RESOLUTION OF THE $\lambda \Phi^4$ PUZZLE
AND A 2 TeV HIGGS BOSON

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ABSTRACT
We argue that massless $(\lambda \Phi^4)_4$ is “trivial” without being entirely trivial. It has a non-trivial effective potential which leads to spontaneous symmetry breaking, but the particle excitations above the broken vacuum are non-interacting. The key to this picture is the realization that the constant background field (the mode with zero 4-momentum) renormalizes differently from the fluctuation field (the finite-momentum modes). This picture reconciles rigorous results and lattice calculations with one-loop and Gaussian-approximation analyses. Because of “triviality”, these latter two methods should be effectively exact. Indeed, they yield the same renormalized effective potential and the same relation $m_h^2 = 8\pi^2 v^2$ between the particle mass and the physical vacuum expectation value. This relation predicts a Higgs mass $m_h \sim 2$ TeV in the standard model. The non-interacting nature of the scalar sector implies, by the equivalence theorem, that Higgs and gauge bosons interact only weakly, through their gauge and Yukawa couplings.
1 INTRODUCTION

The standard model of electroweak interactions makes use of spontaneous symmetry breaking (SSB) to explain the origin of vector-boson masses. The traditional description relies on an essentially classical treatment of a $\lambda \Phi^4$ scalar sector, with perturbative quantum corrections. In this picture the Higgs mass $m_h$ is proportional to the “renormalized coupling” $\lambda_R$, so if the Higgs is heavy ($m_h \geq 0.7$ TeV or so), perturbation theory clearly breaks down. However, in that case it is usually inferred that longitudinal gauge bosons would necessarily become strongly interacting at TeV energies.

However, one should really ask if a full quantum treatment of the scalar sector can give SSB, and if so what the consequences are. Here the whole question is thrown open by the strong evidence that the $\lambda \Phi^4$ theory is “trivial”. Some authors interpret this to mean that the scalar sector of the standard model can only be an effective theory, valid only up to some cutoff scale. Without a cutoff, the argument goes, there would be no scalar self-interactions, and without scalar self-interactions there would be no symmetry breaking. This view also leads to upper bounds on the Higgs mass.

However, symmetry breaking is not incompatible with “triviality”. One could have a non-zero vacuum expectation value for the field, yet find only non-interacting, free-particle excitations above the vacuum. We argue that this is what happens in pure $\lambda \Phi^4$ theory. Our picture reconciles the evidence for “triviality” with the evidence for a non-trivial effective potential.

A non-trivial effective potential with SSB emerged naturally from an analysis of the Gaussian effective potential of $\lambda \Phi^4$ theory. More recently it was realized that the same results emerge from the one-loop effective potential. The key features are “asymptotic freedom” (i.e., a flow of the bare coupling constant $\lambda_B$, as a function of the lattice spacing $a$, in which $\lambda \to 0$ as $a \to 0$, such that a macroscopic correlation length is obtained) and masslessness in the symmetric phase.

Initially, it was thought that the Gaussian-approximation results implied a fully non-trivial, interacting theory. However, since the effective potential contains information only about zero-momentum Green’s functions, it does not by itself provide information about the existence or otherwise of a momentum-dependent renormalized coupling “$\lambda_R(Q^2)$.” Consequently, these results do not necessarily imply the existence of non-trivial interactions in the “shifted” theory. (In fact, the finite-temperature generalization implies that the shifted field is non-interacting, and apparent paradoxes in the extension to the effective action, variously interpreted, may be seen as pointing to the same
conclusion.)

In this paper we elaborate the following picture for pure $\lambda \Phi^4$ theory [10, 11, 20]: Writing the field $\Phi(x)$ as $\phi + h(x)$, we expect the fluctuation field $h(x)$ to be non-interacting, but its mass will be proportional to the background constant field $\phi$. Thus, its vacuum energy is $\phi$ dependent and this, together with the $\lambda \phi^4$ term, produces a non-trivial one-loop effective potential, which should be exact, because there are no $h$-particle interactions. The renormalization requires the bare coupling constant to go to zero in the continuum limit, and involves an infinite re-scaling of the $\phi$ field. The $h(x)$ field, however, is not re-scaled, and this provides a simple way to understand why it is non-interacting. Most of the paper is concerned with pure $\lambda \Phi^4$ theory, but we consider the implications for the electroweak theory in the final section.

2 “TRIVIALITY” AND SSB

Analytical and numerical studies of $(\lambda \Phi^4)_4$ theory [4, 5, 6, 7] have overwhelmingly been driven to the conclusion that it is a “generalized free field theory.” This means that all renormalized Green’s functions of the continuum theory are expressible in terms of the first two moments of a Gaussian distribution [21]:

$$\tau(x) = v,$$

(2.1)

$$\tau(x, y) = v^2 + G(x - y),$$

(2.2)

so that

$$\tau(x, y, z) = v^3 + v(G(x - y) + G(x - z) + G(y - z)),$$

(2.3)

$$\tau(x, y, z, w) = v^4 + v^2(G(x - y) + \text{perm.}) + G(x - y)G(z - w) + \text{perm.},$$

(2.4)

and so on. Here, $v$ is a constant (since we assume that translational invariance is not broken), and $G(x - y)$ is just a free propagator with some mass $m_h$ and residue $Z_h = 1$, since it must satisfy a Källen-Lehmann representation with spectral function $\delta(s - m_h^2)$. The index “$h$” in $Z_h$ and $m_h$ refers to the shifted field $h(x)$ introduced by means of a suitably de-singularized, renormalized field operator $\Phi_R(x)$, such that $\langle \Phi_R(x) \rangle = v$ and $h(x) \equiv \Phi_R(x) - v$. The above equations imply that all connected three- and higher-point Green’s functions of the $h(x)$ field vanish; i.e., “triviality”.

However, although the generalized free-field structure implies a trivially free shifted theory it does not forbid a non-zero value of $v$. Indeed, there are explicit studies of “triviality” in the broken phase ($v \neq 0$) [7]. This suggests that it is possible for the
theory to have a non-trivial effective potential $V_{\text{eff}}$ with SSB minima. Indeed, the lattice calculations of Huang et al \cite{22} do find a non-trivial effective potential, even while finding no sign of particle interactions: they conclude that the theory cannot be entirely trivial \cite{22, 23}. The effective potential, when expanded about the origin, generates the zero-momentum Green’s functions of the symmetric phase \cite{24}. Thus, for the effective potential to be non-trivial and give SSB, there has to be non-triviality in the zero-momentum-mode sector of the symmetric phase, which somehow acts to de-stabilise the symmetric vacuum.

This immediately suggests that we concentrate on massless $\lambda \Phi^4$ theories, for which zero-momentum ($p^\mu = 0$) represents a physical, on-shell point. For a massless theory, a non-trivial effective potential would imply a non-trivial scattering matrix. There is, in fact, good evidence that massless $\lambda \Phi^4$ theory is non-trivial in this sense \cite{25}. [In Ref. \cite{25} the conformal symmetry of the massless $\lambda \Phi^4$ theory is exploited, allowing a mapping of the Minkowski space into the Einstein universe. Asymptotic states are restricted to those which carry “Einstein quantum numbers”. In Minkowski space these are collective states containing an indefinite number of massless particles. As in the well-known Bloch-Nordsieck procedure in QED, such states are needed because of the serious infrared problems. The effective potential also involves a sum of diagrams with an arbitrarily large number of external legs, with infrared divergences cancelling only in the sum \cite{24}.]

It is quite natural to expect that any non-trivial dynamics of the zero-momentum mode would induce SSB: The classical potential of the massless theory is very flat at the origin, and ripe for instability. Moreover, this would be the simplest way for the theory to escape its infrared difficulties. Since the massless theory contains no intrinsic scale, the physical scale, $v$ (with $m_h$ proportional to $v$), must be spontaneously generated by “dimensional transmutation” from a theory with no intrinsic scale in its symmetric phase. This is exactly the philosophy of the classic paper of Coleman and Weinberg \cite{24}.

“Dimensional transmutation” requires ultraviolet divergences to produce a non-zero Callan-Symanzik $\beta$ function. A familiar example is the origin of the $\Lambda$ scale in QCD. However, in that case one can obtain the $\beta$ function perturbatively, from the momentum dependence of the renormalized coupling constant. In $(\lambda \Phi^4)_4$ theory that approach is doomed to failure since “triviality” means that any “renormalized coupling constant”, defined from the 4-point function at non-zero external momenta, must vanish. (It is not surprising, then, that the perturbatively defined $\beta$ function is positive and has no fixed point. This implies an unphysical Landau pole at high energies – which can only be avoided by requiring the renormalized coupling at finite energies to vanish as the ultraviolet
The regulator is removed. In this sense, perturbation theory itself signals “triviality” \[8\]). To extract a more meaningful \(\beta\) function one should start from a quantity that will be finite, and non-vanishing in the infinite-cutoff limit. This is not true for the 4-point function.

However, in our picture, \(V_{\text{eff}}\) is non-trivial and one may extract from it a nonperturbatively defined \(\beta\) function that characterizes the dependence of the bare coupling constant on the ultraviolet regulator. For the picture to be consistent we would expect this \(\beta\) function to be negative, giving asymptotic freedom (\(\lambda_B \to 0\) in the continuum limit), since it seems \[4\] that asymptotic freedom is the only possibility to avoid “entire triviality”.

3 THE EFFECTIVE POTENTIAL

To find \(\beta\) from \(V_{\text{eff}}\) we follow the usual Renormalization-Group (RG) procedure. First, we introduce a generic ultraviolet regulator “\(r\)" (which may be a cutoff \(\Lambda\), or an inverse lattice spacing \(1/a\), or one may identify \(\ln r^2\) with \(2\delta_{d}\) in dimensional regularization, etc.), so that the effective potential depends on \(r\), and on the bare classical field \(\phi_B\) and on the bare mass and coupling parameters:

\[
V_{\text{eff}} = V_{\text{eff}}(\phi_B, \lambda_B, m_B^2, r).
\]  

(3.1)

The bare mass is treated as a counterterm for the quantum theory such that, in any approximation and in any regularization scheme, the condition \[2\]

\[
\left. \frac{\partial^2 V_{\text{eff}}}{\partial \phi_B^2} \right|_{\phi_B=0} = 0
\]

(3.2)

is satisfied. This ensures that the theory has no intrinsic scale in its symmetric phase. (In dimensional regularization this simply requires \(m_B = 0\) and the classical scale invariance of the bare Lagrangian is manifest.) After implementing the masslessness condition, (3.2), the effective potential depends only on \(r\) itself and on the bare parameters \(\phi_B = \phi_B(r)\) and \(\lambda_B = \lambda_B(r)\) whose flow, in the limit \(r \to \infty\), is dictated by the requirement of RG invariance; that is:

\[
\lim_{r \to \infty} \left\{ r \frac{\partial}{\partial r} + \beta \frac{\partial}{\partial \lambda_B} - \gamma \phi_B \frac{\partial}{\partial \phi_B} \right\} V_{\text{eff}}(\phi_B, \lambda_B, r) = 0,
\]

(3.3)

where \(\beta \equiv \frac{r d\lambda_B}{dr}\) and \(\gamma \equiv -\frac{r}{\phi_B} \frac{d\phi_B}{dr}\). This anomalous dimension \(\gamma\) should not be confused with the more conventional quantity associated with the \(p^2\)-dependence of the two-point function. (That quantity would be associated with the cutoff dependence of \(Z_h\), and must
vanish as \( r \to \infty \) since \( Z_h \to 1 \). Here, \( \phi_B \) is a constant background field, and has no dependence on \( p^2 \).

One can easily disentangle \( \beta \) from \( \gamma \) as follows \[16\]. If the effective potential has its non-trivial minimum at a particular \( \phi_B = v_B \); i.e.,

\[
\left. \frac{\partial V_{\text{eff}}}{\partial \phi_B} \right|_{\phi_B=v_B} = 0,
\]

(3.4)

where \( v_B \) will depend on \( r \) and \( \lambda_B \), then the vacuum energy density:

\[
W(\lambda_B, r) = V_{\text{eff}}(v_B(\lambda_B), \lambda_B, r)
\]

(3.5)

will satisfy (from Eq. (3.3))

\[
\lim_{r \to \infty} \left\{ r \frac{\partial}{\partial r} + \beta \frac{\partial}{\partial \lambda_B} \right\} W(\lambda_B, r) = 0.
\]

(3.6)

This equation will determine \( \beta \) and one can then go back to (3.3) to extract \( \gamma \).

Acceptance of “triviality” in the sense of Eqs. (2.1–2.4) means that this procedure represents the only hope for obtaining a not-entirely-trivial continuum limit. (It also means that we may expect a contradiction if we try to use an approximation to the effective potential that is inherently inconsistent with generalized-free-field behaviour. We shall return to this important point in Sect. 6.)

4 THE ONE-LOOP EFFECTIVE POTENTIAL

Now let us see what happens in the simplest approximation scheme. We first write the bare field as

\[
\Phi_B(x) = \phi_B + h_B(x),
\]

(4.1)

where we have separated out the zero-momentum component \( \phi_B = \) constant (demanding, in order for the separation to be unambiguous, that the field \( h_B(x) \) has no Fourier projection on to \( p^\mu = 0 \)). Consider the approximation in which the field \( h_B(x) \) is allowed to interact to all orders in \( \lambda_B \) with \( \phi_B \) but has no non-linear interaction with itself. To this level of approximation \( h_B(x) \) is just a free field whose mass-squared is

\[
\omega^2(\phi_B) \equiv \frac{1}{2} \lambda_B \phi_B^2.
\]

(4.2)

However, through its zero-point energy, it will produce a non-trivial contribution to the effective potential. To compute this we take the shifted, linearized Hamiltonian:

\[
H_o(\phi_B) = \int d^3x \left[ \frac{\lambda_B}{4!} \phi_B^4 + \frac{1}{2} \Pi_h^2 + \frac{1}{2} (\vec{\nabla} h)^2 + \frac{1}{2} \omega^2(\phi_B) h^2 \right],
\]

(4.3)
and find its lowest eigenvalue, as a function of $\phi_B$. This gives the effective potential times a volume factor. Dropping a $\phi_B$-independent constant term, this gives:

\begin{align}
V^{1-\text{loop}}(\phi_B) &= \frac{\lambda_B}{4!} \phi_B^4 + \frac{\omega^4(\phi_B)}{64\pi^2} \left( \ln \frac{\omega^2(\phi_B)}{\Lambda^2} - \frac{1}{2} \right), \\
&= \frac{\lambda_B}{24} \phi_B^4 + \frac{\lambda_B^2 \phi_B^4}{256\pi^2} \left( \ln \frac{\lambda_B \phi_B^2}{2\Lambda^2} - \frac{1}{2} \right),
\end{align}

where we use an ultraviolet cutoff $\Lambda$, as in Ref. [24].

This is, of course, the familiar “one-loop effective potential” [24, 26, 27]. In Ref. [24] it was obtained, using Feynman diagrams of the unshifted theory, from the sum of all one-loop 1PI graphs with external lines $\phi_B$ and massless propagators running round the loop. (This diagrammatic interpretation links the effective potential with the dynamics of the underlying massless theory, as discussed earlier.) Although we use the traditional name, “one-loop”, for this approximation, we stress that it is the above linearization procedure, and not the loop expansion, that is our rationale for it.

Taking this effective potential and following the well-defined procedure described in Eqs. (3.3–3.6) above, one finds straightforwardly [17, 10]:

\begin{align}
\beta &= - \frac{3\lambda_B}{16\pi^2}, \\
\gamma &= - \frac{3\lambda_B}{32\pi^2} = \frac{\beta}{2\lambda_B}.
\end{align}

This result is crucial, and the reader is urged to check it for him- or her-self.

Thus, this $\beta$ is indeed negative. These RG functions allow the limit $\Lambda \to \infty$ to be taken such that the vacuum energy density $W = -m_h^2/(128\pi^2)$ is finite and RG invariant. Hence, the particle mass:

\begin{equation}
m_h^2 \equiv \omega^2(v_B) = \frac{1}{2} \lambda_B v_B^2 = \Lambda^2 \exp(-\frac{32\pi^2}{3\lambda_B}),
\end{equation}

is finite and RG invariant. When re-arranged, this equation shows the asymptotic-freedom property explicitly:

\begin{equation}
\lambda_B = \frac{32\pi^2}{3} \frac{1}{\ln(\Lambda^2/m_h^2)}.
\end{equation}

Eliminating $\Lambda$ in favour of $v_B$, $V^{1-\text{loop}}$ can be expressed as:

\begin{equation}
V^{1-\text{loop}}(\phi_B) = \frac{\lambda_B^2 \phi_B^4}{256\pi^2} \left( \ln \frac{\phi_B^2}{v_B^2} - \frac{1}{2} \right).
\end{equation}
To make the finiteness and RG invariance of $V^{1\text{-}\text{loop}}$ manifest we may re-express it in terms of a renormalized field $\phi_R = Z_\phi^{-\frac{1}{2}} \phi_B$, where an infinite $Z_\phi$ absorbs the cutoff dependence. This $Z_\phi$ must satisfy $\gamma = -\frac{1}{2} \Lambda \frac{d}{d\Lambda} \ln Z_\phi$ where $\gamma$ is the anomalous dimension of Eqs. (3.3, 4.7). The absolute normalization of $Z_\phi$ is fixed by requiring the physical mass $m_h^2$ of the fluctuation field $h_R(x) = h_B(x)$ to be equal to the second derivative, at the minimum, of $V^{1\text{-}\text{loop}}$ with respect to $\phi_R$. (We shall return to justify this condition in a moment.)

As shown in Refs. [10, 12, 11] one obtains

$$Z_{1\text{-}\text{loop}}^{\phi} = \frac{16\pi^2}{\lambda_B} = \frac{3}{2} \ln(\Lambda^2/m_h^2),$$

so that (with $v \equiv v_R = Z_\phi^{-\frac{1}{2}} v_B$)

$$V^{1\text{-}\text{loop}} = \pi^2 \phi_R^4 \left( \ln \frac{\phi_R^2}{v^2} - \frac{1}{2} \right),$$

and

$$m_h^2 = 8\pi^2 v^2,$$

as advertised earlier.

5 THE FIELD RE-SCALING

We now want to discuss two apparently unconventional aspects of this analysis. Firstly, it is crucial to this picture that the $Z_{\phi}^{1/2}$ re-scaling of the constant background field $\phi_B$ is quite distinct from the $Z_h^{1/2} = 1$ re-scaling of the fluctuation field $h(x)$. We can, in fact, express this as a single, overall re-scaling of the whole field, provided that we use a momentum-dependent $Z^{1/2}(p)$:

$$Z^{\frac{1}{2}}(p) = Z_\phi^{\frac{1}{2}} \mathcal{P} + Z_h^{\frac{1}{2}} \overline{\mathcal{P}},$$

where

$$\mathcal{P} \equiv \frac{\delta^4(p)}{\delta^4(0)} \quad \text{and} \quad \overline{\mathcal{P}} = 1 - \mathcal{P}$$

are orthogonal projections ($\mathcal{P}^2 = \mathcal{P}$, $\overline{\mathcal{P}}^2 = \overline{\mathcal{P}}$, $\mathcal{P} \overline{\mathcal{P}} = 0$) which select and remove the $p^\mu = 0$ mode, respectively. [Here $\delta^4(p) \equiv (2\pi)^4 \delta^4(p)$, and $\delta^4(0)$ has the usual interpretation as the spacetime volume.] Note that “$p^\mu = 0$” is a Lorentz-invariant statement so our momentum-dependent $Z(p)$ does not violate any sacred principles. In terms of the Fourier transform
of the field operators it works as follows:

\[ Z^{\frac{1}{2}}(p)\tilde{\Phi}_R(p) = \left( Z^{\frac{1}{2}}_\phi P + Z^{\frac{1}{2}}_h \mathcal{P} \right) \left( \phi_R \delta^4(p) + \tilde{h}_R(p) \right) \]  

(5.3)

\[ = Z^{\frac{1}{2}}_\phi \phi_R \tilde{\delta}^4(p) + Z^{\frac{1}{2}}_h \tilde{h}_R(p) = \phi_B \delta^4(p) + \tilde{h}_B(p) = \tilde{\Phi}_B(p). \]  

(5.4)

(Note that \( P\tilde{h}(p) = 0 \) by definition.) It is easy to check that \( Z^{-1/2}(p) \), \( Z(p) = (Z^{1/2}(p))^2 \), etc., work properly. For consistency, the renormalized momentum-space propagator of the complete \( \tilde{\Phi}_R(p) \) field must be written as:

\[ \phi_R^2 \delta^4(p) + \frac{\mathcal{P}}{p^2 - \omega^2(\phi_R)}, \]  

(5.5)

with \( \omega^2(\phi_R) = 8\pi^2 \phi_R^2 \). (At \( \phi_R = v \) this corresponds to the Fourier transform of Eq. (2.2).)

The \( \mathcal{P} \) projection makes no difference except in the symmetric phase (where \( \omega^2 \) and \( \phi_R^2 \) vanish): the propagator is then formally \( \mathcal{P}/p^2 \), not the free, massless propagator \( 1/p^2 \).

This form may allow not-entirely-free behaviour, while still being compatible with the constraints imposed by scale and conformal invariance (see Ref. [28]).

Secondly, recall that we fixed the absolute normalization of \( Z_\phi \) by requiring the physical mass \( m_h^2 = \frac{1}{2} \lambda_B v_B^2 \) to be equal to the second derivative of \( V^{1-\text{loop}}(\phi_R) \) at its minimum. This arises from the well-known connection between the derivatives of the effective potential and the zero-momentum limit of the 1PI Green’s functions [24]. Here, in the SSB vacuum, the renormalized inverse propagator of the \( h \) field is just \( p^2 - m_h^2 \), which tends to \(-m_h^2\) as \( p^\mu \to 0 \). This must agree with (minus) the second derivative of \( V_{\text{eff}}(\phi_R) \) at the vacuum.

We can re-state the argument in more physical terms: For self-consistency – especially if we believe the \( h \) field to be truly a free field – the effective potential near its minimum should look like a free-field potential \( \frac{1}{2} m_h^2 (\phi_R - v)^2 \) with the same mass as in the propagator. (This is extremely intuitive if one first thinks about the 0+1 dimensional, quantum mechanical case.) Furthermore, because fluctuations of \( h(x) \) that are finite on the scale of \( \phi_B \) are only infinitesimal on the scale of \( \phi_R \), it is quite reasonable that the \( h(x) \) field is only sensitive to the quadratic shape of the renormalized \( V_{\text{eff}} \) in the infinitesimal neighbourhood of the vacuum. Therefore, we can understand why \( h(x) \) behaves as a free field, and thus close the circle of our arguments.

In fact, we may show directly that \( h(x) \) is non-interacting. The bare 3-point and 4-point vertices of the shifted theory are proportional to \( \lambda_B \phi_B \) and \( \lambda_B \), respectively. These are both infinitesimal, of order \( \sqrt{\epsilon} \) and \( \epsilon \), respectively (where \( \epsilon \) is 1/\( \ln \Lambda \) in cutoff regularization, or \( 4 - d \) in dimensional regularization). Any diagram with \( T \) 3-point vertices, \( F \)
4-point vertices, and \( L \) loops can, at most, be of order \((\lambda_B \phi_B)^T (\lambda_B)^F (1/\epsilon)^L\). But it is an identity that \( T/2 + F - L = n/2 - 1 \), where \( n \) is the number of external legs. Thus, all diagrams contributing to the \( n \)-point function are suppressed by a factor of at least \( \epsilon^{n/2-1} \), and hence vanish for \( n \geq 3 \) in the continuum limit. Thus, there are no self-interactions, as a direct consequence of the fact that \( h(x) \), unlike the constant \( \phi \) field, has no infinite re-scaling that can compensate for the infinitesimal strength of \( \lambda_B \). Thus, our results confirm our initial expectation that \( \lambda \Phi^4 \) is a generalized free field theory.

6 BEYOND ONE-LOOP: EXACTNESS CONJECTURE

Now, if the \( h(x) \) field does not self-interact, then the effective action:

\[
\Gamma[\Phi_R] = \int d^4x \left\{ \frac{1}{2} (\partial h_R)^2 - \frac{1}{2} (8\pi^2 \phi_R^2) h_R^2 - \pi^2 \phi_R^4 \left( \ln \frac{\phi_R^2}{v^2} - \frac{1}{2} \right) \right\},
\]

that embodies our results (4.12, 4.13), ought to be exact [29]. The shifted field \( h(x) \) has made a non-trivial contribution to \( V_{\text{eff}} \) through its zero-point energy, but any further modifications would have to be due to its self-interactions – and physically it has none!

This conclusion would not be immediately obvious diagrammatically, because there are indeed vacuum diagrams at all orders that give \( 1/\epsilon \) and finite contributions, as our argument above (for \( n = 0 \)) shows. Similarly, for \( n = 2 \), we see that there are diagrams giving finite contributions to the propagator in all orders. However, since the \( h(x) \) field has no physical interactions, the only reasonable expectation is that the apparent “higher-order corrections” to \( V_{\text{eff}} \) and to \( m_h^2 \) either cancel or can be re-absorbed into the renormalization constants so as to leave the physical results (4.12, 4.13) unchanged.

This conjecture is supported by the observation [10, 11] that exactly the same physical results (4.12, 4.13) for \( V_{\text{eff}} \) and \( m_h^2 \) are obtained in the Gaussian approximation. Discarding terms that will vanish in the limit \( \Lambda \to \infty \), the Gaussian effective potential for the massless case can be expressed as [10, 11]:

\[
V^{\text{Gauss}} = \frac{\lambda_G}{4!} \phi_B^4 + \frac{\Omega^4(\phi_B)}{64\pi^2} \left( \ln \frac{\Omega^2(\phi_B)}{\Lambda^2} - \frac{1}{2} \right),
\]

with \( \Omega^2(\phi_B) = \frac{1}{2} \lambda_G \phi_B^2 \) and \( \lambda_G = \frac{2}{3} \lambda_B \). This has the same structure as the one-loop result, Eq. (4.4). After eliminating \( \Lambda \) in favour of \( v_B \), and then re-scaling the bare vacuum field by \( Z^{\text{Gauss}}_\phi = \frac{16\pi^2}{\lambda_G} = \frac{24\pi^2}{\lambda_B} \), the two approximations are seen to be physically equivalent. There are differences by factors of 2/3 in the unobservable quantities \( \lambda_B \) and \( Z_\phi \), but these differences cancel out in the physically meaningful results (4.12, 4.13) [10, 11].
Notice that both the 1-loop and Gaussian approximations have a variational character: the one-loop potential is the ground-state energy of that part of the Hamiltonian which is quadratic in the shifted field, while the Gaussian approximation corresponds to minimizing the expectation value of the full Hamiltonian in the subspace of Gaussian wavefunctionals. The Gaussian approximation can also be described as a re-summation of an infinite subset, a convergent subseries, of ‘daisy’ and ‘superdaisy’ diagrams \[30, 31\].

Other truncations of the diagrammatic series for the effective potential, if they do not correspond to any variational procedure, and hence do not enjoy a stability property, may give rise to spurious difficulties. Apparent contradictions will inevitably arise if one tries to use an approximation method that is inherently incompatible with generalized-free-field behaviour, Eqs. \[2.1 - 2.4\]. This is the case with the usual loop expansion beyond one loop.

In other words, one should not consider the $\beta$ and $\gamma$ functions in Eqs. \[4.6, 4.7\] as the first terms of a power-series expansion in $\lambda_B$ which can be systematically improved order by order in the loop expansion for the effective potential. Rather, the form of Eqs. \[4.6, 4.7\] will arise in any approximation scheme, no matter how sophisticated, that allows generalized free-field behaviour to be a possible solution. The 2-loop, 3-loop, ... approximations to the effective potential, on the other hand, have a built-in incompatibility with generalized free-field behaviour (think of the spurious $Z_h \neq 1$ effect, starting at the 2-loop level, for instance) and represent merely a perturbative construction.

This point can best be understood in terms of the effective potential for composite operators introduced by Cornwall, Jackiw and Tomboulis \[27\] (CJT), which is based on the rigorous definition of the effective potential through the exact relation

$$\int d^3x V_{\text{eff}}(\phi) = E[\phi, G_o(\phi)], \quad (6.3)$$

where $E[\phi, G] = \min \langle \Psi | H | \Psi \rangle$, minimized over all normalized states $| \Psi \rangle$, subject to the conditions $\langle \Psi | \Phi | \Psi \rangle = \phi$ and $\langle \Psi | \Phi(\vec{x}, t) \Phi(\vec{y}, t) | \Psi \rangle = \phi^2 + G(\vec{x}, \vec{y})$, and where $G_o(\phi)$ is obtained from

$$\frac{\delta E}{\delta G(\vec{x}, \vec{y})}|_{G = G_o(\phi)} = 0. \quad (6.4)$$

Equation \[6.4\] can be solved exactly for $G_o$ in the subspace of Gaussian states and in that case Eq. \[5.3\] leads to the Gaussian effective potential. The problem can also be approached diagrammatically, using CJT’s result that $E[\phi, G]$ has a manifestly covariant expansion containing the one-loop term and the series of 2-particle-irreducible vacuum graphs with propagator $G$ and vertices given by the shifted interaction Lagrangian. However, in an $n$-loop approximation to $E[\phi, G]$ ($n \geq 2$), one cannot exactly solve the resulting
optimization equation, (6.4). One can only solve it in a perturbative sense. Since this does not provide a true stationary solution for $G$ what one obtains is really not an effective potential, except in a perturbative sense. Consequently, one should not attempt to apply the RG equation, (3.3), to the 2-loop, 3-loop, ... approximations to the effective potential. That would be similar to trying to define $\beta$ through the perturbative 4-point function at non-zero external momenta and, as we explained in Sect. 2, that cannot produce a consistent continuum limit.

It is possible, in principle, to consider truncations of the CJT construction that improve upon the one-loop or Gaussian approximations, yet still allow the resulting optimization equation, (6.4), to be solved exactly. For example, one can consider post-Gaussian variational calculations (either Hamiltonian [32] or covariant [33]). In such a calculation, however, we would expect the optimal $G$ to reduce to a free propagator, and our equations (4.12, 4.13) to be unmodified in the continuum limit. Any other result would provide strong evidence that $(\lambda\Phi^4)_4$ is not, in fact, a generalized free field theory. Therefore, if “triviality” is true, as we believe, then Eqs. (4.12, 4.13) should be considered exact.

7 FINITE TEMPERATURE

We briefly discuss finite-temperature effects in order to make two points: (i) our $\lambda\Phi^4$ theory, even though it has no particle scattering, is not entirely trivial; it has non-trivial collective effects, and (ii) our renormalized $v$ has real physical significance.

Since the $h(x)$ field is a non-interacting boson field with mass $\omega^2 = \frac{1}{2} \lambda B \phi_B^2 = 8\pi^2 \phi_R^2$, the only finite-temperature correction to $V_{\text{eff}}$ will be the term [30]:

$$I_1^\beta = \frac{1}{\beta} \int \frac{d^3p}{(2\pi)^3} \ln(1 - \exp\{-\beta(p^2 + \omega^2)^{1/2}\}),$$  \hfill (7.1)

where $\beta = 1/T$, and $T$ is the temperature in units where Boltzmann’s constant is unity. This term, since it modifies a non-trivial effective potential, produces non-trivial effects. In particular, one finds that the theory undergoes a symmetry-restoring phase transition at a critical temperature $T_c$ of order $v$. Note that it is the renormalized $v$, not $v_B$, that sets the scale for the physical observable $T_c$. This is because the depth of the SSB vacuum relative to the symmetric vacuum (which is invariant under re-scalings of $\phi$) is of order $v^4$.

The Gaussian approximation has a very simple generalization to the finite-temperature case, and after renormalization it yields exactly the same result as above, because other would-be contributions are infinitesimal [18]. Thus, we may take over the results from Ref. [18], converting the notation appropriately [34]. The critical temperature is $T_c = \ldots$
0.51394 \((8\pi^2/e)^{1/2} v = 2.7699 v\). The transition is first order with a barrier height that is about a quarter of the original depth of the SSB vacua relative to the origin. Before the phase transition the SSB minima of the finite-temperature effective potential move slightly inward from their zero-temperature positions at \(\pm v\). Because of this there is a slight (5%) decrease in the particle mass between \(T = 0\) and \(T = T_c\), with most of this decrease occurring close to \(T_c\) [18].

8 REMARKS ON THE PERTURBATIVE PICTURE

We have insisted on being able to take the continuum limit. However, if one is prepared to keep the ultraviolet cutoff finite, so that a finite “\(\lambda_R\)” exists, then one can proceed perturbatively, and here we make contact with the classic analysis of Coleman and Weinberg [24]. Their perturbative renormalization \(\lambda_B = \lambda_R(1 + O(\lambda_R))\) leads to perturbative expressions for \(\beta\) and \(\gamma\) [24]:

\[
\beta_{\text{pert}} = + \frac{3\lambda_B^2}{16\pi^2} \quad \text{and} \quad \gamma_{\text{pert}} = O(\lambda_B^2),
\]

which allow a perturbative solution of the RG equation (3.3), for finite ultraviolet cutoff, valid only up to higher order terms in \(\lambda_R\). At one-loop order there is then no wavefunction renormalization, so that \(h_B = h_R\) and \(\phi_B = \phi_R\). Making an “RG improvement” using \(\beta_{\text{pert}}\) produces a re-summation of the “leading log” series in \(x\), defined as

\[
x = \frac{3\lambda_B}{32\pi^2} \ln \frac{2\Lambda^2}{\lambda_B \phi_R^2},
\]

yielding a perturbative, running coupling constant:

\[
\lambda_R(\phi_R^2) = \frac{\lambda_B}{1 + \frac{3\lambda_B}{32\pi^2} \ln \frac{\Lambda^2}{\phi_R^2}}.
\]

(At this level of approximation one may drop the \(\lambda_B/2\) factor in the logarithm; it makes only a sub-leading-log difference.) The resulting “leading-log effective potential” has no SSB minima. However, this leading-log re-summation procedure is questionable and by no means unique. The “leading-log” series has the form \(1-x+x^2-x^3+...\), and converges to \(\frac{1}{1+x}\) only for \(|x| < 1\). However, the SSB minimum of the original one-loop effective potential was precisely at \(x = 1\), so it is not surprising that it can spuriously be made to disappear if one extends the re-summed expression into the region \(x \geq 1\).

Still, one could argue that the re-summation is trustworthy in the region where \(\phi_B\) is not too small and define the “leading-log effective potential” over the whole range of
\( \phi_B \) by analytical continuation. One would start with a very small bare coupling constant \( \frac{1}{16\pi^2} \ll 1 \) and a very large cutoff \( \Lambda \) such that \( x \sim 1 \) but \( \lambda_B x \ll 1 \) so that the important effects are restricted to the leading-log sector. Then, employing \( \beta_{\text{pert}} \) and \( \gamma_{\text{pert}} \) one would find that all the leading logs are contained in the running coupling constant \( \lambda_R(\phi^2_R) \). Consequently, one would end up with a perturbatively renormalized “leading log effective action” that is just the shifted \((\Phi \rightarrow \phi + h(x))\) classical action, with \( \lambda_B \) replaced by \( \lambda_R(\phi^2_R) \). However, if one tried to take the cutoff to infinity with \( \beta_{\text{pert}} \) governing \( \lambda_B \) as a function of \( \Lambda \), one would inevitably be drawn into a region where \( \lambda_B \) is not small. Then \( \lambda_R(\phi^2_R) \) would be driven to zero at any finite \( \phi^2_R \). Alternatively, we can say that the only region with non-vanishing \( \lambda_R(\phi^2_R) \) in which we can trust the leading-log re-summation is at large \( \phi^2_R \), of order \( \Lambda^2 \). In any finite range of \( \phi^2_R \) one cannot justify neglecting “sub-leading” logarithms.

The crux of the matter is the qualitative conflict between the one-loop effective potential itself and its “re-summed” or “RG-improved” form. This unhappy situation can be avoided by going to theories such as scalar electrodynamics. There, as Coleman and Weinberg argue, there is no doubt that the leading-log re-summation is valid, for sufficiently small \( \lambda \) and gauge coupling \( e \). Our type of analysis, though seemingly very different, actually leads to the same physical consequences in this perturbative region \([12, 13]\). However, outside the perturbative region, the relation between \( e^2_B \) and \( \lambda_B \) necessary for renormalizability ceases to be of the form \( e^4_B \) proportional to \( \lambda_B \). In fact, \( e^2_B \) goes back to zero as \( \lambda_B \) approaches the pure-\( \lambda \Phi^4 \)-theory value, Eq. \((4.9)\). Thus, for small \( e_B \) our approach yields two solutions; one perturbative, and one close to pure \( \lambda \Phi^4 \) \([12]\). The perturbative solution yields the 10 GeV Higgs of Coleman and Weinberg, now excluded by experiment.

### 9 IMPLICATIONS FOR ELECTROWEAK THEORY

We now consider the implications of this picture for the standard model of electroweak interactions. This uses four scalar fields in a complex isodoublet:

\[
K(x) = \frac{1}{\sqrt{2}} (\chi_1(x) + i\chi_2(x), v + h(x) + i\chi_4(x)),
\]

where one field \( \chi_3(x) = v + h(x) \) has a non-zero vacuum expectation value. When this form is substituted into the tree-level scalar-vector couplings, it generates mass terms for the gauge fields. Hence, there is a direct relation between \( v \) and the Fermi constant \( G_F \); namely, \( v \sim (\sqrt{2}G_F)^{-1/2} \sim 246 \text{ GeV}. \) Thus, \( v \) is a physical, measurable quantity, and represents the phenomenological vacuum value of the scalar field. The physical origin of
its non-zero value, however, is ascribed to a presently untested part of the theory, namely the “Higgs potential”. In textbook treatments the Higgs potential is treated classically and there is no ‘bare/renormalized’ distinction. However, in our approach the SSB arises only after the full quantum-dynamical content of the scalar sector has been taken into account. Therefore, in our approach, $K(x)$ represents the O(4) extension of our renormalized field $\Phi_R(x)$ (i.e. the Fourier transform of $\Phi_R(p)$ introduced in Sect. 5). This equals $v_R + h(x)$ when evaluated at the minimum of the effective potential. The $v$ in Eq. (9.1) is thus to be identified with $v_R$, the renormalized vacuum expectation value, i.e., the one which includes the full dynamical content of the scalar sector. With $v = v_R$ identified with $(\sqrt{2} G_F)^{-1/2} \approx 246$ GeV our result $m_h^2 = 8\pi^2 v^2$ gives a Higgs mass prediction of 2.19 TeV.

Although the scalar sector must be analyzed and renormalized nonperturbatively, one may continue to treat the gauge and Yukawa sectors perturbatively; that is $g_R = g_B (1 + O(g_B^2))$. Hence, one obtains (up to small corrections from $\mu$-decay) the renormalized relation $M_{W}^2 = \frac{1}{4} g_R^2 v_R^2$, where $g_R$ is the running SU(2) coupling constant evaluated at the $W$ mass scale. In the presence of the gauge and Yukawa couplings, there will be radiative corrections to the effective potential and to the Higgs mass, but these are small provided the top mass is less than 200 GeV [10, 11].

[Note that this attitude is not quite the same as in Refs. [12, 13]. In those papers, which deal only with the effective potential, the gauge coupling was also renormalized in a nonperturbative manner. We believe that there is ultimately a duality between the two approaches. However, for practical purposes – since we know that perturbation theory works well in QED and weak interactions – the present attitude is more useful.]

Our mass prediction $m_h^2 = 8\pi^2 v^2 \approx 2.19$ TeV comes from considering $\lambda \Phi^4$ theory for a single scalar field, whereas the standard model involves the O(4) generalization. In the O(4) $\lambda \Phi^4$ case the one-loop and Gaussian results are not quite identical: one-loop gives $m_h = 1.89$ TeV while the Gaussian approximation gives $m_h = 2.05$ TeV [14, 13]. However, this difference is probably attributable to an inexactness inherent in using “Cartesian-coordinate” fields. Really the Goldstone fields should be described by angular, “polar-coordinate” fields. One can argue that the exact result in the O($N$) case should be just the $N = 1$ result [14, 11]: i.e., that only the radial field affects the shape of the effective potential. This idea is motivated by the approach of Ref. [15] where the functional integral is expressed in polar coordinates; the angular integration gives only a constant term and the non-trivial Jacobian is handled by ghost fields [16].

In any case, one predicts a Higgs mass $m_h \sim 2$ TeV in the Standard Model. More-
over, despite its large mass, the Higgs would be narrow, decaying principally to $t\bar{t}$, and there would be no strong interactions among longitudinal gauge bosons. This follows, by the “equivalence theorem” (see below), from the fact that the scalar sector is non-interacting in the absence of gauge couplings. Now, it might seem at first that “triviality” for broken-phase $O(N)$ theory would be at odds with current algebra, which prescribes definite derivative couplings for Goldstone bosons. However, this conflict can be reconciled by the fact that $v_B$ is infinite, which causes the current-algebra interactions in $O(N)$ $\lambda \Phi^4$ theory to be infinitely suppressed. Thus, both sides of an “Adler-Weisberger” sum rule would be infinitesimal: the scattering cross sections are infinitesimal, reflecting “triviality”, while the other side of the equation is suppressed by a $1/v_B^2$ factor. The scalar self-interactions disappear because $v_B$ is infinite, but it is the finite $v_R$ that sets the scale of the symmetry breaking, and governs $T_c$ and the particle masses.

Our picture is perfectly compatible with the “equivalence theorem” [37, 38], whose physical content, paraphrasing Sect. II of Ref. [38], is the following: For zero gauge coupling(s), $g$, the Goldstone bosons are physical particles while the longitudinal components of the $W$’s are free, unphysical degrees of freedom, included just to maintain manifest Lorentz covariance. We call this the “$g = 0$ theory”. On the other hand, for $g \neq 0$, however small, the longitudinal $W$’s are physical while the Goldstone bosons are now unphysical particles, included just to preserve renormalizability. This situation we call the “$g \to 0$ theory”. The requirement that the physical observables of the two theories are the same in the limit $g \to 0$ implies an equivalence between the physical longitudinal $W$’s of the “$g \to 0$ theory” and the physical Goldstone bosons of the “$g = 0$ theory.” This statement is made precise by the theorem, which is valid to lowest non-trivial order in $g$ and to all orders in the scalar self-interaction [37, 38]. Thus, presumably the theorem remains valid for a nonperturbative scalar sector. In our picture the $g = 0$ theory has non-interacting Higgs and Goldstone particles, so in the small-$g$ theory we expect only weakly interacting Higgs and longitudinal gauge bosons. So, at low energy scales, the only trace of the Higgs particle is represented by the logarithmic one-loop correction to the $\rho$ parameter discovered by Veltman [39].

In our picture there is just no such thing as a “renormalized $\lambda$”, and the quasi-classical relation “$m_h^2 \propto \lambda_R v^2$”, implying a proportionality between the Higgs mass and its self-coupling, is completely misleading. As has been pointed out by Huang [23, 22], the ratio $m_h^2/v^2$ is not a measure of the effective scalar coupling strength.

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References

[1] S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, Proc. 8th Nobel Symp., N. Svartholm ed. (Almqvist and Wicksell Stockholm, 1968) 367.

[2] P. W. Higgs, Phys. Lett. 12, 132 (1964); Phys. Rev. Lett. 13, 508 (1964); F. Englert and R. Brout, *ibid*, 321; G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, *ibid*, 585; T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).

[3] For a complete list of references, see J. F. Gunion, H. E. Haber, G. Kane, and S. Dawson, “The Higgs Hunter’s Guide” (Addison-Wesley, 1990); M. B. Einhorn, editor, “The Standard Model Higgs Boson” (North-Holland, 1991).

[4] J. Fröhlich, Nucl. Phys. B200(FS4), 281 (1982); M. Aizenman, Phys. Rev. Lett. 47, 1 (1981); A. Sokal, Ann. Inst. H. Poincaré, 37, 317 (1982).

[5] For an authoritative review of the current status of rigorous results, see R. Fernández, J. Fröhlich, and A. D. Sokal, *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory* (Springer-Verlag, Berlin, 1992).

[6] K. G. Wilson and J. Kogut, Phys. Rep. C12, 75 (1974); G. A. Baker and J. M. Kincaid, Phys. Rev. Lett. 42, 1431 (1979); B. Freedman, P. Smolensky and D. Weinberger, Phys. Lett. B113, 481 (1982); D. J. E. Callaway and R. Petronzio, Nucl. Phys. B240, 577 (1984); I. A. Fox and I. G. Halliday, Phys. Lett. B 159, 148 (1985); C. B. Lang, Nucl. Phys. B 265, 630 (1986); M. Lüscher and P. Weisz, Nucl. Phys. B 290, 25 (1987).

[7] M. G. do Amaral and R. C. Shellard, Phys. Lett. B171, 285 (1986); M. Lüscher and P. Weisz, Nucl. Phys. 295, 65 (1988).

[8] D. J. E. Callaway, Phys. Rep. 167, 241 (1988).

[9] H. Neuberger, U. M. Heller, M. Klomfass, and P. Vranas, in Proceedings of the XXVIth International Conference on High Energy Physics, Dallas, TX, August 1992.

[10] M. Consoli, in “Gauge Theories Past and Future – in Commemoration of the 60th birthday of M. Veltman”, R. Akhoury, B. de Wit, P. van Nieuwenhuizen and H. Veltman Eds., World Scientific 1992, p. 81; M. Consoli, Phys. Lett. B 305, 78 (1993); “Is There Any Upper Limit on the Higgs Mass?”, preprint INFN Sezione di Catania, July 1992.
[11] V. Branchina, M. Consoli and N. M. Stivala, Z. Phys. C - Particles and Fields, C57, 251 (1993).

[12] R. Ibañez-Meier and P. M. Stevenson, Phys. Lett. B297, 144 (1992).

[13] R. Ibañez-Meier, I. Stancu and P. M. Stevenson, “Gaussian Effective Potential for the U(1) Higgs model”, Rice Preprint DOE/ER/05096-51, July 1992 [hep-ph 9207276].

[14] M. Consoli and A. Ciancitto, Nucl. Phys. B254, 653 (1985).

[15] P. M. Stevenson and R. Tarrach, Phys. Lett. B176, 436 (1986).

[16] P. Castorina and M. Consoli, Phys. Lett. B235, 302 (1990); V. Branchina, P. Castorina, M. Consoli and D. Zappalà, Phys. Rev. D42, 3587 (1990).

[17] V. Branchina, P. Castorina, M. Consoli, and D. Zappalà, Phys. Lett. B274, 404 (1992).

[18] G. A. Hajj and P. M. Stevenson, Phys. Rev. D37, 413 (1988); G. A. Hajj, Ph.D Thesis, Rice University, 1987 (unpublished).

[19] A. Okopińska, Phys. Rev. D38, 2498 (1988); S. Paban and R. Tarrach, Phys. Lett. B213, 48 (1988); J. Soto, Nucl. Phys. B316, 141 (1989); B. Rosenstein and A. Kovner, Phys. Rev. D40, 504 (1989); R. Ibañez-Meier, Phys. Lett. B295, 89 (1992).

[20] We do not consider negative $\lambda_B$, which is a whole other story.

[21] J. Glimm and A. Jaffe, “Quantum Physics: A Functional Integral Point of View” (Springer, New York, 1981).

[22] K. Huang, E. Manousakis, and J. Polonyi, Phys. Rev. D35, 3187 (1987).

[23] K. Huang, Int. J. Mod. Phys. A4, 1037 (1989); in Proceedings of the DPF Meeting, Storrs, CT, 1988.

[24] S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).

[25] J. Pedersen, I. E. Segal and Z. Zhou, Nucl. Phys. B376, 129 (1992).

[26] S. Weinberg, Phys. Rev. D 7, 2887 (1973); R. Jackiw, Phys. Rev. D 9, 1686 (1974).

[27] J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D10, 2428 (1974); R. W. Haymaker, Rivista del Nuovo Cimento 14, 8 (1991).
[28] D. J. Gross and J. Wess, Phys. Rev. D2, 753 (1970).

[29] To be pedantic, the exact effective potential would be the ‘convex envelope’ of $V^{1\text{-loop}}$ (obtained by a double Legendre transform or equivalently by a Maxwell construction). See A. Dannenberg, Phys. Lett. B202, 110 (1980); V. Branchina, P. Castorina, and D. Zappalà, Phys. Rev. D41, 1948 (1990).

[30] L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974).

[31] T. Barnes and G. I. Ghandour, Phys. Rev. D22, 924 (1980).

[32] L. Polley and U. Ritschel, Phys. Lett. B221, 44 (1989); R. Ibañez-Meier, A. Mattingly, U. Ritschel, and P. M. Stevenson, Phys. Rev. D45, 2893 (1992).

[33] R. Ibañez-Meier, L. Polley, and U. Ritschel, Phys. Lett. B279, 106 (1992).

[34] The point that $T_c$ is proportional to the renormalized $v$ was noted in Ref. [12], but the numerical value was quoted incorrectly.

[35] L. Dolan and R. Jackiw, Phys. Rev. D9, 2904 (1974).

[36] Note that the range of integration for the shifted radial field, $-v_B$ to $+\infty$, will effectively become $-\infty$ to $+\infty$ because of the infinite $Z_\phi$ re-scaling. Similarly, the angular fields, defined on $-\pi$ to $\pi$, would become normal massless scalar fields, defined on $-\infty$ to $+\infty$ when re-scaled by $v_B$.

[37] J. Cornwall, D. Levin, and G. Tiktopoulos, Phys. Rev. D10, 1145 (1974); C. Vayonakis, Lett. Nuovo Cimento 17, 383 (1976); B. Lee, C. Quigg, and H. Thacker, Phys. Rev. D16, 1519 (1977); M. Chanowitz and M. K. Gaillard, Nucl. Phys. B261, 379 (1985); Y. Yao and C. Yuan, Phys. Rev. D38, 2237 (1988); H. Veltman, Phys. Rev. D41, 2294 (1990).

[38] J. Bagger and C. Schmidt, Phys. Rev. D41, 264 (1990).

[39] M. Veltman, Acta Phys. Pol. B12, 437 (1981).