Abstract—Given an $N$-relay Gaussian Half-Duplex (HD) diamond network, can we achieve a significant fraction of the capacity of the full network by operating only a subset of $k$ relays? This paper seeks to answer this question, which is also known as the network simplification problem. Building upon the recent result of Cardone et al., this work proves that there always exists a subnetwork of $k = N - 1$ relays that approximately (i.e., up to a constant gap) achieves a fraction $\frac{k}{N}$ of the capacity of the full network. This fraction guarantee is shown to be tight, i.e., there exists a class of Gaussian HD diamond networks with $N$ relays where the best (i.e., the one with the largest capacity) subnetwork of $k = N - 1$ relays has an approximate capacity of $\frac{k}{N}$ of the capacity of the full network. Moreover, it is shown that any optimal schedule of the full network can be used by at least one of the $\binom{N}{k}$ subnetworks of $k$ relays to achieve a worst performance guarantee of $\frac{k}{N}$.

The key step of the proof lies in the derivation of properties of submodular functions, which provide a combinatorial handle of the network simplification problem in Gaussian HD diamond networks.

I. INTRODUCTION

Consider a relay network where $k$ (potentially large) number of relay nodes assist the over-the-air communication between a source and a destination. The wireless network simplification problem seeks to answer the following question: can a significant fraction of the capacity of the full network be achieved by operating only a subset of the available relays?

Wireless network simplification was pioneered by the authors in [1] in the context of Gaussian Full-Duplex (FD) diamond networks. The importance of this problem stems from the several benefits it offers. For example, operating all the available relays might be computationally expensive as the relays must coordinate for transmission and might incur a significant cost in terms of consumed power. Network simplification represents a potential solution to these limiting factors as it promises energy savings and a complexity reduction in the synchronization problem, while ensuring that a significant fraction of the capacity of the full network is achieved.

In this paper, we investigate the network simplification problem for Gaussian Half-Duplex (HD) diamond networks with $N$ relays. Our study is motivated by the fact that currently employed relays operate in HD, unless sufficient isolation between the antennas can be guaranteed or different bands are used for transmission and reception. On the one hand, the problem becomes more challenging due to the intrinsic combinatorial nature of capacity characterization in HD relay networks. A capacity-achieving scheme, in fact, requires an optimization over $2^N$ possible cuts of the network (as in FD) each of which is a combination of $2^N$ possible listen/transmit configuration states. On the other hand, new opportunities arise and can be pursued, such as a simplification in the scheduling problem, since only the selected relays have to be scheduled.

Our goal is to provide a worst-case performance guarantee (in terms of achievable fraction) that holds universally (i.e., independently of the values of the channel parameters) when $k$ relays are selected out of the $N$ possible ones. This work builds upon the recent result in [2], where the authors provided the worst-case guarantee when $k = 1$ and $k = 2$ relays are selected out of the $N = k + 1$ and $N \gg 1$ possible ones. In an attempt to generalize these results to generic values of $N$ and $k$, we here derive a tight worst case fraction guarantee when $k = N - 1$ relays are selected out of the $N$ possible ones.

We prove that there always exists a subnetwork of $k = N - 1$ relays that approximately (i.e., up to a constant gap, which only depends on $N$) achieves a fraction $\frac{N - 1}{N}$ of the capacity of the full network. In particular, we show a surprising result: any optimal schedule of the full network can be used by at least one of the $N$ subnetworks of $k = N - 1$ relays to achieve the worst performance guarantee. This leads to a complexity reduction in the scheduling problem; in fact, it implies that, in order to select an $(N - 1)$-relay subnetwork that approximately achieves a fraction $\frac{N - 1}{N}$ of the full network capacity, there is no need to compute the optimal schedule for each of the $N$ subnetworks. It suffices to only compute the optimal schedule of the full network.

We then show that the worst fraction of $\frac{N - 1}{N}$ is tight, i.e., there exists a class of Gaussian HD diamond networks with

---

1 An $N$-relay diamond network is a two-hop relay network where the source communicates with the destination through $N$ non-interfering relays.
Finally, we generalize the results to generic values of \( k \leq N \), where each of the different subnetworks operates with an optimal schedule of the full network. We prove that an approximate fraction \( \frac{k}{N} \) of the capacity of the full network can always be achieved by selecting \( k \) relays. However, this worst case fraction does not appear to be tight. This result suggests that, when \( k < N - 1 \), forcing the \( k \)-relay subnetworks to operate with the optimal schedule of the full network is too conservative. Thus, in order to provide a tight performance guarantee, it becomes crucial to operate the subnetwork of the \( k \) selected relays with its optimal schedule.

As a result of independent interest, beyond its application in the proof of the worst fraction guarantee, we derive properties of submodular functions, which provide a combinatorial handle of the network simplification problem in Gaussian HD diamond networks.

**Related Work.** The authors in [1] showed that, in any \( N \)-relay Gaussian FD diamond network, there always exists a subnetwork of \( k \) relays that approximately achieves at least a fraction \( \frac{k}{N+1} \) of the capacity of the full network. This result, which is independent of \( N \), is quite promising as it implies that a significant fraction of the capacity can be achieved with reduced complexity, i.e., by performing an optimization only over \( 2^k \) possible cut configurations, instead of over the \( 2^N \) possible ones when all the \( N \) relays are used. Recently, in [2] the authors considered a more general network, namely the Gaussian FD layered network and proved that the worst fraction guarantee for a single path in the network decreases as the number of relays increases. This result suggests that the result in [1] is due to the special structure of the diamond network.

The capacity characterization of HD relay networks is more challenging than the FD counterpart, as in addition to the optimization over the \( 2^N \) cuts, it also requires an optimization over \( 2^N \) listen/transmit configuration states. In an attempt to provide a combinatorial handle to this problem, two natural questions arise.

The first question would be: are all the \( 2^N \) states necessary for capacity characterization? Recently, in [3] the authors proved a surprising result: at most \( N + 1 \) states (out of the \( 2^N \) possible ones) are sufficient for an approximate capacity characterization for a class of HD relay networks, which includes the Gaussian noise network. This result, which implies a significant complexity reduction of the problem, generalizes those in [5], [6], valid only for Gaussian HD diamond relay networks. It is worth noting that these \( N + 1 \) states might still require to operate all the \( N \) relays.

The second question would be: is it possible to provide a fraction guarantee when only a subset of \( k \) relays is selected? The result in [1] implies that in HD we can always guarantee at least a fraction \( \frac{k}{N+1} \) of the capacity of the full network, by simply operating the selected subnetwork half of the time in the broadcast mode (i.e., all the \( k \) relays listen) and half of the time in the MAC mode (i.e., all the \( k \) relays transmit). Although providing a performance guarantee, this result might be too conservative. This is indeed confirmed by the result in [7] where it was proved that, in any Gaussian HD diamond network, there always exists a subnetwork of \( k = 2 \) relays that approximately achieves at least half of the capacity of the full network. Recently, in [4] the authors tightened the result in [7], by showing that for \( k \in \{1, 2\} \) at least a fraction \( \frac{k}{k+1} \) and a fraction \( \frac{2k}{2(k+1)} \) of the full network capacity can be achieved when \( N = k + 1 \) and \( N \gg 1 \), respectively. In [2] it was also proved that these fractions are tight, i.e., there exists a class of Gaussian HD diamond networks where the best \( k \)-relay subnetwork approximately achieves these fractions. This result is quite surprising as it implies that in HD, differently from FD, the worst fraction guarantee decreases as \( N \) increases.

**Paper Organization.** Section II describes the system model and outlines the expressions and notations used throughout the paper. Section III discusses our main results and their implications. Section IV derives properties of submodular functions and diamond networks which are essential for subsequent proofs. The proofs of our main results are outlined in Section V. Finally, Section VI concludes the paper. Some of the proofs can be found in the Appendix.

## II. System Model

Throughout the paper, we denote with \([a:b]\) the set of integers from \( a \) to \( b \geq a \). For all \( x \in \mathbb{R} \), the ceiling and floor functions are denoted by \( \lceil x \rceil \) and \( \lfloor x \rfloor \), respectively. The \( \ell_1 \)-norm of a vector \( \lambda \) is represented by \( \|\lambda\|_1 \).

We consider the Gaussian HD diamond network where a source node wishes to communicate with a destination through \( N \) non-interfering relays operating in HD. Let \( Y_i[t] \) and \( Y_{N+1}[t] \) denote the received signals at time \( t \) by the \( i \)-th relay and the destination, respectively. Similarly, denote by \( X_0[t] \) and \( X_i[t] \) the signals transmitted from the source and the \( i \)-th relay at time \( t \), respectively. The received signals relate to the transmitted signals as

\[
\begin{align*}
Y_i[t] &= (1 - S_i[t]) h_{is} X_0[t] + Z_i[t], \quad \forall i \in [1:N], \\
Y_{N+1}[t] &= \sum_{i=1}^{N} S_i[t] h_{di} X_i[t] + Z_d[t],
\end{align*}
\]

where: (i) \( S_i[t] \) is the binary random variable that represents the state of the \( i \)-th relay (either listening or transmitting) at time \( t \); (ii) \( h_{is} \), \( h_{di} \in \mathbb{C} \) represent the channel coefficients from the source to the \( i \)-th relay and from the \( i \)-th relay to the destination, respectively; (iii) \( Z_i \) and \( Z_d \) denoting the additive white Gaussian noise at the \( i \)-th relay and at the destination, respectively, are independent and identically distributed as \( \mathcal{CN}(0,1) \).

We assume that the transmitted signal from each node satisfies an average power constraint \( \mathbb{E}[|X_i|^2] \leq 1, \forall i \in [0:N] \). We use \( \ell_i, r_i \) to indicate the individual link capacities

\[
\ell_i := \log \left(1 + |h_{is}|^2\right), \quad \forall i \in [1:N],
\]
\[ r_i := \log \left(1 + |h_{di}|^2\right), \quad \forall i \in [1 : N]. \]

With a slight abuse of notation, in the following, we name a network by the set of relays it employs. For example, the full network is referred to as \( \mathcal{N}_F = [1 : N] \).

It follows from the works in \([1, 8]\) that the capacity of the Gaussian HD diamond network \( \mathcal{N}_F \) described in \([1]\) can be approximated to within a constant gap \( G = O(N) \) by

\[
C_{\text{HD}}^{\mathcal{N}_F} = \max_{\lambda \in \Lambda} \min_{A \subseteq [1:N]} \sum_{i \in [0:1]^N} \lambda_i \left( \max_{s \in \mathcal{R}_s} \ell_i + \max_{r \in \mathcal{T}_r} r_i \right),
\]

where: (i) \( \Lambda = \{ \lambda : \lambda \in \mathbb{R}^{2^N}, \lambda \geq 0, \|\lambda\|_1 = 1 \} \) is the set of all possible schedules; (ii) \( \mathcal{R}_s \) (respectively, \( \mathcal{T}_r \)) represents the set of indices of relays listening (respectively, transmitting) in the relaying state \( s \in [0:1]^N \); (iii) \( \mathcal{A}^c = [1 : N] \setminus A \).

In the next section, we discuss the main results of this paper that show that a significant fraction of the approximate capacity \( C_{\text{HD}}^{\mathcal{N}_F} \) can be achieved by smartly selecting \( k \) relays out of the \( N \) possible ones.

### III. Main Results

In this section we summarize and discuss the main results of the paper, which are stated in the following theorem and lemmas.

**Theorem 1.** For any \( N \)-relay Gaussian HD diamond network, there exists a subnetwork of \( N-1 \) relays that achieves, up to a gap, a rate \( C_{\text{HD}}^{N-1,N} \) such that \( C_{\text{HD}}^{N-1,N} \geq \frac{N-1}{N} C_{\text{HD}}^{\mathcal{N}_F} \). Moreover, this bound is tight up to a constant gap.

**Proof:** To prove that the ratio is tight, it suffices to provide an example of an \( N \)-relay network where the best subnetwork of \( N-1 \) relays achieves an approximate capacity, which is exactly the fraction of the full network capacity in Theorem 1. To this end, consider the following structure:

\[
\ell_i = \ell_i \left[ \frac{2i}{N} \right] + \frac{2i}{N}, \quad i \in \left[ 1 : \left\lfloor \frac{N}{2} \right\rfloor \right],
\]

\[
r_i = r_i \left[ \frac{2i}{N} \right] + \frac{N - 2i}{N}, \quad i \in \left[ 1 : \left\lfloor \frac{N}{2} \right\rfloor \right],
\]

if \( N \) is odd:

\[
\ell_N = \infty, \quad r_N = \frac{1}{N}.
\]

Fig. 1 gives a graphical representation of \( C_{\text{HD}}^{N-1,N} \) for \( N \in [2:10] \). It is clear from Fig. 1 that \( \frac{C_{\text{HD}}^{N-1,N}}{C_{\text{HD}}^{\mathcal{N}_F}} = \frac{N-1}{N} \). We defer the proof of the lower bound in Theorem 1 to Section V.

**Remark 1.** The result in Theorem 1 implies that, for any \( N \)-relay Gaussian HD diamond network, smartly removing one relay can reduce the HD capacity of the network by at most \( \frac{1}{N} \) of the full network capacity. We also highlight that the removed relay may or may not be the worst relay (i.e., the one with the smallest single capacity among the \( N \) relays).

For example, consider the network in (3) for \( N = 3 \). It is not difficult to see that for this network: (i) the capacity of the full network \( C_{\text{HD}}^{\mathcal{N}_F} \) is 1; (ii) all the 2-relay subnetworks have a capacity of \( C_{\text{HD}}^{\mathcal{N}_F} = \frac{2}{3} \), with \((i,j) \in [1 : 3]^2 \) and \( i \neq j \); (iii) the single relay capacities are \( C_{\text{HD}}^{\mathcal{N}_F} = \frac{3}{2} \) and \( C_{\text{HD}}^{\mathcal{N}_F} = \frac{1}{2} \). Hence, in this particular network, by removing any of the relays (i.e., the best or the worst), we still retain \( \frac{2}{3} \) of the capacity of the full network.

The proof of the lower bound in Theorem 1 is a direct consequence of the following Lemma 2 which is proved in Section V.

**Lemma 2.** Consider an \( N \)-relay Gaussian HD diamond network \( \mathcal{N}_F \). Denote by \( \mathcal{N}_i \) the subnetwork operating with the \( N-1 \) relays in \( \mathcal{N}_F \setminus \{i\} \). Let \( \lambda \) be a schedule of the full network \( \mathcal{N}_F \) and denote by \( C_{\text{HD}}^{\mathcal{N}_i} \) the approximate achievable rate of \( \mathcal{N}_i \) when operated with the schedule \( \lambda \). Then

\[
\sum_{i=1}^{N} C_{\text{HD}}^{\mathcal{N}_i} \geq (N-1) C_{\text{HD}}^{\mathcal{N}_F},
\]

where \( C_{\text{HD}}^{\mathcal{N}_F} \) is the approximate achievable rate of \( \mathcal{N}_F \) when operated with the schedule \( \lambda \).

**Remark 2.** By having a subnetwork operate with a schedule of the full network, we mean that the subnetwork uses a schedule derived naturally from the full network schedule, as described in the following example.

**Example.** Consider a network of relays in \( \mathcal{N}_F = [1 : 3] \). Let

\[
\lambda = [\lambda_{000}, \lambda_{001}, \lambda_{010}, \lambda_{011}, \lambda_{100}, \lambda_{101}, \lambda_{110}, \lambda_{111}]^T
\]

be a schedule of \( \mathcal{N}_F \). For the subnetwork of relays \( \mathcal{N}_i = \{2, 3\} \), the schedule derived naturally from \( \lambda \) is

\[
\lambda(\mathcal{N}_i) = [\lambda_{000} + \lambda_{100}, \lambda_{001} + \lambda_{101}, \lambda_{010} + \lambda_{110}, \lambda_{011} + \lambda_{111}]^T.
\]
Similarly, for \( N_2 = \{1, 3\} \) and \( N_3 = \{1, 2\} \), we get

\[
\lambda(N_2) = [\lambda_000 + \lambda_{010} \lambda_{001} + \lambda_{101} \lambda_{100} + \lambda_{110} \lambda_{111}]^T,
\]

\[
\lambda(N_3) = [\lambda_000 + \lambda_{001} \lambda_{101} \lambda_{100} + \lambda_{110} \lambda_{111}]^T.
\]

Thus, the achievable rate of a subnetwork (for example \( N_1 \)) when operating with \( \lambda \) is

\[
C_{N_1}^{\text{HD}} = \min_{A \subseteq N_1} \sum_{i \in [0:1]} \lambda_i \left( \max_{i \in A} \ell'_{i,s} + \max_{i \in N_1 \setminus A} r'_{i,s} \right)
\]

\[
= \min_{A \subseteq N_1} \sum_{s \in [0:1]} \lambda_s \left( \max_{i \in A} \ell'_{i,s} + \max_{i \in N_1 \setminus A} r'_{i,s} \right),
\]

(5)

where

\[\ell'_{i,s} = \begin{cases} \ell_i & \text{if } i \in R_s \\ 0 & \text{otherwise} \end{cases}, \quad r'_{i,s} = \begin{cases} r_i & \text{if } i \in T_s \\ 0 & \text{otherwise}. \end{cases}\]

Now, if we focus on the optimal schedule \( \lambda^* \) of the full network, then Theorem 4 and Lemma 2 imply the following.

**Corollary 3.** Let \( \lambda^* \) be an optimal schedule of the full network \( N_F \), then:

1. For any \( N \)-relay Gaussian HD diamond network, there exists a subnetwork \( N_i \) of \( N-1 \) relays such that, when operated with \( \lambda^* \), it satisfies that

\[
C_{N_i}^{\text{HD}} \geq \frac{N - 1}{N} C_{N_F}^{\text{HD}}.
\]

2. There exist \( N \)-relay Gaussian HD diamond networks where \( \lambda^* \) can be used to naturally construct the optimal schedule for each subnetwork of \( N-1 \) relays (see for example, the network in (3)).

**Remark 3.** Corollary 4 implies that, to select a subnetwork of \( N-1 \) relays that guarantees the performance in Theorem 4, it is sufficient to know \( \lambda^* \). In other words, by knowing \( \lambda^* \), there is no need to compute the optimal schedules for each of the \( N \) subnetworks.

Finally, our next result gives a worst performance guarantee for selecting \( k \) relays out of the \( N \) relays of the full network. This is another consequence of Lemma 5 and it is proved in Section V.

**Lemma 4.** Consider an arbitrary \( N \)-relay Gaussian HD diamond network. There exists a subnetwork of \( k \in [1 : N] \) relays that achieves, up to a gap, a rate \( C_{k,N}^{\text{HD}} \) such that

\[
C_{k,N}^{\text{HD}} \geq \frac{k}{N} C_{N_F}^{\text{HD}}. \quad \text{(6)}
\]

The bound also holds if the subnetworks are operated with an optimal schedule of the full network.

**Remark 4.** If we consider the network construction in (3) with \( N \in [3 : 4] \) then, for any optimal schedule of the full network, the best \( k \)-relay subnetwork operated with that optimal schedule approximately achieves \( \frac{k}{N} \) of the HD capacity of the full network. This implies that the second statement of Lemma 4 is tight for \( N = [3 : 4] \). However, when operated with its own optimal schedule, the best subnetwork of \( k < N - 1 \) relays achieves a rate that is strictly greater than \( \frac{k}{N} \) of the HD capacity of the full network. This seems to suggest that when \( k < N - 1 \) the schedule optimization (for the subnetwork) plays a crucial role in the guarantee that can be provided.

**Remark 5.** Lemma 4 provides a different bound from those proved in (2), (7). These bounds can be combined as

\[
C_{k,N}^{\text{HD}} \geq \begin{cases} \max \left\{ \frac{k}{N}, \frac{1}{2} \right\}, & k = 1 \\ \max \left\{ \frac{k}{N}, \frac{1}{3} \right\}, & N \geq k \geq 2 \end{cases} \quad \text{(6)}
\]

From (6), we can see that in some cases (for example when \( N = \{3, 4\} \)), the new bound gives a better guarantee than those already proved in the literature.

**IV. SUBMODULARITY AND CUTS PROPERTIES**

In this section we derive and discuss some properties of submodular functions and diamond networks that form the basis of our proofs in Section V. It is worth noting that, beyond their utilization in the proofs of our main results, these properties might be of independent interest for other applications.

**Definition 1.** For a finite set \( \Omega \), let \( f : 2^\Omega \to \mathbb{R} \) be a set function defined on \( \Omega \). The set function \( f \) is submodular if

\[
\forall \mathcal{A}, \mathcal{B} \subseteq \Omega, \quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \quad \text{(7)}
\]

Building on the definition in (7), we can prove the following property of submodular functions.

**Lemma 5.** Let \( f \) be a submodular function. Then, for any group of \( n \) sets \( \mathcal{A}_i, i \in [1:n] \),

\[
\sum_{i=1}^{n} f(A_i) \geq \sum_{j=1}^{n} f \left( \mathcal{E}^{(n)}_j \right),
\]

where \( \mathcal{E}^{(n)}_j \) is the set of elements that appear in at least \( j \) sets \( A_i, i \in [1:n] \).

**Proof:** The proof relies on the definition of submodular functions and some set-theoretic properties. The detailed proof can be found in Appendix A.

To understand the implications of Lemma 5 consider the following example.

**Example.** Let \( \Omega = [1 : 7] \) and consider the subsets \( A_1 = \{1, 2, 5, 7\}, A_2 = \{2, 4, 5\}, A_3 = \{4, 5, 6\} \). Lemma 5 proves that, for a submodular function \( f \) defined over \( \Omega \), we get

\[
f(\{1, 2, 5, 7\}) + f(\{2, 4, 5\}) + f(\{4, 5, 6\}) \geq f(\{1, 2, 4, 5, 6, 7\}) + f(\{2, 4, 5\}) + f(\{5\}). \quad \text{(8)}
\]

Consider now \( f(A) = \max_{i \in A} \alpha_i \) for \( A \subseteq \Omega \), which is a submodular function. With this, by evaluating (8) we get

\[
\sum_{i=1}^{3} f(A_i) = 7 + 5 + 6 = 18, \quad \sum_{i=1}^{3} f \left( \mathcal{E}^{(3)}_j \right) = 7 + 5 + 5 = 17
\]
\[ \implies \sum_{i=1}^{N} f(A_i) \geq \sum_{i=1}^{N} f\left( \mathcal{E}_j^{(N)} \right) . \]

Next, we use Lemma 5 to prove the following result for diamond networks.

**Lemma 6.** Consider a diamond network \( \mathcal{N}_F \) with \( N \) relays, i.e., \( \mathcal{N}_F = \{1 \ldots N\} \). Let \( \mathcal{N}_i = \mathcal{N}_F \setminus \{i\}, \ i \in [1 : N] \) be a subnetwork of \( N-1 \) relays. Then, for any combination of sets \( A_i \subseteq \mathcal{N}_i \), there exist \( (N-1) \) sets \( A_{F,j} \subseteq \mathcal{N}_F \), \( j \in [1 : N-1] \) such that

\[
\sum_{j=1}^{N} \left( \max_{i \in A_j} \ell_i + \max_{i \in \mathcal{N}_F \setminus A_j} r_i \right) \geq \sum_{j=1}^{N} \left( \max_{i \in A_{F,j}} \ell_i + \max_{i \in \mathcal{N}_F \setminus A_{F,j}} r_i \right). 
\]

Moreover the sets \( A_{F,j} \) do not depend on the values \( (\ell_i, r_i) \).

**Remark 6.** If the network and its subnetworks operate in FD, then Lemma 6 directly relates cuts of the subnetworks \( \mathcal{N}_i \) to cuts of the full network \( \mathcal{N}_F \). Furthermore, by choosing \( A_i \) to be the minimum FD cut of the subnetwork \( \mathcal{N}_i \), we get

\[
\sum_{i=1}^{N} C_{F,j}^{(N)} \geq \sum_{j=1}^{N-1} \left( \max_{i \in A_j} \ell_i + \max_{i \in \mathcal{N}_F \setminus A_j} r_i \right) \geq (N-1)C_{F,j}^{(N)},
\]

where \( C_{F,j}^{(N)} \) (respectively, \( C_{F,j}^{(N)} \)) is the constant gap approximation of the FD capacity of the full network (respectively, the subnetwork \( \mathcal{N}_i \)). This is a different way of proving the result in [1] Theorem 1 for \( k = N-1 \).

We now prove the statement in Lemma 6.

**Proof:** Throughout the proof, we let \( B_j = \mathcal{N}_i \setminus A_j \), \( \forall i \in [1 : N] \), \( f(A) = \max_{i \in A} \ell_i \) and \( g(A) = \max_{i \in A} r_i \), with \( A \subseteq [1 : N] \). It is not difficult to see that \( f, g \) are submodular functions. As a result, we have

\[
\sum_{j=1}^{N} \left( \max_{i \in A_j} \ell_i + \max_{i \in \mathcal{N}_F \setminus A_j} r_i \right) \\
= \sum_{j=1}^{N} [f(A_j) + g(B_j)] \\
\geq \sum_{j=1}^{N} \left[ f\left( \mathcal{E}_j^{(N)} \right) + g\left( \mathcal{F}_j^{(N)} \right) \right] \\
= \sum_{j=1}^{N-1} \left[ f\left( \mathcal{E}_j^{(N)} \right) + g\left( \mathcal{F}_j^{(N)} \right) \right] \\
= \sum_{j=1}^{N-1} \left[ f\left( \mathcal{E}_j^{(N)} \right) + g\left( \mathcal{F}_j^{(N)} \right) \right], \tag{9}
\]

where: (i) the inequality in (a) follows from Lemma 5 with \( \mathcal{E}_j^{(N)} \) (respectively, \( \mathcal{F}_j^{(N)} \)) being the set of elements that appear in at least \( j \) sets \( A_i \), \( i \in [1 : N] \) (respectively, \( B_j \)); (ii) the equality in (b) follows because \( \mathcal{E}_N^{(N)} = \mathcal{F}_N^{(N)} = \emptyset \) since \( \bigcap_{i=1}^{N} \mathcal{N}_i = \emptyset \); (iii) the equality in (c) follows by reordering the sum.

Since \( \forall i \in [1 : N] \), the element \( i \in \mathcal{N}_j \), with \( j \neq i \), and \( A_j \) and \( B_j \) are disjoint \( \forall j \in [1 : N] \) (by definition), then the element \( i \) belongs to exactly \( (N-1) \) sets \( A_j, B_j \). We now claim that \( \mathcal{N}_F \setminus \mathcal{E}_j^{(N)} = \mathcal{F}_N^{(N)} \), \( j \in [1 : N-1] \). Consider an element \( x \in \mathcal{N}_F \); then:

1. Let \( x \in \mathcal{E}_j^{(N)} \), i.e., \( x \) appears in at least \( j \) sets \( A_i \). Since \( x \) appears exactly \( (N-1) \) times in \( A_i, B_i \), this means that \( x \) appears in at most \( (N-1) - j \) sets \( B_i \), i.e., \( x \notin \mathcal{F}_N^{(N)} \). In other words, \( x \notin \mathcal{N}_F \setminus \mathcal{E}_j^{(N)} \). Since this is true \( \forall x \in \mathcal{E}_j^{(N)} \), this implies that \( \mathcal{E}_j^{(N)} \subseteq \mathcal{N}_F \setminus \mathcal{F}_N^{(N)} \) and as a result, \( \mathcal{N}_F \setminus \mathcal{E}_j^{(N)} = \mathcal{F}_N^{(N)} \).

2. Let \( x \notin \mathcal{E}_j^{(N)} \), i.e., \( x \) appears in at most \((j-1)\) sets \( A_i \); since \( x \) in total appears exactly \( (N-1) \) times in \( A_i, B_i \), this means that \( x \) appears in at least \((N-1) - (j-1)\) sets \( B_i \), i.e., \( x \notin \mathcal{F}_N^{(N)} \). Since this is true \( \forall x \in \mathcal{N}_F \setminus \mathcal{E}_j^{(N)} \), this implies that \( \mathcal{N}_F \setminus \mathcal{E}_j^{(N)} \subseteq \mathcal{F}_N^{(N)} \).

Together 1) and 2) imply that \( \mathcal{N}_F \setminus \mathcal{E}_j^{(N)} = \mathcal{F}_N^{(N)} \), \( \forall j \in [1 : N-1] \). Applying this in (9), we obtain

\[
\sum_{j=1}^{N} \left( \max_{i \in A_j} \ell_i + \max_{i \in \mathcal{N}_F \setminus A_j} r_i \right) \\
\geq \sum_{j=1}^{N-1} \left[ f\left( \mathcal{E}_j^{(N)} \right) + g\left( \mathcal{F}_j^{(N)} \right) \right] \\
= \sum_{j=1}^{N-1} \left[ f\left( \mathcal{E}_j^{(N)} \right) + g\left( \mathcal{N}_F \setminus \mathcal{E}_j^{(N)} \right) \right] \\
= \sum_{j=1}^{N-1} \left( \max_{i \in A_j} \ell_i + \max_{i \in \mathcal{N}_F \setminus A_j} r_i \right),
\]

where we let \( A_{F,j} = \mathcal{E}_j^{(N)} \).

Since throughout the proof we made no assumptions on the values of \( (\ell_i, r_i) \), then the sets \( A_{F,j} \) do not depend on the values of \( (\ell_i, r_i) \). This concludes the proof.

Before going into technical details of using these results to prove Theorem 1 and Lemma 2, we state a final remark.

**Remark 7.** By taking note of the link capacities \( (\ell_i, r_i) \), we can prove the lower bound in Lemma 6 with a different construction than the one discussed in the proof above. The key difference for the construction discussed in the proof is that it is independent of \( (\ell_i, r_i) \). This becomes of fundamental importance when we consider HD cuts, as we will see in the proof of Lemma 3 in the next section.

V. PROOFS OF MAIN RESULTS

We now use the results derived in Section IV to prove our main results, presented earlier in Section III.

A. Proof of Lemma 2

Let \( \lambda \) be a schedule of the full network \( \mathcal{N}_F \) with \( N \) relays. Let \( C^{(N)}_{F,j} \) be the approximate HD rate achieved by the subnetwork \( \mathcal{N}_j = \mathcal{N}_F \setminus \{j\} \), \( j \in [1 : N] \), when operated with the schedule constructed from \( \lambda \). Denote by
\( A_i^* \) the minimum cut of the network \( \mathcal{N}_j \) when operated with \( \lambda \). Then from (5), we have
\[
\sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} = \sum_{s \in [0:1]^N} \lambda_s \left[ \sum_{j=1}^{N} \left( \max_{i \in A_j^*} \ell'_{i,s} + \max_{i \in A_j^* \setminus A_j^*} r'_{i,s} \right) \right],
\]
where
\[
\ell'_{i,s} = \begin{cases} 
\ell_i & \text{if } i \in R_s \\
0 & \text{otherwise}
\end{cases}, \quad r'_{i,s} = \begin{cases} 
r_i & \text{if } i \in T_s \\
0 & \text{otherwise}
\end{cases}.
\]
(10)

From the result in Lemma 6 we know that \( \exists \{A_{F_j}\}, j \in [1 : N - 1] \) such that for each \( s \in [0:1]^N \):
\[
\sum_{j=1}^{N} \left( \max_{i \in A_j} \ell'_{i,s} + \max_{i \in A_j \setminus A_j^*} r'_{i,s} \right) \geq \sum_{j=1}^{N-1} \left( \max_{i \in A_{F_j}} \ell'_{i,s} + \max_{i \in A_{F_j} \setminus A_j^*} r'_{i,s} \right),
\]
where \( A_{F_j} \subseteq A_{F_j}, \forall j \in [1 : N - 1] \). Additionally, \( A_{F_j} \) is independent of \( (\ell'_{i,s}, r'_{i,s}) \) and is therefore independent of \( (\ell_i, r_i) \) and the relaying state \( s \). Hence
\[
\sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \geq \sum_{s \in [0:1]^N} \lambda_s \left[ \sum_{j=1}^{N} \left( \max_{i \in A_j} \ell'_{i,s} + \max_{i \in A_j \setminus A_j^*} r'_{i,s} \right) \right] \geq \sum_{s \in [0:1]^N} \lambda_s \left[ \sum_{j=1}^{N-1} \left( \max_{i \in A_{F_j}} \ell'_{i,s} + \max_{i \in A_{F_j} \setminus A_j^*} r'_{i,s} \right) \right] = \sum_{j=1}^{N-1} \sum_{s \in [0:1]^N} \lambda_s \left( \max_{i \in A_{F_j}} \ell'_{i,s} + \max_{i \in A_{F_j} \setminus A_j^*} r'_{i,s} \right) \geq (N - 1) \min_{A \subseteq \mathcal{N}_i} \left\{ \sum_{s \in [0:1]^N} \lambda_s \left( \max_{i \in A} \ell'_{i,s} + \max_{i \in A \setminus A^*_i} r'_{i,s} \right) \right\} = (N - 1) C_{\mathcal{N}_i}^{\text{HD}}.
\]
This completes the proof.

B. Proof of Theorem 7

Denote by \( \lambda^* \) the optimal schedule of the full network \( \mathcal{N}_F \) with \( N \) relays. Let \( C_{\mathcal{N}_i}^{\text{HD}} \) be the approximate HD achievable rate of the subnetwork \( \mathcal{N}_i = \mathcal{N}_F \setminus \{i\} \), with \( i \in [1 : N] \), when operated with \( \lambda^* \).

Since the schedule constructed from \( \lambda^* \) might not be the optimal schedule for the subnetwork, then clearly we have
\[
C_{\mathcal{N}_i}^{\text{HD}} \leq C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} \max_{i \in [1 : N]} C_{\mathcal{N}_i}^{\text{HD}}.
\]
\[
(\text{11a})
\]
This implies that
\[
\sum_{i=1}^{N} C_{\mathcal{N}_F}^{\text{HD}} \geq \sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} \max_{i \in [1 : N]} C_{\mathcal{N}_i}^{\text{HD}}.
\]
\[
(\text{11b})
\]
\[
\sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} C_{\mathcal{N}_i}^{\text{HD}} \leq \sum_{i=1}^{N} \max_{i \in [1 : N]} C_{\mathcal{N}_i}^{\text{HD}}.
\]
\[
(\text{11c})
\]
\[
C_{\mathcal{N}_F}^{\text{HD}} \geq \frac{N - 1}{N} C_{\mathcal{N}_F}^{\text{HD}},
\]
\[
(\text{11d})
\]
From the result above we have
\[
\frac{N - 1}{N} C_{\mathcal{N}_F}^{\text{HD}} \leq C_{\mathcal{N}_{N-k}}^{\text{HD}} \leq C_{\mathcal{N}_F}^{\text{HD}} \quad \iff \quad \frac{C_{\mathcal{N}_{N-k}}^{\text{HD}}}{C_{\mathcal{N}_F}^{\text{HD}}} \geq \frac{k}{N},
\]
since \( |\mathcal{N}_{N-k}| = k \).

VI. CONCLUSIONS

We investigated the network simplification problem in an \( N \)-relay Gaussian HD diamond network. We proved that there always exists a subnetwork of \( k = N - 1 \) relays that approximately achieves at least a fraction \( \frac{N - 1}{N} \) of the capacity of the full network. This result was derived by showing that any optimal schedule of the full network can be used by at least one of the \( N \) subnetworks of \( k = N - 1 \) relays to achieve the worst performance guarantee. Moreover, we provided an example of a class of Gaussian HD diamond networks for which this fraction is tight. We extended this result to generic values of \( k \leq N \) and showed a worst fraction guarantee of \( \frac{k}{N} \), which is too conservative. This observation implies that a tighter bound on the worst fraction has to be derived, which is object of current investigation.

APPENDIX A

PROOF OF LEMMA 5

Let \( f \) be a submodular function. We want to prove that for any group of \( n \) sets \( A_i \),
\[
\sum_{i=1}^{n} f(A_i) \geq \sum_{j=1}^{n} f(C^{(n)}_j).
\]
\[
f(A_{n+1}) + f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k} \left( \bigcap_{i \in I} A_i \right) \right) + \sum_{k=1}^{n-1} f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right)
\]

\[
\geq f \left( \bigcup_{I \subseteq [1:n] \mid |I| = 1} \left( \bigcap_{i \in I} A_i \right) \right) + \sum_{k=1}^{n-1} f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right) + \sum_{k=2}^{n-1} f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k} \left( \bigcap_{i \in I} A_i \right) \right)
\]

\[
\geq \sum_{\ell=1}^{2} f \left( \bigcup_{I \subseteq [1:n+1] \mid |I| = \ell} \left( \bigcap_{i \in I} A_i \right) \right) + \sum_{k=1}^{n-1} \left( \bigcup_{I \subseteq [1:n] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right) + \sum_{k=3}^{n-1} f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k} \left( \bigcap_{i \in I} A_i \right) \right)
\]

\[
\geq \sum_{\ell=1}^{n} f \left( \bigcup_{I \subseteq [1:n+1] \mid |I| = \ell} \left( \bigcap_{i \in I} A_i \right) \right) + f \left( \bigcup_{I \subseteq [1:n] \mid |I| = n} \left( \bigcap_{i \in I} A_i \right) \right) + \sum_{k=1}^{n} f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k} \left( \bigcap_{i \in I} A_i \right) \right)
\]

\[
= \sum_{j=1}^{n+1} f(\mathcal{E}_j^{(n+1)})
\]

where \(\mathcal{E}_j^{(n)}\) is the set of elements that appear in at least \(j\) sets \(A_i, i \in \{1:n\}\). The proof is by induction. For the base case, we appeal to the trivial case that \(f(A_1) = f(\mathcal{E}_1^{(1)})\).

For the proof of the induction step, we prove and use the following property of submodular functions.

**Property 1.** Let \(f \) be a submodular function. Then, \(\forall n > 0\) and \(0 \leq k < n\),

\[
f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k} \left( \bigcap_{i \in I} A_i \right) \right) + f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right) 
\geq f \left( \bigcup_{I \subseteq [1:n+1] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right) + f \left( \bigcup_{I \subseteq [1:n+1] \mid |I| = k} \left( \bigcap_{i \in I} A_i \right) \right)
\]

which can be equivalently rewritten as

\[
\sum_{i=1}^{n} f(A_i) + f(A_{n+1}) \geq \sum_{k=0}^{n-1} f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right)
\]

\[
\geq f(A_{n+1}) + \sum_{k=0}^{n-1} f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right)
\]

\[
= f(\mathcal{E}_1^{(n)}) + \sum_{j=1}^{n+1} f(\mathcal{E}_j^{(n+1)})
\]

The final step in the proof follows by inductively applying Property 1 on the underlined terms with the appropriate \(k\) as in (12), at the top of this page. This concludes the proof.

**A. Proof of Property 1**

To conclude the proof of Lemma 5, we now prove Property 1 using set-theoretic properties.

**Proof:** By using properties of submodular functions and set operations we have

\[
f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k} \left( \bigcap_{i \in I} A_i \right) \right) + f \left( \bigcup_{I \subseteq [1:n] \mid |I| = k+1} \left( \bigcap_{i \in I} A_i \right) \right)
\]
The distributive property of intersection over unions gives
\[
\begin{align*}
&= A_{n+1} \bigcap_{\mathcal{J} \subseteq [1:n]} \left( \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_j \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_i \right) \\
&= \bigcap_{\mathcal{J} \subseteq [1:n]} \left( \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_j \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_i \right).
\end{align*}
\]

Now note that \( \forall \mathcal{I} \subseteq [1:n] \) with \(|\mathcal{I}| = k + 1 \), \( \exists \mathcal{J}_\mathcal{I} \subseteq \mathcal{I} \) with \(|\mathcal{J}_\mathcal{I}| = k \). Then for each \( \mathcal{I} \), we have
\[
\begin{align*}
\left( \bigcap_{i \in \mathcal{I}} \mathcal{A}_i \right) \bigcap \left( \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_j \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_i \right) \\
= \left( \bigcap_{i \in \mathcal{I}} \mathcal{A}_i \right) \bigcap \left( \bigcap_{i \in \mathcal{I}} \mathcal{A}_i \bigcup \left( \bigcup_{\mathcal{L} \subseteq [1:n]} \mathcal{A}_i \bigcup_{\mathcal{L} \subseteq \mathcal{J}_\mathcal{I}} \mathcal{J} \right) \right).
\end{align*}
\]

where the equality (c) follows since \( \mathcal{U} \cap (\mathcal{U} \cup \mathcal{V}) = \mathcal{U} \).

As a consequence we have
\[
\begin{align*}
\left( \bigcap_{i \in \mathcal{I}} \mathcal{A}_i \right) \bigcap \left( \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_j \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_i \right) \\
= \left( \bigcap_{i \in \mathcal{I}} \mathcal{A}_i \right) \bigcap \left( \bigcap_{i \in \mathcal{I}} \mathcal{A}_i \bigcup \left( \bigcup_{\mathcal{L} \subseteq [1:n]} \mathcal{A}_i \bigcup_{\mathcal{L} \subseteq \mathcal{J}_\mathcal{I}} \mathcal{J} \right) \right).
\end{align*}
\]

Finally, by applying (17) for each \( \mathcal{I} \) in (16), we get
\[
S = A_{n+1} \bigcap_{\mathcal{J} \subseteq [1:n]} \left( \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_j \bigcap_{\mathcal{J} \subseteq [1:n]} \mathcal{A}_i \right).
\]

where the last equality follows by using the distributive property of intersection over unions. This proves (15) and concludes the proof of Property [1] \[\blacksquare\]

REFERENCES

[1] C. Nazaroglu, A. Özugr, and C. Fragouli, “Wireless network simplification: The Gaussian N-relay diamond network,” IEEE Trans. Inf. Theory, vol. 60, no. 10, pp. 6329–6341, Oct. 2014.
[2] M. Cardone, C. Fragouli, and D. Tuninetti, “On network simplification for Gaussian half-duplex diamond networks,” to appear in 2016 IEEE Int. Symp. on Inf. Theory (ISIT), arXiv:1601.05766, July 2016.
[3] Y. H. Ezzeldin, A. Sengupta, and C. Fragouli, “Wireless network simplification: Beyond diamond networks,” to appear in 2016 IEEE Int. Symp. on Inf. Theory (ISIT), arXiv:1601.05766, July 2016.
[4] M. Cardone, D. Tuninetti, and R. Knopp, “The approximate optimality of simple schedules for half-duplex multi-relay networks,” in IEEE Inf. Theory Workshop (ITW), 2015, April 2015, pp. 1–5.
[5] H. Bagheri, A. Motahari, and A. Khandani, “On the capacity of the half-duplex diamond channel under fixed scheduling,” IEEE Trans. Inf. Theory, vol. 60, no. 6, pp. 3544–3558, June 2014.
[6] S. Brahma and C. Fragouli, “Structure of optimal schedules in diamond networks,” in 2014 IEEE Int. Symp. on Inf. Theory (ISIT), June 2014, pp. 641–645.
[7] ———, “A simple relaying strategy for diamond networks,” in 2014 IEEE Int. Symp. on Inf. Theory (ISIT), June 2014, pp. 1922–1926.
[8] G. Kramer, “Models and theory for relay channels with receive constraints,” in 42nd Annual Allerton Conference on Communication, Control, and Computing, Sept. 2004, pp. 1312–1321.