Non-asymptotic estimation for Bell function, 
with probabilistic applications

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Abstract

We deduce the non-asymptotical bilateral estimates for moment inequalities
for sums of non-negative independent random variables, based on the correspondent
estimates for the so-called Bell functions and the Poisson distribution.

Key words and phrases: Arbitrary and independent random variables (r.v.),
Bell’s numbers and function, triangle inequality, Grand Lebesgue Spaces (GLS),
Rosenthal estimate, Poisson distribution, asymptotic estimate and expansion, Stir-
ling’s formula, bilateral non-asymptotic estimates, moment generating function
(MGF), optimization, upper and lower evaluate.

Mathematics Subject Classification 2000. Primary 42Bxx, 4202, 68 -01, 62-G05,
90-B99, 68Q01, 68R01; Secondary 28A78, 42B08, 68Q15.

1 Definitions. Notations. Previous results.

Statement of problem.

Definition 1.1. The famous Bell numbers $B(p)$, $p \geq 0$ are defined by means of
the series

\[ B(p) \stackrel{def}{=} e^{-1} \sum_{k=0}^{\infty} \frac{k^p}{k!}. \quad (1.0) \]

These numbers was introduces originally for the integer positive values $p$ by
E.T.Bell [1], [2]; see also Dobinski [6].

They plays a very important role in the combinatorics [21], theory of functions,
asymptotical analysis [4] and especially in the theory probability, in the theory of
summation of independent random variables [3], [7], [8], [9]-[11], [12]-[14], [16], [17],
[19], [24]-[26] etc.

More generally, define the so-called Bell’s function of two variables
\[ B(p, \beta) \overset{\text{def}}{=} e^{-\beta} \sum_{k=0}^{\infty} \frac{k^p \beta^k}{k!}, \quad p \geq 2, \quad \beta > 0, \quad (1.1) \]

so that \( B(p) = B(p, 1) \). On the other words, \( B(p, \beta) \) are the Bell function depending on additional parameter \( \beta \in (0, \infty) \).

Let the random variable (r.v.) \( \tau = \tau[\beta] \), defined on certain probability space \((\Omega, F, \mathbb{P})\) with expectation \( \mathbb{E} \), has a Poisson distribution with parameter \( \beta, \beta > 0 \); write \( \text{Law}(\tau) = \text{Law}\tau[\beta] = \text{Poisson}(\beta) : \)

\[ \mathbb{P}(\tau = k) = e^{-\beta} \frac{\beta^k}{k!}, \quad k = 0, 1, 2, \ldots, \]

It is worth to note that

\[ B(p, \beta) = \mathbb{E}(\tau[\beta])^p, \quad p \geq 0. \quad (1.2) \]

In detail, let \( \eta_j, \ j = 1, 2, \ldots \) be a sequence of non-negative independent random (r.v.); the case of centered or moreover symmetrical distributed r.v. was considered in many works, see e.g. [5], [9]-[11], [15], [17], [19], [23], [24], [25], [26], and so one.

The following inequality holds true

\[ \mathbb{E} \left( \sum_{j=1}^{n} \eta_j \right)^p \leq B(p) \max \left\{ \sum_{j=1}^{n} \mathbb{E} \eta_j^p, \left( \sum_{j=1}^{n} \mathbb{E} \eta_j \right)^p \right\}, \quad p \geq 2, \quad (1.3) \]

where the "constant" \( B(p) \) in (1.2) is the best possible, see [3], [24].

One of the interest applications of these estimates in statistics, more precisely, in the theory of \( U \) statistics may be found in the article [8].

Another application. Let \( n = 1, 2, 3, \ldots; \ a, b = \text{const} > 0; \ p \geq 2, \ \mu = \mu(a, b; p) := a^{p/(p-1)} b^{1/(1-p)}. \) Define the following class of the sequences of an independent non-negative random variables

\[ Z(a, b) \overset{\text{def}}{=} \left\{ \eta_j, \ \eta_j \geq 0, \ \sum_{j=1}^{n} \mathbb{E} \eta_j = a; \ \sum_{j=1}^{n} \mathbb{E} \eta_j^p = b \right\}. \quad (1.4) \]

G. Schechtman in [24] proved that

\[ \sup_{n=1,2,\ldots \{ \eta_j \in Z(a, b) } \mathbb{E} \left( \sum_{j=1}^{n} \eta_j \right)^p = \left( \frac{b}{a} \right)^{p/(p-1)} B(p, \mu(a, b; p)). \quad (1.5) \]

N. G. de Bruijn in the book [4] proved the following asymptotical logarithmical expression for \( B(p) \) as \( p \to \infty : \)

\[ \frac{\ln B(p)}{p} = \ln p - \ln \ln p - 1 + \frac{\ln \ln p}{\ln p} + \frac{1}{\ln p} + \frac{1}{2} \left( \frac{\ln \ln p}{\ln p} \right)^2 + O \left( \frac{\ln \ln p}{\ln^2 p} \right), \quad (1.6) \]
Our aim in this short report is obtaining the bilateral non-asymptotical estimates for introduced before Bell functions, with ”constructive” values of constants.

We refine ones in [3], [4] etc.

Denote as ordinary for arbitrary numerical valued r.v. $\eta$ its Lebesgue-Riesz $L(p)$ norm by $|\eta|_p$:

$$|\eta|_p \overset{\text{def}}{=} \left[\mathbb{E}|\eta|^p\right]^{1/p}, \ p \in [1, \infty).$$ (1.7)

There are also many works about problems raised here, see, e.g. [3], [5], [7], [9]-[11], [12], [15], [19], [23], [24], [25], [26] and other articles mentioned in the references.

2 General approach. Upper and lower estimates.

A. An upper estimate.

Let again $\text{Law}(\tau) = \text{Law}\tau[\beta] = \text{Poisson}(\beta)$. In order to estimate the moments of this variable, we will apply the theory of the so-called Grand Lebesgue Spaces (GLS) [17].

Indeed, let us calculate the moment generating function (MGF) for this variable; it has a form

$$\mathbb{E}e^{\lambda \tau} = e^{-\beta} \sum_{k=0}^{\infty} \frac{\beta^k \exp(\lambda k)}{k!} = \exp \left( \beta \left( e^\lambda - 1 \right) \right).$$ (2.1)

We will use an elementary inequality

$$x^p \leq \left( \frac{p}{\lambda} \right)^p e^{\lambda x}, \ \lambda, x > 0;$$

and have

$$\mathbb{E}\xi^p \leq \left( \frac{p}{\lambda} \right)^p \exp \left[ \beta \left( e^\lambda - 1 \right) \right],$$ (2.2)

or equally

$$|\xi|_p \leq \frac{p}{\lambda} \exp \left\{ \beta \left( e^\lambda - 1 \right)/p \right\}. \quad (2.2a)$$

Let us introduce the following auxiliary function

$$g_\beta(p) \overset{\text{def}}{=} \frac{p}{e} \inf_{\lambda > 0} \left[ \lambda^{-1} \exp \left( \beta \left( e^\lambda - 1 \right) \right) \right]^{1/p}, \ \beta, p > 0. \quad (2.3)$$
As long as the relations (2.2), (2.2a) are true for arbitrary positive value \( \lambda \), one can select as the value \( \lambda \) its optimal value. We obtained actually a following upper estimate for the Bell function.

**Proposition 2.1.**

\[
B^{1/p}(p, \beta) \leq g_\beta(p), \quad p, \beta > 0. \tag{2.4}
\]

**A. A lower estimate.**

We have for the r.v. \( \tau = \tau_\beta : \text{Law } (\tau_\beta) = \text{Poisson}(\beta) \)

\[
\mathbf{E}\tau^p = e^{-\beta} \sum_{k=0}^{\infty} \frac{k^p \beta^k}{k!}, \quad p, \beta > 0.
\]

Let us introduce the following function

\[
h_0(p, \beta) \overset{\text{def}}{=} \sup_{k=1,2,\ldots} e^{-\beta} \left\{ \frac{k^p \beta^k}{k!} \right\}; \tag{2.5}
\]

therefore

\[
B(p, \beta) \geq h_0(p, \beta), \quad p, \beta > 0. \tag{2.6}
\]

The last estimate may be simplified as follows. We will apply the following version of the famous Stirling’s formula [22]

\[
k! \leq \zeta(k), \quad k = 1, 2, \ldots,
\]

where

\[
\zeta(k) \overset{\text{def}}{=} \sqrt{2\pi k} \left( \frac{k}{e} \right)^k e^{1/(12k)}, \quad k = 1, 2, \ldots \tag{2.7}
\]

It is worth to note that the function \( \zeta = \zeta(k) \) may be extended as the function of the real variable \( x \in [1, \infty) : \zeta = \zeta(x) \) by means of the formula (2.7).

Define a new function

\[
h(p, \beta) \overset{\text{def}}{=} \sup_{x \in (1, \infty)} \left\{ e^{1/(6px)} \cdot \left[ \frac{e^{x-\beta} x^{p-1/2}}{\sqrt{2 \pi} x^x} \right]^{1/p} \right\}. \tag{2.8}
\]

We obtained really the following lower estimate for the Bell’s function.

**Proposition 2.2.**

\[
B^{1/p}(p, \beta) \geq h_0(p, \beta), \quad B^{1/p}(p, \beta) \geq h(p, \beta), \quad p, \beta > 0. \tag{2.9}
\]
We concretize further in the next sections the choosing of the values $k_0$, $x_0$, $\lambda$ in order to simplify the estimates (2.4) and (2.9).

3 Main result. Simplification of the upper estimate. The case of one variable.

Suppose here that $\beta = \text{const} > 0$, $p \geq 2\beta$, $p \geq 1$. One can choose in (2.3) the (asymptotically as $p \to \infty$ optimal) value

$$\lambda := \lambda_0 \overset{\text{def}}{=} \ln(p/\beta) - \ln \ln(p/\beta). \quad (3.1)$$

We deduce from the proposition 2.1 after substituting and some cumbersome calculations

**Proposition 3.1.** We assert under our conditions $\beta = \text{const} > 0$, $p \geq 2\beta$, $p \geq 1$

$$B^{1/p}(p, \beta) \leq \frac{p/e}{\ln(p/\beta) - \ln \ln(p/\beta)} \cdot \exp \left\{ \frac{1}{\ln(p/\beta)} - \frac{1}{p/\beta} \right\}. \quad (3.2)$$

Notice that the expression in the right-hand of (3.2) is in strict accordance with the strict asymptotic for Bell’s number (1.6) obtained by N.G. de Bruijn in the book [4], still in the case when $\beta = 1$.

For example,

$$B^{1/p}(p) \leq \frac{p/e}{\ln p - \ln \ln p} \cdot \exp \{1/\ln p - 1/p\}, \ p \geq 2. \quad (3.3)$$

**Remark 3.1.** The obtained estimation (3.2) may be rewritten as follows

$$B^{1/p}(p) \leq \frac{p}{e \ln(p/\beta)} \cdot \left[ 1 + C_1(\beta) \cdot \frac{\ln \ln(p/\beta)}{\ln(p/\beta)} \right], \quad (3.4)$$

where $C_1(\beta) = \text{const} \in (0, \infty)$, and we recall that $p \geq 2\beta$; with "constructive" and absolute value of the "constant" $C_1(\beta)$.

4 Main result. Simplification of the lower estimate. The case of one variable.

Let us return to the relations (2.5), (2.6). One can choose the following value as a capacity of the number $k$; $k := k_0(p, \beta)$
More precisely,

\[ k_0(p, \beta) := \text{Ent} \left[ \frac{p}{\ln(pe/\beta)} \right] + 1, \]  

where \( \text{Ent}[z] \) denotes the integer part of the (positive) number \( z \).

We get again after cumbersome calculations

\[ B^{1/p}(p, \beta) \geq \]  

\[ \beta^{1/\ln(pe/\beta)} \cdot \frac{p}{\ln(pe/\beta)} \cdot \left\{ \exp \left[ \frac{\ln p - \ln(p+\beta)/\beta}{\ln(p+\beta)/\beta} \right] \right\}^{-1}, \]

\[ p, \beta > 0, p/\beta \geq 2. \]  

After simplifications:

\[ B^{1/p}(p, \beta) \geq \frac{p}{e \ln(p/\beta)} \cdot \left[ 1 - C_2(\beta) \cdot \frac{\ln \ln(p/\beta)}{\ln(p/\beta)} \right], \]  

where \( C_2(\beta) = \text{const} \in (0, \infty) \), and we recall that \( p \geq 2\beta \); with "constructive" and absolute value of the "constant" \( C_2(\beta) \in (0, \infty) \).

Notice that the upper and lower bounds almost coincides and almost coincides with the asymptotic expression for Bell’s numbers.

As a slight corollary: under condition (5.2)

\[ \left| \frac{B^{1/p}(p, \beta) - e^{-\frac{p}{\ln(p/\beta)}}}{e^{-\frac{p}{\ln(p/\beta)}}} \right| \leq C_0(\beta) \frac{\ln \ln(p/\beta)}{\ln(p/\beta)}. \]  

### 5 Main result. The case of two variables. Upper bounds.

We give first of all a rough estimate for the Bell function. Namely, let as before the r.v. \( \tau = \tau[\beta] \) has the Poisson distribution with a parameter \( \beta; \beta > 0 : \)

\[ \text{Law}(\tau) = \text{Poisson}(\beta). \]  

Proposition 5.1.
\[ |τ[β]|_p = B^{1/p}(p, β) \leq \beta \frac{p}{e \ln p} \left(1 + C_3 \frac{\ln \ln p}{\ln p}\right), \quad p \geq 2. \quad (5.1) \]

with some absolute constant \(C_3\).

The last estimate in (5.1) is essentially non-improvable at last in the case when \(β = 1\).

**Proof.** We can and will suppose without loss of generality that the number \(β\) is integer. Introduce on an appropriate (sufficiently rich) probability space the sequence \(\{θ(i)\}, i = 1, 2, \ldots\) of independent standard Poisson distributed random variables:

\[ \text{Law}(θ(i)) = \text{Poisson}(1), \Leftrightarrow P(θ(i) = k) = e^{-1/k!}, \quad k = 0, 1, 2, \ldots. \]

The distribution of the sum \(\sum_{i=1}^{β} θ(i)\) coincides with \(τ[β]\); one can assume

\[ τ[β] = \sum_{i=1}^{β} θ(i). \]

One can apply a triangle inequality for the Lebesgue - Riesz norm \(L_p\):

\[ B^{1/p}(p, β) = |τ[β]|_p \leq \sum_{i=1}^{β} |θ(i)|_p = β |θ(1)|_p. \]

It remains to use the proposition (3.1.)

We suppose hereafter that both the variables \(p\) and \(β\) are independent but such that

\[ p \geq 1, \quad β > 0, \quad p/β \leq 2. \quad (5.2) \]

It is this case namely that takes place in the work of G.Schechtman [24], see (1.4). Indeed, suppose for simplicity therein that the non-negative random variables \(\{η_i\}, i = 1, 2, \ldots; \quad η := η_1\), are independent and identical distributed (i.i.d.) and such that for some \(p > 1\)

\[ m_1 := Eη < \infty; \quad m_p := Eη^p < \infty. \]

Then in (1.4)

\[ a = n \ m_1, \quad b = n \ m_p, \]

so that \(β \asymp n, \quad n \to \infty\).

We return to the estimate (2.2a)

\[ |τ[β]|_p \leq \frac{p}{e \lambda} \exp\left\{β (e^λ - 1)/p\right\}. \quad (5.3) \]

But one can now choose in (5.3) under our conditions the value \(λ := p/β\).
**Proposition 5.2.** We get after simple calculations under formulated before in this section conditions (5.2)

\[ B^{1/p}(p, \beta) \leq K_+ \cdot \beta, \]

where

\[ K_+ := \exp \left[ \frac{(e^2 - 3)}{2} \right] \approx 8.9758... \] (5.4)

6 Main result. The case of two variables. Lower bounds.

We retain the conditions (5.2); and we will use also as above the following lower estimate

\[ B(p, \beta) \geq e^{-\beta} \cdot \frac{\beta^p p^p}{\sqrt{2\pi}} k^{-k+1/2} e^{-k+1/(12k)}. \] (6.1)

One can choose in (6.1) the value \( k = k_0 = k_0(p) := p, \) if \( p \) is integer, and \( k_0 := \text{Ent}(p) \) otherwise. We deduce after some calculations:

**Proposition 6.1.** We assert under conditions (5.2)

\[ B^{1/p}(p, \beta) \geq K_- \cdot \beta, \]

where

\[ K_- = (2\pi)^{-1/2} \exp \left[ -1/(2e) + 1/3 \right] \approx 0.6538... \] (6.2)

7 Concluding remarks.

A. It is interest by our opinion to compute the exact value of the constants \( K_\pm \) in the estimates (5.4) and (6.2), as well as to find its "limit" behavior.

B. It is interest also by our opinion to generalize the approximation of the Bell function \( B(p) \) through the so-called Lambert function \( W(\cdot) \), where by definition

\[ W(p) \ e^{W(p)} = p, \ p = 0, 1, 2, \ldots : \]

\[ B(p) \sim \frac{1}{\sqrt{p}} \cdot \left( \frac{p}{W(p)} \right) \cdot \exp \left( \frac{p}{W(p)} - p - 1 \right) \]
onto the more general function $B(p, \beta)$.

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