MIRROR SYMMETRY AND PARTITION FUNCTIONS

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Abstract. Localization methods have produced explicit expressions for the sphere partition functions of (2,2) superconformal field theories. The mirror symmetry conjecture predicts an IR duality between pairs of Abelian gauged linear sigma models, a class of which describe families of Calabi-Yau manifolds realizable as complete intersections in toric varieties. We investigate this prediction for the sphere partition functions and find agreement between that of a model and its mirror up to the scheme-dependent ambiguities inherent in the definitions of these quantities.

1. Introduction

A conformal field theory determines a space of deformations obtained through conformal perturbation theory by defining the deformed $n$-point correlation functions

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle (\lambda) = \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle e^{\sum \lambda^I \int d^2 z \Phi_I(z)}$$

where $\mathcal{O}_i$ are any local operators and $\Phi_I$ are truly marginal operators of dimension $(1,1)$. The integrals lead to divergences requiring regularization but after this is performed the power series in $\lambda$ are believed to be convergent. The two-point functions of the truly marginal operators determine the Zamolodchikov metric

$$g_{IJ} = |x - y|^4 \langle \Phi_I(x) \Phi_J(y) \rangle .$$

This structure was investigated in [1, 2, 3].

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When the theory enjoys \((2, 2)\) superconformal symmetry we have additional structure, and the deformation spaces of such theories have been the subject of detailed study over the past three decades. The \((2, 2)\) superconformal algebra contains a \(U(1)_R \times U(1)_L\) current algebra, the \(R\)-symmetry algebra. Truly marginal supersymmetry-preserving deformations are the top components of chiral or twisted chiral supermultiplets with charges \((1, \pm 1)\) under this symmetry. The deformation space with its metric factors locally as \(M_c \times M_t\) (provided that the supersymmetry is not enhanced beyond \((2, 2)\) \([4]\)).

When the supersymmetry is not enhanced, the \(R\)-symmetry algebra produces a complex structure on each of these factors under which the (restricted) metric is Kähler. Introducing complex local coordinates \(\lambda_a\) on \(M_c\) and \(\tilde{\lambda}_{\tilde{a}}\) on \(M_t\) we have

\[
\begin{align*}
g_{ab} &= \partial_a \partial_{\bar{b}} K_c(\lambda, \bar{\lambda}) \\
g_{\tilde{a}\tilde{b}} &= \partial_{\tilde{a}} \partial_{\bar{\tilde{b}}} K_t(\tilde{\lambda}, \bar{\tilde{\lambda}}).
\end{align*}
\]

The real function \(K_c\) (resp. \(K_t\)) is a Kähler potential, defined locally in the patches of an open cover of \(M_c\) (resp. \(M_t\)). On the overlaps \(U \cap U'\) these functions change by Kähler transformations, acting on \(K_c\) for example as

\[
K_c^{U'} = K_c^U - f_{cU'}(\lambda) - \bar{f}_{cU'}(\bar{\lambda})
\]

for some local holomorphic function \(f_{cU'}\). A compact smooth Kähler manifold typically does not have a global Kähler potential, but the deformation spaces \(M_{t/c}\) are of course typically noncompact.

In 2012, supersymmetric localization methods were applied to \((2, 2)\) gauged linear sigma models \([5, 6]\) to compute the partition function on \(S^2\). Up to a multiplicative factor associated to the conformal anomaly, this partition function is invariant under RG flow and so, for models flowing to \((2, 2)\) superconformal IR fixed points, computes properties of the fixed points. Localization relies on the fact that the supersymmetry algebra on a round \(S^2\) can be embedded in the \((2, 2)\) algebra. This can be done in two ways, producing two partition functions \(Z_{t/c}\) depending on \(\tilde{\lambda}\), respectively \(\lambda\). The authors of \([7]\) conjectured that

\[
Z_{t/c} = (r/r_0)^c/3 e^{-K_{t/c}}
\]

where \(r\) is the radius of \(S^2\) and \(r_0\) is a scheme-dependent constant. \(K_{t/c}\) is a Kähler potential on the moduli space of the IR fixed point to which the UV model flows. Evidence for this conjecture was provided in \([7, 8, 9]\).

The localization calculation requires that some one-loop determinants be regulated, amounting to a choice of renormalization scheme. The Zamolodchikov metric and the complex structure are expected to be scheme-independent properties of the superconformal field theory. This means the partition function, if \((5)\) holds, is determined up to multiplication by the square of a local, possibly scheme-dependent, holomorphic function.

This issue was addressed in \([10]\) from a novel perspective. Working directly with the superconformal theory (unlike the localization methods which use a specific UV completion), this work considered the anomalous dependence of the partition function on Weyl transformations of the spacetime metric while promoting the parameters \(\lambda\) (resp. \(\tilde{\lambda}\)) to chiral (resp.

\(^1\)All three original papers \([5, 6, 7]\) suppressed the dependence on the radius of the two-sphere.

\(^2\)In principle, we might obtain different scheme-dependent constants \(r_{0,c}\) and \(r_{0,t}\), but by adjusting the scheme we may assume (if we wish) that the constants are the same.
twisted chiral) multiplets of (2,2) supersymmetry. Assuming that conformal perturbation
theory can be regularized preserving supersymmetry, these authors demonstrated that

- the conjecture (5) holds.
- the exponentials $e^{-f_{UU}'c}$ of the transition functions in (4) form the transition functions
  of a holomorphic line bundle $L_{K_c}$ on $M_c$. This means the Kähler metric on $M_c$ is
  Hodge. Similar statements hold for $M_t$, whose Kähler metric is also Hodge.

The results of [10] show, essentially, that the effective action for the $\lambda$ multiplets is determined
by a holomorphic section of $L_{K_c}$. The line bundle $L_{K_c}$, if nontrivial, is an obstruction to the
existence of a nowhere-vanishing globally defined partition function. It also means that the
effective action for the chiral multiplets taking values in $M_c$ is not globally defined. If the
parameters were dynamical, a nontrivial $L_{K_c}$ would indicate an inconsistency of the theory;
since they are not, this becomes a characteristic property of the theory.\[3\] Again, similar
statements hold for the twisted chiral multiplets and the line bundle $L_{K_t}$.

The (2,2) superconformal algebra possesses a $\mathbb{Z}_2$ automorphism, the mirror automorphism,
under which the deformation spaces $M_c$ and $M_t$ are exchanged. A mirror pair of quantum
field theories flow to infrared fixed points differing only by this automorphism. One class of
mirror pairs is furnished by non-linear sigma models with Calabi-Yau target space, which
possess (2,2) supersymmetry. Mirror symmetry in this context is the nontrivial statement
that if $X$ and $Y$ are a mirror pair of Calabi-Yau manifolds, the IR dynamics of the two
sigma models is governed by the same SCFT, with the natural mapping induced by the
mirror automorphism [13, 14, 15, 16, 17, 18, 19].

An alternative UV-free model flowing to a (2,2) superconformal field theory is an Abelian
GLSM [20]. For suitable choices of the parameters these can flow to the same IR fixed points
as non-linear sigma models on Calabi-Yau $X$ given by complete intersection subspaces in
toric varieties. For such Calabi-Yau manifolds, the mirror $Y$ is given by a conjecture of
Batyrev and Borisov [22, 23] which includes mirror duals constructed earlier by Greene
and Plesser [17]. This mirror duality, translated to the data defining a GLSM in [26, 27, 28],
implies a corresponding infrared duality of the linear models. This duality can be tested
with the localization results of [5, 6, 29], which enable the computation of same quantity,
the partition functions $Z_{t/c}$, from two different UV descriptions. We perform that test here,
and find agreement between $Z_t$ of the original theory and $Z_c$ of the mirror theory to within
the scheme-dependent ambiguity in their definition.

Predictions of mirror symmetry for the sphere partition functions have been analyzed
previously [5, 8, 29, 30]. However, the form of mirror symmetry that these authors tested
and confirmed was that of Hori and Vafa [31]. The relationship between this mirror symmetry
and that of Batyrev and Borisov is unclear. Our calculation will shed some light on this
relationship, but questions remain.

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3This is analogous to a sigma model anomaly for the non-dynamical scalars, as was pointed out in [11].
4The authors of [10] also considered four-dimensional theories. In that case, it was shown in [12] that
the corresponding line bundle is not trivial in general, and consequently there is more to the anomaly story,
uncovered in [11].
5There are some preliminary results for some non-Abelian GLSMs [21] but for simplicity in this paper we
shall restrict ourselves to the Abelian case.
6Generally, only certain aspects of this duality, mostly topological, have been mathematically proven to
date [24, 25].
This note is structured as follows: in section 2, we will review the structure of Abelian gauged linear sigma models and the mirror map between mirror pairs. Additionally, we will review the results of localization calculations for both partition functions $Z_{t/c}$. In section 3, we will demonstrate the relationship between the partition functions of a model and its dual. Appendix A contains a demonstration that $Z_c$ of the mirror model satisfies a set of system of differential equations shown in [32] to be solved by $Z_t$, a further test of mirror symmetry. Appendix B contains some technical comments on Abelian duality for chiral/twisted chiral multiplets on the sphere.

2. ABELIAN GAUGED LINEAR SIGMA MODELS

An Abelian GLSM is a $(2, 2)$ gauge theory constructed with $n$ chiral multiplets $\Phi_i$ transforming effectively under an Abelian gauge group $G = U(1)^{n-d} \times \Gamma$ for some discrete Abelian group $\Gamma$. The continuous symmetry is gauged by $n-d$ vector multiplets $V_a$ with invariant field strengths $\Sigma_a$. The discrete gauging is implemented as an orbifold. The models of interest also include a superpotential interaction given by a holomorphic gauge invariant polynomial $W(\Phi)$. The action is written in superspace as

$$\mathcal{L} = \int d^4 \theta \left( \sum_i \Phi_i e^{2Q_i V_a} \Phi_i - \frac{1}{4e^2} \sum_a |\Sigma_a|^2 \right)$$

$$+ \mu \int d\theta^+ d\theta^- W(\Phi) + \frac{i}{2\sqrt{2}} \int d\theta^+ d\bar{\theta}^- \tau^a + c.c.$$  \hspace{1cm} (6)

Here $\tau^a = \frac{\theta^a}{2\pi} + i \xi^a$ is a complexified FI term.

A family of GLSMs is characterized by $G$ and the choice of the monomials appearing in $W$, and parameterized by the continuous parameters in (6). These are $\tau^a$ and the coefficients of terms in $W$. In particular, the discrete group $\Gamma$ does not appear in the Lagrangian.

A convenient way to specify these data is to introduce, following [27, 28], an $\tilde{n} \times n$ matrix $P$ of nonnegative integers and a factorization $P = \tilde{T} T$ of this into two integer valued rank-$d$ matrices. The rows of $T$ can be used to construct a collection of Laurent monomials $\Phi^T := \prod_{i=1}^{\tilde{n}} \Phi_i^{T_i}$, and the group $G$ is defined to be the largest subgroup of $H = U(1)^n$ which leaves the monomials $\Phi^T$ invariant. The monomials $\Phi^T := \prod_{i=1}^{\tilde{n}} \Phi_i^{P_i}$ defined by the rows of $P$ are then $G$-invariant by construction, thanks to the relation $P_{i\bar{i}} = \sum_{l} \tilde{T}_{i\bar{l}} \tilde{T}_{i\bar{l}}$. That is, in this language, the gauge charges $Q^i_\alpha$ are a basis for the kernel of $P$. Since $P_{i\bar{i}} \geq 0$ by assumption, we may use these monomials to specify the family of interaction polynomials

$$W(\Phi) := \sum_{i=1}^{\tilde{n}} c_i \Phi^T = \sum_{i=1}^{\tilde{n}} c_i \prod_{i=1}^{n} \Phi_i^{P_{i\bar{i}}},$$  \hspace{1cm} (7)

where $c_i \in \mathbb{C}^*$ parameterize the family. Alternatively, if we are given $G$ and a family of polynomials $W$, it is not difficult to reconstruct the matrices $P$, $\tilde{T}$, and $T$. (Actually, $\tilde{T}$ and $T$ are only well-defined up to ($\tilde{T}, T) \mapsto (\tilde{T} L, L^{-1} T)$, with $L$ an invertible integer matrix.) Conditions on $\tilde{T}$ and $T$ (beyond their rank) ensuring that the generic model in the family is nonsingular were discussed in [33].

The model will flow at energies much smaller than $e$ or $\mu$ to a conformal field theory if the gauge action is such that $\prod_i \Phi_i$ is invariant (implying $\sum_i Q^i_\alpha = 0$), and if there exists
an assignment $\rho_i$ of rational $R$-charges such that $\rho(W) = 2$\footnote{We are referring to the vector $R$-charges.}. The latter is equivalent to $\sum_i P_{\tilde{i}} \rho_i = 2, \forall \tilde{i}$. The central charge of the resulting theory obeys

$$c = \frac{3}{n} \sum_{i=1}^{n} (1 - \rho_i) - (n - d).$$

The $R$-charge assignment $\rho$ will play a role in our discussion. In the localization computation of \cite{5,6,7} this determines the coupling of the GLSM to the curvature of $S^2$ via an embedding of the of the $(2,2)$ rigid supersymmetry algebra on the sphere into the $(2,2)$ superconformal algebra. It is clear that $\rho$ is only defined up to mixing with the gauge symmetry. This mixing has no effect on the IR theory or on the UV theory in the plane, but it does change the UV completion on $S^2$ and thus the renormalization scheme in which the sphere partition function is calculated. The consequences of this observation will factor into the identification of the partition functions of the mirror models.

One advantage of the GLSM is that (some of) the parameters determining the low-energy theory are explicitly clear. The manifest chiral deformations are parameterized by the coefficients $c_i$ of $W$. These can fail to provide global coordinates on $\mathcal{M}_c$ in three ways:

- In general, $c_i$ parameterize the subspace of $\mathcal{M}_c$ representing theories obtainable as low-energy limits of GLSMs of the form \cite{6}, the toric subspace of $\mathcal{M}_c$.
- The $c_i$ can overparameterize the toric subspace. Values of these related by transformations of the form

$$c_i \to \lambda_i^{P_{\tilde{i}}} c_i$$

describe the same models up to the irrelevant field redefinition

$$\Phi_i \to \lambda_i^{-1} \Phi_i, \quad \lambda \in (\mathbb{C}^*)^n. \quad (11)$$

Invariant coordinates are provided by

$$\tilde{q}_\tilde{a} = \prod_i c_i^{P_{\tilde{i}}} \tilde{Q}_{\tilde{i}}^\tilde{a} \in (\mathbb{C}^*)^{\tilde{n} - d}, \quad (12)$$

where $\tilde{Q}_{\tilde{i}}, \tilde{a} = 1, \ldots, \tilde{n} - d$ are a basis for the cokernel of $P$. In general there may be additional identifications on the space of $c_i$. In the cases of interest here these can be “fixed” by setting some of the coefficients to zero \cite{26}, maintaining the rank of $P$.

- The toric subspace of $\mathcal{M}_c$ includes not just $(\mathbb{C}^*)^{\tilde{n} - d}$ but a partial compactification of this space which includes, e.g., Gepner models. Additionally, there is a complex codimension-one subvariety $\Delta_c \subset \mathbb{C}^{\tilde{n} - d}$ (and a corresponding compactification) for which the data do not determine a superconformal fixed point. This might contain some of the coordinate hyperplanes.

Similarly, the exponentiated complexified FI terms

$$q_a = e^{2\pi i \tau a} \in (\mathbb{C}^*)^{n - d} \quad (13)$$

provide coordinates on $\mathcal{M}_t$; more precisely they are holomorphic coordinates in an open neighborhood on the subspace of $\mathcal{M}_t$ describing theories arising as IR limits of GLSMs (the toric subspace). In general some of these may be redundant parameters. A complex codimension one subvariety $\Delta_t \subset (\mathbb{C}^*)^{n - d}$ of these correspond to singular models and do not flow to superconformal fixed points.
The mirror map for GLSMs takes a particularly simple form, anticipated in the notation above [26, 27, 28]. A model constructed with \( \tilde{n} \) chiral multiplets and \( \tilde{n} - d \) vector multiplets, with the gauge representation determined by \( \tilde{Q} \) will flow to the same superconformal fixed point with operators mapped by the mirror automorphism if the parameters \((\tilde{q}_a, \tilde{c}_i)\) are chosen such that

\[
(14) \quad \tilde{q}_a = \tilde{n} \prod_{i=1}^{\tilde{n}} \tilde{c}_i^{\tilde{Q}_i^a}, \quad q_a = n \prod_{i=1}^{n} c_i^{Q_i^a}.
\]

If there is a nontrivial factorization we also exchange \( \tilde{T} \) with \( T^T \). In other words, the dual model exchanges \( P \) for \( P^T \). The toric moduli space for the new model is identical to that of the original, under the exchange of \( M_t \) with \( M_c \). The discriminant \( \Delta_c \) of the resulting model coincides precisely with \( \Delta_t \) for the original, and vice versa.

Equivalently, as noted in [28], one can formulate the mirror model with twisted chiral charged fields coupled to twisted vector multiplets (with chiral field strength). The discussion of parameter spaces above is of course valid in this case as well, replacing chiral by twisted chiral (and vice versa) everywhere. The superconformal theories are in fact identical, and the prediction is that the partition functions must coincide exactly up to the ambiguity in their definition.

We will use this presentation for explicit computations. In other words, we will compare \( Z_t \) for the model built from \( P = \tilde{T} T \) with parameters \((\tilde{q}_a, \tilde{c}_i)\) to \( Z_t \) for the model built from \( P^T = T^T \tilde{T}^T \) and parameters \((\tilde{q}_a, \tilde{c}_i)\) satisfying (14) but composed of twisted chiral charged fields, etc. The latter is equivalent to \( Z_c \) for the model built from the same combinatorial data using chiral charged matter, etc. We will often abuse notation and refer to this as simply \( Z_c \).

2.1. Localization results. To avoid excessive clutter, let us denote by \( g = n - d \) and \( \tilde{g} = \tilde{n} - d \) the ranks of the gauge groups for the original and the dual model, respectively. The \( S^2 \) partition function depending on twisted chiral parameters, \( Z_t \), localizes to an integral of the classical action over the Coulomb branch, with integration measure provided by the 1-loop determinants of quadratic fluctuations around this locus [5, 6]:

\[
(15) \quad Z_t = \left( \frac{r}{r_0} \right)^{\frac{3}{2}} \sum_{m^a \in \mathbb{Z}} \int \frac{d\sigma}{(2\pi)^g} Z_{\text{class}}(\sigma, m) \prod_{i=1}^{n} Z_i(\rho, \sigma, m),
\]

where

\[
(16) \quad Z_{\text{class}} = \exp \left( -4\pi i \xi_a \sigma^a - i \theta_a m^a \right) = \prod_{a=1}^{g} (q_a)^{i \sigma^a - \frac{m^a}{2}} \left( \bar{q}_a \right)^{i \sigma^a + \frac{m^a}{2}}.
\]

and

\[
(17) \quad Z_i = \frac{\Gamma \left( \frac{Q_i^a}{2} - \sum_a Q_i^a (i \sigma^a + \frac{1}{2} m^a) \right)}{\Gamma \left( 1 - \frac{Q_i^a}{2} + \sum_a Q_i^a (i \sigma^a - \frac{1}{2} m^a) \right)}.
\]

The latter are the 1-loop determinants of the matter multiplets around the Coloumb branch.

We will assume that a choice of \( R \)-charges with \( \rho_i > 0 \) has been made, which is always possible, and which implies that the integrand is non-singular over \( \sigma \in \mathbb{R}^r \). The integrand of (15) is meromorphic in each of the \( \sigma^a \) variables and the integral can be evaluated by a
multi-dimensional method of residues, where the contour—and thus which poles contribute—is chosen based on the values of the FI parameters. For example, when there is a single FI parameter, the contour can close in the upper half-plane if \(|q|\) is sufficiently greater than 1 and in the lower half-plane when \(|q|\) is sufficiently less than 1, corresponding to the Landau-Ginzburg and geometric phases, respectively. In the former case, only those poles at \(i\sigma < 0\) contribute, while in the latter case only poles with \(i\sigma > 0\) contribute.

If we make a different choice of \(\rho_i\) by mixing with the gauge symmetry, \(\rho_i \rightarrow \rho_i + \delta a Q^a_i\), it is straightforward to see that the partition function changes only by an overall factor:

\[
Z_t \rightarrow \prod_{a=1}^g |q_a|^{\delta a} Z_t.
\]

As noted previously, this ambiguity in the partition function is an expected scheme-dependent effect. In particular, the choice of an \(R\)-symmetry is needed to define the coupling of the UV theory to the background metric on the sphere, and thus it is needed to define the regularization scheme. However, it has no effect on the scheme-independent quantities derived from the partition function.

The calculation of \(Z_c\) for this model follows from exchanging chiral fields for twisted chiral fields and twisted chiral fields strengths for chiral field strengths, etc., and coupling this model to the same sphere background. The localization for such a model was performed in [8] for a Landau-Ginzburg theory and in [29] for a gauge theory. Both results can be summarized as an integral over the Higgs branch, i.e. the space of orbits under the complexified gauge group of the constant modes of the twisted chiral fields. Note that for a Landau-Ginzburg theory, the space of gauge orbits is the field space itself. In particular,

\[
Z_c = \left(\frac{r}{r_0}\right)^{c/3} \int_{\mathbb{C}^n/(\mathbb{C}^*)^g} d\text{vol} \ e^{\tilde{W}-\tilde{W}},
\]

where \(\tilde{W}\) is the superpotential [5] but written in terms of twisted chiral fields. The measure on the space of gauge orbits, when the quotient is nontrivial, follows from the flat measure on \(\mathbb{C}^n\) after choosing a gauge slice via a finite-dimensional analog of the Fadeev-Popov procedure. To compare to the result of [29], we partially fix the gauge with the standard \(D\)-term constraint:

\[
Z_c = \left(\frac{r}{r_0}\right)^{c/3} \int d^{2n} \tilde{\Phi} \ \det (M^\dagger M) \prod_{a=1}^g \delta (2\mu^a - \xi^a) e^{\tilde{W}-\tilde{W}},
\]

where

\[
(M^\dagger M)_{ab} = \sum_{i=1}^n Q^a_i Q^b_i |\tilde{\Phi}_i|^2
\]

is the Fadeev-Popov measure, and

\[
\mu^a = \frac{1}{2} \sum_{i=1}^n Q^a_i |\tilde{\Phi}_i|^2
\]

If the change in \(\rho_i\) is such that \(\rho_i > 0\) no longer holds, then poles of the integrand will pass through the contour. We define the integral in this case by shifting the contour so that only the same poles contribute as when \(\rho_i > 0\).
is the $D$-term or moment map of the $a$-th $\mathbb{C}^*$ action. Additionally, we can restore the $R$-dependence to the twisted superpotential by scaling the fields

$$\tilde{\Phi}_i \to \left(\frac{r}{r_0}\right)^{\mu_i/2} \tilde{\Phi}_i,$$

recalling that under this scaling, the twisted superpotential has axial $R$-charge 2, and also using (9). The result, apart from irrelevant numerical factors, is that of [29]:

$$Z_c = \left(\frac{r}{r_0}\right)^{n-r} \int d^{2n} \Phi \ \text{det} \ (M^\dagger M) \prod_{a=1}^{g} \delta (2\mu^a - \xi^a) e^{r_0^4 (\tilde{W} - \bar{\tilde{W}})}.$$

The scale, $\mu$, appearing in (6) has been identified with $r^{-1}_0$.

There is no $\rho$-dependent ambiguity in this partition function analogous to that of $Z_t$. This is because a twisted chiral superfield is forced to have vanishing vector $R$-charge while the non-vanishing axial $R$-charge does not affect its coupling to the background metric [34].

However, the partition function does not respect the scaling symmetry of the $c_i$, (10). Instead, under $c_i \to \lambda_i^p c_i$ the fact that $\tilde{W}$ is invariant if this is combined with (11), $\Phi_i \to \lambda_i^{-1} \Phi_i$, shows that $Z_c$ transforms:

$$Z_c \to \prod_{i=1}^{n} |\lambda_i|^{-2} Z_c.$$

This, too, is an expected scheme-dependence. While the superpotential, and therefore the IR fixed point, is invariant under (10) and (11), the UV GLSM is not. Instead, this transformation acts a change of renormalization scheme. The conclusion is that $Z_c$ as calculated from the UV depends not on the invariant coordinates (12) but on the homogeneous coordinates $c_i$. As with the $\rho$ dependence of $Z_t$, IR properties such as the Zamolodchikov metric depend only on the invariant coordinates.

The scheme-dependence of $Z_t$ and that of $Z_c$ are of a different character. One leads to dependence on the choice of $\rho_i$ and the other leads to dependence on the homogeneous coordinates of $\mathcal{M}_c$. Not surprisingly, then, $Z_t$ for a given theory will not be exactly equal to $Z_c$ of its mirror as given by (19), since the former is explicitly a function of the invariant coordinates while the latter is independent of $\rho_i$. This is indeed what we find in the next section.

Before continuing, we remark that the sphere partition functions $Z_{t/c}$ are insensitive to the splitting $P = \tilde{T} T$, and therefore insensitive to any discrete gauge symmetries that result from this splitting, apart from an overall numerical coefficient. This can be argued in a couple of ways. First, the calculation of $Z_t$ proceeds through the Coloumb branch on which the action of these discrete gauge factors is trivial. Alternatively, thinking of the partition function as the two-point function of the identity operator, only untwisted sector states contribute. Twisted sectors do contribute to the partition function on the orbifold of the sphere, or equivalently on the sphere in the presence of defect operators [35, 36] which can create twisted sector states. These may be sensitive to the splitting $P = \tilde{T} T$ and could act as a refined test of mirror symmetry. Similarly, the elliptic genus may be sensitive to this splitting [37, 38]. We will leave such explorations to future work. Therefore, in the following, we will restrict to models with only continuous gauge symmetries.
3. FROM \( Z_t \) TO \( Z_c \)

In the following, we will demonstrate that \( Z_t \) of the model built from \( P \) is equal, up to scheme-dependence, to that of \( Z_c \) for the model built from \( P^T \). Along the way, we will clarify somewhat the relationship between the latter and the Hori-Vafa mirror of the former, a relationship that has not seen much commentary.

To evaluate \( Z_t \), we use the identity
\[
\int_0^\infty dt \ t^n J_\nu(t) = 2^n \frac{\Gamma\left(\frac{1}{2} (\mu + \nu + 1)\right)}{\Gamma\left(\frac{1}{2} (-\mu + \nu + 1)\right)}.
\]

For our purposes, we restrict to \( \nu \in \mathbb{Z} \). In that case, this identity holds when \(-\frac{1}{2} - |\nu| < \text{Re} \ \mu < \frac{1}{2}\) with the first inequality required for convergence near \( t = 0 \) and the second required for convergence as \( t \to \infty \). Setting
\[
\mu_i = \rho_i - 2iQ_i \cdot \sigma - 1, \quad \nu_i = -Q_i \cdot m,
\]
we can apply this identity if we restrict \( 0 < \rho_i < \frac{3}{2} \). Furthermore, since \( \nu_i \in \mathbb{Z} \), we can write the Bessel functions as
\[
J_\nu(t) = \frac{1}{2\pi} \int_{-\pi}^\pi dy \ e^{it \sin y - i\nu y}.
\]

Changing variables \( t_i = 2e^{x_i} \), we have
\[
Z_t = \frac{1}{\pi^n} \left( \frac{x}{r_0} \right)^\frac{1}{2} \int d^n x \int d^n y \sum_{m^a} \int \frac{d^4 \sigma}{(2\pi)^4} \exp \left( \rho_i x_i - 2i\sigma \cdot (Q_i x_i + 2\pi \xi) + im \cdot (Q_i y_i - \theta) + \sum_{i=1}^n e^{x_i + iy_i} - e^{-x_i - iy_i} \right).
\]

In this form, as pointed out in [5, 8], the partition function is that of the Hori-Vafa mirror [31]. Specifically, following (19), it is the sphere partition function of a twisted Landau-Ginzburg theory with fields \( Y_i = x_i + iy_i \) and \( \Sigma_a \) with twisted superpotential
\[
\tilde{W}_{HV} = -i\Sigma_a \left( Q^a Y_i - \log q^a \right) + \sum_{i=1}^n e^{Y_i}.
\]

The imaginary part of a twisted chiral field strength multiplet, such as \( \Sigma_a \), is quantized. The factors of \( e^{\rho_i x_i} = e^{\frac{2\pi}{r} (Y_i + \bar{Y}_i)} \) can be thought of as modifying the measure, changing the variables in terms of which this is flat from \( Y_i \) to \( e^{\frac{2\pi}{r} Y_i} \). The same change of variables was part of the prescription of Hori and Vafa [31] for calculating the periods of compact CICY from an associated, non-compact toric CY. These authors produced indirect arguments for this change of variables, interpreted in flat space as a selection of the universality class of the kinetic terms in the dual model.

In the sphere partition function, they arise naturally. This was understood in [8, 5] as a consequence of abelian duality on the sphere. The sphere partition function depends on the \( R \) charges \( \rho_i \) through a holomorphic dependence on \( \tilde{m} + i\tilde{\xi} \) where \( \tilde{m} \) are twisted masses for the chiral matter fields. The dual of a chiral field with twisted mass was considered in [31], and indeed the effect is to introduce a linear correction to the Hori-Vafa twisted superpotential \( \delta \tilde{W}_{HV} = \tilde{m} Y_i \). This superpotential correction, however, vanishes in the flat space limit \( r \to \infty \).
Another derivation of how these terms appear is described in appendix B. Briefly, dualizing a chiral field with \( R \)-charge \( \rho \) coupled to a background \( R \)-symmetry gauge field leads to a coupling in the dual theory of the form \( \rho Y^\mu \epsilon_\mu^{ij} \), while there is no background \( R \)-symmetry gauge field on the sphere, the supersymmetrization of this term yields the desired linear twisted superpotential.

Continuing with \( Z_t \), recall the integral form of the delta function

\[
\delta (x - y) = \frac{1}{2\pi} \int dk \, e^{ik(x-y)},
\]

where one can view this equality as occurring inside an integral for more rigor. Further, recall the Poisson summation formula

\[
\sum_{\lambda \in \Lambda} F(x + \lambda) = \sum_{\lambda^* \in \Lambda^*} \hat{F}(\lambda^*) \frac{e^{2\pi i \lambda^* (x)}}{\text{vol}(\Lambda^*)},
\]

where \( \Lambda \) is a lattice, \( \Lambda^* \) is its dual, both of which are viewed as subsets of \( \mathbb{R}^D \). Further,

\[
\hat{F}(k) := \int d^Dx \, e^{-2\pi ik(x)} F(x).
\]

Using \( F(x) = \delta(x) \), we have the periodic delta function

\[
\sum_{\lambda \in \Lambda} \delta(x + \lambda) = \sum_{\lambda^* \in \Lambda^*} \frac{e^{2\pi i \lambda^* (x)}}{\text{vol}(\Lambda^*)}.
\]

Applying these formulae,

\[
Z_t = \frac{1}{\pi^d} \left( \frac{r}{r_0} \right)^{\frac{d}{2}} \sum_{m^n} \int d^n x \int_{-\pi}^{\pi} d^n y \ e^{(x_i \rho_i + \sum_i e^{\xi_i + iy_i} - e^{-\xi_i - iy_i})} \prod_{a=1}^g \delta (Q_a^a x_i + 2\pi \xi^a) \, \delta (Q_a^a y_i - \theta^a + 2\pi m^a).
\]

Given any solution, \( a_i \) and \( b_i \), to

\[
\sum_{i=1}^n Q_a^a a_i = -2\pi \xi^a, \quad \sum_{i=1}^n Q_a^a b_i = \theta^a,
\]

we can shift variables

\[
x_i \to x_i + a_i, \quad y_i \to y_i + b_i,
\]

in terms of which

\[
Z_t = \frac{1}{\pi^d} \left( \frac{r}{r_0} \right)^{\frac{d}{2}} \left( \prod_{i=1}^n |\tilde{c}_i|^{\rho_i} \right) \sum_{m^n} \int d^n x \int_{-\pi}^{\pi} d^n y \ e^{(x_i \rho_i + \sum_i \tilde{c}_i e^{\xi_i + iy_i} - \tilde{c}_i e^{-\xi_i - iy_i})} \prod_{a=1}^g \delta (Q_a^a x_i) \, \delta (Q_a^a y_i + 2\pi m^a)
\]

where \( \tilde{c}_i = e^{\alpha_i + ib_i} \). Note that our requirement on \( a_i \) and \( b_i \) translates to

\[
\prod_{i=1}^n \tilde{c}_i^{Q_a^a} = q^a.
\]
which is the monomial-divisor mirror map (14). Further, had we chosen a different solution to (36), \( \tilde{c}_i = \lambda_i \tilde{c}_i \), a shift of variables \( x_i + iy_i \rightarrow x_i + iy_i - \log \lambda_i \) would remove the dependence of \( Z_t \) on \( \lambda_i \). In other words, \( Z_t \) doesn’t depend on which choice of a solution to (36) we use.

Furthermore, we can pause to comment on the the dependence of \( Z_t \) on a choice of \( R \)-charges. Previously, for convergence, we stipulated that the \( R \)-charges lie in the range 0 < \( \rho_i < \frac{\lambda_i}{2} \). However, now it can be seen that a shift of the \( R \)-charges by a linear combination of the gauge charges only multiplies \( Z_t \) by powers of the FI parameters and does not affect convergence. Consider the shift \( \rho_i \rightarrow \rho_i + \delta_a Q_i^a \). The delta function removes the dependence of the integrand on \( \delta_a \). Additionally, the modification of the prefactor amounts to

\[
(40) \quad \prod_{i=1}^{n} |c_i|^{|\rho_i| + \delta_a Q^a_i} = \prod_{a=1}^{g} |q^a|^{|\delta_a|} \prod_{i=1}^{n} |\tilde{c}_i|^{|\rho_i|},
\]

where the monomial divisor map was used. This is the same dependence on \( \delta_a \) as was found from the definition of \( Z_t \) in terms of residues.

The partition function in the form (35) is still in the form of the Hori–Vafa mirror. In [5, 8] it was demonstrated that for Calabi-Yau hypersurfaces in \( \mathbb{P}^n \), the delta function constraints can be solved directly and the resulting partition function is that of an orbifold of a twisted LG theory with twisted superpotential given by the Greene–Plesser (or Batyrev–Borisov) dual to the original model, i.e., it is the dual from [17] in the Landau-Ginzburg phase.

Another way to solve the delta function constraints will more clearly relate the original model, and thus also its Hori-Vafa mirror, to the combinatoric mirror of Batyrev and Borisov. Recall that \( Q^a_i \) span the kernel of the matrix \( P \), therefore the delta functions enforce that \( x \in (\ker P)^\perp \simeq \text{im} P^T \). In turn, this implies there exists \( \tilde{x} \in \mathbb{R}^n \) such that \( x = P^T \tilde{x} \). However, \( \tilde{x} \) is only determined up to \( \ker P^T \), i.e. \( \tilde{x} \) and \( \tilde{x} + \delta_{\tilde{a}} \tilde{Q}^\tilde{a} \) yield the same \( x \), where \( \tilde{Q}^\tilde{a} \) span the kernel of \( P^T \). Similar statements hold for \( y \) modulo 2\( \pi \mathbb{Z} \).

All of that is to say that we can write \( Z_t \) as an integral over the \( \tilde{x} \) and \( \tilde{y} \) in \( \mathbb{R}^n \times T^n \) modulo the action of the ‘gauge’ symmetry: \( \tilde{x} + i\tilde{y} \rightarrow \tilde{x} + i\tilde{y} + (\tilde{\delta}_a + i\tilde{\gamma}_a) \tilde{Q}^\tilde{a} \). The measure on this space follows from the flat measure on \( \mathbb{R}^n \times T^n \) after fixing a gauge slice á la Fadeev-Popov.

\[
(41) \quad Z_t = \frac{1}{2\pi d} \left( \frac{r}{r_0} \right)^{\frac{n}{2}} \prod_{i=1}^{n} |\tilde{c}_i|^{|\rho_i|} \int_{\mathbb{R}^n \times T^n} d\text{vol}(\tilde{x}, \tilde{y}) \exp \left( \sum_{i=1}^{n} \left( \tilde{c}_i e^{\sum_{j}(\tilde{x}_j + i\tilde{y}_j)P_{ji}} - \tilde{c}_i e^{\sum_{j}(\tilde{x}_j - i\tilde{y}_j)P_{ji}} \right) \right).
\]

Here we have used \( P \cdot \rho = (2, 2, \ldots, 2)^T \). The prefactor of \( (2 \sum_{i} \tilde{x}_i) \) can be incorporated into a change of the measure, which will now be flat in terms of the variables \( \tilde{\Phi}_i = e^{\tilde{x}_i + i\tilde{y}_i} \), on which the gauge action is via \((\mathbb{C}^*)^\tilde{y}\):

\[
(42) \quad Z_t = \frac{1}{2\pi d} \left( \frac{r}{r_0} \right)^{\frac{n}{2}} \prod_{i=1}^{n} |\tilde{c}_i|^{|\rho_i|} \int_{\mathbb{C}^n/(\mathbb{C}^*)^\tilde{y}} d\text{vol} \exp \left( \tilde{W} - \tilde{\bar{W}} \right),
\]

with

\[
(43) \quad \tilde{W} = \sum_{i=1}^{n} \tilde{c}_i \prod_{i=1}^{\tilde{n}} \tilde{\Phi}_i^{P_{hi}}.
\]
This is the superpotential of the combinatoric mirror to the original model.

Apart from irrelevant constants (that we have not been especially careful to track and which can be absorbed into a rescaling of \( r_0 \), (42) differs from (19) by the prefactor \( \prod_{i=1}^n |\tilde{c}_i|^{\rho_i} \). Owing to the relationship \( \sum_i P_{ii} \rho_i = 2 \), this prefactor is sufficient to remove the transformation of \( Z_c \) under (10) (using the invariance of \( \tilde{W} \) under this combined with (11)). However, not surprisingly, it introduces the same \( \rho \)-dependent scheme-dependence exhibited by \( Z_t \). Since the disagreement between (42) and (19) is precisely of the expected form, we conclude that \( Z_{t/c} \) is consistent with the mirror symmetry conjecture.

We have made our basic calculation from first principles, using localization, but it would also be interesting to know if our computation could, in the alternative, be based on methods of Givental \[39\] [24], who also used localization to obtain his basic results.

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Appendix A. Demonstration That \( Z_c \) Solves the \( A \)-system

In [32], it is shown that \( Z_t \) satisfies a set of differential equations in the parameters, the \( A \)-system. This equation is most compactly written in terms of an auxiliary function defined as

(44) \[ \Psi_t = \left( \prod_{i=1}^n |\tilde{c}_i|^{-\rho_i} \right) Z_t(q^a, \bar{q}^a). \]

The set of equations is

\[
\prod_{\{i|Q^a_i > 0\}} \left( \frac{\partial}{\partial \tilde{c}_i} \right)^{Q^a_i} \Psi_t = \prod_{\{i|Q^a_i < 0\}} \left( \frac{\partial}{\partial \tilde{c}_i} \right)^{|Q^a_i|} \Psi_t, \quad \forall a,
\]

(45) \[ \sum_{i=1}^n P_{ii} \tilde{c}_i \frac{\partial}{\partial \tilde{c}_i} \Psi_t = -\Psi_t, \quad \forall \tilde{c}_i. \]

The function \( \Psi_t \), we have demonstrated, is precisely \( Z_c \), (19), of the mirror theory. In this presentation, it is quite straightforward to see that \( Z_c \) solves the \( A \)-system, and below we will give the details for completeness. Ignoring numerical factors,

(46) \[ \Psi_t = Z_c(\tilde{c}_i, \bar{c}_i) = \int d\text{vol} \exp \left( \tilde{W} - \bar{W} \right), \quad \text{on } \mathbb{C}^n/(\mathbb{C}^*)^k. \]
with

$$\tilde{W} = \sum_{i=1}^{n} \tilde{c}_i \prod_{i=1}^{\tilde{n}} \tilde{\Phi}_i^{P_i i}.$$ (47)

The only \(\tilde{c}_i\) dependence of \(\Psi_t\) is from \(\tilde{W}\).

$$\frac{\partial}{\partial \tilde{c}_i} \tilde{W} = \prod_{i=1}^{\tilde{n}} \tilde{\Phi}_i^{P_i i}.$$ (48)

Therefore,

$$\left( \frac{\partial}{\partial \tilde{c}_i} \right)^{Q_i^a} e^{\tilde{W}} = e^{\tilde{W}} \prod_{i=1}^{\tilde{n}} \tilde{\Phi}_i^{P_i Q_i^a}.$$ (49)

And,

$$\prod_{\{i|Q_i^a > 0\}} \left( \frac{\partial}{\partial \tilde{c}_i} \right)^{Q_i^a} e^{\tilde{W}} = e^{\tilde{W}} \prod_{i=1}^{\tilde{n}} \tilde{\Phi}_i^{Q_i^a} - \prod_{\{i|Q_i^a < 0\}} \left( \frac{\partial}{\partial \tilde{c}_i} \right)^{|Q_i^a|} e^{\tilde{W}},$$ (50)

where we’ve used \(\sum_i P_i Q_i^a = 0\). This is sufficient to show that the first of (45) holds.

To show the second holds, we observe

$$\sum_{i=1}^{n} P_{i i} \tilde{c}_i \frac{\partial}{\partial \tilde{c}_i} \Psi_t = \int \text{dvol} \tilde{\Phi}_i \frac{\partial}{\partial \tilde{\Phi}_i} \exp \left( \tilde{W} - \tilde{\tilde{W}} \right), \forall i.$$ (51)

After integrating by parts, the result follows.

**Appendix B. Abelian Duality on and off the Sphere**

In this appendix, we argue for the existence on the sphere of the linear twisted superpotential \(\tilde{W} \sim \rho Y\) for fields \(Y\) dual to chiral fields with \(R\)-charge \(\rho\). As stated in the body, this argument has appeared previously in multiple forms. The following is a slight modification of the argument appearing in [21].

Consider a two-dimensional theory of a complex scalar field:

$$S = \int d\phi \wedge * d\phi = \int d\sigma \wedge * d\sigma + \sigma^2 d\theta \wedge * d\theta.$$ (52)

The change of variables from the first equation to the second, \(\phi = \sigma e^{i\theta}\) will be very badly behaved around \(\phi = 0\). Nevertheless, away from this point, we can classically dualize the \(U(1)\) isometry under which \(\theta \to \theta + \epsilon\). To do so, consider instead the following action for a
1-form $c$ and a Lagrange multiplier $\lambda$, ignoring the kinetic action for the $\sigma$ field which plays no part:

$$S = \int \sigma^2 c \wedge *c - 2c \wedge d\lambda. \tag{53}$$

Integrating out $\lambda$ implies $c$ is closed and therefore exact (in $\mathbb{R}^2$ for now and later in $S^2$ also). Therefore, we reproduce the original action. Instead, integrating out $c$ yields

$$*c = \frac{1}{\sigma^2} d\lambda \quad \Rightarrow \quad S = -\int \frac{1}{\sigma^2} d\lambda \wedge *d\lambda. \tag{54}$$

We can repeat this calculation with the current associated to $\theta \to \theta + \epsilon$ coupled to a background gauge field $A$. Our starting point is

$$S = \int \sigma^2 (c - A) \wedge * (c - A) - 2c \wedge d\lambda. \tag{55}$$

Integrating out $c$ yields

$$S = -\int \frac{1}{\sigma^2} d\lambda \wedge *d\lambda + 2\lambda dA. \tag{56}$$

The supersymmetrization of this starts with a chiral superfield $\Phi$ with canonical Kähler potential $|\Phi|^2$. Away from $\Phi = 0$, we may define $\Phi = e^\Pi$, where $\Pi$ is also chiral. To dualize the phase of $\Pi$, analogous to $\theta$ above, we replace $\Pi$ with an unconstrained, real superfield and add a Lagrange multiplier to reinstate the chiral constraint:

$$\mathcal{L} = \int d^4 \theta \ e^{\Pi + \bar{\Pi}} \rightarrow \int d^4 \theta \ e^{2B} - 2B \ (Y + \bar{Y}). \tag{57}$$

Integrating out $Y$ ensures that $B = \Pi + \bar{\Pi}$. Instead, integrating out $B$ we find

$$\mathcal{L} = -\int d^4 \theta \ (Y + \bar{Y}) \log (Y + \bar{Y}). \tag{58}$$

If $\Pi$ is coupled to a background vector supermultiplet, the appropriate supersymmetrization of the coupling of a global, flavor symmetry to a background field, then the above is modified to

$$\mathcal{L} = -\int d^4 \theta \ (Y + \bar{Y}) \log (Y + \bar{Y}) - 2 \ (Y + \bar{Y}) \ V$$

$$\tag{59} = -\int d^4 \theta \ (Y + \bar{Y}) \log (Y + \bar{Y}) - 2 \int d^2 \bar{\theta} Y \Sigma + \text{c.c.}$$

However, if $\Phi$ has $R$-charge $\rho$ and is instead coupled to a background $R$-symmetry gauge field, we still expect a coupling of the dual $Y$ to the field strength of this gauge field from (56), but the supersymmetrization of this coupling will not be $Y \Sigma$ because the $R$-symmetry current is not contained in an ordinary linear multiplet. It is contained in the $R$-multiplet, and so the corresponding gauge field is contained in the gravity multiplet.

The supersymmetric coupling responsible is

$$\rho \int d^2 \bar{\theta} \ \tilde{\mathcal{E}} \tilde{\mathcal{R}} Y + \text{c.c.}, \tag{60}$$

where $\tilde{\mathcal{E}}$ is the twisted supersymmetric density and $\tilde{\mathcal{R}}$ is a twisted chiral curvature superfield containing the Ricci scalar and the curvature of the $U(1)_V$ gauge field, among other terms. When evaluated in the supersymmetric sphere background, this coupling gives precisely the
linear twisted superpotential that effects the change of fundamental variable in agreement with the Hori-Vafa prescription. More details about this coupling can be found in [34, eq. (6.73)] and [9, eq. (3.31)]. One salient feature to note is that this coupling does not survive the flat-space limit, and so it is not present in the original [31].
REFERENCES

[1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, “Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory,” Nucl. Phys. B241 (1984) 333–380. [605(1984)].

[2] A. B. Zamolodchikov, “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,” JETP Lett. 43 (1986) 730–732. [Pisma Zh. Eksp. Teor. Fiz.43,565(1986)].

[3] D. Kutasov, “Geometry on the Space of Conformal Field Theories and Contact Terms,” Phys. Lett. B220 (1989) 153–158.

[4] J. Gomis, Z. Komargodski, H. Ooguri, N. Seiberg, and Y. Wang, “Shortening anomalies in supersymmetric theories,” JHEP 01 (2017) 067. [1611.03101]

[5] F. Benini and S. Cremonesi, “Partition Functions of N = (2, 2) Gauge Theories on S2 and Vortices,” Commun. Math. Phys. 334 (2015), no. 3 1483–1527, 1206.2356.

[6] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, “Exact Results in D= 2 Supersymmetric Gauge Theories,” JHEP 05 (2013) 093. [1206.2606]

[7] J. Gomis and S. Lee, “Exact Kähler Potential from Gauge Theory and Mirror Symmetry,” JHEP 04 (2013) 019. [1210.6022]

[8] E. Gerchkovitz, J. Gomis, and Z. Komargodski, “Sphere Partition Functions and the Zamolodchikov Metric,” JHEP 11 (2014) 001. [1405.7271]

[9] R. Donagi and D. R. Morrison, “Conformal field theories and compact curves in moduli spaces,” JHEP 05 (2018) 021. arXiv:1709.05355 [hep-th]

[10] L. J. Dixon, “Some world-sheet properties of superstring compactifications, on orbifolds and otherwise,” in Superstrings, Unified Theories, and Cosmology 1987 (G. Furlan et. al., eds.), pp. 67–126, World Scientific, Singapore, New Jersey, Hong Kong, 1988.

[11] W. Lerche, C. Vafa, and N. P. Warner, “Chiral rings in N=2 superconformal theories,” Nucl. Phys. B 324 (1989) 427–474.

[12] P. S. Aspinwall, C. A. Lüttken, and G. G. Ross, “Construction and couplings of mirror manifolds,” Phys. Lett. B 241 (1990) 373–380.

[13] P. Candelas, M. Lynker, and R. Schimmrigk, “Calabi–Yau manifolds in weighted P4,” Nucl. Phys. B 341 (1990) 383–402.

[14] B. R. Greene and M. R. Plesser, “Duality in Calabi-Yau Moduli Space,” Nucl. Phys. B338 (1990) 15–57.

[15] P. Candelas, X. C. De la Ossa, P. S. Green, and L. Parkes, “An exactly soluble superconformal theory from a mirror pair of calabi-yau manifolds,” Phys. Lett. B258 (1991) 118–126.

[16] A. B. Givental, “Equivariant Gromov–Witten invariants,” Internat. Math. Res. Notices (1996), no. 13 613–663, arXiv:alg-geom/9603021.

[17] W. Gu and E. Sharpe, “A Proposal for Nonabelian Mirrors,” 1806.04678.

[18] V. V. Batyrev, “Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties,” J. Alg. Geom. 3 (1994) 493–545, alg-geom/9310003.

[19] L. Borisov, “Towards the Mirror Symmetry for Calabi-Yau Complete intersections in Gorenstein Toric Fano Varieties,” arXiv e-prints (Oct., 1993) alg–geom/9310001, alg-geom/9310000.

[20] A. B. Givental, “Equivariant Gromov–Witten invariants,” Internat. Math. Res. Notices (1996), no. 13 613–663, arXiv:alg-geom/9603021.

[21] B. H. Lian, K. Liu, and S.-T. Yau, “Mirror principle. I,” Asian J. Math. 1 (1997), no. 4 729–763, arXiv:alg-geom/9712011.

[22] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, “The Monomial Divisor Mirror Map,” alg-geom/9309007.
[27] P. Candelas, X. de la Ossa, and S. H. Katz, “Mirror symmetry for Calabi–Yau hypersurfaces in weighted $P^4$ and extensions of Landau–Ginzburg theory,” Nucl. Phys. B 450 (1995) 267–292, arXiv:hep-th/9412117.

[28] D. R. Morrison and M. R. Plesser, “Towards mirror symmetry as duality for two-dimensional abelian gauge theories,” Nucl. Phys. Proc. Suppl. 46 (1996) 177–186, hep-th/9508107.

[29] N. Doroud and J. Gomis, “Gauge theory dynamics and Kähler potential for Calabi-Yau complex moduli,” JHEP 12 (2013) 099, 1309.2305.

[30] F. Benini and B. Le Floch, “Supersymmetric Localization in Two Dimensions,” J. Phys. A50 (2017), no. 44 443003, 1608.02955.

[31] K. Hori and C. Vafa, “Mirror Symmetry,” hep-th/0002222.

[32] J. Halverson, V. Kumar, and D. R. Morrison, “New Methods for Characterizing Phases of 2D Supersymmetric Gauge Theories,” JHEP 09 (2013) 143, 1305.3278.

[33] P. S. Aspinwall and M. R. Plesser, “General Mirror Pairs for Gauged Linear Sigma Models,” JHEP 11 (2015) 029, 1507.00301.

[34] C. Closset and S. Cremonesi, “Comments on $\mathcal{N} = (2, 2)$ supersymmetry on two-manifolds,” JHEP 07 (2014) 075, 1404.2836.

[35] K. Hosomichi, “Orbifolds, Defects and Sphere Partition Function,” JHEP 02 (2016) 155, 1507.07650.

[36] K. Hosomichi, S. Lee, and T. Okuda, “Supersymmetric vortex defects in two dimensions,” JHEP 01 (2018) 033, 1705.10623.

[37] F. Benini, R. Eager, K. Hori, and Y. Tachikawa, “Elliptic Genera of Two-Dimensional $\mathcal{N} = 2$ Gauge Theories with Rank-One Gauge Groups,” Lett. Math. Phys. 104 (2014) 465–493, 1305.0533.

[38] F. Benini, R. Eager, K. Hori, and Y. Tachikawa, “Elliptic Genera of 2d $\mathcal{N} = 2$ Gauge Theories,” Commun. Math. Phys. 333 (2015), no. 3 1241–1286, 1308.4896.

[39] A. B. Givental, “Homological geometry and mirror symmetry,” in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), pp. 472–480, Birkhäuser, 1995.