Formulas for the dimensions of some affine Deligne-Lusztig Varieties

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Abstract

Rapoport and Kottwitz defined the affine Deligne-Lusztig varieties $X_{P,\tilde{\omega}}(b\sigma)$ of a quasisplit connected reductive group $G$ over $F = \mathbb{F}_q((t))$ for a parahoric subgroup $P$. They asked which pairs $(b, \tilde{\omega})$ give non-empty varieties, and in these cases what dimensions do these varieties have. This paper answers these questions for $P = I$ an Iwahori subgroup, in the cases $b = 1$, $G = SL_2$, $SL_3$, $Sp_4$. This information is used to get a formula for the dimensions of the $X_{\omega}^k(\sigma)$ (all shown to be non-empty by Rapoport and Kottwitz) for the above $G$ that supports a general conjecture of Rapoport. Here $K$ is a special maximal compact subgroup.

1 Introduction

Let $F$ be $\mathbb{F}_q((t))$ with ring of integers $\mathcal{O}_F$, and let $G$ be a split connected reductive group over $F$. Let $L$ be the completion of the maximal unramified extension of $F$, $\mathbb{F}_q((t))$. Let $\sigma$ be the Frobenius automorphism of $L$ over $F$. Let $\mathcal{B}_n$ be the affine building for $G(E)$ where $E/F$ is the unramified extension of degree $n$ in $L$ (so $E = \mathbb{F}_q^n((t))$), and let $\mathcal{B}_\infty$ be the affine building for $G(L)$. Let $T$ be a split torus in $G$, let $B = UT$ be a Borel subgroup, and let $I$ be an Iwahori in $G(L)$ containing $T(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers of $L$. Let $A_M$ and $C_M$ be the correspondingly specified apartment and alcove, which we assume are in $\mathcal{B}_1$. We will call these the main apartment and the main alcove, respectively. We assume that $C_M$ is in the positive Weyl chamber in $A_M$ specified by $B$. Let $P \supseteq I$ be a parahoric subgroup of $G(L)$. If $b \in G(L)$ then the $\sigma$-conjugacy class of $b$ is $\{x^{-1}b\sigma(x) \mid x \in G(L)\}$. Let $\tilde{W} = N(L)/T(\mathcal{O}_L)$ be the extended affine Weyl group, and let $\tilde{W}_P = N(L) \cap P/T(\mathcal{O}_L)$. Here $N$ is the normalizer of $T$.

If $\tilde{\omega} \in \tilde{W}$, then we define, after Rapoport and Kottwitz, the generalized affine Deligne-Lusztig variety $X_{\omega}^k(\sigma) = \{x \in G(L)/P : inv_P(x, b\sigma(x)) = \tilde{\omega}\}$. Here $inv_P : G(L)/P \times G(L)/P \rightarrow P \backslash G(L)/P = \tilde{W}_P \backslash \tilde{W}/\tilde{W}_P$ is the relative position map associated to $P$. Rapoport asked the question of which pairs $(b, \tilde{\omega})$ give rise to non-empty sets, and for these pairs, what is $\dim(X_{\omega}^k(b\sigma))$? Kottwitz and Rapoport
answered the emptiness/non-emptiness part of this question for \( P = K \), the maximal bounded subgroup of \( G(L) \) associated to some special vertex \( v_M \) of \( C_M \) [7].

In Section 3 of this paper we consider the case \( G = SL_3 \), \( b = 1 \), \( P = I \). Complete results on emptiness/non-emptiness and dimension are shown for this case in Figure 6. In Section 4 we consider \( G = Sp_4 \), \( b = 1 \), \( P = I \). Emptiness/non-emptiness results and dimension results are in Figure 10. The case \( G = SL_2 \), \( b = 1 \), \( P = I \) can be done using an even simpler version of the same methods.

Kottwitz and Rapoport showed in [7] that for general \( G \), \( X^K_w(\sigma) \) is non-empty for any \( \tilde{w} \) corresponding to a dominant cocharacter in the coroot lattice. Rapoport conjectured a specific formula for the dimension of the \( X^K_w(\sigma) \) in [10]. The knowledge of the \( X^I_{\tilde{w}}(\sigma) \) mentioned in the previous paragraph gives knowledge of the \( X^P_w(\sigma) \), so the dimensions of the \( X^K_w(\sigma) \) are computed in Section 5 for \( SL_2 \), \( SL_3 \), and \( Sp_4 \). The result is that \( \dim(X^K_w(\sigma)) = \langle \mu, \rho \rangle \), where \( \mu \in X_*(T) \) dominant corresponds to \( \tilde{w} \in \tilde{W}_K \setminus \tilde{W}/\tilde{W}_K \) and \( \rho \) is half the sum of the positive roots for \( G \). This supports the conjecture of Rapoport in [10]. Preliminary work toward a proof of this conjecture has been done with Kottwitz.

In Section 6 we present a formula that encapsulates part of the results pictured in Figures 5 and 10. The formula also holds for \( SL_2 \). It is too soon to conjecture that this formula holds for general \( G \). Some results on emptiness/non-emptiness for \( b \neq 1 \), \( G = SL_2 \), \( SL_3 \), \( Sp_4 \) can be found in [11]. Section 7 contains a summary of these results.

This work has significance to the study of the reduction modulo \( p \) of Shimura varieties. Interested readers should see the survey article [10] by Rapoport.

## 2 General Methodology

For this and the next two sections we let \( P = I \), so \( \tilde{w} \in \tilde{W}_P \setminus \tilde{W}/\tilde{W}_P = \tilde{W} \), and we let \( X_{\tilde{w}}(\sigma) = X^I_{\tilde{w}}(\sigma) \). For this section, the group \( G \) is simply-connected, so that \( I \) is the stabilizer of \( C_M \). First note that if \( \tilde{w}C_M \cap C_M \) is non-empty, then \( X_{\tilde{w}}(\sigma) \) can be identified with a disjoint union of (non-affine) Deligne-Lusztig varieties, whose structure and dimension are already known [3]. Let \( v_1 \) be a vertex in \( A_M \) and let \( v_2 \) be a vertex in \( B_1 \) in the same \( G(F) \) orbit as \( v_1 \). We require that \( v_1 \notin C_M \). Let \( Q_1 \) be the last alcove in a minimal gallery from \( C_M \) to \( v_1 \), and let \( Q_2 \) be the set of all alcoves \( Q_2 \) containing \( v_2 \) such that \( Q_2 \) and \( \sigma(Q_2) \) have some fixed relative position, \( p_r \). Note that \( Q_1 \) does not depend on the choice of minimal gallery from \( C_M \) to \( v_1 \). We require that \( p_r \) be such that \( Q_2 \cap B_1 = \{ v_2 \} \). We define the \( (v_1, v_2, p_r) \)-piece of \( X_{\tilde{w}}(\sigma) \) (which may be empty) to be all alcoves \( D \subset B_\infty \) such that there exists \( y \in G(L) \) with \( yC_M = D \), \( yQ_1 = Q_2 \) for some \( Q_2 \in Q_2 \) (so \( yv_1 = v_2 \)), and \( inv(D, \sigma(D)) = \tilde{w} \). The dimension of \( X_{\tilde{w}}(\sigma) \) is the supremum of the dimensions of its pieces (this supremum could be infinite, \textit{a priori}, although we will show that it is finite in the cases we will consider).

We define the \( (v_1, v_2, p_r) \)-superpiece to be the collection of all alcoves \( D \subset B_\infty \) such that there exists \( y \in G(L) \) with \( yC_M = D \) and \( yQ_1 = Q_2 \) for some \( Q_2 \in Q_2 \). So the \( (v_1, v_2, p_r) \)-superpiece is the disjoint union, over all \( \tilde{w} \in \tilde{W} \), of the \( (v_1, v_2, p_r) \)-pieces of
the $X_{\bar{w}}(\sigma)$ (many of which will be empty). The approach outlined in this paragraph was suggested by Kottwitz, and is similar to that used in [6].

Note that the structure of the $(v_1, v_2, p_r)$-superpiece does not depend on $v_2$, as long as $v_2$ is some vertex in $B_1$ in the same $G(F)$-orbit as $v_1$. So for each $(v_1, p_r)$-pair, we fix an arbitrary vertex $v_2 \in A_M$ in the same $G(L)$-orbit as $v_1$. So for each $(v_1, p_r)$-pair, we fix an arbitrary vertex $v_2 \in A_M$ in the same $G(L)$-orbit as $v_1$. We will also demonstrate a way of calculating the dimension of each non-empty piece in the $(v_1, v_2, p_r)$-superpiece, again only for $SL_3$ and $Sp_4$. Everything we will do also applies to $SL_2$. Aggregating all this information over all $(v_1, p_r)$-pairs will tell us, for each piece of each $X_{\bar{w}}(\sigma)$, whether it is empty or non-empty, and what its dimension is. This gives the emptiness/non-emptiness and dimension of the $X_{\bar{w}}(\sigma)$ themselves.

3 $SL_3$

In order to carry out the process outlined in Section 2 for $SL_3$, it suffices to consider $v_1$ in the region pictured in Figure 1. All other $v_1$ can be obtained from these by rotating by $120^\circ$ or $240^\circ$ about the center of $C_M$. Further, if $v_1'$ is the rotation of $v_1$ by $\alpha = 120^\circ$ or $240^\circ$ about the center of $C_M$, then it is easy to see that the set $\{\bar{w} \in \hat{W} : \text{the } (v_1', v_2, p_r)\text{-piece of } X_{\bar{w}}(\sigma) \text{ is non-empty}\}$ is the rotation of the set $\{\bar{w} \in \hat{W} : \text{the } (v_1, v_2, p_r)\text{-piece of } X_{\bar{w}}(\sigma) \text{ is non-empty}\}$ by $\alpha$ about the center of $C_M$. Further, the correspondence between these two sets given by rotation by $\alpha$ preserves the dimension of the corresponding pieces.

Given the restriction (mentioned in Section 2) that $p_r$ be such that $Q_2 \cap B_1 = \{v_2\}$, we know that $Q_2$ and $\sigma(Q_2)$ must share exactly one vertex. So for $SL_3$, $p_r$ corresponds to some element of $W$, the finite Weyl group, of length 2 or 3.

Let $\Gamma_{v_1}$ be a minimal gallery from $C_M$ to $v_1$. As mentioned in Section 2, $Q_1$ is the last alcove in $\Gamma_{v_1}$. Let $\Gamma_{(v_1, p_r)}^f$ be $z\Gamma_{v_1}$, where $Q_1$ and $zQ_1$ have relative position $p_r$, $z \in SL_3(L)$, and $z$ sends $A_M$ to $A_M$. Let $\Gamma_{(v_1, p_r)}$ be some fixed minimal connecting


Let $\Omega$ be a gallery in $A_M$ starting at $C_M$, and containing any alcove at most once (so it is non-stuttering, non-backtracking, and does not cross itself). Let $\Omega_1, \Omega_2, \ldots, \Omega_n$ be the alcoves of $\Omega$ in order (so $\Omega_1 = C_M$), and let $e_i$ be the edge between $\Omega_i$ and $\Omega_{i+1}$. Let $j$ be minimal such that $C_M$ and $\Omega_j$ are on opposite sides of the hyperplane $h_j$ in $A_M$ determined by $e_j$. We say $e_j$ is the first choice edge in $\Omega$. If $j$ does not exist then there are no choice edges in $\Omega$. If $j$ does exist, then we define the hard choice at $e_j$ to be the gallery $\Omega_1, \ldots, \Omega_j, \Omega_{j+1}, \ldots, \Omega_n$, and the easy choice at $e_j$ to be the gallery $\Omega_1, \ldots, \Omega_j, f_{h_j}(\Omega_{j+1}), \ldots, f_{h_j}(\Omega_n)$, and we find the minimal $k > j$ such that $h_k$ has $\Omega_k$ and $C_M$ on opposite sides. This is the next choice edge, given the hard choice at $j$, and we can make easy and hard choices here. Given the easy choice at $j$ we consider $\Omega_1, \ldots, \Omega_j, f_{h_j}(\Omega_{j+1}), \ldots, f_{h_j}(\Omega_n)$, and we find the minimal $k$ such that $k > j$, and such that $f_{h_j}(\Omega_k)$ and $C_M$ are on opposite sides of the hyperplane between $f_{h_j}(\Omega_k)$ and $f_{h_j}(\Omega_{k+1})$. This is the next choice edge, given the easy choice at $j$, and we can make either a hard or an easy choice here. In this way we construct a binary tree, $T$, called the choice tree for $\Omega$. Each node in $T$ except the leaves corresponds to a choice edge in $\Omega$. At every node except the leaves, $T$ has a branch corresponding to a hard choice and another corresponding to an easy choice.

One can show that any non-backtracking path from the root node to a leaf of $T$ corresponds to the retraction (onto $A_M$ centered at $C_M$) of some gallery (or galleries) starting at $C_M$ and of the same type at $\Omega$. Such a path is equivalent to the choice of a leaf of $T$, since $T$ is a tree. The gallery $\Omega$ itself corresponds to the path obtained by making all hard choices in $T$. Further, all galleries starting at $C_M$ of the same type as $\Omega$ retract in a way specified by some non-backtracking path from the root.
node of $T$ to a leaf. We define the set of comprehensive folding results of $\Omega$ to be the set of final alcoves of retractions of galleries starting at $C_M$ that have the same type as $\Omega$. By retraction, we always mean the retraction centered at $C_M$ onto $A_M$. So a comprehensive folding result of $\Omega$ can also be thought of as a non-backtracking path $F$ from the root node to a leaf of the choice tree of $\Omega$.

We now observe that the set of comprehensive folding results of $\Omega = \bar{\Gamma}_{(v_1,p_v)}$ contains the set of possible $inv(D,\sigma(D))$ for $D$ in the $(v_1,v_2,p_v)$-superpiece. We claim that this is also an equality of sets. Let $F$ be a non-backtracking path from the root of $T$ to a leaf. We define the cf-dimension of $F$ to be $l(\Gamma_{v_1}) + l(\Gamma_{(v_1,p_v)}) = n_F - 2$, where $n_F$ is the number of hard choices in $F$, and $l$ represents the length of a gallery (the number of alcoves in it). We also claim that the cf-dimension of $F$ is equal to the dimension of the $(v_1,v_2,p_v)$-piece of $X_{wC_M}$, where $wC_M$ is the comprehensive folding result of $\Omega$ corresponding to $F$.

We first note that one can make at most one easy choice for each $\Omega = \bar{\Gamma}_{(v_1,p_v)}$ in Figure 2. Once this choice is made, there are no subsequent choice edges. This can be seen just by analyzing the pictures in Figure 2 on a case by case basis. Also, one can see that $\Gamma_{v_1} \cup \Gamma_{(v_1,p_v)}^c$ is minimal. So the first choice edge in $\Omega$ occurs between two of the alcoves of $\Gamma_{(v_1,p_v)}^f$. Using these facts, one can show that choice edges in $\Omega$ correspond to hyperplanes in $A_M$ that pass between two alcoves of $\Gamma_{(v_1,p_v)}^f$ and that also pass between two alcoves of $\Gamma_{v_1}$. We now seek to prove the claims of the previous paragraph. Given a non-backtracking path $F$ in $T$ from the root node to a leaf, we need to produce some gallery $\Lambda$ such that $y\Gamma_{v_1} = \Lambda$ for some $y \in SL_3\Lambda(L)$ with $yQ_1 = Q_2$ for some $Q_2 \in Q_2$, and such that $\rho_{C_M}(y^{-1}(\Lambda \cup \Lambda^c \cup \sigma(\Lambda)))$ gives the comprehensive folding result determined by $F$. Here $\Lambda^c$ is a minimal gallery from $Q_2$ to $\sigma(Q_2)$ that has the same type as $\Gamma_{(v_1,p_v)}^c$ and $\rho_{C_M}$ is the retraction onto $A_M$ centered at $C_M$.

Note first that $F$ determines the relative position of any two alcoves in $\bar{\Lambda} = \Lambda \cup \Lambda^c \cup \sigma(\Lambda)$. In our $SL_3\Lambda$ case, $F$ is just an indication of the choice edge at which to make the easy choice, if any (since there is at most one easy choice). We will construct $\Lambda$ starting from $\Lambda_n$, the alcove that contains $v_2$. We choose $\Lambda_n = Q_2$. The dimension of the set of choices for this construction is $l(\Gamma_{(v_1,p_v)}^c) - 1$, since the structure of (non-affine) Deligne-Lusztig varieties is known 3. We assume by induction that we have constructed $\Lambda_i, \Lambda_{i+1}, \ldots, \Lambda_n$ (and therefore also $\sigma(\Lambda_n), \sigma(\Lambda_{n+1}), \ldots, \sigma(\Lambda_i)$), and that the dimension of the space of possible such constructions is $l(\Gamma_{(v_1,p_v)}^c) + (n-i) - 1 - n_{(F,i)}$, where $n_{(F,i)}$ is defined as follows. Each choice edge $e$ in $\Omega$ has two corresponding integers $1 \leq \beta_1, \beta_2 \leq n - 1$ such that the hyperplane $h_e$ corresponding to $e$ passes between the $\beta_1^{st}$ and $(\beta_1 + 1)^{th}$ alcoves of $\Gamma_{v_1}$ (where the first alcove of $\Gamma_{v_1}$ is considered to be $C_M$), and between the $\beta_2^{nd}$ and $(\beta_2 + 1)^{th}$ alcoves of $\Gamma_{(v_1,p_v)}^f$ (where the $n^{th}$ alcove of $\Gamma_{(v_1,p_v)}^f$ is considered to be the one containing $v_1$). We define $n_{(F,i)}$ to be the number of choice edges $e$ such that $i \leq \beta_1, \beta_2$, and such that $F$ indicates a hard choice at $e$. Note that if $i = 1$, $l(\Gamma_{(v_1,p_v)}^c) + (n-i) - 1 - n_{(F,i)} = l(\Gamma_{v_1}) + l(\Gamma_{(v_1,p_v)}) - n_F - 2$, and if $i = n$, then $l(\Gamma_{(v_1,p_v)}^c) + (n-i) - 1 - n_{(F,i)} = l(\Gamma_{(v_1,p_v)}^c) - 1$.

Let $A$ be some apartment containing $\Lambda_i$ and $\sigma(\Lambda_i)$. Let $S \subset A$ be the intersection
of all apartments that contain \( \Lambda_i \) and \( \sigma(\Lambda_i) \). Let \( d_{i-1} \) be the edge of \( \Lambda_i \) to which \( \Lambda_{i-1} \) must be attached (this is specified by the requirement that \( \bar{\Lambda} \) and \( \Omega \) be of the same type). Let \( \Lambda_{i-1} \) be the alcove in \( A \) that one gets by reflecting \( \Lambda_i \) about \( d_{i-1} \). Let \( \sigma(\Lambda_{i-1}) \) be the alcove in \( A \) one gets by reflecting \( \sigma(\Lambda_i) \) about \( \sigma(d_{i-1}) \). One can see by considering each of the cases pictured in Figure 2 that either exactly one of \( \Lambda_{i-1} \) and \( \sigma(\Lambda_{i-1}) \) is in \( S \), or neither is in \( S \). Note that the former occurs if and only if \( i - 1 = \min(\beta_1, \beta_2) \) for \( \beta_1, \beta_2 \) the two integers corresponding to some choice edge in \( F \).

Let \( S_{i-1} \) be the intersection of all apartments containing \( S \cup \Lambda_{i-1} \), and let \( S'_{i-1} \) be the intersection of all apartments containing \( S \cup \sigma(\Lambda_{i-1}) \). One can see by considering the cases in Figure 2 that if neither \( \Lambda_{i-1} \) nor \( \sigma(\Lambda_{i-1}) \) is in \( S \), then \( \Lambda_{i-1} \) is not in \( S'_{i-1} \) and \( \sigma(\Lambda_{i-1}) \) is not in \( S_{i-1} \). Therefore, in this case we can choose any \( \Lambda_{i-1} \) adjacent to \( \Lambda_i \) by \( d_{i-1} \). This in turn determines \( \sigma(\Lambda_{i-1}) \) adjacent to \( \sigma(\Lambda_i) \) by \( \sigma(d_{i-1}) \). There is one dimension worth of these choices, so the dimension of the construction down to \( i - 1 \) is \( l(\Gamma_{c(v_1,p_r)}) + (n - i) - 1 - n_{(F,i)} + 1 \). In this case \( n_{(F,i-1)} = n_{(F,i)} \), so \( l(\Gamma_{c(v_1,p_r)}) + (n - i) - 1 - n_{(F,i)} + 1 = l(\Gamma_{c(v_1,p_r)}) + (n - (i - 1)) - 1 - n_{(F,i-1)} \).

We now consider the case in which exactly one of \( \Lambda_{i-1} \) and \( \sigma(\Lambda_{i-1}) \) is in \( S \). We assume \( \Lambda_{i-1} \) is in \( S \). The other case is similar. This means \( i - 1 = \min(\beta_1, \beta_2) \) for \( \beta_1, \beta_2 \) the two integers corresponding to some choice edge \( e \). If \( F \) dictates a hard choice at this point, we must choose \( \Lambda_{i-1} \subset S \). There is only one such choice, causing no increase in the dimension of the construction. If \( F \) dictates an easy choice, we may choose any \( \Lambda_{i-1} \) not in \( A \), but attached to \( \Lambda_i \) via \( d_{i-1} \). There is one dimension worth of such choices, increasing dimension by one. In the former case, \( n_{(F,i-1)} = n_{(F,i)} + 1 \), and in the latter case \( n_{(F,i+1)} = n_{(F,i)} \). In both cases, the dimension of the new structure is \( l(\Gamma_{c(v_1,p_r)}) + (n - (i - 1)) - 1 - n_{(F,i-1)} \). This finishes the proof of the previous claims.

The result of all this is that we can calculate the values of \( \text{inv}(D, \sigma(D)) \) for \( D \) in the \((v_1, v_2, p_r)\)-superpiece, and for each \( \tilde{w} \) in this set we can calculate the dimension of the \((v_1, v_2, p_r)\)-piece of \( X_{\tilde{w}}(\sigma) \). This can all be done through straightforward computations of comprehensive folding results and cf-dimensions. For instance, using \( v_1 \) and \( p_r \) leading to the \( \bar{\Gamma}_{(v_1,p_r)} \) pictured in Figure 3, we get the results pictured in Figure 4. The numbers in Figure 4 are the dimensions of the \((v_1, v_2, p_r)\)-pieces of the \( X_{\tilde{w}}(\sigma) \), with \( \tilde{w} \) corresponding to the alcoves on which the numbers are written. Alcoves with no numbers have empty \((v_1, v_2, p_r)\)-pieces. We did an analogous computation for every \( v_1 \) in the region shown in Figure 1 and for every \( p_r \) for which the corresponding \( w \in W \) has \( l(w) \geq 2 \). We rotated all results about the center of \( C_M \) by 120° and 240°, combining these with the un-rotated results. For any alcove which contained more than one number at that point, we took the maximum (although in all cases for which two numbers occurred in the same alcove, these numbers turned out to be equal). The outcome of this process was Figure 5, which shows the \( X_{\tilde{w}}(\sigma) \) that are non-empty (those corresponding to alcoves that have numbers in them), and the
dimension of these non-empty $X_{\tilde{w}}(\sigma)$. The bold lines in that figure correspond to the
shrunken Weyl chambers that will be discussed in Section 6.

Something observed in the course of the computation is that it never happened
that two different numbers occurred in the same alcove. This means that for any
fixed $\tilde{w}$, the non-empty pieces of $X_{\tilde{w}}(\sigma)$ all have the same dimension. As we will see
in the next section, this may be related to the fact that all vertices in the building
for $SL_3$ are special.

4 $Sp_4$

For $Sp_4$, it suffices to consider $v_1$ in the region pictured in Figure 6. All other $v_1$ can
be obtained from these by reflecting about the line of symmetry of $C_M$. Once results
are obtained for $v_1$ in the region specified, we will have to reflect the results across
the line of symmetry of $C_M$ as well. Note that $v_1$ can be special or non-special for
$Sp_4$, whereas only the special case was possible for $SL_3$.

Given the restriction that $p_r$ be such that $Q_2 \cap B_1 = \{v_2\}$, $Q_2$ and $\sigma(Q_2)$ must
share exactly one vertex, so $p_r$ corresponds to some element of $W$ of length 2, 3, or 4
for $v_1$ special, and some element of length 2 for $v_1$ non-special.

We define $\Gamma_{v_1}$, $\Gamma^{f}_{(v_1,p_r)}$, $\Gamma^{c}_{(v_1,p_r)}$ in the same way as in Section 3. The galleries $\Gamma_{(v_1,p_r)}$
have the general shapes pictured in Figure 7 for the case in which $v_1$ is non-special.
For clarity, only two of the galleries appearing in that figure have all of their parts
labelled. Figure 8 contains general shapes of the $\Gamma_{v_1}$ for $v_1$ special. The 20 different
general shapes of the $\Gamma_{(v_1,p_r)}$ can be deduced from these 4 possible $\Gamma_{v_1}$ by determining
$\Gamma^{f}_{(v_1,p_r)}$ and $\Gamma^{c}_{(v_1,p_r)}$ from each $\Gamma_{v_1}$ using each of the 5 possible $p_r$.

We define choice edges, hard and easy choices, and the choice tree of a non-
stuttering, non-backtracking gallery $\Omega$ in $A_M$ that does not cross itself just as we
did for $SL_3$. Again call the choice tree $T$. As for $SL_3$, any non-backtracking path
Figure 5: Main result in diagram form for $SL_3$

Figure 6: The region containing all vertices $v_1$ that must be considered for $Sp_4$
Figure 7: General shapes of the $\Gamma_{{(v_1,pr)}}$ for $Sp_4$, non-special $v_1$

Figure 8: General shapes of the $\Gamma_{v_1}$ for $Sp_4$, special $v_1$
from the root node of $T$ to a leaf corresponds to the retraction of some gallery starting at $C_M$ of the same type as $\Omega$. All galleries starting at $C_M$ of the same type as $\Omega$ retract in a way specified by some non-backtracking path from the root node of $T$ to a leaf. We define the set of comprehensive folding results of $\Omega$ as before.

The set of comprehensive folding results of $\Omega = \widetilde{\Gamma}_{(v_1, p_r)}$ contains the set of possible $inv(D, \sigma(D))$ for $D$ in the $(v_1, v_2, p_r)$-superpiece. We claim that this is an equality of sets. We define cf-dimensional of a non-backtracking path $F$ from root to leaf as before. We claim that the cf-dimensional of $F$ is equal to the dimension of the $(v_1, v_2, p_r)$-piece of $X_\omega(\sigma)$, where $\tilde{w}C_M$ is the comprehensive folding result of $\Omega$ corresponding to $F$.

We first note that $F$ can contain at most two easy choices. In fact, the maximum number of easy choices that $F$ can contain is $-m + 4$, where $m$ is the length of $p_r$ in $W$. This result is obtained by considering cases. For $SL_3$, the maximum number of easy choices is $-m + 3$. As in the $SL_3$ case, for $Sp_4$, $\Gamma_{v_1} \cup \Gamma^c_{(v_1, p_r)}$ is minimal.

We define a non-primal choice edge to be a node in $T$ that occurs below some easy choice in $T$ (i.e., the non-backtracking path from the root node to the node in question passes through an edge in $T$ corresponding to an easy choice). A primal choice edge is any choice edge that is not non-primal. All choice edges for $SL_3$ are primal. For $SL_4$ and $Sp_4$, all primal choice edges in $\Omega = \widetilde{\Gamma}_{(v_1, p_r)}$ correspond to hyperplanes in $A_M$ that pass between two alcoves of $\Gamma^f_{(v_1, p_r)}$ and that also pass between two alcoves of $\Gamma_{v_1}$.

Given a primal choice edge in $F$, we define the two corresponding integers $1 \leq \beta_1, \beta_2 \leq n - 1$ as in the $SL_3$ case. We will also define $1 \leq \beta_1, \beta_2 \leq n - 1$ for a non-primal choice edge $e$, but in a slightly different way. Since $e$ is non-primal, there is some choice edge $d$ above $e$ in $F$ at which $F$ makes the easy choice. Let $h_d$ be the hyperplane in $A_M$ determined by the edge $d$ in $\Omega$ (so $h_d$ is a hyperplane separating two alcoves of $\Gamma^f_{(v_1, p_r)}$ and also two alcoves of $\Gamma_{v_1}$). Let $f_{h_d}$ be the flip in $A_M$ about $h_d$. Consider the gallery in $A_M$ obtained by applying $f_{h_d}$ to the alcoves in $\Omega$ that occur after $d$ (here $C_M$ is considered to be the first alcove of $\Omega$). Let $\tilde{e} = f_{h_d}(e)$, and let $h_{\tilde{e}}$ be the hyperplane in $A_M$ determined by $\tilde{e}$. Let $\beta_2$ be such that $h_{e}$ passes between the $\beta_2^{th}$ and $(\beta_2 + 1)^{th}$ alcoves of $\Gamma^f_{(v_1, p_r)}$ (here the $n^{th}$ alcove of $\Gamma^f_{(v_1, p_r)}$ is considered to be the one containing $v_1$). If $h_{\tilde{e}}$ passes between two alcoves of $\Gamma_{v_1}$, then let $\beta_1$ be such that $h_{\tilde{e}}$ passes between the $\beta_1^{th}$ and $(\beta_1 + 1)^{th}$ alcoves of $\Gamma_{v_1}$ (here $C_M$ is considered to be the first alcove of $\Gamma_{v_1}$). Otherwise let $\beta_1 = n$.

Now, given a non-backtracking path $F$ from the root node of $T$ to a leaf, we want to produce a gallery $\Lambda$ such that $y\Gamma_{v_1} = \Lambda$ for some $y \in Sp_4(L)$ with $yQ_1 = Q_2$ for some $Q_2 \in Q_2$, and such that $\rho_{C_M}(y^{-1}(\Lambda \cup \Lambda^c \cup \sigma(\Lambda)))$ gives the comprehensive folding result determined by $F$. Here, as before, $\Lambda^c$ is a minimal gallery from $Q_2$ to $\sigma(Q_2)$ that has the same type as $\Gamma^c_{(v_1, p_r)}$.

We choose $\Lambda_n = Q_2$. The dimension of the set of such choices is $l(\Gamma^c_{(v_1, p_r)}) - 1$. We assume by induction that we have constructed $\Lambda_i, \ldots, \Lambda_n$ and $\sigma(\Lambda_n), \ldots, \sigma(\Lambda_i)$, and that the dimension of the space of choices for this construction is $l(\Gamma^c_{(v_1, p_r)}) + (n - i) - n(E,i)$, where $n(E,i)$ is defined to be the number of choice edges $e$ in $F$ such that $i \leq \beta_1, \beta_2$ and such that $F$ indicates a hard choice at $e$. Here $\beta_1, \beta_2$ are the integers
corresponding to \( e \), defined in the new way. If \( i = 1 \), \( l(\Gamma_{(v_1,p_r)}^c) + (n-i) - 1 - n_{(F,i)} = l(\Gamma_{v_1}) + l(\Gamma_{(v_1,p_r)}) - n_F - 2 \), and if \( i = n \) then \( l(\Gamma_{(v_1,p_r)}^c) + (n-i) - 1 - n_{(F,i)} = l(\Gamma_{(v_1,p_r)}^c) - 1 \).

As was the case for \( SL_3 \), let \( A \) be some apartment containing \( \Lambda_i \) and \( \sigma(\Lambda_i) \). Let \( S \subset A \) be the intersection of all apartments that contain both \( \Lambda_i \) and \( \sigma(\Lambda_i) \). Let \( d_{i-1} \) be the edge of \( \Lambda_i \) to which \( \Lambda_{i-1} \) must be attached. Let \( \Lambda_{i-1} \) be the alcove in \( A \) that one gets by reflecting \( \Lambda_i \) about \( d_{i-1} \). Let \( \sigma(\Lambda_{i-1}) \) be the alcove in \( A \) one gets by reflecting \( \sigma(\Lambda_i) \) about \( \sigma(d_{i-1}) \). Let \( S_{i-1} \) be the intersection of all apartments containing \( S \cup \Lambda_{i-1} \) and let \( S'_{i-1} \) be the intersection of all apartments containing \( S \cup \Lambda_{i-1} \).

One can see by considering cases that either 1) \( \Lambda_{i-1} \not\subset S \), \( \Lambda_{i-1} \not\subset S'_{i-1} \), \( \sigma(\Lambda_{i-1}) \not\subset S \), \( \Lambda_{i-1} \subset S'_{i-1} \), \( \sigma(\Lambda_{i-1}) \subset S'_{i-1} \), or 2) \( \Lambda_{i-1} \not\subset S \), \( \Lambda_{i-1} \subset S'_{i-1} \), \( \sigma(\Lambda_{i-1}) \not\subset S \), \( \sigma(\Lambda_{i-1}) \subset S \), \( \Lambda_{i-1} \not\subset S \), or 4) \( \sigma(\Lambda_{i-1}) \subset S \), \( \Lambda_{i-1} \not\subset S \). One can also see by considering cases that \( i = \min(\beta_1, \beta_2) \) (for \( \beta_1, \beta_2 \) the two integers associated to some choice edge \( e \)) if and only if we are in case 2, 3, or 4. In contrast to the \( SL_3 \) case, it is possible for neither \( \Lambda_{i-1} \) nor \( \sigma(\Lambda_{i-1}) \) to be in \( S' \), while still \( \Lambda_{i-1} \subset S'_{i-1} \) and \( \sigma(\Lambda_{i-1}) \subset S_{i-1} \). To see this, consider the case in which \( p_r \) corresponds to an element of \( W \) of length 3 (pictured in Figure 3). This is the only situation in which case 2 arises.

In case 1 we can choose \( \Lambda_{i-1} \) to be any alcove adjacent to \( \Lambda_i \) by \( d_{i-1} \). In this case, the dimension of the space of choices for the construction increases by one, and is therefore equal to \( l(\Gamma_{(v_1,p_r)}^c) + (n - (i - 1)) - 1 - n_{(F,i)} \). We also have \( n_{(F,i)} = n_{(F,i-1)} \).

Cases 3 and 4 only occur when \( p_r \) corresponds to an element of \( W \) of length 2. We address case 3. The other case is similar. We know \( i = \min(\beta_1, \beta_2) \) for \( \beta_1 \) and \( \beta_2 \) the two integers corresponding to some choice edge \( e \). If \( F \) dictates a hard choice at \( e \), we choose \( \Lambda_{i-1} \) in \( S \). In this case there is no increase in the dimension of the space of choices of the construction, and \( n_{(F,i-1)} = n_{(F,i)} + 1 \), so the dimension of the new space of choices is \( l(\Gamma_{(v_1,p_r)}^c) + (n - i) - 1 - n_{(F,i)} = l(\Gamma_{(v_1,p_r)}^c) + (n - (i - 1)) - 1 - n_{(F,i-1)} \). If \( F \) dictates an easy choice at \( e \), we choose \( \Lambda_{i-1} \) to be any alcove attached to \( \Lambda_i \) at \( d_{i-1} \), but not in \( S \). There is one dimension worth of such choices, so the dimension of the space of choices of the construction increases by one. We have \( n_{(F,i-1)} = n_{(F,i)} \), so the new dimension is \( l(\Gamma_{(v_1,p_r)}^c) + (n - i) - 1 - n_{(F,i)} + 1 = l(\Gamma_{(v_1,p_r)}^c) + (n - (i - 1)) - 1 - n_{(F,i-1)} \).

We now consider case 2, which only occurs when \( p_r \) corresponds to an element of \( W \) of length 3. The construction \( \Lambda_1, \ldots, \Lambda_n, \sigma(\Lambda_n), \ldots, \sigma(\Lambda_1) \) is contained in an apartment, and has the general shape pictured in Figure 3. The dashed lines in this figure represent the boundary of \( S \). Any choice of \( \Lambda_{i-1} \) determines a \( g(\Lambda_{i-1}) \) attached to \( \sigma(\Lambda_i) \) via \( \sigma(d_{i-1}) \), just by taking the alcove adjacent to \( \sigma(\Lambda_i) \) via \( \sigma(d_{i-1}) \) in the intersection of all apartments containing \( \Lambda_{i-1} \) and \( S \). We claim that the number of choices of \( \Lambda_{i-1} \) with \( g(\Lambda_{i-1}) = \sigma(\Lambda_{i-1}) \) is non-zero and finite. If this is true and if \( F \) requires a hard choice, take \( \Lambda_{i-1} \) with \( g(\Lambda_{i-1}) = \sigma(\Lambda_{i-1}) \). Then dimension does not increase, and is therefore equal to \( l(\Gamma_{(v_1,p_r)}^c) + (n - i) - 1 - n_{(F,i)} \), which is \( l(\Gamma_{(v_1,p_r)}^c) + (n - (i - 1)) - 1 - n_{(F,i-1)} \), since \( n_{(F,i-1)} = n_{(F,i)} + 1 \). If \( F \) requires an easy choice, take \( \Lambda_{i-1} \) with \( g(\Lambda_{i-1}) \not= \sigma(\Lambda_{i-1}) \). Then dimension increases by one, and is therefore \( l(\Gamma_{(v_1,p_r)}^c) + (n - i) - 1 - n_{(F,i)} + 1 \).
since \( n(F;i-1) = n(F,i) \).

We now prove the claim of the previous paragraph. We can identify the set \( \{ \Lambda_{i-1} \} \) with \( \mathbb{A}^1 \) over \( \bar{F}_q \), where \( F_q \) is the residue field of \( F \). We can identify the set \( \{ \sigma(\Lambda_{i-1}) \} \) with the set \( \{ \Lambda_{i-1} \} \) (and therefore with \( \mathbb{A}^1 \)) using \( g \). The map \( \sigma : \{ \Lambda_{i-1} \} \rightarrow \{ \sigma(\Lambda_{i-1}) \} \) given by the action of \( \sigma \) on \( B_\infty \) therefore gives a map \( \psi : \mathbb{A}^1 \rightarrow \mathbb{A}^1 \). But \( \sigma \) also acts on \( \mathbb{A}^1(\bar{F}_q) \) as the (algebraic) Frobenius, and one can show that if \( \varphi : \mathbb{A}^1 \rightarrow \mathbb{A}^1 \) is defined by \( \psi = \varphi \circ \sigma \), then \( \varphi \) is an algebraic isomorphism of \( \mathbb{A}^1 \). So \( \varphi(x) = ax + b \) with \( a \neq 0 \). The fixed points of \( \psi \) correspond to \( x \in \mathbb{A}^1 \) such that \( a\sigma(x) + b = x \), which has exactly \( q \) solutions since \( \sigma(x) = x^q \).

So now we can compute emptiness/non-emptiness and dimension of the \((v_1, v_2, p_r)\)-piece of \( X_{\bar{w}}(\sigma) \) for each \( \bar{w} \) by doing straightforward computations with cf-dimension. We did this for all \( v_1 \), and we reflected the results across the line of symmetry of \( C_M \). We took maxima whenever two numbers appeared in the same alcove. The results of this process can be found in Figure 10.

In the course of the computation we observed that if the \((v_1, v_2, p_r)\)-piece of \( X_{\bar{w}}(\sigma) \) and the \((v'_1, v'_2, p'_r)\)-piece of \( X_{\bar{w}}(\sigma) \) had different dimensions, then exactly one of \( v_1, v'_1 \) was non-special, and the corresponding piece had the smaller dimension.

### 5 Application to \( \dim(X^K_{\bar{w}}(\sigma)) \)

Let \( \bar{w} \in \hat{W} \) and let \( \mu \) be a dominant cocharacter. Let \( \pi \) be the uniformizer in \( F \). The map \( G(L)/I \rightarrow G(L)/K \) gives a map \( X^I_{\bar{w}}(b\sigma) \rightarrow X^K_{\mu(\pi)}(b\sigma) \) whenever \( I\bar{w}I \subset K\mu(\pi)K \). The non-empty fibers of this map are always \( K/I \), which has dimension equal to the length \( \delta \) of the longest element of the finite Weyl group, \( W \). Further, any point in \( X^K_{\mu(\pi)}(b\sigma) \) is hit by a point in \( X^I_{\bar{w}}(b\sigma) \) for some \( \bar{w} \) with \( I\bar{w}I \subset K\mu(\pi)K \). If \( S_{\mu(\pi)} \subset W \) is defined so that \( \bigsqcup_{\bar{w} \in S_{\mu(\pi)}} I\bar{w}I = K\mu(\pi)K \), then we have \( \dim(X^K_{\mu(\pi)}(b\sigma)) = \max_{\bar{w} \in S_{\mu(\pi)}} (\dim(X^I_{\bar{w}}(b\sigma))) - \delta \). We applied this formula to the cases \( b = 1, G = SL_2, SL_3, Sp_4 \), and found that \( \dim(X^K_{\mu(\pi)}(\sigma)) = \langle \mu, \rho \rangle \), where \( \rho \) is half the sum of the positive roots of \( G \). This result supports Rapoport's Conjecture 5.10 in [10].
Figure 10: Main result in diagram form for $Sp_4$
6 A partial formula for $\dim(X^I_{\tilde{w}}(\sigma))$ for $SL_2$, $SL_3$ and $Sp_4$

Suppose $G$ is a simply-connected group and suppose $\tilde{w} \in \tilde{W}$. Let $\tilde{w} = tw$, where $w \in W$ and $t$ acts on $A_M$ by translation. Let $\eta_2(\tilde{w}) = \alpha \in W$, where $\tilde{w}C_M$ is in the same Weyl chamber as $\alpha C_M$. Let $\eta_1 : \tilde{W} \rightarrow W$ be the quotient map by the subgroup of translations. Let $S$ be the set of simple reflections in $W$, and let $W_T$ be the subgroup of $W$ generated by $T \subset S$.

Let $h_1, \ldots, h_{n+1}$ be the hyperplanes in $A_M$ that contain one of the codimension-one sub-simplices of $C_M$. Here $n$ is the rank of $G$. Let $h_i^{(j)}$ be the hyperplanes in $A_M$ parallel to $h_i$, with $h_i^{(0)} = h_i$. Choose $h_i^{(1)}$ to be as close as possible to $h_i$, but on the other side of $C_M$. We define the union of shrunken Weyl chambers to be the set of all alcoves that are not between $h_i^{(0)}$ and $h_i^{(1)}$ for any $i$.

If $\tilde{w}C_M$ is in the union of shrunken Weyl chambers and if $G = SL_2$, $SL_3$, or $Sp_4$, then $X_{\tilde{w}}(\sigma)$ is non-empty if and only if $\eta_2(\tilde{w})^{-1} \eta_1(\tilde{w}) \eta_2(\tilde{w}) \in W \setminus \cup_{T \subset S} W_T$, and in this case

$$\dim(X^I_{\tilde{w}}(\sigma)) = \frac{l_W(\tilde{w}) + l_W(\eta_2(\tilde{w})^{-1} \eta_1(\tilde{w}) \eta_2(\tilde{w}))}{2}.$$

Here $l_W$ is length in $W$ and $l_{\tilde{w}}$ is length in $\tilde{W}$, as Coxeter groups.

One can examine Figures 5 and 10 to see that the above statement holds for $SL_3$ and $Sp_4$. It also holds for $SL_2$. Note, though, that the new formula says nothing about the dimension or emptiness/non-emptiness of $X_{\tilde{w}}(\sigma)$ for $\tilde{w}$ not in the union of the shrunken Weyl chambers. Figures 5 and 10 give this information for $SL_3$ and $Sp_4$. The complement of the union of the shrunken Weyl chambers for $SL_2$ is just $C_M$, an easy special case.

The above formula might not hold for $SL_4$. One problem is that, for $SL_2$, $SL_3$, and $Sp_4$, there are other ways to specify the set $W \setminus \cup_{T \subset S} W_T$. In particular, $W \setminus \cup_{T \subset S} W_T = \{w \in W : l_W(w) \geq \text{rank}(G)\}$ for these groups, and the $SL_4$ analogues of these two sets are not the same. We think the first formulation is more likely to be appropriate for a general statement.

7 Related results

Some of the emptiness/non-emptiness results of this paper were also obtained using other methods in the author’s Ph.D. thesis [11]. These other methods are more computationally intensive, and do not provide dimension information, but they extend to some extent to $b \neq 1$. Some of the results from [11] can be combined to suggest the below conjecture.

We restrict $G$ to be one of the groups $SL_2$, $SL_3$, and $Sp_4$. Let $D$ be a Weyl chamber in $A_M$, and let $D'$ be the intersection of $D$ with the union of the shrunken Weyl chambers. Then we call $D'$ a shrunken Weyl chamber. Let $b$ be a representative of a $\sigma$-conjugacy class that meets the main torus of $G$. We can choose $b$ so that it
acts on $A_M$ by translation, and such that $bC_M$ is in the main Weyl chamber. We define the $b$-shifted shrunken Weyl chamber associated to $D$ to be $wbw^{-1}D'$, where $w \in W$ is the Weyl group element corresponding to the Weyl chamber $D$.

**Conjecture 7.1** If $b$ and $G$ are restricted as above, and if $\tilde{w}C_M$ is in the union of the $b$-shifted shrunken Weyl chambers, then $X_{\tilde{w}}(b\sigma)$ is non-empty if and only if $\eta_2(\tilde{w})^{-1}\eta_1(\tilde{w})\eta_2(\tilde{w}) \in W \setminus \cup_{T \subset SW} T$.

This conjecture is shown to hold true in [11] for several values of $b$. Information about $\tilde{w}$ not in the $b$-shifted shrunken Weyl chambers is also given in [11] for the same $b$ values, but we have been unable to describe these results with a formula.

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References

[1] Bruhat, F. and Tits, J. *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. 41 (1972), 5–251.

[2] Bruhat, F. and Tits, J. *Groupes réductifs sur un corps local II*, Inst. Hautes Études Sci. Publ. Math. 60 (1984), 197–376.

[3] Deligne, P. and Lusztig, G. *Representations of reductive groups Over finite fields*, Ann. of Math. 103 (1976), 103–161.

[4] Kottwitz, R. *Isocrystals with additional structure*, Compositio Math. 56 (1985), 201–220.

[5] Kottwitz, R. *Isocrystals with additional structure II*, Compositio Math. 109 (1997), 255–339.

[6] Kottwitz, R. * Orbital integrals on $GL_3$*, Amer. J. of Math. 102 (1980), 327–384.

[7] Kottwitz, R. and Rapoport, M. *On the existence of $F$-crystals*, Comment. Math. Helv., to appear, arXiv:math.NT/0202229

[8] Manin, Y. *The theory of commutative formal groups over fields of finite characteristic*, Russian Math. Surveys 18 #6 (1963), 1–81.
[9] Rapoport, M. *A positivity property of the satake isomorphism*, Manuscripta Math. 101 (2000), no. 2 153-166.

[10] Rapoport, M. *A guide to the reduction modulo p of Shimura varieties*, arXiv:math.AG/0205022

[11] Reuman, D. *Determining whether certain affine Deligne-Lusztig sets are empty*, arXiv:math.NT/0211434