STRONG RIGIDITY OF II₁ FACTORS ARISING FROM MALLEABLE ACTIONS OF w-RIGID GROUPS, I

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Abstract. We consider crossed product II₁ factors $M = N \rtimes_\sigma G$, with $G$ discrete ICC groups that contain infinite normal subgroups with the relative property (T) and $\sigma$ trace preserving actions of $G$ on finite von Neumann algebras $N$ that are “malleable” and mixing. Examples are the actions of $G$ by Bernoulli shifts (classical and non-classical), and by Bogoliubov shifts. We prove a rigidity result for isomorphisms of such factors, showing the uniqueness, up to unitary conjugacy, of the position of the group von Neumann algebra $L(G)$ inside $M$. We use this result to calculate the fundamental group of $M$, $\mathcal{F}(M)$, in terms of the weights of the shift $\sigma$, for $G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ and other special arithmetic groups. We deduce that for any subgroup $S \subset \mathbb{R}_+^*$ there exist II₁ factors $M$ (separable if $S$ is countable or $S = \mathbb{R}_+^*$) with $\mathcal{F}(M) = S$. This brings new light to a long standing open problem of Murray and von Neumann.

0. Introduction.

This is the first of a series of papers in which we study rigidity properties of isomorphisms $\theta$ of crossed product II₁ factors $M₀, M$ arising from certain actions of groups on finite von Neumann algebras. We also study isomorphisms between amplifications of such factors. Typically, we assume the “source” factor $M₀$ comes from an action of a group $G₀$ having a large subgroup $H \subset G₀$ with the relative property (T) of Kazhdan-Margulis ($G₀$ is w-rigid), while the “target” factor $M$ comes from an action $(\sigma, G)$ with good “deformation+mixing” properties (a malleable action), e.g. an action by Bogoliubov or Bernoulli shifts (classical and non-classical). The “ideal” type of result we seek to prove, is that any isomorphism

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between such factors comes from a conjugacy of the actions involved. Thus, \( M_0, M \)
can be isomorphic only if they come from identical group+action data.

Such **strong rigidity** results will be obtained in the sequel papers with the same
title \([Po6]\) (the “group measure space” case) and \([Po7]\) (the “non-classical” case).
In the present paper we prove a key preliminary rigidity result needed in this
program, showing that, after a suitable perturbation, any isomorphism \( \theta \) as above
must necessarily take the \( w \)-rigid group \( G_0 \) into the group subalgebra \( L(G) \) of \( M \).
More generally, we prove that any relatively rigid subalgebra \( Q \subset M \) is “swept” by
\( L(G) \), via a canonical (usually inner) automorphism of \( M \).

Besides its rôle in \(([Po6,7]), this result already enables us to calculate here the
fundamental group \( \mathcal{F}(M) \) of crossed product II\(_1\) factors \( M \) coming from (non-classical) Connes-Størmer Bernoulli shift actions of arithmetic groups such as \( G = \mathbb{Z}^2 \rtimes \Gamma \), with \( \Gamma \subset SL(2,\mathbb{Z}) \) a subgroup of finite index, by using results from \(([Po3], [Ga])\). Thus, if \( \{t_i\}_i \) are the weights of \( \sigma \) (see \([CSt]\)), then \( \mathcal{F}(M) \) is equal to
the multiplicative group generated by the ratios \( \{t_i/t_j\}_{i,j} \). As a consequence, we
obtain that any countable subgroup \( S \subset \mathbb{R}_+^* \) can be realized as a fundamental
group of a separable II\(_1\) factor \( M \) (i.e., with \( L^2(M) \) separable, or equivalently \( M \)
countably generated). In fact, by considering Connes Størmer G-Bernoulli shifts
coming from non-separable Araki-Woods factors, we obtain II\(_1\) factors \( M \) with
\( \mathcal{F}(M) \) any uncountable subgroup \( S \subset \mathbb{R}_+^* \) as well, but \( \dim L^2(M) = |S| \).

This brings new light to a longstanding problem of Murray and von Neumann
on the nature of the fundamental group of II\(_1\) factors. They were led to consider
this invariant and to pose this problem by their theory of continuous dimension and
their discovery that one can take “\( t \) by \( t \) matrices” over a II\(_1\) factor \( M \), for any \( t > 0 \).
This one parameter family of II\(_1\) factors, denoted \( M^t \) and called **amplifications**
by \( t \), is used to define the fundamental group of \( M \) by \( \mathcal{F}(M) = \{ t > 0 \mid M^t \simeq M \} \)
([MvN2]). After proving that \( \mathcal{F}(R) = \mathbb{R}_+^* \) (in other words \( R^t \simeq R, \forall t > 0 \))
for the hyperfinite II\(_1\) factor \( R \), they comment: “There is no reason to believe [that
\( \mathcal{F}(M) = \mathbb{R}_+^* \) for all factors \( M \) ]. The general behavior of this invariant remains
an open question.” (see \([MvN2], page 742\)). Variants of this problem were also
mentioned in ([K], [Sa], [J3]).

It took almost 40 years until the first progress in this direction was made, with
Connes’s breakthrough discovery that for group factors \( M = L(G) \), \( \mathcal{F}(L(G)) \) reflects
the rigidity properties of the group \( G \), being countable whenever \( G \) has the
property (T) of Kazhdan. Connes’ idea was further exploited in ([Po2], [GoNe],
[GeGo], [Po5]) to obtain new classes of separable II\(_1\) factors \( M \) with countable
\( \mathcal{F}(M) \), including examples for which \( \mathcal{F}(M) \) contains a prescribed countable set.
But the first exact computation of a fundamental group \( \neq \mathbb{R}_+^* \) was obtained only
recently, in ([Po3]), where it is shown, for instance, that \( \mathcal{F}(L(G)) = \{1\} \) for any
group of the form \( G = \mathbb{Z}^2 \rtimes \Gamma \), with \( \Gamma \) a subgroup of finite index in \( SL(2, \mathbb{Z}) \) (e.g. \( \Gamma = \mathbb{F}_n \)). The result in this paper solves the problem completely, by showing that any subgroup of \( \mathbb{R}^*_+ \) can be realized as a fundamental group of a \( \text{II}_1 \) factor.

Note however that the problem of finding all subgroups of \( \mathbb{R}^*_+ \) that can occur as fundamental groups of separable \( \text{II}_1 \) factors (which is the really interesting case!) is not completely solved. Thus, our result only shows that, besides \( \mathbb{R}^*_+ \) itself, all countable subgroups of \( \mathbb{R}^*_+ \) can appear. However, we conjecture that the only uncountable subgroup of \( \mathbb{R}^*_+ \) that can occur as a fundamental group of a countably generated \( \text{II}_1 \) factor is \( \mathbb{R}^*_+ \). As a supporting evidence, note that there are \( 2^{2^{\aleph_0}} \) many distinct subgroups of \( \mathbb{R}^*_+ \) and only \( 2^{\aleph_0} \) isomorphism classes of countably generated \( \text{II}_1 \) factors, so “most” uncountable subgroups of \( \mathbb{R}^*_+ \) cannot appear as fundamental groups of separable factors.

To state in more details the results in this paper, recall some basic concepts and definitions. Given a finite von Neumann algebra with a trace \( (N, \tau) \) and an action \( \sigma : G \to \text{Aut}(N, \tau) \) of a discrete group \( G \) on \( N \) by \( \tau \)-preserving automorphisms, its associated crossed product von Neumann algebra \( N \rtimes_\sigma G \) is generated by a copy of the group \( G \), \( \{u_g\}_{g \in G} \), and a copy of the algebra \( N \), acting on the Hilbert space \( \mathcal{H} = \bigoplus_g L^2(N, \tau)u_g \) by left multiplication subject to the product rules \( u_g \xi u_h = \sigma_g(\xi) u_{gh}, \forall g, h \in G, \xi \in L^2(N, \tau) \). Thus, the finite sums \( x = \Sigma_g y_g u_g, y_g \in N \) are weakly dense in \( N \rtimes_\sigma G \) and in fact any \( \ell^2 \)-convergent formal sum \( x = \Sigma_g y_g u_g \) with “coefficients” \( y_g \) in \( N \) that satisfies \( x \cdot \mathcal{H} \subset \mathcal{H} \) defines an element in \( N \rtimes_\sigma G \) (as left multiplication operator), and all \( x \in N \rtimes_\sigma G \) are of this form. The trace \( \tau \) on \( N \) extends to all \( N \rtimes_\sigma G \) by \( \tau(\Sigma_g y_g u_g) = \tau(y_e) \).

This construction goes back to Murray and von Neumann ([MvN1,2]). The particular case when \( N = \mathbb{C} \) gives the group von Neumann algebra \( L(G) \) associated to \( G \). Thus, \( L(G) \) is a natural subalgebra in any \( N \rtimes_\sigma G \). It is a \( \text{II}_1 \) factor iff \( G \) is infinite conjugacy class (ICC). In case \( \sigma \) is an action of \( G \) on a probability space \( (X, \mu) \) by measure preserving transformations, it induces an action on the function algebra \( N = L^\infty(X, \mu) \) which preserves \( \tau = \int \cdot d\mu \) and the corresponding crossed product \( L^\infty(X, \mu) \rtimes_\sigma G \) is called the group measure space algebra associated to \( (\sigma, G) \). It is a \( \text{II}_1 \) factor whenever \( \sigma \) is free, ergodic and \( G \) infinite.

An example that we often consider in this paper is the (classical) Bernoulli shift action of \( G \) on product spaces \( (X, \mu) = \Pi_g(Y_0, \nu_0)_g \), with base \( (Y_0, \nu_0) \) a probability space \( \neq \) single point set. These actions are extremely “malleable”, a feature that enables us to detect all “rigid parts” of the group measure space algebra \( L^\infty(X, \mu) \rtimes_\sigma G \). Recall from ([Po3]) that a subalgebra \( Q \) of a \( \text{II}_1 \) factor \( M \) has the relative property (T) (or that \( Q \) is relatively rigid in \( M \)) if any unital, tracial completely positive map \( \phi \) on \( M \) which is close to \( id_M \) on a sufficiently large finite subset of \( M \) (in the Hilbert norm \( \|x\|_2 = \tau(x^* x)^{1/2} \)) is uniformly close to the
identity on the unit ball of $Q$.

0.1. Theorem. Let $\sigma$ be a Bernoulli shift action of an ICC group $G$ and denote $M$ the corresponding group measure space factor. Let $Q \subset M$ be a diffuse von Neumann subalgebra with the relative property (T). If $Q$ is either of type II or its normalizer in $M$ generates a factor, then there exists a unitary element $u \in M$ such that $uQu^* \subset L(G)$. Moreover, if $P$ denotes the von Neumann algebra generated by the normalizer of $Q$ in $M$ then $uPu^* \subset L(G)$.

If $H \subset G_0$ is an inclusion of groups then $L(H) \subset L(G_0)$ has the relative property (T) iff the pair $(G_0, H)$ has the relative property (T) of Kazhdan-Margulis ([Ma]; see also [dHV]). Thus, if $(\sigma_0, G_0)$, $(\sigma, G)$ are actions of groups on finite von Neumann algebras $(N_0, \tau_0)$, $(N, \tau)$, $\theta : M_0 \simeq M$ is an isomorphism of the corresponding crossed product algebras $M_0 = N_0 \rtimes_{\sigma_0} G_0$, $M = N \rtimes_{\sigma} G$ and $H \subset G_0$ is a subgroup with the relative property (T), then $Q = \theta(L(H)) \subset M$ has the relative property (T) in $M$. Thus, if we take the groups $G_0, G$ to be ICC and $G_0$ to be weakly rigid (w-rigid), i.e. to have an infinite normal subgroup with the relative property (T), then Theorem 0.1 shows that any $\theta : M_0 \simeq M$ can be perturbed by an inner automorphism so that to take $L(G_0)$ into $L(G)$ (even onto if $G$ is w-rigid as well).

All we actually need for the proof of the above result is the malleability of $\sigma : G \to \text{Aut}(N, \tau)$ (in the case $N$ is abelian, an even weaker property called sub malleability, is sufficient). This property amounts to the existence of an embedding of $N$ as the core of a von Neumann algebra with discrete decomposition $(N', \varphi)$ ([C2,3], [T1,3]) on which $G$ acts by an extension of $\sigma$, such that there exists a continuous action $\alpha$ of $\mathbb{R}$ on $\hat{N} = N \hat{\otimes} N'$ commuting with the product action $\hat{\sigma}_g = \sigma_g \otimes \sigma_g$ and satisfying $\alpha_1(N \otimes 1) = 1 \otimes N$. We call $\tilde{\sigma}$ a gauged extension for $\sigma$. It comes with a countable multiplicative subgroup $S(\tilde{\sigma}) \subset \mathbb{R}^*_+$, given by the almost periodic spectrum of the discrete decomposition $(N', \varphi)$ (i.e., of the modular group associated with $\varphi$). Since the core of $N' \rtimes G$ is $M = N \rtimes G$, the group $S(\tilde{\sigma})$ is contained in $\hat{\mathcal{P}}(M)$. Even more, for each $\beta \in S(\tilde{\sigma})$, the inclusion $N \rtimes G \subset N' \rtimes G$ gives rise to a family of $\beta$-scaling automorphisms $\text{Aut}_{t\beta}(M; \tilde{\sigma})$, any two of which differ by an inner automorphism of $M$. A stronger version of this property, called s-malleability, requires the existence of an additional period-2 automorphism (“grading”) $\beta$ of $\hat{N}$ that leaves $N'$ pointwise fixed and satisfies $\beta \alpha_t = \alpha_{-t} \beta$, $\forall t$.

A classical Bernoulli shift action $\sigma$ of a group $G$ on the product space $\mathbb{T}^G$ is easily seen to be s-malleable (sub s-malleable for arbitrary base space $(Y_0, \nu_0)$) with $S(\tilde{\sigma}) = \{1\}$ and $\text{Aut}(M; \tilde{\sigma})$ coincides with the inner automorphisms of $M$. A non-classical Connes-Stormer Bernoulli shift action $\sigma$ on the infinite tensor product algebra $(N', \varphi) = \overline{\otimes}_g(M_{k \times k}(\mathbb{C}), \varphi_0)_g$, with $\varphi_0$ a state of weights $\{t_i\}_i$, is also s-malleable. In this case $S(\tilde{\sigma})$ is the multiplicative group generated by the ratios
\[ \{t_i/t_j\}_{i,j} \subset \mathbb{R}_+^* \] (cf. [AW], [P]). By replacing the base \((M_{k \times k}(\mathbb{C}), \varphi_0)\) of the G-Bernoulli shift in the above construction with an arbitrary, possibly non-separable IPTF1 Araki-Woods factor \((N_0, \varphi_0)\), we still get a malleable action. In this case, \(S(\tilde{\sigma})\) is equal to the almost periodic spectrum of \(\varphi_0\), and thus can be taken to be any multiplicative subgroup \(S\) of \(\mathbb{R}_+^*\). Actions by weighted Bogoliubov shifts ([PSt]) satisfy a similar malleability condition. With these notations we have:

0.2. Theorem. Let \(M_i\) be a type II\(_1\) factor of the form \(M_i = N_i \rtimes_{\alpha_i} G_i\), where \(G_i\) is w-rigid ICC, \(\alpha_i : G_i \to \text{Aut}(N_i, \tau_i)\) is a Connes-Størmer Bernoulli shift, \(i = 0, 1\). Assume there exists an isomorphism \(\theta : M_0 \simeq M_1^s\), for some \(s > 0\). Then there exist \(\beta_i \in S(\tilde{\sigma}_i)\) and \(\theta^i_{\beta_i} \in \text{Aut}_{\beta_i}(M_i; \tilde{\sigma}_i)\) such that \(\theta^i_{\beta_i}(\theta(L(G_0))) = L(G_1)^{s\beta_i}\), \(\theta(\theta^0_{\beta_0}(L(G_0))) = L(G_1)^{s\beta_0}\). Moreover, \(\beta_0 = \beta_1\) and \(\theta^i_{\beta_i}\) are unique modulo perturbation by an inner automorphism implemented by a unitary of \(L(G_i)\).

When applied to the case \(G_0 = G_1 = G\), \(\sigma_0 = \sigma_1 = \sigma\) and \(M_0 = M_1 = N \rtimes_{\sigma} G\), with \(G\) w-rigid ICC and \(\sigma\) a Connes-Størmer Bernoulli shift, the above theorem shows that if \(s \in \mathcal{F}(N \rtimes_{\sigma} G)\) then there exists \(\beta \in S(\tilde{\sigma})\) such that \(s\beta \in \mathcal{F}(L(G))\). Hence, if \(\mathcal{F}(L(G)) = \{1\}\) then \(s = \beta^{-1} \in S(\tilde{\sigma})\), implying that \(\mathcal{F}(N \rtimes_{\sigma} G) = S(\tilde{\sigma})\). Thus, if we take \(G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})\) (which is w-rigid), then the calculation \(\mathcal{F}(L(G)) = \{1\}\) in ([Po3]) implies \(\mathcal{F}(N \rtimes_{\sigma} G) = S(\tilde{\sigma})\). More generally, from the results in ([Po3]) on HT factors and their \(\ell^2_{HT}\)-Betti numbers \(\beta^{HT}_n\), for which the fundamental group is trivial whenever there exists some \(n\) with \(\beta^{HT}_n \neq 0, \infty\), one gets:

0.3. Corollary. 1°. Let \(G\) be a w-rigid ICC group and \(\sigma\) a malleable mixing action of \(G\) on \((N, \tau)\), with an extension \(\bar{\sigma}\) having spectrum \(S(\sigma)\). If \(\mathcal{F}(L(G)) = \{1\}\) then \(\mathcal{F}(N \rtimes_{\sigma} G) = S(\bar{\sigma})\). In particular, this is the case if \(L(G)\) is a HT factor and \(\beta^{HT}_n(L(G)) \neq 0, \infty\) for some \(n\).

2°. Let \(G_i\) be w-rigid ICC groups and \(\sigma_i\) Connes-Størmer Bernoulli shifts of \(G_i\) on \((N_i, \tau_i)\), \(i = 0, 1\). Assume \(L(G_i)\) are HT factors with \(\beta^{HT}_n(L(G_0)) = 0\) and \(\beta^{HT}_n(L(G_1)) \neq 0, \infty\) for some \(n\). Then \(N \rtimes_{\sigma_0} G_0\) is not stably isomorphic to \(N_1 \rtimes_{\sigma_1} G_1\).

It is easy to see that for any countable subgroup \(S \subset \mathbb{R}_+^*\) there exists a Connes-Størmer Bernoulli shift with the ratios \(\{t_i/t_j\}_{i,j}\) of its weights \(\{t_i\}_i\) generating \(S\). Thus, any countable subgroup \(S \subset \mathbb{R}_+^*\) can be realized as \(\mathcal{F}(N \rtimes_{\sigma} G)\), with \(G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})\) and \(\sigma\) a Connes-Størmer Bernoulli shift action of \(G\). Noticing that such actions leave invariant a Cartan subalgebra of the hyperfinite II\(_1\) factor, it follows that the factor \(N \rtimes_{\sigma} G\) is naturally isomorphic to the von Neumann algebra of an equivalence relation \(S\) with fundamental group \(\mathcal{F}(S) \overset{\text{def}}{=} \{t > 0 \mid S^t \simeq S\}\).
equal to $\mathcal{F}(N \rtimes_{\sigma} G)$. Moreover, one can take a Connes-Størmer $G$-Bernoulli shift action $\sigma$ on $\mathcal{G}(M_0, \phi_0)_g$, with the base $(N_0, \phi_0)$ a suitable non-separable ITPF1 factor, to get $S(\bar{\sigma})$ to be an arbitrary uncountable subgroup $S \subset \mathbb{R}^*_+$ as well. Thus we have:

0.4. **Corollary.** Let $S$ be an arbitrary subgroup of $\mathbb{R}^*_+$ and let $\Gamma \subset SL(2, \mathbb{Z})$ be a subgroup of finite index (e.g. $\Gamma \simeq \mathbb{F}_n$).

1°. There exist properly outer actions $\sigma$ of $\Gamma$ on an approximately finite dimensional $\text{II}_1$ factor $N$ such that $\mathcal{F}(N \rtimes_{\sigma} \Gamma) = S$. Moreover, if $S$ is countable or equal to $\mathbb{R}^*_+$ then $N$ can be taken to be the hyperfinite $\text{II}_1$ factor (i.e., the unique approximately finite dimensional $\text{II}_1$ factor).

2°. If $S$ is countable, then there exist countable, measurable, measure preserving ergodic standard equivalence relations $S$ of the form $R \rtimes_{\Gamma} \Gamma$, with $R$ ergodic hyperfinite equivalence relation and $\Gamma$ acting outerly on $R$, such that $\mathcal{F}(S) = S$.

Note that if a $\text{II}_1$ factor $M$ comes from an equivalence relation $S$ then $\mathcal{F}(S) \subset \mathcal{F}(M)$. Related to the problem of showing “$\mathcal{F}(M) \neq \mathbb{R}^*_+ \Rightarrow \mathcal{F}(M)$ countable” for all separable $\text{II}_1$ factors $M$, mentioned earlier, it would of course be equally interesting to prove the similar fact for equivalence relations.

The ideas and techniques used in the proof of 0.1 and 0.2 are inspired from ([Po1,3,4]). Thus, the malleability of $\sigma$ combined with the the w-rigidity of $G$ allows a “deformation/rigidity” argument in the algebra $\tilde{M} = \tilde{N} \rtimes_{\tilde{\sigma}} G$. As a result of this argument, we obtain a non-trivial $L(G_0) - L(G)$ Hilbert submodule of $L^2(M, \varphi)$ which is finite dimensional as a right $L(G)$-module ($M$ denotes here the von Neumann algebra $N \rtimes_{\sigma} G$). Using “intertwining subalgebras” techniques similar to ([Po3], A.1), from such a bimodule we get a unitary element $u \in \tilde{M}$ that normalizes $M = N \rtimes G$ and conjugates a corner of $L(G_0)$ onto a corner of $L(G)$. The trace scaling automorphism $\theta_{\beta}$ in Theorem 0.2 is then nothing but $\text{Ad}_u$.

In the case of 0.2, this argument is carried out in the framework of von Neumann algebras with discrete decomposition (thus possibly of type III), whose theory was developed in the early 70’s by A. Connes ([C2,3]). Our work benefits directly or indirectly from these papers, the Tomita-Takesaki theory, Takesaki duality and the work on type III factors in ([CT], [T1]). However, since our states $\varphi$ are almost periodic (thus “almost-like traces”) the formalism simplifies, allowing us to complete most of the proofs by just using “$\text{II}_1$-corners” of $\text{II}_\infty$ factors, their trace scaling automorphisms and the associated crossed product algebras.

The paper is organized as follows: In Section 1 we recall some basic facts about discrete decomposition of von Neumann algebras, introduce definitions and notations related to malleability of actions, and give examples. In Section 2 we prove several equivalent conditions for subalgebras of the core $M = M_\varphi$ of a factor with
discrete decomposition \((\mathcal{M}, \varphi)\) to be conjugate via partial isometries of \(\mathcal{M}\) that normalize \(M\). The effectiveness of these conditions depends on a good handling of relative commutants for subalgebras in \(\mathcal{M} = \mathcal{N} \rtimes \sigma\), and Section 3 proves the necessary such results. In Section 4 we prove the main technical result of the paper: a generalized version of 0.1 showing that if \(\sigma\) is malleable mixing then \(L(G)\) “absorbs” all relatively rigid subalgebras of \(M = N \rtimes \sigma\) \((\text{see } 4.1 \text{ and } 4.4)\).

In Section 5 we prove 0.2-0.4, while in Section 6 we relate the class of factors studied in this paper with the HT factors introduced and studied in ([Po3]), showing the two classes are essentially disjoint. As an application, we prove that \((\text{classical})\) Bernoulli \(\mathbb{F}_n\)-actions cannot be orbit equivalent to the actions of \(\mathbb{F}_n\) considered in ([Po3]), thus providing two new free ergodic actions of \(\mathbb{F}_n\), non-orbit equivalent to the three ones constructed in ([Po3]) and to the one in ([Hj]).

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1. Malleable mixing actions and their gauged extensions

We first recall the definition and properties of von Neumann algebras with discrete decomposition, which play an important role in this paper. Then we consider actions of groups on discrete decompositions and their associated cross product algebras. Following ([Po1]), we next define the notion of gauged extension and the malleability property for actions. We introduce notations, list basic properties and give examples.

1.1. von Neumann algebras with discrete decomposition. A von Neumann algebra with a normal faithful state \((\mathcal{N}, \varphi)\) has a discrete decomposition if the centralizer of the state \(\varphi\), \(N = N_\varphi \overset{\text{def}}{=} \{ x \in \mathcal{N} \mid \varphi(xy) = \varphi(yx), \forall y \in \mathcal{N} \}\), satisfies \(N' \cap \mathcal{N} = Z(\mathcal{N})\) and the modular automorphism group \(\sigma^\varphi\) associated with \(\varphi\) is almost periodic. The almost periodic spectrum of \(\sigma^\varphi\) is called the almost periodic spectrum of the discrete decomposition, and is denoted \(S(\mathcal{N}, \varphi)\). The centralizer algebra \(N\) is called the core of the discrete decomposition.

The discrete decomposition \((\mathcal{N}, \varphi)\) is factorial if \(N = N_\varphi\) is a factor. It is tracial if \(N = N'\) and it is of type III if \(\mathcal{N}\) is a type III von Neumann algebra.

1.1.1. A cross product form for discrete decompositions. The structure of a discrete decomposition \((\mathcal{N}, \varphi)\) can be made more specific as follows:
For each $\beta \in \mathbb{C}$ let $\mathcal{H}^0_\beta = \{ x \in \mathcal{N} \mid \varphi(xy) = \beta \varphi(yx), \forall y \in \mathcal{N} \}$. Then $S(\mathcal{N}, \varphi) = \{ \beta \mid \mathcal{H}^0_\beta \neq 0 \}$ and the vector space $\mathcal{H}^0_\beta$ can alternatively be described as $\{ x \in \mathcal{N} \mid \sigma^\varphi_t(x) = \beta^t x \}$. We have $\mathcal{H}^0_\beta \mathcal{H}^0_{\beta'} = \mathcal{H}^0_{\beta \beta'}$ and $(\mathcal{H}^0_\beta)^* = \mathcal{H}^0_{\beta^{-1}}$, with $\mathcal{H}^0_1 = \mathcal{N}_\varphi = \mathcal{N}$ and with all $\mathcal{H}^0_\beta$ being $\mathcal{N} - \mathcal{N}$ bimodules. In particular, $\Sigma_\beta \mathcal{H}^0_\beta$ is a dense $*$-subalgebra of $\mathcal{N}$, with $L^2(\mathcal{H}^0_\beta) = \overline{\mathcal{H}^0_\beta} \subset L^2(\mathcal{N}, \varphi)$, $\beta \in H(\mathcal{N}, \varphi)$, mutually orthogonal Hilbert spaces and $L^2(\mathcal{N}, \varphi) = \oplus_\beta L^2(\mathcal{H}^0_\beta)$. Thus, if $p_\beta$ denotes the projection onto the eigenspace $L^2(\mathcal{H}^0_\beta)$, then $\Sigma_\beta p_\beta = 1$.

Note that if $E = E_N$ denotes the $\varphi$-preserving conditional expectation of $\mathcal{N}$ onto $\mathcal{N}$, then $E(\mathcal{H}^0_\beta) = 0, \forall \beta \neq 1$. Also, the projection $p_1$ coincides with the projection $\epsilon_N$ implementing the expectation $E$ by $\epsilon_N x \epsilon_N = E(x) \epsilon_N$, $x \in \mathcal{N}$ ([T2]).

1.1.2. The normalizing groupoid. By the above properties, if $x \in \mathcal{H}^0_\beta$, for some $\beta \in H(\mathcal{N}, \varphi)$, and $x = w| x |$ is its polar decomposition, then $| x | \in \mathcal{N}$ and the partial isometry $w$ lies in $\mathcal{H}^0_\beta$. We denote by $\mathcal{G} \mathcal{V}_\beta(\mathcal{N}, \varphi)$ the set of partial isometries in $\mathcal{H}^0_\beta$ and note that if $v \in \mathcal{G} \mathcal{V}_\beta(\mathcal{N}, \varphi)$ then $v^* v, v v^* \in \mathcal{N}$, $\tau(v v^*)/\tau(v^* v) = \beta$, $v N v^* = v v^* N v v^*$, $\tau(v y v^*) = \tau(y)$, $\forall y \in v^* v N v^* v$, where $\tau = \varphi|_N$. Note also that $\mathcal{H}^0_\beta = \text{sp} \mathcal{G} \mathcal{V}_\beta(\mathcal{N}, \varphi)$.

1.1.3. Fundamental group of the core. In case the discrete decomposition $(\mathcal{N}, \varphi)$ is factorial, each non-zero $v \in \mathcal{G} \mathcal{V}_\beta(\mathcal{N}, \varphi)$ can be extended to either an isometry, if $\beta < 1$, or a co-isometry, if $\beta \geq 1$. Also, if $v \in \mathcal{G} \mathcal{V}_\beta(\mathcal{N}, \varphi)$ then $\mathcal{H}^0_\beta = \text{sp} N v N$.

If $(\mathcal{N}, \varphi)$ is both factorial and infinite dimensional, then $\mathcal{N}$ follows a factor of type $\Pi_1$ and any non-zero $v \in \mathcal{G} \mathcal{V}_\beta(\mathcal{N}, \varphi)$ implements an isomorphism $\theta_\beta$ from $\mathcal{N}$ onto the $\beta$-amplification $\mathcal{N}^\beta$ of $\mathcal{N}$, uniquely defined modulo perturbation by inner automorphisms implemented by unitaries in appropriate reduced algebras of $\mathcal{N}^\infty$. Thus, $S(\mathcal{N}, \varphi)$ is included in the fundamental group of $\mathcal{N}$, $\mathcal{F}(\mathcal{N})$. We denote by $\text{Aut}(\mathcal{N}; \mathcal{N})$ the set of such isomorphisms $\sigma_\beta$.

1.1.4. Discrete decompositions from trace scaling actions. All factorial discrete decompositions arise as follows (see [C2,3], [T3]): Let $(\mathcal{N}, \tau)$ be a type $\Pi_1$ factor with its unique trace, $Tr = \tau \otimes Tr_{\mathcal{B}(\ell^2 \mathbb{N})}$ the infinite trace on $\mathcal{N}^\infty = \mathcal{N} \otimes \mathcal{B}(\ell^2 \mathbb{N})$, $S \subset \mathcal{F}(\mathcal{N})$ a countable subgroup of the fundamental group of $\mathcal{N}$ and $\theta$ an action of $S$ on $\mathcal{N}^\infty$ by $Tr$-scaling automorphisms, i.e., such that $Tr \circ \theta_\beta = \beta Tr$, $\forall \beta \in H$. Let $\mathcal{N}^\infty = \mathcal{N}^\infty \rtimes_\theta H$ and $E^\infty$ the canonical conditional expectation of $\mathcal{N}^\infty$ onto $\mathcal{N}^\infty$. Let $q = 1 \otimes q_0$, for some one dimensional projection $q_0$ in $\mathcal{B}(\ell^2 \mathbb{N})$. Then $(\mathcal{N}, \varphi) = (q \mathcal{N}^\infty q, \tau \circ E^\infty|_\mathcal{N})$ has a discrete decomposition and $S = S(\mathcal{N}, \varphi), E = E^\infty|_\mathcal{N}$.

1.1.5. Discrete decomposition for weights. The following more general situation will be needed as well: Let $(\mathcal{C}, \phi)$ be a von Neumann algebra with a normal semifinite
faithful weight. We say that \((C, \phi)\) has a **discrete decomposition** if \(\phi\) is semifinite on the centralizer von Neumann algebra \(C_\phi = \{ x \in C \mid \sigma_t^\phi(x) = x, \forall t \in \mathbb{R} \}\) and if \((p^\mathcal{C}p, \phi(p \cdot p))\) has discrete decomposition for all projections \(p\) in \(C_\phi\) with \(\phi(p) < \infty\).

If this is the case, then we put \(S(C, \phi) \overset{\text{def}}{=} \bigcup_p S(p^\mathcal{C}p, \phi(p \cdot p))\).

We denote by \(H^0 = H^0(C, \phi)\) the Hilbert algebra \(\{ x \in \mathcal{C} \mid \phi(x^*x) < \infty, \phi(xx^*) < \infty \}\) and for each \(\beta \in S(C, \phi)\) we let \(H^0_\beta(C, \phi) \overset{\text{def}}{=} \{ x \in H^0 \mid \phi(xy) = \beta \phi(yx), \forall y \in \mathcal{H}^0 \}\). Note that \(x \in H^0_\beta\) iff \(x \in H^0\) and \(\sigma_t^\phi(x) = \beta^{it}x, \forall t\), and that \(H^0_1\) is a hereditary \(\ast\)-subalgebra of \(C_\phi\), while \(\Sigma^\beta H^0_\beta\) is a dense \(\ast\)-subalgebra of \(C\), with similar multiplicative properties and Hilbert structure as in the case \(\phi\) is a state \(\varphi\).

**1.2. Actions on discrete decompositions.** Let \((N, \varphi)\) be a von Neumann algebra with discrete decomposition and \((N, \tau) = (N_\varphi, \varphi|_{N_\varphi})\) its centralizer algebra, as in Section 1.1. Let \(\sigma : G \to \text{Aut}(N, \varphi)\) be a properly outer action of a discrete group \(G\), whose restriction to \((N, \tau)\) is still denoted \(\sigma\).

Denote \(\mathcal{M} = N \rtimes_\sigma G\) and \(M = N \rtimes_\sigma G\). We regard \(M\) as a subalgebra of \(\mathcal{M}\) in the natural way, with \(\{ u_g \}_{g \in G} \subset M \subset \mathcal{M}\) denoting the canonical unitaries simultaneously implementing the automorphisms \(\sigma_g\) on \(N, N\). We denote by \(L(G) \subset M \subset \mathcal{M}\) the von Neumann subalgebra they generate.

We still denote by \(\varphi\) the canonical extension of \(\varphi\) from \(N\) to \(\mathcal{M}\) and by \(E\) the \(\varphi\)-preserving conditional expectation of \(\mathcal{M}\) onto \(M\). Also, \(E\) will denote the canonical \(\varphi\)-preserving conditional expectation of \(\mathcal{M}\) onto \(N\).

Since the action \(\sigma\) on \(N\) is \(\varphi\)-invariant, it commutes with the corresponding modular automorphism group \(\sigma^\varphi\) on \(N\). Thus, \((\mathcal{M}, \varphi)\) has discrete decomposition, with core \(\mathcal{M}_\varphi = M\). Moreover, \(S(\mathcal{M}, \varphi) = S(N, \varphi), \mathcal{H}^0_\beta(\mathcal{M}) = (\Sigma^\beta \mathcal{H}^0_\beta(N)u_g)^e = \mathcal{P} \mathcal{H}^0_\beta(N)M\) and we have the non-degenerate commuting square:

\[
\begin{array}{c}
M \subset \mathcal{M} \\
\cup \cup E \\
N \subset N
\end{array}
\]

If \(\sigma\) is ergodic on the center of \(N\) (thus on the center of \(N\) too, since \(Z(N) \subset N' \cap N = Z(N)\)), then \(M = N \rtimes_\sigma G\) follows a factor and, by the remarks in 1.1, we have \(S(\mathcal{M}, \varphi) \subset \mathcal{F}(M)\).

**1.3. Basic construction for subalgebras of discrete decompositions.** We recall here the basic construction associated with subalgebras of discrete decompositions and explain how the corresponding extension algebras have discrete decomposition themselves.

**1.3.1. The case of subalgebras of the core.** Let \((\mathcal{M}, \varphi)\) be a von Neumann algebra
with discrete decomposition and let $B \subset M = M_\varphi$ be a von Neumann subalgebra. Let $E_B$ be the unique $\varphi$-preserving conditional expectation of $M$ onto $B$, with $e_B \in \mathcal{B}(L^2(\mathcal{M}, \varphi))$ the orthogonal projection of $L^2(\mathcal{M}, \varphi)$ onto $L^2(B, \varphi)$. Thus, $e_B$ implements $E_B$ on $M$ by $e_BXe_B = E_B(X)e_B$, $\forall X \in M$ ([T2]). We denote by $\langle \mathcal{M}, e_B \rangle$ the von Neumann algebra generated in $\mathcal{B}(L^2(\mathcal{M}, \varphi))$ by $\mathcal{M}$ and $e_B$ and call the inclusions $B \subset \mathcal{M} \subset \langle \mathcal{M}, e_B \rangle$ the basic construction for $B \subset \mathcal{M}$.

Since $e_B \langle \mathcal{M}, e_B \rangle e_B = B e_B$ and $e_B$ has central support 1 in $\langle \mathcal{M}, e_B \rangle$, $\langle \mathcal{M}, e_B \rangle$ is an amplification of $B$, thus being semifinite, and there exists a unique normal semifinite faithful trace $Tr$ on $\langle \mathcal{M}, e_B \rangle$ such that $Tr(be_B) = \varphi(b)$, $\forall b \in B$.

However, the “canonical” weight on $\langle \mathcal{M}, e_B \rangle$ is not $Tr$ but the following: Let $\Phi$ be the normal semifinite faithful operator valued weight of $\langle \mathcal{M}, e_B \rangle$ onto $\mathcal{M}$ determined by $\Phi(xeBy) = xy$, $x, y \in \mathcal{M}$, and denote $\phi = \varphi \circ \Phi$. Then $\phi$ is a normal semifinite faithful weight on $\langle \mathcal{M}, \varphi \rangle$ which satisfies $\phi(e_BY) = \phi(Ye_B)$, $\phi(xY) = \beta\phi(Yx)$ for all $x \in \mathcal{H}^0_{\beta}(\mathcal{M}, \varphi)$, $Y \in \text{sp}\mathcal{M}e_B\mathcal{M}$. Thus, the modular automorphism group $\sigma^e_t$ on $\langle \mathcal{M}, e_B \rangle$ is itself almost periodic and $(\langle \mathcal{M}, e_B \rangle, \phi)$ has discrete decomposition with the eigenspace $\mathcal{H}^0_{\beta}(\langle \mathcal{M}, e_B \rangle, \phi)$ being generated by $\Sigma_{\beta} \mathcal{H}^0_{\beta \phi e_B} \mathcal{H}^0_{\beta - 1}$. Moreover, noticing that $p_{\beta} \in \langle \mathcal{M}, e_B \rangle$, $\forall \beta$, $\phi$ is related to the trace $Tr$ by the formula:

$$\phi(\cdot) = \Sigma_{\beta} \phi(p_{\beta} \cdot p_{\beta}) = \Sigma_{\beta} \phi Tr(p_{\beta} \cdot p_{\beta})$$

1.3.2. The general case. Let now $(\mathcal{B}, \varphi)$ be a discrete decomposition with $\varphi$ a state and let $\mathcal{B}_0 \subset \mathcal{B}$ be a von Neumann subalgebra with the property that there exists a $\varphi$-preserving conditional expectation $E_0$ of $\mathcal{B}$ onto $\mathcal{B}_0$. Let $e_0$ be the orthogonal projection of $L^2(\mathcal{B}, \varphi)$ onto $L^2(\mathcal{B}_0, \varphi)$ and $\mathcal{C} = \langle \mathcal{B}, e_0 \rangle$ the von Neumann algebra generated by $\mathcal{B}$ and $e_0$ on $L^2(\mathcal{B}, \varphi)$. As before, $\Phi(xe_0y) = xy$ defines a normal semifinite faithful operator valued weight of $\mathcal{C}$ onto $\mathcal{B}$, with $\phi = \varphi \circ \Phi$ a normal semifinite faithful weight on $\mathcal{C}$. The inclusion $\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{C} = \langle \mathcal{B}, e_0 \rangle$, with the weight $\phi$ on $\mathcal{C}$, is called the basic construction for $\mathcal{B}_0 \subset \mathcal{B}$.

We have $\text{sp}\mathcal{C}e_0 \subset \mathcal{H}^0(\mathcal{C}, \phi)$ and since $\sigma^e_C = \sigma^e$ for all $\tau$, $x \in \mathcal{H}^0(\mathcal{B}, \varphi)e_0 \mathcal{H}^0(\mathcal{B}, \varphi)$ then $\sigma^e_t(x) = (\beta^t)\tau^t x$. This shows that the centralizer Hilbert-algebra $\mathcal{H}^0(\mathcal{C}, \phi)$ contains $\Sigma_{\beta} \mathcal{H}^0(\mathcal{B}, \varphi)e_0 \mathcal{H}^0(\mathcal{B}, \varphi)$, which has support 1 in $\mathcal{C}$, and thus $(\mathcal{C}, \phi)$ has discrete decomposition with the same almost periodic spectrum $S(\mathcal{C}, \phi)$ as $(\mathcal{B}, \varphi)$.

1.4. Gauged extensions for actions. Let $(N, \tau)$ be a finite von Neumann algebra with a normal, faithful tracial state. Let $G$ be an infinite discrete group and $\sigma : G \to \text{Aut}(N, \tau)$ an action of $G$ on $(N, \tau)$. A gauged extension for $\sigma$ is an action
\( \tilde{\sigma} : G \to \text{Aut}(\mathcal{N} \subset \tilde{N}, \tilde{\varphi}) \) together with a continuous action \( \alpha : \mathbb{R} \to \text{Aut} (\tilde{N}, \tilde{\varphi}) \) satisfying the conditions:

(1.4.1). \( (\mathcal{N} \subset \tilde{N}, \tilde{\varphi}) \) is an inclusion of von Neumann algebras such that if we denote \( \varphi = \tilde{\varphi}|_{\mathcal{N}} \) and \( \tilde{N} = \tilde{\mathcal{N}} \) then \( \mathcal{N} = \mathcal{N}_{\varphi} = \tilde{N} \cap \mathcal{N} \), \( \varphi|_{\mathcal{N}} = \tau \), and both \( (\mathcal{N}, \varphi) \), \( (\tilde{N}, \tilde{\varphi}) \) have discrete decomposition, with \( S(\mathcal{N}, \varphi) = S(\tilde{\mathcal{N}}) \), \( \mathcal{G} \mathcal{V}(\mathcal{N}, \varphi) \subset \mathcal{G} \mathcal{V}(\tilde{\mathcal{N}}, \tilde{\varphi}) \) and \( \tilde{\mathcal{N}} = \mathcal{g}_{\mathcal{G}} \mathcal{V}(\mathcal{N}, \varphi) \).

(1.4.2). \( \tilde{\sigma} : G \to \text{Aut}(\tilde{N}, \tilde{\varphi}) \) is an action such that \( \tilde{\sigma}_{g}(\mathcal{N}) = \mathcal{N} \), \( \forall g \), and \( \tilde{\sigma}|_{\mathcal{N}} = \sigma \).

(1.4.3). \( \alpha \) commutes with \( \tilde{\sigma} \) and satisfies the conditions:

(a) \( \mathcal{sp}^{w}\{ u \in \mathcal{U}(\alpha_{1}(\mathcal{N})) \mid u\mathcal{N}u^{*} = \mathcal{N}, d\varphi(u \cdot u^{*})/d\varphi, d\varphi(u^{*} \cdot u)/d\varphi < \infty \} = \alpha_{1}(\mathcal{N}) \)

(b) \( \mathcal{sp}^{w}\mathcal{N}\alpha_{1}(\mathcal{N}) = \tilde{\mathcal{N}}. \)

(c) \( \tilde{\varphi}(y_{1}\alpha_{1}(x)y_{2}) = \tilde{\varphi}(x)\tilde{\varphi}(y_{1}y_{2}), \forall x, y_{1,2} \in \mathcal{N} \)

We denote by \( S(\tilde{\sigma}) \) the common spectrum \( S(\mathcal{N}, \varphi) = S(\tilde{\mathcal{N}}, \tilde{\varphi}) \) and call it the almost periodic spectrum of the gauged extension \( \tilde{\sigma} \).

The gauged extension \( \tilde{\sigma} \) is tracial if \( \tilde{\mathcal{N}} = \tilde{\mathcal{N}} \) is a finite von Neumann algebra and \( \tilde{\varphi} \) is a trace (a fortiori extending the trace \( \tau \) of \( \mathcal{N} \)), equivalently if \( S(\tilde{\sigma}) = \{ 1 \} \). \( \tilde{\sigma} \) is of type III if \( \mathcal{N}, \tilde{\mathcal{N}} \) are of type III and it is factorial if both \( \mathcal{N}, \tilde{\mathcal{N}} \) are factors.

In Section 4 we will also need gauged extension having an additional symmetry: A graded gauged extension for \( \sigma \) is a gauged extension \( \tilde{\sigma} : G \to \text{Aut}(\mathcal{N} \subset \tilde{N}, \tilde{\varphi}) \), \( \alpha : \mathbb{R} \to \text{Aut}(\tilde{N}, \tilde{\varphi}) \) together with a period 2-automorphism \( \beta \in \text{Aut}(\tilde{N}, \tilde{\varphi}) \) satisfying:

(1.4.4) \( \mathcal{N} \subset \tilde{\mathcal{N}}^{\beta}, \beta \alpha_{t} = \alpha_{-t} \beta, \forall t. \)

Note that if \( \sigma_{i} : G_{i} \to \text{Aut}(N_{i}, \tau_{i}) \) has gauged extension \( (\tilde{\sigma}_{i}, \alpha_{i}) \), \( \forall i \in I \), then \( \otimes_{i} \tilde{\sigma}_{i} \) is a gauged extension for the action \( \otimes_{i} \sigma_{i} \) of \( \times_{i} G_{i} \) on \( (\otimes_{i} N_{i}, \otimes_{i} \tau_{i}) \), with gauge \( (\otimes_{i} \alpha_{i})_{t} = \otimes_{i} \alpha_{i}(t) \). Moreover, \( S(\otimes_{i} \tilde{\sigma}_{i}) \) is the multiplicative subgroup of \( \mathbb{R}_{+}^{*} \) generated by \( S(\tilde{\sigma}_{i}) \), \( i \in I \).

Also, if \( G_{0} \subset G \) is a subgroup and \( \tilde{\sigma} \) is a gauged extension for \( \sigma : G \to \text{Aut}(N, \tau) \), then \( \tilde{\sigma}|_{G_{0}} \) is a gauged extension for \( \sigma|_{G_{0}} \), with \( S(\tilde{\sigma}|_{G_{0}}) = S(\tilde{\sigma}) \).

1.5. Malleability and mixing conditions for actions. An action \( \sigma \) of a group \( G \) on a finite von Neumann algebra \( (N, \tau) \) is malleable (resp. s-malleable) if it has a gauged extension (resp. a graded gauged extension). The action \( \sigma \) is malleable
**mixing** (resp. **s-malleable mixing**) if it is mixing and has a (graded) gauged extension
\[ \tilde{\sigma} : G \to \text{Aut}(\mathcal{N} \subset \tilde{\mathcal{N}}, \tilde{\varphi}) \] with \( \tilde{\sigma} \) mixing, i.e.,
\[ \lim_{g \to \infty} \tilde{\varphi}(x \tilde{\sigma}_g(y)) = \tilde{\varphi}(x) \tilde{\varphi}(y), \forall x, y \in \tilde{\mathcal{N}} \]

By the remarks in 1.2, it follows that if \( \sigma_i : G \to \text{Aut}(N_i, \tau_i), i \in I, \) are malleable (resp. malleable mixing) actions of the same group \( G, \) then the diagonal product action \( \otimes_i \sigma_i : G \to \text{Aut}(\otimes_i N_i, \otimes_i \tau_i), \) defined by \( (\otimes_i \sigma_i)_g = \otimes_i \sigma_{i,g}, \ g \in G, \) is a malleable (resp. malleable mixing) action. Also, note that by ([C1]), if one of the \( \sigma_i \)'s is properly outer then the diagonal product action \( \otimes_i \sigma_i \) is properly outer.

**1.6. Examples of malleable mixing actions.** We show here that the commutative Bernoulli shifts with diffuse base space are s-malleable, while the Connes-Stormer Bernoulli shifts and the Bogoliubov shifts are all malleable mixing (see also [Po1]).

**1.6.1. Commutative Bernoulli shifts.** Let \( (Y_0, \nu_0) \) be a non trivial probability space. Denote \( (X, \mu) = \Pi_g (Y_0, \nu_0)_g \) the infinite product probability space, indexed by the elements of \( G. \) It is trivial to check that \( \sigma \) is mixing and properly outer. Such an action is called a **commutative** (or **classic**) Bernoulli shift.

Let us show that if the base space \( (Y_0, \nu_0) \) is diffuse, i.e. \((Y_0, \nu_0) = (\mathbb{T}, \lambda), \) then \( \sigma \) has a graded gauged extension (so in particular it is malleable mixing). Thus, put \( \mathcal{N} = N = L^\infty (X, \mu), \tau = \int \cdot d\mu \) and still denote by \( \sigma \) the action induced by the above \( \sigma \) on \( (N, \tau). \) Then define \( \tilde{\sigma}_g = \sigma_g \otimes \sigma_g. \) It is clearly an action of \( G \) on \( \tilde{\mathcal{N}} = N \otimes N \) that preserves \( \tilde{\tau} = \tau \otimes \tau. \) We show that it is a tracial graded gauged extension for \( \sigma \) when identifying \( N \) with \( N \otimes C \subset N \otimes N. \)

To see this, we first construct a continuous action \( \alpha : \mathbb{R} \to \text{Aut}(\tilde{\mathcal{N}}, \tilde{\tau}) \) commuting with \( \tilde{\sigma} \) and satisfying \( \alpha(1)(N \otimes \mathbb{C}) = C \otimes N. \) It is in fact sufficient to construct a continuous action \( \alpha_0 : \mathbb{R} \to \text{Aut}(\mathbb{A}_0 \otimes \mathbb{A}_0, \tau_0 \otimes \tau_0) \) such that \( \alpha_0(1)(\mathbb{A}_0 \otimes \mathbb{C}) = C \otimes \mathbb{A}_0, \) where \( (\mathbb{A}_0, \tau_0) = (L^\infty(\mathbb{T}, \lambda), \int \cdot d\lambda). \) Indeed, because then the product action \( \alpha(t) = \otimes_g (\alpha_0(t)_g \end{document}
Let $h \in \tilde{A}_0$ be a self-adjoint element such that $exp(2\pi i h) = u$. It is easy to see that for each $t$, $u$ and $exp(2\pi it) v$ is a pair of Haar unitaries. Denote by $\alpha_0(t)$ the automorphism $u \mapsto u, v \mapsto exp(2\pi it) v$. We then clearly have $\alpha_0(t_1) \alpha_0(t_2) = \alpha_0(t_1 + t_2), \forall t_1, t_2 \in \mathbb{R}$ and $\alpha_0(1) = uv$.

Finally, to construct the grading $\beta$, we let $\beta_0$ act on $\tilde{A}_0$ by $\beta(v) = v, \beta(u) = u^*$. Then clearly $\beta_0$ leaves $A_0$ pointwise fixed and $\beta_0 \alpha_0 t = \alpha_0_{-t} \beta_0$. Thus, if we take $\beta = \otimes_g (\beta_0)_g$ then condition (1.4.4) is satisfied, showing that $\sigma$ is s-malleable.

1.6.2. Non-commutative Bernoulli shifts. Let now $(\mathcal{N}_0, \varphi_0)$ be a von Neumann algebra with discrete decomposition and $(\mathcal{N}, \varphi) = \bigotimes_{g \in G} (\mathcal{N}_0, \varphi_0)_g$. It is easy to see that $(\mathcal{N}, \varphi)$ has itself a discrete decomposition, with spectrum $S(\mathcal{N}, \varphi) = S(\mathcal{N}_0, \varphi_0)$.

If $x = \otimes_g x_g \in \mathcal{N}$ and $h \in G$ then define $\sigma_h(x) = \otimes_g x'_g$, where $x'_g = x_{h^{-1}g}, \forall g$. $\sigma$ is then clearly an action of $G$ on $(\mathcal{N}, \varphi)$, called the $(\mathcal{N}_0, \varphi_0)$-Bernoulli shift. Let further $N = \{ x \in \mathcal{N} \mid \varphi(xy) = \varphi(yx), \forall y \in \mathcal{N} \}$ be the centralizer of the product state $\varphi$. Thus $\sigma_g(N) = N, \forall g$. The restriction of $\sigma$ to $N$, still denoted $\sigma$, is called the Connes-Størmer $(\mathcal{N}_0, \varphi_0)$-Bernoulli shift action of the group $G$.

Like in (1.6.1) the actions $\sigma$ of $G$ on $(\mathcal{N}, \varphi)$ and $N = N_\varphi$ are properly outer and mixing. Moreover, in case $(\mathcal{N}_0, \varphi_0)$ is an IPTF1 factor the action on $N$ follows malleable as well, as shown below (see also [Po1]):

If $(\mathcal{N}_0, \varphi_0) = (M_{k \times k}(\mathbb{C}), \varphi_0)$ for some $2 \leq k \leq \infty$, and $\varphi_0$ is the faithful normal state on $M_{k \times k}(\mathbb{C})$ of weights $\{ t_j \}_j$, then $\sigma$ is called the Connes-Størmer Bernoulli shift of weights $\{ t_j \}_j$. Note that in this case $S(\mathcal{N}, \varphi) = S(\mathcal{N}_0, \varphi_0)$ is equal to the multiplicative subgroup $S = S(\{ t_j \}_j) \subseteq \mathbb{R}^+$ generated by the ratios $\{ t_i/t_j \}_{i,j}$, called the ratio group of $\sigma$. Notice that $S$ is intrinsic to the construction of $\sigma$.

A gauged extension for $\sigma$ can be obtained as follows: Let $(\tilde{\mathcal{N}}, \tilde{\varphi}) = (\mathcal{N}, \varphi) \otimes (\mathcal{N}, \varphi)$ and define $\tilde{\sigma} : G \to \text{Aut}(\tilde{\mathcal{N}}, \tilde{\varphi})$ by $\tilde{\sigma}_g = \sigma_g \otimes \sigma_g$. Let $\{ e_{ij} \}_{i,j}$ be matrix units for $M_{k \times k}(\mathbb{C})$ chosen so that $\varphi_0$ is given by a trace class operator that can be diagonalized in $\text{Alg} \{ e_{ii} \}_i$. It is immediate to check that

$$\alpha_0(t) = \sum_i e_{ii} \otimes e_{ii}$$

$$+ \sum_{i<j} (\cos \pi t/2 (e_{ii} \otimes e_{jj} + e_{jj} \otimes e_{ii}) + \sin \pi t/2 (e_{ij} \otimes e_{ji} - e_{ji} \otimes e_{ij})).$$

defines a continuous action $\alpha_0$ of $\mathbb{R}$ on $M_{k \times k}(\mathbb{C}) \otimes M_{k \times k}(\mathbb{C})$ that leaves $\varphi_0 \otimes \varphi_0$ invariant and satisfies $\alpha_0(1)(M_{k \times k}(\mathbb{C}) \otimes \mathbb{C}) = \mathbb{C} \otimes M_{k \times k}(\mathbb{C})$. Thus, $\alpha_t = \otimes_g (\alpha_0(t))_g$ defines an continuous action $\alpha$ of $\mathbb{R}$ on $(\tilde{\mathcal{N}}, \tilde{\varphi})$ which commutes with $\tilde{\sigma}$ and satisfies $\alpha(1)(\tilde{\mathcal{N}} \otimes \mathbb{C}) = \mathbb{C} \otimes \tilde{\mathcal{N}}$.

It is straightforward to check that $\tilde{\sigma}, \alpha$ verify the conditions (1.4.1)-(1.4.3). Thus, $(\tilde{\sigma}, \alpha)$ is a gauged extension for $\sigma$. Since $\tilde{\sigma}$ is also mixing, $\sigma$ follows malleable mixing,
with $S(\tilde{\sigma}) = S$. Also, note that $\mathcal{N} = \mathcal{N}_\varphi$ is in this case isomorphic to the hyperfinite $\text{II}_1$ factor $R$.

Let now $S \subset \mathbb{R}_+^*$ be an arbitrary subgroup and for each $s \in S$ take $\phi_s$ to be the state on $M_{2 \times 2}(\mathbb{C})$ of weights $s(1 + s)^{-1}, (1 + s)^{-1}$ and let $(\mathcal{N}_0, \phi_0) = \bigotimes_{s \in S}(M_{2 \times 2}(\mathbb{C}), \phi_s)_s$. Then the Connes-Størmer $(\mathcal{N}_0, \varphi_0)$-Bernoulli shift action $\sigma$ of $G$ is malleable mixing and $S(\mathcal{N}_0, \varphi_0) = S$. Indeed, this is because each $(M_{2 \times 2}(\mathbb{C}), \varphi_s)$-Bernoulli shift is malleable mixing, so the observations in 1.5 apply.

1.6.3. Bogoliubov shifts. Following ([PSt]), we consider the following construction: Let $(\mathcal{H}, \pi)$ be a Hilbert space with a representation of $G$ on it. Assume $\pi$ satisfies

(a) $\pi(g) \notin \mathcal{C}1 + \mathcal{K}(\mathcal{H}), \forall g \in G, g \neq e$

(b) $\lim_{g \to \infty} \langle \pi(g)\xi, \eta \rangle = 0, \forall \xi, \eta \in \mathcal{H}$

An example of such a representation is the left regular representation of $G$ on $\mathcal{H} = \ell^2(G)$.

Let $\text{CAR}(\mathcal{H})$ be the CAR algebra associated with $\mathcal{H}$ and $\sigma = \sigma_\pi : G \to \text{Aut}(\text{CAR}(\mathcal{H}))$ be the action on the CAR algebra determined by $\pi$, i.e., $\sigma_g(a(\xi)) = a(\pi(g)(\xi)), \xi \in \mathcal{H}, g \in G$.

For a fixed $0 < t < 1$ denote by $\varphi$ the state on $\text{CAR}(\mathcal{H})$ determined by the constant operator $t1$ on $\mathcal{H}$ and define $(\mathcal{N}, \varphi)$ the von Neumann algebra coming from the GNS representation for $(\text{CAR}(\mathcal{H}), \varphi)$. It is easy to see that $(\mathcal{N}, \varphi)$ has factorial discrete decomposition, with $S(\mathcal{N}, \varphi) = \lambda \mathbb{Z}$, where $\lambda = t/(1 - t)$.

Also, since $\pi$ commutes with the constant operator $t1$, $\sigma$ leaves $\varphi$ invariant and thus can be uniquely extended to an action $\sigma : G \to \text{Aut}(\mathcal{N}, \varphi)$. Condition (a) implies $\sigma$ is outer on $(\mathcal{N}, \varphi)$ and condition (b) assures that $\sigma$ is mixing. Also, take $(\hat{\mathcal{N}}, \hat{\varphi})$ to be the GNS construction for $(\text{CAR}(\mathcal{H} \oplus \mathcal{H}), \varphi)$, where $\varphi$ is given by the same constant operator $t1$, but regarded on $\mathcal{H} \oplus \mathcal{H}$, then take $\tilde{\sigma} = \sigma_{\pi \oplus \pi}$ and $\alpha(t)$ be the action on $(\text{CAR}(\mathcal{H} \oplus \mathcal{H})$ implemented by the unitaries $\cos(\pi it)(e_{11} + e_{22}) + \sin(\pi it)(e_{12} - e_{21})$ on $\mathcal{H} \oplus \mathcal{H}$, where $e_{11}$ is the projection onto $\mathcal{H} \oplus 0$, $e_{22}$ the projection onto $0 \oplus \mathcal{H}$ and $\{e_{ij}\}_{i,j}$ are matrix units in the commutant of $\pi \oplus \pi$. It is straightforward to check that $(\tilde{\sigma}, \alpha)$ is a gauged extension for $\sigma|_\mathcal{N}$, with $\mathcal{N} = \mathcal{N}_\varphi$ isomorphic to the hyperfinite $\text{II}_1$ factor.

1.7. Cross-products associated with gauged extensions. Let $\sigma : G \to \text{Aut}(\mathcal{N}, \tau)$ be a properly outer action and $\tilde{\sigma} : G \to \text{Aut}(\mathcal{N} \subset \hat{\mathcal{N}}, \hat{\varphi})$ a gauged extension for $\sigma$ with gauge $\alpha$. In addition to the cross-product algebras $\mathcal{M} = \mathcal{N} \rtimes_\sigma G$
and \( M = N \times_\sigma G \) considered in 1.2, we let \( \tilde{\mathcal{M}} = \tilde{N} \times_\tilde{\sigma} G \) and regard both \( M, \mathcal{M} \) as subalgebras of \( \tilde{\mathcal{M}} \), with \( \{ u_g g \in G \} \subset M \subset \mathcal{M} \subset \tilde{\mathcal{M}} \) the canonical unitaries and \( L(G) \subset M \subset \mathcal{M} \subset \tilde{\mathcal{M}} \) the von Neumann subalgebra they generate. Also, we still denote by \( \tilde{\phi} \) the canonical extension of \( \phi \) from \( \tilde{N} \) to \( \tilde{\mathcal{M}} \) (thus \( \varphi = \tilde{\phi}|_{\mathcal{M}} \)). Note that since \( \alpha \) commutes with \( \tilde{\sigma} \), it implements a continuous action of \( \mathbb{R} \) on \( (\tilde{\mathcal{M}}, \tilde{\phi}) \), still denoted \( \alpha \).

By 1.2, both \( (\mathcal{M}, \varphi) \) and \( (\tilde{\mathcal{M}}, \tilde{\phi}) \) have factorial discrete decomposition, with cores \( \mathcal{M}_\varphi = M = N \times_\sigma G \) and respectively \( \tilde{\mathcal{M}}_{\tilde{\phi}} = \tilde{M} = \tilde{N} \times_{\tilde{\sigma}} G \), and with \( S(\tilde{\mathcal{M}}, \tilde{\phi}) = S(\mathcal{M}, \varphi) = S(\tilde{\sigma}) \). For each \( \beta \in S(\tilde{\sigma}) \) we denote \( \text{Aut}_\beta(M; \tilde{\sigma}) \) the set of isomorphisms \( \theta : M \cong M^\beta \) implemented by partial isometries in \( \mathcal{G} \mathcal{V}_\beta(\mathcal{M}, \varphi) \).

### 1.8. Basic constructions associated with gauged extensions

By (1.4.3) b), c) it follows that there exists a \( \tilde{\phi} \)-preserving conditional expectation \( \tilde{E}_1 \) of \( \tilde{\mathcal{M}} \) onto \( \mathcal{M}_1 = \alpha_1(\mathcal{M}) \), defined by \( \tilde{E}_1(x_0\alpha_1(x_1)u_g) = \tilde{\phi}(x_0)\alpha_1(x_1)u_g, \forall x_0, x_1 \in N, g \in G \). Also, if \( \tilde{e}_1 \) denotes the orthogonal projection of \( L^2(\tilde{\mathcal{M}}, \tilde{\phi}) \) onto \( L^2(\mathcal{M}_1, \tilde{\phi}) \subset L^2(\tilde{\mathcal{M}}, \tilde{\phi}) \) then \( \tilde{e}_1 X \tilde{e}_1 = \tilde{E}_1(X) \tilde{e}_1, \forall X \in \tilde{\mathcal{M}} \) (cf. [T2]).

Note that \( \tilde{E}_1|_{\mathcal{M}} \) coincides with the unique \( \varphi = \tilde{\phi}|_{\mathcal{M}} \)-preserving conditional expectation \( E_1 \) of \( \mathcal{M} \) onto \( L(G) \). Also, if we denote by \( (\tilde{\mathcal{M}}, \tilde{e}_1) \) the von Neumann algebra generated in \( L^2(\tilde{\mathcal{M}}, \tilde{\phi}) \) by \( \mathcal{M}_1 \subset \tilde{\mathcal{M}} \subset (\tilde{\mathcal{M}}, \tilde{e}_1) \) is a basic construction, in the sense of 1.3.2. Note that the von Neumann algebra \( (\mathcal{M}, \tilde{e}_1) \) defined by \( \tilde{\phi}(x\tilde{e}_1 y) = xy, x, y \in \tilde{\mathcal{M}} \), and \( \tilde{\phi} = \tilde{\phi} \circ \tilde{\Phi} \) then \( \tilde{\Phi}|_{(\mathcal{M}, e_1)} = \Phi \) and we have the following non-degenerate commuting squares:

\[
\begin{align*}
\mathcal{M}_1 & \xrightarrow{\tilde{E}_1} \tilde{\mathcal{M}} \\
\cup & \quad \cup \\
L(G) & \xrightarrow{\tilde{E}_1} \mathcal{M} \subset (\mathcal{M}, e_1)
\end{align*}
\]

Noticing that Takesaki’s criterion ([T2]) is trivially satisfied, it follows that there exists a unique \( \tilde{\phi} \)-preserving conditional expectation \( \mathcal{F} \) of \( (\tilde{\mathcal{M}}, \tilde{e}_1) \) onto \( (\mathcal{M}, e_1) \). The expectation \( \mathcal{F} \) takes \( \tilde{\mathcal{M}} \) (resp. \( \mathcal{M}_1 \)) onto \( \mathcal{M} \) (resp. \( L(G) \)), with \( \mathcal{F}|_{\tilde{\mathcal{M}}} \) (resp. \( \mathcal{F}|_{\mathcal{M}_1} \)) being the unique \( \tilde{\phi} \)-conditional expectation of \( \tilde{\mathcal{M}} \) (resp. \( \mathcal{M}_1 \)) onto \( \mathcal{M} \) (resp. \( L(G) \)).
2. Intertwining subalgebras in factors with discrete decomposition

Let \((\mathcal{M}, \varphi)\) be a von Neumann algebra with discrete decomposition. Thus, \(M = \mathcal{M}_\varphi\) is a finite von Neumann algebra with \(\tau = \varphi|_M\) its faithful trace and \(M' \cap \mathcal{M} = \mathbb{Z}(M)\). Let \(B_0 \subset fMf\), \(B \subset M\) be diffuse von Neumann subalgebras, for some projection \(f \in \mathcal{P}(M)\). We establish in this Section necessary and sufficient conditions for the existence of partial isometries in \(\mathcal{G}\mathcal{V}(\mathcal{M}, \varphi)\) that \(\text{"intertwine"}\) \(B_0\) with subalgebras of \(B\). To this end, we use the notations in 1.3. The proofs are reminiscent of the proofs of (Theorem A.1 in [Po3]) and (Lemmas 4, 5 in [Po4]).

2.1. Theorem. The following conditions are equivalent:

1°. There exists \(a \in B_0' \cap f(\mathcal{M}, e_B)f\), with \(a \geq 0\), \(a \neq 0\) and \(\phi(a) < \infty\).

2°. There exist \(\beta \in H(\mathcal{M}, \varphi)\) and a non-zero projection \(f_0 \in B_0' \cap \langle \mathcal{M}, e_B \rangle\) such that 
\[f_0 \leq p_\beta f\] and \(\text{Tr}(f_0) < \infty\).

3°. There exist \(\beta \in H(\mathcal{M}, \varphi)\), a projection \(q_0 \in B_0\), a non-zero partial isometry \(v \in \mathcal{G}\mathcal{V}_\beta(\mathcal{M}, \varphi)\) and a non-zero \(\xi \in q_0L^2(M)\) such that if we denote \(\xi_0 = \xi v\), then \(q_0B_0q_0\xi_0 \subset \overline{\xi_0B}\).

4°. There exist non-zero projections \(q \in B_0\), \(p \in B\), an (unital) isomorphism \(\psi\) of \(qB_0q\) into \(pBp\) and a non-zero partial isometry \(v_0 \in \mathcal{G}\mathcal{V}_\beta(\mathcal{M}, \varphi)\), for some \(\beta \in H(\mathcal{M}, \varphi)\), such that \(v_0v_0^* \in (qB_0q)\cap qMq\), \(v_0^*v_0 \in \psi(qB_0q)\cap pMp\) and \(xv_0 = v_0\psi(x), \forall x \in qB_0q\).

Proof of 4° \(\Rightarrow\) 1°. If \(v_0\) satisfies 4°, then \(qB_0q\) commutes with \(v_0e_Bv_0^*\), so that if \(v_1, v_2, ..., v_n\) are partial isometries in \(B_0\) with \(v_i^*v_i \leq q\) and \(\Sigma_i v_i^*v_i \in \mathcal{Z}(B_0)\) then \(a = \Sigma_i v_i(v_0e_Bv_0^*)v_i^* \in B_0' \cap \langle \mathcal{M}, e_B \rangle\), while still \(\phi(a) < \infty\) (because \(v_i \in B_0 \subset M\) are in the centerizer of \(\phi\)).

Proof of 1° \(\Rightarrow\) 2°. By 1.3 we have \(p_\beta ap_\beta \in B_0' \cap \langle \mathcal{M}, e_B \rangle_+\), \(\forall \beta \in H = H(\mathcal{M}, \varphi)\), and \(\phi(a) = \Sigma_\beta \phi(p_\beta ap_\beta)\). It follows that there exists \(\beta \in H\) such that \(p_\beta ap_\beta \neq 0\). This implies that all spectral projections \(e_t\) of \(p_\beta ap_\beta\) corresponding to intervals \([t, \infty)\) for \(t > 0\) lie in \(B_0' \cap \langle \mathcal{M}, e_B \rangle_+\) and satisfy \(e_t \leq p_\beta, \text{Tr}(e_t) = \beta^{-1}\phi(e_t) \leq (t\beta)^{-1}\phi(a) < \infty\). Thus, any \(f_0 = e_t \neq 0\) will satisfy 2°.

To prove 2° \(\Rightarrow\) 3° we need the following:

2.2. Lemma. Let \(L\) be a finite von Neumann algebra acting on the Hilbert space \(\mathcal{H}\) and assume its commutant \(L'\) in \(\mathcal{B}(\mathcal{H})\) is also finite. Let \(L_0 \subset L'\) be a von Neumann subalgebra. Then there exists a projection \(q \in L_0\) and \(\xi \in \mathcal{H}\) such that \(\xi = q\xi \neq 0\) and \(qL_0q\xi \subset \overline{L\xi}\).

Proof. If \(L\) is of type I then \(L_0\) and \(L'_0 \cap L'\) are both type I. Let \(q\) be a maximal abelian projection in \(L_0\) and \(q'\) a maximal abelian projection in \((L_0q)' \cap q'L'q\). Then \(qq'q'q'\) is abelian, implying that the commutant of \(Lqq'\) in \(\mathcal{B}(\mathcal{H})\) is abelian. Thus,
$Lqq'$ is cyclic in $qq'H$, i.e., there exists $\xi \in qq'H$ such that $\overline{L\xi} = qq'H$. In particular, $qL_0q\xi \subset L\xi$.

If $L$ has a type I summand, then by cutting with the support projection of that summand we may assume $L$ itself is type I and the first part applies.

If $L$ is of type $\Pi_1$ then let $a \in \mathcal{Z}(L) = \mathcal{Z}(L')$ be the coupling constant between $L$ and $L'$. Thus, $a$ is a positive unbounded operator affiliated with $\mathcal{Z}(L) = \mathcal{Z}(L')$. We have to prove that there exist $q \in \mathcal{P}(L_0)$, $q' \in \mathcal{P}(L_0' \cap L')$ such that $qq' \neq 0$ and $\text{ctr}_{L'}(qq') \leq a^{-1}$, where $\text{ctr}_{L'}$ is the central trace on $L'$. Indeed, for so then the coupling constant of $Lqq'$ on $qq'H$ is $\leq 1$ and $Lqq'$ is cyclic in $qq'H$.

If $e_{[0,1]}(a) \neq 0$ then the statement follows immediately, by taking $q' = e_{[0,1]}(a)$ and $q = 1$. Thus, we may assume $a \geq 1$ and by cutting with a projection in $\mathcal{Z}(L') \subset L_0' \cap L'$ we may also assume $a$ bounded. If $L_0$ (resp. $L_0' \cap L'$) has a type II summand, then by cutting with the support projection of that summand we may assume $L_0$ (resp. $L_0' \cap L'$) is of type $\Pi_1$ and then we can find projections $q \in L_0$ (resp. $q' \in L_0' \cap L'$) of arbitrary scalar central trace in $L_0$ (resp. $L_0' \cap L'$). But then the central trace of $q$ in $L'$ follows equal to that same scalar, thus $\leq a^{-1}$ when chosen sufficiently small.

If both $L_0, L_0' \cap L'$ are type I then, by cutting each one of these algebras by an abelian projection (like in the first part), we may assume both are abelian. This implies $L_0' \cap L'$ is a maximal abelian *-subalgebra of the type $\Pi_1$ von Neumann algebra $L'$. But then $L_0' \cap L'$ has projections of arbitrary scalar central trace in $L'$, by ([K2]). By choosing $q' \in \mathcal{P}(L_0' \cap L')$ of central trace $\leq a^{-1}$, we are done in this case too.

Q.E.D.

**Proof of 2° $\implies$ 3°.** Let $H_0 = f_0(L^2(M, \varphi)) \subset p_\beta(L^2(M, \varphi))$. Since $f_0$ commutes with $B_0$ and $Tr(f_0) < \infty$, it follows that $B_0 H_0 B = H_0$ and $JBJ' \cap B(H_0)$ is a finite von Neumann algebra. By replacing $H_0$ by $q(H_0)$ for some appropriate projection $q$ in $\mathcal{Z}(B_0)$, we may also assume $\dim(H_0)_B < \infty$, i.e., the central valued coupling constant of $JBJ$ in $B(H_0)$ is uniformly bounded.

By Lemma 2.2 there exists a projection $q_0 \in B_0$ and a non-zero vector $\xi_0 \in H_0$ such that $q_0 \xi_0 = \xi_0$ and $q_0 B_0 q_0 \xi_0 \subset \overline{\xi_0 B}$. Since $H_0 \subset L^2(H^0_\beta, \varphi)$, it follows that $\xi_0 = \xi v$ for some $\xi \in L^2(M, \tau)$ and $v \in G\beta(M, \varphi)$.

**Proof of 3° $\implies$ 4°.** Let $\xi_0 = \xi v$ with $\xi \in L^2(M) vv*$ regarded as a square summable operator affiliated with $M$. Note that $E_B(v^* \xi \xi v) \in L^1(B, \tau)_+$ and that $\xi_0 = \xi(v E_B(v^* \xi \xi v)^{-1/2} v^*) v = \xi_0 E_B(\xi_0^* \xi_0)^{-1/2}$ is still in $L^2(M) v$, satisfies $p_0 \overset{\text{def}}{=} E_B((\xi_0^* \xi_0) \xi_0) \in \mathcal{P}(B)$ and

$$q_0 B_0 q_0 \xi_0^* = q_0 B_0 q_0 \xi_0 E_B(\xi_0^* \xi_0)^{-1/2} \subset L^2(\xi_0 B) E_B(\xi_0^* \xi_0)^{-1/2} \overset{\text{def}}{=} L^2(\xi_0^* \xi_0)^{-1/2} B = L^2(\xi_0^* B).$$
Thus, by replacing $\xi_0$ by $\xi'_0$, we may assume $p_0 = E_B(\xi'_0\xi_0)$ is a projection in $B$. Also, if $\xi = \xi_0v^* \in L^2(M)$ then $vE_B(\xi_0\xi_0)v^* = E_B(\xi^*\xi) \in P(B)$.

Let $q \in q_0Bq_0$ be the minimal projection with the property that $(q_0 - q)\xi_0 = 0$. We denote $\psi(x) = E_B(\xi^*_0x\xi_0), x \in qBq_0$, and note that $\psi$ is a unital, normal, faithful, completely positive map from $qBq_0$ into $p_0Bp_0$.

Also, since $x\xi_0 \in L^2(\xi_0p_0Bp_0) = \xi_0L^2(p_0Bp_0)$, it follows that $x\xi_0 = \xi_0\psi(x), \forall x \in qBq_0$. Indeed, for if $x\xi_0 = \xi_0y$, for some $y \in L^2(p_0Bp_0)$, then $\xi_0x\xi_0 = \xi_0^*\xi_0y$ and so

$$\psi(x) = E_B(\xi^*_0x\xi_0) = E_B(\xi^*_0\xi_0y) = E_B(\xi^*_0\xi_0)y = y.$$ 

Thus, for $x_1, x_2 \in qBq_0$ we get $x_1x_2\xi_0 = x_1\xi_0\psi(x_2) = \xi_0\psi(x_1)\psi(x_2)$. Since we also have $(x_1x_2)\xi_0 = \xi_0\psi(x_1x_2)$, this shows that $\psi(x_1x_2) = \psi(x_1)\psi(x_2)$. Thus, $\psi$ is a unital $*$-isomorphism of $qBq_0$ into $p_0Bp_0$.

Thus, since $x\xi_0 = \xi_0\psi(x)$ and $\xi^*_0x = \psi(x)\xi^*_0, \forall x \in qBq_0$, it follows that $[qBq_0, \xi_0\xi^*_0] = 0$. Since $\xi_0 = \xi\psi, [qBq_0, \xi^*_0] = 0$ as well. Thus, if $w = (\xi\xi^*)^{-1/2}\xi$ then $w$ is a partial isometry in $M$ and $xwv = wv\psi(x), \forall x \in qBq_0$. Thus $v_0 = wv$ will do. Q.E.D.

2.3. Corollary. Assume condition 2.1.4° is not satisfied. (Note that this is the case if there exists no embedding $\theta : p_0Bp_0 \hookrightarrow B$, for non-zero $p_0 \in P(B_0)$.) Then we have:

$$(2.3.1) \quad \forall a_1, a_2, \ldots, a_n \in M = M_\varphi, \forall \varepsilon > 0, \exists u \in U(B_0), \|E_B(a_iua_j^*)\|_2 \leq \varepsilon, \forall i, j.$$ 

If in addition $M$ is finite, with $\varphi$ its trace, then conversely, (2.3.1) implies non-2.1.4°.

Proof. Assume by contradiction that there do exist $a_1, a_2, \ldots, a_n \in M$ and $c > 0$ such that $\Sigma_i\|E_B(a_iua_j^*)\|_2^2 \geq c$, $\forall u \in U(B_0)$, and let $b = \Sigma_i a_i^*e_Ba_i \in \text{sp}Me_BM \subset \langle M, e_B \rangle$. We then have the estimates:

$$Tr(bu^*bu) = \Sigma_{i,j} Tr(a_i^*e_Ba_iua_j^*e_B)$$

$$= \Sigma_{i,j} Tr(e_Ba_j^*u^*a_i^*e_Ba_iua_j^*e_B) = \Sigma_{i,j} Tr(E_B(a_j^*u^*a_i^*)E_B(a_iua_j^*))$$

$$= \Sigma_{i,j} \|E_B(a_iua_j^*)\|_2^2 \geq c,$$

for all $u \in U(B_0)$. Let then $a$ be the element of minimal norm $\| \cdot \|_{2,Tr}$ in the weak closure of $\{ubu^* | u \in U(B_0)\}$ in $\langle M, e_B \rangle$. Thus, $0 \leq a \leq 1, Tr(a) \leq Tr(b)$ and $a \in B'_0 \cap \overline{\text{sp}} Me_BM$. Also, $Tr(ba) \geq c > 0$, implying that $a \neq 0$. By $2° \implies 4°$ in Theorem 2.1, it follows that there exists an isomorphism $\theta : p_0Bp_0 \hookrightarrow B$, for some $p_0 \in P(B_0), p_0 \neq 0$, a contradiction.
To prove the converse, note that if 2.1.2 holds true and \( f_0 \in B'_0 \cap \angle M, e_B \) is a finite projection with \( \text{Tr}(f_0) < \infty \) then by (1.4 in [P5]) we may assume \( f_0 = \Sigma_j a_j e_B a_j^* \) for some finite set \( a_1, a_2, \ldots, a_n \in M \), which in turn implies

\[
\Sigma_{i,j} \|E_B(a_i u a_j^*)\|^2_2 = \text{Tr}(f_0 u f_0 u^*) = \text{Tr}(f_0), \forall u \in \mathcal{U}(B_0),
\]

thus contradicting (2.3.1). Q.E.D.

3. Controlling Intertwiners and Relative Commutants

Theorem 2.1 shows the importance of controlling intertwiners and relative commutants of subalgebras of a factor when having to decide whether the subalgebras are conjugate or not. We prove in this section two results along these lines:

3.1. Theorem. Let \((N, \varphi)\) be a von Neumann algebra with discrete decomposition, \(G\) an infinite discrete group and \(\sigma : G \to \text{Aut}(N, \varphi)\) a properly outer mixing action. If \(Q_0 \subset L(G)\) is a diffuse von Neumann subalgebra and \(x \in M = N \rtimes_\sigma G\) satisfies \(Q_0 x \subset \Sigma_i x_i L(G)\), for some finite set \(x_1, x_2, \ldots, x_n \in M\), then \(x \in L(G)\). In particular, \(Q_0' \cap M \subset L(G)\) and if \(v \in M\) is a partial isometry with \([v^* v, Q_0] = 0\) and \(vQ_0 v^* \subset L(G)\) then \(v \in L(G)\).

3.2. Theorem. Let \(\sigma : G \to \text{Aut}(N, \tau)\) be a malleable mixing action with gauge extension \(\tilde{\sigma} : G \to \text{Aut}(\tilde{N} \subset \tilde{N}, \tilde{\varphi})\). Denote \(\tilde{N} = \tilde{N}, M = N \rtimes_\sigma G, \tilde{M} = \tilde{N} \rtimes G\), as in 1.8. Let \(P_0 \subset M\) be a diffuse von Neumann subalgebra such that no corner of \(P_0\) can be embedded (non-unital) into \(N\). Then \(P_0' \cap \tilde{M} \subset M\).

Both these theorems will be derived from a general technical result. To state it we need some notations. Thus, we let \((\mathcal{T}, \varphi)\) be a von Neumann algebra with discrete decomposition, \(G\) an infinite discrete group, \(\sigma : G \to \text{Aut}(\mathcal{T}, \varphi)\) a properly outer action, \(\mathcal{B} = \mathcal{T} \rtimes_\sigma G\) its cross product algebra with canonical state \(\varphi\), as in 1.2. Let \(\mathcal{T}_0 \subset \mathcal{T}\) be a \(\sigma\)-invariant von Neumann subalgebra on which there exists a \(\varphi\)-preserving conditional expectation \(E_0\). Denote \(\mathcal{B}_0 = \mathcal{T}_0 \rtimes_\sigma G \subset \mathcal{B}\) and still denote by \(E_0\) the \(\varphi\)-preserving expectation of \(\mathcal{B}\) onto \(\mathcal{B}_0\) extending the expectation of \(\mathcal{T}\) onto \(\mathcal{T}_0\). Let \(\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{C} = \langle \mathcal{B}, e_0 \rangle\) be the basic construction for \(\mathcal{B}_0 \subset \mathcal{B}\) with its canonical weight \(\phi\), as in 1.3.2.

3.3. Proposition. Assume there exists \(\{b_n\}_n \subset \mathcal{T} \supset \mathcal{T}_0\) such that \(\{1\} \cup \{b_n\}_n\) is an orthonormal basis of \(\mathcal{T}\) over \(\mathcal{T}_0\) with each \(b_n\) in some \(\mathcal{H}_{\beta_n}^0(\mathcal{T}, \varphi)\), \(\forall n\), and such that the following condition is satisfied:

\[
(3.3.1) \lim_{g \to \infty} (\sup\|E_0(b_i^* y \sigma(g)(b_j))\|_\varphi \mid y \in \mathcal{N}_0, \|y\| \leq 1) = 0, \forall i, j.
\]
Let \( P_0 \subset (B_0)_\varphi \) be a diffuse von Neumann subalgebra satisfying the property:

\[(3.3.2) \quad \forall K \subset G \text{ finite, } \forall \delta > 0, \exists u \in \mathcal{U}(P_0) \text{ with } \|E(uu^*_h)\|_\varphi \leq \delta, \forall h \in K.\]

If \( \mathcal{H}_1^0 = \mathcal{H}_1^0(\mathcal{C}, \phi) \) denotes the centralizer Hilbert algebra (see 1.3.2) and \( a \in P'_0 \cap \mathcal{H}_1^0 \) then \( e_0ae_0 = a \).

To prove 3.3, we first need the following:

**3.4. Lemma.** Under the hypothesis of 3.3, for any \( n \) and any \( \varepsilon > 0 \) there exists a finite subset \( K \subset G \) and \( \delta > 0 \) such that if \( u \in \mathcal{U}(B_0) \) satisfies \( \|E(uu^*_h)\|_\varphi \leq \delta, \forall h \in K \), then \( \|E_0(b^*_iub_j)\|_\varphi \leq \varepsilon, \forall i, j \).

**Proof.** Let \( u = \Sigma g_yg_y \), with \( g_y \in T_0 \). Then \( E_0(b^*_iyg_yb_k) = E_0(b^*_iyg_y\sigma(b_j))ug \) implying that

\[
\|E_0(b^*_iub_j)\|_\varphi^2 = \Sigma_g \|E_0(b^*_iyg_y\sigma(b_j))\|_\varphi^2.
\]

By (3.3.1) there exists a finite subset \( K \subset G \) such that \( \sup\{\|E_0(b^*_iyg_y\sigma(b_j))\|_\varphi | y \in N_0, \|y\| \leq 1\} \leq \varepsilon/2, \forall g \in G \setminus K \). On the other hand, since the norm \( \| \cdot \|_\varphi \) implements the strong operator topology on the unit ball of \( M \) and the maps \( N_0 \ni y \mapsto b^*_iy\sigma(b_j) \) are continuous with respect to the strong operator topology, \( \forall i, j, \) it follows that there exists \( \delta > 0 \) such that if \( y \in N_0, \|y\| \leq 1, \|y\|_\varphi \leq \delta \), then \( \|E_0(b^*_iy\sigma(b_j))\|_\varphi \leq (2|K|)^{-1}\varepsilon \). Thus, if \( u \) satisfies \( \|uy\|_\varphi = \|E_0(uu^*_h)\|_\varphi \leq \delta, \forall h \in K \), then

\[
\|E_0(b^*_iub_j)\|_\varphi^2 = \Sigma_g \|E_0(b^*_iyg_y\sigma(b_j))\|_\varphi^2 + \Sigma_h \|E_0(b^*_iyh\sigma(b_j))\|_\varphi^2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \forall i, j.
\]

Q.E.D.

**Proof of Proposition 3.3.** Recall from 1.3.2 that \( \mathcal{H}_1^0 \) is hereditary and contains the \( * \)-algebra \( \Sigma_\beta \mathcal{H}_1^0(B, \varphi)e_0 \mathcal{H}_1^0(B, \varphi)^* \). For \( X \in \mathcal{H}_1^0(\mathcal{C}, \phi) \) (the Hilbert algebra of \( (\mathcal{C}, \phi) \)), as in 1.1.5 and 1.3.2), denote \( \|X\|_{2, \phi} = \phi(X^*X)^{1/2} \).

Since \( \mathcal{H}_1^0 = \mathcal{H}_1^0(\mathcal{C}, \phi) \) is a \( * \)-algebra and \( (1 - e_0)\mathcal{H}_1^0(1 - e_0) \subset \mathcal{H}_1^0 \), it follows that if \( (1 - e_0) \neq 0 \) (resp. \( a(1 - e_0) \neq 0 \)), then by replacing \( a \) by a spectral projection of \( (1 - e_0)aa^*(1 - e_0) \) (resp. \( (1 - e_0)a^*a(1 - e_0) \)) corresponding to some interval \([c, 1]\) with \( c > 0 \), we may assume \( a = f \neq 0 \) is a projection with \( f \leq 1 - e_0 \).

Let \( \varepsilon > 0 \). Since \( \{1\} \cup \{b_n\}_n \) is an orthonormal basis of \( \mathcal{T} \) over \( T_0 \), it is also an orthonormal basis of \( B \) over \( B_0 \). Thus, there exists \( n \) such that the orthogonal projection \( f_0 = \Sigma_{j \leq n} b_j e_0 b_j^* \) of \( L^2(B, \varphi) \) onto the closure of \( \Sigma_{j \leq n} b_j B_0 \) in \( L^2(B, \varphi) \) satisfies \( \|f_0 f - f\|_{2, \varphi} \leq \varepsilon/3 \). Since each \( b_i \) lies in some \( \mathcal{H}_1^0(B, \varphi) \), it follows that \( f_0 \) lies in \( \Sigma_\beta \mathcal{H}_1^0(B, \varphi)e_0 \mathcal{H}_1^0(B, \varphi)^* \subset \mathcal{H}_1^0(\mathcal{C}, \phi) \).
Thus, if \( u \in \mathcal{U}(P_0) \) then
\[
\|uf_0u^*f - f\|_{2,\phi} = \|(uf_0f - f)u^*\|_{2,\phi} = \|f_0f - f\|_{2,\phi} \leq \varepsilon/3.
\]
implying that
\[
\|uf_0u^*f - f_0f\|_{2,\phi} \leq 2\|f_0f - f\|_{2,\phi} \leq 2\varepsilon/3.
\]

Since \( f, f_0 \) are in the centralizer Hilbert algebra \( \mathcal{H}_1^0 \) and \( \phi(uxu^*) = \phi(x) \), \( \forall x \in \mathcal{H}_1^0, u \in P_0 \), by the Cauchy-Schwartz inequality we get
\[
|\phi(f_0uf_0u^*)| = |\phi(f_0uf_0u^*)| \leq \|f_0uf_0u^*\|_{2,\phi} = \|(uf_0f - f_0f)u^*\|_{2,\phi} = \|f_0f - f\|_{2,\phi} = \|f_0f - f\|_{2,\phi} = 2\|f_0f\|_{2,\phi}^2 + 2\phi(f)^{1/2}\phi(f_0uf_0u^*)^{1/2}.
\]

By Lemma 3.4 there exist a finite subset \( K \subset G \) and \( \delta > 0 \) such that if \( u \in \mathcal{U}(P_0) \) satisfies \( \|E(uxu^*)\|_{\phi} \leq \delta, \forall h \in K \), \( \forall 1 \leq i, j \leq n \) where \( \beta = \max_i\{\beta_i \mid 1 \leq i \leq n \} \). By condition (3.3.2) applied for this \( K \) and \( \delta \), there exists \( u \in \mathcal{U}(P_0) \) such that \( \|E(uxu^*)\|_{\phi} \leq \delta, \forall h \in K \).

Thus, if \( \|f\|_{2,\phi} \leq \|f - f_0f\|_{2,\phi} + \|ff_0\|_{2,\phi} \leq \varepsilon/3 + 2\varepsilon/3 = \varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, this shows that \( f = 0 \), a contradiction.

Q.E.D.

**Proof of Theorem 3.1.** Let \( L(G) \overset{E_1}{\to} \mathcal{M} \subset \langle \mathcal{M}, e_1 \rangle \) be the basic construction corresponding to the \( \varphi \)-preserving conditional expectation \( E_1 \) of \( \mathcal{M} \) onto \( L(G) \), like in 1.3.1, with \( E_1 = E_L(G) \) and \( e_1 = e_L(G) \). We also consider the partition of 1 given by the projections \( \{p_\beta\}_\beta \), as in 1.3.1.
Let $f_x$ be the orthogonal projection of $L^2(\mathcal{M}, \varphi)$ onto the closure $\mathcal{H}$ of $Q_0 xL(G)$ in $L^2(\mathcal{M}, \varphi)$. Since $Q_0 \mathcal{H}L(G) = \mathcal{H}$, $f_x \in Q_0 \cap \langle \mathcal{M}, e_1 \rangle$. Also, since $\mathcal{H}$ is contained in the closure of $\Sigma xL(G)$, it follows that $\mathcal{H}$ is a finitely generated right $L(G)$-Hilbert module. Thus $f_x$ lies in the ideal $\mathcal{J}_{e_1}$ generated in $\langle \mathcal{M}, e_1 \rangle$ by $e_1$. In particular $Tr(f_x) < \infty$, where $Tr$ is defined as in 1.3.1.

Since $1 \in L^2(\mathcal{M}, \varphi)$ is a separating vector for $\mathcal{M}$, to prove that $x \in L(G)$ it is sufficient to show that $f_x \leq e_1$.

Assuming $(1 - e_1)f_x(1 - e_1) \neq 0$, it follows that either $a = p_\beta f p_\beta \neq 0$, for some $\beta \neq 1$, or $a = (p_1 - e_1)f(p_1 - e_1) \neq 0$. Since $p_\beta \in \mathcal{M}' \cap \langle \mathcal{M}, e_1 \rangle$, it follows that $a \in Q_0 \cap \langle \mathcal{M}, e_1 \rangle$, $a \in \mathcal{J}_{e_1}$, Tr<sub>$a$</sub> $< \infty$. Also, since under $p_\beta$ the weight $\phi$ defined by $\phi(y_1 e_1^2) = \phi(y_1 y_2)$, $y_1, y_2 \in \mathcal{M}$ is proportional to Tr<sub>$a$</sub> (cf 1.3.1), it follows that $a$ is in the centralizer of $\phi$. Thus, all spectral projections $f$ of $a$ corresponding to intervals of the form $[c, \infty)$ for $c > 0$ will satisfy $f \in Q_0 \cap \langle \mathcal{M}, e_1 \rangle$, $f \in \mathcal{J}_{e_1}$, $\phi(f) < \infty$, $f \leq 1 - e_1$ and $f$ in the centralizer of $\phi$.

Thus, if we apply Proposition 3.3 for $\mathcal{T} = N$, $\mathcal{T}_0 = C$, $\mathcal{T}_1 = N'$ and $\mathcal{B}_0 = B_0 = L(G)$, then we get $f = 0$, a contradiction. Q.E.D.

Proof of Theorem 3.2. Put $\mathcal{T} = \mathcal{N}$, $\mathcal{B} = \mathcal{M}$, $\mathcal{T}_0 = \mathcal{N}$, $\mathcal{T}_0 = N$, $\mathcal{B}_0 = \mathcal{M}$, $\mathcal{B}_0 = M$. Conditions (b) and (c) of (1.4.3) show the existence of a $\varphi$-preserving conditional expectation $E_0$ of $\mathcal{B} = \mathcal{M}$ onto $\mathcal{B}_0 = \mathcal{M}$. On elements of the form $y_1 \alpha_1(y_2)$ with $y_1, y_2 \in \mathcal{N}$, which by (1.4.3) are total in $\mathcal{N}$, it acts by $E_0(y_1 \alpha_1(y_2)) = \varphi(y_2) y_1$.

By Corollary 2.3, condition (3.3.2) is satisfied. Let us show that there exists an orthonormal basis $\{1\} \cup \{b_n\}_n$ of $\mathcal{T} = \mathcal{N}$ over $\mathcal{T}_0 = \mathcal{N}$ verifying (3.3.1). By the definition of $E_0$, any $\{b^0_n\}_n \subset \cup \beta_\alpha \mathcal{H}_0^0(\mathcal{N}, \varphi)$ with $\varphi(b^0_n) = 0$, $\varphi(b^0_n \cdot b^0_n) = 0$, $\forall i, j, n$, and sp\{$\{1\} \cup \{b^0_n\}_n$\} total in $\mathcal{N}$ gives an orthonormal basis of $\mathcal{B}$ over $\mathcal{B}_0$ by letting $b_n = \alpha_1(b^0_n)$. To get such a set $\{b^0_n\}_n$ start with a total subset $1 = a_0, a_1, a_2, \ldots \in \cup \beta \mathcal{H}_0^0(\mathcal{N}, \varphi)$ then apply the Gram-Schmidt algorithm with respect to the scalar product $\langle \cdot, \cdot \rangle_\varphi$.

We next show that $\{b_n\}_n$ this way defined automatically satisfies (3.3.1). By condition (1.4.3), (a) and Kaplanski’s density theorem, there exist some finite, self-adjoint set of unitary elements $v_k \in \mathcal{U}_1 = \{v \in \mathcal{U}(\alpha_1(\mathcal{N})) \mid vNv^* = \mathcal{N}, d\varphi(v \cdot v^*)/d\varphi, d\varphi(v^* \cdot v)/d\varphi < \infty\}$ and scalars $c^2_k$ such that $b^j_j = \Sigma_k c^2_k v_k$ satisfy $\|b^j_j\| \leq \|b_j\|$ and $\|b^j_j\| \varphi \leq \varepsilon(1 + 2\max_j \|b_j\|)^{-1}$. Since $\|E_0(x_1 x_2)\| \varphi \leq \|x_1 x_2\| \varphi \leq \|x_1\| \varphi \|x_2\| \varphi$, $\forall x_1, x_2 \in \mathcal{B}$, for $y \in \mathcal{N}$ with $\|y\| \leq 1$ we get:

$$\|E_0(b^j_j y \sigma_g(b_j))\| \varphi \leq \|E_0(b^j_j y \sigma_g(b^j_j))\| \varphi + \|E_0(b^j_j y \sigma_g(b_j - b^j_j))\| \varphi \leq \Sigma_k c^2_k \|E_0(b^j_j y \sigma_g(v_k))\| \varphi + \|b_j\| \|b_j - b^0_j\| \varphi \leq \Sigma_k c^2_k \|E_0(b^j_j y \sigma_g(v_k))\| \varphi + \varepsilon/2.$$
where $C = \max\{|c_i^j|\}_{j,k}$. But if $v_k' = \sigma_g(v_k)$ then $b_i^r y v_k' = b_i^r v_k' (v_k' y v_k')$ and $v_k' y v_k' \in \mathcal{N}$. Thus, $E_0(b_i^r y v_k') = \tilde{\varphi}(b_i^r v_k') v_k' y v_k'$. Thus, if we denote $c = \max\{|d\varphi(v_k')/d\varphi\}_{k}$ then $|E_0(b_i^r y v_k')|_\varphi \leq c|\tilde{\varphi}(b_i^r v_k')|$ and from the above estimates we get:

$$
\|E_0(b_i^r y b_j)\|_\varphi \leq cC \Sigma_k|\tilde{\varphi}(b_i^r \sigma_g(v_k))| + \varepsilon/2.
$$

But $\tilde{\sigma}$ is mixing on $\tilde{\mathcal{N}}$, in particular on $\alpha_1(\tilde{\mathcal{N}})$. Thus, since $\tilde{\varphi}(b_i) = 0$, we get

$$
\lim_{g \to \infty} \tilde{\varphi}(b_i^r \sigma_g(v_k)) = 0, \forall i, k, showing that (3.3.1) holds true.
$$

Now, since for all $u \in \mathcal{U}(P_0' \cap \tilde{\mathcal{M}})$ we have $ue_0 u^* \in P_0' \cap \mathcal{B}, e_0$ and $ue_0 u^*$ is in the centralizer of $\phi$, we can apply Proposition 3.3 to get $ue_0 u^* \leq e_0$. By the faithfulness of $\phi$ this implies $ue_0 u^* = e_0$, or $ue_0 = e_0 u$. Applying this equality to the separating vector 1 in $L^2(\tilde{\mathcal{M}}, \tilde{\varphi})$, this gives $E_0(u) = u$. But since $E_0$ is $\tilde{\varphi}$-preserving, it takes the centralizer of $\tilde{\varphi}$ into the centralizer of $\varphi$ on $\mathcal{M}$, i.e., $E_0(\mathcal{M}) = \mathcal{M}$. This yields $E_0(P_0' \cap \tilde{\mathcal{M}}) \subset P_0' \cap \mathcal{M}$, thus $u = E_0(u) \in P_0' \cap \mathcal{M}$.

We end this Section with a technical lemma needed in the proof of the main result in the next section.

3.5. Lemma. Let $Q \subset P$ be an inclusion of finite von Neumann algebras and $q \in \mathcal{P}(Q), q' \in \mathcal{P}(Q' \cap P)$.

1° If $Q$ is quasi-regular in $P$ then $qQq'$ is quasi-regular in $qq'Pqq'$ (see 1.4.2 in [Po3] for the definition of quasi-regular subalgebras).

2° If $Q$ is regular in $P$ and $q \in \mathcal{P}(Q)$ satisfies $tr_Q(q) = cz$, for some scalar $c$ and central projection $z \in Z(Q)$, then $qQq$ is regular in $Pq$.

Proof. 1°. We may clearly assume $P$ has a normal faithful trace $\tau$. It is then sufficient to prove that $\forall x \in qN_P(Q)$ and $\varepsilon > 0 \exists z \in Z(Q)$ such that $\tau(1 - z) \leq \varepsilon$ and $qq'zqq'z \in qN_{qq'Pqq'}(qQq)$.

Let $x_1, \ldots, x_n \in P$ be so that $Qx \in \Sigma_i x_i Q$ and $xQ \subset \Sigma_i Q x_i$.

For the given $\varepsilon > 0$ there exists $z \in Z(Q)$ and finitely many partial isometries $v_1, v_2, \ldots, v_m \in Q$ such that $\tau(1 - z) \leq \varepsilon, v_i^* v_j \leq q$ and $\Sigma_j v_i v_j' = z$. If we let $\{y_k\}_k$ be a relabelling of the finite set $\{qq'z x_i v_i q q'\}_{i,j} \cup \{qq' z v_i' x_i q q' z\}_{i,j}$, then we have

$$(qq'q q') (qq'z x q q') \subset \Sigma_k y_k (qq'q q'),$$

$$(qq'z x q q')(qq'Q q q') \subset \Sigma_k (qq'Q q q') y_k.$$

2°. If $q = z \in Z(Q)$ then $\forall u \in N_P(Q), v = zuz$ follows a partial isometry with $vv^*, v^* v \in Z(Q)$ and $vQv^* = vv^* Qv v^*$. The proof of (2.1 in [JPo]) shows that $v$ can be extended to a unitary element in $zPz$ normalizing $Qz$. Thus, $zN_P(Q)z \subset N_{zPz}(Qz)'$, implying that $zPz = \mathcal{P}^w(zN_P(Q)z) \subset N_{zPz}(Qz)'$, i.e., $N_{zPz}(Qz)'' = zPz$. 

If $\text{ctr}_Q(q) = cz$ for some scalar $c$ and $z \in Z(Q)$, then by the first part $Qz$ is regular in $zPz$. This reduces the general case to the case $q \in P$ has scalar central trace in $Q$. But then, if $u \in N_P(Q)$ we have $\text{ctr}(uq^*) = ctr(q)$ so there exists $v \in U(Q)$ such that $vuq^*v^* = q$, implying that $q(vu)q \in qPq$ is a unitary element in the normalizer $N_{qPq}(qQ)$. Thus, $\text{sp}qN_{qPq}(qQ)Q \supset \text{sp}N_P(Q)$.

This yields $\text{sp}qN_{qPq}(qQ)Q \supset qN_{qPq}(qQ)$ and since the right hand term generates $qPq$ while the left hand one is generated by $N_{qPq}(qQ)$, we get $N_{qPq}(qQ)'' = qPq$.

Q.E.D.

4. Rigid embeddings into $N \rtimes_\sigma G$ are absorbed by $L(G)$

In this section we prove a key rigidity result for inclusions of algebras of the form $L(G) \subset M = N \rtimes_\sigma G$, in the case $\sigma : G \to \text{Aut}(N,\tau)$ is a malleable mixing action. Thus, we show that if $Q \subset M$ is a diffuse, relatively rigid von Neumann subalgebra whose normalizer in $M$ generates a factor $P$ (see Section 4 in [Po3] for the definition of relatively rigid subalgebras), then $Q$ and $P$ are “absorbed” by $L(G)$, via automorphisms of $M$ coming from given gauged extensions of $\sigma$.

The proof follows an idea from ([Po1]): Due to malleability, the algebra $M = N \rtimes_\sigma G$ can be perturbed continuously via the gauge action, leaving only $L(G)$ fixed, thus forcing any relatively rigid subalgebra $Q$ of $M$ to sit inside $L(G)$ (modulo some unitary conjugacy). The actual details of this argument will require the technical results from the previous sections. In particular, in order to apply Corollary 3.3 we’ll need $P$ not embeddable into $N$.

An example when $P$ is not embeddable into $N$ is when $Q$ is already rigid in $P$ and $N$ has Haagerup’s compact approximation property (cf. 5.4.1° in [Po3]; for the definition of Haagerup’s property for algebras see [Cho], or 2.0.2 in [Po3]). In particular this is the case if $N$ is approximately finite dimensional (AFD), or if $N = L(\mathbb{F}_n)$ for some $2 \leq n \leq \infty$. Another example of this situation is when $N$ is abelian.

4.1. Theorem. Let $M$ be a factor of the form $M = N \rtimes_\sigma G$, for some malleable mixing action $\sigma$ of a discrete ICC group $G$ on a finite von Neumann algebra $(N,\tau)$. Assume $Q \subset M^\sigma$ is a diffuse, relatively rigid von Neumann subalgebra such that $P = N_{M^\sigma}(Q)''$ is a factor. Assume also that no corner of $P$ can be embedded (non-unitaly) into $N$. If $\tilde{\sigma}$ is a gauged extension for $\sigma$, then there exist a unique $\beta \in H(\tilde{\sigma})$ and a unique $\theta_\beta \in \text{Aut}_\beta(M;\tilde{\sigma})$ such that the isomorphism $\theta_\beta : M^\sigma \simeq M^{s\beta}$ satisfies $\theta_\beta(P) \subset L(G)^{s\beta}$.

N.B.: The uniqueness of $\theta_\beta$ is modulo perturbations from the left by inner automorphisms implemented by unitaries from $L(G)^{s\beta}$.
Proof. The uniqueness is trivial by Theorem 3.1. We split the proof of the existence into seven Steps. For the first six Steps, we assume \( s = 1 \). Then in Step 7, we use the case \( s = 1 \) to settle the general case.

**Step 1.** \( \exists \delta > 0 \) such that \( \forall t > 0, t \leq \delta, \exists w(t) \in \tilde{M}, w(t) \neq 0 \), satisfying \( w(t)y = \alpha_t(y)w(t), \forall y \in Q \).

Let \( \tilde{\sigma} : G \to \text{Aut}(\mathcal{N} \subset \tilde{\mathcal{N}}, \tilde{\varphi}) \) be the given gauged extension with gauge \( \alpha : \mathbb{R} \to \text{Aut}(\tilde{\mathcal{N}}, \tilde{\varphi}) \). With the notations in 1.7, \( \alpha \) implements a continuous action of \( \mathbb{R} \) on the discrete decomposition \( (\tilde{\mathcal{M}}, \tilde{\varphi}) \). In particular, this action implements a continuous action of \( \mathbb{R} \) on the type \( II_1 \) factor \( \tilde{M} \), still denoted \( \alpha \). Thus, \( \lim_{t \to 0} \| \alpha_t(x) - x \|_2 = 0 \), \( \forall x \in \tilde{M} \).

Since \( Q \subset M \) is rigid, \( Q \subset \tilde{M} \) is also rigid, so that there exists \( \delta > 0 \) such that if \( |t| \leq \delta \) then \( \| \alpha_t(u) - u \|_2 \leq 1/2, \forall u \in \mathcal{U}(Q) \). Let \( a(t) \) be the unique element of minimal norm \( \| \cdot \|_2 \) in \( \overline{\mathcal{w}}w^{\oplus} \{ \alpha_t(u)u^* | u \in Q \} \). Since \( \| \alpha_t(u)u^* - 1 \|_2 \leq 1/2, \forall u \in \mathcal{U}(Q) \), we have \( \| a(t) - 1 \|_2 \leq 1/2 \), thus \( a(t) \neq 0 \).

By the uniqueness of \( a(t) \) we have \( \alpha_t(u)a(t)u^* = a(t), \forall u \in \mathcal{U}(Q) \). Thus, \( a(t)y = \alpha_t(y)a(t), \forall y \in Q \), which implies \( a(t)^*a(t) \in Q' \cap \tilde{M} \) and \( a(t)a(t)^* \in \alpha_t(Q') \cap \tilde{M} \).

Thus, if we denote by \( w(t) \) the partial isometry in the polar decomposition of \( a(t), w(t) = a(t)(|a(t)|)^{-1} \), then \( w(t) \neq 0 \) and \( w(t)y = \alpha_t(y)w(t) \).

**Step 2.** If \( t, w = w(t) \neq 0 \) are such that \( wy = \alpha_t(y)w, \forall y \in Q \), then there exists a partial isometry \( w' \in \tilde{M} \) such that \( w'y = \alpha_t(y)w', \forall y \in Q \), and \( \tau(w'w^*) > \tau(w^*)^2/2 \).

To prove this note first that if \( v \in \tilde{M} \) is a unitary element normalizing \( Q \) and if \( \sigma_v \) denotes the automorphism \( v \cdot v^* \) on \( Q \), then \( \alpha_t(v)wv^* \) satisfies

\[
(\alpha_t(v)wv^*)y = \alpha_t(v)ws_v(y)v^*
\]

\[
= \alpha_t(v)\sigma_v(y)wv^* = \alpha_t(v\sigma_v(y))wv^*
\]

\[
= \alpha_t(v\sigma_v(y)v^*)\alpha_t(v)wv^* = \alpha_t(y)(\alpha_t(v)wv^*).
\]

We claim there exists \( v \) in the normalizer \( \mathcal{N}_{\tilde{M}}(Q) \) of \( Q \) in \( \tilde{M} \) such that

\[
\tau(vwv^*v^\alpha_t^{-1}(w^*w)) > \tau(w^*)^2/2.
\]

Indeed, if we would have \( \tau(vwv^*v^\alpha_t^{-1}(w^*w)) \leq \tau(w^*)^2/2, \forall v \in \mathcal{N}_{\tilde{M}}(Q) \), then the element \( h \) of minimal \( \| \cdot \|_2 \) in \( \overline{\mathcal{w}}w^{\oplus} \{ v(wv^*)v^* | v \in \mathcal{N}_{\tilde{M}}(Q) \} \) would satisfy \( 0 \leq h \leq 1, h \in \mathcal{N}_{\tilde{M}}(Q)' \cap \tilde{M}, \tau(h) = \tau(w^*) \) and \( \tau(h\alpha_t^{-1}(w^*w)) \leq \tau(w^*)^2/2 \). But \( \mathcal{N}_{\tilde{M}}(Q)' \cap \tilde{M} \subset P' \cap \tilde{M} \), and by Corollary 3.3 the latter equals \( P' \cap M \). Since
by hypothesis one has \( P' \cap M = \mathbb{C} \), this shows that \( h \in \mathcal{N}_M(Q)' \cap \hat{M} = \mathbb{C} \). Thus we get
\[
\tau(w w^*)^2 = \tau(h)\tau(\alpha_t^{-1}(w^* w)) \leq \tau(w w^*)^2/2,
\]
a contradiction.

For such \( v \in \mathcal{N}_M(Q) \), let \( a = \alpha_t(\alpha_t(v) \overline{w} \overline{w}^*) w \) and note that \( ay = \alpha_{2t}(y) a, \forall y \in Q \).
Moreover, the left support projection of \( a \) has trace \( \tau(v w w^* \overline{v} \overline{w}^* \overline{w}^* w) \) \( \geq \tau(w w^*)^2/2 \). Thus, if we take \( w' \) to be the partial isometry in the polar decomposition of \( a, w' = a|a|^{-1}, \) then \( w'y = \alpha_{2t}(y) w' \), \( \forall y \in Q \). Also, since \( ||a|| \leq 1 \), we have \( \tau(w' w^*) > \tau(w w^*)^2/2 \).

**Step 3.** There exists a non-zero partial isometry \( w_1 \in \hat{M} \) such that \( w_1y = \alpha_1(y) w_1, \forall y \in Q \).

To prove this, let first \( w \) be a non-zero partial isometry in \( \hat{M} \), satisfying \( \tau(w y) = \alpha_{2n}(y) w, \forall y \in Q \), for some large \( n \geq 1 \), as given by **Step 1**. Set \( v_0 = w \). By **Step 2** and induction, there exist partial isometries \( v_k \in \hat{M}, k = 0, 1, 2, \ldots, \) such that \( v_k y = \alpha_{2n+k}(y) v_k, \forall y \in Q \), and \( \tau(v_k v_k^*) > \tau(v_{k-1} v_{k-1}^*)^2/2, \forall k \geq 1 \). Taking \( w_1 = v_n \), it follows that \( w_1y = \alpha_1(y) w_1, \forall y \in Q \) and \( \tau(w_1 w_1^*) > \tau(w w^*)^2/2^{2n-1} \neq 0 \).

**Step 4.** With the notations in 1.8, there exists a positive, non-zero element \( b \in Q' \cap \langle M, e_1 \rangle \) such that \( Tr(b) < \infty \).

Indeed, if \( w_1 \in \hat{M} \) as given by **Step 3**, then \( w_1^* \tilde{e}_1 w_1 \) is a non-zero positive element in \( spM_1 \tilde{e}_1 M_1 \subset \langle \hat{M}_1, \tilde{e}_1 \rangle \) that commutes with \( Q \) and satisfies \( 0 \neq \tilde{\phi}(w_1^* \tilde{e}_1 w_1) < \infty \). Define \( b = \mathcal{F}(w_1^* \tilde{e}_1 w_1) \in \langle \hat{M}, \tilde{e}_1 \rangle \simeq \langle M, e_1 \rangle \). Then \( b \neq 0 \) and \( 0 \leq b \leq 1 \). Since \( \mathcal{F} \) is \( \tilde{\phi} \)-preserving, \( \phi(b) = \tilde{\phi}(b) \leq 1 \).

**Step 5.** There exist projections \( q \in Q, q' \in Q' \cap P \) and a partial isometry \( v_0 \in \mathcal{G} \mathcal{V}_\beta(M, \varphi) \) for some \( \beta \in H(\hat{\sigma}) \), such that \( v_0 v_0^* = qq' \) and \( v_0^* P v_0 \subset L(G) \).

By \( 1^\circ \implies 4^\circ \) in Theorem 2.1, there exist non-zero projections \( q \in Q, p \in L(G), \) an isomorphism \( \psi \) of \( qQQ \) into \( pL(G)p \) and a non-zero partial isometry \( v_0 \in \mathcal{G} \mathcal{V}_\beta(M, \varphi) \), for some \( \beta \in H(\hat{\sigma}) \), such that \( v_0 v_0^* \in (qQQ)' \cap qMq, v_0^* v_0 \in \psi(qQQ)' \cap pMp \) and \( xv_0 = v_0 \psi(x), \forall x \in qQQ \).

Since \( \psi(qQQ) \) is a diffuse von Neumann subalgebra in \( pL(G)p \), by Theorem 3.1 it follows that \( \psi(qQQ)' \cap pMp \subset pL(G)p \), showing that
\[
v_0^* Q v_0 = v_0^* qQQ q' v_0 = \psi(Q)v_0^* v_0 \subset L(G).
\]

But since \( v_0 v_0^* \in (qQQ)' \cap qMq, \) it follows that \( v_0 v_0^* = qq' \) for some \( q' \in Q' \cap M \subset \mathcal{N}_M(Q)' = P \). Thus, \( qq' \in P \) as well. By Lemma 3.7, \( qQQ'q \) is quasi-regular in \( qq'Pqq' \), implying that \( v_0^* Q v_0 \) is quasi-regular in \( v_0^* P v_0 \). By Theorem 3.1, this shows that \( v_0^* P v_0 \subset L(G) \).
Note that, since $P' \cap M = C$ and since we did not use up to now the condition that $G$ is ICC, the rest of the conditions in the hypothesis of 4.1 are sufficient to imply that $G$ has finite radical (see Theorem 4.4 below).

Step 6. End of the proof of the case $s = 1$.

With the notations in Step 5, since $\text{Ad}^*_{\sigma} \in \text{Aut}_{\hat{\beta}}(M; \hat{\sigma})$ and since $P$ and $L(G)$ are factors, it follows that there exists an appropriate amplification $\theta_{\beta}: M \simeq M^\beta$ of $\text{Ad}^*_{\sigma}$ such that $\theta_{\beta}(P) \subset L(G)^\beta$.

Step 7. Proof of the general case.

For general $s$, note first that $P = N_M(Q)'''$ being a factor and $Q$ being diffuse, for any $1 \geq t > 0$ there exists a projection in $q \in Q$ of trace $\tau(q) = t$ and either $q \in Z(Q)$ (in case $Q$ is type I homogeneous) or $\text{ctr}_Q(q) = c_1$ (in case $Q$ is of type II$_1$).

Thus, by replacing $P$ by a factor $P_0$ of the form $M_{n \times n}(qPq) \simeq P^{1/s} \subset (M^s)^{1/s} = M$, for some $t, n$ with $tn = s^{-1}$, and $Q$ by its subalgebra $Q_0 = D_n \otimes qQq$, where $D_n \subset M_{n \times n}(C)$ is the diagonal subalgebra, by Lemma 3.5 we get a subfactor $P_0$ of $M$ with $Q_0 \subset P_0$ a diffuse von Neumann subalgebra such that $N_M(Q_0)'' = P_0$, while $Q_0 \subset M$ still a rigid inclusion (the latter due to 4.4 and 4.5 in [Po2]). The first part applies to get $\theta_{\beta} \in \text{Aut}_{\beta}(M; \hat{\sigma})$ such that $\theta_{\beta}(P_0) \subset L(G)^\beta$ and since $(P \subset M^s) = (P_0 \subset M)^s$, an appropriate $s$-amplification of $\theta_{\beta}$ carries $P$ into $L(G)^s\beta$.

Q.E.D

Note that for commutative Bernoulli shifts (1.6.1) we could only prove malleability in the case the base space $(Y_0, \nu_0)$ has no atoms. To prove that 4.1 holds true for all commutative Bernoulli shifts $\sigma$ and under much weaker conditions on $Q$, we consider the following “malleability-type” condition:

4.2. Definition. Let $(N_1, \tau_1)$ be a diffuse abelian von Neumann algebra and $\sigma_1$ an action of $G$ on $(N_1, \tau_1)$. $\sigma_1$ is sub malleable (resp. sub s-malleable) if it can be extended to a malleable (resp. s-malleable) action $\sigma$ of $G$ on a larger abelian von Neumann algebra $(N, \tau)$ such that there exists an orthonormal basis $\{1\} \cup \{b_i\}_i \subset N$ of $N$ over $N_1$ satisfying

\[
(4.2.1) \quad \lim_{g \to \infty} (\sup\{\|E_{N_1}(b_i^* y \sigma(g)(b_j))\|_\varphi \mid y \in N_1, \|y\| \leq 1\}) = 0, \forall i, j
\]

Recall that by (1.6.1), classical Bernoulli $G$-actions with non-atomic (diffuse) base space are s-malleable mixing. We next show that Bernoulli $G$-actions with arbitrary base are sub s-malleable mixing:
4.3. Lemma. Let \( \sigma_1 \) be the Bernoulli shift action of \( G \) on \( (X, \mu) = \Pi_g(Y_0, \nu_0)_g \), where \((Y_0, \nu_0)\) is an arbitrary (possibly atomic) non-trivial standard probability space. Then the action \( \sigma_1 \) it induces on \( L^\infty(X, \mu) \) is sub s-malleable mixing.

Proof. Denote \( A_0^0 = L^\infty(Y_0, \nu_0) \) and consider the embedding \( A_0^0 \subset A_0^0 \otimes L^\infty(T, \lambda) \simeq L^\infty(T, \lambda) = A_0^0 \). If \( z \in L^\infty(T, \lambda) \) is the Haar generating unitary and \( u = 1 \otimes z \in A_0^0 \) then \( \{u^n\}_n \) is an orthonormal basis of \( A_0^0 \) over \( A_0^0 \). Let \( N_1 = \overline{\otimes}_g(A_0^0)_g, N = \overline{\otimes}_g(A_0^0)_g \) and denote \( \{b_n\}_n \subset N \) the set of elements with \( b_n = \otimes(u^{n_g})_g, n_g \in \mathbb{Z} \) all but finitely many equal to 0.

It is immediate to see that \( \{b_n\}_n \) is an orthonormal basis of \( N \) over \( N_1 \) that checks condition (4.2.1) with respect to the Bernoulli shift action \( \sigma \) of \( G \) on \( N = \overline{\otimes}_g(A_0^0)_g \).

Since \( \sigma \) extends the Bernoulli shift action \( \sigma_1 \) of \( G \) on \( N_1 = \overline{\otimes}_g(A_0^0)_g \) and since by (1.6.1) it has mixing graded gauged extensions, \( \sigma_1 \) follows sub s-malleable mixing. Q.E.D.

4.4. Theorem. Let \( M_1 \) be a factor of the form \( M_1 = N_1 \rtimes_{\tau_1} G \), for some free action \( \tau_1 \) of a discrete group \( G \) on an abelian von Neumann algebra \( (N_1, \tau_1) \). Let \( Q \subset M_1^{\tau_1} \) be a diffuse, relatively rigid von Neumann subalgebra and denote \( P_1 = \mathcal{N}(Q)^{\tau} \) the von Neumann algebra generated by its normalizer in \( M_1^{\tau_1} \).

(i). If \( \tau_1 \) is sub malleable and \( P_1 \) has atomic center then \( G \) has finite radical and for any minimal projection \( p_1 \) in the center of \( P_1 \) there exists a minimal projection \( p \) in the center of \( L(G) \) and a unitary element \( u \in M_1^{\tau_1} \) such that \( u(P_1p_1)^{t_1}u^* \subset (L(G)p)^{st} \), where \( t_1 = \tau(p_1)^{-1}, t = \tau(p)^{-1} \). If moreover \( P_1 \) is a factor and \( G \) is ICC then there exists \( u \in \mathcal{U}(M_1^{\tau_1}) \) such that \( uP_1u^* \subset L(G)^{\tau} \). Also, \( u \) is unique with the above property, modulo perturbation from the left by a unitary in \( L(G)^{\tau} \).

(ii). If \( \tau_1 \) is sub s-malleable, \( Q \) is of type \( \Pi_1 \) and \( G \) is ICC then there exists \( u \in \mathcal{U}(M_1^{\tau_1}) \) such that \( uP_1u^* \subset L(G)^{\tau} \), unique modulo perturbation from the left by a unitary in \( L(G)^{\tau} \).

Proof. Let \( \tau : G \to \text{Aut}(N, \tau) \) be a malleable mixing extension of \( \tau_1 \), which satisfies (4.2.1) and which is either malleable, in the case (i), or s-malleable, in the case (ii). Denote \( M = N \rtimes_{\tau} G \). We first show that in both cases we have \( P_1' \cap M^{\tau} = P_1' \cap M_1^{\tau} \).

One has in fact the following more general result:

4.5. Lemma. Let \((N_1, \tau_1) \subset (N, \tau)\) be an embedding of finite von Neumann algebras for which there exists an orthonormal basis \( \{1\} \cup \{b_n\}_n \) of \( N \) over \( N_1 \) satisfying condition (4.2.1). Let \( \tau : G \to \text{Aut}(N, \tau) \) be a properly outer mixing action leaving \( N_1 \) globally invariant. Denote \( M_1 = N_1 \rtimes G \), regarded as a von Neumann subalgebra of \( M = N \rtimes G \). Assume \( P_1 \subset M_1 \) is so that no corner of \( P_1 \) can be embedded into \( N \). If \( x \in M \) satisfies \( P_1x \subset \Sigma_i x_i M_1 \), for some finite set \( x_1, x_2, \ldots, x_n \in M \), then \( x \in M_1 \). In particular, \( P_1' \cap M = P_1' \cap M_1 \).
Proof. If we put \( T_0 = N_1, T = N, B_0 = M_1, B = M \) then condition (4.2.1) on \( \{ b_n \}_n \) shows that the hypothesis of Proposition 3.3 is satisfied. Thus, if \( E_0 \) denotes the trace preserving conditional expectation of \( M \) onto \( M_1 \) and \( M_1 \underset{E_0}{\subset} M \subset \langle M, e_0 \rangle \) the corresponding basic construction then any \( a \in P'_1 \cap \langle M, e_0 \rangle \) with \( a \geq 0, Tr(a) < \infty \) satisfies \( e_0 a e_0 = a \).

But if \( x \in M \) satisfies the condition in the hypothesis then the orthogonal projection \( f_x \) of \( L^2(M) \) onto the closure \( H \) of \( sp P_1 x M_1 \) in \( L^2(M) \) then \( f_x \in P'_1 \cap \langle M, e_0 \rangle \) (because \( PH M_1 \subset H \)) and \( \dim H M_1 \leq n < \infty \) (because \( H \) is contained in the right \( M_1 \)-Hilbert module \( \Sigma_{j} x_j M_1 \subset L^2(M) \)). Thus, \( Tr(f_x) < \infty \), implying that \( f_x \leq e_0 \), equivalently \( H \subset L^2(M_1) \). In particular \( x \in L^2(M_1) \), thus \( x \in M_1 \). Q.E.D.

Proof of (i). \( N \) being abelian and \( P_1 \) of type II1, it follows that no corner of \( P_1 \) can be embedded into \( N \). By 4.5 this implies \( P'_1 \cap M^* = P'_1 \cap M^*_1 = \mathbb{C} \).

By 3.5.2° a suitable amplification \( P \) of the factor \( P_t P_1 \) is a unitaly embedded into \( M \) and contains a diffuse subalgebra \( Q \subset P \) such that \( Q \subset M \) is rigid and \( N_1(Q)'' = P \). Since \( \sigma \) is malleable mixing and has gauged extension with trivial spectrum (the gauged extension is even abelian), by the first 5 Steps of the proof of 4.2 and the remark at the end of Step 5, \( G \) has finite radical and there exists a non-zero partial isometry \( v_0 \in M \) such that \( v_0^* v_0 \in P, v_0 P v_0^* \subset L(G) \subset M_1 \). A suitable amplification \( u \in M^* \) of \( v_0 \), will then satisfy the conjugacy condition and by 4.5 \( u \in M^*_1 \). The uniqueness is clear by 4.5.

Proof of (ii). Let \( \tilde{\sigma} : G \rightarrow Aut(\tilde{N}, \tilde{\tau}), \alpha : \mathbb{R} \rightarrow Aut(\tilde{N}, \tilde{\tau}), \beta \in Aut(\tilde{N}, \tilde{\tau}), \beta^2 = id \), give a graded gauged extension for \( \sigma \). Denote \( \tilde{M} = \tilde{N} \rtimes_{\tilde{\sigma}} G \).

We first prove that there exists a non-zero partial isometry \( w \in \tilde{M} \) such that \( w^* w \in Q' \cap M, w w^* \in \alpha_1(Q' \cap M), w y = \alpha_1(y) w, \forall y \in Q \).

To this end, note first that Step 1 of the proof of Theorem 4.1, which only used the fact that \( \sigma \) is malleable, shows that for all \( t \) sufficiently small there exists a non-zero partial isometry \( v = v(t) \in \tilde{M} \) such that \( v^* v \in Q' \cap \tilde{M}, v_v^* \in \alpha_t(Q') \cap \tilde{M}, v y = \alpha_t(y) v, \forall y \in Q \). But since \( Q \) is of type II1, no corner of \( Q \) can be embedded into \( N \). Thus, by Theorem 3.2 we have \( Q' \cap \tilde{M} = Q' \cap M \) and so we get:

\[
(4.4.1) \quad v^* v \in Q' \cap M, v v^* \in \alpha_t(Q' \cap M), v y = \alpha_t(y) v, \forall y \in Q.
\]

Assuming now that for some \( 0 < t < 1 \) there exists a partial isometry \( v \in \tilde{M} \) such that \( v^* v \in Q' \cap M, v v^* \in \alpha_t(Q' \cap M) \) and \( v y = \alpha_t(y) v, \forall y \in Q \), we show that there exists a partial isometry \( v' \in \tilde{M} \) satisfying \( \| v' \|_2 = \| v \|_2, v' v^* v' \in Q' \cap M, v' v'^* \in \alpha_t(Q' \cap M) \) and \( v' y = \alpha_{2t}(y) v', \forall y \in Q \). This will of course prove the existence of \( w \), by starting with some \( t = 2^{-n} \) for \( n \) sufficiently large, then proceeding by induction until we reach \( t = 1 \).
Applying \( \beta \) to \( vy = \alpha_t(y)v \) and using that \( \beta(x) = x, \forall x \in M \) and \( \beta\alpha_t = \alpha_{-t}\beta \), we get \( \beta(v)y = \alpha_{-t}(y)\beta(v), \forall y \in Q \). Taking adjoints and plugging in \( y^* \) for \( y \) we further get \( y\beta(v^*) = \beta(v^*)\alpha_{-t}(y), \forall y \in Q \). Thus:

\[
\alpha_t(y)v\beta(v^*) = vy\beta(v^*) = v\beta(v^*)\alpha_{-t}(y), \forall y \in Q.
\]

(4.4.2) \[\alpha_t(y)v\beta(v^*) = vy\beta(v^*) = v\beta(v^*)\alpha_{-t}(y), \forall y \in Q.\]

If we now apply \( \alpha_t \) to the last and first term of (4.4.2) and denote \( v' = \alpha_t(v\beta(v^*)) \), then we get:

\[
v'y = \alpha_{2t}(y)v', \forall y \in Q.
\]

(4.4.3) \[v'y = \alpha_{2t}(y)v', \forall y \in Q.\]

Moreover, since \( v^*v \in Q \subset M \), we have \( \beta(v^*v) = v^*v \) so \( v\beta(v^*) \) is a partial isometry and it has the same range as \( v \). Thus \( v' \) is a partial isometry and \( \|v'\|_2 = \|v\|_2 \). Since \( v' \) is an intertwiner, \( v^*v' \in Q' \cap M = Q' \cap M \) and similarly \( v'v^* \in \alpha_{2t}(Q' \cap M) \).

With the non-zero partial isometry \( w \in \hat{M} \) satisfying \( wQv, \forall y \in Q \) we can apply Steps 4 and 5 in the proof of 4.2 to get a non-zero partial isometry \( v_0 \in M \) such that \( v_0^*v_0 \in Q' \cap M \) and \( v_0Qv_0^* \subset L(G) \).

To end the proof we use a maximality argument. Thus, we consider the set \( \mathcal{W} \) of all families \( \{\{p_i\}, u\} \) where \( \{p_i\} \) are partitions of 1 with projections in \( Q' \cap M \), \( u \in M \) is a partial isometry with \( u^*u = \sum p_i \) and \( u(\Sigma Qp_i)u^* \subset L(G) \). We endow \( \mathcal{W} \) with the order given by \( \{\{p_i\}, u\} \leq \{\{p'_j\}, u'\} \) if \( \{p_i\} \subset \{p'_j\} \), \( u = u'(\Sigma p_i) \).

(\( \mathcal{W}, \leq \)) is clearly inductively ordered.

Let \( \{\{p_i\}, u\} \) be a maximal element. If \( u \) is a unitary element, then we are done. If not, then denote \( q' = 1 - \sum p_i \in Q' \cap M \) and take \( q \in Q \) such that \( \tau(qq') = 1/n \) for some integer \( n \geq 1 \). Denote \( Q_0 = M_{n \times n}(qqq') \) regarded as a von Neumann subalgebra of \( M \), with the same unit as \( M \). By ([Po3]), it follows that \( Q_0 \subset M \) is rigid. Thus, by the first part there exists a non-zero partial isometry \( w \in M \) such that \( w^*w \in Q_0' \cap M \) and \( wQ_0w^* \subset L(G) \). Since \( qq' \in Q_0 \) has scalar central trace in \( Q_0 \), it follows that there exists a non-zero projection in \( w^*wQ_0w^* \) majorised by \( qq' \) in \( Q_0 \).

It follows that there exists a non-zero projection \( q_0 \in qq'Q_0qq' = qqq' \) and a partial isometry \( w_0 \in M \) such that \( w_0^*w_0 = q_0 \) and \( w_0(qqq')w_0^* \subset L(G) \). Moreover, by using the fact that \( Q \) is diffuse, we may shrink \( q_0 \) if necessary so that to be of the form \( q_0 = q_1q' \neq 0 \) with \( q_1 \in \mathcal{P}(Q) \) of central trace equal to \( m^{-1}z \) for some \( z \in \mathcal{Z}(Q) \) and \( m \) an integer. But then \( w_0 \) trivially extends to a partial isometry \( w_1 \in M \) with \( w_1^*w_1 = q'z \in Q' \cap M \) and \( w_1Qw_1^* \subset L(G) \). Moreover, since \( L(G) \) is a factor, we can multiply \( w_1 \) from the left with a unitary element in \( L(G) \) so that \( w_1w_1^* \) is perpendicular to \( uu^* \). But then \( \{\{p_i\}, u + w_1 \} \), where \( u_1 = u + w_1 \), is clearly in \( \mathcal{W} \) and is (strictly) larger than the maximal element \( \{\{p_i\}, u\} \), a contradiction. Q.E.D.
Theorems 4.1, 4.4 allow in fact the control of inclusions \( \theta_\beta(P) \subset L(G)^{s\beta} \) from above as well:

**4.6. Corollary.** Let \( G \) be a discrete ICC group, \( \sigma : G \to \text{Aut}(N, \tau), M = N \rtimes_\sigma G, \) \( s > 0 \) and \( Q \subset M^* \) a diffuse, relatively rigid von Neumann subalgebra, with \( P = N_{M^*}(Q)^{\prime\prime} \) a factor. Assume that either \( \sigma \) is malleable mixing with \( P \) not embeddable into \( N \), or that \( N \) is abelian and \( \sigma \) is sub malleable mixing. If \( P \supset L(G)^s \) then \( P = L(G)^s \), and thus \( Q \subset L(G)^s \) as well.

**Proof.** Since \( P, L(G)^s \) are factors, it is sufficient to prove that there exists a non-zero projection \( p \in \mathcal{P}(L(G)^s) \subset \mathcal{P}(P) \) such that \( pPp \subset L(G) \).

In case \( \sigma \) is malleable mixing, by Theorem 4.1 it follows that if \( \tilde{\sigma} \) is a gauged extension for \( \sigma \) then there exists \( \beta \in S(\tilde{\sigma}) \) and \( \theta_\beta \in \text{Aut}_\beta(M; \tilde{\sigma}) \) such that \( \theta_\beta(P) \subset L(G)^{s\beta} \). In particular, since \( P \supset L(G)^s \) this implies \( \theta_\beta(L(G)^s) \subset L(G)^{s\beta} \). But then Theorem 3.1 implies \( \beta = 1 \) and that \( \theta_\beta \) is actually implemented by a unitary element in \( L(G)^s \). The case \( N \) abelian, \( \sigma \) sub-malleable follows similarly, using 4.4 (a) instead of 4.1. Q.E.D.

5. **Strong rigidity of the inclusions** \( L(G) \subset N \rtimes_\sigma G \)

The “absorption” results in the previous Section allow us to prove here that any isomorphism between amplifications of factors of the form \( N \rtimes_\sigma G \), with \( G \) satisfying a weak rigidity property, \( N \) approximately finite dimensional (or merely having Haagerup’s property) and \( \sigma \) malleable mixing, can be perturbed by an automorphism coming from a given gauged extension so that to carry the subalgebras \( L(G) \) onto each other. We consider the following terminology:

**5.1. Definitions.**

1°. Given a group \( \Gamma \), an infinite subgroup \( \Lambda \subset \Gamma \) is wq-normal in \( \Gamma \) if for any intermediate subgroup \( \Lambda \subset H \subsetneq \Gamma \) there exists \( g \in \Gamma \setminus H \) such that \( gHg^{-1} \cap H \) is infinite. In particular, if there exist subgroups \( \Lambda = \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_n = \Gamma \) with \( \Lambda_i \subset \Lambda_{i+1} \) normal \( \forall 0 \leq i \leq n - 1 \), then \( \Lambda \) is wq-normal in \( \Gamma \). An example of a non wq-normal inclusion of groups is \( \Lambda \subset \Gamma = \Lambda \ast K_0 \) with \( K_0 \) non trivial. However, \( \Lambda \subset (\Lambda \ast K_0) \times K \) is a wq-normal inclusion whenever \( \Lambda, K \) are infinite groups.

2°. A discrete group \( \Gamma \) is w-rigid if it contains an infinite normal subgroup with the relative property (T) of Kazhdan-Margulis ([Ma]; see also [dHV])). Also, we denote by \( wT_0 \) the class of groups \( \Gamma \) having a non virtually abelian wq-normal subgroup \( \Lambda \subset \Gamma \) with the relative property (T).

If an infinite group \( H \) has the property (T) of Kazhdan ([Ka]) then any group \( G \) having \( H \) as a normal subgroup is w-rigid. For instance \( G = H \rtimes K \) for some
arbitrary group \( K \) acting on \( H \) by (possibly trivial) automorphisms. Other examples of w-rigid groups are the groups \( G = \mathbb{Z}^2 \rtimes \Gamma \) with \( \Gamma \subset SL(2, \mathbb{Z}) \) non-amenable (cf. [Ka], [Ma], [Bu]), and \( G = \mathbb{Z}^m \rtimes \Gamma \) with \( \Gamma \) arithmetic lattice in \( SO(n, 1) \) or \( SU(n, 1) \), for suitable \( m \) and suitable actions of \( \Gamma \) on \( \mathbb{Z}^m \) (cf. [Va]). Also, note that the class \( wT_0 \) is closed to normal and finite index extensions and to inductive limits.

5.2. Theorem. Let \( G_i \) be w-rigid ICC groups, \( \sigma_i : G_i \to \text{Aut}(N_i, \tau_i) \) malleable mixing actions with gauged extensions \( \tilde{\sigma}_i \) and \( M_i = N_i \rtimes \sigma_i G_i \), \( i = 0, 1 \). Let \( \theta : M_0 \simeq M_1^s \) be an isomorphism, for some \( s > 0 \). Then there exist unique \( \beta_i \in S(\tilde{\sigma}_i) \) and unique \( \theta_{\beta_i}^1 \in \text{Aut}_{\beta_i}(M_i; \tilde{\sigma}_i) \) such that \( \theta_{\beta_i}^1(\theta(L(G_0))) = L(G_1)^{s\beta_i}, \theta(\theta_{\beta_0}^0(L(G_0))) = L(G_1)^{s\beta_0} \). Moreover, \( \beta_0 = \beta_1 \).

N.B. The uniqueness above is modulo perturbations by inner automorphisms implemented by unitaries from the appropriate amplifications of \( L(G_i) \).

Proof. Let \( H_i \subset G_i \) be infinite wq-normal, relatively rigid subgroups, \( i = 0, 1 \). Thus, \( Q_i = L(H_i) \) is diffuse and the von Neumann algebra \( P_i \) generated by its normalizer \( N_{M_i}(Q_i) \) in \( M_i \) contains \( L(G_i) \) (because \( H_i \) is normal in \( G_i \)). Moreover, by Theorem 3.1, \( P_i \) is contained in \( L(G_i) \). Thus, \( P_i = L(G_i) \).

By applying Theorem 4.1 to \( \theta(Q_0) \subset \theta(P_0) \subset M_1^s \), it follows that there exist unique \( \beta_1 \in S(\tilde{\sigma}_1) \) and \( \theta_{\beta_1}^1 \in \text{Aut}_{\beta_1}(M_1; \tilde{\sigma}_1) \) such that \( \theta_{\beta_1}^1(\theta(P_0)) \subset L(G_1)^{s\beta_1} \). Thus, \( (\theta_{\beta_1}^1 \circ \theta)^{-1}(L(G_1)) \supset P_0 = L(G_0) \) and by Corollary 4.6 we actually have equality. Q.E.D.

We mention in separate statements the case of actions on probability spaces, where due to Theorem 4.4 we can require the actions \( \sigma \) to be merely sub malleable (resp. sub s-malleable). The proof is identical to the one for 5.2 above, using 4.4.(i) (resp. 4.4.(ii)) in lieu of 4.1, and it is thus omitted.

5.3. Theorem. Let \( \sigma : G \to \text{Aut}(N, \tau), \sigma_0 : G_0 \to \text{Aut}(N_0, \tau_0) \) be free ergodic actions of discrete groups on abelian von Neumann algebras \( N, N_0 \). Denote \( M = N \rtimes \sigma G, M_0 = N_0 \rtimes \sigma_0 G_0 \). Let also \( H \subset G_0' \subset G_0 \) be subgroups with \( H \) wq-normal in \( G_0' \). Assume:

(a). \( G \) is ICC; \( \sigma \) is sub malleable mixing.

(b). \( H \) is w-rigid; \( \{hgh^{-1} \mid h \in H\} \) is infinite \( \forall g \in G_0, g \neq e; \sigma_0|_H \) is ergodic.

If \( \theta : M_0 \simeq M^s \) is an isomorphism, then there exists \( u \in U(M^s) \) such that \( u\theta(L(G_0'))u^* \subset L(G)^s \). In particular, if \( (\sigma, G) \) satisfies (a) and \( G_0 \) is w-rigid ICC then \( u\theta(L(G_0))u^* \subset L(G)^s \). Moreover, if both \( G_0, G \) are w-rigid ICC and both \( \sigma_0, \sigma \) sub malleable mixing then \( u\theta(L(G_0))u^* = L(G)^s \).
5.3'. Theorem. Instead of (a), (b) in 5.3, assume
(a'). $G$ is ICC; $\sigma$ is sub $s$-malleable mixing.
(b'). $H$ has the relative property $(T)$ in $G_0$ (i.e. $(G_0, H)$ is a property $(T)$ pair) and is not virtually abelian.

If $\theta : M_0 \simeq M^s$ is an isomorphism, then there exists $u \in U(M^s)$ such that $u\theta(L(G'_0))u^* \subset L(G)^s$. Moreover, if both $G_0, G$ are ICC and in the class $wT_0$ and both $\sigma_0, \sigma$ sub $s$-malleable mixing then $u\theta(L(G_0))u^* = L(G)^s$.

Theorems 4.1, 5.2, 5.3, 5.3' are key ingredients in the proof of strong rigidity results for isomorphisms of cross product factors $N_0 \rtimes_{\sigma_0} G_0, N \rtimes_{\sigma} G$ for $G_0$ w-rigid or in the class $wT_0$ and $\sigma$ commutative or non-commutative Bernoulli shifts, in ([Po6]). This will allow us to classify large classes of factors $N \rtimes_{\sigma} G$, with explicit calculations of various invariants, such as $\mathcal{F}(N \rtimes_{\sigma} G)$.

In this paper we only mention a straightforward application of Theorems 5.2 and of results from ([Po3]): It shows that the fundamental group $\mathcal{F}(N \rtimes_{\sigma} G)$ is equal to $\tilde{S}(\tilde{\sigma})$ for any gauged extension $\tilde{\sigma}$ of a malleable mixing $\sigma$, in the case $G = \mathbb{Z}^2 \rtimes \Gamma$, where $\Gamma$ is a subgroup of finite index in $SL(2, \mathbb{Z})$, or if $G = \mathbb{Z}^N \rtimes \Gamma$ where $\Gamma$ is an arithmetic lattice in either $SU(n,1)$ or $SO(2n,1)$ suitably acting on $\mathbb{Z}^N$, or more generally, if $G$ is a finite product of any of the above groups. Taking $\sigma$ to be Connes-Størmer Bernoulli or Bogoliubov shifts, this result already gives many examples of factors and equivalence relations with arbitrarily prescribed countable fundamental group.

5.4. Corollary. 1°. Let $G$ be a w-rigid ICC group and $\sigma$ a property outer mixing action of $G$ on a AFD von Neumann algebra $(N, \tau)$ (more generally a finite von Neumann algebra with Haagerup’s approximation property). Assume $\sigma$ is malleable mixing (resp. $N$ abelian and $\sigma$ sub malleable mixing). If $\mathcal{F}(L(G)) = \{1\}$ then $\mathcal{F}(N \rtimes_{\sigma} G) = S(\tilde{\sigma})$ for any gauged extension $\tilde{\sigma}$ of $\sigma$ (resp. $\mathcal{F}(N \rtimes_{\sigma} G) = \{1\}$). In particular, this is the case if $L(G)$ is a HT factor and $\beta_n^{HT}(L(G)) \neq 0, \infty$ for some $n$.

2°. Let $G_i$ be w-rigid ICC groups and $\sigma_i$ malleable mixing actions of $G_i$ on AFD algebras $(N_i, \tau_i), i = 0, 1$ (resp. sub malleable mixing, with $N_i$ abelian, $i = 0, 1$). Assume $L(G_i)$ are HT factors with $\beta_n^{HT}(L(G_0)) = 0$ and $\beta_n^{HT}(L(G_1)) \neq 0, \infty$ for some $n$. Then $N_0 \rtimes_{\sigma_0} G_0$ is not stably isomorphic to $N_1 \rtimes_{\sigma_1} G_1$.

Proof. Assume first that $\sigma$ in part 1° and $\sigma_i$ in part 2° are malleable. Under the hypothesis in 2°, denote $M_i = N_i \rtimes_{\sigma_i} G_i$ and assume $\theta : M_0 \simeq M_1^s$ is an isomorphism, for some $s > 0$. Let $\tilde{\sigma}_1$ be a gauged extension for $\sigma_1$. By 5.2 there exists $\beta \in S(\tilde{\sigma}_1)$ and an isomorphism $\theta_\beta : M_1^s \simeq M_1^{s\beta}$ such that $\theta_\beta(\theta(L(G_0))) = L(G_1)^{s\beta}$. 
Thus, if we take $G_i = G$, $N_i = N$, $\sigma_i = \sigma$ as in part 1°, then $\mathcal{F}(L(G)) = \{1\}$ implies $st = 1$, thus $s = 1/t \in S(\tilde{\sigma}_1)$, showing that $\mathcal{F}(N \rtimes \sigma G) \subset S(\tilde{\sigma})$. Since we always have $S(\tilde{\sigma}) \subset \mathcal{F}(N \rtimes \sigma G)$, the equality follows.

The rest of 1° and part 2° are now trivial, by the first part of the proof and ([Po3]).

If we assume $N, N_i$ abelian and $\sigma, \sigma_i$, $i = 0, 1$, sub-malleable, then the proofs are the same, but using 5.3 instead of 5.2. Q.E.D.

### 5.5. Corollary

Let $S$ be a multiplicative subgroup in $\mathbb{R}^+_1$. For each $n \geq 2$ and $k \geq 1$ there exists a properly outer action $\sigma_{n,k}$ of $(\mathbb{F}_n)^k = \mathbb{F}_n \times \ldots \times \mathbb{F}_n$ ($k$ times) on an AFD II$_1$ factor $N$ such that $\mathcal{F}(N \rtimes_{\sigma_{n,k}} \mathbb{F}_n^k) = S, \forall n, k$, and such that for each $n \geq 2$ the factors $\{N \rtimes_{\sigma_{n,k}} (\mathbb{F}_n)^k\}_k$ are non-stably isomorphic. Moreover, $N$ can be taken generated by at most $|S|$ elements as a von Neumann algebra. In particular, if $S$ is countable then $N$ can be taken the hyperfinite II$_1$ factor $R$.

**Proof.** Let $G_{n,k} = (\mathbb{Z}^2)^k \rtimes (\mathbb{F}_n)^k = (\mathbb{Z}^2 \rtimes \mathbb{F}_n)^k$, where the action of $\mathbb{F}_n$ on $\mathbb{Z}^2$ comes from an embedding of finite index $\mathbb{F}_n \subset SL(2, \mathbb{Z})$. Let $\sigma'_{n,k}$ be a Connes-Størmer Bernoulli shifts action of $G_{n,k}$ on an AFD II$_1$ factor $N$, with gauged extension $\tilde{\sigma}'_{n,k}$ satisfying $S(\tilde{\sigma}'_{n,k}) = S$ (cf. last paragraph in 1.6.2). Since $G_{n,k}$ are w-rigid, by 5.3 the factors $M_{n,k} = N \rtimes_{\sigma'_{n,k}} G_{n,k}$ satisfy $\mathcal{F}(M_{n,k}) = S$. Also, for each $n \geq 2 \{M_{n,k}\}_k$ are mutually non stably isomorphic, because by [Po3] we have $\beta_k^{ht}(L(G_{n,k})) = 0$ for all $j \neq k$. But $N \rtimes_{\sigma'_{n,k}} (\mathbb{Z}^2)^k \simeq R$, so denoting by $\sigma_{n,k}$ the corresponding action of $(\mathbb{F}_n)^k$ on $N \rtimes_{\sigma'_{n,k}} (\mathbb{Z}^2)^k$, the first part of the statement follows. The last part is now trivial, since for $S$ countable we can take the base of the shift to be a separable type I factor. Q.E.D.

Note that in the case of Connes-Størmer $(M_{k \times k}, \varphi_0)$-Bernoulli shift actions $\sigma : G \to Aut(N, \tau)$ (1.6.2), the cross product factors $M = N \rtimes \sigma G$ can be realized as a generalized group measure space construction as in ([FM]), i.e., there exists a standard (= countable, ergodic, measure preserving) equivalence relation $\mathcal{R}$ on the standard probability space such that $M = M(\mathcal{R})$. Indeed, this is because $\sigma$ normalizes the “main diagonal” Cartan subalgebra $A$ of $N$, thus normalizing the corresponding hyperfinite equivalence relation $\mathcal{R}_{A \subset N}$ as well, inducing altogether an equivalence relation $\mathcal{R} = \mathcal{R}_{A \subset M}$, with trivial cocycle.

Moreover, in each of these cases the automorphisms in $Aut(\beta(M; \tilde{\sigma}))$, given by the gauged extensions $\tilde{\sigma}$ that come with the construction of $\sigma$, can be chosen to normalize the Cartan subalgebra $A \subset N \subset M$ as well. Thus, in each of these cases one has $S(\tilde{\sigma}) \subset \mathcal{F}(\mathcal{R})$. For non-commutative Bernoulli shifts $\sigma$ this observation has been exploited in [GeGo] to prove the existence of standard equivalence relations.
with the fundamental group countable and containing a prescribed countable set. Due to the obvious inclusion $\mathcal{F}(\mathcal{R}) \subset \mathcal{F}(M(\mathcal{R}))$, Corollaries 5.3 and 5.4 allow us to actually obtain precise calculations of fundamental groups of the corresponding equivalence relations:

5.6. Corollary. Let $S$ be a countable multiplicative subgroup in $\mathbb{R}^+_\times$. There exist standard equivalence relations $\mathcal{R}$ such that $\mathcal{F}(\mathcal{R}) = S$. Moreover, given any $n \geq 2, k \geq 1$, $\mathcal{R}$ can be taken of the form $\mathcal{R} = \mathcal{R}_n \rtimes_{\sigma_{n,k}} (\mathbb{F}_n)^k$, where $\mathcal{R}_n \subset \mathcal{R}$ is an ergodic hyperfinite sub-equivalence relation and $\sigma_{n,k}$ is a free ergodic action of $(\mathbb{F}_n)^k$ on the probability space, which normalizes $\mathcal{R}_n$ and acts outly on it. Also, the equivalence relations $\{\mathcal{R}_n \rtimes (\mathbb{F}_n)^k \}_k$ can be taken mutually non-stably isomorphic.

Proof. In each of the examples used in the proof of 5.4, for fixed $n \geq 2, k \geq 1$, take $A$ to be the “main diagonal” Cartan subalgebra of $N \simeq R$ which is normalized by the action $\sigma'_{n,k}$ of $(\mathbb{Z}^2)^k \rtimes (\mathbb{F}_n)^k$. Thus, the hyperfinite equivalence relation $\mathcal{R}_{AC}R$ is invariant to $\sigma'_{n,k}$. It follows that if we let $\mathcal{R}_n$ be the equivalence relation implemented by $\mathcal{R}_{AC}R$ and $\sigma'_{n,k}|_{(\mathbb{Z}^2)^k}$, and $\mathcal{R}$ be the equivalence relation implemented jointly by $\mathcal{R}_{AC}R$ and the action $\sigma'_{n,k}$ of $(\mathbb{Z}^2)^k \rtimes (\mathbb{F}_n)^k$ then $M = R \rtimes_{\sigma_{n,k}} ((\mathbb{Z}^2)^k \rtimes (\mathbb{F}_n)^k)$ coincides with the factor $M(\mathcal{R})$ associated with $\mathcal{R}$. Thus, $\mathcal{F}(\mathcal{R}) \subset \mathcal{F}(\mathcal{R}) \subset \mathcal{F}(M) = S$. Since we also have $S \subset \mathcal{F}(\mathcal{R})$, we are done. Q.E.D.

5.7. Remarks. 1°. The examples of factors and equivalence relations with prescribed countable fundamental group $S \subset \mathbb{R}^+_\times$ can alternatively be described as follows: Let $(Y_0, \nu_0)$ be an atomic probability space with the property that the ratio set $\{\nu_0(x)/\nu_0(y) \mid x, y \in Y_0\}$ generates the group $S$. Let $G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$. Let $(X, \mu) = \Pi g(Y_0, \nu_0)g$. Denote by $\mathcal{R}_0$ the equivalence relation on $(X, \mu)$ given by $(x_g, y) \sim (y_g, y)$ iff $\exists F \subset G$ finite with $x_g = y_g, \forall g \in G \setminus F$, $\Pi g \in F \mu_0(x_g) = \Pi g \in F \mu_0(y_g)$. Denote $R_0 = M(\mathcal{R}_0)$ the von Neumann algebra associated with $\mathcal{R}_0$, isomorphic to the hyperfinite $\Pi_1$ factor. Note that the Bernoulli shift action $\sigma_0$ of $G$ on $(X, \mu)$ leaves $\mathcal{R}_0$ invariant, thus implementing an action $\sigma$ of $G$ on $R_0$ and an equivalence relation $S = \mathcal{R}_0 \rtimes G$ on $(X, \mu)$. If we let $\mathcal{R} = \mathcal{R}_0 \rtimes \mathbb{Z}^2$, $R = R_0 \rtimes_{\sigma_0} \mathbb{Z}^2$ and $\sigma'$ be the action of $SL(2, \mathbb{Z})$ on $\mathcal{R}$ implemented by $\sigma$, then the action $\sigma_{n,1}$ in the proof of 5.4 and 5.5 coincides with the restriction of $\sigma'$ to a subgroup $\mathbb{F}_n \subset SL(2, \mathbb{Z})$.

2°. In ([Fu]), A. Furman provided examples of ergodic countable measure preserving equivalence relations $\mathcal{R}$ with the property that $\mathcal{R}^t$ is not implementable by a free ergodic action of a countable group, $\forall t > 0$, thus solving a longstanding problem in ([FM]). It seems that the equivalence relations constructed in part 1 of above and in the proof of 5.4 give another class of examples of equivalence relations with this property, whenever $S \neq \{1\}$. However, we were not able to prove this.
6. The factors $N \rtimes_\sigma G$ and the class HT

In this Section we relate the class of cross product factors $N \rtimes_\sigma G$ studied in this paper with the HT factors in ([Po3]), proving that the two classes are essentially disjoint. More precisely, we prove that if $\sigma$ is sub-malleable mixing, then $N \rtimes_\sigma G$ cannot be HT$_{wa}$, a property in between the HT and HT$_s$ conditions considered in ([Po3]). In the process, we obtain two new examples of free ergodic measure preserving actions of $F_n$ on the probability space, $2 \leq n \leq \infty$, non orbit equivalent to the three actions constructed in (8.7 of [Po3]).

In fact, one can show that these two actions are not orbit equivalent to the "residually finite" actions of $F_n$ constructed in ([Hj]) either, thus providing the 5'th and 6'th distinct actions of $F_n$. We will discuss this fact in a forthcoming paper.

In this respect, we should mention that by ([GaPo]) one knows that there exist uncountably many non orbit equivalent free ergodic measure preserving actions of $F_n$ on the probability space, for each $2 \leq n \leq \infty$. However, this is an "existence" result, so it is still quite interesting to prove that certain specific actions of $F_n$ are not orbit equivalent.

6.1 Definition. A Cartan subalgebra $A$ in a $\mathrm{II}_1$ factor $M$ is called a HT$_{ws}$ Cartan subalgebra if $M$ has the property H relative to $A$ and there exists a von Neumann subalgebra $A_0 \subset A$ such that: $A_0' \cap M = A$, $N_M(A_0)' = M$, $A_0 \subset M$ is a rigid inclusion (see [Po3] for the definitions of relative property H and rigid inclusions).

Recalling from ([Po3]) that a Cartan subalgebra $A \subset M$ is HT$_s$ (resp. HT) if $M$ has the property H relative to $A$ and $A \subset M$ is rigid (resp. $\exists A_0 \subset A$ with $A_0' \cap M = A$ and $A_0 \subset M$ rigid), it follows that $A \subset M$ is HT$_s$ implies $A$ is HT$_{ws}$, which in turn implies $A$ is HT. Note also that all concrete examples of HT Cartan subalgebras constructed in ([Po3]) are in fact HT$_{ws}$ (see the proof of Corollary 6.3 below). In particular, this is the case with the examples coming from the actions $\sigma_i, i = 1, 2, 3$, in (5.3.3° and 8.7 of [Po3]).

6.2. Theorem. Let $\Gamma$ be an infinite group with Haagerup’s compact approximation property.

1°. If $\sigma$ is a sub-malleable mixing action of $\Gamma$ on $(N, \tau)$ then $M = N \rtimes_\sigma \Gamma$ contains no diffuse subalgebra $Q \subset M$ with $P = N_M(Q)'$ a factor and $Q \subset P$ rigid. In particular, $M$ is not a HT$_{ws}$ factor.

2°. If $\sigma : G_0 \to \text{Aut}(L^\infty(X, \mu))$ is a free ergodic action such that $A \subset A \rtimes_\sigma G_0$ is a rigid inclusion, then for any ergodic action $\sigma' : G_0 \to \text{Aut}(L^\infty(X', \mu'))$, the diagonal product action $\sigma \otimes \sigma' : G_0 \to \text{Aut}(L^\infty(X \times X', \mu \times \mu'))$ gives rise to a HT$_{ws}$ Cartan subalgebra $L^\infty(X \times X', \mu \times \mu') = A \subset M = A \rtimes_{\sigma \times \sigma'} G_0$.

Proof. 1°. By Theorem 4.4, if $Q \subset M$ is diffuse with $N_M(Q)'' = P$ a factor and
$Q \subset P$ rigid, then there exists $\theta_\beta \in \text{Aut}_\beta(M; \{\tilde{\sigma}_n\})$ with $\theta_\beta(P) \subset L(G)^\beta$, for some $\beta \in S(\{\tilde{\sigma}\}), \{\tilde{\sigma}\}$ being a given sequence of gauged extensions for $\sigma$.

By (4.5.2$^\circ$ in [Po3]), it follows that $\theta_\beta(Q) \subset L(G)^\beta$ is a rigid inclusion. By cutting with an appropriate projection $q \in Q$ of sufficiently small trace, it follows that we have a rigid inclusion $Q_0 \subset pL(G)p$, where $1_{Q_0} = p = \theta_\beta(q) \in L(G)$ and $Q_0 = \theta_\beta(qQq)$.

On the other hand, by (3.1 in [Po3]) $M$ has the property H relative to $N$, so there exists a sequence of unital $N$-bimodular completely positive maps $\Phi_n$ on $M$ such that $\lim_{n \to \infty} \|\Phi_n(x) - x\|_2 = 0, \forall x \in M$, $\tau \circ \Phi_n = \tau$ and $\Phi_n|_{l^2(\Gamma)}$ compact, $\forall n$.

By the rigidity of $Q_0 \subset pMp$ applied to the completely positive maps $p\Phi_n(p \cdot p)p$ on $pMp$, it follows that $\lim_{n \to \infty} \|p\Phi_n(u)p - u\|_2 = 0$ uniformly for $u \in \mathcal{U}(Q_0)$. Since, we also have $\lim_{n \to \infty} \|\Phi_n(p) - p\|_2 = 0$, it follows that there exists $n$ such that $\Phi = \Phi_n$ satisfies

$$
\|\Phi(u) - u\|_2 \leq \tau(p)/2, \forall u \in \mathcal{U}(Q_0)
$$

But $Q_0$ diffuse implies there exists a unitary element $v \in Q_0$ such that $\tau(v^m) = 0, \forall m \neq 0$. Since $\Phi_n$ is compact on $l^2(\Gamma) = L^2(L(\Gamma), \tau)$ and $v^m$ tends weakly to 0, it follows that $\lim_{m \to \infty} \|\Phi(v^m)\|_2 = 0$, and for $m$ large enough, $u = v^m$ contradicts 6.2.1.

2$^\circ$. If we put $A_0 = L^\infty(X, \mu) \subset A$ then by construction we have $A_0 \subset M$ rigid inclusion and $A_0' \cap M = A$. Since $\mathcal{N}_M(A_0)$ contains both $\mathcal{U}(A)$ and the canonical unitaries $\{u_g\}_g$ implementing the cross product, we also have $\mathcal{N}_M(A_0)'' = M$.

Q.E.D.

6.3. Corollary. Let $\Gamma$ be an ICC group with Haagerup’s compact approximation property. Assume $\Gamma$ can act outerly and ergodically on an infinite abelian group $H$ such that the pair $(H \rtimes \Gamma, H)$ has the relative property (T). Denote by $\sigma_1$ the action of $\Gamma$ on $L^\infty(X_1, \mu_1) \simeq L(H)$ implemented by the action of $\Gamma$ on $H$. Let $\sigma_0$ be a (classic) Bernoulli shift action of $\Gamma$ and $\sigma'$ an ergodic but not strongly ergodic action of $\Gamma$ on $L^\infty(X', \mu')$ (cf. [CW]). Let $\sigma_2, \sigma_3$ denote the product actions of $\Gamma$ given by $\sigma_2(g) = \sigma_1(g) \otimes \sigma_0(g), \sigma_3(g) = \sigma_1(g) \otimes \sigma'(g)$. Then $\sigma_i, 0 \leq i \leq 3$, are mutually non orbit equivalent.

Moreover, if $\Gamma$ has an infinite amenable quotient $\Gamma'$, with $\alpha : \Gamma \to \Gamma'$ the corresponding quotient map, and if $\sigma'_0$ denotes a Bernoulli shift action of $\Gamma'$ on the standard probability space, then we can take $\sigma'$ above of the form $\sigma' = \sigma'_0 \circ \alpha$ and the product action $\sigma_4 = \sigma_0 \otimes (\alpha \circ \sigma'_0)$ is not orbit equivalent to $\sigma_i, 0 \leq i \leq 3$. Also, $\sigma_1$ has at most countable Out-group while $\sigma_0, \sigma_2, \sigma_3, \sigma_4$ have uncountable Out-group.
Proof. Note first that the condition on $H \subset H \rtimes \Gamma$ in the hypothesis of the statement is equivalent to $L(H \rtimes \Gamma)$ being a HT$_s$ factor with $L(H)$ its HT$_s$ Cartan subalgebra, in the sense of [Po3].

By 1.6.1, both $\sigma_0$ and $\sigma_4$ implement malleable mixing integral preserving actions on the standard non-atomic abelian von Neumann algebra $A$. By 6.2.2°, $A \rtimes_{\sigma_i} \Gamma$, $i = 1, 2, 3$, are HT$_{ws}$ factors, while by 6.2.1° it follows that $A \rtimes_{\sigma_j} \Gamma$ are not isomorphic to $A \rtimes_{\sigma_i} \Gamma$ for $j = 0, 1, i = 1, 2, 3$. Thus, by ([FM]), $\sigma_j$ are not orbit equivalent to $\sigma_i$, for $j = 0, 1$ and $i = 1, 2, 3$. Also, since $\Gamma'$ is amenable, the action $\sigma'_0$ is not strongly ergodic, so $\sigma_4$ is not strongly ergodic either. Thus $\sigma_0$ and $\sigma_4$ are not orbit equivalent.

Q.E.D.

6.4. Corollary. For each $2 \leq n \leq \infty$, $\{\sigma_i\}_{0 \leq i \leq 4}$ give five free ergodic measure preserving actions of $\mathbb{F}_n$ on the standard probability space that are mutually non orbit equivalent. Moreover, $\sigma_1$ has at most countable Out-group while $\sigma_0, \sigma_2, \sigma_3, \sigma_4$ have uncountable Out-group.

Proof. This is immediate from 6.3, since $\mathbb{F}_n$ does have infinite amenable quotients. Q.E.D.

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