Quasiparticle density of states of $d$-wave superconductors in a disordered vortex lattice

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We calculate the density of states of a disordered inhomogeneous $d$-wave superconductor in a magnetic field. The field-induced vortices are assumed to be pinned at random positions and the effects of the scattering of the quasi-particles off the vortices are taken into account using the singular gauge transformation of Franz and Tešanović. We find two regimes for the density of states: at very low energies the density of states follows a law $\rho(\epsilon) \sim \rho_0 + \beta |\epsilon|^\alpha$ where the exponent is close to 1. A good fit of the density of states is obtained at higher energies, excluding a narrow region around the origin, with a similar power law energy dependence but with $\alpha$ close to 2. Both at low and at higher energies $\rho_0$ scales with the inverse of the magnetic length ($l^{-1} \sim \sqrt{B}$).

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The effect of disorder on the low-energy density of states of $d$-wave superconductors has been a subject of considerable recent interest both for practical and for theoretical reasons [1]. From the practical point of view the presence of disorder pinning mechanisms is important to prevent energy dissipation due to the motion of the vortices in an external field. From the theoretical point of view several conflicting predictions have appeared in the literature. Some progress toward understanding the disparity of theoretical results has been achieved realising that the details of the type of disorder affect significantly the density of states [1]. In contrast to conventional gapped $s$-wave superconductors, the presence of gapless nodes in $d$-wave superconductors leads to power law behavior for the low-$T$ thermodynamic properties and is expected to affect the transport properties. Linearizing the spectrum around the four Dirac-like nodes it has been suggested that the system is critical. It was obtained that the density of states is of the type $\rho(\epsilon) \sim |\epsilon|^\alpha$, where $\alpha$ is a non-universal exponent dependent on the disorder, and that the low energy modes are extended states (critical metal) [2]. Taking into account the effects of inter-nodal scattering (hard-scattering) it has been shown that an insulating state is obtained instead, where the density of states still vanishes at low energy but with an exponent $\alpha = 1$ independent of disorder [3]. The addition of time-reversal breaking creates two new classes designated spin quantum Hall effect I and II, due to their similarities to the usual quantum Hall effect, corresponding to the hard and soft scattering cases, respectively [1]. The proposed formation of a pairing with a symmetry of the type $d + id$ breaks time-reversal symmetry [4] but up to now remains a theoretical possibility. On the other hand applying an external magnetic field naturally breaks time-reversal invariance and therefore it is important to study the density of states in this case.

Moreover, the interaction between the superconductor quasiparticles and the vortices induced by the external magnetic field has also been a subject of considerable debate [5, 6, 7]. In the presence of the vortices the quasiparticles feel the combined effect of the external magnetic field and of the spatially varying field of the chiral supercurrents. Performing a gauge transformation to effectively reduce the system to one in a zero average magnetic field it was shown [8] that the natural low energy modes are Bloch waves rather than the Dirac Landau levels proposed in [5, 6]. These results also showed that the quasiparticles besides feeling a Doppler shift caused by the moving supercurrents [8] (scalar potential) also feel a quantum “Berry” like term due to an Aharonov-Bohm scattering of the quasiparticles by the vortices (vector potential).

In this Letter we will be concerned with the effect of positional vortex disorder on the density of states of a superconductor in an external magnetic field. The density of states of a dirty but homogeneous $s$-wave superconductor in a high magnetic field where the scattering of the quasiparticles off scalar impurities was considered using a Landau level basis [9]. For small amounts of disorder it was found that $\rho(\epsilon) \sim \epsilon^2$ but when the disorder is higher than some critical value a finite density of states is created at the Fermi surface. In the same regime of high magnetic fields but with randomly pinned vortices and no impurities the density of states at low energies increases significantly with respect to the lattice case suggesting a finite value at zero energy [10]. Refs. [11, 12] considered the effects of random and statistically independent scalar and vector potentials on $d$-wave quasiparticles and it was predicted [12] that at low energies $\rho(\epsilon) \sim \rho_0 + a\epsilon^2$, where $\rho_0 \sim B^{1/2}$. To our knowledge the effect of randomly pinned discrete vortices on the spectrum of a $d$-wave superconductor has not been addressed previously.

We will consider the lattice formulation of a disordered $d$-wave superconductor in a magnetic field. We start from the Bogoliubov-de Gennes (BdG) equations $\mathcal{H}\psi = \epsilon\psi$ where $\psi^\dagger(\mathbf{r}) = (u^*(\mathbf{r}), v^*(\mathbf{r}))$ and where the
matrix Hamiltonian is given by
\[ \mathcal{H} = \left( \begin{array}{cc} \hat{h} & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{h}^\dagger \end{array} \right) \]

with
\[ \hat{h} = -t \sum_\delta e^{-\frac{\Delta_0^2}{\delta}} A(\mathbf{r}) \cdot \hat{\mathbf{s}}_{\mathbf{r}+\delta} - \epsilon_F, \]
\[ \hat{\Delta} = \Delta_0 \sum_\delta e^{\frac{\lambda}{\delta} \phi(r)} \hat{\eta}_{\mathbf{r}} e^{\frac{\lambda}{\delta} \phi(r)}. \] (1)

The sums are over nearest neighbors (\( \delta = \pm \mathbf{x}, \pm \mathbf{y} \) on the square lattice): \( A(\mathbf{r}) \) is the vector potential, \( \hat{\mathbf{s}}_{\mathbf{r}} = \mathbf{u}(\mathbf{r} + \delta) \), \( \hat{\eta}_{\mathbf{r}} = 1/4 \) for s-wave pairing and \( \hat{\eta}_{\mathbf{r}} = (-1)^{\hat{n}_{\mathbf{r}}} \hat{\eta}_{\mathbf{r}} \) for d-wave pairing. In eq. (1) we have factorized the phase of the order parameter and have taken the London limit assuming that the amplitude \( \Delta_0 \) is constant everywhere in space, which is valid in the regime of low fields where the size of the vortex cores is negligible. It is convenient to perform a singular gauge transformation to eliminate the phase of the off-diagonal term in the matrix Hamiltonian. In such a way the magnetic field is compensated by an array of opposing half-fluxes. We carry out the unitary transformation \( U \mathcal{H} U^{-1} \), where
\[ U = \left( \begin{array}{cc} e^{i\phi_A(\mathbf{r})} & 0 \\ 0 & e^{-i\phi_B(\mathbf{r})} \end{array} \right) \]

with \( \phi_A(\mathbf{r}) + \phi_B(\mathbf{r}) = \phi(\mathbf{r}) \). The vortices are separated into two groups \( A \) and \( B \) positioned at \{\( \mathbf{r}^A_i \}_{i=1,N_A} \) and \{\( \mathbf{r}^B_i \)\}_{i=1,N_B} such that the vortices \( A \) are only visible to the particles and the vortices \( B \) are only visible to the holes. The phase fields \( \phi_{\mu=A,B} \) are defined through \( \nabla \times \nabla \phi_{\mu}(\mathbf{r}) = 2\pi\sum_\delta \delta(\mathbf{r} - \mathbf{r}_\mu^\delta) \). The resulting system is effectively in a magnetic field \( B_{eff} = B - \phi_{\mu} \sum_\delta \delta(\mathbf{r} - \mathbf{r}_\mu^\delta) \) that vanishes on average if \( N_A = N_B \). Under the unitary transformation the BdG equations convert in the d-wave case to
\[ -t \sum_\delta e^{i\sqrt{2}\lambda}(\mathbf{r} + \delta) - \epsilon_F \mathbf{u}(\mathbf{r}) + \Delta_0 \sum_\delta e^{iA_\delta(\mathbf{r})}(-1)^{\delta_{\mathbf{r}}} \mathbf{v}(\mathbf{r} + \delta) = \epsilon \mathbf{u}(\mathbf{r}) \]
\[ \Delta_0 \sum_\delta e^{iA_\delta(\mathbf{r})}(-1)^{\delta_{\mathbf{r}}} \mathbf{u}(\mathbf{r} + \delta) + t \sum_\delta e^{-i\sqrt{2}\lambda}(\mathbf{r} + \delta) + \epsilon_F \mathbf{v}(\mathbf{r}) = \epsilon \mathbf{v}(\mathbf{r}) \] (2)

where the phase factors are given by
\[ Y^\mu_\delta(\mathbf{r}) = \int_{\mathbf{r}^\delta}^{\mathbf{r}} \mathbf{k}^\mu \cdot d\mathbf{l} \] and \( A_\delta(\mathbf{r}) = \frac{\lambda}{4\pi} \int_{\mathbf{r}}^{\mathbf{r} + \delta} (\mathbf{k}^\mu_{\mathbf{r}} - \mathbf{k}^\mu_{\mathbf{r}^\delta}) \cdot d\mathbf{l} \). Here \( \mathbf{k}^\mu_{\mathbf{r}} = m \mathbf{v}_s^\mu = \hbar \nabla \phi_{\mu} - \frac{\lambda}{2} \mathbf{A} \), where \( \mathbf{k}^\mu_{\mathbf{r}} \) is the superfluid wave vector for the \( \mu \)-supercurrent. The quantity can be calculated for an arbitrary configuration of vortices like
\[ \mathbf{k}^\mu_{\mathbf{r}}(\mathbf{r}) = 2\pi \int \left( \begin{array}{c} \frac{d^2k}{(2\pi)^2} \frac{d\mathbf{k} \times \mathbf{z}}{\hbar^2 + \lambda^2} \sum_{\mu=1}^{N_{\mu}} e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_{\mu})} \end{array} \right). \] (3)

Here \( \lambda \) is the magnetic penetration length and the sum extends over all vortex positions. The BdG equations are then solved taking an arbitrary configuration of vortices and the density of states is calculated in the standard way. The average over disorder is then carried out calculating the density of states for each vortex configuration and then performing the average on the density of states over 100 different configurations.

The situation where the vortices are regularly distributed in a lattice was treated before. Since the average magnetic field vanishes it is possible to solve the BdG equations using a standard Bloch basis since the supercurrent velocities are periodic in space and there is no need to consider the magnetic Brillouin zone. Taking the continuum limit and linearizing the spectrum around each node effectively decouples the nodes. It was shown that the low-energy quasiparticles are then naturally described as Bloch waves and not Dirac-Landau levels as previously proposed. However, it was found that in the linearized problem different assignments of the \( A \) and \( B \) vortices lead to somewhat different spectra, which was unexpected. It was found that taking the theory on the lattice regularized this problem and indeed the system has a manifest internal gauge symmetry such that the spectrum is independent of the \( A-B \) vortex assignments, as it should be. Moreover, the lattice formulation explicitly involves internodal contributions which, as discussed above, are important for the properties of the density of states in the disordered case. In the vortex lattice case, however, it was found that only in special commensurate cases (for the square lattice) the inclusion of the internodal contributions is relevant since only in such cases a gap develops due to the interference terms between the various nodes, estimated to be of the order of \( \sqrt{B} \). In a general incommensurate case the interference is not relevant leading to qualitatively similar spectra. The spectrum is gapless with a linear density of states at low energy. One would therefore expect that in a general disordered vortex case internodal scattering might not be relevant (particularly for high Dirac cone anisotropy \( t/\Delta_0 \)).
We consider a square lattice with lattice constant $\delta$ (taken as 1), where the electron hopping is described by Hamiltonian (1). Application of the external magnetic field generates vortices which are placed at the center of a plaquette (unit cell). The number of vortices in the two-dimensional system of size $L \times L$ is proportional to the quantized magnetic flux piercing through the system. We parametrize the intensity of the magnetic field by the ratio of the number of vortices, $N_\phi$, (divided equally in two groups $A$ and $B$) by the number of lattice sites $B = \frac{N_\phi}{L \times L}$. The $N_\phi$ vortices are distributed randomly over the $L \times L$ plaquettes. We consider then periodic boundary conditions and solve the BdG equations numerically. We consider $\lambda \to \infty$. In this limit the repulsive interaction between vortices is not screened and therefore the vortex distribution is not strictly arbitrary. We assume that the pinning centers are strong enough to overcome the vortex repulsion. We have checked that the case with a random distribution of vortices is qualitatively the same as for the case where the vortex positions are allowed to vary with a radius of a few unit cells around a lattice position. We find that, as the disorder is introduced, the low energy density of states generally increases.

In Figure 1 we show the density of states for the cases of a weak ($B = 1/200$) and of a strong magnetic field ($B = 1/2$; for this high value the density of vortices is high and strictly we are in a regime where the size of the vortex cores can not be neglected). In the case of a strong magnetic field the Landau level structure at high energies is clear and it extends to low energies superimposed by the effects of disorder and the effect of the low-energy modes close to the $d$-wave nodes. At weak fields the density of states is small at low energies having a dip close to zero energy. We have checked for finite size effects on the spectrum. For system sizes larger than $16 \times 16$ the density of states at not very low energies converges and the finite size dependence is negligible. In Figure 2 we show the density of states for a system with size $20 \times 20$ and for various magnetic fields. The density of states at small energies is finite up to quite small energies where there is a dip to a value that decreases as the magnetic field decreases. Only for quite small magnetic
Neglecting the narrow region close to the origin we have fitted the density of states using the power law

$$\rho(\epsilon) = \rho_0 + \beta |\epsilon|^\alpha. \quad (4)$$

In the inset of Figure 2 we show the fits for the various values of the magnetic field. Reasonable fits are obtained taking $\alpha \sim 2$ and we obtain that $\rho_0 \sim B^{1/2}$. In Figure 3 we show the magnetic field dependence of the parameters of the fit for different system sizes $L = 10, 12, 16, 20$. The various system sizes fit in the same universal curve indicating that the finite size effects are negligible. Note that in the lattice case the density of states at low energies is linear $\rho \sim \epsilon$ (this result differs from the behavior obtained by others for a $d$-wave superconductor with no disorder [2, 13]). The finite density of states at zero energy is therefore a consequence of finite disorder.

In Figure 4 we focus on the narrow region close to $\epsilon = 0$ for the same set of parameters considered in Figure 2. Except for the lowest field case $B = 1/200$ the density of states seems to be finite at zero energy. The particular field density of $B = 1/200$ corresponds to only two vortices (since the size of the system is $20 \times 20$). As shown in [14] in this case the spectrum is usually gapped and therefore the density of states vanishes at zero energy. Performing a fit like in eq. (4) we obtain an exponent which is now close to 1. In this regime $\rho_{\text{dip}}$ also scales with $\sqrt{B}$ and the slope scales linearly with $B$. In this low energy regime the finite size effects are still noticeable but the dependence on the magnetic field is common to the various system sizes. At these low energies the density of states for the various system sizes appears to be of the following approximate form $\rho(\epsilon) \sim (1/\omega_H) (1/l^2) F \left( (\epsilon/\omega_H)^{\delta^2/l^2} \right)$, where $\omega_H \sim \sqrt{\Delta H B}$ and $F$ is a universal function. In the left panel of Figure 4 we show $\rho(\epsilon)$ for various fields while in the right panel we illustrate the near scaling at low energies consistent with $F(x) \sim c_1 + c_2 x$ at small $x$. In Figure 5 we show the field dependence of the slope $\beta$ and of the zero energy density of states. Note that $\beta(B \to 0)$ (Fig. 5) appears to be small but finite, consistent with a crossover to a “Dirac node” scaling $(\rho(\epsilon) \sim (1/\omega_H) (1/l^2) F (\epsilon/\omega_H))$ at very low fields.

In summary, we have calculated the density of states of a disordered $d$-wave superconductor in a pinned vortex array. Both the disorder and the magnetic field fill the density of states at low energies. In general we find a finite density of states at zero energy except in the limit of very small magnetic fields. The density of states deviates from the zero energy value by a power law. Excluding a narrow region close to zero energy a good fit is obtained with an exponent close to 2. Performing the same type of fit at very low energies a good fit is obtained with an exponent close to 1. In the very low magnetic field limit the dip of the density of states is more pronounced. The zero energy density of states scales with the inverse of the magnetic length $(1/\sqrt{B})$. Except for the zero energy finite value the energy dependence of the density of states in the case with disorder is similar to the lattice case. This suggests that the vortex disorder does not dramatically

FIG. 4: Low-energy quasiparticle density of states $\rho(\epsilon)$ for different magnetic field $B = 1/200 (\bullet), B = 3/200 (\triangle), B = 5/200 (\square), B = 7/200 (\odot), B = 9/200 (\circ), B = 11/200 (\bigcirc), B = 20/200 (\bullet')$ and $B = 25/200 (\bigcirc')$ in unit of $\hbar c/2e\delta^2$. For each data sets the linear system size is $L = 20$, the same as in Fig. 2. In the left panel the solid lines are linear fits of the dip region below $\epsilon = 0.02\epsilon_0$ of the type $\rho(\epsilon) = \rho_{\text{dip}} + \beta |\epsilon|$. In the right panel we present the near scaling at low energies (see text).

FIG. 5: Slope $\beta$ and density of states $\rho_{\text{dip}}$ (Inset) close to $\epsilon \approx 0$ extracted in the dip region from data with different linear lattice sizes $L = 10 (\bullet), L = 12 (\odot), L = 16 (\square)$ and $L = 20 (\circ)$. Dashed lines are linear fits of $\beta$. In the inset $\rho_{\text{dip}}$ is shown for $L = 20 (\circ)$. The fit of $\rho_{\text{dip}}$ gives a square root dependence in $B$ shown by the solid line.
affect the density of states at low energies. A preliminary analysis of the inverse participation ratio indicates that the states are still extended, as in the lattice case. Our results for the lowest and zero energies are clearly most susceptible to finite size effects — we will report on a detailed study of this region separately. In the gapped s-wave case however the disorder introduces states in the gap thereby changing qualitatively the low energy density of states, as in the high field limit [10]. These results will be presented elsewhere.

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