The multiplicative anomaly for determinants revisited; locality.

Marie-Françoise OUEDRAOGO, Sylvie PAYCHA

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Abstract

Observing that the logarithm of a product of two elliptic operators differs from the sum of the logarithms by a finite sum of operator brackets, we infer that regularised traces of this difference are local as finite sums of noncommutative residues. From an explicit local formula for such regularised traces, we derive an explicit local formula for the multiplicative anomaly of \( \zeta \)-determinants which sheds light on its locality and yields back previously known results.

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Introduction

The determinant on the linear group \( Gl(\mathbb{R}^n) \) reads

\[
\det A = e^{tr(\log A)}
\]

where \( tr \) is the matrix trace. It is independent of the choice of spectral cut used to define the logarithm and is multiplicative as a result of the Campbell-Hausdorff formula and the cyclicity of the trace, namely:

\[
\det(AB) = e^{tr(\log AB)} = e^{tr(\log A + \log B)} = \det A \det B.
\]

In contrast, the \( \zeta \)-determinant

\[
\det_\zeta(A) = e^{-\zeta'(A)(0)},
\]

defined for an admissible elliptic classical pseudodifferential operator \( A \) (with appropriate spectral cut) acting on sections of a vector bundle \( E \) over a closed \( n \)-dimensional manifold \( M \) via the zeta function \( \zeta_A(s) \) associated with \( A \), which corresponds to the

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unique meromorphic extension of the map \( s \mapsto \text{Tr}(A^{-s}) \) given by the \( L^2 \)-trace of \( A^{-s} \) defined on the domain of holomorphicity \( \text{Re}(as) > n \) where \( a \) is the order of \( A \), is not multiplicative. It presents a multiplicative anomaly

\[
\mathcal{M}_\zeta(A, B) = \frac{\det_\zeta(AB)}{\det_\zeta(A) \det_\zeta(B)}
\]

studied independently by K. Okikiolu in \([O2]\) and by M. Kontsevich and S. Vishik in \([KV]\).

The multiplicative anomaly of \( \zeta \)-determinants was expressed in terms of noncommutative residues of classical pseudodifferential operators in the following situations:

- by Wodzicki \([W1]\) (see also \([K]\) for a review) for positive definite commuting elliptic differential operators,
- by Friedlander \([Fr]\) for positive definite elliptic pseudodifferential operators,
- by Okikiolu \([O2]\) for operators with scalar leading symbols,
- by Kontsevich and Vishik \([KV]\) for operators sufficiently close to self-adjoint positive pseudodifferential operators.
- The multiplicative anomaly was further studied by Ducourtioux \([D]\) in the context of weighted determinants also discussed in this paper.

The noncommutative residue \( \text{res} \) (see formula \( \text{(1.6)} \)) originally introduced by Guillemin \([G]\) and Wodzicki \([W1]\), which defines a trace on the algebra \( C\ell(M, E) \) of classical pseudodifferential operators acting on smooth sections of the vector bundle \( E \), is local in so far as it is the integral over \( M \) of a local residue \( \text{res}_x(A) \) which only depends on a finite number of homogeneous components of the symbol of the operator \( A \). Consequently, the multiplicative anomaly is local.

Locality of the multiplicative anomaly for \( \zeta \)-determinants relates to the locality of regularised traces of the difference \( \log(AB) - \log A - \log B \), on which we focus in this paper, investigating their local feature which follows from the vanishing of the residue of \( L(A, B) \).

To see these links, one first observes that regularised traces of \( L(A, B) \) correspond to the multiplicative anomaly of another type of regularised determinants, namely weighted determinants (see \([D]\))

\[
\det_Q(A) = e^{\text{tr}_Q(\log A)},
\]

defined via a regularised trace \( \text{tr}_Q \) (see Definition \([5]\) which uses the regulator \( Q \), called a weight \([\mathbb{W}]\). They differ from \( \zeta \)-determinants by a local expression involving the Wodzicki residue, as can explicitly be seen from the relation (see \([D]\) Proposition III.1.7):

\[
\frac{\det_\zeta(A)}{\det^Q(A)} = e^{-\frac{1}{2\pi i} \text{res} \left( (\log A - \frac{a}{2} \log Q)^2 \right)}.
\]

\footnote{Regularised traces are linear extensions to the algebra \( C\ell(M, E) \) of the ordinary \( L^2 \)-trace on smoothing operators, which are nontracial since the \( L^2 \)-trace does not extend to a trace on the algebra \( C\ell(M, E) \).}

\footnote{A weight is any admissible elliptic operator in \( C\ell(M, E) \) with positive order.}
where $a$ is the order of $A$, $q$ the order of $Q$. Consequently, the multiplicative anomaly for \( \zeta \)-determinants differs from the multiplicative anomaly for weighted determinants by a local expression so that $\log \mathcal{M}_\zeta(A, B) - \text{tr}^Q(L(A, B))$ is local.

On the other hand, one infers the locality of regularised traces $\text{tr}^Q(L(A, B))$ of $L(A, B)$ from the vanishing of the noncommutative residue of $L(A, B)$ (see \( \text{(5.10)} \)), a property shown in [WV] which implies the multiplicativity of the residue determinant. Indeed, since all traces on the algebra of classical pseudodifferential operators on a closed manifold of dimension larger than one are proportional to the noncommutative residue [W1], it follows that $L(A, B)$ is a finite sum of commutators of classical pseudodifferential operators. Combining this with the expression of regularised traces of with positive orders $a$ the noncommutative residue \([W1]\), it follows that indeed, since all traces on the algebra of classical pseudodifferential operators on a closed manifold of dimension larger than one are proportional to the noncommutative residue \([W1]\), it follows that $L(A, B)$ is a finite sum of commutators of classical pseudodifferential operators. Combining this with the expression of regularised traces of brackets in terms on the noncommutative residue (see \( \text{(3.10)} \)), yields the locality of regularised traces $\text{tr}^Q(L(A, B))$ as finite sums of noncommutative residues.

Explicitly, in Theorem 2 we show that for two admissible elliptic operators $A, B$ with positive orders $a$ and $b$, such that the product $AB$ is also admissible, there is an operator $W(\tau)(A, B) \coloneqq \frac{1}{2 \pi i} L(A', A\tau B)$ depending continuously on $\tau$ such that (see \( \text{(5.18)} \))

$$\text{tr}^Q(L(A, B)) = \int_0^1 \text{res} \left( W(\tau)(A, B) \left( \frac{\log(A\tau B)}{a\tau + b} - \frac{\log Q}{q} \right) \right) d\tau. \quad \text{(0.1)}$$

The multiplicative anomaly for weighted determinants derived in Proposition 9 follows in a straightforward manner. From \( \text{(5.4)} \), in Theorem 3 we then derive an explicit local formula for the multiplicative anomaly for operators $A$ and $B$ with positive orders $a$ and $b$ (see equation \( \text{(6.4)} \)):

$$\log \mathcal{M}_\zeta(A, B) = \int_0^1 \text{res} \left( W(\tau)(A, B) \left( \frac{\log(A\tau B)}{a\tau + b} - \frac{\log B}{b} \right) \right) d\tau + \text{res} \left( \frac{L(A, B) \log B}{b} - \frac{\log^2 AB}{2(a + b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b} \right). \quad \text{(0.2)}$$

and similarly with the roles of $A$ and $B$ interchanged. When the operators $A$ and $B$ commute, $L(A, B)$ vanishes and formula \( \text{(0.2)} \) yields back Wodzicki’s formula:

$$\log \mathcal{M}_\zeta(A, B) = -\text{res} \left( \frac{\log^2 AB}{2(a + b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b} \right).$$

The r.h.s in the first line of \( \text{(0.2)} \) comes from a regularised trace $\text{tr}^B(L(A, B))$ described in \( \text{(4.1)} \) with weight $Q = B$. The r.h.s in the second line of \( \text{(0.2)} \), which corresponds to $\log \mathcal{M}_\zeta(A, B) - \text{tr}^Q(L(A, B))$, arises from a combination of two types of local terms: (i) local residues $\text{res}_x(\log^2 AB)$, $\text{res}_x(\log^2 A)$ and $\text{res}_x(\log^2 B)$ arising in formula \( \text{(5.3)} \) for the zeta determinants established in [PS]; (ii) the local residue $\text{res}_x(L(A, B) \log B)$ arising in a “defect formula” for regularised traces \( \text{(5.7)} \) also established in [PS], applied here to the regularised trace $\text{tr}^B(L(A, B))$ with regulator $B$. Since the operator $\frac{L(A, B) \log B}{b} - \frac{\log^2 AB}{2(a + b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b}$ turns out to be classical (see Lemma 4), combining these local residues yields a well-defined noncommutative residue.

Our approach to the multiplicative anomaly of $\zeta$-determinants is inspired by Okikiolu’s in \( \text{(0.2)} \). Before actually computing the multiplicative anomaly, she first
showed \[O1\] that for operators \( A \) and \( B \) with scalar leading symbols,

\[
L(A, B) \simeq \sum_{k=2}^{\infty} C(k)(\log A, \log B),
\]

i.e. that \( L(A, B) - \sum_{k=2}^{n+1} C(k)(\log A, \log B) \) is of order \( < -n \), thus generalising the usual Campbell-Hausdorff formula to classical pseudodifferential operators with scalar leading symbols. Here \( C(k)(\log A, \log B) \) are Lie monomials given by iterated brackets \( C(k)(P, Q) := \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} (\text{Ad} P)^{\alpha_1} (\text{Ad} Q)^{\beta_1} \cdots (\text{Ad} P)^{\alpha_j} (\text{Ad} Q)^{\beta_j} P^{-1} Q \) with the following notational convention: \( (\text{Ad} P)^{\alpha_j} (\text{Ad} Q)^{\beta_j} = 0 \) if \( \beta_j = 0 \) in which case this vanishes if \( \alpha_j > 1 \).

Under the assumption that the operators have scalar leading symbols, the iterated brackets arising in the Campbell-Hausdorff formula have decreasing order, allowing to implement ordinary traces after a certain order. In our more general situation, the leading symbols are not necessarily scalar and the iterated brackets arising in the Campbell-Hausdorff formula hence do not a priori have decreasing order which is why we use regularised traces instead of the ordinary trace and study regularised traces of \( L(A, B) \). Okikiolu’s proof in the case of operators with scalar leading symbols is largely based on the observation that the trace of the operator \( L(A, B) - \sum_{k=2}^{n+1} C(k)(\log A, \log B) \) only depends on the first \( n \) positively homogeneous components of \( A \) and \( B \) where \( n \) is the dimension of the underlying manifold \( M \); this allows her to work with a finite dimensional space of formal symbols. Interestingly, regularised traces of \( L(A, B) \) still only depend on the first \( n \) positively homogeneous components of \( A \) and \( B \). Precisely, given a weight \( Q \) and two admissible operators \( A \) and \( B \) in \( C^\ell(M, E) \) with non negative orders, we show that (see Theorem 1)

\[
\frac{d}{dt} \text{tr}^Q(L(A(1 + tS), B) = \frac{d}{dt} \text{tr}^Q(L(A, B(1 + tS)) = 0,
\]

for any operator \( S \) in \( C^\ell(M, E) \) of order \( < -n \).

The proofs of Theorems 1 and 2 both use the fact that differentiation in \( t \) commutes with regularised traces on differentiable families of constant order, a fact we prove in Proposition 7.

To conclude, this approach sheds light on the locality of multiplicative anomalies for regularised determinants (weighted determinants on the one hand and \( \zeta \)-determinants on the other) in so far as it relates it to the cyclicity of the noncommutative residue and hence to the multiplicativity of the residue determinant via the locality of regularised traces of \( L(A, B) \), which are interesting in their own right.

## 1 The noncommutative residue

We recall a few basic definitions concerning classical pseudodifferential operators on closed manifolds, set some notations and define the noncommutative residue introduced by Wodzicki in \[W1\].

Let \( U \) be an open subset of \( \mathbb{R}^n \). Given \( a \in \mathcal{C} \), the space of symbols \( S^\alpha(U) \) consists

Their precise definition is:

\[
C(k)(P, Q) := \frac{1}{k} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \sum_{\alpha_1 + \beta_1 = k, \alpha_j, \beta_j \geq 0} \frac{(\text{Ad} P)^{\alpha_1} (\text{Ad} Q)^{\beta_1} \cdots (\text{Ad} P)^{\alpha_j} (\text{Ad} Q)^{\beta_j} P^{-1} Q}{\alpha_1! \cdots \alpha_j! \beta_1! \cdots \beta_j!},
\]

with the following notational convention: \((\text{Ad} P)^{\alpha_j} (\text{Ad} Q)^{\beta_j} = 0 \) if \( \beta_j = 0 \) in which case this vanishes if \( \alpha_j > 1 \).
of functions \( \sigma(x, \xi) \) in \( C^\infty(U \times \mathbb{R}^n) \) such that for any compact subset \( K \) of \( U \) and any two multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n \) there exists a constant \( C_{K,\alpha,\beta} \) satisfying for all \( (x, \xi) \in K \times \mathbb{R}^n \)

\[
|\partial_x^\alpha \partial_{\xi}^\beta \sigma(x, \xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{\text{Re}(\alpha)-|\beta|}, \tag{1.1}
\]

where \( \text{Re}(\alpha) \) is the real part of \( \alpha \) and \( |\beta| = \beta_1 + \cdots + \beta_n \).

If \( \text{Re}(a_1) < \text{Re}(a_2) \), then \( S_{a_1}(U) \subset S_{a_2}(U) \).

The product \( \ast \) on symbols is defined as follows: if \( \sigma_1 \in S_{a_1}(U) \) and \( \sigma_2 \in S_{a_2}(U) \),

\[
\sigma_1 \ast \sigma_2(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{\alpha}}{\alpha!} \partial_x^\alpha \sigma_1(x, \xi) \partial_{\xi}^\alpha \sigma_2(x, \xi) \tag{1.2}
\]

i.e. for any integer \( N \geq 1 \) we have

\[
\sigma_1 \ast \sigma_2(x, \xi) - \sum_{|\alpha| < N} \frac{(-i)^{\alpha}}{\alpha!} \partial_x^\alpha \sigma_1(x, \xi) \partial_{\xi}^\alpha \sigma_2(x, \xi) \in S^{a_1+a_2-N}(U).
\]

In particular, \( \sigma_1 \ast \sigma_2 \in S^{a_1+a_2}(U) \).

We denote by \( S^{-\infty}(U) := \bigcap_{a \in \mathbb{R}} S^a(U) \) the algebra of smoothing symbols on \( U \), by \( S(U) := \bigcup_{a \in \mathbb{R}} S^a(U) \) the algebra generated by all symbols on \( U \).

A symbol \( \sigma \) in \( S^a(U) \) is called classical of order \( a \) if there is a smooth cut-off function \( \chi \in C^\infty(\mathbb{R}^n) \) which vanishes for \( |\xi| \leq \frac{1}{2} \) and such that \( \chi(\xi) = 1 \) for \( |\xi| \geq 1 \) such that

\[
\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \sigma_{a-j}(x, \xi) \tag{1.3}
\]

i.e. if for any integer \( N \geq 1 \), we have

\[
\sigma(N)(x, \xi) := \sigma - \sum_{j=1}^{N-1} \chi(\xi) \sigma_{a-j}(x, \xi) \in S^{a-N}(U), \tag{1.4}
\]

where \( \sigma_{a-j}(x, \xi) \) is a positively homogeneous function on \( U \times \mathbb{R}^n \) of degree \( a-j \), i.e. \( \sigma_{a-j}(tx, t\xi) = t^{a-j} \sigma_{a-j}(x, \xi) \) for all \( t \in \mathbb{R}^+ \).

Let \( CS^a(U) \) denote the subset of classical symbols of order \( a \). The symbol product of two classical symbols is a classical symbol and we denote by

\[
CS(U) = \bigcup_{a \in \mathbb{R}} CS^a(U)
\]

the algebra generated by all classical symbols on \( U \).

The noncommutative residue of a symbol \( \sigma \in CS(U) \) at point \( x \) in \( U \) is defined by

\[
\text{res}_x(\sigma) := \int_{S^n_+ U} (\sigma(x, \xi))_{-n} \ d_S \xi, \tag{1.5}
\]

where \( S^n_+ U \subset T^n_+ U \) is the cotangent unit sphere at point \( x \) in \( U \), \( d_S \xi = \frac{1}{(2\pi)^{n/2}} \ d\xi \) is the normalised volume measure on the sphere induced by the canonical volume measure on \( \mathbb{R}^n \) and where as before, \( (\cdot)_{-n} \) denotes the positively homogeneous component degree \( -n \) of the symbol.
Given a symbol $\sigma$ in $S(U)$, we can associate to it the continuous operator $Op(\sigma) : C_c^{\infty}(U) \to C^{\infty}(U)$ defined for $u \in C_c^{\infty}(U)$—the space of smooth compactly supported functions on $U$—by

$$(Op(\sigma)u)(x) = \int e^{ix\cdot \xi} \sigma(x, \xi) \hat{u}(\xi)d\xi,$$

where $d\xi := \frac{1}{(2\pi)^d} d\xi$ with $d\xi$ the ordinary Lebesgue measure on $T^*_xM \simeq \mathbb{R}^n$ and $\hat{u}(\xi)$ is the Fourier transform of $u$.

Since

$$Op(\sigma)u(x) = \int \int e^{i(x-y)\cdot \xi} \sigma(x, \xi)u(y)d\xi dy,$$

$Op(\sigma)$ is an operator with Schwartz kernel given by $k(x,y) = \int e^{i(x-y)\cdot \xi} \sigma(x, \xi)d\xi$, which is smooth off the diagonal.

A pseudodifferential operator $A$ on $U$ is an operator which can be written in the form $A = Op(\sigma) + R$ where $\sigma$ is a symbol in $S(U)$ with compact support and where $R$ is a smoothing operator i.e. $R$ has a smooth kernel. Its symbol $\sigma_A \sim \sigma$ is defined modulo smoothing symbols. If $\sigma$ is a classical symbol of order $a$, then $A$ can be considered as a classical pseudodifferential operator of order $a$.

More generally, let $M$ be a smooth closed manifold of dimension $n$ and $\pi : E \to M$ a smooth finite rank vector bundle over $M$ modelled on a linear space $V$; an operator $A : C^\infty(M, E) \to C^\infty(M, E)$ is a (resp. classical) pseudodifferential operator of order $a$ if given a local trivialising chart $(V, \phi)$ on $M$, for any localisation $A_\nu = \chi^*_\nu A_\nu^1 : C^\infty_c(V) \to C^\infty_c(V)$ of $A$ where $\chi^*_\nu \in C^\infty_c(V)$, the operator $\phi_*(A_\nu) := \phi_\nu A_\nu \phi^{-1}$ from the space $C^\infty_c(\phi(V))$ into $C^\infty(\phi(V))$ is a (resp. classical) pseudodifferential operator of order $a$.

Let $\text{Cl}^a(M, E)$ denote the set of classical pseudodifferential operators of order $a$.

If $A_1 \in \text{Cl}^{a_1}(M, E)$, $A_2 \in \text{Cl}^{a_2}(M, E)$, then the product $A_1A_2$ lies in $\text{Cl}^{a_1+a_2}(M, E)$ and we denote by

$$\text{Cl}(M, E) := \bigcup_{a \in \mathbb{C}} \text{Cl}^a(M, E)$$

the algebra generated by all classical pseudodifferential operators acting on smooth sections of $E$. Let us also introduce the algebra

$$\text{Cl}^{-\infty}(M, E) := \bigcap_{a \in \mathbb{R}} \text{Cl}^a(M, E)$$

of smoothing operators.

Wodzicki proved that any linear form on the algebra $\text{Cl}(M, E)$ which vanishes on operator brackets is proportional to the noncommutative residue $[\text{W}]$. It is built from the noncommutative residue density at point $x$ in $M$ defined by

$$\omega_{\text{res}}(A)(x) := \text{res}_x(\sigma_A) \ dx; \ \text{with} \ \text{res}_x(\sigma_A) := \int_{S^*_xM} \text{tr}_x ((\sigma_A(x, \xi))_{-n}) \ d\xi,$$

where $S^*_xM \subset T^*_xM$ is the cotangent unit sphere at point $x$ in $M$, $d\xi = \frac{1}{(2\pi)^d} d\xi$ is the normalised volume measure on $\mathbb{R}^n$ and where as before, $(\cdot)_{-n}$ denotes the positively homogeneous component.
degree $-n$ of the symbol. It turns out to be a globally defined density on the manifold and gives rise to the noncommutative residue introduced in [W1] and [G]:

$$\text{res}(A) := \int_M \omega_{\text{res}}(A)(x) := \int_M \left( \int_{S^*_x M} \text{tr}_x \left( (\sigma_A(x,\xi))_{-n} \right) d\xi \right) dx.$$

This linear form which only depends on the $-n$ homogeneous part of the symbol of the operator, vanishes on operators of order $< -n$ and is local in the sense that it only depends on a finite number of positively homogeneous components of the symbol of the operator.

By results of Wodzicki [W1] and Guillemin [G] (see also [BG] and [L]), all traces on $C^\ell(M,E)$, i.e. all linear forms which vanish on commutators in $[C^\ell(M,E),C^\ell(M,E)]$ are proportional to the noncommutative residue or equivalently (see also [L])

$$\forall A \in C^\ell(M,E) \quad (\text{res}(A) = 0 \implies A \in [C^\ell(M,E),C^\ell(M,E)]).$$

### 2 Logarithms of operators: $\log(AB) - \log A - \log B$

We review the construction and properties of logarithms of elliptic operators and prove (see Proposition 3) that the expression $\log(AB) - \log A - \log B$ is a finite sum of commutators of zero order classical pseudodifferential operators.

An operator $A \in C^\ell(M,E)$ has principal angle $\theta$ if for every $(x,\xi) \in T^*M - \{0\}$, the leading symbol $(\sigma_A(x,\xi))^L$ has no eigenvalue on the ray $L_\theta = \{re^{i\theta}, r \geq 0 \}$; in that case $A$ is elliptic.

**Definition 1** We call an operator $A \in C^\ell(M,E)$ admissible with spectral cut $\theta$ if $A$ has principal angle $\theta$ and the spectrum of $A$ does not meet $L_\theta = \{re^{i\theta}, r \geq 0 \}$. In particular such an operator is invertible and elliptic. Since the spectrum of $A$ does not meet $L_\theta$, $\theta$ is called an Agmon angle of $A$.

**Remark 1** In applications, an invertible operator $A$ is often obtained from an essentially self-adjoint elliptic operator $B \in C^\ell(M,E)$ by setting $A = B + \pi_B$ using the orthogonal projection $\pi_B$ onto the kernel $\text{Ker}(B)$ of $B$ corresponding to the orthogonal splitting $L^2(M,E) = \text{Ker}(B) \oplus R(B)$ where $R(B)$ is the (closed) range of $B$. Here $L^2(M,E)$ denotes the closure of $C^\infty(M,E)$ w.r. to a Hermitian structure on $E$ combined with a Riemannian structure on $M$.

Let $A \in C^\ell(M,E)$ be admissible with spectral cut $\theta$ and positive order $a$. For $\text{Re}(z) < 0$, the complex power $A^z_\theta$ of $A$ is defined by the Cauchy integral

$$A^z_\theta = \frac{i}{2\pi} \int_{\Gamma_{r,\phi}} \lambda^z_{\theta}(A - \lambda)^{-1} d\lambda$$

where $\lambda^z_{\theta} = |\lambda|^ze^{iz(\arg\lambda)}$ with $\theta \leq \text{arg}\lambda < \theta + 2\pi$. In particular, for $z = 0$, we have $A^0_\theta = I$.

Here

$$\Gamma_{r,\phi} = \Gamma^1_{r,\phi} \cup \Gamma^2_{r,\phi} \cup \Gamma^3_{r,\phi}$$

Note that the biorthogonal (for the dual product) of a subspace $F$ of any vector space coincides with $F$ (see e.g. [E] Chapter II par. 4 n.6 Proposition 10).
where
\[
\Gamma_{r,\theta} = \{ \rho e^{i\theta}, \infty > \rho \geq r \}
\]
\[
\Gamma_{r,\theta}^2 = \{ \rho e^{i(\theta-2\pi)}, \infty > \rho \geq r \}
\]
\[
\Gamma_{r,\theta}^3 = \{ re^{it}, \theta - 2\pi \leq t < \theta \},
\]
is a contour along the ray \(L_\theta\) around the non zero spectrum of \(A\). Here \(r\) is any small positive real number such that \(\Gamma_{r,\theta} \cap \text{Sp}(A) = \emptyset\).

The operator \(A^\theta_{\phi}\) is a classical pseudodifferential operator of order \(a z\) with homogeneous components of the symbol of \(A^\theta_{\phi}\) given by

\[
\sigma_{az-j}(A^\theta_{\phi})(x,\xi) = \frac{i}{2\pi} \int_{\Gamma_{\phi}} \lambda^j b_{-a-j}(x,\xi,\lambda) \, d\lambda
\]

where the components \(b_{-a-j}\) are the positive homogeneous components of the resolvent \((A-\lambda I)^{-1}\) in \((\xi,\lambda^2)\). In particular, its leading symbol is given by \((\sigma_{A_z}(x,\xi))^L = ((\sigma_A)(x,\xi))^L\) and \(A^\theta_{\phi}\) is elliptic.

The definition of complex powers can be extended to the whole complex plane by setting \(A^\theta_k := A^k A^\theta_{-k}\) for \(k \in \mathbb{N}\) and \(\text{Re}(z) < k\); this definition is independent of the choice of \(k\) in \(\mathbb{N}\) and preserves the usual properties, i.e. \(A^z A^z = A^{z+z}, A^0 = I\), for \(k \in \mathbb{Z}\).

Complex powers of operators depend on the choice of spectral cut. Wodzicki \([W1]\) (Ponge in \([Po1]\), see Proposition 4.1, further quotes an unpublished paper by Wodzicki \([W2]\) established the following result.

**Proposition 1** \([W1, W2, Po1]\) Let \(\theta\) and \(\phi\) be two spectral cuts for an admissible operator \(A\) in \(\text{Cl}(M,E)\) such that \(0 \leq \theta < \phi < 2\pi\). The complex powers for these two spectral cuts are related by

\[
A^\theta_k - A^\phi_k = (1 - e^{2i\pi z}) \Pi_{\theta,\phi}(A) A^\theta_k,
\]

where we have set \(\Pi_{\theta,\phi}(A) = A \left( \frac{1}{2\pi} \int_{\Gamma_{\theta,\phi}} \lambda^{-1}(A-\lambda)^{-1} \, d\lambda \right)\) where \(\Gamma_{\theta,\phi}\) is a contour around the cone

\[
A_{\theta,\phi} := \{ \rho e^{it}, \infty > \rho \geq r, \theta < t < \phi \}.
\]

**Remark 2** Formula (2.2) generalises to spectral cuts \(\theta\) and \(\phi\) such that \(0 \leq \theta < \phi + 2k\pi < (2k+1)\pi\) for some non negative integer \(k\) by

\[
A^\theta_k - A^\phi_k = e^{2ik\pi z} I + (1 - e^{2i\pi z}) \Pi_{\theta,\phi}(A) A^\theta_k.
\]

If the cone \(A_{\theta,\phi}\) defined by (2.3) delimited by the angles \(\theta\) and \(\phi\) does not intersect the spectrum of the leading symbol of \(A\), it only contains a finite number of eigenvalues of \(A\) and \(\Pi_{\theta,\phi}(A)\) is a finite rank projection and hence a smoothing operator. In general (see Propositions 3.1 and 3.2 in \([Po1]\)), \(\Pi_{\theta,\phi}(A)\), which is a pseudodifferential projection, is a zero order operator with leading symbol given by \(\pi_{\theta,\phi}(\sigma^L(A))\) defined similarly to \(\Pi_{\theta,\phi}\) replacing \(A\) by the leading symbol \(\sigma^L_A\) of \(A\) so that:

\[
\pi_{\theta,\phi}(\sigma^L_A) = \pi_{\theta,\phi}(\sigma^L_A) := \sigma^L_A \left( \frac{1}{2\pi} \int_{\Gamma_{\theta,\phi}} \lambda^{-1}(\sigma^L_A - \lambda)^{-1} \, d\lambda \right),
\]

where we have set \(\sigma^L_B(x,\xi) = (\sigma_B(x,\xi))^L\) for any \((x,\xi) \in T^*M\) and any \(B \in \text{Cl}(M,E)\).
We are finally ready to define the logarithm. The logarithm of an admissible operator $A$ with spectral cut $\theta$ is defined in terms of the derivative at $z = 0$ of this complex power:

$$\log_\theta(A) = \partial_z A^\theta_{|z=0}.$$

**Remark 3** For a real number $t$, $A$ and $A^t_\theta$ have spectral cuts $\theta$ and $t \theta$; for $t$ close to one, $(A^t_\theta)_{|t=1} = (A^1_\theta)_{|t=1}$ and hence, $(A^t_\theta)_{|t=0} = A^0_\theta$ so that

$$\theta \in \mathbb{R} \implies \log_\theta(A^t) = \partial_z (A^t_\theta)^{\frac{z}{t}} = \partial_z (A^z_\theta)_{|z=0} = t \log_\theta(A).$$

Just as complex powers, the logarithm depends on the choice of spectral cut $O1$. Indeed, differentiating $(2.2)$ w.r.t. to $z$ at $z = 0$ yields for spectral cuts $\theta, \phi$ such that $0 \leq \theta < \phi < 2\pi$ (compare with formula (1.4) in $O1$):

$$\log_\theta(A) - \log_\phi(A) = -2i\pi \Pi_{\theta, \phi}(A).$$

(2.5)

Formula (2.5) generalises to spectral cuts $\theta$ and $\phi$ such that $0 \leq \theta < \phi + 2k\pi < (2k+1)\pi$ for some non negative integer $k$ by

$$\log_\theta(A) - \log_\phi(A) = 2ik\pi I - 2i\pi \Pi_{\theta, \phi}(A).$$

(2.6)

As a result of the above discussion and as already observed in $O1$, when the leading symbol $\sigma^A_\theta$ has no eigenvalue inside the cone $\Lambda_{\theta, \phi}$ delimited by $\Gamma_{\theta, \phi}$ then $\Pi_{\theta, \phi}$ which is a finite rank projection, is smoothing.

Logarithms of classical pseudodifferential operators are not classical anymore since their symbols involve a logarithmic term $\log |\xi|$ as the following elementary result shows.

**Proposition 2** Let $A \in C(\ell(M, E))$ be an admissible operator with spectral cut $\theta$. In a local trivialisation, the symbol of $\log_\theta(A)$ reads:

$$\sigma_{\log_\theta(A)}(x, \xi) = a \log |\xi| I + \sigma_0^A(x, \xi)$$

(2.7)

where $a$ denotes the order of $A$ and $\sigma_0^A$ a symbol of order zero. Moreover, the leading symbol of $\sigma_0^A$ is given by

$$(\sigma_0^A)^L(x, \xi) = \log_\theta \left( \sigma_0^A(x, \frac{\xi}{|\xi|}) \right) \quad \forall (x, \xi) \in T^*M - \{0\}.$$  

(2.8)

In particular, if $\sigma_A$ has scalar leading symbol then so have $\sigma_0^A$ and $\sigma_{\Pi_{\theta, \phi}(A)}$ for any other spectral cut $\phi$. 

**Proof:** Given a local trivialisation over some local chart, the symbol of $A^\theta_{\|z\|=0}$ has the formal expansion $\sigma_{A^\theta_{\|z\|=0}} \sim \sum_{j \geq 0} b_{az-j}^{(z)}$, where $a$ is the order of $A$ and $b_{az-j}^{(z)}$ is a positively homogeneous function of degree $az - j$. Since $\log_\theta(A) = \partial_z A^\theta_{|z=0}$, the formal expansion of the symbol of $\log_\theta(A)$ is $\sigma_{\log_\theta(A)} \sim \sum_{j \geq 0} \partial_z b_{az-j}^{(z)}$. Since $A^\theta_{|z=0} = I$, we have $\sigma_{A^\theta_{|z=0}} \sim I$ where now $I$ stands for the identity on matrices. Thus $b_{az-j}^{(z)}(x, \xi)_{|z=0} = \delta_{0,j} I.$
Suppose that \( \xi \neq 0 \); using the positive homogeneity of the components, we have:

\[
b_{a_{2j}}(x, \xi) = |\xi|^{|2j|} b_{a_{2j}}(x, \frac{\xi}{|\xi|}) \; ;
\]

hence

\[
\partial_\xi b_{a_{2j}}(x, \xi) = a \log |\xi| |\xi|^{|2j|} b_{a_{2j}}(x, \frac{\xi}{|\xi|}) + |\xi|^{|2j|} \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|}).
\]

It follows that

\[
\partial_\xi b_{a_{2j}}(x, \xi)|_{\xi=0} = a \log |\xi| I + \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|})|_{\xi=0}.
\]

Similarly for \( j > 0 \), we have

\[
b_{a_{2j}}(x, \xi) = |\xi|^{a_{2j}} b_{a_{2j}}(x, \frac{\xi}{|\xi|}) \; ;
\]

so that

\[
\partial_\xi b_{a_{2j}}(x, \xi) = a \log |\xi| |\xi|^{a_{2j}} b_{a_{2j}}(x, \frac{\xi}{|\xi|}) + |\xi|^{a_{2j}} \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|}).
\]

Consequently,

\[
\partial_\xi b_{a_{2j}}(x, \xi)|_{\xi=0} = |\xi|^{-j} \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|})|_{\xi=0}.
\]

The terms \( \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|}) \) and \( \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|}) \) are homogeneous functions of degree 0 in \( \xi \). Summing up, we obtain

\[
\sigma_{\text{log}(A)}(x, \xi) = a \log |\xi| I + \sigma_0^A(x, \xi)
\]

where \( \sigma_0^A(x, \xi) = \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|})|_{\xi=0} + \sum_{j>0} |\xi|^{-j} \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|})|_{\xi=0} \). \( \sigma_0^A \) is a symbol of order 0. Its leading symbol reads

\[
(\sigma_0^A(x, \xi))^L = \partial_\xi b_{a_{2j}}(x, \frac{\xi}{|\xi|})|_{\xi=0} = \partial_\xi \left( \sigma_0^A(x, \frac{\xi}{|\xi|}) \right)^L = \log_0 \sigma_0^A(x, \frac{\xi}{|\xi|})
\]

for any \( (x, \xi) \in T^*M - \{0\} \). \( \square \)

This motivates the introduction of log-polyhomogeneous symbols (see e.g. [L]), to which the local noncommutative residue easily extends.

**Definition 2** A symbol \( \sigma \in S(U) \) is called log-polyhomogeneous of order \( a \) and type \( k \) for some non-negative integer \( k \) if there is some smooth function \( \chi \) on \( \mathbb{R}^n \) which vanishes around zero and is identically one outside the unit ball, such that

\[
\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) (\sigma(x, \xi))_{a-j}
\]

where for any non-negative integer \( j \),

\[
(\sigma_A(x, \xi))_{a-j} = \sum_{l=0}^{k} (\sigma_A(x, \xi))_{a-j,l} (x, \xi) \log^l |\xi| \quad \forall (x, \xi) \in T^*U,
\]

with \( \sigma_{a-j,l}, l = 0, \ldots, k \) are positively homogeneous of degree \( a - j \).

The local noncommutative residue at a point \( x \) in \( U \) defined in (2.8) extends to log-polyhomogeneous symbols by:

\[
\text{res}_x(\sigma) := \int_{S^*U} (\sigma(x, \xi))_{a-j} d\sigma\xi.
\]

(2.9)
Powers of the logarithm of a given admissible operator combined with all classical pseudodifferential operators generate the algebra of log-polyhomogeneous operators $[L]$. A log-polyhomogeneous operator $A$ of type $k$ is a pseudodifferential operator whose local symbol $\sigma_A(x, \xi)$ in any local trivialisation asymptotically is log-polyhomogeneous of type $k$.

Let us denote the set of such operators by $\mathcal{Cl}^a_{k}(M, E)$ and its union over all non-negative integers $k$ by $\mathcal{Cl}^a_{\ast}(M, E) = \bigcup_{k=0}^{\infty} \mathcal{Cl}^a_{k}(M, E)$. In particular, a classical pseudodifferential operator is a log-polyhomogeneous operator of log-type 0 and $\mathcal{Cl}^a_{0}(M, E) = \mathcal{Cl}^a_{\ast}(M, E)$. The product of a log-polyhomogeneous operator of type $k$ and a log-polyhomogeneous operator of type $l$ is log-polyhomogeneous operator of type $k+l$ so that, following $[L]$, we can build the algebra $\mathcal{Cl}^a_{\ast}(M, E)$ generated by all log-polyhomogeneous operators.

For an operator $A$ in $\mathcal{Cl}^a_{\ast}(M, E)$, one can define the local noncommutative residue at a point $x$ in $M$ similarly to the case of classical operators by:

$$\text{res}_x(A) := \int_{S^*M} \text{tr}_x(\sigma_A(x, \xi))_{-n} \, d_\xi.$$

However, unlike Lesch's extended noncommutative residue on log-polyhomogeneous operators $[L]$, the locally defined residue density $\text{res}_x(A) \, dx$ is not expected to patch up to a globally defined residue density. However, it does for logarithms of any admissible operator $A$ in $\mathcal{Cl}(M, E)$ and we have $[Sc]$:

$$\text{res}(\log A) = -a \zeta_A(0) \quad (2.10)$$

where $\zeta_A(0)$ is the constant term in the Laurent expansion of the unique meromorphic extension $\zeta_A(s)$ of the map $s \mapsto \text{Tr}(A^{-s})$ given by the $L^2$-trace of $A^{-s}$ defined on the domain of holomorphy $\text{Re}(as) > n$ $[L]$. In $[Sc]$, Scott showed the multiplicativity of the associated residue determinant

$$\det_{\text{res}}(A) := e^{\text{res}(\log A)}.$$

He actually proved a stronger statement, namely that given two admissible operators $A, B$ such that their product $AB$ is also admissible, the following expression

$$L(A, B) := \log(AB) - \log A - \log B$$

has vanishing noncommutative residue.

**Remark 4** Strictly speaking, one should specify the spectral cuts $\theta$ of $A$, $\phi$ of $B$ and $\psi$ of $AB$ in the expression $L(A, B)$ setting instead

$$L^{\theta, \phi, \psi}(A, B) := \log_{\phi}(AB) - \log_{\theta} A - \log_{\phi} B.$$

Then by $[PS]$,

$$L^{\theta, \phi, \psi}(A, B) - L^{\theta', \phi', \psi'}(A, B) = -2i\pi \left( \Pi_{\phi, \psi'}(AB) - \Pi_{\theta, \phi}(A) - \Pi_{\phi, \psi'}(B) \right).$$

Since these have vanishing residue by results of Wodzicki $[W1]$, a change of spectral cut does not affect the residue of $L(A, B)$.

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5This actually is an instance in the case $A(z) = A^{-z}$ of the more general defect formula $[L, PS]$ derived in $[PS]$. 

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Up to a modification of the operators $A$ and $B$, one can actually choose fixed spectral cuts $\theta$ and $\phi$ by the following argument of Okikiolu \cite{O1}:

\[ L^\theta \phi(A, B) = L^\pi \phi(\theta + \phi) \left( e^{i(\pi - \theta)} A, e^{i(\pi - \phi)} B \right). \]

Indeed, if $A, B, AB$ have spectral cut $\theta, \phi, \psi$ respectively, then $A' = e^{i(\pi - \theta)} A$ and $B' = e^{i(\pi - \phi)} B$ have spectral cut $\pi$ and $A' B'$ has spectral cut $\psi + 2\pi - \theta - \phi$. So we can assume that $\theta = \phi = \pi$ without loss of generality.

Keeping in mind these observations, in order to simplify notations we assume that $A$ and $B$ have spectral cuts $\pi$ and drop the explicit mention of the spectral cuts. It follows from (1.7) that

\[ L(A, B) \in [C\ell(M, E), C\ell(M, E)], \quad \text{(2.11)} \]

so that $L(A, B)$ is a finite sum of commutators. The following proposition provides a refinement this statement.

**Proposition 3** Let $A, B$ be two admissible operators, which w.l.o.g. are assumed to have $\pi$ as spectral cut (see Remark \[\text{[3]}\]), such that their product $AB$ is also admissible with spectral cut $\pi$. Then $L(A, B)$ is a finite sum of Lie brackets of operators in $C\ell^b(M, E)$:

\[ L(A, B) \in [C\ell^0(M, E), C\ell^0(M, E)]. \]

**Proof:** Let us check that $L(A, B)$ lies in $C\ell^0(M, E)$. If $\sigma$ has order $a$ and $B$ has order $b$ then $AB$ has order $a + b$, we have

\[
\sigma_{L(A, B)} = \sigma_{\log AB}(x, \xi) - \sigma_{\log A}(x, \xi) - \sigma_{\log B}(x, \xi) \\
= (a + b) \log |\xi| - a \log |\xi|
\]

so that the operator $L(A, B)$ is indeed classical of order $0$ and by \cite{228} it has leading symbol given for any $(x, \xi) \in T^*M/M$ by

\[
(\sigma_{L(A, B)}(x, \xi))_0 = \log \sigma^L_{AB}(x, \xi) - \log \sigma^L_{A}(x, \xi) - \log \sigma^L_{B}(x, \xi) = \Lambda(\sigma^L_{A}, \sigma^L_{B})(x, \xi).
\]

Here as before, $\sigma^L_{C}$ stands for the leading symbol of the operator $C$.

Applying the usual Campbell-Hausdorff formula to the matrices $\sigma^L_{A}(x, \xi)$ and $\sigma^L_{B}(x, \xi)$ and implementing the fibrewise trace $\text{tr}_x$ yields:

\[
\text{tr}_x \left( \log \sigma^L_{AB}(x, \xi) - \log \sigma^L_{A}(x, \xi) - \log \sigma^L_{B}(x, \xi) \right) = \text{tr}_x \left( \Lambda(\sigma^L_{A}, \sigma^L_{B})(x, \xi) \right) = 0.
\]

It follows that any leading symbol trace $\text{Tr}^\Lambda(\sigma(A, B)) := \Lambda \left( (\text{tr}_x(\sigma_C))_0 \right)$ (see e.g. \cite{LP}) on the algebra $C\ell^b(M, E)$ where $\Lambda$ is a current in $C^\infty(S^* M)'$ and the index 0 stands for the positively homogeneous component of degree 0, vanishes on $L(A, B)$:

\[
\text{Tr}^\Lambda(\sigma(A, B)) = \Lambda \left( (\text{tr}_x(\sigma_{L(A, B)}))_0 \right) = 0.
\]

Thus both the noncommutative residue and leading symbol traces vanish on $L(A, B)$. But by the results of \cite{LP}, any trace on $C\ell^b(M, E)$, i.e. any linear form on $C\ell^b(M, E)$ which vanishes on $[C\ell^b(M, E), C\ell^b(M, E)]$, is a linear combination of the noncommutative residue and a leading symbol trace. Consequently (see e.g. \cite{O2} Corollary 4.5), all traces on $C\ell^b(M, E)$ vanish on the operator $L(A, B)$ which therefore lies in $[C\ell^b(M, E), C\ell^b(M, E)]$. \qed

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3 Properties of weighted traces

Since traces on $C^\ell(M,E)$ are proportional to the noncommutative residue which vanishes on smoothing operators, the $L^2$-trace on smoothing operators does not extend to the whole algebra $C^\ell(M,E)$. Instead we use linear extensions which we call weighted traces, of the ordinary $L^2$-trace on smoothing operators to the whole algebra $C^\ell(M,E)$. We review basic properties of weighted traces and prove (see Proposition 7) that the canonical and weighted traces as well as the noncommutative residue commute with differentiation on differentiable families of operators with constant order.

Weighted traces arise as finite parts of canonical traces of holomorphic families of classical pseudodifferential operators. A family $\{f(z)\}_{z \in \Omega}$ in a topological vector space $A$ which is parametrised by a complex domain $\Omega$, is holomorphic at $z_0 \in \Omega$ if the corresponding function $f : \Omega \to A$ admits a Taylor expansion in a neighborhood $N_{z_0}$ of $z_0$

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(z_0) \frac{(z-z_0)^k}{k!} \quad (3.1)$$

which is convergent, uniformly on compact subsets in a neighborhood of $z_0$ (i.e. locally uniformly), with respect to the topology on $A$. The vector space of functions we consider here is $C^\infty(U \times \mathbb{R}^n) \otimes \text{End}(V)$ equipped with the uniform convergence of all derivatives on compact subsets.

**Definition 3** Let $\Omega$ be a domain of $\mathfrak{C}$ and $U$ an open subset of $\mathbb{R}^n$. A family $(\sigma(z))_{z \in \Omega}$ is a holomorphic family of $\text{End}(V)$-valued classical symbols on $U$ parametrised by $\Omega$ when

1. the map $z \mapsto \alpha(z)$ with $\alpha(z)$ the order of $\sigma(z)$, is holomorphic in $z$,
2. $z \mapsto \sigma(z)$ is holomorphic as element of $C^\infty(U \times \mathbb{R}^n) \otimes \text{End}(V)$ and for each $z \in \Omega$, $\sigma(z) \sim \sum_{j=0}^{\infty} \chi(z)\sigma(z)_{\alpha(z)-j}$ (for some smooth function $\chi$ which is identically one outside the unit ball and vanishes in a neighborhood of 0) lies in $CS^{\alpha(z)}(U) \otimes \text{End}(V)$,
3. for any positive integer $N$, the remainder term $\sigma_{(N)}(z) = \sigma(z) - \sum_{j=0}^{N-1} \sigma(z)_{\alpha(z)-j}$ is holomorphic in $z \in \Omega$ as an element of $C^\infty(U \times \mathbb{R}^n) \otimes \text{End}(V)$ and its $k$-th derivative

$$(x,\xi) \mapsto \partial_z^k \sigma_{(N)}(z)(x,\xi) := \partial_z^k \left( \sigma_{(N)}(z)(x,\xi) \right)$$

lies in $S^{s(z)-N+n}(U) \otimes \text{End}(V)$ for all $\epsilon > 0$ locally uniformly in $z$, i.e the $k$-th derivative $\partial_z^k \sigma_{(N)}(z)$ satisfies a uniform estimate \[\square\] in $z$ on compact subsets in $\Omega$.

In particular, for any integer $j \geq 0$, the (positively) homogeneous component $\sigma_{\alpha(z)-j}(z)$ of degree $\alpha(z) - j$ of the symbol is holomorphic on $\Omega$ as an element of $C^\infty(U \times \mathbb{R}^n) \otimes \text{End}(V)$.

It is important to observe that the derivative of a holomorphic family $\sigma(z)$ of classical symbols is not classical anymore since it yields a holomorphic family of symbols $\sigma'(z)$.
of order $\alpha(z)$, the asymptotic expansion of which involves a logarithmic term and reads \cite{PS}:
\[
\sigma'(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \left( \log |\xi| \sigma_{\alpha(z)-j,1}(z)(x, \xi) + \sigma'_{\alpha(z)-j,0}(z)(x, \xi) \right) \forall (x, \xi) \in T^*U
\]
for some smooth cut-off function $\chi$ around the origin which is identically equal to 1 outside the open unit ball and positively homogeneous symbols
\[
\sigma'_{\alpha(z)-j,0}(z)(x, \xi) = |\xi|^j \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right),
\]
\[
\sigma'_{\alpha(z)-j,1}(z)(x, \xi) = \alpha'(z) \sigma_{\alpha(z)-j}(z)(x, \xi)
\]
of degree $\alpha(z) - j$.

The regularised cut-off integral on symbols we are about to introduce is an essential ingredient to build linear extensions of the $L^2$-trace.

The integral $\int_{B^*_x(0,R)} \text{tr}(\sigma(x, \xi)) d\xi$ of the trace of a symbol $\sigma \in CS(U) \otimes \text{End}(V)$ over the ball $B^*_x(0,R)$ of radius $R$ centered at 0 in the cotangent space $T^*_xU$ at a point $x \in U$, has an asymptotic expansion in decreasing powers of $R$ which is polynomial in $\log R$ so that the cut-off integral which corresponds to the constant term in this expansion
\[
\int_{T^*_xU} \text{tr} (\sigma(x, \xi)) d\xi := \lim_{R \to \infty} \int_{B^*_x(0,R)} \text{tr} (\sigma(x, \xi)) d\xi
\]
is well defined. It coincides with the ordinary integral whenever the latter converges. We now recall the properties of cut-off integrals of holomorphic families of symbols.

**Proposition 4**

1. \cite{KV} The cut-off regularised integral $\int_{T^*_xU} \text{tr}_x (\sigma(z)(x, \xi)) d\xi$ of a holomorphic family $\sigma(z)$ of classical pseudodifferential symbols on a neighborhood $U \subset M$ of holomorphic order $\alpha(z)$ is a meromorphic function in $z$ with simple poles. The residue at a pole $z_0$ for which $\alpha'(z_0) \neq 0$ is given by:
   \[
   \text{Res}_{z=z_0} \int_{T^*_xU} \text{tr}_x (\sigma(z)(x, \xi)) d\xi = -\frac{1}{\alpha'(z_0)} \text{res}_x (\sigma(z_0)). \quad (3.2)
   \]

2. \cite{PS} Furthermore, if its holomorphic order is affine and non constant \footnote{Here and in what follows, we assume that the order of the holomorphic family is affine and non constant so that applying the fibrewise trace, formula (1.50) of \cite{PS} boils down to the following one.}, its Laurent expansion has constant term at $z_0$ given by
   \[
   \lim_{R \to \infty} \int_{T^*_xU} \text{tr}_x (\sigma(z)(x, \xi)) d\xi = \int_{T^*_xU} \text{tr}_x (\sigma(z_0)(x, \xi)) d\xi - \frac{1}{\alpha'(z_0)} \text{res}_x (\sigma'(z_0)). \quad (3.3)
   \]

Here we use the local residue extended to log-polynomial symbols (see \cite{PS}) since the derivative $\sigma'(z_0)$ of a holomorphic family of classical symbols $\sigma(z)$ with order $\alpha(z)$ at a point $z_0$ is expected to be logarithmic with same order.

Let us now carry out these constructions to the operator level.

For any $A \in C\ell(M, E)$, for any $x \in M$, the following expression defines a local density:
\[
\omega_{\text{KV}}(A)(x) := \int_{T^*_xM} \text{tr}_x (\sigma_A(x, \xi)) d\xi \quad (3.4)
\]
It patches up to a global density on $M$ whenever the operator $A$ in $\mathcal{C}(M, E)$ has non integer order or has order $< -n$ so that the Kontsevich-Vishik canonical trace $[KV]$

\[ \text{TR}(A) := \int_M \omega_{KV}(A)(x) := \int_M \text{TR}_x(A) \, dx \quad (3.5) \]

makes sense.

The canonical trace can be applied to holomorphic families of classical pseudodifferential operators with varying complex order.

**Definition 4** Let $(A(z))_{z \in \Omega}$ be a family of classical pseudodifferential operators in $\mathcal{C}(M, E)$ with distribution kernels $(x, y) \mapsto K_{A(z)}(x, y)$. The family is holomorphic if

1. the order $\alpha(z)$ of $A(z)$ is holomorphic in $z$,
2. in any local trivialisation of $E$, we can write $A(z)$ in the form $A(z) = \text{Op} \left( \sigma(z) \right) + R(z)$, for some holomorphic family of End$(V)$-valued symbols $(\sigma(z))_{z \in \Omega}$ where $V$ is the model space of the fibres of $E$, and some holomorphic family $(R(z))_{z \in \Omega}$ of smoothing operators i.e. given by a holomorphic family of smooth Schwartz kernels,
3. the (smooth) restrictions of the distribution kernels $K_{A(z)}$ to the complement of the diagonal $\Delta \subset M \times M$, form a holomorphic family with respect to the topology given by the uniform convergence in all derivatives on compact subsets of $M \times M - \Delta$.

**Example 1** Given an admissible operator $A \in \mathcal{C}(M, E)$ with spectral cut $\theta$, a family $z \mapsto A(z)$ is a holomorphic family of $\psi$DOs. In particular, $A'(z) = \partial_z A^{-z}$ is classical, whereas $A'(0) = (\partial_z A^{-z})_{z=0} = -\log \theta A$ is not.

Integrating the formulae in Proposition 4 along the manifold $M$ yields the following result on the level of operators.

**Proposition 5**

1. $[KV]$ The canonical trace $\text{TR}(A(z))$ of a holomorphic family $A(z)$ of classical pseudodifferential operators in $\mathcal{C}(M, E)$ of holomorphic order $\alpha(z)$ is a meromorphic function in $z$ with simple poles and residue at a pole $z_0$ for which $\alpha'(z_0) \neq 0$ is given by:

\[ \text{Res}_{z=z_0} \text{TR}(A(z)) = - \frac{1}{\alpha'(z_0)} \text{res}(A(z_0)). \quad (3.6) \]

2. $[PS]$ Furthermore, if its holomorphic order is affine and non constant, its Laurent expansion has constant term at $z_0$ given by

\[ \text{fp}_{z=z_0} \text{TR}(A(z)) = \int_M dx \left( \text{TR}_x(A(z_0)) - \frac{1}{\alpha'(z_0)} \text{res}_x(A'(z_0)) \right). \quad (3.7) \]

Applying these results to a holomorphic family $A(z) := AQ^{-z}$ where $A$ is any operator in $\mathcal{C}(M, E)$ and $Q$ an admissible operator in $\mathcal{C}(M, E)$ with positive order and spectral cut $\alpha$, we infer (see e.g. $[KV]$, $[P]$, $[CDMP]$ and references therein) that the map $z \mapsto \text{TR}(A Q^{-z})$ is meromorphic with simple poles.
Definition 5 Given an admissible operator $Q$ with positive order, which we call a weight, the $Q$-weighted trace of an operator $A$ in $\text{Cl}(M,E)$ is given by:

$$\text{tr}_Q^\alpha(A) := \text{fp}_{z=0} \text{TR} (AQ^{-z}_\alpha) := \lim_{z \to 0} \left( \text{TR} (AQ^{-z}_\alpha) - \text{Res}_{z=0} \left( \frac{\text{TR} (AQ^{-z}_\alpha)}{z} \right) \right),$$

where $\alpha$ is a spectral cut for $Q$ and where $\text{fp}_{z=0}$ denotes the constant term in the Laurent expansion.

Applying (3.7) to the family $A(z) = AQ^{-z}_\alpha$ yields the following “defect formula” [PS]:

$$\text{tr}_Q^\alpha(A) = \int_M \left( \text{TR}_x(A) - \text{res}_z \left( \frac{\text{TR}(AQ^{-z}_\alpha)}{q} \right) \right) dx,$$

where $q$ stands for the order of $Q$. In particular, for $A = I$ we get back (2.10):

$$\zeta_{Q,\alpha}(0) := \text{fp}_{z=0} \text{TR}(Q^{-z}_\alpha) = -\frac{\text{res} \log_\alpha Q}{q}.$$

Whereas weighted traces are not expected to be local in general since they involve the whole symbol of the operator, the difference of two weighted traces is local in so far as it involves a finite number of homogeneous components of the symbol via the noncommutative residue. Weighted traces depend on the choice of weight and are not cyclic in spite of their name.

Proposition 6 ([CDMP], [MN])

1. Given two weights $Q_1$ and $Q_2$ with common spectral cut $\alpha$ and positive orders $q_1$, $q_2$ we have

$$\text{tr}_Q^{Q_1}(A) - \text{tr}_Q^{Q_2}(A) = \text{res} \left( A \left( \frac{\log_\alpha Q_2}{q_2} - \frac{\log_\alpha Q_1}{q_1} \right) \right),$$

which is a local expression.

2. For any weight $Q$ in $\text{Cl}(M,E)$ with order $q$ and spectral cut $\alpha$, the operators $[A,\log_\alpha Q]$ and $[B,\log_\alpha Q]$ lie in $\text{Cl}(M,E)$ and

$$\text{tr}_Q^\alpha ([A,B]) = \frac{1}{q} \text{res} (A[B,\log_\alpha Q]) = -\frac{1}{q} \text{res} (B[A,\log_\alpha Q]).$$

In particular, if $Q = A$ or $Q = B$, or if the sum of the orders of $A$ and $B$ has real part $<-n$, then $\text{tr}_Q^\alpha ([A,B]) = 0$.

The following technical proposition shows that the canonical and weighted traces as well as the noncommutative residue commute with differentiation on families of operators of constant order, a fact that we will use to derive the multiplicative anomaly of determinants. Differentiable families of symbols and operators are defined in the same way as were holomorphic families in Definitions 3 and 4 replacing holomorphic in the parameter $z$ by differentiable in the parameter $t$.

Proposition 7 Let $A_t$ be a differentiable family of $\text{Cl}(M,E)$ of constant order $\alpha$. 

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1. The noncommutative residue commutes with differentiation

\[
\frac{d}{dt} \text{res}(A_t) = \text{res}(\dot{A}_t),
\]

(3.11)

where we have set \( \dot{A}_t = \frac{d}{dt} A_t \).

2. If the order \( a \) is non integer, the canonical trace commutes with differentiation

\[
\frac{d}{dt} \text{TR}(A_t) = \text{TR}(\dot{A}_t).
\]

(3.12)

3. For any weight \( Q \) with order \( q \) and spectral cut \( \alpha \),

\[
\frac{d}{dt} \text{tr}_t^Q(A_t) = \text{tr}_t^Q(\dot{A}_t).
\]

(3.13)

**Proof:** Using (1.4) we write the symbol \( \sigma_{A_t} \) of \( A_t \) as follows:

\[
\sigma_{A_t}(x, \xi) = \sum_{j=0}^{N-1} \chi(\xi) \left( \sigma_{A_t} \right)_{a-j} (x, \xi) + \left( \sigma_{A_t} \right)_{(N)} (x, \xi).
\]

By assumption, the map \( t \mapsto \sigma_{A_t}(x, \cdot) \) is differentiable leading to a differentiable map \( t \mapsto \int_{S^*_t M} \text{tr}_x \left( \sigma_{A_t} \right)_{-n} \) after integration over the compact set \( S^*_t M \) with derivative: \( t \mapsto \int_{S^*_t M} \text{tr}_x (\dot{\sigma}_{A_t} - n) \), where \( \dot{\sigma}_{A_t} = \sigma_{\dot{A}_t} \) stands for the derivative of \( \sigma_{A_t} \) at \( t \). Thus, the map \( t \mapsto \text{res}(A_t) \) is differentiable with derivative given by (3.11).

2. By (3.4) and (3.5), to prove formula (3.12) we need to check the differentiability of the map \( t \mapsto \int_{T^*_t M} \text{tr}_x \sigma_{A_t} (x, \cdot) \) and to prove that

\[
\frac{d}{dt} \int_{T^*_t M} \text{tr}_x \sigma_{A_t} (x, \cdot) = \int_{T^*_t M} \text{tr}_x \dot{\sigma}_{A_t} (x, \cdot).
\]

The cut-off integral involves the whole symbol which we denote by \( \sigma_t := \sigma_{A_t} \) in order to simplify notations. Since the family \( \sigma_t \) has constant order, \( N \) can be chosen independently of \( t \) in the asymptotic expansion. The corresponding cut-off integral can be computed explicitly (see e.g [PS]):

\[
\int_{T^*_t M} \text{tr}_x (\sigma_t(x, \xi)) \, d\xi = \int_{T^*_t M} \text{tr}_x \left( (\sigma_t)_{(N)} (x, \xi) \right) \, d\xi + \sum_{j=0}^{N-1} \int_{|\xi|\leq 1} \chi(\xi) \text{tr}_x \left( (\sigma_t)_{a-j} (x, \xi) \right) \, d\xi
\]

\[
- \sum_{j=0, a-j+n \neq 0}^{N-1} \frac{1}{a - j + n} \int_{|\xi| = 1} \text{tr}_x \left( (\sigma_t)_{a-j} (x, \omega) \right) \, d\omega.
\]

The map \( t \mapsto \int_{T^*_t M} \text{tr}_x \left( (\sigma_t)_{(N)} (x, \xi) \right) \, d\xi \) is differentiable at any point \( t_0 \) since by assumption the maps \( t \mapsto \text{tr}_x \left( (\sigma_t(x, \xi))_{(N)} \right) \) are differentiable with modulus bounded from above \( \left| \text{tr}_x \left( (\dot{\sigma}_t(x, \xi))_{(N)} \right) \right| \leq C|\xi|^{\text{Re}(a) - N} \) by an \( L^1 \) function.
provided \( N \) is chosen large enough, where the constant \( C \) can be chosen independently of \( t \) in a compact neighborhood of \( t_0 \). Its derivative is given by \( t \mapsto \int_{T^2} \chi \left( (\sigma_t)_{(N)} (x, \xi) \right) d\xi \). The remaining integrals \( \int_{|\xi| \leq 1} \chi(\xi) \left( (\sigma_t)_{a-j} (x, \xi) \right) d\xi \) and \( \int_{|\xi|=1} \left( (\sigma_t)_{a-j} (x, \omega) \right) dS_\omega \) are also differentiable as integrals over compact sets of integrands involving differentiable maps \( t \mapsto \left( (\sigma_t(x, \xi))_{a-j} \right) \).

Their derivatives are given by \( \int_{|\xi| \leq 1} \chi(\xi) \left( (\sigma_t)_{a-j} (x, \xi) \right) d\xi \) and \( \int_{|\xi|=1} \left( (\sigma_t)_{a-j} (x, \omega) \right) dS_\omega \). Thus, \( t \mapsto \int_{T^2} \chi(x, \xi) \left( (\sigma_t)_{a-j} (x, \xi) \right) d\xi \) is differentiable with derivative given by \( \int_{T^2} \chi(x, \xi) \left( (\sigma_t)_{a-j} (x, \xi) \right) d\xi \).

3. By the defect formula \( 3.8 \) we have

\[
\text{tr}^Q_{\alpha}(A_t) = \int_M dx \left( \int_{T^2} \text{tr}_x \sigma_{A_t}(x, \cdot) - \frac{1}{q} \int_{S^1} \text{tr}_x \left( \sigma_{A_t \log_\alpha Q}(x, \cdot) \right)_{-n} \right)
\]

which reduces the proof of the differentiability of \( t \mapsto \text{tr}^Q_{\alpha}(A_t) \) to that of the two maps \( t \mapsto \int_{T^2} \text{tr}_x \sigma_{A_t}(x, \cdot) \) and \( t \mapsto \int_{S^1} \text{tr}_x \left( \sigma_{A_t \log_\alpha Q}(x, \xi) \right)_{-n} \). Differentiability of the first map was shown in the second item of the proof. Let us first investigate the second map. By \( 122 \) we have

\[
(\sigma_{A_t \log_\alpha Q})_{-n} = \sum_{|\alpha| + a - j - k = -n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_x (\sigma_{A_t})_{a-j} \partial^\alpha_x (\sigma_{\log_\alpha Q})_{-k}.
\]

By assumption, the maps \( t \mapsto (\sigma_{A_t})_{a-j} \) are differentiable so that \( t \mapsto \int_{S^1} \text{tr}_x \left( \sigma_{A_t \log_\alpha Q} \right)_{-n} \) is differentiable with derivative

\[
t \mapsto \int_{S^1} \text{tr}_x \left( \sigma_{A_t \log_\alpha Q} \right)_{-n} = \int_{S^1} \text{tr}_x \left( \sigma_{A_t \log_\alpha Q} \right)_{-n}.
\]

Integrating over the compact manifold \( M \) then yields that the map \( t \mapsto \text{tr}^Q_{\alpha}(A_t) \) is differentiable with derivative given by

\[
\int_M dx \left( \int_{T^2} \text{tr}_x \sigma_{A_t}(x, \cdot) - \frac{1}{q} \int_{S^1} \text{tr}_x \left( \sigma_{A_t \log_\alpha Q}(x, \cdot) \right) \right) = \text{tr}^Q_{\alpha}(A_t).
\]

\[
\square
\]

4 Locality of weighted traces of \( L(A, B) \)

Combining \( 2.11 \) with \( 3.10 \) yields the locality of weighted traces \( \text{tr}^Q(L(A, B)) \) as a finite sum of noncommutative residues, independently of the choice of spectral cut.

In this section we show that weighted traces of \( L(A, B) \) only depend on a finite number of homogeneous components of the operators \( A \) and \( B \) (see Theorem \( 1 \)), a fact reminiscent of a similar property observed by Okikiolu in \( 10 \) in the case of operators with scalar leading symbols.
Lemma 1 Let \( A_t \) be a differentiable family of admissible operators in \( \mathcal{C}C(M, E) \) with constant spectral cut \( \alpha \). For any positive integer \( K \) we have

\[
\frac{d}{dt} \log \alpha A_t = \hat{A}_t A_t^{-1} + \sum_{k=1}^{K} \text{ad}^k_{A_t}(\hat{A}_t) A_t^{-(k+1)} + R_K(A_t, \hat{A}_t)
\]

where we have set \( \hat{A}_t := \frac{d}{dt} A_t \) and

\[
R_K(A_t, \hat{A}_t) := \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z \left( (\lambda - A_t)^{-1}, \text{ad}^K_{A_t}(\hat{A}_t) (\lambda - A_t)^{-(K+1)} \right) d\lambda \right)_{\lambda = 0}
\]

(4.14)

since \( A_t \) commutes with \( (\lambda - A_t)^{-k} \).

Here \( \Gamma \) is a contour around the spectrum as in (2.7).

If \( A_t \) commutes with \( \hat{A}_t \) then \( \frac{d}{dt} \log \alpha A_t = \hat{A}_t A_t^{-1} \). If \( A_t \) has constant order \( a \) then \( \frac{d}{dt} \log \alpha A_t \) lies in \( \mathcal{C}C^0(M, E) \).

**Proof:** Since the spectral cut is constant, we drop it in the notation setting \( \log \alpha A_t = \log \alpha A_t \).

Since \( \log \alpha A_t \) has a symbol of the form \( \sigma_t \sim a_t \log |\xi| + \sigma_{0,t}(x, \xi) \) where \( a_t \) is the order of \( A_t \) and \( \sigma_{0,t} \) a symbol of order 0, if \( a_t = a \) is constant, its derivative \( \frac{d}{dt} \log \alpha A_t \), has a symbol of the form \( \frac{d}{dt} \sigma_t \sim \frac{da}{dt} \sigma_{0,t}(x, \xi) \) and is therefore classical of zero order.

Going back to the general situation and assuming differentiability of the family at 0, we derive the formula for the derivative at 0; a change of variable \( t \to t - t_0 \) then yields the corresponding result at any point \( t_0 \) of differentiability. We set \( \Delta_t := \frac{A_t - A_0}{t} \) which tends to \( \hat{A}_0 \) as \( t \) tends to zero. We observe that \((\lambda - A_0)(\lambda - A_t)^{-1} = (\lambda - A_t)(\lambda - A_t)^{-1} + t \Delta_t(\lambda - A_t)^{-1} = 1 + t \Delta_t(\lambda - A_t)^{-1} \), from which we infer that

\[
(\lambda - A_t)^{-1} = (\lambda - A_0)^{-1} + t(\lambda - A_0)^{-1} \Delta_t(\lambda - A_t)^{-1}.
\]

By induction we get

\[
(\lambda - A_t)^{-1} = (\lambda - A_0)^{-1} + \sum_{p=1}^{P} t^p (\lambda - A_0)^{-1} \Delta_t \cdots \Delta_t(\lambda - A_0)^{-1} \Delta_t(\lambda - A_0)^{-1} + t^{P+1} R_P(A_0, \Delta_t, \lambda),
\]

where \( R_P(A_0, \Delta_t, \lambda) \) stands for the remainder term. It follows that

\[
\lim_{t \to 0} \frac{A_t - \lambda)^{-1} - (A_0 - \lambda)^{-1}}{t} = (\lambda - A_0)^{-1} \lim_{t \to 0} \Delta_t (\lambda - A_0)^{-1} = (\lambda - A_0)^{-1} \hat{A}_0(\lambda - A_0)^{-1}.
\]

For \( t \) in a compact neighborhood of 0, we can choose a common contour \( \Gamma \) along a spectral ray around the spectrum of \( A_t \) as in (2.1). Hence,

\[
\frac{d}{dt} \log(A_t) = \lim_{t \to 0} \frac{\log(A_t) - \log(A_0)}{t} = \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z \frac{d}{dt} \log(A_t - \lambda)^{-1} d\lambda \right)_{\lambda = 0}
\]

\[
\frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z (\lambda - A_0)^{-1} \Delta_0(\lambda - A_0)^{-1} d\lambda \right)_{\lambda = 0}.
\]
On the other hand,
\[
\left(\lambda - A_0\right)^{-1} \hat{A}_0 = (\lambda - A_0)^{-1} [A_0, \hat{A}_0] (\lambda - A_0)^{-1} \\
= [A_0, \hat{A}_0] (\lambda - A_0)^{-2} + \text{ad}^2_{\hat{A}_0} (\hat{A}_0) (\lambda - A_0)^{-3} + \left(\lambda - A_0\right)^{-1}, \text{ad}^2_{\hat{A}_0} (\hat{A}_0) (\lambda - A_0)^{-2}
\]
\[
= \sum_{k=1}^{K} \text{ad}^k_{\hat{A}_0} (\hat{A}_0) (\lambda - A_0)^{-(k+1)} + \left(\lambda - A_0\right)^{-1}, \text{ad}^k_{\hat{A}_0} (\hat{A}_0) (\lambda - A_0)^{-K},
\]

Hence,
\[
\frac{d}{dt}_{t=0} \log(A_t) = \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z (\lambda - A_0)^{-1} \hat{A}_0 (\lambda - A_0)^{-1} d\lambda \right)_{|z=0}
\]
\[
= \hat{A}_0 \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z (\lambda - A_0)^{-2} d\lambda \right)_{|z=0}
\]
\[
+ \sum_{k=1}^{K} \text{ad}^k_{\hat{A}_0} (\hat{A}_0) \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z (\lambda - A_0)^{-(k+1)} d\lambda \right)_{|z=0}
\]
\[
+ \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z \left(\lambda - A_0\right)^{-1}, \text{ad}^K_{\hat{A}_0} (\hat{A}_0) \right) (\lambda - A_0)^{-(K+1)} d\lambda \right)_{|z=0}.
\]

Iterated integrations by parts yield
\[
\frac{d}{dt}_{t=\alpha} \log(A_t) = \hat{A}_0 A_0^{-1} + \sum_{k=1}^{K} \text{ad}^k_{\hat{A}_0} (\hat{A}_0) A_0^{-1} (k+1) + R_K(A_0, \hat{A}_0),
\]
where \( R_K(A_0, \hat{A}_0) = \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma} \lambda^z \left[\lambda - A_0\right], \text{ad}^K_{\hat{A}_0} (\hat{A}_0) \right) (\lambda - A_0)^{-(K+1)} d\lambda \right)_{|z=0} \)

is the remainder term. \( \square \)

The following result is reminiscent of an observation made in [O1] (see also [Sc]), namely that only the first \( n \) homogeneous components of the symbols come into play for the derivation of the Campbell-Hausdorff formula for operators with scalar leading symbols; the weighted trace of \( L(A, B) \) presents a similar feature in our more general situation.

**Theorem 1** Given a weight \( Q \) and two admissible operators \( A \) and \( B \) in \( \text{Cl}(M, E) \) with non negative orders, the weighted trace \( \text{tr}^Q(L(A, B)) \) is a local expression as a finite sum of noncommutative residues, which only depends on the first \( n \) homogeneous components of the symbols of \( A \) and \( B \):
\[
\frac{d}{dt} \text{tr}^Q(L(A(1 + tS), B)) = \frac{d}{dt} \text{tr}^Q(L(A, B(1 + tS))) = 0 \ \ \forall S \in \text{Cl}^{< \alpha}(M, E), \quad (4.15)
\]
where \( \text{Cl}^{< \alpha}(M, E) = \cup_{\text{Re}(\alpha) < -n} \text{Cl}^{\alpha}(M, E) \) stands for the algebra of classical operators of order with real part \( < -n \).

**Proof:**

- On the one hand we know that \( L(A, B) \) is a finite sum of commutators of classical pseudodifferential operators \( [P_j, Q_j] \). By [4.11], each weighted trace \( \text{tr}^Q([P_j, Q_j]) \) is proportional to \( \text{res}(Q_j [P_j, \log_{\alpha} Q]) \) so that \( \text{tr}^Q(L(A, B)) \) is indeed a finite sum of noncommutative residues.
• Let us further observe that the proof of (4.15) reduces to the proof at 

Let us check that requirement (4.15) is equivalent to the fact that $\text{tr}^Q(L(A, B))$ only depends on the first $n$ homogeneous components of the symbols of $A$ and $B$.

Given an operator $S$ in $\mathcal{Cl}(M, E)$ of order $<-n$ and an operator $A$ in $\mathcal{Cl}(M, E)$ of order $a$, we first observe that in any local trivialisation the first $n$ homogeneous components of the symbols of $A$ and $A(1+S)$ coincide since $AS$ has order $a-n$.

Conversely, if the first $n$ homogeneous components of the symbols of two classical operators $A$ and $B$ of orders $a$ and $b$ coincide, then $a = b$ and if $B$ is invertible, the first $n$ homogeneous components of the symbol of $B^{-1}$ defined inductively using (1.2) by:

$$(\sigma_{B^{-1}})_{-b} = ((\sigma_B)_b)^{-1},$$

$$(\sigma_{B^{-1}})_{-b-j} = -((\sigma_B)_b)^{-1} \sum_{k+l+|\alpha|=j, l<j} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_x^k (\sigma_B)_{b-k} \partial^{\alpha}_x (\sigma_{B^{-1}})_{b-l},$$

coincide with that of the symbol of $A^{-1}$ since the terms corresponding to $j \leq n$ only involve homogeneous components $(\sigma_B)_{b-k} = (\sigma_A)_{a-k}$ and $(\sigma_{B^{-1}})_{b-l}$ with $k$ and $l$ no larger than $n$. Consequently, by (1.2) it follows that $S = A^{-1}B$ has order $<-n$. Thus, showing that the expression $\text{tr}^Q(L(A, B))$ only depends on the first $n$ homogeneous components of $A$ amounts to showing that $\text{tr}^Q(L(A + S, B)) = \text{tr}^Q(L(A, B))$ for any classical operator $S$ of order $<-n$.

• Let us further observe that the proof of (4.15) reduces to the proof at $t_0 = 0$. Indeed for any real number $t_0$, we have

$$\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q(L(C(1+tT), D) = 0 \quad \forall T \in \mathcal{Cl}^{<n}(M, E), \quad \forall \text{ admissible } C, D$$

$$\implies \frac{d}{dt} \bigg|_{t=t_0} \text{tr}^Q(L(A(1+tS), B) = 0 \quad \forall S \in \mathcal{Cl}^{<n}(M, E), \quad \forall \text{ admissible } A, B.$$}

To check this implication, we set $u = t - t_0$ so that $1 + tS = 1 + t_0S + uS = (1 + uS)(1 + t_0S)^{-1}(1 + t_0S)$. Setting $T = S(1 + t_0S)^{-1}$ which also lies in $\mathcal{Cl}^{<n}(M, E)$, we have

$$L(A(1+tS), B) - L(A(1+uT), (1+t_0S)B)$$

$$= -\log(A(1+uT)(1+t_0S)) - \log(B) + \log A(1+uT) + \log((1+t_0S)B)$$

$$= L(1+t_0S, B) - L(A(1+uT), 1+t_0S),$$

and hence

$$L(A(1+tS), B) = L(A(1+uT), (1+t_0S)B) + L(1+t_0S, B) - L(A(1+uT), 1+t_0S).$$

Differentiating w.r.t. $t$ at $t = t_0$ on the l.h.s boils down to differentiating the r.h.s. at $u = 0$ and the implication (1.12) then easily follows.

• We are therefore left to prove that $\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q(L(A(1+tS), B) = 0$. Applying (3.13) to the operator $A_t := L(A(1+tS), B)$ we have

$$\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q(L(A(1+tS), B) = \text{tr}^Q \left( \frac{d}{dt} \bigg|_{t=0} (L(A(1+tS), B)) \right).$$

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We therefore need to investigate the behaviour of \( \frac{L(A(1+tS), B) - L(A, B)}{t} \) as \( t \to 0 \).

Since

\[ L(A(1+tS), B) - L(A, B) = \log(A(1+tS)B) - \log(AB) - (\log(A(1+tS)) - \log A), \]

let us study the difference \( \log(A(1+tS)C) - \log(AC) \) with \( C \) equal to either \( B \) or the identity operator. Let us apply Lemma 1 to \( A_t := A(1+tS)C \) so that \( A_0 = AC \). When \( t \) varies in a small compact neighborhood of 0, the operators \( A_t \) have a common spectral cut \( c \) which we drop in the notation. Implementing the weighted trace \( \text{tr}^Q \) yields

\[
\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q \left( \log(A(1+tS)C) \right)
\]

\[
= \text{tr}^Q(AC (AC)^{-1}) + \sum_{k=1}^{K} \text{tr}^Q(\text{ad}^k_{AC}(AC)^{-(k+1)}) + \text{tr}^Q(R_K(AC, ASC))
\]

for arbitrary large \( K \) and with remainder term

\[
R_K(AC, ASC) := \frac{d}{dz} \left( \int_{\Gamma_{\alpha}} \lambda^z \left[ (\lambda - AC)^{-1}, \text{ad}^K_{AC}(ASC) \right] (\lambda - AC)^{-(K+1)} d\lambda \right) \bigg|_{z=0}.
\]

But for any positive integer \( k \), by (3.10) we have

\[
\text{tr}^Q(\text{ad}^k_{AC}(ASC)^{-(k+1)}) = \text{tr}^Q \left( \text{ad}_{AC}(\text{ad}^{k-1}_{AC}(ASC)) (AC)^{-(k+1)} \right)
\]

\[
= \text{tr}^Q \left( \text{ad}_{AC} \left( \text{ad}^{k-1}_{AC}(ASC)(AC)^{-(k+1)} \right) \right)
\]

\[
= - \frac{1}{q} \text{res} \left( \text{ad}^{k-1}_{AC}(ASC)(AC)^{-(k+1)} [AC, \log Q] \right)
\]

\[
= 0.
\]

Here we use the fact that the operator \( \text{ad}^{k-1}_{AC}(ASC)(AC)^{-(k+1)} [AC, \log Q] \) has order \( (k-1)(a+c) + a + c + s - (k+1)(a+c) + a + c = s \) (here \( s \) is the order of \( S \), \( a \) the order of \( A \), \( c \) the order of \( C \) with real part smaller than \(-n\)). Thus

\[
\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q \left( \log(A(1+tS)C) \right) = \text{tr}^Q \left( AC (AC)^{-1} \right) + \text{tr}^Q \left( R_K(AC, ASC) \right),
\]

independently of the choice of the integer \( K \). The remainder term \( \text{tr}^Q(R_K(AC, ASC)) \) depends on \( S \) via the iterated brackets \( \text{ad}^K_{AC}(ASC) \) and hence via \( K \). Since it is independent of \( K \), it is also independent of \( S \). Setting \( S = 0 \) which lies in \( C^{t<\alpha}(M, E) \), we infer that \( \text{tr}^Q(R_K(AC, ASC)) \) vanishes for all positive integers \( K \). Thus

\[
\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q \left( \log(A(1+tS)C) \right) = \text{tr}^Q \left( ASC (AC)^{-1} \right) = \text{tr}^Q(ASA^{-1}),
\]

independently of \( C \). Setting back \( C = B \) and \( C = I \) yields

\[
\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q(L(A(1+tS), B)) = \text{tr}^Q \left( \frac{d}{dt} \bigg|_{t=0} \log(A(1+tS)B) \right) - \text{tr}^Q \left( \frac{d}{dt} \bigg|_{t=0} \log(A(1+tS)) \right)
\]

\[
= 0
\]

thus ending the proof of the Theorem.

\[ \square \]
5 A local formula for the weighted trace of \( L(A, B) \)

We derive an explicit local expression for the weighted traces \( \text{tr}^A(L(A, B)) \) and \( \text{tr}^B(L(A, B)) \) of \( L(A, B) \) (see Theorem 2). Our approach is inspired by the Okikiolu’s proof for the Campbell-Hausdorff formula for operators with scalar leading symbols. In the case of operators with scalar leading symbols, as it was noticed and used by Okikiolu, as from a certain order in the Campbell-Hausdorff expansion, one can implement ordinary traces since the iterated brackets have decreasing order. In our more general situation, such a phenomenon does not occur so that we use weighted traces instead.

**Proposition 8** Let \( A \) and \( B \) be two admissible operators with positive orders \( a \) and \( b \) in \( \mathcal{C}(M, E) \) such that their product \( AB \) is also admissible. We have the following identities for weighted traces:

\[
\frac{d}{dt} \bigg|_{t=0} \text{tr}^B(L(A_t, B^\mu)) = 0, \quad \frac{d}{dt} \bigg|_{t=0} \text{tr}^A(L(A^t, B^\mu)) = 0
\]

as well as for the noncommutative residue:

\[
\frac{d}{dt} \bigg|_{t=0} \text{res}(L(A_t, B^\mu)) = 0.
\]

**Proof:** Let us prove the result for the \( B \)-weighted trace; a similar proof yields the result for the \( A \)-weighted trace. By Proposition 7, weighted traces and the residue commute with differentiation on constant order operator so that

\[
\frac{d}{dt} \bigg|_{t=0} \text{tr}^Q(L(A^t, B^\mu)) = \text{tr}^Q \left( \frac{d}{dt} \bigg|_{t=0} L(A^t, B^\mu) \right)
\]

resp.

\[
\frac{d}{dt} \bigg|_{t=0} \text{res}(L(A^t, B^\mu)) = \text{res} \left( \frac{d}{dt} \bigg|_{t=0} (L(A^t, B^\mu)) \right).
\]

But

\[
\frac{d}{dt} \bigg|_{t=0} L(A^t, B^\mu) = \frac{d}{dt} \bigg|_{t=0} \log(A^t B^\mu) - \frac{d}{dt} \bigg|_{t=0} \log A^t.
\]

We therefore apply Lemma 1 to \( A_t := A^t B^\mu \) so that \( A_0 = B^\mu \), including the case \( \mu = 0 \) for which \( A_t = A^t \) and \( A_0 = I \). Since \( A_0 = \log A B^\mu \) and \( A_0 A_0^{-1} = \log A \), implementing the weighted trace \( \text{tr}^B \) yields

\[
\frac{d}{dt} \bigg|_{t=0} \text{tr}^B(\log(A^t B^\mu))
\]

\[
= \text{tr}^B(\log A) + \sum_{k=1}^{K} \text{tr}^B(\text{ad}_{B^\mu}^k(\log A B^\mu) B^{-\mu(k+1)}) + \text{tr}^B(R_K(B^\mu, \log A B^\mu))
\]

for arbitrary large \( K \), with remainder term

\[
R_K(B^\mu, \log A B^\mu) = \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma_\alpha} \lambda^z \left[ (\lambda - B^\mu)^{-1}, \text{ad}_{B^\mu}^K(\log A B^\mu) \right] (\lambda - B^\mu)^{-(K+1)} d\lambda \right)_{z=0}
\]

\[
= \text{ad}_{B^\mu}^K \left( \frac{d}{dz} \left( \frac{i}{2\pi} \int_{\Gamma_\alpha} \lambda^z \left[ (\lambda - B^\mu)^{-1}, \log A B^\mu \right] (\lambda - B^\mu)^{-(K+1)} d\lambda \right)_{z=0} \right).
\]
since $B$ commutes with $B^\mu$.

For any positive integer $k$, by (3.10) we have

$$\text{tr}^B(\text{ad}_{B^\mu}^k (A B^\mu) B^{-\mu(k+1)}) = \text{tr}^B(\text{ad}_{B^\mu} (\text{ad}_{B^\mu}^{k-1} (A B^\mu) B^{-\mu(k+1)})) = \text{tr}^B(\text{ad}_{B^\mu}^{k-1} (A B^\mu) B^{-\mu(k+1)} [B^\mu, \log B]) = 0,$$

since $B$ commutes with $\log B$. A similar computation shows that $\text{tr}^B(R_K(B^\mu, \log A B^\mu)) = 0$. Thus

$$\frac{d}{dt}_{|t=0} \text{tr}^B(\log(A^t B^\mu)) = \text{tr}^B(\log A).$$

It follows that $\frac{d}{dt}_{|t=0} \text{tr}^B(\log(A^t B^\mu)) = \text{tr}^B(\log A)$ independently of $\mu$ so that

$$\frac{d}{dt}_{|t=0} \text{tr}^B(L(A^t, B^\mu)) = 0.$$

Similarly, replacing the weighted trace $\text{tr}^B$ by the noncommutative residue $\text{res}$ and using the cyclicity of the noncommutative residue, yields

$$\frac{d}{dt}_{|t=0} \text{res}(L(A^t, B^\mu)) = 0.$$

The following statement provides a local formula for the multiplicative anomaly of the zeta determinant. It also shows that the residue of $L(A, B)$ vanishes and therefore yields back the multiplicativity of the residue determinant derived in [Sc].

**Theorem 2** For two admissible operators $A, B \in \text{Cl}(M, E)$ with positive orders $a$ and $b$ such that their product $AB$ is also admissible, we have

$$\text{res}(L(A, B)) = 0.$$

Moreover, there is an operator

$$W(\tau)(A, B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B)$$

in $\text{Cl}^0(M, E)$ depending continuously on $\tau$ such that

$$\text{tr}^Q(L(A, B)) = \int_0^1 \text{res}(W(\tau)(A, B) \left( \frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q} \right)) d\tau$$

where $Q$ is any weight of order $q$.

**Proof:** By Proposition we know that $\frac{d}{dt}_{|t=0} \text{res}(L(A^t, B)) = \frac{d}{dt}_{|t=0} \text{tr}^Q(L(A^t, B)) = 0$. We want to compute $\frac{d}{dt}_{|t=0} \text{res}(L(A^t, B)) = \frac{d}{dt}_{|t=0} \text{res}(L(A^{t+\tau}, B))$ and $\frac{d}{dt}_{|t=0} \text{tr}^Q(L(A^t, B)) = \frac{d}{dt}_{|t=0} \text{tr}^Q(L(A^{t+\tau}, B))$. For this we observe that

$$L(AB, D) - L(A, BD) = -\log(AB) - \log(D) + \log A + \log(BD) = L(B, D) - L(A, B)$$
Replacing $A$ by $A^t$, $B$ by $A^r$ and $D$ by $B$, we get
\[ L(A^{t+r}, B) - L(A^t, A^r B) = L(A^r, B) - L(A^t, A^r) = L(A^r, B). \]
Implementing the noncommutative residue, by Proposition 8 we have:
\[
\frac{d}{dt} \big|_{t=0} \text{res}(L(A^t, B)) = \frac{d}{dt} \big|_{t=0} \text{res}(L(A^{t+r}, B)) \\
= \frac{d}{dt} \big|_{t=0} \text{res}(L(A^t, A^r B)) \\
= 0.
\]
Hence
\[
\text{res}(L(A, B)) = \int_0^1 \frac{d}{dt} \big|_{t=\tau} \text{res}(L(A^t, B)) d\tau + \text{res}(L(I, B)) = 0, \tag{5.19}
\]
since $L(I, B) = 0$.
If instead we implement the weighted trace $\text{tr}^Q$, we have:
\[
\frac{d}{dt} \big|_{t=\tau} \text{tr}^Q(L(A^t, B)) = \frac{d}{dt} \big|_{t=0} \text{tr}^Q(L(A^{t+r}, B)) \\
= \frac{d}{dt} \big|_{t=0} \text{tr}^Q(L(A^t, A^r B)) \\
= \frac{d}{dt} \big|_{t=0} \left( \text{tr}^Q(L(A^t, A^r B)) - \text{tr}^{A^r B}(L(A^t, A^r B)) \right)
\]
Since $A$ and $B$ have positive order so has $A^r B$, so that applying Proposition 8 with weighted traces $\text{tr}^{A^r B}$ yields:
\[
\frac{d}{dt} \big|_{t=\tau} \text{tr}^Q(L(A^t, B)) = \frac{d}{dt} \big|_{t=0} \text{tr}^Q(L(A^t, A^r B)) \\
= \frac{d}{dt} \big|_{t=0} \text{tr}^{A^r B}(L(A^t, A^r B)) \\
+ \frac{d}{dt} \big|_{t=0} \left( \text{tr}^Q(L(A^t, A^r B)) - \text{tr}^{A^r B}(L(A^t, A^r B)) \right)
\]
Applying (3.3) to $Q_1 = Q$ and $Q_2 = A^r B$, we infer that
\[
\frac{d}{dt} \big|_{t=0} \left( \text{tr}^Q(L(A^t, A^r B)) - \text{tr}^{A^r B}(L(A^t, A^r B)) \right) \\
= \frac{d}{dt} \big|_{t=0} \text{res} \left( L(A^t, A^r B) \left( \frac{\log(A^r B)}{a + b} - \frac{\log Q}{q} \right) \right) \\
= \text{res} \left( W(\tau)(A, B) \left( \frac{\log(A^r B)}{a + b} - \frac{\log Q}{q} \right) \right),
\]
where $q$ is the order of $Q$ and where we have set $W(\tau)(A, B) := \frac{d}{dt} \big|_{t=0} L(A^t, A^r B)$.
Since $L(I, B) = 0$, we finally find that
\[
\text{tr}^Q(L(A, B)) = \text{tr}^Q(L(A^1, B)) - \text{tr}^Q(L(A^0, B)) \\
= \int_0^1 \text{res} \left( W(\tau)(A, B) \left( \frac{\log(A^r B)}{a + b} - \frac{\log Q}{q} \right) \right) d\tau. \tag{5.20}
\]
6 Multiplicative anomaly for determinants revisited

We first observe that the multiplicative anomaly for weighted determinants studied in [D] has logarithm given by the weighted trace of \( L(A, B) \), as a result of which it is local. We then derive an explicit local formula for the multiplicative anomaly of \( \zeta \) determinants, using the local formula derived previously for weighted traces of \( L(A, B) \).

An admissible operator \( A \in C\ell(M, E) \) with spectral cut \( \theta \) and positive order has well defined \( Q \)-weighted determinant [D] (see also [FrG]) where \( Q \in C\ell(M, E) \) is a weight with spectral cut \( \alpha \):

\[
\det_Q^\alpha(A) := e^{\text{tr}_Q^\alpha(\log \theta A)}.
\]

Here the weighted trace has been extended to logarithms as before, picking out the constant term of the meromorphic map \( z \mapsto \text{TR}(\log \theta A Q^{-z}) \) which can have double poles in contrast to the case of classical operators studied in Section 3.

Remark 5 The weighted determinant, as well as being dependent on the choice of spectral cut \( \theta \), also depends on the choice of spectral cut \( \alpha \).

Since the weighted trace restricts to the ordinary trace on trace-class operators, this determinant, as the \( \zeta \)-determinant, extends the ordinary determinant on operators in the determinant class.

Lemma 2 Let \( 0 \leq \theta < \phi < 2\pi \) be two spectral cuts for the admissible operator \( A \). If there is a cone \( \Lambda_{\theta, \phi} \) (see 2.3) which does not intersect the spectrum of the leading symbol of \( A \) then

\[
\det_Q^\phi(A) = \det_Q^\theta(A).
\]

Proof: Under the assumptions of the proposition, the cone \( \Lambda_{\phi, \theta} \) defined as in Proposition 1 contains only a finite number of points in the spectrum of \( A \) so that \( \log_\phi A - \log_\theta A = 2i\pi \Pi_{\phi, \theta}(A) \) is a finite rank operator and hence smoothing. Hence,

\[
\frac{\det_Q^\phi(A)}{\det_Q^\theta(A)} = e^{\text{tr}_Q^\phi(A - \log_\phi A)} = e^{\text{tr}_Q^\phi(2i\pi \Pi_{\phi, \theta}(A))} = e^{2i\pi \text{tr}(\Pi_{\phi, \theta}(A))} = e^{2i\pi \text{rk}(\Pi_{\phi, \theta}(A))} = 1,
\]

where \( \text{rk} \) stands for the rank. \( \Box \)

The multiplicative anomaly for \( Q \)-weighted determinants of two admissible operators \( A, B \) with spectral cuts \( \theta, \phi \) such that \( AB \) has spectral cut \( \psi \) is defined by:

\[
\mathcal{M}_{\theta, \phi, \psi}^Q(A, B) := \frac{\det_Q^\psi(AB)}{\det_Q^\theta(A) \det_Q^\phi(B)},
\]

which we write \( \mathcal{M}^Q(A, B) \) for simplicity.
Proposition 9 Let $A$ and $B$ be two admissible operators with spectral cuts $\theta$ and $\phi$ in $[0,2\pi]$ such that there is a cone delimited by the rays $L_\theta$ and $L_\phi$ which does not intersect the spectra of the leading symbols of $A$, $B$ and $AB$. Then the product $AB$ is admissible with a spectral cut $\psi$ inside that cone and for any weight $Q$ with spectral cut, dropping the explicit mention of the spectral cuts we have:

$$\log \mathcal{M}^Q(A, B) = \int_0^1 \text{res} \left( W(\tau)(A, B) \left( \frac{\log(A^TB) - \log DET Q}{\alpha \tau + b} \right) \right) d\tau. \quad (6.1)$$

Weighted determinants are multiplicative on commuting operators.

Proof: Since the leading symbol of the product $AB$ has spectrum which does not intersect the cone delimited by $L_\theta$ and $L_\phi$, the operator $AB$ only has a finite number of eigenvalues inside that cone. We can therefore choose a ray $\psi$ which avoids both the spectrum of the leading symbol of $AB$ and the eigenvalues of $AB$. By the above lemma, the weighted determinants $\det Q^Q(A)$, $\det Q^Q(B)$ and $\det Q^Q(AB)$ do not depend on the choices of spectral cuts satisfying the requirements of the proposition.

Since

$$\log \mathcal{M}^Q(A, B) = \log \det Q^Q(AB) - \log \det Q^Q(A) - \log \det Q^Q(B) = \text{tr}^Q(L(A, B)),$$

the logarithm of the multiplicative anomaly for weighted determinants is a local quantity $\log Q$ derived in Theorem 2.

To prove the second part of the statement we observe that

$$[A, B] = 0 \implies L(A, B) = 0. \quad (6.2)$$

Indeed, let $\Gamma$ be a contour as in formula (3.13) along a spectral ray around the spectrum of $A^{t_0}B$ for some fixed $t_0$, then

$$\frac{d}{dt}|_{t=t_0} \log(A^tB) = \frac{i}{2\pi} \int_\Gamma \log \lambda \frac{d}{dt}|_{t=t_0} (A^tB - \lambda)^{-1} d\lambda = \frac{i}{2\pi} \int_\Gamma \log \lambda (A^{t_0}B - \lambda)^{-1} \log A A^{t_0}B (A^{t_0}B - \lambda)^{-1} d\lambda = \log A A^{t_0}B \frac{i}{2\pi} \int_\Gamma \log \lambda (A^{t_0}B - \lambda)^{-2} d\lambda \quad \text{since} \quad [A, B] = 0 = - \log A A^{t_0}B (A^{t_0}B)^{-1} = - \log A.$$

Similarly, we have $\frac{d}{dt}|_{t=t_0} \log(A^t) = - \log A$ so that finally $\frac{d}{dt}|_{t=t_0} \log(A^tB) = \frac{d}{dt}|_{t=t_0} \log(A^tB) - \frac{d}{dt}|_{t=t_0} \log(A^t)$ vanishes. It follows that $L(A, B) = \int_0^1 \frac{d}{dt}|_{t=t_\tau} L(A^t, B) d\tau = 0.$

Since $L(A, B)$ vanishes when $A$ and $B$ commute, weighted determinants are multiplicative on commuting operators. $\square$

Let us now turn to the multiplicative anomaly for $\zeta$-determinants, relating it to weighted traces of $L(A, B)$. An admissible operator $A \in \mathcal{C}(M, E)$ with spectral cut $\theta$ and positive order has well defined $\zeta$-determinant:

$$\det_{\zeta, \theta}(A) := e^{-\zeta_{A, \theta}(0)} = e^{\text{tr}_\theta(\log \sigma A)}$$
since $\zeta_{A,\theta}(z) := \text{TR}(A_\theta^{-z})$ is holomorphic at $z = 0$. In the second equality, the weighted trace has been extended to logarithms as before, picking out the constant term of the meromorphic map $z \mapsto \text{TR}({\log_\theta A}^{Q^{-z}})$ (which can have double poles) with the notations of section 2.

Recall from [PS] that

$$\log \det_{\zeta,\theta}(A) = \int_M dx \left[ \text{TR}_x({\log_\theta A} - \frac{1}{2\pi} \text{res}_x({\log_\theta^2 A})) \right]$$

(6.3)

where $a$ is the order of $A$ and where $\text{res}_x$ is the noncommutative residue density extended to log-polyhomogeneous operators defined previously. This expression corresponds to minus the coefficient in $z$ of the Laurent expansion of $\text{TR}(A^{-z})$.

The $\zeta$-determinant generally depends on the choice of spectral cut. However, it is invariant under mild changes of spectral cut in the following sense.

**Lemma 3** Let $0 \leq \theta < \phi < 2\pi$ be two spectral cuts for the admissible operator $A$. If there is a cone $\Lambda_{\theta,\phi}$ (see 2.3) which does not intersect the spectrum of the leading symbol of $A$ then

$$\det_{\zeta,\theta}(A) = \det_{\zeta,\phi}(A).$$

**Proof:** By (6.3), and since $\log_\phi A - \log_\theta A = 2\pi i \Pi_{\theta,\phi}(A)$ is a finite rank operator and hence smoothing under the assumptions of the proposition, we have

$$\frac{\det_{\zeta,\phi}(A)}{\det_{\zeta,\theta}(A)} = e^{\int_M dx \left[ \text{TR}_x({\log_\phi A} - \frac{1}{2\pi} \text{res}_x({\log_\phi^2 A})) - \int_M dx \left[ \text{TR}_x({\log_\theta A} - \frac{1}{2\pi} \text{res}_x({\log_\theta^2 A})) \right] \right]}$$

$$= e^{\int_M dx \left[ \text{TR}_x({\log_\phi A} - \log_\theta A) - \frac{1}{2\pi} \text{res}_x({\log_\phi^2 A} - {\log_\theta^2 A})) \right]}$$

$$= e^{\int_M dx \left[ \text{TR}_x(2\pi \Pi_{\theta,\phi}(A)) - \frac{1}{2\pi} \text{res}_x((\log_\phi A + \log_\theta A) 2\pi \Pi_{\theta,\phi}(A)) \right]}$$

$$= e^{2\pi i \text{tr}(\Pi_{\theta,\phi}(A))} - \frac{2\pi i}{2\pi} \text{res}_x((\log_\phi A + \log_\theta A) \Pi_{\theta,\phi}(A))$$

$$= e^{2\pi i \text{tr}(\Pi_{\theta,\phi}(A))}$$

$$= 1,$$

where we have used the fact that the noncommutative residue vanishes on smoothing operators on which the canonical trace coincides with the usual trace on smoothing operators. □

The $\zeta$-determinant is not multiplicative. Indeed, let $A$ and $B$ be two admissible operators with positive order and spectral cuts $\theta$ and $\phi$ and such that $AB$ is also admissible with spectral cut $\psi$. The multiplicative anomaly

$$M_{\zeta}^{\theta,\phi,\psi}(A,B) := \frac{\det_{\zeta,\psi}(AB)}{\det_{\zeta,\theta}(A) \det_{\zeta,\phi}(B)},$$

was proved to be local, independently by Okikiolu [O2] for operators with scalar leading symbol and by Kontsevich and Vishik [KV] for operators “close to identity” (see the introduction for a more detailed historical account).

For simplicity, we drop the explicit mention of $\theta, \phi, \psi$ and write $M_{\zeta}(A,B)$.

Even though the operator $\log_\theta^2 A$ is not classical we have the following useful property.

7It was shown in [LP] that all multiplicative determinants on elliptic operators can be built from two basic types of determinants; they do not include the $\zeta$-determinant.

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Lemma 4 Let $A, B$ be admissible operators in $\text{Cl}(M, E)$ with positive orders $a, b$ and spectral cuts $\theta$ and $\phi$ respectively and such that $AB$ (which is elliptic) is also admissible with spectral cut $\psi$. Then

$$K(A, B) := \frac{1}{2(a + b)} \log_2^2 AB - \frac{1}{2a} \log_2^2 A - \frac{1}{2b} \log_2^2 B$$

has a symbol of the form

$$\sigma_K \sim \ln|\xi|(\sigma_0^{AB} - \sigma_0^A - \sigma_0^B) + \sigma_0^K$$

for some zero order classical symbol $\sigma_0^K$ and where we have written $\sigma_{\log A}(x, \xi) = a \ln|\xi|I + \sigma_0^A(x, \xi)$ for an admissible operator $A$ of order $a$.

In particular, both operators $L(A, B) \log_2^a A - K(A, B)$ and $L(A, B) \log_2^b B - K(A, B)$ are classical operators of zero order.

Proof: By formula (2.5), another choice of spectral cut only changes the logarithms by adding an operator in $\text{Cl}^0(M, E)$ so that it will not affect the statement. As usual, we drop the explicit mention of spectral cut assuming the operators have common spectral cuts.

An explicit computation on symbols shows the result. Indeed, since $\sigma_{\log A}(x, \xi) \sim a \ln|\xi| + \sigma_0^A(x, \xi)$, we have

$$\sigma_{\log^2 A}(x, \xi) = \sigma_{\log A} \ast \sigma_{\log A}(x, \xi)$$

$$\sim a^2 \ln^2|\xi|I + 2a \ln|\xi| \sigma_0^A(x, \xi) + \sigma_0^A(x, \xi) \cdot \sigma_0^A(x, \xi)$$

$$+ \sum_{\alpha \neq 0} \frac{(-i)^{\alpha}}{\alpha!} \partial_\xi^{\alpha} \sigma_0^A(x, \xi) \partial_\xi^{\alpha} \sigma_0^A(x, \xi).$$

This yields:

$$\sigma_K(x, \xi) \sim \ln|\xi|(\sigma_0^{AB} - \sigma_0^A - \sigma_0^B)(x, \xi)$$

$$+ \frac{1}{2(a + b)} \sigma_0^{AB}(x, \xi)\sigma_0^B(x, \xi) + \sum_{\alpha \neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_0^{AB}(x, \xi) D_\xi^\alpha \sigma_0^{AB}(x, \xi)$$

$$- \frac{1}{2a} \sigma_0^A(x, \xi)\sigma_0^A(x, \xi) - \sum_{\alpha \neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_0^A(x, \xi) D_\xi^\alpha \sigma_0^A(x, \xi)$$

$$- \frac{1}{2b} \sigma_0^B(x, \xi)\sigma_0^B(x, \xi) - \sum_{\alpha \neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_0^B(x, \xi) D_\xi^\alpha \sigma_0^B(x, \xi)$$

from which the first part of the statement follows.

On the other hand, it follows from (2.12) combined with (2.7) that the operators $L(A, B) \log_2^a A$ and $L(A, B) \log_2^b B$ both have symbols which differ from $\ln|\xi|(\sigma_0^{AB} - \sigma_0^A - \sigma_0^B)(x, \xi)$ by a classical symbol of order zero, from which we infer the second part of the statement. □

The following theorem provides a local formula for the multiplicative anomaly independently of Okikiolu’s assumption that the leading symbols be scalar.

Theorem 3 Let $A$ and $B$ be two admissible operators in $\text{Cl}(M, E)$ with positive orders $a, b$ and with spectral cuts $\theta$ and $\phi$ in $[0, 2\pi]$ such that there is a cone delimited by
the rays $L_{\theta}$ and $L_{\phi}$ which does not intersect the spectra of the leading symbols of $A$, $B$ and $AB$. Then the product $AB$ is admissible with a spectral cut $\psi$ inside that cone and the multiplicative anomaly $M_{\zeta,\theta,\phi,\psi}(A,B)$ is local as a noncommutative residue, independently of the choices of $\theta, \phi$, and $\psi$ satisfying the above requirements. Explicitly, and dropping the explicit mention of the spectral cuts, there is a classical operator $W(\tau)(A,B)$ given by (5.17) of order zero depending continuously on $\tau$ such that:

$$\log M_{\zeta}(A,B) = \int_0^1 \text{res} \left( W(\tau)(A,B) \left( \log(A^TB) - \log B \right) \delta \right) d\tau$$

$$+ \text{res} \left( \frac{L(A,B) \log B}{b} - \log^2 AB \frac{2(a+b) + \log^2 A}{2a} + \log^2 B \frac{2b}{2b} \right)$$

$$= \int_0^1 \text{res} \left( W(\tau)(A,B) \left( \log(A^TB) - \log A \right) \delta \right) d\tau$$

$$+ \text{res} \left( \frac{L(A,B) \log A}{a} - \log^2 AB \frac{2(a+b)}{2a} + \log^2 A \frac{2a}{2a} + \log^2 B \frac{2b}{2b} \right)$$

(6.4)

When $A$ and $B$ commute the multiplicative anomaly reduces to:

$$\log M_{\zeta}(A,B) = -\text{res} \left( \frac{1}{2(a+b)} \log^2 AB - \frac{1}{2a} \log^2 A - \frac{1}{2b} \log^2 B \right)$$

$$= \frac{ab}{2(a+b)} \text{res} \left[ \left( \frac{\log A}{a} - \frac{\log B}{b} \right)^2 \right].$$

(6.5)

**Remark 6** For commuting operators, (6.5) gives back the results of Wodzicki as well as formula (III.3) in [D]:

$$\log M_{\zeta}(A,B) = \frac{\text{res} \left( \log^2(A^TB^{-a}) \right)}{2ab(a+b)}.$$
and weight $B$ with equation (5.18) applied to $Q = B$ we write:

$$
\log \mathcal{M}_\zeta(A, B) = \log \det_\zeta(AB) - \log \det_\zeta(A) - \log \det_\zeta(B)
$$

$$
= \int_M dx \left[ \operatorname{TR}_x(L(A, B)) - \left( \frac{1}{2(a+b)} \res_x(\log^2 AB) - \frac{1}{2a} \res_x(\log^2 A) - \frac{1}{2b} \res_x(\log^2 B) \right) \right]
$$

$$
= \operatorname{tr}^B(L(A, B)) + \int_M dx \left[ \frac{1}{b} \res_x(L(A, B) \log B) - \left( \frac{1}{2(a+b)} \res_x(\log^2 AB) - \frac{1}{2a} \res_x(\log^2 A) - \frac{1}{2b} \res_x(\log^2 B) \right) \right]
$$

$$
= \int_0^1 \res \left( W(\tau)(A, B) \left( \frac{\log(AB)}{\tau} + \frac{\log B}{b} \right) \right) d\tau
+ \res \left( \frac{L(A, B) \log B}{b} - \frac{\log^2 AB}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b} \right),
$$

which proves the first equality in (6.4). The second one can be derived similarly exchanging the roles of $A$ and $B$.

When $A$ and $B$ commute, by (6.2), the operator $L(A, B)$ vanishes so that (6.6) reduces to:

$$
\log \mathcal{M}_\zeta(A, B) = \operatorname{tr}^B(L(A, B)) + \int_M dx \left[ \frac{1}{b} \res_x(L(A, B) \log B) - \left( \frac{1}{2(a+b)} \res_x(\log^2 AB) - \frac{1}{2a} \res_x(\log^2 A) - \frac{1}{2b} \res_x(\log^2 B) \right) \right]
$$

$$
= -\res \left( \frac{\log^2 AB}{2(a+b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b} \right)
$$

$$
= \frac{ab}{2(a+b)} \res \left( \left( \frac{\log A}{a} - \frac{\log B}{b} \right)^2 \right).
$$

\[ \square \]
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Laboratoire de Mathématiques, Complexe des Cézeaux, Université Blaise Pascal, 63 177 Aubière Cedex F. E-mail: sylvie.paycha@math.univ-bpclermont.fr

Département de Mathématiques, Université de Ouagadougou, 03 BP 7021. Burkina Faso. E-mail: marie.oued@univ-ouaga.bf

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