Formation of multiple winding topological defects in the early universe

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Abstract

The formation probability of multiple winding topological defects is calculated by phase and flux distribution analysis based on the Kibble mechanism. The core size of defects is taken into account so that when it is much larger than the correlation length of the Higgs field, high winding configuration can be realized. For example, if the self coupling constant of the Higgs field is smaller than $4 \times 10^{-4}$, the topological inflation may occur at the grand unification energy scale by the multiple winding string produced during the GUT phase transition.

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I. INTRODUCTION

It is believed that topological defects are produced during certain types of cosmological phase transitions and may play an important role in the course of cosmic evolution [1,2]. They can solve some of the unsolved cosmological problems. Particularly topological defects give a natural and attractive model for inflation and may explain the source of baryon asymmetry in our universe.

Inflation is the most promising candidate which can provide the solutions to the problems in the standard Big Bang theory [3] and the realistic model for the inflation is very important subject in the cosmology. Topological inflation has been proposed by Vilenkin [4] and Linde [5]. In this inflation scenario, the energy density which drives the inflationary expansion of the universe is provided by the symmetric state within the defect core where the false vacuum energy is trapped by the topological constraint. If the way of the symmetry breaking in the particle physics theory satisfies the condition for the production of the topological defect, the formation of some kind of defect at the phase transition is inevitable. Then one of the necessary conditions for the inflation that the inflaton field must be in the state which has sufficient vacuum energy density would be realized without any additional constraint. This is the advantage of the topological inflation scenario since in the conventional inflation model, the fine-tuning of initial state selection is required in order that the necessary conditions for the inflation should be satisfied [3]. However, in order to realize the condition that the core length scale is long enough to cover the horizon scale, the symmetry breaking energy scale for the defect formation, \( \eta \), should be larger than the Planck scale as

\[
\eta > \sim M_{pl}.
\]  

Therefore in order to understand the topological inflation in detail, we must work very close to the Planck scale at which our classical field theories will not be valid. When multiple winding defects are employed, however, the energy scale at which the inflation occurs is decreased and the constraint on \( \eta \) is relaxed since the higher winding defect has thicker core length scale than the unit winding defect. The constraint on \( \eta \) when the winding number of the string, \( n \), is included has been derived numerically [7] as

\[
\eta > 0.16 M_{pl} \times n^{-0.56}.
\]  

Hence if the string whose winding number \( n \sim 3 \times 10^3 \) is produced, it is possible that the topological inflation occurs at the GUT scale.

The baryon asymmetry problem is another important subject. Since the sphaleron transition interaction should erase the baryon asymmetry produced before such a process becomes negligible unless the difference between lepton number and baryon number exists, the baryon number generation at the electroweak scale seems to be the most conventional scenario of the baryogenesis at present. The first proposed electroweak baryogenesis scenario relies on having a strong first order phase transition so that the deviation from the thermal equilibrium state, one of the necessary conditions for the baryogenesis, is achieved by the propagation of nucleated bubbles in the plasma [8]. However, it seems that the first order electroweak phase transition is difficult to be established in the standard model [9]. Then the electroweak baryogenesis scenario using electroweak strings was proposed by Brandenberger
and Davis [10]. In this scenario the out-of-equilibrium condition is provided by the collapse of strings. Thus the string baryogenesis has the advantage that it works effectively whether the electroweak phase transition is of first order or not. Since the electroweak string in the standard model is topologically unstable, however, its formation probability is too small to serve for the observational amount of baryon number [11]. Therefore topological defects associated with the electroweak symmetry breaking should be necessary for the electroweak baryogenesis. Recently Soni [12] has suggested a new scenario for the baryon asymmetry generation using the string produced at the electroweak energy scale. He has pointed out that the sphaleron energy in the presence of the string with a few winding number can become negative. Then when the strings within which sphalerons are bounded decay, the baryon number would be produced. In this scenario the thermal equilibrium is violated by the string-sphaleron system which is left below the sphaleron suppression temperature.

In order to know to how much extent the required energy scale for $\eta$ becomes small in the topological inflation scenario, we have to estimate the possibility of multiple winding defect formation. Also in order to determine how much baryon asymmetry is produced in the electroweak baryogenesis scenario by the sphaleron bound state on the string, we have to calculate the formation probability of multiple winding strings. For these reasons, in this Letter we consider the realization of multiple winding topological defect configuration. We employ two kinds of defect, that is, strings and monopoles. For the string case, we investigate the phase distribution of the Higgs field. For the monopole, the distribution of the gauge flux is considered. In both cases, the formation probability of multiple winding defects is estimated quantitatively and the existence of such a defect is confirmed.

II. STRING

First let us consider the breaking of a local $U(1)$-symmetry in the abelian-Higgs model with Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (D_\mu \phi)^\dagger (D_\mu \phi) - \frac{1}{8} \lambda (\phi^\dagger \phi - \eta^2)^2 ,$$

(3)

where $\phi$ is a complex scalar field and the covariant derivative is given by $D_\mu = \partial_\mu - i e A_\mu$ with $A_\mu$ a gauge vector field, $e$ the gauge coupling constant. $F_{\mu\nu}$ is the antisymmetric tensor defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is well known that there is a string solution called the Nielsen-Olesen vortex line [13] in this model. In the Lorentz gauge, the Higgs field configuration far from the core of the string can be written as

$$\phi \simeq \eta e^{in\theta} .$$

(4)

The winding number is a strictly conserved quantity and the total Higgs field phase difference around the string is $2\pi n$. In general, the larger $n$ becomes, the line energy density of the string increases and the core width scale becomes fatter. It has been demonstrated both numerically and analytically that multiple winding strings ($|n| > 1$) are stable when $\lambda/e^2 < 1$ and unstable in the opposite case [14]. In the former case, a number of string may get together and coalesce into one string due to the energetic favorableness. On the other hand, in the latter case, multiple winding strings can break up into $|n|$ pieces of string with unit winding number.
Before estimating the formation probability of multiple winding strings, let us briefly summarize the estimation method for the unit winding string based on the conventional Kibble mechanism. In the thermal phase transition, the field has a typical length scale, that is, the correlation scale, $\xi$. It defines the size of regions within which the values of fields are homogeneous and are independent of those at other regions. When the cosmic temperature decreases sufficiently and the ground state of the Higgs field becomes the true vacuum state, the amplitude of the Higgs field $|\phi|$ should be same almost everywhere, while its phase $\theta$ varies on the correlation scale. Then the physical space is regarded to be divided into correlated volumes and the phase of the Higgs field takes the random value at each correlated region so that the phase distribution has a domain-like structure at the end of cosmological phase transition. Thus in the context of the Kibble mechanism, the formation probability of the string can be estimated as follows [15,16]. For simplicity, the 2-dimensional slice of the 3-dimensional $\theta$ distribution is considered so that the formation of vortices can be analyzed instead of strings.

(A) Divide the plane into 2-dimensional domains whose typical size is equal to the correlation length of the Higgs field $\xi$.

(B) Assign the phase of the Higgs field randomly to one representative point of each domain.

(C) Interpolate the phase between two representative points of neighboring domains so that the gradient energy of the Higgs field takes minimum value (geodesic rule).

Then we can count the total phase change along any closed loop on the plane and calculate how much winding number exists inside the loop so that the formation probability of the vortex can be estimated. Usually it is assumed that at most three different domains meet at the boundary point. It means the closed loop which is used to count the phase change can always be identified to the triangle whose corners correspond to three representative points of these three domains. Thus the above procedure (A) and (B) can be expressed as: divide the plane by regular triangles and assign the phase of the Higgs field randomly to each vertex point where six triangles join. The geodesic rule implies that the difference of the phase between two neighboring points is less than $\pi$. Therefore when the triangle division is imposed, the total difference of the phase along the circumferential edge of the triangle is $2\pi$ at most because of the phase continuity. As a result, the winding number should not exceed the unity and multiple winding vortices never appear in this situation.

However, we cannot say that formation probability of multiple winding vortices equals zero since this estimation is based on too simplified assumptions. In order to construct the method which can detect the existence of multiple winding number, we modify the arrangement of points where the phase of the Higgs field is allocated. Since so long as the plane is divided into triangles the maximum winding number must be only one, the plane should be divided by a polygon other than a triangle so that the total phase difference along its periphery can go beyond $2\pi$. It would be reasonable if we choose the side length of the polygon as $\xi$ since each vertex can be considered to represent the correlated domain in the similar manner to the triangle case. When the diameter scale of the polygon is $R_s$, the total length of the polygon periphery will be $\sim \pi R_s$. Then the number of vertices is $\sim \pi R_s/\xi$. 
and the possible largest phase change is $\sim \pi^2 R_s/\xi$. In the usual triangle case, $\pi R_s/\xi = 3$ so that $R_s \sim \xi$. The revised winding number counting procedure can be expressed as:

(A') Divide the plane into polygons whose diameter scale is equal to $R_s$ and vertex number is $\pi R_s/\xi$.

(B') Assign the phase of the Higgs field randomly to each vertex of the polygon.

The third step is identical to the original version. Then we can count the winding number and estimate the formation probability of multiple winding vortices. In actual calculations, we consider one polygon, assign a phase of the Higgs field ($0 \leq \theta < 2\pi$) randomly to each vertex of this polygon and count the winding number along its periphery. We repeat this process $10^8$ times so that the probability distribution of the string formation for each winding number can be led. The result of calculations for various values of $\pi R_s/\xi$ is shown in FIG. 1.

In our method, the numerical value of the multiple winding vortex formation probability depends on $R_s/\xi$. It would be reasonable that $\xi$ can be obtained by the correlation scale of the Higgs field at Ginzburg temperature $T_G$, when the phase transition terminates and the defect cannot be erased by thermal fluctuations, as

$$ \xi \sim \frac{1}{T_G}, \quad (5) $$

because the correlation length of the massless field corresponding to thermal fluctuations is given by the inverse of temperature [17]. The most natural estimation of $R_s$ would be that it is comparable to the core diameter of the string. Under this assumption, the number of string which can be produced inside each polygon should be one at most since many strings cannot exist within the scale of the string core or in other words all the winding number one polygon contains must belong to a single string. Thus we do not mistake two or more strings for one string if we take $R_s$ to be the core diameter of the string. The string core diameter scale is given by the Compton wavelength of the Higgs field as

$$ R_s = \frac{1}{m_H(T_G)} \sim \frac{1}{\lambda T_G}, \quad (6) $$

at $T_G$ where $m_H$ is the mass of the Higgs field. Therefore the maximum winding number the string can have will be

$$ n_{\text{max}} \sim \left[ \frac{\pi^2 R_s}{2\pi \xi} \right] \sim \left[ \frac{\pi}{2\lambda} \right], \quad (7) $$

where we use the Gauss’s symbol. This result means multiple winding string can be produced when $\lambda$ is smaller than $\pi/4$.

When $\lambda \sim 1$, the formation of multiple winding strings seems to be impossible even in the revised method. There is, however, another possibility that the correlation length of the Higgs field can fluctuate since in reality the size of the domain within which the value of the field is homogeneous may not be constant throughout the universe. As a toy model, we assume that the domain size distributes around its averaged value $\xi$ and
the probability distribution function obeys a Gaussian form whose dispersion is given by $\sigma$. Then we calculate the distribution function of $\pi R_s/\xi$, $P(\pi R_s/\xi)$, and estimate the formation probability of the string for each winding number by summing up that for fixed value of $\pi R_s/\xi$ shown in FIG. 1 with weight $P(\pi R_s/\xi)$. Here, we set $\xi = \sigma = R_s$. Note that the larger $\sigma$ becomes, larger domains which contains no string tend to be produced. The resulting formation probability, $P_s(n)$, of the vortex for $n = 1 - 4$ per one domain is shown in TABLE 1. The larger $n$ becomes, the formation probability of multiple winding vortices decreases exponentially.

### III. MONOPOLE

The formation probability of multiple winding strings depends on the ratio of the circumference of the string to the correlation length of the Higgs field. Also in the case of monopole, a similar consideration can be applied. The formation probability of multiple winding monopoles depends on the ratio of the surface area of monopole to the correlation area of the Higgs field, which is given by $4\pi R_m^2/\pi \xi^2$, where $R_m$ is the core diameter of the monopole. Therefore in principle we can calculate the formation probability of multiple winding monopoles for various values of $4R_m^2/\xi^2$ in a similar manner for strings. However, since it is complicated to interpolate the phase of the Higgs field between two neighboring vertices and count the winding number, here we resort to an easier method.

In the case of local monopole, not only the Higgs field but also the gauge field appears in the theory. Then instead of the phase of the Higgs field, we assign the magnetic flux of the gauge field at each domain. When the total flux which passes through a closed surface is not zero, there must be a monopole or an anti-monopole inside this surface. Thus the gauge flux can play a role similar to the winding number. The simplest monopole solution, the 't Hooft-Polyakov solution [18] for the model in which the $SO(3)$ symmetry is broken to $U(1)$, has the overall magnetic flux whose magnitude is

$$\Phi_B = \frac{4\pi}{e},$$

which corresponds to the unit winding number in the Higgs field case. The 't Hooft-Polyakov monopole with multiple winding number is known to be stable in the Prasad-Sommerfield limit [19]. In actual calculations, we assign only discrete values of magnetic flux as $\pm\pi/e$ for simplicity. Then the total amount of the flux is summed up around the closed surface and the winding number can be calculated. Then the formation probability of the multiple winding monopole is given by

$$P_m(n, l) = \left( i C_{\frac{1}{2} + 2n} + i C_{\frac{1}{2} + 2n + 1} \right) \left( \frac{1}{2} \right)^{l-1},$$

where $n$ is the winding number and $l = 4R_m^2/\xi^2$. We omit the fractional part of the winding number which appears because of the absence of the interpolation process. Since our method of the magnetic flux assignment reproduces the formation probability of a unit winding monopole [16] when $l = 4$, the approximation we have employed should be reasonable. The analytic result for various values of $4R_m^2/\xi^2$ using the equation (8) is shown in FIG. 2.
Similarly to the string case, $R_m$ can be estimated to be $(\lambda T_G)^{-1}$ and $\xi$ equals $T_G^{-1}$. Then the ratio of the surface area of the monopole to the correlation area of the Higgs field is given by

$$\frac{4R_m^2}{\xi^2} \sim \frac{4}{\lambda^2}.$$  \hspace{1cm} (10)

Also in this case the smaller $\lambda$ becomes, the more multiple winding monopoles are produced. Moreover, the spatial variation of the correlation length enables the realization of the high winding configuration in the same way for the string. The resulting formation probability, $P_m(n)$, of the monopole for $n = 1 - 4$ per one domain is shown in TABLE II.

**IV. CONCLUSION**

In the present Letter we have calculated the formation probability of multiple winding topological defects by means of the Kibble mechanism. Since we have taken the fact that the core size scale of the defect can be larger than the correlation scale of fields into account, the formation probabilities depend on the ratio of these two scales. The multiple winding topological defects can be produced when the self coupling constant of the Higgs field, $\lambda$, which defines the defect core thickness is much less than the unity. The smaller $\lambda$ becomes, the formation probability of the multiple winding defects and the maximum winding number the defect can have increase.

This result gives following cosmological implications. First it may be possible that the topological inflation occurs at the GUT energy scale by the multiple winding string produced during the GUT phase transition when $\lambda \lesssim 4 \times 10^{-4}$ which can be calculated from the equations (2) and (7). For the same value of $\lambda$, the maximum winding number the monopole can have is larger than that the string can have. Therefore in the case of monopole, topological inflation may occur for larger value of $\lambda$. Secondly, in the context of the electroweak baryogenesis, the sphaleron bound state scenario using the string whose winding number is $2 \sim 3$ will be promising. Since the formation probability of the double winding string is about one third that of a unit winding string when $\lambda \sim 10^{-1}$, it might be possible to explain the observational amount of the baryon asymmetry. Further quantitative analysis will be needed.

Our method is not the only one which can improve the simple estimation of the defect formation probability. For example, not only the correlation length scale of the Higgs filed but also the core thickness scale of the topological defect would vary since the static and highly symmetric solution of the defect configuration may not always be applied to the actual situation. Another promising modification is the relaxation of the geodesic rule \cite{20}. It is too simplified picture that the value of the field should be interpolated through the shortest path on the vacuum manifold and there is a possibility that larger gradient exists between two regions.

In addition to such revisions, the interaction between strings makes the situation extremely better. We have considered only the moment of string production and neglected the dynamics of the string after its formation when we have calculated the value of $R_s$. In some ranges of the parameter, however, the attractive force operates on strings so that the winding number can accumulate. It corresponds to the case when $R_s/\xi$ is enormous. Maybe
a single string has all the winding number within the horizon scale. Then the topological inflation scenario might successfully work even when $\lambda$ is not so small. Note that also the interaction of the string with the surrounding plasma will affect the dynamical evolution. The drag force can aid the accumulation process because it dissipates the energy of the string pair and allows a bound state to be stable.

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FIG. 1. Formation probability of vortices per one polygon for various values of $\pi R_s/\xi$. $n$ depicts the winding number.
FIG. 2. Formation probability of monopoles per one domain for various values of $\frac{4R_m^2}{\xi^2}$. $n$ depicts the winding number.
TABLES

TABLE I. Formation probability of vortices per one polygon. The fluctuation of the correlation length is taken into account.

| $n$ | $P_s(n)$     |
|-----|--------------|
| 1   | $2.102 \times 10^{-1}$ |
| 2   | $8.36 \times 10^{-4}$  |
| 3   | $4.8 \times 10^{-7}$   |
| 4   | $8 \times 10^{-11}$    |

TABLE II. Formation probability of monopoles per one polyhedron. The fluctuation of the correlation length is taken into account. Here, we set $\xi = \sigma = R_m$.

| $n$ | $P_m(n)$     |
|-----|--------------|
| 1   | $4.285 \times 10^{-2}$ |
| 2   | $7.36 \times 10^{-5}$  |
| 3   | $3.4 \times 10^{-8}$   |
| 4   | $7 \times 10^{-12}$    |