ESTIMATES OF FOURIER COEFFICIENTS OF CUSP FORMS ASSOCIATED TO COFINITE FUCHSIAN SUBGROUPS

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Abstract. Let $\Gamma$ be a cofinite Fuchsian subgroup, and assume that $i \infty$ is a cusp of $\Gamma$. Let $S^k(\Gamma)$ be the complex vector-space of cusp forms of weight-$k$ with respect to $\Gamma$, and let $\{f_1, \ldots, f_d\}$ denote a set of orthonormal basis of $S^k(\Gamma)$ with respect to the Petersson inner-product, where $d_k$ is the dimension of $S^k(\Gamma)$. For $n \geq 1$, and $1 \leq j \leq d_k$, let $a_{n,j}$ denote the $n$-th Fourier coefficient of $f_j$ at $i \infty$. In this article, for $k \in \mathbb{R} > 6$, $n \geq 1$, and for any $\epsilon > 0$, we show that

$$\frac{1}{d_k} \sum_{j=1}^{d_k} |a_{n,j}|^2 = O_X, \epsilon \left(n^{(k-1)/2} + \epsilon\right),$$

where the implied constant depends on the Riemann surface $X$, and on the choice of $\epsilon$. For a fixed $\epsilon > 0$, the implied constant remains stable in covers of hyperbolic Riemann surfaces. Using the above estimate, we prove the Ramanujan-Petersson conjecture for arbitrary cofinite Fuchsian subgroups.

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1. Introduction

1.1. Ramanujan-Petersson conjecture. The study of Fourier coefficients of modular forms is a very active area of research in number theory. Fourier coefficients of modular forms have deep arithmetic and geometric significance. Interest in Fourier coefficients of modular forms has been instigated by Ramanujan’s conjecture on the estimate of the Ramanujan-$\tau$ function.

The Ramanujan-Petersson conjecture is a generalization of Ramanujan’s conjecture to Hecke cusp forms of arbitrary weights. Let $f$ be a Hecke cusp form of weight $k \geq 2$ with respect to the full modular group $SL_2(\mathbb{Z})$, and let $a_n$ be the $n$-th Fourier coefficient of $f$. Then, for any $\epsilon > 0$, the Ramanujan-Petersson conjecture is the following estimate

$$|a_n| = O_{\epsilon} \left(n^{(k-1)/2} + \epsilon\right),$$

where the implied constant depends only on the choice of $\epsilon$. In 1974, in the celebrated article [De74], Deligne proved the above estimate.

Since then, the Ramanujan-Petersson conjecture has been formulated and proved in various cases, and most notably by Harris and Taylor in [HT01]. However, the conjecture is still open for arbitrary Fuchsian subgroups of cofinite volume. With notation as above, the best available result in the context of arbitrary Fuchsian subgroups is the following estimate

(1) $$|a_n| = O_X \left(n^{(k-1)/2 + \frac{1}{6}}\right),$$

where the implied constant depends on the Riemann surface $X$. Estimate (1) was proved by Good in 1981 in [Go81].

1.2. Main result. In this article, we prove a version of the Ramanujan-Petersson conjecture for arbitrary cofinite Fuchsian subgroups. Let $\Gamma \subset PSL_2(\mathbb{R})$ be a Fuchsian subgroup of first kind. We assume that $\Gamma$ admits only one cusp $i \infty$, and that

(2) $$\Gamma_{i \infty} := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \subset \Gamma,$$

where $\Gamma_{i \infty}$ denotes the stabilizer of the cusp $i \infty$ in $\Gamma$. 


For $k \in \mathbb{R}_{>6}$, let $\{f_1, \ldots, f_{d_k}\}$ be an orthonormal basis of $S^k(\Gamma)$ with respect to the Petersson inner-product, which is given by equation (10), and $d_k$ denotes the dimension of the complex vector-space $S^k(\Gamma)$. For $n \geq 1$, let $a_{n,j}$ denote the $n$-th Fourier coefficient of the cusp form $f_j$ at the cusp $i\infty$. Then, for any $\epsilon > 0$, our main result of the article is the following estimate

\begin{equation}
\frac{1}{d_k} \sum_{j=1}^{d_k} |a_{n,j}|^2 = O_{X,\epsilon}(n^{(k-1)+\epsilon}), \tag{3}
\end{equation}

where the implied constant depends on the Riemann surface $X$, and on the choice of $\epsilon$. Furthermore, for a fixed $\epsilon > 0$, the implied constant in the above estimate, remains stable in covers of hyperbolic Riemann surfaces.

For $k \in \mathbb{R}_{>6}$, let $f \in S^k(\Gamma)$ be a cusp form of weight-$k$ with respect to the Fuchsian subgroup $\Gamma$. Furthermore, assume that $f$ is normalized with respect to the Petersson inner-product, and let $a_n$ be its $n$-th Fourier coefficient at the cusp $i\infty$. Then, for any $\epsilon > 0$, our main result (3) implies the following estimate

\begin{equation}
|a_n| = O_{X,\epsilon}(\sqrt{k}n^{\frac{k-1}{2}+\epsilon}), \tag{4}
\end{equation}

where the implied constant depends on the Riemann surface $X$, and on the choice of $\epsilon$. Furthermore, for a fixed $\epsilon > 0$, the implied constant in the above estimate, remains stable in covers of hyperbolic Riemann surfaces, which implies the Ramanujan-Petersson conjecture for arbitrary cofinite Fuchsian subgroups.

Remark 1.1. The proof of estimate (3) relies completely on off-diagonal estimates of the Bergman kernel associated to $S^k(\Gamma)$, which we derived in [AM17] and [AM18]. In [AM17] and [AM18], for notational convenience, we assume that the cofinite Fuchsian subgroup $\Gamma$ admits only one cusp $i\infty$, whose stabilizer $\Gamma_{i\infty}$ is as given in equation (2).

However, in [AM17] and [AM18], we indicate how our estimates of the Bergman kernel extend to the setting of multiple cusps. Hence, our assumption that the Fuchsian subgroup $\Gamma$ consists of only one cusp $i\infty$ does not lead to any loss of generality, and our estimate (3) easily extends to cofinite Fuchsian subgroups with multiple cusps.

Remark 1.2. Our complete analysis relies on estimates of the Bergman kernel from [AM17] and [AM18]. In [AM17] and [AM18], estimates of the Bergman kernel have been derived for $k \in \mathbb{N}_{\geq 6}$. However, arguments that go into the proofs of the Main theorems in [AM17] and [AM18] make no distinction between $k \in \mathbb{N}_{\geq 6}$, and $k \in \mathbb{R}_{\geq 6}$. Hence, in section 2.4, we have stated, and used the extended versions of the Main theorems from [AM17] and [AM18], to $k \in \mathbb{R}_{\geq 6}$.

Remark 1.3. As the estimates of the Bergman kernel from [AM17] and [AM18] are stable in covers of hyperbolic Riemann surfaces, our estimate (3) also remains stable in covers of hyperbolic Riemann surfaces. In fact, our estimate (3) only depends on $r_X$, the injectivity radius of $X$, which is as defined in equation (7).

By working out individual cases, estimates of the Bergman kernel from [AM17] and [AM18] can be extended to $k \in \mathbb{R}_{>2}$. Consequently, our estimate (3) can also be extended to $k \in \mathbb{R}_{>2}$.

Lastly, there is no restriction on $k$ being an integer or even a half-integer, and can be any real number greater than or equal to six.

2. HYPERBOLIC RIEMANN SURFACES, CUSP FORMS, AND BERGMAN KERNELS

In this section, we set up the notation, and state results from literature, which are used to prove estimate (3).
2.1. Hyperbolic upper half-plane and hyperbolic metric. Let
\[ \mathbb{H} := \{ z = x + iy \mid y = \text{Im}(z) > 0 \} \]
denote the hyperbolic upper half-plane. Let \( \mu_{\text{hyp}} \) denote the hyperbolic metric on \( \mathbb{H} \), which is the natural metric on \( \mathbb{H} \), and is of constant negative curvature equal to \(-1\). The hyperbolic metric \( \mu_{\text{hyp}} \) at the point \( z = x + iy \in \mathbb{H} \) is given by the following formula
\[ \mu_{\text{hyp}}(z) := \frac{i}{2} \cdot \frac{dz \wedge d\overline{z}}{y^2} = \frac{dx dy}{y^2}. \] (5)
Let \( d_{\text{hyp}}(z, w) \) denote the natural distance function on \( \mathbb{H} \), which is induced by the hyperbolic metric \( \mu_{\text{hyp}} \). Furthermore, for \( z = x + iy, w = u + iv \in \mathbb{H} \), we have the following relation
\[ \cosh^2 \left( \frac{d_{\text{hyp}}(z, w)}{2} \right) = \frac{|z - w|^2}{4yv}. \] (6)

2.2. Hyperbolic Riemann surface. Let \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) be a cofinite Fuchsian subgroup, and without loss of generality, we assume that \( i\infty \) is the only cusp of \( \Gamma \) with stabilizer
\[ \Gamma_{i\infty} := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \subset \Gamma. \]
Furthermore, we assume that \( \Gamma \) does not admit elliptic elements.

The Fuchsian subgroup \( \Gamma \) acts on \( \mathbb{H} \) via fractional linear transformations, and let \( X := \Gamma \backslash \mathbb{H} \) denote the quotient space. The quotient space \( X \) admits the structure of a hyperbolic Riemann surface of genus \( g > 1 \), and is of finite hyperbolic volume. The cusp \( i\infty \) of \( \Gamma \) corresponds to a puncture of the Riemann surface \( X \). Let \( X_{\Gamma} \) denote a fixed fundamental domain of \( X \).

The hyperbolic metric induces a metric on \( X \), which is compatible with the natural complex structure of \( X \), and we again denote it by \( \mu_{\text{hyp}} \). Furthermore, locally, for \( z, w \in X \), the geodesic distance between the points \( z \) and \( w \) on \( X \), is given by \( d_{\text{hyp}}(z, w) \).

The injectivity radius of \( X \) is given by the following formula
\[ r_X := \inf \left\{ d_{\text{hyp}}(z, \gamma z) \mid z \in \mathbb{H}, \gamma \in \Gamma \backslash \Gamma_{i\infty} \right\}. \] (7)

2.3. Cusp forms and Bergman kernel. For \( k \in \mathbb{R}_{>0} \), let \( \mathcal{S}^k(\Gamma) \) denote the complex vector-space of weight-\( k \) holomorphic cusp forms with respect to the Fuchsian subgroup \( \Gamma \). For \( f \in \mathcal{S}^k(\Gamma) \), and \( z = x + iy \in \mathbb{H} \), the cusp form \( f(z) \) admits the following Fourier expansion at the cusp \( i\infty \)
\[ f(z) = \sum_{n=1}^{\infty} a_n q^n(z), \text{ where } a_n := \int_{0}^{1} f(z) \overline{q^n(z)} dx, \text{ and } q(z) := e^{2\pi iz}. \] (8)
Locally, for \( f \in \mathcal{S}^k(\Gamma) \), the Petersson norm at \( z = x + iy \in \mathbb{H} \) is given by the following formula
\[ |f|_{\text{pet}}^2(z) := y^k |f(z)|^2, \] (9)
which is invariant with respect to the action of \( \Gamma \), and hence, defines a function on \( X \).

The Petersson norm induces an inner-product on \( \mathcal{S}^k(\Gamma) \). For \( f, g \in \mathcal{S}^k(\Gamma) \), the Petersson inner-product is given by the following integral
\[ \langle f, g \rangle_{\text{pet}} := \int_{X_{\Gamma}} y^k f(z) \overline{g(z)} \mu_{\text{hyp}}(z). \] (10)
Let \( \{f_1, \ldots, f_d\} \) denote an orthonormal basis of \( S^k(\Gamma) \) with respect to the Petersson inner-product, where \( d_k \) denotes the dimension of \( S^k(\Gamma) \) as a complex vector space. Then, for \( z, w \in \mathbb{H} \), the Bergman kernel associated to \( S^k(\Gamma) \) is given by the following formula

\[
B^k_X(z, w) := \sum_{j=1}^{d_k} f_j(z) \overline{f_j(w)}.
\]

From Riesz representation theorem, it follows that, the definition of the Bergman kernel \( B^k_X \) is independent of the choice of orthonormal basis for \( S^k(\Gamma) \).

For \( z, w \in \mathbb{H} \), the Bergman kernel \( B^k_X(z, w) \) is a holomorphic cusp form of weight-\( k \) in the \( z \)-variable, and an anti-holomorphic cusp form of weight-\( k \) in the \( w \)-variable. Furthermore, \( B^k_X(z, w) \) is the generating function for the vector-space \( S^k(\Gamma) \), i.e., for \( f \in S^k(\Gamma) \), and \( z = u + iv \in \mathbb{H} \), we have

\[
\int_{X_{v}} i^{k} B^k_X(z, w) f(w) \mu_{\text{hyp}}(w) = f(z).
\]

For \( z, w \in \mathbb{H} \), the Bergman kernel \( B^k_X(z, w) \) can also be represented by the following infinite series (see Proposition 1.3 on p. 77 in [Fr90])

\[
B^k_X(z, w) = \frac{(k-1)(2i)^k}{4\pi} \sum_{\gamma \in \Gamma} \frac{1}{(z - \gamma w)^k} \cdot \frac{1}{j(\gamma, w)^k},
\]

where for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), \( j(\gamma, w) := cw + d \).

The above formula for the Bergman kernel \( B^k_X(z, w) \) given in [Fr90] is missing a factor of \( (2i)^k \), which is taken into account in the above formula.

### 2.4. Estimates of Bergman kernel

In this subsection, we state inequalities and estimates of the Bergman kernel \( B^k_X \) from [AM17] and [AM18], which are later used in section 3 to prove estimate (3).

With notation as above, let \( k \in \mathbb{R}_{\geq 6} \), and locally, let \( z = x + iy, w = u + iv \in X \) with \( \delta := d_{\text{hyp}}(z, w) \). For \( 0 \leq \delta \leq r_{X}/2 \), we have the following estimate from Main theorem of [AM18]

\[
(yv)^{k/2} \cdot |B^k_X(z, w)| \leq C_{r_{X}}^{I} + \frac{(k-1)\Gamma((k-1)/2)}{2\sqrt{\pi} \Gamma(k/2)} \cdot \frac{(4yv)^{k/2}}{(y+v)^{k-1}};
\]

for \( r_{X}/2 < \delta < r_{X} \), we have the following estimate from Main theorem of [AM18]

\[
(yv)^{k/2} \cdot |B^k_X(z, w)| \leq C_{r_{X}}^{2} + \frac{(k-1)\Gamma((k-1)/2)}{2\sqrt{\pi} \Gamma(k/2)} \cdot \frac{(4yv)^{k/2}}{(y+v)^{k-1}};
\]

for \( r_{X} \leq \delta \), we have following estimate from Main theorem of [AM17]

\[
(yv)^{k/2} \cdot |B^k_X(z, w)| \leq C_{r_{X}}^{3} + \frac{(k-1)\Gamma((k-1)/2)}{2\sqrt{\pi} \Gamma(k/2)} \cdot \frac{(4yv)^{k/2}}{(y+v)^{k-1}};
\]

where the constant \( C_{r_{X}}^{I} \) is given by the following formula

\[
C_{r_{X}}^{I} := \frac{k-1}{4\pi} \left( \frac{1}{\cosh^{k}((r_{X}-\delta)/2)} + \frac{32}{\cosh^{k-4}((r_{X}/4))} \right) + \frac{k-1}{\pi(k/2-2)\sinh^{2}(r_{X}/4)} \cdot \frac{1}{\cosh^{k-4}((r_{X}/4))};
\]
the constant $C_{rX}^2$ is given by the following formula

\[
C_{rX}^2 := \frac{k-1}{4\pi} \left( \frac{2}{\cosh^k (\delta/2)} + \frac{16}{\cosh^{k-4} (rX/4)} + \frac{8}{\cosh^{k-3} (rX/2)} \right) + \frac{k-1}{2\pi \sinh^2 (rX/4)} \cdot \left( \frac{1}{(k-2) \cosh^{k-3} (rX/2)} + \frac{1}{(k-2) \cosh^{k-4} (rX/2)} \right);
\]  

(18)

and the constant $C_{rX}^3$ is given by the following formula

\[
C_{rX}^3 := \frac{k-1}{4\pi \sinh(rX)} \left( \frac{\sinh (\delta + rX)}{\cosh^k ((\delta - rX)/2)} + \frac{(k-1) \cosh (rX/4)}{2\pi \sinh (rX/4)} \cdot \frac{\sinh(\delta)}{\cosh^{k-2}(\delta/2)} + \frac{1}{\sinh^2 (rX/4)} \cdot \frac{k-1}{2\pi (k-2) \cosh^{k-2}(\delta/2)} \cdot \frac{1}{\cosh^{k-4}(\delta/2)} \right);
\]  

(19)

Remark 2.1. In [AM17] and [AM18], estimates of $B_X^k$ have been computed for $k \in \mathbb{N}_{\geq 6}$. However, the analysis that went into the proofs of Main theorems of [AM17] and [AM18] holds true for any $k \in \mathbb{R}_{\geq 6}$. Hence, we have stated the extended version of the estimates from [AM17] and [AM18] in equations (14)–(16).

3. Estimates of Fourier coefficients

In this section, using estimates (14)–(16), we prove estimate (3).

Proposition 3.1. With notation as above, for $k \in \mathbb{R}_{\geq 6}$, let $\{f_1, \ldots, f_{d_k}\}$ be an orthonormal basis of $S^k(\Gamma)$ with respect to the Petersson inner-product. For $n \geq 1$, and $1 \leq j \leq d_k$, let $a_{n,j}$ denote the $n$-th Fourier coefficient of the cusp form $f_j$. Then, for $z = x + iy, w = u + iv \in \mathbb{H}$, we have the following estimate

\[
\sum_{j=1}^{d_k} |a_{n,j}|^2 \leq \frac{e^{2\pi n(y-v)}}{(yv)^{k/2}} \times \left( \int_0^1 \int_0^1 C_\delta^1 \, dx + \int_0^1 \int_0^1 C_\delta^2 \, dx + \int_0^1 \int_0^1 C_\delta^3 \, dx + \frac{(k-1)\Gamma((k-1)/2)}{\Gamma(k/2)} \cdot \frac{1}{\sqrt{yv}} \right),
\]

(20)

where the constants $C_\delta^1, C_\delta^2, \text{ and } C_\delta^3$ are given by the following formulæ

\[
C_\delta^1 := \frac{k-1}{4\pi \cosh^k ((rX - \delta)/2)} + \left( 8 + \frac{1}{k/2 - 2} \cdot \frac{1}{\sinh^2 (rX/4)} \right) \cdot \frac{k-1}{\pi \cosh^{k-4}(\delta/2)};
\]  

(21)

\[
C_\delta^2 := \frac{3(k-1)}{\cosh^{k-4} (\delta/4)} + \frac{k-1}{\pi (k-2) \sinh^2 (rX/4)} \cdot \frac{1}{\cosh^{k-3}(\delta/2)};
\]  

(22)

\[
C_\delta^3 := \frac{k-1}{4\pi \sinh(rX)} \cdot \frac{\sinh (\delta + rX)}{\cosh^k ((\delta - rX)/2)} + \frac{k-1}{\pi \sinh (rX/4)} \cdot \frac{1}{\cosh^{k-3}(\delta/2)} + \frac{2}{\pi (k-4) \sinh^2 (rX/4)} \cdot \frac{k-1}{\cosh^{k-4}(\delta/2)};
\]  

(23)

Proof. Let $k \in \mathbb{R}_{\geq 6}$ and $n \geq 1$, and let $z = x + iy, w = u + iv \in \mathbb{H}$ with $\delta = d_{hyp}(z, w)$. From the definitions of the Bergman kernel $B_X^k(z, w)$ and the Fourier coefficient $a_n$, which are given
by equations (11) and (8), respectively, we have
\[
\int_{0}^{1} \int_{0}^{1} B_{\tilde{x}}^{k}(z, w) q^{n}(z) q^{n}(w) dx du = 
\]
(24)
\[
\sum_{j=1}^{d_{k}} \int_{0}^{1} \left( \int_{0}^{1} f_{j}(z) q^{n}(z) dx \right) f_{j}(w) q^{n}(w) du = \sum_{j=1}^{d_{k}} |a_{n,j}|^2.
\]

As \( x, u \) vary from 0 to 1, the hyperbolic distance \( \delta \) between \( z \) and \( w \) varies from 0 to \( \infty \). Put
\[
\mathcal{I}_{1} := \{(x, u) \in [0, 1] \times [0, 1]| 0 \leq \delta \leq r_{X}/2\};
\]
\[
\mathcal{I}_{2} := \{(x, u) \in [0, 1] \times [0, 1]| r_{X}/2 < \delta < r_{X}\};
\]
\[
\mathcal{I}_{3} := \{(x, u) \in [0, 1] \times [0, 1]| r_{X} \leq \delta\}.
\]

So, using equation (24), and combining it with estimates (14), (15), and (16), we derive
\[
\sum_{j=1}^{d_{k}} |a_{n,j}|^2 \leq \int_{0}^{1} \int_{0}^{1} |B_{\tilde{x}}^{k}(z, w)| \cdot |q^{n}(z)| \cdot |q^{n}(w)| dx du \leq \frac{e^{2\pi n(y-v)}}{(yv)^{k/2}} \times \left( \int_{\mathcal{I}_{1}} C_{r_{x}}^{I} dx du + \int_{\mathcal{I}_{2}} C_{r_{x}}^{I} dx du + \frac{(k-1)\Gamma((k-1)/2)}{2\sqrt{\pi}\Gamma(k/2)} \cdot \frac{(4yv)^{k/2}}{(y+v)^{k-1}}. \right)
\]

Combining the above inequality with the fact that \((y+v)^{k-1} \geq (4yv)^{(k-1)/2}\), we arrive at the following inequality
\[
\sum_{j=1}^{d_{k}} |a_{n,j}|^2 \leq \int_{0}^{1} \int_{0}^{1} C_{r_{x}}^{I} dx du + \int_{\mathcal{I}_{3}} C_{r_{x}}^{I} dx du + \frac{(k-1)\Gamma((k-1)/2)}{\Gamma(k/2)} \cdot \sqrt{\frac{yv}{\pi}}.
\]
(25)

We now estimate each of the three integrands on the right-hand-side of the above inequality.

For \( 0 \leq \delta \leq r_{X}/2 \), from the definition of \( C_{r_{x}}^{I} \) given by equation (17), and using the fact that cosh \( (\delta/2) \leq \cosh (r_{X}/4) \), we have the following estimate
\[
C_{r_{x}}^{I} = \frac{k-1}{4\pi} \left( \frac{1}{\cosh^{k}((r_{X}-\delta)/2)} + \frac{32}{\cosh^{k-4}(r_{X}/4)} \right) + \frac{k-1}{\pi(k/2-2)\sinh^{2}(r_{X}/4)} \cdot \frac{1}{\cosh^{k-4}(r_{X}/4)} \leq C_{I}^{I}.
\]
(26)

Similarly, for \( r_{X}/2 < \delta < r_{X} \), from the definition of \( C_{r_{x}}^{I} \) given by equation (18), and using the facts that cosh \( (\delta/4) \leq \cosh (r_{X}/4) \) and cosh \( (\delta/2) < \cosh (r_{X}/2) \), we have the following estimate
\[
C_{r_{x}}^{I} = \frac{k-1}{4\pi} \left( \frac{2}{\cosh^{k}((\delta)/2)} + \frac{16}{\cosh^{k-4}(r_{X}/4)} + \frac{8}{\cosh^{k-3}(r_{X}/2)} \right) + \frac{k-1}{2\pi \sinh^{2}(r_{X}/4)} \cdot \left( \frac{1}{(k-2)\cosh^{k-3}(r_{X}/2)} + \frac{1}{(k-2)\cosh^{k-4}(r_{X}/2)} \right) \leq C_{I}^{I}.
\]
(27)

Similarly, for \( r_{X} \leq \delta \), from the definition of \( C_{r_{x}}^{I} \) given by equation (19), and using the facts that \( \sinh(\delta) \leq 2\cosh^{2}(\delta/2) \) and \( \cosh (r_{X}/4) \leq \cosh (r_{X}/2) \leq \cosh (\delta/2) \), we have the following
estimate
\[
C_{rX}^\beta = \frac{k - 1}{4\pi \sinh(r_X)} \cdot \frac{\sinh(\delta + r_X)}{\cosh^k((\delta - r_X)/2)} + \frac{(k - 1) \cosh(r_X/4)}{2\pi \sinh(r_X/4)} \cdot \frac{\sinh(\delta)}{\cosh^k(\delta/2)} + \frac{\cosh(r_X/2)}{\sinh^2(r_X/4)} \left( \frac{k - 1}{2\pi(k - 2) \sinh(r_X/4)} \right) \frac{\sinh(\delta)}{\cosh^{k-1}(\delta/2)} + \frac{1}{\cosh^{k-2}(\delta/2)} \leq C_\delta^\beta.
\]

Combining estimates (24)–(28), completes the proof of the proposition. \(\square\)

In the ensuing lemma, we estimate each of the terms on the right hand-side of inequality (20). In the following lemma, we estimate the first integral on the right hand-side of inequality (20).

**Lemma 3.2.** With notation as above, for \(k \in \mathbb{R}_{\geq 6}, \) and for \(z = x + iy, w = u + iv \in \mathbb{H}, \) we have the following estimate for the first integral on the right hand-side of inequality (20).

\[
\int_0^1 \int_0^1 C_{rX}^\delta \, dxdu \leq \frac{8(k - 1)\sqrt{yv}}{\pi} \cdot \sinh^{-1}(1/(y + v)) \times \left( e^{rx/2} + 4 + \frac{1}{(k - 4) \sinh^2(r_X/4)} \right).
\]

**Proof.** From the definition of \(C_{rX}^\delta \) given by equation (21), for \(k \in \mathbb{R}_{\geq 6}, \) and for \(z = x + iy, w = u + iv \in \mathbb{H}, \) we have the following estimate

\[
\int_0^1 \int_0^1 C_{rX}^\delta \, dxdu = \int_0^1 \int_0^1 \frac{(k - 1)dxdu}{4\pi \cosh^k((r_X - \delta)/2)} + \int_0^1 \int_0^1 \left( 8 + \frac{1}{(k/2 - 2)} \cdot \frac{1}{\sinh^2(r_X/4)} \right) \frac{(k - 1)dxdu}{\pi \cosh^{k-1}(\delta/2)} \leq
\]

\[
\int_0^1 \int_0^1 \frac{(k - 1)dxdu}{4\pi \cosh((r_X - \delta)/2)} \left( \cosh(r_X/2) - \sinh(r_X/2) \right) \frac{1}{\cosh^2(r_X/2)} \frac{1}{\sinh^2(r_X/4)} \pi \cosh(\delta/2).
\]

We now estimate each of the integrals on the right hand-side of the above inequality. Using the fact that \(\cosh(\delta/2) > \sinh(\delta/2), \) we derive

\[
\int_0^1 \int_0^1 \frac{(k - 1)dxdu}{4\pi \cosh((r_X - \delta)/2)} = \int_0^1 \int_0^1 \frac{(k - 1)dxdu}{4\pi \cosh(\delta/2) \cosh(r_X/2) - \sinh(\delta/2) \sinh(r_X/2)} \leq
\]

\[
\int_0^1 \int_0^1 \frac{(k - 1)dxdu}{4\pi \cosh(\delta/2) \left( \cosh(r_X/2) - \sinh(r_X/2) \right)} = \int_0^1 \int_0^1 \frac{(k - 1)e^{rx/2} dxdu}{4\pi \cosh(\delta/2)}.
\]

Using relation (6), we now compute

\[
\int_0^1 \int_0^1 \frac{dxdu}{\cosh(\delta/2)} = \int_0^1 \int_0^1 \frac{\sqrt{4yv \, dxdu}}{(x - u)^2 + (y + v)^2} = \int_0^1 \int_0^1 \frac{\sqrt{4yv \, dxdu}}{(y + v)} \sqrt{\frac{(x - u)^2}{(y + v)^2} + 1}.
\]
Making the substitution \( \theta := (x - u)/(y + v) \), and using the fact that \( \sinh^{-1} \) is an odd and monotone increasing function on \( \mathbb{R} \), we find that

\[
\int_0^1 \int_0^1 \frac{\sqrt{4yv} \, dx \, du}{(y + v) \sqrt{(x-u)^2 + 1}} = \sqrt{4yv} \int_0^1 \int_{-u/(y+v)}^{(1-u)/(y+v)} \frac{d\theta}{\sqrt{\theta^2 + 1}} =
\]

\[
\sqrt{4yv} \int_0^1 \left( \sinh^{-1} \left( (1-u)/(y+v) \right) - \sinh^{-1} \left( -u/(y+v) \right) \right) \, du =
\]

\[
\sqrt{4yv} \int_0^1 \left( \sinh^{-1} \left( (1-u)/(y+v) \right) + \sinh^{-1} \left( u/(y+v) \right) \right) \, du \leq
\]

\[
4\sqrt{yv} \sinh^{-1} \left( 1/(y+v) \right).
\]

Combining estimates (31), (32), and (33), we arrive at the following estimate for the first integral on the right hand-side of inequality (30)

\[
(34) \quad \int_0^1 \int_0^1 \frac{(k-1) \, dx \, du}{4\pi \cosh \left( (rX - \delta)/2 \right)} \leq \frac{(k-1)e^{rX/2} \sqrt{yv}}{\pi} \cdot \sinh^{-1} \left( 1/(y+v) \right).
\]

Similarly, using computation (33), we arrive at the following estimate for the second integral on the right hand-side of inequality (30)

\[
(35) \quad \frac{4(k-1) \sqrt{yv}}{\pi} \cdot \sinh^{-1} \left( 1/(y+v) \right) \cdot \left( 8 + \frac{2}{k-4} \cdot \frac{1}{\sinh^2 \left( rX/4 \right)} \right).
\]

Combining estimates (30), (34), and (35), completes the proof of the lemma. \( \square \)

We now estimate the second integral on the right hand-side of inequality (20).

**Lemma 3.3.** With notation as above, for \( k \in \mathbb{R}_{>6} \), and for \( z = x + iy \), \( w = u + iv \in \mathbb{H} \), we have the following estimate for the second integral on the right hand-side of inequality (20).

\[
(36) \quad \int_0^1 \int_0^1 C_\delta^2 \, dx \, du \leq 4(k-1)\sqrt{yv} \cdot \sinh^{-1} \left( 1/(y+v) \right) \cdot \left( 6 + \frac{1}{\pi(k-2) \sinh^2 \left( rX/4 \right)} \right).
\]

**Proof.** From the definition of \( C_\delta^2 \) given by (22), for \( k \in \mathbb{R}_{>6} \), and for \( z = x + iy \), \( w = u + iv \in \mathbb{H} \), similar to estimate (20), we have the following estimate

\[
(37) \quad \int_0^1 \int_0^1 C_\delta^2 \, dx \, du \leq \int_0^1 \int_0^1 \frac{3(k-1) \, dx \, du}{\cosh^2 \left( \delta/4 \right)} + \int_0^1 \int_0^1 \frac{k-1}{\pi(k-2) \sinh^2 \left( rX/4 \right)} \cdot \cosh \left( \delta/2 \right) \, dx \, du.
\]

We now estimate each of the integrals on the right hand-side of the above inequality. Using the fact that \( 2 \cosh^2 \left( \delta/4 \right) > \cosh \left( \delta/2 \right) \), we derive

\[
(38) \quad \int_0^1 \int_0^1 \frac{3(k-1) \, dx \, du}{\cosh^2 \left( \delta/4 \right)} \leq \int_0^1 \int_0^1 \frac{6(k-1) \, dx \, du}{\cosh \left( \delta/2 \right)}.
\]

Combining the above estimate with computation (33), we arrive at the following estimate for the first integral on the right hand-side of inequality (37)

\[
(39) \quad \int_0^1 \int_0^1 \frac{3(k-1) \, dx \, du}{\cosh^2 \left( \delta/4 \right)} \leq 24(k-1) \cdot \sqrt{yv} \cdot \sinh^{-1} \left( 1/(y+v) \right).
\]
Similarly, using computation (33), we arrive at the following estimate for the second integral on the right-hand-side of inequality (37):

\[
\int_0^1 \int_0^1 \frac{k - 1}{2} \frac{k - 1}{\pi(k - 2) \sinh^2 \left( \frac{r_X}{4} \right)} \cdot \frac{dxdu}{\cosh \left( \frac{\delta}{2} \right)} \leq \frac{4(k - 1) \sqrt{yv}}{\pi(k - 2) \sinh^2 \left( \frac{r_X}{4} \right)} \cdot \sinh^{-1} \left( \frac{1}{y + v} \right).
\] (40)

Combining estimates (37), (39), and (40), completes the proof of the lemma.

We now estimate the third integral on the right-hand-side of inequality (20).

**Lemma 3.4.** With notation as above, for \( k \in \mathbb{R}_{\geq 6} \), and for \( z = x + iy, w = u + iv \in \mathbb{H} \), we have the following estimate for the third integral on the right-hand-side of inequality (20):

\[
\int_0^1 \int_0^1 C^3 \cdot dxdu \leq \frac{4(k - 1) \sqrt{yv}}{\pi \sinh \left( \frac{r_X}{4} \right)} \cdot \sinh^{-1} \left( \frac{1}{y + v} \right) \cdot \left( e^{3r_X/2} \cosh(r_X) + 1 + \frac{2}{(k - 4) \sinh \left( \frac{r_X}{4} \right)} \right).
\] (41)

**Proof.** From the definition of \( C^3 \) given by (23), for \( k \in \mathbb{R}_{\geq 6} \), and for \( z = x + iy, w = u + iv \in \mathbb{H} \), similar to estimate (30), we have the following estimate:

\[
\int_0^1 \int_0^1 \frac{k - 1}{\pi \sinh \left( \frac{r_X}{4} \right)} \cdot \frac{dxdu}{\cosh \left( \delta/2 \right)} + \int_0^1 \int_0^1 \frac{2(k - 1)}{\pi(k - 4) \sinh^2 \left( \frac{r_X}{4} \right)} \cdot \frac{dxdu}{\cosh \left( \delta/2 \right)}.
\] (42)

We now estimate each of the integrals on the right-hand-side of the above inequality. Using the fact that \( \sinh \left( \delta + r_X \right) \leq 2 \cosh(\delta) \cosh(r_X) \leq 4 \cosh^2(\delta/2) \cosh(r_X) \), similar to estimate (31), we derive

\[
\int_0^1 \int_0^1 \frac{k - 1}{4 \pi \sinh(r_X)} \cdot \frac{\cosh \left( \delta + r_X \right) dxdu}{\cosh^3 \left( (\delta - r_X)/2 \right)} \leq \frac{4(k - 1) \sqrt{yv}}{\pi \sinh \left( \frac{r_X}{4} \right)} \cdot \sinh^{-1} \left( \frac{1}{y + v} \right) \cdot e^{3r_X/2} \cosh(r_X).
\] (43)

Using computation (33), we arrive at the following estimate for the first integral on the right-hand-side of inequality (42):

\[
\int_0^1 \int_0^1 \frac{k - 1}{4 \pi \sinh(r_X)} \cdot \frac{\sinh \left( \delta + r_X \right) dxdu}{\cosh^3 \left( (\delta - r_X)/2 \right)} \leq \frac{4(k - 1) \sqrt{yv}}{\pi \sinh \left( \frac{r_X}{4} \right)} \cdot \sinh^{-1} \left( \frac{1}{y + v} \right) \cdot e^{3r_X/2} \cosh(r_X).
\] (44)

Similarly, using computation (33), we arrive at the following estimate for the second integral on the right-hand-side of inequality (42):

\[
\int_0^1 \int_0^1 \frac{k - 1}{\pi \sinh \left( \frac{r_X}{4} \right)} \cdot \frac{dxdu}{\cosh \left( \delta/2 \right)} \leq \frac{4(k - 1) \sqrt{yv}}{\pi \sinh \left( \frac{r_X}{4} \right)} \cdot \sinh^{-1} \left( \frac{1}{y + v} \right).
\] (45)
Similarly, using computation \((33)\), we arrive at the following estimate for the third integral on the right hand-side of inequality \((42)\):

\[
\int_0^1 \int_0^1 \frac{2(k-1)}{\pi (k-4) \sinh^2 \left( \frac{rX}{4} \right)} \cdot \frac{dx}{\cosh(\delta/2)} \leq \frac{8(k-1)\sqrt{y\nu}}{\pi (k-4) \sinh^2 \left( \frac{rX}{4} \right)} \cdot \sinh^{-1} \left( 1/(y + v) \right) .
\]  

(46)

Combining estimates \((42)\), \((44)\), \((45)\), and \((46)\), completes the proof of the lemma. \(\square\)

In the following theorem, combining the estimate derived in Proposition \((34)\) with the estimates derived in Lemmas \((3.2)\) \((3.4)\), we derive an estimate for the sum of the \(n\)-th Fourier coefficients of any given basis of \(S^k(\Gamma)\).

**Theorem 3.5.** With notation as above, for \(k \in \mathbb{R}_{>6}\), let \(\{f_1, \ldots, f_{d_k}\}\) be an orthonormal basis of \(S^k(\Gamma)\) with respect to the Petersson inner-product. For \(n \geq 1\), and \(1 \leq j \leq d_k\), let \(a_{n,j}\) denote the \(n\)-th Fourier coefficient of \(f_j\). Then, we have the following estimate

\[
\sum_{j=1}^{d_k} |a_{n,j}|^2 \leq \frac{(k-1) \cdot n^{k-1}}{\pi} \cdot \sinh^{-1}(2n) \times
\]

\[
(\log(5n)) \cdot \left( 109 e^{rX}/2 + \frac{8e^5 rX/2}{\sinh \left( rX/4 \right)} + \frac{20}{(k-4) \sinh^2 \left( rX/4 \right)} + \frac{\sqrt{\pi} \Gamma((k-1)/2)}{\Gamma(k/2)} \right).
\]

(47)

**Proof.** For \(k \in \mathbb{R}_{>6}\) and \(n \geq 1\), substituting \(y = v = 1/n\) in estimate \((20)\), and combining it with estimates \((29)\), \((30)\), and \((31)\), we arrive at the following estimate

\[
\sum_{j=1}^{d_k} |a_{n,j}|^2 \leq \frac{(k-1) \cdot n^{k-1}}{\pi} \cdot \sinh^{-1}(2n) \times
\]

\[
\left( e^{rX}/2 + 32 + \frac{8}{\sinh \left( rX/4 \right)} \right) + \frac{24\pi}{(k-2) \sinh^2 \left( rX/4 \right)} + \frac{4}{(k-4) \sinh^2 \left( rX/4 \right)} + \frac{\sqrt{\pi} \Gamma((k-1)/2)}{\Gamma(k/2)} \times
\]

\[
\frac{4e^3 rX/2 \cosh(rX)}{\sinh \left( rX/4 \right)} + \frac{\sqrt{\pi} \Gamma((k-1)/2)}{\Gamma(k/2)}.
\]

(48)

For \(n \geq 1\), observe that

\[
e^{rX}/2 + 32 + 24\pi \leq 109 e^{rX}/2;
\]

\[
\frac{4e^3 rX/2 \cosh(rX)}{\sinh \left( rX/4 \right)} \leq \frac{8e^5 rX/2}{\sinh \left( rX/4 \right)} \leq \frac{8e^5 rX/2}{\sinh \left( rX/4 \right)};
\]

\[
\sinh^{-1}(2n) = \log \left( 2n + \sqrt{4n^2 + 1} \right) \leq \log(5n).
\]

(49) \hspace{1cm} (50) \hspace{1cm} (51)

Combining estimates \((48)\) \((51)\), completes the proof of the theorem. \(\square\)

**Corollary 3.6.** With notation as above, for \(k \in \mathbb{R}_{>6}\) and \(n \geq 1\), and for any \(\varepsilon > 0\), we have the following estimate

\[
\frac{1}{d_k} \sum_{j=1}^{d_k} |a_{n,j}|^2 = O_X \left( n^{(k-1)+\varepsilon} \right),
\]

where the implied constant depends on the Riemann surface \(X\), and on the choice of \(\varepsilon\). Furthermore, for a fixed \(\varepsilon > 0\), the implied constant in estimate \((52)\), remains stable in covers of hyperbolic Riemann surfaces.
**Proof.** From Riemann-Roch theorem or from Selberg trace formula, it is easy to show that $d_k = O(k)$. So from Theorem 3.5, for any $\epsilon > 0$, we have the following estimate

$$\frac{1}{d_k} \sum_{j=1}^{d_k} |a_{n,j}|^2 = O_X(n^{(k-1)+\epsilon}),$$

where the implied constant depends on the Riemann surface $X$, and on the choice of $\epsilon$.

Furthermore, from Proposition 3.1, it follows that estimate (47) depends only on $C^1_{r_X}$, $C^2_{r_X}$, and $C^3_{r_X}$, which are defined in equations (17), (18), and (19), respectively. It has been shown that $C^1_{r_X}$, $C^2_{r_X}$ and $C^3_{r_X}$ are stable in covers of hyperbolic Riemann surfaces in [AM17], [AM18], respectively. This implies that, for a fixed $\epsilon$, the implied constant in estimate (52) also remains stable in covers of hyperbolic Riemann surfaces. \[\square\]

**Corollary 3.7.** With notation as above, for $k \in \mathbb{R}_{\geq 6}$, let $f \in S^k(\Gamma)$ be a cusp form, which is normalized with respect to the Petersson inner-product. For any $n \geq 1$, let $a_n$ denote the $n$-th Fourier coefficient of $f$. Then, for any $\epsilon > 0$, we have the following estimate

$$|a_n| = O_{X,\epsilon}(\sqrt{k n^{\frac{k-1}{2}}} + \epsilon),$$

where the implied constant depends on the Riemann surface $X$, and on the choice of $\epsilon$. Furthermore, for a fixed $\epsilon > 0$, the implied constant in estimate (53), remains stable in covers of hyperbolic Riemann surfaces.

**Proof.** Let $\{f_1, \ldots, f_d\}$ denote an orthonormal basis of $S^k(\Gamma)$ with $f_1 = f$. Then, from Theorem 3.5 and Corollary 3.6 we have

$$|a_n| \leq \sqrt{\sum_{j=1}^{d_k} |a_{n,j}|^2} = O_{X,\epsilon}(\sqrt{k n^{\frac{k-1}{2}}} + \epsilon),$$

where the implied constant depends on the Riemann surface $X$, and on the choice of $\epsilon$. Estimate (54), and similar arguments as in the proof of Corollary 3.6 completes the proof of the corollary. \[\square\]

**Remark 3.8.** Let hypothesis be as an in Corollary 3.7, and let $\Gamma$ be an arithmetic subgroup. Then the hyperbolic Riemann surface $X$ is a finite degree cover of $\text{PSL}_2(\mathbb{Z}) \setminus \mathbb{H}$, which implies that for any $\epsilon > 0$, we have the following estimate

$$\frac{1}{d_k} \sum_{j=1}^{d_k} |a_{n,j}|^2 = O_\epsilon(n^{(k-1)+\epsilon}),$$

where the implied constant depends only on the choice of $\epsilon$, which reproves the Ramanujan-Petersson conjecture for arithmetic subgroups.

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