Balanced pairs on triangulated categories

Xianhui Fu\textsuperscript{a}, Jiangsheng Hu\textsuperscript{b}, Dongdong Zhang\textsuperscript{c} and Haiyan Zhu\textsuperscript{d}\textsuperscript{*}

\textsuperscript{a}School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
\textsuperscript{b}Department of Mathematics, Jiangsu University of Technology, Changzhou 213001, China
\textsuperscript{c}Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
\textsuperscript{d}College of Science, Zhejiang University of Technology, Hangzhou 310023, China

E-mails: fuxianhui@gmail.com, jiangshenghu@jsut.edu.cn, zdd@zjnu.cn and hyzhu@zjut.edu.cn

Abstract

Let $\mathcal{C}$ be a triangulated category. We first introduce the notion of balanced pairs in $\mathcal{C}$, and then establish the bijective correspondence between balanced pairs and proper classes $\xi$ with enough $\xi$-projectives and enough $\xi$-injectives. Assume that $\xi := \xi_X = \xi_Y$ is the proper class induced by a balanced pair $(X, Y)$. We prove that $(\mathcal{C}, \mathcal{E}_\xi, s_\xi)$ is an extriangulated category. Moreover, it is proved that $(\mathcal{C}, \mathcal{E}_\xi, s_\xi)$ is a triangulated category if and only if $X = Y = 0$; and that $(\mathcal{C}, \mathcal{E}_\xi, s_\xi)$ is an exact category if and only if $X = Y = \mathcal{C}$. As an application, we produce a large variety of examples of extriangulated categories which are neither exact nor triangulated.

Keywords: triangulated category; proper class; balanced pair; extriangulated category.

2020 Mathematics Subject Classification: 18G80; 18E10; 18G25; 18G10.

1. Introduction

Relative homological algebra has been formulated by Hochschild in categories of modules and later by Heller, Butler and Horrocks in more general categories with a relative abelian structure. Beligiannis developed in \cite{3} a relative version of homological algebra in triangulated categories in analogy to relative homological algebra in abelian categories, in which the notion of a proper class of exact sequences is replaced by a proper class of triangles. By specifying a class of triangles $\xi$, which is called a proper class of triangles, he introduced the full subcategory $\mathcal{P}(\xi)$ of $\xi$-projective objects and the full subcategory $\mathcal{I}(\xi)$ of $\xi$-injective objects.

The notion of balanced pairs involving two additive full subcategories of an abelian category $\mathcal{A}$ was introduced by Chen \cite{5}, which generalizes projectives and injectives from homological aspects. In general, a balanced pair always shares many similar properties with the projectives and injectives. We refer to Chen \cite{5} for more details. It should be noted that this subject first appeared in Enochs’ work (see \cite{6} for instance), where a pair $(\mathcal{H}, \mathcal{G})$ in $\mathcal{A}$ is balanced if and only if the bifunctor $\text{Hom}_{\mathcal{A}}(-, -)$ is right balanced by $\mathcal{H} \times \mathcal{G}$. For examples of balanced

\textsuperscript{*}Corresponding author. Xianhui Fu was supported by the NSF of China (Grant No. 12071064). Jiangsheng Hu was supported by the NSF of China (Grant Nos. 12171206, 11771212) and Qing Lan Project of Jiangsu Province. Haiyan Zhu was supported by the Natural Science Foundation of Zhejiang Provincial (LY18A010032).
pairs, the reader may refer to [6, Chapter 8]. An interesting and deep result in [13] is that in an abelian category $\mathcal{A}$ with small Ext groups, there exists a one-to-one correspondence between balanced pairs and Quillen exact structures $\xi$ with enough $\xi$-projectives and enough $\xi$-injectives. Motivated by this, it seems natural to introduce the notion of balanced pairs in a triangulated category $\mathcal{C}$, and to establish certain relations connecting balanced pairs with certain classes of triangles in $\mathcal{C}$. Thus, we have the following main result of this paper.

**Theorem 1.1.** Let $\mathcal{C}$ be a triangulated category. The assignments

$$
\Psi : (X, Y) \mapsto \xi_X = \xi^Y \quad \text{and} \quad \Phi : \xi \mapsto (\mathcal{P}(\xi), \mathcal{I}(\xi))
$$

give mutually inverse bijections between the following classes:

1. Balanced pairs $(X, Y)$ in $\mathcal{C}$.
2. Proper classes $\xi$ in $\mathcal{C}$ with enough $\xi$-projectives and enough $\xi$-injectives.

We note that the precise definition of balanced pairs in a triangulated category $\mathcal{C}$ can be seen in Definition 3.8 below. Some examples of balanced pairs are presented Examples 3.9-3.11.

Assume that $E : \mathcal{C}^{op} \times \mathcal{C} \to \text{Ab}$ is an additive bifunctor, where $\mathcal{C}$ is an additive category and $\text{Ab}$ is the category of abelian groups. For any objects $A, C \in \mathcal{C}$, an element $\delta \in E(C, A)$ is called an $E$-extension. Let $s$ be a correspondence which associates an equivalence class $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to an $E$-extension $\delta \in E(C, A)$. This $s$ is called a realization of $E$, if it makes the diagram in [11, Definition 2.9] commutative. A triplet $(\mathcal{C}, E, s)$ is called an extriangulated category if it satisfies the following conditions:

1. $E : \mathcal{C}^{op} \times \mathcal{C} \to \text{Ab}$ is an additive bifunctor;
2. $s$ is an additive realization of $E$; and
3. $E$ and $s$ satisfy certain axioms in [11, Definition 2.12].

Exact categories and extension closed subcategories of an extriangulated category are extriangulated categories. In particular, every triangulated category is an extriangulated category. More precisely, assume that $\mathcal{C}$ is a triangulated category with suspension functor $\Sigma$. For any $\delta \in E(C, A) = \mathcal{C}(C, \Sigma A)$, take a distinguished triangle

$$
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A
$$

and define as $s(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$. With this definition, $(\mathcal{C}, E, s)$ becomes an extriangulated category (see [11, Proposition 3.22(1)]).

**Theorem 1.2.** Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair in a triangulated category $\mathcal{C}$. Then $\xi := \xi_X = \xi^Y$ is a proper class in $\mathcal{C}$. With the notation above, we set $E_\xi := E|_\xi$, that is,

$$
E_\xi(C, A) = \{ \delta \in E(C, A) \mid \delta \text{ is realized as an } E\text{-triangle } A \xrightarrow{x} B \xrightarrow{y} C - \delta \rightarrow \text{ in } \xi \}
$$

for any $A, C \in \mathcal{C}$, and $s_\xi := s|_{E_\xi}$. Hence $(\mathcal{C}, E_\xi, s_\xi)$ is an extriangulated category. Moreover, we have the following
(1) \((C,E_{\xi},s_{\xi})\) is a triangulated category if and only if \(X = Y = 0\); and

(2) \((C,E_{\xi},s_{\xi})\) is an exact category if and only if \(X = Y = C\).

To find more examples of extriangulated categories which are neither exact nor triangulated is an interested topic (see [11], [15] and [8]). Theorem 1.2 provides a systematical way to produce ample examples of extriangulated categories which are neither exact nor triangulated (see Remark 3.14, especially Example 3.11) which include [8, Remark 3.3(1)] as a particular case (see Example 3.9).

The contents of this paper are outlined as follows. In Section 2, we fix notations and recall some definitions and basic facts used throughout the paper. In Section 3, we first introduce and study the balanced pairs in a triangulated category \(C\), and then we give the proof of Theorem 1.1 and Theorem 1.2.

2. Preliminaries

Throughout the paper, we fix a triangulated category \(C = (C, \Sigma, \Delta)\), \(\Sigma\) is the suspension functor and \(\Delta\) is the triangulation.

**Remark 2.1.** Some equivalent formulations for the Octahedral axiom (Tr4), named base change and cobase change, are given in [3, 2.1], which are more convenient to use.

A triangle \((T) : A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \Delta\) is split if \(h = 0\). It is easy to see that if \((T)\) is split, then the morphisms \(f, g\) induce a direct sum decomposition \(B \cong A \oplus C\). The full subcategory of \(\Delta\) consisting of the split triangles will be denoted by \(\Delta_0\).

The following definitions are quoted from [3, Section 2]. A class of triangles \(\xi\) is closed under base change if for any triangle \(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A\) in \(\xi\) and any morphism \(\varepsilon : E \rightarrow C\), one gets from the commutative diagram of triangles

\[
\begin{array}{ccc}
0 & \xrightarrow{\alpha} & M \\
\downarrow & & \downarrow \\
A & \xrightarrow{f'} & G \xrightarrow{g'} E \xrightarrow{h'} \Sigma A \\
\downarrow & \delta & \downarrow \\
A & \xrightarrow{f} & B \xrightarrow{g} C \xrightarrow{h} \Sigma A \\
\downarrow & \varepsilon & \downarrow \\
0 & \xrightarrow{\Sigma M} & \Sigma M \\
\end{array}
\]

that the triangle \(A \xrightarrow{f'} G \xrightarrow{g'} E \xrightarrow{h'} \Sigma A\) belongs to \(\xi\). Dually, one has the notion that a class of triangles \(\xi\) is closed under cobase change. A class of triangles \(\xi\) is closed under suspension if for any triangle \(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A\) in \(\xi\), the triangle

\[
\begin{array}{ccc}
\Sigma^i A & \xrightarrow{(-1)^i \Sigma^if} & \Sigma^i B \\
\downarrow & & \downarrow \\
\Sigma^i C & \xrightarrow{(-1)^i \Sigma^ig} & \Sigma^i+1 A \\
\end{array}
\]
belongs to $\xi$ for all $i \in \mathbb{Z}$. A class of triangles $\xi$ is called saturated if in the situation of base change, whenever the third vertical and the second horizontal triangles are in $\xi$, then the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is in $\xi$.

**Definition 2.2.** (see [3, Definition 2.2]) A full subcategory $\xi \subseteq \Delta$ is called a proper class of triangles if the following conditions hold:

1. $\xi$ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi \subseteq \Delta$.
2. $\xi$ is closed under suspensions and is saturated.
3. $\xi$ is closed under base and cobase change.

There are more interesting examples of proper classes of triangles enumerated in [3, Example 2.3]. Throughout the paper we fix a proper class of triangles $\xi$ in the triangulated category of $C$.

**Definition 2.3.** (see [3, Definition 4.1]) An object $P \in C$ is called $\xi$-projective if for any triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ in $\xi$, the induced sequence of abelian groups

$$0 \rightarrow C(P, A) \rightarrow C(P, B) \rightarrow C(P, C) \rightarrow 0$$

is exact. The triangulated category $C$ is said to have enough $\xi$-projectives provided that for each object $A$ there is a triangle $K \rightarrow P \rightarrow A \rightarrow \Sigma K$ in $\xi$ with $P \in \mathcal{P}(\xi)$.

Dually, one can define $\xi$-injective objects and enough $\xi$-injectives.

It is easy to check that the full subcategory $\mathcal{P}(\xi)$ of $\xi$-projective objects and the full subcategory $\mathcal{I}(\xi)$ of $\xi$-injective objects are full, additive, closed under isomorphisms, direct summands and $\Sigma$-stable, i.e. $\Sigma(\mathcal{P}(\xi)) = \mathcal{P}(\xi)$ and $\Sigma(\mathcal{I}(\xi)) = \mathcal{I}(\xi)$.

A $\xi$-exact complex $X$ is a diagram $\ldots \rightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \rightarrow \ldots$ in $C$ such that there exists triangle $K_{n+1} \rightarrow K_n \rightarrow \Sigma K_n \rightarrow \Sigma K_{n+1}$ in $\xi$ for each integer $n$ and the differential is defined as $d_n = g_{n-1}f_n$ for each $n$.

If $C$ has enough $\xi$-projectives, then for any object in $A \in C$, there exists a $\xi$-exact complex $P \rightarrow A$ such that $P_i \in \mathcal{P}(\xi)$ for all $i \geq 0$, which is said to be a $\xi$-projective resolution of $A$ (see [3, Definition 4.7]).

**Definition 2.4.** (see [3, Section 4]) For any objects $A, B$ of $\mathcal{C}$, choose a $\xi$-projective resolution $P \rightarrow A$ of $A$. For any integer $n \geq 0$, the $\xi$-cohomology groups are defined as $\xi \text{xt}^n_B(A, B) = H^n(\mathcal{C}(P, B))$.

The $\xi$-projective dimension $\xi$-pd $A$ of $A \in C$ is defined inductively. If $A \in \mathcal{P}(\xi)$, then define $\xi$-pd $A = 0$. Next if $\xi$-pd $A > 0$, define $\xi$-pd $A \leq n$ if there exists a triangle

$$K \rightarrow P \rightarrow A \rightarrow \Sigma K$$

in $\xi$ with $P \in \mathcal{P}(\xi)$ and $\xi$-pd $K \leq n - 1$. Finally we define $\xi$-pd $A = n$ if $\xi$-pd $A \leq n$ and $\xi$-pd $A \nleq n - 1$. Of course we set $\xi$-pd $A = \infty$, if $\xi$-pd $A \neq n$ for all $n \geq 0$.

Dually, we can define the $\xi$-injective dimension $\xi$-id $A$ of $A \in C$. 
Definition 2.5. (see [2, Definition 3.2]) A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\xi$ is called $C(-, P(\xi))-exact$ if for any $P \in P(\xi)$, the induced sequence of abelian groups

$$0 \rightarrow C(C, P) \rightarrow C(B, P) \rightarrow C(A, P) \rightarrow 0$$

is exact. A $\xi$-exact complex

$$\cdots \rightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \rightarrow \cdots$$

in $C$ is called $C(-, P(\xi))-exact$ if for any integer $n$, there exists a $C(-, P(\xi))-exact$ triangle in $\xi$

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

and the differential is defined as $d_n = g_{n-1}f_n$ for each $n$.

Definition 2.6. (see [1, Definition 3.6]) A complete $\xi$-projective resolution is a $\xi$-exact complex $P : \cdots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{d_0} P_{-1} \rightarrow \cdots$ in $C$ such that $P$ is $C(-, P(\xi))-exact$ and $P_n$ is $\xi$-projective for each integer $n$. Let $P$ be a $\xi$-exact complex. So for any integer $n$, there exists a triangle $K_{n+1} \xrightarrow{g_n} P_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$ in $\xi$, the object $K_n$ is called $\xi$-$G$-projective.

Dually, one can define complete $\xi$-injective coresolution and $\xi$-$G$-injective object.

Similar to the way of defining $\xi$-projective and $\xi$-injective dimensions, for an object $A \in C$, the $\xi$-$G$-projective dimension $\xi$-$GpdA$ and $\xi$-$G$-injective dimension $\xi$-$GidA$ are defined inductively in [1].

Throughout this paper, the full subcategory of $\xi$-projective (respectively, $\xi$-injective) objects is denoted by $P(\xi)$ (respectively, $I(\xi)$). We denote by $G\mathcal{P}(\xi)$ (respectively, $G\mathcal{I}(\xi)$) the class of $\xi$-$G$-projective (respectively, $\xi$-$G$-injective) objects. It is obvious that $\mathcal{P}(\xi) \subseteq G\mathcal{P}(\xi)$ and $\mathcal{I}(\xi) \subseteq G\mathcal{I}(\xi)$.

3. PROOFS OF THE RESULTS

Assume that $C = (\mathcal{C}, \Sigma, \Delta)$ is a triangulated category with $\Sigma$ the suspension functor and $\Delta$ the triangulation.

Definition 3.1. Let $(T) : A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a triangle in $C$ and $\mathcal{X}$ a full additive subcategory of $C$.

1. The triangle $(T)$ is said to be left $\mathcal{C}(\mathcal{X}, -)-exact$ if the induced map

$$\mathcal{C}(X, f) : \mathcal{C}(X, A) \rightarrow \mathcal{C}(X, B)$$

is a monomorphism for any object $X \in \mathcal{X}$.

2. The triangle $(T)$ is said to be right $\mathcal{C}(\mathcal{X}, -)-exact$ if the induced map

$$\mathcal{C}(X, g) : \mathcal{C}(X, B) \rightarrow \mathcal{C}(X, C)$$

is an epimorphism for any object $X \in \mathcal{X}$.

3. The triangle $(T)$ is said to be $\mathcal{C}(\mathcal{X}, -)-exact$ if it is both left and right $\mathcal{C}(\mathcal{X}, -)-exact$. Dually, we have the notions of left $\mathcal{C}(-, \mathcal{X})$-exact, right $\mathcal{C}(-, \mathcal{X})$-exact and $\mathcal{C}(-, \mathcal{X})$-exact.
Recall that a subcategory \( \mathcal{X} \) is said to be \( \Sigma\text{-stable} \) if \( \Sigma(\mathcal{X}) = \mathcal{X} \). We have

**Lemma 3.2.** Let \( \mathcal{X} \) be a \( \Sigma\text{-stable} \) full additive subcategory of \( \mathcal{C} \). Then the following are equivalent for any triangle \( (T): A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \overset{h}{\longrightarrow} \Sigma A \).

1. The triangle \( (T) \) is left \( \mathcal{C}(\mathcal{X},-)\text{-exact} \).
2. The triangle \( (T) \) is right \( \mathcal{C}(\mathcal{X},-)\text{-exact} \).
3. The triangle \( (T) \) is \( \mathcal{C}(\mathcal{X},-)\text{-exact} \).

**Proof.** It follows from [9, Lemma 2.6]. \( \square \)

In the following, we set \( \xi_\mathcal{X} \) be the class of triangles satisfying the condition of Lemma 3.2. Let \( \mathcal{Y} \) be a full additive subcategory of \( \mathcal{C} \). Recall that a morphism \( f: C \rightarrow Y \) with \( Y \in \mathcal{Y} \) is called a left \( \mathcal{Y}\text{-approximation} \) (or a \( \mathcal{Y}\text{-preenvelope} \)) of \( C \) if \( \mathcal{C}(f, Y'): \mathcal{C}(Y, Y') \rightarrow \mathcal{C}(C, Y') \) is surjective for any object \( Y' \in \mathcal{Y} \). If any \( C \in \mathcal{C} \) admits a left \( \mathcal{Y}\text{-approximation} \), then \( \mathcal{Y} \) is \( \text{covariantly finite} \) in \( \mathcal{C} \). Dually, one has the notions of right \( \mathcal{Y}\text{-approximation} \) (or a \( \mathcal{Y}\text{-precover} \) and \( \text{contravariantly finite} \) subcategory in \( \mathcal{C} \).

**Proposition 3.3.** Let \( \mathcal{X} \) be a \( \Sigma\text{-stable} \) full additive subcategory of \( \mathcal{C} \) which is closed under direct summands. Then \( \xi_\mathcal{X} \) is a proper class of \( \mathcal{C} \). Moreover, \( \mathcal{X} \) is a \( \text{contravariantly finite} \) in \( \mathcal{C} \) if and only if \( \mathcal{C} \) has enough \( \xi_\mathcal{X}\text{-projectives} \) and \( \mathcal{X} = \mathcal{P}(\xi_\mathcal{X}) \).

**Proof.** It is easy to see that \( \xi_\mathcal{X} \) is closed under isomorphisms, finite coproducts and containing all split triangles. Next we claim that \( \xi_\mathcal{X} \) is closed under base change and cobase change. Consider the following commutative diagram of triangles

\[
\begin{array}{c}
0 \rightarrow M' = M \\
\downarrow \alpha \quad \quad \downarrow \lambda \\
A \quad \quad B \\
\quad \quad \downarrow \beta \\
0 \rightarrow \Sigma M \quad = \quad \Sigma M \\
\end{array}
\]

\[
\begin{array}{c}
A \overset{f}{\longrightarrow} G \overset{g'}{\longrightarrow} E \overset{h'}{\longrightarrow} \Sigma A \\
\downarrow \beta \\
A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \overset{h}{\longrightarrow} \Sigma A \\
\downarrow \gamma \\
0 \rightarrow \Sigma M \quad = \quad \Sigma M \\
\end{array}
\]

If the triangle \( A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \overset{h}{\longrightarrow} \Sigma A \) belongs to \( \xi_\mathcal{X} \), then \( \mathcal{C}(\mathcal{X}, f) = \mathcal{C}(\mathcal{X}, \beta)\mathcal{C}(\mathcal{X}, f') \) is monic. Thus \( \mathcal{C}(\mathcal{X}, f') \) is monic, and hence \( \xi_\mathcal{X} \) is closed under base change. Similarly, we can prove that \( \xi_\mathcal{X} \) is closed under cobase change. To prove that \( \xi_\mathcal{X} \) is closed under saturated, we assume that two triangles \( A \overset{f'}{\longrightarrow} G \overset{g'}{\longrightarrow} E \overset{h'}{\longrightarrow} \Sigma A \) and \( M \overset{\lambda}{\longrightarrow} E \overset{\zeta}{\longrightarrow} C \overset{\xi}{\longrightarrow} \Sigma M \) in the diagram (\( \ast \)) belong to \( \xi_\mathcal{X} \). Then \( \mathcal{C}(\mathcal{X}, \varepsilon) \) and \( \mathcal{C}(\mathcal{X}, g') \) are epic. Thus \( \mathcal{C}(\mathcal{X}, g')\mathcal{C}(\mathcal{X}, \beta) = \mathcal{C}(\mathcal{X}, \varepsilon)\mathcal{C}(\mathcal{X}, g') \) is epic, and so is \( \mathcal{C}(\mathcal{X}, g) \), as desired.

Finally, if \( \mathcal{C} \) has enough \( \xi_\mathcal{X}\text{-projectives} \) and \( \mathcal{X} = \mathcal{P}(\xi_\mathcal{X}) \), then it is easy to check that \( \mathcal{X} \) is a \( \text{contravariantly finite} \) in \( \mathcal{C} \). Conversely, assume that \( \mathcal{X} \) is a \( \text{contravariantly finite} \) in \( \mathcal{C} \). It is clear...
Balanced pairs on triangulated categories

that $\mathcal{X} \subseteq \mathcal{P}(\xi_X)$. For the reverse containment, let $P$ be an object in $\mathcal{P}(\xi_X)$. Then there exists a triangle $K \longrightarrow X \longrightarrow P \longrightarrow \Sigma K$ in $\xi_X$ with $X \in \mathcal{X}$. Thus this triangle is split, and hence $P \in \mathcal{X}$. This completes the proof.

The next two results are dual to Lemma 3.2 and Proposition 3.3, we omit the proof.

**Lemma 3.4.** Let $\mathcal{Y}$ be a $\Sigma$-stable full additive subcategory of $\mathcal{C}$. Then the following are equivalent for any triangle $(T): A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.

1. The triangle $(T)$ is left $\mathcal{C}(-,\mathcal{Y})$-exact.
2. The triangle $(T)$ is right $\mathcal{C}(-,\mathcal{Y})$-exact.
3. The triangle $(T)$ is $\mathcal{C}(-,\mathcal{Y})$-exact.

We set $\xi^\mathcal{Y}$ be the class of triangles satisfying the condition of Lemma 3.4.

**Proposition 3.5.** Let $\mathcal{Y}$ be a $\Sigma$-stable full additive subcategory of $\mathcal{C}$ which is closed under direct summands. Then $\xi^\mathcal{Y}$ is a proper class of $\mathcal{C}$. Moreover, $\mathcal{Y}$ is a covariantly finite in $\mathcal{C}$ if and only if $\mathcal{C}$ has enough $\xi^\mathcal{Y}$-injectives and $\mathcal{Y} = \mathcal{I}(\xi^\mathcal{Y})$.

**Definition 3.6.** Let $M$ be an object in $\mathcal{C}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be $\Sigma$-stable full additive subcategories of $\mathcal{C}$.

1. An $\mathcal{X}$-resolution of $M$ is a diagram $X^\bullet \rightarrow M$ such that $X^\bullet := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$ is a complex with $X_i \in \mathcal{X}$ for all $i \geq 0$, and $\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M$ is a $\xi_X$-exact complex. Moreover, the $\mathcal{X}$-resolution $X^\bullet \rightarrow M$ of $M$ is called $\mathcal{C}(-,\mathcal{Y})$-exact if its $\xi_X$-exact complex is $\mathcal{C}(-,\mathcal{Y})$-exact.
2. A $\mathcal{Y}$-coresolution of $M$ is a diagram $M \rightarrow Y^\bullet$ such that $Y^\bullet := 0 \rightarrow Y_0 \rightarrow Y_{-1} \rightarrow \cdots$ is a complex with $Y_i \in \mathcal{Y}$ for all $i \leq 0$, and $M \rightarrow Y_0 \rightarrow Y_{-1} \rightarrow \cdots$ is a $\xi^\mathcal{Y}$-exact complex. Moreover, the $\mathcal{Y}$-coresolution $M \rightarrow Y^\bullet$ of $M$ is called $\mathcal{C}(\mathcal{X},-)\text{-exact}$ if its $\xi^\mathcal{Y}$-exact complex is $\mathcal{C}(\mathcal{X},-)\text{-exact}$.

The next result characterizes when $\mathcal{C}$ has enough $\xi_X$-projectives and $\xi^\mathcal{Y}$-injectives.

**Proposition 3.7.** Assume that $\mathcal{X}$ and $\mathcal{Y}$ are $\Sigma$-stable full additive subcategories of $\mathcal{C}$ which are closed under direct summands. Then the following are equivalent.

1. $\xi_X = \xi^\mathcal{Y}$, $\mathcal{X} = \mathcal{P}(\xi_X)$, $\mathcal{Y} = \mathcal{I}(\xi^\mathcal{Y})$ and every object in $\mathcal{C}$ has enough $\xi_X$-projectives and enough $\xi^\mathcal{Y}$-injectives.
2. The pair $(\mathcal{X},\mathcal{Y})$ satisfies:
   (a) $\mathcal{X}$ is contravariantly finite and $\mathcal{Y}$ is covariantly finite in $\mathcal{C}$.
   (b) For any object $M \in \mathcal{C}$, there is an $\mathcal{X}$-resolution $X^\bullet \rightarrow M$ such that it is $\mathcal{C}(-,\mathcal{Y})$-exact.
   (c) For any object $N \in \mathcal{C}$, there is a $\mathcal{Y}$-coresolution $N \rightarrow Y^\bullet$ such that it is $\mathcal{C}(\mathcal{X},-)\text{-exact}$.

Proof. (1) $\Rightarrow$ (2) follows from Propositions 3.3 and 3.5.
By Lemmas 3.3 and 3.5, it suffices to show \( \xi_X = \xi_Y \). Let \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) be a triangle in \( \xi_X \). By hypothesis, there is an \( \mathcal{X} \)-resolution \( X^\bullet \to C \) of \( C \) such that it is \( \mathcal{C}(-, \mathcal{Y}) \)-exact. Then there exists a \( \xi_X \)-exact complex

\[
\cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} C
\]

in \( \mathcal{C} \) which is \( \mathcal{C}(-, \mathcal{Y}) \)-exact. This gives us a triangle \( K_1 \xrightarrow{g_0} X_0 \xrightarrow{f_0} C \xrightarrow{h_0} \Sigma K_1 \) which is \( \mathcal{C}(-, \mathcal{Y}) \)-exact, and hence we have the following commutative diagram of triangles

\[
\begin{array}{ccc}
K_1 & \xrightarrow{g_0} & X_0 \\
\downarrow{\lambda} & & \downarrow{\beta} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A.
\end{array}
\]

Let \( Y \) be an object in \( \mathcal{Y} \). Applying \( \mathcal{C}(-, Y) \) to the commutative diagram above, we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}(C, Y) & \xrightarrow{\mathcal{C}(g, Y)} & \mathcal{C}(B, Y) & \xrightarrow{\mathcal{C}(f, Y)} & \mathcal{C}(A, Y) \\
\downarrow{\mathcal{C}(\beta, Y)} & & \downarrow{\mathcal{C}(\alpha, Y)} & & \\
0 & \xrightarrow{\mathcal{C}(f_0, Y)} & \mathcal{C}(X_0, Y) & \xrightarrow{\mathcal{C}(g_0, Y)} & \mathcal{C}(K_1, Y) & \xrightarrow{\mathcal{C}(h_0, Y)} & 0.
\end{array}
\]

Note that \( \mathcal{C}(f_0, Y) : \mathcal{C}(C, Y) \to \mathcal{C}(X_0, Y) \) is monic. It follows that \( \mathcal{C}(g, Y) : \mathcal{C}(C, Y) \to \mathcal{C}(B, Y) \) is monic. So \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) is \( \mathcal{C}(-, \mathcal{Y}) \)-exact by Lemma 3.4 and it belongs to \( \xi_Y \). This implies that \( \xi_X \subseteq \xi_Y \). Dually, one can show that \( \xi_Y \subseteq \xi_X \).

Now we introduce the notion of balanced pairs, which parallels [5, Definition 1.1].

**Definition 3.8.** Assume that \( \mathcal{X} \) and \( \mathcal{Y} \) are \( \Sigma \)-stable full additive subcategories of \( \mathcal{C} \) which are closed under direct summands. The pair \( (\mathcal{X}, \mathcal{Y}) \) is called a balanced pair if it satisfies the equivalent conditions of Proposition 3.7.

The following example comes from Krause [10] and Beligiannis [3].

**Example 3.9.** Assume that \( \mathcal{C} \) is a compactly generated triangulated category. Then the class \( \xi \) of pure triangles (which is induced by the compact objects) is proper and \( \mathcal{C} \) has enough \( \xi \)-projectives and \( \xi \)-injectives. Denote by \( \mathcal{PP} \) the class of pure projective objects and by \( \mathcal{PI} \) the class of pure injective objects in \( \mathcal{C} \). It follows that \( (\mathcal{PP}, \mathcal{PI}) \) is a balanced pair in \( \mathcal{C} \).

Recall that \( \mathcal{C} \) is called a \( \xi \)-Gorenstein triangulated category [2, Definition 4.6], if any object of \( \mathcal{C} \) has both \( \xi \)-G-projective and \( \xi \)-G-injective dimension less than or equal to a nonnegative integer \( n \).

**Example 3.10.** Assume that \( \mathcal{C} \) is a \( \xi \)-Gorenstein triangulated category. By [1, Remark 4.4] and its dual, one has that the class \( \mathcal{GP}(\xi) \) of \( \xi \)-G-projective objects is contravariantly finite in \( \mathcal{C} \).
and the class $\mathcal{GI}(\xi)$ of $\xi$-injective objects is covariantly finite in $\mathcal{C}$. So $(\mathcal{GP}(\xi), \mathcal{GI}(\xi))$ is a balanced pair by [12, Theorem 4.3].

Let $R$ be an associative ring with identity. We denote the category of left $R$-modules by $\text{Mod}R$. Assume that $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in $\text{Mod}R$, that is $\mathcal{Y} = \{M : \text{Ext}_{R}^{1}(X, M) = 0 \text{ for every } X \in \mathcal{X}\}$ and $\mathcal{X} = \{N : \text{Ext}_{R}^{1}(N, Y) = 0 \text{ for every } Y \in \mathcal{Y}\}$. We introduce the following classes, which comes from [7]:

1. $\tilde{\mathcal{X}}$ is the class of all exact complexes $X^\bullet$ of left $R$-modules with each cycle $Z_{n}(X^\bullet) \in \mathcal{X}$.

   Similarly, one has the notion of $\mathcal{Y}$.

2. $\text{dg} \mathcal{X}$ is the class of all complexes $X^\bullet$ of left $R$-modules satisfying that $X_{n}^\bullet \in \mathcal{X}$ for any $n \in \mathbb{Z}$ and every chain map $f^\bullet : X^\bullet \to Y^\bullet$ is null homotopic whenever $Y^\bullet \in \mathcal{Y}$.

   Similarly, $\text{dg} \mathcal{Y}$ can be also defined.

**Example 3.11.** Assume that $K(R)$ is the homotopy category of left $R$-modules. Denote by $\mathcal{P}$ the class of projective left $R$-modules and by $\mathcal{I}$ the class of injective left $R$-modules. Then $(\text{dg} \mathcal{P}, \text{dg} \mathcal{I})$ is a balanced pair in $K(R)$.

**Proof.** It is easy to check that $\text{dg} \mathcal{P}$ and $\text{dg} \mathcal{I}$ are $\Sigma$-stable full additive subcategories of $K(R)$. It follows from [4, Lemmas 4.2 and 4.5] that $\text{dg} \mathcal{P}$ is contravariantly finite in $K(R)$ and $\text{dg} \mathcal{I}$ is covariantly finite in $K(R)$. Note that every object $M$ in $K(R)$ has a triangle

$$K \xrightarrow{f} X \xrightarrow{g} M \xrightarrow{h} \Sigma K$$

with $X \in \text{dg} \mathcal{P}$ and $K$ an exact complex by [4, Lemma 4.5]. Then one can construct a $\text{dg} \mathcal{P}$-resolution $X^\bullet \to M$. To prove that it is $K(R)(-, \text{dg} \mathcal{I})$-exact, it suffices to show that any triangle $(T) : K \xrightarrow{f} X \xrightarrow{g} M \xrightarrow{h} \Sigma K$ with $X \in \text{dg} \mathcal{P}$ and $K$ an exact complex is $K(R)(-, \text{dg} \mathcal{I})$-exact. Let $Y$ be an object in $\text{dg} \mathcal{I}$. Then we have the following exact sequence

$$K(R)(M, Y) \longrightarrow K(R)(X, Y) \longrightarrow K(R)(K, Y).$$

Since $K(R)(K, Y) = 0$, it follows that the triangle $(T)$ is $K(R)(-, \text{dg} \mathcal{I})$-exact by Lemma 3.4, as desired. Similarly, one can construct a $\text{dg} \mathcal{I}$-coresolution $M \to Y^\bullet$ which is $K(R)(\text{dg} \mathcal{P}, -)$-exact. This completes the proof.

We are now in a position to prove the main results of this paper.

**3.12. Proof of Theorem 1.1.** Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair in $\mathcal{C}$. Then $\xi_{\mathcal{X}} = \xi^{\mathcal{Y}}$ is the desired proper class such that $\mathcal{X} = \mathcal{P}(\xi_{\mathcal{X}})$ and $\mathcal{Y} = \mathcal{I}(\xi^{\mathcal{Y}})$ by Proposition 3.7. Conversely, assume that $\xi$ is a proper class in $\mathcal{C}$ with enough $\xi$-projectives and enough $\xi$-injectives. We put $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}(\xi), \mathcal{I}(\xi))$. Note that $\Sigma(\mathcal{P}(\xi)) = \mathcal{P}(\xi)$ and $\Sigma(\mathcal{I}(\xi)) = \mathcal{I}(\xi)$. Then $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}(\xi), \mathcal{I}(\xi))$ is a balanced pair by Proposition 3.7.

For any balanced pair $(\mathcal{X}, \mathcal{Y})$, one can check that $\Phi_{\Psi}(\mathcal{X}, \mathcal{Y}) = \Phi(\xi_{\mathcal{X}} = \xi^{\mathcal{Y}}) = (\mathcal{P}(\xi_{\mathcal{X}}), \mathcal{I}(\xi^{\mathcal{Y}})) = (\mathcal{X}, \mathcal{Y})$. On the other hand, assume that $\xi$ is a proper class in $\mathcal{C}$ with enough $\xi$-projectives and enough $\xi$-injectives, it is straightforward to see that $\Psi_{\Phi}(\xi) = \Psi((\mathcal{P}(\xi), \mathcal{I}(\xi))) = \Psi(\mathcal{P}(\xi_{\mathcal{X}}), \mathcal{I}(\xi^{\mathcal{Y}})) = \xi$. This completes the proof.
3.13. Proof of Theorem 1.2. By Theorem 1.1, we have that \( \xi \) is a proper class in \( \mathcal{C} \) with enough \( \xi \)-projectives and enough \( \xi \)-injectives. If follows from [8, Theorem 3.2] that \( (\mathcal{C}, E_\xi, s_\xi) \) is an extriangulated category. Moreover,

1. If \( \mathcal{X} = \mathcal{Y} = 0 \), then \( \xi \) is the class of all triangles by Theorem 1.1. Hence \( (\mathcal{C}, E_\xi, s_\xi) \) is a triangulated category. Conversely, we assume that \( (\mathcal{C}, E_\xi, s_\xi) \) is a triangulated category. Then for any object \( X \) in \( \mathcal{X} \) and any morphism \( f : C \to X \), one has a triangle

\[
(T1) : K \rightarrowtail C \twoheadrightarrow X \rightarrowtail \Sigma K
\]

in \( \xi \). Thus the triangle \( (T1) \) is split, and hence \( X = 0 \) and \( \mathcal{X} = 0 \). Similarly, one can prove that \( \mathcal{Y} = 0 \).

2. If \( \mathcal{X} = \mathcal{Y} = \mathcal{C} \), then \( \xi \) is the class of split triangles by Theorem 1.1. So \( (\mathcal{C}, E_\xi, s_\xi) \) is an exact category by [11, Corollary 3.18]. Conversely, we assume that \( (\mathcal{C}, E_\xi, s_\xi) \) is an exact category. Then any triangle \( (T2) : A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) in \( \xi \) satisfies that \( f \) is monomorphic. Thus the triangle \( (T2) \) is split, and hence \( \xi \) is the class of split triangles. So \( \mathcal{X} = \mathcal{Y} = \mathcal{C} \). This completes the proof.

Remark 3.14. By Theorem 1.2, one can get that the balanced pairs constructed in Examples 3.9-3.11 can induce extriangulated categories which are neither exact nor triangulated.

References

[1] J. Asadollahi and S. Salarian, Gorenstein objects in triangulated categories, J. Algebra 281 (2004) 264-286.
[2] J. Asadollahi and S. Salarian, Tate cohomology and Gorensteinness for triangulated categories, J. Algebra 299 (2006) 480-502.
[3] A. Beligiannis, Relative homological algebra and purity in triangulated categories, J. Algebra 227(1) (2000) 268-361.
[4] W.J. Chen, Z.K. Liu and X.Y. Yang, Recollements associated to cotorsion pairs, J. Algebra Appl. 17 (2018) 1-15.
[5] X.W. Chen, Homotopy equivalences induced by balanced pairs, J. Algebra 324 (2010) 2718-2731.
[6] E.E. Enochs and O.M.G. Jenda, Relative Homological Algebra, Walter de Gruyter, Berlin, New York, 2000.
[7] J. Gillespie, The flat model structure on Ch(R), Trans. Amer. Math. Soc. 356 (2004) 3369-3390.
[8] J.S. Hu, D.D. Zhang, P.Y. Zhou, Proper classes and Gorensteinness in extriangulated categories, J. Algebra 551 (2020) 23-60.
[9] Y.G. Hu, H.L. Yao, X.R. Fu, Tilting objects in triangulated categories, Comm. Algebra 48 (2020) 410-429.
[10] H. Krause, Smashing subcategories and the telescope conjecture: An algebraic approach, Invent. Math. 139 (2000) 99-133.
[11] H. Nakaoka and Y. Palu, Extriangulated categories, Hovey cotorsion pairs and model structures, Cah. Topol. Géom. Différ. Catég. 60 (2019) 117-193.
[12] W. Ren and Z.K. Liu, Gorenstein homological dimensions for triangulated categories, J. Algebra 410 (2014) 258-276.
[13] J.F. Wang, H.H. Li and Z.Y. Huang, Applications of exact structures in abelian categories, Publ. Math. Debrecen 88 (2016) 269-286.
[14] X.Y. Yang, Model structures on triangulated categories, Glasgow Math J, 57 (2015) 263-284.
[15] P.Y. Zhou and B. Zhu, Triangulated quotient categories revisited, J. Algebra 502 (2018) 196-232.