TWISTED CALABI-YAU PROPERTY OF ORE EXTENSIONS

L.-Y. LIU, S.-Q. WANG, AND Q.-S. WU

Abstract. Suppose that $E = A[x; \sigma, \delta]$ is an Ore extension with $\sigma$ an automorphism. It is proved that if $A$ is twisted Calabi-Yau of dimension $d$, then $E$ is twisted Calabi-Yau of dimension $d + 1$. The relation between their Nakayama automorphisms is also studied. As an application, the Nakayama automorphisms of a class of 5-dimensional Artin-Schelter regular algebras are given explicitly.

0. Introduction

In the last twenty years, a lot of research appears on Artin-Schelter regular graded algebras arising from noncommutative projective algebraic geometry, and on Artin-Schelter regular Hopf algebras/quantum groups. Brown and Zhang proved that a noetherian Artin-Schelter regular Hopf algebra is rigid Gorenstein [BZ08], which is called the twisted Calabi-Yau condition in this paper. Such a class of algebras is called twisted Calabi-Yau algebra (see Definition 1.1). Van den Bergh duality [VdB98] holds for any twisted Calabi-Yau algebra. A noetherian Hopf algebra is Artin-Schelter regular if and only if it is twisted Calabi-Yau. In the noetherian connected graded case, an algebra is Artin-Schelter regular if and only if it is graded twisted Calabi-Yau. Associated to a twisted Calabi-Yau algebra, there is an automorphism, called Nakayama automorphism in general, which is unique up to an inner automorphism. A twisted Calabi-Yau algebra is Calabi-Yau in the sense of Ginzburg [Gin07] if and only if its Nakayama automorphism is inner. Calabi-Yau algebra is an algebraic structure arising from the geometry of Calabi-Yau manifolds and mirror symmetry. It has attracted much interest in recent years.

For any finite-dimensional Lie algebra $\mathfrak{g}$, Yekutieli constructed the rigid dualizing complex of $U(\mathfrak{g})$ [Yek00]. In the terminology now, in fact he proved that $U(\mathfrak{g})$ is Calabi-Yau if and only if $\text{tr}(\text{ad} x) = 0$ for all $x \in \mathfrak{g}$. This result is generalized to a more general situation — the PBW deformations of Koszul Calabi-Yau algebras [WZ11]. The quantized enveloping algebra of a complex semisimple Lie algebra is always Calabi-Yau [Che04]. In [BZ08], Brown and Zhang also described the Nakayama automorphism explicitly by using homological integrals for any noetherian Artin-Schelter regular Hopf algebras. Recently, some people are interested in quantum homogeneous spaces, which are right coideal subalgebras of Hopf algebras satisfying some additional conditions. One question is to study when quantum homogeneous spaces are Artin-Schelter regular or twisted Calabi-Yau. The first named and the third named authors studied the twisted Calabi-Yau property of

2010 Mathematics Subject Classification. Primary 16E40, 16S36; Secondary 16E65.

Key words and phrases. Ore extension, twisted Calabi-Yau algebra, Nakayama automorphism, Artin-Schelter regular algebra.
the right coideal subalgebras of a quantized enveloping algebra [LW11]. A class of right coideal subalgebras of a quantized enveloping algebra can be obtained by iterated Ore extensions. This motivates us to study the Nakayama automorphism and the twisted Calabi-Yau property of Ore extensions in this paper. Ore extension is a noncommutative analogue of polynomial extension. If \( E = A[x; \sigma, \delta] \) is a graded Ore extension with \( \sigma \) an automorphism and \( A \) is Artin-Schelter regular, then so is \( E \). This means the twisted Calabi-Yau property of a connected graded algebra is preserved by (graded) Ore extensions. It is natural to ask whether Ore extensions preserve twisted Calabi-Yau property in general situations? The answer is positive when \( \sigma \) is an automorphism.

Let \( E = A[x; \sigma, \delta] \) be an Ore extension with \( \sigma \) an automorphism. There is a short exact sequence of \( E^e \)-modules (see Lemma 2.1)

\[
0 \to E \otimes_A \sigma E \xrightarrow{\partial} E \otimes_A E \xrightarrow{\mu} E \to 0.
\]

Then an \( E^e \)-projective resolution of \( E \) can be constructed by using an \( A^e \)-projective resolution of \( A \). In particular, taking the bar complex of \( A \), the construction is nothing but the construction given by Guccione-Guccione [GG97]. Using this construction, we compute the Hochschild cohomology \( H^\ast(E, E \otimes E) \) and obtain a family of short exact sequences (Theorem 2.7).

**Theorem 0.1.** Let \( A \) be a projective \( k \)-algebra and \( E = A[x; \sigma, \delta] \) be an Ore extension with \( \sigma \) an automorphism. Suppose that \( A \) admits a finitely generated projective resolution as an \( A^e \)-module. Then for any \( n \in \mathbb{N} \),

\[
0 \to H^n(A, E \otimes E) \to H^n(A, E \otimes E^{\sigma^{-1}}) \to H^{n+1}(E, E \otimes E) \to 0
\]

is an exact sequence of \( E^e \)-modules.

We prove that Ore extensions preserve the twisted Calabi-Yau property and describe the relation between the Nakayama automorphisms of \( A \) and \( E \) (Theorem 3.3).

**Theorem 0.2.** Let \( A \) be a projective \( k \)-algebra and \( E = A[x; \sigma, \delta] \) be an Ore extension with \( \sigma \) an automorphism. Suppose that \( A \) is \( \nu \)-twisted Calabi-Yau of dimension \( d \). Then \( E \) is twisted Calabi-Yau of dimension \( d + 1 \), and the Nakayama automorphism \( \nu' \) of \( E \) satisfies that \( \nu'|_A = \sigma^{-1}\nu \) and \( \nu'(x) = ux + b \) for some \( u, b \in A \) with \( u \) invertible.

As an application, we focus on a class of Artin-Schelter regular algebras of dimension 5 which were investigated in detail by the second named and the third named authors [WW12]. Among them, the Nakayama automorphisms of those may constructed by iterated Ore extensions are given explicitly.

The paper is organized as follows. In Section 1, we recall the definitions of twisted Calabi-Yau algebras and Ore extensions, and fix some notations. In Section 2, following [GG97], we study Hochschild cohomology on Ore extensions instead of Hochschild homology. Some exact sequences are obtained and Theorem 0.1 is proved. In Section 3, we prove the main result Theorem 0.2, that is, Ore extensions preserve the twisted Calabi-Yau property if \( \sigma \) is an automorphism. The relation between their Nakayama automorphisms is also described. In Section 4, the main
result is applied to multi-parametric quantum affine spaces and a class of Artin-Schelter regular algebras of dimension 5 which can be constructed by iterated Ore extensions.

1. Preliminaries

1.1. Twisted Calabi-Yau algebras. Throughout, $k$ is a unital commutative ring and all algebras are $k$-algebras. Unadorned $\otimes$ means $\otimes_k$ and $\text{Hom}$ means $\text{Hom}_k$.

Suppose that $A$ is an algebra. Let $A^{\text{op}}$ be the opposite algebra of $A$ and $A^{e} = A \otimes A^{\text{op}}$ be the enveloping algebra of $A$. The term $A^{e}$-modules are used for $A$-$A$-bimodules.

For any two $k$-modules $M, N$, let $\tau_{M, N} : M \otimes N \to N \otimes M$ be the flip map. The subscript is often omitted if there is no confusion. For any $A^{e}$-module $M$ and any endomorphisms $\nu, \sigma$ of $A$, denote by $\nu^{e} M^{e}$ the $A^{e}$-module whose ground $k$-module is $M$ and the action is given by $a \cdot m \cdot b = \nu(a)m\sigma(b)$ for all $a, b \in A$ and $m \in M$. If one of $\nu$ and $\sigma$ is the identity map, then it is usually omitted.

Suppose that $M$ and $N$ are both $A^{e}$-modules. It is easy to see that there are two $A^{e}$-module structures on $M \otimes N$, one is called the outer structure defined by $(a \otimes b) \to (m \otimes n) = am \otimes nb$, and the other is called the inner structure defined by $(m \otimes n) \leftarrow (a \otimes b) = ma \otimes bn$, for any $a, b \in A$, $m \in M$, $n \in N$. Since $A^{e}$ is identified with $A \otimes A$ as a $k$-module, $A \otimes A$ endowed with the outer (resp. inner) structure is nothing but the left (resp. right) regular $A^{e}$-module $A^{e}$. Hence we often say $A^{e}$ has the outer and inner $A^{e}$-module structures. In the following definition, the outer structure on $A^{e}$ is used when computing the homology $\text{Ext}^{i}_{A^{e}}(A, A^{e})$. Thus $\text{Ext}^{i}_{A^{e}}(A, A^{e})$ admits an $A^{e}$-module structure induced by the inner one on $A^{e}$.

**Definition 1.1.** An algebra $A$ is called $\nu$-twisted Calabi-Yau of dimension $d$ for some automorphism $\nu$ of $A$ and for some integer $d \geq 0$ if

1. $A$ is homologically smooth, that is, as an $A^{e}$-module, $A$ has a finitely generated projective resolution of finite length;
2. $\text{Ext}^{i}_{A^{e}}(A, A^{e}) \cong \begin{cases} 0, & i \neq d \\ A^{e}, & i = d \end{cases}$ as $A^{e}$-modules.

Sometimes condition (2) is called the twisted Calabi-Yau condition. In this case, $\nu$ is called the Nakayama automorphism of $A$.

The Nakayama automorphism is unique up to an inner automorphism. A $\nu$-twisted Calabi-Yau algebra $A$ is Calabi-Yau in the sense of Ginzburg [Gin07] if and only if $\nu$ is an inner automorphism of $A$.

Graded twisted Calabi-Yau algebras are defined similarly. Condition (1) is equivalent to that $A$, when viewed as a complex concentrated in degree 0, is a compact object in the derived category $D(A^{e})$ [Nee01], i.e., the functor $\text{Hom}_{D(A^{e})}(A, -)$ commutes with arbitrary coproducts.

1.2. Artin-Schelter regular algebras. In this subsection, $k$ is a field.

**Definition 1.2.** Suppose that $A$ is an algebra with an augmentation map $\varepsilon : A \to k$. Then $A$ is called left Artin-Schelter regular (for short, AS-regular) if

1. $A$ has finite left global dimension $d < \infty$,
Similarly, let \( q^i \) be the k-linear map which is the sum of all the compositions \( \sigma_1 \sigma_2 \cdots \sigma_n \) with \( \sigma_j \) being \( \sigma^{-1} \) or \(-\delta \sigma^{-1} \), and \( \sigma^{-1} \) appearing \( i \) times in each composition. Then for any \( a \in A \) and \( n \geq 1 \),

\[
  ax^n = \sum_{i=0}^{n} x^i q_i^n(a).
\]  

Many ring-theoretic and homological properties are preserved by Ore extensions under certain conditions. We list some of them as follows.

- If \( A \) is an integral domain and \( \sigma \) is injective, then \( E \) is an integral domain.
• If $A$ is a prime ring and $\sigma$ is an automorphism, then $E$ is a prime ring.

• If $A$ has finite right global dimension and $\sigma$ is an automorphism, then $E$ has finite right global dimension, in fact,

$$r.\,gl.\,\dim A \leq r.\,gl.\,\dim E \leq r.\,gl.\,\dim A + 1.$$ 

• If $k$ is a noetherian ring, $A$ is (strongly) right noetherian and $\sigma$ is an automorphism, then $E$ is (strongly) right noetherian.

For the details and other properties of Ore extensions, we refer to [MR87], [GW04], [ASZ99], etc.

Here are some examples of iterated Ore extensions: multi-parameter quantum affine $n$-spaces $O_q(k^n)$, Weyl algebras $A_n(k)$, enveloping algebras $U(g)$ of finite-dimensional nilpotent Lie algebras $g$, the Borel part of quantized enveloping algebras $U_q(g)$ of complex semisimple Lie algebras $g$, and some classes of AS-regular algebras.

1.4. Notations. We fix some notations about complexes and graded modules.

Suppose that $(P,d)$ is a chain complex. The $l$-shift of $P$, denoted by $P[l]$, is defined by $P[l]_n = P_{n-l}$ and $d[l]_n = (-1)^l d_{n-l}$. If $(P',d')$ is another chain complex and $f: P \to P'$ is a morphism of complexes, the mapping cone $\text{cone}(f)$, denoted by $\text{cone}(f)_n = P_{n-1} \oplus P'_n$ and the differential sending $(p,p') \in \text{cone}(f)_n$ to $(d_{n-1}(p), d'_n(p') - f_{n-1}(p))$. Dually, suppose that $(Q,d), (Q',d')$ are cochain complexes and $g: Q \to Q'$ is a morphism of complexes. The $l$-shift of $Q$, denoted by $Q[l]$, is defined by $Q[l]_n = Q^{n+l}$ and $d[l]_n = (-1)^l d^{n+l}$. The mapping cone $\text{cone}(g)$, is defined by $\text{cone}(g)^n = Q^{n+1} \oplus Q^n$ and the differential sending $(q,q') \in \text{cone}(g)_n$ to $(-d^{n+1}(q), d^n(q') + g^{n+1}(q))$. If $f: P \to P'$ is a morphism of $A$-module complexes and $\text{cone}(f,M) \cong \text{Hom}_A(\text{cone}(f),M)[1]$.

For any graded $A$-module $M$, the $n$-shift $M(n)$ of $M$, is defined by $M(n)_i = M_{n+i}$.

We mainly refer to [Lod98] for Hochschild homology and cohomology.

2. Hochschild cohomology on Ore extensions

We investigate the Hochschild cohomology on Ore extensions in this section. From now on, $\sigma$ is always required to be an automorphism.

Lemma 2.1. Let $A$ be an algebra and $E = A[x;\sigma,\delta]$ be an Ore extension. Then the sequence of $E^e$-modules

$$0 \longrightarrow E \otimes_A \sigma^{-1}E \overset{\rho}{\longrightarrow} E \otimes_A E \overset{\mu}{\longrightarrow} E \longrightarrow 0$$

is exact, where $\rho(e \otimes e') = ex \otimes e' - e \otimes xe'$ and $\mu$ is the multiplication.

Proof. First of all, $\rho$ is well-defined since

$$\rho(1 \otimes \sigma^{-1}(a)) = x \otimes \sigma^{-1}(a) - 1 \otimes x\sigma^{-1}(a)$$

$$= x \otimes \sigma^{-1}(a) - 1 \otimes ax - 1 \otimes \delta\sigma^{-1}(a)$$

$$= x\sigma^{-1}(a) \otimes 1 - a \otimes x - \delta\sigma^{-1}(a) \otimes 1$$

$$= ax \otimes 1 - a \otimes x$$

$$= \rho(a \otimes 1).$$
Suppose $\sum_{i=0}^{n} x^i \otimes e_i \in \text{Ker} \, \rho$. Then $\sum_{i=0}^{n} x^{i+1} \otimes e_i - \sum_{i=0}^{n} x^i \otimes xe_i = 0$. Note that $x^{i+1} \otimes e_n$ is the unique term containing $x^{i+1}$ as the first tensor factor. It follows that $e_n = 0$ and so $\sum_{i=0}^{n} x^i \otimes e_i = 0$. Thus $\rho$ is injective.

Now suppose $\sum_{i=0}^{n} x^i \otimes e_i' \in \text{Ker} \, \mu$ with $e_n' \neq 0$. Then

$$\sum_{i=0}^{n} x^i \otimes e_i' = 1 \otimes e_0' + \sum_{i=1}^{n} x^i \otimes e_i'$$

$$= 1 \otimes e_0' + \sum_{i=1}^{n} x^i \otimes e_i' - \sum_{i=1}^{n} x^{i-1} \otimes xe_i' + \sum_{i=1}^{n} x^{i-1} \otimes xe_i'$$

$$= 1 \otimes e_0' + \rho \left( \sum_{i=1}^{n} x^{i-1} \otimes e_i' \right) + \sum_{i=0}^{n-1} x^i \otimes xe_i_{i+1}$$

$$= \sum_{i=0}^{n-1} x^i \otimes e_i'' \pmod{\text{Im} \, \rho}$$

where $e_i'' \in E$ and $e_{n-1}'' \neq 0$. By induction on $n$, we obtain $\text{Ker} \, \mu = \text{Im} \, \rho$.

Therefore, the sequence (2.1) is exact. \qed

**Remark 2.2.** The graded version of Lemma 2.1 is also true. If $\deg(x) = l$, the short exact sequence (2.1) should be modified by

$$0 \rightarrow E \otimes_A \sigma^{-1} E(-l) \xrightarrow{\rho} E \otimes_A E \xrightarrow{\mu} E \rightarrow 0.$$

For any $A^e$-projective resolution $P$ of $A$ with an augmentation map $\varepsilon, E \otimes_A P \otimes_A \sigma^{-1} E$ and $E \otimes_A P \otimes_A E$ are $E^e$-projective resolutions of $E \otimes_A \sigma^{-1} E$ and $E \otimes_A E$ respectively. By the Comparison lemma, $\rho$ can be lifted to a morphism of $E^e$-module complexes from $E \otimes_A P \otimes_A \sigma^{-1} E$ to $E \otimes_A P \otimes_A E$, say $\psi$. Then $\text{cone}(\psi)$ is an $E^e$-projective resolution of $E$ via $\mu(\text{id}_E \otimes \varepsilon \otimes \text{id}_E)$.

Now we start to look at the Hochschild cohomology. Let $P$ be the bar complex of $A$,

$$0 \rightarrow A^{\otimes 2} \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} \cdots A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n+2} \xrightarrow{b'} \cdots$$

where $b': A^{\otimes n+2} \rightarrow A^{\otimes n+1}$ is the map

$$b'(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \cdots a_{n+1}.$$

A lifting map of $\rho$ is constructed in [GG97] as follows.

The two complexes $E \otimes_A P \otimes_A \sigma^{-1} E$ and $E \otimes_A P \otimes_A E$ are $(E \otimes A^e \otimes \sigma^{-1} E, b'_1, \sigma)$ and $(E \otimes A^e \otimes E, b'_{0, \varepsilon})$, respectively, where the differentials are

$$b'_{0, \varepsilon}(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \cdots a_{n+1},$$

$$b'_1(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \cdots a_{n+1} + (-1)^n a_0 \cdots \otimes a_{n-1} \otimes \sigma^{-1}(a_n) a_{n+1}.$$

The lifting map $\{\psi'_n: E \otimes A^{\otimes n} \otimes \sigma^{-1} E \rightarrow E \otimes A^{\otimes n} \otimes E\}_{n \in \mathbb{N}}$ is defined by

$$\psi'_n(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = x \otimes \sigma^{-1}(a_1) \cdots \otimes \sigma^{-1}(a_n) \otimes 1 - 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes x$$
\[
- \sum_{j=1}^{n} 1 \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes \delta \sigma^{-1}(a_j) \otimes \sigma^{-1}(a_{j+1}) \otimes \cdots \otimes \sigma^{-1}(a_n) \otimes 1.
\]

By the above argument, we have

**Lemma 2.3 ([GG97, Propositions 1.1 and 1.2])**. Let \( A \) be an algebra and \( E = A[x; \sigma, \delta] \) be an Ore extension. Then

\[
(2.2) \quad E \otimes \sigma^{-1}E \xrightarrow{\psi_0} E \otimes A \otimes \sigma^{-1}E \xrightarrow{\psi_1} E \otimes A \otimes \sigma^{-1}E \xrightarrow{\psi_2} \cdots
\]

\[
E \otimes E \xrightarrow{\delta_0} E \otimes A \otimes E \xrightarrow{\delta_1} E \otimes A \otimes E \xrightarrow{\delta_2} \cdots
\]

is a commutative diagram of \( E^\ast \)-modules, and

\[
(2.3) \quad \text{cone}(\psi) \xrightarrow{\mu} E \xrightarrow{\varepsilon} 0
\]

is an exact sequence. If further, \( A \) is flat (resp. projective) over \( k \), then (2.3) is a flat (resp. projective) resolution of \( E \) as an \( E^\ast \)-module.

In the following statements, we sometimes write \( f(a_1 \otimes \cdots \otimes a_n) \) as \( f(a_1, \ldots, a_n) \) for convenience.

Let \( M \) be an \( E^\ast \)-module. Applying \( \text{Hom}_{E^\ast}(-, M) \) to (2.2), we have the following commutative diagram

\[
(2.4) \quad \text{Hom}(k, M^{\sigma^{-1}}) \xrightarrow{\theta^{0,0}} \text{Hom}(A, M^{\sigma^{-1}}) \xrightarrow{\theta^{1,0}} \text{Hom}(A \otimes M^{\sigma^{-1}}) \xrightarrow{\theta^{1,1}} \cdots
\]

where the maps are given by, for any \( f \in \text{Hom}(A^{\otimes n}, M), \tilde{f} \in \text{Hom}(A^{\otimes n}, M^{\sigma^{-1}}), \)

\[
\begin{align*}
\theta^{0,n}(f)(a_1, \ldots, a_{n+1}) &= a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
&\quad + (-1)^{n+1} f(a_1, \ldots, a_n a_{n+1}),
\end{align*}
\]

\[
\begin{align*}
\theta^{1,n}(\tilde{f})(a_1, \ldots, a_{n+1}) &= a_1 \tilde{f}(a_1, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \tilde{f}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
&\quad + (-1)^{n+1} \tilde{f}(a_1, \ldots, a_n) \sigma^{-1}(a_{n+1}),
\end{align*}
\]

\[
\begin{align*}
\theta^{0,n}(f)(a_1, \ldots, a_n) &= x f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) - f(a_1, \ldots, a_n) x \\
&\quad - \sum_{j=1}^{n} f(a_1, \ldots, a_{j-1}, \sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n)).
\end{align*}
\]

Obviously, when \( M \) is viewed as an \( A^\ast \)-module, the two rows in the diagram (2.4) is the Hochschild complex \( C^\ast(A, M^{\sigma^{-1}}) \) and \( C^\ast(A, M) \). In general, for any \( A^\ast \)-module \( M \), the differentials of \( C^\ast(A, M) \) and \( C^\ast(A, M^{\sigma^{-1}}) \) are denoted by \( \theta \) and \( \theta_{\sigma^{-1}} \) respectively, if there is no confusion. On the other hand, by Lemma 2.3, we can compute \( H^n(E, M) \) by using \( \text{cone}(\psi) \) or \( \text{cone}(\theta) \).

**Lemma 2.4.** Let \( A \) be a projective \( k \)-algebra and \( E = A[x; \sigma, \delta] \) be an Ore extension and let \( M \) be an \( E^\ast \)-module. For any \( n \in \mathbb{N} \), \( H^n(E, M) \cong H^{n-1}(\text{cone}(\theta)) \).
Proof. By (2.3) and (2.4),
\[ H^n(E, M) = H^n(\text{Hom}_{E^e}(\text{cone}(\psi), M)) \cong H^n(\text{cone}(\theta)[-1]) = H^{n-1}(\text{cone}(\theta)). \]

Now let \( M = E \otimes E \). By the definition of mapping cones, there is a short exact sequence of \( E^e \)-module complexes
\[ 0 \rightarrow C^*(A, E \otimes E^{-1}) \rightarrow \text{cone}(\theta) \rightarrow C^*(A, E \otimes E)[1] \rightarrow 0, \]
where the \( E^e \)-module structure on each complex is induced by the inner structure on \( E \otimes E \). It follows that
\[ \cdots \rightarrow H^{n-1}(C^*(A, E \otimes E^{-1})) \rightarrow H^{n-1}(\text{cone}(\theta)) \rightarrow H^{n-1}(C^*(A, E \otimes E)[1]) \rightarrow \cdots \]
is an exact sequence of \( E^e \)-modules. By Lemma 2.4, the above sequence becomes
\[ (2.5) \]
\[ \cdots \rightarrow H^{n-1}(A, E \otimes E^{-1}) \rightarrow H^n(E, E \otimes E) \rightarrow H^n(A, E \otimes E) \rightarrow \cdots, \]
where the connecting homomorphism \( \partial = H^n(\theta) \).

Since, as \( A^e \)-modules, \( E \otimes E \cong A \otimes A \otimes k[x] \otimes_2, ax^l \otimes x^k b \mapsto a \otimes b \otimes x^l \otimes x^k \), and similarly
\[ E \otimes E^{-1} \cong A \otimes A^{-1} \otimes k[x] \otimes_2 \cong A \otimes A \otimes k[x] \otimes_2, \]
there exist two canonical morphisms of \( k \)-module complexes
\[ C^*(A, A \otimes A^{-1}) \otimes k[x] \otimes_2 \rightarrow C^*(A, E \otimes E^{-1}), \]
\[ C^*(A, A \otimes A) \otimes k[x] \otimes_2 \rightarrow C^*(A, E \otimes E), \]
where the differentials of the left two complexes are \( b_{-1} \otimes \text{id} \otimes_2, b \otimes \text{id} \otimes_2 \), respectively.

We hope to equip the left two complexes with suitable \( E^e \)-module structures such that the above are morphisms of \( E^e \)-module complexes. To this end, for any \( \tilde{f} \in C^n(A, A \otimes A^{-1}) \), define
\[ x \cdot (\tilde{f} \otimes x^l \otimes x^k) = \tilde{f} \otimes x^l \otimes x^{k+1}, \quad \forall k, l \in \mathbb{N}, \]
\[ a \cdot (\tilde{f} \otimes x^l \otimes x^k) = \sum_{i=0}^{k} q_i^k(a) \cdot \tilde{f} \otimes x^l \otimes x^i, \quad \forall a \in A, \]
\[ (\tilde{f} \otimes x^l \otimes x^k) \cdot x = \tilde{f} \otimes x^{l+1} \otimes x^k, \]
\[ (\tilde{f} \otimes x^l \otimes x^k) \cdot a = \sum_{i=0}^{l} \tilde{f} \cdot p_i^l(a) \otimes x^i \otimes x^k, \]
where \( p_i^l \) and \( q_i^k \) are defined in (1.1) and (1.2) respectively, the actions \( q_i^k(a) \cdot \tilde{f} \) and \( \tilde{f} \cdot p_i^l(a) \) are induced from the inner structure on \( A \otimes A^{-1} \).

This makes \( C^*(A, A \otimes A^{-1}) \otimes k[x] \otimes_2 \) be a complex of \( E^e \)-modules and similarly for \( C^*(A, A \otimes A) \otimes k[x] \otimes_2 \).
Lemma 2.5. Suppose that $A$ is a flat $k$-algebra and $E = A[x; \sigma, \delta]$ is an Ore extension. Then there exists a morphism of $E^\ast$-module complexes
\[
\eta: C^\ast(A, A \otimes A) \otimes k[x]^\otimes \to C^\ast(A, A \otimes A^{\sigma^{-1}}) \otimes k[x]^\otimes
\]
such that the following diagram is commutative,
\[
\begin{array}{ccc}
C^\ast(A, A \otimes A^{\sigma^{-1}}) \otimes k[x]^\otimes & \xrightarrow{\eta} & C^\ast(A, E \otimes E^{\sigma^{-1}}) \\
\downarrow & & \downarrow \phi \\
C^\ast(A, A \otimes A) \otimes k[x]^\otimes & \xrightarrow{\theta} & C^\ast(A, E \otimes E).
\end{array}
\]

Proof. For any $y \in A \otimes A^{\sigma^{-1}}$, we use Sweedler’s notation $y = \sum y' \otimes y''$. For any $f \otimes x^l \otimes x^k \in C^n(A, A \otimes A) \otimes k[x]^\otimes$, let
\[
[f, x^l \otimes x^k]: (a_1, \ldots, a_n) \mapsto \sum f(a_1, \ldots, a_n)y'x^l \otimes x^kf(a_1, \ldots, a_n)''
\]
be the corresponding element in $C^n(A, E \otimes E)$. Then
\[
\theta^n([f, x^l \otimes x^k])((x, \ldots, x)) = x([f, x^l \otimes x^k](\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) - ([f, x^l \otimes x^k](a_1, \ldots, a_n))x
\]
\[
- \sum_{j=1}^n [f, x^l \otimes x^k](a_1, \ldots, a_j-1, \delta \sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n))x
\]
\[
= \sum f(x^l \otimes x^k)f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))'x^l \otimes x^kf(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))''
\]
\[
- \sum_{j=1}^n f(a_1, \ldots, a_n)y'x^l \otimes x^kf(a_1, \ldots, a_n)''x
\]
\[
- \sum_{j=1}^n [f, x^l \otimes x^k](a_1, \ldots, a_j-1, \delta \sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n))x
\]
\[
= \sum \sigma(f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))')x^l \otimes x^kf(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))''
\]
\[
+ \sum \delta(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))'x^l \otimes x^kf(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))''
\]
\[
- \sum f(a_1, \ldots, a_n)y'x^l \otimes x^{k+1}\sigma^{-1}(f(a_1, \ldots, a_n))''
\]
\[
+ \sum f(a_1, \ldots, a_n)y'x^l \otimes x^k\delta \sigma^{-1}(f(a_1, \ldots, a_n))''
\]
\[
- \sum_{j=1}^n [f, x^l \otimes x^k](a_1, \ldots, a_j-1, \delta \sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n))x
\]
\[
= [(\sigma \otimes \text{id})f(\sigma^{-1})^\otimes, x^l \otimes x^k](a_1, \ldots, a_n)
\]
\[
- [(\text{id} \otimes \sigma^{-1})f, x^l \otimes x^{k+1}](a_1, \ldots, a_n)
\]
\[
+ [(\delta \otimes \text{id})f(\sigma^{-1})^\otimes, x^l \otimes x^k](a_1, \ldots, a_n)
\]
\[
+ [(\text{id} \otimes \delta \sigma^{-1})f, x^l \otimes x^k](a_1, \ldots, a_n)
\]
\[
- \sum_{j=1}^n [f(\text{id} \otimes \delta^{-1}) \otimes \delta \sigma^{-1} \otimes (\sigma^{-1})^\otimes, x^l \otimes x^k](a_1, \ldots, a_n).
\]

Thus $\eta$ can be defined as follows, so that the diagram (2.6) is commutative. For any $n \in \mathbb{N}$ and $f \in C^n(A, A \otimes A)$,
\[
\eta^n(f \otimes x^l \otimes x^k) = f_1 \otimes x^{l+1} \otimes x^k - f_2 \otimes x^l \otimes x^{k+1} + f_3 \otimes x^l \otimes x^k
\]
It is easy to verify (2.12) $\eta^n$ is $E^n$-linear for all $n$. In fact, it is obvious that $\eta^n(x \cdot (f \otimes x^i \otimes x^k) \cdot x) = x \cdot \eta^n(f \otimes x^i \otimes x^k) \cdot x$. Thus it suffices to show

$$\eta^n(a \cdot (f \otimes 1 \otimes 1)) = a \cdot \eta^n(f \otimes 1 \otimes 1),$$

(2.11)$$\eta^n((f \otimes 1 \otimes 1) \cdot a) = \eta^n(f \otimes 1 \otimes 1) \cdot a. $$

By the definition of $\eta$,

$$\eta^n(a \cdot (f \otimes 1 \otimes 1)) = \eta^n(a \cdot f \otimes 1 \otimes 1) = (a \cdot f)_1 \otimes x \otimes 1 + (a \cdot f)_2 \otimes 1 \otimes x + (a \cdot f)_3 \otimes 1 \otimes 1,$$

and

$$a \cdot \eta^n(f \otimes 1 \otimes 1) = a \cdot (f_1 \otimes x \otimes 1 - f_2 \otimes 1 \otimes x + f_3 \otimes 1 \otimes 1)$$

$$= a \cdot f_1 \otimes x \otimes 1 - \sigma^{-1}(a) \cdot f_2 \otimes 1 \otimes x + \delta \sigma^{-1}(a) \cdot f_2 \otimes 1 \otimes 1 + a \cdot f_3 \otimes 1 \otimes 1$$

$$= a \cdot f_1 \otimes x \otimes 1 - \sigma^{-1}(a) \cdot f_2 \otimes 1 \otimes x + (\delta \sigma^{-1}(a) \cdot f_2 + a \cdot f_3) \otimes 1 \otimes 1.$$ 

It is easy to verify $(a \cdot f)_1 = a \cdot f_1$, $(a \cdot f)_2 = \sigma^{-1}(a) \cdot f_2$. And

$$(a \cdot f)_3(a_1, \ldots, a_n)$$

$$= (\delta \otimes \text{id})(a \cdot f)(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) + (\text{id} \otimes \delta \sigma^{-1})(a \cdot f)(a_1, \ldots, a_n)$$

$$- \sum_{j=1}^n (a \cdot f)(a_1, \ldots, a_{j-1}, \delta \sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n))$$

$$= \sum_{j=1}^n \delta(f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)))' \otimes a f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))^n$$

$$+ \sum_{j=1}^n f(a_1, \ldots, a_n)' \otimes \delta \sigma^{-1}(a f(a_1, \ldots, a_n)^n)$$

$$- \sum_{j=1}^n (a \cdot f)(a_1, \ldots, a_{j-1}, \delta \sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n))$$

$$= \sum_{j=1}^n \delta(f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)))' \otimes a f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))^n$$

$$+ \sum_{j=1}^n f(a_1, \ldots, a_n)' \otimes \delta \sigma^{-1}(a f(a_1, \ldots, a_n)^n)$$

$$+ \sum_{j=1}^n f(a_1, \ldots, a_n)' \otimes a \delta \sigma^{-1}(f(a_1, \ldots, a_n)^n)$$

$$- \sum_{j=1}^n (a \cdot f)(a_1, \ldots, a_{j-1}, \delta \sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n))$$

$$= a \cdot (\delta \otimes \text{id}) f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))$$

$$+ \delta \sigma^{-1}(a) \cdot (\text{id} \otimes \sigma^{-1}) f(a_1, \ldots, a_n)$$

$$+ a \cdot (\text{id} \otimes \delta \sigma^{-1}) f(a_1, \ldots, a_n)$$

with

(2.8)$$f_1 := (\sigma \otimes \text{id}) f(\sigma^{-1})^n$$

(2.9)$$f_2 := (\text{id} \otimes \sigma^{-1}) f$$

(2.10)$$f_3 := (\delta \otimes \text{id}) f(\sigma^{-1})^n + (\text{id} \otimes \delta \sigma^{-1}) f$$

$$- \sum_{j=1}^n f(\text{id} \otimes \delta^{-1} \otimes (\sigma^{-1} \otimes (\sigma^{-1} \otimes \cdots \otimes (\sigma^{-1} \otimes \cdots \otimes (\sigma^{-1} \otimes x)^{n-j}) \cdots)^{n-j})].$$
Lemma 2.6. Suppose that $A$ is a flat $k$-algebra and $E = A[x; \sigma, \delta]$ is an Ore extension. Let $f \in C^n(A, A \otimes A)$ ($n \in \mathbb{N}$) and $f_1, f_2, f_3$ be given by (2.8), (2.9), (2.10). The following are equivalent:

1. $f$ is a cocycle (resp. coboundary) in $C^n(A, A \otimes A)$.
2. $f_1$ is a cocycle (resp. coboundary) in $C^n(A, A \otimes A^{-1})$.
3. $f_2$ is a cocycle (resp. coboundary) in $C^n(A, A \otimes A^{-1})$.

If the above conditions are satisfied, $f_3$ is also a cocycle (resp. coboundary) in $C^n(A, A \otimes A^{-1})$.

Proof. Take $l = k = 1$ in (2.7), then

$$\eta^n(bf \otimes 1 \otimes 1) = (bf)_1 \otimes x \otimes 1 - (bf)_2 \otimes 1 \otimes x + (bf)_3 \otimes 1 \otimes 1$$

$$= b_{\sigma^{-1}}f_1 \otimes x \otimes 1 - b_{\sigma^{-1}}f_2 \otimes 1 \otimes x + b_{\sigma^{-1}}f_3 \otimes 1 \otimes 1.$$

It follows that

$$(bf)_1 = (\sigma \otimes \text{id})(bf)(\sigma^{-1})^{\otimes n+1} = b_{\sigma^{-1}}f_1,$$

$$(bf)_2 = (\text{id} \otimes \sigma^{-1})(bf) = b_{\sigma^{-1}}f_2,$$

and

$$(bf)_3 = b_{\sigma^{-1}}f_3.$$

So $f$ is a cocycle if and only if $f_1$ is a cocycle, if and only if $f_2$ is a cocycle. If any one of $f$, $f_1$ and $f_2$ is a cocycle, then $f_3$ is also a cocycle.

If $f$ is a coboundary, say $f = bg$, then $f_1 = (\sigma \otimes \text{id})(bg)(\sigma^{-1})^{\otimes n} = b_{\sigma^{-1}}g_1$, $f_2 = b_{\sigma^{-1}}g_2$ and $f_3 = b_{\sigma^{-1}}g_3$. Thus $f_1, f_2$ and $f_3$ are all coboundaries.

If either $f_1 = (\sigma \otimes \text{id})(f)(\sigma^{-1})^{\otimes n}$ or $f_2 = (\text{id} \otimes \sigma^{-1})f$ is a coboundary, then $f$ is a cocycle. \qed

Theorem 2.7. Let $A$ be a projective $k$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension. Suppose that $A$ admits a finitely generated projective resolution as an $A^r$-module. Then for any $n \in \mathbb{N},$

$$0 \to H^n(A, E \otimes E) \xrightarrow{\partial} H^n(A, E \otimes E^{\sigma^{-1}}) \to H^{n+1}(E, E \otimes E) \to 0$$

is an exact sequence of $E^r$-modules.

Proof. Since $A$ admits a finitely generated projective resolution as an $A^r$-module, the two parallel arrows in (2.6) are quasi-isomorphisms of $E^r$-module complexes. Thus the sequence (2.5) becomes

$$\cdots \to H^n(A, A \otimes A) \otimes k[x]^{\otimes 2} \xrightarrow{\partial} H^n(A, A \otimes A^{\sigma^{-1}}) \otimes k[x]^{\otimes 2} \to H^{n+1}(E, E \otimes E) \to \cdots$$

where $\partial$ is induced by $\partial$ and $\tilde{\partial} = H^n(\eta)$.

It is sufficient to show $\tilde{\partial}$ is injective.
Proposition 3.1. Let $A$ be a projective $\mathbf{k}$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension. Suppose that $E$ admits a finitely generated $A^e$-projective resolution of finite length, say $P_\cdot$, and

$$\psi: E \otimes_A P_\cdot \otimes_A \mathbf{s}^{-1} E \longrightarrow E \otimes_A P_\cdot \otimes_A E$$

is a morphism lifting $\rho$, then cone($\psi$) is a complex of finitely generated $E^e$-projective modules. Thus the following proposition is concluded immediately.

Proposition 3.2. Let $A$ be a projective $\mathbf{k}$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension. Suppose that

1. $A$ admits a finitely generated projective resolution as an $A^e$-module,
2. $H^i(A, A \otimes A) = 0$ unless $i = d$ for some $d \in \mathbf{N}$.

Then $H^i(E, E \otimes E) = 0$ unless $i = d + 1$.

Let $\omega$, $\omega^e$ and $\Omega$ be the cohomology groups $H^d(A, A \otimes A)$, $H^d(A, A \otimes A^e^{-1})$ and $H^{d+1}(E, E \otimes E)$, respectively. Then $\Omega \cong \omega^e \otimes \mathbf{k}[x]$ and the $E^e$-module structure on $\omega^e \otimes \mathbf{k}[x]$ is given as follows, for any $a \in A$, $[\tilde{f}] \in \omega^e$, $k \in \mathbf{N}$,

$$a \triangleright ([\tilde{f}] \otimes x^k) = \sum_{i=0}^k \eta_i^k(a) [\tilde{f}] \otimes x^i,$$

$$x \triangleright ([\tilde{f}] \otimes x^k) = [\tilde{f}] \otimes x^{k+1},$$

$$([\tilde{f}] \otimes x^k) \triangleright a = [\tilde{f}] a \otimes x^k,$$

$$([\tilde{f}] \otimes x^k) \triangleright x = [f_2] \otimes x^{k+1} - [f_3] \otimes x^k,$$

where $f = (\sigma^{-1} \otimes \text{id}) \tilde{f}(\sigma^e d)$, $f_2$ and $f_3$ are given by (2.9) and (2.10).
Proof. Since $H^i(A, A \otimes A^{-1}) \cong H^i(A, A \otimes^s A)$, by Theorem 2.7, $H^i(E, E \otimes E) = 0$ for all $i \neq d + 1$. And as $E^*$-modules,

$$H^d(A, E \otimes E) \cong \omega \otimes k[x]$, \quad \text{and} \quad H^d(A, E \otimes E^{-1}) \cong \omega' \otimes k[x],$$

where the $E^*$-module structure on $\omega' \otimes k[x] = \oplus_{i \in \mathbb{Z}} \omega' \otimes k[x]^i$ is given by

\[(3.5) \quad x \cdot ([f] \otimes x^i \otimes x^k) = [f] \otimes x^i \otimes x^{k+1}, \quad \forall [f] \in \omega', \quad k, l \in \mathbb{N},\]

\[(3.6) \quad a \cdot ([f] \otimes x^i \otimes x^k) = \sum_{i=0}^{k} q^k_i(a) [f] \otimes x^i \otimes x^l, \quad \forall a \in A,\]

\[(3.7) \quad ([f] \otimes x^i \otimes x^k) \cdot x = [f] \otimes x^{i+1} \otimes x^k,\]

\[(3.8) \quad ([f] \otimes x^i \otimes x^k) \cdot a = \sum_{i=0}^{k} [f] p^k_i(a) \otimes x^i \otimes x^k,

and the $E^*$-module structure on $\omega \otimes k[x] = \oplus_{i \in \mathbb{Z}} \omega \otimes k[x]^i$ is given similarly. By the proof of Theorem 2.7,

$$0 \to \omega \otimes k[x] \to \Omega \to 0$$

is exact.

To show $\Omega \cong \omega' \otimes k[x]$, it suffices to show that $\omega' \otimes k[x]$ is the cokernel of $\partial$.

Now, for any cocycle $\tilde{f} \in C^d(A, A \otimes A^{-1})$, let $f = (\sigma^{-1} \otimes \text{id}) \tilde{f}(\sigma \otimes d)$. Then, by the definition of $\eta$, $f_1 = \tilde{f}$ and

$$\tilde{f} \otimes x^{i+1} \otimes x^k = f_2 \otimes x^i \otimes x^{k+1} - f_3 \otimes x^i \otimes x^k \quad (\text{mod Im } \eta^d).$$

By Lemma 2.6, $f_2$, $f_3$ are also cocycles. If, in particular, $\tilde{f}$ is a coboundary, then so are $f_2$, $f_3$, and vice versa. It follows that for any $l, k \in \mathbb{N},$

$$\tilde{f} \otimes x^i \otimes x^k = \sum_{j=0}^{l} g_j \otimes 1 \otimes x^{j+k} \quad (\text{mod Im } \eta^d) \quad (3.10)$$

for some cocycles $g_j$ in $C^d(A, A \otimes A^{-1})$, and $\tilde{f}$ is a coboundary if and only if all of $g_j$'s are coboundaries.

Obviously, $f = 0$ if and only if $f_1 = 0$. It follows from (2.13) that $\sum_j g_j \otimes 1 \otimes x^j \in \text{Im } \eta^d$ if and only if $g_j = 0$ for all $j$. This implies that the cocycles $g_j$ in (3.10) are unique. Hence there exists a bijection

$$\Phi_1: (\omega' \otimes k[x]) / \text{Im } \partial \to \omega' \otimes k[x].$$

$$[f] \otimes x^i \otimes x^k + \text{Im } \partial \mapsto \sum_{j=0}^{l} [g_j] \otimes x^{j+k}.$$

Therefore, $\Omega \cong \omega' \otimes k[x]$. It follows from (3.5), (3.6), (3.8) that the induced $E^*$-module structure on $\omega' \otimes k[x]$ satisfies (3.1), (3.2), (3.3). By (3.9),

$$\Phi_1([f_1] \otimes x \otimes 1 + \text{Im } \partial) = [f_2] \otimes x - [f_3] \otimes 1. \quad (3.11)$$

Then it follows from (3.2) and (3.7) that $([f] \otimes x^k) \cdot x = [f_2] \otimes x^{k+1} - [f_3] \otimes x^k$, i.e., (3.4) holds.
Theorem 3.3. Let $A$ be a projective $k$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension. Suppose that $A$ is $\nu$-twisted Calabi-Yau of dimension $d$. Then $E$ is twisted Calabi-Yau of dimension $d + 1$ and the Nakayama automorphism $\nu'$ of $E$ satisfies that $\nu'|_A = \sigma^{-1}\nu$ and $\nu'(x) = ux + b$ with $u, b \in A$ and $u$ invertible.

Proof. We still use $\omega$, $\omega'$ and $\Omega$ as above. As $\omega' \cong \sigma \omega$ and $\omega \cong A^\nu$, we may fix a bimodule isomorphism $\varphi : \omega' \to A^\nu$.

It follows from Proposition 3.2 that $\Omega \cong \omega' \otimes k[x] \cong \sigma A^\nu \otimes k[x]$. The $E \otimes A^\text{op}$-module structure on $A^\nu \otimes k[x]$ is induced from (3.1), (3.2) and (3.3). Let us prove $\sigma A^\nu \otimes k[x] \cong E \sigma^{-1} \nu$ as $E \otimes A^\text{op}$-modules.

In fact, the composite
\[
\sigma A^\nu \otimes k[x] \xrightarrow{\sigma^{-1} \otimes \text{id}} A^{\sigma^{-1} \nu} \otimes k[x] \xrightarrow{\text{id}} k[x] \otimes A^{\sigma^{-1} \nu} \xrightarrow{\nu} E^{\sigma^{-1} \nu},
\]
denoted by $\Phi_3$, is an isomorphism of $E \otimes A^\text{op}$-modules.

Clearly, $\Phi_3$ is bijective. For any $a', a \in A$ and $k \in \mathbb{N}$,
\[
\Phi_3((a' \otimes x^k) \cdot a) = \Phi_3(a' \nu(a) \otimes x^k) = x^k \sigma^{-1}(a') \sigma^{-1} \nu(a)
\]
\[
= \Phi_3(a' \otimes x^k) \cdot a,
\]
\[
\Phi_3(x \cdot (a' \otimes x^k)) = \Phi_3(a' \otimes x^{k+1}) = x^{k+1} \sigma^{-1}(a') = x^k \sigma^{-1}(a')
\]
\[
= x \cdot \Phi_3(a' \otimes x^k).
\]

Recall the maps $q^k : A \to A$ in (1.2) such that $ax^k = \sum_{i=0}^{k} x^i q^k_i(a)$,
\[
\Phi_3(a \cdot (a' \otimes x^k)) = \Phi_3\left( \sum_{i=0}^{k} \sigma(q^k_i(a)) a' \otimes x^i \right)
\]
\[
= \sum_{i=0}^{k} x^i q^k_i(a) \sigma^{-1}(a') = ax^k \sigma^{-1}(a')
\]
\[
= a \cdot \Phi_3(a' \otimes x^k).
\]

So $\Omega \cong E^{\sigma^{-1} \nu}$ as $E \otimes A^\text{op}$-modules. There exists an endomorphism $\nu'$ of $E$ such that $\Omega \cong E^{\nu'}$ as $E^e$-modules and $\nu'|_A = \sigma^{-1} \nu$. In such a way, $\Phi_3$ is indeed an isomorphism of $E^e$-modules.

Now we try to decide $\nu'(x)$. Let $\Phi_2 = \varphi \otimes \text{id} : \omega' \otimes k[x] \to A^\nu \otimes k[x]$. Since $\omega \cong A^\nu$ via $\varphi$, there exists a cocycle $\tilde{f} \in C^d(A, A \otimes A^{\sigma^{-1}})$ such that
\[
\Phi_2 \Phi_1 ([\tilde{f}] \otimes 1 \otimes 1 + \text{Im} \tilde{\delta}) = 1_A \otimes 1.
\]
Define $f, h \in C^d(A, A \otimes A)$ by $f = (\sigma^{-1} \otimes \text{id}) \tilde{f}(\sigma \otimes \text{id})$ and $h = (\text{id} \otimes \sigma) \tilde{f}$, respectively. Clearly, $\tilde{f} = f_1 = h_2$. Thus $f$ and $h$ are both cocycles. Then
\[
\nu'(x) = 1_E \cdot x = \Phi_3 \Phi_2 \Phi_1 ([\tilde{f}] \otimes 1 \otimes 1 + \text{Im} \tilde{\delta}) \cdot x
\]
\[
= \Phi_3 \Phi_2 \Phi_1 ([\tilde{f}] \otimes x \otimes 1 + \text{Im} \tilde{\delta})
\]
\[
= \Phi_3 \Phi_2 ([f_2] \otimes x) - \Phi_3 \Phi_2 ([f_3] \otimes 1)
\]
\[
= \Phi_3 (\varphi([f_2]) \otimes x) - \Phi_3 (\varphi([f_3]) \otimes 1)
\]
\[
= x \sigma^{-1} \varphi([f_2]) - \varphi([f_3])
\]
\[
= \varphi([f_2]) x + \delta \sigma^{-1} \varphi([f_2]) - \sigma^{-1} \varphi([f_3]).
\]
Let $u = \varphi([f_2])$ and $b = \delta \sigma^{-1}\varphi([f_2]) - \sigma^{-1}\varphi([f_3])$. Then $\nu'(x) = ux + b$.

On the other hand,
\[
x = x \cdot \Phi_3\Phi_2\Phi_1([\tilde{f}] \otimes 1 \otimes 1 + \text{Im} \tilde{\theta}) = \Phi_3\Phi_2\Phi_1([h_2] \otimes 1 \otimes x + \text{Im} \tilde{\theta})
= \Phi_3\Phi_2\Phi_1([h_1] \otimes x \otimes 1 + \text{Im} \tilde{\theta}) + \Phi_3\Phi_2\Phi_1([h_3] \otimes 1 \otimes 1 + \text{Im} \tilde{\theta})
= \Phi_3\Phi_2\Phi_1([h_1] \otimes 1 \otimes 1 + \text{Im} \tilde{\theta}) \cdot x + \Phi_3\Phi_2\Phi_1([h_3] \otimes 1 \otimes 1 + \text{Im} \tilde{\theta})
= \sigma^{-1}\varphi([h_1]) \cdot x + \sigma^{-1}\varphi([h_3]).
\]

Let $v = \sigma^{-1}\varphi([h_1])$, $c = \sigma^{-1}\varphi([h_3])$. Then
\[
x = v \cdot x + c = v(ux + b) + c = vux + vb + c,
\]
which implies $vu = 1_A$ and $vb + c = 0$.

Since $\nu'|_A$ is an automorphism and $u$ is left invertible, $x \in \text{Im} \nu'$, namely, $\nu'$ is surjective. Suppose that $\nu' (\sum_{i=0}^{n} x^i a_i) = 0$. Then develop $\nu' (\sum_{i=0}^{n} x^i a_i) = \sum_{i=0}^{n} (ux + b)^i \sigma^{-1} \nu(a_i)$ to the form $\sum_{i=0}^{n} x^i a_i'$. It is easy to show that the leading term is $x\sigma^{-1} \nu(u^i) \cdots \sigma^{-2} (u) \sigma^{-1} \nu(a_n)$. So the coefficient is zero. Since $u$ is left invertible and $\sigma, \nu$ are automorphisms, $a_n = 0$. Consequently, $\nu'$ is injective.

Finally, we prove that $u$ is also right invertible. In fact, for any $a \in A$, $xa = \sigma(a)x + \delta(a)$. Under the action of $\nu'$,
\[
(ux + b)\sigma^{-1}\nu(a)
= u(\nu(a)x + \delta \sigma^{-1}\nu(a)) + b \sigma^{-1}\nu(a)
= \sigma^{-1}\nu\sigma(a)(ux + b) + \sigma^{-1}\nu\delta(a).
\]
Comparing the coefficients of $x$, we have $\sigma^{-1}\nu\sigma(a)u = u\nu(a)$ for any $a \in A$. In particular, let $a = \sigma^{-1}\nu^{-1}\sigma(v)$ and so $u$ is also right invertible.

Therefore, by Propositions 3.1, 3.2, $E$ is twisted Calabi-Yau of dimension $d + 1$ and the Nakayama automorphism $\nu'$ satisfies the required conditions. 

\begin{remark}
By the definition of $\eta$ in (2.6), $f_1 = f_2$ if $\sigma = \text{id}$, and $f_3 = 0$ if $\delta = 0$. Thus $\nu'(x) = x + b$ if $\sigma = \text{id}$, and $\nu'(x) = ux$ if $\delta = 0$.
\end{remark}

\section{Applications}
One motivation of studying the twisted Calabi-Yau property of Ore extensions is studying the right coideal subalgebras of the positive Borel part of a quantized enveloping algebra and computing their Nakayama automorphisms [LW11] by the first named and the third named authors. Such algebras can be obtained by iterated Ore extensions. In [LW11], a class of right coideal subalgebras (quantum homogeneous spaces) $C \subseteq U_q(g)$ is proved to be twisted Calabi-Yau, and the Nakayama automorphisms are given explicitly in some cases.

In this section, the base ring $k$ is assumed to be a field.

\subsection{Quantum affine spaces}
As stated in Section 1, multi-parameter quantum affine $n$-spaces $O_q(k^n)$ can be obtained by iterated Ore extensions. Their Nakayama automorphisms can be computed by using Theorem 3.3. Of course, all the results in this subsection are known and can be deduced in some other way.

Let $n \geq 1$ and $q$ be a matrix $(q_{ij})_{n \times n}$ whose entries are in $k$ satisfying $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$. The quantum affine $n$-space $O_q(k^n)$ is defined
to be a \(k\)-algebra generated by \(x_1, \ldots, x_n\) with the relations \(x_jx_i = q_{ij}x_ix_j\) for all \(1 \leq i, j \leq n\).

**Proposition 4.1.** The quantum affine \(n\)-space \(O_q(k^n)\) is twisted Calabi-Yau of dimension \(n\), whose Nakayama automorphism \(\nu\) sends \(x_i\) to \((\prod_{j=1}^{n-1} q_{ji})x_i\).

**Proof.** If \(n = 1\), \(O_q(k) = k[x_1]\). The conclusion is true.

If \(n > 1\), we assume the conclusion holds for \(n - 1\). Let \(q'\) be an \((n - 1) \times (n - 1)\) matrix obtained by deleting the \(n\)th row and the \(n\)th column of \(q\), and \(q''\) by deleting the first row and the first column of \(q\). Now consider the following two quantum \((n - 1)\)-spaces

\[
O_q'(k^{n-1}) = k\langle x_1, \ldots, x_{n-1} \mid x_jx_i = q_{ij}x_ix_j, 1 \leq i, j \leq n - 1 \rangle,
\]

\[
O_q''(k^{n-1}) = k\langle x_2, \ldots, x_n \mid x_jx_i = q_{ij}x_ix_j, 2 \leq i, j \leq n \rangle.
\]

Clearly, \(O_q(k^n) = O_q'(k^{n-1})[x_n; \sigma']\) where \(\sigma'(x_i) = q_{in}x_i\) for \(1 \leq i \leq n - 1\), and \(O_q(k^n) = O_q''(k^{n-1})[x_1; \sigma'']\) where \(\sigma''(x_i) = q_{i1}x_i\) for \(2 \leq i \leq n\).

By the inductive hypothesis, \(O_q'(k^{n-1})\) and \(O_q''(k^{n-1})\) are both twisted Calabi-Yau of dimension \(n - 1\) and their Nakayama automorphisms \(\nu', \nu''\) are given by

\[
\nu'(x_i) = \left(\prod_{j=1}^{n-1} q_{ji}\right)x_i, \quad 1 \leq i \leq n - 1,
\]

\[
\nu''(x_i) = \left(\prod_{j=2}^{n} q_{ji}\right)x_i, \quad 2 \leq i \leq n,
\]

respectively.

Since the invertible elements in \(O_q(k^n)\) are those nonzero scalars in \(k\), the identity map is the only inner automorphism of \(O_q(k^n)\). By Theorem 3.3, \(O_q(k^n)\) is twisted Calabi-Yau of dimension \(n\) whose Nakayama automorphism \(\nu\) satisfies

\[
\nu(x_i) = \sigma'^{-1}\left(\prod_{j=1}^{n-1} q_{ji}\right)x_i = \left(\prod_{j=1}^{n} q_{ji}\right)x_i, \quad 1 \leq i \leq n - 1,
\]

\[
\nu(x_i) = \sigma''^{-1}\left(\prod_{j=2}^{n} q_{ji}\right)x_i = \left(\prod_{j=1}^{n} q_{ji}\right)x_i, \quad 2 \leq i \leq n.
\]

So \(\nu(x_i) = (\prod_{j=1}^{n} q_{ji})x_i\) for \(1 \leq i \leq n\).

Therefore, the proposition holds for all \(n \geq 1\). \(\Box\)

**Remark 4.2.** The same method can be applied to Weyl algebras \(A_n(k)\), \(n \geq 1\). As a consequence, Weyl algebra \(A_n(k)\) is Calabi-Yau of dimension \(2n\).

4.2. A 3-dimensional AS-regular algebra. Let \(A\) be generated by \(x, y, z\) with three relations

\[
yx - xy - x^2, \quad zx - xz, \quad zy - yz - 2xz.
\]

Then \(A\) is a 3-dimensional AS-regular algebra.

Let \(B = k[x, y]/(yx - xy - x^2)\) be the Jordan plane, which is an AS-regular algebra of dimension 2. Obviously, \(B = k[x][y; \delta_1]\) with \(\delta_1(x) = x^2\). It follows that \(B\) is twisted Calabi-Yau, but not Calabi-Yau, with the Nakayama automorphism given by \(\nu(x) = x\) and \(\nu(y) = 2x + y\).
On one hand, $A = B[z; \nu]$ is an Ore extension of Jordan plane. Then $A$ is twisted Calabi-Yau with the Nakayama automorphism $\nu'$ such that $\nu'(x) = x$ and $\nu'(y) = y$.

On the other hand, $A = k[x, z][y; \delta]$ where $\delta$ is given by $\delta(x) = x^2$ and $\delta(z) = -2xz$. So, $\nu'(z) = z$.

It follows that $A$ is Calabi-Yau, which was proved by Berger and Pichereau [BP11].

4.3. A class of AS-regular algebras of dimension 5. Classifying quantum projective spaces $\mathbb{P}^n$ — noncommutative analogues of projective $n$-spaces, is one of the most important questions in noncommutative projective algebraic geometry. An algebraic approach to construct a quantum $\mathbb{P}^n$ is to form the noncommutative projective scheme $\text{Proj} \ A$ [AZ94], where $A$ is a noetherian connected graded AS-regular algebra of global dimension $n + 1$. So the question turns out to be the classification of AS-regular algebras.

Recently, the second named and the third named authors tried to classify quantum $\mathbb{P}^1$’s. In [WW12], AS-regular algebras of dimension 5, generated by two generators of degree 1 with three generating relations of degree 4, are classified under some generic condition. There are nine types such AS-regular algebras in the classification list. Among them, algebras $\mathbf{D}$ and $\mathbf{G}$ can be realized by iterated Ore extensions ([WW12, Proposition 5.7 and Theorem 5.8]).

In this subsection, we compute the Nakayama automorphisms of these two types of algebras. Assume $k$ is a field of characteristic zero. The algebras $\mathbf{D}$ and $\mathbf{G}$ are of the form $k(x, y)/(r_1, r_2, r_3)$.

For algebra $\mathbf{D}$,
\[
\begin{align*}
    r_1 &= x^3 y + px^2 y x + qxyx^2 - p(2p^2 + q)yx^3, \\
    r_2 &= x^2 y^3 - (p(y^2 + q)yx) - q^2 y^3 x^2 + 2(q - p^2)xy^2x + (q - p^2)yx^2y, \\
    r_3 &= xy^3 + pxy^2 + qy^2x - p(2p^2 + q)y^3x,
\end{align*}
\]
where $p, q \in k \setminus \{0\}$ and $2p^2 - p^2q + q^2 = 0$.

For algebra $\mathbf{G}$,
\[
\begin{align*}
    r_1 &= x^3 y + px^2 y x + qxyx^2 + syx^3, \\
    r_2 &= x^2 y^2 + l_2 xyx + l_3 yx + l_4 y^2 x^2 + l_5 xy^2 x + l_6 x^2 y, \\
    r_3 &= xy^3 + pxy^2 + qy^2x + syx^3,
\end{align*}
\]
where
\[
\begin{align*}
    l_2 &= -\frac{s^2(qs - g)}{g(qs + g)}, \quad l_3 = s - \frac{pg(ps - q^2)}{q(qs + g)}, \quad l_4 = -\frac{g^2}{s^2}, \quad l_5 = \frac{ps^2 + sq}{qs + g},
\end{align*}
\]
with $p, q, s, g \in k \setminus \{0\}$, $ps^3 g + qsg^2 + s^5 + g^3 = 0$, $p^3 s = q^3$, $ps \neq q^2$, $q^2 s^2 \neq g^2$ and $s^5 + g^3 \neq 0$.

It is proved that the algebras $\mathbf{D}$ and $\mathbf{G}$ can be obtained as an iterated Ore extension by a unified process [WW12, subsection 5.2]. We give a sketch of the process here.

Let $A = k[y]$ with deg $y = 1$. Let $a, b \in k$ satisfy $ab(a + b)(a^2 + b^2)(a^3 - b^3) \neq 0$.
Define $A_1 = A[z_1; \sigma_1]$ to be the graded Ore extension of $A$ with $\deg z_1 = 3$, where

$$\sigma_1(y) = ay.$$

Define $A_2 = A_1[z_2; \sigma_2, \delta_2]$ to be the graded Ore extension of $A_1$ with $\deg z_2 = 2$, where

$$\sigma_2(y) = by, \quad \sigma_2(z_1) = az_1,$$

$$\delta_2(y) = z_1, \quad \delta_2(z_1) = 0.$$

Define $A_3 = A_2[z_3; \sigma_3, \delta_3]$ to be the graded Ore extension of $A_2$ with $\deg z_3 = 3$, where

$$\sigma_3(y) = a^{-1}b^3y, \quad \sigma_3(z_1) = b^3z_1, \quad \sigma_3(z_2) = az_2,$$

$$\delta_3(y) = z_2^2, \quad \delta_3(z_1) = (a-b)z_2^3, \quad \delta_3(z_2) = 0.$$

Define $A_4 = A_3[x; \sigma_4, \delta_4]$ to be the graded Ore extension of $A_3$ with $\deg x = 1$, where

$$\sigma_4(y) = a^{-1}b^2y, \quad \sigma_4(z_1) = a^{-1}b^3z_1, \quad \sigma_4(z_2) = bz_2, \quad \sigma_4(z_3) = az_3,$$

$$\delta_4(y) = z_2, \quad \delta_4(z_1) = \frac{a^3-b^3}{a(a+b)}z_2^2, \quad \delta_4(z_2) = \frac{a^3-b^3}{a(a+b)}z_3, \quad \delta_4(z_3) = 0.$$

Let $a = p^{-3}q^2, b = -p^{-1}q$, then $A_4 \cong D$. Let $a = s^2g^{-1}, b = -p^{-1}q$, then $A_4 \cong G$. Both isomorphisms send the indeterminants $x, y$ to $A_4$ to the generators $x, y$ of $D$ and $G$, respectively.

Now let us compute the graded Nakayama automorphism $\nu$ of $A_4$.

By Theorem 3.3, $\nu(y) = \sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}(y) = ab^{-6}y$.

Observe that $A_4$ can be also obtained as an iterated Ore extension along the opposite direction, that is, adding $z_3, z_2, z_1, y$ to $k[x]$ successively. The corresponding automorphisms and derivations are determined by $\sigma_i$ and $\delta_i$ ($1 \leq i \leq 4$). We do not give their concrete expressions but only the result $\nu(x) = a^{-1}b^6x$.

Return to the algebras $D$ and $G$. For $D$, $a^{-1}b^6 = p^3q^{-2}p^{-6}q^6 = p^{-3}q^4$, and the Nakayama automorphism $\nu$ is given by

$$\nu(x) = p^{-3}q^4x, \quad \nu(y) = p^3q^{-4}y.$$

For $G$, $a^{-1}b^6 = s^2gp^{-6}q^6 = g$, and the Nakayama automorphism $\nu$ is given by

$$\nu(x) = gx, \quad \nu(y) = g^{-1}y.$$

Thus we have

**Theorem 4.3.** (1) The algebra $D$ is twisted Calabi-Yau with the Nakayama automorphism $\nu$ given by

$$\nu(x) = p^{-3}q^4x, \quad \nu(y) = p^3q^{-4}y.$$

And $D$ is Calabi-Yau if and only if that $p, q$ satisfy the system of equations

$$\begin{cases} p^3 = q^2, \\ 2p^4 - p^2q + q^2 = 0. \end{cases}$$
The algebra $G$ is twisted Calabi-Yau with the Nakayama automorphism $\nu$ given by

$$\nu(x) = gx, \quad \nu(y) = g^{-1}y.$$ 

And $G$ is Calabi-Yau if and only if that $g = 1$.

Acknowledgments

This research is supported by the NSFC (key project 10731070), and STCSM (Science and Technology Committee, Shanghai Municipality, project 11XD1400500), and a training program for innovative talents of key disciplines, Fudan University.

References

[ASZ99] M. Artin, L.W. Small, and J.J. Zhang, Generic flatness for strongly Noetherian algebras, J. Algebra 221 (1999), no. 2, 579–610.

[AZ94] M. Artin and J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (1994), no. 2, 228–287.

[BP11] R. Berger and A. Pichereau, Calabi-Yau algebras viewed as deformations of Poisson algebras, Preprint arXiv:1107.4472 (2011), 42 pp.

[BZ08] K.A. Brown and J.J. Zhang, Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras, J. Algebra 320 (2008), no. 5, 1814–1850.

[Che04] S. Chemla, Rigid dualizing complex for quantum enveloping algebras and algebras of generalized differential operators, J. Algebra 276 (2004), no. 1, 80–102.

[GC97] J.A. Guccione and J.J. Guccione, Hochschild and cyclic homology of Ore extensions and some examples of quantum algebras, K-Theory 12 (1997), no. 3, 259–276.

[Gin07] V. Ginzburg, Calabi-Yau algebras, Preprint arXiv:math/0612139v3 (2007), 79 pp.

[GW04] K.R. Goodearl and R.B. Warfield, An introduction to noncommutative Noetherian rings, second ed., London Mathematical Society Student Texts, vol. 61, Cambridge University Press, Cambridge, 2004.

[Lod98] J.-L. Loday, Cyclic homology, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, 1998.

[LW11] L.-Y. Liu and Q.-S. Wu, Twisted Calabi-Yau property of right coideal subalgebras of quantized enveloping algebras, manuscript, 2011.

[MR87] J.C. McConnell and J.C. Robson, Noncommutative Noetherian rings, Pure and Applied Mathematics (New York), John Wiley & Sons Ltd., Chichester, 1987.

[Nee01] A. Neeman, Triangulated categories, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[VdB97] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra 195 (1997), no. 2, 662–679.

[VdB98] , A relation between Hochschild homology and cohomology for Gorenstein rings, Proc. Amer. Math. Soc. 126 (1998), no. 5, 1345–1348.

[WW12] S.-Q. Wang and Q.-S. Wu, A class of AS-regular algebras of dimension five, J. Algebra 362 (2012), 117–144.

[ZW11] Q.-S. Wu and C. Zhu, Poincaré-Birkhoff-Witt deformation of Koszul Calabi-Yau algebras, Algebra and Representation Theory, DOI 10.1007/s10468-011-9312-4 (2011), pp 16.

[Yek00] A. Yekutieli, The rigid dualizing complex of a universal enveloping algebra, J. Pure Appl. Algebra 150 (2000), no. 1, 85–93.