{-t^{2k}L^*} g(x) \left( \sum_{i \in I} 2^i \left[ \sum_{j_2 \in A_{j_1}} \left| B_{i,1,j_2} \right| \right] \right)^\frac{p}{2} |A_{i,1}(x,t)|
$$

$$
+ 2^i \sum_{j_2 \in A_2} \left| B_{i,2,j_2} \right|^\frac{p}{2} |A_{i,2,j_2}(x,t)|
$$

$$
\lesssim \sum_{i \in I} 2^i \left\{ \sum_{j_2 \in A_{j_1}} \left| B_{i,1,j_2} \right| \right\}^{\frac{1}{2}} \left[ \int_{\mathbb{R}^{n+1}} ^{} \left| G_{k,L^*}(t,x) \right|^2 \frac{dxdt}{t} \right]^{\frac{1}{2}}
$$

$$
\times \left[ \int_{\mathbb{R}^{n+1}} ^{} \left| \bar{A}_{i,1}(x,t) \right|^2 \frac{dxdt}{t} \right]^{\frac{1}{2}}
$$

$$
+ \sum_{j_2 \in A_2} \left| B_{i,2,j_2} \right|^\frac{p}{2} \left[ \int_{\mathbb{R}^{n+1}} ^{} \left| G_{k,L^*}(t,x) \right|^2 \frac{dxdt}{t} \right]^{\frac{1}{2}} \left[ \int_{\mathbb{R}^{n+1}} ^{} \left| A_{i,2,j_2}(x,t) \right|^2 \frac{dxdt}{t} \right]^{\frac{1}{2}}
$$

$$
=: I,
$$

here and in the following formulae, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, we let

$$
G_{k,L^*}(t,x) := (t^{2k}L^*)^M e^{-t^{2k}L^*} g(x).
$$

From this, Proposition 3.7, Definition 2.11, Hölder’s inequality, and the fact that $\omega_C \leq \omega$ and the decreasing property of $\omega$, where $\omega$ and $\omega_C$ are, respectively, as in (3.7) and (3.9), we deduce that

$$
I \lesssim \sum_{i \in I} 2^i |\Omega_i|^{\frac{1}{p}-\frac{1}{2}} \left\{ \sum_{j_2 \in A_{j_1}} \left| B_{i,1,j_2} \right| \right\}^{\frac{1}{2}} \left[ \frac{1}{|\Omega_i|^{\frac{p}{2}-1}} \int_{\mathbb{R}^{n+1}} \left| G_{k,L^*}(t,x) \right|^2 \frac{dxdt}{t} \right]^{\frac{1}{2}}
$$
Let \( g \) be a \((p, \varepsilon, M)\)-molecule, we conclude that

\[
\|f\|_{W^0_H^p(\mathbb{R}^n)} \lesssim \|f\|_{W^0_H^p(\mathbb{R}^n)} \sum_{i \in \mathcal{I}} \omega^i(\Omega_i) \lesssim \|f\|_{W^0_H^p(\mathbb{R}^n)} \sum_{i \in \mathcal{I}} \omega^i_N(2^{-i}|\Omega_i|) \lesssim \|f\|_{W^0_H^p(\mathbb{R}^n)} \int_0^\infty \frac{\omega^i_N(\delta)}{\delta} d\delta
\]

\[
\sim \|f\|_{W^0_H^p(\mathbb{R}^n)} \|g\|_{W^{\frac{1}{p} - 1}_{L^*, N}^\Delta(\mathbb{R}^n)},
\]

where \( \mathcal{I} \) is as in Remark 2.10. This, combined with a density argument, implies that

\[
W^{\frac{1}{p} - 1}_{L^*, N}^\Delta(\mathbb{R}^n) \subset (W^*_{L^p}(\mathbb{R}^n))^*.
\]

Now, we prove the inclusion that \((W^*_{L^p}(\mathbb{R}^n))^* \subset W^{\frac{1}{p} - 1}_{L^*, N}^\Delta(\mathbb{R}^n)\).

Let \( g \in (W^*_{L^p}(\mathbb{R}^n))^* \). From the fact that \( \|m\|_{W^*_{L^p}(\mathbb{R}^n)} \lesssim 1 \), it follows that \( |g(m)| \lesssim 1 \) for all \((p, \varepsilon, M)\)-molecules \( m \). Combining with the fact that, for all \( \varepsilon \in (0, \infty), M \in \mathbb{N} \) and \( m_0 \in \mathcal{M}_\varepsilon^M(\mathbb{R}^n), m_0 \) is a \((p, \varepsilon, M)\)-molecule, we conclude that \( g \in \mathcal{M}_\varepsilon^M(\mathbb{R}^n) \).

Now, we prove that \( \|g\|_{W^{\frac{1}{p} - 1}_{L^*, N}^\Delta(\mathbb{R}^n)} \lesssim \|g\|_{(W^*_{L^p}(\mathbb{R}^n))^*} \). By definition, we first write

\[
\int_0^\infty \frac{\omega^i_N(\delta)}{\delta} d\delta \lesssim \sum_{t \in \mathbb{Z}} \omega^i_N(2^{-t})(2^{-t})
\]

\[
\lesssim \sup_{t \in \mathbb{Z}} \sup_{|\Omega| = 2^{-t}} \sum_{j_1 = 1}^2 \frac{1}{|\Omega|^\frac{2}{p} - 1} \left( \int_{S_t(W_{j_1})} |A_{r_1}g(x)|^2 \, dx + \sum_{j_2 \in \Lambda_2} \int_{S_t(B_{j_2})} |A_{r_2}g(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]
By similarity, we only consider the estimates for the second term of the above quantity. From (3.2), the decreasing property of $\omega_{N_i}$ and the dual norm of $L^2(\mathbb{R}^n)$, we deduce that there exists $\{\varphi_l\}_{l \in \mathbb{Z}} \subset L^2(\mathbb{R}^n)$ such that, for all $l \in \mathbb{Z}$, $\|\varphi_l\|_{L^2(\mathbb{R}^n)} \leq 1$ and

$$
\sum_{l \in \mathbb{Z}} \sup_{|\Omega| = 2^{-l} \in \mathcal{Z}} \sup_{n \in \mathcal{W}_N} \sup_{l_1=1}^{2^{-l} \in \mathcal{W}_N} 2^{-l N_i, (W_j)_{j_1=1}^{2}} \frac{1}{\| \Omega \|^2} \sum_{j_2 \in \Lambda_2} \int_{S_i(B_{2j_2})} |A_{r_2} g(x)|^2 \, dx
\leq \sum_{l \in \mathbb{Z}} \sup_{|\Omega| = 2^{-l} \in \mathcal{Z}} \sup_{n \in \mathcal{W}_N} \sup_{l_1=1}^{2^{-l} \in \mathcal{W}_N} 2^{-l N_i, (W_j)_{j_1=1}^{2}} \frac{C_{2,i}^{1/2}}{\| \Omega \|^2} \sup_{j_2 \in \Lambda_2} \left| \int_{S_i(B_{2j_2})} |A_{r_2} g(x)|^2 \, dx \right|^{1/2}
\leq \sum_{l \in \mathbb{Z}} \sup_{|\Omega| = 2^{-l} \in \mathcal{Z}} \sup_{n \in \mathcal{W}_N} \sup_{l_1=1}^{2^{-l} \in \mathcal{W}_N} 2^{-l N_i, (W_j)_{j_1=1}^{2}} \frac{C_{2,i}^{1/2}}{\| \Omega \|^2} \left[ A_{r_2} \chi_{j_2 \in \Lambda_2} S_i(B_{2j_2}) \varphi_l \right]_{L^2(\mathbb{R}^n)}
=: \sum_{l \in \mathbb{Z}} Q_l,
$$

where $A_{r_2} := (I - e^{-r_2^{2kM}})^M$. Now, for all $l \in \mathbb{Z}$, open sets $\Omega \subset \mathbb{R}^n$, $|\Omega| = 2^{-l}$, $\{W_j\}_{j_1=1}^{2} \in \mathcal{W}_{\Omega}$ and $i \in \mathcal{Z}_+$, let

$$
f_{l,i,(W_j)_{j_1=1}^{2}} := \frac{C_{2,i}^{1/2}}{2^{l N_i, (W_j)_{j_1=1}^{2}} - l \left( \frac{1}{p} - \frac{1}{2} \right)} A_{r_2} \chi_{j_2 \in \Lambda_2} S_i(B_{2j_2}) \varphi_l.
$$

In what follows, for the simplicity of the notation, we write $\mathcal{N}_{i,(W_j)_{j_1=1}^{2}}$ and $f_{l,i,(W_j)_{j_1=1}^{2}}$, respectively, simply by $\mathcal{N}$ and $f_l$. Let $\ell \in \mathcal{Z}_+$ and

$$
S_{\ell}(W_2) := \left( \bigcup_{j_2 \in \Lambda_2} 2^\ell B_{2j_2} \right) \setminus \left( \bigcup_{j_2 \in \Lambda_2} 2^{\ell-1} B_{2j_2} \right).
$$

For all $l \in \mathbb{Z}$, using Chebyshev's inequality and the definition of $\mathcal{W}_\Omega$, we conclude that, for all $\alpha \in (0, \infty)$,

$$
\alpha^p \left| \left\{ x \in \mathbb{R}^n : S_\ell(f_l)(x) > \alpha \right\} \right|$$

Thus, by the $k$-Davies-Gaffney estimates and (3.3), we know that there exists a positive constant $\alpha_6$ such that

\[
\sum_{\ell \in \mathbb{Z}_+} 2^{\ell n(1-\frac{\beta}{2})} C_{2,i}^{p/2} 2^{-iNp} \left\{ \left\| \sum_{\ell \in \mathbb{Z}_+} \int_{0}^{r_2} \left[ \int_{S_\ell(W_2)} \int_{|y-x|<t} \left| t^{2k} L e^{-t^{2k} L} A_{r_2}^* x_{\cup_{j \in \Lambda_2} S(B_{2,j})} \varphi(t) \right|^2 \frac{dy dt dx}{t^{n+1}} \right]^{\frac{1}{2}} \right\}^p 
\]

\[
\lesssim C_{2,i}^{p/2} 2^{-iNp} \sum_{\ell \in \mathbb{Z}_+} 2^{\ell n(1-\frac{\beta}{2})} \left\{ \left\| \sum_{\ell \in \mathbb{Z}_+} \int_{0}^{r_2} \left[ \int_{E_\ell} \left| t^{2k} L e^{-t^{2k} L} A_{r_2}^* x_{\cup_{j \in \Lambda_2} S(B_{2,j})} \varphi(t) \right| \frac{dy dt dx}{t^{n+1}} \right]^{\frac{1}{2}} \right\}^p 
\]

\[
\lesssim C_{2,i}^{p/2} 2^{-iNp} \left[ \sum_{\ell = 0}^{i-1} 2^{\ell n(1-\frac{\beta}{2})} \left\| \varphi(t) \right\|_{L^2(\cup_{j \in \Lambda_2} S(B_{2,j}))}^p \right] 
+ \sum_{\ell = i+2}^{\infty} 2^{\ell n(1-\frac{\beta}{2})} 2^{-\ell p \alpha_6} \left\| \varphi(t) \right\|_{L^2(\cup_{j \in \Lambda_2} S(B_{2,j}))}^p 
\lesssim 1. 
\]

For $i = 0$, using the bounded overlap of $\{B_{2,j}\}_{j \in \Lambda_2}$, we also obtain a similar estimate, the details being omitted.

Similarly, we can show that

\[
\sum_{\ell \in \mathbb{Z}_+} 2^{\ell n(1-\frac{\beta}{2})} C_{2,i}^{p/2} 2^{-iNp} \left\{ \left\| \sum_{\ell \in \mathbb{Z}_+} \int_{S_\ell(W_2)} \int_{|y-x|<t} \left| t^{2k} L e^{-t^{2k} L} A_{r_2}^* x_{\cup_{j \in \Lambda_2} S(B_{2,j})} \varphi(t) \right|^2 \frac{dy dt dx}{t^{n+1}} \right]^{\frac{1}{2}} \right\}^p 
\]

\[
\lesssim 1 
\]

and

\[
\sum_{\ell \in \mathbb{Z}_+} 2^{\ell n(1-\frac{\beta}{2})} C_{2,i}^{p/2} 2^{-iNp} \left\{ \left\| \sum_{\ell \in \mathbb{Z}_+} \int_{S_\ell(W_2)} \int_{|y-x|<t} \left| t^{2k} L e^{-t^{2k} L} A_{r_2}^* x_{\cup_{j \in \Lambda_2} S(B_{2,j})} \varphi(t) \right|^2 \frac{dy dt dx}{t^{n+1}} \right]^{\frac{1}{2}} \right\}^p 
\]
\[
\times \left| t^{2k} L e^{-t^{2k} L} A^*_{r_2 \chi_{j_2 \in A_2 S_i(B_{2,j_2})}} \varphi_l(y) \right|^2 \frac{dy \, dt \, dx}{t^{n+1}} \right|^\frac{1}{q} \right)\frac{1}{p} \lesssim 1.
\]

Combining these estimates, we see that
\[
\alpha^p \left\{ x \in \mathbb{R}^n : \| S_L(f_l)(x) > \alpha \} \right\} \lesssim 1
\]
and hence \( \sup_{l \in \mathbb{Z}} \| f_l \|_{W^p H^{n}({\mathbb{R}^n})} < \infty. \)

Now, let \( F \in \{ f_l \}_{l \in \mathbb{Z}} \) satisfy \( \| F \|_{W^p H^{n}({\mathbb{R}^n})} \sim \sup_{l \in \mathbb{Z}} \| f_l \|_{W^p H^{n}({\mathbb{R}^n})} \). Then, we see that \( \| F \|_{W^p H^{n}({\mathbb{R}^n})} < \infty. \) Moreover, from its definition, we deduce that there exists a positive constant \( \tilde{\alpha} \), independent of \( l \), such that
\[
(3.12) \quad \| F \|_{W^p H^{n}({\mathbb{R}^n})} \sim (\tilde{\alpha}^p \left\{ x \in \mathbb{R}^n : \| S_L(F)(x) > \tilde{\alpha} \} \right\})^{\frac{1}{p}}.
\]

Let \( l_0 \in \mathbb{Z} \) such that \( 2^{l_0} \leq \tilde{\alpha}^p < 2^{l_0+1} \). For all \( l \in \mathbb{Z} \), We first consider the case when \( l \leq l_0 \). By Chebyshev’s inequality, Minkowski’s inequality, Hölder’s inequality and choosing \( q \in (p, 2) \setminus \{1\} \), we conclude that
\[
\left\{ x \in \mathbb{R}^n : \| S_L(F)(x) > \tilde{\alpha} \} \right\} \leq \tilde{\alpha}^{-q} \sum_{l \in \mathbb{Z}_+} \left[ \int_{S_l(W_2)} |S_L(F)(x)|^2 \, dx \right]^{\frac{1}{2}} |S_l(W_2)|^{\frac{1}{2}} \times 2^{l_0(1-\frac{q}{p}) - l(1-\frac{q}{2})}
\]
\[
\leq \tilde{\alpha}^{-q} \sum_{l \in \mathbb{Z}_+} \left[ \int_{S_l(W_2)} |S_L(F)(x)|^2 \, dx \right]^{\frac{1}{2}} \times 2^{l_0(1-\frac{q}{p}) - l(1-\frac{q}{2})}
\]
\[
\leq \tilde{\alpha}^{-q} \left[ \int_{S_l(W_2)} \int_0^\infty \left| \int_{|y-x|<t} t^{2k} L e^{-t^{2k} L} A^*_{r_2 \chi_{j_2 \in A_2 S_i(B_{2,j_2})}} \varphi_l(y) \right|^2 \frac{dy \, dt \, dx}{t^{n+1}} \right]^{\frac{1}{q}}
\]
\[
=: H_1 + H_2.
\]

We first estimate \( H_2 \). To this end, we further write
\[
H_2 \lesssim \tilde{\alpha}^{-q} C^q_{2,i} 2^{-iNq} \sum_{l \in \mathbb{Z}_+} \left[ \int_{S_l(W_2)} \int_0^\infty \left| \int_{|y-x|<t} t^{2k} L e^{-t^{2k} L} A^*_{r_2 \chi_{j_2 \in A_2 S_i(B_{2,j_2})}} \varphi_l(y) \right|^2 \frac{dy \, dt \, dx}{t^{n+1}} \right]^{\frac{1}{q}}
\]
\[
\times \left[ \int_{S_l(W_2)} \int_0^r \int_{|y-x|<t} \left| \int_{r_2} \int_{|y-x|<t} \right| t^{2k} L e^{-t^{2k} L} A^*_{r_2 \chi_{j_2 \in A_2 S_i(B_{2,j_2})}} \varphi_l(y) \right|^2 \frac{dy \, dt \, dx}{t^{n+1}} \right]^{\frac{1}{q}}
\]
\[
+ \left[ \int_{S_l(W_2)} \int_0^{\frac{1}{2} \text{dist}(x, W_2)} \int_{|y-x|<t} \right. \left| \int_{r_2} \int_{|y-x|<t} \right| t^{2k} L e^{-t^{2k} L} A^*_{r_2 \chi_{j_2 \in A_2 S_i(B_{2,j_2})}} \varphi_l(y) \right|^2 \frac{dy \, dt \, dx}{t^{n+1}} \right]^{\frac{1}{q}} \]
=: J_1 + J_2 + J_3.

To estimate J_1, let E_\ell := \{x \in \mathbb{R}^n : \text{dist}(S_\ell(W_2), x) < r_2\} for all \ell \geq 2. From Fubini’s theorem, q > p and the assumption on \varphi_\ell, together with the functional calculus and (3.3) when \ell \in \{2, \ldots, i+1\} or Lemmas 2.3 and 2.4 when \ell \in \{i+2, \ldots\}, we deduce that there exists a positive constant \alpha_7 such that

\begin{equation}
J_1 \lesssim \alpha^{-q} C_{2,i}^{q/2} 2^{-iNq} \sum_{\ell=2}^{\infty} 2^{\ell n(1-\frac{p}{q})} \left\{ \int_{E_\ell} \int_{\mathbb{R}^3} |t|^{2kL} e^{-t^{2kL}} \right\}^q \\
\times \chi_{\cup_{\ell=2}^{i+1} \text{dist}(S_\ell(W_2))} \varphi_\ell(y) \left\{ \int_{E_\ell} \int_{\mathbb{R}^3} |t|^{2kL} e^{-t^{2kL}} \right\}^q \\
\lesssim \alpha^{-q} C_{2,i}^{q/2} 2^{-iNq} \sum_{\ell=2}^{i+1} 2^{\ell n(1-\frac{p}{q})} \left[ 2^{\ell(1/2 - \frac{p}{q})} \|\varphi_\ell\|_2 \right]^{q} \\
+ \alpha^{-q} C_{2,i}^{q/2} 2^{-iNq} \sum_{\ell=i+2}^{\infty} 2^{\ell n(1-\frac{p}{q})} \left[ 2^{\ell(1/2 - \frac{p}{q})} 2^{-\alpha_7 \varphi_\ell} \right]^{q} \\
\lesssim \alpha^{-q} 2^{\ell(1/2 - \frac{p}{q})} \|\varphi_\ell\|_2^{q} \lesssim \alpha^{-q} 2^{\ell(1/2 - \frac{p}{q})},
\end{equation}

here and hereafter, if i = 0, the term \sum_{\ell=2}^{i+1} disappears.

To estimate J_2, for all \ell \in \mathbb{Z}_+, let

\[ F_\ell := \bigcup_{x \in S_\ell(W_2)} \left\{ y \in \mathbb{R}^n : |y - x| < \frac{1}{4} \text{dist}(x, W_2) \right\}. \]

By the triangle inequality, we know that dist(F_\ell, W_2) \gtrsim 2^\ell r_2, which, together with some estimates similar to J_1, implies that there exists a positive constant \alpha_8 such that

\begin{equation}
J_2 \lesssim \alpha^{-q} C_{2,i}^{q/2} 2^{-iNq} \sum_{\ell=2}^{\infty} 2^{\ell n(1-\frac{p}{q})} \left\{ \int_{r_2}^{\infty} \int_{F_\ell} \left( \int_{\mathbb{R}^3} \right)^{M+1} e^{-t^{2kL}} \right\}^q \\
\times \left( 2kL \right)^{-M} \chi_{\cup_{\ell=2}^{i+1} \text{dist}(S_\ell(W_2))} \varphi_\ell(y) \left\{ \int_{r_2}^{\infty} \right\}^q \\
\lesssim \alpha^{-q} C_{2,i}^{q/2} 2^{-iNq} \sum_{\ell=2}^{i+1} 2^{\ell n(1-\frac{p}{q})} \left[ 2^{\ell(1/2 - \frac{p}{q})} \left( \int_{r_2}^{\infty} \right)^{1/4} \right] \|\varphi_\ell\|_2^{q} \\
+ \alpha^{-q} C_{2,i}^{q/2} 2^{-iNq} \sum_{\ell=i+2}^{\infty} 2^{\ell n(1-\frac{p}{q})} \left[ 2^{\ell(1/2 - \frac{p}{q})} \right]^{q} \\
\times \left\{ \int_{r_2}^{\infty} \left( \frac{r_2}{\text{dist}(F_\ell, S_\ell(W_2))} \right) \frac{dt}{4kM+1} \right\}^{q} \lesssim \frac{1}{\alpha^q} 2^{\ell(1/2 - \frac{p}{q})},
\end{equation}
To estimate $J_3$, since $\ell \geq 2$, then, for all $x \in S_\ell(W_2)$, we see that $\frac{1}{4}\text{dist}(x, W_2) > 2^{\ell-3}r_2$. Similar to the estimates of $J_1$ and $J_2$, we write

$$
(3.15) \quad J_3 \lesssim \tilde{\alpha}^{-q}C_{2,i}^{q/2}2^{-iNq} \sum_{\ell=2}^{\infty} 2^{\ell n(1-\frac{q}{p})} \left\{ 2^{\ell(1-\frac{1}{p})} \left[ \int_{2^{\ell-3}r_2}^{\infty} \int_{\mathbb{R}^n} \left| \left( t^{2k}L \right)^{M+1} e^{-t^{2k}L} \right|^{\frac{1}{2}r_2^{2kM}} \right]^{\frac{1}{2}} \right\}^q 
$$

$$
\times \left( \frac{r_2^{2k}L}{r_2^{2kM}} - M \right) A^*_{r_2} X_{j_2 \in \Lambda_2} S_i(\cup j_2 \in \Lambda_2 B_{2,j_2}) \varphi_t(y) \left[ \int_{2^{\ell-3}r_2}^{\infty} \int_{\mathbb{R}^n} dy dt \right]^{\frac{1}{2}} \left\| \varphi_t \right\|_{L^2(\cup j_2 \in \Lambda_2 S_i(B_{2,j_2}))}^q \lesssim \frac{1}{\alpha^q} 2^{q\ell(1-\frac{1}{p})},
$$

which is desired.

We now estimate $H_1$ by further writing

$$
H_1 = \tilde{\alpha}^{-q}C_{2,i}^{q/2}2^{-iNq} \sum_{\ell=0}^{1} 2^{\ell n(1-\frac{q}{p})} 2^{\ell(1-\frac{1}{p})} q \left\{ \left[ \int_{S_\ell(W_2)} \int_{0}^{r_2} \int_{|y-x|<t} \left| t^{2k}L e^{t^{2k}L} A^*_{r_2} X_{j_2 \in \Lambda_2} S_i(B_{2,j_2}) \varphi_t(y) \right|^{2} dy dt dx \right]^{\frac{1}{2}} \right\}^q + \left. \left[ \int_{S_\ell(W_2)} \int_{r_2}^{\infty} \int_{|y-x|<t} \left. \frac{dy dt dx}{t^{p+1}} \right]^{\frac{1}{2}} \right\}^q =: K_1 + K_2.
$$

The estimates for $K_1$ and $K_2$ are similar, respectively, to those for $J_1$ and $J_2$, the details being omitted here.

From these, $g \in (WH^p_L(\mathbb{R}^n))^*$, $(3.12), (3.13), (3.14), (3.15)$ and $q > p$, we deduce that

$$
(3.16) \quad \sum_{l=-\infty}^{l_0} \sum_{l=-\infty}^{l_0} \|g\|_{(WH^p_L(\mathbb{R}^n))^*} \left\{ \tilde{\alpha}^{-q} 2^{q\ell(1-\frac{1}{p})} \right\}^{1/p} \lesssim \|g\|_{(WH^p_L(\mathbb{R}^n))^*},
$$

which is desired.

Now, for $l > l_0$, recalling the definition of $f_l$, by choosing $q \in (0, p)$, similar to the case when $l \leq l_0$, we conclude that

$$
\{|x \in \mathbb{R}^n : S_l(f_l)(x) > \tilde{\alpha} \} \lesssim \frac{1}{\alpha^q} \sum_{\ell \in \mathbb{Z}_+} \left\{ \int_{S_\ell(W_2)} |S_l(f_l)(x)|^2 dx \right\}^{\frac{1}{2}} |S_\ell(W_2)|^{1-\frac{q}{2}} \lesssim \frac{1}{\alpha^q} \sum_{\ell \in \mathbb{Z}_+} C_{2,i}^{q/2}2^{-iNq} \left\{ 2^{\ell(1-\frac{1}{p})} \left[ \int_{S_\ell(W_2)} \int_{0}^{\infty} \int_{|y-x|<t} \left| t^{4k} L^2 e^{t^{2k}L} \right| \right] \right\}^{\frac{1}{2}}
$$
\begin{align*}
&\times A^\ast_{\mathcal{S}_i(B_2, j^2_s)} \varphi(y) \left[ \frac{2}{\mathcal{L}^{n+1}} \int \frac{dy \, dt \, dx}{\mathcal{L}^{n+1}} \right]^q 2^{\mathcal{L}^{n}(1-\frac{q}{2})} \\
&\lesssim \frac{1}{\alpha^q} \sum_{l \in \mathbb{Z}_+} 2^{\mathcal{L}^{n}(1-\frac{q}{2})} C_{2, l} q^{q/2} 2^{-i N e (\frac{1}{2} - \frac{1}{q}) q} \left\{ \int_{S_l(W_2)} \int_{0}^{r_2} \int_{|y-x|<t} \left| t^{4k} L^2 e^{-r_2^k L} \times \mathcal{S}_i(B_2, j^2_s) \varphi(y) \right| \frac{dy \, dt \, dx}{\mathcal{L}^{n+1}} \right\}^q 2^{\mathcal{L}^{n}(1-\frac{q}{2})} \\
&\times A^\ast_{\mathcal{S}_i(B_2, j^2_s)} \varphi(y) \left[ \frac{2}{\mathcal{L}^{n+1}} \int \frac{dy \, dt \, dx}{\mathcal{L}^{n+1}} \right]^q 2^{\mathcal{L}^{n}(1-\frac{q}{2})} \\
&+ \left[ \int_{S_l(W_2)} \int_{r_2}^{\infty} \int_{|y-x|<t} \left| \frac{dy \, dt \, dx}{\mathcal{L}^{n+1}} \right| \right]^q \leq \frac{1}{\alpha^q} 2^{q/2} \left( \frac{1}{p} - \frac{1}{q} \right),
\end{align*}

which, together with \( p > q \) and some estimates similar to those used in (3.10), (3.11) and (3.16), implies immediately that \( \sum_{l \in \mathbb{Z}_+} 2^{\mathcal{L}^{n}(1-\frac{q}{2})} C_{2, l} q^{q/2} 2^{-i N e (\frac{1}{2} - \frac{1}{q}) q} \left\{ \int_{S_l(W_2)} \int_{0}^{r_2} \int_{|y-x|<t} \left| t^{4k} L^2 e^{-r_2^k L} \times \mathcal{S}_i(B_2, j^2_s) \varphi(y) \right| \frac{dy \, dt \, dx}{\mathcal{L}^{n+1}} \right\}^q 2^{\mathcal{L}^{n}(1-\frac{q}{2})} \leq \frac{1}{\alpha^q} 2^{q/2} \left( \frac{1}{p} - \frac{1}{q} \right) \),

This, combined with (3.10), (3.11) and (3.16) again, shows that \( \|g\|_{\mathcal{W} \Lambda_{L^\infty, \mathcal{N}}(\mathbb{R}^n)} \lesssim \|g\|_{(WH_p^s)_{L^\infty, \mathcal{N}}} \), which completes the proof of Theorem 3.6. \( \square \)

**Remark 3.8.** By Theorem 3.6, we see that \( \mathcal{W} \Lambda_{L^\infty, \mathcal{N}}(\mathbb{R}^n) \) for \( \alpha \in [0, \infty) \) is independent of the choice of \( \mathcal{N} \) as in Definition 3.2. Thus, we can write \( \mathcal{W} \Lambda_{L^\infty, \mathcal{N}}(\mathbb{R}^n) \) simply by \( \mathcal{W} \Lambda_{L^\infty}(\mathbb{R}^n) \).

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