THE DEFICIENCY OF BEING A CONGRUENCE GROUP FOR VEECH GROUPS OF ORIGAMIS

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Abstract. We study “how far away” a finite index subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ is from being a congruence group. For this we define its deficiency of being a congruence group. We show that the index of the image of $\Gamma$ in $\text{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is the biggest, if $n$ is the general Wohlfahrt level. We furthermore show that the Veech groups of origamis (or square-tiled surfaces) in $\mathcal{H}_2(2)$ are far away from being congruence groups and that in each genus one finds an infinite family of origamis such that they are “as far as possible” from being a congruence group.

1. Introduction

Teichmüller curves and Veech groups of translation surfaces have been intensively studied within the last ten years since they were introduced in Veech’s famous article [Vee89]. It was already proven by Veech himself that they are discrete subgroups of $\text{SL}_2(\mathbb{R})$ which are never cocompact. Beside this still only few general statements are known. A very special class of translation surfaces are defined by square-tiled surfaces which we also call origamis. In this case the translation surface is especially handsome and the Veech group is always a finite index subgroup of $\text{SL}_2(\mathbb{Z})$. The set of Teichmüller curves coming from origamis is dense in the corresponding moduli space. This makes origamis an interesting class of translation surfaces to study.

Recall that a subgroup of $\text{SL}_2(\mathbb{Z})$ is called a congruence group if it contains one of the principal congruence groups $\Gamma(n)$ (see Section 3). In [Sch05] it was proven that many congruence groups occur as Veech groups. However, as soon as one fixes a stratum in the moduli space $\mathcal{H}_g$ of holomorphic unit area differentials on closed Riemann surfaces of genus $g$, it seems that the Veech group of an origami is more likely to be a non-congruence group. It was e.g. shown by Hubert and Lelièvre in [HL05] that in the stratum $\mathcal{H}_2(2)$ of holomorphic unit area differentials in genus 2 with one zero the Veech groups of all but one origami are non-congruence groups.

In this article we study “how far from being a congruence group” Veech groups of origamis are. For this we define in Section 3 for a general finite index subgroup

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Γ of \( \text{SL}_2(\mathbb{Z}) \) its deficiency of being a congruence group. As a key ingredient we use the Wohlfahrt level \( l \) of Γ (see Section 3). If Γ is a congruence group, this equals the minimal congruence level. In particular, one has that Γ is a congruence group if and only if the level index \( e_\Gamma = [\text{SL}_2(\mathbb{Z}) : \Gamma] \) is equal to the index \( d = [\text{SL}_2(\mathbb{Z}) : \Gamma] \), where \( p : \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/l\mathbb{Z}) \) is the natural projection. In general the number \( e_\Gamma \) can be an arbitrary divisor of \( d \) and we define \( f_\Gamma = d/e_\Gamma \) to be the deficiency with respect to \( l \). Thus the deficiency is 1 if and only if Γ is a congruence group and we may take the deficiency as a measure of how far away Γ is from being a congruence group. We say that Γ is a totally non-congruence group if its deficiency \( f_\Gamma \) is the index \( d \) or equivalently \( e_\Gamma \) is 1, i.e. the projection \( p_l \) is surjective.

We can define the numbers \( e_{\Gamma,l} = e_\Gamma \) and \( f_{\Gamma,l} = f_\Gamma \) in the same way for arbitrary numbers \( l \) possibly different from the Wohlfahrt level. Looking at the case when Γ is a congruence group, one however expects that \( e_{\Gamma,l} \) becomes maximal (or equivalently the deficiency \( f_{\Gamma,l} \) becomes minimal) if \( l \) is the Wohlfahrt level. We show in Section 3 that this is indeed true.

**Theorem 1** (Proof in Section 3). Let Γ be a finite index subgroup of \( \text{SL}_2(\mathbb{Z}) \) and \( l \in \mathbb{N} \). The deficiency \( f_{\Gamma,l} \) becomes minimal if \( l \) is the Wohlfahrt level of Γ.

In particular one has the following conclusion.

**Corollary 1.1.** For a finite index subgroup of \( \text{SL}_2(\mathbb{Z}) \) with Wohlfahrt level \( l \) we have: If Γ is a totally non-congruence group, i.e. \( e_{\Gamma,l} = 1 \), then Γ surjects to \( \text{SL}_2(\mathbb{Z}/n\mathbb{Z}) \) for each natural number \( n \).

Furthermore, one should expect that being a congruence group is an exception and that a general finite index subgroup of \( \text{SL}_2(\mathbb{Z}) \) has a good chance of being a totally non-congruence group. With Theorem 2 we give a handsome criterion for being a totally non-congruence group. For this we consider widths of cusps (see Section 2) at 0 and \( \infty \) of Γ and of a conjugate of Γ.

**Theorem 2** (Proof in Section 3). Let Γ₁ and Γ₂ be conjugated finite index subgroups of \( \text{SL}_2(\mathbb{Z}) \). Suppose that Γ₁ has width \( a_1 \) at the cusp 0 and width \( b_1 \) at the cusp \( \infty \) and Γ₂ has width \( a_2 \) at the cusp 0 and width \( b_2 \) at the cusp \( \infty \). If

\[
(1) \quad n_1 = \text{lcm}(a_1, b_1) \text{ and } n_2 = \text{lcm}(a_2, b_2) \text{ are relatively prime,}
\]

then Γ₁ and Γ₂ are totally non-congruence groups, i.e. they surject onto \( \text{SL}_2(\mathbb{Z}/n\mathbb{Z}) \) for each natural number \( n \).

We then study the deficiency of the Veech groups of origamis in \( \mathcal{H}_2(2) \). Using the criterion from Theorem 2 we show that it is always \( d \) or \( \frac{d}{2} \), i.e. they are totally non-congruence groups or at least far away from being congruence groups.

**Theorem 3.** Let \( O \) be an origami in \( \mathcal{H}_2(2) \) with \( j \) squares and let Γ(\( O \)) be its Veech group. We distinguish the two different cases that \( O \) is in the orbit \( A_j \) or
$B_j$ in the classification of orbits in $\mathcal{H}_2(2)$ by McMullen and Hubert/Lelièvre (see Section 4).

i) If $j$ is odd, $j \geq 5$ and $O$ is in $B_j$, then $\Gamma(O)$ is a totally non-congruence group, i.e. it surjects onto $\text{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{N}$.

ii) If $j$ is even, or $j$ is odd and $O$ is in $A_j$, or $j = 3$, then the deficiency $f_{\Gamma,l}$ with respect to the Wohlfahrt level $l$ of $\Gamma(O)$ is equal to $\frac{4}{3}$. i.e. the index of its image in $\text{SL}_2(\mathbb{Z}/l\mathbb{Z})$ is 3.

**Corollary 1.2.** The Veech groups of all origamis in $\mathcal{H}_2(2)$ form an honest family of non-congruence groups, i.e. an infinite family such that the level index $e_l$ is bounded by a constant. More precisely their level index is bounded by 3.

**Remark:** A similar approach as in Theorem 3 is independently used by Christian Weiss in [Wei12] in the proof of Theorem 5.28, where he calculates the Euler characteristic of twisted Teichmüller curves arising from the non-origami $L$-shaped translation surfaces $L_D$ as defined in [McM03] and [Cal04]. In this case the Veech group is a subgroup of $\text{SL}_2(\mathcal{O}_D)$, where $\mathcal{O}_D$ is a real quadratic order of discriminant $D$ and one can similarly look at groups defined by congruence conditions in $\text{SL}_2(\mathcal{O}_D)$.

The criterion from Theorem 2 is quite general and finally allows us to detect infinite families of origamis whose Veech groups are all totally non-congruence groups also in higher genus.

**Theorem 4.** For each $g \geq 3$, the stratum $\mathcal{H}_g(2g - 2)$ contains an infinite family of origamis whose Veech groups are totally non-congruence groups.

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There is today a wide literature on the topic of translation surfaces, Veech groups and Teichmüller curves. We summarise in the following paragraphs the basics that we need in this article. The reader can find more detailed introductions to these topics with more hints to literature e.g. in [Zor06], [HS07], [Möl09] or [Kap07].

2. Preliminaries

2.1. Translation surfaces, origamis and Veech groups. Recall that a translation surface is a surface $X^*$ together with an atlas $\mu$ such that all transition maps are translations. A translation atlas is in particular holomorphic and thus $(X^*, \mu)$ has the structure of a Riemann surface. We call $(X^*, \mu)$ precompact of genus $g$, if $X^* = X \setminus \{P_1, \ldots, P_k\}$, where $X$ is a closed Riemann surface of genus $g$ and $P_1, \ldots, P_k$ are finitely many cone points on $X$. There is a well-known bijection between precompact translation surfaces of genus $g$ (up to translations) and pairs $(X, \omega)$ (up to isomorphisms preserving $\omega$) where $X$ is a closed Riemann surface of genus $g$ and $\omega$ is a holomorphic differential on $X$ (see e.g. [HS06, Section 1.1.3]). We denote by $H_g(a_1, \ldots, a_r)$ the moduli space of unit area holomorphic differentials on a genus $g \geq 1$ Riemann surface with $r$ zeroes of order $a_1, \ldots, a_r$.

One of the simplest ways to explicitly construct translation surfaces goes as follows: take finitely many copies of the unit square. Glue them along their edges via translations such that each left edge is glued to precisely on right edge, each upper edge to precisely one lower edge and the resulting surface is connected. This way we obtain a closed surface $X$ with a set $\tilde{S}$ of finitely many marked points which come from the vertices of the squares. The Euclidean structure of the plane defines a translation structure on $X^* = X \setminus \tilde{S}$. The points in $\tilde{S}$ become cone-type singularities. We call $X$ a square-tiled surface or - emphasising its combinatorial structure - an origami. If the angle around a point $p$ in $\tilde{S}$ is $2\pi$, we can extend the translation structure into this point. Points in $\tilde{S}$ with total angle bigger than $2\pi$ are called singularities. Straight lines with respect to the translation structure which connect two singularities are called saddle connections. To each saddle connection we associate its developing vector $\vec{v}$ in the Euclidean plane.

In this article we will only work with primitive origamis, which means that the developing vectors of the saddle connections span $\mathbb{Z}^2$ (see [HL06, Lemma 2.1 and paragraph below]).

There are several equivalent ways to combinatorially describe an origami, see [Sch06]. A handsome one, if we want to notate explicit examples, is the description by permutations given as follows. Label the squares by $Q_1, \ldots, Q_d$. Let $\sigma_a$ be the permutation in $S_d$ which assigns $i$ the number of the right neighbour of the square $Q_i$ and $\sigma_b$ the permutation which assigns $i$ the number of the upper neighbour of $Q_i$. The pair $(\sigma_a, \sigma_b)$ completely describes the surfaces up to isomorphism. Renumbering the squares leads to simultaneous conjugation. Since $X$ is connected, the group generated by $\sigma_a$ and $\sigma_b$ acts transitively on $\{1, \ldots, d\}$. 
Conversely each transitive pair \((\sigma_a, \sigma_b)\) of permutations in \(S_d\) defines an origami. A natural description of origamis is via coverings of the torus as described in the following. The square tiling naturally defines a finite covering to the torus \(E\) formed by gluing opposite edges of one square. This covering is ramified over at most one point, namely the one point on \(E\) arising from the vertices of the one square. This leads to the following equivalent definition. An origami is the isomorphism class of a finite connected covering \(p: X \rightarrow E\) ramified at most over one point \(\infty\). Here two coverings \(p_1: X_1 \rightarrow E\) and \(p_2: X_2 \rightarrow E\) are said to be \textit{isomorphic} if there is a map \(f: X_1 \rightarrow X_2\) with \(p_2 \circ f = p_1\).

The point of view of coverings naturally leads us to monodromy. If we are given an origami \(p: X \rightarrow E\), we may remove the point \(\infty\) from \(E\) and all of its preimages in \(X\) and obtain an unramified covering of the once-punctured torus \(E^*\). The fundamental group of \(E^*\) is \(F_2 = F_2(x, y)\), the free group in two generators \(x\) and \(y\). We choose \(x\) and \(y\) to be the simple closed horizontal and vertical curve on \(E^*\), respectively. By the classical theory of coverings this defines a monodromy action \(\rho: F_2 \rightarrow S_d\), where \(d\) is the degree of the covering \(p\) if we choose a base point: The fiber of \(p\) over the base point is a set with \(d\) elements. \(F_2\) acts on it from the right, namely for \(w \in F_2\) and for \(\gamma_w\) the corresponding closed curve on \(E^*\), \(\rho(w)\) maps a point \(q\) in the fiber to the endpoint of the lift of \(\gamma_w\) with starting point \(q\). This depends on the choice of the base point only up to conjugation. Recall that a \textit{monodromy action} is an anti-homomorphism such that its image acts transitively on the set \(\{1, \ldots, d\}\). Observe that we have for the pair of permutations \((\sigma_a, \sigma_b)\) from above that \(\sigma_a = \rho(x)\) and \(\sigma_b = \rho(y)\).

\[\begin{array}{cccccc}
3 & & & & & \\
& 2 & & & & \\
1 & 4 & 7 & 8 & 9 & \\
& & 6 & & & \\
& & & 5 & & \\
\end{array}\]

\textit{Figure 1}: An origami in \(\Omega_3(4)\): Opposite edges are glued.

\textbf{Example 2.1.} Figure 1 shows an origami with 9 squares. Opposite edges are glued. The origami is given by the two permutations:

\[\sigma_a = (1, 4, 7, 8, 9)\] and \(\sigma_b = (1, 2, 3)(4, 5, 6)\).
The marked vertices glue to one point on the surface $X$ of total angle $10\pi$. The unmarked vertices become regular points, i.e. the total angle around them is $2\pi$. A simple Euler characteristic calculation shows that the genus of $X$ is 3.

In [Vee89] Veech introduced a group associated to a translation surface $(X^*, \mu)$ which is today called the Veech group. Let $\text{Aff}^+(X^*, \mu)$ be the group of orientation preserving affine homeomorphisms of $X^*$, i.e. those homeomorphisms which are affine on charts. Thus locally they are of the form $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ with $A \in \text{SL}_2(\mathbb{R})$ and $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. The matrix $A$ is independent of the chart, since the transition maps are translations. We obtain a homomorphism $D: \text{Aff}^+(X, \mu) \to \text{SL}_2(\mathbb{R})$, $f \mapsto A$ called the derivative map. The Veech group $\Gamma(X^*, \mu)$ is the image of $D$. Veech already proved in [Vee89] that $\Gamma(X^*, \mu)$ is a discrete subgroup of $\text{SL}_2(\mathbb{R})$ and that it is never cocompact. One reason why Veech groups were intensively studied in the last fifteen years is that they are closely related to Teichmüller curves (see e.g. [McM03], [HS07]). Namely, each translation surface defines a certain subset of the corresponding moduli space of closed Riemann surfaces of genus $g$. Occasionally, this subset is a complex algebraic curve $C$, then called Teichmüller curve. It can be read off from the Veech group, whether this happens or not. We obtain a Teichmüller curve, if and only if $\Gamma(X^*, \mu)$ has finite volume. In this case one even has that $C$ is birational to $\mathbb{H}/\Gamma(X^*, \mu)$. In this article we will just need the following properties of Veech groups:

- The Veech group of a primitive origami is a finite index subgroup of $\text{SL}_2(\mathbb{Z})$ (see [Sch04, Cor. 2.9] or for a more general discussion [GJ00]).
- $\text{SL}_2(\mathbb{Z})$ acts on the set of isomorphism classes of origamis with $d$ squares and the Veech group is the stabiliser of this action.
- If the translation surface $(X, \mu)$ is defined by an origami, then we have for each rational direction $\vec{v} = \begin{pmatrix} p \\ q \end{pmatrix}$ (with $p, q \in \mathbb{Z}$): The surface decomposes in direction $\vec{v}$ into finitely many cylinders $C_1, \ldots, C_k$. If $m_1, \ldots, m_r$ are the inverse moduli of the cylinders, respectively, and $m$ is the smallest common integer multiple of the $m_i$’s, then

$$A \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} A^{-1}$$

is in the Veech group, where $A$ is an arbitrary matrix in $\text{SL}_2(\mathbb{Z})$ which maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\vec{v}$.

We will explain the second and third statement in more detail in the following two sections.

2.2. Action of $\text{SL}_2(\mathbb{Z})$ on origamis. There is a natural action of the group $\text{SL}_2(\mathbb{R})$ on translation surfaces defined by $A: (X, \mu) \mapsto (X, \mu_A)$, where $\mu_A$ is the translation structure obtained from $\mu$ by composing each chart with the affine
The stabiliser of \((X, \mu)\) (up to isomorphisms) is the Veech group \(\Gamma(X, \mu)\) (see e.g. [EG97, Theorem 1]).

If \((X, \mu)\) comes from a primitive origami \(O\) and \(A \in \operatorname{SL}_2(\mathbb{Z})\), then \(A \cdot (X, \mu)\) is again the translation structure of an origami \(O_\rho = A \cdot O\). More precisely, if \((X, \mu)\) is defined by the covering \(p : X \to E\), then \(A \cdot (X, \mu) = (X, \mu_A)\) is the origami defined by the covering \(p_A := A \circ p : X \xrightarrow{\rho} E \xrightarrow{A} E\). The length of the orbit of the origami \(O\) under this \(\operatorname{SL}_2(\mathbb{Z})\)-action on the set of origamis is the index \([\operatorname{SL}_2(\mathbb{Z}) : \Gamma(X, \mu)]\) of the Veech group in \(\operatorname{SL}_2(\mathbb{Z})\).

In terms of the corresponding monodromy maps, the action expresses as follows: Let \(\rho : F_2 \to S_a\) be the monodromy map corresponding to \(O\), then \(A \cdot \rho = \rho_A = \rho \circ \gamma_A^{-1}\) is the monodromy map of the origami \(A \cdot O\), where \(\gamma_A\) is a preimage of \(A\) under the natural homomorphism \(\operatorname{Out}(F_2) \to \operatorname{Out}(F_2) \cong \operatorname{GL}_2(\mathbb{Z})\).

Finally, in terms of the corresponding pair of permutations the action of \(\operatorname{SL}_2(\mathbb{Z})\) is given as follows: The pair \((\sigma_a^{O_A}, \sigma_b^{O_A})\) with

\[
\sigma_a^{O_A} = \rho(\gamma_A^{-1}(x)) \quad \text{and} \quad \sigma_b^{O_A} = \rho(\gamma_A^{-1}(y))
\]

describes the origami \(O_A\). In particular, for the matrices

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

we obtain the permutations:

\[
\begin{align*}
\sigma_a^{O_T} &= \rho(x) = \sigma_a, & \sigma_b^{O_T} &= \rho(x^{-1}y) = \rho(y)\rho(x)^{-1} = \sigma_b\sigma_a^{-1}, \\
\sigma_a^{O_{T'}} &= \sigma_a^{-1}\sigma_b, & \sigma_b^{O_{T'}} &= \sigma_b, \\
\sigma_a^{O_S} &= \sigma_a^{-1}, & \sigma_b^{O_S} &= \sigma_a.
\end{align*}
\]

and furthermore for the inverse matrices:

\[
\begin{align*}
(\sigma_a^{O_{T^{-1}}}, \sigma_b^{O_{T^{-1}}}) &= (\sigma_a, \sigma_b\sigma_a), & (\sigma_a^{O_{T'^{-1}}}, \sigma_b^{O_{T'^{-1}}}) &= (\sigma_b\sigma_a, \sigma_b) \quad \text{and} \\
(\sigma_a^{O_{S^{-1}}}, \sigma_b^{O_{S^{-1}}}) &= (\sigma_b, \sigma_a^{-1}).
\end{align*}
\]

2.3. Parabolic elements in the Veech group. Recall that a cylinder on a translation surface \((X, \mu)\) is a connected set of simple closed geodesics. If \(w\) is its width (also called circumference) and \(h\) is its height, then the inverse modulus is \(m = \frac{w}{h}\). A cylinder \(C\) has direction \(\vec{v}\), if \(\vec{v}\) is parallel to the developing vector of the simple closed geodesics forming \(C\). Let us suppose for the moment that \(\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). Then \(C\) is obtained from a rectangle of width \(w\) and height \(h\) by gluing the two vertical edges. The affine map

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto T_m \cdot \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad T_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}
\]

defines an affine map on the cylinder which fixes the boundary pointwise. Suppose now that the translation surface \((X, \mu)\) decomposes into horizontal cylinders \(C_1, \ldots, C_k\) of inverse moduli \(m_1, \ldots, m_k\). If \(m\) is an integer multiple of each \(m_i\), then the map \(z \mapsto T_m z\) defines on each cylinder an affine map which fixes the boundary pointwise. These maps glue to an affine map of the whole surface with derivative \(T_m\) which fixes the horizontal saddle connections pointwise. It acts as the product of powers of Dehn twists along the middle lines of the cylinders.

Suppose now that \((X, \mu)\) is an origami given by the permutations \((\sigma_a, \sigma_b)\). Then the surface naturally decomposes into horizontal (not necessarily maximal) cylinders of height 1. The number of cylinders is the number of cycles in \(\sigma_a\). The length of a cylinder is the size of the corresponding cycle. In particular, the matrix \(T^n\) is in the Veech group, where \(n\) is the least common multiple of the cycle lengths of \(\sigma_a\).

For an arbitrary direction \(\vec{v}\) we obtain precisely the same statements if we replace \(T\) by its conjugate \(A T A^{-1}\) with some arbitrary matrix \(A\) which maps \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) to \(\vec{v}\), and the pair of permutations \((\sigma_a, \sigma_b)\) by \(A^{-1} \cdot (\sigma_A, \sigma_B)\) (see [22]). In particular we have that

\[
T'_m = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}
\]

is in the Veech group, if \(m\) is a multiple of the inverse moduli of the vertical cylinders.

**Cusps and widths of cusps.** Let us briefly recall some basics from the theory of Fuchsian groups that will be important in this article. The reader not familiar with this can find a good introduction e.g. in [Kat92]. Suppose that \(\Gamma\) is a finite index subgroup of \(\text{SL}_2(\mathbb{Z})\). We can build a fundamental domain of \(\Gamma\) by taking finitely many images of our favourite fundamental domain of \(\text{SL}_2(\mathbb{Z})\); each image corresponds to one coset of \(\Gamma\) in \(\text{SL}_2(\mathbb{Z})\). The quotient \(\mathcal{C} = \mathbb{H}/\Gamma\) is a Riemann surface of finite type, i.e. it is biholomorphic to a closed surface \(\overline{\mathcal{C}}\) with finitely many points called cusps removed. The word cusp is here used in a dual role. A cusp of \(\Gamma\) (on \(\mathbb{H}\)) is a point \(x\) on the boundary of \(\mathbb{H}\) which is the fixed point of some parabolic element \(A\) in \(\Gamma\). We also call the \(\Gamma\)-orbits of such points cusps, more precisely the cusps of \(\mathbb{H}/\Gamma\). They bijectively correspond to the points in \(\overline{\mathcal{C}}\setminus\mathcal{C}\).

For a fixed cusp \(x\) of \(\Gamma\) on the boundary of \(\mathbb{H}\) the stabiliser of \(x\) in \(\Gamma\) is a cyclic group generated by a parabolic element \(P\). Since \(\Gamma\) is a subgroup of \(\text{SL}_2(\mathbb{Z})\), \(P\) is conjugate to \(T^n\) with \(n \in \mathbb{Z}\) and \(T\) the matrix from [2]. We call the absolute value of \(n\) the width of the cusp \(x\).

The cusps of \(\Gamma\) on \(\mathbb{H}\) bijectively correspond to maximal cyclic parabolic subgroups of \(\Gamma\). Two cusps are in the same \(\Gamma\)-orbit if and only if the corresponding maximal parabolic subgroups are conjugated in \(\Gamma\). Accordingly, the cusps of \(\mathbb{H}/\Gamma\) correspond to the conjugacy classes of maximal cyclic parabolic subgroups of \(\Gamma\). In particular the widths of cusps in the same orbit are equal and we may speak of
the width of cusps of $\mathbb{H}/\text{SL}_2(\mathbb{Z})$. The width of a cusp is the number of copies of the fundamental domain of $\text{SL}_2(\mathbb{Z})$ which lie around it.

3. The deficiency of being a non-congruence group

Recall that a subgroup $\Gamma$ of $\text{SL}(2, \mathbb{Z})$ is called a congruence group of level $l$, if $\Gamma$ contains the principal congruence group $\Gamma(l)$ of level $l$, where

$$\Gamma(l) = \{ A | A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod l \}.$$ 

The minimal congruence level of $\Gamma$ is then the smallest number $l$ such that $\Gamma$ is a congruence group of level $l$. $\Gamma$ is called a congruence group if it is a congruence group of level $l$ for some $l$, otherwise it is called a non-congruence group.

In general we have for an arbitrary finite index subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ and for an arbitrary natural number $m$ the following commutative diagram of exact sequences:

$$1 \longrightarrow \Gamma(m) \longrightarrow \text{SL}_2(\mathbb{Z}) \longrightarrow \text{SL}_2(\mathbb{Z}/m\mathbb{Z}) \longrightarrow 1,$$

where $\Gamma$ is the image of $\Gamma$ in $\text{SL}_2(\mathbb{Z}/m\mathbb{Z})$ by the natural projection $p_m : \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/m\mathbb{Z})$.

Consider the three indices $d := [\text{SL}_2(\mathbb{Z}) : \Gamma]$, $e := [\text{SL}_2(\mathbb{Z}/m\mathbb{Z}) : \Gamma]$ and $f := [\Gamma(m) : \Gamma(m) \cap \Gamma]$. We then have from the diagram that $d = e \cdot f$. Observe that $f = 1$ if and only if $\Gamma$ is a congruence group of level $m$. In general, $f$ is a factor of the index $d$.

**Definition 3.1.** Let $\Gamma$ be a finite index subgroup of $\text{SL}_2(\mathbb{Z})$. The deficiency $f = f_m = f_{\Gamma,m}$ of $\Gamma$ with respect to $m$ is the index $f = [\Gamma(m) : \Gamma(m) \cap \Gamma]$.

**Remark 3.2.** We directly read off from (3) that:

$$f_m = \frac{d}{e_m} \text{ with } d = [\text{SL}_2(\mathbb{Z}) : \Gamma] \text{ and } e_m = [\text{SL}_2(\mathbb{Z}/m\mathbb{Z}) : p_m(\Gamma)].$$

Recall that for a general finite index subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ we can define the general Wohlfahrt level $l$ as follows.

**Definition 3.3.** Let $\Gamma$ be a finite index subgroup of $\text{SL}_2(\mathbb{Z})$. The Wohlfahrt level $l = l(\Gamma)$ of $\Gamma$ is defined by

$$l = l(\Gamma) = \text{lcm}(\{b | b \text{ is the width of a cusp of } \Gamma \}).$$

We will often call $l$ just the level of $\Gamma$. 

The Wohlfahrt level generalises the congruence level. This result, which is crucial for what we do, was proven by Wohlfahrt in \cite{Woh64}.

**Theorem A** (Wohlfahrt). *If \( \Gamma \) is a congruence group, then the Wohlfahrt level equals the congruence level of \( \Gamma \).*

In particular this means for a congruence group that the deficiency \( f_l \) equals 1, if \( l \) is its (Wohlfahrt) level. In this section we will show that for a general finite index subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \) the deficiency \( f_m \) is minimised, if \( m \) equals the Wohlfahrt level \( l(\Gamma) \). Thus let for the rest of this section \( \Gamma \) be a fixed subgroup of \( SL_2(\mathbb{Z}) \) of finite index with Wohlfahrt level \( l = l(\Gamma) \).

Observe first that we immediately obtain from the definition of the Wohlfahrt level the following remark.

**Remark 3.4.** Let \( \Gamma \) and \( \Gamma' \) be two subgroups of \( SL_2(\mathbb{Z}) \). If \( \Gamma \subseteq \Gamma' \), then \( l(\Gamma') \) divides \( l(\Gamma) \).

We will further need the following property of the deficiency.

**Lemma 3.5.** *For a multiple \( k \cdot a \) of a natural number \( a \) we have for the corresponding deficiencies:
\[
  f_{ka} \text{ is a divisor of } f_a.
\]*

**Proof.** Consider the natural projection \( q : SL_2(\mathbb{Z}/ka\mathbb{Z}) \to SL_2(\mathbb{Z}/a\mathbb{Z}) \). Then we have \( q(p_{ka}(\Gamma)) = p_a(\Gamma) \). Thus we obtain for the indices that \( e_a \) divides \( e_{ka} \). Hence the claim follows from Remark 3.2. \( \square \)

We are now ready to prove Theorem \[1\] which we stated in the introduction.

**Proof of Theorem \[1\].** From Lemma 3.5 we have that \( f_m \geq f_{ml} \). We now show that \( f_{ml} = f_l \). Consider the group \( \Gamma' = p_{ml}^{-1}(p_{ml}(\Gamma)) \). It has by its construction the same image in \( SL_2(\mathbb{Z}/lm\mathbb{Z}) \) as \( \Gamma \), i.e. \( p_{lm}(\Gamma) = p_{lm}(\Gamma') \) and \( \Gamma \) is contained in \( \Gamma' \). Furthermore \( \Gamma' \) is by its definition a congruence group. Let \( l' \) be its congruence level which equals the Wohlfahrt level by Theorem A. We obviously have \( \Gamma(l') \subseteq \Gamma' \). Furthermore, by Remark 3.4 we have that \( l' \) divides \( l = l(\Gamma) \) and thus \( \Gamma(l') \subseteq \Gamma(l) \). Altogether we obtain that \( p_{lm}(\Gamma) = p_{lm}(\Gamma') \supseteq p_{lm}(\Gamma(l)) \).

Consider now the following commutative diagram of exact sequences:

\[
\begin{array}{ccc}
1 & \longrightarrow & p_{lm}(\Gamma(l)) \\
\uparrow & & \uparrow \\
SL_2(\mathbb{Z}/lm\mathbb{Z}) & \longrightarrow & SL_2(\mathbb{Z}/l\mathbb{Z}) & \longrightarrow & 1
\end{array}
\]

\[
\begin{array}{ccc}
1 & \longrightarrow & p_{lm}(\Gamma(l)) \cap p_{lm}(\Gamma) \\
\uparrow & & \uparrow \\
p_{lm}(\Gamma(l)) & \longrightarrow & p_{lm}(\Gamma) & \longrightarrow & p_l(\Gamma) & \longrightarrow & 1
\end{array}
\]

Since \( p_{lm}(\Gamma(l)) \) is contained in \( p_{lm}(\Gamma) \), we have that \( p_{lm}(\Gamma(l)) \cap p_{lm}(\Gamma) = p_{lm}(\Gamma(l)) \), i.e. \( [p_{lm}(\Gamma(l)) : p_{lm}(\Gamma(l)) \cap p_{lm}(\Gamma)] = 1 \). Thus we obtain from the diagram

\[
e_{lm} = [SL_2(\mathbb{Z}/lm\mathbb{Z}) : p_{lm}(\Gamma)] = 1 \cdot [SL_2(\mathbb{Z}/l\mathbb{Z}) : p_l(\Gamma)] = e_l.
\]

From this we obtain \( f_{lm} = \frac{d}{e_{lm}} = \frac{d}{e_l} = f_l \) by Remark 3.2. \( \square \)
This result suggests to consider the deficiency of $\Gamma$ with respect to the Wohlfahrt level as a measure of “how far away” $\Gamma$ is from being a congruence group. The bigger it is, the further the group is away from being a congruence group. Equivalently, we may consider the number $e = d_f$. If it is 1, then we are “as far possible” from being a congruence group.

**Definition 3.6.** Let $l$ be the Wohlfahrt level of $\Gamma$ and $f = f_l$, $e = e_l$ as above. We call $f = f(\Gamma)$ the deficiency of $\Gamma$ (of being a congruence group) and $e = e(\Gamma)$ its level index. If $e = 1$, then we say that $\Gamma$ is a totally non-congruence group. For an infinite family $\Gamma_n$, we say it is an honest family of non-congruence groups, if $e(\Gamma_n)$ is bounded by a constant.

Theorem 2 will give us a criterion to show that the Veech group of an origami is a totally non-congruence group. It will be crucial that the Wohlfahrt levels of conjugated groups are equal. One main ingredient will be to consider the width of the cusps $\infty$ and 0. This motivates the following definition.

**Definition 3.7.** Let $\Gamma$ be a subgroup of $\text{SL}_2(\mathbb{Z})$ of finite index which has cusps at $\infty$ and 0 of width $a$ and $b$, respectively. We call the pair $(a, b)$ the normalised cusp-width pair of $\Gamma$.

The main tool for proving Theorem 2 will be the following proposition.

**Proposition 3.8.** Let $a$ and $b$ be the widths of the cusps 0 and $\infty$, respectively, of a finite index subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$. Suppose that we can decompose the Wohlfahrt level $l$ of $\Gamma$ as $l = N \cdot M$ such that

1. $N$ and $M$ are relatively prime and
2. $n = \text{lcm}(a, b)$ divides $N$.

Then we have in $\text{SL}_2(\mathbb{Z}/l\mathbb{Z}) = \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z})$:

$$p_l(\Gamma) \supseteq \{I\} \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z})$$

**Proof.** Write $N = n' \cdot n$ and $n = a \cdot b' = a' \cdot b$ with $n'$, $a'$ and $b'$ in $\mathbb{N}$. Let us first show that the element $(I, T)$ of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z})$ lies in $p_l(\Gamma)$. Since $N$ and $M$ are relatively prime, there are $k$ and $k'$ in $\mathbb{Z}$ such that $k \cdot N + k' \cdot M = 1$ and thus $k \cdot N \equiv 1 \mod M$.

Since $a$ is the width of the cusp $\infty$ with respect to $\Gamma$, we have in particular that $T^a$ and thus also $T^{a \cdot b' \cdot k}$ is in $\Gamma$. Thus

$$p_l(T^{a \cdot b' \cdot k}) = \begin{pmatrix} 1 & Nk \\ 0 & 1 \end{pmatrix} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z})$$

lies in $p_l(\Gamma)$.

Starting with the fact that $T^b$ is in $\Gamma$, we obtain in the same way that

$$\left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$
lies in $p_t(\Gamma)$ and thus we conclude the claim.

\[ \square \]

Proof of Theorem 2. Let $l$ be the common Wohlfahrt level of $\Gamma_1$ and $\Gamma_2$. Let $n_1 = p_1^{k_1} \cdots p_r^{k_r}$ be the prime factorisation. By the definition of the Wohlfahrt level $l$ we have that $n_1$ divides $l$ and thus the prime factorisation of $l$ can be written as

$$ l = p_1^{m_1} \cdots p_r^{m_r} p_r^{m_{r+1}} \cdots p_{r+s}^{m_{r+s}} $$

with $k_1 \leq m_1, \ldots, k_r \leq m_r$. Define $N = p_1^{m_1} \cdots p_r^{m_r}$ and $M = p_r^{m_{r+1}} \cdots p_{r+s}^{m_{r+s}}$. In particular, we have $l = M \cdot N$, $N$ and $M$ are relatively prime and $n_1$ divides $N$. It follows from Proposition 3.8 that $p_t(\Gamma_1)$ contains $I \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z})$. The same is true for $\Gamma_2$ instead of $\Gamma_1$ since these two groups are conjugated, i.e. $p_t(\Gamma_2)$ contains $I \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z})$.

On the other hand, since $n_1$ and $n_2$ are relatively prime, it follows that $\text{gcd}(n_2, N) = 1$. Furthermore we have that $n_2$ divides $l$, as $l$ is the Wohlfahrt level of $\Gamma_2$. Thus $n_2$ in particular divides $M$. We obtain again from Proposition 3.8 that $p_t(\Gamma_2)$ contains $\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \times \{I\}$. Again by conjugation the same is true for $p_t(\Gamma_1)$.

Altogether, we obtain that both groups $p_t(\Gamma_1)$ and $p_t(\Gamma_2)$ contain the full group $\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z}) = \text{SL}_2(\mathbb{Z}/l\mathbb{Z})$.

Motivated by the proof of Theorem 2 we introduce the following notation.

Definition 3.9. Suppose that $n$ and $N$ are two numbers such that $n$ divides $N$. Let $t$ be the largest divisor of $N$ such that $t$ and $n$ have the same prime divisors. We call $t$ the maximal prime divisor equivalent of $n$ in $N$ and denote it by $t = \text{mpde}_N(n)$. Observe that we in particular have that $n$ divides $t$, that $n$ and $t$ have the same prime divisors and that $t$ and $N/t$ are relatively prime.

In Section 4 we will further need a statement similar to Theorem 2 with weaker prerequisites, which we will get from Lemma 3.10.

Lemma 3.10. Suppose that we are in the situation of Theorem 2 except that in (1) it is not given that $n_1$ and $n_2$ are relatively prime. I.e. we have two conjugated groups $\Gamma_1$ and $\Gamma_2$ with normalised cusp-width pairs $(a_1, b_1)$ and $(a_2, b_2)$, respectively, and $n_1 = \text{lcm}(a_1, b_1)$. Define $N = \text{mpde}_N(n_1)$ (see Definition 3.4) and $M = l/N$ as in the proof of the theorem. Let furthermore $g_1 = \text{gcd}(a_2, N)$ and $g_2 = \text{gcd}(b_2, N)$. Then we have:

$$ \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ g_2 & 1 \end{pmatrix} \quad \text{are in } p_t(\Gamma_2). $$

Proof. By its definition $g_1 \equiv k \cdot a_2$ modulo $N$ for some $k \in \mathbb{N}$. Thus it follows from $T^{a_2} \in \Gamma_2$ that $p_t(\Gamma_2)$ contains

$$ \begin{pmatrix} 1 & k \cdot a_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix} \cdot A_2 \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z}) $$

(4)
with some matrix $A_2$ in $\text{SL}_2(\mathbb{Z}/M\mathbb{Z})$. As in the proof of Theorem 2 it follows from Proposition 3.8 that $p_1(\Gamma_1)$ and thus also $p_1(\Gamma_2)$ contains $\{I\} \times \text{SL}_2(\mathbb{Z}/M\mathbb{Z})$. Hence $p_1(\Gamma_2)$ contains in particular $(I, A_2^{-1} \cdot p_M(T^{q_1}))$. From this and (1) we obtain the first part of the claim. The second part follows in the same way.  

4. THE $L$-ORIGAMIS

In this section we show that the family of all Veech groups of origamis in $\mathcal{H}_2(2)$ is an honest family of non-congruence groups (compare Definition 3.6). We furthermore show for a subclass of them that their Veech groups are even totally non-congruence groups.

Recall that we have an explicit classification of $\text{SL}_2(\mathbb{R})$-orbits of origamis in $\mathcal{H}_2(2)$ by Hubert/Lelièvre (for the case that the number of squares is prime) and McMullen (in full generality), see [HL06, Thm.1.1] and [McM05, Cor. 1.2 and Section 6]. They distinguish the orbits only by the number of squares of the origamis and the number of integer Weierstraß points. Recall that in genus 2 the Weierstraß points are the six fixed points of the hyperelliptic involution, which is in the case of an origami the unique affine homeomorphism with derivative $-I$ of order 2 such that the quotient by it has genus 0. Being an integer Weierstraß point means that the point in addition is a vertex of one of the squares of the origami (see Example 4.1 and Example 4.4).

**Theorem B** (Hubert/Lelièvre, McMullen). The set of primitive origamis with $n$ squares form one single $\text{SL}_2(\mathbb{R})$-orbit, if $n$ is even or 3. They form two orbits called $A_n$ and $B_n$, if $n$ is odd and unequal to 3. An origami is in $A_n$, if it has one integer Weierstraß point and in $B_n$, if it has three integer Weierstraß points.

**Example 4.1.** Figure 2 shows two origamis with 5 squares. Opposite edges are glued. Both origamis lie in $\mathcal{H}_2(2)$. The left one has three integer Weierstraß points, the right one only one. Thus the left one lies in the orbit $B_n$ and the right one in the orbit $A_n$.

![Figure 2: The L-shaped origamis $L(3,3)$ and $L(4,2)$ and their Weierstraß points. Opposite edges are glued. Left one is in $B_n$, right one is in $A_n$.](image)

**Remark 4.2.** One easily observes from Theorem B that in $\mathcal{H}_2(2)$ each origami-orbit can be represented by some $L$-shaped origami $L_{a,b}$ $(a, b \geq 2)$ with $n = a + b - 1$ squares, see Figure 2. If $n$ is odd, then $L_{a,b}$ has 1 integer Weierstraß point.
point (i.e. belongs to $A_n$) if $a$ and $b$ are even. It has 3 integer Weierstraß points (i.e. belongs to $B_n$), if $a$ and $b$ are odd.

One furthermore directly observes from the $L$-shape that $\Gamma(L_{a,b})$ contains the parabolic elements $T^a$ and $T^b$ (compare Section 2.3). Moreover, they generate the corresponding maximal cyclic parabolic subgroup of $\Gamma(L_{a,b})$, which can be seen as follows. Any affine homeomorphism of $L_{a,b}$ whose derivative $A \in \Gamma(L_{a,b})$ is parabolic with eigenvector $(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})$ has to permute the horizontal saddle connections. We have three of them. Among them there is a unique one which lies on the boundary of only one cylinder. This property is preserved by the affine homeomorphism. Thus this segment has to be fixed pointwise and with it the full boundary of this cylinder. Hence the affine homeomorphism acts as the power of a Dehn twist on this cylinder and $A$ is a power of $T^a$. Hence $T^a$ generates the corresponding cyclic parabolic subgroup of $\Gamma(L_{a,b})$.

Remark 4.3. The widths of $\Gamma(L_{a,b})$ at the cusps $\infty$ and 0 are $a$ and $b$, respectively.

Example 4.4. Figure 3 shows an other type of origami in $H_2(2)$. The origami $Cr_{2,j}$ (with $j$ squares) is defined by the two permutations $\sigma_a = (1, 2, \ldots, j)$ and $\sigma_b = (1, 2)$. If $j$ is odd, then $Cr_{2,j}$ has 1 integer Weierstraß point and thus belongs to the orbit $A_j$. If $j$ is even, then it has 2 integer Weierstraß points.

![Figure 3: The origamis $Cr_{2,7}$ and $Cr_{2,6}$. Edges with same labels and unlabelled opposite edges are identified. The left one is in $A_7$, the right one has an even number of squares.](image)

One again can directly read off from Figure 3 that $T^j$ and $T^{2j}$ are in the Veech group of $Cr_{2,j}$. Furthermore $Cr_{2,j}$ decomposes into two maximal vertical cylinders, one of height 1 and circumference 2 and one of height $j - 2$ and circumference 1. On the first one we have a vertical saddle connection which lies on the boundary of only this cylinder. Hence a parabolic affine homeomorphism with the vertical direction as eigen direction has to fix this segment and thus has to act on this cylinder as multi Dehn twist. It follows that $T^{2j}$ generates the corresponding maximal cyclic parabolic subgroup of $\Gamma(Cr_{2,j})$. If $j \geq 4$, furthermore a parabolic affine homeomorphism with the horizontal direction as eigen direction has to preserve the unique horizontal saddle connection of length $j - 2$. Hence $T^j$ generates the corresponding maximal cyclic parabolic subgroup of $\Gamma(Cr_{2,j})$. In particular we have that if $j \geq 4$, then the normalised pair of cusp-widths for $\Gamma(Cr_{2,j})$ is $(j, 2)$. 
We now want to study the orbits of origamis in $H_2(2)$. Table 1 shows up to 11 squares for each orbit the index of the Veech group $\Gamma$ in $SL_2(\mathbb{Z})$, the genus and number of cusps of $\mathbb{H}/\Gamma$, the level index of $\Gamma$ and its deficiency. Observe that these data are stable under conjugation of $\Gamma$ and therefore do not depend on the origami but only on the orbit. From the table one guesses the result of Theorem 3 which we prove in the following.

**Proof of Theorem 3 i).** We distinguish the three cases $j \equiv 1$ modulo 4 but not 5, $j \equiv 3$ modulo 4 but not 7 and finally the singular cases $j = 5$ and $j = 7$.

**First Step:** Suppose that $j \equiv 1$ modulo 4 with $j \geq 9$.

This means that $j = 2a - 1$ with $a$ odd. Furthermore $a \geq 5$.

From Hubert/Lelièvre’s and McMullen’s classification (see Theorem B) of the $SL_2(\mathbb{Z})$-orbits of origamis in $H_2(2)$ and Remark 4.2 we know that the origamis $L_{a,a}$ and $L_{a+2,a-2}$ both lie in the orbit $B_j$. Here we need that $a - 2 \geq 2$. Then the Veech groups $\Gamma_1 = \Gamma(L_{a,a})$ and $\Gamma_2 = \Gamma(L_{a+2,a-2})$ are conjugated. Let $l$ be their Wohlfahrt level. By Remark 4.3 we have that the normalised cusp-width pair of $\Gamma_1$ is $(a, a)$ and that of $\Gamma_2$ is $(a+2, a-2)$. Furthermore $n_1 = \text{lcm}(a, a) = a$ and $n_2 = \text{lcm}(a+2, a-2)$ are relatively prime, since $a$ is odd. Thus we are in the situation of Theorem 2 and obtain that $p_l(\Gamma_1) = p_l(\Gamma_2)$ is the full group $SL_2(\mathbb{Z}/l\mathbb{Z})$.

---

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1 calculated with the Software Origami Library, [http://www.math.kit.edu/iag3/seite/ka-origamis/en](http://www.math.kit.edu/iag3/seite/ka-origamis/en)
**Second Step:** Suppose that $j \equiv 3 \text{ modulo } 4$ with $j \geq 11$. Thus $j = 2a - 1$ with $a$ even and $a \geq 6$.

Again we obtain from Hubert/Lelièvre’s and McMullen’s classification that in this case $L_{a+1,a-1}$ and $L_{a+3,a-3}$ both lie in the orbit $B_j$. We use the same arguments as before: We consider the Veech groups $\Gamma_1 = \Gamma(L_{a+1,a-1})$ and $\Gamma_2 = \Gamma(L_{a+3,a-3})$. They are conjugated and the normalised cusp-width pairs are $(a+1, a-1)$ and $(a+3, a-3)$, respectively. In this case we have $n_1 = \text{lcm}(a+1, a-1)$ and $n_2 = \text{lcm}(a+3, a-3)$. Observe that $n_1$ and $n_2$ are relatively prime, since $a$ is even. Thus we are again in the situation of Theorem 2 and obtain the desired statement.

**Third Step:** The remaining cases $j = 7$ and $j = 9$ have been explicitly calculated (see Table I).

We now show the second part of Theorem 3, i.e. that for the other types of orbits, namely the case $j$ even, the case $A_j$ with $j$ odd and the special case $j = 3$, the projection of the Veech group into $\text{SL}_2(\mathbb{Z}/l\mathbb{Z})$ is not surjective, but it is ’almost surjective’. More precisely the index of the image is constantly equal to 3.

**Proof of Theorem 3 ii).** We first show that the level index is at most 3. We distinguish the following cases:

- a) $j$ is odd, $j \equiv 3 \text{ modulo } 4$, $j \geq 7$ and the origami lies in $A_j$,
- b) $j$ is odd, $j \equiv 1 \text{ modulo } 4$, $j \geq 9$ and the origami lies in $A_j$,
- c) $j$ is even and $j \geq 6$,
- d) $j = 3$, $j = 4$ or $j = 5$ and the orbit is $A_5$.

a) Suppose that $j = 2a - 1 \geq 7$ with $a$ even, then $L_a.a$ and $Cr_{2,j}$ lie in the same orbit (see Theorem B, Remark 4.2 Example 4.4). The normalised cusp-width pairs of their Veech groups are $(a, a)$ (see Remark 4.3) and $(j, 2)$ (see Example 4.4 here we need that $j \neq 3$). Define $N = \text{mpde}_1(a)$ (see Definition 3.9). Then $N$ and $a$ have the same prime divisors and thus $2 = \gcd(2, a) = \gcd(2, N)$. Furthermore $j = 2a - 1$ and $a$ are relatively prime, thus $1 = \gcd(j, a) = \gcd(j, N)$. It follows from Lemma 3.10 that

\[ p_t(T) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } p_t(T^{r_2}) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \text{ lie in } p_t(\Gamma(Cr_{2,j})). \]

Since furthermore $-I \in \Gamma(Cr_{2,j})$, we have that the image $p_t(\Gamma(2))$ in $\text{SL}_2(\mathbb{Z}/l\mathbb{Z})$ of the principal congruence group $\Gamma(2)$ is contained in $p_t(\Gamma(Cr_{2,j}))$. Thus its index in $\text{SL}_2(\mathbb{Z}/l\mathbb{Z})$ is at most 6. Since in addition $p_t(T)$ lies in it, the index is at most 3 and we have obtained the desired statement.

b) Let $j = 2a - 1$ with $a$ odd, $a \geq 5$. Then $L_{a-1,a+1}$, $L_{a-3,a+3}$ and $Cr_{2,j}$ are in the same orbit. We have that $2 = \gcd(2, a-1) = \gcd(2, a+1)$ and thus $2 = \gcd(2, N)$, where $N = \text{mpde}_1(\text{lcm}(a - 1, a + 1))$. It follows again by Lemma 3.10 that $p_t(T^{r_2}) \in p_t(\Gamma(Cr_{2,j}))$. 


Furthermore since \( j = 2a - 1 = 2(a - 1) + 1 = 2(a + 1) - 3 \), we have that \( \gcd(a - 1, j) = 1 \) and \( \gcd(a + 1, j) \) divides 3. Thus \( \gcd(N, j) \) is a power \( 3^k \) of 3.

We obtain from Lemma 3.10 that \( p_l(T^{3k}) \in p_l(\Gamma(Cr_{2,j})) \).

Let us now work with \( L_{a-3,a+3} \) instead of \( L_{a-1,a+1} \). We have \( \gcd(a - 3, j) = \gcd(a - 3, 2(a - 3) + 5) \) divides 5 and \( \gcd(a + 3, j) = \gcd(a + 3, 2(a + 3) - 7) \) divides 7. Hence \( \gcd(\text{mpde}_l(\text{lcm}(a - 3, a + 3)), j) \) is of the form \( 5^{k_1} \cdot 7^{k_2} \) and thus by Lemma 3.10 we have that \( p_l(\Gamma(Cr_{2,j})) \) contains \( p_l(T^{5^{k_1}7^{k_2}}) \). Since \( 3^k \) and \( 5^{k_1} \cdot 7^{k_2} \) are relatively prime, we obtain that \( p_l(T) \) is in \( p_l(\Gamma(Cr_{2,j})) \). We conclude the claim as in the previous case.

c) Suppose now \( j = 2a \) with \( a \geq 3 \). Then \( L_{a,a+1}, L_{a-1,a+2} \) and \( L_{2,2a-1} \) are in the same orbit. Let \( N = \text{mpde}_l(\text{lcm}(a, a + 1)) \). We have that \( \gcd(2, N) = 2 \). It follows by Lemma 3.10 that \( p_l(T^2) \) lies in \( p_l(\Gamma(L_{2,2a-1})) \).

Furthermore \( \gcd(2a - 1, a) = 1 \) and \( \gcd(2a - 1, a + 1) = \gcd(2(a + 1) - 3, a + 1) \) divides 3, thus we obtain from Lemma 3.10 that \( p_l(T^{3^{k_1}}) \) is in \( p_l(\Gamma(L_{2,2a-1})) \) for some \( k_1 \in \mathbb{N}_0 \). Let us now work with \( L_{a-1,a+2} \). We have that \( \gcd(2a - 1, a - 1) = \gcd(2(a - 1) + 1, a - 1) = 1 \) and \( \gcd(2a - 1, a + 2) = \gcd(2(a + 2) - 5, a + 2) \) divides 5.

Thus we obtain by Lemma 3.10 that \( p_l(T^{5^{k_2}}) \) is in \( p_l(\Gamma(L_{2,2a-1})) \) for some \( k_2 \in \mathbb{N}_0 \). Again we use that \( 3^{k_1} \) and \( 5^{k_2} \) are relatively prime in order to conclude that \( p_l(\Gamma(L_{2,2a-1})) \) contains \( p_l(T') \). Similarly as in the other cases we conclude the claim.

d) These singular cases have been explicitly calculated, see Table I.

We now show that the level index is at least 3. More precisely, we show that the Veech group is contained in a subgroup of \( SL_2(\mathbb{Z}) \) whose image in \( SL_2(\mathbb{Z}/2\mathbb{Z}) \) is of index 3. Since the level \( l \) is divisible by 2, we obtain the desired property.

Recall that in the cases which we consider the number of integer Weierstraß points of the origami \( O \) is 1 or 2. The Weierstraß points are preimages of the 2-division points \( \infty, A, B \) and \( C \) on the elliptic curve \( E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i) \) with marked point \( \infty = (0, 0) \) (see Figure 4).

\[ \begin{array}{c}
\bullet & \bullet & \bullet \\
\infty & B & A \\
\end{array} \]

Figure 4: The four 2-division points of the elliptic curve \( E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i) \) with marked point \( \infty \).

The integer Weierstraß points are those which are mapped to \( \infty \). Since \( O \) has 1 or 2 integer Weierstraß points, we have 5 or 4 Weierstraß points which are
preimages of $A, B$ or $C$. In particular, one out of the three points $A$, $B$ and $C$ can be distinguished from the other two by the number of Weierstraß points in its preimage. Since affine maps preserve Weierstraß points and each affine map of $X$ descends to $E$, the descend of an affine map on $X$ has to fix this point. Observe that for $P \in \{A, B, C\}$ the subgroup of $\text{SL}_2(\mathbb{Z})$ of elements which fix $P$ is a congruence group of level 2 and has index 3 in $\text{SL}_2(\mathbb{Z})$. This proves the claim. □

5. One Zero strata

Theorem 2 and the methods used in Section 4 are quite general. We exemplary use them to construct infinite families of origamis in each genus $g \geq 3$ whose Veech groups are totally non-congruence groups and prove with this Theorem 4. More precisely, we introduce an explicit family of origamis $O_{g,n}$ ($g \geq 3$, $n \geq 3g-2$) whose Veech groups $\Gamma(O_{g,n})$ are totally non-congruence groups if $n$ is prime to 2,3 and $g-1$.

**Definition 5.1.** Let $O_{g,n}$ for $n \geq 3g-2$ and $g \geq 3$ be the origami given by the permutations

$$
\sigma_a := (1, 4, 7, \ldots, 3g-5, 3g-2, 3g-1, \ldots, n)
$$

$$
\sigma_b := (1, 2, 3)(4, 5, 6)\ldots(3g-5, 3g-4, 3g-3).
$$

The origami is shown in Figure 5. For easier calculations later, we write down $\sigma_a$ and $\sigma_b$ also as functions:

$$
\sigma_a(x) = \begin{cases} 
    x+3, & x \equiv 1 \mod 3 \text{ and } x \leq 3g-5 \\
    x, & x \not\equiv 1 \mod 3 \text{ and } x \leq 3g-3 \\
    x+1, & 3g-2 \leq x \leq n-1 \\
    1, & x = n
\end{cases}
$$

(5)

$$
\sigma_b(x) = \begin{cases} 
    x+1, & x \not\equiv 0 \mod 3 \text{ and } x \leq 3g-3 \\
    x-2, & x \equiv 0 \mod 3 \text{ and } x \leq 3g-3 \\
    x, & x \geq 3g-2
\end{cases}
$$

We directly read off from Figure 5 that $O_{g,n}$ has 1 singularity. The corresponding vertices are shown as marked points in the figure. In addition we have $n-2g+1$ regular vertices. From the Euler characteristic formula we obtain the genus. One further sees from Figure 5 that the saddle connections span $\mathbb{Z}^2$.

**Remark 5.2.** The origami $O_{g,n}$ is primitive, has genus $g$ and lies in the stratum $\mathcal{H}_g(2g-2)$. It has $g$ maximal horizontal cylinders: one of circumference $n-2g+2$ and height 1 and $g-1$ cylinders of circumference 1 and height 2. It further has $g$ maximal vertical cylinders: $g-1$ cylinders with circumference 3 and height 1.
and one cylinder with circumference 1 and height \( n - (3g - 3) \). It follows that the Veech group \( \Gamma(O_{g,n}) \) contains

\[
\begin{pmatrix} 1 & n - 2g + 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}
\]

and the normalised cusp-width pair of \( O_{g,n} \) is \( (a, b) \) with \( a \) divides \( n - 2g + 2 \) and \( b \) divides 3.

We now show that there is an origami \( A \cdot O_{g,n} \) (\( A \in \text{SL}_2(\mathbb{Z}) \)) in the same orbit which has only one cylinder with respect to the horizontal as well as with respect to the vertical direction. Recall from Section 2.2 that if an origami \( O \) is given by a permutation \( (\sigma_a, \sigma_b) \), then \( T^{-1} \cdot O \) and \( T'^{-1} \cdot O \) are given by the pairs

\[
(\sigma_a, \sigma_b \sigma_a) \quad \text{and} \quad (\sigma_b \sigma_a, \sigma_b),
\]

respectively.

Applying these two transformations and their inverse consecutively, we obtain the origamis in the orbit of \( O \).

**Lemma 5.3.** Let \( A = T^{-1}T'^{-1} \) and let \( O_{g,n} \) be the origami from Definition 5.1. If \( g \geq 3 \) and \( n \geq 3g - 2 \), then for the origami \( A \cdot O_{g,n} \) the horizontal and the vertical directions are both one-cylinder directions.

**Proof.** Let \( (\sigma_a, \sigma_b) \) be the pair of permutations from Definition 5.1 describing \( O_{g,n} \). By (6) we have that \( A \cdot O_{g,n} \) is given by the permutations \( (\sigma_b \circ \sigma_a, \sigma_b^2 \circ \sigma_a) \).

We now calculate these permutations. From (6) we obtain:

\[
\sigma_b(\sigma_a(x)) = \begin{cases} 
  x + 4, & x \equiv 1 \pmod{3} \text{ and } x \leq 3g - 8 \\
  x + 1, & x \equiv 2 \pmod{3} \text{ and } x \leq 3g - 4 \\
  x - 2, & x \equiv 0 \pmod{3} \text{ and } x \leq 3g - 3 \\
  3g - 2 = x + 3, & x = 3g - 5 \\
  x + 1, & 3g - 2 \leq x \leq n - 1 \\
  2, & x = n
\end{cases}
\]
Written as a permutation this is:

$$\sigma_b \sigma_a = (1, 5, 6, 4, 8, 9, 7, 11, 12, \ldots, 3g - 8, 3g - 4, 3g - 3, 3g - 5, 3g - 2, 3g - 1, 3g, 3g + 1, \ldots n - 1, n, 2, 3)$$

In particular, this is an $n$-cycle. Furthermore, we have:

$$\sigma_b^2(x) = \begin{cases} 
  x + 2, & x \equiv 1 \mod 3 \text{ and } x \leq 3g - 3 \\
  x - 1, & x \not\equiv 1 \mod 3 \text{ and } x \leq 3g - 3 \\
  x, & x \geq 3g - 2
\end{cases}$$

We then get:

$$\sigma_b^2(\sigma_a(x)) = \begin{cases} 
  x + 5, & x \equiv 1 \mod 3 \text{ and } x \leq 3g - 8 \\
  x - 1, & x \not\equiv 1 \mod 3 \text{ and } x \leq 3g - 3 \\
  3g - 2 = x + 3, & x = 3g - 5 \\
  x + 1, & 3g - 2 \leq x \leq n - 1 \\
  3, & x = n
\end{cases}$$

Written as a permutation this is:

$$\sigma_b^2 \sigma_a = (1, 6, 5, 4, 9, 8, \ldots, 3g - 8, 3g - 3, 3g - 4, 3g - 5, 3g - 2, 3g - 1, 3g, \ldots, n, 3, 2)$$

In particular this is again an $n$-cycle. This finishes the proof. \hfill \Box

We immediately obtain the following corollary.

**Corollary 5.4.** The Veech group $\Gamma(A \cdot O_{g,n})$ contains $T^n$ and $T'^n$. Thus for its normalised cusp-width pair $(a', b')$ we have that $a'$ and $b'$ divide $n$.

We now can conclude Theorem 4 from the following proposition.

**Proposition 5.5.** If $n$ is coprime to 3 and coprime to $2g - 2$, then the Veech group $\Gamma(O_{g,n})$ is a totally non congruence group.

**Proof.** Recall that $\Gamma(O_{g,n})$ and $\Gamma(A \cdot O_{g,n})$ are conjugated. More precisely we have that $\Gamma(A \cdot O_{g,n}) = A \Gamma(O_{g,n}) A^{-1}$. We can apply Theorem 2 for $\Gamma_1 = \Gamma(O_{g,n})$ and $\Gamma_2 = \Gamma(A \cdot O_{g,n})$, since by Remark 5.2 and Corollary 5.4 the least common multiple of the normalised cusp-width pair of $\Gamma_1$ divides lcm$(n - 2g + 2, 3)$ and the least common multiple of the normalised cusp-width pair of $\Gamma_2$ divides $n$. By the assumptions of this proposition it follows that they are coprime. \hfill \Box

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