Symbol Elimination for Parametric Second-Order Entailment Problems
(with Applications to Problems in Wireless Network Theory)

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Abstract. We analyze possibilities of second-order quantifier elimination for formulae containing parameters – constants or functions. For this, we use a constraint resolution calculus obtained from specializing the hierarchical superposition calculus. If saturation terminates, we analyze possibilities of obtaining weakest constraints on parameters which guarantee satisfiability. If the saturation does not terminate, we identify situations in which finite representations of infinite saturated sets exist. We identify situations in which entailment between formulae expressed using second-order quantification can be effectively checked. We illustrate the ideas on a series of examples from wireless network research.

1 Introduction

The main motivation for this work was a study of models for graph classes naturally occurring in wireless network research – in which nodes that are close are always connected, nodes that are far apart from each other are never connected and any other node pairs can, but do not need to be connected. Transformations can be applied to such graphs to make them symmetric; this way we can define further graph classes. When checking inclusion between graph classes described using transformations we need to check entailment of second-order formulae. In addition, many such graph class descriptions are parametric in nature, so the goal is, in fact, to obtain (weakest) conditions on the parameters used in such descriptions that guarantee that graph classes are non-empty or that inclusions hold. This can be achieved by eliminating “non-parametric” constants or function symbols used in the description of such classes.

In this paper we combine methods for general symbol elimination (which we use for eliminating existentially quantified predicates) with methods for property-directed symbol elimination (which we use for obtaining conditions on “parameters” under which formulae are satisfiable or second-order entailment holds). For general second-order quantifier elimination we use a form of ordered resolution similar to that proposed in [18]. For property-directed symbol elimination we use a method we proposed in [41]. The advantage of using such a two-layered approach is that it avoids non-termination that might occur if using only general symbol elimination methods. The main application area we consider in this paper is the analysis of inclusions between graph classes arising in wireless network research. Our main contributions are:
We analyze theories used in modeling graph classes and prove locality of theories of “distances” occurring in this context.

We analyze possibilities of general symbol elimination, using a simple specialization $\text{HRes}_P$ of the hierarchical superposition calculus (a form of ordered resolution) for eliminating a predicate symbol $P$.

If saturation terminates, we analyze possibilities of obtaining weakest constraints on parameters occurring in the clauses which guarantee satisfiability, using methods for property-directed symbol elimination.

If the saturation does not terminate, we study possibilities of representing an (infinite) saturated set as a set of constrained clauses in which the constraints are interpreted in the minimal model of a set of constrained Horn clauses.

We analyze possibilities of effectively checking entailment between formulae expressed using second-order quantification.

We illustrate the ideas on examples related to the study of wireless networks.

Related work. The study of second-order quantifier elimination goes back to the beginning of the 20th century (cf. [10,2,3]). Most of its known applications are in the study of modal logics or knowledge representation [19,22]; in many cases second-order quantifier elimination is proved only for very restricted fragments (cf. e.g. [43]). In [18], Gabbay and Ohlbach proposed a resolution-based algorithm for second-order quantifier elimination which is implemented in the system SCAN. In [5], Bachmair et al. mention that hierarchical superposition (cf. [8,9] for further refinements) can be used for second-order quantifier elimination modulo a theory. In [34,24], Hoder et al. study possibilities of symbol elimination in inference systems (e.g. the superposition calculus and its extension with ground linear rational arithmetic and uninterpreted functions). The main challenge when using saturation approaches for symbol elimination is the fact that the saturated sets might be infinite. Sometimes finite representations of possibly infinite sets of clauses exist: for this, Horbach and Weidenbach introduced a melting calculus [27], later used in [25,26] and [16]. Similar aspects were explored in the study of acceleration for program verification modulo Presburger arithmetic by Boigelot, Finkel and Leroux [14,17], in relationship with array systems by [4], or in the study of constrained Horn clauses (cf. e.g. the survey [12]).

Orthogonal to this direction of study is what we call “property-directed” symbol elimination: There, given a theory $\mathcal{T}$ and a ground formula $G$ satisfiable w.r.t. $\mathcal{T}$, the goal is to derive a (weakest) universal formula $\Gamma$ over a subset of the signature, such that $\Gamma \land G$ is unsatisfiable w.r.t. $\mathcal{T}$. We devised methods for solving such problems in [41] and used them for interpolant computation [41], and invariant generation [36].

We are not aware of other similar approaches to the area of computational (geometric) graph theory. Existing approaches use a logical representation of graphs based on monadic second-order logic (cf. e.g. [15]) or higher-order theorem provers like Isabelle/HOL (cf. e.g. [1]). Our approach is orthogonal; it allows a reduction of many problems to satisfiability modulo a suitable theory.
Structure of the paper. In Section 2 we present the motivation for our research. In Section 3 we introduce the notions on (local) theory extensions needed in the paper and prove the locality of theories of distance functions. In Section 4 we describe (and slightly extend) a method for property-directed symbol elimination we proposed in [41]. In Section 5 we present the $HR_{e}^{L}$ calculus we use for eliminating predicate $P$, and analyze possibilities of giving finite representations for infinite saturated sets and of investigating the satisfiability of the saturated sets. In Section 6 we use these ideas for checking class inclusion. In Section 7 we discuss the way in which we tested the methods we propose on various examples. In Section 8 we present conclusions and plans for future work.

This paper is an extended version of [37] which contains full proofs of the results, a more detailed description of the examples, a description of the systems we used for testing and several examples illustrating how these systems were used.

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2 Motivation

Graph Classes. Graph classes important in wireless network research are: The class $UDG$ of unit disk graphs (two nodes are connected iff they are different and their distance is $\leq 1$); the class $QUDG(r)$ of quasi unit disk graphs, for $r \in (0,1]$ (two distinct nodes with distance $\leq r$ are always connected and nodes with distance $> 1$ are never connected); the class $DTG(r,r)$ of directed transmission graphs for $r > 0$ (every node $v$ has a maximum communication distance $r(v) \leq r$; an edge from $v$ to $w$ exists iff $v \neq w$ and the distance between $v$ and $w$ is $\leq r(v)$).
Many graph classes $C(\overline{\pi})$ (where $\overline{\pi}$ is a sequence of symbols denoting parameters) can be described using inclusion, exclusion and transfer axioms.

The inclusion axioms specify which edges have to exist. For a graph class $C$ the condition under which an edge $E(u, v)$ must exist can be described by a formula $\pi_C^i(u, v)$. Therefore, inclusion axioms have the form:

1. $\forall u, v (\pi_C^i(u, v) \rightarrow E(u, v))$

The exclusion axioms specify which edges are not allowed to exist. For a class $C$ the condition under which an edge $E(u, v)$ is not allowed to exist can be described by a formula $\pi_C^e(u, v)$. Therefore, transfer axioms have the form:

2. $\forall u, v (\pi_C^e(u, v) \rightarrow \neg E(u, v))$

The transfer axioms specify which edges $E(u, w)$ must exist as a consequence of the existence of another edge $E(u, v)$. For a class $C$, we describe these these conditions by a formula $\pi_C^t(u, v, w)$. Therefore, inclusion axioms have the form:

3. $\forall u, v, w \pi_C^i(u, v, w) \land E(u, v) \rightarrow E(u, w)$

If the description of the graph class $C$ depends on parameters $\overline{\pi}$, the formulae $\pi_C^i, \pi_C^e$ and $\pi_C^t$ might contain parameters. We will sometimes indicate this by adding the parameters to the arguments, i.e. writing $\pi_C^i(u, v, \overline{\pi}), \pi_C^e(u, v, \overline{\pi})$ resp. $\pi_C^t(u, v, w, \overline{\pi})$.

We can, e.g., define the classes $\text{MinDG}(r)$, $\text{MaxDG}(r)$ and $\text{CRG}$ using axioms:

- $\text{MinDG}(r)$: axiom (1), where $\pi_C^i(u, v, r)$ is the formula $u \neq v \land d(u, v) \leq r$;
- $\text{MaxDG}(r)$: axiom (2), where $\pi_C^e(u, v, r)$ is the formula $d(u, v) > r$;
- $\text{CRG}$: axiom (3), where $\pi_C^t(u, v, w, r)$ is the formula $u \neq w \land d(u, w) \leq d(u, v)$,

(where $r$ is supposed to be a parameter).

With this notation, the inclusion axiom $\text{MinDG}(r)$ states that if $u \neq v$ and $d(u, v) \leq r$ an edge from $u$ to $v$ must exist; the exclusion axiom $\text{MaxDG}(r)$ states that if $d(u, v) > r$ then we are not allowed to have an edge from $u$ to $v$. The transfer axiom $\text{CRG}$ states that if $u$ and $w$ are different and there is an edge from $u$ to $v$ and $d(u, w) \leq d(u, v)$ then there must exist an edge also from $u$ to $w$.

By combining such axioms we obtain axiomatizations for new graph classes. If the classes $A$ and $B$ of graphs are axiomatized by axioms $Ax_A$ and $Ax_B$ then $Ax_A \land Ax_B$ is an axiomatization for the intersection $A \cap B$.

For instance, the class $\text{UDG} = \text{MinDG}(1) \cap \text{MaxDG}(1)$ is axiomatized by $\text{MinDG}(1) \land \text{MaxDG}(1)$.

We may want to check whether a graph class $C(\overline{\pi})$ has non-empty models, or to determine (weakest) conditions on the parameters $\overline{\pi}$ under which this is the case. This is one of the problems which will be analyzed in this paper.

**Simple transformations on graph classes.** We can define transformations $\gamma$ on graphs that transform the edges and leave the set of vertices unchanged, and form graph classes

$\gamma(C) = \{ \gamma(G) \mid G \in C \}$. 
Two examples of transformations are $\cdot^+$ and $\cdot^-$: Given a graph $G = (V, E)$, we can build the symmetric supergraph $G^+ = (V, E^+)$ resp. symmetric subgraph $G^- = (V, E^-)$, defined by:

$$\forall x, y \ (E^+(x, y) \leftrightarrow (E(x, y) \lor E(y, x)))$$

$$\forall x, y \ (E^-(x, y) \leftrightarrow (E(x, y) \land E(y, x))).$$

We can thus define the classes $C^+ = \{G^+ | G \in C\}$ and $C^- = \{G^- | G \in C\}$. The class of quasi unit disk graphs [7,35] can, for instance, be described as $\text{QUDG}(r) = (\text{MinDG}(r) \cap \text{MaxDG}(1))^-$. We might want to obtain an axiomatization for $\text{QUDG}(r)$ that depends only on the predicates $\pi^r(x, y, r)$ and $\pi^e(x, y, 1)$ or test whether the class is the same as the class described by $(\text{MinDG}(r) \cap \text{MaxDG}(1))^+$. To find an axiomatization of a graph class $\gamma(C)$, where $\gamma$ is a transformation, we need to find a first-order formula equivalent to $\exists E' (N_{E'} \cap \text{Tr}(E', E))$, where $N_{E'}$ is a class of clauses describing class $C$ and $\text{Tr}$ is a formula describing the way the edges of the graph $(V, E) = \gamma(V, E')$ can be obtained from the description of the graph $(V, E')$. We here analyze possibilities of eliminating second-order quantifiers.

**Checking class inclusion.** If we can find such formulae for two graph classes, then we can also check containment (provided the formulae belong to decidable theory fragments). In this paper we analyze situations in which this is possible.

### 3 Theories and local theory extensions

We assume known the basic notions in (many-sorted) first-order logic. We consider signatures of the form $\Pi = (S, \Sigma, \text{Pred})$, where $S$ is a set of sorts, $\Sigma$ is a family of function symbols and $\text{Pred}$ a family of predicate symbols, such that for every function symbol $f$ (resp. predicate symbol $p$) their arity $a(f) = s_1 \ldots s_n \rightarrow s$ (resp. $a(p) = s_1 \ldots s_m$), where $s_1, \ldots, s_n, s \in S$, is specified. If $C$ is a fixed countable set of fresh constants, we denote by $\Pi^C$ the extension of $\Pi$ with constants in $C$. We assume known standard definitions from first-order logic such as $\Pi$-structure, model, satisfiability, unsatisfiability. A $\Pi$-structure is a tuple

$$A = (\{A_s\}_{s \in S}, \{f_A\}_{f \in \Sigma}, \{p_A\}_{p \in \text{Pred}}),$$

where, for every $s \in S$, $A_s$ is a non-empty set (the universe of sort $s$ of the structure), for every $f \in \Sigma$ with arity $s_1 \ldots s_n \rightarrow s$, $f_A : A_{s_1} \times \cdots \times A_{s_n} \rightarrow A_s$, and for every $p \in \text{Pred}$ with arity $s_1 \ldots s_m$, $p_A \subseteq A_{s_1} \times \cdots \times A_{s_m}$.

If $A$ is a $\Pi$-structure, we will denote by $A^A$ the extension of $A$, where we have an additional constant (of sort $s$) for each element $a$ of sort $s$ of $A$ (which we denote with the same symbol) with the natural interpretation mapping the constant $a$ to the element $a$ of $A$. 


If \( \Pi \subseteq \Pi' \) and \( A \) is a \( \Pi' \)-structure, we denote its reduct to \( \Pi \) by \( A |_{\Pi} \).

**Notation.** We will denote with (indexed versions of) \( x, y, z \) variables and with (indexed versions of) \( a, b, c, d \) constants; \( \overline{t} \) will stand for a sequence of variables \( x_1, \ldots, x_n \), and \( \overline{c} \) for a sequence of constants \( c_1, \ldots, c_n \).

**Theories.** Theories can be defined by specifying a set of axioms, or by specifying a class of structures (the models of the theory). If \( F \) and \( G \) are formulae we write \( F \models G \) (resp. \( F \models_T G \) – also written as \( T \cup F \models G \)) to express the fact that every model of \( F \) (resp. every model of \( F \) which is also a model of \( T \)) is a model of \( G \). We denote “falsum” with \( \perp \). \( F \models \perp \) means that \( F \) is unsatisfiable; \( F \models_T \perp \) means that there is no model of \( T \) in which \( F \) is true.

A theory \( T \) over a signature \( \Pi \) allows quantifier elimination (QE) if for every formula \( \phi \) over \( \Pi \) there exists a quantifier-free formula \( \phi^* \) over \( \Pi \) which is equivalent to \( \phi \) modulo \( T \). Examples of theories which allow quantifier elimination are rational and real linear arithmetic (\( \text{LI}(\mathbb{Q}) \), \( \text{LI}(\mathbb{R}) \)), the theory of real closed fields, and the theory of absolutely-free data structures.

Sometimes, in order to define more complex theories we can consider theory extensions and combinations thereof. **Local theory extensions** are a class of theory extensions for which hierarchical reasoning is possible.

### 3.1 Local theory extensions

In what follows, for simplicity we present the main notions in the one-sorted case; the extension to the many-sorted case is immediate.

Let \( \Pi_0 = (\Sigma_0, \text{Pred}) \) be a signature, and \( T_0 \) be a “base” theory with signature \( \Pi_0 \). We consider extensions \( T := T_0 \cup K \) of \( T_0 \) with new function symbols \( \Sigma \) (extension functions) whose properties are axiomatized using a set \( K \) of (universally closed) clauses in the extended signature \( \Pi = (\Sigma_0 \cup \Sigma, \text{Pred}) \), such that each clause in \( K \) contains function symbols in \( \Sigma \). Especially well-behaved are the \( \Psi \)-local theory extensions, i.e. theory extensions \( T_0 \subseteq T_0 \cup K \) as defined above, in which checking ground satisfiability can be done using a finite instantiation scheme described by a suitable closure operator \( \Psi \), without loss of completeness. We express this with the following condition:

\[(\text{Loc}_T^\Psi) \quad \text{For every finite set } G \text{ of ground } \Pi^\mathcal{C} \text{-clauses (for an additional set } \mathcal{C} \text{ of constants) it holds that } T_0 \cup K \cup G \models \perp \text{ if and only if } T_0 \cup K[\Psi_K(G)] \cup G \text{ is unsatisfiable.}\]

where, for every set \( G \) of ground \( \Pi^\mathcal{C} \)-clauses, \( K[\Psi_K(G)] \) is the set of instances of \( K \) in which the terms starting with a function symbol in \( \Sigma \) are in \( \Psi_K(G) = \Psi(\text{est}(K, G)) \), where \( \text{est}(K, G) \) is the set of ground terms starting with a function in \( \Sigma \) occurring in \( G \) or \( K \).

If \( T_0 \) is the pure theory of equality, we obtain the notion of locality [21,20].

**Partial and total models.** In [38] we showed that local theory extensions can be recognized by showing that certain partial models embed into total ones, and
in [30] we established similar results for ψ-local theory extensions and generalizations thereof. We introduce the main definitions here, following mainly the presentation from [30] and [42].

Let \( \Pi = (\Sigma, \text{Pred}) \) be a first-order signature with set of function symbols \( \Sigma \) and set of predicate symbols \( \text{Pred} \). A \emph{partial \( \Pi \)-structure} is a structure \( A = (A, \{ f_A \}_{f \in \Sigma}, \{ p_A \}_{p \in \text{Pred}}) \), where \( A \) is a non-empty set, for every \( n \)-ary \( f \in \Sigma \), \( f_A \) is a partial function from \( A^n \) to \( A \), and for every \( n \)-ary \( p \in \text{Pred} \), \( p_A \subseteq A^n \).

We consider constants (0-ary functions) to be always defined. \( A \) is called a \emph{total structure} if the functions \( f_A \) are all total. Given a (total or partial) \( \Pi \)-structure \( A \) and \( \Pi_0 \subseteq \Pi \) we denote the reduct of \( A \) to \( \Pi_0 \) by \( A|_{\Pi_0} \).

The notion of evaluating a term \( t \) with variables \( X \) w.r.t. an assignment \( \beta : X \rightarrow A \) for its variables in a partial structure \( A \) is the same as for total algebras, except that the evaluation is undefined if \( t = f(t_1, \ldots, t_n) \) and at least one of \( \beta(t_i) \) is undefined, or else \( (\beta(t_1), \ldots, \beta(t_n)) \) is not in the domain of \( f_A \).

**Definition 1** A \emph{weak \( \Pi \)-embedding} between two partial \( \Pi \)-structures \( A \) and \( B \), where \( A = (A, \{ f_A \}_{f \in \Sigma}, \{ p_A \}_{p \in \text{Pred}}) \) and \( B = (B, \{ f_B \}_{f \in \Sigma}, \{ p_B \}_{p \in \text{Pred}}) \) is a total map \( \varphi : A \rightarrow B \) such that

(i) \( \varphi \) is an embedding w.r.t. \( \text{Pred} \cup \{ = \} \), i.e. for every \( p \in \text{Pred} \) with arity \( n \) and every \( a_1, \ldots, a_n \in A \), \( (a_1, \ldots, a_n) \in p_A \) if and only if \( (\varphi(a_1), \ldots, \varphi(a_n)) \in p_B \).

(ii) whenever \( f_A(a_1, \ldots, a_n) \) is defined (in \( A \)), then \( f_B(\varphi(a_1), \ldots, \varphi(a_n)) \) is defined (in \( B \)) and \( \varphi(f_A(a_1, \ldots, a_n)) = f_B(\varphi(a_1), \ldots, \varphi(a_n)) \), for all \( f \in \Sigma \).

**Definition 2** (Weak validity) Let \( A \) be a partial \( \Pi \)-algebra and \( \beta : X \rightarrow A \) a valuation for its variables. \( (\chi, \beta) \) weakly satisfies a clause \( C \) (notation: \( (\chi, \beta) \models_w C \)) if either some of the literals in \( \beta(C) \) are not defined or otherwise all literals are defined and for at least one literal \( L \) in \( C \), \( L \) is true in \( A \) w.r.t. \( \beta \). \( A \) is a weak partial model of a set of clauses \( \mathcal{K} \) if \( (\chi, \beta) \models_w C \) for every valuation \( \beta \) and every clause \( C \) in \( \mathcal{K} \).

**Recognizing \( \Psi \)-local theory extensions.** In [38] we proved that if every weak partial model of an extension \( \mathcal{T}_0 \cup \mathcal{K} \) of a base theory \( \mathcal{T}_0 \) with total base functions can be embedded into a total model of the extension, then the extension is local. In [28] we lifted these results to \( \Psi \)-locality.

Let \( A = (A, \{ f_A \}_{f \in \Sigma \cup \mathcal{C}}, \{ p_A \}_{p \in \text{Pred}}) \) be a partial \( \Pi^C \)-structure with total \( \Sigma \)-functions. Let \( \Pi^A \) be the extension of the signature \( \Pi \) with constants from \( A \). We denote by \( T(A) \) the following set of ground \( \Pi^A \)-terms:

\[
T(A) := \{ f(a_1, \ldots, a_n) \mid f \in \Sigma, a_i \in A, i = 1, \ldots, n, f_A(a_1, \ldots, a_n) \text{ is defined} \}.
\]

Let \( \text{PMd}_{\psi}^w(\Sigma, \mathcal{T}) \) be the class of all weak partial models \( A \) of \( \mathcal{T}_0 \cup \mathcal{K} \), such that \( A|_{\Pi_0} \) is a total model of \( \mathcal{T}_0 \), the \( \Sigma \)-functions are possibly partial, \( T(A) \) is finite and all terms in \( \Psi(\text{est}(\mathcal{K}, T(A))) \) are defined (in the extension \( \Pi^A \) with constants from \( A \)). We consider the following embeddability property of partial algebras:
(Emb\(_{w,f}^\Psi\)) Every \(A \in \text{PMod}_{w,f}^\Psi(\Sigma, T)\) weakly embeds into a total model of \(T\).

We also consider the property (EEmb\(_{w,f}^\Psi\)), which additionally requires the embedding to be elementary, and the property (Comp\(_{w,f}^\Psi\)), which requires that every structure \(A \in \text{PMod}_{w,f}^\Psi(\Sigma, T)\) embeds into a total model of \(T\) with the same support. If \(\Psi\) is the identity, we refer to these properties as (Emb\(_{w,f}\)), (EEmb\(_{w,f}\)) and Comp\(_f\).

When establishing links between locality and embeddability we require that the clauses in \(K\) are flat and linear w.r.t. \(\Sigma\)-functions. When defining these notions we distinguish between ground and non-ground clauses.

**Definition 3** An extension clause \(D\) is flat (resp. quasi-flat) when all symbols below a \(\Sigma\)-function symbol in \(D\) are variables (resp. variables or ground \(\Pi_0\)-terms). \(D\) is linear if whenever a variable occurs in two terms of \(D\) starting with \(\Sigma\)-functions, the terms are equal, and no term contains two occurrences of a variable.

A ground clause \(D\) is flat if all symbols below a \(\Sigma\)-function in \(D\) are constants.

**Definition 4** ([30]) With the above notations, let \(\Psi\) be a map associating with \(K\) and a set of \(\Pi_0\)-ground terms \(T\) a set \(\Psi_K(T)\) of \(\Pi_0\)-ground terms. We call \(\Psi_K\) a term closure operator if the following holds for all sets of ground terms \(T, T'\):

1. \(\text{est}(K, T) \subseteq \Psi_K(T)\),
2. \(T \subseteq T' \Rightarrow \Psi_K(T) \subseteq \Psi_K(T')\),
3. \(\Psi_K(\Psi_K(T)) \subseteq \Psi_K(T)\),
4. for any map \(h : C \rightarrow C\), \(\bar{h}(\Psi_K(T)) = \Psi_{K'}(\bar{h}(T))\), where \(\bar{h}\) is the canonical extension of \(h\) to extension ground terms.

**Theorem 1** ([28,30]) Let \(T_0\) be a first-order theory and \(K\) a set of universally closed flat clauses in the signature \(\Pi\). The following hold:

1. If all clauses in \(K\) are linear and \(\Psi\) is a term closure operator with the property that for every flat set of ground terms \(T, \Psi(T)\) is flat then either of the conditions (Emb\(_{w,f}^\Psi\)) and (EEmb\(_{w,f}^\Psi\)) implies (Loc\(_f^\Psi\)).
2. If the extension \(T_0 \subseteq T = T_0 \setminus K\) satisfies (Loc\(_f^\Psi\)) then (Emb\(_{w,f}^\Psi\)) holds.

The linearity assumption needed to prove that (Emb\(_{w,f}^\Psi\)) implies (Loc\(_f^\Psi\)) can be relaxed if the closure operator \(\Psi\) has additional properties.
Theorem 2 Let $\mathcal{K}$ be a set of $\Sigma$-flat clauses, with the property that every variable occurs only once in every term. Let $\Psi$ be a term closure operator with the property that for every flat set of ground terms $T$, $\Psi(T)$ is flat. Assume that $\mathcal{K}$ and $\Psi$ have the property that for every flat set of ground terms $T$ and every clause $C \in \mathcal{K}$, if $C$ contains terms $f(x_1, \ldots, x_n)$ and $g(y_1, \ldots, y_m)$ (where $f, g \in \Sigma$ are extension functions and $f$ and $g$ are not necessarily different), if $f(t_1, \ldots, t_n), g(s_1, \ldots, s_m) \in \Psi\mathcal{K}(T)$ then $f(t_1, \ldots, s, \ldots, t_n), g(s_1, \ldots, t, \ldots, t_m) \in \Psi\mathcal{K}(T)$. Then ($\text{Emb}^\Psi_{\mathcal{K},f}$) implies ($\text{Loc}^\Psi_f$).

Proof: The proof is included in Appendix B. \qed

Theorem 3 ([39,28]) The following theory extensions have property ($\text{Comp}_f$), hence are local:

(i) The extension of a theory $\mathcal{T}_0$ with uninterpreted function symbols.

(ii) The extension of a theory $\mathcal{T}_0$ containing a predicate $\leq$ which is reflexive with a function $f$ satisfying the axioms $\mathcal{K} = \{ \forall \bar{x} \phi_i(\bar{x}) \rightarrow L_i(\bar{x}) \mid i = 1, \ldots, n \}$, where:

$\phi_i$ are $\mathcal{T}_0$-formulae with $\phi_i(\bar{x}) \land \phi_j(\bar{x}) \rightarrow \bot$ if $i \neq j$.

$L_i(\bar{x})$ has the form (1) $s_i \leq f(\bar{x})$ or (2) $f(\bar{x}) \leq t_i$ or (3) $s_i \leq f(\bar{x}) \leq t_i$

where $s_i, t_i$ are $\Pi_0$-terms and in case (3) $\phi_i \models \sigma_i s_i \leq t_i$.

Hierarchical reasoning. Consider a $\psi$-local theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$. Condition ($\text{Loc}^\Psi_f$) requires that for every finite set $G$ of ground $\Pi^C$-clauses, $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \bot$ iff $\mathcal{T}_0 \cup \mathcal{K}[\psi(G)] \cup G \models \bot$. In all clauses in $\mathcal{K}[\psi(G)] \cup G$ the function symbols in $\Sigma$ only have ground terms as arguments, so $\mathcal{K}[\psi(G)] \cup G$ can be flattened and purified. We thus obtain a set of clauses $\mathcal{K}_0 \cup G_0 \cup \text{Def}$, where $\mathcal{K}_0$ and $G_0$ do not contain $\Sigma$-function symbols and $\text{Def}$ contains clauses of the form $c \approx f(c_1, \ldots, c_n)$, where $f \in \Sigma$, and $c_1, \ldots, c_n$ are constants. This transformation allows us to reduce testing satisfiability w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$ to testing satisfiability w.r.t. $\mathcal{T}_0$.

Theorem 4 ([38]) Let $\mathcal{K}$ be a set of clauses. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ is a $\psi$-local theory extension. For any finite set $G$ of ground $\Pi^C$-clauses, let $\mathcal{K}_0 \cup G_0 \cup \text{Def}$ be obtained from $\mathcal{K}[\psi(G)] \cup G$ by introducing, in a bottom-up manner, new constants $c_i \in C$ for subterms $t \approx f(c_1, \ldots, c_n)$ where $f \in \Sigma$ and $c_i$ are constants, together with definitions $c_i \approx f(c_1, \ldots, c_n)$ (included in $\text{Def}$) and replacing the corresponding terms $t$ with the constants $c_i$ in $\mathcal{K}$ and $G$. Then $\mathcal{T}_1 \cup G \models \bot$ if and only if $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup \text{Con}_0 \models \bot$, where $\text{Con}_0 = \{ \bigwedge_{i=1}^n c_i \approx d_i \rightarrow c \approx d \mid f(c_1, \ldots, c_n) \approx s \in \text{Def} \}$. This method is implemented in the program H-PILoT (Hierarchical Proving by Instantiation in Local Theory Extensions) [29].

1 A similar result can be proved also in the case in which some variables occur several times below a function symbol if $\psi\mathcal{K}$ has the property that if $f(x_1, \ldots, x_1, \ldots, x_n) \in \mathcal{K}$ and $f(t_1, \ldots, x, \ldots, t_n) \in \psi\mathcal{K}(T)$ then $f(t_1, \ldots, t, \ldots, t_n) \in \psi\mathcal{K}(T)$ and $f(t_1, \ldots, s, \ldots, s, t_n) \in \psi\mathcal{K}(T)$. 
3.2 Locality of theories of distances

The theories related to wireless networks used in Section 2 refer to cost or distance functions. We prove that axiomatizations for such functions define local theory extensions. We first formalize the properties of metric spaces \((X, d)\), i.e., sets endowed with a distance function \(d\) satisfying the usual axioms of a metric, and prove a locality property. We then consider variants that contain only some of these axioms.

**Theorem 5** Let \(\mathcal{T}_0\) be the disjoint two-sorted combination of \(\mathcal{E}\), the pure theory of equality (no function symbols), sort \(p\), and \(LI(\mathbb{R})\) (linear real arithmetic), sort \(num\). Let \(\mathcal{T}_d^m\) be the extension of \(\mathcal{T}_0\) with a function \(d\) with arity \(a(d) = p, p \rightarrow num\) satisfying the following set \(\mathcal{K}_d\) of axioms:

\[
\begin{align*}
(d_1) & \forall x, y \ d(x, y) \geq 0 \\
(d_2) & \forall x, y, z \ d(x, y) \leq d(x, z) + d(z, y) \\
(d_3) & \forall x, y \ d(x, y) = d(y, x) \\
(d_4) & \forall x, y \ x = y \rightarrow d(x, y) = 0 \\
(d_5) & \forall x, y \ d(x, y) = 0 \rightarrow x = y
\end{align*}
\]

Let \(\Psi_m\) be defined for every set \(T\) of ground terms by

\[\Psi_m(T) = \{d(a, b) \mid a, b \text{ are constants of sort } p \text{ occurring in } T\}.\]

Then the following hold:

1. \(\Psi_m\) is a closure operator on ground terms.
2. For every finite set \(T\) of ground terms, \(\Psi_m(T)\) is finite.
3. \(\mathcal{T}_d^m\) is a \(\Psi_m\)-local extension of \(\mathcal{T}_0\) satisfying condition \((Comp)^\Psi_T\).

**Proof:**

1. Clearly, for every set \(T\) of ground terms, \(T\) and \(\Psi_m(T)\) contain the same constants of sort \(p\), so \(\Psi_m(\Psi_m(T)) = \Psi_m(T)\). Since the only extension function symbol is \(d\), \(est(K, T) \subseteq \Psi_m(T)\) for every set \(T\) of ground terms. The fact that if \(T_1 \subseteq T_2\) we have \(\Psi_m(T_1) \subseteq \Psi_m(T_2)\) follows from the definition. It is also easy to check that for every map \(h : C \rightarrow C\), \(\overline{h}(\Psi_m(T)) = \Psi_m(\overline{h}(T))\), i.e., \(\Psi_m\) is stable under renaming of constants.

2. If \(T\) is finite, then it contains finitely many constants (say \(n\)). \(\Psi_m(T)\) has then \(n^2\) elements.

3. To prove that \(\mathcal{T}_d^m\) is a \(\Psi_m\)-local extension of \(\mathcal{T}_0\), we prove that it satisfies the embeddability condition \((Comp)^\Psi_{\mathcal{F}}\), i.e., that for every partial model \(\mathcal{P} = (P, \mathbb{R}, d_P)\) of \(\mathcal{T}_d = \mathcal{T}_0 \cup \mathcal{K}_d\) with the properties:

   (i) All function symbols in \(\Sigma_0\) are everywhere defined; \(d\) is partially defined.
   (ii) The set \(T(\mathcal{P}) = \{d(a_1, a_2) \mid a_i \in A_p, d_P(a_1, a_2)\text{ is defined}\}\) is finite, and closed under \(\Psi_m\).

\(d_P\) can be extended to a total function on \(P\) that satisfies the axioms \(\mathcal{K}_d\).

Let \(\mathcal{P} = (P, \mathbb{R}, d_P)\) be a partial model of \(\mathcal{T}_d^m = \mathcal{T}_0 \cup \mathcal{K}_m\) (where \(P\) is the support of sort \(p\), \(\mathbb{R}\) the support of sort \(num\), and \(d_P\) a partial function from \(P \times P\) to \(\mathbb{R}\)) satisfying the conditions above. Then:
We now show how to extend \(d\) defined, \(d(p_1, p_2) \geq 0\), and if \(p_1 = p_2\), \(d(p_1, p_2) = 0\);
- \(d(p, p) = 0\) whenever it is defined;
- if \(d_P(p_1, p_2)\) and \(d_P(p_2, p_1)\) are defined then \(d_P(p_1, p_2) = d_P(p_2, p_1)\); and
- if \(d_P(p_1, p_2), d_P(p_2, p_3), d_P(p_1, p_3)\) are defined then \(d_P(p_1, p_2) \leq d_P(p_2, p_3) + d_P(p_1, p_3)\).

Let \(E = \{(p_1, p_2) \mid d_P(p_1, p_2)\text{ is defined}\}\).

Let \(P_1 = \{p \in P \mid \exists q \in P : d_P(p, q)\text{ defined or }d_P(q, p)\text{ defined}\}\). By the assumption that \(d_P(p_1, p_2)\) is defined only for finitely many tuples \((p_1, p_2)\), \(P_1\) is finite and by condition (ii) above (as \(T(P)\) is closed under \(\mathcal{E}_m\)), \(E = P_1 \times P_2\).

Thus, \(P = P_1 \cup P_2\), such that \(d_1 = d_P|P_1\) is totally defined and \(d\) is nowhere defined on \(P_2\) (for every two different elements \(p_1, p_2 \in P_2\), \(d_P(p_1, p_2)\) is undefined and for every \(p \in P_2\) there is no \(q \in P_1\) such that \(d_P(p, q)\) or \(d_P(q, p)\) is defined).

Since \(P_1\) is finite, the maximum \(m_1 = \max\{d(p, q) \mid p, q \in P_1\}\) exists.

Consider an arbitrary distance function \(d_2\) on \(P_2\) such that \(\sup\{d_2(p_1, p_2) \mid p_1, p_2 \in P_2\}\) is finite (such a function is guaranteed to exist, since the distance axioms are consistent: We can for instance regard all points in \(P_1\) as points in the unit circle and consider the euclidian distances between these points). Thus, the distance function \(d_2\) on \(P_2\) is totally defined and bounded. Let \(m_2\) be such that \(d_2(p, q) \leq m_2\) for all \(p, q \in P_2\).

We now show how to extend \(d\) on \(P_1 \cup P_2\). If \(P_1\) or \(P_2\) are empty we have a total extension of \(d\) already. Assume they are both non-empty. Let \(p_1 \in P_1\) and \(p_2 \in P_2\). We construct a totally defined function \(d : (P_1 \cup P_2)^2 \to \mathbb{R}\) as follows:

\[
d(p, q) = \begin{cases} 
  d_1(p, q) & \text{if } p, q \in P_1 \\
  d_2(p, q) & \text{if } p, q \in P_2 \\
  d_0 + d_1(p, q_1) + d_2(p_2, q) & \text{if } p \in P_1 \text{ and } q \in P_2 \\
  d_0 + d_1(q, q_1) + d_2(p_2, p) & \text{if } p \in P_2 \text{ and } q \in P_1
\end{cases}
\]

where \(d_0 \in \mathbb{R}\), is such that \(d_0 = m + 1\), where \(m = \max(m_1, m_2)\).

We show that \(d\) is a total function that satisfies all the axioms \(K_d\):

- It is clear that \(d\) is a total function and that for all \(x, y \in P_1 \cup P_2, d(x, y) \geq 0\), i.e. it satisfies axiom \((d_4)\).
- Let \(p \in P_1 \cup P_2, p \in P_1\), with \(i = 1\) or \(2\), and since \(d_i\) satisfies axiom \((d_4)\), \(d(p, p) = d_i(p, p) = 0\). Thus \(d\) satisfies axiom \((d_4)\) too.
- Let \(p, q \in P_1 \cup P_2\). If \(p, q \in P_1\) for \(i = 1\) or \(2\), then \(d(p, q) = d_i(p, q) = d_i(q, p) = 0\); the case when \(p \in P_2, q \in P_1\) is similar. Thus \(d\) satisfies axiom \((d_i)\) too.
- We show that \(d\) satisfies the triangle inequality (axiom \((d_2)\)). Let \(p, q, r \in P_1 \cup P_2\). We show that \(d(p, q) \leq d(p, r) + d(r, q)\). We distinguish the following cases:
Case 1: \( p \in P_1, q \in P_2 \). Then \( d(p, q) = d_0 + d_1(p,p_1) + d_2(p_2,q) \).

**Subcase 1.a:** \( r \in P_1 \). Then \( d(p, r) + d(r, q) = d_1(p, r) + d_0 + d_1(r, p_1) + d_2(p_2, r) \geq d_0 + d_1(p, p_1) + d_2(p_2, q) = d(p, q) \).

**Subcase 1.b:** \( r \in P_2 \). Then \( d(p, r) + d(r, q) = d_0 + d_1(p, p_1) + d_2(p_2, q) = d(p, q) \).

Case 2: \( p \in P_2, q \in P_1 \). Then \( d(p, q) = d_0 + d_1(q, p_1) + d_2(p_2, p) \).

**Subcase 2.a:** \( r \in P_2 \). Then \( d(p, r) + d(r, q) = d_2(p_2, r) + d_0 + d_1(r, p_2) + d_2(p_1, q) = d_2(r, p) + d_0 + d_1(q, p_1) + d_2(p_2, r) \geq d_0 + d_1(q, p_1) + d_2(p_2, p) = d(p, q) \).

**Subcase 2.b:** \( r \in P_1 \). Then \( d(p, r) + d(r, q) = d_0 + d_1(p, p_2) + d_2(p_2, r) + d_1(r, q) \geq d_0 + d_1(p, p_1) + d_2(p_2, p) = d(p, q) \).

Case 3: \( p, q \in P_1 \). Then \( d(p, q) = d_1(p, q) \).

**Subcase 3.a:** \( r \in P_1 \). Then \( d(p, q) = d_1(p, r) + d_1(r, q) = d(p, r) + d(r, q) \), since \( d_1 \) satisfies axiom \((d_2)\).

**Subcase 3.b:** \( r \in P_2 \). Then \( d(p, q) = d_1(p, q) \leq d_0 < d(p, r) + d(r, q) \).

Case 4: \( p, q \in P_2 \). Then \( d(p, q) = d_2(p, q) \).

**Subcase 4.a:** \( r \in P_2 \). Then \( d(p, q) = d_2(p, r) + d_2(r, q) = d(p, r) + d(r, q) \), since \( d_2 \) satisfies axiom \((d_2)\).

**Subcase 4.b:** \( r \in P_1 \). Then \( d(p, q) = d_2(p, q) \leq d_0 \leq d(p, r) + d(r, q) \).

In [30] it was proved that condition \((\text{Comp}_1^\Psi)\) for \( T_0 \subseteq T_0 \cup K \) implies \( \Psi \)-locality of the extension if the clauses in \( K \) are flat and linear. The clauses in \( K_m \) are flat, but are not linear. In the proof of the fact that embeddability implies locality linearity is needed in order to ensure that if we have a model \( B \) of \( T_0 \cup K[\Psi(G)] \cup G \) we can define a partial model \( A \) of \( T_0 \cup K \cup G \) and argue that (by \((\text{Comp}_1^\Psi)\)) this model embeds into a total model of \( T_0 \cup K \cup G \). We construct \( A \) as follows: Its universe(s) are the same as for \( B \), and \( f(a_1, \ldots, a_n) \) is defined in \( A \) if there exists constants \( c_1, \ldots, c_n \) which interpret in \( A \) as \( a_1, \ldots, a_n \) and \( f(c_1, \ldots, c_n) \) occurs in \( \Psi(G) \). This definition is used to associate with every valuation in \( A \) in which all terms in a clause \( C \) are defined a substitution \( \sigma \) such that \( C \sigma \in K[\Psi(G)] \).

If the clause \( C \) is linear the substitution can be defined without problems. If \( C \) contains a variable in different terms, it might be difficult to define \( \Sigma \) because for different occurrences of \( x \) we might find different suitable terms.

This problem does not occur here because of the fact that \( \Psi_m \) adds all necessary instances that allow to define \( \sigma \) without problems.

Alternatively, it can be easily checked that all assumptions in Theorem 2 hold in this case, so in this case embeddability entails locality.

We can still obtain local theory extensions if we leave out some of the metric axioms. Below we consider, for instance, extensions with a function \( d \) in which all the axioms of a metric except for the triangle inequality hold.

**Theorem 6** Let \( T_0 \) be the disjoint two-sorted combination of the theory \( E \) of pure equality (no function symbols), sort \( P \), and \( LI(\mathbb{R}) \) (linear real arithmetic),
sort num. Let $\mathcal{T}_d^a$ be the extension of $\mathcal{T}_0$ with a function $d$ with arity $a(d) = p, p \to \text{num}$ satisfying the following set $\mathcal{K}_n$ of axioms:

\[(d_1) \forall x, y. d(x, y) \geq 0\]
\[(d_2) \forall x, y. d(x, y) = d(y, x)\]
\[(d_3) \forall x, y. x = y \to d(x, y) = 0\]
\[(d_4) \forall x, y. d(x, y) = 0 \to x = y\]

Let $\Psi_n$ be defined for every set $T$ of ground terms by

$$
\Psi_n(T) = T \cup \{d(t_2, t_1) \mid d(t_1, t_2) \in T\} \cup \{d(a, a) \mid a \text{ is a constant of sort } p \text{ occurring in } T\}
$$

Then $\mathcal{T}_d^a$ is a $\Psi_n$-local extension of $\mathcal{T}_0$.

Proof: To prove locality we have to show that every partial model of $\mathcal{T}_d^a = \mathcal{T}_0 \cap \mathcal{K}_n$ which is closed under $\Psi_n$ can be extended to a total model. Let $\mathcal{P} = (P, \mathbb{R}, d_P)$ be a partial model of $\mathcal{T}_d = \mathcal{T}_0 \cup \mathcal{K}_d$ (where $P$ is the support of sort $p$, $\mathbb{R}$ the support of sort num, and $d_P$ a partial function from $P \times P$ to $\mathbb{R}$) satisfying the conditions above. We construct a total function $d : P \times P \to \mathbb{R}$ as follows:

$$
d(p, q) = \begin{cases} 
d_P(p, q) & \text{if } d_P(p, q) \text{ is defined} \\
0 & \text{if } d_P(p, q) \text{ is not defined and } p = q \\
1 & \text{if } d_P(p, q) \text{ is not defined and } p \neq q
\end{cases}
$$

It is easy to check that $d$ satisfies all the axioms in $\mathcal{K}_n$. The considerations in the previous proof (or Theorem 2) can be used also in this case to show that embeddability implies locality in spite of the non-linearity due to the choice of the closure operator. \qed

**Theorem 7** Let $\mathcal{T}_0$ be the disjoint combination of the theory of pure equality (sort $p$) and linear real arithmetic (sort $\text{num}$). The following extensions of $\mathcal{T}_0$ with a function $d$ (sort $p \times p \to \text{num}$) are $\Psi$-local, with $\Psi$ being the identity function.

(i) $\mathcal{T}_d^a$, the extension of $\mathcal{T}_0$ with an uninterpreted function $d$.
(ii) $\mathcal{T}_d^a$, the extension of $\mathcal{T}_0$ with a function $d$.

The extension $\mathcal{T}_d^a = \mathcal{T}_0 \cup \mathcal{K}_s$, where $\mathcal{K}_s = \forall x, y. d(x, y) = 0$ is $\Psi_s$-local, where $\Psi_s(T) = T \cup \{d(a, b) \mid d(b, a) \in T\}$.

Proof. (i) and (ii) are a direct consequence of Theorem 3; the locality proof for $\mathcal{T}_d^a$ is similar to the one for $\mathcal{T}_d^a$. \qed

We present all the results together in the following theorem:

**Theorem 8** Let $\mathcal{T}_0$ be the disjoint combination of the theory $\mathcal{E}$ of pure equality (sort $p$) and linear real arithmetic (sort $\text{num}$). The following extensions of $\mathcal{T}_0$ with a function $d$ (sort $p \times p \to \text{num}$) are $\Psi$-local for a suitable closure operator $\Psi$:

(1) $\mathcal{T}_d^a = \mathcal{T}_0 \cup \mathcal{K}_m$, where $\mathcal{K}_m$ are axioms of a metric, is $\Psi_m$-local, where $\Psi_m(T) = \{d(a, b) \mid a, b \text{ constants of sort } p \text{ occurring in } T\}$. 

Algorithm 1 Symbol elimination in theory extensions [40,41]

**Input:** Theory extension $T_0 \cup K$ with signature $II = I_{00} \cup (\Sigma \cup \Sigma_{par})$
where $\Sigma_{par}$ is a set of parameters
Set $T$ of ground $II$-terms

**Output:** $\forall \forall \forall T_2(\forall)$ (universal $I_{00} \cup \Sigma_{par}$-formula)

1. **Step 1** Purify $K[T]\cup G$ as described in Theorem 4 (with set of extension symbols $\Sigma_1$). Let $K_0 \cup G_0 \cup \text{Con}_0$ be the set of $II$-clauses obtained this way.
2. **Step 2** Let $G_1 = K_0 \cup G_0 \cup \text{Con}_0$. Among the constants in $G_1$, we identify
   (i) the constants $c_f, f \in \Sigma_{par}$, where $c_f$ is a constant parameter or $c_f$ is introduced by a definition $c_f \approx f(c_1, \ldots, c_k)$ in the hierarchical reasoning method,
   (ii) all constants $\tau_p$ occurring as arguments of functions in $\Sigma_{par}$ in such definitions. Replace all the other constants $\tau$ with existentially quantified variables $\tau$ (i.e. replace $G_1(\tau_p, \tau, \tau)$ with $\exists \exists G_1(\tau_p, \tau, \tau)$).
3. **Step 3** Construct a formula $G_1(\tau_p, \tau, \tau)$ equivalent to $\exists \exists G_1(\tau_p, \tau, \tau)$ w.r.t. $T_0$ using a method for quantifier elimination in $T_0$.
4. **Step 4** Replace each constant $c_f$ introduced by definition $c_f = f(c_1, \ldots, c_k)$ with the term $f(c_1, \ldots, c_k)$ in $G_1(\tau_p, \tau, \tau)$. Let $G_3(\tau_p)$ be the formula obtained this way. Replace $\tau_p$ with existentially quantified variables $\tau$.
5. **Step 5** Let $\forall G_2(\forall)$ be $\forall G_2(\forall)$.

(2) $T^p = T_0 \cup K_p$, where $K_p$ contains all axioms of a metric except for the triangle inequality, is $\Psi_{\alpha}$-local, where $\Psi_{\alpha}(T) = T \cup \{d(b, a) \mid d(a, b) \in T\} \cup \{d(a, a) \mid a \text{ a constant of sort } p \}$ occurring in $T$.

(3) $T^p$, the extension of $T_0$ with an uninterpreted function $d$, and $T^p = T_0 \cup K_p$, where $K_p = \forall x, y \ d(x, y) \geq 0$, are $\Psi_\alpha$-local, where $\Psi(T) = T$.

4 Property-directed symbol elimination and locality

In [41] we proposed a method for property-directed symbol elimination described in Algorithm 1. We present a slight generalization.

**Theorem 9** ([40,41]) Let $T_0$ be a $I_{00}$-theory allowing quantifier elimination\(^2\) $\Sigma_{par}$ be a set of parameters (function and constant symbols) and $II = (S, \Sigma, \text{Pred})$ be such that $\Sigma \cap (\Sigma_0 \cup \Sigma_{par}) = \emptyset$. Let $K$ be a set of clauses in the signature $I_{00} \cup \Sigma_{par} \cup \Sigma$ in which all variables occur also below functions in $\Sigma_1 = \Sigma_{par} \cup \Sigma$. Assume $\forall T \subseteq T_0 \cup K$ satisfies condition (Comp\(_\forall^\Psi\)) for a suitable closure operator $\Psi$ with $\text{est}(G) \subseteq \Psi_{\forall_k}(G)$ for every set $G$ of ground $II$-clauses. Then, for $T = \Psi_{\forall_k}(G)$, Algorithm 1 yields a universal $I_{00} \cup \Sigma_{par}$-formula $\forall \forall \forall T_2(\forall)$ such that $T_0 \cup \forall \forall \forall T_2(\forall) \cup K \cup G \models \bot$ which is entailed by every universal formula $\Gamma$ with $T_0 \cup \Gamma \cup K \cup G \models \bot$.

\(^2\) If $T_0$ does not allow QE but has a model completion $T^*_0$ which does, and if we use QE in $T^*_0$ in Algorithm 1, $T_0 \land \forall \forall \forall T_2(\forall) \cup G \models \bot$, but $\forall \forall \forall T_2(\forall)$ might not be the weakest universal formula $\Gamma$ with the property that $T_0 \cup \Gamma \cup K \models \bot$. 
Consider theories \( T_0 \cup K \cup \forall x \Gamma_T(x) \) that are flat and linear then the extension is \( \Psi \)-local. Let \( T = \Psi_K(G) = \Psi(\text{est}(K,G)) \).

By \( \Psi \)-locality, \( T_0 \cup \Gamma \cup K \cup G \) is unsatisfiable if and only if \( T_0 \cup \Gamma \cup \text{K}_T \cup G \) is unsatisfiable, if and only if (with the notations in Steps 1–5 of Algorithm 1) \( T_0 \cup \Gamma \cup K_0 \cup G_0 \cup \text{Con}_0 \cup \text{Def} \) is unsatisfiable. Let \( A \) be a model of \( T_0 \cup \Gamma \). Then in \( A \) there are no possible values for the constants \( G_1(\overline{p}_1, \overline{q}_1, \overline{r}) = K_0 \cup G_0 \cup \text{Con}_0 \cup \text{Def} \) for which \( G_1(\overline{p}_1, \overline{q}_1, \overline{r}) \) is true in \( A \). Hence, \( A \not \models \exists \overline{p} G_1(\overline{p}_1, \overline{q}_1, \overline{r}) \). So (with the notation used when describing Steps 1–5) \( A \not \models \exists \overline{p} G_1(\overline{p}_1, \overline{q}_1, \overline{r}) \). It follows that \( A \models \forall \overline{p} \Gamma_T(\overline{r}) \).

This reduction method was implemented in sehpilot (for details cf. Section 7).

## 5 Second-order quantifier elimination

Let \( T \) be a theory with signature \( II = (S, \Sigma, \text{Pred}) \) and \( P_1, \ldots, P_n, Q_1, \ldots, Q_m \) be predicate symbols which are not in \( \text{Pred} \). Let \( II' = (S, \Sigma, \text{Pred} \cup \{ P_1, \ldots, P_n \}) \) and \( II'' = (S, \Sigma, \text{Pred} \cup \{ Q_1, \ldots, Q_m \}) \). \( F \) be a \( II' \)-formula and \( G \) a \( II'' \)-formula. Let \( \Pi \)-structure \( A \) is a model of \( \exists P_1 \ldots P_n F \) (notation: \( A \models \exists P_1 \ldots P_n F \)) if there exists an \( II' \)-structure \( B \) such that \( B \models F \) and \( B\upharpoonright_T = A \). We say that \( \exists P_1 \ldots P_n F \) entails \( \exists Q_1 \ldots Q_m G \) w.r.t. \( T \) (and use the notation: \( \exists P_1 \ldots P_n F \models_T \exists Q_1 \ldots Q_m G \)) iff for every \( \Pi \)-structure \( A \) which is a model of \( T \), if \( A \models \exists P_1 \ldots P_n F \) then \( A \models \exists Q_1 \ldots Q_m G \).

If there exists a first-order formula \( F_0 \) over the signature \( II \) such that for every model \( A \) of \( T \), \( A \models F_0 \) iff \( A \models \exists P_1 \ldots P_n F \), we say that \( F_0 \) and \( \exists P_1 \ldots P_n F \) are equivalent w.r.t. \( T \) (and write \( F_0 \equiv_T \exists P_1 \ldots P_n F \)).

We consider here only the elimination of one predicate; for formulae of the form \( \exists P_1 \ldots P_n F \) the process can be iterated. Let \( T \) be a theory with signature \( II = (S, \Sigma, \text{Pred}) \) and let \( II' = (S, \Sigma, \text{Pred} \cup \{ P \}) \), where \( P \not \in \text{Pred} \).

Let \( F \) be a universal first-order \( II' \)-formula. Our goal is to compute, if possible, a first-order \( II' \)-formula \( G \) such that \( G \equiv_T \exists P F \). We adapt the hierarchical superposition calculus proposed in [8,9] to this case.

We consider theories \( T \) over many-sorted signatures \( II = (S, \Sigma, \text{Pred}) \), where the set of sorts \( S = S_I \cup S_u \) consists of a set \( S_I \) of interpreted sorts and a set \( S_u \) of uninterpreted sorts. The models of the theories are \( II' \)-structures \( A = \{ \{ A_s \}_{s \in S_I}, \{ f_A \}_{f \in \Sigma}, \{ p_A \}_{p \in \text{Pred}} \} \), where each support of interpreted sort is considered to be fixed. Following the terminology used in [8,9], we will refer to elements in the fixed domain of sort \( s \in S_I \) as domain elements of sort \( s \).

Let \( F \) be a universal first-order formula over signature \( II' = (S, \Sigma, \text{Pred} \cup \{ P \}) \).

We can assume, without loss of generality, that \( F \) is a set of clauses of the form \( \forall \overline{x} D(\overline{x}) \lor C(\overline{x}) \), where \( D(\overline{x}) \) is a clause over the signature \( II \) and \( C(\overline{x}) \) is...
a clause containing literals of the form \( (\neg)P(x_1, \ldots, x_n) \), where \( x_1, \ldots, x_n \) are variables. Such clauses can also be represented as constraint clauses in the form \( \forall x \phi(x) \| C(x) \), where \( \phi(x) := (\neg)D(x) \). We will refer to clauses of this form as constrained \( P \)-clauses.

Let \( \succ \) be a strict, well-founded ordering on terms that is compatible with contexts and stable under substitutions. As in [9] we assume that \( \succ \) has the following properties:

(i) \( \succ \) is total on ground terms,
(ii) \( t \succ d \) for every domain element \( d \) of interpreted sort \( s \) and every ground term \( t \) that is not a domain element.

Let \( HRes^P \) be the calculus containing the following ordered resolution and factorization rules for constrained \( P \)-clauses:

\[
\begin{align*}
\phi_1 \| P(x) \lor C & \quad \phi_2 \| \neg P(y) \lor D \\
\phi_1 \land \phi_2 \| \sigma & \| (C \lor D) \sigma \\
\phi \| P(x) \lor P(y) \lor C & \quad \phi \| (P(x) \lor C) \sigma \\
\phi \| (P(x) \lor C) \sigma & \| (P(x) \land P(y)) \sigma
\end{align*}
\]

where

(i) \( \sigma = \text{mgu}(P(x), P(y)) \)
(ii) \( P(x)\sigma \) is strictly maximal in \( (P(x) \lor C)\sigma \)
(iii) \( \neg P(y)\sigma \) is maximal in \( (\neg P(y) \lor D)\sigma \).

**Redundancy.** The inference rules are supplemented by a redundancy criterion \( R = (R_c, R_t) \) meant to specify:

- a set \( R_c \) of redundant clauses (which can be removed), and
- a set \( R_t \) of redundant inferences (which do not need to be computed).

We say that a set of clauses \( N^* \) is saturated up to \( R \)-redundancy w.r.t. \( HRes^P \) if every \( HRes^P \) inference with premises in \( N^* \) is redundant (i.e. in \( R_t \)).

The following notion of redundancy \( R_c^0 \) for clauses is often used: A (constrained) clause is redundant w.r.t. a set \( N \) of clauses if all its ground instances are entailed w.r.t. \( T \) by ground instances of clauses in \( N \) which are strictly smaller w.r.t. \( \succ \). We will use the following notion of redundancy for inferences: If \( R_c \) is a redundancy criterion for clauses, we say that an inference \( \iota \) on ground clauses is redundant w.r.t. \( N \) if either one of its premises is redundant w.r.t. \( N \) and \( R_c \) or, if \( C_0 \) is the conclusion of \( \iota \) then there exist clauses \( C_1, \ldots, C_n \in N \) that are strictly smaller w.r.t. \( \succ \) than the maximal premise of \( \iota \) and \( C_1, \ldots, C_n \models C_0 \).

A non-ground inference is redundant if all its ground instances are redundant.

**Example 1 (Semantic \( T \)-entailment; redundancy criterion \( R_T \))** We say that a constrained \( P \)-clause \( \forall y \phi(y) \| C(y) \) is \( T \)-semantically entailed by \( \forall y \psi(y) \| D(y) \) if the following conditions hold:

---

3. We can bring the clauses to this form using variable abstraction.
4. These conditions are satisfied by an LPO with an operator precedence in which the predicate symbol \( P \) (which can be regarded as function symbol with output sort \text{bool}) is larger than the other operators and domain elements are minimal w.r.t. \( \succ \) which is supposed to be well-founded on the domain elements.
(i) \( C = D \).
(ii) \( \mathcal{T} \models \forall \mathcal{T} \phi(\mathcal{T}) \rightarrow \psi(\mathcal{T}) \) and
(iii) \( \neg \phi(\mathcal{T}) \sigma \supset \neg \psi(\mathcal{T}) \sigma \) for every valuation \( \sigma \).

We say that a clause \( C \) is \( \mathcal{R}_\mathcal{T} \)-redundant w.r.t. a set \( N \) of clauses if it is \( \mathcal{T} \)-semantically entailed by a clause in \( N \).

Note that if \( C_1 = (\phi(\mathcal{T}) \mid |C(\mathcal{T})|) \) is \( \mathcal{T} \)-semantically entailed by \( C_2 = (\psi(\mathcal{T}) \mid |D(\mathcal{T})|) \) then \( C_1 \sigma \supset C_2 \sigma \) and \( C_2 \sigma \models \mathcal{T} C_1 \sigma \) for every ground substitution \( \sigma \), so \( \mathcal{R}_\mathcal{T} \)-redundant clauses are \( \mathcal{R}_0 \)-redundant.

We call the notion of redundancy induced on inferences also \( \mathcal{R}_\mathcal{T} \)-redundancy. ■

Let \( \mathcal{R} = (\mathcal{R}_s, \mathcal{R}_0) \) be a redundancy criterion with \( \mathcal{R}_s \subseteq \mathcal{R}_0 \). We want to prove that if \( N \) is a set of constrained \( P \)-clauses over background theory \( \mathcal{T} \), \( N^* \) its saturation (up to \( \mathcal{R} \)-redundancy) under \( HRes^* \), and \( N_0^* \) the set of clauses in \( N^* \) not containing \( P \), then for every model \( \mathcal{A} \) of \( \mathcal{T} \), \( \mathcal{A} \) is a model of \( N_0^* \) if and only if there exists a \( \mathcal{H} \)-structure \( B \) with \( B \models N \) and \( B_{\mathcal{H}} = \mathcal{A} \). The proof of this fact is very similar to the proof of the completeness of hierarchical superposition. Since our goal is different, we present here all the details just for the sake of completeness. (The results are probably known, already in [5] it was mentioned that hierarchical superposition can be used for second-order quantifier elimination.) We start with a lemma.

**Lemma 10** Let \( \mathcal{H} = (S, \Sigma, \text{Pred}), \mathcal{T} \) be a theory with signature \( \mathcal{H} \) and let \( \mathcal{A} = (\{A_s\}_{s \in S}, \{f_s\}_{f \in \Sigma}, \{p_A\}_{p \in \text{Pred}}) \) be a \( \mathcal{H} \)-structure which is a model of \( \mathcal{T} \). For every element \( a \in A_s \) we add a new constant of sort \( s \) (which we denote \( a \)). Let \( C_A = \bigcup_{s \in S} A_s \) be the set of all constants introduced this way, and \( \mathcal{A}^A \) be the extension of \( \mathcal{A} \) with constants from \( C_A \) which are interpreted in the usual way. Let \( N \) be a set of clauses over signature \( \mathcal{H} \). Then the following are equivalent:

1. \( \mathcal{A} \) is a model of \( N \).
2. \( \mathcal{A}^A \) is a model of the set \( N_A \) of all ground instances of \( N \) in which the variables are replaced with constants in \( C_A \).

**Proof:** (1) \( \Rightarrow \) (2): Assume that \( \mathcal{A} \) is a model of \( N \). Let \( C \) be a clause in \( N_A \). Then \( C \) is obtained from a clause \( C' \in N \) by replacing every variable \( x \) with a constant \( a_x \in C_A \) (such that if the variable \( x \) has sort \( s \) then \( a_x \in A_s \)). Let \( \beta : X \rightarrow \mathcal{A} \) be defined by \( \beta(x) = a_x \) for every \( x \in X \) occurring in \( C' \) and defined arbitrarily for all other variables. Since \( \mathcal{A} \) is a model of \( N \), the clause \( C' \) is true in \( \mathcal{A} \) in the valuation \( \beta \). But \( \mathcal{A}(\beta(C')) \) is obtained by evaluating the function and predicate symbols as in \( \mathcal{A} \) and every variable \( x \) occurring in \( C' \) as \( a_x \). This is exactly the value of \( C' \) in \( \mathcal{A}^A \), thus \( \mathcal{A}^A \) is a model of \( C \).

(2) \( \Rightarrow \) (1): Assume that \( \mathcal{A}^A \) is a model of \( N_A \). Let \( C \in N \) and let \( \beta : X \rightarrow \mathcal{A} \) be a valuation. For every \( x \in X \), let \( a_x := \beta(x) \). As discussed before, the value of \( C \) in \( \mathcal{A} \) w.r.t. \( \beta \) is the same as the value of \( C' = C_\sigma \) in \( \mathcal{A}^A \), where \( \sigma \) is the substitution that associates with every variable \( x \) the constant \( a_x \in C_A \). Since \( \mathcal{A}^A \) is a model of the set \( N_A \) and \( C' \in N_A \), \( \mathcal{A}^A \) is also a model of \( C' \). It follows that \( C \) is true in \( \mathcal{A} \) w.r.t. \( \beta \) for every valuation \( \beta \), i.e. that \( \mathcal{A} \) is a model of \( N \). □
Theorem 11. Let $N$ be a set of constrained $P$-clauses over background theory $T$, $N^*$ its saturation (up to $R$-redundancy) under $\text{HRes}^E_P$, and $N^*_0$ the set of clauses in $N^*$ not containing $P$. For every model $A$ of $T$ the following are equivalent:

1. $A$ is a model of $N^*_0$.
2. There exists a $\Pi'$-structure $B$ with $B \models N$ and $B \models_{\Pi'} = A$.

Proof: First, note that for constrained $P$-clauses the hierarchical superposition calculus specializes to $\text{HRes}^E_P$: With the terminology used in [5,8,9], the background signature is $\Pi$; the only foreground symbol is $P$. Since there are no “background”-sorted terms starting with a “foreground” function symbol, sets of $P$-clauses are sufficiently complete. (Even if we regard predicates as functions with values in the domain \{0, 1\}, since predicates can only take values 0 or 1 sufficient completeness is guaranteed.) Since in the special case we consider there are no foreground terms, in this case all substitutions are simple as well.

(2) $\Rightarrow$ (1) follows from the soundness of the hierarchical superposition calculus.

(1) $\Rightarrow$ (2) is proved with a model construction similar to the one used for proving completeness of hierarchical superposition. Let $A$ be a model of $N^*_0$, and $\Pi$ the set of all instances of $N^*_0$ in which the variables are replaced with constants in $C_A$.

Since $A$ is a model of $N^*_0$, by Lemma 10, $A^A = A$ is a model of the set $N^*_0$ of instances of $N^*_0$ in which the variables are replaced with constants in $C_A$.

By Zermelo’s theorem, there exists a total well-founded strict order $\succ$ on the set of all constants in $\Pi$, and starting with this ordering we can obtain a total well-founded strict ordering (which we denote again with $\succ$) on the set of all ground terms over $\Pi^A$, which can be extended in the usual way to the set of all ground clauses over the signature $\Pi^A = (S, \Sigma \cup C_A, \text{Pred} \cup \{P\})$, such that the literals containing the predicate symbol $P$ are larger than the literals not containing $P$.

Consider the clauses in the set $N^*_0$ ordered increasingly according to the clause ordering induced by the atom ordering. Since $N^*$ is saturated w.r.t. $\text{HRes}^E_P$, $N^*_0$ is also saturated w.r.t. $\text{HRes}^E_P$. We construct a model for $N^*_0$ using a canonical model construction, similar to the one usually used for proving completeness of ordered resolution. We sketch the construction here:

We start with an interpretation in which all atoms in the set $\{P(a_1, \ldots, a_k) \mid i \in \{1, \ldots, n\}, a_j \in C_A \text{ constant of suitable sort, for all } j \in \{1, \ldots, k\}\}$ are false. The clauses in $(N^*_0)_A$ are smaller than the clauses containing the predicate symbol $P$ and by assumption are all true in $A$, hence also in the interpretation that we start with (and will remain true in the process of constructing $B$). We therefore only need to consider the set $N^*_P$ of all constrained $P$-clauses containing the predicate symbol $P$. When considering a clause $C$, we assume that we already constructed a partial interpretation $B_{<C}$ that makes true all clauses strictly smaller than $C$.

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5 This is compatible with regarding $P$ as a function symbol with output sort \text{bool} and using an ordering in which $P$ is the largest function symbol.
If $C$ is true in the partial interpretation $B_{<C}$ nothing needs to be done.
If $C$ is false in the partial interpretation $B_{<C}$ we need to change $B_{<C}$ such that $C$ becomes true (such that the clauses smaller than $C$ remain true).

We proceed as follows: If $C$ is false in $B_{<C}$ and contains exactly one maximal literal which is positive (which needs to start with $P$ and is for instance of the form $P(\pi)$), we change the interpretation of $P$ such that it contains the tuple $\pi$, i.e., such that $P(\pi)$ becomes true. We denote this by setting $\Delta_C := \{P(\pi)\}$. Otherwise we do not change the interpretation, i.e., $\Delta_C := \emptyset$.

The candidate model is the limit of all these changes, $B := A \cup \bigcup_{C \in N_i} \Delta_C$.

We can show that the expansion $B^A$ of $B$ with the constants in $C_A$ (with the usual interpretation) is a model of $(N^*_P)_A$ in the usual way: Assume that there exists a clause in $(N^*_P)_A$ which is not true in $B^A$. Since the ordering on $(N^*_P)_A$ is well-founded, we consider without loss of generality the smallest clause in $(N^*_P)_A$ which is false in $B^A$. We can show in the usual way that using a resolution or a factorization step we can produce a smaller clause false in $B^A$, which is either in $(N^*_P)_A$ or in $(N^*_A)_s$, so in both cases we obtain a contradiction.

Since $B^A$ is a model of $N^*_A$, and $B_s = A$, for every sort $s$ (hence $B^A = B^B$), it follows that $B$ is a model of $N^* = N^*_0 \cup N^*_P$.

**Corollary 12** Let $T$ be a theory, $N$ be a set of constrained $P$-clauses and $N^*$ be a set of constrained $P$-clauses obtained by saturating $N$ in $HRes^P_\infty$ up to redundancy. Let $A$ be a $\Pi'$-structure which is a model of $T$. Then $A$ is a model of $N^*_0$ if and only if there exists a $\Pi'$-structure $B$ with $B \models N$ and $B\models_T = A$.

**Proof:** Follows from the fact that in the proof of Theorem 11 the clauses which are redundant are entailed (w.r.t. $T$) by clauses that are smaller hence cannot be minimal counterexamples and cannot influence the way model $B$ is built because every redundant clause $C$ is true in $B_{<C}$.

### 5.1 Case 1: Saturation is finite

If the saturation $N^*$ of $N$ under $HRes^P_\infty$ (up to $\mathcal{R}$-redundancy) is finite and $N^*_0$ is the set of clauses in $N^*$ not containing $P$ then, by Theorem 11, the universal closure of the conjunction of the clauses in $N^*_0$ is equivalent to $\exists P \, N$.

**Example 2** Consider a class of graphs described by the following set $N$ of constrained $E$-clauses:

$\{ (1)\pi^t(u,v)|E(u,v), \ (2)\pi^t(u,w,v)|E(u,v) \rightarrow E(u,w), \ (3)\pi^e(u,v)|\neg E(u,v) \}$

For arbitrary predicates $\pi^t, \pi^e$ and $\pi^t$ we can generate with $HRes^E_\infty$ an infinite set of clauses including, e.g., all clauses of the form:

$(4_n) \pi^t(u,v) \land \pi^t(u,w_1,v) \land \pi^t(u,w_2,w_1) \land \cdots \land \pi^t(u,w_n, w_{n-1}) \mid E(u,w_n)$

If we assume that $\pi^t, \pi^e, \pi^t$ satisfy the additional axioms defining a theory $T_\pi$:
then all inferences by resolution between clauses (1) and (2), (2) and (3) are \( T_\pi \)-redundant. The inferences between (2) and (2) are also free

\[ \pi_c \]

where \( c \) reduction in Theorem 4 to check that (4) of pure equality, for a suitable closure operator. We can use the hierarchical

\[ \Psi \]

model which might satisfy additional properties (for instance positivity or symmetry).

which can be, for instance, the Euclidean metric, or a concrete cost function – which might be actually described in a detailed way. Let

\( C(r_1, r_2) \) be a graph class described by the set \( N \) of axioms in Example 2, where \( \pi_i, \pi_c, \pi^t \) are defined by the axioms:

\[
\text{Def}_\pi(r_1, r_2) = \{ \forall u, v \pi_i(u, v) \leftrightarrow u \neq v \land d(u, v) \leq r_1(u), \]

\[
\forall u, v \pi^t(u, v) \leftrightarrow d(u, v) > r_2(u), \]

\[
\forall u, v, w \pi_c(u, w, v) \leftrightarrow u \neq w \land d(u, w) \leq d(u, v) \}\]

so \( N^* = N \cup \{4\} \) is saturated up to \( T_\pi \)-redundancy. By Theorem 11, \( N \) is satisfiable if \( \forall u, v (\pi(u, v) \land \pi^t(u, v) \rightarrow \bot) \) is satisfiable w.r.t. \( T_\pi \).

When modelling concrete situations, the predicates \( \pi_i, \pi_c \) and \( \pi^t \) might not be arbitrary, but might have definitions using other symbols with given properties.

**Example 3** The theory \( T_\pi \) might be actually described in a detailed way. Let \( C(r_1, r_2) \) be a graph class described by the set \( N \) of axioms in Example 2, where \( \pi_i, \pi^t, \pi^s \) are defined by the axioms:

\[
\text{Def}_\pi(r_1, r_2) = \{ \forall u, v \pi_i(u, v) \leftrightarrow u \neq v \land d(u, v) \leq r_1(u), \]

\[
\forall u, v \pi^t(u, v) \leftrightarrow d(u, v) > r_2(u), \]

\[
\forall u, v, w \pi_c(u, w, v) \leftrightarrow u \neq w \land d(u, w) \leq d(u, v) \}\]

where \( d \) is a distance or cost function. We can regard the theory extension \( T \subseteq T_\pi = T \cup \text{Free}(r_1, r_2) \cup \text{Def}_\pi \), where \( T \) is one of the theories \( T_d^\pi, T_d^\pi \) or \( T_d \), introduced in Theorem 8, and \( \text{Free}(r_1, r_2) \) the theory in which \( r_1, r_2 \) are regarded as uninterpreted unary function symbols. Therefore, \( T \) can be represented as a \( \Psi \)-local extension of the disjoint combination of a theory of real numbers and of pure equality, for a suitable closure operator. We can use the hierarchical reduction in Theorem 4 to check that (c1), (c2) and (c3) are valid w.r.t. \( T \).

In applications we might not be interested in checking the satisfiability of \( N \) or the satisfiability of \( \forall u, v (\pi_i(u, v) \land \pi^t(u, v) \rightarrow \bot) \) w.r.t. \( T_\pi \), but in a specific model \( A \) satisfying \( T_\pi \) (we refer to it as “canonical model”).

This is the case, for instance, in the applications in wireless network theory analyzed in Section 2: The vertices of the graphs considered in this context are very often points in the Euclidian space, and the distance is a concrete function which can be, for instance, the Euclidean metric, or a concrete cost function – which might satisfy additional properties (for instance positivity or symmetry).
If we want to analyze such graph classes in full generality, we might assume that some of the properties of some of the parameters are not fully specified.

Let $A$ be a model of a theory $T$ describing properties of function symbols in a set $\Sigma$ we want to model. We assume that $\Sigma$ contains a set of “parameters” $\Sigma_{\text{par}}$ (function symbols whose properties are “underspecified” in $T$). In some situations, if we are given a set $N$ of constrained clauses, we might be interested in obtaining (weakest) universal conditions $\Gamma$ on $\Sigma_{\text{par}}$ such that for every fixed model $A$ of $T$ which also satisfies $\Gamma$, there exists an interpretation for $P$ in $A$ for which $N$ is satisfied, i.e. $A \models \exists P \ N$. We present a situation in which this is possible.

**Theorem 13** Let $T$ be a theory with signature $\Pi = (S, \Sigma, \text{Pred})$, $N$ a set of constrained $P$-clauses. Assume that the saturation $N^*$ of $N$ up to $T$-redundancy w.r.t. HRes$_P$ is finite; let $N^*_0$ be the set of clauses in $N^*$ not containing $P$.

Let $\Sigma_{\text{par}} \subseteq \Sigma$ be a set of parameters. Assume that one of the following conditions holds:

(i) $T$ allows quantifier elimination or
(ii) $T_0 \subseteq T = T_0 \cup K$ is a local theory extension satisfying condition $(\text{Comp}_f^\Psi)$ for a suitable term closure operator $\Psi$ and $T_0$ allows quantifier elimination.

If (i) holds, we can use quantifier elimination and if (ii) holds then we can use Algorithm 1 to obtain a (weakest) universal formula $\Gamma$ over the signature $\Sigma_{\text{par}}$ with the property that $T \cup \Gamma \cup \exists x \neg N^*_0(x)$ is unsatisfiable, i.e. $A \models \exists P \ N$.

**Proof:** By the completeness of the hierarchical superposition calculus, $N$ is satisfiable iff the set $N^*_0$ of clauses in $N^*$ which do not contain $P$ is satisfiable. We denote by $\forall x N^*_0(x)$ the formula represented by the set of clauses $N^*_0$. Let $A$ be a model of $T$. Assume that $A$ is not a model for $\forall x N^*_0(x)$. Then $\exists x \neg N^*_0(x)$ is true in $A$. In particular, it follows that $T \land \exists x \neg N^*_0(x)$ is satisfiable.

We can apply Algorithm 1 to construct a weakest universal formula $\Gamma$ over the signature $\Sigma_{\text{par}}$ with the property that $T \cup \Gamma \cup \exists x \neg N^*_0(x)$ is unsatisfiable, i.e. with the property that $T \cup \Gamma \models \forall x N^*_0(x)$. Then every model $B$ of $T$ which also satisfies the constraints in $\Gamma$ is a model of $\forall x N^*_0(x)$.

**Example 4** Consider again the situation described in Example 3. We show how one can use Theorem 9 and Algorithm 1 to derive constraints $\Gamma$ on the parameters $r_1, r_2$ under which for every model $A$ of $T_d^u$ the formula

$$\pi^1(u, v) \land \pi^2(u, v) = (u \neq v \land d(u, v) \leq r_1(u)) \land (d(u, v) > r_2(u))$$

is unsatisfiable in $A$ (we consider the case in which $d$ is an uninterpreted function; other axioms for $d$ can be analyzed as well).

Note that the formula above is unsatisfiable in any model $A$ of $T_d^u$ whose support $A_p$ of sort $p$ has cardinality 1. If we only consider models $A$ with $|A_p| \geq 2$ then we can proceed as follows:
Step 1: We purify the formula by introducing new constants: \( c_d := d(u, v), c_1 = r_1(u), c_2 = r_2(u) \) and obtain: \( u \neq v \wedge c_d \leq c_1 \wedge c_d > c_2 \).

Step 2: We quantify existentially all constants not denoting terms starting with \( r \) or used as arguments of \( r \) or used as arguments of \( r \). We find an axiomatization for the graph class \( C \).

Step 3: After quantifier elimination in a combination of \( \Sigma_2 \).

Step 4: We replace the constants \( c_1, c_2 \) with the terms they denote and quantify the arguments existentially and obtain: \( \exists u(r_2(u) < r_1(u)) \).

Step 5: We negate this condition and obtain: \( \forall u(r_1(u) \leq r_2(u)) \).

Example 5 We find an axiomatization for the graph class \( C^- = \{ G^- \mid G \in C \} \), when class \( C \) is described by the set \( N \) of constrained clauses in Example 2 and \( \pi, \pi^* \) and \( \pi^* \) satisfy conditions \( (c1), (c2), (c3) \). Let \( N^* = N \cup \{(4)\} \) be obtained by saturating \( N \) under \( HRes^E \) up to redundancy. A graph \( H = (V, E) \in C^- \) iff there exists a graph \( G = (V, E) \in C \) such that \( H = G^- \). This condition can be described by \( M = N^* \cup Tr(E, F) \), where \( Tr(E, F) = \{ \forall x, y (F(x, y) \leftrightarrow E(x, y) \wedge E(y, x)) \} \), which can be written in the form of constrained clauses as:

\[
\begin{align*}
\forall u, v, x, y, z & \quad \pi^*(u, v) \wedge \pi^*(u, x, y) \wedge \pi^*(u, x, v) \\
\forall u, v, x, y, z & \quad \pi^*(u, x, y) \wedge \pi^*(u, v, z) \\
\forall u, v, x, y, z & \quad \pi^*(u, x, y) \wedge \pi^*(u, x, v) \wedge \pi^*(u, v, z)
\end{align*}
\]

To find an axiomatization for the class \( \{ (V, F) \mid \exists (V, E) \in C \) with \( (V, F) = (V, F)^- \) \} we need to eliminate the second-order quantifier from the formula \( \exists E(N^* \cup Tr(E, F)) \).

The base theory is \( \Sigma \cup UIF_F \), the extension of \( \Sigma \) with the uninterpreted function symbol \( F \), with signature \( \Pi_F = (\Sigma, Pred \cup \{F\}) \).

Since the background theory in this case is not arithmetic, and since the method for second-order quantifier elimination implemented in SCAN [18] is very similar to \( HRes^E \), we used SCAN on the clause set \( N^* \):

\[(c1) \quad \forall u, v, w, x \quad \pi^*(u, w, v) \wedge \pi^*(u, x, v) \rightarrow \pi^*(u, x, v)
(c2) \quad \forall u, v, w \quad \pi^*(u, v) \wedge \pi^*(u, w, v) \rightarrow \pi^*(u, w)
(c3) \quad \forall u, v, w \quad \pi^*(u, w) \wedge \pi^*(u, w, v) \rightarrow \pi^*(u, v)

(1) \quad \forall u, v \quad \pi^*(u, v) \rightarrow E(u, v)
(2) \quad \forall u, v \quad \pi^*(u, v) \rightarrow \neg E(u, v)
(3) \quad \forall u, v, w \quad \pi^*(u, w, v) \wedge E(u, v) \rightarrow E(u, w)
(T1) \quad \forall u, v \quad F(u, v) \rightarrow E(u, v)
(T2) \quad \forall u, v \quad F(u, v) \rightarrow E(v, u)
(T3) \quad \forall u, v \quad F(u, v) \wedge E(v, u) \rightarrow F(u, v)
\]

and obtained a set of clauses representing the formula containing axioms \( N_T = \{ (c1), (c2), (c3), (4) \} \), where

\[(4) \quad \forall u, v \quad \pi^*(u, v) \wedge \pi^*(u, v) \rightarrow \perp\]

\(^6\) We can consider only models \( A \) whose support of sort \( p \) is infinite. The theory that formalizes this is the model completion of the theory \( E \) of pure equality which allows quantifier elimination. We can then use the method for quantifier elimination in combinations of theories with \( QE \) described in [30].
and axioms $N^*_F = \{(F_1), \ldots, (F_6)\) :
\begin{align*}
(F_1) & \forall x, y \quad F(x, y) \rightarrow F(y, x) \\
(F_2) & \forall x, y \quad F(y, x) \rightarrow \neg \pi^x(x, y) \\
(F_3) & \forall x, y \quad F(x, y) \rightarrow \neg \pi^y(x, y) \\
(F_4) & \forall x, y \quad \pi^i(x, y) \wedge \pi^j(y, x) \rightarrow F(y, x) \\
(F_5) & \forall x, y, z \quad \pi^i(x, y) \wedge \pi^j(y, x, z) \wedge F(y, z) \rightarrow F(x, y) \\
(F_6) & \forall x, y, z, u \quad \pi^i(x, y, z) \wedge \pi^j(y, x, u) \wedge F(y, u) \rightarrow F(y, x)
\end{align*}

The universal closure $G$ of the conjunction of these clauses is equivalent w.r.t. $T \cup UIF_F$ to the formula $\exists E(N^* \cup Tr(E, F))$, and thus axiomatizes $C^*$.

\[\blacksquare\]

5.2 Case 2: Finite representation of possibly infinite saturated sets

The saturation of a set $N$ of constrained $P$-clauses up to redundancy under $HRes^*_P$ might be infinite. We here consider a very special case under which a finite set of constrained $P$-clauses $N = \{\phi_i(\overline{x}) \parallel C_i(\overline{z}) \mid i = 1, \ldots, n\}$ can have a saturation that can be finitely described: The situation in which the set of clauses $\{C_1, \ldots, C_n\}$ can be finitely saturated under ordered resolution.

**Theorem 14** Let $N = \{\phi_i(\overline{x}) \parallel C_i(\overline{z}) \mid i = 1, \ldots, n\}$ be a finite set of constrained $P$-clauses and $N^*_P = \{C_1, \ldots, C_n\}$.

Assume that the saturation of $N^*_P$ under ordered resolution is finite, $N^*_A = \{C_1, \ldots, C_n, C_{n+1}, \ldots, C_{n+k}\}$, and the set $I_P$ of all possible inferences used for deriving these clauses is finite and can be effectively described. If $\perp \not\subseteq N^*_P$, then $\exists P \ N \equiv T \top$. Assume now that $\perp \subseteq N^*_P$. Let $A$ be a model of $T$, $T_A$ the theory with $A$ as canonical model (i.e. $T_A = Th(A)$). Let $N_A$ be the set of all instances of $N$ in which the variables are replaced with elements in $A$ (seen as constants). Then:

1. The saturation $N^*_A$ of $N_A$ up to $R_T$-redundancy can be described as $N^*_A = \{\mu^t_*(\overline{x}) \parallel C_t(\overline{z}) \mid i = 1, \ldots, n + k, t$ elements of $A\}$, where $\mu^t_i = \{\overline{x}, \ldots, \overline{z}\}$ are given by the minimal model of the constrained Horn clauses\footnote{The definitions are presented in Appendix C.} $CH_N$ w.r.t. $T_A$:

\begin{align*}
CH_N &= \{\phi_i(\overline{x}) \Rightarrow \mu_i(\overline{z}) \mid i \in \{1, \ldots, n\}\} \\
&\cup \{ (\mu_i(\overline{x}) \wedge \mu_j(\overline{y})) \Rightarrow \mu_k(\overline{z}) \mid C_k(\overline{z}) \text{ is obtained by a resolution inference in } I_P \text{ from } C_i(\overline{x}) \text{ and } C_j(\overline{y}) \text{ with m.g.u. } \sigma\} \\
&\cup \{ \mu_i(\overline{x}) \Rightarrow \mu_k(\overline{z}) \mid C_k(\overline{z}) \text{ is obtained by a factorization inference in } I_P \text{ from } C_i(\overline{x}) \text{ with m.g.u. } \sigma\}\}
\end{align*}

2. Let $A^\mu$ be the extension of $A$ with predicates $(\mu_i)_i$, whose interpretation is given by $(\mu^t_i)_i$. Let $j$ be such that $\perp = C_j$. Then $A \models \exists P N$ iff $CH_N \cup \{\neg \mu_j(\overline{x})\}$ is satisfiable w.r.t. $T_A$.\[\blacksquare\]
Proof: Obviously, \( CH_N \) is satisfiable (it has one trivial model, in which all predicate symbols \( \mu_i \) are true). If \( A \) is a model of \( T \), in this theorem we consider constrained Horn clauses over an assertion language that has one canonical model, namely \( A \), i.e. w.r.t. the corresponding theory \( T_A \). In [12] it is shown – using a canonical model construction – that every set \( H \) of constrained Horn clauses over an assertion language that has canonical models has a unique least model. This model is defined inductively by taking

\[ I_0 := \emptyset \] and

\[ I_{i+1} := \{ r(\vec{a}) \mid \text{body}(x) \rightarrow r(x) \in H, I_i \models_A \text{body}(\vec{a}), \vec{a} \text{ is a tuple of constants in } A \}. \]

The construction stabilizes at the first limit ordinal \( \omega \) with an interpretation \( I_\omega \); so the set of constrained Horn clauses \( CH_N \) has a unique least model w.r.t. \( T_A \).

This construction of this unique least model parallels the saturation process for the set of ground instances of the clauses \( N_A^* \); the saturated set is:

\[ N_A^* = \bigcup_{n=0}^{\infty} \{ \xi_n(\vec{a}) \mid (\forall k \in \mathbb{N}, \vec{a} \text{ is a sequence of elements in } A) \} \]

where \( \xi_n(\vec{a}) \) is a clause in \( N_A \) or are obtained (in a finite number of steps) from clauses in \( N_A \) using resolution and/or factorization. If we allow for potentially infinite disjunctions it can be described as:

\[ N_A^* = \{(\forall k \in \mathbb{N}, \xi_n(\vec{a})) \mid (\forall i \in N, \vec{a} \text{ is a sequence of elements in } A) \} \]

and models for \( \xi_i = \bigvee_{k \in \mathbb{N}} \xi_k \) can be built in a similar way to the way the interpretations for \( \mu_i \) in the minimal model for \( CH_N \) are built.

To prove (2) note that the following are equivalent:

(i) \( A \models \exists P \ N \);
(ii) There exists a \( \mathcal{H}' \)-structure \( B \) with \( B \models N \) and \( B_{|A} = A \);
(iii) \( A \models N_0^* \);
(iv) \( A^A \models (N_0^*)_A = (N_A^*)_0 \);
(v) \( A^A \models \neg \bigvee_{k \in \mathbb{N}} \xi_i(\vec{a}) \) for every sequence \( \vec{a} \) of elements in \( A \);
(vi) \( A^A \models \neg \mu_j(\vec{a}) \) for every sequence \( \vec{a} \) of elements in \( A \);
(vii) \( A^\mu \models \forall \vec{a} \neg \mu_j(\vec{a}) \);
(viii) \( CH_N \cup \{ \neg \mu_j(\vec{a}) \} \) is satisfiable w.r.t. \( T_A \);

where \( N_0^* \) is the set of all clauses in \( N^* \) which do not contain \( P \).

(i) and (ii) are equivalent by definition; (ii) and (iii) by Theorem 11; (iii) and (iv) by Lemma 10; (iv) and (v) by the fact that the conjunction of all clauses of \( (N_0^*)_0 \) can be succinctly represented by taking a possibly infinite disjunction in the constraint in front of \( \bot \); (v) and (vi) are equivalent due to (1); (vi) and (vii) by definition.

(vii) \( \Rightarrow \) (viii): By assumption (vii), \( A^\mu \models \forall \vec{a} \neg \mu_j(\vec{a}) \). Since \( (\mu_i^A(\vec{a}))_i \) are the interpretations of \( \mu_i \) in the least model for \( CH_N \) w.r.t. \( T_A \), it follows that \( A^\mu \models CH_N \cup \{ \forall \vec{a} \neg \mu_j(\vec{a}) \} \), hence \( CH_N \cup \{ \forall \vec{a} \neg \mu_j(\vec{a}) \} \) is satisfiable w.r.t. \( T_A \).

(viii) \( \Rightarrow \) (vii): Assume now that \( CH_N \cup \{ \forall \vec{a} \neg \mu_j(\vec{a}) \} \) is satisfiable w.r.t. \( T_A \), i.e., there exists an expansion \( B \) of \( A \) with interpretations for the predicates \( \mu_i \) such that \( B \models CH_N \cup \{ \forall \vec{a} \neg \mu_j(\vec{a}) \} \). Let \( M^A \) be the least model of \( CH_N \cup \{ \neg \mu_j(\vec{a}) \} \). It can be constructed with the canonical construction explained before, by considering the set \( (CH_N \cup \{ \neg \mu_j(\vec{a}) \})_A \) of instances of clauses in \( CH_N \cup \{ \neg \mu_j(\vec{a}) \} \) with
constants in \( A \) and marking \( \mu_1(\overline{\tau}) \) as true if we have a rule \( \text{body}(x) \rightarrow \mu_i(x) \).

Note that \( \{ \neg \mu_j(\overline{\tau}) \} \) does not contribute to this model building process. This means that the least model of \( CH_N \cup \{ \neg \mu_j(\overline{\tau}) \} \) is actually the least model of \( CH_N \), hence in the least model of \( CH_N \) the formula \( \forall \overline{\tau} \neg \mu_j(\overline{\tau}) \) is true, which means that \( A^\mu = \forall \overline{\tau} \neg \mu_j(\overline{\tau}) \).

If \( T \) has only one (canonical) model and is supported by \( \mu Z \) \([23]\), we can use \( \mu Z \) for checking whether \( N \) is satisfiable.

**Example 6** Consider the set \( N \) consisting of the following constrained \( P \)-clauses:

1. \( x = y \mid P(x, y) \),
2. \( y = x + 1 \mid P(y, z) \rightarrow P(x, z) \),
3. \( n(x, y) \mid \neg P(x, y) \)

over the theory of integers without multiplication with model \( Z \). Saturating \( N \) without any simplification strategy yields the infinite set \( N^* \) consisting of:

\[
(1)\bigwedge_{i=1}^k x_i = x_{i-1} + 1 \mid P(x_0, x_k) \\
(2)\bigwedge_{i=1}^k x_i = x_{i-1} + 1 \mid P(x_k, z) \rightarrow P(x_0, z) \\
(3)\ n(x_0, y) \land \bigwedge_{i=1}^k x_i = x_{i-1} + 1 \mid \neg P(x_k, y) \\
(4)\ \bigwedge_{i=1}^k x_i = x_{i-1} + 1 \land n(x_0, x_k) \mid \bot, k \in \mathbb{N}
\]

(i) We first show how Theorem 14 can be used in this case. Let \( N_P = \{ C_1, C_2, C_3 \} \), where \( C_1 = P(x_1, y_1) \), \( C_2 = P(y_2, z_1) \rightarrow P(x_1, z_1) \), and \( C_3 = \neg P(x_3, y_3) \). We can saturate \( N_P \) as follows: From \( C_1 \) and \( C_3 \) we can derive \( C_4 = \bot \); from \( C_1 \) and \( C_2 \) we can derive a clause of type \( C_1 \), from \( C_2 \) and \( C_2 \) a clause of type \( C_2 \) and from \( C_2 \) and \( C_3 \) a clause of type \( C_3 \). We obtain \( N^*_P = \{ C_1, C_2, C_3, C_4 \} \). By Theorem 14, the saturation of \( N \) in \( N^* \):

\[
\{ \mu_1(x, y) \mid P(x, y), \mu_2(x, y, z) \mid P(y, z) \rightarrow P(x, z), \mu_3(x, y) \mid \neg P(x, y), \mu_4(x, y) \mid \bot \},
\]

where \( \mu_1, \mu_2, \mu_3, \mu_4 \) are given by the minimal model \( M \) of \( CH_N \):

\[
CH_N = \{ x = y \rightarrow \mu_1(x, y), \ y = x + 1 \rightarrow \mu_2(x, y, z), \ n(x, y) \rightarrow \mu_3(x, y), \\
\mu_2(x, y) \land \mu_2(u, x, y) \rightarrow \mu_1(u, y), \ \mu_3(x, y) \land \mu_2(x, u, y) \rightarrow \mu_3(u, y), \\
\mu_2(x, y, z) \land \mu_2(u, x, z) \rightarrow \mu_2(u, y, z), \ \mu_1(x, y) \land \mu_3(x, y) \rightarrow \mu_4(x, y) \}
\]

\( \mu Z \) cannot check whether this set of Horn constraints is satisfiable because of the parameter \( n \). If we replace \( n(x, y) \) with \( x > y \) \( \mu Z \) yields the following solution:

\[
\mu_1(x, y) = x \leq y, \ \mu_2(x, y, z) = (y > z) \lor (x < z), \ \mu_3(x, y) = x > y, \ \mu_4 = \bot.
\]

(ii) Alternatively, note that if we use the fact that \( \exists x_1 \ldots x_{k-1} \bigwedge_{i=1}^k x_i = x_{i-1} + 1 \equiv \tau \)

\[
x_k = x_0 + k \text{ we obtain an infinite set of clauses consisting of:}
\]

\[
(1)_k y = x + k \mid P(x, y) \\
(2)_k y = x + k \mid P(y, z) \rightarrow P(x, z) \\
(3)_k n(x, y) \land z = x + k \mid \neg P(z, y) \\
(4)_k y = u + k \land n(u, y) \mid \bot, k \in \mathbb{N}
\]

\[\text{If the set } N \text{ of constrained } P \text{-clauses (hence the set of constrained Horn clauses } CH_N) \text{ contains at least one parameter then } \mu Z \text{ often returns "unknown". In addition, if } \mu Z \text{ can prove satisfiability of } CH_N \cup \{ \neg \mu_j(\overline{\tau}) \} \text{ for a non-parametric problem, the model it returns is not guaranteed to be minimal in general, and cannot be used for representing the saturated set of clauses. By Theorem 14 (2), satisfiability of } CH_N \cup \{ \neg \mu_j(\overline{\tau}) \} \text{ is sufficient for proving the satisfiability of } N \text{ in this case.} \]

---

\[\text{Symbol Elimination and Applications 25}\]
If we regard $k$ in each clause as a universally quantified variable (with additional condition $k \geq 0$) we obtain:

$$N' = \{ y = x + k \land k \geq 0 \} \mathbf{P}(x, y), \quad y = x + k \land k \geq 0 \} \mathbf{P}(y, z),$$

$$n(x, y) \land z = x + k \land k \geq 0 \} \mathbf{\neg P}(z, y), \quad y = u + k \land k \geq 0 \land n(u, y)\} \mathbf{\bot}.$$  

If $A = (\mathbb{Z}, n_A)$, $A \models \exists P N'$ iff $A \models \forall u, y, k \ (k \geq 0 \land y = u + k \to \neg n(u, y))$.

**Remark:** In linear integer arithmetic the interpretations of $(\mu_i)_{1 \leq i \leq 4}$ in the minimal model of $CHN$ w.r.t. the model $A = (\mathbb{Z}, n_A)$, for a fixed interpretation of $n$ (say as $n_A(x, y) = (x > y)$) are:

- $\mu_1(x, y) = \mu_2(x, y, z) = \exists k (k \geq 0 \land y = x + k)$,
- $\mu_3(x, y) = \exists z \exists k (n(z, y) \land x = z + k)$ and $\mu_4(x, y) = \mu_1(x, y) \land \mu_3(x, y)$.

Example 6(ii) uses acceleration techniques, in particular the following result:

**Theorem 15 ([14,17])** Let $N$ be a set of constrained clauses of the form:

$$N = \{ \phi(\overline{x}) \ | \ R(\overline{x}), \ \phi(\overline{x}) \land \overline{y} = M : \overline{x} + \overline{y} \ | \ R(\overline{x}) \rightarrow R(\overline{y}) \}$$

where $\overline{x}, \overline{y}$ describe vectors of $n$ variables, $\overline{x}$ a vector of $n$ constants in $\mathbb{Z}$, $\phi_0$ is a condition expressible in Presburger arithmetic and $M = (m_{i,j})_{1 \leq i,j \leq n}$ is a $n \times n$ matrix over $\mathbb{Z}$, and $\phi(x_1, \ldots, x_n) = \bigwedge_{i=1}^k (\sum_{j=1}^n a_{ij} x_j \leq b_i)$, where $a_{ij}, b_i \in \mathbb{Z}$.

The interpretation of $\overline{R}$ in the minimal model of $N$ is Presburger definable if $\langle M \rangle = \{ M^n \ | \ n \in \mathbb{N} \}$ is finite. If $\phi = \top$ then the interpretation of $\overline{R}$ in the minimal model of $N$ is Presburger definable if and only if $\langle M \rangle = \{ M^n \ | \ n \in \mathbb{N} \}$ is finite.

Acceleration techniques have been investigated e.g. for fragments of theories of arrays with read and write in the presence of iterators and selectors in [4]. Similar ideas are used in the superposition calculus in [16, 27], and in approaches which combine superposition and induction [31] or use solutions for recurrences in loop invariant generation [33, 32]. We plan to analyze such aspects in future work.

### 6 Checking Entailment

Let $\mathcal{T}$ be a theory with signature $\Pi = (S, \Sigma, \text{Pred})$, and let $\overline{P}_1 = P_1^1, \ldots, P_{n_1}$ and $\overline{P}_2 = P_2^1, \ldots, P_{n_2}$ be finite sequences of different predicate symbols with $P_j \notin \text{Pred}$, and $\Pi_i = (\Sigma, \text{Pred} \cup \{ P_j \mid 1 \leq j \leq n_i \})$ for $i = 1, 2$.

Let $F_1$ be a universal $\Pi_1$-formula and $F_2$ be a universal $\Pi_2$-formula. We analyze the problem of checking whether “$\exists \overline{P}_1 F_1$ entails $\exists \overline{P}_2 F_2$ w.r.t. $\mathcal{T}$” holds.

**Example 7** Such questions arise in the graph-theoretic problems discussed in Section 2. Let $A$ be a class of graphs described by axioms $\mathbf{Ax}_A$ and $B$ be a class of graphs described by axioms $\mathbf{Ax}_B$. Let $\mathcal{T}$ be a theory used for expressing these axioms. Consider the $\rightarrow$ and $\leftarrow$ transformations described in Section 2. Then $A^+ \subseteq B^-$ (i.e. for every graph $H = (V, F) \in A^+$ we have $H \in B^-$) if and only if $\exists E_A (\mathbf{Ax}_A \land Tr^+(E_A, F)) \models_\mathcal{T} \exists E_B (\mathbf{Ax}_B \land Tr^-(E_B, F))$.  

\[ \mathbb{Q} \]
Assume that there exist $\Pi$-formulae $G_1$ and $G_2$ such that $G_1 \equiv_{\mathcal{T}} \exists F_1$ and $G_2 \equiv_{\mathcal{T}} \exists F_2$. Such formulae can be found either by saturation\(^9\) by successively eliminating $P_1, \ldots, P_n$, or by using acceleration techniques or other methods. In this case, $\exists F_1 \models_{\mathcal{T}} \exists F_2$ iff $G_1 \models_{\mathcal{T}} G_2$ (which is the case if $G_1 \wedge \neg G_2 \models_{\mathcal{T}} \bot$).

The problem of checking whether $G_1 \wedge \neg G_2 \models_{\mathcal{T}} \bot$ in general undecidable, even if $G_1$ and $G_2$ are universal formulae and $\mathcal{T}$ is the extension of Presburger arithmetic or real arithmetic with a new function or predicate symbol (cf. \[44\]).

If $G_1 \wedge \neg G_2$ is in a fragment of $\mathcal{T}$ for which checking satisfiability is decidable, then we can effectively check whether $\exists F_1 \models_{\mathcal{T}} \exists F_2$. This is obviously the case when $\mathcal{T}$ is a decidable theory. We will show that a similar condition can be obtained for local theory extensions of theories allowing quantifier elimination if $G_1$ and $G_2$ are universal formulae and the extensions satisfy a certain “flatness property” which allows finite complete instantiation and that in both cases we can also generate constraints on “parameters” under which entailment holds.

**Theorem 16** Assume that there exist $\Pi$-formulae $G_1$ and $G_2$ such that $G_1 \equiv_{\mathcal{T}} \exists F_1$ and $G_2 \equiv_{\mathcal{T}} \exists F_2$. If $\mathcal{T}$ is a decidable theory then we can effectively check whether $\exists F_1 \models_{\mathcal{T}} \exists F_2$. If $\mathcal{T}$ has quantifier elimination and the formulae $F_1, F_2$ contain parametric constants, we can use quantifier elimination in $\mathcal{T}$ to derive conditions on these parameters under which $\exists F_1 \models_{\mathcal{T}} \exists F_2$.

**Theorem 17** Assume that there exist universal $\Pi$-formulae $G_1$ and $G_2$ such that $G_1 \equiv_{\mathcal{T}} \exists F_1$ and $G_2 \equiv_{\mathcal{T}} \exists F_2$, and that $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$, where $\mathcal{T}_0$ is a decidable theory with signature $\Pi_0 = (S_0, \Sigma_0, \text{Pred}_0)$ where $S_0$ is a set of interpreted sorts and $\mathcal{K}$ is a set of (universally quantified) clauses over $\Pi = (S_0 \cup S_1, \Sigma_0 \cup \Sigma_1, \text{Pred}_0 \cup \text{Pred}_1)$, where (i) $S_1$ is a new set of uninterpreted sorts, (ii) $\Sigma_1, \text{Pred}_1$ are sets of new function, resp. predicate symbols which have only arguments of uninterpreted sort $\in S_1$, and all function symbols in $\Sigma_1$ have interpreted output sort $\in S_0$. Assume, in addition, that all variables and constants of sort in $\Sigma_1$ in $\mathcal{K}$, $G_1$ and $\neg G_2$ occur below function symbols in $\Sigma_1$. Then:

1. We can use the decision procedure for $\mathcal{T}_0$ to effectively check whether $G_1 \wedge \neg G_2 \models_{\mathcal{T}} \bot$ (hence if $\exists F_1 \models_{\mathcal{T}} \exists F_2$).
2. If $\mathcal{T}_0$ allows quantifier elimination and the formulae $F_1, F_2$ (hence also $G_1, G_2$) contain parametric constants and functions, we can use Algorithm 1 for obtaining constraints on the parameters under which $\exists F_1 \models_{\mathcal{T}} \exists F_2$.

**Proof:** Let $C$ be the set of constants of uninterpreted sort $s \in S_1$ occurring in $\mathcal{K}, G_1$ and $\neg G_2$. Note that $G_1 \wedge \neg G_2$ is satisfiable w.r.t. $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ iff $(\mathcal{K} \wedge G_1)^{\{C\}} \wedge \neg G_2$ is satisfiable, where $(\mathcal{K} \wedge G_1)^{\{C\}}$ is the set of all instances of $\mathcal{K} \wedge G_1$ in which the variables of sort $s \in S_1$ are replaced with constants of sort

\[^9\] We can iterate the application of $HRes^C$ for variables $P_1, \ldots, P_n$ (in this order). This corresponds to a variant of ordered resolution which we denote by $HRes^{P_1, \ldots, P_n}$; if saturation terminates the conjunction of clauses not containing $P_1, \ldots, P_n$ is equivalent to $\exists P_1, \ldots, P_n \exists F_i$, where $\exists F_i$ is the clause form of $F_i$. 

We obtain the axiomatization $G$ and the axiomatization $G_{\lnot}$. We want to check whether $G_{\lnot}$ is satisfiable for $\mathcal{T}$.

Example 8 Let $\text{QUDG}(r) = (\text{MinDG}(r) \cap \text{MaxDG}(1))^\ast$, be axiomatized by $\text{MinDG}(r) \cup \text{MaxDG}(1) \cup \text{Tr}^{-}(E, F)$, where:

$\text{MinDG}(r) : \forall x, y \, \pi^i(x, y, r) \rightarrow E(x, y)$ where $\pi^i(x, y, r) = x \neq y$ and $d(x, y) \leq r(x)$

$\text{MaxDG}(1) : \forall x, y \, \pi^e(x, y) \rightarrow \neg E(x, y)$ where $\pi^e(x, y) = d(x, y) > 1$

$\text{Tr}^{-}(E, F) : \forall x, y \, (F(x, y) \leftrightarrow E(x, y) \land E(y, x))$.

We want to check whether $A(r) \subseteq B(r)$, where $A(r) = \text{QUDG}(r)$ and $B(r) = (\text{MinDG}(r) \cap \text{MaxDG}(1))^\ast$ is described by $\text{MinDG}(r) \cup \text{MaxDG}(1) \cup \text{Tr}^{-}(E, F)$.

We obtain the axiomatization $G_1$ by eliminating $E$ from

$$\exists E(\text{MinDG}(r) \land \text{MaxDG}(1) \land \text{Tr}^{-}(E, F))$$

and the axiomatization $G_2$ by eliminating $E$ from

$$\exists E(\text{MinDG}(r) \land \text{MaxDG}(1) \land \text{Tr}^{+}(E, F)).$$

We check whether $G_1 \models T G_2$, i.e. whether $G_1 \land \lnot G_2$ is unsatisfiable w.r.t. $\mathcal{T}$, where $\lnot G_2$ is the conjunction of the following ground formulae (we ignore the negation of the first clause obviously implied by $G_1$):

$(g_1) \, \pi^e(a, b) \land \pi^e(b, a) \land F(b, a)$

$(g_2) \, \pi^e(a, a) \land F(a, a)$

$(g_3) \, F(a, b) \land \lnot F(b, a)$

$(g_4) \, \pi^e(a, b) \land \lnot F(b, a)$

$(g_5) \, \pi^e(a, b) \land \lnot F(b, a)$

By Theorem 17 (2), we can consider the set of all instances of $G_1$ in which the variables of sort $p$ are replaced with the constants $a, b$, then use a method for checking ground satisfiability of $G_1[T] \land g_i$ w.r.t. $\mathcal{T}_d$ (d is uninterpreted), $\mathcal{T}^+_{d}$ (d is positive), $\mathcal{T}^-_{d}$ (d is symmetric) and $\mathcal{T}^{mp}_{d}$ (d is a metric). For this, we use H-PLL and H-PLL [29] to test whether $G_1[T] \land g_i$ is satisfiable for $i \in \{1, 2, 3\}$, then use Algorithm 1, sephilot to derive conditions on parameters under which $G_1[T] \land g_i$ is unsatisfiable. We give here two examples:
(1) We consider $d$ and $r$ to be parameters, i.e. we eliminate only $F$ from $G_1[T] \wedge g_i$. For $T_d^m$ we get the condition
\[ C_{d,r}^m = \forall x, y (x \neq y \wedge d(x, y) \leq 1 \wedge d(x, y) \leq r(x) \rightarrow d(y, x) \leq r(y)). \]

(2) We consider only $r$ to be a parameter, i.e. we eliminate the symbols $F$ and $d$. For $T_d^r$ we obtain the condition
\[ C_r = \forall x, y (r(y) < 1 \wedge x \neq y \rightarrow r(y) \geq r(x)). \]

This condition holds e.g. if $r(x) = r(y)$ for all $x, y$, i.e. if $r$ is a constant function. Adding this as an additional condition we get unsatisfiability of $G_1[T] \wedge g_i$ with $i \in \{4, 5\}$ for $T_d^m$ and $T_d^r$, but not for $T_d^u$ and $T_d^p$.

Checking the other inclusion We now check whether $B(r) \subseteq A(r)$, where $A(r) = \text{QUDG}(r)$ and $B(r) = (\text{MinDG}(r) \cap \text{MaxDG}(1))^+$. We have the axiomatizations $G_1, G_2$ for the two classes.

We check whether $G_2 \models_T G_1$, i.e. whether $G_2 \land \neg G_1$ is unsatisfiable w.r.t. $T$, where $\neg G_1$ is the disjunction of the following ground formulae (we ignore the negation of the first clause obviously implied by $G_2$):

\[ (g_1) \quad \pi'(a, b) \land \pi'(b, a) \land \neg F(b, a) \]
\[ (g_2) \quad \pi'(a, b) \land F(a, b) \quad \pi'(a, b) \land \neg F(b, a) \]
\[ (g_3) \quad F(a, b) \land \neg F(b, a) \]

We use H-PiLoT for checking ground satisfiability of $G_2[T] \land g_i$ w.r.t. $T \in \{T_u^p, T_u^p, T_d^p, T_d^p, T_d^m\}$. For $T_u$ and $T_m$ we obtain unsatisfiability of $G_2[T] \land g_i$ for $i \in \{1, 2, 3, 4\}$, thus we have proved that the inclusion holds for these two theories. For $T_m$ and $T_u$ we get satisfiability for cases 2 and 3. We use Algorithm 1 to obtain conditions on parameters such that $G_2[T] \land g_2$ and $G_2[T] \land g_3$ is unsatisfiable.

If we consider $d$ and $r$ to be parameters, i.e. we eliminate only $F$ from $G_1[T] \land g_i$, we obtain the condition
\[ C_{d,r} = \forall x, y (d(y, x) > 1 \lor d(x, y) \leq 1 \lor d(x, y) \leq r(x) \lor x = y). \]

It is easy to see that this condition holds if $d$ is symmetric.

7 Tests

We tested the methods we proposed on several examples. We used various tools for solving the various types of symbol elimination considered in this paper.

Second-order quantifier elimination. Since the implementations of the hierarchical superposition calculus we are aware of have as background theory linear arithmetic and in our examples we had more complex theories, we used a form of abstraction first: We renamed the constraints over more complex theories with new predicate symbols, and used SCAN [18] for second-order quantifier elimination. SCAN performs second-order quantifier elimination in first-order logic. It takes as input a formula of the form $F(P_1, \ldots, P_n)$ containing predicate symbols $P_1, \ldots, P_n$ and applies a clause form transformation, ordered resolution and de-Skolemization on this formula. In case of termination and if de-Skolemization is
possible, it returns a first-order formula equivalent to $\exists P_1 \ldots \exists P_n F(P_1, \ldots, P_n)$, which does not contain the predicate symbols $P_1, \ldots, P_n$.

**Satisfiability checking and property-directed symbol elimination.** For satisfiability checking we used H-PILoT [29] (after preparing the input such that the instances that have to be used are clear for the prover). H-PILoT carries out a hierarchical reduction to the base theory. Standard SMT provers or specialized provers can be used for testing the satisfiability of the formulae obtained after the reduction. H-PILoT uses eager instantiation and the hierarchical reduction, so provers like CVC4 [6] or Z3 [13,11] are in general faster in proving unsatisfiability. The advantage of using H-PILoT is that knowing the instances needed for a complete instantiation allows us to correctly detect satisfiability (and generate models) in situations in which e.g. CVC4 returns “unknown”, and use property-directed symbol elimination to obtain additional constraints on parameters which ensure unsatisfiability.

For obtaining the constraints on parameters we used the method described in Algorithm 1 proposed in [41] which was implemented in sehpilot for the case in which the base theory is the theory of real-closed fields. sehpilot (Symbol Elimination with H-PILoT) receives a list of parameters as a command line (and possibly a list of already existing constraints on these parameters) and uses H-PILoT for the hierarchical reduction to a problem in the base theory (Step 1 in Algorithm 1) and for generating a corresponding REDLOG file. The constants are classified as required in Step 2 of Algorithm 1 and the REDLOG file is changed accordingly such that only those symbols that are not a parameter or argument of a parameter are considered to be existentially quantified. Redlog is used for quantifier elimination (Step 3 of Algorithm 1); then the constants contained in the obtained formula are replaced back with the terms they represent (Step 4). Finally, the formula obtained this way is negated (Redlog is used for further simplifications).

The way we used these tools is illustrated on some tests in Appendix A.

8 Conclusions

In this paper, we analyzed possibilities of combining general second-order symbol elimination and property-directed symbol elimination. For eliminating existentially quantified predicates from universal first-order formulae we used a constrained resolution calculus (obtained from specializing the hierarchical superposition calculus). We analyzed situations in which saturation terminates and two possibilities of obtaining finite representations also in cases in which saturation might not terminate: (i) Using an encoding of the constraints of the saturated set of clauses as smallest fixpoints of certain families of constrained Horn clauses and (ii) using acceleration. For checking the satisfiability of families of constrained Horn clauses we used the fixpoint package of Z3 [23].

If the saturation terminates, or the infinite saturated set of clauses has a finite representation, we can use the obtained set of clauses for checking entailment. We proved a $\Psi$-locality property for a class of formulae; this allowed us to use
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the prover H-PILoT (after preparing the input such that the instances that have to be used are clear) for analyzing the satisfiability of formulae w.r.t. models in a theory $\mathcal{T}$ and for checking entailment between formulae. Property-based symbol elimination proved useful for obtaining (weakest) constraints $\Gamma$ on “parameters” used in the description of the theory $\mathcal{T}$ such that satisfiability or entailment is guaranteed in models satisfying $\Gamma$.

In future work we would like to find possibilities of identifying situations in which second-order quantifier elimination using resolution terminates and study possibilities of using (and generalizing) methods based on constrained Horn clauses or acceleration for obtaining finite representations of potentially infinite clause sets. We would also like to analyze possibilities of checking entailment when the second-order quantifier elimination method returns a fixpoint and not a formula. (The main obstacle when working on this problem was that $\mu Z$ returns “unknown” in the presence of parameters.)

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A Tests

We here present some of the tests we made for the examples in the paper. We show how we used the tools on these examples including the corresponding input and output files.

A.1 Tests for Example 3

We used H-PILoT to check that \((c_1), (c_2)\) and \((c_3)\) from Example 2 are valid w.r.t. \(\mathcal{T} = \mathcal{T}_{dm}^n \cup \text{Free}(r_1, r_2)\). For this we show, one after the other, that the negation of each of these formulae is unsatisfiable w.r.t. \(\mathcal{T}\). We start with \((c_1)\).

In the input file for H-PILoT we have the axioms for a metric specified under Clauses and the negation of \((c_1)\) is the Query.

\[
\begin{align*}
\text{Base functions:} & = \{ (+, 2), (-, 2), (\ast, 2) \} \\
\text{Extension functions:} & = \{(r_1, 1, 1), (r_2, 1, 1), (d, 2, 1) \} \\
\text{Relations:} & = \{(\leq, 2), (<, 2), (>\leq, 2), (>\), 2) \} \\
\text{Clauses :=} & = (\text{FORALL } x, y). \ d(x, y) = 0 \implies x = y; \quad (\text{FORALL } x, y). \ x = y \implies d(x, y) = 0; \quad (\text{FORALL } x, y). \ d(x, y) = d(y, x); \quad (\text{FORALL } x, y, z). \ d(x, y) \leq d(x, z) + d(z, y); \\
\text{Query :=} & = \neg (u = v); \quad d(u, v) \leq r_1(u); \quad \neg (u = w); \quad d(u, w) \leq d(u, v); \quad \text{OR}(u = w, d(u, w) > r_1(u));
\end{align*}
\]

H-PILoT performs the hierarchical reduction described in Theorem 10, then hands the reduced problem over to a prover to check for satisfiability. We here used Z3, which is also the default prover used by H-PILoT. We obtain the following output from H-PILoT:

\[
\begin{align*}
\text{Reduced problem written to} & = \text{example4-check_redundancy_1.smt}. \\
\text{unsat} & = \text{H-PILoT spent 0.013423s on the problem.} \\
\text{unsat} & = \text{The prover needed 0.012652s for the problem.} \\
\text{Total running time:} & = 0.026075s.
\end{align*}
\]

The answer is “unsat” (unsatisfiable), so we have proved that \((c_1)\) is valid w.r.t. \(\mathcal{T} = \mathcal{T}_{dm}^n \cup \text{Free}(r_1, r_2)\). In the same way we can prove validity of \((c_2)\) and \((c_3)\) w.r.t. \(\mathcal{T}_{dm}^n \cup \text{Free}(r_1, r_2)\), and also the validity of the three formulae w.r.t. \(\mathcal{T}_d^p \cup \text{Free}(r_1, r_2)\), \(\mathcal{T}_d^p \cup \text{Free}(r_1, r_2)\) and \(\mathcal{T}_d^n \cup \text{Free}(r_1, r_2)\).

Remark. The encoding in H-PILoT presented above did not use two different sorts \(p\) and \(\text{num}\) as described in the theoretical considerations. Since in our case the sort \(p\) can be considered to be uninterpreted and there are no function
symbols of arity \( p^k \rightarrow p \), the following holds: Let \( G \) be a flat ground formula over a signature containing a binary function \( d \) and a unary function \( r_1 \) with the property that the only constraints on the constants used as arguments for \( d \) and \( r_1 \) are equalities and disequalities. Then the following are equivalent:

1. \( G \) is satisfiable w.r.t. the extension of \( \mathbb{R} \) with a binary function \( d \) satisfying the metric axioms and a unary free function symbol \( r_1 \).
2. \( G \) is satisfiable w.r.t. the extension of the two-sorted theory \( T_d^m \) with the free function \( r_1 \).

Indeed, from every model \( A = (R, d_A : R^2 \rightarrow \mathbb{R}, r_{1A} : R \rightarrow \mathbb{R}) \) of \( G \) w.r.t. the extension of \( \mathbb{R} \) with a binary function \( d \) satisfying the metric axioms and a unary free function symbol \( r_1 \), we can define a model

\[ B = (B_p, \mathbb{R}, d_B : B_p^2 \rightarrow \mathbb{R}, r_{1B} : B_p \rightarrow \mathbb{R}) \]

of \( T_d^m \cup \text{Free}(r_1) \) as follows:

- Take as \( B_p \) an isomorphic copy (via isomorphism \( i \)) of the set \( \{c_A | c \text{ constant of sort } p \text{ occurring in } G\} \),
- Define \( d_B, r_{1B} \) as follows:
  \[ d_B(i(c_A), i(d_A)) := d_A(c_A, d_A); \]
  \[ r_{1B}(i(c_A)) := r_{1A}(c_A). \]

The converse implication is analogous, with the only difference that we first construct a partial algebra \( A = (\mathbb{R}, d_A : R^2 \rightarrow \mathbb{R}, r_{1A} : \mathbb{R} \rightarrow \mathbb{R}) \) by considering an injective map from the support of sort \( p \) in \( \mathbb{R} \) and then we use the locality property of \( T_d^m \cup \text{Free}(r_1) \) to prove the existence of a total model with support \( \mathbb{R} \).

### A.2 Tests for Example 4

We use sehpiot (an implementation of Algorithm 1) to derive constraints \( \Gamma' \) on the parameters \( r_1, r_2 \) such that

\[ \pi^t(u, v) \land \pi^e(u, v) = (u \neq v \land d(u, v) \leq r_1(u)) \land (d(u, v) > r_2(u)) \]

is unsatisfiable (we consider the case in which \( d \) is an uninterpreted function). Since sehpiot first uses H-PiLoT for the hierarchical reduction (and afterwards Redlog for quantifier elimination), the input file is in H-PiLoT syntax:

```plaintext
Base_functions:=\{(+ , 2), (- , 2), (*) , 2)\}
Extension_functions:=\{(r1 , 1 , 1 ), (r2 , 1 , 1 ), (d , 2 , 1 )\}
Relations:=\{(<= , 2 ), (< , 2 ), (>= , 2 ), (> , 2 )\}
Query :=
NOT(u = v);
d(u , v) <= r1(u);
d(u , v) > r2(u);
```
Note that when using sehpilot the user has to specify which symbols have to be eliminated.

Assume that \( r_1 \) and \( r_2 \) are parameters (and thus should not be eliminated). Since the variable \( u \) occurs as an argument of \( r_1 \) and \( r_2 \) (which are parameters), \( u \) should not be eliminated. We have to eliminate the remaining symbols, i.e. \( v \) and \( d \). We obtain the following output (in verbose mode) from sehpilot:

```plaintext
\[
\begin{align*}
\text{symbol elimination and applications} & \quad 37 \\
\text{Note that when using sehpilot the user has to specify which symbols have to be eliminated.} \\
\text{Assume that } r_1 \text{ and } r_2 \text{ are parameters (and thus should not be eliminated).} \\
\text{Since the variable } u \text{ occurs as an argument of } r_1 \text{ and } r_2 \text{ (which are parameters),} \\
\text{\( u \) should not be eliminated. We have to eliminate the remaining symbols, i.e. } v \text{ and } d. \\
\text{We obtain the following output (in verbose mode) from sehpilot:}
\end{align*}
\]```
One can for example see which new constants are introduced in the hierarchical reduction \((e_1, e_2, \text{and } e_3)\) and which terms they represent. The output also shows the result obtained directly after the elimination \((e_2 > e_3)\), the negation of this result \((e_2 \leq e_3)\), and finally the universally quantified formula with the constants replaced back with the corresponding terms \((\forall u(r_1(u) \leq r_2(u)))\).

Remark: The current implementation of \textsf{sehpilot} assumes that the problems are expressed in a local extension of the theory of real closed fields and a reduction to quantifier elimination in the theory of real-closed fields is performed. For the examples we considered this does not lead to loss of generality because the constraints on constants of sort \(p\) are only equalities and disequalities. If variables initially of sort \(p\) are eliminated, they do not occur below any parameter. Such variables occur separately from the variables of original sort \(\text{num}\) in the quantifier elimination problem. This means that the quantifier elimination problem is of the form

\[
\exists x_1, \ldots, x_n \exists y_1, \ldots, y_m C_p(x_1, \ldots, x_n) \land C_{\text{num}}(y_1, \ldots, y_m)
\]

where \(x_1, \ldots, x_n\) are variables of sort \(p\) and \(y_1, \ldots, y_m\) are variables of sort \(\text{num}\), which is equivalent to:

\[
\exists x_1, \ldots, x_n C_p(x_1, \ldots, x_n) \land \exists y_1, \ldots, y_m C_{\text{num}}(y_1, \ldots, y_m).
\]

Quantifier elimination in the theory of real-closed fields can be used for the formula \(\exists y_1, \ldots, y_m C_{\text{num}}(y_1, \ldots, y_m)\).

If we consider theories whose models of sort \(p\) contain infinitely many elements, then – since the constraint \(C_p\) contains only equalities and disequalities – the method for quantifier elimination in the theory of infinite sets can be simulated by the method for quantifier elimination in real closed fields. This is the reason why for this type of problems we can use quantifier elimination in the theory of real closed fields without problems.

### A.3 Tests for Example 8

In order to check whether the class containment

\[
\text{QUDG}(r) = (\text{MinDG}(r) \cap \text{MaxDG}(1))^\sim \subseteq (\text{MinDG}(r) \cap \text{MaxDG}(1))^+ \]

holds we have to check whether \(G_1 \land g_i\) is unsatisfiable for all \(i \in \{1, 2, 3, 4, 5\}\) (where \(G_1\) is the axiomatization for \(\text{QUDG}(r)\) and the \(g_i\) are the ground formulae obtained from the negation of \(G_2\), the axiomatization of the other class; cf. Example 8).

We assume that \(d\) is a metric. Using \textsf{H-PILoT} we can show that \(G_1 \land g_i\) is unsatisfiable w.r.t. \(T^m_d\) for \(i \in \{1, 2, 3\}\). We here only show the test for the case \(G_1 \land g_4\) in detail (the case \(G_1 \land g_5\) is similar and yields the same results).

We check satisfiability of \(G_1 \land g_4\) w.r.t. \(T^m_d\) using \textsf{H-PILoT}. We have the following input file:
Symbol Elimination and Applications

\textbf{Base functions:} := \{(+,2), (−,2), (\ast,2)\}

\textbf{Extension functions:} := \{(r, 1, 1), (d, 2, 1), (F, 2, 1)\}

\textbf{Relations:} := \{(\leq, 2), (<, 2), (\geq, 2)\}

\textbf{Clauses :=}

% axioms for G1
\begin{align*}
(FORALL x,y) & . d(x,y) \leq r(x) \land d(x,y) > 1 \\
\quad & \implies x = y, \ 0 = 1; \\
(FORALL x,y) & . d(x,y) \leq r(x) \land d(y,x) \leq r(y) \\
\quad & \implies x = y, \ F(y,x) = 1; \\
(FORALL x,y) & . d(x,y) > 1 \implies F(x,y) = 0; \\
(FORALL x,y) & . d(x,y) > 1 \implies F(y,x) = 0; \\
(FORALL x,y) & . F(x,y) = 1 \implies F(y,x) = 1;
\end{align*}

% axioms for d being a metric
\begin{align*}
(FORALL x,y) & . d(x,y) \geq 0; \\
(FORALL x,y) & . d(x,y) = 0 \implies x = y; \\
(FORALL x,y) & . x = y \implies d(x,y) = 0; \\
(FORALL x,y) & . d(x,y) = d(y,x); \\
(FORALL x,y,z) & . d(x,y) \leq d(x,z) + d(z,y);
\end{align*}

% F is either 0 or 1
\begin{align*}
(FORALL x,y) & . \implies F(x,y) = 0, F(x,y) = 1;
\end{align*}

\textbf{Query :=}

% g4 of (not G2)
\begin{align*}
\text{NOT}(a = b); \\
d(a,b) \leq r(a); \\
F(a,b) = 0;
\end{align*}

% needed for instantiation
\begin{align*}
F(a,b) & = F(a,b); \\
F(b,a) & = F(b,a); \\
F(a,a) & = F(a,a); \\
F(b,b) & = F(b,b);
\end{align*}

\begin{align*}
d(a,b) & = d(a,b); \\
d(b,a) & = d(b,a); \\
d(a,a) & = d(a,a); \\
d(b,b) & = d(b,b);
\end{align*}

\begin{align*}
r(a) & = r(a); \\
r(b) & = r(b);
\end{align*}

Note that the trivial equalities at the end of the file are used to ensure that H-PILoT computes sufficiently many instances. We obtain the following output from H-PILoT:
Reduced problem written to example8-tn-4.smt.

Unknown. Prover says 'sat' but this can only be trusted for local extensions and this problem is not known to be local.

H-PILoT spent 0.208608 s on the problem.
The prover needed 0.01677 s for the problem.
Total running time: 0.225378 s.

Since we know that $T_m^{su} \cup \text{Free}(r_1, r_2)$ is a $\Psi$-local theory extension and we ensured that H-PILoT computes sufficiently many instances, we know that $G_1 \land g_4$ is satisfiable. This means that the class inclusion does not hold in general. We use sehpilot to derive (weakest) conditions $\Gamma$ on parameters such that unsatisfiability of $G_1 \land \Gamma \land g_4$ is guaranteed.

We first consider $r$ to be the only parameter, i.e. we tell sehpilot to eliminate $F$ and $d$ ($a$ and $b$ appear as arguments of parameter $r$ and are therefore not eliminated). The input file is the same file that was used for checking satisfiability with H-PILoT. We get the following output (using verbose mode) from sehpilot:

\begin{verbatim}
[2021-02-20 12:55:56.356 | INFO] convert prefix notation
{'e_10': ' r(b)', 'e_9': ' r(a)', 'e_8': ' d(b, a)',
' e_7': ' d(b, b)', 'e_6': ' d(a, b)',
' e_5': ' d(a, a)', 'e_4': ' F(b, b)',
' e_3': ' F(b, a)', ' e_2': ' F(a, b)',
' e_1': ' F(a, a)'}

[2021-02-20 12:55:56.356 | INFO] assignments of new variables
\begin{align*}
& e_{10} = r(b) \\
& e_9 = r(a) \\
& e_8 = d(b, b) \\
& e_7 = d(b, a) \\
& e_6 = d(a, b) \\
& e_5 = d(a, a) \\
& e_4 = F(b, b) \\
& e_3 = F(b, a) \\
& e_2 = F(a, b) \\
& e_1 = F(a, a)
\end{align*}

[2021-02-20 12:55:56.356 | INFO] reduced assignments
\begin{align*}
& e_{10} = r(b) \\
& e_9 = r(a) \\
& e_8 = d(b, b) \\
& e_7 = d(b, a) \\
& e_6 = d(a, b) \\
& e_5 = d(a, a) \\
& e_4 = F(b, b) \\
& e_3 = F(b, a) \\
& e_2 = F(a, b) \\
& e_1 = F(a, a)
\end{align*}

[2021-02-20 12:55:56.356 | INFO] variables that will be eliminated by REDLOG:
\begin{align*}
& e_{10}, e_9, e_8, e_7, d, e_6, d_5, e_4, F, e_2
\end{align*}

[2021-02-20 12:55:56.357 | INFO] extension functions declared in loc-file
\{( 'r', 1, 1), ('d', 2, 1), ('F', 2, 1)\}
\end{verbatim}
Redlog does not simplify the results of the quantifier elimination very well, so in many cases one obtains long formulae, which sometimes can be simplified. In this case the constraint computed by sehpilot can be simplified to

\[ C' = \forall x, y (r(y) < 1 \land x \neq y \rightarrow r(y) \geq r(x)). \]

We could also choose different parameters, e.g. we could assume \( d \) and \( r \) to be parameters and then tell sehpilot to eliminate only \( F \). In this case the computed constraint will be:

\[ C^{d,r} = \forall x, y (x \neq y \land d(x, y) \leq 1 \land d(x, y) \leq r(x) \rightarrow d(y, x) \leq r(y)). \]
B Proof of Theorem 2

**Theorem 2.** Let $\mathcal{K}$ be a set of $\Sigma$-flat clauses, with the property that every variable occurs only once in every term. Let $\Psi$ be a term closure operator with the property that for every flat set of ground terms $T$, $\Psi(T)$ is flat.

Assume that $\mathcal{K}$ and $\Psi$ have the property that for every flat set of ground terms $T$ and for every clause $C \in \mathcal{K}$, if $C$ contains terms $f(x_1, \ldots, x_n)$ and $g(y_1, \ldots, x, \ldots, y_m)$ (where $f, g \in \Sigma$ are extension functions and $f$ and $g$ are not necessarily different), if $f(t_1, \ldots, t_n), g(s_1, \ldots, s, \ldots, s_m) \in \Psi\mathcal{K}(T)$ then $f(t_1, \ldots, s, \ldots, t_n), g(s_1, \ldots, t, \ldots, s_m) \in \Psi\mathcal{K}(T)$. Then $\{\text{Emb}_{w,f}^\Psi\}$ implies $\{\text{Loc}_{w,f}^\Psi\}$.

**Proof:** Assume that $\mathcal{T}_0 \cup \mathcal{K}$ is not a $\Psi$-local extension of $\mathcal{T}_0$. Then there exists a set $\mathcal{G}$ of ground clauses (with additional constants) such that $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{G} \models \bot$ but $\mathcal{T}_0 \cup \mathcal{K}[\mathcal{P}_\mathcal{K}(\mathcal{G})] \cup \mathcal{G}$ has a weak partial model $P$ in which all terms in $\mathcal{P}_\mathcal{K}(\mathcal{G})$ are defined. We assume w.l.o.g. that $G = G_0 \cup G_1$, where $G_0$ contains no function symbols in $\Sigma$ and $G_1$ consists of ground unit clauses of the form $f(c_1, \ldots, c_n) \approx c$ where $c_i, c$ are constants in $\Sigma_0 \cup \Sigma_\mathcal{c}$ and $f \in \Sigma$.

We construct another structure, $\mathcal{A}$, having the same support as $P$, which inherits all relations in $\text{Pred}$ and all maps in $\Sigma_0 \cup \Sigma_\mathcal{c}$ from $P$, but on which the domains of definition of the $\Sigma$-functions are restricted as follows: for every $f \in \Sigma$, $f_A(a_1, \ldots, a_n)$ is defined if and only if there exist constants $c_1, \ldots, c^n$ such that $f(c_1, \ldots, c^n)$ is in $\mathcal{P}_\mathcal{K}(\mathcal{G})$ and $c_i = c_i^P$ for all $i \in \{1, \ldots, n\}$. In this case we define $f_A(a_1, \ldots, a_n) := f_P(c_1^P, \ldots, c_n^P)$. The reduct of $\mathcal{A}$ to $(\Sigma_0 \cup \Sigma_\mathcal{c}, P, \text{Pred})$ coincides with that of $P$. Thus, $\mathcal{A}$ is a model of $\mathcal{T}_0 \cup G_0$. By the way the operations in $\Sigma$ are defined in $\mathcal{A}$ it is clear that $\mathcal{A}$ satisfies $G_1$, so $\mathcal{A}$ satisfies $\mathcal{G}$.

We now show that $\mathcal{A} \models_w \mathcal{K}$. Let $D$ be a clause in $\mathcal{K}$. If $D$ is ground then all its terms are defined, and all terms starting with an extension function are contained in $\mathcal{P}_\mathcal{K}(\mathcal{G})$, i.e. $D \in \mathcal{K}[\mathcal{P}_\mathcal{K}(\mathcal{G})]$, so $D$ is true in $P$, hence it is also true in $\mathcal{A}$.

Now consider the case in which $D$ is not ground. Let $\beta : X \rightarrow \mathcal{A}$ be an arbitrary valuation. Again, if there is a term $t$ in $D$ such that $\beta(t)$ is undefined, we immediately have that $\beta$ weakly satisfies $D$. So let us suppose that for all terms $t$ occurring in $D$, $\beta(t)$ is defined. We associate with $\beta$ a substitution $\sigma$ as follows: Let $x$ be a variable. We have the following possibilities:

**Case 1:** $x$ does not occur below any extension function. This case is unproblematic. We can define $\sigma(x)$ arbitrarily.

**Case 2:** $x$ occurs in a unique term $t = f(...x...y...)$ (which may occur more than once) and $x$ occurs only once in $t$. From the fact that $\beta(t)$ is defined, we know that there are ground terms which we will denote by $t_x, t_y, \ldots$ such that $\beta(x) = (t_x)_P, \beta(y) = (t_y)_P, \ldots$ Since $\beta(t) = f_A(...(t_x)_P \ldots (t_y)_P \ldots)$ is defined, $f(\ldots, t_x, \ldots, t_y, \ldots) \in \mathcal{P}_\mathcal{K}(\mathcal{G})$. We can define $\sigma(x) = t_x$.

**Case 3:** $x$ occurs in two or more terms of the form $f_k(x_1^k, \ldots, x_{n_k}^k), 1 \leq k \leq p, p \geq 2$, but occurs at most once in any term of $C$, where $f_1, \ldots, f_n$ are


function symbols, not necessarily different (but in terms starting with the same function symbols $x$ occurs on different positions).

From the fact that $\beta(f_k(x_1^k, \ldots, x_i^k))$ is defined, we know that there are ground terms which we will denote by $t_{x_1^k}, t_{x_i^k}$ such that for every $k$ with $1 \leq k \leq p$:

\[ \beta(x) = (t_{x_1^k})_p, \beta(x_i^k) = (t_{x_i^k})_p \text{ for } 1 \leq k \leq p \text{ and } 1 \leq i \leq n_k, \text{ and} \]

\[ \beta(f_k(x_1^k, \ldots, x_i^k)) = f_A((t_{x_1^k})_p, \ldots, (t_{x_i^k})_p, p), \]

i.e. $f(t_{x_1^k}, \ldots, t_{x_i^k}, \ldots) \in \Psi_K(G)$.

We know that $\Psi_K$ has the property that for every clause $C \in K$, if $C$ contains terms $f_i(x_1^i, \ldots, x_{n_i}^i)$ and $f_k(x_1^k, \ldots, x_{n_k}^k)$ and if

$f_i(t_{x_1^i}, \ldots, t_{x_{n_i}^i}) \in \Psi_K(G)$ and $f_k(t_{x_1^k}, \ldots, t_{x_{n_k}^k}) \in \Psi_K(G)$

then also $f_i(t_{x_1^i}, \ldots, t_{x_{n_i}^i}, x_i) \in \Psi_K(G)$ and $f_k(t_{x_1^k}, \ldots, t_{x_{n_k}^k}) \in \Psi_K(G)$.

This means that we can define $\sigma(y) = t_y$ for every linear variable: for every variable $x$ which occurs in different terms, let $t_x$ be one of the terms obtained as before (say $t_x = t_{x_2^k}$) and define $\sigma(x) = t_x$.

Thus, we can construct a substitution $\sigma$ with $\sigma(D) \in K[G]$ and $\beta \circ \sigma = \beta$. As $(P, \beta) \models_w \sigma(D)$ we can infer $(A, \beta) \models_w D$.

We now show that $D(A) = \{ f(a_1, \ldots, a_n) \mid f_A(a_1, \ldots, a_n) \text{ defined} \}$ is closed under $\Psi_K$. By definition, $f(a_1, \ldots, a_n) \in D(A)$ iff there exist constants $c_1, \ldots, c_n$ with $c_i A = a_i$ for all $i$ and $f(c_1, \ldots, c_n) \in \Psi_K(G)$. Thus,

\[ D(A) = \{ f(a_1, \ldots, a_n) \mid f_A(a_1, \ldots, a_n) \text{ defined} \} = \{ f(c_1 A, \ldots, c_n A) \mid c_i \text{ constants with } f(c_1, \ldots, c_n) \in \Psi_K(G) \} = \overline{\Psi_K(\overline{f}(\Psi_K(G)))} \]

where $h(c_i) = a_i$ for all $i$.

As $A \models_w K$, $A$ weakly embeds into a total algebra $B$ satisfying $T_0 \cup K$. But then $B \models G$, so $B \models \overline{T_0 \cup K \cup G}$, which is a contradiction.

**Remark:** A similar result can be proved also in the case in which some variables occur several times below a function symbol if $\Psi_K$ has the property that if $f(x_1, \ldots, x_i, \ldots, x_n) \in K$ and $f(t_1, \ldots, s_i, \ldots, t_n) \in \Psi_K(T)$ then $f(t_1, \ldots, t_i, \ldots, t_n) \in \Psi_K(T)$ and $f(t_1, \ldots, s_i, \ldots, s_n) \in \Psi_K(T)$.

C Constrained Horn Clauses: Definitions

We give the definitions of constrained Horn clauses, mainly following the presentation in [12].
Definition 5 ([12]) Conjunctions $\Pi$ of constrained Horn clauses are constructed as follows:

$$\Pi ::= \text{chc} \land \Pi | \top$$

$$\text{chc} ::= \forall \text{var. chc} | \text{body} \rightarrow \text{head}$$

$$\text{pred} ::= \text{upred} | \phi$$

$$\text{head} ::= \text{pred}$$

$$\text{body} ::= \top | \text{pred} | \text{body} \land \text{body} | \exists \text{var. body}$$

$$\text{upred} ::= \text{an uninterpreted predicate applied to terms}$$

$$\phi ::= \text{a formula whose terms and predicates are interpreted over } A$$

$$\text{var} ::= \text{a variable}$$

A clause where the head is a formula $\phi$ is called a query or a goal clause. The terminology “fact clause” is used for a clause whose head is an uninterpreted predicate and body is a formula $\phi$.

It is easy to see that in Theorem 14, if we guarantee that the formulae $\phi_i$ are formulae whose terms and predicates are interpreted over $A$ then all clauses of the form

$$\phi_i(\overline{x}) \rightarrow \mu_i(\overline{x})$$

are constrained Horn clauses, hence:

$$CH_N = \{ \phi_i(\overline{x}) \rightarrow \mu_i(\overline{x}) | i \in \{1, \ldots, n\} \}$$

$$\cup \{ (\mu_i(\overline{x}) \land \mu_j(\overline{y}))\sigma \rightarrow \mu_k(\overline{z}) | C_k(\overline{z}) \text{ is obtained by a resolution}$$

inference in $I_P$ from $C_i(\overline{x})$ and $C_j(\overline{y})$ with m.g.u. $\sigma \}$$

$$\cup \{ \mu_i(\overline{x})\sigma \rightarrow \mu_k(\overline{z}) | C_k(\overline{z}) \text{ is obtained by a factorization}$$

inference in $I_P$ from $C_i(\overline{x})$ with m.g.u. $\sigma \}$$

is a set of constrained Horn clauses.