Integrable solutions of a generalized mixed-type functional integral equation

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Abstract

In this work, we prove the existence of integrable solutions for the following generalized mixed-type nonlinear functional integral equation

\[ x(t) = g\left(t, (Tx)(t)\right) + f\left(t, \int_0^t k(t,s)u(t,s,(Qx)(s))\, ds\right), \quad t \in [0, \infty). \]

Our result is established by means of a Krasnosel'skii type fixed point theorem proved in [M.A. Taoudi: Integrable solutions of a nonlinear functional integral equation on an unbounded interval, Nonlinear Anal. 71 (2009) 4131-4136]. In the last section we give an example to illustrate our result.

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1. Introduction

Consider the following mixed-type nonlinear functional integral equation

\[ x(t) = g\left(t, (Tx)(t)\right) + f\left(t, \int_0^t k(t,s)u(t,s,(Qx)(s))\, ds\right), \quad t \in [0, \infty). \] (1.1)
where \( f, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}, u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) and \((Tx)(t), (Qx)(t)\) are given while \( x(t) \) is an unknown function.

In [1], the authors studied the existence of integrable solutions of the following special case of the equation (1.1)

\[
x(t) = f \left( t, \int_0^t k(t,s)u(s,x(s)) \, ds \right), \quad t \in [0, \infty).
\]

The following generalization of this equation

\[
x(t) = g(t,x(t)) + f \left( t, \int_0^t k(t,s)u(s,x(s)) \, ds \right), \quad t \in [0, \infty),
\]

has been studied by [2] with the presence of the perturbation term \( g \).

In this paper, we are going to study the existence of integrable solutions of the more general form (1.1). A classical point of view for solving Eq. (1.1) is to write the equation in the form

\[
Ax + Bx = x, \tag{1.2}
\]

where \( A \) and \( B \) are two nonlinear operators.

Fixed point theory seems to be one of the most natural and powerful tools in studying the solvability of integral equations in the form (1.2). In [3], Krasnosel’skii established a fixed point theorem which was frequently used to solve some special integral equations in the form (1.2), see [4, 5]. Krasnosel’skii combined the famous Banach contraction principle of [6] and the classical Schauder fixed point theorem of [7] to prove that \( A + B \) has a fixed point in a nonempty closed convex subset \( \mathcal{M} \) of a real Banach space \( \mathcal{X} \) if \( A \) and \( B \) satisfy the following conditions (see [3, 8]):

- \( A \) is continuous and compact;
- \( B \) is a strict contraction;
- \( AM + BM \subseteq \mathcal{M}. \)

Generalizations and improvements of such a result have been made in several directions, we refer for example to the papers [9–19] and the references therein.
A deep variant of Krasnosel’skii type fixed point theorems is established by Latrach and Taoudi in [14]. In [2], Taoudi established an improvement of this variant. The most important advantage of [14, Theorem 2.1] and [2, Theorem 3.7] is that the operator $A$ is not assumed to be compact. Let us recall the Krasnosel’skii type fixed point theorem [2, Theorem 3.7].

**Theorem 1.1.** Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a Banach space $X$. Suppose that $A : \mathcal{M} \to X$ and $B : \mathcal{M} \to X$ such that:

1. $A$ is (ws)-compact;
2. There exists $\gamma \in [0, 1]$ such that $\mu(AS + BS) \leq \gamma \mu(S)$ for all $S \subseteq \mathcal{M}$; here $\mu$ is an arbitrary measure of weak noncompactness on $X$;
3. $B$ is a separate contraction;
4. $A\mathcal{M} + B\mathcal{M} \subseteq \mathcal{M}$.

Then there is $x \in \mathcal{M}$ such that $Ax + Bx = x$.

Our aim is to prove the existence of solutions of Eq. (1.1) in the space $L^1(\mathbb{R}^+)$ of Lebesgue integrable functions on the real half-axis $\mathbb{R}^+ = [0, \infty)$. Theorem 1.1 plays a crucial role in establishing our result in Theorem 3.1. A technique of measures of weak noncompactness used in [1] will be implemented. The result obtained in this paper generalizes the result of [2] to a more general equation such as Eq. (1.1) and extends the technique used in [1] to our more general context under special assumptions.

The outline of this paper is as follows. In Section 2, we recall some notations, definitions and basic tools which will be used in our investigations. Section 3 is devoted to state our main result and to prove some preliminary results. In Section 4, we prove our main result. In the last section we construct a nontrivial example illustrating our result.

### 2. Preliminaries

In this section we recall without proofs some of useful facts on Lebesgue space $L^1(\mathbb{R}^+)$, the superposition operator, contractions, (ws)-compact operators and measures of weak noncompactness.
2.1. The Lebesgue Space

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+$ be the interval $[0, +\infty)$. For a fixed Lebesgue measurable subset $I$ of $\mathbb{R}$, let $\text{meas}(I)$ be the Lebesgue measure of $I$ and denote by $L^1(I)$ the space of Lebesgue integrable functions on $I$, equipped with the standard norm

$$\|x\|_I = \|x\|_{L^1(I)} = \int_I |x(t)| \, dt.$$  

In the case when $I = \mathbb{R}_+$ the norm $\|x\|_{L^1(\mathbb{R}_+)}$ will be briefly denoted by $\|x\|$. Now, let us recall the following criterion of weak noncompactness in the space $L^1(\mathbb{R}_+)$ established by Dieudonne [20]. It will be frequently used in our discussions.

**Theorem 2.1.** A bounded set $X$ is relatively weakly compact in $L^1(\mathbb{R}_+)$ if and only if the following two conditions are satisfied:

(a) for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\text{meas}(D) \leq \delta$ then $\int_D |x(t)| \, dt \leq \epsilon$ for all $x \in X$.

(b) for any $\epsilon > 0$ there exists $\tau > 0$ such that $\int_{\tau}^{\infty} |x(t)| \, dt \leq \epsilon$ for any $x \in X$.

2.2. The Superposition Operator

For a fixed interval $I \subset \mathbb{R}$, bounded or not, consider a function $f : I \times \mathbb{R} \to \mathbb{R}$. The function $f = f(t, x)$ is said to satisfy the Carathéodory conditions if it is Lebesgue measurable in $t$ for every fixed $x \in \mathbb{R}$ and continuous in $x$ for almost every $t \in I$. The following theorem due to Scorza Dragoni [21] explains the structure of functions satisfying Carathéodory conditions.

**Theorem 2.2.** Let $I$ be a bounded interval and let $f : I \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Carathéodory conditions. Then, for each $\epsilon > 0$ there exists a closed subset $D_\epsilon$ of the interval $I$ such that $\text{meas}(I \setminus D_\epsilon) \leq \epsilon$ and $f|_{D_\epsilon \times \mathbb{R}}$ is continuous.

The superposition operator (or Nemytskii operator) associated with $f$ is defined as follows.
Definition 2.3. Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. The superposition operator generated by $f$ is the operator $N_f$ which assigns to each real measurable function on $I$ the real function $(N_f x)(t) = f(t, x(t))$, $t \in I$.

Appell and Zabrejko \cite{appell1985} realized a thorough study of superposition operators in which several types of results are presented. Among them, we recall the following fundamental theorem which gives a necessary and sufficient condition ensuring that $N_f$ maps continuously $L^1(I)$ into itself when $I$ is an unbounded interval. The authors generalized the result proved by Krasnosel’skii \cite{krasnosel1968} when $I$ is a bounded interval.

Theorem 2.4. The superposition operator $N_f$ generated by the function $f$ maps the space $L^1(I)$ continuously into itself if and only if

$$|f(t, x)| \leq a(t) + b|x|,$$

for all $t \in I$ and all $x \in \mathbb{R}$, where $a \in L^1(I)$ and $b \geq 0$ is a constant.

2.3. Contractions

Let $(X, d)$ be a metric space. It is well known that a mapping $B : X \to X$ is called a strict contraction if there exists $k \in (0, 1)$ such that $d(Bx, By) \leq kd(x, y)$ for every $x, y \in X$. In the following definition, we recall the notion of a large contraction introduced by Burton in \cite{burton1974}.

Definition 2.5. Let $(X, d)$ be a metric space. We say that $B : X \to X$ is a large contraction if $d(Bx, By) < d(x, y)$ for $x, y \in X$ with $x \neq y$ and if $\forall \epsilon > 0$ there exists $\delta < 1$ such that $[x, y \in X, d(x, y) \geq \epsilon] \Rightarrow d(Bx, By) \leq \delta d(x, y)$.

Now, we recall the notion of a separate contraction introduced by Liu and Li \cite{liu1983} which is weaker than the strict contraction and large contraction in the sense that every strict contraction is a separate contraction and every large contraction is a separate contraction. The same authors gave in \cite{liu1983} an example of a separate contraction which is not a strict contraction and another example of a separate contraction which is not a large contraction.
Definition 2.6. Let \((X,d)\) be a metric space. We say that \(B : X \to X\) is a separate contraction if there exist two functions \(\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying:

1. \(\psi(0) = 0\), \(\psi\) is strictly increasing,
2. \(d(Bx, By) \leq \varphi(d(x,y))\),
3. \(\psi(r) + \varphi(r) \leq r\) for \(r > 0\).

2.4. \((ws)\)-compact Operators

We recall the following definition from [27].

Definition 2.7. Let \(M\) be a subset of a Banach space \(X\). A continuous map \(A : M \to X\) is said to be \((ws)\)-compact if for any weakly convergent sequence \((x_n)_{n \in \mathbb{N}}\) in \(M\) the sequence \((Ax_n)_{n \in \mathbb{N}}\) has a strongly convergent subsequence in \(X\).

From this definition it immediately follows that a map \(A\) is \((ws)\)-compact if and only if it maps relatively weakly compact sets into relatively compact ones.

2.5. Measures of weak noncompactness

We recall some basic facts concerning measures of weak noncompactness, see [28]. Let us assume that \(E\) is an infinite dimensional Banach space with norm \(\|\cdot\|\) and zero element \(\theta\). Denote by \(\mathcal{M}_E\) the family of all nonempty and bounded subsets of \(E\) and by \(\mathcal{N}_E^w\) the subset of \(\mathcal{M}_E\) consisting of all relatively weakly compact sets. For a subset \(X\) of \(E\), the symbol \(\text{Conv}X\) will denote the convex hull with respect to the norm topology. Finally, we denote by \(B(x,r)\) the ball centered at \(x\) and of radius \(r\). We write \(B_r\) instead of \(B(\theta, r)\).

Definition 2.8. A mapping \(\mu : \mathcal{M}_E \to \mathbb{R}_+\) is said to be a measure of weak noncompactness in \(E\) if it satisfies the following conditions:

1. The kernel of \(\mu\) defined by \(\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}\) is nonempty and \(\ker \mu \subset \mathcal{N}_E^w\).
2. \(X \subset Y \Rightarrow \mu(X) \leq \mu(Y)\).
3. \(\mu(\text{Conv}X) = \mu(X)\).
4. \(\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y)\) for \(\lambda \in [0, 1]\).
(5) If \((X_n)_{n\geq 1}\) is a sequence of nonempty, weakly closed subsets of \(E\) with \(X_1\) bounded and \(X_{n+1} \subset X_n\) for \(n = 1, 2, 3, \ldots\) and if \(\lim_{n \to \infty} \mu(X_n) = 0\) then \(\bigcap_{n=1}^{\infty} X_n\) is nonempty.

The first important measure of weak noncompactness in a concrete Banach space \(E\) was defined by De Blasi [29] as follows:

\[ \beta(X) = \inf \{ \varepsilon > 0 : \text{there is a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + B_{\varepsilon} \} \]

The De Blasi measure of weak noncompactness \(\beta\) plays a significant role in nonlinear analysis and has many applications, [28–31]. It is worthwhile to mention that it is rather difficult to express \(\beta\) with a convenient formula in a concrete Banach space \(E\). Our problem under consideration (1.1) will be studied in the Banach space \(E = L^1(\mathbb{R}^+)\). In [32], Banaś and Knap constructed a useful measure of weak noncompactness \(\mu\) in the space \(L^1(\mathbb{R}^+)\). The following construction of \(\mu\) is based on the criterion of weak noncompactness given in Theorem 2.1 due to [20]: For a bounded subset \(X\) of \(L^1(\mathbb{R}^+)\) we define

\[ \mu(X) = c(X) + d(X), \] (2.1)

where

\[ c(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[ \int_{\Omega} |x(t)| \, dt : \Omega \subset \mathbb{R}^+, \text{meas}(\Omega) \leq \varepsilon \right] \right\} \right\}, \] (2.2)

and

\[ d(X) = \lim_{T \to \infty} \left\{ \sup \left[ \int_{T}^{\infty} |x(t)| \, dt : x \in X \right] \right\}. \] (2.3)

3. Assumptions, statement of results

In this section, we state the existence of solutions to the functional integral equation (1.1) in the space \(L^1(\mathbb{R}^+)\). First, observe that the problem (1.1) may
be written in the form

$$Ax + Bx = x,$$  \hspace{1cm} \text{(3.1)}

where $B = N_g T$ and $A = N_f U Q$, where $N_g$ and $N_f$ are the superposition operators associated to $g$ and $f$ respectively (see Definition \ref{def:superposition}) and $U$ is the operator defined by

$$(Ux)(t) = \int_0^t k(t, s)u(t, s, x(s)) \, ds.$$  \hspace{1cm} \text{(3.2)}

Our aim is to prove that $A + B$ has a fixed point in $L^1(\mathbb{R}_+)$ by applying Theorem \ref{thm:fixed_point}. We consider Eq. \ref{eq:main_eq} under the following assumptions:

(A1) The functions $g, f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions and there are constants $b, b_1 > 0$ and functions $a, a_1 \in L^1(\mathbb{R}_+)$ such that

$$|g(t, x)| \leq a(t) + b|x|,$$
$$|f(t, x)| \leq a_1(t) + b_1|x|,$$

for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(A2) The operator $Q$ maps continuously the space $L^1(\mathbb{R}_+)$ into itself and there are constants $\rho_1, \rho_2 > 0$, functions $\gamma_1, \gamma_2 \in L^1(\mathbb{R}_+)$ and increasing functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$, absolutely continuous such that

$$|(T x)(t)| \leq \gamma_1(t) + \rho_1 |x(\phi(t))|,$$
$$|(Q x)(t)| \leq \gamma_2(t) + \rho_2 |x(\psi(t))|,$$

for $x \in L^1(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$. Moreover, there are constants $m, M > 0$ such that $\phi'(t) \geq m$ and $\psi'(t) \geq M$ for almost all $t \geq 0$.

(A3) The mapping $B = N_g T$ is a separate contraction.
The function $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, i.e., the function $t \to u(t, s, x)$ is measurable on $\mathbb{R}_+$ for every fixed $(s, x) \in \mathbb{R}_+ \times \mathbb{R}$ and the function $(s, x) \to u(t, s, x)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$ for almost every $t \in \mathbb{R}_+$.

There are constants $\beta, \lambda > 0$, functions $\alpha, \gamma \in L^1(\mathbb{R}_+)$ and a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ which is measurable with $\lim_{\delta \to 0} h(\delta) = 0$, such that

$$|u(t, s, x)| \leq \alpha(s) + \beta|x|,$$

for all $t, s \in \mathbb{R}_+, x \in \mathbb{R}$, and

$$|u(t, s, x) - u(t + \delta, s, x)| \leq h(\delta) \left[\gamma(s) + \lambda|x|\right],$$

for any $t, s \in \mathbb{R}_+, x \in \mathbb{R}$ and $\delta$ small.

The function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is measurable such that the linear Fredholm integral operator

$$(Kx)(t) = \int_0^t |k(t, s)| x(s) \, ds$$

is a continuous map from $L^1(\mathbb{R}_+)$ into itself.

$\gamma = b \rho_1 m^{-1} + b_1 \rho_2 \beta M^{-1} \|K\| < 1$, where $\|K\|$ denotes the norm of the operator $K$.

Now, we are in a position to present our existence result.

**Theorem 3.1.** Let assumptions (A1)-(A7) be satisfied. Then, equation [1.1] has at least one solution $x \in L^1(\mathbb{R}_+)$. 

**Remark 3.2.** Under the assumption (A2), the operators $T$ and $Q$ map $L^1(\mathbb{R}_+)$ into itself. Moreover, the operator $Q$ is assumed to be continuous. However, the operator $T$ is not assumed to be continuous.

To prove the main result, we demonstrate the continuity of $A$ and we establish $L^1(I)$-estimates for any nonempty measurable subset $I$ of $\mathbb{R}_+$. 

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Proposition 3.3. Suppose that assumptions (A1)-(A4), (A6) and the inequality (3.3) from (A5) hold. Then

(a) The operator \( A \) maps \( L^1(\mathbb{R}^+) \) continuously into itself.

(b) For any nonempty measurable subset \( I \) of \( \mathbb{R}^+ \) and \( x \in L^1(\mathbb{R}^+) \) we have the following estimations

\[
\|Ax\|_I \leq \|a_1\|_I + b_1 \|K\| \left[ \|\alpha\|_I + \beta \|\gamma_2\|_I + \beta M^{-1} \rho_2 \|x\|_{\psi(t)} \right] \tag{3.4}
\]

and

\[
\|Bx\|_I \leq \|a\|_I + b \left[ \|\gamma_1\|_I + \rho_1 m^{-1} \|x\|_{\phi(t)} \right] \tag{3.5}
\]

Proof. First we prove that the operator \( U \) given by (3.2) maps \( L^1 = L^1(\mathbb{R}^+) \) continuously into itself. In view of the inequality (3.3) from assumption (A5) and the assumption (A6), it is easy to observe that the operator \( U \) transforms \( L^1 \) into itself. Now, let \( \{x_n\} \) be a sequence in \( L^1 \) which converges to \( x \) in \( L^1 \). We show that \( \{Ux_n\} \) converges to \( Ux \) in \( L^1 \). For every \( \tau > 0 \), in view of our assumptions we have

\[
\|Ux_n - Ux\|_{L^1} \leq \int_0^\tau |(Ux_n)(t) - (Ux)(t)| \, dt + \int_\tau^\infty |(Ux_n)(t)| \, dt \leq \delta_n + \int_\tau^\infty \int_0^\tau |k(t, s)| (\alpha(s) + \beta |x_n(s)|) ds \, dt + \int_\tau^\infty \|Ux(t)\| \, dt = \delta_n + \|K\alpha\|_{L^1([\tau, \infty))} + \beta \|Kx_n\|_{L^1([\tau, \infty))} + \|Ux\|_{L^1([\tau, \infty))}
\]

where \( \delta_n = \int_0^\tau |(Ux_n)(t) - (Ux)(t)| \, dt. \)

Now, since \( Ux \in L^1 \) and \( K\alpha \in L^1 \) we deduce that terms \( \|K\alpha\|_{L^1([\tau, \infty))} \) and \( \|Ux\|_{L^1([\tau, \infty))} \) are arbitrarily small provided \( \tau \) is taken sufficiently large.

On the other hand, from continuity of the operator \( K \) we conclude that \( Kx_n \) converges to \( Kx \) in \( L^1 \). Then, the sequence \( \{Kx_n\} \) is relatively compact. In
view of Theorem 2.1, we infer that the terms of the sequence \(\{\|Kx_n\|_{L^1([\tau,\infty))}\}\) are arbitrarily small provided the number \(\tau\) is large enough.

Keeping in mind the inequality (3.3) from assumption (A5) and applying the so-called majorant principle (see [5, 33]), we conclude that the operator \(U|_{L^1([0,\tau])}: L^1([0,\tau]) \to L^1([0,\tau])\) is continuous. Then, we deduce that \(\delta_n\) converges to 0 as \(n\) goes to infinity.

Therefore \(Ux_n\) converges to \(Ux\) in \(L^1\). This means that the operator \(U\) maps \(L^1(\mathbb{R}_+)\) continuously into itself. From this fact, assumption (A1), Theorem 2.4 and the continuity of \(Q\) we conclude that the operator \(A = NfUQ\) is continuous on the space \(L^1\).

In the sequel, we prove the estimation (3.4). For any nonempty measurable subset \(I\) of \(\mathbb{R}_+\) and \(x \in L^1\) we have

\[
\int_I |Ax(t)|dt \leq \int_I a_1(t)dt + b_1 \int_I \left| \int_0^t k(t,s)u(t,s, (Qx)(s))ds \right| dt
\]

\[
\leq \int_I a_1(t)dt + b_1 \left( \int_I \int_0^t |k(t,s)| (\alpha(s) + \beta |(Qx)(s)|) ds dt \right)
\]

\[
\leq \int_I a_1(t)dt + b_1 \left( \int_I \int_0^t |k(t,s)| (\alpha(s) + \beta \gamma_2(s)) ds dt \right)
\]

\[
+ b_1 \beta \rho_2 \int_I \left( \int_0^t |k(t,s)| (|x(\psi(s))|) ds \right) dt
\]

\[
= \|a_1\|_I + b_1 \|K\|_I + b_1 \beta \|K\|_I + b_1 \beta \rho_2 \|Kx(\psi)\|_I
\]

\[
\leq \|a_1\|_I + b_1 \|K\|_I \|\alpha\|_I + b_1 \beta \|K\|_I \|\gamma_2\|_I
\]

\[
+ b_1 \beta \rho_2 \|K\|_I \int_I |x(\psi(t))| dt
\]

\[
\leq \|a_1\|_I + b_1 \|K\|_I \|\alpha\|_I + b_1 \beta \|K\|_I \|\gamma_2\|_I
\]

\[
+ b_1 \beta \rho_2 \|K\|_I M^{-1} \int_I |x(\psi(t))| \psi'(t) dt
\]

\[
\leq \|a_1\|_I + b_1 \|K\|_I \|\alpha\|_I + b_1 \beta \|K\|_I \|\gamma_2\|_I L^1(I)
\]

\[
+ b_1 \beta \rho_2 M^{-1} \|K\|_I \int_{\psi(I)} x(\theta) d\theta
\]

Hence, we obtain the estimation (3.4). In the same way, the estimation (3.5) is obtained using our assumptions on \(g\) and \(T\).
Also, we need the following lemma to prove our main result. In the proof of this lemma, we implement a technique used in [3].

**Lemma 3.4.** Let $Z$ be a nonempty, bounded and relatively weakly compact set of $L^1(\mathbb{R}^+)$ and $I_\tau = [0, \tau]$, where $\tau > 0$. For any $\epsilon > 0$ there exists a closed subset $D_\epsilon$ of the interval $I_\tau$ with $\text{meas}(I_\tau \setminus D_\epsilon) \leq \epsilon$ such that the set $A(Z)$ is relatively compact in the space $C(D_\epsilon)$.

**Proof.** Let $\epsilon > 0$. In view of Theorem 2.2 we can find a closed subset $D_\epsilon$ of the interval $I_\tau = [0, \tau]$ such that $\text{meas}(D_\epsilon') \leq \epsilon$ (where $D_\epsilon' = I_\tau \setminus D_\epsilon$) and such that $f|_{D_\epsilon \times \mathbb{R}}$ and $k|_{D_\epsilon \times I_\tau}$ are continuous. In the sequel, we show that $A(Z)$ is equibounded and equicontinuous in the space $C(D_\epsilon)$ in order to apply Ascoli-Arzelà theorem. Let us take an arbitrary $x \in Z$. Then for every $t \in D_\epsilon$, we have

\[
|UQx(t)| = \left| \int_0^t k(t, s)u(t, s, (Qx)(s)) \, ds \right|
\]
\[
\leq \int_0^t |k(t, s)| \alpha(s) \, ds
\]
\[
\leq \int_0^t |k(t, s)| \alpha(s) + \beta \gamma_2(s) + \beta \rho_2 \psi(s) \, ds
\]
\[
\leq \kbar \left( \|\alpha\| + \beta \gamma_2 \right) + \beta \rho_2 \int_0^t \psi(s) \, ds
\]
\[
\leq \kbar \left( \|\alpha\| + \beta \gamma_2 \right) + \beta \rho_2 \int_0^t \psi(s) \, ds
\]
\[
= \kbar \left( \|\alpha\| + \beta \gamma_2 \right) + \beta \rho_2 \int_0^t \psi(s) \, ds
\]
\[
\leq \kbar \left( \|\alpha\| + \beta \gamma_2 \right) + \beta \rho_2 \int_0^t \psi(s) \, ds
\]
\[
\leq \kbar \left( \|\alpha\| + \beta \gamma_2 \right) + \beta \rho_2 \int_0^t \psi(s) \, ds
\]

where $\kbar = \max \{|k(t, s)| : (t, s) \in D_\epsilon \times I_\tau\}$ and $\pibar = \sup \{|x| : x \in Z\}$. In the sequel, we will denote by $U_\epsilon$ the quantity

\[
U_\epsilon := \kbar \left( \|\alpha\| + \beta \gamma_2 \right) + \beta \rho_2 \pibar M^{-1}.
\]
Then, using the assumption (A1) we obtain

\[ |(Ax)(t)| \leq a_1(t) + b_1 |(UQx)(t)| \leq \overline{\sigma}_1 + b_1 U \]

for every \( t \in D_\varepsilon \), where \( \overline{\sigma}_1 = \sup \{ a_1(t) : t \in D_\varepsilon \} \). This proves that the set \( A(Z) \) is equibounded on the set \( D_\varepsilon \).

Now, let us consider \( t_1, t_2 \in D_\varepsilon \), \( t_1 \leq t_2 \) and \( \delta = t_2 - t_1 \). For a fixed \( x \in Z \), we denote \( U_{t_1,t_2}^x = (UQx)(t_2) - (UQx)(t_1) \). Then, in view of our assumptions, we have

\[
|U_{t_1,t_2}^x| = \left| \int_{0}^{t_2} k(t_2, s)u(t_2, s, (Qx)(s)) \, ds - \int_{0}^{t_1} k(t_1, s)u(t_1, s, (Qx)(s)) \, ds \right|
\]

\[
\leq \int_{0}^{t_1} |k(t_2, s)u(t_2, s, (Qx)(s)) - k(t_1, s)u(t_1, s, (Qx)(s))| \, ds
\]

\[
+ \int_{t_1}^{t_2} |k(t_2, s)u(t_2, s, (Qx)(s))| \, ds
\]

\[
\leq \int_{0}^{t_1} |k(t_2, s)u(t_2, s, (Qx)(s)) - k(t_1, s)u(t_1, s, (Qx)(s))| \, ds
\]

\[
+ \int_{0}^{t_1} |k(t_1, s)u(t_2, s, (Qx)(s)) - k(t_1, s)u(t_1, s, (Qx)(s))| \, ds
\]

\[
+ \int_{t_1}^{t_2} |k(t_2, s)u(t_2, s, (Qx)(s))| \, ds
\]

\[
= \int_{0}^{t_1} |k(t_2, s) - k(t_1, s)| \cdot |u(t_2, s, (Qx)(s))| \, ds
\]

\[
+ \int_{0}^{t_1} |k(t_1, s)| \cdot |u(t_2, s, (Qx)(s)) - u(t_1, s, (Qx)(s))| \, ds
\]

\[
+ \int_{t_1}^{t_2} |k(t_2, s)| \cdot |u(t_2, s, (Qx)(s))| \, ds
\]

\[
\leq \int_{0}^{t_1} |k(t_2, s) - k(t_1, s)| \left[ \alpha(s) + \beta \gamma_2(s) + \beta \rho_2 |x(\psi(s))| \right] \, ds
\]

\[
+ \int_{0}^{t_1} |k(t_1, s)| h(\delta) \left[ \gamma(s) + \lambda \gamma_2(s) + \lambda \rho_2 |x(\psi(s))| \right] \, ds
\]

\[
+ \int_{t_1}^{t_2} |k(t_2, s)| \left[ \alpha(s) + \beta \gamma_2(s) + \beta \rho_2 |x(\psi(s))| \right] \, ds,
\]

if \( \delta \) is taken small enough. Now, we denote by \( w^\tau(k, \cdot) \) the modulus of continuity of the function \( k \) on the set \( D_\varepsilon \times I_\tau \) given by
\[ w^\tau (k, \delta) = \sup \{ |k(t, s) - k(t, \sigma)| : t \in D, \text{ and } s, \sigma \in I_\tau \text{ with } |s - \sigma| \leq \delta \}. \]

Therefore, we obtain

\[
|\mathcal{U}^{t_1,t_2}_x| \leq w^\tau (k, \delta) \left( \int_0^\tau |\alpha(s) + \beta \gamma_2(s) + \beta \rho_2 |\psi(s)|\, ds \right) \\
+ \overline{K} h(\delta) \int_0^\tau [\gamma(s) + \lambda \gamma_2(s) + \lambda \rho_2 |\psi(s)|] \, ds \\
+ \overline{K} \int_{t_1}^{t_2} [\alpha(s) + \beta \gamma_2(s) + \beta \rho_2 |\psi(s)|] \, ds
\]

\[
\leq w^\tau (k, \delta) \left( ||\alpha|| + \beta \|\gamma_2\| + \beta \rho_2 M^{-1} \int_0^\tau |\psi(s)| \, ds \right) \\
+ \overline{K} h(\delta) \left( ||\gamma|| + \lambda \|\gamma_2\| + \lambda \rho_2 M^{-1} \int_0^\tau |\psi(s)| \, ds \right) \\
+ \overline{K} \left( \int_{t_1}^{t_2} (\alpha(s) + \beta \gamma_2(s)) \, ds + \beta \rho_2 M^{-1} \int_{t_1}^{t_2} |\psi(s)| \, ds \right) \\
= w^\tau (k, \delta) \left( ||\alpha|| + \beta \|\gamma_2\| + \beta \rho_2 M^{-1} \int_0^\tau |\psi(s)| \, ds \right) \\
+ \overline{K} h(\delta) \left( ||\gamma|| + \lambda \|\gamma_2\| + \lambda \rho_2 M^{-1} \int_0^\tau |\psi(s)| \, ds \right) \\
+ \overline{K} \left( \int_{t_1}^{t_2} (\alpha(s) + \beta \gamma_2(s)) \, ds + \beta \rho_2 M^{-1} \int_{\psi(t_1)}^{\psi(t_2)} |\psi(s)| \, ds \right)
\]

Keeping in mind that \( k|_{D \times I_\tau} \) is uniformly continuous, we conclude that \( w^\tau (k, \delta) \) is arbitrarily small provided that the number \( \delta \) is small enough.

Absolute continuity of \( \psi \) ensures that \( \psi(t_2) - \psi(t_1) \) is small enough when we take \( \delta \) small enough. Considering the fact that \( Z \) is bounded, relatively weakly compact and using Theorem 2.11 we obtain that the elements of the set

\[
\left\{ \int_{\psi(t_1)}^{\psi(t_2)} |\psi(s)| \, ds : x \in Z \right\},
\]

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are uniformly arbitrarily small provided the number $\delta$ is small enough.

Similarly, the number $\int_{t_1}^{t_2} (\alpha(s) + \beta \gamma_2(s)) \, ds$ is arbitrarily small provided the number $\delta$ is small enough.

The number $h(\delta)$ is arbitrarily small provided the number $\delta$ is small enough, thanks to hypothesis $\lim_{\delta \to 0} h(\delta) = 0$ from assumption (A5).

This proves that the set $U(Z)$ is equicontinuous in the space $C(D_\epsilon)$. Hence, uniform continuity of $f|_{D_\epsilon \times [0, U_\epsilon]}$, where $U_\epsilon$ is given by (3.6), implies that the set $A(Z)$ is equicontinuous in the space $C(D_\epsilon)$.

Therefore, $A(Z)$ is equibounded and equicontinuous in the space $C(D_\epsilon)$. Then, by Ascoli-Arzelà theorem we obtain that $A(Z)$ is a relatively compact set in the space $C(D_\epsilon)$.

4. Proof of Theorem 3.1

Proof of Theorem 3.1 We prove that the operators $A$ and $B$ from Eq. (3.1) satisfy the hypothesis of Theorem 1.1.

Step 1: First we prove that there exists a positive number $r > 0$ such that

$$A(B_r) + B(B_r) \subseteq B_r.$$  

Let $x, y \in L^1 = L^1(\mathbb{R}_+)$. From estimations (3.4) and (3.5), we get

$$\|Ax + By\| \leq \|a_1\| + b_1 \|K\| \left(\|\alpha\| + \beta \|\gamma_2\| + \beta M^{-1} \rho_2 \|x\|\right) + \|a\| + b \left(\|\gamma_1\| + \rho_1 m^{-1} \|y\|\right) \leq C + b_1 \rho_2 \beta M^{-1} \|K\| \|x\| + b \rho_1 m^{-1} \|y\|, \quad (4.1)$$

where $C = \|a_1\| + \|a\| + b_1 \|K\| (\|\alpha\| + \beta \|\gamma_2\|) + b \|\gamma_1\|$. Let

$$\gamma = b \rho_1 m^{-1} + b_1 \rho_2 \beta M^{-1} \|K\|, \quad (4.2)$$

and $r$ be the real number defined by $r = \frac{C}{1 - \gamma}$. Thanks to hypothesis (A7), we have $r > 0$. Clearly if $x, y \in B_r$, then the estimation (4.1) becomes
\|Ax + By\| \leq C + \gamma r = r.

**Step 2:** Now, we show that there exists \(\gamma \in [0, 1]\) such that \(\mu(AS + BS) \leq \gamma \mu(S)\) for every bounded subset \(S\) of \(L^1\), where \(\mu\) is the measure of weak noncompactness defined by (2.1). Let us fix a nonempty subset \(S\) of \(L^1\).

From estimations (3.4) and (3.5), for every nonempty measurable subset \(I\) of \(\mathbb{R}_+\) and for any \(x \in S\) we have

\[\|Ax + Bx\|_I \leq \|a\|_I + b_1 \|K\| \left(\|\alpha\|_I + \beta \|\gamma_2\|_I + \beta M^{-1} \rho_2 \|x\|_{\psi(t)}\right) + \|a\|_I + b \left(\|\gamma_1\|_I + \rho_1 m^{-1} \|x\|_{\phi(t)}\right),\]

(4.3)

Observe that the set consisting of one element of \(L^1\) is weakly compact. Then, from Theorem 2.1 we conclude that

\[\lim_{\varepsilon \to 0} \{\sup \left[\|h\|_I : \text{meas}(I) \leq \varepsilon\right]\} = 0,\]

for \(h \in \{a_1, \gamma_1, \gamma_2, \alpha, a\}\).

Therefore, using the definition (2.2) and taking into account that the functions \(\psi\) and \(\phi\) are supposed to be absolutely continuous, from the inequality (4.3) we obtain

\[c(AS + BS) \leq \gamma c(S),\]

(4.4)

where \(\gamma\) is given by (4.2). From assumption (A7) we have \(\gamma < 1\).

Now, consider an arbitrary \(\tau > 0\). Taking \(I = [\tau, \infty)\), the inequality (4.3) becomes

\[
\int_{\tau}^{\infty} |(Ax)(t) + (Bx)(t)| \, dt \leq \int_{\tau}^{\infty} a_1(t) \, dt + \int_{\tau}^{\infty} a(t) \, dt \\
+ b_1 \|K\| \left(\int_{\tau}^{\infty} \alpha(t) \, dt + \beta \int_{\tau}^{\infty} \gamma_2(t) \, dt\right) \\
+ b_1 \rho_2 \beta M^{-1} \|K\| \int_{\phi(\tau)}^{\infty} |x(t)| \, dt \\
+ b \int_{\tau}^{\infty} \gamma_1(t) \, dt + b_1 \rho_1 \beta^{-1} \int_{\phi(\tau)}^{\infty} |x(t)| \, dt.
\]
Theorem 2.1 ensures that \( d(h) = 0 \) for \( h \in \{ a_1, \gamma_1, \gamma_2, \alpha, a \} \), where \( d \) is defined by (2.3). Then, the last estimate leads to

\[
d(AS + BS) \leq \gamma d(S).
\] (4.5)

Combining estimations (4.4) and (4.5) with the definition (2.1) we obtain

\[
\mu(AS + BS) \leq \gamma \mu(S).
\]

**Step 3:** \( A \) is \( ws \)-compact. From Proposition 3.3, the operator \( A \) is continuous. Now, consider a weakly convergent sequence \( \{x_n\} \) in \( B_r \) and fix a number \( \epsilon > 0 \). Applying Theorem 2.1 for the relatively compact set \( \{ Ax_n : n \in \mathbb{N} \} \), we deduce that there exist \( \tau > 0 \) and \( \delta > 0 \) such that for any \( n \in \mathbb{N} \) we have

\[
\int_\tau^\infty |(Ax_n)(t)| \, dt \leq \frac{\epsilon}{8},
\] (4.6)

and

\[
\int_D |(Ax_n)(t)| \, dt \leq \frac{\epsilon}{4},
\] (4.7)

for each subset of \( \mathbb{R} \) such that \( meas(D) \leq \delta \).

By using Lemma 3.4 for \( Z = \{ x_n : n \in \mathbb{N} \} \), for every \( p \in \mathbb{N} \), there exists a closed subset \( D_p \) of the interval \( I_\tau = [0, \tau] \) with \( meas(D'_p) \leq \frac{1}{p} \) such that \( \{ Ax_n : n \in \mathbb{N} \} \) is relatively compact in the space \( C(D_p) \). Passing to subsequences if necessary we can assume that \( \{ Ax_n \} \) is a Cauchy sequence in \( C(D_p) \), for every \( p \in \mathbb{R} \). Then, we can choose \( p_0 \) large enough such that \( meas(D'_{p_0}) \leq \delta \) and for every \( m, n \geq p_0 \)

\[
\| Ax_n - Ax_m \|_{C(D_{p_0})} = \max_{t \in D_{p_0}} |Ax_n(t) - Ax_m(t)| \leq \frac{\epsilon}{4(meas(D'_{p_0}) + 1)}.
\] (4.8)

Now, using (4.7) with (4.8) we obtain

\[
\int_0^\tau |Ax_n(t) - Ax_m(t)| \, dt = \int_{D_{p_0}} |Ax_n(t) - Ax_m(t)| \, dt
\]
\[
\int_{D_{p_0}} |Ax_n(t) - Ax_m(t)| \, dt \\
\leq \frac{\epsilon \text{meas}(D_{p_0}')}{4(\text{meas}(D_{p_0}) + 1)} + \int_{D_{p_0}} |Ax_n(t)| \, dt \\
+ \int_{D_{p_0}} |Ax_m(t)| \, dt \\
\leq \frac{3\epsilon}{4}. \tag{4.9}
\]

Finally, by combining (4.6) and (4.9) for \(m, n \geq p_0\) we obtain

\[
\|Ax_n - Ax_m\|_{L^1} = \int_0^\infty |Ax_n(t) - Ax_m(t)| \, dt \\
\leq \int_0^\tau |Ax_n(t) - Ax_m(t)| \, dt + \int_\tau^\infty |Ax_n(t) - Ax_m(t)| \, dt \\
\leq \int_0^\tau |Ax_n(t) - Ax_m(t)| \, dt + \int_\tau^\infty |Ax_n(t)| \, dt \\
+ \int_\tau^\infty |Ax_m(t)| \, dt \\
\leq \epsilon.
\]

This proves that \(\{Ax_n\}\) is a Cauchy sequence in the Banach space \(L^1(\mathbb{R}_+)\).

Then \(\{Ax_n\}\) has a strongly convergent subsequence in \(L^1(\mathbb{R}_+)\).

Now, by applying Theorem 1.1 with \(M = B_r\) we obtain the existence of an integrable solution for the problem (1.1).

\[\square\]

5. Example

Consider the mixed-type functional integral equation

\[
x(t) = \frac{t}{t^3 + 1} + \frac{1}{4} \ln \left[ 1 + \left( \frac{x^3(2t)}{1 + x^2(2t)} + e^{-t} \int_0^\infty e^{-\tau} \frac{x(\tau)}{1 + x^2(\tau)} \, d\tau \right)^2 \right] \\
+ \frac{1}{2} \arctan \left[ \int_0^t (t + s)e^{-t}u(t, s, (Qx)(s)) \, ds \right]^2,
\]  \tag{5.1}

where

\[
u(t, s, x) = \frac{1 + t + s}{2 + (1 + t + s)^3} + \frac{ts(t + \sqrt{3} \sin x)}{4(s + 1)(t^2 s^2 + 1)},
\]
and

\[(Qx)(t) = \frac{x^2(t)}{1 + |x(t)|} \int_0^t e^{-(t+\tau)} \frac{x(\tau)}{1 + x^2(\tau)} d\tau.\]

The equation (5.1) is of the form (1.1) with

\[g(t, x) = \frac{t}{t^3 + 1} + \frac{1}{4} \ln(1 + x^2),\]
\[f(t, x) = \frac{1}{2} \arctan x^2,\]
\[k(t, s) = (t + s)e^{-t},\]

\[(Tx)(t) = \frac{x^3(2t)}{1 + x^2(2t)} + e^{-t} \int_0^\infty e^{-\tau} \frac{x(\tau)}{1 + x^2(\tau)} d\tau.\]

Next, we prove that assumptions (A1) – (A7) are fulfilled.

(A1) Taking into account that \(\arctan x^2 \leq 2x\) for \(x \geq 0\) and \(\ln(1 + x^2) \leq x\), it is easy to see that \(g(t, x)\) and \(f(t, x)\) satisfy assumption (A1) with \(a(t) = t/(t^3 + 1)\), \(b = 1/4\), \(a_1(t) = 0\) and \(b_1 = 1\).

(A2) For \(x \in L^1(\mathbb{R}_+)\), it is easy to see the inequalities

\[|(Tx)(t)| \leq e^{-t} + |x(2t)|\quad \text{and}\quad |(Qx)(t)| \leq |x(t)|.\]

We take

\[\gamma_1(t) = e^{-t}, \rho_1 = 1, \phi(t) = 2t, m = 2,\]

and

\[\gamma_2(t) = 0, \rho_2 = 1, \psi(t) = t, M = 1,\]

Now, we will prove that \(Q\) is continuous on \(L^1(\mathbb{R}_+)\). Let \(\{x_n\}\) be a sequence in \(L^1(\mathbb{R}_+)\) which converges in \(L^1(\mathbb{R}_+)\) to a function \(x \in L^1(\mathbb{R}_+)\). Denoting \(\rho_n = \|Qx_n - Qx\|\), we have

\[\rho_n \leq \int_0^\infty \left| \frac{x_n^2(t)}{(1 + |x_n(t)|)} - \frac{x^2(t)}{1 + |x(t)|} \right| \left( \int_0^t e^{-(t+\tau)} \frac{|x_n(\tau)|}{1 + x_n^2(\tau)} d\tau \right) dt\]

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\[
+ \int_0^\infty \frac{x^2(t)}{1 + |x(t)|} \left( \int_0^t e^{-(t+\tau)} \left| \frac{x_n(\tau)}{1 + x_n^2(\tau)} - \frac{x(\tau)}{1 + x^2(\tau)} \right| d\tau \right) dt
\leq \int_0^\infty |x_n(t) - x(t)| \left( \int_0^\infty e^{-\tau} d\tau \right) dt
+ \int_0^\infty |x(t)| \left( \int_0^t |x_n(\tau) - x(\tau)| d\tau \right) dt
\leq \int_0^\infty |x_n(t) - x(t)| dt + \int_0^\infty |x(t)| \left( \int_0^\infty |x_n(\tau) - x(\tau)| d\tau \right) dt
= (1 + ||x||) \|x_n - x\|.
\]

This proves that \(Q\) is continuous.

(A3) Let \(x, y \in L^1(\mathbb{R}_+)\). Using the Mean Value Theorem, we have

\[
|g(t, (Tx)(t)) - g(t, (Ty)(t))| = \frac{1}{4} |\ln \left( 1 + ((Tx)(t))^2 \right) - \ln \left( 1 + ((Ty)(t))^2 \right)|
\leq \frac{1}{4} |(Tx)(t) - (Ty)(t)|
\leq \frac{1}{4} \left| \frac{x^3(2t)}{1 + x^2(2t)} - \frac{y^3(2t)}{1 + y^2(2t)} \right|
+ \frac{1}{4} e^{-t} \int_0^\infty \left| \frac{x(\tau)}{1 + x^2(\tau)} - \frac{y(\tau)}{1 + y^2(\tau)} \right| d\tau
\leq \frac{1}{2} |x(2t) - y(2t)| + \frac{1}{4} e^{-t} \int_0^\infty |x(\tau) - y(\tau)| d\tau.
\]

Hence, we get

\[
\|g(t, (Tx)(t)) - g(t, (Ty)(t))\| = \int_0^\infty |g(t, (Tx)(t)) - g(t, (Ty)(t))| dt
\leq \frac{1}{2} \int_0^\infty |x(2t) - y(2t)| dt
+ \frac{1}{4} \left( \int_0^\infty e^{-t} dt \right) \int_0^\infty |x(\tau) - y(\tau)| d\tau
= \frac{1}{2} \|x - y\|.
\]

Therefore \(B\) is a strict contraction.

(A4) Obviously, the function \(u\) satisfies Carathéodory conditions.
Taking into account that the function \( h_1(z) = \frac{z}{2 + z^3} \) is nonincreasing for \( z \geq 1 \) and the maximum of the function \( h_2(z) = \frac{z^2 + \sqrt{3}z}{1 + z^2} \) is \( 3/2 \), for every \( t, s \geq 0 \) and \( x \in \mathbb{R} \) we have

\[
|u(t, s, x)| \leq \frac{1 + s}{2 + (1 + s)^3} + \frac{t^2 s^2 + \sqrt{3} t s}{4(s + 1)(t^2 s^2 + 1)} |x|
\]

\[
\leq \frac{1 + s}{2 + (1 + s)^3} + \frac{3}{8} |x|
\]

We take \( \alpha(s) = \frac{1 + s}{2 + (1 + s)^3} \) and \( \beta = 3/8 \).

Now, for every \( t, w, s \geq 0 \) and \( x \in \mathbb{R} \) it can be easily observed that

\[
|u(t, s, x) - u(w, s, x)| \leq \frac{1 + t + s}{2 + (1 + t + s)^3} - \frac{1 + w + s}{2 + (1 + w + s)^3}
\]

\[
+ \frac{1}{4(s + 1)} \left| \frac{t^2 s^2}{(t^2 s^2 + 1)} - \frac{w^2 s^2}{(w^2 s^2 + 1)} \right| x
\]

\[
+ \frac{\sqrt{3}}{4(s + 1)} \left| \frac{ts}{(t^2 s^2 + 1)} - \frac{ws}{(w^2 s^2 + 1)} \right| x \sin x
\]

(5.2)

Applying the Mean Value Theorem for the function \( h_1(z) = \frac{z}{2 + z^3} \) between \( z_1 = 1 + t + s \) and \( z_2 = 1 + w + s \) and taking into account that \( |h'_1(z)| \leq \frac{2}{2 + z^3} \leq \frac{2}{2 + (1 + s)^3} \) for every \( z \geq 1 + s \) we obtain

\[
\frac{1 + t + s}{2 + (1 + t + s)^3} - \frac{1 + w + s}{2 + (1 + w + s)^3} \leq \frac{2}{2 + (1 + s)^3} |t - w|.
\]

Now, substituting \( z = ts \) and \( y = ws \) into the inequalities \( \left| \frac{z^2}{1 + z^2} - \frac{y^2}{1 + y^2} \right| \leq |z - y| \) and

\[
|z - y| \leq \left| \frac{z}{1 + z^2} - \frac{y}{1 + y^2} \right| \leq |z - y|
\]

We obtain from (5.2) the inequality

\[
|u(t, s, x) - u(w, s, x)| \leq \frac{2}{2 + (1 + s)^3} |t - w| + \frac{|t - w| s}{4(s + 1)} |x|
\]

\[
+ \frac{\sqrt{3} |t - w| s}{4(s + 1)} |x|
\]

\[
\leq |t - w| \left( \frac{2}{2 + (1 + s)^3} + \frac{1 + \sqrt{3}}{4} \right) |x|
\]

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We take $h(\delta) = |\delta|$, $\gamma_0(s) = \frac{2}{2 + (1 + s)^3}$ and $\lambda = (1 + \sqrt{3})/4$.

(A6) Since the kernel $k(t, s) = (t + s)e^{-t}$ is nonnegative we have $(Kx)(t) = \int_0^t k(t, s)x(s) \, ds$. It is proved in [1] that $\|K\| = 2/\sqrt{e}$.

(A7) We have $b = 1/4$, $b_1 = \rho_1 = \rho_2 = M = 1$, $\beta = 3/8$, $m = 2$ and $\|K\| = 2/\sqrt{e}$. Therefore

$$\gamma = b \rho_1 m^{-1} + b_1 \rho_2 \beta M^{-1} \|K\| = \frac{1}{8} + \frac{3}{4\sqrt{e}} < 1.$$ 

Therefore, Theorem 3.1 ensures that Eq. (1.1) has at least one solution in the space $L^1(\mathbb{R}_+)$. 

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