Reshetikhin–Turaev invariants of Seifert 3–manifolds for classical simple Lie algebras, and their asymptotic expansions

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Abstract

We derive explicit formulas for the Reshetikhin–Turaev invariants of all oriented Seifert manifolds associated to an arbitrary complex finite dimensional simple Lie algebra $\mathfrak{g}$ in terms of the Seifert invariants and standard data for $\mathfrak{g}$. A main corollary is a determination of the full asymptotic expansions of these invariants for lens spaces in the limit of large quantum level. Our results are in agreement with the asymptotic expansion conjecture due to J. E. Andersen [1], [2].

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1 Introduction

In 1988 E. Witten [55] proposed new invariants $Z^G_k(X, L) \in \mathbb{C}$ of an arbitrary 3–manifold $X$ with an embedded colored link $L$ by quantizing the Chern–Simons field theory associated to a simple and simply connected compact Lie group $G$, $k$ being an arbitrary positive integer called the (quantum) level. (Here and in the following a 3–manifold is a closed oriented 3–manifold. In particular a Seifert manifold is an oriented Seifert manifold.) The invariant $Z^G_k(X, L)$ is given by a Feynman path integral over the (infinite dimensional) space of gauge equivalence classes of connections in a $G$ bundle over $X$. This integral should be understood in a formal way since, at the moment of writing, it seems that no mathematically rigorous definition is known, cf. [27, Sect. 20.2.A]. We will call the invariants $Z^G_k$ for the quantum $G$–invariants or the Witten invariants associated to $G$.

N. Reshetikhin and V. G. Turaev [44] were the first to give a rigorous construction of quantum invariants of 3–manifolds with embedded knots. In fact they constructed invariants $\tau^g_{r, \mathbb{C}}(X, L) \in \mathbb{C}$ of the pair $(X, L)$ by combinatorial means using surgery presentations of $(X, L)$ and irreducible representations of the quantum deformations of $\mathfrak{sl}_2(\mathbb{C})$ at certain roots of unity, $r$ being an integer $\geq 2$ associated to the order of the root of unity. Later quantum invariants $\tau^g_r(X, L) \in \mathbb{C}$ associated to other complex simple Lie algebras $\mathfrak{g}$ were constructed using representations of the quantum deformations of $\mathfrak{g}$ at ‘nice’ roots of unity, see [53]. We call $\tau^g_r$ for the quantum $\mathfrak{g}$–invariants or the RT–invariants associated to $\mathfrak{g}$.

Both in Witten’s approach and in the approach of Reshetikhin and Turaev the invariants are part of a family of topological quantum field theories (TQFT’s). This implies that the invariants are defined for compact oriented 3–dimensional cobordisms (perhaps with some extra structure on the boundary), and satisfy certain cut-and-paste axioms, see [4], [11], [42], [52]. The TQFT’s of Reshetikhin and Turaev can from an algebraic point of view be given a more general formulation by using so-called modular (tensor) categories [52]. The representation theory of the quantum deformations of $\mathfrak{g}$ at certain roots of unity, $\mathfrak{g}$ an arbitrary finite dimensional complex simple Lie algebra, induces such modular categories, see e.g. [32], [9], [35].

The invariants of Witten are defined by means of a path integral as stated above. A natural way in physics to obtain information about quantities defined by means of such path integrals is to study their perturbative (or asymptotic) expansion for large level $k$. In fact, by using stationary phase approximation
techniques together with path integral arguments Witten was able [55] to express the leading large $k$ asymptotics (or the so-called semiclassical approximation) of $Z_k^G(X)$ as a sum over the set of stationary points for the Chern–Simons functional. The terms in this sum are expressed by such topological/geometric invariants as Chern–Simons invariants, Reidemeister torsions and spectral flows, so here we see a way to extract topological information from the invariants. A full asymptotic expansion of Witten’s invariant is expected on the basis of a full perturbative analysis of the Feynman path integral, see [7], [8], [6].

The first rigorous verifications of the conjectured formula for the semiclassical approximation were given, partly by Freed and Gompf [13] presenting a large amount of computer calculations for the $SU(2)$–invariants of lens spaces and some 3–fibered Seifeit manifolds, and about the same time by Jeffrey [26] and Garoufalidis [14] who independently gave exact calculations of the semiclassical approximation of the $SU(2)$–invariants of lens spaces. Jeffrey also verified parts of the conjecture for the semiclassical approximation of $Z_k^G(X)$ for $G$ arbitrary and $X$ belonging to a class of mapping tori of the torus.

It is generally believed that the family of TQFT’s of Reshetikhin and Turaev is a mathematical realization of Witten family of TQFT’s. This belief has together with the above works on the perturbative expansion of Witten’s invariants led to a detailed conjecture, the asymptotic expansion conjecture (AEC), which specifies the asymptotic behaviour of the RT–invariants. The AEC was proposed by Andersen in [1], where he proved it for mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least two using the gauge theoretic approach to the quantum invariants.

Let us give an outline of the results obtained in this paper. Firstly, we determine formulas for the invariants $\tau_r^g$ of all Seifert manifolds in terms of the Seifert invariants and standard data for $\mathfrak{g}$, $\mathfrak{g}$ being an arbitrary complex finite dimensional simple Lie algebra, cf. Theorem 4.3. Theorem 4.3 is a generalization of [17, Theorem 8.4]. For a certain subclass of the Seifert manifolds, containing all the Seifert fibered integral homology spheres, we simplify the expression for the invariants considerable, cf. Theorem 4.8. This result is a generalization of [33, Formula (4.2)]. Secondly, we analyse more carefully the invariants $\tau_r^g(X)$ for $X$ any lens space. Theorem 5.1 gives the result for any lens space and any of the invariants $\tau_r^g$. Proposition 5.2 gives more compact expressions for $\tau_r^g(L(p,q))$ in case $r$ and $p$ are coprime. A main corollary of Theorem 5.1, Corollary 5.5, is a determination of the large $r$ asymptotics of the quantum $\mathfrak{g}$–invariants of the lens spaces. The result is in agreement with the AEC, and leads together with the AEC to a Conjecture 5.6 for the Chern–Simons invariants of the flat $G$ connections on any lens space, $G$ being an arbitrary simply connected, compact
simple Lie group. All the results for lens spaces are generalizations of results in [26], which considered the \( g = \mathfrak{sl}_2(\mathbb{C}) \) case. To be precise Theorem 5.1 generalizes [26, Theorem 3.4], Proposition 5.2 generalizes [26, Theorem 3.7], and Corollary 5.5 is a generalization of [26, Formula (5.7)].

We have via recent private communication learned that J. E. Andersen has for the groups \( G = SU(n) \) proved the asymptotic expansion conjecture for all closed 3–manifolds via the gauge theoretic approach. The proof involves asymptotics of Hitchin’s connection over Teichmüller space, approximations to all orders, of the boundary states of handle bodies and techniques similar to the ones presented in [3]. Where Andersen works with the gauge theoretic definition of the quantum invariant, we work with the definition of Reshetikhin and Turaev and our proof of the AEC for lens spaces is very different from Andersen’s general proof in the \( SU(n) \)–case.

A major part of the paper is concerned with studying a certain family of finite dimensional complex representations \( R^\theta_{g,r} \) of \( SL(2,\mathbb{Z}) \), one representation for each pair \( g,r \). These representations are known from the study of theta functions and modular forms in connection with the study of affine Lie algebras, cf. [29], [28, Sect. 13]. They also play a fundamental role in conformal field theory and (therefore) in the Chern–Simons TQFT’s of Witten, see e.g. [15], [54], [55].

In case \( g = \mathfrak{sl}_2(\mathbb{C}) \), Jeffrey [25], [26] has determined a nice formula for \( R^\theta(U) \) in terms of the entries in \( U \in SL(2,\mathbb{Z}) \) and the integer \( r \). Theorem 2.6 is a generalization of Jeffrey’s result to arbitrary \( g \), compare with [26, Sect. 2]. The representations \( R^\theta_{g,r} \) are of interest when calculating the RT–invariants of the Seifert manifolds since certain matrices, which can be expressed through these representations, enter into the formulas of the invariants.

The paper is organized as follows. In Sect. 2 we derive formulas for the representations \( R^\theta_{g,r} \). Sect. 3 is a short section intended to introduce notation for the Seifert manifolds. Moreover, we recall surgery presentations for these manifolds due to Montesinos [39]. In Sect. 4 we recall the formulas for the RT–invariants of the Seifert manifolds for an arbitrary modular category. These formulas are then used together with the results in Sect. 2 to calculate the \( g \)–invariants of the Seifert manifolds. In Sect. 5 we analyse the case of lens spaces more carefully. In the final Sect 6 we state a rational surgery formula for the quantum \( g \)–invariants, specializing a rational surgery formula [17, Theorem 5.3] for the RT–invariants associated to an arbitrary modular category. Besides an appendix is added presenting two related proofs of a reciprocity formula for Gauss sums, Proposition 2.2, which plays a vital role in this paper.

After having finished this work, we learned via private communication that Marino has obtained a similar result as ours Theorem 4.8 in case the Siefert
manifold has base $S^2$ and the Lie algebra $\mathfrak{g}$ is simply laced, cf. [38, Formula (4.11)]. It seems that the result Theorem 2.6 (in the form of the first formula in Corollary 2.7) has been known in the mathematical physics literature for some time at least for the simply laced case, cf. [38, Formula (2.5)], [45, Formula (1.6)]. We have, however, not been able to find any proof of this result in the literature.

This paper is an extensive expansion of the paper [19] and gives also the details left out in that paper, in particular the proof of the main Lemma 2.4.

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2 Formulas for the representations of $\text{SL}(2, \mathbb{Z})$.

In this section we analyse a family of unitary representations of $\text{SL}(2, \mathbb{Z})$ associated with a complex finite dimensional simple Lie algebra $\mathfrak{g}$. First let us fix some notation for $\mathfrak{g}$. (For details about standard material for Lie algebras we refer to [22].) Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\mathfrak{h}^*_\mathbb{R}$ be the $\mathbb{R}$–vector space spanned by the roots. We let $X$ and $Y$ be the weight lattice and the root lattice respectively. Let $\langle \cdot, \cdot \rangle$ be the standard inner product in $\mathfrak{h}^*_\mathbb{R}$ normalized such that the long roots have length $\sqrt{2}$. (That is, $\langle \cdot, \cdot \rangle$ is proportional to the inner product induced by the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$.) Then the short
roots have length $\sqrt{2/m}$, where $m = 1$ if $\mathfrak{g}$ is simply laced, i.e. belongs to the series $A$, $D$ or $E$, $m = 2$ if $\mathfrak{g}$ belongs to the series $B$ or $C$ or is of type $F_4$, and $m = 3$ if $\mathfrak{g}$ is of type $G_2$. In the following, the symbol $m$ will be reserved for this number. For $x \in \mathfrak{h}_\mathbb{R}^* \setminus \{0\}$, we let $x^\vee = 2x/\langle x, x \rangle$. If $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ is an arbitrary set of simple (basis) roots and if $\{\lambda_1, \lambda_2, \ldots, \lambda_l\} \subseteq \mathfrak{h}_\mathbb{R}^*$ is the set of fundamental dominant weights relative to $\Pi$, i.e. $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, then $X$ is the $\mathbb{Z}$-lattice generated by $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ and $Y$ is the $\mathbb{Z}$-lattice generated by $\Pi$. Let $\Delta$ be the set of roots. Then $\{\alpha^\vee \mid \alpha \in \Delta\}$ are the so-called dual roots or coroots (relative to our inner product $\langle \cdot, \cdot \rangle$). (So in this paper a coroot is in $\mathfrak{h}_\mathbb{R}^*$ and not in $\mathfrak{h}$.) The coroot lattice $Y^\vee$ is the $\mathbb{Z}$-lattice generated by $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ for an arbitrary set of simple roots $\{\alpha_1, \ldots, \alpha_l\}$. Recall that

\[
X = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in Y^\vee\},
\]

We note that $X$ and $Y^\vee$ are dual to each other. It is obvious from the above, that the Weyl group $W$ preserves the lattices $X$, $Y$, and $Y^\vee$. Let us also note the following facts: For all $x, y \in Y^\vee$ we have

\[
\langle x, y \rangle \in \mathbb{Z}, \quad \langle x, x \rangle \in 2\mathbb{Z}.
\]

There exists a (least) positive integer $D$ such that we for all $\mu, \xi \in X$ have

\[
\langle \mu, \xi \rangle \in \frac{1}{D}\mathbb{Z}, \quad \langle \mu, \mu \rangle \in \frac{2}{D}\mathbb{Z}.
\]

Let us fix a set $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ of simple roots in the following and let $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ be the set of fundamental dominant weights relative to $\Pi$. The (open) fundamental Weyl chamber (relative to $\Pi$) is the set

\[
\text{FWC} = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, \alpha_i \rangle > 0, i = 1, \ldots, l\}.
\]

We let $C = \overline{\text{FWC}}$ be the topological closure of FWC. For a positive integer $k$, the $k$-alcove (relative to $\Pi$) is the (closed) set

\[
C_k = \{x \in C \mid \langle x, \theta \rangle \leq k\},
\]

where $\theta$ is the highest root of $\mathfrak{g}$ (relative to $\Pi$). (Note that $\theta$ is a long root). We will also need the following sets:

\[
Q_k = \{c_1\lambda_1 + \ldots + c_l\lambda_l \mid c_1, \ldots, c_l \in [0, k]\},
\]

\[
P_k = \{c_1\alpha_1^\vee + \ldots + c_l\alpha_l^\vee \mid c_1, \ldots, c_l \in [0, k]\}.
\]

Let $W_k^\text{aff} = W \ltimes kY^\vee$ be the affine Weyl group acting on $\mathfrak{h}_\mathbb{R}^*$ in the usual sense ($kY^\vee$ acting by translations). It follows that $P_k$ is a fundamental domain of the group $kY^\vee$. Moreover, it is well-known (see [28, Sect. 6] for a proof) that
For fixed \( \lambda \) and we put \(| \cdot |_{\lambda} \) such that \( M \) is precisely tiled by \( k \)-alcoves. We let \( \Delta_+ \) be the set of positive roots (relative to \( \Pi \)). For \( \alpha \in \Delta_+ \) and \( n \in \mathbb{Z} \) we let
\[
H^k_{\alpha,n} = \{ x \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle x, \alpha \rangle = nk \},
\]
and we put
\[
H^k = \cup_{\alpha \in \Delta_+, n \in \mathbb{Z}} H^k_{\alpha,n} = \{ x \in \mathfrak{h}^*_{\mathbb{R}} \mid \exists \alpha \in \Delta_+ : \langle x, \alpha \rangle \in k\mathbb{Z} \}.
\]

For fixed \( \lambda_0, \lambda_1 \in X \) and fixed integers \( b \) and \( a \neq 0 \) we let
\[
g^{a,b,k}_{\lambda_0,\lambda_1}(\lambda) = \sum_{\mu \in Y^\vee/aY^\vee, w,w' \in W} \det(ww') \exp \left( \frac{\pi \sqrt{-1} b}{ak} \langle \lambda + k\mu \rangle^2 \right)
+ 2\pi \sqrt{-1} \frac{a}{k} \langle \lambda + k\mu, -w(\lambda_0) - aw'(\lambda_1) \rangle
\]
for \( \lambda \in X \), where \( | \cdot | \) is the norm associated to \( \langle \cdot, \cdot \rangle \). (Note that by (1) and (2) the summand in the expression of \( g^{a,b,k}_{\lambda_0,\lambda_1} \) only depends on \( \mu \) (mod \( aY^\vee \)).) In particular,
\[
g^{1,b,k}_{\lambda_0,\lambda_1}(\lambda) = \sum_{w,w' \in W} \det(ww') \exp \left( \frac{\pi \sqrt{-1}}{k} (b|\lambda|^2 + \langle \lambda, -2w(\lambda_0) - 2w'(\lambda_1) \rangle) \right).
\]

By using similar arguments as in [26, Sect. 4] (see also [35]), we have

**Proposition 2.1** The map \( g^{a,b,k}_{\lambda_0,\lambda_1} : X \to \mathbb{C} \) is invariant under the action of the affine Weyl group \( W^a_k \). Moreover, \( g^{a,b,k}_{\lambda_0,\lambda_1}(\lambda) = 0 \) for any \( \lambda \in X \cap H^k \).

**Proof** The invariance under the action by an element \( u \in W \) follows by the fact that \( u \) is orthogonal together with the identity \( \det(ww') = \det(w^{-1}ww^{-1}w') \) and the fact that \( W \) preserves \( Y^\vee/aY^\vee \) (since it preserves \( Y^\vee \)). The invariance under the action by an element \( kx, x \in Y^\vee, \) is obvious. (For \( |a| = 1 \), use (1) and (2).) To prove the last claim, let \( \lambda \in X \cap H^k_{\alpha,\mu} \) for a positive root \( \alpha \) and an integer \( n \), and let \( s_\alpha \) be the reflection in \( \alpha \). Fix a \( w \in W \) and get
\[
\langle \lambda + k\mu, -w(\lambda_0) - as_\alpha^{-1}w'(\lambda_1) \rangle = \langle \lambda + k\mu, -w(\lambda_0) - w'(\lambda_1) \rangle + a\langle \lambda + k\mu, \alpha \rangle \langle \alpha^\vee, w'(\lambda_1) \rangle,
\]
where \( \langle \alpha^\vee, w'(\lambda_1) \rangle \in \mathbb{Z}, \) and \( \langle \lambda + k\mu, \alpha \rangle \in k\mathbb{Z}, \) so
\[
\sum_{w' \in W} \det(w') \exp \left( \frac{2\pi \sqrt{-1}}{ak} \langle \lambda + k\mu, -w(\lambda_0) - aw'(\lambda_1) \rangle \right) = 0.
\]
In the calculations to follow, a multi-dimensional reciprocity formula for Gauss sums plays a crucial role. Let $V$ be a real vector space of dimension $l$ with inner product $\langle \cdot, \cdot \rangle$, $\Lambda$ a lattice in $V$ and $\Lambda^*$ the dual lattice. For an integer $r$, a self-adjoint automorphism $f: V \to V$, and an element $\psi \in V$, we assume
\[
\frac{r}{2} \langle \lambda, f(\lambda) \rangle, \quad \ell \langle \lambda, f(\eta) \rangle, \quad r \langle \lambda, \psi \rangle \in \mathbb{Z}, \quad \forall \lambda, \eta \in \Lambda, \quad (5)
\]
and $f(\Lambda^*) \subseteq \Lambda^*$. Then we have

**Proposition 2.2 (Reciprocity formula for Gauss sums)**
\[
\operatorname{vol}(\Lambda^*) \sum_{\lambda \in \Lambda/r \Lambda} \exp \left( \frac{\pi \sqrt{-1}}{r} \langle \lambda, f(\lambda) \rangle \right) \exp \left( 2\pi \sqrt{-1} \langle \lambda, \psi \rangle \right) = \left( \det \frac{f}{\sqrt{-1}} \right) r^{l/2} \sum_{\mu \in \Lambda^*/f(\Lambda^*)} \exp \left( -\pi r \sqrt{-1} \langle \mu + \psi, f^{-1}(\mu + \psi) \rangle \right). \]

For a proof, see [25, Sect. 2]. For the convenience of the reader, we sketch Jeffrey's proof in the appendix. Moreover, we present in the appendix a slightly alternative proof. Both arguments rely on the Poisson resummation formula.

Below we will use the reciprocity formula, Proposition 2.2, with $\Lambda = X$, the dual lattice being the coroot lattice $X^\vee$.

The group $\text{SL}(2, \mathbb{Z})$ is generated by two matrices
\[
\Xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (6)
\]
We note that
\[
\Xi^2 = (\Xi \Theta)^3 = -1. \quad (7)
\]
For a tuple of integers $C = (m_1, \ldots, m_t)$ we let $C_k = (m_1, \ldots, m_k)$ and
\[
B_k^C = \begin{pmatrix} a_k^C & b_k^C \\ c_k^C & d_k^C \end{pmatrix} = \Theta^{m_k} \Xi \Theta^{m_{k-1}} \Xi \ldots \Xi \Theta^{m_1} \Xi \quad (8)
\]
for $k = 1, 2, \ldots, t$, and let $B^C = B_t^C$. Moreover, we put
\[
a_0^C = d_0^C = 1, \quad b_0^C = c_0^C = 0.
\]
We say that $C$ has length $|C| = t$. If it is clear from the context what $C$ is we write $a_k$ for $a_k^C$ etc. By [26, Proposition 2.5] the elements $a_i, b_i, c_i, d_i$ satisfy the recurrence relations
\[
a_k = m_k a_{k-1} - c_{k-1}, \quad c_k = a_{k-1}, \quad b_k = m_k b_{k-1} - d_{k-1}, \quad d_k = b_{k-1} \quad (9)
\]
for \( k = 1, 2, \ldots, t \). Moreover,
\[
\frac{b_k}{a_k} = -\left( \frac{1}{a_1} + \frac{1}{a_2 a_1} + \cdots + \frac{1}{a_k a_{k-1}} \right)
\] (10)
and \((m_1, \ldots, m_k)\) is a continued fraction expansion of \( a_k/b_k, \ k = 1, 2, \ldots, t\), i.e.
\[
a_k \frac{b_k}{a_k} = m_k - \frac{1}{m_{k-1} - \frac{1}{\cdots - \frac{1}{m_1}}}
\] (11)

Let in the following \( \kappa \in \mathbb{Z}_{>0} \) be fixed. We have a representation \( \mathcal{R} = \mathcal{R}^0_\kappa \) of \( \text{SL}(2, \mathbb{Z}) \) given by
\[
\mathcal{R}(\Xi)_{\lambda \mu} = \sqrt{-1}^{\frac{|\Delta_+|}{\kappa/2}} \left| \frac{\text{vol}(X)}{\text{vol}(Y^\vee)} \right|^{1/2} \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa} \langle w(\lambda), \mu \rangle \right),
\]
\[
\mathcal{R}(\Theta)_{\lambda \mu} = \delta_{\lambda \mu} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} |\lambda|^2 - \frac{\pi \sqrt{-1}}{\hbar^\vee} |\rho|^2 \right)
\] (12)
for \( \lambda, \mu \in \text{int}(C_\kappa) \cap X \). Here \( \rho \) is half the sum of positive roots. In the following we also write \( \tilde{U} \) for \( \mathcal{R}(U) \). Note that the matrices \( \mathcal{R}(\Xi) \) and \( \mathcal{R}(\Theta) \) are symmetric (in fact \( \mathcal{R}(\Theta) \) is diagonal).

**Remark 2.3** It is well-known that the representations \( \mathcal{R} \) are unitary. Let us here give a few references to this fact. Unitarity of \( \mathcal{R}(\Theta) \) follows immediately from (12). By [9, Theorem 3.3.20] we have
\[
\mathcal{R}(\Xi)_{\lambda \mu} = s_{\lambda \mu},
\] (13)
where \( s \) is defined in [9, Formula (3.1.16)], and \( \bar{\cdot} \) is complex conjugation. (In the expression for \( s_{\lambda \mu} \) in [9, Theorem 3.3.20] there is a minor mistake; one has to replace \( i^{|\Delta_+|} \) by \( i^{-|\Delta_+|} \).) By the proof of [9, Theorem 3.3.20] it follows that
\[
{s_{\lambda \mu}^2 = \delta_{\lambda \mu}^*}. \]
(14)
Here \( \lambda^* = -w_0(\lambda - \rho) + \rho = -w_0(\lambda) \), where \( w_0 \) is the longest element in \( W \) (relative to our set of simple roots \( \Pi \)) and where we use that \( w_0(\rho) = -\rho \), see [23, Sect. 1.8]. By [23, Sect. 1.8] we have \( \det(w_0) = (-1)^{|\Delta_+|} \). By this and (12) we get
\[
\mathcal{R}(\Xi)_{\lambda^* \mu} = \overline{\mathcal{R}(\Xi)_{\lambda \mu}}.
\] (15)
Now unitarity of \( \mathcal{R}(\Xi) \) follows from (13), (14), and (15). The original reference for the unitarity of \( \mathcal{R} \) seems to be [15, Appendix].
One should note that the expressions for the entries of $R(Ξ)$ and $R(Θ)$ are well-defined for all $λ, µ ∈ X$. Note also that if $λ$ or $µ$ is an elements of $X$ belonging to the boundary of $C_κ$, then $R(Ξ)_{λµ} = 0$ (use the same argument as in the proof of the final claim of Proposition 2.1). This observation allows us to shift between int($C_κ$) ∩ $X$ and $C_κ$ ∩ $X$ as summation index set in formulas below. Following Jeffrey [25, Sect. 2], [26, Sect. 2] we consider

$$T_{λ_0,λ_{t+1}}^C = \sum_{λ_1,...,λ_t ∈ C_κ ∩ X} \tilde{Ξ}_{λ_{t+1}+λ_t} Θ_{λ_t} m_{λ_{t+1}-1} Θ_{λ_{t-1}} m_{λ_{t-2}} · · · Θ_{λ_1} m_{λ_0}$$

for $λ_0, λ_{t+1} ∈ X$, where we write $Θ_λ$ for $Θ_{λλ}$. Then we have the following generalization of [26, Lemma 2.6]:

**Lemma 2.4** Assume that $C = (m_1, ..., m_t)$ is a sequence of integers such that $a_k$ is nonzero for $k = 1, ..., t$. Then

$$T_{λ_0,λ_{t+1}}^C = K_{λ_0}^C ∑_{w ∈ W} det(w) ∑_{µ ∈ Y'/a_1 Y'} exp \left( -\frac{π\sqrt{-1}c_t}{a_t κ} |λ_{t+1} + κ µ + \frac{w(λ_0)}{c_t}|^2 \right)$$

for all $λ_0, λ_{t+1} ∈ X$, where

$$K_{λ_0}^C = \frac{\sqrt{-1}^{|t+1|∆_+|}}{(κ|a_t|)^{1/2} vol(Y')} \ ζ^t D_t \ exp \left( -\frac{π\sqrt{-1}}{h'} (\sum_{i=1}^t m_i)|ρ|^2 \right)$$

$$× \ exp \left( -\frac{π\sqrt{-1}}{κ} \left( \sum_{i=1}^{t-1} \frac{1}{a_i a_{i-1} d_1} \right) |λ_0|^2 \right).$$

Here $ζ = exp \frac{π\sqrt{-1}}{2}$ and $D_t = \text{sign}(a_0 a_1) + · · · + \text{sign}(a_{t-1} a_t)$.

**Proof** We prove the proposition by induction on the length of $C$. First consider $T_{λ_0,λ_2}^C$, $λ_0, λ_2 ∈ X$. Since $Ξ$ is symmetric we have

$$T_{λ_0,λ_2}^C = \sum_{λ_1 ∈ C_κ ∩ X} \tilde{Ξ}_{λ_2 λ_1} Θ_{λ_1} m_{λ_1} = \frac{(-1)^{|Δ_+|}}{κ'} \ \frac{vol(X)}{vol(Y')} \ \ exp \left( -\frac{π\sqrt{-1}}{h'} m_1 |ρ|^2 \right) \ \ ∑_{λ ∈ C_κ ∩ X} g_{λ_0,λ_2}^{m_1,κ} (λ),$$

where $g_{λ_0,λ_2}^{m_1,κ}$ is given by (4). The closure $Q_m$ of $Q_m$ is precisely tiled by 1–alcoves. Moreover, if $Q_N$ is tiled by $n$ 1–alcoves, then $Q_{kN}$ is tiled by $n$ $k$–alcoves. We also have that if $Q_N$ is tiled by $n$ 1–alcoves, then $Q_{kN}$ is tiled by $k^n$ 1–alcoves. Note that vol($Q_N$) = $N'$ vol($X$) and vol($C_1$) = vol($P_1$)/|W|. We use here that $C_1$ is a fundamental domain for the action of $W_1^{aff}$, while $P_1$
is a fundamental domain for the action of $Y^\vee$. Therefore, if $\bar{Q}_N$ is precisely tiled by $n$ 1–alcoves, then $n = \text{vol}(\bar{Q}_N)/\text{vol}(C_1) = N^t|W|/\text{vol}(X)/\text{vol}(Y^\vee)$. Let $N = mD$, where $D$ is the integer from (3). By Proposition 2.1 and the above we get

\[
\sum_{\lambda \in X/N\kappa X} g_{\lambda_0,\lambda_2}^{1, m_1, \kappa}(\lambda) = N^t|W|/\text{vol}(X)/\text{vol}(Y^\vee) \sum_{\lambda \in C_0 \cap X} g_{\lambda_0,\lambda_2}^{1, m_1, \kappa}(\lambda).
\]

Therefore

\[
T_{\lambda_0,\lambda_2}^{C_1} = \frac{(-1)^{|\Delta_+|}}{\kappa^t N^t|W|} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1|\rho|^2 \right) \sum_{w,w' \in W} \det(ww') \\
\times \sum_{\lambda \in X/N\kappa X} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} (m_1|\lambda|^2 + \langle \lambda, -2w(\lambda_0) - 2w'(\lambda_2) \rangle) \right).
\]

Let $f = m_1 \text{Nid}_{b_0^\vee}$, $\psi = \psi(w,w') = -\frac{1}{\kappa}(w(\lambda_0) + w'(\lambda_2))$, $w, w' \in W$, and $r = N\kappa$. Then, by (1), (2), and (3), the assumptions (5) are satisfied, so by Proposition 2.2 we get

\[
T_{\lambda_0,\lambda_2}^{C_1} = \frac{(-1)^{|\Delta_+|}}{r^t|W|} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1|\rho|^2 \right) \sum_{w,w' \in W} \det(ww') \\
\times \sum_{\lambda \in X/N\kappa X} \exp \left( \frac{\pi \sqrt{-1}}{r} \langle \lambda, f(\lambda) \rangle \right) \exp \left( 2\pi \sqrt{-1} \langle \lambda, \psi \rangle \right)
\]

\[
= \text{vol}(Y^\vee)^{-1} \left( \frac{(-1)^{|\Delta_+|}}{r^t|W|} \exp \left( \frac{f}{\sqrt{-1}} \right) \right)^{-1/2} \frac{r^{1/2}}{\sqrt{-1}} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1|\rho|^2 \right) \\
\times \sum_{w,w' \in W} \det(ww') \sum_{\mu \in Y^\vee/f(Y^\vee)} \exp \left( -\pi \sqrt{-1} r \langle \mu + \psi, f^{-1}(\mu + \psi) \rangle \right)
\]

\[
= \frac{(-1)^{|\Delta_+|}}{(\kappa N)^{1/2} \text{vol}(Y^\vee)|W|} \left( \frac{\sqrt{-1}}{m_1 N} \right)^{1/2} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1|\rho|^2 \right) \\
\times \sum_{w,w' \in W} \det(ww') \sum_{\mu \in Y^\vee/m_1 NY^\vee} \exp \left( -\frac{\pi \sqrt{-1}}{m_1} |\mu + \psi|^2 \right).
\]

Here $\exp \left( -\frac{\pi \sqrt{-1}}{m_1} |\mu + \psi|^2 \right)$ is invariant under $\mu \mapsto \mu + m_1 \alpha$, $\mu, \alpha \in Y^\vee$, by (1) and (2), so

\[
T_{\lambda_0,\lambda_2}^{C_1} = \frac{(-1)^{|\Delta_+|}}{(\kappa)^{1/2} \text{vol}(Y^\vee)|W|} \left( \frac{\sqrt{-1}}{m_1} \right)^{1/2} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1|\rho|^2 \right) \\
\times \sum_{w,w' \in W} \det(ww') \sum_{\mu \in Y^\vee/m_1 Y^\vee} \exp \left( -\frac{\pi \sqrt{-1} \kappa}{m_1} |\mu + \psi|^2 \right)
\]

\[11\]
\[ T_{\lambda_0, \lambda_{t+1}} = \sum_{\lambda_t \in C_t \cap X} \tilde{\Theta}_{\lambda_{t+1}, \lambda_t} \tilde{z}_{\lambda_{t+1}} \theta_{\lambda_t} t_{\lambda_0, \lambda_t}^{C_t-1} \]

Assume next by induction that the lemma is true for all sequences of length \( t - 1 \). Then we get for \( \lambda_0, \lambda_{t+1} \in X \) that

\[ \begin{aligned} T_{\lambda_0, \lambda_{t+1}} &= \frac{(-1)^{|\Delta_+|}}{\kappa^{t/2} \text{vol}(Y^\vee) |W|} \left( \frac{\sqrt{-1}}{m_1} \right)^{t/2} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1 |\rho|^2 \right) \sum_{w, w' \in W} \det(w w') \\
&\times \sum_{\mu \in Y^\vee / m_1 Y^\vee} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1 |\rho|^2 \right) \\
&\times \sum_{\mu \in Y^\vee / m_1 Y^\vee} \exp \left( -\frac{\pi \sqrt{-1}}{m_1 \kappa} |\mu - w' - 1 w(\lambda_0) - \lambda_2|^2 \right) \\
&= \frac{(-1)^{|\Delta_+|}}{\kappa^{t/2} \text{vol}(Y^\vee) |W|} \left( \frac{\sqrt{-1}}{m_1} \right)^{t/2} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_1 |\rho|^2 \right) \\
&\times \sum_{w, w' \in W} \det(w w') \\
&\times \sum_{\mu \in Y^\vee / m_1 Y^\vee} \exp \left( -\frac{\pi \sqrt{-1}}{m_1 \kappa} |\mu - w(\lambda_0) - \lambda_2|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} \left| \sum_{i=0}^{t-1} a_i \left( m_i |\lambda_i|^2 + \langle \lambda_i, -2w(\lambda_{t+1}) \rangle \right) \right| \right) \\
&\times \sum_{w \in W} \det(w) \sum_{\mu \in Y^\vee / m_1 Y^\vee} \exp \left( -\frac{\pi \sqrt{-1}}{m_1 \kappa} |\mu - w(\lambda_0) - \lambda_2|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} \left| \sum_{i=0}^{t-1} a_i \left( m_i |\lambda_i|^2 + \langle \lambda_i, -2w(\lambda_{t+1}) \rangle \right) \right) \right). \]

Finally we replace \( \mu \) by \(-\mu\), and use that \( m_1 = a_1, c_1 = 1 \).
By (1) and (2) we have

\[
\sum_{w, w' \in W} \det(ww') \sum_{\lambda \in C_\kappa \cap X} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} (\langle \lambda_t, m_t \lambda_t \rangle + \langle \lambda_t, -2w'(\lambda_{t+1}) \rangle) \right) \\
\times \sum_{\mu \in Y^\vee / a_{t-1}Y^\vee} \exp \left( -\frac{\pi \sqrt{-1} c_{t-1}}{a_{t-1} \kappa} |\lambda_t + \kappa \mu + \frac{w(\lambda_0)}{c_{t-1}}|^2 \right) \\
= \sum_{w, w' \in W} \det(ww') \sum_{\lambda \in C_\kappa \cap X} \sum_{\mu \in Y^\vee / a_{t-1}Y^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} (m_t |\lambda_t + \kappa \mu|^2 + \langle \lambda_t + \kappa \mu, -2w'(\lambda_{t+1}) \rangle) \right) \\
\times \exp \left( -\frac{\pi \sqrt{-1} c_{t-1}}{a_{t-1} \kappa} |\lambda_t + \kappa \mu|^2 \right) \\
+ \frac{2\pi \sqrt{-1}}{a_{t-1} \kappa} (\lambda_t + \kappa \mu, -w(\lambda_0)) - \frac{\pi \sqrt{-1}}{a_{t-1} c_{t-1} \kappa} |\lambda_0|^2 \right) \\
= \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} c_{t-1} \kappa} |\lambda_0|^2 \right) \sum_{\lambda \in C_\kappa \cap X} g_{\lambda_0, \lambda_{t+1}}(\lambda_t).
\]

We want to alter the sum \( \sum_{\lambda \in C_\kappa \cap X} g_{\lambda_0, \lambda_{t+1}}(\lambda_t) \) by applying the reciprocity formula, Proposition 2.2. Before we can do this we need to alter this sum using some symmetries. By Proposition 2.1 and the fact that \( C_\kappa \) and \( P_\kappa \) are fundamental domains of respectively \( W^\text{aff}_\kappa \) and \( \kappa Y^\vee \), we get

\[
|W| \sum_{\lambda \in C_\kappa \cap X} g_{\lambda_0, \lambda_{t+1}}(\lambda) = \sum_{P_\kappa \cap X} g_{\lambda_0, \lambda_{t+1}}(\lambda).
\]

Since the map \( Y^\vee / a_{t-1}Y^\vee \times (P_\kappa \cap X) \to P_{a_{t-1} \kappa} \cap X \), \((\mu, \lambda) \mapsto \lambda + \kappa \mu\), defines a bijection, we therefore get

\[
\sum_{\lambda \in C_\kappa \cap X} g_{\lambda_0, \lambda_{t+1}}(\lambda) = \frac{1}{|W|} \sum_{\lambda \in P_{a_{t-1} \kappa} \cap X} h_{\lambda_0, \lambda_{t+1}}(\lambda),
\]

where

\[
h_{\lambda_0, \lambda_{t+1}}(\lambda) = \sum_{w, w' \in W} \det(ww') \exp \left( \frac{\pi \sqrt{-1} a_{t-1} \kappa}{a_{t-1} \kappa} |\lambda|^2 \right) \\
\times \exp \left( \frac{2\pi \sqrt{-1}}{a_{t-1} \kappa} (\lambda_t - w(\lambda_0) - a_{t-1} w'(\lambda_{t+1})) \right). \]

Exactly as in the proof of Proposition 2.1 we get that \( h_{\lambda_0, \lambda_{t+1}} : X \to \mathbb{C} \) is invariant under the action of \( W^\text{aff}_{a_{t-1} \kappa} \), and, moreover, that \( h_{\lambda_0, \lambda_{t+1}}(\lambda) = 0 \) for
all $\lambda \in X \cap H^\kappa$. Now let $N = mD$ as in the first part of the induction and get that

$$\sum_{\lambda \in X/\kappa a_{t-1}N} h_{\lambda_0, \lambda_{t+1}} = \frac{N^t|W| \vol(X)}{\vol(Y)} \sum_{\lambda \in C_{t-1} \cap X} h_{\lambda_0, \lambda_{t+1}}(\lambda),$$

where we use that $Q_{\kappa a_{t-1}N}$ is precisely tiled by $\kappa a_{t-1}$-alcoves, in fact by $N^t|W| \vol(X)/\vol(Y)$ of these alcoves. Therefore

$$\sum_{\lambda \in C_{t} \cap X} g_{\lambda_0, \lambda_{t+1}} = \frac{\vol(Y)}{N^t|W| \vol(X)} \sum_{\lambda \in X/\kappa a_{t-1}N} h_{\lambda_0, \lambda_{t+1}}(\lambda).$$

Putting everything together we get

$$T_{\kappa a_{t-1}, \kappa a_{t}} = \frac{R_{\kappa a_{t-1}} \sqrt{\frac{1}{a_{t-1} a_{t}}}}{K_{\kappa a_{t-1}} \sqrt{\frac{1}{a_{t-1} a_{t}}}} \frac{\vol(Y \vee)}{\vol(X)} \frac{\vol(Y)^{1/2}}{\vol(X)^{1/2}} \exp \left( -\pi \sqrt{\frac{-1}{a_{t-1} a_{t}}} |\lambda_0|^2 \right) \sum_{\lambda \in X/\kappa a_{t-1}N} h_{\lambda_0, \lambda_{t+1}}(\lambda).$$

We are now in a position where we can use the reciprocity formula. If we let $f = na_0\id_{N^t}$, $r = \kappa a_{t-1}N$, and $\psi = \psi(w, w') = -\frac{1}{\kappa a_{t-1}}(w(\lambda_0) + a_{t-1} a_{t} w'_{\kappa a_{t-1}}(\lambda_{t+1}))$, $w, w' \in W$, then

$$\sum_{\lambda \in X/\kappa a_{t-1}N} h_{\lambda_0, \lambda_{t+1}} = \sum_{w, w' \in W} \det(w w')$$

$$\times \sum_{\lambda \in X/\kappa a_{t-1}N} \exp \left( \frac{\pi \sqrt{-1}}{r} \lambda_0 \right) \exp \left( 2\pi \sqrt{-1} \langle \lambda, \psi \rangle \right).$$

By (1), (2), and (3) the assumptions (5) are satisfied, so by Proposition 2.2 we obtain

$$\sum_{\lambda \in X/\kappa a_{t-1}N} h_{\lambda_0, \lambda_{t+1}} = \vol(Y \vee)^{-1} \left( \frac{f}{\sqrt{-1}} \right)^{-1/2} r^{1/2} \sum_{w, w' \in W} \det(w w')$$

$$\times \sum_{\mu \in Y \vee / f(Y \vee)} \exp \left( -\pi r \sqrt{-1} \langle \mu + \psi, f(\mu + \psi) \rangle \right)$$

$$= \left( \frac{\sqrt{-1}}{a_t} \right)^{1/2} \frac{\kappa a_{t-1}}{\vol(Y \vee)} \sum_{w, w' \in W} \det(w w')$$

$$\times \sum_{\mu \in Y \vee / a_t N Y \vee} \exp \left( -\pi \kappa a_{t-1} \sqrt{-1} \langle \mu + \psi, f(\mu + \psi) \rangle \right).$$
Here \( \exp \left( -\frac{\pi \kappa a_{t-1} \sqrt{a_t}}{\kappa} |\mu + \psi|^2 \right) \) only depends on \( \mu \mod (a_t Y^\vee) \), so in total we get that

\[
\mathcal{T}_{\lambda_0, \lambda_{t+1}}^C = K_{\lambda_0}^{c_{t-1}} \mathcal{V}_{\lambda_{t-1}}^{1/2} \frac{1/\kappa}{|W|} \exp \left( -\frac{\pi \sqrt{-1}}{h Y^\vee} m_t |\rho|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} c_{t-1} \kappa} |\lambda_0|^2 \right)
\]

\[
\times \left( \sqrt{-1} \right)^{1/2} \frac{1/\kappa}{a_t} (\kappa a_{t-1})^{1/2} \sum_{w \in W} \det(w) \text{vol}(w)
\]

\[
\times \sum_{\mu \in Y^\vee/a_t Y^\vee} \exp \left( -\frac{\pi \kappa a_{t-1} \sqrt{a_t}}{\kappa \mu} |\mu + \psi|^2 \right),
\]

where we use that \( \text{vol}(X) = \text{vol}(Y^\vee)^{-1} \). In a similar way as in the first step of the induction we finally get

\[
\mathcal{T}_{\lambda_0, \lambda_{t+1}}^C = K_{\lambda_0}^{c_t} \sum_{w \in W} \det(w)
\]

\[
\times \sum_{\mu \in Y^\vee/a_t Y^\vee} \exp \left( -\frac{\pi c_t \sqrt{a_t}}{\kappa \mu} |\mu + \psi|^2 \right),
\]

where we use that \( a_{t-1} = c_t \). Here

\[
K_{\lambda_0}^{c_t} = K_{\lambda_0}^{c_{t-1}} \sqrt{-1} \frac{1/\kappa}{|W|} \exp \left( -\frac{\pi \sqrt{-1}}{h Y^\vee} m_t |\rho|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} c_{t-1} \kappa} |\lambda_0|^2 \right)
\]

\[
\times \left( \sqrt{-1} \right)^{1/2} \frac{1/\kappa}{a_t} (\kappa a_{t-1})^{1/2} \sum_{w \in W} \det(w) \text{vol}(w)
\]

\[
\times \exp \left( -\frac{\pi \sqrt{-1}}{h Y^\vee} \sum_{i=1}^t m_i |\rho|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} \left( \sum_{i=1}^{t-2} \frac{1}{a_{i-1} a_{i}} \right) |\lambda_0|^2 \right)
\]

\[
\times \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} c_{t-1} \kappa} |\lambda_0|^2 \right) \left( \frac{|a_{t-1}|}{|a_t|} \right)^{1/2} \kappa^{1/2} \sum_{i=1}^t \text{sign}(a_{t-1})
\]

\[
= \sqrt{-1} \frac{1/\kappa}{|W|} \exp \left( -\frac{\pi \sqrt{-1}}{h Y^\vee} \sum_{i=1}^t m_i |\rho|^2 \right)
\]

\[
\times \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} c_{t-1} \kappa} |\lambda_0|^2 \right) \left( \frac{|a_{t-1}|}{|a_t|} \right)^{1/2} \kappa^{1/2} \sum_{i=1}^t \text{sign}(a_{t-1})
\]

\[
= \sqrt{-1} \frac{1/\kappa}{|W|} \exp \left( -\frac{\pi \sqrt{-1}}{h Y^\vee} \sum_{i=1}^{t-1} \frac{1}{a_{i-1} a_{i}} |\lambda_0|^2 \right),
\]

where we use that \( c_{t-1} = a_{t-2} \).
We want to use the above result to find a simple expression for the entries of \( R(U) \) in terms of the entries of \( U \) and data for the Lie algebra \( \mathfrak{g} \). There is, however, a small hurdle to overcome because of the assumption on the \( a_k \)'s in Lemma 2.4. To this end we need

**Lemma 2.5** Let \( U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{\pm 1\} \) with \( c \neq 0 \). Then we can write

\[
U = V \Theta^n
\]

where \( n \in \mathbb{Z} \) and \( V \) is given in the following way: If \( a = 0 \) then \( V = \Xi \); if \( a \neq 0 \) then there exists a sequence of integers \( C \) such that \( V = B^C \) and such that \( a_k^C \neq 0, \ k = 1, 2, \ldots, |C| \).

**Proof** If \( a = 0 \) then \( 1 = \det(U) = -bc \) so \( c = -b = \pm 1 \). As element of PSL(2, \( \mathbb{Z} \)) we therefore have

\[
U = \begin{bmatrix} 0 & -1 \\ 1 & d' \end{bmatrix} = \Xi \Theta^{d'}
\]

where \( d' \in \{ \pm d \} \). Now assume that \( a \neq 0 \). By (7) we can find a tuple of integers \( C' = (m_1, \ldots, m_t) \) such that \( U = B^{C'} \). If \( a_i = a_i^{C'} \neq 0, \ i = 1, 2, \ldots, t \), we let \( C = C' \) and \( n = 0 \). Therefore assume this is not the case. Let

\[
i = \max \{ j \in \{1, 2, \ldots, t\} \mid a_j = 0 \}.
\]

Since \( a_i = a \neq 0 \) we have \( i < t \). As in the case \( a = 0 \) we have that \( B^{C_i} = \Xi \Theta^j \) for some \( j \in \mathbb{Z} \). If \( i = t - 1 \) then

\[
U = \Theta^{m_t} \Xi \Theta^j = \Theta^{m_t + j} = \begin{bmatrix} 1 & m_t + j \\ 0 & 1 \end{bmatrix}
\]

contradicting the fact that \( c \neq 0 \). Therefore \( i < t - 1 \) and \( U = W \Theta^{n'} \) with \( n' = m_{i+1} + j \) and

\[
W = \Theta^{m_1} \Xi \cdots \Theta^{m_{t-1}} \Xi.
\]

Let \( n_k = m_{i+k+1}, \ k = 1, 2, \ldots, t - i - 1 \), let \( C'' = (n_1, \ldots, n_{t-i-1}) \), and let \( a'_k = a_k^{C''} \) etc. Then \( W = B^{C''} \), and

\[
\begin{bmatrix} a_{i+k+1} & b_{i+k+1} \\ c_{i+k+1} & d_{i+k+1} \end{bmatrix} = B^{C'}_{i+k+1} = B^{C''}_k \Theta^{n'} = \begin{bmatrix} a'_k & b'_k \\ c'_k & d'_k \end{bmatrix} \begin{bmatrix} 1 & n' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a'_k & n'a'_k + b'_k \\ c'_k & n'c'_k + d'_k \end{bmatrix},
\]

so in particular \( a'_k = a_{i+k+1} \neq 0, \ k = 1, 2, \ldots, t - i - 1 \), by the maximality of \( i \). Therefore we can let \( n = n' \) and \( C = C'' \) \( \square \).
For the next theorem we need the Rademacher Phi function $\Phi$, which is defined on $\text{PSL}(2, \mathbb{Z})$ by

$$\Phi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} \frac{a+d}{b} - 12(\text{sign}(c))s(d, c) & , c \neq 0, \\ \frac{b}{c} & , c = 0. \end{cases} \quad (16)$$

Here, for $c \neq 0$, the Dedekind sum $s(d, c)$ is given by

$$s(d, c) = \frac{1}{4|c|} \sum_{j=1}^{|c|-1} \cot \frac{\pi j}{c} \cot \frac{\pi dj}{c}. \quad (17)$$

for $|c| > 1$ and $s(d, \pm 1) = 0$. If $A_i = \left( \begin{array}{cc} a_i & b_i \\ c_i & d_i \end{array} \right) \in \text{SL}(2, \mathbb{Z})$ such that $A_3 = A_1A_2$ we have

$$\Phi(A_3) = \Phi(A_1) + \Phi(A_2) - 3\text{sign}(c_1c_2c_3). \quad (18)$$

We refer to [43] for a comprehensive description of the Rademacher Phi function. We will also need

$$|\rho|^2 = \frac{\dim g}{12} = \frac{2|\Delta_+| + l}{12}, \quad (19)$$

where the first identity is Freudenthal’s strange formula. If $c = 0$ then $U = \epsilon \Theta^b$ for some $b \in \mathbb{Z}$ and $\epsilon \in \{ \pm 1 \}$. By (12) we immediately get

$$\mathcal{R}(\Theta^b)_{\lambda\mu} = \mathcal{R}(\Theta)_{\lambda\mu}^b = \delta_{\lambda\mu} \exp \left( b \left( \frac{\pi\sqrt{-1}}{\kappa} |\mu|^2 - \frac{\pi\sqrt{-1}}{h^\vee} |\rho|^2 \right) \right)$$

for $\lambda, \mu \in \text{int}(C_\kappa) \cap X$. For the case $U = -\Theta^b = \Xi^2 \Theta^b$ (see (7)) we use the identity $\mathcal{R}(\Xi^2)_{\lambda\mu} = \delta_{\lambda\mu^*}$, which follows by the unitarity of $\mathcal{R}$ and (15) (alternatively use (13) and (14) directly). This gives

$$\mathcal{R}(-\Theta^b)_{\lambda\mu} = \mathcal{R}(\Theta)_{\lambda^*\mu}^b = \delta_{\lambda^*\mu} \exp \left( b \left( \frac{\pi\sqrt{-1}}{\kappa} |\mu|^2 - \frac{\pi\sqrt{-1}}{h^\vee} |\rho|^2 \right) \right)$$

for $\lambda, \mu \in \text{int}(C_\kappa) \cap X$. For the case $c \neq 0$ we have

**Theorem 2.6** Let $U = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z})$ with $c \neq 0$, and let $\lambda, \mu \in \text{int}(C_\kappa) \cap X$. Then

$$\mathcal{R}(U)_{\lambda\mu} = \frac{(\sqrt{-1} \text{sign}(c))^{|\Delta_+|}}{(\kappa|c|)^{l/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi\sqrt{-1}}{h^\vee} \Phi(U) |\rho|^2 \right) \times \exp \left( \frac{\pi\sqrt{-1} d}{\kappa c} |\mu|^2 \right) \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi\sqrt{-1}}{\kappa c} |\lambda + \kappa \nu|^2 \right) \times \sum_{w \in W} \det(w) \text{exp} \left( -\frac{2\pi\sqrt{-1}}{\kappa c} \langle \lambda + \kappa \nu, w(\mu) \rangle \right).$$
If \( a \neq 0 \) we also have

\[
\mathcal{R}(U)_{\lambda \mu} = \frac{(\sqrt{-1} \text{sign}(c))^{\mid \Delta \mid}}{(\kappa |c|)^{1/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} \Phi(U) |\rho|^2 \right) \\
\times \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \frac{b}{a} |\mu|^2 \right) \sum_{\nu \in W} \text{det}(w) \\
\times \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \frac{a}{c} (\lambda + \kappa \nu - \frac{w(\mu)}{a})^2 \right).
\]

**Proof**  According to the previous lemma there exists an integer \( n \), a sign \( \epsilon \in \{ \pm 1 \} \) and a \( V \in \text{SL}(2, \mathbb{Z}) \) as in the Lemma 2.5 such that \( U = \epsilon V \Theta^n \). Let us first assume that \( \epsilon = 1 \). Assume, moreover, that \( a \neq 0 \) and that \( n = 0 \), i.e. assume that \( U = B^C \), where \( \mathcal{C} = (m_1, \ldots, m_t) \) and \( a_k \neq 0, k = 1, 2, \ldots, t \). Let \( \mathcal{C}' = (m_1, \ldots, m_{t-1}) \). Then by Lemma 2.4

\[
\mathcal{R}(U)_{\lambda \mu} = \tilde{\Theta}_{\lambda \lambda}^{m_1} T_{\mu, \lambda}^{C'} \\
= K_\mu^C \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_t |\lambda|^2 \right) \sum_{w \in W} \text{det}(w) \\
\times \sum_{\nu \in Y^\vee/\epsilon Y^\vee} \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} \kappa} |\lambda + \kappa \nu + \frac{w(\mu)}{a_{t-1}}|^2 \right) \\
= K_\mu^C \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_t |\lambda|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} \kappa} |\mu|^2 \right) \\
\times \sum_{w \in W} \text{det}(w) \sum_{\nu \in Y^\vee/\epsilon Y^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} m_t |\lambda + \kappa \nu|^2 \right) \\
\times \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} \kappa} |\lambda + \kappa \nu|^2 \right) \\
\times \exp \left( -\frac{2\pi \sqrt{-1}}{a_{t-1} \kappa} (\lambda + \kappa \nu, w(\mu)) \right) \\
= K_\mu^C \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} m_t |\lambda|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{a_{t-1} a_{t-2} \kappa} |\mu|^2 \right) \\
\times \exp \left( -\frac{\pi \sqrt{-1}}{a_t a_{t-1} \kappa} |\mu|^2 \right) \sum_{w \in W} \text{det}(w) \\
\times \sum_{\nu \in Y^\vee/\epsilon Y^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \frac{a}{c} (\lambda + \kappa \nu - \frac{w(\mu)}{a})^2 \right).
\]
where we have used (9). Let us calculate the factor in front of the sum. By Lemma 2.4 we get

\[ K := K_{\mu}^C \exp \left( -\frac{\sqrt{-1}}{h^\vee} m_t |\rho|^2 \right) \times \exp \left( -\frac{\sqrt{-1}}{a_t a_t - 2 \kappa} |\mu|^2 \right) \exp \left( -\frac{\sqrt{-1}}{a_t a_t - 2 \kappa} |\mu|^2 \right) \]

\[ = \frac{\sqrt{-1}^{|\Delta_+|} \zeta t D_{t-1}}{(\kappa |c|)^{t/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\sqrt{-1}}{h^\vee} (\sum_{i=1}^t m_i |\rho|^2) \right) \times \exp \left( -\frac{\sqrt{-1}}{\mu} (\sum_{i=1}^t \frac{1}{a_t a_t - 2 \kappa} |\mu|^2) \right). \]

By [26, Formula (2.20)] we have

\[ \Phi(U) = \sum_{i=1}^t m_i - 3 \sum_{i=1}^{t-1} \text{sign}(a_t a_t). \]

This together with (10) gives

\[ K = \frac{\sqrt{-1}^{|\Delta_+|} \zeta t D_{t-1}}{(\kappa |c|)^{t/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\sqrt{-1}}{h^\vee} (\Phi(U) + 3 D_{t-1}) |\rho|^2 \right) \]

\[ \times \exp \left( -\frac{\sqrt{-1}}{\mu} b |\mu|^2 \right). \]

By (19) we then have

\[ \sqrt{-1}^{-(t-1)|\Delta_+|} \zeta t D_{t-1} \exp \left( -\frac{\sqrt{-1}}{h^\vee} (\Phi(U) + 3 D_{t-1}) |\rho|^2 \right) \]

\[ = \sqrt{-1}^{-(t-1)|\Delta_+|} \exp \left( \left( \frac{l \pi \sqrt{-1}}{4} - \frac{(2|\Delta_+| + l) \pi \sqrt{-1}}{4} \right) D_{t-1} \right) \]

\[ \times \exp \left( -\frac{\sqrt{-1}}{h^\vee} \Phi(U) |\rho|^2 \right) \]

\[ = \sqrt{-1}^{-(t-1)-D_{t-1})|\Delta_+|} \exp \left( -\frac{\sqrt{-1}}{h^\vee} \Phi(U) |\rho|^2 \right). \]

Moreover,

\[ t - 1 - D_{t-1} = t - 1 - \sum_{i=1}^{t-1} \text{sign}(a_t a_t) = \sum_{i=1}^{t-1} (1 - \text{sign}(a_t a_t)) \]

and

\[ \sqrt{-1}^{1-\text{sign}(a_t a_t)} = \text{sign}(a_t a_t), \]
so $\sqrt{-1}^{k-1-D_{k-1}} = \text{sign}(a_{k-1}) = \text{sign}(c)$ since $a_0 = 1$. This proves the last given formula for the entries of $\mathcal{R}(U)$. The first formula follows easily from this by observing that

$$\frac{b}{a} + \frac{1}{ac} = \frac{d}{c}.$$ 

Next assume that $U = B^C \Theta^n$ with $n \neq 0$, where $C$ is as above. Then

$$B^C = U \Theta^{-n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -na + b \\ c & -nc + d \end{pmatrix},$$

and since the theorem is valid for $U = B^C$ (by the above) we get

$$\mathcal{R}(U)_{\lambda\mu} = \mathcal{R}(B^C)_{\lambda\nu} \mathcal{R}(\Theta)_{\nu\mu}^{n} \exp\left( -\frac{\pi \sqrt{-1}}{h} \frac{1}{|\kappa|} \Phi(B^C + n) |\rho|^2 \right) \times \sum_{\nu \in Y^\vee / cY^\vee} \exp\left( \frac{\pi \sqrt{-1}}{\kappa} \frac{a}{c} |\mu|^2 \right) \exp\left( \frac{\pi \sqrt{-1}}{\kappa} \frac{d}{c} |\mu|^2 \right) \times \sum_{w \in W} \det(w) \exp\left( -\frac{2\pi \sqrt{-1}}{\kappa} \langle \lambda + \kappa \nu, w(\mu) \rangle \right).$$

By (18), $\Phi(B^C \Theta^n) = \Phi(B^C) + \Phi(\Theta^n) = \Phi(B^C) + n$ and the result follows. Next consider the case where $a = 0$ (and $\epsilon = 1$) so $U = \Xi \Theta^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the result follows directly by inserting the formulas for the entries of $\mathcal{R}(\Xi)$ and $\mathcal{R}(\Theta)$ into

$$\mathcal{R}(U)_{\lambda\mu} = \mathcal{R}(\Xi)_{\lambda\nu} \mathcal{R}(\Theta)_{\nu\mu}^{n}$$

and by using that $\Phi(\Xi \Theta^n) = \Phi(\Xi) + \Phi(\Theta^n) = n$ (use (18)).

Let us finally consider the case where $\epsilon = -1$, so $U = -V \Theta^n$ ($V$ being as in Lemma 2.5). By the remarks just before the theorem we get

$$\mathcal{R}(U)_{\lambda\mu} = \mathcal{R}(-U)_{\lambda^*\mu^*}.$$ 

Since the theorem is valid for $-U$ (by the above) and since $\lambda^* = -w_0(\lambda)$ and $\det(w_0) = (-1)^{\Delta_{+1}}$, see Remark 2.3, we get

$$\mathcal{R}(U)_{\lambda\mu} = \frac{\sqrt{-1} \text{sign}(c)^{\mid \Delta_{+1} \mid}}{(\kappa |c|)^{1/2} \text{vol}(Y^\vee)} \exp\left( -\frac{\pi \sqrt{-1}}{h} \Phi(U) |\rho|^2 \right) \exp\left( \frac{\pi \sqrt{-1}}{\kappa} \frac{d}{c} |\mu|^2 \right) \times \sum_{\nu \in Y^\vee / cY^\vee} \exp\left( \frac{\pi \sqrt{-1}}{\kappa} \frac{a}{c} |w(\lambda) - \kappa w_0(\nu)|^2 \right) \times \sum_{w \in W} \det(w)^{-1} \exp\left( \frac{2\pi \sqrt{-1}}{\kappa c} \langle -w_0(\lambda) + \kappa w_0(\nu), w(\mu) \rangle \right).$$

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by which the theorem follows. 

Since \( R \) is a unitary representation we have

\[
\mathcal{R}(U)_{\lambda\mu} = \mathcal{R}(U^{-1})_{\mu\lambda}
\]

for any \( U \in \text{SL}(2,\mathbb{Z}) \) and all \( \lambda, \mu \in \text{int}(C_{\kappa}) \cap X \). By this and the facts that \( \Phi(U^{-1}) = -\Phi(U) \) (use (18)) and \( U^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \) we get

**Corollary 2.7** Let \( U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{Z}) \) with \( c \neq 0 \), and let \( \lambda, \mu \in \text{int}(C_{\kappa}) \cap X \). Then

\[
\mathcal{R}(U)_{\lambda\mu} = \frac{(\sqrt{-1} \text{sign}(c))^{\Delta_+}}{(\kappa|c|)^{1/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi \sqrt{-1} \Phi(U) |\rho|^2}{h^\vee} \right) \\
\times \exp \left( \frac{\pi \sqrt{-1} a}{\kappa c} |\lambda|^2 \right) \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi \sqrt{-1} d}{\kappa c} |\mu + \kappa \nu|^2 \right) \\
\times \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa c} \langle \lambda + \kappa \nu, w(\mu) \rangle \right).
\]

If \( d \neq 0 \) we also have

\[
\mathcal{R}(U)_{\lambda\mu} = \frac{(\sqrt{-1} \text{sign}(c))^{\Delta_+}}{(\kappa|c|)^{1/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi \sqrt{-1} \Phi(U) |\rho|^2}{h^\vee} \right) \\
\times \exp \left( \frac{\pi \sqrt{-1} b}{\kappa d} |\lambda|^2 \right) \sum_{w \in W} \det(w) \\
\times \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi \sqrt{-1} d}{\kappa c} |\mu + \kappa \nu - \frac{w(\lambda)}{d}|^2 \right).
\]
We are particularly interested in expressions for $R(U)_{\lambda\mu}$ in case $\lambda$ or $\mu$ is equal to $\rho$. Note that since $\rho^* = \rho$, $\Xi^2 = -1$ and $R(\Xi^2)_{\lambda\mu} = \delta_{\lambda\mu}$, then
\[ R(-U)_{\lambda\rho} = R(U)_{\lambda\rho}, \quad R(-U)_{\rho\lambda} = R(U)_{\rho\lambda} \tag{21} \]
for all $\lambda \in \text{int}(C_\kappa) \cap X$, so these entries are in fact functions of $\text{PSL}(2, \mathbb{Z})$. By the Weyl denominator formula we have
\[
\sum_{w \in W} \det(w) \exp \left\{ -\frac{2\sqrt{-1}}{\kappa c} (\lambda + \kappa \nu, w(\rho)) \right\} = \prod_{\alpha \in \Delta_+} 2\sqrt{-1} \sin \left( \frac{\pi}{\kappa c} (\lambda + \kappa \nu, \alpha) \right),
\]
so we get the alternative formula
\[
R(U)_{\lambda\rho} = \frac{(2 \text{ sign}(c))^{|\Delta_+|}}{(\kappa |c|)^{1/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} \Phi(U)|\rho|^2 \right) \times \exp \left( \frac{\pi \sqrt{-1}}{c} |\rho|^2 \right) \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa c} |\lambda + \kappa \nu|^2 \right) \times \prod_{\alpha \in \Delta_+} \sin \left( \frac{\pi}{\kappa c} (\lambda + \kappa \nu, \alpha) \right),
\]
for $U \in \text{SL}(2, \mathbb{Z})$ as in Theorem 2.6. By using the Weyl denominator formula and the first expression in Corollary 2.7 we get
\[
R(U)_{\rho\mu} = \frac{(2 \text{ sign}(c))^{|\Delta_+|}}{(\kappa |c|)^{1/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} \Phi(U)|\rho|^2 \right) \times \exp \left( \frac{\pi \sqrt{-1}}{c} |\rho|^2 \right) \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa c} |\mu + \kappa \nu|^2 \right) \times \prod_{\alpha \in \Delta_+} \sin \left( \frac{\pi}{\kappa c} (\mu + \kappa \nu, \alpha) \right),
\]
for $U \in \text{SL}(2, \mathbb{Z})$ as in Theorem 2.6. By using the Weyl denominator formula and the first expression in Corollary 2.7 we get
\[
R(U)_{\lambda\mu} = \frac{(2 \text{ sign}(c))^{|\Delta_+|}}{(\kappa |c|)^{1/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} \Phi(U)|\rho|^2 \right) \times \exp \left( \frac{\pi \sqrt{-1}}{c} |\rho|^2 \right) \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa c} |\lambda + \kappa \nu|^2 \right) \times \prod_{\alpha \in \Delta_+} \sin \left( \frac{\pi}{\kappa c} (\lambda + \kappa \nu, \alpha) \right).
\]
We end this section with some symmetry considerations needed elsewhere. As mentioned already below Remark 2.3 the expressions for $R(\Xi)_{\lambda\mu}$ and $R(\Theta)_{\lambda\mu}$ in (12) are well-defined for all $\lambda, \mu \in X$. If $U \in \text{SL}(2, \mathbb{Z})$ we can find a tuple of integers $C = (m_1, \ldots, m_t)$ such that $U = B^C$. We can therefore use the formula
\[
R(U)_{\lambda\mu} = \frac{(2 \text{ sign}(c))^{m_t}}{(\kappa |c|^t)^{1/2} \text{vol}(Y^\vee)} \exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} \Phi(U)|\rho|^2 \right) \times \exp \left( \frac{\pi \sqrt{-1}}{c} |\rho|^2 \right) \sum_{\nu \in Y^\vee/cY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{\kappa c} |\lambda + \kappa \nu|^2 \right) \times \prod_{\alpha \in \Delta_+} \sin \left( \frac{\pi}{\kappa c} (\lambda + \kappa \nu, \alpha) \right).
\]
to extend $(\lambda, \mu) \mapsto R(U)_{\lambda\mu}$ to all of $X \times X$. Here $T_{\mu,\lambda}^{C_t,1}$ is defined above Lemma 2.4. We can also make such an extension directly by using the expressions in Theorem 2.6. An easy inspection of the proof of Theorem 2.6 shows that these two extensions coincide.
Lemma 2.8 Let $U \in \text{SL}(2, \mathbb{Z})$. Then $(\lambda, \mu) \mapsto \mathcal{R}(U)_{\lambda \mu}$, $X \times X \to \mathbb{C}$, is invariant under the action of $\kappa Y^\vee$ on each factor. Moreover,

$$\mathcal{R}(U)_{w(\lambda)w'(\mu)} = \det(w) \det(w') \mathcal{R}(U)_{\lambda \mu}.$$ 

Finally, $\mathcal{R}(U)_{\lambda \mu} = 0$ if $\lambda$ or $\mu$ belongs to $X \cap H^\kappa$.

Proof Since $\Xi^4 = 1$ we have $U = \Xi B = C \Xi$, where $B = \Xi^3 U$ and $C = U \Xi^3$. The lemma then follows by the fact that it is true for $U = \Xi$. □

Since $C_\kappa$ is a fundamental domain for the action of $W^\kappa_{\text{aff}}$ it follows by Lemma 2.8 that (20) is valid for all $\lambda, \mu \in X$. Therefore we have that the expressions for $\mathcal{R}(U)_{\lambda \mu}$ stated in Corollary 2.7 are valid for all $\lambda, \mu \in X$. Note that (22) is valid for all $\lambda \in X$ and (23) is valid for all $\mu \in X$.

3 Seifert manifolds

For Seifert manifolds we will use the notation introduced by Seifert in his classification results for these manifolds, see [49], [50], [17, Sect. 2]. That is, $(\varepsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ is the Seifert manifold with orientable base of genus $g \geq 0$ if $\varepsilon = o$ and non-orientable base of genus $g > 0$ if $\varepsilon = n$ (where the genus of the non-orientable connected sum $\#^k \mathbb{R}P^2$ is $k$). (In [49], [50] $(\varepsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ is denoted $(O, \varepsilon; g \mid b; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n)$, but we leave out the $O$ since we are only dealing with oriented Seifert manifolds.) The pair $(\alpha_j, \beta_j)$ of coprime integers is the (oriented) Seifert invariant of the $j$'th exceptional (or singular) fiber. We have $0 < \beta_j < \alpha_j$. The integer $-b$ is equal to the Euler number of the Seifert fibration $(\varepsilon; g \mid b)$ (which is a locally trivial $S^1$–bundle). The sign is chosen so that the Euler number of the spherical (or unit) tangent bundle over an orientable surface $\Sigma$ is equal to the Euler characteristic of $\Sigma$, see [39, Chap. 1 and 4], [48, Sect. 3]. More generally the Seifert Euler number of $(\varepsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ is $E = -\left(b + \sum_{j=1}^n \beta_j/\alpha_j\right)$.

We note that lens spaces are Seifert manifolds with base $S^2$ and zero, one or two exceptional fibers. According to [39, Fig. 12 p. 146], the manifold $(\varepsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ has a surgery presentation as shown in Fig. 1 if $\epsilon = o$ and as shown in Fig. 2 if $\epsilon = n$. The $g$ indicate $g$ repetitions.

For completeness we will also state the results in terms of the non-normalized Seifert invariants due to W. D. Neumann, see [24]. For a Seifert manifold $X$ with non-normalized Seifert invariants $(\varepsilon; g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ the invariants $\varepsilon$ and $g$ are as above. The $(\alpha_j, \beta_j)$ are here pairs of coprime integers with
$\alpha_j > 0$ but not necessarily with $0 < \beta_j < \alpha_j$. These pairs are not invariants of $X$, but can be varied according to certain rules. In fact, $X$ has a surgery presentation as shown in Fig. 1 with $b = 0$ if $\epsilon = o$ and as shown in Fig. 2 with $b = 0$ if $\epsilon = n$. The Seifert Euler number of $X$ is $-\sum_{j=1}^{n} \beta_j/\alpha_j$ (which is an invariant of the Seifert fibration $X$). For more details we refer to [24, Sect. I.1].

4 The RT–invariants of the Seifert 3–manifolds

In this section we will use Corollary 2.7 together with results in [17] to derive expressions for the quantum $g$–invariants of all Seifert manifolds, $g$ being an arbitrary complex finite dimensional simple Lie algebra. The formulas for the invariants will be expressed in terms of the Seifert invariants together with
Theorem 4.1

Fraction expansion of $\alpha$

We put $\alpha = \frac{a}{b} = (a, b)$, and $\alpha$ is coprime. Moreover, let $\tau = \tau_\alpha$ be the involution in $I$ determined by the condition that $V_i$ is isomorphic to the dual of $V_i$. An element $i \in I$ is called self-dual if $i = i^*$. For such an element we have a $K$-module isomorphism $\text{Hom}_V(V \otimes V, I) \cong K$, $V = V_i$. The map $x \mapsto x(i_0 \otimes \theta_V)c_{V,V}$ is a $K$-module endomorphism of $\text{Hom}_V(V \otimes V, I)$, so is a multiplication by a certain $\varepsilon_i \in K$. By the definition of the braiding and twist we have $(\varepsilon_i)^2 = 1$. In particular $\varepsilon_i \in \{\pm 1\}$ if $K$ is a field. We have a distinguished element $0 \in I$ such that $V_0 = I$.

The $S$- and $T$-matrices of $V$ are the matrices $S = (S_{ij})_{i,j \in I}$, $T = (T_{ij})_{i,j \in I}$ given by $S_{ij} = \text{tr}(c_{V_i,V_j} \circ c_{V_j,V_i})$ and $T_{ij} = \delta_{ij} v_i$, where $\text{tr}$ is the categorical trace of $V$, $\delta_{ij}$ is the Kronecher delta equal to 1 if $i = j$ and zero elsewhere, and $v_i \in K$ such that $\theta_{V_i} = v_i \text{id}_{V_i}$.

Assume that $V$ has a rank $D$, i.e. an element of $K$ satisfying

$$D^2 = \sum_{i \in I} \dim(i)^2,$$

where $\dim(i) = \dim(V_i) = \text{tr}(i_0 \text{id}_{V_i})$. We let

$$\Delta = \sum_{i \in I} v_i^{-1} \dim(i)^2.$$

Moreover, let $\tau = \tau_{(\mathcal{V}, \mathcal{D})}$ be the RT-invariant associated to $(\mathcal{V}, \mathcal{D})$, cf. [52, Sect. II.2]. For a tuple of integers $\mathcal{C} = (m_1, \ldots, m_l)$, let

$G^\mathcal{C} = T^m S \cdots T^m S$.

We put $a_0 = 2$ and $a_n = 1$. Moreover, let $b_{ij}^{(n)} = 1$ and $b_{ij}^{(n)} = \delta_{jj'}$, $j \in I$. Given pairs $(\alpha_j, \beta_j)$ of coprime integers we let $C_j = (a_1^{(j)}, \ldots, a_{m_j}^{(j)})$ be a continued fraction expansion of $\alpha_j / \beta_j$, $j = 1, 2, \ldots, n$.

**Theorem 4.1** ([17]) Let $M = (\epsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, $\epsilon = 0, n$. Then

$$\tau(M) = (\Delta D^{-1})^a \mathcal{D}^{a_0g - 2 - \sum_{j=1}^n m_j} \times \sum_{j \in I} (\varepsilon_j)^{a_0g} \frac{1}{j_1 \cdots j_n} \dim(j)^{2-n-a_0g} \left( \prod_{i=1}^n (SG^\mathcal{C}_i)_{j_0} \right),$$

25
\[ \sigma_{\epsilon} = (a_{\epsilon} - 1) \text{sign}(E) + \sum_{j=1}^{n} \text{sign}(\alpha_j \beta_j) + \frac{1}{3} \sum_{j=1}^{n} \left( \sum_{k=1}^{m_j} a_{k(j)} - \Phi(B_{C_j}) \right). \] (24)

Here \( E = -\left( b + \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} \right) \) is the Seifert Euler number.

The RT–invariant \( \tau(M) \) of the Seifert manifold \( M \) with non-normalized Seifert invariants \( \{ \epsilon; g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \} \), is given by the same expression with the exceptions that the factor \( v - b_j \) has to be removed and \( E = -\sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} \).

The theorem is also valid in case \( n = 0 \). In this case one just has to put all sums \( \sum_{j=1}^{n} \) equal to zero and all products \( \prod_{i=1}^{n} \) equal to 1. Note that \( \epsilon_g^j = 1 \) if \( g \) is even and \( \epsilon_g^j = \epsilon_j \) if \( g \) is odd since \( \epsilon_j^2 = 1 \). The sum \( \sum_{j=1}^{n} \text{sign}(\alpha_j \beta_j) \) is of course equal to \( n \) for normalized Seifert invariants.

Let us next consider the lens spaces. For \( p, q \) a pair of coprime integers, recall that \( L(p, q) \) is given by surgery on \( S^3 \) along the unknot with surgery coefficient \( -\frac{p}{q} \). In the following corollary we include the possibilities \( L(0, 1) = S^1 \times S^2 \) and \( L(1, q) = S^3, \quad q \in \mathbb{Z} \).

**Corollary 4.2** ([17]) Let \( p, q \) be a pair of coprime integers. If \( q \neq 0 \) we let \( (a_1, \ldots, a_{n-1}) \) be a continued fraction expansion of \( -\frac{p}{q} \). If \( q = 0 \), put \( n = 3 \) and \( a_1 = a_2 = 0 \). Then

\[ \tau(L(p, q)) = (\Delta \Phi)^{\sigma} \Phi^{-n} G_{00}^C, \] (25)

where \( C = (a_1, \ldots, a_{n-1}, 0) \) and \( \sigma = \frac{1}{3} \left( \sum_{l=1}^{n-1} a_l - \Phi(B_{C}) \right) \).

**The RT–invariants of the Seifert manifolds for the classical Lie algebras** Let \( \mathfrak{g} \) be a complex finite dimensional simple Lie algebra and let

\[ q = e^{\pi \sqrt{-1} r}, \quad r = m \kappa, \quad \kappa \text{ is an integer} \geq h^\vee. \]

Let \( U_q(\mathfrak{g}) \) be the quantum group associated to these data as defined by Lusztig, see [37, Part V]. We follow [9, Sect. 1.3 and 3.3] here but will mostly use notation from [52] for modular categories as above. (Note that what we denote \( U_q(\mathfrak{g}) \) here is denoted \( U_q(\mathfrak{g})|_{q = e^{\pi \sqrt{-1} m \kappa}} \) in [9].) Let \( (V^g_r, \{ V_i \}_{i \in I}) \) be the modular category induced by these data, cf. [9, Theorem 3.3.20]. In particular the index set for the simple objects is \( I = \text{int}(C_\pm) \cap X \). We use here the shifted indexes (shifted by \( p \)) (contrary to [9]). Normally the irreducible modules of \( U_q(\mathfrak{g}) \) (of type 1) \( q \) a formal variable are indexed by the cone of dominant integer weights \( X_+ \). Here we denote the irreducible module associated with \( \mu \in X_+ \) by \( V_{\mu + p} \). For
q a root of unity as above $V_\lambda$ is an irreducible module of $U_q(g)$ of non-zero dimension if $\lambda \in I$. The involution $I \to I$, $\lambda \mapsto \lambda^*$ is given by $\lambda^* = -w_0(\lambda)$, where $w_0$ is the longest element in $W$, see also Remark 2.3. The distinguished element $0 \in I$ is equal to $\rho$. According to [9, Theorem 3.3.20] we can use

$$D = \kappa^{l/2} \left| \frac{\text{vol}(Y^\vee)}{\text{vol}(X)} \right|^{1/2} \prod_{\alpha \in \Delta_+} 2 \sin \left( \frac{\pi \langle \alpha, \rho \rangle}{\kappa} \right)$$

(26)
as a rank of $V_\lambda$. According to the same theorem we have that

$$\Delta D^{-1} = \omega^{-3},$$

(27)where

$$\omega = e^{2\pi \sqrt{-1} h^\vee / 2} = \exp \left( \frac{\pi \sqrt{-1} |\rho|^2}{h^\vee} \right) \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} |\rho|^2 \right),$$

(28)where as usual $c = \frac{\kappa - h^\vee}{\kappa} \dim(g)$ is the central charge and where the last equality follows from Freudenthal’s strange formula (19).

Let $\tilde{s}$ be as defined in [9, Sect. 3.1] and let $S$ be the $S$–matrix of $V_\lambda^g$. Observe that in the terminology of [52], $\tilde{s}$ is the $S$–matrix of the mirror of $V_\lambda^g$. This implies that

$$S_{\lambda\mu} = \tilde{s}_{\lambda^* \mu} = \tilde{s}_{\lambda \mu^*}$$

for $\lambda, \mu \in I$. By [9, Formula (3.1.16)] we have $\tilde{s} = Ds$, where $s$ is the matrix also considered in Remark 2.3. The matrix $t$ in [9] is equal to the $T$–matrix of $V_\lambda^g$. It follows then from [9, Theorem 3.3.20] and (12) that $T = \omega \tilde{\Theta}$. By these remarks and (13) and (15) we conclude that

$$S_{\lambda\mu} = D \tilde{\Xi}_{\lambda\mu}, \quad T_{\lambda\mu} = \omega \tilde{\Theta}_{\lambda\mu}$$

(29)for $\mu, \lambda \in I$. Let $C = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and let $k \in \{0,1\}$. By (29) we immediately get

$$(S^k G^C)_{\lambda\rho} = D^{k+n} \omega \prod_{j=1}^n a_j \left( \tilde{\Xi}^k \tilde{\Theta} a_n \check{\Xi} a_1 \tilde{\Xi} \right)_{\lambda\rho}$$

(30)for $\lambda \in I$. We also need the explicit formula for $\dim(\lambda)$, $\lambda \in I$. In fact

$$\dim(\lambda) = S_{\lambda\rho} = D \tilde{\Xi}_{\lambda\rho} = D \kappa^{-1/2} \left| \frac{\text{vol}(X)}{\text{vol}(Y^\vee)} \right|^{1/2} \prod_{\alpha \in \Delta_+} 2 \sin \left( \frac{\pi \langle \alpha, \lambda \rangle}{\kappa} \right)$$

(31)see also [9, Formulas (3.3.2) and (3.3.5)].

Let $\tau_r^g = \tau_{(V_\rho^g, D)}$ be the RT–invariant associated with $(V_\rho^g, D)$. Given a pair of coprime integers $(\alpha, \beta)$, $\alpha > 0$, we let $\beta^*$ be the inverse of $\beta$ in the multiplicative group of units of $\mathbb{Z}/\alpha \mathbb{Z}$. For integers $a, b \neq 0$ we let

$$S(a/b) = 12 \text{sign}(b) s(a, b),$$

(32)
where \( s(a, b) \) is given by (17). This is the so-called Dedekind symbol. (In particular, the right-hand side of (32) only depends on the rational number \( a/b \).) Then we have the following generalization of [17, Theorem 8.4]:

**Theorem 4.3** Let \( M = (\epsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)), \epsilon \in \{o, n\} \). Then we have the following generalization of [17, Theorem 8.4]:

\[
\tau^g_r(M) = \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \left[ 3(a_\epsilon - 1) \text{sign}(E) - E - \sum_{j=1}^{n} S \left( \frac{\beta_j}{\alpha_j} \right) \right] |\rho|^2 \right) 
\times \frac{\sqrt{-1} |\Delta_+| |\kappa|^l(a_g/2 - 1)}{2^l \Delta_+(n + a_g - 2) \text{vol}(Y^\vee)^2 - a_g \mathcal{A}^{l/2}} 
\times \exp \left( \frac{3 \pi \sqrt{-1}}{h^\vee} (1 - a_\epsilon) \text{sign}(E)|\rho|^2 \right) Z^g_r(M; r),
\]

where \( \mathcal{A} = \prod_{j=1}^{n} \alpha_j \) and

\[
Z^g_r(M; r) = \sum_{\lambda \in I} \int_{\lambda}^{a_g} \left( \prod_{\alpha \in \Delta_+} \sin^{2 - n - a_g} \left( \frac{\pi \langle \lambda, \alpha \rangle}{\kappa} \right) \right) \exp \left( \frac{\pi \sqrt{-1}}{\kappa} |\lambda|^2 \right) 
\times \sum_{w_1, \ldots, w_n \in W} \sum_{\nu_1 \in Y^\vee/\alpha_1 Y^\vee} \ldots \sum_{\nu_n \in Y^\vee/\alpha_n Y^\vee} \left( \prod_{j=1}^{n} \det(w_j) \right) 
\times \exp \left( -\pi \sqrt{-1} \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} (\kappa |\nu_j|^2 + 2 \langle w_j(\rho), \nu_j \rangle) \right) 
\times \exp \left( -\frac{2 \pi \sqrt{-1}}{\kappa} \langle \lambda, \sum_{j=1}^{n} \frac{\kappa \nu_j + w_j(\rho)}{\alpha_j} \rangle \right).
\]

The RT-invariant \( \tau^g_r(M) \) of the Seifert manifold \( M \) with non-normalized Seifert invariants \( \{\epsilon; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\} \) is given by the same expression.

The theorem is also valid in case \( n = 0 \). In this case one just has to put the sum \( \sum_{w_1, \ldots, w_n \in W} \sum_{\nu_1 \in Y^\vee/\alpha_1 Y^\vee} \ldots \sum_{\nu_n \in Y^\vee/\alpha_n Y^\vee} \) in \( Z^g_r(M; r) \) equal to 1, \( \epsilon = o, n \), and put \( \mathcal{A} = 1 \) and \( \sum_{j=1}^{n} S(\beta_j/\alpha_j) = 0 \).

**Proof** The proof follows very closely the proof of [17, Theorem 8.4]. Let \( M = (\epsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)), \epsilon \in \{o, n\} \). Choose tuples of integers \( C_j = (a_{j,1}, \ldots, a_{jm,j}) \) such that \( B^{C_j} = \begin{pmatrix} \alpha_j & \rho_j \\ \beta_j & \sigma_j \end{pmatrix} \), \( j = 1, 2, \ldots, n \). By Theorem 4.1
and (11) we have
\[
\tau_\gamma^\theta(M) = (\Delta D^{-1})^{\sigma_\gamma} D^{M, g - 2 - \sum_{j=1}^m m_j} \times \sum_{\lambda \in \ell} b_\lambda^{(e)} \varepsilon_\lambda^{a_g} \nu_\lambda^{b} \dim(\lambda)^{2 - n - a_g} \left( \prod_{i=1}^n (SG_i)^{\lambda_\rho} \right),
\]
where \(\sigma_\gamma\) is given by (24). By using (27), (29), (30), and (31) we get
\[
\tau_\gamma^\theta(M) = \alpha_\epsilon(\kappa) \omega^{-3\epsilon_g - b + \sum_{j=1}^n \sum_{\lambda = 1}^{m_j} a_j^{(j)}} \times \prod_{\lambda \in \ell} \sin^{2 - n - a_g} \left( \frac{\pi(\alpha_\lambda, \lambda)}{\kappa} \right) \prod_{i=1}^n (\tilde{N}_i)^{\lambda_\rho},
\]
where \(N_i = \Xi B_i = \left( \frac{-\beta_i}{\alpha_i} \right), \frac{-\sigma_i}{\rho_i} \) and
\[
\alpha_\epsilon(\kappa) = \kappa^{(n + a_g - 2)/2}\Delta_+ (2 - n - a_g) \ vol(Y^\vee)^{n + a_g - 2}.
\]
By Corollary 2.7 we get
\[
\prod_{i=1}^n (\tilde{N}_i)^{\lambda_\rho} = \beta_\epsilon(\kappa) \sum_{w_1, \ldots, w_n \in W} \left( \prod_{j=1}^n \det(w_j) \right) \sum_{\nu_1 \in Y^\vee/\alpha_1 Y^\vee} \ldots \sum_{\nu_n \in Y^\vee/\alpha_n Y^\vee} \times \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \sum_{j=1}^n \frac{\beta_j}{\alpha_j} \rho + \kappa \nu_j \right)^2 \times \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa} \sum_{j=1}^n \frac{1}{\alpha_j} (\rho + \kappa \nu_j, w_j(\lambda)) \right),
\]
where
\[
\beta_\epsilon(\kappa) = \frac{\sqrt{-1}^{n|\Delta_+|}}{\kappa^{n/2} \Delta^{1/2} \ vol(Y^\vee)^{n}} \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} \sum_{j=1}^n \Phi(N_j) \right) \left| \rho \right|^2 \times \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} \sum_{j=1}^n \frac{\beta_j}{\alpha_j} \right) \left| \lambda \right|^2.
\]
In particular we have

\[
\alpha_{\epsilon}(\kappa)\beta_{\epsilon}(\kappa)\hat{\Theta}_{\lambda\lambda}^{-b} = \frac{\sqrt{-1}^n|\Delta_+|^{l(a_g/2-1)}}{2^{|\Delta_+|/2} \kappa(l(a_g-2)/2)} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} E|\lambda|^2 \right) \times \exp \left( \frac{\pi \sqrt{-1}}{h'} \left( b - \sum_{j=1}^n \Phi(N_j) \right) |\rho|^2 \right).
\]

By (18), \( \Phi(N_i) = \Phi(B_i^C) - 3\text{sign}(\alpha_i\beta_i) \). By this and (24) we get

\[
3\sigma_{\epsilon} = 3(a_{\epsilon} - 1) \text{sign}(E) - \sum_{j=1}^n \Phi(N_j) + \sum_{j=1}^n \sum_{k=1}^m a_{k}^{(j)}.
\]

Therefore

\[
\tau_{\epsilon}^g(M) = \frac{\sqrt{-1}^n|\Delta_+|^{l(a_g/2-1)}}{2^{|\Delta_+|/2} \kappa(l(a_g-2)/2)} \frac{1}{\text{vol}(Y^\vee)^{2-a_g}} \sum_{j=1}^n \Phi(N_j) - 3(a_{\epsilon} - 1) \text{sign}(E) - b
\]

\[
\times \exp \left( \frac{\pi \sqrt{-1}}{h'} \left( b - \sum_{j=1}^n \Phi(N_j) \right) |\rho|^2 \right)
\]

\[
\times \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \left( \sum_{j=1}^n \rho_j \alpha_j \right) |\rho|^2 \right)
\]

\[
\times \sum_{\lambda \in I} \varepsilon_{\lambda} \left( \prod_{\alpha \in \Delta_+} \sin^{2-\alpha_g} \left( \frac{\pi \langle \lambda, \alpha \rangle}{\kappa} \right) \right) \exp \left( \frac{\pi \sqrt{-1}}{\kappa} E|\lambda|^2 \right)
\]

\[
\times \sum_{w_1, \ldots, w_n \in W_{\nu_1 \in Y^\vee/\alpha_1 Y^\vee}} \cdots \sum_{\nu_n \in Y^\vee/\alpha_n Y^\vee} \left( \prod_{j=1}^n \det(w_j) \right)
\]

\[
\times \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \sum_{j=1}^n \rho_j \alpha_j (\kappa |\nu_j|^2 + 2\langle w_j(\rho), \nu_j \rangle) \right)
\]

\[
\times \exp \left( - \frac{2\pi \sqrt{-1}}{\kappa} \langle \lambda, \sum_{j=1}^n \kappa \nu_j + w_j(\rho) \rangle \right).
\]
By (28) we get

$$\omega \sum_{j=1}^{n} \Phi(N_j) - 3(a_\epsilon - 1) \sign(E) - b \exp \left( \frac{\pi \sqrt{-1}}{h \nu} \left( b - \sum_{j=1}^{n} \Phi(N_j) \right) |\rho|^2 \right)$$

$$\times \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \left( \sum_{j=1}^{n} \frac{\rho_j}{\alpha_j} \right) |\rho|^2 \right)$$

$$= \exp \left( \frac{3\pi \sqrt{-1}}{h \nu} (1 - a_\epsilon) \sign(E)|\rho|^2 \right)$$

$$\times \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \left[ 3(a_\epsilon - 1) \sign(E) + b + \sum_{j=1}^{n} \left( \frac{\rho_j}{\alpha_j} - \Phi(N_j) \right) \right] |\rho|^2 \right).$$

The theorem now follows by using (16) together with the facts that $s(a, b) = s(a', b)$ if $a' a \equiv 1 \pmod{b}$ and $s(-a, b) = -s(a, b)$, cf. [43, Chap. 3]. (The identity $s(-a, b) = -s(a, b)$ follows immediately from (17).) The case with non-normalized Seifert invariants follows as above by letting $b$ be equal to zero everywhere.

By the above proof we get the following compact formula for the invariant of the Seifert manifold $M = (\epsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$:

$$\tau^g(M) = \gamma_\epsilon(\kappa) \sum_{\lambda \in I} \lambda^{(e) \alpha,g} \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} b |\lambda|^2 \right)$$

$$\times \left( \tilde{\Xi}_{\lambda \rho} \right)^{2-n-a_\epsilon g} \left( \prod_{i=1}^{n} (\tilde{N}_i)_{\lambda \rho} \right),$$

where $N_i \in SL(2, \mathbb{Z})$ with first column equal to $\left( \begin{array}{c} -\beta_i \\ \alpha_i \end{array} \right)$, $i = 1, \ldots, n$, and

$$\gamma_\epsilon(\kappa) = \exp \left( \frac{3\pi \sqrt{-1}}{h \nu} \left[ (1 - a_\epsilon) \sign(E) + \sum_{j=1}^{n} \Phi(N_j) \right] |\rho|^2 \right)$$

$$\times \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \left[ 3(a_\epsilon - 1) \sign(E) + b - \sum_{j=1}^{n} \Phi(N_j) \right] |\rho|^2 \right).$$

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The coprime case In this section we will derive a particularly nice expression for the quantum $g$–invariants of the Seifert manifolds $M = (\epsilon; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ with the integers $\alpha_1, \ldots, \alpha_n$ mutually coprime. The subclass of these Seifert manifolds additionally satisfying that $g = 0$ contains the Seifert fibered integral homology spheres, cf. [24, Corollary 6.2 and pp. 36–37]. The results in this subsection generalize the results obtained in [33, Sect. 4.1], where the case $g = sl_2(C)$ is considered. Let us first observe that by (33) and Lemma 2.8 we have

$$\tau^g(M) = \frac{\gamma_\epsilon(\kappa)}{|W|} \sum_{\lambda \in \bar{P} \cap X} b^{(\epsilon)}_\lambda \epsilon_\lambda^g \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} b|\lambda|^2 \right) \times \left( \Xi_{\lambda \rho} \right)^{2-n-a \cdot g} \left( \prod_{i=1}^{n} (\bar{N}_i)_{\lambda \rho} \right),$$

where $\bar{P}_\kappa = P_\kappa \setminus H^\kappa$. To establish this identity we also extended the function $g: I \to \mathbb{C}$, $g(\lambda) = b^{(\epsilon)}_\lambda \epsilon_\lambda^g$, to all of $X$ by letting it be constantly 1 if $\epsilon = 0$ and by forcing it to satisfy the following symmetry properties if $\epsilon = n$:

$$g(w(\lambda)) = \det(w)^{a \cdot g} g(\lambda), \quad w \in W, \lambda \in X,$n$$

$$g(\lambda + x) = g(\lambda), \quad x \in \kappa Y^\vee, \lambda \in X,$$

$$g(\lambda) = 0, \quad \lambda \in X \cap \bigcup_{\alpha \in \Delta_+, b \in \mathbb{Z}} H^\kappa_{\alpha, b}.$$n

Remark 4.4 If $\epsilon = n$ we have

$$D^{-2} \sum_{i \in I} \dim(i) F(L_{i\lambda}) = b^{(n)}_\lambda \epsilon_\lambda,$$

by [17, Lemma 4.2], where $L_{i\lambda}$ is a certain 2–component link, one component colored by $V_i$ and the other by $V_\lambda$. Recall that $b^{(n)}_\lambda = \delta_{\lambda \lambda^*}$. In [34] it was observed that one in a natural way can extend the invariant $F(L_{i\lambda})$ to be defined for all $\lambda \in X$ in such a way that $F(L_{iw(\lambda)}) = \det(w) F(L_{i\lambda})$, $w \in W$. The thus extended function $\lambda \to F(L_{i\lambda})$ is automatically invariant under a change $\lambda \to \lambda + x, x \in \kappa Y^\vee$. (One has to be a little careful here. In [34], $F(L_{i\lambda})$ is denoted $J_L(i, \lambda)$. It is shown in [34] that this function is component-wise invariant under the action by the affine Weyl group $W^\text{aff}_\kappa$ up to a sign coming from the sign of the Weyl group element, see the refined first symmetry principle [34, Theorem 2.11]. The lattice denoted $Y'$ in [34] is equal to $\frac{1}{m} Y^\vee$ here ($m$ is denoted $d$ in [34]). Since $r$ in [34] is the same as $r$ here, i.e. $r = m\kappa$, we have that $W \ltimes r Y'$ in [34] is equal to $W^\text{aff}_\kappa$ here.) We see, that the symmetries (34) actually are consequences of the symmetry results in [34].
By Theorem 4.3 and (34) (use also $E = -b - \sum_{j=1}^{n} \beta_j / \alpha_j$) we have

$$Z^g_{\epsilon}(M; r) = \alpha(\kappa) \sum_{\lambda \in \mathcal{P}_\lambda \cap \mathcal{X}} b^{(\epsilon)}_{\lambda} \lambda^{a_{\lambda}g} \left( \prod_{\alpha \in \Delta_+} \sin^{2-n-a_{\alpha}} g \left( \frac{\pi(\lambda, \alpha)}{\kappa} \right) \right)$$

\[ \times \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} b|\lambda|^2 \right) \sum_{\text{w}_1, \ldots, \text{w}_n \in \mathcal{W}} \left( \prod_{j=1}^{n} F_j(\lambda, \nu_j, \text{w}_j) \right), \]

where

$$\alpha(\kappa) = \frac{1}{|W|} \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} \left( \sum_{j=1}^{n} \beta_j^* \alpha_j \right) |\rho|^2 \right)$$

and

$$F_j(\lambda, \nu, w) = \det(w) \times \exp \left( -\frac{\pi \sqrt{-1}}{\kappa \alpha_j} \left[ \beta_j |\lambda|^2 + 2(\lambda, \kappa \nu + w(\rho)) + \beta_j^* |\kappa \nu + w(\rho)|^2 \right] \right).$$

By using (1) and (2) one finds the following symmetry result:

**Lemma 4.5** Let $j \in \{1, 2, \ldots, n\}$. The map $F_j: X \times Y^\vee \times W \to \mathbb{C}$ is invariant under the transformations

(a) $(\lambda, \nu, w) \to (\lambda \pm \kappa x, \nu \mp \beta_j x, w),$

(b) $(\lambda, \nu, w) \to (\lambda, \nu \pm \alpha_j x, w),$

(c) $(\lambda, \nu, w) \to (\lambda \pm \kappa \alpha_j x, \nu, w),$

for any $x \in Y^\vee$. \hfill \Box

By (1) we get that $\prod_{\alpha \in \Delta_+} \sin^{2-n-a_{\alpha}} g \left( \frac{\pi(\lambda, \alpha)}{\kappa} \right)$ is invariant under a change $\lambda \to \lambda + \kappa x$, $x \in Y^\vee$.

**Corollary 4.6** For an arbitrary but fixed $j_0 \in \{1, 2, \ldots, n\}$ and an arbitrary $x \in Y^\vee$, the expression

$$H(\lambda, \nu_1, \ldots, \nu_n) := b^{(\epsilon)}_{\lambda} \lambda^{a_{\lambda}g} \left( \prod_{\alpha \in \Delta_+} \sin^{2-n-a_{\alpha}} g \left( \frac{\pi(\lambda, \alpha)}{\kappa} \right) \right)$$

\[ \times \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} b|\lambda|^2 \right) \sum_{\text{w}_1, \ldots, \text{w}_n \in \mathcal{W}} \left( \prod_{j=1}^{n} F_j(\lambda, \nu_j, \text{w}_j) \right) \]

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is invariant under the transformation
\[
\lambda \rightarrow \lambda \pm \kappa \frac{A}{\alpha_j} x,
\]
\[
\nu_j \rightarrow \nu_j \mp \beta_j \frac{A}{\alpha_j} \delta_{jj_0} x, \quad j = 1, 2, \ldots, n,
\]
where \( A = \prod_{j=1}^n \alpha_j \).

Choose integers \( k_1, \ldots, k_n, l_1, \ldots, l_n, a_1, \ldots, a_n, \) and \( b_1, \ldots, b_n, \) such that
\[
k_j \beta_j + l_j \alpha_j = 1,
\]
\[
a_j \alpha_j + b_j \frac{A}{\alpha_j} = 1
\]
for \( j = 1, 2, \ldots, n \). Let \( (\nu_1, \ldots, \nu_n) \in Y^\vee / \alpha_j Y^\vee \times \ldots Y^\vee / \alpha_n Y^\vee \). Let us show, that for any \( \lambda \in \tilde{P}_\kappa \cap X \) there exists a \( \lambda' \in X \) such that \( H(\lambda, \nu_1, \ldots, \nu_n) = H(\lambda', 0, \ldots, 0) \). To see this, let \( x \in Y^\vee \) and \( j \in \{1, 2, \ldots, n\} \) be arbitrary but fixed. First observe that by Lemma 4.5, \( H \) is invariant under the transformation \( \nu_j \rightarrow \nu_j \pm \beta_j \alpha_j x, \lambda \) and \( \nu_k, k \neq j, \) unchanged. By Corollary 4.6, \( H \) is invariant under the transformation \( \nu_j \rightarrow \nu_j \pm (\beta_j A / \alpha_j) x, \lambda \rightarrow \lambda \mp (\kappa A / \alpha_j) x, \nu_k, k \neq j, \) unchanged. If we use the first transformation \( a_j \) times and the second transformation \( b_j \) times we see that \( H \) is invariant under the transformation
\[
\lambda \rightarrow \lambda \mp (\kappa b_j A / \alpha_j) x,
\]
\[
\nu_k \rightarrow \nu_k \pm \beta_j \delta_{jk} x, \quad k = 1, 2, \ldots, n.
\]
By using this transformation \( k_j \) times and by using the transformation \( \nu_k \rightarrow \nu_k \pm \alpha_j \delta_{jk} x, k = 1, 2, \ldots, n, \lambda \) unchanged, \( l_j \) times we see that \( H \) is invariant under the transformation
\[
\lambda \rightarrow \lambda \mp (\kappa k_j b_j A / \alpha_j) x,
\]
\[
\nu_k \rightarrow \nu_k \pm \delta_{jk} x, \quad k = 1, 2, \ldots, n.
\]
In particular we can change \( \nu_j \) to 0 and keep \( H \) unchanged if we at the same time change \( \lambda \) to \( \lambda + (\kappa k_j b_j A / \alpha_j) \nu_j \), so \( H \) is invariant under the transformation
\[
\lambda \rightarrow \lambda + \kappa \sum_{j=1}^n k_j b_j \frac{A}{\alpha_j} \nu_j,
\]
\[
\nu_k \rightarrow 0, \quad k = 1, 2, \ldots, n.
\]
By using the above result, and the fact that \( H \) is invariant under the transformation \( \lambda \rightarrow \lambda + \kappa A x, x \in Y^\vee \), the \( \nu_k \) unchanged, we can always arrange it so that \( \lambda \) is an element of
\[
J := \tilde{P}_\kappa \cap X + \kappa \left\{ \sum_{i=1}^l m_i \alpha_i^\vee \mid m_1, \ldots, m_l \in \{0, 1, \ldots, A-1\} \right\}.
\]
We have a bijection \((\tilde{P}_\kappa \cap X) \times Y^\vee/\alpha Y^\vee \to P_{\kappa A} \cap X\), \((\lambda, \mu) \mapsto \lambda + \kappa \mu\), and \(J\) is equal to the image of \((\tilde{P}_\kappa \cap X) \times Y^\vee/\alpha Y^\vee\) which can also be identified with \((P_{\kappa A} \setminus H^\kappa) \cap X = X/\kappa A Y^\vee \setminus H^\kappa\).

**Lemma 4.7** Let \(b_j\) and \(k_j\) be as above. The map

\[
(\tilde{P}_\kappa \cap X) \times Y^\vee/\alpha_j Y^\vee \times \ldots \times Y^\vee/\alpha_n Y^\vee \to J
\]

\[
(\lambda, \nu_1, \ldots, \nu_n) \mapsto \lambda + \kappa \sum_{j=1}^n k_j b_j \frac{A}{\alpha_j} \nu_j + \kappa A \sum_{i=1}^l n_i \alpha_i^\vee
\]

is a bijection, where the \(n_i\) are the unique integers such that the right-hand side is an element of \(J\).

**Proof** Assume that

\[
\lambda + \kappa \sum_{j=1}^n k_j b_j \frac{A}{\alpha_j} \nu_j + \kappa A \sum_{i=1}^l n_i \alpha_i^\vee = 0,
\]

where \(\nu_j \in Y^\vee/\alpha_j Y^\vee\), \(j = 1, 2, \ldots, n\), and \(\lambda\) is the difference of two elements in \(\tilde{P}_\kappa \cap X\). Then we immediately get that \(\lambda = 0\) and

\[
\sum_{j=1}^n k_j b_j \frac{A}{\alpha_j} \nu_j + A \sum_{i=1}^l n_i \alpha_i^\vee = 0.
\]

Write \(\nu_j = \sum_{i=1}^l m_i^{(j)} \alpha_i^\vee\), \(m_i^{(j)} \in \{0, 1, \ldots, \alpha_j - 1\}\), and get

\[
\sum_{j=1}^n k_j b_j \frac{A}{\alpha_j} m_i^{(j)} + A n_i = 0, \quad i = 1, 2, \ldots, l,
\]

since \(\alpha_1^\vee, \ldots, \alpha_l^\vee\) is a basis for \(h^*_R\). Now let \(j_0\) be arbitrary but fixed. Then we have

\[
k_{j_0} b_{j_0} \frac{A}{\alpha_{j_0}} m_i^{(j_0)} = - \sum_{j=1, j \neq j_0}^n k_j b_j \frac{A}{\alpha_j} m_i^{(j)} - A n_i
\]

for all \(i \in \{1, 2, \ldots, l\}\). But \(\alpha_{j_0}\) is a divisor of the right-hand side so is also a divisor of \(m_i^{(j_0)}\), since \(\alpha_{j_0}\) and \(k_{j_0} b_{j_0} A/\alpha_{j_0}\) are coprime. Therefore \(m_i^{(j_0)} = 0\), \(i = 1, 2, \ldots, l\), so \(\nu_{j_0} = 0\). It follows that \(\nu_j = 0\) for all \(j \in \{1, 2, \ldots, n\}\) (and \(\sum_{i=1}^l n_i \alpha_i^\vee = 0\), so \(n_1 = \ldots = n_l = 0\)). The surjectivity follows now by the fact that \(J\) and \((\tilde{P}_\kappa \cap X) \times Y^\vee/\alpha_j Y^\vee \times \ldots \times Y^\vee/\alpha_n Y^\vee\) contain the same number of elements, namely \(A^l |\tilde{P}_\kappa \cap X|\) elements.  

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By the above results we get that $Z^g(M; r)$ is given by an expression obtained in the following way: Take the original expression for $Z^g(M; r)$ as stated in Theorem 4.3, divide it by $|W|$, replace the summation index set $I$ by $J$, remove the sum $\sum_{\nu_1 \in Y^\vee/\alpha_1 Y^\vee} \ldots \sum_{\nu_n \in Y^\vee/\alpha_n Y^\vee}$ and put all $\nu_1, \ldots, \nu_n$ equal to zero, i.e.

$$Z^g_\epsilon(M; r) = \frac{1}{|W|} \sum_{\lambda \in J} b^\epsilon_{\lambda} \left( \prod_{\alpha \in \Delta_+} \sin^{2-n-a_\epsilon g} \left( \frac{\pi \langle \lambda, \alpha \rangle}{\kappa} \right) \right) \times \exp \left( \frac{\pi \sqrt{-1} E |\lambda|^2}{\kappa} \right) \sum_{w_1, \ldots, w_n \in W} \prod_{j=1}^{n} \det(w_j) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa \alpha_j} \langle \lambda, w_j(\rho) \rangle \right).$$

Here

$$\sum_{w_1, \ldots, w_n \in W} \left( \prod_{j=1}^{n} \det(w_j) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa \alpha_j} \langle \lambda, w_j(\rho) \rangle \right) \right)$$

$$= \prod_{j=1}^{n} \left( \sum_{w_j \in W} \det(w_j) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa \alpha_j} \langle \lambda, w_j(\rho) \rangle \right) \right)$$

$$= \prod_{j=1}^{n} \prod_{\alpha \in \Delta_+} 2\sqrt{-1} \sin \left( -\frac{\pi}{\kappa \alpha_j} \langle \lambda, \alpha \rangle \right),$$

so we have shown

**Theorem 4.8** Let $M = (\epsilon; g | b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)), \epsilon \in \{o, n\},$ and assume that the $\alpha_j$ are mutually coprime. Then

$$\tau^g_\epsilon(M) = \exp \left( \frac{\pi \sqrt{-1}}{\kappa} \left[ 3(a_\epsilon - 1) \text{sign}(E) - E - \sum_{j=1}^{n} S \left( \frac{\beta_j}{\alpha_j} \right) \right] |\rho|^2 \right)$$

$$\times \frac{\kappa^{l(a_\epsilon g/2-1)}}{2^{\Delta_+ |(a_\epsilon g - 2)\omega(Y^\vee)|W^2} \sqrt{\text{vol}(Y^\vee)^2 - a_\epsilon g |W| \mathcal{A}^{l/2}}}$$

$$\times \exp \left( \frac{3\pi \sqrt{-1}}{h^\vee} (1-a_\epsilon) \text{sign}(E) |\rho|^2 \right) W^g_\epsilon(M; r),$$

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where $A = \prod_{j=1}^{n} \alpha_j$ and

$$W^g_r(M; r) = \sum_{\lambda \in J} b^g_{\lambda} \epsilon^\alpha(\lambda, \alpha) \prod_{\alpha \in \Delta_+} \sin\left(2n - a\epsilon_g(\lambda, \alpha)\right) \prod_{j=1}^{n} \prod_{\alpha \in \Delta_+} \sin\left(\pi \left\langle \lambda, \alpha \right\rangle \kappa\right) \exp\left(\pi \sqrt{-1} \kappa E|\lambda|^2\right) \prod_{j=1}^{n} \prod_{\alpha \in \Delta_+} \sin\left(\pi \kappa \alpha_j \left\langle \lambda, \alpha \right\rangle\right).$$

The $RT$–invariant $\tau^g_r(M)$ of the Seifert manifold $M$ with non-normalized Seifert invariants $\{\epsilon, g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ is given by the same expression.

5 The case of lens spaces

In this section we present different expressions for the invariant $\tau^g_r(L(p, q))$, $p, q$ being an arbitrary but fixed pair of coprime integers. Let $b, d$ be any integers such that $U = \begin{pmatrix} q & b \\ p & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. Assume $q \neq 0$, let

$$V = -\Xi U = \begin{pmatrix} p & d \\ -q & -b \end{pmatrix},$$

and let $C' = (a_1, a_2, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$ such that $B^{C'} = V$. Then $C'$ is a continued fraction expansion of $-p/q$ and $U = \Xi V = B^C$ where $C = (a_1, a_2, \ldots, a_{n-1}, 0)$. By Corollary 4.2, (27) and (30) we therefore get

$$\tau^g_r(L(p, q)) = \omega^{\Phi(U)} \tilde{U}_{\rho \rho},$$

(35)

where $\omega$ is given by (28). If $q = 0$ we have $p = 1$ and $L(p, q) = S^3$. In this case we have $\tau^g_r(L(p, q)) = D^{-1}$. We also have $U = \Xi \Theta^d$ so by using (12), (26) and (28) we find that the right-hand side of (35) is also equal to $D^{-1}$. The identity (35) coincides with [26, Formula (3.7)] for $g = \mathfrak{sl}_2(\mathbb{C})$, see also [17, Formula (49)].

By elaborating on the expression (35) along the same lines as in [26, Sect. 3] we can now easily derive an explicit expression for $\tau^g_r(L(p, q))$. If $q \neq 0$ we get
by Theorem 2.6 that
\[
R(U)_{\rho \rho} = \left( \sqrt{-1} \text{sign}(p) \right)^{\Delta_+} \frac{(\kappa |p|)^{l/2} \text{vol}(Y^\vee)}{\exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} \Phi(U) |\rho|^2 \right)} \times \exp \left( \frac{\pi \sqrt{-1} b}{\kappa} |q| \right) \sum_{w \in W} \text{det}(w) \\
\times \sum_{\nu \in Y^\vee / pY^\vee} \exp \left( \frac{\pi \sqrt{-1} q}{pq \kappa} |\nu| \right) \exp \left( \frac{2\pi \sqrt{-1}}{p} \langle \nu, q\rho - w(\rho) \rangle \right).
\]

By (28) and (35) we then get
\[
\tau_p^0(L(p, q)) = \left( \sqrt{-1} \text{sign}(p) \right)^{\Delta_+} \frac{(\kappa |p|)^{l/2} \text{vol}(Y^\vee)}{\exp \left( -\frac{\pi \sqrt{-1}}{h^\vee} \Phi(U) |\rho|^2 \right)} \times \exp \left( \frac{\pi \sqrt{-1} b}{\kappa} |q| \right) \sum_{w \in W} \text{det}(w) \\
\times \sum_{\nu \in Y^\vee / pY^\vee} \exp \left( \frac{\pi \sqrt{-1} q}{pq \kappa} |\nu| \right) \exp \left( \frac{2\pi \sqrt{-1}}{p} \langle \nu, q\rho - w(\rho) \rangle \right).
\]

Here
\[
\exp \left( \frac{\pi \sqrt{-1} b}{\kappa} |q| \right) \exp \left( \frac{\pi \sqrt{-1} q}{pq \kappa} |\nu| \right) \exp \left( \frac{2\pi \sqrt{-1}}{p} \langle \nu, q\rho - w(\rho) \rangle \right)
\]

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\[
\begin{aligned}
&= \exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} \left( pb|\rho|^2 + q^2|\rho|^2 + |\rho|^2 - 2q\langle \rho, w(\rho) \rangle \right) \right) \\
&= \exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} \left( d|\rho|^2 + q|\rho|^2 - 2\langle \rho, w(\rho) \rangle \right) \right),
\end{aligned}
\]

where we use that \( pb + 1 = qd \). Moreover,

\[
\exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} \left( d|\rho|^2 + q|\rho|^2 - 2\langle \rho, w(\rho) \rangle \right) \right)
= \exp \left( -\frac{2\pi\sqrt{-1}}{pq\kappa} \langle \rho, w(\rho) \rangle \right) \exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} \left( d + q - \Phi(U) \right) |\rho|^2 \right).
\]

Here

\[
\Phi(U) = \frac{d + q}{p} - 12 \text{sign}(p)s(d, p),
\]

\( qd \equiv 1 \pmod{p} \)

so

\[
\exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} \left( d + \frac{q}{p} - \Phi(U) \right) |\rho|^2 \right) = \exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} 12 \text{sign}(p)s(q, p)|\rho|^2 \right).
\]

By putting all the pieces together and using (32) we get

**Theorem 5.1** The RT–invariants associated to \( g \) of the lens space \( L(p, q) \), \( p \neq 0 \), are given by

\[
\tau_{r}^{g}(L(p, q)) = \left( \frac{\sqrt{-1} \text{sign}(p)}{\kappa|p|^{1/2} \text{vol}(Y^\vee)} \right)^{|\Delta_{+}|} \exp \left( \frac{\pi\sqrt{-1}}{\kappa} S \left( \frac{q}{p} \right) |\rho|^2 \right) \\
\quad \times \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi\sqrt{-1}}{pq\kappa} \langle \rho, w(\rho) \rangle \right) \\
\quad \times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} q|\nu|^2 \right) \exp \left( 2\pi\sqrt{-1} \frac{q}{p} \langle \nu, q\rho - w(\rho) \rangle \right). 
\]

We also have

\[
\tau_{r}^{g}(L(p, q)) = \left( \frac{2 \text{sign}(p)}{\kappa|p|^{1/2} \text{vol}(Y^\vee)} \right)^{|\Delta_{+}|} \exp \left( \frac{\pi\sqrt{-1}}{\kappa} S \left( \frac{q}{p} \right) |\rho|^2 \right) \\
\quad \times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \frac{\pi\sqrt{-1}}{pq\kappa} q|\nu|^2 \right) \exp \left( 2\pi\sqrt{-1} \frac{q}{p} \langle \nu, \rho \rangle \right) \\
\quad \times \prod_{\alpha \in \Delta_{+}} \sin \left( \frac{\pi}{pq\kappa} \langle \rho + \kappa\nu, \alpha \rangle \right).
\]
The second formula simply follows by using the Weyl denominator formula on
the first formula. (A direct check shows, that the above theorem is also true
for \( q = 0 \), in which case \( p = \pm 1 \) and \( L(p, q) = S^3 \). If \( p = 0 \), then \( q = \pm 1 \) and \( L(p, q) = S^1 \times S^2 \), and \( \tau^\theta(S^1 \times S^2) = 1 \) in the normalization used here.)

**The coprime case** In this subsection we consider the coprime case, i.e. the
case \( (r, p) = 1 \), \( r = m\kappa \). In particular \( p \neq 0 \). From Theorem 5.1 we have

\[
\tau^\theta(L(p, q)) = \frac{\sqrt{-1} \text{sign}(p)|\Delta|}{(|p|)l/2 \text{vol}(Y^\vee)} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} S \left( \frac{q}{p} \right) |\rho|^2 \right) \Sigma,
\]

where

\[
\Sigma = \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{p\kappa} \langle \rho, w(\rho) \rangle \right) \exp \left( -\frac{\pi \sqrt{-1}}{pq\kappa} \left| \frac{q\rho - w(\rho)}{q\kappa} \right|^2 \right) \times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \pi \sqrt{-1} \frac{q}{p} |\nu + \frac{q\rho - w(\rho)}{q\kappa}|^2 \right).
\]

First assume that \( p \) is odd. Let \( \rho_w = q\rho - w(\rho) \). Since \( p \) and \( 4qr \) are coprime, there exist integers \( c \) and \( a \) such that \( pc + 4qra = 1 \). By definition of \( \rho \), we have \( \rho \in \frac{1}{2}Y \). Moreover, \( mY \subseteq Y^\vee \), so \( 2m\rho_w \in Y^\vee \). Therefore

\[
\Sigma = \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{p\kappa} \langle \rho, w(\rho) \rangle \right) \exp \left( -\frac{\pi \sqrt{-1}}{pq\kappa} |\rho_w|^2 \right) \times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \pi \sqrt{-1} \frac{q}{p} |\nu + \rho_w (4am + \frac{pc}{q\kappa})|^2 \right)
\]

\[
= \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{p\kappa} \langle \rho, w(\rho) \rangle \right) \exp \left( -\frac{\pi \sqrt{-1}}{pq\kappa} |\rho_w|^2 \right) \times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \pi \sqrt{-1} \frac{q}{p} |\nu + \rho_w \frac{pc}{q\kappa}|^2 \right)
\]

\[
= \Sigma' \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \pi \sqrt{-1} \frac{q}{p} |\nu|^2 \right),
\]

where

\[
\Sigma' = \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{p\kappa} \langle \rho, w(\rho) \rangle \right) \exp \left( \frac{\pi \sqrt{-1}}{pq\kappa} |\rho_w|^2 (p^2c^2 - 1) \right).
\]
We used here (1) and the fact that $\rho_w \in X$. Since $p^2c^2 - 1 = 4qra(4qra - 2)$ we get

$$
\Sigma' = \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi\sqrt{-1}}{p\kappa} (\rho, w(\rho)) \right) \\
\times \exp \left( \frac{\pi\sqrt{-1}}{p} 4am(-2 + 4qra)[(q^2 + 1)|\rho|^2 - 2q(\rho, w(\rho))] \right) \\
= \sum_{w \in W} \det(w)e^\beta e^w,
$$

where

$$
e^\beta = \exp \left( -\frac{4\pi\sqrt{-1}}{p} am(1 + pc)(q^2 + 1)|\rho|^2 \right),
$$

$$
e^w = \exp \left( -\frac{2\pi\sqrt{-1}}{p\kappa} [1 + 4qra(4qra - 2)] \langle \rho, w(\rho) \rangle \right).
$$

Since $\rho \in X$ and $2m\rho \in Y^\vee$ we have $2m|\rho|^2 \in \mathbb{Z}$ so

$$
e^\beta = \exp \left( -\frac{4\pi\sqrt{-1}}{p} am(q^2 + 1)|\rho|^2 \right) = \exp \left( -\frac{2\pi\sqrt{-1}}{p} 4^*(q + q^*)r^*2m|\rho|^2 \right),
$$

where $n^*$ is the inverse of $n \pmod{p}$ for any integer $n$ coprime to $p$. Moreover,

$$
e^w = \exp \left( -\frac{2\pi\sqrt{-1}}{\kappa} pc^2 \langle \rho, w(\rho) \rangle \right) = \exp \left( -\frac{2\pi\sqrt{-1}}{\kappa} c\langle \rho, w(\rho) \rangle \right),
$$

where we used $pc^2 = c - 4qrac$ and the fact that $2m\langle \rho, w(\rho) \rangle \in \mathbb{Z}$. We thus obtain

$$
\tau_r^\phi(L(p, q)) = \frac{(\sqrt{-1}\text{sign}(p))^{\Delta_+}}{(\kappa|p|)^{t/2}\text{vol}(Y^\vee)} \exp \left( \frac{\pi\sqrt{-1}}{\kappa} S \left( \frac{q}{p} \right) |\rho|^2 \right) \\
\times \exp \left( -\frac{2\pi\sqrt{-1}}{p} 4^*(q + q^*)r^*2m|\rho|^2 \right) \\
\times \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi\sqrt{-1}}{\kappa} c\langle \rho, w(\rho) \rangle \right) \\
\times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \pi\sqrt{-1}q_p\kappa|\nu|^2 \right).
$$
Here
\[
\sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa} c(\rho, w(\rho)) \right) = \prod_{\alpha \in \Delta_+} 2\sqrt{-1} \sin \left( -\frac{\pi c}{\kappa} \langle \rho, \alpha \rangle \right)
\]
by the Weyl denominator formula. Note that \(c\) is the inverse of \(p \pmod{4r}\).

Next we assume that \(p\) is even. Then \(q\) is odd and there exist two integers \(c\) and \(a\) such that \(4pc + qra = 1\). We put \(\rho_w = \frac{1}{2}(q\rho - w(\rho)) \in \frac{1}{2}Y\), so \(2m\rho_w \in Y^\vee\). Moreover, \(2\rho_w \in Y \subseteq X\). We therefore find
\[
\Sigma = \Sigma'' \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \pi \sqrt{-1} \frac{q}{p}\kappa |\nu|^2 \right),
\]
where
\[
\Sigma'' = \sum_{w \in W} \det(w) \exp \left( \frac{2\pi \sqrt{-1}}{pk} (\rho, w(\rho)) \right)
\]
\[
\times \exp \left( \frac{\pi \sqrt{-1}}{pq\kappa} (16p^2c^2 - 1)(2\rho_w)^2 \right)
\]
\[
= \sum_{w \in W} \det(w) \exp \left( \frac{2\pi \sqrt{-1}}{pk} (\rho, w(\rho)) \right)
\]
\[
\times \exp \left( \frac{am\pi \sqrt{-1}}{p} (qra - 2) \left[ (q^2 + 1)|\rho|^2 - 2q(\rho, w(\rho)) \right] \right)
\]
\[
= \sum_{w \in W} \det(w)e^\beta e^w,
\]
where
\[
e^\beta = \exp \left( \frac{am\pi \sqrt{-1}}{p} (qra - 2)(q^2 + 1)|\rho|^2 \right),
\]
\[
e^w = \exp \left( \frac{2amq\pi \sqrt{-1}}{p} (2 - qra)(\rho, w(\rho)) - \frac{2\pi \sqrt{-1}}{pk} (\rho, w(\rho)) \right).
\]
Here \(qra - 2 = -1 - 4pc\) so
\[
e^\beta = \exp \left( \frac{2\pi \sqrt{-1}}{4p} a(q^2 + 1)2m|\rho|^2 \right)
\]
\[
= \exp \left( \frac{2\pi \sqrt{-1}}{4p} (q + q^*)r^*2m|\rho|^2 \right),
\]
where \(n^*\) is the inverse of \(n \pmod{4p}\) for any integer \(n\) coprime to \(p\). Moreover,
\[
e^w = \exp \left( \frac{2\pi \sqrt{-1}}{pk} (aqr - 1)(\rho, w(\rho)) \right) = \exp \left( \frac{2\pi \sqrt{-1}}{\kappa} 4c(\rho, w(\rho)) \right).
\]
We have thus shown
Proposition 5.2 Let $r = m\kappa$ be coprime to $p$. If $p$ is odd we have

$$
\tau_p^g(L(p, q)) = \frac{\sqrt{-1}\text{sign}(p)^{\lfloor \Delta_+ \rfloor}}{(\kappa|p|)^{t/2} \text{vol}(Y^\vee)} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} S \left( \frac{q}{p} \right) |\rho|^2 \right) \\
\times \exp \left( -\frac{2\pi \sqrt{-1}}{p} 4^*(q + q^*)r^*2m|\rho|^2 \right) \\
\times \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa} c(\rho, w(\rho)) \right) \\
\times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{p} q|\nu|^2 \right).
$$

Alternatively we have

$$
\tau_p^g(L(p, q)) = \frac{\sqrt{-1}\text{sign}(p)^{\lfloor \Delta_+ \rfloor}}{(\kappa|p|)^{t/2} \text{vol}(Y^\vee)} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} S \left( \frac{q}{p} \right) |\rho|^2 \right) \\
\times \exp \left( -\frac{2\pi \sqrt{-1}}{p} 4^*(q + q^*)r^*2m|\rho|^2 \right) \\
\times \left( \prod_{\alpha \in \Delta_+} \sin \left( \frac{\pi c}{\kappa} \langle \rho, \alpha \rangle \right) \right) \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{p} q|\nu|^2 \right).
$$

Here $n^*$ is the inverse of $n \pmod p$ for any integer $n$ coprime to $p$, and $c$ is the inverse of $p \pmod {4r}$.

If $p$ is even we have

$$
\tau_p^g(L(p, q)) = \frac{\sqrt{-1}\text{sign}(p)^{\lfloor \Delta_+ \rfloor}}{(\kappa|p|)^{t/2} \text{vol}(Y^\vee)} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} S \left( \frac{q}{p} \right) |\rho|^2 \right) \\
\times \exp \left( -\frac{2\pi \sqrt{-1}}{4p} (q + q^*)r^*2m|\rho|^2 \right) \\
\times \sum_{w \in W} \det(w) \exp \left( -\frac{2\pi \sqrt{-1}}{\kappa} 4c(\rho, w(\rho)) \right) \\
\times \sum_{\nu \in Y^\vee/pY^\vee} \exp \left( \frac{\pi \sqrt{-1}}{p} q|\nu|^2 \right).
$$

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Alternatively we have

\[ \tau^B_r(L(p, q)) = \frac{(2 \text{sign}(p))^{\Delta + 1}}{(|\kappa| p)^{1/2} \text{vol}(Y)} \exp \left( \frac{\pi \sqrt{-1}}{\kappa} S \left( \frac{q}{p}, |\rho|^2 \right) \right) \times \exp \left( -\frac{2\pi \sqrt{-1}}{4p} (q + q^*) r^* 2m |\rho|^2 \right) \times \left( \prod_{\alpha \in \Delta_+} \sin \left( \frac{4\pi c}{\kappa} \langle \rho, \alpha \rangle \right) \right) \sum_{\nu \in Y \lor /pY} \exp \left( \pi \sqrt{-1} \frac{q}{p} |\nu|^2 \right). \]

Here \( n^* \) is the inverse of \( n \) (mod \( 4p \)) for any integer \( n \) coprime to \( p \), and \( 4c \) is the inverse of \( p \) (mod \( r \)).

**Remark 5.3**

a) By a direct check using Theorem 5.1 one finds that \( L(64, 9) \) and \( L(64, 25) \) are distinguished by the \( sl_4(C) \)–invariant, in fact by \( \tau^e_{sl_4(C)} \). On the other hand these lens spaces can not be distinguished by the LMO invariant. This follows by [10, Proposition 5.1] and the fact that \( S(9/64) = S(25/64) = -63/32 \), cf. [30, Remark (4.14)]. Since the perturbative invariant \( \tau^P_{sl_4(C)} \) can be recovered from the LMO invariant, cf. [41], we see that \( \tau^P_{sl_4(C)} \) does not separate \( L(64, 9) \) and \( L(64, 25) \). The perturbative invariant \( \tau^P_{sl_4(C)} \) is determined by the family of quantum \( P_{sl_4(C)} \)–invariants \( \tau^P_{sl_4(C)} \), see [36]. Oppositely it is expected that the perturbative invariant \( \tau^P_{sl_4(C)} \) dominates the invariants \( \tau^P_{sl_4(C)} \), cf. [36, Conjecture 1.8]. From the explicit formulas for the \( P_{sl_4} \)–invariants \( \tau^P_{sl_4(C)} \) of the lens spaces in [51], it follows that these invariants can not distinguish between \( L(64, 9) \) and \( L(64, 25) \) for \( r \) a prime. It would be interesting to examine if this is also the case for the non-prime \( r \) for which \( \tau^P_{sl_4(C)} \) is defined.

b) In a forthcoming paper [20] we make detailed calculations of the Gaussian sums \( \sum_{\nu \in Y \lor /pY} \exp \left( \pi \sqrt{-1} \frac{q}{p} |\nu|^2 \right) \), thereby obtaining more detailed separation results.

**The asymptotic expansion conjecture and lens spaces**

In this section we calculate the large \( r \) asymptotics of \( r \mapsto \tau^B_r(L(p, q)) \), \( r = m \kappa \). Let us begin by some introductory remarks. Therefore, let \( X \) be an arbitrary closed oriented 3–manifold and let \( G \) be a simply connected compact simple Lie group with complexified Lie algebra \( g \). We are interested in the behaviour of the complex function \( \kappa \mapsto \tau^B_r(X) \) in the limit of large \( \kappa \), i.e. \( \kappa \to \infty \). It is believed that Witten’s TQFT associated with \( G \) and \( k \) coincides with the TQFT of
Reshetikhin and Turaev associated with $\mathfrak{g}$ and $k + h^\vee$. In particular it is conjectured that Witten’s semiclassical approximation for $Z_k^G(X)$ should be valid for the function $\kappa \mapsto \tau^\mathfrak{g}_\kappa(X)$, and furthermore that this function should have a full asymptotic expansion in the limit $\kappa \to \infty$. The precise formulation of this is stated in the following conjecture, called the asymptotic expansion conjecture (AEC).

**Conjecture 5.4** (J. E. Andersen [1], [2]) Let $\{\alpha_1, \ldots, \alpha_M\}$ be the set of values of the Chern–Simons functional of flat $G$ connections on a closed oriented 3–manifold $X$. Then there exist $d_j \in \mathbb{Q}$, $\tilde{I}_j \in \mathbb{Q}/\mathbb{Z}$, $b_j \in \mathbb{R}^+$ and $c_n^{(j)} \in \mathbb{C}$ for $j = 1, \ldots, M$ and $n = 1, 2, 3, \ldots$ such that we for all $N = 0, 1, 2, \ldots$ have

$$\tau^\mathfrak{g}_{\text{max}}(X) = \sum_{j=1}^{M} b_j e^{2\pi \sqrt{-1} \kappa \alpha_j} e^{\pi \sqrt{-1} \tilde{I}_j/4} \left(1 + \sum_{n=1}^{N} c_n^{(j)} \kappa^{-n}\right) + o(\kappa^{d-N})$$

(36)
in the limit $\kappa \to \infty$, where $d = \max\{d_1, \ldots, d_M\}$.

Here $f(\kappa) = o(\kappa^{d-N})$ means as usual that $f(\kappa)/\kappa^{d-N} \to 0$ as $\kappa \to \infty$. The AEC was proposed by Andersen in [1], where he proved it for the mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least 2 and for general $\mathfrak{g}$ using the gauge theoretic approach to the quantum invariants. These manifolds are Seifert manifolds with orientable base and Seifert Euler number equal to zero, see [1, Sect. 4]. Note that the semiclassical approximation is given by putting $N = 0$ in the sum $\sum_{j=1}^{M} \ldots$ in (36). This part of the AEC and some of the conjectures concerning the topological interpretation of the different parts of the asymptotic formula, see [2] for details, are in fact inspired by the works of Witten, Freed and Gompf, Jeffrey and Rozansky on the semiclassical approximation of Witten’s invariants.

As already stated, Jeffrey [25], [26] and Garoufalidis [14] made completely rigorous calculations of the semiclassical approximation of the $SU(2)$–invariants of lens spaces. Actually these calculations contain a complete verification of the AEC for the lens spaces and $\mathfrak{g} = sl_2(\mathbb{C})$. Jeffrey’s calculations also contain a proof of the AEC for a certain class of mapping tori over the torus for an arbitrary complex finite dimensional simple Lie algebra.

In [46], [47] L. Rozansky calculated the Witten $SU(2)$–invariants of all Seifert manifolds with orientable base and carried through a rather technical analysis leading to a candidate for the full asymptotic expansion of these invariants (for the Seifert manifolds with Seifert Euler number different from zero). As shown in [17, Sect. 8] the invariants calculated by Rozansky are equal to the
RT–invariants associated to $\mathfrak{sl}_2(\mathbb{C})$. However, to actually prove that his formula gives the asymptotic expansion of the invariants one has to incorporate estimations of error terms in the calculations. This was left out in Rozansky’s calculations. In [16], [18] the first author has supplemented the calculations of Rozansky by making the necessary error estimates thereby proving, that Rozansky’s formula is really the asymptotic expansion of these invariants. In [18] the calculations are carried through for a big class of functions including the $\mathfrak{sl}_2(\mathbb{C})$–invariants of all oriented Seifert manifolds with orientable base or non-orientable base with even genus (also the ones with Seifert Euler number equal to zero). Based on results of D. Auckly [5] the Chern–Simons invariants can be identified in the asymptotic formula thereby proving the AEC, Conjecture 5.4, for these Seifert manifolds and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. It should be mentioned that Rozansky and Lawrence have calculated the asymptotics of the $\mathfrak{sl}_2(\mathbb{C})$–invariants of a subclass of the Seifert manifolds with base $S^2$ by a method which avoids the rather technical analysis of error terms, cf. [33]. However, it seems that their method does not work for arbitrary oriented Seifert manifolds.

From Theorem 5.1 it is obvious, that the large $\kappa$ asymptotics of $\tau_\mathfrak{g}^\theta(L(p,q))$ is on a form as predicted by the asymptotic expansion conjecture, Conjecture 5.4. We note that $L(p,q)$ and $L(p',q')$ are homeomorphic if and only if $p = p'$ and

$$q \equiv \pm q' \pmod{p} \quad \text{or} \quad qq' \equiv \pm 1 \pmod{p}.$$

A homeomorphism preserves orientation if and only if the relevant sign is $+$. If $q^*$ denotes the inverse of $q$ in the group of units of $\mathbb{Z}/p\mathbb{Z}$ we therefore have that there is an orientation preserving homeomorphism between $L(p,q)$ and $L(p,q^*)$. In particular, we can exchange $q$ and $q^*$ in any of the formulas for $\tau_\mathfrak{g}^\theta(L(p,q))$. For the following discussion this seems to be an advantage. We have

$$\sin \left( \frac{\pi}{p\kappa} (\rho + \kappa \nu, \alpha) \right) = \sin \left( \frac{\pi}{p\kappa} \langle \rho, \alpha \rangle \right) \cos \left( \frac{\pi}{p} \langle \nu, \alpha \rangle \right) + \cos \left( \frac{\pi}{p\kappa} \langle \rho, \alpha \rangle \right) \sin \left( \frac{\pi}{p} \langle \nu, \alpha \rangle \right).$$

If we let

$$\mathcal{M}_j = \{ \nu \in Y^\vee/pY^\vee \mid \langle \nu, \alpha \rangle \in p\mathbb{Z} \text{ for exactly } j \text{ elements } \alpha \text{ in } \Delta_+ \}$$

for $j = 0, 1, \ldots, |\Delta_+|$, then we get

**Corollary 5.5** There exists a family of complex numbers $c_n^{(\nu)}$, $n = 1, 2, \ldots$,
\( \nu \in Y^\vee/pY^\vee \), depending directly on \( p \) but only on \( q \) through \( S(q/p) \), such that

\[
\tau_s^p(L(p,q)) = \frac{(2 \text{sign}(p))^{[\Delta_+]}}{(\kappa|p|)^{1/2} \text{vol}(Y^\vee)} \sum_{j=0}^{[\Delta_+]} \left( \frac{\pi}{p\kappa} \right)^j \sum_{\nu \in M_j} b_\nu \exp \left( 2\pi \sqrt{-1} \kappa \frac{q^*}{2p} |\nu|^2 \right) \times \exp \left( 2\pi \sqrt{-1} \frac{q^*}{p} \langle \nu, \rho \rangle \right) \left( 1 + \sum_{n=1}^\infty c_n^{(\nu)} \kappa^{-n} \right),
\]

for all \( \kappa \in \mathbb{Z}_{\geq \sqrt{h^\vee}} \), where

\[
b_\nu = \prod_{\alpha \in \Delta_+: \langle \nu, \alpha \rangle \in p\mathbb{Z}} (-1)^{\langle \nu, \alpha \rangle/p} \langle \rho, \alpha \rangle \prod_{\alpha \in \Delta_+: \langle \nu, \alpha \rangle \notin p\mathbb{Z}} \sin \left( \frac{\pi}{p} \langle \nu, \alpha \rangle \right).
\]

Note here that \( S(q/p) = S(q^*/p) \). The infinite power series in \( 1/\kappa \) present in the above corollary are convergent for all \( \kappa \in \mathbb{Z}_{\geq \sqrt{h^\vee}} \). Note also that \( 0 \in M_{[\Delta_+]} \), so this set is non-empty. Let us in some greater detail look at a few examples.

First assume that \( g = sl_2(\mathbb{C}) \). In this case we have that \( \Delta_+ \) contains one element \( \alpha \) of length \( \sqrt{2} \) and \( Y^\vee = Y = \text{Span}_\mathbb{Z}\{\alpha\} \) so \( \text{vol}(Y^\vee) = \sqrt{2} \). For \( n \in \{0, 1, \ldots, |p| - 1\} \) we have

\[
\langle n\alpha, \alpha \rangle = 2n
\]

so if we identify \( n\alpha \) by \( n \) we have

\[
\mathcal{M}_1 = \begin{cases} 
\{0\}, & \text{if } p \text{ is odd} \\
\{0, |p|/2\}, & \text{if } p \text{ is even}
\end{cases}
\]

and \( \mathcal{M}_0 = \{0, 1, \ldots, |p| - 1\} \setminus \mathcal{M}_1 \). For \( p \) odd we therefore have

\[
\tau_{sl_2(\mathbb{C})}^p(L(p,q)) = \frac{\pi}{|p|\kappa} \sqrt{\frac{2}{|p|\kappa}} \left( 1 + \sum_{l=1}^\infty c_l^{(0)} \kappa^{-l} \right) + \text{sign}(p) \sqrt{\frac{2}{|p|\kappa}} \sum_{n=1}^{\frac{|p|-1}{2}} \exp \left( 2\pi \sqrt{-1} \kappa \frac{q^*}{p} n^2 \right) \exp \left( 2\pi \sqrt{-1} \frac{q^*}{p} n \right) \times \sin \left( \frac{2\pi n}{|p|} \right) \left( 1 + \sum_{l=1}^\infty c_l^{(n)} \kappa^{-l} \right).
\]

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For $p$ even we find that
\[
\tau_{r}^{sL(C)}(L(p, q)) = \frac{\pi}{|p|\kappa} \sqrt{\frac{2}{|p|\kappa}} \left( 1 - (-1)^{q^{*}p/2} + \sum_{l=1}^{\infty} \left( c_{l}^{(0)} + c_{l}^{|p|/2} \right) \kappa^{-l} \right) \\
+ \text{sign}(p) \sqrt{\frac{2}{|p|\kappa}} \sum_{n \in \{1, 2, \ldots, |p| - 1\} : n \neq \frac{|p|}{2}} \exp \left( 2\pi \sqrt{-1} \frac{q^{*}n^{2}}{p} \right) \times \exp \left( 2\pi \sqrt{-1} \frac{q^{*n}}{p} \right) \sin \left( \frac{2\pi q^{*}n}{p} \right) \left( 1 + \sum_{l=1}^{\infty} c_{l}^{(n)} \kappa^{-l} \right).
\]

For all $p$ we find that the leading large $\kappa$ asymptotics of $\tau_{r}^{sL(C)}(L(p, q))$ is
\[
\sqrt{-1} \text{sign}(p) \sqrt{\frac{2}{|p|\kappa}} \sum_{n=1}^{|p|-1} \exp \left( 2\pi \sqrt{-1} \frac{q^{*}n^{2}}{p} \right) \sin \left( \frac{2\pi q^{*}n}{p} \right) \sin \left( \frac{2\pi n}{p} \right),
\]
where we use that
\[
\sum_{n=1}^{|p|-1} \exp \left( 2\pi \sqrt{-1} \frac{q^{*}n^{2}}{p} \right) \cos \left( 2\pi q^{*}n \right) \sin \left( 2\pi n \right) = 0.
\]

This result coincides with [26, Formula (5.7)]. It is well-known, see e.g. [26, Formula (5.3)], that the set of values of the Chern–Simons functional of flat $SU(2)$ connections on $L(p, q)$ is \footnote{There seems to be a problem with signs here. The set $S$ of Chern–Simons invariants has been calculated by Kirk and Klassen, cf. [31, Theorem 5.1]. According to their result all the above stated Chern–Simons values have to be multiplied by $-1$. Note that $L(p, q)$ in [31] is equal to $L(p, -q)$ here.}
\[
S = \left\{ \frac{q^{*}n^{2}}{p} \text{ (mod 1)} \mid n = 0, 1, \ldots, |p| - 1 \right\},
\]
so here we see the reason for replacing $q$ by $q^{*}$. (Note that $q^{*}(p-n)^{2}/p \equiv q^{*}n^{2}/p$ (mod 1).) If we e.g. have $p = k^{2}$, where $k$ is a positive integer, then
\[
\frac{q^{*}n^{2}}{p} \equiv 0 \text{ (mod 1)}
\]
for $n = k$, so we see that two elements belonging to different sets $\mathcal{M}_{j}$ can have the same Chern–Simons value.

Let us also examine the type $A_{2}$. Here $Y^{\vee} = Y = \text{Span}_{\mathbb{Z}}\{\alpha_{1}, \alpha_{2}\}$, where $\alpha_{i}$ have length $\sqrt{2}$, $i = 1, 2$. We have $\Delta_{+} = \{\alpha_{1}, \alpha_{2}, \alpha_{1} + \alpha_{2}\}$, and in particular
$|\Delta_\nu| = 3$. If we identify $\nu = k\alpha_1 + n\alpha_2 \in Y^\vee/pY^\vee$ with $(k,n) \in \{0,1,\ldots,|p|-1\}^2$ one finds by an elementary analysis that

$$
\mathcal{M}_3 = \begin{cases} 
\{(0,0),(l,2l),(2l,l)\}, & \text{if } |p| = 3l, \ l \in \mathbb{Z} \setminus \{0\}, \\
\{(0,0)\}, & \text{otherwise,}
\end{cases}
$$

and $\mathcal{M}_2 = \emptyset$.

and $\mathcal{M}_0 = \{0,1,\ldots,|p|-1\}^2 \setminus (\mathcal{M}_1 \cup \mathcal{M}_3)$, where $\mathcal{M}_1 = \emptyset$ for $|p| = 3$ and

$$
\mathcal{M}_1 = \left\{(k,n) \in \{0,1,\ldots,|p|-1\}^2 \begin{array}{l}
\text{exactly one of the identities} \\
\text{is satisfied}
\end{array} \begin{array}{l}
k = 2n, \ n = 2k, \\
k = 2n - |p|, \ n = 2k - |p| \\
k + n = |p|\end{array} \right\}
$$

otherwise. For $|p| = 1$ we have $\mathcal{M}_3 = \{(0,0)\}$ and $\mathcal{M}_j = \emptyset$, $j = 0,1,2$, and for $|p| = 2$ we have $\mathcal{M}_3 = \{(0,0)\}$, $\mathcal{M}_1 = \{(0,1),(1,0),(1,1)\}$ and $\mathcal{M}_0 = \mathcal{M}_2 = \emptyset$.

Conjecture 5.5 leads together with Conjecture 5.4 immediately to the following conjecture (use the uniqueness property of asymptotic expansions of the form (36), i.e. the fact that an arbitrary function $f: \mathbb{Z}_{>0} \rightarrow \mathbb{C}$ has at most one asymptotic expansion of the form (36) if the $\alpha_j$’s are mutually different and rational).

**Conjecture 5.6** The set of values of the Chern–Simons functional of flat $G$ connections on $L(p,q)$ is given by

$$
\left\{ \frac{q^n}{2p^2} | \nu |^2 \pmod{\mathbb{Z}} \begin{array}{l}
\nu \in Y^\vee/pY^\vee
\end{array} \right\},
$$

for any simply connected, compact simple Lie group $G$.

Proving this conjecture will together with Corollary 5.5 give a proof of the AEC for the invariants $\tau_g^G(L(p,q))$. In fact, by the uniqueness property of asymptotic expansions of the form (36), Conjecture 5.6 should for $G = SU(n)$ be a corollary of Corollary 5.5 and the recent result of Andersen, mentioned in the introduction. For $G = SU(n)$ Conjecture 5.6 should also follow from results in [40].

### 6 A rational surgery formula for the invariant $\tau^G_f$

In this final section we derive a rational surgery formula for the invariant $\tau^G_f$. The result follows easily from the surgery formula derived in [17] in the setting

\[\text{perhaps with the opposite signs dependend on the choice of conventions}\]
of a general modular tensor category. By rational surgery we mean rational surgery on an arbitrary closed oriented 3–manifold $M$ along a framed link inside $M$. Let us briefly recall the situation from [17]. Let $(V, \{V_i\}_{i \in I})$ be a fixed modular category with a fixed rank $D$, and let $\tau$ be the RT–invariant associated to these data. For a link $L$ we let col($L$) be the set of mappings from the set of components of $L$ to the index set $I$. Before giving the result in the general case, let us first consider rational surgery along links in $S^3$. If $L \subseteq S^3$ is framed and oriented and $\lambda \in$ col($L$) we let $\Gamma(L, \lambda)$ be the colored ribbon graph induced by $L$ with the $i$’th component $L_i$ of $L$ colored by $V_{\lambda(L_i)}$.

**Theorem 6.1** ([17]) Let $L$ be a link in $S^3$ with $n$ components and let $M$ be the 3–manifold given by surgery on $S^3$ along $L$ with surgery coefficient $p_i/q_i \in \mathbb{Q}$ attached to the $i$'th component, $i = 1, 2, \ldots, n$ (so we assume $q_i \neq 0$, $i = 1, 2, \ldots, n$, see the comments to (37)). Moreover, let $\Omega$ be a colored ribbon graph in $M$ (also identified with a colored ribbon graph in $S^3 \setminus L$). Let $L_0$ be $L$ considered as a framed link with all components given the framing 0 and with an arbitrary chosen but fixed orientation. Finally, let $C_i = (a_{i1}, \ldots, a_{im_i})$ be a continued fraction expansion of $p_i/q_i$, $i = 1, 2, \ldots, n$. Then

$$
\tau(M, \Omega) = (\Delta D^{-1})^{\sigma+\sum_{i=1}^n c_i D^{-\sum_{i=1}^n m_i}} \times \sum_{\lambda \in \text{col}(L)} \tau(S^3, \Gamma(L_0, \lambda) \cup \Omega) \left( \prod_{i=1}^n G_{\lambda(L_i)0}^{C_i} \right),
$$

where $c_i = \frac{1}{3} \left( \sum_{j=1}^{m_i} a_{ij} - \Phi(B^{C_i}) \right)$, $i = 1, \ldots, n$, and $\sigma$ is the signature of the linking matrix of $L$ (with the surgery coefficients $p_1/q_1, \ldots, p_n/q_n$ on the diagonal).

In the case of surgery on arbitrary closed oriented 3–manifolds along framed links we do not have a preferred framing as in the case of surgery on $S^3$ (or on another integral homology sphere), i.e. we cannot identify the framing of a link component with an integer in a canonical way, see [17, Appendix B]. Here, by a framed link in a closed oriented 3–manifold $M$, we mean a pair $(L, Q)$, where $Q = \prod_{i=1}^n Q_i$: $\Pi_{i=1}^n (B^2 \times S^1) \to M$ is an embedding (or more precisely an isotopy class of such embeddings) and $L$ is the image by $Q$ of $\Pi_{i=1}^n (0 \times S^1)$. For other definitions of framed links in 3–manifolds and how these relate to this definition we refer to [17, Appendix B]. The following result is sensitive to a choice of orientations. We will use the following conventions.

**Conventions 6.2** The space $B^2 \times S^1$ is the standard solid torus in $\mathbb{R}^3$ with the orientation induced by the standard right-handed orientation of $\mathbb{R}^3$. Here
$S^1$ is the standard unit circle in the $xz$–plane with centre 0 and oriented counterclockwise, i.e., $e_3$ is a positively oriented tangent vector in the tangent space $T_{e_1}S^1 \subseteq \mathbb{R}^3$, $e_i$ being the $i$'th standard unit vector in $\mathbb{R}^3$, see Fig. 3. For a framed link $(L,Q)$ as above we will always assume that each copy of $B^2 \times S^1$ is this oriented standard solid torus, and that $Q$ is orientation preserving after giving the image of $Q$ the orientation induced by that of $M$ (we can always obtain this by composing some of the $Q_i$ by $g \times \text{id}_{S^1}$ if necessarily, where $g : B^2 \rightarrow B^2$ is an orientation reversing homeomorphism). Moreover, we orient $L$ so that $Q_i$ restricted to $S^1 \times \{0\}$ is orientation preserving for each $i$. The oriented meridian $\alpha$ and longitude $\beta$, see Fig. 3, represent a basis (over $\Lambda$) of $H_1(\Sigma_{(1)}; \Lambda) = \Lambda \oplus \Lambda$, $\Lambda = \mathbb{Z} \oplus \mathbb{R}$, $\Sigma_{(1)} = S^1 \times S^1$. (For the notation $\Sigma_{(1)}$, see [52, Chap. IV].)

\[\begin{figure}[h]
\centering
\begin{tikzpicture}
  \draw[->] (0,0) -- (4,0) node[anchor=north] {$x$};
  \draw[->] (0,0) -- (0,4) node[anchor=east] {$z$};
  \draw (2,0) circle (1cm);
  \draw (-1,1) arc (90:270:1cm);
  \draw (1,-1) arc (-90:90:1cm);
  \draw[dashed] (1,-1) arc (270:90:1cm);
  \draw[<-] (1,-1) arc (270:90:1cm) node[anchor=south] {$\alpha$};
  \draw[<-] (1,1) arc (-90:90:1cm) node[anchor=south] {$\beta$};
\end{tikzpicture}
\caption{Figure 3}
\end{figure}\]

Let us recall the notion of rational surgery on $M$ along $(L,Q)$. Therefore, let $U_i = Q_i(B^2 \times S^1)$ and let $l_i = Q_i(e_1 \times S^1)$ oriented so that $[l_i] = [L_i]$ in $H_1(U_i; \mathbb{Z})$, where $L_i = Q_i(0 \times S^1)$. Moreover, let $\mu_i = Q_i(\partial B^2 \times 1)$ oriented so that $(\partial Q_i)_*([\alpha]) = [\mu_i]$ in $H_1(\partial U_i; \mathbb{Z})$, where $\partial Q_i$ is the restriction of $Q_i$ to $\partial B^2 \times S^1 = \Sigma_{(1)}$. Let $(p_i, q_i)$ be pairs of coprime integers, let $h_i : \partial U_i \rightarrow \partial U_i$ be homeomorphisms such that

\[(h_i)_*([\mu_i]) = \pm (p_i[\mu_i] + q_i[l_i]) \quad (37)\]

in $H_1(\partial U_i; \mathbb{Z})$, let $h$ be the union of the $h_i$, and let $U = \bigcup_{i=1}^n U_i$ be the image of $Q$. Then the 3–manifold $M' = (M \setminus \text{int}(U)) \cup_h U$ is said to be the result of doing surgery on $M$ along the framed link $(L,Q)$ with surgery coefficients $\{p_i/q_i\}_{i=1}^n$. If $q_i = 0$ so $p_i = \pm 1$ we just write $\infty$ for $p_i/q_i$. Such surgeries do not change the manifold (up to an orientation preserving homeomorphism). If, in (37), $p_i = 0$ and $q_i = \pm 1$ for all $i$, i.e. all surgery coefficients are 0, then we call $M'$ the result of doing surgery on $M$ along the framed link $(L,Q)$. We
equip $M'$ with the unique orientation extending the orientation in $M \setminus \text{int}(U)$. The above generalizes ordinary rational surgery along links in $S^3$. We call a homeomorphism $h$ satisfying (37) an attaching map for the surgery. We can and will always choose an orientation preserving attaching map. Up to an orientation preserving homeomorphism the result of doing surgery on $M$ along the framed link $(L, Q)$ with surgery coefficients $\{p_i/q_i\}_{i=1}^n$ is well defined, independent of the choices of representative $Q$ and attaching map $h$.

For $\lambda \in \text{col}(L)$ we let $\Gamma(L, \lambda) = \bigcup_{i=1}^n \Gamma(L_i, \lambda(L_i))$, where $\Gamma(L_i, j)$ is the colored ribbon graph equal to the directed annulus $Q_i(\{-1/2, 1/2\} \times 0) \times S^1$ with oriented core $L_i$ and color $V_j, j \in I$.

**Theorem 6.3** ([17]) Let $C_i = (a_1^{(i)}, \ldots, a_m^{(i)}) \in \mathbb{Z}^{m_i}$ be a continued fraction expansion of $p_i/q_i, i = 1, \ldots, n$. Moreover, let $\Omega$ be a colored ribbon graph in $M'$ (also identified with a colored ribbon graph in $M \setminus L$). Then
\[
\tau(M', \Omega) = (\Delta^{D}^{-1})^\mu + \sum_{\lambda \in \text{col}(L)} \tau(M, \Gamma(L, \lambda) \cup \Omega) \left( \prod_{i=1}^n G_{\lambda(L_i)0}^{C_i} \right),
\]
where $\mu$ is a sum of signs given by (38) and $c_i = \frac{1}{3} \left( \sum_{j=1}^{m_i} a_j^{(i)} - \Phi(B^{C_i}) \right)$,

\[i = 1, \ldots, n.\]

The theorem is proved by using the machinery of the $2+1$–dimensional TQFT of Reshetikhin and Turaev induced by $(\mathcal{V}, \{V_i\}_{i \in I}, \mathcal{D})$, see [17, Sect. 5]. The integer $\mu$, present in the above theorem, is given by the following sum of Maslov indices
\[
\mu = \sum_{i=1}^n \mu((\partial Q_i)_*(\lambda_0), (\partial Q_i)_*(\lambda_i), N_i).
\]

We refer to [52, Sect. IV.3] or [17, Sect. 5.2] for the definitions of Maslov index and Lagrangian subspace. The spaces $\lambda_0, \lambda_i$, and $N_i$ are Lagrangian subspaces of $H_1(\Sigma_{(1)}; \mathbb{R})$ given by $\lambda_0 = \text{Span}_\mathbb{R}\{[\alpha]\}$, $\lambda_i = \text{Span}_\mathbb{R}\{p_i[\alpha] + q_i[\beta]\}$, and $N_i$ equal to the kernel of the inclusion homomorphism $H_1(\partial U_i; \mathbb{R}) \to H_1(M_{i-1} \setminus \text{int}(U_i); \mathbb{R})$, where $M_i$ is the manifold obtained by doing surgery on $M$ along $\left( \bigsqcup_{j=1}^i L_j, \bigsqcup_{j=1}^i Q_j \right)$ with surgery coefficients $\{p_j/q_j\}_{j=1}^i, i = 1, 2, \ldots, n$, and $M_0 = M$.

We have the following corollary to Theorem 6.1 and 6.3:
Corollary 6.4  Let the situation be as in Theorem 6.3 and let \( B_i \in \text{SL}(2, \mathbb{Z}) \) with first column equal to \( \pm \left( \begin{array}{c} p_i \\ q_i \end{array} \right) \), \( i = 1, 2, \ldots, n \). Then

\[
\tau^B(M', \Omega) = \left( \exp \left( \frac{\pi \sqrt{-1}}{h} |\rho|^2 \right) \exp \left( -\frac{\pi \sqrt{-1}}{\kappa} |\rho|^2 \right) \right)^{\sum_{i=1}^n \Phi(B_i) - 3\mu} \times \sum_{\lambda \in \text{col}(L)} \tau^B(M, \Gamma(L, \lambda) \cup \Omega) \left( \prod_{i=1}^n (\tilde{B}_i \lambda(L_i) \rho) \right),
\]

where \( \mu \) is given by (38). If all the \( q_i \) are different from 0, a similar formula holds with \( M', M, L \) and \( \mu \) replaced by \( M, S^3, L_0, \) and \( \sigma \) respectively, where \( M, L_0 \) and \( \sigma \) are as in Theorem 6.1.

To see this simply choose tuples of integers \( C_i \) such that \( B_i = B^C_i \) and use (27), (28), and (30) and the fact that \( C_i \) is a continued fraction expansion of \( p_i/q_i \), \( i = 1, 2, \ldots, n \).

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Appendix

In the proof of the main Lemma 2.4, the reciprocity formula, Proposition 2.2, played a crucial role. The idea of using the reciprocity formula stems from Jeffrey’s calculations in [25], [26] as already stated in the introduction. Jeffrey’s proof of the reciprocity formula, which is a verbatim generalization of the argument presented in the proof of [12, Chap. IX Theorem 1] can be found in her thesis [25].

In this appendix we will first sketch Jeffrey’s proof, which builds on a limiting case of the Poisson summation formula applied to Gaussian functions. A main ingredient is the Fourier transformation of a Gaussian function, which rely on an analytic continuation argument for complex functions of several variables. Next we will present a slightly different argument which avoids the direct use of this Fourier transform result.
Jeffrey’s proof of Proposition 2.2  According to the Poisson summation formula we have
\[ \sum_{n \in \mathbb{Z}^l} \hat{\phi}(an) = (2\pi/a)^l \sum_{n \in \mathbb{Z}^l} \phi(2\pi n/a), \quad a > 0 \]
for \( \phi \in \mathcal{S}(\mathbb{R}^l) \), see [21, p. 178]. Here \( \mathcal{S}(\mathbb{R}^l) \), is the Schwartz space of smooth \( (C^\infty) \) functions that are rapidly decreasing at infinity, and \( \hat{\phi} \) is the Fourier transformation of \( \phi \). Recall that
\[ \hat{\phi}(\xi) = \int_{\mathbb{R}^l} e^{-i(x,\xi)} \phi(x)dx, \quad \phi \in L^1(\mathbb{R}^l) \]
for \( \xi \in \mathbb{R}^l \), where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^l \). (Below \( \langle \cdot, \cdot \rangle \) will also be used to denote the inner product in \( V \), but the meaning of \( \langle \cdot, \cdot \rangle \) will always be clear from the context.) We recall that \( \mathcal{S}(\mathbb{R}^l) \subseteq L^1(\mathbb{R}^l) \).

Let \( V_C = V \otimes_{\mathbb{R}} \mathbb{C} \) and extend \( \langle \cdot, \cdot \rangle \) to a hermitian product in \( V_C \) in the usual way. Let \( \tau \in \text{End}_C(V_C) \) such that the imaginary part of \( \tau \) is positive definite. By assumption the sum
\[ \sum_{\lambda \in \Lambda} \exp(\pi\sqrt{-1}(\tau(\lambda),\lambda)) \exp(2\pi\sqrt{-1}(\lambda,\psi)) \]
is absolutely convergent. In general, if \( \Omega \) is a symmetric complex \( l \times l \)-matrix with positive definite imaginary part and \( z \in \mathbb{C}^l \) and if \( \phi: \mathbb{R}^l \to \mathbb{C} \) is given by
\[ \phi(x) = \exp(\pi\sqrt{-1}x^t\Omega x + 2\pi\sqrt{-1}x^t z) \]
then
\[ \hat{\phi}(2\pi \xi) = \left( \det \left( \frac{\Omega}{\sqrt{-1}} \right) \right)^{-1/2} \exp \left( -\pi\sqrt{-1}(\xi - z)^t\Omega^{-1}(\xi - z) \right). \]
Here the square root is positive on the positive real axis with a cut along the negative real axis. To see this, first assume that \( \Omega^{-1}z \in \mathbb{R}^l \) and complete the square and use [21, Theorem 7.6.1]. The general case then follows by analytic continuation.

By using this result together with the Poisson summation formula we get
\[ \Sigma_{\lambda \in \Lambda} \exp(\pi\sqrt{-1}(\tau(\lambda),\lambda)) \exp(2\pi\sqrt{-1}(\lambda,\psi)) \]
\[ = \text{vol}(\Lambda)^{-1} \left( \det \left( \frac{\tau}{\sqrt{-1}} \right) \right)^{-1/2} \sum_{\mu \in \Lambda*} \exp \left( -\pi\sqrt{-1}(\tau^{-1}(\mu + \psi),\mu + \psi) \right) \]
if we furthermore e.g. assume that \( \tau \) can be represented by a symmetric matrix w.r.t. a basis of \( V_C \) of the form \( \{w_1 \otimes 1, \ldots, w_l \otimes 1\} \), where \( \{w_1, \ldots, w_l\} \) is some basis of \( V \). This is always the case in the situations where we will use (39) below, since \( f: V \to V \) is self-adjoint. Let us use the identity (39) with \( \tau = \frac{1}{\epsilon} f_C + \sqrt{-1}\varepsilon \text{id}_C \), \( \varepsilon > 0 \), where \( f_C = f \otimes \text{id}_C \). For \( \lambda, \alpha \in \Lambda \) we have \( F(\lambda + r\alpha) = F(\lambda) \) by (5), where
\[ F(\lambda) = \exp \left( \frac{\pi\sqrt{-1}}{r}(f(\lambda),\lambda) \right) \exp(2\pi\sqrt{-1}(\lambda,\psi)), \]
(40)
so the left-hand side of (39) becomes

\[ \text{LHS}(\varepsilon) = \sum_{\Lambda/r\Lambda} F(\lambda) \sum_{\alpha \in \Lambda} \exp(-\pi\varepsilon|\lambda + r\alpha|^2). \]

Here

\[ \sum_{\alpha \in \Lambda} \exp(-\pi\varepsilon|\lambda + r\alpha|^2) = \text{vol}(\Lambda^*) \left( \frac{1}{\varepsilon r^2} \right)^{1/2} \times \sum_{\beta \in \Lambda^*} \exp \left( -\pi \langle \beta, \frac{1}{\varepsilon r^2} \beta \rangle \right) \exp \left( 2\pi\sqrt{-1}\langle \beta, \frac{\lambda}{r} \rangle \right) \]

by using (39) with the roles of $\Lambda$ and $\Lambda^*$ reversed and with $\tau^{-1} = -\sqrt{-1}\varepsilon r^2 \text{id}_{V_C}$ (so $\tau = \sqrt{-1}\frac{1}{\varepsilon r^2} \text{id}_{V_C}$ has positive definite imaginary part). We first observe that

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{l/2} \text{LHS}(\varepsilon) = r^{-l} \text{vol}(\Lambda^*) \sum_{\Lambda/r\Lambda} F(\lambda). \]

This follows by the fact that

\[ \lim_{\varepsilon \to 0^+} \sum_{\beta \in \Lambda^*} \exp \left( -\pi \langle \beta, \frac{1}{\varepsilon r^2} \beta \rangle \right) \exp \left( 2\pi\sqrt{-1}\langle \beta, \frac{\lambda}{r} \rangle \right) = 1, \]

which follows by the fact that

\[ \left| \sum_{\beta \in \Lambda^* \setminus \{0\}} \exp \left( -\pi \langle \beta, \frac{1}{\varepsilon r^2} \beta \rangle \right) \exp \left( 2\pi\sqrt{-1}\langle \beta, \frac{\lambda}{r} \rangle \right) \right| \]

\[ \leq \sum_{\beta \in \Lambda^* \setminus \{0\}} \exp \left( -\frac{\pi}{2\varepsilon r^2} |\beta|^2 \right) \leq \exp \left( \frac{\pi}{2\varepsilon r^2} c^2 \right) \sum_{\beta \in \Lambda^*} \exp \left( -\frac{\pi}{2\varepsilon r^2} |\beta|^2 \right) \]

for $\varepsilon \in [0, 1]$, $c = \min\{ |\beta| : \beta \in \Lambda^* \setminus \{0\} \} > 0$.

Hereafter we calculate $\lim_{\varepsilon \to 0^+} \varepsilon^{l/2} \text{RHS}(\varepsilon)$, where $\text{RHS}(\varepsilon)$ is the right-hand side of (39) with $\tau = \frac{1}{r}f_C + \sqrt{-1}\varepsilon \text{id}_{V_C}$. Note that

\[ \tau^{-1} = r f_C^{-1} - \sqrt{-1}\varepsilon (r f_C^{-1})^2 (\text{id}_{V_C} + \sqrt{-1}\varepsilon r f_C^{-1})^{-1}. \]

By using this we find that $\text{RHS}(\varepsilon)$ is equal to

\[ \text{vol}(\Lambda)^{-1} \left( \frac{\tau}{\sqrt{-1}} \right)^{-1/2} \sum_{\mu \in \Lambda^*/f(\Lambda^*)} \exp(-\pi\sqrt{-1}(\mu + \psi, r f^{-1}(\mu + \psi))) \]

\[ \times \sum_{\beta \in \Lambda^*} \exp(-\pi\varepsilon r^2 ((\text{id} + \sqrt{-1}\varepsilon r f_C^{-1})^{-1} (f^{-1}(\mu + \psi) + \beta), f^{-1}(\mu + \psi) + \beta)), \]

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where we use that \( G(\mu + f(\beta)) = G(\mu) \) for \( \mu, \beta \in \Lambda^* \), where \( G(\mu) = \exp(-\pi \sqrt{-1}(\mu + \psi, rf^{-1}(\mu + \psi))) \). The sum \( \sum_{\beta \in \Lambda^*} \) can now be calculated by using (39) once more with \( \psi \) replaced by \( f^{-1}(\mu + \psi) \) and \( \tau = \frac{1}{\sqrt{\varepsilon r}^2} (\id + \sqrt{-1}r f c) \). By doing this we get

\[
\text{RHS}(\varepsilon) = \left( \det\left( \frac{1}{\sqrt{-1}} fc + \varepsilon id \psi \right) \right)^{-1/2} \left( \frac{1}{\varepsilon r^2} \right)^{1/2} \left( \det(\id + \sqrt{-1}r f c) \right)^{1/2} \times \sum_{\mu \in \Lambda^*/f(\Lambda^*)} \exp \left( -\pi \sqrt{-1}(\mu + \psi, rf^{-1}(\mu + \psi)) \right) \times \exp \left(\frac{\pi}{\varepsilon r^2} |\lambda|^2 \right) \left( -\frac{\pi \sqrt{-1}}{r} (f(\lambda), \lambda) \right) \times \exp \left( 2\pi \sqrt{-1}(f^{-1}(\mu + \psi), \lambda) \right).
\]

As before we find that the sum over \( \Lambda \) converges to 1 as \( \varepsilon \to 0_+ \). Therefore

\[
\lim_{\varepsilon \to 0_+} \varepsilon^{l/2} \text{RHS}(\varepsilon) = \left( \det\left( \frac{f}{\sqrt{-1}} \right) \right)^{-1/2} r^{-1/2} \times \sum_{\mu \in \Lambda^*/f(\Lambda^*)} \exp \left( -\pi \sqrt{-1}(\mu + \psi, rf^{-1}(\mu + \psi)) \right).
\]

Since \( \lim_{\varepsilon \to 0_+} \varepsilon^{l/2} \text{RHS}(\varepsilon) = \lim_{\varepsilon \to 0_+} \varepsilon^{l/2} \text{LHS}(\varepsilon) \) the result follows.

**A second proof of Proposition 2.2** This proof builds on the following periodicity result:

**Lemma 6.5** Let \( \Lambda \) be a lattice in \( V \) and let \( h : V \to V \) be a linear map such that \( h(\Lambda) \subseteq \Lambda \). Moreover, let \( g_\varepsilon : V \to V, \varepsilon \in [0, a] \), be a curve of self-adjoint positive definite maps such that \( g_\varepsilon \to g_0 \) in \( \text{End}_R(V) \) as \( \varepsilon \to 0 \), where \( g_0 \) is positive definite, \( a \) being a fixed positive number. Finally, let \( v_0 \in V \) be fixed but arbitrary and let \( F : V \to \mathbb{C} \) be a map such that

\[
F(\lambda + h(\alpha)) = F(\lambda)
\]

for all \( \lambda, \alpha \in \Lambda \). Then

\[
\sum_{\lambda \in \Lambda/h(\Lambda)} F(\lambda) = \text{vol}(\Lambda) |\text{det}(h)| \sqrt{\text{det}(g_0)} \times \lim_{\varepsilon \to 0_+} \varepsilon^{l/2} \sum_{\lambda \in \Lambda} e^{-\pi \varepsilon (\lambda + v_0, g_\varepsilon (\lambda + v_0))} F(\lambda).
\]

**Proof** By assumption we have

\[
\sum_{\lambda \in \Lambda} e^{-\pi \varepsilon (\lambda + v_0, g_\varepsilon (\lambda + v_0))} F(\lambda) = \sum_{\lambda \in \Lambda/h(\Lambda)} F(\lambda) \sum_{\alpha \in \Lambda} e^{-\pi \varepsilon (\lambda + v_0 + h(\alpha), g_\varepsilon (\lambda + v_0 + h(\alpha)))}.
\]

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The lemma will therefore follow if we can show that
\[ \text{vol}(\Lambda)|\det(h)|\sqrt{\det(g_0)} \lim_{\varepsilon \to 0+} \varepsilon^{l/2} \sum_{\alpha \in A} e^{-\pi \varepsilon(v+h(\alpha),g_k(v+h(\alpha)))} = 1 \]
for any \( v \in V \). This is done by changing the sum \( \sum_{\alpha \in A} \) to a sum over \( \mathbb{Z}^l \) (using coordinates) and then use the Poisson summation formula to this sum. Note that if \( \mathcal{V} = \{v_1, \ldots, v_l\} \) is a basis for \( V \) such that \( A \) is generated by this set over the integers and if \( \mathcal{W} = \{w_1, \ldots, w_l\} \) is an orthonormal basis for \( V \) then \( \text{vol}(\Lambda) = \det(k) \), where \( k: V \to V \) is the linear isomorphism given by \( k(w_j) = v_j, j = 1, 2, \ldots, l \). \( \square \)

Now let \( F: \Lambda \to \mathbb{C} \) be given by (40), and let \( h = \text{rid}_V: V \to V \). Then \( F(\lambda + h(\alpha)) = F(\lambda) \) for \( \alpha, \lambda \in \Lambda \) by (5), so by the above lemma we get
\[
\sum_{\lambda \in \Lambda} F(\lambda) = \text{vol}(\Lambda)r^l \lim_{\varepsilon \to 0+} \varepsilon^{l/2} \sum_{\lambda \in \Lambda} e^{-\pi \varepsilon|\lambda|^2} F(\lambda).
\]
To continue we use coordinates. Let \( \mathcal{V} \) and \( \mathcal{W} \) be bases for \( V \) as in the proof of Lemma 6.5, and let \( C \) be the matrix of \( f \) w.r.t. \( \mathcal{W} \). Moreover, let \( D = (d_{ij})_{i,j=1}^l \) such that \( v_j = \sum_{i=1}^l d_{ij}w_i \). Then
\[
\sum_{\lambda \in \Lambda} e^{-\pi \varepsilon|\lambda|^2} F(\lambda) = \sum_{\lambda \in \Lambda} e^{-\pi \varepsilon(Dn,Dn)} \exp \left( \frac{\pi \sqrt{-1}}{r}(Dn,Cn) \right) \exp \left( 2\pi \sqrt{-1}(Dn,y) \right),
\]
where \( y \) are the coordinates of \( \psi \) w.r.t. the basis \( \mathcal{W} \). By the Poisson summation formula we get
\[
\sum_{\lambda \in \Lambda} e^{-\pi \varepsilon|\lambda|^2} F(\lambda) = |\det(D)|^{-1} \sum_{m \in \mathbb{Z}^l} \int_{\mathbb{R}^l} e^{2\pi \sqrt{-1}(m,D^{-1}x)} \times e^{-\pi \varepsilon(x,x)} \exp \left( \frac{\pi \sqrt{-1}}{r}(x,Cx) \right) \exp \left( 2\pi \sqrt{-1}(x,y) \right) dx.
\]
The summands are all Gaussian integrals and can be calculated by diagonalizing \( C \). In fact, if we choose an orthogonal matrix \( Q \) such that \( Q^{-1}CQ = \text{Diag}(\lambda_1, \ldots, \lambda_l) \) and let \( \eta = \varepsilon r^2 \), then we arrive at the following identity
\[
\sum_{\lambda \in \Lambda} F(\lambda) = r^{l/2} \left( \det \left( \frac{f}{\sqrt{-1}} \right) \right)^{-1/2} \lim_{\eta \to 0+} \eta^{l/2} \sum_{m \in \mathbb{Z}^l} \exp \left( -\pi \eta(m + D^t y, D^{-1}C^{-1}QH(\eta)Q^{-1}C^{-1}(D^{-1})^t(m + D^t y)) \right) \times \exp \left( -\pi \sqrt{-1}r(D^{-1}QH(\eta)Q^{-1}C^{-1}(D^{-1})^t(m + D^t y), m + D^t y) \right),
\]
where \( H(\eta) = \text{Diag}(f_1(\eta), \ldots, f_l(\eta)) \), where \( f_j(\eta) = \left( 1 + \left( \frac{\eta}{\lambda_j} \right)^2 \right)^{-1} \to 1 \) as \( \eta \to 0+ \). Next we use the following technical but straightforward
Lemma 6.6 Let \( a > 0 \) and let \( A : [0, a] \to \mathrm{GL}(l, \mathbb{R}) \) be a curve of positive definite symmetric matrices such that \( A(\varepsilon) \to A_0 \) as \( \varepsilon \to 0_+ \), where \( A_0 \) is a symmetric positive definite matrix. Moreover, let \( B : [0, a] \to \mathrm{GL}(l, \mathbb{R}) \) be a curve and \( B_0 \) a fixed matrix such that
\[
\lim_{\varepsilon \to 0_+} \varepsilon^{-1/2} \sum_{n \in \mathbb{Z}^l} e^{-\varepsilon(n + x_1, A(\varepsilon)(n + x_1))} \exp \left( \sqrt{-1} \langle n + x_2, B(\varepsilon)(n + x_2) \rangle \right) = 0.
\]
for any \( x_1, x_2 \in \mathbb{R}^l \) in the sense that if one of the two limits exists then does the other and they are equal.

By this lemma (and Lemma 6.5) we get that
\[
\sum_{\lambda \in \Lambda/\Gamma\Lambda} F(\lambda) = \eta^{l/2} \lim_{\eta \to 0_+} \left( \frac{f}{\sqrt{-1}} \right)^{-1/2} \exp \left( -\pi \eta \langle m + D^t y, D^{-1} C^{-1} (D^{-1})^t (m + D^t y) \rangle \right)
\]
for any \( x_1, x_2 \in \mathbb{R}^l \) in the sense that if one of the two limits exists then does the other and they are equal.