On Jannsen’s conjecture for Hecke characters of imaginary quadratic fields

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Abstract

We present a collection of results on a conjecture of Jannsen about the \( p \)-adic realizations associated to Hecke characters over an imaginary quadratic field \( K \) of class number 1.

The conjecture is easy to check for Galois groups purely of local type (§1). In §2 we define the \( p \)-adic realizations associated to Hecke characters over \( K \). We prove the conjecture under a geometric regularity condition for the imaginary quadratic field \( K \) at \( p \), which is related to the property that a global Galois group is purely of local type. Without this regularity assumption at \( p \), we present a review of the known situations in the critical case §3 and in the non-critical case §4 for these realizations. We relate the conjecture to the non-vanishing of some concrete non-critical values of the associated \( p \)-adic \( L \)-function of the Hecke character.

Finally, in §5 we prove that the conjecture follows from a general conjecture on Iwasawa theory for almost all Tate twists.

1 The Jannsen conjecture on local type Galois groups

Jannsen’s conjecture \cite{9} predicts the vanishing of a second Galois cohomology group for the \( p \)-adic realization of almost all Tate twists of a pure Chow motive. It also specifies the Tate twists where this cohomology group could not vanish. Without this specification the conjecture is a generalization of the classical weak Leopoldt conjecture.

We refer to Jannsen’s original paper \cite{9} and Perrin-Riou’s paper \cite[Appendix B]{14} for the relations with other conjectures and for general results.

Let \( F \) be a number field with algebraic closure \( \overline{F} \). Let \( X \) be a smooth, projective variety of pure dimension \( d \) over \( F \). Let \( p \) be a prime number, and \( S \) a finite set of places of \( F \), containing all places above \( \infty \) and \( p \), and all primes where \( X \) has bad reduction. Let \( G_S \) be the Galois group over \( F \) of the maximal \( S \)-ramified (unramified outside \( S \)) extension of \( F \), that we call \( F_S \).

Conjecture 1.1 (Jannsen). If \( \overline{X} = X \times_F \overline{F} \), then

\[
H^2(G_S, H^i_{\text{et}}(\overline{X}, \mathbb{Q}_p(n))) = 0 \text{ if } \begin{cases} \hspace{2cm} (a) \ i + 1 < n, \hspace{6cm} \text{ or} \\ (b) \ i + 1 > 2n. \hspace{6cm} \end{cases}
\]

This conjecture can be verified also from the étale cohomology with \( \mathbb{Z}_p \) or \( \mathbb{Q}_p/\mathbb{Z}_p \) coefficients.

Lemma 1.2 (lemma 1 \cite{9}). The following statements are equivalent:

1. \( H^2(G_S, H^i_{\text{et}}(\overline{X}, \mathbb{Q}_p(n))) = 0. \)

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2. \(H^2(G_S, H^1_{et}(\overline{X}, \mathbb{Z}_p(n)))\) is finite.

3. \(H^2(G_S, H^1_{et}(\overline{X}, Q_p/\mathbb{Z}_p(n)))\) is finite.

4. (if \(p \neq 2\) or \(F\) is totally imaginary) \(H^2(G_S, \tilde{H}^1_{et}(\overline{X}, Q_p/\mathbb{Z}_p(n))) = 0\) where \(\tilde{H}^1_{et}(\overline{X}, Q_p/\mathbb{Z}_p(n))\) is the \(p\)-divisible part of the group \(H^1_{et}(\overline{X}, Q_p/\mathbb{Z}_p(n))\).

There is an analogous conjecture if one replaces \(G_S\) by \(\text{Gal}(\overline{F}_p/F_p)\), the absolute Galois group of the local field \(F_p\), where \(F_p\) is the completion at \(p\) of \(F\).

**Conjecture 1.3** (Jannsen). \(H^2(\text{Gal}(\overline{F}_p/F_p), H^1_{et}(\overline{X}, Q_2(n))) = 0\) if \(i + 1 < n\) or \(i + 1 > 2n\).

Lemma [12] is also valid in this situation, i.e. replacing \(G_S\) by \(\text{Gal}(\overline{F}_p/F_p)\).

Denote by \(G_S(p)\) the Galois group \(\text{Gal}(F_S(p)/F)\), where \(F_S(p)\) means the maximal \(p\)-extension of \(F\) inside \(F_S\). Let \(S_p\) be the set of primes of \(F\) above \(p\).

Let us denote by \(G_v\) the Galois group of the maximal \(p\)-extension of the local field \(F_v\) (with \(v \in S\)) over \(F_v\); we write also \(G_v = \text{Gal}(F_v(p)/F_v)\). We have the natural surjective map

\[i_{v,S}: G_v \rightarrow G_v\]

where \(G_v\) denotes the \(p\)-part of the decomposition group at \(v\) of \(G_S\). Let \(\varphi_v : G_v \rightarrow G_S(p)\) be the natural inclusion.

**Definition 1.4.** The Galois group \(G_S(p)\) is purely of local type iff the map \(\varphi_v \circ i_{v,S}\) is an isomorphism.

It is easy to see that this definition is equivalent to the one given in [13] (B.2.6).

**Lemma 1.5.** Let \(M(j)\) be a \(p\)-primary divisible \(G_v = \text{Gal}(F_v(p)/F_v)\)-module of cofinite type. If \(\mu_p \notin F_v\) then \(H^2(G_v, M(j)) = 0\), where \(\mu\) denotes a primitive \(p\)-root of unity. If \(\mu_p \in F_v\), then \(H^2(G_v, M(j)) = M(j-1)_{G_v}\).

Finally, let \(M(j)\) be any \(p\)-primary divisible \(\text{Gal}(\overline{F}_v/F_v)\)-module of cofinite type. Then \(H^2(\text{Gal}(\overline{F}_v/F_v), M(j)) = M(j-1)_{\text{Gal}(\overline{F}_v/F_v)}\).

**Proof.** If \(\mu_p \notin F_v\) then \(G_v\) is free, hence \(H^2(G_v, M(j)) = 0\). Otherwise it is a Poincaré group of dimension two with dualizing module \(Q_p/\mathbb{Z}_p(1)\). Using local Tate duality we get \(H^2(G_v, M(j)) = M(j-1)_{G_v}\).

The last statement follows from local Tate duality. \(\square\)

**Lemma 1.6.** Let \(S'\) be a subset of \(S\) which contains \(S_p\), the places of \(F\) above \(p\). Suppose that the Galois group \(\text{Gal}(F_{S'}(p)/F)\) is purely of local type for a place \(w \in S'\) of \(F\). Furthermore let \(M\) be a \(p\)-primary divisible \(\text{Gal}(F_{S'}(p)/F)\)-module of cofinite type such that \(M(j-1)_{\text{Gal}(\overline{F}_v/F_v)} = 0\) for all \(v \in S' \setminus \{w\}\). Then

\[H^2(G_S, M(j)) = 0.\]

**Proof.** Let us consider part of Soulé's exact sequence recalled in [8] §3,(13):

\[H^2(G_{S'}, M(j)) \rightarrow H^2(G_S, M(j)) \rightarrow \oplus_{v \in S' \setminus \{w\}} M(j-1)_{\text{Gal}(\overline{F}_v/F_v)} \rightarrow 0,\]
where the last term of this sequence is zero by hypothesis. Observe that
\[ H^2(G_{w'}, M(j)) = H^2(G_{w',}(p), M(j)) \]
by a result of Neumann [13 Theorem 1]. Since \( G_{w',}(p) \) is purely of local type,
\[ H^2(G_{w',}(p), M(j)) = H^2(G_w, M(j)). \]
If \( p \nmid F_w \) then \( H^2(G_w, M(j)) = 0 \). Otherwise \( H^2(G_w, M(j)) = (j - 1)_{\text{Gal}(F_w(p)/F_w)}. \) But, in our case, the action of \( \text{Gal}(\overline{F}_w/F_w) \) factors through the pro-\( p \)-quotient \( G_w \), and so \( M(j - 1)_{\text{Gal}(F_w(p)/F_w)} = 0 \) by hypothesis. \( \square \)

**Remark 1.7.** If \( X \) has good reduction at \( p \) and \( p \nmid \mathfrak{p} \), the vanishing of the Galois group \( H^2(F_p, \mathcal{H}_{ct}(X, \mathbb{Q}_p(j))) \) with \( i \neq 2(j - 1) \) follows from lemma 1.3.5 and weights arguments (see [13, lemma 3]). As usual \( H^j(F_v, -) \) means \( H^j(\text{Gal}(\overline{F}_v/F_v), -) \). This result can also be proved when \( X \) has potentially good reduction at \( p \) [13, lemma 12].

When \( p \mid \mathfrak{p} \), Soulé proves the vanishing of this group for \( j > i + 1 \) or \( j < 1 \) [13, corollary 5, lemma 11]. Moreover if \( p \) is unramified in \( F/Q \) and \( X \) has good reduction at \( p \) then \( H^2(F_p, \mathcal{H}_{ct}(X, \mathbb{Q}_p(j))) = 0 \) if \( i \neq 2(j - 1) \) [13, corollary 6]. In this situation if \( i - 2j \neq 2 \) and \( i - 2j \neq 0 \) we obtain:
\[ \dim_{\mathbb{Q}_p} H^1(F_p, \mathcal{H}_{ct}(X, \mathbb{Q}_p(j))) = -\chi(H^1_{ct}(\overline{X}, \mathbb{Q}_p(j))) = [F_p : \mathbb{Q}_p] \dim_{\mathbb{Q}_p}(H^1_{ct}(\overline{X}, \mathbb{Q}_p(j))), \]
where \( \chi \) is the Euler characteristic in local Galois cohomology. The last equality follows from [13, 7.3.8].

## 2 The geometric regularity condition.

We keep once and for all the notations of §1. Let \( \mathfrak{p} \) be a prime of \( F \) such that \( \mathfrak{p} \mid \mathfrak{p} \) and the inertia group at \( \mathfrak{p} \) acts trivially on \( M \), or on \( \overline{X} \).

Let \( E \) be a fixed elliptic curve defined over an imaginary quadratic field \( K \), and with CM by \( \mathcal{O}_K \), the ring of integers of \( K \). This hypothesis implies that \( d(K) = 1 \). Associated to this elliptic curve there is a Hecke character \( \varphi \) of the imaginary quadratic field \( K \) with conductor \( \mathfrak{f} \), an ideal of \( \mathcal{O}_K \) which coincides with the conductor of the elliptic curve \( E \).

Consider the category of Chow motives \( \mathcal{M}_Q(K) \) over \( K \) with morphisms induced by graded correspondences in Chow theory tensored with \( Q \). Then, the motive of the elliptic curve \( E \) has a canonical decomposition \( h(E)_Q = h^0(E)_Q \oplus h^1(E)_Q \oplus h^2(E)_Q \). The motive \( h^1(E)_Q \) has a multiplication by \( K [5] \). For \( w \) a strictly positive integer, let us consider the motive \( \otimes^w_Q h^1(E)_Q \), which has multiplication by \( T_w := \otimes^w_Q K \). Observe that \( T_w \) has a decomposition \( \prod T_\theta \) as a product of fields \( T_\theta \), where \( \theta \) runs through the \( \text{Aut}(\mathbb{C}) \)-orbits of \( \mathbb{T}^w = \text{Hom}(T_w, \mathbb{C}) \), where \( \mathbb{T} = \text{Hom}(K, \mathbb{C}) \). This decomposition defines some idempotents \( e_\theta \) and gives also a decomposition of the motive and its realizations. Let us fix once and for all an immersion \( \lambda : K \to \mathbb{C} \) as in [5, p.135].

The \( L \)-function associated to the motive \( e_\theta(\otimes^w h^1(E)_Q) \) corresponds to the \( L \)-function associated to the CM-character \( \psi_\theta = e_\theta(\otimes^w \varphi) : A^*_K \to K^* [5, \S 1.3.1] \) (for equivalent definitions of Hecke characters over an imaginary quadratic field,
see [7 §2.2]). With respect to the fixed embedding $\lambda$, this CM-character corresponds to $\varphi^w$, where $a, b \geq 0$ are integers such that $w = a + b$. The pair $(a, b)$ is the infinite type of $\varphi_q$. We note that there are different $\theta$ with the same infinite type. Every $\theta$ gives two elements of $\Upsilon^w$, one given by the infinite type $\vartheta \in \Hom_K(T_w, \mathbb{C})$ and the other one coming from the other embedding. When $\theta$ corresponds to $\vartheta = (\lambda_1, \ldots, \lambda_w) \in \Hom_K(T_w, \mathbb{C})$ with type $(a, b)$, we denote by $\overline{\vartheta}$ the one that corresponds to the element $\overline{\vartheta} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_w) \in \Hom_K(T_w, \mathbb{C})$, where $\lambda_i$ are $\mathbb{Q}$-immersions of $K$ into $\mathbb{C}$ and the bar denotes complex conjugation. Observe that $\overline{\vartheta}$ has type $(b, a)$.

Let us denote by $\otimes^w h^1(E)$ the integer Chow motive (similar to $\otimes^w h^1(E)_{\mathbb{Q}}$ but without tensoring by $\mathbb{Q}$ the correspondences). The Chow motive

$$M_\theta := e_\theta(\otimes^w h^1(E)),$$

is a Chow motive with coefficients in $\mathcal{O}_K[1/D_K]$ where $D_K$ is the discriminant of $K$, $e_\theta \in \mathcal{O}_K[1/D_K]$ when $w > 1$.

The $p$-adic realization of the motive $M_\theta(n)$:

$$M_{\theta \mathbb{Q}_p}(n) := H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Q}_p(n)) \cong e_\theta(\otimes^w (T_p E \otimes \mathbb{Q}_p))(n - w),$$

is unramified outside any set $S$ which contains the finite primes of $K$ which divide $p_\theta$, where $p_\theta$ is the conductor of $\varphi_q$. We have also a natural lattice associated to it, corresponding for $p \nmid D_K$ to $H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(n)) \cong e_\theta(\otimes^w T_p E)(n - w)$ and we denote it by

$$M_{\theta \mathbb{Z}_p}(n) := H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(n)).$$

We impose once and for all that $p \nmid D_K$.

Since $e_\theta(\otimes^w h^1(E)) \subseteq h^w(E^w)$ Jannsen’s conjecture for $h^w(E^w)$ implies the following conjecture for Hecke characters.

**Conjecture 2.1.** Let $p$ be a prime such that $E$ has good reduction at the primes over $p$ in $K$. Let $S$ be a set of primes of $K$ which contains the primes of $K$ over $p$ and the primes of $\mathfrak{f}$. Then,

$$H^2(\Gal(K_S/K), M_{\theta \mathbb{Q}_p}(n)) = 0,$$

for $n > w + 1$ or $w + 1 > 2n$.

Consider $\mathcal{E} \subseteq K[E[p]]$ with $K \subseteq \mathcal{E}$. We suppose also $p > 3$. Denote by $\mathcal{F} = K[E[p)]$. We know that $E \times_K \mathcal{F}$ has good reduction everywhere.

We introduce the notion of regularity for $p$ at $\mathcal{E}$, which turns out to be closely related to the condition that a global Galois group is purely of local type.

Let us assume in this section once for all that $p$ splits in $K$ with $p = pp^*$ where $p, p^*$ are different primes of $K$.

**Definition 2.2.** Suppose that $p > 3$, $(p) = pp^*$ in $K$, $p \neq p^*$ and $E$ has good reduction at $p$ and $p^*$. Let $S_p$ be the set of primes of $\mathcal{E}$ that divide $p$. The prime $p$ is called regular for $E$ and $\mathcal{E}$ if $E_{S_p}(p)$ is a $\mathbb{Z}_p$-extension of $\mathcal{E}$. We say that $p$ is regular for $E$ and $\mathcal{E}$ if $p$ and $p^*$ are regular for $E$ and $\mathcal{E}$.
Remark 2.3. The above regularity condition admits an analogue of Kummer’s criterion as in the classical notion of regularity at \( p \) [2, B.1.2,1.7,2.16-17]. One of the equivalent notions of regularity is related to a critical value of the \( L \)-function associated to some concrete Hecke characters of the form \( \varphi^w \) [17].

Proposition 2.4 (Wingberg [17]). Let \( F \) be a number field between \( K \) and \( F \). The prime \( p \) is regular for \( E \) and \( F \) if and only if \( \text{Gal}(\mathbb{F}_p, \mathbb{F}) \) is purely of local type with respect to \( p^* \), where \( S_p \) denotes the places of \( F \) above \( p \).

Remark 2.5. Thus, for this notion of regular primes the Grunwald-Wang theorem holds: the maximal \( p \)-extension of \( F_p \) coincides with the completion at \( p^* \) of the maximal \( p \)-extension of \( F \) (use [2, corollary B.2.6] with proposition [2,4]).

Let us write a consequence of proposition 2.4 (the case \( w = 1 \) is a result of Wingberg [17]).

Corollary 2.6. Let \( p \) be a regular prime for \( E \) and \( F = K(E[p]) \). Let \( F \) be an extension of \( K \) inside \( F \). Denote by \( S^* \) a finite set of primes of \( F \) containing the primes above \( p \) and those where \( E \times_K F \) has bad reduction. Then,

\[
H^2(\text{Gal}(\mathbb{F}_S, \mathbb{F}), H^w(M_{\vartheta}, \mathbb{Q}_p/\mathbb{Z}_p(j + w))) = 0
\]

for any integer \( w = a + b \) with \( w + 2(j - 1) \neq 0 \).

Proof. By Kummer theory, we have that

\[
H^1_{et}(E, \mathbb{Q}_p/\mathbb{Z}_p(1)) = E[p^\infty] = E[p^\infty] \oplus E[(p^*)^\infty]
\]

where \( E[a^\infty] = \lim_{\to} E[a^n] \) is the inductive limit of the \( a^n \)-torsion points of the elliptic curve \( E \). Thus \( M(w + j) := H^w(M_{\vartheta} \otimes_K \mathbb{F}, \mathbb{Q}_p/\mathbb{Z}_p(w + j)) \cong e_{\vartheta}(\otimes_p E[p^\infty])(j) \). Since \( K \subseteq F \subseteq F \), we need only to prove

\[
H^2(\text{Gal}(\mathbb{F}_S, \mathbb{F}), M(w + j)) = 0,
\]

because \( F \subseteq F \) is unramified outside \( S^* \) [2, II.1.8]. Since \( F/F \) is prime to \( p \) it is enough to show

\[
H^2(\text{Gal}(\mathbb{F}_S, \mathbb{F}), M(w + j)) = 0.
\]

Now, \( M(w + j) \) is a \( \text{Gal}(\mathbb{F}_S, \mathbb{F}) \)-module because the Galois action on \( M(w + j) \) factors through \( \text{Gal}(\mathbb{F}[p^\infty])/\mathbb{F} \) and \( E \times_K F \) has good reduction [6, 1.9,(i),(ii)].

Using lemma [1.6] it is enough to prove the vanishing of some coinvariant modules in local Galois groups.

Since \( E \times_K F \) has good reduction everywhere, in particular in the places of \( S^* \setminus S_p \), the same is true for \( E^w = E \times_K E \times_K \cdots \times_K E \) over the field \( F \). Then, by the proved Weil conjectures on \( H^w(E^w, \mathbb{Q}_p(n)) \), the Frobenius at \( v \) over \( F \) does not act as identity in any subspace of the above cohomology group for \( w - 2n \neq 0 \). By local Tate duality \( M(w + j - 1)_{\text{Gal}(\mathbb{F}_S/\mathbb{F}_S)} \) is dual to \( M_{\text{Gal}(\mathbb{F}_S/\mathbb{F}_S)} \) and this module vanishes for \( w - 2(w + j - 1) \neq 0 \).
By hypothesis $p^*$ is regular in $F$, hence (Theorem 2.4)

$$\text{Gal}(\mathcal{F}_{S_p}(p)/F) \cong \text{Gal}(\mathcal{F}_p(p)/\mathcal{F}_p).$$

Thus, it remains only to show that

$$M(w + j - 1)_{\text{Gal}(\mathcal{F}_p(p)/\mathcal{F}_p)} = M(w + j - 1)_{\text{Gal}(\mathcal{F}_p(p)/\mathcal{F}_p)} = 0.$$

The $\text{Gal}(\mathcal{F}_p/\mathcal{F}_p)$-action on $M(w + j - 1)$ factors though the pro-$p$-quotient by the regularity and the condition $\mu_p \subseteq \mathcal{F}_p$. By [9] proof of Lemma 3.2.5] these coinvariants are always zero unless $a - (j - 1 + w) = 0$ and $b - (j - 1 + w) = 0$. If both are zero, we have $0 = a - (j - 1 + w) + b - (j - 1 + w) = w - 2j + 2 - 2w = -(w + 2j - 2)$ which is impossible by our hypothesis.

$$\square$$

**Remark 2.7.** The above result, and corollaries 2.8 and 2.10 below, are true assuming only that $S^*$ is a finite set of places which contains the primes above $p$ and the primes of $\mathcal{F}$ such that the inertia acts non-trivially in $M_{\theta} \times_K \mathcal{F}$ (use standard arguments like in the proof of [11, 2.2.16]).

**Corollary 2.8** (Jannsen’s conjecture under regularity). Let $p$ be a regular prime for $E$ and $F = K(E[p])$. Let $\mathcal{F}$ be an extension of $K$ inside $F$. Let $S^*$ be a finite set of primes containing the primes above $p$ and the primes where the elliptic curve $E \times_K \mathcal{F}$ has bad reduction. Let $M_{\theta}$ be $e_0(\otimes^w h^1(E))$. Then

$$H^2(\text{Gal}(\mathcal{F}_{S^*}/\mathcal{F}), M_{\theta}(n)) = 0$$

if $w + 2(n - w - 1) \neq 0$.

**Proof.** Is a direct consequence of the above corollary and lemma 1.2 $\square$

**Corollary 2.9.** Let $p$ be a regular prime for $E$ and $F = K(E[p])$. Let $\mathcal{F}$ be an extension of $K$ inside $F$. Let $S^*$ be a finite set of primes of $\mathcal{F}$ containing the primes above $p$ and the primes where the elliptic curve $E \times_K \mathcal{F}$ has bad reduction. Fix an integer $w \geq 1$. Then

$$H^2(\text{Gal}(\mathcal{F}_{S^*}/\mathcal{F}), (\otimes^w h^1(E)(1))_{\text{et}}(j)) = 0$$

for any integer $j$ such that $w + 2(j - 1) \neq 0$, where $(\otimes^w h^1(E)(1))_{\text{et}}(j)$ denotes the étale cohomology group $(\otimes^w H^1_e(E \times_K \mathcal{F}, \mathbb{Q}_p(1)))(j)$ which is the $p$-adic realization of the motive $(\otimes^w h^1(E))(w + j)$.

**Proof.** The integer idempotents $e_0$ give us a decomposition of the above motive

$$(\otimes^w h^1(E)(1))(j) = \prod_0 e_0((\otimes^w h^1(E)(1))(j)),$$

thus a decomposition of the $p$-adic realization

$$H^2(\text{Gal}(\mathcal{F}_{S^*}/\mathcal{F}), (\otimes^w h^1(E)(1))_{\text{et}}(j)) =$$

$$\otimes_0 H^2(\text{Gal}(\mathcal{F}_{S^*}/\mathcal{F}), e_0(\otimes^w h^1(E)(1))_{\text{et}}(j))$$

which is zero by Corollary 2.8 $\square$
Corollary 2.10. Let $p$ be a regular prime for $E$ and $\mathcal{F}$ and let be $w - 2n \neq 0$ and $2n - w - 2 \neq 0$. Let $S$ be a finite set of primes of $K$ containing the primes over $p$ and those where $E$ has bad reduction. Then

$$\dim_{\mathbb{Q}} H^1(\text{Gal}(KS/K), M_{\mathbb{Q}_p}(n)) + \dim_{\mathbb{Q}} H^1(\text{Gal}(KS/K), M_{\mathbb{Q}_p}(w + 1 - n)) = [K : \mathbb{Q}] \dim_{\mathbb{Q}} M_{\mathbb{Q}_p} = 4.$$ 

Moreover,

$$\dim_{\mathbb{Q}} H^1(\text{Gal}(KS/K), M_{\mathbb{Q}_p}(w + l + 1)) = 2 = \dim_{\mathbb{Q}} H^1(\text{Gal}(KS/K), M_{\mathbb{Q}_p}(-l))$$

for $-l \leq \min(a, b)$ if $a \neq b$(and $-l < a$ if $a = b$).

Proof. By the conditions on the weights and the regularity assumption we obtain $H^0(G_S, M_{\mathbb{Q}_p}(m)) = 0$ and $H^2(G_S, M_{\mathbb{Q}_p}(m)) = 0$, with $m = n$ and $m = w + 1 - n$ (and the same statements for $\bar{\theta}$ instead of $\theta$). Thus, the first equality holds by [9] corollary 1, proof of lemma 2. Observe that $M_{\mathbb{Q}_p} \cong e_\theta(\otimes^w T_{\mathbb{Q}_p}E(-1)) \otimes \mathbb{Q}_p$ has $\mathbb{Q}_p$-rank equal to 2 (see for example [1, \S 2]); therefore the second equality follows. The last statement follows from the Beilinson conjecture for these characters (proved by Deninger [5]) and [9] Lemma 2 at the twist $w + l + 1$. 

Observe $M_\varnothing(n)$ is a submotive of $h^w(E^w)(n)$, therefore using Soulé’s result (see remark 1.7) we obtain, without the regularity assumption:

Corollary 2.11. Let $p$ be a prime of $K$ such that $p|p$. Suppose $w - 2n \neq -2$ if $p$ is unramified, otherwise $n > w + 1$ or $n < 1$. Then,

$$H^2(K_p, M_{\mathbb{Q}_p}(n))) = 0.$$ 

Moreover if also $w - 2n \neq 0$, then

$$\dim_{\mathbb{Q}_p} H^1(K_p, M_{\mathbb{Q}_p}(n)) = [K_p : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} M_{\mathbb{Q}_p}(n).$$

3 The conjecture in the critical situation

We follow the notations of \S 2, but now $p$ does not necessarily split in $K$. The critical situation corresponds to realizations of the motive

$$M_\varnothing(n)$$

where $\theta$ has infinity type $(a, b$) and $n$ satisfies:

$$\min(a, b) < n \leq \max(a, b).$$

Let us impose that the weight is $\leq -3$; this means $a + b - 2n \leq -3$.

Tsuji in [16, \S 9, \S 10] proves the Jannsen conjecture in this case, $p$ inert and $\theta$ of infinite type $(k - j, 0)$ with $\vartheta_\varnothing = (\lambda, \ldots, \lambda)$, $n = k$ and $0 \leq -j < k$. One obtains also the case $p$ inert and $\theta$ of infinite type $(0, j - k)$, $n = j$ and $0 \leq -k < j$ by complex conjugation. The $p$-adic realization corresponds to:

$$M_{(k-\varnothing,0)\mathbb{Q}_p}(k) \cong V_p(E)^{\otimes k} \otimes \overline{V_p(E)}^{\otimes j};$$

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where \( \otimes \) is \( \otimes_{\mathcal{O}_K} \otimes_{\mathcal{O}_p} \mathbb{K} \). \( \mathcal{O}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K \) and the bar means complex conjugation.

Observe then that any \( M_p(n) \) in the critical situation has a \( p \)-adic realization (we write the situation \( b = \min(a, b) \)) isomorphic to

\[
V_p(E)^{\otimes n-b} \otimes \overline{V_p(E)}^{\otimes n-a}.
\]

Thus, if we write \( k = n-b \) and \( j = n-a \) we need to study Jannsen’s conjecture only for the \( p \)-adic realizations \( M_{(k-j,0)}(k) \) (the situation \( a = \min(a, b) \)) corresponds by complex conjugation to study the motives \( M_{(a-k-j)}(j)) \).

We have the following isomorphism,

\[
(M_{\mathbb{Q}_p}(n))^* (1) \cong M_{\mathbb{Q}_p}(-n+w+1).
\]

After all the above considerations we have in the critical situation:

**Theorem 3.1** (Tsuji, theorem 9.1 and prop.10.2 [16]). Let \( p \) be a prime \( \geq 5 \) which is inert in \( K \) and such that \( E \) has good reduction at the prime above \( p \) in \( K \). Consider \( M_p(n) \) with \( \min(a,b) < n \leq \max(a, b) \) (\( a \neq b \)) satisfying \( a+b-2n \leq -3 \), then

1. \( H^2(G_S, M_{\mathbb{Q}_p}(n)) = 0 \), and
2. \( H^2(G_S, M_{\mathbb{Q}_p}(1-n+w)) = 0 \), where \( G_S = \text{Gal}(K_S/K) \) with \( S \) a finite set of primes of \( K \) containing the primes dividing \( pf \).

**Theorem 3.2** (Tsuji, lemma 10.1 [16]). Let \( \min(a,b) < n \leq \max(a, b) \) (\( a \neq b \)) and \( a+b-2n \leq -3 \). Suppose \( p \) is inert in \( K \) and let \( p \) be the prime of \( K \) above \( p \).

1. Let \( q \) be any non-zero prime ideal of \( \mathcal{O}_K \) different to \( p \). Then,
   \[ H^i(K_q, M_{\mathbb{Q}_p}(n)) = 0 \] and \[ H^i(K_q, M_{\mathbb{Q}_p}(1-n+w)) = 0 \] for all \( i \).
2. \( H^i(K_p, M_{\mathbb{Q}_p}(n)) = 0 \) and \( H^i(K_p, M_{\mathbb{Q}_p}(1-n+w)) = 0 \) for \( i = 0 \) and 2.

**Remark 3.3.** The main points needed to obtain theorem [16] are [16] Theorem 6.1] and the main Iwasawa conjecture proved by Rubin. Both results are also known in the split situation by B.Han [8, §5.1] and Rubin [15] respectively. Suppose now that \( p \) splits in \( K, p = \mathfrak{p}^w \). In order to obtain [16] for the split case one could try to write in detail the second part of Tsuji’s paper [16] II replacing \( p \) by \( p = \mathfrak{p}^w \). Let us indicate some steps: rewrite [8] Theorem 4§5.1 with the unit \( e_{1,S,p} \) in the notation of [16] §5] to obtain [16] Theorem 6.1] (one will need a result similar to [11] Lemma 3.3]), replace \( \mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p \) in §8 and §9 of [16] and check that the arguments follow (remember that in our case \( L = K \) and \( A_p = A_p = \mathcal{O}_p \)).

4 The conjecture in the non-critical situation

Let us recall that we have fixed an imaginary quadratic field \( K \) with \( cl(K) = 1 \) and \( E \) has good reduction at the places above \( p \) in \( K \).

From the specialization of the elliptic polylogarithm [11], one can prove the equality between the image by the Soulé regulator map \( r_p \) of a module \( R_\theta = \alpha_\theta \mathcal{O}_K \) in \( K \)-theory for the motive \( e_\theta(\otimes h^i(E))(w+l+1) \) (see [3] 3.3]
for the precise definition of $\alpha$) and the image of the elliptic units module by a Soulé map $e_p$ (see [3, 4.6] for the definition of $e_p$) with $w - 2(w + l + 1) \leq -3$ with $-l \leq \min(a, b)$ (we refer to [3, cor.5.9] for this equality). This result is obtained, under some restrictions, from the main conjecture of Iwasawa theory for imaginary quadratic fields.

Let us fix in this section the following assumptions: let $p \geq 5$ be a prime with $p > N_{K/Q}$ and $\theta$ with infinite type $(a, b)$ such that $a \neq b(\text{mod}\{O_K^*\})$. Suppose also that $\mathcal{O}_K^* \to (O_K/\langle p \rangle)^*$ is injective and the $\Delta = \text{Gal}(K(E[p])/K)$-representation of $\text{Hom}_{\mathcal{O}_p}(M_{2 \mathbb{Z}_p}(w + l), \mathcal{O}_p)$ satisfies the Rubin’s theorem on the Iwasawa main conjecture [15], where $\mathcal{O}_p$ denotes $\mathcal{O}_K \otimes \mathbb{Z}_p$.

**Proposition 4.1.** Suppose that $H^2(\text{Gal}(K_S/K), M_{2 \mathbb{Z}_p}(w + l + 1)) = 0$ with $S$ finite set of primes of $K$ which contains the ones dividing $p$. Then $\tau_p(\mathcal{R}_p \otimes \mathbb{Z}_p)$ has $\mathbb{Q}_p$-rank equal to 2, in particular $e_p$ is injective.

**Proof.** The same arguments of the proof of [11, Prop.5.2.5] can be used with the diagram [3, Lemma 4.11] to obtain the result. $\square$

As observed by Tsuji [16, (11.12)], Jannsen’s conjecture can be obtained from the non-vanishing of the map $e_p$ into local cohomology of the elliptic units module if [16, Question 11.15] has a positive answer in the split case, and in this case one can show the same result of Tsuji by using the arguments of remark [3, 5]. But since we do not want to write all the technical details, we make the following

**assumption:** (I) the result [10, (11.12)] is true also when $p$ splits in $K$.

In his thesis [7], Geisser studies this Soulé map $e_p$ locally, i.e., with values in the local cohomology. In this study appear the values at non-critical values of the $p$-adic $L$-functions associated to $\psi_{\Omega_i}$ for $i = 1, 2$ (up to twist by cyclotomic character). The $\psi_{\Omega_i}$ comes from the Hecke character $\psi_{\theta}$ associated to $\theta$ as follows: $\psi_{\theta} : \mathbb{Q}_p \to \psi_{\Omega_1} \otimes \psi_{\Omega_2}$ (see for an extended explanation [7] or [1, §3]). Let us denote by $\iota : H^1(\text{Gal}(\mathbb{Q}_p/K), M_{2 \mathbb{Z}_p}(w + l + 1)) \to H^1(\text{Gal}(\mathbb{Q}_p/K), M_{2 \mathbb{Z}_p}(w + l + 1))$ the restriction map.

**Proposition 4.2** (3.4 in [1]). Let $p$ be a prime that splits in $K$.

Suppose furthermore $w = a + b \geq 1$, $a + l > 0$, $b + l > 0$ and $p > 3w + 2l + w + 1$. Then the length of the coimage of $\iota \circ \tau_p(\mathcal{R}_p)$ in $H^1(K_p, M_{2 \mathbb{Z}_p}(w + l + 1))$ is equal to the $p$-adic valuation of

$$G(\psi_{\Omega_1}, \kappa^l, u_1^{-a_0 - 1} - 1, u_2^{-b_0 - 1} - 1) G(\psi_{\Omega_2}, \kappa^l, u_1^{-b_0 - 1} - 1, u_2^{-a_0 - 1} - 1),$$

where $\kappa$ denotes the cyclotomic character of $G = \text{Gal}(K(E[p^\infty])/K)$ and $G$ denote the $p$-adic $L$-functions.

**Corollary 4.3.** Let us assume (I) and the hypothesis of proposition [4.2]. If the non-critical values obtained in the $p$-adic $L$-functions $G(\psi_{\Omega_1}, \kappa^l, u_1^{-a_0} - 1, u_2^{-b_0} - 1 - 1)$ and $G(\psi_{\Omega_2}, \kappa^l, u_1^{-b_0} - 1, u_2^{-a_0} - 1 - 1)$ do not vanish, then Jannsen’s conjecture is true for $M_{2 \mathbb{Z}_p}(-l)$, i.e.

$$H^2(G_2, M_{2 \mathbb{Z}_p}(-l)) = 0,$$

and the conclusions with $n = w + l + 1$ of corollary [2, 10] are also satisfied.
Proof. By the local Jannsen conjecture we obtain from 2.11 that
\[ \text{rank}_{\mathbb{Z}_p} H^1(K_p, M_{\theta \mathbb{Z}_p}(w + l + 1)) = 2. \]
By the non-vanishing hypotheses and 4.2, we see that the module \( \iota(r_p(R\theta)) \) has finite index in \( H^1(K_p, M_{\theta \mathbb{Z}_p}(w + l + 1)) \); therefore \( \dim_{\mathbb{Q}_p}(r_p(R\theta \otimes \mathbb{Q}_p)) = \dim_{\mathbb{Q}_p}(r_p(R\theta \otimes \mathbb{Q}_p)) = 2 \) and the Soulé map \( r_p \) does not vanish in \( R\theta \). This is equivalent by [16, (11.12)] to Jannsen’s conjecture for \( H^2(G_S, M_{\theta \mathbb{Q}_p}(-l)) \) (using the Tate-Poitou long exact sequence).

Remark 4.4. Moreover by [17, p.171,(11.12)] (assuming (I)), the Jannsen conjecture for \( M_{\theta \mathbb{Q}_p}(-l) \) implies the non-vanishing of the product of the non-critical values of the \( p \)-adic \( L \)-functions which appear in 4.3.

5 The weak Jannsen conjecture

We want to consider now the following weak form of Jannsen’s conjecture, which claims for our concrete realizations that: the Galois cohomology groups \( H^2(G_S, M_{\theta \mathbb{Q}_p}(n)) \) vanish for almost all Tate twist \( n \). This weak Jannsen conjecture is equivalent to the weak Leopoldt conjecture [14].

Proposition 5.1. Let us suppose that the cyclotomic \( \mu \)-invariant of every abelian extension of \( K \) is zero. Let \( S \) be a set of primes of \( K \) containing the primes of \( K \) dividing \( p^f \). Then for almost all \( n \) we have
\[ H^2(G_S, M_{\theta \mathbb{Q}_p}(n)) = 0. \]
Proof. Since we have a Hecke character the Galois group \( \text{Gal}(K'/K) \) fixing the \( p \)-torsion of \( M_{\theta \mathbb{Q}_p}(n)/M_{\theta \mathbb{Z}_p}(n) \) is an abelian extension of \( K \). By [14, B.2 Corollary (i)] (here we use the hypothesis that \( \mu \) vanishes for the cyclotomic extension of the abelian field \( K' \) over \( K \)) we obtain that
\[ e_{\tilde{\chi}} H^2(\text{Gal}(K_S/K(\mu_{p^\infty})), M_{\theta \mathbb{Z}_p}(n)) \]
is a \( \mathbb{Z}_p[[\text{Gal}(K(\mu_{p^\infty})/K)]]^{\tilde{\chi}} \)-torsion module, where \( e_{\tilde{\chi}} \) are the idempotents associated to characters \( \tilde{\chi} \) of \( \text{Gal}(K(\mu_{p^\infty})/C_\infty) \), with \( C_\infty \) the cyclotomic extension of \( K \). This fact is equivalent to
\[ H^2(\text{Gal}(K_S/K(\mu_{p^\infty})), M^*_{\theta \mathbb{Q}_p}/M^*_{\theta \mathbb{Z}_p}) = 0, \]
where \( * \) means \( \text{Hom}(, \mathbb{Q}_p) \) or \( \text{Hom}(, \mathbb{Z}_p) \) respectively [13, prop. 1.3.2]. Now use [9, Lemma 8] to obtain the result.

Remark 5.2. The work of Kato for CM modular forms [17, §15] gives a proof of the weak Jannsen conjecture [2.7] without any assumption on the \( \mu \) invariant. Moreover in [4] we obtain a different proof of proposition [2.7] without the \( \mu \) vanishing assumption using Iwasawa modules in two variables, imposing that \( p + 1 | b - a | = a - b \) if \( p \) is inert in \( K \), or \( p - 1 | b - a | = a - b \) if \( p \) splits.
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