ON A CLASS OF ALGEBRAS ASSOCIATED TO DIRECTED GRAPHS

ISRAEL GELFAND, VLADIMIR RETAKH\(^1\),
SHIRLEI SERCONEK\(^2\) AND ROBERT LEE WILSON

Abstract

To any directed graph we associate an algebra with edges of the graph as generators
and with relations defined by all pairs of directed paths with the same origin and terminus.
Such algebras are related to factorizations of polynomials over noncommutative algebras.
We also construct a basis for our algebras associated to layered graphs.

0. Introduction

Factorizations of noncommutative polynomials play an important role in many areas of
mathematics such as operator theory, integrable systems, and Yang-Baxter equations (see,
for example, \([O, V]\)). In this paper we use directed graphs and algebras associated with
those graphs as a natural framework for studying such factorizations.

Let \(R\) be an associative ring with unit and \(P(\tau) \in R[\tau]\) a polynomial over \(R\), where \(\tau\)
is a central variable. Assume that \(P(\tau)\) is a monic polynomial (i.e. the leading coefficient
in \(P(\tau)\) equals 1). When \(R\) is a (commutative) field there exists at most one (up to
rearrangement of the factors) factorization of a monic polynomial \(P(\tau)\) of degree \(n\) into a
product of linear polynomials:

\[
P(\tau) = (\tau - y_n)(\tau - y_{n-1}) \cdots (\tau - y_1)
\]

When the ring \(R\) is not commutative, there may exist many factorizations of type (0.1).

In \([GGRSW]\) the elements \(y_1, y_2, \ldots, y_n\) in formula (0.1) where called the pseudoroots
of the polynomial \(P(\tau)\). The element \(y_1\) is a right root of \(P(\tau)\) and the element \(y_n\) is a left
root of \(P(\tau)\).

Let \(X = \{x_1, x_2, \ldots, x_n\} \subseteq R\) be a generic set of right roots of \(P(\tau)\) (meaning that if
\(k \geq 2\) and \(\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subseteq X\), the corresponding Vandermonde matrix is invertible).
It was shown in \([GR1]\) (see also \([GR2, GGRW]\)) that the right roots \(x_1, x_2, \ldots, x_n\) define a

\(^1\)partially supported by NSA
\(^2\)partially supported by CNPq/PADCT

Typeset by \(\text{\LaTeX}\)
set of pseudoroots $x_{A,i}$ of $P(\tau)$. Here $A \subseteq \{1, 2, \ldots, n\}$, $i \in \{1, 2, \ldots, n\} \setminus A$, and $x_{\emptyset,j} = x_j$ for all $j$.

According to [GR1] (see also [GR2, GGRW]) for any ordering $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$ the pseudoroots $y_k = x_{\{i_1,\ldots,i_{k-1}\},i_k}$, $k = 1, 2, \ldots, n$ define a factorization (0.1).

The pseudoroots $x_{A,i}$ satisfy the following identities for any $i, j \not\in A$

\[(0.2a)\quad x_{A\cup\{i\},j} + x_{A,i} = x_{A\cup\{j\},i} + x_{A,j}\]

\[(0.2b)\quad x_{A\cup\{i\},j} x_{A,i} = x_{A\cup\{j\},i} x_{A,j}\]

The paper [GRW] introduces and studies the algebra $Q_n$, called the universal algebra of pseudoroots, generated by elements $x_{A,i}$ satisfying the identities (0.2).

The study of $Q_n$ arose from the theory of quasideterminants and, specifically, from the noncommutative version of the Viète Theorem and its relations to the theory of noncommutative symmetric functions (see, for example, [GGRW]). In particular, the algebra $Q_n$ is quadratic, Koszul, and its dual algebra $Q'_n$ has finite dimension (see [GGRSW, SW]).

A natural description of the algebra $Q_n$ (and, therefore, factorizations of noncommutative polynomials) can be given by using directed graphs. Let $\Gamma_n$ be the Hasse graph corresponding to the lattice of subsets of the set $\{1, 2, \ldots, n\}$. Vertices of $\Gamma_n$ are subsets $A \subseteq \{1, 2, \ldots, n\}$. Edges of $\Gamma_n$ are defined by pairs $(A,i)$ where $i \in \{1, 2, \ldots, n\} \setminus A$. Such edges go from the vertex $A \cup \{i\}$ to the vertex $A$.

The graph $\Gamma_n$ possesses several properties. It is a hypercube with $2^n$ vertices. It is a layered graph: each vertex $A$ has a level $|A| = \text{card } A$. There is only one vertex $\{1, 2, \ldots, n\}$ of level $n$ and only one vertex $\ast$ of level 0 and any edge $a$ goes from a vertex of level $r$ to a vertex of level $r-1$.

One can describe the relations in the algebra $Q_n$ by using geometric properties of the directed graph $\Gamma_n$. Fix a field $F$. Let $T$ be the free associative algebra over $F$ generated by the edges of $\Gamma_n$. Then the algebra $Q_n$ is a quotient algebra of $T$ modulo relations (0.2).

To every directed path $\pi = (a_1, a_2, \ldots, a_m)$ in $\Gamma_n$ (where $a_1, \ldots, a_m$ are edges of $\Gamma_n$) there is a corresponding polynomial $P_\pi(\tau) \in T[\tau]$, $P_\pi(\tau) = (\tau - a_1)(\tau - a_2)\ldots(\tau - a_m)$. Denote the image of $P_\pi(\tau)$ in $Q_n[\tau]$ by $\tilde{P}_\pi(\tau)$. If two paths $\pi_1$ and $\pi_2$ both go from a vertex $v$ to a vertex $v'$ then

\[(0.3)\quad \tilde{P}_{\pi_1}(\tau) = \tilde{P}_{\pi_2}(\tau)\]

in $Q_n[\tau]$.

The identity (0.3) defines relations in $Q_n$. Note that the defining relations (0.2) arise from those pairs of paths forming a diamond, i.e. a path $\pi_1$ consisting of edges going from $A \cup \{i, j\}$ to $A \cup \{i\}$ and from $A \cup \{i\}$ to $A$, and a path $\pi_2$ consisting of edges going from $A \cup \{i, j\}$ to $A \cup \{j\}$ and from $A \cup \{j\}$ to $A$. 

2
In this paper we introduce and study a class of algebras associated to certain directed graphs as a natural generalization of the geometric definition of the algebra $Q_n$. It is more convenient for us to use formulas similar to (0.3) by introducing a new variable $t = \tau^{-1}$.

Let $\Gamma = (V, E)$ be a directed graph with vertices $V$ and edges $E$. Assume that $\Gamma$ is layered and that the maximum level is $n$. Fix a field $F$ and denote by $T(E)$ the free associative algebra over $F$ generated by all edges. To any path $\pi = (e_1, e_2, \ldots, e_k)$ in $\Gamma$ corresponds a polynomial
\[
P_\pi(t) = (1 - te_1)(1 - te_2) \cdots (1 - te_k) \in T(E)[t]/(t^{n+1}).
\]
We define the algebra $A(\Gamma)$ to be the quotient algebra of $T(E)$ modulo relations defined by the equalities implied by
\[
(0.4) \quad P_{\pi_1}(t) = P_{\pi_2}(t)
\]
where the paths $\pi_1$ and $\pi_2$ both go from a vertex $v$ to a vertex $v'$.

The algebra $A(\Gamma_n)$ coincides with the algebra $Q_n$.

Our main examples of directed graphs are the Hasse graphs associated with lattices of subsets and subspaces, abstract polytopes, complexes, and partitions.

We have already mentioned that the algebra $Q_n$, its subalgebras, quotient algebras, and dual algebras possess many interesting properties (see [GRW, GGR, GGRSW, SW, Pi]). We believe that the same is true for algebras associated to other directed graphs. In particular, we are planning to relate the structure of the algebra $A(\Gamma)$ with the geometry of the graph.

As a first step in these directions, we construct in this paper a linear basis for the algebra $A(\Gamma)$ for a large class of directed graphs $\Gamma$. This basis has a “geometric” nature. In particular, our result simplifies the construction of the basis for $Q_n$ given in [GRW].

We now briefly describe this basis. For each vertex $v \in V$ with level $|v| > 0$, we choose (arbitrarily) an edge $e_v$ beginning at the vertex $v$. This defines a unique path $\pi_v$ from $v$ to $\ast$. In $A(\Gamma)$ write $\tilde{e}$ for the image of $e \in E$ and
\[
P_{\pi_v}(t) = \sum_{k=0}^{|v|} \tilde{e}(v, k)t^k.
\]
We say that two pairs $(v, k), (w, m) \in V \times \mathbb{Z}_{\geq 0}$ can be composed (and write $(v, k) \triangleright (w, m)$) if there is a path of length $k - m$ from $v$ to $w$. Let $\mathcal{B}(\Gamma)$ be the set of all sequences
\[
b = ((b_1, m_1), (b_2, m_2), ..., (b_k, m_k))
\]
where $k \geq 0, b_1, b_2, ..., b_k \in V, 1 \leq m_i \leq |b_i|$ for $1 \leq i \leq k$, and $(b_i, m_i) \neq (b_{i+1}, m_{i+1})$ for $1 \leq i < k$.

For $b = ((b_1, m_1), ..., (b_k, m_k)) \in \mathcal{B}(\Gamma)$ set
\[
\tilde{e}(b) = \tilde{e}(b_1, m_1)...\tilde{e}(b_k, m_k).
\]
Theorem 4.3. Let $\Gamma = (V, E)$ be a layered graph, $V = \bigcup_{i=0}^{n} V_i$, and $V_0 = \{\ast\}$ where $\ast$ is the unique minimal vertex of $\Gamma$. Then $\{\bar{e}(b) \mid b \in B(\Gamma)\}$ is a basis for $A(\Gamma)$.

An equivalent formulation of this theorem may be obtained by replacing each "generating function coefficient" $\bar{e}(v, k)$ by the "monomial" $\tilde{e}_{v(0)} \tilde{e}_{v(1)} \ldots \tilde{e}_{v(k-1)}$ where $v(0), v(1), \ldots, v(k) = \ast$ are the vertices of the path $\pi_v$. (This monomial is the leading term of $\bar{e}(v, k)$ in an appropriate filtration.) This "monomial" formulation of our basis theorem, when specialized to the graph $\Gamma_n$, gives the basis theorem of [GRW] for $Q_n$.

1. The directed graph $\Gamma = (V, E)$

Let $\Gamma = (V, E)$ be a directed graph. That is, $V$ is a set (of vertices), $E$ is a set (of edges), and $t : E \rightarrow V$ and $h : E \rightarrow V$ are functions. ($t(e)$ is the tail of $e$ and $h(e)$ is the head of $e$.)

We will assume throughout the remainder of the paper that $\Gamma = (V, E)$ is a layered graph with $V = \bigcup_{i=0}^{n} V_i$, and $V_0 = \{\ast\}$ where $\ast$ is the unique minimal vertex of $\Gamma$ (i.e., for every $v \in V, v \neq \ast$, there exists $e \in E$ with $t(e) = v$).

For each $v \in \bigcup_{i=1}^{n} V_i$ we will fix, arbitrarily, some $v_e \in E$, with $t(e) = v$. If $v \in V_i$ we write $|v| = i$ and say that $v$ has level $i$. Similarly, if $e \in E_i$ we write $|e| = i$ and say that $e$ has level $i$.

If $v, w \in V$, a path from $v$ to $w$ is a sequence of edges $\pi = \{e_1, e_2, \ldots, e_k\}$ with $t(e_1) = v$, $h(e_k) = w$ and $t(e_{i+1}) = h(e_i)$ for $1 \leq i < k$. We write $v = t(\pi)$, $w = h(\pi)$. We also write $v > w$ if there is a path from $v$ to $w$.

Let $l(\pi)$, the length of $\pi$, denote $k$, and let $|\pi|$, the level of $\pi$, denote $|e_1| + \ldots + |e_k|$.

If $\pi_1 = \{e_1, \ldots, e_k\}, \pi_2 = \{f_1, \ldots, f_l\}$ are paths with $h(\pi_1) = t(\pi_2)$ then $\{e_1, \ldots, e_k, f_1, \ldots, f_l\}$ is a path; we denote it by $\pi_1 \pi_2$.

For $v \in V$, write $v^{(0)} = v$ and define $v^{(i+1)} = h(v^{(i)})$ for $0 \leq i < |v|$. Then $v^{(|v|)} = \ast$ and $\pi_v = \{v^{(0)}, \ldots, v^{(|v| - 1)}\}$ is a path from $v$ to $\ast$.

2. The filtered algebra $T(E)$

Let $T(E)$ denote the free associative algebra on $E$ over a field $F$. Define

$$T(E)_i = \text{span}\{e_1 \ldots e_r \mid r \geq 0, |e_1| + \ldots + |e_r| \leq i\}.$$ 

If $a \in T(E)_i$, $a \notin T(E)_{i-1}$, write $|a| = i$. For a path $\pi = \{e_1, e_2, \ldots, e_k\}$ define

$$P_\pi(t) = (1 - te_1) \ldots (1 - te_k) \in T(E)[t]/(t^{n+1}).$$

4
Note that $P_{\pi_1\pi_2}(t) = P_{\pi_1}(t)P_{\pi_2}(t)$ if $h(\pi_1) = t(\pi_2)$. Write

$$P_{\pi}(t) = \sum_{k=0}^{l(\pi)} (-1)^k e(\pi, k) t^k.$$  

Set $e(\pi, k) = 0$ if $k > l(\pi)$. For $v \in \cup_{i=1}^n V_i$, set $P_v(t) = P_{\pi_v}(t)$ and $e(v, k) = e(\pi_v, k)$. Also, set $P_\ast(t) = 1$ and $e(\ast, k) = 0$ if $k > 0$.

**Definition 2.1** Let $R$ be the ideal in $T(E)$ generated by

$$\{ e(\pi_1, k) - e(\pi_2, k) \mid t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2), 1 \leq k \leq l(\pi_1) \}.$$  

Note that this implies

$$P_{\pi_1}(t) \equiv P_{\pi_2}(t) \mod R[t].$$  

Now assume $v > u$, so there is a path $\pi$ from $v$ to $u$. Then

$$t(\pi_{\pi u}) = v = t(\pi_v),$$  

$$h(\pi_{\pi u}) = * = h(\pi_v)$$  

and so

$$P_{\pi_{\pi u}}(t) \equiv P_{\pi_v}(t) \mod R[t].$$  

But

$$P_{\pi_{\pi u}}(t) = P_{\pi}(t)P_{\pi_u}(t) = P_{\pi}(t)P_{\pi_u}(t),$$  

$$P_{\pi_u}(t) = P_v(t)$$  

so

$$P_{\pi}(t) \equiv P_v(t)P_{\pi_u}(t)^{-1} \mod R[t].$$  

Noting that $P_{\pi}(t)$ is a polynomial of degree $l(\pi) = |v| - |u|$ and writing $(1 - a)^{-1} = 1 + a + a^2...$ for a nilpotent element $a$, we obtain

$$P_{\pi}(t) \equiv \sum_{r \geq 0, i_0 \geq 0, i_1, \ldots, i_r \geq 1} (-1)^{i_0 + \ldots + i_r + r} e(v, i_0)e(u, i_1)e(u, i_1)\ldots e(u, i_r) t^{i_0 + \ldots + i_r}$$  

$$\equiv \sum_{j=0}^{l(\pi)} \left( \sum_{r \geq 0, i_0 \geq 0, i_1, \ldots, i_r \geq 1, i_0 + \ldots + i_r = j} (-1)^{i_0 + \ldots + i_r} e(v, i_0)e(u, i_1)e(u, i_1)\ldots e(u, i_r) t^j \right) \mod R[t].$$  

5
Let

\[ H(v, u, j) = \sum_{r, i_0, \ldots, i_r \geq 0, i_0 + \ldots + i_r = j} (-1)^{j+r} e(v, i_0) e(u, i_1) \cdots e(u, i_r) \]

so that

\[ P_{\pi}(t) \equiv \sum_{j=0}^{l(\pi)} H(v, u, j) t^j \mod R[t]. \]

It follows that, modulo \( R[t] \),

\[ P_{v}(t) \equiv P_{\pi}(t)P_{u}(t) \]

\[ \equiv \left( \sum_{j=0}^{l(\pi)} H(v, u, j) t^j \right) \left( \sum_{i_{r+1}=0}^{|u|} (-1)^{i_{r+1}} e(u, i_{r+1}) t^{i_{r+1}} \right) \]

\[ \equiv \sum_{r, i_0, i_{r+1} \geq 0, i_0 + \ldots + i_r \leq |v| - |u|, i_0 + \ldots + i_{r+1} \leq |v|} (-1)^{i_0 + \ldots + i_{r+1} + r} e(v, i_0) e(u, i_1) \cdots e(u, i_{r+1}) t^{i_0 + \ldots + i_{r+1}} \mod R[t]. \]

Setting \( k = |v| - |u| \) and comparing coefficients of \( t^{k+l} \) gives

\[ e(v, k + l) \equiv e(v, k) e(u, l) + \]

(2.1) \[ \sum_{r, i_0, i_{r+1} \geq 0, i_1, \ldots, i_r \geq 1, i_0 < k, i_0 + \ldots + i_r \leq k, i_0 + \ldots + i_{r+1} = k + l} (-1)^r e(v, i_0) e(u, i_1) \cdots e(u, i_{r+1}) \mod R. \]

Writing

\[ E(v, u, k, l) = \sum_{r, i_0, i_{r+1} \geq 0, i_1, \ldots, i_r \geq 1, i_0 < k, i_0 + \ldots + i_r \leq k, i_0 + \ldots + i_{r+1} = k + l} (-1)^r e(v, i_0) e(u, i_1) \cdots e(u, i_{r+1}) \]

we obtain

\[ e(v, k + l) - e(v, k) e(u, l) \equiv E(v, u, k, l) \mod R \]

when \( v > u, |v| - |u| = k \).

We also have
Lemma 2.1. $E(v, u, 1, l) = -e(u, 1)e(u, l) + e(u, l + 1)$.

Proof: Setting $k = 1$ in the sum defining $E(v, u, k, l)$ gives $i_0 = 0, r = 0$ (hence $i_1 = l + 1$) or $i_0 = 0, r = 1$ (hence $i_1 = l, i_2 = l$). These two choices give the two terms on the right-hand side.

Note that $|e(v, k)| = k|v| - k(k - 1)/2$, and so, if $k = |v| - |u|$, $|e(v, k)e(u, l)| = |e(v, k)| + |e(u, l)|$

$= k|v| - k(k - 1)/2 + l|u| - l(l - 1)/2$

$= (k + l)|v| - (k + l)(k + l - 1)/2$

$= |e(v, k + l)|$.

We also have

Lemma 2.2. If $|v| - |u| = k$, $|E(v, u, k, l)| < |e(v, k + l)|$.

Proof: It is sufficient to show that each summand in the expression for $E(v, u, k, l)$ belongs to $T(E)|_e(v, k + l)| = 1$. Thus we must show

$i_0|v| + (k + l - i_0)|u| - i_0(i_0 - 1)/2 - ... - i_r + 1(i_r + 1 - 1)/2 < (k + l)|v| - (k + l)(k + l - 1)/2$

where $r \geq 0, i_0, i_r + 1 \geq 0, i_1, ..., i_r \geq 1, i_0 + ... + i_r < k$, $i_0 < k$ and $i_0 + ... + i_r + 1 = k + l$.

Since $|u| = |v| - k$ this is equivalent to

$-(k + l - i_0)k - (i_0^2 + ... + i_r^2 - k - l)/2 < -(k + l)(k + l - 1)/2$

which may be simplified to $(k - i_0)^2 + ... + i_r^2 > l^2$. This holds since $i_r + 1 \leq l$ and $i_0 < k$.

Lemma 2.3. a) If $f \in E$ then $f - e(t(f), 1) + e(h(f), 1) \in R$.
b) If $v > u$, $|u| = |v| - 1$ then $e(v, 1)e(u, k) - e(v, k + 1) + e(u, k + 1) - e(u, 1)e(u, k) \in R$.

Proof: a) $P_{t(f)}(t) \equiv P_{f \pi_{h(f)}}(t) = (1 - tf)P_{h(f)}(t) \mod R[t]$, so $e(t(f), 1) \equiv f + e(h(f), 1) \mod R$.
b) Since $v > u$ and $|u| = |v| - 1$ there is $f \in E$ with $t(f) = v, h(f) = u$. Then

$P_v(t) \equiv P_{f \pi_{h(f)}}(t) = (1 - tf)P_u(t)$

$\equiv (1 + te(v, 1) - te(u, 1))P_u(t) \mod R[t]$.

Thus

$e(v, k + 1) \equiv e(u, k + 1) - e(v, 1)e(u, k) + e(u, 1)e(u, k) \mod R$.

7
Lemma 2.4. \( e(v, 1)E(u, w, k, l) - e(u, 1)E(u, w, k, l) \equiv E(v, w, k+1, l) - E(u, w, k+1, l) \mod R. \)

**Proof:** By the definition, the left-hand side (LHS) is
\[
\sum (-1)^r e(v, 1)e(u, i_0)e(w, i_1)\ldots e(w, i_{r+1}) - \sum (-1)^r e(u, 1)e(u, i_0)e(w, i_1)\ldots e(w, i_{r+1}),
\]
where the sums are over \( r \geq 0, i_0, i_{r+1} \geq 0, i_1, \ldots, i_r \geq 1, i_0 < k, i_0 + \ldots + i_r \leq k \) and \( i_0 + \ldots + i_{r+1} = k + l. \)

By Lemma 2.3, \( e(v, 1)e(u, i_0) - e(u, 1)e(u, i_0) \equiv e(v, i_0 + 1) - e(u, i_0 + 1) \mod R. \) Thus the LHS is congruent to
\[
\sum (-1)^r e(v, i_0)e(w, i_1)\ldots e(w, i_{r+1}) - \sum (-1)^r e(u, i_0)e(w, i_1)\ldots e(w, i_{r+1}),
\]
where the summation is over all \( r \geq 0, i_0 \geq 1, i_{r+1} \geq 0, i_1, \ldots, i_r \geq 1, i_0 < k + 1, i_0 + \ldots + i_r \leq k + 1 \) and \( i_0 + \ldots + i_{r+1} = k + l + 1. \)

But this is equal to the same expression where the sum is over all \( r \geq 0, i_0 \geq 0, i_{r+1} \geq 0, i_1, \ldots, i_r \geq 1, i_0 < k + 1, i_0 + \ldots + i_r \leq k + 1 \) and \( i_0 + \ldots + i_{r+1} = k + l + 1 \) (since \( e(v, 0) = e(u, 0) = 1 \).) This is the right hand side of the asserted congruence.

Lemma 2.5. \( R \) is generated by all \( e(\pi_1, k) - e(\pi_2, k), t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2) = *. \)

**Proof:** Let \( S \) be the ideal generated by all such elements. Thus, for such \( \pi_1, \pi_2, \)
\[
P_{\pi_1}(t) \equiv P_{\pi_2}(t) \mod S[t].
\]

Let \( P_{\pi_3}(t), P_{\pi_4}(t) \in T(E)[t]/(t^{n+1}) \) satisfying \( t(\pi_3) = t(\pi_4), h(\pi_3) = h(\pi_4) = w. \) Then \( t(\pi_3 w) = t(\pi_3) = t(\pi_4) = t(\pi_4 w), h(\pi_3 w) = h(\pi_4 w) = * = h(\pi_4 \pi_w) \) and
\[
P_{\pi_3}(t)P_{\pi_w}(t) = P_{\pi_3 \pi_w}(t) = P_{\pi_4}(t)P_{\pi_w}(t) \mod S[t].
\]

Then since \( P_{\pi_w}(t) \) is invertible, \( P_{\pi_3}(t) \equiv P_{\pi_4}(t) \mod S[t]. \) Since the coefficients of all \( P_{\pi_3}(t) - P_{\pi_4}(t) \) generate \( R, \) we have the result.

Lemma 2.6. Let \( S_1 = \{ f - e(t(f), 1) + e(h(f), 1) | f \in E \} \) and
\[
S_2 = \{-e(v, 1)e(u, k) - e(v, k+1) + e(u, k+1) + e(u, 1)e(u, k) | u, v \in V, v > u, |u| = |v| - 1 > 0 \}.
\]

Then \( S_1 \cup S_2 \) generates \( R. \)

**Proof:** By Lemma 2.3, \( S_1 \cup S_2 \subseteq R. \) Let \( v \in V, |v| > 0 \) and let \( \pi = \{ e_1, e_2, ..., e_{|v|} \} \) be a path from \( v \) to \(*. \) By lemma 2.5, it is sufficient to show that \( P_{\pi}(t) \equiv P_{\pi}(t) \mod (S_1 \cup S_2)[t]. \)
We proceed by induction on \( |v| \). If \( |v| = 1 \), then \( \pi = \{ e_1 \} \) where \( t(e_1) = v, h(e_1) = \ast \). Then \( P_\pi(t) = 1 - te_1 \equiv 1 + e(v, 1) - e(\ast, 1) \equiv 1 - te_0 \equiv P_v(t) \mod S_1[t] \).

Now assume \( |v| > 1 \) and write \( \pi = e_1\pi' \) with \( t(e_1) = v, h(e_1) = u \). Then \( |u| < |v| \) and so by induction we have \( P_u(t) \equiv P_{\pi'}(t) \mod (S_1 \cup S_2)[t] \). Since

\[
P_v(t) \equiv (1 - te(v, 1) + te(u, 1))P_u(t) \mod S_2[t]
\]

and

\[
1 - te(v, 1) + te(u, 1) \equiv 1 - te_1 \mod S_1[t]
\]

we have

\[
P_v(t) \equiv (1 - te_1)P_u(t) \equiv (1 - te_1)P_{\pi'}(t) \equiv P_\pi(t) \mod (S_1 \cup S_2)[t],
\]

as required.

### 3. The algebra \( A(\Gamma) \)

Let

\[
A(\Gamma) = T(E)/R
\]

and

\[
A(\Gamma)_i = (T(E)_i + R)/R.
\]

This gives \( A(\Gamma) \) the structure of a filtered algebra. Let \( \tilde{\cdot} \) denote the canonical homomorphism \( \tilde{\cdot}: T(E) \to T(E)/R = A(\Gamma) \) and write \( \tilde{e}(v, k), \tilde{E}(v, u, k, l), \) etc., for the images of the elements \( e(v, k), E(v, u, k, l) \), etc. If \( a \in A(\Gamma)_i, a \notin A(\Gamma)_{i-1} \), write \( |a| = i \).

Note that

\[
\{ \tilde{e}(v, k) \mid v \in \bigcup_{i=1}^n V_i, \ k \leq |v| \}
\]

generates \( A(\Gamma) \), since by Lemma 2.3a), \( f = \tilde{e}(t(f), 1) - \tilde{e}(h(f), 1) \) for any \( f \in E \).

We now develop some notation for products of the \( \tilde{e}(v, k) \).

We say that a pair \((v, k), v \in V, 0 \leq k \leq |v|\), can be composed with the pair \((u, l)\), \( u \in V, 0 \leq l \leq |u| \), if \( v > u \) and \( |u| = |v| - k \). If \((v, k)\) can be composed with \((u, l)\) we write \((v, k) \models (u, l)\).

Let \( B_1(\Gamma) \) be the set of all sequences

\[
b = ((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k))
\]

where \( k \geq 0, b_1, b_2, \ldots, b_k \in V, 0 \leq m_i \leq |b_i| \) for \( 1 \leq i \leq k \). Let \( \emptyset \) denote the empty sequence. Define \( |b| \), the level of \( b \), by

\[
|b| = \sum_{i=1}^{k} \{ m_i |b_i| - \frac{m_i(m_i - 1)}{2} \}.
\]
If \( 1 \leq s \leq k \) write \( b^s = ((b_s, m_s), \ldots, (b_k, m_k)) \). Write \( b^{k+1} = \emptyset \).

If \( b = ((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k)) \) and \( c = ((c_1, n_1), (c_2, n_2), \ldots, (c_s, n_s)) \in B_1(\Gamma) \) define
\[
  b \circ c = ((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k), (c_1, n_1), (c_2, n_2), \ldots, (c_s, n_s)).
\]

Let
\[
  B(\Gamma) = \{ b = ((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k)) \in B_1(\Gamma) | (b_i, m_i) \not\in (b_{i+1}, m_{i+1}), \ 1 \leq i < k \}.
\]

For \( b = ((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k)) \in B_1(\Gamma) \) set
\[
  \bar{e}(b) = \bar{e}(b_1, m_1) \ldots \bar{e}(b_k, m_k).
\]

Then \( |\bar{e}(b)| = |b| \) and \( \bar{e}(b \circ c) = \bar{e}(b) \bar{e}(c) \). Clearly \( \{\bar{e}(b) | b \in B_1(\Gamma)\} \) spans \( A(\Gamma) \). The following lemma is immediate from (2.1) and the definition of \( E(v, u, k, l) \).

**Lemma 3.1.** If \( (v, k) \models (u, l) \) then \( \bar{e}(v, k) \bar{e}(u, l) = \bar{e}(v, k + l) - \bar{E}(v, u, k, l) \).

For \( b = ((b_1, m_1), (b_2, m_2), \ldots, (b_k, m_k)) \in B(\Gamma) \) define
\[
  z(v, k, b) = \min\{ j | (v, k + 1 + \ldots + m_{j-1}) \not\models (b_j, m_j) \} \cup \{k + 1\}.
\]

**Lemma 3.2.** Let \( b \in B(\Gamma) \). Then \( \bar{e}(v, k) \bar{e}(b) = \)
\[
  \bar{e}((v, k + m_1 + \ldots + m_{z(v, k, b) - 1})) \circ b^{z(v, k, b) - 1}
  - \sum_{j=1}^{z(v, k, b) - 1} \bar{E}(v, b_j, k + m_1 + \ldots + m_{j-1}, m_j) \bar{e}(b^{j+1}).
\]

**Proof:** For \( 1 \leq i \leq z(v, k, b) - 1 \), we have \( (v, k + 1 + \ldots + m_{i-1}) \models (b_i, m_i) \) and so, by Lemma 3.1, \( \bar{e}(v, k + 1 + \ldots + m_{i-1}) \bar{e}(b^i) = \bar{e}((v, k + 1 + \ldots + m_{i-1} + m_i) \circ b^{i+1}) - E(v, b_i, k + \ldots + m_{i-1}, m_i) \). It follows, by induction on \( i \), that for \( 1 \leq i \leq z(v, k, b) - 1 \)
\[
  \bar{e}(v, k) \bar{e}(b) = \bar{e}((v, k + 1 + \ldots + m_{i-1} + m_i) \circ b^{i+1})
  - \sum_{j=1}^{i} \bar{E}(v, b_j, k + 1 + \ldots + m_{j-1}, m_j) \bar{e}(b^{j+1}).
\]

Taking \( i = z(v, k, b) - 1 \) gives the lemma.
Corollary 3.3. \( S = \{ \tilde{e}(b) \mid b \in B(\Gamma) \} \) spans \( A(\Gamma) \).

Proof: Since \( 1 \in S \), it is sufficient to show that \( \text{span} S \) is invariant under multiplication by \( \tilde{e}(v, k) \), hence sufficient to show that \( \tilde{e}(v, k) \tilde{e}(b) \in \text{span} S \), if \( b \in B(\Gamma) \). We prove this by induction on \( |\tilde{e}(v, k) \tilde{e}(b)| \). If \( |\tilde{e}(v, k) \tilde{e}(b)| = 0 \), then \( \tilde{e}(v, k) \tilde{e}(b) \in F1 \subseteq \text{span} S \). Now assume that \( \tilde{e}(w, l) \tilde{e}(c) \in \text{span} S \) whenever \( c \in B(\Gamma) \) and \( |\tilde{e}(w, l) \tilde{e}(c)| < |\tilde{e}(v, k) \tilde{e}(b)| \).

In view of Lemma 2.2, this implies \( \tilde{E}(v, b_1, k + m_1 + ... + m_{j-1}, m_j) \tilde{e}(b^{j+1}) \in S \) for \( 1 \leq j \leq z(v, k, b) - 1 \). But by Lemma 3.2,

\[
\tilde{e}(v, k) \tilde{e}(b) = \\
\tilde{e}((v, k + m_1 + ... + m_{z(v, k, b)} - 1) \circ b^{z(v, k, b)}) \\
- \sum_{j=1}^{z(v, k, b)-1} \tilde{E}(v, b_j, k + m_1 + ... + m_{j-1}, m_j) \tilde{e}(b^{j+1}).
\]

Since \( (v, k + m_1 + ... + m_{z(v, k, b)} - 1) \circ b^{z(v, k, b)} \in B(\Gamma) \), every summand on the right-hand side of this expression belongs to \( S \), so \( \tilde{e}(v, k) \tilde{e}(b) \in S \), as required.

4. Independence Theorem

Define \( B \) to be the vector space over \( F \) with basis \( B(\Gamma) \). Let \( B_i = \text{span}\{ b \mid |b| \leq i \} \).

We will define a linear transformation \( \mu : T(E) \otimes B \to B \) giving \( B \) the structure of a \( T(E) \)-module.

First define \( \mu_1 : E \times B(\Gamma) \to B \) by

\[
\mu_1 : (f, \emptyset) \mapsto e(f, 1)
\]

for \( f \in E_1 \) (where \( \emptyset \) denotes the empty sequence) and

\[
\mu_1 : E_i \times B_h \to B
\]

if \( i + h > 1 \). Then, as \( T(E) \) is the free algebra on the set \( E \), \( \mu_1 \) extends to a linear transformation, again denoted \( \mu_1 \) from \( T(E) \otimes B \) to \( B \) giving \( B \) the structure of a \( T(E) \)-module.
Now assume \( s > 1 \) and that we have defined maps

\[
\mu_j : E \times B(\Gamma) \to B
\]

for \( 1 \leq j \leq s - 1 \) such that

\[
\mu_j : E_i \times B(\Gamma)_h \to B_{i+h}
\]

for all \( 1 \leq i \leq n \) and \( h \geq 0 \). Assume also that

\[
\mu_j : E_i \times B(\Gamma)_h \to 0
\]

whenever \( i + h \geq j \) and that

\[
\mu_{j'}|_{E_i \times B(\Gamma)_h} = \mu_{j''}|_{E_i \times B(\Gamma)_h}
\]

whenever \( j' \geq j'' \geq i + h \). As in the case of \( \mu_1 \), each \( \mu_j \) extends to a linear transformation, again denoted \( \mu_j \) from \( T(E) \otimes B \) to \( B \) giving \( B \) the structure of a \( T(E) \)-module.

We now define

\[
\mu_s : E \times B(\Gamma) \to B
\]

by

\[
\mu_s|_{E_i \times B(\Gamma)_h} = \mu_{s-1}|_{E_i \times B(\Gamma)_h}
\]

for \( i + h \neq s \), and

\[
\mu_s : (f, b) \mapsto (v, 1 + m_1 + ... + m_{z(v,1,b)-1}) \circ b^{z(v,1,b)}
\]

\[
(z(v,1,b)-1) \sum_{j=1}^{z(v,1,b)-1} E(v, b_j, 1 + m_1 + ... + m_{j-1}, m_j) b^{j+1}
\]

\[
- (u, 1 + m_1 + ... + m_{z(u,1,b)-1}) \circ b^{z(u,1,b)}
\]

\[
+ \sum_{j=1}^{z(u,1,b)-1} E(u, b_j, k + m_1 + ... + m_{j-1}, m_j) b^{j+1}
\]

for \( b = ((b_1, m_1), ..., (b_k, m_k)) \), \( t(f) = v, h(f) = u, |f| + |b| = s \). (Note that, as \( e(*, j) = 0 \) for \( j > 0 \), the last two summands vanish when \( |f| = 1 \).)

Thus we have inductively defined \( \mu_j \) for all \( j \). Define

\[
\mu : T(E) \otimes B \to B
\]

by

\[
\mu|_{T(E)_i \otimes B_h} = \mu_{i+h}|_{T(E)_i \otimes B_h}.
\]

To simplify the notation we write \( fb \) for \( \mu(f \otimes b) \).
**Lemma 4.1.** (a) \(e(v, k)b = (v, k + m_1 + \ldots + m_{z(v, k, b)-1}) \circ b^{z(v, k, b)} - \sum_{j=1}^{z(v, k, b)-1} E(v, b_j, k + m_1 + \ldots + m_{j-1}, m_j)b^{j+1}.\)

(b) \(RB = (0)\)

**Proof:** Note that (a) is clear if \(k = 0\) (for \(e(v, 0) = 1\) and \(z(v, 0, b) = 1\)). Also (4.1) shows that (a) holds when \(k = 1\) (as \(e(v, 1) = e_{v(0)} + e_{v(1)} + \ldots + e_{v(|\Gamma| - 1)}\) and so the expression given by (4.1) for \(e(v, 1)b\) is a telescoping series). Thus (a) holds whenever \(|e(v, k)| + |b| \leq 1.\)

Let \(R_i = R \cap A(\Gamma)_i\) and note that \(R_0 = (0)\) and, by Lemma 2.6, \(R_1\) is spanned by \(\{f - e(t(f), 1)|f \in E_1\}\). Hence \(R_1B_0 = (0)\) whenever \(i + h < 1.\)

Now assume \(s > 1\) and that

\[e(w, l)c = (w, k + n_1 + \ldots + n_{z(w, l, c)-1}) \circ c^{z(w, l, c)} - \sum_{j=1}^{z(w, l, c)-1} E(w, c_j, l + n_1 + \ldots + n_{j-1}, n_j)c^{j+1}\]

whenever \(c = ((c_1, n_1), \ldots, (c_p, c_p)) \in B(\Gamma)\). Assume also that \(|e(w, l)| + |c| < s\) and \(R_iB_h = (0)\) whenever \(i + h < s.\) We will show that if \(|e(v, k)| + |b| = s\), then (a) holds and that if \(i + h = s\) then \(R_iB_h = (0)\), thus proving the lemma by induction.

By the expression obtained for \(e(v, 1)b\), we have

\[(f - e(t(f), 1) + e(h(f), 1))b = 0\]

for all \(f \in E, b \in B(\Gamma)\). Thus, by Lemma 2.6, to show \(R_iB_h = (0)\) whenever \(i + h = s\) it is sufficient to show that

\[(-e(v, 1)e(u, k) - e(v, k + 1) + e(u, k + 1) + e(u, 1)e(u, k))b = 0\]

whenever \(|e(v, k)| + |b| = s.\)

Now assume that \(|e(v, k)| + |b| = s.\) Since \(|e(u, k-1)| < |e(v, k)|\) and \(|e(u, k)| < |e(v, k)|\), the induction assumption gives values for \(e(u, k-1)b\) and \(e(u, k)b\). Thus

\[e(v, 1)e(u, k-1)b + e(u, k)b - e(u, 1)e(u, k-1)b = e(v, 1)(u, k - 1 + m_1 + \ldots + m_{z(u, k-1, b)-1}) \circ b^{z(u, k-1, b)}\]
\[ z(u, k-1, b) - 1 \]
\[- \sum_{j=1}^{z(u, k-1, b) - 1} e(v, 1) E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j) b^{j+1} \]
\[ + (u, k + m_1 + \ldots + m_{z(u, k-1, b) - 1}) \circ b^{z(u, k, b)} \]
\[- \sum_{j=1}^{z(u, k-1, b) - 1} e(v, 1) E(u, b_j, k + m_1 + \ldots + m_{j-1}, m_j) b^{j+1} \]
\[- e(u, 1)(u, k - 1 + m_1 + \ldots + m_{z(u, k-1, b) - 1}) \circ b^{z(u, k-1, b)} \]
\[ + \sum_{j=1}^{z(u, k-1, b) - 1} e(u, 1) E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j) b^{j+1} \]

By the definition of the actions of \( e(v, 1) \) and \( e(u, 1) \) this becomes

\[ (v, k + m_1 + \ldots + m_{z(v, k, b) - 1}) \circ b^{z(u, k-1, b)} \]
\[- E(v, u, 1, k - 1 + m_1 + \ldots + m_{z(v, k, b) - 1}) b^{z(u, k-1, b)} \]
\[- \sum_{j=z(v, k, b) - 1}^{z(u, k-1, b) - 1} E(v, b_j, k + m_1 + \ldots + m_{j-1}, m_j) b^{j+1} \]
\[- \sum_{j=1}^{z(u, k-1, b) - 1} e(v, 1) E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j) b^{j+1} \]
\[ + (u, k + m_1 + \ldots + m_{z(u, k, b) - 1}) \circ b^{z(u, k, b)} \]
\[- \sum_{j=1}^{z(u, k, b) - 1} E(u, b_j, k + m_1 + \ldots + m_j) b^{j-1} \]
\[- (u, 1) \circ (u, k - 1 + m_1 + \ldots + m_{z(u, k-1, b) - 1}) \circ b^{z(u, k-1, b)} \]
\[ + \sum_{j=1}^{z(u, k-1, b) - 1} e(u, 1) E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j) b^{j+1}. \]

Set

\[ G = -e(v, 1) E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j) \]
\[ + e(u, 1) E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j) \]

14
\[ +E(v, b_j, k + m_1 + \ldots + m_{j-1}, m_j) \]
\[-E(u, b_j, k + m_1 + \ldots + m_j). \]

By Lemma 2.6, \( G \in R \) and by Lemma 2.2 \( |Gb^{j+1}| < s \). Hence by the induction assumption, \( Gb^{j+1} = 0 \). Thus we may replace

\[ e(v, 1)E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j)b^{j+1} \]
\[ -e(u, 1)E(u, b_j, k - 1 + m_1 + \ldots + m_{j-1}, m_j)b^{j+1} \]

by

\[ E(v, b_j, k + m_1 + \ldots + m_{j-1}, m_j)b^{j+1} \]
\[ -E(u, b_j, k + m_1 + \ldots + m_j)b^{j+1}. \]

Thus our expression becomes

\[ (v, k + m_1 + \ldots + m_{z(v,k,b)-1}) \circ b^{z(u,k-1,b)} \]
\[ -E(v, u, 1, k - 1 + m_1 + \ldots + m_{z(v,k,b)-1})b^{z(u,k-1,b)} \]
\[ - \sum_{j=1}^{z(v,k,b)-1} E(v, b_j, k + m_1 + \ldots + m_{j-1}, m_j)b^{j+1} \]
\[ + (u, k + m_1 + \ldots + m_{z(u,k,b)-1}) \circ b^{z(u,k,b)} \]
\[ -(u, 1) \circ (u, k - 1 + m_1 + \ldots + m_{z(u,k-1,b)-1}) \circ b^{z(u,k-1,b)} \]

But by Lemma 2.1

\[ E(v, u, 1, k - 1 + m_1 + \ldots + m_{z(v,k,b)-1}) \]
\[ = -e(u, 1)e(u, k - 1 + m_1 + \ldots + m_{z(v,k,b)-1}) + e(u, k + m_1 + \ldots + m_{z(v,k,b)-1}). \]

Since

\[ |E(v, u, 1, k - 1 + m_1 + \ldots + m_{z(v,k,b)-1})| + |b^{z(u,k-1,b)}| < s, \]

by Lemma 2.2, and (as \( |u| < |v|)\)

\[ |e(u, 1)e(u, k - 1 + m_1 + \ldots + m_{z(v,k,b)-1})| + |b^{z(u,k-1,b)}| < s, \]

and

\[ |e(u, k + m_1 + \ldots + m_{z(v,k,b)-1})| + |b^{z(u,k-1,b)}| < s, \]
the induction assumption allows us to replace
\[ E(v, u, 1, k - 1 + m_1 + ... + m_{z(v, k, b) - 1})b^{z(u, k - 1, b)} \]
by
\[ -(u, 1) \circ (u, k - 1 + m_1 + ... + m_{z(u, k - 1, b) - 1}) \circ b^{z(u, k - 1, b)} \]
\[ +(u, k + m_1 + ... + m_{z(u, k, b) - 1}) \circ b^{z(u, k, b)}. \]

Making this substitution gives
\[ e(v, 1)e(u, k - 1)b + e(u, k)b - e(u, 1)e(u, k - 1)b \]
(4.2)
\[ = (v, k + m_1 + ... + m_{z(v, k, b) - 1}) \circ b^{z(v, k, b)} \]
\[ - \sum_{j=1}^{z(v, k, b) - 1} E(v, b_j, k + m_1 + ... + m_{j-1}, m_j)b^{j+1}. \]

Since \( P_v(t) = (1 + te(v, 1) - te(v^{(1)}, 1))P_{v^{(1)}}(t) \) we have
\[ e(v, k) = e(v, 1)e(v^{(1)}, k - 1) + e(v^{(1)}, k) - e(v^{(1)}, 1)e(v^{(1)}, k - 1). \]
Thus setting \( u = v^{(1)} \) in (4.2) gives (a). Since the right-hand side of (4.2) is independent
of \( u \) we also obtain
\[ e(v, k)b = e(v, 1)e(u, k - 1)b + e(u, k)b - e(u, 1)e(u, k - 1)b \]
for all \( u \) with \( v > u, |u| = |v| - 1 \). As noted above, this completes the proof of (b).

**Corollary 4.2.** There is an action of \( A(\Gamma) \) on \( B \) satisfying
\[ \tilde{e}(v, k)b = (v, k + m_1 + ... + m_{z(v, k, b) - 1}) \circ b^{z(v, k, b)} \]
\[ - \sum_{j=1}^{z(v, k, b) - 1} \tilde{E}(v, b_j, k + m_1 + ... + m_{j-1}, m_j)b^{j+1}. \]

Consequently, for \( b \in B(\Gamma) \), \( e(b)1 = b \). Since \( B(\Gamma) \subseteq B \) is linearly independent, \( \{\tilde{e}(b) \mid b \in B(\Gamma)\} \) is linearly independent. Therefore, we have proved the following theorem:
Theorem 4.3. Let \( \Gamma = (V, E) \) be a layered graph, \( V = \bigcup_{i=0}^{n} V_i \), and \( V_0 = * \) where * is the unique minimal vertex of \( \Gamma \). Then \( \{ \tilde{e}(b) \mid b \in B(\Gamma) \} \) is a basis for \( A(\Gamma) \).

Note that

\[
\{ \tilde{e}(b) \mid |\tilde{e}(b)| \leq i \}
\]

is a basis for \( A(\Gamma)_i \). Therefore, writing \( \tilde{e}(b) + A(\Gamma)_{i-1} \in \text{gr } A(\Gamma) \) where \( |\tilde{e}(b)| = i \) we have:

Corollary 4.4. Let \( \Gamma = (V, E) \) be a layered graph, \( V = \bigcup_{i=0}^{n} V_i \), and \( V_0 = * \) where * is the unique minimal vertex of \( \Gamma \). Then \( \{ \tilde{e}(b) \mid b \in B(\Gamma) \} \) is a basis for \( \text{gr } A(\Gamma) \).

Also, if \( |\tilde{e}(v, k)| = i \), we have

\[
\tilde{e}(v, k) + A(\Gamma)_{i-1} = \tilde{e}_{v(0)} \tilde{e}_{v(1)} \ldots \tilde{e}_{v(k-1)} + A(\Gamma)_{i-1}.
\]

Write

\[
\tilde{e}(v, k) = \tilde{e}_{v(0)} \tilde{e}_{v(1)} \ldots \tilde{e}_{v(k-1)}
\]

and, for \( b = ((v_1, k_1), \ldots, (v_s, k_s)) \in B(\Gamma) \), write

\[
\tilde{e}(b) = \tilde{e}(v_1, k_1) \ldots \tilde{e}(v_s, k_s).
\]

Then, if \( |\tilde{e}(b)| = i \), it follows that

\[
\tilde{e}(b) + A(\Gamma)_{i-1} = \tilde{e}(b) + A(\Gamma)_{i-1}
\]

and so we have:

Corollary 4.5. Let \( \Gamma = (V, E) \) be a layered graph, \( V = \bigcup_{i=0}^{n} V_i \), and \( V_0 = * \) where * is the unique minimal vertex of \( \Gamma \). Then \( \{ \tilde{e}(b) \mid b \in B(\Gamma) \} \) is a basis for \( A(\Gamma) \).

Note that, if \( \Gamma = \Gamma_n \), the basis for \( Q_n = A(\Gamma_n) \) given by Corollary 4.5 is the basis constructed in [GRW].

References

[GR1] I. Gelfand, V. Retakh, Gelfand Mathematical Seminars 1993-95, Birkhauser Boston, 1996, pp. 93-100.

[GR2] I. Gelfand, V. Retakh, Quasideterminants I, Selecta Math. (N.S.) 3 (1997), 517-546.

[GGRSW] I. Gelfand, S. Gelfand, V. Retakh, S. Serconek, and R. Wilson, Hilbert series of quadratic algebras associated with decompositions of noncommutative polynomials, J. Algebra 254 (2002), 279–299.

[GGRW] I. Gelfand, S. Gelfand, V. Retakh, R. Wilson, Quasideterminants, Advances in Math. 193 (2005), 56-141.

[GRW] I. Gelfand, V. Retakh, and R. Wilson, Quadratic-linear algebras associated with decompositions of noncommutative polynomials and Differential polynomials, Selecta Math. (N.S.) 7 (2001), 493–523.
[O] A. Odesskii, Set-theoretical solutions to the Yang-Baxter relation from factorization of matrix polynomials and $\theta$-functions, Mosc. Math. J. 3 (2003), no. 1, 97–103, 259.

[Pi] D. Piontkovski, Algebras associated to pseudo-roots of noncommutative polynomials are Koszul, math.RA/0405375.

[SW] S. Serconek and R. L. Wilson, Quadratic algebras associated with decompositions of noncommutative polynomials are Koszul algebras, J. Algebra 278 (2004), 473–493.

[V] A. Veselov, Yang-Baxter maps and integrable dynamics, Phys. Lett. A 314 (2003), no. 3, 214–221.

I.G., V.R., R.W.: Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019
S.S: IME-UFG CX Postal 131 Goiania - GO CEP 74001-970 Brazil
E-mail address:
igelfand@math.rutgers.edu, vretakh@math.rutgers.edu,
serconek@math.rutgers.edu, rwilson@math.rutgers.edu