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One-Dimensional Traps, Two-Body Interactions, Few-Body Symmetries

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Abstract This article identifies the configuration space and kinematic symmetry groups for \( N \) identical fermions or bosons in one-dimensional traps with Galilean-invariant two-body interactions. Asymmetric, symmetric and harmonic traps are considered. These symmetries explain degeneracies in the few-body spectrum and demonstrate how tuning the trap shape and the particle interactions can manipulate these degeneracies. The additional symmetries that emerge in the non-interacting limit and in the unitary limit of an infinitely strong contact interaction provide insight into the relationship between universality and symmetry in few-body systems.

Keywords One-dimensional traps · Few-body symmetries · Unitary limit of contact interaction

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1 Introduction

The focus of this article is the non-relativistic, one-dimensional, few-body Hamiltonian with the following characteristics: (1) Each particle has the same mass and experiences the same trapping potential. (2) There is a two-body interaction term for each pair that depends only on the distance between particles. (3) Each particle has a finite number of internal levels that do not participate directly in the trap or two-body interactions. Without loss of generality, call these internal levels the spin components.

Putting these conditions together, the total Hamiltonian for the system can be expressed as

$$\hat{H}^N = \sum_{i=1}^N \hat{H}^1_i + \sum_{i<j}^N \hat{V}_{ij}. \quad (1a)$$

Denoting each canonical pair of particle observables by $[\hat{Q}_i, \hat{P}_j] = i\delta_{ij}$ and choosing natural units, the one-body Hamiltonian for particle $i$ is

$$\hat{H}^1_i = \frac{1}{2} \hat{P}_i^2 + V^1(\hat{Q}_i). \quad (1b)$$

The two-body interaction term has the property $\hat{V}_{ij} = V^2(|\hat{Q}_i - \hat{Q}_j|)$. Particular attention is focused on the contact interaction, expressed in particle coordinates $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ as

$$V_{ij}(\mathbf{q}) = g\delta(q_i - q_j). \quad (1c)$$

This model Hamiltonian has a long history inspired by applications to atomic, molecular, nuclear and condensed matter physics. Going back to the beginnings of quantum mechanics, various subfields have ascribed different names (e.g. Stoner Hamiltonian, Tonks-Girardeau gas, no-core shell model) to particular instances of the model and its higher dimensional generalizations. There is also a large mathematical physics literature on the one-dimensional case, and certain special cases of $\hat{H}^N$ are exemplars of solvability in few-body and many-body systems [1; 2; 3; 4]. All these applications and results are not reviewed here, but the increasingly precise preparation, control and measurement of ultracold trapped atomic systems in effectively one-dimensional traps is driving another surge of theoretical interest in this model, e.g. [5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15; 16; 17; 18; 19; 20; 21; 22; 23; 24; 25; 26; 27; 28; 29; 30; 31; 32].

Although group theory has a long history of being productive in quantum mechanics, the “Gruppenpest” can be so frustrating that it is customary to begin with an explanation of why all this

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1 See the Introduction to [33] for a discussion of the Gruppenpest.
mathematical apparatus is worth the effort. The essential claim is that the symmetry classifications provided in this article can be exploited for qualitative, analytic and numeric studies of few-body systems trapped in one dimension. It can help solve or simplify numerous questions about the spectrum, degeneracy and dynamics, including the following:

- identical particle symmetrization
- perturbation theory from the non-interacting to the weak interaction limit
- perturbation theory from the unitary limit of the contact interaction to the nearly-unitary limit
- methods of exact diagonalization in truncated Hilbert spaces
- perturbation theory for not-quite identical particles
- adiabatic or non-adiabatic particle dynamics under variation of interaction parameters or trap shape.

Some examples of these are provided in this article, and many applications of the representation theory of the symmetric group exist in the recent literature \[3, 8, 14, 18, 27, 29\]. The focus of this article is to see how much more solvability is provided by additional configuration space and kinematic symmetries inherited from the trap shape and Galilean invariance of the interactions. The experimental tunability of few-body symmetries and the close connection between finite groups and integrability \[35, 36\] suggest novel possibilities for embodying mathematical structures in ultracold atomic systems.

Symmetry can also aid in the study of “universal” few body phenomena, a term used (with some local variation) to describe dynamical effects that do not depend strongly on the particular details of the constituent few body systems or on the nature of their interactions. See \[2, 37, 38\] for discussions of universality in one-dimension. Universal properties established in atomic systems could also reveal themselves in few-body systems at the chemical or nuclear scale. Universality can also drive the dynamics of coherence, entanglement and equilibration in certain many-body systems. One approach to universality is to figure out how much about the few-body system can be inferred from the symmetries of \( \hat{H}^N \) without specific knowledge of the trap or the interaction. The relationships among trap shape, interaction and permutation symmetry for identical particles partially determine algebraic solvability. The degree to which a few-body system possesses solvability is at least some component of universal-
ity. The best example is provided by the unitary limit of the contact interaction, which has enough symmetry to be exactly solved for any \( N \in \{5, 6, 7, 8, 9, 11, 12, 13, 14, 27, 29\}.

One more application of symmetry methods is that they provide geometrical insight into the highly-abstract interplay of trap shape, interaction, spin and particle symmetrization. Especially for low particle numbers, symmetries can be pictured and manipulated in the mind. To a large extent, the geometrical constructions and geometrical methods applied in the works \([8; 15; 20; 23; 32; 39]\) motivated this work.

The second section of this article explains what configuration space and kinematic symmetries are possible for one particle in the three kinds of traps considered here: asymmetric, symmetric and harmonic. The third section gives an overview of the symmetric group \( S_N \) and its representations. The \( S_N \) representation space called the permutation module is shown to be especially useful for the analysis of \( N \) identical particles. The fourth and longest section begins by identifying the minimal symmetries for \( N \) non-interacting particles inherited from their composition and describing the geometric realization of particle permutations and other symmetries in configuration space. The irreducible representation for the minimal kinematic symmetry group are derived and the notion of state permutation symmetry, as distinct from particle permutation symmetry, is introduced. A final result of this section is the isomorphism between the bosonic non-interacting spectrum and the fermionic non-interacting spectrum.

The fifth section classifies the symmetries for particles interacting via two-body Galilean invariant potentials. The symmetries of two-body matrix elements are derived and state permutation symmetry makes another appearance. The sixth section discusses the additional symmetries that emerge in the unitary limit of the contact interaction, specifically ordering permutation symmetry. The seventh and concluding section describes some possible extensions and applications of this work.

2 One-Particle Symmetries

Denote the configuration space symmetry group by \( C_1 \) and the kinematic symmetry group by \( K_1 \). These one-particle symmetry groups are the building blocks of the multi-particle analysis. This section describes these groups for asymmetric, symmetric and harmonic traps. They are basic groups familiar to most readers, and this section establishes notation and conventions necessary for subsequent sections.
Consider one particle in a one-dimensional trap and denote its spatial Hilbert space $\mathcal{K}$. Assume the energy spectrum $\sigma^{1} = \{\epsilon_{0}, \epsilon_{1}, \ldots\}$ of the one-particle Hamiltonian $\hat{H}^{1}$ is discrete, countably-infinite and non-degenerate so that

$$\mathcal{K} = \bigoplus_{n=0}^{\infty} \mathcal{K}_{n}$$

(2)

where $\mathcal{K}_{n}$ is a one-dimensional subspace where time evolution is represented as $\exp(-i\epsilon_{n}t)$. An energy spectrum with this form excludes finite wells, infinite lattices and probably some interesting pathological cases. However, it includes double-wells, multiple-wells and all the greatest hits of one-dimensional solvability like the harmonic well, infinite square well, Pölsch-Teller potential, Morse potential, etc.²

Eigenstates of $\hat{H}^{1}$ are denoted by kets containing the spectral index

$$\hat{H}^{1}|n\rangle = \epsilon_{n}|n\rangle.$$  

(3)

and the corresponding wave functions are

$$\phi_{n}(q) = \langle q|n\rangle.$$  

(4)

No functional dependence of $\epsilon_{n}$ on $n$ is implied, although algebraic and transcendental expressions certainly exist for specific solvable potentials. One convenience of this notation is that for symmetric one-dimensional wells the quantum number $n$ also determines the parity

$$\hat{\Pi}|n\rangle = (-1)^{n}|n\rangle.$$  

(5)

For convenience, sometimes the one-particle eigenstates will be denoted by state labels $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, etc. with wave functions $\phi_{\alpha}(q)$ and (for symmetric traps) parities $\pi_{\alpha}$.

For one particle in one dimension the configuration space is $\mathcal{Q} \sim \mathbb{R}^{1}$. The configuration space symmetry $C_{1}$ is the group of all transformations of $\mathcal{Q}$ realized by operators that commute with the one-particle Hamiltonian $\hat{H}^{1}$. For an asymmetric trap, no such operators exist and $C_{1} \sim I$ is the trivial group of just the identity. For a symmetric trap, there is a single point about which reflections are a symmetry and $C_{1} \sim O(1)$ is the parity group.³

² See [40] for an interesting discussion of symmetries and partial symmetries of lattice-like multi-well potentials.

³ Although not a trap, for the case of a constant potential (e.g. no potential) the group $C_{1}$ is the Euclidean group in one dimension $E_{1} = O(1) \ltimes T_{q}$, where $T_{q} \sim \mathbb{R}^{1}$ is the group of spatial translations in $\mathcal{Q}$ and $\ltimes$ denotes the semidirect product.
The one-particle kinematic symmetry group $K_1$ is the group of all operators that commute with $\hat{H}^1$, and therefore necessarily contains $C_1$. Consider the three cases:

- For asymmetric traps, the kinematic group is $K_1 = T_t$, where $T_t \sim \mathbb{R}^1$ is the time translation group generated by exponentiation of $\hat{H}^1$. This gives nothing new.

- For symmetric traps, parity is included and $K_1 = O(1) \times T_t$, parity is a good quantum number, and $K = K_+ \oplus K_-$ can be decomposed into sectors of fixed parity

$$K_+ = \bigoplus_{k=0}^{\infty} K_{2k} \text{ and } K_- = \bigoplus_{k=0}^{\infty} K_{2k+1}.$$  

- For harmonic traps $K_1 \sim U(1) \times T_t$. Here $U(1)$ is the group of transformations that changes the phase of the ladder operators $\hat{a}$ and $\hat{a}^\dagger$. Define a unitary representation of $U(1)$ by operators $\hat{U}(\phi)$ for $\phi \in [0, 2\pi)$ such that

$$\hat{b} = \hat{U}(\phi)\hat{a}\hat{U}^\dagger(\phi) = \exp(i\phi)\hat{a},$$

This transformation leaves $H^1$ invariant and can be thought of as rotations in two-dimensional phase space.

Note that for all three kind of traps, $C_1$ and $K_1$ are abelian groups. For abelian groups, irreducible representations are one-dimensional, and this is consistent with the assumption of a non-degenerate, discrete one-particle spectrum $\sigma^1$. More generally, the true kinematic group for a Hamiltonian should have irreps with the same dimension as the degeneracy of the spectrum. If it does not, then some symmetry has been missed.

Finally, if the single-particle Hamiltonian $\hat{H}^1$ is spin independent and there are $J$ spin components, then there is also a factor group of $U(J)$ to the kinematic symmetry. The total Hilbert space for the one particle system is the tensor product of the spin Hilbert space and the spatial Hilbert space.

$$H = \mathcal{S} \otimes K \sim \mathbb{C}^J \otimes L^2(\mathbb{R}). \quad (6)$$

Any unitary operator that acts only on $\mathcal{S} \sim \mathbb{C}^J$ certainly commutes with $\hat{H}^1$. Further, if the internal components really are spin components of a particle with spin $s$, then $J = 2s + 1$ and the spin operators $\hat{S}^2$ and $\hat{S}_z$ form a complete set of commuting operators that commute with the Hamiltonian.
3 The Symmetric Group

For every kind of trap and for any Galilean invariant interactions, the configuration space symmetry group and the kinematic symmetry group has $S_N$ as a subgroup. The properties of $S_N$ and its irreducible representations (irreps) are well-known (c.f. [33; 41; 42; 43]) and frequently applied in few-body physics.

The first subsection establishes the local notation and definitions for $S_N$ elements and irreps. Experts in the symmetric group could probably skip this section; novices may find it too brief to be useful. It is aimed at the audience between those two extremes. The second subsection defines compositions and permutation modules, and the third discusses how to incorporate spin and spatial degrees of freedom by taking direct products of the spatial Hilbert space $\mathcal{K}$ and the spin Hilbert space $\mathcal{S}$.

3.1 $S_N$ Basics

Elements of $S_N$ can be denoted by permutations $p = \{i_1 \ldots i_N\}$ or cycles $c = (ij \ldots k)$. For example, the permutation $p = \{1324\}$ and cycle $c = (23)$ realize the same elements of $S_4$. All elements with the same cycle structure form a conjugacy class. There is a conjugacy class of $S_N$ for each partition $[\mu] = [\mu_1 \mu_2 \ldots \mu_r]$ of $N$. A partition of $N$ is a set of non-negative integers that sum to $N$; denote the set of partitions of $N$ by $P(N)$. As an example, the five partitions and conjugacy classes for $S_4$ are:

- $[4^1]$: 4 one-cycles, i.e. the identity $e = (1)(2)(3)(4) = \{1234\}$;
- $[2^1 1^1 1^1]$: 1 two-cycle and 2 one-cycles, i.e. the six transpositions $(12) = \{2134\}, (13), (14), (23), (24), \text{ and } (34)$;
- $[2^2]$: 2 two-cycles and 2 one-cycles, i.e. the four disjoint, double transpositions $(12)(34) = \{2143\}, (13)(24), \text{ and } (14)(23)$;
- $[3^1 1^1]$: 1 three-cycles and 1 one-cycles, i.e. the eight permutations $(123) = \{2314\}, (132), (124), (142), (134), (143), (234), \text{ and } (243)$;
- $[4^1]$: 1 four cycle, i.e. the six permutations $(1234) = \{2341\}, (1243), (1324), (1342), (1423), (1432)$.

For each partition of $N$ there is also an irrep of $S_N$. There is a canonical ‘lexicographic’ ordering of irreps: the lowest partition $[N]$ is the one-dimensional, totally symmetric irrep and highest partition $[1^N]$ is the one-dimensional, totally antisymmetric irrep; other partitions correspond to multi-dimensional
representations with mixed symmetry under permutations. Irreps can be depicted as Ferrers diagrams, i.e. $r$ rows of boxes with the $i$th row having $\mu_i$ boxes (also called Young diagram). The Ferrers diagrams for $N = 4$ in order from least to greatest are

The notation $[\mu]^\top$ indicates the conjugate irrep of $[\mu]$. A conjugate irrep is the partition of $N$ that has the Ferrers diagram $[\mu]$ with rows and columns reversed, e.g. $[31]^\top = [21^2]$, and $[N]^\top = [1^N]$. Some partitions are self-conjugate, like $[2^2]$ and $[31^2]$.

The finite-dimensional vector space that carries the irrep $[\mu]$ is denoted $M^{[\mu]}$. Denote the dimension of $S_N$ irreps by $y^{[\mu]}$. For $N = 4$, these dimensions are $y^{[4]} = 1$, $y^{[31]} = 3$, $y^{[2^2]} = 2$, $y^{[21^2]} = 3$, and $y^{[1^4]} = 1$. The irrep dimension $y^{[\mu]}$ can be calculated using the Frobenius formula, the hook-length method, or by counting the number of standard Young tableaux that are possible for a given Ferrers diagram, e.g. for irrep $[21^2]$ the standard Young tableaux are

The irrep matrices $D^{[\mu]}$ depend on the basis chosen for $M^{[\mu]}$ and there are several methods for selecting bases and generating these matrices. These are not necessary for the results of this article, but are required for many applications.

### 3.2 Compositions and Permutation Modules

Consider a sequence of $N$ non-negative integers $\mathbf{n} = \langle n_1, n_2, \ldots, n_N \rangle$, where $n_i$ is an element of the sequence. The composition $(\mathbf{n})$ of $\mathbf{n}$ describes the numbers $n_i$ that appear in $\mathbf{n}$ and their degeneracies $\nu_i$ without regard to sequence order. One notation for a composition is $(\nu) = (\nu_0, \nu_1, \nu_2, \ldots)$, with trailing zeros dropped. For example, the sequence $\langle 2, 0, 1, 4, 1 \rangle$ has the composition $(1, 2, 1, 0, 1)$ because the number 1 appears twice, the numbers 0, 2, and 4 appear once, and the number 3 and numbers greater than 4 do not appear. Instead of numbers, more more general symbols can be used, e.g. the composition of a sequence of 5 symbols $\langle \alpha \beta \alpha \beta \gamma \rangle$ is denoted $(\nu) = (\alpha^2 \beta^2 \gamma)$. Note that $\sum \nu_i = N$ and the shape
[ν] of a composition (ν) is also a partition [ν] ∈ P(N), e.g. if (ν) = (1, 2, 1, 0, 1) then [ν] = [21^3] and if (ν) = (α^2 β^2 γ) then [ν] = [2^21].

The set of all sequences with the same composition (ν) forms a basis for a representation space of S_N called a permutation module ℳ[ν]. The action of p ∈ S_N on a basis sequence is

\[ p \cdot (n_1, n_2, \cdots, n_N) = (n_{p_1}, n_{p_2}, \cdots, n_{p_N}). \]  

(7)

The dimension \( d[ν] \) of \( ℳ[ν] \), or equivalently the number of sequences with composition (ν), depends only on the shape [ν] of the composition (ν). The formula for \( d[ν] \) is

\[ d[ν] = \frac{N!}{ν_1! ν_2! \cdots ν_r!}. \]  

(8)

For example, if (ν) = (1, 2, 1, 0, 1) then \( d[ν] = 60 \) and if (ν) = (α^2 β^2 γ) then \( d[ν] = 30 \).

Except for \( ℳ[4] = ℳ[31] \), the permutation modules are reducible with respect to S_N:

\[ ℳ[ν] = \bigoplus_{[μ] \leq [ν]} K_{[μ][ν]} ℳ[μ] \]  

(9)

where \([μ] \leq [ν]\) means that the partition [μ] is lower than [ν] in lexicographic ordering and \( K_{[μ][ν]} \) is the Kostka number describing the number of times the irrep [μ] appears in \( D[ν] \). Methods for calculating the Kostka numbers are well-established (for example, using characters [41], using combinatoric methods [42], using using the intrinsic group of the composition [33]). As an example, for \( N = 4 \) there are five permutation modules \( ℳ[ν] \) with the following reductions

\[ ℳ[4] = ℳ[4] \]

\[ ℳ[31] = ℳ[4] ⊕ ℳ[31] \]

\[ ℳ[2^2] = ℳ[4] ⊕ ℳ[31] ⊕ ℳ[2^2] \]

\[ ℳ[21^2] = ℳ[4] ⊕ 2 ℳ[31] ⊕ ℳ[2^2] ⊕ ℳ[21^2] \]

\[ ℳ[1^4] = ℳ[4] ⊕ 3 ℳ[31] ⊕ 2 ℳ[2^2] ⊕ 3 ℳ[21^2] ⊕ ℳ[1^4] \]  

(10)

The first permutation module \( ℳ[4] \) corresponds to a composition where all \( n_i \) are the the same; this is the trivial, totally symmetric representation of S_4. The second module \( ℳ[31] \) is the representation of S_4 when the composition (ν) has three \( n_i \) the same; it is called the defining representation of S_4. Notice that the permutation modules \( ℳ[21^2] \) and \( ℳ[1^4] \) are not simply reducible; multiple copies of the same
S₄ irrep appear. Different copies of the same irrep \( M^{[\mu]} \) can be distinguished by semi-standard Weyl tableaux, e.g. for the composition \( (\nu) = (\alpha, \beta^2, \gamma) \) with shape \( [\nu] = [21^2] \), there are two Weyl tableaux with shape \( [\mu] = [31] \)

\[
\begin{array}{cccc}
\alpha & \beta & \beta \\
\gamma & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
\alpha & \beta & \gamma \\
\beta & \\
\end{array}
\] (11)

The regular representation also has state permutation symmetry and plays a special role in the case of the unitary limit of the contact interaction, as shown below.

3.3 Symmetrization of Identical Particles with and without Spin

The \( N \)-particle Hamiltonian (interacting or non-interacting) always has \( S_N \) symmetry, and so the spatial Hilbert space \( K \) can be decomposed into sectors corresponding to irreps \( [\mu] \in P(N) \)

\[
K = \bigoplus_{[\mu] \in P(N)} K_{[\mu]};
\] (12)

For trapped particles, each sector \( K_{[\mu]} \) is isomorphic to an infinite tower of irreps spaces \( M^{[\mu]} \). Each particular copy of \( M^{[\mu]} \) is an energy eigenspace and there may be multiple copies of \( M^{[\mu]} \) corresponding to the same energy.

For one-component fermions and bosons, the spin Hilbert space \( S \) is trivial and the total Hilbert space is just \( \mathcal{H} \sim K \). Only states in the totally symmetric sector \( K_{[N]} \) can be populated by one-component bosons, and only the totally antisymmetric sector \( K_{[1^N]} \) is available for one-component fermions. For fermions and bosons with \( J > 1 \) components, the spin Hilbert space \( S \sim \mathbb{C}^J \) can ‘carry’ some of the symmetry or antisymmetry required for bosons or fermions and so the other sectors of \( K \) are relevant for identical bosons and fermions.

One way to treat this case is to reduce \( S \) into \( S_N \) irrep spaces

\[
S = \bigoplus_{[\mu] \in P(N)} S_{[\mu]};
\] (13)

using standard techniques (c.f. [33; 41; 44]) and then reduce the tensor product \( \mathcal{H} = S \otimes K \) into irreps using the Clebsch-Gordan series for \( S_N \). If the internal components really are spin and \( J = 2s + 1 \),
then the total spin operator $\hat{S}^2 = \sum_i \hat{S}^2_i$ and total spin component $\hat{S}^z = \hat{S}^z_i$ are invariant under $S_N$ and can be diagonalized along with the $S_N$ irreps.

Explaining this topic exceeds the ambitions of the present article, but an important result is this: for each irrep $[\nu]$ in the decomposition of $S$, there will be a single bosonic state for each copy of $M^{[\nu]}$ in the sector $K^{[\nu]}$, and for each irrep $[\nu]$ in the decomposition of $S$ there is a single fermionic state for each copy of $M^{[\nu]T}$ in the in the sector $K^{[\nu]T}$. Constructing explicit spin-spatial states is a technical challenge that increases with complexity as $N$ and $J$ get bigger, but it is an algebraically solvable problem. See [27; 34] for recent applications of these methods to trapped particles with spin.

In summary, reducing the spatial Hilbert space into irreps of $S_N$ is useful for symmetrizing identical particles, as well as understanding the degeneracy of energy eigenstates and how the energy levels split and combine as the trap and interaction are changed. The rest of this article shows how additional symmetries of the interaction and the trap enrich this structure.

4 Non-Interacting Particles

For the non-interacting $N$-particle system, denote the configuration space symmetry group as $C_0^N$ and the kinematic symmetry group as $K_0^N$. The total non-interacting system inherits a minimal configuration space symmetry group and a minimal kinetic symmetry group from its construction out of one-particle systems:

$$C_0^N \geq C_{0,\text{min}}^N \equiv S_N \ltimes C_1^{\times N},$$

$$K_0^N \geq K_{0,\text{min}}^N \equiv S_N \ltimes K_1^{\times N},$$

(14)

where $G^{\times N}$ means the group constructed from $N$-fold direct product of $G$ with itself and the symmetric group $S_N$ acts via a semidirect product $\ltimes$ on the abelian, normal subgroups of $C_1^{\times N}$ or $K_1^{\times N}$ by rearranging terms in the direct product.

Before diving into representation theory, let us physically motivate this construction. In the case of a symmetric well, each particle’s individual parity operator $\hat{\Pi}_i$ commutes with all the other parity operators and with the total Hamiltonian

$$\hat{H}_0^N = \sum_{i=0}^N \hat{H}_i^1.$$ 

(15)
Therefore the configuration space symmetry $C_0^N$ must at least have an abelian subgroup $O(1)^\times N$ generated by those $N$ commuting parity operators. Similarly, each individual particle’s Hamiltonian $\hat{H}_i^1$ commutes with the total Hamiltonian $\hat{H}_0$. This implies that each particle’s individual time evolution is still a good symmetry; the clocks of individual particles are not synchronized unless there are interactions. Therefore the kinematic group $K_0^N$ must at least have $T_0^\times N$ as a subgroup. Of course particle exchanges commute with the total Hamiltonian, so $S_N$ must also be subgroup of both $C_0^N$ and $K_0^N$. However particle exchanges do not commute with the single-particle parities and Hamiltonians. Instead they act as permutations on the $N$-fold product subgroups $C_1^\times N$ and $K_1^\times N$. The semidirect product in (14) captures that structure of the minimal subgroups of $C_0^N$ and $K_0^N$ the same way the semidirect product is useful for describing groups of affine transformations.

In the subsections below, the groups $C_0^N$ and $K_0^N$ and their irreps are explored in more detail. See Table I for a summary of results for $N = 2, 3$ and 4.

4.1 Configuration Space Symmetry Group $C_0^N$

The configuration space of $N$ particles in one dimension $Q \sim \mathbb{R}^N$ is isomorphic to one particle in $N$ dimensions, and therefore low $N$ situations can be visualized and described using the terms and techniques of geometry. Additionally, for most asymmetric and symmetric wells, the configuration space symmetry group is a finite point group, a class of groups that is completely characterized and classified in all dimensions (see [45]). Point groups in two and three dimensions are familiar to many physicists from applications in molecular and solid state physics.

For the asymmetric well, the one-particle configuration symmetry group is trivial $C_1 \sim I$ and the configuration space symmetry $C_0^N$ is isomorphic to the permutation group $S_N$. Each element in $S_N$ is realized by a geometrical transformation of configuration space $Q$, for example:

- Two-cycles $(ij)$ are reflections across the $(N-1)$-dimensional hyperplane $V_{ij} \subset Q$ defined by $q_i = q_j$.

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4 This observation may seem obvious, but it was a remark to this effect in [47] discussing the similarly separable problem of a particle in a cubic box that allowed me to understand the connection between permutation modules and $K_0^N$. 
Three-cycles \((ijk)\) are generated by two overlapping two-cycles \((ij)(jk)\), and they are simple rotations by \(\pm 2\pi/3\) in the plane perpendicular to the \((N-2)\)-dimensional hyperplane \(V_{ij} \cap V_{jk}\).

Double two-cycles \((ij)(kl)\) are generated by two non-overlapping two-cycles \((ij)(kl)\). These are double reflections across orthogonal hyperplanes \(V_{ij}\) and \(V_{kl}\), and they are equivalent to a simple rotation by \(\pm \pi\) in the plane perpendicular to the \((N-2)\)-dimensional hyperplane \(V_{ij} \cap V_{kl}\).

Four-cycles \((ijkl)\) are simple rotoreflections (or improper reflections). Specifically, each is a rotation by \(\pm \pi/2\) in the plane perpendicular to \(V_{ik} \cap V_{jl}\) and followed by reflection across the same plane.

Longer cycle structures correspond to compound reflections, compound rotations and compound rotoreflections, equivalence classes of orthogonal transformations that exist in dimensions greater than three.

The geometrical realization of \(S_N\) is equivalent to the point symmetry group of a regular \(N\)-simplex: for \(N = 2\) the digon, for \(N = 3\) the triangle, for \(N = 4\) the tetrahedron, for \(N = 5\) the pentachoron, etc. Note also that conjugacy classes of \(S_N\) are associated to equivalence classes of \(O(N-1)\), e.g. elements of \(S_4\) are reflections, simple rotations, and simple rotoreflections, and these are the three equivalence classes of \(O(3)\) and not \(O(4)\). This restriction to equivalence classes of \(O(N-1)\) is because the one-dimensional manifold defined by \(q_1 = q_2 = \cdots = q_N\) is invariant under the geometric realization of every \(p \in S_N\).

For a symmetric well, the configuration space symmetry \(C_0^N\) must contain \(S_N \ltimes O(1)^{\times N}\), the semidirect product of the permutation group \(S_N\) on the \(N\)-fold tensor product of the reflection group \(O(1)\). Each individual particle parity operator \(\hat{\Pi}_i\) is realized by reflections across the \((N-1)\)-dimensional hyperplane given by \(q_i = 0\). The total parity \(\hat{\Pi}\) is the product of all the individual parities

\[
\hat{\Pi} = \hat{\Pi}_1 \hat{\Pi}_2 \cdots \hat{\Pi}_N.
\]

The group \(S_N \ltimes O(1)^{\times N}\) has order \(2^N N!\), it is isomorphic to the wreath product \(S_2 \wr S_N\), and it is also known as the hyperoctahedral group \[48\]. The \(N = 2\) and \(N = 3\) dimensional examples are the point groups of a square and cube, respectively and their irreps and the reductions of those irreps into irreps of the subgroup \(S_N\) are well-known. For \(N = 2\) the group is called \(D_4\) and it has five irreps, four one-dimensional and one two-dimensional. All four of the one-dimensional reps have even total parity. The two-dimensional irrep has odd total parity and can be reduced into the \(S_2\) irreps \([2] \oplus [1^2]\). For the
three dimensional case, the group is called $O_h$ and there are ten irreps, half with even total parity and half with odd; see [47] for a recent discussion. The higher dimensional point groups for the hypercubes are less well-known in physics but their irreps can be found by induction from the normal subgroup $O(1)^\times N$. That is how the irrep dimension and multiplicity was calculated in Table 1 for $N = 4$.

For a harmonic well, $C_{N0}^N$ is larger than minimal symmetry inherited from the construction [14]. There is full rotational and reflectional symmetry in $Q$ and so $C_{N0}^N \sim O(N)$ is the full orthogonal group in $N$ dimensions. The hyperspherical representation of this group is well-known: irreps are labeled by $\lambda$ and have the dimension [46]

$$d(\lambda, N) = \frac{(N + 2\lambda - 2)(N + \lambda - 3)!}{\lambda!(N - 2)!}$$ (17)

for $N > 2$. This formula gives the familiar results $d(\lambda, 3) = 2\lambda + 1$ and $d(\lambda, 4) = (\lambda + 1)^2$. This case has been examined in more detail in [29] and further applications will appear in another article in preparation.

Note that if $C_{N0}^N$ were the only symmetry of the non-interacting Hamiltonian $\hat{H}_{N0}^N$, then we would expect that the energy levels would have degeneracies corresponding to the dimensions of those irreps. However, they certainly do not and the explanation of spectral degeneracies requires the consideration of the kinematic symmetry group $K_{N0}^N$.

4.2 Kinematic Symmetry Group $K_{N0}^N$

Denote the non-interacting $N$-particle spectrum by $\sigma_{N0}^N = \{E_0, E_1, E_2, \cdots \}$, which is still discrete but no longer non-degenerate. The spatial Hilbert space is decomposable into energy eigenspaces

$$\mathcal{K} = \bigoplus_k \mathcal{K}_k$$

where $\hat{H}_{N0}^N \mathcal{K}_k = E_k \mathcal{K}_k$. If the kinematic symmetry group $K_{N0}^N$ has been completely and correctly identified, then each energy eigenspace $\mathcal{K}_k$ will carry an irreducible representation of $K_{N0}^N$. Therefore, the dimensions of the irreps of $K_{N0}^N$ correspond to the degeneracies of energies $E_k \in \sigma_{N0}^N$. This section demonstrates that unless there are accidental degeneracies or emergent few-body symmetries, then the minimal kinematic group $K_{N0,\text{min}}^N = S_N \ltimes K_{1}^\times N$ is sufficient to explain the degeneracies of $E_k \in \sigma_{N0}^N$. The
Each sequence of three numbers determines a composition \((\nu)\), e.g. \((000)\) is (3), \((001)\) is \((2,1)\), \((013)\) is \((1,1,0,1)\). The superscript denotes the number of sequences with that composition, or equivalently the dimension of \(K_\nu \sim M^{(\nu)}\). Unless the specific values of \(\epsilon_i \in \sigma_1\) are known, only the partial ordering of composition energies \(E_\nu\) that is given by the arrows is defined. The boxed sequences have the shape \([1]^3\) and therefore their composition subspaces \(K_\nu\) carry the regular representation of \(S_3\) and have state permutation symmetry. There is a one-component bosonic state in every composition, but there are only one-component fermionic states in the boxed compositions. Note that if the sequence \((012)\) is added element-wise to each of the original sequences (e.g. \((002) + (012) = (014)\)) the chart will have the same form, giving a one-to-one mapping from bosonic states to fermionic states.

Irreps of \(K_{0,\min}^N\) are isomorphic to the permutation modules described in Section 3 and their reduction into \(S_N\) irreps is algebraically solvable.

The elements of \(\sigma_0^N\) and their degeneracies can be determined by forming compositions of one-particle energies \(\epsilon_n\) in the single particle spectrum \(\sigma^1\). The energy level with composition \((\nu) = (\nu_0, \nu_1, \ldots)\) has energy \(E_{(\nu)} = \nu_0 \epsilon_0 + \nu_1 \epsilon_1 + \cdots\). Note that only a partial ordering of \(\sigma_0^N\) is possible unless the specific one-particle energies \(\epsilon_n \in \sigma_1\) are known. For example, with two particles \(E_0 = E_{(2)} = 2 \epsilon_0\) and \(E_1 = E_{(11)} = \epsilon_0 + \epsilon_1\), but \(E_2\) could equal \(E_{(101)} = \epsilon_0 + \epsilon_2\) or \(E_{(02)} = 2 \epsilon_1\) (or both in the case of a harmonic well). See Fig. 1 for an example of partial ordering with three particles.

The tensor product of \(N\) non-interacting basis states is compactly denoted by

\[
|n\rangle \equiv |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_N\rangle,
\]
or alternatively $|\alpha\beta\cdots\rangle$. Call this basis where each particle has a definite state the “particle basis”.

Note that the multi-particle wave functions

$$\Phi_n(q) = \langle q|n\rangle = \prod_{i=1}^{N} \phi_{n_i}(q_i)$$

(20)

can always be chosen as real functions for the trapped system. Each $N$-particle tensor product basis vector $|n\rangle$ is an eigenvector $\hat{H}_0^N$ with energy $E_{(n)}$ given by the composition $(n)$. The degeneracy $d[\nu]$ of the energy level $E_{(\nu)}$ is the number of particle basis vectors with that composition, and as explained in the previous section, this number is determined by the shape $[\nu]$ of the composition $(\nu)$, in other words, by the pattern of degeneracies in the composition without regard to specific quantum numbers.

The spatial Hilbert space is decomposable into subspaces $\mathcal{K}_{(\nu)}$ spanned by particle basis vectors with composition $(\nu)$

$$\mathcal{K} = \bigoplus_{(\nu)} \mathcal{K}_{(\nu)}. \quad (21)$$

Note that this decomposition is not the same as (13). The sectors $\mathcal{K}_{[\mu]}$ are infinite towers of $S_N$ irrep spaces $M_{[\mu]}$ each with dimension $y_{[\mu]}$ whereas $\mathcal{K}_{(\nu)}$ are the composition subspaces with dimension $d[\nu]$. Each $\mathcal{K}_{(\nu)}$ is isomorphic to a permutation module $M_{[\nu]}$, as is demonstrated in the next section by explicit construction of the $d[\nu]$-dimensional irrep $D_{[\nu]}$ of the minimal kinematic symmetry group $S_N \ltimes T_{t}^{\times N}$ that acts on $\mathcal{K}_{(\nu)} \sim M_{[\nu]}$.

### 4.2.1 Irreps of $K_{0,\text{min}}^N$

An element $(p, t)$ of $S_N \ltimes T_{t}^{\times N}$ is a pair formed by $p \in S_N$ and $t = \{t_1, t_2, \ldots, t_N\} \in T_{t}^{\times N}$. The group multiplication rule in $S_N \ltimes T_{t}^{\times N}$ is

$$(p'; t') \cdot (p; t) = (p'p; t + O^\top(p)t')$$

(22)

such that

$$(p, t) = (p, 0) \cdot (e, t), \quad (23)$$

where $e$ is the identity in $S_N$ and $0$ is the identity in $T_{t}^{\times N}$. The $N \times N$ matrix $O(p)$ permutes the components of an $N$-dimensional vector $x$:

$$O(p)x = O(p)\{x_1, x_2, \ldots, x_N\}$$

$$= \{x_{p_1}, x_{p_2}, \ldots, x_{p_N}\}. \quad (24)$$
The matrix $O^\top(p) = O^{-1}(p) = O(p^{-1})$ is orthogonal and in each row and column of all matrix elements are zero except a single one. The choice of the transpose in (22) is for later convenience.

The unitary representation $\hat{U}(p, t)$ of $S_N \rtimes T^\times N$ has a natural realization on the particle basis $|n\rangle$:

\[
\hat{U}(p, 0)|n_1 n_2, \ldots, n_N\rangle = |n_{p_1}, n_{p_2}, \ldots, n_{p_N}\rangle = |O(p)n\rangle
\]

\[
\hat{U}(\epsilon, t)|n\rangle = \exp(-i\epsilon \cdot t)|n\rangle
\]

\[
\hat{U}(p, t)|n\rangle = \exp(-i\epsilon \cdot t)(O(p)n),
\]

where $\epsilon = \{\epsilon_{n_1}, \epsilon_{n_2}, \ldots, \epsilon_{n_N} \}$. The orbit of any $N$ particle basis vector $|n\rangle$ with composition $(\nu)$ under the representation (25) spans the composition space $K_{(\nu)}$. The unitary matrix representation $D^{[\nu]}(p, t)$ on $K_{(\nu)}$ defined by

\[
\hat{U}(p, t)|n\rangle = \sum_{n' \in (n)} D^{[\nu]}_{n'n}(p, t)|n'\rangle
\]

is $d[\nu]$ dimensional. The action of the subgroup $D^{[\nu]}(p, 0)$ on $K_{(\nu)}$ is unitarily equivalent to the action of the symmetric group on permutation module $M^{[\nu]}$.

An alternate derivation of the irreps of $S_N \rtimes T^\times N$ uses induced representation theory. The character of the normal, abelian subgroup $T^\times N$ is $\exp(-i\epsilon \cdot t)$ and it is determined by a sequence of energies $\epsilon$, or equivalently by a sequence $n$. The distinct orbits of $n$ under $S_N$ are classified by the shape of the composition $[\nu]$. The ‘little group’ of a composition $(\nu)$, i.e. the group of transformations that leaves a canonical representative $\tilde{n} \in (\nu)$ invariant, is $S_{[\nu]} = S_{v_1} \times S_{v_2} \times \cdots \times S_{v_r}$. The degeneracy $d[\nu]$ is the order of the coset $S_N/S_{[\nu]}$. Continuing a previous example, when $\nu = (1, 1, 2, 0, 1)$, one choice for the representative is $\tilde{n} = \{0, 1, 2, 2, 4\}$. The partition $[\nu] = \{21^3\}$ has little group $S_{21^3} \sim S_2$ with two elements $e$ and (34). This construction guarantees that the irreps are irreducible.

4.2.2 State Permutation Symmetry

Note that only the permutation module $M^{[2^N]}$ contains a totally antisymmetric irrep of $S_N$ and therefore has an energy level that could be populated by one-component fermions. Because of this

5 Note that $O$ is not an irreducible representation of $S_N$. The representation $O$ is called the defining representation of $S_N$. By a similarity transformation $J^\top O J$ it is decomposed into $S_N$ irreps $[N] \oplus [N - 1, 1]$. The set of similarity transformations $J$ that reduce the defining representation is the equivalence class of normalized Jacobi coordinate systems for $N$ one-dimensional particles.
spatial antisymmetry, compositions ($\nu$) with shape $[1^N]$ play an important role in the case of contact interactions, as discussed below. Subspaces $K_{(\nu)}$ corresponding to compositions shapes $[\nu] = [1^N]$ have an additional symmetry called state permutation symmetry. State permutation symmetry occurs when a composition space $K_{(\nu)}$ and the corresponding irrep $D[\nu]$ are unchanged by exchange of states in the composition. State permutation symmetry is not a symmetry of $\hat{H}_0^N$; generally, exchanging states changes the composition and changes the energy.

An an example, consider the case of three particles and $(\mu) = (\alpha\beta\gamma)$. Permutation symmetry acts on the particle basis as defined above, e.g. the $S_3$ elements $(12)$ and $(123)$ act as

$$\hat{U}(12)|\alpha\beta\gamma\rangle = |\beta\alpha\gamma\rangle; \quad \hat{U}(12)|\alpha\gamma\beta\rangle = |\gamma\alpha\beta\rangle;$$

$$\hat{U}(123)|\alpha\beta\gamma\rangle = |\gamma\alpha\beta\rangle; \quad \hat{U}(123)|\alpha\gamma\beta\rangle = |\beta\alpha\gamma\rangle.$$ 

In contrast, state permutations exchange state labels, not particles. Introduce operators the $\hat{p}_{\alpha\beta}$ and $\hat{p}_{\alpha\beta\gamma}$ that act on state labels in the following way:

$$\hat{p}_{\alpha\beta}|\alpha\beta\gamma\rangle = |\beta\alpha\gamma\rangle; \quad \hat{p}_{\alpha\beta}|\alpha\gamma\beta\rangle = |\beta\gamma\alpha\rangle;$$

$$\hat{p}_{\alpha\beta\gamma}|\alpha\beta\gamma\rangle = |\beta\gamma\alpha\rangle; \quad \hat{p}_{\alpha\beta\gamma}|\alpha\gamma\beta\rangle = |\alpha\beta\gamma\rangle.$$ 

The group of transformations on $K_{(\alpha\beta\gamma)}$ that is generated by the state permutations $\hat{p}_{\alpha\beta}$ and $\hat{p}_{\beta\gamma}$ is isomorphic to $S_3$ but it is different from the realization of $S_3$ provided by particle exchanges. Both realizations of $S_3$ are distinct subgroups of $S_6$, i.e. the permutations of the particle basis of $K_{(\alpha\beta\gamma)}$. See Ref. [33] for a description of how state permutation symmetry can be used to distinguish the multiple copies of $S_N$ irreps $[\mu]$ that appear in permutation modules $\mathcal{M}[\nu]$.

4.2.3 Bosonic and Fermionic Spectral Isomorphism

Select the unique sequence in the composition $(\nu)$ with state labels arranged in increasing order. For example, this is the sequence $\langle 01124 \rangle$ from $(\nu) = (1, 2, 1, 0, 1)$. By adding 0 to the first element of the sequence, 1 to the next element and so on, a new composition $(\nu')$ is produced which has the shape $[\nu'] = [1^N]$, e.g. the sequence $\langle 01124 \rangle$ with shape $[21^3]$ becomes the sequence $\langle 12348 \rangle$ which has shape $[1^5]$. This means that for every totally symmetric state (because there is one copy of $\mathcal{M}[N]$ in every composition subspace $K_{(\nu)}$), there is a partner antisymmetric state in $K_{(\nu')}$, and the relationship is
one-to-one. This mapping is a generalization of the result of Crescimanno \[49\] that the bosonic and fermionic non-interacting spectrum have the same structure for the harmonic oscillator, just shifted by a constant value. It is also related to the famous Girardeau fermionization mapping of identical one-component bosons \[50\]. See Fig. \[1\] for another example with \( N = 3 \).

4.2.4 The Group \( K_0^N \) for Symmetric Traps

For the asymmetric trap, \( K_0^N = S_N \ltimes T_t^N \) and the characterization of its irreps is complete. Symmetric traps have the larger one-particle kinematic symmetry group \( K_1 = O(1) \ltimes T_t \), but this does not complicate the irrep structure much. It does mean however that not all \( \mathcal{K}_{(\nu)} \) subspaces transform under the same \( K_0^N \) irrep. Even though two compositions have the same shape, they may not transform under the single-particle parities \( \Pi_i \) the same way.

Consider the case of three particles. There are three composition shapes: \([3]\), \([21]\) and \([1^3]\). The spectrum composition shape \([3]\) corresponds to compositions of a single particle state like \( |\alpha\alpha\alpha\rangle \) in one-dimensional composition subspace \( \mathcal{K}_{(\alpha^3)} \). There are two types of \( K_0^N \) irreps then, one with \( \pi_\alpha = 1 \) and positive total parity \( \pi = \pi_\alpha^3 = 1 \) and one with \( \pi_\alpha = -1 \) and negative total parity. For spectrum compositions like \((\alpha^2\beta)\) with shape \([21]\), there are four distinct parity compositions:

\[
\pi_\alpha = \pi_\beta = +1; \quad \pi_\alpha = -\pi_\beta = +1; \quad \pi_\alpha = -\pi_\beta = -1; \quad \pi_\alpha = \pi_\beta = -1.
\]

Each of these corresponds to a different three-dimensional irrep of \( K_0^N \), two with even total parity and two with odd. Finally, for spectral combinations like \((\alpha\beta\gamma)\) there are also four distinct parity compositions:

\[
\pi_\alpha = \pi_\beta = \pi_\gamma = +1; \quad \pi_\alpha = \pi_\beta = -\pi_\gamma = +1;
\]
\[
\pi_\alpha = -\pi_\beta = -\pi_\gamma = +1; \quad \pi_\alpha = \pi_\beta = \pi_\gamma = -1.
\]

The other parity compositions are isomorphic by state permutation symmetry.

Note also that the groups \( K_1 \) and \( K_1^* \) are abelian normal subgroups of \( K_0^N \), and so the induced representation construction for \( S_N \ltimes T_t^N \) described at the end of section 4.2.1 can also be used with minimal modification. The reduction of composition subspaces \( \mathcal{K}_{(\nu)} \) into \( S_N \) irreps is unchanged, there
are just additional quantum numbers. Note that the total number of distinct irreps of the minimally-constructed \( C_N^0 \) and \( K_N^0 \) are always the same although the distribution of irrep dimensions is not.

### 4.2.5 Emergent and Accidental Symmetries

When \( K_N^0 \) is equal to the minimal kinematic symmetry group \( S_N \ltimes K_1^N \), each composition subspace \( K(\nu) \) is uniquely related to an energy level \( E(\nu) \in \sigma_0^N \). The reduction of the spatial Hilbert space into irreps of \( K_N^0 \) is the same as the decomposition into composition subspaces (21). However, if there are coincident energy levels, i.e. two or more compositions lead to the same total energy, then there are additional kinematic symmetries.

The first case is accidental degeneracies, like the Pythagorean degeneracies that occur in the infinite square well and its higher dimensional generalizations \[ \text{[47; 51]} \]. These accidental degeneracies can be formulated as an ad hoc kinematic symmetry by defining operators that act as the identity in most energy subspaces but diagonalize the accidentally degenerate composition subspaces. In such a formulation, each accidental degeneracy requires the addition of a new operator that commutes with the Hamiltonian. Thinking of this as a kinematic symmetry is therefore not productive, because these symmetry operators must be inferred from the degeneracies and not the other way around. Accidental degeneracies of this sort will not be considered further here.

The other reason for coincident energy levels is that there is an emergent few-body symmetry, i.e. a symmetry beyond what is inherited from the one-particle symmetries. The harmonic well is the most famous example. Its energy levels have a degeneracy larger than can be explained by \( K_N^0 \). For the energy level \( \hbar \omega (X + N/2) \) with total excitation \( X = \sum_i n_i \), the degeneracy is \[ \text{[52; 53]} \].

\[
d(X, N) = \frac{(X + N - 1)!}{X!(N - 1)!}.
\]

This degeneracy can be derived from combinatorics, or it can be explained by the fact that \( K_N^0 \sim U(N) \), the group of unitary transformations in \( N \) dimensions. As in the one-particle case, the \( U(N) \) symmetry can be thought of either as the group of symplectic, orthogonal transformations in phase space or as the group of unitary transformations of the \( N \) annihilation operators: \( \text{Sp}(2N) \cap \text{O}(2N) \sim U(N) \). The irreps
of U(N) can be labeled by total excitation $X$ and the irrep space is the direct sum of all composition spaces with the same total excitation.

5 Interacting Particles

The introduction of Galilean-invariant two-body interactions among the identical particles breaks the symmetry encapsulated by the subgroups $C_1^{\times N}$ and $K_1^{\times N}$. The levels in the non-interacting energy spectrum $\sigma_0^N$ therefore split and degeneracies reduced when two-body interactions are turned on. However, the permutation symmetry subgroup is preserved, as well as any transformation in $C_1^{\times N}$ or $K_1^{\times N}$ that also commutes with the interaction operator

$$\hat{V}^N = \sum_{i<j}^N \hat{V}_{ij}. \tag{28}$$

Denote the symmetry groups that remain after interaction by $C^N$ and $K^N$. They are subgroups of $C_0^N$ and $K_0^N$, respectively, and the reduction of $K_0^N$ irreps into $C_N^N$ irreps is essentially the weak perturbation problem.

See Table 2 for information about $C^N$ and $K^N$ and their irreps for low particle numbers. Before classifying them for the three types of traps, we take a brief detour into the symmetries of the two-body matrix elements.

5.1 Symmetries of the Two-Body Matrix Elements

The two body matrix elements

$$\langle \mathbf{n}|\hat{V}_{12}|\mathbf{n}'\rangle = v^{n_1 n_2 \ldots n_N}_{n_1' n_2' \ldots n_N'} \delta_{n_3 n_3'} \ldots \delta_{n_N n_N'} \tag{29}$$

have the property

$$v^{\alpha \beta}_{\gamma \zeta} = v^{\gamma \zeta}_{\alpha \beta} \tag{30}$$

by the hermiticity of $\hat{V}_{ij}$ and remembering that the stationary states of single trapped particles can always be chosen as real so that $v^{\alpha \beta}$ is also real. Galilean invariance constrains the position representation of the two-body interaction to have the form

$$\langle \mathbf{q}|\hat{V}_{ij}|\mathbf{q}'\rangle = V_2(|q_i - q_j|)\delta^N(\mathbf{q} - \mathbf{q}'). \tag{31}$$
and so $\hat{V}_{ij} = \hat{V}_{ji}$ and
\[
v_{\alpha\beta} = v_{\beta\alpha}. \tag{32}
\]
Putting these together, the following four two-body matrix elements are equivalent for any Galilean-invariant two-body interaction potential $\hat{V}_{ij}$:
\[
v_{\alpha\beta} = v_{\beta\alpha} = v_{\gamma\alpha} = v_{\alpha\gamma}. \tag{33}
\]
Additionally, for the contact interaction the two-body matrix elements also have the property
\[
v_{\alpha\beta} = v_{\beta\alpha} = v_{\gamma\alpha}. \tag{34}
\]
because
\[
\langle \alpha\beta | \hat{V}_{12} | \gamma\zeta \rangle = g \int dq_1 dq_2 \phi_\alpha^*(q_1) \phi_\beta^*(q_2) \delta(q_1-q_2)
\times \phi_\gamma(q_1) \phi_\zeta(q_2)
= g \int dq \phi_\alpha^*(q) \phi_\beta^*(q) \phi_\gamma(q) \phi_\zeta(q). \tag{35}
\]
Combining (33) and (34), for contact interactions the two-body matrix elements $v_{\gamma\zeta}$ with all of the 24 possible permutations of the state labels take the same value. This state permutation invariance of the contact interaction is not shared by the interaction-free Hamiltonian $\hat{H}_N^0$ but it provides an alternate explanation for why the states in totally antisymmetric representations of $S_N$ are unperturbed by the contact interaction. Since $\hat{V}^N$ is symmetric under exchange of particles and the two-body matrix elements $v_{\gamma\zeta}$ are symmetric under exchange of states, $\hat{V}^N$ annihilates states in the antisymmetric sector $K_{[1N]}$.

5.2 Configuration Space Symmetry Group $C^N$

The operator $\hat{V}^N$ has the same configuration space symmetry as the coincidence manifold $V_N$ of all configurations in which at least two particles coincide in position $\hat{V}^N = \bigcup_{i<j} V_{ij}$.
where \( V_{ij} \) is the \((N-1)\)-dimensional hyperplane with \( q_i = q_j \). The coincidence manifold divides configuration space \( \mathcal{Q} \) into \( N! \) identical sections. By Galilean invariance, the operator \( \hat{V}^N \) and manifold \( \mathcal{V}_N \) are invariant under permutation of particles and under total inversion \( \hat{H} \). Additionally, Galilean invariance implies that \( \hat{V}^N \) commutes with the total momentum \( \hat{P} = \sum \hat{P}_i \), and therefore \( \mathcal{V}_N \) is invariant under with translations in, or inversion of, the center-of-mass coordinate \( R \sim \sum q_i \). This configuration space symmetry of the coincidence manifold \( \mathcal{V}_N \) is therefore isomorphic to

\[
C^N_{\mathcal{V}} = S_N \times T_R \times O(1)^{\times 2},
\]

where \( T_R \) is the translation group along center-of-mass ‘direction’ in configuration space, one copy of \( O(1) \) is total inversion \( \hat{H} \) and one copy of \( O(1) \) is center-of-mass inversion \( \hat{H}_R \). Relative inversion, i.e. inversion of the relative coordinates but preserving the orientation of the center-of-mass, is defined \( \hat{H}_r \equiv \hat{H}_R \hat{H} = \hat{H} \hat{H}_R \) and is therefore also in \( C^N_{\mathcal{V}} \).

Excluding the translation symmetry (which will be broken by any trap) the remaining symmetry group \( S_N \times O(1)^{\times 2} \) is isomorphic to the point group of a \( N \)-dimensional prism with end faces that are dual \( N \)-simplices, e.g. a hexagonal prism for \( N = 3 \) or an octahedral prism for \( N = 4 \). Note for \( N = 2 \) the operator \( \hat{H}_r \) and the two-cycle (12) are realized by the same transformation in \( \mathcal{Q} \). However, for \( N > 2 \) the transformation \( \hat{H}_r \) is distinct from any element of \( S_N \).

The configuration space symmetry group \( C^N \) of the total interacting Hamiltonian \( \hat{H}^N \) must contain the intersection of the transformations in \( C^N_{\mathcal{V}} \) and \( C^N_{\mathcal{V}^0} \). Consider the three cases:

- Asymmetric well \( C^N \sim S_N \): The interacting Hamiltonian has the same configuration space symmetry as the non-interacting Hamiltonian, namely just \( S_N \). As described above, particle exchanges are realized in \( \mathcal{Q} \) by \( N! \) orthogonal transformations that leave the center-of-mass axis and orientation invariant.

- Symmetric well \( C^N \sim S_N \times O(1) \): Total inversion \( \hat{H} \) is a symmetry transformation and the \( 2N! \) elements of \( C^N \) are realized by orthogonal transformations that leave the center-of-mass axis (but not necessarily the orientation) invariant. There is a doubling of irreps compared to the previous case, one set of \( [\mu] \in P(N) \) with even parity and one set with odd parity. Note that even for

\[ * \] This is an essential difference between one dimension and higher dimensions: in higher dimensions the particles can slip past each other without the configuration passing through the configuration manifold.
symmetric traps relative parity is not a good quantum number for \( N > 2 \) interacting particles unless the external single-particle potential is quadratic in position.

- Harmonic well \( C^N \sim S_N \times O(1)^{\times 2} \): For quadratic trapping potentials like the harmonic well, the center-of-mass and relative coordinates are separable. The total inversion \( \hat{I} \) and center-of-mass inversion \( \hat{I}_R \) (or equivalently \( \hat{H}_R \) and relative inversion \( \hat{H}_r \)) are independently good symmetry transformations. The group \( S_N \times O(1)^{\times 2} \) has order \( 4N! \). The four irreps for every partition \( [\mu] \in P(N) \) correspond to the four possible combinations of total and relative parity. Note that for the harmonic well, spectral compositions (\( \mu \)) have definite total parity but such compositions need not have definite relative parity (or center-of-mass quantum number, see below). Since there are multiple compositions with the same energy for the harmonic well, these other observables must be separately diagonalized \[29\].

### 5.3 Kinematic Symmetry Group \( K^N \)

Unless there are accidental or emergent symmetries, the kinematic symmetry group for interacting particles is just \( C^N \times T_F \), where in this case the time translation group generated by \( \hat{H}^N \) is just a phase in each energy eigenspace. The clocks of the individual particles are synchronized by the interaction and are no longer symmetries. The additional time translation symmetry has no effect on the irrep structure of \( K^N \), so it is the same as \( C^N \) for the asymmetric and general asymmetric trap.

For the harmonic trap, there is again an emergent kinematic symmetry due to the separability of center-of-mass and relative coordinates. The center-of-mass degree of freedom in phase space is unperturbed by the interaction and so there is a realization of the \( U(1) \) symmetry of a one-dimensional harmonic oscillator for that degree of freedom. The kinematic symmetry for the harmonic potential is therefore \( K^N = S_N \times O(1) \times U(1) \times T_1 \). The \( U(1) \) symmetry gives each irrep of this group an additional quantum number. This is exploited in Ref. \[18\] to further decompose the kinematic Hilbert space into non-interacting sectors for more efficient exact diagonalization and in Ref. \[29\] to construct adiabatic mappings, both for the case of contact interactions.
6 Unitary Limit of Contact Interactions

In the unitary limit $g \rightarrow \infty$ of the contact interaction in one-dimension, the particles cannot get past each other. Classically, the particles rattle back and forth in the trap, bouncing with perfectly elastic collisions off each other or rebounding from the edge of the trap potential. The order of the particles is stable under these dynamics. In fact, if the particles are distinguishable, each of the $N!$ particle orderings can be thought of as an independent, decoupled system. The spatial Hilbert space $\mathcal{K}$ can be decomposed into ‘ordering sectors’ $\mathcal{K}_p$

$$\mathcal{K} = \bigoplus_{p \in S_N} \mathcal{K}_p$$  \hspace{1cm} (38)

The energy spectrum within each of these independent sectors $\mathcal{K}_p$ can be deduced from the one-particle spectrum $\sigma^1$ quite simply: whenever there is a fermionic state in the $N$-particle non-interacting spectrum $\sigma^N_0$, there is a stationary state in each $\mathcal{K}_p$. This is because the boundary condition imposed by contact interactions is ‘automatically’ solved by the non-interacting fermionic states due to antisymmetry. At the unitary limit, wave functions must vanish on the coincidence manifold $\mathcal{V}_N$. All ordering sectors $\mathcal{K}_p$ are identical and therefore the $N$-particle spectrum $\sigma^N_\infty$ in the unitary limit is composed of $N!$-degenerate energy levels. From a given one-particle spectrum $\sigma^1$, the spectrum $\sigma^N_\infty$ and many features of the states at the unitary limit can be determined universally, including the incorporation of identical particles with spin $\{5, 6, 7, 8, 9, 11, 12, 13, 14, 27, 29\}$.

The rest of this section elaborates upon these points, highlighting the interplay of the dual symmetries of particle permutation and ordering permutation. The inclusion of parity and additional kinematic and dynamic symmetries is also discussed.

6.1 Snippet Basis

Define a sector of configuration space $Q_p \subset Q$ by the condition $q_{p_1} < q_{p_2} < \cdots < q_{p_N}$. The sector $Q_p$ is bounded by the $(N-1)$ hyperplanes $V_{p_1p_2}$, $V_{p_2p_3}$, $\ldots$, and $V_{p_{N-1}p_N}$. In the unitary limit of the contact interaction, the wave functions must vanish at the edges of the sectors $Q_p$, but inside the sector they satisfy the non-interacting Hamiltonian $\tilde{H}_0^N$. Denote the Hilbert space of wave functions on $Q_p$ satisfying the proper boundary conditions at $Q$ by $L^2(Q_p)$.
All the $K_p$ ordering subspaces are equivalent and the decomposition into ordering sectors means the configuration space realization of the spatial Hilbert space is equivalent to

$$K \sim \mathbb{C}^{N!} \otimes L^2(Q_e)$$

where $Q_e$ is the configuration space section with canonical ordering $q_1 < q_2 < \cdots < q_N$. The configuration space symmetries $C^N$ described in the previous section are realized by orthogonal transformations of $\mathbb{C}^{N!}$.

One basis for $K$ is provided by the snippet basis $[6; 14]$. Denote by $|[\nu][1^N]; p\rangle$ the single, totally antisymmetric state in a composition space $K^{(\nu)}$ that has composition shape $[\nu] = [1^N]$. Then the snippet basis vectors are the $N!$ states denoted $|([\mu][1^N]; p)\rangle$ with the property that

$$\langle q | ([\nu][1^N]; p) \rangle = \begin{cases} \sqrt{N!} \langle q |([\nu][1^N]) \rangle & q \in Q_p \\ 0 & q \not\in Q_p \end{cases}$$

(40)

Each spatial Hilbert space sector $K_p$ is spanned by the infinite tower of states $|([\nu][1^N]; p)\rangle$, one for all compositions $\nu$ with shape $[1^N]$. Remembering that every composition $\nu$ with arbitrary shape $[\nu]$ is associated with another composition $(\nu')$ with shape $[1^N]$ by adding the sequence $\langle 12 \cdots N \rangle$ to the representative ordered sequence $\vec{n} \in \nu$, then there is an element $|([\nu'][1^N]; p)\rangle \in K_p$ for every composition $\nu'$ and there is a one-to-one between spectrum of states at the unitary limit and the bosonic states.

6.2 Ordering Permutation Symmetry

The symmetric group $S_N$ acts naturally on the snippet basis. Define a representation $\hat{U}(p)$ of $p \in S$ by

$$\hat{U}(p)|([\nu][1^N]; p)\rangle = |([\nu'][1^N]; pp')\rangle.$$ 

(41)

Since this representation of $S_N$ is $N!$-dimensional it must be reducible. In fact it is the regular representation of $S_N$ and the representation space is isomorphic $\mathcal{M}(1^N)$.

The regular representation appeared in the non-interacting case when the composition $([\mu]$ had state permutation symmetry, i.e. when $[\mu] = [1^N]$ and therefore the $N$ states in the composition $([\mu]$ were equivalent under exchange. Here a similar role is played by order permutation symmetry. To
demonstrate the contrast with four particles, particle permutations such as \((12) = \{2134\}\) exchange particles 1 and 2 in the snippet basis

\[
\hat{U}(\{2134\}; \{1234\}) = |(\nu)[1^N]; \{2134\}\rangle
\]

\[
\hat{U}(\{2134\}; \{1234\}) = |(\nu)[1^N]; \{3124\}\rangle.
\]

On the other hand, the ordering permutation \(\sigma_{(12)}\) exchanges the first and second particles in the order, no matter which particles those are

\[
\sigma_{(12)}|(\nu)[1^N]; \{1234\}\rangle = |(\nu)[1^N]; \{2134\}\rangle
\]

\[
\sigma_{(12)}|(\nu)[1^N]; \{3214\}\rangle = |(\nu)[1^N]; \{2314\}\rangle.
\]

Like state permutation symmetry in the regular representation \(M^{[1^N]}\), the set of ordering transformations form a group isomorphic to \(S_N\) but distinct from the particle permutations group. Unlike state permutation symmetry, ordering permutations are a symmetry of the total Hamiltonian \(\hat{H}^N\) that emerges in the unitary limit of the contact interaction, and not just a symmetry of a composition subspace. Although each energy eigenspace of \(\hat{H}^N\) has the full \(U(N!)\) symmetry available in \(C^{N!}\), the subgroup formed by one copy of \(S_N\) for particle permutations and another copy for ordering permutations is sufficient to explain and diagonalize the observed degeneracy in energy levels at the unitary limit.

### 6.3 Incorporating Trap Symmetries

For the case of the symmetric well, one element of the ordering permutation group is equivalent to reversing the order of the particles. For example, for three particles that element is \(\sigma_{(13)}\) and for four particles that element is \(\sigma_{(14)(23)}\). The parity operator is realized on the snippet basis by

\[
\hat{H}|(\nu)[1^N]; \{(p_1p_2\cdots p_{N-1}p_N)\} = \pi(\nu)\sigma_{(1N)(2N-1)}\cdots|(\nu)[1^N]; \{(p_1p_2\cdots p_{N-1}p_N)\} = \pi(\nu)|(\nu)[1^N]; \{(p_{N}p_{N-1}\cdots p_{2p1})\},
\]

where \(\pi(\nu) = \pi_{\nu_1}\pi_{\nu_2}\cdots\pi_{\nu_r}\) is the total parity of the composition \((\nu)\). This representation of the parity operator can be diagonalized to decompose \(C^{N!}\) into \(S_N\) irreps \(M^{[\mu]}\) with a given parity. Note that not all of these irreps will have the same parity as the original fermionic composition. This diagonalization
is is discussed for the harmonic case in [29] and tables for $N = 3$, $N = 4$ and $N = 5$ are provided. In the harmonic case, relative parity and the center-of-mass excitation are also good quantum numbers. They are not necessarily commensurate with composition subspaces, but they can be diagonalized simultaneously with the particle and ordering permutations symmetries.

7 Conclusion

Despite its length, this article has left out many relevant topics, like the SO(2, 1) symmetry of the contact potential in a harmonic trap, lattice symmetries, supersymmetric potentials in one dimensions, and the possibility of interaction symmetries that depend on the internal structure of the particles. Further, some of these symmetry classifications could be generalized to higher dimensions, although symmetry is less constraining as the number of degrees of freedom grows. The effect of intrinsic three or higher few-body interactions on the spectrum could also be incorporated.

However, without adding to the complication of the one-dimensional trap, two-body interaction model, there is still much work to be done. Efficient methods of state construction are required for perturbation theory and exact diagonalization and for calculating reduced density matrices, correlation functions, and entanglement spectra among particles and between spin and spatial observables. For the infinite well, the contact interacting is solvable for any interaction strength via the Bethe ansatz. The two particle contact interaction is solvable, too, for any interaction strength and harmonic potentials. One intriguing avenue for future work is to use these symmetry methods to better understand how solvability breaks down as the number of particles is increased and the potential shape is changed, although that is known to be a hard problem.

A final possible avenue for future work is to exploit the close connection between finite groups and number theory. Perhaps there are practical protocols for simulations of number theory problems that employ combinations of adiabatic tunings and diabatic quenches of the trap shape and and interaction strength to manipulate states.

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This table provides information about the kinematic symmetry group $K_N^0$ and configuration space symmetry group $C_N^0$ for $N = 2$, 3 and 4 identical non-interacting particles in asymmetric, symmetry and harmonic traps. See text to understand notation for structure of $C_N^0$ and $K_N^0$. For each group, the dimensions of the unitary irreducible representations (irreps) is provided, and the number of inequivalent irreps with that dimension is indicated by the superscript. The symbol $\lambda \in \mathbb{N}_0$ labels inequivalent $O(N)$ irreps and $X \in \mathbb{N}_0$ labels inequivalent $U(N)$ irreps. Note that for $N = 2$, there is an infinite tower of inequivalent irreps of $O(2)$ for $\lambda > 0$. The Coxeter notation and order is given for the finite configuration space groups, and the point group notation is also given for $N = 2$ and 3.

|       | Asym. Trap | Sym. Trap | Harm. Trap |
|-------|------------|-----------|------------|
| $C_2^0$ | $S_2$       | $S_2 \times O(1)^{\times 2}$ | $O(2)$ |
| Irreps | $1^2$       | $1^4, 2$   | 1, 2       |
| Point  | $D_1$       | $D_4$      | $O(2)$     |
| Coxeter| $A_1 \sim [\ ]$ | $B C_2 \sim [4]$ | $\text{NR}$ |
| Order  | 2           | 8          | $\infty$  |
| $K_2^0$ | $S_2 \times T_1^{\times 2}$ | $S_2 \times (O(1) \times T_1)^{\times 2}$ | $U(2)$ |
| Irreps | 1, 2        | $1^2, 2^3$ | $(X + 1)$  |

|       | Asym. Trap | Sym. Trap | Harm. Trap |
|-------|------------|-----------|------------|
| $C_3^0$ | $S_3$       | $S_3 \times O(1)^{\times 3}$ | $O(3)$ |
| Irreps | $1^2, 2$   | $1^4, 2^2, 3^4$ | $2\lambda + 1$ |
| Point  | $C_{3v}$   | $O_h$      | $O(3)$     |
| Coxeter| $A_2 \sim [3]$ | $B C_3 \sim [4, 3]$ | $\text{NR}$ |
| Order  | 6          | 48         | $\infty$  |
| $K_3^0$ | $S_3 \times T_1^{\times 3}$ | $S_3 \times (O(1) \times T_1)^{\times 3}$ | $U(3)$ |
| Irreps | 1, 3, 6    | $1^2, 3^4, 6^4$ | $(X + 2)!$ |
|       |            |            | $2^{X^2}$  |

|       | Asym. Trap | Sym. Trap | Harm. Trap |
|-------|------------|-----------|------------|
| $C_4^0$ | $S_4$       | $S_4 \times O(1)^{\times 4}$ | $O(4)$ |
| Irreps | $1^2, 2, 3^2$ | $1^4, 2^2, 3^4, 4^4, 6^4, 8^2$ | $(\lambda + 1)^2$ |
| Coxeter| $A_3 \sim [3, 3]$ | $B C_4 \sim [4, 3, 3]$ | $\text{NR}$ |
| Order  | 24         | 384        | $\infty$  |
| $K_4^0$ | $S_4 \times T_1^{\times 4}$ | $S_4 \times (O(1) \times T_1)^{\times 4}$ | $U(4)$ |
| Irreps | 1, 4, 6, 12, 24 | $1^2, 4^4, 6^4, 12^8, 24^{16}$ | $(X + 3)!$ |
|       |            |            | $6^{X^2}$  |
Table 2 This table provides information about the kinematic symmetry group $K^N$ and configuration space symmetry group $C^N$ for $N = 2, 3$ and 4 identical non-interacting particles in asymmetric, symmetry and harmonic traps. For each group, the dimensions of the unitary irreducible representations (irreps) is provided, and the number of inequivalent irreps with that dimension is indicated by the superscript. The Coxeter notation and order is given for the finite configuration space groups, and the point group notation is also given for $N = 2$ and 3.

|       | Asym. Trap | Sym. Trap | Harm. Trap |
|-------|------------|-----------|------------|
| $C^2$ | $S_2$      | $S_2 \times O(1)$ | $S_2 \times O(1)$ |
| Irreps| $1^2$      | $1^4$     | $1^4$      |
| Point | $D_1$      | $D_2$     | $D_2$      |
| Coxeter | $A_1 \sim [\ ]$ | $A_1^2 \sim [2]$ | $A_1^2 \sim [2]$ |
| Order | 2          | 4         | 4          |
| $K^2$ | $S_2 \times T_T$ | $S_2 \times (O(1) \times T_T)$ | $S_2 \times (U(1) \times T_T)$ |
| Irreps| $1^2$      | $1^4$     | $1^4$      |
| $C^3$ | $S_3$      | $S_3 \times O(1)$ | $S_3 \times O(1)^{\times 2}$ |
| Irreps| $1^2, 2$  | $1^4, 2^2$ | $1^8, 2^4$ |
| Point | $C_{3v}$  | $D_{3d}$  | $D_{3h}$  |
| Coxeter | $A_2 \sim [3]$ | $[[3]]$ | $[[3], 2]$ |
| Order | 6          | 12        | 24         |
| $K^3$ | $S_3 \times T_T$ | $S_3 \times (O(1) \times T_T)$ | $S_3 \times (O(1) \times U(1) \times T_T)$ |
| Irreps| $1^2, 2$  | $1^4, 2^2$ | $1^8, 2^4$ |
| $C^4$ | $S_4$      | $S_4 \times O(1)$ | $S_4 \times O(1)^{\times 2}$ |
| Irreps| $1^2, 2, 3^2$ | $1^4, 2^2, 3^4$ | $1^8, 2^4, 3^8$ |
| Coxeter | $A_3 \sim [3, 3]$ | $[[3, 3]]$ | $[[3, 3], 2]$ |
| Order | 24         | 48        | 96         |
| $K^4$ | $S_4 \times T_T$ | $S_4 \times (O(1) \times T_T)$ | $S_4 \times (O(1) \times U(1) \times T_T)$ |
| Irreps| $1^2, 2, 3^2$ | $1^4, 2^2, 3^4$ | $1^8, 2^4, 3^8$ |
