Robustness of Greedy Approval Rules

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Abstract

We study the robustness of GreedyCC, GreedyPAV, and Phragmén’s sequential rule, using the framework introduced by Bredereck et al. [6] for the case of (multiwinner) ordinal elections and adopted to the approval setting by Gawron and Faliszewski [15]. First, we show that for each of our rules and every committee size $k$, there are elections in which adding or removing a certain approval causes the winning committee to completely change (i.e., the winning committee after the operation is disjoint from the one before the operation). Second, we show that the problem of deciding how many approvals need to be added (or removed) from an election to change its outcome is $\text{NP}$-complete for each of our rules. Finally, we experimentally evaluate the robustness of our rules in the presence of random noise.

1 Introduction

We study the extent to which perturbing the input of several approval-based multiwinner voting rules affects their outcome. We focus on GreedyCC, GreedyPAV, and Phragmén rules, whose common feature is that they choose members of the winning committee in a sequential, greedy way.

In a multiwinner approval election, each voter indicates which candidates he or she finds appealing—i.e., which ones he or she approves—and a voting rule provides the winning committee (i.e., a fixed-size group of candidates). For example, the approval voting rule (AV) chooses committees of individually excellent candidates (i.e., the most approved ones), the proportional approval voting rule (PAV) ensures proportional representation of the voters, and the Chamberlin-Courant rule (CC) seeks a diverse committee. Unfortunately, while AV can be computed in polynomial time, finding the winning committees under the other two rules is intractable. Luckily, there are many workarounds for this issue. For example, instead of CC we can use its approximate variant GreedyCC, and instead of PAV we can either use GreedyPAV or the Phragmén rule (see the overview of Lackner and Skowron [17] for a discussion of these rules and their properties). While there is robustness analysis of AV, CC, and PAV [15], analogous results are missing for these greedy rules and our goal is to fill this hole.

We use the robustness framework of Bredereck et al. [6], as adopted to the case of approval elections by Gawron and Faliszewski [15]. This framework consists of the following elements:

*See https://github.com/Project-PRAGMA/Greedy-Robust-EUMAS-2022 for the source code of the experiments performed in this paper.
1. Evaluating the extent to which introducing a single small change may affect the outcome of a rule. For example, we say that a rule has $\text{ADD}$-robustness level equal to $\ell$ if adding a single approval results in, at most, replacing $\ell$ committee members. $\text{REMOVE}$-robustness level is defined analogously, but for the case of removing a single approval. Robustness levels of a rule describe its worst-case behavior under minimal perturbations of the input.

2. Establishing the complexity of the $\text{ROBUSTNESS-RADIUS}$ problem, which asks if a given number of basic operations (such as adding or removing approvals) suffices to change the election outcome. Solving this problem for various elections would measure a rule’s robustness to targeted attacks on a per-instance basis. However, since $\text{ROBUSTNESS-RADIUS}$ is $\text{NP}$-complete for many rules, neither Bredereck et al. [6] nor Gawron and Faliszewski [15] carried out such experiments and we follow them in this respect.

3. Computing for various elections how many randomly selected basic operations are needed, on average, to change their outcomes. This measures the rules’ robustness to random noise.

Gawron and Faliszewski [15] considered AV, SAV (a rule similar in spirit to AV), CC, and PAV. They have shown that AV has $\{\text{ADD}, \text{REMOVE}\}$-robustness levels equal to 1, while the other rules have them equal to the committee size (although there are some intricacies for the case of SAV). Further, they have shown that $\text{ROBUSTNESS-RADIUS}$ is in P for AV and SAV, but is $\text{NP}$-hard for CC and PAV. Given hardness of computing CC and PAV, this last result is not very surprising, but Gawron and Faliszewski have also shown fixed-parameter tractable algorithms for the respective problems. Unfortunately, Gawron and Faliszewski [15] did not pursue experimental studies (as some of their rules are $\text{NP}$-hard, even computing robustness to random noise would require nontrivial computing resources).

Our Contribution. We complement the work of Gawron and Faliszewski [15] by considering GreedyCC, GreedyPAV, and Phragmén. We show that their $\{\text{ADD}, \text{REMOVE}\}$-robustness levels are equal to the committee size and we show that the $\text{ROBUSTNESS-RADIUS}$ problem is $\text{NP}$-complete for each of them. Since our rules are polynomial-time to compute, this result is not as immediate as in the case of CC or PAV. Finally, we experimentally evaluate the robustness of our rules, and of AV, to random noise.

Related Work. In addition to the works of Bredereck et al. [6] and Gawron and Faliszewski [15], our results are closely related to the line of work on the complexity of bribery in elections. In a bribery problem, we are given an election and we ask if a certain outcome—such as including a certain candidate among the winners (in the constructive setting) or precluding a certain candidate from winning (in the destructive setting)—can be achieved by modifying the preferences of the voters with operations of a certain cost. The study of bribery was initiated by Faliszewski, Hemaspaandra, and Hemaspaandra [11] and was continued by many others (see the overview of Faliszewski and Rothe [12]). The $\text{ROBUSTNESS-RADIUS}$ problem can be seen as a variant of destructive bribery. $\text{SWAP-BRIbery}$, introduced by Elkind, Faliszewski, and Slinko [10], was used to study the robustness of single-winner voting rules by Shiryaev et al. [21], Baumeister and Hogreve [3], and Boehmer et al. [5]. Magrino et al. [18], Cary [8], and Xia [25] used variants of destructive bribery to study robustness.

Whenever we speak of “robustness levels” without indicating whether we mean the $\text{ADD}$ or $\text{REMOVE}$ variant, we collectively refer to both.
margin of victory under various single-winner voting rules. The main difference between the studies of robustness and margin of victory is that in the former, the authors typically use fine-grained bribery variants that allow for making small modifications in the votes (in our case, these mean adding or removing single approvals), whereas in the latter the authors typically use coarse-grained bribery variants that allow operations that change the whole votes arbitrarily.

Our work is closely related to that of Faliszewski, Skowron and Talmon [14], who study bribery of approval-based multiwinner rules under adding, removing, and swapping approvals. The main difference between our work and theirs is that they focus on the constructive setting and we study the destructive one.

2 Preliminaries

We write \( \mathbb{N}_+ \) to denote the set \{1, 2, \ldots \} and for each integer \( t \), by \([t]\) we mean the set \{1, \ldots, t\}. We use the Iverson bracket notation, i.e., given a logical expression \( P \), we write \([P]\) to mean 1 if \( P \) is true and to mean 0 otherwise.

2.1 Approval Elections and Multiwinner Rules

An election is a pair \( E = (C, V) \), where \( C = \{c_1, \ldots, c_m\} \) is a set of candidates and \( V = \{(v_1, \ldots, v_n)\} \) is a collection of voters. Each voter \( v_i \) has a set \( A(v_i) \subseteq C \) of candidates that he or she approves. The approval score of a candidate is the number of voters that approve him or her.

A multiwinner voting rule \( R \) is a function that given an election \( E \) and committee size \( k \) outputs a family of size-\( k \) winning committees (i.e., a family of size-\( k \) subsets of \( C \)). If a rule always outputs a unique committee, then we say that it is resolute (in practice, non-resolute rules require tie-breaking rules, but we disregard this issue). For example, the approval voting rule (the AV rule) selects committees of \( k \) candidates with the highest approval scores. AV belongs to the class of Thiele rules, which are defined as follows: Consider an election \( E = (C, V) \) and a nonincreasing function \( \lambda: \mathbb{N}_+ \to [0, 1] \), such that \( \lambda(1) = 1 \) (we will refer to functions satisfying these conditions as OWA functions\(^2\)). We define the \( \lambda \)-score of a set \( S \subseteq C \) of candidates as:

\[
\lambda\text{-score}_E(S) = \sum_{v \in V} \left( \sum_{t=1}^{\lfloor S \cap A(v) \rfloor} \lambda(t) \right).
\]

Given an election \( E \) and committee size \( k \), the \( \lambda \)-Thiele rule outputs those size-\( k \) committees \( W \) that have the highest \( \lambda \)-score. For example, the AV rule uses the constant function \( \lambda_{AV}(i) = 1 \). This rule is meant to choose committees of individually excellent candidates, hence it considers the candidates with the highest individual approval scores. We are also interested in the Chamberlin–Courant rule (the CC rule) and the proportional approval voting rule (the PAV rule), which use functions \( \lambda_{CC}(i) = [i = 1] \) and \( \lambda_{PAV}(i) = 1/i \), respectively. Under CC, a voter assigns score 1 to a committee exactly if he or she approves at least one of its members, and assigns score 0 otherwise. This rule was introduced by Chamberlin and Courant [9] in the context of ordinal elections and was adapted to the approval setting by Procaccia et al. [19] and Betzler et al. [4]. Its purpose is to find diverse committees, so that as many voters as possible feel represented by at least one of the committee members. The PAV rule was introduced by Thiele [24] and its more elaborate scoring system is designed to ensure proportional representation of the voters [1] [7].

\(^2\)The name refers to the class of order-weighted operators (OWA operators), introduced by Yager [26] and used by Skowron et al. [22] to define a class of rules closely related to the Thiele ones.
Both CC and PAV are NP-hard to compute \cite{19, 22, 2} and we are mostly interested in the rules defined by their greedy approximation algorithms. These algorithms run as follows (let $E = (C, V)$ be the input election, $k$ be the committee size, and $\lambda$ be the OWA function used):

We start with an empty committee $W = \emptyset$ and perform $k$ iterations, where in each iteration we extend $W$ with a single candidate $c$ that maximizes the value $\lambda$-score$_E(W \cup \{c\}) - \lambda$-score$_E(W)$. If several candidates satisfy this condition then we break the tie according to a given tie-breaking order. We output $W$ as the unique winning committee.

We refer to the incarnations of this algorithm for $\lambda_{\text{CC}}$ and $\lambda_{\text{PAV}}$ as GreedyCC and GreedyPAV, respectively. When analyzing an $i$-th iteration of these algorithms, for each candidate $c$ we refer to the value $\lambda$-score$_E(W \cup \{c\}) - \lambda$-score$_E(W)$ as the score of $c$. For GreedyCC, we imagine that as soon as a candidate is included in the committee, all the voters that approve him or her are removed (indeed, these voters would not contribute positive score to any further candidates).

We are also interested in the Phragmén rule (or, more specifically, in the Phragmén’s sequential rule, but we use the shorter name in this paper). The Phragmén rule proceeds according to the following scheme ($E = (C, V)$ is the input election and $k$ is the committee size):

Initially, we have committee $W = \emptyset$. The voters start with no money, but they receive it at a constant rate (so, at each time point $t \in \mathbb{R}$, $t \geq 0$, each voter has in total received money of value $t$). At every time point for which there is a candidate $c$ not included in $W$ who is approved by voters that jointly have one unit of money, this candidate is “purchased.” That is, candidate $c$ is added to $W$ and the voters that approve him or her have all their money reset to 0 (i.e., they pay for $c$). If several candidates can be purchased at the same time, we consider them in a given tie-breaking order. The process continues until $W$ reaches size $k$ or all the remaining candidates have approval score zero (in which case we extend $W$ according to the tie-breaking order). We output $W$ as the unique winning committee.

Similarly to PAV, Phragmén provides committees that ensure proportional representation of the voters \cite{20}. For a detailed discussion of these rules (including an alternative definition of Phragmén), we point the reader to the survey of Lackner and Skowron \cite{17}. Faliszewski et al. \cite{13} offer a general overview of multiwinner voting. Note that GreedyCC, GreedyPAV, and Phragmén are resolute.

### 2.2 Robustness of Multiwinner Voting Rules

We use the robustness framework introduced by Bredereck et al. \cite{6} for the ordinal setting and adapted to the approval one by Gawron and Faliszewski \cite{15}. In particular, we consider the ADD and REMOVE operations, where the former means adding a single approval to some vote and the latter means removing a single approval from a vote. Let us fix committee size $k$. For an operation $\text{OP} \in \{\text{ADD}, \text{REMOVE}\}$, we say that a multiwinner voting rule $\mathcal{R}$ is $\ell$-$\text{OP}$-robust (or, that its $\text{OP}$-robustness level is $\ell$) if $\ell$ is the smallest integer such that for every election $E = (C, V)$, where $|C| \geq 2k$\footnote{This is mostly a technical assumption, to ensure that there are enough candidates so that all members of a committee can be replaced with non-members.} and every election $E'$ obtained from $E$ with a single operation of type $\text{OP}$, the following holds:

For each committee $W \in \mathcal{R}(E, k)$ there is a committee $W' \in \mathcal{R}(E', k)$ such that $|W \cap W'| \geq k - \ell$ (i.e., for every winning committee of $E$ there is a winning committee of $E'$ that differs in at most $\ell$ candidates).
Intuitively, if a rule is 1-ADD-robust then adding a single approval in an election held according to this rule may, at most, lead to replacing a single member of the winning committee. On the other hand, if a rule is k-ADD-robust, then adding a single approval sometimes leads to replacing the whole committee. Gawron and Faliszewski [15] have shown that AV is 1-{ADD,REMOVE}-robust, whereas both CC and PAV are k-{ADD,REMOVE}-robust (they also considered the SWAP operation, which means moving an approval from one candidate to another within a vote, and obtained analogous results for it).

Following Bredereck et al. [6] and Gawron and Faliszewski [15], we also study the ROBUSTNESS-RADIUS problem. Intuitively, in this problem we are interested in the smallest number of operations required to change the election result. The more are necessary, the more robust is the result on the given election.

**Definition 1.** Let \( R \) be a multiwinner voting rule and let \( \text{Op} \) be either \( \text{ADD} \) or \( \text{REMOVE} \). In the \( R \)-\( \text{Op} \)-ROBUSTNESS-RADIUS problem we are given an election \( E \), a committee size \( k \), and a nonnegative integer \( B \) (referred to as the budget). We ask if it is possible to perform up to \( B \) operations of type \( \text{Op} \) so that for the resulting election \( E' \) it holds that \( R(E,k) \neq R(E',k) \).

### 3 Robustness Level

The results of Bredereck et al. [6] and Gawron and Faliszewski [15] give some intuitions regarding robustness levels that we may expect from multiwinner rules. On the one hand, simple, polynomial-time computable rules that focus on individual excellence tend to have robustness levels equal to 1 (this includes, e.g., AV in the approval setting, and a number of rules in the ordinal one). Indeed, Bredereck et al. [6, Theorem 6] have shown that if a rule selects a committee with the highest score, this score is easily computable, and the rule’s robustness level is bounded by a constant, then some winning committee can be computed in polynomial time. On the other hand, more involved rules that focus on proportionality or diversity—in particular those NP-hard to compute—tend to have robustness levels equal to the committee size. However, regarding rules that form the committee sequentially, so far there was only one data point: Bredereck et al. [6] have shown that single transferable vote (STV; a well-known rule for the ordinal setting) has robustness levels equal to the committee size. We provide three more such examples by showing that GreedyCC, GreedyPAV, and Phragmén also have robustness levels equal to the committee size.

First, we consider the relationship between ADD-ROBUSTNESS and REMOVE-ROBUSTNESS for resolute rules.

**Proposition 1.** Let \( R \) be a resolute multiwinner voting rule, and let \( \ell \) be a positive integer. \( R \) is \( \ell \)-ADD-robust if and only if it is \( \ell \)-REMOVE-robust.

**Proof.** Let us fix committee size \( k \) and a resolute multiwinner rule \( R \). Further, assume that \( R \) is \( \ell \)-ADD-robust for some integer \( \ell \). We will show that it also is \( \ell \)-REMOVE-robust. To see this, let us consider two elections, \( E \) and \( E' \), where \( E' \) is obtained from \( E \) by removing an approval and both elections contain at least \( 2k \) candidates. Let \( W \) be the unique winning committee in \( R(E,k) \) and \( W' \) be the unique winning committee in \( R(E',k) \). Since \( R \) is \( \ell \)-ADD-robust, we know that \( W \) and \( W' \) differ by at most \( \ell \) candidates (it suffices to apply the definition of \( \ell \)-ADD-robustness, but with the roles of \( E \) and \( E' \) reversed). Similarly, we see that there are two such elections whose winning committees differ by exactly \( \ell \) candidates. By applying analogous reasoning, we see that if \( R \) is \( \ell \)-REMOVE-robust then it is also \( \ell \)-ADD-robust. \( \square \)
Next we show that GreedyCC, GreedyPAV, and Phragmén are $k$-\{ADD, REMOVE\}-robust.

**Theorem 1.** Let $k$ be the committee size. For each multiwinner rule $R$ in \{GreedyCC, GreedyPAV, Phragmén\}, $R$ is both $k$-ADD-robust and $k$-REMOVE-robust.

**Proof.** Let us fix committee size $k$. Since our rules are resolute, by Proposition [1] it suffices to show their $k$-ADD-robustness. To this end, we will form two elections, $E = (C, V)$ and $E' = (C, V')$, where $E'$ can be obtained from $E$ by adding a single approval, such that for each of our rules the unique winning committee for $E$ is disjoint from the one for $E'$.

We let the candidate set be $C = A \cup B$, where $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, and we set the tie-breaking order to be:

$$a_1 \succ \cdots \succ a_k \succ b_1 \succ \cdots \succ b_k.$$ 

The voter collection of election $E$ is as follows:

1. We have $k - 1$ voters approving $\{a_1, b_1\}$.
2. For each $i \in \{2, \ldots, k\}$ we have a single voter approving $\{a_i, b_i\}$.
3. For each $i \in \{2, \ldots, k\}$ we have a single voter approving $\{a_i, b_i\}$.
4. For each $i \in \{2, \ldots, k\}$ we have $2k - 3$ voters approving $\{a_i, b_i\}$.
5. We have voter $v_0$ with empty approval set.

As the reader can verify, every candidate from $C$ is approved by exactly $2(k - 1)$ voters. Election $E'$ is defined in the same way, except that voter $v_0$ approves $b_1$. For each of our rules we will show that committee $A$ wins in election $E$ and committee $B$ wins in election $E'$.

We first consider GreedyCC and election $E$. At the beginning of the first iteration, each candidate has score $2(k - 1)$ and, due to the tie-breaking order, GreedyCC chooses $a_1$. As a consequence, in the second iteration the score of $b_1$ decreases by $k - 1$ points and the scores of candidates $b_2, \ldots, b_k$ decrease by 1 point each. As there is no voter who approves two different candidates from $A$, each of the candidates in $\{a_2, \ldots, a_k\}$ still has $2(k - 1)$ points. Hence $a_2$ is selected. The same reasoning applies to the following $k - 2$ iterations, during which the remaining members of $A$ are chosen.

On the other hand, for election $E'$ GreedyCC outputs committee $B$. To see this, note that in the first iteration $b_1$ has score higher by one point than every other candidate and, so, is selected irrespective of the tie-breaking order. Then the scores of candidates in $A$ decrease below $2(k - 1)$, but the scores of candidates in $\{b_2, \ldots, b_k\}$ remain equal to $2(k - 1)$. Hence these candidates are selected in the following iterations. Since GreedyCC outputs committee $A$ for election $E$ and committee $B$ for election $E'$, we see that GreedyCC is $k$-ADD-robust.

The case of GreedyPAV is analogous to that of GreedyCC. Indeed, the only difference between the operation of GreedyPAV and GreedyCC on elections $E$ and $E'$ is that when under GreedyCC the score of some candidate drops by $x$, the score of the same candidate drops by $x/2$ under GreedyPAV (naturally, this is not a general feature of these rules, but one that is specific to our two elections). As a consequence, both rules choose the same committees for $E$ and $E'$ and, so, GreedyPAV is $k$-ADD-robust.

Finally, we consider the Phragmén rule. Since in election $E$ each candidate is approved by $2(k - 1)$ voters, the first moment when a group of voters can purchase a candidate is $1/2(k - 1)$. Due
to the tie-breaking order, they first buy $a_1$, followed by $a_2$ and all the other members of $A$ (as no two members of $A$ are approved by the same voter, for each member of $A$ there is a group of voters with sufficient funding). Thus the rule outputs committee $A$. In election $E'$, candidate $b_1$ has $2k - 1$ approvals and is purchased at time $1/2k - 1$. As a consequence, all voters who approve $b_1$ have their budgets reset to 0. The next time when there is a group of voters that can purchase a candidate is $1/2(k-1)$. One can verify that at this point for each candidate in $\{b_2, \ldots, b_k\}$ there is a disjoint group of voters that has a unit of money, whereas there is no such group for any member of $A$. Hence, Phragmén outputs committee $B$. As in the previous two cases, this means that Phragmén is $k$-ADD-robust.

The reader may worry that the above results hold due to the fact that our rules are resolute, but this is not the case. For example, if one used parallel-universes tie-breaking (where a rule outputs all the committees that could win for some way of resolving the internal ties) then the result would still hold. For example, for GreedyCC it would suffice to add one more voter approving both $a_1$ and $b_1$ to elections $E$ and $E'$. Then, GreedyCC with parallel-universes tie-breaking would output both $A$ and $B$ as the winning committees for $E$, but for $E'$ it would output only $B$. This would show its $k$-ADD-robustness (from the point of view of committee $A$).

4 Robustness Radius: Complexity Results

In this section we show that the ROBUSTNESS-RADIUS problem is NP-complete for each of our rules, for both adding and removing approvals. We observe that for each of our rules and operation type, the respective ROBUSTNESS-RADIUS problem is clearly in NP. Indeed, it suffices to non-deterministically guess which approvals to add/remove, compute the winning committees before and after the change (since our rules are resolute, in each case there is exactly one), and verify that they are different. Hence, in our proofs we will focus on showing NP-hardness. To this end, we give reductions from the following variant of the X3C problem (it is well known that this variant of the problem remains NP-complete [16]; note that in the standard variant of X3C one does not assume that each member of $U$ belongs to exactly three sets).

Definition 2. An instance of RX3C consists of a universe set $U = \{u_1, \ldots, u_{3n}\}$ and a family $S = \{S_1, \ldots, S_{3n}\}$ of three-element subsets of $U$, such that each member of $U$ belongs to exactly three sets from $S$. We ask if there is a collection of $n$ sets from $S$ whose union is $U$ (i.e., we ask if there is an exact cover of $U$).

All our reductions follow the same general scheme: Given an instance of RX3C we form an election where the sets are the candidates and the voters encode their content. Additionally, we also have candidates $p$ and $d$. Irrespective which operations we perform (within a given budget), all the set candidates are always selected, but by performing appropriate actions we control the order in which this happens. If the order corresponds to finding an exact cover, then additionally candidate $p$ is selected. Otherwise, our rules select $d$.

We first focus on adding approvals and then argue why our proofs adapt to the case of removing approvals.

Theorem 2. GreedyCC-ADD-ROBUSTNESS-RADIUS is NP-complete.

Proof. We give a reduction from the RX3C problem. Our input consists of the universe set $U = \{u_1, \ldots, u_{3n}\}$ and family $S = \{S_1, \ldots, S_{3n}\}$ of three-element subsets of $U$. We know that each
Prior to any bribery, each candidate \( S_i \in \mathcal{S} \). We introduce two integers, \( T = 10n^5 \) and \( t = 10n^3 \) and we interpret both as large numbers, with \( T \) being significantly larger than \( t \). We form an election \( E = (C, V) \) with candidate set \( C = \{ S_1, \ldots, S_{3n} \} \cup \{ p, d \} \), and with the following voters:

1. For each \( S_i \in \mathcal{S} \), there are \( T \) voters that approve candidate \( S_i \).
2. For each two sets \( S_i \) and \( S_j \), there are \( T \) voters that approve candidates \( S_i \) and \( S_j \).
3. There are \( 2nT + 4nt \) voters that approve \( p \) and \( d \).
4. For each \( u_\ell \in U \), there are \( t \) voters that approve \( d \) and those candidates \( S_i \) that correspond to the sets containing \( u_\ell \).
5. There are \( n \) voters who do not approve any candidates.

The committee size is \( k = 3n + 1 \) and the budget is \( B = n \). We assume that the tie-breaking order among the candidates is:

\[
S_1 > S_2 > \cdots > S_{3n} > p > d.
\]

Prior to any bribery, each candidate \( S_i \) is approved by \( 3nT + 3t \) voters, \( p \) is approved by \( 2nT + 4nt \) voters, and \( d \) is approved by \( 2nT + 7nt \) voters.

Let us now consider how GreedyCC operates on this election. Prior to the first iteration, all the set candidates have the same score, much higher than that of \( p \) and \( d \). Due to the tie-breaking order, GreedyCC chooses \( S_1 \). As a consequence, all the voters that approve \( S_1 \) are removed from consideration and the scores of all other set candidates decrease by \( T \) (or by \( T + t \) or \( T + 2t \), for the sets that included the same one or two elements of \( U \) as \( S_1 \)). GreedyCC acts analogously for the first \( n \) iterations, during which it chooses a family \( \mathcal{T} \) of \( n \) set elements (we will occasionally refer to \( \mathcal{T} \) as if it really contained the sets from \( \mathcal{S} \), and not the corresponding candidates).

After the first \( n \) iterations, each of the remaining \( 2n \) set candidates either has \( 2nT \), \( 2nT + t \), \( 2nT + 2t \), or \( 2nT + 3t \) approvals (depending how many sets in \( \mathcal{T} \) have nonempty intersection with them). Let \( x \) be the number of elements from \( U \) that do not belong to any set in \( \mathcal{T} \). Candidate \( p \) is still approved by \( 2nT + 4nt \) voters, whereas \( d \) is approved by \( 2nT + 4nt + xt \) voters. Thus at this point there are two possibilities. Either \( x = 0 \) and, due to the priority order, GreedyCC selects \( p \), or \( x > 0 \) and GreedyCC selects \( d \). In either case, in the following \( 2n \) iterations it chooses the remaining \( 2n \) set candidates (because after the \( n + 1 \)-st iteration the score of that among \( p \) and \( d \) who remains drops to zero or nearly zero). If candidate \( p \) is selected without any bribery, then it means that we can find a solution for the RX3C instance using a simple greedy algorithm. In this case, instead of outputting the just-described instance of GreedyCC-Add-Robustness-RADIUS, we output a fixed one, for which the answer is yes. Otherwise, we know that without any bribery the winning committee is \( \{ S_1, \ldots, S_{3n}, d \} \). We focus on this latter case.

We claim that it is possible to ensure that the winning committee changes by adding at most \( n \) approvals if and only if there is an exact cover of \( U \) with \( n \) sets from \( \mathcal{S} \). Indeed, if such a cover exists, then it suffices to add a single approval for each of the corresponding sets in the last group of voters (those that originally do not approve anyone). Then, by the same analysis as in the preceding paragraph, we can verify that the sets forming the cover are selected in the first \( n \) iterations, followed by \( p \), followed by all the other set candidates.

For the other direction, let us assume that after adding some \( t \) approvals the winning committee has changed. One can verify that irrespective of which (up-to) \( n \) approvals we add, in the first \( n \)
iterations GreedyCC still chooses \( n \) set candidates. Thus, at this point, the score of \( p \) is at most \( 2nT + 4nt + n \) and the score of \( d \) is at least \( 2nT + 4nt + xt - n \) (where \( x \) is the number of elements from \( U \) not covered by the chosen sets; we subtract \( n \) to account for the fact that \( n \) voters that originally approved \( d \) got approvals for the candidates selected in the first \( n \) iterations). If at this point \( d \) is selected, then in the following \( 2n \) iterations the remaining set candidates are chosen and the winning committee does not change. This means that \( p \) is selected. However, this is only possible if \( x = 0 \), i.e., if the set candidates chosen in the first \( n \) iterations correspond to an exact cover of \( U \).

A very similar proof also works for the case of GreedyPAV. The main difference is that now including a candidate in a committee does not allow us to forget about all the voters that approve him or her (the proof is in the appendix).

**Theorem 3.** GreedyPAV-ADD-ROBUSTNESS-RADIUS is NP-complete.

The proof for the case of Phragmén-ADD-ROBUSTNESS-RADIUS is similar in spirit to the preceding two, but requires careful calculation of the times when particular groups of voters can purchase respective candidates.

**Theorem 4.** Phragmén-ADD-ROBUSTNESS-RADIUS is NP-complete.

**Proof.** We give a reduction from RX3C. As input, we get a universe set \( U = \{u_1, \ldots, u_{3n}\} \) and a family \( S = \{S_1, \ldots, S_{3n}\} \) of size-3 subsets of \( U \). Each element of \( U \) appears in exactly three sets from \( S \). We ask if there is a collection of \( n \) sets that form an exact cover of \( U \).

Our reduction proceeds as follows. First, we define two numbers, \( T = 900n^{12} \) and \( t = 30n^5 \). The intuition is that both numbers are very large, \( T \) is significantly larger than \( t^2 \), and \( t \) is divisible by \( 6n \) (the exact values of \( T \) and \( t \) are not crucial; we did not minimize them but, rather, used values that clearly work and simplify the reduction). We form an election \( E = (C, V) \) with candidate set \( C = \{S_1, \ldots, S_{3n}\} \cup \{p, d\} \) and the following voters:

1. For each \( S_i \in S \), there are \( T \) voters that approve candidate \( S_i \). We refer to them as the \( S \)-voters.
2. For each \( u_\ell \in U \), there are \( t^2 \) voters that approve those candidates \( S_i \) that correspond to the sets containing \( u_\ell \). We refer to them as the universe voters. For each \( u_\ell \in U \), \( \tfrac{t}{3n} \) of \( u_\ell \)'s universe voters additionally approve candidate \( d \). We refer to them as the \( d \)-universe voters.
3. There are \( T + 3t^2 - 2t \) voters that approve both \( p \) and \( d \). We refer to them as the \( p/d \)-voters.
4. There are \( \tfrac{T}{6n} \) voters that approve \( p \). We refer to them as the \( p \) voters.
5. There are \( n \) voters who do not approve any candidate, and to whom we refer as the empty voters.

The committee size is \( k = 3n + 1 \) and the budget is \( B = n \). The tie-breaking order is:

\[ S_1 > S_2 > \cdots > S_{3n} > d > p. \]

In this election, each candidate \( S_i \) is approved by exactly \( T + 3t^2 \) voters, \( d \) is approved by \( (T + 3t^2 - 2t) + t \) voters, and \( p \) is approved by \( (T + 3t^2 - 2t) + \tfrac{T}{6n} \) voters.
Let us consider how Phragmén operates on this election (we encourage the reader to consult Figure 1 while reading the following text). First, we observe that when we reach time point $D = \frac{1}{T + \frac{t^2}{3}}$ then all the not-yet-selected set candidates (for whom there still is room in the committee) are selected. Indeed, at time $D$ the $S$-voters collect enough funds to buy them. On the other hand, the earliest point of time when some voters can afford to buy a candidate is $A = \frac{1}{T + \frac{t^2}{3}} + 3t$. Specifically, at time $A$ set voters and universe voters jointly purchase up to $n$ set candidates (selected sequentially, using the tie-breaking order and taking into account that when some candidate is purchased then all his or her voters spend all their so-far collected money). Let us consider some candidate $S_i$ that was not selected at time point $A$. Since $S_i$ was not chosen at $A$, at least $t^2$ of the $3t^2$ universe voters that approve $S_i$ paid for another candidate at time $A$. Thus the earliest time when voters approving $S_i$ might have enough money to purchase him or her is $C$, such that:

$$C = \frac{1}{T + \frac{t^2}{3}} + \frac{(C - A)^2}{(T + \frac{t^2}{3})} = 1.$$  

Simple calculations show that $C = \frac{1 + At^2}{T + 3t^2}$. Noting that $A = \frac{1}{T + \frac{t^2}{3}}$, we have that $C = A + A^2t^2$. However, prior to reaching time point $C$, either candidate $p$ or candidate $d$ is selected. Indeed, at time point $B_{pd} = \frac{1}{T + \frac{t^2}{3} - 2t}$ the $p/d$ voters alone would have enough money to buy one of their candidates: We show that $B_{pd} < C$, or, equivalently, that $\frac{1}{B_{pd}} > \frac{1}{C}$. It holds that $\frac{1}{B_{pd}} = T + 3t^2 - 2t$ and:

$$\frac{1}{C} = \frac{1}{A + At^2} = \frac{1}{A} \cdot \frac{1}{1 + At^2} = \frac{T + 3t^2}{1 + \frac{t^2}{T + 3t^2}} = \frac{(T + 3t^2)^2}{T + 4t^2}.$$  

By simple transformations, $\frac{1}{B_{pd}} > \frac{1}{C}$ is equivalent to:

$$(T + 3t^2 - 2t)(T + 4t^2) > (T + 3t^2)^2.$$  

The left-hand side of this inequality can be expressed as:

$$((T + 3t^2 - 2t)(T + 3t^2) + t^2) = ((T + 3t^2)^2 + (t^2 - 2t)(T + 3t^2) - 2t^3,$$

positive because $t^2 - 2t > 2t$ due to our assumptions and, hence, our inequality holds. All in all, we have $A < B_{pd} < C < D$.

It remains to consider which among $p$ and $d$ is selected. If $p$ were to be selected, then it would happen at time point $B_p = \frac{1}{T^2 + 3t^2 - 2t + \frac{1}{7t}}$. This is when the $p/d$- and $p$ voters would collect enough
money to purchase $p$ (assuming the former would not spend it on $d$ earlier). Now, if at time $A$ fewer than $n$ set candidates were selected, then at least $\frac{t}{3n}$ of the $d$-universe voters would retain their money and, hence, $d$ would be selected no later than at time point $B_d = \frac{1}{T^2 + 3t^2 - 2t + \frac{t}{6n}} < B_p$. On the other hand, if at time point $A$ exactly $n$ set candidates were selected (who, thus, would have to correspond to an exact cover of $U$) then all the $d$-universe voters would lose their money and voters who approve $d$ would not have enough money to buy him or her before time $B_p$. Indeed, in this case the money accumulated by voters approving $d$ would at time $B_p$ be:

$$X = \frac{T + 3t^2 - 2t}{T + 3t^2 - 2t + \frac{t}{6n}} + t \left( \frac{1}{T + 3t^2 - 2t + \frac{t}{6n}} - \frac{1}{T + 3t^2} \right)$$

money of the $p/d$ voters

money collected by the $d$-universe voters between time points $A$ and $B_p$

We claim that $X < 1$, which is equivalent to the following inequality (where we replace $T + 3t^2$ with $M$; note that $M = \frac{1}{A}$):

$$\frac{M - t}{M - 2t + \frac{t}{6n}} < 1 + \frac{t}{M} = \frac{M + t}{M}$$

By simple transformations, this inequality is equivalent to:

$$0 < \frac{Mt + \frac{t^2}{6n} - 2t^2}{6n}$$

which holds as $t > 6n$ and $M > 2t^2$. To conclude, if at time point $A$ there are $n$ set candidates selected for the committee, then $p$ is selected for the committee as well.

Finally, we observe that irrespective of which among $p$ and $d$ is selected for the committee, the voters that approve the other one do not collect enough money to buy him or her until time $D$. Thus the winning committee either consists of all the set candidates and $d$, or of all the set candidates and $p$, where the latter happens exactly if at time $A$ candidates corresponding to an exact cover of $U$ are selected.

If at point $A$ Phragmén would choose candidates corresponding to an exact cover of $U$ then our reduction outputs a fixed yes-instance of Phragmén-ADD-ROBUSTNESS-RADIUS (as we have just found that an exact cover exists). Otherwise we output the formed election with committee size $k = 3n + 1$ and budget $B = n$. To see why this reduction is correct, we make the following three observations:

1. By adding $n$ approvals, we cannot significantly modify any of the time points $A$, $B_d$, $B_p$, $B_{pd}$, $C$, and $D$ from the preceding analysis, except that we can ensure which (up to) $n$ sets are first considered for inclusion in the committee just before time point $A$.

2. If there is a collection of $n$ sets in $\mathcal{S}$ that form an exact cover of $U$, then—by the above observation—we can ensure that these sets are selected just before time point $A$ (by adding one approval for each of them among $n$ distinct empty voters). Hence, if there is an exact cover then—by the preceding discussions—we can ensure that the winning committee changes (to consist of all the set candidates and $p$).
3. If there is no exact cover of $U$, then no matter where we add (up to) $n$ approvals, candidate $d$ gets selected and, so, the winning committee does not change (in particular, even if we add $n$ approvals for $p$).

Since the reduction clearly runs in polynomial time, the proof is complete.

It remains to argue that REMOVE-ROBUSTNESS-RADIUS also is NP-complete for each of our rules. This, however, is easy to see. In each of the three proofs above, we had budget $B = n$ and $n$ voters with empty approval sets. We were using these $n$ voters to add a single approval for each of the $n$ sets forming an exact cover, leading to the selection of $p$ instead of $d$. For the case of removing approvals, it suffices to replace the $n$ empty voters with $3n$ ones, such that each set candidate is approved by exactly one of them, and to set the budget to $B = 2n$. Now we can achieve the same result as before by deleting approvals for those set candidates that do not form an exact cover. Hence the following holds.

**Corollary 1.** Let $R$ be one of GreedyCC, GreedyPAV and Phragmén. $R$-REMOVE-ROBUSTNESS-RADIUS is NP-complete.

## 5 Robustness to Random Noise: Experimental Results

Let us now move on to an experimental analysis of our rules’ robustness to random noise. The main idea of the experiment is as follows: First, we generate a number of elections from a given distribution and, for each of them, we compute its winning committee. Then, we perform a given number of random operations, such as adding or removing approvals (specified via a perturbation level, described below), and we compute the proportion of elections that change their outcome and the average number of committee members that get replaced. We do so for each of our rules (including AV), for several distributions, and for a range of perturbation levels. Our main observation is that the results for AV, PAV, and Phragmén are quite similar to each other, but those for CC stand out. Further, the results may quite strongly depend on the distribution of votes. Below we describe our setup and present the results in more detail.

**Generating Elections.** To generate synthetic elections, we use the resampling model recently introduced by Szufa et al. [23]. This model is parameterized by two numbers, $p, \phi \in [0, 1]$, and to generate an election with candidate set $C = \{c_1, \ldots, c_m\}$ and voter collection $V = (v_1, \ldots, v_n)$, it proceeds as follows: First, we choose a central approval set $A$ of $\lfloor p \cdot m \rfloor$ candidates from $C$ (uniformly at random from all subset of $C$ of this cardinality). Then, for each voter $v_i$ we set his or her initial approval set $A(v_i)$ to be equal to $A$. Finally, for each voter $v_i$ and each candidate $c_j$, with probability $\phi$ we remove $c_j$ from the approval set of $v_i$ (if it were there) and, then, we let $v_i$ approve $c_j$ with probability $p$. In other words, initially all voters have the central approval set, but for each candidate we resample its approval with probability $\phi$. For example, for $\phi = 0$ each voter has identical approval set, which includes $\lfloor p \cdot m \rfloor$ candidates, whereas for $\phi = 1$ each voter approves each candidate independently, with probability $p$. The closer $\phi$ is to 0, the more similar are the votes, and the closer it is to 1, the more diverse they are.

**Perturbation Levels.** Given an election $E = (C, V)$, a perturbation level $\ell \in [0, 1]$ specifies how many operations of adding or removing approvals we are supposed to perform. For the ADD
Figure 2: Probabilities of changing elections results for the resampling model with $p = 0.1$ (blue lines) and $p = 0.3$ (orange lines) for different rules (columns of the plot) and different values of $\phi$ (rows of the plot) and different perturbation levels ($x$ axis). Each data point corresponds to 200 elections with 100 candidates, 100 voters, and committee size 10. Wide light blue and light orange lines represent standard deviation.

In our election $E$, there are $X = \sum_{v \in V} |A(v)|$ approvals in total, but if each voter approved each candidate then there would be $|C| \cdot |V|$ approvals. Thus the number of not appearing approvals is $|C| \cdot |V| - X$. For the REMOVE operation, perturbation level $\ell$ means removing an $\ell$ fraction of the approvals in the election, chosen uniformly at random.

Performing the Experiment. To perform our experiment for a given multiwinner rule $\mathcal{R}$, we consider values of $p \in \{0.1, 0.3\}$, values of $\phi \in \{0.25, 0.5, 0.75, 1\}$, perturbation levels $\ell$ between
Figure 3: Average number of replaced committee members (the setup of the plot is analogous to the one in Figure 2).

0 and 0.95, with a step of 0.05 (but also including perturbation level 0.01), and operations $\text{Op} \in \{\text{Add}, \text{Remove}\}$. For each combination of these parameters we generate 200 elections with 100 candidates and 100 voters from the resampling model with parameters $p$ and $\phi$. For each of these elections we compute its $R$ winning committee of size $k = 10$, apply operations $\text{Op}$ as specified by the perturbation level, and compute the winning committee of the resulting election (of the same size). We report the fraction of elections for which the two committees differ and the average number of candidates by which they differ. We show the results in Figures 2 and 3.

**Analysis.** The results in Figures 2 and 3 show several interesting patterns. Most strikingly, the results for AV, PAV, and Phragmén are very similar to each other (to the point that it is often quite difficult to distinguish respective plots), whereas those for CC stand out sharply. This suggests that
the nature of choosing diverse committees, as done by CC, is quite different from that of choosing individually excellent ones (as done by AV) or proportional ones (as done by PAV and Phragmén).

Second observation is that it is much easier to affect the results of elections where the votes approve, on average, $p = 0.3$ fraction of the candidates (orange lines in Figures 2 and 3) than those where they approve, on average, $p = 0.1$ fraction of them (blue lines in Figures 2 and 3). This is somewhat counterintuitive. For example, in AV one would expect that with fewer approvals in total it would be easier to push some non-winning candidate into the committee by, say, adding approvals, because the bar for entering the committee should be low. On the other hand, the added approvals come from a wider set of possibilities.

The next observation is that the higher the value of $\phi$, the easier it is to affect the output committees. This is intuitive as for small values of $\phi$ the votes are highly correlated, whereas for larger $\phi$ the votes are more random and more fragile to adding noise.

6 Summary

We have complemented the results of Bredereck et al. [6] and Gawron and Faliszewski [15] by considering the robustness of GreedyCC, GreedyPAV, and Phragmén. We have found that their robustness levels are equal to the committee size (which means that even a minimal change to the votes can lead to completely replacing the winning committee), that the problems of deciding if modifying their input to a certain extent may change their outcomes are NP-complete, and we have observed how these rules react to random noise.

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A Proof of Theorem 3

Proof. The proof is analogous to that of Theorem 2 and we only modify some details of the argument. We reduce from RX3C and the input instance consists of the universe set $U = \{u_1, \ldots, u_{3n}\}$ and family $\mathcal{S} = \{S_1, \ldots, S_{3n}\}$ of three-element subsets of $U$. Each member of $U$ belongs to exactly three sets from $\mathcal{S}$.

We have two integers, $T = 10n^5$ and $t = 10n^3$, interpreted as two large numbers, with $T$ significantly larger than $t$. We form an election $E = (C, V)$ with candidate set $C = \{S_1, \ldots, S_{3n}\} \cup \{p, d\}$, and with the following voters:

1. For each $S_i \in \mathcal{S}$, there are $T$ voters that approve candidate $S_i$.
2. For each two sets $S_i$ and $S_j$, there are $T$ voters that approve candidates $S_i$ and $S_j$.
3. There are $2nT + 0.5nT + 4nt$ voters that approve $p$ and $d$.
4. For each $u_\ell \in U$, there are $t$ voters that approve $d$ and those candidates $S_i$ that correspond to the sets containing $u_\ell$.
5. There are $1.5nt$ voters who approve $p$.
6. There are $n$ voters who do not approve any candidates.

The committee size is $k = 3n + 1$ and the budget is $B = n$. We assume that the tie-breaking order among the candidates is:

$S_1 \succ S_2 \succ \cdots \succ S_{3n} \succ p \succ d$.

Prior to adding any approvals and running the rule, each candidate $S_i$ has score $3nT + 3t$, $p$ has score $2nT + 0.5nT + 4nt + 1.5nt$, and $d$ has score $2nT + 0.5nT + 4nt + 3nt$. 

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Let us now consider how GreedyPAV operates on this election. Prior to the first iteration, all the set candidates have the same score, much higher than that of $p$ and $d$. Due to the tie-breaking order, GreedyPAV chooses $S_1$. As a consequence, the scores of all other set candidates decrease to $(3n - 1)T + 0.5T$ plus some number of points from the fourth group of voters (but this part of the score is much smaller than that from the first two groups of voters). GreedyPAV acts analogously during the first $n$ iterations and it chooses a family $T$ of $n$ set elements.

After the first $n$ iterations, each of the remaining $2n$ set candidates has $2nT + 0.5nT$ points from the first two groups of voters and at most $3nt$ points from the fourth group (in fact less, but this bound suffices). On the other hand, both $p$ and $d$ have at least $2nT + 0.5nT + 4nt$ points and, so, in the next iteration the algorithm chooses one of them. Specifically, $p$ has score $2nT + 0.5nT + 4nt + 1.5nt$ and $d$ has score $2nT + 0.5nT + 4nt + 1.5nt + x$, where the value of $x$ is as follows. If $T$ corresponds to an exact cover of $U$, then $x = 0$ because for each set candidate that GreedyPAV adds during the first $n$ iteration, $d$ loses $1.5t$ points from the fourth group of voters. However, if $T$ does not correspond to an exact cover of $U$, then for at least one added set candidate the score of $d$ does not decrease by $1.5t$ but by at most $t + (\frac{1}{2} - \frac{1}{3})t$. Thus $x$ is at least $\frac{t}{3}$. So, if $T$ corresponds to an exact cover of $U$ then $p$ is selected and otherwise $d$ is. In either case, the score of the unselected one drops by at least $1.25nT$, which means that in the following $2n$ iterations the remaining set candidates are selected (because each of them has score at least $1.5nT$).

If candidate $p$ is selected without adding any approvals, then it means that we can find a solution for the RX3C instance using a simple greedy algorithm. In this case, instead of outputting the just-described instance of GreedyPAV-ADD-ROBUSTNESS-RADIUS, we output a fixed one, for which the answer is yes. Otherwise, we know that without any bribery the winning committee is \{$S_1, \ldots, S_{3n}, d$\}. We focus on this latter case.

If there is an exact cover of $U$ by sets from $S$ then it suffices to add a single approval for each of the corresponding sets in the last group of voters (those that originally do not approve anyone). Then, by the same analysis as above, we can verify that the sets forming the cover are selected in the first $n$ iterations, followed by $p$, followed by all the other set candidates.

For the other direction, let us note that no matter which $n$ approvals we add, it is impossible to modify the general scenario that GreedyPAV follows: It first chooses $n$ set candidates, then either $p$ or $d$ is selected (where the former can happen only if the first $n$ set candidates correspond to an exact cover of $U$), and finally the remaining $2n$ set candidates are chosen. This is so, because adding $n$ voters results in modifying each of the scores by a value between $-n$ and $n$ and such changes do not affect our analysis from the preceding paragraphs. This completes the proof. \qed