INTERSECTIONS OF $\ell^p$ SPACES IN THE BOREL HIERARCHY

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ABSTRACT. We show that if $Y$ is one of the spaces $\ell^q$, $c_0$, $\ell^\infty$ or $\bigcap_{p>b} \ell^p$, where $0 < q, b < \infty$, and the Fréchet space $\bigcap_{p>a} \ell^p$ is contained in $Y$ properly, then $\bigcap_{p>a} \ell^p$ first shows up in the Borel hierarchy of $Y$ at the multiplicative class of the third level. In particular $\bigcap_{p>a} \ell^p$ is neither an $F_\sigma$ nor a $G_\delta$ subset of $Y$. This answers a question by Nestoridis. This result provides a natural example of a set in the third level of the Borel hierarchy and with its help we also give some examples in the fourth level.

1. Introduction and Results

It is frequent in analysis to encounter $\subseteq$-chains of topological vector spaces $(X_i)_{i \in \mathcal{I}}$ and ask questions about their relation with respect to the structure that they carry. For example the questions whether $X_p$ in infinite dimensional closed subspace (spaceability), where $X$, the vector space of all real sequences, not receive notable attention; see for example [1, 4, 3, 2] and [6].

In this article we are concerned with the chain of $\ell^p$ spaces, $p \in (0, \infty)$, together with $c_0$ and $\ell^\infty$ under the topological aspect. As it was shown by Nestoridis the space $\ell^p$ is an $F_\sigma$ meager subset of $\ell^q$, where $0 < p < q < \infty$; in fact the same is true if instead of $\ell^q$ we have one of the following spaces $c_0$, $\ell^\infty$ or $\bigcap_{q>b} \ell^q$ for $b > p$, see [6, Proposition 1]. Consequently, if $Y$ is one of the latter spaces and $a \in [0, \infty)$ is such that $\bigcap_{p>a} \ell^p$ is contained in $Y$ properly, then the intersection $\bigcap_{p>a} \ell^p$ is contained in a meager $F_\sigma$ subset of $Y$; but it is left open in [6] (see the comments following the proof of Proposition 1) if the latter intersection remains $F_\sigma$ in $Y$.

We prove that if $Y$ is one of the spaces $\ell^q$, $c_0$, $\ell^\infty$ or $\bigcap_{p>b} \ell^p$ where $0 < q, b < \infty$, then the intersection $\bigcap_{p>a} \ell^p$ first shows up in the Borel hierarchy of $Y$ at the multiplicative class of the third level, where $0 \leq a < q$ if $Y = \ell^q$, $0 \leq a < b$ if $Y = \bigcap_{p>b} \ell^p$, and $0 \leq a < \infty$ if $Y = c_0$, $\ell^\infty$. In particular $\bigcap_{p>a} \ell^p$ is not an $F_\sigma$ or a $G_\delta$ subset of $Y$; this answers Nestoridis’ question. This result provides a natural example of a set in the third level of the Borel hierarchy and with its help we also give some examples in the fourth level.

We proceed with a brief review of the necessary notions. First we clarify that we include 0 in the set of natural numbers and consequently all our sequences (unless stated otherwise) have a 0-th term. The sequential space $\ell^p$, where $0 < p < \infty$ is the vector space of all real sequences $(x_n)_{n \in \mathbb{N}}$ for which $\sum_{n=0}^{\infty} |x_n|^p < \infty$. As it is

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well-known the spaces $\ell^p$, $p > 0$, are increasing on $p$ and in fact for all $0 < p < q$ and all sequences $(x_n)_{n \in \mathbb{N}} \in \ell^p$ we have $(\sum_{n=0}^{\infty} |x_n|^q)^{1/q} \leq (\sum_{n=0}^{\infty} |x_n|^p)^{1/p}$.

When $p \geq 1$ the space $\ell^p$ admits the norm

$$
\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p}, \quad x = (x_n)_{n \in \mathbb{N}} \in \ell^p.
$$

Then $(\ell^p, \| \cdot \|_p)$ is a separable Banach space, $1 \leq p < \infty$.

If $0 < p < 1$ we define the metric $d_p$ on $\ell^p$ by

$$
d_p(x, y) = \sum_{n=0}^{\infty} |x_n - y_n|^p, \quad x = (x_n)_{n \in \mathbb{N}}, \; y = (y_n)_{n \in \mathbb{N}} \in \ell^p.
$$

Then $(\ell^p, d_p)$ is a complete separable metric space.

We are also concerned with spaces of the form $\bigcap_{q>b} \ell^q$ where $0 \leq b < \infty$. First we fix once and for all sequences $(p_i^b)_{i \in \mathbb{N}}$, $b \geq 0$, of positive real numbers such that $p_i^0 = b$; if $b < 1$ we assume that $p_i^0 < 1$. We define the metric $d_{>b}$ on $\bigcap_{q>b} \ell^q$ depending on the value of $b$. If $b \geq 1$ we define

$$
d_{>b}(x, y) = \sum_{i=0}^{\infty} 2^{-(i+1)} \cdot \frac{\|x - y\|_{p_i^b}}{1 + \|x - y\|_{p_i^b}}, \quad x, y \in \ell^b,
$$

(notice that $p_i^b > 1$ so that $\| \cdot \|_{p_i^b}$ is defined).

If $b < 1$ we define

$$
d_{>b}(x, y) = \sum_{i=0}^{\infty} 2^{-(i+1)} \cdot \frac{d_{p_i^b}(x, y)}{1 + d_{p_i^b}(x, y)}, \quad x, y \in \ell^b,
$$

where $d_{p_i^b}$ is as above (notice that $p_i^b < 1$).

Then $(\bigcap_{q>b} \ell^q, d_{>b})$ is a complete separable metric space.

The sequence space $\ell^\infty$ is the vector space of all bounded real sequences and is equipped with the supremum norm, $\|x\|_\infty = \sup\{|x_n| \mid n \in \mathbb{N}\}$, $x = (x_n)_{n \in \mathbb{N}} \in \ell^\infty$. Then $(\ell^\infty, \| \cdot \|_\infty)$ is a non-separable Banach space.

The space $c_0$ consists of all real sequences that converge to 0 and with the (restriction of the) supremum norm $\| \cdot \|_\infty$ it is a separable Banach space.

Next we employ some tools from descriptive set theory. Given a metric space $X$ we denote by $\Sigma^0_n(X)$ the family of all open subsets of $X$ and by $\Pi^0_n(X)$ the family of all closed subsets of $X$. Recursively we define $\Sigma^0_{n+1}(X)$ to be the family of all countable unions of $\Pi^0_n(X)$ sets and $\Pi^0_{n+1}(X)$ to be their complements in $X$. For example $\Sigma^0_0(X)$ is the family of all $F_\sigma$ subsets of $X$, $\Pi^0_0(X)$ is the family of all $G_\delta$ subsets of $X$ and so on. Instead of $A \in \Sigma^0_n(X)$ we will say that $A$ is a $\Sigma^0_n$ subset of $X$, or simpler that $A$ is a $\Sigma^0_n$ set when $X$ is understood from the context. The classes $\Sigma^0_n$ and $\Pi^0_n$ are also known as the additive and the multiplicative classes of the $n$-th level in the Borel hierarchy. It is easy to see that the pre-image of a $\Sigma^0_n$ subset $B$ of $Y$ under a continuous function $f : X \to Y$ is a $\Sigma^0_m$ subset of $X$; obviously the same holds for $\Pi^0_n$.

It is well-known that $\Sigma^0_0(X) \cup \Pi^0_0(X) \subseteq \Sigma^0_{n+1}(X) \cap \Pi^0_{n+1}(X)$, i.e., every $\Sigma^0_0(X)$ (and consequently every $\Pi^0_0(X)$) is both $\Sigma^0_{n+1}(X)$ and $\Pi^0_{n+1}(X)$. If $X$ is an uncountable complete separable metric space then $\Sigma^0_0(X) \neq \Pi^0_0(X)$.
Recall that a topological space is Polish if it generated by a complete separable metric space. It is clear that the preceding definitions can be given in the context of Polish spaces and it is irrelevant which accompanying complete metric we choose.

In the sequel we will employ the Baire space $\mathbb{N}^\mathbb{N}$ with the product topology. This is a Polish space and a basis for its topology is given by the family of all sets of the form

$$N(s_0, \ldots, s_n) = \{ \alpha \in \mathbb{N}^\mathbb{N} \mid \alpha(i) = s_i \text{ for all } i = 0, \ldots, n \},$$

where $s_0, \ldots, s_n \in \mathbb{N}$. Evidently these sets are clopen and therefore $\mathbb{N}^\mathbb{N}$ has a basis consisting of clopen sets, i.e., it is a zero-dimensional Polish space. The Cantor space is $\{0, 1\}^{\mathbb{N}} \equiv 2^{\mathbb{N}}$, and is a closed subspace of the Baire space.

A continuous reduction of a set $A \subseteq Z$ to a set $B \subseteq X$ is a continuous function $f : Z \to X$ such that $A = f^{-1}[B]$. If a non-$\Sigma^0_\alpha$ set $A$ continuously reduces to a given set $B \subseteq X$, then $B$ cannot be a $\Sigma^0_\alpha$ subset $X$, since the class $\Sigma^0_\alpha$ is closed under continuous pre-images. This is a standard technique for showing that a given set $B$ is not $\Sigma^0_\alpha$: we start with a known non-$\Sigma^0_\alpha$ set $A$ and we show that it continuously reduces to $B$. In fact we usually show a slightly stronger property that is worth mentioning:

A set $P \subseteq X$ is $\Pi^0_n$-complete if it is a $\Pi^0_n$ subset of $X$ and every $\Pi^0_n(Z)$ set $A$, where $Z$ is a zero-dimensional Polish space, continuously reduces to $P$.

Analogously one defines the notion of $\Sigma^0_n$-completeness, and it is clear that $P \subseteq X$ is $\Pi^0_n$-complete exactly when $X \setminus P$ is $\Sigma^0_n$-complete.

It is easy to see that a $\Pi^0_n$-complete set $P \subseteq X$ cannot be $\Sigma^0_n(X)$: since $\Sigma^0_n(2^\mathbb{N}) \neq \Pi^0_n(2^\mathbb{N})$ we can find some $A \in \Pi^0_n(2^\mathbb{N}) \setminus \Sigma^0_n(2^\mathbb{N})$; the set $A$ continuously reduces to $P$, so if $P$ were $\Sigma^0_n$ then $A$ (being the continuous pre-image of a $\Sigma^0_n$ set) would be a $\Sigma^0_n$ subset of $2^\mathbb{N}$, a contradiction.

Moreover it is clear that if a $\Pi^0_n$-complete set $P$ continuously reduces to a $\Pi^0_n$ set $B$ then $B$ is $\Pi^0_n$-complete as well. We can now state our main result.

**Theorem 1.** For all $a, q$ with $0 \leq a < q < \infty$ the intersection $\bigcap_{p > a} \ell^p$ is a $\Pi^0_3$-complete subset of $\ell^q$. Moreover the continuous reductions can be chosen to take values in the closed unit ball of $\ell^q$.

It follows that $\bigcap_{p > a} \ell^p \neq \Sigma^0_3(\ell^q)$ and therefore $\bigcap_{p > a} \ell^p$ is neither an $F_\sigma$ nor a $G_\delta$ subset of $\ell^q$.

Using the preceding result we can see that for every $\delta > 0$ the continuous reductions in the latter can be chosen to take values in the closed $\delta$-ball of $\ell^q$ centered at 0. This is because $\bigcap_{p > a} \ell^p$ is a linear space and hence for every function $f : Z \to \ell^q$ that reduces some $Q$ to $\bigcap_{p > a} \ell^p$ we will have

$$z \in Q \iff f(z) \in \bigcap_{p > a} \ell^p \iff \delta \cdot f(z) \in \bigcap_{p > a} \ell^p,$$

for all $z \in Z$. In other words the function $\delta \cdot f$ remains a reduction of $Q$ to $\bigcap_{p > a} \ell^p$.

**Corollary 2.** Suppose that $0 \leq a < b < \infty$ and let $Y$ be one of the spaces $\bigcap_{p > b} \ell^q$, $c_0$ or $\ell^\infty$. Then the intersection $\bigcap_{p > a} \ell^p$ is a $\Pi^0_3$-complete subset of $Y$, and in particular $\bigcap_{p > a} \ell^p$ is neither an $F_\sigma$ nor a $G_\delta$ subset of $Y$.

**Proof.** From the the comments following the proof of Proposition 1 in [6] it follows that $\bigcap_{p > a} \ell^p$ is a $\Pi^0_3$ subset of $Y$.

For the continuous reductions first we apply Theorem 1 with $\ell^b$ in the place of $\ell^q$. Next we notice that the identity $id : \ell^b \to Y$ is continuous. When $Y = c_0, \ell^\infty$
corollary 3. For all $a, q \in [0, \infty)$ with $a < q$ the following are $\Sigma^0_q$- and $\Pi^0_q$-complete subsets of $\ell^q$:  

\[ A = \{(x_{m,n})_{m,n \in \mathbb{N}} \in \ell^q \mid \exists m \in \mathbb{N} \cup \{0\} \forall n \in \mathbb{N} \cup \{0\} \|^q \} \]  

\[ B = \{(x_{m,n})_{m,n \in \mathbb{N}} \in \ell^q \mid \forall m \in \mathbb{N} \cup \{0\} \exists n \in \mathbb{N} \cup \{0\} \|^q \} \],

where a double sequence is identified with a usual one using a diagonal enumeration as above.

The analogous statement holds if we replace $\ell^q$ with $\bigcap_{p > a} \ell^p$ for $b > a, c_0$ or $\ell^c$.

Proof. Since $A$ is the complement of $B$ it is enough to prove that the latter is a $\Pi^0_q$-complete subset of $\ell^q$. First, we show that $B$ is a $\Pi^0_q$ set.

For all $m$ we define $h_m : \ell^q \to \ell^q : (x_{k,n})_{k,n \in \mathbb{N}} \mapsto (x_{m,n})_{m,n \in \mathbb{N}}$; notice that, for all $q \geq 1$, \( h_m^{-1}(x_{k,n})_{k,n \in \mathbb{N}} = (x_{m,n})_{m,n \in \mathbb{N}} \|_{\ell^q} \) for $q \geq 1$, where $(x_{k,n})_{k,n}, (y_{k,n})_{k,n} \in \ell^p$. The similar assertion holds for $d_q$ if $q \in (0, 1)$. Therefore $h_m$ is well-defined and 1-Lipschitz.

We also define the set $B_m = \{(x_{k,n})_{k,n \in \mathbb{N}} \in \ell^q \mid (x_{m,n})_{m,n \in \mathbb{N}} \notin \bigcap_{p > a} \ell^p \}$ so that $B_m = h_m^{-1}(\ell^q \setminus \bigcap_{p > a} \ell^p)$ for $m \in \mathbb{N}$. The set $\ell^q \setminus \bigcap_{p > a} \ell^p$ is $\Sigma^0_q$ and hence $B_m$ is also $\Sigma^0_q$ as the pre-image of the former set under a continuous function. Hence $B = \bigcap_{m \in \mathbb{N}} B_m$ is a $\Pi^0_q$ subset of $\ell^q$.

Suppose now that $Z$ is a zero-dimensional Polish space and that $P \subseteq Z$ is $\Pi^0_q$. Write $P = \bigcap_{m \in \mathbb{N}} P_m$ where $P_m$ is a $\Sigma^0_q$ subset of $Z$. From Theorem 1 and its subsequent remarks there exists for each $m$ a continuous function $f_m : Z \to \ell^q$ such that $P_m = f_m^{-1}(\ell^q \setminus \bigcap_{p > a} \ell^p)$ and $\|f_m(z)\|^q \leq 2^{-m}$.

Define $f : Z \to (\ell^q)^N : f(z) = (f_m(z))_{m \in \mathbb{N}}$. We write $f(z)$ as a double sequence $(x_{m,n})_{m,n \in \mathbb{N}}$ so that $f_m(z) = (x_{m,n})_{m,n \in \mathbb{N}}$ for all $m$. From the fact that $\|f_m(z)\|^q \leq 2^{-m}$ it is evident that $f(z) \in \ell^q$. Moreover the function $f$ is continuous; this is because $\|f_m(z)\|^q \leq 2^{-m}$ for all $z$ and the continuity of each $f_m$. Finally for all
the function $H_Y$ we have
\[ z \in P \iff (\forall m)[z \in P_m] \iff (\forall m)[f_m(z) \notin \bigcap_{p > a} \ell^p] \]
\[ \iff (\forall m)[\langle x_{m,n} \rangle_{n \in \mathbb{N}} \notin \bigcap_{p > a} \ell^p] \iff f(z) \in B. \]

Now we assume that $Y$ is one of $\bigcap_{q > b} \ell^q$, $c_0$ or $\ell^\omega$. First we notice that for all $m$ the function $H_m : Y \to Y : (x_{k,n})_{k,n \in \mathbb{N}} \mapsto (x_{m,n})_{n \in \mathbb{N}}$ is well-defined and 1-Lipschitz: this is clear when $Y = c_0$, $\ell^\omega$ since $H_m$ maps a sequence to a subsequence; when $Y = \bigcap_{q > b} \ell^q$ the assertion follows as above with $\ell^q$ using also that the function $f(t) = t/(1 + t)$, $t > -1$, is strictly increasing. Then the set $C = \bigcap_{m \in \mathbb{N}} H_m^{-1}[Y \setminus \bigcap_{p > a} \ell^p]$ is a $\Pi^0_4$ subset of $Y$.

Regarding the continuous reductions to $C$ suppose that $P$ is a $\Pi^0_4$ subset of a zero-dimensional Polish space $Z$. We apply the first part of the proof with $b$ in the place of $q$ (if $Y = c_0$, $\ell^\omega$ we choose $b = q > a$) and we get a continuous function $f : Z \to \ell^b : z \mapsto \langle x_{m,n} \rangle_{m,n}$ such that
\[ z \in P \iff (\forall m)[\langle x_{m,n} \rangle_{n \in \mathbb{N}} \notin \bigcap_{p > a} \ell^p], \]
for all $z$. Now as in the proof of Corollary 2 the function $id : \ell^b \to Y$ is continuous and $id \circ f : Z \to Y$ is the required reduction of $P$ to $C$.

2. The proof Theorem 1

We know from the comments following the proof of Proposition 1 in [6] that $\bigcap_{p > a} \ell^p$ is a $\Pi^0_4$ subset of $\ell^q$. To carry out the proof it suffices to pick a $\Pi^0_4$-complete set $P_3 \subseteq 2^\mathbb{N}$ and show that it reduces to $\bigcap_{p > a} \ell^p$ via a continuous function $f : 2^\mathbb{N} \to \ell^q$ which takes values in the closed unit ball of $\ell^q$. There is in fact a canonical choice for such $P_3$.

In the sequel we establish some terminology to be used in the proof. A finite sequence on a set $X$ is a function on $\{i \in \mathbb{N} : i < n\}$ to $X$, where $n \in \mathbb{N}$. We allow $n = 0$ in which case we mean the empty sequence $\emptyset$. In general a finite sequence $\sigma$ will be denoted by $(\sigma(0), \ldots, \sigma(n - 1))$. The preceding $n$ is the length of $\sigma$ and is denoted by $\text{lh}(\sigma)$, so that $\sigma(i)$ is defined exactly when $i < \text{lh}(\sigma)$, and $\sigma = (\sigma(0), \ldots, \sigma(\text{lh}(\sigma) - 1))$.

By $X^{< \mathbb{N}}$ we mean the set of all finite sequences of $X$. Given $\sigma, \tau \in X^{< \mathbb{N}}$ we define the concatenation $\sigma * \tau$ of $\sigma$ and $\tau$ to be the finite sequence that is obtained if we put $\sigma$ and $\tau$ together (starting with the former),
\[ \sigma * \tau = (\sigma(0), \ldots, \sigma(\text{lh}(\sigma) - 1)), \tau(0), \ldots, \tau(\text{lh}(\tau) - 1)). \]

We say that $\sigma$ is an initial segment of $\tau$ or that $\tau$ extends $\sigma$ and write $\sigma \subseteq \tau$ if $\text{lh}(\sigma) \leq \text{lh}(\tau)$ and for all $i < \text{lh}(\sigma)$ we have $\sigma(i) = \tau(i)$. We will write $\sigma \sqsubseteq \tau$ when $\tau$ extends $\sigma$ properly.

We fix the bijective function $[\cdot] : \mathbb{N}^2 \to \mathbb{N}$ that “moves diagonally upwards”:
\[
\begin{align*}
0 &= [0, 0] & 2 &= [0, 1] & 5 &= [0, 2] & 9 &= [0, 3] & \cdots \\
1 &= [1, 0] & 4 &= [1, 1] & 8 &= [1, 2] & \cdots & \cdots \\
3 &= [2, 0] & 7 &= [2, 1] & \cdots & \cdots & \cdots & \cdots \\
6 &= [3, 0] & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{align*}
\]
Figure 1. The diagonal arrangement of a finite sequence $\sigma$ when $\text{lh}(\sigma) = 8$

Row 0: $\sigma([0,0]) \quad \sigma([0,1]) \quad \sigma([0,2])$
Row 1: $\sigma([1,0]) \quad \sigma([1,1])$
Row 2: $\sigma([2,0]) \quad \sigma([2,1])$
Row 3: $\sigma([3,0])$

The canonical $\Pi^0_3$-complete set that we will use in our proof is

$$P_3 = \{ \alpha \in 2^\omega \mid (\forall i)(\exists j_0)(\forall j \geq j_0)[\alpha([i,j]) = 0] \}.$$  

(This is up to homeomorphism the same set as the $P_3$ in [5] p. 179, where it is proved that the latter set is $\Pi^0_3$-complete.)

Using the preceding pairing function we can arrange diagonally in an array every finite sequence $\sigma$ on a set $X$, see for example Figure 1.

The depth $\text{dpth}(\sigma)$ of a non-empty finite sequence $\sigma$ is the row with the largest number that is reached by $\sigma$ (we enumerate the rows starting with 0), i.e.,

$$d(\sigma) = \max \{ i \mid (\exists k)[\sigma[i,k] \neq \emptyset] \}.$$  

According to Figure 1 $\text{dpth}(\sigma) = 3$ when $\text{lh}(\sigma) = 8$. For technical reasons we define $\text{dpth}(\emptyset) = -1$.

The level $\text{lvl}(\sigma)$ of a non-empty finite sequence $\sigma$ is the row where $\sigma$ obtains its last value, i.e.,

$$\text{lvl}(\sigma) = \{ \text{the unique } i \text{ for which } [i,k] \text{ is the greatest element} \}$$

in the domain of $\sigma$ for some $k$.

According to Figure 1 $\text{lvl}(\sigma) = 2$ when $\text{lh}(\sigma) = 8$.

The following properties regarding the depth and the level of a non-empty sequence are easy to see:

$$\text{lvl}(\sigma) \leq \text{dpth}(\sigma)$$
$$\sigma \subseteq \sigma' \implies \text{dpth}(\sigma) \leq \text{dpth}(\sigma')$$
$$\text{dpth}(\sigma * (s)) \leq \text{dpth}(\sigma) + 1$$
$$\text{lvl}(\sigma) = 0 \implies \text{lvl}(\sigma * (s)) = \text{dpth}(\sigma * (s)) = \text{dpth}(\sigma) + 1$$
$$\text{lvl}(\sigma) > 0 \implies \text{lvl}(\sigma * (s)) = \text{lvl}(\sigma) - 1 \text{ and } \text{dpth}(\sigma * (s)) = \text{dpth}(\sigma)$$

where $\sigma \in X^{<\omega}$ is non-empty and $s \in \omega$.

For reasons of exposition we will make a slight abuse of the notation and denote $\sum_{n=0}^{\omega} \|x_n\|^p$ by $\|x\|^p$ where $p > 0$ and $x = (x_n)_{n \in \omega} \in \ell^p$. Of course $\| \cdot \|_p$ is a norm only when $p \geq 1$ in which case $\|x\|^p$ is the $p$-th power of $\|x\|_p$; if $p < 1$ the expression $\|x\|^p$ is just another name for $d_p(x,0) = \sum_{n=0}^{\omega} |x_n|^p$.

The notions of depth and level will be utilized for $X = \{0,1\}$, but on the other hand we will also be dealing with finite sequences of real numbers. We will regularly identify a finite sequence $u \in \mathbb{R}^{<\omega}$ with the infinite one $u \ast \vec{0} = u \ast (0,0,\ldots) \in c_{00}$, so that when we write $\|u - v\|^p$ for $u, v \in \mathbb{R}^{<\omega}$ we mean $\|u \ast \vec{0} - v \ast \vec{0}\|^p$. It is then
clear that
\[ \|u\|_p^p = \sum_{n \in \text{lh}(u)} |u(n)|^p. \]

Notice that
\[ \|u \ast v - u\|_p^p = \|v\|_p^p. \]

**Claim:** Suppose that \( q > p_0 > p_1 > \cdots > p_k > p_{k+1} > 0, r_0, \ldots, r_k, M, \varepsilon > 0 \) and that \( u \in \mathbb{R}^{\mathbb{N}} \) is such that \( \|u\|_{p_i} < r_i \) for all \( i = 0, 1, \ldots, k \). Then there exists a non-empty finite sequence \( v \in [0, \infty)^{\mathbb{N}} \) such that
\[ \|v\|^q_q < \varepsilon, \]
\[ \|u \ast v\|_{p_i} < r_i, \quad i = 0, 1, \ldots, k \]
\[ \|u \ast v\|_{p_{k+1}} > M. \]

**Proof of the Claim.**

Put \( \delta = \min\{r_i - \|u\|_{p_i} \mid i = 0, \ldots, k\} > 0 \) and consider a sequence \((x_n)_{n \in \mathbb{N}}\) of non-negative real numbers which is a member of \( \ell^p \setminus \ell^{p_{k+1}} \).

Then there is \( n_0 \) such that \( \sum_{n=n_0}^{\infty} x_{n}^{p_{k}} < \min\{\delta, \varepsilon\} \) and \( x_n < 1 \) for all \( n \geq n_0 \). It follows that
\[ \sum_{n=n_0}^{\infty} x_{n}^{p_{i}} \leq \sum_{n=n_0}^{\infty} x_{n}^{p_{k}} < \delta \]
for all \( i = 0, \ldots, k \). Further
\[ \sum_{n=n_0}^{\infty} x_{n}^{q} \leq \sum_{n=n_0}^{\infty} x_{n}^{p_{k}} < \varepsilon. \]

Since \((x_n)_{n \in \mathbb{N}} \notin \ell^{p_{k+1}}\) we have \( \sum_{n=n_0}^{\infty} x_{n}^{p_{k+1}} = \infty \). Therefore there is some \( n_1 \geq n_0 \) such that \( \sum_{n=n_0}^{n_1} x_{n}^{p_{k+1}} > M. \)

Take \( v = (x_{n_0}, \ldots, x_{n_1}). \) Clearly \( \|v\|^q_q = \sum_{n=n_0}^{n_1} x_{n}^{q} \leq \sum_{n=n_0}^{\infty} x_{n}^{q} < \varepsilon. \) For all \( i = 0, \ldots, k \) we have
\[ \|u \ast v\|_{p_i} = \sum_{n \in \text{lh}(u)} |u(n)|^{p_i} + \sum_{n=n_0}^{n_1} x_{n}^{p_i} \]
\[ \leq \|u\|_{p_i} + \sum_{n=n_0}^{\infty} x_{n}^{p_i} \]
\[ < \|u\|_{p_i} + \delta_0 \]
\[ \leq r_i. \]

Moreover
\[ \|u \ast v\|_{p_{k+1}} \geq \sum_{n=n_0}^{n_1} x_{n}^{p_{k+1}} > M. \]

This concludes the proof of the claim.

In the sequel we fix a sequence of real numbers \((p_i)_{i \in \mathbb{N}}\) with \( p_i \downarrow a \), so that \( \bigcap_{p > a} \ell^p = \bigcap_{i \in \mathbb{N}} \ell^{p_i} \), and with \( p_0 < q \). For example \((p_i)_{i \in \mathbb{N}}\) could be a shift of \((p_i^0)_{i \in \mathbb{N}}\).
The main construction.} We show that for every non-empty \( \sigma \in 2^{< \mathbb{N}} \) there is a non-empty \( \varphi(\sigma) \in [0, +\infty)^{< \mathbb{N}} \) and natural numbers \( M_i(\sigma), 0 \leq i \leq \text{dpth}(\sigma) \) with the following properties:

1. \( \sigma' \subseteq \sigma \implies \varphi(\sigma') \subseteq \varphi(\sigma) \)
2. \( \|\varphi(\sigma * (s)) - \varphi(\sigma)\|_q < 2^{-(\text{lh}(\sigma) + 1)} \) and \( \|\varphi((s))\|_q < 2^{-1}, s = 0, 1 \)
3. \( \|\varphi(\sigma)\|_{p_i} < M_i(\sigma), i = 0, \ldots, \text{dpth}(\sigma) \)
4. \( M_i(\sigma * (0)) = M_i(\sigma), i = 0, \ldots, \text{dpth}(\sigma) \)
5. \( M_i(\sigma * (1)) = M_i(\sigma), i = 0, \ldots, \text{lvl}(\sigma * (1)) - 1 \) if \( \text{lvl}(\sigma * (1)) > 0 \)
6. \( \|\varphi(\sigma * (1))\|_{p_i} > \text{lh}(\sigma) + 1, \) where \( i = \text{lvl}(\sigma * (1)) \)

As to (4) notice that \( i \leq \text{dpth}(\sigma) \leq \text{dpth}(\sigma * (0)) \), so that \( M_i(\sigma * (0)) \) is defined. Similarly in (5) we have \( \text{lvl}(\sigma * (1)) - 1 \leq \text{dpth}(\sigma * (1)) - 1 \leq \text{dpth}(\sigma) + 1 - 1 = \text{dpth}(\sigma) \), so that \( M_i(\sigma) \) and \( M_i(\sigma * (1)) \) are both defined.

The idea in order to show the above assertions is roughly as follows. When we extend \( \sigma \) by 0 we also extend \( \varphi(\sigma) \) by 0; the norms remain the same and hence the \( M_i \)'s can remain the same for \( i \leq \text{dpth}(\sigma) \). We might need to add one more \( M_i \), namely \( M_{\text{dpth}(\sigma) + 1}(\sigma * (0)) \) if the depth of \( \sigma * (0) \) increases by 1 from \( \text{dpth}(\sigma) \), but this poses no problems. The interesting case is when we extend by 1. Then we extend \( \varphi(\sigma) \) so that for all \( i < \text{lvl}(\sigma * (1)) \) the \( p_i \)-norm remains below \( M_i(\sigma) \) and hence we can take \( M_i(\sigma * (1)) \) to be the same as \( M_i(\sigma) \); on the other hand we make a substantial increase on the \( p_i \)-norm for \( i = \text{lvl}(\sigma * (1)) \). This is possible from the preceding Claim. The \( M_i \)'s for \( i = \text{lvl}(\sigma * (1)), \ldots, \text{dpth}(\sigma * (1)) \) are easily arranged.

Formally we define functions

\[
\varphi : 2^{< \mathbb{N}} \rightarrow [0, +\infty)^{< \mathbb{N}} \quad \text{and} \quad \psi : 2^{< \mathbb{N}} \rightarrow \mathbb{N}^{< \mathbb{N}} : \psi(\sigma) = (M_0(\sigma), \ldots, M_{\text{dpth}(\sigma)}(\sigma))
\]

which satisfy the required properties. The definition is done by recursion on \( \text{lh}(\sigma) \) starting with \( \text{lh}(\sigma) = 0 \).

Put \( \varphi(\varnothing) = \psi(\varnothing) = \varnothing \). Assume that for some \( n \in \mathbb{N} \) we have the following:

\( D^* \): \( \varphi(\sigma), \psi(\sigma) \) are defined and \( \text{lh}(\psi(\sigma)) = \text{dpth}(\sigma) + 1 \) for all \( \sigma \) with length at most \( n \),

\( 1^* \): property (1) holds for all \( \sigma, \sigma' \) with length at most \( n \),

\( 2^* \): the first part of (2) holds for all \( \sigma \) with \( \text{lh}(\sigma * (s)) \leq n \) and if \( n = 1 \) the second part of (2) holds as well,

\( 3^* \): property (3) holds for all \( \sigma \neq \varnothing \) with length at most \( n \),

\( 4^* \): property (4) holds for all non-empty \( \sigma \) with \( \text{lh}(\sigma * (0)) \leq n \),

\( 5^* \): property (5) holds for all non-empty \( \sigma \) with \( \text{lh}(\sigma * (1)) \leq n \), and

\( 6^* \): property (6) holds for all possibly empty \( \sigma \) with \( \text{lh}(\sigma * (1)) \leq n \).

We show now that the properties \( D^*, (1^*)-(6^*) \) hold for \( n + 1 \). First we define \( \varphi(\tau) \) and \( \psi(\tau) \) when \( \text{lh}(\tau) = n + 1 \). There are two cases: a) \( \tau = \sigma * (0) \) and b) \( \tau = \sigma * (1) \) for some \( \sigma \) with \( \text{lh}(\sigma) = n \).

In the case of a) we define \( \varphi(\tau) = \varphi(\sigma * (0)) = \varphi(\sigma) * (0) \).

As for \( \psi(\sigma) \) suppose first that \( \sigma \neq \varnothing \) so that \( \text{dpth}(\sigma) \geq 0 \). We define \( M_i(\tau) = M_i(\sigma * (0)) = M_i(\sigma) \) for \( i = 0, \ldots, \text{dpth}(\sigma) \). If \( \text{dpth}(\tau) = \text{dpth}(\sigma) \) this completes the definition. Else \( \text{dpth}(\tau) = \text{dpth}(\sigma) + 1 \) and we define further \( M_{\text{dpth}(\tau)}(\tau) \) to be the least natural greater than \( \|\varphi(\tau)\|_{p_{\text{dpth}(\tau)}} \). If \( \sigma = \varnothing \) then \( \text{dpth}(\tau) = \text{dpth}((0)) = 0 \) and we define \( M_0(\tau) = 1 \).
In the case of b) we apply the Claim from above. Assume first that $\text{lvl}(\tau) = \text{lvl}(\sigma \ast (1)) > 0$, so in particular $\sigma \neq \emptyset$. Put $k = \text{lvl}(\tau) - 1 = \text{lvl}(\sigma \ast (1)) - 1 \leq \text{dpth}(\sigma)$. From the induction hypothesis $\|\varphi(\sigma)\|_{P_i}^p < M_i(\sigma)$ for all $i = 0, \ldots, k$. So from the preceding claim there exists some $v \in [0, \infty)^{<N}$ such that $\|v\|_q^q < 2^{-(\text{lh}(\sigma) + 1)}$, $\|\varphi(\sigma) \ast v\|_{P_i}^p < M_i(\sigma)$ for all $i = 0, \ldots, k$ and $\|\varphi(\sigma) \ast v\|_{P_{k+1}}^p > \text{lh}(\sigma) + 1$. We define
\[
\varphi(\tau) = \varphi(\sigma \ast (1)) = \varphi(\sigma) \ast v \quad \text{and} \quad M_i(\tau) = M_i(\sigma \ast (1)) = M_i(\sigma), \quad i = 0, \ldots, k.
\]
We define further $M_i(\tau)$ to be least natural greater than $\|\varphi(\sigma) \ast v\|_{P_i}^p$ for $i = \text{lvl}(\tau), \ldots, \text{dpth}(\tau)$. The remaining sub-case is when $\text{lvl}(\tau) = \text{lvl}(\sigma \ast (1)) = 0$. (This includes the case $\sigma = \emptyset$.) According to the Claim above there exists some $v \in [0, \infty)^{<N}$ such that $\|v\|_q^q < 2^{-(\text{lh}(\sigma) + 1)}$ and $\|\varphi(\sigma) \ast v\|_{P_0}^p > \text{lh}(\sigma) + 1$. We define
\[
\varphi(\tau) = \varphi(\sigma \ast (1)) = \varphi(\sigma) \ast v
\]
and $M_i(\tau)$ to be the least natural greater than $\|\varphi(\sigma) \ast v\|_{P_i}^p$ for $i = 0, \ldots, \text{dpth}(\tau)$. This settles $(D^*)$ for $n + 1$.

It is not hard to verify that the properties $(1^*)$-$(6^*)$ hold for $n + 1$ and the inductive step is done.

The properties $(1)$-$(6)$ are immediate from $(1^*)$-$(6^*)$ for sufficiently large $n$.

The definition of the reduction. We take the function $\varphi : 2^{<N} \to [0, +\infty)^{<N}$ and the naturals $M_i(\sigma)$, $0 \leq i \leq \text{dpth}(\sigma)$, $\sigma \in 2^{<N}$, as above. From (1) the sequences $\varphi((\alpha(0), \ldots, \alpha(k)))$ for $k \in \mathbb{N}$ are compatible and their union forms an infinite sequence of real numbers.

We define $f : 2^N \to \mathbb{R}_N$ such that $f(\alpha)$ is the unique infinite sequence of real numbers that is formed by $\varphi((\alpha(0), \ldots, \alpha(k)))$ for $k \in \mathbb{N}$. In other words
\[
f(\alpha)(n) = \varphi((\alpha(0), \ldots, \alpha(k)))(n)
\]
for all large $k$, where $n \in \mathbb{N}$.

It is helpful to take the sequence $(u_k^\alpha)_{k \in \mathbb{N}}$ of members of $[0, +\infty)^{<N}$ with $u_k^\alpha = \varphi(\alpha(0))$ and $\varphi((\alpha(0), \ldots, \alpha(k))) \ast u_{k+1}^\alpha = \varphi((\alpha(0), \ldots, \alpha(k), \alpha(k + 1)))$, so that
\[
f(\alpha) = u_0^\alpha \ast u_1^\alpha \ast \ldots \ast u_k^\alpha \ast \ldots
\]
Moreover it is clear that
\[
\varphi((\alpha(0), \ldots, \alpha(k))) = u_0^\alpha \ast \ldots \ast u_k^\alpha.
\]
So if $\sigma = (\alpha(0), \ldots, \alpha(k))$,
\[
\|\varphi(\sigma \ast (\alpha(k + 1))) - \varphi(\sigma)\|_p^p = \|\varphi(\sigma) \ast u_{k+1}^\sigma - \varphi(\sigma)\|_p^p = \sum_{n < \text{lh}(u_{k+1}^\sigma)} |u_{k+1}^\sigma(n)|^p
\]
for all $p \geq 1$; in particular from (2) we have
\[
\sum_{n < \text{lh}(u_k^\sigma)} |u_k^\sigma(n)|^q < 2^{-(k+1)}
\]
for all $\alpha \in 2^N$ and all $k \in \mathbb{N}$. (The case $k = 0$ follows directly from the second inequality of (2)).
The function $f$ takes values in the closed unit ball of $\ell^q$. For all $\alpha \in 2^N$ and all $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that the sequence $u^n_0 \ldots u^n_k$ extends $(f(\alpha)(0), \ldots, f(\alpha)(n))$; so from (7),
\[
\sum_{n=0}^m |f(\alpha)(n)|^q \leq \sum_{t=0}^k \sum_{n < \text{lh}(\alpha^n_t)} |u^n_t(n)|^q \leq \sum_{t=0}^k 2^{-(t+1)} \leq 1.
\]
It follows that $f(\alpha) \in \ell^q$ and $\|f(\alpha)\|_q^q \leq 1$.

The function $f$ is continuous. Applying (7) again we observe that
\[
\|f(\alpha) - u^n_0 \ldots u^n_k\|_q^q = \sum_{t=k+1}^\infty \sum_{n < \text{lh}(\alpha^n_t)} |u^n_t(n)|^q \leq \sum_{t=k+1}^\infty 2^{-(t+1)} = 2^{-(k+1)}
\]
for all $\alpha \in 2^N$ and all $q \in \mathbb{N}$.

Given $\alpha \in 2^N$ and $\varepsilon > 0$ we choose $k$ large enough so that $2^{-(k+1)}/q < \varepsilon/2$ if $q \geq 1$ and $2^{-(k+1)} < \varepsilon/2$ if $0 < q < 1$. If $\beta$ agrees with $\alpha$ up to $k$, i.e., if $\beta \in N(\alpha(0), \ldots, \alpha(k))$ we have $u^n_\alpha = u^n_\beta$ for all $i = 0, \ldots, k$.

Case $q \geq 1$:
\[
\|f(\alpha) - f(\beta)\|_q \leq \sum_{n = 0}^\infty |(f(\alpha) - u^n_0 \ldots u^n_k)|^q + \|u^n_\beta \ldots u^n_k - f(\beta)\|_q^q \leq 2^{-(k+1)} + 2^{-(k+1)/q} < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Case $0 < q < 1$:
\[
d_q(f(\alpha), f(\beta)) \leq d_q(f(\alpha), u^n_0 \ldots u^n_k) + d_q(u^n_\beta \ldots u^n_k, f(\beta)) \leq \|f(\alpha) - u^n_0 \ldots u^n_k\|_q + \|u^n_\beta \ldots u^n_k - f(\beta)\|_q^q < 2^{-(k+1)} + 2^{-(k+1)} < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

The function $f$ is a reduction between the required sets. We fix an $\alpha \in 2^N$. We need to show that $\alpha \in P_3 \iff f(\alpha) \in \bigcap_{p > \alpha} \ell^p$.

Assume first that $\alpha \notin P_3$, so that for some $i \in \mathbb{N}$ we have $\alpha([i, j]) = 1$ for infinitely many $j$. We can then find a strictly increasing sequence $(j_m)_{m \in \mathbb{N}}$ of positive naturals such that $\alpha([i, j_m]) = 1$ for all $m$. We fix $m \in \mathbb{N}$ and we put $\sigma = (\alpha(0), \ldots, \alpha([i, j_m] - 1))$, so that $\alpha([0, j_m]) = \sigma * (1)$. Clearly $\text{lvl}(\sigma * (1)) = i$, so from (6),
\[
\sum_{n=0}^\infty |f(\alpha)(n)|^p = \sum_{n < \text{lvl}(\sigma * (1))} |\varphi(\sigma * (1))(n)|^p > i, j_m + 1.
\]
Since $\lim_{m \to \infty} [i, j_m] = \infty$ we have that $f(\alpha) \notin \ell^p$, and so $f(\alpha) \notin \bigcap_{p > \alpha} \ell^p$.

Finally assume that $\alpha \in P_3$ and we show that $f(\alpha) \in \ell^p$ for all $i$. We take some $i \in \mathbb{N}$; since $\alpha \in P_3$ there some $j_0$ such that for all $j \geq j_0$ and all $i' \leq i$ we have $\alpha([i', j]) = 0$.

We put $\sigma_0 = (\alpha(0), \ldots, \alpha([i, j_0]))$. Evidently $i = \text{lvl}(\sigma_0) \leq \text{dpth}(\sigma_0)$ and so $M_i(\tau)$ is defined for all $\tau$ extending $\sigma_0$. The idea is to show that $M_i(\sigma_0)$ controls $M_i(\tau)$ as $\tau$ extends $\sigma_0$ along $\alpha$. This is clear by (4) when adding 0; when adding 1, due to the choice of $j_0$, the level of our sequence must be below $i$, see Figure 2, and therefore we are covered by (5).
Figure 2. The diagonal arrangement of $\alpha \in P_3$

To make the latter precise we claim that

\[(8) \quad M_i(\sigma_0) = M_i(\tau) \text{ for all } \tau = (\alpha(0), \ldots, \alpha(k)), \text{ where } k \geq [i, j_0].\]

We prove this by induction on $k$. If $k = [i, j_0]$ then $\tau = \sigma_0$ and the assertion is trivial. Assume that (8) holds for some $k \geq [i, j_0]$. We put $\sigma = (\alpha(0), \ldots, \alpha(k))$ and $\tau = (\alpha(0), \ldots, \alpha(k), \alpha(k+1))$, so that $\tau = \sigma \ast (\alpha(k+1))$ and $\sigma_0 \subseteq \sigma$.

From the induction hypothesis $M_i(\sigma) = M_i(\sigma_0)$. If $\alpha(k+1) = 0$ then $\tau = \sigma \ast (0)$ and from (4), since $i \leq \text{dpth}(\sigma_0) \leq \text{dpth}(\sigma)$ we have $M_i(\tau) = M_i(\sigma) = M_i(\sigma_0)$. Now we assume $\alpha(k+1) = 1$ and so $\tau = \sigma \ast (1)$. Since $\alpha([i', j]) = 0$ for all $j \geq j_0$ and all $i' \leq i$ the level of the finite sequence $\tau$ is at least $i + 1$ (see Figure 2). Hence $\text{lvl}(\sigma \ast (1)) - 1 = \text{lvl}(\tau) - 1 \geq i \geq 0$, so (5) is applicable to $i$ and $\tau = \sigma \ast (1)$. From the latter it follows $M_i(\tau) = M_i(\sigma \ast (1)) = M_i(\sigma) = M_i(\sigma_0)$ and the induction step is complete.

From (3) and (8) we have for all $k \geq [i, j_0]$,

$$
\sum_{i=0}^{k} \sum_{n \in \text{lin}(u_i^p)} \|u_i^p(n)\|_{P_i}^p = \|u_0^p \ast \ldots \ast u_k^p\|_{P_i}^p = \|\varphi((\alpha(0), \ldots, \alpha(k))\|_{P_i}^p < M_i((\alpha(0), \ldots, \alpha(k))) = M_i(\sigma_0).
$$

It follows that $\sum_{n=0}^{\infty} |f(\alpha(n))|^p \leq M_i(\sigma_0)$ and therefore $f(\alpha) \in \ell^p$.

References

[1] F. Bayart, K.-G. Grosse-Erdmann, V. Nestoridis, and C. Papadimitropoulos. Abstract theory of universal series and applications. Proc. Lond. Math. Soc. (3), 96(2):417–463, 2008.
[2] Luis Bernal-González, Daniel Pellegrino, and Juan B. Seoane-Sepúlveda. Linear subsets of nonlinear sets in topological vector spaces. Bull. Amer. Math. Soc. (N.S.), 51(1):71–130, 2014.
[3] Karl-Goswin Grosse-Erdmann. Universal families and hypercyclic operators. Bull. Amer. Math. Soc. (N.S.), 36(3):345–381, 1999.
[4] Jean-Pierre Kahane. Baire’s category theorem and trigonometric series. J. Anal. Math., 80:143–182, 2000.
[5] Alexander S. Kechris. Classical Descriptive Set Theory, volume 156 of Graduate Texts in Mathematics. Springer-Verlag, 1995.
[6] Vassili Nestoridis. A project about chains of spaces, regarding topological and algebraic genericity and spaceability. https://arxiv.org/abs/2005.01023, 2020.

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