Quantum or classical perception according to the Imaging Theorem.

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An assessment is given as to the extent to which pure unitary evolution, as distinct from environmental decohering interaction, can provide the transition necessary for an observer to interpret perceived quantum dynamics as classical. This has implications for the interpretation of quantum wavefunctions as a characteristic of ensembles or of single particles and the related question of wavefunction “collapse”.

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I. INTRODUCTION

A. Particle or wave or particle ensemble?

In the scattering of electromagnetic waves around a sharp material object, the nature of the perceived output depends upon the resolution and sensitivity of the instrument. For visible light, in the case of the human eye usually a sharp outline of the object, ascribable to a ray description of the light, would be inferred. However, from measurement with an instrument able to resolve at sub-wavelength accuracy, a blurred outline corresponding to a diffraction pattern and ascribable to the wave nature of the light would be inferred. Continuing the improvement in measurement sensitivity would lead to the detection of individual particles i.e. photons. Their trajectories appear in a seemingly arbitrary pattern until enough photons are counted. Then the statistical distribution gradually assumes the structured form expected on the basis of the wave picture of electromagnetism. This is the wave-particle duality of light. With increasing resolution and sensitivity of the measurement, there are three levels of perception, classical ray trajectory, the wave picture and the ensemble of (quantum) particles picture. Note that in the first two cases an ensemble of many photons are registered, essentially simultaneously. In the third case individual photons are detected but a wave pattern only emerges after a sufficient number of photons are registered.

A similar situation arises in the wave-particle duality of matter. For an ensemble of identical particles with a very large mass on an atomic scale one assigns to their motion a unique classical trajectory so long as the resolution of, say position detection, is not itself on the atomic scale. When the mass of the particles is on the atomic scale it is necessary to calculate their motion from the wave picture of quantum mechanics. Increasing the sensitivity and resolution to detect individual particles leads to a seemingly arbitrary pattern until the statistics are sufficient that a pattern predicted by the wave description emerges [1]. Again there are three levels of perception of the ensemble, unique classical trajectory, many particles registered as a wave pattern or the statistical pattern from individual quantum particles. Which description is appropriate depends upon the the resolution and sensitivity of the measurement.

Note that for light, the first two descriptions, ray and wave pictures both belong to the classical domain and quantisation is necessary only for the statistical photon picture. For material particles, only the first picture is classical and the second two are both quantum. Of course this is because the Maxwell wave equation for light is a part of classical electromagnetism whereas the wave equation for matter is the quantum Schrödinger equation.

A further key element of quantum mechanics is the interpretation of the modulus squared of the wavefunction as a statistical probability. Here, two points of view have emerged. The first, to be called the ensemble picture, is that the probability describes the percentage of members of an ensemble of identical, and identically-prepared, particles having a particular value of a dynamical variable. The second, to be called the single-particle (SP) picture, considers that it is the probability with which an individual particle exhibits the given value. The difference is that in the ensemble interpretation, only measurements on the whole ensemble are meaningful. In the SP picture meaning is assigned to a measurement on a single particle.

Here it is argued that only the ensemble interpretation of the wavefunction is tenable. However, it should be made clear from the outset that “ensemble” refers to an ensemble of $N$ measurements, not necessarily $N$ particles. The initial conditions have to be identical and the wavefunction gives statistical information on the outcomes. The measurements can be simultaneous or sequential. In the case of $N$ particles the particles must be indistinguishable. This specification of ensembles of measurements is necessary since, unthinkable to the founders of quantum mechanics, experiments today can be made on trapped single electrons, atoms or molecules. Then the wavefunction gives only statistical information on a sequence of measurements in which the same particle initially is brought into the same state e.g. experiments on quantum jumps. The classic case of an ensemble of $N$ identical particles is a scattering experiment where a beam of $N$ particles, all in the same state and with a
density so low that they do not interact, scatter successively from the same target. However, were it possible to perform the scattering experiment by re-cycling a single particle, brought again into its initial state \(N\) times, quantum mechanics would predict the same statistical distribution of measured outcomes.

The object of this work is to re-appraise, in the light of the SP and ensemble pictures, the transition from quantum to classical mechanics by emphasising the role of the “Imaging theorem” (IT) [2–6] in determining what an observer perceives as a consequence of the experimental resolution. The IT shows that any system of particles emanating from microscopic separations describable by quantum dynamics will acquire characteristics of classical trajectories simply through unitary propagation to the macroscopic separations at which measurements are made. The IT bears some similarity to Ehrenfest’s theorem (ET) in that classical mechanics is defined through the wavefunction. However, it is much more powerful in that, whereas the ET is concerned with averages only, in the IT the actual coordinates of the quantum wavefunction obey classical mechanics. Furthermore, the IT has meaning only within the ensemble interpretation of the wavefunction.

On the basis of the IT, it emerges that whether one ascribes

a) classical dynamics (a single trajectory analogous to a light ray)

b) a quantum wave description of the ensemble as a whole or

c) single particles registered separately whose statistical distribution corresponds to a wave,

to the movement of material particles depends upon the precision and extent to which the dynamical variables of position and momentum are determined by the measurement. This is equally true for many-particle entangled wavefunctions as it is for ensembles described by single particle wavefunctions. In the former case it is the composite of many particles which is to be viewed as a member of the quantum ensemble, not the individual particles.

Although historically the hallmark of quantum mechanics has been taken to be the non-commutivity of classically conjugate variables (so much so that Born has the commutation relation carved on his gravestone) in recent years, with the birth of the field loosely called “quantum information”, emphasis has shifted to the wave nature of the equation. This linearity leads to the phenomena known as eigenstate superposition, entanglement (for states of more than one particle) and interference arising from the wave nature of the equation.

The elimination of the overtly quantum effects of entanglement and coherence as a prerequisite for the transition to classical mechanics has been ascribed to interaction with the environment \([11]\). This theory is part of the wider study of open quantum systems and various amplifications of the original scheme have been proposed e.g. the “continuous spontaneous localization” (CSL) model \([11]\). These approaches usually involve propagation of the quantum density matrix.

The principal feature of such models is that the interaction with an environment leads to an elimination of the off-diagonal elements of the density matrix, which is a key element of a transition to classical behaviour.

A theory closer to the wavefunction evolution considered in this paper involves a stochastic Schrödinger equation (see Ref.\([12]\) and references therein) where terms arising from interaction with the environment cause wavefunction changes leading to classicality. These theories will be referred to here under the general name of “decoherence theory” (DT).

Without doubt DT can explain many features of the quantum to classical transition but, according to DT, unitary evolution in the system Hamiltonian alone does not contribute to this transition. The main aim here is to show that this is not the case. Generically, according to the IT, a quantum system wavefunction or corresponding density matrix propagating in time without environmental interaction will develop such that the position and momentum coordinates change according to classical mechanics. In particular, the off-diagonal density matrix elements acquire oscillatory phase factors such that, except under a high-resolution measurement, they average to zero. In this sense, the IT does not negate any predictions of DT, rather it is complementary to it. However the unitary propagation transition occurs over time and position increments which are still of atomic dimensions and thus largely obviate any additional changes to the density matrix ascribable to the environment.

An exhaustive discussion of DT with an honest appraisal of its notable successes but also its limitations is given in the reviews of Schlosshauer \([9, 10]\). It is clear that this theory is anchored firmly in the SP interpretation of the wavefunction since wavefunction collapse plays a prominent role.

DT can be employed also to describe certain aspects of the measurement process. In this paper discussion is confined to detection as in modern multi-particle detectors where the quantum particles, following autonomous unitary propagation through vacuum, interact, essentially instantaneously and certainly irreversibly, with a classical detector. Hence the precise details of the measurement of position are of no concern.

In section II the properties and interpretation assigned to the wavefunction in this work are stated. One key element is the ensemble picture and this is sufficient to interpret all collision experiments. Then in section III the IT is presented and shown to be compatible only with the ensemble interpretation of the wavefunction as a field of information.

The question of the ensemble and SP pictures of the wavefunction is discussed further in section IV where the history of these concepts is described in some detail. The
relevance of this question to current theories of wavefunction collapse and the environmental decoherence explanation of the quantum to classical transition is the subject of section V. Examples of how the IT provides an autonomous quantum to classical transition and the manifestations of interference and entanglement within the IT are given. Section VI contains the Conclusions. Appendix A contains mathematical details of the time propagation of the density matrix according to IT and Appendix B a corresponding treatment of two-particle entanglement in the IT limit.

In this work only continuum quantum states are considered of relevance in the transition to classical mechanics. Bound states and quantised internal degrees of freedom (e.g. intrinsic spin) are viewed as wholly quantum features. Furthermore, there is no discussion of the measurement process itself. In the particle detectors employed in modern experiments, the quantum particle (not a wave !) is intercepted by a macroscopic detector involving an enormous number of atomic degrees of freedom, giving a completely irreversible transformation. The particle energy is absorbed through ionisation or photon emission in the detector and amplified to give a recorded signal.

II. INTERPRETATION OF THE WAVEFUNCTION

Here a simple but sufficient interpretation of the wavefunction is adopted. This is the minimum of supposition required to explain modern multi-hit coincident detection of particles emanating from complexes of atomic dimension. In this view many of the apparent difficulties with the meaning and properties of the quantum wavefunction are avoided. The following rules are adopted in connection with the detection of moving particles.

1) The wavefunction always describes a statistical ensemble of identically-prepared particles. No meaning can be ascribed to the wavefunction of a single particle.

2) The wavefunction \( \psi(r) \) is an “information field” of the ensemble whose extent can be infinite or spatially confined.

3) The quantity \( |\psi(r)|^2 dr \) gives the probability to detect a given particle from the ensemble at position \( r \) with a resolution \( dr \) (Born’s rule \([13]\)). The quantity \( |\psi(p)|^2 dp \), where \( \psi(p) \) is the wavefunction in momentum space, gives the probability to detect a given particle from the ensemble with momentum \( p \).

4) When information, either partial or total, is extracted by a measurement, the corresponding part of the quantum wavefunction has been utilised and no further information can be extracted.

The view of the wavefunction as an information field has been emphasised particularly by Solovev \([14]\) and, consequent on this view, the popular expression that a particle can also behave as a wave is redundant. What is detected is always a particle. The wavefunction simply assigns a probability amplitude that a particle from an ensemble of identical particles will be detected to have particular values of the dynamical variables. The only difference from a classical ensemble is the propagation of the wavefunction according to the Schrödinger equation. This simple but sufficient interpretation of the wavefunction is inherent in the IT where unitary evolution of an ensemble according to the Schrödinger equation in semi-classical approximation results in a wavefunction whose co-ordinates (either space or momentum) evolve according to classical mechanics.

As will be shown in the following, the IT provides many of the features of wavefunction propagation ascribed to decoherence due to environmental interaction. However, it largely obviates the need to introduce decoherence at all. A further widely-accepted aspect of quantum mechanics, often attributed also to decoherence, is the concept that measurement leads to a wavefunction collapse. This concept is peculiar to the SP picture. One consequence of this interpretation is the apparent difficulty that, for a many-particle wavefunction, collapse implies instantaneous action at a distance.

III. THE IMAGING THEOREM

The result known as the imaging theorem can be expressed in a few equations. Details of the original proof for free motion can be found in the book of Kemble \([2]\) and its generalisation for arbitrary motion in Ref. \([3]\). The propagation of a wavefunction in space and time can be expressed as an integral equation

\[
\Psi(r_f, t_f) = \int \tilde{K}(r_f, t_f; p, t_i) \tilde{\Psi}(p, t_i) \, dp, \tag{1}
\]

where \( \tilde{K}(r_f, t_f; p, t_i) = \langle r_f | U(t_f, t_i) | p \rangle \) is the mixed coordinate-momentum propagator and \( U(t, t_i) \) is the time-development operator. The times \( t_f \) and \( t_i \) are the final and initial times respectively. For \( n \) particles, the operator \( p \) denotes the initial \( 3n \)-dimensional momentum and correspondingly \( r_f \) represent the final position of the \( n \) particles. The IT rests on the asymptotic large \( r_f \) limit when the action becomes much greater than \( \hbar \) and the propagator can be approximated by its semi-classical form. Then evaluation of the integral in stationary-phase approximation leads to the result (see Ref. \([3]\) for details)

\[
\Psi(r_f, t_f) \approx (-i)^{3n/2} \left( \frac{dp}{dr_f} \right)^{1/2} \exp \left( \frac{i}{\hbar} S_\epsilon(r_f, t_f; r_i, t_i) \right) \tilde{\Psi}(p_i, t_i) \tag{2}
\]
where $r_f$, $r_i$, and $p_i$ are $n$-dimensional position and momentum vectors connected by classical mechanics according to the classical action function $S_c$. The probability density for detection of a particle of the ensemble then is given by

$$|\Psi(r_f(t_f))|^2 \approx \frac{dp_i}{dr_f} |\tilde{\Psi}(p_i,t_i)|^2. \tag{3}$$

Since the coordinates of the wavefunctions now conform to classical mechanics, this form has a wholly classical, statistical interpretation. An ensemble of particles with probability density $|\tilde{\Psi}(p_i,t_i)|^2$ of initial momentum $p_i$ move on classical trajectories and hence the ensemble members evolve to the position probability density $|\Psi(r_f(t_f))|^2$.

The factor $dp_i/dr_f$ is the classical trajectory density of finding the system in the volume element $dr_f$ given that it started with a momentum $p_i$ in the volume element $dp_i$ (see Gutzwiller [15, chap. 1]). Quantum mechanics provides the initial ensemble momentum distribution located at a microscopic distance $r_i \approx 0$. Each element of the initial momentum wavefunction is then mapped onto the spatial wavefunction at large distance $r_f$, where the coordinates are related by classical mechanics.

From Eq. (3) one has also the asymptotic equality of probabilities in initial momentum space and final position space, i.e.

$$|\Psi(r_f(t_f))|^2 \, dr_f = |\tilde{\Psi}(p_i,t_i)|^2 \, dp_i. \tag{4}$$

That is, the loci of points of equal probability of particle detection are classical trajectories. Nevertheless, according to Eq. (2), the wavefunction remains intact.

Clearly, the IT can only be interpreted in the ensemble picture. The wavefunction spreading corresponds to the natural divergence of an ensemble of classical trajectories of differing initial momentum emanating from a microscopic volume and being detected after traversing a macroscopic distance. Nevertheless, estimates of the $r_f$ and $t_f$ values at which the semi-classical approximation becomes valid (Ref. [3]) show that this occurs for values which are still microscopic, typically only tens of atomic units, the precise value dependent upon particle masses and energies.

The IT states that for a wavepacket of arbitrary width the wavefunction coordinates of the ensemble particles evolve along different classical trajectories asymptotically. The spreading of the quantum wavepacket mirrors the identical spreading of a classical ensemble with the same initial momentum distribution. However, only in the limit that the spatial wavepacket remains much narrower than the size of the particles, does each member of the ensemble follow the same classical trajectory as is required in the SP picture.

It is to be emphasised that the IT describes classical evolution of the wavefunction variables and the transition to this property arises from unitary quantum propagation i.e. the transition to classical behaviour is autonomous; external interactions are unnecessary. This has the consequence that experimentalists can trace, from the measured location of detection, the classical trajectory back to the quantum reaction zone. This is valid even for light particles such as electrons.

## IV. HISTORICAL CONTEXT

The question of the transition from quantum to classical mechanics in the motion of particles is as old as wave mechanics itself. Schrödinger, immediately following his invention of wave mechanics in a sequence of papers in 1926, investigated the classical limit of wave mechanics. In a paper [16] entitled "On the continuous transition from micro- to macro-mechanics" he gave an example of how a packet of waves describing the harmonic oscillator can move in such a way that the displacement of the wavepacket as a whole follows the well-known classical dynamics of the one-dimensional harmonic oscillator. In this calculation Schrödinger repeatedly draws the analogy of superpositions of oscillator eigenfunctions to wavepackets formed from classical normal modes on an oscillating string. The important point to note here is that Schrödinger was seeking, through the wave equation, to represent a single particle as a packet of quantum waves which is so localised in space that it can be perceived as a classical particle. Nevertheless he recognized the limitations of his model, pointing out, for example, that a non-dispersive packet can only be built from bound eigenfunctions and any admixture of continuum states will result in an expanding wavepacket as in the optical case.

This latter point was taken up by Heisenberg [17] in a lengthy paper on the interpretation of the new quantum mechanics and its relation to classical mechanics. In a section also called “the transition from micro- to macro-mechanics”, Heisenberg criticised the relevance of bound states in connection with classical mechanics and gently rebukes Schrödinger (p.185) in suggesting that he has forgotten Bohr’s objections to such an interpretation. To illustrate the difficulty with continuum motion Heisenberg showed that an initial gaussian wavepacket moving freely will spread in space as a function of time and so cannot represent a single material particle.

The ensemble viewpoint has been amplified particularly in the work of Ballentine [18, 19]. He defines the classical limit as corresponding to an ensemble of material particles moving along classical trajectories with a distribution in position and momentum. This picture is the basis of modern semi-classical quantum mechanics and emerges quite naturally from the IT. Also it is one important aspect of “Bohmian” mechanics [20] where “quantum” trajectories are assigned to particles. However, as pointed out above, both Schrödinger and Heisenberg blurred the distinction between the particle and ensemble pictures.

The emphasis on ensembles can be traced back to 1927.
in a paper by Kennard \cite{21} discussed below. Kennard showed that the centroid of quantum “probability packets” move according to classical mechanics. This result was extended by Ehrenfest \cite{22} and apparently, the clarification of the connection of quantum to classical mechanics received an enormous boost with this publication. The paper appeared also in 1927, only one year after Schrödinger’s papers introducing wave mechanics. Ehrenfest proved the theorem showing that quantum position and momentum \textit{expectation values} obey a law similar to Newton’s law of classical mechanics. The paper is entitled “A remark on the approximate validity of classical mechanics within quantum mechanics”. Mindful of Heisenberg’s proof of free wavepacket spreading, Ehrenfest is careful to stress that, within the particle picture, the motion of the mean value according to Newtonian mechanics is only meaningful “for a small wavepacket which remains small (mass of the order of 1gm.)”. Clearly he was thinking of a single particle described by a small wavepacket. The ideas that narrow wavepackets and Ehrenfest’s theorem (ET) embody the nature of the quantum to classical transition for a single particle, pervade most elementary text books on quantum mechanics even today.

However, there is only one situation where this is justified at all. That is the case where a particle becomes macroscopically massive and the ensemble of such particles is represented by a wavepacket whose width, the variation around the centre-of-mass position of the ensemble, is very small compared to the spatial extent of each particle. Then, as Ehrenfest recognised \cite{22}, for a massive particle, the spreading of the wavepacket is negligible even over cosmic time scales. Accordingly, all particles of an ensemble would be assigned, to normal macroscopic accuracy, exactly the same trajectory. In this case a single particle has identical dynamics to all other members of the ensemble and it is adequate to talk of the wavepacket as describing the motion of a single particle. The ensemble and SP pictures merge into one. The localisation then implies that the particle position, identical with the wavepacket centre-of-mass, obeys Ehrenfest’s theorem and hence follows Newton’s equation (up to quadratic potentials) or that the deviations are negligible. It is this feature of the large mass limit that probably leads the authors of some text books to assign a wavefunction to a single particle even where the wavefunction is not localised.

In practice, since almost all observations on quantum systems require detection of very many identically-prepared particles, not simultaneously but over a given time, it is understandable that the probability is assigned as if it were a property of a single particle. Statistically, often one obtains the same answer whether a varying property distribution is attributed to an ensemble, each member of which has a fixed value, or whether one assumes that one and the same member of the ensemble appears at different times with a different value of the property.

A. The particle and ensemble pictures

In this section a summary of some of the important papers leading to the interpretation of the wavefunction and the quantum to classical transition are given.

1. Ehrenfest 1927

In discussing the ET, Ehrenfest’s original notation will be used. Specifically, he sought to emphasise the connection to classical mechanics by defining average quantities with the usual names given to the position and momentum variables of Hamilton theory. He defined the “variables”

\begin{equation}
Q(t) = \int_{-\infty}^{+\infty} dx \, x \, \Psi^* \\
P(t) = \frac{i}{\hbar} \int_{-\infty}^{+\infty} dx \, \frac{\partial \Psi^*}{\partial x}.
\end{equation}

Note that Ehrenfest used the symbol \( h \) for what we would now call \( \hbar \). Using these definitions of average position and momentum and the time-dependent Schrödinger equation (TDSE) for a particle of mass \( m \) in a potential \( V(x) \), one can derive the pair of equations

\begin{equation}
\frac{dQ}{dt} = \frac{1}{m} P
\end{equation}

and

\begin{equation}
m \frac{d^2Q}{dt^2} = \frac{dP}{dt} = \int dx \, \Psi \Psi^* \left( -\frac{\partial V}{\partial x} \right).
\end{equation}

Ehrenfest comments upon this result: “whenever the width of the (probability) wavepacket is fairly small, the acceleration (of the centre-of-mass \( Q \)) of the wavepacket corresponds, in the sense of Newton’s equation, to the force operating at the position of the wavepacket \((-\partial V/\partial x)\)”.

A rather inconclusive comment, although Ehrenfest does show that a free wavepacket remains small for macroscopic times if the mass is large enough. Of course, in order to truly correspond to classical mechanics the average value of the force must be replaced by the force at the position \( Q \), i.e.,

\begin{equation}
m \frac{d^2Q}{dt^2} = \left( -\frac{\partial V}{\partial Q} \right).
\end{equation}

Unfortunately perhaps, since it has led to an unjustified importance being attributed to the ET, it turns out that for the cases \( V = a, V = ax \) and \( V = ax^2 \), where \( a \) is a real constant, this equation is true. For these simple but important cases the average position of a wavepacket follows exactly Newton’s law. This feature is often presented as the proof of the transition to classical mechanics. The spreading of wavepackets remains a problem however, and only in special, non-typical cases does the
wavepacket remain narrow. Also, for all other potentials with terms higher than quadratic one does not have motion according to Newton’s law. Hence, for these two reasons and despite its appealing form, in general Ehrenfest’s theorem cannot be considered as describing the transition to classical mechanics, as emphasised by Ballentine [18, 19].

2. Kennard 1927

The 1927 paper of Kennard [21] is a very important landmark in the development of the meaning of the wavefunction. Interestingly, this is one of the last papers to utilise predominantly the Born, Heisenberg, Jordan [23] theory of matrix mechanics. Kennard clearly ascribes to the ensemble viewpoint and defines a “probability amplitude” $M(q)$ for a variable $q$ in matrix mechanics, which is later shown to be equivalent to the Schrödinger wavefunction $\psi(q)$. Despite this, Kennard attributes the statistical interpretation to Pauli and not to Born, although the latter’s paper interpreting the Schrödinger $|\psi|^2$ statistically was published already in 1926.

Kennard’s paper pre-dates that of Ehrenfest by a few months. In it he considers the motion of the centroid of “probability packets” and shows that, for the cases of free motion or constant electric or magnetic fields, the centroid obeys classical mechanics. As explained above, this is due to the simplicity of the potentials but Kennard viewed it an important attribute of what we would now call the “Heisenberg picture” of quantum mechanics.

As perhaps the first to emphasise the ensemble picture, Kennard shows that Heisenberg’s “proof” of the uncertainty principle is properly formulated as the statistical spread of momentum and position measured on an ensemble of identical systems. The spread, for the particular case of a free wavepacket, is calculated using the probability $MM^*$ $dq$ which is identical to Born’s probability interpretation of the Schrödinger wavefunction.

As mentioned above, the case of free motion had been solved already by Heisenberg [12] who showed that a Schrödinger free wavepacket spreads in time. Kennard, although he shows that his probability amplitude $M$ is the same as a Schrödinger wavefunction $\psi$, uses this as an argument against the superiority of Schrödinger wave mechanics with respect to matrix mechanics (Kennard was in Copenhagen at the time and in close contact to Heisenberg, hence the predilection for matrix mechanics). He argues against the use of the Schrödinger wave equation by pointing out that a spreading wavefunction of an electron must correspond to a spreading of charge density. Note that here, in interpreting the $\psi$ wavefunction, Kennard is assuming the SP picture applies to the wavefunction. Then he points out that a detection of the electron must localise the full charge at a point. Hence, because of the measurement, the original diffuse wavepacket “loses any further physical meaning” and must be replaced by “a new, smaller wavepacket”. Kennard is using the necessity, in the particle picture, to invoke a “collapse of the wavefunction” as an argument against the use of a Schrödinger wavefunction. Apparently, he reserves the ensemble interpretation for the results of matrix mechanics.

Following this objection to the collapse scenario, Kennard then advances the ensemble interpretation of the probability amplitude of matrix mechanics. He writes “the wavepacket spreads, for example, like a charge of shot, in which each pellet describes a trajectory dependent upon its initial position and motion and the whole charge spreads in time as a consequence of differences in these initial conditions”. In the ensemble picture, as distinct from the SP picture, there is no problem with the spreading of the wavepacket. Classically particles with different initial momenta will spread as they move from micro- to macroscopic distances. In retrospect, Kennard probably deserves recognition for the “Ehrenfest” theorem, but perhaps this is denied him since he couched his proof in the language of matrix mechanics, whereas Ehrenfest used wave mechanics.

3. After 1927

Although it appears that the ET and associated narrow wavepacket particle picture dominate elementary text books, reminders have been given continually since 1927 of the problems involved with this picture and the essential interpretation of a wavefunction as representing an ensemble and not a single particle. In 1929 Slater [24] gave a layman interpretation in terms of classical statistical distributions as analogy to quantum probability distributions. Kemble in 1935 [25], in a letter on the celebrated EPR paper [26], comments that the interpretation of quantum mechanics “asserts that the wavefunctions of Schrödinger theory have meaning primarily as descriptions of the behaviour of (infinite) assemblages of identical systems similarly prepared”. Writing in 1970, Ballentine [18] advances several arguments “in favour of considering the quantum state description to apply only to an ensemble of similarly prepared systems, rather than supposing as is often done, that it exhaustively represents a single physical system”. In a scholarly essay in 1980, on the “Probability interpretation of quantum mechanics”, Newton [27] emphasizes that “the very meaning of probability implies the ensemble interpretation”.

In 1994, Ballentine et.al. [14] examined the Ehrenfest theorem from the point of view of the quantum/classical transition and concluded that “the conditions for the applicability of Ehrenfest’s theorem are neither necessary nor sufficient to define the classical regime.” Furthermore, in connection with the ensemble or SP pictures they concluded that “the classical limit of a quantum state is an ensemble of classical orbits, not a single clas-
cical orbit.”

Holland [21] has given a full account of Bohmian mechanics in which the wavefunction itself represents a distribution of “quantum” trajectories, such that the ensemble interpretation arises automatically in this presentation of quantum mechanics.

The SP picture lies at the heart of most debates on the meaning of quantum mechanics right up to the present day in an enormous number of papers. It leads to the concepts of wavefunction collapse and action at a distance. Its assumption can be traced not only to the first work of Schrödinger and Heisenberg but led to the EPR paper and all subsequent discussions of the questions raised there. These questions do not arise in the ensemble picture, although of course the origin of the statistical behaviour predicted by the Schrödinger wavefunction remains unexplained.

V. IMAGING THEOREM AND DECOHERENCE THEORY: IT AND DT

As stated in the Introduction, the suppression of state superposition, entanglement and interference can be seen as a requirement on the way to a classical limit of quantum mechanics and has come to be known as “decoherence theory” (DT). In the following the transition from quantum to classical perception is discussed first. Then the quantum features of wavefunction collapse, entanglement and interference are considered.

A. The transition to classical mechanics.

There is an enormous literature on DT and alternative models such as lead to “spontaneous localisation” due to stochastic interaction. They are connected, directly or indirectly, with the so-called “measurement problem” and the questions of wavefunction collapse and “action at a distance”. Popularly these effects are called “spooky” or manifestations of “quantum weirdness”. Such fanciful epithets arise usually in connection with the SP and are largely unnecessary in the ensemble picture. Space does not permit a discussion of the many and varied aspects of this debate, so here consideration is given to those aspects relevant to the quantum to classical transition embodied in the IT and to the SP or ensemble interpretation of the wavefunction. This implies that attention is focussed almost exclusively on quantum states of continuum motion. Internal bound states are not considered explicitly.

The essence of DT is given in the now famous paper of Zurek [8] and in more detail in the reviews of Schlosshauer [4,10]. A more exhaustive treatment with discussion of the ħ dependence of the environmental interaction terms is to be found in the stochastic Schrödinger equation approach [12]. Here the simpler original density matrix version of Zurek [8] is sufficient as illustration.

The basic mechanism of DT by which certain quantum aspects are eliminated is quite opaque, accounting for the universality of this phenomenon. In the simplest case presented in Ref. [8], a one-dimensional two-state quantum system $S$, with wavefunctions $\psi_n$, is assumed to become entangled with an “environment” with corresponding wavefunctions $E_n$. Limiting to two-state quantum systems, the ensuing entangled state vector is

$$|\Psi\rangle = \alpha|\psi_1\rangle|E_1\rangle + \beta|\psi_2\rangle|E_2\rangle$$

and gives a total density matrix $\rho = |\Psi\rangle\langle\Psi|$. According to Ref. [8], “the statistics of all possible local measurements on $S$ are exhaustively encoded in the reduced density matrix $\rho_S$”, given by

$$\rho_S = Tr_{E} \rho = \alpha^2|\psi_1\rangle\langle\psi_1| + \beta^2|\psi_2\rangle\langle\psi_2| + \alpha^*\beta|\psi_1\rangle\langle\psi_2| (E_2|E_1\rangle + \alpha\beta^*|\psi_2\rangle\langle\psi_1| (E_1|E_2\rangle).$$

(10)

Then a measurement of the particle’s position is given by the diagonal element,

$$\rho(x,x) = |\alpha|^2|\psi_1(x)|^2 + |\beta|^2|\psi_2(x)|^2 + 2Re[\alpha^*\beta^*\psi_1(x)\psi_2^*(x)(E_2|E_1\rangle)]$$

(11)

where “the last term represents the interference contribution”. The assumption of DT is that in general the states of the environment are orthogonal and so the interference term disappears. More importantly, from Eq. (10) the off-diagonal terms disappear and one has a diagonal density matrix only. From Eq. (11) this has two “classical” terms interpreted as classical probabilities.

A slightly different model is adopted in Ref. [8] in that the two states comprising the system $S$ are taken as two spatially-separated Gaussian wavefunctions. The corresponding system density matrix exhibits four peaks. This density matrix is propagated in time subject to a temperature-dependent environment interaction. The result is to give a density matrix of diagonal form with only two spatially-separated Gaussian peaks along the diagonal. In this case the decoherence reduces the off-diagonal elements to zero and the diagonal term does not contain the “interference” contribution since the Gaussians do not overlap. This removal of coherence between different spatial parts of the wavefunction is considered to correspond to the emergence of classicality.

There are several problems with this interpretation, discussed in some detail in the reviews [8,10]. One is the question of the degree of the transition to classicality. The reduced density matrix, although diagonal, is still a quantum density matrix. As Schlosshauer remarks [4] there are two probabilities on the diagonal of the density matrix so one has a mixed state rather than a pure state. However, in the SP picture the question of which of the states the system collapses into is not determined.
In connection with the classical transition he writes \[10\] “the interaction between a macroscopic system and its environment will typically lead to a rapid approximate diagonalisation of the reduced density matrix in position space and thus to spatially localised wavepackets that follow (approximately) Hamiltonian trajectories”. This aspect of the emergence of classicality appears to be an assumption. Implicit is the SP picture in which the diagonal elements represent narrow wavepackets giving classical behaviour via Ehrenfest’s theorem. The ultimate spreading of these wavepackets is not considered, although suitable environmental interaction can lead to the wavepackets remaining narrow. In short, the transition to classicality is viewed as an elimination of quantum coherence effects and the vital aspect of the emergence of classical dynamics according to Newton not treated in detail.

In appendix A, following the example of Ref. \[8\], the free unitary propagation of two, initially narrow, gaussian wavepackets within the IT is calculated. It is shown that, under low detector resolution, the density matrix also assumes the diagonal form

\[
\rho(x,x',t) = \frac{1}{\sqrt{\pi\eta(t)}} \left( e^{-\left(x-X_1\right)^2/\eta^2} + e^{-\left(x-X_2\right)^2/\eta^2} \right) \tag{12}
\]

where \(X_1, X_2\) are the centres of the wavepackets and the time-dependent width is \(\eta = \tilde{\sigma}t/\mu\), for initial width \(\tilde{\sigma}\) and particle mass \(\mu\). Hence, the intrinsic spreading of the wavepacket with time emerges as expected in the ensemble picture. In this picture there is no problem of interpretation of the two probabilities: 50% of the ensemble members will be detected near to \(X_1\) and 50% near to \(X_2\). Wavefunction collapse is unnecessary. Most important however, in the IT, the propagation of the coordinates of the diagonal density matrix is according to classical mechanics. Nevertheless, if the resolution is on the microscopic scale then interference and manifestations of quantum propagation resulting from finite off-diagonal elements can be detected. Just as in optics, the perception of particle trajectory (ray) or wave is decided by the sharpness of vision.

That a spin system can exhibit classical behaviour under low observation resolution (coarse measurement) has been noted also recently and proposed as a cause of the quantum to classical transition ancillary to DT \[28, 29\]. However, this is simply another example of the unitary transition to classical mechanics, in this case for motion involving angular momentum. Indeed, in the case of the discrete quantisation of the angular momentum of continuum particles, the passage to classical mechanics is as old as scattering theory itself. It finds its simplest expression in the textbook example of potential scattering. In a full quantum theory the projectile wavefunction is expanded in a superposition of partial waves of fixed integer angular momentum \(l\), in action units of \(\hbar\).

At low velocities only a few partial waves contribute to the scattering and quantum theory is needed. However, the semi-classical limit is now very simple; it is given by \(l \gg 1\) corresponding to the action much greater than \(\hbar\). In this limit of high energy where very many \(l\) values contribute, the sum over \(l\) is replaced by an integral and leads, again in stationary phase approximation, to a classical cross-section. This is completely analogous to the treatment of Fraunhofer diffraction in optics.

In more general scattering cases the transition to classical mechanics is considered in ion-atom or ion-molecule collisions \[30\] or, in nuclear physics for example in the Coulomb excitation of nuclei \[31\]. Then one replaces the quantum heavy-ion projectile motion by a completely classical prescribed trajectory. Note that in this high-energy limit the motion is well-approximated as classical in all regions, whereas the IT gives classical behaviour of coordinates only outside the quantum reaction zone but in principle for particles of all energy and mass.

These treatments are examples of the general field of semi-classical quantum mechanics, which is of course the basis of the IT. Early discussions giving great understanding of the connection between quantum and classical mechanics are given by Berry and Mount \[32\] and by Miller \[33\], for example. Major contributions made by Gutzwiller are to be found in Ref. \[15\].

**B. Decoherence and quantum gravity**

The aim of quantum gravity is to quantise space-time as well as matter. Although still a subject of some controversy, one version of quantum gravity uses functional differential equations of form similar to the Schrödinger equations of non-relativistic quantum theory. Hence there is interest in the semi-classical forms of these equations and the transition to classical mechanics, which implies the Einstein field equations of general relativity. It has been proposed \[34, 35\] that the transition to classical mechanics, as in non-relativistic quantum mechanics, is caused by decoherence. Halliwell writes \[34\] “We show........that the density matrix of the Universe will decohere if the long-wavelength modes of an inhomogeneous massless scalar field are traced out.” That is, the decohering of the density matrix of quantum gravity occurs exactly as above by tracing over unobserved elements. The decoherence transition in quantum gravity has been discussed extensively by Kiefer \[35\] and its role is assessed in a very recent paper by Rugh and Zinkernagel \[36\]. However, here also the operation of the IT under unitary time propagation has not been considered. Although details must be worked out, it is clear that the IT will also lead to the appearance of classical characteristics as in normal quantum mechanics. Then, when the Hilbert-Einstein action (see \[35\]) of quantum gravity greatly exceeds \(\hbar\), the propagation of quantum probabilities will be along classical “trajectories” described by Einstein’s field equations, exactly analogous to Eq. \[9\].
C. Wavefunction collapse

Another aspect emphasised in DT, but which has far wider acceptance in the interpretation of quantum mechanics, is the phenomenon of wavefunction collapse peculiar to the SP picture. However, the difference between SP and ensemble pictures is less pronounced for bound (discrete eigenvalues) than for continuum (continuous eigenvalues) states. For bound states, one can interpret a wavefunction in a linear superposition

\[ \Psi = \sum_n c_n \phi_n \]  

in two ways, which for statistical purposes lead to essentially the same results. Then, to which picture one adheres is largely a matter of personal taste. In the ensemble picture the wavefunction \( \Psi \) is assigned to the ensemble as a whole. Each member is in a well-defined state and the probability \( |c_n|^2 \) is the percentage of members of the ensemble occupying a given state \( \phi_n \) which typically is an eigenstate of some dynamical observable. In the single-particle picture, the complete wavefunction \( \Psi \) is assigned to each particle which is considered to be in a linear superposition of eigenstates. Upon measurement the wavefunction collapses with a probability \( |c_n|^2 \) into one particular eigenstate.

The divergence of the pictures is far more pronounced for continuum states of, in principle, infinite extent where wavefunction collapse in position space implies localisation of the wave. This is the situation criticised by Kennard in 1927. Again, for some purposes it is immaterial whether one interprets \( |\Psi(x)|^2 \, dx \) as the probability that a given particle from a large ensemble is to be found at position \( x \) or that a single particle is found at \( x \) rather than somewhere else with a different probability. However, again, the latter single-particle picture requires, upon measurement, a collapse of the wavefunction to a form whose limit is a delta function at position \( x \). This transition, say from a delocalised plane-wave form of macroscopic dimensions, is hardly imaginable, as Kennard noted.

Indeed, the experiments supporting the ensemble picture have already been performed. Using electron diffraction through a pair of slits, it has been shown \[ \text{[1]} \] that the wave interference pattern is built up slowly by registering many hundreds of hits of individual electrons on a detector screen. The density of probability of hits at particular points on the screen is decided by the wavefunction of the ensemble.

The SP picture is the wavefunction, extending over macroscopic distance, represents the potential detection position of each single electron. The detector is required to instigate decoherence leading to instantaneous wavefunction collapse (from macroscopic to microscopic extent) and the electron being registered at a single localised point on the detector. Again, one is faced with the dilemma of Kennard in understanding such a transition.

D. Entanglement and interference

That superposition applies to an ensemble is made clear by the process of radioactive decay, for example. Although usually thought of in the time domain, the stationary picture is simpler. An ensemble of nuclei is described by a superposition of the state of a bound nucleus and a state of two product nuclei separated but at the same total energy. The intrusion of a measuring device simply determines in which state does a given member of the ensemble appear. The absence of a signal in a measuring device denotes undecayed state and a signal denotes a decay. The half-life is interpreted from a sequence of measurements on the ensemble. It is not a property of a single nucleus, although colloquially the half-life is often so ascribed. This aspect is emphasised particularly in the very clear exposition of Rau \[ \text{[37]} \].

The study of collision complexes in nuclear, atomic and molecular physics has long been concerned with the questions of measurement of interference and entanglement effects \[ \text{[38, 39]} \]. Coincidence detection of collision fragments in entangled states are performed with increasing sophistication (see, for example, Ref. \[ \text{[40]} \]). Such detectors often involve external extracting fields under whose influence, classical motion of the collision fragments is assumed and shown to be appropriate. Nevertheless quantum coherence is preserved showing that environmental decoherence does not occur in such experiments. In atomic physics effects of interference and entanglement are usually called “coherence and correlation” effects. Unfortunately, to a large extent the implications of such studies for questions such as the observation of entanglement and transition to classical mechanics have not been appreciated by the community of quantum information and decoherence. Furthermore, although decoherence is suggested as “an omnipresent phenomenon” such that “Even microscopic systems such as large molecules are rapidly decohered by the interaction with thermal radiation on a time scale that is much shorter than any practical observation could resolve”, \[ \text{[2]} \] this is patently not true in coincidence experiments where quantum correlation is maintained to macroscopic distances even in the presence of external electromagnetic fields.

The difference between SP and ensemble pictures is even more stark when one considers entanglement, or coherence and correlation in collision experiments. In DT, if the environment is simplified to consist of only one quantum degree of freedom, the system, say an ensemble of single particles, can become entangled with it. Then, mere non-observance of the environment will lead to decoherence of the quantum system. This is shown clearly in Ref. \[ \text{[3]} \] and is known as environment-induced decoherence. Mathematically this is similar to
the reduced density matrix of Eq. (10). At the level of a single degree of freedom the decoherence is reversible by a monitoring of the environment wavefunction. However, environments typically involve very many degrees of freedom leading to practical irreversibility and loss of interference effects.

Interestingly, the effect of an entangled single-degree-of-freedom environment is the same as that of any particle entangled with the quantum system. Non-observance of it leads to decoherence. This effect has long been studied in coincidence experiments involving many-particle detection [38]. A particularly clear example where the connection to decoherence has been noted is in the two-electron entangled state resulting from photoionisation of the $H_2$ molecule [41]. This is an analogue of the case of helium considered in Appendix B. The degree of decoherence of a many-body entangled state depends upon which particles are not observed or even which dynamical properties are observed and which are not. Coherence can be fully or partially removed according to the experiment. In the language of the experimentalist, either one registers the “coincidence” spectrum or the “singles” spectrum.

The classical simple example of entanglement is that of two particles, either their spatial entanglement as considered in the EPR paper for linear momentum states, or the entanglement of two particles forming a total spin singlet. In Appendix B a simple example which has been well-studied in fragmentation experiments is discussed in detail. This is the full fragmentation of the helium atom by a single photon. Although not new from the point of view of entanglement, this example is given since it comprises both spatial and spin entanglement. Furthermore, from the IT, the electrons can be assigned classical trajectories within the two-electron quantum wavefunction. This is not a “Gedankenexperiment” but a real measured system.

The two electrons emerging can be detected in coincidence and occupy a $^1P_0$ two-electron continuum state (this means their state is a spin singlet, has total orbital angular momentum one unit and odd parity). A selection rule [42] says that electrons of the same energy cannot be ejected back-to-back i.e at 180°. That is, the two-electron state has a node for this configuration. In Appendix B it is shown that, if one of the electrons is left undetected a counter will register electrons of a given energy at a particular angle. However, if a detector diametrically opposed is switched on to detect electrons of the same energy in coincidence, the counts in both detectors will be zero. This coherent state can be made incoherent by switching off one of the detectors when electrons will be measured again. The essence is that this pure effect of wavefunction entanglement is evident, even though according to the IT, the electrons are moving on classical trajectories after they exit the reaction zone with well-defined momenta.

In interpreting the wavefunction it is crucial that the ensemble is viewed as an ensemble of two-electron systems. This two-electron wavefunction is the single quantum entity and it is essential that it be transmitted to the macroscopic detection zone unchanged (environmental decoherence would not allow this). Then there is no problem with the ensemble picture. The wavefunction node says that the total ensemble simply has zero probability that a given member (pair of electrons) will be emitted in the forbidden configuration. The coincidence detection of both position and momentum extracts the information from the wavefunction of the ensemble of two-electron states (spin is undetected). The non-coincident detection of electrons extracts information on the ensemble of single electrons. The effect of entanglement is non-local simply because the two-electron wavefunction is non-local. The SP picture has the usual problem with non-locality (about which countless papers have been written). How can the non-detection of one electron at a given point (requiring a collapse of the wavefunction) imply that also no electrons are detected at a point opposite and a macroscopic distance away?

The explanation of interference patterns in terms of semi-classical wavefunctions and the underlying classical trajectories has been given in great detail by Kleber and co-workers [43] and will not be repeated here. Based upon the IT (see eq.(1) of Kleber [44]), their theory is used to interpret experiments such as those of Blondel et.al. [45]. Here the “photoionisation microscope” exhibits interference rings of electrons ionised from a negative ion in the presence of an extracting electric field. In the semi-classical explanation electrons can occupy two classical trajectories. Either they proceed directly to the detector or, initially they are ejected moving away from the detector but are turned around in the electric field. The imaging of the spatial wavefunction squared is obtained by detection on a fixed flat screen i.e. the position only of electrons is detected. Then an interference pattern from the two trajectories is observed. However, were the vector position and vector momentum of the electrons to be observed, that would correspond to a “which way” determination and the perception would be of two distinct classical trajectories. Interestingly, as distinct from entanglement, in this case it is a lack of information which gives rise to wave perception. Blondel et.al. [45] remark also that for ionisation from neutral atoms the interference rings are too small to be detected, again showing that perception depends upon resolution.

VI. CONCLUSIONS

The imaging theorem corresponds only to the ensemble interpretation. According to the IT, an initial momentum distribution decides the spatial wavefunction at macroscopic distance. This corresponds to an ensemble
of classical particles with the same initial momentum distribution. Each particle appears to move along a classical trajectory to be registered at well-defined position at a distant screen. The locus of points of equal probability is the classical trajectory but the probability is given by the quantum position wavefunction.

It has been shown that;

1) The IT preserves the quantum wavefunction but the momentum and position coordinates change in time according to classical mechanics.

2) as a result of the IT, unitary evolution of quantum systems, even over microscopic distances, leads to perception of an ensemble of particles as following classical trajectories.

3) Standard measurement techniques, either on single or multiple particles, can lead to perception or otherwise of the quantum properties of interference and entanglement according to the information registered. The inference of classical or quantum behaviour depends ultimately upon the resolution and detail of the measurement performed.

Without environment influence, within the IT, unitary evolution of quantum systems results in effective decohering effects of an autonomous nature which lead to the emergence of classical characteristics. The decohering effects are of a more subtle nature than in DT. They occur due to cancellation of oscillating terms of different phase. On the one hand they arise in the stationary phase evaluation of the integral appearing in Eq. (1) leading to the IT itself. On the other hand they lead to non-resolution of oscillatory terms in the propagation of the density matrix to macroscopic times, as in Eq. (A7) and Eq. (A8). Hence, lack of sufficient resolution results in effective decoherence although paradoxically it arises from the very terms, oscillatory phase factors, which are the hallmark of quantum coherence in the wavefunction.

The preservation of the wavefunction can lead to interference. However, the perception of non-classical interference patterns, or not, again depends upon the resolution of the measurement performed. Somewhat paradoxically, the observation of interference patterns implies that, although resolution is high, incomplete information as to the different trajectories encoded in the wavefunction variables is extracted by the measurement. That is, a “which way” detection is not performed. Then, whether one perceives quantum or classical dynamics depends simply upon the precision of the measurement performed and the amount of information extracted from the wavefunction, in close analogy to the optical case.

In the case of the detection of the effects of particle entanglement it is essential to treat the ensemble entity as that corresponding to the many-particle wavefunction. Incomplete extraction of the information encoded in the ensemble wavefunction corresponds to an effective decoherence, although for unitary propagation this is reversible. Although the evolution of classical characteristics occurs independently of an environment, if there is external coupling to environmental degrees of freedom then the additional effects of DT will be evident also.

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Appendix A: The density matrix

As in the discussion of decoherence by Zurek [3] the time development of a one-dimensional single-particle ensemble wavepacket is considered. The wavepacket is composed of two gaussians centred at $x = X_1$ and $x = X_2$ with width such that there is essentially no overlap at $t = 0$. The initial state is then

$$\Psi(x, t = 0) = (\pi \sigma^2)^{-1/4} \sum_{i=1,2} e^{-x_i^2/(2\sigma^2)} \quad (A1)$$

where $x_i \equiv x - X_i$. For $t > t_0$ this initial wavefunction propagates freely in time and has the exact form

$$\Psi(x, t) = (\sigma^2/\pi)^{1/4} \left(\sigma^2 + \frac{ith}{\mu}\right)^{-1/2} \times \sum_{i=1,2} \exp \left[ -\frac{x_i^2}{2 \left( \sigma^2 + \frac{ith}{\mu} \right) } \right] \quad (A2)$$

The IT condition emerges in the limit of large times and distances. Large times corresponds to $\hbar t/\mu \gg \sigma^2$. Then the spatial wavefunction assumes the IT form,

$$\Psi(x, t) \approx \left( \frac{\sigma^2}{\pi} \right)^{1/4} \left( \frac{\mu}{i\hbar t} \right)^{1/2} \times \sum_{i=1,2} e^{-(\mu x_i \sigma/\sqrt{2\hbar t})^2} e^{i\mu x_i^2/(2\hbar t)} \quad (A3)$$

The IT limit giving the classical trajectory is such that $x_i$ and $t$ both become large but the ratio is a constant classical velocity. To emphasise this we introduce the momenta $p_i = \mu x_i / t$. We also define, as the width of the Gaussian in momentum space, $\tilde{\sigma} \equiv \hbar / \sigma$. Then we can simplify the asymptotic spatial wavefunction using

$$(\mu x_i \sigma/\sqrt{2\hbar t})^2 \equiv p_i^2/(2\tilde{\sigma}^2) \quad (A4)$$

and the energy phases

$$\mu x_i^2/(2\hbar t) = p_i^2 t/(2\mu \hbar) \quad (A5)$$
The asymptotic spatial wavefunction is then,

\[ \Psi(x, t) \approx \left( \frac{\mu}{i\sqrt{\pi \sigma t}} \right)^{1/2} \sum_{i=1,2} e^{-p_i^2/(2\sigma^2)} \sim t e^{-p_i^2/(2\sigma^2)} \]  

which looks exactly like a pair of free momentum gaussians propagating in time and corresponds to the 1D form of the IT of Eq. (2), with \( dp_i/dx_i = \mu/t \) for free motion.

The diagonal element of the density matrix is defined as \( \rho(x, x, t) = \Psi^*(x, t)\Psi(x, t) \) and is

\[ \rho(x, x, t) = \frac{\mu}{\sqrt{\pi \sigma t}} \sum_{i=1,2} e^{-p_i^2/\sigma^2} \]

\[ + 2 \cos \left[ \left( p_1^2 - p_2^2 \right) t / (2\mu\hbar) \right] e^{-\left( p_1^2 + p_2^2 \right) / (2\sigma^2)} \]  

(A7)

The off-diagonal density matrix is defined as \( \rho(x, x', t) = \Psi^*(x, t)\Psi(x', t) \) and consists of four terms,

\[ \rho(x, x', t) = \frac{\mu}{\sqrt{\pi \sigma t}} \sum_{i,j=1,2} e^{-\left( p_i^2 + p_j^2 \right) / (2\sigma^2)} e^{-i(p_i - p_j)^2 t / (2\mu\hbar)} \]

(A8)

At \( t = 0 \) this gives rise to four gaussian peaks, as in Ref. [8]. It reduces to the diagonal element when \( p_i = p'_i \), i.e. \( x = x' \) as it should.

One sees that the diagonal matrix element shows two peaks at \( p_i = 0, p_2 = 0 \) or equivalently \( x = X_1, x = X_2 \). There is also an interference term. In the off-diagonal element there are four peaks, with the two additional peaks at \( x' = X_1 \) and \( x' = X_2 \). These also contain oscillatory phase factors giving interference.

Clearly, to observe interference effects the temporal resolution must typically be less than one oscillation, i.e. \( t < 4\mu\pi / (p_1^2 - p_2^2) \). If we take the two peaks to be separated by 1a.u., then in atomic units we have \( t < 4\pi \approx 10^{-16} \) secs.. However, typical resolutions are nanosec., that is seven orders of magnitude larger than this. If the resolution is \( \delta t \equiv \tau \) then the measurement must be integrated over this time period. Typically the oscillatory terms will then give, omitting constants

\[ \int_{-\tau/2}^{\tau/2} e^{i(p_1^2 - p_2^2) t} dt \approx \delta(p_1^2 - p_2^2) \]  

(A9)

and similarly for the off-diagonal element when \( p_i \) is replaced by \( p'_i \). In other words, the oscillations will average to zero under low resolution of measurement on an atomic time scale. From Eq. (A7) this implies that the density matrix will exhibit only two diagonal gaussian peaks for such measurements,

\[ \rho(x, x, t) = \frac{\mu}{\sqrt{\pi \sigma t}} \left( e^{-p_1^2/\sigma^2} + e^{-p_2^2/\sigma^2} \right) \]  

(A10)

with \( p_i = \mu(x - X_i) / t \). For the off-diagonal elements, from Eq. (A8), all the terms will average to zero under normal time resolution to give zero off-diagonal elements. This is exactly the limit, elimination of off-diagonal density matrix elements, given by Zurek [8] as the classical limit and resulting from time propagation in the presence of an interacting environment. However, we emphasise again that the wavepackets on the diagonal are spreading and only in the limit that particles are macroscopically massive can this be ignored to give localised single particles as envisaged in [8]. The particle picture is assumed.

By contrast the IT proves classicality emerges from unitary Hamiltonian propagation under low temporal resolution, in that the density matrix then has only two diagonal peaks. Quantum coherence is lost except where the temporal and spatial resolution are extremely high. The peaks represent an ensemble of classical particles moving on classical trajectories, centred around the two centre-of-mass trajectories, and distributed according to the initial gaussian momentum wavefunction.

**Appendix B: Entanglement and two-particle wavefunctions**

To illustrate the main features an extremely simple case of two-particle entanglement is considered. This is the ionization of both electrons from the ground state of the helium atom by absorption of a single photon. These experiments are described in detail in Ref. [46]. In this process the two electrons in the continuum occupy an entangled two-particle state of pure \( 1P^0 \) symmetry. The essence of the theory can be demonstrated by assuming the simplest model of the continuum, namely that, upon ionization, the two electrons occupy plane-wave states of fixed momentum. We define the wavevectors \( k_1 \equiv p_1 / \hbar \). Then the T-matrix element in dipole approximation for ionisation of both electrons can be written [16],

\[ T(k_1, k_2) = \langle k_1, k_2 | e. (\nabla_1 + \nabla_2) | \psi(r_1, r_2) \rangle \]  

(B1)

where \( e \) is the photon polarisation vector. The matrix element is easily evaluated to give

\[ T(k_1, k_2) = e. (k_1 + k_2) \hat{\psi}(k_1, k_2) \]  

(B2)

where \( \hat{\psi}(k_1, k_2) \) is the Fourier transform of the ground state space wavefunction \( \psi(r_1, r_2) \).

We specify further that the scalar momenta are equal \( k_1 = k_2 \). Then a more general derivation [16] shows that the forefactor \( \hat{\psi} \) corresponds to a two-particle entangled angular state represented by a bipolar harmonic reflecting the \( 1P^0 \) symmetry.

Let us consider specific measurements. The most general is a coincident measurement of both electrons. If we fix the detector measuring electron 2 at an angle \( (\hat{e}, \hat{k}_2) \), then it is clear that a detector for electron 1 positioned diametrically opposite and accepting electrons of the same energy, will register zero counts due to the factor \( (k_1 + k_2) \) being zero for \( k_1 = -k_2 \). This is a direct consequence of the entangled state of the two electrons. From the IT one can relate the T-matrix element
at time $t_0$, when the electrons leave the collision zone to the position wave function at later times

$$
\Psi(r_1, r_2, t) = \left( \frac{1}{it} \right)^{3/2} e^{-i(k_1^2 + k_2^2)t/2} T(k_1, k_2, t_0). \quad (B3)
$$

where in atomic units we have put the electron mass and \( \hbar \) equal to unity. Clearly, if the detectors are simply moved further away at the same angular positions there will be no change in the angular distribution since, from the IT, the position coordinates of the two electrons are given by the classical relation \( r_i = k_it \) so that \( r_1 + r_2 = (k_1 + k_2)t \). In particular the zero for emission in opposite directions remains.

Now consider leaving detector one in position but simply switching off detector two which is diametrically opposite. Then one must integrate over all possible emission directions of electron 2. From Eq\( \ref{B2} \) this gives a signal in detector one which is proportional to (we put \( t_0 = 0 \)),

$$
\int |\Psi(r_1, r_2)|^2 \, dr_2 \propto \langle \epsilon, k_1 \rangle^2 \int |\tilde{\psi}(k_1, k_2)|^2 \, dk_2
+ 2 \langle \epsilon, k_1 \rangle \int \langle \epsilon, k_2 \rangle |\tilde{\psi}(k_1, k_2)|^2 \, dk_2
+ \int \langle \epsilon, k_2 \rangle |\tilde{\psi}(k_1, k_2)|^2 \, dk_2.
$$

(B4)

which is plainly not zero in general. Now electrons will be registered in detector one. In fact, if one takes to good approximation the ground state wavefunction as independent of angular factors, then the term involving $\epsilon, k_2$ averages to zero and one has the simple form

$$
\int |\Psi(r_1, r_2)|^2 \, dr_2 \propto A \cos^2 \theta_1 + B \quad (B5)
$$

of the angular distribution of electron 1, where $A$ and $B$ are constants.

Hence, the mere switching-off of one detector appears to have destroyed the entanglement of the two-particle wavefunction. From this point of view the effect of a reduced measurement is the same as that ascribed to decoherence through interaction with the environment. However, in this case the electron 2 acts as environment for electron 1. This is seen when one substitutes the IT relation $k_i = mr_i/(\hbar t)$, to write the detection probability in the form

$$
\int |\Psi(r_1, r_2)|^2 \, dr_2
\propto \int (\epsilon, \hat{r}_1 + \epsilon, \hat{r}_2)^2 |\tilde{\psi}(mr_1/(\hbar t), mr_2/(\hbar t))|^2 \, dr_2
$$

(B6)

which is nothing more than the trace of the two-electron density matrix over the space of electron 2. In decoherence theory, see Section VA, quantum effects are considered to be lost by tracing over the unobserved environment and the interaction with the environment is irreversible. However, in this case, the entangled two-electron wavefunction is recovered simply by switching on detector 2 again, when no counts will be observed in detector one.

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