Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation

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Abstract

In this paper, compact finite difference schemes for the modified anomalous fractional sub-diffusion equation and fractional diffusion-wave equation are studied. Schemes proposed previously can at most achieve temporal accuracy of order which depends on the order of fractional derivatives in the equations and is usually less than two. Based on the idea of weighted and shifted Gr"{u}nwald difference operator, we establish schemes with temporal and spatial accuracy order equal to two and four respectively.

Keywords: Modified anomalous fractional sub-diffusion equation, Fractional diffusion-wave equation, Compact difference scheme, Weighted and shifted Gr"{u}nwald difference operator

1 Introduction

Since fractional differential equations turn out to model many physical processes more accurately than the classical ones, in the past decades, increasing attentions have been made on these equations. Readers can refer to the books [1, 2] for theoretical results on fractional differential equations. This paper concerns with methods for obtaining accurate numerical approximations to the solutions of fractional sub-diffusion equations and fractional diffusion-wave equations. A fractional sub-diffusion equation is an integro-partial differential equation obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of order between zero and one. When the time derivative is of order between one and two, we get a fractional diffusion-wave equation. Fractional derivatives of order between zero and one are widely used in describing anomalous diffusion processes [3], while fractional diffusion-wave equations have applications in modeling universal electromagnetic, acoustic, and mechanical responses [4, 5].

Numerical methods for the modified anomalous fractional sub-diffusion equation and the diffusion-wave equation have been considered by many authors, one may refer to [6–17] and the references therein. We point out here that one of the main tasks for developing accurate finite

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difference scheme of fractional differential equations is to discretize the fractional derivatives. Noticing that fractional derivatives are defined through integrals, one can approximate the derivatives by interpolating polynomials [18]. Using this idea, Sousa and Li developed a second order discretization for the Riemann-Liouville fractional derivative [19].

We note that there are alternative ways to tackle the problem. In [20], a shifted Grünwald formula was proposed by Meerschaert and Tadjeran to approximate fractional derivatives of order $\alpha \in (1, 2)$ for fractional advection-dispersion flow equations. It is also worth to mention that stability of forward-Euler scheme and weighted averaged difference scheme based on Grünwald-Letnikov approximation were analyzed in [21, 22] for time fractional diffusion equations. Very recently, accurate finite difference schemes based on weighted and shifted Grünwald difference operator were developed for solving space fractional diffusion equations in [23, 24].

Inspired by their work on the weighted and shifted Grünwald difference operator, in this paper, we establish high order schemes for the fractional diffusion-wave equation and the modified anomalous fractional sub-diffusion equation, which were proposed and studied in [8, 9] and [25–27], respectively. We remark that the order of temporal accuracy of schemes proposed previously can at most be a fraction depending on the fractional derivatives in the equations and is usually less than two. We show that the schemes proposed in this paper are of order $\tau^2 + h^4$, where $\tau$ and $h$ are the temporal and spatial step sizes respectively.

This paper is organized as follows. Some preliminaries will be given in the next section. Compact schemes are proposed and studied in Section 3 and 4 for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation respectively. In the last section, numerical tests are carried out to justify the theoretical results.

2 Preliminaries

We first recall that the Caputo fractional derivative of order $\gamma \in (1, 2)$ for a function $f(t)$ is defined as

$$ C^\gamma_a D^\gamma_t f(t) = \frac{1}{\Gamma(2-\gamma)} \int_a^t \frac{f''(s)}{(t-s)^{\gamma-1}} ds, $$

with $\Gamma(\cdot)$ being the gamma function and, for $\alpha \in (0, 1)$, the Riemann-Liouville fractional derivative of order $\alpha$ for $f$ is defined as

$$ a D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha} ds. $$

Closely related to the fractional derivatives of a function is the Riemann-Liouville fractional integral which is given by

$$ a I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds. $$

In order to develop second order approximation of the Riemann-Liouville fractional derivative, we consider the shifted Grünwald approximation [20] to the Riemann-Liouville fractional derivative given by

$$ A^\alpha_{\tau,r} f(t) = \tau^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(t - (k - r)\tau), $$

where
where \( g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} \) for \( k \geq 0 \). Inspired by [28], we similarly introduce the shifted operator to the Riemann-Liouville fractional integral defined by

\[
B_{\tau,r}^{\alpha} f(t) = \tau^\alpha \sum_{k=0}^{\infty} \omega_k^{(\alpha)} f(t - (k - r)\tau),
\]

where \( \omega_k^{(\alpha)} = (-1)^k \binom{-\alpha}{k} \) for \( k \geq 0 \).

The following lemma was given in [29].

**Lemma 2.1** Suppose \( \alpha > 0 \), \( f(t) \in L^p(\mathbb{R}) \), \( p \geq 1 \). The Fourier transform of the Riemann-Liouville fractional integral satisfy the following:

\[
\mathcal{F}[\mathcal{I}_t^{\alpha} f(t)] = (i\omega)^{-\alpha} \hat{f}(\omega),
\]

where \( \hat{f}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} f(t) dt \) denotes the Fourier transform of \( f(t) \).

**Lemma 2.2**

(i) Let \( f(t) \in L^1(\mathbb{R}) \), \( -\infty D_t^{\alpha+2} f \) and its Fourier transform belong to \( L^1(\mathbb{R}) \), and define the weighted and shifted Grünwald difference operator by

\[
D_{\tau,p,q}^{\alpha} f(t) = \frac{\alpha - 2q}{2(p - q)} A_{\tau,p}^{\alpha} f(t) + \frac{2p - \alpha}{2(p - q)} A_{\tau,q}^{\alpha} f(t),
\]

then we have

\[
D_{\tau,p,q}^{\alpha} f(t) = -\infty D_t^{\alpha} f(t) + O(\tau^2)
\]

for \( t \in \mathbb{R} \), where \( p \) and \( q \) are integers and \( p \neq q \).

(ii) Let \( f(t) \), \( -\infty I_t^{\alpha} f(t) \) and \( (i\omega)^{2-\alpha} \mathcal{F}[f](\omega) \) belong to \( L^1(\mathbb{R}) \). Define the weighted and shifted difference operator by

\[
I_{\tau,p,q}^{\alpha} f(t) = \frac{2q + \alpha}{2(q - p)} B_{\tau,p}^{\alpha} f(t) + \frac{2p + \alpha}{2(p - q)} B_{\tau,q}^{\alpha} f(t),
\]

then we have

\[
I_{\tau,p,q}^{\alpha} f(t) = -\infty I_t^{\alpha} f(t) + O(\tau^2)
\]

for \( t \in \mathbb{R} \), where \( p \) and \( q \) are integers and \( p \neq q \).

**Proof.** The proof of (i) can be found in [23]. The proof of (ii) is similar to that of (i) but we give it here for the completeness of our presentation.

Referring to the definition of \( B_{\tau,r}^{\alpha} \), we let

\[
I_{\tau,p,q}^{\alpha} f(t) = \tau^\alpha \left[ \mu_1 \sum_{k=0}^{\infty} \omega_k^{(\alpha)} f(t - (k - p)\tau) + \mu_2 \sum_{k=0}^{\infty} \omega_k^{(\alpha)} f(t - (k - q)\tau) \right].
\]

Taking Fourier transform on (2), we get

\[
\mathcal{F}[I_{\tau,p,q}^{\alpha} f(t)](\omega) = \tau^\alpha \sum_{k=0}^{\infty} \omega_k^{(\alpha)} \left[ \mu_1 e^{-i\omega(k-p)\tau} + \mu_2 e^{-i\omega(k-q)\tau} \right] \mathcal{F}[f](\omega)
\]

\[
= \tau^\alpha \left[ \mu_1 (1 - e^{-i\omega\tau})^{-\alpha} e^{i\omega p} + \mu_2 (1 - e^{-i\omega\tau})^{-\alpha} e^{i\omega q} \right] \mathcal{F}[f](\omega)
\]

\[
= (i\omega)^{-\alpha} \left[ \mu_1 W_p(i\omega\tau) + \mu_2 W_q(i\omega\tau) \right] \mathcal{F}[f](\omega),
\]

where

\[
W_p(i\omega\tau) = \sum_{k=0}^{\infty} \omega_k^{(\alpha)} e^{-i\omega(k-p)\tau}
\]

and

\[
W_q(i\omega\tau) = \sum_{k=0}^{\infty} \omega_k^{(\alpha)} e^{-i\omega(k-q)\tau}.
\]
In order to achieve second order accuracy, we let the coefficients \( \mu_1 \) and \( \mu_2 \) satisfy the following system:

\[
\begin{cases}
\mu_1 + \mu_2 = 1, \\
(p + \frac{\alpha}{2})\mu_1 + (q + \frac{\alpha}{2})\mu_2 = 0,
\end{cases}
\]

which implies that \( \mu_1 = \frac{2q+\alpha}{2(q-p)} \) and \( \mu_2 = \frac{2p+\alpha}{2(p-q)} \).

Denote \( \hat{g}(\omega, \tau) = \mathcal{F}[I_{\tau, p,q}^\alpha f](\omega) - \mathcal{F}[-\infty I_{t}^\alpha f](\omega) \), then by \([3], [11] \) and Lemma \([2], [11] \) we have

\[
|\hat{g}(\omega, \tau)| \leq C \tau^2 |i\omega|^{2-\alpha} |\mathcal{F}[f](\omega)|.
\]

Thus

\[
|I_{\tau, p,q}^\alpha f - -\infty I_{t}^\alpha f| = |g| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\omega, \tau)| d\omega \leq C \|(i\omega)^{2-\alpha} \mathcal{F}[f](\omega)\|_{L^1} \tau^2 = O(\tau^2).
\]

We are now ready to establish our high order compact schemes in the next two sections.

### 3 A compact scheme for the fractional sub-diffusion equation

Consider the following modified anomalous fractional sub-diffusion equation

\[
\frac{\partial u(x,t)}{\partial t} = \left( \kappa_1 \frac{\partial^\alpha}{\partial t^\alpha} + \kappa_2 \frac{\partial^\beta}{\partial t^\beta} \right) \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right] + f(x,t), \quad 0 \leq x \leq L, \quad 0 < t \leq T, \quad (5)
\]

subject to

\[
\begin{align*}
&u(x,0) = 0, \quad 0 \leq x \leq L, \\
&u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad 0 < t \leq T,
\end{align*} \quad (6)
\]

where \( 0 < \alpha, \beta < 1, \kappa_1, \kappa_2 \geq 0 \). We have used \( \frac{\partial^\alpha}{\partial t^\alpha} \) and \( \frac{\partial^\beta}{\partial t^\beta} \) to denote the Riemann-Liouville fractional operators \( _0D_t^\alpha \) and \( _0D_t^\beta \) with respect to the time variable \( t \).

**Remark 3.1** (i) We note that, in \([25], [26], [27] \), the fractional derivatives in the equation are of order \( 1-\alpha \) and \( 1-\beta \) with \( \alpha, \beta \in (0, 1) \). We have changed the notations here in order to match the presentation for the two types of equations discussed in this paper.

(ii) Without loss of generality, we have assumed the initial condition \( u(x,0) = 0 \). If \( u(x,0) = \psi(x) \), one may consider the equation for \( v(x,t) = u(x,t) - \psi(x) \) instead.

To develop a finite difference scheme for the problem \([5], [6] \), we let \( h = \frac{L}{M} \) and \( \tau = \frac{T}{N} \) be the spatial and temporal step sizes respectively, where \( M \) and \( N \) are some given positive integers. For \( i = 0, 1, \ldots, M \) and \( k = 0, 1, \ldots, N \), denote \( x_i = ih, \quad t_k = k\tau \). For any grid function \( u = \{u_i^k|0 \leq i \leq M, \quad 0 \leq k \leq N\} \), we introduce the following notations:

\[
\delta_x u_{i-\frac{1}{2}}^k = \frac{1}{h}(u_{i}^k - u_{i-1}^k), \quad \delta_x^2 u_{i}^k = \frac{1}{h}(\delta_x u_{i+\frac{1}{2}}^k - \delta_x u_{i-\frac{1}{2}}^k),
\]

\[ \mathcal{H}u_i = \begin{cases} 
(1 + \frac{h^2}{12}\delta^2_x)u_i = \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \leq i \leq M - 1, \\
u_i, & i = 0 \text{ or } M. 
\end{cases} \]

With this notations, we study the problem under the following inner product and norms:
\[ \langle u, v \rangle = h \sum_{i=1}^{M-1} u_iv_i, \quad \langle \delta_x u, \delta_x v \rangle = h \sum_{i=0}^{M-1} \left( \delta_x u_{i+\frac{1}{2}} \right) \left( \delta_x v_{i+\frac{1}{2}} \right), \]
\[ \|u\|^2 = \langle u, u \rangle, \quad \|u\|_\infty = \max_{0 \leq i \leq M} |u_i|. \]

It is easy to check that, if \( v \in \mathcal{V} = \{ w | w = (w_0, w_1, \ldots, w_M), w_0 = w_M = 0 \} \), the following identity holds
\[ \langle \delta_x^2 u, v \rangle = -\langle \delta_x u, \delta_x v \rangle, \]
which plays important role in our analysis.

Note that one can continuously extend the solution \( u(x,t) \) to be zero for \( t < 0 \). By choosing \((p,q) = (0,-1)\) in (i) of Lemma 2.2 we get \( \frac{\alpha-2q}{2(p-q)} = 2+\alpha, \quad \frac{2p-\alpha}{2(p-q)} = -\frac{\alpha}{2} \) in (1), which gives
\[ \frac{\partial^n}{\partial x^n}[u_{xx}(x_i, t_{n+1})] = \tau^{-\alpha} \left( \frac{2 + \alpha}{2} \sum_{k=0}^{n+1} g_k^{(\alpha)} \delta_x^2 u_{i}^{n+1-k} - \alpha \sum_{k=0}^{n} g_k^{(\alpha)} \delta_x^2 u_{i}^{n-k} \right) + O(\tau^2 + h^2) \]
\[ = \tau^{-\alpha} \sum_{k=0}^{n+1} \lambda_k^{(\alpha)} \delta_x^2 u_{i}^{n+1-k} + O(\tau^2 + h^2), \]
where
\[ \lambda_0^{(\alpha)} = \frac{2 + \alpha}{2} g_0^{(\alpha)}, \quad \lambda_k^{(\alpha)} = \frac{2 + \alpha}{2} g_k^{(\alpha)} - \alpha g_{k-1}^{(\alpha)}, \quad k \geq 1. \quad (7) \]

We can therefore consider an weighted Crank-Nicolson type discretization for equation (5) given by
\[ \frac{u_{i}^{n+1} - u_{i}^{n}}{\tau} = \frac{\kappa_1 \tau^{-\alpha}}{2} \left( \sum_{k=0}^{n+1} \lambda_k^{(\alpha)} \delta_x^2 u_{i}^{n+1-k} + \sum_{k=0}^{n} \lambda_k^{(\alpha)} \delta_x^2 u_{i}^{n-k} \right) \]
\[ + \frac{\kappa_2 \tau^{-\beta}}{2} \left( \sum_{k=0}^{n+1} \lambda_k^{(\beta)} \delta_x^2 u_{i}^{n+1-k} + \sum_{k=0}^{n} \lambda_k^{(\beta)} \delta_x^2 u_{i}^{n-k} \right) + \frac{1}{2} (f_i^n + f_i^{n+1}). \]

In order to raise the accuracy in the spatial direction, we need the following lemma:

**Lemma 3.1** (30) Denote \( \zeta(s) = (1-s)^3[5-3(1-s)^2] \). If \( f(x) \in C^6[x_{i-1}, x_{i+1}], 1 \leq i \leq M - 1 \), then it holds that
\[ \frac{1}{12} [f''(x_{i-1}) + 10f''(x_i) + f''(x_{i+1})] = \frac{1}{h^2} [f(x_{i-1}) - 2f(x_i) + f(x_{i+1})] \]
\[ + \frac{h^4}{360} \int_0^1 [f^{(6)}(x_i - sh) + f^{(6)}(x_i + sh)] \zeta(s) ds. \]
Based on Lemma 3.1, we therefore propose the following compact scheme:

\[ \mathcal{H}(u_i^{n+1} - u_i^n) = \frac{\kappa_1 \tau^{1-\alpha}}{2} \left( \sum_{k=0}^{n+1} \lambda_k^{(\alpha)} \delta_x^2 u_i^{n+1-k} + \sum_{k=0}^{n} \lambda_k^{(\alpha)} \delta_x^2 u_i^{n-k} \right) \]

\[ + \frac{\kappa_2 \tau^{1-\beta}}{2} \left( \sum_{k=0}^{n+1} \lambda_k^{(\beta)} \delta_x^2 u_i^{n+1-k} + \sum_{k=0}^{n} \lambda_k^{(\beta)} \delta_x^2 u_i^{n-k} \right) + \frac{\tau}{2} \mathcal{H}(f_i^n + f_i^{n+1}), \]

0 ≤ n ≤ N − 1, 1 ≤ i ≤ M − 1,

\[ u_0^n = \varphi_1^n, \quad u_M^n = \varphi_0^n, \quad 1 ≤ n ≤ N, \]

\[ u_i^n = 0, \quad 0 ≤ i ≤ M. \]

(8)

(9)

It is easy to see that at each time level, the difference scheme is a linear tridiagonal system with strictly diagonal dominant coefficient matrix, thus the difference scheme has a unique solution.

The following lemmas are critical for establishing the convergence of the proposed scheme.

**Lemma 3.2** Let \( \lambda_n^{(\alpha)} \) be defined as in (7), then for any positive integer \( k \) and real vector \((v_1, v_2, \ldots, v_k)^T \in \mathbb{R}^k\), it holds that

\[ \sum_{n=0}^{k-1} \left( \sum_{p=0}^{n} \lambda_p^{(\alpha)} v_{n+1-p} \right) v_{n+1} ≥ 0. \]

**Proof.** For simplicity of presentation, in this proof, we denote \( g_p = g_p^{(\alpha)}, \lambda_p = \lambda_p^{(\alpha)} \) without ambiguity. One can easily check that, to prove the above quadratic form is nonnegative is equivalent to proving the symmetric Toeplitz matrix \( T \) is positive semi-definite, where

\[
T = \begin{pmatrix}
\lambda_0 & \frac{\lambda_1}{2} & \frac{\lambda_2}{2} & \cdots & \frac{\lambda_{k-1}}{2} \\
\frac{\lambda_1}{2} & \lambda_0 & \frac{\lambda_2}{2} & \cdots & \frac{\lambda_{k-1}}{2} \\
\frac{\lambda_2}{2} & \frac{\lambda_1}{2} & \lambda_0 & \cdots & \frac{\lambda_{k-1}}{2} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\frac{\lambda_{k-1}}{2} & \cdots & \frac{\lambda_{k-2}}{2} & \frac{\lambda_{k-1}}{2} & \lambda_0 
\end{pmatrix}.
\]

Notice that the generating function (see [31]) of \( T \) is given by

\[ f(\alpha, x) = \lambda_0 + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k e^{ikx} + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k e^{-ikx} \]

\[ = \frac{2 + \alpha}{2} g_0 + \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{2 + \alpha}{2} g_k - \frac{\alpha}{2} g_{k-1} \right) e^{ikx} + \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{2 + \alpha}{2} g_k - \frac{\alpha}{2} g_{k-1} \right) e^{-ikx} \]

\[ = \frac{2 + \alpha}{4} (1 - e^{ix})^\alpha + \frac{2 + \alpha}{4} (1 - e^{-ix})^\alpha - \frac{\alpha}{4} e^{ix}(1 - e^{ix})^\alpha - \frac{\alpha}{4} e^{-ix}(1 - e^{-ix})^\alpha. \]

As mentioned in [23], we only need to consider the principal value of \( f(\alpha, x) \) on \([0, \pi]\) which gives

\[ f(\alpha, x) = \frac{2 + \alpha}{4} [2i \sin(\frac{x}{2}) e^{ix}]^\alpha + \frac{2 + \alpha}{4} [2i \sin(\frac{x}{2}) e^{-ix}]^\alpha \]

\[ - \frac{\alpha}{4} e^{ix}[2i \sin(\frac{x}{2}) e^{ix}]^\alpha - \frac{\alpha}{4} e^{-ix}[2i \sin(\frac{x}{2}) e^{-ix}]^\alpha \]

\[ = [2 \sin(\frac{x}{2})]^\alpha \left\{ \frac{2 + \alpha}{2} \cos[\frac{\alpha}{2}(\pi - x)] - \frac{\alpha}{2} \cos[\frac{\alpha}{2}(\pi - x) - x] \right\}. \]
Let \( h(\alpha, x) = \frac{2 + \alpha}{2} \cos[\frac{\alpha}{2} (\pi - x)] - \frac{\alpha}{2} \cos[\frac{\alpha}{2} (\pi - x) - x] \), then one can easily check that
\[
h_x(\alpha, x) = \frac{\alpha(2 + \alpha)}{2} \sin(\frac{x}{2}) \cos[\frac{\alpha}{2} (\pi - x) - \frac{x}{2}] \geq 0.
\]
Therefore \( h(\alpha, x) \) is nondecreasing with respect to \( x \) and \( h(\alpha, x) \geq h(\alpha, 0) = \cos(\frac{\alpha}{2} \pi) \geq 0 \), which implies that \( f(\alpha, x) \geq 0 \). The lemma now follows as a result of the Grenander-Szeg"o Theorem \[31\]. □

**Remark 3.2** We note here that the function \( h(\alpha, x) \) can not be proved to be nonnegative by considering differentiation with respect to \( \alpha \) as in \[23\].

**Lemma 3.3** (Grownall’s inequality \[32\]) Assume that \( \{k_n\} \) and \( \{p_n\} \) are nonnegative sequences, and the sequence \( \{\phi_n\} \) satisfies
\[
\phi_0 \leq g_0, \quad \phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l \phi_l, \quad n \geq 1,
\]
where \( g_0 \geq 0 \). Then the sequence \( \{\phi_n\} \) satisfies
\[
\phi_n \leq \left( g_0 + \sum_{l=0}^{n-1} p_l \right) \exp \left( \sum_{l=0}^{n-1} k_l \right), \quad n \geq 1.
\]

With all the preparation, we can now show the convergence and stability of the compact finite difference scheme \[8\]–\[9\].

**Theorem 3.1** Assume that \( u(x,t) \in C^{2,1}_x([0,L] \times [0,T]) \) is the solution of \[5\]–\[6\] and \( \{u_j^k|0 \leq j \leq M, 0 \leq k \leq N\} \) is the solution of the finite difference scheme \[8\]–\[9\], respectively. Denote
\[
e_k^i = u(x_i, t_k) - u_j^k, \quad 0 \leq i \leq M, \quad 0 \leq k \leq N.
\]
Then there exists a positive constant \( \tilde{c}_1 \) such that
\[
\|e_k^i\| \leq \tilde{c}_1(\tau^2 + h^4), \quad 0 \leq k \leq N.
\]

**Proof.** We can easily get the error equation:
\[
H(e_{k+1}^i - e_k^i) = \frac{\eta_1^{1-\alpha}}{2} \sum_{l=0}^{k} \lambda_l^{(\alpha)} \delta_x^2(e_{k+1-l}^i + e_{k-l}^i) + \frac{\eta_2^{1-\beta}}{2} \sum_{l=0}^{k} \lambda_l^{(\beta)} \delta_x^2(e_{k+1-l}^i + e_{k-l}^i) + \tau R_{k}^{i+1}
\]
\[
0 \leq k \leq N-1, \quad 1 \leq i \leq M-1,
\]
\[
e_0^0 = e_M^0 = 0, \quad 1 \leq k \leq N
\]
\[
e_0^0 = 0, \quad 0 \leq i \leq M,
\]
where \( |R_{k}^{i+1}| \leq c_1(\tau^2 + h^4) \).
Denote $C = \frac{1}{12} tr[i[1, 10, 1]$, multiplying (11) by $h(e_{i+1} + e_1^k)$ and summing in $i$, we obtain

$$h(e_{i+1} + e_1^k)^T C (e_{i+1} - e_1^k) = \frac{\kappa_1 \tau^{1-\alpha}}{2} \sum_{l=0}^{k} \lambda_{l}^{(\alpha)} \langle \delta_x^{2} (e_{i+1-l}^k + e_{i-1}^k), e_{i+1}^k + e_1^k \rangle$$

$$+ \frac{\kappa_2 \tau^{1-\beta}}{2} \sum_{l=0}^{k} \lambda_{l}^{(\beta)} \langle \delta_x^{2} (e_{i+1-l}^k + e_{i-1}^k), e_{i+1}^k + e_1^k \rangle$$

$$+ \tau h(e_{i+1} + e_1^k)^T R_{i+1}$$

$$= -\frac{\kappa_1 \tau^{1-\alpha}}{2} \sum_{l=0}^{k} \lambda_{l}^{(\alpha)} \langle \delta_x^{2} (e_{i+1-l}^k + e_{i-1}^k), \delta_x (e_{i+1}^k + e_1^k) \rangle$$

$$- \frac{\kappa_2 \tau^{1-\beta}}{2} \sum_{l=0}^{k} \lambda_{l}^{(\beta)} \langle \delta_x^{2} (e_{i+1-l}^k + e_{i-1}^k), \delta_x (e_{i+1}^k + e_1^k) \rangle$$

$$+ \tau h(e_{i+1} + e_1^k)^T R_{i+1}.$$ (11)

Summing up for $0 \leq k \leq n - 1$ and noticing that

$$h(e_{i+1} + e_1^k)^T C (e_{i+1} - e_1^k) = h(e_{i+1}^T C e_{i+1} - e_1^T C e_1)$$

$$h e^T C e_n \geq \frac{2}{3} \| e_n^2 \|,$$

and

$$\tau h(e_{i+1} + e_1^k)^T R_{i+1} \leq \frac{\tau}{3} (\| e_{i+1}^2 \| + \| e_1^2 \| + \frac{3 \tau}{2} \| R_{i+1}^2 \|,$$

we get, by Lemma 3.2 that

$$\frac{2}{3} \| e_n^2 \| \leq \frac{\kappa_1 \tau^{1-\alpha}}{2} \sum_{k=0}^{n} \sum_{l=0}^{k} \lambda_{l}^{(\alpha)} \langle \delta_x^{2} (e_{i+1-l}^k + e_{i-1}^k), \delta_x (e_{i+1}^k + e_1^k) \rangle$$

$$- \frac{\kappa_2 \tau^{1-\beta}}{2} \sum_{k=0}^{n} \sum_{l=0}^{k} \lambda_{l}^{(\beta)} \langle \delta_x^{2} (e_{i+1-l}^k + e_{i-1}^k), \delta_x (e_{i+1}^k + e_1^k) \rangle$$

$$+ \tau \| e_{i+1}^2 \| + \| e_1^2 \| + \frac{3 \tau}{2} \sum_{k=0}^{n-2} \| R_{i+1}^2 \| + \tau h(e_n + e_{n-1})^T R_n$$

$$\leq \frac{1}{3} \| e_n^2 \| + \| e_{n-1}^2 \| + \frac{3 \tau^2}{4} \| R_n^2 \| + \frac{3 \tau}{4} \| R_{n-1}^2 \|$$

$$+ \frac{\tau}{3} \sum_{k=1}^{n-1} \| e_k^2 \|^2 + \frac{\tau}{3} \sum_{k=1}^{n-2} \| e_k^2 \|^2 + \frac{3 \tau}{2} \sum_{k=0}^{n-2} \| R_{k+1}^2 \|^2$$

$$\leq \frac{1}{3} \| e_n^2 \| + \frac{3 \tau^2}{4} \| R_n^2 \| + \frac{2 \tau}{3} \sum_{k=1}^{n-1} \| e_k^2 \| + \frac{3 \tau}{2} \sum_{k=0}^{n-1} \| R_{k+1}^2 \|$$

which gives

$$\| e_n^2 \| \leq \frac{9 \tau^2}{4} \| R_n^2 \| + 2 \tau \sum_{k=1}^{n-1} \| e_k^2 \| + \frac{9 \tau}{2} \sum_{k=0}^{n-1} \| R_{k+1}^2 \|$$

$$\leq 2 \tau \sum_{k=1}^{n-1} \| e_k^2 \|^2 + c (\tau^2 + h^4)^2.$$

then the desired result follows by Lemma 3.3. □
Remark 3.3 By using similar techniques, we can show that the compact scheme \([8–10]\) is stable for \(0 < \alpha, \beta < 1\). In fact, suppose that \(\{v_i^k\}\) is the solution of

\[
\mathcal{H}(v_i^{k+1} - v_i^k) = \frac{\kappa_1 \tau^{-\alpha}}{2} \left( \sum_{l=0}^{k+1} \lambda_l^{(\alpha)}(\delta_x^2 v_i^{k+1-l} + \sum_{l=0}^{k} \lambda_l^{(\alpha)} \delta_x^2 v_i^{k-l}) \right)
+ \frac{\kappa_2 \tau^{-\beta}}{2} \left( \sum_{l=0}^{k+1} \lambda_l^{(\beta)}(\delta_x^2 v_i^{k+1-l} + \sum_{l=0}^{k} \lambda_l^{(\beta)} \delta_x^2 v_i^{k-l}) \right) + \frac{\tau}{2} \mathcal{H}(f_i^k + f_i^{k+1}),
\]

where \(\rho \in \mathcal{V}\). Then, by subtracting \([8–10]\) from \([12–13]\), we obtain the following equations for \(\varepsilon_i^k = v_i^k - u_i^k - \rho_i\),

\[
\mathcal{H}(\varepsilon_i^{k+1} - \varepsilon_i^k) = \frac{\kappa_1 \tau^{-\alpha}}{2} \left( \sum_{l=0}^{k+1} \lambda_l^{(\alpha)}(\delta_x^2 \varepsilon_i^{k+1-l} + \sum_{l=0}^{k} \lambda_l^{(\alpha)} \delta_x^2 \varepsilon_i^{k-l}) \right) + \frac{\kappa_1 \tau^{-\alpha}}{2} \left( \sum_{l=0}^{k+1} \lambda_l^{(\alpha)} \delta_x^2 \varepsilon_i^{k} \right) + \frac{\tau}{2} \mathcal{H}(\varepsilon_i^k + \varepsilon_i^{k+1}),
\]

0 \leq k \leq N - 1, \ 1 \leq i \leq M - 1,

\[
v_0^i = \varphi^i_1, \quad v_M^i = \varphi^i_2, \quad 1 \leq k \leq N, \quad v_0^i = \rho_i, \quad 0 \leq i \leq M,
\]

where \(\rho \in \mathcal{V}\). Notice that \([13-14]\), \(g_l^{(\alpha)}\) and \(g_l^{(\beta)}\) are less than 0 for \(l \geq 1\) and

\[
\tau^{-\alpha} \sum_{l=0}^{k} g_l^{(\alpha)} = \frac{1}{\Gamma(1 - \alpha)} + O(\tau), \quad \tau^{-\beta} \sum_{l=0}^{k} g_l^{(\beta)} = \frac{1}{\Gamma(1 - \beta)} + O(\tau).
\]

We can therefore obtain

\[
\frac{\tau^{-\alpha}}{2} \left( \sum_{l=0}^{k+1} \lambda_l^{(\alpha)} + \sum_{l=0}^{k} \lambda_l^{(\alpha)} \right) = \frac{\tau^{-\alpha}}{2} \left[ \left( 1 + \frac{\alpha}{2} \right) \sum_{l=0}^{k+1} g_l^{(\alpha)} + \sum_{l=0}^{k} g_l^{(\alpha)} - \frac{\alpha}{2} \sum_{l=0}^{k} g_l^{(\alpha)} \right]
= \frac{1}{\Gamma(1 - \alpha)} + O(\tau),
\]

and

\[
\frac{\tau^{-\beta}}{2} \left( \sum_{l=0}^{k+1} \lambda_l^{(\beta)} + \sum_{l=0}^{k} \lambda_l^{(\beta)} \right) = \frac{1}{\Gamma(1 - \beta)} + O(\tau).
\]

Now multiplying \([13]\) by \(h(\varepsilon_i^{k+1} + \varepsilon_i^k)\) and summing in \(i\) and \(k\), by arguments similar to that for the proof of Theorem 3.1, we can get that

\[
\|\varepsilon^k\|^2 \leq 2\tau \sum_{l=0}^{k-1} \|\varepsilon^l\|^2 + 5c_2^2 \|\delta_x^2 \rho\|^2,
\]

where \(c_2 = 1 + \frac{\kappa_1}{\Gamma(1 - \alpha)} + \frac{\kappa_2}{\Gamma(1 - \beta)}\), it then follows from Lemma 3.3 that \(\|\varepsilon^k\|^2 \leq 5c_2^2 e^{2T} \|\delta_x^2 \rho\|^2\). Finally we can conclude that \(\|v^k - u^k\| \leq \|v^k - u^k - \rho\| + \|\rho\| \leq \sqrt{5c_2} e^{2T} \|\delta_x^2 \rho\| + \|\rho\|\).
Before the numerical experiments, we first turn to the study on the fractional diffusion-wave equation.

4 A Compact scheme for the fractional diffusion-wave equation

In this section, we consider the following time fractional diffusion-wave equation

\[ \frac{\partial}{\partial t} D_t^\gamma u = \kappa \frac{\partial^2 u}{\partial x^2} + g(x, t), \quad 0 \leq x \leq L, \quad 0 < t \leq T, \quad 1 < \gamma < 2, \]  
(15)

subject to

\[ u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = \phi(x), \quad 0 \leq x \leq L, \]
\[ u(0, t) = \varphi_1(t), \quad u(L, t) = \varphi_2(t), \quad 0 < t \leq T, \]  
(16)

where \( \kappa \) is a positive constant, \( D_t^\gamma u \) is the Caputo fractional derivative of \( u \) with respect to time variable \( t \).

However, instead of solving (15) directly, we follow the technique in [33], where the problem is equivalently changed to the following:

\[ \frac{\partial u(x, t)}{\partial t} = \phi(x) + \frac{\kappa}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \frac{\partial^2 u(x, s)}{\partial x^2} ds + f(x, t), \quad 0 \leq x \leq L, \quad 0 < t \leq T, \]  
(17)

where \( 0 < \alpha = \gamma - 1 < 1 \), \( f(x, t) = \alpha I_t^\alpha g(x, t) \), and \( \alpha I_t^\alpha \) is the Riemann-Liouville fractional integral operator with respect to \( t \).

Once again, we choose \((p, q) = (0, -1)\) in (ii) of Lemma 2.2 yielding \( \mu_1 = 1 - \frac{\alpha}{2} \), \( \mu_2 = \frac{\alpha}{2} \) in [2], and

\[ \alpha I_t^\alpha u_{xx}(x, t_{n+1}) = \tau^\alpha \left[ \left(1 - \frac{\alpha}{2}\right) \sum_{k=0}^{n+1} \lambda_k \omega_k^{(\alpha)} u_{i}^{n+1-k} + \frac{\alpha}{2} \sum_{k=0}^{n} \lambda_k \omega_k^{(\alpha)} u_{i}^{n-k} \right] + O(\tau^2 + h^2) \]
(18)

where

\[ \lambda_0 = (1 - \frac{\alpha}{2}) \omega_0^{(\alpha)}, \quad \lambda_k = (1 - \frac{\alpha}{2}) \omega_k^{(\alpha)} + \frac{\alpha}{2} \omega_{k-1}^{(\alpha)}, \quad k \geq 1. \]

We can therefore introduce the compact scheme of (16)–(17) as

\[ \mathcal{H}(u_{i}^{n+1} - u_{i}^{n}) = \tau \mathcal{H} \phi_i + \frac{\kappa \tau^{\alpha+1}}{2} \left( \sum_{k=0}^{n+1} \lambda_k \delta_x^{2} u_{i}^{n+1-k} + \sum_{k=0}^{n} \lambda_k \delta_x^{2} u_{i}^{n-k} \right) + \frac{\tau}{2} \mathcal{H}(f_{i}^{n} + f_{i}^{n+1}), \]  
(19)

\[ 0 \leq n \leq N - 1, \quad 1 \leq i \leq M - 1, \]
\[ u_{0}^{n} = \varphi_1^n, \quad u_{M}^{n} = \varphi_2^n, \quad 1 \leq n \leq N, \]
\[ u_{i}^{0} = 0, \quad 0 \leq i \leq M. \]  
(20)

Due to the same reasons stated in the last section, the difference scheme (19)–(20) has a unique solution.

Similar to the role of Lemma 3.2 to Theorem 3.1, convergence of the scheme (19)–(20) is established using the following lemma:
Lemma 4.1 Let \( \{\lambda_n\}_{n=0}^{\infty} \) be defined as in (18), then for any positive integer \( k \) and real vector \((v_1, v_2, \ldots, v_k)^T \in \mathbb{R}^k\), it holds that
\[
\sum_{n=0}^{k-1} \left( \sum_{p=0}^{n} \lambda_p v_{n+1-p} \right) v_{n+1} \geq 0.
\]

Proof. The generating function of the Toeplitz matrix for the corresponding sequence is given by
\[
f(\alpha, x) =\lambda_0 + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k e^{ikx} + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k e^{-ikx}
= (1 - \frac{\alpha}{2})\omega_0(\alpha) + \frac{1}{2} \sum_{k=1}^{\infty} \left( (1 - \frac{\alpha}{2})\omega_k(\alpha) + \frac{\alpha}{2} \omega_{k-1}(\alpha) \right) e^{ikx} + \frac{1}{2} \sum_{k=1}^{\infty} \left( (1 - \frac{\alpha}{2})\omega_k(\alpha) + \frac{\alpha}{2} \omega_{k-1}(\alpha) \right) e^{-ikx}
= 2 - \alpha \frac{4}{e^{ix}-e^{-ix}} - \frac{2}{4} (1 - e^{-ix}) - \alpha + \frac{\alpha}{4} e^{ix}(1 - e^{-ix}) - \alpha + \frac{\alpha}{4} e^{-ix}(1 - e^{-ix}) - \alpha
= [2\sin(x/2)]^{-\alpha} \left\{ (1 - \frac{\alpha}{2}) \cos(\frac{\alpha}{2}(\pi - x)) + \frac{\alpha}{2} \cos[x + \frac{\alpha}{2}(\pi - x)] \right\}.
\]

Let \( g(\alpha, x) = (1 - \frac{\alpha}{2}) \cos(\frac{\alpha}{2}(\pi - x)) + \frac{\alpha}{2} \cos[x + \frac{\alpha}{2}(\pi - x)] \), then we can easily compute
\[
g_{\alpha} = -\sin(x/2) \sin(\frac{x}{2} + \frac{\alpha}{2}(\pi - x)) - (1 - \frac{\alpha}{2}) \sin(\frac{\alpha}{2}(\pi - x)) \pi - x - \frac{\alpha}{2} \sin[x + \frac{\alpha}{2}(\pi - x)] \pi - x \leq 0.
\]
Since \( g(\alpha, x) \) is nonincreasing with respect to \( \alpha \), we have \( g(\alpha, x) \geq g(1, x) = 0 \). This yields \( f(\alpha, x) \geq 0 \) and the result follows. \( \square \)

Remark 4.1 (i) We point out that interpolating polynomials have been used in [34] to approximate the Riemann-Liouville fractional integral with second order accuracy. However, one can easily test that the corresponding coefficients do not satisfy Lemma 4.1 in general.

(ii) The Grenander-Szegö Theorem is proved for continuous generating function in [31]. However, one can check that the arguments used in the proof of the Grenander-Szegö Theorem can still be applied here to conclude Lemma 4.1.

With Lemma 4.1, we can obtain the following convergence result of our compact difference scheme (19)–(20). Since the proof is similar to that of Theorem 3.1, we therefore skip the details.

Theorem 4.1 Assume that \( u(x, t) \in C^6_{x,t}([0, L] \times [0, T]) \) is the solution of (16)–(17) and \( \{u^k_i\}_{0 \leq i \leq M, 0 \leq k \leq N} \) is the solution of the finite difference scheme (19)–(20), respectively. Denote
\[
e_k^i = u(x_i, t_k) - u^k_i, \quad 0 \leq i \leq M, \quad 0 \leq k \leq N.
\]
Then there exists a positive constant \( \tilde{c}_2 \) such that
\[
\|e^k\| \leq \tilde{c}_2(\tau^2 + h^4), \quad 0 \leq k \leq N.
\]
Remark 4.2 Following the ideas of the proof for Theorem 3.1 and Remark 3.3, one can show that the proposed compact scheme (19)–(20) is stable in the sense that if \( \{v^l_i\} \) is the solution of

\[
\mathcal{H}(v^{k+1}_i - v^k_i) = \tau \mathcal{H}(\phi_i + \tilde{\rho}_i) + \frac{\kappa \tau^{\alpha+1}}{2} \left( \sum_{l=0}^{k+1} \lambda_l \delta_x^2 v^{k+1-l}_i + \sum_{l=0}^{k} \lambda_l \delta_x^2 v^{k-l}_i \right) + \frac{\tau}{2} \mathcal{H}(f^k_i + f^{k+1}_i),
\]

\[0 \leq k \leq N - 1, \quad 1 \leq i \leq M - 1,\]

\[v^k_0 = \varphi^k_1, \quad v^k_M = \varphi^k_2, \quad 1 \leq k \leq N,\]

\[v^0_i = \rho_i, \quad 0 \leq i \leq M,\]

with \( \rho, \tilde{\rho} \in \mathcal{V} \), then

\[\|v^k - u^k\| \leq \sqrt{5e^{\tau T} \left( \frac{\kappa}{\Gamma(\alpha + 1)} + 1 \right) \delta_x^2 \rho + \tilde{\rho}} + \|\rho\|.
\]

5 Numerical experiments

In this section, we carry out numerical experiments using the proposed finite difference schemes to illustrate our theoretical statements. All our tests were done in MATLAB. We suppose \( L = T = 1 \). The maximum norm errors and 2-norm errors between the exact and the numerical solutions

\[E_\infty(h, \tau) = \max_{0 \leq k \leq N} \|U^k - u^k\|_\infty, \quad E_2(h, \tau) = \max_{0 \leq k \leq N} \|U^k - u^k\|_2\]

are shown. Furthermore, the temporal convergence order, denoted by

\[Rate_{1\infty} = \log_2 \left( \frac{E_{\infty}(h, 2\tau)}{E_{\infty}(h, \tau)} \right), \quad Rate_{12} = \log_2 \left( \frac{E_2(h, 2\tau)}{E_2(h, \tau)} \right)\]

for sufficiently small \( h \), and the spatial convergence order, denoted by

\[Rate_{2\infty} = \log_2 \left( \frac{E_{\infty}(2h, \tau)}{E_{\infty}(h, \tau)} \right), \quad Rate_{22} = \log_2 \left( \frac{E_2(2h, \tau)}{E_2(h, \tau)} \right)\]

when \( \tau \) is sufficiently small, are reported. The numerical results given by these examples justify our theoretical analysis.

Example 5.1 The following problem was studied in [27]:

\[
\frac{\partial u(x, t)}{\partial t} = \left( \frac{\partial^{1-\tilde{\alpha}}}{\partial t^{1-\tilde{\alpha}}} + \frac{\partial^{1-\tilde{\beta}}}{\partial t^{1-\tilde{\beta}}} \right) \left[ \frac{\partial^2 u(x, t)}{\partial x^2} \right] + g(x, t),
\]

\[u(0, t) = 0, \quad u(1, t) = t^{1+\tilde{\alpha}+\tilde{\beta}} \sin(1), \quad 0 \leq t \leq 1,\]

\[u(x, 0) = 0, \quad 0 \leq x \leq 1,\]

where

\[g(x, t) = \sin(x) \left[ (1 + \tilde{\alpha} + \tilde{\beta}) t^{\tilde{\alpha}+\tilde{\beta}} + \frac{\Gamma(2 + \tilde{\alpha} + \tilde{\beta})}{\Gamma(1 + 2\tilde{\alpha} + \tilde{\beta})} t^{2\tilde{\alpha}+\tilde{\beta}} + \frac{\Gamma(2 + \tilde{\alpha} + \tilde{\beta})}{\Gamma(1 + \tilde{\alpha} + 2\beta)} t^{\tilde{\alpha}+2\tilde{\beta}} \right].\]
In terms of the notations in this paper, we consider

\[ \frac{\partial u(x,t)}{\partial t} = \left( \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial^\beta}{\partial t^\beta} \right) \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right] + f(x,t), \]

\[ u(0,t) = 0, \quad u(1,t) = t^{3-\alpha-\beta} \sin(1), \quad 0 \leq t \leq 1, \]

\[ u(x,0) = 0, \quad 0 \leq x \leq 1, \]

with

\[ f(x,t) = \sin(x) \left[ (3-\alpha-\beta)t^{2-\alpha-\beta} + \frac{\Gamma(4-\alpha-\beta)}{\Gamma(4-2\alpha-\beta)} t^{2-\alpha-\beta} + \frac{\Gamma(4-\alpha-\beta)}{\Gamma(4-\alpha-2\beta)} t^{3-\alpha-2\beta} \right]. \]

It is easy to check that the exact solution is \( u(x,t) = \sin(x)t^{3-\alpha-\beta} \).

Table 1: Numerical convergence orders in temporal direction with \( h = \frac{1}{30} \), \( \alpha = 0.35 \), \( \beta = 0.05 \) for Example 5.1.

| \( \tau \) | \( E_\infty(h,\tau) \) | \( \text{Rate}_{1\infty} \) | \( E_2(h,\tau) \) | \( \text{Rate}_{12} \) |
|-----------|-----------------|-----------------|-----------------|-----------------|
| 1/5       | 8.2083e-4       | *               | 5.9004e-4       | *               |
| 1/10      | 1.9894e-4       | 2.0447          | 1.4192e-4       | 2.0557          |
| 1/20      | 5.0072e-5       | 1.9903          | 3.5877e-5       | 1.9840          |
| 1/40      | 1.2567e-5       | 1.9944          | 9.0135e-6       | 1.9929          |
| 1/80      | 3.1502e-6       | 1.9961          | 2.2593e-6       | 1.9962          |
| 1/160     | 7.8883e-7       | 1.9976          | 5.6574e-7       | 1.9976          |

Figure 1 shows the exact solution (left) and numerical solution (right) for Example 5.1 when \( \alpha = 0.2 \), \( \beta = 0.7 \), \( h = \tau = \frac{1}{50} \). Meanwhile, we list, in Table 1 and Table 2, the convergence order in temporal direction with \( h = \frac{1}{30} \) and, in Table 3 and Table 4, the convergence order in...
Table 2: Numerical convergence orders in temporal direction with $h = \frac{1}{30}$, $\alpha = 0.2$, $\beta = 0.7$ for Example 5.1.

| $\tau$ | $E_\infty(h, \tau)$ | Rate$_{1\infty}$ | $E_2(h, \tau)$ | Rate$_{12}$ |
|--------|----------------------|-------------------|----------------|-------------|
| 1/5    | 7.3332e-4 *          | 5.2626e-4 *       |                |             |
| 1/10   | 1.8938e-4 1.9532     | 1.3595e-4 1.9527  |                |             |
| 1/20   | 4.8407e-5 1.9680     | 3.4765e-5 1.9674  |                |             |
| 1/40   | 1.2305e-5 1.9759     | 8.8421e-6 1.9752  |                |             |
| 1/80   | 3.1336e-6 1.9734     | 2.2500e-6 1.9745  |                |             |
| 1/160  | 8.0171e-7 1.9667     | 5.7465e-7 1.9692  |                |             |

Table 3: Numerical convergence orders in spatial direction with $\tau = \frac{1}{8000}$, $\alpha = 0.15$, $\beta = 0.35$ for Example 5.1.

| $h$    | $E_\infty(h, \tau)$ | Rate$_{2\infty}$ | $E_2(h, \tau)$ | Rate$_{22}$ |
|--------|----------------------|-------------------|----------------|-------------|
| 1/2    | 1.1910e-5 *          | 9.8357e-6 *       |                |             |
| 1/4    | 8.6322e-7 4.0102     | 6.3362e-7 3.9563  |                |             |
| 1/8    | 5.4954e-8 3.9734     | 3.9832e-8 3.9916  |                |             |
| 1/16   | 3.7617e-9 3.8688     | 2.7001e-9 3.8829  |                |             |

Spatial direction with $\tau = \frac{1}{8000}$. The convergence order of the numerical results matches that of the theoretical one.

We next consider a special case of the modified anomalous fractional sub-diffusion equation:

**Example 5.2** Consider the following example (35, 36).

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1-\tilde{\alpha}}}{\partial t^{1-\tilde{\alpha}}} \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right] + \left[ 2t + \frac{8\pi^2 t^{1+\tilde{\alpha}}}{\Gamma(2+\tilde{\alpha})} \right] \sin(2\pi x),$$

$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq 1,$

$u(x,0) = 0, \quad 0 \leq x \leq 1.$

The problem can be written as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\alpha}{\partial t^\alpha} \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right] + \left[ 2t + \frac{8\pi^2 t^{2-\alpha}}{\Gamma(3-\alpha)} \right] \sin(2\pi x),$$

$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq 1,$

$u(x,0) = 0, \quad 0 \leq x \leq 1,$

with the exact solution given by $u(x,t) = t^2 \sin(2\pi x)$.

Convergence order of the proposed scheme in temporal direction with $h = \frac{1}{100}$ is reported in Table 5, while in Table 6, the convergence order in spatial direction with $\tau = \frac{1}{4000}$, $\alpha = 0.5$ is listed. In this particular example, since the convergent rates of the maximum norm and the 2-norm coincide for the digits listed, they are reported as Rate1 and Rate2 in the tables. Once again, both tables confirm the theoretical result.
Table 4: Numerical convergence orders in spatial direction with $\tau = \frac{1}{8000}$, $\alpha = 0.75$, $\beta = 0.15$ for Example 5.1

| $h$   | $E_\infty(h,\tau)$ | Rate2$_\infty$ | $E_2(h,\tau)$ | Rate2$_2$ |
|-------|---------------------|----------------|---------------|-----------|
| 1/2   | 1.4250e-5           | *              | 1.0077e-5     | *         |
| 1/4   | 8.8434e-7           | 4.0103         | 6.4856e-7     | 3.9576    |
| 1/8   | 5.6182e-8           | 4.0749-8       | 3.9924        |
| 1/16  | 3.8214e-9           | 2.7425-9       | 3.8932        |

Table 5: Numerical convergence orders in temporal direction with $h = \frac{1}{100}$ for Example 5.2

| $\alpha$ | $\tau$ | $E_\infty(h,\tau)$ | $E_2(h,\tau)$ | Rate1 |
|----------|--------|---------------------|---------------|-------|
| $\alpha = 0.3$ | 1/5    | 5.9575e-3           | 4.2126e-3     | *     |
|          | 1/10   | 1.4796e-3           | 1.0463e-3     | 2.0095|
|          | 1/20   | 3.6392e-4           | 2.5733e-4     | 2.0235|
|          | 1/40   | 9.1188e-5           | 6.4480e-5     | 1.9967|
|          | 1/80   | 2.2869e-5           | 1.6171e-5     | 1.9954|
|          | 1/160  | 5.7679e-6           | 4.0785e-6     | 1.9873|
| $\alpha = 0.5$ | 1/5    | 1.0436e-2           | 7.3790e-3     | *     |
|          | 1/10   | 2.6295e-3           | 1.8594e-3     | 1.9886|
|          | 1/20   | 6.5613e-4           | 4.6395e-4     | 2.0028|
|          | 1/40   | 1.6477e-4           | 1.1651e-4     | 1.9935|
|          | 1/80   | 4.1396e-5           | 2.9272e-5     | 1.9929|
|          | 1/160  | 1.0424e-5           | 7.3708e-6     | 1.9896|
| $\alpha = 0.7$ | 1/5    | 1.4850e-2           | 1.0500e-2     | *     |
|          | 1/10   | 3.8141e-3           | 2.6970e-3     | 1.9610|
|          | 1/20   | 9.6191e-4           | 6.8018e-4     | 1.9874|
|          | 1/40   | 2.4234e-4           | 1.7136e-4     | 1.9889|
|          | 1/80   | 6.1157e-5           | 4.3245e-5     | 1.9865|
|          | 1/160  | 1.5448e-5           | 1.0923e-5     | 1.9851|

Our next two examples are for fractional diffusion-wave equations:

**Example 5.3** Consider the problem:

\[
\begin{align*}
&C^0 \mathcal{D}_t^\gamma u = \frac{\partial^2 u}{\partial x^2} + e^x [\Gamma(\gamma + 2) t - t^{\gamma+1}], \quad 0 \leq x \leq 1, \quad 0 < t \leq 1, \quad 1 < \gamma < 2, \\
u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \leq x \leq 1, \\
u(0,t) = t^{1+\gamma}, \quad u(1,t) = e t^{1+\gamma}, \quad 0 < t \leq 1,
\end{align*}
\]

the problem can be equivalently changed to

\[
\begin{align*}
&\frac{\partial u(x,t)}{\partial t} = \mathcal{I}_t^\alpha u_{xx}(x,t) + e^x \left[ (\alpha + 2) t^{\alpha+1} - \frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)} t^{2\alpha+2} \right], \quad 0 \leq x \leq 1, \quad 0 < t \leq 1, \\
u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \leq x \leq 1, \\
u(0,t) = t^{2+\alpha}, \quad u(1,t) = e t^{2+\alpha}, \quad 0 < t \leq 1,
\end{align*}
\]

where $\alpha = \gamma - 1$. The exact solution for this problem is $u(x,t) = e^{x t^{\alpha+2}}$. 

Table 6: Numerical convergence orders in spatial direction with $\tau = 1/4000$ when $\alpha = 0.5$ for Example 5.2.

| $h$  | $E_\infty(h, \tau)$ | $E_2(h, \tau)$ | Rate2 |
|------|---------------------|----------------|--------|
| 1/4  | 2.7020e-2           | 1.9106e-2      | *      |
| 1/8  | 1.5651e-3           | 1.1067e-3      | 4.1097 |
| 1/16 | 9.6041e-5           | 6.7911e-5      | 4.0265 |
| 1/32 | 5.9906e-6           | 4.2360e-6      | 4.0029 |
| 1/64 | 3.8961e-7           | 2.7549e-7      | 3.9426 |

Table 7: Numerical convergence orders in temporal direction with $h = 1/30$ for Example 5.3.

| $\alpha$ | $\tau$ | $E_\infty(h, \tau)$ | Rate1$\infty$ | $E_2(h, \tau)$ | Rate12 |
|----------|--------|---------------------|----------------|----------------|--------|
| $\alpha = 0.3$ | 1/5    | 5.2645e-3           | *              | 3.9527e-3      | *      |
|          | 1/10   | 1.3874e-3           | 1.9239         | 9.8966e-4      | 1.9978 |
|          | 1/20   | 3.7394e-4           | 1.8916         | 2.6259e-4      | 1.9141 |
|          | 1/40   | 9.8549e-5           | 1.9239         | 6.9307e-5      | 1.9218 |
|          | 1/80   | 2.5742e-5           | 1.9367         | 1.8173e-5      | 1.9312 |
|          | 1/160  | 6.6877e-6           | 1.9445         | 4.7249e-6      | 1.9435 |
| $\alpha = 0.6$ | 1/5    | 8.6638e-3           | *              | 5.9115e-3      | *      |
|          | 1/10   | 2.2702e-3           | 1.9322         | 1.5390e-3      | 1.9415 |
|          | 1/20   | 5.7192e-4           | 1.9890         | 3.8707e-4      | 1.9914 |
|          | 1/40   | 1.4463e-4           | 1.9835         | 9.7399e-5      | 1.9906 |
|          | 1/80   | 3.6248e-5           | 1.9963         | 2.4487e-5      | 1.9919 |
|          | 1/160  | 9.0979e-6           | 1.9943         | 6.1472e-6      | 1.9940 |
| $\alpha = 0.9$ | 1/5    | 1.3142e-2           | *              | 8.7666e-3      | *      |
|          | 1/10   | 3.4817e-3           | 1.9163         | 2.2332e-3      | 1.9729 |
|          | 1/20   | 8.8458e-4           | 1.9767         | 5.5942e-4      | 1.9971 |
|          | 1/40   | 2.2304e-4           | 1.9877         | 1.3978e-4      | 2.0008 |
|          | 1/80   | 5.5750e-5           | 2.0003         | 3.4943e-5      | 2.0001 |
|          | 1/160  | 1.3919e-5           | 2.0019         | 8.7372e-5      | 1.9998 |

Table 7 and Table 8 justify the accuracy of the scheme proposed in Section 4.

In the last example, we consider a problem where the exact solution cannot be found readily. We note that, in this example, $u_t$ is not identically equal to zero initially.

Example 5.4 Consider the problem:

$$\frac{D_t^{\gamma} u}{\partial x^2} + \sin(2\pi x) \left[ \frac{\Gamma(\gamma+3)}{2} t^2 + 4\pi^2 t^2+\gamma \right].$$

$$0 \leq x \leq 1, \quad 0 < t \leq 1, \quad 1 < \gamma < 2,$$

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0.1 \sin(2\pi x), \quad 0 \leq x \leq 1,$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t \leq 1.$$
Table 8: Numerical convergence orders in spatial direction with $\tau = \frac{1}{5000}$ when $\alpha = 0.5$ for Example 5.3.

| $h$ | $E_\infty(h, \tau)$ | Rate2$\infty$ | $E_2(h, \tau)$ | Rate22 |
|-----|---------------------|---------------|----------------|--------|
| 1/2 | 3.6421e-5           | *             | 2.5753e-5      | *      |
| 1/4 | 2.2865e-6           | 3.9936        | 1.7004e-6      | 3.9208 |
| 1/8 | 1.4229e-7           | 4.0062        | 1.0616e-7      | 4.0016 |
| 1/16| 8.5477e-9           | 4.0572        | 5.9751e-9      | 4.1511 |

By noting $\alpha = \gamma - 1$, the problem can be equivalently changed to
\[
\frac{\partial u(x, t)}{\partial t} =_0 I^\alpha_t u_{xx}(x, t) + \sin(2\pi x) \left[ (\alpha + 3)t^{\alpha+2} + 4\pi^2 \frac{\Gamma(\alpha + 4)}{\Gamma(2\alpha + 4)} t^{2\alpha+3} \right], \quad 0 \leq x \leq 1, \quad 0 < t \leq 1,
\]
\[
u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0.1 \sin(2\pi x), \quad 0 \leq x \leq 1,
\]
\[
u(0, t) = u(1, t) = 0, \quad 0 < t \leq 1.
\]

We take the numerical solution with $M = 400$, $N = 4000$ as the ‘true’ solution when computing the errors. From the last two tables, we can see that the scheme still works properly in this situation.

Table 9: Numerical convergence orders in temporal direction with $h = \frac{1}{50}$ for Example 5.4.

| $\alpha$ | $\tau$ | $E_\infty(h, \tau)$ | $E_2(h, \tau)$ | Rate1 |
|---------|-------|---------------------|----------------|-------|
| $\alpha = 0.3$ | $1/10$ | 2.9448e-3        | 2.0864e-3      | *     |
|         | $1/20$ | 8.1395e-4        | 5.7669e-4      | 1.8551 |
|         | $1/40$ | 2.1283e-4        | 1.5079e-4      | 1.9353 |
|         | $1/80$ | 5.3775e-5        | 3.8100e-5      | 1.9847 |
|         | $1/160$ | 1.2898e-5       | 9.1384e-6      | 2.0598 |
| $\alpha = 0.5$ | $1/10$ | 4.6281e-3        | 3.2791e-3      | *     |
|         | $1/20$ | 1.3211e-3        | 9.3602e-4      | 1.8087 |
|         | $1/40$ | 3.4933e-4        | 2.4750e-4      | 1.9191 |
|         | $1/80$ | 8.9065e-5        | 6.3103e-5      | 1.9717 |
|         | $1/160$ | 2.1877e-5       | 1.5500e-5      | 2.0254 |
| $\alpha = 0.7$ | $1/10$ | 6.4028e-3        | 4.5364e-3      | *     |
|         | $1/20$ | 1.6342e-3        | 1.1579e-3      | 1.9701 |
|         | $1/40$ | 4.2105e-4        | 2.9832e-4      | 1.9565 |
|         | $1/80$ | 1.0677e-4        | 7.5650e-5      | 1.9794 |
|         | $1/160$ | 2.6331e-5       | 1.8655e-5      | 2.0197 |

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Table 10: Numerical convergence orders in spatial direction with $\tau = \frac{1}{2000}$ when $\alpha = 0.5$ for Example 5.4.

| $h$   | $E_{\infty}(h, \tau)$ | Rate$_{2, \infty}$ | $E_2(h, \tau)$ | Rate$_{2, 2}$ |
|-------|------------------------|---------------------|----------------|---------------|
| 1/5   | 9.0588e-3              | *                   | 6.7352e-3      | *             |
| 1/10  | 5.4015e-4              | 4.0679              | 4.0160e-4      | 4.0679        |
| 1/20  | 3.4978e-5              | 3.9488              | 2.4733e-5      | 4.0212        |
| 1/40  | 2.0755e-6              | 4.0749              | 1.4676e-6      | 4.0749        |

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