Optical lattice for tripod-like atomic level structure

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Standard optical potentials use off-resonant laser standing wave induced AC-Stark shift. In a recent development [Phys. Rev. Lett. 117, 233001 (2016)] a three-level scheme in a configuration coupled coherently by resonant laser fields was introduced leading to an effective lattice with sub-wavelength potential peaks. Here, as an extension of that work, a four level atomic setup in the tripod configuration is used to create spin 1/2-like two-dimensional dark-space with 1D motion and the presence of external gauge fields. Most interestingly for possible applications, the lifetime for a dark subspace motion is up to two orders of magnitude larger than for a similar A system. The model is quite flexible leading to lattices with significant nearest, next-nearest, or next-next-nearest hopping amplitudes, J1, J2, J3 opening up new intriguing possibilities to study, e.g. frustrated systems. The characteristic Wannier functions lead also to a new type of inter-site interactions not realizable in typical optical lattices.

I. INTRODUCTION

AC-Stark shift based optical potentials induced by far-detuned laser standing waves has enabled to implement discrete lattice models [1] linking ultra-cold atomic physics with condensed matter physics. Or rather enriching the latter with bosonic systems such as e.g. Bose-Hubbard (BH) model. The experimental demonstration of a quantum phase transition between the superfluid and Mott insulating phases [2] was followed by intensive investigations in different, more complex schemes [3–5] involving spinor lattice gases, long range interactions, disordered systems, or an implementation of topological insulators.

The standing wave optical potentials proved to be very versatile allowing to create, typically with the application of additional Raman lasers, interesting coupling between sites, e.g. leading to a construction of artificial gauge fields or spin-orbit coupling as reviewed in [5]. The atomic ground state sublevels could serve as an additional synthetic dimension [6–8], allowing, e.g. to extend the Hall physics to four dimensions [6]. Still the standing wave optical potential has some drawbacks. The typical \( \cos^2(k_L x) \) spatial dependence (with \( k_L \) being the laser light wavevector) leads to the dominance of nearest neighbor tunneling over hops involving separated sites. Similarly on-site interactions dominate over inter-site terms making investigations of interaction related physics for spinless fermions in optical lattices difficult.

Recently an alternative scheme for creating optical potentials has been proposed [10,11]. It relies on a resonant dipole-coupling of three atomic levels with the position-dependent Rabi frequencies involving a common atomic excited state. This differs from the standard approach where two-photon resonant lasers are far detuned from a single-photon transition [12,13]. The resulting Lambda system is characterized by a position-dependent dark state. The dynamics of atoms constrained to the dark state is that of a particle moving in the presence of a scalar potential which features evenly spaced subwavelength peaks. Early Yb experiments [14,15] confirmed the expected band structure, but the system lifetimes were, disappointingly, at least one order of magnitude lower than for AC-Stark potentials.

In this work we present the tripod system [10,17], with four resonantly coupled levels that is an interesting extension of the A system. It features two degenerate dark states implementing spin 1/2-like physics providing at the same time a new possible realization of spin-orbit coupling in a one-dimensional lattice supplementing the existing schemes [18,19]. In Section II we adapt the derivation of the A system dark-state description to the tripod scheme. We detail the resulting periodic spin 1/2-like model for the movement of the particle in the gauge field and discuss the Bloch theory including lifetime computation of the dark state bands. The tight-binding description of atoms populating low lying bands is discussed in Section IV. As it turns out, the model leads, in a natural way, to a quite peculiar extended Hubbard model with significant hopping not only to the nearest sites (nn) but also to the next nearest neighbors (nnn) as well as to next-next nearest neighbors (nnnn). Such a highly interesting and unusual property is due to the shape of the Wannier basis functions corresponding to the non-standard lattice felt by the atoms. This opens up a possibility of frustration related studies in the model. The conclusions are drawn and future perspectives involving the study of interacting particles are discussed in Sec. V.

II. THE HAMILTONIAN

We consider a gas of ultracold atoms whose motion is restricted to one dimension for example by a strong external optical potential of the form \( V(y, z) = m_\omega^2(y^2 + z^2)/2 \). The atoms populate three ground state configuration atomic states \( |g_1\rangle, |g_2\rangle, |g_3\rangle \) that are coupled to an excited state \( |e\rangle \). The dipole coupling of each of \( |g_i\rangle \) to \( |e\rangle \) is characterized by a Rabi frequency \( \Omega_i(x) \). The wave-
length of the three lasers is assumed to be equal to $\lambda_L$. In a rotating frame, after neglecting rapidly oscillating terms, the Hamiltonian of the system considered reads

$$H = -\hbar^2 \frac{\partial^2}{2m_a \partial x^2} + H_\alpha(x),$$

where $m_a$ is the atomic mass and

$$H_\alpha(x) = \begin{pmatrix} -\Delta - i\Gamma_e/2 & \Omega_1(x) & \Omega_2(x) & \Omega_3(x) \\ \Omega_1(x) & 0 & 0 & 0 \\ \Omega_2(x) & 0 & 0 & 0 \\ \Omega_3(x) & 0 & 0 & 0 \end{pmatrix}.$$ (2)

Here $\Delta$ is a possible common detuning of all three lasers and $\Gamma_e$ is spontaneous emission rate of the excited state. We consider Rabi frequencies of the form

$$\Omega_1(x) = \Omega_1 \sin(k_L x),$$

$$\Omega_2(x) = \Omega_2 \sin(k_L x + a),$$

$$\Omega_3(x) = \Omega_3,$$ (3)

where $k_L = 2\pi/\lambda_L$. The $\lambda_L$-periodicity defines a natural energy scale – the recoil energy equal to $E_R = \hbar^2 k_L^2 / (2m_a)$.

The matrix $H_\alpha(x)$, in Eq. (2), for $\Gamma_e \neq 0$, is non-Hermitian. It is diagonalized by finding a biorthogonal set of right and left eigenvectors. When using bra/ket notation, the bra vector always refers to a proper left set of right and left eigenvectors. Hermitian. It is diagonalized by finding a biorthogonal set of right and left eigenvectors. When using bra/ket notation, the bra vector always refers to a proper left eigenvector, a part of biorthogonal set. There exist two linearly independent right eigenvectors $|D_1(x)\rangle$, $|D_2(x)\rangle$ corresponding to energy $E(x) = 0$ – two dark states. They are of the form

$$|D_i(x)\rangle = \sum_{n=1}^{3} u_{i,n} |g_i\rangle, \quad \Omega(x) \perp u_i,$$ (4)

They are not affected by value of $\Delta, \Gamma_e$.

The Hamiltonian matrix [2] has also two (right) bright states $|B_\pm(x)\rangle$ with energies

$$E_\pm(x) = -\bar{\Delta}/2 \pm \sqrt{(\bar{\Delta}/2)^2 + \|\Omega(x)\|^2},$$

where $\bar{\Delta} = \Delta + i\Gamma/2$. These states read

$$|B_\pm(x)\rangle = \mathcal{N}_\pm(x) \left( E_\pm(x)|e\rangle + \sum_{i=1}^{3} \Omega_i(x)|g_i\rangle \right),$$ (6)

where

$$\mathcal{N}_\pm(x) = \frac{1}{\sqrt{(E_\pm(x))^2 + \|\Omega\|^2}},$$

$$\|\Omega\| = \sqrt{|\Omega_1(x)|^2 + |\Omega_2(x)|^2 + |\Omega_3(x)|^2}$$ (7)

are the normalization factors. The left eigenvectors, that together with (6) form a biorthonormal set, are given by:

$$\langle B_\pm(x)| = \mathcal{N}_\pm(x) \left( E_\pm(x)|e\rangle + \sum_{i=1}^{3} \Omega^*_i(x)|g_i\rangle \right).$$ (8)

For $\Gamma_e \neq 0$ the states $|B_\pm(x)\rangle$ undergo a spontaneous emission with the rate comparable to that of an excited state $|e\rangle$. In this work we are interested in the case when atoms populate primarily the stable channels $|D_1(x)\rangle, |D_2(x)\rangle$ and the energy scale set by $\Omega_i$ dominates the kinetic energy of atoms. This also ensures that a phenomenological description of losses from largely unpopulated $|e\rangle$ via $i\Gamma_e$ is justified $\Gamma_e \ll \|\Omega\|$ [20].

The gap $\Delta E$ between the dark and the bright state channels, for $\Delta = 0$, is given by $\min E_{\pm}(x)$. For $a = 0$ we have $\Delta E = \|\Omega_3\|$. For $a > 0$ the gap $\Delta E$ increases until $a = \pi/2$ where $\Delta E = \min\{\sqrt{\Omega_1^2 + \Omega_2^2}, \sqrt{\Omega_3^2 + \Omega_2^2}\}$. The dependence of $E_\pm$ on $x$ for few selected values of $a$ is shown in Fig 1.

It is worth stressing that we focus on non-interacting bosons or physics of ultracold spinless fermions, where direct collisions are suppressed. If this is not the case one may wonder whether the collision of two atoms in dark state may not lead to one particle in each bright state as $E_+(x) + E_-(x) = -\Delta$. We leave this question open for the interacting case study, here let us mention that the process may be suppressed by taking a sufficient detuning $\Delta$ still in the limit $\Delta \ll \|\Omega\|$.

The Hamiltonian (1) may be addressed using a Born-Oppenheimer type transformation [10] [16] applying

$$\mathcal{B}(x) = \{ |D_1(x)\rangle, |D_2(x)\rangle, |B_+(x)\rangle, |B_-(x)\rangle\}$$ (9)

as a position-dependent basis. Writing an arbitrary wavefunction in this basis as:

$$|\psi(x)\rangle = d_1(x) |D_1(x)\rangle + d_2(x) |D_2(x)\rangle +$$

$$+ b_-(x) |B_-(x)\rangle + b_+(x) |B_+(x)\rangle,$$ (10)

yields the following Hamiltonian matrix:

$$H_B = \left[\frac{P - A(x)}{2m_a}\right] + \text{diag}(0, 0, E_+(x), E_-(x)),$$ (11)
or
\[ H_B = \frac{1}{2m_a} \left[ P^2 - 2A(x)P + \Phi(x) \right] + \text{diag}[0, 0, E_+(x), E_-(x)]. \] (12)
The $H_B$ now acts on vectors of the form:
\[ \psi(x) = \begin{pmatrix} d_1(x) \\ d_2(x) \\ b_+(x) \\ b_-(x) \end{pmatrix}. \] (13)
The operator $P = -i\hbar \partial_x \otimes 1_4$. The $A(x)$ is given by
\[ A_{MN}(x) = i\hbar \langle M(x) | \partial_x | N(x) \rangle, \quad M(x), N(x) \in \mathcal{B}(x). \] (14)
and
\[ \Phi(x) = A(x)^2 + i\hbar \partial_x A(x). \] (15)

A. The dark state subspace

The large diagonal terms $E_-(x), E_+(x)$ in Eq. (12) allow for the separation of the dark-state physics in Hamiltonian (12) by the following dark-state projection:
\[ H_2 = QH_BQ. \quad Q = |D_1(x)\rangle\langle D_1(x)| + |D_2(x)\rangle\langle D_2(x)| \] (16)
This will be evident from the upcoming numerical analysis.

The states $|D_1(x)\rangle, |D_2(x)\rangle$ are energy-degenerate and thus not uniquely defined. Different choices of basis lead to equivalent description of the model. We favor those leading to simple, well-behaved and intuitive potentials $A(x)$. We consider only $\lambda_L$-periodic $|D_1(x)\rangle$. We also opt to work with $|D_1(x)\rangle$ with real coefficients, which automatically implies $A_{11}(x) = A_{22}(x) = 0$.

Let us take
\[ |D_1(x)\rangle \propto |\xi\rangle \times \Omega(x), \] (17)
and
\[ |D_2(x)\rangle \propto |D_1(x)\rangle \times \Omega(x). \] (18)
This assures their mutual orthogonality and by Eq. (14) such vectors are indeed dark. The vector $|\xi\rangle$ cannot be parallel to $\Omega(x)$ but otherwise can be arbitrary. We choose:
\[ |\xi\rangle = \Omega_1 |g_1\rangle + \Omega_2 |g_2\rangle. \] (19)
We will later discuss the advantages of the above choice for $|\xi\rangle$, namely good analytic properties in the limit $a \to 0$. Using Eq. (19) we find
\[ |D_1(x)\rangle = N_1(x) \begin{pmatrix} \Omega_2 \Omega_3 \\ -\Omega_1 \Omega_3 \\ \Omega_1 \Omega_2 \sin(a + k_L x) - \sin(k_L x) \end{pmatrix} \] (20)
and
\[ |D_2(x)\rangle = N_2(x) \begin{pmatrix} \Omega_1 \left[ \Omega_2^2 f(k_L x) + \Omega_2 \right] \\ \Omega_2 \left[ \Omega_1^2 f(\pi/2 + k_L x - a/2) + \Omega_1 \right] \\ -\Omega_3 \left[ \Omega_2^2 \sin(a + k_L x) + \Omega_1^2 \sin(k_L x) \right] \end{pmatrix}, \] (21)
where
\[ f(y) = \sin(a + y)[\sin(a + y) - \sin(y)]. \] (22)
When $a = 0$ the state $|D_1(x)\rangle$ becomes position-independent
\[ |D_1(x)\rangle = \frac{\Omega_2}{\sqrt{\Omega_1^2 + \Omega_2^2}} |g_1\rangle - \frac{\Omega_1}{\sqrt{\Omega_1^2 + \Omega_2^2}} |g_2\rangle. \] (23)
and
\[ D_2(x) = N_2(x)[\Omega_1 \Omega_2 |g_1\rangle + \Omega_2 \Omega_3 |g_2\rangle - (\Omega_1^2 + \Omega_2^2) \sin(k_L x) |g_3\rangle]. \] (24)
The latter can be written as
\[ |D_2(x)\rangle = \frac{1}{\sqrt{\Omega_1^2 + \Omega_2^2}} [\Omega_1 a - \Omega_2 \sin(k_L x) |g_3\rangle \] (25)
where $|a\rangle = \cos \beta |g_1\rangle + \sin \beta |g_2\rangle$, $\tan \beta = \Omega_2/\Omega_1$, $\Omega_3 = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_p^2} = \sqrt{\Omega_3}$. The above form is formally identical to the form of a single dark state in the $\Lambda$ system configuration (10).

At this point let us briefly comment why $|\xi\rangle = \Omega_1 |g_1\rangle + \Omega_2 |g_2\rangle$ is a good choice for the vector that generates $|D_1(x)\rangle, |D_2(x)\rangle$ by Eqs. (17) and (18). Let us for example consider $|\xi\rangle = |g_3\rangle$. It leads to
\[ |D_1(x)\rangle \sim -\Omega_2 \sin(k_L x + a) |g_1\rangle + \Omega_1 \sin k_L x |g_2\rangle \] (26)
\[ |D_2(x)\rangle \sim \Omega_1 \Omega_3 \sin k_L x |g_1\rangle + \Omega_2 \Omega_3 \sin(k_L x + a) |g_2\rangle \]
\[ - [\Omega_1^2 \sin^2 k_L x + \Omega_2^2 \sin^2(k_L x + a)] |g_3\rangle \] (27)
which is apparently analytically simpler than the previous results (20-21). In fact it gives a simple, analytic calculations of coefficients of $A(x)$ and $A^2(x)$ [see Appendix B]. However, the limit $a \to 0$ agrees with Eq. (23) only up to a sign, namely both $|g_1\rangle$ and $|g_2\rangle$ components flip their signs when $k_L x = n\pi$, $n$ – integer making them only piecewise constant and, in particular, discontinuous. For that reason, the derivatives in the definition of $A(x)$, Eq. (14) are ill-defined as $a \to 0$.

B. Gauge potentials

Having chosen the dark state basis $|D_i(x)\rangle$’s (20-21), one finds the gauge potential $A(x)$ with Eq. (14). For the dark-state projected Hamiltonian $H_2$ one restricts $A(x)$ and $A^2(x)$ to the upper left $2 \times 2$ block. The coefficients $A_{ij}(x)$ for $i, j < 3$ clearly do not depend on the choice of
FIG. 2. The spatial dependence of elements of matrices $A(x)$ and $A^2(x)$ describing the couplings within the dark state subspace for $\Omega_1 : \Omega_2 : \Omega_3 = 50: 20 : 5$. The value of the phase shift $a$ is indicated in each of the panels. Panel a) shows the limiting case of $a = 0$ for $(A^2)_{ij}(x)$ [see also Eq. (29)]. Panel b) shows $A_{12}(x), (A^2)_{11}$ and $(A^2)_{12}$ for a small, but nonzero $a = 0.05$. Coefficient $(A^2)_{22}(x)$ for this $a$ is similar to one in a). Panel c) shows $(A^2)_{22}(x)$ for a larger $a = 0.435$, while smaller coefficients: $A_{12}(x), (A^2)_{11}$ and $(A^2)_{12}$ (x) are depicted on a separate panel d). The dependence of peak height of $(A^2)_{11}(x)$ and $(A^2)_{22}(x)$ on the phase $a$ is shown in panel e).

the bright states phase, as evident from [14]. The same

holds for $(A^2)_{ij}(x)$ [16]:

$$(A^2)_{ij}(x) = -\hbar^2 \sum_{M(x)\in B(x)} \langle M’(x)|M|D’(x)\rangle =$$

$$= \hbar^2 \langle D’(x)|D’(x)\rangle$$

Let us first consider a special case of $a = 0$. With the position-independent $|D_1(x)\rangle$ and $|D_2(x)\rangle$ given by [24], the coefficients of $2 \times 2$ projections of $A(x)$ and $A^2(x)$ included in [10] are all zero except for the $(A^2)_{22}(x)$ which is equal to:

$$(A^2)_{22}(x) = \left( -\frac{\epsilon \cos k_L x}{\epsilon^2 + \sin^2 k_L x} \right)^2,$$  \hspace{1cm} (29)

with $\epsilon = \Omega_3 / \sqrt{\Omega_1^2 + \Omega_2^2} = \Omega_p / \Omega_c$, in analogy to $\Lambda$ system. The Hamiltonian $H_2$ is then a direct sum of two decoupled $D_1, D_2$ channels. The Hamiltonian [12] for the particle in the $D_1$ channel is that of a freely moving particle

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}.$$  \hspace{1cm} (30)

with running waves eigenfunctions of the form $d_1(x) = \exp(iqx)$. The Hamiltonian $H_2$ for the particle in the $D_2$ channel reduces to the movement in the scalar potential given by $(A^2)_{22}(x)$:

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + (A^2)_{22}(x).$$  \hspace{1cm} (31)

The above potential, given by (29), is precisely the subwavelength comb potential which appears for $\Lambda$ system construction [19]. It is shown in Fig. 2b).

For $a \neq 0$ the coefficients $A_{12}(x)$ and $(A^2)_{12}(x)$ are non-zero, and the two channels $D_1$, $D_2$ are coupled. Fig. 2b) shows the spatial dependence of $A(x)$ and $A^2(x)$ for a small $a = 0.05$ value (for that a value, the $(A^2)_{22}(x)$ resembles that for $a = 0$). The potentials change for a larger value of $a$ as shown in Fig. 2b) in panels (c) and (d) for $a = 0.435$. The potentials $(A^2)_{11}$ and $(A^2)_{22}$ are in the form of a comb, with $(A^2)_{22}$ being much larger than $(A^2)_{11}$ (the remnant of vanishing $(A^2)_{11}$ in the limiting case $a = 0$). This is also evident from Fig. 2b) which shows the maximum height of $(A^2)_{11}$ and $(A^2)_{22}$ as the function of $a$. All the above potentials are clearly $\lambda_L / 2$-periodic implying that the dark state-only model $H_2$ has period $\lambda_L / 2$ – half of the period of the full model $H$.

The potential shapes depend obviously on the choice of the basis in the dark subspace. In the Appendix this issue is discussed further.

III. THE BAND STRUCTURE

A. General considerations

We discuss the band structure of the full model [1]. In numerical calculations we work directly with the
FIG. 3. Band structure of the model (1) for $\Omega_1 : \Omega_2 : \Omega_3 = 50 : 20 : 5$. Panel a) shows the spectrum for $\Omega_1 = 1000E_R$ and $a = 0.435$. Panel b) $\Omega_1 = 5000E_R$ and $a = 0.435$. Panel c) shows the dark-state only limit of the band structure for $\Omega_1 = 5000E_R$ and $a = 0$. While in A0-c) $\Gamma_x = 0$ in panel d) we show the band structure for $\Gamma_x = 1000E_R$ and $\Omega_1 = 5000E_R$. The black line within bands shows the Re $E_\alpha$ and the red region is given by the Re $E_\alpha \pm 50\text{Im} E_\alpha$ curves, denoting $q$-dependent losses due to spontaneous emission. For clarity the bright state eigenvalues have been removed from this panel.

full Hamiltonian (1). The BO decomposition into dark and bright states, and in particular Hamiltonian $H_2$ in Eq. (16) is instrumental for the interpretation of the results.

We look for the quasiperiodic Bloch eigenstates of the $\lambda_L$-periodic model (1) directly in $|g_i\rangle, |e\rangle$ basis:

$$B_q(x) = e^{iqx} \left( \sum_{i=1}^3 b_{g_i}(x)|g_i\rangle + b_e(x)|e\rangle \right),$$

$$\equiv e^{iqx} [b_{g_1}(x), b_{g_2}(x), b_{g_3}(x), b_e(x)]^T. \tag{33}$$

The period of the Hamiltonian and the Bloch theory guarantee that $b_e(x)$ are $\lambda_L$-periodic functions and $q$ is the quasimomentum $q \in BZ_1 = [-\pi/\lambda_L, \pi/\lambda_L]$, where $BZ_1$ is the Brillouin zone. Looking at Eq. (1) and spatial dependence of $\Omega_1$’s one finds an extra parity symmetry: coefficients $b_{g_1}, b_{g_2}$ are actually all $\lambda_L$-periodic or $\lambda_L$-2-antiperiodic, and if $b_{g_1}, b_{g_2}$ are $\lambda_L$-2-periodic then $b_e, b_{g_3}$ are $\lambda_L$-2-antiperiodic and vice versa.

For the numerical formulation of the eigenproblem for Hamiltonian (1) the Fourier series expansion of $b_e(x)$ is used. It puts the Hamiltonian, $H$, in a sparse matrix form, which is diagonalized using standard numerical packages [21]. The different eigenvalues $E_\alpha^q$ are indexed by $\alpha$ for each value of $q \in BZ_1$.

Consider first $\Gamma_x = 0$ case – the corresponding band structure obtained for some particular values of $\Omega_i$ is depicted in Fig. 3. When $||\Omega||$ dominates other energy scales, the full spectrum contains energy levels that can be traced back to: $E_+$ bright states, $E_-$ bright states and dark states $D_1, D_2$, effectively described by Eq. (16).

Fig. (3a) shows the section of the band structure at low energies above zero that feature a series of bands and two nearly vertical lines of eigenvalues intersecting them. We identify the bands with the dark subspace, as they can be reproduced with identical Bloch bands computation for dark-state-projected model $H_2$. The vertical lines originate from $E_-$ bright states and are modelled by the Hamiltonian:

$$H_{B-} = -\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} + E_-(x). \tag{34}$$

Energy levels that can be traced to $E_+(x) \gg 0$ are absent in the figure. This channel does not have any energy levels at energies close to 0, however, its eigenstates can be coupled to dark states off-resonantly.

The dark subspace bands are two-valued. This comes from the fact that coefficients of $A(x), A^2$ as in (12) are of period $\lambda_L/2$ and the Bloch theory applied to Fig. 3 assumes twice larger lattice period of $\lambda_L$, natural period of Eq. (1). Complementary results from $H_2$ model could be obtained from Bloch theory with lattice period $\lambda_L/2$ and a larger Brillouin zone $BZ_2 = [-2\pi/\lambda_L, 2\pi/\lambda_L]$. This would yield dark only Bloch eigenfunctions of the form:

$$(B_D)_q^\alpha = e^{iqx} (b_{D_1}(x)|D_1(x)\rangle + b_{D_2}(x)|D_2(x)\rangle) \equiv e^{iqx} [b_{D_1}(x), b_{D_2}(x)]^T \tag{35}$$

where $q \in BZ_2$ and $b_\alpha$ are $\lambda_L/2$-periodic. We note that the above vector, when re-expressed in the $|g_1\rangle, |g_2\rangle, |g_3\rangle, |e\rangle$ is only $\lambda_L$-periodic just as $|D_1(x)\rangle, |D_2(x)\rangle$. The 4-channel computation for large $||\Omega||$ yields good approximation of the above. For quasi-momenta $q, q' \in BZ_2, q \in BZ_1$ such that $|q - q'| = 2\pi/\lambda_L$ the $\lambda_L$-periodic Bloch theory treatment of full $H$ ascribes them both to a single $q \in BZ_1$. Such folding has already appeared for a special case of a $\Lambda$ system (see 19) and is not unique to the tripod configuration, it is a simple consequence of the mismatch between the dark-state lattice constant and the period of the model.

Couplings due to $A(x)$ between dark states and the resonant $E_-$ states lead to small avoided crossings [clearly visible in Fig. 3a]. Fig. 3b) illustrates the fact that for larger $\Omega_1$ the avoided crossings with the $E_-$ bright state get narrower indicating an even better isolation of the dark subspace. The improving separation between the dark subspace and the $E_-$ bright band with increasing $\Omega_1$ is easily understood from the model (16). The avoided crossings appear between low lying dark-states and the highly excited $E_-$ with the same $q \in BZ_1$. Increasing $||\Omega||$ pushes the $E_-$ manifold towards more negative energies. As a result for larger $||\Omega||$ the wave vector describing freely moving $E_-$ with approximately zero energy is
more and more oscillating. This reduces the coupling to dark state Bloch vector via \( A(x) \).

For a comparison we show also bands corresponding to \( a = 0 \) case in Fig. [3]). Here the dark state \( |D_1\rangle \) is position independent [see (23)], the spectrum is a sum of that of a freely moving particle in channel \( D_1 \) [see Fig. [3]) and a \( D_2 \)-particle feeling the presence of the potential (29). The two spectra intersect each other with no avoided crossings forming between them. The bright state line also does not couple to position-independent \( D_1 \) through \( A(x) \) (14). The avoided crossing prominent in Fig. [3] is between \( B_- \) and \( D_2 \) channel bands.

B. Dark-bands lifetime

Let us consider now the lifetime of different bands. When \( \Gamma_e \neq 0 \), in the Hamiltonian (2), the energies \( E_{\pm}(x) \) of the bright state channels (5) that appear in (11), (12) are complex, in particular \( \text{Im} E_{\pm}(x) = -\Gamma_e/4 \) for \( \alpha = 0 \). The diagonalization of the model (2) focusing on low-lying dark state band reveals strong \( q \)-dependence of \( \text{Im} E^2_{\pm}(x) \) as shown in Fig. [3]). Here we assume \( \Gamma_e = 1000E_R \) and red shaded areas \( \text{Im} E^0_q \) are multiplied by 50 to make them more visible. The \( q \)-dependence for the \( \Lambda \) system was already described in [10] and indirectly observed experimentally [14].

![Graph showing the \( q \)-averaged decay rate \( \Gamma_e \) for \( \Omega_1 = 5000E_R, \Omega_2 = 2000E_R, \Omega_3 = 500E_R \) as the function of \( \Gamma_e \) for 2nd band. Panel b) shows the same as the function of \( ||\Omega|| \) for same relative ratio of \( \Omega_i \)’s and \( \Gamma_e = 1000E_R \). The panel c) shows \( \Gamma_e \) as the function of \( a \) for \( \Gamma_e = 1000E_R \) and two sets of \( \Omega_2 = (5000E_R, 2000E_R, 500E_R) \), for four first bands. Black (darker) lines correspond to \( \Omega_2 = 2000E_R \), while red (lighter) lines to \( \Omega_2 = 1000E_R \). The bands are depicted subsequently as solid, dashed, dashed-dotted and dotted lines.](image)

The total lifetime of the gas populating a certain band is approximated by the inverse of the \( q \)-averaged decay rate

\[
\bar{\Gamma}_e = \frac{2}{\text{vol}(BZ_1)} \int_{q \in BZ_1} \text{Im} E^0_q \, dq. \tag{36}
\]

Figure (4) shows the numerically obtained dependence of \( \bar{\Gamma}_e \) on \( \Omega_1 = 5000E_R, \Omega_2 = 2000E_R, \Omega_3 = 500E_R \). Panel b) shows the dependence on \( ||\Omega|| \) assuming a fixed ratio \( \Omega_1 : \Omega_2 : \Omega_3 = 50 : 20 : 5 \). Just like in the \( \Lambda \) system [10], one observes that \( \bar{\Gamma}_e \propto \Gamma_e \) and \( ||\Omega||^{-2} \).

A good approximation for \( \text{Im} E^0_q \) follows from the second-order perturbation theory arguments. The imaginary contribution to the energy of the dark state Bloch function \( (B_D)^0_q(x) \) is

\[
\text{Im} \Delta E^0_q = -\sum_{\beta} \sum_{\sigma \in \pm} \Gamma_e \frac{|\langle (B_D)^0_q|H_c|E_{\beta,\sigma}^\pm \rangle|^2}{2 E^0_q - \text{Re}(E_{\sigma}^\beta)^2 + \frac{\Delta^2}{4}} \tag{37}
\]

where \( E_{\sigma}^\beta \) refer to bright state eigenvectors in potentials \( E_\sigma(x), \sigma \in \pm \) with the same quasimomentum \( q \). \( H_c \) in (37) contains all the non-diagonal terms in Hamiltonian (16). For the vast majority of states indexed by \( \beta \) the \( \Gamma^2 \) term in the denominator may be neglected. Moreover, the sum is dominated by by bright states \( E_\pm(x) \) with energy close to \( \max E_\pm(x) \propto ||\Omega|| \), and bright states of \( E_\pm(x) \) with energy close to \( \min E_\pm(x) \propto ||\Omega|| \). This qualitatively explains the observed dependence on \( \Gamma_e \) and \( ||\Omega|| \).

The coupling in the numerator of (37) depends on \( A(x) \) terms in (12). It is greatly increased if the coefficients of \( A \) responsible for the coupling of the dark state to the bright states are sharply peaked and large as it happens in the \( a = 0 \) limit leading then to larger losses as shown in Fig. [4]). Already \( a \approx 0.435 \) offers order of magnitude longer lifetime than \( \Lambda \) system case, \( a = 0 \). We also note that the ratio of \( \Omega_1 : \Omega_2 : \Omega_3 \) strongly affects the expected lifetime, particularly for large \( a \). Fig. [4]) presents the simulated \( \bar{\Gamma}_e \) for first four bands for \( \Omega = (5000E_R, 2000E_R, 500E_R) \) (black lines) and \( \Omega = (5000E_R, 1000E_R, 500E_R) \) (red lines). Reducing the ratio \( \Omega_2/\Omega_3 \) from 4 to 2 results in the order of magnitude shorter lifetime for large \( a \approx \pi/2 \).

C. Spin decomposition of Bloch bands

Let us discuss the decomposition of Bloch eigenvectors into atomic states \( |g_1\rangle, |g_2\rangle, |g_3\rangle, \) \( |e\rangle \). We use again our exemplary set of parameters \( \Omega = (5000E_R, 2000E_R, 500E_R) \) for an illustration, taking also \( \Gamma_e = 0 \). Figure [5] shows the averages \( \bar{g}_i = \int_0^{\Lambda e} |(B^0_D|g_i \rangle|^2 \) for different quasimomenta within first two bands of \( a = 0 \) and \( a = 0.435 \) systems.

For \( a = 0 \) the lowest “band” is actually a portion of the parabolic energy dependence of a freely moving particle. It forms a closed band as soon as \( a \neq 0 \). The \( \bar{g}_i \) are in that case constant and given by the constant coefficients of Eq. (23). For the first excited band, that is
the lowest band in the $D_2$ channel, the dependence on $q$ is very small [panel b]]. When $a \neq 0$ is increased towards the final value $a = 0.435$ the well defined bands are formed, as in Fig. 3b. Different parts of each band intersect other energy levels, experiencing avoided crossings in different ways. For example the $D_1$ and $D_2$ channels at energy close to $1E_R$ for $q = 0$ in Fig. 3c experience transition through the avoided crossing before well separated bands in Fig. 3b are formed. At the same time energy levels at $q = \pm \pi / \lambda_L \in BZ_1$ remain nearly unaffected by other energy levels. This is the reason for the observed strong dependence on $q$ of decompositions $\tilde{g}_i$ in Fig. 5c and d) for $a = 0.435$. Specifically, one can observe that decompositions into $\tilde{g}_i$ of the first excited band for $q = 0 \in BZ_2$ resembles the decomposition of the $D_2$ channel in Fig. 3b), and for $q = \pm 2\pi / \lambda_L \in BZ_2$ that of $D_1$ in Fig. 3c). At the same time the lowest band shows strong dependence of $\tilde{g}_i$ on $q$ which however does not approximate $D_1$ or $D_2$ for any $q$.

FIG. 5. Decomposition of Bloch states in $BZ_2$ into $|g_i\rangle$, where solid, dashed and dot-dashed lines correspond to, respectively, $i = 1, 2, 3$. Top row a), b) corresponds to $a = 0$, the bottom row c), d) $a = 0.435$. Left column a), c) shows lowest bands, while the right column b), d) – first excited bands. The inset in b) presents a weak dependence of $|g_i\rangle$ on this panel using $|g_3\rangle$ as an example.

The variation of the overlap of Bloch vectors on $|g_1\rangle, |g_2\rangle, |g_3\rangle$ within the same band is directly observable. Consider a bosonic, non-interacting gas cooled down to the least energetic state of the band. Denote the quasimomentum of such a state by $q_0$. If an extra potential $H_{\text{tilt}} = -F x$ is added (see 22) to the Hamiltonian 1 then a steady drift of the quasimomentum $q(t) = q_0 + Ft/\hbar$ occurs, allowing one to reach the desired value of $q$ by controlling the application time of $H_{\text{tilt}}$. The value of $F \lambda_L$ should be much smaller than the energy gap to other bands, to prevent populating them. The spin decomposition can be studied by turning off the lasers responsible for Rabi frequencies $\Omega$, and splitting the atomic cloud in $|g_1\rangle, |g_2\rangle, |g_3\rangle$ components by a magnetic field gradient.

The response of the tripod system to gradient $H_{\text{tilt}}$ would form a coherent, time-dependent transfer of populations of atomic states $|g_1\rangle, |g_2\rangle, |g_3\rangle$, a feature which is easily measurable.

### IV. THE TIGHT BINDING MODEL

A tight-binding model conveniently describes movement of the particles populating a particular band. We describe first the construction of the Wannier functions in analogy to the textbook Wannier function calculation for a cosine-squared optical lattice 23 25.

#### A. The construction of Wannier functions in two-dimensional dark subspace

We start with the basis of all the dark-state Bloch functions $$\{(B_D)_q^\alpha : q \in BZ_2\}$$ for a particular band $\alpha$ of $H_2$. Then the Wannier function can be expressed as:

$$W_n^\alpha(x) = N \int_{q \in BZ_2} (B_D)^q(x) e^{i\theta_{q,n}} dq, \quad (38)$$

the index $n$ denotes localization over $n$-th lattice site, $x_n = x_0 + n \lambda_L / 2$, and $N$ ensures that $\int_{BZ} |W_n^\alpha(x)|^2 dx = 1$. The functional dependence of phases $\theta_{q,n}$ on $q$ has to be chosen to localize the $W_n^\alpha$. To find it, we adapt the method by Kivelson 24. The $H_2$ Hamiltonian is considered under periodic boundary conditions in a box of a sufficient total length $L$. This discretizes the Brillouin Zone $BZ_2 \to [0, 2\pi / L, \ldots, 4\pi / \lambda_L]$. We construct the $L \times L$ matrix

$$M_{q,q'} = \langle (B_D^q_0 \mid e^{2\pi i x/L} (B_D^q_0) \rangle, \quad q,q' \in BZ_2. \quad (39)$$

Its eigenvalues are complex phases of the type $\exp[(2\pi i x_n)/L]$ – that determines $x_0$. The corresponding eigenvector then defines the values of $\theta_{q,n}$ that localize $W_n^\alpha$ around the location $x_n$. We have verified that the obtained Wannier functions are exponentially localized around $x_n$, as expected for this procedure 24.

It is also worth noting that using this method Wannier functions can be computed directly within $\Lambda_L$-periodic Bloch theory ($q \in BZ_1$) by including both branches of a folded band while computing 39 matrix elements.

For a single-channel problem with a periodic potential, one can calculate a single Wannier function e.g. $W_0^1(x)$ and then use a discrete translation $W_n^1(x) := W_0^1(x - (x_n - x_0))$ to complete the basis. For the tripod system this is also a possibility but a special care should be taken when applying translation to $W_0^1(x)$ given by Eq. (38) when expressed in $|g_3\rangle, |e\rangle$ basis. Indeed if we expand $W_n^\alpha(x)$ in $|D_1(x)\rangle$ basis:

$$W_0^\alpha(x) = w_1(x)|D_1(x)\rangle + w_2(x)|D_2(x)\rangle \quad (40)$$
then one can shift the $w_i(x)$ functions only and resum
the $W_n(x)$ to obtain
\[ W_n^\alpha(x) = w_1(x-n\lambda_L/2)|D_1(x)\rangle + w_2(x-n\lambda_L/2)|D_2(x)\rangle. \] (41)
Instead one might attempt to translate the entire $W_n^\alpha(x)$ obtaining
\[ w_1(x-n\lambda_L/2)|D_1(x-n\lambda_L/2)\rangle + \]
\[ + w_2(x-n\lambda_L/2)|D_2(x-n\lambda_L/2)\rangle \neq W_n^\alpha(x). \] (42)

As $|D_i(x)\rangle$ are only $\lambda_L$-periodic, both approaches are not
equivalent for odd $n$. The former approach leading to
\[ \text{(41)} \] is a proper one, as it corresponds to a shift of the
coefficients of the Wannier function by the Hamiltonian
\[ \text{(41)} \]
where:
\[ \text{obtaining} \]
\[ \text{The second equality is true assuming that the global} \]
\[ \text{Wannier functions are discussed further in Section \[IV.D\], after} \]
\[ \text{Section \[IV.D\],} \]
\[ \text{constructing the tight binding} \]
\[ \text{Hamiltonian description.} \]

B. Hopping amplitudes

The Hamiltonian $H_2$, restricted to band $\alpha$ when ex-
pressed in the basis $W_n^\alpha(x)$ transforms to
\[ H_{\text{hopp},\alpha} = - \sum_{n,m} J_{nm}^\alpha \hat{a}_n^\dagger \hat{a}_m^\alpha + H.c. \] (43)
where:
\[ J_{nm}^\alpha = - \int dx \ (W_n^\alpha)^\dagger(x)H_2(x)W_m^\alpha(x) = \]
\[ = - \frac{1}{\text{vol}(BZ_2)} \int_{BZ_2} e^{i(m-n)q\lambda_L/2} E_q^\alpha dq. \] (44)
The second equality is true assuming that the global
phase factors of $W_n^\alpha(x)$ are defined by \[ \text{(41)}. \]
The hopping amplitudes depend only on the distance between
sites $n, m$ so one can simplify the notation by defining:
\[ J_{n-m}^\alpha := J_{n,m}^\alpha. \] (45)

The hopping amplitudes (referred simply as hoppings
later on) may be directly calculated from their defini-
tion \[ (44) \] using previously determined Wannier functions.
Their exponential tail requires, however, a special care for
accurate determination of $J_i^\alpha$, important, in particular,
for $i > 1$. However, the hoppings $J_i^\alpha$ can be read from
the band energies, for $q \in BZ_2$:
\[ E_q^\alpha = E_0 - 2J_i^\alpha \cos(q\lambda_L/2) - 2J_i^\alpha \cos(2q\lambda_L/2) - \ldots \] (46)

Calculation directly in $|g_i\rangle, |e\rangle$ basis followed by band
unfolding suffices to determine $J_{i}^\alpha$ as well. As the
$J_i$-s are
defined with respect to \( q \in BZ_2 \), a mistake in the unfolding of \( q \in BZ_1 \) would lead to a sign flip of \( J_i \)-s with an odd \( i \). The Bloch vectors obtained in the 4-channel calculation can be projected back onto the \( |D_1(x)|, |D_2(x)| \) space. The quasi-periodicity of the coefficients

\[
b_{D_i}(x + \lambda L/2) = \exp(iq\lambda L/2)b_{D_i}(x)
\]

allows to distinguish the two Bloch states \( q, q' \in BZ_2 \), \( |q - q'| = 2\pi/\lambda L \) correspond to the same point in \( BZ_1 \).

Unfortunately, the band unfolding by assigning of \( q', q \in BZ_2 \) is gauge-dependent. Applying Eq. (47) uses a particular gauge during projection on \( |D_1(x)| \). As a result when two eigenvectors for a particular \( q \in BZ_1 \) are being relabelled by \( q, q' \in BZ_2 \), \( |q - q'| = 2\pi/\lambda L \) the assignment of \( q, q' \) is reverse for \( D_1 \) as in Eqs. (20), (21) and \( D_1 \) as in Eqs. (20'), (21'). This means that the dependence on \( q \) of the quasienergy \( E_q^\alpha \) present in Eq. (46) differs by a translation by \( 2\pi/\lambda L \), and a sign flip in \( J_n \): 

\[ J_n^\alpha \rightarrow J_n^\alpha(-1)^\alpha \].

The ambiguity of signs of \( J_n^\alpha \) is not in conflict with definition of \( J_n^\alpha \) by means of the Wannier functions, Eq. (44). It is fully recovered when the \( J_n^\alpha \) are computed from Eq. (44) in both gauges.

We follow the gauge choice given by Eqs. (20), (21) and calculate the tunnelings for low lying (and long living) bands. Again we discuss similar parameter values as before, i.e. \( \Omega = (5000, 2000, 500)E_R \). Consider first the lowest band taking a familiar from the standard Bose-Hubbard model form – compare Fig. 3. Not surprisingly \( |J_2| \ll |J_1| \) for most of values of the phase shift parameter \( a \) thus nearest neighbor hopping dominates. Interestingly, however, \( J_1 \) changes sign when \( a \) is varied – compare Fig. 6(c) which allows for realization of frustration as discussed in Section IV C.

In Fig. 6(b) and c) we show the values of \( J_n^\alpha \)’s as the function of \( a \) for the first excited, almost flat, band (compare Fig. 3) that results in an unusual relation between \( J_1 \) and longer distance hopping amplitudes. For \( a < 0.2 \) or \( a > 0.45 \), for this band, amplitudes for long-distance hopping \( J_i \) with \( i > 3 \) are non-negligible indicating that the tight binding approach may be not a best choice in such a case. However for \( a \in [0.2, 0.45] \) only \( J_1, J_2, J_3 \) can be considered for an accurate tight-binding model. The next nearest hopping \( J_2 \) is larger then nearest-neighbor amplitude \( J_1 \) and larger than next-next-nearest neighbor amplitude \( J_3 \). Only for \( a \approx 0.2 \) do \( J_1 \) and \( J_2 \) become comparable but are of opposite signs. Around \( a = 0.435 \) a special situation occurs as \( J_1 \approx 0 \). This is in agreement with band structures in Fig. 3 where the second band seems to be “single valued” at the scale of the figure.

The Fig. 6(d) shows a similar calculation of hopping amplitudes for the excited band, but for the ratio of \( \Omega_1 : \Omega_2 : \Omega_3 = 50 : 40 : 5 \). The same configurations of amplitudes \( J_i \) occur in a different range of the phase shift \( a \) parameter. The corresponding interval for \( a \) is \([0.1, 0.22]\). Its location and size depends approximately linearly on the ratio \( \Omega_3/\Omega_2 \), as long as \( \Omega_2 < \Omega_1 \) and \( \Omega_2/\Omega_3 \gg 1 \). When \( \Omega_2/\Omega_3 \approx 1 \) the region of interest does not exist (naive interpolation puts it for \( a > \pi/2 \)).

C. Frustration

It is well known that a 1D spin-1/2 chain with \( J_1 \) (nn) and \( J_2 \) (nnn) interactions may exhibit frustration [26] – a situation in which it is not easy to satisfy energetically minimalization of all the possible bonds [27]. Such a chain maps into a nonpartite triangular ladder in which for negative tunnelings \( J_1, J_2 \) kinetic frustration occurs [28] reviews the physics of periodically driven systems that enable a change of sign of the tunneling matrix elements, see also [29]. In our situation the sign of \( J_1 \) (or any \( J_i \) odd \( i \)) can be inverted by a gauge transformation (reverting the sign of the every second Wannier function). It is thus more interesting that for the lowest band \( J_2 \) becomes negative (antiferromagnetic). For most \( a \) values \( J_1 \) dominates making frustration difficult to observe. However, since \( J_1 \) changes sign [around \( a = 0.275 \) for the chosen values of \( \Omega_i \), see Fig. 6(c)] it becomes small and comparable to \( J_2 \) for nearby \( a \) values leading to quite standard frustrating system. Note that a change of sign of \( J_1 \) in triangular lattices was realized via periodic lattice shaking (see [23, 24] and references therein) – here no additional shaking is needed and frustrating conditions are are realised by changing the phase mismatch \( a \).

Situation is equally interesting for the first excited band. Here, compare Fig. 6 both the nn, J1 and nnn, J3 may change sign depending on \( a \) value while \( J_2 \) remains positive and large. Consider first the simplest situation when we adjust \( a \) such that \( J_3 \) vanishes. The system maps to a triangular ladder with \( J_2 \) positive and regardless of the sign of \( J_1 \) no frustration occurs. This is again a manifestation of the fact that the change of sign of every second Wannier functions is just a gauge transformation that changes the sign of \( J_{2i+1} \) leaving the physics unaltered.

In the presence of \( J_3 \) the models becomes less obvious. The mapping on the triangular ladder does not work anymore. Instead one can map a 1D chain into a three dimensional tetrahedral linear compound as depicted in Fig. 7. Such a representation allows us for a better visualization of a competition between different hopping terms. Now it is easy to see that if the signs of \( J_1 \) and...
$J_3$ are different the system will frustrate as one cannot minimize energetically the $i, i+1$ and $i, i+3$ bonds. On the other hand, in the interval of $a$ values where $J_1$ and $J_3$ are of the same sign, no kinetic frustration occurs.

D. The properties of Wannier functions

In the light of highly non-standard relations between $J_1, J_2, J_3$ hopping amplitudes [Section IV] it is instructive to inspect spatial profiles of Wannier functions of the tripod system. Again assume $\Omega = (5000E_R, 2000E_R, 500E_R)$ as an example. Figure 8 shows the total density $||W_n|\alpha|^2$ for the Wannier functions. Panels a) and c) show the Wannier function for the lowest dark state band for $a = 0.435$ and $a = 1.3$ respectively. The notable feature is a non-vanishing overlap of densities of neighboring Wannier functions (they remain of course orthogonal to each other). For a large $a$ additional modulation shows indicating poorer confinement which corresponds well with large values of long range hopping $J_{1,2,3}$. For the first excited band panels Fig. 8b) and d) show again $a = 0.435$ and $a = 1.3$ cases. The second band shows Wannier functions that are bimodal and have a total width $\sim \lambda_L$. Despite that the $2\lambda_L$-dispersed Wannier functions are mutually orthogonal. This is possible only because the Wannier functions can alter decomposition into separate $g_1, g_2, g_3$ in a position dependent way. When calculating the inner product of $W_n^\alpha$ and $W_{n+1}^\alpha$ the result is zero only after the summation over $\sigma \in \{g_1, g_2, g_3\}$. This is not possible in a scalar Wannier function for a standard optical potential.

Standard integrals describing two-particle interaction in Hubbard-type models are often of the type:

$$U = \int W_n(x)\bar{W}_m(x')V(x-x')W_{\alpha}(x)W_{\beta}(x)dx dx'$$  \hspace{1cm} (48)

For the first excited band the Wannier functions $W_n$ and $W_{n+1}$ significantly overlap allowing the above integral to yield large value of $U$ even for non-onsite processes. This potentially makes construction of discrete models with long-range interaction much easier without need to use, e.g. dipolar interactions. This correlates well with with non-standard relations between $J_1^{\alpha}, J_2^{\alpha}, J_3^{\alpha}$ which allow also for long range hopping.

V. SUMMARY AND OUTLOOKS

In this work a four-level system in tripod configuration hosting two dark state is presented. The position-dependence of the dark state is set by position-dependence of Rabi frequencies $\Omega_i(x)$’s. This constraint creates periodic gauge-field like potentials that give rise to a band structure with well separated bands. In contrast to the conceptually similar case of three-level system in $\Lambda$ configuration, the lifetime of the gas population the bands can be substantially increased. The controlling parameter is $a$ – the phase difference between two lasers implementing the $\Omega_1$ and $\Omega_2$ couplings. The band structure is characterized by a highly-nonstandard relations between hopping amplitudes to nearest-, next-nearest-, next-next-nearest-neighbor lattice sites, allowing for efficient long range hopping. This is reflected in non-standard shape of Wannier functions for the appropriate band that are supported at two neighboring unit cells.

When this work was close to completion we became aware of a recent preprint of E. Gvozdiakov, P.rackauskas and G. Juzeliunas [30] that also considered the similar tripod configuration for optical lattice creation. We believe that our choice of parameters minimizes the population of bright states (which makes such a lattice more stable than for the choice in [30]).

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Appendix A: A different gauge choice

It is natural to ask how other choices of the dark state basis compare to the one discussed above.

A general different possible basis choice for the dark state subspace is given by a position-dependent two di-
mensional unitary transformation
\[(|D_1(x)\rangle, |D_2(x)\rangle) \rightarrow U_2(x)(|D_1(x)\rangle, |D_2(x)\rangle), \quad (A1)\]
by angle \(\alpha(x)\). Such a transformation preserves the overall Hamiltonian form with \(A(x)\) transformed as:
\[A \rightarrow U(x)AU^\dagger(x) - i\hbar \frac{\partial U(x)}{\partial x} U^\dagger(x). \quad (A2)\]
where \(U(x)\) is a four-dimensional extension of \(U_2(x)\) including the transformation of the bright states (by phase factors). To preserve \(A_{ii} = 0, i < 3\) we can use \(U_2(x) = \exp[i\sigma_3\alpha(x)].\) Extra freedom to pick \(\alpha(x)\) does not unfortunately allow for nullification of non-diagonal terms of \(A(x)\) [see Appendix B].

Features such as the height and location of \((A^2)_{11}, (A^2)_{22}\) peaks and spatial dependence of terms \((A^2)_{12}, (A^2)_{22}\) are strongly gauge-dependent. To illustrate this we consider an alternative gauge choice. Instead of using dark states in Eq. (20) and (21) let us use Eq. (26) and (27) as a basis for determination of potential \(A(x)\). Figure 9 shows the gauge potentials in that case. Due to analytical simplicity of Eq. (26) and (27), one can work out the formulas for the \(A(x)\). They are:
\[A_{11}(x) = A_{22}(x) = 0 \]
\[A_{12}(x) = -ik_L\frac{\Omega_1\Omega_2\Omega_3\sin a}{\Xi_1^2(x)\Xi_2^2(x)}, \]
\[(A^2)_{11}(x) = k_L^2\left(\frac{\Omega_1\Omega_2\sin a}{\Xi_1(x)}\right)^2, \]
\[(A^2)_{12}(x) = -k_L^2\frac{\Omega_1\Omega_2\Omega_3\chi(x)\sin a}{\Xi_1^2(x)\Xi_2^2(x)} , \]
\[(A^2)_{22}(x) = k_L^2\left[\left(\frac{\Omega_1\Omega_2\Omega_3\sin a}{\Xi_1(x)\Xi_2^2(x)}\right)^2 + \left(\frac{\Omega_3\chi(x)}{\Xi_1^2(x)\Xi_2(x)}\right)^2\right] , \quad (A3)\]
where
\[2\Xi_1^2(x) = (\Omega_1^2 + \Omega_2^2) - C\cos(2k_Lx + b) \]
\[\Xi_2^2(x) = \Omega_3^2 + \Xi_2^2(x), \]
\[2\chi(x) = C\sin(2k_Lx + b), \]
\[C = \sqrt{\Omega_1^4 + \Omega_2^4 + 2\Omega_1^2\Omega_2^2\cos 2a}, \]
\[\sin b = \frac{\Omega_3^2\sin 2a}{C} . \quad (A4)\]

In this case the gauge potentials differ qualitatively from the ones in Fig. 2. First the potential maxima of \((A^2)_{11}\) and \((A^2)_{22}\) coincide and \((A^2)_{22}\) is nearly far from potential peaks. In the gauge choice defined by Eq. (20) and (21) the potential \((A^2)_{11}\) features a series of narrow peaks that are located between the peaks of \((A^2)_{22}\), shifted by \(\lambda_L/4\).

Another distinct feature of the choice in Eq. (20) and (27) is divergence of height of peaks of \((A^2)_{11}\) as \(a \to 0\) [see Fig. 9(3)]. This is in stark contrast to the case of Fig. 2 where \((A^2)_{11} \to 0\). This straightforwardly follows from Eq. (26). This is because for \(x = 0\) and \(x = -a\) the \(|D_1(x)\rangle\) is respectively \(|g_2\rangle\) and \(|g_1\rangle\). This implies rapid variation of \(|D_1(x)\rangle\) and in turn divergent \(A_{11}(x)\) – see Eq. (28).
Appendix B: Failure to zero $A_{12}$ by the gauge transformation

Using the gauge freedom, one could hope to vanish all elements of $A_{KL}$ in the dark subspace. Then, the dark-subspace Hamiltonian would simplify to

$$H = \frac{P^2}{2m} + \frac{(A')_{2\times2}}{2m}.$$  \hspace{1cm} (B1)

We consider a system, where $\Delta = 0$ and $\Omega_i$ are real giving $A_{11} = A_{22} = 0$. We choose an arbitrary, position-dependent basis $d_1(x)$, $d_2(x)$. The convenient choice is

$$d_1(x) = \hat{\Omega} \times \frac{\Omega'}{||\Omega||}, \hspace{1cm} (B2)$$
$$d_2(x) = -\frac{\hat{\Omega}'}{||\Omega||},$$

where $\hat{\Omega} = \Omega/||\Omega||$. The equation $A_{12} = 0$ implies

$$D_1' \cdot \Omega = 0, \hspace{1cm} (B3)$$
$$D_2' \cdot \Omega = 0.$$

We see, that there is still residual gauge freedom – we can rotate the basis by any position-independent rotation matrix. This freedom is well captured when we reformulate our problem. We express $D_i$ as a position-dependent rotation of $d_i$:

$$\begin{pmatrix} D_1(x) \\ D_2(x) \end{pmatrix} = \begin{pmatrix} \cos(\alpha(x)) - \sin(\alpha(x)) \\ \sin(\alpha(x)) \cos(\alpha(x)) \end{pmatrix} \begin{pmatrix} d_1(x) \\ d_2(x) \end{pmatrix}. \hspace{1cm} (B4)$$

After some algebra, we can write the final equation for $\alpha(x)$:

$$\alpha'(x) = -\frac{\Omega \cdot (\Omega' \times \hat{\Omega}'')}{||\hat{\Omega}||^2} = \frac{\det(\hat{\Omega}, \hat{\Omega}', \hat{\Omega}'')}{||\hat{\Omega}'||^2}. \hspace{1cm} (B5)$$

The solution is unique up to a constant, which reflects the residual gauge freedom. In this gauge

$$A = \frac{i\hbar}{\sqrt{2}}\frac{\Omega}{||\Omega||} = \begin{pmatrix} 0 & 0 & \sin \alpha & \sin \alpha \\ -\sin \alpha & -\cos \alpha & -\cos \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \end{pmatrix}, \hspace{1cm} (B6)$$

and

$$A^2 = \hbar^2 \left| \hat{\Omega}' \right|^2 = \begin{pmatrix} \sin^2 \alpha & \sin \alpha \cos \alpha & 0 & 0 \\ \sin \alpha \cos \alpha & \cos^2 \alpha & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}. \hspace{1cm} (B7)$$

There is, however, one problem – although $\alpha'(x)$ is essentially periodic for periodic $\Omega$, the solution $\alpha(x)$ may not be, when $\int_{\text{period}} d\alpha' = \frac{\pi}{n}$, $n \in \mathbb{Z}$. Then, for $\Omega(x) = \Omega(x + \lambda)$, equation $\hspace{1cm} (B4)$ implies that $D_1(x)$ and $D_2(x)$ are aperiodic functions of $x$. This aperiodicity translates back to remaining terms of $A(x)$ via $\hspace{1cm} (B6)$ which makes it impossible to directly apply Bloch theory.

[1] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Physical Review Letters 81, 3108 (1998).
[2] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, nature 415, 39 (2002).
[3] M. Lewenstein, A. Sanpera, and V. Ahufinger, Ultracold Atoms in Optical Lattices: Simulating quantum many-body systems (Oxford University Press, 2012).
[4] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D.-S. Luehmann, B. A. Malomed, T. Sowiński, and J. Zakrzewski, Rep. Prog. Phys. 78, 066001 (2015).
[5] N. Cooper, J. Dalibard, and I. Spielman, Reviews of modern physics 91, 015005 (2019).
[6] O. Boada, A. Celi, J. I. Latorre, and M. Lewenstein, Phys. Rev. Lett. 108, 133001 (2012).
[7] A. Celi, P. Massignan, J. Ruseckas, N. Goldman, I. B. Spielman, G. Juzeliūnas, and M. Lewenstein, Phys. Rev. Lett. 112, 043001 (2014).
[8] D. Suszalski and J. Zakrzewski, Phys. Rev. A 94, 033602 (2016).
[9] M. Lohse, C. Schweizer, H. M. Price, O. Zilberberg, and I. Bloch, Nature 553, 55 (2018).
[10] M. Łącki, M. Baranov, H. Pichler, and P. Zoller, Physical review letters 117, 233001 (2016).
[11] F. Jendrzejewski, S. Eckel, T. Tiecke, G. Juzeliūnas, G. Campbell, L. Jiang, and A. Gorshkov, Physical Review A 94, 063422 (2016).
[12] J. Larson and J.-P. Martikainen, Physical Review A 80, 033605 (2009).
[13] J. Larson and J.-P. Martikainen, Physical Review A 80, 033605 (2009).
[14] Y. Wang, S. Subhankar, P. Bienias, M. Łącki, T.-C. Tsui, M. A. Baranov, A. V. Gorshkov, P. Zoller, J. V. Porto, S. L. Rolston, et al., Physical review letters 120, 083601 (2018).
[15] T.-C. Tsui, Y. Wang, S. Subhankar, J. V. Porto, and S. L. Rolston, Phys. Rev. A 101, 041603 (2020).
[16] J. Ruseckas, G. Juzeliūnas, P. Öhberg, and M. Fleischhauer, Physical review letters 95, 010404 (2005).
[17] J. Dalibard, F. Gerber, G. Juzeliūnas, and P. Öhberg, Reviews of Modern Physics 83, 1523 (2011).
[18] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Nature 471, 83 (2011).
[19] C. Hamner, Y. Zhang, M. A. Khanehchi, M. J. Davis, and P. Engels, Phys. Rev. Lett. 114, 070401 (2015).

[20] In general, the more complete treatment of losses would be by the Lindblad master equation approach.

[21] In this work we have used standard scipy diagonalization function eigs.

[22] M. B. Dahan, E. Peik, J. Reichel, Y. Castin, and C. Salomon, Physical Review Letters 76, 4508 (1996).

[23] W. Kohn, Physical Review 115, 809 (1959).

[24] S. Kivelson, Physical Review B 26, 4269 (1982).

[25] N. Marzari, A. A. Mostofi, J. R. Yates, I. Souza, and D. Vanderbilt, Reviews of Modern Physics 84, 1419 (2012).

[26] H.-J. Mikeska and A. K. Kolezhuk, Lect. Notes Phys. 645, 1 (2004).

[27] H. T. Diep, Frustrated spin systems (World Scientific, Singapore, 2004).

[28] A. Eckardt, Rev. Mod. Phys. 89, 011004 (2017).

[29] K. Sacha, K. Targonska, and J. Zakrzewski, Phys. Rev. A 85, 053613 (2012).

[30] E. Gvozdiovas, P. Rackauskas, and G. Juzeliunas, “Optical lattice with spin-dependent sub-wavelength barriers,” (2021), arXiv:2105.15148 [quant-ph].