Subclasses in Mixing Correlation-Growth Processes with Randomness

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We show that in the construction of continuum equations for competitive growth processes that are a mixture of random deposition and a correlation process, a distinction must be made within a *single universality class* between depositions that do and do not create voids in the bulk. Within these subclasses the bulk morphology is reflected in the surface roughening via *nonuniversal* prefactors in the universal scaling of the surface width.

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Many complex systems are studied by mapping onto a suitable nonequilibrium surface-growth problem [1, 2]. The dynamics of the buildup of the correlations in a system can then be explored with surface-growth methodologies. Large-scale properties are described within a continuum model by universal stochastic growth equations and tested with simulation models. The trouble is, simple atomistic models are often not adequate to reproduce the complex physics of the observed surface phenomena that may involve contributions from several universal processes, but the continuum description of such multicomponent growth has not yet been developed. A representative example comes from an applied model in computer science [2] when the asynchronous dynamics of conservative updates in a system of parallel processors is modeled as a virtual-time surface that represents variations in scale-dilatation observed in RD+X models [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In our analysis we use as an example universal RD+KPZ growth processes in (1 + 1) dimensions, and generalize our approach to other processes in (1 + n) dimensions.

A scaling hypothesis for competitive RD+X processes [7] states that if a correlation growth X occurs with a constant probability \( p \), its continuum equation must be invariant under the scaling

\[ x \to x, \quad t \to t/f(p), \quad h \to h/g(p), \]

where \( g(p) \) and \( f(p) \) are arbitrary suitable functions of \( p \in (0; 1] \). This invariance implies that \( f(p) = p^2 \), and when \( X = \text{KPZ} \) it leads to the KPZ equation [3] for the RD+KPZ mix [7]:

\[ h_t = \nu_0 f(p) h_{xx} + (\lambda_0/2) f^{3/2}(p) h_x^2 + \eta(x, t), \]

where \( h \equiv h(x, t) \) is the height field; \( x \) and \( t \) are the spacial and time coordinates, respectively; \( \nu_0 \) and \( \lambda_0 \) are constants. When \( \lambda_0 = 0 \), Eq. (2) is the Edwards-Wilkinson (EW) equation [14] when \( X = \text{EW} \). When \( \nu_0 = \lambda_0 = 0 \), Eq. (2) defines universal RD dynamics. Many simulation models of RD+EW and RD+KPZ growth processes [7, 10] suggest \( g(p) = p^\beta \) in Eq. (1), which leads to the Family-Vicsek universal scaling [15] of the average surface width \( w(p, t) \) [7]:

\[ w(p, t) = \frac{L^\alpha}{p^\delta} F \left( p^{2\delta} \frac{t}{L^\gamma} \right). \]

For substrates of size \( L \), \( F(y) \) describes two limit-regimes of evolution: \( F(y) \sim y^{\alpha/2} \) if \( y \ll 1 \) (growth); and, \( F(y) \sim \text{const} \) if \( y \gg 1 \) (saturation). In Eq. (3), \( \alpha \) and

\[ \beta, \gamma, \delta, \theta, \nu, \lambda \]
$z$ are the universal roughness and dynamic exponents, respectively, of the universality class of the correlation growth $X$. The scale-dilatation exponent $\delta$ in scaling prefactors in Eq. (4), however, is nonuniversal. It has been observed that in some models $\delta \approx 1$ across universality classes, and in some other models $0 < \delta \lesssim 1$ within a single universality class. Also, there are models where prefactors in Eq. (4) do not at all obey a power law in $p$ [10]. Here, we shall establish that this variation is not accidental, but rather reflects the properties of the bulk of the deposited material.

Consider aggregations where particles fall onto a sub-strate of $L$ sites, where they may be accepted in accordance to a rule that generates correlations among the sites. This correlation growth occurs with probability $p$ and competes with RD growth that occurs with probability $q = 1 - p$. When a particle is accepted at a site, the site increases its height by $\Delta h$. If, e.g., component 1 is RD, and component 2 is a correlation growth in the KPZ universality class, their corresponding growth equations are

\begin{align}
h_{1,t} &= \eta_1(x,t), \\
h_{2,t} &= \nu_0 h_{2,xx} + (\lambda_0/2) h_{2,xx}^2 + \eta_2(x,t),
\end{align}

where $h_n(x,t)$, $n = 1, 2$, is the column height at $x$ after time $t$ when the component $n$ acts alone. Assume for simplicity that the noise terms are of the same strength, i.e., $\eta_1 = \eta_2$. In two-component growth, when both components act simultaneously, the column height $h(x,t)$ is incremented due to either of the components with their corresponding probabilities $\tilde{p}$ and $\tilde{q}$, $\tilde{p} + \tilde{q} = 1$:

$$\Delta h(x,t) = \tilde{p}\Delta h_2(x,t) + \tilde{q}\Delta h_1(x,t).$$

Here, probability $\tilde{p}$ (or $\tilde{q}$) is the fraction of contributions to $h$ from component 2 (or 1). For some processes this fraction is identical to a fraction of times when $h(x)$ is increased due to component 2 (or 1) for the times from 0 to $t$. However, as explained later, this is not so for all processes. In Eq. (6), $\Delta h_n$ is understood as “being incremented due to the process $n$,” $n = 1, 2$. In this statistical sense, Eq. (6) expresses a simplectic decomposition of $\Delta h(x,t)$ into its vertex-components $\Delta h_n(x,t)$. Dividing Eq. (6) by $\Delta t$, and taking the limit $\Delta t \to 0$, we obtain the equation for time rates, $h_t = \nu_0 \tilde{p}h_{2,xx} + \tilde{q}h_{1,1t}$, to which we substitute Eqs. (4)–(5):

$$h_t = \nu_0 \tilde{p}h_{2,xx} + (\lambda_0/2) \tilde{p}h_{2,xx}^2 + (\tilde{p} + \tilde{q})\eta(x,t).$$

In Eq. (7), $h(x,t)$ is the column height that rises at $x$ as the result of two processes acting simultaneously from the beginning to time $t$. Here, $h_2(x,t)$ is the part of $h(x,t)$ that was created by the component 2 in this time. The other part was created by component 1. In other words, $h_2(x,t)$ is so far an unknown fraction of $h(x,t)$.

To find a relation between $h$ and $h_2$, one must consider nonuniversal properties of aggregation processes.

We distinguish between the following two groups of surface growths. In one group we place all simple absorption processes with conserved flux that do not create voids in the bulk of the deposited material. We call this group absorption-bulk-compact (ABC) growths. The other group, which we call dense-or-lace-bulk (DOLB) growths, contains processes that are not ABC-type. The DOLB group includes desorption processes that may lead to a dense bulk as well as absorptions that lead to the formation of voids. Note, RD processes are ABC growth processes. As we show in the next paragraph, when component 2 is of the ABC-type, $\tilde{p}$ and $\tilde{q}$ in Eq. (6) express fractional contributions to $h$ in terms of times, and then $h_2(x,t) = ph(x,t)$. This is not true when component 2 is a DOLB growth.

Consider a discrete representation of events at coordinate $x$. Suppose, there are $t$ deposition events in total, with $t_1$ events due to component 1, and $t_2$ events due to component 2, $t = t_1 + t_2$. In ABC growth, after $t$ events, the total column height is $h = t\Delta h$, where contributions from components 1 and 2 are, respectively, $h_1 = t_1\Delta h$ and $h_2 = t_2\Delta h$. Thus, $h_1/h = t_1/t = q$ and $h_2/h = t_2/t = p$. Therefore, in ABC growth $h_2 = ph$, and in Eqs. (6)–(7) $\tilde{p} = p$ and $\tilde{q} = q$.

Next consider that the component 2 is a DOLB growth that creates voids. Now, an individual deposition event due to component 2 not only increases $h$ by $\Delta h$, but may also result in the creation of voids. The net effect is as though component 2 deposited $\Delta h$ and the voids. Therefore, in $t_2$ events, its contribution to the column height is $h_2 = (t_2 + m)\Delta h$, where $m\Delta h$ reflects the increase in height due to the presence of voids. The component 1 is RD, i.e., ABC-type, and $h_1 = t_1\Delta h$. After $t$ events, the net column height is $h = h_1 + h_2 = (t + m)\Delta h$. Thus, $h_1/h = t_1/(t + m) < t_1/t = q$ and $h_2/h = (t_2 + m)/(t + m) > t_2/t = p$. Fractions $q_{\text{corr}} \equiv h_1/h$ and $p_{\text{corr}} \equiv h_2/h$ are the effective probabilities of deposition events due to components 1 and 2, respectively, as they would result from measuring the column height. For some types of two-component growth with RD, the probability $p_{\text{corr}}$ can be expressed approximately as $p_{\text{corr}} = p^\delta$ [10]. For DOLB growths with voids $\delta < 1$ (because $p_{\text{corr}} > p$). When the component 2 is a DOLB growth with desorption, in the above reasoning one should change $m \to -m$. This will give $q_{\text{corr}} > q$ and $p_{\text{corr}} < p$, and $p_{\text{corr}} = p^\delta$ with $\delta > 1$. The value of $\delta$ is specific to the particular deposition process of component 2. Therefore, in DOLB growth $h_2 = p_{\text{corr}}h$, and in Eqs. (6)–(7) $\tilde{p} = p_{\text{corr}}$ and $\tilde{q} = 1 - p_{\text{corr}}$.

In general, $h_2(x,t) = p_{\text{corr}}h(x,t)$ and $\tilde{p} \equiv p_{\text{corr}}$, where $p_{\text{corr}} = p$ if the correlation component is an ABC growth. When the correlation component is a DOLB growth, and when the effective probability is well approximated by a power law $p^\delta$, the result can be summarized as $p_{\text{corr}} = p^\delta$. 
where $\delta = 1$ for ABC growths and $\delta \neq 1$ for DOLB growths. This result is combined with Eq. (7) to give the continuum equation for the RD+KPZ mix:

$$h_t = \nu_0 p^{2\delta} h_{xx} + (\lambda_0/2)p^{3\delta} h_x^2 + \eta(x,t). \tag{8}$$

When in Eq. (5) $\lambda_0 \equiv 0$, the analogous reasoning gives the RD+EW dynamic:

$$h_t = \nu_0 p^{2\delta} h_{xx} + \eta(x,t). \tag{9}$$

Both results, Eqs. (8)-9, are in accord with our former derivation that lead to Eq. (2) 7. Matching Eq. (8) with Eq. (2) gives $f(p) = p^{2\delta}$, which form of $f(p)$ was used formerly to derive the approximate prefactors in Eq. (3). The inverse of the scaling (1) when applied to Eqs. (3) transforms them to continuum equations for a "pure" correlation processes of $p = 1$. Explicitly, it collapses all evolution curves $w(p,t)$ (for all $L$ and $p$) either onto $w(1,t)$ or onto a neighborhood of $w(1,t)$ 7, following Eq. (3), provided the effective probabilities $p_{\text{eff}}$ can be approximated by the power-law $p^\delta$. When such a fit is not possible Eq. (8) is obeyed but then the scaling prefactors must be expressed directly in terms of effective probabilities. This is because the factor $p^\delta$ in the coefficients of Eqs. (8)-9 is only a fit to the effective probability $p_{\text{eff}}$. In fig. 4 we give an example of the exact scaling where nonlinear prefactors in Eq. (9) are directly expressed by $p_{\text{eff}}$ via the substitution $p^\delta \rightarrow p^\delta_{\text{eff}}(p) = \sqrt{f(p)}$ for the RD+BD model when BD is the NN sticking rule. Here, the effective probability depends on both $p$ and the mean compactness $c(p)$ of the bulk formed in the RD+BD process: $p_{\text{eff}} = 1 - qc(p)$. The perfect data collapse in the full range of $p \in (0;1)$, seen in fig. 4, can be contrasted with fig. 5 of Ref. 7 that shows only an approximate data collapse for the same system with the best fit exponent $\delta \approx 0.41$ in Eq. (6). It needs to be said explicitly that the scaling where $\delta = 1/2$ in Eq. (8), proposed in Refs. 4 13 for RD+BD models, does not produce data collapse at all. The RD+BD model when BD is the NNN sticking rule provides an example where $p_{\text{eff}}(p)$, and thus the nonlinear prefactors $f(p)$ and $g(p)$ in Family-Vicsek universal scaling, cannot be expressed by a power-law $p^\delta$. In this system the surface roughening obeys power laws in effective probability that incorporates either the compactness or the voidness of the bulk, which gives excellent data collapse of $w(p,t)$, similar to that seen in fig. 4.

The approach introduced here by the example of KPZ processes, can be applied to a broad range of stochastic growth models RD+X, where component 2 can be any isotropic growth in $(1+n)$ dimensions:

$$h_{2,t}(x,t) = G(h_2) + \eta_2(x,t), \tag{10}$$

where $\vec{x}$ is $n$ dimensional, and the operator $G$ represents only local interactions 15. In the general case, Eq. (7) is written as $h_t = p_{\text{eff}} h_{2,t} + q_{\text{eff}} h_{1,t}$, and combined with Eqs. (4) and (10), to find for the competitive growth

$$h_t(\vec{x},t) = p_{\text{eff}} G(p_{\text{eff}} h) + \eta(\vec{x},t), \tag{11}$$

where $\eta = (1 - p_{\text{eff}}) \eta_1 + p_{\text{eff}} \eta_2$, and the noise strengths may be different. Eqs. (11-14) represent the same universality class since the multiplication by $p_{\text{eff}}$ does not modify local interactions: $p_{\text{eff}}$ affects the noise strength and the gradient of the height field, but it does not generate new terms other than those already given by operator $G$. Hence, if a correlation growth belongs to a given universality class, its mix with RD will remain in the same class. Elementary calculations show that Eq. (11) is invariant under the scaling $g(p)h(\vec{x},t) = h'(\vec{x},t') = f(p)t$. If $g(p) = p_{\text{eff}}(p)$ and $f(p) = p_{\text{eff}}^2(p)$, and if the noise strengths are the same, this scaling maps the universal dynamics of RD+X onto the universal dynamics of X. In this case the invariance implies $g(p)w(p,t) = w'(f(p)t)$, where $w'()$ has universal scaling properties of the process X. When X is either in the KPZ or in the EW universality class, and if additionally $p_{\text{eff}} \approx p^\delta$, we recover Eq. (11).

When both the RD and the correlation component 2 have deposits of unit height, when $p_{\text{eff}} \approx p^\delta$, we have $\delta = 1$ if component 2 is of the ABC-type; and, $\delta \neq 1$ if it is of the DOLB-type. In the latter case, the value of the exponent $\delta$ is specific to component 2. When $p_{\text{eff}}$ incorporates explicitly bulk properties, the scaling is $g(p) = p_{\text{eff}}^\delta(p)$, where the new scale-dilatation exponent $\delta$ is obtained from the slope of $\ln w^2(p)$ plotted vs $\ln p_{\text{eff}}(p)$ at saturation. In DOLB growth with voids, $p_{\text{eff}}$ can be determined by measuring the mean density of voids in the bulk (Fig. 1). Similarly, in DOLB growth with description $p_{\text{eff}}$ is connected to the mean fraction of the removed material 16.

The analysis presented here explains scaling results of the following mixed-growth models in $(1+1)$ dimensions: Model A 9-11; component 2 is RD with surface relaxation. Model B 7: component 2 simulates a deposition of a sticky non-granular material of variable droplet
size. Model C is component 2 is the NN sticking rule of BD. Model D is a deposition of Poisson-random numbers to the local surface minima. Models A and B are ABC growths in the EW universality class, where \( p_{\text{eff}} = p \) and \( \delta = 1 \) (Fig.2). Models C and D belong to the KPZ universality class. Model C is an example of DOLB growth with voids, with a 53.2% void density in the bulk when \( p = 1 \), and in this case \( \delta \approx 0.41 < 1 \). Model D is a DOLB-type growth that produces a compact bulk but component 2 is flux non-conserving, and here \( \delta \approx 1 \). Extensions of Models A and C to \((1 + n)\) dimensions \([10]\), \( n = 2, 3 \), yield results that conform to our theoretical predictions of \( p_{\text{eff}} \approx p^\delta \) with \( \delta \neq 1 \) for mixing RD with DOLB processes, and \( \delta = 1 \) for mixing RD with ABC processes. Additional examples include cases when component 2 is a restricted Kim-Kosterlitz solid-on-solid model \([11]\), and when it represents the Villain-Lai-Das Sarma universality class \([12]\).

The extension of the approach presented here to other competitive growth processes may provide a tool to understand the observed dynamics of surface growth. Realistic systems may involve many component-processes, some of which may be dominant. Within our formalism a departure point may be a generalization of Eq. \([6]\):

\[
\Delta h(x,t) = \sum_k p_{\text{eff}}^{(k)} \Delta h^{(k)}(x,t),
\]

where the summation is over contributing processes, and \( \Delta h^{(k)} \) is the column-height increment due to the \( k \)th process. In first approximation component-processes are not explicitly correlated. Each process is encountered with probability \( p_k \), \( \sum_k p_k = 1 \), and contributes to the growth with an effective probability \( p_{\text{eff}}^{(k)} \), \( \sum_k p_{\text{eff}}^{(k)} = 1 \). In the trivial case of all components being ABC type models with unit mean deposit height \( p_{\text{eff}}^{(k)} = p_k \). For a DOLB growth \( p_{\text{eff}}^{(k)} \) will have to be determined. Depending on the model, this can be done by analyzing the growth when process \( k \) acts alone, and measuring either the mean bulk density or the mean fraction of the detached material or both \([14]\). Simplectic decompositions like the one proposed in Eq. \([12]\) have a long history of applications in many diverse fields.

In summary, the derived continuum equations and the resulting scaling show that model-dependent prefactors in universal scaling laws can be determined from bulk structures. This necessitates the distinction between the absorption-bulk-compact and the dense-or-lace-bulk growth processes in the analysis of competitive mixed-growth models.

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