On the Law of Transformation of Affine Connection and its Integration.
Part 1. Generalization of the Lame equations

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Abstract

The law of transformation of affine connection for n-dimensional manifolds as the system of nonlinear equations on local coordinates of manifold is considered. The extension of the Darboux-Lame system of equations to the spaces of constant negative curvature is demonstrated. Geodesic deviation equation as well as the equations of geodesics are presented in the form of the matrix Darboux-Lame system of equations.

1 Introduction

It is well-known that the integrable equations have numerous applications in geometry. The Korteveg-de Vries equation, sin-Gordon equation, Tzitzeika equation, Kadomtzev-Petviashvili equation, Zakharov-Manakov system of equations and others are the most famous examples of such type of equations. This can be explained by the fact that the mentioned above equations have the Lax pair representation, which is equivalent to the condition of zero curvature for suitable connections. From this we infer that the law of transformation of affine connection is a key to understanding the nature of such type of equations and their integrability.

The matrix Zakharov-Manakov system of equations discovered in context of formal generalization of Inverse Scattering Transform Method in multidimensional case [1] after its geometrical interpretation as the Darboux-Lame system [2–6] gives an example of such relation between equations

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and geometry because it corresponds to the simplest case of the law of transformation of the flat affine connection. For the spaces of non-zero curvature the law of transformation of connection leads to generalization of Zakharov-Manakov and Darboux-Lame systems of equations.

The problem of integration of equations of geodesics has a great significance in geometry. We show here that in a certain case this problem is connected with the matrix Darboux-Lame system of equations and its generalizations.

The equation of geodesic deviation is also important in geometry, in the theory of Riemann spaces. For surfaces it coincides, in essence, with the Gauss equation in geodesic coordinates while in the three- and higher-dimensional Riemann spaces it carries essential information about these spaces. Solutions of the geodesic deviations equation (Jacoby fields) and their properties (e.g., the existence of conjugate points) are related to various important characteristics of Riemann spaces.

In this paper we also consider applications of the matrix Darboux-Lame system to the study of the geodesic deviation equation.

2 Equation of the law of transformation of affine connection

Here we will present for convenience basic formulas, which will be used in what follows. Let \( M^n \) be the space endowed with the affine connection. This means that the components of connections \( \Gamma^i_{jk}(x^l) \) and \( \bar{\Gamma}^i_{jk}(y^l) \) in two various systems of coordinates \((x^l)\) and \((y^l)\) are connected by the relations

\[
\frac{\partial^2 y^k}{\partial x^i \partial x^j} = \Gamma^l_{ij}(x^s) \frac{\partial y^k}{\partial x^l} - \bar{\Gamma}^k_{lm}(y^s) \frac{\partial y^l}{\partial x^i} \frac{\partial y^m}{\partial x^j}.
\]

We can consider these relations as the system of partial differential equations for \( n \) unknown functions \( y^k \) of variables \( x^k \). In general case this is a compatible system of equations and its properties depend on the curvature tensor

\[
R^i_{klm} = \left[ \frac{\partial \Gamma^i_n}{\partial x^l} - \frac{\partial \Gamma^i_l}{\partial x^n} + \Gamma^i_l \Gamma^l_n - \Gamma^l_n \Gamma^i_l \right]_k,
\]

which has the law of transformation as the four rank tensor

\[
R^i_{klm} = \frac{\partial x^i}{\partial y^s} \frac{\partial y^r}{\partial x^k} \frac{\partial y^q}{\partial x^l} \frac{\partial y^p}{\partial x^n} R^s_{rqp},
\]
where $\Gamma_l$ are matrices with components $\Gamma^i_{lk}$ and $R^i_{kln}$ are components of curvature tensor of the connection $\Gamma_l$.

The system for manifolds of dimension three is the main object of our consideration.

3 Triply-orthogonal systems of surfaces, flat coordinates and the Darboux-Lame system of equations

The above system of equations in three-dimensional case is the system of 18 equations for three unknown functions $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$, depending on three variables $(x, y, z)$. It contains 18 coefficients of affine connection of the manifold and their solutions depend on the specific choice of coefficients of connection. With the help of coordinates $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ the geometry of the 3-dim space is described. Let us consider the simplest of examples.

It corresponds to the flat space, i.e. when the coefficients of connection $\bar{\Gamma}^i_{jk} \equiv 0$, or the tensor of curvature of the connection $\Gamma^i_{jk}$ is equal to zero:

The initial system takes the form

$$\frac{\partial^2 y^k}{\partial x^i \partial x^j} = \Gamma^l_{ij}(x^s) \frac{\partial y^k}{\partial x^l}. \quad (3)$$

For 3-dim space we have the following system of equations

$$\frac{\partial^2 u}{\partial x \partial y} = \Gamma^1_{12} \frac{\partial u}{\partial x} + \Gamma^2_{12} \frac{\partial u}{\partial y} + \Gamma^3_{12} \frac{\partial u}{\partial z}, \quad \frac{\partial^2 u}{\partial x \partial z} = \Gamma^1_{13} \frac{\partial u}{\partial x} + \Gamma^2_{13} \frac{\partial u}{\partial y} + \Gamma^3_{13} \frac{\partial u}{\partial z},$$

$$\frac{\partial^2 u}{\partial y \partial z} = \Gamma^1_{23} \frac{\partial u}{\partial x} + \Gamma^2_{23} \frac{\partial u}{\partial y} + \Gamma^3_{23} \frac{\partial u}{\partial z}, \quad \frac{\partial^2 u}{\partial x^2} = \Gamma^1_{11} \frac{\partial u}{\partial x} + \Gamma^2_{11} \frac{\partial u}{\partial y} + \Gamma^3_{11} \frac{\partial u}{\partial z},$$

$$\frac{\partial^2 u}{\partial y^2} = \Gamma^1_{22} \frac{\partial u}{\partial x} + \Gamma^2_{22} \frac{\partial u}{\partial y} + \Gamma^3_{22} \frac{\partial u}{\partial z}, \quad \frac{\partial^2 u}{\partial z^2} = \Gamma^1_{33} \frac{\partial u}{\partial x} + \Gamma^2_{33} \frac{\partial u}{\partial y} + \Gamma^3_{33} \frac{\partial u}{\partial z},$$

and corresponding equations for the coordinates $v(x, y, z)$ and $w(x, y, z)$.

The system of equations is very well known in classical Differential Geometry. This is the Lame system of equations for the triply orthogonal curvilinear coordinate systems in a flat Euclidean 3-dim space. At present the half of this system containing only mixed derivatives has appeared in an non-evident general matrix form in work [1] in context of the multidimensional generalization of integrable differential equations but without any
applications. An explicit form of this system of equations with its clear geometrical sense, simplest solutions and applications was presented by the author (see [2-6]).

Here I briefly review some results of this approach. For the Riemann 3-dim space in orthogonal metric
\[ ds^2 = A^2(x, y, z)dx^2 + B^2(x, y, z)dy^2 + C^2(x, y, z)dz^2 \]
the condition
\[ R_{ijkl} = \kappa (g_{ik}g_{jl} - g_{il}g_{jk}) \]
leads to six equations, three equations of which do not contain parameter \( \kappa \).

\[
A_{zy} = (C_y/C)A_z + (B_z/B)A_y,  \\
B_{zx} = (C_x/C)B_z + (A_z/A)B_x,  \\
C_{xy} = (A_y/A)C_x + (B_x/B)C_y.
\]

These equations named the Darboux system can be considered as the scalar reduction of the general matrix Zakharov-Manakov system of equations [1] and they can be integrated by the Inverse Scattering Method using the following linear problem:

\[
\Phi_{zy} = (C_y/C)\Phi_z + (B_z/B)\Phi_y,  \\
\Phi_{zx} = (C_x/C)\Phi_z + (A_z/A)\Phi_x,  \\
\Phi_{xy} = (A_y/A)\Phi_x + (B_x/B)\Phi_y.
\]

The partial case of the Darboux system of equations is connected with theory of the normal Riemann space. The notion of normal Riemann space was introduced by Eisenhart.

**Definition 1** The n-dimensional Riemmanian space with local coordinates \( u^i \) is normal when the conditions on main curvatures \( K_i \) is fulfilled

\[
\frac{\partial K_i}{\partial u^l} = 3\lambda_l + 3\mu_l K_l,  \\
\frac{\partial K_i}{\partial u^i} = \lambda_i + \mu_i K_i, \quad i \neq l,  \\
\frac{\partial \ln g_{ij}}{\partial u^l} = \frac{2}{K_l - K_i} \frac{\partial K_i}{\partial u^l} \quad i \neq l,
\]

where \( \lambda \) and \( \mu \) are some functions of coordinates \( u^i \).
Remark 1 The values $K_1, K_2, \cdots, K_n$ are given the name of the principal curvatures relatively to some symmetrical tensor $b_{ij}$ of $n$-dimensional Riemannian space $M^n$ with metric

$$ds^2 = g_{ij}du^i du^j$$

if they are the roots of algebraic equation

$$|b_{ij} - Kg_{ij}| = 0.$$  

The inherent vectors $\xi^i_h$ of main directions of the tensor $b_{ij}$ from the equations

$$(b_{ij} - K_h g_{ij})\xi^i_h = 0$$

are defined. They are orthogonal

$$g_{ij}\xi^i_p \xi^j_q = 0 \quad p \neq q,$$

and satisfy to the conditions

$$b_{ij}\xi^i_p \xi^j_q = 0, \quad p \neq q.$$  

According to [7] the system of equations for the principal curvatures in the 3-dim case looks as

$$(K_2 - K_3)K_{1x} + 3(K_3 - K_1)K_{2x} + 3(K_1 - K_2)K_{3x} = 0,$$

$$3(K_2 - K_3)K_{1y} + (K_3 - K_1)K_{2y} + 3(K_1 - K_2)K_{3y} = 0,$$

$$3(K_2 - K_3)K_{1z} + (K_3 - K_1)K_{2z} + (K_1 - K_2)K_{3z} = 0.$$  

Using the relations

$$\frac{A_y}{A} = \frac{K_{1y}}{K_2 - K_1}, \quad \frac{A_z}{A} = \frac{K_{1z}}{K_3 - K_1}, \quad \frac{B_x}{B} = \frac{K_{2x}}{K_1 - K_2},$$

$$\frac{B_z}{B} = \frac{K_{2z}}{K_3 - K_2}, \quad \frac{C_x}{C} = \frac{K_{3x}}{K_1 - K_3}, \quad \frac{C_y}{C} = \frac{K_{3y}}{K_2 - K_3},$$

we get

$$K_{1xy} + \frac{A_y}{A} K_{1x} + \left(\frac{B_x}{B} + \frac{A_x}{A} - \frac{A_{xy}}{A_y}\right) K_{1y} = 0,$$

$$K_{1xz} + \frac{A_z}{A} K_{1x} + \left(\frac{C_x}{C} + \frac{A_x}{A} - \frac{A_{xz}}{A_z}\right) K_{1z} = 0.$$  

\[ K_{1zy} + \left( \frac{A_y}{A} - \frac{C_y A_z}{C A_y} \right) K_{1y} + \left( \frac{C_y}{C} + \frac{A_y}{A} - \frac{A_{zy}}{A_z} \right) K_{1z} = 0, \]

or

\[ K_{1zy} + \left( \frac{A_y}{A} - \frac{B_z A_y}{B A_z} \right) K_{1z} + \left( \frac{B_z}{B} + \frac{A_z}{A} - \frac{A_{zy}}{A_z} \right) K_{1y} = 0. \]

By analogic way the system of equations for \( K_2 \) and \( K_3 \) have been written.

For the construction of partial solutions of this system we can use the linear system of equations

\[
\Phi_{zy} + \frac{1}{2(z-y)} [\Phi_z - \Phi_y] = 0,
\]

\[
\Phi_{zx} + \frac{1}{2(z-x)} [\Phi_z - \Phi_x] = 0,
\]

\[
\Phi_{xy} + \frac{1}{2(x-y)} [\Phi_x - \Phi_y] = 0.
\]

If \( \varphi \) and \( \chi \) are two solutions of the system (4) connected by the relation:

\[
\frac{\partial \varphi}{\partial u_i} = \chi/2 - u_i \frac{\partial \chi}{\partial u_i},
\]

where \( u_i = (x, y, z) \), then the equations (4) are the conditions of compatibility for (3).

So, if the function \( \chi_h \) is a solution of the system (4), then the function \( \varphi_h \) also will be a solution. As result we have the following relations between the solutions

\[
\frac{\partial \chi_{h+1}}{\partial u_i} = \chi_h/2 - u_i \frac{\partial \chi_h}{\partial u_i},
\]

As example, begining from the trivial solution \( \chi_0 = 0 \) we can obtain the solutions in the form of symmetrical functions of variables \( (x, y, z) \) [5, 7].

Recently the Darboux and the full Lame system of equations have been integrated by the IST-method [8,9,10].

**Remark 2** The full Lame system of equations can be applied to integration of the Einstein equations. The simplest illustration of that is the result of E.Kasner on the representation of the Schwarzschild solution of the Einstein equations

\[
ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2 \theta d\varphi^2
\]
as six squares of differentials in 6-dim flat space

\[ ds^2 = -dx^2 - dy^2 - dz^2 + dX^2 + dY^2 - dZ^2, \]

where \( X, Y, Z \) are defined by equations:

\[
Z = \int \sqrt{1 + \frac{256m^4}{(R^2 + 16m^2)^3}} dR, \quad X = \frac{R \sin t}{\sqrt{R^2 + 16m^2}},
\]

\[
Y = \frac{R \cos t}{\sqrt{R^2 + 16m^2}}, \quad R = \sqrt{8m(r - 2m)}.
\]

Using this result we can obtain the Schwarzschild solutions of the Einstein equations from the Lame system for the flat six-dimensional space.

It is apparent that arbitrary 4-dim metric can be presented as embedded in a flat space of suitable dimension and use then the theory of the Lame equations for integration of the Einstein equations.

Remark 3 The theory of the Lame system of equations is connected with the Euler-Picard system of partial differential equations, having important applications to number theory. The equations have the form

\[
\frac{\partial^2 F}{\partial t_i \partial t_j} + \frac{l}{m(t_j - t_i)} \left( \frac{\partial F}{\partial t_i} - \frac{\partial F}{\partial t_j} \right) = 0,
\]

\[
\frac{\partial^2 F}{\partial t_k^2} + \frac{l}{n(t_k - 1)t_k} \left\{ \sum_{i=1, i \neq k}^{r} \frac{t_i^2 - t_i}{t_i - t_k} \frac{\partial}{\partial t_i} - \sum_{i=1, i \neq k}^{r} \frac{(t_k - 1)t_i}{t_i - t_k} - (r + 2)(t_k - 1)(t_k - 1) \right\} \frac{\partial}{\partial t_k} + \frac{l(r + 2) - n}{n} F = 0,
\]

where \( l \) and \( n \) are natural numbers such that \( 0 < l < n \) and the integral functions

\[
\int_{\alpha} \frac{dx}{y}
\]

of variables \( t = t_1...t_r \) and of the family of cycles \( \alpha_t \) on compact Riemann surfaces of planar equations of the type

\[ Y^n = (X - 1)(X - t_1)(X - t_2)...(X - t_r), \]

are solutions of the Euler-Picard system [11].
4 On equations for the coordinates of distorted spaces

The simplest generalization of the Lame system of equations is connected with the manifolds of constant negative curvature. They have the metric

\[ ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2} \]

and for such a metric the matrices of the Christoffel’s symbols are

\[
\begin{bmatrix}
0 & 0 & -\frac{1}{w} \\
0 & 0 & 0 \\
\frac{1}{w} & 0 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{1}{w} \\
\frac{1}{w} & 0 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
-\frac{1}{w} & 0 & 0 \\
0 & -\frac{1}{w} & 0 \\
0 & 0 & -\frac{1}{w} \\
\end{bmatrix},
\]

From the fundamental system of equations (1) for the coordinates we get

\[ u_{xy} = (\Gamma_{12}^1 + \frac{w_y}{w})u_x + (\Gamma_{12}^2 + \frac{w_x}{w})u_y, \]

\[ u_{xz} = (\Gamma_{13}^1 + \frac{w_z}{w})u_x + (\Gamma_{13}^2 + \frac{w_x}{w})u_z, \]

\[ u_{yz} = (\Gamma_{23}^1 + \frac{w_z}{w})u_y + (\Gamma_{23}^2 + \frac{w_y}{w})u_z, \]

\[ v_{xy} = (\Gamma_{12}^1 + \frac{w_y}{w})v_x + (\Gamma_{12}^2 + \frac{w_x}{w})v_y, \]

\[ v_{xz} = (\Gamma_{13}^1 + \frac{w_z}{w})v_x + (\Gamma_{13}^2 + \frac{w_x}{w})v_z, \]

\[ v_{yz} = (\Gamma_{23}^1 + \frac{w_z}{w})v_y + (\Gamma_{23}^2 + \frac{w_y}{w})v_z, \]

where the coefficients \( \Gamma_{12}^3, \Gamma_{12}^2, \) and \( \Gamma_{12}^1 \) can be reduced to zero using the special choice of system of coordinates.

The first three of above equations have the form of the Darboux system of equations and the equations for the coordinate \( w(x, y, z) \) can be written as follows:

\[ w_{xy} + \frac{w_x w_y}{w} = \frac{A_y}{A} w_x + \frac{B_y}{B} w_y - \frac{u_x u_y + v_x v_y}{w}, \]

\[ w_{xz} + \frac{w_x w_z}{w} = \frac{A_x}{A} w_x + \frac{C_x}{C} w_z - \frac{u_x u_z + v_x v_z}{w}, \]

\[ w_{yz} + \frac{w_y w_z}{w} = \frac{B_z}{B} w_y + \frac{C_y}{C} w_z - \frac{u_y u_z + v_y v_z}{w}. \]
After the change of variable $w = \sqrt{R}$ we obtain the following compatible system of equations

\[
R_{zy} = \frac{B_z}{B} R_y + \frac{C_y}{C} R_z - 2(u_y u_z + v_y v_z),
\]

\[
R_{xx} = \frac{A_z}{A} R_x + \frac{C_x}{C} R_z - 2(u_x u_z + v_x v_z),
\]

\[
R_{xy} = \frac{A_y}{A} R_x + \frac{B_y}{B} R_y - 2(u_x u_y + v_x v_y).
\]

The solutions of this system of equations can be used to study problems of the theory of 3-dimensional manifolds, the knots theory and so on.

It is apparent that in an analogous way one can investigate the Einstein spaces and other more general affine-connected spaces.

**Remark 4** It can be shown that the fundamental system of equations (6) can be integrated (using some modification!) with the help of representation of the functions $u_i$ in the form

\[
u_i(x, y, z) = U_i(x, y, z) \exp \left[ \frac{x}{\lambda + 1} + \frac{y}{\lambda - 1} + \frac{z}{\lambda + 1} \right].
\]

Corresponding solutions for connection coefficients and coordinates $u_i$ of space have important applications to different problems of Geometry.

## 5 On geodesics of the space of affine connection

The problem of integration of equations of geodesics is very important in Differential Geometry. The simplest case of such type of equations is connected with the second order ODE

\[
y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0.
\]

It turns out that the theory of the matrix Laplace equation is connected with this problem.

The general equations of geodesics of the affine connected spaces with coefficients $\Gamma_{ij}^k$ are:

\[
\frac{d^2 x^i}{ds^2} + \Gamma_{ij}^k \frac{dx^k}{ds} \frac{dx^j}{ds} = 0.
\]
Let us present them in the following form

\[
\frac{d^2x}{ds^2} + \Gamma^i_1 \left( \frac{dx}{ds} \right)^2 + 2\Gamma^i_1 \frac{dx}{ds} \frac{dy}{ds} + \Gamma^i_2 \left( \frac{dy}{ds} \right)^2 + 2\Gamma^i_1 \frac{dx}{ds} \frac{dz^j}{ds} + \\
2\Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + \Gamma^k_j \frac{dz^j}{ds} = 0,
\]

\[
\frac{d^2y}{ds^2} + \Gamma^i_1 \left( \frac{dx}{ds} \right)^2 + 2\Gamma^i_1 \frac{dx}{ds} \frac{dy}{ds} + \Gamma^i_2 \left( \frac{dy}{ds} \right)^2 + 2\Gamma^i_1 \frac{dx}{ds} \frac{dz^j}{ds} + \\
2\Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + \Gamma^k_j \frac{dz^j}{ds} = 0,
\]

\[
\frac{d^2z^i}{ds^2} + \Gamma^i_1 \left( \frac{dx}{ds} \right)^2 + 2\Gamma^i_1 \frac{dx}{ds} \frac{dy}{ds} + \Gamma^i_2 \left( \frac{dy}{ds} \right)^2 + 2\Gamma^i_1 \frac{dx}{ds} \frac{dz^j}{ds} + \\
2\Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + \Gamma^k_j \frac{dz^j}{ds} = 0,
\]

writing the coordinates \(x^i\) in the form \(x^i = (x, y, z^i)\).

Then we will consider the coordinates \(z^i\) as the functions of variables \(x, y\). So the following relations are fulfilled:

\[
\frac{dz^i}{ds} = z^i_x \frac{dx}{ds} + z^i_y \frac{dy}{ds}
\]

and

\[
\frac{d^2z^i}{ds^2} = z^i_xx \left( \frac{dx}{ds} \right)^2 + 2z^i_x \frac{dx}{ds} \frac{dy}{ds} + z^i_y \left( \frac{dy}{ds} \right)^2 + z^i_x \frac{d^2x}{ds^2} + z^i_y \frac{dy}{ds} \frac{d^2y}{ds^2}.
\]

Putting these relations in above formulas one obtains the system of equations for the functions \(z^i(x, y)\) (after making equal to zero the expressions at the derivatives \(\left( \frac{dx}{ds} \right)^2\), \(\left( \frac{dy}{ds} \right)^2\) and \(\frac{d^2x}{ds^2}\), \(\frac{d^2y}{ds^2}\)).

\[
z^i_{xx} = [\Gamma^i_1 \frac{d^2x}{ds^2} + 2\Gamma^i_1 \frac{dx}{ds} \frac{dy}{ds} + \Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + 2\Gamma^i_1 \frac{dx}{ds} \frac{dz^j}{ds} + 2\Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + \Gamma^k_j \frac{dz^j}{ds}]
\]

\[
z^i_{xy} = [\Gamma^i_1 \frac{d^2x}{ds^2} + 2\Gamma^i_1 \frac{dx}{ds} \frac{dy}{ds} + \Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + 2\Gamma^i_1 \frac{dx}{ds} \frac{dz^j}{ds} + 2\Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + \Gamma^k_j \frac{dz^j}{ds}]
\]

\[
z^i_{yy} = [\Gamma^i_1 \frac{d^2x}{ds^2} + 2\Gamma^i_1 \frac{dx}{ds} \frac{dy}{ds} + \Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + 2\Gamma^i_1 \frac{dx}{ds} \frac{dz^j}{ds} + 2\Gamma^i_2 \frac{dy}{ds} \frac{dz^j}{ds} + \Gamma^k_j \frac{dz^j}{ds}]
\]

So one obtains the following statement...
Proposition 1 There is one-to-one correspondence between the second order ODE

\[ y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0. \]

and two-dimensional surfaces of the affine connected space \( A^n \).

The following relations between the coefficients \( a_i(x, y) \) of the equation, the coordinates \( z^k(x, y) \) of the space \( A^n \) and the coefficients of the connections \( \Gamma_{ij}^k(x, y) \) are true

\[
\begin{aligned}
y'' &- \left[ \Gamma_{22}^1 + 2\Gamma_{2j}^1 z^j_x + \Gamma_{kj}^1 z^k_x z^j_y \right] y'^3 + \left[ \Gamma_{22}^2 - 2\Gamma_{12}^1 + \Gamma_{kj}^2 z^k_x z^j_y \right] y'^2 + \left[ 2\Gamma_{2j}^2 z^j_x - \Gamma_{kj}^1 z^k_x z^j_y \right] y' + \\
&+ \left[ 2\Gamma_{12}^2 - \Gamma_{11}^1 \right] y + 2\Gamma_{1j}^2 - 2\Gamma_{1j}^1 z^j_x - \Gamma_{kj}^1 z^k_x z^j_y = 0.
\end{aligned}
\]

The above mentioned system of equations for coordinates \( z^i(x, y) \) in general case is nonlinear generalization of the matrix Laplace equations.

Restricting our consideration to the linear part of this system of equations we get

\[
\begin{aligned}
Z_{xx} &= A(x, y)Z_x + B(x, y)Z_y, \\
Z_{xy} &= C(x, y)Z_x + D(x, y)Z_y, \\
Z_{yy} &= E(x, y)Z_x + F(x, y)Z_y,
\end{aligned}
\] (7)

where \( Z = z^i(x, y) \) is a vector-function, \( A, B, C, D, E, F \) are matrices.

From the conditions of compatibility the relations follow:

\[
\begin{aligned}
A_y - C_x &= [C, A] + DC - BE, \\
D_y - F_x &= [FD] + EB - CD, \\
D_x - B_y &= AD + BF - CB + D^2, \\
C_y - E_x &= EA + FC - DE - C^2.
\end{aligned}
\] (8)

Let us consider some particular cases.

1. \( N = 3 \). The linear system is:

\[
Z_{xx} = (\Gamma_{11}^1 - 2\Gamma_{13}^3) Z_x + \Gamma_{11}^2 Z_y,
\]
\[ Z_{xy} = (\Gamma_{12}^1 - \Gamma_{23}^3)Z_x + (\Gamma_{12}^2 - \Gamma_{13}^3)Z_y, \]
\[ Z_{yy} = \Gamma_{22}^1 Z_x + (\Gamma_{22}^2 - 2\Gamma_{23}^3)Z_y. \]

From the condition of compatibility \( Z_{xxy} = Z_{xyx} \) and \( Z_{yyx} = Z_{xyy} \) we get:

\[ \Gamma_{13}^3 = \frac{1}{3}(\Gamma_{11}^1 + \Gamma_{12}^2), \quad \Gamma_{23}^3 = \frac{1}{3}(\Gamma_{12}^1 + \Gamma_{22}^2), \quad \Gamma_{23}^2 = \Gamma_{13}^1 \]
and as result we obtain the system of equations

\[ Z_{xx} = \frac{1}{3}(\Gamma_{11}^1 - 2\Gamma_{12}^2)Z_x + \Gamma_{11}^1 Z_y, \]
\[ Z_{xy} = \frac{1}{3}(2\Gamma_{12}^1 - \Gamma_{22}^2)Z_x + \frac{1}{3}(2\Gamma_{12}^2 - \Gamma_{11}^1)Z_y, \]
\[ Z_{yy} = \Gamma_{22}^1 Z_x + \frac{1}{3}(\Gamma_{22}^2 - 2\Gamma_{12}^1)Z_y. \]

The corresponding equations of geodesics are

\[ y'' - \Gamma_{22}^1 y'^3 + (\Gamma_{22}^2 - 2\Gamma_{12}^1)y'^2 + (2\Gamma_{12}^2 - \Gamma_{11}^1)y' + \Gamma_{12}^1 = 0. \quad (9) \]

From the conditions of compatibility one obtains the equations

\[ a_{1x} - a_{2y} = 2a_3a_1 - 2a_2^2, \quad a_{2x} - a_{3y} = a_1a_4 - a_2a_3, \]
\[ a_{4y} - a_{3x} = -2a_2a_4 + 2a_3^2. \]

These conditions for coefficients \( a_i \) correspond to the ODE (9) with the projective flat of connection, i.e. the components of its curvature tensor are equal to zero [12,13].

2. \( N=4 \) In linear approximation we obtain

\[ z_{xx}^3 = (\Gamma_{11}^1 - 2\Gamma_{13}^3)z_x^3 + \Gamma_{11}^1 z_y^3 - 2\Gamma_{14}^3z_x^4, \]
\[ z_{xy}^3 = (\Gamma_{12}^1 - \Gamma_{23}^3)z_x^3 + (\Gamma_{12}^2 - \Gamma_{13}^3)z_x^3 - \Gamma_{24}^3z_x^4 - \Gamma_{14}^3z_y^4, \]
\[ z_{yy}^3 = \Gamma_{22}^1 z_x^3 + (\Gamma_{22}^2 - 2\Gamma_{23}^3)z_x^3 - 2\Gamma_{24}^3z_y^4, \]
\[ z_{xx}^4 = (\Gamma_{11}^1 - 2\Gamma_{14}^3)z_x^4 + \Gamma_{11}^1 z_y^4 - 2\Gamma_{13}^3z_x^4, \]
\[ z_{xy}^4 = (\Gamma_{12}^1 - \Gamma_{24}^3)z_x^4 + (\Gamma_{12}^2 - \Gamma_{14}^3)z_x^4 - \Gamma_{13}^3z_y^4 - \Gamma_{23}^3z_y^4, \]
\[ z_{yy}^4 = \Gamma_{22}^1 z_x^4 + (\Gamma_{22}^2 - 2\Gamma_{24}^3)z_x^4 - 2\Gamma_{23}^3z_y^4. \]
Or in the matrix form

\[
Z_{xx} = \begin{pmatrix}
\Gamma_{11}^1 - 2\Gamma_{13}^3 & -2\Gamma_{14}^3 \\
-2\Gamma_{13}^4 & \Gamma_{11}^1 - 2\Gamma_{14}^3
\end{pmatrix} Z_x + \begin{pmatrix}
\Gamma_{11}^2 & 0 \\
0 & \Gamma_{11}^2
\end{pmatrix} Z_y,
\]

\[
Z_{xy} = \begin{pmatrix}
\Gamma_{12}^1 - \Gamma_{23}^3 & -\Gamma_{24}^3 \\
-\Gamma_{23}^4 & \Gamma_{12}^1 - \Gamma_{24}^3
\end{pmatrix} Z_x + \begin{pmatrix}
\Gamma_{12}^2 - \Gamma_{13}^3 & -\Gamma_{14}^3 \\
-\Gamma_{13}^4 & \Gamma_{12}^2 - \Gamma_{14}^3
\end{pmatrix} Z_y,
\]

\[
Z_{yy} = \begin{pmatrix}
\Gamma_{22}^1 & 0 \\
0 & \Gamma_{22}^1
\end{pmatrix} Z_x + \begin{pmatrix}
\Gamma_{22}^2 - 2\Gamma_{23}^3 & -2\Gamma_{24}^3 \\
-2\Gamma_{23}^4 & \Gamma_{22}^2 - 2\Gamma_{24}^3
\end{pmatrix} Z_y.
\]

Let us consider the surfaces in 4-dim space corresponding to the equations \( y'' = f(x, y) \). According to (9) one get

\[ y'' + \Gamma_{11}^2 = 0, \]

and the following relations are true:

\[ \Gamma_{22}^1 = 0, \quad \Gamma_{22}^2 - 2\Gamma_{12}^1 = 0, \quad 2\Gamma_{12}^2 - \Gamma_{11}^1 = 0, \quad \Gamma_{11}^2 \neq 0. \]

From these relations we get

\[ F = 2C, \quad A = 2D, \quad E = 0, \]

and the system takes the form

\[
C_x - 2D_y + 2CD - DC = 0, \quad D_x - B_y + CB - 2BC - D^2 = 0,
\]

\[ D_y - 2C_x + 2DC - CD = 0, \quad C_y = C^2. \]

So from the first and third equations we have

\[ C_x - D_y + CD - DC = 0, \]

and the matrices \( C \) and \( D \) can be presented in the form

\[ C = \Theta_x \Theta^{-1}, \quad D = \Theta_x \Theta^{-1}, \]

where \( \Theta(x, y) \) is matrix satisfying the compatible system of equations

\[ \Theta_{xy} = \Theta_x \Theta^{-1} \Theta_y + \Theta_y \Theta^{-1} \Theta_x \]

\[ \Theta_{yy} = 2 \Theta_y \Theta^{-1} \Theta_y. \]
Then the following relations are fulfilled:

\[ y'' - \Gamma_{22}^1 y^3 + (\Gamma_{22}^2 - 2\Gamma_{12}^1) y' + (2\Gamma_{12}^2 - \Gamma_{11}^1) y' + \Gamma_{11}^2 + \\
+ 2\Gamma_{13}^2 z' + \Gamma_{33}^3 z'^2 + 2(\Gamma_{23}^2 - \Gamma_{13}^1) y' z' - 2\Gamma_{23}^1 y'^2 z' - \Gamma_{33}^1 y' z'^2 = 0, \\
\]

\[ z'' - \Gamma_{33}^1 z^3 + (\Gamma_{33}^2 - 2\Gamma_{13}^1) z'^2 + (2\Gamma_{13}^2 - \Gamma_{11}^1) z' + \Gamma_{11}^3 + \\
+ 2\Gamma_{12}^3 y' + \Gamma_{22}^3 y'^2 + 2(\Gamma_{23}^2 - \Gamma_{12}^1) y' z' - 2\Gamma_{23}^1 y'^2 z' - \Gamma_{22}^1 y'^2 z' = 0. \]

From these relations the matrix Darboux-Lame system has appeared by natural way.

In fact, the coordinates \( x^i \) can be written in the form \( x, y, z, q^i(x, y, z) \). Then the following relations are fulfilled:

\[ \frac{dq^i}{ds} = \dot{q}^i = q^i_x \dot{x} + q^i_y \dot{y} + q^i_z \dot{z} \]

and

\[ \frac{d^2 q^i}{ds^2} = \ddot{q} = q^i_{xx} (\dot{x})^2 + 2q^i_{xy} \dot{x} \dot{y} + q^i_{yy} (\dot{y})^2 + 2q^i_{xz} \dot{x} \dot{z} + 2q^i_{yz} \dot{y} \dot{z} + q^i_{zz} (\dot{z})^2 + q^i_x \ddot{x} + q^i_y \ddot{y} + q^i_z \ddot{z}. \]

Putting these relations in the general equations of geodesics:

\[ \frac{d^2 x^i}{ds^2} + \Gamma_{k j}^i \frac{dx^k}{ds} \frac{dx^j}{ds} = 0. \]

we obtain the system of equations for the functions \( q^i(x, y) \) (after making equal to zero the expressions at the derivatives \( \frac{dx}{ds} \), \( \frac{dy}{ds} \), \( \frac{dz}{ds} \), and \( \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \)).

It has the form:

\[ q^i_{xx} = [\Gamma_{11}^1 + 2\Gamma_{1j}^i q^j_x + \Gamma_{k j}^i q^k_x q^j_x] q^i_x + \\
+ [\Gamma_{11}^2 + 2\Gamma_{1j}^i q^j_x + \Gamma_{k j}^i q^k_x q^j_x] q^i_y + \\
+ [\Gamma_{11}^3 + 2\Gamma_{1j}^i q^j_x + \Gamma_{k j}^i q^k_x q^j_x] q^i_z - 2\Gamma_{1j}^i q^j_x - \Gamma_{k j}^i q^k_x q^j_x - \Gamma_{11}^i, \]

\[ q^i_{xy} = [\Gamma_{12}^1 + \Gamma_{1j}^i q^j_y + \Gamma_{2j}^i q^j_x + \frac{1}{2} \Gamma_{k j}^i (q^k_x q^j_y + q^k_y q^j_x)] q^i_x + \\
+ [\Gamma_{12}^2 + \Gamma_{2j}^i q^j_y + \Gamma_{2j}^i q^j_x + \frac{1}{2} \Gamma_{k j}^i (q^k_x q^j_y + q^k_y q^j_x)] q^i_y + \\
+ [\Gamma_{12}^3 + \Gamma_{2j}^i q^j_y + \Gamma_{2j}^i q^j_x + \frac{1}{2} \Gamma_{k j}^i (q^k_x q^j_y + q^k_y q^j_x)] q^i_z. \]
where

\[ q^i_{xz} = \left[ (\Gamma^1_{13} + \Gamma^1_{3j} q^j_z + \Gamma^1_{j3} q^j_x + \frac{1}{2} \Gamma^2_{kj} (q^k_x q^j_y + q^k_y q^j_x)) q^i_x + \right. \]

\[ + \left. (\Gamma^2_{12} + \Gamma^2_{3j} q^j_x + \Gamma^2_{j2} q^j_z) q^i_y \right] q^i_z - \frac{1}{2} \Gamma^i_{kj} (q^k_x q^j_y + q^k_y q^j_x) - \Gamma^i_{12}, \]

\[ q^i_{yz} = \left[ (\Gamma^1_{23} + \Gamma^1_{3j} q^j_z + \Gamma^1_{j3} q^j_y + \frac{1}{2} \Gamma^2_{kj} (q^k_y q^j_x + q^k_x q^j_y)) q^i_x + \right. \]

\[ + \left. (\Gamma^2_{23} + \Gamma^2_{3j} q^j_x + \Gamma^2_{j2} q^j_z) q^i_y \right] q^i_z - \frac{1}{2} \Gamma^i_{kj} (q^k_y q^j_x + q^k_x q^j_y) - \Gamma^i_{23}, \]

\[ q^i_{yy} = \left[ (\Gamma^1_{22} + 2 \Gamma^1_{3j} q^j_y + \Gamma^1_{j3} q^j_y q^j_y) q^i_x + \right. \]

\[ + \left. \frac{1}{2} \Gamma^2_{kj} (q^k_y q^j_y + q^k_y q^j_y) q^i_x - \frac{1}{2} \Gamma^i_{kj} (q^k_y q^j_y + q^k_y q^j_y) - \Gamma^i_{22}, \right] \]

\[ q^i_{zz} = \left[ (\Gamma^1_{33} + 2 \Gamma^1_{j3} q^j_z + \Gamma^1_{k3} q^k z^j q^j_z) q^i_x + \right. \]

\[ + \left. \frac{1}{2} \Gamma^2_{kj} (q^k z^j q^j_z + q^k z^j q^j_z) q^i_x - 2 \Gamma^i_{kj} (q^k z^j q^j_z + q^k z^j q^j_z) - \Gamma^i_{33} \right]. \]

So, in linear approximation we get the matrix Lame system of equations on coordinates \( q^i(x, y, z) \).

6 Equation of geodesic deviation and the matrix Darboux-Lame system

The general equation of geodesic deviation is

\[ \frac{D^2 \eta^i}{ds^2} + R^{i}_{kjm} \frac{dx^k}{ds} \frac{dx^m}{ds} \eta^j = 0, \]

where \( R^{i}_{kjm} \) is the tensor of curvature of manifold.

We transform this equation into an easy-to-use form. According to the rule of covariant differentiation we obtain the relation

\[ \frac{D \eta^i}{ds} = \frac{d \eta^i}{ds} + \Gamma^{i}_{jk} \eta^j \frac{dx^k}{ds} = \rho^i \]
and then
\[ \frac{D^2 \eta^i}{ds^2} = \frac{D\rho^i}{ds} = \frac{d\rho^i}{ds} + \Gamma^i_{lm} \frac{dx^m}{ds} = \]
\[ = \frac{d^2 \eta^i}{ds^2} + \frac{\partial \Gamma^i_{jk}}{\partial \eta^j} \frac{dx^k}{ds} + \frac{\partial \rho^i}{ds} \frac{dx^k}{ds} + \frac{\partial \Gamma^i_{jk}}{\partial \eta^j} \frac{d^2 x^k}{ds^2} + \Gamma^i_{lm} \frac{d \eta^l}{ds} + \frac{\partial \eta^i}{ds} + \frac{\partial \rho^i}{ds} \frac{dx^k}{ds} \frac{dx^m}{ds}. \]

Furthermore, using the equations of geodesics
\[ \frac{d^2 x^k}{ds^2} + \Gamma^i_{pq} \frac{dx^p}{ds} \frac{dx^q}{ds} = 0, \]
we can get the relation:
\[ \frac{d^2 \eta^i}{ds^2} + 2\Gamma^i_{lm} \frac{dx^m}{ds} \frac{d \eta^l}{ds} + \left[ \frac{\partial \Gamma^i_{jk}}{\partial x^l} \frac{dx^k}{ds} \frac{dx^m}{ds} + \Gamma^i_{lm} \frac{d \eta^l}{ds} + \frac{\partial \eta^i}{ds} + \frac{\partial \rho^i}{ds} \frac{dx^k}{ds} \right] \eta^j = 0. \]

Putting here the expression for the curvature tensor
\[ R^i_{klm} = \frac{\partial \Gamma^i_{jm}}{\partial x^j} - \frac{\partial \Gamma^i_{kjm}}{\partial x^m} + \Gamma^i_{nj} \Gamma^j_{km} - \Gamma^i_{nm} \Gamma^n_{kj}, \]
we find the relation
\[ \frac{d^2 \eta^i}{ds^2} + 2\Gamma^i_{lm} \frac{dx^m}{ds} \frac{d \eta^l}{ds} + \left[ \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{nl} \Gamma^n_{jk} - \Gamma^i_{jm} \Gamma^m_{kj} + \Gamma^i_{nj} \Gamma^k_{jm} - \Gamma^i_{nm} \Gamma^k_{nj} \right] \frac{dx^k}{ds} \frac{dx^l}{ds} \eta^j, \]
whence after rearrangement the desired formula for the equation of geodesic deviation follows
\[ \frac{d^2 \eta^i}{ds^2} + 2\Gamma^i_{lm} \frac{dx^m}{ds} \frac{d \eta^l}{ds} + \frac{\partial \Gamma^i_{kl}}{\partial x^j} \frac{dx^k}{ds} \frac{dx^l}{ds} \eta^j = 0. \]

This form of equation of geodesic deviation is convenient in applications (see [14]). Let us consider some of them.

In three-dimensional case one obtains
\[ \frac{d \eta^i}{ds} = \eta^i \dot{x} + \eta^i \dot{y} + \eta^i \dot{z} \]
and

\[
\frac{d^2 \eta^i}{ds^2} = \ddot{\eta}^i = \eta^i_{xx}(\dot{x})^2 + 2\eta^i_{xy}\dot{x}\dot{y} + \eta^i_{yy}(\dot{y})^2 + 2\eta^i_{xz}\dot{x}\dot{z} + 2\eta^i_{yz}\dot{y}\dot{z} + \eta^i_{zz}(\dot{z})^2 + \eta^i_{xx}\ddot{x} + \eta^i_{yy}\ddot{y} + \eta^i_{zz}\ddot{z}.
\]

Putting these relations in (10) and using the equations of geodesics we get the matrix Lamé system of equations

\[
\begin{align*}
\eta^i_{xx} + 2\Gamma^i_{11}\eta^j_x - \Gamma^i_{11}\eta^j_x - \Gamma^i_{11}\eta^j_y - \Gamma^i_{11}\eta^j_z + \frac{\partial \Gamma^i_{11}}{\partial x^j}\eta^j &= 0, \\
\eta^i_{yy} + 2\Gamma^i_{22}\eta^j_y - \Gamma^i_{22}\eta^j_x - \Gamma^i_{22}\eta^j_y - \Gamma^i_{22}\eta^j_z + \frac{\partial \Gamma^i_{22}}{\partial x^j}\eta^j &= 0, \\
\eta^i_{zz} + 2\Gamma^i_{33}\eta^j_z - \Gamma^i_{33}\eta^j_x - \Gamma^i_{33}\eta^j_y - \Gamma^i_{33}\eta^j_z + \frac{\partial \Gamma^i_{33}}{\partial x^j}\eta^j &= 0, \\
\eta^i_{xy} + \Gamma^i_{11}\eta^j_y + \Gamma^i_{12}(\eta^j_x + \eta^j_l) + \Gamma^i_{13}\eta^j_z + \Gamma^i_{23}\eta^j_z - \Gamma^i_{12}\eta^j_y - \Gamma^i_{12}\eta^j_y - \Gamma^i_{12}\eta^j_y + \frac{\partial \Gamma^i_{12}}{\partial x^j}\eta^j &= 0, \\
\eta^i_{xz} + \Gamma^i_{11}\eta^j_z + \Gamma^i_{13}(\eta^j_z + \eta^j_l) + \Gamma^i_{12}\eta^j_x + \Gamma^i_{23}\eta^j_z - \Gamma^i_{13}\eta^j_y - \Gamma^i_{13}\eta^j_y - \Gamma^i_{13}\eta^j_y + \frac{\partial \Gamma^i_{13}}{\partial x^j}\eta^j &= 0, \\
\eta^i_{yz} + \Gamma^i_{12}\eta^j_z + \Gamma^i_{23}(\eta^j_z + \eta^j_l) + \Gamma^i_{13}\eta^j_z + \Gamma^i_{22}\eta^j_z - \Gamma^i_{23}\eta^j_z - \Gamma^i_{23}\eta^j_z - \Gamma^i_{23}\eta^j_z + \frac{\partial \Gamma^i_{23}}{\partial x^j}\eta^j &= 0.
\end{align*}
\]

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