Partitioning Problems with Splitting and Interval Targets

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Abstract
We consider a variant of the \( n \)-way number partitioning problem, in which some fixed number \( s \) of items can be split between two or more bins. We show a two-way polynomial-time reduction between this variant and a second variant, in which the maximum bin sum must be within a pre-specified interval. We prove that the second variant can be solved in polynomial time if the length of the allowed interval is at least \((n - 2)/n\) times the maximum item size, and it is NP-hard otherwise. Using the equivalence between the variants, we prove that number-partitioning with \( s \) split items can be solved in polynomial time if \( s \geq n - 2 \), and it is NP-hard otherwise.

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1 https://cstheory.stackexchange.com/q/42275/9453
2 https://cs.stackexchange.com/a/141322.
3 https://cs.stackexchange.com/a/149567
1 Introduction

1.1 Partition with Splitting

In optimization problems, it is usually determined in advance whether each decision variable is discrete (in $\mathbb{Z}$) or continuous (in $\mathbb{R}$). For example, when we partition items into bins, we can either assume that each item must be put completely in a single bin (— discrete decision variables), or we can assume that each item can be split between different bins (— continuous decision variables).

In practice, one can often decide at “runtime” whether to consider a decision variable discrete or continuous. For example, consider two heirs who inherited three houses and have to divide them fairly. The house values are $100, 200, 400$. If all of them are discrete, then an equal division is not possible. If all of them are continuous, then an equal division is easy to attain by giving each heir 50% of every house, but it is very inconvenient since all houses must be jointly managed. A solution most often used in practice is to choose a single house that will be shared between the heirs. In this case, by sharing the house with value 400, one can attain a division in which each heir receives the same value of 350.\footnote{By “sharing” we mean that the ownership of the shared item is split. In the above example, the house worth 400 is split such that one heir gets 7/8 of it, and the other gets 1/8 of it (plus the 100 and 200).}

Finding such allocations requires the flexibility to choose, based on the input, a single decision-variable to treat as continuous.

This paper studies the following problem, where $n \geq 2$ and $s \geq 0$ are fixed integers.

$\text{PartitionWithSplitItems}(n, s)$: given a set of items with positive values, decide if there exists a partition into $n$ bins with equal sums, in which at most $s$ items are split between two or more bins.

It is easy to see that, when $s \geq n - 1$, the answer is always “yes”: put the items on a line, cut the line into $n$ pieces with an equal total value, and put each piece in a bin. Since $n - 1$ cuts are made, at most $n - 1$ items need to be split between pieces. In contrast, when $s = 0$, the problem is equivalent to the NP-hard problem Partition. So for $n = 2$, the problem is well-understood. The main goal of this paper is to analyze the runtime complexity of the problem for any integers $n \geq 3$ and $1 \leq s \leq n - 2$.

1.2 Partition with Interval Target

Approximate algorithms for optimization problems usually try to find solutions whose objective value lies in a small interval around the optimal solution. For example, given a minimization problem with the objective function $f$ and optimal value $\text{OPT}$, and an approximation accuracy $\epsilon > 0$, an approximation algorithm aims to find a solution $x$ with

$$\text{OPT} \leq f(x) \leq (1 + \epsilon) \cdot \text{OPT}.$$ 

If the run-time is polynomial in the problem size and $1/\epsilon$, then it is called an FPTAS.

On our way to solve $\text{PartitionWithSplitItems}(n, s)$, we first solve the following problem, where $n \geq 2$ is an integer and $t > 0$ is a rational number:

$\text{PartitionWithIntervalTarget}(n, t)$: given a set of $m$ items with positive integer sizes, where the sum of all sizes is $nS$, decide if there exists a partition into $n$ bins with sizes $b_1, \ldots, b_n$ such that

$$S \leq \max(b_1, \ldots, b_n) \leq (1 + t) \cdot S.$$
Note that $S \leq \max(b_1, \ldots, b_n)$ always holds by the pigeonhole principle. Writing it explicitly highlights the similarity between $\text{PartitionWithIntervalTarget}(n, t)$ and finding an FPTAS to the problem of minimizing $\max(b_1, \ldots, b_n)$, which is the famous makespan minimization problem. We prove that, with $n = 2$ bins, $\text{PartitionWithIntervalTarget}(n, t)$ can be solved in time polynomial in $m$ and $1/t$ (Section 3.1), which strengthens this similarity.

But with $n \geq 3$ bins, the similarity breaks apart. Denote the largest item size by $n \cdot M$. We prove that, for any constant $n \geq 3$, $\text{PartitionWithIntervalTarget}(n, t)$ can be solved in polynomial time when $t \geq (n-2) \frac{M}{S}$, and is NP-hard otherwise. In other words, given a rational number $d \geq 0$, we define

$$\text{PartitionWithDIntervalTarget}(n, d):$$

given a set of items with positive values, where the sum of all items is $nS$ and the largest item is $nM$, decide if there exists a partition into $n$ bins with sums $b_1, \ldots, b_n$ such that

$$S \leq \max(b_1, \ldots, b_n) \leq S + d \cdot M.$$

We show that, for any constant $n \geq 3$, it can be solved in polynomial time for any $d \geq n-2$ (Section 3.2), and is NP-hard for any $d < n-2$ (Section 3.3).

1.3 Two-way reduction

We tie the knots by proving a two-way polynomial-time reduction between the problems $\text{PartitionWithSplitItems}$ and $\text{PartitionWithDIntervalTarget}$ (Section 4). Using this reduction, we prove that, for any fixed integer $n \geq 3$, $\text{PartitionWithSplitItems}(n, s)$ can be solved in polynomial time for any integer $s \geq n-2$, and is NP-hard for any integer $s \leq n-3$ (Section 4).

1.4 Variant

There is another way to measure the amount of splitting. Instead of considering the number of split items, one can consider the number of times each item is split. The number of splittings is at least the number of split items, but might be larger: one item split into $k$ different bins counts as $k-1$ splittings. This variant is called $\text{PartitionWithSplittings}(n, s)$, it is similar to the problem $\text{PartitionWithSplitItems}(n, s)$ at the exception that $s$ denotes the number of splittings. Here, too, the answer is always “yes” when $s \geq n-1$. We prove that it is NP-hard for all $s \leq n-2$. The full proof details are in the Appendix C.

Note that at [14] a weaker result was presented: that $\text{PartitionWithSplittings}(n, s)$ is NP-Hard for all $s \leq n-3$.

2 Related work

2.1 Splitting in fair division

The idea of finding fair allocations with a bounded number of split items originated from Brams and Taylor [3][4]. They presented the Adjusted Winner (AW) procedure for allocating items among two agents with possibly different valuations. AW finds an allocation that is envy-free (no agent prefers the bundle of another agent), equitable (both agents receive the same subjective value), and Pareto-optimal (there is no other allocation where some agent gains and no agent loses). At most one item is split between the agents; hence, AW can be seen as a generalization of $\text{PartitionWithSplitItems}(2, 1)$ for agents with different
valuations. AW was applied (at least theoretically) to division problems in divorce cases and international disputes \cite{6,10} and was studied empirically \cite{13,7}.

The Adjusted Winner procedure is designed for two agents. For \(n \geq 3\) agents, the number of splitting was studied in an unpublished manuscript of Wilson \cite{16} using linear programming techniques. He proved the existence of an egalitarian allocation of goods (— an allocation in which all agents have a largest possible equal utility \cite{11}) — with at most \(n - 1\) split items; this can be seen as a generalization of PartitionWithSplitItems\((n,n-1)\).

Goldberg et al. \cite{8} studied the problem of consensus partitioning. In this problem, there are \(n\) agents with different valuations, and the goal is to partition a set of items into some \(k\) subsets (where \(k\) and \(n\) may be different), such that each agent values each subset at exactly \(1/k\). They prove that a consensus partitioning with at most \(n(k - 1)\) split items can be found in polynomial time.

Most similar to our paper is the recent work of Sandormirskiy and Segal-Halevi \cite{12}. Their goal is to find an allocation among \(n\) agents with different valuations, which is both fair and fractionally Pareto-optimal (fPO) — a property stronger than Pareto-optimality. This is a very strong requirement: when \(n\) is fixed, and the valuations are generic (— for every two agents, no two items have the same value-ratio), the number of fPO allocations is polynomial in \(m\), and it is possible to enumerate all such allocations in polynomial time. Based on this observation, they present an algorithm that finds an allocation with the smallest number of split items, among all allocations that are fair and fPO. In contrast, in our paper, there is no requirement for fractional-PO. Dropping the fPO requirement may allow allocations with fewer split items, but the number of potential allocations becomes exponential, so enumerating them all is no longer feasible.

Recently, Bei et al. \cite{1} studied an allocation problem where some items are divisible and some are indivisible. In contrast to our setting, in their setting the distinction between divisible and indivisible items is given in advance — the algorithm can only divide items that are pre-determined as divisible.

### 2.2 Splitting in other settings

Fractional relaxations of integer linear programs are very common in approximation algorithm design. However, most works do not consider the number of fractional variables.

One similar notion is related to preemption in machine-scheduling. Preemption means that a job can be split into parts, each of which can be scheduled independently, either on the same or on a different machine, with the additional constraint that no two parts of a job are scheduled on different machines at the same time. We found two works in which the amount of preemption is bounded: Soper and Strusevich \cite{15} allow at most one preemption, while Liu and Cheng \cite{9} consider a variant in which there is a penalty for each preemption. Note that the preemption penalty is incurred even if a job is split on the same machine.

Bourjolly and Pulleyblank \cite{2} study similar notions in the context of vertex cover in graphs. They present an algorithm for finding a minimum fractional vertex cover, in which the number of fractional vertices is as small as possible.

It is promising to see that many different computer science fields can integrate the concept of bounded splitting into their problems.

### 3 Partition with Interval Target

There are \(m\) items with positive integer sizes, that should be partitioned into \(n\) bins. The sum of item sizes is \(n \cdot S\) for some integer \(S\). We represent an \(n\)-partition by a vector \(b_1, \ldots, b_n\),
where \( b_i \) is the sum of item sizes in bin \( i \). Given a number \( t > 0 \), we define a \( t \)-feasible partition as a partition for which

\[
S \leq \max(b_1, \ldots, b_n) \leq (1 + t) \cdot S.
\]

The problem \( \text{PartitionWithIntervalTarget}(n, t) \) is to find a \( t \)-feasible partition, or decide that a \( t \)-feasible partition does not exist.

The problem is closely related to the problem of makespan minimization:

\[
\begin{aligned}
\text{Minimize} & \quad \max(b_1, \ldots, b_n) \\
\text{such that} & \quad b_1, \ldots, b_n \text{ are bin sums of a partition.}
\end{aligned}
\]

For any fixed \( n \), problem \( (1) \) has an FPTAS \cite{17}: it finds a partition with

\[
OPT \leq \max(b_1, \ldots, b_n) \leq (1 + \epsilon) \cdot OPT,
\]

in time polynomial in \( m \) and \( 1/\epsilon \). Despite the similarity between these problems, an FPTAS for \( (1) \) does not immediately lead to a polynomial-time algorithm for \( \text{PartitionWithIntervalTarget} \), but we will use it as a subroutine in our algorithms.

### 3.1 Algorithm for \( n = 2 \) bins

We start with a warm-up algorithm for two bins.

\begin{itemize}
\item [Theorem 1.] For any \( t > 0 \), \( \text{PartitionWithIntervalTarget}(2, t) \) can be solved in time \( O(\text{poly}(m, 1/t)) \), where \( m \) is the number of input items.
\end{itemize}

**Proof.** First, run the FPTAS for problem \( (1) \), taking \( \epsilon = t/2 \). If it finds a solution with value at most \( (1 + t) \cdot S \), then this is a \( t \)-feasible partition, so we return “yes”.

Otherwise, we know that the optimal value of \( (1) \) is larger than \( (1 + t) \cdot S/(1 + \epsilon) \). So in any partition, \( \max(b_1, b_2) > (1 + t) \cdot S/(1 + \epsilon) \). Let us call a bin with sum larger than \( (1 + t) \cdot S/(1 + \epsilon) \) an almost-full bin. So if the FPTAS to \( (1) \) does not find a \( t \)-feasible partition, then any \( t \)-feasible partition must have at least one almost-full bin.

Next, consider the following problem.

\[
\begin{aligned}
\text{Minimize} & \quad b_2 \\
\text{such that} & \quad b_1, b_2 \text{ are bin sums of a partition} \\
& \quad b_1 \leq (1 + t) \cdot S.
\end{aligned}
\]

This problem has an FPTAS too; this follows from Section 6 in \cite{17}, since the problem has a dynamic program which is \( \text{DP-benevolent under critical-coordinate} \) (\( b_1 \) is the critical coordinate). We run this FPTAS with the same \( \epsilon = t/2 \). If it finds a solution to \( (2) \) with value most \( (1 + t) \cdot S \), then it is a \( t \)-feasible partition, so we return “yes”.

Otherwise, we know that the optimal value of \( (2) \) is larger than \( (1 + t) \cdot S/(1 + \epsilon) \). So in any partition in which \( b_1 \leq (1 + t) \cdot S \), we must have \( b_2 > (1 + t) \cdot S/(1 + \epsilon) \), that is, bin 2 is almost-full. In particular, this is true for any \( t \)-feasible partition in which \( b_1 \) is the largest sum. This implies that, in any \( t \)-feasible partition, \( \text{both} \) bins must be almost-full. But this is impossible, since \( \epsilon = t/2 \) implies \( (1 + t) \cdot S/(1 + \epsilon) > S \) while \( b_1 + b_2 = 2S \). Hence, at this point, our algorithm returns “no \( t \)-feasible partition exists”.

The run-time of the algorithm is dominated by the run-time of the two FPTAS-s, which is in \( O(\text{poly}(m, 1/\epsilon)) = O(\text{poly}(m, 1/t)) \).
Remark 2 (Cardinality constraints). Consider the variant of PartitionWithInterval-Target where the goal is to decide if there exists a \( t \)-feasible partition in which each bin contains at most \( k \) items, for some integer \( k \geq m/2 \).

The algorithm in the present section can be adapted to solve this variant too. The same dynamic program that solves problems (1) and (2), can be adapted to also count the number of items in each bin. This number can be recorded exactly since its value is at most \( m \).

Therefore, the technique of [17] can be used to obtain an FPTAS for this case too. The rest of the algorithm works unchanged.

The dynamic program is formally written in Appendix A.

3.2 Algorithm for \( n \geq 3 \) bins

For \( n \geq 3 \), the run-time depends on the size of the largest item in the input. Denote this size by \( n \cdot M \). Given a number \( d \geq 0 \), a \( d \)-possible partition is a partition in which

\[ S \leq \max(b_1, \ldots, b_n) \leq S + dM. \]

So a \( d \)-possible partition is a \( t \)-feasible partition with \( t = dM/S \). The problem PartitionWithDIntervalTarget(\( n, d \)) is to find a \( d \)-possible partition, or decide it does not exist.

Theorem 3. For any integer \( n \geq 3 \) and rational number \( d \geq n - 2 \), PartitionWithDIntervalTarget(\( n, d \)) can be solved in time \( O(\text{poly}(m)) \).

The rest of this subsection is devoted to proving Theorem 3.

Our goal is to find a \( d \)-possible partition. We define \( t := dM/S \), so that a \( d \)-possible partition is a \( t \)-feasible partition. Note that \( d \geq n - 2 \) implies \( t \geq (n - 2)/m \geq 1/m \).

We run the FPTAS to problem (1) with \( \epsilon = t/4m^2 \). Note that \( 1/\epsilon \in O(m^2) \), so the FPTAS runs in time \( O(\text{poly}(m)) \). If it finds a solution with value at most \( (1 + t) \cdot S \), then it is \( t \)-feasible, so we return “yes”.

Otherwise, we know that the optimal value of (1) is larger than \( (1 + t) \cdot S/(1 + \epsilon) \), so any \( t \)-feasible partition must have at least one almost-full bin.

Next, consider the following problem.

Minimize \( \max(b_2, b_3, \ldots, b_n) \) such that \( b_1, b_2, b_3, \ldots, b_n \) are bin sums of a partition and \( b_1 \leq (1 + t) \cdot S \).

Similarly to problem (2), this problem is DP-benevolent with the critical-coordinate \( b_1 \), so it has an FPTAS. We run this FPTAS with the same \( \epsilon = t/4m^2 \). If it finds a solution to (3) with value at most \( (1 + t) \cdot S \), then it is a \( t \)-feasible partition, so we return “yes”.

Otherwise, we know that the optimal value of (3) is at least \( (1 + t) \cdot S/(1 + \epsilon) \). So any \( n \)-partition with \( b_1 \leq (1 + t) \cdot S \) must have \( \max(b_2, b_3, \ldots, b_n) \geq (1 + t) \cdot S/(1 + \epsilon) \). In particular, this is true for any \( t \)-feasible partition in which \( b_1 \) is the largest sum. So any \( t \)-feasible partition must have at least two almost-full bins.

The natural next step would be to consider the following problem:

Minimize \( \max(b_1, \ldots, b_n) \) such that \( b_1, b_2, b_3, \ldots, b_n \) represents a partition and \( b_1 \leq (1 + t) \cdot S \) and \( b_2 \leq (1 + t) \cdot S \).
However, this problem probably does not have an FPTAS even for \( n = 3 \). Note that for \( n = 3 \) the problem is equivalent to

\[
\text{Maximize } b_1 + b_2 \\
\text{such that } b_1 \leq (1 + t) \cdot S \text{ and } b_2 \leq (1 + t) \cdot S,
\]

which is known as the multiple-subset-sum problem. This problem does not have an FPTAS unless \( P = NP \) — by a reduction from Partition [6].

Therefore, we take a detour from the algorithm and prove some existential results about partitions with two or more almost-full bins. We assume that there are more items than bins, that is, \( m > n \). Then, \( \epsilon = t/4m^2 < t/4n^2 \). This assumption is without loss of generality, since if \( m \leq n \) the problem is trivial.

### 3.2.1 Structure of partitions with two or more almost-full bins

Our main structural lemma is;

- **Lemma 4.** Suppose \( d \geq n - 2 \) and \( t = dM/S \) and the following hold.
  
  (a) There is no \( t \)-feasible partition with at most 1 almost-full bin;
  
  (b) There is a \( t \)-feasible partition with at least 2 almost-full bins.

  Then, there is a \( t \)-feasible partition with the following properties.

  (c) Exactly two bins (w.l.o.g. bins 1 and 2) are almost-full.

  (d) The sum of every not-almost-full bin \( i \in \{3, \ldots, n\} \) satisfies

  \[
  \left( 1 - \frac{2}{n} t - 2\epsilon \right) \cdot S \leq b_i \leq \left( 1 - \frac{2}{n} t + (n - 1)2\epsilon \right) \cdot S.
  \]

  (e) Every almost-full bin contains only big-items — items with size greater than \( nS \left( \frac{t}{n-2} - 2\epsilon \right) \).

  (f) Every not-almost-full bin contains either big-items that are larger than every item in bin 1,2 or small-items that are smaller than \( 2nS\epsilon \).

  (g) Every not-almost-full bin contains the same number of big-items, say \( \ell \), where \( \ell \) is an integer (and may contain any number of small-items).

  (h) Every almost-full bin contains \( \ell + 1 \) big-items (and no small-items). So the total number of big-items is \( n\ell + 2 \).

This lemma allows us to separate the problem into two sub-problems: dividing the largest and smallest items between bins 3, \ldots, \( n \), and dividing the remaining items between bins 1, 2.

To prove Lemma 4, we start with an arbitrary \( t \)-feasible partition with some \( k \geq 2 \) almost-full bins 1, \ldots, \( k \), and transform it using a sequence of transformations to another \( t \)-feasible partition satisfying the properties. Note that the transformations are not a part of our algorithm — they are used only to prove the lemma.

- For (d), if there are \( k \geq 3 \) almost-full bins, we move an item from bins 3, \ldots, \( k \) to a not-almost-full bin. We prove that, as long as \( k \geq 2 \), the target bin remains not-almost-full. We repeat until \( k = 2 \) and only bins 1 and 2 remain almost-full.

- For (e), for the lower bound, if there is \( i \in \{3, \ldots, n\} \) for which \( b_i \) is smaller than the lower bound, we move an item from bins 1, 2 to bin \( i \). We prove that bin \( i \) remains not-almost-full, so by (d) bins 1, 2 remain almost-full. We repeat until \( b_i \) satisfies the lower bound. Once all bins satisfy the lower bound, the upper bound is automatically satisfied.

- For (f), if bin 1 or 2 contains an item that is not big, we move it to some bin \( i \in \{3, \ldots, n\} \). We prove that bin \( i \) remains not-almost-full, so bins 1, 2 remain almost-full. We repeat until bins 1 and 2 contain only big-items.
For (f), some bin $i \in \{3, \ldots, n\}$ contains an item bigger than $2nS\epsilon$, and smaller than any item in bin 1 or bin 2, we exchange the items. We prove that, after the exchange, $b_i$ remains not-almost-full so bins 1, 2 remain almost-full. We repeat until bins 1, 2 contain only the smallest big-items. Note that transformations (d), (e), (f) increase the sum in the not-almost-full bins, so eventually the process must end.

For (g) we use the fact that, by (d), the difference between two not-almost-full bins is at most $2nS\epsilon$, and show that it is too small to allow a difference of a whole big-item.

For (h), because by (f) bins 1 and 2 contain the smallest big-items, whereas their sum is larger than bins $3, \ldots, n$, they must contain at least $\ell + 1$ big-items. We prove that, if they contain $\ell + 2$ big-items, then their sum is larger than $(1 + t)S$, which contradicts $t$-feasibility.

The full proof details are in Appendix B.

3.2.2 Back to the algorithm

We are now ready to present the complete algorithm.

1. Run the FPTAS for problem (1). If it finds a $t$-feasible partition, return “yes”.
2. Run the FPTAS for problem (3). If it finds a $t$-feasible partition, return “yes”.
3. Let $B$ be the set of big-items.
4. If $|B|$ is not of the form $n\ell + 2$ for some integer $\ell$, return “no”.
5. Let $B_{12}$ be the set of $2\ell + 2$ smallest items in $B$ (break ties arbitrarily). Put all other items (big and small) in $B_{3n}$.
6. Partition the items in $B_{3n}$ into $n - 2$ bins $b_3, \ldots, b_n$ using the FPTAS for the following problem [17]:

   Minimize $\max(b_3, \ldots, b_n)$
   such that $b_3, \ldots, b_n$ are bin sums of a partition of $B_{3n}$.

   If in the returned partition $\max(b_3, \ldots, b_n) > (1 + t)S$, return “no”.
7. Partition the items in $B_{12}$ into two bins, $b_1, b_2$, as explained in Section 3.2.3. If it fails, return the partition $b_1, b_2, b_3, \ldots, b_n$.

► Lemma 5. If there is a $t$-feasible partition, then the above algorithm finds it.

Proof. Suppose there exists a $t$-feasible partition. As explained in Section 3.2, if the partition has at most one almost-full bin, then it is found by the FPTAS in step 1 or 2 of the algorithm. Otherwise, the partition must have two almost-full bins and satisfy all the properties of Lemma 4.

By Lemma 4, property (m) holds, so the algorithm does not return “no” in step 4.

By properties (c), (i) there exists a partition of $B_{3n}$ into $n - 2$ bins $3, \ldots, n$ which are not almost-full, so $\max(b_3, \ldots, b_n) < (1 + t - 2\epsilon)S$. Therefore, the FPTAS in step 6 finds a partition with $\max(b_3, \ldots, b_n) < (1 + \epsilon)(1 + t - 2\epsilon)S < (1 + t)S$.

The final step 7 is justified by the next subsection.

3.2.3 Algorithm to divide items between bins 1 and 2

At this point, we have a set $B_{12}$ containing the $2\ell + 2$ smallest big-items, and we have to decide if they can be partitioned into two subsets of $\ell + 1$ items with a sum at most $(1 + t)\cdot S$. Denote the sum of items in $B_{12}$ by $S_{12}$. 

Denote by $nU$ the largest item in $B_{12}$. Construct a new set $\overline{B}_{12}$ by replacing each item $x \in B_{12}$ by its “inverse”, defined by $\overline{x} := nU - x$. Note that, by property 4, all inverses are between 0 and $2nS\epsilon$. Denote the sum of inverses by $2\overline{S}_{12}$. Given a $t$-feasible partition of $B_{12}$ with sums $b_1, b_2$, denote the sums of the corresponding partition of $\overline{B}_{12}$ by $\overline{b}_1, \overline{b}_2$. Note that, since both bins contain $\ell + 1$ items,

\[
\overline{b}_i = (\ell + 1) \cdot nU - b_i \quad \text{for } i \in \{1, 2\}
\]

\[
\overline{S}_{12} = (\ell + 1) \cdot nU - S_{12}
\]

Our goal can be stated as

\[
b_i \leq (1 + t)S \quad \text{for } i \in \{1, 2\}
\]

\[
\iff \overline{b}_i \geq (\ell + 1)nU - (1 + t)S \quad \text{for } i \in \{1, 2\}
\]

\[
\iff \overline{b}_i \leq 2\overline{S}_{12} - (\ell + 1)nU + (1 + t)S \quad \text{for } i \in \{1, 2\}
\]

\[
= \overline{S}_{12} + (S + tS - S_{12})
\]

which is equivalent to the problem of finding a $\overline{t}$-feasible partition of $\overline{B}_{12}$, where we define

\[
\overline{t} := (S + tS - S_{12})/\overline{S}_{12}.
\]

By our result for two bins (Section 3.1 and Remark 2), this problem can be decided in time $O(\text{poly}(m, 1/\overline{t}))$. It remains to prove that $1/\overline{t}$ is polynomial in $m$.

We know that, in each of the not-almost-full bins, there are $\ell$ items that are at least as large as $nU$ (in addition to some small-items). Therefore:

\[
nS \geq 2\overline{S}_{12} + (n - 2) \cdot \ell \cdot nU
\]

\[
\implies S - S_{12} \geq -\frac{n - 2}{n} S_{12} + \frac{n - 2}{n} \ell \cdot nU
\]

We also know that, since $d \geq n - 2$:

\[
tS = dM \geq (n - 2)M \geq (n - 2)U.
\]

Summing up these two inequalities gives:

\[
S - S_{12} + tS \geq \frac{n - 2}{n} \left( \ell \cdot nU - S_{12} + nU \right)
\]

\[
= \frac{n - 2}{n} \left( (\ell + 1) \cdot nU - S_{12} \right)
\]

\[
= \frac{n - 2}{n} \cdot \overline{S}_{12}.
\]

Therefore, $\overline{t} \geq \frac{n - 2}{n}$, so $1/\overline{t} \in O(1)$, and the sub-problem can be decided in time $O(\text{poly}(m))$.

### 3.3 Hardness for $n \geq 3$ bins and $d < n - 2$

To complement the results of the previous subsection, we prove

> **Theorem 6.** For any integer $n \geq 3$ and positive rational number $d < n - 2$, \textsc{Partition-WithDIntervalTarget}(n, d) is NP-hard.
Proof. The proof is by reduction from the Balanced Partition problem: given a set of integers, decide if they can be partitioned into two subsets with the same sum and the same cardinality.

Given an instance $I_1$ of Balanced Partition, denote the number of items in it by $2m'$. Define $M$ to be the sum of numbers in $I_1$ divided by $2n(1 - \frac{d}{n-2})$, so that the sum is $2n(1 - \frac{d}{n-2})M$. We can assume w.l.o.g. that all items are at most $n(1 - \frac{d}{n-2})M$, since if some item is larger than half the sum, the answer is necessary “no”.

Construct an instance $I_2$ by replacing each item $x$ in $I_1$ by $nM - x$. So $I_2$ contains $2m'$ items between $n(\frac{d}{n-2})M$ and $nM$. Their sum, which we denote by $2S'$, satisfies

$$2S' = 2m' \cdot nM - 2n \left(1 - \frac{d}{n-2}\right) M = 2n \left(m' - 1 + \frac{d}{n-2}\right) M.$$ 

Clearly, $I_1$ has an equal-sum equal-cardinality partition (with the sum $n \left(1 - \frac{d}{n-2}\right) M$) if and only if $I_2$ has an equal-sum equal-cardinality partition (with sum $S' = n \left(m' - 1 + \frac{d}{n-2}\right) M$).

Construct an instance $I_3$ of $d$-possible partition by adding $(n-2)(m'-1)$ items of size $nM$. Note that $nM$ is indeed the largest item size in $I_3$, as in the definition of $\text{PartitionWithDIntervalTarget}$. Denote the sum of item sizes in $I_3$ by $nS$. Then

$$nS = 2S' + (n-2)(m'-1) \cdot nM$$

$$= n \left(2(m'-1) + \frac{2d}{n-2} + (n-2)(m'-1)\right) M$$

$$= n \left(n(m'-1) + \frac{2d}{n-2}\right) M;$$

$$S + dM = \left(n(m'-1) + \frac{2d}{n-2} + d\right) M$$

$$= \left(n(m'-1) + \frac{nd}{n-2}\right) M$$

$$= S',$$

so a partition is $d$-possible iff all $n$ sums are at most $S + dM = S'$.

If $I_2$ has an equal-sum partition, then the items of $I_2$ can be partitioned into two bins of sum $S'$, and the additional $(n-2)(m'-1)$ items can be divided into $n-2$ bins of $m-1$ items each. Note that their sum is

$$(m'-1) \cdot nM = n(m'-1)M$$

$$= S - \frac{2}{n-2} dM < S + dM = S',$$

so the resulting partition is a $d$-possible partition of $I_3$.

Conversely, suppose $I_3$ has a $d$-possible partition. Let us analyze its structure.

- Since the partition is $d$-possible, the sum of every two bins is at most $2(S + dM)$.
- So the sum of every $n-2$ bins is at least $nS - 2(S + dM) = (n-2)S - 2dM$.
- Since the largest $(n-2)(m'-1)$ items in $I_3$ sum up to exactly $(n-2)S - 2dM$ by (4), every $n-2$ bins must contain at least $(n-2)(m'-1)$ items.
- Since $I_3$ has $(n-2)(m'-1) + 2m'$ items overall, $n-2$ bins must contain exactly $(n-2)(m'-1)$ items, their size must be $nM$, and their sum must be $(n-2)S - 2dM$.
- The other two bins contain together $2m'$ items with sum $2(S + dM)$, so each of these bins must have a sum of exactly $S + dM$. Since $(m'-1) \cdot nM < S + dM$ by (4), each of these two bins must contain exactly $m'$ items.
These latter two bins are an equal-sum equal-cardinality partition for $I_2$. This completes the reduction.

Note that the proof is valid even when $d$ depends on $m$, e.g. when $d = (n - 2) - 1/m$.

4 Partition with Split Items

We slightly generalize the problem of partitioning with split items, to allow for optimization. Given integers $n \geq 2$, $s \geq 0$ and rational number $t \geq 0$, define:

\begin{align*}
\text{PartitionWithSplitItems}(n,s,t) & : \text{given a set of items with positive integer sizes and total sum } nS, \text{ decide if there exists a partition into } n \text{ bins, where } s \text{ items may be split between two or more bins, such that } \\
& \quad \max(b_1, \ldots, b_n) \leq (1 + t)S.
\end{align*}

In the introduction, for simplicity, we presented the case $t = 0$. The following lemma shows that, without loss of generality, we can consider for splitting only the largest items.

\begin{lemma}
For every integer $s \geq 1$, for every partition with $s$ split items and bin sums $b_1, \ldots, b_n$, there exists a partition with the same bin sums $b_1, \ldots, b_n$ in which only the $s$ largest items are split.
\end{lemma}

\begin{proof}
Consider a partition in which some item $x$ is split between two or more bins, whereas some item $y > x$ is allocated entirely to some bin $i$. Construct a new partition as follows:

\begin{itemize}
\item Move item $x$ to bin $i$;
\item Remove from bin $i$, a fraction $\frac{x}{y}$ of item $y$;
\item Split that fraction of item $y$ among the other bins, in the same proportions as the previous sharing of item $x$.
\end{itemize}

All bin sums remain the same. Repeat the argument until only the largest items are split.
\end{proof}

\begin{theorem}
For any integers $n \geq 2$ and $d \geq 0$, there is a polynomial-time reduction from $\text{PartitionWithDIntervalTarget}(n,d)$, to $\text{PartitionWithSplitItems}(n,d,0)$.
\end{theorem}

\begin{proof}
Given an instance $I$ of the $\text{PartitionWithDIntervalTarget}(n,d)$ problem, we construct an instance $I'$ of $\text{PartitionWithSplitItems}(n,d,0)$ by adding $d$ items of size $nM$, where $nM$ is the size of the biggest item in $I$.

Assume first that $I$ has a $d$-possible partition. Then there are $n$ bins with a sum at most $S + dM$. Take the $d$ added items of size $nM$ and add them to the bins, possibly splitting some items between bins, such that the sum of each bin becomes exactly $S + dM$. This is possible because the sum of the items in $I'$ is $nS + dnM = n(S + dM)$. We have a 0-feasible partition of $I'$ with at most $d$ split items.

Assume second that there $I'$ has a 0-feasible partition. Then there are $n$ bins with sum $S + dM$. By Lemma 7 we can assume the split items are the largest ones, which are the $d$ added items of size $nM$. Remove these items to get a partition of $I$. Of course, each new bin sum is at most $S + dM$, so the partition is $d$-possible.
\end{proof}

Combining Theorem 8 with Theorem 6 gives:

\begin{corollary}
$\text{PartitionWithSplitItems}(n,s,0)$ is NP-hard for any integers $n \geq 3$ and $s \in \{0,1,\ldots,n - 3\}$.
\end{corollary}

\begin{theorem}
For any integers $n \geq 2$, $s \geq 0$ and rational $t \geq 0$, there is a polynomial-time reduction from $\text{PartitionWithSplitItems}(n,s,t)$, to $\text{PartitionWithDIntervalTarget}(n,d)$, for some $d \geq s$.
\end{theorem}
Proof. Given an instance $I$ of \textsc{PartitionWithSplitItems}(n, s, t), denote the sum of all items by $nS$ and the largest item size by $nM$. Construct an instance $I'$ of \textsc{PartitionWithDIntervalTarget}(n, d) by removing the $s$ largest items from $I$. Denote the sum of remaining items by $nS'$ for some $S' \leq S$, and the largest remaining item size by $nM'$ for some $M' \leq M$. Note that the size of every removed item is between $nM'$ and $nM$, so $sM' \leq S - S' \leq sM$.

Set $d := (S + tS - S')/M'$, so $S' + dM' = S + tS$. Note that $d \geq (S - S')/M' \geq s$.

Assume first that $I$ has a $t$-feasible partition with $s$ split items. By Lemma 7, we can assume that only the $s$ largest items are split. Therefore, removing the $s$ largest items results in a partition of $I'$ with no split items, where the sum in each bin is at most $S + tS = S' + dM'$. This is a $d$-possible partition of $I'$.

Assume second that $I'$ has a $d$-possible partition. In this partition, each bin sum is at most $S' + dM' = S + tS$, so it is a $t$-feasible partition of $I'$. To get a $t$-feasible partition of $I$, take the $s$ previously-removed items and add them to the bins, possibly splitting some items between bins, such that the sum in each bin remains at most $S + tS$. This is possible because the sum of all items is $nS \leq n(S + tS)$.

Combining Theorem 10 with Theorem 3 gives:

\begin{itemize}
  \item Corollary 11. \textsc{PartitionWithSplitItems}(n, s, t) is polynomial-time solvable for any integers $n \geq 3$, $s \geq n - 2$ and rational $t \geq 0$.
\end{itemize}

Using binary search, we can solve in weakly-polynomial time the optimization problem: find an $n$-partition with $s$ split items in which the largest sum is minimized, whenever $s \geq n - 2$. We do not know whether the problem can be solved in strongly-polynomial time.

5 Conclusion

We presented two variants of the multiway number partitioning problem. In the language of machine scheduling, the variants are: (a) finding a schedule that minimizes the makespan on $n$ identical machines when $s$ jobs can be split between machines; (b) finding a schedule in which the makespan is in a given interval. It may be interesting to study these two variants for scheduling on different machines. The equivalence between these variants (Section 4) does not necessarily hold, so these are two different problems.

In the language of fair item allocation, we have solved the problem of finding a fair allocation among $n$ agents with identical valuations, when some $s$ items may be split between agents. With identical valuations, "fair" simply means that each agent receives the same sum of values. When agents may have different valuations, there are various generalizations to this fairness notion, such as proportionality, envy-freeness or equitability. It may be interesting to develop algorithms for finding such allocations with a bounded number of split items.

Our analysis shows the similarities and differences between these two variants and the more common notion of FPTAS. One may view our results as introducing an alternative kind of approximation: instead of keeping the variables discrete and allowing a bounded deviation in the objective value, we keep the objective value optimal and allow a bounded number of continuous variables. It may be interesting to study this approximation notion in other optimization problems.
APPENDIX

A  Proof of Remark 2

Proof of Remark 2. We use the terminology of Woeginger [17].

The dynamic program. For $k = 1, \ldots, n$ define the input vector $X_k = [p_k]$, where $p_k$ is the size of item $k$. A state $S = [s_1, s_2, l_1, l_2]$ in $S_k$ encodes a partial allocation for the first $k$ items $I_1, \ldots, I_k$, where $s_i$ is the sum of items on bin $i \in \{1, 2\}$ in the partial allocation and $l_i$ is the number of items on bin $i$. The set $F$ contains two transition functions $F_1$ and $F_2$:

\[
F_1(p_k, s_1, s_2, l_1, l_2) = [s_1 + p_k, s_2, l_1 + 1, l_2]
\]
\[
F_2(p_k, s_1, s_2, l_1, l_2) = [s_1, s_2 + p_k, l_1, l_2 + 1]
\]

Intuitively, the function $F_i$ corresponds to putting item $k$ on bin $i$. Finally, set the minimization objective to

\[
G(s_1, s_2, l_1, l_2) = \begin{cases} \max\{s_1, s_2\}, & \text{if } l_1 = l_2 \\ \infty, & \text{otherwise} \end{cases}
\]

The initial state space $S_0$ is set to $\{[0, 0, 0, 0]\}$. This defines a Simple DP, with $\alpha = 1$ (the size of the input vectors) and $\beta = 4$ (the size of the state vectors).

Benevolence. We show that our problem is ex-benevolent, as defined at [17] Section 4. This requires defining a degree vector $D$ of size $\beta$, which determines how much each state is allowed to deviate from the optimal state. The sets $F$, $G$ and the vector $D$ should satisfy several conditions denoted by C.3(i), C.4(ii,iii,iv) and C.5.

We define the degree vector as $D = [1, 1, 0, 0]$. Note that the third and fourth coordinates correspond to the number of items in each bin, for which we need an exact number and not an approximation.

Condition C.3(i) is satisfied with $g = 1$: if a state $[s_1, s_2, l_1, l_2]$ is $[D, \Delta]$-close to another state $[s_1', s_2', l_1', l_2']$ (where $\Delta$ is a factor determined by the required approximation accuracy $\epsilon$), then we must have $l_1 = l_1'$ and $l_2 = l_2'$ because the degree of coordinates 3 and 4 is 0. Therefore, $G(s_1, s_2, l_1, l_2)$ is $[D, \Delta]$-close to $G(s_1', s_2', l_1', l_2')$.

Condition C.4 (i) holds since all functions in $F$ can be evaluated in polynomial time. C.4 (ii) holds since the cardinality of $F$ is a constant. C.4 (iii) holds since the cardinality of $S_0$ is a constant. C.4(iv) is satisfied for coordinates 1, 2 since their value is upper bounded by the sum of the items $nS$, so their logarithm is bounded by the size of the input. For coordinates 3, 4 (whose degree is 0), the condition is satisfied since their value is upper bounded by the number of items $n$.

Condition C.5 holds since, if a state $[s_1, s_2, l_1, l_2]$ is $[D, \Delta]$-close to another state $[s_1', s_2', l_1', l_2']$, by definition of $[D, \Delta]$-close [Section 2](2.1), we must have $l_1 = l_1'$ and $l_2 = l_2'$ because the degree of coordinates 3 and 4 is 0. At coordinates 1 and 2, both transition functions are polynomials of degree 1, so the condition holds by Lemma 4.1(i) of [17].

Hence, our problem is ex-benevolent. By the main theorem of [17], it has an FPTAS.

B  Proof of Lemma 4: Properties of instances with at least two almost-full bins

Proof of Lemma 4 property [c]. Pick a $t$-feasible partition with some $k \geq 3$ almost-full bins. Necessarily $k \leq n - 1$, since the sum of $n$ almost-full bins would be larger than $nS$.
It is sufficient to show that there exists a \( t \)-feasible partition with \( k - 1 \) almost-full bins; then we can proceed by induction down to 2. Assume w.l.o.g. that the almost-full bins are \( 1, \ldots, k \) and the not-almost-full bins are \( k + 1, \ldots, n \). By definition,

\[
b_1, \ldots, b_k \geq (1 + t) \cdot S / (1 + \epsilon) > ((1 + t) \cdot S)(1 - \epsilon) > (1 + t - 2\epsilon) \cdot S,
\]

so,

\[
b_{k+1} + \cdots + b_n = nS - b_1 - \cdots - b_k < nS - kS \cdot (1 + t - 2\epsilon) < (n - k - kt + 2\epsilon) \cdot S.
\]

Denote by \( b_{\min} \) the smallest bin sum (breaking ties arbitrarily). By the pigeonhole principle,

\[
b_{\min} < \frac{(n - k - kt + 2\epsilon) \cdot S}{n - k} = \left(1 - \frac{k}{n - k} t + \frac{k}{n - k} 2\epsilon \right) \cdot S.
\]

By assumption, all item sizes are at most \( nM \). Suppose we take one item from bin \( k \), and move it to \( b_{\min} \). Then after the move,

\[
b_{\min} < \left(1 - \frac{3}{n - 3} t + \frac{n - 1}{n - (n - 1)} 2\epsilon \right) \cdot S + nM = \left(1 - \frac{3}{n - 3} t + \frac{n - 1}{n - (n - 1)} 2\epsilon \right) \cdot S + ntS/d
\]

(since \( 3 \leq k \leq n - 1 \) and \( dM = tS \))

\[
\leq \left(1 - \frac{3}{n - 3} t + \frac{2n\epsilon - 2\epsilon}{n - 2} \right) \cdot S + \frac{n}{n - 2} tS
\]

(since \( d \geq n - 2 \))

\[
< \left(1 - \frac{2}{n - 2} t - 2\epsilon \right) \cdot S + \frac{n}{n - 2} tS
\]

(by definition of \( \epsilon \))

\[
< (1 + t - 2\epsilon) \cdot S,
\]

so \( b_{\min} \) remains not-almost-full.

If bin \( k \) is still almost-full, then we are in the same situation: we have exactly \( k \) almost-full bins. So we can repeat the argument, and move another item from bin \( k \) to the (possibly different) bin \( b_{\min} \). Since the number of items in bin \( k \) decreases with each step, eventually, it becomes not-almost-full. \( \triangleright \)

Now, we have a partition with exactly two almost-full bins, which we assume to be bins 1 and 2. The not-almost-full bins are bins \( 3, \ldots, n \). We now transform the partition so that the sum in each bin \( 3, \ldots, n \) is bounded within a small interval.

**Proof of Lemma 4, property (d).** For the lower bound, suppose there is some \( i \in \{3, \ldots, n\} \) with \( b_i < \left(1 - \frac{2}{n - 2} t - 2\epsilon \right) \cdot S \). Move an item from bin 1 to bin \( i \). Its new sum satisfies

\[
b_i < \left(1 - \frac{2}{n - 2} t - 2\epsilon \right) \cdot S + nM = \left(1 - \frac{2}{n - 2} t - 2\epsilon \right) \cdot S + ntS/d
\]

\[
= \left(1 + \left(\frac{2 - 2}{n - 2} + \frac{n}{d}\right) t - 2\epsilon \right) \cdot S
\]

\[
\leq \left(1 + \left(\frac{2 - 2}{n - 2} + \frac{n}{n - 2}\right) t - 2\epsilon \right) \cdot S
\]

\[
= (1 + t - 2\epsilon) \cdot S,
\]

so bin \( i \) is still not almost-full. Assumption (a) implies that \( b_1, b_2 \) must still be almost-full. So we are in the same situation and can repeat the argument until the lower bound is satisfied.
The upper bound on \( b_i \) is proved by simply subtracting the lower bounds of the other \( n - 1 \) bins from the sum of all items, \( nS \):

\[
b_i \leq nS - (b_1 + b_2) - (n - 3) \left( 1 - \frac{2}{n - 2} t - 2\epsilon \right) \cdot S
\]

\[
< nS - 2((1 + t - 2\epsilon) \cdot S) - (n - 3) \left( 1 - \frac{2}{n - 2} t - 2\epsilon \right) \cdot S
\]

\[
= \left( n - 2(1 + t - 2\epsilon) - (n - 3) \left( 1 - \frac{2}{n - 2} t - 2\epsilon \right) \right) \cdot S
\]

\[
= \left( 1 - \frac{2}{n - 2} t + (n - 1)2\epsilon \right) \cdot S,
\]

so property (6) is satisfied.

A corollary of property (d) is that the difference between the sums of two almost-full bins is at most \( 2nS \cdot 2\epsilon = nS \cdot t/2m^2 \).

Next, we transform the partition such that the almost-full bins 1 and 2 contain only “big” items, which we define as items larger than \( nS \left( \frac{t}{n - 2} - 2\epsilon \right) \).

**Proof of Lemma 4, property (f).** Suppose there exists some \( i \in \{3, \ldots, n\} \) for which \( b_i \) contains an item \( x_i \) such that \( x_i > 2nS\epsilon \), but \( x_i < x_{12} \) for some item \( x_{12} \) in bin 1 or 2. We exchange \( x_i \) and \( x_{12} \),

\[
b_i < \left( 1 - \frac{2}{n - 2} t + (n - 1)2\epsilon \right) \cdot S + nM - 2nS\epsilon
\]

\[
< \left( 1 - \frac{2}{n - 2} t + (n - 1)2\epsilon \right) \cdot S + \frac{ntS}{d} - 2nS\epsilon \quad \text{(since } dM = tS)\]

\[
< \left( 1 - \frac{2}{n - 2} t + (n - 1)2\epsilon \right) \cdot S + \frac{nS}{n - 2} - 2nS\epsilon \quad \text{(since } d \leq n - 2)\]

\[
< (1 + t - 2\epsilon) \cdot S,
\]

then, \( b_i \) is not almost-full. So by (f), bin 1 (or 2) must remain almost-full, and we can repeat the argument. Each move increases the sum in bins 3, \ldots, \( n \), so the process must end.
Proof of Lemma 4 property (g). Assume for contradiction that two not-almost-full bins contain a different number of big-items; say bin $j$ has at least one big-item more than bin $i$.

We first claim that, for any two big-items $x_i, x_j$, the difference $|x_j - x_i| < 2nS\epsilon$. W.l.o.g $x_j > x_i$. For any item, $x_j \leq nM$. For any big-item, $x_i > nS\left(\frac{t}{n-2} - 2\epsilon\right)$. So

$$x_j - x_i < nM - nS\left(\frac{t}{n-2} - 2\epsilon\right)$$

$$= \frac{nSt}{d} - \frac{nSt}{n-2} + 2nS\epsilon$$

(since $dM = tS$)

$$\leq 2S\epsilon n.$$  

(since $d \geq n-2$)

So even if all the big-items in $j$ are smaller than all big-items in $i$, and all the (at most $m$) small-items are in $i$, the difference between them satisfies

$$b_j - b_i > nS\left(\frac{t}{n-2} - 2\epsilon\right) - m \cdot 2n\epsilon S - m \cdot 2n\epsilon S$$

$$= nS\left(\frac{t}{n-2} - 2\epsilon\right) - 2m \cdot 2n\epsilon S$$

$$= nS\left(\frac{t}{n-2} - 2\epsilon(1 + 2m)\right)$$

$$\geq \frac{nS}{m-2} \frac{2t}{m}$$

(by $\epsilon$ definition.)

$$= nS\left(\frac{2t}{(m-2)m}\right)$$

$$> 2nSt/m^2.$$  

But by property (h), the difference is at most $2nS\epsilon = nSt/2m^2$. ▶

Proof of Lemma 4 property (h). The size of every item is at most $nM \leq ntS/(n-2)$. By property (g), each not-almost-full bin contains $\ell$ big-items. Then by the previous conditions:

$$b_i \leq \ell \cdot ntS/(n-2) + m \cdot 2n\epsilon S$$

(7)

We focus on bin 1; the proof for bin 2 is the same. If bin 1 contains fewer than $\ell + 1$ items, then $b_1 \leq b_i$, which contradicts (h).

Assume for contradiction that bin 1 contains more than $\ell + 1$ items. Then:

$$b_1 \geq (\ell + 2) \cdot \left(nS\left(\frac{t}{n-2} - 2\epsilon\right)\right)$$

(by big-item size)

$$= \ell ntS/(n-2) + m \cdot 2n\epsilon S - m \cdot 2n\epsilon S - \ell ntS2\epsilon + 2ntS/(n-2) - 4nS\epsilon$$

$$\geq b_i - m \cdot 2n\epsilon S - \ell ntS2\epsilon + 2ntS/(n-2) - 4nS\epsilon$$

(by 7)

$$\geq \left(1 - \frac{2}{n-2} t - 2\epsilon\right) S - m \cdot 2n\epsilon S - \ell ntS2\epsilon + 2ntS/(n-2) - 4nS\epsilon$$

(by 5)

$$= S + \left(\frac{2n - 2}{n-2}\right) tS - m \cdot 2n\epsilon S - \ell ntS2\epsilon - 4nS\epsilon - 2\epsilon S$$

$$= S + tS + tS((n/n-2) - m \cdot 2\epsilon/t - \ell ntS2\epsilon - 4nS\epsilon/t - 2\epsilon/t)$$

$$> S + tS + tS(1 - 1/2 - \ell/2m - 1/m - 1/2nm)$$

(since $\epsilon < t/4nm$)

$$> S + tS,$$  

(since $0 < \ell < m/3$)

which contradicts the assumption that $(b_1, b_2, \ldots, b_n)$ represent a $t$-feasible allocation. ▶
Proof of Theorem 12: Hardness of Partitioning with Splittings

Theorem 12. \textsc{PartitionWithSplittings}(n,s) is NP-Hard for all $s \leq n - 2$.

Proof of Theorem 12. The proof is by reduction from the subset sum problem. Given an instance $I_1$ of the subset sum problem with $m$ items summing to $S$ with target sum $T < S$, we build an instance $I_2$ of the problem \textsc{PartitionWithSplittings}(n,s), by adding two items, $x_1, x_2$ such that $x_1 = S + T$ and $x_2 = 2S(s + 1) - T$ and $n - 2 - s$ auxiliary items of size $2S$. Notice that the sum of the items in $I_2$ equals

$$S + (S + T) + 2S(s + 1) - T + 2S(n - 2 - s)$$
$$= 2S + 2S(s + 1) + 2S(n - 2 - s)$$
$$= 2S \cdot (1 + s + 1 + n - 2 - s)$$
$$= 2Sn.$$

The goal is to partition the items into $n$ bins with a sum of $2S$ per bin, with at most $s$ splittings.

Assume first that there is a subset $W_1$ in $I_1$ with sum equal to $T$. Define a set $W_2$ that contains all items in $I_1$ that are not in $W_1$, plus $x_1$. The sum in $W_2$ is

$$(S - T) + x_1 = S + T + S - T = 2S,$$

Assign the items of $W_2$ to bin #1. Assign each auxiliary item to a different bin. There are $n - (n - 2 - s + 1) = s + 1$ bins left. The sum of the remaining items is $2S(s + 1)$. As explained in the introduction, they can be partitioned into $s + 1$ bins of equal sum $2S$, with at most $s$ splittings. All in all, there are $n$ bins with a sum of $2S$ per bin, and the total number of splittings is $s$.

Assume second that there exists an equal partition for $n$ bins with $s$ splittings. Since $x_2 = 2S(s + 1) - T = 2S \cdot s + (2S - T) > 2S \cdot s$, this item must be split between $s + 1$ bins, which makes the total number of splittings at $s$. Also, the auxiliary items must be assigned without splittings into $n - 2 - s$ different bins. There is $n - s - 1 - n + 2 + s = 1$ bin remaining, say bin $i$, containing only whole items, not containing any part of $x_2$, and not containing any auxiliary item. Bin $i$ must contain $x_1$, otherwise its sum is at most $S$ (sum of items in $I_1$). Let $W_1$ be the items of $I_1$ that are not in bin $i$. The sum of $W_1$ is $S - (2S - x_1) = x_1 - S = T$, so it is a solution to $I_1$. ▶
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