Supplemental Materials for:
“The Likelihood Ratio Test in High-Dimensional Logistic Regression Is Asymptotically a Rescaled Chi-Square”

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Abstract
This document presents the proof of Lemma 6(ii) given in the paper [1]: “The Likelihood Ratio Test in High-Dimensional Logistic Regression Is Asymptotically a Rescaled Chi-Square”.

1 Proof of Lemma 6(ii)
We shall prove that \( V(\tau^2) < \tau^2 \) whenever \( \tau^2 \) is sufficiently large. Before proceeding, we recall from the main text and [2, Proposition 6.4] that

\[
V(\tau^2) := \frac{1}{\kappa} \mathbb{E} \left[ \Psi^2(\tau Z; b(\tau)) \right] = \frac{1}{\kappa} \mathbb{E} \left[ \left( b(\tau) \rho' \left( \text{prox}_{b(\tau)}(\tau Z) \right) \right)^2 \right],
\]

(1)

where \( b(\tau) \) obeys

\[
\kappa = \mathbb{E} \left[ \Psi'(\tau Z; b(\tau)) \right] = 1 - \mathbb{E} \left[ \frac{1}{1 + b(\tau) \rho'' \left( \text{prox}_{b(\tau)}(\tau Z) \right)} \right].
\]

(2)

In what follows, we study the logistic and probit models separately.

1.1 The logistic case
Consider the bivariate functions

\[
h(h, \tau) : = \mathbb{E} \left[ \frac{1}{1 + b h'' \left( \text{prox}_{h \rho} (\tau Z) \right)} \right],
\]

\[
w(b, \tau) = \mathbb{E} \left[ \left( \rho' \left( \text{prox}_{h \rho} (\tau Z) \right) \right)^2 \right],
\]

which plays a central role in [1] and [2]. In the sequel, we will first analyze these two functions for any \( b \) obeying

\[
b = c_0 \tau
\]

(3)

for some constant \( c_0 > 0 \). The result is this:

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Lemma 1. For any constant $c_0 > 0$, one has

$$\lim_{\tau \to \infty} h(c_0 \tau, \tau) = \mathbb{P} \{ Z < 0 \text{ or } Z > c_0 \};$$

$$\lim_{\tau \to \infty} w(c_0 \tau, \tau) = \mathbb{P} \{ Z > c_0 \} + \frac{1}{c_0^2} \mathbb{E} \left[ Z^2 \mathbf{1}_{\{0 < Z < c_0\}} \right].$$

Recall that $0 < \kappa < 1/2$. One can easily find two constants $c_0 > \tilde{c}_0 > 0$ such that

$$\mathbb{P} \{ Z < 0 \text{ or } Z > c_0 \} < 1 - \kappa < \mathbb{P} \{ Z < 0 \text{ or } Z > \tilde{c}_0 \}.$$

In view of Lemma 1, for any sufficiently large $\tau > 0$ one has

$$h(c_0 \tau, \tau) < 1 - \kappa = h(b(\tau), \tau) < h(\tilde{c}_0 \tau, \tau).$$

According to Lemma 5], $h(b, \tau)$ is a monotonic function in $b$ for any given $\tau > 0$, thus implying that

$$b(\tau) \in [\tilde{c}_0 \tau, c_0 \tau];$$

that said, $b(\tau)$ scales linearly in $\tau$ as $\tau \to \infty$. Furthermore, since $b(\tau)$ is the solution to $h(b, \tau) = 1 - \kappa$, one has

$$\lim_{\tau \to \infty} \mathbb{P} \left\{ Z < 0 \text{ or } Z > \frac{b(\tau)}{\tau} \right\} = 1 - \kappa,$$

which leads to the closed-form expression

$$\lim_{\tau \to \infty} \frac{b(\tau)}{\tau} = \Phi^{-1}(\kappa + 0.5).$$

We are now ready to characterize the variance map. Note that when $\tau$ is sufficiently large,

$$\frac{\mathcal{V}(\tau^2)}{\tau^2} = \frac{b^2(\tau)}{\tau^2} \cdot \frac{\mathbb{E} \left[ h' \left( \text{prox}_{b(\tau)\rho} (\tau Z) \right) \right]^2}{1 - \mathbb{E} \left[ \frac{1}{1+\nu(\tau)\rho(\text{prox}_{b(\tau)\rho} (\tau Z))} \right]}$$

$$= (1 + o(1)) \frac{b^2(\tau)}{\tau^2} \left\{ \mathbb{P} \left\{ Z > \frac{b(\tau)}{\tau} \right\} + \mathbb{E} \left[ \tau^2 \mathbb{P} \left\{ Z > \frac{b(\tau)}{\tau} \right\} \right] \right\}$$

$$= (1 + o(1)) \frac{\mathbb{E} \left[ Z^2 \mathbf{1}_{\{0 < Z < \frac{b(\tau)}{\tau}\}} \right]}{\mathbb{P} \left\{ 0 < Z < \frac{b(\tau)}{\tau} \right\}}.$$

This together with the expression of $\frac{b(\tau)}{\tau}$ in \cite{1} gives

$$\lim_{\tau \to \infty} \frac{\mathcal{V}(\tau^2)}{\tau^2} = \frac{\mathbb{E} \left[ Z^2 \mathbf{1}_{\{0 < Z < \frac{b(\tau)}{\tau}\}} \right]}{\mathbb{P} \left\{ 0 < Z < \frac{b(\tau)}{\tau} \right\}}_{x = \Phi^{-1}(\kappa + 0.5)}.$$

In order to prove that $\mathcal{V}(\tau^2) \leq \tau^2$ for large $\tau$, it suffices to show that the function

$$g(x) := x^2 \mathbb{P} \{ Z > x \} + \mathbb{E} \left[ Z^2 \mathbf{1}_{\{0 < Z < x\}} \right] - \mathbb{P} \left\{ 0 < Z < x \right\}$$

obeys $g(x) < 0$ for all $x > 0$. To this end, some algebra gives

$$g(x) = x^2 \int_x^\infty \phi(z) \, dz + \int_0^x z^2 \phi(z) \, dz - \int_0^x \phi(z) \, dz$$

$$= x^2 \int_x^\infty \phi(z) \, dz - x \phi(x) \bigg|_0^\infty + \int_0^x \phi(z) \, dz - \int_0^x \phi(z) \, dz$$

$$= x \left( x \int_x^\infty \phi(z) \, dz - \phi(x) \right) < 0,$$
where (10) comes from integration by parts, and the last inequality follows from \( \int_{x}^{\infty} \phi(z) \, dz < \frac{1}{x} \phi(x) \). This establishes that \( \mathcal{V}(\tau^2) \leq \tau^2 \) for any sufficiently large \( \tau > 0 \).

Finally, we prove Lemma 1

**Proof of Lemma 1** Take \( \varepsilon > 0 \) to be an arbitrarily small constant. We study \( \frac{1}{1+b \rho''(\prox_{b \rho}(\tau Z))} \) and \((\rho' (\prox_{b \rho}(\tau Z)))^2\) in three separate cases.

- **Case 1:** \( Z \leq -\varepsilon \). Recall that \( \prox_{b \rho}(\tau Z) \) is the solution to

\[
\frac{b}{e^t} \frac{e^t}{1+e^t} + t = \tau Z, \tag{11}
\]

which implies that

\[
\prox_{b \rho}(\tau Z) = \tau Z - b \frac{e^t}{1+e^t} \bigg|_{t=\prox_{b \rho}(\tau Z)} < \tau Z \leq -\varepsilon. \tag{12}
\]

When \( \tau \to \infty \), this yields

\[
0 \leq b \rho''(\prox_{b \rho}(\tau Z)) = b \frac{e^t}{(1+e^t)^2} \bigg|_{t=\prox_{b \rho}(\tau Z)} \leq b e^t \bigg|_{t=\prox_{b \rho}(\tau Z)} \leq c_0 \varepsilon e^{-\varepsilon\tau} \to 0,
\]

or equivalently,

\[
1 - \frac{1}{1+b \rho''(\prox_{b \rho}(\tau Z))} \to 0 \quad \text{as} \quad \tau \to \infty.
\]

Similarly, one can derive

\[
(\rho' (\prox_{b \rho}(\tau Z)))^2 = \frac{e^{2t}}{(1+e^t)^2} \bigg|_{t=\prox_{b \rho}(\tau Z)} \leq e^{2\prox_{b \rho}(\tau Z)} \leq e^{-2\varepsilon\tau} \to 0,
\]

where (a) follows from (12).

- **Case 2:** \( Z \geq \frac{b}{\tau} + \varepsilon \). In this case, it holds that

\[
\prox_{b \rho}(\tau Z) = \tau Z - b \frac{e^t}{1+e^t} \bigg|_{t=\prox_{b \rho}(\tau Z)} \geq \tau \left( \frac{b}{\tau} + \varepsilon \right) - b = \varepsilon \tau.
\]

Applying a similar argument as in the previous case, we see that as \( \tau \to \infty \),

\[
1 - \frac{1}{1+b \rho''(\prox_{b \rho}(\tau Z))} \to 0 \quad \text{and} \quad (\rho' (\prox_{b \rho}(\tau Z)))^2 \to 1.
\]

- **Case 3:** \( \varepsilon < Z < \frac{b}{\tau} - \varepsilon \). We can first rule out the possibility of \( |\prox_{b \rho}(\tau Z)| \geq \tau \). In fact, if \( |\prox_{b \rho}(\tau Z)| \geq \tau \) and \( \prox_{b \rho}(\tau Z) \geq 0 \), then

\[
\frac{b}{e^t} \bigg|_{t=\prox_{b \rho}(\tau Z)} + \prox_{b \rho}(\tau Z) \geq \frac{b}{1+e^{\prox_{b \rho}(\tau Z)}} \geq \frac{b}{1+e^{\prox_{b \rho}(\tau Z)}} \geq \frac{b}{1+e^{\prox_{b \rho}(\tau Z)}} = b - \frac{b}{1+e^{\prox_{b \rho}(\tau Z)}} \geq \frac{b}{1+e^{\prox_{b \rho}(\tau Z)}} \geq \frac{b - c_0 \tau}{e^{\Theta(\tau)}} \geq b - \varepsilon \tau > \tau Z,
\]

where (b) follows from the assumptions \( b_0 = c \tau \) and \( |\prox_{b \rho}(\tau Z)| \geq \tau \), and (c) holds when \( \tau \) is sufficiently large. This violates the identity (11). Similarly, if \( |\prox_{b \rho}(\tau Z)| \geq \tau \) and \( \prox_{b \rho}(\tau Z) < 0 \), then

\[
\frac{b}{e^t} \bigg|_{t=\prox_{b \rho}(\tau Z)} + \prox_{b \rho}(\tau Z) < b \frac{e^{\prox_{b \rho}(\tau Z)}}{1+e^{\prox_{b \rho}(\tau Z)}} = c_0 \tau e^{-|\prox_{b \rho}(\tau Z)|} \leq \varepsilon \tau \leq \tau Z,
\]

(d)
where (d) follows when \( \tau \) is sufficiently large. This inequality contradicts (11) as well. As a result, we reach
\[
|\text{prox}_{b\rho}(\tau Z)| = o(\tau)
\]
in this case, which combined with (11) gives
\[
\frac{b e^t}{1 + e^t} \bigg|_{t = \text{prox}_{b\rho}(\tau Z)} = (1 + o(1)) \tau Z. \tag{13}
\]
Additionally, (13) leads to
\[
\frac{1}{1 + b\rho''(\text{prox}_{b\rho}(\tau Z))} = (1 + o(1)) \left(1 - \frac{\tau Z}{b}\right), \tag{14}
\]
which is bounded away from 0 in this case. Taken together, (13) and (14) yield
\[
\frac{1}{1 + b\rho''(\text{prox}_{b\rho}(\tau Z))} = \frac{1}{1 + (1 + o(1)) \tau Z (1 - \frac{\tau Z}{b})} \to 0
\]
and
\[
(\rho'(\text{prox}_{b\rho}(\tau Z)))^2 = \left(\frac{e^t}{1 + e^t}\right)^2 \bigg|_{t = \text{prox}_{b\rho}(\tau Z)} = (1 + o(1)) \tau^2 Z^2 \frac{b^2}{b^2}.
\]
Putting the above cases together and applying dominated convergence gives
\[
\lim_{\tau \to \infty} \left\{ E \left[ \frac{1}{1 + b\rho''(\text{prox}_{b\rho}(\tau Z))} \right] - E \left[ \frac{1}{1 + b\rho''(\text{prox}_{b\rho}(\tau Z))} \mathbb{1}_{\{|Z| \leq \varepsilon \text{ or } |Z - b/\tau| \leq \varepsilon\}} \right] \right\}
= \lim_{\tau \to \infty} \left\{ E \left[ \mathbb{1}_{\{Z < -\varepsilon\}} + E \left[ \mathbb{1}_{\{Z > b/\tau - \varepsilon\}} \right] \right] \right\} = \lim_{\tau \to \infty} \mathbb{P} \left\{ Z < -\varepsilon \text{ or } Z > \frac{b}{\tau} + \varepsilon \right\}
\]
when \( b = c_0 \tau \) for some constant \( c_0 > 0 \). Recognizing that
\[
E \left[ \frac{1}{1 + b\rho''(\text{prox}_{b\rho}(\tau Z))} \mathbb{1}_{\{|Z| \leq \varepsilon \text{ or } |Z - b/\tau| \leq \varepsilon\}} \right] \leq E \left[ \mathbb{1}_{\{|Z| \leq \varepsilon \text{ or } |Z - b/\tau| \leq \varepsilon\}} \right] \leq 4\varepsilon
\]
and
\[
\mathbb{P} \left\{ -\varepsilon \leq Z \leq 0 \text{ or } \frac{b}{\tau} \leq Z \leq \frac{b}{\tau} + \varepsilon \right\} \leq 2\varepsilon,
\]
we arrive at
\[
\lim_{\tau \to \infty} \mathbb{E} \left[ \frac{1}{1 + b\rho''(\text{prox}_{b\rho}(\tau Z))} \right] - \lim_{\tau \to \infty} \mathbb{P} \left\{ Z < 0 \text{ or } Z > \frac{b}{\tau} \right\} \leq 6\varepsilon.
\]
Since \( \varepsilon > 0 \) can be arbitrarily small, we have
\[
\lim_{\tau \to \infty} \mathbb{E} \left[ \frac{1}{1 + b\rho''(\text{prox}_{b\rho}(\tau Z))} \right] = \lim_{\tau \to \infty} \mathbb{P} \left\{ Z < 0 \text{ or } Z > \frac{b}{\tau} \right\} \tag{15}
\]
when \( b = c_0 \tau \). Similarly,
\[
\lim_{\tau \to \infty} \mathbb{E} \left[ (\rho'(\text{prox}_{b\rho}(\tau Z)))^2 \right] = \lim_{\tau \to \infty} \left\{ \mathbb{P} \left\{ Z > \frac{b}{\tau} \right\} + \frac{\tau^2}{b^2} \mathbb{E} \left[ Z^2 \mathbb{1}_{\{0 < Z < \frac{b}{\tau}\}} \right] \right\}.
\]
1.2 The probit case

The proof proceeds with the following 3 steps:

(i) Show that for any $b > 0$ and $\epsilon > 0$, there exist constants $c_{1,b}, c_{2,b}, c_3, c_4 > 0$, depending on $\epsilon$, such that

\[
\begin{align*}
\sup_{z > c_{1,b}} \left| \text{prox}_{\nu b}(z) - \frac{z}{b+1} \right| & \leq \epsilon, \\
\sup_{z < -c_{2,b}} \left| \text{prox}_{\nu b}(z) - z \right| & \leq \epsilon, \\
\sup_{z > c_{3}} \left| \nu''(z) - 1 \right| & \leq \epsilon, \\
\sup_{z < -c_{4}} \left| \nu''(z) \right| & \leq \epsilon.
\end{align*}
\]

(16)

In particular, one can take

\[
c_{1,b} := \max \left\{ b\nu'(\sqrt{2}) + \sqrt{2}, \ 2\sqrt{2}b, \ \frac{4b}{\epsilon} \right\} \quad \text{and} \quad c_{2,b} := \max \left\{ 2b\nu'(0), \ \sqrt{8 \log \frac{b}{\epsilon}} \right\}.
\]

(17)

(ii) Show that for any constant $\eta > 0$, for all $\tau$ sufficiently large, one has

\[
\left| 1 - \frac{1}{1 + b(\tau) + 1} - 2\kappa \right| \leq \eta.
\]

(18)

(iii) Show that for any constant $0 < \eta < 1 - 2\kappa$ and for $\tau$ sufficiently large, one has

\[
\left| \frac{\nu(\tau^2)}{\tau^2} - 2\kappa \right| \leq \eta.
\]

(19)

In the sequel, we elaborate on each of these three steps.

**Step (i).** Recall that for any $x > 0$, one has $\frac{\phi(x)}{x} \left( 1 - \frac{1}{x} \right) \leq 1 - \Phi(x) \leq \frac{\phi(x)}{1 - x}$, since $\nu'(x) = \frac{\phi(x)}{1 - \Phi(x)}$, this gives

\[
\left| \nu'(x) - x \right| \leq \frac{1}{x - x^2} \leq \frac{2}{x}, \quad x \geq \sqrt{2}.
\]

(20)

We start with the first inequality in (16). From the definition of $\text{prox}(\cdot)$, we have the defining relation

\[
\nu'(\text{prox}_{\nu b}(z)) + \text{prox}_{\nu b}(z) = z.
\]

(21)

Therefore, if we take $z_{b,1} := \nu'(\sqrt{2}) + \sqrt{2}$, then this identity (21) indicates that $\text{prox}_{\nu b}(z_{b,1}) = \sqrt{2}$. Moreover, $\text{prox}_{\nu b}(z)$ is monotonically increasing in $z$ (see [2, Eqn. (56)]), which tells us that

\[
\text{prox}_{\nu b}(z) \geq \text{prox}_{\nu b}(z_{b,1}) = \sqrt{2}, \quad \forall z > z_{b,1}.
\]

(22)

Rearranging the identity (21) and combining it with (20) and (22), we obtain

\[
\left| \frac{z}{b+1} - \text{prox}_{\nu b}(z) \right| = \frac{b}{b+1} \left| \nu'(\text{prox}_{\nu b}(z)) - \text{prox}_{\nu b}(z) \right| \leq \frac{2b/(b+1)}{\nu' \left( \text{prox}_{\nu b}(z) \right)} \leq \frac{\sqrt{2b}}{b+1}, \quad \forall z > z_{b,1}.
\]

(23)

(24)

This inequality provides a lower bound on $\text{prox}_{\nu b}(z)$:

\[
\text{prox}_{\nu b}(z) \geq \frac{z - \sqrt{2b}}{b+1} \geq \frac{z}{2(b+1)}
\]

for all $z$ obeying $z > z_{b,1}$ and $z > 2\sqrt{2}b$. Substitution into (23) once again gives

\[
\left| \frac{z}{b+1} - \text{prox}_{\nu b}(z) \right| \leq \frac{2b/(b+1)}{\nu' \left( \text{prox}_{\nu b}(z) \right)} \leq \frac{4b}{\epsilon}, \quad \forall z > \max \left\{ z_{b,1}, \ 2\sqrt{2b}, \ \frac{4b}{\epsilon} \right\}.
\]
establishing the first bound in (16).

We now turn to the second result in (16). Similarly, it is seen from (21) that $\prox_{b\rho}(z_{h,2}) = 0$ with $z_{h,2} := b\rho'(0) > 0$. The monotonicity of $\prox_{b\rho}(\cdot)$ implies that

$$\prox_{b\rho}(z) \leq \prox_{b\rho}(z_{h,2}) = 0, \quad \forall z < z_{h,2}. $$

Recognizing that $\rho'(x) > 0$ and $\rho''(x) > 0$ for any $x$ and using the relation (21), we arrive at

$$|z - \prox_{b\rho}(z)| = b\rho'(\prox_{b\rho}(z)) \leq b\rho'(0), \quad \forall z < z_{h,2},$$

thus indicating that

$$\prox_{b\rho}(z) \leq z + b\rho'(0) \leq z/2, \quad \forall z < -2z_{h,2} < 0.$$ Substituting it into (25) and using the fact that $\rho'(x) = \frac{\phi(x)}{1 - \Phi(x)} \leq 2\phi(x) \leq e^{-x^2/2}$ for all $x < 0$, we get

$$|z - \prox_{b\rho}(z)| = b\rho'(\prox_{b\rho}(z)) \leq \rho'(z/2) \leq be^{-z^2/8}, \quad \forall z < -2z_{h,2} < 0,$$

where (a) follows since $\rho''(x) > 0$. The upper bound (26) will not exceed $\epsilon > 0$ as long as $z < -\max\left\{2z_{h,2}, \sqrt{8 \log \frac{1}{\epsilon}}\right\}$. This establishes the second bound of (16).

The remaining two inequalities regarding $\rho''$ are rather straightforward and the proofs are thus omitted.

**Step (ii).** Recognizing that $\Psi'(z;b) = \frac{b\rho'(z)}{1 + b\rho''(z)}|_{z = \prox_{b\rho}(z)}$, we see that $b(\tau)$ is the solution to

$$1 - \kappa = E[g(\tau Z, b)] \quad \text{with} \quad g(x, b) := \frac{1}{1 + b\rho''(\prox_{b\rho}(x))}. \quad (27)$$

As a result, everything boils down to quantifying $E[g(\tau Z, b)]$.

Consider any sufficiently small $\epsilon > 0$. We first obtain an approximation of $E[g(\tau Z, b)]$. Specifically, we claim that taking $c_\epsilon := \frac{1}{2} \tau^2$ leads to

$$E \left[ g(\tau Z, b) 1_{|\tau Z| > c_\epsilon} \right] \leq E[g(\tau Z, b)] \leq E \left[ g(\tau Z, b) 1_{|\tau Z| < c_\epsilon} \right] + \epsilon. \quad (28)$$

The lower bound is trivial since $0 \leq g(x, b) \leq 1$. To see why the upper bound holds, we invoke Cauchy-Schwarz to derive

$$E \left[ g(\tau Z, b) 1_{|\tau Z| \leq c_\epsilon} \right] \leq \sqrt{E \left[ g^2(\tau Z, b) \right]} \sqrt{P\left(|\tau Z| \leq \frac{c_\epsilon}{\tau}\right)} \leq \sqrt{E \left[ g^2(\tau Z, b) \right]} \leq \sqrt{2} \frac{c_\epsilon}{\tau} = \epsilon, \quad (29)$$

where (b) arises since $0 \leq g(x, b) \leq 1$. This inequality (29) matches the upper bound in (28). In short, we see that $E \left[ g(\tau Z, b) 1_{|\tau Z| > c_\epsilon} \right]$ is a reasonably tight approximation of $E \left[ g(\tau Z, b) \right]$, and it suffices to look at

$$E \left[ g(\tau Z, b) 1_{|\tau Z| < c_\epsilon} \right] = E \left[ g(\tau Z, b) 1_{\{\tau Z < -c_\epsilon\}} \right] + E \left[ g(\tau Z, b) 1_{\{\tau Z > c_\epsilon\}} \right]. \quad (30)$$

We first control the second term in the right-hand side of (30). Suppose for the moment that

$$c_\epsilon > \max\{c_{1, b}, (c_3 + \epsilon)(b + 1), c_{2, b}, c_4 + \epsilon\}. $$

According to (16), on the event $\{\tau Z > c_\epsilon\}$ one has

$$\frac{\tau Z}{b + 1} - \epsilon \leq \prox_{b\rho}(\tau Z) \leq \frac{\tau Z}{b + 1} + \epsilon \quad \text{and} \quad 1 - \epsilon \leq \rho''(\prox_{b\rho}(\tau Z)) \leq 1 + \epsilon,$$

where the second inequality holds since $\prox_{b\rho}(\tau Z) \geq \frac{\tau Z}{b + 1} - \epsilon > \frac{c_\epsilon}{b + 1} - \epsilon \geq c_3$. Plugging these inequalities into (27) gives

$$\frac{1}{1 + b(1 + \epsilon)} \leq g(\tau Z, b) \leq \frac{1}{1 + b(1 - \epsilon)}. $$
In addition, similar to (29) we get
\[
\frac{1}{2} \geq P(\tau Z > c_\epsilon) = P(\tau Z < -c_\epsilon) = \frac{1}{2} \left( 1 - P\left( |Z| \leq \frac{c_\epsilon}{\tau} \right) \right) \geq \frac{1}{2} \left( 1 - \frac{2c_\epsilon}{\tau} \right) = \frac{1}{2} (1 - \epsilon^2). \]

The above bounds taken collectively reveal that
\[
\frac{1}{1 + b(1 + \epsilon)} \cdot \frac{1}{2} (1 - \epsilon^2) \leq \mathbb{E} \left[ g(\tau Z, b) 1_{\{\tau Z > c_\epsilon\}} \right] \leq \frac{1}{1 + b(1 - \epsilon)} \cdot \frac{1}{2}. \tag{31}
\]

We can employ similar arguments to control the first term in the right-hand side of (28) as well. Since \( c_\epsilon > \max\{c_{2, b}, c_4 + \epsilon\}, \) on the event \( \{\tau Z < -c_\epsilon\} \) we have
\[
\tau Z - \epsilon \leq \text{prox}_{b\rho}(\tau Z) \leq \tau Z + \epsilon \quad \text{and} \quad -\epsilon \leq \rho''(\text{prox}_{b\rho}(\tau Z)) \leq \epsilon,
\]
a direct consequence of (16). This implies that
\[
\frac{1}{1 + b \epsilon} \leq g(\tau Z, b) \leq \frac{1}{1 - b \epsilon}
\]
and, therefore,
\[
\frac{1}{1 + b \epsilon} \cdot \frac{1}{2} (1 - \epsilon^2) \leq \mathbb{E} \left[ g(\tau Z, b) 1_{\{\tau Z < -c_\epsilon\}} \right] \leq \frac{1}{1 - b \epsilon} \cdot \frac{1}{2}. \tag{32}
\]
Combining (28), (31) and (32), we conclude that for any \( \epsilon > 0, \)
\[
1 - \epsilon^2 \left\{ \frac{1}{1 + b(1 + \epsilon)} + \frac{1}{1 + b \epsilon} \right\} \leq \mathbb{E} \left[ g(\tau Z, b) \right] \leq \frac{1}{2} \left\{ \frac{1}{1 + b(1 - \epsilon)} + \frac{1}{1 - b \epsilon} \right\} + \epsilon,
\]
as long as \( c_\epsilon = \frac{1}{2} \tau \epsilon^2 > \max\{c_{1, b}, (c_3 + \epsilon)(b + 1), c_{2, b}, c_4 + \epsilon\}, \) or equivalently,
\[
\tau > \frac{2 \max\{c_{1, b}, (c_3 + \epsilon)(b + 1), c_{2, b}, c_4 + \epsilon\}}{\epsilon^2},
\]
where the lower bound is on the order of \( b/\epsilon^2. \) Effectively, we have established that for any given \( b \) and any sufficiently small \( \epsilon > 0 \) (so that \( b \epsilon < 1 \) and \( \epsilon < 1 \)), if \( \tau \) is sufficiently large (as specified above) one has
\[
\left| \mathbb{E} \left[ g(\tau Z, b) \right] - \frac{1}{2} \left( \frac{1}{1 + b} + 1 \right) \right| \leq \tilde{c}_4 (\epsilon + b \epsilon) \tag{33}
\]
for some universal constant \( \tilde{c}_4 > 0 \) independent of \( b, \epsilon, \tau. \)

We can then combine this result (33) with the constraint (27) to derive an estimate on \( b(\tau). \) Fix any \( \eta > 0. \) Let \( b_1 \) and \( b_2 \) be two constants such that
\[
\frac{1}{2} \left( \frac{1}{1 + b_1} + 1 \right) = 1 - \kappa - \frac{\eta}{4}, \quad \frac{1}{2} \left( \frac{1}{1 + b_2} + 1 \right) = 1 - \kappa + \frac{\eta}{4}.
\]
Picking \( \epsilon > 0 \) sufficiently small so that \( \max\{\tilde{c}_4(1 + b_1)\epsilon, \tilde{c}_4(1 + b_2)\epsilon\} < \eta/4 \) and \( \tau \gg \max\{b_1, b_2\} / \epsilon^3, \) we can ensure that
\[
\mathbb{E} \left[ g(\tau Z, b_1) \right] < 1 - \kappa < \mathbb{E} \left[ g(\tau Z, b_2) \right].
\]
Recall that for any \( \tau > 0, \) the function \( G(b) := 1 - \mathbb{E} \left[ g(\tau Z, b) \right] \) is strictly increasing in \( b \) (see [1] Lemma 5) and, hence,
\[
b_2 \leq b(\tau) \leq b_1, \quad \Rightarrow \quad \frac{1}{2(1 + b_1)} \leq \frac{1}{2(1 + b(\tau))} \leq \frac{1}{2(1 + b_2)}.
\]
Combining these together, we obtain
\[
\left\| \left( 1 - \frac{1}{b(\tau) + 1} \right) - 2\kappa \right\| \leq \eta, \tag{34}
\]
for any \( \eta > 0 \) with the proviso that \( \tau \) is sufficiently large. This finishes Step (ii). In particular, this yields

\[
\lim_{\tau \to \infty} b(\tau) = \frac{2\kappa}{1 - 2\kappa}.
\]  

**Step** (iii). Now we move on to the variance map

\[
\mathcal{V}(\tau^2) = \frac{b(\tau)^2}{\kappa} \mathbb{E} \left[ \rho'(\text{prox}_{b(\tau)}(\tau Z))^2 \right].
\]  

For notational convenience, we set

\[ h(x) := \rho'(\text{prox}_{b(\tau)}(x))^2, \]

a key mapping in the definition (36). Before proceeding, we remark that from the properties of \( \rho' \), for any \( \epsilon > 0 \), there exist constants \( c_5, c_6 > 0 \), depending on \( \epsilon \), such that

\[
\sup_{z > c_5} |\rho'(z) - z| \leq \epsilon, \quad \sup_{z < -c_6} |\rho'(z)| \leq \epsilon.
\]

As before, we decompose the function \( \mathcal{V}(\tau^2) \) as follows:

\[
\mathcal{V}(\tau^2) - \frac{b(\tau)^2}{\kappa} \mathbb{E} \left[ h(\tau Z) \mathbf{1}_{\{|\tau Z| > \alpha_e\}} \right] = \frac{b(\tau)^2}{\kappa} \mathbb{E} \left[ h(\tau Z) \mathbf{1}_{\{|\tau Z| \leq \alpha_e\}} \right]
\]

for some point \( \alpha_e > 0 \) to be specified later. This gives

\[
\mathbb{E}[h(\tau Z) \mathbf{1}_{\{|\tau Z| < \alpha_e\}}] \leq \sqrt{\mathbb{E}[h^2(\tau Z) \mathbf{1}_{\{|\tau Z| < \alpha_e\}}]} \sqrt{\mathbb{P}(|\tau Z| \leq \alpha_e)} \leq C(\alpha_e, b) \sqrt{2\Phi \left( \frac{\alpha_e}{\tau} \right) - 1},
\]

where

\[ C(\alpha_e, b) = \rho'(\text{prox}_{b(\alpha_e)}(\alpha_e))^2. \]

The last inequality of (38) holds since (1) \( \rho'(z) \geq 0 \) is an increasing function of \( z \); (2) \( \text{prox}_{b(\tau)}(x) \) is an increasing function of \( x \) (see [2, Eqn. (56)]). For any given \( \epsilon > 0 \), one can pick \( \tau \) sufficiently large so that the above bound \( C(\alpha_e, b) \sqrt{2\Phi \left( \frac{\alpha_e}{\tau} \right) - 1} \) is below \( \epsilon \). The particular choice of \( \tau \) will be made clear later. Under these conditions,

\[
\mathbb{E}[h(\tau Z) \mathbf{1}_{\{|\tau Z| > \alpha_e\}}] \leq \mathbb{E}[h(\tau Z)] \leq \mathbb{E}[h(\tau Z) \mathbf{1}_{\{|\tau Z| < \alpha_e\}}] + \mathbb{E}[h(\tau Z) \mathbf{1}_{\{|\tau Z| > \alpha_e\}}] + \epsilon.
\]

We first control the second term in the right-hand side of (39). To this end, we choose

\[ \alpha_e > \max \{ c_1, b, c_2, (c_5 + \epsilon)(b + 1), 6c_6 + 2\epsilon \} \]

as before. Then from (10) and (37), on the event \( \{ \tau Z > \alpha_e \} \) we have

\[
\frac{\tau Z}{b + 1} - \epsilon \leq \text{prox}_{b(\tau)}(\tau Z) \leq \frac{\tau Z}{b + 1} + \epsilon \quad \text{and} \quad \frac{\tau Z}{b + 1} - 2\epsilon \leq \rho'(\text{prox}_{b(\tau)}(\tau Z)) \leq \frac{\tau Z}{b + 1} + 2\epsilon.
\]

This yields

\[
\left( \frac{\tau Z}{b + 1} - 2\epsilon \right)^2 \leq h(\tau Z) \leq \left( \frac{\tau Z}{b + 1} + 2\epsilon \right)^2
\]

on the event \( \{ \tau Z > \alpha_e \} \), and hence

\[
\mathbb{E} \left[ \left( \frac{\tau Z}{b + 1} - 2\epsilon \right)^2 \mathbf{1}_{\{\tau Z > \alpha_e\}} \right] \leq \mathbb{E}[h(\tau Z) \mathbf{1}_{\{|\tau Z| > \alpha_e\}}] \leq \mathbb{E} \left[ \left( \frac{\tau Z}{b + 1} + 2\epsilon \right)^2 \mathbf{1}_{\{\tau Z > \alpha_e\}} \right].
\]

Similarly for the first term in the right-hand side of (39), as \( \alpha_e > \max \{ c_2, 6c_6 + 2\epsilon \} \), on the event \( \{ \tau Z < -\alpha_e \} \), we have

\[
\tau Z - \epsilon \leq \text{prox}_{b(\tau)}(\tau Z) \leq \tau Z + \epsilon \quad \text{and} \quad -\epsilon \leq \rho'(\text{prox}_{b(\tau)}(\tau Z)) \leq \epsilon.
\]
that

\[\mathbb{P}(\tau Z > \alpha_e) = \mathbb{P}(\tau Z < -\alpha_e) = \frac{1}{2}(1 - \delta_e) \text{ for some } \delta_e \text{ small which is a function of } \epsilon \text{ and which vanishes as } \epsilon \to 0.\]  

This yields

\[0 \leq \mathbb{E}[h(\tau Z)1_{\{\tau Z < -\alpha_e\}}] \leq \frac{\epsilon^2}{2} (1 - \delta_e). \quad (41)\]

Combining the relations (39), (40) and (41) we obtain that

\[\frac{b^2}{\kappa} \mathbb{E} \left[ \left( \frac{\tau Z}{b+1} - 2\epsilon \right)^2 1_{\{\tau Z > \alpha_e\}} \right] \leq \mathcal{V}(\tau^2) \leq \frac{b^2}{\kappa} \left\{ \mathbb{E} \left[ \left( \frac{\tau Z}{b+1} + 2\epsilon \right)^2 1_{\{\tau Z > \alpha_e\}} \right] + \frac{\epsilon^2}{2} (1 - \delta_e) + \epsilon \right\}. \quad (42)\]

We still need to evaluate \(\mathbb{E} \left[ \left( \frac{\tau Z}{b+1} - 2\epsilon \right)^2 1_{\{\tau Z > \alpha_e\}} \right].\) To this end, we define two quantities

\[\alpha_1 := \mathbb{E} [Z 1_{\{\tau Z > \alpha_e\}}] \quad \text{and} \quad \alpha_2 := \mathbb{E} [Z^2 1_{\{\tau Z > \alpha_e\}}].\]

Using the properties of the normal CDF, one can show that

\[\frac{\tau}{\sqrt{2\pi}} - \alpha_e \leq \tau \alpha_1 \leq \frac{\tau}{\sqrt{2\pi}} \quad \text{and} \quad \frac{\tau^2}{2} - \frac{\alpha_1^2}{2} \leq \tau^2 \alpha_2 \leq \frac{\tau^2}{2}. \quad (43)\]

Using the above relations and rearranging, the bounds in (42) can be rewritten as

\[\mathcal{V}(\tau^2) \geq \frac{b^2}{\kappa} \left[ \frac{\tau^2}{2(b+1)^2} - \frac{\alpha_1^2}{2(b+1)^2} - \frac{4\epsilon}{\sqrt{2\pi}(b+1)} + 2\epsilon^2 (1 - \delta_e) \right];\]

\[\mathcal{V}(\tau^2) \leq \frac{b^2}{\kappa} \left[ \frac{\tau^2}{2(b+1)^2} + \epsilon \left( \frac{4\tau}{\sqrt{2\pi}(b+1)} + 1 \right) + \frac{5}{2} \epsilon^2 (1 - \delta_e) \right].\]

Finally, observing that \(b \geq 0,\) we arrive at

\[\left| \mathcal{V}(\tau^2) - \frac{b^2}{2\kappa (b+1)^2} \right| \leq \frac{b^2}{\kappa} \left\{ \epsilon \left( \frac{8\tau}{\sqrt{2\pi}} + 1 \right) + \frac{\delta_e \alpha_1^2}{2} + \frac{\epsilon^2}{2} (1 - \delta_e) \right\},\]

which is equivalent to

\[\left| \frac{\mathcal{V}(\tau^2)}{\tau^2} - \frac{1}{2\kappa} \left( 1 - \frac{1}{b+1} \right) \right|^2 \leq \frac{b^2}{\kappa} \left\{ \epsilon \left( \frac{8\tau}{\sqrt{2\pi}} + 1 \right) + \frac{\delta_e \alpha_1^2}{2\tau^2} + \frac{\epsilon^2}{2\tau^2} (1 - \delta_e) \right\}. \quad (44)\]

Note that in the bound above \(\alpha_e\) also depends on \(b.\) Henceforth we denote \(\alpha_e \) as \(\alpha_e(b).\) Next, we invoke the result from Step (ii) to ensure that \(b(\tau)\) is bounded for all sufficiently large values of \(\tau.\)

Fix \(\eta' > 0\) such that \(0 < \eta' < 1 - 2\kappa.\) Let \(\tau_0\) be the threshold above which for all values of \(\tau\) the relation (34) holds with \(\eta = \eta'/2.\) Then \(\forall \tau \geq \tau_0,\) one has

\[b(\tau) \leq \frac{2\kappa + \eta'}{1 - 2\kappa - \eta'} =: a(\eta').\]

For all \(\tau \geq \tau_0,\) we have

\[\left| \frac{\mathcal{V}(\tau^2)}{\tau^2} - \frac{1}{2\kappa} \left( 1 - \frac{1}{b+1} \right) \right|^2 \leq \frac{a(\eta')^2}{\kappa} \left\{ \epsilon \left( \frac{8\tau}{\sqrt{2\pi}} + 1 \right) + \frac{\delta_e \alpha_e(a(\eta))^2}{2\tau^2} + \frac{\epsilon^2}{2\tau^2} (1 - \delta_e) \right\},\]

where \(\alpha_e(a(\eta))\) is any constant above \(\max\{c_1 a(\eta), c_2 a(\eta), c_5 + \epsilon (a(\eta) + 1), c_6 + 2\epsilon\}.\) We choose \(\tau > \tau_0\) so that \(C(\alpha_e(a(\eta)), a(\eta)) \sqrt{2\Phi(\alpha_e)} - 1\) is below \(\epsilon,\) and the above bound in the RHS is below \(\eta = \eta'/2.\) This gives

\[\left| \frac{\mathcal{V}(\tau^2)}{\tau^2} - 2\kappa \right| \leq \left| \frac{\mathcal{V}(\tau^2)}{\tau^2} - \frac{1}{2\kappa} \left( 1 - \frac{1}{b+1} \right) \right|^2 + \left| 2\kappa - \frac{1}{2\kappa} \left( 1 - \frac{1}{b+1} \right) \right|^2 \leq \eta'.\]
Hence, for any such $\tau$

$$\frac{V(\tau^2)}{\tau^2} \leq 2\kappa + \eta' < 1,$$

from the choice of $\eta'$. In particular, we have established that

$$\lim_{\tau \to \infty} \frac{V(\tau^2)}{\tau^2} = 2\kappa.$$

**Remark** 1. In fact, the above analysis works for a broader class of link functions beyond the probit case. Specifically, more general sufficient conditions for the above result to hold are the following: in addition to conditions mentioned in [1, Section 2.3.3].

- $\rho'(x) \to 0$ when $x \to -\infty$, and $\rho'(x)/x \to 1$, when $x \to \infty$; further, $|\rho'(x) - x| \leq f(x)$ for all $x$ positive, where $f(x)$ is some function obeying $f(x) \to 0$ when $x \to \infty$.
- $\rho''$ is bounded, converges to 1 when $x \to \infty$ and converges to 0 when $x \to -\infty$. $-\infty$ are swapped.
- In addition, for any given $z$, $b\rho''(\text{prox}_{b\rho}(z)) \to \infty$ when $b \to \infty$.

**References**

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[2] David Donoho and Andrea Montanari. High dimensional robust M-estimation: Asymptotic variance via approximate message passing. *Probability Theory and Related Fields*, pages 1–35, 2013.