FROM CUBES TO TWISTED CUBES
VIA GRAPH MORPHISMS IN TYPE THEORY

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Abstract. Cube categories are used to encode higher-dimensional categorical structures. They have recently gained significant attention in the community of homotopy type theory and univalent foundations, where types carry the structure of such higher groupoids. Bezem, Coquand, and Huber [5] have presented a constructive model of univalence using a specific cube category, which we call the BCH category.

The higher categories encoded with the BCH category have the property that all morphisms are invertible, mirroring the fact that equality is symmetric. This might not always be desirable: the field of directed type theory considers a notion of equality that is not necessarily invertible.

This motivates us to suggest a category of twisted cubes which avoids built-in invertibility. Our strategy is to first develop several alternative (but equivalent) presentations of the BCH category using morphisms between suitably defined graphs. Starting from there, a minor modification allows us to define our category of twisted cubes. We prove several first results about this category, and our work suggests that twisted cubes combine properties of cubes with properties of globes and simplices (tetrahedra).

1. Introduction and Motivation

A cube category is a category whose objects are (usually) finite-dimensional cubes, and whose morphisms are mappings of some sort between these cubes. There are many different cube categories [1, 3, 5, 6, 11], and they are used to encode higher categorical structures.

Homotopy type theory [21] is a variation of Martin-Löf’s intensional type theory. The characteristic and novel view adapted in Homotopy type theory is that types carry the structure of higher categories, or, to be precise, higher groupoids (i.e. all morphisms are invertible). This view supports Voevodsky’s univalence principle which should seen as a central concept of homotopy type theory. The first model of such a type theory, given by Voevodsky [22] (see also the presentation by Kapulkin and Lumsdaine [10]), uses simplicial sets. However, it is still an open question how simplicial sets can be used to build a constructive model of type theory with univalent universes. Instead, this has been achieved by Bezem, Coquand, and Huber [5] using cubical sets. Starting from there, cubes have gathered a lot of attention in the type theory community, leading to various cubical type theories which have univalence not as an axiom but as a built-in derivable principle [7, 2, 4, 16]. Many different cube categories have been considered in this context.

The important cube category used by Bezem, Coquand, and Huber [5] (from now on referred to as the BCH cube category) uses finite sets of variable names as objects, and a morphism from a set I to a set J is a function \( f : I \to J \cup \{0, 1\} \) which is “injective on the left part”, i.e. \( f(i_1) = f(i_2) = j \) with \( j : J \) implies \( i_1 = i_2 \). One goal of this paper is to develop several alternative presentations of this category, mainly using graph morphisms. We have two main motivations to do this. The first is that, as we hope, our alternative and intuitive
(but equivalent) definitions enable new views on the category and facilitate the discovery of further observations. The second motivation is that a minor change in the definition will allow us to construct a new cube category, the twisted cubes from the title. We will come back to this in a moment.

The standard way to create models (of both higher categories and type theories) using simplicial or cubical index categories is to take presheaves and equip them with certain Kan-filling conditions. These filling conditions encode composition of morphisms as well as associativity and all higher coherence laws that one needs. A typical such Kan-filling condition for the 2-cube says that, given the “partial square” of three solid edges on the right, one can always find the dashed edge (together with an actual filler for the square).

One important observation here is that, in the case of the BCH cube category and other cube categories, invertibility of morphisms is built-in. Consider the partial square on the left, where two of the three solid edges are identities and the third is an actual non-trivial morphism (or equality) \( p \) from \( x \) to \( y \). Using the Kan filling operation described above, we get a morphism from \( y \) to \( x \), which serves as the inverse of \( p \).

The invertibility of morphisms is useful for most forms of type theory, where equality is symmetric. This however is not always the case, cf. the proposals for directed type theories by Nuyts [15], Riehl and Shulman [18], North [14], and others. Their aim is to generalise type theory by replacing (higher) groupoids by general (higher) categories. In a nutshell, this means that “equality” (or whatever takes the place of equality) is not necessarily invertible. This happens naturally in the universe, since not every function is invertible. We think that a very valuable long-term goal would be to make the connection of directed with cubical type theories and create some sort of directed cubical type theory. This is at the moment certainly out of reach, and we do not know how such a type theory could be built. Nevertheless, it motivates us to explore variations of the BCH category which do not have the described built-in equality.

To avoid invertibility, we “twist” the left-most edge of the 2-dimensional cube, as shown on the right, to ensure that the construction from before becomes impossible. Using our graph morphisms that we develop for the BCH cube category, it becomes very easy to define this twisting for cubes of all dimensions. Figure 1 and Figure 2 show the twisted 3- and 4-dimensional cube, from two different projections. The construction can be roughly described as follows:

Naturally, the faces of a twisted \( n \)-cube have to be twisted \((n - 1)\)-cubes. Looking at Figure 1, we see that the twisted 3-cube can be constructed by taking a twisted 2-cube and “thickening” it. Thickening means that we multiply it with the interval (the 1-cube), i.e. take the “cylinder object”, in order to create a new dimension; then, we reverse all the edges in the “domain” copy of the 2-cube that we started with. This construction works for all \( n \geq 1 \), and of course, the twisted 0-cube is simply a point.

Twisted cubes do not only remove the discussed source of invertibility, but they also make the composition of morphisms somewhat more natural. The filling of a “standard” square can be interpreted as saying that the composition of two edges equals the composition of the other two edges, and if we want to see the lid as the composite of the three other edges, then one has to be inverted. In contrast, in the twisted square, the lid can be seen directly as the single composite of the three other edges. The right half of Figure 1 shows the projection of the twisted 3-cube, and the biggest square (011-001-101-111) is the lid. As for the square,
Figure 1. The 3-dimensional twisted cube using parallel and perspective projections. In both cases, the lid (i.e. the last face which can be recovered by filling) is marked. In the right picture, this face is the big square. The lid should be seen as the composite of the other faces.

Figure 2. The 4-dimensional twisted cube using parallel and perspective projections. The lid is shadowed on the left. It is the biggest cube on the right.

this lid should be seen as the composite of the other (here five) faces. Intuitively, one starts with the small inner square, composes it with the top and the bottom squares, and extends it to the left and the right. Figure 2 shows the similar situation for the 4-dimensional twisted cube where one starts with the inner 3-cube, then extends to the front and the back, the top and the bottom, and the left and the right.

The “twisting” pattern also appears in the twisted arrow category [13], also known as the category of factorisations [12]. However, it is unclear how to generalise this idea to more than squares; it has been developed to solve a different problem.

In the main body of the paper, we first introduce the framework of graph morphisms for standard (non-twisted) cubes. We consider the properties of meet/join and dimension preservation of graph morphisms, and conclude that both of these are suitable refinements to ensure that the category of graph morphisms matches the BCH category. The proof of this is the main result of Section 2. We use this development to introduce and examine twisted cubes in Section 3. We will see that they have many characteristic properties that standard cubes are lacking. Some of them, such as a Hamiltonian path through the cube and the fact that vertices are totally ordered, as well as the category of twisted cubes is a Reedy category, are more familiar from simplicial structures but not from cubical ones. Another interesting feature, neither familiar from cubical nor from simplicial structures but globular structure,
is that surjective maps are unique (i.e. there is only one way to degenerate a twisted cube). These and other observations allow us to define a further representation of the category of twisted cubes which does not make use of graphs.

**Setting** We use a standard version of Martin-Löf’s dependent type theory as our meta-language. We assume function extensionality, but we do not require other axioms or features since we mostly work with finite sets, which are extremely well-behaved by default (in particular, it does not matter for us whether UIP/Axiom K is assumed or not).

**Summary of Contributions** Our main contributions are as follows:

- We give several alternative but equivalent presentations of the BCH cube category.
- We introduce twisted cubes, a variation of the BCH cube category which allows for filling conditions without built-in invertibility.
- We show several results about twisted cubes. These include connections to simplices (a unique Humiliation path and the property of being Reedy category) and to globes (unique surjective maps and degeneracies).

## 2. A Standard Cube Category

In this section, we discuss various representations of the cube category $\square_{\text{BCH}}$. This category was used by Bezem, Coquand, and Huber to present a constructive model of univalence [5]. In Section 3, we will see how minimal modifications lead to a category of twisted cubes.

Keeping in mind that we use type theory as the language in which the results are presented (i.e. as our meta-theory), we use the following notations: $\mathbb{N}$ are the natural numbers, including 0. For $n : \mathbb{N}$, the set $\underline{n}$ is the finite set with elements $\{0, 1, \ldots, n - 1\}$. In particular, $\underline{2}$ is the set of booleans. As usual, $\underline{m}^\underline{n}$ is simply the function set $\underline{m} \rightarrow \underline{n}$. We denote elements of $\underline{2}^\underline{n}$ by binary sequences as in $0 \cdot 1 \cdot 1 \cdot 0$. This means a function $f$ is denoted by $f(0) \cdot f(1) \cdot f(2) \ldots f(n - 1)$. If there is no risk of confusion, we omit the $\cdot$ and simply use juxtaposition as in $0110$.

In several situations, we want to consider a type of functions into a coproduct which is injective “on the left part of the codomain”. To make this precise, we introduce a notation:

**Definition 1** ($\leftarrow$). Assume $A$, $B$, and $C$ are given types. For a function $f : A \rightarrow (B \cup C)$, we say that $f$ is injective on the left part if

\[
\text{left-inj}(f) \equiv \Pi(x, y : A, z : B). (f(x) = \text{inl}(z)) \rightarrow (f(y) = \text{inl}(z)) \rightarrow x = y.
\]  

We write the type of functions which are injective on the left part as

\[
(A \leftarrow B \cup C) \equiv \Sigma(f : A \rightarrow (B \cup C)). \text{left-inj}(f).
\]

In the next lemma, a function $f : A \rightarrow B \cup 1$ is called a partial function, with $1$ being the “undefined” part.\(^1\) The following simple but useful (and well-known) result will be necessary. It could be formulated in higher generality, but a version which is sufficient for us is this:

\(^1\)Technically, these are of course only the partial functions from $A$ to $B$ with decidable support. Since we only work with finite types, it is not surprising that we only need to consider the decidable case.
Lemma 2. Given \( m, n : \mathbb{N} \), injective partial functions from \( m \) to \( n \) are in bijection with injective partial functions from \( n \) to \( m \). In other words, we have an equivalence
\[
\left( m \xrightarrow{\text{left}} n + 1 \right) \simeq \left( n \xrightarrow{\text{left}} m + 1 \right). \tag{3}
\]

Proof. The equivalence can be constructed directly. Given an \( f : m \xrightarrow{\text{left}} n + 1 \), we have to construct a function \( g : n \xrightarrow{\text{left}} m + 1 \). For \( i : n \), we can decide whether there is a \( k \) such that \( f(k) = \text{inl}(i) \). If so, then this \( k \) is unique due to injectivity, and we set \( g(i) := \text{inl}(k) \); otherwise, we set \( g(i) := \text{inr}(0) \). Checking that this is an equivalence is routine. \( \blacksquare \)

The presentation of the cube category in question that we start with is the one given by Bezem, Coquand, and Huber [5] (which is the same as in Huber’s PhD thesis [9]). Since it is sufficient for our purposes, we use a skeletal variation: our objects are not finite sets but rather natural numbers.

Definition 3 (category \( \square_{\text{BCH}} [5, 9] \)). The category \( \square_{\text{BCH}} \) has natural numbers as objects and, for \( m, n : \mathbb{N} \), a morphism in \( \square_{\text{BCH}}(m, n) \) is a function \( f : m \rightarrow n + 2 \) which is injective on the \( n \)-part. In type-theoretic notation:
\[
\text{obj}(\square_{\text{BCH}}) := \mathbb{N} \quad \square_{\text{BCH}}(m, n) := m \xrightarrow{\text{left}} n + 2 \tag{4}
\]

Composition \( g \circ f \) is defined to be the set-theoretic composition \( (g + \text{id}_2) \circ f \).

What we will need is the opposite of this category, \( \square_{\text{BCH}}^{\text{op}} \). While the above definition is short and abstract, a description closed to the intuitive idea of cubes is helpful for our later developments. Let us consider graphs \( G = (V, E) \) of nodes (vertices) and edges, where \( V \) is a set with decidable equality and \( E \) is a subset of \( V \times V \). A standard way to implement this is to let \( E \) be a family of “mere propositions”, indexed twice over \( V \), However, we write \( (s, t) : E \) for \( E(s, t) \) and assume that \( E \) is given in the “total space” formulation. Furthermore, in our cases \( E \) will always be a decidable subset.

\( E \) being a subset means that our graphs do not have multiple parallel edges, i.e. for any pair of vertices, there is at most one edge between them, and it is decidable whether there is an edge between two given vertices.

Given a graph, we construct a new graph as follows. Note that the “total space” of the edges of the new graph is \( E + E + V \), but in order to make clear which vertices these new edges connect, we use “set theory style” notation:

Definition 4. Given \( G = (V, E) \), the graph-prism of \( G \), denoted as \( \text{prism}(G) := (\text{prism}(V), \text{prism}(E)) \) is another graph where
\[
\text{prism}(V) := 2 \times V \quad \text{prism}(E) := \{ \ ((0, \ s), \ (0, \ t)) \mid (s, t) : E \} \cup \{ \ ((1, \ s), \ (1, \ t)) \mid (s, t) : E \} \cup \{ \ ((0, \ v), \ (1, \ v)) \mid v : V \}. \tag{6}
\]

This allows us to define the standard cube as a graph:\(^2\)

Definition 5. Given \( n : \mathbb{N} \), the standard cube \( C_n \) is defined as follows:
\[
C_0 := (1, \ \{(0, 0)\}) \quad C_{n+1} := \text{prism}(C_n) \tag{7}
\]

\(^2\)Most of graphs in this paper are reflexive graphs to support degeneracies as graph morphisms.
Another way of defining $C_n$, without recursion, is the following. Here, we give the “total space” of edges $\text{edges}(C_n)$ together with functions $\text{src}, \text{trg} : \text{edges}(C_n) \to \text{nodes}(C_n)$.

**Definition 6.** In the following, our convention is that $-1$ is empty (i.e. the same as 0):

\[
\begin{align*}
\text{nodes}(C_n) & := 2^n & (8) \\
\text{edges}(C_n) & := 2^n + \left( n \times 2^{n-1} \right) & (9) \\
\text{src}(\text{inl}(v)) & := \text{trg}(\text{inl}(v)) & (10) \\
\text{src}(\text{inr}(i, x_0 x_1 \ldots x_{n-2})) & := x_0 x_1 \ldots x_{i-1} 0 x_i \ldots x_{n-2} & (11) \\
\text{trg}(\text{inr}(i, x_0 x_1 \ldots x_{n-2})) & := x_0 x_1 \ldots x_{i-1} 1 x_i \ldots x_{n-2} & (12)
\end{align*}
\]

In Definition 6, the left part ($2^n$) are the “identities” (one for each node), while the right part ($n \times 2^{n-1}$) represents the non-trivial edges. Figure 3 shows drawings for $C_0$ to $C_3$.

**Lemma 7.** Definition 5 and Definition 6 define isomorphic graph structures.

This observation allows us to use whichever is more convenient in any given situation.

![Figure 3](image-url)  

**Figure 3.** An illustration of $C_n$ for $n \leq 3$. The labels on the vertices and edges are in accordance with (8) and (9). The identity loops are hidden to tidy up the diagrams. This allows us to unambiguously hide the constructor $\text{inr}$ as well.

A **graph morphism** from $G = (V, E)$ to $G' = (V', E')$ is, as usual, a function between the node types which preserves the edges:

\[
\text{grp-hom} \left( (V, E), (V', E') \right) := \Sigma(f : V \to V').\Pi(v_0, v_1 : V).E(v_0, v_1) \to E'(f(v_0), f(v_1)) \quad (13)
\]

We can now consider the following category:

**Definition 8 (category $\Box_{\text{grp}}$).** The category $\Box_{\text{grp}}$ has natural numbers as objects. A morphism between $m$ and $n$ is a graph morphism from $C_m$ to $C_n$, as in:

\[
\begin{align*}
\text{obj}(\Box_{\text{grp}}) & := \mathbb{N} \\
\Box_{\text{grp}}(m, n) & := \text{grp-hom} \left( C_m, C_n \right)
\end{align*}
\]

Composition is composition of graph morphisms.

The category $\Box_{\text{grp}}$ has more morphisms than $\Box_{\text{BCH}}^{\text{op}}$. One example would be the morphism in $\text{grp-hom} \left( C_2, C_1 \right)$ which maps the three nodes $00$, $01$, $10$ all to $0$ and $11$ to $1$. Another example is the morphism which maps $00$ to $0$, and $01$, $10$, $11$ all to $1$. Both of these morphisms do not have analogues in $\Box_{\text{BCH}}^{\text{op}}$. In other words, $\Box_{\text{gr}}$ has connections. We do not want these since the category $\Box_{\text{BCH}}^{\text{op}}$ that we are trying to find alternative definitions for does not have them. In order to remedy this, we refine the definition of the morphisms in $\Box_{\text{gr}}$. Let us formulate the following auxiliary definitions.
Given a graph morphism \( g : \text{grp-hom}(C_m, C_n) \), it is easy to define what it means that it preserves binary meets resp. joins:

\[
\text{pres-meet}(g) := \Pi(u, v : 2^m).g(u \sqcap v) = g(u) \sqcap g(v) \quad (16)
\]

\[
\text{pres-join}(g) := \Pi(u, v : 2^m).g(u \sqcup v) = g(u) \sqcup g(v) \quad (17)
\]

Note that preserving meets and joins is a property (a “mere proposition”) of morphisms. For general morphisms between graphs which might not have all meets or joins, the definition is more subtle but still straightforward; one can always define the property of being a meet (join) and then say that any vertex which has this property is mapped to one which also has it. We omit the precise type-theoretic formulation.

The two mentioned examples of morphisms which are “too much” in \( \square_{\text{grp}} \) do not preserve binary meets resp. joins.

**Definition 10** (category \( \square_{\text{cont}} \)). The category \( \square_{\text{cont}} \) has \( \mathbb{N} \) as objects and, as morphisms, graph morphisms between standard cubes which preserve meets and joins (\( \text{cont} \) for continuous):

\[
\text{obj} \left( \square_{\text{cont}} \right) := \mathbb{N} \quad (18)
\]

\[
\square_{\text{cont}}(m, n) := \Sigma(g : \text{grp-hom}(C_m, C_n)) \text{pres-meet}(g) \times \text{pres-join}(g) \quad (19)
\]

This gives us a category which is indeed equivalent (in fact isomorphic) to \( \square_{\text{BCH}}^\text{op} \):

**Theorem 11.** The categories \( \square_{\text{BCH}}^\text{op} \) and \( \square_{\text{cont}} \) are isomorphic. The isomorphism on the object part is the identity, i.e. the equivalence is given by a family \( e \) as in:

\[
e : \Pi(m, n : \mathbb{N}).\square_{\text{BCH}}^\text{op}(m, n) \simeq \square_{\text{cont}}(m, n). \quad (20)
\]

Before giving a proof, we formulate the following:

**Lemma 12.** Consider the full subgraph of \( C_n \) which has exactly \((n+1)\) vertices, namely the “origin” \( 00...0 \) and the “base vectors” which have exactly one 1. We call this subgraph \( B_n \), where the \( B \) stands for “base”, and it comes with the inclusion \( i : B_n \hookrightarrow C_n \). For any \( m \), “forgetting” the property of preserving the joins and composing with \( i \) as in

\[
\lambda g. i \circ (\text{proj}_1(g)) : (\Sigma(g : \text{grp-hom}(C_n, C_m)) \text{pres-join}(g)) \rightarrow \text{grp-hom}(B_n, C_m) \quad (21)
\]

is an equivalence. Moreover, \( g \) preserves meets if and only if \( i \circ (\text{proj}_1(g)) \) does.
Proof. The only binary joins that $B_n$ has are trivial, so every morphism $\text{grp-hom}(B_n, C_m)$ is join-preserving. Thus, the first claim of the lemma is that every such morphism can be extended in a unique way as shown in the diagram to the right. Every node of $C_n$ which is not in $B_n$, i.e. every node which is not the origin or a base vector, can be written as a join of base vectors. Since we need to preserve joins, it is therefore determined where the node has to be sent to. The map defined in this way preserves all binary joins, and it preserves binary meets if and only if the input does. □

Proof of Theorem 11. We first give the overview of the argument as a chain of equivalences, we then justify each step.

\[
\square_{\text{cont}}(m,n) \equiv \Sigma(g : \text{grp-hom}(C_m, C_n)).\text{pres-meet}(g) \times \text{pres-join}(g)
\]

[Step 1] \equiv \Sigma(g : \text{grp-hom}(B_m, C_n)).\text{pres-meet}(g)

[Step 2] \equiv \Sigma(z : 2^n, d : m \xrightarrow{\text{left}} n + 1).\Pi(i : m, j : n).(d(i) = \text{inl}(j)) \rightarrow (z(j) = 0)

[Step 3] \equiv \Sigma(z : 2^n, e : n \xrightarrow{\text{left}} m + 1).\Pi(i : m, j : n).(e(j) = \text{inl}(i)) \rightarrow (z(j) = 0)

[Step 4] \equiv \Sigma(z : 2^n, e : n \xrightarrow{\text{left}} (m + 1)).\text{left-inj}(f) \times \Pi(i : m, j : n).(e(j) = \text{inl}(i)) \rightarrow (z(j) = 0)

[Step 5] \equiv \Sigma(\alpha : \Pi(j : g).\Sigma(e : m + 1, z : 2).\Pi(i : m).(e = \text{inl}(i)) \rightarrow z = 0).\text{left-inj}(\text{proj}_1 \circ \alpha)

[Step 6] \equiv \Sigma(\alpha : \Pi(j : g).\Sigma(e : m + 1, z : 2)).\text{left-inj}(\alpha)

\equiv \square_{\text{BCH}}^{\text{op}}(m, n)

Step 1 holds by Lemma 12. Let us look at Step 2. Giving a graph homomorphism between $B_m$ and $C_n$ corresponds to choosing where the origin is mapped to, and choosing where each (non-trivial) edge of $B_m$ is mapped to. For the origin, we use the component $z : 2^m$. There are $m$ non-trivial edges in $B_m$, and $z$ is an endpoint of $n$ non-trivial edges and one trivial edge in $C_n$. This gives us up to $m \rightarrow n + 1$ possible functions, but since we only consider meet-preserving morphisms, every function needs to be injective on the left part, leading to $d : m \xrightarrow{\text{left}} n + 1$. Moreover, if $d(i) = \text{inl}(j)$ for some $i, j$, then the image of the origin must be the starting point of the edge in dimension $j$, i.e. $z(j) = 0$. Step 3 is an application of Lemma 2 (essentially, it swaps the roles of $m$ and $n$). Step 4 only unfolds the definition $\xrightarrow{\text{left}}$.

In Step 5, the usual distributivity between $\Sigma$ and $\Pi$ (under the propositions-as-types view referred to as the “axiom of choice”) is used: $z, e$, and the unnamed last component can all be seen as (dependent) functions with domain $n$. The dependent function $\alpha$ combines them into a single dependent function with domain $n$ and a codomain that consists of multiple components which, again, are called $e, z$, and unnamed. Only the component expressing the “injectivity on the left part”-property cannot be seen as a function in $n$. In Step 6, we massage the codomain of $\alpha$: We have $e : m + 1$ and also $z : 2$, but the condition says that $z$ is determined unless $e = \text{inr}(0)$; thus, the type is equivalent to $m + 2$.

We omit the calculation which shows that the constructed equivalence preserves composition of morphisms in the categories. □

In Section 3, we will switch from standard cubes to twisted cubes. The directions of some edges will be reversed. It is therefore an advantage to formulate a condition similar to the one above meets and joins without referring to the direction of edges. This is indeed possible:
We say that a morphism \( f : \text{grp-hom}(C_m, C_n) \) is dimension-preserving if \( f \) maps edges of the same dimension to edges of the same dimension,

\[
\dim\text{-pres}(f) : \equiv \Pi(e_1, e_2 : \text{edges}(C_n)).(\dim(e_1) = \dim(e_2)) \rightarrow (\dim(f(e_1)) = \dim(f(e_2))).
\]  

The category \( \Box_{\dim} \) makes use of these concepts:

\[
\text{obj}(\Box_{\dim}) : \equiv \mathbb{N} \quad \Box_{\dim}(m, n) : \equiv \Sigma(g : \text{grp-hom}(C_m, C_n)).\dim\text{-pres}(g)
\]

As \( \text{pres-meet}(g) \) and \( \text{pres-join}(g) \), preserving the dimension as in (24) is a proposition in the sense of homotopy type theory (has at most one proof).

**Remark 14.** For a graph morphism \( f \) as in the definition above, the following condition says that \( f \) is “injective on dimensions” (on the non-trivial part):

\[
\dim\text{-inj}(f) : \equiv \Pi(e_1, e_2 : \text{edges}(C_m), j : n).((\dim(f(e_1))) = \dim(f(e_2))) \equiv (\dim(e_1) = \dim(e_2)).
\]

However, note that this follows directly from \( \dim\text{-pres}(f) \): Assume \( e_1, e_2 \) are edges such that \( \dim(f(e_1)) \) and \( \dim(f(e_2)) \) are equal and non-trivial. If \( e_1 \) and \( e_2 \) are not “parallel” (i.e. not in the same dimension), then we can find \( e'_1 \) in the same dimension as \( e_1 \) such that \( e'_1 \) and \( e_2 \) are adjacent (i.e. the endpoint of one is the starting point of the other). It is clear that \( f(e'_1) \) and \( f(e_2) \) cannot go into the same non-trivial direction, since we can only go one step into a given direction before going back.

The connection to meet- and join-preserving is given by the following result:

**Lemma 15.** A morphisms \( f : \text{grp-hom}(C_m, C_n) \) is join-and-meet-preserving exactly if it is dimension-preserving.

**Proof.** This follows easily by going via morphisms \( \text{grp-hom}(B_m, C_n) \) as in Lemma 12. The graph \( B_m \) has exactly one edge for every non-trivial dimension, and the proof is analogous to the one of Lemma 12.

**Corollary 16** (Section summary). The categories \( \Box_{\BCH}^{\circ}, \Box_{\text{cont}}, \) and \( \Box_{\dim} \) are isomorphic.

### 3. A Category of Twisted Cubes

As discussed in the introduction, we build on our framework of graph morphisms to define a category of twisted cubes. A small change of Definition 4 gives us these twisted cubes:

**Definition 17.** Given a graph \( G = (V, E) \), the twisted graph-prism of \( G \), denoted as \( \text{tw-prism}(G) :\equiv (\text{tw-prism}(V), \text{tw-prism}(E)) \) is the graph defined by

\[
\text{tw-prism}(V) :\equiv \mathbb{Z} \times V
\]

\[
\text{tw-prism}(E) :\equiv \{ ((0, t), (0, s)) \mid (s, t) : E \}
\]

\[
\cup \{ ((1, s), (1, t)) \mid (s, t) : E \}
\]

\[
\cup \{ ((0, v), (1, v)) \mid v : V \}.
\]
We then define:

**Definition 18.** Given \( n : \mathbb{N} \), the twisted cube \( T_n \) is defined as follows:

\[
T_0 \equiv (1, \{(0, 0)\}) \\
T_{n+1} \equiv \text{tw-prism}(T_n)
\] (30)

Alternatively, we can tweak Definition 5 to get a non-recursive definition. As before, the convention is that \(-1\) is empty.

**Definition 19.** The non-recursive definition of \( T_n \) is as follows:

\[
\begin{align*}
\text{nodes}(T_n) & \equiv 2^n \\
\text{edges}(T_n) & \equiv 2^n + (n \times 2^{n-1}) \\
\text{src}(\text{inl}(v)) & \equiv \text{trg}(\text{inl}(v)) \\
& \equiv v \\
\text{src}(\text{inr}(i, x_0 x_1 ... x_{n-2})) & \equiv x_0 x_1 ... x_{i-1} \cdot b \cdot x_i ... x_{n-2} \\
\text{trg}(\text{inr}(i, x_0 x_1 ... x_{n-2})) & \equiv x_0 x_1 ... x_{i-1} \cdot (1 - b) \cdot x_i ... x_{n-2}
\end{align*}
\] (31-35)

where \( b = 1 \) if the total number of zeros in \( x_0 x_1 ... x_{i-1} \) is odd, and \( b = 0 \) otherwise.

This means that an edge is reversed (compared to the standard cubes discussed before) exactly if the number of zeros in dimensions that come before the edge is odd (note that the condition talks about \( x_{i-1} \), not \( x_{n-2} \)). The twisted cubes of dimension up to 3 are illustrated in Figure 4; see also Figures 1 and 2 in the introduction.

**Lemma 20.** Definition 18 and Definition 19 define isomorphic graph structures. ■

\[ \epsilon \quad 0 \xrightarrow{(0, \cdot)} 1 \]

\[
\begin{array}{c}
01 \\
\downarrow (0, 1) \\
00 \xrightarrow{(0, 0)} 10
\end{array}
\]

\[
\begin{array}{c}
011 \\
\uparrow (0, 11) \\
010 \xrightarrow{(0, 10)} 110 \\
\downarrow (0, 10) \\
000 \xrightarrow{(0, 00)} 100 \\
\uparrow (0, 01) \\
001 \xrightarrow{(0, 01)} 101
\end{array}
\]

**Figure 4.** An illustration of \( T_n \) where \( n \leq 3 \).

\( T_n \) has an interesting property that the standard cube \( C_n \) does not have: Its free preorder \( T_n^* \) is isomorphic to the total order on \( 2^n \) elements. This observation was originally suggested by Paolo Capriotti and Jakob von Raumer in a discussion with the first author of this paper. Note that this observation should not be misunderstood to mean that \( T_n \) itself is uninteresting. Its edges give it a unique structure, as visualised in Figure 5.

The idea behind this result is that \text{tw-prism} preserves the property of having a preorder that is total. To elaborate on this, if \( G^* \) is a total order, then \((\text{tw-prism} G)^*\) consists of two copies of \( G^* \), where the first copy is turned around. One of the edges added in (29) links the largest node in the first copy to the smallest node in second copy, thus every element of the second copy is larger than all the elements of the first.

**Theorem 21.** For all \( n : \mathbb{N} \), the preorder \( T_n^* \) is isomorphic to the total order \((2^n, <)\).
Note that Theorem 21 is a property which one usually expects for simplicial structures, but not for cubical ones.

Another related observation is that we can find a path from the smallest vertex to the largest vertex of $T_n$ which respects the direction of the edges, and which visits each vertex exactly once. Recall that such a path is called a Hamiltonian path. We record this:

**Theorem 22.** For all $n : \mathbb{N}$, there is exactly one Hamiltonian path through $T_{n+1}$. This path contains exactly one edge in the first dimension (i.e. the one which is added when going from $T_n$ to $T_{n+1}$). Moreover, this single edge in the new dimension connects the Hamiltonian paths through the two copies of $T_n$ of which $T_{n+1}$ consists by definition, cf. (26).

**Proof of Theorem 21 and Theorem 22.** As before, we denote elements of $2^n$ as sequences such as 00101 (binary representation with most significant bit first) or, for clarity, by $0 \cdot 0 \cdot 1 \cdot 0 \cdot 1$. We use the endofunction $\text{rev}$ on $2^n$, which simply replaces each 0 in a sequence by a 1 and vice versa; i.e. it sends the number $i$ to $2^n - 1 - i$ (note that $\text{rev}$ does not reverse the sequence, but the ordering on $2^n$).

Let us define endofunctions $f_n$ and $g_n$ on $2^n$, by induction on $n$. Note that, at this point, we do not talk about graph morphisms but only about functions between sets. The base cases of the induction are uniquely determined. We define $f$ and $g$ by

\[
\begin{align*}
  f_{n+1}(0 \cdot \bar{x}) &\equiv 0 \cdot f_n(\text{rev}(\bar{x})) & g_{n+1}(0 \cdot \bar{x}) &\equiv 0 \cdot \text{rev}(g_n(\bar{x})) \\
  f_{n+1}(1 \cdot \bar{x}) &\equiv 1 \cdot f_n(\bar{x}) & g_{n+1}(1 \cdot \bar{x}) &\equiv 1 \cdot g_n(\bar{x}).
\end{align*}
\]

(36)

It is easy to calculate that, by induction, $f$ and $g$ are inverse to each other. We want to show that they extend to morphisms between preorders,

\[
\begin{align*}
  \hat{f}_n : (2^n, <) &\to T_n^* & \hat{g}_n : T_n^* &\to (2^n, <).
\end{align*}
\]

(38)

To construct $\hat{f}_n$ and the Hamiltonian path through the cube, it suffices to show: for $x, y : 2^n$ with $x + 1 = y$, we have an edge $f_n(x) \to f_n(y)$.

We do induction on $n$. For $n = 0$, this is vacuously true (such $x, y$ do not exist). For $n = n' + 1$, there are multiple cases:

- case $x = 0 \cdot x'$ and $y = 0 \cdot y'$: Then, the assumption gives us $x' + 1 = y'$ and we have to find an edge $0 \cdot f_n(\text{rev}(x')) \to 0 \cdot f_n(\text{rev}(y'))$. Looking at Definition 17, we can get this if we have $f_n(\text{rev}(y')) \to f_n(\text{rev}(x'))$. This holds by induction, since $\text{rev}$ reverses the order which gives us $\text{rev}(y') + 1 = \text{rev}(x')$.

- case $x = 1 \cdot x'$ and $y = 1 \cdot y'$: Similar to the previous case, but nothing gets reversed.

- case $x = 0 \cdot x'$ and $y = 1 \cdot y'$: In this case, we have $x = 0111 \ldots$ and $y = 1000 \ldots$ We need to find an edge $0 \cdot f(\text{rev}(111 \ldots)) \to 1 \cdot f(000 \ldots)$, which simplifies to $0 \cdot f(000 \ldots) \to 1 \cdot f(000 \ldots)$. This edge is directly given in (29).

- case $x = 1 \cdot x'$ and $y = 0 \cdot y'$: Contradicts with the assumption $x + 1 = y$.

This shows that there is a Hamiltonian path, and it is given by $\hat{f}_n$. The definition of $f$ as in (36,37) also shows that $f_{n+1}$ consists of two copies of $f_n$, implying the last claim of Theorem 22. In order to prove Theorem 21, we need to construct $\hat{g}_n$. It is enough to show that, for an edge from $u$ to $v$ in $T_n$, we have $g(u) \leq g(v)$. This follows by straightforward induction, going through the edges in Definition 17. But Theorem 21 implies that there is at most one Hamiltonian path. ■
Remark 23. Note that every vertex \( v \) in \( T_n \) is an endpoint of \( n \) non-trivial edges. The number of zeros in the binary representation in the “order number” of \( v \) (i.e. the value \( g_n(v) \) in the proof of Theorem 21) equals the number of outgoing edges. Figure 5 shows this.

Analogously to Definition 8, we can now define the category of twisted graph morphisms:

**Definition 24 (category ¹grp).** The category ¹grp has natural numbers as objects, and morphisms from \( m \) to \( n \) are graph morphisms between twisted cubes:

\[
\text{obj}(¹grp) \equiv \mathbb{N} \\
¹grp(m, n) \equiv \text{grp-hom}(T_m, T_n)
\]

It is easy to see that the category ¹grp has a version of connections. Since we are looking for a “twisted analogue” of ²BCH, we need to refine it further. In Section 2, we have discussed the restriction to (meet and join)-preserving morphisms, and to dimension-preserving morphisms. It follows directly from Theorem 21 that every morphism in ¹grp preserves all binary meets and joins, so this condition becomes trivial; it does not avoid connections. However, preserving dimensions is still a non-trivial condition which does avoid connections. The definition of equation (24) still works.

**Definition 25 (category ¹dim).** The category ¹dim has dimension-preserving maps between twisted cubes as morphisms:

\[
\text{obj}(¹dim) \equiv \mathbb{N} \\
¹dim(m, n) \equiv \Sigma(g : \text{grp-hom}(T_m, T_n)).\text{dim-pres}(g)
\]

Note that the explanation of Remark 14 holds for the twisted cube category as well.

A consequence of Theorem 21 is that morphisms in ¹dim cannot “swap dimensions”. But an even stronger result holds, namely that surjective morphisms are unique:

**Theorem 26.** There is exactly one surjective morphism in ¹dim(m, n) for \( m \geq n \).

(Clearly, there is none if \( m < n \).)

**Proof.** The key to the proof is Theorem 22. Clearly, the Hamiltonian path in \( T_m \) goes through all vertices. Due to surjectivity, its image has to go through all vertices of \( T_n \). In other words, the \( T_m \)-Hamiltonian path has to be mapped to the \( T_n \)-Hamiltonian path. Since the graph morphisms that we consider preserve the dimension, the only edge in the \( T_m \)-path which can be mapped to the single edge in the first dimension in the \( T_n \)-path is just this single edge in the first dimension in the \( T_m \)-path; i.e. the middle edge has to be mapped to the middle edge. From here, it follows by induction that there can only be at most one surjective graph morphism.
What is left to show is that there actually is a surjective graph morphism if $m \geq n$. It is enough to construct a surjective graph morphism $f : \mathbb{M}_{\text{dim}}(n + 1, n)$, from where we get any other by $(m - n)$-fold composition (0-fold composition is the identity). Such a graph morphism is given by

$$f(x_0 \ldots x_{n-1}x_n) \equiv (x_0 \ldots x_{n-1}). \quad (41)$$

Since the directions of the edges do not depend on the very last dimension, this works (cf. Definition 19).

An important consequence of the above result is that there is a unique way to degenerate a twisted cube. We do not go into this here (but see the conclusions at the end of the paper). Here, we go into a different direction.

Let us write $\text{intv}$ ("interval") for the finite set $\{0, 1, *\}$. Of course, $\text{intv}$ is isomorphic to $\mathbb{Z}$, but referring to the last element as $*$ helps the intuition, we hope.

**Definition 27.** A face of the twisted $n$-cube $T_n$ is a function $f : n \to \text{intv}$. The dimension of a face, written $\text{dim}(f)$, equals the number of times $f$ takes $*$ as value (i.e. the size of $f^{-1}(*)$). The type of faces of dimension $k$ is written as $\text{faces}(n, k)$.

The face $f : n \to \text{intv}$ represents the full subgraph of $T_n$ of vertices on which $f$ “matches” (a vertex $x_0x_1 \ldots x_{n-1}$ is matched if, for every $i$, we have $f(i) = x_i$ or $f(i) = *$).

**Lemma 28.** The image of $f : \mathbb{M}_{\text{dim}}(m, n)$ is a face.

**Proof.** This follows from the property of preserving the dimension. $\blacksquare$

**Lemma 29.** The $m$-faces are the only injective maps $\mathbb{M}_{\text{dim}}(m, n)$:

$$\text{faces}(n, m) \simeq \Sigma(f : \mathbb{M}_{\text{dim}}(m, n)).\text{-inj}(f). \quad (42)$$

**Proof.** Every face gives rise to a canonical injective dimension-preserving morphism, as dictated by the inclusion of the full subgraph that the face represents into $T_n$. The fact that these are the only ones follows from Theorem 21 (we cannot “swap dimensions”) and Lemma 28. $\blacksquare$

As with Theorem 21 before, Lemma 29 is a result which is usually found in simplicial structures, but not in cubical ones. In any case, we now easily get:

**Lemma 30** (factorisation of dimension preserving morphisms). Given a morphism $f : \mathbb{M}_{\text{dim}}(m, n)$, there is exactly one way to write it as the composition $f = \text{inj}(f) \circ \text{surj}(f)$ of a surjective dimension preserving graph morphism followed by an injective one. This means that the map

$$(\Sigma(k : \mathbb{N}).(\Sigma(h : \mathbb{M}_{\text{dim}}(k, n)).\text{-inj}(h)) \times (\Sigma(g : \mathbb{M}_{\text{dim}}(m, k)).\text{-surj}(g))). \to \mathbb{M}_{\text{dim}}(m, n) \quad (43)$$

$$(k, (h, i), (g, s)) \mapsto h \circ g \quad (44)$$

is an equivalence. Moreover, morphisms $\mathbb{M}_{\text{dim}}(m, n)$ are in 1-to-1 correspondence with faces of $T_n$ of dimension $\leq m$.

**Proof.** A consequence of Lemma 28 is that the factorisation on the level of sets of vertices works. The second claim follows from the first: In (43), the $k$ and the surjective map are uniquely determined (i.e. contractible components) by Theorem 26. By Lemma 29, injective maps correspond to faces. $\blacksquare$
Remark 31. It follows from Lemma 30 and the proof of Theorem 26 that all the non-empty fibres of a dimension-preserving morphism between twisted cubes have the same size. The reverse is the case as well: a morphism between twisted graphs where all non-empty fibres have the same size is dimension-preserving.

Another consequence of the above results is that $\boxtimes_{\text{dim}}$ can be given the structure of a Reedy category (cf. [8]). Recall that a Reedy category is a category $R$ with a degree function $d : \text{obj}(R) \rightarrow \mathbb{N}$ and two subcategories $R^+$ and $R^-$, such that:

- both subcategories are wide, i.e. contain all the objects of $R$;
- every nonidentity morphism in $R^+$ raises the degree;
- every nonidentity morphism in $R^-$ lowers the degree;
- and every morphism of $R$ can be written as a morphisms in $R^-$ followed by a morphism in $R^+$ in a unique way.

The reason why Reedy categories are interesting is that they enable certain inductive constructions. In the setting of type theory, they have been discussed by Shulman [19].

Theorem 32. The category $\boxtimes_{\text{dim}}$ is a Reedy category where the degree of an object is the object itself (recall that objects are natural numbers). $\boxtimes_{\text{dim}}^+$ is the subcategory of injective morphisms, and $\boxtimes_{\text{dim}}^-$ is the subcategory of surjective morphisms.

Proof. The first three properties are clear, and the factorisation is given by Lemma 30. ■

Another feature of $\boxtimes_{\text{dim}}$ is its monoidal structure. It mimics the Cartesian product of standard cubes, but we have to take care of the twisting:

Definition 33 (monoidal product of $\boxtimes_{\text{dim}}$). $\boxtimes_{\text{tw}} \cdot : \boxtimes_{\text{dim}} \times \boxtimes_{\text{dim}} \rightarrow \boxtimes_{\text{dim}}$ is the bifunctor given on objects by $m \boxtimes_{\text{tw}} n \equiv m + n$. Let $b$ be 1 if the number of zeroes in the sequence $x_0 \ldots x_{m-1}$ is odd, and 0 if it is even; and define $c$ analogously for the sequence $f(x_0 \ldots x_{m-1})$. For a single bit $z$ and a sequence $s$, we write $z \boxtimes_{\text{dist}} s$ for the result of applying "\boxtimes z" on every element of the sequence.

Then, for $f : \boxtimes_{\text{dim}}(m, m')$ and $g : \boxtimes_{\text{dim}}(n, n')$, the bifunctor is defined by

$$(f \boxtimes_{\text{tw}} g)(x_0 \ldots x_{m-1} \cdot x_m \ldots x_{m+n-1}) \equiv f(x_0 \ldots x_{m-1}) \cdot (c \boxtimes_{\text{dist}} s)(b \boxtimes_{\text{dist}} x_m \ldots x_{m+n-1}). \quad (45)$$

It is easy to see that $\boxtimes_{\text{tw}} \cdot$ preserves graph homomorphisms and the dimension preserving condition. Therefore, $(\boxtimes_{\text{dim}}, \boxtimes_{\text{tw}} \cdot, 0)$ is a (strict) monoidal category.

Finally, let us record an alternative representation of the category $\boxtimes_{\text{dim}}$ which does not go via graph morphisms.

Definition 34 (ternary notation: category $\boxtimes_{\text{tri}}$). The category $\boxtimes_{\text{tri}}$ has natural numbers as objects, and a morphism from $m$ to $n$ is a function $\mathfrak{g} : m \rightarrow \text{intv}$ which takes $*$ at most $m$ times as image:

$$\text{obj}(\boxtimes_{\text{tri}}) \equiv \mathbb{N} \quad \boxtimes_{\text{tri}}(m, n) \equiv \Sigma(f : m \rightarrow \text{intv}).f^{-1}(\ast) \leq m \quad (46)$$

The identity morphisms are the functions that are constantly $\ast$. To define the composition of $f : \boxtimes_{\text{tri}}(k, m)$ and $g : \boxtimes_{\text{tri}}(m, n)$, we need to define a function $g \circ f : m \rightarrow \text{intv}$ (which is $\ast$ at most $k$ times). We define $(g \circ f)(i)$ by recursion on $i$, simultaneously with the values $i'$

\footnote{Degrees can more generally be arbitrary ordinals, but $\mathbb{N}$ is sufficient in our case.}
and $b_i$, as follows:

$$(g \circ f)(i) \equiv \begin{cases} 
  g(i) & \text{if } g(i) \in \{0, 1\} \\
  (f(i')) \text{xor } b_i & \text{if } g(i) = \star \text{ and } f(i') \in \{0, 1\} \\
  \star & \text{if } g(i) = \star \text{ and } f(i') = \star
\end{cases} \quad (47)$$

where

- $i'$ is the number of occurrences of $\star$ in the sequence $g(0), g(1), \ldots, g(i-1)$;
- $b_i$ is 1 if the number of zeros in the sequence $(g \circ f)(0), (g \circ f)(1), \ldots, (g \circ f)(i-1)$ is odd, and 0 if it is even.

Note that a morphism in $\tri_{\dim}(m, n)$ can be represented as a sequence such as $01\star0\star10$ of length $n$ which contains the symbol $\star$ at most $m$ times, which is why we refer to it as ternary notation.

**Remark 35.** There is a category of twisted semi-cubes, denoted by $\tri_{\dim}^+$, which is exactly the same as $\tri_{\tri}$ except that the number of $\star$ in the sequence must be exactly $m$, i.e. “$\leq$” is changed to “$=$” in the definition of $\tri_{\tri}(m, n)$. This category is equivalent to the subcategory of $\dim_{\dim}$, denoted as $\dim_{\dim}^+$, which consists of injective dimension-preserving graph homomorphism. Note that this injectivity condition is equivalent to removing the reflexive edges from Definition 18.

If we remove the expression $(\text{xor } b_i)$ in the definition of morphisms of $\tri_{\dim}^+$, then the category becomes equivalent to the category of standard cubes but without degeneracies and swapping dimensions. In other words, the expression $(\text{xor } b_i)$ characterises “twisted-ness”.

**Theorem 36.** The categories $\dim_{\dim}$ and $\tri_{\tri}$ are isomorphic, with the object part being the identity. In particular, we have:

$$\dim_{\dim}(m, n) \simeq \tri_{\tri}(m, n) \quad (48)$$

**Proof.** As the following chain of equivalences:

$\dim_{\dim}(m, n)$

[Lemma 30] $\simeq \Sigma(k : \mathbb{N}). (\Sigma(h : \dim_{\dim}(k, n)). \text{is-inj}(h)) \times (\Sigma(g : \dim_{\dim}(m, k)). \text{is-surj}(g))$

[Theorem 26] $\simeq \Sigma(k : \mathbb{N}). (\Sigma(h : \dim_{\dim}(k, n)). \text{is-inj}(h)) \times (k \leq m)$

[Lemma 29] $\simeq \Sigma(k : \mathbb{N}). \text{faces}(n, k) \times (k \leq m)$

[simplification] $\simeq \Sigma(f : n \to \text{intv}). f^{-1}(\star) \leq m$

$\equiv \tri_{\tri}(m, n)$

When transported along this isomorphism, the composition of $\dim_{\dim}$ gets mapped to the composition of $\tri_{\tri}$, as required. $\blacksquare$

### 4. Conclusions and Future Work

In this paper, we have introduced and proved multiple results about twisted cube categories. In future work, we plan to examine them further, e.g. algebraic presentation via generators and relations. Such presentations exist for many different cube categories in the literature. As far as we are aware, such a definition has not been suggested for the BCH category, but the presentations by Antolini [3] and Newstead [11] are easy to adapt to that category. Interestingly, further adapting the generators to the twisted setting simplifies them
significantly, which mirrors the fact that morphisms between twisted cubes cannot swap dimensions. Moreover, our Theorem 26 implies that degeneracies are unique: there is only one single way in which a twisted $n$-cube can be degenerated to get a twisted $(n+1)$-cube. A consequence is that we do not need to impose relations between different degeneracies.

This, we hope, will help us to develop the higher categorical structures that can be encoded as presheaves on the category of twisted cubes. One ultimate goal would be to model some form of directed cubical type theory mirroring the model by Bezem, Coquand, and Huber [5]. This however seems currently out of reach.

Another direction which we want to explore is to not consider set-valued presheaves, but type-valued presheaves instead. To facilitate this, we can consider the category of twisted semi-cubes mentioned on Remark 35. From there, type-valued presheaves can be encoded as Reedy-fibrant diagrams in a known style [20]. We can then add a condition reminiscent of Rezk’s \textit{Segal-condition} [17] by stating that the projection from twisted semi-cubical types to the sequence of types along the Hamiltonian path is an equivalence. It seems that this is promising for a construction of composition and higher coherences, although it remains to be worked out.

\textbf{Acknowledgements} We would like to thank Paolo Capriotti and Jakob von Raumer. Both offered many suggestions during interesting discussions. In particular, the initial observation on which Theorem 21 is based was suggested by them, and the idea of considering graph morphisms was found in one of our many interesting discussions. We are also grateful to the participants of TYPES’19 in Oslo and the summer school on HTT/UF in Leeds, in particular to Emily Riehl, Christian Sattler, and Steve Awodey, for helpful comments and discussions.

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