Mass fluctuations in random average transfer process in open set-up

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Abstract. We define a new mass transport model on a one-dimensional lattice of size $N$ with continuous masses at each site. The lattice is connected to mass reservoirs of different ‘chemical potentials’ at the two ends. The mass transfer dynamics in the bulk is equivalent to the dynamics of the gaps between particles in the random average process. In the non-equilibrium steady state, we find that the multi-site arbitrary order cumulants of the masses can be expressed as an expansion in powers of $1/N$ where at each order the cumulants have a scaling form. We introduce a novel operator approach which allows us to compute these scaling functions at different orders of $1/N$. Moreover, this approach reveals that, to express the scaling functions for higher order cumulants completely one requires all lower order multi-site cumulants. This is in contrast to the Wick’s theorem in which all higher order cumulants are expressed solely in terms of two-site cumulants. We support our results with evidence from Monte Carlo simulations.

Keywords: non-equilibrium statistical physics, random average process, exact solution, statistical physics, steady state

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1. Introduction

Characterization of the steady-state and dynamical properties of systems which are
driven out of thermal equilibrium has been a subject of intense theoretical and exper-
imental studies for many years. Unlike their equilibrium counterparts, no general the-
oretical framework exists within which properties of non-equilibrium systems can be
analysed. More precisely, the stationary distribution of a system in a non-equilibrium
steady state (NESS) is often not known unlike the equilibrium case where the Gibbs–
Boltzmann distribution is known to be correct. To characterise and describe such out-of-
equilibrium systems, one often studies the fluctuations and correlations among different
degrees of freedom or among some coarse-grained degrees of freedom of the system.

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Mass transfer models have played a paradigmatic role in the formulation of the theory of non-equilibrium systems. The zero range process \([1]\), for example, has played an important role in studying non-equilibrium phase transitions and hydrodynamics. Other examples include chipping models \([2, 3]\) the misanthrope process \([4, 5]\), the simple exclusion process \([6, 7]\) and the random average process (RAP) \([8–12]\). In the last decade, a general formulation called macroscopic fluctuation theory \([13]\) has been developed which describes mass transfer models with short range transfer and obeying ‘gradient condition’ \([14, 15]\). In this formulation such systems exhibit (fluctuating) diffusive hydrodynamics of a conserved density field \(\rho(x,t)\), characterised solely by the diffusivity \(D(\rho)\) and the mobility \(\sigma(\rho)\). To obtain the density dependence of the transport coefficients \(D(\rho)\) and \(\sigma(\rho)\), one usually starts from a microscopic description and identifies the coarse-grained density, corresponding to the microscopic conserved quantity of the dynamics, and the associated current. One calculates \(D(\rho)\) and \(\sigma(\rho)\) by computing the variance of the local density and its response to a small external field \([13, 16]\). The transport coefficients calculated in this way only contain information about the first two moments of the mass transfer statistics. It is assumed that higher moments of the mass transfer distribution scale out under coarse-graining and thus do not matter in the large system-size limit.

In this paper, we study a mass transport model which is the dual of the RAP \([11]\). This model is defined on a one dimensional lattice of size \(N\), with each site having positive mass \(g_i\) where \(i = 1, 2, \ldots, N\) is the site index. A random fraction \(\eta\) of the mass from site \(i\) goes with equal probability to the neighbouring sites \(i \pm 1\) at unit rate, with the distribution of \(\eta\), \(R(\eta)\), being arbitrary. In this model, one can investigate the effect of higher moments of the mass-transfer distribution systematically by tuning the distribution \(R(\eta)\). Although only the first two moments of \(R(\eta)\) appear in the hydrodynamic description of the RAP \([8, 17]\) we show that all higher moments appear in the continuum limit of this process, as one studies cumulants of higher orders in the site masses \(g_i\). Thus, we go beyond the hydrodynamic description to provide a microscopic continuum limit of the process.

We study the process in an ‘open system’ setting, where the boundary sites are connected to two reservoirs of different ‘chemical potentials’. As a result the system reaches a NESS with a net current across the system. In this paper we study this NESS by computing various cumulants of the masses \(g_i\). In a recent study it has been observed that in the NESS the two point correlations \(\langle g_i g_j \rangle_c = \langle g_i g_j \rangle - \langle g_i \rangle \langle g_j \rangle\) as well as the single site variance \(\langle g_i^2 \rangle_c = \langle g_i^2 \rangle - \langle g_i \rangle^2\) possess scaling behaviour in the large \(N\) limit with determinable scaling forms \([12]\). We show that the particular structure of the RAP process allows one to extend this scaling behaviour to multi-site cumulants of arbitrary order. Employing a novel operator method we compute these scaling functions in terms of the scaling functions of lower order correlation functions.

2. Definition of the model and summary of the results

The mass transfer process is defined on a line of \(N\) sites, with the masses \(g_i, i \in \{1, 2, \ldots, N\}\) at each site being continuous positive variables. The system is connected to
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**Figure 1.** A schematic diagram of the random average mass transfer system. The system is connected to two different reservoirs at the two boundaries. In time $dt$, a random fraction $\eta$ [chosen from distribution $R(\eta)$] of the mass at a site can get transferred to the nearest neighbour sites on the left or right with equal probability $\frac{dt}{2}$. The masses of the reservoirs on the left and on the right are maintained at given distributions $P_L(g)$ and $P_R(g)$ with means $\bar{g}_L$ and $\bar{g}_R$, respectively.

Mass reservoirs at site 1 and $N$. A schematic diagram of the system is given in figure 1. The mass from site 1 ($N$) can get transferred to the reservoirs on the left (right). Similarly, mass from the reservoir on the left (right) can come to site 1 ($N$). The dynamics at any site in the bulk is given by:

- In a small time interval $dt$, a random fraction $\eta$ of mass $g_i$ from the site $i$ gets transferred to either of its neighbours ($i-1$) or ($i+1$) with equal probability $\frac{dt}{2}$.
- The random variable $\eta \in (0,1)$ is chosen from a given probability distribution $R(\eta)$.

Mathematically, for a given pair of sites say, $\{i,i+1\}$, in the bulk, the dynamics can be written as

$$\left\{g_i(t + dt), g_{i+1}(t + dt)\right\} = \begin{cases} 
\{g_i(t)(1-\eta), g_{i+1} + \eta g_i\} & \text{with prob } R(\eta)d\eta \frac{dt}{2} \\
\{g_i(t), g_{i+1}(t)\} & \text{with prob } 1 - \frac{dt}{2}.
\end{cases} \quad \text{for } 2 \leq i \leq N - 1, \quad (1)
$$

Note that the dynamics in the bulk is mass-conserving and homogeneous under uniform re-scaling of all the masses. The sites 1 and $N$ exchange mass with the reservoirs at site 0 and $N + 1$ respectively. The reservoirs are described as follows. The distribution of the mass $g_0$ at the left reservoir at site 0, is given by a specified distribution $P_L(g_0)$. Similarly the distribution of the mass $g_{N+1}$ at the right reservoir at site $N + 1$ is given by $P_R(g_{N+1})$. These two distributions are externally controlled and remain unchanged with time even as the reservoirs are exchanging mass with the system. The dynamics at the edge sites 1 and $N$ is defined as:

- In every time step $dt$, site 1 transfers a random fraction $\eta$ of its mass to the right to site 2 with probability $dt/2$ or to the left boundary of the system, with probability

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dt/2, where it disappears from the system. Also, in a time dt, with probability dt/2, a mass \( \eta g_0 \) can get transferred to site 1.

- Site \( N \) behaves similarly, either losing mass \( \eta g_N \) to the right boundary with probability \( dt/2 \) or gaining a mass \( \eta g_{N+1} \) with probability \( dt/2 \) from the right boundary. It also exchanges mass with site \( N - 1 \) in accordance with the bulk dynamics.

We take the distribution of the mass in the reservoirs to be the same as the steady state distribution of masses when there is no drive [11].

We call this mass transfer process the random average transfer process (RATP) as this process is closely related to the RAP. This connection is best illustrated in the ring geometry. The RAP process [8, 9] on a ring is defined with \( N \) single file particles moving on a 1D ring, which can not overtake each other. In the random-sequential version of the process, in a small time step \( dt \), a particle, say the \( i \)th, jumps from its current position \( x_i \) either to the left or to the right with equal probability \( dt/2 \). Whenever the particle jumps in one direction, it jumps by a random fraction of the space available till the next particle in that direction. As a result, they maintain their initial order of sequence and do not overtake each other. Now, corresponding to the \( N \) particles in the RAP, consider a periodic one dimensional lattice of \( N \) sites performing mass transfer where the mass at the \( i \)th site is exactly the gap \( g_i = x_{i+1} - x_i \) between the \( i \)th and \((i + 1)\)th particle in the RAP picture. Thus particles from the RAP picture are mapped to the links between lattice sites in the RATP picture. Now, the jump made by the \( i \)th particle towards the \((i + 1)\)th particle in the RAP picture corresponds to mass transfer from the \( i \)th site to \((i - 1)\)th site in the RATP picture. This establishes the connection between the single-file process and the mass transfer process and also justifies the name RATP.

The state of the RATP system at any time \( t \) is given by the masses \( \{g_i\} \) at the sites \( 1 \leq i \leq N \). Let the probability distribution of this configuration be denoted by \( P(\{g_i\}, t) \). This distribution evolves according to the following master equation

\[
\frac{d}{dt} P(\{g_i\}) = \frac{1}{2N} \left[ \sum_{i=1}^{N-1} \int_0^\infty dg_i \int_0^\infty dg_{i+1} \int_0^1 d\eta R(\eta) P(\{g_i\}) \times \delta(g_i - g_i' + \eta g_i') \times \delta(g_{i+1} - g_{i+1}' - \eta g_i) + \delta(g_{i+1} - g_{i+1}' + \eta g_{i+1} - \eta g_i) \right] + \int_0^\infty d\eta R(\eta) P(\{g_i\}) P_L(g_0) \delta(g_i - g_0' + \eta g_i') \\
+ \int_0^\infty d\eta R(\eta) P(\{g_i\}) P_R(g_{N+1}) \delta(g_N - g_{N+1}' - \eta g_N) \\
\times \{ \delta(g_N - g_N' + \eta g_N') + \delta(g_{N} - g_{N+1}' - \eta g_{N+1}) \}, \tag{2}
\]

where \( \{g_i'\} \) is the configuration \( \{g_i\} \) with \( g_i, g_{i+1} \) replaced with \( g_i', g_{i+1}' \), and \( P_L(g) \) is a distribution with mean \( \tilde{g}_L \) and \( P_R(g) \) has mean \( \tilde{g}_R \). When \( \tilde{g}_L \neq \tilde{g}_R \), the system is driven by the particle reservoirs at the two ends. As a result we expect the system to reach a non-equilibrium steady state with a nonzero mass current flowing from
the boundary with the higher mass to the boundary with the lower mass. In this paper we are interested in understanding the properties of this steady state distribution by computing various multi-site and multi-order correlations of the masses \( \{ g_i \} \) of the form

\[
C_{(i_1,i_2,...,i_k)}^{(m_1,m_2,...,m_k)} = \langle g_{i_1}^{m_1} g_{i_2}^{m_2} \ldots g_{i_k}^{m_k} \rangle_c,
\]

with \( k \in \{1,2,3,\ldots,N\} \), \( i_k \in \{1,2,\ldots,N\} \) and \( m_k \in \{1,2,3,\ldots\} \). Here the subscript \( c \) stands for the connected correlation (cumulant). We call \( \sum m_i = M \) the order of the cumulant. Our aim is to compute these correlations for large \( N \). We show that cumulants have a scaling expansions in terms of the continuum limit of the process. To demonstrate our results, we first discuss the scaling structure of connected correlations up to order \( M = 3 \), for large \( N \) and then we present general case of arbitrary order with multiple sites.

For example, we show that connected correlations up to \( M = 3 \) have the following scaling forms for large \( N \) (for \( M = 4 \) see appendix A):

- **Correlations of order \( M = 1 \):**
  \[
  C_i^{(1)} = \langle g_i \rangle_c \approx C_0^{(1)}(x), \quad \text{where} \quad x = \frac{i}{N+1}.
  \]

- **Correlations of order \( M = 2 \):**
  \[
  C_{ij}^{(1,1)} = \langle g_i g_j \rangle_c \approx \frac{1}{N} C_1^{(1,1)}(x,y),
  \]
  \[
  C_{ii}^{(1,1)} = \langle g_i^2 \rangle_c \approx C_0^{(2)}(x) + \frac{1}{N} C_1^{(2)}(x),
  \]

- **Correlations of order \( M = 3 \):**
  \[
  C_{ijk}^{(1,1,1)} = \langle g_i g_j g_k \rangle_c \approx \frac{1}{N^2} C_2^{(1,1,1)}(x,y,z),
  \]
  \[
  C_{ii}^{(1,1,1)} = \langle g_i^3 \rangle_c \approx C_0^{(3)}(x) + \frac{1}{N} C_1^{(3)}(x) + \frac{1}{N^2} C_2^{(3)}(x),
  \]

where \( x = \frac{i}{N+1}, \ y = \frac{j}{N+1} \) and \( z = \frac{k}{N+1} \).

We then show the recursive operator structure, that allows us to determine the terms in the scaling expansion above in terms of lower-order scaled cumulants. For the scaling functions defined above this recursive structure for the scaling functions gives the results

\[
C_0^{(1)}(x) = \bar{g}_L + (\bar{g}_R - \bar{g}_L) \ x,
\]

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\[ C_0^{(2)}(x) = \frac{\mu_2}{\mu_1 - \mu_2} [C_0^{(1)}(x)]^2 \]  \hspace{1cm} (9)

\[ C_1^{(2)}(x) = \frac{\mu_1}{\mu_1 - \mu_2} C_1^{(1,1)}(x,x) \]  \hspace{1cm} (10)

\[ C_1^{(2,1)}(x,y) = \frac{2\mu_2}{\mu_1 - \mu_2} C_0^{(1)}(x) C_1^{(1,1)}(x,y) \]  \hspace{1cm} (11)

\[ C_2^{(2,1)}(x,y) = \frac{\mu_1}{\mu_1 - \mu_2} C_2^{(1,1,1)}(x,x,y) \]  \hspace{1cm} (12)

\[ C_0^{(3)}(x) = \frac{2\mu_2^2}{(\mu_1 - \mu_2)^2} [C_0^{(1)}(x)]^3 \]  \hspace{1cm} (13)

\[ C_1^{(3)}(x) = \frac{6\mu_1\mu_2}{(\mu_1 - \mu_2)^2} C_0^{(1)}(x) C_1^{(1,1)}(x,x) \]  \hspace{1cm} (14)

\[ C_2^{(3)}(x) = \frac{\mu_1(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2} C_1^{(1,1,1)}(x,x,x), \]  \hspace{1cm} (15)

where \( \mu_k = \int_0^1 d\eta \eta^k R(\eta) \) are the moments of the distribution \( R(\eta) \). \( C_0^{(1)}(x) \) represents the average mass profile, which is seen to be linearly interpolating between the densities of the reservoirs of the two ends.

We now state the results in a more general form. Recall that \( \sum_i m_i = M \) is the order of the cumulant defined in equation (3). A cumulant of order \( k \) involving \( k \) distinct sites is what we call the ‘max-site’ correlation function. For \( k = 2 \), the max-site cumulant is the correlation function \( \langle g_ig_j \rangle \), with \( i \neq j \). We demonstrate that for large \( N \) all cumulants have the following expansion in scaling forms, in orders of \( 1/N \),

\[ C_i^m = \langle g_i^{m_1} g_i^{m_2} \cdots g_i^{m_k} \rangle \approx \frac{1}{N^{k-1}} C_{k-1}^m(x) + \frac{1}{N^k} C_k^m(x) \]

\[ + \cdots + \frac{1}{N^{M-1}} C_{M-1}^m(x), \quad \text{with} \quad x_j = \frac{i_j}{N+1}, \]  \hspace{1cm} (16)

where, \( i = (i_1,i_2,\ldots,i_k) \), \( m = (m_1,m_2,\ldots,m_k) \), and \( x = (x_1,x_2,\ldots,x_k) \).

Here \( C_i^m(x) \) is the scaling correlation function of continuous variables \( x_j \in [0,1] ; \ j = 1,2,\ldots,k \) at order \( 1/N^\ell \). We show that all higher order \( (M > k) \) connected correlations of \( k \) sites can be expressed in terms of the ‘max-site’ scaling correlation functions of the type \( C_{\ell-1}^{(1,1,\ldots,1)}(x_1,x_2,\ldots,x_\ell) \) with \( \ell = 1,2,\ldots,M \).

Our main purpose in this paper is to describe an operator recursion method which allows us to do this. Then, using the same operator recursion along with the evolution equation (2) we also show that the ‘max-site’ scaled correlation functions satisfy Poisson equations inside unit hypercube of dimension \( \ell \) with source ‘charges’ distributed.
appropriately,
\[
(\partial^2_{x_1} + \partial^2_{x_2} + \cdots + \partial^2_{x_\ell}) C^{(1,1,\ldots,1)}_{\ell-1}(x_1, x_2, \ldots, x_\ell) = S_{\ell-1}(x_1, x_2, \ldots, x_\ell),
\]
for \(0 \leq x_i \leq 1, \ i = 1, 2, \ldots, \ell,\) \hspace{1cm} (18)

with boundary conditions
\[
C^{(1,1,\ldots,1)}_{\ell-1}(x_1, x_2, \ldots, x_j, \ldots, x_\ell)|_{x_j=0, \text{ or } 1} = 0, \ \text{for } j = 1, 2, \ldots, \ell. \hspace{1cm} (19)
\]

The source term \(S_{\ell-1}(x_1, x_2, \ldots, x_\ell)\) depends on the lower order max-site scaling correlation functions, i.e. on \(C^{(1,1,\ldots,1)}_{\ell-1}(x_1, x_2, \ldots, x_l)\) for \(0 \leq l \leq \ell - 1.\)

The paper is organised as follows. In section 3, we show using a direct approach for the open RATP system that cumulants of order 2 can be derived using microscopic methods. However, this approach becomes cumbersome for cumulants of higher orders, and in section 4, we describe an operator expansion method that allows us to write higher-order cumulants in terms of lower-order ones. In section 5, we test the results of the operator method against simulations. In section 6, we derive equations for the max-site correlation functions, which cannot be reduced further using the operator method. These scaled correlation functions obey Poisson equations, which we derive for the two-site and three-site correlation functions, showing that the operator method highly simplifies the derivation of these correlation functions. In section 7, we conclude with some discussion on future directions.

3. A direct approach

In the steady state, putting \(dP(\{g_i\})/dt = 0,\) we find that the master equation (2) becomes

\[
\left[ \sum_{i=1}^{N-1} \int_0^\infty dg^i_1 \int_0^\infty dg^i_{l+1} \int_0^1 d\eta R(\eta) P(\{g^i_1\}) \times \left\{ \delta(g^i_1 - g^i_{l+1} + \eta g^i_{l+1}) \delta(g^i_{l+1} - g^i_{l+1} + \eta g^i_{l+1}) \delta(g^i_l - g^i_{l} - \eta g^i_{l+1}) \right\} + \int_0^\infty dg^i_1 \int_0^\infty dg_0 \int_0^1 d\eta R(\eta) P(\{g^i_1\}) P_L(g_0) \left\{ \delta(g^i_1 - g^i_{l+1} + \eta g^i_{l+1}) \delta(g^i_l - g^i_{l} - \eta g^i_{l+1}) \right\} \right] = 0. \hspace{1cm} (20)
\]

From this equation one can get the equations satisfied by the correlations functions at different order \(M.\) For example, at order \(M = 1\) one obtains

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are given by

\[ \langle g_{i+1} \rangle_c - 2 \langle g_i \rangle_c + \langle g_{i-1} \rangle_c = 0, \quad \text{for} \quad i = 1,2,..,N, \quad \text{with} \quad \langle g_0 \rangle = \bar{g}_L, \]

and \( \langle g_{N+1} \rangle = g_R. \) \hfill (21)

This is a simple equation and one can solve this equation exactly. The solution is given by

\[ C_i^{(1)} = \langle g_i \rangle_c = \bar{g}_L + (\bar{g}_R - \bar{g}_L) \frac{i}{N+1}, \quad \text{for} \quad i = 1,2,..,N. \] \hfill (22)

In a similar way one can get the equations satisfied by the two-point correlations that appear at order \( M = 2. \) Starting again from equation (20), one computes \( C_{i,j}^{(1,1)} = \langle g_i g_j \rangle_c \) and obtains the following equations

\[ \mu_1 \left[ C_{i+1,j}^{(1,1)} + C_{i-1,j}^{(1,1)} + C_{i,j+1}^{(1,1)} + C_{i,j-1}^{(1,1)} - 4C_{i,j}^{(1,1)} \right] = 0, \quad \text{for} \quad 1 \leq (i,j) \leq N, \]

and \( |i - j| \geq 2, \)

\[ \mu_1 \left[ C_{i+1,i+1}^{(1,1)} + C_{i-1,i+1}^{(1,1)} + C_{i,i+2}^{(1,1)} + C_{i,i+1}^{(1,1)} - 4C_{i,i+1}^{(1,1)} \right] - \mu_2 \left[ C_{i,i}^{(1,1)} + C_{i+1,i+1}^{(1,1)} \right]
\]

\[ = \mu_2 \left[ (C_{i}^{(1)})^2 + (C_{i+1}^{(1)})^2 \right], \quad \text{for} \quad 1 \leq i \leq N - 1, \]

\[ \mu_1 \left[ C_{i+1,i+1}^{(1,1)} + C_{i-1,i+1}^{(1,1)} + C_{i,i}^{(1,1)} - 4C_{i,i}^{(1,1)} \right] + \mu_2 \left[ C_{i-1,i-1}^{(1,1)} + C_{i+1,i+1}^{(1,1)} + 2C_{i,i}^{(1,1)} \right]
\]

\[ = -\mu_2 \left[ (C_{i-1}^{(1)})^2 + (C_{i+1}^{(1)})^2 + 2(C_{i}^{(1)})^2 \right], \quad \text{for} \quad 1 \leq i \leq N, \]

with boundary conditions \( C_{0,j}^{(1,1)} = C_{N+1,j}^{(1,1)} = C_{j,0}^{(1,1)} = C_{N+1,j}^{(1,1)} = 0 \) for \( 1 \leq j \leq N. \) Note that the equations for two-point functions closes into themselves as they do not involve higher order or higher point correlations. Also remember that \( C_i^{(1)} = \langle g_i \rangle \) is the average mass at site \( i. \) One can solve the above equations exactly in this case also and the solutions are given by

\[ C_{i,j}^{(1,1)} = \begin{cases} A \frac{(\bar{g}_R - \bar{g}_L)^2}{N+1} \frac{i}{N+1} \left( 1 - \frac{j}{N+1} \right), & \text{for} \quad 1 \leq i < j \leq N, \\ A \frac{(\bar{g}_R - \bar{g}_L)^2}{N+1} \frac{j}{N+1} \left( 1 - \frac{i}{N+1} \right), & \text{for} \quad 1 \leq j < i \leq N, \end{cases} \]

where \( A = \frac{\mu_1}{\mu_1 - \mu_2}, \)

\[ \text{for} \quad 1 \leq i \leq N, \quad i = j, \]

where \( C_i^{(1)} \) is provided in equation (22). Notice that the solutions in equations (22) and (25) are in the scaling forms as in equations (4) and (5) with the scaling functions given explicitly as

\[ C_0^{(1)}(x) = \bar{g}_L + (\bar{g}_R - \bar{g}_L) x \] \hfill (26)
\[
C^{(1,1)}_{1}(x,y) = \begin{cases} \frac{\mu_2}{\mu_1 - \mu_2}(\bar{g}_R - \bar{g}_L)^2 x(1-y), & \text{for } 0 < x < y < 1 \\ \frac{\mu_1}{\mu_1 - \mu_2}(\bar{g}_R - \bar{g}_L)^2 y(1-x), & \text{for } 0 < y < x < 1 \end{cases}
\]

\[
C^{(2)}_0(x) = \frac{\mu_2}{\mu_1 - \mu_2} [\bar{g}_L + (\bar{g}_R - \bar{g}_L) x]^2, \text{ for } 0 < x < 1 \text{ and,}
\]

\[
C^{(2)}_1(x) = \frac{\mu_1}{\mu_1 - \mu_2} \frac{\mu_2}{\mu_1 - \mu_2} (\bar{g}_R - \bar{g}_L)^2 x(1-x), \text{ for } 0 < x < 1.
\] (27)

Let us now look at correlations of order \(M = 3\). At this order the correlation that involves maximum three distinct sites is \(C^{(1,1,1)}_{i,j,k} = \langle g_i g_j g_k \rangle_c \) (with \(i \neq j \neq k\)). Once again starting from the steady state master equation (20), one can write the equations satisfied by this correlation as well as other two correlations \(C^{(1,1,1)}_{i,j,i} \) and \(C^{(1,1,1)}_{i,i,i} \) at order \(M = 3\), which once again close into themselves as their equations do not involve higher order correlations. This closing property can be observed at every order \(M\) of correlations. This is due to linear and locally independent properties of the mass mixing process that defines the RATP. One can try to solve these equations at every order microscopically as done for \(M = 1\) and \(M = 2\). But it can be easily realised that the microscopic procedure soon becomes quite cumbersome as the order of the correlations \(M\) increases.

Instead, in the next section, we present a new method to compute solutions in the scaling form assuming that scaling similar to that shown for orders \(M = 1\) and 2 holds at every order. The assumption of the existence of scaling structure as in equation (16) at orders \(M > 2\) can be observed numerically, and we later present numerical support for this assumption. To find the correlations in the scaling forms, in the next section, we employ a novel operator structure.

4. An operator expansion method

In the previous section we have shown that the correlations at order \(M = 1\) and 2 have expansions in powers of \(1/N\) where coefficients have well defined scaling forms in equations (4) and (5) with scaling functions given explicitly in equation (27). We have obtained these scaling functions from the exact microscopic solutions. In this section we will discuss an alternate method through which one can compute all the cumulants up to a given order \(M = n\), given that all possible max-site scaling correlation functions are known up to order \(n\). For example, if \(C^{(1)}_{0}(x)\), \(C^{(1,1)}_{1}(x,y)\) and \(C^{(1,1,1)}_{2}(x,y,z)\) are known, then all other cumulants up to order \(M = 3\) can be computed. This is because higher order cumulants in the steady state can be expressed in terms of lower order cumulants by breaking the cumulants into disconnected cumulants of lower orders. This procedure can be elegantly expressed in terms of operator recursion relations which we derive below.

Let us first focus on the second order \((M = 2)\) cumulants at a single site \(\langle g_i^2 \rangle\). We consider the evolution of the correlation function \(\langle g_i g_{i+1} \rangle\) which can be easily obtained from the time dependent master equation (2) as
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\[ \frac{d}{dt} \langle g_i g_{i+1} \rangle = \frac{1}{2} \mu_1 \left( \langle g_i^2 \rangle + \langle g_{i-1}^2 \rangle + \langle g_i g_{i+2} \rangle - 4 \langle g_i g_{i+1} \rangle \right) \]

\[ - \frac{1}{2} \mu_2 (\langle g_i^2 \rangle + \langle g_{i+1}^2 \rangle) \]

(28)

since the LHS is zero in the steady-state we have

\[ \langle g_i^2 \rangle + \langle g_{i+1}^2 \rangle = \frac{\mu_1}{\mu_1 - \mu_2} \left( 4 \langle g_i g_{i+1} \rangle - \langle g_{i-1} g_{i+1} \rangle - \langle g_i g_{i+2} \rangle \right) . \]

(29)

Now using the scaling forms in equations (4) and (5) in the above equations it is easy to see that, in the large \( N \) limit, \( \langle g_i^2 \rangle_c \approx \frac{\mu_2}{\mu_1 - \mu_2} (C_0^{(1)})^2 \) at order \( N^0 \), while at order \( N^{-1} \), it is equal to \( \frac{\mu_1}{\mu_1 - \mu_2} C_1^{(1,1)}(x,y) \). This suggests the following operator recursion relation

\[ g_i^2 \equiv \frac{\mu_1}{\mu_1 - \mu_2} g_i g_i, \text{ where } g_i g_i \equiv \lim_{j \rightarrow i} g_i g_j . \]

(30)

As will soon become clear, this operator method turns out to be a very efficient method to compute cumulants at different orders. The meaning of this operator equation is best understood by specifying its action under steady state averages. Inside the angular brackets the ‘separated’ product operator, \( \langle g_i g_i \rangle_c \) gives the scaling correlation function \( C_1^{(1,1)}(x,y) \) at order \( 1/N \). Let us now demonstrate how this operator equation is applied to obtain \( \langle g_i^2 \rangle_c \). Taking a steady state average on both sides of equation (30)

\[ \langle g_i^2 \rangle = \frac{\mu_1}{\mu_1 - \mu_2} \langle g_i g_i \rangle = \frac{\mu_1}{\mu_1 - \mu_2} \lim_{j \rightarrow i} \langle g_i g_j \rangle , \]

(31)

here the limit \( j \rightarrow i \) is understood in the sense that the corresponding scaling variables approach each other, i.e. \( y \rightarrow x \) in the large \( N \) limit where \( x = \frac{i}{N+1} \) and \( y = \frac{j}{N+1} \). Now expand the averages in terms of the cumulants and connected correlation functions as

\[ \langle g_i \rangle^2 + \langle g_i^2 \rangle_c = \frac{\mu_1}{\mu_1 - \mu_2} \lim_{j \rightarrow i} \langle g_i g_i \rangle + \langle g_i g_j \rangle_c \]

(32)

\[ \langle g_i^2 \rangle_c = \frac{\mu_2}{\mu_1 - \mu_2} \langle g_i \rangle^2 + \frac{\mu_1}{\mu_1 - \mu_2} \lim_{j \rightarrow i} \langle g_i g_j \rangle_c \]

(33)

\[ \langle g_i^2 \rangle_c = \frac{\mu_2}{\mu_1 - \mu_2} \langle g_i \rangle^2 + \frac{1}{N} \frac{\mu_1}{\mu_1 - \mu_2} \lim_{y \rightarrow x} \langle C_1^{(1,1)}(x,y) \rangle \]

(34)

\[ \langle g_i^2 \rangle_c = \frac{\mu_2}{\mu_1 - \mu_2} \langle C_1^{(1)}(x) \rangle^2 + \frac{1}{N} \frac{\mu_1}{\mu_1 - \mu_2} \langle C_1^{(1,1)}(x,x) \rangle \]

(35)

as announced in equation (5). Because of the locality of the evolution equations for the RATP, the recursion equation (30) can be used to compute higher order and multiple site cumulants of the form \( \langle g_i^2 \rangle_c \) involving \( g_i^2 \). We now demonstrate the use of this procedure to derive various other cumulants:
Computation of $\langle g_i^2g_k \rangle_c$ where $k \neq i$:

$$
\langle g_i^2g_k \rangle = \frac{\mu_1}{\mu_1 - \mu_2} \langle g_i,g_i,g_k \rangle = \frac{\mu_1}{\mu_1 - \mu_2} \lim_{j \to i} \langle g_i,g_j,g_k \rangle. 
$$

(36)

Expanding both sides in cumulants, we find

$$
\langle g_i^2g_k \rangle = \frac{\mu_1}{\mu_1 - \mu_2} \langle g_i,g_i,g_k \rangle, 
$$

(37)

$$
\langle g_i^2g_k \rangle_c + 2\langle g_i \rangle \langle g_i,g_k \rangle_c + \langle g_i^2 \rangle_c \langle g_k \rangle + \langle g_i \rangle \langle g_i \rangle \langle g_k^2 \rangle_c + 2\langle g_i \rangle \langle g_i \rangle \langle g_k \rangle + \langle g_i \rangle \langle g_i \rangle \langle g_k \rangle
$$

$$
= \frac{\mu_1}{\mu_1 - \mu_2} \lim_{j \to i} \left[ \langle g_i \rangle \langle g_k \rangle + \langle g_i \rangle \langle g_i \rangle \langle g_k \rangle_c + \langle g_i \rangle \langle g_i \rangle \langle g_k \rangle_c + \langle g_i \rangle \langle g_i \rangle \langle g_k \rangle_c + \langle g_i \rangle \langle g_i \rangle \langle g_k \rangle_c \right], 
$$

(38)

$$
\langle g_i^2g_k \rangle_c = \frac{1}{\mathcal{N}} \frac{2\mu_2}{\mu_1 - \mu_2} C^{(1)}_0(x)C^{(1,1)}_1(x,z) + \frac{1}{\mathcal{N}^2} \frac{\mu_1}{\mu_1 - \mu_2} C^{(2,1,1)}_2(x,x,z),
$$

as stated in equation (6).

(39)

Computation of $\langle g_i^2g_k \rangle_c$ where $k \neq i$:

$$
\langle g_i^2g_k \rangle_c = \frac{\mu_1^2}{(\mu_1 - \mu_2)^2} \langle g_i,g_i,g_k,g_k \rangle, \text{ expanding both sides in connected correlations we get,}
$$

(40)

$$
\langle g_i^2g_k \rangle_c + 2\langle g_i \rangle \langle g_i,g_k \rangle_c + \langle g_i \rangle \langle g_i \rangle \langle g_k^2 \rangle_c + \langle g_i \rangle \langle g_i \rangle \langle g_k \rangle + \langle g_i \rangle \langle g_i \rangle \langle g_k \rangle
$$

$$
= \frac{\mu_1^2}{(\mu_1 - \mu_2)^2} \lim_{j \to k} \left[ \langle g_i,g_j,g_k,g_l \rangle_c + \langle g_i \rangle \langle g_j,g_i,g_l \rangle_c + \langle g_j \rangle \langle g_k,g_j,g_l \rangle_c \right]
$$

$$
+ \langle g_i \rangle \langle g_j \rangle \langle g_i \rangle + \langle g_i \rangle \langle g_j \rangle \langle g_j \rangle \langle g_k \rangle + \langle g_i \rangle \langle g_j \rangle \langle g_j \rangle \langle g_k \rangle + \langle g_i \rangle \langle g_j \rangle \langle g_j \rangle \langle g_k \rangle + \langle g_i \rangle \langle g_j \rangle \langle g_i \rangle \langle g_j \rangle \langle g_j \rangle \langle g_k \rangle
$$

$$
+ \langle g_j \rangle \langle g_j \rangle \langle g_j \rangle \langle g_i \rangle \langle g_i \rangle \langle g_j \rangle \langle g_j \rangle \langle g_k \rangle. 
$$

(41)

Now we use the scaling forms of $\langle g_i \rangle$, $\langle g_i,g_j \rangle_c$, $\langle g_i^2 \rangle_c$, $\langle g_i^2g_k \rangle_c$ given in equations (4), (5) and (39) and, the following max-site correlations

$$
\langle g_i,g_j,g_k \rangle_c = \frac{1}{\mathcal{N}^2} C^{(1,1,1)}_2(x,y,z), \text{ for } i \neq j \neq k,
$$

with $x = \frac{i}{N+1}$, $y = \frac{j}{N+1}$, $z = \frac{k}{N+1}$, and

$$
\langle g_i,g_j,g_k,g_l \rangle_c = \frac{1}{\mathcal{N}^3} C^{(1,1,1,1)}_3(x,y,z,w), \text{ for } i \neq j \neq k \neq l,
$$

additionally with $w = \frac{l}{N+1}$.

(42)
on both sides of equation (41). After carrying out some straightforward manipulations and simplifications, we get

\[
\langle g_i^2 g_{k_2} \rangle_c \approx \frac{1}{N} \frac{4\mu_2^2}{(\mu_1 - \mu_2)^2} C_0^{(1)}(x) C_0^{(1)}(z) C_1^{(1)}(x, z) + \frac{1}{N^2} \frac{2\mu_1 \mu_2}{(\mu_1 - \mu_2)^2} \times \left[ C_0^{(1)}(x) C_2^{(1,1)}(x, z) + C_0^{(1)}(z) C_2^{(1,1)}(x, x, z) \right] + \frac{1}{N^3} \frac{\mu_1^2}{(\mu_1 - \mu_2)^2} C_3^{(1,1,1)}(x, x, z, z).
\]

(43)

- Generalisation to higher order and higher point correlations of the form \(\langle g_i^2 g_{k_2} \cdots g_{k_n} \rangle_c\) where none of the \(k_j \neq i\) for all \(j = 1, 2, \ldots, n\).

\[
\langle g_i^2 g_{k_2} \cdots g_{k_n} \rangle_c = \frac{\mu_1}{\mu_1 - \mu_2} \langle g_i g_{k_2} \cdots g_{k_n} \rangle_c
\]

(44)

\[
\langle g_i^2 g_{k_2} \cdots g_{k_n} \rangle_c + 2 \langle g_i \rangle \langle g_i g_{k_2} \cdots g_{k_n} \rangle_c + \langle g_i \rangle^2 \langle g_{k_2} \cdots g_{k_n} \rangle_c + \langle g_i \rangle^2 \langle g_{k_2} \cdots g_{k_n} \rangle_c
\]

\[
= \frac{\mu_1}{\mu_1 - \mu_2} \lim_{j \to i} \left[ \langle g_i \rangle^2 \langle g_{k_2} \cdots g_{k_n} \rangle_c + \langle g_{k_2} \cdots g_{k_n} \rangle_c \langle g_i g_j \rangle_c + \langle g_i \rangle \langle g_{k_2} \cdots g_{k_n} \rangle_c + \langle g_i \rangle \langle g_j g_{k_2} \cdots g_{k_n} \rangle_c \right].
\]

(45)

Note that the operator equation (30) can not be used to compute correlations which involve \(g_i^3\), for example, \(\langle g_i^3 \rangle_c, \langle g_i^3 g_j \rangle_c\), etc. For that we need an operator recursion equation for \(g_i^3\). Following the same procedure as was used to derive equation (30), we find

\[
g_i^3 = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} g_i^2 g_i.
\]

(46)

Using this operator equation one can straightforwardly work out the following cumulants involving \(g_i^3\):

- \(\langle g_i^3 \rangle_c\):

\[
\langle g_i^3 \rangle_c = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} \langle g_i^2 g_i \rangle_c
\]

expanding both sides in connected correlations we get,

\[
\langle g_i^3 \rangle_c + 3 \langle g_i \rangle \langle g_i^2 \rangle_c + \langle g_i \rangle^3 = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} \lim_{j \to i} \left[ \langle g_i \rangle^2 \langle g_j \rangle_c + 2 \langle g_i \rangle \langle g_i g_j \rangle_c + \langle g_i^2 \rangle_c \langle g_j \rangle_c + \langle g_i^2 g_j \rangle_c \right].
\]

(48)

Now once again we use the scaling forms for \(\langle g_i \rangle, \langle g_i g_j \rangle_c, \langle g_i^2 \rangle_c, \langle g_i^2 g_j \rangle_c\) given in equations (4), (5) and (39) on both sides of the above equation. After simplifying we get
This equation can be solved in closed form for certain special cases of $R$ where $f$ that in the operator form one can write terms of the max-site cumulants. To do this we first observe (from previous examples) In this section we give general expressions for all multi-site cumulants of higher order in $4.1. General equations for the higher order cumulants

\begin{equation}
\langle g_i^3 \rangle_c = \frac{2\mu_2}{(\mu_1 - \mu_2)^2} \left( C_0^{(1)}(x) \right)^3 + \frac{1}{N} \frac{6\mu_1\mu_2}{(\mu_1 - \mu_2)^2} C_0^{(1)}(x) C_1^{(1,1)}(x,x) + \frac{1}{N^2} \frac{\mu_1(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2} C_2^{(1,1,1)}(x,x,x). \tag{49}
\end{equation}

• $\langle g_i^3 g_k \rangle_c$ for $k \neq i$:

\begin{equation}
\langle g_i^3 g_k \rangle = \frac{\mu_1(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2} (g_i \cdot g_i \cdot g_i \cdot g_k). \tag{50}
\end{equation}

Expanding both sides in cumulants, and using the scaling forms for $\langle g_i \rangle$, $\langle g_i^2 \rangle_c$, $\langle g_i^2 g_k \rangle_c$, $\langle g_i^3 \rangle_c$ as derived above, we get

\begin{equation}
\langle g_i^3 g_k \rangle = \frac{1}{N} \frac{6\mu_2 - \mu_1\mu_2}{(\mu_1 - \mu_2)^2} C_0^{(1)}(x) C_0^{(1)}(y) C_1^{(1,1)}(x,y) + \frac{1}{N} \frac{12\mu_1\mu_2 - 6\mu_2^2}{(\mu_1 - \mu_2)^2} C_0^{(1)}(x) C_1^{(1,1)}(x,y) + \frac{1}{N^2} \frac{6\mu_2^2}{(\mu_1 - \mu_2)^2} C_0^{(1)}(x) C_2^{(1,1,1)}(x,x,y) + \frac{1}{N^3} \frac{\mu_1(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2} C_3^{(1,1,1,1)}(x,x,y). \tag{51}
\end{equation}

One can generalize the above operator method to compute cumulants of arbitrary powers using the following operator recursion relation $g_i^n$

\begin{equation}
g_i^n = \frac{1}{(1 - (1 - \eta)^n)} - \mu_n \sum_{k=1}^{n-1} \binom{n}{k} \mu_k g_i^{n-k} \cdot g_k, \quad n = 1, 2, \ldots \tag{52}
\end{equation}

which is derived in appendix B.

4.1. General equations for the higher order cumulants

In this section we give general expressions for all multi-site cumulants of higher order in terms of the max-site cumulants. To do this we first observe (from previous examples) that in the operator form one can write

\begin{equation}
g_i^n = f_n g_i \cdot g_i \cdot \ldots \tag{53}
\end{equation}

where $f_n$ are constants independent of the site index, and are functions of the moments of $R(\eta)$. Inserting this form into equation (52), we get a recursion relation for $f_n$,

\begin{equation}
f_n = \frac{1}{(1 - (1 - \eta)^n)} - \mu_n \sum_{k=1}^{n-1} \binom{n}{k} \mu_k f_{n-k} f_k. \tag{54}
\end{equation}

This equation can be solved in closed form for certain special cases of $R(\eta)$ e.g. uniform distribution. However for arbitrary $R(\eta)$, this can be solved recursively to determine $f_n$.
for all orders. It also follows that the $f_n$s up to order $n$ are functions only of the first $n$ moments of $R(n)$. The first few $f_n$s are explicitly given by

$$
\begin{align*}
  &f_1 = 1, \\
  &f_2 = \frac{\mu_1}{\mu_1 - \mu_2}, \\
  &f_3 = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} f_1 f_2 = \frac{\mu_1 (\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2}, \\
  &f_4 = \left[ \frac{2(\mu_1 + \mu_3)f_3 + 3\mu_2 f_2^2}{2(\mu_1 + \mu_3) - 3\mu_2 - \mu_4} \right] = \frac{\mu_1}{(\mu_1 - \mu_2)^2} \frac{2(\mu_1 + \mu_3)(\mu_1 + \mu_2) + 3\mu_1 \mu_2}{2(\mu_1 + \mu_3) - 3\mu_2 - \mu_4}.
\end{align*}
$$

Once the sequence $f_n$ is determined, all the cumulants of $g_i$ till order $n$ can be computed in terms of the lower order cumulants and $f_m$'s with $1 \leq m \leq n$ and the max-site cumulants of order $k \leq n$. To see this one needs to take expectation on both sides of equation (53). On the left hand side one expands $\langle g_i^n \rangle$ in terms of its lower order cumulants as

$$
\langle g_i^n \rangle = n! \sum_{\{p_m\}} \prod_{m=0}^{n} \frac{1}{p_m! (m!)^{p_m}} \langle g_i^{p_m} \rangle_c \delta_{n, \sum_{m=0}^{n} m p_m},
$$

for example, $\langle g_i^3 \rangle = (g_i^3)_c + 3 \langle g_i \rangle (g_i^2)_c + (g_i)^3$. In the above equation $\delta_{a,b}$ represents Kronecker delta and $p_m$'s take non-negative integer values. Similarly, one can expand the right hand side as

$$
\langle g_i \ldots \rangle \left\{ \lim_{j \to i} \ldots \right\} = \lim_{j \to i} \langle g_i g_j g_k \ldots \rangle = \lim_{j \to i} \left[ (\langle g_i g_j g_k \ldots \rangle_c + \langle g_i \rangle \langle g_j g_k \ldots \rangle + \langle g_j \rangle \langle g_i g_k \ldots \rangle + \ldots + \langle g_i \rangle \langle g_j \rangle \langle g_k \ldots \rangle_c + \ldots \right].
$$

Again for example, $\langle g_i \cdot g_i \cdot g_i \rangle = (g_i)^3 + 3 \langle g_i \rangle (g_i^2)_c |_{j \to i} + (g_i g_j g_k)_c |_{k \to j \to i}$, which when used for $n = 3$ gives equation (49).

Similarly, for $n = 4$ one finds

$$
\langle g_i^4 \rangle = f_4 \left( C_0^{(1)}(x)^4 + \frac{6}{N} C_0^{(1)}(x)^2 C_1^{(1,1)}(x,x) + \frac{3}{N^2} C_1^{(1,1)}(x,x)^2 + \frac{4}{N^2} C_0^{(1)}(x) C_2^{(1,1,1)}(x,x,x) + \frac{1}{N^3} C_3^{(1,1,1,1)}(x,x,x,x) \right).
$$

Expanding the moment $\langle g_i^4 \rangle$ on LHS in terms of the cumulants and after some algebraic manipulations one finds
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\[
\langle g_4 \rangle_c = \left( f_4 - \frac{6\mu_1^2 + 4\mu_1\mu_2 + \mu_2^2}{(\mu_1 - \mu_2)^2} \right) C_0^{(1)}(x)^4 + \frac{1}{N} \left( 6f_4 - \frac{24\mu_1\mu_2 + 6\mu_1^2}{(\mu_1 - \mu_2)^2} \right) C_0^{(1)}(x)^2 C_1^{(1,1)}(x,x)
\]
\[
+ \frac{1}{N^2} \left( 3f_4 - \frac{3\mu_1^2}{(\mu_1 - \mu_2)^2} \right) C_1^{(1,1)}(x,x)^2 + \frac{1}{N^3} 4 \left( f_4 - \frac{\mu_1 + \mu_2}{(\mu_1 - \mu_2)^2} \right) C_0^{(1)}(x)^3
\]
\[
\times C_2^{(1,1,1)}(x,x,x) + \frac{1}{N^3} f_4 C_2^{(1,1,1,1)}(x,x,x,x),
\]

(59)

where explicit expression of \( f_4 \) is given in equation (55). Thus, once again we observe that any cumulant at order \( n \), can be expressed in terms of all the max-site scaling correlation functions \( C_{j-1,...,1}^{(1)}(x_1,x_2,...,x_j) \) of order \( j \) till \( n \), i.e. \( \forall \ j \) such that \( 1 \leq j \leq n \).

In section 6, we turn to the question is how do we find these max-site scaling correlation functions at a given order, where we find that at any order \( n \) the max-site scaling correlation function satisfies a Poisson equation inside a unit cube of dimension \( n \) with source ‘charges’ distributed appropriately. In the next section, we first present numerical verification of the results of this section using Monte Carlo simulations of the RATP.

5. Numerical verification of the theoretical predictions

We now test the validity of our operator method by numerically verifying the results obtained in the previous section using the operator method and which are difficult to obtain via the direct method discussed in section 3. As mentioned while describing the model in section 2, a random fraction \( \eta \) (chosen from distribution \( R(\eta) \)) of the mass from one site gets transferred to two neighbouring sites with equal probability at unit rate. The masses at the boundary sites are externally controlled to maintain the (time independent) mass distribution at the left and right boundaries to be \( P_L(g) \) and \( P_R(g) \) respectively. We simulate this dynamics using the Monte Carlo method. In our simulation we choose the following jump distribution

\[
R(\eta) = \Theta(1-\eta)\Theta(\eta) \left[ 3 \Theta \left( \frac{1}{4} - \eta \right) + \frac{1}{3} \Theta \left( \eta - \frac{1}{4} \right) \right]
\]

(60)

where \( \Theta(x) \) is the Heaviside theta function. For this distribution we get

\[
\mu_1 = \frac{1}{4}, \ \mu_2 = \frac{1}{8}, \ \mu_3 = \frac{11}{128} \quad \text{and} \quad \mu_4 = \frac{43}{640}.
\]

(61)

At the boundaries, we consider the following mass distributions

\[
P_{L,R}(g) = \frac{1}{g_{L,R}} e^{-\frac{g}{g_{L,R}}}
\]

(62)

with \( g_L = 2 \) and \( g_R = 1 \). Below we provide numerical verification of our results for cumulants of the first few orders, which can be reduced to the functions \( C_0^{(1)}(x) \) and \( C_1^{(1,1)}(x,y) \) by the operator method. For the functions \( C_0^{(1)}(x) \) and \( C_1^{(1,1)}(x,y) \), we use the expressions in equations (22) and (25).
Figure 2. Comparison of simulation results (filled circles) with theory (solid lines) of the expectation value of \( g_i \) on a lattice of size \( N = 49 \). Solid points are obtained from simulation using the jump distribution given in equation (60). The parameters used in the simulation are \( \bar{g}_L = 2 \) and \( \bar{g}_R = 1 \).

Figure 3. Comparison of simulation results (filled circles) with theory (solid line) of (a) the correlation function \( \langle g_i g_j \rangle \) along the line \( j = N - i \), and (b) the cumulant \( \langle g_i^2 \rangle_c \), with the \( O(1) \) contribution, \( \mu_2 \langle g_i \rangle^2 / (\mu_1 - \mu_2) \), subtracted. The parameters used in the simulation are the same as figure 2.

(a) Mean \( \langle g_i \rangle \): in figure 2 we verify our first results on the mean mass \( \langle g_i \rangle \). As announced in equations (22) and (26), we indeed observe that the average mass decreases linearly as one goes from the left to the right end of the lattice.

(b) Correlation function \( \langle g_i g_j \rangle_c \): the scaling behaviour of the two point connected correlation \( \langle g_i g_j \rangle_c |_{i \neq j} \) as announced in equations (27) is verified in figure 3(a), where we plot the correlation along the line \( i + j = N \).

(c) The variance \( \langle g_i^2 \rangle_c \): the behaviour of the variance as expected from equation (35) is shown in figure 3(b), where we have subtracted the \( O(1) \) contribution \( \mu_2 \mu_1 / (\mu_1 - \mu_2) C_0^{(1)}(x)^2 \). These results which follow from the operator method were also obtained through the exact calculation in section 3. We now move to higher cumulants, which cannot be easily derived from the microscopic method, but expressions for which were derived using the operator method in section 4.
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Figure 4. Comparison of simulation results (filled circles) with theory (solid line) of the two-site cumulants (a) \( \langle g_i^2 g_j \rangle_c \) and (b) \( \langle g_i^2 g_j^2 \rangle_c \), both along the line \( j = N - i \). The parameters used in the simulation are the same as figure 2. The deviation from the scaling behaviour for points near the diagonal \( i = j \) is due to the fact that these are affected by finite size effects because of the different scaling behaviour of the cumulant on the diagonal (see equation (59)).

(d) The third-order cumulant on two sites, \( \langle g_i^3 g_j \rangle_c \): next we consider the third-order cumulant \( \langle g_i^3 g_j \rangle_c \) given in equation (39). Note that this is the first non-trivial cumulant that we have calculated using the operator method. In figure 4(a), we plot the leading order contribution to \( \langle g_i^3 g_j \rangle_c \) given by the first term on the right hand side of equation (39) along the line \( i + j = N \). The excellent agreement between the theoretical prediction and simulation results validates our result.

(e) The fourth-order cumulant on two sites, \( \langle g_i^2 g_j^2 \rangle_c \): in figure 4(b) we plot the simulation results for cumulant \( \langle g_i^2 g_j^2 \rangle_c \) along the line \( i + j = N \) and compare it with the predictions from the operator method given in equation (43). We observe excellent agreement once again, except near the diagonal \( i = j \), where the scaling shows strong finite-size effects because of the different scaling of the diagonal cumulant \( \langle g_i^4 \rangle_c \). We examine this cumulant separately below.

(f) The one-site third-order cumulant, \( \langle g_i^3 \rangle_c \): as given in equation (49), the \( O(1) \) contribution to \( \langle g_i^3 \rangle_c \) for the jump distribution in equation (60) is equal to \( 2C_0^{(1)}(x)^2 \). Subtracting this contribution from \( \langle g_i^3 \rangle_c \), we compare the \( O(\frac{1}{N}) \) correction with simulation results in figure 5(a). This verifies the validity of the operator method for higher-order cumulants.

(g) The one-site fourth-order cumulant, \( \langle g_i^4 \rangle_c \): we finally compare the next-to-leading-order prediction (i.e, the term at \( O(\frac{1}{N}) \)) for the fourth cumulant \( \langle g_i^4 \rangle_c \), as given in equation (59) with simulation results in figure 5(b). Once again the excellent agreement shows that the operator method captures higher cumulants very well.
Figure 5. Comparison with theory (solid line) of next-to-leading order \((O(\frac{1}{N}))\) contributions to the one-site cumulants (a) \(\langle g_1^3 \rangle\) and (b) \(\langle g_1^4 \rangle\), with the \(O(1)\) contributions subtracted. In (b), we use the numerical values \(a_1 = \frac{4588}{499}\) and \(a_2 = \frac{45492}{499}\) obtained from equations (59) and (61). The parameters used in the simulation are the same as figure 2.

6. Moment generating functions and derivation of the Poisson equations for the max-site correlations

In this section we look at the moment generating function (MGF) of the masses, defined as

\[
\left\langle e^{\sum_{i=1}^{N} \lambda_i g_i} \right\rangle = \sum_{m=0}^{\infty} \frac{1}{m!} (\lambda_1 g_1 + \lambda_2 g_2 + \ldots + \lambda_N g_N)^m = \sum_{m=0}^{\infty} \sum_{m_1=0}^{m} \sum_{m_2=0}^{m} \ldots \sum_{m_N=0}^{m} \frac{1}{m_1! m_2! \ldots m_N!} \lambda_1^{m_1} \lambda_2^{m_2} \ldots \lambda_N^{m_N} \langle g_1^{m_1} g_2^{m_2} \ldots g_N^{m_N} \rangle. \tag{63}
\]

For mass at single site, say \(i\)th, the MGF is

\[
\left\langle e^{\lambda g_i} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \langle g_i^n \rangle, \tag{64}
\]

which using the operator equation (53) can be written as

\[
\left\langle e^{\lambda g_i} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n f_n \left\langle \underbrace{g_i \cdot g_i \cdot \ldots \cdot g_i}_{n \text{ times}} \right\rangle, \tag{65}
\]

Defining the function

\[
G(\lambda) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \lambda^n \tag{66}
\]
it is easy to see that the MGF in equation (65) can be written as

\[ \langle e^{\lambda g_i} \rangle = \langle G(\lambda g_i \cdot) \rangle, \quad \text{where} \quad G(\lambda g_i \cdot) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \lambda^n g_i \cdot g_i \cdot \ldots \cdot g_i, \]  

(67)

represents the MGF operator. We now take the extra step of assuming that the operator equation (53) is valid to all orders in $1/N$. This means that the above equation is assumed to be exact. This is warranted because, as we will see below, the terms in the steady-state equations for the cumulant generating function (CGF) which remain after the operator recursion is satisfied can be incorporated into source terms for the Poisson equations ‘max-site’ correlation functions. In the following, we demonstrate this explicitly for the 2-point and 3-point correlation functions. In other words, we conjecture that in the continuum limit, the RATP steady-state is fully described by the operator recursion equations (53) and Poisson equations for the correlation functions, equations (18) and (19).

For convenience, we now define a new operator $(\delta_i \cdot)$, such that $g_i \cdot = m_i + \delta_i \cdot$ where $m_i = \langle g_i \rangle$, one gets

\[ \langle e^{\lambda g} \rangle = \langle G(\lambda (m_i + \delta_i \cdot)) \rangle, \quad \text{and for the joint CGF} \]

\[ \langle e^{\sum_{i=1}^{N} \lambda g_i} \rangle = \langle G(\lambda_1 (m_1 + \delta_1 \cdot) G(\lambda_2 (m_2 + \delta_2 \cdot)) \ldots G(\lambda_N (m_N + \delta_N \cdot)) \rangle. \]

(69)

The advantage of separating the mean $m_i$ from $g_i$ is that the average of the combinations of $\delta_i \cdot$ directly represents the cumulants and hence the scaling correlation functions in the large $N$ limit e.g.

\[ \langle \delta_i \rangle = \langle g_i \rangle_c = 0, \]

(70)

\[ \langle \delta_i \delta_j \rangle_{i \neq j} = \langle g_i g_j \rangle_c|_{i \neq j} \rightarrow \frac{1}{N} C^{(1,1)}_1(x, y), \quad \text{with} \quad x = \frac{i}{N}, \ y = \frac{j}{N}, \]

(71)

\[ \langle \delta_i \delta_j \delta_k \rangle_{i \neq j \neq k} = \langle g_i g_j g_k \rangle_c|_{i \neq j \neq k} \rightarrow \frac{1}{N^2} C^{(1,1,1)}_2(x, y, z), \]

\[ \text{with} \quad x = \frac{i}{N}, \ y = \frac{j}{N} \text{ and } z = \frac{k}{N}. \]

(72)

If the CGFs were known explicitly then one can expand both sides of equations (68) and (69) in powers of $\lambda$s. For example, the right hand side (RHS) of one-point CGF can be expanded as

\[ \langle G(\lambda (m_i + \delta_i \cdot)) \rangle = G(\lambda m_i) + \frac{1}{2} \lambda^2 G''(\lambda m_i) \langle \delta_i \cdot \delta_i \rangle + \ldots, \]

(73)

whereas the expansion of the left hand side (LHS) is given in equation (64). In the above we have used $G''(z) = d^2 G(z)/dz^2$ and $\langle \delta_i \rangle = 0$. For two-point CGF also, the RHS can be expanded similarly and one has

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that this equation can be rewritten in terms of the addition to that over the distribution of $g$
provide the Poisson equation (mentioned in equation (18)) satisfied by the max-site correlation at that order. In the following we demonstrate how does one get such a Poisson equation at order $\ell = 2$ and $\ell = 3$.

6.1. Derivation of the Poisson equation satisfied by $C_1^{(1,1)} (x,y)$

6.1.1. MGF equation of $g_i$. To proceed, starting from the master equation (2) we obtain the following equation satisfied by the MGF $\langle e^{\lambda g_i} \rangle$ in the steady state:

$$4\langle e^{\lambda g_i} \rangle = \left( e^{\lambda (g_{i+1} + \eta g_{i+1})} \right) + \left( e^{\lambda (g_i + \eta g_{i+1})} \right) + 2 \left( e^{\lambda (1-\eta) g_i} \right).$$  \hspace{1cm} (75)

In the above equation, $\langle \ldots \rangle$ on terms involving $\eta$ also represents average over $\eta$ in addition to that over the distribution of $g_i$, i.e. $\langle e^{\lambda (1-\eta) g_i} \rangle = \int_0^1 d\eta R(\eta) \langle e^{\lambda (1-\eta) g_i} \rangle$. Note that this equation can be rewritten in terms of the $G$ function, introduced in equation (66), as

$$4\langle G(\lambda g_i) \rangle - 2\langle G(\lambda (1-\eta) g_i) \rangle = \langle G(\lambda g_i) G(\lambda \eta g_{i+1}) \rangle + \langle G(\lambda g_i) G(\lambda \eta g_{i-1}) \rangle.$$  \hspace{1cm} (76)

Now, we show that the recursion relations (54) get translated to the operator equation

$$2\langle G(\lambda g_i) \rangle - \langle G(\lambda (1-\eta) g_i) \rangle = \langle G(\lambda g_i) G(\lambda \eta g_i) \rangle.$$  \hspace{1cm} (77)

This can be seen by using equation (65) to get

$$2 \sum_n \frac{1}{n!} \lambda^n f_n - \sum_n \frac{1}{n!} \lambda^n f_n (1-\eta)^n = \sum_i \sum_j \frac{1}{i! j!} \lambda^{i+j} \langle \eta^j \rangle f_i f_j.$$  \hspace{1cm} (78)

Equating coefficients of $\lambda^n$ on both sides, we get

$$f_n (2 - (1-\eta)^n) = f_0 f_n + \sum_{k=1}^{n-1} \binom{n}{k} \mu_k f_k f_{n-k} + \mu_n f_0 f_n$$  \hspace{1cm} (79)

which, recalling that $f_0 = 1$, is equivalent to equation (54).
We now demand that the operator equation (76) is valid in all orders of $\lambda$, that is, 
\[ \langle \Delta G \rangle = 0, \] 
where, 
\[ \Delta G = G(\lambda g_i \cdot G(\lambda \eta g_{i+1}) + G(\lambda g_{i-1} \cdot G(\lambda \eta g_i) - 2G(\lambda g_i \cdot G(\lambda \eta g_{i+1}) \right. \]
for large $N$. Using the expansion of $G(\lambda g_i)$ given in equation (66) in equation (80) and then equating terms of different orders of $\lambda$ on the LHS to zero, we get relations among various cumulants as observed in section 4.

- $O(\lambda^0)$: At this order we get the identity $2 = 2$.
- $O(\lambda)$: At this order we get 
\[ \langle g_{i+1} \rangle - 2\langle g_i \rangle + \langle g_{i-1} \rangle = 0 \] 
which simplifies to 
\[ \nabla^2 m = \Rightarrow \partial^2 \mathcal{C}^{(1)}(x) = 0, \] 
where $x = \frac{i}{N}$ with $\mathcal{C}^{(1)}(0) = \bar{g}_L$, and $\mathcal{C}^{(1)}(1) = \bar{g}_R$. (82)

Here $\nabla^2$ denotes the discrete second difference operator, i.e. $\nabla^2 f_i = f_{i+1} - 2f_i + f_{i-1}$, and $\partial^2 \equiv N^2 \Delta^2$ represent the continuum partial derivative of second order. It is easy to check from equation (26) that the above equation is indeed true.

- $O(\lambda^2)$: Collecting terms at $O(\lambda^2)$ from both sides and equating we get
\[ \mu_1 f_1 [\langle g_i \cdot g_{i+1} \rangle + \langle g_{i+1} \cdot g_i \rangle + \langle g_i \cdot g_{i-1} \rangle + \langle g_{i-1} \cdot g_i \rangle - 4\langle g_i \cdot g_i \rangle] + \mu_2 f_2 [\langle g_{i+1} \cdot g_{i+1} \rangle + \langle g_{i-1} \cdot g_{i-1} \rangle - 2\langle g_i \cdot g_i \rangle] = 0, \] 
which, after using $g_i = m_i + \delta_i$ simplifies to
\[ \left[ \langle \delta_i \cdot \delta_{i+1} \rangle + \langle \delta_{i+1} \cdot \delta_i \rangle + \langle \delta_i \cdot \delta_{i-1} \rangle + \langle \delta_{i-1} \cdot \delta_i \rangle - 4\langle \delta_i \cdot \delta_i \rangle \right] = -\frac{\mu_2}{(\mu_1 - \mu_2)} \left[ \nabla^2 m^2 + \nabla^2 \langle \delta_i \cdot \delta_i \rangle \right], \] 
where $\nabla^2$ is defined above.

6.1.2. Joint MGF equations of $g_i$ and $g_{i\pm 1}$. Till now we have looked at how the correlations $\langle \delta_i \cdot \delta_j \rangle$ near the diagonal $i = j$ behave. Let us now look at the equation satisfied by the joint MGF of $g_i$ and $g_{i+1}$, which can again be obtained from the master equation (2):
\[ 6 \langle e^{\lambda_1 g_i + \lambda_2 g_{i+1}} \rangle = \left\langle e^{\lambda_1 g_i + \lambda_2 (\lambda_1 - \lambda_2) g_{i+1}} \right\rangle + \left\langle e^{[\lambda_1 - \eta(\lambda_1 - \lambda_2)] g_i + \lambda_2 g_{i+1}} \right\rangle + \left\langle e^{\lambda_1 g_i + \lambda_2 g_{i+1} + \lambda_2 g_{i+1}} \right\rangle + \left\langle e^{\lambda_1 (1-\eta) g_i + \lambda_2 g_{i+1}} \right\rangle + \left\langle e^{\lambda_1 g_i + \lambda_2 (1-\eta) g_{i+1}} \right\rangle. \]
Expressing this equation in terms of the $G$ function, introduced in equation (66), and simplifying using equations (77) and (80), we get

\[
8 \langle G(\lambda_1 g_i) G(\lambda_2 g_{i+1}) \rangle = \langle G(\lambda_1 g_i) G(\lambda_1 \eta g_{i+1}) G(\lambda_2 g_{i+1}) \rangle \\
+ \langle G(\lambda_1 g_i) G(\lambda_1 \eta g_{i-1}) G(\lambda_2 g_{i+1}) \rangle \\
+ \langle G(\lambda_1 g_i) G(\lambda_2 g_{i+1}) G(\lambda_2 \eta g_{i+1}) \rangle \\
+ \langle G(\lambda_1 g_i) G(\lambda_2 g_{i+1}) G(\lambda_2 \eta g_{i}) \rangle \\
+ 2 \langle G(\lambda_1 [1 - \eta] g_i) G(\lambda_2 g_{i+1}) \rangle \\
+ 2 + \langle G(\lambda_1 g_i) G(\lambda_2 [1 - \eta] g_{i+1}) \rangle.
\]

Following a similar procedure as done above, one finds that equations at $O(\lambda_1)$, $O(\lambda_2)$, $O(\lambda_1^2)$ and $O(\lambda_2^2)$ are already obtained in the section 6.1.1. We obtain a new equation only by equating the coefficient of $O(\lambda_1 \lambda_2)$ term to zero. \( \langle g_i \cdot g_{i+2} \rangle + \langle g_i \cdot g_j \rangle + \langle g_{i+1} \cdot g_{i+1} \rangle - 4 \langle g_i \cdot g_{i+1} \rangle = 0 \) which, after using the form \( g_i = m_i + \delta_i \) simplifies to

\[
\langle \delta_i \cdot \delta_{i+2} \rangle + \langle \delta_i \cdot \delta_{i+1} \rangle + \langle \delta_{i+1} \cdot \delta_{i+1} \rangle - 4 \langle \delta_i \cdot \delta_{i+1} \rangle = 0.
\]

\[
\text{(87)}
\]

6.1.3. **Joint MGF equations of \( g_i \) and \( g_j \) with \( \lvert i - j \rvert \geq 2 \).** To see how the correlations \( \langle \delta_i \delta_j \rangle \) in the bulk, i.e. \( i \neq j \) behave, we consider from the following two-point MGF equation

\[
8 \left< e^{\lambda_1 g_i + \lambda_2 g_{i+1}} \right> = \left< e^{\lambda_1 (g_i + g_{i+1}) + \lambda_2 g_{i+1}} \right> + \left< e^{\lambda_1 (g_i + g_{i-1}) + \lambda_2 g_i} \right> + \left< e^{\lambda_1 g_i + \lambda_2 (g_{i+1} + g_{i+1})} \right> \\
+ \left< e^{\lambda_1 g_i + \lambda_2 (g_{i+1} + g_{i-1})} \right> + 2 \left< e^{\lambda_1 (g_i - g_{i+1}) + \lambda_2 g_{i+1}} \right> + 2 \left< e^{\lambda_1 g_i + \lambda_2 (g_i - g_{i+1})} \right>.
\]

\[
\text{(88)}
\]

In this case also we do not get any new equations from $O(\lambda_1)$, $O(\lambda_2)$, $O(\lambda_1^2)$ and $O(\lambda_2^2)$ as expected. The only new equation we get from $O(\lambda_1 \lambda_2)$ is \( \langle g_i \cdot g_{i+1} \rangle + \langle g_i \cdot g_{i-1} \rangle + \langle g_{i+1} \cdot g_j \rangle + \langle g_{i-1} \cdot g_j \rangle - 4 \langle g_i \cdot g_j \rangle = 0 \), which, after using the form $g_i = m_i + \delta_i$ simplifies to

\[
\langle \delta_i \cdot \delta_{i+1} \rangle + \langle \delta_i \cdot \delta_{i-1} \rangle + \langle \delta_{i+1} \cdot \delta_i \rangle + \langle \delta_{i-1} \cdot \delta_i \rangle - 4 \langle \delta_i \cdot \delta_j \rangle = 0.
\]

\[
\text{(89)}
\]

Combining equations (84), (87) and (89), appropriately, we get

\[
D_{ij} \langle \delta_i \cdot \delta_j \rangle = -\delta_{ij} \frac{\mu_2}{\mu_1 - \mu_2} \left( \nabla_i^f m_i \right)^2 \left[ \nabla_i^2 m_i^2 + \nabla_i^2 \langle \delta_i \cdot \delta_i \rangle \right], \quad \text{where,}
\]

\[
\text{(90)}
\]

\[
D_{ij} \langle \delta_i \cdot \delta_j \rangle = \langle \delta_i \cdot \delta_{i+1} \rangle + \langle \delta_i \cdot \delta_{i-1} \rangle + \langle \delta_{i+1} \cdot \delta_j \rangle + \langle \delta_{i-1} \cdot \delta_j \rangle - 4 \langle \delta_i \cdot \delta_j \rangle
\]

\[
\text{(91)}
\]

and $\delta_{ij}$ is Kronecker delta. We have used the notation $\nabla_i^f$ to denote the forward difference operator, $\nabla_i^f f_i \equiv f_{i+1} - f_i$. Our next task is to go to the continuum limit in the $N \to \infty$ limit where we replace
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\[ \delta_{ij} \to \frac{1}{N} \delta(x-y) \]
\[ (\nabla_i^f m_i)^2 \to \frac{2}{N^2} (\partial_x C_0^{(1)}(x))^2 \]
\[ D_{ij} \langle \delta_i \cdot \delta_j \rangle \to \frac{1}{N^2} \nabla^2 C_1^{(1,1)}(x,y) \]

where \( \nabla^2 = \partial_x^2 + \partial_y^2 \) denotes the continuum Laplacian. Inserting these continuum forms in equation (90) we in the leading order \( O(1/N^3) \) get,

\[ \nabla^2 C_1^{(1,1)}(x,y) = -\delta(x-y) \frac{2\mu_2}{\mu_1 - \mu_2} (\partial_x C_0^{(1)}(x))^2. \]  

(93)

One needs to solve this equation inside a the unit square \( (x,y) \in [0,1] \times [0,1] \) with the correlation function vanishing at the boundaries, i.e. \( C_1^{(1,1)}(x,y)|_{(x,y) \in \partial} = 0 \). This is because the fluctuations of the masses at the boundaries are independent of the bulk masses. The solution to this equation is easily found to be

\[ C_1^{(1,1)}(x,y) = \begin{cases} 
\frac{\mu_2}{\mu_1 - \mu_2} (\bar{g}_R - \bar{g}_L)^2 y(1-x) & \text{if } x > y \\
\frac{\mu_2}{\mu_1 - \mu_2} (\bar{g}_R - \bar{g}_L)^2 x(1-y) & \text{if } y > x 
\end{cases} \]  

(94)

as already announced in equation (27).

6.2. Derivation of the Poisson equation satisfied by \( C_2^{(1,1,1)}(x,y,z) \)

We now outline the derivation of the Poisson equation satisfied by the three-point scaled correlation function. We first consider the case when two of the points are coincident, and using the equation for the cumulant \( \langle g_k e^{\lambda g_i} \rangle \) at \( O(\lambda^2) \), with points \( i \) and \( k \) far away from each other. Then we consider when all three points are coincident, and thus consider \( \langle e^{\lambda m_k} \rangle \) at \( O(\lambda^3) \).

6.2.1. Two coincident points, \( y = z \). Consider the evolution equation for \( \langle g_k e^{\lambda g_i} \rangle \), where \( |i - k| \geq 2 \),

\[ \frac{d}{dt} \langle g_k e^{\lambda g_i} \rangle = \langle (\nabla_i^f g_k + \nabla_i^b g_k) e^{\lambda g_i} \rangle + \langle g_k (\Delta G)_i \rangle \]

(95)

where \( \Delta G \) is the operator defined previously in equation (80), and the subscript \( i \) denotes that it is taken at the site \( i \). We also use the notation \( \nabla_i^f \) and \( \nabla_i^b \) for forward and backward difference operators, \( \nabla_i^f f_i \equiv f_{i+1} - f_i \), \( \nabla_i^b f_i \equiv f_{i-1} - f_i \). Setting the LHS to 0 we get the steady-state values. Further, using \( g_k = m_k + \delta_k \), with \( \nabla^2_k m_k = 0 \) and \( m_k((\Delta G)_i) = 0 \), we get

\[ \langle \eta \nabla^2_k \delta_k e^{\lambda g_i} \rangle + \langle \delta_k (\Delta G)_i \rangle = 0. \]

(96)
We can now expand in orders of $\lambda$ to get equations for the correlation functions. At $O(\lambda)$ we get $D_{ki}(\delta_k \delta_i)$, as previously attested. At $O(\lambda^2)$, the non-zero contribution from the first term is

$$\mu_1 \frac{f_2}{2} \langle (\nabla^2_k \delta_k)(2m_i\delta_i + \delta_i^2) \rangle$$

and that from the second term is

$$\left\langle \delta_k \eta \left[ f_2^2 + f_2 \eta \right] g_i \left[ \nabla^2_i g_i + \nabla^b g_i \right] + \frac{f_2}{2} \eta^2 \left[ \langle \nabla^2_i g_i \rangle^2 + \langle \nabla_i^b g_i \rangle^2 \right] \right\rangle.$$  \hspace{1cm} (98)

After taking the continuum limit for large $N$, as done in equation (92), we get,

$$\nabla^2 C(x,y,z)|_{y=z} = -\frac{2\mu_1}{3\mu_1 - 2\mu_2} (\partial_y M(y)) (\partial_y C(x,y)).$$ \hspace{1cm} (99)

6.2.2. Three coincident points, $x = y = z$. For all three points coincident, the derivation involves keeping terms in $\langle \Delta G \rangle = 0$ to order $\lambda^3$. At $O(\lambda^3)$, the terms in $\Delta G$ are

$$\left[ \left( \frac{\mu_1}{2} + \mu_2 \right) f_2 + \frac{1}{3} f_3 \mu_3 \right] \left[ 2m_i \langle \delta_i \nabla^2_i \delta_i \rangle + \langle \delta_i^2 \nabla^2_i \delta_i \rangle \right]$$

$$+ \left[ \frac{1}{2} \mu_2 f_2 + \frac{1}{3} \mu_3 f_3 \right] \left[ 2m_i \langle \nabla^2_i (m_i) \rangle^2 + \langle (m_i + \delta_i) \rangle \left( \langle \nabla^2_i (m_i + \delta_i) \rangle^2 + \langle \nabla^b_i (m_i + \delta_i) \rangle^2 \right) \right]$$

$$+ \left[ \frac{1}{6} \mu_3 f_3 \right] \left[ \langle \left( \langle \nabla^2_i (m_i + \delta_i) \rangle^2 + \langle \nabla^b_i (m_i + \delta_i) \rangle^2 \right) \right].$$ \hspace{1cm} (100)

In the following we use the below expressions in the continuum limit,

$$\langle \delta_i \nabla^2_i \delta_i \rangle \rightarrow \frac{1}{N^3} \frac{1}{2} \nabla^2 C(x,y)|_{x=y}$$ \hspace{1cm} (101)

$$\langle \left( \langle \nabla^2_i \delta_i \rangle^2 + \langle \nabla^b_i \delta_i \rangle^2 \right) \rangle \rightarrow -\frac{1}{N^3} \nabla^2 C(x,y)|_{x=y}$$ \hspace{1cm} (102)

$$\langle \delta_i^2 \nabla^2_i \delta_i \rangle \rightarrow \frac{1}{N^4} \frac{1}{3} \nabla^2 C(x,y,z)|_{x=y=z}$$ \hspace{1cm} (103)

$$\langle \delta_i (\langle \nabla^2_i \delta_i \rangle^2 + \langle \nabla^b_i \delta_i \rangle^2) \rangle \rightarrow -\frac{1}{N^4} \nabla^2 C(x,y,z)|_{x=y=z}$$ \hspace{1cm} (104)

$$\langle \left( \langle \nabla^2_i \delta_i \rangle^3 + \langle \nabla^b_i \delta_i \rangle^3 \right) \rangle \rightarrow O \left( \frac{1}{N^5} \right).$$ \hspace{1cm} (105)

We also use that $f_1 = 1$, $f_2 = \frac{\mu_1}{\mu_2 - \mu_1}$ and $f_3 = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} f_2$. Simplifying, we get

$$\nabla^2 C(x,y,z)|_{x=y=z} = k_3 \partial_x^3 (M(x)^3)$$ \hspace{1cm} (106)
where

\[ k_3 = \frac{\mu_2}{\mu_1 - \mu_2} \frac{2\mu_1 - \mu_3(\mu_1 + \mu_2)}{\mu_1(\mu_1 - \mu_2) - \mu_3(\mu_1 + \mu_2)}. \]  

(107)

Thus, collecting all of the possible source terms, we finally get the following equation for the three-point correlation function,

\[
\nabla^2 C(x,y,z) = -\frac{2\mu_1}{3\mu_1 - 2\mu_2} \left[ (\partial_y M(y))(\partial_y C(x,y))\delta(y-z) \right. \\
+ (\partial_x M(x))(\partial_x C(x,z))\delta(x-y) + (\partial_z M(z))(\partial_z C(y,z))\delta(x-z) \bigg] \\
+ \frac{\mu_2}{\mu_1 - \mu_2} \frac{2\mu_1 - \mu_3(\mu_1 + \mu_2)}{\mu_1(\mu_1 - \mu_2) - \mu_3(\mu_1 + \mu_2)} \partial_x^2 (M(x)^3)\delta(x-y)\delta(y-z). 
\]  

(108)

7. Conclusions

In this paper we have studied a mass transfer model defined on a one dimensional lattice of size \( N \) driven by two reservoirs of different ‘chemical potentials’ at the two ends. We have shown that in the steady state, correlations involving an arbitrary set of sites (say \( k \) sites out of \( N \)) and arbitrary order, say \( M \), have expansions in powers of \( 1/N \) starting at order \( O(1/N^{k-1}) \) to order \( O(1/N^{M-1}) \). We found that at each order, the correlation functions possess scaling forms which can in principle be expressed explicitly in terms of lower order max-site correlations. This is in contrast to Wick’s theorem where any higher order correlations can be expressed in terms of two point correlation functions. The important difference is that the scaling function of a \( k \)-point correlation of arbitrary order involves all the max-site correlations \( \langle g_{i_1}g_{i_2}\cdots g_{i_m}\rangle_c \) up to order \( k \), i.e. with \( 1 \leq m \leq k \). To establish this result we employed a novel operator method, valid in the large \( N \) limit. The intriguing recursive structure of the operator method allows us to determine the scaling functions at different order explicitly if the max-site correlation functions up to that order are known. These max-site correlation functions satisfy Poisson equations inside a hypercube. The operator method also provides a simple way to derive these Poisson equations for the two point and three point correlations. The operator method can in principle be employed to derive the explicit form of Poisson equation for higher point correlations systematically.

Our study of multi-site correlations of masses at arbitrary order in the NESS can be extended in different directions. One immediate question is whether this special structure suggests exact solvability of this model. Another interesting question is whether a similar structure exists for correlations in non-stationary regimes as well. We also expect a similar operator method to hold in higher dimensions, and it would then be interesting to extend the results of this paper to two- and three-dimensional versions of the RATP. Finally, another interesting direction would be to explore the possibility of the existence of such an operator method in other mass transfer models.
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Appendix A. Hierarchy of the correlations \( \langle g_{i_1}^{m_1} g_{i_2}^{m_2} \ldots g_{i_k}^{m_k} \rangle_c \) of order \( M = 4 \)

In the following, we use the notations \( x \equiv \frac{i}{N+1} \), \( y \equiv \frac{j}{N+1} \), \( z \equiv \frac{k}{N+1} \) and \( w \equiv \frac{l}{N+1} \). For the hierarchy of correlations up to order \( M = 3 \), see equations (4)–(6).

- Correlations of order \( M = 4 \):
  \[
  \langle g_i g_j g_k g_l \rangle \approx \frac{1}{N^3} C_3^{(1,1,1,1)}(x,y,z,w) \\
  \langle g_i^2 g_j g_k \rangle \approx \frac{1}{N^2} C_2^{(2,1,1)}(x,y,z) + \frac{1}{N^3} C_3^{(2,1,1)}(x,y,z) \\
  \langle g_i^2 g_j^2 \rangle \approx \frac{1}{N} C_1^{(2,2)}(x,y) + \frac{1}{N^2} C_2^{(2,2)}(x,y) + \frac{1}{N^3} C_3^{(2,2)}(x,y) \\
  \langle g_i^3 g_j \rangle \approx \frac{1}{N} C_1^{(3,1)}(x,y) + \frac{1}{N^2} C_2^{(3,1)}(x,y) + \frac{1}{N^3} C_3^{(3,1)}(x,y) \\
  \langle g_i^4 \rangle \approx C_0^{(4)}(x) + \frac{1}{N} C_1^{(4)}(x) + \frac{1}{N^2} C_2^{(4)}(x) + \frac{1}{N^3} C_3^{(4)}(x).
  \]

Appendix B. Recursion relations for the operator \( g_i^n \)

Starting from the master equation in (2), one finds

\[
\frac{d}{dt} \langle g_i^n \rangle = \int_0^1 d\eta \ R(\eta) \ [ \langle (g_i + \eta g_{i+1})^n \rangle + \langle (g_i + \eta g_{i-1})^n \rangle + 2(1-\eta)^n \langle g_i^n \rangle - 4\langle g_i^n \rangle ] \quad \text{(B1)}
\]

Thus, in the steady-state, expanding the brackets we have

\[
2\langle g_i^n \rangle = \sum_{k=1}^n \binom{n}{k} \mu_k \ [ \langle g_i^{n-k} (g_{i+1}^k + g_{i-1}^k) \rangle ] + 2 \int_0^1 d\eta \ R(\eta) (1-\eta)^n \langle g_i^n \rangle \quad \text{(B2)}
\]

which simplifies to

\[
\langle (1 - (1-\eta)^n) \rangle \langle g_i^n \rangle - \frac{\mu_n}{2} \langle (g_{i-1}^n) + g_{i+1}^n \rangle = \sum_{k=1}^{n-1} \binom{n}{k} \mu_k \ [ \langle g_i^{n-k} (g_{i+1}^k + g_{i-1}^k) \rangle ] \quad \text{(B3)}
\]
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Taking the scaling limit (i.e. \(N \to \infty\) limit) implies \(\frac{1}{2}(g_{i-1}^k + g_{i+1}^k) \to g_i^k + O(N^{-2})\), which finally provides the operator equation

\[
g_i^n = \frac{1}{\langle 1 - (1 - \eta)^n \rangle - \mu_n} \sum_{k=1}^{n-1} \binom{n}{k} \mu_k g_i^{n-k} \cdot g_i^k.
\]

(B4)

It can be checked that for the cases \(n = 2, n = 3\) and \(n = 4\), this gives equations (35), (49) and (59) respectively.

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