RATIONAL POINTS AND NON-ANTICANONICAL HEIGHT FUNCTIONS

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Abstract. A conjecture of Batyrev and Manin predicts the asymptotic behaviour of rational points of bounded height on smooth projective varieties over number fields. We prove some new cases of this conjecture for conic bundle surfaces equipped with some non-anticanonical height functions. As a special case, we verify these conjectures for the first time for some smooth cubic surfaces for height functions associated to certain ample line bundles.

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1. Introduction

1.1. The Batyrev–Manin conjecture. This paper is concerned with counting rational points of bounded height on algebraic varieties. Let $X$ be a smooth projective variety over a number field $k$ with $X(k) \neq \emptyset$ and let $D$ be a divisor on $X$. Recall from the theory of heights that to each choice of adelic metric on the line bundle $\mathcal{O}_X(D)$ one can associate a choice of height function $H$. (Note that this theory works equally well for $\mathbb{Q}$-divisors, see e.g. [CLT02, §2].) If $D$ is big, then such height functions have the important property that the cardinality

$$N(U, H, B) = \#\{x \in U(k) : H(x) \leq B\}$$

is finite for some open dense subset $U \subset X$ and all $B > 0$. If $X$ is a Fano variety, or a variety which is close to being Fano, then a conjecture of Batyrev and Manin [BM90] predicts an asymptotic formula of the shape

$$N(U, H, B) \sim c_{U, H} B^{a(D)} (\log B)^{b(D)},$$

(1.1)

for some $c_{U, H} > 0$ and for certain exponents $a(D)$ and $b(D)$ defined in terms of the geometry of $D$ (we recall the definitions of $a(D)$ and $b(D)$ in §4).
Much emphasis has been placed on the special case $D = -K_X$, i.e. on anticanonical height functions. Here $a(-K_X) = 1$ and $b(-K_X) = \text{rank} \, \text{Pic} \, X$. In this case the asymptotic formula (1.1) has been verified in many special cases, but is still open in general. For example for smooth cubic surfaces the best known upper bound over $\mathbb{Q}$ is $N(U, H, B) \leq \epsilon B^{1/3+\epsilon}$ due to Heath-Brown [HB97], under the assumption that $X$ contains 3 coplanar lines and one takes $U$ to be the complement of all the lines in $X$.

One of the observations in this paper is that one can sometimes obtain better results for non-anticanonical heights. (Note that the harmonic analysis approach to counting rational points of bounded height [CLT02, STBT07] usually works for all choices of height function, rather than just anticanonical heights.) If $X$ is Fano with $\text{Pic} \, X = \mathbb{Z}$ then all heights are rational powers of anticanonical heights. So to obtain non-trivial non-anticanonical heights, one requires extra geometric structure. In this paper we take this to come from a conic bundle structure (see §2 for definitions).

1.2. Del Pezzo surfaces and conic bundle surfaces. There are some “easy” non-anticanonical heights which one can work with.

Example 1.1. Let $X$ be a smooth cubic surface given as a blow-up $\pi : X \to \mathbb{P}^2_k$ in 6 rational points in general position. Let $U \subset X$ be the complement of the lines in $X$ and let $H$ be a height function associated to the divisor $\pi^*(-K_{\mathbb{P}^2})$. Then, due to the functoriality of heights, we have

$$N(U, H, B) = \# \{ x \in \pi(U) : H_{-K_{\mathbb{P}^2}}(x) \leq B \},$$

which one can of course asymptotically estimate using Schanuel’s theorem. However, the divisor $\pi^*(-K_{\mathbb{P}^2})$ is not ample, which is reflected in the fact that one is really just counting rational points on $\mathbb{P}^2_k$ in this case.

Our first result concerns del Pezzo surfaces with a conic bundle structure. Here we are able to deal with ample line bundles, so that the counting problem does not come from a simpler variety.

Theorem 1.2. Let $X$ be a del Pezzo surface of degree $d$ over a number field $k$ with a conic bundle structure $\pi : X \to \mathbb{P}^1$. Let $U \subset X$ be the complement of the singular fibres of $\pi$ and assume that $U(k) \neq \emptyset$. Let $\alpha > 1$ if $d \geq 3$ and $\alpha > 2$ if $d = 2, 1$. Let $H$ be a choice of height function associated to the $\mathbb{Q}$-divisor $-K_X + \alpha F$, where $F$ is the class of a fibre of $\pi$. Then

$$N(U, H, B) \sim c_{U,H}B, \quad \text{as} \quad B \to \infty,$$

for some $c_{U,H} > 0$.

In Theorem 1.2 and throughout the rest of this paper, we take $\alpha$ to be a rational number. For $\alpha \geq 0$, the $\mathbb{Q}$-divisors $-K_X + \alpha F$, being the sum of an ample divisor and a semi-ample divisor, are ample. Theorem 1.2 agrees with the Batyrev–Manin conjecture (see §4) and applies, for example, to cubic surfaces with a line. It proves, for the first time, a case of the Batyrev–Manin conjecture for smooth cubic surfaces.
with respect to a height function associated to some ample line bundle. (Facts about del Pezzo surfaces with a conic bundle structure can be found in [FLS16 §5].)

We also obtain results which apply to more general conic bundles. Note that for a conic bundle surface \( \pi : X \to \mathbb{P}^1 \), in general the anticanonical divisor \(-K_X\) won’t be big. However, if \( F \) is a fibre of \( \pi \), then the \( \mathbb{Q} \)-divisors \(-K_X + \alpha F\) will be big for sufficiently large \( \alpha \), and these provide us with a natural class of height functions satisfying the Northcott property on some open subset. Our result is as follows.

**Theorem 1.3.** Let \( \pi : X \to \mathbb{P}^1 \) be a conic bundle surface with anticanonical divisor \(-K_X\) and fibre \( F \). Let \( \alpha > (8 - K_X^2)/3 \) and let \( H \) be a choice of height function associated to the \( \mathbb{Q} \)-divisor \(-K_X + \alpha F\). There exists a proper closed subset \( E \subset X \) such that for all open dense subsets \( U \subset X \setminus E \) with \( U(k) \neq \emptyset \) we have

\[
N(U, H, B) \sim c_{U,H} B, \quad \text{as } B \to \infty,
\]

for some \( c_{U,H} > 0 \).

One needs to avoid a subset \( E \subset X \) in Theorem 1.3 as there can accumulating subvarieties in general (the Northcott property may even fail on some curves for our height function). One can make very explicit which curves need to be removed; see Theorem 3.1 and Example 3.2.

The leading constant \( c_{U,H} \) in Theorem 1.3 is the sum of the Peyre constants of the smooth fibres of \( \pi \) (see Theorem 3.11 for an explicit equation). We explain in §4 how this is agrees with the conjectural constant proposed by Batyrev and Tschinkel in [BT98]. However, we show that Conjecture 3.5.1 from loc. cit., concerning the distribution of the Tamagawa measures in the family, is in fact false in our case (the first counter-examples to this were found by Derenthal and Gagliardi [DG16]).

### 1.3. Higher dimensional conic bundles.

Our results also apply to some other higher dimensional conic bundles over a number field \( k \). Our most general result in higher dimensions is Theorem 3.1; for simplicity we state some special cases here. For example, we can handle conic bundles in biprojective spaces.

**Theorem 1.4.** Let \( X \subset \mathbb{P}^n \times \mathbb{P}^2 \) be a smooth biprojective hypersurface over \( k \) of bidegree \((e,2)\) for some \( e \) and let \( \pi : X \to \mathbb{P}^n \) be the natural projection. Let \( \mathcal{O}_X(F) = \mathcal{O}_X(1,0) \) and let \( H \) a choice of height associated to \(-K_X + \alpha F\) for \( \alpha > e \). Let \( U \subset X \) be an open dense subset which does not meet any singular fibre of \( \pi \) and satisfies \( U(k) \neq \emptyset \). Then

\[
N(U, H, B) \sim c_{U,H} B, \quad \text{as } B \to \infty,
\]

for some \( c_{U,H} > 0 \).

Theorem 1.4 applies to the family of diagonal plane conics

\[
y_0x_0^2 + y_1x_1^2 + y_2x_2^2 = 0 \subset \mathbb{P}^2 \times \mathbb{P}^2,
\]

and to the family of all plane conics

\[
y_0x_0^2 + y_01x_0x_1 + y_02x_0x_2 + y_11x_1^2 + y_12x_1x_2 + y_22x_2^2 = 0 \subset \mathbb{P}^5 \times \mathbb{P}^2.
\]
Le Boudec [LB15] has proved upper and lower bounds of the correct order of magnitude for the anticanonical height function for (1.2), but an asymptotic formula for the anticanonical height is still unknown in this case.

We can also handle conic bundles arising from cubic hypersurfaces.

**Theorem 1.5.** Let $X \subset \mathbb{P}^{n+2}$ be a smooth cubic hypersurface of dimension $n + 1$ over $k$ with a line $L \subset X$. Let $\tilde{X}$ be the blow-up of $L$ and $\pi: \tilde{X} \to \mathbb{P}^n$ the associated conic bundle morphism. Let $F$ be the pull-back of the hyperplane class on $\mathbb{P}^n$, $\alpha > 2$ and $H$ a choice of height associated to $-K_{\tilde{X}} + \alpha F$. Let $U \subset \tilde{X}$ be an open dense subset with $U(k) \neq \emptyset$ which does not meet any singular fibre of $\pi$. Then

$$N(U, H, B) \sim c_{U, H} B, \quad \text{as } B \to \infty,$$

for some $c_{U, H} > 0$.

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## 2. Conic bundles and projective bundles

### 2.1. Conic bundles

**Definition 2.1.** A conic bundle over a field $k$ is a morphism $\pi: X \to Y$ of smooth projective varieties over $k$ whose fibres are isomorphic to plane conics.

The anticanonical bundle $\omega_X^{-1}$ induces the anticanonical bundle on each smooth fibre of $\pi$. The pushforward $\mathcal{E} := \pi_* (\omega_X^{-1})$ is a vector bundle of rank 3 which induces an embedding $X \hookrightarrow \mathbb{P} (\mathcal{E})$ such that $\pi$ is compatible with the natural projection $\mathbb{P}(\mathcal{E}) \to Y$. (This follows from an application of [Ben12, Prop. 1.1.6, Lem. 1.1.8], for example.) We follow the Grothendieck convention regarding projective bundles, namely that $\mathbb{P}(\mathcal{E})$ denotes the space of 1-dimensional quotients of $\mathcal{E}$.

### 2.2. Projective bundles

We work with special choices of projective bundles in order to make the above set-up and the resulting height functions explicit. We work over a field $k$, assumed to not have characteristic 2 for simplicity. The theory presented here is just a mild generalisation to higher dimensions of [FLS16, §2].

Let $a_0, a_1, a_2 \in \mathbb{Z}$. We consider the following $\mathbb{P}^2$-bundles over $\mathbb{P}^n$:

$$\mathbb{F}_n(a_0, a_1, a_2) := \mathbb{P} (\mathcal{E}_{\mathbb{P}^n}(a_0) \oplus \mathcal{E}_{\mathbb{P}^n}(a_1) \oplus \mathcal{E}_{\mathbb{P}^n}(a_2)).$$

(2.1)

Note that permuting the $a_i$ or replacing $(a_0, a_1, a_2)$ by $(a_0 + f, a_1 + f, a_2 + f)$ gives an isomorphic $\mathbb{P}^2$-bundle. We let $\mathcal{M}$ be the class of the relative hyperplane bundle and $F$ the pull-back of the hyperplane class on $\mathbb{P}^n$ in the Picard group. The bundle $\mathbb{F}_n(a_0, a_1, a_2)$ can be constructed as an explicit quotient of the space $(\mathbb{A}^{n+1}\setminus 0) \times (\mathbb{A}^3\setminus 0)$ by the following action of $G^2_m$:

$$(\lambda, \mu) \cdot (y_0, \ldots, y_n; x_0, x_1, x_2) = (\lambda y_0, \ldots, \lambda y_n; \lambda^{-a_0} \mu x_0, \lambda^{-a_1} \mu x_1, \lambda^{-a_2} \mu x_2).$$

(2.2)

We therefore obtain well-defined coordinates $(y_0 : \cdots : y_n; x_0 : x_1 : x_2) = (y; x)$ on $\mathbb{F}_n(a_0, a_1, a_2)$ which are bihomogenous with respect to the action (2.2).
A hypersurface of bidegree \((2, e)\) in \(\mathbb{F}_n(a_0, a_1, a_2)\) has an equation of the shape
\[
\sum_{0 \leq i, j \leq 2} f_{i,j}(y)x_ix_j = 0.
\] (2.3)
Throughout this paper, we follow the convention that \(f_{i,j} = f_{j,i}\). The natural projection \(\pi : X \to \mathbb{P}^n\) equips \(X\) with the structure of a conic bundle. Bihomogeneity implies that the degrees of the \(f_{i,j}\) are given by the following matrix
\[
\begin{pmatrix}
2a_0 + e & a_0 + a_1 + e & a_0 + a_2 + e \\
a_0 + a_1 + e & 2a_1 + e & a_1 + a_2 + e \\
a_0 + a_2 + e & a_1 + a_2 + e & 2a_2 + e
\end{pmatrix}.
\] (2.4)

**Lemma 2.2.** Let \(X\) be a smooth hypersurface of bidegree \((2, e)\) in \(\mathbb{F}_n(a_0, a_1, a_2)\). Let \(\Delta \in k[y_0, \ldots, y_n]\) be the associated discriminant polynomial, i.e. the determinant of the matrix of the quadratic form defining (2.3). Then

1. \(-K_X = M + ((n + 1) - a_0 - a_1 - a_2 - e)F\).
2. The discriminant \(\Delta\) is a squarefree homogeneous polynomial with \(\deg \Delta = 2(a_0 + a_1 + a_2) + 3e\).

**Proof.** By [Har77, Ex. III.8.4], the canonical divisor of \(\mathbb{F}_n(a_0, a_1, a_2)\) is \((- (n + 1) + a_0 + a_1 + a_2)F - 3M\). Part (1) therefore follows from the adjunction formula.

For Part (2), the degree of \(\Delta\) is calculated by noting that it is simply the trace of the matrix (2.4). To prove that \(\Delta\) is squarefree, we may work locally around each divisor in \(\mathbb{P}^n\). Let \(R\) be the local ring at some codimension 1 point of \(\mathbb{P}^n\) and consider \(X_R \to \text{Spec } R\); this is regular as \(X\) is regular. As \(R\) is a discrete valuation ring, the vector bundle \(\mathcal{O}_{\mathbb{P}^n}(a_0) \oplus \mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \mathcal{O}_{\mathbb{P}^n}(a_2)\) trivialises over \(R\). Moreover, as \(R\) is a local ring with \(2 \in R^*\), we may diagonalise the equation of \(X_R\) over \(R\) [MH73, Cor. I.3.4] to find that
\[
X_R : \ r_0x_0^2 + r_1x_1^2 + r_2x_2^2 = 0 \subset \mathbb{P}^2_R,
\]
where \(r_i \in R\). In particular the base change \(\Delta_R \in R\) of \(\Delta\) is given by \(\Delta_R = ur_0r_1r_2\) for some \(u \in R^*\). However, as \(X_R\) is regular, a calculation shows that the valuation of \(r_0r_1r_2\) is at most 1. Applying this to each codimension 1 point proves the claim. \(\square\)

### 3. Proof of results

#### 3.1. Statement

We begin by considering higher dimensional conic bundles. The results from \([1,3]\) will be proved using the following result on conic bundles in the \(\mathbb{P}^2\)-bundles over \(\mathbb{P}^n\) from (2.1).

**Theorem 3.1.** Let \(a_0 \leq a_1 \leq a_2 \in \mathbb{Z}\) and let \(X \subset \mathbb{F}_n(a_0, a_1, a_2)\) be a smooth hypersurface of bidegree \((e, 2)\) over a number field \(k\), for some \(e \in \mathbb{Z}\). Let \(\pi : X \to \mathbb{P}^n\) be the natural projection and \(F\) the pull-back of the hyperplane class on \(\mathbb{P}^n\). Let \(U \subset X\) be an open dense subset with \(U(k) \neq 0\) which does not meet any singular fibre of \(\pi\) and which does not meet the hypersurface \(x_2 = 0\). Let \(H\) be a choice of height associated to \(-K_X + \alpha F\) for some \(\alpha > e + 2(a_0 + a_1 + a_2)/3\). Then
\[
N(U, H, B) \sim c_{U, H}B, \quad \text{as } B \to \infty,
\]
for some \( c_{U,H} > 0 \).

We choose our \( U \) as in the statement of Theorem 3.1 as the singular fibres and the hyperplane \( x_2 = 0 \) are accumulating subvarieties in general. (The hyperplane \( x_2 = 0 \) defines a degree 2 multisection of \( \pi \).) That one needs to remove the singular fibres is clear, however that one also needs to remove \( x_2 = 0 \) is illustrated by the following example (note that \( x_2 \) is special only because we stipulated that \( a_2 \geq \max\{a_1, a_0\} \).)

**Example 3.2.** Let \( a \in \mathbb{Z} \) satisfy \( 3 \mid a \) and \( a/3 > 4 \). Take \( \alpha = 2a/3 + 1 \) and \( D = -K_X + \alpha F \). Let \( f \) be a squarefree binary form of degree 2a and consider the smooth surface
\[
X : \quad x_0^2 - x_1^2 = f(s,t)x_2^2 \quad \subset \mathbb{P}_1(0,0,a),
\]
equipped with its natural conic bundle structure \( \pi : X \to \mathbb{P}^1 \). Note that \( \alpha \) satisfies the assumptions of Theorem 3.1, with \( e = 0 \).

However, let \( C \) be the curve given by \( x_0 + x_1 = x_2 = 0 \) (this is a section of \( \pi \)). Lemma 2.2 shows that \( D = -K_X + \alpha F = M + (3-a/3)F \). However \( C \cdot M = 0 \), hence \( C \cdot D = 3 - a/3 < 0 \). Thus \( C \) contains infinitely many rational points of height less than any given \( B > 0 \) with respect to \( D \) (the failure of the Northcott property on \( C \) can also be verified using the explicit description of the height function given in Lemma 3.3). Thus \( C \) must be removed for the conclusion of Theorem 3.1 to hold.

**Remark 3.3.** For conic bundles inside the special \( \mathbb{P}^2 \)-bundles \( \mathbb{P}(\mathcal{E}) \) with \( \mathcal{E} \) a direct sum of three line bundles, one can make the height functions and equations of the conic bundle explicit (see \( \S 3.3 \) and \( \S 3.4 \)). From a highbrow perspective, this is because \( \mathbb{P}(\mathcal{E}) \) is toric in this case, as reflected in its description (2.2) as a quotient of an open in an affine space.

To generalise our method to general rank 3 vector bundles \( \mathcal{E} \) on \( \mathbb{P}^n \), one requires an explicit description of the Cox ring of \( \mathbb{P}(\mathcal{E}) \). However, it does not seem even to be known whether this ring is always finitely generated when rank \( \mathcal{E} = 3 \) (i.e. whether \( \mathbb{P}(\mathcal{E}) \) is a Mori dream space). Special cases where finite generation is known include the tangent bundle of \( \mathbb{P}^n \) [HS10, Thm. 5.9]. It would be interesting to study conic bundles inside other projective bundles. (Note that subtleties only arise when \( n > 1 \), as every vector bundle on \( \mathbb{P}^1 \) is a direct sum of lines bundles [Har77, Ex. V.2.6]).

3.2. **Proof strategy of Theorem 3.1.** Each conic \( Q \) in the family has \( c_Q B + o_Q(B) \) points of height at most \( B \) for some \( c_Q \geq 0 \). We will show that the sum over these contributions is convergent via the dominated convergence theorem. To achieve this we require a uniform upper bound for the number of rational points on each conic; this is provided by the following result due to Browning and Swarbrick Jones.

We fix, once and for all, a set \( \mathcal{C} \) of integral representatives of the ideal classes of \( k \). All implied constants in the paper are allowed to depend on \( \mathcal{C} \). We let \( \tau_k(\cdot) \) be the divisor function on ideals of \( \mathcal{O}_k \) and \( \Omega_\infty \) be the set of archimedean places of \( k \).

**Lemma 3.4 ([BSJ14, Theorem 2.3]).** Let \( A \in M_3(\mathcal{O}_k) \) be a symmetric matrix which is invertible over \( k \) and let \( Q \) be the associated ternary quadratic form over \( \mathcal{O}_k \). Let
$B_1, B_2, B_3 \in [0, \infty)^{\Omega_x}$, with $B_i = (B_{i,v})_{v \in \Omega_x}$. Let $N(A, B_1, B_2, B_3)$ be the number of all $x \in \mathbb{P}^2(k)$ that have a representative $x = (x_0, x_1, x_2) \in \mathcal{O}_k^3$ with $Q(x) = 0$, $x_0 \mathcal{O}_k + x_1 \mathcal{O}_k + x_2 \mathcal{O}_k \in \mathcal{C}$, and $|x_i|_v \leq B_{i,v}$ for all $1 \leq i \leq 3$ and all $v \in \Omega_x$.

Write $\Delta(A) := \det A$ and let $\Delta_0(A)$ be the ideal of $\mathcal{O}_k$ generated by the $(2 \times 2)$-minors of $A$. Then

$$N(A, B_1, B_2, B_3) \leq \tau_k(\Delta(A)) \left( \frac{\mathfrak{n}(\Delta_0(A))^{3/2} \prod_{v \in \Omega_x} (B_{1,v}B_{2,v}B_{3,v})^{m_v}}{|N_{k/q}(\Delta(A))|} \right)^{1/3} + 1 \right).$$

Here $\mathfrak{n}$ denotes the ideal norm and $N_{k/q} : k \to \mathbb{Q}$ the field norm. Observe that the implied constant in the above result does not depend on the matrix $A$.

### 3.3. Heights

To implement our strategy, we need to make the height functions explicit. Denote by $(y, x)$ the coordinates in $\mathbb{A}^{n+1} \times \mathbb{A}^3$ and by $(y; x) = (y_0 : \cdots : y_n; x_0 : x_1 : x_2)$ the corresponding coordinates on $\mathbb{P}_n(a_0, a_1, a_2)$. Note that we prove Theorem 3.1 for completely general choices of height function $H$ associated to $-K_X + \alpha F$. We do this by relating $H$ to a “standard” choice of height $H^*(y; x)$.

Let $m = [k : \mathbb{Q}]$. For a place $v$ of $k$ we let $m_v = [k_v : \mathbb{Q}_w]$ where $w$ is the unique place of $\mathbb{Q}$ below $v$. We similarly let $| \cdot |_v$ be the absolute value on $k_v$ extending the standard absolute value $| \cdot |_w$ on $\mathbb{Q}_w$.

**Lemma 3.5.** Let $A = n + 1 + \alpha - (a_0 + a_1 + a_2 + e)$ and define $H^*(y; x) = \prod_v H^*_v(y, x)^{m_v}$ with local factors

$$H^*_v(y, x) = \max_i \{ |y_i|_v \}^A \max_j \max_i \{ |y_i|_v \}^{a_j} |x_j|_v \}.$$  

Then $H^*$ is a height on $X$ associated to $-K_X + \alpha F$. In particular, there exist $c_1, c_2 > 0$ such that for all $(y; x) \in X(k)$ we have

$$c_1 H^*(y; x) \leq H(y; x) \leq c_2 H^*(y; x).$$

**Proof.** The construction of the height follows from Weil’s height machine as follows. Let $M$ be the relative hyperplane class. The line bundle $\mathcal{O}_X(M)$ is generated by the global sections $y_i^a x_j$. In particular, a choice of height function is given by

$$H_M(y, x) = \prod_v \max_j \max_i \{ |y_i|_v \}^{a_j} |x_j|_v \}^{m_v}.$$ 

Similarly, the bundle $\mathcal{O}_X(F)$ is generated by the global sections $y_i$. By Lemma 2.2 we know that $-K_X = M + ((n+1)-a_0-a_1-a_2-e)F$, thus $-K_X + \alpha F = M + AF$ in Pic $X$. Hence a choice of height is given by $H_M \cdot H^A_F$, which is exactly the height $H^*$. The second part of the lemma follows from standard properties of heights. \qed

For $y \in \mathbb{P}^n(k)$ we denote by $H(y)$ its usual $\mathcal{O}(1)$-height, and for $y \in \mathcal{O}_k^{n+1}$ we write $H_\infty(y) := \prod_{v|\infty} \max_i \{ |y_i|_v \}^{m_v}$. 
3.4. Fibration. As a hypersurface of bidegree \((e, 2)\) in \(\mathbb{P}(a_0, a_1, a_2)\), the conic bundle \(X\) is cut out by a bihomogeneous form

\[
Q_y(x) = Q(y, x) = \sum_{0 \leq i, j \leq 2} f_{i,j}(y)x_ix_j,
\]

with \(f_{i,j}(y) \in \mathcal{O}_k[y]\) a form in \(n + 1\) variables of degree \(\deg f_{i,j} = a_i + a_j + e\) over \(\mathcal{O}_k\). We assume that \(f_{i,j} = f_{j,i}\) for all \(i, j\). Recall that \(\pi : X \to \mathbb{P}^n\) is the conic bundle morphism \((y; x) \mapsto y\). Let \(\Delta(y) = \det(f_{i,j}(y))_{i,j}\) be the discriminant of \(\pi\), a form in \(n + 1\) variables over \(\mathcal{O}_k\). For every \(y \in \mathcal{O}_k^{n+1}\), we let \(\Delta_0(y) \subseteq \mathcal{O}_k\) be the ideal generated by all \((2 \times 2)\)-minors of the matrix \((f_{i,j}(y))_{i,j}\).

Let \(y \in k^{n+1} \setminus \{0\}\). Then the plane conic \(C_y\) defined in \(\mathbb{P}^2\) by the ternary quadratic form \(Q_y\) is isomorphic to the fibre \(X_y\) above \(y \in \mathbb{P}^n(k)\) via \(x \mapsto (y; x)\). The restriction of \(H\) to a smooth fibre \(X_y\) is an anticanonical height on \(X_y\), which pulls back to an anticanonical height \(H_y(x) = H(y; x)\) on \(C_y\).

**Lemma 3.6.** For any \(y \in \mathcal{O}_k^{n+1}\) we have \(\mathfrak{N} \Delta_0(y)^3 \ll |N_{k/Q}(\Delta(y))|^2\).

**Proof.** Denote the \((2 \times 2)\)-minors of \(M(y) := (f_{i,j}(y))_{i,j}\) by \(m_{i,j}(y)\), for \(0 \leq i, j \leq 2\). Let \(M^*(y) := ((-1)^{i+j}m_{i,j}(y))_{i,j}\) be the adjugate matrix, then \(\det M^*(y) \in \Delta_0(y)^3\). From Cramer’s rule, we get \(M(y)M^*(y) = \Delta(y)I\), where \(I\) is the \((3 \times 3)\)-identity matrix. Taking determinants, we end up with \(\det M^*(y) = \Delta(y)^2\). Thus, as ideals of \(\mathcal{O}_k\) we have \(\Delta_0(y)^3 \mid \Delta(y)^2\), and the lemma follows upon taking norms.

**Lemma 3.7.** Let \(y \in \mathcal{O}_k^{n+1}\) with \(y_0\mathcal{O}_k + \cdots + y_n\mathcal{O}_k \in \mathcal{C}\) and assume that \(C_y\) is smooth. Then

\[
N(C_y, H_y, B) \leq \tau_k(\Delta(y)) \left( \frac{B \mathfrak{N}(\Delta_0(y))^{1/2}}{H_x(y)^{|a_0+a_1+a_2|/3+1} |N_{k/Q}(\Delta(y))|^{1/3} + 1} \right).
\]

**Proof.** Write \(H^*_v(y, x) := \prod_{v \mid x} H^*_v(y, x)^{m_v}\). Every rational point \(x \in C_y(k)\) has a representative \(x = (x_0, x_1, x_2) \in \mathcal{O}_k^3\) with \(Q_y(x) = 0\), which satisfies moreover

\[
x_0\mathcal{O}_k + x_1\mathcal{O}_k + x_2\mathcal{O}_k \in \mathcal{C}, \quad \text{and} \quad (3.3)
\]

\[
H^*_v(y, x) = H^*_v(y, x)^{1/m} \quad \text{for all} \quad v \mid \infty.
\]

To obtain (3.4), we used Dirichlet’s unit theorem (cf. [Ser97 §13.4]). If \(H(y; x) \leq B\), then the representative \(x\) satisfies

\[
H^*_\infty(y, x) \ll H^*(y; x) \ll H(y; x) \ll B,
\]

where the first estimate holds due to (3.3) and the analogous assumption for \(y\), and the second estimate is due to Lemma 3.5. With (3.4), this yields

\[
|x_j|_v \leq \frac{H^*_v(y, x)}{\max_i \{|y_i|_v\}^{A+a_j}} \leq \frac{B^{1/m}}{\max_i \{|y_i|_v\}^{A+a_j}}, \quad j = 0, 1, 2.
\]

With this observation, the desired bound now follows from Lemma 3.4.

**Lemma 3.8.** Let \((y; x) \in U(k)\) satisfy \(H(y; x) \leq B\). Then \(H(y) \leq B^{1/(A+a_2)}\).
Proof. Since $x_2 \neq 0$ on $U$, this follows immediately from Lemma \[3.5\] \hfill \Box

Lemma 3.9. Let $y \in \mathcal{O}_k$ with $y_0 \mathcal{O}_k + \cdots + y_n \mathcal{O}_k \in \mathcal{C}$, such that the fibre $X_y$ is smooth. Let $\varepsilon > 0$. Then
\[
\frac{N(X_y \cap U, H, B)}{B} \leq \varepsilon \frac{1}{H(y)^{(a_0+a_1+a_2)/3+A-\varepsilon}} + \frac{1}{H(y)^{A+a_2-\varepsilon}}.
\]

Proof. The estimate clearly holds when $N(X_y \cap U, H, B) \leq 0$. Thus, let us assume that $(y; x) \in U(k)$ with $H(y; x) \leq B$. Then $H(y) = H_x(y)$ and $B \geq H(y)^{A+a_2}$ by Lemma \[3.8\] The lemma is now an immediate consequence of the bounds in Lemma \[3.6\] and Lemma \[3.7\] the isomorphism $C_y \cong X_y$, and the fact that, for all $\varepsilon > 0$,
\[
\tau_k(\Delta(y)) \leq \varepsilon |N_{k/Q}(\Delta(y))|^{\varepsilon} = \prod_{v \mid \infty} |\Delta(y)|_{v}^{\varepsilon v} \leq \varepsilon H_{k}(y)^{\varepsilon \deg \Delta}. \hfill \Box
\]

Lemma 3.10. There exists $\varepsilon > 0$ such that the infinite series
\[
\sum_{y \in \mathbb{P}^n(k)} \frac{1}{H(y)^{A+(a_0+a_1+a_2)/3-\varepsilon}} \quad \text{and} \quad \sum_{y \in \mathbb{P}^n(k)} \frac{1}{H(y)^{A+a_2-\varepsilon}}
\]
are convergent.

Proof. Recalling the definition of $A$ given in Lemma \[3.5\] and our assumptions on $\alpha$ in Theorem \[3.1\] we have
\[
A + a_2 \geq A + (a_0 + a_1 + a_2)/3 = n + 1 + \alpha - e - 2(a_0 + a_1 + a_2)/3 > n + 1.
\]
Choosing $\varepsilon > 0$ sufficiently small, we therefore arrange the exponents of $H(y)$ to be strictly larger than $n + 1$. Hence the result follows from Schanuel’s theorem \[Sch79\] and partial summation. \hfill \Box

Theorem 3.11. Under the same assumptions of Theorem \[3.1\] the following hold.

(1) For $y \in \mathbb{P}^n(k)$ with $X_y$ smooth, we have
\[
N(X_y \cap U, H, B) = c_y B(1 + o_y(1)), \text{ as } B \to \infty,
\]
where $c_y \geq 0$ is Peyre’s constant for $X_y$ (or $c_y = 0$ if $X_y \cap U = \emptyset$).

(2) The sum
\[
c_{U,H} := \sum_{y \in \mathbb{P}^n(k)} c_y
\]
is convergent.

(3) We have
\[
N(U, H, B) = c_{U,H} B(1 + o(1)), \text{ as } B \to \infty.
\]
Proof. If \( X_y \cap U \) is not empty, then it differs from \( X_y \) in only finitely many points. Since the restriction of \( H \) to \( X_y \) is an anticanonical height, the asymptotical formula in (1) is just Manin’s conjecture for \( \mathbb{P}^1 \), proved in [Pey95, Cor. 6.2.18].

For (2) and (3), we choose representatives \( y \) for the points \( y \in \mathbb{P}^n(k) \) as in Lemma 3.9. Then

\[
\sum_{y \in \mathbb{P}^n(k), X_y \text{ smooth}} c_y = \sum_{y \in \mathbb{P}^n(k), X_y \text{ smooth}} \lim_{B \to \infty} \frac{N(X_y \cap U, H, B)}{B}.
\]

The upper bound in Lemma 3.9 is independent of \( B \), and summable by Lemma 3.10. Thus, the dominated convergence theorem yields (2) and moreover allows us to exchange sum and limit, giving (3).

\[ \square \]

This completes the proof of Theorem 3.1.

3.5. Conic bundle surfaces. In the case of conic bundle surfaces we can obtain a slightly stronger result than Theorem 3.1.

**Theorem 3.12.** Let \( a_0 < a_1 \leq a_2 \in \mathbb{Z} \) and let \( X \subset \mathbb{P}_1(a_0, a_1, a_2) \) be a smooth hypersurface of bidegree \( (e, 2) \) over a number field \( k \), for some \( e \in \mathbb{Z} \). Let \( \pi : X \to \mathbb{P}^1 \) be the natural projection and \( F \) the pull-back of the hyperplane class on \( \mathbb{P}^1 \). Let \( U \subset X \) be an open dense subset with \( U \cap k \neq \emptyset \) which does not meet any singular fibre of \( \pi \) and which does not meet the hypersurface \( x_2 = 0 \). Let \( H \) be a choice of height associated to \( -K_X + \alpha F \) for some \( \alpha > a_0 + a_1 + e \). Then

\[
N(U, H, B) \sim c_{U,H} B, \quad \text{as } B \to \infty,
\]

for some \( c_{U,H} > 0 \).

3.5.1. Proof of Theorem 3.12. The proof is a minor variant of the proof of Theorem 3.1. We achieve this by performing a more careful analysis of the factors \( \Delta_0(y) \) and \( \Delta(y) \). We keep the notation from the proof of Theorem 3.1. The following is well known, and follows from a minor variant of the proof of [Bro01, Lemma 7].

**Lemma 3.13.** Let \( (y_0, y_1) \in \mathcal{O}_k^2 \) with \( y_0 \mathcal{O}_k + y_1 \mathcal{O}_k \in \mathcal{C} \). Then \( \mathfrak{m} \Delta_0(y_0, y_1) \leq 1 \).

Note that Lemma 3.13 is specific to the case \( n = 1 \); the bound \( \mathfrak{m} \Delta_0(y_0, \ldots, y_n) \leq 1 \) need not hold in general if \( n > 1 \).

**Lemma 3.14.** Let \( y_0, y_1 \in \mathcal{O}_k \) with \( y_0 \mathcal{O}_k + y_1 \mathcal{O}_k \in \mathcal{C} \), such that the fibre \( X_y \) is smooth. Let \( \varepsilon > 0 \). Then

\[
\frac{N(X_y \cap U, H, B)}{B} \leq \varepsilon \left( \frac{1}{H(y)^{(a_0 + a_1 + a_2)/3 + A - \varepsilon}} \left| N_{k/\mathbb{Q}}(\Delta(y)) \right|^{1/3} + \frac{1}{H(y)^{A + a_2 - \varepsilon}} \right).
\]

Proof. In light of Lemma 3.13 the proof is a minor modification of the proof of Lemma 3.9. \( \square \)
Lemma 3.15. For each $y \in \mathbb{P}^1(k)$, we choose a fixed representative $y = (y_0, y_1) \in \mathcal{O}_k^2$ with $y_0 \mathcal{O}_k + y_1 \mathcal{O}_k \in \mathcal{C}$. Then, for small enough $\varepsilon$, the infinite series

$$\sum_{y \in \mathbb{P}^1(k)} \frac{1}{H(y)^{(a_0 + a_1 + a_2)/3} + A - \varepsilon} |N_{K/Q}(\Delta(y))|^{1/3}$$

and

$$\sum_{y \in \mathbb{P}^1(k) \big| H(y) \leq B \bigg| \Delta(y) \neq 0} \frac{1}{H(y)^{A + a_2 - \varepsilon}}$$

converge.

Proof. The second series converges as $A + a_2 = 2 + \alpha - (a_0 + a_1 + \varepsilon) > 2$, by the choice of $\alpha$.

As for the first series, the summand is invariant under multiplication of $(y_0, y_1)$ by elements of $\mathcal{O}_k^\times$. By a standard argument using Dirichlet’s unit theorem (see e.g. [Ser97, §13.4]), we may thus assume that the fixed representative $(y_0, y_1)$ for each point in $\mathbb{P}^1(k)$ satisfies

$$\max\{|y_0|_v, |y_1|_v\}^{m_v} = H_\infty(y)^{1/|\Omega_x|} = H(y)^{1/|\Omega_x|} \text{ for all } v \mid \infty.$$

Then it follows from [BSJ14, Theorem 2.4] and a dyadic splitting of the sum that

$$\sum_{y \in \mathbb{P}^1(k) \big| H(y) \leq B \bigg| \Delta(y) \neq 0} \frac{1}{|N_{K/Q}(\Delta(y))|^{1/3}} \ll \varepsilon \; B^{2 + \deg \Delta + \varepsilon}.$$

The lemma follows from this by partial summation and Lemma [3.2] using the observation that

$$\frac{(a_0 + a_1 + a_2)}{3} + A + \frac{\deg \Delta}{3} = A + a_0 + a_1 + a_2 + e > 2,$$

since $A + a_2 > 2$ and $a_0 + a_1 + e = \deg f_{0,1} \geq 0$ (cf. (2.4)). \(\square\)

Thus, replacing Lemma 3.9 and Lemma 3.10 in the proof of Theorem 3.11 by Lemma 3.12 and Lemma 3.15 respectively, we see that the conclusions of Theorem 3.11 remain valid in case $n = 1$ under the weaker assumptions of Theorem 3.12. This concludes our proof of Theorem 3.12. \(\square\)

3.6. Proof of Theorem 1.3. Let $X$ be as in Theorem 1.3. As every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles [Har77, Ex. V.2.6], we may choose equations for $X$ inside some $\mathbb{P}^1(a_0, a_1, a_2)$ with $0 \leq a_0 \leq a_1 \leq a_2$ as a smooth hypersurface of bidegree $(e, 2)$, for some $e \geq 0$. We are thus in the setting of Theorem 3.12. On noting that $8 - K_X^2 = 2(a_0 + a_1 + a_2) + 3e$ [FLS16, Prop. 2.5], we therefore have

$$\frac{8 - K_X^2}{3} = \frac{2(a_0 + a_1 + a_2)}{3} + e \geq a_0 + a_1 + e.$$

Thus $\alpha > (8 - K_X^2)/3$ implies that $\alpha > a_0 + a_1 + e$, so applying Theorem 3.12 with $E$ the union of the singular fibres and the hypersurface $x_2 = 0$, gives the result. \(\square\)
3.7. Proof of Theorem 1.2. As above we embed $X$ inside $\mathbb{F}_1(a_0, a_1, a_2)$, for some $0 \leq a_0 \leq a_1 \leq a_2$, as a smooth hypersurface of bidegree $(e, 2)$. For $d \geq 6$ we may apply Theorem 1.3. For $5 \leq d \leq 3$ and $d < 3$ it is shown in [FLS16, Thm. 5.6] that the invariants may be chosen to satisfy $a_0 + a_1 + e = 1$ and $a_0 + a_1 + e = 2$, respectively. Thus we may apply Theorem 3.12 in these cases.

It remains to show that $S: x_2 = 0$ is not an accumulating subvariety. Let $C$ be an irreducible component of $S$. As $S$ is a multisection of $\pi$, we see that $C$ is also a multisection of $\pi$. It follows that $C \cap F \neq \emptyset$. Moreover, as $-K_X$ is ample, we have $C \cdot (-K_X) \geq 1$. We deduce that

$$C \cdot (-K_X + \alpha F) \geq 1 + \alpha > 2.$$  

Standard results for counting rational points on curves (see e.g. [Ser97, §9.7]) show that $C$ contains $O_{C}(B^{2/(1+\alpha)})$ rational points of bounded height, hence $C$ does not effect the main term of the asymptotic formula, as claimed. □

3.8. Proof of Theorem 1.4. This follows immediately from Theorem 3.1, on noting that the contribution from the rational points in $x_2 = 0$ is negligible. (The coordinate $x_2$ is not special in this case as $a_0 = a_1 = a_2 = 0$). □

3.9. Proof of Theorem 1.5. Let $X \subset \mathbb{P}^{n+2}$ be a smooth cubic hypersurface over $k$ with a line $L \subset X$. Let $P \to \mathbb{P}^{n+2}$ be the blow-up of $\mathbb{P}^{n+2}$ in $L$; this is isomorphic to the $\mathbb{P}^2$-bundle over $\mathbb{P}^n$ given by $\mathbb{F}_n(0, 0, 1)$ [EH16, Prop. 9.11]. Moreover, the strict transform of $X$ inside $P$ is exactly the blow-up $\tilde{X}$ of $X$ in $L$ and the induced map $\pi: \tilde{X} \to \mathbb{P}^n$ is the associated conic bundle morphism.

We claim that $\tilde{X}$ has bidegree $(1, 2)$ in $\mathbb{F}_n(0, 0, 1)$. To verify this, we may assume that $L: z_2 = \cdots = z_{n+2} = 0$, where the $z_i$ are coordinates on $\mathbb{P}^{n+2}$. The blow-up map is then given by

$$\mathbb{F}_n(0, 0, 1) \to \mathbb{P}^{n+2}, \quad (y; x) \mapsto (x_0 : x_1 : y_0 x_2 : \cdots : y_n x_2).$$

Using this, one easily sees that the strict transform of $X$ has the claimed bidegree.

Using Theorem 3.1 it suffices to show that the contribution from the hypersurface $x_2 = 0$ is negligible in this case. (Note that this is the exceptional divisor of the blow-up). Let $H^*$ be the height from Lemma 3.5. Then

$$\sharp \{(y; x) \in \tilde{X}(k) : H^*(y; x) \leq B, x_2 = 0\} \leq 2\sharp \{y \in \mathbb{P}^n(k) : H(y)^{n+1+\alpha-2} \leq B\} = o(B),$$

since $\alpha > 2$ by assumption. The result therefore follows from Theorem 3.1. □

4. Compatibility with conjectures

We now explain the compatibility of our results with the Batyrev–Manin conjecture [BM90] and Batyrev–Tschinkel’s conjecture [BT98] for the leading constant.
4.1. **Batyrev–Manin.** Let $X$ be a smooth projective rationally connected variety over a field $k$ of characteristic 0 and $D$ a big $\mathbb{Q}$-divisor on $X$. Let $\Lambda_{\text{eff}}(X)$ be the pseudo-effective cone of $X$, i.e. the closure of the cone of effective divisors on $X$. We recall that the constants $a(D)$ and $b(D)$ from (1.1) are conjecturally given by

$$a(D) = \inf\{a \in \mathbb{R} : aD + K_X \in \Lambda_{\text{eff}}(X)\}$$

and $b(D)$ is the codimension of the minimal face of $\Lambda_{\text{eff}}(X)$ which contains the adjoint divisor $a(D)D + K_X$.

**Lemma 4.1.** Let $\pi : X \to \mathbb{P}^n$ be a conic bundle over a field $k$ of characteristic 0. Let $F$ be the pull back of the hyperplane class and let $\alpha \in \mathbb{Q}_{>0}$ be such that $D = -K_X + \alpha F$ is big. Then

$$a(D) = 1, \quad b(D) = 1.$$ 

**Proof.** As $D + K_X = \alpha F \in \Lambda_{\text{eff}}(X)$, we clearly have $a(D) \leq 1$. So let $\varepsilon > 0$ and assume that the $\mathbb{R}$-divisor

$$P := (1 - \varepsilon)D + K_X$$

is pseudo-effective. Then we have

$$\alpha F = D + K_X = \varepsilon D + P.$$ 

As the sum of a big divisor and a pseudo-effective divisor is big, this implies that $\alpha F$ is big (this follows from the fact that the big cone is the interior of the pseudo-effective cone). However $\alpha F$ is clearly not big, as the map $\pi$ is not birational; contradiction.

The adjoint divisor is thus $a(D)D + K_X = \alpha F$. To calculate $b(D)$ we use [HTT15 Prop. 18]. As $X$ is rationally connected [GHS03 Cor. 1.3] we have Pic $X = \text{NS} \ X$. Thus it follows from [HTT15 Prop. 18] that

$$b(D) = \text{rank Pic } X - \text{rank Pic}_\pi X,$$ 

where Pic$_\pi X \subset$ Pic $X$ is the sublattice of $\pi$-vertical divisors, i.e. classes of divisors $E \subset X$ such that $\pi(E) \neq \mathbb{P}^n$. We claim that there is an exact sequence

$$0 \to \text{Pic}_\pi X \to \text{Pic } X \to \text{Pic } X_\eta \to 0,$$ 

where $X_\eta$ denotes the generic fibre of $\pi$. Exactness on the left is clear, whereas exactness on the right follows from simply taking the closure in $X$ of any divisor on $X_\eta$. For exactness in the middle, let $E$ be a divisor whose restriction to the generic fibre is principal, i.e. there is a rational function $f$ on $X_\eta$ such that $E|_{X_\eta} = \text{div } X_\eta f$. But $f$ is equally well a rational function on $X$, hence we have $E - \text{div } X f \in \text{Pic}_\pi X$. It follows that $[E] \in \text{Pic}_\pi X$, which shows that (4.2) is indeed exact.

As $X_\eta$ is just a conic, we have rank Pic $X_\eta = 1$. Therefore (4.1) and the exactness of (4.2) imply that $b(D) = 1$, as required. \hfill $\square$

Lemma 4.1 shows that the asymptotic formulae we obtain in this paper agree with the conjecture (1.1).
4.2. Batyrev–Tschinkel. The leading constant in Theorem 3.11 is equal to the sum of the Peyre constants of the smooth fibres of \( \pi : X \to \mathbb{P}^n \). This is in agreement with the conjectural constant proposed by Batyrev and Tschinkel in [BT98 §3.5].

Namely, we are in the situation of Case 1 of [BT98 §3.5], and, as explained there, the leading constant should be given as the sum of the leading constants of each of the smooth fibres (in the terminology of [BT98], our variety \( X \) is not "strongly \( \mathcal{L} \)-saturated" and not "\( \mathcal{L} \)-primitive", but the map \( \pi \) is an "\( \mathcal{L} \)-primitive fibration").

However, in [BT98 Conj. 3.5.1] is stated a related conjecture, which turns out to fail to hold in our case. This conjecture is fairly general; we make it explicit in the case of conic bundles considered in this paper.

**Conjecture 4.2** (Batyrev-Tschinkel). Let \( \pi : X \to \mathbb{P}^n \) be a conic bundle over a number field \( k \) and let \( F \) be the pull back of the hyperplane class. Let \( \alpha \in \mathbb{Q}_{>0} \) be such that \( -K_X + \alpha F \) is big and choose an adelic metric on \( \mathcal{O}_X(D) \). Let \( H \) be the usual \( \mathcal{O}(1) \)-height on \( \mathbb{P}^n \). Then there exists \( c_2 > c_1 > 0 \) and \( U \subset \mathbb{P}^n \) dense open such that for all \( y \in U(k) \) we have

\[
\frac{c_1}{H(y)^{n+1+\alpha}} \leq \tau(X_y) \leq \frac{c_2}{H(y)^{n+1+\alpha}}.
\]

Here \( \tau(X_y) \) denote the Tamagawa measure of the fibre \( X_y \) with respect to the adelic metric induced by \( \mathcal{O}_X(D) \).

We illustrate the failure of both inequalities in this conjecture for the hypersurface \( X \) over \( \mathbb{Q} \) defined by

\[
Q(s, t, x_0, x_1, x_2) = x_0^2 + x_1^2 - stx_2 \subset \mathbb{F}_1(0, 0, 1),
\]

with respect to the \( \mathcal{O}_X(D) \)-height \( H^\star \) from Lemma 3.5.

The method we present can be generalised without too much difficulty to the more general set up of Conjecture 4.2. Counter-examples to the upper bound in a different setting have been found by Derenthal and Gagliardi [DG16], however counter-examples to the lower bound appear here for the first time.

We consider the fibres \( X_t \) above points \((1 : t) \in \mathbb{P}^1(\mathbb{Q})\), which are isomorphic to the plane conics \( C_t \) defined by \( x_0^2 + x_1^2 = tx_2^2 \). First note that if \( t \) is prime and \( t \equiv 3 \mod 4 \) then the lower bound of the conjecture clearly fails: the corresponding conic has no rational point, so the Tamagawa measure of the fibre is 0 but the lower bound in Conjecture 4.2 is positive. As for the upper bound, we have the following.

**Lemma 4.3.** Let \( t \) be a positive squarefree integer whose prime divisors are all 1 mod 4. Then

\[
\tau(X_t) = \frac{\pi}{t^{2+\alpha}} \prod_{p|t} 2 \left( 1 - \frac{1}{p} \right) \prod_{p|t^2} \left( 1 - \frac{1}{p^2} \right).
\]

In particular, for such \( t \) we have

\[
\tau(X_t) \geq \frac{\pi (8/5)^{\omega(t)}}{t^{2+\alpha}}
\]

where \( \omega(t) \) denotes the number of prime factors of \( t \).
Proof. The conic $X_t$ has a rational point in this case, by Fermat’s theorem. The height on $X$ pulls back along the isomorphism $C_t \cong X_t$ to the height $H = \prod_v H_v$, where

\[ H_\infty(x) = t^{\alpha+1} \max\{|x_0|, |x_1|, |tx_2|\}, \quad \text{and} \quad H_p(x) = \max\{|x_0|, |x_1|, |x_2|\} \text{ for prime } p. \]

The Tamagawa number has the form

\[ \tau(X_t) = \tau(C_t) = \sigma_\infty \prod_p \sigma_p, \]

with $\sigma_\infty, \sigma_p$ the local densities. We compute $\sigma_\infty$ using [Pey95, Lem. 5.4.4]. We may restrict ourselves to the open subsets $U_\circ := \{(x_0 : x_1 : 1) \mid \pm x_1 > 0\} \subseteq C_t(\mathbb{R})$, since the complement of their union is finite. With the obvious charts $\rho_\pm : U_\pm \to (-\sqrt{t}, \sqrt{t}), (x_0 : x_1 : 1) \mapsto x_0$, we get $\sigma_\infty = \sigma_+ + \sigma_-$, where

\[ \sigma_\pm = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{dx}{H_\infty(\rho_\pm^{-1}(x)) \left| (\partial Q/\partial x_1)(1, t, \rho_\pm^{-1}(x)) \right|} = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{dx}{t^{2+\alpha}2\sqrt{t-x^2}} = \frac{\pi}{2t^{2+\alpha}}. \]

Thus, $\sigma_\infty = \pi/t^{2+\alpha}$. For the $p$-adic densities, we let

\[ N(p^n) := \sharp\{x \bmod p^n \mid x \not\equiv 0 \bmod p, \, x_0^2 + x_1^2 \equiv tx_2^2 \bmod p^n\}. \]

Then it is well known that (cf. [PT01, Cor. 3.5])

\[ \sigma_p = \lim_{n \to \infty} \frac{N(p^n)}{p^{2n}}. \]

Let $p$ be an odd prime. If $p \nmid t$ then it is well-known that $\sigma_p = (1 - p^{-2})$. If $p \mid t$, the proof of [FLS16, Prop. 3.6] shows that

\[ \sigma_p = 2(1 - p^{-1}) \geq 8/5. \]

For $p = 2$, an application of Hensel’s lemma shows that $N(2^n) = 2^{2n-6}N(8)$ for all $n \geq 3$, so $\sigma_2 = N(8)/64$. One can verify by direct calculations that $N(8) = 64$ in both possible cases $t \equiv 1, 5 \bmod 8$. Thus, $\sigma_2 = 1$. \hfill \Box

This shows the failure of the upper bound in Conjecture 4.2. However the problem just lies with the non-archimedean densities, and in our case we have

\[ \tau(X_y) \ll_\varepsilon \frac{H(y)^\varepsilon}{H(y)^{2+\alpha}} \]

for all $\varepsilon > 0$. So the upper bound in Conjecture 4.2 does not fail “too badly”. It is for this reason that the sum of Peyre constants in Theorem 3.11 is still convergent.
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