Abstract. Fix an integer $t \geq 2$ and a primitive $t$th root of unity $\omega$. We consider the specialized skew hook Schur polynomial $h_{\lambda/\mu}(X, \omega X, \ldots, \omega^{t-1}X, Y, \omega Y, \ldots, \omega^{t-1}Y)$, where $\omega^k X = (\omega^k x_1, \ldots, \omega^k x_n)$, $\omega^k Y = (\omega^k y_1, \ldots, \omega^k y_m)$ for $0 \leq k \leq t - 1$. We characterize the skew shapes $\lambda/\mu$ for which the polynomial vanishes and prove that the nonzero polynomial factorizes into smaller skew hook Schur polynomials. Then we give a combinatorial interpretation of $h_{\lambda/\mu}(1, \omega^d, \ldots, \omega^{d(tm-1)/t}, \omega^d, \ldots, \omega^{d(tm-1)})$, for all divisors $d$ of $t$, in terms of ribbon supertableaux. Lastly, we use the combinatorial interpretation to prove the cyclic sieving phenomenon on the set of semistandard supertableaux of shape $\lambda/\mu$ for odd $t$. Using a similar proof strategy, we give a complete generalization of a result of Lee–Oh (arXiv: 2112.12394, 2021) for the cyclic sieving phenomenon on the set of skew SSYT conjectured by Alexandersson–Pfannerer–Rubey–Uhlin (Forum Math. Sigma, 2021).

1. Introduction

Hook Schur functions (Schur supersymmetric functions) are supersymmetric functions indexed by integer partitions. They are the characters of irreducible covariant tensor representations of $\text{gl}(m/n)$ introduced by Berele and Regev [BR87] in their study of Lie superalgebras. They also form a $\mathbb{Z}$-basis of the ring of supersymmetric functions, generalizing Schur functions. Additionally, skew hook Schur functions are indexed by the skew shape partitions and generalize skew Schur functions. For background, see [Moe07].

The starting point for this work is to provide a factorization result for the skew hook Schur functions considered in $tn/tm$ variables, evaluated at $(\exp(2\pi i k/t)x_j)_{0 \leq k \leq t-1, 1 \leq j \leq n}$, $/(\exp(2\pi i k/t)y_j)_{0 \leq k \leq t-1, 1 \leq j \leq m}$, for $t \geq 2$ a fixed positive integer. This is stated as [Theorem 3.2].

This generalizes the factorization of Schur polynomials considered by Littlewood [Lit06] and independently by Prasad [Pra16]. We note in passing that the result is generalized to other classical group characters [AK22].

We then study the cyclic sieving phenomenon introduced by V. Reiner, D. Stanton and D. White [RSW04]. Let SSYT$_k(\lambda/\mu)$ of semistandard Young tableaux of shape $\lambda/\mu$ filled with numbers in $\{1, \ldots, k\}$ and $C_k$ be the cyclic group of order $k$. Rhoades [Rho10] showed that if $\lambda$ is a rectangular partition of length at most $k$, then the triple 

$$(\text{SSYT}_k(\lambda), C_k, q^{-m(\lambda)}s_\lambda(1, q, \ldots, q^{k-1})),$$

exhibits the cyclic sieving phenomenon, where $m(\lambda) = \sum_{i=1}^{k} (i-1)\lambda_i$ and $s_\lambda(1, q, \ldots, q^{k-1})$ is the principle specialization of the Schur polynomial. Such a result was further generalized for partition $\lambda$ of any shape with $\text{gcd}(|\lambda|, k) = 1$ [OP19].

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Alexandersson, Pfannerer, Rubey and Uhlin proposed the following conjecture [APRU21, Conjecture 50] generalizing Rhoades’s result. There exists an action of the cyclic group $C_t$ of order $t$ on $SSYT_k(t\lambda/t\mu)$ such that the triple

$$(SSYT_k(t\lambda/t\mu), C_t, s_{t\lambda/t\mu}(1, q, \ldots, q^{k-1})),$$

exhibits the cyclic sieving phenomenon. Here $t\lambda/t\mu$ is the stretched Young diagram of $\lambda/\mu$ by $t$. If $t$ does not divide $k$, then the conjecture is false [LO21]. But the conjecture is true if $k$ is divisible by $t$ [LO21, Theorem 1.1]. More precisely, it can be reformulated as follows: let $t\lambda/t\mu$ be the stretched Young diagram of $\lambda/\mu$ by $t$. If $t$ does not divide $k$, then the conjecture is false [LO21]. But the conjecture is true if $k$ is divisible by $t$. More precisely, it can be reformulated as follows: let $t\lambda/t\mu$ be a skew partition. If $\lambda_i - \mu_i$ is divisible by $t$ for all $i \geq 1$, then there exists an action of the cyclic group $C_t$ of order $t$ such that the triple

$$(SSYT_{tn}(\lambda/\mu), C_t, s_{\lambda/\mu}(1, q, \ldots, q^{tn-1})),$$

exhibits the cyclic sieving phenomenon. In this work, we generalize the result for all skew shapes $\lambda/\mu$. More specifically, we show that the above triple,

$$(SSYT_{tn}(\lambda/\mu), C_t, s_{\lambda/\mu}(1, q, \ldots, q^{tn-1})),$$

exhibits the cyclic sieving phenomenon if and only if the sign of certain permutations associated to $\lambda$ and $\mu$ are equal. Lastly, we give a similar cyclic sieving phenomenon on the set of supertableaux. These are stated as Theorem 4.4 and Theorem 4.5.

The plan for the rest of the paper is as follows. We give all the definitions and preliminary results in Section 2. We prove the hook Schur factorization result in Section 3. As a corollary, we get the factorization of specialized skew Schur functions. In Section 3, we also give a combinatorial description of hook Schur polynomial evaluated at the roots of unity. We prove the cyclic sieving phenomenon on the set of tableaux and supertableaux in Section 4.

2. Preliminaries

Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of nonnegative integers. The length of a partition $\lambda$, denoted $\ell(\lambda)$ is the number of nonzero parts of $\lambda$. A partition $\lambda$ can be represented pictorially as a Young diagram, whose $i^{th}$ row contains $\lambda_i$ left-justified boxes. For partitions $\lambda$ and $\mu$ such that $\mu \subseteq \lambda$ ($\mu_i \leq \lambda_i$, for all $i \geq 1$), the skew shape $\lambda/\mu$ is the set theoretic difference $\lambda \setminus \mu$. Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_m)$ be tuples of commuting indeterminates.

We consider the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in $n$ independent variables $x_1, \ldots, x_n$ with integer coefficients. A polynomial in this ring is symmetric if it is invariant under the action of permuting the variables. The elementary symmetric function and the complete symmetric function indexed by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are:

$$e_\lambda(X) = \prod_{i=1}^{n} e_{\lambda_i}(X), \quad e_r(X) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1}x_{i_2} \cdots x_{i_r}, \quad r \geq 1,$$

(2.1)

$$h_\lambda(X) = \prod_{i=1}^{n} h_{\lambda_i}(X), \quad h_r(X) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} x_{i_1}x_{i_2} \cdots x_{i_r} \text{ for } r \geq 1.$$
We note that \( e_0(X) = h_0(X) = 1 \). It is convenient to define \( e_r(X) \) and \( h_r(X) \) to be zero for \( r < 0 \). We write the generating functions for the symmetric functions \( e_r(X) \) and \( h_r(X) \).

\[
(2.3) \quad \sum_{r \geq 0} e_r(X)q^r = \prod_{i=1}^{n} (1 + x_iq) \quad \text{and} \quad \sum_{r \geq 0} h_r(X)q^r = \prod_{i=1}^{n} \frac{1}{(1 - x_iq)}.
\]

Recall that a semistandard tableau or tableau of shape \( \lambda/\mu \) with entries \( 1 < \cdots < n \) is a filling of \( \lambda/\mu \) such that entries increase weekly along rows and strictly along columns. If entries increase strictly both along its columns and rows, then we call it a standard tableau. If \( \mu \subset \lambda (\mu_i \leq \lambda_i, i \geq 1) \), then define skew Schur functions \( s_{\lambda/\mu} \) as

\[
(2.4) \quad s_{\lambda/\mu}(X) = \sum_T \text{wt}(T), \quad \text{wt}(T) := \prod_{i=1}^{n} x_i^{n_i(T)},
\]

where the sum is taken over all tableaux of shape \( \lambda/\mu \) and \( n_i(T), i \in [n] \) is the number of occurrences of \( i \) in \( T \). Otherwise \( s_{\lambda/\mu}(X) = 0 \). If \( \mu = \emptyset \), then \( s_{\lambda}(X) \) is the Schur function. See [Mac15] for more details and background. The Jacobi-Trudi formula gives \( s_{\lambda/\mu}(X) \) in terms of complete symmetric functions.

\[
(2.5) \quad s_{\lambda/\mu}(X) = \det \left( h_{\lambda_i - \mu_j - i+j}(X) \right)_{1 \leq i,j \leq n}.
\]

Now we will give a supersymmetric analogue of the symmetric functions defined above. We consider the ring \( \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_m] \) of polynomials in \( n + m \) independent variables \( x_1, \ldots, x_n, y_1, \ldots, y_m \) with integer coefficients. A polynomial \( f(X,Y) \) in this ring is doubly symmetric if it is symmetric in both \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_m) \). Moreover, if substituting \( x_n = t \) and \( y_m = -t \) results in an expression independent of \( t \), then we call \( f(X,Y) \) a supersymmetric function. The elementary supersymmetric function and the complete supersymmetric function indexed by \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) are:

\[
(2.6) \quad E_{\lambda}(X/Y) = \prod_{i=1}^{n} E_{\lambda_i}(X/Y),
\]

where \( E_r(X/Y) = \sum_{j=0}^{r} e_j(X)h_{r-j}(Y), \quad r \geq 1 \).

\[
(2.7) \quad H_{\lambda}(X/Y) = \prod_{i=1}^{n} H_{\lambda_i}(X/Y),
\]

where \( H_r(X/Y) = \sum_{j=0}^{r} h_j(X)e_{r-j}(Y), \quad r \geq 1 \). We note that \( E_0(X/Y) = H_0(X/Y) = 1 \). It is convenient to define \( E_r(X/Y) \) and \( H_r(X/Y) \) to be zero for \( r < 0 \).

**Definition 2.1.** A supertableau (semistandard supertableau) \( T \) of shape \( \lambda/\mu \) with entries

\[
1 < 2 < \cdots < n < 1' < 2' < \cdots < m'
\]

is a filling of the shape with these entries satisfying the following conditions:

- entries increase weakly along rows and columns
- the unprimed entries strictly increase along rows
- the primed entries strictly increase along columns
We use the shorthand notation \([n] \cup [m]\) to denote the ordered set \(\{1, \ldots, n, 1', \ldots, m'\}\) such that \(1 < \cdots < n < 1' < \cdots < m'\). The weight of a supertableau is given by

\[
\text{wt}(T) := \prod_{i=1}^{n} x_{i}^{n_{i}(T)} \prod_{j=1}^{m'} y_{j}^{n_{j'}(T)},
\]

where \(n_{k}(T), k \in [n] \cup [m]\) is the number of occurrences of \(k\) in \(T\). Denote the set of supertableaux of skew shape \(\lambda/\mu\) with entries in \([n] \cup [m]\) by \(\text{SSYT}_{n/m}(\lambda/\mu)\). For integer partitions \(\mu \subset \lambda\), the skew hook Schur polynomial, denoted \(\text{hs}_{\lambda/\mu}(X/Y)\) is given by:

\[
(2.8) \quad \text{hs}_{\lambda/\mu}(X/Y) := \sum_{T \in \text{SSYT}_{n/m}(\lambda/\mu)} \text{wt}(T).
\]

If \(\mu = \emptyset\), then \(\text{hs}_{\lambda}(X/Y)\) is the hook Schur polynomial.

**Example 2.2.** Let \(n = 2\) and \(m = 1\) and consider the skew shape \((2, 2)/(1)\). Then we have the following supertableaux in \(\text{SSYT}_{2/1}((2, 2)/(1))\):

\[
\begin{array}{c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 1 & 1' & 2 & 1' \\
1 & 1' & 2 & 1' & 1 & 1' & 2 & 1'
\end{array}
\]

\[
\text{hs}_{(2,2)/(1)}(x_1, x_2/y_1) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 y_1 + 2 x_1 x_2 y_1 + x_2^2 y_1 + x_1 y_1^2 + x_2 y_1^2.
\]

The Jacobi-Trudi formula gives \(\text{hs}_{\lambda/\mu}(X/Y)\) in terms of the complete supersymmetric functions [PT92].

\[
(2.9) \quad \text{hs}_{\lambda/\mu}(X/Y) = \det (H_{\lambda-i, \mu-i+j}(X/Y))_{1 \leq i, j \leq n}.
\]

If \(\mu = \emptyset\), then the hook Schur polynomial \(\text{hs}_{\lambda}(X/Y)\) is nonzero if and only if \(\lambda_{n+1} \leq m\).

A **border strip** or **ribbon** of size \(t\) is a connected subdiagram of \(t\) boxes of the Young diagram of \(\lambda\), which contains no \(2 \times 2\) block of squares. Therefore, successive rows and columns of a border strip overlap by exactly one box. The **height** of a border strip is one less than the number of rows it has. For example,

\[
\begin{array}{c|c|c|c|c|c}
& & & & & \\
\end{array}
\]

is a border strip of length 6 and height 3.

**Definition 2.3.** The **\(t\)-core** of the partition \(\lambda\), denoted \(\text{core}_{t}(\lambda)\), is the partition obtained by successively removing border strips of size \(t\) from the Young diagram of \(\lambda\).

It is a nontrivial fact that the resulting partition is independent of the order of removal; see [Loe11, Theorem 11.16]. For example, the only 2-cores are staircase shapes, i.e. of the form \((k, k-1, \ldots, 1, 0), k \in \mathbb{N}\).
Definition 2.4. A ribbon tableau (resp. ribbon supertableau) is a tableau (resp. supertableau) of shape $\lambda/\mu$ such that the shape determined by the entries labelled $i$, for each $i$, is a $t$-ribbon. Such a tableau is called a standard ribbon tableau (resp. standard ribbon supertableau) if the entries are distinct in different ribbons.

For a ribbon tableau or supertableau $S$, let $\text{Rib}(S)$ be the set of its ribbons and for $\xi \in \text{Rib}(S)$, we define position of $\xi$ in the shape of $S$ as

$$
pos(\xi) = \max\{j - i | (i, j) \in \xi\}. $$

For a partition $\lambda$ and an integer $\ell$ such that $\ell(\lambda) \leq \ell$, define the beta-set of $\lambda$ by $\beta(\lambda) \equiv \beta(\lambda, \ell) = (\beta_1(\lambda, \ell), \ldots, \beta_{t}(\lambda, \ell))$ by $\beta_i(\lambda, \ell) = \lambda_i + \ell - i$. Furthermore, for $0 \leq i \leq t - 1$, let $n_i(\lambda) \equiv n_i(\lambda, \ell)$ be the number of parts of $\beta(\lambda)$ congruent to $i$ (mod $t$) and $\beta_j^{(i)}(\lambda)$, $1 \leq j \leq n_i(\lambda)$ be the $n_i(\lambda)$ parts of $\beta(\lambda)$ congruent to $i$ (mod $t$) in decreasing order. We will write $\beta(\lambda)$ and $n_i(\lambda)$ whenever $\ell$ is clear from the context. Macdonald defines $t$-core and $t$-quotient of a partition using the beta-set and we recall the construction.

Proposition 2.5 ([Mac15, Example I.1.8]). Let $\lambda$ be a partition with $\ell(\lambda) \leq \ell$.

1. The $\ell$ numbers $t_j + i$, where $0 \leq j \leq n_i(\lambda) - 1$ and $0 \leq i \leq t - 1$, are all distinct. Arrange them in descending order, say $\tilde{\beta}_1 > \cdots > \tilde{\beta}_t$. Then the $t$-core of $\lambda$ has parts $(\text{core}_t(\lambda))_i = \tilde{\beta}_i - \ell + i$. Thus, $\lambda$ is a $t$-core if and only if these $\ell$ numbers $t_j + i$, where $0 \leq j \leq n_i(\lambda) - 1$ and $0 \leq i \leq t - 1$ form its beta-set $\beta(\lambda)$.

2. The parts $\beta_j^{(i)}(\lambda)$ may be written in the form $t_i \tilde{\beta}_j^{(i)} + i$, $1 \leq j \leq n_i(\lambda)$, where $\tilde{\beta}_1^{(i)} > \cdots > \tilde{\beta}_n^{(i)}(\lambda) \geq 0$. Let $\lambda_j^{(i)} = \tilde{\beta}_j^{(i)} - n_i(\lambda) + j$, so that $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_{n_i(\lambda)}^{(i)})$ is a partition. Then the $t$-quotient $\text{quo}_t(\lambda)$ of $\lambda$ is a cyclic permutation of $\lambda^* = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(t - 1)})$. The effect of changing $\ell \geq \ell(\lambda)$ is to permute the $\lambda^{(j)}$ cyclically, so that $\lambda^*$ should perhaps be thought of as a ‘necklace’ of partitions.

Remark 2.6. For partitions $\lambda$ and $\mu$ of length at most $tn$, core$_t(\lambda)/$core$_t(\mu)$ is empty if and only if $n_i(\lambda) = n_i(\mu)$ for all $i \in [0, t - 1]$.

For a partition of length at most $tn$, let $\sigma_\lambda \in S_n$ be the permutation that rearranges the parts of $\beta(\lambda)$ such that

$$
\beta_{\sigma_\lambda(j)}(\lambda) \equiv q \pmod{t}, \quad \sum_{i=0}^{q-1} n_i(\lambda) + 1 \leq j \leq \sum_{i=0}^{q} n_i(\lambda),
$$

arranged in decreasing order for each $q \in \{0, 1, \ldots, t - 1\}$. For the empty partition, $\beta(\emptyset, tn) = (tn - 1, tn - 2, \ldots, 0)$ with $n_q(\emptyset, tn) = n$, $0 \leq q \leq t - 1$ and
in one line notation with $\text{sgn}(\sigma) = (-1)^{\sum_{T \in \text{Rib}(\lambda/\mu)} \text{ht}(T)}$.

Lemma 2.8. \cite[Chapter I.1, Example 8(a)]{Mac15} Let $\lambda, \mu$ be partitions of length at most $\ell$ such that $\mu \subset \lambda$, and such that the set difference of Young diagrams $\lambda \setminus \mu$ is a border strip of length $t$. Then $\beta(\mu)$ can be obtained from $\beta(\lambda)$ by subtracting $t$ from some part $\beta_i(\lambda)$ and rearranging in descending order.

Lemma 2.8. \cite[Proposition 3.3.1]{vL99} Let $\lambda$ and $\mu$ be partitions such that core$_t(\lambda)/\text{core}_t(\mu)$ is empty. Suppose Rib($\lambda/\mu$) be the set of $t$-ribbon tableau of shape $\lambda/\mu$. Then the parity of $\sum_{T \in \text{Rib}(\lambda/\mu)} \text{ht}(T)$ is independent of $T$, where $\text{ht}(T) = \sum_{\xi \in \text{Rib}(T)} \text{ht}(\xi)$.

Proposition 2.9. Let $\lambda$ and $\mu$ be partitions of length at most $tn$ such that core$_t(\lambda)/\text{core}_t(\mu)$ is empty. Then

$$\text{sgn}(\sigma_\lambda) \text{sgn}(\sigma_\mu) = (-1)^{\sum_{B \in \text{Rib}(\lambda/\mu)} \text{ht}(B)}.$$ 

Proof. Suppose $\mu$ is obtained from $\lambda$ by removing a border strip $\xi$. By Lemma 2.7, we see that $\beta(\mu)$ can be obtained from $\beta(\lambda)$ by subtracting $t$ from some part $\beta_i(\lambda)$ and rearranging in descending order. The height of $\xi$ is precisely the number of shifted transpositions we applied. Proceeding inductively completes the proof.

3. Skew hook Schur polynomial

In this section, we consider the specialized skew hook Schur polynomials. Recall, $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_m)$ and $\omega$ is a primitive $t$‘th root of unity. We denote our indeterminates by $(X^t/Y^t) := (X_\omega X, \omega^2 X, \ldots, \omega^{t-1} X/Y, \omega Y, \omega^2 Y, \ldots, \omega^{t-1} Y)$.

Theorem 3.1. For $k \geq 0$, the complete supersymmetric function $H_k(X^t/Y^t)$ is given by

$$H_k(X^t/Y^t) = \begin{cases} 0 & \text{if } k \not\equiv 0 \pmod{t}, \\ H_+^t(X^t/(-1)^{t-1}Y^t) & \text{otherwise}. \end{cases}$$

Proof. By \cite{2.7}, the required complete supersymmetric function is

$$H_k(X^t/Y^t) = \sum_{l=0}^{k} h_l(X^t) e_{k-l}(Y^t).$$

By the generating function identities in \cite{2.3}, we have

$$\sum_{r \geq 0} e_r(Y^t) q^r = \prod_{i=1}^m (1 + y_i q)(1 + y_i \omega q) \ldots (1 + y_i \omega^{t-1} q)$$

and

$$\sum_{r \geq 0} h_r(X^t) q^r = \prod_{i=1}^n \frac{1}{(1 - x_i q)(1 - x_i \omega q) \ldots (1 - x_i \omega^{t-1} q)} = \sum_{b \geq 0} h_b(X^t) q^{bt}.$$
On comparing the coefficients, we see that \( e_r(Y^{(\omega)}) \) and \( h_r(X^{(\omega)}) \) are nonzero if and only if \( t \) divides \( r \). In that case

\[
(3.2) \quad e_r(Y^{(\omega)}) = e_r^{(t)}((-1)^{t-1}Y^t) \quad \text{and} \quad h_r(X^{(\omega)}) = h_r^{(t)}(X^t).
\]

If \( t \) does not divide \( k \), then for each \( l \in [0,k] \), either \( h_l(X^{(\omega)}) \) or \( e_{k-l}(Y^{(\omega)}) \) is zero. This implies \( H_k(X^{(\omega)}/Y^{(\omega)}) = 0 \). And if \( t \) divides \( k \), then substituting the values (3.2) in (3.1), we have

\[
H_k(X^{(\omega)}/Y^{(\omega)}) = \sum_{l=0}^{t} h_l(X^t)e_{k-l}((-1)^{t-1}Y^t) = H_k^{(t)}(X^t/(-1)^{t-1}Y^t).
\]

This completes the proof. □

**Theorem 3.2.** Let \( \lambda \) and \( \mu \) be partitions of length at most \( tn \). Then the skew hook Schur polynomial \( hs_{\lambda/\mu}(X^{(\omega)}/Y^{(\omega)}) \) is given by

1. If \( \text{core}_1(\lambda)/\text{core}_1(\mu) \) is non-empty, then
   \[
   hs_{\lambda/\mu}(X^{(\omega)}/Y^{(\omega)}) = 0.
   \]

2. If \( \text{core}_1(\lambda)/\text{core}_1(\mu) \) is empty, then
   \[
   hs_{\lambda/\mu}(X^{(\omega)}/Y^{(\omega)}) = \text{sgn}(\sigma_\lambda)\text{sgn}(\sigma_\mu)\prod_{i=0}^{t-1} hs_{\lambda(i)/\mu(i)}(X^t/(-1)^{t-1}Y^t).
   \]

**Proof.** By the Jacobi-Trudi type identity (2.9) for the skew hook Schur polynomials, we see that the required skew hook Schur polynomial is

\[
(3.3) \quad hs_{\lambda/\mu}(X^{(\omega)}/Y^{(\omega)}) = \det(H_{\lambda_i-\mu_j-i+j}(X^{(\omega)}/Y^{(\omega)})) = \det(H_{\beta_j(\lambda)-\beta_j(\mu)}(X^{(\omega)}/Y^{(\omega)})).
\]

Permuting the rows and columns of the determinant by \( \sigma_\lambda \) and \( \sigma_\mu \) respectively, defined in (2.10), we see that the skew hook Schur polynomial is

\[
\text{sgn}(\sigma_\lambda)\text{sgn}(\sigma_\mu)\det\left(\begin{array}{cccc}
H_{\beta_{\lambda(1)}(\lambda)-\beta_{\mu(1)}(\mu)} & H_{\beta_{\lambda(2)}(\lambda)-\beta_{\mu(1)}(\mu)} & \cdots & H_{\beta_{\lambda(t)}(\lambda)-\beta_{\mu(1)}(\mu)} \\
H_{\beta_{\lambda(1)}(\lambda)-\beta_{\mu(2)}(\mu)} & H_{\beta_{\lambda(2)}(\lambda)-\beta_{\mu(2)}(\mu)} & \cdots & H_{\beta_{\lambda(t)}(\lambda)-\beta_{\mu(2)}(\mu)} \\
\vdots & \vdots & \ddots & \vdots \\
H_{\beta_{\lambda(t)}(\lambda)-\beta_{\mu(1)}(\mu)} & H_{\beta_{\lambda(t)}(\lambda)-\beta_{\mu(2)}(\mu)} & \cdots & H_{\beta_{\lambda(t)}(\lambda)-\beta_{\mu(t)}(\mu)}
\end{array}\right),
\]

where \( H_j = H_j(X^{(\omega)}/Y^{(\omega)}) \) for all \( j \in \mathbb{Z} \). By Theorem 3.1, we have \( H_j = 0 \) if \( j \equiv 0 \pmod{t} \) and \( j > 0 \). Also, \( H_j = 0 \) if \( j < 0 \). Substituting these values in the above determinant, we see that the skew hook Schur polynomial is

\[
\text{sgn}(\sigma_\lambda)\text{sgn}(\sigma_\mu)\det\left(\begin{array}{cccc}
\hat{S}_0 & \cdots & 0 \\
\hat{S}_1 & \ddots & \vdots \\
0 & \cdots & \hat{S}_{t-1}
\end{array}\right),
\]

where

\[
\hat{S}_p = \left(H_{\beta_{\lambda(j)}(\lambda)-\beta_{\lambda(j)}(\mu)}\right)_{1 \leq i \leq \bar{n}_p(\lambda), 1 \leq j \leq \bar{n}_p(\mu)}, p \in [0, t-1].
\]
If \(\text{core}_t(\lambda)/\text{core}_t(\mu)\) is non-empty, then by Remark 2.6 \(n_i(\lambda) \neq n_i(\mu)\) for some \(i \in [0, t - 1]\), then the \((i + 1)\)th diagonal block is not a square block. So, \(hs_{\lambda/\mu}(X(\omega)/Y(\omega)) = 0\). If \(\text{core}_t(\lambda)/\text{core}_t(\mu)\) is empty, then again by Remark 2.6 \(n_i(\lambda) = n_i(\mu)\) for all \(0 \leq i \leq t - 1\). Finally, by Proposition 2.5 and (2.9),

\[
de(\hat{S}_p) = hs_{\lambda(p)}(X^t/(-1)^{(t-1)}Y^t).
\]

we get the desired result. \(\square\)

Since \(hs_{\lambda/\mu}(X/\emptyset) = s_{\lambda/\mu}(X)\), we have the following corollary.

**Corollary 3.3.** Let \(\lambda\) and \(\mu\) be partitions of length at most \(tn\). Then the skew Schur polynomial \(s_{\lambda/\mu}(X(\omega))\) is given by

1. If \(\text{core}_t(\lambda)/\text{core}_t(\mu)\) is non-empty, then
   \[s_{\lambda/\mu}(X(\omega)) = 0\.
   \]

2. If \(\text{core}_t(\lambda)/\text{core}_t(\mu)\) is empty, then
   \[s_{\lambda/\mu}(X(\omega)) = \text{sgn}(\sigma_{\lambda}) \text{sgn}(\sigma_{\mu}) \prod_{i=0}^{t-1} s_{\lambda^{(i)}/\mu^{(i)}}(X^t).
   \]

In [vL99, Proposition 3.1.2], if we take \(A = [n] \cup [m]\), then we have the following result.

**Theorem 3.4.** Let \(\lambda\) and \(\mu\) are partitions of length at most \(tn\) such that \(\text{core}_t(\lambda)/\text{core}_t(\mu)\) is empty and \(n_i(\lambda, tn) = n_i(\mu, tn) = n_i\) for all \(i \in [0, t - 1]\). Then there is a bijection between the set of standard \(t\)-ribbon supertableaux \(S\) of shape \(\lambda/\mu\) with entries in \([n] \cup [m]\), and the set of \(t\)-tuples \((S_0, S_1, \ldots, S_{t-1})\) of standard supertableaux, where \(S_i\) has shape \(\lambda^{(i)}/\mu^{(i)}\), and the sets of entries of the \(S_i\) are mutually disjoint. If \(x\) is a square of \(\lambda^{(i)}/\mu^{(i)}\), then \(S_i(x) = S(\xi)\) for a \(\xi \in \text{Rib}(S)\) with \(\text{pos}(\xi) = t(\text{pos}(x) + n_i - n) + i\).

For a bijection similar to the above theorem involving semistandard supertableaux, without the condition of disjointness on entries, we define a standardisation of semistandard supertableau. The standardisation of a semistandard supertableau is a standard supertableau obtained from it by renumbering its entries such that the relative order of distinct entries is preserved, and equal unprimed and primed entries are made increasing from left to right and top to bottom respectively. It is well defined since the ribbons with same entries have distinct positions and ordering them by increasing position (for unprimed entries) and decreasing position (for primed entries) gives a valid standard supertableau. See Figure 2 for an example of standardisation. Therefore, we have the following generalization of [vL99, Proposition 3.2.2]

![Figure 2. semistandard 3-ribbon supertableau and its standardization](image)
Remark 3.7. \[ \text{is the number of empty. If } \]
\[ \text{pertableaux, with } \]
\[ \text{semistandard. For invertibility we need to order all the occurrences of the same entry in any of the tableaux } T_i, \text{ in order to determine the } S_i; \text{ makes clear that these occurrences } T_i(x) \text{ should be ordered by increasing value of } t(\text{pos}(x) + n_i - n) + i \text{ if } T_i(x) \text{ are unprimed, and by decreasing value of } t(\text{pos}(x) + n_i - n) + i \text{ if } T_i(x) \text{ are primed.} \]

As an example, the semistandard 3-ribbon tableau and its standardisation displayed in Figure 2 corresponds to

\[
\begin{array}{cccc}
1 & 3' & 1' & 2' \\
2 & 3' & 3 & 2 \\
3 & & & \\
\end{array}
\quad
\begin{array}{cccc}
X & 1 & 3' & 1' & 2' \\
3 & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3' & 1' & 2' \\
2 & 4' & 3 & 2 \\
4 & & & \\
\end{array}
\]

Corollary 3.6. Let \( \lambda \) and \( \mu \) be partitions of length at most \( tn \). Then for all divisors \( d|t \),
\[ \prod_{i=0}^{t-1} \text{hs}_{\lambda(i)/\mu(i)}(X^t/Y^t) = \sum_R \text{wt}(R), \]
where \( R \) runs over the set of \( \frac{t}{d} \)-ribbon supertableaux of shape \( \lambda/\mu \) filled with entries \([dn]\cup[dm]\].

Remark 3.7. Let \( \lambda \) and \( \mu \) be partitions of length at most \( tn \) such that \( \text{core}_t(\lambda)/\text{core}_t(\mu) \) is empty. If \( \text{sgn}(\sigma_\lambda) = \text{sgn}(\sigma_\mu) \), then by Corollary 3.3 and Corollary 3.6 at \( X = (1, \ldots, 1) \), \( Y = \emptyset \), for all divisors \( d|t \), \( s_{\lambda/\mu}(1,\omega^d,\ldots,\omega^{d(tn-1)}) \) is the number of \( \frac{t}{d} \)-ribbon tableaux of shape \( \lambda/\mu \) filled with entries \([dn]\). Moreover, if \( t \) is odd, then \( \text{hs}_{\lambda/\mu}(1,\omega^d,\ldots,\omega^{d(tn-1)};1,\omega^d,\ldots,\omega^{d(tn-1)}) \) is the number of \( \frac{t}{d} \)-ribbon supertableaux of shape \( \lambda/\mu \) filled with entries \([dn]\cup[dm]\), for all \( d|t \).

Corollary 3.8. Suppose \( t \) is prime. Let \( \phi_{\lambda/\mu}(r,s) \) be the number of \( r \)-ribbon tableaux of shape \( \lambda/\mu \) filled with entries in \([s]\). Then \( \phi_{\lambda/\mu}(1,tn) - \phi_{\lambda/\mu}(t,n) \) is a multiple of \( t \).

Proof. If \( \text{core}_t(\lambda) \neq \text{core}_t(\mu) \), then by Definition 2.4, \( \phi_{\lambda/\mu}(t,n) = 0 \). By (2.4), we have
\[ s_{\lambda/\mu}(1,\omega,\ldots,\omega^{tn-1}) = \sum_{T\in\text{SSYT}_{tn}(\lambda/\mu)} \omega^{\sum_{i=0}^{n-1}(n_{2+i}(s)+2n_{3+i}(s)+\cdots+(t-1)n_{(i+1)}(s))} = 0, \]
where the second equality uses Corollary 3.3 at \( X = (1, \ldots, 1) \). Since \( \sum_{i=0}^{t-1}a_i\omega^i = 0 \) for some \( a_i \in \mathbb{Z} \) implies \( a_i = c \) for all \( i \), \( \phi_{\lambda/\mu}(1,tn) \) is a multiple of \( t \) and the corollary holds. If \( \text{core}_t(\lambda) = \text{core}_t(\mu) \), then by Remark 3.7, \( \phi_{\lambda/\mu}(1,tn) - \phi_{\lambda/\mu}(t,n) \)
\[ = \sum_{T\in\text{SSYT}_{tn}(\lambda/\mu)} \left(1 - \text{sgn}(\sigma_\lambda)\text{sgn}(\sigma_\mu)\omega^{\sum_{i=0}^{n-1}(n_{2+i}(s)+2n_{3+i}(s)+\cdots+(t-1)n_{(i+1)}(s))}\right). \]
Since \( \phi_{\lambda/\mu}(1, tn) - \phi_{\lambda/\mu}(t, n) \) is an integer, and \( \sum_{i=0}^{t-1} a_i \omega^i = 0 \) for some \( a_i \in \mathbb{Z} \) implies \( a_i = c \) for all \( i \), \( \phi_{\lambda/\mu}(1, tn) - \phi_{\lambda/\mu}(t, n) \) is a multiple of \( t \). This completes the proof. \( \square \)

**Remark 3.9.** A result similar to Corollary 3.8 holds for the number of super tableaux. Suppose \( t \) is an odd prime. Let \( \psi_{\lambda/\mu}(r, s/u) \) be the number of \( r \)-ribbon super tableaux of shape \( \lambda/\mu \) filled with entries in \([s] \cup [u]\). Then \( \psi_{\lambda/\mu}(1, tn/tm) - \psi_{\lambda/\mu}(t, n/m) \) is a multiple of \( t \).

### 4. Cyclic sieving phenomenon

Let \( C_t \) be the cyclic group of order \( t \) acting on a finite set \( X \) and \( f(q) \) a polynomial with nonnegative integer coefficients. Then the triple \( (X, C_t, f(q)) \) is said to exhibit the cyclic sieving phenomenon (CSP) if, for any integer \( k \geq 0 \),

\[
|\{ x \in X \mid \sigma \cdot x = x \} | = f(\omega^k),
\]

where \( \sigma \) is a generator of \( C_n \) and \( \omega \) is a primitive \( t \)-th root of unity.

**Theorem 4.1.** [Sag11, Theorem 11.1] The triple

\[
(\text{SSYT}_{tn}(k)), C_t, h_k(1, q, \ldots, q^{tn-1})
\]

exhibits the cyclic sieving phenomenon. If, in addition, \( t \) is odd then the triple

\[
\left( \binom{[tn]}{k}, C_t, e_k(1, q, \ldots, q^{tn-1}) \right)
\]

where \( \binom{[tn]}{k} \) is the set of \( k \)-element subsets of \([tn]\), exhibits the cyclic sieving phenomenon.

By [LO21, Remark 3.3] and (2.7), we have the following corollary.

**Corollary 4.2.** If \( t \) is odd, then

\[
(\text{SSYT}_{tn/tm}(k)), C_t, H_k(1, q, \ldots, q^{tn-1}/1, q, \ldots, q^{tm-1})
\]

exhibits the cyclic sieving phenomenon.

**Lemma 4.3.** [AA19, Theorem 2.7] Suppose \( f(q) \in \mathbb{Z}_{\geq 0}[q] \) and \( f(\omega^j) \in \mathbb{Z}_{\geq 0} \), for each \( j \in \{1, \ldots, t\} \). Let \( X \) be any set of size \( f(1) \). Then there exists an action of the cyclic group \( C_t \) of order \( t \) on \( X \) such that \( (X, C_t, f(q)) \) exhibits the cyclic sieving phenomenon if and only if for each \( d \mid t \),

\[
f(\omega^d) = \sum_{j \mid d} j c_j,
\]

for some nonnegative integers \( c_j \).

Recall the definition of \( \sigma_\lambda \) from (2.10).

**Theorem 4.4.** Let \( \lambda \) and \( \mu \) be partitions of length at most \( tn \) such that \( \text{sgn}(\sigma_\lambda) = \text{sgn}(\sigma_\mu) \). Then there exists an action of the cyclic group \( C_t \) of order \( t \) such that the triple

\[
(\text{SSYT}_m(\lambda/\mu)), C_t, s_{\lambda/\mu}(1, q, \ldots, q^{tm-1})
\]

exhibits the cyclic sieving phenomenon.
Proof. Let $f(q) = s_{\lambda/\mu}(1, q, \ldots, q^{n-1})$. Since $f(q)$ is given by (2.4), $f(q) \in \mathbb{Z}_{\geq 0}[q]$. By Lemma 4.3, it is sufficient to show for each $d|t$ there exists $c_d \geq 0$ such that (1.2) holds.

We prove this by induction on $t$. If $t = 2$, then take $c_1 = \phi_{\lambda/\mu}(2, a)$ and $2c_2 = \phi_{\lambda/\mu}(1, 2a) - \phi_{\lambda/\mu}(2, a) \geq 0$, as derived in Corollary 3.8 at $t = 2$. Assume that the result holds for all positive integers less than $t$. Fix $t$. If $\lambda/\mu = (k)$, then by Theorem 4.1 and Lemma 4.3, for all $d|t$,  
\[
h_k(1, \omega^d, \ldots, \omega^{d(tn-1)}) = \sum_{j|d} ja^k_j,
\]
for some non negative integers $a^k_j$. Since \((\sum_{j|d} jp_j) (\sum_{j|d} jq_j) = \sum_{j|d} jr_j\), where $r_j = \sum_{i|d, i<j} i(p_i q_j + p_j q_i)$, by the Jacobi Trudi identity (2.5), for all $d|t$, we see that  
\[
f(\omega^d) = \det \left( \sum_{j|d} ja^{\lambda_i-\mu_i-i+j} \right) = \sum_{j|d} jc_j.
\]
Therefore,  
\[
f(q) \equiv \sum_{j|t} c_j (1 + q^j + \cdots + q^{j(j-1)}) \pmod{q^t - 1}.
\]
Since $f(q) \in \mathbb{Z}_{\geq 0}[q]$, $c_t \geq 0$. Fix $d < t$. Then by Corollary 3.3 at $X = (1, \ldots, 1)$,  
\[
f(\omega^d) = \begin{cases} 
0, & \text{core}_d^\lambda(\lambda) \neq \text{core}_d^\mu(\mu), \\
\prod_{i=0}^{t-1} s_{\lambda(i)/\mu(i)}(1, \ldots, 1)_{\text{dn}} & \text{core}_d^\lambda(\lambda) = \text{core}_d^\mu(\mu).
\end{cases}
\]
Since $d < t$, by inductive argument, for all $e|d$ and $i \in [0, \frac{t}{d} - 1]$,  
\[
s_{\lambda(i)/\mu(i)}(1, \omega^{te/d}, \ldots, \omega^{te(dn-1)/d}) = \sum_{j|e} jd^{(i)}_j,
\]
for some nonnegative integers $d^{(i)}_j$. If $\text{core}_d^\lambda(\lambda) = \text{core}_d^\mu(\mu)$, then take $e = d$ in (4.6) and substitute in (4.5) to get  
\[
f(\omega^d) = \prod_{i=0}^{t-1} \left( \sum_{j|d} jd^{(i)}_j \right) = \sum_{j|d} jc_j,
\]
where the last equality uses (4.4). The uniqueness of $c_j$ implies $c_j \geq 0$ for all $j|d$. This completes the proof. \qed

We now state our final result and give a sketch of the proof following similar ideas as in the proof of Theorem 4.4.

Theorem 4.5. Suppose $t$ is odd. Let $\lambda$ and $\mu$ be partitions of length at most $tn$ such that $\text{sgn}(\sigma_\lambda) = \text{sgn}(\sigma_\mu)$. Then there exists an action of the cyclic group $C_t$ of order $t$ such that the triple  
\[
(\text{SSYT}_{tn/tm}(\lambda/\mu), C_t, \text{hs}_{\lambda/\mu}(1, q, \ldots, q^{tn-1}/1, q, \ldots, q^{tm-1}))
\]
exhibits the cyclic sieving phenomenon.

Proof. Let $f(q) = h_{\lambda/\mu}(1, q, \ldots, q^{t-1})$. Since $f(q)$ is given by (2.8), $f(q) \in \mathbb{Z}_{\geq 0}[q]$. We apply Lemma 4.3 to prove the result.

The proof proceeds by induction on $t$. If $t = 3$, then take $c_1 = \psi_{\lambda/\mu}(3, a)$ and $3c_3 = \psi_{\lambda/\mu}(1, 3a) - \psi_{\lambda/\mu}(3, a) \geq 0$, as in Remark 3.9 at $t = 3$. Assume that the result holds for all odd integers less than $t$. Fix $t$. If $\lambda/\mu = (k)$, then by Corollary 4.2 and Lemma 4.3, for all $d|t$,

$$H_k(1, \omega^d, \ldots, \omega^{d(t-1)}/1, \omega^d, \ldots, \omega^{d(t-1)}) = \sum_{j|d} j a_j^k;$$

for some non-negative integers $a_j^k$. Then by the Jacobi-Trudi identity (2.9), for all $d|t$, we see that

$$f(\omega^d) = \det \left( \sum_{j|d} j a_j^{\lambda_i - \mu_j - i + j} \right) = \sum_{j|d} j c_j.$$

Therefore,

$$f(q) \equiv \sum_{j|t} c_j (1 + q^{\frac{j}{2}} + \cdots + q^{\frac{j}{2}(j-1)}) \pmod{q^t - 1}.$$

Since $f(q) \in \mathbb{Z}_{\geq 0}[q]$, $c_t \geq 0$. Fix $d < t$. Then by Theorem 3.2, at $X = (1, \ldots, 1)$ and $Y = (1, \ldots, 1)$, we have

$$f(\omega^d) = \begin{cases} 0 & \text{core}_{\frac{d}{e}}(\lambda) \neq \text{core}_{\frac{d}{e}}(\mu), \\ \prod_{i=0}^{\frac{t}{e}-1} h_{\lambda(i)/\mu(i)} \left( \underbrace{1, \ldots, 1}_{dn}, \underbrace{1, \ldots, 1}_{dm} \right) & \text{core}_{\frac{d}{e}}(\lambda) = \text{core}_{\frac{d}{e}}(\mu). \end{cases}$$

(4.8)

Since $d < t$ and $d$ is odd, by inductive argument, for all $e|d$ and $i \in [0, \frac{t}{e} - 1]$,

$$h_{\lambda(i)/\mu(i)} \left( 1, \omega^{te/d}, \ldots, \omega^{te(dn-1)/d}/1, \omega^{te/d}, \ldots, \omega^{te(dm-1)/d} \right) = \sum_{j|e} j d_j^{(i)};$$

(4.9)

for some non-negative integers $d_j^{(i)}$. If $\text{core}_{\frac{d}{e}}(\lambda) = \text{core}_{\frac{d}{e}}(\mu)$, then take $e = d$ in (4.9) and substitute in (4.8) to get

$$f(\omega^d) = \prod_{i=0}^{\frac{t}{d}-1} \left( \sum_{j|d} j d_j^{(i)} \right) = \sum_{j|d} j c_j,$$

where the last equality uses (4.7). The uniqueness of $c_j$ implies $c_j \geq 0$ for all $j|d$. This completes the proof. \[\Box\]

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