ON THE VANISHING CYCLES OF A MEROMORPHIC FUNCTION ON THE COMPLEMENT OF ITS POLES

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Abstract. We study vanishing cycles naturally attached to a meromorphic function with isolated singularities, in both local and global settings.

1. Introduction

In the local setting, we consider a meromorphic function germ $p_x/q_x: (\mathcal{Z}, x) \rightarrow \mathbb{P}^1(\mathbb{C})$, where $(\mathcal{Z}, x)$ is the germ of a complex manifold and $p_x$ and $q_x$ are holomorphic germs at the point $x$. By definition, $p_x'/q_x'$ is equal to $p_x/q_x$ if and only if there exists a holomorphic germ $u$ such that $u(x) \neq 0$ and that $p_x = u p'_x$ and $q_x = u q'_x$.

In the global setting, a meromorphic function $p/q: \mathcal{Z} \rightarrow \mathbb{P}^1(\mathbb{C})$ is defined as the ratio of two sections, $p$ and $q$, of a holomorphic line bundle $L \rightarrow \mathcal{Z}$ over a connected compact complex manifold $\mathcal{Z}$. We consider only examples of projective $\mathcal{Z}$, which case insures the existence of global meromorphic functions, by Kodaira embedding theorem.

Our constructions and results in the two settings are completely similar and parallel. This is why we shall adopt in this paper a unique notation for both situations: $p/q: \mathcal{Z} \rightarrow \mathbb{P}^1(\mathbb{C})$ can alternatively mean a meromorphic germ or a global meromorphic function.

The meromorphic function induces a holomorphic function $f: X \rightarrow \mathbb{C}$, on the complement of $X := \mathcal{Z} \setminus \{q = 0\}$ of the pole locus $\{q = 0\}$. Then we call vanishing homology the relative homology $H_*(X, F; \mathbb{Z})$, where $F$ denotes a general fiber of $f: X \rightarrow \mathbb{C}$. Since $f$ is a non-proper function, vanishing cycles may appear not only because of the critical points of $f$, but also because of a certain non-regular behavior of the fibers of $f$ in the neighbourhood of the pole locus $\{q = 0\}$, which is more subtle to detect. We interpret the later phenomenon in terms of singularities of the meromorphic function, to which we give a precise meaning. Our study concerns the class of meromorphic functions with isolated singularities, including possible singularities occuring in the indeterminacy locus. This is a class of non-generic pencils, far beyond the class of generic pencils that is currently considered in Lefschetz theory. We send the reader to [Ti4] and [Ti3] for comments on the relations to Lefschetz theory.

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While the homology of $X$ and that of $F$ might be very complicated, it turns out (Theorem 2.8) that the vanishing homology $H_\ast(X, F; \mathbb{Z})$ is concentrated in dimension $n = \dim X$ and moreover that the space $X/F$ has the homotopy type of a bouquet of spheres $\bigvee S^n$. The proof needs new technical ingredients, due to the general notion of singularity that we use here. Proposition 3.2 and Lemma 3.3 are crucial in this respect.

We prove that the polar Milnor number which we attach to an isolated $G$-singularity is the number of vanishing cycles which are “concentrated” at this singularity (Proposition 2.7 and Corollary 3.7). Alternatively, the polar Milnor number $\lambda_\xi$ at some point $\xi$ coincides with the jump of the usual local Milnor number of the pencil of hypersurfaces. Vanishing cycles can now be detected by a multibranch jump formula (Theorem 4.1).

We also show that the global vanishing homology decomposes into a direct sum of vanishing homologies at the atypical fibres, with localization in case of $G$-isolated singularities (Proposition 2.7 and Corollary 3.7). We prove a general global Picard type formula for the monodromy (Proposition 5.1). Significant examples and particular cases are treated in §4.

The vanishing homology in the context of meromorphic functions extends the one of local proper holomorphic functions, which has been initiated by Milnor [Mi] and developed in many aspects ever since. The meromorphic setting also generalizes the study of the topology of polynomial functions via their singularities at infinity and the study of one-parameter families of non compact hypersurfaces, developed in the last decade. (We send the reader to [Ti3] for more details on these connections.) New phenomena may occur: unlike the polynomial case, where singularities at infinity “stay at the same place” for all fibers, in the meromorphic case singularities can split or even disappear (see Example 4.4).

The results addressed here are based on our 1999 preprint [ST2], which has not been published. Ever since, there appeared several papers which quote it, like [DN2], [GLM2], [Ti3]. The recent survey [Ti3] treats aspects of the topology of meromorphic functions on singular spaces, reviewing some of the results presented in the preprint [ST2] and in this paper. Connectivity results of Lefschetz type via nongeneric pencils (i.e. global meromorphic functions) are proved in [Ti4]. From a quite different point of view, meromorphic germs are discussed in [GLM1,2], where the main interest is the zeta-function of the monodromy. Classifications aspects have been explored by Arnold [Ar], who found the list of simple germs of meromorphic functions under natural equivalence relations.

2. SINGULARITIES OF $f$ ALONG THE INDETERMINACY LOCUS

The definitions in this section naturally extend the ones used in case of polynomials and certain classes of regular functions on affine varieties, see [Ti1] [Ti3] and also [Ti4]. According to our convention in the Introduction, we treat in parallel the local and global settings, using a common notation.
Definition 2.1. We call completed space the global hypersurface $X$ of $\mathcal{Z} \times \mathbb{P}^1$ defined by:

$$sp(z) - tq(z) = 0,$$

where $[t : s] \in \mathbb{P}^1$. Then $X$ is the analytic closure in $\mathcal{Z} \times \mathbb{P}^1$ of the graph of $f : X \to \mathbb{C}$. We denote by $\pi : X \to \mathbb{P}^1$ the projection on $\mathbb{P}^1$. This is a proper holomorphic function which extends $f : X \to \mathbb{C}$. The space $\mathcal{X}^{\text{pol}} := X \cap \{(p = q = 0) \times \mathbb{P}^1\}$ is a divisor of $X$ and $X$ is identified with $X \setminus (\mathcal{X}^{\text{pol}} \cup X_\infty)$, where $X_\infty := \pi^{-1}\{s = 0\}$.

We endow $X$ with a locally finite Whitney stratification $\mathcal{W}$ such that $X$ is a stratum. In case of a germ at $x \in \mathcal{Z}$ of a meromorphic function one considers the germ of such a Whitney stratification at the line $\{x\} \times \mathbb{P}^1 \subset X$. In both situations, the local finiteness of the strata implies, by using Thom-Mather Isotopy Lemma [1] à la Verdier [Ve], the following finiteness result.

Proposition 2.2. The stratified projection $\pi : X \to \mathbb{P}^1$ with respect to $\mathcal{W}$ is locally topologically trivial over $\mathbb{P}^1 \setminus \Lambda_f$, for some finite set $\Lambda_f \subset \mathbb{P}^1$. In particular, the restriction $\pi|_X = f$ is a locally trivial $C^\infty$-fibration over $\mathbb{C} \setminus \Lambda_f$. $\square$

We shall call the set of atypical values, and denote it by $\Lambda_f$, the minimal set $\Lambda_f$ which satisfies Proposition 2.2. Notice that, if we take the meromorphic function $q/p$ instead of $p/q$, then we get the same space $X$ (even if $X$ will be $Z \setminus \{p = 0\}$ instead of $Z \setminus \{q = 0\}$). In particular, $\Lambda_f$ can be identified with $\Lambda_f$ by the isomorphism $[t : s] \mapsto [s : t]$.

For any subset $A \subset \mathbb{P}^1$, we denote $X_A := \pi^{-1}(A)$, $F_A := f^{-1}(A)$ and the general fibre $F_t := F_t = X \cap f^{-1}(t)$, for some $t \notin \Lambda_f$. Let $D$ be a small disc centered at $a \in \Lambda_f$, such that $D \cap \Lambda_f = \{a\}$.

A crucial problem in investigating the topology of the fibres of $f$ is how to detect and to control the change of topology. This is an open problem, in general, even for holomorphic germs or polynomial functions, but well understood in case the singularities are isolated.

Definition 2.3. Let $Z, x) \to \mathbb{P}^1$ be a germ of a meromorphic function. To every $a \in \mathbb{P}^1$, one associates the germ $\pi : (X, (x, a)) \to \mathbb{P}^1$. By restriction to $X = X \setminus (\mathcal{X}^{\text{pol}} \cup X_\infty)$, this defines the germ $f_t : (X, (x, a)) \to \mathbb{C}$, which we denote by $f_{x,a}$.

For some fixed $x \in \{q = p = 0\}$, one has a one-parameter family of germs $(X, (x, a))$, indexed over $a \in \mathbb{P}^1 \setminus \{s = 0\}$. Unlike the case of holomorphic germs, here the point $(x, a)$ is not in $X$ but in its closure. So the germ $f_{x,a}$ is uniquely determined by the determination of the point $a = [p(x) : q(x)]$.

Furtheron, take a Whitney stratification $\mathcal{W}$ of $X$ which has $X$ as open stratum, remarking that $\text{Sing} X \subset \mathcal{X}^{\text{pol}}$. For all small enough radii $\varepsilon$ of a ball $B_\varepsilon(x, a) \subset \mathcal{Z} \times \mathbb{P}^1$ centered at $(x, a)$, the sphere $S_\varepsilon = \partial B_\varepsilon(x, a)$ intersects transversally all the finitely many strata in the neighbourhood of $(x, a)$. This defines a Milnor-Lê fibration (cf [Le2]), i.e. a locally trivial fibration $\pi : \mathcal{X}^{\text{pol}} \cap B_\varepsilon(x, a) \to D^*$ over a small enough
punctured disk $D^* \subset \mathbb{P}^1$ centered at $a$, which restricts to a locally trivial fibration on the complement of $X^{\text{pol}}$, namely:

\begin{equation}
(2.1) \quad f_1 : F_{D^*} \cap B_\varepsilon(x, a) \to D^*.
\end{equation}

This fibration will be called the Milnor-Lê fibration of the function germ $f_{x,a}$ at the point $(x, a) \in X$. It depends on the point $(x, a)$. In particular, the radius $\varepsilon$ of the ball depends on the point $a$. From Proposition 2.2 it follows that, since $\pi$ is stratified-transversal to $X$ over $\mathbb{P}^1 \setminus \Lambda$, the fibration $f_1 : F_\varepsilon \cap B_\varepsilon(x, a) \to D$ is a trivial fibration, for all but a finite number of values of $a \in \mathbb{P}^1$, where $x \in \{p = q = 0\}$ is fixed.

We now endow $X$ with a “partial Thom stratification”, cf [111]. Suppose that $X$ is endowed with a complex stratification $\mathcal{G} = \{\mathcal{G}_\alpha\}_{\alpha \in S}$ such that $X$ is a stratum and $X^{\text{pol}}$ is a union of strata. If $\mathcal{G}_\alpha \cap \overline{\mathcal{G}_\beta} \neq \emptyset$ then, by definition, $\mathcal{G}_\alpha \subset \overline{\mathcal{G}_\beta}$ and in this case we write $\mathcal{G}_\alpha < \mathcal{G}_\beta$.

Let $(x, a) \in X^{\text{pol}} \setminus X_{\infty}$. We consider on $X$ the Thom $(a_q)$ regularity condition at $(x, a)$, see e.g. [GWPL, ch. 1] for the definition. In terms of the relative conormal (see [Le], [HMS]), the condition $(a_q)$ at $\xi := (x, a)$ for the strata $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ translates to the inclusion: $T^{*}_{\mathcal{G}_\alpha} \supset \{(T^{*}_{\mathcal{G}_\beta})\xi\}$. It is known that this condition is independent on $q$, up to multiplication by a unit, see e.g. [111, Prop. 3.2]. We therefore may and shall refer to this as Thom regularity condition relative to $X^{\text{pol}}$, at $(x, a)$.

**Definition 2.4.** We say that $\mathcal{G}$ is a $\partial\tau$-stratification (partial Thom stratification) relative to $X^{\text{pol}}$ if at any point $\xi \in X$, any two strata $\mathcal{G}_\alpha < \mathcal{G}_\beta$ with $\xi \in \mathcal{G}_\alpha \subset X^{\text{pol}}$ and $\mathcal{G}_\beta \subset X \setminus X_{\infty}$ satisfy the Thom regularity condition relative to $X^{\text{pol}}$.

The Whitney stratification $\mathcal{W}$ of $X$ considered in Proposition 2.2 is an example of $\partial\tau$-stratification relative to $X^{\text{pol}}$. This follows from [BMM, Théorème 4.2.1] or [111, Theorem 3.9]. Nevertheless the $\partial\tau$-stratifications are less demanding than Whitney stratifications and than Thom stratifications.

**Definition 2.5.** We consider the singular locus of $\pi$ with respect to some $\partial\tau$-stratification relative to $X^{\text{pol}}$, denoted by $\mathcal{G}$ and we say that the following closed subset of $X \setminus X_{\infty}$:

\[
\text{Sing}_{\mathcal{G}}f := (X \setminus X_{\infty}) \cap \cup_{\mathcal{G}_\alpha \in \mathcal{G}} \text{closure}(\text{Sing}_{\pi|_{\mathcal{G}_\alpha}})
\]

is “the singular locus of $f$” with respect to $\mathcal{G}$. We say that $f$ has isolated singularities with respect to $\mathcal{G}$ if $\dim \text{Sing}_{\mathcal{G}}f \leq 0$. We say that $f$ has isolated singularities at $a \in \mathbb{C}$ (or at the fibre $X_a$) if $\dim X_a \cap \text{Sing}_{\mathcal{G}}f \leq 0$.

The space $X$ is nonsingular and consists of one stratum. The set $X \cap \text{Sing}_{\mathcal{G}}f$ of $\mathcal{G}$-singularities on $X$ is just the usual singular set $\text{Sing}f \subset Z \setminus \{q = 0\}$. The singularities of the new type are $X^{\text{pol}} \cap \text{Sing}_{\mathcal{G}}f$.

We show that singularities of this type are manageable when they are isolated. In this case one may localize the variation of topology of fibres, which phenomenon has been observed before in the case of holomorphic germs, by Milnor [Mi], and in case

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1 One may compare to [GLM1,2], where different definitions have been used.
of polynomial and regular functions [ST1], [Ti1]. Actually the proof for meromorphic functions follows the arguments of [Ti1, Theorem 4.3] and we leave it to the reader.

For instance, when \( \dim \mathcal{Z} = 2 \), the pencil \( p/q \) has isolated \( \mathcal{G} \)-singularities, relative to the coarsest partial Thom stratification \( \mathcal{G} \), if and only if the fibres of \( f \) are reduced.

**Proposition 2.6.** Let \( f \) have isolated singularities with respect to \( \mathcal{G} \) at \( a \in \mathbb{C} \) and let \( X_a \cap \text{Sing}_\mathcal{G} f = \{ a_1, \ldots, a_k \} \). Then the variation of topology of the fibres of \( f \) at \( F_a \) is localizable at the points \( a_i \). \( \square \)

The localization result implies that the vanishing cycles are concentrated at the isolated singularities, as follows:

**Proposition 2.7.** Let \( f \) have isolated singularities with respect to \( \mathcal{G} \) at \( a \in \mathbb{C} \) and let \( X_a \cap \text{Sing}_\mathcal{G} f = \{ a_1, \ldots, a_k \} \). Let \( D \subset \mathbb{C} \) be a small enough closed disc centered at \( a \) and let \( s \in \partial D \). Then, for any small enough balls \( B_i \subset \mathbb{C} \times \mathbb{C} \) centered at \( a_i \), we have:

(a) \( H^*(F_D, F_s) \simeq \oplus_{i=1}^k \tilde{H}^{2n-1-*}(B_i \cap X_s) \).

(b) \( H_*(B_i \cap F_D, B_i \cap F_s) \simeq \tilde{H}^{2n-1-*}(B_i \cap X_s), \forall i \in \{1, \ldots, k\} \).

**Proof.** Note first that in the local setting we have just one singular point, i.e. \( k = 1 \).

(a). A general Lefschetz duality result (see e.g. [Br, Prop. 5.2]) says that, since we work with triangulable spaces, we have:

\[
H^*(F_D, F_s) \simeq H^{2n-*}(X_D, X_s).
\]

Next, the cohomology group splits, through excision, into local contributions, by Proposition 2.6:

\[
H^*(X_D, X_s) = \oplus_{i=1}^k H^*(B_i \cap X_D, B_i \cap X_s) = \oplus_{i=1}^k \tilde{H}^{*-*}(B_i \cap X_s),
\]

where the second equality holds because \( B_i \cap X_s \) is contractible, for small enough ball \( B_i \).

(b). The same Lefschetz duality result may be applied locally to yield:

\[
H_*(B_i \cap F_D, B_i \cap F_s) \simeq \tilde{H}^{2n-*}(B_i \cap X_D, B_i \cap X_s).
\]

Note that the decomposition \( H_*(F_D, F_s) \simeq \oplus_{i=1}^k H_*(B_i \cap F_D, B_i \cap F_s) \) also follows from Proposition 2.6. \( \square \)

Our main result concerning vanishing cycles at the level of homotopy type is the following.

**Theorem 2.8.** Let \( f \) have isolated singularities with respect to some \( \partial \tau \)-stratification \( \mathcal{G} \) relative to \( X^{\text{pol}} \). Let \( F \) be a general fibre of \( f \) and let \( D \subset \mathbb{C} \) be a small enough open disc centered at some \( a \in \Lambda_f \). Then the space \( X \), resp. \( F_D \), is obtained from \( F \) to which one attaches a number of cells of real dimension \( n = \dim \mathcal{Z} \). In particular we have the following homotopy equivalences:

(a) \( X/F \simeq \bigvee S^n \), in the global setting.

(b) \( F_D/F \simeq \bigvee S^n \), in the local or global setting.
In the local setting, by $F_D$ and $F$ we mean the intersections with some small sphere $B_\varepsilon$, so (b) should read: $B_\varepsilon \cap F_D / B_\varepsilon \cap F \simeq \sqrt{S^n}$.

We shall give the proof in [3] after introducing a few technical ingredients. The number of spheres will be discussed in Corollary 3.7. In the local setting, Theorem 2.8 extends Milnor’s bouquet theorem [Mi], whereas in the global setting, it extends the bouquet result for polynomial functions [ST1, Theorem 3.1] and is of similar flavor as [Ti1, Theorem 4.6].

3. Polar curves and Milnor numbers at the indeterminacy locus

We show first that an isolated $G$-singularity at a point of $X_{p}ol$ is detectable by the presence of a certain local polar locus, which we define in the following.

**Definition 3.1.** Let $\xi = (x, a) \in X_{p}ol \setminus X_\infty$ and consider a small neighbourhood $V \subset Z$ of $x$. Let $\text{Sing} f$, respectively $\text{Sing} (f, q)$, denote the singular locus of the restriction $f : X \cap (V \times \mathbb{C}) \to \mathbb{C}$, respectively $(f, q) : X \cap (V \times \mathbb{C}) \to \mathbb{C}^2$.

The polar locus $\Gamma_\xi(f, q)$ is the germ at $\xi$ of the analytic space:

$$\text{closure}\{\text{Sing} (f, q) \setminus (\text{Sing} f \cup X_{p}ol)\} \subset X.$$  

From the definition we get the isomorphism $\Gamma_\xi(f, qu) \simeq \Gamma_\xi(p, q)$. The polar locus depends on the multiplicative unit $u$, i.e. $\Gamma_\xi(f, qu)$ is different from $\Gamma_\xi(f, q)$, meanwhile we shall prove that it induces well defined local invariants.

**Proposition 3.2.** Let $\xi = (x, a) \in X_{p}ol \setminus X_\infty$. Let $f$ have an isolated $G$-singularity at $\xi$. Then:

(a) The polar locus $\Gamma_\xi(f, qu)$ is either void or $\dim \Gamma_\xi(f, qu) = 1$, and this does not depend on the multiplicative unit $u$.

(b) The intersection multiplicity $\text{mult}_\xi(\Gamma_\xi(f, qu), X_a)$ is independent on the unit $u$.

**Proof.** For (a), we only give the rough idea and send for details to [ST2, Prop. 4.2] and [Ti3, Prop. 3.4]. The first claim follows by usual arguments, as in [Ti1]. The independence on $u$ is a consequence of the independence of the relative conormal $PT^{*}_{qu}$ proved in [Ti1, Prop. 3.2].

If the polar locus is void, the multiplicity in (b) is zero. Suppose next that $\Gamma_\xi(f, q)$ has dimension 1. Consider a small enough ball $B \subset Z \times \mathbb{C}$ centered at $\xi$, to fit in the Milnor-Lê fibration of the function $\pi$ at $\xi$:

$$\pi| : B \cap X_{D^*} \to D^*,$$

where $D \subset \mathbb{C}$ is centered at $a$. The notation $\Gamma(f, q)$ will stay for the representative in $B$ of the germ $\Gamma_\xi(f, q)$. We may choose $D$ so small, that for all $s \in \partial D$, those intersection points $X_s \cap \Gamma(f, q)$ which tend to $\xi$ when $s$ tends to $a$, are inside $B$. This is possible because $\Gamma(f, q)$ is a curve which cuts $X_{p}ol$ at $\xi$.

We shall compute the homology $H_\ast(B \cap X_s)$ of the Milnor fibre of the fibration (3.1). Inside $B$, the restriction of the function $q$ to $B \cap X_s$ has a finite number of isolated singularities, which are precisely the points of intersection $B \cap X_s \cap \Gamma(f, q)$. 


Lemma 3.3. Let $f$ have isolated $G$-singularities at $\xi$. Let $B$ be a small enough ball at $\xi$ such that the sphere $S := \partial B$ cuts transversely all those finitely many strata of $G$ which have $\xi$ in their closure and does not intersect other strata.

Then, there exist small enough discs $D$ and $\delta$ such that $(\pi, q)^{-1}(\nu)$ is transverse to $S$, for all $\nu \in D \times \delta^\circ$.

Proof. If the statement was not true, then there would exist a sequence of points $\eta_i \in S \cap (\mathbb{X} \setminus \mathbb{X}^{\text{pol}})$ tending to a point $\eta \in S \cap \mathbb{X}_a \cap \mathbb{X}^{\text{pol}}$, such that the intersection of tangent spaces $T_{\eta_i}f^{-1}(f(\eta_i)) \cap T_{\eta_i}q^{-1}(q(\eta_i))$ is contained in $T_{\eta_i}(S \cap X)$. Assuming, without loss of generality, that the following limits exist, we get the inclusion:

$$\lim T_{\eta_i}f^{-1}(f(\eta_i)) \cap \lim T_{\eta_i}q^{-1}(q(\eta_i)) \subset \lim T_{\eta_i}(S \cap X).$$

Let $G_\alpha \subset \mathbb{X}^{\text{pol}}$ be the stratum containing $\eta$. Remark that dim $G_\alpha$ $\geq 2$, since $G_\alpha \ni \xi$ and $\pi \cap G_\alpha$. This implies that dim $G_\alpha \cap X_\alpha \geq 1$.

We have, by the definition of the stratification $G$, that $\lim T_{\eta_i}q^{-1}(q(\eta_i)) \supset T_\eta G_\alpha$ and obviously $T_{\eta_i}(G_\alpha \cap X_\alpha) \subset T_\eta G_\alpha$. On the other hand, $\lim T_{\eta_i}f^{-1}(f(\eta_i)) \supset T_\eta (G_\alpha \cap X_\alpha)$, since $\pi \cap G_\alpha$. In conclusion, the intersection in (3.2) contains $T_\eta (G_\alpha \cap X_\alpha)$. But, since $S \cap G_\alpha$, the limit $\lim T_{\eta_i}(S \cap X)$ cannot contain $T_\eta (G_\alpha \cap X_\alpha)$ and this gives a contradiction.

Let $\hat{\delta}$ be so small that $B \cap X_s \cap q^{-1}(\hat{\delta}) \cap \Gamma(f, q) = \emptyset$. By the Lemma 3.3 above and by choosing appropriate $D$ and $\delta$, the map $q : B \cap X_s \cap q^{-1}(\hat{\delta}) \rightarrow \hat{\delta}^\circ$ is a locally trivial fibration. Therefore $B \cap X_s \cap q^{-1}(\hat{\delta})$ is homotopy equivalent, by retraction, to the central fibre $B \cap X_s \cap \mathbb{X}^{\text{pol}}$. This proves our claim.

We now remark that the central fibre $B \cap X_s \cap \mathbb{X}^{\text{pol}}$ is just the complex link at $\xi$ of the space $\mathbb{X}^{\text{pol}}$. The space $\mathbb{X}^{\text{pol}}$ is a product $\{q = 0\} \cap \{p = 0\} \times \mathbb{C}$ at $\xi$, along the projection axis $\mathbb{C}$, hence its complex link is contractible, and so is $B \cap X_s \cap q^{-1}(\hat{\delta})$.

Pursuing the proof of Proposition 3.2, we observe that $B \cap X_s$ is homotopy equivalent to $B \cap X_s \cap q^{-1}(\tilde{\delta})$, for $D$ and $\delta$ like in Lemma 3.3 and, in addition, the radius of $D$ much smaller than the radius of $\delta$. This supplementary condition is meant to insure that $\Gamma(f, q) \cap B \cap X_s = \Gamma(f, q) \cap B \cap X_s \cap q^{-1}(\hat{\delta})$.

Now, the total space $B \cap X_s \cap q^{-1}(\hat{\delta})$ is built up by attaching to the space $B \cap X_s \cap q^{-1}(\tilde{\delta})$, which is contractible, a finite number of cells of dimension $n - 1$, which correspond to the Milnor numbers of the isolated singularities of the function $q$ on $B \cap X_s \cap q^{-1}(\delta \setminus \hat{\delta})$. The sum of these numbers is, by definition, the intersection multiplicity $\text{mult}_\xi(\Gamma(f, q), X_\alpha)$.

We have proven that:

$$(3.3) \quad \dim H_{n-1}(B \cap X_s) = \text{mult}_\xi(\Gamma(f, q), X_\alpha) \quad \text{and} \quad \tilde{H}_i(B \cap X_s) = 0, \text{ for } i \neq n - 1.$$ 

When replacing all over in our proof the function $q$ by $qu$, we get the same relation (3.3), with $qu$ instead of $q$. This concludes our proof of 3.2. \qed
The above proof shows that \( B \cap X_s \) is, homotopically, a ball to which one attaches a certain number of \((n-1)\)-cells. We therefore get:

**Corollary 3.4.** Let \( f \) have an isolated \( G \)-singularity at \( \xi \). The fibre \( B \cap X_s \) of the local fibration \((3.1)\) is homotopy equivalent to a bouquet of spheres \( \bigvee S^{n-1} \).

\[ \□ \]

**Definition 3.5.** We denote the number of spheres by \( \lambda_\xi := \dim H_{n-1}(B \cap X_s) \) and call it the polar Milnor number at \( \xi \). We say that \( f \) has vanishing cycles at \( \xi \) if \( \lambda_\xi > 0 \).

In the global setting, then we denote by \( \lambda_a \) the sum of the polar Milnor numbers at singularities on \( X_a \cap X_{pol} \) and also denote \( \lambda = \sum_{a \in A} \lambda_a \).

3.1. **Proof of Theorem 2.8.** We take back the notations of Theorem 2.8. Since \( \text{Sing}_G f \subset X \setminus X_{\infty} \) is a finite set of points, the variation of topology of the fibres of \( f \) is localizable at those points (cf. Proposition 2.6). Let \( X_a \cap \text{Sing}_G f = \{a_1, \ldots, a_k\} \), with \( k = 1 \) in the local setting.

For some point \( a_i \in X \cap \text{Sing}_G f \), by Milnor’s classical result for holomorphic functions with isolated singularity \([Mi]\), it follows that the pair \((B_{\varepsilon,i} \cap F_{D_a}, B_{\varepsilon,i} \cap F_s)\) is \((n-1)\)-connected, where \( s \in D^*_a \).

In case \( a_i \in X_{pol} \cap \text{Sing}_G f \), we may invoke the following lemma, which is a version of a result by Hamm and Lê \([HL, \text{Corollary 4.2.2}]\) for our partial Thom stratification:

**Lemma 3.6.** (\([Ti1, \text{Cor. 2.7}]\)) The pair \((B_{\varepsilon,i} \cap F_{D_a}, B_{\varepsilon,i} \cap F_s)\) is \((n-1)\)-connected, where \( s \in D^*_a \). \[ \□ \]

We conclude that the space \( X \) is built up starting from a fibre \( F \), then moving it within a fibration with a finite number of isolated singularities. By the above connectivity result and by Switzer’s result \([Sw, \text{Proposition 6.13}]\), at each singular point one has to attach a number of cells of dimensions \( \geq n \). In fact the cells to be attached are of dimension precisely \( n \), by the following reason. We may apply the duality Proposition 2.7(b) and invoke Corollary 3.4, which show that the relative homology \( H_s(F_{D_a}, F_s) \) is concentrated in dimension \( n \).

Then one can map a bouquet of \( n \) spheres into \( F_{D}/F_s \) such that this map is an isomorphism in homology. This implies, by Whitehead’s theorem, that the map induces an isomorphism of homotopy groups. (Remark that \( F_{D}/F_s \) is simply connected whenever \( n \geq 2 \).) Since we work with analytic objects, therefore triangulable, the space \( F_{D}/F_s \) is a CW-complex. For CW-complexes, weak homotopy equivalence coincides with homotopy equivalence.

Let us remark that the total number of \( n \)-cells is the sum of the local Milnor numbers, resp. the polar Milnor numbers. This ends the proof of our theorem.

As consequence, we get the relative Betti numbers (see Proposition 2.7). This may be compared to similar formulas in case of polynomial functions \([ST1]\).

**Corollary 3.7.** Let \( f \) have isolated \( G \)-singularities at \( a \in \mathbb{C} \) with respect to some \( \partial \tau \) stratification \( G \). Then \( H_j(F_{D}, F_s) = 0 \) for \( j \neq n \) and:

\[ b_n(F_D, F_s) = (-1)^n \chi(F_D, F_s) = \mu_a + \lambda_a, \]
where $\mu_a$ is the sum of the Milnor numbers of the singularities of $F_a$ and $\lambda_a$ denotes the sum of the polar Milnor numbers at $X_a \cap X_{pol}$.

In particular, if $f$ has isolated $G$-singularities at all fibres, then:

$$b_n(X, F) = (-1)^n \chi(X, F) = \mu + \lambda, \quad H_j(X, F) = 0, \text{ for } j \neq n,$$

where $\mu$ is the total Milnor number of the singularities of $f$ on $Z \setminus \{q = 0\}$ and $\lambda$ is the total polar Milnor number at $X_{pol} \setminus X_{\infty}$.

4. Vanishing cycles in special cases and examples

The singular locus $\text{Sing} X \subset Z \times \mathbb{C}$ is contained in $X_{pol}$ and can be complicated. We have $\text{Sing} X \setminus X_{\infty} = \cup_{t \in \mathbb{C}} (\text{Sing} X_t) \cap X_{pol}$. However, $X_{pol} \setminus \text{Sing} X$ is a Whitney stratum and $\text{Sing} X$ is a union of Whitney strata, in the canonical Whitney stratification $\mathcal{W}$ of $X$ which has $X$ as a stratum.

We shall consider here a $\partial \tau$-stratification $G$ which may be coarser than $\mathcal{W}$ (and which exists, by Definition 2.4 and the remark following it). Then $\text{Sing}_G f \cap X_{pol} \subset \text{Sing} X$. Indeed, this follows from the fact that the space $X_{pol} \setminus X_{\infty}$ is locally a product $\{q = p = 0\} \times \mathbb{C}$ and the projection $\pi$ is transversal to it off $\text{Sing} X$.

In particular, for $n = 2$, $f$ has isolated $G$-singularities at $a \in \mathbb{C}$ if and only if $F_a$ is reduced.

Let $\xi = (x, a) \in X_{pol} \setminus X_{\infty}$. We assume in the following that $\dim_\xi \text{Sing} X_a = 0$. This implies that $\dim_\xi \text{Sing}_G f \leq 0$ and that the germ $(\text{Sing} X, \xi)$ is either a curve or just the point $\xi$. If it is a curve, then it can have several branches and its intersection with $X_s$ is, say, $\{\xi_1(s), \ldots, \xi_k(s)\}$, for any $s \in D^*$, where $D \subset \mathbb{C}$ is a small enough disc at $a$.

The germs $(X_s, \xi_i(s))$ are germs of hypersurfaces with isolated singularity. Let $\mu_i(s)$ denote the Milnor number of $(X_s, \xi_i(s))$. Then $\sum_{i=1}^k \mu_i(s) \leq \mu(a)$. Equality may hold only if $k = 1$, by the well known non-splitting result of Lê D.T [Lê1]. In general, we have:

**Theorem 4.1.** Let $\dim_\xi \text{Sing} X_a = 0$ and $\dim_\xi \text{Sing} X = 1$. Then:

$$\lambda_\xi = \mu(a) - \sum_{i=1}^k \mu_i(s).$$

**Proof.** The hypothesis implies that the germ of $\text{Sing}_G f$ at $\xi$ is the point $\xi$ or it is void. For any $s \in D$ small enough, the germ $(X_s, \xi_i(s))$ is locally defined by the function:

$$h = p - tq : (Z \times \mathbb{C}, \xi_i(s)) \to \mathbb{C}.$$

We have that, locally at $\xi$, the singular locus $\text{Sing} h$ is equal to $\text{Sing} X$, in particular included into $X_{pol}$. Consider the map $(h, t) : (Z \times \mathbb{C}, \xi_i(s)) \to \mathbb{C}^2$. Note that the polar locus $\Gamma_\xi(h, t)$ is a curve or it is void, since $\xi$ is an isolated $G$-singularity. Following [Lê2], there is a fundamental system of privileged polydisc neighbourhoods of $\xi$ in $Z \times \mathbb{C}$, of the form $(P_a \times D'_a)$, where $D'_a \subset \mathbb{C}$ is a disc at $a$ and $P_a$ is a polydisc at $x \in Z$ such
that the map

\[(h, t) : (\mathbb{Z} \times \mathbb{C}) \cap (P_\alpha \times D'_\alpha) \cap (h, t)^{-1}(D_\alpha \times D'_\alpha) \to D_\alpha \times D'_\alpha\]

is a locally trivial fibration over \((D^*_\alpha \times D'_{\alpha}) \setminus \text{Im}(\Gamma(h, t))\). We chose \(D_\alpha\) and \(D'_\alpha\) such that \(\text{Im}(\Gamma(h, t)) \cap \partial(D^*_\alpha \times D'_{\alpha}) = \text{Im}(\Gamma(h, t)) \cap (D^*_\alpha \times \partial(D'_{\alpha}))\). Let \(s \in \partial D_\alpha\). Observe that \((P_\alpha \times D'_{\alpha})\) is contractible, since it is the Milnor fibre of the linear function \(h, t\) on a smooth space. This is obtained, up to homotopy type, by attaching to \((P_\alpha \times D'_{\alpha})\) a certain number \(r\) of \(n\)-cells, equal to the sum of the Milnor numbers of the function \(h_1 : (P_\alpha \times D'_{\alpha}) \to D_\alpha\). Since we have the homotopy equivalence \((h, t)^{-1}(0, s) \cap (P_\alpha \times D'_{\alpha}) \simeq B \cap \mathbb{X}_s\), we get, by Corollary 3.4 and Definition 3.5, that \(r = \lambda_\xi\).

Now \((h, t)^{-1}(\eta, s) \cap (P_\alpha \times D'_{\alpha})\) is homotopy equivalent to the Milnor fibre of the germ \((\mathbb{X}_\alpha, \xi)\), which has Milnor number \(\mu(a)\). The space \((h, t)^{-1}(s) \cap (P_\alpha \times D'_{\alpha})\) is obtained from \((h, t)^{-1}(\eta, s) \cap (P_\alpha \times D'_{\alpha})\) by attaching exactly \(r\) cells of dimension \(n\) (coming from the polar intersections) and of a number of \(n\)-cells coming from the intersections with \(\text{Sing}\ h\). This number of cells is, by definition, \(\sum_{i=1}^{k} \mu_i(s)\). We get the equality \(\mu(a) = r + \sum_{i=1}^{k} \mu_i(s)\). Since \(r = \lambda_\xi\), our proof is done. \(\square\)

**Remark 4.2.** If in the hypothesis of Theorem 4.1 the dimension of \(\text{Sing}\ \mathbb{X}\) is not 1 but 0, then the result still holds, with the remark that in this case \(\mu_i(s) = 0\), \(\forall i\) and \(\forall s \in D^*\). Hence \(\lambda_\xi = \mu(a)\).

We give in the remainder three examples, one on \(\mathbb{P}^2(\mathbb{C})\) with no singularities in the complement of \(\mathbb{X}_{\text{pol}}^0\) and two on a nonsingular quadratic surface in \(\mathbb{P}^3(\mathbb{C})\).

**Example 4.3.** \(E_{p,q}^{a,b}: f = \frac{x(z^{a+b} + x^a y^b)}{y^p z^q}\), with \(a + b + 1 = p + q\) and \(a, b, p, q \geq 1\).

This defines a meromorphic function on \(\mathbb{P}^2(\mathbb{C})\). For some \(t \in \mathbb{C}\), the space \(\mathbb{X}_t\) is given by:

\[(4.1)\]

\[x(z^{a+b} + x^a y^b) = ty^p z^q\]

We have \(\mathbb{X}_{\text{pol}}^0 \setminus \mathbb{X}_\infty = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} \times \mathbb{C}\). According to Theorem 4.1, we look for jumps in the Milnor number within the family of germs (4.1):

(a) at \([1 : 0 : 0]\), chart \(x = 1\). No jumps, since uniform Brieskorn type \((b, a + b)\).

(b) at \([0 : 1 : 0]\), chart \(y = 1\). For \(t \neq 0\), Brieskorn type \((a+1, q)\), with \(\mu(t) = a(q - 1)\).

If \(t = 0\), then we have \(x^{a+1} + xz^{a+b} = 0\) with \(\mu(0) = a^2 + ab + b\) and the jump at \(\xi = ([0 : 1 : 0], 0)\) is \(\lambda_\xi = a^2 + ab + b - a(q - 1) = b + ap\), by Theorem 4.1.

(c) at \([0 : 0 : 1]\), chart \(z = 1\). No jumps, since type \(A_0\) for all \(t\).

We get the total jump \(\lambda = b + ap\). A straightforward computation shows that \(\mu = 0\). The fibres of \(f\) can be described as follows. If \(t = 0\), we have \(c + 1\) disjoint copies of \(\mathbb{C}^*\), where \(c = \gcd(a, b)\); hence \(\chi(F_0) = 0\). If \(t \neq 0\), we compute \(\chi(F) = -(b + ap)\), by a branched covering argument. The vanishing homology is concentrated in dimension 2. Taking \(X = \mathbb{C}^2 \setminus \{y = 0\}\), we get the Betti number \(b_2(X, F) = \chi(X, F) = \chi(X) - \chi(F) = -b - ap\).
\[ \chi(F) = 0 + (b + ap) = b + ap. \] It follows \( b_2(X, F) = \lambda + \mu \), which agrees with Corollary 3.7.

**Example 4.4.** Let \( Z \subset \mathbb{P}^3 \) be the nonsingular hypersurface given by \( h = x^2 + z^2 + yw = 0 \) and consider the meromorphic function \( y/x \). It has its axis \( \{x = y = 0\} \) tangent to \( h = 0 \) at \([0:0:0:1]\). By computations we get \( \mu = 0, \lambda = 1 \) (jump \( A_0 \rightarrow A_1 \), at \( t = 0 \)). The general fibre \( F \) is contractible, the special fibre \( F_0 \) is \( \mathbb{C} \cup \mathbb{C} \) and \( X \) is homotopy equivalent to \( S^2 \). All the connected components of fibres are contractible, however the global vanishing homology is generated by a relative 2-cycle. We may remark here that the jump \( A_0 \rightarrow A_1 \) cannot occur in case of polynomial functions at infinity.

**Example 4.5.** Consider the meromorphic function \( f = \frac{x(z^2 + xy)}{z^3} \) on the nonsingular hypersurface \( Z \subset \mathbb{P}^3 \) given by \( h = yw + x^2 - z^2 = 0 \). Then \( \mathbb{X}_\text{pol} \setminus \mathbb{X}_\infty = [0:0:0:1] \times \mathbb{C} \cup [0:1:0:0] \times \mathbb{C} \), where \([x:y:z:w]\) are the homogeneous coordinates in \( \mathbb{P}^3 \).

Along \([0:0:1:0] \times \mathbb{C} \), in the chart \( w = 1 \) and coordinates \( x \) and \( z \) on \( X \), we have the family of curves (germs of \( \mathbb{X}_t \)):

\[ (4.2) \quad x(z^2 - x^3 + xz^2) = tz^3. \]

For all \( t \), this is a \( D_3 \) singularity, so no jumps.

Along \([0:1:0:0] \times \mathbb{C} \), in the chart \( y = 1 \) and, again, \( x \) and \( z \) as coordinates on \( X \), we have the family of curves (germs of \( \mathbb{X}_t \)):

\[ (4.3) \quad x(z^2 + x) = tz^3. \]

This has type \( A_2 \) if \( t \neq 0 \) and \( A_3 \) if \( t = 0 \). Thus the jump at \( \xi := ([0:1:0:0], 0) \) is \( \lambda_\xi = 1 \) and the total jump is \( \lambda = 1 \).

By simple computations, we get \( \mu = 2 \), since there are two singular fibres, \( F_{\pm 1} \), with \( A_1 \)-singularities. There are 3 atypical fibres: \( F_0 \simeq \mathbb{C}^* \cup \mathbb{C}^* \), \( F_{\pm 1} \simeq \mathbb{C}^* \) and the general fibre \( F \simeq \mathbb{C}^* \). Since \( X \simeq S^2 \), we get \( b_2(X, F) = 2 - (-1) = 3 \) global vanishing cycles, \( X/F \simeq \bigvee_3 S^2 \).

5. Monodromy fibration and a global Picard phenomenon

We continue to consider the local and global settings in the same time. Let \( \hat{D} \subset \mathbb{C} \) be a closed disc, big enough such that \( \hat{D} \supset \Lambda_f \setminus \infty \), where \( \infty \) denotes the point \([1:0] \in \mathbb{P}^1 \).

Let \( D_i \subset \hat{D} \) be a small enough closed disc at \( a_i \in \Lambda_f \), such that \( D_i \cap \Lambda_f = \{a_i\} \). Take a point \( s \) on the boundary of \( \hat{D} \) and, for each \( i \), a path \( \gamma_i \subset \hat{D} \) from \( s \) to some fixed point \( s_i \in \partial D_i \), with the usual conditions: the path \( \gamma_i \) has no self intersections and does not intersect any other path \( \gamma_j \), except at the point \( s \). By Proposition 2.2 the fibration \( f : X \setminus f^{-1}(\Lambda_f) \rightarrow \mathbb{C} \setminus \Lambda_f \) is locally trivial, hence we may use excision in the pair \((F_D, F_s)\) and get an isomorphism (induced by the inclusion of pairs):

\[ (5.1) \quad \oplus_{a_i \in \Lambda_f} H_*(F_{D_i}, F_{s_i}) \rightarrow H_*(X, F_s), \]
This shows that each inclusion \((F_{D_i}, F_{s_i}) \subset (X, F_{s_i})\) induces an injection in homology \(H_\ast(F_{D_i}, F_{s_i}) \hookrightarrow H_\ast(X, F_{s_i})\). We also get by excision the following split exact sequence:

\[
0 \to H_\ast(F_{D_i}, F) \to H_\ast(X, F) \to \bigoplus_{a_j \in \Lambda_f, j \neq i} H_\ast(F_{D_j}, F) \to 0.
\]

We next consider the monodromy \(h_i\) around an atypical value \(a_i \in \Lambda_f\). This is induced by a counterclockwise loop around the small circle \(\partial D_i\). The monodromy acts on the pair \((X, F)\) and we denote its action in homology by \(T_i\).

The following sequence of maps:

\[
\begin{align*}
H_{q+1}(X, F) & \xrightarrow{\partial} H_q(F) \xrightarrow{w} H_{q+1}(F_{\partial D_i}, F) \xrightarrow{i} H_{q+1}(X, F),
\end{align*}
\]

where \(w\) denotes the Wang map (which is an isomorphism, by the Künneth formula), gives, by composition, the map \(T_i - \text{id} : H_{q+1}(X, F) \to H_{q+1}(X, F)\).

This overlaps the first two maps in the following sequence:

\[
H_q(F) \xrightarrow{w} H_{q+1}(F_{\partial D_i}, F) \xrightarrow{i} H_{q+1}(X, F) \xrightarrow{\partial} H_q(F).
\]

The last arrow in the sequence (5.2) fits in the commutative diagram:

\[
\begin{array}{ccc}
H_{q+1}(F_{\partial D_i}, F) & \xrightarrow{i} & H_{q+1}(X, F) \\
\downarrow & & \downarrow \\
H_{q+1}(F_{D_i}, F) & & \\
\end{array}
\]

where all three arrows are induced by inclusion.

It follows that the submodule of “anti-invariant cycles” \(\mathcal{I}_\ast(T_i) := \text{Im}(T_i - \text{id} : H_\ast(X, F) \to H_\ast(X, F))\) is contained in the direct summand \(H_\ast(F_{D_i}, F)\) of \(H_\ast(X, F)\). If \(\mathcal{I}_\ast\) denotes the submodule generated by \(\mathcal{I}_\ast(T_i)\), for all \(a_i \in \Lambda_f\), then:

\[
\mathcal{I}_\ast = \bigoplus_{a_i \in \Lambda_f} \mathcal{I}_\ast(T_i).
\]

Using Picard’s decomposition of the monodromy, Lefschetz has proven the famous relation for the monodromy around a simple nodal singularity on a nonsingular ambient space, wellknown as Picard-Lefschetz formula. The following result describes a global Picard phenomenon.

**Proposition 5.1.** Identify \(H_\ast(X, F)\) to \(\bigoplus_{a_i \in \Lambda_f} H_\ast(F_{D_i}, F)\) by the isomorphism (5.1). Then, for \(\omega \in H_\ast(X, F)\), we have:

\[
T_i(\omega) = \omega + \psi_i(\omega),
\]

for some \(\psi_i(\omega) \in H_\ast(F_{D_i}, F)\). \(\square\)

In the global setting, when specializing to a homologically contractible total space \(X\), the natural \(\partial\)-map \(H_\ast(X, F) \to H_{\ast-1}(F)\) becomes an isomorphism and we get: \(\tilde{H}_\ast(F) = \bigoplus_{a_i \in \Lambda_f} H_{\ast+1}(F_{D_i}, F)\). This occurs for instance in case of a polynomial function \(g : \mathbb{C}^n \to \mathbb{C}\), for which \(X = \mathbb{C}^n\). Results on invariant cocycles were obtained in [NNT]. In our more general setting, these results can also be proved by dualizing from homology to cohomology. One obtains in this way statements about invariant cocycles \(\text{Ker}(T^i - \text{id} : H^\ast(X, F) \to H^\ast(X, F))\) instead of anti-invariant cycles.
In the particular case of polynomial functions, the above Picard formula (extracted from our preprint [ST2]) was independently noticed in [NN2] and [DN1]. We end by an easy consequence, remarked in the special case of polynomial functions in [DN2], which holds in our more general setting of local and global meromorphic functions.

**Corollary 5.2.** Assume that the number of paths is \(l\) and the paths \(\gamma_1, \ldots, \gamma_l\) are counterclockwise ordered. Then the Coxeter element \(T_\infty := T_l \circ \cdots \circ T_1\) determines the generators \(T_i, \forall i \in \{1, \ldots, l\}\).

**Proof.** Use the direct sum decomposition (5.3) together with the following adapted decomposition of \(T_\infty - \text{id}\), where we couple two-by-two consecutive terms: \(T_\infty - \text{id} = (T_l \circ \cdots \circ T_1 - T_{l-1} \circ \cdots \circ T_1) + \cdots + (T_1 - \text{id})\).

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