Abstract. Let \( X \) be a complex smooth quasi-projective variety with a fixed epimorphism \( \nu: \pi_1(X) \rightarrow \mathbb{Z} \). In this paper, we consider the asymptotic behaviour of invariants such as Betti numbers with all possible field coefficients and the order of the torsion part of singular integral homology associated to \( \nu \), known as the \( L^2 \)-type invariants. At homological degree one, we give concrete formulas for these limits by the geometric information of \( X \) when \( \nu \) is orbifold effective. The proof relies on a study about cohomological degree one jump loci of \( X \). We extend part of Arapura’s result for cohomological degree one jump loci of \( X \) with complex field coefficients to the one with positive characteristic field coefficients. As an application, when \( X \) is a hyperplane arrangement complement, a combinatoric upper bound is given for the number of parallel positive dimensional components in cohomological degree one jump loci with complex coefficients. Another application is that we give a positive answer to a question posed by Denham and Suciu for hyperplane arrangement.

1. Introduction

There is a general principal to consider a classical invariant of a finite CW complex \( X \) and define its analog for some covering space of \( X \). This leads to the \( L^2 \)-type invariants. Atiyah [Ati76] introduced the notion of \( L^2 \)-Betti numbers in the context of a regular covering of a closed Riemannian manifold. After that, there has been vast literature for the \( L^2 \)-invariant theory, see [Luc02]. The \( L^2 \)-invariant theory has attracted considerable interests and has relations with many other fields, such as operator theory and Algebraic K-theory. A particular important result is Lück’s approximation theorem [Luc94], which states that the \( L^2 \)-Betti numbers of the universal cover of a finite polyhedron can be found as limits of normalised Betti numbers of finitely sheeted normal coverings.

1.1. \( L^2 \)-type invariants. Assume that \( X \) is a connected finite CW-complex with a fixed group epimorphism \( \nu: \pi_1(X) \rightarrow \mathbb{Z} \). Let \( X^\nu \) denote the corresponding infinite cyclic covering space of \( X \). For any positive integer \( N \), let \( X^{\nu,N} \) denote the covering space of \( X \) associated to the following composition of maps

\[ \pi_1(X) \xrightarrow{\nu} \mathbb{Z} \xrightarrow{N} \mathbb{Z}. \]
One can study $X^{\nu}$ by its approximation $X^{\nu,N}$. Consider the following two limits

\begin{align}
(1) \quad & \lim_{N \to \infty} \frac{\dim H_i(X^{\nu,N}, \mathbb{K})}{N} \\
(2) \quad & \lim_{N \to \infty} \frac{\log |H_i(X^{\nu,N}, \mathbb{Z})_{\text{tor}}|}{N},
\end{align}

for some field $\mathbb{K}$ and

where $|H_i(X^{\nu,N}, \mathbb{Z})_{\text{tor}}|$ denote the cardinality of the torsion part of $H_i(X^{\nu,N}, \mathbb{Z})$. In prior we do not know if these two limits exist.

When $\mathbb{K} = \mathbb{C}$, the well known Lück’s approximation theorem [Luc94] shows that the first limit exists and it coincides with the $L^2$-Betti number defined by the Von Neumann algebra. (Lück’s approximation theorem is proved with much more generality.) Meanwhile, Abért, Jaikin-Zapirain and Nikolov [AJN11, Theorem 17] showed that the first limit also exists when $\text{char}(\mathbb{K}) > 0$. We set

$$
\alpha_i(X^{\nu}, \mathbb{K}) := \lim_{N \to \infty} \frac{\dim H_i(X^{\nu,N}, \mathbb{K})}{N}.
$$

Once we know this limit exists, one can compute it by homology jump loci, see e.g. [DS14, Theorem 2.5] and Proposition 2.5.

**Proposition 1.1.** Let $\mathbb{K}$ be an algebraically closed field. With the assumptions and notations as above, for any $i \geq 0$ we have

$$
\alpha_i(X^{\nu}, \mathbb{K}) = \dim H_i(X, L_\rho) \quad \text{for } \rho \in \mathbb{K}^* \text{ being general}.
$$

Here $L_\rho$ is the rank one $\mathbb{K}$-local system on $X$ corresponding to $\rho$ pulling back by $\nu$. In particular, $\alpha_i(X^{\nu}, \mathbb{K})$ is always an integer.

When $X$ is a complex smooth projective variety and $\mathbb{K} = \mathbb{C}$, similar formulas for Betti numbers and Hodge numbers can be found in [DL19]. When $X$ is a complex smooth quasi-projective variety and $\mathbb{K} = \mathbb{C}$, there are a lot of works to study the polynomial periodicity for the Betti numbers and Hodge numbers of congruence covers, see [Suc01, Section 5] and [Bud09]. However we do not pursue this direction in this paper.

On the other hand, the limit in (2) also exists, e.g. see [Le14, Theorem 5], and it can be computed by the Mahler measure (see Definition 2.7) of the $i$-th integral Alexander polynomial $\Delta_i(X^{\nu}) \in \mathbb{Z}[t, t^{-1}]$ (see Definition 2.3). We set

$$
M_i(X^{\nu}) := \lim_{N \to \infty} \frac{\log |H_i(X^{\nu,N}, \mathbb{Z})_{\text{tor}}|}{N}.
$$

The above discussion shows that these limits are determined by jump loci and integral Alexander polynomials. When $X$ is a smooth complex quasi-projective variety, under certain assumptions for $\nu$ we give concrete formulas to compute these limits at homological degree one (namely $i = 1$) by geometric information of $X$. The formulas rely on a detailed study about the cohomological degree one jump loci of $X$ that we discuss in next subsection.

**1.2. Main results.**
1.2.1. Cohomology jump loci and orbifold maps. Let $X$ be a smooth complex quasi-projective variety with $\pi_1(X) = G$ and $K$ be an algebraically closed field. The group of $K$-valued characters, $\text{Hom}(G, K^*)$, is a commutative affine algebraic group. Each character $\rho \in \text{Hom}(G, K^*)$ defines a rank one $K$-local system on $X$, denoted by $L_\rho$.

**Definition 1.2.** The cohomology jump loci of $X$ are defined as

$$V^i_k(X, K) := \{ \rho \in \text{Hom}(G, K^*) \mid \dim_K H^i(X, L_\rho) \geq k \}.$$  

When $k = 1$, we simply write $V^i(X, K)$.

When $K = \mathbb{C}$, the cohomology jump loci of complex smooth quasi-projective variety have been intensively studied. In particular, the well-known structure theorem (see Theorem 3.3) for $V^i(X, \mathbb{C})$ put strong constraints for the homotopy type of complex smooth quasi-projective variety. If we focus on cohomological degree one jump loci, a celebrity result due to Beauville [Bea92], Arapura [Ara97] and Artal Bartolo, Cogolludo-Agustín and Matei [ACM13] puts even stronger constraints for its fundamental group.

To explain their results, we recall the definition of orbifold maps. Let $\Sigma_{g,r}$ be a smooth Riemann surface of genus $g \geq 0$ with $r \geq 0$ points removed. Assume $b_1(\Sigma_{g,r}) = 2g + r - 1 > 0$, i.e., $\Sigma_{g,r} \neq \mathbb{CP}^1, \mathbb{C}^*$.

**Definition 1.3.** Let $X$ be a complex smooth quasi-projective variety. An algebraic map $f : X \to \Sigma_{g,r}$ is called an orbifold map, if $f$ is surjective and has connected generic fiber. There exists a maximal Zariski open subset $U \subset \Sigma_{g,r}$ such that $f$ is a fibration over $U$. Say $B = \Sigma_{g,r} - U$ (could be empty) has $s$ points, denoted by $\{ q_1, \ldots, q_s \}$. We assign the multiplicity $\mu_j$ of the fiber $f^*q_j$ (the gcd of the coefficients of the divisor $f^*q_j$) to the point $q_j$. Such orbifold map $f$ is called of type $(g, r, \mu)$ with $\mu = (\mu_1, \ldots, \mu_s)$ and we say $f$ is

- large if $2g + r - 2 > 0$,
- small if $2g + r - 2 = 0$ and $\prod_{j=1}^s \mu_j > 1$,
- null if $2g + r - 2 = 0$ and $\prod_{j=1}^s \mu_j = 1$.

Here when $B = \emptyset$, $\prod_{j=1}^s \mu_j = 1$ by convention. If $\prod_{j=1}^s \mu_j = 1$, we say $f$ has no multiple fiber. When $\Sigma_{g,r}$ is clear in the context, we simply write $\Sigma$. The orbifold group associated to these data is defined as

$$\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu) := \pi_1(\Sigma_{g,r} \setminus \{ q_1, \ldots, q_s \})/\langle \gamma_j^{\mu_j} = 1 \text{ for all } 1 \leq j \leq s \rangle,$$

where $\gamma_j$ is a meridian of $q_j$.

Our definition of orbifold maps is a bit of different from the classical one, e.g. see [DS14, Section 4.1], as we also include null orbifold maps. For example, if $f : \mathbb{C}^n \to \mathbb{C}$ is a reduced weighted homogeneous polynomial, then $f : \mathbb{C}^n \setminus f^{-1}(0) \to \mathbb{C}^*$ gives the Milnor fibration, hence a null orbifold map.

An orbifold map $f : X \to \Sigma$ of type $(g, r, \mu)$ induces a surjective map on the fundamental group level (see e.g. [CKO03, Lemma 3])

$$f_* : \pi_1(X) \twoheadrightarrow \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu),$$
which induces an embedding
\[ V^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), \mathbb{K}) \to V^1(X, \mathbb{K}), \]
see e.g. [Suc14B, Proposition A.1]. The following theorem shows that every positive dimensional component of \( V^1(X, \mathbb{C}) \) can be realized by the orbifold map in this way. This idea appeared in Beauville’s work [Bea92] for the projective case and was extended to the quasi-projective case by Arapura [Ara97]. Further properties were found by Dimca [Dim07, Dim09], Dimca, Papadima and Suciu [DPS09], Artal Bartolo, Cogolludo-Agustín and Matei [ACM13], etc. We recall the theorem here and adjust it for our needs.

**Theorem 1.4.** [Ara97, ACM13, Dim07] Let \( X \) be a complex smooth quasi-projective variety. Then the following two statements hold.

(a) We have
\[ V^1(X, \mathbb{C}) = \bigcup_f f^*V^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)), \mathbb{C}) \cup Z, \]
where \( Z \) is a finite set of torsion points and the union runs over all large and small orbifold maps \( f: X \to \Sigma_{g,r} \). Moreover, \( f^*V^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), \mathbb{C}) \) has \( \prod_{j=1}^s \mu_j \) many parallel components when \( f \) is large and has \( (\prod_{j=1}^s \mu_j - 1) \) many parallel components when \( f \) is small.

(b) Given an orbifold map \( f: X \to \Sigma \) of type \((g, r, \mu)\) and any \( \rho \in \text{Hom}(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), \mathbb{C}^*) \), we have
\[ \dim H^1(X, f^* L_{\rho}) \geq \dim H^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), L_{\rho}) \]
and equality holds with finitely many (torsion) exceptions.

In this paper, we consider the above theorem with arbitrary algebraically closed field coefficients. Theorem 1.4(a) is generalized by Delzant as follows using Bieri-Neumann-Strebel invariants when \( X \) is smooth projective.

**Theorem 1.5.** [Delz08] Let \( X \) be a complex smooth projective variety with \( b_1(X) > 0 \) and \( \mathbb{K} \) be an algebraically closed field. Then we have
\[ V^1(X, \mathbb{K}) = \bigcup_f f^*V^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)), \mathbb{K}) \cup Z', \]
where \( Z' \) is a finite set of points (depends on \( \mathbb{K} \)) and the union runs over all large and small orbifold maps \( f: X \to \Sigma_{g,r} \).

We prove the following generalization of Theorem 1.4(b).

**Theorem 1.6.** Let \( X \) be a complex smooth quasi-projective variety with \( b_1(X) > 0 \) and \( \mathbb{K} \) be an algebraically closed field. Given an orbifold map \( f: X \to \Sigma \) of type \((g, r, \mu)\) and any \( \rho \in \text{Hom}(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), \mathbb{K}^*) \), we have
\[ \dim H^1(X, f^* L_{\rho}) \geq \dim H^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), L_{\rho}) \]
and equality holds with finitely many exceptions.
For any \( \rho \in \text{Hom}(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), \mathbb{K}^*) \), \( \dim H^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu), L_\rho) \) is computed in subsection 3.2. Theorem 1.6 together with Theorem 1.5 indeed gives a complete description of all positive dimensional component of \( \mathcal{V}_k^1(X, \mathbb{K}) \) for all \( k \) when \( X \) is smooth projective.

**Remark 1.7.** Theorem 1.4 (see e.g. [Suc14A, Theorem 9.3]) and Theorem 1.5 ([Del74]) also work for compact Kähler manifold. Correspondingly, one can also prove Theorem 1.6 for compact Kähler manifold by the same proof presented in this paper.

1.2.2. **Formulas for \( L^2 \)-type invariants.** Next we connect the epimorphism \( \nu \) to the orbifold maps.

**Definition 1.8.** Let \( X \) be a complex smooth quasi-projective variety with an epimorphism \( \nu: \pi_1(X) \twoheadrightarrow \mathbb{Z} \). We say that \( \nu \) is orbifold effective if there is an orbifold map \( f: X \rightarrow \Sigma_{g,r} \) such that \( \nu \) factors through \( f^* \) as follows:

\[
\begin{aligned}
\pi_1(X) & \xrightarrow{\nu} \mathbb{Z}, \\
\pi_1^{\text{orb}}(\Sigma_{g,r}, \mu) & \xrightarrow{f^*} \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu) \\
\end{aligned}
\]

We say that \( \nu \) is orbifold effective by \( f \) and call \( \nu \) being of type \((g, r, \mu)\).

**Remark 1.9.** This definition is well-defined, see Remark 4.3. Note that there are examples where \( \nu \) is not orbifold effective, see Example 4.5. On the other hand, if \( H^1(X, \mathbb{Q}) \) has pure Hodge structure of weight \((1, 1)\) (i.e. \( X \) has a smooth compactification \( \overline{X} \) such that \( H^1(\overline{X}, \mathbb{Q}) = 0 \)), then any epimorphism \( \nu: \pi_1(X) \twoheadrightarrow \mathbb{Z} \) is orbifold effective. (This is why we include null orbifold maps in our definition.) Typical examples are hyperplane arrangement complement, see Remark 4.4.

**Theorem 1.10.** Let \( X \) be a complex smooth quasi-projective variety and \( \mathbb{K} \) be a field with \( \text{char}(\mathbb{K}) = p \geq 0 \). Consider an epimorphism \( \nu: \pi_1(X) \twoheadrightarrow \mathbb{Z} \). Then we have the following.

(a) If \( \nu \) is orbifold effective of type \((g, r, \mu)\), we have

\[
\alpha_1(X^\nu, \mathbb{K}) = 2g + r - 2 + \# \{ j \mid p \text{ divides } \mu_j \}
\]

and

\[
M_1(X^\nu) = \sum_{j=1}^{s} \log \mu_j.
\]

In particular, \( \Delta_1(X^\nu) \) is a product of \( \prod_{j=1}^{s} \mu_j \) with some cyclotomic polynomials.

(b) Assume that \( X \) is either smooth projective or \( H^1(X, \mathbb{Q}) \) has pure Hodge structure of weight \((1, 1)\). If \( \alpha_1(X^\nu, \mathbb{K}) > 0 \) for some field \( \mathbb{K} \) or \( M_1(X^\nu) > 0 \), \( \nu \) is orbifold effective by a large or small orbifold map.

The above formula for \( \alpha_1(X^\nu, \mathbb{K}) \) follows from Theorem 1.6 directly and the one for \( M_1(X^\nu) \) follows by a similar proof to Theorem 1.6. Theorem 1.10(b) follows from Theorem 1.5 and Remark 1.9.
1.3. **Application to hyperplane arrangements.** Let $X$ be the complement of a hyperplane arrangement in $\mathbb{P}^n$. Given an epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$ and a field $\mathbb{K}$, we denote $\nu_Z$ and $\nu_k$ the corresponding element in $H^1(X, \mathbb{Z})$ and $H^1(X, \mathbb{K})$, respectively. Note that $\nu_Z \cup \nu_Z = 0$, hence $\nu_k \cup \nu_k = 0$. Then we get two chain complexes by cup product

\[(H^*(X, \mathbb{Z}), \cdot \nu_Z): \quad H^0(X, \mathbb{Z}) \xrightarrow{\nu_Z} H^1(X, \mathbb{Z}) \xrightarrow{\nu_Z} H^2(X, \mathbb{Z}) \rightarrow \cdots.\]

\[(H^*(X, \mathbb{K}), \cdot \nu_k): \quad H^0(X, \mathbb{K}) \xrightarrow{\nu_k} H^1(X, \mathbb{K}) \xrightarrow{\nu_k} H^2(X, \mathbb{K}) \rightarrow \cdots.\]

**Definition 1.11.** With the above notations, we define the $i$-th Aomoto Betti number with $\mathbb{K}$-coefficients as

$$\beta_i(X, \nu_k) := \dim_{\mathbb{K}} H^i(H^*(X, \mathbb{K}), \nu_k).$$

and the $i$-th Aomoto torsion number as

$$\tau_i(X, \nu_Z) := |H^{i+1}(H^*(X, \mathbb{Z}), \cdot \nu_Z)_{\text{tor}}|.$$  

Here the shift by 1 is due to the Universal Coefficient Theorem.

Since it is well-known that the cohomology ring $H^*(X, \mathbb{Z})$ are combinatorially determined, so are $\beta_i(X, \nu_k)$ and $\tau_i(X, \nu_Z)$ once $\nu$, considered as an element in $H^1(X, \mathbb{Z})$, is fixed. For any field $\mathbb{K}$, Papadima and Suciu [PS10] showed that

$$\alpha_i(X, \nu_k) \leq \beta_i(X, \nu_k).$$

On the other hand, we give the following combinatorial upper bound for $M_i(X^\nu)$.

**Proposition 1.12.** Let $X$ be the complement of a hyperplane arrangement. For any epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$ and any $i \geq 0$, we have

$$\exp(M_i(X^\nu)) \mid \tau_i(X, \nu_Z).$$

In particular, if $\nu$ is orbifold effective of type $(g, r, \mu)$, then

$$\prod_{j=1}^s \mu_j \mid \tau_1(X, \nu_Z).$$

(3)

Based on Theorem 1.4(a), we give a different interpretation of (3) as follows.

**Corollary 1.13.** Let $X$ be a hyperplane arrangement complement. Let $V$ be a positive dimensional irreducible component of $\mathcal{V}^1(X, \mathbb{C})$ and say it has $m$ many parallel components (including itself) in $\mathcal{V}^1(X, \mathbb{C})$. An epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$ induces an embedding $\nu^*: \mathbb{C}^* \to \text{Hom}(\pi_1(X), \mathbb{C}^*)$. If there exists some $\rho \in \text{Hom}(\pi_1(X), \mathbb{C}^*)$ such that $\text{inv}^* \subset \rho \cdot V$, we have the following:

- If $\dim V \geq 2$, then $m$ divides $\tau_1(X, \nu_Z)$.
- If $\dim V = 1$, then $m + 1$ divides $\tau_1(X, \nu_Z)$. 

Remark 1.14. It is a long-standing open question for hyperplane arrangement if $\mathcal{V}(X, C)$ is combinatorially determined. By the tangent cone equality it is known that the positive dimension components of $\mathcal{V}(X, C)$ passing through the origin are combinatorially determined (see e.g. [Dim17, Theorem 6.1]) and they can be described by the multinet structure due to Falk and Yuzvinsky [FY07]. Then the above corollary gives a combinatorial upper bound for the number of its parallel components, which is new up to our knowledge. Under certain assumptions for the multinet (see Assumption 5.8), we show that $\mathcal{V}(X, C)$ has no parallel 2-dimensional components associated to this multinet.

On the other hand, there is an approach, called pointed multinet structure, to find the translated positive dimension component of $\mathcal{V}(X, C)$, which is introduced by Suciu, e.g., see [DS14, Definition].

Since $H_*(X, \mathbb{Z})$ is always torsion free for hyperplane arrangement complement $X$, people was wondering if the Milnor fiber of a central hyperplane arrangement also exhibits the same property. In [DS14], Denham and Suciu provided a complete answer to this question by showing that for any prime number $p \geq 2$ there is a central hyperplane arrangement whose Milnor fiber has non-trivial $p$-torsion in homology. Their results hold under the assumption that $p$ does not divide the number of hyperplanes and they asked if this assumption can be dropped [DS14, Question 8.12]. As an application of the $L^2$-type results, we follow Denham and Suciu’s construction in [DS14] and answer their question positively.

Theorem 1.15. For every prime $p \geq 2$, there is a central hyperplane arrangement whose Milnor fiber has non-trivial $p$-torsion in homology and $p$ divides the number of hyperplanes in the arrangement.

1.4. Structure of the paper. In section 2, we introduce the Alexander modules and Alexander polynomials for the pair $(X, \nu)$ and study their relations with the $L^2$-type invariants. Section 3 is devoted to the cohomology jump loci and detailed calculations of these invariants for the orbifold group. In section 4, we give the proof for Theorem 1.6 and Theorem 1.10. Section 5 is devoted to hyperplane arrangements. In section 5.1, we prove Proposition 1.12 and compute some examples; while in section 5.2, we show that if the hyperplane arrangement admits certain kind of combinatoric structure, then $\mathcal{V}(X, C)$ has no parallel 2-dimensional components for the multinet. In section 5.3, we prove Theorem 1.15.

1.5. Notations.

(1) $\mathbb{K}$ is assumed to be an algebraically closed filed in the rest of paper and char($\mathbb{K}$) denotes the characteristic of the field $\mathbb{K}$.

(2) For any prime number $p \geq 2$, let $\overline{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$ with $p$ elements.

(3) Let $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the collection of positive integers and non-negative integers, respectively.

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2. Alexander and $L^2$-type invariants

In this section, we always assume that $X$ is a connected finite CW-complex with a fixed group epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$. Denote by $X^\nu$ and $X^\nu,N$ the corresponding covering spaces of $X$ as introduced in Section 1.1. We study various topological properties of $X^\nu$ and its approximation $X^\nu,N$.

2.1. Algebraic Preliminaries. We start with some basic commutative algebra facts. Let $R$ be a Noetherian UFD and $M$ be a finitely generated $R$-module. The rank of $M$, denoted by $\text{rank } M$, is defined to be $\dim Q \otimes_R M$, where $Q$ is the fraction field of $R$. Consider a finite presentation of $M$

$$R^p \xrightarrow{\partial} R^q \longrightarrow M \longrightarrow 0,$$

where $\partial$ is a $(p \times q)$ matrix in $R$. Assume that the matrix $\partial$ has rank $r$. Let $I(M)$ denote the ideal generated by all possible $(r \times r)$-minors of $\partial$. Two finite presentations of $M$ can be related by a sequence of elementary operations, so this ideal does not depend on the choice of presentation. Since $R$ is a UFD, we can define the greatest common divisor of an ideal and set

$$\Delta(M) := \gcd(I(M)).$$

Then $\Delta(M)$ is defined uniquely up to a multiplication with a unit of $R$. When $\partial = 0$, $\Delta(M) = 1$ by convention.

Lemma 2.1. [Tur01, Lemma 4.9] With the assumptions and notations as above, we have

$$\Delta(M) = \Delta(M_{\text{tor}}),$$

where $M_{\text{tor}}$ denote the torsion sub-module of $M$.

Lemma 2.2. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated $R$-modules. Then we have

$$\Delta(M) | \Delta(M') \cdot \Delta(M'').$$

Moreover, if $\text{rank } M = \text{rank } M''$ (i.e. $\text{rank } M' = 0$), we have

$$\Delta(M) = \Delta(M') \cdot \Delta(M'').$$

Proof. The first claim follows from Horseshoe Lemma [Wei94, Horseshoe Lemma 2.2.8]. In fact, for a finite presentation of $M'$ with matrix $\partial'$ and one for $M''$ with matrix $\partial''$, Horseshoe Lemma gives a way to construct a finite presentation for $M$ with matrix

$$\begin{pmatrix} \partial' & * \\ 0 & \partial'' \end{pmatrix}.$$
Since localization is an exact functor, we get \( \text{rank} M = \text{rank} M' + \text{rank} M'' \), which implies
\[
\text{rank} \begin{pmatrix} \partial' & \ast \\ 0 & \partial'' \end{pmatrix} = \text{rank} \partial' + \text{rank} \partial''.
\]

Then it is easy to see \( \Delta(M) \mid \Delta(M') \cdot \Delta(M'') \).

For the second claim, we have the following commutative diagram:
\[
\begin{array}{ccc}
M & \longrightarrow & M'' \\
\downarrow & & \downarrow \\
M \otimes_R Q & \longrightarrow & M'' \otimes_R Q
\end{array}
\]

where \( Q \) is the fraction field of \( R \). The additional assumption \( \text{rank} M = \text{rank} M'' \) implies that the bottom horizontal map is an isomorphism. We claim that the induced map \( M_{\text{tor}} \rightarrow M_{\text{tor}}'' \) is surjective. For any \( x \in M_{\text{tor}}'' \), there exists \( y \in M \) maps to \( x \). Note that \( x = 0 \) in \( M'' \otimes_R Q \), hence so is \( y \), which implies \( y \in M_{\text{tor}} \). Note that taking torsion part is a left exact functor. Putting all together we get a short exact sequence of finitely generated \( R \)-modules:
\[
0 \rightarrow M_{\text{tor}}' \rightarrow M_{\text{tor}} \rightarrow M_{\text{tor}}'' \rightarrow 0.
\]

It implies \( \Delta(M_{\text{tor}}) = \Delta(M_{\text{tor}}') \cdot \Delta(M_{\text{tor}}'') \), e.g. see [SW77, Lemma 2.1 (i)]. Then the claim follows from Lemma 2.1 directly.

2.2. **Alexander modules and Alexander polynomials.** Recall that \( X \) is a connected finite CW-complex with a fixed group epimorphism \( \nu : \pi_1(X) \rightarrow \mathbb{Z} \). The group of covering transformations of \( X' \) is isomorphic to \( \mathbb{Z} \) and acts on it. By choosing lifts of the cells of \( X \) to \( X' \), we obtain a free basis for the cellular chain complex (with \( R \)-coefficients) of \( X' \) as \( R[t, t^{-1}] \)-modules, where \( R[t, t^{-1}] = R[\mathbb{Z}] \). So the cellular chain complex of \( X' \), \( C_*(X', R) \), is a bounded complex of finitely generated free \( R[t, t^{-1}] \)-modules:
\[
\cdots \rightarrow C_{i+1}(X', R) \overset{\partial^R_i}{\rightarrow} C_i(X', R) \overset{\partial^R_{i-1}}{\rightarrow} C_{i-1}(X', R) \overset{\partial^R_{i-2}}{\rightarrow} \cdots \overset{\partial^R_0}{\rightarrow} C_0(X', R) \rightarrow 0.
\]

With the above free basis for \( C_*(X', R) \), \( \partial^R_i \) can be written down as a matrix with entries in \( R[t, t^{-1}] \). Note that \( R[t, t^{-1}] \) is also a Noetherian UFD.

**Definition 2.3.** The \( i \)-th homology group \( H_i(X', R) \) of \( C_*(X', R) \), regarded as a finitely generated \( R[t, t^{-1}] \)-module, is called the \( i \)-th homology Alexander module of the pair \( (X, \nu) \) with \( R \)-coefficients. \( \Delta(H_i(X', R)) \) is called the \( i \)-th Alexander polynomial of \( (X, \nu) \) with \( R \)-coefficients, denoted by \( \Delta_i(X', R) \).

Consider the case \( R = \mathbb{Z} \). Note that \( \Delta_i(X', \mathbb{Z}) \) is only defined up to multiplication by a unit in \( \mathbb{Z}[t, t^{-1}] \). But there is a unique representative of the associate class with no negative powers of \( t \), non-zero constant term and positive coefficient for the leading term. We simply use the notation \( \Delta_i(X', \mathbb{Z}) \) to denote this representative. In particular, \( \Delta_i(X') \) is an integer valued polynomial.
Note that the homology Alexander modules and Alexander polynomials are homotopy invariants for the pair \((X, \nu)\). In particular, \(\Delta_1(X^\nu, R)\) depends only on \(\pi_1(X)\) and \(\nu\). One can also define \(\Delta_i(X^\nu, R)\) from the map \(\partial_i^R\) in (4) directly.

**Proposition 2.4.** Suppose that the \(R[t, t^{-1}]\)-module map \(\partial_i^R\) in (4) has rank \(r\). Then \(\Delta_i(X^\nu, R)\) equals to the greatest common divisor of all possible \((r \times r)\)-minors of \(\partial_i^R\).

*Proof.* Note that we have the following short exact sequence of \(R[t, t^{-1}]\)-modules
\[
0 \to H_i(X^\nu, R) \to C_i(X^\nu, R)/\text{im}\partial_i^R \to \text{im}\partial_{i-1}^R \to 0,
\]
where \(\text{im}\) denotes the image functor. Since \(\text{im}\partial_{i-1}^R\) is torsion free, we get
\[
H_i(X^\nu, R)_{\text{tor}} \cong (C_i(X^\nu, R)/\text{im}\partial_i^R)_{\text{tor}}.
\]
Then Lemma 2.1 implies that
\[
\Delta_i(X^\nu, R) = \Delta(H_i(X^\nu, R)_{\text{tor}}) = \Delta((C_i(X^\nu, R)/\text{im}\partial_i^R)_{\text{tor}}) = \Delta(C_i(X^\nu, R)/\text{im}\partial_i^R).
\]
Consider the finite presentation of \(C_i(X^\nu, R)/\text{im}\partial_i^R\)
\[
C_{i+1}(X^\nu, R) \xrightarrow{\partial_i^R} C_i(X^\nu, R) \xrightarrow{\partial_i^R} C_i(X^\nu, R)/\text{im}\partial_i^R \to 0
\]
Then the claim follows by definition. \(\square\)

### 2.3. \(L^2\)-type invariants.

#### 2.3.1. \(L^2\)-Betti number. As mentioned in Section 1.1, for any field \(\mathbb{K}\) the following limit
\[
\alpha_i(X^\nu, \mathbb{K}) = \lim_{N \to \infty} \frac{\dim H_i(X^\nu, \mathbb{K})}{N}
\]
exists. Note that Betti number with \(\mathbb{K}\)-coefficients only depend on \(\text{char}(\mathbb{K})\), not on the specific choice of the field \(\mathbb{K}\). So without loss of generality, we only need to consider the case where \(\mathbb{K}\) is algebraically closed. In this subsection, we give a direct proof for the existence of the limit and relate these limits with Alexander modules.

**Proposition 2.5.** With the assumptions and notations as above, for any \(i \geq 0\) the limit
\[
\lim_{N \to \infty} \frac{\dim H_i(X^\nu, \mathbb{K})}{N}
\]
exists. Moreover, we have
\[
\alpha_i(X^\nu, \mathbb{K}) = \text{rank} H_i(X^\nu, \mathbb{K}) = \dim H_i(X, L_\rho)
\]
where \(\rho \in \mathbb{K}^*\) is general and \(L_\rho\) is the corresponding rank one \(\mathbb{K}\)-local system on \(X\) pulling back by \(\nu\).

For the proof of this proposition, we need the following lemma.
Lemma 2.6. Given a nonzero polynomial \( h(t) = t^d + \cdots + a_1 t + a_0 \in \mathbb{K}[t] \), we denote by \( J_N \) the following \( N \times N \) \((N \geq d)\) matrix

\[
J_N := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

Then there exists a positive integer \( c \) such that for \( N \) large enough we have

\[
N - c \leq \text{rank}(J_N) \leq N,
\]

where \( c \) does not depend on \( N \).

Proof. Set \( g(t) = \gcd(t^N - 1, h(t)) \). Then there exist \( u_1(t), u_2(t) \in \mathbb{K}[t] \) such that

\[
g(t) = u_1(t)(t^N - 1) + u_2(t)h(t).
\]

Note that \( J_N^N = \text{Id} \), where \( \text{Id} \) is the \((N \times N)\) identity matrix. It gives that

\[
g(J_N) = u_1(J_N)(J_N^N - \text{Id}) + u_2(J_N)h(J_N) = u_2(J_N)h(J_N),
\]

hence \( \text{rank}(J_N) \leq \text{rank}(J_N) \). On the other hand, since \( g(t) \mid h(t) \), we have \( \text{rank}(J_N) \geq \text{rank}(J_N) \). Hence

\[
\text{rank}(J_N) = \text{rank}(J_N).
\]

There exists a positive integer \( c \), which does not depend on \( N \), such that \( g(t) \mid t^c - 1 \). In fact, since \( \mathbb{K} \) is algebraically closed, we can write down \( h(t) = \prod_{j=1}^d (t - \alpha_j) \). Collect all \( \alpha_j \) such that there exists some positive integer \( n_j \) such that \( \alpha_j^{n_j} = 1 \). When \( \text{char}(\mathbb{K}) = 0 \), \( c \) can be taken as the product of such \( n_j \); while when \( \text{char}(\mathbb{K}) = p > 0 \), \( c \) can be taken as the product of such \( n_j \) with some power of \( p \).

Note that for any field \( \mathbb{K} \) the dimension of \( \ker(J_N^N - \text{Id}) \) is \( \gcd(c, N) \). Hence for \( N \) large enough we have

\[
\text{rank}(J_N) = \text{rank}(J_N) \geq \text{rank}(J_N^N - \text{Id}) \geq N - c
\]

The other part of the inequality is obvious. \( \square \)

Proof of Proposition 2.5. Recall that \( C_*(X^n, \mathbb{K}) \) is the cellular chain complex of \( X^n \) with the field coefficient \( \mathbb{K} \). Each \( C_*(X^n, \mathbb{K}) \) is a free \( \mathbb{K}[t^{\pm 1}] \)-module with finite rank. Since \( \mathbb{K}[t^{\pm 1}] \) is a PID, by choosing suitable basis we can assume that \( \partial^\mathbb{K}_i \) has the following form

\[
\begin{pmatrix}
h_1 \\
\vdots \\
h_{m_i}
\end{pmatrix}
\]

with \( m_i = \text{rank}\partial^\mathbb{K}_i \). In particular,

\[
\text{rank}H_i(X^n, \mathbb{K}) = \text{rank}C_i(X^n, \mathbb{K}) - m_i - m_{i-1}.
\]
Let us recall the definition of Mahler measure based on Jensen’s formula
\[
\log \prod_{j=1}^d \max \{1, |\alpha_j|\}.
\]
We say that \( h \) is of cyclotomic type, if all its roots \( \alpha_j \) are roots of unity. In particular, in this case \( \mathcal{M}(h) = \log |a_d| \).
Note that the Mahler measure of the Alexander polynomial $\Delta_i(X^\nu)$ is well defined since the units of $\mathbb{Z}[t, t^{-1}]$ are the monomials $\pm t^k$ for some $k \in \mathbb{Z}$.

**Theorem 2.8.** [Le14, Theorem 5] With the above assumptions and notations, for any $i \geq 0$ the limit $\lim_{N \to \infty} \log \frac{|H_i(X^{\nu,N}, \mathbb{Z})_{\text{tor}}|}{N}$ exists and we have that

$$
\lim_{N \to \infty} \log \frac{|H_i(X^{\nu,N}, \mathbb{Z})_{\text{tor}}|}{N} = M(\Delta_i(X^\nu)).
$$

Based on the above theorem, we set

$$
M_i(X^\nu) := \lim_{N \to \infty} \log \frac{|H_i(X^{\nu,N}, \mathbb{Z})_{\text{tor}}|}{N} = M(\Delta_i(X^\nu)).
$$

### 2.3.3. Universal Coefficients Theorem

By the Universal Coefficients Theorem,

$$
\dim H_i(X, \mathbb{C}) \leq \dim H_i(X, \mathbb{K})
$$

for any field coefficient $\mathbb{K}$. Moreover, if $\text{char}(\mathbb{K}) = p > 0$ and the inequality is strict, then either $H_i(X, \mathbb{Z})$ or $H_{i-1}(X, \mathbb{Z})$ has non-trivial $p$-torsion. On the other hand, if $H_i(X, \mathbb{Z})$ has non-trivial $p$-torsion, then

$$
\dim H_i(X, \mathbb{C}) < \dim H_i(X, \mathbb{F}_p) \text{ and } \dim H_{i+1}(X, \mathbb{C}) < \dim H_{i+1}(X, \mathbb{F}_p)
$$

**Proposition 2.9.** Let $X$ be a connected finite CW-complex with a fixed group epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$. Then we have $\alpha_i(X^{\nu}, \mathbb{C}) \leq \alpha_i(X^{\nu}, \mathbb{F}_p)$. Moreover, if $\alpha_i(X^{\nu}, \mathbb{C}) < \alpha_i(X^{\nu}, \mathbb{F}_p)$, then we have the following

1. For $N$ large enough either $H_i(X^{\nu,N}, \mathbb{Z})$ or $H_{i-1}(X^{\nu,N}, \mathbb{Z})$ has non-trivial $p$-torsion.
2. $\Delta_i(X^\nu) = 0$ or $\Delta_{i-1}(X^\nu) = 0$ in $\mathbb{F}_p[t, t^{-1}]$, hence $M_i(X^\nu) > 0$ or $M_{i-1}(X^\nu) > 0$, respectively.

**Proof.** (1) Set $\epsilon = |\alpha_i(X, \mathbb{C}) - \alpha_i(X, \mathbb{F}_p)|/2$. For $N$ large enough, we have

$$
\frac{\dim_{\mathbb{C}} H_i(X^{\nu,N}, \mathbb{C})}{N} < \alpha_i(X^{\nu}, \mathbb{C}) + \epsilon = \alpha_i(X^{\nu}, \mathbb{F}_p) - \epsilon < \frac{\dim_{\mathbb{F}_p} H_i(X^{\nu,N}, \mathbb{F}_p)}{N}.
$$

Hence $\dim_{\mathbb{C}} H_i(X^{\nu,N}, \mathbb{C}) < \dim_{\mathbb{F}_p} H_i(X^{\nu,N}, \mathbb{F}_p)$ and the conclusion follows.

(2) $\alpha_i(X^{\nu}, \mathbb{C}) < \alpha_i(X^{\nu}, \mathbb{F}_p)$ is equivalent to

$$
\text{rank} C_i(X^{\nu}, \mathbb{C}) - \text{rank} \partial^C_i - \text{rank} \partial^E_{i-1} < \text{rank} C_i(X^{\nu}, \mathbb{F}_p) - \text{rank} \partial^E_i - \text{rank} \partial^E_{i-1},
$$

which implies either $\text{rank} \partial^E_i = \text{rank} \partial^C_i > \text{rank} \partial^E_{i-1}$ or the same inequality for degree $i - 1$. Note that $\text{rank} \partial^E_i > \text{rank} \partial^E_{i-1}$ if and only if $p \mid \Delta_i(X^\nu)$ (i.e., $\Delta_i(X^\nu) = 0$ in $\mathbb{F}_p[t, t^{-1}]$). In particular, $M_i(X^\nu) \geq \log p$. Then the claim follows. $\square$

**Remark 2.10.** If $\Delta_i(X^\nu)$ is of cyclotomic type and $M_i(X^\nu) \neq 0$, then there exists a prime number $p$ such that $\Delta_i(X^\nu) = 0$ in $\mathbb{F}_p[t, t^{-1}]$, hence

$$
\alpha_i(X^{\nu}, \mathbb{C}) < \alpha_i(X^{\nu}, \mathbb{F}_p) \text{ and } \alpha_{i+1}(X^{\nu}, \mathbb{C}) < \alpha_{i+1}(X^{\nu}, \mathbb{F}_p).
$$
2.4. **Aomoto complex.** Given an epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$, we denote $\nu_\mathbb{Z}$ the corresponding element in $H^1(X, \mathbb{Z})$. By obstruction theory, there is a map $g: X \to S^1$ and a class $\omega \in H^1(S^1, \mathbb{Z})$ such that $\nu_\mathbb{Z} = g^*(\omega)$. Hence we have

$$\nu_\mathbb{Z} \cup v_\mathbb{Z} = f^*(\omega \cup \omega) = 0.$$ 

For any field $\mathbb{K}$, there is a corresponding class $\nu_\mathbb{K} \in H^1(X, \mathbb{K})$ and we have $\nu_\mathbb{K} \cup v_\mathbb{K} = 0$. Then one gets the following two Aomoto complexes by cup product

\[
(H^*(X, \mathbb{Z}), \cdot) : \quad H^0(X, \mathbb{Z}) \xrightarrow{\nu_\mathbb{Z}} H^1(X, \mathbb{Z}) \xrightarrow{\nu_Z} H^2(X, \mathbb{Z}) \longrightarrow \cdots
\]

and

\[
(H^*(X, \mathbb{K}), \cdot) : \quad H^0(X, \mathbb{K}) \xrightarrow{\nu_\mathbb{K}} H^1(X, \mathbb{K}) \xrightarrow{\nu_\mathbb{K}} H^2(X, \mathbb{K}) \longrightarrow \cdots
\]

**Definition 2.11.** With the above assumptions and notations, we define the $i$-th Aomoto Betti number with $\mathbb{K}$-coefficients as

$$\beta_i(X, \nu_\mathbb{K}) := \dim_\mathbb{K} H^i(H^*(X, \mathbb{K}), \cdot_\mathbb{K}).$$

and the $i$-th Aomoto torsion number as

$$\tau_i(X, \nu_\mathbb{Z}) := |H^{i+1}(H^*(X, \mathbb{Z}), \cdot_\mathbb{K})_{\text{tor}}|.$$

Here the shift by 1 is due to the Universal Coefficient Theorem.

The following nice theorem due to Papadima-Suciu [PS10] gives the relation between $\alpha_i(X^\nu, \mathbb{K})$ and $\beta_i(X, \nu_\mathbb{K})$. In fact, they constructed a spectral sequence converges to $H_*(X^\nu, \mathbb{K})$ ([PS10, Proposition 8.1]) with the first page being the Aomoto complex [PS10, Corollary 8.3].

**Theorem 2.12.** Given the pair $(X, \nu)$ and any field $\mathbb{K}$, we have that

$$\alpha_i(X^\nu, \mathbb{K}) \leq \beta_i(X, \nu_\mathbb{K}).$$

Next we recall the following theorem due to Papadima-Suciu [PS10, Theorem 12.6] specialised to our case.

**Theorem 2.13.** Let $X$ be a connected finite minimal CW-complex with a fixed epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$. Then the linearization of the equivariant cochain complex of the cover $X^\nu$, with coefficients in $\mathbb{Z}$ or a field $\mathbb{K}$, coincides with the Aomoto complex $(H^*(X, \mathbb{Z}), \cdot_\mathbb{Z})$ or $(H^*(X, \mathbb{K}), \cdot_\mathbb{K})$, respectively.

As an application of this theorem, we have the following result.

**Proposition 2.14.** If $X$ is a connected finite minimal CW complex with a fixed epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$ such that $\alpha_i(X^\nu, \mathbb{C}) = \beta_i(X, \nu_\mathbb{C})$, then we have

$$\Delta'_i(X^\nu)(1) | \tau_i(X, \nu_\mathbb{Z}),$$

where $\Delta'_i(X^\nu)$ is obtained from $\Delta_i(X^\nu)$ by taking out $t - 1$ factors. The same conclusion also holds at homological degree $i - 1$. 
Proof. When \( X \) is a minimal CW complex, up to homotopy and under suitable basis we can assume \( \partial_i^X (1) = 0 \) and write \( \partial_i^X \) in matrix form \((t - 1)A_i\). Under the same basis, the above theorem shows that the map \( H^i(X, \mathbb{Z}) \xrightarrow{\cong} H^{i+1}(X, \mathbb{Z}) \) can be identified with \( A_i(1) \) (up to dual operation). Note that

\[
\alpha_i(X^\nu, \mathbb{C}) = b_i(X) - \text{rank}\partial_i^Z - \text{rank}\partial_{i-1}^Z
\]

and

\[
\beta_i(X, \nu_\mathbb{C}) = b_i(X) - \text{rank}A_i(1) - \text{rank}A_{i-1}(1).
\]

Since \( \text{rank}\partial_i^Z = \text{rank}(t - 1)A_i = \text{rank}A_i \), \( \alpha_i(X^\nu, \mathbb{C}) = \beta_i(X, \nu_\mathbb{C}) \) implies

\[
\text{rank}A_i = \text{rank}A_i(1) \quad \text{and} \quad \text{rank}A_{i-1} = \text{rank}A_{i-1}(1).
\]

Let \( r_i \) denote the rank of \( A_i \). For a \((r_i \times r_i)\)-minor of \( A_i \), say \( g \), it may happen \( g(1) = 0 \). Hence \( \Delta(A_i) \) taking values at 1 divides the greatest common divisor of all possible \((r_i \times r_i)\)-minors of \( A_i(1) \), which coincides with \( \tau_i(X, \nu_\mathbb{Z}) \). In particular, \( \Delta(A_i) \) taking value at 1 is not 0. Then the conclusion follows by noting that \( \Delta(A_i) = \Delta_i(X^\nu) \). The claim for homological degree \( i - 1 \) follows by the same argument. \( \square \)

Hyperplane arrangement complement satisfies the assumptions in the above proposition. For more details, see Section 5.

3. Cohomology jump loci and orbifold groups

3.1. Cohomology jump loci. Let \( X \) be a connected finite CW-complex with \( \pi_1(X) = G \) and \( \mathbb{K} \) be an algebraically closed field, e.g. \( \mathbb{C}, \overline{\mathbb{F}}_p \). The group of \( \mathbb{K} \)-valued characters, \( \text{Hom}(G, \mathbb{K}^*) \), is the moduli space of rank one \( \mathbb{K} \)-local system on \( X \). The cohomology jump loci \( \mathcal{V}_k^i(X, \mathbb{K}) \) of \( X \) are defined as in Definition 1.2. Cohomology jump loci are closed sub-varieties of \( \text{Hom}(G, \mathbb{K}^*) \) and homotopy invariants of \( X \). For cohomological degree one, \( \mathcal{V}_1^i(X, \mathbb{K}) \) depends only on \( \pi_1(X) \) (e.g. see [Suc11, Section 2.2]). So for any finitely presented group \( G \), \( \mathcal{V}_k^1(G, \mathbb{K}) \) is well defined.

Remark 3.1. One can also define the homology jump loci of \( X \) as follows

\[
\mathcal{W}_i^k(X, \mathbb{K}) := \{ \rho \in \text{Hom}(G, \mathbb{K}^*) \mid \dim_{\mathbb{K}} H_i(X, L_\rho) \geq k \}.
\]

Let \( \rho^{-1} \) denote the inverse of \( \rho \) in \( \text{Hom}(G, \mathbb{K}^*) \). Then we have the following isomorphism between \( \mathbb{K} \)-vector spaces [Dim04, p. 50]

\[
H^i(X, L_{\rho^{-1}}) \cong \text{Hom}_{\mathbb{K}}(H_i(X, L_\rho), \mathbb{K}),
\]

which gives

\[
\mathcal{V}_i^k(X, \mathbb{K}) = \{ \rho \in \text{Hom}(G, \mathbb{K}^*) \mid \rho^{-1} \in \mathcal{W}_i^k(X, \mathbb{K}) \}.
\]

So \( \mathcal{V}_k^1(X, \mathbb{K}) \) and \( \mathcal{W}_i^1(X, \mathbb{K}) \) share the same information.

Remark 3.2. If \( X \) is a finite connected CW complex, then \( \mathcal{V}_0^0(X, \mathbb{K}) = \{ \mathbb{K}_X \} \) consists of just one point, the trivial rank-one local system, and \( \mathcal{V}_k^0(X, \mathbb{K}) = \emptyset \) for \( k > 1 \).
Chomology jump loci can be viewed as generalizations of $\alpha_*(X^\nu, K)$. In fact, consider an epimorphism $\nu: \pi_1(X) \twoheadrightarrow \mathbb{Z}$. It induces an injective map 

$$\nu^*: K^* \to \text{Hom}(G, K^*)$$

By Remark 3.1 and Proposition 2.5, it is easy to see that $\alpha_i(X^\nu, K) = k$ if and only if $\text{im} \nu^* \subseteq V^i_k(X, K)$ and $\text{im} \nu^* \nsubseteq V^i_{k+1}(X, K)$.

The corresponding generalizations of $\beta_*(X, \nu_K)$ are the resonance varieties. For its definition and properties, see e.g. [Suc11].

The following structure theorem for cohomology jump loci of complex smooth quasi-projective varieties are contributed by many people and we name a few here: Green and Lazarsfeld [GL91], Simpson [Sim93], Arapura [Ara97], Dimca, Papadima [DP14], Dimca, Papadima and Suciu [DPS09], etc. It is finalized by Budur and Wang in [BW15, BW20].

**Theorem 3.3.** [BW15, BW20] If $X$ is a complex smooth quasi-projective variety, then $V^i_k(X, \mathbb{C})$ is a finite union of torsion translated sub-tori of $\text{Hom}(G, \mathbb{C}^*)$.

The following question about the cohomology jump loci for arbitrary field coefficients are still open [Suc01, Section 3.5]:

**Question 3.4.** Does the structure theorem hold for $V^i_k(X, K)$ of a complex smooth quasi-projective variety $X$ with char($K$) > 0?

Theorem 3.3 implies the following property for the Alexander polynomial associated to the pair $(X, \nu)$.

**Proposition 3.5.** Let $X$ be a complex smooth quasi-projective variety. For any epimorphism $\nu: \pi_1(X) \twoheadrightarrow \mathbb{Z}$, the $i$-th Alexander polynomial $\Delta_i(X, \nu)$ is of cyclotomic type for any $i$. Hence $\exp(M_i(X^\nu))$ is a positive integer.

**Proof.** The structure theorem for cohomology jump loci implies that [BLW18, Proposition 1.4] all the roots of $\Delta_i(X^\nu, \mathbb{C})$ are roots of unity. Since $\Delta_i(X^\nu, \mathbb{C})$ and $\Delta_i(X^\nu)$ only differ by multiplication with a non-zero constant integer, the claim follows. □

**Remark 3.6.** Note that $\exp(M_i(X^\nu))$ is not always an integer for general $X$. For example, the integral Alexander polynomial of the complement $X$ of the knot $4_1$ in $S^3$ [SW02, p. 981] is given by

$$\Delta_1(X^\nu) = t^2 - 3t + 1,$$

hence $\exp(M_1(X^\nu)) = \frac{3 + \sqrt{5}}{2}$.

3.2. **Orbifold Groups.** In this subsection, we compute the degree 1 cohomology jump loci and $L^2$-type invariants for the orbifold group.

Let $\Sigma_{g,r}$ be a Riemann surface of genus $g \geq 0$ and with $r \geq 0$ points removed. To have $b_1(X) > 0$, we always assume either $g > 1$ or $g = 0, r > 1$, i.e., $X \neq \mathbb{C}\mathbb{P}^1, \mathbb{C}$. Consider $\Sigma_{g,r}$ with $s$ marked points $\{q_1, \ldots, q_s\}$ and a weight vector $\mu = (\mu_1, \ldots, \mu_s)$, where $\mu_i \in \mathbb{Z}_{>0}$,
as in [Dim07, ACM13, Suc14A]. The orbifold group $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)$ associated to these data is defined as

$$\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu) := \pi_1(\Sigma_{g,r}\backslash\{q_1, \dots, q_s\})/\langle \gamma_j^{\mu_j} = 1 \text{ for all } 1 \leq j \leq s\rangle,$$

where $\gamma_j$ is a meridian of $q_j$. If $\mu_j = 1$ for some $j$, we get the same orbifold group by omitting $q_j$ and $\mu_j$. So without loss of generality, we assume $\mu_j > 1$ for all $1 \leq j \leq s$.

### 3.2.1. Non-compact case

If $\Sigma_{g,r}$ is not compact (i.e., $r > 0$), then $\pi_1(\Sigma_{g,r})$ is a free group with rank $n = 2g + r - 1$, hence $\Gamma \cong F_n \ast \mathbb{Z}_{\mu_1} \ast \cdots \ast \mathbb{Z}_{\mu_s}$. Then we have

$$\text{Hom}(\Gamma, \mathbb{K}^*) := \{(t_1, \dots, t_n, \lambda_1, \dots, \lambda_s) \in (\mathbb{K}^*)^{n+s} | \lambda_j^{\mu_j} = 1 \text{ for all } 1 \leq j \leq s\}.$$

When $\text{char}(\mathbb{K}) = 0$, we have the following short exact sequence as in [ACM13, Proposition 2.7]:

$$1 \to (\mathbb{K}^*)^n \to \text{Hom}(\Gamma, \mathbb{K}^*) \to \oplus_{j=1}^s C_{\mu_j} \to 1,$$

where $C_{\mu_j}$ is the cyclic multiplicative subgroup of $\mathbb{K}^*$ with order $\mu_j$. Hence $\text{Hom}(\Gamma, \mathbb{K}^*)$ has $\prod_{j=1}^s \mu_j$ connected components, with every connected component isomorphic to $(\mathbb{K}^*)^n$.

On the other hand, when $\text{char}(\mathbb{K}) = p > 0$, let $\mu_j'$ be the largest positive integer such that $\gcd(p, \mu_j') = 1$ and $\mu_j' \mid \mu_j$ ($\mu_j'$ could be 1). Then we have the following short exact sequence:

$$1 \to (\mathbb{K}^*)^n \to \text{Hom}(\Gamma, \mathbb{K}^*) \to \oplus_{j=1}^s C_{\mu_j'} \to 1,$$

where $C_{\mu_j'}$ is the cyclic multiplicative subgroup of $\mathbb{K}^*$ with order $\mu_j'$. Hence $\text{Hom}(\Gamma, \mathbb{K}^*)$ has $\prod_{j=1}^s \mu_j'$ connected components, with every connected component isomorphic to $(\mathbb{K}^*)^n$. In particular, if $\mu_j$ is a power of $p$ for all $1 \leq j \leq s$ (i.e., $\prod_{j=1}^s \mu_j' = 1$), $\text{Hom}(\Gamma, \mathbb{K}^*) \cong (\mathbb{K}^*)^n$.

In both cases, for any $\rho = (t_1, \dots, t_n, \lambda_1, \dots, \lambda_s) \in \text{Hom}(\Gamma, \mathbb{K}^*)$, we set $\ell_{\mathbb{K}}(\rho, \mu)$ as the number of trivial coordinates $\lambda_j = 1$ such that $\text{char}(\mathbb{K}) = p \nmid \mu_j$. When $\rho$ is the trivial character, we simply write $\ell_{\mathbb{K}}(\mu)$.

**Proposition 3.7.** With the above notations, we have

$$\dim H^1(\Gamma, L_\rho) = \begin{cases} n + s - \ell_{\mathbb{K}}(\rho, \mu) - 1, & \text{if } L_\rho \neq \mathbb{K}_\Gamma, \\ n + s - \ell_{\mathbb{K}}(\mu), & \text{if } L_\rho = \mathbb{K}_\Gamma. \end{cases}$$

Here $\mathbb{K}_\Gamma$ denotes the trivial character on $\Gamma$.

**Proof.** Consider the Eilenberg–MacLane space $K(\pi_1, 1)$ of $\Gamma$, say $Y$. The homology group $H_1(\Gamma, L_\rho)$ can be computed by the chain complex $C_*(Y, L_\rho)$, see [Dim04, p. 50]. By Fox
calculus $\partial_2^\rho$ has the form of a $(n + s) \times s$ matrix as follows

$$
\begin{pmatrix}
\frac{\lambda_j^{s-1}}{\lambda_j-1} & 0 & \ldots & 0 \\
0 & \frac{\lambda_j^{s-1}}{\lambda_j-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{\lambda_j^{s-1}}{\lambda_j-1} \\
0 & 0 & \ldots & 0
\end{pmatrix}.
$$

(5)

Note that for any $1 \leq j \leq s$, $\frac{\lambda_j^{s-1}}{\lambda_j-1} \neq 0$ in $\mathbb{K}$ if and only if $\lambda_j = 1$ and $p \nmid \mu_j$. Then we see that the above matrix has rank $\ell_{\mathbb{K}}(\rho, \mu)$. If $L_\rho$ is not the constant sheaf, then the rank of $\partial_1^\rho$ is 1, hence

$$
\dim H_1(Y, L_\rho) = n + s - \ell_{\mathbb{K}}(\rho, \mu) - 1.
$$

On the other hand, if $L_\rho$ is the constant sheaf, then the rank of $\partial_1^\rho$ is 0, hence

$$
\dim H_1(Y, \mathbb{K}) = n + s - \ell_{\mathbb{K}}(\mu).
$$

Then the claim follows from Remark 3.1. \qed

3.2.2. Compact case. If $\Sigma_{g,r}$ is compact (i.e., $r = 0$), then

$$
\Gamma \cong (x_1, \ldots, x_g, y_1, \ldots, y_g, \gamma_1, \ldots, \gamma_s) \prod_{i=1}^g [x_i, y_i] \prod_{j=1}^s \gamma_j = 1, \gamma_j^{\mu_j} = 1 \text{ for all } 1 \leq j \leq s,
$$

hence

$$
\text{Hom}(\Gamma, \mathbb{K}^*) = \{(t_1, \ldots, t_{2g}, \lambda_1, \ldots, \lambda_s) \mid (\mathbb{K}^*)^{2g+s} | \lambda_1 \cdots \lambda_s = 1, \lambda_j^{\mu_j} = 1 \text{ for } 1 \leq j \leq s\}.
$$

Since we assumed $\mu_j > 1$ for all $1 \leq j \leq s$, we get $s \geq 2$.

As in the non-compact case, when $\text{char}(\mathbb{K}) = p > 0$, we set $\mu'_j$ to be the largest positive integer such that $\gcd(p, \mu'_j) = 1$ and $\mu'_j \mid \mu_j$. To unify the notations, we set $\mu'_j = \mu_j$ when $\text{char}(\mathbb{K}) = 0$. Then in both cases, we have the following short exact sequence:

$$
1 \to (\mathbb{K}^*)^{2g} \to \text{Hom}(\Gamma, \mathbb{K}^*) \to (\bigoplus_{j=1}^s C_{\mu'_j}) / C_{\mu'} \to 1,
$$

where $\mu' \coloneqq \text{lcm}(\mu'_1, \ldots, \mu'_s)$ and the last term is the cokernel of the natural mapping $C_{\mu'} \to \bigoplus_{j=1}^s C_{\mu'_j}$. Hence $\text{Hom}(\Gamma, \mathbb{K}^*)$ has $\prod_{j=1}^s \mu'_j$ connected components, with every connected component being isomorphic to $(\mathbb{K}^*)^{2g}$.

For any $\rho = (t_1, \ldots, t_{2g}, \lambda_1, \ldots, \lambda_s) \in \text{Hom}(\Gamma, \mathbb{K}^*)$, we set $\ell_{\mathbb{K}}(\rho, \mu)$ as the number of trivial coordinates $\lambda_j = 1$ such that $\text{char}(\mathbb{K}) = p \nmid \mu_j$. When $\rho$ is the trivial character, we simply write $\ell_{\mathbb{K}}(\mu)$. 
Proposition 3.8. With the notations above, we have
\[
\dim H^1(\Gamma, L_\rho) = \begin{cases} 
2g + s - 2 - \ell_\mathbb{K}(\rho, \mu), & \text{if } L_\rho \neq \mathbb{K}_\Gamma, \\
2g + s - 1 - \ell_\mathbb{K}(\mu), & \text{if } L_\rho = \mathbb{K}_\Gamma, \ell_\mathbb{K}(\mu) < s \\
2g, & \text{if } L_\rho = \mathbb{K}_\Gamma, \ell_\mathbb{K}(\mu) = s.
\end{cases}
\]

Proof. Consider the Eilenberg–MacLane space \(K(\pi_1, 1)\) of \(\Gamma\), say \(Y\). By Fox calculus \(\partial_2^0\) has the form of a \((2g + s) \times (s + 1)\) matrix as follows
\[
\begin{pmatrix}
\lambda_{s1}^{-1} & 0 & \cdots & 0 & 1 \\
0 & \lambda_{s2}^{-1} & \cdots & 0 & \lambda_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_{ss}^{-1} & \prod_{j=1}^{s-1} \lambda_j \\
0 & 0 & \cdots & 0 & y_1 - 1 \\
0 & 0 & \cdots & 0 & 1 - x_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & y_g - 1 \\
0 & 0 & \cdots & 0 & 1 - x_g
\end{pmatrix}
\]
(6)

If \(L_\rho\) is not the constant sheaf, this matrix has rank \(\ell_\mathbb{K}(\rho, \mu) + 1\) and \(\partial_2^0\) has rank 1, hence
\[
\dim H_1(Y, L_\rho) = 2g + s - 2 - \ell_\mathbb{K}(\rho, \mu).
\]

While for the constant sheaf case, if \(\ell_\mathbb{K}(\mu) = s\), the above matrix has rank \(s\); and if \(\ell_\mathbb{K}(\mu) < s\) (i.e., there exists some \(\mu_j\) such that \(\text{char}(\mathbb{K}) = p | \mu_j\)), the above matrix has rank \(\ell_\mathbb{K}(\mu) + 1\). Then we have
\[
\dim H_1(Y, \mathbb{K}) = \begin{cases} 
2g + s - \ell_\mathbb{K}(\mu) - 1, & \text{if } \ell_\mathbb{K}(\mu) < s \\
2g, & \text{if } \ell_\mathbb{K}(\mu) = s.
\end{cases}
\]

Then the claim follows from Remark 3.2.3. \qedsymbol

3.2.3. \(L^2\)-type invariants for orbifold group. Let \(\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)\) be the orbifold group associated to the data \((g, r, \mu)\). As mentioned before, any epimorphism \(\nu: \Gamma \to \mathbb{Z}\) gives a \(\mathbb{Z}\)-cover of the space \(K(\Gamma, 1)\). We set
\[
\Delta_1(\Gamma^\nu) := \Delta_1(K(\Gamma, 1)^\nu) \text{ and } \alpha_1(\Gamma^\nu, \mathbb{K}) := \alpha_1(K(G, 1)^\nu, \mathbb{K}).
\]

Proposition 3.9. With the above notations, for any epimorphism \(\nu: \Gamma \to \mathbb{Z}\), we have
\[
(1) \quad \alpha_1(\Gamma^\nu, \mathbb{K}) = -\chi(\Sigma_{g,r}) + \# \{j \mid \text{char}(\mathbb{K}) = p \text{ divides } \mu_j\}.
\]
\[
(2) \quad \Delta_1(\Gamma^\nu) = \begin{cases} 
\prod_{j=1}^s \mu_j, & \text{if } r > 0 \\
(\prod_{j=1}^s \mu_j) \cdot (t - 1), & \text{if } r = 0.
\end{cases}
\]
Proof. (1) Note that for any \( \rho \in \mathbb{K}^* \), the pulled character \( \nu^* \rho \) always have \( \ell_{\mathbb{K}}(\nu^* \rho, \mu) = \ell_{\mathbb{K}}(\mu) \), since the torsion part of the abelianization of \( \Gamma \) has to map to zero by \( \nu \). Then the claim follows by computation.

(2) If \( r > 0 \), \( \lambda_j = 1 \) implies \( \lambda_j - 1 = \mu_j \). Then the claim follows from (5). If \( r = 0 \), \( x_i \) and \( y_i \) in (6) will be replaced by some power of \( t \). \( \nu \) being epimorphism implies that the greatest common divisor of the non-zero powers is \( \pm 1 \). Then the claim follows. \( \Box \)

4. Proof of the main results

The following well known result will play a crucial role in the proofs of Theorem 1.6 and Theorem 1.10, e.g. see [CKO03, Lemma 3].

**Theorem 4.1.** Consider an orbifold map \( f : X \to \Sigma \) of type \((g,r,\mu)\). Let \( F \) denote the generic fiber of \( f \). Then we have a short exact sequence of groups

\[
\pi_1(F) \to \pi_1(X) \xrightarrow{f} \pi_1^{\text{orb}}(\Sigma_{g,r},\mu) \to 1,
\]

where the first map is induced by the inclusion from \( F \) to \( X \). In particular the kernel of \( f_* \) is finitely generated.

4.1. **Proof of Theorem 1.6.** By Remark 3.1, we only need to prove the theorem for the homology version. By Theorem 4.1, the orbifold map \( f \) gives the following group extension

\[
1 \to K \to G \xrightarrow{f} \Gamma \to 1,
\]

where \( G = \pi_1(X) \), \( \Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r},\mu) \) and the kernel \( K \) of \( f_* \) is a finitely generated normal subgroup of \( G \).

Given a character \( \rho \in \text{Hom}(\Gamma, \mathbb{K}^*) \), it gives a rank one \( \mathbb{K} \)-local system \( L_{\rho} \) of \( \Gamma \). Let \( f^*L_{\rho} \) denote the rank one local system pulling back to \( G \). Then we have the Hochschild-Serre spectral sequence:

\[
E^2_{s,t} = H_s(\Gamma, H_t(K, f^*L_{\rho})) \Rightarrow H_{s+t}(G, f^*L_{\rho}).
\]

Next we explain how to review \( H_1(K, f^*L_{\rho}) \) as a \( \Gamma \)-module. Let \( P_\bullet \) be a free \( G \)-resolution of \( \mathbb{Z} \), then the \( \Gamma \)-action on \( H_*(K, f^*L_{\rho}) = H_*(P_\bullet \otimes_K f^*L_{\rho}) \) is induced by the tensor product of the \( G \)-action on \( f^*L_{\rho} \) and \( \Gamma \)-action on \( P_\bullet \). Since \( K \) is the kernel of \( f_* \), \( f^*L_{\rho} \) taking restriction over \( K \) is the constant sheaf. Then we have

\[
H_*(K, f^*L_{\rho}) = H_*(P_\bullet \otimes_K f^*L_{\rho}) = H_*(P_\bullet)_K \otimes \mathbb{K} = H_*(K, \mathbb{Z}) \otimes L_{\rho}
\]
as \( \Gamma \)-modules. The \( \Gamma \)-action on \( H_*(K, \mathbb{Z}) \otimes L_{\rho} \) is the tensor product of the \( \Gamma \)-conjugation on \( H_*(K, \mathbb{Z}) \) and the \( \Gamma \)-action on \( \mathbb{K} \) induced by \( \rho \). In particular, we have \( E^2_{1,0} = H_1(\Gamma, L_{\rho}) \).

Then the spectral sequence gives us the following short exact sequence

\[
0 \to E^2_{0,1}/\text{im}d_2 \to H_1(G, f^*L_{\rho}) \to H_1(\Gamma, L_{\rho}) \to 0,
\]

where \( d_2 : E^2_{2,0} \to E^2_{0,1} \) is the differential map on \( E^2 \)-page. Hence

\[
\dim H_1(G, f^*L_{\rho}) \geq \dim H_1(\Gamma, L_{\rho})
\]

and the equality holds when \( E^2_{0,1} = 0 \).
Since $K$ is a finitely generated group, $H_1(K, \mathbb{Z}) \otimes L_\rho$ is a finite dimensional $\mathbb{K}$-vector space, say denoted by $W$. For any $\gamma \in \Gamma$, let $c(\gamma)$ denote the corresponding linear transformation on $W$ induced by the $\Gamma$-conjugation on $H_1(K, \mathbb{Z})$. Note that the linear transformation $c(\gamma)$ is independent with the character $\rho$. Therefore

$$E^2_{0,1} = H_0(\Gamma, H_1(K, \mathbb{Z}) \otimes L_\rho) \cong W/(c(\gamma) \otimes \rho(\gamma) - \text{Id})(w)|w \in W, \gamma \in \Gamma).$$

Recall the moduli space $\text{Hom}(\Gamma, \mathbb{K}^*)$ as in section 3.2. In the non-compact case, for any $\rho = (t_1, \cdots, t_n, \lambda_1, \cdots, \lambda_s) \in \text{Hom}(\Gamma, \mathbb{K}^*)$, $E^2_{0,1} = 0$ if we choose $t_i$ not equal to the eigenvalues of $c(x_i)$ for some $1 \leq i \leq n$. Here $x_i$ is the $i$-th generator of the group $\Gamma = F_1 \ast Z_{\mu_1} \ast \cdots \ast Z_{\mu_s}$. Then the claim follows since $c(x_i)$ has only finitely many eigenvalues and there are only finitely many choices for $\lambda_j$. The compact case follows by a similar proof.

4.2. **Proof of Theorem 1.10.** The formula for $\alpha_1(X^\nu, \mathbb{K})$ follows from Theorem 1.6 and the computations in subsection 3.2 directly. The rest part is devoted to the proof for $M_1(X^\nu)$.

By Definition 1.8, one indeed gets the following sequence of epimorphisms:

$$G \xrightarrow{f} \Gamma \twoheadrightarrow \mathbb{Z}.$$

The map $\Gamma \twoheadrightarrow \mathbb{Z}$ gives a rank one local system $\mathcal{L}$ of $\Gamma$ with stalk $\mathbb{Z}[t^\pm]$. Let $f^*\mathcal{L}$ denote the rank one local system pulling back to $G$. In particular,

$$H_1(\Gamma, \mathcal{L}) \cong H_1(\Gamma^\nu, \mathbb{Z}) \text{ and } H_1(G, f^*\mathcal{L}) \cong H_1(X^\nu, \mathbb{Z}).$$

Applying the Hochschild-Serre spectral sequence, we get:

$$E^2_{s,t} = H_s(\Gamma, H_t(K, f^*\mathcal{L})) \Rightarrow H_{s+t}(G, f^*\mathcal{L}),$$

where $K$ is the kernel of $f_*$. Similarly to the proof of Theorem 1.6, we have $E^2_{1,0} = H_1(\Gamma, \mathcal{L})$ and a short exact sequence of finitely generated $\mathbb{Z}[t^\pm]$-modules

$$0 \to E^2_{0,1}/\text{im}d_2 \to H_1(G, f^*\mathcal{L}) \to H_1(\Gamma, \mathcal{L}) \to 0.$$

On the other hand, we claim that $E^2_{0,1} = H_0(\Gamma, H_1(K, f^*\mathcal{L}))$ is a finitely generated abelian group. In fact,

$$H_0(\Gamma, H_1(K, f^*\mathcal{L})) \cong H_1(K, \mathbb{Z}) \otimes \mathbb{Z}[t^\pm]/\langle (c(\gamma) \otimes \nu(\gamma) - \text{Id})(x)|x \in H_1(K, \mathbb{Z}) \otimes \mathbb{Z}[t^\pm], \gamma \in \Gamma \rangle$$

where $c(\gamma)$ is the $\Gamma$-conjugation on $H_1(K, \mathbb{Z})$. Then the claim follows since $\Gamma \twoheadrightarrow \mathbb{Z}$ is surjective.

$E^2_{0,1}$ being a finitely generated abelian group implies that the rank of $E^2_{0,1}$ as a $\mathbb{Z}[t^\pm]$-module is 0, hence so is $E^2_{0,1}/\text{im}d_2$. By Lemma 2.2 we have

$$\Delta_1(G^\nu) = \Delta_1(\Gamma^\nu) \cdot \Delta(E^2_{0,1}/\text{im}d_2).$$

By Proposition 3.5, $\Delta_1(G^\nu)$ is of cyclotomic type, hence so is $\Delta(E^2_{0,1}/\text{im}d_2)$ and $\Delta_1(\Gamma^\nu)$. Then $M(\Delta(E^2_{0,1}/\text{im}d_2))$ only depends on the leading coefficient of $\Delta(E^2_{0,1}/\text{im}d_2)$. This leading coefficient is indeed 1. Otherwise, there exists a prime number $p$ which divides the leading coefficient. Since $\Delta(E^2_{0,1}/\text{im}d_2)$ is cyclotomic type, $p$ divides $\Delta(E^2_{0,1}/\text{im}d_2)$. By the definition of Alexander polynomial, this implies that rank of $E^2_{0,1}/\text{im}d_2 \otimes \mathbb{F}_p$ as a
$\mathbb{F}_p[t^{\pm}]$-module is at least one. In particular, $E_{0,1}^2/\text{im}d_2$ is not a finitely generated abelian group. This contradicts with the fact that $E_{0,1}^2$ is a finite generated abelian group.

Putting all together, we have that

$$M_1(X^\nu) = M_1(\Gamma^\nu) = \sum_{j=1}^s \log \mu_j,$$

where the last equality follows from Proposition 3.9.

### 4.3. Some remarks.

We close this section with some remarks for orbifold maps.

**Remark 4.2.** Given two orbifold maps $f_i: X \to \Sigma_i$ for $i = 1, 2$. We say that they are equivalent if there is an isomorphism $g: \Sigma_1 \to \Sigma_2$ such that $\mu_2(g(B_1)) = \mu_1(B_1)$ and the following diagram is commutative

$$\begin{array}{ccc}
X & \xrightarrow{f_1} & \Sigma_1 \\
\downarrow{f_2} & & \downarrow{g} \\
\Sigma_2 & \xrightarrow{\quad} & \Sigma_2
\end{array}$$

Then Theorem 1.4 indeed implies that there are at most finitely many equivalent large or small orbifold maps for $X$ [Ara97, Theorem 1.6], since $V^1(X, \mathbb{C})$ is algebraic. For an orbifold map $f: X \to \Sigma$, the image of the induced embedding $\text{Hom}(\pi_1(\Sigma), \mathbb{C}^*) \hookrightarrow \text{Hom}(\pi_1(X), \mathbb{C}^*)$ is called the shadow of $f$ as in [ACM13]. Then for any two either large or small orbifold maps $f_i: X \to \Sigma_i$ with $i = 1, 2$, their shadows are either equal or intersect at the trivial character [ACM13, Lemma 6.4].

**Remark 4.3.** Assume that $\nu: \pi_1(X) \rightarrow \mathbb{Z}$ is orbifold effective. Note that for the last claim in Definition 1.8 it is possible that $\nu$ factors through two different orbifold maps.

If the associated orbifold map $f$ in Definition 1.8 is either large or small, $f$ is unique under the equivalence relationship given in the above remark. In fact, $\nu$ and $f$ induces two injective maps

$$\nu_C^*: \mathbb{C}^* \hookrightarrow \text{Hom}(\pi_1(X), \mathbb{C}^*) \quad \text{and} \quad f_C^*: \text{Hom}(\pi_1(\Sigma), \mathbb{C}^*) \hookrightarrow \text{Hom}(\pi_1(X), \mathbb{C}^*).$$

In particular, $\text{im}\nu_C^* \subseteq \text{im}f_C^*$. Remark 4.2 shows that $f$ is unique, hence it makes sense to call $\nu$ of type $(g, r, \mu)$.

On the other hand, when $f$ is null, $\Sigma$ is either $\mathbb{C}^*$ or an Elliptic curve and $\mu$ is trivial. In particular, $\nu$ is of type $(0, 2, 1)$ or $(1, 0, 1)$, respectively.

**Remark 4.4.** Consider $\nu$ as an element in $H^1(X, \mathbb{Z})$. Under certain Hodge structure assumptions for $\nu$, it is always orbifold effective.

1. $\nu: \pi_1(X) \rightarrow \mathbb{Z}$ is induced by an algebraic map $h: X \rightarrow \mathbb{C}^*$ if and only if, when considered as an element in $H^1(X, \mathbb{Z})$, $\nu$ is of Hodge type $(1, 1)$, i.e., $\nu \in F^1H^1(X, \mathbb{C}) \cap F^1H^1(X, \mathbb{C})$. Here $F$ stands for the Hodge filtration. This follows from Deligne’s theory of 1-motives (cf. [Del74, (10.I.3)]). Then $\nu$ is orbifold effective in this case.
In fact, projectivising and resolving indeterminacy, we get a map \( \overline{h} : \overline{X} \to \mathbb{CP}^1 \).

Using Stein factorization, we get the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \overline{X} \\
\downarrow^{h} & & \downarrow^{h'} \\
\text{im}(f) & \xrightarrow{h''} & S \\
\downarrow & & \downarrow \\
\mathbb{C}^* & \to & \mathbb{CP}^1,
\end{array}
\]

where \( h'' \) is a finite map, \( h' \) has connected fiber and \( f := h'|_X : X \to \text{im}(f) \). [Dim07, Lemma 2.2] shows that \( f \) has connected generic fiber. Hence \( f \) is an orbifold map and it makes \( \nu \) orbifold effective.

In particular, if \( H^1(X, \mathbb{Q}) \) has pure Hodge structure of weight \((1, 1)\), then any epimorphism \( \nu : \pi_1(X) \to \mathbb{Z} \) is orbifold effective. This assumption is equivalent to that \( X \) has a smooth compactification \( \overline{X} \) such that \( H^1(\overline{X}, \mathbb{Q}) = 0 \). Typical examples are hyperplane arrangement complement, toric arrangement complement and the complement of some algebraic curves in \( \mathbb{CP}^2 \). See [Dim07, Example 2.3] for more examples.

(2) \( \nu : \pi_1(X) \to \mathbb{Z} \) factors through the induced map on the fundamental group level by an algebraic map \( f : X \to E \) with \( E \) being an Elliptic curve, if and only if, \( \nu \), considered as an element in \( H^1(X, \mathbb{Z}) \), is contained in a dimension 2 weight 1 pure sub-Hodge structure of \( H^1(X, \mathbb{Q}) \). This also follows from Deligne’s theory of 1-motives (cf. [Del74, (10.I.3)]). Using a similar proof as above, \( \nu \) is also orbifold effective in this case.

(3) Let \( \text{alb} : X \to \text{Alb}_X \) denote the Albanese map of \( X \), where \( \text{Alb}_X \) is a semi-abelian variety. Since the epimorphism \( \nu \) only concerns the fundamental group, using Lefschetz hyperplane section theorem, we can assume that \( X \) has complex dimension 2. Then \( \text{imalb} \) has dimension either 1 or 2. If \( \text{dimimalb} = 1 \), then any \( \nu : \pi_1(X) \to \mathbb{Z} \) is orbifold effective by a similar proof as above.

Example 4.5. In general \( \nu \) is not necessarily always orbifold effective. Consider \( X = E \times \mathbb{C}^* \) with \( E \) being an Elliptic curve. Then we have \( \pi_1(X, \mathbb{Z}) \cong \pi_1(E, \mathbb{Z}) \oplus \pi_1(\mathbb{C}^*, \mathbb{Z}) \cong \mathbb{Z}^3 \).

Take a corresponding basis for this direct sum. Say \( \nu : \mathbb{Z}^3 \to \mathbb{Z} \) is given by three integers \((a, b, c)\). Then \( \nu \) is orbifold effective, if and only if, either \( c = 0 \) or \( a = 0 = b \). In fact, let \( f : X \to \Sigma \) be an orbifold map. Then \( f \) induces an injective map \( H^1(\Sigma, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \) of mixed Hodge structures. Then we have the following three possible cases:

- If the mixed Hodge structure on \( H^1(\Sigma, \mathbb{Q}) \) has pure weight one, then \( c = 0 \).
- If the mixed Hodge structure on \( H^1(\Sigma, \mathbb{Q}) \) has pure weight two, then \( a = 0 = b \).
• $H^1(\Sigma, \mathbb{Q})$ is isomorphic to $H^1(X, \mathbb{Q})$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \Sigma \\
\downarrow & & \downarrow \\
\text{Alb}_X & \longrightarrow & \text{Alb}_\Sigma,
\end{array}
$$

where the left vertical map and the bottom horizontal map are both isomorphisms. This is impossible since $\Sigma$ has complex dimension one.

5. Hyperplane arrangement

In this section, we specialize to the case of hyperplane arrangements. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$, defined by a product of degree 1 homogeneous polynomials

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} f_H.$$ 

This also gives a central hyperplane arrangement in $\mathbb{C}^{n+1}$, denoted by $\overline{\mathcal{A}}$. Set

$$X(\mathcal{A}) := \mathbb{P}^n - \mathcal{A} \text{ and } M(\mathcal{A}) := \mathbb{C}^{n+1} \setminus \overline{\mathcal{A}}.$$ 

Note that $M(\mathcal{A}) = X(\mathcal{A}) \times \mathbb{C}^*$, hence $\mathcal{V}^1(X(\mathcal{A}), \mathbb{K}) = \mathcal{V}^1(M(\mathcal{A}), \mathbb{K})$. When the context is clear, we simply write $X(\mathcal{A})$ as $X$.

5.1. Some properties and examples. We first list some facts for hyperplane arrangement complement $X$.

1. $X$ is homotopy equivalent to a minimal CW complex [DP03, Ran02], hence $H^i(X, \mathbb{Z})$ is a free abelian group for any $0 \leq i \leq n$.
2. The cohomology ring $H^*(X, \mathbb{Z})$ is determined by the combinatorial data [Dim17, Chapter 3].
3. For any epimorphism $\nu: \pi_1(X) \twoheadrightarrow \mathbb{Z}$, $\alpha_i(X^\nu, \mathbb{C}) = \beta_i(X, \nu|_C)$ for any $i$. This follows from the tangent cone equality between $\mathcal{V}_i^1(X, \mathbb{C})$ and the corresponding resonance varieties of $X$, see e.g. [CO00, Theorem 3.7]

Next we give an application of Proposition 2.14. For a positive integer $k > 1$, let $\Phi_k(t)$ be the irreducible cyclotomic polynomial of primitive $k$-th root of unity. It is known that

$$\Phi_k(1) = \begin{cases} 
1 & \text{if } k \text{ is not a power of any prime number}, \\
 p & \text{if } k = p^r \text{ for some prime number } p \text{ and some positive integer } r.
\end{cases}$$

For any epimorphism $\nu: \pi_1(X) \twoheadrightarrow \mathbb{Z}$, by Proposition 3.5 $\Delta_i(\nu)$ is of cyclotomic type. Let $\Delta'_i(X^\nu)$ denote the polynomial obtained from $\Delta_i(X^\nu)$ by taking out $(t - 1)$ factors and we write it down as

$$\Delta'_i(X^\nu) = c_i \prod_{k \geq 1} \Phi_k(t)^{e_{i,k}}.$$
Proposition 5.1. With the above assumptions and notations for the hyperplane arrangement complement $X$, for any degree $i > 0$, we have
\begin{equation}
    c_i \left( \prod_p p^{e_{i,p} + e_{i,p^2} + \cdots} \right) \mid \tau_i(X, \nu_Z),
\end{equation}
where the product runs over all prime numbers $p > 0$. Moreover, we have
\[
    \exp(M_i(X^\nu)) \mid \tau_i(X, \nu_Z).
\]
In particular, say $\nu$ is orbifold effective of type $(0, r, \mu)$. Then we have
\[
    \prod_{j=1}^{s} \mu_j \mid \tau_1(X, \nu_Z).
\]

Proof. Since $\Delta_i(X^\nu)$ is of cyclotomic type, we get $M_i(X^\nu) = \log c_i$, where $c_i$ is the leading coefficient of $\Delta_i(X^\nu)$. By the properties listed above, hyperplane arrangement complement satisfies the assumptions in Proposition 2.14, hence the first claim follows. It implies that $c_i \mid \tau_i(X, \nu_Z)$. Combined with Theorem 1.10, the third claim follows. \hfill \Box

One may compare (7) in the above proposition with the following result due to Cohen and Orlik [CO00, Theorem 1.3] (also see [PS10] for its generalization).

Theorem 5.2. Let $X$ be a hyperplane arrangement complement. Then for any epimorphism $\nu: \pi_1(X) \to \mathbb{Z}$ and any $\lambda \in \mathbb{C}^*$ with order being some power of a prime number $p$, we have
\[
    \dim H^i(X, L_\lambda) \leq \beta_i(X, \nu_{\mathbb{Z}_p}).
\]

Next two examples show that both inequalities in Theorem 2.12 and Proposition 5.1 could happen.

Example 5.3. Let $\mathcal{A} := \{ [x, y] \in \mathbb{P}^1 | x^d + y^d = 0 \}$. Take $\nu: \pi_1(M(\mathcal{A})) \to \mathbb{Z}$ as the epimorphism that sends the meridian for $H_\ell$ to an integer $n_\ell$, where $\gcd \{ n_\ell : 1 \leq \ell \leq d \} = 1$. It is easy to see that for any field $K$
\[
    \alpha_1(M(\mathcal{A})^\nu, K) = \begin{cases} 0, & \text{if } \sum_{\ell=1}^{d} n_\ell \neq 0, \\ d - 2, & \text{if } \sum_{\ell=1}^{d} n_\ell = 0, \end{cases}
\]
hence $M_1(M(\mathcal{A})^\nu) = 0$.

We have a basis $\{ a_1, \ldots, a_d \}$ for $H^1(M(\mathcal{A}), \mathbb{Z})$, where $a_\ell$ corresponds to the $\ell$-th hyperplane, hence $\nu_{\mathbb{Z}} = \sum_{\ell=1}^{d} n_\ell a_\ell$. $H^2(M(\mathcal{A}), \mathbb{Z})$ is generated by $\{ a_{ij} := a_i \cup a_j \}_{1 \leq i < j \leq d}$ with the relation $a_{ij} = a_{1j} - a_{1i}$. Hence $H^2(M(\mathcal{A}), \mathbb{Z})$ has a basis $\{ a_{\ell\ell} \}_{2 \leq \ell \leq d}$. Let us write down the matrix of the map $H^1(M(\mathcal{A}), \mathbb{Z}) \overset{\nu_{\mathbb{Z}}}{\to} H^2(M(\mathcal{A}), \mathbb{Z})$ under these basis:
\[
    \begin{pmatrix}
        -n_2 & n_3 & n_4 & \cdots & n_d \\
        - \sum_{i \neq 2} n_i & n_3 & n_4 & \cdots & n_d \\
        n_2 & - \sum_{i \neq 3} n_4 & \cdots & n_d \\
        \vdots & \vdots & \vdots & \vdots & \vdots \\
        n_2 & n_3 & n_4 & \cdots & - \sum_{i \neq d} \end{pmatrix}
\]
When $\sum_{\ell=1}^{d} n_{\ell} \neq 0$, this matrix has rank $d - 1$ and $\tau_1(M(A), \nu_{\mathbb{Z}}) = (\sum_{\ell=1}^{d} n_{\ell})^{d-2}$ (since $\gcd\{n_{\ell}: 1 \leq \ell \leq d\} = 1$). On the other hand, when $\sum_{\ell=1}^{d} n_{\ell} = 0$, it has rank 1 and the matrix can be changed to \[
abla^{d-1} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix} \] by elementary column operation since $\gcd\{n_{\ell}: 1 \leq \ell \leq d\} = 1$, hence $\tau_1(M(A), \nu_{\mathbb{Z}}) = 1$. Then we have 
\[
\beta_1(M(A), \nu_{\mathbb{K}}) = \begin{cases} d - 2, & \text{if } \sum_{\ell=1}^{d} n_{\ell} = 0, \\ d - 2, & \text{if } p \mid \sum_{\ell=1}^{d} n_{\ell} \neq 0, \text{ where } p = \text{char}(\mathbb{K}). \\ 0, & \text{if } p \nmid \sum_{\ell=1}^{d} n_{\ell} \neq 0 \end{cases}
\]
Hence if $p \mid \sum_{\ell=1}^{d} n_{\ell} \neq 0$, $\alpha_1(M(A)^{\nu}, \mathbb{C}) < \beta_1(M(A), \nu_{\mathbb{F}_p})$ and $\exp(M_1(M(A)^{\nu})) = 1 < (\sum_{\ell=1}^{d} n_{\ell})^{d-2} = \tau_1(M(A), \nu_{\mathbb{Z}}).

Under suitable basis $\partial^{d}_{\mathbb{P}^2}$ can be written as form of matrix of size $d \times (d - 1)$
\[
\begin{pmatrix}
1 - t^{n_1} & 1 - t^{n_2} & \ldots & 1 - t^{n_{d-1}} \\
 \frac{t}{t^{\sum_{\ell=1}^{d} n_{\ell}}} - 1 & 0 & \ldots & 0 \\
0 & \frac{t}{t^{\sum_{\ell=1}^{d} n_{\ell}}} - 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{t}{t^{\sum_{\ell=1}^{d} n_{\ell}}} - 1
\end{pmatrix}
\]

One can get
\[
\Delta_1(M(A)^{\nu}) = \begin{cases} (t - 1)(t^{\sum_{\ell=1}^{d} n_{\ell}} - 1)^{d-2}, & \text{if } \sum_{\ell=1}^{d} n_{\ell} \neq 0, \\ t - 1, & \text{if } \sum_{\ell=1}^{d} n_{\ell} = 0. \end{cases}
\]
In particular, if $\sum_{\ell=1}^{d} n_{\ell} \neq 0$, $\left(1 + t + \ldots + t^{(\sum_{\ell=1}^{d} n_{\ell})-1}\right)^{d-2} \big|_{t=1} = (\sum_{\ell=1}^{d} n_{\ell})^{d-2}$, which coincides with $\tau_1(M(A), \nu_{\mathbb{Z}})$.

**Example 5.4.** Let $B$ be the deleted $B_3$-arrangement in $\mathbb{P}^2$ with defining equations
\[
zxy(x - y)(x - z)(y - z)(x - y - z)(x - y + z).
\]
Order the hyperplanes as the factors of the defining polynomial. Let $X(A)$ be the complement of the arrangement. It was first discovered by Suciu [Suc02] that $\mathcal{V}^1(X(A), \mathbb{C})$ has a transalted component
\[
V := \rho \circ \{t, t^{-1}, 1, t^{-1}, t, t^2, t^{-2} \mid t \in \mathbb{C}^*\}
\]
with $\rho = (1, -1, -1, -1, 1, 1, 1) \in \left(\mathbb{C}^*\right)^7$, e.g. see [Dim17, Example 6.16]. Here we take $\{z = 0\}$ as the hyperplane at infinity. This component is induced by the orbifold map $f: X \to \mathbb{C}^*$ as following
\[
f([x, y, z]) = \frac{x(y - z)(x - y - z)^2}{y(x - z)(x - y + z)^2}
\]
and \( f \) is of type \((0, 2, 2)\). This suggests us to consider \( \nu = (1, -1, 0, -1, 1, 2, -2) \in H^1(X, \mathbb{Z}) \) induced by \( f \). Then \( \alpha_1(X^\nu, \mathbb{K}) = \begin{cases} 1, & \text{if } \text{char} (\mathbb{K}) = 2, \\ 0, & \text{if } \text{char} (\mathbb{K}) \neq 2, \end{cases} \) and \( M_1(X^\nu) = \log 2 \).

On the other hand, the Aomoto complex for \( \nu_Z \) and \( \nu_K \) can be computed by the formula given in [Dim17, p.119]. A direct computation shows that

\[
\tau_1(X, \nu_Z) = 4, \quad H^2(H^*(X, \mathbb{Z}), \cdot \nu_Z)_{\text{tor}} \cong \mathbb{Z}/4\mathbb{Z} \quad \text{and} \quad \beta_1(X, \nu_K) = \begin{cases} 1, & \text{if } \text{char} (\mathbb{K}) = 2, \\ 0, & \text{if } \text{char} (\mathbb{K}) \neq 2. \end{cases}
\]

Using the formula given in [Suc02, Example 4.1] (be careful that in [Suc02, Example 4.1] hyperplanes are ordered different from ours), we get \( \Delta_1(X^\nu) = 2(t+1) \), hence \( \Delta_1(X^\nu)(1) = 4 \), which coincides with \( \tau_1(X, \nu_Z) \).

It would be interesting to have an example where \( \Delta_1(X^\nu)(1) \neq \tau_1(X, \nu_Z) \).

5.2. Multinet. We first recall the definition of multinet.

**Definition 5.5.** Let \( \mathcal{A} \) be an essential hyperplane arrangement in \( \mathbb{P}^2 \). A multinet on \( \mathcal{A} \) is a partition of \( \mathcal{A} \) into \( k \geq 3 \) subsets \( \mathcal{A}_1, \ldots, \mathcal{A}_k \), together with an assignment of multiplicities, \( n: \mathcal{A} \to \mathbb{Z}_{>0} \), and a subset \( \mathcal{X} \) of intersection points in \( \mathcal{A} \) satisfying the following conditions:

\( \nu \in \mathcal{A} \) and \( \nu' \in \mathcal{A}_j \) with \( i \neq j, H \cap H' \in \mathcal{X} \);

\( \nu \) and \( \nu' \in \mathcal{X} \), the sum \( \sum_{H \in \mathcal{X}, x \in H} n_H \) is independent of \( i \);

such that \( H_{j-1} \cap H_j \not\in \mathcal{X} \) for \( 1 \leq j \leq s \).

Such multinet is called a \((k, \kappa)\)-multinet.

By results of Pereira-Yuzvinsky [PY08] and Yuzvinsky [Yuz09], \( k = 3 \) or \( 4 \) if \( |\mathcal{X}| > 1 \). It was conjectured by Yuzvinsky that \( k = 4 \) only happens for Hessian arrangement. As shown by Falk, Pereira and Yuzvinsky [FY07, PY08], every \((k, \kappa)\)-multinet determines a large orbifold map \( f: M(\mathcal{A}) \to \Sigma_{0,k} \subset \mathbb{P}^1 \) by

\[
f(x) := \left[ \prod_{H \in \mathcal{A}_1} f^{n_{H}}(x), \prod_{H \in \mathcal{A}_2} f^{n_{H}}(x) \right].
\]

**Definition 5.6.** Let \( \mathcal{A} \) be an essential hyperplane arrangement in \( \mathbb{C}\mathbb{P}^2 \). We call an orbifold map \( f: M(\mathcal{A}) \to \Sigma_{0,k} \) non-local, if it is not the restriction of another orbifold map over the complement of a sub-arrangement of \( \mathcal{A} \).

**Theorem 5.7.** Let \( \mathcal{A} \) be an essential hyperplane arrangement in \( \mathbb{P}^2 \). Then every non-local orbifold map \( f: M(\mathcal{A}) \to \Sigma_{0,k} \) satisfies either

- \( k = 2 \) and \( f \) is a small orbifold map,
- \( k = 3, 4 \) and \( f \) is obtained from a \((k, \kappa)\)-multinet for some \( \kappa \).
As \( k = 4 \) conjecturally only happens for Hessian arrangement, we consider a multinet on \( \mathcal{A} \) with \( k = 3 \). Under the following assumptions, we will show that the corresponding orbifold map has no multiple fiber.

**Assumption 5.8.** Given a \((3, \kappa)\)-multinet on \( \mathcal{A} \) in \( \mathbb{P}^2 \), we assume that

(a) \( n_H = 1 \) for any \( H \in \mathcal{A}_1 \) or \( \mathcal{A}_2 \).
(b) If \( H, H' \in \mathcal{A}_1 \) and \( H \cap H' \notin \mathfrak{X} \), then there is no third hyperplane in \( \mathcal{A} \) passing through \( H \cap H' \). So is \( \mathcal{A}_2 \).
(c) Consider \( \mathcal{A}_3 \) and \( \mathfrak{X} \). If there exists a point \( x \in \mathfrak{X} \) such that \( x \) is contained in precisely one hyperplane \( H \in \mathcal{A}_3 \), one replaces \( \mathcal{A}_3 \) and \( \mathfrak{X} \) by deleting the hyperplane \( H \) and the point \( x \), respectively. We assume that one can run this procedure for \( \mathcal{A}_3 \) and \( \mathfrak{X} \) until there is no hyperplane left in \( \mathcal{A}_3 \).

**Proposition 5.9.** With the above assumptions, \( V^1(\mathcal{X}(\mathcal{A}), \mathbb{C}) = V^1(M(\mathcal{A}), \mathbb{C}) \) does not have translated two dimensional component corresponding to this multinet, i.e., the corresponding orbifold map has no multiple fiber.

**Proof.** We take \( \nu \) which maps the meridian of \( H \) to 1 for \( H \in \mathcal{A}_1 \) or \( \mathcal{A}_2 \) and \( -2n_H \) for \( H \in \mathcal{A}_3 \). Then the claim follows if we can show that \( \tau_1(M(\mathcal{A})) = 1 \).

Consider the map \( H^1(M(\mathcal{A}), \mathbb{Z}) \overset{\nu}{\rightarrow} H^2(M(\mathcal{A}), \mathbb{Z}) \). Let \( \mathfrak{Y}_i \) denote the intersection points in the sub-arrangement \( \mathcal{A}_i \). Due to Brieskorn Decomposition [Dim17, Theorem 3.2],

\[
H^2(M(\mathcal{A}), \mathbb{Z}) \cong \bigoplus_{x \in \mathfrak{X}, \mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3} H^2(M(\mathcal{A}_x), \mathbb{Z}),
\]

where \( \mathcal{A}_x \) is the central line arrangement for the point \( x \). For any \( x \in \mathfrak{X} \), Definition 5.5(c) and the choice of \( \nu \) give

\[
\sum_{H \in \mathcal{A}_1, x \in H} n_H + \sum_{H \in \mathcal{A}_2, x \in H} n_H - \sum_{H \in \mathcal{A}_3, x \in H} 2n_H = 0,
\]

and Definition 5.5(c) together with Assumption 5.8(a) implies that one can find \( H \in \mathcal{A}_1 \) with \( n_H = 1 \) and \( x \in H \). Then the computations in Example 5.3 show that one can replace the sub-matrix corresponding to \( \mathcal{A}_i \times \mathfrak{X} \) by the column obtained from the incidence relation between the hyperplanes and \( \mathfrak{X} \). We have the following matrix

\[
\begin{pmatrix}
\mathfrak{X} & \mathfrak{Y}_1 & \mathfrak{Y}_2 & \mathfrak{Y}_3 \\
\mathcal{A}_1 & C_1 & B_1 & 0 \\
\mathcal{A}_2 & C_2 & 0 & B_2 \\
\mathcal{A}_3 & C_3 & 0 & 0 & B_3
\end{pmatrix}
\]

Here \( C_i \) is the \( |\mathcal{A}_i| \times |\mathfrak{X}| \) incidence matrix, where the entry \( (H, x) \) is 1 precisely when \( x \in H \). The incidence matrix \( \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \) has rank \( d - 2 \) (see [LY00, Theorem 3.2]), where \( d = |\mathcal{A}| \). Together with Definition 5.5(a), one gets that \( C_3 \) is full row rank. Assumption 5.8(c) implies that \( C_3 \) can be changed to \( \begin{pmatrix} \text{Id} & 0 \end{pmatrix} \) by elementary column operations. Here \( \text{Id} \) is the identity matrix.
Next we compute the matrix $B_1$. Consider a graph, whose vertices correspond to the hyperplanes in $\mathcal{A}_1$ and two vertices has an unordered edge if the corresponding two hyperplanes intersect at a point in $\mathcal{Y}_1$. Definition 5.5(d) is equivalent to say that this graph is connected. Assumption 5.8(b) implies that there is a one to one correspondence between the edges and $\mathcal{Y}_1$. Choose a spanning tree in this graph and consider the submatrix corresponding to the vertices and the edges in it. By Assumption 5.8(a),(b), we get a $\kappa \times (\kappa - 1)$ matrix, where $\kappa = |\mathcal{A}_1|$ and every column in this matrix has 1 and $-1$ both for precisely one entry and 0 for the rest. By easy computations, we get that $B_1$ has rank $\kappa - 1$ and has a $(\kappa - 1) \times (\kappa - 1)$ minor being $\pm 1$. The same computation works for $B_2$.

Since $\beta_i(M(\mathcal{A}),\nu_C) = \alpha_i(M(\mathcal{A})^\nu, \mathbb{C}) = -\chi(\Sigma_{0,2}) = 1$, the matrix (8) has rank $d - 2$. Putting all the computations for $C_3$, $B_1$ and $B_2$ together, we find a $(d - 2) \times (d - 2)$ minor being $\pm 1$. Then the proof is done.

**Example 5.10.** We give a list of examples which satisfies Assumption 5.8: Figure 1(a), Figure 2, Figure 3(a),(b) in [FY07] with $A, B, C$ corresponding to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and Figure 1 in [Suc14B]. It is also easy to find examples which do not satisfy Assumption 5.8, see Figure 1(b) in [FY07] or the monomial (alias Ceva) arrangement (with $m \geq 3$) in [Dim17, Example 6.11].

### 5.3 A question by Denham and Suciu.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$, defined by a product of degree 1 homogeneous polynomials

$$Q(\mathcal{A}) := \prod_{1 \leq \ell \leq d} f_{\ell}.$$ 

This also gives a central hyperplane arrangement in $\mathbb{C}^{n+1}$, denoted by $\overline{\mathcal{A}}$. Let

$$F(\mathcal{A}) := \{x \in \mathbb{C}^{n+1} \mid Q(\mathcal{A})(x) = 1\}$$

denote the Milnor fiber of the arrangement $\mathcal{A}$. Then $F(\mathcal{A})$ admits a finite cover to $X(\mathcal{A})$.

For the proof of Theorem 1.15, we use the same construction introduced by Denham and Suciu in [DS14]. Their construction focuses entirely on ‘non-modular torsion’: that is, $p$-torsion in the homology of cyclic covers, where $p$ does not divide the order of the cover, hence part of results in [DS14] are proved under this assumption. We modify all the related results in [DS14] without this assumption and list them as follows.

The next theorem, corresponding to [DS14, Theorem 4.3], follows from Theorem 1.10 and Proposition 2.9.

**Theorem 5.11.** Let $X$ be a complex smooth quasi-projective variety. Suppose that there is an orbifold fibration $f: X \to \Sigma$ of type $(g, r, \mu)$ with $\prod_{j=1}^s \mu_j > 1$. Take a prime number $p$ dividing $\prod_{j=1}^s \mu_j$. Then, for all sufficiently large integer $N$, there exists a regular, $N$-fold cyclic cover $X^N \to X$ such that $H_1(X^N, \mathbb{Z})$ has $p$-torsion.

**Proof.** Let $\Gamma$ be the corresponding orbifold group. Take any epimorphism $\nu: \Gamma \to \mathbb{Z}$ and consider the covering space $X^\nu$ for the composed map

$$\pi_1(X) \xrightarrow{f_*} \Gamma \xrightarrow{\nu} \mathbb{Z}.$$
Let \( X^{\nu,N} \) denote the corresponding \( N \)-fold cover. Theorem 1.10 shows \( \alpha_1(X^{\nu},\mathbb{C}) < \alpha_1(X^{\nu},\overline{\mathbb{F}}_p) \). Then the claim follows from Proposition 2.9 by taking \( i = 1 \).

For a technical reason, we need to allow multiplicities on the hyperplanes: if \( \mathbf{m} := (m_1, \ldots, m_d) \in \mathbb{Z}_{>0}^d \) is a positive lattice vector, then the pair \((\mathcal{A}, \mathbf{m})\) is called a multiarrangement. A defining polynomial for the multiarrangement is the product

\[
Q(\mathcal{A}, \mathbf{m}) := \prod_{\ell=1}^d f_{\ell}^{m_\ell}
\]

and

\[
F(\mathcal{A}, \mathbf{m}) := \{ x \in \mathbb{C}^{n+1} | Q(\mathcal{A}, \mathbf{m})(x) = 1 \}
\]

is called the Milnor fiber of the multiarrangement \((\mathcal{A}, \mathbf{m})\). Set \( m := \sum_{\ell=1}^d m_\ell \). Then \( F(\mathcal{A}, \mathbf{m}) \) admits a regular \( m \)-fold cover of \( X(\mathcal{A}) \) given by the homomorphism

\[
\delta: \pi_1(X(\mathcal{A})) \to \mathbb{Z}/m\mathbb{Z}, \gamma_\ell \mapsto m_\ell \mod m,
\]

where \( \gamma_\ell \) is the meridian for \( H_\ell \) in \( X(\mathcal{A}) \).

The following lemma corresponds to [DS14, Proposition 6.7].

**Proposition 5.12.** Let \( \mathcal{A} \) be a hyperplane arrangement and let \( X^N \to X \) be a connected regular \( \mathbb{Z}_N \)-cover of \( X = X(\mathcal{A}) \) given by an epimorphism \( \chi: \pi_1(X) \to \mathbb{Z}/N\mathbb{Z} \). Then there exists infinitely many multiplicity vectors \( \mathbf{m} \in \mathbb{Z}_{>0}^d \) for which the covering projection \( F(\mathcal{A}, \mathbf{m}) \to X \) factors through \( X^N \).

Moreover, for any prime \( p > 0 \), we may choose \( \mathbf{m} \) such that \( \frac{m}{N} \) is not divisible by \( p \), where \( m = \sum_{\ell=1}^d m_\ell \).

**Proof.** For any \( 1 \leq \ell \leq d \), say \( \chi \) sends \( \gamma_\ell \) to \( \chi_\ell \mod N \), where \( \chi_\ell \in [0, N-1] \) is an integer. In particular, \( N \) divides \( \sum_{\ell=1}^d \chi_\ell \). Choose any multiplicity vector \( (q_1, \ldots, q_d) \in \mathbb{Z}_{>0}^d \) such that \( m_\ell = \chi_\ell + N \cdot q_\ell > 0 \) and \( \frac{m}{N} \) is not divisible by \( p \), where \( m = \sum_{\ell=1}^d m_\ell \). Then it is clear \( F(\mathcal{A}, \mathbf{m}) \to X \) factors through \( X^N \). \( \square \)

**Example 5.13.** [DS14, Example 6.12] Fix an integer \( \mu \geq 2 \). Let \( \mathcal{A}_\mu \) be the deleted monomial arrangement, where its defining equations in \( \mathbb{P}^2 \) are \( yz(x^\mu - y^\mu)(x^\mu - z^\mu)(y^\mu - z^\mu) \). Ordering the hyperplanes as the factors of the defining polynomial. Its projective complement \( X(\mathcal{A}_\mu) \) admits an orbifold map \( X(\mathcal{A}_\mu) \to \mathbb{C}^* \) given by

\[
\frac{z^\mu(x^\mu - y^\mu)}{y^\mu(x^\mu - z^\mu)}.
\]
which is of type $(0, 2, \mu)$. Moreover, $\mathcal{V}(\mathcal{M}(A), \mathbb{C})$ has a translated component

$$V := \rho \otimes \{ t^\mu, t^{-\mu}, t, \ldots, t^{-1}, \ldots, t, 1, \ldots, 1 \mid t \in \mathbb{C}^* \}$$

with $\rho = (1, \zeta^{-1}, \ldots, \zeta^{-1}, 1, \ldots, 1, \zeta, \ldots, \zeta)$. Here $\zeta$ is a primitive $\mu$-th root of unity.

For any fixed prime number $p$ and large enough $N$, we can choose the multiplicity vector as

$$m = (\mu, kN - \mu, 1, \ldots, 1, N - 1, \ldots, N - 1, N, \ldots, N)$$

and $m = (2\mu + k)N$ such that $p$ is coprime with $2\mu + k$. Then this multiplicity vector satisfies the requirement in Proposition 5.12.

Next we go to the construction considered in [DS14]. For $m_1 > 2$, by [DS14, Definition 7.8] we consider the parallel connection of the multiarrangement $(A, m)$ with a central line arrangement in $\mathbb{C}^2$, denoted by $A \circ H_1 \mathcal{P}_{m_1}$, where the line arrangement has exactly $m_1$ hyperplanes. Up to change of coordinates, we can assume that $H_1$ for $A$ is defined by $\{x_1 = 0\}$ and the defining equations for the line arrangement are $\prod_{j=0}^{m_1-1} (y_1 - j \cdot y_2)$. Then $A \circ H_1 \mathcal{P}_{m_1}$ can be viewed as a central multiarrangement in $\mathbb{C}^{n+2}$ with the following defining equations by the coordinates $(x, y_2) \in \mathbb{C}^{n+2}$

$$x_1 \cdot \left( \prod_{\ell=2}^{d} f^{m_\ell}_\ell(x) \right) \cdot \prod_{j=1}^{m_1-1} (x_1 - j \cdot y_2).$$

In particular, $X(A \circ H_1 \mathcal{P}_{m_1}) \cong X(A) \times \Sigma_{0,m_1}$, where $\Sigma_{0,m_1} := \mathbb{CP}^1 \setminus \{m_1 \text{ points}\}$. If $m_1 = 1$, then $A \circ H_1 \mathcal{P}_{m_1}$ is simply the original arrangement $A$.

**Proposition 5.14.** With the above notations, we have a pullback diagram between the respective Milnor covers,

$$\begin{array}{ccc}
F(A, m) & \rightarrow & F(A \circ H_1 \mathcal{P}_{m_1}) \\
\downarrow & & \downarrow \\
X(A) & \rightarrow & X(A \circ H_1 \mathcal{P}_{m_1})
\end{array}$$

where $j$ is the map given by choosing a base point in $\mathcal{P}_{m_1}$. Moreover, if $m_1 \geq 3$ and $i \geq 1$, there exists a surjective group homomorphism

$$H_{i+1}(F(A \circ H_1 \mathcal{P}_{m_1}), \mathbb{Z})_{\text{tor}} \rightarrow H_i(F(A, m), \mathbb{Z})_{\text{tor}}.$$

**Proof.** The first claim is already proved in [DS14, Lemma 8.4] and we only need to prove the second one.

$\Sigma_{0,m_1}$ has a simple cell structure with a single 0-dimensional cell, say $a$, and $(m_1 - 1)$ many 1-dimensional cell, say $\{b_1, \ldots, b_{m_1-1}\}$. The product structure $X(A \circ H_1 \mathcal{P}_{m_1}) \cong X(A) \times \Sigma_{0,m_1}$ gives a cell structure for $X(A \circ H_1 \mathcal{P}_{m_1})$. Note that $F(A, m)$ and $F(A \circ H_1 \mathcal{P}_{m_1})$ are $\mathbb{Z}/m\mathbb{Z}$-fold covers of $X(A)$ and $X(A \circ H_1 \mathcal{P}_{m_1})$, respectively. By choosing fixed lifts of the cells of $F(A, m)$ and $F(A \circ H_1 \mathcal{P}_{m_1})$ respectively, we obtain free basis for the chain complex.
$C_*(F(A, m), \mathbb{Z})$ and $C_*(F(A \circ_{H_1} P_{m_1}), \mathbb{Z})$ as $R$-modules, where $R = \mathbb{Z}[t]/(t^m - 1) \cong \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$. The $R$-module map between finitely generated free $R$-modules

$$C_i(F(A, m), \mathbb{Z}) \xrightarrow{\partial} C_{i-1}(F(A, m), \mathbb{Z})$$

can be written down as a matrix, say $A$, with entries in $\mathbb{Z}[t]/(t^m - 1)$. By the product structure $X(A \circ_{H_1} P_{m_1}) \cong X(A) \times \sum_{a, m_1}$, we have the following matrix for the map $C_{i+1}(F(A \circ_{H_1} P_{m_1}), \mathbb{Z}) \to C_i(F(A \circ_{H_1} P_{m_1}), \mathbb{Z})$:

$$
\begin{pmatrix}
(i - 1, b_1) & (i - 1, b_2) & \cdots & (i - 1, b_{m_1 - 1}) & (i, a) \\
(i, b_1) & A & 0 & \cdots & 0 \\
(i, b_2) & 0 & A & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(i, b_{m_1 - 1}) & 0 & 0 & \cdots & A \\
(i + 1, a) & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
$$

Here for example $(i, b_1)$ stands for the basis $C_i(X(A), R) \otimes_R \langle b_1 \rangle$, where $\langle b_1 \rangle$ is the free $R$-module generated by $b_1$. When $m_1 \geq 3$, by elementary operations the above matrix becomes

$$
\begin{pmatrix}
A & 0 & \cdots & 0 & (t - 1)\text{id} \\
0 & A & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & A & 0 \\
0 & 0 & \cdots & 0 & * \\
\end{pmatrix}
$$

Consider $\mathbb{Z}[t]/(t^m - 1) \cong \mathbb{Z}^m$ as an abelian group. Then $A$ can be also viewed as a matrix (with different size) with entries in $\mathbb{Z}$ and the torsion part of $H_i(F(A, m), \mathbb{Z})$ is determined by $A$. Then the above computation gives the claim.

Denham and Suciu iterated the parallel connections of the multiarrangement $(A, m)$ with a collection of line arrangements and obtained the following definition.

**Definition 5.15.** [DS14, Definition 8.1] The polarization of a multiarrangement $(A, m)$, denoted by $\mathcal{A}||m$, is the arrangement of $m = \sum_{\ell=1}^d m_\ell$ hyperplanes given by

$$
\mathcal{A}||m = A \circ_{H_1} P_{m_1} \circ_{H_2} P_{m_2} \circ_{H_3} \cdots \circ_{H_d} P_{m_d}.
$$

Note that the polarization of a multiarrangement is a reduced hyperplane arrangement with $m = \sum_{\ell=1}^d m_\ell$ many hyperplanes. Let $F(\mathcal{A}||m)$ denote the Minor fiber for the arrangement $(\mathcal{A}||m)$. A similar diagram to the one presented in Proposition 5.14 is obtained in [DS14, Lemma 8.4].

**Proof of Theorem 1.15.** We take $\mathcal{A}_\mu$ to be the arrangement considered in Example 5.13 with $X$ being its complement. There exists an orbifold map $f$ of type $(0, 2, \mu)$ for $X$. By choosing $\mu$ such that $p$ divides $\mu$, for any integer $N$ big enough Theorem 5.11 gives $X^N$, a $\mathbb{Z}/N\mathbb{Z}$-cover of $X$, such that $H_1(X^N, \mathbb{Z})$ has non-trivial $p$-torsion. Take the multiplicity...
vector given in Example 5.13. Since \( F(\mathcal{A}_\mu, \mathbf{m}) \) is a \( \frac{m}{N} \)-fold cover of \( X^N \) and \( p \) does not divide \( \frac{m}{N} \), by [DS14, Lemma 2.4] \( H_1(F(\mathcal{A}_\mu, \mathbf{m}), \mathbb{Z}) \) also has non-trivial \( p \)-torsion. Using Proposition 5.14 repetitively, we get that \( H_{2\mu+3}(F(\mathcal{A}_\mu|\mathbf{m}), \mathbb{Z}) \) has non-trivial \( p \)-torsion (since there are exactly \( \mu \) many \( m_\ell = 1 \)).

Then we take \( \mathcal{B} = \mathcal{A}_\mu|\mathbf{m} \), which is a reduced hyperplane arrangement with \( |\mathcal{B}| = m \).

For the additional requirement \( p \) dividing \( |\mathcal{B}| \), we can choose \( N \) such that \( p \) divides \( N \), hence \( p \) divides \( m \). Then the claim follows.

\[ \square \]

**Remark 5.16.** As we used the same construction as in [DS14], for any prime number \( p \geq 2 \), the Milnor fiber of the central arrangement we construed can have non-trivial \( p \)-torsion in homology at degree \( 2p + 3 \), same as in [DS14, Corollary 8.8]. Denham and Suciu asked if there is a hyperplane arrangement \( \mathcal{A} \) whose Milnor fiber \( F(\mathcal{A}) \) has torsion in \( H_1(F(\mathcal{A}), \mathbb{Z}) \) [DS14, Question 8.10]. Yoshinaga gave the first example where \( H_1(F(\mathcal{A}), \mathbb{Z}) \) has non-trivial 2-torsion, see [Yos20].

**References**

[AJN11] M. Abért, A. Jaikin-Zapirain, N. Nikolov, *The rank gradient from a combinatorial viewpoint*. Groups Geom. Dyn. 5 (2011), no. 2, 213-230.

[Ahl66] L. V. Ahlfors, *Complex analysis: An introduction of the theory of analytic functions of one complex variable*. Second edition McGraw-Hill Book Co., New York-Toronto-London 1966 xiii+317 pp.

[Ara97] D. Arapura, *Geometry of cohomology support loci for local systems. I*. J. Algebraic Geom. 6 (1997), no. 3, 563-597.

[Ati76] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*. Astérisque, 32-33:43-72, 1976.

[ACM13] E. Artal Bartolo, J. I. Cogolludo-Agustín, D. Matei, *Characteristic varieties of quasi-projective manifolds and orbifolds*. Geom. Topol. 17 (2013), no. 1, 273-309.

[Beauville90] A. Beauville, *Annales du H^1 pour les fibrés en droites plats*. Complex algebraic varieties (Bayreuth, 1990), 1-15, Lecture Notes in Math., 1507, Springer, Berlin, 1992.

[Bud09] N. Budur, *Unitary local systems, multiplier ideals, and polynomial periodicity of Hodge numbers*. Adv. Math. 221 (2009), no. 1, 217-250.

[BLW18] N. Budur, Y. Liu, B. Wang, *The monodromy theorem for compact Kähler manifolds and smooth quasi-projective varieties*. Math. Ann. 371 (2018), no. 3-4, 1069-1086.

[BW15] N. Budur, B. Wang, *Cohomology jump loci of quasi-projective varieties*. Ann. Sci. Éc Norm. Supér. (4) 48 (2015), no. 1, 227-236.

[BW20] N. Budur, B. Wang, *Absolute sets and the decomposition theorem*. Ann. Sci. Éc Norm. Supér. (4) 53 (2020), no. 2, 469-536.

[CKO03] F. Catanese, J. Keum, K. Oguiso, *Keiji, Some remarks on the universal cover of an open K3 surface*. Math. Ann.325(2003), no.2, 279–286.

[CO00] C.D. Cohen, P. Orlik, Peter, *Arrangements and local systems*. Math. Res. Lett. 7 (2000), no. 2-3, 299-316.

[DS14] G. Denham, A. Suciu, *Multinets, parallel connections, and Milnor fibrations of arrangements*. Proc. Lond. Math. Soc. (3) 108 (2014), no. 6, 1435-1470.

[Del74] P. Deligne, *Théorie de Hodge. III*. Inst. Hautes Études Sci. Publ. Math. (44):5-77, 1974.

[Delz08] T. Delzant, *Trees, valuations and the Green-Lazarsfeld set*. Geom. Funct. Anal. 18 (2008), no. 4, 1236-1250.
[DL19] L. F. Di Cerbo and L. Lombardi. $L^2$-Betti numbers and convergence of normalized Hodge numbers via the weak generic Nakano vanishing theorem. arXiv:1906.06279. to appear in Annales de l’Institut Fourier. 2

[Dim04] A. Dimca, Sheaves in Topology. Universitext, Springer-Verlag, Berlin, 2004. 15, 17

[Dim07] A. Dimca, Characteristic varieties and constructible sheaves. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 18 (2007), no. 4, 365-389. 4, 17, 23

[Dim09] A. Dimca, On admissible rank one local systems. J. Algebra 321 (2009), no. 11, 3145-3157. 4

[Dim17] A. Dimca, Hyperplane arrangements. An introduction. Universitext. Springer, Cham, 2017. xii+200 pp. ISBN: 978-3-319-56220-9; 978-3-319-56221-6. 7, 24, 26, 27, 28, 29

[DP03] A. Dimca, S. Papadima, Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements. Ann. of Math. (2) 158 (2003), no. 2, 473-507. 24

[DP14] A. Dimca, S. Papadima, Non-abelian cohomology jump loci from an analytic viewpoint. Commun. Contemp. Math. 16 (2014), no. 4, 1350025, 47 pp. 16

[DPS09] A. Dimca, S. Papadima, A. I. Suciu, Topology and geometry of cohomology jump loci. Duke Math. J. 148 (2009) 405-457. 4, 16

[FY07] M. Fulk, S. Yuzvinsky, Multinets, resonance varieties, and pencils of plane curves. Compositio Math. 143 (2007), no. 4, 1069-1088. 7, 27, 29

[GL91] M. Green, R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles. J. Amer. Math. Soc. 4 (1991), no. 1, 87-103. 16

[Le14] T. Le, Homology torsion growth and Mahler measure. Comment. Math. Helv. 89 (2014), no. 3, 719-757. 2, 13

[LY00] A. Libgober, S. Yuzvinsky, Cohomology of the Orlik-Solomon algebras and local systems. Compositio Math. 121 (2000), no. 3, 337-361. 28

[Luc94] W. Lück, Approximating $L^2$-invariants by their finite-dimensional analogues. Geom. Funct. Anal. 4 (1994), no. 4, 455-481. 1, 2

[Luc02] W. Lück, $L^2$-invariants: theory and applications to geometry and $K$-theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. 44. Springer-Verlag, Berlin, 2002. xvi+595 pp. ISBN: 3-540-43566-2. 1

[Mah60] K. Mahler, An application of Jensen’s formula to polynomials. Mathematika 7 (1960), 98-100. 12

[Mah62] K. Mahler, On some inequalities for polynomials in several variables. J. London Math. Soc. 37 (1962), 341-344. 12

[PS10] S. Papadima, A. I. Suciu, The spectral sequence of an equivariant chain complex and homology with local coefficients. Trans. Amer. Math. Soc. 362 (2010), no. 5, 2685-2721. 6, 14, 25

[PY08] J. V. Pereira, S. Yuzvinsky, Completely reducible hypersurfaces in a pencil. Adv. Math. 219 (2008), no. 2, 672-688. 27

[Ran02] R. Randell, Morse theory, Milnor fibers and minimality of hyperplane arrangements. Proc. Amer. Math. Soc. 130 (2002), no. 9, 2737-2743. 24

[SW02] D. S. Silver, S. G. Williams, Mahler measure, links and homology growth. Topology 41 (2002), no. 5, 979-991. 16

[Sim93] C. Simpson, Subspaces of moduli spaces of rank one local systems. Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 3, 361-401. 16

[Suc01] A. I. Suciu, Fundamental groups of line arrangements: Enumerative aspects. in: Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), pp. 43-79, Contemp. Math., vol 276, Amer. Math. Soc., Providence, RI, 2001. 2, 16

[Suc02] A. I. Suciu, Translated tori in the characteristic varieties of complex hyperplane arrangements. Topology Appl. 118 (2002), no. 1-2, 209-223. 26, 27
[Suc11] A. I. Suciu, *Fundamental groups, Alexander invariants, and cohomology jumping loci. (English summary)* Topology of algebraic varieties and singularities. 179-223, Contemp. Math., 538, Amer. Math. Soc., Providence, RI, 2011.

[Suc14A] A. I. Suciu, *Characteristic varieties and Betti numbers of free abelian covers.* Int. Math. Res. Not. 2014, no. 4, 1063–1124.

[Suc14B] A. I. Suciu, *Hyperplane arrangements and Milnor fibrations.* Ann. Fac. Sci. Toulouse Math. (6) 23 (2014), no. 2, 417-481.

[SW77] D. W. Sumners, J. M. Woods, *The monodromy of reducible plane curves.* Invent. Math. 40 (1977), no. 2, 107-141.

[Tur01] V. Turaev, *Introduction to Combinatorial Torsions.* Lectures in Mathematics. Springer, 2001. xii+91 pp.

[Wei94] C. A. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, Cambridge, 1994.

[Yos20] M. Yoshinaga, *Double coverings of arrangement complements and 2-torsion in Milnor fiber homology.* Eur. J. Math. 6 (2020), no. 3, 1097-1109.

[Yuz09] S. Yuzvinsky, *A new bound on the number of special fibers in a pencil of curves.* Proc. Amer. Math. Soc. 137 (2009), no. 5, 1641-1648.

Fenglin Li: School of Mathematical Sciences, University of Science and Technology of China, 96 Jinzhai Road, Hefei Anhui 230026 China

Email address: fl0125@mail.ustc.edu.cn

Yongqiang Liu: The Institute of Geometry and Physics, University of Science and Technology of China, 96 Jinzhai Road, Hefei Anhui 230026 China

Email address: liuyq@ustc.edu.cn