Adversarial Radar Inference. From Inverse Tracking to Inverse Reinforcement Learning of Cognitive Radar

Vikram Krishnamurthy, School of Electrical & Computer Engineering, Cornell University, vikramk@cornell.edu

1 Introduction

Cognitive sensing refers to a reconfigurable sensor that dynamically adapts its sensing mechanism by using stochastic control to optimize its sensing resources. For example, cognitive radars are sophisticated dynamical systems; they use stochastic control to sense the environment, learn from it relevant information about the target and background, then adapt the radar sensor to satisfy the needs of their mission. The last two decades have witnessed intense research in cognitive/adaptive radars [15, 32, 33, 21].

This chapter discusses addresses the next logical step, namely inverse cognitive sensing. By observing the emissions of a sensor (e.g. radar or in general a controlled stochastic dynamical system) in real time, how can we detect if the sensor is cognitive (rational utility maximizer) and how can we predict its future actions? The scientific challenges involve extending Bayesian filtering, inverse reinforcement learning and stochastic optimization of dynamical systems to a data-driven adversarial setting. Our methodology transcends classical statistical signal processing (sensing and estimation/detection theory) to address the deeper issue of how to infer strategy from sensing. The generative models, adversarial inference algorithms and associated mathematical analysis will lead to advances in understanding how sophisticated adaptive sensors such as cognitive radars operate. As will be discussed below, the framework involves the interaction of two DDDAS (dynamic data driven applications systems).

Figure 1: Schematic of Adversarial Inference Problem. “Our” side is a drone/UAV or electromagnetic signal that probes the enemy’s cognitive sensor (e.g. radar). Given our state sequence \(\{x_k\}\) and observed enemy’s actions \(\{a_k\}\), our aims are to: (i) Estimate the enemy’s belief (posterior) \(\pi_k\) and its sensor accuracy (ii) Optimally probe the enemy with \(\{x_k\}\) to minimize the variance of our estimate of its accuracy (iii) Determine if the enemy is cognitive (rational utility maximizer), and estimate its utility function (iv) Track time varying equilibria in a game theoretic setting.
1.1 Objectives

The central theme involves an adversarial signal processing problem comprised of “us” and an “adversary”. “Us” refers to a drone/UA V or electromagnetic signal that probes an “adversary” cognitive sensing system. Figure 1 shows the schematic setup. A cognitive sensor observes our kinematic state \( x_k \) in noise as the observation \( y_k \). It then uses a Bayesian tracker to update its posterior distribution \( \pi_k \) of our state \( x_k \) and chooses an action \( u_k \) based on this posterior. We observe the sensor’s action in noise as \( a_k \). Given knowledge of “our” state sequence \( \{x_k\} \) and the observed actions \( \{a_k\} \) taken by the enemy’s sensor, this project focuses on the following interrelated aspects; each topic below is backed by extensive preliminary research and results:

1. **Inverse filtering and Estimating the Enemy’s Sensor Gain.** How to estimate the enemy’s estimate of us? Suppose the enemy observes our state in noise; updates its posterior distribution \( \pi_k \) of our state \( x_k \) and then chooses an action \( u_k \) based on this posterior. Given knowledge of “our” state and sequence of enemy’s actions observed in noise \( \{a_k\} \), how can the enemy’s posterior distribution (random measure) be estimated? We will develop computationally efficient filtering algorithms with performance bounds.

A related question is: How to remotely estimate the enemy’s sensor observation likelihood when it is estimating us? This is important because it tells us how accurate the enemy’s sensor is; in the context of Figure 1 it tells us, how accurately the enemy tracks our drone. The data we have access to is our state (probe signal) \( \{x_k\} \) and measurements of the enemy’s radar actions \( \{a_k\} \). Estimating the enemy’s sensor accuracy is non-trivial with several challenges. First, even though we know our state and state dynamics model (transition law), the enemy does not. The enemy needs to estimate our state and state transition law based on our trajectory; and we need to estimate the enemy’s estimate of our state transition law. Second, computing the maximum likelihood estimate of the enemy’s sensor gain involves inverse filtering; as shown in our preliminary work, the convergence rate of numerical algorithms to compute the estimate is substantially slower than classical maximum likelihood estimation. We will also study identifiability issues and consistency properties of the resulting maximum likelihood estimator.

2. **Optimal Probing of Adversary.** Referring to Figure 1 how can we optimally probe the enemy’s sensor to minimize the covariance of our estimate of the enemy’s observation likelihood? “Our” state can be viewed as a probe signal which causes the adversary’s radar to act; so choosing the optimal state sequence is an input/experimental design problem. We will consider two approaches. The first approach is analytical – using stochastic dominance, we will establish a partial ordering amongst probe transition matrices which results in a corresponding ordering of our SNR of the adversary’s actions. The second approach is numerical – we will construct stochastic gradient algorithms to optimize the probe transition matrix.

3. **Revealed Preferences and Inverse Reinforcement Learning.** Our working assumption is that a cognitive sensor satisfies economic rationality, i.e., a cognitive sensor is a utility maximizer that optimizes its actions \( a_k \) subject to physical level (Bayesian filter) constraints. How can we detect this utility maximization behavior? Nonparametric detection of utility maximization behavior is the central theme of revealed preferences in microeconomics. A remarkable result is Afriat’s theorem: it provides a necessary and sufficient condition for a finite dataset to have originated from a utility maximizer. We will develop constrained utility estimation methods in a Bayesian framework that account for signal processing constraints introduced by the Bayesian tracker. These impose nonlinear constraints involving the spectrum (eigenvalues) of the achieved covariance of
the tracker; such spectral revealed preferences require generalizing the classical Afriat theorem.

When the enemy’s actions are observed in noise, how can a statistical detector be constructed to test for utility maximization behavior? We will also address the related issue of: How can our state (probe signal) be chosen by us to minimize the Type 2 error of detecting if the adversary is deploying an economic rational strategy, subject to a constraint on the Type 1 detection error?

1.2 Perspective and Context. Relevance to DDDAS

The adversarial dynamics considered in this chapter fit naturally within the Dynamic Data Driven Applications Systems (DDDAS) paradigm. The adversary’s radar senses, adapts and learns from us. In turn we adapt sense and learn from the adversary. So in simple terms we are modeling and analyzing the interaction of two DDDAS. In this context this chapter has two major themes: inverse filtering which is a Bayesian framework for interacting DDDAS, and inverse cognitive sensing which is a non-parametric approach for utility estimation for interacting DDDAS.

Here are some unusual features that motivate the ideas in this chapter:

(i) Slow Learning: Inverse filtering (estimating the enemy’s estimate of us) has slow convergence - given \( n \) data points the covariance decreases as \( O(n^{-1/3}) \), whereas the classical Kalman filter covariance decreases as \( O(n^{-1}) \). The change in convergence speed exhibits a remarkable phase transition.

(ii) Data driven approach: The revealed preference approach is fundamentally different to the model-centric theme that is widely used in signal processing where one postulates an objective function (typically convex) and then proposes optimization algorithms. In contrast, the revealed preference approach is data centric - given a dataset, we wish to determine if is consistent with utility maximization, i.e., if the sensor satisfies economics-based rationality.

(iii) Classical inverse reinforcement learning aims to estimate the utility function of a decision maker by observing its decisions and input state [43]; the existence of a utility function (rationality) is assumed implicitly. The revealed preference approach considered in this chapter addresses a deeper and more fundamental question: does a utility function exist that rationalizes the given data (with signal processing constraints) and if yes, estimate it. As discussed below, inverse reinforcement learning is a trivial case of revealed preferences.

(iv) Finally, this chapter is an early step in understanding how to design stealthy cognitive sensors whose cognitive functionality is difficult to detect by an observer. (In simple terms, how to design a smart sensor that acts dumb?). This generalizes the physics based low-probability of intercept (LPI) requirement of radar (which requires low power emission) to the systems-level issue: How should the sensor choose its actions in order to avoid detection of its cognition? This work is also motivated by the design of counter-autonomous systems: given measurements of the actions of an autonomous adversary, how can our counter-autonomous system estimate the underlying belief of the adversary, predict future actions and therefore guard against these actions. Specifically, [37] places counter unmanned autonomous systems at a level of abstraction above the physical sensors/actuators/weapons and datalink layers; and below the human controller layer.
2 Inverse Filtering and Estimating Enemy’s Sensor

This section discusses inverse filtering in an adversarial system, see Figure 2 for the schematic setup. Our main ideas involve estimating the enemy’s estimate of us and estimating the enemy’s sensor observation likelihood.

2.1 Background and Preliminary Work

(a) Schematic of Adversarial Inference Problem. Our side is a drone/UAV or electromagnetic signal that probes the adversary’s cognitive radar system.

(b) Graphical representation of inverse filtering problem (1). The shaded nodes are known to us, while the white nodes are unknown. This is different to classical filtering where \( y_1, \ldots, y_k \) are known and \( \pi_k \) is to be estimated.

Figure 2: Schematic and Graphical representation of adversarial inverse filtering problem considered in this chapter.

The problem formulation involves two players; “us” and “adversary”. With \( k = 1, 2, \ldots \) denoting discrete time, the model has the following dynamics:

\[
x_k \sim P_{x|x_k-1}, \quad x_0 \sim \pi_0
\]
\[
y_k \sim B_{x,y}, \quad \pi_k = T(\pi_{k-1}, y_k)
\]
\[
a_k \sim G_{\pi_k}, \quad a_k = p(a|\pi_k)
\]

Let us explain the notation in (1):

- \( x_k \in \mathcal{X} \) is our Markovian state with transition kernel \( P_{x|x_k-1} \), prior \( \pi_0 \) and state space \( \mathcal{X} \).
- \( y_k \) is the adversary’s noisy observation of our state \( x_k \); with observation likelihoods \( B_{x,y} \).
- \( \pi_k = p(x_k|y_{1:k}) \) is the adversary’s belief (posterior) of our state \( x_k \) where \( y_{1:k} \) denotes the sequence \( y_1, \ldots, y_k \). The operator \( T \) in (1) is the classical Bayesian optimal filter

\[
T(\pi, y)(x) = \frac{B_{x,y} \int_{\mathcal{X}} P_{\xi|x} \pi(\xi) d\xi}{\int_{\mathcal{Y}} \int_{\mathcal{X}} B_{x,y} \int_{\mathcal{X}} P_{\xi|x} \pi(\xi) d\xi dx}
\]

Let \( \Pi \) denote the space of all such beliefs. When the state space \( \mathcal{X} \) is finite, then \( \Pi \) is the unit \( X - 1 \) dimensional simplex of \( X \)-dimensional probability mass functions.
• $a_k$ denotes our measurement of the adversary’s action based on its current belief $\pi_k$. More explicitly, the adversary chooses an action $u_k$ as a deterministic function of $\pi_k$ and we observe $u_k$ in noise as $a_k$. We encode this as $G_{\pi_k,a_k}$; this is the conditional probability of observing action $a_k$ given the adversary’s belief $\pi_k$.

Figure 2 displays a schematic and graphical representation of the model (1). The schematic model shows “us” and the adversary’s variables. In the graphical model, the shaded nodes are known to us, while the white nodes are computed by the adversary and unknown to us. This is in contrast to classical filtering where $y_k$ is known to us and $x_k$ is to be estimated, i.e, $\pi_k$ is to be computed.

**Aim:** Referring to model (1) and Figure 2, our aims are:

1. How to estimate the adversary’s belief given measurements of its actions (which are based on its filtered estimate of our state)? In other words, assuming probability distributions $P, B, G$ are known, we aim to estimate the adversary’s belief $\pi_k$ at each time $k$, by computing posterior $p(\pi_k | \pi_0, x_0:k, a_1:k)$.

2. How to estimate the enemy’s observation kernel $B$, i.e its sensor gain? This tells us how accurate the enemy’s sensor is.

From a practical point of view, estimating the enemy’s belief and sensor parameters allows us to calibrate its accuracy and predict (in a Bayesian sense) future actions of the enemy.

**Related Works.** In our recent works [39, 40, 38], the mapping from belief $\pi$ to enemy’s action $a$ was assumed deterministic. Specifically, [39] gives a deterministic regression based approach to estimate the adversary’s model parameters in a Hidden Markov model. In comparison, our proposed research here assumes a probabilistic map between $\pi$ and $a$ and we develop Bayesian filtering algorithms for estimating the posterior along with MLE algorithms for $\theta$. Estimating/reconstructing the posterior given decisions based on the posterior is studied in microeconomics under the area of social learning [14, 41] and game-theoretic generalizations [6]. There are strong parallels between inverse filtering and Bayesian social learning [14], [29, 28, 27]; the key difference is that social learning aims to estimate the underlying state given noisy posteriors, whereas our aim is to estimate the posterior given noisy measurements of the posterior and the underlying state. Recently, [24] used cascaded Kalman filters for LQG control over communication channels.

### 2.2 Inverse Filtering Algorithms

*How to estimate the enemy’s posterior distribution $\pi_k$ of us?*

Here we discuss inverse filtering for special cases of the model (1). Define the posterior distribution $\alpha_k(\pi_k) = p(\pi_k | a_{1:k}, x_{0:k})$ of the adversary’s posterior distribution given our state sequence $x_{0:k}$ and actions $a_{1:k}$. Note that the posterior $\alpha_k(\cdot)$ is a random measure since it is a posterior distribution of the adversary’s posterior distribution (belief) $\pi_k$. By using a discrete time version of Girsanov’s theorem and appropriate change of measure [17] (or a careful application of Bayes rule) we can derive the following functional recursion for $\alpha_k$:

$$
\alpha_{k+1}(\pi) = \frac{G_{\pi,a_{k+1}} \int B_{x_{k+1},y_{k+1},\pi_k} \alpha_k(\pi_k)d\pi_k}{\int G_{\pi,a_{k+1}} \int B_{x_{k+1},y_{k+1},\pi_k} \alpha_k(\pi_k)d\pi_k d\pi} 
$$

---

1This chapter deals with discrete time. Although we will not pursue it here, our recent paper [35] uses a similar continuous time formulation. This yields interesting results involving Malliavin derivatives and stochastic calculus.
Here \( y_{\pi_k} \) is the observation such that \( \pi = T(\pi_k, y) \) where \( T \) is the adversary’s filter \(^2\). We call \(^3\) the *optimal inverse filter* since it yields the Bayesian posterior of the adversary’s belief given our state and noisy measurements of the adversary’s actions.

**Example: Inverse Kalman Filter**

We consider a second special case of \(^3\) where the inverse filtering problem admits a finite dimensional characterization in terms of the Kalman filter. Consider a linear Gaussian state space model

\[
\begin{align*}
  x_{k+1} &= Ax_k + w_k, \quad x_0 \sim \pi_0 \\
  y_k &= Cx_k + v_k
\end{align*}
\]

where \( x_k \in \mathcal{X} = \mathbb{R}^X \) is “our” state with initial density \( \pi_0 \sim N(\hat{x}_0, \Sigma_0) \), \( y_k \in \mathcal{Y} = \mathbb{R}^Y \) denotes the adversary’s observations, \( w_k \sim N(0, Q_k) \), \( v_k \sim N(0, R_k) \) and \( \{w_k\}, \{v_k\} \) are mutually independent i.i.d. processes.

Based on observations \( y_{1:k} \), the adversary computes the belief \( \pi_k = N(\hat{x}_k, \Sigma_k) \) where \( \hat{x}_k \) is the conditional mean state estimate and \( \Sigma_k \) is the covariance; these are computed via the classical Kalman filter equations\(^2\)

\[
\begin{align*}
  \Sigma_{k+1|k} &= A \Sigma_k A' + Q \\
  S_{k+1} &= C \Sigma_{k+1|k} C' + R \\
  \hat{x}_{k+1} &= A \hat{x}_k + \Sigma_{k+1|k} C' (y_{k+1} - CA \hat{x}_k) \\
  \Sigma_{k+1} &= \Sigma_{k+1|k} - \Sigma_{k+1|k} C' S_k^{-1} C \Sigma_{k+1|k}
\end{align*}
\]

The adversary then chooses its action as \( \bar{a}_k = \phi(\Sigma_{k}) \hat{x}_k \) for some pre-specified function\(^3\) \( \phi \). We measure the adversary’s action as

\[
a_k = \phi(\Sigma_k) \hat{x}_k + \epsilon_k, \quad \epsilon_k \sim \text{iid } N(0, \sigma^2_{\epsilon})
\]

The Kalman covariance \( \Sigma_k \) is deterministic and fully determined by the model parameters. So to estimate the belief \( \pi_k = N(\hat{x}_k, \Sigma_k) \) we only need to estimate \( \hat{x}_k \) at each time \( k \) given \( a_{1:k}, x_{0:k} \). Substituting \(^4\) for \( y_{k+1} \) in \(^7\), we see that \(^7\), \(^8\) constitute a linear Gaussian system with unobserved state \( \hat{x}_k \), observations \( a_k \), and known exogenous input \( x_k \):

\[
\begin{align*}
  \hat{x}_{k+1} &= (I - \psi_{k+1} C) A \hat{x}_k + \psi_{k+1} x_{k+1} + \psi_{k+1} C \hat{x}_k + \psi_{k+1} C x_{k+1} \\
  a_k &= \phi(\Sigma_k) \hat{x}_k + \epsilon_k, \quad \epsilon_k \sim \text{iid } N(0, \sigma^2_{\epsilon}) \\
  \text{where } \psi_{k+1} &= \Sigma_{k+1|k} C' S_k^{-1}
\end{align*}
\]

\(^2\)For localization problems, we will use the information filter form:

\[
\begin{align*}
  \Sigma_{k+1}^{-1} &= \Sigma_{k+1|k}^{-1} + C' R^{-1} C, \quad \psi_{k+1} = \Sigma_{k+1} C' R^{-1}
\end{align*}
\]

Similarly, the inverse Kalman filter in information form reads

\[
\begin{align*}
  \hat{\Sigma}_{k+1}^{-1} &= \hat{\Sigma}_{k+1|k}^{-1} + \hat{C}_{k+1} R^{-1} \hat{C}_{k+1}, \quad \hat{\psi}_{k+1} = \hat{\Sigma}_{k+1} \hat{C}'_{k+1} R^{-1}
\end{align*}
\]

\(^3\)This choice is motivated by linear quadratic Gaussian control where the action (control) is chosen as a linear function of the estimated state \( \hat{x}_k \) weighted by the covariance matrix.
ψ_{k+1} is called the Kalman gain.

To summarize, our filtered estimate of the adversary’s filtered estimate \( \hat{\chi}_k \) given measurements \( a_{1:k}, x_{0:k} \) is achieved by running “our” Kalman filter on the linear Gaussian state space model (9), where \( \hat{\chi}_k, \psi_k, \Sigma_k \) in (9) are generated by the adversary’s Kalman filter. Therefore, our Kalman filter uses the parameters

\[
\begin{align*}
\tilde{A}_k &= (I - \psi_{k+1} C) A, \quad \tilde{F}_k = \psi_{k+1} C, \quad \tilde{C}_k = \phi(\Sigma_k), \\
\tilde{Q}_k &= \psi_{k+1} \psi_{k+1}', \quad \tilde{R} = \sigma_e^2.
\end{align*}
\]

The equations of our inverse Kalman filter are:

\[
\begin{align*}
\Sigma_{k+1|k} &= \tilde{A}_k \Sigma_k \tilde{A}_k' + \tilde{Q}_k \\
\tilde{s}_{k+1} &= \tilde{C}_{k+1} \Sigma_{k+1|k} \tilde{C}_{k+1}' + \tilde{R} \\
\tilde{\chi}_{k+1} &= \tilde{A}_k \tilde{\chi}_k + \tilde{C}_{k+1} \Sigma_{k+1|k} \tilde{C}_{k+1}' \\
&\quad \times \left[ a_{k+1} - \tilde{C}_{k+1} \left( \tilde{A}_k \tilde{\chi}_k + \tilde{F}_k x_{k+1} \right) \right] \\
\Sigma_{k+1} &= \Sigma_{k+1|k} - \Sigma_{k+1|k} \tilde{C}_{k+1}' \Sigma_{k+1|k} \tilde{C}_{k+1} \Sigma_{k+1|k}
\end{align*}
\]

Note \( \tilde{\chi}_k \) and \( \Sigma_k \) denote our conditional mean estimate and covariance of the adversary’s conditional mean \( \hat{\chi}_k \). The computational cost of the inverse Kalman filter is identical to the classical Kalman filter, namely \( O(X^2) \) computations at each time step.

**Remarks:**

1. As discussed in [36], inverse Hidden Markov model (HMM) filters and inverse particle filters can be derived. For example, the inverse HMM filter deals with the case when \( \pi_k \) is computed via a Hidden Markov model (HMM) filter and the estimates of the HMM filter are observed in noise. In this case the inverse filter has a computational cost that grows exponentially with the size of the observation alphabet.

2. In filtering it is important to determine conditions under which the filter forgets its initial condition geometrically fast (this is often called stability of the filter). In future work, it is worthwhile obtaining minorization conditions for the geometric ergodicity of the inverse HMM filter, namely, conditions on our transition matrix, enemy’s observation matrix and our observation of the enemy’s action matrix so that the inverse filter forgets its initial condition geometrically fast.

3. A general approximation solution for (3) involves sequential Markov chain Monte-Carlo (particle filtering). In particle filtering, cases where it is possible to sample from the so called optimal importance function are of significant interest [44] [12]. In inverse filtering, our paper [36] shows that the optimal importance function can be determined explicitly due to the structure of the inverse filtering problem. Specifically, in our case, the “optimal” importance density is \( \pi^* = p(\pi_k | y_k, \pi_{k-1}, y_{k-1}, x_k, a_k) \). Note that in our case

\[
\pi^* = p(\pi_k | \pi_{k-1}, y_k) p(y_k | x_k, a_k) = \delta(\pi_k - T(\pi_{k-1}, y_k)) \ p(y_k | x_k)
\]

is straightforward to sample from. There has been a substantial amount of recent research in finite sample concentration bounds for the particle filter. We will use these results to evaluate the so called sample complexity of the resulting particle filter.
2.3 Slow Learning: Cascaded Kalman Filters and Phase-Transition

How to quantify slow learning in cascaded Bayesian filters?

In simple terms, inverse filtering discussed above involves cascaded optimal filters where the second filter uses the estimates generated by the first filter. We now outline our plans to study a remarkable property involving two cascaded Kalman filter where the first Kalman filter feeds its estimate to the second Kalman filter, and the second Kalman filter feeds its estimate to the first Kalman filter. Below we will discuss an adversarial example involving localizing a drone.

![Figure 3: Slow Learning with Two Kalman Filters in Series](image)

To illustrate the main idea, consider a Gaussian random variable scalar state \( x_0 \in \mathbb{R} \) (instead of a random process). The first Kalman filter obtains Gaussian measurements \( y_k \) of \( x_0 \). It uses \( y_k \) together with the prior \( \hat{x}_{k-1} \) from the second Kalman filter to compute the estimate \( \hat{x}_k \) at time \( k \). The second Kalman filter observes \( \hat{x}_k \) from the first Kalman filter in Gaussian noise as \( a_k = \hat{x}_k + \epsilon_k \) where \( \epsilon_k \) is an iid Gaussian sequence with zero mean and variance \( \sigma^2_\epsilon \). The second Kalman filter then uses \( a_k \) to estimate \( x_0 \). Call the estimate generated by the second Kalman filter as \( \hat{\hat{x}}_k \). This estimate \( \hat{\hat{x}}_k \) is then fed back to the first Kalman filter.

Two remarkable properties of the above setup are (see also [51, 52, 14]):

- The covariance decreases with time \( k \) as \( \Sigma_k = O(k^{-1/3}) \) instead of \( O(k^{-1}) \) in the standard Kalman filter. This depicts “slow learning” of the inverse Kalman filter (in Fig.3).

- This rate of slow learning is independent of the variance \( \sigma^2_\epsilon \) as long as \( \sigma^2_\epsilon \) is strictly positive. Note that if \( \sigma^2_\epsilon = 0 \), then \( \Sigma_k = O(k^{-1}) \). This shows that a cascaded model of two Kalman filters where the second Kalman filter observes the first Kalman has a phase change in behavior: when the estimates of the first Kalman filter are corrupted even by small amounts of noise, the convergence rate slows down drastically to \( O(k^{-1/3}) \).

Example. Sequential Localization Game

Thus far we assumed that \( x_0 \) is known to us and our aim was to estimate the adversary’s estimate. Consider now the following localization game. Suppose that neither us nor the adversary know the underlying state \( x_0 \). For example, \( x_0 \) could denote the threat level of an autonomous drone hovering relative to a specific location as shown in Figure 4. Second, there is feedback: the adversary’s action affects our estimate and our action affects the adversary’s estimate. The setup is naturally formulated in game-theoretic terms. The adversary and us play a sequential game to localize (estimate) \( x_0 \).
1. At odd time instants $k = 1, 3, \ldots$, the adversary takes active measurements of the target and makes decisions based on its estimate. We eavesdrop on the adversary’s decisions and apply Bayes rule to estimate (localize) the target’s state $x_0$. The details are as follows:

(i) The adversary observes our action $u_{k-1} = \hat{x}_{k-1}$ in noise as $a_{k-1} = \hat{x}_{k-1} + \varepsilon_{k-1}$ where $\varepsilon_{k-1} \sim N(0, \sigma^2\varepsilon)$. The adversary assumes that our state estimate is $a_{k-1}$.

(ii) The adversary takes a direct measurement $y_k$ of $x_0$ and uses its Kalman filter (7) to update its state estimate:

$$\hat{x}_k = (1 - \psi_k)a_{k-1} + \psi_kv_k,$$

where $\psi_k$ is the associated covariance update.

(iii) The adversary then takes action $u_k = \hat{x}_k$. This action minimizes the mean square error of the estimate.

(iv) We eavesdrop on (observe) the adversary’s action in noise as $a_k = \hat{x}_k + \varepsilon_k$. Assume $\varepsilon_k \sim N(0, \sigma^2\varepsilon)$ iid. So

$$a_k = (1 - \psi_k)a_{k-1} + \psi_kv_k + \psi_kx_0 + \varepsilon_k.$$

Since $a_{k-1} = u_{k-1} + \varepsilon_{k-1}$, we can define an observation $z_k$ that is informationally equivalent to $a_k$ as

$$z_k \overset{\text{defn}}{=} \frac{a_k}{\psi_k} - \frac{1 - \psi_k}{\psi_k}u_{k-1} = \frac{1 - \psi_k}{\psi_k} \varepsilon_{k-1} + v_k + \frac{\varepsilon_k}{\psi_k} + x_0 \quad (12)$$

We know $z_k$; $u_{k-1}$ was our action at time $k - 1$, and $a_k$ is our measurement of the adversary’s action at time $k$.

To summarize, at odd time instants, $z_k$ specified in (12) is our effective observation of the unknown state $x_0$ purely based on sensing the adversary’s actions.

2. At even time instants $k = 2, 4, \ldots$, we take active measurements $y_k$ of the target’s state $x_0$. Then we update our estimate of our target using the Kalman filter.

The above setup constitutes a social learning sequential game [42]. Since the decisions are made myopically each time (to minimize the mean square error), the strategy profile at each time constitutes a Nash equilibrium. More importantly [42, Theorem 5], the asymptotic behavior (in time) is captured by a social learning equilibrium (SLE).

**Theorem 2.1** [36]. Consider the sequential game formulation for localizing the random variable state $x_0$. Then for large $k$, the covariance of our estimate of the state $x_0$ is $\Sigma_k = 2k^{-1}$ (which is twice the covariance for classical localization).

**Remark.** Theorem 2.1 has two interesting consequences.
1. The asymptotic covariance is independent of $\sigma^2_\epsilon$ (i.e. SNR of our observation of the adversary’s action) as long as $\sigma^2_\epsilon > 0$. If $\sigma^2_\epsilon = 0$, there is a phase change and the covariance $\Sigma_k = k^{-1}$ as in the standard Kalman filter.

2. In the special case where only (12) is observed at each time (and the $\epsilon_{k-1}$ term is omitted), the formulation reduces to the Bayesian social learning problem of [14, Chapter 3]. In that case, [14, Proposition 3.1] shows that $\Sigma_k = O(k^{-1/3})$. The remarkable property is that this rate of slow learning is independent of the variance $\sigma^2_\epsilon$ as long as $\sigma^2_\epsilon$ is strictly positive [51, 52]. As discussed in [14], this result implies that models where one assumes that the enemy’s action is observed perfectly are not robust; any noise in the observation process drastically reduces the rate of convergence of the estimator.

To summarize, Theorem 2.1 confirms the intuition that sequentially learning the state indirectly from the adversary’s actions (and the adversary learning the state from our actions) slows down the convergence of the localization estimator.

The above setup constitutes a social learning sequential game [42]. Since the decisions are made myopically each time (to minimize the mean square error), the strategy profile at each time constitutes a Nash equilibrium. More importantly [42, Theorem 5], the asymptotic behavior (in time) is captured by a social learning equilibrium (SLE).

2.4 Estimating the Enemy’s Sensor Gain

Thus far we have discussed Bayesian estimation of the adversary’s belief state. We now discuss how to estimate the enemy’s sensor observation kernel $B$ in (1) which quantifies the accuracy of the adversary’s sensors. We assume that $B$ is parametrized by an $M$-dimensional vector $\theta \in \Theta$ where $\Theta$ is a compact subset of $\mathbb{R}^M$. Denote the parametrized observation kernel as $B^{\theta}$. Assume that both us and the adversary know $P$ (state transition kernel) and $G$ (probabilistic map from adversary’s belief to its action). Then given our state sequence $x_{0:N}$ and adversary’s action sequence $a_{1:N}$, our aim is to compute the maximum likelihood estimate (MLE) of $\theta$. That is, with $L_N(\theta)$ denoting the log likelihood, the aim is to compute

$$
\theta^* = \arg\max_{\theta \in \Theta} L_N(\theta), \quad L_N(\theta) = \log p(x_{0:N}, a_{1:N} | \theta).
$$

The likelihood can be evaluated from the un-normalized inverse filtering recursion [3]

$$
L_N(\theta) = \log \int_\Pi q_N^\theta(\pi) d\pi, \quad q_k^\theta(\pi) = G_{\pi, a_{k+1}} \int_\Pi B_{x_k+1, y_k, \pi}^\theta q_{k+1}^\theta(\pi_k) d\pi_k
$$

Given (13), a local stationary point of the likelihood can be computed using a general purpose numerical optimization algorithm.

Remark. EM Algorithm is not useful: The Expectation Maximization (EM) algorithm is a popular numerical algorithm for computing the MLE especially when the Maximization (M) step can be computed in closed form. Unfortunately, for estimating the adversary’s observation model, due to the time evolving dynamics in the inverse Kalman filter, the EM algorithm is not useful since the M-step involves a non-convex optimization that cannot be done in closed form. There is no obvious way of choosing the latent variables to avoid this.

As mentioned earlier, otherwise the adversary estimates $P$ as $\hat{P}$ and we need to estimate the adversary’s estimate as $\hat{\hat{P}}$. This makes the estimation task substantially more complex.
Example. Estimating Adversary’s Gain in Linear Gaussian case

Consider the setup in Sec. 2.2 where our dynamics are linear Gaussian and the adversary observes our state linearly in Gaussian noise (4). The adversary estimates our state using a Kalman filter, and we estimate the adversary’s estimate using the inverse Kalman filter (9). Using (9), (10), the log likelihood for the adversary’s observation gain matrix $\alpha = C$ based on our measurements is

$$L_N(\theta) = \text{const} - \frac{1}{2} \sum_{k=1}^{N} \log |\tilde{S}_k^\alpha| - \frac{1}{2} \sum_{k=1}^{N} t_k'(\tilde{S}_k^\alpha)^{-1} t_k$$

$$t_k = a_k - \bar{C}_k^\alpha \bar{A}_k^\alpha \tilde{x}_{k-1} - \bar{F}_k^\alpha \bar{x}_{k-1}$$  \hspace{1cm} (14)

where $t_k$ are the innovations of the inverse Kalman filter (11). In (14), our state $x_{k-1}$ is known to us and therefore is a known exogenous input. Also note from (10) that $\bar{A}_k, \bar{F}_k$ are explicit functions of $C$, while $\bar{C}_k$ and $\bar{Q}_k$ depend on $C$ via the adversary’s Kalman filter.

Using (13), the log likelihood for the adversary’s observation gain matrix $\alpha = C$ can be evaluated.

To provide insight, Figure 5 displays the log-likelihood versus adversary’s gain matrix $C$ in the scalar case for 1000 data points. The four sub-figures correspond to true values of $C^o = 1.5, 3$ respectively.

Each sub-figure in Figure 5 has two plots. The plot in red is the log-likelihood of $\hat{C} \in (0, 10]$ evaluated based on the adversary’s observations using the standard Kalman filter. (This is the classical log-likelihood of the observation gain of a Gaussian state space model.) The plot in blue is the log-likelihood of $C \in (0, 10]$ computed using our measurements of the adversary’s action using the inverse Kalman filter (where the adversary first estimates our state using a Kalman filter) - we call this the inverse case.

Figure 5 shows that the log likelihood in the inverse case (blue plots) has a less pronounced maximum compared to the standard case (red plots). Therefore, numerical algorithms for computing the MLE of the enemy’s gain using our observations of the adversary’s actions (via the inverse
Kalman filter) will converge much more slowly than the classical MLE (based on the adversary’s observations). This is intuitive since our estimate of the adversary’s parameter is based on the adversary’s estimate of our state and so has more noise.

**Cramer Rao (CR) bounds.** Is it instructive to compare the CR bounds for MLE of $C$ for the classic model versus that of the inverse Kalman filter model. Table 1 displays the CR bounds (reciprocal of expected Fisher information) for the four examples considered above evaluated using via the algorithm in [13]. It shows that the covariance lower bound for the inverse case is substantially higher than that for the classic case. This is consistent with the intuition that estimating the adversary’s parameter based on its actions (which is based on its estimate of us) is more difficult than directly estimating $C$ in a classical state space model.

**Remarks:**

1. **Consistency of MLE.** The above example shows that the likelihood surface of $L_N(\theta) = \log p(x_{0:N}, a_{1:N}|\theta)$ is flat and hence compute the MLE numerically can be difficult. Even in the case when we observe the enemy’s actions perfectly, our NIPS paper [39] shows that non-trivial observability conditions need to be imposed on the system parameters.

   For the linear Gaussian case where we observe the enemy’s Kalman filter in noise, consistency of the MLE for the for the adversary’s gain matrix $C$, the strong consistency can be shown fairly straightforwardly. Specifically, if we assume that state matrix $A$ is stable, and the state space model is an identifiable minimal realization, then the enemy’s Kalman filter variables converge to steady state values geometrically fast in $k$ [5] implying that asymptotically the inverse Kalman filter system is stable linear time invariant. Then, the MLE $\theta^*$ for the adversary’s observation matrix $C$ is unique and a strongly consistent estimator [9].

2. **Estimating the enemy’s estimate of our dynamics (transition matrix)** It is of interest to develop algorithms to estimate the enemy’s estimate of our transition matrix and analyze their performance. The adversary estimates $P$ as $\hat{P}$ and we need to estimate the adversary’s estimate as $\hat{\hat{P}}$. In future work we will examine conditions under which the MLE is identifiable and consistent.

**Table 1: Comparison of Cramer Rao bounds for $C$ - classical model vs inverse Kalman filter model**

| $C^\circ$ | Classic | Inverse |
|-----------|---------|---------|
| 0.5       | $0.24 \times 10^{-3}$ | $5.3 \times 10^{-3}$ |
| 1.5       | $1.2 \times 10^{-3}$   | $37 \times 10^{-3}$ |
| 2         | $2.1 \times 10^{-3}$   | $70 \times 10^{-3}$ |
| 3         | $4.6 \times 10^{-3}$   | $336 \times 10^{-3}$ |
3 Inverse Cognitive Radar – Revealed Preferences and Inverse Reinforcement Learning

The previous section was concerned with estimating the enemy’s posterior belief and sensor accuracy. This section discusses detecting utility maximization behavior and estimating the enemy’s utility function in the context of cognitive radars.

Cognitive radars [20] use the perception-action cycle of cognition to sense the environment, learn from it relevant information about the target and the background, then adapt the radar sensor to optimally satisfy the needs of their mission. A crucial element of a cognitive radar is optimal adaptivity: based on its tracked estimates, the radar adaptively optimizes the waveform, aperture, dwell time and revisit rate. In other words, a cognitive radar is a constrained utility maximizer. This section is motivated by the next logical step, namely, inverse cognitive radar. The enemy’s cognitive radar observes our state in noise; it uses a Bayesian estimator (target tracking algorithm) to update its posterior distribution of our state and then chooses an action based on this posterior. From the intercepted emissions of an enemy’s radar:

A: Are the enemy sensor’s actions consistent with optimizing a monotone utility function (i.e., is the cognitive sensor behavior rational in an economics sense)? If so how can we estimate a utility function of the enemy’s cognitive sensor that is consistent with its actions?

B: How to construct a statistical detection test for utility maximization when we observe the enemy sensor’s actions in noise?

C: How can we optimally probe the enemy’s sensor by choosing our state to minimize the Type 2 error of detecting if the radar is deploying an economic rational strategy, subject to a constraint on the Type 1 detection error? “Our” state can be viewed as a probe signal which causes the enemy’s sensor to act.

D: How to detect utility maximization of a Bayesian adversary that is rationally inattentive? Rational inattention (from behavioral economics) introduces an information theoretic cost to the accuracy of an observation.

The main synthesis/analysis framework we will use is that of revealed preferences [50, 19, 16] from microeconomics which aims to determine preferences by observing choices. The results presented below are developed in detail in our recent paper [30].

Classical Inverse Reinforcement Learning (IRL) vs Revealed Preferences. Since we focus on revealed preferences, it is worthwhile comparing it with inverse reinforcement learning.

- Inverse reinforcement learning (IRL) aims to estimate the cost vector $c$ (which depends only on the state) of an infinite horizon Markov decision process (MDP) by observing the decisions and knowing the controlled transition probabilities [43]. The idea in [43] is simple: If the decisions $a$ are chosen from optimal policy $\mu^*$, then the infinite horizon cumulative cost satisfies $L(\mu^*, c) \leq L(\mu, c)$ for any policy $\mu$. Also $L(\mu, c)$ is a linear function of cost vector $c$, i.e., $L(\mu, c) = H(\mu)c$ where matrix $H(\mu)$ depends on the transition matrix and discount factor. Since there are only a finite number of possible policies $\mu$ (as the state and action space are finite), clearly the cost vector $c$ satisfies the finite number of linear inequalities:

$$[H(\mu^*) - H(\mu)]c \leq 0, \quad \mu = 1, 2, \ldots, U$$

(15)

Note also that in IRL, the existence of a utility/cost function is assumed implicitly.
The revealed preference approach that we consider addresses a deeper and more fundamental question: *does a utility function exist that rationalizes the given data with budget constraints? If so, estimate this utility function.* Indeed IRL \(^{15}\) is a trivial case of the revealed preference framework. It is surprising that despite powerful revealed preference methods being developed since the 1960s (e.g. Afriat’s theorem), and numerous advances subsequently, the machine learning community is completely oblivious to this. Finally we note that the first result in IRL was developed by Kalman \(^{25}\) in the 1964 under the terminology “inverse optimal control”.

### 3.1 Background. Revealed Preferences and Afriat’s Theorem

Nonparametric detection of utility maximization behavior is the central theme in the area of revealed preferences in microeconomics. We briefly outline a key result.

**Definition 3.1** (\(^{2,3}\)). A system is a utility maximizer if for every probe \(\alpha_n \in \mathbb{R}^m\), the response \(\beta_n \in \mathbb{R}^m\) satisfies

\[
\beta_n \in \arg\max_{\alpha_n' \beta \leq 1} U(\beta)
\tag{16}
\]

where \(U(\beta)\) is a monotone utility function.

In economics, \(\alpha_n\) is the price vector and \(\beta_n\) the consumption vector. Then \(\alpha_n' \beta \leq 1\) is a natural budget constraint\(^5\) for a consumer with 1 dollar. Given a dataset of price and consumption vectors, the aim is to determine if the consumer is a utility maximizer (rational) in the sense of (16).

The key result is the following remarkable theorem due to Afriat \(^{16,3,2,50,48}\).

**Theorem 3.2** (Afriat’s Theorem \(^{2}\)). *Given a data set \(D = \{(\alpha_n, \beta_n), n \in \{1, 2, \ldots, N\}\}, the following statements are equivalent:*

\(^5\)The budget constraint \(\alpha_n' \beta \leq 1\) is without loss of generality, and can be replaced by \(\alpha_n' \beta \leq c\) for any positive constant \(c\). A more general nonlinear budget incorporating spectral constraints will be discussed below.
1. The system is a utility maximizer and there exists a monotonically increasing, continuous, and concave utility function by satisfies (16).

2. For \( u_t \) and \( \lambda_t > 0 \) the following set of inequalities has a feasible solution:

\[
u_s - u_t - \lambda_t \alpha'_t (\beta_s - \beta_t) \leq 0 \quad \forall t, s \in \{1, 2, \ldots, N\}\.
\]

(17)

3. Explicit monotone and concave utility functions that rationalize the dataset by satisfying (16) are given by:

\[
U(\beta) = \min_{t \in \{1, 2, \ldots, N\}} \{u_t + \lambda_t \alpha'_t (\beta - \beta_t)\}
\]

(18)

where \( u_t \) and \( \lambda_t \) satisfy the linear inequalities (17).

4. The data set \( D \) satisfies the Generalized Axiom of Revealed Preference (GARP) also called cyclic consistency, namely for any \( t \leq N, \alpha'_t \beta_t \geq \alpha'_t \beta_{t+1} \quad \forall t \leq k - 1 \implies \alpha'_k \beta_k \leq \alpha'_k \beta_1 \).

Afriat’s theorem tests for economics-based rationality; its remarkable property is that it gives a necessary and sufficient condition for a system to be a utility maximizer based on the system’s input-output response. The feasibility of the set of inequalities (17) can be checked using a linear programming solver; alternatively GARP can be checked using Warshall’s algorithm with \( O(N^3) \) computations [49] [47].

The recovered utility using (18) is not unique; indeed any positive monotone increasing transformation of (18) also satisfies Afriat’s Theorem; that is, the utility function constructed is ordinal. This is the reason why the budget constraint \( \alpha'_n \beta_1 \leq 1 \) is without generality; it can be scaled by an arbitrary positive constant and Theorem 3.2 still holds. In signal processing terminology, Afriat’s Theorem can be viewed as set-valued system identification of an \( \text{argmax} \) system; set-valued since (18) yields a set of utility functions that rationalize the finite dataset \( D \).

### 3.2 Framework. Cognitive Radar as a Utility Maximizer

*Our working assumption is that a cognitive sensor satisfies economics-based rationality; that is, a cognitive sensor is a utility maximizer in the sense of (18) with a possibly nonlinear budget constraint. The main question then is:

*How to use revealed preferences as a systematic method to detect cognitive radars?*

We briefly discuss the key ideas in a simplified setting. Let \( k = 1, 2, \ldots \) denote discrete time (fast time scale) and \( n = 1, 2, \ldots \) denote epoch (slow time scale). The framework is as follows, see Fig 7:

1. Our state \( x_k \) is parametrized by our probe signal \( \alpha_n \). In a classical linear Gaussian state space model used in target tracking [8], our probe \( \alpha_n \) parametrizes the state noise covariance \( Q(\alpha_n) \) which models acceleration maneuvers of our drone.

6We use monotone and local non-satiation interchangeably. Afriat’s theorem was originally stated for a non-satiated utility function which is slight generalization of monotone.

7As pointed out in Varian’s influential paper [47], a remarkable feature of Afriat’s theorem is that if the dataset can be rationalized by a monotone utility function, then it can be rationalized by a continuous, concave, monotonic utility function. Put another way, continuity and concavity cannot be refuted with a finite dataset.
2. The enemy’s radar controller chooses an action $\beta_n$ (e.g. radar waveform) by optimizing a utility function $U(\beta)$ subject to a possibly nonlinear constraint wrt $(\alpha_n, \beta)$. In a classical target tracking model, $\beta_n$ parametrizes the observation noise covariance $R(\beta_n)$. From a practical point of view, when a radar adapts its waveform function $\beta_n$, in effect it adapts the measurement noise covariance $R(\beta_n)$.

3. Given the above linear Gaussian state space model with state noise covariance matrix $Q(\alpha_n)$ and observation noise covariance matrix $R(\beta_n)$, the enemy’s cognitive radar records measurements $y_k$ of our kinematic state $x_k$. The enemy uses a Kalman filter tracking algorithm to compute the filtered posterior density $\pi_k = N(\hat{x}_k, \Sigma_k)$ where $\hat{x}_k$ is the conditional mean state estimate and $\Sigma_k$ is the covariance of the estimate.

It is well known that under the assumption that the linear Gaussian state space model parameters satisfy detectability and stabilizability [5], the asymptotic covariance $\Sigma_{k+1|k}$ as $k \to \infty$ is the unique non-negative definite solution of the algebraic Riccati equation (ARE):

$$\mathcal{A}(\alpha, \beta, \Sigma) \overset{\text{defn}}{=} -\Sigma + A(\Sigma - \Sigma C' [C \Sigma C' + R(\beta)]^{-1} C \Sigma) A' + Q(\alpha) = 0$$

(19)

where $\alpha_n$ and $\beta_n$ are the probe and response signals of the radar at epoch $n$. Since $\Sigma$ is parametrized by $\alpha, \beta$, we write the solution of the ARE at epoch $n$ as $\Sigma^*_n(\alpha, \beta)$. We will use the ARE (19) to describe our research plans in spectral revealed preferences below.

**Example. Testing for Cognitive Radar: Spectral Revealed Preferences with Linear Budget**

We now show that Afriat’s theorem (Theorem 3.2) can be used to determine if a radar is cognitive. The assumption here is that the utility function $U(\beta)$ maximized by the radar is a monotone function (unknown to us) of the predicted covariance of the target. Our main task is to formulate and justify a linear budget constraint $\alpha_n' \beta \leq 1$ in Afriat’s theorem. Specifically, suppose

---

8For details of ambiguity functions and waveform design we refer to [26]. It is here that important practicalities regarding actual radar operation become important. As mentioned earlier, our collaborator Dr. Rangaswamy at AFRL will provide us with a sophisticated test bed to emulate radar waveforms in a realistic environment.
1. Our probe $\alpha_n$ that characterizes our maneuvers, is the vector of eigenvalues of the positive definite matrix $Q$

2. The radar response $\beta_n$ is the vector of eigenvalues of the positive definite matrix $R^{-1}$.

Then the cognitive radar chooses its waveform parameter $\beta_n$ at each slow time epoch $n$ to maximize a utility $U(\cdot)$:

$$\beta_n \in \arg\max_{\alpha_n, \beta \leq 1} U(\beta)$$

(20)

where $U$ is a monotone increasing function of $\beta$.

Then Afriat’s theorem (Theorem 3.2) can be used to detect utility maximization and construct a utility function that rationalizes the response of the radar. Recall that the 1 in the right hand side of the budget $\alpha_n^T \beta \leq 1$ can be replaced by any non-negative constant.

It only remains to justify the linear budget constraint $\alpha_n^T \beta \leq 1$ in (20). The $i$-th component of $\alpha$, denoted as $\alpha(i)$, is the incentive for considering the $i$-th mode of the target; $\alpha(i)$ is proportional to the signal power. The $i$-th component of $\beta$ is the amount of resources (energy) devoted by the radar to this $i$-th mode; a higher $\beta(i)$ (more resources) results in a smaller measurement noise covariance, resulting in higher accuracy of measurement by the radar. So $\alpha_n^T \beta$ measures the signal to noise ratio (SNR) and the budget constraint $\alpha_n^T \beta \leq 1$ is a bound on the SNR. A rational radar maximizes a utility $U(\beta)$ that is monotone increasing in the accuracy (inverse of noise power) $\beta$.

However, the radar has limited resources and can only expend sufficient resources to ensure that the precision (inverse covariance) of all modes is at most some pre-specified precision $\bar{\Sigma}^{-1}$ at each epoch $n$. We can then justify the linear budget constraint as follows:

Lemma 3.3. The linear budget constraint $\alpha_n^T \beta \leq 1$ implies that solution of the ARE (19) satisfies $\Sigma_n^{-1}(\alpha_n, \beta) \preceq \bar{\Sigma}^{-1}$ for some symmetric positive definite matrix $\bar{\Sigma}^{-1}$.

The proof of Lemma 3.3 follows straightforwardly using the information Kalman filter formulation \[5\], and showing that $\Sigma_n^{-1}$ is increasing in $\beta$. Afriat’s theorem requires that the constraint $\alpha_n^T \beta \leq 1$ is active at $\beta = \beta_n$. This holds in our case since $\Sigma_n^{-1}$ is increasing in $\beta$.

To summarize, we can use Afriat’s theorem (Theorem 3.2) with $\alpha_n$ as the spectrum of $Q$ and $\beta_n$ as the spectrum of $R$, to test a cognitive radar for utility maximization (20). Moreover, Afriat’s theorem constructs a set of utility functions (18) that rationalize the decisions of the cognitive sensor.

Summary: The above approach of spectral revealed preferences, i.e., choosing the probe signal $\alpha$ as the spectrum of $Q^{-1}$ and the response signal $\beta$ as the spectrum of $R$, yields a novel class of methods for detecting utility maximization behavior in a radar. To our knowledge, such methods have not been studied in the context of adaptive sensing or inverse reinforcement learning. Also, Statement 4 of the theorem is constructive; it yields explicit utility functions that rationalize the decisions of the cognitive sensor. From a practical point of view, estimating the enemy sensor’s utility function, allows us to predict (in a Bayesian sense) its future actions and therefore guard against these actions.

Finally, although we will not elaborate here (due to lack of space), the above framework sets the stage for future work that addresses the following deeper questions:

*How to design cognitive sensors that barely pass Afriat’s theorem (or generalization)?*

*Equivalently: How to design smart (economic rational) sensors that appear dumb?*

*Are there useful cognitive radars that are not economic rational?*
3.3 Detecting Cognitive Radars in a Noisy Setting

If the sensor’s response $\beta_n$ is observed in noise, then violation of Afriat’s theorem could be either due to measurement noise or absence of utility maximization. We will construct statistical detection tests to decide if the sensor is a utility maximizer and therefore “cognitive”. Our proposed research builds on stochastic revealed preferences [18, 1, 4], and our previous works in [34, 7, 23]. Our plan proceeds as follows. To highlight the main ideas, assume additive measurement errors:

$$z_n = \beta_n + \varepsilon_n,$$

where $\varepsilon_n \sim \mathcal{N}(0, \sigma^2)$ are independent and identically distributed (iid) (21). Here $z_n$ is the noisy measurement of response (action) $\beta_n$ of the cognitive sensor.

Given the noisy data set $D_{\text{obs}} = \{(\alpha_n, z_n) : n \in \{1, \ldots, N\}\}$, how can a statistical test be designed for testing utility maximization (16) in a dataset due to measurement errors? Let $H_0$ denote the null hypothesis that the data set $D_{\text{obs}}$ satisfies utility maximization. Similarly, let $H_1$ denote the alternative hypothesis that the data set does not satisfy utility maximization. Then:

**Type-I errors:** Reject $H_0$ when $H_0$ is valid.

**Type-II errors:** Accept $H_0$ when $H_0$ is invalid. (22)

We will design statistical tests to detect if an agent is seeking to maximize a utility function. Also statistical tests will be constructed jump changes in the utility function as in our earlier work [7]. These tests evaluate the probability of the minimum perturbation required for the observed dataset to fall inside the convex polytope specified by the revealed preferences inequalities (17).

Finally, we will derive analytical expressions for a lower bound on the false alarm probability.

3.4 Optimal Probing to Detect Utility Maximization Behavior

What choice of our probe signal yields the smallest Type II error [22] in detecting if the enemy sensor is a utility maximizer, subject to maintaining the Type I error within a specified bound?

The framework above guarantees that if we observe the radar response in noise, then the probability of Type-I errors (deciding that the radar is not a utility maximizer when it is) is less then a threshold $\gamma$ for the decision test. The goal is to enhance the statistical test by adaptively optimizing the probe vectors $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ to reduce the probability of Type-II errors (deciding that the radar is cognitive when it is not). Our proposed framework is shown in Figure 8.

We will formulate the optimal probing problem as follows: Estimate optimal probe signals $\alpha$ as

$$\arg\min_{\alpha \in \mathbb{R}^{m \times N}} J(\alpha) = \mathbb{P}(\text{Error}(\alpha, \beta, \varepsilon) > \gamma | \{\alpha, \beta(\alpha)\} \in \mathcal{A}).$$

(23)

Here $\mathbb{P}(\cdot | \cdot)$ denotes the conditional probability that the statistical test accepts $H_0$, given that $H_0$ is false. $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N]$ is the noise variable defined in (21), and $\gamma$ is the significance level. The set $\mathcal{A}$ contains all the elements $\{\alpha, \beta(\alpha)\}$, with $\beta(\alpha) = [\beta_1, \beta_2, \ldots, \beta_N]$, where $\{\alpha, \beta\}$ does not satisfy the GARP inequalities of Afriat’s Theorem 3.2.

Since Error($\cdot$) is not known explicitly, (23) is a simulation based stochastic optimization problem. To estimate a local minimum value of $J(\alpha)$, we will use stochastic optimization algorithms such as the simultaneous perturbation stochastic gradient (SPSA) algorithm [46].
3.5 Bayesian Revealed Preferences and Rational Inattention

We now briefly discuss an extension that involves Bayesian revealed preferences for detecting cognitive sensors. Suppose the enemy is a Bayesian cognitive sensor that chooses its action at each time instant to maximize an expected utility function based on the noisy measurement of an underlying state. Assume that the Bayesian agent is rationally inattentive, that is, obtaining accurate measurements are expensive. This information acquisition cost affects the observation chosen by the agent and hence affects the action chosen by the agent. We observe the dataset of actions of the Bayesian agent and know the underlying state (our state). Our research aims to address the question:

*How to estimate the utility function and rational inattention cost of the Bayesian enemy?*

Bayesian revealed preferences have been studied in the context of human decision making with rational inattention in [10, 11]. Unlike Afriat’s Theorem 3.2, the utility function in the Bayesian set-up is discrete valued; still one can construct a utility function that rationalizes the data. Rational inattention by Bayesian agents was pioneered by Sims [45]. Rational inattention is a form of bounded rationality - the key idea is that a sensor’s resources for information acquisition are limited and can be modeled in information theoretic terms as a Shannon capacity limited communication channel. The action $a$ and information acquisition policy $\mu$ of a rationally inattentive Bayesian agent are obtained by optimizing the following utility

$$
\max_{\mu} \mathbb{E}_y \left\{ \max_a \mathbb{E} \{ U(x, a) | y, \mu \} - \lambda \left( H(\pi) - H(T(\pi, y, \mu)) \right) \right\}, \quad \lambda \in \mathbb{R}_+ \tag{24}
$$

The first term in (24) is standard Bayesian utility maximization, namely, choosing action $a$ to optimize the conditional expectation of a utility function given noisy measurements $y$. But the measurement $y$ depends on the sensing mode $\mu$ used by the agent. It is here that the second term in (24) comes in. The second term is the rational inattention cost: it is the difference in the entropies of the prior and posterior when choosing an attention strategy $\mu$. Clearly the more accurate the posterior, the lower its entropy and hence the higher the cost. The motivation in behavioral economics by Sims is that human attention is expensive and translating data into a
decision is constrained by finite Shannon capacity to process information. In cognitive sensing, the motivation for rational inattention is that the enemy’s sensor incurs a higher measurement cost when it deploys a more accurate sensing mode.

The aim is devise revealed preference methods to detect if the enemy is a rationally inattentive Bayesian utility maximizer; and then estimate the utility $U$ in (24). This involves the so called NIAS and NIAC conditions (with suitable modifications using a Bayesian tracker) described in [10], which can be viewed as a Bayesian generalization of GARP in the classical Afriat’s theorem. One can consider the more general Renyi entropy and divergence; which allows for more flexibility in modeling sensor information (Renyi entropy is a generalization of Shannon entropy); see [1]. As another extension, it is worthwhile consider stopping time problems where the horizon of decision making is not one step but an integer valued random variable. Optimal stopping time formulations are widely used in adaptive radars in the context of stochastic control [32, 33, 15]. An important example is inverse quickest change detection; by observing the response of a rationally inattentive cognitive sensor and detector, we will devise Bayesian revealed preference methods to estimate the delay penalty and false alarm cost of the enemy’s detection algorithm. To our knowledge, these have not been studied in the literature. Another extension is to estimate the costs involved in multiple stopping time problems [31] given the response of the agent.

Summary and extensions. The ideas developed in this section constitute a principled framework for inverse reinforcement learning of cognitive sensing systems. As mentioned earlier, revealed preferences addresses a deeper and more fundamental question than classical inverse reinforcement learning: does a utility function exist that rationalizes the given data (with signal processing constraints) and if so, estimate it. The discussion presented above include several novel features including spectral revealed preferences, detecting utility maximization behavior in noise, optimally probing the enemy, and finally detecting Bayesian rationally inattentive behavior.

To keep the page length short, we have omitted two other ideas that we are currently pursuing: (i) revealed preferences to detect play from the equilibrium of a potential game [22]; i.e., how to determine if a dataset is consistent with play from the Nash equilibrium of a potential game between the enemy and us. (ii) revealed preferences when the utility functions are only partially orderable (on a lattice). This arises if the utility function as matrix-valued; e.g. covariance matrix of the enemy’s Bayesian estimator. This involves estimating preferences on Hausdorff spaces.

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