On Lipschitz continuity and smoothness up to the boundary of solutions of hyperbolic Poisson’s equation

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Abstract

We solve the Dirichlet problem \( u|_{\partial B^n} = \varphi \), for hyperbolic Poisson’s equation \( \Delta_h u = \mu \) where \( \varphi \in L_1(\partial B^n) \) and \( \mu \) is a measure that satisfies a growth condition.

Next we present a short proof for Lipschitz continuity of solutions of certain hyperbolic Poisson’s equations, previously established at [3].

In addition, we investigate some alternative assumptions on hyperbolic Laplacian, which are connected with Riesz’s potential. Also, local Hölder continuity is proved for solution of certain hyperbolic Poisson’s equations.

We show that, if \( u \) is hyperbolic harmonic in the upper half-space, then \( \frac{\partial u}{\partial y}(x_0, y) \to 0, y \to 0^+ \), when boundary function \( f \) of the functions \( u \) is differentiable at the boundary point \( x_0 \). As a corollary, we show \( C^1_c(H^n) \) smoothness of a hyperbolic harmonic function, which is reproduced from the \( C^1_c(R^{n-1}) \) boundary values.

1 Introduction

First, we give some notation. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), and by \( |x| \) we denote Euclidean norm of vector \( x \), and \( xy \) denotes the scalar product \( \sum_{j=1}^{n} x_j y_j \).

For \( R > 0 \), by \( B(a, R) \) and \( S(a, R) \) we denote the ball and the sphere in \( \mathbb{R}^n \) with center at \( a \) of radius \( R \). By \( B(R) \) and \( S(R) \) we denote \( B(0, R) \) and \( S(0, R) \). We use \( B^n \) and \( S^{n-1} \) for \( B(1) \) and \( S(1) \).

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \). We use standard notation for function spaces on \( \Omega \) (see [GT]).

- \( C^k(\Omega) \): the set of functions having all derivatives of order less then or equal to \( k \) continuous in \( \Omega \).
- \( C^k(\overline{\Omega}) \): the set of functions in \( C^k(\Omega) \) all of whose derivatives of order less than or equal to \( k \) have continuous extensions to \( \overline{\Omega} \).
- \( \text{supp} \ u \): the support of \( u \), the closure of the set on which \( u \neq 0 \).
- \( C^k_c(\Omega) \): the set of functions in \( C^k(\Omega) \) with compact support in \( \Omega \).

Let \( x_0 \in D \), where \( D \) is a bounded subset of \( \mathbb{R}^n \) and \( f \) is a function defined on \( D \). For

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0 < \alpha < 1$, we say that $f$ is H"{o}lder continuous with exponent $\alpha$ at $x_0$ if $f$ is continuous at $x_0$ and
\[
\sup_{x \in D, x \neq x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|^{\alpha}} < +\infty.
\]
When $\alpha = 1$, we say that $f$ is Lipschitz-continuous at $x_0$.

Suppose that $D$ is not necessarily bounded. We say that $f$ is uniformly H"{o}lder continuous with exponent $\alpha$ in $D$ if
\[
\sup_{x, y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < +\infty, \quad 0 < \alpha < 1.
\]

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $k$ a non-negative integer. The H"{o}lder spaces $C^{k,\alpha}(\Omega)(C^{k,\alpha}(\overline{\Omega}))$ are defined as the subspaces of $C^k(\Omega)(C^k(\overline{\Omega}))$ consisting of functions whose $k$-th order partial derivatives are uniformly H"{o}lder continuous (H"{o}lder continuous) with exponent $\alpha$ in $\Omega$. $\text{Lip}(\Omega)$ denotes class of function which are Lipshitz continuous on the set $\Omega$.

### 1.1 Möbius transformation in space

**Definition 1.1.** Inversion with respect to unit sphere in $\mathbb{R}^n$ is defined as
\[
J(a) = a^* = \frac{a}{|a|^2} \quad \text{for } a \neq 0 \quad \text{and} \quad J(0) = \infty, J(\infty) = 0.
\]

Following [5], the reflection (inversion) with respect to sphere $S(a, R)$ is given by
\[
\sigma_a x = a^* + R(a)^2(x - a^*)^*.
\]

The following matrix represents orthogonal projection in $\mathbb{R}^n$ onto the one-dimensional subspace spanned by $x \in \mathbb{R}^n$, $x \neq 0$.
\[
Q(x)_{ij} = \frac{x_i x_j}{|x|^2}.
\]

Let $r_a : y \mapsto y'$ be the reflection with respect to the hyper plane orthogonal to $a \neq 0$ which contains the origin. It is easy to check that $r_a = \text{Id} - 2Q(a)$. Indeed,
\[
y' = (\text{Id} - 2Q(a))y = y - 2\frac{(ya)a}{|a|^2}.
\]

Define $T_a = r_a \circ \sigma_a = (\text{Id} - 2Q(a))\sigma_a$. Check
\[
T_a x = \frac{(1 - |a|^2)(x - a) - |x - a|^2a}{|x, a|^2},
\]
where $|x, a| = |x||x^* - a| = |a||x - a^*|$ and $|x, a|^2 = 1 + |x|^2|a|^2 - 2xa$. It is easy to check that $T_a a = 0$, $T_a 0 = -a$ and $T_a$ maps $[-\bar{a}, \bar{a}]$ onto itself. Also,
\[
|T_a x| = \frac{1 - |y|^2}{|x, y|^2}, \quad \text{and in particular} \quad |T_a x| = \frac{1 - |y|^2}{|0, y|^2} = 1 - |y|^2.
\]

Denote by $\widehat{M}(\mathbb{B}^n)$ the set of all Möbius transformations in $\mathbb{B}^n$. For more informations about the Möbius transformations in $\mathbb{B}^n$, see [5]. It is well known that
\[
1 - |T_a x|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{|x, a|^2},
\]
and if $\gamma \in \widehat{M}(\mathbb{B}^n)$, where $\widehat{M}(\mathbb{B}^n)$ is . then
\[
\frac{|\gamma x - \gamma y|}{||\gamma x, \gamma y||} = \frac{|x - y|}{||x, y||}.
\]

In the literature authors often use $-T_a$ instead of $T_a$. To avoid possible confusion we denote $-T_a$ with $\varphi_a$. 

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1.2 Hyperbolic Poisson’s equation

Recall that hyperbolic Laplace operator in the $n$-dimensional hyperbolic ball $B^n$ for $n \geq 2$ is defined as
\[
\Delta_h u(x) = (1 - |x|^2)^2 \Delta u(x) + 2(n-2)(1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}(x).
\]

In terms of the mapping $\varphi_a$, the hyperbolic metric $d_h$ in $B^n$ is given by
\[
d_h(a,b) = \log \left( \frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|} \right)
\]
for all $a,b \in B^n$.

For all $\varphi \in \hat{M}(B^n)$, in [5] it is proved that:
\[
\Delta_h (u \circ \varphi) = \Delta_h u \circ \varphi.
\]

We say that $u : B^n \to \mathbb{R}$ satisfies hyperbolic Laplace equation if $\Delta_h u = 0$ in $B^n$. Non-homogenous hyperbolic Laplace equation, i.e. $\Delta_h u = \psi, \psi \not\equiv 0$ is called hyperbolic Poisson’s equation.

1.3 Hyperbolic Poisson’s kernel and Hyperbolic Green function

Set $0 < r < 1$ and define
\[
g(r) := \int_0^1 \frac{(1-t^2)^{n-2}}{t^{n-1}} \, dt.
\]

Then hyperbolic Green function is given by
\[
g(x,y) = G_h(x,y) = g(|T_y x|) = g \left( \frac{|x-y|}{|x,y|} \right).
\]

For $n = 2$ holds $g(r) = \log \frac{1}{r}$ and for $n > 2$ holds $g(r) \sim \frac{1}{n-2}r^{2-n}$ if $r \to 0$, and $g(r) = O((1-r)^{n-1})$ if $r \to 1$.

Let $d\sigma$ be the $(n-1)$-dimensional Lebesgue measure normalized so that $\sigma(S^{n-1}) = 1$.

The Poisson-Szego kernel $P_h$ for hyperbolic laplacian $\Delta_h$ is given by
\[
P_h(x,t) = \left( \frac{1 - |x|^2}{1 - |t|^2} \right)^{n-1} \sigma^{-1},
\]
which satisfies
\[
\int_{S^{n-1}} P_h(x,t) \, d\sigma(t) = 1.
\]

Let us define hyperbolic Poisson’s integral
\[
P_h[f](x) = \int_{S^{n-1}} P_h(x,t) f(t) \, d\sigma(t).
\]

for $f \in L_1(S^{n-1})$ and, also, the hyperbolic Green integral
\[
G_h[\psi](x) = \int_{B^n} G_h(x,y) \psi(y) \, d\tau(y),
\]
for appropriate functions $\psi$ and
\[
d\tau(x) = \frac{d\nu(x)}{(1 - |x|^2)^n}
\]
where $\nu$ is the $n$-dimensional Lebesgue volume measure normalized so that $\nu(B^n) = 1$. 3
1.4 Main results

Dirichlet problem is well understood for smooth metrics. For example, one can see Chapter IX of [17]. It turns out that this problem for hyperbolic metric on the unit ball with boundary data on the unit sphere is very interesting and it is considered recently in [3]. Here the metric density goes to $\infty$ near the boundary. Among the other things, J. Chen, M. Huang, A. Rasila and X. Wang used nice properties of Möbius transformation and hyperbolic Green function of the unit ball (described for example in [3],[6]) and integral estimate. Precisely, the authors show that if $n \geq 3$ and $u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}}^n, \mathbb{R}^n)$ is a solution to the hyperbolic Poisson equation, then it has a representation

$$u = P_h[\phi] - G_h[\psi],$$

provided that

$$u \mid_{S^{n-1}} = \phi \quad \text{and} \quad \int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| \, d\tau(x) < \infty.$$ 

Here $P_h$ and $G_h$ denote Poisson and Green integrals with respect to $\Delta_h$, respectively. Furthermore, they prove that functions of the form $u = P_h[\phi] - G_h[\psi]$ are Lipschitz continuous.

This can be stated as follows:

Let us consider the following Dirichlet boundary problem

$$\begin{cases}
  u(x) = \phi(x), & \text{if } x \in S^{n-1}, \\
  (\Delta_h)u(x) = \psi, & \text{if } x \in \mathbb{B}^n.
\end{cases}$$

**Theorem A.** [3] Suppose that $u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}}^n, \mathbb{R}^n)$ for $n \geq 3$ and

$$\int_{\mathbb{B}^n} (1-|x|^2)^{n-1} |\psi(x)| \, d\tau(x) \leq \mu_1,$$

where $\mu_1 > 0$ is a constant.

If $u$ satisfies (1.10), then

1. $u = P_h[\phi] - G_h[\psi]$ and
2. $U = u \circ \varphi_x = P_h[\phi \circ \varphi_x] - G_h[\psi \circ \varphi_x], \ x \in \mathbb{B}^n.$

In [3] the following result is also established:

**Theorem B.** [3] Let $n \geq 3$. Suppose that

1. $u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}}^n, \mathbb{R}^n)$ is of the form (1.9);
2. there is a constant $L \geq 0$ such that $|\phi(\xi) - \phi(\eta)| \leq L|\xi - \eta|$ for all $\xi, \eta \in S^{n-1};$
3. there is a constant $M \geq 0$ such that $|\psi(x)| \leq M(1 - |x|^2)$ for all $x \in \mathbb{B}^n.$

Then, there is a constant $N = N(n, L, M)$ such that for $x, y \in \mathbb{B}^n$,

$$|u(x) - u(y)| \leq N|x - y|,$$

where the notation $N = N(n, L, M)$ means that the constant $N$ depends only on the quantities $n, L$ and $M$.

In fact, they prove more general result:

1. if function $\phi$ satisfies (1) then $\Phi = P_h[\phi]$ is Lipshitz and
2. if function $\psi$ satisfies (3) then $\Psi = G_h[\psi]$ is Lipshitz.

Let us note that statement corresponding to (A) is not valid for the euclidean Poisson’s integral. For more details, see below.

In Proposition [4.1],[4.2] we give a brief proof of the above theorem. In [3] Lemma 5.2 the authors used some hypergeometric series techniques for proving inequalities (4.1) and (4.2). Instead of this approach we used Proposition [4] and some basic inequalities, which are modification of
some estimates from planar Hardy space theory (for details see Lemma 4.1, 4.4 and Lemma 4.2), where the simple change of variables
\[ \theta = (1-r)u, \]
in the integral
\[ \int_0^\infty \frac{\theta^{n-2} \theta}{(1-r)^2 + \frac{4r}{\pi^2} \theta^2} d\theta, \]
gives optimal growth estimates, for our purposes.

In Theorem 3.1, it is proved, that local \( \alpha \)-Hölder continuity (\( 0 < \alpha \leq 1 \)) at the boundary point \( x \) of \( \mathbb{B}^n \) implies Hölder continuity along the whole radius of that point. This result is analogous to [MSS, Theorem 6.2], except for the case \( \alpha = 1 \). The case of Lipshitz boundary function \( \phi \) for harmonic functions is investigated by the first author with M. Arsenović and V. Mbojlović in [18]. Under additional condition that \( P[\phi] \) is \( K \)-quasiregular, they proved that \( P[\phi] \) is Lipshitz on \( \mathbb{B}^n \).

In [8] the authors proved that condition (3) can’t be excluded from the statement of the Theorem B. In subsection 3.1 we introduced some alternative assumptions on hyperbolic Laplacian \( \psi \) in order to get Hölder or Lipshitz continuity of hyperbolic Green potential of \( \psi \). Euclidean Green potential, among other things, is investigated by the first author in [9].

In Section 5, we proved, when we assume \( C^1_c \) smoothness of boundary functions \( f \), that all partial derivatives of hyperbolic Poisson’s integral of \( f \) can be continuously extended to the boundary of \( \mathbb{H}^n \). At first, we show that, if \( u \) is hyperbolic harmonic in the upper half-space, then \( \frac{\partial u}{\partial y}(x_0, y) \to 0, y \to 0^+ \), when boundary function \( f \) of the functions \( u \) is differentiable at the boundary point \( x_0 \). Generally, this is not true for (Euclidean) harmonic functions. In [13] we can see an example of \( g \in C^1(\mathbb{R}) \) such that (Euclidean) harmonic extension \( u \) of \( g \) in \( \mathbb{H}^2 \) satisfies \( \limsup_{y \to 0^+} \frac{\partial u}{\partial y}(0, y) = +\infty \).

## 2 Dirichlet problem for Hyperbolic Poisson’s equation

For Harmonic and Subharmonic Function Theory on the Hyperbolic Ball we refer the interested reader to Stoll [6].

**Definition A** (Definition 4.3.1 [6]). Let \( D \) be a subset of \( \mathbb{R}^n \). A function \( f : D \to [-\infty, +\infty) \) is upper semicontinuous at \( x_0 \in D \) if for every \( \alpha \in \mathbb{R}^n \) with \( \alpha > f(x_0) \) there exists a \( \delta > 0 \) such that
\[ f(x) < \alpha \text{ for all } x \in D \cap B(x_0, \delta). \]

**Definition B** (Definition 4.3.3 [6]). Let \( \Omega \) be an open subset of \( \mathbb{B}^n \). An upper semicontinuous function \( f : \Omega \to [-\infty, +\infty) \), with \( f \neq -\infty \), is \( \mathcal{H} \)-subharmonic on \( \Omega \) if
\[ f(a) \leq \int_{S^{n-1}} f(\varphi_a(rt)) d\sigma(t) \]
for all \( a \in \Omega \) and all \( r \) sufficiently small.

**Theorem C** (Theorem 4.6.3 [6]). If \( f \) is a \( \mathcal{H} \)-subharmonic on \( \mathbb{B}^n \), then there exists a unique regular Borel measure \( \mu_f \) on \( \mathbb{B}^n \) such that
\[ \int_{\mathbb{B}^n} \Delta_h \psi \, d\mu_f = \int_{\mathbb{B}^n} f \Delta_h \psi \, d\tau \]
for all \( f \in C^2_c(\mathbb{B}^n) \).
Proof. Let \( \int \) Using Proposition A we get for almost every \( t \) 9.4.1 from [6].

For the second part of this proof use Theorem 8.3.3 subsection 5.9 Theorem 2 and Theorem 9.4.1 from [6].
3 Hyperbolic harmonic functions in the unit ball and local Hölder continuity

Firstly, we refer to the paper [MSS], specifically, to the Proposition 5.10. In this proposition we use the following notation: $\sigma_{n-1}$ is the surface area of the sphere $S^{n-1}$ and $\varphi$ is an angle between radius vector of point $\eta \in S^{n-1}$ and radius vector of the point $\hat{x}$.

Let us define $\sigma_s(n) = \frac{\sigma_{n-2}}{\sigma_{n-1}}$. Using formula $\sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ we get $\sigma_s(n) = \frac{1}{\sqrt{\pi r^{1-1/2^2}}}$.

**Proposition B.** (Proposition 5.10 [MSS]) If $f$ is a function on $S^{n-1}$, then

$$\int_{S^{n-1}} f(\eta) d\sigma(\eta) = \sigma_{n-2} \int_0^\pi f(\varphi) \sin^{n-2} \varphi d\varphi.$$

We will prove the following theorem, which is analogous to [MSS, Theorem 6.2].

**Theorem 3.1.** Suppose that $0 < \alpha \leq 1$, $h$ is a hyperbolic harmonic mapping from $\mathbb{H}^n$ which is continuous on $\mathbb{H}^n$, and

(h1) let $x_0 \in S^{n-1}$ and $|h(x) - h(x_0)| \leq M|x - x_0|^{\alpha}$ for $x \in S^{n-1}$.

Then there is a constant $M_n$ such that

$$(1-r)^{1-\alpha}|h'(r;x_0)| \leq M_n, \quad 0 \leq r < 1.$$

**Proof.** Let $h_b$ denote the restriction of $h$ on $S^{n-1}$. Since $h$ is hyperbolic harmonic on $\mathbb{H}^n$ and continuous on $\mathbb{H}^n$, then

$$h(x) = \int_{S^{n-1}} P_h(x, \eta) h_b(\eta) d\sigma(\eta) \quad (3.1)$$

for every $x \in \mathbb{H}^n$. Set $d := d(x) = 1 - |x|^2$. By computation

$$\partial_{x_k} P_h(x, t) = -2(n-1) \left( \frac{x_k}{|x-t|^2} + d(x) \frac{x_k - t_k}{|x-t|^4} \right) \left( \frac{1 - |x|^2}{|x-t|^2} \right)^{n-2}.$$  

Hence, if $d \leq |x-t|$, then

$$|\partial_{x_k} P_h(x, t)| \leq c_1 \frac{(1 - |x|^2)^{n-2}}{|x-t|^{2(n-1)}}. \quad (3.2)$$

Let $x = re_n$ and let $\theta$ be the angle between $t$ and $e_n$. Then $s := |x-t|^2 = 1 - 2r \cos \theta + r^2$ depends only on $\theta$ for fixed $x$. Next, since $\int_{S^{n-1}} \partial_t P_h(x, t) h(e_n) d\sigma(t) = 0$, we find

$$\partial_{x_k} h(x) = \int_{S^{n-1}} \partial_t P_h(x, t) (h(t) - h(e_n)) d\sigma(t). \quad (3.3)$$

Hence by (3.2) and the hypothesis (h1), we get

$$|\partial_{x_k} h(x)| \leq c_2 (1 - |x|^2)^{n-2} \int_{S^{n-1}} \frac{|e_n - t|^\alpha}{|x-t|^{2(n-1)}} d\sigma(t). \quad (3.4)$$

Therefore the proof of Theorem 3.1 is reduced to the proof of the following proposition. \(\Box\)

**Proposition 3.2.** Suppose that $n \geq 3$ and $0 < \alpha \leq 1$ and $x = re_n$ for $0 < r < 1$. Then

$$I_\alpha(re_n) := (1 - |x|^2)^{n-2} \int_{S^{n-1}} \frac{|e_n - t|^\alpha}{|x-t|^{2(n-1)}} d\sigma(t) \leq c \frac{1}{(1-r)^{1-\alpha}},$$

where $c = c(\alpha, n)$ is a positive constant which depends only on $n$ and $\alpha$.  

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Using similar approach, if \( \omega \) is a majorant and \( \delta_r = 1 - r^2 \), one can prove

\[
I_\omega (r e_n) := (1 - |x|^2)^{n-2} \int_{S^{n-1}} \frac{\omega(|e_n - t|)}{|x - t|^{2(n-1)}} \, d\sigma(t) \leq c \cdot \frac{\omega(\delta_r)}{\delta_r}.
\]

Proof. We use spherical cups \( S^\theta \) defined by \( t_n > \cos \theta \) and integration with parts. Since for a fixed \( \theta \in [0, \pi], |e_n - t| \leq \theta \) for \( t \in S^\theta \), by an application of Proposition 3.2 to \( f(t) = \frac{|e_n - t|^\alpha}{|x - t|^{2(n-1)}} \), we get

\[
I_\alpha (r e_n) \leq c_3 (1 - |x|^2)^{n-2} \int_0^\pi \frac{|\theta|^{n-2} |\theta|^\alpha}{((1 - r)^2 + \frac{4r}{\pi} \theta^2)^{n-1}} \, d\theta.
\]

Next using \( (1 + \frac{4r}{\pi} u^2)^{-1} \leq c_5 (1 + u^2)^{-1} \) for \( \frac{1}{2} \leq r < 1 \) and a change of variables

\[ \theta = (1 - r) u, \]

we find

\[
I_\alpha (r e_n) \leq c_6 (1 - r)^{n-1} \int_0^\infty \frac{u^{\alpha+n-2}}{(1 + u^2)^{n-2}} \, du.
\]

Denote by \( J(\alpha) \) the last expression on the right hand side of the previous formula. Since \( g(u) = \frac{u^{\alpha+n-2}}{(1+u^2)^{n-1}} \sim u^{\alpha-n} \) for \( u \to +\infty \) and by hypothesis \( 0 < \alpha \leq 1 \) (and therefore \( \alpha - n < -1 \)), the integral \( J(\alpha) \) converges and we have

\[
I_\alpha (r e_n) \leq c_7 (1 - r)^{\alpha-1} \quad \text{for} \quad \frac{1}{2} \leq r < 1.
\]

\( (1 - r)^{1-\alpha} A(r) \) is continuous on \([0,1/2]\) and attains a maximum \( c_8 \), that is

\[
I_\alpha (r e_n) \leq c_9 (1 - r)^{\alpha-1} \quad \text{for} \quad 0 \leq r \leq \frac{1}{2},
\]

where \( c_9 = c_3 c_8 \). Hence from (3.8) and (3.9) with \( c = \max\{c_7, c_9\} \) the proof of Proposition follows.

On this point, it is interesting to observe that case \( \alpha = 1 \), in other words, Lipschitz continuity of boundary function, does not imply Lipschitz continuity of its Euclidean harmonic extension. However, in the case of hyperbolic harmonic extension, under the assumptions of Theorem 3.1 we can conclude that, when \( \alpha = 1 \), the statement of this Theorem holds.

Combining Proposition 3.2 and (3.4) we get the proof of Theorem.

4 Growth of partial derivatives for the hyperbolic Green potential

In this section, we consider Hölder continuity and Lipschitz continuity of hyperbolic Green potential on the unit ball in \( \mathbb{R}^n \).

We need the following statement.
Lemma 4.1. Let \( n \geq 3 \) and \( r = |x| \), where \( 0 < r < 1 \). If
\[
A(r, \rho) = \int_{S^{n-1}} \frac{d\sigma(\xi)}{\sqrt{1 - 2\rho(x, \xi) + \rho^2|x|^2}} \quad \text{and} \quad B(r, \rho) = \int_{S^{n-1}} \frac{d\sigma(\xi)}{1 - 2\rho(x, \xi) + \rho^2|x|^2},
\]
then we have the following conclusions.

(i) There exists \( C_1, C_2 > 0 \) such that for all \( 0 < \rho < 1 \) and \( 1/2 < r < 1 \) we have that \( A(r, \rho) \leq \frac{C_1(n)}{\sqrt{\rho}} \) if \( n \geq 3 \) and \( B(r, \rho) \leq \frac{C_2(n)}{\rho} \) if \( n > 3 \).

(ii) If \( n = 3 \) then there exist \( C_3, C_4 > 0 \) such that \( B(r, \rho) \leq C_3 - C_4 \log(1 - \rho) \) for \( 0 < \rho < 1 \) and \( 1/2 < r < 1 \).

Proof. By using Proposition [3] we can check that
\[
A(r, \rho) = \sigma_s(n) \int_0^\pi \frac{\sin^{n-2} \theta}{\sqrt{(1-\rho^2r^2 + 4\rho r^2 \sin^2\theta)^2}} d\theta, \quad B(r, \rho) = \sigma_s(n) \int_0^\pi \frac{\sin^{n-2} \theta}{(1-\rho^2r^2 + 4\rho r^2 \sin^2\theta)^2} d\theta,
\]
where \( \theta \) is an angle between vectors \( x \) and \( \xi \). Using \( \sin x \geq \frac{2}{\pi} x \) for \( x \in (0, \frac{\pi}{2}) \), we get
\[
A(r, \rho) \leq \frac{\tilde{C}_1}{\sqrt{\rho^2}} \int_0^\pi \frac{\theta^{n-2}}{\theta} d\theta, \quad B(r, \rho) \leq \frac{\tilde{C}_2}{\rho^2} \int_0^\pi \frac{\theta^{n-2}}{\theta^2} d\theta.
\]
The first and the second integral converges in case \( n \geq 3 \) and \( n \geq 4 \), respectively, which gives us (i). In order to prove (ii) let us assume \( n = 3 \). Then, we have
\[
B(r, \rho) \leq \sigma_s(3) \int_0^\pi \frac{\theta d\theta}{(1-\rho^2r^2 + 4\rho r^2 \sin^2\theta)^2}
\]
In the case of \( 0 < \rho < 1/2 \) we have \( B(r, \rho) \leq 2\pi^2 \sigma_s(3) \). Let \( 1/2 < r, \rho < 1 \) and let’s use the change of variables \( \theta = (1 - \rho)u \). Then we have
\[
B(r, \rho) \leq \sigma_s(3) \int_0^\pi \frac{u du}{1 + \frac{u^2}{\pi^2}} = \sigma_s(3) \left( \int_0^1 \frac{u du}{1 + \frac{u^2}{\pi^2}} + \int_1^\frac{\pi}{\rho} \frac{u du}{1 + \frac{u^2}{\pi^2}} \right)
\]
\[
\leq C_3 + \frac{\tilde{C}_4}{\rho^2} \int_1^\frac{\pi}{\rho} \frac{u du}{u^2}
\]
\[
\leq C_3 - C_4 \log(1 - \rho).
\]

\( \square \)

Proposition 4.2. Suppose \( n \geq 3, \psi \in C(\mathbb{B}^n, \mathbb{R}^n) \) and \( |\psi(x)| \leq M(1 - |x|^2) \) in \( \mathbb{B}^n \), where \( M \) is a constant. Let
\[
I_{2,k}(x) = \int_{\mathbb{B}^n} \left| \frac{\partial}{\partial x_k} G_h(x, y) \psi(y) \right| d\tau(y).
\]
Then there is a constant \( \beta = \beta(n, M) \) such that
\[
I_{2,k}(x) \leq \beta_1, \quad \text{for all } x \in \mathbb{B}^n.
\]

Proof. Formula (5.3) from [3] gives \( \frac{\partial}{\partial x_k} G_h(x, y) = (D_k G_h)_1(x, y) + (D_k G_h)_2(x, y) \), where
\[
(D_k G_h)_1(x, y) := -\frac{\frac{(x_k - y_k)(1 - |x|^2)^{n-1}(1 - |y|^2)^{n-1}}{n|x - y|^n [x, y]^n}}
\]
\[
(D_k G_h)_2(x, y) = \frac{\partial}{\partial x_k} G_h(x, y).
\]

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and 
\[(D_k G_h)_2(x, y) := \frac{x_k (1 - |x|^2)^{n-2} (1 - |y|^2)^{n-1}}{n |x - y|^{n-2} [x, y]^n}.\]

Also, from [3] we have 
\[I_{2,k}(x) \leq \frac{1}{n} (I_{3,k} + I_{4,k}),\]
where 
\[I_{3,k}(x) = \int_{\mathbb{S}^n} \frac{|x_k - y_k| (1 - |x|^2)^{n-1} (1 - |y|^2)^{n-1}}{|x - y|^n [x, y]^n} |\psi(y)| d\tau(y),\]
and 
\[I_{4,k}(x) = \int_{\mathbb{S}^n} \frac{|x_k| (1 - |x|^2)^{n-2} (1 - |y|^2)^{n-1}}{|x - y|^n [x, y]^n} |\psi(y)| d\tau(y).\]

After applying condition (h3), and introducing change of variable \(y = \varphi_x(w)\), having in mind (1.2) and (1.3), we get 
\[|I_{3,k}(x)| \leq J_{3,k}(x) := M \int_{\mathbb{S}^n} \frac{d\nu(w)}{|x, w|^{n-1}} \quad \text{and} \quad |I_{4,k}(x)| \leq J_{4,k}(x) := M \int_{\mathbb{S}^n} \frac{d\nu(w)}{|x, w|^2 |w|^{n-2}}.\]

(For more details, see [3, Claim 5.3, 5.4]). Finally, using Lemma 4.1 we get 
\[J_{3,k}(x) = n M \int_{\mathbb{S}^n-1} \frac{d\sigma(\xi)}{|x, \rho \xi|^{n-1}} d\rho = \int_0^1 A(r, \rho) d\rho < +\infty \quad (4.1)\]
and 
\[J_{4,k}(x) = n M \int_0^1 \rho d\rho \int_{\mathbb{S}^n-1} \frac{d\sigma(\xi)}{|x, \rho \xi|^2} = \int_0^1 \rho B(r, \rho) d\rho < +\infty. \quad (4.2)\]

Here, we used that \(|x, \rho \xi| = \sqrt{1 - 2\rho(x, \xi) + \rho^2 |x|^2}|.\]

Let \(\Omega \subset \mathbb{R}^n\) and \(F, G: \Omega \to \mathbb{R}^+ := [0, +\infty)\). We use notations \(F(x) \leq G(x), x \in \Omega\) which means that there is \(C > 0\) such that \(F(x) \leq CG(x), x \in \Omega\).

**Lemma 4.3.** Suppose that \(n \geq 3\). Let \(I_m = \int_{\mathbb{S}^n-1} \frac{d\sigma(\xi)}{|x - \xi|^m}\), where \(x \in \mathbb{R}^n\) and \(r = |x|\). When \(r \to 1^-\), we have 
(i) \(I_m\) is bounded for \(0 < m < n - 1\).
(ii) \(I_{n-1} \leq \log \frac{1}{1-r}\).
(iii) \(I_m \leq \frac{1}{(1-r)^{n-1}}\) for \(m > n - 1\).

**Proof.** Using Proposition [3] we can conclude that,
\[I_m = \sigma_*(n) \int_0^\pi \sin^{n-2} \theta \frac{\sin^{n-2} \theta}{(1-r)^2 + 4r \sin^2 \frac{\theta}{2}} \frac{d\theta}{\theta}.\]

Now we can rewrite integral \(I_m\) in the form
\[I_m = \sigma_*(n) \left( \int_{\pi}^{\frac{\pi}{2}} \frac{\sin^{n-2} \theta}{((1-r)^2 + 4r \sin^2 \frac{\theta}{2})^{\frac{m+n}{2}}} d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin^{n-2} \theta}{((1-r)^2 + 4r \sin^2 \frac{\theta}{2})^{\frac{m+n}{2}}} d\theta \right).\]

Since \(\sin x \sim x\) for \(x \in (0, \pi/2)\) we have that, when \(r \to 1^-\), there is \(c_3 > 0\) such that 
\[I_m \leq c_3 \int_{0}^{\frac{\pi}{2}} \frac{\theta^{-2}}{\theta^m} d\theta,\]
which gives us conclusion (i). Also, there exists \(c_4, c_5 > 0\) such that 
\[I_m = c_4 \int_{0}^{\frac{\pi}{2}} \frac{\sin^{n-2} \theta}{((1-r)^2 + 4r \sin^2 \frac{\theta}{2})^{\frac{m+n}{2}}} d\theta + c_5.\]
In order to prove \((ii)\), we use change of variables \(\theta = (1 - r)u\). Then we have
\[
I_m \lesssim \frac{1}{(1 - r)^{m-n+1}} \int_0^{\pi(1-r)} u^{n-2} \frac{du}{(1 + u^2)^{n-2}},
\]
which gives us our conclusion.

Now, let us assume that \(m = n - 1\). We have
\[
I_{n-1} \lesssim \int_0^{\pi(1-r)} u^{n-2} \frac{du}{(1 + u^2)^{n-2}}
= \left( \int_0^1 u^{n-2} \frac{du}{(1 + u^2)^{n-2}} \right) + \int_1^{\pi(1-r)} u^{n-2} \frac{du}{(1 + u^2)^{n-2}}
\lesssim \log \frac{1}{1 - r}.
\]

Before the next result, let us prove the following

**Lemma 4.4.** If \(r = |x|, 0 \leq \rho < 1/2 \leq r < 1\) and \(B(r, \rho)\) is defined as in Lemma 4.1 we have that
\[
\int_0^1 B(r, \rho) \, d\rho < +\infty \quad (4.3)
\]

**Proof.** By using Proposition 4.3 we can check that
\[
B(r, \rho) = \sigma_\star(n) \int_0^\pi \frac{\sin^{n-2} \theta}{(1 - \rho^2 + 4\rho \sin^2 \frac{\theta}{2})^2} \, d\theta.
\]
If \(0 < \rho \leq 1/2\), we have
\[
B(r, \rho) \lesssim \int_0^\pi \frac{\theta^{n-2}}{1/4} \, d\theta = \text{const.}
\]
For \(1/2 < \rho < 1\) we can use
\[
B(r, \rho) \lesssim \frac{1}{\rho} \int_0^\pi \frac{\theta^{n-2}}{\theta^2} \, d\theta \lesssim \frac{1}{\rho}, \text{ for } n > 3.
\]
This means that, for \(n > 3\) we have our result. In the case \(n = 3\) we have that condition \((ii)\) from Lemma 4.1 the conclusion.

**Proposition 4.5.** Suppose \(n \geq 3, \psi \in C(\mathbb{B}^n, \mathbb{R}^n)\) and \(|\psi(x)| \leq M(1 - |x|^2)\) in \(\mathbb{B}^n\), where \(M\) is a constant. Let \(1 \leq k \leq n\) and \(u(x) = \int_{\mathbb{B}^n} G_h(x, y)\psi(y) \, d\tau(y)\). Then
\[
\frac{\partial}{\partial x_k} u(x) = \int_{\mathbb{B}^n} \frac{\partial}{\partial x_k} G_h(x, y)\psi(y) \, d\tau(y),
\]
and there exists a constant \(\beta_1 = \beta_1(n, M)\) such that
\[
\left| \frac{\partial}{\partial x_k} u(x) \right| \leq \beta_1, \text{ for all } x \in \mathbb{B}^n.
\]
Proof. Let us fix \( x \in \mathbb{B}_n \) and denote \( V^x = u \circ \varphi_x \). Then, by using invariance of \( \Delta_h \) we have
\[
\Delta_h V^x(y) = \Delta_h (u \circ \varphi_x)(y) = \Delta_h u (\varphi_x(y)) = \psi (\varphi_x(y)) \quad \text{and}
\]
\[
V^x(0) = \int_{\mathbb{B}_n} G_h(0,y) \Delta_h V^x(y) \, d\tau(y) = \int_{\mathbb{B}_n} G_h(0,y) \psi (\varphi_x(y)) \, d\tau(y).
\]
This means that
\[
\frac{\partial}{\partial x_k} V^x(0) = \int_{\mathbb{B}_n} \frac{\partial}{\partial x_k} G_h(0,y) \psi (\varphi_x(y)) \, d\tau(y).
\]
Formula (5.3) from [3] gives us
\[
\frac{\partial}{\partial x_k} G_h(0,y) = \frac{y_k (1 - |y|^2)^{n-1}}{n|y|^n}.
\]
Now, we have
\[
\left| \frac{\partial}{\partial x_k} V^x(0) \right| \leq \frac{1}{n} \int_{\mathbb{B}_n} \frac{(1 - |y|^2)^{n-1} |\psi (\varphi_x(y))|}{|y|^{n-1}} \, d\tau(y) \leq \int_{\mathbb{B}_n} \frac{(1 - |y|^2)^{n-1}}{|y|^{n-1}} (1 - |\varphi_x(y)|^2) \, d\tau(y).
\]
Since \( 1 - |\varphi_x(y)|^2 = \left( \frac{(1 - |y|^2)(1 - |x|^2)}{|x, y|^2} \right) \), we get
\[
\left| \frac{\partial}{\partial x_k} V^x(0) \right| \leq \int_{\mathbb{B}_n} \frac{d\nu(y)}{|y|^{n-1} |x, y|^2} (1 - |x|^2) = \int_0^1 dp \int_{\mathbb{S}^{n-1}} \frac{d\sigma(\xi)}{1 - 2p(x, \xi) + p^2|x|^2} (1 - |x|^2) \leq 1 - |x|^2,
\]
for \( 1/2 \leq |x| < 1 \), by Lemma 4.1. Using (cf. [3])
\[
|\nabla V^x(0)| \approx |\nabla u(x)|(1 - |x|^2),
\]
we find that \( \nabla u \) is bounded, q.e.d. \( \square \)

4.1 Green function and Riesz potential

At the end of this section, let us investigate some alternative assumptions, which can be stated, instead of condition (h3).

Let us introduce the following condition.

(h3)-1 Function \( \psi \) belongs to \( C(\mathbb{B}^n) \) and \( (1 - |y|^2)^{-2} \psi \) belongs \( L^p(\mathbb{B}^n, \nu(y)) \) for some \( p > n \).

Let \( \mu \in (0,1) \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). We define Riesz potential, as the following operator \( V_\mu \) on \( L^1(\Omega, d\nu) \)
\[
V_\mu(x) = \int_{\Omega} |x - y|^{n(\mu - 1)} f(y) \, d\nu(y).
\]
Now, let us state [GT] Lemma 7.12.

**Lemma A.** The operator \( V_\mu \) maps \( L^p(\Omega, d\nu) \) continuously into \( L^q(\Omega, d\nu) \) for any \( q, 1 \leq q \leq +\infty \), satisfying
\[
0 \leq \delta = \delta(p, q) = \frac{1}{p} - \frac{1}{q} < \mu.
\]
Since
\[
|(D_k G_h)_1(x,y)| \leq K_0(x,y) := \frac{(1 - |x|^2)^{n-1}(1 - |y|^2)^{n-1}}{|x - y|^{n-1} |x|^n}
\]
and \( (1 - |x|^2)^{n-1} \leq [x, y]^n \), we find
\[
|(D_k G_h)_1(x,y)| \leq K_1(x,y) := \frac{(1 - |y|^2)^{n-2}}{|x - y|^{n-1}} \quad \text{and} \quad |(D_k G_h)_2(x,y)| \leq K_2(x,y) := \frac{(1 - |y|^2)^{n-3}}{|x - y|^{n-2}}.
\]
Set
Now, let us introduce the hyperbolic Poisson’s kernel on \( \mathbb{H} \) is the hyperbolic harmonic extension given by (5.1). Then the Poisson’s integral formula for the upper half-space asserts that if \( \psi \) with the set \( \{ x \mid y^2 < 1 \} \) boundary behaviour of the hyperbolic harmonic function on \( \mathbb{R} \) function on \( \mathbb{R} \) function on \( \mathbb{R} \) function on \( \mathbb{R} \) function on \( \mathbb{R} \) function on \( \mathbb{R} \) satisfies (h1) and \( \Delta_h u \) satisfies (4.6). Then we have the following conclusion:

If \( u \) is \( \alpha \)-Hölder on \( \mathbb{H} \), then \( u \) is \( \alpha \)-Hölder on \( \mathbb{H} \), for any \( 0 < \alpha \leq 1 \).

5 Boundary behaviour of the hyperbolic harmonic function on the upper half space

For \( z \in \mathbb{R}^n \) let us write \( z = (x, y) \) with \( x = (x_1, \ldots, x_{n-1}) \). Let us also identified \( \mathbb{R}^{n-1} \) with the set \( \{ z \in \mathbb{R}^n : y = 0 \} \) and denote upper half-space as \( \mathbb{H}^n = \{ (x, y) \in \mathbb{R}^n : y > 0 \} \), and \( V \) is the \( n \)-dimensional Lebesgue volume measure on \( \mathbb{R}^n \).

Now, let us introduce the hyperbolic Poisson’s kernel on \( \mathbb{H}^n \) as

\[
P_h(x, y) = \frac{2}{n \omega_n} \left( \frac{y}{|x|^2 + y^2} \right)^{n-1}.
\]

The Poisson’s integral formula for the upper half-space asserts that if \( f \) is bounded continuous function on \( \mathbb{R}^{n-1} \), then the function \( u \) defined by

\[
u(x, y) = \frac{2}{n \omega_n} \int_{\mathbb{R}^{n-1}} P_h(x - t, y) f(t) dV(t)
\]

is hyperbolic harmonic for \( y > 0 \). The constant \( \omega_n \) in the formula denotes the volume of the unit ball in \( \mathbb{R}^n \). For more details, see [3] Theorem 5.6.2. In this case, we will shortly write \( u = P_h[f] \) in \( \mathbb{H}^n \).

Now we will state the lemma, proof is analogous to the proof of [3] Lemma 6.1.

Lemma 5.1. Let \( f \) be a continuous function with compact support in \( \mathbb{R}^{n-1} \), and \( u \) be its hyperbolic harmonic extension given by (5.1). Then \( u \) satisfies

\[
\lim_{y \to 0} \frac{\partial u}{\partial x_i}(x, y) = 0 \quad \text{for} \quad 1 \leq i \leq n - 1, \quad \text{and} \quad \lim_{y \to 0} \frac{\partial u}{\partial y}(x, y) = 0,
\]

and the convergence is uniform in \( x \in \mathbb{R}^{n-1} \). If, in addition, \( f \in C^1 \) then
\[ \frac{\partial u}{\partial x_i}(x,y) = \frac{2}{n\omega_n} \int_{\mathbb{R}^{n-1}} y^{n-1} \frac{\partial f}{\partial t_i}(t) \, dV(t) \]

for all \( 1 \leq i \leq n-1 \), where \( t = (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} \). Hence \( \frac{\partial u}{\partial x_i} \) is continuous up to the boundary given by \( y = 0 \) for \( 1 \leq i \leq n-1 \). Moreover,

\[ \lim_{y \to 0} \frac{\partial u}{\partial x_i}(x,y) = \frac{\partial f}{\partial x_i}(x). \]

Next, let us state the following condition:

(i) \( f \) belongs to \( C^1(\mathbb{R}^{n-1}) \) and has compact support.

Before we state the main result of this section, let us introduce the following

**Lemma 5.2.** If we denote

\[ I_s^\alpha(y) = \int_{\mathbb{R}^{n-1}} \frac{|x|^\alpha}{(|x|^2 + y^2)^{\frac{n-1}{2}}} \, dV(x), \]

we have that \( I_s^\alpha(y) \leq \frac{1}{y^{n-1}} \), for \( s > n-1 \) and \( 0 < \alpha \leq 1 \).

Special case of Lemma 5.2 gives that there exist \( M_1(n) > 0 \) such that

\[ y^{n-2} \int_{\mathbb{R}^{n-1}} \frac{|x|}{(|x|^2 + y^2)^{n-1}} \, dV(x) \leq M_1(n), \quad y > 0. \]

**Lemma B.** (\[\text{(14)}\]) If \( f \) is a continuous and radial function (i.e. \( f(x) = \tilde{f}(|x|) \)) on the closed ball of radius \( R \) in \( \mathbb{R}^n \), centered at the origin, then

\[ \int_{B(R)} f(x) \, dV(x) = \sigma_{n-1} \int_0^R \tilde{f}(r)r^{n-1} \, dr. \]

Now, we can formulate the next Theorem, which will be important for proof of the main result of this section

**Theorem 5.3.** Let \( f \) satisfies condition (i) and, let, in addition, function \( f \) be differentiable at the point 0. If \( u \) is defined as \[\text{(5.1)}\], then

\[ \lim_{y \to 0} \frac{\partial u}{\partial y}(0,y) = 0. \]

**Proof.** After an easy computation, we can get

\[ \frac{\partial}{\partial y} P_h(x,y) = \frac{2(n-1)}{n\omega_n} \frac{y^{n-2}}{(|x|^2 + y^2)^{n-1}} \frac{|x|^2 - y^2}{|x|^2 + y^2}. \]

If we denote \( q(x,y) := \frac{|x|^2 - y^2}{|x|^2 + y^2} \), we can see that \( |q(z)| \leq 1 \) on \( \mathbb{H}^n \). This gives us that

\[ \left| \frac{\partial}{\partial y} P_h(x,y) \right| \leq K_n(z) := \frac{y^{n-2}}{(|x|^2 + y^2)^{n-1}} \text{ for every } z \in \mathbb{H}^n. \quad (5.2) \]

Let \( M_1(n) \) be as in Lemma 5.2. By the assumption, for every \( \epsilon = \epsilon(0) > 0 \) there exists \( \delta = \delta(0) > 0 \) such that

\[ f(x) - f(0) = a_1 x_1 + \ldots + a_{n-1} x_{n-1} + \epsilon(x)|x|, \text{ where } |\epsilon(x)| < \frac{\epsilon}{2M_1(n) \omega_n} \text{ if } |x| < \delta. \]

Similarly, as in the proof of Theorem \[\text{(4.1)}\] we have
\[ \frac{\partial}{\partial y}u(0, y) = \frac{2}{n\omega_n} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial y} P_h(x, y)(f(x) - f(0)) \, dV(x) = \]
\[= \frac{2}{n\omega_n} \int_{B(\delta)} \frac{\partial}{\partial y} P_h(x, y)(a_1x_1 + \ldots + a_{n-1}x_{n-1} + \epsilon(x|x|) \, dV(x) + \]
\[+ \frac{2}{n\omega_n} \int_{B(\delta)^c} \frac{\partial}{\partial y} P_h(x, y)(f(x) - f(0)) \, dV(x). \]

Since \( x_k \frac{\partial}{\partial y} P_h(x, y) \) is odd function of the variable \( x_k \) and ball \( B(\delta) \) is symmetric domain with respect to hyperplane \( x_k = 0 \), for all \( 1 \leq k \leq n-1 \), we have
\[ \frac{\partial}{\partial y}u(0, y) = \frac{2}{n\omega_n} \left( \int_{B(\delta)} \frac{\partial}{\partial y} P_h(x, y)\epsilon(x)|x| \, dV(x) + \int_{B(\delta)^c} \frac{\partial}{\partial y} P_h(x, y)(f(x) - f(0)) \, dV(x) \right). \]

If we define
\[ I_{1,n}(y) := \int_{B(\delta)} \left| \frac{\partial}{\partial y} P_h(x, y) \right| \epsilon(x)|x| \, dV(x), \quad I_{2,n}(y) := \int_{B(\delta)^c} \left| \frac{\partial}{\partial y} P_h(x, y) \right| |f(x) - f(0)| \, dV(x), \]
we have that \( \left| \frac{\partial}{\partial y}u(0, y) \right| \leq I_{1,n}(y) + I_{2,n}(y) \). Now, using inequality \ref{eq:2} we get
\[ I_{1,n}(y) \leq \epsilon y^{n-2} \int_{\mathbb{R}^{n-1}} \frac{|x|}{(|x|^2 + y^2)^{n-1}} \, dV(x) < \frac{\epsilon n\omega_n}{2}. \]

Also, we have that \( I_{2,n}(y) \leq y^{n-2} \int_{|x| \geq \delta} \frac{\, dV(x)}{(|x|^2 + y^2)^{n-1}} \). Let us denote \( J_{\delta,n}(y) := \int_{|x| \geq \delta} \frac{\, dV(x)}{(|x|^2 + y^2)^{n-1}} \). After introducing change of variables \( x = yt \), we get that
\[ J_{\delta,n}(y) = \frac{1}{y^{n-1}} \int_{|t| \geq \frac{\delta}{y}} \frac{\, dV(t)}{(1+t^2)^{n-1}} = \frac{\sigma_{n-1}}{y^{n-1}} \int_{\frac{\delta}{y}}^{+\infty} \frac{r^{n-2}}{(1+r^2)^{n-1}} \, dr = \frac{\sigma_{n-1}}{y^{n-1}} \int_{0}^{\frac{\delta}{y}} \frac{\rho^{n-2}}{(1+\rho^2)^{n-1}} \, d\rho. \]

Since \( 1 + \rho^2 \geq 1 \) we have that \( J_{\delta,n}(y) \leq \frac{1}{y^{n-1}} \). This means that
\[ I_{2,n}(y) \leq M_2(f, n) y^{n-2} \frac{\epsilon n\omega_n}{2}, \]
if \( 0 < y < \left( \frac{\epsilon \sigma_{n-1} \delta \omega_n}{2 M_2(f, n)} \right)^{1/(n-2)} \). Finally, we have
\[ \left| \frac{\partial}{\partial y}u(0, y) \right| < \epsilon, \text{ for } 0 < y < \delta_0. \]

**Lemma 5.4.** Let \( f \) and \( u \) be functions defined as in Theorem 5.3 and let \( f \) be differentiable in the neighbourhood of the point 0 and let every partial derivative of function \( f \) be continuous at point 0. Then, for every \( \epsilon > 0 \) there exist \( \eta, \delta_0 > 0 \) such that,
\[ \left| \frac{\partial}{\partial y}u(x, y) \right| < \epsilon, \text{ for every } |x| < \eta, 0 < y < \delta_0. \]

In fact, \( \lim_{|z| \to 0} \frac{\partial}{\partial y}u(z) = 0 \).

**Proof.** First, for every \( \epsilon > 0 \) there exist \( \delta, \eta > 0 \), such that
\[ f(x+h) - f(x) = \frac{\partial}{\partial x_1}f(x)h_1 + \ldots + \frac{\partial}{\partial x_{n-1}}f(x)h_{n-1} + \epsilon(x, h)|h|, \]
and \( |\epsilon(x, h)| < \epsilon \) if \( |x| < \eta, |h| < \delta \).

This can easily be proved, using the Mean Value Theorem \[7\] Corollary 10.2.9. Namely, we use that, there are \(\eta > 0\) and \(\delta > 0\) (which does not depend of \(s, t\)) such that
\[
|f(t + h_i e_i) - f(t) - \frac{\partial}{\partial x_i} f(0) h_i| < \frac{\epsilon}{2(n - 1)} |h_i| \quad \text{for} \ |t| < \eta, |h_i| < \delta, 1 \leq i \leq n - 1.
\]
Note that, in this case, we need that
\[
\left| \frac{\partial}{\partial x_i} f(s) - \frac{\partial}{\partial x_i} f(0) \right| < \frac{\epsilon}{2(n - 1)}, \quad \text{for all} \ s < \eta + \delta.
\]
Here, \(e_1, e_2, \ldots, e_{n-1}\) is the standard orthonormal base of \(\mathbb{R}^{n-1}\). Now, we are left only to use the fact that
\[
f(x + h) - f(x) = (f(x + h) - f(x + h - h_1 e_1)) +
(f(x + h - h_1 e_1) - f(x + h - h_1 e_1 - h_2 e_2)) + \ldots +
(f(x + h_{n-1} e_{n-1} - f(x)).
\]
and,
\[
|f(x + h) - f(x) - \frac{\partial}{\partial x_1} f(x) h_1 - \ldots - \frac{\partial}{\partial x_{n-1}} f(x) h_{n-1}| \leq
\frac{1}{|h|} \left\{ |f(x + h) - f(x + h - h_1 e_1) - \frac{\partial}{\partial x_1} f(0) h_1| + |\frac{\partial}{\partial x_1} f(0) - \frac{\partial}{\partial x_1} f(x)| |h_1| +
|f(x + h - h_1 e_1) - f(x + h - h_1 e_1 - h_2 e_2) - \frac{\partial}{\partial x_2} f(0) h_2| +
|\frac{\partial}{\partial x_2} f(0) - \frac{\partial}{\partial x_2} f(x)| |h_2| + \ldots +
|f(x + h_{n-1} e_{n-1}) - f(x) - \frac{\partial}{\partial x_{n-1}} f(0) h_{n-1}| +
|\frac{\partial}{\partial x_{n-1}} f(0) - \frac{\partial}{\partial x_{n-1}} f(x)| |h_{n-1}| \right\} \leq
\leq \epsilon (|h_1| + \ldots + |h_{n-1}|) \leq \epsilon, \quad \text{for} \ |x| < \eta, |h| < \delta.
\]
Now, we use that in Equation \[5.4\] \(\delta_0\) depends of \(\delta\), which has been chosen from the definition of differentiability of function \(f\) at \(0\). Hence, we found an \(\eta\)-neighbourhood of point \(0\) in which \(\delta(x)\), from definition of differentiability of \(f\) at \(x, \) actually, does not depend of \(x\), we can use the proof of Theorem \[5.3\] to get our result.

From the Lemma \[5.4\] we can conclude that

**Theorem 5.5.** Let \(f \in C^1_c(\mathbb{R}^{n-1})\) and \(u = P_h[f]\). Then \(u \in C^1(\mathbb{R}^n)\).

6 Appendix

6.1 General kernels \(P_{\alpha,\beta}\) of Poisson’s type

In \[10\] author introduced special Riemannian metrics on unit ball and upper half-space in \(\mathbb{R}^n\) and corresponding Laplace-Beltrami operators. Following methods described in this article, one can investigate boundary behaviour of solutions to corresponding Dirichlet’s problems. Namely, Poisson’s kernel for this kind of operators, may be written in the form
\[ P_{\alpha,\beta}(x, y) = \frac{(1 - |x|^2)^\alpha}{|x - y|_{2\beta}^{\beta}}, \]

for \( \alpha, \beta \in \mathbb{R}, \beta > 0 \). If we wish to emphasize that \( x \) is a fixed for a moment, sometime we write \( x_0 \) instead of \( x \). Set \( P = P_{\alpha,\beta}, d = d(x) = 1 - |x|^2, E = E(x, y) = |x - y|^2, p = d_{\alpha} = d^{\alpha}, q = E_{\beta} = E^\beta \) and \( X = X(x_0, v^0, y) = \langle \nabla x P_{\alpha,\beta}(x_0, y), v^0 \rangle \), where \( x_0 \in \mathbb{B}_p \) and \( v^0 \in T_{x_0}\mathbb{B}_p \) is the unit vector.

Check that \( X(x, y) = 2PY \), where

\[ Y = Y(x, y) = \beta \langle y, v^0 \rangle - a_0 - \alpha \frac{a_0}{d} = \beta \langle y, v^0 \rangle - b_0, \]

\( a_0 = a_0(x_0, v^0) = \langle x_0, v^0 \rangle \) and \( b_0 = \left( \frac{\beta}{E} + \frac{\alpha}{d} \right) a_0 \). In particular, \( a_0(0, v^0) = 0, b_0(0, v^0) = 0 \).

In harmonic case, \( \alpha = 1, \beta = n/2 \) and for \( y \in \mathbb{S}, P(0, y) = 1, Y(0, y, l) = \frac{n}{2}(y, l) \) and therefore if \( v \) is harmonic bounded on \( \mathbb{B}, D_1v(0) = \langle \nabla x v(0), l \rangle = n \int_0^l y, v^s d\sigma(y) \).

In hyperbolic harmonic case, \( \alpha = \beta = n - 1 \).

In \cite{11} Geller introduced a family of differential operators,

\[ \Delta_{\alpha,\beta} = (1 - |z|^2) \sum_{i,j} \delta_{ij} - z_i \overline{z}_j D_i D_j + \alpha R + \beta \overline{R} - \alpha \beta, \]

where \( R = \sum_z D_i \). \( \varphi_z \) is the automorphism of the ball that maps \( z \) to 0, and such that \( \varphi_z^2 = \text{Id} \) (see \cite{12} pg 297.)

\[ d\lambda(z) = \frac{1}{(1 - |z|^2)^{n+1}} dV(z) \quad (6.1) \]

For Dirichlet problem see formula (3.5) in \cite{11}:

\[ u(z) = \int_{\mathbb{B}^n} u(\zeta) P_{\alpha,\beta}(z, \zeta) dA(\zeta) + \int_{\mathbb{B}^n} \Delta_{\alpha,\beta} u(\omega)(1 - \overline{z}\omega)^\alpha (1 - z\overline{\omega})^\beta (1 - |\omega|^2)^{-\alpha - \beta} d\lambda(\omega) \quad (6.2) \]

For instance, it holds if \( u \in C^2(\mathbb{B}^n) \cap C(\overline{\mathbb{B}^n}) \) and

\[ \int_{\mathbb{B}^n} |\Delta_{\alpha,\beta} u(\omega)| \frac{dV(\omega)}{1 - |\omega|} < +\infty. \quad (6.3) \]

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