A LOWER BOUND ON THE CANONICAL HEIGHT FOR POLYNOMIALS

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Abstract. We prove a lower bound on the canonical height associated to polynomials over number fields evaluated at points with infinite forward orbit. The lower bound depends only on the degree of the polynomial, the degree of the number field, and the number of places of bad reduction.

1. Introduction

The canonical height associated to a rational morphism \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree at least two over a number field \( K \) is a function \( \hat{h}_\phi : \mathbb{P}^1(K) \to \mathbb{R} \) satisfying
\[
\hat{h}_\phi(\phi(\alpha)) = \deg(\phi)\hat{h}_\phi(\alpha) \quad \text{and} \quad \hat{h}_\phi(\alpha) = h(\alpha) + O(1),
\]
where \( h \) is the Weil logarithmic height \([5]\). One basic property of \( \hat{h}_\phi \) is that \( \hat{h}_\phi(\alpha) = 0 \) if and only if \( \phi^i(\alpha) = \phi^j(\alpha) \) for some \( i \neq j \). A natural question then arises: how small can \( \hat{h}_\phi(\alpha) \) be under the assumption that \( \alpha \in K \) has infinite forward orbit under \( \phi \)? We will address how the minimal possible nonzero value of \( \hat{h}_\phi(\alpha) \) grows with the height of \( \phi \) in the moduli space \( M_d \) of degree \( d \) dynamical systems.

The canonical height attached to \( \phi \) has an analogue in the setting of elliptic curves. Lang conjectured that for an elliptic curve \( E \) over a number field \( K \) with minimal discriminant \( D_{E/K} \),
\[
\hat{h}(P) \geq c_1 \log |\text{Norm}_{K/Q} D_{E/K}| + c_2
\]
for any non-torsion point \( P \in E(K) \), where \( c_1 > 0 \) and \( c_2 \) depend only on \( K \) \([11\text{, p. } 92]\). In \([16]\), Silverman gives a partial solution to this conjecture, showing that it holds for constants \( c_1 > 0 \) and \( c_2 \) depending on \([K : \mathbb{Q}]\) and on the number of primes at which \( E/K \) has split multiplicative reduction.

In the dynamical setting, Silverman has made the following conjecture \([15\text{, } \S 4.11]\).

Conjecture 1.1. Let \( h_{M_d} \) be the height function associated to an embedding of the space of degree \( d \geq 2 \) rational maps \( M_d \) into projective space, and let \( K \) be a number field. There exists a positive constant \( c \) depending only on \( K \), \( d \), and the choice of \( h_{M_d} \), such that for all rational maps \( \phi \in K(z) \) of degree \( d \),
\[
\hat{h}_\phi(P) \geq c \max\{h_{M_d}(\langle \phi \rangle), 1\}
\]
for any non-preperiodic point \( P \in \mathbb{P}^1(K) \).

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In this article, we prove a weaker version of this claim in the case of polynomials, analogously to the aforementioned result of Silverman.

**Theorem 1.2.** Let \( h_{M_d} \) be the height function associated to an embedding of the space of degree \( d \geq 2 \) rational maps \( M_d \) into projective space, and let \( K \) be a number field. Suppose \( \phi \in K[z] \) is a polynomial of degree \( d \) and has \( s \) or fewer places of bad reduction. There exist constants \( \kappa_1 > 0 \) and \( \kappa_2 \) depending only on \( d, s, [K : \mathbb{Q}] \), and the choice of \( h_{M_d} \) such that

\[
\hat{h}_\phi(\alpha) \geq \kappa_1 h_{M_d}(\langle \phi \rangle) + \kappa_2
\]

for any \( \alpha \in K \) having infinite forward orbit under \( \phi \).

In particular, we recover full uniformity across rational maps conjugate to a polynomial having everywhere good reduction. We remark that in [8], Hindry and Silverman prove that the full statement of Lang’s Conjecture follows from the abc-Conjecture. One might wonder whether a similar assumption would give a proof of Conjecture 1.1 for polynomials.

The strategy behind the proof of Theorem 1.2 draws its inspiration from the central ideas of [10], which concerns the map \( \phi(z) = z^d + c \). Ingram shows that if many iterates \( \phi^i(\alpha) \) have small local height at some given valuation \( v \), then any pre-image of these \( \phi^i(\alpha) \) under \( \phi \) must be \( v \)-adically very close to some root of \( \phi \). Moreover, the degree of closeness increases suitably as the local height of \( c \) grows. It follows from the pigeon-hole principle that some explicit proportion of the \( \phi^i(\alpha) \) must be very close together in the \( v \)-adic metric, in a way reminiscent of Lemmas 3 and 4 of [16]. The product formula then yields a lower bound on the global canonical height.

For degree \( d \) polynomials not of the form \( z^d + c \), this approach fails: one cannot conclude that all pre-images of low local height iterates \( \phi^i(\alpha) \) lie close to a root of \( \phi \). Instead, working with representatives of \( \langle \phi \rangle \in M_d \) in a certain normal form \( f_c \), any pre-image under \( f_c^3 \) of a low local height point \( f_c^i(\alpha) \) necessarily lies near some root of \( f_c^3 \). The key to this argument is an analysis of the equipotential curves of the local Green’s functions of \( f_c \). Assuming at least some critical point of \( f_c \) has \( v \)-adically unbounded orbit, the level sets corresponding to points in the grand orbit of the fastest escaping critical point form the boundaries of adjacent annuli. These sequences of annuli are endowed with a natural a tree structure, which was analyzed in depth in [4], [6], and [7]. By tracking how the moduli of the annuli grow as the local height of \( f_c \) grows, we argue that any \( f_c^i(\alpha) \) with low local height at \( v \) must have some pre-image under \( f_c^3 \) that is \( v \)-adically close to a root of \( f_c^3 \). We are able to implement this idea in both the archimedean and non-archimedean settings, allowing us to incorporate non-archimedean places of bad reduction in the statement of Theorem 1.2.

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2. Background

Let \( \text{Rat}_d \) be the set of degree \( d \) rational maps \( \phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \), where \( d \geq 2 \). Under its natural embedding into \( \mathbb{P}^{2d+1}(\mathbb{C}) \), it is the complement of the hypersurface given by the
resultant locus, and hence has the structure of an affine variety. The action of conjugation by Möbius transformations over $\mathbb{C}$ determines a quotient map

$$\pi : \text{Rat}_d \to \mathcal{M}_d,$$

where $\mathcal{M}_d$ has the structure of an affine variety defined over $\mathbb{Q}$, and $\pi$ can be defined over $\mathbb{Q}$ [15, §4.4]. The space $\mathcal{M}_d$ can thus be viewed as a moduli space of degree $d$ rational maps of $\mathbb{P}^1(\mathbb{C})$. Embedding the variety $\mathcal{M}_d$ into projective space $\mathbb{P}^N$ for a suitable $N$ yields an associated height function $h_{\mathcal{M}_d}$.

For a number field $K$, let $M_K$ denote the set of places of $K$, each giving rise to a distinct absolute value, and normalized so that each $v$ restricts to the usual $v$-adic absolute value on $\mathbb{Q}$. Fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, and let $v_0 \in M_K$ be the corresponding valuation. Let $M_K^\circ$ denote the archimedean places in $M_K$, and let $M_K^0$ denote the non-archimedean places. If $\phi(z) = a_dz^d + a_{d-1}z^{d-1} + \cdots + a_1z + a_0$ and $v \in M_K^0$, then we say $\phi$ has bad reduction at $v$ if either $|a_d|_v \neq 1$, or $|a_i|_v > 1$ for some $0 \leq i \leq d-1$ (see [15, §2.5] for a definition that applies to rational maps). For $v \in M_K$, let $\mathbb{C}_v$ denote the completion of an algebraic closure of $K_v$. For $\phi(z) \in \mathbb{C}_v[z]$ and $z \in \mathbb{C}_v$, let

$$G_{\phi,v}(z) = \lim_{n \to \infty} \frac{1}{d^n} \log \max\{1, |\phi^n(z)|_v\}$$

be the standard $v$-adic escape-rate function, and let

$$M_v(\phi) = \max\{G_{\phi,v}(c_i) \mid \phi'(c_i) = 0\}.$$

(See §3.4, 3.5 of [15] for a proof that the limit defining $G_{\phi,v}(z)$ exists.) When $v = v_0$, we will denote $M_{v_0}(\phi)$ by simply $M(\phi)$, and similarly for $G_{\phi,v_0}$. When $v$ is archimedean, the escape-rate function is harmonic wherever it is non-zero; consequently, the level sets corresponding to non-zero values are finite unions of closed curves [13]. Following convention, we will use the alternate notation $\hat{\lambda}_v$ instead of $G_{\phi,v}$ whenever $\hat{\lambda}_v = G_{\phi,v}$ is being referred to in its role as a local height function. Note that $G_{\phi,v}(z)$ obeys the transformation rule

$$G_{\phi,v}(\phi(z)) = dG_{\phi,v}(z)$$

for all $z \in \mathbb{C}_v$.

If $\phi \in K[z]$, then $\phi$ has good reduction at $v \in M_K^0$ only if $\hat{\lambda}_v(\alpha) = \log \max\{1, |\alpha|_v\}$ for all $\alpha \in K$ [2, Proposition 1.4]. This fact will be used in the proof of Lemma 5.4. The canonical height $h_{\phi}(\alpha)$ of $\alpha \in K$ is defined as

$$\hat{h}_{\phi}(\alpha) = \lim_{n \to \infty} \frac{1}{d^n} h(\phi^n(\alpha)).$$

It can also be expressed as

$$\hat{h}_{\phi}(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} n_v \hat{\lambda}_v(\alpha),$$

where $n_v = [K_v : \mathbb{Q}_v]$. The canonical height is invariant under conjugation: in other words, if $\phi = \mu^{-1} \circ \psi \circ \mu$ for a Möbius transformation $\mu \in \overline{\mathbb{Q}}(z)$, then $\hat{h}_{\phi}(\alpha) = \hat{h}_{\psi}(\mu^{-1}(\alpha))$.

**Lemma 2.1.** [9, Lemma 2.1] Let $v \in M_K$, let $d \geq 2$, and let

$$\phi(z) = a_dz^d + a_{d-1}z^{d-1} + \cdots + a_1z + a_0 \in K[z].$$
If \( z \in \mathbb{C}_v \) satisfies
\[
|z|_v > C_{\phi,v} := |2d|_v \max_{0 \leq i \leq d} \left\{ 1, \left| \frac{a_i}{a_d} \right|_v^{1/(d-i)} \right\},
\]
then
\[
G_{\phi,v}(z) = \log |z|_v + \frac{1}{d-1} \log |a_d|_v + \epsilon(\phi,v,z)
\]
where
\[
-\log 2 \leq \epsilon(\phi,v,z) \leq \log \frac{3}{2}
\]
if \( v \in M^\infty_K \), and \( \epsilon(\phi,v,z) = 0 \) if \( v \in M^0_K \).

In [9], Ingram relates \( C_{\phi,v} \) to \( M_v(\phi) \) for polynomials \( \phi \in \overline{\mathbb{Q}}[z] \) in a particular normal form, which we now introduce. For \( c = (c_1, \ldots, c_{d-1}) \in \mathbb{A}^{d-1}(\overline{\mathbb{Q}}) \) and \( d \geq 2 \), set
\[
f_c(z) = \frac{1}{d} z^d - \frac{1}{d-1} (c_1 + \cdots + c_{d-1}) z^{d-1} + \cdots + (-1)^{d-1} c_1 c_2 \cdots c_{d-1} z,
\]
so that
\[
f'_c(z) = \prod_{i=1}^{d-1} (z - c_i).
\]

**Lemma 2.2.** [9] Let \( C_{f_c,v} \) be as in Lemma 2.1. There is a constant \( \xi_v \) depending only on \( d \) and on \( v \) such that
\[
\log C_{f_c,v} \leq M_v(f_c) + \xi_v
\]
for all \( f_c \). Moreover, \( \xi_v = 0 \) for all but finitely many \( v \in M_K \).

**Proof.** The proof follows immediately from Lemmas 2.2 and 2.5 of [9]. \( \square \)

Finally, we show that any \( \phi \in K[z] \) has a conjugate of the form \( f_c \), with field of definition having degree bounded in terms of \( d \) and \([K : \mathbb{Q}]\).

**Lemma 2.3.** Every \( \phi \in \overline{\mathbb{Q}}[z] \) of degree \( d \geq 2 \) is affine conjugate to a polynomial of the form \( f_c \) for some \( c \in \mathbb{A}^{d-1}(\overline{\mathbb{Q}}) \). If \( \phi \) is defined over a number field \( K \), then \( f_c \) is defined over a number field \( L \) such that \([L : K] \leq d(d-1)\).

**Proof.** Let \( \phi(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \), and let \( \gamma \) be a fixed point of \( \phi \). If \( \mu = (a_d d)^{-1/(d-1)} z + \gamma \), then \( g = \mu^{-1} \circ \phi \circ \mu \) is of the form \( f_c \). Indeed, it is easy to check that the derivative \( g' \) is monic, and that \( g \) has a fixed point at 0. But the form of \( f_c \) is characterized by exactly these two properties. Finally, \( f_c \) is defined over \( L = K(\gamma, (a_d d)^{-1/(d-1)}) \), and \([L : K] \leq d(d-1)\). \( \square \)

3. **Local height bounds: Archimedean case**

We remind the reader of standard facts from complex analysis, which will be invoked in the proof of Proposition 3.3. By the Poincaré-Koebe uniformization theorem, any doubly connected domain in \( \mathbb{C} \) is biholomorphic to an open round annulus. We will thus refer to any such domain as an *annulus*. If an annulus \( A_1 \) is biholomorphic to the bounded round annulus \( A_2 = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \} \), we define the *modulus* of \( A_1 \) (and \( A_2 \)) to be
\[
\text{mod}(A_1) = \frac{1}{2\pi} \log \left( \frac{r_2}{r_1} \right).
\]
where the modulus is infinite if \( r_1 = 0 \). Finally, we say an annulus \( A_1 \) is essentially embedded in an annulus \( A_2 \) if \( A_1 \subset A_2 \), and the bounded complementary component of \( A_2 \) is contained in the bounded complementary component of \( A_1 \).

**Lemma 3.1.** ([14, §9.3.5]) If \( f : A_1 \to A_2 \) is a degree \( k \) holomorphic covering map of annuli in \( \mathbb{C} \), then

\[
\text{mod}(A_1) = \frac{1}{k} \text{mod}(A_2).
\]

**Lemma 3.2** ([13], Corollary B.6). If \( A_1, A_2, \ldots, A_m \) are disjoint essentially embedded sub-annuli of the annulus \( A \subset \mathbb{C} \), then

\[
\text{mod}(A_1) + \cdots + \text{mod}(A_m) \leq \text{mod}(A).
\]

We now use these properties of annuli and holomorphic covering maps between them to prove a key proposition.

**Proposition 3.3.** Let \( \phi \in \mathbb{C}[z] \) be of degree \( d \geq 2 \), and assume \( M = M(\phi) > 0 \). Let \( C_2 \) be a connected component of the level set \( G_\phi(z) = M/d^2 \). Then the annulus bounded by \( C_2 \) and the curve \( G_\phi(z) = dM \) has modulus at least \( \frac{1}{2\pi}(d + \frac{1}{d-1})M \).

**Proof.** On the set \( \{ z \in \mathbb{C} \mid G_\phi(z) > M \} \), the function \( G_\phi(z) \) equals \( \log \| \tau_\phi \| \), where \( \tau_\phi \) is the Böttcher coordinate near \( \infty \). Therefore \( \tau_\phi \) takes the region \( M < G_\phi(z) < dM \) to the region \( M < \log |z| < dM \). As \( \tau_\phi \) is an isomorphism, the annulus \( A_0 \) bounded by the curves \( G_\phi(z) = dM \) and \( G_\phi(z) = M \) has modulus \( \frac{1}{2\pi}(d - 1)M \). Removing from \( A_0 \) the level curves of any elements in the grand orbit of a critical point of \( \phi \), we obtain disjoint fundamental subannuli \( A_{0,1}, \ldots, A_{0,j} \) of \( A_0 \). Again using the Böttcher coordinate, we observe that

\[
\sum_{i=1}^{j} \text{mod}(A_{0,i}) = \text{mod}(A_0).
\]

Now let \( C_2 \) be any connected component of the level curve \( G_\phi(z) = M/d^2 \), and let \( C_1 \) be the unique connected component of the level curve \( G_\phi(z) = M/d \) which bounds a region in \( \mathbb{C} \) containing \( C_2 \). Let \( A_2 \) be the annulus with boundaries \( C_1 \) and \( C_2 \). Within \( A_2 \), there is a collection \( A_{2,1}, \ldots, A_{2,j} \) of essentially embedded subannuli, where for each \( 1 \leq i \leq j \), \( A_{2,i} \) is mapped by \( \phi^2 \) onto the fundamental subannuli \( A_{0,1}, \ldots, A_{0,j} \) respectively. By Lemma 3.1 we have

\[
\text{mod}(A_{2,i}) \geq \frac{1}{(d - 1)^2} \text{mod}(A_{0,i})
\]

for all \( 1 \leq i \leq j \). Combining (3) with (4) and applying Lemma 3.2 we get

\[
\text{mod}(A_2) \geq \frac{1}{2\pi} \frac{M}{d - 1}.
\]

A similar argument shows that \( \text{mod}(\phi(A_2)) \geq \frac{1}{2\pi} M \). We conclude that the modulus of the annulus bounded by \( G_\phi(z) = dM \) and \( C_2 \) is at least \( \frac{1}{2\pi} \left( (d - 1) + 1 + \frac{1}{d - 1} \right) M = \frac{1}{2\pi} \left( d + \frac{1}{d - 1} \right) M \). \( \square \)

Proposition 3.3 implies a corollary about pre-images of low height points under \( f_c^3 \). We first make the following definition.
Definition. For $C_{f_e,v}$ as in Lemma 2.1 let
\[ B_{f_e,v}(\infty) = \mathbb{C}_v \setminus D(0, C_{f_e,v}), \]
where $D(0, C_{f_e,v}) = \{ z \in \mathbb{C}_v : |z|_v \leq C_{f_e,v} \}.$

We use the notation $C_{f_e}$ and $B_{f_e}(\infty)$ when considering $v = v_0,$ the restriction to $\overline{\mathbb{Q}}$ of the standard complex absolute value. (Recall that we have fixed an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}.$)

Corollary 3.4. Let $\alpha \in \mathbb{C},$ and let $d \geq 2.$ There exists a constant $\delta = \delta(d)$ depending only on $d,$ so that for every $c \in \mathbb{A}^{d-1}(\overline{\mathbb{Q}}),$ $\alpha \notin B_{f_e}(\infty)$ implies that every $y \in f_e^{-3}(\alpha)$ satisfies
\[ \min_{\beta \in f_e^{-3}(0)} |y - \beta| \leq -\frac{1}{d - 1} M(f_e) + \delta. \]

Proof. By Lemmas 2.1 and 2.2 we see that for all $f_e$ with $M(f_e)$ sufficiently large,
\[ \{ z \in \mathbb{C} \mid G_{f_e}(z) > dM(f_e) \} \subset B_{f_e}(\infty). \]

Lemma 2.1 then implies that for all $f_e$ with $M = M(f_e)$ sufficiently large, $\{ z \in \mathbb{C} \mid G_{f_e}(z) \leq dM(f_e) \}$ is contained in a disk of radius comparable to $e^{dM(f_e)}.$ If $\alpha \notin B_{f_e}(\infty),$ then by (2), any $y \in f_e^{-3}(\alpha)$ is contained in the closed region $D_1$ bounded by some connected component $C_y$ of the curve $G_{f_e}(z) = M/d^2.$ Let $D_2 = \{ z \in \mathbb{C} \mid G_{f_e}(z) \leq dM \},$ and let $A(f_e^3)$ be the annulus in $\mathbb{C}$ bounded by $G_{f_e}(z) = dM$ and $C_y.$ Every annulus $A \subset \mathbb{C}$ of sufficiently large modulus contains an essentially embedded round annulus of modulus differing by at most a constant from the modulus of $A.$ Applying Proposition 3.3, it follows that for $M(f_e)$ sufficiently large, there is an essentially embedded round annulus $A'(f_e^3)$ contained in $A(f_e^3)$ with outer boundary of radius comparable to $e^{dM(f_e)}$ and modulus at least $\frac{1}{2^d}(d + \frac{1}{d-1})M(f_e) - O(1).$ We now observe that $f_e^3$ maps $D_1$ onto $D_2,$ and that $0 \in D_2.$ From this, we conclude that there exist an $m$ and a $C$ such that for all degree $d$ polynomials $f_e \in \mathbb{Q}[z]$ with $M(f_e) \geq m,$ and for all $\alpha \in \overline{\mathbb{Q}},$ $\alpha \notin B_{f_e}(\infty)$ implies that each $y \in f_e^{-3}(\alpha)$ satisfies
\[ |y - \beta| \leq \frac{C}{\exp(\frac{1}{d-1} M(f_e))} \]
for some root $\beta \in \overline{\mathbb{Q}}$ of $f_e^3.$ On the other hand, if $M(f_e) < m,$ then by Lemma 2.2
\[ \log C_{f_e} \leq m + \xi_{v_0}, \]
so by the triangle inequality,
\[ \log |y - \beta| < \log 2 + m + \xi_{v_0} \leq \log 2 + \delta'. \]
As $m$ and $\xi_{v_0}$ depend only on $d,$ it follows that $\delta'$ can be chosen to depend only on $d.$ \[ \square \]

4. Local height bounds: non-Archimedean case

For a number field $K$ and $v \in M_K^0,$ recall that $\mathbb{C}_v$ denotes the completion of an algebraic closure $\overline{K}_v$ of $K_v.$ A disk in $\mathbb{C}_v$ is a set of the form $D = \{ z \in \mathbb{C}_v \mid |z - a|_v < r \}$ for some $a \in \mathbb{C}_v,$ where $r > 0.$ For such a disk $D,$ we write $\overline{D}$ to denote the set $\{ z \in \mathbb{C}_v \mid |z - a|_v \leq r \}.$ An annulus in $\mathbb{C}_v$ is a set of the form $\{ z \in \mathbb{C}_v \mid r_1 < |z - a|_v < r_2 \}$ for some $a \in \mathbb{C}_v,$ and some non-negative $r_1 < r_2.$ We define the modulus of such an annulus to be $\frac{1}{2\pi} \log (r_2/r_1),$ by analogy with the archimedean case. We have the following proposition, which shows that moduli of annuli transform the same way under covering maps as in the archimedean setting.
Proposition 4.2. Let $D \subset \mathbb{C}_v$ be a disk, and let $\phi \in \mathbb{C}_v[z]$ be a polynomial of degree $d \geq 1$. Then $\phi^{-1}(D)$ is a disjoint union $D_1 \cup \cdots \cup D_m$ of disks, with $1 \leq m \leq d$. Moreover, for each $i = 1, \ldots, m$, there is an integer $1 \leq d_i \leq d$ such that every point in $D$ has exactly $d_i$ pre-images in $D$, and that $d_1 + \cdots + d_m = d$.

Propositions 4.1 and 4.2 imply a non-archimedean counterpart of Corollary 3.4.

Proposition 4.3. Let $v \in M^0_K$, let $\alpha \in \mathbb{C}_v$, and let $d \geq 2$. There exists a constant $\delta_v$ depending only on $d$ and on $v$ such that for every $f_c(z) \in \mathbb{C}_v[z]$ of degree $d$, $\alpha \notin B_{f_c,v}(\infty)$ implies that every $y \in f_c^{-3}(\alpha)$ satisfies

$$
\min_{\beta \in f_c^{-3}(0)} |y - \beta|_v \leq -\frac{1}{d-1}M_v(f_c) + \delta_v.
$$

Proof. Lemma 2.1 implies that there exists an $m_v$ such that for all $f_c$ with $M = M_v(f_c) \geq m_v$, the level set $G_{f_c,v}(z) = dM$ is in $B_{f_c,v}(\infty)$, and the distortion factor $\epsilon(f_c, v, z)$ is $0$. As Lemma 2.2 specifies that there are finitely many non-zero $\xi_v$, this $m_v$ can be chosen to be independent of $v \in M^0_K$. We denote it by $m_0$.

From Lemma 2.1 it follows that when $M_v(f_c) \geq m_0$, the level set $G_{f_c,v}(z) = dM$ is a set of the form $|z|_v = R$. For any connected component (disk) pre-image $D_1$ of $D_2 := \{z \in \mathbb{C}_v \mid G_{f_c,v}(z) < dM\}$ under $f_c$, $D_2 - D_1$ is an annulus $A(f_c^3)$. A similar proof as the proof of Proposition 3.3 shows that the modulus of $A(f_c^3)$ is at least $\frac{1}{2\pi}(1 - \frac{1}{d-1})M_v(f_c)$ (note that this annulus is round, as non-archimedean disks are centerless, so we do not need to invoke anything akin to the Grötzsch inequality). From Proposition 4.2 we know that $f_c^3$ maps $D_1$ onto $D_2$. Thus, if $\alpha \notin B_{f_c,v}(\infty)$ where $M_v(f_c) \geq m_0$, then for every $y \in f_c^{-3}(\alpha)$, we have

$$
(5) \quad \min_{\beta \in f_c^{-3}(0)} |y - \beta|_v \leq \exp \left(-\frac{1}{d-1}M_v(f_c)\right).
$$

On the other hand, when $M_v(f_c) < m_0$, we have

$$
\log C_{f_c,v} < m_0 + \xi_v,
$$

so by the ultrametric inequality, if $\alpha \notin B_{f_c,v}(\infty)$, then every $y \in f_c^{-3}(\alpha)$ satisfies

$$
(6) \quad \log |y - \beta|_v < m_0 + \xi_v \leq \delta_v
$$

for some $\delta_v$ depending only on $d$ and on $v$. Combining (5) and (6) completes the proof. \qed

5. Key Inequalities

In this section, as well as in [110] the main ideas are inspired by those used by Ingram in [10]. Let $d \geq 2$. For $\phi(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \in K[z]$ and $v \in M_K$, let

$$
\lambda_v(\phi) = \max\{\lambda_v(a_i) \mid 0 \leq i \leq d\},
$$

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where \( \lambda_v(a_i) = \log \max \{1, |a_i|_v \} \), let \( n_v = [K_v : \mathbb{Q}_v] \), and let
\[
h(\phi) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in MK} n_v \lambda_v(\phi).
\]
Let \( \phi \in K[z] \) be conjugate to \( f_e \in L[z] \), where \( L \) is as in Lemma 2.3 From this point onward, we denote \( f_e \) by \( f \) for ease of notation.

**Lemma 5.1** ([9]). For any \( v \in M_L \), we have
\[
\lambda_v(f) \leq dM_v(f) + \eta
\]
for some constant \( \eta \) depending only on \( d \).

**Proof.** This results from combining Lemmas 2.1, 2.2 and 2.5 of [9]. \( \square \)

**Lemma 5.2.** Let \( v \in M_L \). Let \( \alpha \in B_{f,v}(\infty) \cap L \). Then there exists a constant \( C_1 \) depending only on \( d \) such that if \( j > i \), then
\[
\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq d^{i+3} \hat{\lambda}_v(\alpha) + C_1.
\]

**Proof.** From the definition of \( C_{f,v} \), we have
\[
\log C_{f,v} \geq \log (2d) + \frac{1}{d} \lambda_v(f).
\]
Thus,
\[
\alpha \in B_{f,v}(\infty) \iff \lambda_v(\alpha) > \lambda_v(C_{f,v}) \geq \log (2d) + \frac{1}{d} \lambda_v(f).
\]
If \( \alpha \in B_{f,v}(\infty) \cap L \), then, this yields

\[
\hat{\lambda}_v(\alpha) \geq \log (2d) + \frac{1}{d} \lambda_v(f) - \log 2
\]
by Lemma 2.1. Moreover, by Lemma 2.1 and the triangle inequality,
\[
(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq (d-1)(\log 2 + \hat{\lambda}_v(f^j(\alpha)) + \log 2)
\]
\[
= (d-1)(d^j \hat{\lambda}_v(\alpha) + 2 \log 2).
\]
Therefore, rearranging (7) and combining with (8) gives
\[
\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq d \hat{\lambda}_v(\alpha) + d \log 2 - d \log (2d)
\]
\[
+ (d-1)(d^j \hat{\lambda}_v(\alpha) + 2 \log 2)
\]
\[
\leq (d^j+1)(d-1) + d \hat{\lambda}_v(\alpha) + C_1
\]
\[
\leq d^j+3 \hat{\lambda}_v(\alpha) + C_1
\]
whenever \( \alpha \in B_{f,v}(\infty) \). \( \square \)

**Proposition 5.3.** Let \( \alpha \in L \) have infinite orbit under \( f \), let \( X \) be a finite set of positive integers, and let \( v \in M_L \). Then there exists \( Y \subset X \) containing at least \( \#X - 3 \) \( \frac{d^3 + 1}{d^3} \) values such that for all \( i, j \in Y \) with \( j > i \), we have
\[
\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq d^j \hat{\lambda}_v(\alpha) + C_2
\]
for some constant \( C_2 \) depending only on \( d \).
Proof. Suppose that at least \( \frac{#X-3}{d^3+1} \) values \( k \in X \) have \( f^k(\alpha) \in B_{f,v}(\infty) \), and let \( Y \) be the set of such values. If \( i, j \in Y \) with \( j > i \), then Lemma \([5.2]\) applied to \( f^i(\alpha) \) implies
\[
\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^{j-i}(\alpha)| \leq d^{j-i+3} \lambda_v(f^i(\alpha)) + C_1
\]
\[
\phantom{=} = d^{j+3} \lambda_v(\alpha) + C_1.
\]

Now suppose that fewer than \( \frac{#X-3}{d^3+1} \) values \( k \in X \) satisfy \( f^k(\alpha) \in B_{f,v}(\infty) \). Then there are more than \( \frac{d^3(#X-3)}{d^3+1} + 3 \) values \( k \in X \) such that \( f^k(\alpha) \not\in B_{f,v}(\infty) \), so more than \( \frac{d^3}{d^3+1} (#X - 3) \) such that \( f^{k+3}(\alpha) \not\in B_{f,v}(\infty) \). By Corollary \([3.4]\), Proposition \([4.3]\) and the pigeon-hole principle, there is a \( \beta \in \mathbb{Q} \) with \( f^3(\beta) = \beta \) and
\[
\log |f^k(\alpha) - \beta|_v \leq \frac{-1}{d-1} M_v(f) + \max\{\delta, \delta_v\}
\]
for at least \( \frac{#X-3}{d^3+1} \) values \( k \in X \). Applying Lemma \([5.1]\) to the right-hand side, this gives
\[
\log |f^k(\alpha) - \beta|_v \leq \frac{1}{d(d-1)} (\eta - \lambda_v(f)) + \max\{\delta, \delta_v\}
\]
\[
= \frac{-\lambda_v(f)}{d(d-1)} + \frac{\eta}{d(d-1)} + \max\{\delta, \delta_v\}
\]
for at least \( \frac{#X-3}{d^3+1} \) values \( k \in X \). If \( i \) and \( j \) are two of these values, then by the triangle inequality,
\[
\log |f^i(\alpha) - f^j(\alpha)|_v \leq \frac{-\lambda_v(f)}{d(d-1)} + \frac{\eta}{d(d-1)} + \max\{\delta, \delta_v\} + \log 2
\]
where \( \delta, \eta, \) and \( \delta_v \) depend only on \( d \). Hence
\[
\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq \eta + d(d-1)(\max\{\delta, \delta_v\} + \log 2).
\]
Setting
\[
C_2 = C_1 + \eta + d(d-1)(\max\{\delta, \delta_v\} + \log 2)
\]
completes the proof. \( \square \)

**Lemma 5.4.** Let \( v \) be a non-archimedean place of \( L \) such that \( f \in L[z] \) has good reduction at \( v \). Then for all \( \alpha \in L \) with \( h_f(\alpha) \neq 0 \), and all \( j > i \), we have
\[
\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq (d-1)d^{j+1} \lambda_v(\alpha).
\]

**Proof.** Since \( f \) has good reduction at \( v \), \( \lambda_v(f) = 0 \), so it suffices to show that
\[
d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq (d-1)d^{j+1} \lambda_v(\alpha).
\]
But this is clear: \( f \) having good reduction at \( v \) implies that \( \hat{\lambda}_v(\alpha) = \lambda_v(\alpha) \) for all \( \alpha \in L \), and so
\[
\log |f^i(\alpha) - f^j(\alpha)|_v \leq \log \max\{|f^i(\alpha)|_v, |f^j(\alpha)|_v, 1\} = \max\{\hat{\lambda}_v(f^i(\alpha)), \hat{\lambda}_v(f^j(\alpha))\} = d^{j+1} \lambda_v(\alpha)
\]
for all \( j > i \). \( \square \)
6. Proof of Main Theorem

Proof of Theorem 1.2. Let \( \phi \in K[z] \) be affine conjugate to \( f = f_c \in L[z] \), where \( L \) is as in Lemma 2.3 and let \( \alpha \in L \) be such that \( \hat{h}_f(\alpha) \neq 0 \). Let
\[
N = \frac{3}{d^3}(d^3 + 3)^{r+s+1} - 1,
\]
where \( r \) is the number of distinct archimedean absolute values of \( L \), and \( s \) is the number of \( v \in M_L^1 \) of bad reduction. Denote the union of the set of places of bad reduction of \( f_c \) and the archimedean places of \( f \) by \( M_L^1 \). Let \( X = \{1, 2, \ldots, N\} \), and let \( v \in M_L^1 \). It is easy to show that
\[
\frac{\#X - 3}{d^3 + 1} \geq \frac{3}{d^3}(d^3 + 3)^{r+s} - 1.
\]
Hence, by Proposition 5.3, we can choose a subset \( X' \subset X \) such that
\[
\#X' \geq \frac{\#X - 3}{d^3 + 1} \geq \frac{3}{d^3}(d^3 + 3)^{r+s} - 1
\]
and such that for all \( i, j \in X' \) with \( j > i \), we have
\[
(9) \quad \lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v \leq d^{i+3} \hat{\lambda}_v(\alpha) + C_2
\]
for some constant \( C_2 \) depending only on \( d \). By induction on the size of the resulting subsets, we obtain a set \( Y \subset X \) with
\[
\#Y \geq \frac{3}{d^3}((d^3 + 3)^{r+s} - 1) > 3
\]
such that (9) holds for every \( v \in M_L^1 \) and for all \( j > i \in Y \). From this, we obtain
\[
(10) \quad \sum_{v \in M_L^1} n_v[\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v] \leq \sum_{v \in M_L^1} n_v[d^{i+3} \hat{\lambda}_v(\alpha) + C_2]
\]
for \( j > i \in Y \), and \( n_v = [L_v : \mathbb{Q}_v] \). From Lemma 5.4 we also have
\[
(11) \quad \sum_{v \in M_L^1 - M_L^1} n_v[\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v] \leq n_v d^{i+3} \sum_{v \in M_L^1 - M_L^1} \hat{\lambda}_v(\alpha).
\]
Summing (10) and (11) and applying the product formula, we deduce that
\[
[L : \mathbb{Q}]h(f) = \sum_{v \in M_L} n_v(\lambda_v(f) + d(d-1) \log |f^i(\alpha) - f^j(\alpha)|_v)
\]
\[
\leq \sum_{v \in M_L} n_v(d^{i+3} \hat{\lambda}_v(\alpha)) + \sum_{v \in M_L^1} n_v C_2
\]
\[
\leq [L : \mathbb{Q}]d^{i+3} \hat{h}_f(\alpha) + [L : \mathbb{Q}](r + s)C_2
\]
\[
\leq [L : \mathbb{Q}]d^{N+3} \hat{h}_f(\alpha) + [L : \mathbb{Q}](r + s)C_2
\]
where \( C_2 \) depends only on \( d \). In particular,
\[
(12) \quad h(f) << \min\{\hat{h}_f(\alpha) \mid \alpha \in L, \hat{h}_f(\alpha) \neq 0\}.
\]
Now let
\[
h_{min}(\langle f \rangle) = \min\{h(g) \mid g \sim f, g \in \mathbb{Q}[z]\},
\]
where \( g \sim f \) if and only if \( g \) is conjugate to \( f \) by a Möbius transformation over \( \mathbb{Q} \). By the proof of [17, Lemma 6.32], we have

\[
h_{\text{min}}(\langle f \rangle) \preceq h_{\mathcal{M}_d}(\langle f \rangle).
\]

Thus (12) implies

\[
h_{\mathcal{M}_d}(\langle \phi \rangle) = h_{\mathcal{M}_d}(\langle f \rangle) \ll \min\{\hat{h}_f(\alpha) \mid \alpha \in L, \hat{h}_f(\alpha) \neq 0\}
\]

\[
\leq \min\{\hat{h}_\phi(\alpha) \mid \alpha \in K, \hat{h}_\phi(\alpha) \neq 0\},
\]

where the implied constants depend only on \( d \), \([L : \mathbb{Q}]\), and the number of places of bad reduction of \( f = f_c \). (Recall that we fixed a choice of \( h_{\mathcal{M}_d} \) in advance.) Finally, we note that the number of places of bad reduction of \( f = f_c \) is at most the number of places of bad reduction of \( \phi \) plus the number of primes dividing \( d \). This can be seen from the conjugation taking \( \phi \) to \( f_c \), and from the fact that \( \phi(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \) has bad reduction at all primes dividing \( a_d \). From Lemma 2.3 we had \([L : K] \leq d(d-1)\), so the implied constants depend only on \( d \), \([K : \mathbb{Q}]\), and the number of places of bad reduction of \( \phi \). This completes the proof.

\[\square\]

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