Pair production in a plane wave due to a thermal background

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Abstract. Ever since Schwinger’s publication [J. Schwinger, Phys. Rev. 82, 664 (1951)], the maxim that there can be no pair creation from vacuum in a plane wave has been often cited. We put forward an analysis showing that in any real situation, where thermal effects are present, in a single plane-wave field, even in the limit of zero frequency (a constant crossed field), pair creation can indeed occur. Interestingly, we find that the pair-production rate depends non-perturbatively on both the temperature and the amplitude of the constant crossed field.

1. Introduction
One immediate consequence of Dirac’s relativistic equation for the electron is that, given sufficient energy density, massive particles can be generated by electromagnetic radiation, in the form of particle-antiparticle pairs. Early results by Sauter [1], Heisenberg and Euler [2] and Weisskopf [3] were followed by an approach based on effective field theory by Schwinger [4] who derived the rate of pair creation per unit volume in a uniform and constant electric field of strength $E$. This rate has a non-perturbative dependency on the combination $E/E_{cr}$, where $E_{cr}$ is the so-called “critical” electric field $E_{cr} = m^2c^3/\hbar = 1.3 \times 10^{16}$ Vcm$^{-1}$, equivalent to a critical peak intensity of $I_{cr} = 4.6 \times 10^{29}$ Wcm$^{-2}$, where $e$ is the positron charge and $m$ is its mass. More than a decade later, an intensive campaign was begun to investigate quantum electrodynamics in intense external fields, given the impetus by key initial results such as the rate of photon decay into electron-positron pairs [5, 6], which also depends non-perturbatively on the external field through the parameter $\chi = e\hbar\sqrt{(k_\mu F^{\mu\nu})^2}/m^3c^4$, with photon four-vector $k_\mu$ and background field $F^{\mu\nu}$. Several factors have led to a heightened and sustained interest in better understanding and devising ways of detecting such phenomena. This is in part due to the planning of upcoming laser facilities, such as ELI (Extreme Light Infrastructure) [7], XCELS (eXawatt Center for Extreme Light Studies) [8] and HiPER (High Power laser Energy Research) [9], where it is intended that electromagnetic fields will be generated orders of magnitude larger than the current record of around $10^{22}$ Wcm$^{-2}$ [10], and even up to a percent of the critical value. It is also due in part to recent theoretical advances, both analytical [11, 12, 13, 14, 15] and from numerical simulation [16, 17, 18], that strongly indicate that pair creation could be observed at orders of magnitude well below the critical field (a recent review of strong-field QED effects can be found in [19]).
Pair creation is predicted to occur in various contexts. For example, possible terrestrial laboratory experiments in which two intense electromagnetic plane waves propagate against each other with [20] and without [21, 22, 23] the presence of a nucleus modelled by a Coulomb field; in an astrophysical context, such as the decay of photons in the strong magnetic field of magnetars [24]; and also in constant electric fields catalysed with a thermal background, having already been calculated in different formalisms [25, 26, 27, 28]. However, Schwinger famously derived a “no-go theorem” for pair creation, stating in [4], that “there are no nonlinear vacuum phenomena for a single plane wave, of arbitrary strength and spectral composition”. First, this is forbidden as in a plane wave, photons propagate parallel to one another and so cannot physically interact with one another. Second, the only two unique electromagnetic relativistic invariants vanish in a single plane wave.

In the current paper, we do not challenge that a single plane wave induces no pair creation in vacuum, rather we explain how, when one takes into account the inevitable presence of some background heat radiation, pair creation can indeed proceed in single plane waves in all realisable scenarios, a point also demonstrated in [29]. Specifically, this will be shown to be the case for a constant crossed field, which can be reached by taking the zero-frequency limit of an infinite plane-wave. It will turn out that the “thermal” rate for pair creation by a photon gas at a given temperature \( T \) in a constant crossed field of strength \( E \) will be non-perturbative in both variables. In addition to the qualitative result and the interesting appearance of a thermodynamic variable in the pair-creation exponent, these results could be of relevance when describing heavy-ion collisions, relativistic plasmas and pair production processes near neutron stars.

2. Thermal photon gas in an external plane wave

The case we are considering is an intense external electromagnetic plane wave in thermal equilibrium with a photon gas at temperature \( T \). The intensity can be quantified with the classical non-linearity parameter, often simply referred to as the “intensity-parameter” \( \xi = (m/\omega_l)(E/E_{\omega_l}) \gg 1 \), where \( \omega_l \) is the angular frequency of the photons of the external field, and we have already set here and subsequently \( \hbar = c = k_B = 1 \) (\( k_B \) is the Boltzmann constant). The non-linearity parameter also qualifies the type of pair-creation. \( \xi \ll 1 \) refers to a multi-photon process, whereas \( \xi \gg 1 \) refers to a tunneling process, which is characteristically frequency-independent, tending to this limit with increasing \( \xi \) [30]. This corresponds to the so-called “constant crossed field” when applied to a plane wave, for which the field is crossed \((E \cdot B = 0, E \cdot E = B \cdot B)\) and constant. As modern laser systems have demonstrated values of \( \xi \) as large as around \( 10^2 \) [10], we are motivated to focus our study on constant crossed fields interacting with the thermal photon gas.

The process of lowest-order pair-creation can be calculated from the polarisation operator, shown in Fig. 1. The optical theorem relates the imaginary part of the forward scattering amplitude to the process acquired by cutting a symmetric diagram in half. In Fig. 1, we show two ways that this can be achieved, which we have calculated agree with one another to leading order. One can either calculate the scattering of free photons in the external field and sum over a thermal Bose-Einstein distribution (the left-hand diagram), or one can calculate the polarisation of the virtual pairs due to an external field when they exchange a thermal photon (the right-hand diagram). A vacuum polarised by a field \( F^{\mu \nu}(x) \), with four-co-ordinate \( x \), which varies slowly over the reduced Compton wavelength \( \lambda_C = 1/m \), can in general be investigated via the effective Lagrangian formulation of QED [31]. Schwinger’s comment referred to this object being identically zero for all plane-wave backgrounds. In a purely plane wave and by extension, a constant crossed field, the only two unique relativistic invariants that one can construct: \(- F^{\mu \nu} F_{\mu \nu} = 2(E^2 - B^2) \) and \(- F^{\mu \nu} F_{\mu \nu} = 4(E \cdot B) \), where \( F^{\mu \nu} = \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}/2 \) and \( \epsilon_{\mu \nu \rho \sigma} \) is the rank-four Levi-Civita symbol (antisymmetric in all indices and \( \epsilon^{0123} = 1 \)), both
Two different approaches to calculating pair-creation by thermal photons (wavy lines) in a plane wave (in which the virtual electron-positron loop is dressed). Using the optical theorem to split the diagrams in half, summing pair creation by a single photon over a thermal ensemble (left) gives the same leading-order rate as splitting the two-loop polarisation operator with a thermalised internal photon.

vanish identically. Suppose we specialise the discussion to constant fields $F^\mu_\nu(x) = F^\mu_\nu$. Then when a thermal bath of photons is present, whose constituent photons have on-shell wavevectors $k^\mu = (\omega, k)$, one can form a third unique relativistic invariant $(k^\mu F^\mu_\nu)^2$, which is in general non-zero. In addition to the classical non-linearity parameter $\xi$, which diverges for a constant crossed field, QED phenomena in the strong constant fields can also be described with the quantum non-linearity parameter $\chi = e\sqrt{|(k^\mu F^\mu_\nu)^2|}/m^3$, directly related to this third, non-zero, relativistic invariant. $\chi$ will therefore be the microscopic quantity defining the macroscopic interaction between external field and thermal bath.

We limit ourselves to the most accessible parameter range for experiments, namely $\chi \ll 1$. If the thermal photons have an energy approximately equal to the temperature, then this condition on $\chi$ corresponds to $\delta = (T/m)\chi_E = (T/m)(E/E_{cr}) \ll 1$. Throughout our analysis, we make two key assumptions: i) processes involving larger numbers of thermal photons can be neglected (i.e. the perturbative expansion in thermal photons is valid) and ii) pair creation due to thermal photon-photon inelastic scattering is much less likely than due to thermal photons inelastically scattering with the external-field. The first assumption is valid if $|e|T/m \ll 1$ (in [32] it was shown that a perturbative expansion for the interaction between the photon and electron-positron field is valid when $\alpha\chi^{2/3} \ll 1$ where $\alpha = e^2$ is the fine-structure constant). This limit also allows us to assume the formation region of each pair is much smaller than the average region between pair-generation events, so that an incoherent sum over the thermal ensemble is a good approximation. The validity of the second condition has been already investigated in [29], which we mention here, is fulfilled when $\sqrt{\delta} \gg T/m$.

3. Thermal pair creation rate via sum over ensemble
Before proceeding with the calculations, we note that, although we demand a thermal equilibrium for our result to be exact, the approach of summing over the thermal ensemble can be used to approximate pair-creation rates for physically-relevant scenarios. One such example is the irradiation of solid-state targets such as thin foils by ultra-intense lasers. In a recent experiment, prolific positron creation was observed [33, 34] by irradiating target foils of various proton numbers using an intense laser for which $\xi \gg 1$, allowing processes to be locally approximated with the constant crossed field rates. It was shown that hot photons with energies up to the pair-creation threshold and beyond were generated, allowing a distribution with a high “effective” temperature to also be considered, albeit out of equilibrium. Integration of the pair creation rate over this ensemble could then be used as an approximation to the non-equilibrium rate. It should be mentioned however, that an accurate calculation for pair creation rates in a laser-irradiated relativistic plasma must take into account the many competing channels such as
pair production from photon-electron collisions and from the Bethe-Heitler process (photon-ion collisions), which can play a substantial role in the plasma’s evolution.

We now turn to the main calculation of this article, pair creation in the interaction of a constant crossed field with a pure thermal gas. In light of the above discussion, we stress that since our analysis assumes a thermal equilibrium, as soon as the first pair is created, the vacuum fermion state no longer exists and the result is no longer exact. Instead, one would have to perform a calculation that took into account effects such as Pauli blocking. Therefore, the word “rate” should more precisely refer to the probability per unit volume per unit time for pair creation to begin, rather than the number of pairs produced per unit volume per unit time. Proceeding via the optical theorem, which for a single photon interacting with a virtual normalisation 1 for pair creation to begin, rather than the number of pairs produced per unit volume per unit time.

By using the following integral identities, the remaining integrations can be analytically further justified in the discussion, to give:

\[ R_\alpha(k, \chi E) = -\frac{2\alpha m^2}{3\omega} \int_1^\infty dv \frac{2v + 1 + (1-\alpha)^a}{v \sqrt{v(v-4)}} \frac{\text{Ai}'(z)}{z}, \] (1)

where \( z = (v/\chi)^{3/2} \), \( \text{Ai}() \) is the Airy function of the first kind (defined as in [36], with normalisation 1/\( \pi \)).

Since we are interested in cases where \( \delta \ll 1 \), we can assume that \( \chi \ll 1 \) is satisfied by the vast majority of photons in the distribution. This allows us to expand Eq. (1), which will be further justified in the discussion, to give:

\[ R_\alpha(k, \chi E) \sim \alpha m \sqrt{\frac{3}{2}} \frac{m}{\omega} e^{-\frac{z}{3\chi}}. \] (2)

By summing Eq. (2) over the Bose-Einstein distribution, one can then obtain the probability of a density of pairs being created in the interaction of thermal photons and external field \( n_{th}(T, \chi E) = dN_{th}/dV dt \), well-approximated by an integration:

\[ n_{th}(T, \chi E) = \sum_a \int \frac{d^2k}{(2\pi)^3} \frac{1}{e^{\omega/T} - 1} R_\alpha(k, \chi E). \] (3)

Inserting the asymptotic rate Eq. (2) into Eq. (3), with the substitutions \( \phi = \omega/T \) and \( y = 1 - \cos \theta \), with \( \cos \theta = k \cdot \mathbf{x}/\omega \), one can see that at small \( \chi \), only large values of \( \phi \) contribute to the integral, giving eventually:

\[ n_{th}(T, \chi E) \sim \frac{3\sqrt{3}T^3 \alpha m \chi E}{32\sqrt{2}\pi^2} \int_0^\infty d\phi \int_0^2 dy \phi^2 y \frac{e^{-\phi}}{\phi^{3/2}y^{3/2}}. \] (4)

By using the following integral identities, the remaining integrations can be analytically performed,

\[ \int_0^\infty dz z^2e^{-z-a^2/z} = 2a^3K_3(2a), \] (5)

\[ \int_0^2 dy y^{-1/2}K_3(b/\sqrt{y}) = 4b^{-1}K_2(b/\sqrt{2}), \] (6)

where \( K_n() \) is the modified Bessel function of the second kind, giving:

\[ n_{th}(T, \chi E) \sim \frac{\alpha T^2 m^2}{\pi^2} \sqrt{\frac{3}{2}} K_2 \left( \sqrt{\frac{16}{3\delta}} \right). \] (7)
The analysis shows that this result is valid at $\sqrt{\delta} \ll 1$ (see [29]), therefore, by using the expansion of the modified Bessel function for large arguments:

$$K_2(a/\sqrt{\delta}) \sim \sqrt{\frac{\pi \sqrt{\delta}}{2a}} e^{-\frac{a}{\sqrt{\delta}}},$$

for $a, \delta > 0$, one then acquires [29]:

$$n_{\text{th}}(T, \chi E) \sim \frac{3^{3/4} \alpha}{4 \pi^{3/2}} m^4 \left(\frac{T}{m}\right)^2 \delta^{1/4} e^{-\frac{1}{\sqrt{3\delta}}}, \quad \delta = \frac{T}{m \chi E}. \tag{9}$$

This result for the density of created pairs demonstrates an interesting dependency on environmental variables. It is non-perturbative in both the external field and the temperature, although only lowest-order thermal photons were included. The non-perturbative dependency in the pre-exponent and exponent is also quite different to other results for thermal pair creation. For example, the thermal part of the rate of pair creation at finite temperature in a constant electric field, for the relevant case of $T/m, \chi E \ll 1$, has been shown to be [31, 25]

$$n_E(T, \chi E) = (\alpha \pi^2/90)m^4(T/m)^4 \chi E^{-2} \exp(-\pi/\chi E).$$

For a constant electric field, the temperature enters only in a perturbative sense, in the pre-exponent. In contrast, here it occurs on the same footing as the external field — indeed the exponent is the geometric average of the purely thermal pair-creation exponent $(2m/T, \text{see [29]})$ and the external-field part of the photon-seeded pair-creation exponent $(8/3 \chi E)$. One reason why in a constant crossed field, the temperature could be expected to play such an important role is that when the temperature is zero, the pair creation rate is identically zero. The logarithm of the rate $\log_{10} n_{\text{th}}(T, \chi E)$ calculated from Eq. (9) for a space-time four-volume $\lambda_4^4 = m^{-4} = 7.4 \times 10^{-53} \text{ cm}^4 \text{ s}$ is plotted in Fig. 2. Above the dashed line, the expected number of pairs created in an optical strong laser beam of typical four-volume $\Omega_L = \tau_l V_l$ with duration $\tau_l = 10 \text{ fs}$ and spatial extent $V_l = \pi \times (0.8 \mu m)^2 \times c \tau_l = 6 \times 10^{-12} \text{ cm}^3$ (where we temporarily reinstate the speed of light, $c$). The parameters used in the plot have been chosen so that the background process of purely thermal pair creation is negligible compared to the external-field process [29]. We can justify using the expression for single-photon pair creation asymptotically small in $\chi$, by comparing these results with a numerical integration of the full single-photon pair creation rate over the thermal ensemble (i.e. inserting Eq. (1) in Eq. (3)). In Fig. 3 we see that the relative difference in using the asymptotic rate remains less than ten percent for values of $\delta = 0.03$. We conclude that Eq. (9) is a good approximation for parameters of interest. That this deviation is a function only of $\delta$ can be seen by performing the variable substitution $\phi = \omega/T$ in Eq. (3), leading to the combination $|\Delta n_{\text{th}}(T, \chi E)|/n_{\text{th}}(T, \chi E)$ depending only on $\delta$.

4. *Thermal pair creation rate via effective action*

As displayed in Fig. 1, there are two different approaches one can use to acquire the thermal pair creation rate. Here we concentrate on the second diagram — use of the optical theorem on the two-loop polarisation operator with thermalised internal photons. This is equivalent to averaging the one-loop polarisation tensor over a thermal photon bath, and brings additional insight into the result Eq. (9). Since our calculations are valid in thermal equilibrium, we can use the Matsubara formalism. If we assume that only the photon propagator need be thermalised (this will be justified later), the Matsubara rate $n_M$ is given by:

$$n_M(T, \chi E) = -T \text{Im} \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \frac{1}{k_n^2} g_{\mu\nu} \Pi^{\mu\nu}(k_n, \chi E), \tag{10}$$
Figure 2. In the plot is the logarithm of the expected number of pairs created in the fundamental space-time volume of dimension reduced Compton wavelength $\lambda_4 = 7.4 \times 10^{-53} \text{ cm}^3 \text{ s}$ when a thermal photon gas is polarised by a constant crossed field. The number of created pairs is calculated using Eq. (9). The parameters in the plot have been chosen so that the background process of zero-field pure thermal pair creation is negligible compared that due to the external field. The expected number of pairs created by this process in a typical strong optical laser four-volume $\Omega_l = 6 \times 10^{-26} \text{ cm}^3 \text{ s}$, rises above one for regions above the dashed line.

where $k_n = (2\pi i n T, k)$ and $\Pi^{\mu\nu}(k, \chi_E)$ is the polarization operator in a constant crossed field [37]. If one transforms the sum into a complex integral using an integral relation (see Eq. (4.11) in [38]), then it can be shown that the thermal average rate in the previous section Eq. (3) and the Matsubara rate Eq. (10) exactly coincide. We can use the two-loop effective in the Matsubara formalism for a general constant-field background [25, 31], which can be written as a sum over an integer $n$ of a triple propertime integral. If one takes the crossed-field limit and performs a saddle point expansion of the propertime integrals, Eq. (9) is recovered when the condition $\gamma T = T^2/m^2 \delta \ll 1$ is fulfilled. This restriction on $\gamma T$ agrees with assumptions made in the previous section.

For increasing temperature, one should in principle also take into account the thermal fluctuations of the fermion loop. However, it has been shown [31, 39, 40] that for the temperatures range of interest $T \ll m$, the thermal correction to the electrons and positrons is exponentially suppressed by a factor $\sim \exp(-m/T)$. This suppression justifies our reasoning throughout this paper that thermalising only the photon should be a good approximation when $T \lesssim m$. Moreover, our results add to the general trend, that the thermal contribution toward pair creation generally exceeds that of the zero-temperature contribution, in the limit of weak fields (see also [31]). Here it is particularly evident, as the zero-temperature pair creation rate in a constant crossed field is identically zero.

5. Non-dissipative interaction

The interaction between the external field and thermal photon gas is mediated by the polarised virtual pairs of the vacuum, depicted in the polarisation tensor in Fig. 1. In addition to the dissipative process of pair creation, this interaction also provides the non-dissipative process of
Figure 3. This log-log-plot compares the accuracy of the thermal rate calculated using the asymptotic single-photon pair creation formula for $\chi \ll 1$ compared to using the exact single-photon formula (using Eq. (9) and Eq. (1) in Eq. (3) respectively). The relative difference between the two approaches, $|\Delta n_{th}(T, \chi_E)|/n_{th}(T, \chi_E)$ is plotted as a function of its only dependent variable, $\delta$.

thermal photon scattering in the external field. One can derive [37] an altered dispersion relation for a thermal photon with on-shell four-momentum $k^\mu_0 = (\omega_0, \mathbf{0})$ in vacuum:

$$\omega_{1,2}(\chi) = \omega_0 - \frac{\alpha 11 \pm 3 m^2}{180\pi} \frac{\omega_0^2}{\omega_0 \chi_0^2},$$

(11)

with $\chi_0 = e \sqrt{||k_\mu F^{\nu\mu}||^2/m^3} \ll 1$. Here we make the following observation. Common thermodynamical quantities can be derived from the partition function $Z$:

$$Z = \text{Tr} \ e^{-\hat{H}/T}$$

(12)

and when the Hamiltonian operator $\hat{H}$ is expanded in thermal photon energy eigenstates, which are modified in the plane-wave background, thermodynamical quantities such as the photon gas pressure, would seem to also, as a consequence, be modified. This hints a new possibility for searching for vacuum polarisation effects - by reading the signatures in thermodynamic changes due to microscopic strong-field QED effects. Here, as the change in refractive index is perturbative in the external field, so are the modifications in such thermodynamical quantities. More details can be found in [41].

6. Discussion and conclusion

We have shown how, when one acknowledges the inevitable presence of photons from a thermal and by extension non-thermal background, in all realistic physical scenarios, pair creation can ensue even in a constant crossed field and hence a single plane wave. This does not contradict
Schwinger’s maxim, but demonstrates the contrast of what is predicted to occur in an experiment compared to ideal, theoretical, conditions. Moreover, the probability for this new process of thermal pair creation in a constant crossed field has an interesting and non-trivial form. It is non-perturbative in the external field strength as is the case for photon-seeded pair creation in a constant crossed field, but also non-perturbative in temperature, which enters on the same footing as the external field. Contrasted with non-perturbative pair-creation rates in just an external field, or just a finite-temperature background, the rate when both are present is related to their geometric average. In addition, we have noted that the presence of the external field and the elastic scattering of thermal photons it invokes, will alter the value of thermodynamical variables such as the equilibrium pressure of a photon gas.

References

[1] Sauter F 1931 Z. Phys. 69 742
[2] Heisenberg W and Euler H 1936 Z. Phys. 98 714
[3] Weisskopf V 1936 Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 14 6
[4] Schwinger J 1951 Phys. Rev. 82 664–679
[5] Brown L S and Kibble T W B 1964 Phys. Rep. 133 A705–A719
[6] Nikishov A I and Ritus V I 1964 Sov. Phys. JETP 19 529–541
[7] 2012 Extreme Light Infrastructure http://www.extreme-light-infrastructure.eu
[8] 2012 eXawatt Center for Extreme Light Studies http://www.xcels.iapras.ru
[9] 2012 High Power Energy Research http://www.hiperlaser.org
[10] Yanovsky V et al. 2008 Opt. Express 16 2109
[11] Narozhny N B et al. 2004 Phys. Lett. A 330 1–6
[12] Bell A R and Kirk J G 2008 Phys. Rev. Lett. 101 200403
[13] Fedotov A M et al. 2010 Phys. Rev. Lett. 105 080402
[14] Sokolov I V et al. 2010 Phys. Rev. Lett. 105 195005
[15] Bulanov S S et al. 2010 Phys. Rev. Lett. 105 220407
[16] Nerush E N et al. 2011 Phys. Rev. Lett. 106 035001
[17] Elkina N V et al. 2011 Phys. Rev. ST Accel. Beams 14(5) 054401
[18] Ridgers C P et al. 2012 Phys. Rev. Lett. 108(16) 165006
[19] Di Piazza A et al. 2012 Rev. Mod. Phys. 84 1177–1228
[20] Di Piazza A et al. 2009 Phys. Rev. Lett. 103 170403
[21] Brezin E and Itzykson C 1970 Phys. Rev. D 2 1191
[22] Popov V S 1971 Sov. Phys. JETP 13 185
[23] Dunne G V, Gies H and Schützhold R 2009 Phys. Rev. D 80 111301
[24] Thompson C 2008 Astrophys. J. 2 1258–1281
[25] Gies H 2000 Phys. Rev. D 61 085021
[26] GavriloS P and Gitman D M 2008 Phys. Rev. D 78 045017
[27] Kim S P, Lee H K and Yoon Y 2010 Phys. Rev. D 82 025016
[28] Monin A K and Zayakin A V 2008 JETP Lett. 87 709
[29] King B, Gies H and Di Piazza A 2012 Phys. Rev. D 86
[30] Ritus V I 1985 J. Russ. Laser Res. 6 497–617
[31] Dittrich W and Gies H 2000 Probing the Quantum Vacuum (Berlin: Springer-Verlag)
[32] Narozhny N B 1980 Phys. Rev. D 21(4) 1176–1183 URL http://link.aps.org/doi/10.1103/PhysRevD.21.1176
[33] Chen H et al. 2009 Phys. Rev. Lett. 102 105001
[34] Chen H et al. 2009 Phys. Plasmas 16 122702
[35] Baler V N, Mil’shtein A I and Strakhoverenko V M 1976 Sov. Phys. JETP 42 961–965
[36] Olver F W J 1997 Asymptotics and Special Functions (A K Peters Ltd., 63 South Avenue, Natick, MA 01760: AKP Classics)
[37] Ritus V I 1971 Ann. Phys. 69 555–582
[38] Quirós M 2007 Acta Phys. Pol. B 38 3061–3703
[39] Elmfors P and Skagerstam B S 1995 Phys. Lett. B 348 141–148
[40] Gies H 1999 Phys. Rev. D 60 105002
[41] King B 2010 Vacuum Polarisation Effects in Intense Laser Fields Ph.D. thesis University of Heidelberg
http://www.ub.uni-heidelberg.de/archiv/10846