EXACT CONTROLLABILITY FOR THE LAMÉ SYSTEM

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Abstract. In this article, we prove an exact boundary controllability result for the isotropic elastic wave system in a bounded domain Ω of $\mathbb{R}^3$. This result is obtained under a microlocal condition linking the bicharacteristic paths of the system and the region of the boundary on which the control acts. This condition is to be compared with the so-called Geometric Control Condition by Bardos, Lebeau and Rauch [3]. The proof relies on microlocal tools, namely the propagation of the $C^\infty$ wave front and microlocal defect measures.

1. Introduction. Let $\Omega$ be a bounded open domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We consider the following isotropic elastic wave system

$$
\begin{aligned}
Pu := \partial_t^2 u - \Delta_e u &= 0 &\text{in } \mathbb{R}_+ \times \Omega, \\
\quad u &= 0 &\text{on } \mathbb{R}_+ \times \partial\Omega, \\
\quad u(0,x) &= \varphi_1 \quad \text{and} \quad \partial_t u(0,x) = \varphi_2.
\end{aligned}
$$

(1.1)

Here, $\Delta_e$ denotes the elasticity operator, which is the $3 \times 3$ matrix-valued differential operator defined by

$$
\Delta_e u = \mu \Delta u + (\lambda + \mu) \nabla \text{div} u, \quad u = (u_1, u_2, u_3)^T,
$$

(1.2)

and we assume that the Lamé constants $\lambda$ and $\mu$ satisfy the conditions

$$
\mu > 0, \quad \lambda + \mu > 0.
$$

(1.3)

We define the energy spaces $H_1$ and $H_0$ by

$$
H_1(\Omega) = (L^2(\Omega))^3 \times (L^2(\Omega))^3 \quad \text{and} \quad H_0(\Omega) = (L^2(\Omega))^3 \times (H^{-1}(\Omega))^3,
$$

equipped with their natural norms.

The aim of this paper is to study the boundary exact controllability for system (1.1) under a microlocal condition. Concerning the scalar wave equation, this question has been intensively studied during the last decades. Until the end of 2010 Mathematics Subject Classification. Primary: 93C20, 93B05; Secondary: 74B05, 35L51, 35L05.

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the 80’s, most of the positive results of exact controllability were established under (global) geometrical assumptions, namely the so-called Γ-condition by J.-L. Lions (see [17]), essentially based on a multiplier method. Later on, in [3], Bardos, Lebeau and Rauch established boundary observability inequalities (and as a consequence the boundary controllability) under a microlocal condition (i.e., a property in the cotangent bundle $T^* (\mathbb{R} \times \Omega)$), the so-called geometric control condition (GCC in short), linking the set on which the control is acting and the generalized bicharacteristics of the wave operator. More precisely, an open subset $\Sigma$ of $\partial \Omega$ satisfies the GCC at time $T$ if every generalized bicharacteristic, starting from $\Omega$ at $t = 0$ and traveling at speed of waves, intersects the set $\Sigma$ at a non diffractive point, before the time $T$. In addition, taking into account the work of Burq-Gérard [9], it is now classical that exact controllability (with stability with respect to $(\Sigma, T)$) is equivalent to the GCC. Finally, let us notice that the proofs of all these results are based on microlocal tools, namely, the propagation of wave front sets and microlocal defect measures.

Now, in what concerns the elastic wave system, the situation is somewhat different, and only a few facts are known. First, exact controllability was proved by J.-L. Lions in [17], under the Γ-condition. Then, following [3], Bardos, Masrour and Tatout established in a series of notes [4], [5], [6], the same result under a microlocal condition. We are going to state a controllability result under a geometrical control condition weaker than that of [4], see Remark 4. Our geometric control condition is weaker because our definition of nondiffractive points is less restrictive than in [4], see Definition 2.3. Moreover, here we are going to use the propagation of the $C^\infty$-wave front borrowed from K. Yamamoto [19] for the Lamé system, while in [4], [5], [6], the observability inequality relies on the propagation of the $H^1$-wave front for the solutions of the Lamé system. This result is well established for the wave equation, but, to the best of the authors’ knowledge, it is not proved in the literature for the solutions to the Lamé system. We still refer to Remark 4 for additional comments.

In [10], Burq and Lebeau introduced the microlocal defect measures attached to sequences of solutions of the Lamé system and proved a propagation result when the energy of the longitudinal component goes to zero. These measures were used in [11] to prove the local energy decay for solutions of the Lamé system outside obstacles. In the same spirit, we can also quote the work of Lebeau-Zuazua [16] on the thermoelasticity system, and finally the result of Bellassoued [7] on the Lamé system with Neumann boundary conditions.

The result we present here is to be compared with the works of [3] and [11]. Indeed, considering the notion of bicharacteristic path introduced in [11] (see Definition 2.2), and assuming a well adapted Geometric Control Condition for the Lamé system, GCCL in short (see Definition 2.5), we are in a general geometrical setting very close to the one in [3]. Moreover, thanks to [19], where the author established the propagation of singularities for the Lamé system in presence of a boundary, we have at our disposal all the microlocal machinery to be able to follow a fully microlocal strategy. The last ingredient we need is a first lifting lemma for the wave front in the spirit of [3, Theorem 2.2], and a second one for the microlocal defect measures analogous to the one stated in the scalar case in [8, Lemme 2.2]. These results are proved in Section 3.2.

In the present paper, we have tried to present very detailed proofs, with self
2. Statement of the main result. We first make the following geometrical assumption:

\((H)\) The domain \(\Omega\) has no contact of infinite order with its tangents.

We start this section by some geometrical concepts and properties related to the boundary value problem of elasticity system. Then we recall the definition of a generalized bicharacteristic path, and finally, we state our main theorem. All the material below is borrowed from [11]. We have chosen to follow word by word Section 2 of this paper in order to make the present work self-contained.

First, we notice that the determinant of the principal symbol of the operator \(P = \partial_t^2 - \Delta_e\) is given by (see [20])

\[
p(t, x; \tau, \xi) = (\mu |\xi|^2 - \tau^2) \left( (\lambda + 2\mu)|\xi|^2 - \tau^2 \right).
\]

This leads to two bicharacteristic families in \(\text{Char}^P\), the characteristic set of \(P\), namely those of the symbols

\[
p_L(t, x; \tau, \xi) = c_L^2 |\xi|^2 - \tau^2 \quad \text{and} \quad p_T(t, x; \tau, \xi) = c_T^2 |\xi|^2 - \tau^2,
\]

where \(c_L = \sqrt{\lambda + 2\mu}\), and \(c_T = \sqrt{\mu}\) are the longitudinal and transversal velocities of propagation.

We shall use the notation \(p_{L,T}\) and \(c_{L,T}\) for statements which are true both for longitudinal and transversal waves.

Let us set \(M = \mathbb{R} \times \Omega\). In the interior, i.e. in \(T^*(\mathbb{R} \times \Omega)\), wavefront sets propagate independently along the null bicharacteristics of each one of the two families. However, at the boundary \(\partial M = \mathbb{R} \times \partial \Omega\), one has to consider the inverse images of the characteristic points, in \(\{p = 0\}\), with respect to the projection

\[
j: T^* (\mathbb{R} \times \Omega) |_{\partial M} \longrightarrow T^* (\partial M).
\]

We are going to illustrate what happens at the boundary point \((t, x) \in \partial M\). Let \((\tau, \eta) \neq (0, 0)\) be a tangential direction to \(\partial M\) at \((t, x)\), i.e. \(\eta \cdot n(x) = 0\), \(n(x)\) being the unit normal to \(\partial \Omega\) at \(x\), exterior normal to \(\Omega\). We can consider \((\tau, \eta)\) as an element of \(T^*_{(t, x)}(\partial M)\), and looking for its inverse image in both characteristic sets means looking for \(\alpha_{L,T} \in \mathbb{R}\) such that

\[
p_{L,T}(t, x; \tau, \eta + \alpha_{L,T} n(x)) = 0,
\]

that is

\[
p_{L,T}(t, x; \tau, \eta + \alpha_{L,T} n(x)) = c_{L,T}^2 (|\eta|^2 + \alpha_{L,T}^2) - \tau^2,
\]

with

\[
\alpha_L = \pm \sqrt{\frac{\tau^2}{c_L^2} - |\eta|^2} \quad \text{and} \quad \alpha_T = \pm \sqrt{\frac{\tau^2}{c_T^2} - |\eta|^2}.
\]

Hence, for the existence of such reals \(\alpha_T\) and \(\alpha_L\), one of the two inequalities

\[
r_L := \tau^2 - c_L^2 |\eta|^2 \geq 0 \quad \text{or} \quad r_T := \tau^2 - c_T^2 |\eta|^2 \geq 0
\]

must be fulfilled. From a geometrical point of view, there are five possibilities for a tangential direction \((\tau, \eta) \neq (0, 0)\), with different numbers of inverse images with
respect to the projection (2.1). We can now introduce the transversal characteristic manifold 
$$\text{Char}T = (\text{Char}T)_\Omega \cup (\text{Char}T)_{\partial \Omega},$$
where
$$\begin{align*}
(\text{Char}T)_\Omega &= \{(t, x, \tau, \xi) \mid x \in \Omega, t > 0, c_T^2|\xi|^2 - \tau^2 = 0\}, \\
(\text{Char}T)_{\partial \Omega} &= \{(t, y, \tau, \eta) \mid y \in \partial \Omega, t > 0, r_T \geq 0\},
\end{align*}$$
and the longitudinal characteristic manifold 
$$\text{Char}L = (\text{Char}L)_\Omega \cup (\text{Char}L)_{\partial \Omega},$$
where
$$\begin{align*}
(\text{Char}L)_\Omega &= \{(t, x, \tau, \xi) \mid x \in \Omega, t > 0, c_L^2|\xi|^2 - \tau^2 = 0\}, \\
(\text{Char}L)_{\partial \Omega} &= \{(t, y, \tau, \eta) \mid y \in \partial \Omega, t > 0, r_L \geq 0\}.
\end{align*}$$

The characteristic manifold of the Lamé system is
$$\text{Char}P = (\text{Char}P)_\Omega \cup (\text{Char}P)_{\partial \Omega}.$$
and by the assumption (1.3) on the Lamé moduli ($\mu > 0$, $\lambda + \mu > 0$), one obtains
$$\text{Char}P \subset \text{Char}L \text{ and } (\text{Char}P)_\Omega \subset (\text{Char}T)_\Omega \cup (\text{Char}L)_\Omega.$$

Definition 2.1. Let $\rho = (t, x, \tau, \eta)$ belong to $T^*(\partial M)\setminus 0$. We say that
1. $\rho$ is elliptic (or $\rho \in E$) iff $\rho \notin (\text{Char}P)_{\partial \Omega}.$
2. $\rho$ is hyperbolic for the longitudinal wave (or $\rho \in H_L$) iff $r_L > 0.$
3. $\rho$ is glancing for the longitudinal wave (or $\rho \in G_L$) iff $r_L = 0.$
4. $\rho$ is hyperbolic for the transversal wave (or $\rho \in H_T$) iff $r_T > 0.$
5. $\rho$ is glancing for the transversal wave (or $\rho \in G_T$) iff $r_T = 0.$

We recall that each one of the sets $\text{Char}L$ and $\text{Char}T$ is endowed with a generalized bicharacteristic flow. Now, we give a brief description borrowed from [16] (see also [15, Section 24.3, p.441]) of the flow associated to $\text{Char}T$ (for instance). Under assumption (H), each ray is constituted of segments in $\Omega$ described by
$$\begin{align*}
t(s) &= -2\tau, \\
x(s) &= 2c_T^2\xi,
\end{align*}$$
$\tau$ and $\xi$ being constant since we deal with constant coefficients. Moreover, every segment intersects the boundary $\mathbb{R} \times \partial \Omega$ in one of the following two ways:
- Transversally, i.e. $x(s_0) \in \partial \Omega$ for some $s_0$, $c_T|\eta(s_0)| < \tau(s_0)$, i.e. $r_T(s_0) > 0$.
At these points we have
$$\lim_{s \to s_0^+} \xi(s) = \eta(s_0) - \frac{\sqrt{r_T}}{c_T} n(x(s_0)), \quad \lim_{s \to s_0^-} \xi(s) = \eta(s_0) + \frac{\sqrt{r_T}}{c_T} n(x(s_0)).$$
- Tangentially at a diffractive point, i.e., $x(s_0) \in \partial \Omega$ for some $s_0$, and there exists $\varepsilon > 0$ small enough such that $x(s) \in \Omega$ when $0 < |s - s_0| < \varepsilon$. Moreover, $c_T|\eta(s_0)| = |\tau(s_0)|$, i.e. $r_T(s_0) = 0$.
In addition, these segments may be connected to arcs of curves
$$\begin{align*}
\tilde{t}(s) &= (t(s), x(s), \tau(s), \xi(s)), s \in [a, b], \text{ such that } x(s) \in \partial \Omega, \\
r_T(s) &= 0, \tilde{\tau}(s) = 0, \tilde{t}(s) = -2\tau, \tilde{x}(s) = 2c_T\eta(s) \text{ and } D\eta(s) = 0,
\end{align*}$$
where $D$ denotes the covariant derivative over $\partial \Omega$.

In other words, $s \in [a, b] \rightarrow x(s) \in \partial \Omega$ is a geodesic curve such that $\dot{x}(s) = 2c_T\eta(s)$.
Now, we are going to make a description of a generalized bicharacteristic path associated to the Lamé operator $P$, and we refer to [19] for more details. The generalized bicharacteristic flow lives in $\text{Char}P \subset T^*M$ and, for $\rho \in \text{Char}P$, we denote by $\Gamma(s, \rho)$ the generalized bicharacteristic path starting from $\rho$. Since $\text{Char}P$ is the disjoint union of $(\text{Char}P)_\Omega$, $\mathcal{H}_T$, and $\mathcal{G}_T$, we shall consider separately the
case where $\rho$ belongs to each one of these sets. Moreover all the description below holds for $|s|$ small.

**Case 1.** $\rho \in (\text{Char } P)_{\Omega}$.

Here $\rho = (t, x, \tau, \xi)$ where $x \in \Omega$, $t \in [0, T]$, $p(t, x, \tau, \xi) = 0$. Then for small $|s|$ we have

$$\Gamma(s, \rho) = (t(s), x(s), \tau, \xi) \subset T^*(\mathbb{R} \times \Omega),$$

where $(x(s), \xi)$ is the bicharacteristic starting from the point $(x, \xi)$ of $p_L$ if $\rho \in \text{Char } L_{\Omega}$, and of $p_T$ if $\rho \in \text{Char } T_{\Omega}$.

**Case 2.** $\rho \in (\text{Char } P)_{\partial \Omega}$, i.e. $0 \leq r_T$.

Here $\rho = (t, x, \tau, \eta)$ where $x \in \partial \Omega$, $t \in [0, T]$ and the equation $p(t, x, \tau, \eta + \xi_n) = 0$ has roots $\xi_n = \alpha_{L,T}(x)$ described in (2.3).

For $s > 0$ (resp. $s < 0$), let $\Gamma^+(s, \rho) = (t(s), x^+(s), \tau, \xi^+)$ (resp. $\Gamma^-(s, \rho) = (t(s), x^-(s), \tau, \xi^-)$) be the outgoing (resp. incoming) bicharacteristic of $p$. The generalized bicharacteristic path is such that $\Gamma(0, \rho) = \rho$ and

$$\Gamma(s, \rho) = \begin{cases} \Gamma^+(s, \rho), & 0 < s < \varepsilon, \\ \Gamma^-(s, \rho), & -\varepsilon < s < 0. \end{cases}$$

In the sequel, we describe the four possibilities that may occur. In all the cases 2.1, 2.2.a, 2.2.b, 2.2.c below, we assume that no part of any generalized bicharacteristic stays in the boundary (i.e. $x(s) \notin \partial \Omega$ for $0 < |s| < \varepsilon$). This is the case when the point $\Gamma(0, \rho) = \rho$ is hyperbolic or at least strictly diffractive.

**2.1.** The bicharacteristic $(x^\pm(s), \xi^\pm)$ obeys

$$\begin{cases} x^+(s) = x + 2c^2_L s \xi^+, & 0 < s < \varepsilon, \\ x^-(s) = x + 2c^2_L s \xi^-, & -\varepsilon < s < 0, \end{cases}$$

where $\xi^+ = \eta - \frac{\sqrt{r_T}}{c_T} n(x)$ and $\xi^- = \eta + \frac{\sqrt{r_T}}{c_T} n(x)$.

In particular, as represented in the figure below, if $0 < r_T$, one has $x(s) \in \Omega$ for small $|s| \neq 0$.

![Figure 1.](image-url)

**2.2.** If $0 \leq r_L$ (i.e. $\eta \in G_L \cup \mathcal{H}_L \subset \mathcal{H}_T$), we have the following possibilities

2.2.a.

$$\begin{cases} x^+(s) = x + 2c^2_L s \xi^+, & 0 < s < \varepsilon, \\ x^-(s) = x + 2c^2_L s \xi^-, & -\varepsilon < s < 0, \end{cases}$$
where $\xi^+ = \eta - \frac{\sqrt{r_T}}{c_L} n(x)$ and $\xi^- = \eta + \frac{\sqrt{r_T}}{c_L} n(x)$.

2.2.b.

\[
\begin{align*}
    & \{ x^+(s) = x + 2c_T^2 s \xi^+, \quad 0 < s < \varepsilon, \\
    & x^-(s) = x + 2c_T^2 s \xi^-, \quad -\varepsilon < s < 0,
\end{align*}
\]

where $\xi^+ = \eta - \frac{\sqrt{r_T}}{c_T} n(x)$ and $\xi^- = \eta + \frac{\sqrt{r_T}}{c_T} n(x)$.

2.2.c.

\[
\begin{align*}
    & \{ x^+(s) = x + 2c_T^2 s \xi^+, \quad 0 < s < \varepsilon, \\
    & x^-(s) = x + 2c_T^2 s \xi^-, \quad -\varepsilon < s < 0,
\end{align*}
\]

where $\xi^+ = \eta - \frac{\sqrt{r_T}}{c_T} n(x)$ and $\xi^- = \eta + \frac{\sqrt{r_T}}{c_T} n(x)$.

Remark 1.

1. If $\Gamma(0, \rho) = \rho$ is a glancing point and one half bicharacteristic, say for instance $\gamma_L^+$, stays in the boundary for $s > 0$, then the description above is given by (2.4), where $r_T$ and $c_T$ have to be replaced by $r_L$ and $c_L$. In this case, we have the following figure:

![Figure 2](image_url)

2. If $r_L < 0 \leq r_T$, then only the case 2.1 may occur, and if $0 \leq r_L < r_T$ then the four cases 2.1, 2.2.a, 2.2.b and 2.2.c may occur.

3. In the cases 2.2.b and 2.2.c, the nature of the generalized bicharacteristic path changes when hitting the boundary, since it moves from $(\text{Char} L)_\Omega$ to $(\text{Char} T)_\Omega$ in 2.2.b and conversely, from $(\text{Char} T)_\Omega$ to $(\text{Char} L)_\Omega$ in case 2.2.a.

Definition 2.2. We shall call generalized bicharacteristic path any curve which consists of generalized bicharacteristics of $P$, with possibility of switching from a characteristic manifold to another one, at each point of $\partial M$, in the way described above.

The projection of such a generalized bicharacteristic path on $\Omega$ will be called a generalized geodesic path.

Remark 2. A generalized geodesic path is constituted of segments living in $\Omega$, that intersect the boundary transversally (at hyperbolic points for $p_L$ or $p_T$), or tangentially (at diffractive points). These segments may be connected to arcs of curves, living in $\partial \Omega$, which are projections of glancing rays associated to $p_L$ or $p_T$.

Before introducing the geometric control condition for the Lamé system, we need the following definition. Here we denote by $\pi$ the canonical projection from the cotangent bundle onto the basis.
Definition 2.3. A point \( \rho \in T^*(\partial M) \setminus 0 \) is said to be nondiffractive for the longitudinal wave if \( \rho \in \mathcal{H}_L \) or if \( \rho \in \mathcal{G}_L \), and the free bicharacteristic, \( \exp s H_{pl} \rho \), passes over the complement of \( M \) for arbitrarily small values of \( s \), where \( \rho \) is the unique point in \( (\text{Char} L) \cap j^{-1}(\rho) \) (see (2.1) for the definition of \( j \)).

Similarly, the point \( \rho \) is said to be nondiffractive for the transversal wave if it satisfies the definition above where we replace ‘\( L \)’ by ‘\( T \)’.

In other words, in each case, the projection on the \( (t, x) \)-space of the free bicharacteristic ray, \( \gamma_L \) or \( \gamma_T \), crosses the boundary \( \partial M \) at the point \( \pi(\rho) \).

This is nothing more than the classical definition introduced in the work [3].

Now, we define the nondiffractive points for the Lamé system.

Definition 2.4. The point \( \rho \in T^*(\partial M) \setminus 0 \) is nondiffractive if one of the following properties holds true:
1. \( \rho \in \mathcal{H}_T \cap (\mathcal{H}_L \cup \mathcal{E}_L) \),
2. \( \rho \in \mathcal{G}_L \) and is nondiffractive for the longitudinal wave,
3. \( \rho \in \mathcal{G}_T \) and is nondiffractive for the transversal wave.

Remark 3. Notice that, due to the assumption on the velocities \( (c_L > c_T) \), \( \rho \in \mathcal{H}_T \) in case 2, and \( \rho \in \mathcal{E}_L \) in case 3 of the above definition.

Definition 2.5. Let \( T > 0 \) and \( \Sigma \) be an open subset of the boundary \( \partial \Omega \). We say that \((\Sigma, T)\) satisfies the geometric control condition \((\text{GCCL})\) in brief, if every generalized bicharacteristic path, starting from \( \overline{\Omega} \) at time \( t = 0 \), intersects the region \( \Sigma \) at a nondiffractive point before the time \( T \).

We are now able to state our observability result. For this, we consider the homogeneous adjoint system:

\[
\begin{align*}
\partial_t^2 v - \mu \Delta v - (\lambda + \mu) \nabla \text{div} v &= 0 \quad \text{in } [0, T] \times \Omega, \\
v &= 0 \quad \text{on } [0, T] \times \partial \Omega, \\
(v(0), \partial_t v(0)) &\in H_1(\Omega). 
\end{align*}
\] (2.5)

Proposition 1. Assume that \((\Sigma, T)\) satisfies the geometric control condition \((\text{GCCL})\). Then there exists a constant \( c > 0 \) such that

\[
\|(v(0), \partial_t v(0))\|_{H_1(\Omega)}^2 \leq c \int_0^T \int_{\Sigma} \left| \frac{\partial v}{\partial n} \right|^2 d\sigma dt, \quad (2.6)
\]

for every solution \( v \) of the homogeneous system (2.5).

By the HUM method, this estimate naturally leads to the following exact controllability result.

Theorem 2.6. Assume that \((\Sigma, T)\) satisfies the geometric control condition \((\text{GCCL})\). Then for every initial data \((\varphi_1, \varphi_2) \in H_0(\Omega) = (L^2(\Omega))^3 \times (H^{-1}(\Omega))^3\), there exists \( g \in (L^2([0, T] \times \partial \Omega))^3 \) such that the solution \( u \) of

\[
\begin{align*}
\partial_t^2 u - \Delta_x u &= 0 \quad \text{in } (0, T) \times \Omega, \\
u &= g \quad \text{on } (0, T) \times \partial \Omega, \\
u(0, \cdot) &= \varphi_1 \quad \text{and} \quad \partial_t u(0, \cdot) = \varphi_2 \quad \text{in } \Omega, 
\end{align*}
\] (2.7)
satisfies \( u(T, \cdot) = \partial_t u(T, \cdot) = 0 \).
Remark 4.
1. As pointed out in the introduction, this result has been proved in the book of J.-L Lions [17] under the so-called geometrical Γ-condition, using the multiplier method. Here, we invoke a microlocal condition linking the generalized bicharacteristic paths and the region of the boundary on which the control acts. This condition is to be compared to the so-called geometric control condition introduced by Bardos, Lebeau and Rauch in [3] for boundary observability and control of the scalar wave equation.

2. As we wrote in the introduction, in [6], the authors set up a microlocal approach for the observability/control of the Lamé system in bounded domains, with zero-stress boundary condition. In this case, the situation is slightly different since besides the transversal and longitudinal waves, Rayleigh waves appear along the boundary. However, two remarks have to be addressed here:

- The microlocal condition they introduce is more restrictive than condition $(GCCL)$ we give here. For instance, a generalized bicharacteristic path which hits the boundary at hyperbolic points seems to be excluded from the geometric control condition given [6].

- Their proof is essentially based on the propagation of the $H^1$-wave front set for solutions of the Lamé system, and this fact is not clearly proved in the existing literature. As a matter of fact, the propagation of the $H^1$-wave front set could be established by using the propagation of the $C^\infty$ wave front proved by [19] and the forward $L^2$-wellposedness of this system (see also [3, Theorem 3.3] and [16, Appendix B]). But this is not straightforward and, as far as we know, it is not written in the existing literature.

3. Actually, we chose to prove our result by using microlocal defect measures.

4. With the same tools one can also prove exact controllability with internal control. More precisely, let $\omega$ be an open subdomain of $\Omega$ and $T > 0$. In the spirit of [3] and analogously to the $(GCCL)$ condition stated above, we say that $(\omega,T)$ satisfies the geometric control condition if every generalized geodesic path starting from $\Omega$ at $t = 0$ intersects the set $\omega$ before the time $T$.

Theorem 2.7. Assume that $(\omega,T)$ satisfies the $(GCCL)$. Then for every initial data $(\varphi_1, \varphi_2) \in H^1_0(\Omega) = (H^1_0(\Omega))^3 \times (L^2(\Omega))^3$, there exists $g \in (L^2([0,T] \times \Omega))^3$, with $\text{supp}(g) \subset [0,T] \times \omega$, such that the solution $u$ of

$$
\begin{cases}
\partial_t^2 u - \Delta_x u = g & \text{in } (0,T) \times \Omega, \\
 u = 0 & \text{on } (0,T) \times \partial \Omega, \\
 u(0,\cdot) = \varphi_1 & \text{in } \Omega, \\
 \partial_t u(0,\cdot) = \varphi_2 & \text{in } \Omega,
\end{cases}
$$

satisfies $u(T,\cdot) = \partial_t u(T,\cdot) = 0$.

3. Microlocal analysis. In this section, we introduce the microlocal material we need in the proof of the observability estimate (2.6). Essentially, we first recall the notion of wave front set up to the boundary, and next the notion of microlocal defect measures associated to sequences of solutions of (2.5) with their properties of support localization and propagation. We also give some lifting lemma, inspired from [3] and [8] where the scalar wave equation is considered.

But first of all, we recall the classical ‘direct’ estimate giving the hidden regularity of the solution.
Lemma 3.1. There exists a constant $c > 0$ such that
\[
\int_0^T \int_{\partial\Omega} \left| \frac{\partial v}{\partial n} \right|^2 \, d\sigma dt + \int_0^T (\text{div} \, v)^2 \, d\sigma dt \leq c \|(v(0), \partial_t v(0))\|_{H^1(\Omega)}^2,
\] (3.1)
for every solution $v$ of system (2.5).

Proof. The proof is classical and follows closely the method used in the scalar case. Indeed, if $\sum_{j=1}^3 a_j(x) \partial_{x_j}$ is a smooth vector field defined in a neighborhood of $\overline{\Omega}$, equal to the outward normal derivative on $\partial\Omega$, it suffices to take the inner product of the system by the vector $H(x, \partial_x)v = \sum_{j=1}^3 a_j(x) \partial_{x_j} v$ and integrate by parts. □

Let us now recall some concepts of microlocal analysis and fix some notation. The local and microlocal regularity of a distribution in the free space are well known. If a distribution $v$ is in $C^\infty$ locally near $m \in \overline{M}$ (resp. $H^s$), we will write $v \in C^\infty(m)$ (resp. $v \in H^s(m)$). In what follows, we recall some classical facts near the boundary.

For $m \in \partial M$, it is well known that one can find a local system of geodesic coordinates near $m$, given by $y = (y_0, y_1, y_2, y_3) = (y_0, y')$ such that $M = \{y_0 > 0\}$, $\partial M = \{y_0 = 0\}$ and the wave equation $\Box u = (\partial^2_t - \Delta_{y}) u = 0$ takes locally the following form
\[
\Box u = (\partial^2_{y_0} + R(y, D_{y'})) u = 0.
\] (3.2)
Here $R(y, D_{y'})$ is a second order tangential elliptic differential operator with real principal symbol.

Let us fix $r > 0$ sufficiently small, $\rho \in T^* (\partial M) \backslash \{0\}$, and $v$ an extendable vector-distribution defined in $B_r(\pi(\rho)) \cap M$, solution of $(\partial^2_t - \Delta_{y}) v = 0$. With this notation, we recall the notion of microlocal regularity up to the boundary (see for instance [15], [16]).

Definition 3.2. 1. We say that $\rho \notin WF_b v$ if one can find a tangential pseudodifferential operator $Q(y, D_{y'})$, elliptic at $\rho$, such that $Q(y, D_{y'}) v \in C^\infty(B_r(\pi(\rho)) \cap M)$.

2. For $s \in \mathbb{R}$, we define the Sobolev microlocal regularity by $\rho \notin WF^s_b v$ if there exists a $0$-order tangential pseudodifferential operator $Q(y, D_{y'})$, elliptic at $\rho$, such that $Q(y, D_{y'}) v \in H^s(B_r(\pi(\rho)) \cap M)$.

3.1. Microlocal defect measures. First of all, from [10, Lemme 4.1], it is well known that one can decompose the solution $v$ of system (2.5) into
\[
v = v_L + v_T,
\] (3.3)
where the longitudinal wave $v_L$ and the transversal wave $v_T$ respectively satisfies a wave system
\[
\begin{aligned}
(\partial_t^2 - c_L^2 \Delta) v_L &= 0, & \text{rot } v_L &= 0, \\
(\partial_t^2 - c_T^2 \Delta) v_T &= 0, & \text{div } v_T &= 0,
\end{aligned}
\] (3.4)
with $c_L = \sqrt{\lambda + 2\mu}$, and $c_T = \sqrt{\mu}$. Moreover, if $(v^k)_{k}$ is a bounded sequence of solutions of (2.5) weakly converging to $0$ in $H^1_{\text{loc}}((0, T) \times \Omega)$, the sequences $(v^k_L)$ and $(v^k_T)$ are also of bounded energy and weakly converging to 0 in $H^1_{\text{loc}}((0, T) \times \Omega)$.

In this way, according to [10], we can attach to $(v^k_L)$ (resp. $(v^k_T)$) a microlocal defect measure $\nu_L$ (resp. $\nu_T$). These measures are orthogonal in the sense of measure theory (see [10] or [13, Lemme 3.30]). In addition, $\nu_L$ is supported in the characteristic set $\text{Char} \mathcal{L} = (\text{Char} \mathcal{L})_\Omega \cup (\text{Char} \mathcal{L})_{\partial \Omega} \text{ and } \nu_T$ is supported in $\text{Char} \mathcal{T}$.
This property is known as the elliptic regularity theorem for the microlocal defect measures.

Let us now analyze the propagation properties of the measures $\nu_L$ and $\nu_T$. We will use the notation of section 2. In the interior, i.e. in $T^*(\mathbb{R}_t \times \Omega)$, we are in presence of two waves which propagate independently, so we have at our disposal the classical measure propagation Theorem of P. Gérard [14]. Near the boundary $\partial M$, we have to take into account the nature of the bicharacteristics hitting $\partial M$. Take $\rho$ in $\text{Char} P_{\partial \Omega}$. For $r_{L,T} = r_{L,T}(\rho) \geq 0$, we denote $\gamma_{L,T}^-$ (resp. $\gamma_{L,T}^+$) the longitudinal/transversal incoming (resp. outgoing) bicharacteristic to (resp. from) $\rho$ (these half bicharacteristics do not contain $\rho$).

**Proposition 2.** With the above notation, the following statements are true.

1. If $r_L < 0$, then $\rho$ is an elliptic point for the longitudinal wave. Hence, $\nu_L = 0$ near $\rho$ and
   (i) $\nu_T = 0$ near $\rho$ if $r_T < 0$,
   (ii) The support of $\nu_T$ propagates from $\gamma_T^-$ to $\gamma_T^+$ if $0 \leq r_T$.
2. If $0 < r_L \leq r_T$, then $\rho$ is a hyperbolic point for the longitudinal and the transversal wave. In this case, if $\gamma_{L,T}^- \cap \text{support}(\nu_{L,T}) = \emptyset$, then the support of $\nu_{T,L}$ propagates from $\gamma_{T,L}^-$ to $\gamma_{T,L}^+$.
3. If $0 = r_L < r_T$, then $\rho$ is a glancing point for the longitudinal wave. In this case, if $\gamma_L^- \cap \text{support}(\nu_L) = \emptyset$, then the support of $\nu_T$ propagates from $\gamma_T^-$ to $\gamma_T^+$. 

As a consequence, using the conservation of the total mass (see [10, Proof of Theorem 4]), we obtain:

**Corollary 1.** For $0 \leq r_L$, we have the following equivalence

$$
(\gamma_L^- \cap \text{support}(\nu_L)) \cup (\gamma_T^- \cap \text{support}(\nu_T)) = \emptyset
$$

if and only if

$$
(\gamma_L^+ \cap \text{support}(\nu_L)) \cup (\gamma_T^+ \cap \text{support}(\nu_T)) = \emptyset.
$$

**Remark 5.**

1. If one of the two statements of this equivalence holds true, then we get $\nu_{T,L} = 0$ near $\rho$.
2. This result is the analogue for measures of the $C^\infty$-wave front propagation theorem of K.Yamamoto [20].

**Theorem 3.3.**

$$(\gamma_L^- \cup \gamma_T^-) \cap WF_0(u) = \emptyset \quad \text{if and only if} \quad (\gamma_L^+ \cup \gamma_T^+) \cap WF_0(u) = \emptyset.$$ 

3.2. **Some lifting lemma.** It is well known that one of the key points in the proof of observability inequality in [3] is the lifting lemma [3, Theorem 2.2]. It provides a gain of microlocal regularity for the invisible solutions, i.e. the solutions of (2.5) satisfying

$$
v|_{(0,T) \times \Sigma} = 0, \quad \frac{\partial v}{\partial n}|_{(0,T) \times \Sigma} = 0,
$$

which, in turn, proves that the set of such solutions is reduced to the singleton $\{0\}$. Moreover, in this argument, the authors also make use of the propagation of the Sobolev wave front, and this fact is not clear in our context (see [21]). However, we by-pass this difficulty by setting a lifting lemma in the $C^\infty$ framework, and by using the propagation of the $C^\infty$ wave front established in [19].
Lemma 3.4. Let \( \rho \) be a non diffractive point of \( T^*(\partial M) \setminus 0 \) and \( v \) be a vector distribution defined near \( \pi(\rho) \) in \( M \), solution of
\[
\partial_t^2 v - \Delta_v v \in C^\infty(\pi(\rho)), \quad \left. v \right|_{\partial M} \in C^\infty(\pi(\rho)), \quad \left. \frac{\partial v}{\partial n} \right|_{\partial M} \in C^\infty(\pi(\rho)).
\] (3.5)
Then \( \rho \notin WF_b(v) \).

To prove this result, we decompose the solution \( v \) into \( v = v_L + v_T \), and apply the \( C^\infty \)-lifting Lemma of [2] to each one of the distributions \( \text{div} \ v \) and \( \text{rot} \ v \). Actually, we will prove the more precise following statement which is the local version of [3, Theorem 2.2] adapted to the framework of the Lamé system.

Lemma 3.5. Let \( s \) belong to \( \mathbb{R} \), \( \rho \) be a non diffractive point of \( T^*(\partial M) \setminus 0 \) and let \( v \) be a vector distribution defined near \( \pi(\rho) \) in \( M \), solution of
\[
\partial_t^2 v - \Delta_v v \in C^\infty(\pi(\rho)), \quad \left. v \right|_{\partial M} \in H^s(\pi(\rho)), \quad \left. \frac{\partial v}{\partial n} \right|_{\partial M} \in H^{s-1}(\pi(\rho)).
\] (3.6)
Then \( \rho \notin WF_b^s(v) \).

Proof. First, without loss of generality, we may assume that \( v \in H^s(\pi(\rho)) \) for some \( s' < s - 1 \). Moreover, in the local geodesic coordinate system, the divergence can be expressed as
\[
\text{div} \ v = \sum_j \alpha_j(y) \frac{\partial v_j}{\partial n} + X_j(y, \partial y') v_j,
\] (3.7)
where, for \( j = 1, 2, 3 \), \( X_j(y, \partial y') \) are tangential vector fields and \( \alpha_j \) are smooth functions. Of course, a similar formula holds true for the vector \( \text{rot} \ v \). From (3.7), equation (3.6) and the related boundary data, we deduce that
\[
\left. \frac{\partial^2 v}{\partial n^2} \right|_{\partial M} \in H^{s-2}(\pi(\rho)).
\]
Then taking the divergence and the rotationnal of system (3.6), we easily see that the function \( \text{div} \ v \) and the vector \( \text{rot} \ v \) respectively satisfy
\[
\begin{cases}
(\partial_t^2 - c_L^2 \Delta)(\text{div} \ v) \in C^\infty(\pi(\rho)), \\
\left. \text{div} \ v \right|_{\partial M} \in H^{s-1}(\pi(\rho)), \quad \left. \frac{\partial(\text{div} \ v)}{\partial n} \right|_{\partial M} \in H^{s-2}(\pi(\rho)),
\end{cases}
\] (3.8)
and
\[
\begin{cases}
(\partial_t^2 - c_L^2 \Delta)(\text{rot} \ v) \in C^\infty(\pi(\rho)), \\
\left. \text{rot} \ v \right|_{\partial M} \in H^{s-1}(\pi(\rho)), \quad \left. \frac{\partial(\text{rot} \ v)}{\partial n} \right|_{\partial M} \in H^{s-2}(\pi(\rho)).
\end{cases}
\] (3.9)
Now we use the fact that the point \( \rho \in T^*(\partial M) \setminus 0 \) is non diffractive. First, in each case of Definition 2.4, we can apply the lifting result of [3] to system (3.9). We can also do this with system (3.8) in cases 1 and 2 of Definition 2.4. Finally, the third case is easier since the point \( \rho \) is elliptic for the longitudinal wave. It follows that \( \rho \notin WF_b^{s-1}(\text{div} \ v) \cup WF_b^{s-1}(\text{rot} \ v) \). Therefore, one can find a 0-order tangential pseudodifferential operator \( Q_0(y, D y') \), with symbol \( q_0(y, y') \), supported near \( \rho \), equal to 1 in a small conical neighborhood of \( \rho \), such that
\[
Q_0(y, D y')(\text{div} \ v) \in H^{s-1}(\pi(\rho)) \quad \text{and} \quad Q_0(y, D y')(\text{rot} \ v) \in H^{s-1}(\pi(\rho)).
\] (3.10)
Then, we obtain
\[
\begin{align*}
\text{div}(Q_0(y, D_{y'})v) &= Q_0(y, D_{y'}) (\text{div} v) + [\text{div}, Q_{0}](y, D_{y'}) v \in H^{s'}(\pi(\rho)), \\
\text{rot}(Q_0(y, D_{y'})v) &= Q_0(y, D_{y'}) (\text{rot} v) + [\text{rot}, Q_{0}](y, D_{y'}) v \in H^{s'}(\pi(\rho)).
\end{align*}
\] (3.11)

With these equations and the boundary data in (3.6), we obtain
\[
\begin{align*}
\Delta_y Q_0(y, D_{y'}) v &\in H^{s'-1}(\pi(\rho)), \\
(Q_0(y, D_{y'}) v)_{|\partial M} &\in H^{s}(\pi(\rho)),
\end{align*}
\] (3.12)

which leads to
\[
Q_0(y, D_{y'}) v \in H^{s'+1}(\pi(\rho)),
\] (3.13)

(remind that the operator \((Q_0(y, D_{y'})\) is compactly supported near \(\pi(\rho)\).) And this precisely means that \(\rho / \notin WF_{b}^{s'+1}(v)\). The result then follows by repeating this argument. For convenience, we just detail the first step of the induction. First, we notice that in (3.10) and (3.13), one can replace the 0-order tangential pseudodifferential operator \(Q_0(\cdot, D_{y'})\), by any other 0-order tangential pseudodifferential operator \(Q_1(\cdot, D_{y'})\) with symbol \(q_1(y, \eta')\), such that \(\supp(q_1) \subset \{(y, \eta'), q_0(y, \eta') = 1\}\).

Thanks to (3.10) and (3.13), this operator satisfies
\[
\text{div}(Q_1 v) = Q_1 (\text{div} v) + [\text{div}, Q_1] v \in H^{r}(\pi(\rho)),
\]
with \(r = \min(s - 1, s' + 1)\), and similarly
\[
\text{rot}(Q_1 v) = Q_1 (\text{rot} v) + [\text{rot}, Q_1] v \in H^{r}(\pi(\rho)).
\]
Hence, we obtain \(Q_1 v \in H^{r'}(\pi(\rho))\), with \(r' = \min(s, s' + 2)\).

\[\square\]

**Remark 6.** Notice that this lifting lemma can be proved under microlocal assumptions, as in [3, Theorem 2.2]: \(\rho / \notin WF^{s}(v_{|\partial M})\) and \(\rho / \notin WF^{s-1}(\partial v / \partial M)\).

Now we come to the lifting lemma for the microlocal defect measures.

**Lemma 3.6.** Let \(\rho / \) be a non diffractive point of \(T^{*}(\partial M)\backslash 0\), and \((v^k)\) be a sequence of (vector) functions weakly converging to 0 in \((H^1(0, T[\times \Omega]))^3\), and satisfying
\[
\begin{align*}
\frac{\partial^2 v^k}{\partial t^2} - \Delta_v v^k &= 0 \quad \text{in } \{0, T[\times \Omega), \\
v^k_{|\partial M} &= 0 \quad \text{on } \{0, T[\times \partial \Omega, \\
\frac{\partial v^k}{\partial n}_{|\partial M} &\to 0 \quad \text{in } L^2(\pi(\rho)).
\end{align*}
\] (3.14)

In addition, let \(v^k = v_{L,T}^k + v_{L,T}^T\) be its decomposition into longitudinal and transversal waves. Then if \(\mu_{L,T}\) is a microlocal defect measure attached to the sequence \((v_{L,T}^k)\), one has \(\rho / \notin \supp(\mu_{L,T})\).

**Remark 7.** In this framework, the notion of support of the microlocal defect measure has to be understood in the sense of a “wave front set”. In particular, the statement “\(\rho / \notin \supp(\mu_{L,T})\)” means that there exists a 0-order tangential pseudodifferential operator \(Q(y, D_{y'})\), defined near \(\rho / \) and elliptic at this point, such that
\[
Q(y, D_{y'}) v_{L,T}^k \to 0 \text{ in } H^1(\pi(\rho)).
\] (3.15)
Proof. Proof of Lemma 3.6. Here we apply the strategy of the previous proof. We establish that
\begin{equation}
\begin{cases}
Q(y, D_y) \text{div } v^k \to 0 \text{ in } L^2(\pi(\rho)), \\
Q(y, D_y) \text{rot } v^k \to 0 \text{ in } L^2(\pi(\rho)),
\end{cases}
\end{equation}
where \(Q(y, D_y)\) is as above, and the result will follow. We recall that the longitudinal and the transversal parts of \(v^k\), \(v^k_L\) and \(v^k_T\) satisfy
\begin{equation}
\begin{cases}
\text{div } v^k_L = \text{div } v^k \quad \text{and} \quad \text{rot } v^k_L = 0, \\
\text{rot } v^k_T = \text{rot } v^k \quad \text{and} \quad \text{div } v^k_T = 0.
\end{cases}
\end{equation}

We only examine the sequence \((\text{div } v^k_L)\), the proof for \((\text{rot } v^k_T)\) will be similar. For that we set \(h^k = \text{div } v^k_L\). Hence \(h^k\) weakly converges to 0 in \(L^2([0, T] \times \Omega)\), and (3.14) yields
\begin{equation}
\begin{cases}
\partial_t^2 h^k - c_1^2 \Delta h^k = 0 \quad \text{in } [0, T] \times \Omega, \\
h^k|_{\partial M} \to 0 \quad \text{in } L^2(\pi(\rho)), \\
\partial h^k/\partial n|_{\partial M} \to 0 \quad \text{in } H^{-1}(\pi(\rho)).
\end{cases}
\end{equation}

The goal is then to prove that
\begin{equation}
Q(y, D_y) h^k \to 0 \text{ in } L^2(\pi(\rho)),
\end{equation}
for some 0-order tangential pseudodifferential operator \(Q(y, D_y)\) defined near \(\rho\) and elliptic at this point. Actually, this is the lifting lemma for the measures proved by [8] or [1] at the \(H^1\)-level. Therefore, everything we need to do here is to check that the same argument runs at the \(L^2\)-level. First of all, without loss of generality, we may slightly precise the assumptions and assume that the point \(\rho\) is glancing non-diffusive (in particular, the set \(j^{-1}(\rho)\) is reduced to a singleton). Indeed, near hyperbolic points, the situation is easier to treat thanks to [18] or [10, Appendix A.2]. Now, we follow word by word the argument of [8]. Let us denote by \(\tilde{h}^k\) the extension of \(h^k\) by 0 outside \(\Omega\). We have
\begin{equation}
(\partial_t^2 - c_1^2 \Delta) \tilde{h}^k = \frac{\partial h^k}{\partial n}|_{\partial M} \otimes \delta|_{\partial M} - h^k|_{\partial M} \otimes \delta'|_{\partial M} = R_k
\end{equation}
and one gets
\begin{equation}
R_k \to 0 \text{ in } H^{-3/2-\varepsilon} (V_{\pi(\rho)}) \quad \text{for all } \varepsilon > 0,
\end{equation}
where \(V_{\pi(\rho)}\) is a small neighborhood of \(\pi(\rho)\) in \(\mathbb{R}^{1+3}\). Let \(C(y, D_y)\) be a 0-order compactly supported pseudodifferential operator with symbol \(c = c(y, \eta)\) supported near \(\tilde{\rho} = j^{-1}(\rho)\), and denote by \(p\) the symbol of the operator \(\partial_t^2 - c_1^2 \Delta\) (in the \((y, \eta)\) coordinate system). Let \(\tilde{a}\) be the solution of
\begin{equation}
\frac{1}{i} \{p, \tilde{a}\} = |c|^2,
\end{equation}
supported near the half-bicharacteristic (in \(T^*\mathbb{R}^{1+3}\)) issued from the (glancing non-diffusive) point \(\rho\), and leaving \(\Omega\). Obviously \(\tilde{a}\) is of order \((-1)\) and purely imaginary. Furthermore, one can find \(\Psi \in C_0^\infty(\mathbb{R}^{1+3})\) such that, with \(a = \Psi \tilde{a} \Psi\), the symbol \(\frac{1}{i} \{p, a\} - |c|^2\) is supported in \(\{y_0 < -\varepsilon\}^c\) for some \(\varepsilon > 0\). Then, let \(A\) be a compactly supported pseudodifferential operator of order \((-1)\) with principal symbol equal to \(a\). The symbolic calculus gives
\begin{equation}
(\partial_t^2 - c_1^2 \Delta) A - A (\partial_t^2 - c_1^2 \Delta) = C^* C + T_{-1} + T_0,
\end{equation}
where $T_j$ is of order $-j$ and $T_0$ is supported in $\{y_0 < -\varepsilon\}$. As we claimed from the beginning, once we have constructed these various pseudodifferential symbols, the argument is similar to that of [8], the only difference concerns the orders of the operators $C$ and $A$ which are now respectively 0 and $(-1)$. 

The Malgrange theorem provides the following division formula

$$ A = A_{-1} + A_{-2}\partial_n + B_{-3}(\partial_l^2 - c_l^2\Delta) + B_{-2} \tag{3.23} $$

where $A_{-j}$ is a tangential pseudodifferential operator of order $(-j)$ and $B_{-j}$ is a pseudodifferential operator of order $(-j)$. Taking into account (3.19), (3.22), and the fact that $h^k \Delta$ is bounded in $L^2$, we have

$$ \left\|Ch^k\right\|^2_{L^2} = 2Re((\partial_l^2 - c_l^2\Delta)h^k, A_k^h)_{L^2} - ((\partial_l^2 - c_l^2\Delta)h^k, (A + A^*)h^k)_{L^2} + \alpha_k, $$

where $\alpha_k$ denotes any sequence satisfying

$$ \alpha_k \to 0 \text{ as } k \to \infty. $$

Now, $A + A^*$ is of order $(-2)$ since the principal symbol of $A$ is pure imaginary. Therefore,

$$ ((\partial_l^2 - c_l^2\Delta)h^k, (A + A^*)h^k)_{L^2} = (R_k, (A + A^*)h^k)_{L^2} \to 0, $$

in view of (3.20). So

$$ \left\|Ch^k\right\|^2_{L^2} = 2Re(R_k, (A_{-1} + A_{-2}\partial_n)h^k)_{L^2} + 2Re(R_k, B_{-3}R_k + B_{-2}h^k)_{L^2} + \alpha_k \tag{3.24} $$

Furthermore, we notice that the symbol $b_{-j}$ of $B_{-j}$ is supported in a conical neighborhood of $\tilde{\rho} = j^{-1}(\rho)$, i.e. in some set of the form $\{|b_0| < c|y'|\}$. This allows us to drop the $\varepsilon$ in (3.20). Consequently, the second term of the RHS in (3.24) goes to zero as $k \to \infty$. Let us focus on the first term. We have

$$ (R_k, (A_{-1} + A_{-2}\partial_n)h^k)_{L^2} = (\partial_n h^k_{\partial M}, A_{-1}(h^k)_{\partial M})_{L^2} + (\partial_n h^k_{\partial M}, A_{-2}(\partial_n h^k)_{\partial M})_{L^2} $$

$$ - (h^k_{\partial M}, A_{-1}(\partial_n h^k)_{\partial M})_{L^2} + (h^k_{\partial M}, A_{-2}(\partial_n h^k)_{\partial M})_{L^2} + \alpha_k. $$

Here, it is easy to see that the first three terms of the RHS of the above identity go to zero, since the supports of the symbols of the tangential operators $A_{-j}$ intersect the boundary $\partial M$ on a small neighborhood of $\pi(\rho)$ where the assumptions on the traces given by (3.17) are fulfilled. As for the last term, it suffices to use the equation represented in the local geodesic coordinates (3.2). Consequently, we obtain the limit

$$ C(y, D_y)h^k \to 0 \text{ in } L^2(0, T]\times \mathbb{R}^3). \tag{3.25} $$

Now we finish the proof by establishing (3.18). To this end, we consider a 0-order pseudodifferential partition of unity near $\pi(\rho)$:

$$ h^k = \sum_j \chi_j(y, D_y)h^k + \alpha_k. \tag{3.26} $$

If the essential support of $\chi_j(y, \eta)$ does not intersect $\text{Char} \mathcal{L}$, the microlocal elliptic regularity for the microlocal defect measures tells us that

$$ \chi_j(y, D_y)h^k \to 0 \text{ in } L^2(0, T]\times \mathbb{R}^3), \tag{3.27} $$
and consequently,
\[
Q(y, D_{y'}) \chi_j(y, D_y) h^k \to 0 \text{ in } L^2([0, T] \times \mathbb{R}^3)
\]  
(3.28)
for any 0-order tangential pseudodifferential operator \(Q(y, D_{y'})\). On the other hand, since the point \(\rho\) is glancing, we can assume that the essential support of the other \(\chi_j\) does not intersect the conormal manifold of \(\partial M\). Hence, \(Q(y, D_{y'}) \chi_j(y, D_y) \in OpS^0(\mathbb{R}^3 \times \mathbb{R}^4)\) that is a pseudodifferential operator in \(y\) with essential support contained in \((j^*)^{-1}(\text{ess supp } (Q))\) (see [15, Theorem 18.1.35]). Choosing \(Q\) with support in a sufficiently small neighborhood of \(\rho\), the points of ess supp \(Q \chi_j \cap \text{Char} \mathcal{L}\) will lie in the elliptic set of \(C\). Then \((y, D_y)h^k = \alpha_k\) together with (3.28) yields
\[
Q(y, D_{y'}) \chi_j h^k \to 0 \text{ in } L^2([0, T] \times \mathbb{R}^3).
\]  
(3.29)
Then, summing on \(j\), we obtain (3.18), i.e.:
\[
Q(y, D_{y'}) v^k \to 0 \text{ in } L^2(\pi(\rho)),
\]
since \(v = 0\) outside \(\Omega\). Now, to finish the proof, we argue as in (3.11) and (3.12).
\[
\begin{align*}
\text{div}(Q(y, D_{y'}) v^k_L) &= Q(y, D_{y'}) (\text{div } v^k) + |\text{div } Q(y, D_{y'}) v^k_L| \to 0 \text{ in } L^2(\pi(\rho)), \\
\text{rot}(Q(y, D_{y'}) v^k_L) &= |\text{rot } Q(y, D_{y'}) v^k_L| \to 0 \text{ in } L^2(\pi(\rho)).
\end{align*}
\]  
(3.30)
In addition, we remind once again that the pseudodifferential operator \(Q_0(y, D_{y'})\) is compactly supported near the point \(\pi(\rho) \in \partial M\). Consequently, using the equation and the convergence of boundary data, we obtain
\[
\begin{align*}
\Delta_y Q(y, D_{y'}) v^k_L &\to 0 \text{ in } H^{-1}(\pi(\rho)), \\
(Q(y, D_{y'}) v^k_L)|_{\partial M} &\to 0 \text{ in } L^2(\pi(\rho)),
\end{align*}
\]  
(3.31)
which gives
\[
Q(y, D_{y'}) v^k_L \to 0 \text{ in } H^1(B_r(\pi(\rho)));
\]  
(3.32)
by integration by parts. \(\square\)

4. Proof of Theorem 2.6. In this section, we prove the observability estimate (2.6).

Proof. We argue by contradiction and we consider a sequence of solutions \((v^k)_{k \geq 1}\) of (2.5) satisfying
\[
\begin{align*}
\int_0^T \int_\Sigma \left| \frac{\partial v^k}{\partial n} \right|^2 \leq 1/k, \\
\|(v^k(0), \partial_t v^k(0))\|_{H^1(\Omega)} = 1.
\end{align*}
\]  
(4.1)
Extracting eventually a suitable subsequence, we may assume that there exists a function \(v \in L^\infty(0, T; (H^1_0(\Omega))^3) \cap (H^1((0, T) \times \Omega))^3\) such that
\[
v_k \rightharpoonup v \text{ weakly in } H_1((0, T) \times \Omega).
\]
This weak limit \(v\) satisfies
\[
\begin{align*}
\partial_t v - \mu \Delta v - (\mu + \lambda) \nabla \text{div } v &= 0 \text{ in } (0, T) \times \Omega, \\
\text{div } v &= \frac{\partial v}{\partial n} = 0 \text{ on } (0, T) \times \Sigma, \\
v &= 0 \text{ on } (0, T) \times \partial \Omega.
\end{align*}
\]  
(4.2)
Proposition 3. The unique solution of system (4.2) above is the trivial one \( v = 0 \).

Proof. The main idea of the proof is given in [3], but to be complete, we briefly recall it. We identify the function \( v \) solution of system (4.2) with its initial data \( \varphi \in H_1(\Omega) \), and we consider the space \( G = \{ \varphi \in H_1(\Omega) \mid \text{solution of (4.2)} \} \). First, thanks to the lifting Lemma 3.4, every non diffractive point \( \rho \in T^*((0, T) \times \Sigma) \) is not in \( WF_b(v) \). Therefore, according to the geometric control condition and the result of [19] on propagation of singularities, \( G \) is constituted of smooth functions. Moreover, in view of the direct estimate (3.1), \( G \) is obviously closed in \( H_1 \) and we deduce that it is of finite dimension.

To prove the proposition, we argue by contradiction. We assume that system (4.2) admits a solution \( v \neq 0 \). Since \( \partial_t \) operates on \( G \), it admits an eigenvalue \( \lambda \), and there exists a non-zero vector-function \( v_0(x) \) on \( \Omega \), such that \( \Delta_x v_0 = \lambda^2 v_0, \partial v_0 / \partial n = 0 \) on \( \Sigma, v_0 = 0 \) on \( \partial \Omega \). Then, extending \( v_0 \) by 0 outside \( \Omega \), in a small neighborhood of \( \Sigma \), by unique continuation property of \( \Delta_x \), see for instance [12], we have \( v_0 = 0 \) which gives a contradiction. The proof of the proposition is complete. \( \square \)

Now, we continue the proof of the observability inequality (2.6). The weak limit of the sequence \( (v^k)_k \) being 0, we consider the decomposition (3.3), \( v^k = v^k_L + v^k_T \), and we analyze the microlocal defect measures \( \mu_L, T \) respectively attached to each one of these sequences. Let \( q \in T^*_\Sigma((0, T) \times \Omega) \) be such that \( q \in \text{support}(\nu_L) \cup \text{support}(\nu_T) \).

Two possibilities may occur:

(a) \( q \in T^*((0, T) \times \Omega) \), i.e. \( q \) is an interior point.
(b) \( q \in T^*((0, T) \times \partial \Omega) \), \( q \) is on the boundary.

Step 1. Construction of the forward bicharacteristic.

Case (a). Here one can find a bicharacteristic \( (\gamma_L \text{ or } \gamma_T) \) issued from \( q \), traced forward in time and contained in the support of the associated measure (i.e. \( \gamma_L \subset \text{support}(\nu_L) \text{ or } \gamma_T \subset \text{support}(\nu_T) \)). If this ray intersects the boundary before the time \( t = T \), then we are in case (b).

Case (b). This case is divided into three sub-cases. The point \( q \) may be a hyperbolic, elliptic or glancing point for the longitudinal wave.

If \( q \) is a hyperbolic point for the longitudinal wave, i.e. if \( r_L(q) > 0 \), then it is also hyperbolic for the transversal wave, and thanks to the propagation theorem and its corollary, we see that at least one of the two outgoing bicharacteristics at \( q \) is locally contained the support of the associated measure (i.e. \( \gamma_L^+ \subset \text{support}(\nu_L) \text{ or } \gamma_T^+ \subset \text{support}(\nu_T) \)). Thus, by considering any point of the outgoing charged bicharacteristic \( q' \in \gamma_L^+ \text{ or } q' \in \gamma_T^+ \), we are reduced to the case (a) above.

If \( q \) is an elliptic point for the longitudinal wave, i.e. if \( r_L(q) < 0 \), then it is hyperbolic or glancing for the transversal wave, and we know that the measure is carried by the transversal component and its support propagates from \( \gamma_T^- \) to \( \gamma_T^+ \). In particular, if \( r_T(q) > 0 \), we are in the previous sub-case. Moreover, if \( r_T(q) = 0 \), notice that as long as \( \gamma_T^+ \) lies in the boundary \( (0, T) \times \partial \Omega \), it is contained in \( \text{support}(\nu_T) \). In addition, if this ray leaves the boundary at some point \( q_1 = \gamma_T^+(s_1) \), we know that the outgoing internal bicharacteristic \( \gamma_T^+(s > s_1) \) issued from \( q_1 \) is charged, since \( q_1 \) is still elliptic for the longitudinal wave. Hence, we are again in the situation of case (a) above.

Finally, we consider the case where \( q \) is a glancing point for the longitudinal wave. Here, we notice that as long as the outgoing ray \( \gamma_L^+ \) issued from \( q \) is glancing, we have hyperbolic points for the transversal wave (see Figure 2). In addition, setting
\[ \gamma = \gamma_L(s_1), \] and assuming that for some \( s_2 > s_1 \) the arc of curve \([\gamma_L(s_1), \gamma_L(s_2)]\) lies in the boundary, if \( \gamma_L^+ \) leaves the support of the measure \( \nu_L \) at some point \( q = \gamma_L(s) \), \( s_1 < s \leq s_2 \), say \( q' = \gamma_L(s) \notin \text{support}(\nu_L) \), we know that the hyperbolic internal outgoing ray \( \gamma_L^+ \) issued from \( q' \) is charged, i.e. \( \gamma_L^+ \in \text{support}(\nu_L) \).

Summarizing the above argument, we conclude that we can construct a forward bicharacteristic path \( \Gamma_1 \) issued from \( q \) (the union of all these successive rays \( \gamma_L \) or \( \gamma_T \) charged by the measure \( \nu_L \) or \( \nu_T \)) contained in \( \text{support}(\nu_L) \cup \text{support}(\nu_T) \).

**Step 2. Construction of the backward bicharacteristic.** We achieve the same type of construction, starting from \( q \), to obtain a backward bicharacteristic path \( \Gamma_2 \), and we set \( \{q_0\} = \Gamma_2 \cap \{t = 0\} \).

**Step 3. Use of GCCCL.** The bicharacteristic path \( \Gamma = \Gamma_1 \cup \Gamma_2 \) starting from \( q_0 \) and traced forward in time is charged, i.e. it is completely included in \( \text{support}(\nu_L) \cup \text{support}(\nu_T) \). But according to the geometric control condition, it intersects the subset \( \Sigma \) of the boundary before the time \( T \), at a nondiffractive point \( \rho \). And this is in contradiction with (4.1), since the lifting Lemma 3.6 asserts that, in this situation, \( \rho \notin \text{supp}(\nu_L, T) \). Finally, we get the strong convergence \( v^k \to 0 \) in \( (H^1((0, T) \times \Omega))^3 \), and this contradicts the fact that \( \|v^k(0)\|_{H^1_0}^2 + \|\partial_t v^k(0)\|_{L^2}^2 = 1 \). The proof is complete.

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