Hopf bifurcation in the full repressilator equations

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In this paper, we prove that the full repressilator equations in dimension six undergo a supercritical Hopf bifurcation. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

Oscillatory networks are a particular kind of regulatory molecular networks, that is, collections of interacting molecules in a cell. The regulatory oscillators can be used to study abnormalities of a process in the cell, from sleep disorders to cancer. So, they attract significant attention among biologists and biophysicists. There are many implementations of artificial oscillatory networks (see, e.g., [1–7]). One of them is the repressilator [8]. Its genetic implementation uses three proteins that cyclically repress the synthesis of one another. The following system of DEs describes the behavior of the repressilator:

\[
\begin{align*}
\dot{m}_1 &= -m_1 + \frac{\alpha}{1 + v^n} + \epsilon a, \\
\dot{m}_2 &= -m_2 + \frac{\alpha}{1 + w^n} + \epsilon a, \\
\dot{m}_3 &= -m_3 + \frac{\alpha}{1 + u^n} + \epsilon a, \\
\dot{u} &= -\beta(u - m_1), \\
\dot{v} &= -\beta(v - m_2), \\
\dot{w} &= -\beta(w - m_3)
\end{align*}
\]  

(1)

Here, \(u, v,\) and \(w\) are proportional to the protein concentration, while \(m_i\) are proportional to the concentration of mRNA corresponding to those proteins. The nonlinear function \(f(x) = \frac{x^n}{1 + x^n}\) reflects synthesis of the mRNAs from the DNA controlled by regulatory elements. The parameter \(\alpha_0\) represents uncontrolled part of the mRNA synthesis, and it is usually small. The explicit inclusion of the mRNA concentration variables into the model is given by \(\beta\). Given that in general \(\beta \ll 1\) and \(\alpha_0\) is very small, we consider \(\alpha_0 = \epsilon a\) and \(\beta = \epsilon b\), where \(a\) and \(b\) are positive constants and \(\epsilon > 0\) is sufficiently small. So, system (1) becomes

\[
\begin{align*}
\dot{m}_1 &= -m_1 + \frac{\alpha}{1 + v^n} + \epsilon a, \\
\dot{m}_2 &= -m_2 + \frac{\alpha}{1 + w^n} + \epsilon a, \\
\dot{m}_3 &= -m_3 + \frac{\alpha}{1 + u^n} + \epsilon a, \\
\dot{u} &= -\epsilon b(u - m_1), \\
\dot{v} &= -\epsilon b(v - m_2), \\
\dot{w} &= -\epsilon b(w - m_3)
\end{align*}
\]  

(2)

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In the papers [9, 10], the authors consider a reduced system of dimension three. The reduction assumes that the three excluded variables, that is, \( m_i \), evolve an order of magnitude faster than the other three. In [9], the authors prove that in the reduced system exhibits a supercritical Hopf bifurcation. The existence of a Hopf bifurcation in the reduced system does not imply that the full system, in dimension six, also has a supercritical Hopf bifurcation. It just gives an indication about its existence. Here, in this work, we consider the full system in dimension six, and we extend the results of [9, 10] on the supercritical Hopf bifurcation to the six-dimensional differential system (2). Our main result is the following one.

**Theorem 1**

Let \( n > 2 \) be an integer. The point \((r_0, r_0, r_0, r_0, r_0, r_0)\) with

\[
r_0 = \sqrt{\frac{2}{n - 2}} + \varepsilon \frac{a - 3b_n \sqrt{\frac{2}{n - 2}}}{n - 2} + \mathcal{O}(\varepsilon^2)
\]

is an equilibrium of the differential system (2) with

\[
\alpha = \alpha_{\text{bif}} = \frac{n}{n - 2} \sqrt{\frac{2}{n - 2}} + \varepsilon \frac{n}{(n-2)^2} \left(-a(n-5) - 9b_n \sqrt{\frac{2}{n - 2}}\right) + \mathcal{O}(\varepsilon^2)
\]

The eigenvalues of the linear part of system (2) at this equilibrium are \(\left\{ \pm \varepsilon \sqrt{3}b + \mathcal{O}(\varepsilon^2), -1 + \varepsilon \left(-1 \pm \sqrt{3}\right)b + \mathcal{O}(\varepsilon^2), -\varepsilon 3b + \mathcal{O}(\varepsilon^2), -1 + \varepsilon 2b + \mathcal{O}(\varepsilon^2) \right\}\). Moreover, there is a single supercritical Hopf bifurcation at \(\alpha = \alpha_{\text{bif}}\), and there exists a small \(\varepsilon_0 > 0\), such that for \(\alpha_{\text{bif}} < \alpha < \alpha_{\text{bif}} + \varepsilon_0\), the system (2) has a stable limit cycle.

The repressilator differential system depends on five parameters \(a, b, \alpha, \varepsilon, n\) and on six variables \(m_1, m_2, m_3, u, v, w\). From Theorem 1, it follows that the supercritical Hopf bifurcation that such system exhibits takes place when \(n > 2\) on the hypersurface

\[
\alpha = \left(\frac{n}{n - 2} + 1\right) \sqrt{\frac{2}{n - 2}} + \varepsilon \frac{n}{(n-2)^2} \left(-a(n-5) - 9b_n \sqrt{\frac{2}{n - 2}}\right) + \mathcal{O}(\varepsilon^2)
\]

inside the five-dimensional parameter space, recall that \(\varepsilon\) is a small parameter. We must remark that the reduced repressilator differential system studied in [3,4] depends only on two parameters \(a, n\) and on three variables \(m_1, m_2, m_3\). From Proposition 1 of [3], we obtain that the supercritical Hopf bifurcation that such reduced system exhibits takes place when \(n > 2\) on the curve

\[
\alpha = \left(\frac{n}{n - 2} + 1\right) \sqrt{\frac{2}{n - 2}} + \varepsilon \frac{n}{(n-2)^2} \left(-a(n-5) - 9b_n \sqrt{\frac{2}{n - 2}}\right) + \mathcal{O}(\varepsilon^2)
\]

in the plane of parameters.

**2. Proof of Theorem 1**

It is clear that system (2) has the equilibrium \(p_0 = (r_0, r_0, r_0, r_0, r_0, r_0)\) where \(r_0\) is solution of the equation

\[
\frac{\alpha}{1 + \varepsilon} = r - \varepsilon a
\]

From (3), we have that \(\alpha = (r_0 - \varepsilon a) (1 + r_0^3)\). So, substituting \(\alpha\) in the linear part of system (2) at the equilibrium \(p_0\), we obtain

\[
A = \begin{pmatrix}
-1 & 0 & 0 & 0 & \Delta & 0 \\
0 & -1 & 0 & 0 & \Delta & 0 \\
0 & 0 & -1 & 0 & \Delta & 0 \\
\varepsilon b & 0 & 0 & -\varepsilon b & 0 & 0 \\
0 & \varepsilon b & 0 & 0 & -\varepsilon b & 0 \\
0 & 0 & \varepsilon b & 0 & 0 & -\varepsilon b \\
\end{pmatrix}
\]

where \(\Delta = -\frac{m_0^{-1} + n (r_0 - \varepsilon a)}{1 + r_0^3}\). The eigenvalues of \(M\) are

\[
\varepsilon \frac{(-2 + (n-2) \sqrt{3}n) \varepsilon b}{2(1 + r_0^3)} + \mathcal{O}(\varepsilon^2), \quad -1 + \varepsilon \frac{(-1 + (1+n) \varepsilon b)}{2(1 + r_0^3)} + \mathcal{O}(\varepsilon^2),
\]

\[
-1 + \varepsilon \frac{m_0 \varepsilon b}{1 + r_0^3} + \mathcal{O}(\varepsilon^2), \quad -\varepsilon \frac{1 + (1+n) \varepsilon b}{1 + r_0^3} + \mathcal{O}(\varepsilon^2)
\]

We impose that the real part of the eigenvalues \(\varepsilon \frac{(-2 + (n-2) \sqrt{3}n) \varepsilon b}{2(1 + r_0^3)} + \mathcal{O}(\varepsilon^2)\) is zero, and we obtain

\[
r_0 = \sqrt{\frac{2}{n - 2}} + \varepsilon \frac{a - 3b_n \sqrt{\frac{2}{n - 2}}}{n - 2} + \mathcal{O}(\varepsilon^2)
\]
Substituting (4) in (3), we obtain

$$\alpha_{bi} = \frac{n}{n-2} \left( \frac{2}{n-2} - \frac{n}{(n-2)^2} \right) \left[ a(n-5) + 9b \sqrt{\frac{2}{n-2}} \right] + \mathcal{O} (\varepsilon^2) \quad (5)$$

Substituting (4) in $M$ and computing the eigenvalues, we obtain $\pm \varepsilon \sqrt{3b} + O(\varepsilon^2)$. When $\varepsilon > 0$, the point $p_0$ is a weak focus of system (2) restricted to the central manifold of $p_0$, and the limit cycle that emerges from $p_0$ is stable.

The linearization of (2) at $p_0$ has a pair of conjugate purely imaginary eigenvalues, and the other four eigenvalues have negative real part. This is the setting for a Hopf bifurcation. We can expect to see a small-amplitude limit cycle branching from the fixed point $p_0$. It remains to compute the first Lyapunov coefficient $\ell_1(p_0)$ of (2) near $p_0$. When $\varepsilon > 0$, the point $p_0$ is a weak focus of system (2) restricted to the central manifold of $p_0$, and the limit cycle that emerges from $p_0$ is stable.

Here, we use the following result presented on page 180 of the book by Kuznetsov [11] for computing $\ell_1(p_0)$.

**Lemma 2**

Let $x = F(x)$ be a differential system having $p_0$ as an equilibrium point. Consider the third-order Taylor approximation of $F$ around $p_0$ given by $F(x) = Ax + B(x, x) + C(x, x, x) + O(|x|^3)$. Assume that $A$ has a pair of purely imaginary eigenvalues $\pm \lambda i$. Let $q$ be the eigenvector of $A$ corresponding to the eigenvalue $\lambda i$, normalized so that $\bar{q} \cdot q = 1$, where $\bar{q}$ is the conjugate vector of $q$. Let $p$ be the adjoint eigenvector such that $A^T p = -\lambda p$ and $\bar{q} \cdot q = 1$. If $I$ denotes the $6 \times 6$ identity matrix, then

$$\ell_1(p_0) = \frac{1}{2\lambda} \left( \text{Re} \left[ \bar{q} \cdot C(q, q, \bar{q}) - 2 \bar{q} \cdot B(q, A^{-1} B(q, \bar{q})) + \bar{q} \cdot B(\bar{q}, (2\lambda i - A)^{-1} B(q, q)) \right] \right) + \mathcal{O}(\varepsilon^2)$$

In our case, the linear part of system (2) at the equilibrium $p_0$ is

$$A = \begin{pmatrix}
-1 & 0 & 0 & -2 + \varepsilon \sigma & 0 \\
0 & -1 & 0 & 0 & -2 + \varepsilon \sigma \\
0 & 0 & -1 & -2 + \varepsilon \sigma & 0 \\
b e & 0 & 0 & -b e & 0 \\
0 & b e & 0 & 0 & -b e \\
0 & 0 & b e & 0 & 0 -b e
\end{pmatrix} + \mathcal{O}(\varepsilon^2)$$

where

$$\sigma = 6b + \frac{1}{(n-2)^2} \left( b^2(51 - 39n) + ba 12 \sqrt{\frac{n-2}{2}} - a^2 \sqrt{\left( \frac{n-2}{2} \right)^2 - 1} \right)$$

We have that $A$ has an eigenvalue $\pm \sqrt{3}b + \mathcal{O}(\varepsilon^2)$. Now, we compute the bilinear and trilinear functions $B$ and $C$. Considering the vector field $(f_1, f_2, f_3, f_4, f_5, f_6)$ associated to the differential system (2), we observe that all second and third derivatives vanish except $\frac{\partial f_1}{\partial w}, \frac{\partial f_1}{\partial v}, \frac{\partial f_1}{\partial w}, \frac{\partial f_1}{\partial w}, \frac{\partial f_1}{\partial w}, \frac{\partial f_1}{\partial w}$, and $\frac{\partial f_1}{\partial w}$. Computing these derivatives, taking into account (4) and (5), we obtain that

$$\frac{\partial^2 f_1}{\partial w^2}(p_0) = \frac{\partial^2 f_1}{\partial v^2}(p_0) = \frac{\partial^2 f_1}{\partial u^2}(p_0) = \gamma,$$

$$\frac{\partial^2 f_1}{\partial w^3}(p_0) = \frac{\partial^2 f_1}{\partial v^3}(p_0) = \frac{\partial^2 f_1}{\partial u^3}(p_0) = \delta$$

where

$$\gamma = -2(n-5) \sqrt{\frac{n-2}{2}} + \varepsilon \sqrt{\frac{2}{(n-2)^2}} \left( a(5n-13) + 3b \sqrt{\frac{2}{n-2}} \left( n^2 - 12n + 23 \right) \right) + \mathcal{O}(\varepsilon^2)$$

and

$$\delta = -2 \varepsilon \sqrt{\frac{n-2}{n-2}} \left( n^2 - 15n + 38 \right) + \varepsilon \frac{2}{n-2} \left( \frac{n-2}{2} \right)^{3/2} \times \left( 2a(7n^2 - 57n + 98) + 3b \sqrt{\frac{2}{n-2}} \left( n^3 - 31n^2 + 182n - 272 \right) \right) + \mathcal{O}(\varepsilon^2)$$
So, the bilinear function $B$ is given by

$$B((x_1, y_1, z_1, u_1, v_1, w_1), (x_2, y_2, z_2, u_2, v_2, w_2)) = (\gamma v_1 v_2, \gamma w_1 w_2, \gamma u_1 u_2, 0, 0, 0)$$

and the trilinear function $C$ is given by the expression

$$C((x_1, y_1, z_1, u_1, v_1, w_1), (x_2, y_2, z_2, u_2, v_2, w_2), (x_3, y_3, z_3, u_3, v_3, w_3)) = (\delta v_1 v_2 v_3, \delta w_1 w_2 w_3, \delta u_1 u_2 u_3, 0, 0, 0)$$

Computing the normalized eigenvector $q$ of $A$, associated to the eigenvalue $\epsilon \sqrt{3} bi + \mathcal{O}(\epsilon^2)$, we obtain

$$q = \left( \frac{1 - \sqrt{3} i}{\sqrt{15}}, \frac{-1 - \sqrt{3} i}{2 \sqrt{15}}, \frac{1 + \sqrt{3} i}{2 \sqrt{15}}, \frac{-1 + \sqrt{3} i}{2 \sqrt{15}}, \frac{1}{\sqrt{15}} \right) + \epsilon \left( \frac{30 \sqrt{3} i}{30 \sqrt{5}}, \frac{30 \sqrt{3} i}{30 \sqrt{5}}, \frac{5 \sqrt{3} b}{30 \sqrt{5}} + \mathcal{O}(\epsilon^2) \right)$$

The normalized adjoint eigenvector of the transpose matrix $A$ with the eigenvalue $-\epsilon \sqrt{3} bi$ is

$$p = \left( 0, 0, 0, \frac{-\sqrt{15} - 3 \sqrt{3} i}{6}, \frac{-\sqrt{15} + 3 \sqrt{3} i}{6} \right) + \epsilon \left( \frac{-5 - 5 \sqrt{3} i b}{2 \sqrt{15}}, \frac{-5 + 5 \sqrt{3} i b}{2 \sqrt{15}}, \frac{5b + (1 + \sqrt{3} i) \sigma}{\sqrt{15}}, \frac{-5 (1 + \sqrt{3} i) b + (1 - \sqrt{3} i) \sigma}{\sqrt{15}}, \frac{-5 (1 - \sqrt{3} i) b - 2 \sigma}{\sqrt{15}} \right) + \mathcal{O}(\epsilon^2)$$

According to Lemma 2, in order to compute $\ell_1(p_0)$, we need to compute first $A^{-1}$ and $(2 \sqrt{3} bi \epsilon - A)^{-1}$. We have that $A^{-1} = \frac{1}{\epsilon} A_{-1} + A_0 + \epsilon A_1 + \mathcal{O}(\epsilon^2)$,

$$A_{-1} = \frac{1}{96} \begin{pmatrix} 0 & 0 & 0 & 8 & 2 & -4 \\ 0 & 0 & 0 & -4 & 8 & 2 \\ 0 & 0 & 0 & 2 & -4 & 8 \\ 0 & 0 & 0 & -1 & 2 & -4 \\ 0 & 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 2 & 4 & -1 \end{pmatrix}$$

$$A_0 = \frac{1}{276} \begin{pmatrix} -3b & 6b - 12b & -4a & 5a & -4a \\ -12b & -3b & 6b - 4a & -4a & 5a \\ 6b - 12b & -3b & 5a & -4a & -4a \\ -3b & 6b - 12b & -4a & 5a & -4a \\ -12b & -3b & 6b & -4a & 4a \\ 6b - 12b & -3b & 5a & -4a & -4a \end{pmatrix}$$

$$A_1 = \frac{\sigma}{816} \begin{pmatrix} -12b & 15b - 12b & -10a & 8a & -\sigma \\ -12b & -12b & 15b & -\sigma & -10a & 8a \\ 15b & -12b & -12b & 8a & -\sigma & -10a \\ -12b & 15b - 12b & -10a & 8a & -\sigma \\ -12b & -12b & 15b & -\sigma & -10a & 8a \\ 15b & -12b & -12b & 8a & -\sigma & -10a \end{pmatrix}$$

and the expression of $(2 \sqrt{3} bi \epsilon - A)^{-1}$ is very large, and we present it in the Appendix. The first, second, and third terms of $\ell_1(p_0)$ given in Lemma 2, respectively, are

$$\text{Re}(\sigma \cdot C(q,q,q)) = \frac{1}{15} \sqrt{(n-2)^2(n^2 - 15n + 38)b\epsilon + \mathcal{O}(\epsilon^2)}.$$
and
\[ \text{Re}\left(\vec{p} \cdot B(q, A^{-1} B(q, \bar{q}))\right) = -\frac{4}{45} \sqrt{\left(\frac{n-2}{2}\right)^2 \left(n^2 - 10n + 25\right)} b \varepsilon + \mathcal{O}(\varepsilon^2) \]

Consequently, we obtain
\[ \text{Re}\left(\vec{p} \cdot B(q, (2\lambda\bar{l} - A)^{-1} B(q, q))\right) = 0 + \mathcal{O}(\varepsilon^2) \]

As we said before, \( \ell_1(p_0) < 0 \) implies that we have a supercritical Hopf bifurcation at \( \alpha = \alpha_{\text{bif}} \), so there exists \( \varepsilon_0 > 0 \), such that for \( \alpha_{\text{bif}} < \alpha < \alpha_{\text{bif}} + \varepsilon_0 \), the system (2) has a stable limit cycle.

### 3. Conclusions

The repressilator model is an implementation of an artificial oscillatory network used for studying the collections of interacting molecules in a cell. The model is given by a six-dimensional differential system. Buse et al. published two nice papers [9, 10] analyzing a reduced system of dimension three. In this reduced system, they show the existence of a supercritical Hopf bifurcation. Because the reduction is reasonable, we may expect that such supercritical Hopf bifurcation must also occur in the actual six-dimensional differential system. We prove that this is the case.

### Appendix

The matrix \( \left(2\sqrt{3}b\varepsilon il - A\right)^{-1} \) is given by
\[
\left(2\sqrt{3}b\varepsilon il - A\right)^{-1} = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{pmatrix}
\]

where \( C_{ij} = A_{ij} + B_{ij} \), for \( i, j = 1, 2, \ldots, 6 \).

\[
\begin{align*}
A_{11} &= \frac{1}{189} \left(213 - 16i\sqrt{3}\right), \\
A_{12} &= \frac{2}{189} \left(-9 + 34i\sqrt{3}\right), \\
A_{13} &= \frac{4}{189} \left(-15 - 4i\sqrt{3}\right), \\
A_{14} &= \frac{4}{81} \left(12 \left(\sqrt{3} - 124i\right) b e - \left(124\sqrt{3} + 3i\right) \sigma\varepsilon + 72\sqrt{3} + 18i\right), \\
A_{15} &= \frac{3156\sqrt{3} b e + 6144i b e + 512\sqrt{3} \sigma\varepsilon - 789i\sigma\varepsilon - 864\sqrt{3} + 666i}{810\sqrt{3} b e + 2187 b e}, \\
A_{16} &= \frac{4}{81} \left(12 \left(50 + 87i\sqrt{3}\right) b e + \left(261 - 50i\sqrt{3}\right) \sigma\varepsilon + 18i\sqrt{3} - 225\right), \\
A_{21} &= \frac{4}{81} \left(12 \left(50 + 87i\sqrt{3}\right) b e + \left(261 - 50i\sqrt{3}\right) \sigma\varepsilon + 18i\sqrt{3} - 225\right), \\
A_{22} &= \frac{1}{189} \left(213 - 16i\sqrt{3}\right), \\
A_{23} &= \frac{1}{189} \left(213 - 16i\sqrt{3}\right),
\end{align*}
\]
\[
A_{24} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{25} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{26} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{31} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{32} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{33} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{34} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{35} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{36} = \frac{4 \left( 12 \left( 50 + 87i \sqrt{3} \right) \beta e + \left( 261 - 50i \sqrt{3} \right) \sigma e + 18i \sqrt{3} - 225 \right)}{81 \left( 27 - 10i \sqrt{3} \right) \beta e},
\]
\[
A_{41} = \frac{1}{189} \left( 9 - 34i \sqrt{3} \right),
\]
\[
A_{42} = \frac{2}{189} \left( 15 + 4i \sqrt{3} \right),
\]
\[
A_{43} = \frac{4}{189} \left( -3 + 2i \sqrt{3} \right),
\]
\[
A_{44} = \frac{\left( -912 \sqrt{3} \beta e + 480i \beta e + 40 \sqrt{3} \sigma e - 228 \sigma e - 228 \sqrt{3} + 333i \right)}{810 \sqrt{3} \beta e + 2187i \beta e},
\]
\[
A_{45} = \frac{\left( 1188 \sqrt{3} \beta e - 984i \beta e - 82 \sqrt{3} \sigma e - 297 \sigma e + 36 \sqrt{3} + 450i \right)}{810 \sqrt{3} \beta e + 2187i \beta e},
\]
\[
A_{46} = \frac{\left( 4 \left( \sqrt{3} + 44i \right) \beta e + \left( 44 \sqrt{3} - 3i \right) \sigma e - 9 \left( 4 \sqrt{3} + i \right) \right)}{81 \left( 10 \sqrt{3} + 27i \right) \beta e},
\]
\[
A_{51} = \frac{4}{189} \left( -3 + 2i \sqrt{3} \right),
\]
\[
A_{52} = \frac{4}{189} \left( -3 + 2i \sqrt{3} \right),
\]
\[
A_{53} = \frac{2}{189} \left( 15 + 4i \sqrt{3} \right),
\]
\[
A_{54} = \frac{\left( 4 \left( \sqrt{3} + 44i \right) \beta e + \left( 44 \sqrt{3} - 3i \right) \sigma e - 9 \left( 4 \sqrt{3} + i \right) \right)}{81 \left( 10 \sqrt{3} + 27i \right) \beta e},
\]
\[
A_{55} = \frac{\left( -912 \sqrt{3} \beta e + 480i \beta e + 40 \sqrt{3} \sigma e - 228 \sigma e - 228 \sqrt{3} + 333i \right)}{810 \sqrt{3} \beta e + 2187i \beta e},
\]
\[
A_{56} = \frac{\left( 1188 \sqrt{3} \beta e - 984i \beta e - 82 \sqrt{3} \sigma e - 297 \sigma e + 36 \sqrt{3} + 450i \right)}{810 \sqrt{3} \beta e + 2187i \beta e},
\]
\[
A_{61} = \frac{2}{189} \left( 15 + 4i \sqrt{3} \right),
\]
\begin{align*}
A_{62} &= \frac{4}{189} \left( -3 + 2i\sqrt{3} \right), \\
A_{63} &= \frac{1}{189} \left( 9 - 34i\sqrt{3} \right), \\
A_{64} &= 1188\sqrt{3}be - 984ibe - 82\sqrt{3}ae - 297\sigma e + 36\sqrt{3} + 450i, \\
A_{65} &= \frac{810\sqrt{3}be + 2187ibe}{1188\sqrt{3}be - 984ibe - 82\sqrt{3}ae - 297\sigma e + 36\sqrt{3} + 450i}, \\
A_{66} &= \frac{810\sqrt{3}be + 2187ibe}{1188\sqrt{3}be - 984ibe - 82\sqrt{3}ae - 297\sigma e + 36\sqrt{3} + 450i}, \\
B_{11} &= \frac{2e \left( -4800 - 5921i\sqrt{3} \right) b + \left( -362 + 316i\sqrt{3} \right) \sigma}{3969}, \\
B_{12} &= \frac{\epsilon \left( 8 \left( 215\sqrt{3} + 42i \right) b - \left( 2\sqrt{3} + 259i \right) \sigma \right)}{81 \left( 4\sqrt{3} + i \right)}, \\
B_{13} &= \frac{\epsilon \left( 8 \left( 215\sqrt{3} + 42i \right) b - \left( 2\sqrt{3} + 259i \right) \sigma \right)}{81 \left( 4\sqrt{3} + i \right)}, \\
B_{14} &= \frac{2e}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 96 \left( 444\sqrt{3} - 131i \right) b^2 - 52 \left( 41\sqrt{3} + 324i \right) \sigma + \left( -516\sqrt{3} + 271i \right) \sigma^2 \right), \\
B_{15} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 24 \left( -1760\sqrt{3} + 5807i \right) b^2 + 2 \left( 9247\sqrt{3} + 5952i \right) \sigma + \left( -14\sqrt{3} - 215i \right) \sigma^2 \right), \\
B_{16} &= \frac{4e}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( -48 \left( 1058\sqrt{3} + 2825i \right) b^2 + 16 \left( -1021\sqrt{3} + 1362i \right) \sigma + \left( 758\sqrt{3} + 1259i \right) \sigma^2 \right), \\
B_{17} &= \frac{2e \left( -4800 - 5921i\sqrt{3} \right) b + \left( -362 + 316i\sqrt{3} \right) \sigma}{3969}, \\
B_{18} &= \frac{\epsilon \left( 8 \left( 215\sqrt{3} + 42i \right) b - \left( 2\sqrt{3} + 259i \right) \sigma \right)}{81 \left( 4\sqrt{3} + i \right)}, \\
B_{19} &= \frac{\epsilon \left( 8 \left( 215\sqrt{3} + 42i \right) b - \left( 2\sqrt{3} + 259i \right) \sigma \right)}{81 \left( 4\sqrt{3} + i \right)}, \\
B_{20} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( -48 \left( 1058\sqrt{3} + 2825i \right) b^2 + 16 \left( -1021\sqrt{3} + 1362i \right) \sigma + \left( 758\sqrt{3} + 1259i \right) \sigma^2 \right), \\
B_{21} &= \frac{2e}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 96 \left( 444\sqrt{3} - 131i \right) b^2 - 52 \left( 41\sqrt{3} + 324i \right) \sigma + \left( -516\sqrt{3} + 271i \right) \sigma^2 \right), \\
B_{22} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 24 \left( -1760\sqrt{3} + 5807i \right) b^2 + 2 \left( 9247\sqrt{3} + 5952i \right) \sigma + \left( -14\sqrt{3} - 215i \right) \sigma^2 \right), \\
B_{23} &= \frac{\epsilon \left( 8 \left( 215\sqrt{3} + 42i \right) b - \left( 2\sqrt{3} + 259i \right) \sigma \right)}{81 \left( 4\sqrt{3} + i \right)}, \\
B_{24} &= \frac{2e}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 96 \left( 444\sqrt{3} - 131i \right) b^2 - 52 \left( 41\sqrt{3} + 324i \right) \sigma + \left( -516\sqrt{3} + 271i \right) \sigma^2 \right), \\
B_{25} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 24 \left( -1760\sqrt{3} + 5807i \right) b^2 + 2 \left( 9247\sqrt{3} + 5952i \right) \sigma + \left( -14\sqrt{3} - 215i \right) \sigma^2 \right), \\
B_{26} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 24 \left( -1760\sqrt{3} + 5807i \right) b^2 + 2 \left( 9247\sqrt{3} + 5952i \right) \sigma + \left( -14\sqrt{3} - 215i \right) \sigma^2 \right), \\
B_{27} &= \frac{\epsilon \left( 8 \left( 215\sqrt{3} + 42i \right) b - \left( 2\sqrt{3} + 259i \right) \sigma \right)}{81 \left( 4\sqrt{3} + i \right)}, \\
B_{28} &= \frac{2e \left( -4800 - 5921i\sqrt{3} \right) b + \left( -362 + 316i\sqrt{3} \right) \sigma}{3969}, \\
B_{29} &= \frac{2e \left( -4800 - 5921i\sqrt{3} \right) b + \left( -362 + 316i\sqrt{3} \right) \sigma}{3969}, \\
B_{30} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( 24 \left( -1760\sqrt{3} + 5807i \right) b^2 + 2 \left( 9247\sqrt{3} + 5952i \right) \sigma + \left( -14\sqrt{3} - 215i \right) \sigma^2 \right), \\
B_{31} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( -48 \left( 1058\sqrt{3} + 2825i \right) b^2 + 16 \left( -1021\sqrt{3} + 1362i \right) \sigma + \left( 758\sqrt{3} + 1259i \right) \sigma^2 \right), \\
B_{32} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( -48 \left( 1058\sqrt{3} + 2825i \right) b^2 + 16 \left( -1021\sqrt{3} + 1362i \right) \sigma + \left( 758\sqrt{3} + 1259i \right) \sigma^2 \right), \\
B_{33} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( -48 \left( 1058\sqrt{3} + 2825i \right) b^2 + 16 \left( -1021\sqrt{3} + 1362i \right) \sigma + \left( 758\sqrt{3} + 1259i \right) \sigma^2 \right), \\
B_{34} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( -48 \left( 1058\sqrt{3} + 2825i \right) b^2 + 16 \left( -1021\sqrt{3} + 1362i \right) \sigma + \left( 758\sqrt{3} + 1259i \right) \sigma^2 \right), \\
B_{35} &= \frac{\epsilon}{81 \left( 2\sqrt{3} - 3i \right)^3} \left( -48 \left( 1058\sqrt{3} + 2825i \right) b^2 + 16 \left( -1021\sqrt{3} + 1362i \right) \sigma + \left( 758\sqrt{3} + 1259i \right) \sigma^2 \right).
\end{align*}
\[
B_{36} = \frac{2e}{81(2\sqrt{3} - 3i)^3 b} \left( 96 \left( 444\sqrt{3} - 131i \right) b^2 - 52 \left( 41\sqrt{3} + 324i \right) b\sigma + \left( -516\sqrt{3} + 271i \right) \sigma^2 \right), \\
B_{41} = \frac{2e}{81(4\sqrt{3} + i)} \left( -259\sqrt{3}b + 6ib + 8\sqrt{3}\sigma + 22i\sigma \right), \\
B_{42} = -\frac{e}{81(4\sqrt{3} + i)} \left( 8 \left( -23\sqrt{3} + 96i \right) b + \left( 40\sqrt{3} + 19i \right) \sigma \right), \\
B_{43} = \frac{4e}{81(4\sqrt{3} + i)} \left( 2 \left( 35\sqrt{3} + 54i \right) b + \left( 6\sqrt{3} - 13i \right) \sigma \right), \\
B_{44} = \frac{2e}{81(2\sqrt{3} - 3i)^3 b} \left( 96 \left( 14\sqrt{3} - 215i \right) b^2 - 52 \left( 53\sqrt{3} + 6i \right) b\sigma + \left( 2\sqrt{3} + 259i \right) \sigma^2 \right), \\
B_{45} = \frac{e}{81(2\sqrt{3} - 3i)^3 b} \left( 24 \left( 758\sqrt{3} + 1259i \right) b^2 + 2 \left( 1627\sqrt{3} - 3810i \right) b\sigma - 8 \left( 32\sqrt{3} + 23i \right) \sigma^2 \right), \\
B_{46} = \frac{e}{81(2\sqrt{3} - 3i)^3 b} \left( 48 \left( -516\sqrt{3} + 271i \right) b^2 + 16 \left( 131\sqrt{3} + 576i \right) b\sigma + \left( 252\sqrt{3} - 253i \right) \sigma^2 \right), \\
B_{51} = \frac{4e}{81(4\sqrt{3} + i)} \left( 2 \left( 35\sqrt{3} + 54i \right) b + \left( 6\sqrt{3} - 13i \right) \sigma \right), \\
B_{52} = \frac{2e}{81(4\sqrt{3} + i)} \left( -259\sqrt{3}b + 6ib + 8\sqrt{3}\sigma + 22i\sigma \right), \\
B_{53} = -\frac{e}{81(4\sqrt{3} + i)} \left( 8 \left( -23\sqrt{3} + 96i \right) b + \left( 40\sqrt{3} + 19i \right) \sigma \right), \\
B_{54} = \frac{e}{81(2\sqrt{3} - 3i)^3 b} \left( 48 \left( -516\sqrt{3} + 271i \right) b^2 + 16 \left( 131\sqrt{3} + 576i \right) b\sigma + \left( 252\sqrt{3} - 253i \right) \sigma^2 \right), \\
B_{55} = \frac{2e}{81(2\sqrt{3} - 3i)^3 b} \left( 96 \left( 14\sqrt{3} - 215i \right) b^2 - 52 \left( 53\sqrt{3} + 6i \right) b\sigma + \left( 2\sqrt{3} + 259i \right) \sigma^2 \right), \\
B_{56} = \frac{e}{81(2\sqrt{3} - 3i)^3 b} \left( 24 \left( 758\sqrt{3} + 1259i \right) b^2 + 2 \left( 1627\sqrt{3} - 3810i \right) b\sigma - 8 \left( 32\sqrt{3} + 23i \right) \sigma^2 \right), \\
B_{61} = -\frac{e}{81(4\sqrt{3} + i)} \left( 8 \left( -23\sqrt{3} + 96i \right) b + \left( 40\sqrt{3} + 19i \right) \sigma \right), \\
B_{62} = \frac{4e}{81(4\sqrt{3} + i)} \left( 2 \left( 35\sqrt{3} + 54i \right) b + \left( 6\sqrt{3} - 13i \right) \sigma \right), \\
B_{63} = \frac{4e}{81(4\sqrt{3} + i)} \left( 2 \left( 35\sqrt{3} + 54i \right) b + \left( 6\sqrt{3} - 13i \right) \sigma \right), \\
B_{64} = \frac{e}{81(2\sqrt{3} - 3i)^3 b} \left( 24 \left( 758\sqrt{3} + 1259i \right) b^2 + 2 \left( 1627\sqrt{3} - 3810i \right) b\sigma - 8 \left( 32\sqrt{3} + 23i \right) \sigma^2 \right), \\
B_{65} = \frac{e}{81(2\sqrt{3} - 3i)^3 b} \left( 48 \left( -516\sqrt{3} + 271i \right) b^2 + 16 \left( 131\sqrt{3} + 576i \right) b\sigma + \left( 252\sqrt{3} - 253i \right) \sigma^2 \right), \\
B_{66} = \frac{2e}{81(2\sqrt{3} - 3i)^3 b} \left( 96 \left( 14\sqrt{3} - 215i \right) b^2 - 52 \left( 53\sqrt{3} + 6i \right) b\sigma + \left( 2\sqrt{3} + 259i \right) \sigma^2 \right)
\]
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