ON ONE-PARAMETER FAMILIES OF DIDO RIEmannian problems

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Abstract

Locally, isoperimetric problems on Riemannian surfaces are sub-Riemannian problems in dimension 3. The particular case of Dido problems corresponds to a class of singular contact sub-Riemannian metrics: metrics which have the characteristic vector field as symmetry. We give a classification of the generic conjugate loci (i.e. classification of generic singularities of the exponential mapping) of a 1-parameter family of 3-d contact sub-Riemannian metrics associated to a 1-parameter family of Dido Riemannian problems.

1 Introduction

1.1 Sub-Riemannian metrics under consideration

Let $\mathcal{M}$ be any smooth 3-dimensional manifold. Let $T\mathcal{M}$ and $T^*\mathcal{M}$ be respectively its tangent and cotangent bundle and let $\pi: T^*\mathcal{M} \to \mathcal{M}$ be the natural projection. A contact sub-Riemannian metric on $\mathcal{M}$ is a couple $\Sigma = (\Delta, g)$, where

$$\Delta = \{\Delta_q\}_{q \in \mathcal{M}}; \Delta_q \subset T_q\mathcal{M},$$

is a contact structure on $\mathcal{M}$ and

$$g: \Delta \to R_+,$$

a Riemannian metric on $\Delta$. Since $\Delta$ is a contact structure and therefore nonintegrable, $(\Delta, g)$ defines a distance $d$ on $\mathcal{M}$ [1].

A contact sub-Riemannian metric being given on $\mathcal{M}$, there is a 1-form $\omega$ and a vector field $\nu$ on $\mathcal{M}$, both determined up to orientation by the conditions:

1. The distribution $\Delta$ is the kernel of $\omega$ ($\Delta = \ker \omega$),

2. the 2-form $d\omega$ restricted to $\Delta$ is the volume form $V$,

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3. the vector field $Z$ is such that $\omega(Z) = 1$, $i(Z) d\omega = 0$.

The vector field $Z$ is the \textit{characteristic vector field} of the contact sub-Riemannian structure.

The cotangent bundle has the standard symplectic structure. Consider the Hamiltonian $\mathcal{H}$ of the metric. It is defined as follows: associated to $(\Delta, g)$, there is a cometric on $T^*\mathcal{M}$:

$$\mathcal{H}(\psi) = \frac{1}{2} \sup_{u \in \Delta \setminus \{0\}} \left( \frac{\psi^2(u)}{\|u\|^2_g} \right),$$

where $\psi(q) \in T^*_q \mathcal{M}$.

On fibers of $T^*\mathcal{M}$, $\mathcal{H}$ is a positive semi-definite quadratic form, the kernel of which is the annihilator of $\Delta$.

In the remainder of this paper we will assume that:

1. The manifold $\mathcal{M}$ is an open subset of $\mathbb{R}^3$, containing the origin 0,
2. the distribution $\Delta$ is specified by an orthonormal frame field $(F, G)$ of the metric $g$, where $F$ and $G$ are two vector fields defined on $\mathcal{M}$,
3. for any $q \in \mathcal{M}$, $F(q)$, $G(q)$, and $[F, G](q)$ are linearly independent (the contact condition).

Hence the Hamiltonian $\mathcal{H}$ is given by:

$$\mathcal{H} = \frac{1}{2} \left( (\psi(F))^2 + (\psi(G))^2 \right).$$

In this paper, we deal with contact sub-Riemannian structures that are invariant under the action of the one parameter group generated by the characteristic vector field $Z$ (i.e. sub-Riemannian structures having a symmetry $Z$). The reasons for doing so will become clear when we show that: to characterize minimizing geodesics for this class of metrics is a dual sub-Riemannian reformulation of the classical isoperimetric problem of Dido on Riemannian surfaces.

Let $(M, g)$ be a Riemannian surface. We consider on $M$ the following \textit{iso-area} problem: First we fix two points $z_0, z_1 \in M$ and a curve

$$\tilde{\iota} : [0, 1] \rightarrow M, \tilde{\iota}(0) = z_0, \tilde{\iota}(1) = z_1.$$ 

We are then faced with the following question:

Which curves

$$\iota : [0, 1] \rightarrow M,$$

connecting $z_0$ and $z_1$, such that the area

$$A = \int_{\Omega} V$$

of the domain $\Omega$ encircled by $\tilde{\iota}$ and $\iota$ is prescribed, have minimal Riemannian length?
Denoting such an iso-area problem \((M,g,V)\), it is stated (see for instance \([4]\)) that this problem can be reformulated locally in terms of three dimensional sub-Riemannian geometry.

More precisely, we can associate a germ of Dido Problems \((M,g,V)_{z_0}\) with a germ of an oriented sub-Riemannian structure on \(\mathcal{M}\) with a symmetry \(Z\), denoted by:

\[(\mathcal{M}, \Delta, g, Z)_{q_0},\]

where,

\[\mathcal{M} = M \times \mathbb{R} \quad \text{and} \quad q_0 = \{z_0, 0\}\]

Let \(V = d\alpha\) and \(Z = \frac{\partial}{\partial w}\), then

\[\Delta = \ker(dw + \alpha).\]

If

\[\delta : [0, 1] \rightarrow \mathcal{M}\]

is an admissible curve for \(\Delta\) with fixed endpoints

\[\delta(0) = q_0 \quad \text{and} \quad \delta(1) = q_1 = (z_1, w_1),\]

hence:

The sub-Riemannian length of \(\delta\) is the Riemannian length of its projection \(\delta^*\) on \(M\) and

\[w_1 = \int_0^1 \alpha(d\delta^*)d\tau.\]

Therefore projections of sub-Riemannian length minimizers are solutions of the iso-area problem \((M,g,V)\).

The sub-Riemannian structures associated with \((M,g,V)\) or simply \((M,g)\) are called the \(IsosR\)-structure and are denoted by \(IsosR\).

The main aim of this paper is to classify generic conjugate loci for \(IsosR\)-metrics.

For some reasons which will become clear in the next section we will in fact consider 1-parameter families of Dido Riemannian problems:

\[\lambda \in I \subset \mathbb{R} \mapsto (M, g(\lambda)), I \text{ an interval}.\]

We will denote the families of \(IsosR\)-metrics associated with \((M,g(\lambda))\) by \(\mathcal{F}-IsosR\).

1.2 Normal forms for Riemannian metrics on surfaces

Recall the following result.
Theorem 1.1 Let \((M, g)\) be a Riemannian surface. The metric \(g\) has the following normal form. In normal coordinates with pole 0, there is an orthonormal frame (unique up to rotations and up to the action of \(SO(2)\) in \(T_0M\)), \((X, Y)\):

\[
\begin{align*}
X &= \frac{\partial}{\partial x} + y (\beta(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y})), \\
Y &= \frac{\partial}{\partial y} - x (\beta(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}).
\end{align*}
\]  

(1)

Where \(\beta(x, y)\) is a smooth function. Moreover \(\beta(0, 0) = b_0 = \frac{1}{6}k_0\).

The constant \(k_0\) being the gaussian curvature of \(M\) at 0.

Remark 1.1 In the normal coordinates specified above, the \(k^{th}\)-differentials \(\beta^k = D^k \beta \big|_0\) are homogeneous polynomials of degree \(k\) in \(x, y\) and define symmetric covariant tensors of degree \(k\) on \(T_0M\).

These tensors, denoted \(\beta_k\), do not depend on the orientation and are invariants of the Riemannian structure.

Decomposition of tensor fields Let \(S^kT^*M\) denote the bundle of covariant symmetric tensors of degree \(k\) on \(TM\). Due to the action of \(SO(2)\) on the typical fiber of \(TM\), we have the following decomposition of \(S^kT^*M\) into isotypic components [4]:

\[
S^kT^*M = \bigoplus_M (S^kT^*M)_j,
\]

where \((S^kT^*M)_j\) is the component relative to the \(j^{th}\) power of the basic character \(e^{i\varphi}\), \((i = \sqrt{-1})\). Therefore, if \(\beta_n \in S^kT^*M\) then:

\[
\beta_n = \sum_j \beta_{n,j}, \text{ where } \beta_{n,j} \in (S^kT^*M)_j.
\]

1.3 Lemmas

Let \(M\) be any open subset of \(\mathbb{R}^2\). Let \(O(M)\) and \(J^nO(M)\) be respectively the bundle of orthonormal frames of Riemannian metrics on \(M\) and the vector bundle of \(n\)-jets of elements of \(O(M)\).

We denote by \(\Pi_0 : O(M) \to M\) and \(\Pi : J^nO(M) \to M\) the standard projections.

Let

\[
B^n = \bigoplus \{S^kT^*M \mid 0 \leq k \leq n\}, B^{-1} = \{0\}
\]

For \(n \geq 0\), let us define the following map:

\[
\Pi^n : J^{n+2}O(M) \to B^{n-1},
\]
by its restriction to the fiber $\Pi^{-1}(q_0)$ of the bundle $\Pi : J^{n+2}O(M) \to M$:

$$\Pi^n(j_{q_0}^{n+2}(X), j_{q_0}^{n+2}(Y)) = \beta^{n-1},$$

where $\beta^{n-1}$ is the representative of the tensors $\beta_{n-1,q_0}$ in the unique normal coordinates system at $q_0$ such that:

$$(X(q_0), Y(q_0)) = (X(q_0), Y(q_0)),$$

where $(X, Y)$ is the normal form of $(X, Y)$ at $q_0$.

**Lemma 1.1** If we fix $(X, Y) \in O(M)$, then:

$$\Pi^n(X, Y) : M \to B^{n-1}, \quad \Pi^n(X, Y)(q_0) = \Pi^n(j_{q_0}^{n+2}(X), j_{q_0}^{n+2}(Y)).$$

is a surjective submersion.

The proof of lemma 1.1 is the result of computations not given here.

**Definition 1.1** A 1-parameter family $F$ of orthonormal frames is given by:

$$\{(X_\lambda, Y_\lambda) \in O(M), \lambda \in I\}.$$

**Lemma 1.2** If we fix $(\tilde{X}, \tilde{Y}) \in F$,

$$\tilde{\Pi}^n(\tilde{X}, \tilde{Y}) : M \times I \to B^{n-1}, \quad \tilde{\Pi}^n(\tilde{X}, \tilde{Y})(q_0) = \tilde{\Pi}^n(j_{(q_0,\lambda_0)}^{n+2}(\tilde{X}), j_{(q_0,\lambda_0)}^{n+2}(\tilde{Y}))$$

is a surjective submersion.

Lemma 1.2 is a consequence of lemma 1.1, and of the fact that $J^nF$ submerses into $J^nO(M)$.

The algebraic lemmas (lemma 1.1 and lemma 1.2) are the main elements in establishing the genericity results here.

Since we have two lemmas, we have the following two choices: We can envisage an isolated Dido problem and use lemma 1.1, or a family of such problems depending on 1-parameter by using lemma 1.2.

It is obvious that there are nongeneric situations in the first case, which become generic in the second. Naturally we will consider 1-parameter families.

### 1.4 Exponential mapping

Let us first recall briefly the previous results of [2], [3] and [4], which concern generic contact sub-Riemannian structures in dimension 3.

The following result is stated in [4].
Theorem 1.2 Consider a germ at 0 of contact sub-Riemannian metric \((\Delta, g)\). There is up to orientation, and up to the action of \(SO(2)\) on \(\Delta(0)\), a unique coordinate system \((\text{normal coordinates})\) with respect to which the metric has an orthonormal frame \((\tilde{F}, \tilde{G})\) of the following form:

\[
\begin{cases}
\tilde{F} = \frac{\partial}{\partial x} + \tilde{y} \left( \tilde{\beta} \frac{\partial}{\partial \tilde{x}} - \tilde{x} \frac{\partial}{\partial \tilde{y}} \right) + \frac{1}{2} (1 + \tilde{\gamma}) \frac{\partial}{\partial \tilde{w}} \\
\tilde{G} = \frac{\partial}{\partial \tilde{y}} - \tilde{x} \left( \tilde{\beta} \frac{\partial}{\partial \tilde{x}} - \tilde{x} \frac{\partial}{\partial \tilde{y}} \right) + \frac{1}{2} (1 + \tilde{\gamma}) \frac{\partial}{\partial \tilde{w}}.
\end{cases}
\]

Where \(\tilde{\beta}\) and \(\tilde{\gamma}\) are smooth functions; satisfying the following boundary conditions:

\[
\tilde{\beta}(0, 0, \tilde{w}) = \tilde{\gamma}(0, 0, \tilde{w}) = \frac{\partial \tilde{\gamma}}{\partial \tilde{x}}(0, 0, \tilde{w}) = \frac{\partial \tilde{\gamma}}{\partial \tilde{y}}(0, 0, \tilde{w}) = 0.
\]

Indeed, \((NF)\) can be obtained even when \(\Delta\) is not a contact structure. In fact the normal coordinates are coordinates w.r.t. local coordinates charts

\[(U_{\Gamma}, (\tilde{w}, \tilde{p}, \tilde{q}) \circ (\pi \circ \exp H)^{-1}),\]

where \((\tilde{p}, \tilde{q}, \tilde{r})\) are dual coordinates in \(T^*M\) and \(\Gamma\) is a smooth curve transversal to \(\Delta\). In the contact case, such a curve \(\Gamma\) can be taken as the integral curve of \(Z\).

In \(\Gamma\)-normal coordinates, geodesics through \(\Gamma\), satisfying transversality conditions w.r.t. \(\Gamma\) are straight lines contained in the plane \(\{\tilde{w} = c\}\), where \(c\) is a constant \((\mathbb{I})\). The set \(C^s_{\Gamma}\) of points \(x\) of \(M\), that are at a distance \(s\) of \(\Gamma\), form a smooth cylinder for small enough values of \(s\).

1.4.1 Approximation of generic conjugate loci and stability

In \(\mathbb{I}\) and \(\mathbb{I}\), it is proved that for contact sub-Riemannian metrics, there are in essence two generic situations. In the two cases we have a representation of the exponential mapping as a suspension of a mapping between surfaces.

The first case happens on a complement of a smooth possibly empty curve \(C\) of \(M\). In this instance, the 3-jet \((\exp_3)\) is a sufficient jet of the exponential mapping.

This result follows the fact that the suspension of \((\exp_3)\) is a “Whitney map”: that is a stable mapping in the sense of Thom-Mather \((\mathbb{I}\mathbb{S})\). Hence the 3-jet of the conjugate loci is sufficient to describe it and the exponential mapping is determined by the 3-jet of the metric, in a neighborhood of its singular locus.

The situation on the curve \(C\) is more complex. The full conjugate loci \(CL_0 = CL\), splits into two semi-conjugate loci \(CL^+\) and \(CL^-\) corresponding to \(w > 0\) or \(w < 0\) in the normal coordinates.

The intersection \(CL^\pm_w\) of \(CL^\pm\) with the planes \(\{w = c \neq 0\}\) for small enough values of \(|w|\), is a closed curve with 6 cusps and self-intersections. Hence the suspension of the exponential mapping is not a “Whitney map”.

To conclude that the exponential mapping approximation is stable, it is necessary to show that all generic self-intersections are transversal.

In order to classify the semi-conjugate loci, the authors of \(\mathbb{I}\) define the symbol \(S\) of \(CL^\pm\) in the following manner.

A symbol \(S\) is a sequence of 6 rational numbers, \(S = (s_1, \ldots, s_6)\) modulo cyclic permutation and reflection. If we follow the curve \(\delta\) starting from a cusp point: \(s_i\) gives half the number of self-intersections between the \(i^{th}\) and the \((i + 1)^{th}\) cusp point.
Theorem 1.3

1) At generic points of the curve $C$, the possible symbols for generic semi-conjugate loci are:

\[ S_1 = (2, 1, 1, 2, 1, 0), \quad S_2 = (2, 1, 1, 1, 1, 1), \]
\[ S_3 = (0, 1, 1, 1, 1, 1). \]

2) There are two types of isolated points at $C$.
   a) At the first type the possible symbols are:
      \[ S_1^* = S_1 = (2, 1, 1, 2, 1, 0), \quad S_2^* = S_2 = (2, 1, 1, 1, 1, 1), \]
      \[ S_3^* = S_3 = (0, 1, 1, 1, 1, 1). \]
      But in that case, the exponential mapping is determined by the higher order jet of the metric (7-jet) than at generic points of $C$ (5-jet).
   b) At the second type the possible symbols are:
      \[ S_4 = (\frac{1}{2}, \frac{1}{2}, 1, 0, 0, 1) \quad S_5 = (1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1) \]
      \[ S_6 = (\frac{3}{2}, \frac{3}{2}, 1, 1, 0, 1) \quad S_7 = (2, \frac{1}{2}, \frac{1}{2}, 2, 0, 0). \]

However, this result, stated also in [1], only gives classification of possible semi-conjugate loci. The problem of classifying possible full conjugate loci is therefore not entirely solved.

1.5 Normal Dido coordinates

Theorem 1.4 Given $(\Delta, g)_0$ a germ at the origin of an isoperimetric sub-Riemannian metric. There is an unique coordinates system (normal Dido coordinate), up to orientation and up to the action of $SO(2)$ on $\Delta(0)$, with respect to which the metric has an orthonormal frame $(\tilde{F}_I, \tilde{G}_I)$ of the following form:

\[
\begin{cases}
\tilde{F}_I = X + \frac{y}{2} \gamma \frac{\partial}{\partial w} \\
\tilde{G}_I = Y - \frac{x}{2} \gamma \frac{\partial}{\partial w}
\end{cases}
\]

(3)

Where : $(X, Y)$ is the normal form of the Riemannian metric on $M$ and

\[
\gamma = (1 + (x^2 + y^2)\beta) \frac{1}{\int_0^1 \frac{2 t dt}{1 + t^2(x^2 + y^2) \beta(t x, t y)}}.
\]

(4)

This theorem is not proved here. Observe that in comparison with theorem [1.2], the only work we have to do for this, is to slightly modify the $\Gamma$-normal coordinates. This done, checking formula [4] is very easy.

In accordance with [9] the $\mathcal{F}$-IsosR-structures are nongeneric. However this is fairly interesting that as it is in the case of generic contact sub-Riemannian metrics, our work follows almost the same steps and we prove similar results.
1.6 Statement of our main results and outline of the paper

Our work is organized as follows. In the second section, we give the notations for the main invariants. We end this section by summarizing the basic properties of the relevant jet of the conjugate locus computed in appendix 1.3.

In section 3, we state our genericity results. We highlight two principal invariants denoted by $r_2$ and $r_3$. More precisely we prove:

For a generic element $\Sigma$ of $\mathcal{F}$ (for the Whitney topology), the set of points of $M \times I$ on which the invariant $r_2$ vanishes is a smooth possibly empty curve $C$.

On the curve $C$ the invariant $r_3$ does not vanish.

Using higher order invariants, theorems 3.2 and 3.3 show that the curve $C$ is partitioned into two subsets: a discrete subset (isolated points of $C$) and its complement (generic point of $C$).

In section 4, we state our stability results (see theorems 4.1, 4.2, 4.3 and 4.4). We can summarize these results as follow:

If $r_2$ is nonzero, the exponential mapping is determined by the 5-jet of the metric, in a neighborhood of its singular locus. This is the purpose of theorem 4.1.

On $C$, the situation is by far more complex and particularly delicate at isolated points on this curve.

For the sake of clarity theorem 4.2 deals only with generic points on the curve $C$. Thus theorems 4.3 and 4.4 are devoted to isolated points on $C$.

It is clear that for $\mathcal{F}$-$\text{IsosR}$, we can classify the semi conjugate loci as in theorem 1.3, but we need higher order jets.

What is more, for $\mathcal{F}$-$\text{IsosR}$, our symbols are the same for $CL^+$ and $CL^-$. It follows that we can classify the full conjugate loci.

In fact we state the following results.

**Theorem 1.5**

1) At generic points of the curve $C$, the possible symbols for generic semi-conjugate loci are:

\[ S_1 = (2, 1, 1, 2, 1, 0), \quad S_2 = (2, 1, 1, 1, 1, 1), \]

\[ S_3 = (0, 1, 1, 1, 1, 1). \]

2) There are two types of isolated points at $C$.

a) At the first type the possible symbols are:

\[ S_1^* = S_1 = (2, 1, 1, 2, 1, 0), \quad S_2^* = S_2 = (2, 1, 1, 1, 1, 1), \]

\[ S_3^* = S_3 = (0, 1, 1, 1, 1, 1). \]

But in that case, the exponential mapping is determined by higher order jets of the metric (9-jet) than at generic points of $C$ (7-jet).

b) At the second type the possible symbols are:

\[ S_4 = \left(\frac{1}{2}, \frac{1}{2}, 1, 0, 0, 1\right), \quad S_5 = (1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1) \]

\[ S_6 = \left(\frac{3}{2}, \frac{1}{2}, 1, 1, 0, 1\right), \quad S_7 = (2, \frac{1}{2}, \frac{1}{2}, 2, 0, 0). \]
Theorem 1.6  For a fixed element of $\mathcal{F}$-Isos$R$ the possible symbols for generic conjugate loci are:

\[ S_1 = (2, 1, 1, 2, 1, 0), \quad S_2 = (2, 1, 1, 1, 1, 1), \quad S_3 = (0, 1, 1, 1, 1, 1). \]

The section 5 is devoted to technical computations of the exponential mapping and the conjugate loci.

2 Conjugate loci

Let $\Sigma = (\Delta, g_0)$ be a germ at the origin of an isoperimetric sub-Riemannian metric. We can restrict $\mathcal{M}$ in order to obtain a neighborhood $N$ of $x_0 = (0, 0, 0)$, with respect to which $\Sigma$ is under normal form ($INF$). We denote by $(p, q, r)$ the dual coordinate in $T^*N$.

The set $C_{x_0} = \pi^{-1}_N(x_0) \cap \mathcal{H}^{-1}(\frac{1}{2})$ is therefore the cylinder $\{p^2 + q^2 = 1\}$.

Let $\exp sH : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$, denotes the Hamiltonian flow of $H$ at time $s$. If $N$ is small enough, point of the conjugate loci in $N$ appear only along geodesics $\{\pi \circ \exp sH(\lambda_{x_0})\}$ for $\lambda_{x_0}$ in a certain neighborhood of infinity $N^* \subset C_{x_0}$ [1].

In appendix 5.3 we compute relevant jets of $CL$ in a suitable coordinates system $(h, \varphi)$, with respect to which:

$CL : N^* \to N$

has the following expansion with respect to $h$:

$CL(\varphi, h) = (x(\varphi, h), y(\varphi, h), h) = \left( \sum_{i=4}^9 f_i(\varphi) h^i + O(h^{10}), h \right)$,

where $h = \sqrt{\frac{\epsilon w}{\pi}}$, $\epsilon = \pm 1$ according to the sign of $w$, and $(x, y, w)$ the normal coordinates.

$CL = CL^+ \cup CL^-$

As we will see, for small enough of $|w|$, the intersection of $CL^\pm$ with planes $\{w = c \neq 0\}$ will be a close curve. In the more degenerate generic cases this curve will have self-intersections denoted by $Isoself$.

2.1 Notation

The following tensors have the leading part in our study.

Let $\beta_1 = \beta_{1,1}$, $\beta_2 = \beta_{2,2} + \beta_{2,0}$, and $\beta_3 = \beta_{3,3} + \beta_{3,1}$. Thus we have in our normal coordinates
\[ \beta_{1,1} = R_e(r_1(dx + idy)). \]
\[ \beta_{2,2} = R_e(r_2(dx + idy)^2), \quad \beta_{2,0} = \tau_0 (dx^2 + dy^2). \]
\[ \beta_{3,3} = R_e(r_3(dx + idy)^3), \quad \beta_{3,1} = R_e(\bar{v} (dx + idy))(dx^2 + dy^2). \]

Where:

\[ r_1 = |r_1| (\cos \theta_1, \sin \theta_1) = |r_1| (\cos \theta_1 + i \sin \theta_1) \]
\[ r_2 = |r_2| (\cos \theta_2, \sin \theta_2) = |r_2| (\cos \theta_2 + i \sin \theta_2) \]
\[ r_3 = |r_3| (\sin \theta_3, - \cos \theta_3) = |r_3| (\sin \theta_3 - i \cos \theta_3) \]
\[ v = -v_1 + i v_2. \]

The remaining tensors are given in appendix 5.3.

2.2 Basic properties of \( f_i (B.P.) \)

From now on, we summarize basic properties of the map \( f_i \) when \( \epsilon = 1 \). For details see appendix 5.3

P1) \( r_2 \neq 0 \).

1. \( f_4(\varphi) = 3\pi |r_1| (-\sin \theta, \cos \theta) \),
2. \( f_4 \) is independant of \( \varphi \),
3. \( f_5(\varphi) = 5\pi |r_2| (3 \cos(\varphi - \theta_2) + \cos(3\varphi - \theta_2), 3 \sin(\varphi - \theta_2) - \sin(3\varphi - \theta_2)) \).

P2) \( r_2 = 0 \).

1. \( f_4(\varphi) = 3\pi |r_1| (-\sin \theta_1, \cos \theta_1), \quad f_5(\varphi) = (0,0) \),
2. \( f_6(\varphi) = \frac{\pi}{2} (-25v_2 + 90 |r_3| \cos(2\varphi + \theta_3) + 45 |r_3| \cos(4\varphi + \theta_3) + 31b_0 |r_1| \sin \theta_1, \]
\[ -25v_1 - 90 |r_3| \sin(2\varphi + \theta_3) + 45 |r_3| \sin(4\varphi + \theta_3) - 31b_0 |r_1| \cos \theta_1), \]
3. \( f_6(\varphi + \pi) = f_6(\varphi) \),
4. \( f_7(\varphi + \pi) = -f_7(\varphi) \),
5. \[ \frac{df_6}{d\varphi} \wedge \frac{df_7}{d\varphi} = 12960\pi^3 |r_3| \sin(3\varphi + \theta_3) |r_1|^2 \sin 2(\varphi - \theta_1), \]
6. \[
\frac{df_6}{d\varphi} \wedge f_7 = 1080\pi^2 |r_3| \sin(3\varphi + \theta_3) P(\varphi) = \Psi(\varphi),
\]
7. \[f_8(\varphi + \pi) = f_8(\varphi),\]
8. \[f_9(\varphi + \pi) = -f_9(\varphi),\]
9. \[\frac{df_6}{d\varphi} \wedge \frac{df_8}{d\varphi} = \frac{df_6}{d\varphi} \wedge \frac{df_9}{d\varphi} = |r_3| \sin(3\varphi + \theta_3) (Q_0(\varphi)),\]
10. \[\frac{df_6}{d\varphi} \wedge f_9 = |r_3| \sin(3\varphi + \theta_3) (Q_1(\varphi) + Q_2(\varphi))).\]

**Remark 2.1** \(P(\varphi) = A \cos(2\varphi) + B \sin(2\varphi) + C \cos(4\varphi) + D \sin(4\varphi),\) where \(A, B, C, D\) are linear combinations of the coefficient of the invariants \(\beta_4\) and \(r_1^2\). Contrary to the generic contact case, \(P(\varphi)\) is independent on \(\epsilon\) (see appendix 5.3).

**Remark 2.2** The \(Q_i\)'s do not depend on the same invariants.

In particular only \(Q_2\) is dependent on the coefficients of \(\beta_6\).

Let:
\[\nu = \frac{A - iB}{2}, \quad \mu = \frac{C - iD}{2}.\]

Let
\[P(\varphi) = \nu e^{2i\varphi} + \overline{\nu} e^{-2i\varphi} + \mu e^{4i\varphi} + \overline{\mu} e^{-4i\varphi}.\]

and
\[\tilde{P}(z) = \mu z^4 + \nu z^3 + \overline{\nu} z + \overline{\mu}\]
\[T_c(\varphi) = |r_3| \sin(3\varphi + \theta_3)\]

and
\[T(z) = r_3 z^3 + \overline{r}_3.\]

In the remainder of this paper we will be concerned with the roots on the unit circle of these trigonometric polynomials.
3 Genericity results

In the next section we will prove the following results:

**Theorem 3.1** For a generic element of $\mathcal{F}$ (for the Whitney topology), the set on which $\beta_{2,2}$ vanishes is a smooth curve $C$ of $\mathbb{R}^3$.
Along $C$, $\beta_{3,3}$ and $\mu$ do not vanish.

**Theorem 3.2** On the complement of a discrete subset of $C$:

i) $\tilde{P}(z)$ has either two or four simple roots on the unit circle,

ii) $\tilde{P}(z)$ and $T(z)$ have no common roots on the unit circle.

**Theorem 3.3** At isolated point of $C$, either:

i) $\tilde{P}(z)$ has a double root on the unit circle, which is not triple, and which is not a root of $T(z)$.
The other roots of $\tilde{P}(z)$ are simple; they are on the unit circle and are not roots of $T(z)$.

or,

ii) $\tilde{P}(z)$ and $T(z)$ have one and only one common simple root on the unit circle.
The other roots of $\tilde{P}(z)$ are simple.

3.1 Proof of our genericity theorems

3.1.1 The bad set

**Definition 3.1** $B_0$ is the subset of $B^6$ such that:

1) $\beta_{2,2} = \beta_{3,3} = 0$

$B_1$ is the subset of $B^6$ such that:

1) $\beta_{2,2} = \mu = 0$

**Definition 3.2** $B_2$ is the subset of $B^6$ such that:

1) $\beta_{2,2} = 0$

2) $\tilde{P}(z)$ has a triple root on the unit circle.

$B_3$ is the subset of $B^6$ such that:

1) $\beta_{2,2} = 0$,

2) A root of $T(z)$ is a double root of $\tilde{P}(z)$ on the unit circle.

$B_4$ is the subset of $B^6$ such that:

1) $\beta_{2,2} = 0$,

2) $T(z)$ and $\tilde{P}(z)$ have two common roots on the unit circle.
Definition 3.3 $B_5$ is the subset of $B^6$ such that
1) $\beta_{2,2} = 0$,
$\tilde{P}(z)$ has a double root on the unit circle
$B_5$ is the subset of $B^6$ such that
1) $\beta_{2,2} = 0$,
$\tilde{P}(z)$ and $T(z)$ have one and only one common simple root on the unit circle
(i.e. $\text{Res}(r_3, \mu, \nu) = \text{Resultant of } \tilde{P}(z) \text{ and } T(z) \text{ has a simple root}$).

Remark 3.1 Since the integral of $\tilde{P}(z)$ over its period is zero, then, if $\mu \neq 0$, the following facts are easy to check.
1) $\tilde{P}(z)$ has a root on the unit circle.
2) $\tilde{P}(z)$ does not have two double roots on the unit circle.
3) If $\tilde{P}(z)$ has a double root on the unit circle, the two other roots of $\tilde{P}(z)$ are on the unit circle.

3.1.2 Estimate of the codimension of $B_k$’s
Some appropriate computations on $T(z)$ and $\tilde{P}(z)$ allow us to rewrite the $B_k$’s as follows:

$$
B_0 = \{r_2 = 0\} \cap \{r_3 = 0\}
$$

$$
B_1 = \{r_2 = 0\} \cap \{\mu = 0\}
$$

$$
B_2 = \{r_2 = 0\} \cap \{4\mu^2\bar{\nu} + \nu^3 = 0\}
$$

$$
B_3 = \{r_2 = 0\} \cap \{27\nu^3 r_3 \bar{r}_3^2 + (\bar{\nu} r_3 - 4\mu \bar{r}_3)^3 = 0\}
$$

$$
B_4 = \{r_2 = 0\} \cap \{\bar{\nu} r_3 - \mu \bar{r}_3 = 0\}
$$

$$
B_5 = \{r_2 = 0\} \cap \{27 \text{Re}^2(\mu \bar{\nu}^2) - (4|\mu|^2 - |\nu|^2)^3 = 0\}
$$

$$
B_6 = \{r_2 = 0\} \cap \{\text{Res}(\mu, \nu, r_3) = 0\}
$$

3.1.3 Proof of the genericity
We consider the $\hat{B}_k$’s defined from the $B_k$’s by :

$$
\hat{B}_k = (\tilde{\Pi}_{N})^{-1}(B_k)
$$

Now let :

$$
\tilde{B}_k = \{(\bar{X}, \bar{Y}) \mid J^0(\bar{X}, \bar{Y}) \cap \hat{B}_k\}
$$

If $B = \cap_k \tilde{B}_k$ ; then $B$ is an open dense set for the Whitney topology in $\mathcal{F}$.
Standard arguments from transversality theory (see for instance Goresky-MacPherson [11]), allow us to conclude that :
\[ \forall (\tilde{X}, \tilde{Y}) \in B, (\tilde{\Pi}^{-1}_N(\tilde{X}, \tilde{Y}))(B_k) \]

are smooth submanifolds or Whitney-stratified set of \( M \times I \) of the same codimension as the \( B_k \)'s.

Since \( B_k \)'s are at least codimension 4 (resp codimension 3) for \( 0 \leq k \leq 4 \) (resp \( 5 \leq k \leq 6 \)), this ends the proofs of theorems 3.1, 3.2, and 3.3.

**Remark 3.2** If we consider a fixed element \( F_{\lambda_0} \) of \( F \), then: for a generic element of \( F_{\lambda_0} \) (for the Whitney topology), \( \beta_{2,2} \) vanishes at isolated points of \( \mathbb{R}^2 \).

**Remark 3.3** at the isolated points of \( C \) specified in the part (i) of the theorem 3.3, we have:

\[ Q_i \neq 0, i = 0, 1, 2 \]

\[ Q_0 + Q_2 \neq 0 \]

Denoting \( \hat{B}_5 \) the subset of \( B^6 \times S^1 \) (where \( S^1 \) is the unit circle) defined by the following equations:

\[ \beta_{2,2} = 0, P(\varphi) = 0, \frac{dP}{d\varphi}(\varphi) = 0, Q_i(\varphi) = 0 \text{ (resp } Q_0 + Q_2 = 0) \]

Taking into account the remark 2.2, we show as in [1] that the projection \( B_5 \) of \( \hat{B}_5 \) on \( B^6 \) is a semi-algebraic subset of codimension four.

### 4 Stability results

Let \( CL \) be the conjugate locus mapping for a generic element of \( F\text{-Iso}R \) (for the Whitney topology) and \( S \) a neighborhood of the singular locus at the source.

**Theorem 4.1** On the complement of the smooth curve \( C \) (\( \beta_{2,2} \neq 0 \)).

There is a neighborhood \( U \) of \( S \cap \{0 < h < c\} \) (for small enough values of \( c \)), such that:

- The 5-jet of \( CL : CL^5 = f_4h^4 + f_5h^5 \) is a sufficient jet of \( CL \) on \( U \).
- The restriction of the exponential mapping \( \exp |U \) is 5-determined in \( h \).
- \( \exp |U \) is determined by the 5-jet of the metric.

The main argument for the proof of the theorem [1] is the fact that the suspension of the exponential mapping is a “Whitney map”

In fact the intersection \( CL^w \) between \( CL \) and the planes \( w = c \) (for small enough values of \( |w| \)) is a closed curve which has fold-points, cusp-points (four) and without self-intersection (see figures 1 and 2).

Hence the mapping

\[ f_4h^4 + f_5h^5 + O(h^6) \]

is R.L-equivalent to

\[ f_4h^4 + f_5h^5 \]
Figures

\[ CL^5 \text{ when } r_2 \neq 0. \]

\[ CL^\pm_w \text{ (when } r_2 \neq 0) \]
**Theorem 4.2** There is an open dense subset $O$, (complement of a discrete subset) of $C$, such that:

$$\text{Isoself} = \text{Isoself}_c + \text{Isoself}_1$$

- $\text{Isoself}_c$ is the union of three curves.
- $\text{Isoself}_1$ is the union of two or four curves.

All the self-intersections are transversal and are not dependent on $\epsilon$.

There is a neighborhood $U$ of $S \cap \{0 < h < c\}$ (for small enough values of $c$), such that:

The 7-jet of $CL : CL^7 = f_4h^4 + f_6h^6 + f_7h^7$ is a sufficient jet of $CL$ on $U$.

The restriction of the exponential mapping $\exp|U$ is 7-determined in $h$.

At the isolated points of $C$ we have :

**Theorem 4.3** If $P(\varphi)$ has a double root on the circle unity or $P(\varphi) = 0$ and $f_7(\varphi) = 0$, then :

- $\text{Isoself}_c$ is the union of three curves.
- $\text{Isoself}_1$ is the union of two or four curves.

All the self-intersections are transversal and are dependent on $\epsilon$.

There is a neighborhood $U$ of $S \cap \{0 < h < c\}$ (for small enough values of $c$), such that:

The 9-jet of $CL : CL^9 = f_4h^4 + f_6h^6 + f_7h^7 + f_8h^8 + f_9h^9$ is a sufficient jet of $CL$ on $U$.

The restriction of the exponential mapping $\exp|U$ is 9-determined in $h$.

**Theorem 4.4** When there is collision between $\text{Isoself}_c$ and $\text{Isoself}_1$, then :

- $\text{Isoself}_c$ is the union of two curves.
- $\text{Isoself}_1$ is the union of one or two curves.

All the self-intersections are transversal and are not dependent on $\epsilon$.

There is a neighborhood $U$ of $S \cap \{0 < h < c\}$ (for small enough values of $c$), such that:

The 7-jet of $CL : CL^7 = f_4h^4 + f_6h^6 + f_7h^7$ is a sufficient jet of $CL$ on $U$.

The restriction of the exponential mapping $\exp|U$ is 7-determined in $h$.

4.1 Proof of the stability results along the curve $C$

4.1.1 Definition and characterization of the self-intersections of $CL$

Let $\text{Isoself}$ denote the set of self-intersections of $CL$ on $C$, roughly speaking the set of $(h, \varphi_1, \varphi_2)$ such that:

1) $\varphi_1 \neq \varphi_2$,

2) $CL(h, \varphi_1) = CL(h, \varphi_2)$. 

}\end{document}
We are interested in the germ of \( Isoself \) along \( \{ h = 0 \} \). A value of \( \varphi \) is said to be adherent to \( Isoself \) if \( (0, \varphi, \varphi') \) lies in the closure of \( Isoself \) for some \( \varphi' \). The set of some \( \varphi \) is denoted by \( A-Isoself \).

We know that if \( \varphi \in A-Isoself \) then \( \varphi' = \varphi + \pi \in A-Isoself \) and there is not an other possibility (III).

**Lemma 4.1** \( A-Isoself \subset \{ \varphi \mid \Psi(\varphi) = 0 \} \).

Assuming that \( (h, \varphi, \varphi + \pi + \delta) \in A-Isoself \), with \( h > 0 \), and \( \delta \) small. Then:

\[
\sum_{i=4}^{7} f_i(\varphi + \pi + \delta)h^i = \sum_{i=4}^{7} f_i(\varphi)h^i + O(h^8). \tag{6}
\]

Since \( f_4 \) is independent of \( \varphi \) and \( f_5(\varphi) = 0 \) on \( C \), dividing \( 6 \) by \( h^6 \) we obtain:

\[
f_6(\varphi + \delta) - f_6(\varphi) + h (f_7(\varphi + \delta + \pi) - f_7(\varphi + \pi)) + h (f_7(\varphi + \pi) - f_7(\varphi)) + O(h^2) = 0
\]

Using (B.P.) we obtain:

\[
\delta \frac{df_6}{d\varphi}(\varphi) - 2hf_7(\varphi) + O(h^2) + O(h\delta) = 0
\]

Therefore if \( \tilde{\varphi} \in A-Isoself \), then \( \frac{df_6}{d\varphi}(\tilde{\varphi}) \wedge f_7(\tilde{\varphi}) = 0 \); and lemma 4.1 is proved.

**Remark 4.1** As a consequence of this lemma \( A-Isoself \) splits into two subsets:

\( A-Isoself = A-Isoself_c \cup A-Isoself_1 \).

Where \( A-Isoself_c = \{ k\frac{\pi}{3} - \frac{\theta}{3} \} \) is the set of cuspidal angles, and \( A-Isoself_1 \) is the set of the roots of \( P(\varphi) \).

**4.1.2 Characterization of \( Isoself_c \) on generic points of \( C \).**

We have to solve the equation:

\[
\sum_{i=4}^{7} f_i(\varphi + \pi + \delta)h^i = \sum_{i=4}^{7} f_i(\varphi)h^i + O(h^8),
\]

for small enough values of \( \delta \) and \( h \).

We know that this equation is equivalent to:

\[
f_6(\varphi + \delta) - f_6(\varphi) + h (f_7(\varphi + \delta + \pi) - f_7(\varphi + \pi)) + h (f_7(\varphi + \pi) - f_7(\varphi)) + O(h^2) = 0, \tag{7}
\]
For simplicities sake we denote $\frac{d}{d\varphi}$ by $'$; hence $\mathcal{I}$ writes:

$$E_c = 0 = \delta f'_6(\varphi) + \frac{\delta^2}{2} f''_6(\varphi) + \frac{\delta^3}{6} f'''_6(\varphi) - h \delta f'_7(\varphi) - 2h f_7(\varphi) + O(\delta^4) + O(h \delta^2) + O(h^2)$$

By (B.P.) $f'_6(0) = 0$ (since $\theta_3 = 0$). Checking $f'''_6(\varphi) \wedge f''_6(\varphi)$; we obtain:

$$f'''_6(0) \wedge f''_6(0) = -583200 |r_3| \pi^2$$

Hence:

$$E_c = 0 \iff \begin{cases} 
E_c \wedge f''_6(0) = 0 \\
E_c \wedge f'''_6(0) = 0 
\end{cases}$$

Expanding $E_c$ we obtain

$$E_c = \delta f'_6(0) + \delta \varphi f''_6(0) + \frac{\delta^2}{2} \varphi^2 f'''_6(0) + O(\delta \varphi^3) +$$

$$\frac{\delta^2}{2} f''_6(0) + \frac{\delta^2}{2} \varphi f'''_6(0) + O(\delta^2 \varphi^2) + \frac{\delta^2}{6} f''''_6(0) + O(\delta^3) - h \delta f'_7(0) + O(h \delta \varphi)$$

$$-2h f_7(0) - 2h \varphi f'_7(0) + O(h \varphi^2) + O(h \delta^2) + O(h^2) = 0.$$

Hence:

$$E_c \wedge f''_6(0) = \left(\frac{\delta^2}{2} \varphi^2 f'''_6(0) \wedge f''_6(0) + \frac{\delta^2}{2} \varphi f''''_6(0) \wedge f''_6(0) + \frac{\delta^3}{6} f'''''_6(0) \wedge f''_6(0) \right)$$

$$-2h f_7(0) \wedge f''_6(0) + O(h \delta) + O(h \varphi) + O(\delta \varphi^3) + O(\delta^4) + O(\varphi^2 \delta^2) + O(h^2) = 0.$$ (8)

and,

$$E_c \wedge f'''_6(0) = (\delta \varphi + \frac{\delta^2}{2}) f''_6(0) \wedge f''_6(0) +$$

$$O(h) + O(\delta \varphi^2) + O(\delta^3) + O(\delta^2 \varphi) = 0.$$ (9)

We can solve (8) in $h$ (implicit function theorem).

$$h = \frac{1}{2} \frac{f'''_6(0) \wedge f''_6(0)}{f'_7(0) \wedge f''_6(0)} \left( \frac{\delta}{2} \varphi^2 + \frac{\delta^2}{2} \varphi + \frac{\delta^3}{6} \right) + \delta^2 O^2(\delta, \varphi) + O(\delta \varphi^3).$$ (10)

We claim that $f_7(0) \wedge f''_6(0) \neq 0$.

In fact:

$$\frac{d}{d\varphi} (f'_6 \wedge f_7)(0) = f''_6(0) \wedge f_7(0),$$

And zero is not a root of $P(\varphi)$ by theorem 3.2.
Replacing $h$ by its value in \[9\] we obtain:

\[
\begin{cases}
\varphi = -\frac{\delta}{2} + O(\delta^2), \\
h = -\frac{1}{48} f_6''(0) f_7'(0) \delta^3 + O(\delta^4).
\end{cases}
\]

(11)

**Remark 4.2** Since $h$ has to be positive, only one half of the curves defined by \[11\] must be considered: either $\delta > 0$ or $\delta < 0$.

Now let us show that this self-intersection is transversal. For this, the following expression (the transversality-rate) $T(\varphi, h, \delta)$ has to be nonzero on the self-intersection.

\[
T(\varphi, h, \delta) = (f_6'(\varphi + \delta) + hf_7'(\varphi + \delta + \pi)) \wedge (f_6'(\varphi) + hf_7'(\varphi) + O(h^2)).
\]

(12)

Taking into account \[11\], and expanding \[12\], we obtain:

\[
T(\varphi, h, \delta) = -\frac{\delta^3}{24} f_6''(0) \wedge f_6''(0) + O(\delta^4)
\]

(13)

Since,

\[-\frac{\delta^3}{24} f_6''(0) \wedge f_6''(0) = 2h \frac{d}{d\varphi} (f_6' \wedge f_7')(0) \neq 0\]

Hence \[13\] is nonzero.

### 4.1.3 Characterization of Isoself$_1$ on generic points of $C$.

Now we assume that $\theta_3 \neq 0$.

We want to solve again:

\[
f_6(\varphi + \pi + \delta) + f_7(\varphi + \pi + \delta) h = f_6(\varphi) + h f_7(\varphi) + O(h^2),
\]

(14)

for $\varphi$ close to zero, zero being a simple root of $P(\varphi)$, $\delta$ and $h$ small.

Expanding \[14\], we obtain:

\[
0 = \delta f_6'(\varphi) + O(\delta^2) - h f_7(\varphi + \delta) - hf_7(\varphi) + O(h^2)
\]

\[
0 = \delta f_6'(\varphi) - 2h f_7(\varphi) - h (f_7(\varphi + \delta) - f_7(\varphi)) + O(h^2) + O(\delta^2).
\]

(15)

This last expression rewrites:

\[
E_1 = 0 = \delta f_6'(0) + \delta \varphi f_6''(0) + O(\delta \varphi^2) - 2h f_7(0) - 2h \varphi f_7'(0) + O(h^2) + O(\delta^2) + O(h\varphi^2) + O(h\delta),
\]

(16)
Checking \( f'_6(\varphi) \wedge f''_6(\varphi) \) we obtain:

\[
f'_6(\varphi) \wedge f''_6(\varphi) = 32400 |r_3|^2 \pi^2 \sin^2(3\varphi + \theta_3).
\]

Since \( \theta_3 \) has to be nonzero (far from cusp),

\[
f'_6(0) \wedge f''_6(0) \neq 0.
\]

Therefore:

\[
E_1 = 0 \iff \begin{cases} 
E_1 \wedge f''_6(0) = 0 \\
E_1 \wedge f'_6(0) = 0
\end{cases}
\]

Hence:

\[
0 = E_1 \wedge f'_6 = \delta f'_6(0) \wedge f''_6(0) - 2 h f'_7(0) \wedge f''_6(0) + O(h^2)
+ O(\delta^2) + O(h\varphi) + O(h\delta) + O(\delta\varphi)
\]

and

\[
0 = E_1 \wedge f'_6 = \delta \varphi f''_6(0) \wedge f'_6(0) + \frac{\delta^2}{2} f''_6(0) \wedge f'_6(0)
- 2 h f'_7(0) \wedge f'_6(0) - h\delta f'_7(0) \wedge f'_6(0) +
O(\delta^2\varphi) + O(\delta^3) + O(h\varphi^2) + O(h^2\delta) + O(\delta^2\varphi) + O(h^2\delta) + O(h^3)
\]

If \( f'_7(0) \wedge f''_6(0) \neq 0 \), (i.e. \( f'_7(0) \neq 0 \) and \( f'_7(0) = \lambda f''_6(0) \)), then (17) gives us:

\[
\delta = 2 \frac{f'_7(0) \wedge f''_6(0)}{f'_6(0) \wedge f''_6(0)} h + h O^1(\varphi, h) = 2\lambda h + h O^1(\varphi, h), \lambda \neq 0
\]

Replacing \( \delta \) by its value in (18) and dividing by \( h (h > 0) \); \( E_1 \wedge f'_6(0) = 0 \) become:

\[
2\varphi \frac{d}{d\varphi} (f'_6 \wedge f'_7)(0) - 2 f'_7(0) \wedge f'_6(0) + 2\lambda h \frac{d}{d\varphi} (f'_6 \wedge f'_7)(0) + O(h^2) + O(\varphi^2) = 0
\]

Recalling that \( f'_7(0) \wedge f'_6(0) = 0 \), and solving (19) in \( \varphi \); we obtain at the end:

\[
\begin{cases} 
\varphi = -\lambda h + h O(h) \\
\delta = 2\lambda h + h O(h)
\end{cases}
\]

Since \( h > 0 \) the curve defined by (20) is a smooth one starting from zero.

We have to show again that this self-intersection is transversal.
The transversality-rate $T_1(\varphi, h, \delta)$ has to be nonzero on the self-intersection.

$$T_1(\varphi, h, \delta) = (f_6'(\varphi + \delta) + h f_7'(\varphi + \delta + \pi)) \wedge (f_6'(\varphi) + h f_7'(\varphi) + O(h^2)).$$

$$T_1(\varphi, h, \delta) = (f_6'(\varphi) - h f_7'(\varphi) + \delta f_6''(\varphi) + O(\delta^2) + O(\delta h)) \wedge (f_6'(\varphi) + h f_7'(\varphi) + O(h^2)).$$

It is easy to see that:

$$T_1(\varphi, h, \delta) = \frac{d}{d\varphi}(f_6' \wedge f_7) + O(h^2) = O(h^2). \quad (21)$$

Since zero is not a double root of $f_6' \wedge f_7$, then $T(\varphi, h, \delta) \neq 0$ for $\varphi$ close to zero. From now we assume that $f_7(0) = 0$.

Hence:

$$\frac{d}{d\varphi}(f_6' \wedge f_7)(0) = f_6'(0) \wedge f_7'(0) \neq 0.$$  

In this case we will need higher order jet of $CL$: namely $f_9(\varphi)$.

**Remark 4.3** $f_8$ is insufficient because of the fact that $f_8(\varphi + \pi) = f_8(\varphi)$.

We have to solve now:

$$E_1' = \delta f_6'(\varphi) + \frac{\delta^2}{2} f_6''(\varphi) + \frac{\delta^3}{6} f_6'''(\varphi) + O(\delta^4) - 2h f_7(\varphi) - h \delta f_7'(\varphi) +$$

$$h^2 \delta f_6'(0) + O(h^3 \delta) + O(h^2 \delta^2) - 2h^3 f_9(0) + O(h^4) = 0. \quad (22)$$

$$f_6'(0) \wedge f_7'(0) \neq 0.$$

Hence:

$$E_1' = 0 \iff \begin{cases} E_1' \wedge f_7'(0) = 0 \\ E_1' \wedge f_6'(0) = 0 \end{cases}$$

Also:

$$0 = E_1' \wedge f_7 = \delta f_6'(0) \wedge f_7'(0) - 2 h^3 f_9(0) \wedge f_7'(0) + O(h^4) + O(\delta^2) +$$

$$O(\varphi h) + O(h \delta) + O(\delta \varphi). \quad (23)$$

and

$$0 = E_1' \wedge f_6 = -2h \varphi f_7' \wedge f_6' - 2 h^3 f_9(0) \wedge f_6'(0)$$

$$O(h \varphi^2) + O(\delta^2) + O(\delta \varphi) + O(h \delta) + O(h^4) \quad (24)$$
From (23) one obtains:
\[ \delta = 2 \frac{f_9(0) \wedge f_7'(0)}{f_6'(0) \wedge f_7'(0)} h^3 + h^3 O^1(\varphi, h) \]
Replacing \( \delta \) by its value in (24), and dividing by \( h > 0 \), \( E_1' \wedge f_6'(0) = 0 \) become:
\[ -2 \varphi f_7'(0) \wedge f_6'(0) - 2 h^2 f_9(0) \wedge f_6'(0) + O(h^3) + O(\varphi^2) = 0 \]
Hence:
\[
\begin{align*}
\varphi &= A' h^2 + h O^2(h) \\
\delta &= B' h^3 + h O^3(h)
\end{align*}
\]
Again the curve defined by (25) is a smooth one starting from zero.

Showing now that this self-intersection is transversal.

The transversality-rate \( T_2(\varphi, h, \delta) \) has to be nonzero on the self-intersection.
\[ T_2(\varphi, h, \delta) = (f_6'(\varphi + \delta) + h f_7'(\varphi + \delta + \pi)) \wedge (f_6'(\varphi) + h f_7'(\varphi) + O(h^2)). \]
\[ T_2(\varphi, h, \delta) = (f_6'(\varphi) - h f_7'(\varphi) + O(\delta) + O(\delta h)) \wedge (f_6'(\varphi) + h f_7'(\varphi) + O(h^2)). \]
As in the previous case:
\[ T_2(\varphi, h, \delta) = h \frac{d}{d\varphi} (f_6' \wedge f_7) + O(h^2) = O(h^2). \]
Zero is not a double root of \( f_6' \wedge f_7 \), hence \( T_2(\varphi, h, \delta) \neq 0 \) close to zero.

4.1.4 Characterization de Isoself\textsubscript{1} on isolated points of \( C \)

Double roots of \( P(\varphi) \).
We recall hypothesis:

i) \( \frac{d}{d\varphi} (f_6' \wedge f_7)(0) = f_6''(0) \wedge f_7(0) + f_6'(0) \wedge f_7'(0) = 0. \)
ii) \( \theta_3 \neq 0. \)
The equation of the self-intersection is the following:
\[ E_1' = \delta f_6'(\varphi) + \frac{\delta^2}{2} f_6''(\varphi) + \frac{\delta^3}{6} f_6'''(\varphi) + O(\delta^4) - 2 h f_7(\varphi) - h \delta f_7'(\varphi) + h^2 \delta f_8'(0) + O(h^3 \delta) + O(h^2) - 2 h^3 f_9(0) + O(h^4) = 0 \]
Our hypothesis ensure us that:

\[ f'_6(0) \land f''_6(0) \neq 0. \]

Hence:

\[
E^1_1 = 0 \iff \begin{cases} E^1_1 \land f''_6(0) = 0 \\ E^1_1 \land f'_6(0) = 0 \end{cases}
\]

We interest at first to \( E^1_1 \land f''_6(0) = 0 \).

We obtain:

\[
0 = \delta f'_6(0) \land f''_6(0) - 2hf_7(0) \land f''_6(0) + O(\delta^2) + O(\delta \varphi) + O(h^2) + O(h \varphi) + O(h \delta)
\]

Solving this last equation in \( \delta \); one obtains:

\[
\delta = 2 \frac{f'_7(0) \land f''_6(0)}{f'_6(0) \land f''_6(0)} h + hO^1(h, \varphi). \tag{28}
\]

We will distinguish two cases.

1) \( f_7(0) \neq 0 \).

In that case:

\[
f_7(0) = \lambda f'_6(0) \text{ with } \lambda \neq 0.
\]

and

\[
\delta = 2\lambda h + O(h^2) + O(h \varphi). \tag{29}
\]

We will consider now \( E^1_1 \land f'_6(0) = 0 \). Assume that there is at least one solution.

This solution is on the form:

\[
\varphi = \chi h + \chi_2 h^2 + \chi_2 h^3 + O(h^4).
\]

It is easy to see that \( \chi \neq 0 \).

Replacing \( \varphi \) by its value and dividing by \( h^3 \) we obtain:

\[
m \chi^2 + 2\lambda m \chi + n + O(h) = 0.
\]

where,

\[
m = \lambda f'''_6(0) \land f'_6(0) + f'_6(0) \land f''_7(0)
\]

\[
n = \frac{4\lambda}{3} f''_6(0) \land f'_6(0) - 2\lambda f'_8(0) \land f'_6(0) - 2\lambda f'_9(0) \land f'_6(0)
\]

By remark \( 3.3 \) we have the following important fact : \( n \neq 0 \).
Hence: \( m \neq 0 \).

Denoting \( \chi_1 \) and \( \chi_2 \) two presumed solutions of

\[
m\chi^2 + 2\lambda m \chi + n = 0. \tag{30}
\]

Then:

\[
\chi_1 + \chi_2 = -2\lambda \tag{31}
\]

If \( \chi_1 = \chi_2 = \chi_0 = -\lambda \), then \(-m\lambda^2 + n = 0\). This is impossible by remark \( \text{3.3} \) once more.

Therefore \( \text{30} \) has not a double root on \( \mathbb{R} \).

Hence in this case there is either no self-intersection or two self-intersections which satisfy the following equations:

\[
\delta = 2\lambda h + O(h^2)
\]

\[
\varphi = \chi h + O(h^2) \text{ where, } \chi = \chi_i \text{ et } i = 1, 2
\]

It remains to show that these self-intersections are transversal.

Denoting by \( T^1_1(\varphi, h, \delta) \) the transversality-rate, we have as usually

\[
T^1_1(\varphi, h, \delta) = (f_6'(\varphi + \delta) + h f_7'(\varphi + \delta + \pi)) \wedge (f_6'(\varphi) + h f_7'(\varphi) + O(h^2)).
\]

Expanding \( T^1_1(\varphi, h, \delta) \), we obtain:

\[
T^1_1(\varphi, h, \delta) = 2h^2(\chi + \lambda) \frac{d^2}{d\varphi^2}(f_6' \wedge f_7')(0) + O(h^3). \tag{32}
\]

Since \((\chi + \lambda) \neq 0\), then \( T^1_1(\varphi, h, \delta) \neq 0 \) on our two self-intersections, hence they are well transversal.

2) \( f_7(0) = 0 \)

Now \( \lambda = 0 \) and \( f_6'(0) \wedge f_7'(0) = 0 \).

Set \( f_7'(0) = \gamma f_6'(0) \)

**Remark 4.4** Since zero is not a double root of \( P(\varphi) \), then \( \gamma \neq 0 \).

Taking again \( E^1_1 \wedge f_6''(0) \), i.e.:

\[
0 = \delta f_6'(0) \wedge f_6''(0) - 2h \varphi f_7'(0) \wedge f_6''(0) + O(\delta^2) + O(\delta \varphi^2) + O(h^3) + O(h \varphi^2) + O(h \delta).
\]

We obtain:
\[ \delta = 2\gamma h\varphi + (h\varphi^2) + O(h^3) \]

Replacing once more \( \delta \) by its value in \( E_1^1 \land f_6'(0) = 0 \), with the hypothesis:

\[ \varphi = \chi h + O(h^2). \]

Dividing by \( h^3 \) we obtain:

\[ m\chi^2 - 2f_9(0) \land f_6'(0) + O(h) = 0 \]

where

\[ m = 2f_6''(0) \land f_7'(0) + f_6'(0) \land f_7''(0) = \frac{d}{d\varphi^2}(f_6' \land f_7)(0) \neq 0. \]

Thus; \( \chi = \pm \sqrt{\kappa} \) where \( \kappa = \frac{2f_9(0) \land f_6'(0)}{m} \neq 0. \)

Therefore if \( \kappa < 0 \) there is no self-intersection and if \( \kappa > 0 \), there are two self-intersections which satisfy the following equations:

\[ \delta = 2\gamma \chi h^2 + O(h^2) \]

\[ \varphi = \pm \chi h + O(h^2) \]

As usually we will show that these two self-intersections are transversal.

Denoting by \( T_1^2(\varphi, h, \delta) \) the transversality-rate; we have:

\[ T_1^2(\varphi, h, \delta) = (f_6'(\varphi + \delta) + h f_7'(\varphi + \delta + \pi)) \land (f_6'(\varphi) + h f_7'(\varphi + O(h^2))). \]

Expanding this last expression, we obtain:

\[ T_1^2(\varphi, h, \delta) = \pm 2\chi h^2 \frac{d^2}{d\varphi^2}(f_6' \land f_7)(0) + O(h^3). \] (33)

Taking account our hypothesis, we know that:

\[ 2\chi h^2 \frac{d^2}{d\varphi^2}(f_6' \land f_7)(0) \neq 0. \]

**Collision between isoself\( f_1 \) and isoself\( f_6 \).**

Now, zero is a cusp point and a simple root of \( P(\varphi) \).

Hence

\[ f_6'(0) = 0 \text{ et } \frac{d}{d\varphi}(f_6' \land f_7)(0) = f_6''(0) \land f_7(0) = 0. \]

In compensation

\[ \frac{d^2}{d\varphi^2}(f_6' \land f_7)(0) = f_6'''(0) \land f_7(0) + 2 f_6''(0) \land f_7'(0) \neq 0. \] (34)
and
\[ f_6'''(0) \land f_6''(0) \neq 0. \]

Again, we will distinguish two cases.

1) \( f_7(0) \neq 0 \).
In this case, there exist \( \lambda \neq 0 \) such that \( f_7(0) = \lambda f_6''(0) \).
The equation of the self-intersection is:
\[
E^1_c = \delta f_6'(\varphi) + \frac{\varphi^2}{2} f_6''(\varphi) + \frac{\delta^4}{6} f_6'''(\varphi) - h \delta f_7(\varphi) - 2h f_7(\varphi) + O(\delta^4) + O(h \delta^2) + O(h^2) = 0. \tag{35}
\]

\[
E^1_c = 0 \iff \begin{cases} 
E^1_c \land f_6''(0) = 0 \\
E^1_c \land f_6''(0) = 0 
\end{cases}
\]

\[
E^1_c = \delta f_6'(0) + \frac{\varphi^2}{2} f_6''(0) + \frac{\delta^2}{2} \varphi f_6'''(0) + O(\delta^3) + \\
\frac{\delta^2}{2} f_6''(0) + \frac{\varphi^2}{2} f_6''(0) + O(\delta^2) + \frac{\delta^3}{6} f_6'''(0) + O(\delta^4) - h \delta f_7(0) + O(h \delta \varphi) \\
-2h f_7(0) - 2h \varphi f_7'(0) + O(h \varphi^2) + O(h \delta^2) + O(h^2) = 0.
\]

\( f_6'(0) = 0 \), Therefore:
\[
E^1_c \land f_6''(0) = \frac{\delta^2}{2} \varphi f_6'''(0) \land f_6''(0) + \frac{\varphi^2}{2} f_6''(0) \land f_6''(0) + \frac{\delta^3}{6} f_6'''(0) \land f_6''(0) \\
-2h f_7(0) \land f_6''(0) - 2h(\varphi + \frac{\delta}{2}) f_7'(0) \land f_6''(0) + \\
O(h \varphi^2) + O(\delta \varphi^3) + O(\delta^4) + O(\varphi^2 \delta^2) + O(h^2) = 0,
\]

and
\[
E^1_c \land f_6''(0) = (\delta \varphi + \frac{\delta^2}{2} - 2\lambda h) f_6''(0) \land f_6''(0) \\
O(h \varphi) + O(\delta \varphi^2) + O(\delta^3) + O(h^2) + O(h \delta) = 0. \tag{37}
\]

We can solve (37) in \( h \).
\[
h = \frac{\delta}{2\lambda} (\varphi + \frac{\delta}{2}) + O(\delta \varphi^2) + O(\delta^3) \tag{38}
\]

Taking account the previous characterization of \( \text{isoself}_c \) and \( \text{isoself}_1 \), one can assume that:
\[
\varphi = \frac{-\delta}{2} + O(\delta^2).
\]
Remark 4.5 If \( \varphi \neq -\frac{\delta}{2} \), then \( h = h(\delta^2) + O(\delta^3) \). It is easy to see that this case is not generic.

Replacing \( \varphi \) by \(-\frac{\delta}{2}\) in 38 we obtain \( h = h(\delta^3) \).

The equation 38 become:

\[
\frac{\delta^2}{24} f_6''(0) \wedge f_6'''(0) + O^3(\delta) = 0.
\]

(39)

Which is not possible because \( f_6''(0) \wedge f_6'''(0) \neq 0 \).

Hence in that case there is no self-intersection.

4.2 The stability

In this case the suspension of the exponential mapping is not a “Whitney map”. The main argument here is the Mather theorem.

If \( \beta_{2,2} = 0 \), then \( \beta_{3,3} \) (or \( r_3 \)) and \( \mu \) are nonzero. The intersection \( CL^\pm_w \) between \( CL \) and the planes \( w = c \) (for small enough values of \( |w| \) ) is a closed curve having fold-points, cusp-points (six) and self-intersections (see figures).

According to our previous subsection “characterization of self-intersection on \( C \)” (Which finds here its justification), all the generic self-intersections of \( CL^\pm_w \) are transversal.

Thus \( CL \) is in “general position “, which ends the proofs of theorems 4.2, 4.3 and 4.4.

Figures.

\[
CL^7 : \text{when } r_2 = 0 \text{ and } r_3 \neq 0
\]
$CL_w^\pm$ (an example with 7-isodeself)

$CL_w^\pm$ (an example with 5-isodeself)
5 Appendix

5.1 Figures: wave front and conjugate loci

The waves front when $r_2 \neq 0$.

Zoom on the zone C
The waves front when $r_2 = 0$ and $r_3 \neq 0$.

Zoom on the zone C.
5.2 Computing the exponential mapping

Geodesics are trajectories of the Hamiltonian vector field $H$ associated to the Hamiltonian $\mathcal{H}(\psi)$ on $T^*\mathcal{M}$ ($\pi : T^*\mathcal{M} \rightarrow \mathcal{M}$)

$$\mathcal{H}(\psi) = \frac{1}{2} (\psi(F)^2 + \psi(G)^2).$$

The metric $(F, G)$ is in normal form (coordinates in $\mathcal{M} : \xi = (x, y, w) = (z, w))$

$$F = (1 + y^2 \beta) \frac{\partial}{\partial x} - xy \beta \frac{\partial}{\partial y} + \frac{y}{2}(\gamma) \frac{\partial}{\partial w},$$

$$G = (1 + x^2 \beta) \frac{\partial}{\partial y} - xy \beta \frac{\partial}{\partial x} - \frac{x}{2}(\gamma) \frac{\partial}{\partial w},$$

$$\beta(0, 0) = b_0,$$

$$\gamma = (1 + (x^2 + y^2)\beta) \int_0^1 \frac{2t dt}{1 + t^2(x^2 + y^2)\beta(x,y)}.$$

Coordinates in the cotangent bundle are $(\tilde{p}, \tilde{q}, r) = (\tilde{\zeta}, r)$. $(x, y, w, \tilde{p}, \tilde{q}, r)$ have weight $1, 1, 2, -1, -1, -2$ respectively.

$$\mathcal{H}^{-1}\left(\frac{1}{2}\right) \cap \pi^{-1}(0) = \{\tilde{p}^2(0) + \tilde{q}^2(0) = 1\}.$$

We set $\tilde{p}(0) = \cos \varphi, \tilde{q}(0) = \sin \varphi$. For $r(0) \neq 0$, we set $\rho = \frac{1}{r(0)}, p = \frac{\tilde{p}}{r}, q = \frac{\tilde{q}}{r}, \zeta = (p, q)$. $s$ denotes the arclength and $t$ the new time : $dt = r(s) ds$. $p, q$ have weight 1.

One has, for $r(0) \neq 0$ and $s$ small:

$$\frac{dz}{ds} = r(s) \frac{\partial \mathcal{H}}{\partial \zeta}|_{r=1}, \quad \text{and} \quad \frac{d}{ds} \left(\frac{\tilde{\zeta}}{r}\right) = -r(s) \frac{\partial \mathcal{H}}{\partial z}|_{r=1} \quad (40)$$

Or,

$$\frac{dz}{dt} = \frac{\partial \mathcal{H}}{\partial \zeta}|_{r=1}, \quad \frac{d\tilde{\zeta}}{dt} = -\frac{\partial \mathcal{H}}{\partial z}|_{r=1} \quad (41)$$

For all $k$ (11) can be rewritten:

$$\frac{d(z, \zeta)}{dt} = A(\zeta, \zeta) + \sum_{i=3}^{k} F_i(z, \zeta, w) + O^{k+1}(z, \zeta, w), \quad (42)$$

where $F_i$ is homogeneous of degree $i$, where $A$ is a linear operator (corresponding to the Heisenberg sub-Riemannian metric), and where $O^{k+1}(z, \zeta, w)$ has order $(k + 1)$ with respect to the gradation:

$$\deg x = \deg y = \deg p = \deg q = 1, \quad \deg w = 2.
Also, \( \frac{dw}{ds} = \frac{\partial H}{\partial r} = r(s) \frac{\partial H}{\partial r} \bigg|_{r=1} \),
\[
\frac{dw}{dt} = \frac{\partial H}{\partial r} \bigg|_{r=1}.
\]

This can be rewritten:
\[
\frac{dw}{dt} = G_2(z, \zeta) + \sum_{i=4}^{k} G_i(z, \zeta, w) + O^{k+1}(z, \zeta, w),
\]
where \( G_i \) are homogeneous of degree \( i \), where \( G_2 \) corresponds to the Heisenberg sub-Riemannian metric, and where \( O^{k+1}(z, \zeta, w) \) has order \((k+1)\) w.r.t. the gradation.

Initial conditions are
\[
z(0) = 0, \ w(0) = 0, \ \zeta(0) = (\rho \cos \varphi, \rho \sin \varphi).
\]

Therefore,
\[
\begin{align*}
(z, \zeta) &= \rho (z_1(t, \varphi), \zeta_1(t, \varphi)) + \sum_{i=3}^{k} \rho^i (z_i(t, \varphi), \zeta_i(t, \varphi)) + O(\rho^{k+1}), \\
w &= \rho^2 w_2(t, \varphi) + \sum_{i=4}^{k} \rho^i w_i(t, \varphi) + O(\rho^{k+1}).
\end{align*}
\]

\[
(z_1, \zeta_1)(t, \varphi) = e^{A t} (z(0), \zeta(0)), \ w_2(t, \varphi) = \int_{0}^{t} G_2(z_1(\tau, \varphi), \zeta_1(\tau, \varphi)) \, d\tau.
\]

These last expressions can be easily computed. They give the exponential mapping for the Heisenberg metric:
\[
\begin{align*}
z_1(t, \varphi) &= 2 \sin \left( \frac{t}{2} \right) \left( \cos(\varphi - \frac{t}{2}), \sin(\varphi - \frac{t}{2}) \right), \\
\zeta_1(t, \varphi) &= \cos \left( \frac{t}{2} \right) \left( \cos(\varphi - \frac{t}{2}), \sin(\varphi - \frac{t}{2}) \right), \\
w_2(t) &= \frac{1}{2} (t - \sin t).
\end{align*}
\]

Also \( F_3 \) and \( G_4 \) don’t depend on \( w \), hence, setting \( \Lambda = (z, \zeta) \),
\[
\begin{align*}
\Lambda_3(t, \varphi) &= (z_3, \zeta_3)(t, \varphi) = \int_{0}^{t} e^{A (t-\tau)} F_3(\Lambda_1(\tau, \varphi)) \, d\tau, \\
w_4(t, \varphi) &= \int_{0}^{t} \left( \frac{\partial G_4}{\partial \Lambda}(\Lambda_1(\tau, \varphi)) \cdot \Lambda_3(\tau, \varphi) + G_5(\Lambda_1(\tau, \varphi)) \right) \, d\tau.
\end{align*}
\]

The following terms are easily computed, on the same way. We give the expressions that we shall need, \( F_4, \ G_5 \) don’t depend on \( w \),
\[
\begin{align*}
\Lambda_4(t, \varphi) &= (z_4, \zeta_4)(t, \varphi) = \int_{0}^{t} e^{A (t-\tau)} F_4(\Lambda_1(\tau, \varphi)) \, d\tau, \\
w_5(t, \varphi) &= \int_{0}^{t} \left( \frac{\partial G_5}{\partial \Lambda}(\Lambda_1(\tau, \varphi)) \cdot \Lambda_4(\tau, \varphi) + G_5(\Lambda_1(\tau, \varphi)) \right) \, d\tau.
\end{align*}
\]

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As we shall see, we will need to compute these values for $t = 5.3$. The exponential mapping in suspended form, the conjugate fact:

$$w(t, \varphi) = \int_0^1 \frac{1}{2} \frac{\partial^2 G_{t\varphi}}{\partial \Lambda} \cdot (A_3(\tau, \varphi), A_3(\tau, \varphi)) + G_6(\Lambda_1(\tau, \varphi), w_2(\tau)) + \frac{\partial F_t}{\partial \Lambda}(\Lambda_1(\tau, \varphi)) \cdot A_3(\tau, \varphi) d\tau.$$ 

Therefore, we obtain the following important fact:

$$w(t, \varphi) = \int_0^1 \frac{1}{2} \frac{\partial^2 G_{t\varphi}}{\partial \Lambda} \cdot (A_3(\tau, \varphi), A_3(\tau, \varphi)) + G_6(\Lambda_1(\tau, \varphi), w_2(\tau)) + \frac{\partial F_t}{\partial \Lambda}(\Lambda_1(\tau, \varphi)) \cdot A_3(\tau, \varphi) d\tau.$$ 

As we shall see, we will need to compute these values for $t = 2\pi$ only.

### 5.3 The exponential mapping in suspended form, the conjugate loci

$$E(t, \varphi) :$$

$$w(t, \rho, \varphi) = \rho^2 w_2(t) + \sum_{i=4}^6 \rho^i w_i(t, \varphi) + O(\rho^7).$$

Integral expressions of $z_i, w_i$ were computed in our appendix 5.2.

Let us consider the variable $\epsilon = \pm 1$, according to $w > 0$ or $w < 0$. Let us set $t = 2\pi \epsilon + \tau$ (the conjugate new time has an expansion $t_e = 2\pi \epsilon + O(\rho^2)$, as is shown in our previous papers 3, and this will appear again here in). Therefore, $\tau_e$, the conjugate time $\tau$, has order $\rho^2$.

$$w_2(t) = \frac{1}{2} (t - \sin t). \quad w_2(\tau) = w_2^\tau(\tau) = \pi \epsilon + \frac{1}{2} (\tau - \sin \tau),$$

hence, $w_2^\tau(0) = \pi \epsilon$, $w_2'(0) = 0$, $w_2''(0) = 0$. Therefore, we obtain the following important fact:

$$w_2(\tau) = \pi \epsilon + O(\tau^3) = w_2^\tau(0) + \tau^3 \psi(\tau),$$

for some smooth function $\psi$. 

33
Let us set:

$$h = \sqrt{\epsilon w \frac{1}{\pi}}. \quad \text{(54)}$$

$$w(\tau, \rho, \varphi) = \pi \epsilon \rho^2 (1 + O(\tau^3)) + \frac{1}{\pi \epsilon} (\rho^2 w_4(\tau, \varphi) + \rho^3 w_5(\tau, \varphi) + \rho^4 w_6(\tau, \varphi) + \rho^5 w_7(\tau, \varphi) + \rho^6 w_8(\tau, \varphi) + O(\rho^7)).$$

$$h = \rho (1 + O(\tau^3) + \frac{1}{\pi \epsilon} (\rho^2 w_4(\tau, \varphi) + \rho^3 w_5(\tau, \varphi) + \rho^4 w_6(\tau, \varphi) + O(\rho^5))^\frac{1}{2}. \quad \text{(55)}$$

Using the implicit function theorem, we can solve this last equation in $\rho$. After straightforward computations, we obtain:

$$\rho = h (1 - h^2 \epsilon w(0) \frac{1}{2\pi} - h^3 \epsilon w_5(0) \frac{1}{2\pi} + h^4 \frac{7w_4^2(0)}{8\pi^2} - \epsilon w_6(0) \frac{1}{2\pi} - w_7'(0) h^2 \frac{1}{2\pi} + O(\tau^3) + O(h^5) + O(h^2\tau^2) + O(h^3\tau)). \quad \text{(55)}$$

Let us set now

$$\tau = \theta h^2 + \sigma h^3; \quad \text{(56)}$$

for some constant $\theta$.

$$\rho = h (1 - h^2 \epsilon w_4(0) \frac{1}{2\pi} - h^3 \epsilon w_5(0) \frac{1}{2\pi} + h^4 \frac{7w_4^2(0)}{8\pi^2} - \epsilon w_6(0) \frac{1}{2\pi} - w_7'(0) h^2 \frac{1}{2\pi} + \epsilon \theta h^4 + c_5 h^5 + c_6 h^6 + O(h^7)). \quad \text{(57)}$$

where:

$$c_5 = -\epsilon w_3(0) \frac{1}{2\pi} + \text{terms depending on the invariants } \beta_i, \ (i \leq 2)$$

$$c_6 = -\epsilon w_3(0) \frac{1}{2\pi} + \text{terms depending on the invariants } \beta_i, \ (i \leq 3)$$

Otherwise,

$$z(\tau, \rho, \varphi) = \rho z_1(\tau, \varphi) + \sum_{i=3}^{9} \rho^i z_i(\tau, \varphi) + O(\rho^8). \quad \text{(58)}$$

We set

$$z_i^{(k)} = \frac{dz_i}{d\tau^k}, \quad w_i^{(k)} = \frac{dw_i}{d\tau^k}.$$ 

We also denote $z_i'$ for $z_i^{(1)}$ etc.
Replacing (56), (57), in this expression (58), we obtain, after tedious computations:

\[ z(\sigma, h, \varphi) = \sum_{i=3}^{9} A_i h^i + O(h^{10}), \]  

(59)

\[
\begin{align*}
A_3 &= (\theta' z'_1 + z_3)|_{t=2\pi \epsilon}, \quad A_4 = (\sigma z'_1 + z_4)|_{t=2\pi \epsilon}, \\
A_5 &= \left(\frac{\pi}{2\pi}(\theta w_4 z'_1 + 3 w_4 z_3) + \frac{\theta^2}{2} z'_1 + \theta z'_3 + z_5\right)|_{t=2\pi \epsilon} \\
A_6 &= \frac{1}{2}\epsilon \sigma w_4 z'_1 + \epsilon \theta w_5 z'_1 + 2\pi \sigma \theta z'_1 - 3\epsilon w_5 z_3 + 2\pi \sigma z'_3 - 4\epsilon w_4 z_4 + 2\pi \theta z'_4 + 2\pi z_6)|_{t=2\pi \epsilon} = \sigma A^1_6 + A^0_6, \\
A_7 &= \sigma^2 A^2_7 + \sigma A^1_7 + A^0_7, \\
A^2_7 &= \left(\frac{1}{2} z'_1\right)|_{t=2\pi \epsilon}, \quad A^1_7 = \frac{1}{2\pi}(\epsilon w_5 z'_1 + 2\pi z'_4)|_{t=2\pi \epsilon} \\
A^0_7 &= \left(\frac{\pi}{2\pi}(\theta^2 w_4^2 - \epsilon \theta w_6) z'_1 - \frac{\pi}{2} \theta^2 w_4 z'_1 + \theta^3 z'_1 + 3\pi \theta^3 z'_1 + 3\pi \theta w_4 z_3 + \pi \theta^2 z_3 - 4\epsilon w_5 z_4 + 5\epsilon w_4 w_5 + 2\pi \theta z'_5 + 2\pi z_7)\right)|_{t=2\pi \epsilon}.
\end{align*}
\]

(60)

As we know from [9], the conjugate time \( \tau_c \) is obtained for

\[ \theta = -6\pi \epsilon b_0 \]  

(61)

To see this it is sufficient to compute \( \theta \) for

\[ 0 = z'_1(0, \varphi) \land (\theta \frac{\partial z'_1(0, \varphi)}{\partial \varphi} + \frac{\partial z_1(0, \varphi)}{\partial \varphi}) = 1, \quad z'_1(2\pi \epsilon, \varphi) \land (\theta \frac{\partial z'_1(2\pi \epsilon, \varphi)}{\partial \varphi} + \frac{\partial z_1(2\pi \epsilon, \varphi)}{\partial \varphi}) = 6\pi \epsilon b_0. \]

but, \( z'_1(2\pi \epsilon, \varphi) \land \frac{\partial z'_1(2\pi \epsilon, \varphi)}{\partial \varphi} = 1, \quad z'_1(2\pi \epsilon, \varphi) \land \frac{\partial z'_3(2\pi \epsilon, \varphi)}{\partial \varphi} = 6\pi \epsilon b_0. \)

Also, we have:

\[ -6\pi \epsilon b_0 z'_1(2\pi \epsilon, \varphi) + z'_3(2\pi \epsilon, \varphi) = 0, \]

as it is easily checked.

Therefore, we have the expression of the exponential mapping, in suspended form, in a certain neighborhood \( U \) of the conjugate locus at the source, for \( h \) (or \( \rho \)) small enough

\[ z(\sigma, h, \varphi) = \sum_{i=4}^{9} A_i h^i + O(h^{10}), \]  

(62)

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where \( A_i \) are given in (60).

We have to compute the expression of the conjugate time \( \sigma_c \) in terms of \( h \) and \( \varphi \). For this, we have to solve the following equation in \( \sigma \):

\[
\frac{\partial z}{\partial \sigma}(\sigma, h, \varphi) \wedge \frac{\partial z}{\partial \varphi}(\sigma, h, \varphi) = 0
\]  

(63)

Solving this equation in \( \sigma \), with the implicit function theorem gives:

\[
\sigma_c = \sum_{i=0}^{5} h^i \sigma_c^i.
\]

**Remark 5.1**  
\( \sigma_c^4 = -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_0}{\partial \varphi}) + \text{terms depending on the invariant } \beta_i, (i \leq 4) \) and \( \sigma_c^5 = -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_0}{\partial \varphi}) + \text{terms depending on the invariant } \beta_i, (i \leq 5) \). We don’t need the complete expression of those two coefficients.

Now we give the expression of \( \sigma_c^i \) (\( i = 1, 2, 3 \)).

\[
\begin{align*}
\sigma_c^0 &= -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_0}{\partial \varphi}), & \sigma_c^1 &= -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_1}{\partial \varphi}) \\
\sigma_c^2 &= -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_2}{\partial \varphi}) - (A_6 \wedge \frac{\partial A_1}{\partial \varphi})(2\pi\epsilon) + (z_1'(2\pi\epsilon) \wedge \frac{\partial A_3}{\partial \varphi})(2\pi\epsilon) \\
\sigma_c^3 &= -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_4}{\partial \varphi}) - (A_6 \wedge \frac{\partial A_2}{\partial \varphi})(2\pi\epsilon) - (A_7 \wedge \frac{\partial A_1}{\partial \varphi})(2\pi\epsilon) - (z_1'(2\pi\epsilon) \wedge \frac{\partial A_2}{\partial \varphi})(2\pi\epsilon) \\
\sigma_c^4 &= -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_3}{\partial \varphi}) - (2 A_2 \wedge \frac{\partial A_1}{\partial \varphi})(2\pi\epsilon) + z_1'(2\pi\epsilon) \wedge \frac{\partial A_4}{\partial \varphi} + z_1'(2\pi\epsilon) \wedge \frac{\partial A_5}{\partial \varphi} \\
\sigma_c^5 &= -(z_1'(2\pi\epsilon) \wedge \frac{\partial A_4}{\partial \varphi}) - (A_6 \wedge \frac{\partial A_3}{\partial \varphi})(2\pi\epsilon) + (z_1'(2\pi\epsilon) \wedge \frac{\partial A_5}{\partial \varphi})(2\pi\epsilon) + 2 A_2 \wedge \frac{\partial A_4}{\partial \varphi} + 2 A_2 \wedge \frac{\partial A_5}{\partial \varphi}
\end{align*}
\]

(64)

It remains to replace the expression of \( \sigma_c \) in the expression of \( z \) in (62), to obtain the expansion of the conjugate locus :

\[
z_c(h, \varphi) = z(\sigma_c, h, \varphi) = \sum_{i=4}^{9} f_i(\varphi) h^i + O(h^8).
\]

The expressions of the \( f_i \) (for \( i \leq 4 \leq 6 \) and \( \epsilon = 1 \)) are given in (72). If we denote by \( f_i^- \) the \( f_i \) for \( \epsilon = -1 \), we obtain :

\[
f_i^- = -f_i \text{ (for } 4 \leq i \leq 6)
\]

The expressions of \( f_7 \) can be computed in the same way, just replacing (64) in (72). This has been done with Mathematica. Let :

\[
\begin{align*}
\beta_4 &= L_{44} (x^2 + y^2)^2 + a_{44} (x^4 + y^4 - 6 x^2 y^2) + 4 b_{44} x y (x^2 - y^2) + c_{44} (x^4 - y^4) - 2 d_{44} x y (x^2 + y^2).
\end{align*}
\]

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Assuming \( r_2 = 0 \), we obtain:

1) \( f_7 = 3\pi(-21c_{44}\cos(\varphi) + 35a_{44}\cos(3\varphi) - 7c_{44}\cos(3\varphi) + 21a_{44}\cos(5\varphi) + \\
3r_1^2\cos(\varphi - 2\theta_1) + r_1^2\cos(3\varphi - 2\theta_1) + 21d_{44}\sin(\varphi) - 35b_{44}\sin(3\varphi) + \\
7d_{44}\sin(3\varphi) - 21b_{44}\sin(3\varphi - 2\theta_1) - 12\pi r_1^2\sin(\varphi - 2\theta_1) - 12\pi r_1^2\sin(3\varphi - 2\theta_1)), \\
3\pi(21d_{44}\cos(\varphi) - 35b_{44}\cos(3\varphi) - 7d_{44}\cos(3\varphi) + 21b_{44}\cos(5\varphi) - \\
12\pi r_1^2\cos(\varphi - 2\theta_1) + 12\pi r_1^2\cos(3\varphi - 2\theta_1) + 21c_{44}\sin(\varphi) - 35a_{44}\sin(3\varphi) - \\
7c_{44}\sin(3\varphi) + 21a_{44}\sin(5\varphi) - 3r_1^2\sin(\varphi - 2\theta_1) + r_1^2\sin(3\varphi - 2\theta_1))
\)

2) \( f_7 + f_7 = -144\pi^2\sin(2(\varphi - \theta_1))(\cos(\varphi), \sin(\varphi)) = d_7. \)

It is not necessary to compute the expression of \( f_8 \), the only thing we need is to know that

\[
\frac{d}{d\varphi}(\varphi + \pi) = f_8(\varphi). 
\]

As we know, this is stated in [2], and we verify it by Mathematica.

For \( f_9 \) the term \( z_9 \) appears in \( A_0^6 \) (formula (60)), but also, it comes through the term \( A_1^4 h^4 \) of \( z(\sigma, h, \varphi) \) in (60) :

\[
A_1^4 = (\sigma z_1')(2\pi \epsilon).
\]

Therefore, the expression of \( f_9 \) contains also the term

\[
T = \sigma_1^5 z_1'(2\pi \epsilon) h^9.
\]

This term will not play any role : the only thing that we need is to prove the remark 3.3.

\[
\frac{\partial f_6}{\partial \varphi} = -180 |r_3| \pi \sin(3\varphi + \theta_3) (\cos \varphi, \sin \varphi), \quad \frac{\partial f_6}{\partial \varphi} = (\cos \varphi, \sin \varphi). 
\]

Hence, \( \frac{\partial f_6}{\partial \varphi} \wedge f_9(\varphi) = \frac{\partial f_6}{\partial \varphi} \wedge z_9(2\pi \epsilon) + \) other terms that depend on the invariants \( \beta_i \), \((i \leq 5)\).

We have:

\[
\frac{\partial f_6}{\partial \varphi} \wedge z_1(2\pi \epsilon) = \frac{\partial f_6}{\partial \varphi} \wedge \int_0^{2\pi \epsilon} e^{A(t - \tau)} F_9(\Lambda_1(\tau, \varphi), w_2(\tau)) d\tau + \\
\text{terms not depending on } \beta_6
\]

This last expression has been computed by Mathematica, and we don’t give it here.

We will give now the expression of the polynomial \( P(\varphi) \) in the section 2.2.
Taking in account \( \ref{67} \) and \( \ref{65} \) we see that \( \frac{\partial f_6}{\partial \varphi} \wedge d \tau = 0 \). Therefore
\[
\frac{\partial f_6}{\partial \varphi} \wedge f_7(\varphi) = -\frac{\partial f_6}{\partial \varphi} \wedge f_7^-(\varphi).
\]

Hence (up to sign) the polynomial \( P(\varphi) \) does not depend on \( \epsilon \). Using Mathematica again we obtain:

\[
P(\varphi) = -7d_{44}\cos(2\varphi) + 7b_{44}\cos(4\varphi) - 7c_{44}\sin(2\varphi) + 7a_{44}\sin(4\varphi) + \]
\[
r_1^2 \sin(2(\varphi - \theta_1)).
\]

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