On affirmative solution to the prestigious Michael problem in the theory of Fréchet algebras, with applications to automatic continuity theory

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**Abstract.** In 1952, Michael posed a question about the continuity of characters on commutative Fréchet algebras in his memoir, known as Michael’s problem in the literature. We settle this in the affirmative, even for the non-commutative case. Indeed, we continue our recent works, and develop two approaches to directly attack on the problems. The first approach is to show that the test case for this problem – the Fréchet algebra $\mathcal{U}$ of all entire functions on $\ell^\infty$ – is, in fact, a Fréchet algebra $\mathcal{C}[[X]]$, if there exists a discontinuous character on $\mathcal{U}$. Discussions along these lines always skirt the Dales-McClure’s problem (1977), solved affirmatively by Dales, Read and the author in 2010, as well as the Loy’s problem (1974), solved affirmatively in the Fréchet case by the author recently. The elementary, but crucial, idea is that to express the test algebra as a weighted Fréchet symmetric algebra over a Banach space $\ell^1$.

In the second approach, the existence of a discontinuous character on $\mathcal{U}$ would allow us to generate another Fréchet algebra topology $\tau$, inequivalent to the usual Fréchet algebra topology $\tau_0$, by applying the Read’s method (he used this method to show that the famous Singer-Wermer conjecture (1955) does not hold in the Fréchet case, and we have recently constructed two Fréchet algebras, admitting countably many mutually inequivalent Fréchet
algebra topologies by this method; such examples are not known even in the Banach case).

In both approaches, an important tool is a topological version of the (symmetric) tensor algebra over a Banach space; we use the notion of “tensor product by rows”, introduced by Read, in the second approach.

Several mathematicians worked on Michael problems since 1952, giving affirmative solutions for special classes of Fréchet algebras under various conditions, or discussing various test cases, or discussing various approaches, or discussing various other equivalent forms, or deriving other important automatic continuity results such as the uniqueness of the Fréchet algebra topology for commutative Fréchet algebras by alternate, difficult, lengthy methods. We summarize effect of our affirmative solutions on these attempts in addition to give various new applications in automatic continuity theory.
1 Introduction

An important subject in the theory of Fréchet algebras is certain questions from automatic continuity theory, which may have applications to commutative rings and algebras, theory of SCV and complex dynamical systems; e.g., the uniqueness of the Fréchet algebra topology on certain Fréchet algebras, the continuity property of certain homomorphisms between certain Fréchet algebras as well as that of (higher) derivations on Fréchet algebras. So far, we have established the uniqueness of the Fréchet algebra topology for Fréchet algebras of power series in one indeterminate [P1], and in several indeterminates [P3]. Dales, Read and the author have also obtained an affirmative solution to the Dales-McClure problem (1977), and have also obtained other interesting automatic continuity results, including a reduction of Michael problem (discussed below) [DPR]. Recently, we have used the discontinuity of derivations to give other inequivalent Fréchet algebra topology to certain Fréchet algebras, in order to solve the Loy’s problem (1974) [P4]. We have also constructed two (maiden) examples of Fréchet algebras admitting countably many mutually inequivalent Fréchet algebra topologies to discuss the famous Singer-Wermer conjecture in Fréchet algebras [P5]. All this work has some connection with the prestigious Michael problem; in fact,
all this work has turned out stepping stones for the complete solution to the problem as we shall see below.

In 1952, Michael posed a question about the continuity of characters on Fréchet algebras in his memoir [M], known as Michael’s problem in the literature. It is likely that the question was already discussed by Mazur in Warsaw around 1937 [DPR]. Several significant analysts worked on this problem; especially, Arens [Ar], Shah [Sh], Clayton [Cl], Akkar [Ak], Carpenter [Cl], Craw [Cr], Goldmann [Go1], Schottenloher [Sco], Muro [Mu], Mujica [Mj], Dixon and Fremlin [DF], Dixon and Esterle [DE], Esterle [E1, E2, E3], Ephraim [EPH], Forster [For], Markoe [Ma], Stensones [St], Želazko [Z1], and Dales, Read and the author [DPR]. Despite a lot of efforts by various mathematicians to solve Michael problem, it seems that only six significant ideas appeared in the literature since 1952 [Ar, Sh, Cl, DE, E2, DPR]. The strong partial result was obtained by Arens in 1958 [Ar]; he showed that finitely (resp., rationally) generated Fréchet algebras are functionally continuous. As far as we know, the most latest effort was made by Dales, Read and the author in 2010 [DPR]; we showed that a well known test case for this problem – the Fréchet algebra $\mathcal{U}$ of entire functions on $\ell^\infty$ [Cl, DE, E2] – is, in fact, a Fréchet algebra of power series [DPR], and so, the natural
question of whether characters are automatically continuous on a Fréchet algebra which is a Fréchet algebra of power series, is the prestigious Michael problem itself in disguise. Incidentally, the author asked Želazko in 2004 whether the Michael problem has an affirmative solution for ALL Fréchet algebras of power series [Z3]. Obviously, an affirmative answer to the author’s question would extend the Arens result for the singly generated case to the non-singly generated case within the class of Fréchet algebras of power series, and would, of course, solve Michael problem in view of Thm. 10.1 of [DPR].

Since 1958, it was not clear to functional analysts how to extend the Arens result from the finitely generated case to the countably generated case, in order to obtain functional continuity of the test algebra $U$; we remark that the Arens result, which is the only consistent general partial result about this problem, was based itself upon the Mittag-Leffler theorem (an essential ingredient in the proof). Also, one strongly believes that there must be some ways to apply appropriate methods within automatic continuity theory as the problem falls in this theory. So, we represent two approaches here. In both the approaches, an important tool is a topological version of the (symmetric) tensor algebra over a Banach space [DM, DPR]; we use the notion of “tensor product by rows”, introduced by Read [R], in the second approach.
We take this opportunity to give honor to all this work, especially the Dixon and Esterle approach [DE] and the Esterle approaches [E1, E2]. We hope that the present work will encourage some people to invest some time and energy in order to make progress on the associated problem in the theory of SCV (by showing that these two problems are, indeed, equivalent); in particular, Dixon and Esterle may like to revisit their approaches, and Foraess and Stensones (and their team) may like to study the associated problem in the theory of SCV in the light of the present work [BH, FS, Fo1, Fo2, Gl, Ki, St].

Throughout the paper, “algebra” will mean a non-zero, complex, (non)-commutative algebra with identity unless otherwise specified. We recall that a Fréchet algebra is a complete, metrizable locally convex algebra $A$ whose topology $\tau$ may be defined by an increasing sequence $(p_m)_{m \in \mathbb{N}}$ of submultiplicative seminorms. The basic theory of Fréchet algebras was introduced in [Go, M]. The principal tool for studying Fréchet algebras is the Arens-Michael representation, in which $A$ is given by an inverse limit of Banach algebras $A_m$.

A Fréchet algebra $A$ is called a uniform Fréchet algebra if for each $m \in \mathbb{N}$ and for each $x \in A$, $p_m(x^2) = p_m(x)^2$. Let $k \in \mathbb{N}$ be fixed. We
write $\mathcal{F}_k$ for the algebra $\mathcal{C}[[X_1, X_2, \ldots, X_k]]$ of all formal power series in $k$ commuting indeterminates $X_1, X_2, \ldots, X_k$, with complex coefficients. A fuller description of this algebra is given in [D, §1.6], and for the algebraic theory of $\mathcal{F}_k$, see [ZS, Ch. VII]; we briefly recall some notations, which will be used throughout the paper. Let $k \in \mathbb{N}$, and let $J = (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^+$.

Set $|J| = j_1 + j_2 + \cdots + j_k$; ordering and addition in $\mathbb{Z}^+$ will always be component-wise. A generic element of $\mathcal{F}_k$ is denoted by

$$\sum_{J \in \mathbb{Z}^+} \lambda_J X^J = \sum \{ \lambda_{(j_1, j_2, \ldots, j_k)} X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} : (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^+ \}.$$

The algebra $\mathcal{F}_k$ is a Fréchet algebra when endowed with the weak topology $\tau_c$ defined by the coordinate projections $\pi_I : \mathcal{F}_k \to \mathcal{C}', I \in \mathbb{Z}^+$, where $\pi_I(\sum_{J \in \mathbb{Z}^+} \lambda_J X^J) = \lambda_I$. A defining sequence of seminorms for $\mathcal{F}_k$ is $(p'_m)$, where $p'_m(\sum_{J \in \mathbb{Z}^+} \lambda_J X^J) = \sum_{|J| \leq m} |\lambda_J| \quad (m \in \mathbb{N})$. A Fréchet algebra of power series in $k$ commuting indeterminates is a subalgebra $A$ of $\mathcal{F}_k$ such that $A$ is a Fréchet algebra containing the indeterminates $X_1, X_2, \ldots, X_k$ and such that the inclusion map $A \hookrightarrow \mathcal{F}_k$ is continuous (equivalently, the projections $\pi_I, I \in \mathbb{Z}^+$, are continuous linear functionals on $A$) [P3]. We defined a Fréchet algebra with a power series generator and discussed several examples of power series generated Fréchet algebras of power series in [BP]. One extends the notion of Fréchet algebra with a power series generator
in finitely many indeterminates case appropriately, and discusses analogous examples of power series generated Fréchet algebras of power series in $\mathcal{F}_k$; e.g., $\mathcal{F}_k$, the Beurling-Banach (Fréchet) algebras in $\mathcal{F}_k$ (called the Beurling-Fréchet algebras of (semi)weight types in [P3]), $\text{Hol}(U)$, $U$ an open unit disc in $\mathcal{O}^k$, $\text{Hol}(\mathcal{O}^k)$, $A^\infty(\Gamma^k)$, $\Gamma^k$ a unit circle in $\mathcal{O}^k$.

We write $\mathcal{F}_\infty$ for the algebra $\mathcal{C}[[X_1, X_2, \ldots]]$ of all formal power series in countably many commuting indeterminates $X_1, X_2, \ldots$, with complex coefficients. We shall also require the non-commutative version of $\mathcal{F}_\infty$, denoted by $\mathcal{B} = \mathcal{C}_{nc}[[X_1, X_2, \ldots]]$. A fuller description of these algebras is given in [DPR, §9]; see also [E2] for $\mathcal{F}_\infty$. Both are Fréchet algebras under the usual topology $\tau_c$ of coordinatewise convergence. A Fréchet algebra of power series in countably many commuting indeterminates is a subalgebra $A$ of $\mathcal{F}_\infty$ such that $A$ is a Fréchet algebra containing the indeterminates $X_1, X_2, \ldots$ and such that the inclusion map $A \hookrightarrow \mathcal{F}_\infty$ is continuous (equivalently, the projections $\pi_I$, $I \in (\mathbb{Z}^+)^{<\omega}$, are continuous linear functionals on $A$); also a Fréchet algebra of power series in countably many non-commuting indeterminates is a subalgebra $A$ of $\mathcal{B}$ such that $A$ is a Fréchet algebra containing the indeterminates $X_1, X_2, \ldots$ and such that the inclusion map $A \hookrightarrow \mathcal{B}$ is continuous (equivalently, the projections $\pi_I$, $I \in S_{nc}$ (the free semigroup...
in countably many (non-commuting) elements $X_1, X_2, \ldots$), are continuous linear functionals on $A$). We extend the notion of Fréchet algebra with a power series generator in countably many indeterminates case appropriately, and discuss analogous examples of power series generated Fréchet algebras of power series in $\mathcal{F}_\infty$; e.g., $\mathcal{F}_\infty$, the Beurling-Banach (Fréchet) algebras in $\mathcal{F}_\infty$ (e.g., the Banach algebras $U_m$, for each $m \in \mathbb{N}$, and the test case $U$ as we shall see below).

We remark that we shall consider these algebras in the indeterminates $X_0, X_1, \ldots$, depending on our requirements, but this change will make no difference on their algebraic/topological structures, except the notational freedom that we want to avail (e.g., in §4). Some remarks on $\mathcal{F}_\infty$ are in order. The algebra $\mathcal{F}_\infty$ is a graded algebra which is not noetherian, as the ideal generated by $X_1, X_2, \ldots$, is not closed [DPR] (cf. also [Z2]); the algebra $\mathcal{B}$ is also a graded algebra which is not noetherian for the same reason [DPR] (Żelazko posed a question whether Thm. 5 holds true in the non-commutative case [Z2]). In [E2], Esterle remarks that $\mathcal{F}_\infty$ is an integral domain, all principal ideals in $\mathcal{F}_\infty$ are closed, but that he does not know whether or not all finitely generated ideals in $\mathcal{F}_\infty$ are closed. Here, we remark that the algebra $\mathcal{F}_\infty$ is a Fréchet algebra of finite type (introduced by Kopp
in [K]) and so, all finitely generated ideals are closed in $\mathcal{F}_{\infty}$ [K, Remark, p. 222]. It was shown that the algebra $\mathcal{F}_{\infty}$ is embedded into $\mathcal{F}_2$ as an extension of Thm. 2.2 of [DPR]; it was also shown that the algebra $\mathcal{F}_k$ cannot be embedded in $\mathcal{F}$ for each $k \geq 2$ [DPR, Thm. 2.6]. Below, we show that the algebra $\mathcal{F}_{\infty}$ cannot be embedded in $\mathcal{F}$, extending Thm. 2.6 of [DPR].

**Theorem 1.1** There is no embedding of $\mathcal{F}_{\infty}$ into $\mathcal{F}$.

**Proof.** Assume towards a contradiction that $\theta : \mathcal{F}_{\infty} \to \mathcal{F}$ is an embedding. Then $\theta$ is not a surjection, for this would imply that $\mathcal{F}_{\infty} \cong \mathcal{F}$, and this is impossible because $\mathcal{F}$ is noetherian, but $\mathcal{F}_{\infty}$ is not. Thus, by [DPR, Thm. 11.2], this embedding is continuous (one also arrives at the continuity of $\theta$, using the notion of weighted order of $r$; see proof of Thm. 9.1 of [DPR] which we shall again consult in Cor. 5.10 below) and so, $\mathcal{F}_{\infty}$ is a Fréchet subalgebra of $\mathcal{F}$ containing the indeterminate $X$ (cf. [DPR, Thm. 10.1]). By Cor. 11.3 of [DPR], $\mathcal{F}_{\infty}$ is a Fréchet algebra of power series and so, $\mathcal{F}_{\infty} = \mathcal{F}$, because if $\mathcal{F}_{\infty}$ is a proper subalgebra of $\mathcal{F}$, then the Fréchet algebra topology of $\mathcal{F}_{\infty}$ can be given by an increasing sequence $(p_m)$ of norms by [P1, Thm. 3.3], a contradiction to the fact that the usual Fréchet algebra topology $\tau_c$ on $\mathcal{F}_{\infty}$ is defined by an increasing sequence of proper seminorms (Read defined another inequivalent Fréchet algebra topology which is also given by
an increasing sequence of proper seminorms (see [R, Def. 1.12]); further, the
author gave countably many inequivalent Fréchet algebra topologies on $\mathcal{F}_\infty$, they all are defined by increasing sequences of proper seminorms (see [P5])). One also arrives at a contradiction due to the fact that any proper Fréchet subalgebra of $\mathcal{F}$ contains Banach algebras of power series in its Arens-Michael representation, or does satisfy the equicontinuity condition (E) ([P1, Thm. 3.6]), but the algebra $\mathcal{F}_\infty$ is a Fréchet algebra of finite type [K] as well as it does not satisfy the equicontinuity condition (E) because it has countably many inequivalent Fréchet algebra topologies, generated by sequences of proper seminorms, as discussed above [P3, Thm. 4.4, P5]. Now that $\mathcal{F}_\infty = \mathcal{F}$, a contradiction because it is neither noetherian (algebraic point of view) nor it has a unique Fréchet algebra topology (topological point of view). Hence there is no embedding of $\mathcal{F}_\infty$ into $\mathcal{F}$. \hfill $\Box$

We shall also require in a future proof the 'averaging map' and symmetrizing map $\tilde{\sigma}$ on $\mathcal{B}$, and symmetric elements of $\mathcal{B}$ [DPR]. There is a product in $\mathcal{B}_{\text{sym}}$, and with this product $\vee$, it is a commutative, unital algebra and $\mathcal{B}_{\text{sym}} = \epsilon(\mathcal{F}_\infty)$, where $\epsilon : \mathcal{F}_\infty \to \mathcal{B}$ is a continuous linear embedding [DPR]; in fact, it is a continuous, injective homomorphism as we shall see below.

In this paper, we shall be concerned with the affirmative solution to
Michael problems (commutative case as well as non-commutative case). We give two approaches. We shall briefly discuss some general theory of topological (symmetric) tensor algebras over a Banach space in the next section, in order to establish notation that will be used throughout the paper. We shall also discuss Arens-Michael representations and spectra of such algebras. In particular, we are mainly interested in certain semigroup (Banach/Fréchet) algebras over two specific semigroups, which are graded subalgebras of $\mathcal{F}_{\infty}$ (respectively, of $\mathcal{B}$ in the non-commutative case). As we shall see, these algebras over a Banach space $\ell^1$ is the technical main-spring of the paper.

The first approach is discussed in Section 3. The essential ingredient is to describe the test algebra $U$ in terms of weighted Fréchet symmetric algebra $\hat{\bigvee}_WE$ over the Banach space $\ell^1$ (this was the example, discussed in [P4], that motivated the ideas of the present paper), and then, to show that the existence of a discontinuous character on $U$ leads us to produce a non-degenerate, totally discontinuous higher point derivation ($d_n$) on $U$ at this discontinuous character. This would imply that $U = \mathcal{F}$ by Thms. 10.1 and 11.2 of [DPR], a contradiction. As consequences, we have an affirmative solution to the second problem of Michael; we extend certain results about functional continuity of Stein algebras, established by Forster [For], Markoe
[Ma] and Ephriam [EPH]. We also discuss implications of our solution in view of [E2, Thm. 2.7], and give further interesting remarks in automatic continuity theory.

The second approach is discussed in Section 4. Although we shall show that both the non-commutative case and the commutative case are dependent on each other (cf. [DE] and Thm. 4.1 below), we shall work along the Read’s method to give another inequivalent Fréchet algebra topology to the non-commutative test case $\mathcal{U}_{nc}$. This would admit that the commutative test case $\mathcal{U}$ also admits two inequivalent Fréchet algebra topologies, a contradiction to the fact that $\mathcal{U}$, being a commutative, semisimple Fréchet algebra of power series (even in $\mathcal{F}_\infty$), has a unique Fréchet algebra topology [C1, P1, P3]. Our argument for this section is kept short because it uses the key ideas involved in [R].

In Section 5, we summarize the previous developments on Michael problems in view of our affirmative solutions; e.g., we shall show that the Dixon and Fremlin result holds in the non-commutative case; we shall discuss Cor. 6 of Carpenter’s result [C1] and shall also discuss Michael problem in view of (dis) continuity of the derivation $\partial/\partial X_0$ on $\mathcal{U}$ (resp., on $\mathcal{U}_{nc}$) [C2]. It is a known fact that another equivalent form of Michael problem is the long-
standing problem of continuity of a homomorphism from a Fréchet algebra $B$ into a semisimple Fréchet algebra $A$ (an affirmative answer would give us the shortest proof of a uniqueness of the Fréchet algebra topology for commutative, semisimple Fréchet algebras). We shall establish the continuity of this homomorphism. In fact, it is a surprising consequence that an attempt to solve the non-commutative analogue of this problem (as well as of continuity of derivations on non-commutative, semisimple Fréchet algebras) leads us to certain automatic continuity results as well as the third approach to affirmatively answer Michael problems (even for more general complete, metrizable topological algebras (in commutative case as well as non-commutative case)) by extending Esterle’s result [E5], many thanks to Thomas Lem. 1.1a [T1]. We also extend some automatic (dis) continuity results of [DPR, §12], in order to complete the circle of ideas.

2 Topological tensor algebras over a Banach space

In the next sections, our important tool is a topological version of the (symmetric) tensor algebra over a vector space [Gr, Chapter III]. The topological
version of the tensor algebra over a Banach space appear in [Co, Le]. For a
general information about topological tensor products, we refer to [Gro, Tr].
However, a fuller description of what we require is given in [DM, D]; we briefly
recall some notation, which will be used throughout the paper. Let $E$ be a
Banach space. For each $p \geq 2$, write $\hat{\otimes}^p E$ for the completion of $\otimes^p E$ with
respect to the equicontinuous tensor product norm (respectively, the projective
tensor product norm). When it is unnecessary to distinguish, we may
write $\hat{\otimes}^p E$ for either completion, and $\| \cdot \|$ for the specified norm. Then $\hat{\otimes} E$
is a unital, non-commutative Fréchet tensor algebra over $E$ (if $\dim E > 1$)
with respect to the coordinatewise convergence topology defined by an in-
creasing sequence $(\| \cdot \|_m)$ of seminorms, where $\| \sum_p u_p \|_m = \sum_{p=0}^m \| u_p \|$. We
refer to the subspaces $\hat{\otimes}^p E$ as the homogeneous subspaces of $\hat{\otimes} E$.

We shall also need in a future proof a commutative analogue $\hat{\bigvee} E$ of $\hat{\otimes} E$,
and a closed, linear subspace $\hat{\bigvee}^p E$ of $\hat{\otimes}^p E$. Recall that $\hat{\bigvee} E$ is a commutative,
unital Fréchet symmetric algebra over $E$ with respect to the same topology
defined by an increasing sequence $(\| \cdot \|_m)$ above, but with a different product.

We remark that if $\tilde{\sigma} = \bigoplus \tilde{\sigma}_p$ is the continuous, symmetrizing epimorphism
$\tilde{\sigma} : \hat{\otimes} E \to \hat{\bigvee} E$, then $\ker \tilde{\sigma}$ is the closed, two-sided ideal of $\hat{\otimes} E$ generated
by $\{ u \otimes v - v \otimes u : u, v \in \bigotimes^1 E = E \}$, so that the algebras in [Co] and [Le]
and the algebras $\hat{\bigvee}E$ are topologically isomorphic.

There are some important Banach subalgebras of the above examples. Let $\omega$ be a weight on $\mathbb{Z}^+$ and let $A$ be one of the algebras $\hat{\bigotimes}E$ and $\hat{\bigvee}E$. Define $\hat{\bigotimes}_\omega E$ and $\hat{\bigvee}_\omega E$, respectively, as

$$\{ u = (u_p) \in A : \|u\|_\omega := \sum_{p=0}^{\infty} \|u_p\|_p \omega(p) < \infty \}.$$  

We obtain two unital Banach subalgebras, with $\hat{\bigvee}_\omega E$ being commutative. These algebras are called weighted Banach tensor algebras and weighted Banach symmetric algebras, respectively. The algebra $\hat{\bigotimes}_\omega E$, where $\omega \equiv 1$ on $\mathbb{Z}^+$, is the Banach tensor algebra which is the starting point for the constructions in [Co] and [Le]. We recall the following theorem from [DM] (also, cf. [Le, Satz 2 and Satz 3]), describing the maximal ideal space and semisimplicity of $\hat{\bigvee}_\omega E$.

**Theorem 2.1**

(i) The space of characters on $\hat{\bigvee}_\omega E$ is homeomorphic with the ball

$$\{ \lambda \in E' : \|\lambda\| \leq \inf_p \omega(p)^{\frac{1}{p}} \equiv \omega_{\infty} \} \ (w^*\text{ - topology}).$$

(ii) $\hat{\bigvee}_\omega E$ (w.r.t. the equicontinuous tensor norm) is semisimple if and only if $\omega_{\infty} > 0$. 

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(iii) \( \bigvee_\omega E \) (w.r.t. the projective tensor norm) is semisimple if and only if 
\( \omega_\infty > 0 \) and \( E \) has the approximation property.

Along the same lines, for an increasing sequence \( W = (\omega_m) \) of weights on \( \mathbb{Z}^+ \), we define \( \widehat{\otimes}_WE \) and \( \bigvee_W E \), respectively, as
\[
\{ u = (u_p) \in A : p_m(u) := \sum_{p=0}^\infty \|u_p\|_p \omega_m(p) < \infty \text{ for all } m \in \mathbb{N} \}.
\]
We obtain two unital Fréchet subalgebras, with \( \bigvee_W E \) being commutative. These algebras are called weighted Fréchet tensor algebras and weighted Fréchet symmetric algebras, respectively. Clearly, their Arens-Michael representations are given by
\[
\bigvee_W E = \bigcap_{m \in \mathbb{N}} \bigvee_{\omega_m} E \quad \text{and} \quad \widehat{\otimes}_WE = \bigcap_{m \in \mathbb{N}} \widehat{\otimes}_{\omega_m} E.
\]

Next, we discuss the Arens-Michael representation of \( \bigvee E \). For each \( m \), \( \bigvee E/\ker(\| \cdot \|_m) \) is a Banach algebra with respect to the norm \( \| \cdot \|_m \), and is isomorphic with
\[
(\bigvee E)_m = \{ \sum_{p=0}^m u_p : u_p \in \bigvee^p E \},
\]
and the product being
\[
(\sum_{p=0}^m u_p) \vee (\sum_{p=0}^m v_p) = \sum_{p=0}^m (\sum_{i+j=p} u_i \vee v_j).
\]
Similarly, we also have the Arens-Michael representation of \( \widehat{\otimes}E \).

We shall also require to study certain semigroup (Banach or Fréchet) algebras from [DPR, §9, 10], which are graded subalgebras of \( F_\infty \) in the
commutative case (respectively, of $B$ in the non-commutative case). Mainly, we consider these algebras on the semigroup $S = (\mathbb{Z}^+)^{<\omega}$ in the commutative case, and on the free semigroup $S_{nc}$ in countably many non-commuting elements $X_1, X_2, \ldots$ in the non-commutative case; we shall discuss these examples, required for our approaches to solve Michael problem affirmatively, in Section 3. Most importantly, these algebras may be viewed as weighted Banach tensor algebras, weighted Banach symmetric algebras, and the test cases for Michael problem as their Fréchet analogues.

Let $E = \ell^1(\mathbb{Z}^+)$, a Banach space. Then, as in [DPR, §10], for each $p \in \mathbb{N}$, $\hat{\otimes}^p E$ can be identified with $\ell^1((\mathbb{Z}^+)^p)$ as a Banach space. This Banach space can also be viewed as the space of absolutely summable functions on $\mathbb{N}^p$ (equivalently, the space of functions in $A(D^p)$ with absolutely convergent Taylor series). Here, we remark that they are even identified as Banach algebras as follows. By [D, 1.3.11], $\ell^1 \otimes \ell^1$ is a commutative algebra with identity. Then, it is easy to see that it is a normed algebra with respect to the projective tensor norm (note that $\ell^1$-norm is submultiplicative on $\ell^1$). Now $\ell^1 \hat{\otimes} \ell^1$ is, indeed, a Banach algebra by noticing that the product can be extended to $\ell^1 \hat{\otimes} \ell^1$ by [DM]. The main point of this identification should be emphasized. The projective tensor product of two singly generated
Banach algebras is identified with a doubly generated Banach algebra, but, unfortunately, we cannot extend this to $\mathcal{N}$-fold tensor product.

As above, $\bigvee^p E$ is a a closed, linear subspace of $\bigotimes^p E$, consisting of the symmetric elements. Recall that $\bigvee^p E$ is the range of the symmetrizing map $\bar{\sigma}_p$ (equivalently, the projection of norm 1) on $\bigotimes^p E$. This Banach space can also be viewed as the space of absolutely summable symmetric functions on $\mathcal{N}^p$ (equivalently, the space of symmetric functions in $A(D^p)$ with absolutely convergent Taylor series). Then, $\bigotimes E$ is a non-commutative, unital Fréchet tensor algebra over $E$, and $\bigvee E$ it’s commutative analogue (but with different product), consisting of the symmetric elements. Recall that $\bigvee E$ is the range of the symmetrizing map $\bar{\sigma}$ on $\bigotimes E$. The algebra $\bigotimes E$ is naturally identified with a graded subalgebra of $\mathcal{B}$, and $\bigvee E$ is naturally identified with a graded subalgebra of $(\mathcal{B}_{\text{sym}}, \vee)$, called the unital Fréchet symmetric algebra over $E$. The latter algebra can also be viewed as the algebra of locally absolutely summable symmetric functions on $\mathcal{N}^\mathcal{N}$ (equivalently, the algebra of symmetric functions in $A(D^{\mathcal{N}})$ with locally absolutely convergent Taylor series), and the former algebra can also be viewed as the algebra of locally absolutely summable functions on $\mathcal{N}^{\mathcal{N}}$ (equivalently, the algebra of functions in $A(D^{\mathcal{N}})$ with locally absolutely convergent Taylor series).
Let $m \in \mathbb{N}$ be fixed. We discuss the maximal ideal space and semisimplicity of a Banach algebra $[\hat{\bigvee} E]_m$ in the following theorem, we sketch the proof here for the reader’s convenience (cf. proof of Thm. 1.2 of [DM]).

**Theorem 2.2** (i) The space of characters on $[\hat{\bigvee} E]_m$ is homeomorphic with the closed unit ball $D^m$ in $\mathbb{C}^m$ ($w^*$-topology).

(ii) $[\hat{\bigvee} E]_m$ (w.r.t. the equicontinuous tensor norm) is semisimple.

(iii) $[\hat{\bigvee} E]_m$ (w.r.t. the projective tensor norm) is semisimple if $E$ has the approximation property.

**Proof.** (i) Let $\lambda \in D^m$. Then we can consider $\lambda \in \ell^\infty (= E')$ by taking components 0 for all $n \geq m + 1$. Obviously, $\|\lambda\| \leq 1$. For each $p = 1, 2, \ldots, m$, take $\lambda_p$ to be the bounded linear functional defined on $[\hat{\bigvee} E]$ by the $p$-tuple $(\lambda, \lambda, \ldots, \lambda) \in (\ell^\infty)^p$. Then $\|\lambda_p\| \leq 1$, so that $\phi_\lambda(\sum_{p=0}^m u_p) = \sum_{p=0}^m \lambda_p(u_p)$ defines a bounded linear functional on $[\hat{\bigvee} E]_m$. Clearly, $\phi_\lambda$ is a character on $[\hat{\bigvee} E]_m$, and that the map $\lambda \mapsto \phi_\lambda$ is an injection from $D^m$ into the character space of $[\hat{\bigvee} E]_m$. Now suppose that $\phi$ is a character on $[\hat{\bigvee} E]_m$, and define $\phi|\hat{\bigvee} E = \lambda_p$ for each $p \leq m$. Then $\lambda_1 \equiv \lambda \in \ell^\infty$. Take $\lambda_1 \equiv \lambda \in D^m$ by cutting the tail of $\lambda$. Then it is easy to see that $\lambda_p$ are associated with $\lambda$ exactly as above. Thus $\lambda \mapsto \phi_\lambda$ is a bijection. It is clear
that the inverse map $\phi \mapsto \lambda(= \phi|_{\ell^1})$ is continuous; the equivalence of the $w^*$-topology follows.

(ii) and (iii) could be proved by making trivial modifications in the proof of (ii) and (iii) of Thm. 1.2 of [DM]. We note that the character space of the range algebra (which is contained in $\ell^1$) contains the closed unit disc in $\mathcal{A}'$ and that the range algebra is semisimple. \hfill $\square$

**Corollary 2.3** The Fréchet algebra $(\hat{\bigvee}E, (\| \cdot \|_m))$ is semisimple. \hfill $\square$

**Remarks.** 1. There is a non-commutative analogue of the above theorem, describing the (same) maximal ideal space and semisimplicity of a Banach algebra $[\hat{\bigotimes}E]_m$, and so the Fréchet algebra $(\hat{\bigotimes}E, (\| \cdot \|_m))$ is semisimple, being inverse limit of semisimple Banach algebras [M].

2. As commutative Banach algebras,

$$[\hat{\bigvee}E]_m \cong ([\hat{\bigvee}E]_m)_\omega = \{ \sum_{p=0}^{m} u_p \in [\hat{\bigvee}E]_m : \| \sum_{p=0}^{m} u_p \| = \sum_{p=0}^{m} \| u_p \| \},$$

where $\omega$ is a weight on the finite semigroup $(\mathbb{Z}^+)_m = \{0, 1, \ldots, m\}$ with semigroup operation addition modulo $m+1$ (cf. [BP, p. 144]), defined by $\omega(k) = 1$, $k \in (\mathbb{Z}^+)_m$. Similarly, there is a non-commutative analogue of this identification.

Next, we describe the character space of a Fréchet algebra $\hat{\bigvee}E$ in the
Theorem 2.4  The character space $M(\hat{V}E)$ is a hemicompact $k$-space which is homeomorphic with $\ell^\infty_{[1]}$, the closed unit ball of $\ell^\infty$ ($w^*$-topology).

Proof. By Thm. 2.1 above, the character space of a Banach algebra $\hat{V}_\omega E$, where $\omega \equiv 1$ on $\mathbb{Z}^+$, is homeomorphic with the closed unit ball of $\ell^\infty$ in the $w^*$-topology, and this Banach algebra is, indeed, a dense Banach subalgebra of a Fréchet algebra $\hat{V} E$, so it is clear that the character space of $\hat{V} E$ is continuously embedded in the closed unit ball of $\ell^\infty$ ($w^*$-topology) by [Go, Lem. 3.2.5]. Moreover, from the general theory of Féchet algebras, it is clear that $M(\hat{V} E)$ is a hemicompact, $k$-space, since $M(\hat{V} E) = \bigcup_{m \in \mathbb{N}} M([\hat{V} E]_m) \cong \bigcup_{m \in \mathbb{N}} D^m$ by Thm. 2.2 above. To show that it is homeomorphic with the closed unit ball of $\ell^\infty$, let $\lambda \in \ell^\infty$ with $\|\lambda\| \leq 1$. Then, for each fixed $m \in \mathbb{N}$, take $\lambda_m \in D^m$ by cutting the tail of $\lambda$. By Thm. 2.2 (i) above, $\lambda_m = \phi_{\lambda_m} \in M([\hat{V} E]_m) \cong D^m$. So, $\lambda = \lim_m \lambda_m \in \lim_m M([\hat{V} E]_m) = \bigcup_{m \in \mathbb{N}} M([\hat{V} E]_m)$, since $D^m \subset D^{m+1}$ for each fixed $m \in \mathbb{N}$. Thus the injection is, indeed, a bijection between $M(\hat{V} E)$ and $\ell^\infty_{[1]}$. The equivalence of the $w^*$-topology follows.  

Remarks. 1. For each fixed $m \in \mathbb{N}$, $M([\hat{V} E]_m) \cong D^m$ is continuously embedded in the closed unit ball of $\ell^\infty$ in the $w^*$-topology by [Go, Lem.
3.2.5] (we have a continuous, dense embedding from \( \hat{V}_\omega E \), where \( \omega \equiv 1 \) on \( \mathbb{Z}^+ \), into \( [\hat{V} E]_m \)).

2. There is an alternate proof as follows (cf. proof of Thm. 1.2 of [DM]): Let \( \lambda \in \ell_\infty^{[1]} \). Take \( \lambda_0 = 1 \in \mathcal{C} \), and for each \( p = 1, 2, \ldots \), take \( \lambda_p \) to be the unique bounded linear functional defined on \( \hat{V}^p E \) by the \( p \)-tuple \( (\lambda, \lambda, \ldots, \lambda) \in (\ell_\infty)^p \). Then \( \|\lambda_p\| \leq 1 \), so that \( \phi_\lambda(\sum_{p=0}^{\infty} u_p) = \sum_{p=0}^{\infty} \lambda_p(u_p) \) defines a bounded linear functional on \( \hat{V} E \) (if required, one extends \( \phi_\lambda \) from \( \hat{V}_\omega E \), where \( \omega \equiv 1 \) on \( \mathbb{Z}^+ \), to \( \hat{V} E \)). Clearly, \( \phi_\lambda \) is a character on \( \hat{V} E \), and that the map \( \lambda \mapsto \phi_\lambda \) is an injection from \( \ell_\infty^{[1]} \) into \( M(\hat{V} E) \). Now suppose that \( \phi \) is a character on \( \hat{V} E \), and define \( \phi|\hat{V}^p E = \lambda_p \) for each \( p \in \mathbb{N} \).

Then \( \lambda_1 \equiv \lambda \in \ell_\infty \). Then it is easy to see that the \( \lambda_p \) are associated with \( \lambda \) exactly as above. Thus \( \lambda \mapsto \phi_\lambda \) is a bijection. It is clear that the inverse map \( \phi \mapsto \lambda (= \phi|\ell^1) \) is continuous; the equivalence of the \( w^* \)-topology follows.

3. There is also an alternate (but longer) proof of the fact that the algebra \( \hat{V} E \) is semisimple as follows (cf. proof of Thm. 1.2 of [DM]); we describe it as it shares some useful information about the algebra: For each \( p > 1 \), the linear functionals from \( (\ell_\infty)^p \) separates the points of \( \hat{\otimes}^p E \). On \( \hat{V}^p E \), the linear functionals determined by \( (\ell_\infty)^p \) can be described in terms of \( p \)-tuples \( (\lambda, \lambda, \ldots, \lambda), \lambda \in \ell_\infty \). Thus, the \( p \)-tuples \( (\lambda, \lambda, \ldots, \lambda) \) separate the
points of $\hat{V}^pE$. Now suppose that $\lambda \in \ell^\infty_1$. Take $\lambda_p$ as in 2. above. Then the map $\sum_{p=0}^{\infty} u_p \mapsto \sum_{p=0}^{\infty} \lambda_p(u_p)X^p$ is a homomorphism from $\hat{V}E$ into a proper subalgebra of $F$ (which contains $\ell^1(I^+)$; one may extend continuously the homomorphism, discussed in proof of (ii) of Thm. 1.2 of [DM], to $\hat{V}E$, and so, the range of this extended homomorphism contains $\ell^1(I^+))$. Since $(\lambda, \lambda, \ldots, \lambda)$ ($\lambda \in \ell^\infty$) separate the points of $\hat{V}^pE$, it is clear that $\sum_{p=0}^{\infty} \lambda_p(u_p)X^p \neq 0$ for some $\lambda$ if $\sum_{p=0}^{\infty} u_p \neq 0$. So the map is injective. Thus, $\hat{V}E$ is (isometrically isomorphic to) a Fréchet algebra of power series in $F$. Certainly, $\hat{V}E$ is properly contained in $F_\infty$, because if both are equal, then it contradicts to the fact that $F_\infty$ cannot be embedded in $F$ as a Fréchet algebra of power series by Thm. 1.1 above. Since the range algebra contains $\ell^1(I^+)$, it is surely properly contained in $F$ due to [K] (the Arens-Michael representation of $\hat{V}E$ contains infinite-dimensional Banach algebras, whereas $F$ is the only Fréchet algebra of power series, which is also of finite type [P1]). Since $\hat{V}E$ is a Fréchet algebra of power series, it has a unique Fréchet algebra topology (and so, $\hat{\otimes}E$ also has a unique Fréchet algebra topology). Further, $\hat{V}E$ does satisfy the equicontinuity condition (E) and each $[\hat{V}E]_m$ is a Banach algebra of power series in $F$. Recall that $\hat{V}E$ is the algebra of locally absolutely summable symmetric functions on $\mathcal{N}^N$ (equivalently, the
algebra of symmetric functions in $A(\ell_1^{\infty})$ with locally absolutely convergent Taylor series), and so it is semisimple. Similarly, $\hat{\otimes}E$ is also semisimple.

4. The algebra $\hat{\bigvee}_\omega E$, where $\omega \equiv 1$ on $\mathbb{Z}^+$, is isometrically isomorphic with $\ell^1((\mathbb{Z})^{<\omega})$ [DPR], so it may be viewed as the algebra of functions in $A(\ell_1^{\infty})$ with absolutely convergent Taylor series (equivalently, the algebra of absolutely summable symmetric functions on $\mathbb{R}^N$). It is interesting to note that the proof of showing this algebra a Banach algebra of power series, could drastically be shortened by noticing that the homomorphism, discussed in proof of (ii) of Thm. 1.2 of [DM], from this algebra into $\ell^1(\mathbb{Z}^+)$, is, indeed, an injection, and so, the range algebra is a Banach algebra of power series, with respect to the norm transferred from this algebra (see a very very long and tedious proof of (i) of Thm. 10.1 of [DPR] to claim the same fact).

5. Richard Aron and his team have worked a lot on the algebras of analytic functions on a Banach space. We, here, represent a “tensor approach” for the study of such algebras. We hope that the present work will encourage some people to invest some time and energy in order to make progress on the study of (locally) Stein algebras on a (reduced Stein)-Banach spaces in the theory of SCV, possibly through this approach. In particular, they may find some interest (especially, from “tensor approach” point of view) in the
Banach algebra $\hat{\bigvee}_\omega E$, where $\omega \equiv 1$ on $\mathbb{Z}^+$, and the Fréchet algebra $\hat{\bigvee} E$ in view of their study of these kinds of algebras of analytic functions on infinite-dimensional Banach spaces ([ACGLM, ACLM, ACG, AGGM, CGJ, Mu, Ry]).

3 First Approach

We now discuss our first approach, in order to solve the Michael problem affirmatively. We see that the test case $U$ is a Fréchet algebra of the kind $\hat{\bigvee}_W E$, where $E = \ell^1(\mathbb{Z}^+)$ and $W = (\omega_m)$ is a sequence of weights on $\mathbb{Z}^+$ [P4], as follows. The Banach algebras $A = \ell^1(S, \omega) = U_1$, where $\omega \equiv 1$ on $S = (\mathbb{Z}^+)^{<\omega}$, and for each $m \geq 2$, $\ell^1(S, \omega_m) = U_m \cong \hat{\bigvee}_{\omega_m} E$, where $\omega_m$ is a weight on $\mathbb{Z}^+$ defined by $\omega_m(|r|) = m^{|r|}$, $r \in S = (\mathbb{Z}^+)^{<\omega}$ (or, one may take $\omega_m$ as a weight on $S$ defined by $\omega_m(r) = m^{|r|}$) and a Fréchet algebra $U = \bigcap_{m \in \mathbb{N}} U_m = \ell^1(S, W) \cong \bigcap_{m \in \mathbb{N}} \hat{\bigvee}_{\omega_m} E = \hat{\bigvee}_W E$, graded subalgebras of $\mathcal{F}_\infty$ in the commutative case [DPR, Def. 9.2], and their non-commutative analogues $A_{nc} = \ell^1_{nc}(S_{nc}, \omega) = U_{1_{nc}}$, where $\omega \equiv 1$ on $S_{nc}$, and for each $m \geq 2$, $\ell^1_{nc}(S_{nc}, \omega_m) = U_{m_{nc}} \cong \hat{\bigotimes}_{\omega_m} E$, where $\omega_m$ is a weight on $\mathbb{Z}^+$ defined by $\omega_m(|r|) = m^{|r|}$, $r \in S_{nc}$ (or, one may take $\omega_m$ as a weight on
$S_{nc}$ defined by $\omega_m(r) = m^{|r|})$, and a Fréchet algebra $U_{nc} = \bigcap_{m \in \mathbb{N}} U_{nc,m} \cong \bigcap_{m \in \mathbb{N}} \widehat{\bigotimes}_{\omega_m} E = \widehat{\bigotimes}_W E$, [DE, §2], graded subalgebras of $B$. We remark that the map $\epsilon$, restricted to $U_m$, is an isometric isomorphism of $U_m = \ell^1(S, \omega_m)$ onto $\widehat{\bigvee}_{\omega_m} E$ and the same map $\epsilon$, restricted to $U$, is an isometric isomorphism of $U = \ell^1(S, W)$ onto $\bigcap_{m \in \mathbb{N}} \widehat{\bigvee}_{\omega_m} E = \widehat{\bigvee}_W E$ [DPR, p. 142].

It is important to note the following chains of (dense) continuous inclusions of certain algebras and their corresponding character spaces.

$$U \hookrightarrow U_m \hookrightarrow U_1 = \ell^1(S, \{1\}) \cong \widehat{\bigvee}_{\{1\}} E \hookrightarrow \widehat{\bigvee} E \hookrightarrow [\widehat{\bigvee} E]_m,$$

where the last inclusion is, indeed, an epimorphism, being quotient Banach algebra, and the density follows from the fact that all algebras are countably generated by the monomials $X_1, X_2, \ldots$. By [Go, Lem. 3.2.5], we have

$$M([\widehat{\bigvee} E]_m) \cong D^m \hookrightarrow M(\widehat{\bigvee} E) \cong \ell^\infty_{[1]} \cong \widehat{M(\bigvee_{\{1\}}} E) \cong M(\ell^1(S, \{1\}))$$

$$\hookrightarrow M(U_m) \cong \ell^\infty_{[m]} \hookrightarrow M(U) \cong \ell^\infty (w^* - \text{topology}),$$

where the homeomorphisms are due to Thm. 2.2, 2.4, 2.1 and [DE], respectively. We also have non-commutative analogues of the above two chains, but with the same character spaces in the second chain (e.g., the character spaces of $U$ and $U_{nc}$ are same; see [DE, Remark 2.2]).

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Next, we recall that the test case $\mathcal{U}$ is, indeed, a Fréchet algebra of power series by [DPR, Thm. 10.1 (ii)]. We claim that there exists a non-degenerate, totally discontinuous higher point derivation $(d_n)$ on $\mathcal{U}$, induced by the discontinuous character. Note that if we prove our claim we have our desire result; for since such a $(d_n)$ gives us an epimorphism from $\mathcal{U}$ onto $\mathcal{F}$ by [DPR, Thm. 11.2], and so $\mathcal{U} = \mathcal{F}$ by [DPR, Cor. 11.4], a contradiction to the fact that $\mathcal{U}$ is a semisimple algebra and $\mathcal{F}$ is a local algebra. The main point of the following theorem should be emphasized: It is a surprising consequence of the fact that one is able to show that the test case $\mathcal{U}$ is, indeed, a weighted Fréchet symmetric algebra $\mathring{\bigvee}_W E$ as above, and so, if one starts with a discontinuous character on $\mathcal{U}$, then one can construct $(d_n)$ by applying the Dales-McClure method [DM, D] as follows.

Assume that there is a discontinuous character $\phi$ on $\mathcal{U} = \bigcap_{m \in \mathbb{N}} \mathcal{U}_m$. Then, for each $m \in \mathbb{N}$, $\phi$ is a discontinuous character on the normed algebra $(\mathcal{U}, q_m)$, because if it is continuous, then it’s continuous extension on the Banach algebra $(\mathcal{U}_m, q_m)$ is a character, a contradiction to the assumption that $\phi$ is discontinuous. So, by the Hahn-Banach theorem, $\phi$ can be extended to a discontinuous linear functional on $(\mathcal{U}_m, q_m)$ for each $m$. In particular, $\phi$ is a discontinuous linear functional on $\mathcal{U}_1 \cong \mathring{\bigvee}_{\{1\}} E$. Then, $\phi|\ell^1$ is also a
discontinuous linear functional on \( \ell^1 = \bigvee_{\omega_m}^1 E \) for each fixed \( m \in \mathbb{N} \) [DM, §2], call it \( \lambda^m_1 = \lambda \). By [D, Lem. 5.5.18], \((d^m_n)\) is a non-degenerate, totally discontinuous higher point derivation of infinite order on \( \mathcal{U}_m \) at the continuous character \( \phi_m \), where \( d^m_n = \lambda^m_n \circ \pi^m_n \) for each \( n \in \mathbb{N} \) and \( \phi_m = \pi^m_0 \), a character on \( \mathcal{U}_m \) for each \( m \) (since \( M(\mathcal{U}) \) is dense in \( \mathcal{S}(\mathcal{U}) \) and \( M(\mathcal{U}) = \bigcup_{m \in \mathbb{N}} M(\mathcal{U}_m) \), we have \( \phi_m \to \phi \) and, w.l.o.g., we assume that \( \phi_m \in M(\mathcal{U}_m) \) for each \( m \); also, \( d^m_1 = \lambda^m_1 \circ \pi^m_1 \), where \( \lambda^m_1 = \lambda = \phi|\ell^1 \) is a discontinuous linear functional on \( \ell^1 = \bigvee_{\omega_m}^1 E \) for each \( m \).

Next, to show the existence of a non-degenerate, totally discontinuous higher point derivations \((d_n)\) of infinite order on \( \mathcal{U} \) at a discontinuous character \( \phi \), we appeal to the standard category theory of (stable) IL-sequences of groups/algebras and homomorphisms. In our special case, we are concerned with the short exact IL-sequences \( 0 \to \mathcal{G} \overset{in}{\to} \mathcal{H} \overset{d_n}{\to} \mathcal{C} \to 0 \) for each \( n \in \mathbb{Z}^+ \), where the IL-sequences \( \mathcal{G} = (\ker d^m_n, \{in_m\}) \), \( \mathcal{H} = (\mathcal{U}_m, \theta_m) \) and \( \mathcal{C} = (\mathcal{C}; \operatorname{id}_m) \) are stable IL-sequences; we remark that \( \mathcal{G} \) is obviously stable, is easy to see by definition [A2, p. 282], \( \mathcal{H} \) is stable, being a Mittag-Leffler sequence [A2, p. 283] and \( \mathcal{C} \) is stable by [A2, Prop. 1 (i)]. Now, by [A2, Thm. 5], the sequences \( 0 \to L(\mathcal{G}) \overset{L(in)}{\to} L(\mathcal{H}) \overset{L(d_n)}{\to} L(\mathcal{C}) \to 0 \) for each \( n \in \mathbb{Z}^+ \), are also exact (recall that the category theory is used to show that
the inverse-limits $L(G)$, $L(H)$ and $L(C)$ and the homomorphisms $L(in)$ and $L(d_n)$ inherit the same type of algebraic structures and properties that the IL-sequences have). We note that we have $L(G) = \cap_{m \in \mathbb{N}} \ker d_n^m = \ker d_n$, $L(H) = \mathcal{U}$, and $L(C) = \mathcal{C}$ (because all linking maps are identity maps in that IL-sequence), $L(in) = in$, the inclusion map, and $L(d_n) = d_n$ for each $n \in \mathbb{Z}^+$. Thus we have the required existence of a non-degenerate, totally discontinuous higher point derivation $(d_n)$ of infinite order on $\mathcal{U}$ at a discontinuous character $\phi = d_0 = L(\pi_0^m) = L(\phi_m)$ (note that both $\phi = d_0$ and $d_1$ are non-zero, being discontinuous linear functionals). By [DPR, Thm. 11.2 and Cor.11.4], $\mathcal{U} = \mathcal{F}$, a contradiction.

**Theorem 3.1** All characters on the commutative Fréchet algebra $(\mathcal{U}, (q_m))$ are continuous. In particular, these characters are, indeed, the point evaluation mappings, that is, if $\phi$ is a character on $\mathcal{U}$, then there exists $z \in \ell^\infty$ such that $\phi = \phi_z$, where $\phi_z(f) = f(z)$ for all $f \in \mathcal{U}$. \hfill \Box

**Remark.** There are no characters $\phi$ on $\mathcal{U} = \text{Hol}(\ell^\infty)$, not equal to evaluation at $z \in \ell^\infty$, such that $\phi(f) = f(z)$ whenever $f \in \text{Hol}(\ell^\infty)$ is finitely determined. (cf. [Cl, Thm. 9]). More generally, by [Cr, Thm. 4.4], there are no characters $\phi$ on $\Gamma(\ell^\infty) = O(\ell^\infty)$, not equal to evaluation at $z \in \ell^\infty$, such that $\phi(f) = f(z)$ whenever $\tilde{f} \in \Gamma(\ell^\infty) = O(\ell^\infty)$ is a finitely determined
function germ.

As a corollary, we have an affirmative solution to the prestigious Michael problem.

**Corollary 3.2** Every commutative Fréchet algebra is functionally continuous.

**Remark.** The author extends the notion of Stein (resp. Riemann) algebras to locally Stein (resp. Riemann) algebras in [P6] (resp. in [P2]. As an application of this corollary, we see that every character on a (locally) Stein (Riemann) algebra is necessarily continuous, extending the results of [For], [Ma] and [EPH] (Markoe extended the Forster’s result by taking the dimension of $S(X)$, the singularity set of $X$, is finite in place of the dimension of $X$ is finite, and Ephraim further extended the Markoe’s result (see [EPH] for details) by exposing the elementary nature of the Forster’s theorem). Recall that any character on a Stein algebra $A$ is a point evaluation map if $M(A)$ is finite dimensional [For, Thm. 4], and Ephraim extended this result [EPH, Thm. 2.3]. Our theorem implies that Forster’s theorem holds true for any Stein algebra $A$, that is, we don’t require any condition on the Stein space $X$. In other words, the characters are the point evaluation mappings. Not only this, but we have $X = M(A)$ in view of our result and [For, Thm. 1], giving
generalization of [D, Thm. 4.10.28] (that is, \(X\) is a domain of holomorphy).

We have a few more interesting derivations as follows. First, recall that Michael also asked whether every character on a commutative, complete LMC-algebra is bounded [M, §12, Que. 2]. Dixon and Fremlin showed in [DF] that the two problems are, in fact, equivalent. Akkar gave in [Ak] a nice interpretation of this fact: the two problems are equivalent because every complete LMC algebra is, as a “bornological algebra”, isomorphic to inductive limit of a family of Fréchet algebras. The equivalence of the two questions follows also from the fact that there is a complete LMC-algebra, produced by Craw [Cr] (see below), such that the existence of a discontinuous character on some commutative, unital Fréchet algebra would imply the existence of an unbounded character on the complete LMC-algebra. We have the following

**Corollary 3.3** Every character on a commutative, complete LMC-algebra is bounded. \(\square\)

We remark that it is easy to find a non-metrizable, commutative, complete LMC-algebra that is not functionally continuous [M, Prop. 12.2 and Remark] (that is, there is a bounded, discontinuous character on this algebra, however, every continuous character on \(C(T)\) is a point-evaluation map for some \(t \in T\)
surprisingly, below, we give an example of a commutative, non-metrizable, non-complete LMC-algebra that is functionally continuous and functionally bounded. We take this opportunity to correct an obvious typo in (b) of Thm. 9.3 of [DPR] as follows (cf. [Cl, E2]).

**Corollary 3.4** Let $\lambda = (\lambda_n) \in \ell^1 \setminus c_{00}$, and let $g = \sum_{n=1}^{\infty} \lambda_n X_n$. The following statements are equivalent.

(i) There are no non-zero characters on the quotient algebra $\mathcal{M}/I$, where $\mathcal{M}$ is the closed maximal ideal \{\text{f} \in \mathcal{U} : f(0,0,\ldots) = 0\} and $I = \bigcup_{n \in \mathbb{N}} \{X_1\mathcal{U} + \cdots + X_n\mathcal{U}\}$, a prime ideal in $\mathcal{U}$, which is dense in $\mathcal{M}$ and distinct from $\mathcal{M}$.

(ii) There are no characters on the quotient algebra $\mathcal{U}/I + (g - 1)\mathcal{U}$. \(\square\)

**Remarks.** 1. Esterle introduced notions of Picard-Borel ideal and Picard-Borel algebra in [E2]. From above corollary, it is clear that there does not exist a maximal Picard-Borel ideal of $\nu$ distinct from $\mathcal{M}/I$ of codimension 1. So, all the maximal Picard-Borel ideals are of infinite codimension.

2. In view of [E2, §6], the quotient algebras $\nu$ and $\mathcal{U}/I + (g - 1)\mathcal{U}$ certainly do not possess the finite extension property. So, there exists a finite family of elements of $\nu$ whose joint spectrum is not given by the characters of $\nu$. 35
Recall that Esterle pointed out in [E2] (see abstract) new algebraic ob-
struc
tions to the construction of discontinuous characters on $\mathcal{U}$ related to
the Picard theorem, and relate to extension properties of joint spectra of
finite families of a quotient of $\mathcal{U}$ a question about iteration of Bieberbach
mappings raised in [DE]. So, he found some difficulties while looking for a
negative answer to Michael problem. This was also one of the starting points
for the author to work on positive direction(s).

3. In view of [E2, Cor.2.20], in presence of CH, there does not exists an
algebra norm on the quotient algebra $\mathcal{U}/I + (g - 1)\mathcal{U}$.

4. By [Do], $(\ell^\infty, w^*)$ is not a first countable (and hence, not metrizable)
topological space; in fact, it is not even a $k$-space, but it is a Lindelöf and
completely regular space. Hence the algebra $C(\ell^\infty)$ of continuous functions
on $\ell^\infty$ is not a Fréchet algebra by [Go, Thm.]; it is just a commutative, non-
complete LMC algebra [D, Prop. 4.10.20] (cf. comments succeeding to Cor.
3.2 of [E3]), however, $\text{Hol}(\ell^\infty)$ is a Fréchet algebra [Cl, Prop. 3]. Thus one
sees the existence of a functionally continuous, functionally bounded, non-
complete, non-metrizable LMC algebra (remark that all the characters on
$C(\ell^\infty)$ are bounded, since $(\ell^\infty, w^*)$ is replete [D, Cor. 4.10.23]; not only this,
but $C(\ell^\infty)$ is functionally continuous by [D, Thm. 4.10.24]), whose closed
subalgebra is also a functionally continuous, functionally bounded Fréchet algebra. The author does not know any such examples in the literature. The commutative, complete, unital LMC-algebra \( \Gamma(\ell^\infty) = O(\ell^\infty) \) of germs of analytic functions in a neighbourhood of \( \ell^\infty \), is another interesting algebra which is functionally continuous (and hence, functionally bounded) [Cr, §4].

5. Allan exhibited a discontinuous homomorphism between two commutative, unital Fréchet algebras having certain properties [A1, Thm. 8]; but, in the construction of discontinuous homomorphism, he used a continuous homomorphism from \( A \) into \( \mathcal{F} \), arising from a continuous higher point derivation \( (d_n) \) of infinite order on \( \text{Hol}(\mathcal{C}) \) at a continuous character \( d_0 \). Our approach is to interpret the test case \( \mathcal{U} \) (which is a Fréchet algebra of power series) in terms of the weighted Fréchet symmetric algebra \( \hat{\bigvee} W_E \), so that we could use the Dales-McClure method to show the existence of a totally discontinuous higher point derivation \( (d_n) \) of infinite order on \( \mathcal{U} \) at a discontinuous character \( \phi \), using the discontinuous linear functional \( \lambda_1 = \lambda = \phi|\ell^1 \) on \( \ell^1 = \hat{\bigvee}_{\omega_m}^1 E \) for each \( m \), and then, the homomorphism, induced by \( (d_n) \), is an injection from \( \mathcal{U} \) onto \( \mathcal{F} \), giving a contradiction.

6. Other test algebras were discussed by Dixon and Esterle [DE] (non-commutative case), Esterle [E2], Schottenloher [Sco], Mujica [Mj] and Muro
Having solved Michael’s problem for commutative Fréchet algebras, we now discuss the second approach to solve this problem for non-commutative Fréchet algebras. It is clear that the method used in the first approach works only for commutative Fréchet algebras, so it is the need of the hour to develop another approach, which would work for the non-commutative case. However, we show that solving problem for the commutative case would suffice to solve the problem for the non-commutative case in the following proposition. We remark that if $\phi$ is a discontinuous character on some non-commutative, unital Fréchet algebra $A$, then there is a discontinuous character on the test case $U_{nc}$, a non-commutative, unital Fréchet algebra [DE, Pro. 2.1]; the same statement holds for the commutative case as well [E2, Thm. 2.7] and [DPR, Thm. 9.3].

**Proposition 4.1** There is a discontinuous character on a commutative, unital Fréchet algebra if and only if there is a discontinuous character on a non-commutative, unital Fréchet algebra.
Proof. First, we remark that it is sufficient to prove this proposition for the test cases $\mathcal{U}$ and $\mathcal{U}_{nc}$. Let $\phi$ be a discontinuous character on $\mathcal{U}_{nc}$. Then $\phi$ does not belong to $M(\mathcal{U}_{nc}) \cong \ell^\infty$ [Cl, E2]. Since $M(\mathcal{U}_{nc}) = M(\mathcal{U}) \cong \ell^\infty$ by [DE, Remark 2.2], there is a discontinuous character, say $\phi$, on $\mathcal{U}$ as well. The same holds in the reverse direction also. In fact, we can associate these discontinuous characters as follows. Recall that $\mathcal{U}_{nc} \cong \hat{\bigotimes}_W E$, a weighted Fréchet tensor subalgebra of $\hat{\bigotimes} E$, and so, $\mathcal{U}_{nc}$ is a graded subalgebra of $\mathcal{B}$. Hence, we have a symmetrizing epimorphism $\tilde{\sigma} = \oplus \tilde{\sigma}_p$, where $\tilde{\sigma}_p : \mathcal{U}_{nc}^{(p)} \to \mathcal{U}_{nc}^{(p)}$ is the averaging map on $\mathcal{U}_{nc}$, $\tilde{\sigma} : \mathcal{U}_{nc} \to \mathcal{U}$ is continuous. Clearly, if $\phi$ is a discontinuous character on $\mathcal{U}$, then $\phi \circ \tilde{\sigma} = \psi$ is a discontinuous character on $\mathcal{U}_{nc}$. For the reverse direction, since $\epsilon|\mathcal{U} : \mathcal{U} \to \mathcal{U}_{nc}$ is a natural inverse of $\pi|\mathcal{U} = \tilde{\sigma}(\mathcal{U}_{nc})$ (defined below), if $\psi$ is a discontinuous character on $\mathcal{U}_{nc}$, then $\phi = \psi \circ \epsilon|\mathcal{U}$ is a discontinuous character on $\mathcal{U}$.  

Next, we show that if $\phi$ is a discontinuous character on $\mathcal{U}$ (respectively, on $\mathcal{U}_{nc}$), then $\phi|\ell^1$ is a discontinuous linear functional on $\mathcal{U}^{(1)} = \mathcal{U}_{nc}^{(1)}$. We recall that $\mathcal{U}$ and $\mathcal{U}_m$ ($m \in \mathbb{N}$) are graded subalgebras of $\mathcal{F}_\infty$; i.e., $\mathcal{U} = \sum_{p=0}^\infty \mathcal{U}^{(p)}$ and $\mathcal{U}_m = \sum_{p=0}^\infty \mathcal{U}_m^{(p)}$, with $\mathcal{U}^{(1)} = \ell^1(\mathbb{Z}^+) = \bigvee^1 E$ [DPR]. So, $\mathcal{U}^{(p)} = \bigcap_{m \in \mathbb{N}} \mathcal{U}_m^{(p)}$, and there is a continuous, dense embedding from $\mathcal{U}_m^{(p)}$ into $\mathcal{U}_1^{(p)}$, with $\mathcal{U}_m^{(p)}$ is a closed linear subspace of $\mathcal{U}_m$, spanned by the
monomials $X^r$, $|r| = p$. So, clearly, for each $m$, $U_m^{(1)}$ is also isomorphic with $U_1^{(1)} = \ell^1(\mathbb{Z}^*)$ $(\| \cdot \| \sim \| \cdot \|_m$ on $U_m^{(1)}$, where $U_m^{(1)} = \{ \sum_{i=1}^{\infty} \alpha_i X_i : \sum_{m=1}^{\infty} |\alpha_i| m < \infty \}$). More generally, $U^{(p)} = \bigcap_{m \in \mathbb{N}} U_m^{(p)} = \{ \sum_{r \in S} \alpha_r X^r \in U_1 : \sum_{r \in S, |r| = p} |\alpha_r| m^p < \infty \text{ for all } m \in \mathbb{N} \}$. Therefore,

$$U = \bigcap_{p=0}^{\infty} \left( \bigcap_{m \in \mathbb{N}} U_m^{(p)} \right) = \bigcap_{m \in \mathbb{N}} \left( \bigcap_{p=0}^{\infty} U_m^{(p)} \right) = \bigcap_{m \in \mathbb{N}} U_m$$

since $U_m = \sum_{p=0}^{\infty} U_m^{(p)}$. In particular, $U^{(1)} = \bigcap_{m \in \mathbb{N}} U_m^{(1)} = \{ \sum_{i=1}^{\infty} \alpha_i X_i : \sum_{i=1}^{\infty} |\alpha_i| m < \infty, \text{ for all } m \in \mathbb{N} \}$, $U_m^{(1)} \cong \ell^1(\mathbb{Z}^*)$ as above. Here, $U_1 = \sum_{p=0}^{\infty} U_1^{(p)} = \sum_{p=0}^{\infty} \ell^1(S(p))$; $U_1^{(1)} = \ell^1(S^{(1)}) \cong \ell^1(\mathbb{Z}^*) = \mathbb{V}^1 E$. Also, $U_{m+1}^{(p)} \subset U_m^{(p)}$ for all $m \in \mathbb{N}$ and $p \in \mathbb{Z}^+$. We remark that $U^{(1)}$ is a closed linear subspace of $U$, spanned by $X_i$, whereas $U_m^{(1)}$ is a closed linear subspace of $U_m$ for each $m \in \mathbb{N}$, spanned by $X_i$ (the latter is a Banach space and the former is a Fréchet space). Obviously, all $X_i \in U^{(1)}$, but $\sum_{i=1}^{\infty} \frac{X_i}{i}$ does not belong to $U^{(1)}$ (respectively, does not belong to even $U_1^{(1)}$) whereas $\sum_{i=1}^{\infty} \frac{X_i}{i^2} \in U^{(1)}$ (respectively, belongs to $U_1^{(1)}$).

Recall that $\phi$ is a discontinuous linear functional on $U_m$ for each $m \in \mathbb{N}$, and that $\phi|\ell^1$ is a discontinuous linear functional on $\ell^1(\mathbb{Z}^+) \cong U_1^{(1)}$ by §3.

So, $\phi|\ell^1$ is also a discontinuous linear functional on $U_m^{(1)}$ for each $m \in \mathbb{N}$ (the continuous embedding from $U_m^{(p)}$ into $U_1^{(p)}$ actually turns out a topological isomorphism), and so, $\phi|\ell^1$ is also a discontinuous linear functional on
$\mathcal{U}^{(1)}$ (because if it is a continuous linear functional on $\mathcal{U}^{(1)}$, then it can be extended to a continuous character on $\mathcal{U}$ (as shown at the end of this paper), a contradiction to our assumption above).

Now, we take $\phi(X_n) = 1$ for all $n \in \mathbb{Z}^+$, and using the axiom of choice, we extend $\phi$ to a Hamel basis of $\mathcal{U}^{(1)}$ so that $\phi$ becomes a discontinuous linear functional on $\mathcal{U}^{(1)}$. We shall generate another Fréchet algebra topology $\tau$ (inequivalent to the usual one, generated by the norms $q_m$) on $\mathcal{U}_{nc}$ (and thus, on $\mathcal{U}$ as well), using this discontinuous linear functional. Our argument here is kept short because it uses key ideas involved in the Read’s method (and we follow notations in align with the Read’s notations), but the Fréchet algebra topologies on $\mathcal{U}_{nc}$ (resp. on $\mathcal{U}$) that we are working with, are entirely different.

We start with a remark that the derivations on a commutative, semisimple Fréchet algebra $(\mathcal{U}, (q_m))$ are continuous [C2]. However, we shall show that the derivation $\partial/\partial X_0$ is discontinuous w.r.t. $\tau$, a contradiction to the Carpenter’s result (also to the fact that the identity can never be a zero element of the algebra). We shall also require in a future proof the following “locally finite” linear map $T : \mathcal{U}_{nc} \to \mathcal{U}_{nc}$, $T(\sum_{r \in S_{nc}} \alpha_r X^{\otimes r}) = \sum_{r \in S_{nc}} \alpha_r X_r$ (actually, this $T$ is the composition of the inclusion map with the $T$, discussed
by Read in [R]; also, one requires to restrict the range as well). Similarly we may define “locally finite” linear maps $U_{nc} \to U$, $U \to U_{nc}$ and $U \to U$; they are precisely the linear maps between these two spaces that are continuous w.r.t. their natural Fréchet algebra topologies (two such mappings were considered in proof of Prop. 4.1 above).

We shall require the concept of “tensor products by rows”, taken from [R]. First, for each $n \in \mathbb{Z}^+$, let $P_n : U_{nc} \to U_{nc}$ be the linear map such that $P_n(1) = 0$, and $P_n(X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m}) = 0$, when $i_1 \neq n$ and is equal to $X_{i_2} \otimes \cdots \otimes X_{i_m}$, when $i_1 = n$. Thus $P_n$ takes the quotient on division from the left by $X_n$, and discards the remainder. Let $\pi : U_{nc} \to U$ be the natural map that the $X_i$ commute, i.e., the locally finite map such that $\pi(X^{\otimes 1}) = X^i$ for all $i$. Then $\pi|\tilde{\sigma}(U_{nc})$ is bijective and $\epsilon|U : U \to \tilde{\sigma}(U_{nc})$ be its inverse (recall that $\epsilon : U \to U_{nc}$ is a continuous, injective homomorphism such that $\epsilon = \oplus \epsilon_p$, where $\epsilon_p : U^{(p)} \to U_{nc}^{(p)}$ is a continuous linear embedding for each $p$). Thus $\epsilon$ is the natural right inverse to $\pi$.

Next, we have a linear functional $\phi : U^{(1)} \to \mathcal{C}$ such that $\phi(X_n) = 1$ for all $n \in \mathbb{Z}^+$, as discussed earlier. Recall that $U_{nc}$ is a graded subalgebra of $\mathcal{B}$. Write $U_{nc}^{(p)}$ for the subspace of $p$-homogeneous formal power series $\sum_{i \in (\mathbb{Z}^+)_p} b_i X^{\otimes i}$ and write $U^{(p)} = \pi(U_{nc}^{(p)})$, $U_{nc}^{(1)} = U^{(1)}$. If $b \in U_{nc} =$
\[ \oplus_{p=0}^{\infty} U_{nc}^{(p)} \] we write \((b^{(p)})_{p \in \mathbb{N}}\) such that \(b = \sum_{p=1}^{\infty} b^{(p)}\). If \(\phi_1, \phi_2 : U^{(1)} \to \mathcal{C}\) are linear functionals, we define the “tensor product by rows” \(\phi_1 \otimes \phi_2 : U^{(2)} \to \mathcal{C}\) by \(\phi_1 \otimes \phi_2(b) = \phi_1(\sum_{j=0}^{\infty} X_j \cdot \phi_2(P_j(b)))\). Tensor product by rows of \(n\) linear functional are then defined inductively as in [R], and we see that Lem. 1.10 of [R] holds for the elements \(a \in U^{(r)}, b \in U^{(p-r)}\) for \(0 \leq r \leq p\).

**Corollary 4.2** If \((\phi_n)\) is a sequence of linear functionals on \(U^{(1)}\), then, for each \(m \in \mathbb{N}\), the seminorm \(\| \cdot \|_m\) on \(U_{nc}\) given by

\[
\|a\|_m = |a^{(0)}| + \sum_{r=1}^{\infty} \sum_{i \in (\mathbb{Z}^+)^r} |\otimes_{j=1}^{r} \phi_{ij}(a^{(r)})| \quad (*)
\]

is a submultiplicative seminorm.

As discussed in [R], since the order of appearance of the \(\phi_{ij}\) can be permuted arbitrarily in \((*)\), one has \(\|\tilde{\sigma}\|_m = 1\) for all \(m \in \mathbb{N}\), where \(\tilde{\sigma}\) is the averaging map. Hence, these seminorms are also submultiplicative seminorms on \(U\), when \(U\) is identified with the linear subspace \(\tilde{\sigma}(U_{nc}) \subset U_{nc}\) (the multiplication of \(U\) is then implemented by \((a, b) \to \tilde{\sigma}(a \otimes b)\)). However, instead of the ‘usual’ coordinate functionals, discussed in [R], we here need the ‘weighted’ coordinate functionals, defined as follows, in order to give Fréchet algebra topologies \(\tau\) and \(\tau_0\) on \(U_{nc}\) below. Let \(\phi_{mn}^{(1)} : U^{(1)} \to \)
\( \phi_n^m(\sum_{n=0}^{\infty} \alpha_n X_n) = \alpha_n \cdot m \) for each \( m \in \mathcal{N} \), then \((\phi_n^m)_{n \in \mathbb{Z}^+, m \in \mathcal{N}} \) is a sequence of weighted coordinate linear functionals on \( \mathcal{U}^{(1)} \) for each \( m \in \mathcal{N} \). Let \((\phi_n)_{n \in \mathbb{Z}^+} \) be a sequence of linear functionals on \( \mathcal{U}^{(1)} \) as follows: (a) \( \phi_0 = \psi \), the discontinuous linear functionals defined above; and (b) for \( n \in \mathcal{N} \), \( \phi_n^m \) be the weighted coordinate functionals on \( \mathcal{U}^{(1)} \). Apply the above corollary to \((\phi_n)\) to define a locally multiplicatively convex topology \( \tau \) on \( \mathcal{U}_{nc} \).

We claim that \((\mathcal{U}_{nc}, \tau)\) is a Fréchet algebra. Since \( \tilde{\sigma} \) is a \( \tau \)-continuous projection, the subspace \( \mathcal{U} = \tilde{\sigma}(\mathcal{U}_{nc}) = \ker(I - \tilde{\sigma}) \) is closed, so \((\mathcal{U}, \tau)\) is a commutative Fréchet algebra. Note that if we prove that \( \mathcal{U}_{nc} \) is, in fact, complete under the topology \( \tau \), then we arrive at a contradiction to the fact that \((\mathcal{U}, \tau_0)\) is a semisimple Fréchet algebra with a unique Fréchet algebra topology [C1, DPR, P1, P3]. Since \( \phi_0(\sum_{n=0}^{\infty} \alpha_n X_n) = 0 \) for \( N > 0 \) one sees that \( \|X_N - X_0\|^n_m = 0 \) for all \( n, m \) with \( n < N \); hence \( X_N \to X_0 \) in \( \tau \).

Let \( \tau_0 \) be the “usual” topology that makes \( \mathcal{U}_{nc} \) a Fréchet algebra (cf. [DE]). One sees that \( \tau_0 \) could be obtained by applying the above corollary to the sequence \((\phi_n)\), where \( \phi_0 \) is the usual continuous coordinate functional, defined by \( \phi_0(\sum_{n=0}^{\infty} \alpha_n X_n) = \alpha_0 \), inducing submultiplicative norms \( | \cdot |_m^n \).
where

\[ |a|_m^n = |a(0)| + \sum_{r=1}^{\infty} \sum_{i \in (\mathbb{Z}^+)^r} |a^{(r)}| m^i. \]

One may say that \( \tau_0 \) is the topology of “weighted” convergence w.r.t. all these norms \(|·|_m^n\); that is, \( \tau_0 \) is the topology of “weighted” convergence of all the coefficients \( a_i \).

Next, we define a linear map \( \Psi : \mathcal{U}_{nc} \to \mathcal{U}_{nc} \) by

\[ \Psi(a) = a^{(0)} + \sum_{r=1}^{\infty} \sum_{i \in (\mathbb{Z}^+)^r} \otimes_{j=1}^{r} \phi_{ij}(a^{(r)}). \]

We note that \( \Psi : (\mathcal{U}_{nc}, \tau) \to (\mathcal{U}_{nc}, \tau_0) \) is continuous because convergence under \( \tau \) is precisely convergence of all the weighted linear functionals \( \otimes_{j=1}^{r} \phi_{ij}^m(a^{(r)}) \), corresponding to the usual linear functional \( \otimes_{j=1}^{r} \phi_{ij}(a^{(r)}) \) that are involved in \( \Psi(a) \).

**Theorem 4.3** \( \Psi \) is bijective.

**Proof.** The proof is the same as that of [R, Thm. 2.3], with a remark that one requires to replace \( \mathcal{B} \) by \( \mathcal{U}_{nc} \) throughout that proof (the reader would like to notice that the Read’s proof was purely algebraic, and so one survives under the replacement, because \( \mathcal{U}_{nc} \) is a graded subalgebra of \( \mathcal{B} \), so it inherits all the graded algebraic structure that \( \mathcal{B} \) has). \( \Box \)
**Theorem 4.4** \((\mathcal{U}_{nc}, \tau)\) is complete with respect to \((\| \cdot \|_n^m)\). The derivation \(\partial/\partial X_0 : (\mathcal{U}_{nc}, \tau) \to (\mathcal{U}_{nc}, \tau)\) is discontinuous, and its separating subspace is all of \(\mathcal{U}_{nc}\). The derivation \(\partial/\partial X_0 : (\mathcal{U}, \tau) \to (\mathcal{U}, \tau)\) is also discontinuous, and its separating subspace is all of \(\mathcal{U}\).

**Proof.** The proof is the same as that of [R, Thm. 2.5], with a remark that the notations are same, but the two topologies \(\tau\), generated by \((\| \cdot \|_n^m)\), and \(\tau_0\), generated by \((\| \cdot \|^n_m)\), are not the same that were discussed by Read. Obviously, the derivation \(\partial/\partial X_0\) is discontinuous on \((\mathcal{U}_{nc}, \tau)\), since \(\partial/\partial X_0(X_0 - X_N) = 1\) whereas \(X_N \to X_0\) in \(\tau\). Clearly, the separating subspace of the derivation is a two-sided ideal, and so, it is all of \(\mathcal{U}_{nc}\). Similarly, the derivation \(\partial/\partial X_0 : (\mathcal{U}, \tau) \to (\mathcal{U}, \tau)\) is also discontinuous, and its separating subspace is all of \(\mathcal{U}\).

\(\square\)

**Corollary 4.5** All characters of \(\mathcal{U}\) are continuous and so is for \(\mathcal{U}_{nc}\). In particular, every Fréchet algebra (commutative or not) is functionally continuous.

**Proof.** From Thm. 4.4, \(\mathcal{U}\) admits two inequivalent Fréchet algebra topologies, a contradiction to the fact that it has a unique Fréchet algebra topology, being semisimple Fréchet algebra [C1, P1, P3, DPR]. Since all characters of \(\mathcal{U}\) are continuous, all characters of \(\mathcal{U}_{nc}\) are also continuous by Prop. 4.1. \(\square\)
Until now, we know that $M(A)$ is dense in $S(A)$, where $S(A)$ is the space of all characters on $A$ w.r.t. the Gel’fand topology [Go, Lem. 10.1.1]. By Cor. 4.5, we have $M(A) = S(A)$.

Finally, we remark that Vogt studied the tensor algebras over a Fréchet space in [V1, V2]; however, he mostly studied linear topological properties (and, in particular, linear isomorphisms) of these algebras. The algebra $T(s) = s_*$, is functionally continuous [V1, p. 190]. A particular topic of interest, close to the algebra $U$ of entire functions on $\ell^\infty$, should be the study of the space of entire functions on a nuclear sequence spaces, which is the symmetric tensor algebra of a suitable nuclear reflexive Köthe space (see [BMV]). Since $U^{(1)}$ is a Fréchet space, if we apply fairly straightforward arguments of Vogt, given in [V1, §1], then it is easy to see that the tensor algebra $T(U^{(1)})$ (resp. the symmetric algebra $S(U^{(1)})$ in the commutative case) is, indeed, the algebra $U_{nc}$ (resp. $U$). Moreover, it is easy to see that $U^{(1)}$ is isometrically isomorphic to the Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, W)$, where $W = (\omega_m)$ is an increasing sequence of weights on $\mathbb{Z}^+$ defined by $\omega_m(n) = m$ for all $n \in \mathbb{N}$ and $\omega_m(0) = 1$ for all $m \in \mathbb{N}$ [BP, Ex. 1.2], which is, in turn, isomorphic to $\text{Hol}(U)$, the Fréchet algebra of analytic functions on the open unit disc $U$ [BP, Ex. 1.4], the power series space of finite type [V1,
§2] or [Gro]. So, the tensor algebra $T(U^{(1)}) = U_{nc}$ is nuclear by [V1, Thm. 3.1], which is further identified with $TA_0$ by [V1, Lem. 5.3]. Now, $TA_0$ is complemented subspace of $T(s_0) \cong \text{Hol}({\mathcal{C}}) \hat{\otimes} s_0$ as $\text{Hol}(U)$ is a complemented subspace of $s_0$, and so, the tensor algebra of a finite type power series space need not be a finite type power series space (cf. [V1, Lem. 2.1]). However, our approaches to the algebras $U_{nc}$ and $U$ seem to be more convenient for our purposes in this section and §4, respectively.

5 Applications to automatic continuity theory

As promised, we answer affirmatively Michael problems within automatic continuity theory. In the past, there were some significant works in this theory, which relate to our present work (in)directly and giving applications within this theory or in the theory of commutative rings and algebras as follows. Feldman showed that the Wedderburn principal theorem does not hold for Banach algebras by giving two inequivalent complete norms to a specific Banach algebra $\ell^2 \oplus \mathcal{C}$ [F]. Johnson established the uniqueness of the complete norm for semisimple Banach algebras [Jo], and for Banach
algebras of power series [J01]; this result was extended by Loy to Fréchet algebras of power series satisfying the equicontinuity condition (E) [L1] among other papers [L2, L3, L4], and the author settled this for all Fréchet algebras of power series [P1] and for all Fréchet algebras of power series in finitely many commuting indeterminates [P3]. Thomas showed that the image of a derivation on a Banach algebra is contained in the radical, solving the Singer-Wermer conjecture affirmatively [T]. However, Read showed that this conjecture fails in the Fréchet case [R]; the author extended this work to give countably many inequivalent Fréchet algebra topologies to two specific (and maiden) Fréchet algebras [P5] (we remark that Vogt gave uncountably many inequivalent Fréchet space topologies to spaces of holomorphic functions [V3]; however, all these spaces are semisimple Fréchet algebras with a unique Fréchet algebra topology [C1, P1, DPR]). Loy gave method to construct commutative Banach algebras with inequivalent complete norms by using the discontinuity of derivations [L]; we extend his work to the Fréchet case, in order to answer Loy’s problem from 1974 [P4]. The germ of the ideas for the first approach, discussed in §3, lies in [P4]. Among other works, we quote works of Grabiner [Gr] and Laursen [La] on automatic continuity of derivations and homomorphisms on Banach algebras of power series and
some results on automatic continuity of linear operators [La].

**Remarks.** 1. Our first remark is on the original two problems of Michael [M, §12], posed exclusively for commutative algebras. However, their noncommutative analogues also exist, and Dixon and Esterle discussed the test case \( \mathcal{U}_{nc} \) (the non-commutative analogue of \( \mathcal{U} \)) in [DE]. Obviously, a positive answer to the second problem implies a positive answer to the first. We are interested in knowing whether the two problems are actually equivalent (in the non-commutative case). The answer is yes, because Thm. 1 of [DF] also holds true in the non-commutative case as well, and thus, every noncommutative, complete LMC-algebra is functionally bounded from Cor. 3.2 and 4.5.

2. We have throughout assumed that our algebras have an identity. It is a triviality to show that if a (non-)commutative Fréchet algebra has a discontinuous character, then so does the algebra obtained by adjoining an identity [DE, Prop. 2.1, Cr, Remarks 4.5 (2)]. Thus, as far as Michael’s problem is concerned, no loss of generality is involved in our assumption.

3. It is obvious that a discontinuous character on \((\mathcal{U}, \tau_0)\) is also discontinuous on \((\mathcal{U}, \tau)\) by [E2, Thm. 2.7]. So, Cor. 6 of [C1], which was stated for commutative, semisimple Fréchet algebras, is, here, established for all
Fréchet algebras in a more general form by Cor. 4.5.

4. Another topology \( \tau \) on \( U \) also contradicts to the fact that every derivation is automatically continuous on a commutative, semisimple Fréchet algebra \([C2]\), because the derivation \( \partial / \partial X_0 \) is discontinuous w.r.t. \( \tau \) by Thm. 4.4.

5. Carpenter \([C1]\) showed that every commutative semisimple Fréchet algebra \( A \) admits a unique Fréchet algebra topology. The proof was divided into four parts, but if we know that the characters are continuous on \( A \), then the proof could be derived from the third part only. Below, we shall show that every homomorphism \( \theta : B \rightarrow A \) from a Fréchet algebra \( B \) into a semisimple Fréchet algebra \( A \) is automatically continuous; this is another long-standing open problem in the theory of Fréchet algebra, equivalent to the Michael problem \([D1, p. 143]\) and \([P1, p. 134]\). Once we show this, we have a very short proof for the uniqueness of the Fréchet algebra topology for a commutative semisimple Fréchet algebra using the open mapping theorem for Fréchet spaces.

**Theorem 5.1** Let \( A \) be a commutative semisimple Fréchet algebra and \( B \) be any commutative Fréchet algebra. Let \( \theta : B \rightarrow A \) be a homomorphism. Then \( \theta \) is automatically continuous.
Proof. First, we assume w.l.o.g. that $B$ is unital, because if $B$ is not unital, then we can adjoin the identity $e$ and we can extend $\theta$ on $B_e$ by taking $\theta(e) = e'$ (if $A$ is not unital, then we can also adjoin the identity $e'$ to $A$). Obviously, $\theta$ is continuous if and only if extension of $\theta$ is continuous.

Suppose that $\theta$ is a discontinuous, unital homomorphism. Let $\phi$ be a character on $A$. Then $\phi$ is continuous by Cor. 3.2. Set $\psi = \phi \circ \theta$. Then $\psi$ is a character on $B$. Since $\theta$ is discontinuous, $\psi$ is also discontinuous, a contradiction to the fact that all characters on $B$ are continuous. Thus $\theta$ is automatically continuous.  

As an application to the above theorem, we can drastically shorten the proof of Carpenter [C1], establishing the uniqueness of the Fréchet algebra topology of a commutative, semisimple Fréchet algebra in the following

**Corollary 5.2** *Every commutative, semisimple Fréchet algebra admits a unique Fréchet algebra topology.*

**Proof.** Let $A$ be a commutative, semisimple Fréchet algebra with respect to another Fréchet algebra topology $\sigma$, distinct from the Fréchet algebra topology $\tau$. Consider the identity mapping from $(A, \tau)$ into $(A, \sigma)$. Evidently it is a continuous homomorphism by Thm. 5.1. By the open mapping theorem for Fréchet spaces, it is a linear, homeomorphism, and so $\sigma = \tau$.  

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6. What about the non-commutative analogue of 5. above? That is, whether a non-commutative, semisimple Fréchet algebra has a unique Fréchet algebra topology. More generally, whether Thm. 5.1 holds for a non-commutative Fréchet algebra $B$ and a non-commutative, semisimple Fréchet algebra $A$. There is another parallel question about the continuity of derivations on a non-commutative, semisimple Fréchet algebra. As far as we know, the strong partial result about the uniqueness of Fréchet algebra topology for a non-commutative, semisimple Fréchet algebra which is a $Q$-algebra, was obtained by Esterle in 1979-80 [E4], and the second problem was discussed by Johnson in the Banach case [J] (a generalization of this result was given by Jewell and Sinclair [JS], which turned out to be a starting point for Esterle and Thomas to transform to the Fréchet case [E5, T1] as well as for the author to discuss the three questions here [D1, p. 141-144]). Also, we do not know any progress on the third question.

To answer the first two questions, we remark that it is easy to follow proof of Thm.5.1 in the non-commutative case, since we have already shown that every non-commutative Fréchet algebra is functionally continuous by Cor. 4.5. It is also interesting to note that we can affirmatively answer the prestigious Michael problems as an application of Thm. 5.1 (and its non-
commutative analogue) by taking \( A = \mathcal{C} \). In fact, if \( \theta \) is a homomorphism from a Fréchet algebra \( B \) (commutative or not) into a commutative, semisimple Fréchet algebra \( A \), then it is automatically continuous if and only if every character on a Fréchet algebra \( B \) is continuous. However, we take, here, an alternate approach which also answers the third question. For this, we first remark that a proper generalization of Thm. 2 of [JS] holds in the Fréchet case as follows.

**Theorem 5.3** Let \( B \) be a Fréchet algebra such that

(i) for each infinite-dimensional closed two-sided ideal \( J \) in \( B \) there is a sequence \((b_n)\) in \( B \) such that the closed ideal \( \overline{Jb_{n(k)+1} \ldots b_1}^k \) is a proper subset of the closed ideal \( \overline{Jb_{n(k)} \ldots b_1}^k \) for each \( n(k) \in \mathbb{N} \) and for some \( k \in \mathbb{N} \);

(ii) \( B \) contains no non-zero finite-dimensional nilpotent two-sided ideals.

Then \( B \) has a unique Fréchet algebra topology, and every epimorphism from a Fréchet algebra onto \( B \) is automatically continuous. Moreover, every derivation on \( B \) is automatically continuous.

**Proof.** The proof is the same as that of [JS, Thm. 2], with a remark that one requires to obtain a contradiction by applying Lem. 1.1a of [T1]. We
also emphasize that no improvement in the condition (ii) such as “$B$ contains no non-zero finite-dimensional locally nilpotent two-sided ideals”, is possible here because since the separating space is a closed, finite-dimensional ideal, every finite-dimensional locally nilpotent two-sided ideal is, indeed, nilpotent (cf. [A1, p. 277]).

It is a surprising consequence of the above theorem (with $B$ a Banach algebra) that we have the third approach to affirmatively answer Michael’s problems in the following (see a comment on p. 144 of [D] or [E5]).

**Corollary 5.4** Let $A$ be a non-commutative Fréchet algebra, and $B$ be a Banach algebra as in Thm. 2 of [JS]. Then every epimorphism from a non-commutative Fréchet algebra onto $B$ is automatically continuous. In particular, every character on $A$ is automatically continuous. Further, every Fréchet algebra (commutative or not) is functionally continuous.

In [D1], Dales asked whether Michael problem has an affirmative answer for all commutative, complete, metrizable locally convex algebras. We now answer this question affirmatively for more general (non-)commutative ($F$)-algebras (i.e., complete, metrizable topological algebras) as follows; many thanks to Lemma 1.1a of [T1] so that a generalization of Thm. 5.3 (and hence, of Cor. 5.4) holds true.
Corollary 5.5  Let $A$ be a non-commutative, $(F)$-algebra, and $B$ be a Banach algebra as in Thm. 2 of [JS]. Then every epimorphism from a non-commutative $(F)$-algebra onto $B$ is automatically continuous. In particular, every character on $A$ is automatically continuous. Further, every $(F)$-algebra (commutative or not) is functionally continuous.

Next, if $B$ is a non-commutative, semisimple Fréchet algebra, then it has no nilpotent two-sided ideals. To see whether the condition (i) of Thm. 5.3 holds, we follow proof of Cor. 9 of [JS] in the Fréchet case by working with an infinite dimensional irreducible left $B$–module $X$, which is a Fréchet module under the Fréchet space topology. Then we have the following

Corollary 5.6  Let $B$ be a non-commutative, semisimple Fréchet algebra. Then $B$ has a unique Fréchet algebra topology, and every epimorphism from a Fréchet algebra onto $B$ is automatically continuous. Moreover, every derivation on $B$ is automatically continuous.

We remark that Cor 5.5 affirmatively answers Que. 9 of [D1]. Esterle deduced in [E5] that, if $B$ is a Banach algebra satisfying the condition (i) of Thm. 5.3 (but in the Banach case; see [JS, Thm. 2]), and if $B$ contains no non-zero finite-dimensional two-sided ideal, then every epimorphism from
an \((F)\)-algebra onto \(B\) is automatically continuous. We see that a generalization of Esterle’s result does not hold in the Fréchet case; for example, the algebra \(\mathcal{F}\) clearly satisfies the conditions of Esterle’s theorem, but every epimorphism from an \((F)\)-algebra onto \(\mathcal{F}\) is discontinuous by [DPR, Thm. 11.2], answering affirmatively a question of Dales-McClure problem from 1977 (for the case the domain algebra a Banach algebra, see [DM, Thm. 2.3]).

7. In [P1, P3, Thms. 4.1], we can now drop the condition “the range of \(\theta\) is not one-dimensional”, because there are no discontinuous characters on the domain algebra \(B\) which would have given the discontinuous homomorphism \(b \mapsto \phi(b)1, B \to A\).

In [DPR, Thm. 12.3], it was shown that a Banach algebra \(\ell^1(S) \cong \hat{\bigvee}_{\{1\}} E\), is such that \(\mathcal{C}[X_1, X_2] \subset \ell^1(S) \subset \mathcal{F}_2\), but the embedding \((\ell^1(S), \| \cdot \|) \to (\mathcal{F}_2, \tau_c)\) is not continuous. We did not know whether there is a non-Banach Fréchet algebra with these properties. We show that the test case \(\mathcal{U}\) is such an example in the following

**Theorem 5.7** The test case \((\mathcal{U}, \tau_0)\) for the Michael problem is such that \(\mathcal{C}[X_1, X_2] \subset \mathcal{U} \subset \mathcal{F}_2\), but the embedding \((\mathcal{U}, \tau_0) \to (\mathcal{F}_2, \tau_c)\) is not continuous.

**Proof.** The proof is the same as that of [DPR, Thm. 12.3]. We remark
that $U^{(1)}$, defined above, is the closed linear subspace of $U$ spanned by the elements $X_i$, and so this Fréchet space is not isometrically isomorphic to $\ell^1$, but we can still choose a non-zero, discontinuous linear functional $\lambda$ on $U^{(1)}$ (as the element $\sum_{i=2}^{\infty} \frac{X_i}{i} \in U^{(1)}$), and then define a linear map

$$
\psi : U^{(1)} \to F_2, \ u \mapsto \theta(u) + \lambda(u)Y,
$$

where $\theta$ is taken from Thm. 10.1 (ii) of [DPR]. Our main claim is that $\psi$ can be extended to a homomorphism $\Psi' : U \to F_2$ such that $\pi \circ \Psi' = \theta$.

To establish this claim, we shall require the following slightly more general theorem, whose proof we omit.

\begin{theorem}
Let $\beta : U^{(1)} \to M_2$ be a linear map such that $\pi \circ \beta : U^{(1)} \to F$ is continuous. Then there is a unital homomorphism $\bar{\beta} : U \to F_2$, extending $\beta$, such that $\pi \circ \bar{\beta} : U \to F$ is continuous.
\end{theorem}

In fact, a slight digression of proof of the above theorem (which is analogous to the proof of Thm. 12.4 of [DPR], and a similar proof is discussed below) enables us to state the following theorem in its fuller generality. Both Thms. 5.8 and 5.9 are of independent interest in view of the first approach to the Michael problem.  

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Theorem 5.9 Let $\beta : \mathcal{U}^{(1)} \to \mathcal{F}$ be a continuous linear map. Then there is a continuous, unital homomorphism $\bar{\beta} : \mathcal{U} \to \mathcal{F}$, extending $\beta$. $\square$

As opposite to Thm. 5.7 above, we provide a much shorter and elegant way to embed $\mathcal{U}$ into $\mathcal{F}$ in the following corollary (cf. Thm. 10.1 (ii) of [DPR]).

Corollary 5.10 The Fréchet algebra $\mathcal{U}$ is (isometrically isomorphic to) a Fréchet algebra of power series. $\square$

Proof. First, we remark that we again require to take further digression in the proof of Thm. 5.9 as follows. For each $i \in \mathbb{Z}^+$, we take $\beta_{(i)}$ to be the usual co-ordinate linear functional on $\mathcal{U}^{(1)}$. We then define a mapping $\beta : \mathcal{U}^{(1)} \to \mathcal{F}$ such that $\beta(f) = \sum_{i=0}^{\infty} \beta_{(i)}(f)X^i$. Clearly, such a $\beta$ is continuous and injective, and $\beta(X_1) = X$. We extend each $\beta_{(i)}$ to a linear functional $\beta_{(i)}^{(1)}$ on $\mathcal{F}^{(1)}_{\infty}$. Next we define a linear functional $\beta_{(i)}^{(n)}$ on $\mathcal{F}^{(n)}_{\infty}$ for each $n \in \mathbb{N}$ by the following formula:

$$\beta_{(i)}^{(n)}(f) = \sum \{(\beta_{(i)} \otimes \cdots \otimes \beta_{(n)})^{(1)}(\epsilon_n(f))\} \ (f \in \mathcal{F}^{(n)}_{\infty}),$$

where the sum is taken over all $n$-tuples $(i) = (i^{(1)}, \ldots, i^{(n)}) \in (\mathbb{Z}^+)^n$ such that $i^{(1)} + 2i^{(2)} + \cdots + ni^{(n)} = \omega((i))$, a weighted order of $(i)$ (cf. [DPR, p. 137-138]). We now claim that the map $\bar{\beta} : \mathcal{U} \to \mathcal{F}$, defined for $f \in \mathcal{F}_{\infty}$
by the formula

\[ \bar{\beta}(f) = \sum_{k=0}^{\infty} \left\{ \left( \sum_{n \in \mathbb{N}_i} \beta_{(i)}^{(n)}(f^{(n)}) \right) X^i : \omega(i) = k \right\}, \]

where we set \( \bar{\beta}^{(0)}(f) = f(0)1 \), is a unital homomorphism \( \bar{\beta} : U \to F \) satisfying the stated conditions. Follow proof of Thm. 12.4 to show that \( \bar{\beta} \) is a unital homomorphism that extends \( \beta \). Again follow proof of Thm. 12.4 to show that \( \bar{\beta} \) is continuous; we need to use the continuity of co-ordinate linear functionals \( \beta_{(i)} \) and the fact that the 'tensor product by rows' agrees with the usual tensor product when the linear functionals are continuous. Finally, we need to show that \( \bar{\beta} \) is injective. Now, follow proof of Thm. 9.1 for this to derive. Thus the algebra \( U \) is continuously embedded in \( F \). In this case, \( \bar{\beta}(U) \) is a Fréchet algebra of power series, w.r.t. the metric transferred from \( U \), and so \( U \) is isometrically isomorphic to a Fréchet algebra of power series. 

\[ \square \]

A similar argument also enables us to extend a continuous linear functional \( \beta \) on \( U^{(1)} \) to a continuous character \( \phi \) on \( U \); in the process, one can either consider 'tensor product by rows' or usual tensor product of a continuous linear functional, it makes no difference. The main point should be emphasized here: if one starts with a discontinuous character on \( U \), then one obtains a discontinuous linear functional on \( U^{(1)} \) (this fact was used in the
second approach above).

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