Matrix Models of Two-Dimensional Gravity and Discrete Toda Theory

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Abstract

Recursion relations for orthogonal polynomials, arising in the study of the one-matrix model of two-dimensional gravity, are shown to be equivalent to the equations of the Toda-chain hierarchy supplemented by additional Virasoro constraints. This is the case without the double scaling limit. A discrete time variable to the matrix model is introduced. The discrete time dependent partition functions are given by $\tau$ functions which satisfy the discrete Toda molecule equation. Further the relations between the matrix model and the discrete time Toda theory are discussed.

Key Words: discrete Toda molecule equation, two-dimensional gravity, $\tau$-function
1 Introduction

Matrix models have attracted a lot of attention as possible models for nonperturbative string theory and two-dimensional gravity. In particular the matrix model of two-dimensional gravity can be solved exactly using the orthogonal polynomial theory.[1] One of the most intriguing features is the connection with the theory of integrable systems. It has been shown that the partition function of a matrix model is the \( \tau \)-function. Thus we see that an important object in the theory of integrable equations appears naturally from the matrix model.

Discrete-time integrable system is currently a focus of an intensive study. It is interesting to note that most of the well-known features of continuous-time integrable systems such as Lax-pair, Bäcklund transformation, Painlevé property, etc., carry over to the case of discrete-time integrable systems. The Toda molecule equation with discrete time (hereafter, the discrete Toda molecule equation) can be utilized for the numerical computation of matrix eigenvalues. Hirota et al. have suggested similarities between their method and the LR factorization method.[2]

Let us introduce the discrete Toda molecule equation. Let \( I_0, \cdots, I_{N-1} \) and \( V_0, \cdots, V_{N-2} \) be dynamical variables, and \( l \) be discrete “time” variable. The discrete-time Toda molecule equation is given by,[4]

\[
I_k(l+1) - I_k(l) = V_k(l) - V_{k-1}(l+1),
\]

\[
I_k(l+1)V_k(l+1) = I_{k+1}(l)V_k(l).
\]

The boundary condition \( V_{-1} = V_{N-1} = 0 \) is imposed for any time \( l \). By a transformation,

\[
I_k(l) = \frac{\tau_k(l+1)\tau_{k-1}(l)}{\tau_k(l)\tau_{k-1}(l+1)},
\]

\[
V_k(l) = \frac{\tau_{k+1}(l)\tau_{k-1}(l+1)}{\tau_k(l)\tau_k(l+1)},
\]

the equations of motion, (1.1) and (1.2), are cast into the following bilinear equation,

\[
\tau_k(l)^2 - \tau_k(l-1)\tau_k(l+1) + \tau_{k+1}(l-1)\tau_{k-1}(l+1) = 0.
\]

One easily see that under (1.3), (1.2) holds identically and (1.1) reduces to (1.4).

The outline of this letter is as follows. In section 2 we first consider a theory of discrete-time dependent orthonormal polynomials (DTOP). Then we see that a case of DTOP can be formulated by the discrete Toda theory. The Lax-pair of the discrete Toda molecule equation is derived using the properties of the orthonormal polynomials. In section 3 we show our \( \tau \) function in section 2 is the partition function of the scaler product model. This model can be obtained in the limit of the Kontsevich integral. In section 4
we extend Hirota’s discrete Toda theory and apply the discrete Toda theory to DTOP. In section 5 we show a way from DTOP to string equation. In section 6 we consider about continuous and discrete Toda theories. The last section is devoted to concluding remarks.

2 Discrete Toda Theory and Orthogonal Polynomials

Let us begin by associating a time variable $l$ to the orthogonal polynomial theory.\[3,\[4] The distribution function $\rho(\lambda;l)$ or the weight function $w(\lambda;l) = d\rho(\lambda;l)/d\lambda$ uniquely defines the orthonormal polynomial $\varphi_n(\lambda;l)$. According to Schmidt’s diagonalization method, we have

$$\varphi_n(\lambda;l) = \frac{1}{\sqrt{\tau_n \tau_{n-1}}} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & \lambda & \cdots & \lambda^n \end{vmatrix},$$

where $s_n$’s are the moments defined by $s_n(l) = \int \lambda^nd\rho(\lambda;l)$ and the tau function $\tau_n$ is defined by

$$\tau_n(l) = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}.$$ (2.2)

Here we consider the case $w(\lambda;l+1) = \lambda w(\lambda;l)$. Remark that the determinant in (2.1) is a Hankel-Hadamard-type determinant. $\varphi_n(\lambda;l)$’s satisfy the orthonormal relations,

$$\int \varphi_n(\lambda;l)\varphi^*_m(\lambda;l)d\rho(\lambda;l) = \delta_{mn}, \quad l = 1, 2, \ldots.$$ (2.3)

In matrix models based on the hermitian matrices the integral in (2.3) is along the real axis, and therefore $\varphi^*_m = \varphi_m$ are assumed hereafter. These are basic results of the orthogonal polynomial theory, except that we have introduced a discrete-time (“zero-time”) variable $l$. This tau function is precisely the same tau function as in Hirota-Sato’s soliton theory. The tau function $\tau_n(l)$ satisfies the discrete Toda molecule equation (1.4), which is known to be a completely integrable system.

We derive the Lax-pair of this system using the orthogonal polynomials. First, we know from the orthogonal polynomial theory that the three-term relationship holds:

$$a_{n+1}(l)\varphi_{n+1}(\lambda;l) + b_n(l)\varphi_n(\lambda;l) + a_n(l)\varphi_{n-1}(\lambda;l) = \lambda\varphi_n(\lambda;l),$$

where $a_n$’s and $b_n$’s are given by

$$a_n(l) = \int \lambda\varphi_n(\lambda;l)\varphi_{n-1}(\lambda;l)d\rho(\lambda;l),$$

$$b_n(l) = \int \lambda\varphi^2_n(\lambda;l)d\rho(\lambda;l).$$

(2.5)
It is a half (space part) of the Lax-pair. By using (2.3) and the condition \(w(\lambda; l + 1) = \lambda w(\lambda; l)\) the other half (time part) is obtained,

\[
\alpha_n(l) \varphi_n(\lambda; l) = \alpha_n(l + 1) \varphi_n(\lambda; l + 1) + V_{n-1}(l) \alpha_{n-1}(l + 1) \varphi_{n-1}(\lambda; l + 1),
\]

(2.6)

where

\[
\frac{\alpha_{n+1}(l)}{\alpha_n(l)} = a_{n+1}(l),
\]

(2.7)

\[
a_n(l) = I_{n-1}(l) V_{n-1}(l),
\]

(2.8)

\[
b_n(l) = I_n(l) + V_{n-1}(l).
\]

(2.9)

As a compatibility condition of (2.4) and (2.6), we obtain the equations of motion (1.1) and (1.2) for field variables \(I_n\) and \(V_n\).

From (1.3) and (2.8) we have

\[
a_{n+1}(l) = \frac{\tau_{n+1}(l) \tau_{n-1}(l)}{\tau_n^2(l)}.
\]

(2.10)

Note that (2.10) is the same for the continuous-time Toda molecule equation. A useful formula which expresses \(\tau_n\) in terms of \(a_k\)'s is derived by inverting (2.10),

\[
\tau_n = s_0^{n+1}(a_1^n)(a_2^n)^{n-1} \cdots (a_n^1).
\]

(2.11)

This formula is called a Barners-type formula.

3 Matrix Model and Discrete Toda Theory

For \(p \times p\) hermitian matrices \(X\) and \(K\), let us consider the Kontsevich integral,\(^5\),\(^6\),\(^7\)

\[
\mathcal{F}_{V,p}\{K\} = \int_{p \times p} dXE^{-\text{tr}V(X) + \text{tr}KX}.
\]

(3.1)

The eigenvalues of matrices \(X\) and \(K\) are respectively denoted as \(x_\gamma\) and \(k_\delta\). Then the Kontsevich integral (3.1) is calculated as\(^8\)

\[
\mathcal{F}_{V,p}\{K\} = (2\pi)^{p(p-1)/2} \frac{\det_{\gamma \delta} \hat{\phi}_\gamma(k_\delta)}{\Delta(k)},
\]

(3.2)

where

\[
\hat{\phi}_\gamma(k) = \int dx x^{\gamma-1} e^{-V(x)+kx}, \quad \gamma = 1, 2, \ldots, p
\]

(3.3)

and \(\Delta(k)\) is the “van der Monde determinant”, \(\Delta(k) = \det_{i,j} k_i^{j-1} = \Pi_{i>\gamma} (k_i - k_j)\). Note that if the “zero-time” \(l\) is introduced, then

\[
\mathcal{F}_{V,p}\{l|K\} = \mathcal{F}_{V(X)-\log X,p}\{K\} = (2\pi)^{p(p-1)/2} \frac{\det_{\gamma \delta} \hat{\phi}_\gamma(k_\delta, l)}{\Delta(k)},
\]

(3.4)
where
\[ \hat{\phi}_\gamma(k, l) = \int dx x^{l+\gamma-1} e^{-V(x)+kx}. \] (3.5)

We introduce one matrix model of the form:
\[ Z_p = c_p \int_{p \times p} dX e^{-\text{Tr} V(X)}, \] (3.6)
where \( c_p \) is a constant.

Taking \( V = V(X) - l \log X \), we get,
\[ Z_p(l) = c_p \int_{p \times p} dX e^{-\text{Tr} V(X)} = \lim_{K \to 0} \mathcal{F}_{V,p} \{ l | K \} = \lim_{(k_j) \to 0} \frac{\text{Det}_{ij} \hat{\phi}_i^{\{V\}}(k_j, l)}{\Delta(k)}. \] (3.7)

Note that \( \hat{\Phi}_\gamma \{ V \} (l) = \int dx x^{l} e^{-V(x)+kx} \). The formula (3.7) is the same as (2.2) in the case \( d \rho(\lambda; l) \).

Then, one find that the partition function of the one matrix model is nothing but the tau function. In the special case \( l = 0 \) the one matrix is the normal one-matrix model.

4 Extended Discrete Toda Theory

We choose the orthogonal polynomials as follows:
\[ \Phi_n(\lambda; l) \equiv e^{\phi_n(l)/2} \varphi_n(\lambda; l) = \alpha_n(l) \varphi_n(\lambda; l), \] (4.1)
where
\[ a_n(l) = e^{\phi_n(l)/2-\phi_{n-1}(l)/2}. \] (4.2)
\( \Phi_n(\lambda; l) \) satisfy a trivial scalar product,
\[ \int \Phi_n(\lambda; l) \Phi_m(\lambda; l) d\rho(\lambda; l) = e^{\phi_n(l)} \delta_{nm}. \] (4.3)

Recalling a Barners-type formula (2.11), we see that the partition function of the one matrix model is given by
\[ Z_p(l) = \prod_{i=0}^{p-1} e^{\phi_i(l)}. \] (4.4)

We can rewrite representations (2.4) and (2.6) as
\[ \lambda \Phi_n(\lambda; l) = \Phi_{n+1}(\lambda; l) + \{ I_n(l) + V_{n-1}(l) \} \Phi_n(\lambda; l) + V_{n-1}(l) I_{n-1}(l) \Phi_{n-1}(\lambda; l), \] (4.5)
\[ \Phi_n(\lambda; l) = \Phi_n(\lambda; l+1) + V_{n-1}(l) \Phi_{n-1}(\lambda; l+1), \] (4.6)
for $n = 0, 1, 2, \cdots, N - 1$. In fact we have to consider $N$ in the limit $N \to \infty$ hereafter. However for convinience’s sake, we use $N$. From (4.3) and (4.6) we have

$$\lambda \Phi_n(\lambda; l + 1) = I_n(l) \Phi_n(\lambda; l) + \Phi_{n+1}(\lambda; l).$$

(4.7)

Note that (4.6) and (4.7) reduces to Hirota’s operators [2] if we set $\lambda = 1$. Let $|l\rangle$ be an $N$-dimensional vector:

$$|l\rangle = [\Phi_0(\lambda; l), \Phi_1(\lambda; l), \cdots, \Phi_{N-1}(\lambda; l)]^T,$$

(4.8)

where the superscript $T$ denotes the transposition of matrix, and $L(l)$ and $R(l)$ be $N \times N$ matrices:

$$L(l) = \begin{pmatrix} 1 & 0 \\ V_0(l) & 1 \\ V_1(l) & 1 \\ \vdots & \vdots \\ V_{N-2}(l) & 1 \\ 0 & 0 \end{pmatrix},$$

(4.9)

$$R(l) = \begin{pmatrix} I_0(l) & 0 \\ I_1(l) & 1 \\ I_2(l) & \cdots \\ \vdots & \vdots \\ I_{N-1}(l) & 1 \\ 0 & 0 \end{pmatrix}.$$

(4.10)

In terms of the matrices $L(l)$ and $R(l)$, (4.6) and (4.7) are expressed as

$$|l\rangle = L(l)|l + 1\rangle,$$

(4.11)

$$\lambda |l + 1\rangle = R(l)|l\rangle.$$  

(4.12)

Let us introduc a matrix $A(l)$ by

$$A(l) = L(l)R(l).$$

(4.13)

From (4.11)-(4.13), we have

$$\lambda^k |l\rangle = A(l)^k |l\rangle.$$  

(4.14)

We define new operators

$$L^{(k)}(l) = L(l)L(l+1)\cdots L(l+k-1),$$

(4.15)

$$R^{(k)}(l) = R(l+k-1)R(l+k-2)\cdots R(l).$$

(4.16)

From (4.11)-(4.13), we also have

$$L^{(k)}(l)|l + k\rangle = |l\rangle.$$  

(4.17a)
\[ R^{(k)}(l)|l\rangle = \lambda^{k}|l + k\rangle, \quad (4.17b) \]
\[ (L^{(k)}(l))^{-1}L^{(k)}(l) = E, \quad (4.17c) \]

where \( E \) is an identity operator.

The compatibility condition of (4.14) and (4.17b) yields a matrix equation:
\[ A(l + k)^k = (L^{(k)}(l))^{-1}A(l)^kL^{(k)}(l). \quad (4.18) \]

In the particular case \( k = 1 \), (4.18) becomes
\[ R(l)L(l) = L(l + 1)R(l + 1), \quad (4.19) \]
which gives the discrete Toda molecule equation.

We may use the bra-ket expression for scalar product,
\[ \langle \Phi_n(\lambda; l)|\Phi_m(\lambda; k)\rangle = \int \Phi_n(\lambda; l)|\Phi_m(\lambda; k)d\rho(0). \quad (4.20) \]

Remark that the distribution function is \( \rho(0) \equiv \rho(0;l) \) in all \( l \). Then, the following relations hold,
\[ e^{\phi_n(l)} = \int \Phi_n(\lambda; l)^2d\rho(\lambda; l) = \langle \Phi_n(\lambda; l)|\lambda^l|\Phi_n(\lambda; l)\rangle = \langle \Phi_n(\lambda; l)|A^l|\Phi_n(\lambda; l)\rangle. \quad (4.21) \]

### 5 String equation

In this section we consider the one matrix model
\[ Z_p(l, t) \equiv c_p\int_{p\times p} dH (\det H)^l e^{-\sum_{k=0}^{\infty} t_k \text{Tr}H^k}, \quad (5.1) \]
where \( H \) is a \( p\times p \) hermitian matrix. For this model the string equation is as follows:[8, 9, 10]
\[ L_q(l)Z_p(l) = 0, \quad q \geq -1, \quad (5.2) \]
with
\[ L_q(l) \equiv \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+q}} + \sum_{k=0}^{q} \frac{\partial^2}{\partial t_k \partial t_{q-k}} - l \frac{\partial}{\partial t_q} \quad q \geq 0, \quad (5.3) \]

Note that \( \partial/\partial t_{-1} \) and \( \partial/\partial t_0 \) are defined:
\[ \frac{\partial}{\partial t_{-1}} e^{\phi_n(l)} = \langle \Phi_n(\lambda; l)|\lambda^{l-1}|\Phi_n(\lambda; l)\rangle, \quad (5.4) \]
\[
\frac{\partial}{\partial t_0} e^{\phi_n(l)} = \langle \Phi_n(\lambda; l) | A^l | \Phi_n(\lambda; l) \rangle = e^{\phi_n(l)}. \tag{5.5}
\]

Using (5.5), we have
\[
\frac{\partial}{\partial t_0} Z_p(l) = pZ_p(l). \tag{5.6}
\]
It is easy to see that \( L_n \)'s form a closed Virasoro algebra:
\[
[L_n(l), L_m(l)] = (n - m)L_{n+m}(l), \quad n, m \geq 0. \tag{5.7}
\]
They are identified with the Virasoro constraints for a normal one-matrix model without \( l \).

Let us discuss the string equation from the theory orthonormal of polynomials. We can show that the weight function is satisfied if
\[
\frac{d\rho(\lambda; l)}{d\lambda} = \lambda^l \exp(-\sum_{k=0}^{\infty} t_k \lambda^k). \tag{5.8}
\]
First we have, from the orthonormality,
\[
0 = \int \frac{\partial \varphi_n(\lambda; l)}{\partial \lambda} \varphi_n(\lambda; l) d\rho(\lambda; l) = -\int \varphi_n^2(\lambda; l) \frac{\partial}{\partial \lambda} d\rho(\lambda; l)
\]
\[
= -\sum_{k=0}^{\infty} k t_k \int \lambda^k \varphi_n^2(\lambda; l) d\rho(\lambda; l) + l \int \frac{\varphi_n^2(\lambda; l)}{\lambda} d\rho(\lambda; l). \tag{5.9}
\]
Summing (5.9) over \( n \) from 0 to \( p - 1 \), we get
\[
L_{-1}(l) Z_p(l) = 0. \tag{5.10}
\]
This is the lowest string equation.

Second from the orthonormality, we also have
\[
n = \int \lambda \frac{\partial \varphi_n(\lambda; l)}{\partial \lambda} \varphi_n(\lambda; l) d\rho(\lambda; l), \tag{5.11}
\]
which can be partially integrated as
\[
n = -\int \varphi_n(\lambda; l) \frac{\partial}{\partial \lambda} (\lambda \varphi_n(\lambda; l) d\rho(\lambda; l))
\]
\[
= -\int \varphi_n(\lambda; l) \{ \varphi_n(\lambda; l) + \lambda \frac{\partial \varphi_n(\lambda; l)}{\partial \lambda} \} d\rho(\lambda; l) - \sum_{k=0}^{\infty} k t_k \lambda^k \varphi_n(\lambda; l) \} d\rho(l). \tag{5.12}
\]
Using again the orthonormality, we get
\[
2n + 1 + l = \sum_{k=0}^{\infty} k t_k \int \lambda^k \varphi_n^2(\lambda; l) d\rho(\lambda; l). \tag{5.13}
\]
If we sum over (5.13) with respect to \( n \) from 0 to \( p - 1 \), we obtain
\[
L_0(l) Z_p = 0. \tag{5.14}
\]
This is the string equation for \( q = 0 \).
6 Continuous Toda and Discrete Toda Theories

Operators $A(l)^k$ are not symmetric, but one can rewrite it in a symmetric form:

$$A(l)^k = T(l)A(l)^kT(l)^{-1}$$  \hspace{1cm} (6.1)

where $T(l)$ is

$$
\begin{pmatrix}
  e^{\phi_0(l)/2} & 0 \\
  e^{\phi_1(l)/2} & \ddots \\
  0 & \ddots & e^{\phi_{N-1}(l)/2}
\end{pmatrix}.
$$  \hspace{1cm} (6.2)

In fact, $A(l)$ is a matrix representation of (2.4):

$$
\begin{pmatrix}
  b_0 & a_1 \\
  a_1 & b_1 & a_2 \\
  \vdots & \vdots & \ddots & \ddots \\
  a_{N-2} & b_{N-2} & a_{N-1} \\
  0 & a_{N-1} & b_{N-1}
\end{pmatrix}.
$$  \hspace{1cm} (6.3)

From the operator $A(l)^\infty$ the continuous Toda hierarchy is obtained. The Toda vertex operator is defined by:

$$\tilde{X}(\lambda; l) = \exp(-\sum t_i \lambda^i) \exp(2 \sum \lambda^{-1} \frac{\partial}{\partial t_i}).$$  \hspace{1cm} (6.4)

Using this operator we can relate $Z_p(l)$ and $Z_{p-1}(l)$ as

$$Z_p(l) = \tilde{X}(l)Z_{p-1}(l),$$  \hspace{1cm} (6.5)

where

$$\tilde{X}(l) = \int \lambda^{2p-2-l} \tilde{X}(l)d\lambda.$$  \hspace{1cm} (6.6)

On the other hand, we can construct relation between $Z_p(l)$ and $Z_p(l+k)$ as follows. First we write $Z_p(l)$ in the bra-ket expression,

$$Z_p(l) = \prod_{i=0}^{p-1} |\Phi_i(\lambda; l)\rangle \langle \Phi_i(\lambda; l)|A(l)^l.$$  \hspace{1cm} (6.7)

We next define operator by

$$\tilde{T}^{(k)}(l) = R^{(k)}(l) \bigotimes (L^{(k)}(l))^{-1} = (L^{(k)}(l))^{-1} \bigotimes R^{(k)}(l).$$  \hspace{1cm} (6.8)

Since $w(\lambda; l+k) = \lambda^k w(\lambda; l)$ the relation between $Z_p(l)$ and $Z_p(l+k)$ can be expressed as

$$Z_p(l+k) = \tilde{T}^{(k)}(l)Z_p(l)$$  \hspace{1cm} (6.9)
7 Concluding Remarks

In the present letter we have put forward to a point of view that matrix models are related to the discrete Toda theory.

By introducing a discrete time variable to the theory of orthogonal polynomials, we have found that the whole theory corresponds to the discrete Toda theory. We have also shown that the $\tau$ functions of the orthogonal polynomials are equal to the partition functions of the matrix model. This matrix model is obtained in the limit of the Kontsevich integral and in the case $l = 0$ becomes the normal one-matrix model. To summarize, the theory of the matrix model can be described by using the discrete Toda theory.

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