Abstract

Making use of the exact solutions of the $N = 2$ supersymmetric gauge theories we construct new classes of superconformal field theories (SCFTs) by fine-tuning the moduli parameters and bringing the theories to critical points. In the case of SCFTs constructed from pure gauge theories without matter $N = 2$ critical points seem to be classified according to the A-D-E classification as in the two-dimensional SCFTs.
Recently there have been some major advancements in our understanding of the strong coupling dynamics of 4-dimensional supersymmetric gauge theories \[1\],\[2\],\[3\],\[4\]. In the case of \(N = 2\) supersymmetry exact results for the low-energy effective Lagrangians have been obtained for a large class of gauge groups and matter couplings \[1\],\[2\], \[5\]-\[15\]. It turned out that the prepotential of the effective theory develops singularities in the strong coupling region when some of the solitons become massless. The behavior of the theory around the strong coupling singularities can be determined by rewriting the theory in terms of the dual ‘magnetic’ variables. Information on the strong coupling behavior of the theory together with its known behavior in the weak coupling region leads to a complete determination of the prepotential in the whole range of the moduli space.

In this paper we make use of these exact solutions of \(N = 2\) supersymmetric gauge theories and construct systematically new classes of \(N = 2\) supercoformal field theories (SCFTs) in 4-dimensions. We use the approach of refs.\[16\], \[17\] where the parameters of the moduli space of the theory (expectation values of the scalar field of the \(N = 2\) vector multiplet) and masses of matter hypermultiplets are adjusted so that massless solitons with mutually non-local charges coexist. When solitons of mutually non-local charges are present, the system is necessarily at a critical point and one obtains a superconformal field theory.

In the following we first describe our recent work \[18\] and discuss in detail the \(SU(N)\) gauge theories coupled with matter hypermultiplets and find new classes of non-trivial SCFTs. We locate superconformal points and determine the critical exponents of scaling operators. We shall see that the nature of the SCFTs is controlled by the unbroken sub-group of \(SU(N)\) and the global flavor symmetry. We then study SCFTs based on \(SO(N), Sp(2N)\) gauge theories.

It turns out that in the case of pure gauge theories without matter hypermultiplets SCFTs constructed from \(SU(N + 1), SO(2N + 1)\) and \(Sp(2N)\) gauge theories are identical and form a single universality class. On the other hand, SCFTs constructed from \(SO(2N)\) are distinct and form another universality class. We call these as \(A_N\)-type and \(D_N\)-type SCFTs, respectively. Critical exponents of their scaling fields are expressed by a formula \(2(e_i + 1)/(h + 2), i = 1, 2, \cdots, N\) where \(e_i\) and \(h\) are Dynkin exponents and dual-Coxeter numbers of \(A_N\) and \(D_N\) algebras, respectively.

This suggests the possibility of the \(A − D − E\) classification of \(N = 2\) four-
dimensional SCFTs. As for the case of the $E_N$ gauge theories, we will analyze their critical behaviors by making use of the string-theoretic construction given recently in [21] which reproduces elliptic curves and differential forms of known $N = 2$ gauge theories starting from $K_3$-fibered Calabi-Yau manifolds. In the case of SCFTs based on $E_N$ groups we again find the formula for their critical exponents $2(e_i + 1)/(h + 2)$ where $e_i$ and $h$ represent the Dynkin exponents and dual-Coxeter numbers of $E_6, E_7$ and $E_8$ algebras.

Thus we conjecture that there exists a $A − D − E$ classification behind the $N = 2$ 4-dimensional SCFTs: the classification originates from that of the degeneration of $K_3$ surfaces which appear in the $K_3$-fibered Calabi-Yau manifolds in the study of heterotic-type II string duality [20, 22, 23].

SCFTs from gauge theories coupled to matter

Let us first briefly recall the results of ref. [17] on $SU(2)$ gauge theory. In the case of the gauge group $SU(2)$ it is possible to introduce matter hypermultiplets (in the vector representation) up to $N_f = 4$ without losing the asymptotic freedom. In [17] authors considered the case of a common mass $m = m_i$ $(i = 1, \cdots, N_f)$ for all $N_f$ flavors. This is the case when the highest criticality is reached for each value of $N_f$. Parameters of the theory are then given by $u = \frac{1}{2} \text{Tr} \phi^2$ and $m$ where $\phi$ denotes the scalar field of the $N = 2$ vector multiplet.

Let us discuss the case of $N_f = 2$ for the sake of illustration. We first recall that the exact solution of the theory is described using an elliptic curve

$$\mathcal{C} : \quad y^2 = (x^2 - u + \frac{\Lambda^2}{8})^2 - \Lambda^2(x + m)^2, \quad (1)$$

where $\Lambda$ is the dynamical mass scale of the theory. The discriminant of the curve is given by

$$\Delta = \frac{1}{16} \Lambda^4 (8m^2 - 8u + \Lambda^2)^2 \Delta_m, \quad (2)$$

$$\Delta_m = (8u - 8\lambda m + \Lambda^2)(8u + 8\lambda m + \Lambda^2). \quad (3)$$

The power 2 of the factor $8m^2 - 8u + \Lambda^2$ in $\Delta$ means that the singularity at $u = u^* = m^2 + \Lambda^2/8$ has a multiplicity 2 and belong to the 2 representation of the flavor symmetry group $SU(N_f = 2)$. When the mass $m$ becomes large, this zero of the discriminant moves out to $\infty$ as $u \approx m^2$ and becomes the
massless squark singularity with its bare mass ±m being canceled by the vacuum value \( a = \pm m \) of the scalar field \( \phi \). Let us call such a singularity as the squark singularity although in the strong coupling region it represents a massless solitonic state carrying a magnetic charge.

In order to locate the superconformal point one first sets the value of \( u \) at the squark singularity \( u^* \). Then the curve and \( \Delta_m \) become

\[
y^2 = (x + m)^2(x - m - \Lambda)(x - m + \Lambda), \quad \Delta_m = 4\Lambda^4(2m + \Lambda)^2(2m - \Lambda)^2. \tag{4}
\]

One then adjusts the value of \( m \) at \( m^* = \pm \Lambda/2 \) so that \( \Delta_m \) vanishes. Then the squark singularity collides with the singularity of the monopole (or dyon) and we generate a critical behavior

\[
y^2 = (x \mp \frac{\Lambda}{2})^3 (x \mp \frac{3\Lambda}{2}). \tag{5}
\]

It is straightforward to analyze perturbations around this critical point and determine scaling dimensions \([u], [m]\) of the parameters \( u, m \) given by \([m] = 2/3, [u] = 4/3\). Results of ref. [17] are summarized in Table 1.

| \( N_f \) | \( m \) | \( u \) | \( C_2 \) | \( C_3 \) |
|---|---|---|---|---|
| 1 | 4/5 | 6/5 |   |   |
| 2 | 2/3 | 4/3 | 2  |   |
| 3 | 1/2 | 3/2 | 2  | 3  |

Table 1: Universality Classes of \( N = 2 \) SCFTs based on \( SU(2) \) Gauge Theory

We note that in the cases \( N_f \geq 2 \) there appear Casimir operators \( C_j \) associated with the global flavor symmetry group \( SU(N_f) \) with the dimensions \([C_j] = j\).

Let us now turn to the case of the \( SU(N_c) \) theory and start presenting our results. We consider the case of \( N_f \) matter hypermultiplets in vector representations with a common mass \( m \). (We may add extra flavors with different masses, however, at critical points this amounts only to shifting the rank \( N_c \) of the group). The curve is given by \([8]\)

\[
C : \quad y^2 = C(x)^2 - G(x), \tag{6}
\]

\[
C(x) = x^{N_c} + s_2 x^{N_c-2} + \cdots + s_{N_c} + \frac{\Lambda^{2N_c-N_f}}{4} \sum_{i=0}^{N_f-N_c} x^{N_f-N_c-i} \binom{N_f}{i} m^i,
\]
\[ G(x) = \Lambda^{2N_c-N_f}(x+m)^{N_f}, \] (7)

where the terms proportional to \( \Lambda^{2N_c-N_f} \) in \( C(x) \) are absent in the case \( N_f < N_c \). The meromorphic 1-form is given by

\[ \lambda = xd \log \frac{C-y}{C+y}. \] (9)

Expectation values of the scalar field \( \phi \) are obtained by integrating the 1-form around suitable homology cycles of the curve.

When \( N_f \geq 2 \), it is possible to show that the discriminant of the curve \( C \) has a factorized form

\[ \Delta = \Delta_s \Delta_m, \] (10)
\[ \Delta_s = (C(x = -m))^{N_f} \] (11)

((10),(11) may be shown in the following way. In the case of even flavors \( N_f = 2n \) we can split the curve as \( y^2 = y_+(x)y_-(x) \) with \( y_\pm(x) = C(x) \pm \Lambda^{N_c-n}(x+m)^n \). Then the discriminant becomes \( \Delta = (\text{resultant}(y_+, y_-))^2 \times \text{resultant}(y_+, y'_+) \times \text{resultant}(y_-, y'_-). \) Here \( ' \) means the derivative in \( x \). If one uses the formula \( \text{resultant}(y_+, y_-) = C(-m)^n \), one recovers (11). \( N_f=\text{odd} \) case may be treated in a similar way). The factor \( \Delta_s \) carries the power \( N_f \) and represents the squark singularity. In our search for superconformal points let us first set \( \Delta_s = 0 \). This fixes the value of \( s_{N_c} \) at \( s_{N_c}^* = -m^{N_c} - s_2(-m)^{N_c-2} \cdots \). The function \( C(x) \) becomes divisible by \( x+m \) and expressed as \( C(x) = (x+m)C_1(x) \) with a polynomial \( C_1(x) \) of order \( N-1 \). The curve becomes \( y^2 = (x+m)^2(C_1(x)^2 - \Lambda^{2N_c-N_f}(x+m)^{N_f-2}) \). It turns out that the value of \( \Delta_m \) at \( s_N = s_N^* \) factors as \( \Delta_1 \Delta_2m \) where \( \Delta_1 \) (a power of) the resultant of \( C_1(x) \) and \( x+m \). We next set \( \Delta_1 = 0 \) by adjusting the parameter \( s_{N-1} \) to its critical value \( s_{N-1}^* \). \( C_1(x) \) then becomes divisible by \( x+m \) and is written as \( C_1(x) = (x+m)C_2(x) \) with a polynomial \( C_2(x) \) of order \( N-2 \). The curve now has a 4-th order degeneracy at \( x = -m \), \( y^2 = (x+m)^4(C_2(x)^2 - \Lambda^{2N_c-N_f}(x+m)^{N_f-4}) \) and describes a critical theory.

We can iterate this procedure. We extract powers of \( x+m \) from \( C(x) \) by adjusting parameters \( s_N, s_{N-1}, s_{N-2}, \cdots \) successively and bring the curve to higher criticalities. As far as the extracted power \( \ell \) of \( x+m \) from \( C(x) \)
does not exceed $N_f/2$, the order of degeneracy of $C(x)^2$ is lower than that of $G(x)$ and the curve acquires a degeneracy of order $2\ell, y^2 \approx (x + m)^{2\ell}$.

When $\ell$ becomes greater than $N_f/2$, the order of degeneracy of $C(x)^2$ exceeds that of $G(x)$ and one can not necessarily increase the criticality of the curve by extracting higher powers out of $C(x)$. As we shall see below, when $N_f=\text{odd}$, the highest criticality of the curve is given by $N_f, y^2 \approx (x + m)^{N_f}$ while in the case of even flavor $N_f = 2n$ it is given by $N_c + n, y^2 \approx (x + m)^{N_c+n}$.

We classify critical points of $SU(N_c)$ theories into 4 groups;

1. $y^2 \approx (x + m)^{2\ell}; \begin{cases} \quad 2\ell \leq 2n - 2, & N_f = 2n \\ 2\ell \leq 2n, & N_f = 2n + 1 \end{cases}$

2. $y^2 \approx (x + m)^{N_f}; \quad N_f = 2n + 1$ (13)

3. $y^2 \approx (x + m)^{N_f}; \quad N_f = 2n$ (14)

4. $y^2 \approx (x + m)^{p+N_f}; \quad 0 < p \leq N_c - n, \quad N_f = 2n$ (15)

It turns out that theories of the class 1 above are free field theories. In order to construct non-trivial theories we have to bring the criticality of the curve at least as high as $N_f$ as in (13),(14),(15). We first analyze the SCFTs of the class 1 above and then turn to the discussion of the non-trivial SCFTs given by the classes 2, 3 and 4.

**Class 1**

In these theories the $G(x)$ term in the curve (6) has a higher criticality than $C(x)^2$ and may be ignored when we analyze the theory at the critical point. We may also ignore terms with higher power of $x + m$ in $C(x)$ than $(x + m)\ell$. Then $y^2 \approx C(x)^2 = (x + m)^{2\ell}$. We apply perturbations to this critical point as

$$C(x) = (x + m)^\ell - t_j(x + m)^j, \quad 0 \leq j \leq \ell - 1. \quad (16)$$

Perturbation splits the $\ell$-fold zeros of $C(x)$ at $x = -m$ into $j$-fold zeros. In order to describe the removal of the degeneracy we make a change of variable

$$x = -m + t_j \frac{1}{t_j^{j-1}} z. \quad (17)$$
Then
\[ C(x) = t_j \overline{\ell} (z^\ell - z^j). \tag{18} \]

Integration contours of the 1-form are given by the paths connecting \((\ell - j)\)-th roots of unity in the complex \(z\)-plane. It is possible to show that the term \(d \log(C - y)/(C + y)\) in \(\lambda \) (9) does not produce a factor dependent on \(t_j\). Only a power of \(t_j\) appears from the factor \(x\) in front of \(d \log(C - y)/(C + y)\) under the change of variable (16). Thus periods behave as \(a_i, a^D_i \approx t_j \overline{\ell} \). By requiring that the dimensions of the periods to be unity [17], we find that \([t_j] = \ell - j\). Integral values of the dimensions indicate that this is a free field theory.

In addition to the above perturbations removing the degeneracy of the roots of \(C(x)\) we may consider perturbations which remove the degeneracy of the masses \(m_i, i = 1, \ldots, N_f\) of the hypermultiplets. Under perturbation \(G(x)\) is replaced by
\[ \Lambda^{2N_c - N_f} \left( (x + m)^{N_f} + \sum_{i=2}^{N_f} C_i(x + m)^{N_f-i} \right). \tag{19} \]

Proceeding as in the previous case we can easily determine the exponents of the fields \(C_i(x + m)^{N_f-i}\) as
\[ [C_i] = i, \quad i = 2, \ldots, N_f. \tag{20} \]

These are in fact the Casimir operators of the \(SU(N_f)\) flavor symmetry.

A basic feature of the superconformal points of the class 1 is that the value of the mass \(m\) is left arbitrary at the critical point and the critical values of the tuned parameters \(s_{N_c}^*, s_{N_c-1}^*, \ldots, s_{N_c-\ell+1}^*\) become simultaneously large as the mass \(m\) is increased. Values of the other parameters \(s_2, \ldots, s_{N_c-\ell}\) are not fixed at the critical point and we may also take them to be large. Thus the critical points of class 1 stretch out to the “exterior region” of the moduli space. When the values of the moduli parameters are all much larger than \(\Lambda\), we may adopt the semi-classical reasoning. If we ignore instanton effects and put \(\Lambda = 0\), the curve becomes classical \(y^2 = C(x)^2 = (x^{N_c} + s_2 x^{N_c-2} + \cdots + s_N)^2\) and its discriminant is given by a classical expression. Then the condition of degeneracy of the function \(C(x) \approx (x + m)^\ell\) becomes the condition of the degeneracy of the eigenvalues of the field \(\phi\), i.e. \(\ell\) of the
eigenvalues of $\phi$ have to coincide. This implies that the $SU(\ell)$ sub-group of $SU(N_c)$ is left unbroken at class 1 superconformal points.

Thus near the region of the critical points of the class 1 theories we have effectively an $SU(\ell)$ gauge theory coupled with $N_f$ hypermultiplets. Since $2\ell < N_f$, the theory is in the asymptotically non-free regime. We expect that class 1 theories are at trivial fixed points. Let us denote the class 1 theory with the behavior $y^2 \approx (x + m)^{2\ell}$ as $M_{2\ell}^{N_f}$.

Class 2

Let us now turn to the class 2 theories which are intrinsic to the odd flavor case $N_f = 2n + 1$. Class 2 theories are obtained by further extracting powers of $x + m$ from the function $C(x)$. Once the extracted power $\ell$ exceeds $n$, $C(x)^{2\ell}$ term becomes irrelevant at the critical point and the curve is dominated by the term $G(x)$, $y^2 \approx (x + m)^{N_f}$. The perturbations on the eigenvalues of $\phi$ are given by

$$y^2 = ((x + m)^\ell - t_j (x + m)^j)^2 - \Lambda^{2N_c - N_f} (x + m)^{N_f}$$

$$\approx t_j^2 (x + m)^{2j} - \Lambda^{2N_c - N_f} (x + m)^{N_f}, \quad 0 \leq j < N_f/2. \quad (21)$$

$$y^2 \approx t_j^2 (x + m)^{2j} - \Lambda^{2N_c - N_f} (x + m)^{N_f}, \quad 0 \leq j < N_f/2. \quad (22)$$

By making a change of variable as

$$x = -m + t_j^{N_f - 2j} z \quad (23)$$

and using the fact that $C \approx y$, we find that again the only power of $t_j$ comes from the factor $x$ in front of the 1-form. The scaling dimensions are given by

$$[t_j] = \frac{N_f}{2} - j, \quad j = 0, 1, \cdots, n. \quad (24)$$

These fields have half-integral dimensions. If one restricts oneself to relevant fields, there exist only two $[t_n] = 1/2, [t_{n-1}] = 3/2$.

There also exist Casimir operators associated with the $SU(N_f)$ symmetry and the operators carry integral dimensions

$$[C_j] = j \quad (25)$$

as in the class 1 case.
The special feature of the class 2 theories is that these superconformal points do not depend on $N_c$ so far as $2N_c > N_f$. In fact the dimensions of the relevant operator $1/2, 3/2$ are exactly the same as in the case of $N_c = 2$ (see Table [1]). Thus they represent a universality class of $N = 2$ SCFTs with the global $SU(N_f = \text{odd})$ symmetry. We denote this universality class as $M_{2n+1}^2$.

As in class 1 theories class 2 conformal points extend to the semi-classical regions of the moduli space, i.e. all the adjusted parameters grow as $m$ increases. (The case $N_f = 2N_c - 1$ is an exception. In this case the value of $m$ is fixed to be in the strong coupling region $m^* \approx \Lambda$). The same argument as in the class 1 theories applies and we have effectively an $SU(\ell)$ gauge theory interacting with $N_f$ matter multiplets. In the present case, however, $2\ell > N_f$ and the system is in the asymptotically free regime. Thus the class 2 theories are non-trivial and are interacting superconformal models.

**Class 3**

Let us now go to class 3 theories of even flavors $N_f = 2n$. These theories are obtained by further adjusting the parameters of class 1 theories so that a power $(x + m)^n$ is extracted from $C(x)$. $C(x)$ is written as $C(x) = (x + m)^nC_n(x)$ with an $(N_c - n)$-th order polynomial $C_n(x)$ and the curve becomes

$$y^2 = (x + m)^{2n}(C_n(x)^2 - \Lambda^{2N_c - N_f}).$$

(26)

Without further tuning parameters the curve has the behavior $y^2 \approx (x + m)^{2n}$.

Class 3 theories also extend to the semi-classical regions $\{s_i\}, m \gg \Lambda$. At the class 3 critical point the unbroken subgroup is $SU(n)$ which is exactly the gauge group whose beta function vanishes in the presence of $N_f = 2n$ flavors. Class 3 theories are therefore expected to be in the same universality class as the known $N = 2$ SCFT [2, 9, 11, 12] of $SU(n)$ gauge theory with $2n$ massless matter multiplets. In fact we may decompose each of the squark superfields $Q^a, a = 1, 2, \cdots, 2n$ (belonging to the vector representation of $SU(N_c)$) into a sum of vector and singlet representations of $SU(n)$. Then the bare masses of the squark superfields of the vector representation of $SU(n)$ are exactly canceled by the vacuum expectation values of the field $\phi$ (which has $n$ degenerate eigenvalues $-m$). Thus there exist $2n$ massless squarks belonging to the vector representation of $SU(n)$. Hence the class 3 theories
belong to the same universality class as the massless $N_c = n$, $N_f = 2n$ theory. We denote this universality class as $M_{2n}^2$.

**Class 4**

Let us now turn to class 4 theories of even flavors $N_f = 2n$. We start from the curve of the class 3 theory (26) and enhance its criticality by adjusting the parameters of $C_n(x)$. We first note that the right-hand-side of (26) is given by a product of factors, $(x + m)^{2n}$ and $C_n(x)^2 - \Lambda^{2N_c - N_f}$. The first factor describes the curve of the $M_{2n}^2$ theory and the second factor describes that of the pure Yang-Mills theory of gauge group $SU(N_c - n)$ (without matter fields). Thus class 4 theories are interpreted as the coupled model of $M_{2n}^2$ and pure Yang-Mills theories.

Let us rewrite the curve as

$$y^2 = (x + m)^{2n}(C_n(x) + \Lambda^{N_c-n})(C_n(x) - \Lambda^{N_c-n})$$

and expand $C_n(x)$ in powers of $(x + m)$,

$$C_n(x) = (x + m)^{N_c-n} - Nm(x + m)^{N_c-n-1} + s_2(x + m)^{N_c-n-2} + \cdots + s_{N_c-n}$$

We can successively adjust the parameters as $s_{N_c-n}^* = -\Lambda^{N_c-n}$, $s_{N_c-n-1}^* = 0$ etc. and bring the curve to higher criticalities. The number of available parameters is $N_c - n$ and hence the highest criticality is given by $y^2 \approx (x + m)^{N_c+n}$. The parameters of the most singular curve are all fixed inside the strong coupling region and hence it represents an inherently strongly coupled field theory (at a lower criticality there are parameters which are left undetermined). We denote the universality class represented by a curve $y^2 \approx (x + m)^{p+2n}$ ($0 < p \leq N_c - n$) as $M_{2n+p}^2$.

Let us next analyze the perturbations of the most critical theory $M_{N_c+n}^{2n}$. Properties of perturbations of other theories are similar. The critical value of the mass $m^*$ vanishes in $M_{N_c+n}^{2n}$ and we may set $m = 0$ from the start. (The case $N_f = 2n - 2$ is an exception. In this case $m^*$ is non-zero and is of the order of $\Lambda$). Then it is easy to locate the critical point

$$s_{N_c}^* = 0, s_{N_c-1}^* = 0, \cdots, s_{N_c-n}^* = \pm \Lambda^{N_c-n}, \cdots, s_3^* = 0, s_2^* = 0.$$ 

(If $N_f > N_c$, $s_{2N_c-N_f}^* = -\Lambda^{2N_c-N_f}/4$). The curve reads as $y^2 = x^{N_c+n}(x^{N_c-n} - 2\Lambda^{2N_c-N_f})$. 

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We then apply perturbations of the form \( t_j x^j \) to \( C(x) \),

\[
y^2 \approx (x^{N_c} - t_j x^j)x^n, \quad 0 \leq j \leq N_c - 1. \tag{29}
\]

If we make a change of variable

\[
x = t_j^{N_c - j} z,
\]

we find \( y \approx t_j^{(N_c+n)/2(N_c-j)} \sqrt{(z^{N_c} - z^j)z^n}, C \approx t_j^{n/(N_c-j)}z^n \) and \(|y| \ll |C|\). The 1-form behaves as

\[
\lambda = xd \log (1 - y/C)/(1 + y/C) \approx -2xd(y/C) \approx t_j^{(N_c+2-n)/2(N_c-j)} . \tag{31}
\]

Hence the dimensions are given by

\[
[t_j] = \frac{2(N_c - j)}{N_c + 2 - n}, \quad 0 \leq j \leq N_c - 1 . \tag{32}
\]

Class 4 theories are strongly coupled conformal theories and the dimensions of the scaling fields (32) could become arbitrary small as \( N_c \) is increased.

It is easy to repeat the computation in the case of lower critical points \( M_{2n+p}^2 \) with \( p < N_c - n \) and we find that the dimensions of the perturbations are given by (32) with \( N_c - n \) being replaced by \( p \). In fact the lower critical point of an \( SU(N_c) \) gauge theory corresponds exactly to the most critical one of the gauge theory of a lower rank \( SU(N'_c = n + p) \).

In summary, we have obtained the list of universality classes of Table 3.

In the second row of dimensions in each universality class exponents of the Casimir operators of the global flavor symmetry are listed. Note that a part of the scaling dimensions of the \( M_{2n+p}^2 \) theory (\( \frac{2r}{p+2}, j = 2, \ldots, p \)) agree with those given by the pure \( SU(p) \) gauge theory.

Let us next discuss SCFTs based on \( N = 2 \) gauge theories with gauge groups \( SO(N_c) \) and \( Sp(2N_c) \) coupled with matter in vector representations. We first recall that the curves of \( SO(N_c) \) and \( Sp(2N_c) \) theories are given by \[7, 8, 12, 13\]

\[
SO(2r) : \quad y^2 = C(x)^2 - \Lambda^{2(2r-2-N_f)}x^4(x^2 - m^2)^{N_f}, \tag{33}
\]

\[
C(x) \equiv x^{2r} + s_2 x^{2r-2} + \cdots + s_{2r-2} x^2 + s_r^2,
\]
Table 2: Universality Classes of $N = 2$ SCFTs based on $SU(N_c)$ Gauge Theory

| class | name | dimensions |
|-------|------|------------|
| 1     | $M_{2\ell}^{N_f}$ | $2\ell < N_f$ | $\{1, 2, 3, \cdots, \ell\}$, $\{2, 3, \cdots, N_f\}$ |
| 2     | $M_{2n+1}^{2n+1}$ | $\{\frac{1}{2}, \frac{3}{2}, \cdots, n + \frac{1}{2}\}$, $\{2, 3, \cdots, 2n + 1\}$ |
| 3     | $M_{2n}$ | $\{1, 2, 3, \cdots, n\}$, $\{2, 3, \cdots, 2n\}$ |
| 4     | $M_{2n+p}^{2n+p}$ | $0 < p \leq N_c - n$ | $\left\{\frac{2j}{p+2}, j = 1, 2, \cdots, n + p\right\}$, $\left\{\frac{2(p+j)}{p+2}, j = 2, 3, \cdots, 2n\right\}$ |

(There is an extra term $P(x)$ which is a polynomial of order $2N_f - (N_c - 2)$ in $x$ and $m$ for $N_f \geq N_c/2 - 2$ ($N_f \geq N_c/2 - 3/2$) in $SO(N_c = \text{even})$ ($SO(N_c = \text{odd})$) theories). The 1-forms read as

\[
\lambda = xd\log \frac{C - y}{C + y}; \quad SO(N_c), \quad \text{(36)}
\]

\[
\lambda = xd\log \frac{C - xy}{C + xy}; \quad Sp(2N_c). \quad \text{(37)}
\]

We can again adjust the moduli parameters and extract powers of $x^2 - m^2$ from $C(x)$ and bring theories to superconformal points. Depending on the power $\ell$ of $(x^2 - m^2)^\ell$ extracted from $C(x)$ and the parity of $N_f$ we can construct the analogues of the class 1-4 theories.

It is easy to argue generally that the SCFTs built on $SO(N_c), Sp(2N_c)$ gauge theories with $m^* \neq 0$ are identical to those we have just constructed for $SU(N_c)$ gauge symmetry. In fact at the critical point $x^2 \approx m^{*2} \neq 0$
extra factors of $x^4$ and $x^2$ in (47)-(49) become irrelevant and the curves and the 1-forms become exactly the same as those of the $SU(N_c)$ case. Thus the scaling dimensions of the theories are identical to those listed in Table 2.

We may also note that when the $m^* 
eq 0$, the global flavor symmetry of the theory is $SU(N_f)$ irrespective of the gauge groups. Moreover, when $C(x)$ is divisible by $(x^2 - m^2)\ell$, the unbroken subgroup of the gauge group is given by $SU(\ell)$. (The scalar field $\phi$ possesses $\ell$ pairs of degenerate eigenvalues $m, -m$ which breaks the gauge groups $SO(2r), Sp(2r)$ down to $SU(\ell)$, $\ell < r$).

Thus the flavor group and the effective gauge group coincide with those of the $SU(N_c)$ theory and the SCFTs built on $SO(N_c)$, $Sp(2N_c)$ agree with those of $SU(N_c)$. Note that, however, SCFTs based on $SO(N_c)$ and $Sp(2N_c)$ gauge theories with $m^* = 0$ have flavor symmetries $Sp(2N_f)$ and $SO(2N_f)$, respectively and belong to different universality classes.

SCFTs constructed from pure gauge theories

So far we have assumed that $N_f \geq 2$ so that we can distinguish squarks from other singularities. Let us now turn to the discussion of the pure gauge case $N_f = 0$ and its universality classes. As we shall see below, $SO(2r+1)$ and $Sp(2r)$ theories have critical points at $x = x^* \neq 0$ and their SCFTs belong to the same universality as the $SU(r+1)$ theory. On the other hand, $SO(2r)$ theories have critical points at $x = x^* = 0$ and provide new universality classes.

In the case of $SU(r+1)$ and $SO(2r)$ one can easily locate the highest criticality of pure $N = 2$ Yang-Mills theories: $y^2 = x^{r+1}$ for $SU(r+1)$ and $y^2 = x^{2r+2}$ for $SO(2r)$. The moduli parameters are tuned to be of the order $\Lambda$ and these are strongly coupled SCFTs. Let us denote their universality classes as $MA_r$ and $MD_r$, respectively. Scaling dimensions are given by $2j/(r+3)$ ($j = 2, 3, \cdots, r+1$) for $MA_r$ and $j/r$ ($j = 2, 4, \cdots, 2r-2, r$) for $MD_r$, respectively. On the other hand, in the case of groups $SO(2r+1)$ and $Sp(2r)$ it is not easy to locate the highest criticality explicitly. By counting the number of parameters, however, we find that the singularity is of the form $y^2 = (x^2 - b^2)^{r+1}$ ($b \neq 0$ is of order $\Lambda$) and their SCFTs belong to the same universality class as $MA_r$. In summary, in Table 3 we present a list of universality classes in $N = 2$ pure Yang-Mills theories with classical gauge groups.

We explicitly write down the dimensions for lower rank theories:
Table 3: SCFTs based on $N = 2$ pure Yang-Mills theories

| name | gauge groups | dimensions |
|------|--------------|------------|
| $MA_r$ | $SU(r+1), SO(2r+1), Sp(2r)$ | $2\frac{e + 1}{h + 2}$, $e = 1, 2, \cdots, r$ |
| $MD_r$ | $SO(2r)$ | $2\frac{e + 1}{h + 2}$, $e = 1, 3, \cdots, 2r - 3, r - 1$ |

Note that there exist unique universality classes in rank 2 and 3 theories and they coincide with the $SU(2)$ gauge theory with $N_f = 1$ and $N_f = 2$ flavors, respectively (see Table 1). At rank 4, there appear two universality classes and one of them, $MD_4$, coincides with the $N_f = 3$, $SU(2)$ theory.

**$E_n$ Gauge Theories**

In the above we have seen that the SCFTs based on pure gauge theories with gauge groups $SU(r+1), SO(2r+1), Sp(2r)$ coincide and give a universality class $MA_r$ of SCFTs while those based on $SO(2r)$ are different and provide another universality class $MD_r$. Critical exponents of both series are expressed as

$$2\frac{e_i + 1}{h + 2}, \quad i = 1, 2, \cdots, r$$

(38)

where $e_i$ and $h$ are the Dynkin exponents and dual-Coxeter numbers of the algebras $A_r$ and $D_r$. This result suggests the possibility of an $A-D-E$ type classification of $N = 2$ SCFTs.

We would like to now discuss critical behaviors of $E_n$ gauge theories in order to test the idea of the A-D-E classification. At the moment an explicit solution of $E_n$ gauge theories is not known although a method for their construction has been suggested in [19]. In the following we adopt a different
approach making use of a string-theoretic derivation of the 4-dimensional Yang-Mills theories [20, 21]. In [21] authors have streamlined the derivation of (hyperelliptic) curves and differential forms of the \( N = 2 \) Yang-Mills theory by considering the Calabi-Yau manifolds with a \( K_3 \) fibration [20, 22, 23]. For instance, we start from a Calabi-Yau hypersurface in a weighted projective space \( WP_{5,1,2,8,12} \)

\[
W = \frac{1}{24} (x_1^{24} + x_2^{24}) + \frac{1}{12} x_3^{12} + \frac{1}{3} x_4^{12} + \frac{1}{2} x_5^2 + \psi_0(x_1x_2x_3x_4x_5)
\]
\[
+ \frac{1}{6} \psi_1(x_1x_2x_3)^6 + \frac{1}{12} (x_1x_2)^{12} = 0.
\]

(39) is known to describe the \( SU(3) \) pure gauge theory in the field theory limit [20]. By introducing new parameters \( a = -\frac{\psi_0^6}{\psi_1}, \ b = \psi_2^2, \ c = -\psi_2/\psi_1^2 \) and change of variables \( x_1/x_2 = \zeta^{1/2}b^{1/2}, \ x_1^2 = x_0^2\zeta^{1/2}, W \) is rewritten as

\[
W(\zeta, a, b, c) = \frac{1}{24}(\zeta + b \zeta + 2)x_0^{12} + \frac{1}{12} x_3^{12} + \frac{1}{3} x_4^{12} + \frac{1}{2} x_5^2 + \frac{1}{6}\sqrt{c} (x_0x_3)^6 + (\frac{a}{\sqrt{c}})^2 x_0x_3x_4x_5.
\]

(40)

For each fixed value of \( \zeta, W = 0 \) describes a \( K_3 \) surface.

Discriminant of the Calabi-Yau manifold (40) is given by

\[
\Delta \approx (b - 1)((1 - c)^2 - bc^2)((1 - a)^2 - c^2) - bc^2
\]

(41)

The field theory limit is achieved by taking

\[
a = -2\epsilon u^{3/2}/3\sqrt{3}, \ b = \epsilon^2 \Lambda^6, \ c = 1 - \epsilon(-2u^{2/3}/3\sqrt{3} + v)
\]

(42)

and letting \( \epsilon \equiv (\alpha')^{3/2} \to 0 \). Here \( u, v \) are gauge invariant Casimirs of \( SU(3) \).

After a suitable change of variables \( W \) takes the form (we choose a patch \( x_0 = 1 \))

\[
W = z + \frac{\Lambda^6}{z} + 2C_{A_2}(x, u, v) + s^2 + w^2
\]

(43)

where \( C_{A_2}(x, u, v) = x^3 - ux - v \).

Now one considers the period integral of the holomorphic 3-form on the Calabi-Yau manifold

\[
\omega = \int \Omega = \int \frac{dz}{z} \wedge \frac{ds \wedge dx}{\partial W/\partial w \mid_{W=0}}
\]

(44)
In the case of the $A_2$ singularity

$$\frac{\partial W}{\partial w}\bigg|_{W=0} = 2\sqrt{z + \Lambda^6/z + 2C_{A_2}(x, u, v) + s^2}. \quad (45)$$

The integral over $s$ is trivial and equals $2\pi$. The boundary of the $s$-integration is given by

$$z + \frac{\Lambda^6}{z} + 2C_{A_2}(x, u, v) = 0 \quad (46)$$

which describes the spectral curve [24, 13]. If one shifts $z \to y - C_{A_2}$, one recovers the hyperelliptic curve

$$y^2 = C_{A_2}(x, u, v)^2 - \Lambda^6. \quad (47)$$

The period is rewritten as

$$\omega = \pi \int \frac{dz}{z} \wedge dx = \pi \oint x \frac{d(y - C_{A_2})}{(y - C_{A_2})}, \quad y^2 = C_{A_2} - \Lambda^6 \quad (48)$$

which reproduces (9).

In the above we have considered the case when $K_3$ surface (ALE space) degenerates to have an $A_2$-type singularity. We may generalize this construction and consider the case when $K_3$ surface degenerates into more general types of ADE singularities. In the case of $E_n$ singularities one can no longer perform the $s$-integral and reduce $\omega$ to an integral over a Riemann surface. It turns out, however, we can still analyze the critical behavior of $\omega$ and determine the exponents of the $E_n$ gauge theories.

Let us first discuss the $E_6$ theory,

$$W = z + \frac{\Lambda^{24}}{z} + 2C_{E_6}(x, s) + w^2,$$

$$C_{E_6}(x, s) = x^4 + s^3 + u_1x^2s + u_4xs + u_5x^2 + u_7s + u_8x + u_{11} \quad (49)$$

where $u_i$ are the perturbations of the $E_6$ singularity. The period is given by

$$\omega = \int \frac{dz}{z} \wedge \frac{ds \wedge dx}{\sqrt{z + \frac{\Lambda^{24}}{z} + 2C_{E_6}(x, s)}} \quad (50)$$
The critical point is located at
\[ x^* = 0, \quad s^* = 0, \quad z^* = \Lambda^{12}, \quad u_{11}^* = -\Lambda^{12}. \] (51)

Let us first determine the exponent of the parameter \( u_1 \). By perturbing away from the critical point
\[ z = \Lambda^{12} + t^{1/2} \Lambda^6 \bar{z}, \quad x = t^{1/4} \bar{x}, \quad s = t^{1/3} \bar{s}, \quad u_1 = t^{1/6} \bar{u}_1 \] (52)
we have
\[ \omega \approx t^{1/2 - 1/2 + 1/4 + 1/3} \int d\bar{z} \wedge \frac{d\bar{s} \wedge d\bar{x}}{\sqrt{\bar{z}^2 + \bar{x}^4 + \bar{s}^4 + \bar{u}_1 \bar{x}^2 \bar{s}}} \approx t^{7/12} \] (53)

By introducing the unit of mass \( \mu \) we find
\[ t \approx \mu^{12/7}, \quad u_1 \approx \mu^{2/7}. \] (54)

Thus the perturbation \( u_1 \) has the exponent 2/7. We can similarly determine the exponents of the parameters \( u_i \) (\( i = 4, 5, 7, 8, 11 \)). They read as
\[ \frac{2(e_i + 1)}{14}, \quad i = 1, 4, 5, 7, 8, 11 \] (55)
and hence again fit to the formula \( 2(e_i + 1)/(h + 2) \) where the dual-Coxeter number \( h \) of \( E_6 \) is 12.

We may also compute exponents for the \( E_7 \) and \( E_8 \) gauge theories. Singularities are described by the polynomials
\[ C_{E_7}(x, s) = x^3 + x s^3 + u_1 s^4 + u_5 s^3 + u_7 x s + u_9 s^2 + u_{11} x \
+ u_{13} s + u_{17}, \] (56)
\[ C_{E_8}(x, s) = x^5 + s^3 + u_1 x^3 s + u_7 x^2 s + u_{11} x^3 + u_{13} x s + u_{17} x^2 \
+ u_{19} s + u_{23} x + u_{29}. \] (57)

We find that their critical exponents are given by
\[ \frac{2(e_i + 1)}{20}, \quad i = 1, 5, 7, 9, 11, 13, 17 \quad \text{for } E_7 \] (58)
\[ \frac{2(e_i + 1)}{32}, \quad i = 1, 7, 11, 13, 17, 19, 23, 29 \quad \text{for } E_8 \] (59)
It is easy to check that the above construction reproduces the exponents of Table 3 in the case of $A_n$ and $D_n$ singularities. Thus we have some considerable evidence for the A-D-E classification of SCFTs originating from pure $N = 2$ gauge theories. The pattern of the classification follows from that of the degeneration of $K_3$ surfaces which appear in the heterotic-type II duality based on $K_3$-fibered Calabi-Yau manifolds. More details will be discussed elsewhere.

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