Quantization of semi-classical twists and noncommutative geometry

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Abstract

A problem of defining the quantum analogues for semi-classical twists in $U(g)[[t]]$ is considered. First, we study specialization at $q = 1$ of singular coboundary twists defined in $U_q(g)[[t]]$ for $g$ being a nonexceptional Lie algebra, then we consider specialization of noncoboundary twists when $g = sl_3$ and obtain $q$–deformation of the semi-classical twist introduced by Connes and Moscovici in noncommutative geometry.

Keywords: Noncommutative geometry, Hopf algebras

1 Introduction

Hopf algebras play an increasingly important role in noncommutative geometry and Quantum Field Theory. One of the sources for producing new types of Hopf algebras is twisting, a deformation of the coalgebraic structure of a given Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$ preserving the algebraic structure $(H, \mu, \eta)$. Such deformations are generated by the twisting elements (twists) $F \in (H \otimes H)$ satisfying the conditions

$$\begin{align*}
F_{12}(\Delta \otimes \text{id})(F) &= F^{23}(\text{id} \otimes \Delta)(F) \\
(\epsilon \otimes \text{id})(F) &= (\text{id} \otimes \epsilon)(F) = 1
\end{align*}$$

(1)

that guarantee $H^F \equiv (H, \mu, \eta, \text{Ad} F \circ \Delta, \epsilon, S)$ is a new Hopf algebra. In fact, when $H$ is not finite dimensional, $F$ is usually defined in some completion of the tensor product and $H$ is understood to be a topological Hopf algebra.

In this article we consider two types of twists: the semi-classical ones if $H = U(g)[[t]]$ and the quantum ones if $H = U_q(g)[[t]]$. Some of the semi-classical deformations such as those defined by the Jordanian twists [5] [10] appear as the limiting cases of the quantum ones (in the sense that specialization

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at $q = 1$ is extended to work for topological Hopf algebras). It was a motivation for us to study the quantum twists as many computational problems involved into a direct check of (1) drastically resolve when one works with $U_q(\mathfrak{g})[[t]]$ instead of $U(\mathfrak{g})[[t]]$ and thus the quantum twists is a source for many universal deformation formulas in the sense of [5].

The work is organized as follows. After preliminary section intended to fix notations, we show that if $\mathfrak{g}$ is a nonexceptional simple Lie algebra, then a quantum analogue of the Jordanian twist can be taken to be a coboundary twist in $U_q(\mathfrak{g})[[t]]$:

$$\mathcal{J}(e_\lambda) := (W \otimes W)\Delta(W^{-1}), \text{ where } W = \exp_{q_\lambda} \left( \frac{t}{1 - q_\lambda} e_\lambda \right); \quad q_\lambda := q^{(\lambda, \lambda)}$$

with $e_\lambda$ being a quantum highest root generator in some quantum Cartan-Weyl basis. We prove that $\mathcal{J}(e_\lambda)$ is nonsingular and specializes to a nontrivial twisting of $U(\mathfrak{g})[[t]]$. As an application of the Jordanian twists [5, 10] to noncommutative geometry [1], we prove that there is a homomorphism of the Connes-Moscovici Hopf algebra $\iota : H_1 \to U^F(\mathfrak{sl}_3)[[t]]$, with $F$ being a Jordanian twist, where $H_1$ has the following structure

$$\begin{align*}
[Y, X] &= X, \quad [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad [\delta_k, \delta_l] = 0, \quad k, l \geq 1 \\
\Delta(Y) &= Y \otimes 1 + 1 \otimes Y, \quad \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1 \\
\Delta(X) &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y.
\end{align*}
$$

Through factoring $H'_1 := H_1 / < \delta_2 - \frac{1}{2} \delta^2_2 >$, one obtains in fact an embedding

$$\iota : H'_1 \hookrightarrow U^F(\mathfrak{sl}_3)[[t]]$$

and the twist found in [5]:

$$F = \sum_{n \geq 0} t^n \sum_{k=0}^n \frac{S(X)^k}{k!}(2Y + k)_{n-k} \otimes \frac{X^{n-k}}{(n-k)!}(2Y + n - k)_k$$

where $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$ and $S(X) = -X + \delta_1 Y$, can be obtained as a pullback $F = \iota_* \Phi$ of a semi-classical twist $\Phi$ in $U^F(\mathfrak{sl}_3)[[t]]$. In section ?? we show that $\iota$ can be "quantized", thus leading to a quantum analogue of $H'_1$:

$$\begin{align*}
kxk^{-1} &= q^2 x, \quad kzk^{-1} = q^2 z, \quad q^2 xz - zx = -tz^2 \\
\Delta(k) &= k \otimes k, \quad \Delta(z) = z \otimes k + 1 \otimes z \\
\Delta(x) &= x \otimes k^{-1} + 1 \otimes x + tkz \otimes \frac{(k - k^{-1})}{1 - q^2}.
\end{align*}$$

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2 Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra with the set of simple roots $\pi = \{\alpha_1, \ldots, \alpha_N\}$ and the Cartan matrix $(A)_{ij} = a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. By definition, a Hopf algebra $U_\mathfrak{g}(\mathfrak{g})$ is generated by $\{e_i, f_i, k_i^{\pm1}\}_{1 \leq i \leq N}$ over $\mathbb{C}(q)$ which are subject to the following relations

$$k_i e_j k_i^{-1} = q^{(\alpha_i, \alpha_j)} e_j, \quad k_i f_j k_i^{-1} = q^{-(\alpha_i, \alpha_j)} f_j$$  \hspace{1cm} (4)

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}$$  \hspace{1cm} (5)

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \binom{1-a_{ij}}{n} q_i e_j f_j e_j^{-1} = 0 \text{ for } i \neq j$$  \hspace{1cm} (6)

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \binom{1-a_{ij}}{n} q_i f_j f_j^{-1} = 0 \text{ for } i \neq j$$  \hspace{1cm} (7)

where $q_i = q^{\frac{\alpha_i, \alpha_i}{2}}$ and

$$\left[ \frac{m}{n} \right]_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}$$

$$[k]_q! = [1]_q [2]_q \cdots [k]_q, \quad [k]_q! = (q^k - q^{-k})/(q - q^{-1})$$

The Hopf algebra structure is defined uniquely by fixing the values of the coproduct on the Chevalley generators

$$\Delta(k_i) = k_i \otimes k_i$$  \hspace{1cm} (8)

$$\Delta(e_i) = k_i^{-1} \otimes e_i + e_i \otimes 1, \quad \Delta(f_i) = f_i \otimes k_i + 1 \otimes f_i$$  \hspace{1cm} (9)

$$S(k_i) = k_i^{-1}, \quad S(e_i) = -k_i e_i, \quad S(f_i) = -f_i k_i^{-1}$$  \hspace{1cm} (10)

$$\varepsilon(k_i) = 1, \quad \varepsilon(e_i) = 0, \quad \varepsilon(f_i) = 0.$$  \hspace{1cm} (11)

Letting $q_h = e^h$ and $K_i := q_i^h s_i$ in (4)-(11), we come to definition of $U_h(\mathfrak{g})[[h]]$, the topological Hopf algebra over $\mathbb{C}[[h]]$.

One introduces a linear ordering on the set of positive roots $\Delta_+$ by fixing the reduced decomposition of the longest element in the Weyl group $w_0 = s_{i_1} s_{i_2} \cdots s_{i_M}$. Then the linear ordering read from the left to the right is the following

$$\Delta_+ = \{\alpha_{i_1}, s_{i_1} \alpha_{i_2}, \ldots, s_{i_1} s_{i_2} \cdots s_{i_{M-1}} \alpha_{i_M}\}.$$  \hspace{1cm} (12)

An ordering (12) is normal, namely for each $\alpha, \beta \in \Delta_+$ such that $\alpha + \beta \in \Delta_+$ and $\alpha \prec \beta$, we have $\alpha \prec \alpha + \beta \prec \beta$. There is one-to-one correspondence between the reduced decompositions of the longest element in the Weyl group and the normal orderings given by (12). Following [8], one defines the generators corresponding to the composite roots. For a chosen normal ordering on $\Delta_+$ let $\alpha, \beta, \gamma \in \Delta_+$
be pairwise noncollinear roots, such that $\gamma = \alpha + \beta$. Let $\alpha$ and $\beta$ are taken so that there are no other roots $\alpha'$ and $\beta'$ with the property $\gamma = \alpha' + \beta'$. Then if $e_{\pm\alpha}$ and $e_{\pm\beta}$ have already been constructed, we set

$$e_{\gamma} = e_{\alpha}e_{\beta} - q^{-(\alpha,\beta)}e_{\beta}e_{\alpha}, \quad e_{-\gamma} = e_{-\beta}e_{-\alpha} - q^{-(\beta,\alpha)}e_{-\alpha}e_{-\beta}.$$ 

For any root $\gamma \in \Delta_+$ define

$$\hat{R}_\gamma := \exp_{q'}(-(q - q^{-1})a^{-1}_\gamma e\gamma \otimes k^{-1}_\gamma f\gamma),$$

where $q'_\gamma = q^{(\gamma,\gamma)}$ and

$$\exp_{q'}(x) := \sum_{n \geq 0} \frac{x^n}{(n)_{q'}}$$

with factors $a_\gamma$ coming from the relations

$$[e_\gamma, e_{-\gamma}] = a_\gamma \frac{k_\gamma - k^{-1}_\gamma}{q - q^{-1}}.$$
where $C$ is regarded as an $\mathcal{A}$ module ($q$ acts as 1). By construction of the quantum Cartan-Weyl basis we have the property
$$\Delta(e_\beta) \in \hat{\mathcal{U}}_q(\mathfrak{g}) \otimes \hat{\mathcal{U}}_q(\mathfrak{g})$$
but, in fact, one can deduce from (13) more restrictive property
$$\Delta(e_\beta) - k_\beta^{-1} \otimes e_\beta - e_\beta \otimes 1 \in (q - q^{-1}) \hat{\mathcal{U}}_q^+(\mathfrak{g}) \otimes \hat{\mathcal{U}}_q^+(\mathfrak{g})$$  (14)
where $\hat{\mathcal{U}}_q^+(\mathfrak{g})$ is generated by \{\(e_i, k_i^{-1}, k_i - 1 / q - 1\)\}. In what follows we are usually working rather with completions $\hat{\mathcal{U}}_q(\mathfrak{g})[[t]]$ and $\hat{\mathcal{U}}_q(\mathfrak{g}) \otimes \hat{\mathcal{U}}_q(\mathfrak{g})[[t]]$ in which the twists are to be defined. Let us formulate the following simple result which is of value for further application

**Proposition 1.** Let $\mathcal{F} \in \hat{\mathcal{U}}_q(\mathfrak{g}) \otimes \hat{\mathcal{U}}_q(\mathfrak{g})[[t]]$ be a twist in $U_q(\mathfrak{g})[[t]]$, then its specialization $\mathcal{F}$, obtained by order-wise specialization of its coefficients from $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ at $q = 1$, is still a twist in $U(\mathfrak{g})[[t]]$.

**Proof.** Indeed, representing $\mathcal{F}$ as a series
$$\mathcal{F} = 1 \otimes 1 + \mathcal{F}_1 t + \mathcal{F}_2 t^2 + \cdots$$
we see that (1) is equivalent to an infinite set of identities and each of them after specializing $q = 1$ remain valid. Thus
$$\mathcal{F} = 1 \otimes 1 + \mathcal{F}_1 t + \mathcal{F}_2 t^2 + \cdots$$
is a twist in $U(\mathfrak{g})[[t]]$.

## 3 Quantum Jordanian twists

We restrict ourselves to consideration of nonexceptional Lie algebra $\mathfrak{g}$ and define the quantum Jordanian twists as those specializing to semi-classical ones which define quantization of skew-symmetric extended Jordanian $\tau$-matrices:

$$r_\lambda = H_\lambda \wedge E_\lambda + 2 \sum_{\gamma_1 < \gamma_2, \gamma_1 + \gamma_2 = \lambda} E_{\gamma_1} \wedge E_{\gamma_2}$$

by the rule
$$\mathcal{R} = \mathcal{F}_{\lambda 21} \mathcal{F}_{\lambda}^{-1} = 1 \otimes 1 + t \ r_\lambda \text{ mod } t^2,$$
where we have denoted by $H_\lambda, E_\lambda$ the elements of the classical Cartan-Weyl basis.

Let us fix some normal ordering on $\Delta_+$ and define a generator $e_\lambda \in U_q(\mathfrak{g})$ corresponding to the highest root $\lambda$ according to the recipe from the previous section. Then nonexceptional root systems are remarkable by the following property:
Proposition 2. Let $\mathfrak{g}$ be a non exceptional Lie algebra, then there is such a normal ordering "≺" on $\Delta_+$ so that

$$[e_\gamma, e_\lambda]_{q^{-(\gamma, \lambda)}} = 0$$

for any $\gamma \prec \lambda$, and $e_\gamma, e_\lambda \in U_q(\mathfrak{g})$.

Proof. The proof is based on the expansion [8, 9]:

$$e_\gamma e_\lambda - q^{-(\gamma, \lambda)} e_\lambda e_\gamma = \sum_{\gamma_1 \prec \cdots \prec \gamma_j} c_{l, \gamma, \lambda} e_{\gamma_1} \cdots e_{\gamma_j}$$

(15)

where $c_{l, \gamma, \lambda} \in \mathbb{C}[q, q^{-1}]$. Non zero terms in the sum are subject to condition

$$\gamma + \lambda = l_1 \gamma_1 + l_2 \gamma_2 + \cdots + l_j \gamma_j.$$  

(16)

In $A_N$ we choose a normal ordering as

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \alpha_2 + \cdots + \alpha_N \succ \alpha_2 \succ \cdots \succ \alpha_{N-1} + \alpha_N \succ \alpha_N$$

and $\lambda = \alpha_1 + \cdots + \alpha_N$. If $\lambda \succ \gamma \succ \alpha_N$ we can satisfy (15) only with zero coefficients.

In $B_N$ we have

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-1} + 2\beta \succ \alpha_2 \succ \alpha_2 + \alpha_3 \succ \cdots \succ \beta$$

and $\lambda = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-1} + 2\beta$.

In $C_N$ we fix the following ordering

$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \cdots \prec \alpha_1 + \cdots + \alpha_{N-1} \prec 2(\alpha_1 + \cdots + \alpha_{N-1}) + \beta \prec \alpha_1 + \cdots + \alpha_{N-1} + \beta \prec \cdots \prec \alpha_2 + \alpha_3 \prec \cdots \prec \beta,$$

and $\lambda = 2(\alpha_1 + \cdots + \alpha_{N-1}) + \beta$. This ordering eliminates all non zero terms on the r.h.s of (15).

In $D_N$ we have quite a similar situation

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \cdots + \alpha_{N-1} \succ \alpha_1 + \cdots + \alpha_{N-1} + \beta \succ \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-2} + \alpha_{N-1} + \beta \succ$$

$$\alpha_2 \succ \alpha_2 + \alpha_3 \succ \cdots \succ \beta,$$

and $\lambda = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-2} + \alpha_{N-1} + \beta$.

As a direct consequence of Proposition 2 we obtain
Proposition 3. A \( q \)-commutation holds
\[
(e_\lambda \otimes 1)(\Delta(e_\lambda) - e_\lambda \otimes 1) = q_\lambda (\Delta(e_\lambda) - e_\lambda \otimes 1)(e_\lambda \otimes 1) \text{ where } q_\lambda = q^{(\lambda, \lambda)}.
\]

Proof. The statement follows from Proposition 2 and (13) if one notices that
\[
\hat{R}_{\lambda}(e_\lambda \otimes 1) = (e_\lambda \otimes 1)\hat{R}_{\lambda}.
\]

Let us define a coboundary twist
\[
\mathcal{J}(e_\lambda) = (W \otimes W)\Delta(W^{-1}), \text{ where } W = \exp_{q_\lambda} \left( \frac{t}{1-q_\lambda} e_\lambda \right)
\]
Then the following is true

Proposition 4. \( \mathcal{J}(e_\lambda) \) is nonsingular and defines a nontrivial twisting of \( U(g)[[t]] \) in the limit \( q \to 1 \).

Proof. By Proposition 3 we have
\[
\mathcal{J}(e_\lambda) = \exp_{q_\lambda} \left( \frac{t}{1-q_\lambda} 1 \otimes e_\lambda \right) \exp_{q_\lambda^{-1}} \left( -\frac{t}{1-q_\lambda} (\Delta(e_\lambda) - e_\lambda \otimes 1) \right).
\]
The latter representation is nonsingular and from \( \hat{U}_q(g) \otimes \hat{U}_q(g)[[t]] \), which is obvious if one uses the Campbell-Hausdorff formula after applying the dilogarithmic representation of \( q \)-exponent [4]:
\[
\exp_{q_\lambda} \left( \frac{t}{1-q_\lambda} x \right) = \exp \left( \sum_{n \geq 1} \frac{t^n}{n(1-q_\lambda^n)} x^n \right)
\]
along with the properties [14] and
\[
[e_\lambda, \hat{U}_q^+(g)] \in (q-1) \hat{U}_q^+(g).
\]

Note, that to be self-consistent one can directly verify that [14] satisfies [20], considering [20] as a functional equation for the function \( \text{Li}_2(t \cdot x, q) : \text{Li}_2(t \cdot x, q) = \ln(\exp_q(\frac{t}{1-q} x)) \). Finally, \( \mathcal{J} \) is a twist by Proposition 1.

Example 1. Consider \( g = sl_{N+1} \). We have the following formula for the co-product associated with the chosen normal ordering
\[
\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \alpha_2 + \cdots + \alpha_N \succ \alpha_2 \succ \cdots \succ \alpha_{N-1} + \alpha_N \succ \alpha_N
\]
given by
\[
\Delta(e_{\epsilon_1-\epsilon_N+1}) = k_{\epsilon_1-\epsilon_N+1}^{-1} \otimes e_{\epsilon_1-\epsilon_N+1} + e_{\epsilon_1-\epsilon_N+1} \otimes 1 + (1-q^2) \sum_{i=1}^{N-1} e_{\epsilon_i-\epsilon_{i+1}} k_{\epsilon_{i+1}-\epsilon_N+1}^{-1} \otimes e_{\epsilon_i-\epsilon_{i+1}}
\]
where $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

To calculate specialization $\mathcal{J}(e_{\epsilon_{1}-\epsilon_{N+1}})$, we represent $\mathcal{J}(e_{\epsilon_{1}-\epsilon_{N+1}})$ in the following form

$$\mathcal{J}(e_{\epsilon_{1}-\epsilon_{N+1}}) = \exp_q(-\frac{t}{1-q^2} e_{\epsilon_{1}-\epsilon_{N+1}}) \cdot \exp_q(-\frac{t}{1-q^2} e_{\epsilon_{1}-\epsilon_{N+1}}),$$

where

$$C_{1,N+1} = \exp_q(-\frac{t}{1-q^2} e_{\epsilon_{1}-\epsilon_{N+1}}) \cdot \exp_q(-\frac{t}{1-q^2} e_{\epsilon_{1}-\epsilon_{N+1}})$$

and

$$\mathcal{J}_1 = \exp_q(-\frac{t}{1-q^2} e_{\epsilon_{1}-\epsilon_{N+1}}) \cdot \exp_q(-\frac{t}{1-q^2} e_{\epsilon_{1}-\epsilon_{N+1}}).$$

Calculation of (18) - (19) is based on the Heine’s formula from [6]:

$$1 + \sum_{n \geq 1} t^n \frac{(\alpha)_q \cdots (\alpha + n - 1)_q}{(n)_q!} x^n = \exp_q(\frac{t}{1-q} x) \exp_q(-\frac{t}{1-q} x).$$

Note that (20) can be recast so that to hold in $\hat{U}_q(g) \otimes \hat{U}_q(g)[[t]]$:

$$1 \otimes 1 + \sum_{n \geq 1} \frac{t^n}{(n)_q!} \left( \frac{k_{\epsilon_{1}-\epsilon_{N+1}} - 1}{q^2 - 1} \right) \cdots \left( \frac{k_{\epsilon_{1}-\epsilon_{N+1}} q^{2(n-1)} - 1}{q^2 - 1} \right) \otimes e_{\epsilon_{1}-\epsilon_{N+1}} = \mathcal{J}_1$$

as one checks $\frac{k_{\epsilon_{1}-\epsilon_{N+1}} q^{2(n-1)} - 1}{q^2 - 1} \in \hat{U}_q(g)$. Applying the specialization map

$$a \mapsto \overline{a} := a \otimes 1$$

to each of the tensor factors in $\mathcal{J}(e_{\epsilon_{1}-\epsilon_{N+1}})$ we come to a formula of [5, 10]:

$$\mathcal{J}(e_{\epsilon_{1}-\epsilon_{N+1}}) =$$

$$\exp(-t \sum_{i=1}^{N-1} E_{1,i+1} \otimes E_{i+1,N+1} e^{-\frac{t}{2} \sigma_{1,N+1}}) \cdot$$

$$\left( 1 \otimes 1 + \sum_{n \geq 1} (-1)^n t^n \frac{H_{1,N+1}(H_{1,N+1} - 1) \cdots (H_{1,N+1} - n + 1)}{2^n n!} \otimes E_{1,N+1} \right)$$

where

$$\sigma_{1,N+1} = \ln(1 - t E_{1,N+1}) = - \sum_{n \geq 1} \frac{t^n}{n} E_{1,N+1}$$

and

$$E_{i,j} = e_{\epsilon_{i}-\epsilon_{j}}, \quad H_{1,N} = \frac{k_{\epsilon_{1}-\epsilon_{N+1}} - 1}{q - 1}.$$
4 The Cremmer-Gervais twist and its specialization at $q \to 1$

In this section we consider nontrivial quantum twists in $U_q(sl_3)[[h]]$ and their semi-classical limits $q \to 1$. As is known from the classification of Belavin-Drinfeld triples, there are two possible Belavin-Drinfeld triples for $sl_3$. The first one, the empty triple, is accounted for the Drinfeld-Jimbo deformation itself, while the second is associated with another deformation which can be called the Cremmer-Gervais quantization and there is a solution to defining the twisting element providing a possibility to deform $U_h(sl_3)[[h]]$ further. To be self-consistent we first recall the construction of this twist from [1] and then study different possibilities to define specialization $q \to 1$, unveiling a surprising connection with the Connes-Moscovici algebra $H_1$.

**Proposition 5 ([1]).** An element

$$J_{CG} = \Phi_{CG} \cdot K = \exp_{q_h^{-1}}(\xi \cdot e_{32} \otimes e_{12}) \cdot q_h^{h_{w_2} \otimes h_{w_1}}, \quad \xi \in h \cdot \mathbb{C}[[h]],$$

where

$$h_{w_1} = \frac{2}{3} e_{11} - \frac{1}{3} (e_{22} + e_{33}), \quad h_{w_2} = \frac{1}{3} (e_{11} + e_{22}) - \frac{2}{3} e_{33} \tag{22}$$

with $w_{1,2}$ being the fundamental weights, is a twist.

**Proof.** It is clear that $K = q_h^{h_{w_2} \otimes h_{w_1}}$ defines an abelian twist of $U_{q_h}(sl_3)[[h]]$. It leads to the following new Hopf algebra $U_{q_h}^K(sl_3)[[h]]$ with the same algebra structure as for $U_{q_h}(sl_3)[[h]]$ and the new deformed coproducts:

$$\Delta_K(e_{12}) = q_h^{-2} h_{1, -1} \otimes e_{12} + e_{12} \otimes 1, \quad \Delta_K(e_{23}) = q_h^{h_{1, -2}} \otimes e_{23} + e_{23} \otimes q_h^{h_{1, 0}}$$

$$\Delta_K(e_{21}) = e_{21} \otimes q_h^{h_{2, -1}} + q^{-h_{0, 1}} \otimes e_{21}, \quad \Delta_K(e_{32}) = e_{32} \otimes q_h^{-2 h_{1, -1}} + 1 \otimes e_{32}$$

where $h_{m,n} := m h_{w_1} + n h_{w_2}$. We are done if we prove that $\Phi_{CG}$ is a twist for $U_{q_h}(sl_3)[[h]]$. Indeed, explicitly (1) reads as the following

$$\exp_{q_h^{-1}}(\xi \cdot e_{32} \otimes e_{12} \otimes 1) \cdot \exp_{q_h^{-1}}(\xi \cdot e_{32} \otimes q_h^{-2 h_{1, -1}} + 1 \otimes e_{32}) \otimes e_{12}) =$$

$$\exp_{q_h^{-1}}(\xi \cdot 1 \otimes e_{32} \otimes e_{12}) \cdot \exp_{q_h^{-1}}(\xi \cdot e_{32} \otimes (q_h^{-2 h_{1, -1}} \otimes e_{12} + e_{12} \otimes 1))$$

and by the characteristic property of $q-$exponent

$$\exp_q(x + y) = \exp_q(y) \exp_q(x); \quad xy = q \cdot yx \tag{1}$$

(1) holds.

Consider now the problem of defining specialization of $J_{CG}$. To do so, we introduce from the beginning a $\mathbb{C}(q)$ analogue of $U_{q_h}^K(sl_3)[[h]]$, which we denote by $U_q'(sl_3)$. As an algebra $U_q'(sl_3)$ is an extension of $U_q(sl_3)$ obtained by
attaching elements $L_i$ (the maximal lattice), so that $K_j = \prod_{i=1}^{3} L_i^{a_{ij}}$. On the other hand, as a coalgebra $U_q'(\mathfrak{sl}_3)$ has a new coproduct fixed uniquely by its values on the Chevalley generators

$$
\Delta(L_i) = L_i \otimes L_i
$$

$$
\Delta(e_1) = L_1^{-2} L_2^2 \otimes e_1 + e_1 \otimes 1,
\Delta(e_2) = L_1 L_2^{-2} \otimes e_2 + e_2 \otimes L_1
$$

$$
\Delta(f_1) = f_1 \otimes L_1^2 L_2^{-1} + L_2^{-1} \otimes f_1,
\Delta(f_2) = f_2 \otimes L_1 L_2^{-2} + 1 \otimes f_2,
$$

where $L_i$ are invertible, pairwise commuting and satisfying

$$
L_i e_j L_i^{-1} = q^{\delta_{ij}} e_j,
L_i f_j L_i^{-1} = q^{-\delta_{ij}} f_j.
$$

In the sense of [?] define the regular form $\hat{U}_q'(\mathfrak{sl}_3)$ as an algebra such that there is an isomorphism

$$
\hat{U}_q'(\mathfrak{sl}_3) \otimes \mathbb{C}(q) \approx U_q'(\mathfrak{sl}_3).
$$

$\hat{U}_q'(\mathfrak{sl}_3)$ is generated over $\mathbb{A}$ by the following set of generators

$$
\left\{ L_i^{-1}, L_i - \frac{1}{q-1}, e_i, f_i \right\}
$$

If we denote by the "barred" generators the images of generators under specialization map

$$
a \mapsto \bar{a} := a \otimes \mathbb{A} 1,
$$

then the generators of $\hat{U}_q'(\mathfrak{sl}_3)$ specialize to the classical Chevalley generators of $U(\mathfrak{sl}_3)$ as the following:

$$
\bar{e}_1 = E_{12}, \quad \bar{e}_2 = E_{23}, \quad \frac{L_1 - 1}{q-1} = h_{w_1}
$$

$$
\bar{f}_1 = E_{21}, \quad \bar{f}_2 = E_{32}, \quad \frac{L_2 - 1}{q-1} = h_{w_2}
$$

(see [22]). Let us additionally make completion $U_q'(\mathfrak{sl}_3)[[t]]$ then we can formulate a $q$–analogue of Proposition 5.

**Proposition 6.** An element

$$
\hat{\Phi}_{CG} = \exp_{q^{-2}}(\zeta f_2 \otimes e_1), \quad \zeta \in t \cdot \mathbb{A}[[t]]
$$

is a twist in $\hat{U}_q'(\mathfrak{sl}_3)[[t]]$.

On the other hand apart from $\hat{\Phi}_{CG}$ we can construct the twists in $U_q'(\mathfrak{sl}_3)[[t]]$ which restrict to $U'_q(\mathfrak{sl}_3)[[t]]$ only after a suitable change of basis which is implemented by some coboundary twist, namely we can prove
**Proposition 7.** An element

\[ \mathcal{F}_{CG} = (V \otimes V) \exp_{q^{-2}}(\frac{q^{-2} \cdot \zeta \cdot t}{1 - q^2} f_2 \otimes e_1) \Delta(V^{-1}), \quad \zeta \in t \cdot \mathcal{A}[[t]], \]

where

\[ V = \exp_{q^{-2}}(-\frac{\zeta}{1 - q^2} e_1 L_1^2 L_2^{-2}) \cdot \exp_{q^2}(\frac{t}{1 - q^2} f_2), \]

restricts to a twist of \( \hat{U}'_q(\mathfrak{s}l_3)[[t]] \).

**Proof.** By the form of the coproducts

\[ \Delta(e_1 L_1^2 L_2^{-2}) = e_1 L_1^2 L_2^{-2} \otimes L_1^2 L_2^{-2} + 1 \otimes e_1 L_1^2 L_2^{-2}, \quad \Delta(f_2) = f_2 \otimes L_1^{-2} L_2^2 + 1 \otimes f_2 \]

we have explicitly

\[ \mathcal{F}_{CG} = \left( V \otimes \exp_{q^{-2}}(-\frac{\zeta}{1 - q^2} e_1 L_1^2 L_2^{-2}) \right) \exp_{q^{-2}}(\frac{q^{-2} \cdot \zeta \cdot t}{1 - q^2} f_2 \otimes e_1) \exp_{q^{-2}}(-\frac{t}{1 - q^2} f_2 \otimes L_1^{-2} L_2^2) \Delta(\exp_{q^2}(\frac{\zeta}{1 - q^2} e_1 L_1^2 L_2^{-2})). \]

Using the five terms relation, \( \mathbb{H} \):
If \( [u, [u, v]]_{q^{-2}} = [v, [u, v]]_{q^{-2}} = 0 \) then

\[ e_{q^2}(u) \cdot e_{q^2}(v) = e_{q^2}(v) \cdot e_{q^2}(\frac{1}{1 - q^2} [u, v]) \cdot e_{q^2}(u) \quad \text{(24)} \]

we can simplify

\[ \exp_{q^{-2}}(-\frac{t}{1 - q^2} f_2 \otimes L_1^{-2} L_2^2) \exp_{q^{-2}}(-\frac{\zeta}{1 - q^2} 1 \otimes e_1 L_1^2 L_2^{-2}) = \]

\[ \exp_{q^{-2}}(-\frac{\zeta}{1 - q^2} 1 \otimes e_1 L_1^2 L_2^{-2}) \exp_{q^{-2}}(\frac{q^{-2} \cdot \zeta \cdot t}{1 - q^2} f_2 \otimes e_1) \exp_{q^{-2}}(-\frac{t}{1 - q^2} f_2 \otimes L_1^{-2} L_2^2) \]

and thus \( \mathcal{F}_{CG} \) transforms to the following form

\[ \mathcal{F}_{CG} = \exp_{q^{-2}}(-\frac{t}{1 - q^2} (f_2 \otimes L_1^{-2} L_2^2 + e_1 L_1^2 L_2^{-2} \otimes 1)) \exp_{q^2}(\frac{t}{1 - q^2} (f_2 \otimes 1 + L_1^2 L_2^{-2} e_1 \otimes L_1^2 L_2^{-2})) \]

and finally by the Heine’s formula we have

\[ \mathcal{F}_{CG} = 1 \otimes 1 + \sum_{n \geq 1} \frac{1}{(n)_{q^2!}} \left( 1 \otimes \frac{L_1^{-2} L_2^2 - 1}{q^2 - 1} \cdots \frac{L_1^{-2} L_2^2 q^{2(n-1)} - 1}{q^2 - 1} \right) (t \cdot f_2 \otimes 1 + \zeta \cdot L_1^2 L_2^{-2} e_1 \otimes L_1^2 L_2^{-2})^n \]

the latter expression restricts to \( \hat{U}'_q(\mathfrak{s}l_3)[[t]] \) as we have

\[ \frac{L_1^{-2} L_2^2 q^{2(n-1)} - 1}{q^2 - 1} \in \hat{U}'_q(\mathfrak{s}l_3) \]
5 Semi-classical twists and noncommutative geometry

Proposition 8. There is a semi-classical twist \( F \) and a homomorphism

\[
\iota : \mathcal{H}_1 \rightarrow U^F(\mathfrak{sl}_3)[[t]]
\]

Proof. We solve more general problem of obtaining quantization \( \mathcal{H}'_{1,q} \) in the sense that there is an embedding

\[
\iota_q : \mathcal{H}'_{1,q} \rightarrow U'_q(\mathfrak{sl}_3)[[t]]
\]

where \( \mathcal{H}'_{1,q} \) denotes an appropriately defined \( q \)-deformation of \( \mathcal{H}'_1 \). Consider the following coboundary twist in \( U'_q(\mathfrak{sl}_3)[[t]] \):

\[
J = (W \otimes W) \Delta(W^{-1}), \quad W = \exp_{q^{-2}}(-\frac{t}{1-q^2} e_{1+2}L_1)
\]

where

\[
e_{1+2}L_1 := (e_1 e_2 - q e_2 e_1)L_1
\]

has the following coproduct

\[
\Delta(e_{1+2}L_1) = e_{1+2}L_1 \otimes L_1^2 + 1 \otimes e_{1+2}L_1 + (1 - q^2) e_1 L_1^2 L_2^{-2} \otimes e_2 L_1.
\]

By the properties

\[
(1 \otimes e_{1+2}L_1)(e_{1+2}L_1 \otimes L_1^2 + (1 - q^2) e_1 L_1^2 L_2^{-2} \otimes e_2 L_1) = qu^{-2}(e_{1+2}L_1 \otimes L_1^2 + (1 - q^2) e_1 L_1^2 L_2^{-2} \otimes e_2 L_1)(1 \otimes e_{1+2}L_1),
\]

\[
(e_1 L_1^2 L_2^{-2} \otimes e_2 L_1)(e_{1+2}L_1 \otimes L_1^2) = q^{-2}(e_{1+2}L_1 \otimes L_1^2)(e_1 L_1^2 L_2^{-2} \otimes e_2 L_1)
\]

\( J \) is nonsingular, the reasoning is same as in Proposition 4, and can be represented in the following form

\[
\underbrace{J_1}_{\text{J}_1 = \text{Ad} \exp_{q^{-2}}(-\frac{t}{1-q^2} e_{1+2}L_1 \otimes 1)(\exp_q(t e_1 L_1^2 L_2^{-2} \otimes e_2 L_1))}
\]

\[
\underbrace{J_2}_{\exp_q(-\frac{t}{1-q^2} e_{1+2}L_1 \otimes 1) \exp_q(t \frac{1}{e_{1+2}L_1 \otimes L_1^2})}
\]

Let us check that for both factors we have

\[
J_{1,2} \in U'_q(\mathfrak{sl}_3) \otimes U'_q(\mathfrak{sl}_3)[[t]].
\]

Indeed, it follows from explicit form of the factors \( J_{1,2} \):

\[
J_1 = \exp_q(t e_1 L_1^2 L_2^{-2} \frac{1}{1-t e_{1+2}L_1} \otimes e_2 L_1),
\]
\[
\mathcal{J}_2 = \exp_{q^2} \left( \frac{q^{-2} t}{1 - q^{-2}} e_{1+2} L_1 \otimes 1 \right) \left( \exp_{q^2} \left( \frac{q^{-2} t}{1 - q^{-2}} e_{1+2} L_1 \otimes L_1 \right) \right)^{-1} = \\
= 1 \otimes 1 + \sum_{n \geq 1} t^n \frac{(-1)^n}{(n)_{q^{-2}}!} (e_{1+2} L_1)^n \otimes \frac{(L_1^2 - 1)}{q^2 - 1} \frac{(L_1^2 q^{-2} - 1)}{q^2 - 1} \cdots \frac{(L_1^2 q^{-2(n-1)} - 1)}{q^2 - 1}.
\]

Calculating specialization at \( q = 1 \) we come to

\[
\overline{\mathcal{J}} = \exp(t \ E_{12} \ \frac{1}{1 - t \ E_{13}} \otimes E_{23}) \left( 1 \otimes 1 + \sum_{n \geq 1} (-1)^n t^n E_{13}^n \otimes \frac{h_{w_1}(h_{w_2} - 1) \cdots (h_{w_1} - n + 1)}{n!} \right)
\]

\( \overline{\mathcal{J}} \) defines a noncoboundary deformation of \( U(\mathfrak{sl}_3)[[t]] \) as it follows from \( \mathcal{J}_{21} \neq \overline{\mathcal{J}} \).

On the other hand its quantum counter part \( \mathcal{J} \) is coboundary in \( U'_q(\mathfrak{sl}_3)[[t]] \) and amounts to switching to another basis of \( U'_q(\mathfrak{sl}_3)[[t]] \). In particular, the subset of generators

\[
\{ L_1 L_2^{-1}, e_1, f_2 \}
\]

is changing in the following way

\[
L_1^2 L_2^{-2} \mapsto W(L_1^2 L_2^{-2}) W^{-1} = L_1^2 L_2^{-2}, \quad e_1 \mapsto W(e_1) W^{-1} = e_1 \frac{1}{1 - t e_{1+2} L_1}
\]

\[
f_2 \mapsto W(f_2) W^{-1} = f_2 - \frac{t}{1 - q^2} q^{-1} L_1^2 L_2^{-2} e_1 \frac{1}{1 - t e_{1+2} L_1}
\]

and we can form a Hopf subalgebra \( D_q \subset U'_q(\mathfrak{sl}_3)[[t]] \) generated by the set of generators

\[
\left\{ k := L_1^2 L_2^{-2}, x := f_2, z := q^{-1} L_1^2 L_2^{-2} e_1 \frac{1}{1 - t e_{1+2} L_1} \right\}
\]

with the following structure:

\[
k x k^{-1} = q^2 x, \quad k z k^{-1} = q^2 z, \quad q^2 x z - z x = -t z^2
\]

\[
\Delta(k) = k \otimes k, \quad \Delta(z) = z \otimes k + 1 \otimes z
\]

\[
\Delta(x) = x \otimes k^{-1} + 1 \otimes x + t z \otimes \frac{(k - k^{-1})}{1 - q^2}.
\]

The structure of its specialization \( D_1 \approx D_q \hat{\otimes} A_1 \) is the following:

\[
\begin{align*}
\Delta(y) &= y \otimes 1 + 1 \otimes y, & \Delta(z) &= z \otimes 1 + 1 \otimes z
\end{align*}
\]
\[ \Delta(\bar{x}) = \bar{x} \otimes 1 + 1 \otimes \bar{x} - t \bar{z} \otimes y \]

where
\[ y := \frac{k - 1}{q - 1}. \]

Finally we obtain the stated map
\[ \iota : \mathcal{H}_1' \rightarrow D_1 \subset \mathcal{U}^{-\mathcal{T}}(\mathfrak{sl}_3)[[t]] \]

by fixing its values on the generators
\[ \iota(X) = -\frac{1}{2}x, \quad \iota(Y) = \frac{1}{2}y, \quad \iota(Z) = tz. \]

where the generators \( X, Y, Z \) fulfills the relations
\[ [Y, X] = X, \quad [Y, Z] = Z, \quad [X, Z] = \frac{1}{2}Z^2 \]

and the coproducts are obtained from (2) by substitution \( Z \) for \( \delta_1 \). Next, \( \iota \) is an isomorphism as \( \iota \) maps the basis of \( \mathcal{H}_1' \):
\[ \{ x^k y^l z^m \}_{k,l,m \geq 0} \]

onto the basis of \( D_1 \):
\[ \{ x^k y^l z^m \}_{k,l,m \geq 0} \]

Now we can obtain

**Proposition 9.** An element

\[ F_q = (WV \otimes WV) \exp_{q^{-2}} \left( \frac{q^{-3}}{1 - q^2} t^3 f_2 \otimes e_1 \right) \Delta(V^{-1})(W^{-1} \otimes W^{-1}), \]

where
\[ V = \exp_{q^{-2}} \left( -\frac{q^{-1}}{1 - q^2} e_1 L_1^2 L_2^{-2} \right) \cdot \exp_{q^2} \left( \frac{t}{1 - q^2} f_2 \right), \]

restricts to a twist in \( \hat{U}^{-\mathcal{T}}_q(\mathfrak{sl}_3)[[t]] \).

**Proof.** It is convenient to introduce notation
\[ \tilde{F}_{CG} = (V \otimes V) \exp_{q^{-2}} \left( \frac{q^{-3}}{1 - q^2} t^3 f_2 \otimes e_1 \right) \Delta(V^{-1}). \]

Then by the reasoning of Proposition 8 we can prove that
\[ \tilde{F}_{CG} = 1 \otimes 1 + \sum_{n \geq 1} \frac{1}{(n)_q!} \left( 1 \otimes \frac{L_1^{-2} L_2 - 1}{q^2 - 1} \cdots \frac{L_1^{-2} L_2^{2(n-1)} - 1}{q^2 - 1} \right) (t \cdot f_2 \otimes 1 + \frac{q^{-1} t^2}{1 - q^2} L_1^2 L_2^{-2} e_1 \otimes L_1^2 L_2^{-2})^n \]

Next, the following element
\[ \tilde{F}_{CG}^W := (W \otimes W) \tilde{F}_{CG} \Delta(W^{-1}) \]

is a twist in \( U_q'\mathfrak{sl}_3[[t]] \) and respectively
\[ F_q = (W \otimes W) \tilde{F}_{CG}(W^{-1} \otimes W^{-1}) \]

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defines a twist in $U_q^{\text{qJ}}(\mathfrak{sl}_3[[t]])$. Explicitly

$$F_q = 1 \otimes 1 + \sum_{n \geq 1} \frac{1}{(n)_q^n} \left( 1 \otimes \frac{L_1^{-2}L_2^{-1} \cdots L_1^{-2}L_2^{-1}q^{2(n-1)} - 1}{q^2 - 1} \right) \cdot \left( t \cdot f_2 \otimes 1 - t^2 \cdot q^{-1} + e_1 \frac{1}{1 - t e_{1+2L_1} \otimes \frac{L_1^{-2}L_2^{-2} - 1}{q^2 - 1}} \right)^n.$$ 

again similarly to Proposition 6 we check that $F_q$ restricts to a twist in $\hat{U}_q^{\text{qJ}}(\mathfrak{sl}_3)[[t]]$. If we specialize $q = 1$ and apply Proposition 8 we obtain

$$F_1 = 1 \otimes 1 + \sum_{n \geq 1} \frac{t^n}{n!} (1 \otimes Y(Y - 1) \cdots (Y - n + 1))(2X \otimes 1 + Z \otimes Y)^n.$$ 

Note that a formula for $F_1$ was obtained in [7] by a direct check in the study of the twists for $\mathfrak{sl}_2$ Yangian. To make correspondence with [9] we additionally twist $F_1$ by a coboundary:

$$F_1' = (\exp(t XY) \otimes \exp(t XY)) F_1 \Delta(\exp(-t XY))$$

expanding in $t$ we have

$$F_1' = 1 \otimes 1 + t \cdot (X \otimes Y - Y \otimes X + ZY \otimes Y) + \cdots$$

Thus $F_1'$ is equivalent to [9].

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