ABSTRACT

New status in quantum mechanics is connected with recent achievements in the inverse problem. With its help instead of about ten exactly solvable models which serve as a basis of the contemporary education there are infinite (!) number, even complete sets of such models. So, the whole quantum mechanics is embraced by them. They correspond to all possible variations of spectral parameters which determine all properties of quantum systems. There appears a possibility to change at wish quantum objects by variation of these parameters as control levers and examine quantum systems in different thinkable situations. As a result, we acquire a vision of the intrinsic logic of behavior of any thinkable system, including real ones. The regularities revealed by computer visualization of these models were reformulated into unexpectedly simple universal rules of arbitrary transformations and what is more, their elementary constituents were discovered (new breakthrough). Of these elementary "bricks" it is possible in principle to construct objects with any given properties. This book of inverse problem quantum pictures is utmost intelligible and recommended to any physicists, chemists, mathematicians, biologists from students to professors who are interested in laws of the microworld.
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Preface

"The method of the inverse problem is without any hesitation one of the most beautiful discoveries in the mathematical physics of XX century"

Novikov S.P., Zakharov V.E., Manakov S.V., Pitaevsky L.P.

Quantum mechanics has recently celebrated 100 years of the notion of quant (M.Plank) and 75 years of the Schrödinger equation which determines from the given potential the wave function and hence the properties of the investigated object, the direct problem. Only in a quarter of a century there appeared the inverse problem (IP) equations of Gelfand-Levitan-Marchenko [?, ?]. This achievement of Russian mathematicians gives us the all penetrating vision. There are no tools like our eyes with microscopes for examining details of such small objects as atomic and nuclear systems. For this we need "illumination" with such ultra-short waves which cannot be operated as the visible light. It appears that the formalism of IP replace our eyes here. This is a wonderful present of mathematicians, which for a long time was not properly realized and used by physicists.

Quantum mechanics like the Moon was known only from one side (direct problem). This absolutely unsatisfactory situation with quantum education remains till now. So we need the global program of raising the level of professional qualification and this book is intended for the physical community from students to professors. With its help they will acquire the new knowledge with the minimal efforts.

Standing "on the shoulders of the giants" (being based on the results of the fathers-creators of the IP) we can succeed with computer visualization of exactly solvable models (ESM) in widening the horizon of quantum literacy. So, for us the whole comprehension of the contemporary wave mechanics was significantly transmuted. These were the fascinating lessons on quantum intuition which we would like to share with readers.

The suggested monograph is not to serve as substitution for the already published textbooks on quantum mechanics, but as important addition. The reader will be acquainted with the algorithms of elementary and universal transformations: the potential perturbations which change any chosen spectral parameter keeping all other unperturbed.

So, the notion is acquired about the simple "building blocks" of which as with children toy constructor set can be composed quantum systems with desired properties. So, it is possible (at least theoretically) to transform gradually one quantum system into any other one. This illustrates the methods of the theory of spectral, scattering and decay control. This is such a degree of quantum mastery about which it was difficult even to dream in the direct problem approach. And it is done following the discovered in IP very simple and exact formulae of the infinite number, even (complete sets! of exactly solvable models (ESM), these "milestones of the cognition". Meanwhile the most recent textbooks give the information about hardly ten such models (rectangular, oscillator, Coulomb, etc. potentials).
To the first successes on this way were devoted our books [?]: ”Lessons on Quantum Intuition”, 1996 and ”New ABC of Quantum Mechanics in pictures”, 1997. The best results from them was included in this book, but with completely renewed illustrations, which were selected from the thousands we have got during the recent years.

As a principle of exposition we have used the comparisons of the direct and inverse problems, of peculiarities of wave motion over the discrete lattices and continuum variables, of the one- and multichannel processes, of the solutions for the second and higher order equations. This book is based on the lecture courses [?] delivered at the leading Moscow universities (MSU, Phys.-Tech., MEPHI) and reports in ∼ 150 centers of the world.

It is desired that the reading of this book will be maximally creative process. Authors are particularly interested in cooperation with the readers in different forms. Please write us (zakharev@thsun1.jinr.ru; chabanov@thsun1.jinr.ru) your comments on aspects you will like in this book and any criticism and your suggestions, too. You are invited to our homepage: [http://thsun1.jinr.ru/~zakharev/].
INTRODUCTION

"I wanted most to give you some appreciation of the wonderful world and the physicist’s way of looking at it, which, I believe, is a major part of the true culture of modern times. (there are probably professors of other subjects who would object, but I believe that they are completely wrong). Perhaps you will not only have some appreciation of this culture; it is even possible that you may want to join in the greatest adventure that the human mind has ever begun" Richard P. Feynman

Infinite examples of solving the Schrödinger equation during several decades from its creation have convinced physicists that arbitrary variations of potentials $V$ lead to perturbation of almost all spectral characteristics ($S$). This witnessed also about complicated nonlinear connections of interactions $V$ with observables ($S$) in spite of linearity of the Schrödinger equation itself. It seemed that in this situation even a suspicion about the possibility of independent variations of separate $S$ parameters could not appear. Really, till now the common reaction of an audience to information about such delicate spectral control (including the authors of textbooks on quantum mechanics) is used to be expressed in emotional words of type: "is it possible ??!".

Such possibility to change the chosen spectral parameter with this sniper’s accuracy would be fantastically great because the spectral parameters could be the levers of flexible and precise control of the physical properties. This was discovered by using the inverse problem (IP) formalism, where $S$ are the input parameters. Traditionally, IP was considered as restoration $S \rightarrow V$ of a concrete potential from experimental data $S$. We ourselves have not understood for a long time that it is incomparably more interesting to use potential transformations $\tilde{V} \rightarrow V$ for construction of arbitrary quantum systems.

This new status of IP is promoted by existence of exact analytical expressions for the transformed potentials and the wave functions (exactly solvable models – ESM) which correspond to the changes of separate spectral parameters $S$. The complete set of $S$-parameters determines the relevant potential uniquely. So, the sequence of ESM corresponding to variations of the picked out parameters allow one in principle to approximate gradually with an arbitrary precision any imaginable object, see, for example, Fig.4.1.

All these innumerable ESM of the potential transformations $\Delta V = V - \tilde{V}$ are simple building blocks in themselves. The computer visualization allowed us to clarify that they are in their turn combined of the simplest previously unknown fundamental constituents. There were revealed corresponding elementary universal algorithms of constructing quantum systems.

Nothing like this we have expected to discover in quantum mechanics. This gives a more deep understanding of quantum theory (its essence, the relations $S \rightarrow V$). The particular value of these algorithms is their bif qualitative predictive power, which has already helped us to find new effects, for example, those listed below in
this introduction. Without this, their search would be like a blind hope to find “a needle in a hay stack”. It is often possible to succeed in revealing the key points of the new effects even without computer. Such a quantum intuition allows one to economize the efforts, memory and time. Here is a **rare combination of the simplicity with the mathematical precision**.

The corresponding mathematical formalism will be also presented in a maximal accessible form. The IP is introduced as generalization of the widely known procedure of Gramm-Schmidt orthogonalization to the case of infinite, and what is more, continuous "number" of vectors. The reader will be acquainted with the methods of supersymmetry (SUSYQ). The IP ESM’s turn out to be a subset of ESM generated by SUSYQ. This increases still more the possibilities of the quantum control. For example, sometimes with SUSYQ the energy level shifts becomes infinitely more simple procedure.

For most physicists, the IP is still "the thing in itself". So we try to share our discoveries which transform the **antiintuitive aspects of the theory into obvious ones** with everybody who is interested in new quantum literacy.

We shall briefly enumerate some "wonders" of the quantum design to give a notion about most essential points in the book contents. In order to illustrate corresponding examples we shall refer most impatient readers to selected important figures scattered throughout the book. Here, it is necessary to realize that each of these wonders is not only a separate point in the space of quantum knowledge, but a bundle of rays elucidating significant regularities hidden before, and all together this points compose the supporting manifold of notions renovating our quantum intuition.

◊ **We begin with shifting** arbitrary separate bound state energy level $E_n$, **keeping all other spectral data unchanged**. Fulfilling an infinite number of these additional conditions by choosing the proper potential in direct problem approach was a hopeless thing. In IP and SUSYQ approaches it is one of simplest standard operations. The following qualitative rule was discovered for the shifts up (down) of a level. **Each bump of the chosen state must be pushed by the potential barrier (well) in the vicinity of the bump extremum, and for conservation the positions of other levels there must be compensating wells (barriers) in the vicinity of the knots of the chosen state.** See the examples for one bump in Figs.1.1, 1.2 and exact formulae (??).

◊ **The mutual coincidence of spectral parameters in two different systems (1,2) for lowest bound states brings together the profiles of the corresponding potentials $V_1$, $V_2$ in the proper energy region.** This is a qualitative illustration of the theorem about a unique correspondence between the potential and its spectral data. In a particular case, it can be achieved by the energy level shifts of bound states in the above-mentioned way, as is shown in Figs. 4.1, 4.2.

◊ **The same algorithm of the level shifts in the infinite deep rectangular well can be applied to the control of spectral zones of the periodical systems.** The corresponding potential perturbation must be periodically continued from the given interval. As a result, the chosen zone of the band spectrum can be shifted over the energy scale (see, e.g., Fig.6.22). Really, each energy level of a system in a separate
finite interval is expanded into the allowed zone after the periodical continuation (appearance of the spectral band structure). So, the shift of one such level give rise to the shift of the spectral zone, too. We can join in this way the neighbor allowed zones, which means the **liquidation of one lacunae** (forbidden zone).

- The method analogous to the energy level shift allows one to **tear from the continuous spectrum its lowest state, lower boundary** \((E = 0)\), **transforming it into bound state** \((E_n < 0)\), see Fig.1.6. So, with the initial system of a free wave motion \((\dot{V}(x) = 0)\) we get **absolutely transparent** potentials. This is because the considered elementary transformations do not violate the spectral properties of other states of the continuous spectrum of free initial motion.

- The **shift of the chosen bound state into the complex energy plane**, transforms the corresponding initial stationary state into the decaying one. **Only in contrast with the usual Gamov decaying states** these states have not unrestricted growth in any direction.

- It is possible to construct still unknown **periodical potentials without(!) spectral lacunae** with such shifts \(E \rightarrow E + i\gamma\) of the continuum spectrum states.

- You will also see how to **shift the localization of the wave function** \(\Psi_n(x)\) of the chosen \(n^{th}\) state. It means the control of the probability density distribution via the change of the **spectral weight factor** \(n\) (SWF). For example, the wave function can be pressed to the origin, see Fig.2.1. Variations of different SWF \(c_n\) together with the shifts of energy levels compose a **complete set** of arbitrary potential transformations. It unexpectedly appears possible to understand how the individual bumps of the wave functions are transformed by variation of a SWF and what simplest potential perturbations are needed for that. **The block well-barrier (or barrier-well) shifts the corresponding bump to the left (right).** Simultaneously, all other states undergo some recoil in the opposite direction, so that there occurs separation of the chosen state, see Figs 2.1, 2.2. This is an important element of the quantum literacy, which has never before been mentioned.

- Remarkable itself, the same algorithm appears to be suitable also for **removing the given energy level or creation of a new level at the prescribed place (position)** of the bound states discrete spectrum, keeping all other energy levels untouched. When the SWF \(c_n\) is changed to zero, see eqs. (??) and (??), the \(n^{th}\) bound state is removed to infinity by a carrier-potential, see Fig.2.5.

- Analogously, the creation of bound states can be considered as carrying them from "behind the horizon" to the desired place.

- The same algorithm of shifting states to the left (right) in the infinite rectangular well can be used for **tearing the continuous spectrum by creation of lacuna, the forbidden spectral zone of the periodic system** at the given place on the energy scale. For this, it is needed to continue periodically the potential perturbation for shifting the state with the energy \(E_n\). Really, the symmetry violation of the eigenfunction \(\Psi_n(x)\) pressed to one of the ends of the separate interval results in divergent solutions for the periodical structure. This is because smooth matching of the unsymmetrical solutions on the adjacent intervals (with periodical potential perturbation continuation) is possible only with exponentially
increasing oscillations. Such behavior is characteristic of solutions in the forbidden zone. For the well-known analogy it can be compared with the exponential increase in solutions under the potential barriers. Since we can arbitrary change the value of SWF $c$ for arbitrary auxiliary eigenvalue problems, the degree of forbiddenness in an arbitrary energy point of the spectrum is under our control.

There are exact solutions corresponding to finite gap potentials ($N < \infty$ lacunae) [1]. In the above-mentioned approach induced by finite interval IP eigensolutions with spectral parameters $E_\nu; c_\nu$ we achieved a significant extension of the ESM classes with $N = \infty$.

◊ If two neighbor energy levels are shifted to one another in symmetrical potentials $V(x) = V(-x)$, there occurs splitting of the corresponding states which parts are going from one another until they disappear at $\pm \infty$ (effective annihilation). It is caused by the mutual incompatibility of orthogonal states, see Figs 5.1, 5.7. For nonsymmetric potentials there can be annihilation of only one of the two degenerating states.

The phenomenon of annihilation of the spectral zones of periodical structures can occur also when they are shifted to one another, see Fig.6.22.

It will be also shown how to force the more deep quasibound state behind the potential barrier to decay more intensively than the upper ones by using $c$-shifts of space localization of the wave functions. The example of such a decay (resonance widths) control is shown in Fig.6.7.

◊ It is also possible to transform the scattering state into the bound state embedded into the continuum at the same energy point $E$. It is done by an infinite number of the above-mentioned potential well-barrier blocks. It seems that the wave with the positive energy must fly away from the region of initial concentration. Really, due to interference it appears to be confined even at the energy above the barriers, see Fig.2.14.

◊ With the potentials confining waves at the origin on the half axis, we constructed potential on the whole axis $x$ with 100% reflection at the given energies $E_\nu$ and even above barriers [?]. Such a resonance reflection has not yet been considered in wave mechanics. It is significant that the parameters of these resonances (their widths, positions and number) are under control, see Figs 6.13, 6.14.

◊ We shall give the simple original explanation of resonance tunneling (total transparency at $E = E_\nu$ under barriers). It will be shown how an additional flux coming from the opposite direction can help or hinder the tunneling of initially incident waves which is often not suspected (see section ??).

The rules of bound state creation at $E > 0$ and $E < 0$ can be generalized to the forbidden and allowed zones of periodical structures.

The effects enumerated above and those to be considered further are not independent facts, but islands of understanding the new quantum regularities, which are gradually merged to continents of knowledge allowing to turn the ”wonders” of the new science into a natural manifestation of the microwave world.

◊ It seems paradoxical that the waves on lattices (in the case of discrete space coordinate) can be hold as bound states on the smooth potential slope: they cannot fly to infinity although no barriers hinder this, see Fig 7.13. On the lattices
bound states can live even inside the "up side down" wells, above the states of the continuum spectrum, see Fig 7.6. The tunneling through the up side down barriers hanging from above forbidden zone is also possible on lattice, see Fig.7.26.

The behavior of waves on lattices appears to be valuable like the model for investigations of periodical structures and still hardly understandable nonlocal interactions. So, the minimal nonlocal perturbations allow one to squeeze and broaden the spectral zones in different configurational space regions, to produce the inversion of spectrum when, for example, the number of signs changes ("knots") of bound state wave functions is decreasing (!) with the excitation energy, see Figs 7.21, 7.22. This instructive exotic character of the discrete quantum mechanics is supplemented by the generalized "Schrödinger equations" of higher (4th and more) order, which opens new possibilities for description of a simultaneous motion of waves of different kinds (with different frequencies) in the same direction.

The following remark will help understanding of the discrete problem specificity. In some sense, the continuous problem spectrum corresponds to only a lower half of the spectrum band $E < E_{sc}$ ($E_{sc}$ denotes the center of the spectral zone) in the problems with the discrete coordinate (in the limit of vanishing lattice step, the second half as if goes beyond the infinity $E_{sc} \to \infty$). Generally, discrete quantum mechanics is richer than the ordinary continuous one and includes the latter as a particular limiting case.

What was said in the first sections of the book about the simplest one-dimensional systems can be extended to complicated many-dimensional objects which are conveniently described by the systems of coupled Schrödinger equations: multichannel formalism. Only there, instead of scalar wave functions, potentials and spectral weight factors, the corresponding vector- and matrix-valued analogues are considered. It turns out, many wonderful discoveries (first chapters) not only find their own matrix-valued generalizations, but there opens a free range for finding new phenomena having no one-channel analogues. So the interactions were found which provide absolute transparency at any energies of the continuous spectrum of the systems even with barriers (!), see Fig.8.14, and without bound states at all, see Fig.8.15.

Recently, new mechanisms of total resonance transparency and absolute nonpenetrability were found, which consist in the accumulation of waves in the closed channel and their subsequent decay into the entrance channel, which results in suppression of either reflected or transmitted waves. Analogous resonances were revealed in systems described by differential equations of higher ($\geq 4$) order. It was found that the same system has a paradoxical ability to combine incompatible, one would think, features to be transparent (100%) and simultaneously completely reflective at one energy point, and even retain the waves in a bound state – confinement in the continuum. Such freaks of wave interference are possible at different boundary conditions. The acquaintance with such peculiarities of quantum systems raises to a new level the understanding of the possibilities of the quantum design.

You will see how, increasing one of the components of the spectral weight
vector (generalization of the factor $c$), the waves of a multichannel system can be gathered in one place not only in the configurational space, but also in the channel space, see Fig. 8.3. In doing so, other states as if undergo the recoil to the opposite side in both the spaces (the phenomenon of state separation).

In multichannel systems, the spectrum can branch out. The algorithms of wave packet movement and spectral branch control will be demonstrated.

◊ A notion about interchannel oscillations of a two-component wave packet constructed from near spaced states of a narrow doublet will be introduced. There is analogy with one-dimensional equal potential wells divided by barrier, through which the wave packet is tunneling hither and thither. In the interchannel case the part of the separating barrier plays the weak coupling of channels and the motion is considered over the discrete channel variable. This is a significant addition to the algorithms of packet control.

In systems of several bodies the connected particles assist one another to penetrate barriers. This effect does not reveal itself in the limiting cases (when the particles tunnel through the barrier being independent or compressed to one point), but becomes distinctly apparent in the intermediate case of internal oscillating motions of coupled particles. There is also symmetry violation of tunneling in the opposite directions through the nonsymmetrical barriers in contrast with the one channel case.

All this enriches our quantum intuition and allows one to look into mechanisms of the wave motion along labyrinths of the channels. These secrets of quantum multichannel ”kitchen” were hidden before inside the black box of computer calculations.

One can say that you keep in your hands a book of pictures-snapshots from the reverse, invisible in the ”old” approach, side of quantum mechanics, which supplements known books of quantum pictures by Brandt and Dahmen [?], Popov et al [?] on the traditional theory of direct problem.

We supplied the text with exercises and comments which will help a reader to learn contents of the book.
Chapter 1 Main notions about shifting positions of bound state energy levels (by potential transformation preserving symmetry)

How to deform the potential in order to change in the desired way the disposition of separate energy levels \( E_\nu \), the most important spectral characteristics? Such elementary transformations reveals connections between observables and the related forces acting in quantum systems. Here are explained the simple qualitative rules, discovered by us, of lowering and raising an individual energy level \( E_\nu \). It would be difficult to guess how to do it within the traditional approach of the direct problem. In fact, this possibility has not hitherto been mentioned in most up-to-date manuals; though the acquaintance with these samples of ‘control’ of quantum waves is the best way of comprehending the microworld.

The qualitative essence of the majority of quantum system transformation algorithms, to be discussed here, can be distinctly revealed by the simplest models. So we will start with the infinite rectangular potential well of width \( \pi \) and look how its relief changes under variations of a selected energy level value \( E_\nu \). The potential perturbations are especially visual against the background of the flat bottom of the chosen initial well.

At first, it is better to restrict ourselves to the potential transformations which do not violate the symmetry of the initial potential \( \hat{V}(x) = V(-x) \). In this case, only the purely discrete energy levels \( E_n \) form the complete set of spectral parameters which uniquely determine the potential shape. Really, for determination of a symmetric potential it is sufficient to have one its half because another one is the same. So, although for the arbitrary infinite deep potential well all the energy levels are only the half of spectral data, they are enough to determine a symmetric system exactly. In the unsymmetric case \( V(x) \neq V(-x) \), there will also be needed spectral weights of each bound state, but this will be discussed in the next sections of this book.

How should the potential bottom be deformed to shift the only chosen energy level (spectral ”brick”) while keeping the other ones at their previous positions? It is very important to understand just these elementary transformations to have a notion about the methods of constructing arbitrary systems with the given properties. Let us at first move the lowest energy level with the most simple wave function. Here the following question is helpful. In which place of the well is the ground state most sensitive to the potential perturbations? Of course, where the probability of finding the particle has a maximal absolute value. For the ground state of the infinite rectangular well it is its central part (see the unperturbed wave function \( \Psi_1 \) which has the only bump of \( \sin(kx) \), in Fig. 1.1a,b). Hence, for instance, to shift down the first level it is necessary to shift down the bottom of the potential just at the center, as is shown in Fig. 1.1a,c [?]. The nearer to the ends of the wave function bump (to its knots at the walls of the initial well) the weaker is the sensitivity. At the boundary points (at the walls) the function vanishes, it completely ceases to respond to potential perturbations there.
However, the potential well perturbation is not sufficient for our task. If the potential perturbation is purely attractive, it will shift down all the levels; but our goal is to consider the simplest, elementary transformations changing only one spectral parameter and leaving all remaining parameters at their previous places.

It is clear that we have to add a compensating repulsion, the potential hills at the walls of the initial well, see Fig. 1.1a,c. They will weakly influence the ground state, because inside the region of their action the wave function is small (near the knots at the walls). The different situation is for the excited states: there the absolute values of their wave functions are somewhat bigger than those of the shifted ground state and vice versa at the center. In the process of lowering of the ground state $E_1$ the central well in $\Delta V(x)$ is deepening, and the barriers from both sides become higher, see Fig. 1.1.

This picture is very instructive because it demonstrates the qualitative features of the universal, elementary transformation of $\tilde{V}$ and $\tilde{\Psi}_1$ on the interval of one bump. This allows one to understand the rule of transformation of potentials and waves for any shifted states with several bumps and in arbitrary potentials, see the following figures. It should be emphasized that although all levels $E_n \neq 1$, except the chosen one, $E_1$ are conserved, the wave functions of all states $\Psi_n(x)$ undergo distortions.

The exact form of the potential perturbation is determined by the formalism of supersymmetry in quantum mechanics SUSYQ. For the concrete case of the initial rectangular well the transformations have the following exact expression (the interval $[-\pi/2, \pi/2]$)

$$V(x) = \tilde{V}(x) + 2t \frac{d}{dx}\left\{\sin(\sqrt{1+tx})\cos(x)/[\sin(\sqrt{1+tx})\sin(x) + \sqrt{1+t}\cos(\sqrt{1+tx})\cos(x)]\right\} \quad (1)$$

with the expression for the normalized eigenfunction at the new eigenvalue $E = 1 + t$ is (see the general case see in (2)):

$$\Psi(x, 1 + t) = \cos(\sqrt{1+ta})/[\sin(\sqrt{1+tx})\sin(x) + \sqrt{1+t}\cos(\sqrt{1+tx})\cos(x)]\}.$$

We shall postpone the derivation of analogous expressions in the general case for any potentials till chapter 3. For the present, it is important to know that the initial $\tilde{V}$ and $\tilde{\Psi}$ serve as the building material for the new transformed potentials $V$ and functions $\Psi$(in our case, the usual sines and their derivatives).

Pay attention to the simplicity of exact expression (1). It is wonderful, because one could hardly suppose that after many tens of years since the creation of quantum mechanics there would appear a possibility of achieving, with so elementary tools, the result which seemed impossible. Really, this is analogous to a virtuous surgery on transplantation of a level to a new place without any "injuring" the positions of all other levels. So, it proves to be feasible to satisfy the infinite(!) number of additional conditions. To facilitate the believe in that, let us remind that all the
eigenstates are orthogonal and, in some sense, absolutely unlike one another. This circumstance plays a principal role in the IP formalism.

It is of great importance to inform as soon as possible broad physicist circles, particularly the lecturers on quantum mechanics, about the revelation of such constituent elements of fundamental spectral transformations.

The first our task is to concentrate one’s attention on the qualitative understanding, "from the first principles", of quantum design. Further, it will make easier to master the mathematical formalism: after clarifying the physical essence it will be easier to follow the details of the corresponding mathematical expressions. It is not sufficient to know the mathematics of the inverse problem, its equations, formulæ of exact models, etc. It is necessary to have a physical intuition allowing one to heuristically (without computers and analytical mathematical apparatus) foresee the algorithms of how to transform a potential to get the desired properties of the considered system.

The significance of energy level position can be illustrated, for example, by the fact that "a quantum of the light exciting the chlorophyll molecule transfers one electron on a special level. And in a fraction of second it returns into the initial state triggering the mechanism of any life activity on the earth. Though only a few organisms (plants and some bacteria) contain chlorophyll providing the photosynthesis and transforming the energy of light, but owing to them it becomes available for all others including the mankind” (P.Raven, R.Evert, S.Eichhorn, Modern botany). So, our own existence depends on the position of one quantum energy level. Although we still cannot change in practice the spectrum of chlorophyll in the desired way, but it is important to acquire the understanding of the principle of the relation $\Delta V(x) \leftrightarrow \Delta E_\nu$. At the same time, the (nano-) technology which can be used for creation of the desired effective potentials in electronics (superlattices, quantum optics with the variable refraction coefficients, etc.) is intensively developing. However, in the theory, the possibility of influencing the spectral parameters provides a deeper insight into the internal logic of constitution and behavior of any quantum system, even if we do not intend to change real objects.

It is wonderful that all this can be seen using the exactly solvable models which appeared to be so numerous (even complete sets !) that they can in principle to approximate arbitrary cases. As in the first example all will be clear in this chapter on model pictures without formulæ.

So, it is natural to suppose that for shifting up the first energy level $E_1$, the bottom of the potential must be lifted in the center, as is shown in Fig.1.2. [?].

However, as in the previous case, if the potential perturbation is purely repulsive, it shifts all the levels of the spectrum. Our goal here is to keep other levels (except the chosen one) at their initial places. Let us repeat the following: for a deeper understanding of the spectral engineering one should learn most elementary transformations. Hence, it is necessary to create compensating attraction, i.e. potential wells must be introduced at the places where the sensitivity of the ground state to the potential perturbations is weakened: in the vicinity of knots near the walls of the initial well, see Fig.1.2a,b. Equation (1) gives just such a special form of the barriers and wells to shift the energy $E_1$ of the bound state up while keeping the lev-
els of exited states fixed. For them the potential barrier is completely compensated by additional wells as in Fig.1.2a. In particular, it is evident even without formulae that these wells pool down the first exited state $|\Psi_2(x)|$, having its two bumps apart from the center, more strongly than the $\Psi_1(x)$ whose absolute value is smaller near the walls.

As the ground state level $E_1$ is lifted to its neighbor $E_2$, the central barrier in Fig.1.2b increases and the wells at the edges become deeper. The perturbations shown in Fig.1.2b keep the second level from shifting due to a mutual balance between attraction and repulsion.

The profiles of perturbations $\Delta V(x)$ in Fig.1.2a,b are like the upside-down potential curves in Fig.1.1,c. for the case of lowering the ground state. The same ”inversion” appears also for upward-downward shifts of other states and in the cases of different initial potentials.

As the energy levels come close together, the absolute values of the corresponding wave functions $|\Psi_1(x)|$ and $|\Psi_2(x)|$ become more alike. It is shown in Fig.1.2c how the wave function of the ground state is changed: it gradually splits into two hills in separate potential wells. The value of $|\Psi_1(x)|$ in the central part decreases (barrier forces it out); and in order to push up this two-humped ground state, the two-humped barrier is needed, see the minimum beginning to show in the center of the barrier in Fig.1.2b. Several curves in Fig.1.2b., corresponding to the evolution of the potential and the wave functions allow one to imagine qualitatively the whole continuous manifold of their intermediate forms. All the curves shown belong to the class of the exactly solvable models, see section 3.3.4. The unexpected ”annihilation” of states in the limit of total coincidence (degeneration) of levels is shown in Fig.5.1. This incompatibility at one energy of several one-dimensional quantum bound states is the bright characteristic of eigenfunction properties.

Now when the qualitative explanations of $\Psi, V$ transformations are found they seem so natural, but it was before difficult to guess them because it seemed that there was no hope at all for such a gift of quantum theory. The generalization of these rules to all the states of the discrete spectrum and arbitrary symmetric potentials, as will be shown further, was even more unexpected. The revealing of these rules suddenly dawned upon us. Before this, it even did not come to our mind that behind rather exotic bends of $\Delta V()$, appearing on the PC screen as a result of millions of numerical operations, there could be hidden clear, very significant and simple physics. This is an example of the everlasting magic of comprehending the truth: turning the inconceivable into the evidence.

Let us compare the forms of the potential curves $\Delta V_1(x)$ and $\Delta V_2(x)$, shifting upward and downward the ground state in Figs 1.1, 1.2 and the exited one in Figs 1.3, 1.4. There is a simple regularity in repeating influence of force on separate bumps of the corresponding functions. Now it is easy to explain the shape of potential perturbation for the upward (downward) shift, for example, of only the second level in the infinite rectangular well. The corresponding unperturbed wave function represents one period of $\sin(kx)$ with two bumps and one knot in the center. So, for shifting downward (upward) the second level there must be in the perturbation potential, already two minima (maxima) of attraction (repulsion) at the places where
the $|\Psi_2(x)|$ has maxima as shown in Fig.1.3, 1.4. This can be seen from Eqs (??) and (??). Three barriers (wells) near three knots of $|\Psi_2(x)|$ keep other levels from moving (down-) upward. For the inverse problem such a control is a simple procedure because (as mentioned before) in that approach the level positions are the entrance parameters. Here the absolute precision of the level control is possible again due to the mutual orthogonality of all the states of the spectrum, the difference of their bumps and their displacement.

It is remarkable that all this is performed with only the local transformations of the interaction $\Delta V(x)$. In the direct problem there were already known shifts of separate energy levels, but it was done by nonlocal (separable) potentials often considered as less convenient for description of quantum systems from a physical point of view.

In the process of shifting the levels $E_2$ to $E_1$ ($E_3$), the form of the absolute value $|\Psi_2(x)|$ bears a greater resemblance to the $|\Psi_1(x)|$ ($|\Psi_3(x)|$). This will be more clearly shown further when considering a very interesting transition to the limit of the complete coincidence of $E_2$ with $E_1$ ($E_3$) and their effective annihilation, see Figs 5.1, 5.6.

The above considerations completely apply to the case of wells with nonvertical walls. See for example, the deformations of the harmonic oscillator potential in Fig.1.5-e [?]. Pay attention to the analogy of potential deformations there and in Fig.1.2. In Fig.1.5c-e the shapes of $V, \Delta V$, lifting the ground states in the rectangular and oscillator wells are compared.

We can detach the lowest state with zero energy $E = 0$ from the free motion continuous spectrum and transform it into the bound state of the soliton-like reflectionless potential well, see Fig.1.6. For this purpose, we shall use "the intuition" we have acquired in shifting down the state of discrete spectrum. Though the initial systems seem to differ very much from one another, they become alike in the limiting case when the walls of the rectangular well go to $\pm \infty$: transition to the infinite wide well. The form of the potential perturbation $V_1$ (the soliton-like well in Fig.1.6) we search for, can be explained in the following way. It occurs as a result of lowering the "scattering" state with the energy $E = 0$. The wave length of this state is infinite and the unperturbed wave function represents a horizontal line corresponding to the single bump of sine with the infinitely small frequency. Shifting down the state with one bump must be performed according to the rules with only one well. The compensating repulsive barriers are "not seen" because the knots of a bump with zero wave number move to $\pm \infty$ and disappear along with these barriers. When detaching the lowest state from the continuum, it is possible to pull down it to the arbitrary depth in the negative energy region. In the case of the discrete spectrum, the shift of one level leaves all other states unaltered. Here are conserved the characteristics of the continuous spectrum, the values of reflection coefficient at different energies. So, the property of free waves having no reflection is inherent in the soliton-like wells. After the detachment from the continuum of one bound state, there will remain, as the lowest scattering state $E = 0$, a wave with one knot and two infinitely long bumps. For tearing also this state into the discrete spectrum, there will be needed, according to the same logic, two wells and
one barrier of potential perturbation $V_2$, see Fig.1.6b. Meanwhile, the new wave function with $E = 0$ gets the second knot.

If one takes, as an initial potential, a hill which gives considerable reflection, then creating a single bound state by symmetric potential perturbation [[?, 1985] will give a well with barriers at both sides with the same reflection as for the initial barrier, see Fig.1.7. The same can be achieved with a well being aside of the potential hill, so one obtains a non-symmetrical $V(x)$. Another example of shifting down a level represents a $\delta$-well with the only bound state, see Fig. 1.8, when compensating potential barriers are not required.

As an additional confirmation of the rules formulated above, the transformations of the oscillator well are shown in Fig.1.9a,b: (a) the 20th and (b) the 30th level are shifted upward. Here $\Delta V(x)$ has (a) 20 and (b) (30) hills and 21 (31) wells. It is remarkable that lower states practically are not changed. This is because of a strong inertia of low-lying states with respect to frequent oscillations of the perturbative potential. Their influences almost cancel each other when averaging. Only the high states are significantly changed near the shifted 20th (30th) level. There is a conformity between the wave functions and the perturbation $\Delta V(x)$ in that energy region, due to their commensurable frequencies (some kind of "resonance").

The shifts of levels $E_n$ can be performed not only over the real energy scale, but the chosen levels $E_n$ can be moved into the complex energy plane. The resulting unusual non-Gamov decaying states will be considered in section ??.

Many aspects of quantum design have opened for us in the process of the intensive work on this book. We think that, thereby, we have to some extent met the requirements of the nontedious writing declared by L. Levitsky in his diary which we have recently found by chance: "The sense of book writing consists in that it first of all opens something to its author. I am sure that good books differ from the bad ones in that the authors of the first ones, beginning to write them, have cleared up something for themselves, while authors of the others have beforehand known everything". So, in particular, while writing this book there appeared an interesting hypothesis. It was later confirmed by exact calculations. It appeared that using the above-mentioned potential perturbations like (??), but continued periodically, we can control the position of allowed and forbidden zones of band spectra, see section ???. The zones can be moved apart or drawn together up to their merging (disappearance of lacuna dividing them). Later we shall show a set of the corresponding examples.
FIGURE CAPTIONS

Fig.1.1 a) Potential perturbation of the infinite rectangular well which causes lowering (shown by the arrow $\Delta E_1$) of only one ground state level $E_1 = 1 \rightarrow E_1 = -4$. The thick black arrow directed down shows the well (dashed-dotted line $\Delta V(x)$) that influences the most sensitive central part of the wave function and pulls the level down. Black arrows directed up near the walls of initial well, i.e. near the knots of $\Psi_1(x)$ where it is the least sensitive to potential variations, point to the barriers which keep fixed all other levels $E_{i\neq1}$. The thin dashed lines show the transformations of the wave functions $\Psi_1 \rightarrow \Psi_{1.2}$. All this demonstrates the qualitative features of universal, elementary transformation for a single bump. This will allow one to understand the rule of transformation for arbitrary states with many bumps and for arbitrary potentials.

b) Evolution of the function $\Psi_1$ for $\Delta E = -1, -3$

c) Evolution of the potential for $\Delta E = -1, -2, -3$.

Do not confuse b,c) with real motions in some quantum system: these are variations of $\Psi_1, V$ during gradual transitions from one system to another.

Fig.1.2 a) Deformations (dashed-dotted line) of the rectangular well ($V(x) = 0$) raising the ground state level $E_1$ to $E_2$. Pay attention to the form of transformed potentials $V(x)$ in this and previous figures: they are qualitatively similar up to the change of the sign. The barrier in $\Delta V$ in the sensitive center of the wave function $\Psi_1$ pushes $E_1$ up (shown by thick black arrow). There are wells from both sides pointed by thick black arrows down, where $\Psi_1$ is less sensitive. They weakly influence on $E_1$, but neutralize the barrier influence on all other levels $E_{i>1}$ keeping them fixed. The dashed lines show the energy levels and the deformed wave functions $\Psi_1, \Psi_2$. This is another example of level control for state with only one bump. It will allow to clarify the general case of shifts of arbitrary levels in arbitrary potentials.

b,c) Evolution of potential $V(x)$ and function $\Psi_1$ for $\Delta E = 1, 2, 2.5$. Pay attention to the deformation of $\Psi_1$. Its shape approaches to absolute value of $\Psi_2$. This is due to a tendency to lowering the central barrier in the middle in Fig.1.2.

Fig.4.1 Demonstration of approximation of the (a) linear, (b) oscillator and (c) rectangular infinite potential wells by the reflectionless wells of finite depth when the lower parts of their discrete spectra coincide (eight lower bound state energy levels) [?].

Fig.4.2 Stepwise well of finite depth with 8 equidistant bound states energy levels approximates the lower part of the oscillator well in the energy region of spectral coincidence (T.Stroh) although in the upper parts their spectra are quite different (discrete and continuous).

Fig.6.22 a) Shift of the upper boundary of the second allowed zone for "Dirac comb", periodic $\delta$-barriers. This boundary coincides with the second energy level of the auxiliary rectangular well of widths equal to period and with the unpenetrable walls instead of $\delta$-peaks. Shift of this level up by $\Delta V(x)$ in the partial well will shift the whole corresponding zone in periodically continued potential perturbation $\Delta V(x)$. The level pools its zone while the lower boundary of the neighbor zone becoming wider comes down to meet the zone below. This motion leads to
disappearance of the gap between the second and the third zones at \( \Delta E = 1 \). So the second level of the auxiliary well becomes also the lower boundary of the upper zone. Then the zones go apart \( \delta E = 2, 3, 3.9 \), but the shifted level is "sticked" to the other zone. It squeezes this zone till annihilation because the upper boundary \( E = 9 \) is fixed according to the algorithm of shifting the only one energy level. b) Shift down of the upper boundary \( E = 4 \) of the allowed zone in Dirac comb. This boundary pushes its zone down until its lower boundary reaches the lower allowed zone. Then its fixed upper boundary \( E = 1 \) becomes also the lower boundary of the lowering zone. After the merging, the lower zone is teared off the upper zone and goes down and the upper zone is squeezed by its boundaries: the fixed \( E = 1 \) and the lowering second auxiliary energy level. c) Shift up of the lower boundary of the second allowed zone of the \( \delta \) wells comb (of the ground state level of the auxiliary eigenvalue problem on the period). This boundary pushes its zone until its upper boundary reaches the position of the second fixed energy level \( E = 4 \) and the lacuna between zones disappeared. After that the zone squeezing begins as in the case b). Then the upper zone splits from the lower one and goes up restoring the disappeared lacuna. The annihilation of the auxiliary energy levels leads to disappearance of all allowed zones.

Fig.1.6 a) The lowest state of the continuous spectrum of the free motion with \( E = 0 \) is splitted down and becomes a bound state \( E_1 < 0 \). The perturbation \( \Delta V(x) \) shown by the arrow is a soliton-like well \( V(x) \). It has no barriers unlike the perturbation in Fig.1.1. It is because the "knots" of the \( E = 0 \) state disappear at \( \pm \infty \) and because it must not change the zero reflection of the initial free waves, although all wave functions undergo transformation. For example, the new state with \( E = 0 \), which does not increase at infinity, has a knot and horizonal asymptotics. b) Two bound state levels are teared from the continuous spectrum (arrows \( \Delta E_1, \Delta E_2 \)).

Fig.2.1 The \((a,b)\) transformation of the infinite rectangular well \( \hat{V} \rightarrow V \) and eigenfunctions \( \hat{\Psi}_1 (x) \) \((a,d)\), \( \hat{\Psi}_2 (x) \) \((a,c)\) by increasing SWF \( \hat{c}_1 \rightarrow c_1 \), the derivative \( \hat{\Psi}_1 (x = 0) \) at the left wall. The scales of the functions \( \Psi_1, \Psi_2 \) \((a)\) are shifted up to the corresponding energy levels \( E_1, E_2 \). In \((b,c,d)\) the evolution of the potential and functions with increasing \( c_1 \) 2, 5, 10, 20 times is demonstrated. Meanwhile all energy levels \( E_n \) and all SWF, except one \( c_n \neq 1 \), remain unchanged, as is seen for \( \Psi_2 \) \((a,c)\). The parameter SWF \( c_1 \) controls the localization of the wave function \( \Psi_1(x) \) \((a,d,e)\) in space: by increasing \( c_1 \) the ground state is pressed to the left potential wall, as is shown by arrows in \((a,d)\). This is performed by the potential barrier in \((a,b)\) on the right which shifts the function \( \Psi_1(x) \) to the left, and the potential well on the left which simultaneously compensates the influence of the barrier on the energy levels and keeps them all at the same places. All wave functions, except the ground state, undergo some recoil in the opposite direction which is demonstrated by \( \Psi_2(x) \) \((a,c)\). So there is separation of the bound state from others. Compare with Fig.2.2, where the SWF \( c_2 \) was changed. (f) Transformation of the oscillator potential (barrier+well) by shift of the ground state to the right.

Fig.2.2 Deformation of the rectangular potential and functions by increasing SWF \( c_2 \), the derivative \( \Psi_2 \) at the left edge of the well. \((b,c,d)\) evolution of the
potential and functions with increase in SWF $c_2$ 2,3,6 times. The scales of the functions $\Psi_2, \Psi_3$ (a) are shifted up to the corresponding energy levels $E_2, E_3$. The position of all energy levels $E_\lambda$ and SWF $\lambda \neq 2$ (b,d) is not changed. The potential perturbations $\Delta V(x)$ (a,c) consist of two blocks ”well-barrier”, each for one of two bumps of $\Psi_2$. Compare with Fig.2.1. The influence of these blocks reveals in shifting the maxima of each bump in $\Psi_2$ to the left. The relative values of these bumps also increase to the left due to a smooth connection of $\Psi_2$ at knots with violated symmetry of derivatives $\Psi'_2$ at knots. Pay attention to that the central knot of $\Psi_2$ remains at the same place.

Fig.2.5 a) ”Pressing out” the ground state of the finite rectangular well $\tilde{V}$ by decreasing SWF $c_1 = \Psi'_1(x = 0)$. This shift is performed by a universal potential block barrier+well $V(x)$. In this case, the soliton-like reflectionless well and an additional potential peak at the edge of the initial well compensating the smoothing of the potential step, provide the invariance of all other spectral parameters of discrete and continuous states. The bound state wave function $\Psi_1$ concentrates in the carrier well shifted to the right where are the conditions for constructive interference of multiply reflected waves in a soliton like well at $E = E_1$. At the place of the initial rectangular well $\Psi_1$ is negligibly small due to a destructive interference of waves multiply reflected from the walls of the narrowed well. Wave functions of other states are somewhat recoiled to the origin. For instance, $\Psi_2$ in contrast with $\Psi_1$ is concentrated near $x = 0$, as if it becomes the ground state in the left well and its second bump inside the right well (it is shown with 1000 times enlargement) is negligibly small because of destructive interference.

b) Comparison of the initial and transformed functions $\tilde{\Psi}_2$, $\Psi_2$ of the second bound state. The curved arrow shows how the probability distribution undergoes recoil to the origin without changing SWF $c_2 = \Psi'_2(x = 0)$. The second bump of the function $\tilde{\Psi}_2$ is almost pumped into the first one of the transformed wave $\Psi_2$ and the knot disappears in the limit $c_1 \rightarrow 0$.

Fig.5.1 The graduate deformation of the infinite rectangular well and functions of bound states when $E_2$ approaches $E_3$, and the effective annihilation (they are pressed into the potential walls and disappear behind the infinite potential barriers) of the degenerated states, compare with Fig.1.4. The states above the degenerated ones gradually lose the first and the last bumps with the corresponding knots. Inside the new potential these states have numbers lowered by 2.

Fig.5.7 Symmetrical deformation (b,c,d) of the oscillator potential (a) $\tilde{V}(x) = x^2$ during coming together of bound state levels $E_3, E_4$ (b,c) in the limit of their degeneration (d). First 8 wave functions are shown. The waves $\Psi_3(x)$ and $\Psi_4(x)$, each teared into two parts, are concentrated inside special carrier wells and moved in the limit of total degeneration to $\pm \infty$ (annihilation). The wave functions of the states below the degenerated doublet are not radically changed. However, $\Psi_5(x)$ and others $\Psi_m(x), m > 5$ have decreasing first and last bumps (b – d) disappearing in the limit, see $\Psi_8(x)$ (b $\rightarrow$ c $\rightarrow$ d), compare with Fig.5.3.

Fig.6.7. Demonstration of the control of resonance widths. The delay time of scattered waves inside the potential in which the first bump of the lowest quasibound
state is shifted inside the soliton-like well through the potential barrier to its outer edge to make its decay easier. Compare with Fig.2.5. As a result, the width of the first resonance becomes broader than for the second one.

Fig.2.14 Creation of a bound state at the negative energy (a) and in the continuous spectrum (b-e) [?

Fig.6.13. On the right side \((x \geq 0)\) the potential \(V_{BSEC}(x)\) is shown which transforms the free wave (sinus) into BSEC \(\Psi_{BSEC}(x)\) confined at the origin. The axes for \(\Psi_{BSEC}(x)\) and \(V_{BSEC}(x)\) are shifted to avoid their overlap. Pay attention to coincidence of bumps and knots of the BSEC wave function with blocks ”well-barrier” and even knots of the perturbation potential. Just this correlation provides the confinement of waves near the origin of the pinned out point \(E_{BSEC} = 1\) of the energy scale. At energy points different from \(E_{BSEC}\) the physical solutions are not decreasing in asymptotics. On the whole axis \(x\) there is a total reflection of waves (symbolical arrows on the left side of Fig.) by the potential \(V_c\) continued to the left \(x\) half-axis as zero and coinciding on the right half-axis with \(V_{BSEC}(x)\). Only one solution at \(E = E = 1; c = 1\) is decreasing when \(x \to \infty\) (the other independent one is nonphysical and increases).

Fig.6.14. The absolute value of the reflection coefficient \(|R(E)|\) for the potential with BSEC on the half axis \(0 \leq x < \infty\) at energy \(E_{BSEC} = 10\) with resonance reflection (100 % at this point) for waves on the whole axis. With increasing SWF of BSEC \(c\) the width of the resonance of \(|R(E)|\) near \(E_{BSEC} = 10\) becomes greater. For \(c \to 0\) this width converges to zero: the dashed line corresponds to this limiting \(R(E)\).

Fig.7.13 The bound state wave functions on the linear slopes (, b, c): \(V(\alpha) = \alpha; \alpha/2; \alpha/4c\) of the discrete variable \(\alpha\) are the Bessel functions \(J_{\alpha}\) of index equal to whole numbers. Do not confuse with the usual argument of Bessel functions \(kr\) which is fixed in our case (!). The values of the functions depending on the discrete variable \(\alpha\) are connected by the dashed line to make the function easier to imagine. Here the parameter \(z\) does not play the part of a coordinate, but regulates the steepness of the potential slope. The less steep is the slope, the more dense becomes the equidistant bound state spectrum \(E_n\). The functions of all these states for the fixed \(V(\alpha)\) have the same form, but with the argument shifted by the whole number. For the non-whole number \(\alpha\), but with the whole value of the lattice step, the Bessel functions are nonphysical states with increasing absolute values in the direction to the left.

Fig.7.6. Bound state in the upside down well above the continuous spectrum of the allowed zone has the characteristic change of the sign. Nevertheless the enveloping function of the upper bound state is similar to the function of the ordinary bound state in a usual well (in our case of the ground state). As in the case of continuous variables, there is enough arbitrary small deepening of the upper boundary of the allowed zone into the upper forbidden zone for the appearance of at least one bound state. Compare with the form of the wave functions in the infinite rectangular well on the lattice, see Fig.7.21.

Fig.7.26 The local potential \(V(x)\) bends the allowed zone in the plane \((E,n)\) without changing its width. Compare with the influence of the minimal nonlocal part of the potential which controls the width of the zone, as is shown in Fig.7.25.
The discrete values of the functions are connected for clarity by the solid lines.

Fig.7.25 Potential \( u(n) \), which determines the mutual influence of the neighbor points (minimal nonlocality), controls the change of the width of the allowed zone (b) at different points \( n \).

Fig.8.14 The transparent interaction matrix with one bound state at the energy \( E_b = -0.5 \) with SWFs \( M_1 = 1, M_2 = 0.001 \) and different thresholds \( \epsilon_1 = 0, \epsilon_2 = 1 \), at which the channels become open. Besides the soliton-like well \( V_{11}(x) \) on the right there is a part of \( V_{\alpha\beta}(x) \) on the left (see Fig. 8.15) which strongly mixes the components of the channel waves. The barrier in \( V_{11}(x) \) shown by the solid line reflects waves which are suppressed by the waves “decaying” from the “quasibound state” concentrated inside the well \( V_{22}(x) + \epsilon_2 \) shown by the dashed line. At small energies, the waves which have to tunnel through the barrier in \( V_{11}(x) \) go round it through the second channel using the channel coupling

\[ V_{12}(x) = V_{21}(x) \] (dotted-line) The same as in (a) only the wave function components \( \Psi_1; \Psi_2 \) of the bound state are shown. The small component \( \Psi_2 \) is shown 800 times enlarged. (b) The violation of conservation of the partial wave fluxes \( J_\alpha \) occurs only inside the left part of the interaction matrix and outside the fluxes return to their previous unperturbed values. The case is shown (d) of the incident wave only in the first channel which is transferred into the second channel, but then completely (!) returns to the first one even with the energy \( E > \epsilon_2 \) when both the channels are open.

Fig.8.15 The two-channel interaction matrix with nonequal thresholds \( \epsilon_2 = 1 \) completely transparent at all energies of the continuous spectrum and having no bound states. So, this \( V_{\alpha\beta}(x) \) has no analogue in the one-channel case where there are no reflectionless potentials without bound states. The remarkable peculiarity of this interaction matrix is the barrier in \( V_{11}(x) \) which does not violate the transparency of the system although intensively reflects waves. However, the reflected waves are suppressed by the waves coming from the second channel due to the coupling \( V_{12} \) and going backward with the same amplitude and the opposite phase as the reflected waves (destructive interference). It is interesting that this interaction matrix is a constituent part of the matrix with the bound state shown in Fig.8.15.