Progressive Rate-Filling: A Framework for Agile Construction of Multilevel Polar-Coded Modulation

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Abstract—We propose the progressive rate-filling method as a framework to study agile construction of multilevel polar-coded modulation. We show that the bit indices within each component polar code can follow a fixed, precomputed ranking sequence (e.g., the Polar sequence in the 5G standard) while their allocated rates (i.e., the number of information bits of each component polar code) can be fast computed by exploiting the target sum-rate approximation and proper rate-filling methods. In particular, we propose two rate-filling methods based on the capacity and the rate considering the finite blocklength effect. The proposed construction methods can be performed independently of the actual channel condition with $O(m)$ ($m$ denotes the modulation order) complexity and robust to diverse modulation and coding schemes (MCSs) in the 5G standard, which is a desired feature for practical systems.

Index Terms—Polar-coded modulation, code construction, progressive rate-filling, target sum-rate approximation.

I. INTRODUCTION

POLAR codes, invented by Arıkan [1], show competitive performance compared to state-of-the-art error-correcting codes for a wide range of code lengths and have been adopted in 5G new radio (NR) [2]. Shortly after polar codes were invented, the underlying channel polarization phenomenon has been found universal in many other signal processing problems [3], [4]. Particularly, in order to improve spectral efficiency, Seidl has introduced a 2$^m$-ary polar-coded modulation scheme [3]. Taking the dependencies among the bits that are mapped to a modulation symbol as a special kind of channel transform, polar-coded modulation (PCM) is derived under the framework of two-stage channel polarization, i.e., the modulation partition and the binary partition. The whole structure of PCM follows the multilevel coding (MLC) scheme [5], however, the novel two-stage channel transform concatenated manner accommodates a joint design of polar coding and modulation which allows one to describe the two techniques in a unified context rather than a simple combination.

In practical wireless communication systems, adaptive modulation and coding (AMC) is one of the most critical link adaptation techniques [6]. AMC works by dynamically adjusting the modulation and coding scheme (MCS), including modulation order and coding rate, to be aligned with the instantaneous channel quality, thus the link throughput can be maximized [7]. A critical issue in practical coded modulation systems is the code construction, which is expected to be agile and robust to diverse MCSs. Similar to that of standard polar codes, the construction of an MLC-PCM system can also be implemented through density evolution (DE) [8] or Gaussian approximation (GA) [9] algorithm. The reliabilities of bit subchannels of all component polar codes are calculated and sorted to select the information indices set. Despite their high accuracy, these online construction methods usually rely on specific channel parameters and involve high computational complexity that scales linearly with the blocklength. Worse still, these methods also introduce sorting complexity.

Recently, the polarization weight (PW) sequence [10] provides a new idea to construct polar codes without dependence on transmission channels. Inspired by the PW sequence, the Polar sequence has been employed for polar code construction in the 5G standard [2]. However, the PW/Polar sequence can only be employed for index-ranking within each component code in the MLC-PCM system. To take advantage of the PW/Polar sequence, the critical step is to accurately allocate the number of information bits for each component code. Related works and efficient solutions are still missing. Furthermore, the high computational and sorting complexities incurred by the DE/GA online construction methods [11] depending on channel conditions are highly impractical for AMC systems with varying parameters. The expected construction scheme is universal, in the sense that it can be performed on-the-fly and efficiently at ideal $O(1)$ cost as in 5G NR, also it should be robust to any channel parameter in the AMC system.

In this letter, we aim to set up a new framework for agile and robust construction of MLC-PCM. We analyze the properties of MLC-PCM structure and show that the bit indices within each component polar code can follow a fixed, precomputed ranking sequence, e.g., the PW sequence or the Polar sequence in the 5G standard, while their allocated rates (i.e., the number of information bits of each component polar code) can be fast computed with the target sum-rate approximation and proper rate-filling methods. The proposed code construction methods need only a one-shot operation with $O(m)$ complexity, which provides an answer to the MLC-PCM construction question of practical interest. In particular, we propose two rate-filling methods. One is based on the subchannel capacities under the first-stage modulation partition, where the reference signal-to-noise-ratio (SNR) is autonomously obtained by finding an equivalent channel whose capacity just equals the target sum-rate of total $m$ component polar codes. By this means, the code construction is realized agilely for arbitrary MCS.
and independently of the actual channel condition. Also, it is proven to be capacity-achieving as the blocklength goes to infinity. The other rate-filling method considers the rate with the finite blocklength effect [12], which can further improve the rate-filling accuracy by slightly increasing computational complexity. The proposed rate-filling methods are carried out in a progressive manner that ensures the summation of allocated information bits of all component polar codes strictly equals the total number of information bits.

Notational Conventions: In this letter, the calligraphic characters, such as $\mathcal{X}$, are used to denote sets. Let $|\mathcal{X}|$ denote the cardinality of the set $\mathcal{X}$. We write lowercase letters (e.g., $x$) to denote scalars. The notation $v^N_k$ denotes a $N$-dimensional row vector $(v_1, v_2, \ldots, v_N)$, we use $v^1_1$ to denote a subvector $(v_1, v_1+1, \ldots, v_{y})$ of $v^1_N$, and $v$ denotes the $i$-th element in $v^1_N$. The bold letters, such as $\mathbf{X}$, stand for matrices. Especially, for positive integer $N$, $[N] \triangleq \{1, 2, \ldots, N\}$. The function $\lceil \cdot \rceil$ denotes the ceiling operation.

II. MULTILEVEL POLAR-CODED MODULATION

Let $W : \mathcal{X} \rightarrow \mathcal{Y}$ denote a discrete memoryless channel (DMC) with input symbol modulation $x \in \mathcal{X}$ (alphabet size $|\mathcal{X}| = 2^m$), output symbols $y \in \mathcal{Y}$ from an arbitrary alphabet $\mathcal{Y}$, and the mutual information is marked as $I(X; Y)$. Given the $m$-bit sequence $b^m_k$ to modulation symbol mapping rule $\varphi : \{0, 1\}^m \rightarrow \mathcal{X}$, the order-$m$ modulation partition (MP) can be characterized as $\varphi : W \rightarrow (W_1, W_2, \ldots, W_m)$. It transforms $W$ to an ordered set of binary-input memoryless channels (BMCs) $\{W_k\}$ with $k = 1, 2, \ldots, m$, and they are referred as modulation subchannels. Following [3], each $W_k$ of an $m$-MP is assumed to have the knowledge of the output of $W$ and the transmitted bits over these modulation subchannels $W_{k'}$ with smaller indices $k' = 1, 2, \ldots, k-1$, i.e.,

$$W_k(y, b_1^{k-1}|b_k) = \frac{1}{2^{m-1}} \sum_{b_1^{k-1} \in \{0,1\}^{m-k}} W(y|x = \varphi(b_1^m)).$$ \hspace{1cm} (1)

Hence, from the perspective of mutual information, we have

$$\sum_{k=1}^m I(W_k) = \sum_{k=1}^m I(B_k; Y|B_1, B_2, \ldots, B_{k-1}) = I(X; Y),$$ \hspace{1cm} (2)

where $B_k$ denotes the random variable corresponding to $b_k$ in $b^m_k$. Accordingly, the MP preserves the capacity.

The second stage performs the conventional N-dimensional binary channel polarization transform $\mathbf{G}_N = \mathbf{F}_2^\otimes n$ [1], [2] on each of these $m$ bit synthesized subchannels $W_k$, where $N = 2^n$, $n = 1, 2, \ldots, \mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\mathbf{F}_2^\otimes n$ stands for the $n$-th Kronecker power of $\mathbf{F}_2$. The resulting BMCs $\{W_k^{(i)}\}$ with $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, m$ are referred as bit subchannels. The two-stage channel transform is shown in Fig. 1. Followed by [1], the transition probability of each bit subchannel is written as

$$W_k^{(i)}(y_1^N, u_k^{k-1}|u_n) = \sum_{u_k^{k-1} \in \{0, 1\}^{N-k}} \left( \frac{1}{2^{N-k}} \prod_{i'=1}^{N} W_k(y_{i'}, b_k^{k-1}(i')|b_k(i')) \right).$$ \hspace{1cm} (3)

where $a = (k-1)N + i$, the bit vector $b_k^{k-1}(i')$ corresponding to the $i'$-th symbol is $b_k^{k-1}(i') = (y_{i'}, y_{i'+N}, \ldots, y_{i'+(m-1)N})$, and $u_k^{kN} = (k-1)N+1 = u_k^{kN}$. The resulting BMCs $\{W_k^{(i)}\}$ is based on the DE/GA algorithm [4], [5]. Given the channel parameters, it first calculates the capacity of each modulation subchannel under the first-stage MP, i.e.,

$$I(W_k) = \sum_{b_1^N} \sum_{y \in \mathcal{Y}} \frac{\Pr(y|b_1^N)}{2^k} \cdot \log_2 \frac{\Pr(y|b_1^N)}{\Pr(y|b_1^{k-1})},$$ \hspace{1cm} (4)

where the transition probability is

$$\Pr(y|b_1^N) = \frac{1}{2^{m-k}} \sum_{b_k^{m-k} \in \{0,1\}^{m-k}} W(y|x = \varphi(b_1^m)).$$ \hspace{1cm} (5)

Then, each $W_k$ is approximated by a binary-input additive white Gaussian noise (BI-AWGN) channel $W_k^{(i)}$ whose capacity equals $I(W_k)$. The reliabilities of $mn$ bit subchannels are computed by using surrogate channels $\{W_k^{(i)}\}$ and the DE/GA algorithm [4], [11]. The information indices $\{A_k\}$ are obtained by ranking all these bit subchannels with their reliabilities.

III. PROGRESSIVE RATE-FILLING CONSTRUCTION

In this section, we propose two rate-filling methods for a fast and low-complexity construction of MLC-PCM. Note that each component polar code corresponds to one modulation subchannel $W_k$, thus their $N$ bit indices can be sorted with the PW sequence [10] or the Polar sequence in the 5G standard [2]. Actually, there is no need to sort all $mn$ bit subchannels like that in the online DE/GA construction, thus the main task is to compute their allocated rates, i.e., the number of information bits of each component polar code $|A_k|$.

Fig. 1. The two-stage channel transform in MLC-PCM [3].
A. Rate-Filling Based on Capacity

To find out the number of allocated information bits \( K_k = |A_k| \) for each component polar code, we propose a rate-filling method with \( R_k = K_k/N \). Note that we know the total number of information bits \( K \), the target transmission rate is \( R_T \triangleq K/N = mR \) bits per symbol, which is also referred to as the sum-rate of component polar codes. Hence, the key is adjusting the parameters of \( W \), e.g., the SNR of \( W \) under the AWGN channel, to find an equivalent channel \( \overline{W} \) whose capacity just equals the target sum-rate \( R_T \), i.e.,

\[
R_T = I(\overline{W}) \Rightarrow mR
\]

\[
= \sum_{x \in X} \sum_{y \in Y} \frac{1}{2m} \cdot \overline{W}(y|x) \cdot \log_2 \frac{\overline{W}(y|x)}{\sum_{x' \in X} \frac{1}{2m} \cdot \overline{W}(y|x')} \quad (6)
\]

For this identical sum-rate approximated channel \( \overline{W} \), we can also compute the modulation subchannel capacities \( \{I(\overline{W}_k)\} \) under the MP as (4).

**Proposition 1:** The rate-filling for each component polar code is performed as

\[
R_k \leftarrow mR \cdot \frac{I(\overline{W}_k)}{I(\overline{W})} = I(\overline{W}_k) \quad (7)
\]

for \( k = 1, 2, \ldots, m \).

A special note is that the code construction is intended for polar codes, which means all the source bits along with the attached bits, e.g., cyclic-redundancy-check (CRC) bits [13] etc., should be counted in the number of information bits \( K \). In practical AMC systems, this rate-filling is independent of blocklength \( N \), which enables the code construction to be robust to varying parameters. The channel condition, e.g., SNR, determines the code construction in an implicit way that selects the MCS, including \( m \) and \( R \), to be aligned with the instantaneous channel state in AMC systems.

Next, we prove that the proposed rate-filling construction scheme is capacity-achieving. First, we give the definition of consistence property for modulation channels.

**Definition 1:** Given two modulation channels \( W' \) and \( W'' \), suppose \( W' \) is upgraded with respect to \( W'' \) [14], i.e., \( W' \supseteq W'' (I(W') \geq I(W'')) \), if their split modulation subchannels satisfy \( W'_k \supseteq W''_k \) for any \( k = 1, 2, \ldots, m \), we define this as the consistence property.

For example, given the modulation scheme and modulation order \( m \), two AWGN modulation channels with different SNRs satisfy the consistence property.

**Theorem 1:** For any modulation channel satisfying the consistence property, the proposed rate-filling strategy is capacity-achieving as the blocklength \( N \) goes to infinity.

**Proof:** Applying the channel polarization theorem [1] to MLC-PCM, for each \( N \)-length component polar code, the subchannels \( \{W_{k,i}(n)\} \) polarize in the sense that, for any fix \( \delta (\in [0,1]) \), as \( N \) goes to infinity, the fraction of indices \( i (\in [N]) \) for which \( I(W_{k,i}(n)) \approx (1-\delta)I(W_k) \), and the fraction for which \( I(W_{k,i}(n)) \approx 0 \) goes to \( 1-I(W_k) \). The good bit indices set \( S(W_k) \) for \( W_k \), i.e., those with systematic capacity near 1) satisfy \( \lim_{N \to +\infty} \frac{|S(W_k)|}{N} = I(W_k) \).

Given the total coding rate \( R \) and the modulation scheme with order \( m \) in MLC-PCM, the proposed rate-filling construction is performed as (7). Under this modulation configuration,

it is sufficient to prove that for any channel \( W \) which is upgraded with respect to \( \overline{W} \) in (6), i.e., \( W \supseteq \overline{W} (I(W) \geq I(\overline{W}) = mR) \), suppose \( W \) and \( \overline{W} \) satisfy the consistence property, we have the system error probability \( \epsilon \to 0 \) as \( N \to +\infty \). Since \( W_k \supseteq \overline{W}_k \) for any \( k = 1, 2, \ldots, m \), due to the nesting property of polar codes [14], it can be derived that good bit indices \( S(\overline{W}_k) \) for \( \overline{W}_k \) (i.e., those with systematic capacity near 1) must be a subset of good bit indices \( S(W_k) \) for \( W_k \), i.e., \( S(\overline{W}_k) \subset S(W_k) \). Note that \( S(\overline{W}_k) \) is indeed the selected information set \( A_k \) as \( N \) goes to infinity, thus all the information-carrying subchannels are of capacity 1 such that the system error probability \( \epsilon \to 0 \) in this sense, the capacity-achieving property of the proposed rate-filling code construction method is proven.

In practical implementation, giving modulation and coding parameters, the proposed rate-filling scheme should strictly ensure the summation of allocated information bits of each component polar code equals the total number of information bits, i.e., \( \sum_{k=1}^{m} K_k = K \). To this end, the proposed rate-filling is performed in a progressive manner from the modulation subchannel of the highest capacity to the lowest capacity. The whole procedure is summarized as Algorithm 1.

**Algorithm 1:** Progressive Rate-Filling Code Construction

1. Initialize system parameters: modulation order \( m \), symbol blocklength (component code length) \( N \), total number of information bits \( K \), symbol mapping rule \( \varphi : \{0,1\}^m \leftrightarrow X \).
2. Find an equivalent channel \( \overline{W} \) as (6);
3. Calculate the modulation subchannel capacities \( \{I(\overline{W}_k)\} \);
4. Sort the \( m \) modulation subchannels such that \( I(\overline{W}_{k_1}) \geq I(\overline{W}_{k_2}) \geq \cdots \geq I(\overline{W}_{k_m}) \);
5. for \( i = 1, 2, \ldots, m \) do
6. Compute the number of information bits for the \( k_i \)th component polar code as
7. \[ K_{k_i} = \left( \frac{K - \sum_{t'=1}^{i-1} K_{k_{t'}}}{\sum_{t'=1}^{m} I(\overline{W}_{k_{t'}})} \right) \cdot I(\overline{W}_{k_i}) \]
8. return The information sets of each component polar code \( A_1, A_2, \ldots, A_m \).

B. Rate-Filling Based on Finite Blocklength Rate

Note that the above rate-filling scheme based on capacity does not take into account the blocklength \( N \) of each component polar code, this can well match most cases especially for large \( N \). However, for some cases, the accuracy of rate-filling can be further improved by considering the rate with the finite blocklength effect [12]. For the \( k \)-th component code, given the blocklength \( N \), the maximal rate achievable with the error probability \( \epsilon_k \) is closely approximated as

\[
M(W_k, N, \epsilon_k) = I(W_k) - \sqrt{\frac{V_k}{N}} Q^{-1}(\epsilon_k),
\]
where $Q$ is the complementary Gaussian cumulative distribution function, and the channel dispersion $V_k$ is computed as
\[ V_k = \sum_{b_i \in \{0,1\}} \sum_{y \in \mathcal{Y}} \Pr(y|b^{i}_k) \left( \log_2 \frac{\Pr(y|b^{i}_1)}{\Pr(y|b^{k-1}_1)} \right)^2 - I^2(W_k), \] (9)
where $I(W)$ and the transition probabilities are defined in (4) and (5) [12]. Given the target system error probability $\epsilon$, it is still difficult to determine the error probability $\epsilon_k$ of each component code. However, note that the rate-filling process may allocate more information bits to component codes of higher capacity, the error probabilities $\epsilon_k$ tend to be close with each other. Therefore, we can approximate $\epsilon_k = \tau$ for any $k = 1,2,\ldots, m$, thus given the target block error ratio (BLER) $\epsilon$, we can derive that
\[ \epsilon = 1 - \prod_{k=1}^{m} (1 - \epsilon_k) \Rightarrow 1 - \epsilon = (1 - \tau)^m \]
\[ \Rightarrow \epsilon_k = \tau = 1 - (1 - \epsilon)^{\frac{1}{m}}. \] (10)
In this way, given the system target BLER $\epsilon$, one can compute the error probability $\epsilon_k$ for each component code. To allocate the coding rates for MLC-PCM, we first adjust the parameters of $W$ to find an equivalent channel $\tilde{W}$ whose achievable sum-rate at finite blocklength $N$ equals $R_T$, i.e.,
\[ R_T = mR = \sum_{k=1}^{m} M(\tilde{W}_k, N, \epsilon_k), \] (11)
where $M(\tilde{W}_k, N, \epsilon_k)$ is calculated as (8) and the target BLER $\epsilon$ can be chosen with the practical requirements, e.g., $10^{-1}$ [7], etc., such that $\epsilon_k = \tau$ is computed as (10). With the surrogate channel $\tilde{W}$, the rate-filling procedure is given as (12).

**Proposition 2**: The rate-filling for each component polar code is performed as
\[ R_k \leftarrow mR \cdot \frac{M(\tilde{W}_k, N, \epsilon_k)}{\sum_{k'=1}^{m} M(\tilde{W}_{k'}, N, \epsilon_{k'})} = M(\tilde{W}_k, N, \epsilon_k), \] (12)
for $k = 1,2,\ldots, m$.

Apparently, the proposed rate-filling scheme in Proposition 2 is an enhanced version of that in Proposition 1. It improves the accuracy of rate-filling with slightly increased computational complexity in (8). In practical implementation, the rate-filling should also be carried out in a progressive manner, and the whole procedure is similar to that in Algorithm 1 where capacity terms $\{I(\tilde{W}_k)\}$ are replaced with finite blocklength rate terms $\{M(\tilde{W}_k, N, \epsilon_k)\}$.

**IV. PERFORMANCE EVALUATION**

We make a comprehensive performance evaluation of the proposed rate-filling code construction methods. The performance of MLC-PCM constructed by state-of-the-art GA methods [3], [4] (i.e., the “LM-DGA” method in [11]) and that of the LDPC-coded modulation in the 5G standard [2] are also provided as comparisons. For ease of exposition, the proposed rate-filling method based on capacity in Section III-A is abbreviated as “RF-I”, and the proposed rate-filling method based on the finite blocklength rate in Section III-B is abbreviated as “RF-II”. In addition, the target BLER $\epsilon$ in (10) of the RF-II construction method is set to $10^{-1}$ for all the subsequent simulation cases, which follows the practical BLER requirement in 4G and 5G wireless systems [7]. The CRC-aided successive cancellation list (CA-SCL) decoding [13] is used for MLC-PCM, where the list size is 32 and the 16-bit CRC sequence in the 5G standard [2] is adopted. The sum-product algorithm (SPA) with 25 iterations and layered scheduling is used for LDPC decoding, which presents comparable complexity with polar decoding [4].

In Fig. 2, we compare the BLER performance of various coded modulation schemes under the AWGN channel. For all cases, the code rate is $R = 0.5$ and the symbol blocklength is set to $N = 512$ which corresponds to component blocklength in the MLC-PCM. The modulation scheme employs the quadrature amplitude modulation (QAM) [2]. Given the modulation order of QAM $m \in \{4,6,8\}$, the blocklength of LDPC code is $mn$. It should be noted that the online GA construction in MLC-PCM is executed individually for each evaluated SNR value, but the proposed rate-filling construction method is a one-shot operation that is independent with the evaluated SNR.

Clearly, the MLC-PCM constructed by RF-I can achieve almost identical performance as the GA construction under 16QAM. With modulation order increasing, the performance of RF-I becomes worse. That is because the number of component polar codes increase with the modulation order, and thus the accuracy of RF-I without considering the blocklength decreases for high-order modulation schemes, e.g., 64QAM and 256QAM, etc. However, the performance of MLC-PCM constructed by RF-II is aligned well with the GA construction for diverse modulation orders, which validates the effect of considering finite blocklength. Therefore, the proposed RF-II construction method is more robust with a slight increase of computation complexity in (8). Compared to the LDPC-coded modulation in the 5G standard [2], the MLC-PCM shows stable performance gain.

In Fig. 3, we show the minimum required SNR to achieve BLER = $10^{-1}$ under AWGN and Rayleigh fast fading channels with the symbol blocklength $N = 256$. The employed MCSs (including modulation order $m$ and code rate $R$) follow the 5G standard [6, Table 5.1.3.1-2], and the corresponding code rate range is marked in each subfigure. Clearly, under both AWGN and fading channels, the proposed two rate-filling construction methods are aligned well with the conventional GA construction, which outperforms the 5G LDPC-coded
modulation schemes. For high modulation orders, the results of RF-I do not show obvious loss like these in Fig. 2. That indicates the robustness of the proposed rate-filling methods for practical configurations as the 5G standard [2].

Fig. 4 shows the link throughput comparison result under the block-fading channel, where the symbol blocklength is \( N = 256 \). The AMC mechanism adaptively selects MCS to maximize the link throughput under BLER constraint \( \leq 10^{-1} \) [7], which is indeed employed in the 5G NR system. Clearly, the MLC-PCM constructed by the proposed RF-II method shows stable gain with respect to the LDPC-coded modulation scheme in the 5G standard [2]. That validates the agility and robustness of the proposed code construction method.

Regarding the complexity comparison, we do not count the complexity of capacity calculation since it can be formulated as a lookup table in practicality, and these information theory metric computations are also required in the DE/GA construction. It is intuitive to see that each component code is of \( O(1) \) complexity by using the readily available Polar/PW sequence so that the whole progressive rate-filling procedure involves \( O(n) \) complexity without sorting operation, which is much lower than the \( O(mN) \) computational complexity of the GA method that also needs sorting operation.

V. Conclusion

In this letter, we propose two progressive rate-filling methods to realize a robust and channel-independent construction of MLC-PCM. Simulation results have shown that the proposed construction method is robust to diverse MCSs.

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