PAIRS TRADING UNDER DRIFT UNCERTAINTY AND RISK PENALIZATION

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Abstract. In this work, we study the dynamic portfolio optimization problem related to the pairs trading, which is an investment strategy that matches a long position in one security with a short position in another security with similar characteristics. The relation between pairs, called spread, is modeled by a Gaussian mean-reverting process whose drift rate is modulated by an unobservable continuous-time finite state Markov chain. Using the classical stochastic filtering theory, we reduce this problem with partial information to the one with complete information and solve it for the logarithmic utility function, where the terminal wealth is penalized by the riskiness of the portfolio according to the realized volatility of the wealth process. We characterize optimal dollar-neutral strategies as well as optimal value functions under both full and partial information and show that the certainty principle holds for the optimal portfolio strategy. Finally, we provide a numerical analysis for a simple example with a two-state Markov chain.

Keywords: pairs trading, regime-switching, utility maximization, risk penalization and partial information.

1. Introduction

Pairs trading is an investment strategy that attempts to capitalize on market inefficiencies arising from imbalances between two or more stocks, or deviations from “equilibrium” so to speak. Simply, it involves a long position and a short position in a pair of similar stocks that have moved together historically. Examples of such pairs can be given as ExxonMobil and Royal Dutch Shell for the oil industry or Pfizer and Glaxo-SmithKline for the pharmaceutical industry. The underlying rationale of pairs trading is to buy the underperformer and sell the overperformer in an anticipation that the security that has relatively performed badly will make up in the coming periods, perhaps even overperform the other, and vice-versa. For this reason, it is also classified as a convergence or mean-reversion strategy. The pair of stocks is selected in a way that it forms a mean-reverting portfolio, referred to as the spread. By forming an appropriate spread, pairs traders try to limit the directional risk that is arising from market’s up or down movements by simultaneously going long on one stock and short in another. Since the market risk is mitigated, profits depend only on the price changes between the two stocks and they can be realized through a net gain on the spread. Therefore, one can also see the pairs trading in the class of market-neutral trading strategies. To achieve market neutrality, traders can choose corresponding strategies so that the...
resulting portfolio has zero (CAPM) beta, hence beta-neutral. Alternatively one can use dollar-neutral strategies, that is investing equal dollar amount in each stock. However, we should remark that market neutrality does not imply either risk-free return or an arbitrage in the classical sense. Risk inherited in pair strategies are just different than the risk in investing strategies involving only long or short position in a specific stock or market. Indeed, pairs trading is a form of a statistical arbitrage, which can be defined broadly as a long horizon trading strategy that generates riskless profits asymptotically (see Hogan et al. [17] for the definition of the statistical arbitrage and Gönçü and Akyıldırım [15] for the existence of statistical arbitrage for the pairs trading strategies). As it is empirically documented by Gatev et al. [14], coupled with a simple pairs selection algorithm, such statistical arbitrage strategies may yield average annualized excess returns of up to 11 percent, which stays still profitable after compensated by the most conservative transaction costs.

In this work, we consider the portfolio optimization problem of a trader with a logarithmic utility from risk penalized terminal wealth investing in a pair of assets whose dynamics have a certain dependence structure in a Markov regime-switching model. More precisely, we model the spread process (log-price differential) as an Ornstein–Uhlenbeck process with a partially observable Markov modulated drift. Our motivation for modeling the drift of the spread, and drifts of both assets, as a function of an unobservable finite-state Markov chain, has couple of advantages. Firstly, drifts of financial assets are hardly constant and observable especially if we think of the convergence-type investment strategies that are usually valid for longer periods. Secondly, although pairs are selected in such a way that they have similar characteristics, the spread dynamics between them might be prone to different regimes. For example, if one leg of the pair is selected to be listed in an index such as S&P 500 while the other is not, this might increase the demand for the one that is listed. Hence that would eventually increase the level of the spread, at least until the one listed in the index is deleted from the index or the other leg of the pair is also added. Moreover, in reality, it is difficult to observe or characterize both microstructure (market-based) or macrostructure (economy-wide) state variables changing accordingly to certain regimes. That would also necessitate using a partial information framework to model such state processes. Numerous studies analyze portfolio selection problem in a complete or partial information and/or Markov regime-switching framework (see, for example, Zhou and Yin [35], Bäuerle and Rieder [2], and Sotomayor and Cadenillas [30] for the complete information case with Markov regime switching or Bäuerle and Rieder [3], Frey et al. [12] and Björk et al. [4] for the partial information case). However, to the best of our knowledge, identification of optimal pairs trading strategies in a Markov-modulated drift under partial information framework is new.

To be precise, our proposed model is an extended version of the model given by Mudchanatongsuk et al. [27], who found the optimal pairs trading strategies in a dollar-neutral setting for an investor with the power utility. Although investing equal dollar amount (as proportions of wealth) in pairs seems to be restrictive, it is meaningful when
CAPM betas of the selected stocks are very close to each other. Our model extends the work of Mudchanatongsuk et al. [27] by allowing a partially observed Markov-modulated drifts both for the price processes and the spread hence enabling them to change by the changing conditions. As the second extension, to find the optimal trading strategies, we use a risk penalized terminal wealth as it is suggested by Rogers [28, Section 2.22]. By penalizing the terminal wealth according to the realized volatility of the wealth process, the investor hopes to prevent the pairs trader pursuing risky strategies. Using risk penalization seems to be appropriate in pairs trading as most of them are executed by hedge funds and proprietary trading houses which engage in high-risk transactions on behalf of investors. This effectively increases the risk aversion of the trader and make her take a less risky position. Apart from certain mathematical convenience, our choice of logarithmic utility function can be justified in several financial grounds. Firstly, although an investor can choose any utility function, reflecting its risk tolerance, a repetitive situation such as the one reflected in mean-reversion type trading strategies, tends to force the utility function into the one that is close to the logarithm. For instance (see Luenberger [26, Chp. 15]), in the power utility case it can be shown in a very simple example that too aggressive or too conservative choices for risk-aversion parameter imply unrealistic preferences for the investor such as betting on strategies that have large losses with high probability and hence not suitable if investor is focused in a long sequence of repeated trials. This can only be alleviated when the risk-aversion parameter $\gamma$ in power utility is close to zero, behaving more like the logarithmic utility. Therefore, we can argue that utility functions that are close to logarithm are appropriate for our setting. Secondly, by penalizing the terminal wealth with the realized volatility of the portfolio and using logarithmic utility, we are able to capture the intertemporal risk factor in our model more easily with just one parameter.

Although both empirical and theoretical literature on pairs trading is growing, published research on optimal portfolio problem is rather limited. Mudchanatongsuk et al. [27] solve the stochastic control problem for pairs trading under power utility for terminal wealth. Tourin and Yan [32] develop an optimal portfolio strategy to invest in two risky assets and the money market account, assuming that log-prices are co-integrated as in the option pricing model of Duan and Pliska [6]. Cartea and Jaimungal [5] extend Tourin and Yan [32] to allow the investor to trade in multiple co-integrated assets and provide an explicit closed-form solution of the dynamic trading strategy while assuming that the drift of asset returns consists of an idiosyncratic and a common drift component. Lee and Papanicolaou [22] solve the optimal pairs trading problem with in a power utility setting, where the drift uncertainty is modeled by a continuous mean-reverting process. It is also worth mentioning here the work of Elliott et al. [9] that proposes a pairs trading strategy based on stochastic filtering of a mean-reverting Gaussian Markov chain for the spread, which is observed in Gaussian noise.

\footnote{$U(x) = \frac{x^\gamma}{\gamma}$ for $\gamma \leq 1$.}
Apart from identification of optimal trading strategies through utility maximization from terminal wealth, there is also a recent literature on optimal liquidation and optimal (entry-exit) timing strategies related to pairs trading. For example, studies by Ekström et al. [7], Larsson et al. [21], and Zeng and Lee [34] focus on how to optimally liquidate a pairs trade by incorporating stop-loss thresholds. Moreover, Leung and Li [24] study an optimal double stopping problem to analyze the timing to start and subsequently liquidate the position subject to transaction costs and Lei and Xu [23] analyze a multiple entry-exit problem of a pair of co-integrated assets. An extensive list of references and a literature review on pairs trading and statistical arbitrage can be found in the recent survey paper by Krauss [20].

To summarize our contributions; first, we characterize the optimal dollar-neutral strategies both in full and partial information settings with risk-penalized terminal wealth for a log-utility trader and show that optimal strategies are dependent on both correlation between two assets and the mean-reverting spread. We show that the effect of risk-penalization on optimal strategies is an increase in the risk-aversion uniformly in a constant proportion that is not dependent on time. Furthermore, we characterize the optimal value functions both by giving a stochastic representation and using dynamic programming, in which verification results are provided for both partial and full information cases. We also provide filtering equations via innovations approach that are necessary to reduce the problem with partial information to one that is under full information. Comparing the optimal strategies under full and partial information, we show that the so-called certainty equivalence principle holds, i.e. the optimal portfolio strategy in the latter case can be obtained replacing the unobservable state variable with its filtered estimate. However, this is not valid for the optimal value function. Although, both value functions are quadratic in the current value of the spread, for the full information case coefficients are characterized by a system of ordinary differential equations, whereas in partial information case, it is characterized by a system of partial differential equations with respect to time and state of the filter. Moreover, we present numerical results for a toy example with a two-state Markov chain in both full and partial information settings. Our analysis shows that average data do not contain sufficient information to obtain the optimal value for the pairs trading problem for logarithmic utility preferences. This result is in contrast with the one for the classical portfolio optimization problem with Markov modulation (see Bäuerle and Rieder [2, Section B]). Furthermore we show that there is always a gain from using filtering.

The remainder of the paper is organized as follows. Section 2 introduces the model. In Section 3 we analyze the portfolio optimization problem in a full information setting. In Section 4 we solve the utility maximization problem under partial information. In Section 5 we provide the numerical analysis of the toy example with a two-state Markov chain.
2. Model

We consider a finite time interval $[0, T]$ and a continuous-time finite-state Markov chain $Y$ defined on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, where $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the global filtration that satisfies the usual conditions; all processes we consider here are assumed to be $\mathbb{G}$-adapted. Suppose $Y$ has the state space $\mathcal{E} = \{e_1, e_2, \ldots, e_K\}$ where, without loss of generality, we assume that $e_k$ is the basis column vector of $\mathbb{R}^K$. $Y$ has the intensity matrix $Q = (q^{ij})_{i,j \in \{1, \ldots, K\}}$ and its initial distribution is denoted by $\Pi = (\Pi_1, \ldots, \Pi^K)$. We now introduce the jump measure of the Markov chain $Y$. Let $\{T_n\}_{n \geq 1}$ be the sequence of jump times of $Y$ and denote by $m$ the jump measure of $Y$, which is given by

$$m([0, t] \times \{j\}) := \sum_{n \geq 1} 1_{\{Y_{T_n} = j\}} 1_{\{T_n \leq t\}}, \quad j \in \{1, \ldots, K\}, \ t \in [0, T],$$

with the compensator

$$\nu([0, t] \times \{j\}) = \int_0^t \sum_{i \neq j} q^{ij} 1_{\{Y_s = i\}} \, ds, \quad j \in \{1, \ldots, K\}, \ t \in [0, T].$$

This allows to write the semimartingale decomposition of $Y$ as

$$Y_t = Y_0 + \int_0^t (j - Y_{s-}) q^{Y_{s-} j} \, ds + \int_0^t \sum_{j=1}^K (j - Y_{s-})(m - \nu)(ds \times \{j\}), \ t \in [0, T].$$

We consider a risk-free asset with the dynamics $dS_t^{(0)} = r S_t^{(0)} \, dt$, $S_0^{(0)} \in \mathbb{R}_+$, where $r > 0$ is the risk-free rate. We have two stocks $S^{(1)}$ and $S^{(2)}$ and the price of the first stock is assumed to follow a Markov-modulated diffusion given by

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = \mu(Y_t) \, dt + \sigma \, dW_t^{(1)}, \quad S_0^{(1)} \in \mathbb{R}_+ \quad (1)$$

with $\sigma > 0$ and where $W^{(1)}$ is a $\mathbb{G}$-Brownian motion independent of $Y$. Since the Markov chain takes values in a finite state space we have that for every $t \in [0, T]$, $\mu(Y_t) = \mu Y_t$ with $\mu = (\mu_1, \ldots, \mu_K)^\top$ and $\mu_i = \mu(e_i), \ i \in \{1, \ldots, K\}$.

It is assumed that the spread $S_t = \log S_t^{(1)} - \log S_t^{(2)}, \ t \in [0, T]$, follows a Markov-modulated Ornstein–Uhlenbeck process:

$$dS_t = \kappa(\theta(Y_t) - S_t) \, dt + \eta \, dW_t, \quad S_0 \in \mathbb{R}, \quad (2)$$

where $\kappa > 0$ and $\eta > 0$, $W$ is a $\mathbb{G}$-Brownian motion with $\langle W^{(1)}, W \rangle_t = \rho t$ and $\theta(Y_t) = \theta(Y_t), \ t \in [0, T]$ with $\theta = (\theta_1, \ldots, \theta_K)^\top$ and $\theta_i = \theta(e_i), \ i \in \{1, \ldots, K\}$. It follows from (1) and (2) that

$$\frac{dS_t^{(2)}}{S_t^{(2)}} = \left(\mu(Y_t) - \kappa(\theta(Y_t) - S_t) + \frac{1}{2} \eta^2 - \rho \sigma \eta\right) \, dt + \sigma \, dW_t^{(1)} - \eta \, dW_t, \quad S_0^{(2)} \in \mathbb{R}_+.$$
Let $X$ be the value of a self-financing portfolio and let $h^{(1)}$ and $h^{(2)}$ denote fractions of the wealth invested in $S^{(1)}$ and $S^{(2)}$, respectively.

**Admissible Investment Strategies.** We consider dollar-neutral pairs trading strategies, that is, we work under the standing assumption that

$$A1 \ h_t^{(1)} = -h_t^{(2)}, \text{ for all } t \in [0, T].$$

In the sequel we are going to use the notation $h = h^{(1)}$. Note that $h_t \in \mathbb{R}$ for every $t \in [0, T]$ and the portfolio weight on the risk-free asset is always 1. In order to ensure that the wealth process is well defined, we consider investment strategies that satisfy

$$A2 \ \mathbb{E} \left[ \int_0^T h_u^2 \, du \right] < \infty.$$

A $\mathbb{G}$-adapted self-financing investment strategy which satisfy $A1$ and $A2$ is called an admissible investment strategy. We denote the set of admissible strategies by $\mathcal{A}$. Accordingly, the dynamics of the pairs-trading portfolio is given by

$$\frac{dX_t^h}{X_t^h} = \left( h_t \left( \kappa(\theta(Y_t) - S_t) - \frac{\eta^2}{2} + \rho \sigma \eta \right) + r \right) \, dt + h_t \eta \, dW_t, \ X_0^h \in \mathbb{R}_+.$$

Notice that for a given $h \in \mathcal{A}$, $X^h$ is a controlled process. In what follows, for the sake of notational simplicity we suppress $h$ dependency and write $X$ instead of $X^h$. The objective of the trader is to maximize expected utility from terminal wealth. In the risk-penalized setting, see Rogers [28, Section 2.22], however, the idea is to prevent the trader from pursuing risky strategies at the expenses of the investor. The investor agrees to pay the trader at time $T$ the risk-penalized amount

$$Z_T = X_T \exp \left( -\frac{1}{2} \epsilon \int_0^T \eta^2 h_s^2 \, ds \right), \ \epsilon > 0.$$

Hence the terminal value of the wealth process is ‘discounted’ by its realized volatility. It follows from Itô’s formula that the dynamics of $Z$ is given by:

$$\frac{dZ_t}{Z_t} = \left( h_t \left( \kappa(\theta(Y_t) - S_t) - \frac{\eta^2}{2} + \rho \sigma \eta \right) + r - \frac{\epsilon \eta^2 h_t^2}{2} \right) \, dt + h_t \eta \, dW_t, \ Z_0 \in \mathbb{R}_+. \ (3)$$

In the sequel we study the optimization problem for a trader who is endowed with a logarithmic utility, in case of regime switching and risk penalization. First we consider the situation where the trader may observe the Markov chain $Y$ that influences the dynamics of the assets and the spread, and subsequently we assume that the Markov chain is not observable and solve the optimization problem under partial information.

### 3. Optimization problem under full information

In this section we suppose that the trader can observe all sources of randomness in the market. Formally we have the following problem

$$\max \ \mathbb{E}^t,z,s,i \left[ \log Z_T \right], \ (4)$$
where $E^{t,z,s,i}$ denotes the conditional expectation given $Z_t = z$, $S_t = s$ and $Y_t = i$. The goal is to characterize the optimal strategy and the corresponding value function. In this case it is possible to obtain the optimal portfolio strategy by pointwise maximization. To this, by Itô’s Lemma we can explicitly characterize the portfolio value at time $T$ as

$$Z_T = z \exp \left\{ \int_t^T \left( h_u \left( \kappa(\theta_u) - S_u \right) - \frac{\eta^2}{2} + \rho \sigma \eta \right) \, du \right. \right.$$

$$\left. + \left. \int_t^T h_u \eta \, dW_u \right\}. \tag{5}$$

Assumption $A2$ guarantees that the stochastic integral in the above expression is a true martingale and hence has a zero expected value. Then computing the expectation in (4), we get

$$E^{t,z,s,i}[\log Z_T] = \log(z) + r(T - t) - E^{t,z,s,i} \left[ \int_t^T h_u \left( \kappa(\theta_u) - S_u \right) - \frac{\eta^2}{2} + \rho \sigma \eta \right] \, du \left. \right].$$

Hence, pointwise maximization in (5) yields to the following optimal strategy,

$$h^*(t, s, i) = \frac{1}{1 + \varepsilon} \left( \frac{\kappa(\theta_i - s)}{\eta^2} + \frac{\rho \sigma}{\eta} - \frac{1}{2} \right).$$

By inserting the optimal strategy into (5), we get the following stochastic representation for the optimal value function,

$$\log(z) + r(T - t) + \frac{1}{2\eta^2(1 + \varepsilon)} E^{t,z,s,i} \left[ \int_t^T \left( \kappa(\theta_u) - S_u \right) - \frac{\eta^2}{2} + \rho \sigma \eta \right]^2 \, du \right].$$

Note that optimal value function is always positive provided that $z > 1$, and the expectation above can be evaluated by computing the first and second moments of the Markov-modulated Ornstein–Uhlenbeck process. This can be achieved, for example as given in Huang et al. [18], by solving a non-homogeneous linear system of differential equations.

The optimal portfolio strategy $h^*$ has three components. First component is the fixed fraction of wealth, that is $\frac{1}{2}$. This is intuitively clear considering “non-pairs” in the sense that there is no correlation or any co-integrated spread between two stocks. The other two components are arising from the dependence structure between two stocks and hence really due to pairs trading. To wit, suppose now that the current spread is equal to the long-term mean of the current regime, that is $\kappa(\theta_i - s) = 0$, then the optimal strategy for a given $\varepsilon > 0$ is determined by only the correlation $\rho$ between first stock and spread scaled by the ratio of volatilities of both. On the other hand, if $\rho$ is zero, it is only determined by the spread dynamics.
We will show above results also by dynamic programming, which makes it possible to identify the value function as the unique solution of the corresponding Hamilton–Jacobi–Bellman (HJB) equation. This type of representation provides a way to examine the value function of the control problem in more detail. We also mention that in the current setting the market is incomplete implying that, for instance, we can not rely on the martingale approach.

We define the value function of the trader by

\[ V(t, z, s, i) = \sup_{h \in A} \mathbb{E}^{t, z, s, i} \left[ \log Z_T \right] . \]

In the sequel, we use the following notation for the partial derivatives: for every function \( g : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \), we write, for instance, \( \frac{\partial g}{\partial t} = g_t \).

Define \( b(s, i) = \kappa (\theta_i - s) - \frac{\eta^2}{2} + \rho \sigma \eta \) and suppose that the value function \( V(\cdot, i) \in C^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R}) \) for every \( i \in \{1, \ldots, K\} \), then it solves the HJB equation given by

\[
0 = \sup_{h \in A} \left\{ V_t(t, z, s, i) + \frac{1}{2} \left( h^2 \eta^2 z^2 V_{zz}(t, z, s, i) + \eta^2 V_{ss}(t, z, s, i) + 2h \eta^2 z V_{zs}(t, z, s, i) \right) + \left( - \frac{\varepsilon \eta^2 h^2}{2} + b(s, i) h + r \right) z V_z(t, z, s, i) + \kappa (\theta_i - s) V_s(t, z, s, i) + \sum_{j=1}^{K} V(t, z, s, j) q_{ij} \right\},
\]

for every \( i \in \{1, \ldots, K\} \), subject to the terminal condition \( V(T, z, s, i) = \log(z) \), for all \((z, s) \in \mathbb{R}_+ \times \mathbb{R}\) and \(i \in \{1, \ldots, K\}\).

Theorem below provides a characterization of the value function and the optimal strategy for the current optimization problem.

**Theorem 3.1.** Suppose that the trader has a logarithmic utility function, then the optimal portfolio strategy is

\[
h^*(t, s, i) = \frac{1}{1 + \varepsilon} \left( \frac{\kappa (\theta_i - s)}{\eta^2} + \frac{\rho \sigma}{\eta} - \frac{1}{2} \right).
\]

The value function is given by

\[ V(t, z, s, i) = \log(z) + d(t) s^2 + c(t, i) s + f(t, i), \]

where the function \( d(t) \) is given by

\[ d(t) = \frac{\kappa^2}{2\eta^2(1 + \varepsilon)} \left( 1 - e^{-2\kappa(T-t)} \right), \]
and the functions $c(t,i)$ and $f(t,i)$ for $i \in \{1, \ldots, K\}$ solve the following system of ordinary differential equations

$$c_t(t,i) - \kappa c(t,i) + 2\kappa \theta_i d(t) - \frac{\kappa^2 \theta_i - \kappa \eta^2 + \kappa \rho \sigma \eta}{\eta^2(1+\varepsilon)} + \sum_{j=1}^{K} c(t,j)q^{ij} = 0, \quad (7)$$

$$f_t(t,i) + d(t)\eta^2 + \kappa \theta_i c(t,i) + \frac{(\kappa \theta_i - \frac{1}{2} \eta^2 + \rho \sigma \eta)^2}{2\eta^2(1+\varepsilon)} + r + \sum_{j=1}^{K} f(t,j)q^{ij} = 0 \quad (8)$$

with boundary conditions $d(T) = 0$, $c(T,i) = 0$, and $f(T,i) = 0$ for all $i \in \{1, \ldots, K\}$.

**Proof. Existence:** Consider the HJB equation (6). By the first order condition, the candidate for an optimal strategy is given in the feedback form

$$h^*(t,z,s,i) = -\frac{\eta^2 V_{zs}(t,z,s,i) + b(s,i)V_z(t,z,s,i)}{\eta^2(z V_{zz}(t,z,s,i) - \varepsilon V_z(t,z,s,i))}. \quad (9)$$

It follows from the form of the utility function that for all $i \in \{1, \ldots, K\}$ the value function can be rewritten as $V(t,z,s,i) = \log(z) + u(t,s,i)$, for some function $u(t,s,i)$ such that $u(T,s,i) = 0$. Since $V(t,z,s,i)$ is concave and increasing in $z$, the second order condition, given by $zV_{zz} - \varepsilon V_z < 0$, holds true for $\varepsilon > 0$ and therefore (9) is the maximizer and hence the optimal portfolio strategy.

Inserting the ansatz for the value function in equations (9) and (6) leads to

$$0 = u_t(t,s,i) + \frac{b^2(s,i)}{2\eta^2(1+\varepsilon)} + r + \frac{1}{2} \eta^2 u_{ss}(t,s,i) + \kappa (\theta_i - s) u_s(t,s,i) + \sum_{j=1}^{K} u(t,s,j)q^{ij}, \quad (10)$$

and the optimal strategy becomes

$$h^*(t,z,s,i) = \frac{b(s,i)}{\eta^2(1+\varepsilon)}. \quad (11)$$

Suppose now that the function $u(t,s,i)$ is of the form $u(t,s,i) = d(t)s^2 + c(t,i)s + f(t,i)$. Computing the derivatives and plugging into (11) we get the following equation

$$0 = r + c_t(t,i)s + d_t(t)s^2 + f_t(t,i) + \eta^2 d(t) + \frac{(\kappa (\theta_i - s) - \frac{\eta^2}{2} + \rho \sigma \eta)^2}{2\eta^2(1+\varepsilon)} + \kappa (\theta_i - s)(c(t,i) + 2d(t)s) + \frac{(\kappa (\theta_i - s) - \frac{\eta^2}{2} + \rho \sigma \eta)^2}{2\eta^2(1+\varepsilon)} + \sum_{j=1}^{K} (sc(t,i) + f(t,j))q^{ij}. \quad (12)$$

Collecting together the terms with $s^2$, $s$ and the remaining ones we get that the function $d(t)$ solves

$$d_t(t) - 2\kappa d(t) + \frac{\kappa^2}{2\eta^2(1+\varepsilon)} = 0,$$
and for every $i \in \{1, \ldots, K\}$, $c(t, i)$ and $f(t, i)$ solve the system of ODEs in (7) and (8), respectively (see e.g., Teschl [31, Theorem 3.19]). Therefore $V(\cdot, i) \in C^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R})$ for every $i \in \{1, \ldots, K\}$.

**Verification:** In order to conclude that $V$ is the value function, we need to show a verification result. Let $v(t, z, s, i)$ be a solution of the HJB equation (6), and denote by $L$ the generator of the process $(Z, S, Y)$, that is

$$LF(t, z, s, i) = F_t(t, z, s, i) + \sum_{j=1}^{K} F(t, z, s, j)q_{ij}$$

$$+ \frac{1}{2} \left(h^2 \eta^2 z F_{zz}(t, z, s, i) + \eta^2 F_{ss}(t, z, s, i) + 2\eta h F_{zs}(t, z, s, i)\right) + \left(b(s, i)h - \frac{\epsilon \eta^2 h^2}{2} + r\right) z F_z(t, z, s, i) + \kappa(\theta_i - s) F_s(t, z, s, i),$$

for every function $F(\cdot, i) \in C^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R})$, for every $i \in \{1, \ldots, K\}$. Given an admissible control $h \in \mathcal{A}$, let $Z^h$ be the solution to equation (3). Applying Itô’s formula we get

$$v(T, Z^h_T, S_T, Y_T) = v(t, z, s, i) + \int_t^T L v(r, Z^h_r, S_r, Y_r) \, dr$$

$$+ \int_t^T \left( v_z(r, Z^h_r, S_r, Y_r) Z^h_r h_r \eta + v_s(r, Z^h_r, S_r, Y_r) \eta \right) \, dW_r$$

$$+ \int_t^T \sum_{j=1}^{K} v(r, Z^h_r, S_r, j) - v(r, Z^h_r, S_r, Y_r^{-}) (m - \nu)(dr \times \{j\}),$$

where we have express the term $\sum_{t \leq r \leq T} v(r, Z^h_r, S_r, Y_r) - v(r, Z^h_r, S_r, Y_r^{-})$ as an integral with respect to the jump measure of $Y$ as

$$\sum_{t \leq r \leq T} v(r, Z^h_r, S_r, Y_r) - v(r, Z^h_r, S_r, Y_r^{-}) =$$

$$\int_t^T \sum_{j=1}^{K} v(r, Z^h_r, S_r, j) - v(r, Z^h_r, S_r, Y_r^{-}) (m - \nu)(dr \times \{j\})$$

$$+ \int_t^T \sum_{j=1}^{K} v(r, Z^h_r, S_r, j) - v(r, Z^h_r, S_r, Y_r^{-}) q_{Y_r^{-} j} \, dr.$$
Since \( v \) satisfies equation (6) we get
\[
v(T, Z^h_T, S_T, Y_T) \leq v(t, z, s, i)
\]
\[
+ \int_t^T \left( v_z(r, Z^h_r, S_r, Y_r) Z^h_r h_r \eta + v_s(r, Z^h_r, S_r, Y_r) \eta \right) dW_r
\]
\[
+ \int_t^T \sum_{j=1}^K v_z(r, Z^h_r, S_r, j) - v_z(r, Z^h_r, S_r, Y_r)(m - \nu)(dr \times \{j\}).
\] (11)

The form of \( v \), together with the fact that \( h \) is an admissible strategy provides the following integrability conditions
\[
\mathbb{E} \left[ \int_0^T (v_z(r, Z^h_r, S_r, Y_r) Z^h_r h_r \eta)^2 \right] \leq \mathbb{E} \left[ \int_0^T h_r^2 \eta^2 \right] < \infty,
\]
\[
\mathbb{E} \left[ \int_0^T (v_s(r, Z^h_r, S_r, Y_r) \eta)^2 \right] \leq C_1 \mathbb{E} \left[ \int_0^T (1 + S_r^2) \eta^2 \right] < \infty,
\]
\[
\mathbb{E} \left[ \int_0^T \sum_{j=1}^K |v(r, Z^h_r, S_r, j) - v(r, Z^h_r, S_r, Y_r)| q^{Y_j} \eta \right] \leq C_2 \mathbb{E} \left[ \int_0^T (1 + S_r^2) \right] < \infty,
\]
which ensure that the \((\mathbb{F}, \mathbb{P})\)-local martingales
\[
\int_0^T \left( v_z(r, Z^h_r, S_r, Y_r) Z^h_r h_r \eta + v_s(r, Z^h_r, S_r, Y_r) \eta \right) dW_r, \quad t \in [0, T]
\]
\[
\int_0^T \sum_{j=1}^K v(r, Z^h_r, S_r, j) - v(r, Z^h_r, S_r, Y_r)(m - \nu)(dr \times \{j\}), \quad t \in [0, T]
\]
are true martingales. Taking the expectation on both sides of the inequality (11) we get that
\[
V(t, z, s, i) \leq v(t, z, s, i).
\]

If \( h \) is a maximizer of equation (6) then we obtain the equality in the expression above.

Remark 3.1. Instead of a dollar-neutral strategy, suppose the trader wants to use a beta-neutral trading strategy, that is a strategy of the form \( \beta_1 h^{(1)} + \beta_2 h^{(2)} = 0 \), where \( \beta_1 \) and \( \beta_2 \) denote CAPM betas of \( S^{(1)} \) and \( S^{(2)} \), respectively. Then the optimal strategy is given by
\[
h^*(t, s, i) = \frac{1}{1 + \varepsilon} \left( \mu_i \beta_2 (\beta_2 - \beta_1) + \beta_1 \beta_2 \kappa (\theta_i - s) - \beta_1 \beta_2 \frac{\rho^2}{2} + \beta_1 \beta_2 \rho \sigma \eta \right),
\]
and the value function has the similar structure as in the dollar-neutral case given above.
4. Optimization Problem under Partial Information

We assume now that the state process $Y$ is not directly observable by the trader. Instead, she observes the price process $S^{(1)}$ and the spread $S$ and she knows the model parameters. Hence, information available to the trader is carried by the filtration

$$\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \quad \mathcal{F}_t = \sigma\{S_u, S^{(1)}_u, 0 \leq u \leq t\}, \quad \mathcal{F}_t \subset \mathcal{G}_t.$$ 

**Admissible Investment Strategies.** Decisions of the trader should depend only on the information available to her at time $t$. That is, we consider self-financing investment strategies such that $h_t$ is $\mathcal{F}_t$-measurable for every $t \in [0, T]$. Then we say that an $\mathcal{F}$-adapted self-financing investment strategy $h$ that satisfies $A1$ and $A2$ is an $\mathcal{F}$-admissible investment strategy. We denote the set of $\mathcal{F}$-admissible strategies by $\mathcal{A}_{\mathcal{F}}$.

The partially informed trader aims to maximize the expected utility $E[\log Z_T]$, over the class $\mathcal{A}_{\mathcal{F}}$. In this case, we naturally end up with an optimal control problem under partial information. In the next part, to solve such a problem we will derive an equivalent control problem under full information via the so-called reduction approach (see, e.g. Fleming and Pardoux [11]). This requires the derivation of the filtering equation for the unobservable state variable. After reduction the corresponding control problem can be interpreted as one with smooth transition governed by the dynamics of filtered probabilities. In Section 5 we discuss this characteristics in the case of a two-state Markov chain.

4.1. The filtering equation. The observations process is given by

$$(dS^{(1)}_t, dS_t) = A(t, Y_t)dt + \Sigma dB_t,$$

where $B = (W^{(1)}, W^{(2)})$ is a 2-dimensional $\mathcal{G}$-Brownian motion and

$$A(t, Y_t) = \left(\begin{array}{c} \mu(Y_t) \\ \kappa(\theta(Y_t) - S_t) \end{array} \right), \quad \Sigma = \left(\begin{array}{cc} \sigma & 0 \\ \rho \eta & \sqrt{1 - \rho^2} \eta \end{array} \right), \quad t \in [0, T].$$

For any function $f$, we denote by $\hat{f}(Y_t)$ the optional projection with respect to filtration $\mathcal{F}$, that is $\hat{f}(Y_t) = E[f(Y_t)|\mathcal{F}_t]$, for every $t \in [0, T]$. This gives the filter for $f(Y)$. By the finite state property of the Markov chain we get that

$$\hat{f}(Y_t) = \sum_{j=1}^{K} f(e_j)p^j_t, \quad t \in [0, T],$$

where $p^j_t = E[\mathbf{1}_{\{Y_t = e_j\}}|\mathcal{F}_t] = P(Y_t = e_j|\mathcal{F}_t)$, $t \in [0, T]$. Then, in order to characterize the filter, it is sufficient to derive the dynamics of the processes $p^j, j \in \{1, \ldots, K\}$. To this, we are going to use the so-called *innovations approach*. This method is based on finding a suitable $\mathcal{F}$-adapted process that drives the dynamics of the filter, see e.g. Wonham [32] and Elliott et al. [8] for more details. We define the 2-dimensional process
Pairs trading under drift uncertainty and risk penalization

\[ I = (I^{(1)}, I^{(2)})^\top \] by

\[ I_t = B_t - \int_0^t \Sigma^{-1}(A(u, Y_u) - A(u, \hat{Y}_u)) \, du, \quad t \in [0, T]. \]

Explicitly we have

\[ I^{(1)}_t = W^{(1)}_t - \int_0^t \frac{\mu(Y_u) - \hat{\mu}(Y_u)}{\sigma} \, du, \]
\[ I^{(2)}_t = W^{(2)}_t - \int_0^t \frac{\sigma \kappa(\theta(Y_u) - \hat{\theta}(Y_u)) - \rho \eta (\mu(Y_u) - \hat{\mu}(Y_u))}{\sigma \eta \sqrt{1 - \rho^2}} \, du, \]

for every \( t \in [0, T]. \)

**Remark 4.1.** The process \( I \) is called innovation process. By Liptser and Shiryaev [25, Theorem 9.1], \( I \) is an \((F, \mathbb{P})\)-Brownian motion.

Note that, since the signal \( Y \) and the Brownian motion \( B \) driving the observation process are independent, the filtration \( F \) coincides with the natural filtration of the innovation process, see Allinger and Mitter [1, Theorem 1]. Then, by Jacod and Shiryaev [19, Theorem III.4.34 (a)] every \((\mathbb{P}, F)\)-local martingale \( M \) admits the following decomposition:

\[ M_t = M_0 + \int_0^t H_u \, dI_u, \quad t \in [0, T], \tag{12} \]

for some \( F \)-adapted 2-dimensional process \( H \) such that

\[ \int_0^T \|H_u\|^2 \, du < \infty \quad \mathbb{P} \text{-a.s.} \]

We recall the notation \( \mu = (\mu_1, \ldots, \mu_K)^\top \), where \( \mu_i = \mu(e_i) \), and \( \theta = (\theta_1, \ldots, \theta_K)^\top \), where \( \theta_i = \theta(e_i) \). Also introduce \( f = (f_1, \ldots, f_K)^\top \), where \( f_i = f(e_i) \). Next theorem provides the filter dynamics.

**Theorem 4.1.** For every \( i \in \{1, \ldots, K\} \), the filter process \( p^i \) satisfies

\[ p^i_t = p^i_0 + \int_0^t \sum_{j=1}^K q^{ji} p^j_u \, du + \frac{1}{\sigma} \int_0^t p^i_u (\mu^i - \mu^\top p_u) \, dI^{(1)}_u \]
\[ + \frac{1}{\sigma \eta \sqrt{1 - \rho^2}} \int_0^t p^i_u (\sigma \kappa (\theta_i - \hat{\theta}^\top p_u) - \eta \rho (\mu_i - \mu^\top p_u)) \, dI^{(2)}_u, \tag{13} \]

for every \( t \in [0, T] \).

**Proof.** Consider the semimartingale decomposition of \( f(Y) \) given by

\[ f(Y_t) = f(Y_0) + \int_0^t \langle Qf, Y_u \rangle \, dt + M^{(1)}_t, \quad t \in [0, T], \]
where $M^{(1)}$ is a $(\mathbb{G}, \mathbb{P})$-martingale. Now, projecting over $\mathcal{F}_t$ leads to

$$\tilde{f}(Y_t) - \tilde{f}(Y_0) - \int_0^t \langle Qf, \tilde{Y}_u \rangle \, du = M^{(2)}_t, \quad t \in [0, T],$$

where $M^{(2)}$ is an $(\mathbb{F}, \mathbb{P})$-martingale. Using the martingale representation in (12) we get

$$\tilde{f}(Y_t) - \tilde{f}(Y_0) - \int_0^t \langle Qf, \tilde{Y}_u \rangle \, dt = M_0 + \int_0^t H_u \, dI_u, \quad t \in [0, T].$$

Let $m_t = I_t + \int_0^t \Sigma^{-1} A(u, Y_u) \, du$, for every $t \in [0, T]$. Computing the product $f(Y) \cdot m$ and projecting on $\mathcal{F}_t$, we obtain

$$f(Y_t) \cdot m_t = \int_0^t m_u \langle Qf, \tilde{Y}_u \rangle \, du + \int_0^t \Sigma^{-1} f(Y_u) A(u, Y_u) \, du + M^{(3)}_t, \quad t \in [0, T],$$

for some $(\mathbb{F}, \mathbb{P})$-martingale $M^{(3)}$. We now compute the product $\tilde{f}(Y) \cdot m$ as

$$\tilde{f}(Y_t) \cdot m_t = \int_0^t m_u \langle Qf, \tilde{Y}_u \rangle \, du + \int_0^t \Sigma^{-1} \tilde{f}(Y_u) A(u, Y_u) \, du + \int_0^t H_u \, du + M^{(4)}_t.$$

for every $t \in [0, T]$, where $M^{(4)}$ is an $(\mathbb{F}, \mathbb{P})$-martingale. Comparing the finite variation terms in (14) and (15), we get

$$H^{(1)}_t = \frac{\tilde{f}(Y_t) \mu(Y_t) - \tilde{f}(Y_t) \mu(Y_t)}{\sigma},$$

$$H^{(2)}_t = \frac{\sigma k f(Y_t) \theta(Y_t) - \tilde{f}(Y_t) \theta(Y_t)}{\sigma \eta \sqrt{1 - \rho^2}},$$

for every $t \in [0, T]$. Finally choosing $f(Y_t) = 1_{\{Y_t = x_1\}}$, we obtain the result. \qed

**Remark 4.2.** Here notice that the drift and diffusion coefficients in (13) are continuous, bounded and locally Lipschitz. This implies that $p$ is the unique strong solution of the filtering equation (13).

### 4.2. Reduction of the Optimal Control Problem

The semimartingale decomposition of $Z$ and $S$ with respect to the observation filtration are given by

$$Z_t = Z_0 + \int_0^t Z_u \left( h_u \left( \kappa (\theta^\top p_u - S_u) - \frac{\eta^2}{2} + \rho \sigma \eta \right) + r - \frac{\varepsilon \eta^2 h_u^2}{2} \right) \, du$$

$$+ \eta \int_0^t h_u Z_u \left( \rho \, dI_u^{(1)} + \sqrt{1 - \rho^2} \, dI_u^{(2)} \right), \quad t \in [0, T],$$

and

$$S_t = S_0 + \int_0^t \kappa (\theta^\top p_u - S_u) \, du + \eta \int_0^t \left( \rho \, dI_u^{(1)} + \sqrt{1 - \rho^2} \, dI_u^{(2)} \right), \quad t \in [0, T].$$

Thanks to uniqueness of the solution of the filtering equation we can consider the $(K + 2)$-dimensional process $(Z, S, p)$ as the state process and introduce the equivalent
optimal control problem under complete information, see, e.g., Fleming and Pardoux [11]. We have

\[
\max \mathbb{E}^{t,z,s,p}[\log Z_T],
\]

where \( \mathbb{E}^{t,z,s,p} \) denotes the conditional expectation given \( Z_t = z, S_t = s \) and \( p_t = p \), where \((z, s, p) \in \mathbb{R}^+ \times \mathbb{R} \times \Delta_K \), with \( \Delta_K \) denoting the \((K - 1)\)-dimensional simplex. To obtain the optimal control strategy it is possible to apply point-wise maximization which also leads to a stochastic representation for the value function. Here, as in the case of full information, we aim to characterize the value function as the solution of the corresponding HJB equation.

**Theorem 4.2.** Suppose that the trader has a logarithmic utility function. Then the optimal portfolio strategy under partial information is given by

\[
h^*(t, s, p) = \frac{1}{1 + \varepsilon} \left( \frac{\kappa (\theta^\top p - s)}{\eta^2} + \frac{\rho \sigma}{\eta} - \frac{1}{2} \right),
\]

and the optimal value is given by

\[
V(t, z, s, p) = \log(z) + d(t)s^2 + c(t, p)s + f(t, p),
\]

where the function \( d(t) \) is given by

\[
d(t) = \frac{\kappa^2}{2\eta^2(1 + \varepsilon)} \left( 1 - e^{-2\kappa(T-t)} \right),
\]

and the functions \( c(t, p) \) and \( f(t, p) \) solve the following system of partial differential equations:

\[
c_t(t, p) + \frac{1}{2} \sum_{i,j=1}^{K} \tilde{\alpha}^{ij}(p)c_{p^i p^j}(t, p) + \sum_{i,j=1}^{K} c_{p^i}(t, p)q^{ij}p^j + \kappa (2d(t)\theta^\top p - c(t, p)) - \gamma(p) = 0,
\]

\[
f_t(t, p) + \frac{1}{2} \sum_{i,j=1}^{K} \tilde{\alpha}^{ij}(p)f_{p^i p^j}(t, p) + \sum_{i,j=1}^{K} f_{p^i}(t, p)q^{ij}p^j + \eta \sum_{i=1}^{K} c_{p^i}(t, p)\tilde{\beta}^i(p)
\]

\[
+ c(t, p)\kappa\theta^\top p + \eta^2 d(t) + \frac{(\kappa\theta^\top p - \frac{1}{2}\eta^2 + \rho \sigma \eta)^2}{2\eta^2(1 + \varepsilon)} + r = 0,
\]

\[\]
with boundary conditions \(d(T) = 0, c(T, p) = 0, f(T, p) = 0\) for every \(p \in \Delta_K\), and where

\[
\tilde{\alpha}_{i,j}(p) = H^{(1,1)}(p)H^{(j,1)}(p) + H^{(i,2)}(p)H^{(j,2)}(p), \quad i, j \in \{1, \ldots, K\},
\]

\[
\tilde{\beta}_i(p) = \rho H^{(i,1)}(p) + \sqrt{1 - \rho^2}H^{(i,2)}(p), \quad i \in \{1, \ldots, K\},
\]

\[
H^{(1,1)}(p) = p^\top(\mu_i - \mu^\top) \frac{\sigma}{\eta} \quad \text{and} \quad H^{(i,2)}(p) = p^\top(\sigma \kappa (\theta_i - \theta^\top) - \eta \rho (\mu_i - \mu^\top)) \frac{\sqrt{1 - \rho^2}}{\eta \sigma},
\]

\[
\gamma(p) = \frac{\kappa}{\eta^2(1 + \varepsilon)} (\theta^\top p - \frac{1}{2} \eta^2 + \rho \eta).
\]

Proof. Existence: Denote by \(b(s, p) = \kappa(\theta^\top p - s) - \frac{\eta^2}{2} + \rho \sigma \eta\). For the current setting we have the following HJB equation

\[
0 = \sup_{h \in A^p} \left\{ V_t(t, z, s, p) + \left( -\frac{\varepsilon \eta^2 h^2}{2} + b(s, p)h + r \right) zV_z(t, z, s, p) \right. \\
+ \kappa(\theta^\top p - s) V_s(t, z, s, p) + \sum_{i,j=1}^K q_{ij} p_j V_{pp}(t, z, s, p) \\
+ \frac{1}{2} \left( h^2 \eta^2 zV_{zz}(t, z, s, p) + \eta^2 V_{ss}(t, z, s, p) + 2h \eta^2 zV_{zs}(t, z, s, p) \right) \\
+ \sum_{i=1}^K V_{sp}(t, z, s, p) \eta \tilde{\beta}_i(p) + \sum_{i,j=1}^K \tilde{\alpha}_{ij}(p) V_{pp}(t, z, s, p) + \sum_{i=1}^K h \eta \tilde{\beta}_i(p) V_{zp}(t, z, s, p) \left. \right\},
\]

subject to the terminal condition \(V(T, z, s, p) = \log(z)\), for all \(z > 0, s \in \mathbb{R}\) and for every \(p \in \Delta_K\).

From the first order condition, the candidate for an optimal strategy is given in the feedback form

\[
h^*(t, z, s, p) = -\frac{\eta^2 V_z(t, z, s, p) + b(s, p) V_z(t, z, s, p) + \sum_{i=1}^K \eta \tilde{\beta}_i(p) V_{zp}(t, z, s, p)}{\eta^2(zV_{zz}(t, z, s, p) - \varepsilon V_z(t, z, s, p))}.
\]

It follows from the form of the utility function that the value function can be rewritten as \(V(t, z, s, p) = \log(z) + u(t, s, p)\), for some function \(u(t, s, p)\) such that \(u(T, s, p) = 0\) for all \((s, p) \in (\mathbb{R} \times \Delta_K)\). Since \(V(t, z, s, p)\) is concave and increasing in \(z\), the second order condition, given by \(zV_{zz} - \varepsilon V_z < 0\), holds true for \(\varepsilon > 0\) and therefore (20) is the maximizer and the optimal portfolio strategy.

Here, we choose \(u\) of the form \(u(t, s, p) = d(t)s^2 + c(t, p)s + f(t, p)\). Inserting this ansatz in equations (19) and (20) leads to the system of linear partial differential
equations in (17)-(18) and the following linear ordinary differential equation
\[
d_t(t) - 2\kappa d(t) + \frac{\kappa^2}{2\eta^2(1+\varepsilon)} = 0.
\]

Note that the system (17) and (18) admits a unique solution (see e.g., Friedman [13, Chp. 9]). This implies that \( V \in C^{1,2,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \Delta K) \).

Verification: To conclude that \( V \) is the value function, we show a verification result. Let \( v(t, z, s, p) \) be a solution of (19) with the boundary condition \( v(T, z, s, p) = \log(z) \). Let \( h \in \mathcal{AF} \) be an \( \mathcal{F} \)-admissible control, let \( Z^h \) the solution to equation (16). Denote by \( \mathcal{L}^\mathcal{F} \) the \( (\mathcal{F}, \mathbb{P}) \)-generator of the process \((Z, S, p)\), that is for every \( F \in C^{1,2,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \Delta K) \) we have
\[
\mathcal{L}^\mathcal{F} F(t, z, s, p) = F_t(t, z, s, p) \\
+ \frac{1}{2} \left( h^2 \eta^2 z^2 F_{zz}(t, z, s, p) + \eta^2 F_{ss}(t, z, s, p) + 2h\eta^2 z F_{zs}(t, z, s, p) \right) \\
+ \left( -\frac{\varepsilon \eta^2 h^2}{2} + b(s, p)h + r \right) z F_z(t, z, s, p) \\
+ \kappa \left( \theta^T p - s \right) F_s(t, z, s, p) \\
+ \sum_{i,j=1}^{K} q_{ij} p^j F_{p^i}(t, z, s, p) + \sum_{i,j=1}^{K} \tilde{a}^{ij}(p) F_{p^i p^j}(t, z, s, p) \\
+ \sum_{i=1}^{K} F_{sz}(t, z, s, p) \eta \tilde{\beta}^i(p) + \sum_{i=1}^{K} h\eta \tilde{\beta}^i(p) F_{zp^i}(t, z, s, p).
\]

Applying Itô’s formula we get
\[
v(T, Z^h_T, S_T, p_T) = v(t, z, s, p) + \int_t^T \mathcal{L}^\mathcal{F} v(u, Z^h_u, S_u, p_u) \, du \\
+ \int_t^T \rho \eta \left( v_z(u, Z^h_u, S_u, p_u) Z^h_u h_u + v_s(u, Z^h_u, S_u, p_u) \right) \, dI_u^{(1)} \\
+ \int_t^T \sqrt{1 - \rho^2 \eta} \left( v_z(u, Z^h_u, S_u, p_u) Z^h_u h_u + v_s(u, Z^h_u, S_u, p_u) \right) \, dI_u^{(1)} \\
+ \int_t^T \sum_{i=1}^{K} H^{(i,1)}(p_u) v_{p^i}(u, Z^h_u, S_u, p_u) \, dI_u^{(1)} + \int_t^T \sum_{i=1}^{K} H^{(i,2)}(p_u) v_{p^i}(u, Z^h_u, S_u, p_u) \, dI_u^{(2)}.
\]
By equation (19) we get
\[ v(T, Z^h_T, S_T, p_T) \leq v(t, z, s, p) \]
\[ + \int_t^T \rho \eta \left( v_z(u, Z^h_u, S_u, p_u) Z^h_u h_u + v_s(u, Z^h_u, S_u, p_u) \right) \, dI_u^{(1)} \]
\[ + \int_t^T \sqrt{1 - \rho^2} \eta \left( v_z(u, Z^h_u, S_u, p_u) Z^h_u h_u + v_s(u, Z^h_u, S_u, p_u) \right) \, dI_u^{(1)} \]
\[ + \int_t^T \sum_{i=1}^{K} H^{(i,1)}(p_u) v_{p_i}(u, Z^h_u, S_u, p_u) \, dI_u^{(1)} + \int_t^T \sum_{i=1}^{K} H^{(i,2)}(p_u) v_{p_i}(u, Z^h_u, S_u, p_u) \, dI_u^{(2)}. \]

(21)

Note that both stochastic integrals with respect to \( I^{(1)} \) and \( I^{(2)} \) are true martingales. Indeed, by the form of the solution of the HJB equation \( v(t, z, s, p) = \log(z) + d(t) s^2 + c(t, p) s + f(t, p) \), the fact that \( h \) is an \( \mathbb{F} \)-admissible strategy and boundedness of the functions \( d(t), c(t, p), f(t, p) \) and their derivatives over the compact interval \([0, T] \times \Delta K\), we get that
\[ \mathbb{E} \left[ \int_t^T (h^2_u + c^2(u, p_u) + d^2(u) S^2_u) \, du \right] < \infty, \]
\[ \mathbb{E} \left[ \sum_{i=1}^{K} \left( c^2_{p_i}(u, p_u) S^2_u + f^2_{p_i}(u, p_u) \right) \left( (H^{(i,1)}(p_u))^2 + (H^{(i,2)}(p_u))^2 \right) \, du \right] < \infty. \]

Then taking the expectation on both sides of inequality (21) implies that \( V(t, z, s, p) \leq v(t, z, s, p) \). Moreover if \( h^* \) is a maximizer of equation (19), then we obtain the equality \( V(t, z, s, p) = v(t, z, s, p) \). \( \square \)

**Comments and discussion.** As Theorem 3.1 and Theorem 4.2 show, optimal strategies depend on both correlation between two assets and the mean-reverting spread. Moreover, they do not depend on the risk-free rate \( r \) due to the fact that a priori we restrict ourselves to the dollar-neutral pairs trading strategies. Comparing optimal strategies under full and partial information, we can say that the so-called *certainty equivalence principle* holds, i.e., the optimal portfolio strategy in the latter case can be obtained by replacing the unobservable state variable with its filtered estimate.

The effect of risk-penalization on optimal strategies is to increase the risk-aversion uniformly in a constant proportion that is not dependent on time. This effectively decreases the proportion of wealth invested in pairs and increases the proportion of wealth invested in the risk-free asset. Considering the optimal value functions, in both cases, they are quadratic functions of the current value of the spread. However in both cases, coefficients (factor loadings) on the quadratic term, \( s^2 \), depend only on time. This result is worth to mention since it means that the trader does not really consider...
the effect of the partial information on the quadratic level of the current spread. Finally
note that similar results holds true for beta-neutral strategies.

5. Toy Example: Two-State Markov Chain

In this section, we give a toy example of our proposed model, where the unobservable
Markov chain has only two states. During our analysis, we set $z = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.6$,
$\mu_1 = 0.2$ and $\mu_2 = 1$. In the first step we consider the full information case, where the
trader knows the state of the Markov chain. Then, we investigate the case with the
partial information.

5.1. The full information case. In this part of our analysis, we employ Theorem 3.1
and solve corresponding system of ODEs numerically. In the following, since we set
$z = 1$ we suppress the dependence of the value function on $z$ and write $V(t, s, i)$ for
$i \in \{1, 2\}$.

In Figure 1 we illustrate the value function with respect to time to maturity for
different values of initial spread: $s = 0.1$, $s = 0.3$ and $s = 0.7$. Naturally, for all $s$
and $i \in \{1, 2\}$, the optimal value increases in time to maturity. However, there is no
monotone relation between value functions $V(\cdot, s, 1)$ and $V(\cdot, s, 2)$. Namely, for $s = 0.1$,
$V(t, s, 2) > V(t, s, 1)$, for all $t$ whereas for $s = 0.7$ we observe the opposite behavior. On
the other hand, for $s = 0.3$, this relation depends on the remaining time to maturity.
Here, the intersection point of the two functions $V(\cdot, s, 1)$ and $V(\cdot, s, 2)$ depends on the
transition intensities of the Markov chain $q^{12}$ and $q^{21}$. In particular, for this example,
fixing all other parameters, the intersection point moves to the right as $q^{12}$ gets larger.

Figure 1. Value function for different values of $s$ for initial state given is $\epsilon_1$ (dashed line) and for initial state given is $\epsilon_2$ (solid line): $(T - t) = 3$
years, $z = 1$, $r = 0.01$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $\kappa = 1$, $\rho = 0.9$, $\sigma = 0.2$,
$\eta = 0.2$, $\varepsilon = 0.3$, $q^{12} = 0.7$ and $q^{21} = 0.2$. Value function for $s = 0.1$
(Left panel); Value function for $s = 0.3$ (Middle panel); Value function
for $s = 0.7$ (Right panel).
Next, we compare the value of the optimal portfolio problem with the value function computed using the *average data*. In this way we intend to see whether the knowledge of average data is sufficient for obtaining the value function. To this, we set $q^{12} = 1$ and $q^{21} = 2$, and compute the stationary distribution of the Markov chain $Y$ as $(\pi, 1 - \pi)$ where $\pi = \frac{q^{21}}{q^{12} + q^{21}} = 0.67$. Then, we get the average data $\overline{\theta} = \pi \theta_1 + (1 - \pi) \theta_2 = 0.27$. Denote by $\nabla(t, s)$ the value function assuming that there is no Markov modulation and the mean-reversion level of the spread is given by $\overline{\theta}$. In Figure 2 we plot $\nabla(t, s)$ versus $\mathbb{E}^\pi[V(t, s, Y_t)] = \pi V(t, s, 1) + (1 - \pi)V(t, s, 2)$.

![Value function](image)

**Figure 2.** Value from Markov modulated case (solid line) vs value from average data (dashed line): $z = 1, r = 0.01, \theta_1 = 0.1, \theta_2 = 0.6, \kappa = 1, \rho = 0.9, \sigma = 0.2, \eta = 0.2, \varepsilon = 0.5, q^{12} = 1$ and $q^{21} = 2$. Value for different initial spread values (Left panel) for $T - t = 0.1$ years; value for different times to maturity (Right panel) for $s = 0.3$.

We observe that $\mathbb{E}^\pi[V(t, s, Y_t)] > \nabla(t, s)$. This result is in contrast with the one for the classical portfolio optimization problem with Markov modulation in the case of logarithmic utility preferences (see Bäuerle and Rieder [2, Section B]). It shows that average data do not contain sufficient information to obtain the optimal value for the pairs trading problem. We attribute this to the mean-reverting nature of the underlying state variables.

Figure 3 depicts the behavior of the value function with respect to the mean-reversion speed $\kappa$, for correlation values $\rho = 0.1$ and $\rho = 0.9$. Here we observe that higher values of $\kappa$ not necessarily lead to larger portfolio values in general. For instance, in the left panel, for the second state (solid line) the value function does not have a monotone behavior with respect to $\kappa$. Overall, the impact of correlation depends on the given initial state of the chain, the value of $\kappa$ and the volatility terms $\eta$ and $\sigma$.

5.2. **The partial information case.** In the partially observable setting, having only two states enables us to reduce the number of state variables for our filtered control problem since $p := p^1 = 1 - p^2$. Then we only need the dynamics of $p$, given, after
Figure 3. Impact of $\kappa$ on value function for $\rho = 0.1$ (grey line) and for $\rho = 0.9$ (black line) for initial state given is $e_1$ (dashed line) and for initial state given is $e_2$ (solid line): $T - t = 3$ years, $z = 1$, $r = 0.01$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $\eta = 0.9$, $\varepsilon = 0.3$, $q^{12} = 0.7$ and $q^{21} = 0.2$, $\sigma = 0.2$ (Left panel); $\sigma = 0.9$ (Right panel).

arrangement, by

$$d p_t = (q^{12} + q^{21}) \left( \frac{q^{21}}{q^{12} + q^{21}} - p_t \right) dt + \sqrt{\nu_1^2 + \nu_2^2 p_t (1 - p_t)} d I_t^{(3)}, \quad (22)$$

where $\nu_1 = \frac{(\mu_1 - \mu_2)}{\sigma}$ and $\nu_2 = \frac{\sigma(\theta_1 - \theta_2) - \eta(\mu_1 - \mu_2)}{\sigma \eta \sqrt{1 - p^2}}$, and $I_t^{(3)} = \frac{\nu_1}{\sqrt{\nu_1^2 + \nu_2^2}} I_t^{(1)} + \frac{\nu_2}{\sqrt{\nu_1^2 + \nu_2^2}} I_t^{(2)}$ is an $\mathbb{F}$-Brownian motion. We can write the semimartingale representation of wealth and spread processes with respect to filtration $\mathbb{F}$ as

$$d Z_t = Z_t \left( h_t \left( \kappa (\theta_2 + (\theta_1 - \theta_2)) p_t - S_t \right) - \frac{\eta^2}{2} + \rho \sigma \eta \right) + r - \frac{\varepsilon \eta^2 h_t^2}{2} \right) dt \quad (23)$$

$$+ \eta h_t Z_t d \tilde{I}_t,$$

and

$$d S_t = \kappa (\theta_2 + (\theta_1 - \theta_2)) p_t - S_t \right) dt + \eta d \tilde{I}_t, \quad (24)$$

where $\tilde{I}$ is a $\mathbb{F}$-Brownian motion with $\langle \tilde{I}, I^{(3)} \rangle_t = \frac{\nu_1 \rho + \nu_2 \sqrt{1 - p^2}}{\sqrt{\nu_1^2 + \nu_2^2}} t$.

Note that one can interpret the reduced control problem with state variables $(Z, S, p)$ given by (23), (24) and (22) as a pairs trading model with smooth transitions. That is, one can see $p$ as a state variable process governing smooth transitions between two regimes with different long-term means for the spread, that is, $\theta_1$ and $\theta_2$. The dynamics of $p$ is also very similar to a mean-reverting Jacobi-type (or Wright–Fisher) diffusion used in population genetics to model allele frequencies \cite{2}, see e.g., \cite{10}, \cite{29} or \cite{16}.

\footnote{For the Jacobi or Wright–Fisher diffusion, the diffusion coefficient is given by $\sqrt{p(1 - p)}$.}
In this case the value function can be written as $V(t, z, s, p^1, p^2) = \tilde{V}(t, z, s, p)$, and, as in Theorem 4.2, the optimal value is given by $\tilde{V}(t, z, s, p) = \log(z) + d(t) s^2 + \tilde{c}(t, p) s + \tilde{f}(t, p)$, where the function $d(t)$ is given by

$$d(t) = \frac{\kappa^2}{2\eta^2(1 + \varepsilon)} \left(1 - e^{-2\kappa(T-t)}\right),$$

and the functions $\tilde{c}(t, p)$ and $\tilde{f}(t, p)$ solve the following system of partial differential equations:

$$\begin{align*}
\tilde{c}_t(t, p) - &\frac{\kappa^2(\theta_2 + (\theta_1 - \theta_2)p) - \kappa(-\frac{\eta^2}{1 + \varepsilon} + \rho \eta)}{\eta^2(1 + \varepsilon)} - \kappa \tilde{c}(t, p) + 2\kappa(\theta_2 + (\theta_1 - \theta_2)p)d(t) \\
&+ (q^{12} + q^{21}) \left(\frac{q^{21}}{q^{12} + q^{21}} - p\right) \tilde{c}_p(t, p) + \frac{1}{2} (\nu_1^2 + \nu_2^2)p^2(1 - p)^2 \tilde{c}_{pp}(t, p) = 0, \\
\tilde{f}_t(t, p) + &\frac{\kappa^2(\theta_2 + (\theta_1 - \theta_2)p)^2 + (\rho \sigma \eta - \frac{\eta^2}{2})^2 + 2\kappa(\theta_2 + (\theta_1 - \theta_2)p)(\rho \sigma \eta - \frac{\eta^2}{2})}{2\eta^2(1 + \varepsilon)} \\
&+ \eta^2 d(t) + r + \kappa(\theta_2 + (\theta_1 - \theta_2)p) \tilde{c}(t, p) + (q^{12} + q^{21}) \left(\frac{q^{21}}{q^{12} + q^{21}} - p\right) \tilde{f}_p(t, p) \\
&+ \frac{1}{2} (\nu_1^2 + \nu_2^2)p^2(1 - p)^2 \tilde{f}_{pp}(t, p) + \kappa(\theta_1 - \theta_2)p(1 - p) \tilde{c}_p(t, p) = 0.
\end{align*}$$

(25)

with boundary conditions $d(T) = 0$, $\tilde{c}(T, p) = 0$, $\tilde{f}(T, p) = 0$ for every $p \in [0, 1]$.

We use an explicit finite-difference method to solve the system of PDEs given in (25) numerically. In order to guarantee the positivity of the scheme we use forward-backward approximation for the first order derivatives. The value function in the partial information case has a similar behavior with respect to the parameters as the one in the full information case. However, we stress that in the partial information setting, also the drift parameters $\mu_1$ and $\mu_2$ play a role. In particular, the relative values of $\mu_1$, $\mu_2$ and the noise parameters $\sigma$ and $\eta$ control for the precision of the filtered probability estimates.

In Figure 4 we illustrate that the trader benefits from using filtered estimates instead of average data. In line with the full information case, gains from filtering increase in time to maturity. On the other hand, they get smaller as $p$ moves towards $\frac{1}{2}$, which represent the most uncertain situation.

6. CONCLUSION

In this paper we consider the pairs trading for a trader with logarithmic utility preferences and risk penalized terminal wealth. By penalizing the terminal wealth with the realized volatility of the portfolio, we can capture the intertemporal risk factor more easily with just one parameter, that is $\varepsilon$. We assume that the mean-reversion level of the spread is Markov switching and study the utility maximization problem under full
and partial information, corresponding to the cases where the trader may or may not observe directly the state of the Markov chain.

In the full information setting, we compute the optimal strategy and characterize the value function as the unique (classical) solution of the HJB equation that reduces, after arrangement, to a system of ODEs. In the partial information case we first derive the filter dynamics and then study the optimization problem corresponding with underlying state processes, where the unobservable state of the Markov chain is replaced by its filtered estimate. We address the problem by dynamic programming and we represent the value function in terms of the solution of a system of PDEs.

In the last part of the paper, we present a numerical analysis in the case where the Markov chain has two possible states. In the full information setting we study the behavior of the value function with respect to several parameters. An interesting result is that the value of the optimal portfolio is always strictly larger than the value function computed when the regime-switching, mean-reversion level of the spread is replaced by the average mean-reversion level (with respect to the stationary distribution of the Markov chain). This suggests that the knowledge of the average data is not sufficient to obtain the optimal portfolio value.

In the partial information case, we observe that the trader always benefits from using filtered estimates for the state of the Markov chain instead of the average data. Gains are larger in more certain situations, that is when probabilities of being in one state are close to zero or one, and lower in more uncertain settings, equivalently, when probabilities are close to 0.5.
Acknowledgments

The authors thank the participants of 14th Europt Workshop on Advances in Continuous Optimization for discussions. Sühan Altay gratefully acknowledges financial support from the Austrian Science Fund (FWF) under grant P25216. The work on this paper was completed while Zehra Eksi was visiting the Department of Economics, University of Perugia as a part of the ACRI Young Investigator Training Program (YITP). The support of the Association of Italian Banking Foundations and Savings Banks (ACRI) is greatly acknowledged.

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