FROM GENERALIZED PERMUTAHEDRA TO GROTHENDIECK POLYNOMIALS VIA FLOW POLYTOPES

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Abstract. We prove that for permutations \( 1\pi' \) where \( \pi' \) is dominant, the Grothendieck polynomial \( G_{1\pi'}(x) \) is a weighted integer-point transform of its Newton polytope with all weights nonzero. We also show that the Newton polytopes of the homogeneous components of \( G_{1\pi'}(x) \) are generalized permutahedra. Moreover, the Schubert polynomial \( S_{1\pi'}(x) \) for dominant \( \pi' \) equals the integer-point transform of a generalized permutahedron. These results imply recent conjectures of Monical, Tokcan and Yong regarding the supports of Schubert and Grothendieck polynomials for the special case of permutations \( 1\pi' \), where \( \pi' \) is dominant. We connect Grothendieck polynomials and generalized permutahedra via a family of dissections of flow polytopes obtained from the subdivision algebra. We naturally label each simplex in a dissection by a sequence, called a left-degree sequence, and show that the left-degree sequences arising from simplices of a fixed dimension in our dissections of flow polytopes are exactly the integer points of generalized permutahedra. This connection of left-degree sequences and generalized permutahedra together with the connection of left-degree sequences and Grothendieck polynomials established in earlier work of Escobar and the first author reveal the beautiful relation between generalized permutahedra and Grothendieck polynomials.

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1. Introduction

The flow polytope \( \mathcal{F}_G \) associated to a directed acyclic graph \( G \) is the set of all flows \( f : E(G) \to \mathbb{R}_{\geq 0} \) of size one. Flow polytopes are fundamental objects in combinatorial optimization \cite{16}, and in the past decade they were also uncovered in representation theory \cite{1, 11}, the study of the space of diagonal harmonics \cite{7, 12}, and the study of Schubert and Grothendieck polynomials \cite{4, 5}. In this paper we establish the deep connection between flow polytopes and generalized permutahedra and use this connection to prove that for certain
permutations, the supports of Schubert polynomials as well as the homogeneous components of Grothendieck polynomials are integer points of generalized permutahedra.

A natural way to analyze a convex polytope is to dissect it into simplices. The relations of the subdivision algebra, developed in a series of papers [8, 9, 10], encode dissections of a family of flow (and root) polytopes (see Section 2 for details). The key to connecting flow polytopes and generalized permutahedra lies in the study of the dissections of flow polytopes obtained via the subdivision algebra:

1. **How are the dissections of a flow polytope obtained via the subdivision algebra related to each other?**

In Theorem A we give a full characterization of the left-degree sequences (Definition 2.5) of any dissection of a flow polytope obtained via the subdivision algebra, and we show that while the dissections themselves are different their left-degree sequences are the same. That the left-degree sequences do not depend on the dissection was previously proved in special cases by Escobar and the first author [4], and independently from the authors, Grinberg [6] recently showed it for arbitrary graphs in his study of the subdivision algebra. Our characterization of the left-degree sequences of any reduction tree of any graph serves as the cornerstone of the rest of the work in this paper.

Since by Theorem A the left-degree sequences are an invariant of the underlying flow polytope and do not depend on the choice of dissection, it is natural to ask:

2. **What is the significance of the left-degree sequences associated to a flow polytope \( F_G \)?**

The answer to this question is both inspiring and revealing. In Theorem B we prove that left-degree sequences of \( F_G \) with fixed sums are exactly lattice points of generalized permutahedra, which were introduced by Postnikov in his beautiful paper [14]. Moreover, we show that the left-degree polynomial \( L_G(t) \) (Definition 4.2) has polytopal support (Definition 4.1).

In earlier work of Escobar and the first author [4], it was shown that some left-degree polynomials are Grothendieck polynomials. This brings us to:

3. **What does the answer to (2) imply about Schubert and Grothendieck polynomials?**

In Theorem C we conclude that for all permutations 1\( \pi' \) where \( \pi' \) is dominant, the Grothendieck polynomial \( \mathcal{G}_{1\pi'}(x) \) is a weighted integer-point transform of its Newton polytope, with all weights nonzero. Moreover, the homogeneous components of \( \mathcal{G}_{1\pi'}(x) \) are weighted integer-point transforms of their Newton polytopes, which are all generalized permutahedra. For the homogeneous component corresponding to the Schubert polynomial \( \mathcal{G}_{1\pi'}(x) \), something more is true: it equals the integer-point transform of its Newton polytope, which is a generalized permutahedron. Theorem C implies in particular that the recent conjectures of Monical, Tokcan, and Yong [13, Conjecture 5.1 & 5.5] are true for permutations 1\( \pi' \), where \( \pi' \) is a dominant permutation.

The outline of this paper is as follows. Sections 2 covers the necessary background. Sections 3, 4, and 5 answer questions (1), (2) and (3) from above respectively. For ease of reading Sections 3, 4, and 5 are phrased for simple graphs. In Section 6 we show that our techniques extend to generalize all results to all graphs.
2. Background information

In this section, we summarize definitions, notations, and results that we use later. Throughout this paper, by **graph** we mean a loopless directed graph where multiple edges are allowed, as described below. Although we sometimes refer to edges by their endpoints, we keep in mind that $E(G)$ is a multiset. We also adopt the convention of viewing each element of a multiset as being distinct, so that we may speak of subsets, though we will use the word submultiset interchangeably to highlight the multiplicity. Due to this convention, all unions in this paper are assumed to be disjoint multiset unions.

For any integers $m$ and $n$, we will frequently use the notation $[m,n]$ to refer to the set \( \{m, m+1, \ldots, n\} \) and $[n]$ to refer to the set $[1,n]$.

**Flow polytopes.** Let $G$ be a loopless graph on vertex set $[0,n]$ with edges directed from smaller to larger vertices. For each edge $e$, let $\text{in}(e)$ denote the smaller (initial) vertex of $e$ and $\text{fin}(e)$ the larger (final) vertex of $e$. Imagine fluid moving along the edges of $G$. At vertex $i$ let there be an external inflow of fluid $a_i$ (outflow of $-a_i$ if $a_i < 0$), and call $a = (a_0, \ldots, a_n)$ the netflow vector. Formally, a flow on $G$ with netflow vector $a$ is an assignment $f : E(G) \to \mathbb{R}_{\geq 0}$ of nonnegative values to each edge such that fluid is conserved at each vertex. That is, for each vertex $i$

$$\sum_{\text{in}(e) = i} f(e) - \sum_{\text{fin}(e) = i} f(e) = a_i.$$ 

The **flow polytope** $F_G(a)$ is the collection of all flows on $G$ with netflow vector $a$. Alternatively, let $M_G$ denote the incidence matrix of $G$, that is let the columns of $M_G$ be the vectors $e_i - e_j$ for $(i, j) \in E(G)$, $i < j$, where $e_i$ is the $(i+1)$-th standard basis vector in $\mathbb{R}^{n+1}$. Then,

$$F_G(a) = \{ f \in \mathbb{R}_n^\geq : M_G f = a \}.$$ 

From this perspective, note that the number of integer points in $F_G(a)$ is exactly the number of ways to write $a$ as a nonnegative integral combination of the vectors $e_i - e_j$ for edges $(i, j)$ in $G$, $i < j$, that is the **Kostant partition function** $K_G(a)$. For brevity, we write $F_G := F_G(1,0,\ldots,0,-1)$, and we refer to $F_G$ as the flow polytope of $G$, since in this paper our primary focus is on studying these particular flow polytopes.

The following milestone result giving the volume of flow polytopes was shown by Postnikov and Stanley in unpublished work:

**Theorem 2.1 (Postnikov-Stanley).** Given a loopless connected graph $G$ on vertex set $\{0,1,\ldots,n\}$, let $d_i = \text{indeg}_G(i) - 1$ for each vertex $i$, where $\text{indeg}_G(i)$ is the number of edges incoming to vertex $i$ in $G$. Then, the normalized volume of the flow polytope of $G$ is

$$\text{Vol } F_G = K_G \left( 0, d_1, \ldots, d_n, -\sum_{i=1}^n d_i \right).$$

Baldoni and Vergne [1] generalized this result for flow polytopes with arbitrary netflow vectors. Theorem 2.1 beautifully connects the volume of the flow polytope of any graph to an evaluation of the Kostant partition function. We note that since the number of integer points of a flow polytope is already given by a Kostant partition function evaluation, the volume of any flow polytope is given by the number of integer points of another.
Recall that two polytopes $P_1 \subseteq \mathbb{R}^{k_1}$ and $P_2 \subseteq \mathbb{R}^{k_2}$ are **integrally equivalent** if there is an affine transformation $T : \mathbb{R}^{k_1} \rightarrow \mathbb{R}^{k_2}$ that is a bijection $P_1 \rightarrow P_2$ and a bijection $\text{aff}(P_1) \cap \mathbb{Z}^{k_1} \rightarrow \text{aff}(P_2) \cap \mathbb{Z}^{k_2}$. Integrally equivalent polytopes have the same face lattice, volume, and Ehrhart polynomial. We write $P_1 \equiv P_2$ to denote integral equivalence.

While simple to prove, the following lemma is important. We leave its proof to the reader. For the rest of the paper, given a graph $G$ and a set $S$ of its edges, we use the notation $G/S$ to denote the graph obtained from $G$ by contracting the edges in $S$ (and deleting loops) and we use the notation $G\setminus S$ to denote the graph obtained from $G$ by deleting the edges in $S$.

For a set $V$ of vertices of $G$, we also use the notation $G\setminus V$ to denote the graph obtained from $G$ by deleting the vertices in $V$ and all edges incident to them. When $S$ or $V$ consists of just one element, we simply write $G/e$ or $G/v$.

**Lemma 2.2.** Let $G$ be a graph on $[0,n]$. Assume vertex $j$ has only one outgoing edge $e$ and netflow $a_j \geq 0$. If $e$ is directed from $j$ to $k \in [n]$, then

$$F_G(a_0, \ldots, a_n) \text{ and } F_{G/e}(a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k + a_j, a_{k+1}, \ldots, a_n)$$

are integrally equivalent. An analogous result holds if $j$ has only one incoming edge and $a_j \leq 0$.

**Dissections of flow polytopes.** For graphs with a special source and sink, there is a systematic way to dissect the flow polytope $F_G$ studied in [10]. Let $G$ be a graph on $[0,n]$, and define $\tilde{G}$ on $[0,n] \cup \{s,t\}$ with $s$ being the smallest vertex and $t$ the biggest vertex by setting $E(\tilde{G}) = E(G) \cup \{(s,i),(i,t) : i \in [0,n]\}$. The systematic dissections can be expressed in the language of the subdivision algebra or equivalently in terms of reduction trees [8, 9, 10]. We use the language of reduction trees in this paper.

Let $G_0$ be a graph on $[0,n]$ with edges $(i,j)$ and $(j,k)$ for some $i < j < k$. By a reduction on $G$, we mean the construction of three new graphs $G_1$, $G_2$ and $G_3$ on $[0,n]$ given by:

$$E(G_1) = E(G) \setminus \{(j,k)\} \cup \{(i,k)\}$$
$$E(G_2) = E(G) \setminus \{(i,j)\} \cup \{(i,k)\}$$
$$E(G_3) = E(G) \setminus \{(i,j),(j,k)\} \cup \{(i,k)\}$$

We say $G_0$ reduces to $G_1$, $G_2$ and $G_3$. We also say that the above reduction is at vertex $j$, on the edges $(i,j)$ and $(j,k)$.

**Proposition 2.3.** Let $G_0$ be a graph on $[0,n]$ which reduces to $G_1$, $G_2$ and $G_3$ as above. Then for each $m \in [3]$, there is a polytope $Q_m$ integrally equivalent to $F_{\tilde{G}_m}$ such that $Q_1$ and $Q_2$ subdivide $F_{\tilde{G}_0}$ and intersect in $Q_3$. That is, the polytopes $Q_1$, $Q_2$, and $Q_3$ satisfy

$$F_{\tilde{G}_0} = Q_1 \cup Q_2 \text{ with } Q_1 \cap Q_2^c = \emptyset \text{ and } Q_1 \cap Q_2 = Q_3.$$ 

Moreover, $Q_1$ and $Q_2$ have the same dimension as $F_{\tilde{G}_0}$ and $Q_3$ has dimension one less.

**Proof.** Let $r_1$ and $r_2$ denote the edges of $G_0$ from $i$ to $j$ and from $j$ to $k$ respectively that were used in the reduction. Viewing $\mathbb{R}^{\#E(\tilde{G}_0)}$ as functions $f : E(\tilde{G}_0) \rightarrow \mathbb{R}$, cut $F_{\tilde{G}_0}$ with the hyperplane $H$ defined by the equation $f(r_1) = f(r_2)$. Let $Q_1$ be the intersection of $F_{\tilde{G}_0}$ with the positive half-space $f(r_1) \geq f(r_2)$, let $Q_2$ be the intersection of $F_{\tilde{G}_0}$ with the negative half-space $f(r_1) \leq f(r_2)$, and let $Q_3$ be the intersection of $F_{\tilde{G}_0}$ with the hyperplane $H$. See Figure 1 for an illustration of the integral equivalence between $Q_m$ and $F_{\tilde{G}_m}$. Notice that
since we are doing the reductions on the edges of $G_0$ (as opposed to on the edges incident to the source or sink in $\tilde{G}_0$), it follows that the hyperplane $H$ meets $\mathcal{F}_{\tilde{G}_0}$ in its interior, giving the claims on the dimensions of $Q_m$, $m \in [3]$. □

Figure 1. An illustration of the integral equivalence between $Q_m$ and $\mathcal{F}_{\tilde{G}_m}$ for $m \in [3]$ used Proposition 2.3.

Iterating this subdivision process produces a dissection of $\mathcal{F}_{\tilde{G}_0}$ into simplices. This process can be encoded using a reduction tree. A reduction tree of $G$ is constructed as follows. Let the root node of the tree be labeled by $G$. If a node has any children, then it has three children obtained by performing a reduction on that node and labeling the children with the graphs defined in (2.1). Continue this process until the graphs labeling the leaves of the tree cannot be reduced. See Figure 2 for an example.

Fix a reduction tree $\mathcal{R}(G)$ of $G$. Let $L$ be a graph labeling one of the leaves in $\mathcal{R}(G)$. Lemma 2.2 implies that $\mathcal{F}_L$ is a simplex, so the flow polytopes of the graphs labeling the leaves of $\mathcal{R}(G)$ dissect $\mathcal{F}_{\tilde{G}}$ into simplices. All dissections we consider in this paper will be dissections into simplices. By full-dimensional leaves of $\mathcal{R}(G)$, we mean the leaves $L$ with $\#E(L) = \#E(G)$. By lower-dimensional leaves we mean all other leaves $L$ of $\mathcal{R}(G)$. Note that the full-dimensional leaves correspond to top-dimensional simplices in the dissection of $\mathcal{F}_{\tilde{G}}$, and the lower-dimensional leaves index intersections of the top-dimensional simplices. Since all simplices above are unimodular, it follows that:

**Corollary 2.4.** The normalized volume of $\mathcal{F}_{\tilde{G}}$ equals the number of full-dimensional leaves in any reduction tree of $G$. Moreover, the number of leaves with a fixed number of edges is independent of the reduction tree.

**Left-degree sequences.** Let $G$ be a graph on $[0,n]$, and let $\mathcal{R}(G)$ be a reduction tree of $G$. Denote by $\text{indeg}_G(i)$ the number of edges directed into vertex $i$. For each leaf $L$ of
Figure 2. A reduction tree for a graph on three vertices. The edges involved in each reduction are shown in bold. The left-degree sequences of the leaves are shown in blue.

\[ R(G), \text{consider the left-degree sequence } (\text{indeg}_L(1), \text{indeg}_L(2), \ldots, \text{indeg}_L(n)). \] By full-dimensional sequences we will mean left-degree sequences of full-dimensional leaves of \( R(G) \). The following definition is central to this paper.

**Definition 2.5.** Denote by \( \text{LD}(G) \) the multiset of left-degree sequences of leaves in a reduction tree of \( G \).

Although the actual leaves of a reduction tree are dependent on the individual reductions performed, we prove in Theorem A that \( \text{LD}(G) \) is independent of the particular reduction tree considered.

3. Triangular Arrays and Left-Degree Sequences

In this section, we expand the technique described in \([10]\) that characterized left-degree sequences of full-dimensional leaves in a specific reduction tree of a graph. We give a characterization of the left-degree sequences of all leaves of this reduction tree, not just the full dimensional ones. This enables us to relate the left-degree sequences to generalized permutahedra in Section 4 and to use left-degree sequences and generalized permutahedra to show in Section 5 that the Schubert and Grothendieck polynomials have polytopal support. The main theorem of this section is the following. The independence of \( \text{LD}(G) \) on the reduction tree was first proved independently by Grinberg \([6]\) in his study of the subdivision algebra.

**Theorem A.** For any graph \( G \) on \([0, n]\) the multiset of left-degree sequences \( \text{LD}(G) \) in any reduction tree of \( G \) equals the first columns of \( \text{Sol}_G(F) \) over all \( F \subseteq E(G \setminus 0) \), also denoted by \( \text{InSeq}(T(G)) \). In particular, \( \text{LD}(G) \) is independent of the choice of reduction tree.
For simplicity, throughout this section we restrict to the case where $G$ is a simple graph on the vertex set $[0, n]$. The set $\text{Sol}_G(F)$ is defined in Definition 3.6 for simple graphs. We address the general case in Section 6 where we also prove Theorem A.

We start by generalizing [10, Lemma 3] to include the descriptions of the lower dimensional leaves of reductions performed at a special vertex $v$. The proof is a straightforward generalization of that of [10, Lemma 3] illustrated in Figure 3. The key to the proof is the special reduction order, whereby we always perform a reduction on the longest edges possible that are incident to the vertex at which we are reducing (the length of an edge being the absolute value of the difference of its vertex labels). We leave the details of the proof to the interested reader.

Lemma 3.1. Assume $G$ has a distinguished vertex $v$ with $p$ incoming edges and one outgoing edge $(v, u)$. If we perform all reductions possible which involve only edges incident to $v$ in the special reduction order, then we obtain graphs $H_i$, $i \in [p + 1]$, and $K_j$, $j \in [p]$, with $(\text{indeg}_{H_i}(v), \text{indeg}_{H_i}(u)) = (p + 1 - i, i)$ and $(\text{indeg}_{K_j}(v), \text{indeg}_{K_j}(u)) = (p - j, j)$.

We now construct a specific reduction tree $T(G)$ and characterize the left-degree sequences of its leaves. Denote by $I_i$ the set of incoming edges to vertex $i$ in $G$. Let $V_i$ be the set of vertices $k$ with $(k, i) \in I_i$, and let $G[0, i]$ be the restriction of $G$ to the vertices $[0, i]$. For any reduction tree $R(G)$, by $\text{InSeq}(R(G))$ we mean the multiset of left-degree sequences of the leaves of $R(G)$. Since we will build $T(G)$ inductively from $T(H)$ for smaller graphs $H$, it is convenient to let $\text{InSeq}^n(R(H))$ denote the multiset $\text{InSeq}(R(H))$ with each sequence padded on the right with zeros to have length $n$.

We proceed using the following algorithm, analogous to the one described in [10]:

- For the base case, define the reduction tree $T(G[0, 1])$ to be the single leaf $G[0, 1]$. Hence, $\text{InSeq}(T(G[0, 1])) = \{(\text{indeg}_{G}(1))\}$.
- Having built $T(G[0, i])$, construct the reduction tree $T(G[0, i + 1])$ from $T(G[0, i])$ by appending the vertex $i + 1$ and the edges $I_{i+1}$ to all graphs in $T(G[0, i])$ and then
performing reductions at each vertex in \( V_{i+1} \) on all graphs corresponding to the leaves of \( \mathcal{T}(G[0, i]) \) in the special reduction order as described below.

- Let \( V_{i+1} = \{i_1 < i_2 < \cdots < i_k\} \) and let \((s_1, \ldots, s_n)\) be one of the sequences in \( \text{InSeq}^n(\mathcal{T}(G[0, i])) \). Applying Lemma [3.1](#) to each of the vertices \( i_1, \ldots, i_k \), we see that the leaves of \( \mathcal{T}(G[0, i+1]) \) which are descendants of the graph with \( n \)-left-degree sequence \((s_1, \ldots, s_n)\) in \( \mathcal{T}(G[0, i]) \) will have \( n \)-left-degree sequences exactly given by

\[
(s_1, \ldots, s_n) + v^{i+1}[i_1] + \cdots + v^{i+1}[i_k]
\]

where \( v^{i+1}[i] \in S_1(i) \cup S_2(i) \) and \( S_1, S_2 \) are given by:

- \( S_1(i) = \{(c_1, \ldots, c_n) : c_i = 0 \text{ for } i \notin \{i_t, i + 1\}, c_i = s_{i_t} - s \text{ for } s \in [s_{i_t} + 1]\} \),
- \( S_2(i) = \{(c_1, \ldots, c_n) : c_i = 0 \text{ for } i \notin \{i_t, i + 1\}, c_i = s_{i_t} - 1 - s \text{ and } c_{i+1} = s \text{ for } s \in [s_{i_t}]\} \).

**Definition 3.2.** For a simple graph \( G \) on \([0, n]\), denote by \( \mathcal{T}(G) \) the specific reduction tree constructed using the algorithm described above.

**Definition 3.3.** To each leaf \( L \) of \( \mathcal{T}(G) \), associate the triangular array of numbers \( \text{Arr}(L) \) given by

\[
a_{n,1} \quad a_{n-1,1} \quad \cdots \quad a_{3,1} \quad a_{2,1} \quad a_{1,1} \\
a_{n,2} \quad a_{n-1,2} \quad \cdots \quad a_{3,2} \quad a_{2,2} \\
\vdots \quad \vdots \quad \ddots \\
a_{n,n-1} \quad a_{n-1,n-2} \\
a_{n,n}
\]

where \((a_{i,1}, a_{i,2}, \ldots, a_{i,n})\) is the left-degree sequence of the leaf of \( \mathcal{T}(G[0, i]) \) preceding (or equaling if \( i = n \)) \( L \) in the construction of \( \mathcal{T}(G) \).

**Theorem 3.4 ([10], Theorem 4).** The arrays \( \text{Arr}(L) \) for full-dimensional leaves \( L \) of \( \mathcal{T}(G) \) are exactly the nonnegative integer solutions in the variables \( \{a_{i,j} : 1 \leq j \leq i \leq n\} \) to the constraints:

- \( a_{1,1} = \#E(G[0, 1]) \)
- \( a_{i,j} \leq a_{i-1,j} \) if \((j, i) \in E(G)\)
- \( a_{i,j} = a_{i-1,j} \) if \((j, i) \notin E(G)\)
- \( a_{i,i} = \#E(G[0, i]) - \sum_{k=1}^{i-1} a_{i,k} \)

**Example 3.5.** If \( G \) is the graph on vertex set \([0, 4]\) with \( E(G) = \{(0, 1), (0, 2), (1, 2), (2, 3), (2, 4), (3, 4)\} \), then from Theorem [3.4](#) we obtain the constraints:

\[
0 \leq a_{4,1} = a_{3,1} = a_{2,1} \leq a_{1,1} = 1 \\
0 \leq a_{4,2} \leq a_{3,2} \leq a_{2,2} = 3 - a_{2,1} \\
0 \leq a_{4,3} \leq a_{3,3} = 4 - a_{3,1} - a_{3,2} \\
0 \leq a_{4,4} = 6 - a_{4,1} - a_{4,2} - a_{4,3}
\]

The solutions to these constraints yield the full-dimensional left-degree sequences \((a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4})\) of \( G \).
Given a graph $G$, we write the constrains specified in Theorem 3.8 in the form shown in Example 3.5 and call them the **triangular constraint array** of $G$. We proceed by generalizing triangular constraint arrays to encode the lower-dimensional leaves of $T(G)$ as well.

**Definition 3.6.** Denote by $\text{Tri}_G(\emptyset)$, or when the context is clear, by $\text{Tri}(\emptyset)$, the triangular constraint array of $G$. For each subset $F \subseteq E(G\setminus\emptyset)$ (recall that $G$ is simple in this section), define a constraint array $\text{Tri}(F)$ by modifying $\text{Tri}(\emptyset)$ as follows: for each $(j, i) \in F$ and each ordered pair $(m, j)$ with $n \geq m \geq i$, replace each occurrence of $a_{m,j}$ by $a_{m,j} + 1$ and add 1 to the constant at the leftmost edge of row $j$. Denote by $\text{Sol}_G(F)$, or when the context is clear, by $\text{Sol}(F)$, the collection of all integer solution arrays to the constraints $\text{Tri}(F)$.

**Example 3.7.** With $G$ as in Example 3.5 and $F = \{(2, 3), (2, 4), (3, 4)\}$, we have

$$\text{Tri}(F) : \begin{align*}
0 &\leq a_{4,1} = a_{3,1} = a_{2,1} \leq a_{1,1} = 1 \\
2 &\leq a_{4,2} + 2 \leq a_{3,2} + 1 \leq a_{2,2} = 3 - a_{2,1} \\
1 &\leq a_{4,3} + 1 \leq a_{3,3} = 3 - a_{3,1} - a_{3,2} \\
0 &\leq a_{4,4} = 3 - a_{4,1} - a_{4,2} - a_{4,3}
\end{align*}$$

The characterization of $\text{InSeq}(T(G))$ given in the construction of $T(G)$ implies the following theorem.

**Theorem 3.8.** The leaves of $T(G)$ are in bijection with the multiset union of solutions to the arrays $\text{Tri}(F)$, that is

$$\{\text{Arr}(L) : L \text{ is a leaf of } T(G)\} = \bigcup_{F \subseteq E(G\setminus\emptyset)} \text{Sol}_G(F).$$

In particular, $\text{InSeq}(T(G))$ is the (multiset) image of the right-hand side under the map that takes a triangular array to its first column $(a_{n,1}, \ldots, a_{n,n})$.

Since Theorem A shows that $\text{InSeq}(R(G)) = \text{LD}(G)$ for any reduction tree $R(G)$ of $G$, we can now state the following important definition.

**Definition 3.9.** For any $F \subseteq E(G\setminus\emptyset)$, denote by $\text{LD}(G,F)$ the submultiset of $\text{LD}(G)$ consisting of sequences occurring as the first column of an array in $\text{Sol}(F)$.

As a consequence of Theorem 3.8

$$\text{InSeq}(T(G)) = \bigcup_{F \subseteq E(G\setminus\emptyset)} \text{LD}(G,F).$$

Combinatorially, we can think of $\text{LD}(G,F)$ in the following way. Construct the reduction tree $T(G)$ of $G$. Take any graph $H$ appearing as a node of $T(G)$. Let $H$ have descendants $H_1$, $H_2$ and $H_3$ in $T(G)$ obtained by the reduction on edges $(i, j)$ and $(j, k)$ in $H$ with $i < j < k$, so that $H_3$ has edge set $E(H)\setminus\{(i, j), (j, k)\} \cup \{(i, k)\}$. Label the edge in $T(G)$ between $H$ and $H_3$ by $(j, k)$. To each leaf $L$ of $T(G)$, associate the set of all labels on the edges of the unique path from $L$ to the root $G$ of $T(G)$. The left-degree sequences of leaves assigned a set $F$ in this manner are exactly the elements of the multiset $\text{LD}(G,F)$.

To understand the multisets $\text{Sol}(F)$ and $\text{LD}(G,F)$, we study the constraint arrays $\text{Tri}(F)$. We begin by investigating the case where $G = K_{n+1}$ is the complete graph on $[0, n]$. Given $F \subseteq E(K_{n+1}\setminus\emptyset)$, consider the numbers
(3.1) \[ f_{i,j} = \#\{(j, k) \in F : k \leq i\}. \]

Observe that for each \( F \subseteq E(K_{n+1\setminus 0}) \), \( \text{Tri}(F) \) is obtained from \( \text{Tri}(\emptyset) \) by replacing \( a_{i,j} \) in \( \text{Tri}(\emptyset) \) by \( a_{i,j} + f_{i,j} \) and replacing the 0 in the leftmost spot of row \( j \) by \( f_{n,j} \). Also note that \( f_{j,j} = 0 \) for each \( j \). Modify \( \text{Tri}(F) \) to obtain a new constraint array denoted \( A_{K_{n+1}}(F) \) with the same solutions by subtracting \( f_{n,j} \) from each term in row \( j \) for each \( j \), so that the leftmost column becomes all zeros. For notational compactness, let \( b_{i,j} = a_{i,j} + f_{i,j} \).

\( A_{K_{n+1}}(F) \) is given by

\[
\begin{align*}
0 & \leq b_{n,1} - f_{n,1} \\
0 & \leq b_{n,2} - f_{n,2} \\
\vdots & \quad \vdots \\
0 & \leq b_{n,n} - f_{n,n} \end{align*}
\]

\( 0 \leq a_{2,1} \leq a_{1,1} = 1 \)
\( 0 \leq a_{2,2} = 2 - a_{2,1} \)

\( 0 \leq a_{2,1} + 1 \leq a_{1,1} = 1 \)
\( 0 \leq a_{2,2} = 2 - a_{2,1} \)

Note that the real solution set in variables \( \{a_{i,j}\} \) to \( A_{K_{n+1}}(F) \) is a polytope in \( \mathbb{R}^{(n+1)^2} \).

We first show that it is a flow polytope. For any constraint array \( A \), denote by \( \text{Poly}(A) \) the polytope defined by the inequalities in \( A \).

**Lemma 3.10.** Let \( K_{n+1} \) be the complete graph on \([0,n] \). Fix \( F \subseteq E(K_{n+1\setminus 0}) \) and let \( Q = \text{Poly}(A_{K_{n+1}}(F)) \). Then, there exists a graph denoted \( \text{Gr}(K_{n+1}) \) and a netflow vector \( a_{K_{n+1}}^F \) such that \( Q \) is integrally equivalent to \( \mathcal{F}_{\text{Gr}(K_{n+1})}(a_{K_{n+1}}^F) \).

**Figure 4.** A small example demonstrating Theorem 3.8. In general, \( \text{Sol}_{\mathcal{T}}(F) \) will be empty for many \( F \).
Proof. For \( \{(i, j) : 1 \leq j < i \leq n\} \), we introduce slack variables \( z_{i,j} \) to convert the inequalities in \( A_{K_{n+1}}(F) \) into equations \( Y_{i,j} \) via

\[
Y_{i,j} : \begin{cases}
   a_{i,j} + f_{i,j} + z_{i,j} = a_{i-1,j} + f_{i-1,j} & \text{if } i > j \\
   \sum_{k=1}^{i} a_{i,k} = \#E(K_{n+1}[0,i]) & \text{if } i = j.
\end{cases}
\]

Define an equivalent system of equations \( \{Z'_{i,j}\} \) by setting

\[
Z'_{i,j} : \begin{cases}
   Y_{i,j} & \text{if } i > j \text{ or } i = j = 1 \\
   Y_{i,j} - Y_{i-1,j-1} - \sum_{k=1}^{j-1} Y_{j,k} & \text{if } i > j > 1.
\end{cases}
\]

We then modify each equation \( Z'_{i,j} \) by rearranging negated terms to get equations \( Z_{i,j} \) given by

\[
Z_{i,j} : \begin{cases}
   a_{i,j} + z_{i,j} = a_{i-1,j} + f_{i-1,j} - f_{i,j} & \text{if } i > j \\
   a_{i,j} = \mathrm{indeg}_{K_{n+1}}(1) & \text{if } i = j = 1 \\
   a_{i,j} = \mathrm{indeg}_{K_{n+1}}(j) + \sum_{k=1}^{j-1} z_{j,k} & \text{if } i > j > 1
\end{cases}
\]

where we use that \( \#E(K_{n+1}[0,j]) - \#E(K_{n+1}[0,j-1]) = \mathrm{indeg}_{K_{n+1}}(j) \).

We now construct the graph \( \mathrm{Gr}(K_{n+1}) \). Let the vertices of \( \mathrm{Gr}(K_{n+1}) \) be

\[ \{v_{i,j} : 1 \leq j \leq i \leq n\} \cup \{v_{n+1,n+1}\} \]

with the ordering \( v_{1,1} < v_{2,1} < \cdots < v_{n,1} < v_{2,2} < \cdots < v_{n,n} < v_{n+1,n+1} \).

Let the edges of \( \mathrm{Gr}(K_{n+1}) \) be labeled suggestively by the flow variables \( a_{i,j} \) and \( z_{i,j} \). Set \( E(\mathrm{Gr}(K_{n+1})) = E_a \cup E_z \) where

- \( E_a \) consists of edges \( a_{i,j} : v_{i,j} \to v_{i+1,j} \) for \( 1 \leq j \leq i \leq n \) and
- \( E_z \) consists of edges \( z_{i,j} : v_{i,j} \to v_{i,i} \) for \( 1 \leq j < i \leq n \)

and we take indices \( (n+1, j) \) to refer to \( (n+1, n+1) \).

To define the netflow vector \( a_{K_{n+1}}^F \), we assign netflow \( \mathrm{indeg}_{K_{n+1}}(j) \) to vertices \( v_{j,j} \) with \( j < n+1 \), we assign netflow

\[-\#E(K_{n+1}) + \sum_{k=1}^{n-1} f_{n,k}\]

to \( v_{n+1,n+1} \), and we assign netflow \( f_{i-1,j} - f_{i,j} \) to each remaining vertex \( v_{i,j} \).

The netflow vector \( a_{K_{n+1}}^F \) is given by reading each row of the triangular array

\[
\begin{array}{cccccccc}
   f_{n-1,1} - f_{n,1} & f_{n-2,1} - f_{n-1,1} & \cdots & f_{1,1} - f_{2,1} & \mathrm{indeg}_{K_{n+1}}(1) \\
   f_{n-1,2} - f_{n,2} & \cdots & f_{2,2} - f_{3,2} & \mathrm{indeg}_{K_{n+1}}(2) \\
   \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
   \mathrm{indeg}_{K_{n+1}}(n) & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{array}
\]

\]
right to left starting with the first row, moving top to bottom, and then appending $-\#E(K_{n+1}) + \sum_{k=1}^{n-1} f_{n,k}$ at the end.

By construction, the flow equation at vertex $v_{i,j}$ in $Gr(K_{n+1})$ is exactly the equation $Z_{i,j}$ for $(i, j) \neq (n + 1, n + 1)$. At $v_{n+1,n+1}$, the flow equation is $Y_{n,n}$, which follows from the equations $Z_{i,j}$ and adds no additional restrictions. \qed

We now generalize Lemma 3.10 to any simple graph $G$ on $[0, n]$. Note that for $F \subseteq E(G\setminus 0)$, $\text{Tri}_G(F)$ can be obtained from $\text{Tri}_{K_{n+1}}(F)$ by turning certain inequalities into equalities and changing all occurrences of $\#E(K_{n+1}[0,j])$ to $\#E(G[0,j])$ for each $j$. In the language of the proof of Lemma 3.10, this amounts to setting $z_{i,j} = 0$ whenever $(j, i) \notin E(G)$. Relative to the graph $Gr(K_{n+1})$, this is equivalent to deleting the edges labeled $z_{i,j}$ for $(j, i) \notin E(G)$. Thus, we have the following extension of the construction given in the proof of Lemma 3.10.

Definition 3.11. For a simple graph $G$ on $[0, n]$ define a graph $Gr(G)$ on vertices

$$\{v_{i,j} : 1 \leq j \leq i \leq n\} \cup \{v_{n+1,n+1}\}$$

ordered $v_{1,1} < v_{2,1} < \cdots < v_{n,1} < v_{2,2} < \cdots < v_{n,n} < v_{n+1,n+1}$ and with edges $E_a \cup E_z$ where

- $E_a$ consists of edges $a_{i,j} : v_{i,j} \rightarrow v_{i+1,j}$ for $1 \leq j \leq i \leq n$ and
- $E_z$ consists of edges $z_{i,j} : v_{i,j} \rightarrow v_{i,i}$ for $(j, i) \in E(G\setminus 0)$.

For any $F \subseteq E(G\setminus 0)$, we define a netflow vector $\mathbf{a}_G^F$ for $Gr(G)$ by reading each row of the triangular array

\[
\begin{array}{cccccc}
    f_{n-1,1} - f_{n,1} & f_{n-2,1} - f_{n-1,1} & \cdots & f_{1,1} - f_{2,1} & \text{indeg}_G(1) \\
    f_{n-1,2} - f_{n,2} & \cdots & f_{2,2} - f_{3,2} & \text{indeg}_G(2) \\
    \vdots & \ddots & \ddots & \ddots \\
    \text{indeg}_G(n) \\
\end{array}
\]

right to left starting with the first row, moving top to bottom, and then appending $-\#E(G) + \sum_{k=1}^{n-1} f_{n,k}$ at the end, where $f_{i,j} = \#\{(j, k) \in F : k \leq i\}$.

Proposition 3.12. Let $G$ be a simple graph on $[0, n]$ and $F \subseteq E(G\setminus 0)$. Then, $\text{Poly}(\text{Tri}_G(F))$ is integrally equivalent to $\mathcal{F}_{Gr(G)}(\mathbf{a}_G^F)$. In particular, the multiset of solutions $\text{Sol}(F)$ to $\text{Tri}(F)$
consists precisely of the projections of integral flows on \( \text{Gr}(G) \) with netflow \( \mathbf{a}_G^F \) onto the edges labeled \( \{a_{i,j}\} \).

**Example 3.13.** Let \( G \) be the graph on vertex set \([0, 4]\) with edge set 
\( E(G) = \{(0, 1), (0, 2), (1, 2), (2, 3), (2, 4), (3, 4)\} \) and \( F = \{(2, 3)\} \). The graph \( \text{Gr}(G) \) and netflow \( \mathbf{a}_G^F \) are:

![Diagram](image)

Observe that contracting the edges \( \{a_{1,1}, a_{2,1}, a_{3,1}, a_{2,2}, a_{3,2}, a_{3,3}\} \) yields the graph below, which is exactly \( \tilde{G}\backslash\{s, 0\} \). The next result shows that this occurs in general.

![Diagram](image)

For a graph \( G \) and a subset \( F \subseteq E(G \backslash 0) \), view \( F \) as a subgraph of \( G \) on the same vertex set. Note that for each \( j \),
\[
f_{n,j} = \#\{(j, k) \in F : k \leq n\} = \text{outdeg}_F(j)
\]
and the number
\[
-\#E(G) + \sum_{k=1}^{n-1} f_{n,k}
\]
appearing as the last entry of \( \mathbf{a}_G^F \) equals \( -\#E(G \backslash F) \).

**Theorem 3.14.** Let \( G \) be a simple graph on \([0, n]\) and \( F \subseteq E(G \backslash 0) \). Then, the flow polytopes
\[
\mathcal{F}_{\text{Gr}(G)} \left( \mathbf{a}_G^F \right) \text{ and } \mathcal{F}_{\tilde{G}\backslash\{s, 0\}} \left( d_1^F, d_2^F, \ldots, d_n^F, -\#E(G \backslash F) \right)
\]
are integrally equivalent, where \( d_j^F = \text{indeg}_G(j) - \text{outdeg}_F(j) \) for \( j \in [n] \).

**Proof.** First, note that in \( \text{Gr}(G) \), the edges \( \{a_{i,j} : i < n\} \) are each the only edges incoming to their target vertex. Contracting these edges via Lemma 2.2 identifies vertices \( v_{i,j} \) and \( v'_{i,j} \). Label the representative vertices \( v_{j,j} \) by \( j \) for \( j \in [n] \) and \( v_{n+1,n+1} \) by \( t \). The remaining edges are
\[
z_{i,j} : j \rightarrow i \text{ for } (j, i) \in E(G) \text{ and } a_{n,j} : j \rightarrow t \text{ for } j \in [n],
\]
which, are exactly the edges of $\tilde{G} - \{s, 0\}$.

Viewing the netflow vector $a^F_G$ as the array

$$
\begin{align*}
&f_{n-1,1} - f_{n,1} & f_{n-2,1} - f_{n-1,1} & \cdots & f_{1,1} - f_{2,1} & \text{indeg}_G(1) \\
&f_{n-1,2} - f_{n,2} & \cdots & f_{2,2} - f_{3,2} & \text{indeg}_G(2) \\
&\vdots & \ddots & \vdots & \vdots & \vdots \\
&\text{indeg}_G(n) & \ldots & \text{indeg}_G(n) & -\#E(G \setminus F),
\end{align*}
$$

Lemma 2.2 implies the entries of the netflow vector after contracting are given by reading the sums of each row from top to bottom.

Recall from Definition 3.9 that $LD(G, F)$ is the multiset of left-degree sequences occurring as the first column $(a_{n,1}, \ldots, a_{n,n})$ of an array in $Sol(F)$.

**Corollary 3.15.** Let $G$ be a simple graph on $[0, n]$ and $F \subseteq E(G \setminus 0)$. If $b^F_G$ is the vector

$$b^F_G = (\text{indeg}_G(1) - \text{outdeg}_G(1), \ldots, \text{indeg}_G(n) - \text{outdeg}_G(n), -\#E(G \setminus F))$$

and $\psi$ is the map that takes a flow on $\tilde{G} \setminus \{s, 0\}$ to the tuple of its values on the edges $\{(j, t) : j \in [n]\}$, then $LD(G, F)$ equals the (multiset) image under $\psi$ of all integral flows on $\tilde{G} \setminus \{s, 0\}$ with netflow vector $b^F_G$.

In particular, $LD(G, F)$ is in bijection with integral flows on $\tilde{G} \setminus \{s, 0\}$ with netflow $b^F_G$.

We note that the preceding result implies a formula for the Ehrhart polynomial of flow polytopes of graphs with special source and sink vertices. In particular, a special case of Theorem 2.1 follows readily.

**Theorem 3.16.** Let $G$ be a simple graph on $[0, n]$ and let $d_i = \text{indeg}_G(i)$. Then, the normalized volume of the flow polytope on $\tilde{G}$ is

$$\text{Vol } F_{\tilde{G}} = K_{\tilde{G} \setminus \{s, 0\}}(d_1, \ldots, d_n, -\#E(G)).$$

Moreover, the Ehrhart polynomial of $F_{\tilde{G}}$ is

$$Ehr(F_{\tilde{G}}, t) = (-1)^d \sum_{i=0}^{d} (-1)^i \left( \sum_{F \subseteq E(G \setminus 0)} K_{\tilde{G} \setminus \{s, 0\}}(b^F_G) \right) \left( t + i \right),$$

where $b^F_G = (\text{indeg}_G(1) - \text{outdeg}_G(1), \ldots, \text{indeg}_G(n) - \text{outdeg}_G(n), -\#E(G \setminus F))$.

**Proof.** From the dissection of $F_{\tilde{G}}$ obtained via the reduction tree $T(G)$, it follows that $\text{Vol } F_{\tilde{G}}$ is the number of full-dimensional left-degree sequences. By Corollary 3.15, these are in bijection with the integer points in the flow polytope $F_{\tilde{G} \setminus \{s, 0\}}(d_1, \ldots, d_n, -\#E(G))$, proving (3.2).

To prove (3.3) note that $F^o_{\tilde{G}} = \bigcup_{\sigma^o \in D_{T(G)}} \sigma^o$, where $D_{T(G)}$ is the set of open simplices corresponding to the leaves of the reduction tree $T(G)$. Then, $Ehr(F^o_{\tilde{G}}, t) = \sum_{\sigma^o \in D_{T(G)}} Ehr(\sigma^o, t)$. Since all simplices in $D_{T(G)}$ are unimodular, it follows that for a $k$-dimensional simplex
\[\sigma^o \in D_{T(G)}, \quad \text{Ehr}(\sigma^o, t) = \text{Ehr}(\Delta^o, t), \quad \text{where } \Delta \text{ is the standard } k\text{-simplex. By [3, Theorem 2.2] } \text{Ehr}(\Delta^o, t) = \binom{t-1}{k}. \quad \text{Thus, } \text{Ehr}(\mathcal{F}_G^o, t) = \sum_{i=0}^{\infty} f_i \binom{t-1}{i}, \quad \text{where } f_i \text{ is the number of } i\text{-simplices in } D_{T(G)}. \]

If we let \(d = \#E(\tilde{G}) - \#V(\tilde{G}) + 1\), which is the dimension of \(\mathcal{F}_G\), then for \(i \in [0, d]\),

\[f_i = \sum_{F \subseteq E(G) \setminus \{0\}} K_{\tilde{G}\setminus\{s, 0\}}(b_G^F) \quad \text{for } i \in [0, d].\]

Corollary 3.13 then implies

\[f_i = \sum_{F \subseteq E(G) \setminus \{0\}} K_{\tilde{G}\setminus\{s, 0\}}(b_G^F) \quad \text{for } i \in [0, d].\]

Therefore,

\[\text{Ehr}(\mathcal{F}_G^o, t) = \sum_{i=0}^{d} \left( \sum_{F \subseteq E(G) \setminus \{0\}} K_{\tilde{G}\setminus\{s, 0\}}(b_G^F) \right) \binom{t-1}{i}.\]

From the Ehrhart-Macdonald reciprocity [3, Theorem 4.1]

\[\text{Ehr}(\mathcal{F}_G^o, t) = (-1)^d \text{Ehr}(\mathcal{F}_G^o, -t),\]

it follows that

\[\text{Ehr}(\mathcal{F}_G^o, t) = (-1)^d \sum_{i=0}^{d} \left( \sum_{F \subseteq E(G) \setminus \{0\}} K_{\tilde{G}\setminus\{s, 0\}}(b_G^F) \right) \binom{-t-1}{i} \]

\[= (-1)^d \sum_{i=0}^{d} (-1)^i \left( \sum_{F \subseteq E(G) \setminus \{0\}} K_{\tilde{G}\setminus\{s, 0\}}(b_G^F) \right) \binom{t+i}{i}.\]

\[
\square
\]

4. **Newton polytopes of left-degree polynomials**

In this section, we study the Newton polytopes of polynomials \(L_G(t)\) built from left-degree sequences (see Definition 4.2). We first show that each of these polynomials have polytopal support (Definition 4.1). Then, we investigate the Newton polytopes of their homogeneous components and certain homogeneous subcomponents and prove that these Newton polytopes are generalized permutahedra. We can summarize some of our results as:

**Theorem B.** Let \(G\) be a graph on \([0, n]\). Then the left-degree polynomial \(L_G(t)\) has polytopal support, and the Newton polytope of each homogeneous component \(L_G^k(t)\) of \(L_G(t)\) of degree \(\#E(G) - k\) is a generalized permutahedron.

Theorems 4.1, 4.9 and 4.21 imply Theorem B and also contain a lot more detail regarding the players in Theorem B.
Definition 4.1. Recall that for a polynomial \( f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha t^\alpha \), the **Newton polytope** is
\[
\text{Newton}(f) = \text{Conv } \{ \alpha : c_\alpha \neq 0 \}.
\]
We say the polynomial \( f \) has **polytopal support** if \( c_\alpha \neq 0 \) whenever \( \alpha \in \text{Newton}(f) \), that is whenever the integer points of \( \text{Newton}(f) \) are exactly the exponents of monomials appearing in \( f \) with nonzero coefficients.

The question of when a polynomial has polytopal support is a very natural one, and has recently been investigated for various polynomials from algebra and combinatorics by Monical, Tokcan and Yong in [13], who refer to this notion as the SNP property (saturated Newton polytope property).

Recall from Definition 3.9 that for a simple graph \( G \) and a subset \( F \subseteq E(G\setminus 0) \), \( \text{LD}(G, F) \) denotes the submultiset of \( \text{LD}(G) \) consisting of sequences occurring as the first column of an array in \( \text{Sol}(F) \). Just as in Section 3, for the remainder of this section we add the simplifying assumption that \( G \) has no multiple edges. All of the results of this section are also valid for graphs with multiple edges, with similar proof and notation modifications to those described in Section 6.

Definition 4.2. Let \( G \) be a graph on \([0,n]\). For \( \alpha \in \text{LD}(G) \), let \( \text{codim}(\alpha) = \#E(G) - \sum_{i=1}^{n} \alpha_i \). Define the **left-degree polynomial** \( L_G(t) \) in variables \( t = (t_1, t_2, \ldots, t_n) \) by
\[
L_G(t) = \sum_{\alpha \in \text{LD}(G)} (-1)^{\text{codim}(\alpha)} t^\alpha.
\]
Similarly, for \( F \subseteq E(G\setminus 0) \), define \( L_{G,F}(t) \) by
\[
L_{G,F}(t) = \sum_{\alpha \in \text{LD}(G,F)} (-1)^{\text{codim}(\alpha)} t^\alpha = \sum_{\alpha \in \text{LD}(G,F)} (-1)^{\#F} t^\alpha.
\]

Note that the \((-1)^{\text{codim}(\alpha)}\) in Definition 4.2 has no effect on the Newton polytope. It is present so the definition of the left-degree polynomial agrees with the definition of right-degree polynomials utilized in [4] that we address in Section 5.

Restating Theorem 3.8 in terms of left-degree sequences gives the multiset union decomposition
\[
\text{LD}(G) = \bigcup_{F \subseteq E(G\setminus 0)} \text{LD}(G, F).
\]
Relative to Newton polytopes, this implies
\[
(4.1) \quad \text{Newton}(L_G(t)) = \text{Conv} \left( \bigcup_{F \subseteq E(G\setminus 0)} \text{Newton}(L_{G,F}(t)) \right).
\]

We first study the polytope \( \text{Newton}(L_G(t)) \) and then the component pieces \( \text{Newton}(L_{G,F}(t)) \).

To start, we define a new constraint array.

Definition 4.3. Let \( G \) be a simple graph on \([0,n]\). Proceed as follows:

- Start with the triangular constraint array \( \text{Tri}_G(\emptyset) \) of \( G \) as in Theorem 3.4.
- Replace the zero on the left of row \( j \) by \( y_{n,j} + y_{n-1,j} + \cdots + y_{j+1,j} \) for \( j \in [n-1] \), so the zero on the left in row \( n \) is left unchanged.
• For each \((i, j)\) with \(n \geq i > j \geq 1\), replace all occurrences of \(a_{i,j}\) in the array by \(a_{i,j} + \sum_{k=j+1}^{i} y_{k,j}\).

• For every \((j, i) \notin E(G \setminus 0)\), set \(y_{i,j} = 0\) throughout.

We refer to this array as the augmented constraint array of \(G\) and view it as having variables \(a_{i,j}\) and \(y_{i,j}\) subject to the additional constraints that for all \(1 \leq j < i \leq n\),

\[0 \leq y_{i,j} \leq 1.\]

Example 4.4. If \(G\) is the graph on vertex set \([0, 4]\) with \(E(G) = \{(0, 1), (0, 2), (1, 2), (2, 3), (2, 4), (3, 4)\}\), then we start with the constraints:

\[
\begin{align*}
0 & \leq a_{4,1} = a_{3,1} = a_{2,1} = a_{1,1} = 1 \\
0 & \leq a_{4,2} \leq a_{3,2} \leq a_{2,2} = 3 - a_{2,1} \\
0 & \leq a_{4,3} \leq a_{3,3} = 4 - a_{3,1} - a_{3,2} \\
0 & \leq a_{4,4} = 6 - a_{4,1} - a_{4,2} - a_{4,3}
\end{align*}
\]

After performing the modifications, we arrive at:

\[
\begin{align*}
 y_{2,1} & \leq a_{4,1} + y_{2,1} = a_{3,1} + y_{2,1} = a_{2,1} + y_{2,1} \leq a_{1,1} = 1 \\
y_{4,2} + y_{3,2} & \leq a_{4,2} + y_{4,2} + y_{3,2} \leq a_{3,2} + y_{3,2} \leq a_{2,2} = 3 - a_{2,1} - y_{2,1} \\
y_{4,3} & \leq a_{4,3} + y_{4,3} \leq a_{3,3} = 4 - a_{3,1} - y_{2,1} - a_{3,2} - y_{3,2} \\
0 & \leq a_{4,4} = 6 - a_{4,1} - y_{2,1} - a_{4,2} - y_{4,2} - y_{3,2} - a_{4,3} - y_{4,3}
\end{align*}
\]

Theorem 4.5. Let \(A\) denote the augmented constraint array of \(G\) and \(\text{Poly}(A)\) the polytope defined by the real valued solutions to \(A\) with the additional constraints \(0 \leq y_{i,j} \leq 1\) for all \(i, j\) with \(1 \leq i, j \leq n\). If \(\rho\) is the projection that maps a solution of \(A\) to its values \((a_{n,1}, \ldots, a_{n,n})\), then

\[
\text{Newton}(L_G(t)) = \rho(\text{Poly}(A)).
\]

Furthermore, each integer point in the right-hand side is in \(LD(G)\), so \(L_G\) has polytopal support.

For the proof of Theorem 4.5 and later Theorem 4.20, it is convenient to replace \(\text{Poly}(A)\) by an integrally equivalent flow polytope using the proof techniques from Lemma 3.10 and Theorem 3.14. Begin with the case where \(G\) is a complete graph. By introducing slack variables \(z_{i,j}\) for the inequalities in the augmented constraint array (not \(0 \leq y_{i,j} \leq 1\)), we get equations \(Y_{i,j}\) given by

\[
Y_{i,j} : \begin{cases} 
  a_{i,j} + y_{i,j} + z_{i,j} = a_{i-1,j} & \text{if } i > j \\
  a_{i,j} = #E(G[0, i]) & \text{if } i = j = 1 \\
  \sum_{k=1}^{i} a_{i,k} + \sum_{m=2}^{i-1} \sum_{k=1}^{m} y_{m,k} = #E(G[0, i]) & \text{if } i = j > 1 
\end{cases}
\]
Applying the exact same transformation used in the proof of Lemma 3.10, we get equivalent equations $Z_{i,j}$ given by

$$Z_{i,j} : \begin{cases} a_{i,j} + y_{i,j} + z_{i,j} = a_{i-1,j} & \text{if } i > j \\ a_{i,j} = \text{indeg}_G(1) & \text{if } i = j = 1 \\ a_{i,j} = \text{indeg}_G(i) + \sum_{k=1}^{i-1} z_{i,k} & \text{if } i > 1 \end{cases}$$

To move from the complete graph to any simple graph, just set $y_{i,j} = 0$ and $z_{i,j} = 0$ whenever $(j,i) \notin E(G)$. We can realize the solutions to the $Z_{i,j}$ as points in a flow polytope of some graph. However, to account for the additional restrictions $0 \leq y_{i,j} \leq 1$, we view it as a capacitated flow polytope. This is for convenience and is not mathematically significant since any capacitated flow polytope is integrally equivalent to an uncapacitated flow polytope.

**Definition 4.6.** Define the augmented constraint graph $\text{Gr}^{\text{aug}}(G)$ to have vertex set \{v_{i,j} : 1 \leq j \leq i \leq n\} with the ordering $v_{1,1} < v_{2,1} < \cdots < v_{n,1} < v_{2,2} < \cdots < v_{n,n} < v_{n+1,n+1}$ and edge set $E_a \cup E_z \cup E_y$ labeled by the variables $a_{i,j}$, $z_{i,j}$, and $y_{i,j}$ respectively, where

- $E_a$ consists of edges $a_{i,j} : v_{i,j} \rightarrow v_{i+1,j}$ for $1 \leq j \leq i \leq n$,
- $E_z$ consists of edges $z_{i,j} : v_{i,j} \rightarrow v_{i,i}$ for $(j,i) \in E(G \setminus 0)$,
- $E_y$ consists of edges $y_{i,j} : v_{i,j} \rightarrow v_{n+1,n+1}$ for $(j,i) \in E(G \setminus 0)$,

and we take indices $(n+1, j)$ to refer to $(n+1, n+1)$. Define a netflow vector $a^\text{aug}_G$ by reading each row of the array

$$\begin{array}{ccccccc} 0 & 0 & \cdots & 0 & \text{indeg}_G(1) \\ 0 & 0 & \cdots & 0 & \text{indeg}_G(2) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \text{indeg}_G(n) & \cdots & \cdots & \cdots & \text{#}_E(G) \end{array}$$

from right to left and reading the rows from top to bottom.

Denote by $\mathcal{F}_{\text{Gr}^{\text{aug}}(G)}(a^\text{aug}_G)$ the capacitated flow polytope of the graph $\text{Gr}^{\text{aug}}(G)$ with netflow $a^\text{aug}_G$ and with the capacity constraints $0 \leq y_{i,j} \leq 1$ for all $1 \leq j < i \leq n$. By construction, the points in $\mathcal{F}_{\text{Gr}^{\text{aug}}(G)}(a^\text{aug}_G)$ are exactly the solutions to the augmented constraint array of $G$.

**Definition 4.7.** Similar to Theorem 3.14, contracting the edges \{a_{i,j} : 1 \leq j \leq i < n\} of $\text{Gr}^{\text{aug}}(G)$ and relabeling the representative vertices $v_{n,j}$ by $j$ and $v_{n+1,n+1}$ by $t$, we obtain a graph called the augmented graph of $G$. This graph is denoted $G^{\text{aug}}$ and is defined on vertices $[n] \cup \{t\}$ with labeled edges $E_a \cup E_z \cup E_y$ where

- $E_a$ consists of edges $a_{n,j} : j \rightarrow t$ for $j \in [n]$;
- $E_z$ consists of edges $z_{i,j} : j \rightarrow i$ for $(j,i) \in E(G \setminus 0)$;
- $E_y$ consists of edges $y_{i,j} : j \rightarrow t$ for $(j,i) \in E(G \setminus 0)$.
Example 4.8. For $G$ the complete graph on $[0, 3]$, the graphs $Gr_{\text{aug}}(G)$ and $G_{\text{aug}}$ are shown below.

Before proceeding to the proof of Theorem 4.5, recall that
\[ b_G^F = (\text{indeg}_G(1) - \text{outdeg}(1), \ldots, \text{indeg}_G(n) - \text{outdeg}(n), -\#E(G' \setminus F)) \]
for any $F \subseteq E(G' \setminus 0)$. Denote by $F_{\text{aug}}^\mathbb{Z}$ the capacitated flow polytope of the graph $G_{\text{aug}}$ with netflow $b_G^0$ and the capacity constraints $0 \leq y_{i,j} \leq 1$ for all $1 \leq j < i \leq n$.

Proof of Theorem 4.5. By the constructions of Definitions 4.6 and 4.7, we have integral equivalences of capacitated flow polytopes
\[ \text{Poly}(A) \equiv F_{G_{\text{aug}}}^\mathbb{Z}(a_G^{\text{aug}}) \equiv F_{G_{\text{aug}}}^c(b_G^0). \]

Thus, it suffices to prove
\[ \text{Newton}(L_G(t)) = \psi(F_{G_{\text{aug}}}^c(b_G^0)). \]
where $\psi$ is the projection that takes a flow on $F_{G_{\text{aug}}}^c(b_G^0)$ to its values on the edges labeled \{a_{n,j} : j \in [n]\}. This is accomplished in Theorem 4.9. \qed

Theorem 4.9. For $G$ a graph on $[0, n]$, the Newton polytope of the left-degree polynomial $L_G(t)$ and the capacitated flow polytope $F_{G_{\text{aug}}}^c(b_G^0)$ satisfy
\[ \text{Newton}(L_G(t)) = \psi(F_{G_{\text{aug}}}^c(b_G^0)), \]
where where $\psi$ is the projection that takes a flow on $F_{G_{\text{aug}}}^c(b_G^0)$ to its values on the edges labeled \{a_{n,j} : j \in [n]\}.

Proof. Let $\alpha \in \text{LD}(G, F)$ for $F \subseteq E(G' \setminus 0)$. Consider the set of integer flows on $G_{\text{aug}}$ such that each edge $y_{i,j}$ has flow 1 if $(j, i) \in F$ and zero otherwise. By the construction of $G_{\text{aug}}$, these are in bijection with the integer flows on $\tilde{G}\setminus\{s, 0\}$ with netflow vector $b_G^F$, which in turn are in bijection to $(\text{LD}(G, F))$ (Corollary 3.15). Thus $\alpha$ is the projection of a capacitated flow on $G_{\text{aug}}$ with netflow $b_G^F$.

Conversely, let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \psi(F_{G_{\text{aug}}}^c(b_G^F))$ be an integer point. Then, there exists some flow $f$ (not necessarily integral) on $G_{\text{aug}}$ with netflow $b_G^0$, having the integer values $\alpha_j$ on the edges $(j, t)$. If we remove these edges and modify the netflow vector accordingly, the new flow polytope we get is the (integrally capacitated) flow polytope of a graph with an integral netflow vector. Any such polytope has integral vertices [16] Theorem 13.1]. Thus, we can choose $f$ to be an integral flow.

Since the edges labeled $y_{i,j}$ are constrained between 0 and 1, $f$ takes value 0 or 1 on these edges. If we let $F = \{(j, i) \in E(G' \setminus 0) : f$ takes value 1 on the edge labeled by $y_{i,j}\}$, then $f$ induces a flow on $\tilde{G}\setminus\{s, 0\}$ with netflow vector $b_G^F$, so $\alpha \in \text{LD}(G, F)$. \qed
We now analyze the component polytopes \( \text{Newton}(L_{G,F}(t)) \) and show that they are generalized permutahedra. We first briefly recall the relevant definitions from [14].

A \textbf{generalized permutahedron} is a deformation of the usual permutahedron obtained by parallel translation of the facets. Generalized permutahedra are parameterized by real numbers \( \{z_I\}_{I \subseteq [n]} \) with \( z_\emptyset = 0 \) and satisfying the supermodularity condition

\[
z_{I \cup J} + z_{I \cap J} \geq z_I + z_J \quad \text{for any } I, J \subseteq [n].
\]

For a choice of parameters \( \{z_I\}_{I \subseteq [n]} \), the associated generalized permutahedron \( P_n^z(\{z_I\}) \) is defined by

\[
P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \text{ and } \sum_{i=1}^n t_i = z_{[n]} \right\}.
\]

There is a subclass of generalized permutahedra given by Minkowski sums of dilations of the faces of the standard \((n-1)\)-simplex. For \( I \subseteq [n] \), let \( \Delta_I = \text{Conv}(\{e_i : i \in I\}) \), where \( e_i \) is the \( i \)th standard basis vector in \( \mathbb{R}^n \) and \( \Delta_\emptyset \) is the origin. Given a set \( \{y_I\}_{I \subseteq [n]} \) of nonnegative real numbers with \( y_\emptyset = 0 \), denote by \( P_n^y(\{y_I\}) \) the polytope

\[
P_n^y(\{y_I\}) = \sum_{I \subseteq [n]} y_I \Delta_I.
\]

\textbf{Proposition 4.10} ([14], Proposition 6.3). \textit{For nonnegative real numbers \( \{y_I\}_{I \subseteq [n]} \), the polytope \( P_n^y(\{y_I\}) \) is a generalized permutahedron \( P_n^z(\{z_I\}) \) with \( z_I = \sum_{J \subseteq I} y_J \).}

We now return to left-degree polynomials. For \( F \subseteq E(G \setminus \{0\}) \), recall the numbers \( f_{i,j} \) given by

\[
f_{i,j} = \#\{ (j,k) \in F : k \leq i \}.
\]

By Corollary 3.15 (Theorem 6.3 for the general case), \( \text{LD}(G,F) \) is in bijection with integral flows on the graph \( \tilde{G} \setminus \{s,0\} \) with the netflow vector \( b^F_G \) defined by

\[
b^F_G = (\text{indeg}_G(1) - \text{outdeg}_F(1), \ldots, \text{indeg}_G(n) - \text{outdeg}_F(n), -\#E(G \setminus F))
\]

via projection onto the edges \((i,t)\). To each \( I \subseteq [n] \), associate the integer \( z_I^F \) given by

\[
(4.2) \quad z_I^F = \min \left\{ \sum_{i \in I} f(i,t) : f \text{ is a flow on } \tilde{G} \setminus \{s,0\} \text{ with netflow vector } b^F_G \right\}.
\]

\textbf{Definition 4.11}. For a collection of vertices \( I \) of a graph \( G \), define the outdegree \( \text{outdeg}_G(I) \) to be the number of edges from vertices in \( I \) to vertices not in \( I \).

Note that the parameters \( z_I^F \) of (4.2) satisfy the supermodularity condition since \( z_I^F = z_{I'}^F \) where \( I' \) is the largest subset of \( I \) satisfying \( \text{outdeg}_G(I') = 0 \).

Our goal is to show that

\[
\text{Newton}(L_{G,F}(t)) = P_n^z(\{z_I^F\}_{I \subseteq [n]}).
\]

The proof relies on the following fact about flow polytopes, which readily follows from the max-flow min-cut theorem.
Lemma 4.12. Let $G$ be a graph on $[0, n]$ and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$. Then $\mathcal{F}_G(\alpha)$ is nonempty if and only if

$$
(4.3) \quad \sum_{i=0}^{n} \alpha_i = 0 \text{ and } \sum_{i \in S} \alpha_i \leq 0 \text{ for all } S \subseteq [0, n] \text{ with outdeg}_G(S) = 0.
$$

Proof. Observe that the conditions (4.3) are necessary in order for $\mathcal{F}_G(\alpha)$ to be nonempty. We now show they are also sufficient. For this, we rephrase the problem as a max-flow problem on another graph. Let $G' = (V(G) \cup \{s, t\}, E(G) \cup \{(s, i) \mid i \in [0, n], \alpha_i > 0\} \cup \{(i, t) \mid i \in [0, n], \alpha_i < 0\})$ with edges directed from smaller to larger vertices, where $s$ is the smallest and $t$ is the largest vertex. Let the upper capacity of the edges $(s, i)$, with $i \in [0, n], \alpha_i > 0$, be $\alpha_i$ and the upper capacity of the edges $(i, t)$, with $i \in [0, n], \alpha_i < 0$, be $-\alpha_i$. All edges have a lower capacity of 0 and the edges also belonging to $G$ all have the upper capacity $\sum_{i \in [0, n], \alpha_i > 0} \alpha_i$. If the maximum flow on $G'$ saturates the edges incident to $s$ (equivalently, to $t$) then $\mathcal{F}_G(\alpha)$ is nonempty. We thus proceed to show that if $\alpha$ satisfies (4.3) with the given $G$, then the maximum flow on $G'$ saturates the edges incident to $s$. In other words, if $\alpha$ satisfies (4.3) with the given $G$, then the value of the maximum flow on $G'$ is $\sum_{i \in [0, n], \alpha_i > 0} \alpha_i$.

Recall that by the max-flow min-cut theorem [16, Theorem 10.3] the maximum value of an $s-t$ flow on $G'$ subject to the above capacity constraints equals the minimum capacity of an $s-t$ cut in $G'$. For the cut $\{(s), V(G) \setminus \{s\}\}$ the capacity is $\sum_{i \in [0, n], \alpha_i > 0} \alpha_i$, and we show that this is the minimum capacity of an $s-t$ cut in $G'$. If the cut contains any edge not incident to $s$ or $t$, then the capacity of that edge is already $\sum_{i \in [0, n], \alpha_i > 0} \alpha_i$. On the other hand, if the cut does not contain any edge not incident to $s$ or $t$, the partition of vertices is of the form $\{(s) \cup S, S^c \cup \{t\}\}$, where $S \subseteq [0, n]$ with outdeg$_G(S) = 0$ and $S^c = [0, n] \setminus S$. Thus, by (4.3) we have $\sum_{i \in S} \alpha_i \leq 0$. The capacity of the cut $\{(s) \cup S, S^c \cup \{t\}\}$ is $\sum_{i \in S^c, (s,i) \in G'} \alpha_i - \sum_{i \in S, (i,t) \in G'} \alpha_i$. Note that $\sum_{i \in S^c, (s,i) \in G'} \alpha_i - \sum_{i \in S, (i,t) \in G'} \alpha_i \geq \sum_{i \in [0, n], \alpha_i > 0} \alpha_i$ since it is equivalent to $0 \geq \sum_{i \in S, \alpha_i > 0} \alpha_i + \sum_{i \in S, \alpha_i < 0} \alpha_i = \sum_{i \in S} \alpha_i$. In other words, the capacity of any cut is at least $\sum_{i \in [0, n], \alpha_i > 0} \alpha_i$, and we saw that this is achieved. Thus, the value of the maximum flow on $G'$ is $\sum_{i \in [0, n], \alpha_i > 0} \alpha_i$, as desired. \hfill \Box

Theorem 4.13. For a simple graph $G$, $F \subseteq E(G\setminus\emptyset)$, and $\{z^F_i\}$ the parameters defined by (4.2), Newton($L_{G,F}(t)$) is the generalized permutohedron

$$
\text{Newton}(L_{G,F}(t)) = \text{Conv}(\text{LD}(G, F)) = P_n^z\{z^F_i\}_{i \in [n]}.
$$

Furthermore, each integer point of $P_n^z\{z^F_i\}$ is in LD($G, F$), so Newton($L_{G,F}(t)$) has polytopal support.

Proof. Since LD($G, F$) equals the projection of integral flows on $\tilde{G}\setminus\{s, 0\}$ with netflow $b^F_G$ onto the edges $\{(j, t)\}_{j \in [n]}$, Conv(LD($G, F$)) $\subseteq P_n^z\{z^F_i\}$.

For the reverse direction, let $d$ denote the truncation of $b^F_G$ by its last entry, that is let $d = (d_1, \ldots, d_n)$ where

$$
d_i = \text{indeg}_G(i) - \text{outdeg}_F(i).
$$

We must show that each point $x = (x_1, \ldots, x_n) \in P_n^z\{z^F_i\}$, the assignment $a_{n,j} = x_j$ in $\tilde{G}\setminus\{s, 0\}$ can be extended to a flow on $\tilde{G}\setminus\{s, 0\}$. This is equivalent to showing

$$
\mathcal{F}_{G\setminus\emptyset}(d - x) \neq \emptyset.
$$
By Lemma 4.12 it suffices to note that
\[ \sum_{i \in S} d_i - x_i \leq 0 \text{ for all } S \subseteq [n] \text{ with } \text{outdeg}_G(S) = 0. \]
However, since outdeg\(_G\)(S) = 0, we have
\[ \sum_{i \in S} x_i \geq z_S = \sum_{i \in S} d_i. \]

We further show that Newton\((L_{G,F}(t))\) can be written as \(P^n_i\{y_I\}\) for some parameters \(y_I\). Let \(L = \{ J \subseteq [n] : \text{outdeg}_H(J) = 0 \}\). Then \(L\) is a lattice, so consider the set \(Q\) of join irreducible elements of \(L\). For \(J \subseteq [n]\), define
\[
y_J^F = \begin{cases} \\text{indeg}_G(k) & \text{if } J \in Q, J \text{ covers } J' \text{ in } L, J \setminus J' = \{k\} \\ 0 & \text{if } J \notin Q \end{cases}
\]

**Proposition 4.14.** \(P^n_i\{y_I^F\} = P^n_i\{z_I^F\}\)

**Proof.** Note that \(z_I^F = z_{I_1}^F\) where \(I_1\) is the largest element of \(L\) contained in \(I\). Thus,
\[
z_I^F = z_{I_1}^F = \sum_{k \in I_1} b_k = \sum_{J \subseteq Q} \sum_{J \subseteq I_1} y_J^F = \sum_{J \subseteq I} y_J^F.
\]

Apply Proposition 4.10.

From (4.4), we can read off the \(\{y_I^F\}\) decomposition of Newton\((L_{G,F}(t))\). Let \(\delta(i)\) denote all the vertices of \(G\) that can be reached from \(i\) by an increasing path (including \(i\) itself). Then,
\[
(4.5) \quad \text{Newton}(L_{G,F}(t)) = \sum_{i=1}^{n} \text{indeg}_G(i)\Delta_{\delta(i)}.
\]

**Example 4.15.** For a simple graph \(G\), recall that the transitive closure of \(G\) is the simple graph formed by adding edges \((i, j)\) to \(E(G)\) whenever the vertices \(i\) and \(j\) are connected by an increasing path in \(G\). If \(G\) is a simple graph on \([0, n]\) such that the transitive closure of \(G'\{0\}\) is complete, then for each \(F \subseteq E(G'\{0\})
\[
\text{Newton}(L_{G,F}(t)) = \Pi_n(\text{indeg}_G(1) - \text{outdeg}_F(1), \ldots, \text{indeg}_G(n) - \text{outdeg}_F(n))
\]
where \(\Pi_n(x)\) is the Pitman-Stanley polytope as defined in \([17]\), but shifted up one dimension in affine space, that is
\[
\Pi_n(x) = \left\{ t \in \mathbb{R}_{\geq 0}^n : \sum_{p=1}^{k} t_p \leq \sum_{p=1}^{k} x_p \text{ for } k \in [n-1], \text{ and } \sum_{p=1}^{n} t_p = \sum_{p=1}^{n} x_p \right\}
\]
\[
= x_n\Delta_{(n)} + x_{n-1}\Delta_{(n-1,n)} + \cdots + x_1\Delta_{[n]}
\]

**Proposition 4.16.** If \(T\) is a tree on \([0, n]\), then Newton\((L_{T,F}(t))\) is a simple polytope.

**Proof.** By the Cone-Preposet Dictionary for generalized permutahedra, \((15)\), Proposition 3.5) it is enough to show that each vertex poset \(Q_v\) is a tree-poset, that is, its Hasse diagram has no cycles. To show this, let \(I \subseteq [n]\) and consider the normal fan \(N(\Delta_I)\) of the simplex \(\Delta_I\). By (4.5), the normal fan of Newton\((L_{G,F}(t))\) is the refinement of normal fans \(N(\Delta_I)\).
Thus, a maximal cone of the normal fan of Newton($L_{G,F}(t)$) is given by an intersection of maximal cones in each $N(\Delta_I)$ for $I = \delta(j)$, $j \in [n]$, $\text{indeg}_T(j) > 0$. A maximal cone in $N(\Delta_I)$ gives the vertex poset relations $x_i > x_j$ for all $j \in I$ and any chosen $i \in I$. Thus, relations in the Hasse diagram of a vertex poset lift to undirected paths in $T$.

If some $Q_v$ has a cycle $C$, then we can lift the relations to get two different paths in $T$ between two vertices. This subgraph will contain a cycle, contradicting that $T$ is a tree. □

The Newton polytopes of the homogeneous components of $L^k_G(t)$ are also generalized permutahedra.

**Definition 4.17.** For each $k \geq 0$ let $L^k_G(t)$ denote the degree $\#E(G) - k$ homogeneous component of $L_G(t)$, that is

$$L^k_G(t) = \sum_{F \in E(G) \backslash 0} L_{G,F}(t)$$

For a simple graph $G$ on $[0, n]$, the proof of Theorem 4.5 showed that the augmented graph $G^{\text{aug}}$ of Definition 4.7 has the property that the projection of integral flows on $G^{\text{aug}}$ with netflow

$$b_G^\theta = (\text{indeg}_G(1), \ldots, \text{indeg}_G(n), -\#E(G))$$

and capacitance $0 \leq y_{i,j} \leq 1$ for all $1 \leq j < i \leq n$ onto the edges labeled $a_{n,j}$ for $j \in [n]$ is exactly LD$(G)$. The following construction is a variation on this theme designed so its integral flows will only project to left-degree sequences whose entries have a particular sum.

**Definition 4.18.** Given a simple graph $G$ on $[0, n]$ and $k \geq 0$, let $G^{(k)}$ be the graph on $[1, n+1] \cup \{t\}$ with labeled edges $E_a \cup E_z \cup E_y$ where

- $E_a$ consists of edges $a_{n,j} : j \to t$ for $j \in [n]$;
- $E_z$ consists of edges $z_{i,j} : j \to i$ for $(j, i) \in E(G \backslash 0)$;
- $E_y$ consists of edges $y_{i,j} : j \to n+1$ for $(j, i) \in E(G \backslash 0)$.

The flow polytope $F_{G^{(k)}}(b_G^{(k)})$ is the flow polytope of $G^{(k)}$ with netflow vector $b_G^{(k)} = (\text{indeg}_G(1), \ldots, \text{indeg}_G(n), -k, k - \#E(G))$ and capacities 1 on the edges $y_{i,j}$.

**Example 4.19.** For $G$ the complete graph on $[0, 3]$, $G^{(k)}$ is shown below alongside $G^{\text{aug}}$ for comparison.

Note that capacitated integral flows on $G^{(k)}$ with netflow $b_G^{(k)}$ are in bijection with capacitated integral flows on $G^{\text{aug}}$ with netflow $b_G^\theta$, where exactly $k$ edges $y_{i,j}$ have flow 1, and the bijection preserves the values on the edges $\{a_{n,j} : j \in [n]\}$.
Theorem 4.20. For $k \geq 0$, if $\psi$ is the projection that takes a flow on $F_{G}^{c}(b_{G}^{(k)})$ to the tuple of its values on the edges labeled $a_{n,j}$ for $j$ in $[n]$, then
\[ \text{Newton}(L_{G}^{k}(t)) = \psi \left( F_{G}^{c}(b_{G}^{(k)}) \right). \]
Furthermore, each integer point in the right-hand side is a left-degree sequence with components that sum to $\#E(G) - k$, so $L_{G}^{k}$ has polytopal support.

Proof. Let $\alpha$ be an integer point in $\text{Newton}(L_{G}^{k}(t))$, so $\alpha \in \text{LD}(G,F)$ for $F \subseteq E(G \setminus \emptyset)$ with $\#F = k$. Then, $\alpha$ corresponds to a capacitated integral flow on $G^{\text{aug}}$ with netflow $b_{G}^{0}$, which in turn corresponds to a capacitated integral flow on $G^{(k)}$ with netflow $b_{G}^{(k)}$ that $\psi$ takes to $\alpha$.

Conversely, let $\alpha$ be an integer point in $\psi \left( F_{G}^{c}(b_{G}^{(k)}) \right)$. Lift $\alpha$ to an integral flow $f$ on $G^{(k)}$. The flow $f$ corresponds to an integral flow on $G^{\text{aug}}$, so if $F = \{(j,i) : y_{k,j} = 1 \text{ in } f\}$, then $\#F = k$ and $\alpha \in \text{LD}(G,F)$. \qed

Similar to the proof of Theorem 4.13 for $k \geq 0$ and $I \subseteq [n]$, define parameters $z_{I}^{(k)}$ by
\[ z_{I}^{(k)} = \min \left\{ \sum_{i \in I} f(i,t) : f \text{ is a flow on } G^{(k)} \text{ with netflow vector } b_{G}^{(k)} \right\}. \]

Theorem 4.21. For $k \geq 0$ and $\{z_{I}^{(k)}\}$ the parameters defined by (4.6), $\text{Newton}(L_{G}^{k}(t))$ is the generalized permutahedron
\[ \text{Newton}(L_{G}^{k}(t)) = P_{n}^{z} \{ z_{I}^{(k)} \}_{I \subseteq [n]}. \]
Furthermore, each integer point of $P_{n}^{z} \{ z_{I}^{(k)} \}$ is a left-degree sequence, so $\text{Newton}(L_{G,F}(t))$ has polytopal support. Additionally, if $G$ is an acyclic graph, then $L_{G}^{0}(t)$ is the integer-point transform of its Newton polytope.

Proof. The proof of the first two statements is analogous to that of Theorem 4.13.

To prove the third statement we must show that if $G$ is an acyclic graph, all nonzero coefficients of $L_{G}^{0}$ are 1. It follows from Corollary 3.15 (Theorem 6.3) that $\text{LD}(G,\emptyset)$ equals the multiset of projections of integral flows on $\tilde{G} \setminus \{s,0\}$ with the netflow vector $b_{G}^{0}$. Then, the multiplicity of any particular $\alpha \in \text{LD}(T,\emptyset)$ is the number of flows on $G \setminus \emptyset$ with netflow $b_{G}^{0} - \alpha$. However, acyclic graphs admit at most one flow for any given netflow vector, so every element of $\text{LD}(G,\emptyset)$ has multiplicity 1. This implies all coefficients in $L_{G}^{0}$ are 0 or 1. \qed

Theorems 4.5 and 4.21 imply:

Corollary 4.22. Given a graph $G$ on the vertex set $[0,n]$ with $m$ edges, we have that
\[ \text{Newton}(L_{G}(t)) \cap \{(x_{1}, \ldots, x_{n}) \in \mathbb{R}^{n} | \sum_{i=1}^{n} = m - k\} = P_{n}^{z} \{ z_{I}^{(k)} \}_{I \subseteq [n]}, \]
for the parameters $\{z_{I}^{(k)}\}$ given in (4.6).

Proof. We have that $\text{Newton}(L_{G}(t)) \cap \{(x_{1}, \ldots, x_{n}) \in \mathbb{R}^{n} | \sum_{i=1}^{n} = m - k\} = \text{Newton}(L_{G}^{k}(t))$, which by Theorem 4.21 equals $P_{n}^{z} \{ z_{I}^{(k)} \}_{I \subseteq [n]}$. \qed
Theorems 3.16 and 4.21 imply:

**Corollary 4.23.** If $G$ is an acyclic graph on $[0, n]$, then the normalized volume of the flow polytope of $\tilde{G}$ is

$$\text{Vol } \mathcal{F}_{\tilde{G}} = \text{Ehr}(P_0^G, 1),$$

where $P_0^G := \text{Newton}(L_0^G(t))$ is the generalized permutahedron specified in Theorem 4.21.

Corollary 4.23 is of the same flavor as Postnikov’s following beautiful result; for the details of the terminology used in this theorem refer to [14].

**Theorem 4.24.** [14, Theorem 12.9] For a bipartite graph $G$, the normalized volume of the root polytope $Q_G$ is

$$\text{Vol } Q_G = \text{Ehr}(P_0^G, 1),$$

where $P_0^G$ is the trimmed generalized permutahedron.

Root polytopes and flow polytopes are closely related, as can be seen by contrasting the techniques and results in the papers [8, 9, 10, 11, 14]. It is thus reasonable to expect that Corollary 4.23 and Theorem 4.24 are related mathematically. We invite the interested reader to investigate their relationship.

5. **Newton polytopes of Schubert and Grothendieck polynomials**

In this section, we discuss the connection between left-degree sequences, Schubert polynomials, and Grothendieck polynomials discovered in [4] and relate it to their Newton polytopes. Our main theorem is as follows:

**Theorem C.** Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ where $\pi'$ is a dominant permutation of $\{2, 3, \ldots, n+1\}$. Then the Grothendieck polynomial $G_\pi$ has polytopal support and the Newton polytope of each homogeneous component of $G_\pi$ is a generalized permutahedron. In particular, the Schubert polynomial $S_\pi$ has polytopal support and $\text{Newton}(S_\pi)$ is a generalized permutahedron. Moreover, $S_\pi$ is the integer-point transform of its Newton polytope.

Theorem C implies that the recent conjectures of Monical, Tokcan, and Yong [13, Conjecture 5.1 & 5.5] are true for permutations $1\pi'$, where $\pi'$ is a dominant permutation. The following conjecture, discovered jointly with Alex Fink, is a strengthening of [13, Conjecture 5.5]. We have tested it for all $\pi \in S_n$, for $n \leq 8$.

**Conjecture 5.1.** The Grothendieck polynomial $G_\pi$ has polytopal support and the Newton polytope of each homogeneous component of $G_\pi$ is a generalized permutahedron.

Since [4] uses right-degree sequences and right-degree polynomials instead of their left-degree counterparts, we will adopt this convention throughout this section. To simplify notation, all graphs in this section will be on the vertex set $[n+1]$. Note the following easy relation between right-degree and left-degree.

Given a graph $G$ on vertex set $[n+1]$, let $G^*$ be the mirror image of the graph $G$ with vertex set shifted to $[0, n]$. More formally, let $G^*$ be the graph on vertices $[0, n]$ with edges

$$E(G^*) = \{(n+1-j, n+1-i) : (i, j) \in E(G)\}.$$

The right-degree sequences of $G$ are exactly the left-degree sequences of $G^*$ read backwards. We can then define the **right-degree multiset** $\text{RD}(G)$ as the multiset of right-degree sequences of leaves in any reduction tree of $G$, and $\text{RD}(G, \emptyset)$ the submultiset of sequences whose components sum to $\#E(G)$ (notation consistent with $\text{LD}(G, F)$ in Definition 3.9).
Definition 5.2. For any graph $G$ on $[n+1]$, define the right-degree polynomial $R_G$ by
\[ R_G(t_1, t_2, \ldots t_n) = \sum_{\alpha \in RD(G)} (-1)^{\text{codim}(\alpha)} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \]
where $\text{codim}(\alpha) = \#E(G) - \sum_{i=1}^n \alpha_i$.

For $k \geq 0$, let $R_G^k(t)$ denote the degree $\#E(G) - k$ homogeneous component of $R_G(t)$.

Define the reduced right-degree polynomial $\tilde{R}_G$ as follows: If $\{v_{i_1}, \ldots, v_{i_k}\}$ are the vertices of $G$ with positive outdegree, then $R_G$ is a polynomial in $t_{i_1}, \ldots, t_{i_k}$. Obtain $\tilde{R}_G$ by relabeling the variables $t_{i_m}$ by $t_m$ for each $m$. Note that $R_G^0$ (resp. $\tilde{R}_G^0$) is the top homogeneous component of $R_G$ (resp. $\tilde{R}_G$), and is given by
\[ R_G^0(t_1, \ldots, t_n) = \sum_{\alpha \in RD(G, \emptyset)} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \]

The following statement collects the right-degree analogues of Theorem 4.5 and Theorem 5.21 from the previous section.

Theorem 5.3. Let $G$ be a graph on $[n+1]$. Then, $R_G(t)$ has polytopal support, and the Newton polytope of each homogeneous component $R_G^k$ is a generalized permutahedron. Additionally, if $G$ is an acyclic graph, then $R_G^0(t)$ is the integer-point transform of its Newton polytope.

Recall that for a polytope $P \subseteq \mathbb{R}^m$, the integer-point transform of $P$ is
\[ L_P(x_1, \ldots, x_m) = \sum_{p \in P \cap \mathbb{Z}^m} x^p. \]

We now recall the definition of pipe dreams of a permutation and the characterization of Schubert and Grothendieck polynomials in terms of pipe dreams.

Definition 5.4. A pipe dream for $\pi \in S_{n+1}$ is a tiling of an $(n+1) \times (n+1)$ matrix with two tiles, crosses $\bigm|$ and elbows $\dashv$, such that
1. all tiles in the weak south-east triangle are elbows, and
2. if we write $1, 2, \ldots, n+1$ on the top and follow the strands (ignoring second crossings among the same strands), they come out on the left and read $\pi$ from top to bottom.

A pipe dream is reduced if no two strands cross twice.

Figure 6. A reduced pipe dream for $\pi = 2143$. All tiles not shown are elbows.

For $\pi \in S_{n+1}$ let $PD(\pi)$ denote the collection of all pipe dreams of $\pi$ and $RPD(\pi)$ the collection of all reduced pipe dreams of $\pi$. For $P \in PD(\pi)$, define the weight of $P$ by
\[ wt(P) = \prod_{(i,j) \in \text{cross}(P)} t_i. \]
Recall that for any $\pi \in S_{n+1}$, the Grothendieck polynomial $G_\pi$ can be represented in terms of pipe dreams of $\pi$ by:

$$G_\pi(t_1, \ldots, t_n) = \sum_{P \in \mathrm{PD}(\pi)} wt(P)$$

and the Schubert polynomial $G_\pi$ is the lowest degree homogeneous component of the Grothendieck polynomial:

$$G_\pi(t_1, \ldots, t_n) = \sum_{P \in \mathrm{RPD}(\pi)} wt(P).$$

In [4], it is proved that $\mathrm{RD}(T)$ is independent of the reduction tree for $T$ a tree, and the following connection to Grothendieck polynomials is shown.

**Theorem 5.5** ([4], Theorem 5.3). Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ where $\pi'$ is a dominant permutation of $\{2, 3, \ldots, n+1\}$. Then, there is a tree $T(\pi)$ and nonnegative integers $g_i = g_i(\pi')$ such that

$$\tilde{R}_{T(\pi)}(t) = \left( \prod_{i=1}^{n} t_i^{g_i} \right) G_\pi(t_1, \ldots, t_n).$$

Explicitly, if $C(\pi)$ denotes the set core($\pi$) $\cup$ \{(1,1)\}, then $g_i(\pi)$ is the number of boxes in column $i$ of $C(\pi)$.

In terms of Newton polytopes, Theorem 5.3 implies

$$\text{Newton}(G_\pi) = \varphi\left(\text{Newton}(\tilde{R}_{T(\pi)}(t))\right)$$

and

$$\text{Newton}(G_\pi) = \varphi\left(\text{Newton}(\tilde{R}_{T(\pi)}^0(t))\right)$$

where $\varphi$ is the integral equivalence

$$(x_1, \ldots, x_n) \mapsto (g_1 - x_1, \ldots, g_n - x_n).$$

**Proof of Theorem 5.5**. By Theorem 5.3 right-degree polynomials $R_G(t)$ have polytopal support. Since $\text{Newton}(\tilde{R}_{T(\pi)}(t))$ is the image of $\text{Newton}(R_{T(\pi)}(t))$ by a projection forgetting coordinates that are always zero, it follows from Theorem 5.5 that $G_\pi$ has polytopal support.

Theorem 5.3 and Theorem 5.5 also yield that each homogeneous component of $G_\pi$ has polytopal support and that their Newton polytopes are generalized permutahedra. In particular, this holds for the Schubert polynomial. Since by [4] the Schubert polynomial of $\pi = 1\pi'$, where $\pi'$ is a dominant permutation, has 0,1 coefficients, the last statement also follows.

From the proof of Theorem 5.5 in [4], one can infer the following new transition rule for Schubert polynomials of permutations of the form $1\pi'$ with $\pi'$ dominant.

**Lemma 5.6. (Transition rule for Schubert polynomials.)** Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ with $\pi'$ a dominant permutation of $\{2, \ldots, n+1\}$. Let $\pi'$ have diagram given by the partition $\lambda(\pi') = (\lambda_1, \ldots, \lambda_{z})$ with $\lambda_z = k$. For $0 \leq l \leq k$, let $w_l$ be the permutation on $\{2, \ldots, n+1\}$ whose diagram is the partition $(\lambda_1 - (k-l), \ldots, \lambda_{z-1} - (k-l))$. Then

$$G_\pi(x) = \sum_{l=0}^{k} \left( \prod_{m=1}^{l} x_m \right) \left( \prod_{p=l+2}^{k+1} x_p^2 \right) G_{1w_l}(x_{\phi_l})$$
where \( \mathbf{x} = (x_1, x_2, \ldots) \), \( x_{\phi_l} = (x_{\phi_l(1)}, x_{\phi_l(2)}, \ldots) \), and \( \phi_l(i) = \begin{cases} i & \text{if } i \leq l + 1 \\ i + k - l & \text{if } i \geq l + 2 \end{cases} \)

We illustrate the above transition rule in the following example.

**Example 5.7.** Let \( \pi = 14523 \). Then, \( \pi' = 4523 \), so \( \lambda(\pi') = (2, 2) \). For \( 0 \leq l \leq 2 \), the permutation \( w_l \) will have diagram given by the partition \( (l) \). These permutations are \( w_0 = 2345, w_1 = 3245 \), and \( w_2 = 3425 \). Hence, the terms in the transition rule are

\[
\begin{align*}
(1)(x_2^2x_3^2)\mathcal{G}_{1w_0}(x_1, x_4, x_5, x_6) &= x_2^2x_3^2 \\
(x_1)(x_3^2)\mathcal{G}_{1w_1}(x_1, x_2, x_4, x_5) &= x_1^2x_3^2 + x_1x_2x_3^2 \\
(x_1x_2)(1)\mathcal{G}_{1w_2}(x_1, x_2, x_3, x_4) &= x_1^2x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_2x_3^2 + x_2x_3^2.
\end{align*}
\]

Adding these terms together gives the expected polynomial

\[ \mathcal{G}_\pi(x_1, x_2, x_3, x_4) = x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_3^2 + x_1x_2x_3 + x_1x_2x_3^2 + x_2x_3^2. \]

6. **Left-degree sequences are invariants of the graph**

In this section we generalize the results of the Section 3 to any graph \( G \), not necessarily simple. Similar accommodations can be made to generalize Sections 4 and 5. We also prove Theorem A, which characterizes the left-degree sequences of the leaves of a reduction tree of \( G \), and concludes that they are independent of the choice of reduction tree, and are therefore an invariant of \( G \) itself. To deal with multiple edges in \( E(G) \), we view each element of \( E(G) \) as being distinct. Formally, we may think of assigning a distinguishing number to each copy of a multiple edge. In this way, we may speak of subsets \( F \subseteq E(G\setminus0) \) in the usual sense.

For \( G \) any graph on the vertex set \([0, n]\), we can still construct the reduction tree \( \mathcal{T}(G) \) using the same algorithm as before in Definition 3.2. As in the case of simple graphs, the leaves of this specific reduction tree can be encoded as solutions to some constraint arrays. The key is using a generalized version of Lemma 3.1 with multiple incoming and outgoing edges at vertex \( v \). This generalization is derived the same way and is not harder, but far more technical. The arrays we obtain are no longer necessarily triangular, but rather they may be staggered. This is explained below and demonstrated in Examples 6.1 and 6.2. We leave the proofs to the interested reader; they are straightforward generalizations of those in the previous section.

**Triangular arrays** \( \text{Tri}_G(\emptyset) \) for arbitrary \( G \). For the case of full-dimensional degree sequences, replace each \( a_{i,j}^{(1)} \) by \( a_{i,j}^{(1)} \) in Definition 3.3 and Theorem 3.4, and add variables \( a_{i,j}^{(k)} \) with \( k > 1 \) for each additional copy of the edge \((j, i)\) appearing in \( G \). When there are \( k > 1 \) copies of the edge \((j, i) \in E(G) \), also replace \( a_{i,j}^{(1)} \leq a_{i-1,j}^{(1)} \) in the constraint array by \( a_{i,j}^{(1)} \leq a_{i,j}^{(2)} \leq \cdots \leq a_{i,j}^{(k)} \leq a_{i-1,j}^{(1)} \). The following example demonstrates these changes.

**Example 6.1.** Following Example 3.5 if \( G \) is the graph on vertex set \([0, 4]\) with

\[ E(G) = \{(0, 1), (0, 1), (0, 2), (1, 2), (1, 2), (2, 3), (2, 4), (3, 4), (3, 4)\}, \]
we obtain the constraints:

\[0 \leq a_{4,1}^{(1)} = a_{3,1}^{(1)} = a_{2,1}^{(1)} \leq a_{2,1}^{(2)} \leq a_{1,1}^{(1)} = 2\]
\[0 \leq a_{4,2}^{(1)} \leq a_{3,2}^{(1)} \leq a_{2,2}^{(1)} = 5 - a_{2,1}^{(1)}\]
\[0 \leq a_{4,3}^{(1)} \leq a_{4,3}^{(2)} \leq a_{3,3}^{(1)} = 6 - a_{3,1}^{(1)} - a_{3,2}^{(1)}\]
\[0 \leq a_{4,4}^{(1)} = 9 - a_{4,1}^{(1)} - a_{4,2}^{(1)} - a_{4,3}^{(1)}\]

**Triangular arrays** \( \text{Tri}_G(F) \) **for arbitrary** \( G \). Similarly, we can encode all left-degree sequences by introducing the arrays \( \text{Tri}(F) \) used in Theorem 3.8. To do this we view \( E(G) \) as a multiset, so we formally view each copy of a multiple edge \((j, i)\) as a distinct element. Let \( F \) vary over subsets of \( E(G) \setminus \emptyset \), and define \( \text{Tri}_G(F) \) from (the general version of) \( \text{Tri}_G(\emptyset) \) as before using the numbers \( f_{i,j} \) of (3.1) and treating each \( a_{i,j}^{(m)} \) identically for different \( m \).

**Example 6.2.** With \( G \) as in Example 6.1 and \( F = \{(1, 2), (1, 2), (2, 3)\} \), the array \( \text{Tri}(F) \) is given by

\[2 \leq a_{4,1}^{(1)} + 2 = a_{3,1}^{(1)} + 2 = a_{2,1}^{(1)} + 2 \leq a_{2,1}^{(2)} \leq 2 \leq a_{1,1}^{(1)} = 2\]
\[1 \leq a_{4,2}^{(1)} + 1 \leq a_{3,2}^{(1)} + 1 \leq a_{2,2}^{(1)} = 3 - a_{2,1}^{(1)}\]
\[0 \leq a_{4,3}^{(1)} \leq a_{4,3}^{(2)} \leq a_{3,3}^{(1)} = 3 - a_{3,1}^{(1)} - a_{3,2}^{(1)}\]
\[0 \leq a_{4,4}^{(1)} = 6 - a_{4,1}^{(1)} - a_{4,2}^{(1)} - a_{4,3}^{(1)}\]

Using the definition of \( \text{Tri}_G(F) \) for arbitrary graphs \( G \), we can extend the definitions of \( \text{Sol}_G(F) \) and \( \text{LD}(G, F) \) from simple graphs to arbitrary graphs \( G \). As in Proposition 3.12 for each \( F \subseteq E(G) \setminus \emptyset \) the polytope \( \text{Poly} (\text{Tri}_G(F)) \) is integrally equivalent to the flow polytope of a graph \( \text{Gr}(G) \), a straightforward generalization of Definition 3.11. The proofs of Theorem 3.14 and its Corollaries then go through with minor changes. In particular, we have the following crucial result.

**Theorem 6.3.** Let \( G \) be a graph on \([0, n]\), \( \rho \) be the map that takes a triangular array in any \( \text{Sol}_G(F) \) to its first column \( \left(a_{n,1}^{(1)}, \ldots, a_{n,n}^{(1)}\right) \), and \( \psi \) be the map that takes a flow on \( \tilde{G} \setminus \{s, 0\} \) to the tuple of its values on the edges \( \{(j, t) : j \in [n]\} \). For \( F \subseteq E(G) \setminus \emptyset \), recall the netflow vector

\[b_G^F = (\text{indeg}_{G}(1) - \text{outdeg}_{G}(1), \ldots, \text{indeg}_{G}(n) - \text{outdeg}_{G}(n), -\#E(G \setminus F))\].

Then for each \( F \subseteq E(G) \setminus \emptyset \),

\[
\text{LD}(G, F) = \rho \left( \text{Sol}_G(F) \right) = \psi \left( F_{\tilde{G} \setminus \{s, 0\}} \left( b_G^F \cap \mathbb{Z}^{\#E(\tilde{G} \setminus \{s, 0\})} \right) \right), \quad \text{so}
\]
\[
\text{InSeq} \left( T(G) \right) = \bigcup_{F \subseteq E(G) \setminus \emptyset} \text{LD}(G, F)
\]
\[
= \bigcup_{F \subseteq E(G) \setminus \emptyset} \rho \left( \text{Sol}_G(F) \right)
\]
\[
= \bigcup_{F \subseteq E(G) \setminus \emptyset} \psi \left( F_{\tilde{G} \setminus \{s, 0\}} \left( b_G^F \cap \mathbb{Z}^{\#E(\tilde{G} \setminus \{s, 0\})} \right) \right)
\]
In the proof of Theorem 6.3 below, it will be more convenient to use an equivalent formulation of Theorem 6.3: Instead of considering flows on \( \widetilde{G} \setminus \{s, 0\} \) with netflow vector \( \mathbf{b}_G^F \), consider flows on \( \widetilde{G} \setminus \{s\} \) with netflow vector \((0, \mathbf{b}_G^F)\), where

\[
(0, \mathbf{b}_G^F) = (0, \text{indeg}_G(1) - \text{outdeg}_F(1), \ldots, \text{indeg}_G(n) - \text{outdeg}_F(n), -\#E(G \setminus F)).
\]

Next, we use Theorem 6.3 to prove that for all graphs \( G \) on \([0, n]\), \( \text{LD}(G) \) depends only on \( G \) and not on the choice of reduction tree of \( G \) as stated in Theorem 6.3. Before proceeding with the proof, we first recall the relevant notation introduced previously. For a graph \( G \) on \([0, n]\), let \( \mathcal{R}(G) \) be any reduction tree of \( G \) and \( T(G) \) the specific reduction tree whose leaves are encoded by the arrays \( \text{Sol}_G(F) \) (constructed in Definition 3.2). Recall that \( \text{InSeq}(\mathcal{R}(G)) \) denotes the multiset of left-degree sequences of the leaves of \( \mathcal{R}(G) \). Since \( \text{LD}(G) \) was defined as the left-degree sequences of leaves in any reduction tree of \( G \), to show this definition is valid it suffices to prove that \( \text{InSeq}(\mathcal{R}(G)) = \text{InSeq}(T(G)) \).

**Proof of Theorem 6.3** We proceed by induction on the maximal depth of a reduction tree of \( G \). For the base case, the only reduction tree possible is the single leaf \( G \). For the induction, perform a single reduction on \( G \) using fixed edges \( r_1 = (i, j) \) and \( r_2 = (j, k) \) with \( i < j < k \) to get graphs \( G_1, G_2, \) and \( G_3 \), with notation as in (2.1). Note that we are selecting particular edges \( r_1 \) and \( r_2 \) even if there are multiple edges \((i, j)\) or \((j, k)\). Let \( r_3 \) denote the new edge \((i, k)\) in \( G_m \) for each \( m \in [3] \). Let \( \mathcal{R}(G_m) \) be the reduction tree of \( G_m \), \( m \in [3] \), induced from \( \mathcal{R}(G) \) by restriction to the node labeled by \( G_m \) and all of its descendants.

By the induction assumption, \( \text{InSeq}(\mathcal{R}(G_m)) \) is exactly \( \text{InSeq}(T(G_m)) \), so

\[
\text{InSeq}(\mathcal{R}(G)) = \bigcup_{m \in [3]} \text{InSeq}(\mathcal{R}(G_m)) = \bigcup_{m \in [3]} \text{InSeq}(T(G_m)).
\]

Thus, we need to show that

\[
\bigcup_{m \in [3]} \text{InSeq}(T(G_m)) = \text{InSeq}(T(G))
\]

regardless of the choice of \( r_1 \) and \( r_2 \). However, if \( \rho \) is the map that takes an array to its first column, then Theorem 6.3 implies the disjoint union decompositions

\[
\text{InSeq}(T(G)) = \bigcup_{F \subseteq E(G\setminus 0)} \rho(\text{Sol}_G(F)),
\]

and for each \( m \in [3] \),

\[
\text{InSeq}(T(G_m)) = \bigcup_{F \subseteq E(G_m \setminus 0)} \rho(\text{Sol}_{G_m}(F)).
\]

Thus, to prove (6.1), it suffices to show

\[
\bigcup_{F \subseteq E(G\setminus 0)} \rho(\text{Sol}_G(F)) = \bigcup_{m \in [3]} \bigcup_{F \subseteq E(G_m \setminus 0)} \rho(\text{Sol}_{G_m}(F)).
\]

To show (6.2), to each \( F \subseteq E(G\setminus 0) \), we associate a tuple \((F_m)_{m \in I(F, r_1, r_2)}\) with \( I(F, r_1, r_2) \subseteq [3] \) and \( F_m \subseteq E(G_m \setminus 0) \), \( m \in [3] \), such that each subset of any \( E(G_m \setminus 0) \) is in exactly one
tuple and for each \( F \subseteq E(G \setminus 0) \),

\[
\rho (\text{Sol}_G(F)) = \bigcup_{m \in I(F,r_1,r_2)} \rho (\text{Sol}_{G_m}(F_m)).
\]

By Theorem 6.3, we verify the equivalent condition

\[
\psi \left( \mathcal{F}_{\tilde{G}\setminus\{s\}}^{E(\tilde{G}\setminus\{s\})} \right) = \bigcup_{m \in I(F,r_1,r_2)} \psi \left( \mathcal{F}_{\tilde{G}_m\setminus\{s\}}^{E(\tilde{G}_m\setminus\{s\})} \right).
\]

To make the notation more compact, let \( H = \tilde{G}\setminus\{s\} \) and \( H_m = \tilde{G}_m\setminus\{s\} \) for \( m \in [3] \). We proceed in several cases depending on \( F, r_1, r_2 \). In each case, the argument is very similar to the proof of Proposition 2.3.

I. Suppose that \( r_1 \) is not incident to vertex 0. The following four cases deal with this case.

Case 1: \( r_1, r_2 \not\in F \): Associate to \( F \) the tuple \( (F_1, F_2) \) with

\[
F_1 = F \quad \text{and} \quad F_2 = F.
\]

Let \( h \) be an integral flow on \( H \) with netflow vector \( (0, b^F_G) \). For \( m \in [3] \), we define integral flows on \( H_m \) with netflow \( (0, b^F_{G_m}) \) having the same image under \( \psi \).

- If \( h(r_1) \geq h(r_2) \), define \( h_1 \) on \( H_1 \) with netflow \( b^F_{G_1} \) by

  \[
h_1(e) = \begin{cases} 
    h(r_1) & \text{if } e = r_3 \\
    h(r_1) - h(r_2) & \text{if } e = r_1 \\
    h(e) & \text{otherwise}
  \end{cases}
\]

- If \( h(r_1) < h(r_2) \), define \( h_2 \) on \( H_2 \) with netflow \( b^F_{G_2} \) by

  \[
h_2(e) = \begin{cases} 
    h(r_1) & \text{if } e = r_3 \\
    h(r_2) - h(r_1) - 1 & \text{if } e = r_2 \\
    h(e) & \text{otherwise}
  \end{cases}
\]

For the inverse map, given integral flows \( h_m \) on \( H_m \) with netflow \( b^F_{G_m} \) for \( m \in [2] \), define flows \( h^{(m)} \) on \( H \) by

\[
h^{(1)}(e) = \begin{cases} 
    h_1(r_1) + h_1(r_3) & \text{if } e = r_1 \\
    h_1(r_3) & \text{if } e = r_2 \\
    h_1(e) & \text{otherwise}
  \end{cases}
\]

and

\[
h^{(2)}(e) = \begin{cases} 
    h_2(r_3) & \text{if } e = r_1 \\
    h_2(r_2) + h_2(r_3) + 1 & \text{if } e = r_2 \\
    h_2(e) & \text{otherwise}
  \end{cases}
\]

Case 2: \( r_1 \in F, r_2 \not\in F \): Associate to \( F \) the tuple \( (F_1, F_2) \) with

\[
F_1 = F \setminus \{r_1\} \cup \{r_3\} \quad \text{and} \quad F_2 = F \setminus \{r_1\} \cup \{r_3\}.
\]

Use the same maps on flows given in Case 1.

Case 3: \( r_1 \not\in F, r_2 \in F \): Associate to \( F \) the tuple \( (F_1, F_2, F_3) \) with

\[
F_1 = F \setminus \{r_2\} \cup \{r_1\}, \quad F_2 = F, \quad \text{and} \quad F_3 = F \setminus \{r_2\}.
\]

Let \( h \) be an integral flow on \( H \) with netflow vector \( (0, b^F_G) \). For \( m \in [3] \), we define integral flows on \( H_m \) with netflow \( (0, b^F_{G_m}) \) having the same image under \( \psi \).
• If \( h(r_1) > h(r_2) \), define \( h_1 \) on \( H_1 \) with netflow \( b_{F_1}^{E_1} \) by

\[
h_1(e) = \begin{cases} 
  h(r_2) & \text{if } e = r_3 \\
  h(r_1) - h(r_2) - 1 & \text{if } e = r_1 \\
  h(e) & \text{otherwise}
  \end{cases}
\]

• If \( h(r_1) < h(r_2) \), define \( h_2 \) on \( H_2 \) with netflow \( b_{F_2}^{E_2} \) by

\[
h_2(e) = \begin{cases} 
  h(r_1) & \text{if } e = r_3 \\
  h(r_2) - h(r_1) - 1 & \text{if } e = r_2 \\
  h(e) & \text{otherwise}
  \end{cases}
\]

• If \( h(r_1) = h(r_2) \), define \( h_3 \) on \( H_3 \) with netflow \( b_{F_3}^{E_3} \) by

\[
h_3(e) = \begin{cases} 
  h(r_1) & \text{if } e = r_3 \\
  h(e) & \text{otherwise}
  \end{cases}
\]

Given integral flows \( h_m \) on \( H_m \) with netflows \( b_{G_m}^{E_m} \) for \( m \in [3] \), construct the inverse map by defining flows \( h^{(m)} \) on \( H \) for \( m \in [3] \). Let \( h^{(2)} \) be the same as in Case 1, and define

\[
h^{(1)}(e) = \begin{cases} 
  h_1(r_1) + h_1(r_3) + 1 & \text{if } e = r_1 \\
  h_1(r_3) & \text{if } e = r_2 \\
  h_1(e) & \text{otherwise}
  \end{cases}
\]

and

\[
h^{(3)}(e) = \begin{cases} 
  h_3(r_3) & \text{if } e = r_1 \\
  h_3(r_3) & \text{if } e = r_2 \\
  h_3(e) & \text{otherwise}
  \end{cases}
\]

Case 4: \( r_1, r_2 \in F \): Associate to \( F \) the tuple \((F_1, F_2, F_3)\) with

\[
F_1 = F \setminus \{r_2\} \cup \{r_3\}, \quad F_2 = F \setminus \{r_1\} \cup \{r_3\}, \quad \text{and} \quad F_3 = F \setminus \{r_1, r_2\} \cup \{r_3\}.
\]

Use the maps on flows given in Case 3.

A straightforward check shows that every \( F \subseteq E(G_m \setminus 0) \) for \( m \in [3] \) is reached exactly once by cases 1-4.

II. Suppose that \( r_1 \) is incident to vertex 0. The following two cases deal with this case.

Case 1’: \( r_2 \notin F \): Associate to \( F \) the tuple \((F_1, F_2)\) with

\[
F_1 = F \quad \text{and} \quad F_2 = F.
\]

Use the maps on flows given in Case 1.

Case 2’: \( r_2 \in F \): Associate to \( F \) the tuple \((F_2, F_3)\) with

\[
F_2 = F \quad \text{and} \quad F_3 = F \setminus \{r_2\}
\]

Use the maps on flows for \( H_2 \) and \( H_3 \) given in Case 3.

A straightforward check shows that every \( F \subseteq E(G_m \setminus 0) \) for \( m \in [3] \) is reached exactly once by cases 1’-2’.

\[\square\]

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