Simplicial approach to derived differential manifolds

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Abstract

Derived differential manifolds are constructed using the usual homotopy theory of simplicial rings of smooth functions. They are proved to be equivalent to derived differential manifolds of finite type, constructed using homotopy sheaves of homotopy rings (D.Spivak), thus preserving the classical cobordism ring. This reduction to the usual algebraic homotopy can potentially lead to virtual fundamental classes beyond obstruction theory.

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1 Introduction

The objective of this paper is to simplify the theory of derived smooth manifolds, developed in [Sp10], so that one can work with derived manifolds using just simplicial $C^\infty$-rings, instead of homotopy sheaves of homotopy $C^\infty$-rings.

This complete elimination of sheaf theoretic techniques is possible because the usual softness of structure sheaves on classical manifolds remains
true after deriving: every derived differential manifold of finite type\(^1\) (as defined in \([Sp10]\)) is locally weakly equivalent to a derived manifold, whose structure sheaf is a simplicial diagram of soft sheaves.

In \([Sp10]\) it is proved, that derived differential manifolds, constructed there, have the same cobordism ring as classical manifolds, with intersections being performed by taking homotopy limits.

Our reduction preserves this property, and thus we provide a model for intersection theory of differential manifolds, using the usual simplicial closed model structure on the category of simplicial \(C^\infty\)-rings. Intersections are obtained by taking the usual homotopy colimits.

This ability to express intersections homotopically correctly and functorially, by using a single simplicial ring, instead of a homotopy sheaf of such rings, potentially allows one to go beyond obstruction theories in working with virtual fundamental classes.

Here is our approach: we use \(C^\infty\)-rings as the basis for everything we do in smooth geometry. Such rings are just algebras over a particular algebraic theory (\([La63]\), we recall the definition at the beginning of Section \(^2\). The theory of \(C^\infty\)-rings is well developed (e.g. \([MR91]\), \([Du81]\), \([GS03]\), \([Jo11a]\), and many others).

Homotopy theory is defined on the category of simplicial \(C^\infty\)-rings, i.e. simplicial diagrams of \(C^\infty\)-rings. These rings inherit a simplicial closed model structure from the category of simplicial sets (\([Qu67]\)), and it is this homotopy theory that we use. We recall the definition in Section \(^3\).

In \([Sp10]\) homotopy is based on the notion of local weak equivalence. Given a topological space \(X\), one considers the category of sheaves of simplicial \(C^\infty\)-rings, and defines local weak equivalence to be a map that induces an isomorphism between sheaves of the corresponding homotopy groups.

This requirement to have an isomorphism on the level of sheaves of homotopy groups and not merely pre-sheaves, is responsible for the local nature of local weak equivalences. In the simplicial setting this notion was defined in \([Ja87]\). To work with this homotopy theory in terms of closed model categories, one needs hypercovers \([DHI04]\), and they are used in \([Sp10]\), resulting in the notion of a homotopy sheaf of homotopy \(C^\infty\)-rings. We recall the definition in Section \(^4\).

Different from algebraic geometry, local weak equivalences in differential geometry can be treated in a very simple manner. Using softness of structure

\(^1\) A derived manifold is of finite type, if it is possible to embed its classical part into \(\mathbb{R}^n\) for some \(n \geq 0\).
sheaves, we prove that every derived manifold of finite type is locally weakly equivalent to a derived manifold, whose presheaves of homotopy groups are already sheaves. This implies that the functor of global sections maps local weak equivalences to weak equivalences, and so we can go from sheaves to simplicial $C^\infty$-rings without losing any homotopical structure.

In [Sp10], instead of pre-sheaves of $C^\infty$-rings, pre-sheaves of homotopy $C^\infty$-rings are used, i.e. one requires that structure equations are not literally satisfied, but only up to homotopy. One can rectify such homotopy rings into honest $C^\infty$-rings, and it can be done in terms of a Quillen equivalence ([Ba02], [Be06]). Thus we get the following diagram of adjunctions:

\[
\begin{array}{c}
\text{Sheaves of simplicial } C^\infty\text{-rings} \\
\downarrow^\Gamma \\
\text{Simplicial } C^\infty\text{-rings}
\end{array} \xleftarrow{\text{Spivak’s theory}} \xrightarrow{\text{adjunctions}} \xrightarrow{\text{ functor of global sections}} \xrightarrow{\Gamma} \xrightarrow{\text{Spivak’s theory}} \xrightarrow{\text{adjunctions}} \xrightarrow{\Gamma} \xrightarrow{\text{Spivak’s theory}} \xrightarrow{\text{adjunctions}} \xrightarrow{\Gamma} \xrightarrow{\text{Spivak’s theory}} \xrightarrow{\text{adjunctions}} \xrightarrow{\Gamma} \xrightarrow{\text{Spivak’s theory}}
\]

where the categories in the first row come with local weak equivalences, the third category has the usual weak equivalences. Assuming all manifolds are of finite type, each functor maps (local) weak equivalences to (local) weak equivalences. Moreover, all units and counits are (local) weak equivalences.

Going over to simplicial localizations of these categories, we conclude that these adjunctions induce weak equivalences. In [Sp10] derived manifolds are defined by gluing affine derived manifolds, which are just intersections, computed as homotopy limits. Weak equivalences of simplicial localizations, given by (1), show that all of this can be done in the category of simplicial $C^\infty$-rings, using the usual closed model structure.

Here is the plan of the paper: in Section 2 we describe the correspondence between $C^\infty$-spaces and $C^\infty$-rings. Using softness of the structure sheaves, one proves that this is an equivalence of categories. This material is standard, and we provide it mostly to fix the notation.

In Section 3 we extend the classical results from Section 2 to the simplicial case. We show that the standard model category of simplicial $C^\infty$-rings is a model for the category of simplicial $C^\infty$-spaces, where gluing is performed by local weak equivalences.

In Section 4 we recall the construction from [Sp10], and (assuming finite type) prove that it provides a model for simplicial $C^\infty$-spaces. Thus [Sp10] is reduced, via the category of simplicial $C^\infty$-spaces, to the usual construction, involving homotopy colimits in the category of simplicial $C^\infty$-rings. This correspondence, restricted to good fibrant replacements on the side of [Sp10], is just the functor of global sections.
We would like to indicate some differences between the approach to derived geometry adopted here, and other places. When one glues by weak equivalences, one can choose the underlying topological spaces to be spectra of the 0-th homotopy group or the 0-th component of the structure sheaf. We (and [Sp10]) use the former, while [CK01], [CK02], [TV05], [TV08] use the latter.

In [Jo11b] another approach to derived manifolds is used, where instead of sheaves of simplicial $C^\infty$-rings, one uses sheaves of presentations of $C^\infty$-rings. We believe that constructions of [Jo11a] can be obtained from ours by truncating simplicial sets from level 2 and up. In particular, this would allow one to reformulate the theory of [Jo11b] in terms of $C^\infty$-rings, and their modules. We will pursue this elsewhere.

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2 $C^\infty$-rings and $C^\infty$-spaces

Let $C^\infty R$ be the category of product preserving functors

$$A : C^\infty \longrightarrow \text{Set},$$

where $C^\infty$ has $\{\mathbb{R}^n\}_{n \geq 0}$ as objects, and smooth maps as morphisms. Clearly, any such $A$ is determined (up to a unique isomorphism) by the set $A(\mathbb{R})$ and the action of $\{C^\infty(\mathbb{R}^n)\}_{n \geq 0}$ on $A(\mathbb{R})$, making it into a $C^\infty$-ring. We will write $A$ to mean both the functor and the corresponding $C^\infty$-ring.

As an example consider a smooth manifold $X$, i.e. a Hausdorff, second countable space with smooth Euclidean atlas of bounded dimension. The set of smooth functions on $X$ is a $C^\infty$-ring, moreover the assignment $X \mapsto C^\infty(X)$ is a full and faithful functor (e.g. [MR91]).

The forgetful functor $C^\infty R \rightarrow \text{Set}$, defined by $A \mapsto A(\mathbb{R})$, has a left adjoint

$$S \longrightarrow C^\infty(\mathbb{R}^S),$$

where $C^\infty(\mathbb{R}^S)$ is the ring of smooth finite functions on $\mathbb{R}^S := \text{Hom}_{\text{Set}}(S, \mathbb{R})$, i.e. functions that factor through a projection $\mathbb{R}^S \rightarrow \mathbb{R}^F$, with $F \subseteq S$ finite, and a smooth function $\mathbb{R}^F \rightarrow \mathbb{R}$. 
A $C^\infty$-ring $A$ is **local**, if $A$ has a unique maximal ideal $m \subset A$, and $A/m \cong \mathbb{R}$. A typical example of a local $C^\infty$-ring is the ring of germs of smooth functions at the origin of $\mathbb{R}^n$. A $C^\infty$-ring $A$ is finitely generated if $A$ is a quotient of $C^\infty(\mathbb{R}^n)$. Following [MR91], we will denote the full subcategory of $C^\infty\mathcal{R}$, consisting of finitely generated $C^\infty$-rings, by $\mathcal{L}$. For any smooth manifold (second countable and of finite dimension), the $C^\infty$-ring of smooth functions is finitely generated (e.g. [MR91]).

A $C^\infty$-space is a pair $(X, \mathcal{O}_X)$, where

1. $X$ is a Hausdorff topological space,
2. $\mathcal{O}_X$ is a soft sheaf of finitely generated $C^\infty$-rings on $X$,
3. $\forall p \in X$, the stalk $(\mathcal{O}_X)_p$ is a local $C^\infty$-ring.

A morphism of $C^\infty$-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a pair $(\phi, \phi^*)$, where $\phi : X \to Y$ is a continuous map, and $\phi^* : \mathcal{O}_Y \to \phi_*(\mathcal{O}_X)$ is a morphism of sheaves of $C^\infty$-rings. We will denote the category of $C^\infty$-spaces by $\mathcal{G}$. Note that we require the structure sheaf to be a sheaf of finitely generated $C^\infty$-rings, i.e. our $C^\infty$-spaces are of finite type. This requirement is equivalent to demanding that a given space is embeddable into some $\mathbb{R}^n$, or, in the case of manifolds, that the dimension is finite.

There is an obvious functor

$$\Gamma : \mathcal{G} \to \mathcal{L}^{op}, \quad (X, \mathcal{O}_X) \mapsto \Gamma(X, \mathcal{O}_X).$$

(4)

This functor has a right adjoint ([Du81], theorem 8), that we now construct. Let $A \in \mathcal{L}$, spectrum of $A$ is the $C^\infty$-space $(Sp(A), \mathcal{O}_{Sp(A)})$, where

$$Sp(A) := Hom_{\mathcal{L}}(A, \mathbb{R}),$$

(5)
equipped with Zariski topology, and $\mathcal{O}_{Sp(A)}$ is given by localization. Here is the explicit description: a basic open subset of $Sp(A)$ is

$$U_a := \{ p : A \to \mathbb{R} \text{ s.t. } p(a) \neq 0 \}, \quad a \in A.$$  

(6)

Clearly $U_{a_1} \cap U_{a_2} = U_{a_1 a_2}$, and hence $\{U_a\}_{a \in A}$ is indeed a basis of a topology. Choosing a presentation $A \cong C^\infty(\mathbb{R}^n)/\mathfrak{a}$, one can identify

$$Sp(A) \cong \{ p : C^\infty(\mathbb{R}^n) \to \mathbb{R} \text{ s.t. } p(\mathfrak{a}) = 0 \},$$

(7)

Note, that the corresponding morphisms between stalks are automatically local, since they are morphisms of local $\mathbb{R}$-algebras, that have $\mathbb{R}$ as the residue field.
i.e. $\text{Sp}(A)$ consists of the zeroes of $A$ in $\mathbb{R}^n$. Every open subset of $\mathbb{R}^n$ has a characteristic function ([MR91], lemma I.1.4), therefore, Zariski topology on $\text{Sp}(A)$ coincides with the topology, induced from $\mathbb{R}^n$, and every open subset of $\text{Sp}(A)$ is of the form ($\mathbb{B}$).

For each $a \in A$, let $A\{a^{-1}\}$ be the smooth localization of $A$ at $a \neq 0$, i.e. it is obtained by universally inverting (in the category of $C^\infty$-rings) every $a' \in A$, s.t. $\forall p \in \text{Sp}(A) \ p(a) \neq 0 \Rightarrow p(a') \neq 0$. We define $\mathcal{O}_{\text{Sp}(A)}$ to be the sheaf, associated to $U \mapsto A\{a^{-1}\}$.

There is another description of $\mathcal{O}_{\text{Sp}(A)}$. Let $\mathcal{O}_{\mathbb{R}^n}$ be the sheaf of $C^\infty$-functions on $\mathbb{R}^n$, and let $a \subseteq \mathcal{O}_{\mathbb{R}^n}$ be subsheaf of ideals, defined as follows:

$$f \in \Gamma(U, a) \text{ if and only if } \forall p \in U, f_p \in \mathfrak{A}_p.$$  \hfill (8)

Denoting by $\iota : \text{Sp}(A) \subseteq \mathbb{R}^n$ the inclusion, given by $C^\infty(\mathbb{R}^n) \to A$, it is easy to see that

$$\mathcal{O}_{\text{Sp}(A)} \cong \iota^*(\mathcal{O}_{\mathbb{R}^n})/\iota^*(a),$$  \hfill (9)

and hence $\mathcal{O}_{\text{Sp}(A)}$ is soft, and its stalks are local $C^\infty$-rings.

Finally, being a subspace of $\mathbb{R}^n$, $\text{Sp}(A)$ is clearly Hausdorff. Therefore $(\text{Sp}(A), \mathcal{O}_{\text{Sp}(A)}) \in \mathcal{G}$, and universal property of localization gives us a functor

$$\text{Sp} : \mathcal{C}^{op} \to \mathcal{G}, \quad A \mapsto (\text{Sp}(A), \mathcal{O}_{\text{Sp}(A)}).$$  \hfill (10)

In general, it is not true that $A \cong \Gamma(\text{Sp}(A), \mathcal{O}_{\text{Sp}(A)})$. Consider the following example: let $A := C^\infty(\mathbb{R}^2)/\mathfrak{A}$, where $\mathfrak{A}$ consists of functions that vanish in some product neighborhood of the $y$-axis. Then $\Gamma(\text{Sp}(A), \mathcal{O}_{\text{Sp}(A)}) = C^\infty(\mathbb{R}^2)/\overline{\mathfrak{A}}$, where $\overline{\mathfrak{A}}$ is the ideal of functions that vanish in some (arbitrary) neighborhood of the $y$-axis. Clearly $\mathfrak{A} \neq \overline{\mathfrak{A}}$.

This example can be generalized. An ideal $\mathfrak{A} \subseteq C^\infty(\mathbb{R}^n)$ is called germ determined$^3$ if

$$\forall f \in C^\infty(\mathbb{R}^n) - \mathfrak{A}, \exists p \in \mathbb{R}^n \text{ s.t. } f_p \notin \mathfrak{A}_p.$$  \hfill (11)

A $C^\infty$-ring $A$ is germ determined, if $A \cong C^\infty(\mathbb{R}^n)/\mathfrak{A}$, with $\mathfrak{A}$ being germ determined. We will denote the full subcategory of $\mathcal{L}$, consisting of germ determined $C^\infty$-rings, by $\mathcal{G}$. The inclusion $\mathcal{G} \subseteq \mathcal{L}$ has a left adjoint, that we will denote by $A \mapsto \tilde{A}$. Explicitly: $\tilde{A} \cong C^\infty(\mathbb{R}^n)/\overline{\mathfrak{A}}$, where $\overline{\mathfrak{A}}$ is the germ determined ideal, generated by $\mathfrak{A}$. Proof of the following proposition is straightforward (use (9)).

$^3$In [Du81] such ideals are called ideals of local character, we adopt the terminology from [MR91].
Proposition 1 Let $A \in \mathcal{L}$, then $\forall a \in A$, $\Gamma(U_a, O_{Sp(A)}) \cong A\{a^{-1}\}$.

In particular, if $A$ is germ determined, $\Gamma(Sp(A), O_{Sp(A)}) \cong A$. Actually, this is part of an equivalence of categories.

Proposition 2 ([Du81]) Let $(X, \mathcal{O}_X) \in \mathcal{G}$, then $\Gamma(X, \mathcal{O}_X) \in \mathcal{G}$. Moreover,

$$\text{Sp} : \mathcal{G}^{op} \rightleftarrows \mathcal{G} : \Gamma$$

is an equivalence of categories.

Proof: Since objects of $\mathcal{G}$ are locally ringed spaces, standard technique shows that $\Gamma : \mathcal{G} \to \mathcal{L}^{op}$ is left adjoint to $\text{Sp} : \mathcal{L}^{op} \to \mathcal{G}$. Let $(X, \mathcal{O}_X) \in \mathcal{G}$, using this adjunction and Proposition 1 we see that the identity map on $\Gamma(X, \mathcal{O}_X)$ factors through $\text{Sp}(\Gamma(X, \mathcal{O}_X))$, and therefore $\Gamma(X, \mathcal{O}_X) \in \mathcal{G}$.

Let $A := \Gamma(X, \mathcal{O}_X)$. To prove that the adjunction $\Gamma : \mathcal{G} \rightleftarrows \mathcal{G}^{op} : \text{Sp}$ is an equivalence, we need to show that $(\iota, \iota^*) : (X, \mathcal{O}_X) \to (Sp(A), O_{Sp(A)})$, corresponding to the identity on $(X, \mathcal{O}_X)$, is an isomorphism. Suppose there was $p : \Gamma(X, \mathcal{O}_X) \to \mathbb{R}$, that did not correspond to a point in $X$. Let $m_p$ be the kernel of $p$. Since $\Gamma(X, \mathcal{O}_X)$ is finitely generated, all of its maximal ideals, having $\mathbb{R}$ as the residue field, are finitely generated, and hence there are $\alpha_1, \ldots, \alpha_n \in m_p$ generating $m_p$. By assumption there is no point in $X$, where all $\alpha_i$'s vanish, therefore $\alpha_1^2 + \ldots + \alpha_n^2 \neq 0$ on $X$, and hence it is invertible, contradiction to existence of $p$. This means that $\iota$ is surjective. Since $X$ is Hausdorff and $\mathcal{O}_X$ is soft, it is standard to prove, that $\iota$ is also injective and its inverse is continuous.

Now we identify $X = Sp(A)$. Since $\mathcal{O}_X$ is soft, $\forall p \in X$, $A = \Gamma(X, \mathcal{O}_X) \to (\mathcal{O}_X)_p$ is surjective, and hence $(\mathcal{O}_{Sp(A)})_p \to (\mathcal{O}_X)_p$ is surjective as well. Let $a \in A$, s.t. $a_p \mapsto 0_p \in (\mathcal{O}_X)_p$, then there is an open $p \in U \subseteq X$, s.t. $\forall q \in U$ $a_q \mapsto 0_q \in (\mathcal{O}_X)_p$. Let $b \in A$, s.t. $b_p = 1_p \in (\mathcal{O}_{Sp(A)})_p$, $b_q = 0_q \in (\mathcal{O}_{Sp(A)})_q \forall q \in X - U$. Then, clearly, $(ab)_q \mapsto 0_q \in (\mathcal{O}_X)_q \forall q \in X$, i.e. $ab \mapsto 0 \in \Gamma(X, \mathcal{O}_X)$. Therefore $ab = 0 \in A$, and hence $a_p = 0 \in (\mathcal{O}_{Sp(A)})_p$, i.e. $\mathcal{O}_{Sp(A)} \to \mathcal{O}_X$ is an isomorphism. $\blacksquare$

From Proposition 2 it is clear, that for a finitely generated $C^\infty$-ring, being germ determined is equivalent to being geometric, i.e. being the ring of smooth functions on a $C^\infty$-space. The requirement to be finitely generated is essential, since even the free $C^\infty$-ring $C^\infty(\mathbb{R}^S)$, with $S$ infinite, is not isomorphic to $\Gamma(Sp(C^\infty(\mathbb{R}^S)), O_{Sp(C^\infty(\mathbb{R}^S)))}.$

Yet, one has the notion of a germ determined $C^\infty$-ring also in the infinitely generated case ([Bo11]). In general, for $A$ to be germ determined is equivalent to $A \to \Gamma(Sp(A), O_{Sp(A)})$ being injective.
Surjectivity is more complicated. If $Sp(A)$ is paracompact, it is enough for $A$ to be locally complete\footnote{In [Jo11a] this is called being complete with respect to locally finite sums.} i.e. if a family $\{a_i\} \subseteq A$ is s.t. $\forall p \in Sp(A)$ only finitely many of $\{(a_i)_q\}$ are not 0 in a neighborhood of $p$, then $\exists a \in A$, s.t. $\forall p \in Sp(A) \ a_p = \Sigma(a_i)_p$. Any finitely generated, germ determined $C^\infty$-ring is complete in this sense. This is not always true for the infinitely generated ones.

These notions can be extended to modules over $C^\infty$-rings.\footnote{Modules over $C^\infty$-rings are just modules over the underlying commutative $R$-algebras.} An $A$-module $M$ is germ determined and locally complete if for any family $\{t_i\} \subseteq M$, s.t $\forall p \in Sp(A)$ almost all of $\{(t_i)_q\}$ are 0 in a neighborhood of $p$, there is a unique $t \in M$, s.t. $\forall p \Sigma(t_i)_p = t_p$. We will denote the category of such $A$-modules by $\text{Mod}_A$. Since sheaves of modules over soft sheaves of rings are themselves soft, we have the following conclusion.

Proposition 3 Let $A$ be a finitely generated $C^\infty$-ring. The functor of global sections defines an equivalence between the category of sheaves of $O_{Sp(A)}$-modules and $\text{Mod}_A$.

3 Simplicial $C^\infty$-rings and simplicial $C^\infty$-spaces

A simplicial $C^\infty$-ring is a simplicial diagram

$$A_\bullet := (A_k, \{\partial_{k+1,i} : A_{k+1} \to A_k\}_{i=0}^{k+1}, \{\delta_{k,j} : A_k \to A_{k+1}\}_{j=0}^{k})_{k \geq 0} \quad (13)$$

in the category of $C^\infty$-rings. We don’t require any $A_k$ to be finitely generated. Morphisms are the usual natural transformations between simplicial diagrams. We will denote the category of simplicial $C^\infty$-rings by $sC^\infty R$.

From [Qu67] we know that $sC^\infty R$ is a simplicial closed model category, where weak equivalences and fibrations are those morphisms that define respectively weak equivalences and fibrations between the underlying simplicial sets. Since each simplicial $C^\infty$-ring is a simplicial $R$-module, it is enough to calculate homotopy groups at one chosen point (e.g. 0), and, moreover,

$$\pi_k(A_\bullet, 0) \cong H_k(M(A_\bullet)), \quad (14)$$

where $M(A_\bullet) = (\bigoplus A_k, \Sigma(-1)^i \partial_{k+1,i})$ is the Moore complex of $A_\bullet$. We will write $\pi_k(A_\bullet)$ instead of $\pi_k(A_\bullet, 0)$.
Boundaries \( \partial_{1,0}, \partial_{1,1} : A_1 \to A_0 \) define a \( C^\infty \)-congruence on \( A_0 \), and \( \pi_0(A_\bullet) \) is the quotient of \( A_0 \) by this congruence. Therefore \( \pi_0(A_\bullet) \) is a \( C^\infty \)-ring. Moreover, since each \( A_k \) is in particular a commutative \( \mathbb{R} \)-algebra, \( \forall k \geq 1 \pi_k(A_\bullet) \) is a module over \( \pi_0(A_\bullet) \).

Let \( X \) be a topological space, and let \( O_{\bullet,X} \) be a sheaf of \( C^\infty \)-rings on \( X \). We will denote by \( \pi_k(O_{\bullet,X}) \), \( k \geq 0 \) the sheaves associated to presheaves

\[
U \longrightarrow \pi_k(\Gamma(U, O_{\bullet,X})), \quad k \geq 0.
\]

Clearly, \( \pi_0(O_{\bullet,X}) \) is a sheaf of \( C^\infty \)-rings, and \( \{\pi_k(O_{\bullet,X})\}_{k \geq 1} \) are sheaves of \( \pi_0(O_{\bullet,X}) \)-modules.

A simplicial \( C^\infty \)-space is a pair \( (X, O_{\bullet,X}) \), where \( X \) is a topological space, and \( O_{\bullet,X} := \{O_k, X\}_{k \geq 1} \) is a sheaf of simplicial \( C^\infty \)-rings on \( X \), s.t. \((X, \pi_0(O_{\bullet,X}))\) is a \( C^\infty \)-space (Section 2), i.e. \( X \) is Hausdorff, and \( \pi_0(O_{\bullet,X}) \) is a soft sheaf of finitely generated \( C^\infty \)-rings, whose stalks are local \( C^\infty \)-rings.

A morphism \( (X, O_{\bullet,X}) \to (Y, O_{\bullet,Y}) \) is given by a pair \( (\phi, \phi^\sharp) \), where \( \phi : X \to Y \) is a continuous map, and \( \phi^\sharp := \{\phi_k^\sharp : O_k, Y \to \phi_*(O_k, X)\}_{k \geq 0} \) is a morphism of sheaves of simplicial \( C^\infty \)-rings. We will denote the category of simplicial \( C^\infty \)-spaces by \( sG \).

### 3.1 Local weak equivalences

Given \( (X, O_{\bullet,X}) \in sG \), we have a \( C^\infty \)-space \((X, \pi_0(O_{\bullet,X}))\) and a sequence \( \{\pi_k(O_{\bullet,X})\}_{k \geq 1} \) of sheaves of modules on it. We will denote this \( C^\infty \)-space, together with the sheaves of modules, by \( \pi_\bullet(X, O_{\bullet,X}) \). Every morphism \((\phi, \phi^\sharp) : (X, O_{\bullet,X}) \to (Y, O_{\bullet,Y})\) in \( sG \) induces a morphism

\[
\pi_\bullet(\phi, \phi^\sharp) : \pi_\bullet(X, O_{\bullet,X}) \longrightarrow \pi_\bullet(Y, O_{\bullet,Y}).
\]

We call \((\phi, \phi^\sharp)\) a local weak equivalence, if \( \pi_\bullet(\phi, \phi^\sharp) \) is an isomorphism.

Let \( sL \subset sC^\infty R \) be the full subcategory, consisting of \( A_\bullet \), s.t. \( \pi_0(A_\bullet) \) is a finitely generated \( C^\infty \)-ring.\(^6\) There is an obvious functor

\[
\Gamma : sG \to sL^{op}, \quad (X, O_{\bullet,X}) \mapsto \Gamma(X, O_{\bullet,X}).
\]

In general \( \Gamma \) does not map local weak equivalences in \( sG \) to weak equivalences in \( sL^{op} \). It does so for a particular kind of simplicial \( C^\infty \)-spaces, that we are going to consider next.

\(^6\)Note that we do not require any \( A_k \) to be finitely generated, only \( \pi_0(A_\bullet) \).
A localized simplicial $C^\infty$-space is a $C^\infty$-space $(X, \mathcal{O}_{\bullet,X})$, s.t. stalks of $\mathcal{O}_{0,X}$ are local $C^\infty$-rings. We will denote by $s\underline{\mathcal{G}} \subset s\mathcal{G}$ the full subcategory, consisting of localized simplicial $C^\infty$-spaces. The following proposition explains why $\Gamma$, restricted to $s\underline{\mathcal{G}}$, maps local weak equivalences to weak equivalences.

**Proposition 4** Let $(X, \mathcal{O}_{\bullet,X}) \in s\underline{\mathcal{G}}$, then $\mathcal{O}_{0,X}$ is a soft sheaf.

**Proof:** Since $(X, \pi_0(\mathcal{O}_{\bullet,X}))$ is a $C^\infty$-space, $X$ is paracompact. Therefore ([Go60], th. II.3.7.2), to prove that $\mathcal{O}_{0,X}$ is soft, it is enough to show that it is locally soft, i.e. $\forall p \in X$ there is a neighborhood $U \ni p$, s.t. $\forall V_1, V_2 \subseteq U$ closed, with $V_1 \cap V_2 = \emptyset$, there is $\alpha \in \Gamma(U, \mathcal{O}_{0,X})$, s.t. $\alpha V_1 = 0, \alpha V_2 = 1$.

Let $p \in X$, and let $\{f_i\}_{1 \leq i \leq n}$ be a set of generators of $\Gamma(X, \pi_0(\mathcal{O}_{\bullet,X}))$ as a $C^\infty$-ring. Since $(\mathcal{O}_{0,X})_p \to (\pi_0(\mathcal{O}_{\bullet,X}))_p$ is surjective, there is a neighborhood $W \ni p$, s.t. $\{f_i|_W\}_{i=1}^n$ are in the image of $\Gamma(W, \mathcal{O}_{0,X}) \to \Gamma(W, \pi_0(\mathcal{O}_{\bullet,X}))$. Choose a closed neighborhood $U \ni p$, s.t. $U \subseteq W$, we claim that $U$ satisfies the conditions of theorem II.3.7.2 in [Go60].

Indeed, since $\pi_0(\mathcal{O}_{\bullet,X})$ is soft, $\Gamma(X, \pi_0(\mathcal{O}_{\bullet,X})) \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is surjective, and hence $\Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is generated by the images of $\{f_i\}_{i=1}^n$. Therefore $\Gamma(U, \mathcal{O}_{0,X}) \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is surjective. We can choose a finitely generated $C^\infty$-subring $A \subseteq \Gamma(U, \mathcal{O}_{0,X})$, s.t. $A \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is surjective. We claim that $U \cong Sp(A)$ as topological spaces.

Suppose not, surjectivity of $A \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ implies that $U \subseteq Sp(A)$ as a closed subspace, hence $Sp(A) - U$ is non-empty and open. Therefore, $\exists a \in A$, s.t. $a \neq 0, a_U = 0$, when considered as a section of $\mathcal{O}_{Sp(A)}$. Since $A$ is finitely generated, $\exists b \in A$, s.t. $ab = 0$ and $b(q) = 1 \forall q \in U$. Since $(X, \mathcal{O}_{\bullet,X})$ is localized, $b$ is invertible in $\Gamma(U, \mathcal{O}_{0,X})$, and therefore $a = 0$ as a section of $\mathcal{O}_{0,X}$, contradiction to $A$ being a $C^\infty$-subring. So $U \cong Sp(A)$.

From $U \cong Sp(A)$ and $A$ being finitely generated, we conclude that $\forall V_1, V_2 \subseteq U$ closed, s.t. $V_1 \cap V_2 = \emptyset$, there is $\alpha \in A$, s.t. $\alpha V_1 = 0, \alpha V_2 = 1$, where we consider $\alpha$ as a section of $\mathcal{O}_{Sp(A)}$. Since $(X, \mathcal{O}_{\bullet,X})$ is localized, $\alpha$ has the same properties, when considered as a section of $\mathcal{O}_{0,X}$. ■

Let $(X, \mathcal{O}_{\bullet,X}) \in s\underline{\mathcal{G}}$. Since $\mathcal{O}_{0,X}$ is soft, all sheaves of boundaries in $\mathcal{M}(\mathcal{O}_{\bullet,X})$ are soft as well, and hence (see e.g. [Go60], theorem II.3.5.2)

$$\forall k \geq 0 \quad \Gamma(X, \pi_k(\mathcal{O}_{\bullet,X})) \cong \pi_k(\Gamma(X, \mathcal{O}_{\bullet,X})).$$ (18)

This immediately implies the following proposition.

**Proposition 5** The functor of global sections $\Gamma : s\underline{\mathcal{G}} \to s\mathcal{C}^{op}$ maps local weak equivalences to weak equivalences.
It might appear that localized simplicial $C^\infty$-spaces are very special. However, the following proposition shows that every simplicial $C^\infty$-space can be localized, s.t. the result is locally weak equivalent to the original space.

**Proposition 6** The inclusion $\mathfrak{s}\mathcal{G} \subset \mathcal{G}$ has a right adjoint. Unit of this adjunction is an isomorphism, and counit consists of local weak equivalences.

**Proof:** Let $(X, \mathcal{O}_{\bullet,X}) \in \mathcal{G}$, and let $p \in X$. The stalk $(\mathcal{O}_{0,X})_p$ does not have to be local, but $(\pi_0(\mathcal{O}_{\bullet,X}))_p$ is local. Therefore, since $(\pi_0(\mathcal{O}_{\bullet,X}))_p$ is a quotient of $(\mathcal{O}_{0,X})_p$, we get a distinguished $p : (\mathcal{O}_{0,X})_p \longrightarrow \mathbb{R}$. (19)

Let $(\mathcal{O}_{0,X})_p$ be the localization of $(\mathcal{O}_{0,X})_p$ at $p$, i.e. it is obtained by universally inverting every $f_p \in (\mathcal{O}_{0,X})_p$, whose value at $p$ is not 0. As usual with $C^\infty$-rings, the natural map $(\mathcal{O}_{0,X})_p \longrightarrow (\mathcal{O}_{0,X})_p$ is surjective. We will denote kernel of (20) by $t_p$. For an open $U \subseteq X$, we will call a section $\alpha \in \Gamma(U, \mathcal{O}_{\bullet,X})$ trivial, if $\forall p \in U \alpha_p \in t_p$. It is clear that all trivial sections together comprise a sheaf of ideals $t \subset \mathcal{O}_{0,X}$. It is easy to check that $\forall p \in X \ t_p$ is exactly the set of germs of $t$ at $p$.

We define $\mathcal{O}_{\bullet,X} := \mathcal{O}_{\bullet,X}/t\mathcal{O}_{\bullet,X}$, (21)

where the r.h.s. is taken in the category of sheaves of simplicial $C^\infty$-rings, i.e. we first divide, and then sheafify. Proposition 10 gives the middle isomorphism in

$\forall p \in X, \ (\pi_0(\mathcal{O}_{\bullet,X})_p) \cong (\pi_0((\mathcal{O}_{\bullet,X})_p)) \cong (\pi_0((\mathcal{O}_{\bullet,X})_p)) \cong (\pi_0(\mathcal{O}_{\bullet,X}))_p$, (22)

and hence $(X, \mathcal{O}_{\bullet,X}) \in \mathcal{G}$. Since stalks of $\mathcal{O}_{\bullet,X}$ are local $C^\infty$-rings, we have that in fact $(X, \mathcal{O}_{\bullet,X}) \in \mathfrak{s}\mathcal{G}$. It is clear that $(X, \mathcal{O}_{\bullet,X}) \Rightarrow (X, \mathcal{O}_{\bullet,X})$ extends to a functor $\mathcal{G} \rightarrow \mathfrak{s}\mathcal{G}$, and this functor is right adjoint to $\mathfrak{s}\mathcal{G} \subset \mathcal{G}$.

If $(X, \mathcal{O}_{\bullet,X})$ is localized, there are no non-zero trivial sections, and hence unit of the adjunction is an isomorphism. Using Proposition 10 again, we conclude that the map $\mathcal{O}_{\bullet,X} \rightarrow \mathcal{O}_{\bullet,X}$ is a weak equivalence stalk-wise, i.e. counit of the adjunction consists of local weak equivalences. ■

Let $\Gamma : \mathfrak{s}\mathcal{G} \rightarrow \mathcal{S}\mathcal{L}^{op}$ be composition of the localization functor $\mathfrak{s}\mathcal{G} \rightarrow \mathfrak{s}\mathcal{G}$ and the functor of global sections $\Gamma : \mathfrak{s}\mathcal{G} \rightarrow \mathcal{S}\mathcal{L}^{op}$. Propositions 5 and 6 imply that $\Gamma$ maps local weak equivalences to weak equivalences. Our next objective is to define a functor, going in the opposite direction.
3.2 Spectrum of a simplicial $C^\infty$-ring

Let $A_\bullet \in sL$, i.e. $A_\bullet$ is a simplicial $C^\infty$-ring, s.t. $\pi_0(A_\bullet)$ is finitely generated as a $C^\infty$-ring. We define

$$Sp(A_\bullet) := Sp(\pi_0(A_\bullet)).$$

(23)

Let $U \subseteq Sp(A_\bullet)$ be open. For any $p \in U$ we have an evaluation

$$p : A_0 \longrightarrow \pi_0(A_\bullet) \overset{p}{\longrightarrow} \mathbb{R}. \quad (24)$$

Let $U := \{a \in A_0 \text{ s.t. } \forall p \in U, p(a) \neq 0\}$, we define $A_\bullet[U^{-1}]$ to be the simplicial $C^\infty$-ring, obtained by universally inverting every $a \in U$, and its degeneracies in $A_{\geq 1}$. It is clear that $U \mapsto A_\bullet[U^{-1}]$ is a presheaf of simplicial $C^\infty$-rings on $Sp(A_\bullet)$. Let $O_{\bullet,Sp(A_\bullet)}$ be the associated sheaf.

**Proposition 7** Let $(Sp(A_\bullet), O_{\bullet,Sp(A_\bullet)})$ be defined as above. Then

$$\pi_0(O_{\bullet,Sp(A_\bullet)}) \cong \mathcal{O}_{Sp(\pi_0(A_\bullet))}. \quad (25)$$

**Proof:** Recall that $\mathcal{O}_{Sp(\pi_0(A_\bullet))}$ is sheafification of the presheaf

$$U \mapsto \pi_0(A_\bullet)[[U^{-1}], \quad U \subseteq Sp(\pi_0(A_\bullet)),$$

(26)

where $[[U]] \subseteq \pi_0(A_\bullet)$ consists of those elements, that do not vanish at any $p \in U$, i.e. $U \subseteq A_0$ is the pre-image of $[U]$. Therefore we have a canonical $A_0[U^{-1}] \rightarrow \pi_0(A_\bullet)[[U^{-1}], and it obviously factors through $\pi_0(A_\bullet[U^{-1}])$.

Using universal properties of localization and sheafification, we arrive at

$$\pi_0(O_{\bullet,Sp(A_\bullet)}) \rightarrow \mathcal{O}_{Sp(\pi_0(A_\bullet))}. \quad (27)$$

To prove that (27) is an isomorphism, we need to look at the stalks. Let $p \in Sp(A_\bullet)$, then $(\mathcal{O}_{Sp(\pi_0(A_\bullet))})_p \cong \pi_0(A_\bullet)/m^p_p, (O_{\bullet,Sp(A_\bullet)})_p \cong A_\bullet/m^O_p A_\bullet$, where $m^O_p \subset \pi_0(A_\bullet), m^p_p \subset A_0$ consist of elements that have 0 germs at respectively $p$ and $p$ (see Proposition 9). Let $t$ be the kernel of $A_0 \rightarrow \pi_0(A_\bullet)$. It is easy to see that the l.h.s. of (27) is $A_0/t + m^O_p$, while the r.h.s. is $(A_0/t)/m^p_p$. It is straightforward to see that $A_0/t + m^O_p = (A_0/t)/m^p_p$.  

The previous proposition implies that $(Sp(A_\bullet), O_{\bullet,Sp(A_\bullet)}) \in s\mathcal{G}$. Since construction of $O_{\bullet,Sp(A_\bullet)}$ involves only inverting elements, it is clearly functorial. Moreover, it is obvious that stalks of $O_{0,Sp(A_\bullet)}$ are local $C^\infty$-rings, so we have a functor

$$Sp : sL^{op} \rightarrow s\mathcal{G}, \quad A_\bullet \mapsto (Sp(A_\bullet), O_{\bullet,Sp(A_\bullet)}). \quad (28)$$
In general, it is not true that $A_\bullet$ is weakly equivalent to $\Gamma(Sp(A_\bullet), O_{Sp(A_\bullet)})$. Even in the simple case when $A_\bullet$ is a discrete $C^\infty$-ring, i.e. a constant simplicial diagram $\Delta^{op} \to X \in C^\infty R$, if $A$ is not germ determined, then $A \not\cong \Gamma(Sp(A), O_{Sp(A)})$.

A simplicial $C^\infty$-ring $A_\bullet$ is geometric if it is homotopically finitely generated and germ determined, i.e. if

1. $\pi_0(A_\bullet)$ is a finitely generated, germ determined $C^\infty$-ring,

2. $\forall k \geq 1, \pi_k(A_\bullet)$ is a germ determined, locally complete $\pi_0(A_\bullet)$-module (see Section 2 for definition of local completeness).

Let $sG \subset sL$ be the full subcategory, consisting of geometric simplicial $C^\infty$-rings. If $(X, O_{\bullet,X}) \in sG$, from [13] we know, that for any $k \geq 0 \pi_k(\Gamma(X, O_{\bullet,X})) \cong \Gamma(X, \pi_k(O_{\bullet,X}))$, and hence $\pi_0(\Gamma(X, O_{\bullet,X}))$ is finitely generated, germ determined, and locally complete $\pi_0(\Gamma(X, O_{\bullet,X}))$-modules. Therefore, the funtor of global sections $\Gamma: sG \to sG^{op}$. 

**Proposition 8** The functor of global sections $\Gamma: \overline{sG} \to sG^{op}$ is left adjoint to $Sp: sG^{op} \to \overline{sG}$. Unit of this adjunction is an isomorphism, the counit consists of weak equivalences.

**Proof:** Let $(X, O_{\bullet,X}) \in \overline{sG}, A_\bullet \in sG$. Any morphism $f_\bullet : A_\bullet \to \Gamma(X, O_{\bullet,X})$ induces $\pi_0(f_\bullet) : \pi_0(A_\bullet) \to \pi_0(\Gamma(X, O_{\bullet,X})) \cong \Gamma(X, \pi_0(O_{\bullet,X}))$, and hence a continuous map $\phi : X \to Sp(A_\bullet)$. Since stalks of $O_{0,X}$ are local, universal property of localization implies that $f_\bullet$ defines $\phi_\bullet : O_{\bullet,Sp(A_\bullet)} \to \phi_\bullet(O_{\bullet,X})$.

Let $\overline{A_\bullet}$ be obtained by universally inverting every $a \in A_0$, s.t. $p(a) \neq 0$ for any $p \in Sp(\pi_0(A_\bullet))$. From Proposition [9] we know that $A_\bullet \to \overline{A_\bullet}$ is surjective, and hence, using universal property of sheafification and proceeding as in the proof of Proposition [2] we conclude that $f_\bullet \mapsto (\phi, \phi_\bullet)$ is a bijective correspondence. It is clearly functorial, i.e. $Sp$ is right adjoint to $\Gamma$.

Suppose that $A_\bullet = \Gamma(X, O_{\bullet,X})$. Since $\pi_0(A_\bullet) \cong \Gamma(X, \pi_0(O_{\bullet,X}))$, obviously $X \cong Sp(A_\bullet)$ as topological spaces. Then, since $O_{0,X}$ is soft, and its stalks are local, one proves that $O_{\bullet,Sp(A_\bullet)} \cong O_{\bullet,X}$ as in the discrete case (Proposition [2]). So unit of the adjunction is indeed an isomorphism.

Now we compare $A_\bullet$ and $\Gamma(Sp(A_\bullet), O_{Sp(A_\bullet)})$. From Proposition [10] we know that $A_\bullet \to \overline{A_\bullet}$ is a weak equivalence. It remains to show that sheafification preserves this homotopy type, i.e. that

$$\gamma : \overline{A_\bullet} \to \Gamma(Sp(A_\bullet), O_{Sp(A_\bullet)}) \tag{29}$$

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is a weak equivalence. We can assume that $A_\bullet = \overline{A}_\bullet$. Let $\alpha \in M(A_\bullet)$ be a cycle, s.t. $\gamma(\alpha)$ is a boundary. This implies that $\forall p \in Sp(A_\bullet), \gamma(\alpha)_p$ is a boundary in $(O_\bullet,Sp(A_\bullet))_p$. From Proposition 9 we conclude that $(O_\bullet,Sp(A_\bullet))_p \cong A_\bullet/m^2_p A_\bullet$, therefore we can construct $\beta \in A_0$, s.t. $\alpha \beta$ is homologous to 0, yet $p(\beta) \neq 0$. This implies that $\alpha \mapsto 0$ in $H_\bullet(M(A_\bullet))_p$. Since this happens for every $p$, and we assume $\pi_k(A_\bullet)$ to be germ determined $\forall k \geq 0$, $\alpha$ has to be a boundary itself.

Now let $\xi \in \Gamma(\text{Sp}(A_\bullet),O_\bullet,\text{Sp}(A_\bullet))$ be a cycle. Let $p \in \text{Sp}(A_\bullet)$, then $\xi_p \in (O_\bullet,\text{Sp}(A_\bullet))_p \cong A_\bullet/m^2_p A_\bullet$ is a cycle. It is easy to see that for any $\alpha \in A_0$, whose support is contained in a small open set around $p$, $\alpha \xi_p$ extends to a cycle in $A_\bullet$. Therefore, using partition of unity, we can find a family of cycles $\{\beta_i\} \subseteq A_\bullet$, s.t. $\{\gamma(\beta_i)\}$ is locally finite and

$$\forall p \in Sp(A_\bullet), \quad \sum_i (\beta_i)_p = \xi_p.$$  

(30)

Since $\forall k \geq 0$, $\pi_k(A_\bullet)$ is locally complete, the corresponding family of homology classes $\{[\beta_i]\}$ adds to one class $[\beta] \in H_\bullet(M(A_\bullet))$. It is clear that $\gamma(\beta)$ is homologous to $\xi$. ■

We have started with the category $sG$, and we were interested in the simplicial localization of $sG$ with respect to local weak equivalences. We have constructed two adjunctions:

$$sG \rightleftarrows sG^{\text{op}}, \quad sG \rightleftarrows sG.$$ 

(31)

In each case the unit and counit consist of local weak equivalences (in the case of $sG$, $\overline{sG}$), or weak equivalences (in the case of $sG$). Therefore simplicial localization of $sG^{\text{op}}$, with respect to weak equivalences, is weakly equivalent to simplicial localization of $sG$ with respect to local weak equivalences.

We finish this section with technical results, that were used in some of the propositions above.

**Proposition 9** Let $A \in \mathcal{C}^\infty R$, and let $\mathfrak{I} \subseteq A$ be an ideal, s.t. $A/\mathfrak{I}$ is finitely generated. Let $V \subseteq \text{Sp}(A/\mathfrak{I})$ be closed, and let

$$\Lambda := \{a \in A \text{ s.t. } \forall p \in V \quad p(a) \neq 0\}$$

(32)

Let $A_V$ be obtained by inverting every $a \in \Lambda$. Let $S \subseteq A$ be a set of generators. Then

$$A_V \cong \mathcal{C}^\infty(R^S)/\mathfrak{I} + m^Q_S,$$ 

(33)

7Here $p$ is the composition $A \rightarrow A/\mathfrak{I} \xrightarrow{\text{p}} R$. 

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where $\mathfrak{A} \subset C^\infty(\mathbb{R}^S)$ is the kernel of $C^\infty(\mathbb{R}^S) \to A$, and $m^p_V \subset C^\infty(\mathbb{R}^S)$ consists of functions, that have 0 germ at $C^\infty(\mathbb{R}^S) \to A \to A/\mathfrak{I} \to \mathbb{R}$ for any $p \in V$.

**Proof:** Since $\mathfrak{A} \subset \mathfrak{A} + m^0_V$, there is a canonical $\phi : A \to C^\infty(\mathbb{R}^S)/\mathfrak{A} + m^0_V$. First we prove that $\forall a \in \Lambda \phi(a)$ is invertible. Let $\tilde{a} \in C^\infty(\mathbb{R}^S)$ be any pre-image of $a$. Since $A/\mathfrak{I}$ is finitely generated, we can choose a finite $F \subseteq S$, s.t. $C^\infty(\mathbb{R}^F) \hookrightarrow C^\infty(\mathbb{R}^S) \to A/\mathfrak{I}$ is surjective, and $\tilde{a} \in C^\infty(\mathbb{R}^F)$. Let $V_F$ be the image of $V$ in $\mathbb{R}^F$. Since $\tilde{a}(p) \neq 0 \forall p \in V$, using partition of unity on $\mathbb{R}^F$, we can find $\tilde{b} \in C^\infty(\mathbb{R}^F)$, s.t. $\tilde{a}\tilde{b} - 1 = 0$ in a neighborhood of $V_F$. This implies that $\tilde{a}\tilde{b} = 1$, considered as an element of $C^\infty(\mathbb{R}^S)$, has 0 germ in a neighborhood of every $p \in V$, i.e. $\tilde{a}\tilde{b} - 1 \in m^0_V$, and hence $\phi(a)$ is invertible in $C^\infty(\mathbb{R}^S)/\mathfrak{A} + m^0_V$.

Since $\phi$ is obviously surjective, for any $A \to B$, that inverts every $a \in \Lambda$, there is at most one factorization through $\phi$. To prove that at least one such factorization exists, we construct $\chi : C^\infty(\mathbb{R}^S)/\mathfrak{A} + m^0_V \to A_V$, s.t. the following diagram is commutative.

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & C^\infty(\mathbb{R}^S)/\mathfrak{A} + m^0_V \\
\downarrow{\psi_0} & & \downarrow{\chi} \\
A_V & \xleftarrow{\chi} & C^\infty(\mathbb{R}^S)/\mathfrak{A} + m^0_V \\
\end{array}
$$

(34)

To construct this $\chi$, it is enough to show that kernel of $C^\infty(\mathbb{R}^S) \to A_V$ contains $m^0_V$. Let $\tilde{a} \in m^0_V$, and choose a finite $F \subseteq S$, s.t. $\tilde{a} \in C^\infty(\mathbb{R}^F)$, and $C^\infty(\mathbb{R}^F) \to A/\mathfrak{I}$ is surjective, let $V_F$ be the image of $V$ in $\mathbb{R}^F$. By assumption, there is an open $U_F \supseteq V_F$, s.t. $\tilde{a} = 0$ on $U_F$. Using paracompactness of $\mathbb{R}^F$, we can find $\tilde{b} \in C^\infty(\mathbb{R}^F)$, s.t. $\tilde{b}(p) \neq 0 \forall p \in V_F$ and $supp(\tilde{b}) \subseteq U_F$. This means that $\tilde{a}\tilde{b} = 0$, yet the image of $\tilde{b}$ in $A_V$ is invertible. Hence $\tilde{a} \mapsto 0 \in A_V$. ■

**Proposition 10** Let $A_\bullet$ be a simplicial $C^\infty$-ring, s.t. $\pi_0(A_\bullet)$ is finitely generated and germ determined. Let $\overline{A}_\bullet$ be obtained by universally inverting every $a \in A_0$, s.t. $p(a) \neq 0$, $\forall p \in Sp(\pi_0(A_\bullet))$. Then the natural map $\phi_\bullet : A_\bullet \to \overline{A}_\bullet$ is a weak equivalence.

**Proof:** From Proposition 9 we know that $\overline{A}_0 \cong A_0/m$, for some ideal $m \subseteq A_0$. It is easy to see that $\overline{A}_\bullet \cong A_\bullet/mA_\bullet$.

Clearly $mM(A_\bullet)$ is a subcomplex, and hence to prove the proposition it is enough to show that $mM(A_\bullet)$ is acyclic. Let $k \geq 0$, and let $\beta = \sum_{1 \leq i \leq n} b_i\beta_i$, $\beta_i \in \overline{A}_k$.
where $b_i \in m$ and $\beta_i \in A_k$. Since $\pi_0(A_\bullet)$ is finitely generated, and there are only finitely many of $b_i$'s, one can use partition of unity to find $a \in A_0$, s.t. $a\beta = 0$, yet $p(a) \neq 0 \ \forall p \in V$. The latter implies that the image of $a$ in $\pi_0(A_\bullet)$ is invertible ($\pi_0(A_\bullet)$ is assumed to be germ determined), and hence, if $\beta$ is a cycle, it is necessarily a trivial one.

Now assume there is $\alpha \in A_{k+1}$, s.t. $d\alpha = \beta$. Using partition of unity again, we can find $b \in m$, s.t. $bb_i = b_i$ for all $1 \leq i \leq n$. Then obviously $b\alpha \in mA_{k+1}$ and $d(b\alpha) = b\beta = \beta$. ■

### 4 Spivak’s construction

In this section we show that construction of derived manifolds, done in [Sp10], can be equivalently performed in the category $s\mathcal{L}$ of simplicial $C^\infty$-rings $A_\bullet$, s.t. $\pi_0(A_\bullet)$ is finitely generated.

The meeting point of Spivak’s construction and the usual homotopy theory of simplicial $C^\infty$-rings is the category $s\mathcal{G}$ of simplicial $C^\infty$-spaces, defined in Section 3. We start with recalling (and somewhat reformulating) the constructions of [Sp10].

Let $\mathcal{CG}$ be the category of compactly generated Hausdorff spaces. We define a category $\mathbf{RS}$ as follows:

1. objects are pairs $(X, O_{\bullet,X})$, where $X \in \mathcal{CG}$ and $O_{\bullet,X}$ is a functor

\[
\text{Open}(X)^{op} \times C^\infty \longrightarrow SSet,
\]  \hspace{1cm} \text{(35)}

2. morphisms are pairs $(\phi, \phi^\ast)$, where $\phi : X \to Y$ is a continuous map, and $\phi^\ast$ is a natural transformation

\[
\text{Open}(Y)^{op} \times C^\infty \xrightarrow{\phi^{-1} \times Id} \text{Open}(X)^{op} \times C^\infty \xrightarrow{\phi^\ast} SSet
\] \hspace{1cm} \text{(36)}

For any fixed $X \in \mathcal{CG}$, we will denote by $\mathbf{RS}(X) \subset \mathbf{RS}$ the full subcategory consisting of pre-sheaves on $X$. We equip each $\mathbf{RS}(X)$ with the injective
closed model structure, where $O_{\bullet,X} \rightarrow O'_{\bullet,X}$ is a weak equivalence or cofibration, if $O_{\bullet,X}(U, \mathbb{R}^n) \rightarrow O'_{\bullet,X}(U, \mathbb{R}^n)$ is respectively a weak equivalence or cofibration of simplicial sets, $\forall U \in \text{Open}(X), \forall n \geq 0$.

Since we have not required $O_{\bullet,X}(U, -) : C^\infty \rightarrow SSet$ to be product preserving, the natural maps

$$
O_{\bullet,X}(U, \mathbb{R}^{m+n}) \rightarrow O_{\bullet,X}(U, \mathbb{R}^m) \times O_{\bullet,X}(U, \mathbb{R}^n) \quad (37)
$$

are not required to be isomorphisms, and not even weak equivalences, i.e. $O_{\bullet,X}$ is not necessarily a pre-sheaf of $C^\infty$-rings. Similarly, having a hypercover $\{U_i\}$ of $U$, the natural map

$$
O_{\bullet,X}(U, \mathbb{R}^k) \rightarrow \text{holim}(O_{\bullet,X}(U_i, \mathbb{R}^k)) \quad (38)
$$

does not have to be a weak equivalence, i.e. $O_{\bullet,X}(-, \mathbb{R}^k)$ is not necessarily a homotopy sheaf of simplicial sets.

Using the fact that $\text{RS}(X)$ is a left proper, cellular, simplicial closed model category, one can perform a left localization of $\text{RS}(X)$ with respect to (37) and (38) (e.g. [Hi09]). The result is a left proper, simplicial closed model category, that we will denote by $\text{Shv}(X)$. Moreover, any continuous map $\phi : X \rightarrow Y$ induces a Quillen adjunction

$$
\phi^* : \text{Shv}(Y) \leftrightharpoons \text{Shv}(X) : \phi_* \quad (39)
$$

Homotopy sheaves of homotopy simplicial $C^\infty$-rings on $X$ are the fibrant objects in $\text{Shv}(X)$. Every $O_{\bullet,X} \in \text{Shv}(X)$ is cofibrant, and we will denote by $O_{\bullet,X}$ a chosen functorial fibrant replacement. For any $U \in \text{Open}(X)$, $O_{\bullet,X}(U, -)$ is a homotopy $C^\infty$-ring, i.e. (37) is a weak equivalence $\forall m, n \geq 0$, and therefore we have a well defined sheaf of $C^\infty$-rings $\pi_0(O_{\bullet,X})$, and a sequence of sheaves of $\pi_0(O_{\bullet,X})$-modules $\{\pi_k(O_{\bullet,X})\}_{k \geq 1}$.

A weak equivalence in $\text{RS}$ is a morphism $(\phi, \phi^\sharp) : (X, O_{\bullet,X}) \rightarrow (Y, O_{\bullet,Y})$, s.t. $\phi$ is a homeomorphism, and $\phi^\sharp : O_{\bullet,Y} \rightarrow \phi_* (O_{\bullet,X})$ is a weak equivalence in $\text{Shv}(Y)$. Equivalently, we can demand that $O_{\bullet,Y} \rightarrow \phi_* (O_{\bullet,X})$ is a local weak equivalence, i.e. it induces isomorphisms

$$
\pi_k(O_{\bullet,Y}) \rightarrow \phi_* (\pi_k(O_{\bullet,X})), \quad \forall k \geq 0. \quad (40)
$$

We will denote by $\text{RS}$ the simplicial localization of $\text{RS}$ with respect to these weak equivalences. Presence of simplicial closed model structure on each $\text{Shv}(X)$ makes computing $\text{RS}$ easier than usual.
Proposition 11  The category $\text{RS}$, together with the subcategory of weak equivalences, admits a homotopy calculus of fractions.

Proof: Let $(\phi, \phi^\sharp) : (X, O\cdot X) \rightarrow (Y, O\cdot Y)$ be a weak equivalence in $\text{RS}$, we will say that $(\phi, \phi^\sharp)$ is a trivial cofibration or trivial fibration, if correspondingly $\phi^\sharp$ is a trivial fibration or a trivial cofibration. Using closed model structure on $\text{Shv}(Y)$, it is obvious that every weak equivalence in $\text{RS}$ can be written as a composition of a trivial cofibration, followed by a trivial fibration.

Consider diagrams

\[
\begin{array}{ccc}
(X, O\cdot X) & \xrightarrow{(\alpha, \alpha^\sharp)} & (Y, O\cdot Y) \\
\downarrow (\phi, \phi^\sharp) & & \downarrow (\psi, \psi^\sharp) \\
(X', O\cdot X') & \xrightarrow{(\beta, \beta^\sharp)} & (Z', O\cdot Z')
\end{array}
\] (41)

where $(\phi, \phi^\sharp)$ is a trivial cofibration, and $(\psi, \psi^\sharp)$ is a trivial fibration. It is easy to see that

\[
(Y, O\cdot Y \prod_{\alpha^\sharp (O\cdot X)} (\alpha^{-1})_* (O\cdot X')), \quad (W, O\cdot W \prod_{\beta^\sharp (O\cdot Z')} (\psi^{-1})_* (O\cdot Z))
\] (42)

are respectively colimit and limit of (41). Since right Quillen functors preserve trivial fibrations, and left Quillen functors preserve trivial cofibrations, it is clear that pushout of $(\phi, \phi^\sharp)$ is a trivial cofibration, and pullback of $(\psi, \psi^\sharp)$ is a trivial fibration. Using the 2-out-of-3 property, we see that, if in addition $(\alpha, \alpha^\sharp), (\beta, \beta^\sharp)$ are weak equivalences, their pushout and pullback are weak equivalences as well. Use [DK80a], proposition 8.2. ■

The same argument shows that any subcategory of $\text{RS}$, defined by putting conditions on the sheaves of homotopy groups, also admits a homotopy calculus of fractions. We are interested in the following two subcategories:

- let $\text{LRS} \subset \text{RS}$ be the full subcategory, consisting of pairs $(X, O\cdot X)$, s.t. stalks of $\pi_0(O\cdot X)$ are local $C^\infty$-rings,
- let $\text{LRS}_{fgs} \subset \text{LRS}$ be the full subcategory of pairs $(X, O\cdot X)$, s.t. $\pi_0(O\cdot X)$ is a soft sheaf of finitely generated $C^\infty$-rings.

As with $\text{RS}$, we will denote by $\text{LRS}, \text{LRS}_{fgs}$ the simplicial localizations with respect to local weak equivalences. Because $\text{RS}$ admits a homotopy
calculus of fractions, we know that $\text{Hom}_{\text{RS}}((X, O_\bullet, X), (Y, O_\bullet, Y))$ is weakly equivalent to simplicial set of hammocks of the following form:

\[
\begin{array}{c}
(X', O_\bullet, X') & \rightarrow & (Y', O_\bullet, Y') \\
\downarrow & & \downarrow \\
(X'', O_\bullet, X'') & \rightarrow & (Y'', O_\bullet, Y'') \\
\downarrow & & \downarrow \\
(X, O_\bullet, X) & \rightarrow & (Y, O_\bullet, Y) \\
\downarrow & & \downarrow \\
(X^{(n)}, O_\bullet, X^{(n)}) & \rightarrow & (Y^{(n)}, O_\bullet, Y^{(n)}) \\
\end{array}
\]

where vertical arrows, and arrows going to the left are weak equivalences. The same is true for $LRS_{fgs}$. Since $LRS, LRS_{fgs}$ are full subcategories of $\text{RS}$, defined by a condition on weak equivalence classes, we immediately have the following result.

**Proposition 12** The inclusions $LRS_{fgs} \subset LRS \subset \text{RS}$ induce weak equivalences on the spaces of morphisms.

So far we have used only presence of a closed model structure on each $\text{Shv}(X)$. Now we will use the fact that these closed model structures are simplicial. For any $(X, O_\bullet, X), (Y, O_\bullet, Y) \in \text{RS}$ we have a simplicial set

\[
\coprod_{\phi \in \text{Hom}_{\text{CG}}(X, Y)} \prod_{k \geq 0} \text{Hom}_{\text{Shv}(Y)}(O_\bullet, Y \otimes \Delta[k], \phi_*(O_\bullet, X)),
\]

which we will denote by $\text{Hom}_{\text{RS}}((X, O_\bullet, X), (Y, O_\bullet, Y))$.

**Proposition 13** For any $(X, O_\bullet, X), (Y, O_\bullet, Y) \in \text{RS}$ there is a weak equivalence of simplicial sets:

\[
\text{Hom}_{\text{RS}}((X, O_\bullet, X), (Y, O_\bullet, Y)) \simeq \text{Hom}_{\text{RS}}((X, O_\bullet, X), (Y, O_\bullet, Y)).
\]

**Proof:** Recall from the proof of Proposition $\prod$ that $(\phi, \phi^\sharp) : (X, O_\bullet, X) \rightarrow (Y, O_\bullet, Y)$ is a trivial fibration, if $\phi$ is a homeomorphism, and $\phi^\sharp : O_\bullet, Y \rightarrow \phi^*(O_\bullet, X)$ is a trivial cofibration. Since cofibrations in each $\text{Shv}(X)$ are just
injective maps, it is easy to see that closing trivial fibrations in $\mathcal{RS}$ with respect to the 2-out-of-3 property, produces all weak equivalences. Hence $\mathcal{RS}$ can be computed as simplicial localization of $\mathcal{RS}$ with respect to trivial fibrations.

Proceeding as in the proof of Proposition 11, one sees that $\mathcal{RS}$, together with trivial fibrations, admits a calculus of homotopy right fractions ([DK80a], proposition 8.1). Therefore, $\text{Hom}_{\mathcal{RS}}((X, O_{\bullet}X), (Y, O_{\bullet}Y)$ is weakly equivalent to simplicial set of hammocks of the following form:

$$
\begin{align*}
(Y', O_{\bullet}Y') & \Rightarrow (Y''', O_{\bullet}Y'''') \\
(Y''', O_{\bullet}Y''') & \Rightarrow (X, O_{\bullet}X) \\
(X, O_{\bullet}X) & \Rightarrow (Y, O_{\bullet}Y) \\
(Y, O_{\bullet}Y) & \Rightarrow (Y^{(n)}, O_{\bullet}Y^{(n)})
\end{align*}
$$

(46)

where vertical arrows, and arrows going to the left are trivial fibrations. Moreover, since nerves of equivalent categories are weakly equivalent, we can assume that $Y^{(k)} = Y \forall k \geq 1$.

It is easy to see that in each such hammock, every path from $X$ to $Y$ has the same underlying continuous map $\phi : X \to Y$, and hence we have a decomposition of simplicial sets

$$
\text{Hom}_{\mathcal{RS}}((X, O_{\bullet}X), (Y, O_{\bullet}Y)) = \coprod_{\phi} \text{Hom}_{\mathcal{RS}}((X, O_{\bullet}X), (Y, O_{\bullet}Y))_{\phi}. \quad (47)
$$

Fix a $\phi$, using functorial fibrant replacement in each $\text{Shv}(X)$, we can assume that all pre-sheaves in [46] are fibrant. Pushing forward everything to $Y$, we see that [46] becomes a hammock between $(Y, \phi_{\bullet}(O_{\bullet}X))$ and $(Y, O_{\bullet}Y)$ in $\text{Shv}(Y)$, and this correspondence is bijective, i.e. we have

$$
\text{Hom}_{\mathcal{RS}}((X, O_{\bullet}X), (Y, O_{\bullet}Y)) \simeq \text{Hom}_{\text{Shv}(Y)}(O_{\bullet}Y, \phi_{\bullet}(O_{\bullet}X)), \quad (48)
$$

where $\text{Shv}(Y)$ is the simplicial localization of $\text{Shv}(Y)$ with respect to trivial fibrations.
cofibrations. Finally, since $\mathcal{O}_{\bullet,Y}$ is cofibrant, and $\phi_*(\mathcal{O}_{\bullet,X})$ is fibrant, we have

$$\text{Hom}_{\text{Shv}(Y)}(\mathcal{O}_{\bullet,Y}, \phi_*(\mathcal{O}_{\bullet,X})) \simeq \text{Hom}_{\text{Shv}(Y)}(\mathcal{O}_{\bullet,Y} \otimes \Delta[k], \phi_*(\mathcal{O}_{\bullet,X}))$$ (49)

Here we use [DK80b], proposition 5.2, corollary 4.7. ■

In [Sp10] $\text{LRS}$ is equipped with simplicial structure, which we will denote by $\hat{\text{LRS}}$. We have seen now that $\hat{\text{LRS}}$ is weakly equivalent to $\text{LRS}$, that we have constructed here. Therefore, all constructions involving homotopy limits (e.g. derived manifolds), that one can perform in $\hat{\text{LRS}}$, can be equivalently performed in $\text{LRS}$.

Of course, the advantage of Spivak’s construction is that it is more manageable, than simplicial localization. Now we show that there is another model, which is more manageable still.

Recall ([Sp10]) that an affine derived manifold is a homotopy limit (in $\hat{\text{LRS}}$) of a diagram

$$\begin{array}{ccc}
\mathbb{R}^0 & \xrightarrow{} & \mathbb{R}^n \\
\downarrow & & \downarrow \\
\mathbb{R}^m & \to & \mathbb{R}^n
\end{array} (50)$$

Since $\text{LRS} \simeq \hat{\text{LRS}}$, we can use $\text{LRS}$ instead. Moreover, if $(X, \mathcal{O}_{\bullet,X})$ is the homotopy pullback of (50), then $\pi_0(\mathcal{O}_{\bullet,X})$ is a soft sheaf of finitely generated $C^\infty$-rings. Therefore affine derived manifolds lie in the full subcategory $\text{LRS}_{fgs} \subset \text{LRS}$.

Arbitrary derived manifolds in [Sp10] are defined by gluing affine ones. So if we restrict to derived manifolds of finite type, i.e. $(X, \mathcal{O}_{\bullet,X})$ such that $\pi_0(\mathcal{O}_{\bullet,X})$ is a sheaf of finitely generated $C^\infty$-rings, we are still inside $\text{LRS}_{fgs}$.

There is an obvious (full) inclusion

$$sG \subset \text{LRS}_{fgs}.$$ (51)

We claim that (51) induces a weak equivalence of simplicial localizations (with respect to local weak equivalences). Indeed, any functor $C^\infty \to SSet$ can be rectified to a simplicial $C^\infty$-ring, i.e. it can be changed into a product preserving functor. Moreover, this process is functorial, and it preserves homotopy type ([Be06]).

Let $< sG > \subset \text{LRS}$ be the full subcategory, consisting of $sG$ and all pairs $(X, \mathcal{O}_{\bullet,X})$, s.t. $\mathcal{O}_{\bullet,X}$ is weakly product preserving and cofibrant in the projective closed model structure on $\text{RS}(X)$. Then $sG \subset < sG >$ induces a
weak equivalence between simplicial localizations with respect to local weak equivalences ([Lu09], lemma 5.5.9.9). On the other hand, using cofibrant replacement functor $\mathbf{LRS} \to \mathbf{LRS}$ (with respect to the projective model structures on $\mathbf{RS}(X)$), we conclude that $< sG > \subset \mathbf{LRS}_{fgs}$ also induces a weak equivalences between simplicial localizations.

Altogether, using results of Section 3, we now know that taking homotopy limit of (50) in $\mathbf{LR\hat{s}}$ is equivalent to doing the same in $sG^{op}$, or equivalently in $s\mathcal{L}^{op}$, which has the usual (projective) closed model structure. Therefore:

**Theorem 1** The simplicial category of derived manifolds of finite type, as defined in [Sp10], is weakly equivalent to the full subcategory of $sG^{op}$, consisting of objects, that locally are homotopy limits (taken in $s\mathcal{L}$) of (50).

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