ON DYNAMICAL FINITENESS PROPERTIES OF ALGEBRAIC GROUP SHIFTS

BY

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ABSTRACT

Let $G$ be a group and let $V$ be an algebraic group over an algebraically closed field. We introduce algebraic group subshifts $\Sigma \subset V^G$ which generalize both the class of algebraic sofic subshifts of $V^G$ and the class of closed group subshifts over finite group alphabets. When $G$ is a polycyclic-by-finite group, we show that $V^G$ satisfies the descending chain condition and that the notion of algebraic group subshifts, the notion of algebraic group sofic subshifts, and that of algebraic group subshifts of finite type are all equivalent. Thus, we obtain extensions of well-known results of Kitchens and Schmidt to cover the case of many non-compact group alphabets.

1. Introduction

The main goal of this paper is to extend several well-known results on the descending chain condition and the finiteness property of closed group subshifts whose alphabets are compact Lie groups to the context of algebraic group subshifts whose alphabets are algebraic groups over an algebraically closed field. By results of Tanaka and Chevalley (cf. [8]), the category of compact Lie groups is a certain subcategory of the category of $\mathbb{R}$-algebraic groups. We remark that there exist many non-compact algebraic groups, e.g., nontrivial complex linear algebraic groups.

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In order to state the results, we introduce some notation and basic definitions. The set of integers and the set of real numbers are denoted by $\mathbb{Z}$ and $\mathbb{R}$ respectively. Let $A, B$ be sets. Denote by $A^B$ the set of all maps from $B$ into $A$. Let $C \subset B$. If $x \in A^B$, the restriction $x|_C \in A^C$ is given by $x|_C(c) = x(c)$ for all $c \in C$. If $X \subset A^B$, the restriction of $X$ to $C$ is defined as 

$$X_C := \{x|_C : x \in X\} \subset A^C.$$ 

Fix an algebraically closed field $K$. An algebraic variety over $K$ is a reduced $K$-scheme of finite type and is identified with the set of $K$-points (cf. [13, Corollaire 6.4.2]). An algebraic group means a group that is also an algebraic variety with group operations given by algebraic morphisms (cf. [21]). Algebraic subvarieties are Zariski closed subsets and algebraic subgroups are abstract subgroups which are also algebraic subvarieties.

We continue with some elementary symbolic dynamics. Let $A$ be a set, called the alphabet, and let $G$ be a group, called the universe. Elements of $A^G = \prod_{g \in G} A$ are called configurations over the group $G$ and the alphabet $A$. The shift action of the group $G$ on $A^G$ is defined by $(g, x) \mapsto gx$, where 

$$gx(h) := x(g^{-1}h)$$

for all $x \in A^G$ and $g, h \in G$. A configuration $x \in A^G$ is periodic if its $G$-orbit is finite in $A^G$. If $A$ is a group, then $A^G$ admits a natural group structure induced by a pointwise group operation on each factor.

The prodiscrete topology on $A^G$ is the product topology where each factor $A$ is equipped with the discrete topology. With respect to this topology, $A^G$ is compact if and only if $A$ is finite. To avoid confusion, when $A$ is a topological space and not merely a set, the Tychonoff topology on $A^G$ will mean the product topology induced by the original topology on each factor $A$. By Tychonoff’s theorem, $A^G$ is compact with respect to the Tychonoff topology if $A$ is compact.

A group $G$ is polycyclic-by-finite if there exist subgroups 

$$G = G_n \supseteq G_{n-1} \supseteq \cdots \supseteq G_0 = \{1\}$$

such that for every $1 \leq k \leq n$, $G_{k-1}$ is a normal subgroup of $G_k$ and $G_k/G_{k-1}$ is cyclic or finite (cf. [25]).

In [17], [31, Theorem 4.2], Kitchens and Schmidt established the following important descending chain property (see, e.g, [31] for various applications):
Theorem 1.1 (cf. [31]): Let $G$ be a polycyclic-by-finite group and let $A$ be a compact Lie group. Let $(\Sigma_n)_{n \geq 0}$ be a descending sequence of $G$-invariant subgroups of $A^G$. Suppose that for all $n \geq 0$, $\Sigma_n$ is closed in $A^G$ with respect to the Tychonoff topology. Then the sequence $(\Sigma_n)_{n \geq 0}$ eventually stabilizes.

The main object studied in this paper is the class of algebraic group subshifts defined as follows. We remark here that for every algebraic group $V$ and every finite set $E$, the fiber product $V^E$ is naturally an algebraic group.

Definition 1.2: Let $G$ be a group and let $V$ be an algebraic group over an algebraically closed field. A $G$-invariant subset $\Sigma \subset V^G$ is called an **algebraic group subshift** of $V^G$ if it is closed in $V^G$ with respect to the prodiscrete topology and if the restriction $\Sigma_E \subset V^E$ is an algebraic subgroup for any finite subset $E \subset G$.

With the above notations, an algebraic group subshift of $V^G$ is automatically an abstract subgroup (cf. Proposition 4.1).

As in the classical theory of symbolic dynamics, the closedness property in the full shift $A^G$, where $A$ is a set and $G$ is a group, with respect to the prodiscrete topology of $A^G$ is the weakest closedness condition we must require for $G$-invariant subsets of $A^G$ to avoid pathologies.

Our first main result (cf. Theorem 7.1 and Proposition 6.1) is the following extension of Theorem 1.1 to algebraic group subshifts.

Theorem 1.3: Let $G$ be a polycyclic-by-finite group. Let $V$ be an algebraic group over an algebraically closed field. Let $(\Sigma_n)_{n \geq 0}$ be a descending sequence of algebraic group subshifts of $V^G$. Then the sequence $(\Sigma_n)_{n \geq 0}$ eventually stabilizes.

Now let $G$ be a group and let $A$ be a set. A $G$-invariant subset $\Sigma \subset A^G$ is called a **subshift** of $A^G$. We remark that no condition on the closedness of such subshift $\Sigma$ in $A^G$ is required. Given subsets $D \subset G$ and $P \subset A^D$, we define the following subshift of $A^G$:

\[
\Sigma(A^G; D, P) := \{x \in A^G: (g^{-1}x)|_D \in P \text{ for all } g \in G\}.
\]

Such a set $D$ is called a **defining window** of $\Sigma(A^G; D, P)$. If $D$ is finite, $\Sigma(A^G; D, P)$ is clearly closed in $A^G$ with respect to the prodiscrete topology and it is called the subshift of **finite type** of $A^G$ associated with $D$ and $P$.
Subshifts of finite type can be regarded as generalizations of higher dimensional topological Markov shifts. They are fundamental objects in various areas such as information theory and smooth dynamical systems (see, e.g., [1], [18, Ch. 7], [29], [30], [20], [11] and the references therein).

In [17], [31, Theorem 4.2], the following remarkable finiteness result on closed group subshifts with compact Lie group alphabets is established:

**Theorem 1.4 (cf. [31]):** Let $G$ be a polycyclic-by-finite group and let $A$ be a compact Lie group. Let $\Sigma \subset A^G$ be a $G$-invariant subgroup which is closed in $A^G$ with respect to the Tychonoff topology. Then $\Sigma$ is a subshift of finite type of $A^G$.

It turns out that an analogous result also holds for algebraic group subshifts. Before giving the statement, we recall a strong finiteness condition on algebraic group subshifts introduced in [6].

**Definition 1.5 (cf. [6]):** Let $G$ be a group and let $V$ be an algebraic group over an algebraically closed field. A subset $\Sigma \subset V^G$ is an **algebraic group subshift of finite type** if there is a finite subset $D \subset G$ and an algebraic subgroup $W \subset V^D$ such that $\Sigma = \Sigma(A^G; D, W)$.

With the above notations, it is obvious from Definition 1.5 and the definition in (1.1) that algebraic group subshifts of finite type are indeed subshifts of finite type so that they are closed with respect to the prodiscrete topology.

We establish the following result (cf. Theorem 7.1) which extends Theorem 1.4 to notably cover the case of non-compact algebraic group alphabets.

**Theorem 1.6:** Let $G$ be a polycyclic-by-finite group. Let $V$ be an algebraic group over an algebraically closed field. Let $\Sigma$ be an algebraic group subshift of $V^G$. Then $\Sigma$ is an algebraic group subshift of finite type of $V^G$.

Given sets $A, B$ and a group $G$, following the pioneering work of John von Neumann [23], a map $\tau : B^G \to A^G$ is called a **cellular automaton** if there exist a finite subset $M \subset G$ called **memory set** and a map $\mu : B^M \to A$ called **local defining map** such that

(1.2) \[ \tau(x)(g) = \mu((g^{-1}x)|_M) \quad \text{for all} \quad x \in B^G \quad \text{and} \quad g \in G. \]

Clearly, a cellular automaton $\tau : B^G \to A^G$ is uniformly continuous and $G$-equivariant (cf. [3]). The converse also holds by the Curtis-Hedlund theorem [15] when $A, B$ are finite and in the general case by the result of [2].
Now let $U, V$ be algebraic groups over an algebraically closed field $K$. Recall the following definition introduced in [6] (see also [5], [26] and [12]). A cellular automaton $\tau : U^G \to V^G$ is an algebraic group cellular automaton if $\tau$ admits a memory set $M$ whose associated local defining map $\mu : U^M \to V$ is a homomorphism of $K$-algebraic groups. Given subshifts $\Sigma_1 \subset U^G$ and $\Sigma_2 \subset V^G$, a map $\tau : \Sigma_1 \to \Sigma_2$ is an algebraic group cellular automaton if it is the restriction of some algebraic group cellular automaton $U^G \to V^G$.

We denote by $\text{Hom}_{U,V,G}^\text{algr}(\Sigma_1, \Sigma_2)$ the set of all algebraic group cellular automata $\Sigma_1 \to \Sigma_2$. When $U = V$ and $\Sigma_1 = \Sigma_2 = \Sigma$, we denote

$$\text{End}_{V,G}^\text{algr}(\Sigma) := \text{Hom}_{V,V,G}^\text{algr}(\Sigma, \Sigma).$$

**Definition 1.7** (cf. [6]): Let $G$ be a group and let $V$ be an algebraic group over an algebraically closed field. A subset $\Sigma \subset V^G$ is an algebraic group sofic subshift if it is the image of an algebraic group subshift of finite type $\Sigma' \subset U^G$, where $U$ is a $K$-algebraic group, under an algebraic group cellular automaton $U^G \to V^G$.

When the universe $G$ is countable, algebraic group sofic subshifts are indeed algebraic group subshifts (cf. Theorem 4.5). Thus we obtain the following direct consequence of Theorem 1.6:

**Corollary 1.8:** Let $G$ be a polycyclic-by-finite group. Let $V$ be an algebraic group over an algebraically closed field. Let $\Sigma$ be an algebraic group sofic subshift of $V^G$. Then $\Sigma$ is an algebraic group subshift of finite type of $V^G$.

Moreover, we obtain in this paper some first finiteness results in the category of algebraic group subshifts (cf. Section 8).

**Theorem 1.9:** Let $G$ be a polycyclic-by-finite group. Let $U, V$ be algebraic groups over the same algebraically closed field. Let $\Sigma_1$ and $\Sigma_2$ be respectively algebraic group subshifts of $U^G$ and $V^G$. Let $\tau \in \text{Hom}_{U,V,G}^\text{algr}(U^G, V^G)$. Then $\tau^{-1}(\Sigma_2)$ and $\tau(\Sigma_1)$ are algebraic group subshifts of finite type of $U^G$ and $V^G$ respectively.

Given a map $f : X \to X$ from a set $X$ to itself, the limit set $\Omega(f)$ of $f$ is defined as the intersection of the images of its iterates, i.e., $\Omega(f) := \bigcap_{n \geq 1} f^n(X)$, where $f^n := f \circ \cdots \circ f$ ($n$-times). The map $f$ is said to be stable if

$$f^{n+1}(X) = f^n(X)$$

for some integer $n \geq 1$. 
The notion of limit sets is introduced by Wolfram [32] and were subsequently studied notably in [22], [9] [16], [14], [6].

We give another direct application of our main results on the dynamics and the limit sets of endomorphisms of algebraic group subshifts.

**Corollary 1.10:** Let $G$ be a polycyclic-by-finite group and let $V$ be an algebraic group over an algebraically closed field. Let $\Sigma$ be an algebraic group subshift of $V^G$ and let $\tau \in \text{End}_{V,G,\text{algr}}(\Sigma)$. Then the limit set $\Omega(\tau)$ is an algebraic group subshift of finite type of $V^G$ and $\tau$ is stable, i.e.,

$$f^{n+1}(X) = f^n(X)$$

for some integer $n \geq 1$.

**Proof.** Theorem 4.5 shows that the iterated images $\tau^n(\Sigma)$ are algebraic group subshifts of $V^G$. Then Theorem 5.1 on the intersection of algebraic group subshifts implies that

$$\Omega(\tau) := \bigcap_{n \geq 1} \tau^n(\Sigma)$$

is also an algebraic group subshift of $V^G$. Consequently, $\Omega(\tau)$ is an algebraic group subshift of finite type of $V^G$ by Theorem 1.6. Hence, it follows from [6, Theorem 1.3.(iv)] that $\tau$ must be stable. \hfill \blacksquare

It turns out that the proofs of our main theorems can be applied verbatim to obtain generalizations (cf. Section 9) to the so called **admissible group subshifts** (cf. Definition 9.11). For example, we obtain the following finiteness result (cf. Theorem 9.13 and Example 9.12):

**Theorem 1.11:** Let $G$ be a polycyclic-by-finite group. Let $A$ be an Artinian group (resp. an Artinian module). Then every closed subshift of $A^G$ which is also an abstract subgroup (resp. a submodule) is a subshift of finite type.

Here, a group $\Gamma$ is **Artinian** if every descending sequence of subgroups of $\Gamma$ eventually stabilizes.

It is not known whether any of the above finiteness results still holds for some universe $G$ which is not a polycyclic-by-finite group. Note that if a group $G$ admits a non-finitely-generated subgroup, then there exist a finite group $A$ and a closed subshift $\Sigma \subset A^G$ which is also an abstract subgroup but $\Sigma$ is not a subshift of finite type (cf. [24] and [28]).
The paper is organized as follows. Section 2 provides some lemmata on the window change of subshifts and recalls the machinery of inverse systems. In Section 3, we show that the closedness in the prodiscrete topology of a subset in the full shift is stable under restriction to arbitrary subsets under mild algebraic hypotheses (Lemma 3.1, Lemma 3.2). We then define in Section 4 the class of algebraic subshifts which generalizes algebraic sofic subshifts introduced in [6]. Proposition 4.1 shows that algebraic group subshifts are automatically abstract subgroups and all algebraic group sofic subshifts are algebraic group subshifts. Theorem 5.1 shows that arbitrary intersections of algebraic group subshifts are also algebraic group subshifts.

We obtain in Section 6 the equivalence between the descending chain condition and the finite type property of algebraic group subshifts (Proposition 6.1). This leads us to extend the notion of groups of Markov type in [31, Definition 4.1] to introduce the class of groups of algebraic Markov type (Definition 6.2) which is stable by taking subgroups (Proposition 6.3) and by extensions by cyclic groups (Corollary 7.7).

Section 7 contains the main technical result Theorem 7.3 whose proof requires new ideas and techniques to overcome the absence of the compactness hypothesis on the alphabets. We prove in Theorem 7.1 that every polycyclic-by-finite group is of algebraic Markov type which implies both Theorem 1.3 and Theorem 1.6. Without the compactness property, it is the Noetherian property of the Zariski topology that allows the extension from compact Lie groups to algebraic groups to be made. Section 8 studies inverse images of homomorphisms of algebraic group subshifts and proves Theorem 1.9. We then introduce admissible Artinian group structures (Definition 9.1) that are groups equipped with a certain collection of subgroups satisfying the descending chain condition. A common generalization of the finiteness results in the Introduction is then obtained in Theorem 9.13 for the class of admissible group subshifts whose alphabets are admissible Artinian group structures.

Finally, Section 10 gives some applications on the density of periodic configurations for certain algebraic subshifts (Theorem 10.2) and for admissible group subshifts (Corollary 10.3, Corollary 10.5). Another application of the techniques developed in this paper can be found in [27] where it is shown that an intrinsic shadowing property is always satisfied for the valuation action of group cellular automata on an admissible group subshift.
2. Preliminaries

To fix the notations, suppose that $G$ is a group and let $E, F$ be subsets of $G$. Then we write $EF := \{gh : g \in E, h \in F\}$.

2.1. Window change lemmata for subshifts. For subshifts described by the formula (1.1), we have the following easy but useful observation.

**Lemma 2.1:** Let $G$ be a group and let $A$ be a set. Let $D \subset G$ be a subset and let $P \subset A^D$. Let $\Sigma = \Sigma(A^G; D, P) \subset A^G$. Then, for every subset $E \subset G$ such that $D \subset E$, we have $\Sigma = \Sigma(A^G; E, \Sigma_E)$.

**Proof.** The proof is similar to [6, Lemma 5.1].

Now suppose that $H$ is a subgroup of a group $G$. Let $E \subset G$ be a subset such that $Hk_1 \neq Hk_2$ for all $k_1, k_2 \in E$ such that $k_1 \neq k_2$. Let $A$ be a set. Denote $B := A^E$. Then for every subset $F \subset H$, we have a canonical bijection $B^F = A^{FE}$ defined as follows. For every $x \in B^F$, we associate an element $y \in A^{FE}$ given by

$$y(hk) := (x(h))(k)$$

for every $h \in F$ and $k \in E$. The obtained bijection is clearly functorial with respect to inclusions $F \subset F'$ of subsets of $H$ in a trivial way.

**Lemma 2.2:** With the above notations and hypotheses, let $D \subset H$ and let $P \subset A^{DE} = B^D$ be subsets. Then the following equality between subshifts of $A^G$ holds:

$$\Sigma(A^G; HE, \Sigma(B^H; D, P)) = \Sigma(A^G; DE, P).$$

**Proof.** Let us denote $\Sigma := \Sigma(A^G; HE, \Sigma(B^H; D, P))$.

Let $x \in \Sigma$ and let $g \in G$. Then we have $(g^{-1}x)|_{HE} \in \Sigma(B^H; D, P)$. Since $DE \subset HE$, we deduce from the canonical bijection $A^{DE} = B^D$ and the definition of $\Sigma(B^H; D, P)$ that $(g^{-1}x)|_{DE} \in P$. Thus, we find that $\Sigma \subset \Sigma(A^G; DE, P)$.

Conversely, let $x \in \Sigma(A^G; DE, P)$ and let $g \in G$. Then it follows that $(g^{-1}x)|_{DE} \in P$. Since $DE \subset HE$, we have

$$((g^{-1}x)|_{HE})|_{DE} \in P.$$ 

Therefore, $(g^{-1}x)|_{HE} \in \Sigma(B^H; D, P)$. Hence, we deduce that $x \in \Sigma$ so that $\Sigma(A^G; DE, P) \subset \Sigma$ and the conclusion follows.
2.2. Restriction, Induction and Closedness. Let \( A \) be a set. Let \( H \) be a subgroup of a countable group \( G \) and let \( \Lambda \subset V^H \) be a subshift. Let \( \Sigma := \Sigma(V^G; H, \Lambda) \) (cf. (1.1)). We call \( \Sigma \) the induction subshift associated with \( \Lambda \) and the subgroup \( H \) of \( G \).

The following lemma will be used to prove Lemma 4.2 and Proposition 6.3.

**Lemma 2.3:** Let the notations and hypotheses be as above. Then we have \( \Sigma_H = \Lambda \). Moreover, with respect to the prodiscrete topology, \( \Sigma \) is closed in \( A^G \) if and only if \( \Lambda \) is closed in \( A^H \).

**Proof.** We have a canonical factorization \( \Sigma = \prod_{c \in G/H} \Sigma_c \) where each \( x \in \Sigma \) is identified with \( (x|_c)_{c \in G/H} \in \prod_{c \in G/H} \Sigma_c \) (cf. [6, Section 2.5]). Recall that \( \Sigma_c := \{z|_c : z \in \Sigma\} \) for every left coset \( c \in G/H \).

For every \( c \in G/H \), we choose \( g_c \in c \subset G \) and thus obtain a homeomorphism \( \phi_c : \Sigma_c \to \Sigma_H \) given by

\[
\phi_c(y)(h) := y(g_ch)
\]

for all \( y \in \Sigma_c \). If \( c = H \), we choose \( g_c = 1_G \) so that \( \phi_H = \text{Id}_{\Sigma_H} \).

By definition of \( \Sigma \), we have \( \Sigma_H \subset \Lambda \). Conversely, let \( z \in \Lambda \); then it is the restriction to \( H \) of the configuration \( x = (x_c)_{c \in G/H} \in \Sigma \) defined by \( x_c = g_cz \) for \( c \in G/H \). Thus \( \Lambda = \Sigma_H \) and \( \phi_c(\Sigma_c) = \Lambda \) for every \( c \in G/H \).

As \( G \) is countable, there exists an increasing sequence \( (E_n)_{n \geq 0} \) of finite subsets of \( G \) such that \( G = \bigcup_{n \geq 0} E_n \). Then for every \( c \in G/H \), the sets \( E_n \cap c \) are finite and form an increasing sequence whose union is \( c \).

Suppose first that \( \Lambda \) is closed in \( A^H \). Let \((y_n)_{n \geq 0}\) be a sequence in \( \Sigma \) which converges to some \( z \in A^G \). Then for \( c \in G/H \), the sequence \((y_n|_c)_{n \geq 0}\) in \( \Sigma_c \) converges to \( z|_c \in A^c \). Since \( \Lambda \) is closed, so is \( \Sigma_c = \phi_c^{-1}(\Lambda) \) and thus \( z|_c \in \Sigma_c \). Hence,

\[
z = (z|_c)_{c \in G/H} \in \prod_{c \in G/H} \Sigma_c = \Sigma
\]

so \( \Sigma \) is closed in \( A^G \).

Conversely, suppose that \( \Sigma \) is closed in \( A^G \). Let \((z_n)_{n \geq 0}\) be a sequence in \( \Lambda \) which converges to some \( z \in A^H \). For every \( n \geq 0 \), define \( y_n \in \Sigma \) by setting \((y_n)_c = g_cz_n \in A^c \) for \( c \in G/H \). Let \( y \in A^G \) be given by \( y_c = g_cz \in A^c \) for all \( c \in G/H \). Then it is clear that \((y_n)_{n \geq 0}\) converges to \( y \) in \( A^G \). Since \( \Sigma \) is closed, it follows that \( y \in \Sigma \) and thus \( z = y_H \in \Sigma_H = \Lambda \). We conclude that \( \Lambda \) is closed in \( A^H \). ■
2.3. **Inverse limits of closed algebraic inverse systems.** Let \((I, \prec)\) be a directed set, i.e., a partially ordered set with respect to the binary relation \(\prec\) in which every pair of elements has a common upper bound. An **inverse system** of sets indexed by \(I\) consists of a set \(X_i\) for each \(i \in I\) and a **transition map** \(\varphi_{ij}: X_j \to X_i\) for all \(i, j \in I\) such that \(i \prec j\). We require that transition maps satisfy the following compatibility conditions:

\[
\varphi_{ii} = \text{Id}_{X_i} \quad \text{for all } i \in I, \\
\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \quad \text{for all } i, j, k \in I \text{ with } i \prec j \prec k.
\]

When the index set and the transition maps are clear, we simply say an inverse system \((X_i)_{i \in I}\).

The **inverse limit** of an inverse system \((X_i)_{i \in I}\) with transition maps \(\varphi_{ij}: X_j \to X_i\) is defined as the following subset of \(\prod_{i \in I} X_i\):

\[
\lim_{\leftarrow i \in I} (X_i, \varphi_{ij}) = \lim_{\leftarrow i \in I} X_i \subset \prod_{i \in I} X_i
\]

which consists of all \((x_i)_{i \in I}\) such that \(\varphi_{ij}(x_j) = x_i\) for all \(i \prec j\).

In this paper, we shall use repeatedly the following lemma which gives a sufficient condition for the nonemptiness of an inverse system.

**Lemma 2.4:** Let \(K\) be an algebraically closed field. Let \((X_i, f_{ij})\) be an inverse system indexed by a countable directed set \((I, \prec)\), where each \(X_i\) is a nonempty \(K\)-algebraic variety and each transition map \(f_{ij}: X_j \to X_i\) is an algebraic morphism such that \(f_{ij}(X_j) \subset X_i\) is a Zariski closed subset for all \(i \prec j\). Then

\[
\lim_{\leftarrow i \in I} X_i \neq \emptyset.
\]

**Proof.** The statement is proved in [26, Proposition 4.2]. The hypothesis that each \(X_i\) is a \(K\)-algebraic variety is essential. It implies that each \(X_i\) is quasi-compact with respect to the Zariski topology. Hence, the image in \(X_i\) under every algebraic morphism \(X_j \to X_i\) of the intersection of every decreasing sequence of closed subsets of \(X_j\) is equal to the intersection of the images of these closed subsets in \(X_i\). This allows us to construct an induced inverse subsystem of \((X_i, f_{ij})\) satisfying the Mittag-Leffler condition, whence the nonemptiness of \(\lim_{\leftarrow i \in I} X_i\). \(\blacksquare\)
2.4. **Inverse limits and closed subsets in the prodiscrete topology.**

Let \( G \) be a countable set and let \((E_n)_{n \geq 0}\) be an increasing sequence of finite subsets of \( G \) such that \( \bigcup_{n \geq 0} E_n = G \). Let \( A \) be a set and let \( \Sigma \subset A^G \) be a subset. For every \( m \geq n \geq 0 \), the inclusion \( E_n \subset E_m \) induces a canonical projection \( A^{E_m} \to A^{E_n} \) which in turn induces a well-defined map \( \pi_{nm}: \Sigma_{E_m} \to \Sigma_{E_n} \).

Thus, we obtain an inverse system \((\Sigma_{E_n})_{n \geq 0}\) with transition maps \( \pi_{nm} \) for \( m \geq n \geq 0 \). We have the following useful approximation result (cf. [6, Section 4]):

**Lemma 2.5:** With the above notations, suppose that \( \Sigma \) is closed in \( A^G \) with respect to the prodiscrete topology. Then we have a canonical bijection

\[
\lim_{\leftarrow n \geq 0} \Sigma_{E_n} = \Sigma.
\]

**Proof.** For every \( x \in \Sigma \), we have \( x_n := x|_{E_n} \in \Sigma_{E_n} \) and we can clearly associate an element \( (x_n)_{n \geq 0} \in \lim_{\leftarrow n \geq 0} \Sigma_{E_n} \).

Conversely, let \( x = (x_n)_{n \geq 0} \in \lim_{\leftarrow n \geq 0} \Sigma_{E_n} \). Then \( x_n \in \Sigma_{E_n} \) for every \( n \geq 0 \).

As the transition maps \( \pi_{nm} \) are simply projection maps and as \( \bigcup_{n \geq 0} E_n = G \), we can associate a well-defined configuration \( x \in A^G \) given by

\[
x(g) := x_n(g)
\]

for every \( g \in G \) and every \( n \geq 0 \) large enough such that \( g \in E_n \). Thus

\[
x|_{E_n} = x_n \in \Sigma_{E_n}
\]

for every \( n \geq 0 \). Since every finite subset of \( G \) is contained in some set \( E_n \) for some \( n \geq 0 \), we deduce that \( x \) belongs to the closure of \( \Sigma \) in \( A^G \) with respect to the prodiscrete topology. The closedness of \( \Sigma \) then implies that \( x \in \Sigma \).

It is obvious that the above operations are mutually inverse. The proof is completed. \( \blacksquare \)

3. **Restriction and closedness property**

In this section, we show that under suitable algebraic assumptions, closedness in the prodiscrete topology is stable under restriction to arbitrary subsets.

Suppose that \( G \) is a countable set. Let \( A \) be a set and let \( \Sigma \) be a closed subset of \( A^G \) with respect to the prodiscrete topology. Let \( K \) be an algebraically closed field.
Lemma 3.1: With the above notations and hypotheses, suppose in addition that $\Sigma$ satisfies the following property:

(P) for every finite subset $E \subset G$, $\Sigma_E$ is a $K$-algebraic group and for all finite subset $F \subset E$, the induced projection $\pi_{F,E}: \Sigma_E \to \Sigma_F$ is a homomorphism of $K$-algebraic groups.

Then for every subset $H \subset G$, the restriction $\Sigma_H$ is also closed in $A^H$ with respect to the prodiscrete topology.

Proof. Since $G$ is countable, there exists an increasing sequence $(E_n)_{n \geq 0}$ of finite subsets of $G$ such that $G = \bigcup_{n \geq 0} E_n$. For every $n \geq 0$ denote $F_n := H \cap E_n$, then clearly $H = \bigcup_{n \geq 0} F_n$.

Let $d \in A^H$ belong to the closure of $\Sigma_H$ in $A^H$ with respect to the prodiscrete topology. Then, for every $n \geq 0$, we have $d|_{F_n} \in \Sigma_{F_n}$. But since $F_n \subset E_n$ and $\Sigma_{F_n} = (\Sigma_{E_n})_{F_n}$, we obtain for every $n \geq 0$ a nonempty subset of $\Sigma_{E_n}$ as follows:

$$Z_n := \{ x \in \Sigma_{E_n} : x|_{F_n} = d|_{F_n} \}.$$ 

It is clear that for every $m \geq n \geq 0$, the restriction of $\pi_{E_n,E_m}$ to $Z_m$ induces a well-defined map $p_{nm}: Z_m \to Z_n$. Since $\pi_{E_n,F_n}: \Sigma_{E_n} \to \Sigma_{F_n}$ is a homomorphism of $K$-algebraic groups by the hypothesis (P), it follows that $Z_n = \pi_{E_n,F_n}^{-1}(d|_{F_n})$ is a translate of an algebraic subgroup of $\Sigma_{E_n}$. Therefore, for every $m \geq n \geq 0$, the transition map $p_{nm}$ of the inverse system $(Z_n)_{n \geq 0}$ is a morphism of algebraic varieties and $p_{nm}(Z_m)$ is Zariski closed in $Z_n$.

Hence, Lemma 2.4 applied to the inverse system $(Z_n)_{n \geq 0}$ tells us that there exists $x \in \varprojlim_{n \geq 0} Z_n \subset A^G$. We infer from the definition of the sets $Z_n$ that

$$x|_{F_n} = d|_{F_n}$$

for every $n \geq 0$. This shows that $x|_H = d \in A^H$. As $\Sigma$ is closed in $A^G$ and $\bigcup_{n \geq 0} E_n = G$, we find that

$$\varprojlim_{n \geq 0} Z_n \subset \varprojlim_{n \geq 0} \Sigma_{E_n} = \Sigma$$

(cf. Lemma 2.5). It follows that $x \in \Sigma$. Since $d = x|_H$, we conclude that $d \in \Sigma_H$ and thus $\Sigma_H$ is closed in $A^H$ with respect to prodiscrete topology. 

Using standard properties of complete algebraic varieties, the above proof can be easily adapted to show the following similar result for algebraic subshifts.
Lemma 3.2: With the above notations and hypotheses, assume moreover that $\Sigma$ satisfies the following condition:

(Q) for every finite subset $E \subset G$, $\Sigma_E$ is a complete $K$-algebraic variety and for all finite subset $F \subset E$, the projection $\pi_{F,E}: \Sigma_E \to \Sigma_F$ is a morphism of $K$-algebraic varieties.

Then for every subset $H \subset G$, the restriction $\Sigma_H$ is also closed in $A^G$ with respect to the prodiscrete topology. 

4. Algebraic subshifts and algebraic group subshifts

We begin with an observation that all algebraic group subshifts (cf. Definition 1.2) are automatically abstract subgroups.

Proposition 4.1: Let $G$ be a countable group and let $V$ be an algebraic group over an algebraically closed field. Let $\Sigma$ be an algebraic group subshift of $V^G$. Then $\Sigma$ is an abstract subgroup of $V^G$.

Proof. Since $G$ is countable, we can find an increasing sequence $(E_n)_{n \geq 0}$ consisting of finite subsets of $G$ such that $\bigcup_{n \geq 0} E_n = G$.

Let $\varepsilon \in V$ be the neutral element and we denote multiplicatively the group law of $V$. It follows from the definition of algebraic group subshifts that for every $n \geq 0$, the restriction $\Sigma_{E_n}$ is a subgroup of $V^{E_n}$. Hence, $\varepsilon^{E_n} \in \Sigma_{E_n}$ for every $n \geq 0$. Since $\Sigma$ is closed in $V^G$ with respect to the prodiscrete topology, we deduce immediately from Lemma 2.5 that

$$\varepsilon^G = (\varepsilon^{E_n})_{n \geq 0} \in \lim_{\leftarrow} \Sigma_{E_n} = \Sigma.$$

Now let $x, y \in \Sigma$. Then clearly $(x^{-1}y)|_{E_n} = (x^{-1})|_{E_n} y|_{E_n} \in \Sigma_{E_n}$ for every $n \geq 0$. Then as above, Lemma 2.5 also implies that $x^{-1}y \in \Sigma$. We conclude that $\Sigma$ is indeed an abstract subgroup of $V^G$. 

The next lemma shows that inductions of algebraic group subshifts are also algebraic group subshifts.

Lemma 4.2: Let $H$ be a subgroup of a group $G$. Let $V$ be an algebraic group over an algebraically closed field and let $\Lambda$ be an algebraic group subshift of $V^H$. Then $\Sigma := \Sigma(V^G, H, \Lambda)$ is an algebraic group subshift of $V^G$. 

Proof. With respect to the prodiscrete topology, as $\Lambda$ is closed in $A^H$, Lemma 2.3 implies that $\Sigma$ is also closed in $A^G$.

As in Lemma 2.3, we have $\Lambda = \Sigma_H$ and there is a canonical factorization

$$\Sigma = \prod_{c \in G/H} \Sigma_c, \quad x \mapsto (x|_c)_{c \in G/H}. \quad (4.1)$$

Let $E \subset G$ be a finite subset. Then (4.1) induces a factorization

$$\Sigma_E = \prod_{c \in G/H} \Sigma_{c \cap E},$$

For each $c \in G/H$, choose $h_c \in G$ such that $h_cc = H$. Observe that whenever $c \cap E \neq \emptyset$,

$$h_{c^{-1}}\Sigma_{c \cap E} = \Sigma_{h_c(c \cap E)} = (\Sigma_H)_{h_c(c \cap E)} = \Lambda_{h_c(c \cap E)} \subset V^{h_c(c \cap E)}.$$

Since $h_c(c \cap E)$ is a finite subset of $H$ and since $\Lambda$ is an algebraic group subshift, we deduce that $\Lambda_{h_c(c \cap E)}$ is an algebraic subgroup of $V^{h_c(c \cap E)}$ thus $\Lambda_{c \cap E}$ is also an algebraic subgroup of $V^{c \cap E}$. This implies that $\Sigma_E = \prod_{c \in G/H} \Sigma_{c \cap E}$, where $c$ runs over $G/H$ such that $c \cap E \neq \emptyset$, is indeed an algebraic subgroup of $V^E$. The conclusion follows.

The following results are direct generalizations of [6, Theorem 10.1], [6, Theorem 8.1], and [6, Theorem 7.1] in the context of algebraic group subshifts over a countable universe.

**Theorem 4.3:** Let $G$ be a countable group. Let $V$ be an algebraic group over an algebraically closed field and let $\Sigma \subset V^G$ be a subshift. Consider the following properties:

(a) $\Sigma$ is a subshift of finite type;
(b) $\Sigma$ is an algebraic group subshift of finite type;
(c) every descending sequence of algebraic group subshifts of $V^G$

$$\Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_n \supset \Sigma_{n+1} \supset \cdots$$

such that $\bigcap_{n \geq 0} \Sigma_n = \Sigma$ eventually stabilizes.

Then we have (b) $\implies$ (a) $\implies$ (c). Moreover, if $\Sigma \subset A^G$ is an algebraic group subshift, then (a) $\iff$ (b) $\iff$ (c).

**Proof.** The proof is the same, mutatis mutandis, as the proof given in [6, Theorem 10.1]. First, one remarks that the key local result [6, Theorem 7.1] now becomes part of the definition of algebraic group subshifts. Second, the only
modification needed to weaken the finite generation condition on the group $G$ in [6, Theorem 7.1] by the countability assumption on $G$ is the following. Since $G$ is countable, we can find an increasing sequence $(M_i)_{i \geq 0}$ of finite subsets of $G$ such that $1_G \in M_0$ and $\bigcup_{i \geq 0} M_i = G$. Then it suffices to replace everywhere the finite subset $M^i$ in the proof of [6, Theorem 10.1] by the finite subset $M_i$ for every $i \geq 0$.

Likewise, the fundamental closed image property of an algebraic cellular automaton also holds for endomorphisms of algebraic group subshifts.

**Theorem 4.4:** Let $G$ be a countable group. Let $U, V$ be algebraic groups over an algebraically closed field. Let $\Sigma$ be an algebraic group subshift of $U^G$ and let $\tau \in \text{Hom}_{U,V,G,\text{-alg}}(U^G, V^G)$. Then $\tau(\Sigma)$ is closed in $V^G$ with respect to the prodiscrete topology.

**Proof.** With a similar modification indicated in the proof of Theorem 4.3, the proof of Theorem 4.4 is, *mutatis mutandis*, the same as the proof of [6, Theorem 8.1]. We leave the details to the readers.

A similar and straightforward extension to a countable universe of [6, Theorem 7.1] together with Theorem 4.4 give us the following result.

**Theorem 4.5:** Let $G$ be a countable group and let $V$ be an algebraic group over an algebraically closed field. Suppose that $\Sigma$ is an algebraic group sofic subshift of $V^G$. Then $\Sigma$ is an algebraic group subshift of $V^G$.

We now introduce algebraic subshifts which will be necessary for the statement of Theorem 10.2. The definition is analogous to Definition 1.2 of algebraic group subshifts given in the Introduction.

**Definition 4.6:** Let $G$ be a group and let $V$ be an algebraic variety over an algebraically closed field. A closed subshift $\Sigma \subset V^G$ is called an algebraic subshift of $V^G$ if for every finite subset $E \subset G$, the restriction $\Sigma_E \subset V^E$ is an algebraic subvariety.

It is clear that all algebraic group subshifts are algebraic subshifts. Moreover, it follows from [6, Theorem 7.1] and Theorem 4.4 that whenever all the alphabets involved are complete algebraic varieties over the same algebraically closed field, all algebraic sofic subshifts and thus all algebraic subshifts of finite type defined
in [6] are algebraic subshifts. As for algebraic group subshifts discussed above, the universe can be taken to be countable and not necessarily finite generated.

We remark that there exist algebraic subshifts which are not of finite type, e.g., the even subshift (cf. [19, Example 3.14]).

We have the following useful lemma which tells us that algebraic group subshifts and certain algebraic subshifts satisfy the properties (P) and (Q) introduced in Lemma 3.1 and in Lemma 3.2.

**Lemma 4.7:** Let $G$ be a group and let $V$ be an algebraic group (resp. a complete algebraic variety) over an algebraically closed field. Let $\Sigma$ be an algebraic group subshift (resp. an algebraic subshift) of $V^G$. Let $E \subset F$ be finite subsets of $G$. Then the restriction to $\Sigma_F$ of the canonical projection $V^F \to V^E$ induces a well-defined homomorphism of algebraic groups (resp. morphism of algebraic varieties) $\Sigma_F \to \Sigma_E$.

**Proof.** The verifications are straightforward using Definition 4.6. We only recall the fact that if a map between algebraic groups is a homomorphism of abstract groups which is also a morphism of algebraic varieties, then it is actually a homomorphism of algebraic groups. \hfill \blacksquare

5. Intersection of algebraic group subshifts

It is a straightforward verification that the intersection of finitely many algebraic group subshifts is also an algebraic group subshift. When the universe is a countable group, we will show that the intersection of every descending sequence of algebraic group subshifts is again an algebraic group subshift.

**Theorem 5.1:** Let $G$ be a countable group. Let $V$ be an algebraic group over an algebraically closed field. Suppose that $(\Sigma_n)_{n \geq 0}$ is a descending sequence of algebraic group subshifts of $V^G$. Then $\Sigma := \bigcap_{n \geq 0} \Sigma_n$ is an algebraic group subshift of $V^G$.

**Proof.** Since $\Sigma_n$ is a closed subshift of $V^G$ for all $n \geq 0$, so is the intersection $\Sigma$. Let $E \subset G$ be a finite subset. By definition, $(\Sigma_n)_E \subset V^E$ is an algebraic subgroup for every $n \geq 0$. We clearly have $\Sigma_E \subset (\Sigma_n)_E$ for every $n \geq 0$ so $\Sigma_E \subset \bigcap_{n \geq 0} (\Sigma_n)_E$. 
For the converse conclusion, let \( z \in \bigcap_{n \geq 0} (\Sigma_n)_E \). Since \( G \) is countable, we can find an increasing sequence \( (M_n)_{n \geq 0} \) of finite subsets of \( G \) such that

\[
\{1_G\} \cup E \subset M_0
\]

and \( \bigcup_{n \geq 0} M_n = G \).

Consider the inverse system \((\Sigma_{ij})_{i,j \geq 1}\) defined by \( \Sigma_{ij} := (\Sigma_j)_{M_i} \subset V^{M_i} \) for every \( i, j \geq 1 \). Note that \( \Sigma_{i,j+1} \subset \Sigma_{ij} \) since \( \Sigma_{j+1} \subset \Sigma_j \) for all \( i, j \geq 1 \). Moreover, every \( \Sigma_{ij} \) is an algebraic subgroup of \( V^{M_i} \) since every \( \Sigma_j \) is an algebraic group subshift of \( V^G \).

The **unit horizontal transition maps** of the inverse system are the canonical homomorphisms of algebraic groups \( p_{ij} : \Sigma_{i+1,j} \to \Sigma_{ij} \) defined by

\[
p_{ij}(x) = x|_{M_i}
\]

for every \( x \in \Sigma_{i+1,j} \) as \( M_i \subset M_{i+1} \) (cf. Lemma 4.7). The **unit vertical transition maps** \( q_{ij} : \Sigma_{i,j+1} \to \Sigma_{ij} \) are simply defined as the inclusion homomorphisms.

The directed set \( I = \{1, 2, \ldots\}^2 \) is partially ordered by \((u, v) \prec (i, j)\) if and only if \( u \leq i \) and \( v \leq j \). The transition maps of the inverse system \((\Sigma_{ij})_{(i,j) \in I}\) are the compositions of the unit transition maps and they are well-defined as it can be checked easily that for all \((i, j) \in I:\)

\[
q_{ij} \circ p_{i,j+1} = p_{ij} \circ q_{i+1,j}
\]

(see also [6, Section 4]).

As previously said, \( p_{ij} \) and \( q_{ij} \) are homomorphisms of algebraic groups. Hence the transition maps of the inverse system \((\Sigma_{ij})_{(i,j) \in I}\) are homomorphisms of algebraic groups.

Consider the inverse subsystem \((X_{ij})_{(i,j) \in I}\) of \((\Sigma_{ij})_{(i,j) \in I}\) defined by

\[
X_{ij} := \{x \in (\Sigma_j)_{M_i} : x|_E = z\} \subset \Sigma_{ij}
\]

with \( z \in \bigcap_{n \geq 0} (\Sigma_n)_E \) chosen at the beginning of the proof. Therefore, for every \((i, j) \in I\), we can find \( z_j \in \Sigma_j \) such that \((z_j)|_E = z\). It follows that \((z_j)|_{M_i} \in X_{ij}\) and \( X_{ij} \) is thus nonempty. Consider the homomorphism of algebraic groups \( \pi_{ij} : \Sigma_{ij} \to \Sigma_E \) which is the restriction to \( \Sigma_{ij} \) of the canonical projection \( V^{M_i} \to V^E \). Then clearly \( X_{ij} = z_j|_{M_i} \Ker(\pi_{ij}) \) is a translate of an algebraic subgroup of \( \Sigma_{ij} \) for every \((i, j) \in I\). Hence, it follows from the above paragraph that the transition maps of the inverse subsystem \((X_{ij})_{(i,j) \in I}\) have Zariski closed images.
Therefore, Lemma 2.4 applied to the inverse subsystem \((X_{ij})_{(i,j) \in I}\) tells us that \(\lim_{(i,j) \in I} X_{ij}\) is nonempty.

For every fixed \(j \geq 1\), Lemma 2.5 implies that

\[
\lim_{i \geq 1} X_{ij} \subseteq \lim_{i \geq 1} \Sigma_{ij} = \Sigma_j.
\]

Hence, we can easily deduce that

\[
\lim_{(i,j) \in I} X_{ij} = \lim_{j \geq 1} \lim_{i \geq 1} X_{ij} = \lim_{j \geq 1} \Sigma_j = \bigcap_{j \geq 1} \Sigma_j =: \Sigma.
\]

It follows that there exists \(y \in \Sigma\) such that \(y \in \lim_{(i,j) \in I} X_{ij}\). It is immediate from the construction that \(y|_E = z\). Therefore,

\[
\bigcap_{n \geq 0} (\Sigma_n)_E \subset \Sigma_E
\]

and we have \(\Sigma_E = \bigcap_{n \geq 1} (\Sigma_n)_E \subset V^E\).

Since \(((\Sigma_n)_E)_{n \geq 1}\) is a descending sequence of algebraic subgroups of \(V^E\), it eventually stabilizes by the Noetherianity of the Zariski topology. Hence, \(\Sigma_E\) is also an algebraic subgroup of \(V^E\). This proves that \(\Sigma\) is indeed an algebraic group subshift of \(V^G\). The proof of Theorem 5.1 is completed.

6. Groups of algebraic Markov type

In this section, the equivalence between the descending chain condition for the full shifts and the finite type property of algebraic group subshifts is established when the universe is countable (Proposition 6.1). This leads us to define groups of algebraic Markov type (Definition 6.2) as an extension of the class of groups of Markov type [31, Definition 4.1].

6.1. Descending chain condition and finite type property. As an application of Theorem 5.1, we obtain the following result similar to [31, Theorem 3.8]:
Proposition 6.1: Let $G$ be a countable group and let $V$ be an algebraic group over an algebraically closed field. The following are equivalent:

(a) every algebraic group subshift of $V^G$ is an algebraic group subshift of finite type;

(b) every descending sequence of algebraic group subshifts of $V^G$ eventually stabilizes.

Proof. Suppose first that (a) is verified. Let $(\Sigma_n)_{n \geq 0}$ be a descending sequence of algebraic group subshifts of $V^G$. Let $\Sigma := \bigcap_{n \geq 0} \Sigma_n$. By Theorem 5.1, we know that $\Sigma$ is an algebraic group subshift of $V^G$. The condition (a) then implies that $\Sigma$ is an algebraic group subshift of finite type of $V^G$. By Theorem 4.3, it follows that the sequence $(\Sigma_n)_{n \geq 0}$ eventually stabilizes. This shows that $(a) \implies (b)$.

Conversely, assume that (b) is satisfied. Let $\Sigma$ be an algebraic group subshift of $V^G$. As $G$ is countable, there exists an increasing sequence $(E_n)_{n \geq 0}$ of finite subsets of $G$ such that $G = \bigcup_{n \geq 0} E_n$.

Then for every $n \geq 0$, we find that $\Sigma_{E_n}$ is an algebraic subgroup of $V^{E_n}$. Since $E_n$ is also finite, we deduce that $\Sigma_n := \Sigma(V^G; E_n, \Sigma_{E_n})$ is an algebraic group subshift of finite type of $V^G$.

It is clear that $\Sigma \subset \Sigma_{n+1} \subset \Sigma_n$ for every $n \geq 0$. In particular, $\Sigma \subset \bigcap_{n \geq 0} \Sigma_n$. On the other hand, let $z \in \bigcap_{n \geq 0} \Sigma_n$. Then by definition of the subshifts $\Sigma_n$, we have $z|_{E_n} \in \Sigma_{E_n}$ for every $n \geq 0$. By Lemma 4.7, we can regard $(\Sigma_{E_n})_{n \geq 0}$ as an inverse system whose transition maps are given by the canonical homomorphisms $\Sigma_{E_m} \to \Sigma_{E_n}$ for every $m \geq n \geq 0$. Since $\Sigma$ is closed, Lemma 2.5 implies that $z \in \varprojlim_{n \geq 0} \Sigma_{E_n} = \Sigma$ and hence $\bigcap_{n \geq 0} \Sigma_n \subset \Sigma$.

We conclude that $\Sigma = \bigcap_{n \geq 0} \Sigma_n$. But then (b) implies that the descending sequence $(\Sigma_n)_{n \geq 0}$ must stabilize. Hence, there exists $N \geq 0$ such that $\Sigma = \bigcap_{n \geq 0} \Sigma_n = \Sigma_N$ which is an algebraic group subshift of finite type of $V^G$. Hence $(b) \implies (a)$ and the proof is completed.

6.2. The Class of Groups of Algebraic Markov Type. By analogy with the definition of groups of Markov type given in [31, Definition 4.1], we introduce the class of groups of algebraic Markov type as follows.

Definition 6.2: A countable group $G$ said to be of algebraic Markov type if for every algebraic group $V$ over an algebraically closed field, the full shift $V^G$ satisfies one of the equivalent conditions of Proposition 6.1.
By the Noetherianity of the Zariski topology, it is clear from the above definition that every finite group is a group of algebraic Markov type. We are going to show that the class of groups of algebraic Markov type is stable under taking subgroups.

**Proposition 6.3:** Let $G$ be a group of algebraic Markov type. Then every subgroup of $G$ is also a group of algebraic Markov type.

**Proof.** Let $H$ be a subgroup of $G$ and let $V$ be an algebraic group over an algebraically closed field. Let $(\Lambda_n)_{n \geq 0}$ be a descending sequence of algebraic group subshifts of $V^H$.

For every $n \geq 0$, we consider the induction subshift $\Sigma_n := \Sigma(V^G; H, \Lambda_n)$ which is an algebraic group subshift of $V^G$ by Lemma 4.2. Since $\Lambda_{n+1} \subset \Lambda_n$, it follows that $\Sigma_{n+1} \subset \Sigma_n$ for every $n \geq 0$. Hence, we obtain a descending sequence $(\Sigma_n)_{n \geq 0}$ of algebraic group subshifts of $V^G$.

Since $G$ is a group of algebraic Markov type by hypothesis, the sequence $(\Sigma_n)_{n \geq 0}$ eventually stabilizes. Since $(\Sigma_n)_H = \Lambda_n$ for every $n \geq 0$ by Lemma 2.3, it follows that the sequence $(\Lambda_n)_{n \geq 0}$ must also stabilize. By Proposition 6.1 and Definition 6.2, we can thus conclude that $H$ is a group of algebraic Markov type.

In Section 7, we will see that the extensions of cyclic groups by groups of algebraic Markov type are groups of algebraic Markov type (cf. Theorem 7.3 and Proposition 7.2).

7. Dynamical finiteness of algebraic group subshifts

Our goal in this section is to give a proof of the following result which is analogous to [31, Theorem 4.2].

**Theorem 7.1:** Every polycyclic-by-finite group is of algebraic Markov type.

By definition of groups of algebraic Markov type, we deduce immediately the main theorems mentioned in the Introduction.

**Proof of Theorem 1.3 and Theorem 1.6.** By Definition 6.2, it is a direct consequence of Theorem 7.1 and Proposition 6.1.

The proof of Theorem 7.1 will occupy the rest of the present section. It results from Theorem 7.3 in Section 7.2 and Proposition 7.2 in Section 7.1 below.
7.1. The case of extension by finite groups. The following proposition is a direct application of Lemma 2.2.

**Proposition 7.2:** Let \( 0 \to H \to G \xrightarrow{\varphi} F \to 0 \) be an extension of countable groups. Suppose that \( H \) is a group of algebraic Markov type and \( F \) is finite. Then the group \( G \) is also of algebraic Markov type.

**Proof.** Let \( V \) be an algebraic group over an algebraically closed field and let \( \Sigma \) be an algebraic group subshift of \( V^G \). We must show that \( \Sigma \) is an algebraic group subshift of finite type of \( V^G \).

Since \( \varphi \) is surjective, we can choose a finite subset \( E \subset G \) such that \( |E| = |F| \) and \( \varphi(E) = F \). Equivalently, \( E \) is a complete set of representatives of cosets of \( H \) in \( G \).

Let \( U = V^E \); then \( U \) is an algebraic group. Then by the discussion before Lemma 2.2, we have a canonical bijection \( U^H = V^G \): to each \( x \in U^H \), we associate an element \( y \in V^{HE} \) given by \( y(hk) := (x(h))(k) \) for every \( h \in H \) and \( k \in E \). It is clear that the above bijection commutes with the shift actions of the group \( H \).

Therefore, \( \Sigma \) can be regarded as an algebraic group subshift of \( U^H \) with the shift action of the group \( H \). Since \( H \) is of Markov type, we deduce that \( \Sigma \) is an algebraic group subshift of finite type of \( U^H \). Thus, there exists a finite subset \( D \in H \) and an algebraic subgroup \( P \subset U^D \) such that \( \Sigma = \Sigma(U^H; D, P) \).

By Lemma 2.2, we find that

\[
\Sigma = \Sigma(V^G; G, \Sigma) = \Sigma(V^G; HE, \Sigma(U^H; D, P)) = \Sigma(V^G; DE, P).
\]

Since \( DE \) is finite and \( P \) is an algebraic subgroup of \( U^D = V^{DE} \), we conclude that \( \Sigma \) is an algebraic group subshift of finite type of \( V^G \).

7.2. The case of infinite cyclic extension. We will now prove the main technical result of the paper which is an extension of [31, Lemma 4.4]. We remark that the proof of [31, Lemma 4.4] relies in a crucial way on the compactness of the alphabets and thus the compactness of the induced Tychonoff topology on the full shifts. However, in our setting, the full shift is equipped with the prodiscrete topology and is never compact unless when the underlying alphabet is finite.
Theorem 7.3: Let $0 \to H \to G \xrightarrow{\phi} Z \to 0$ be an extension of countable groups. Suppose that $H$ is a group of algebraic Markov type. Then the group $G$ is also of algebraic Markov type.

Proof. Let $V$ be an algebraic group over an algebraically closed field. Let $\Sigma \subset V^G$ be an algebraic group subshift. We must show that $\Sigma \subset V^G$ is an algebraic group subshift of finite type.

We denote by $\varepsilon$ the neutral element of $V$. The group operations on $V$ are written multiplicatively. Let $0_H = 0_G$ be the neutral element of the groups $H$ and $G$ whose group operations are denoted additively.

Since $\phi$ is surjective, we can fix $a \in G$ such that $\phi(a) = 1$. Then we have a decomposition of $G$ into disjoint cosets of $H$ in $G$:

$$G = \bigsqcup_{n \in \mathbb{Z}} Ha^n, \quad (7.1)$$

which defines a bijection $\Phi: H \times \mathbb{Z} \to G$ given by

$$\Phi((h, n)) = ha^n$$

for every $h \in H$ and $n \in \mathbb{Z}$. We regard $\Phi$ as a coordinate function for $G$.

Since $H$ is countable, there exists an increasing sequence $(F_n)_{n \geq 1}$ of finite subsets of $H$ such that $0_H \in F_1$ and $H = \bigcup_{n \geq 1} F_n$.

For every integer $n \geq 1$, let us denote $G_n := \{ -n, \ldots, 0 \} \subset \mathbb{Z}$, $G^*_n := \{ -n, \ldots, -1 \} \subset \mathbb{Z}$ and $I_n := \{ -n, \ldots, n \} \subset \mathbb{Z}$.

We define

$$X_n := \{ x |_H : x \in \Sigma, x|_{\Phi(H \times G^*_n)} = \varepsilon^{\Phi(H \times G^*_n)} \} \subset V^H. \quad (7.2)$$

It is straightforward from the definition that $X_{n+1} \subset X_n$ and $X_n$ is an $H$-invariant subset of $V^H$ for every $n \geq 1$.

The steps of the proof of Theorem 7.3 can be summarized as follows:

**Step 1**: First, we will show (cf. Lemma 7.6) that the sequence $(X_n)_{n \geq 1}$ is a sequence of algebraic group subshifts of $V^H$. Then since $H$ is of algebraic Markov type, Theorem 4.3 implies that $(X_n)_{n \geq 1}$ stabilizes, i.e., there exists $N \geq 1$ such that $X_n = X_N$ for all $n \geq N$.

**Step 2**: Next, we prove that $\Sigma := \Sigma(V^G; \Omega, \Sigma_\Omega)$ for $\Omega := \Phi(H \times I_N)$ $\subset G$.

**Step 3**: To conclude, we use the assumption that $H$ is of algebraic Markov type to find a finite subset $D \subset H$ such that $\Sigma = \Sigma(V^G; \bar{D}, P)$ where $\bar{D} = \Phi(D \times I_N) \subset \Omega$ and $P = \Sigma_{\Phi(D \times I_N)}$. 
Lemma 7.4: Let $E \subset H$ be a finite subset. Then $(X_n)_E$ is an algebraic subgroup of $V^E$ for every $n \geq 1$.

Proof. Fix an integer $n \geq 1$. Since $E$ is finite, there exists an integer $k_0 \geq n$ such that $E \subset F_{k_0}$. Consider the following subset $Y \subset \Sigma_E$ defined by

$$(7.3) \quad Y := \bigcap_{k \geq k_0} Y_k,$$

where $Y_k := \{x|E: x \in \Sigma, x|_{\Phi(F_k \times G^*_n)} = \varepsilon^{\Phi(F_k \times G^*_n)}\}$.

Let $k \geq k_0$ be an integer. Since $\Sigma$ is an algebraic group subshift, $\Sigma_{\Phi(F_k \times G_n)}$ is an algebraic subgroup of $V^{\Phi(F_k \times G_n)}$. Let $\pi^* : \Sigma_{\Phi(F_k \times G_n)} \to \Sigma_{\Phi(F_k \times G^*_n)}$ and $\pi_E : \Sigma_{\Phi(F_k \times G_n)} \to \Sigma_E$ be the canonical homomorphisms induced respectively by the inclusions $\Phi(F_k \times G^*_n) \subset \Phi(F_k \times G_n)$ and $E \subset \Phi(F_k \times G_n)$ (cf. Lemma 4.7).

Then we find that $Y_k = \pi_E(\text{Ker}(\pi^*))$ which is clearly an algebraic subgroup of $\Sigma_E$. Therefore, (7.3) implies that $Y$ is the intersection of a descending sequence of algebraic subgroups of $\Sigma_E$ and thus of $V^E$. By the Noetherianity of the Zariski topology on $V^E$, it follows that $Y$ is an algebraic subgroup of $V^E$.

We are going to prove that $(X_n)_E = Y$. The inclusion $(X_n)_E \subset Y$ is immediate. For the converse inclusion, let $y \in Y$. We must show that there exists $x \in X_n$ such that $x|_E = y$. For every $k \geq k_0$, consider the following subset of $\Sigma_{\Phi(F_k \times I_k)}$:

$$(7.4) \quad Y_k(y) := \{x|_{\Phi(F_k \times I_k)}: x \in \Sigma, x|_E = y, x|_{\Phi(F_k \times G^*_n)} = \varepsilon^{\Phi(F_k \times G^*_n)}\}.$$

Since $y \in Y = \bigcap_{k \geq k_0} Y_k$, we can find for every $k \geq k_0$ a configuration $x_k \in \Sigma$ such that $y = x_k|_E$ and $x_k|_{\Phi(F_k \times G^*_n)} = \varepsilon^{\Phi(F_k \times G^*_n)}$. This shows that $x_k|_{\Phi(F_k \times I_k)} \in Y_k(y)$ for every $k \geq k_0$.

For every integer $k \geq k_0$, consider the canonical homomorphisms of algebraic groups $\psi^*_k : \Sigma_{\Phi(F_k \times I_k)} \to \Sigma_{\Phi(F_k \times G^*_n)}$ and $\phi_k : \Sigma_{\Phi(F_k \times I_k)} \to \Sigma_E$ (cf. Lemma 4.7). Then it is not hard to see that

$$(7.5) \quad Y_k(y) = x_k|_{\Phi(F_k \times I_k)}(\text{Ker}(\psi^*_k) \cap \text{Ker}(\phi_k)).$$
Hence, we find that $Y_k(y)$ is a translate by $x_k|\Phi(F_k \times I_k)$ of the algebraic subgroup $\text{Ker}(\psi_k^\ast) \cap \text{Ker}(\phi_k)$ of $\Sigma_F(F_k \times I_k)$ for every $k \geq k_0$.

The sequence $(Y_k(y))_{k \geq k_0}$ forms an inverse system of nonempty sets whose transition maps $Y_m(y) \to Y_k(y)$, where $m \geq k \geq k_0$, are the restrictions of the canonical homomorphisms $\Sigma_F(F_m \times I_m) \to \Sigma_F(F_k \times I_k)$ induced by the inclusions $\Phi(F_k \times I_k) \subset \Phi(F_m \times I_m)$.

Since $Y_m(y)$ is a translate of an algebraic subgroup of $\Sigma_F(F_m \times I_m)$, the transition maps $Y_m(y) \to Y_k(y)$ have Zariski closed images for all $m \geq k \geq k_0$.

Therefore, Lemma 2.4 implies that there exists $x \in \lim_{\leftarrow k \geq k_0} Y_k(y)$. By the construction of the set $Y_k(y)$, we find that $x|_E = y$ and that $x|\Phi(F_k \times G_n^\ast) = \varepsilon \Phi(F_k \times G_n^\ast)$ for every $k \geq k_0$.

Since $H = \bigcup_{k \geq k_0} F_k$, we deduce that $x|\Phi(H \times G_n^\ast) = \varepsilon \Phi(H \times G_n^\ast)$.

Note that

$$\lim_{\leftarrow k \geq k_0} Y_k(y) \subset \lim_{\leftarrow k \geq k_0} \Sigma_F(F_k \times I_k)$$

since $Y_k(y) \subset \Sigma_F(F_k \times I_k)$ for every $k \geq k_0$. On the other hand,

$$\lim_{\leftarrow k \geq k_0} \Sigma_F(F_k \times I_k) = \Sigma$$

by the closedness of $\Sigma$ in $V^G$ with respect to the prodiscrete topology and as

$$\bigcup_{k \geq n} \Phi(F_k \times I_k) = G$$

(cf. Lemma 2.5). It follows that $x \in \Sigma$.

Hence, by definition of $X_n$ (cf. (7.2)), it follows that $x \in X_n$. Since $x|_E = y$ as well, we deduce that $Y \subset (X_n)_E$.

We can thus conclude that $(X_n)_E = Y$ is an algebraic subgroup of $V^H$. This proves Lemma 7.4.

**Lemma 7.5:** For every integer $n \geq 1$, the subshift $X_n$ is closed in $V^H$ with respect to the prodiscrete topology.

**Proof.** Let us fix an integer $n \geq 1$. For every $k \geq n \geq 1$, let

$$X_{n,k} := \{x|\Phi(F_k \times I_k) : x \in \Sigma, x|\Phi(F_k \times G_n^\ast) = \varepsilon \Phi(F_k \times G_n^\ast)\} \subset \Sigma_F(F_k \times I_k).$$

In other words, $X_{n,k}$ is the kernel of the canonical homomorphism of algebraic groups $\Sigma_F(F_k \times I_k) \to \Sigma_F(F_k \times G_n^\ast)$ (cf. Lemma 4.7). It follows that $X_{n,k}$ is an algebraic subgroup of $\Sigma_F(F_k \times I_k)$ and thus of $V^\Phi(F_k \times I_k)$. 

For all $m \geq k \geq n$, the inclusion $\Phi(F_k \times I_k) \subset \Phi(F_m \times I_m)$ induces a projection $\pi_{km}: V^{\Phi(F_m \times I_m)} \subset V^{\Phi(F_k \times I_k)}$. If $x \in V^{\Phi(F_m \times I_m)}$ satisfies

$$x|_{\Phi(F_m \times G_n^\ast)} = \varepsilon^{\Phi(F_m \times G_n^\ast)}$$

then clearly $\pi_{km}(x)|_{\Phi(F_k \times G_n^\ast)} = \varepsilon^{\Phi(F_k \times G_n^\ast)}$. Hence, the restriction of $\pi_{km}$ to $X_{n,m}$ defines a homomorphism of algebraic groups $p_{km}: X_{n,m} \to X_{n,k}$.

We thus obtain an inverse system $(X_{n,k})_{k \geq n}$ whose transition maps $p_{km}$ are homomorphisms of algebraic groups for $m \geq k \geq n$.

Now suppose that $z \in V^H$ belongs to the closure of $X_n$ in $V^H$ with respect to the prodiscrete topology. Hence, by definition of $X_n$, there exists for each $k \geq n$ a configuration $y_k \in \Sigma$ such that $y_k|F_k = z|F_k$ and that $y_k|\Phi(H \times G_n^\ast) = \varepsilon^{\Phi(H \times G_n^\ast)}$.

For every $k \geq n$, consider the following subset of $\Sigma_{\Phi(F_k \times I_k)}$:

$$(7.6) \quad X_{n,k}(z) := \{x|_{\Phi(F_k \times I_k)}: x \in \Sigma, x|_{\Phi(F_k \times G_n^\ast)} = \varepsilon^{\Phi(F_k \times G_n^\ast)}, x|_{F_k} = z|F_k\}.$$  

Observe that $y_k|_{\Phi(F_k \times I_k)} \in X_{n,k}(z)$ for every $k \geq n$. A similar argument as in the proof of (7.5) shows that $X_{n,k}(z)$ is a translate of an algebraic subgroup of $\Sigma_{\Phi(F_k \times I_k)}$.

Therefore, we obtain an inverse subsystem $(X_{n,k}(z))_{k \geq n}$ of $(X_{n,k})_{k \geq n}$ where every $X_{n,k}(z)$ is nonempty for $k \geq n$ and all the transition maps have Zariski closed images. Then, again by Lemma 2.4, there exists

$$x \in \lim_{k \geq n} X_{n,k}(z) \subset \lim_{k \geq n} \Sigma_{\Phi(F_k \times I_k)} = \Sigma.$$  

By construction, we find that $x|_{\Phi(F_k \times G_n^\ast)} = \varepsilon^{\Phi(F_k \times G_n^\ast)}$ and that $x|_{F_k} = z|F_k$ for every $k \geq n$. Thus, by letting $k \to \infty$, we obtain

$$x|_{\Phi(H \times G_n^\ast)} = \varepsilon^{\Phi(H \times G_n^\ast)} \quad \text{and} \quad x|_{H} = z.$$  

Hence, $z = x|_{H} \in X_n$ and this proves that $X_n$ is closed in $V^H$ with respect to the prodiscrete topology.

The proof of Lemma 7.5 is completed.  

Combining Lemma 7.4, Lemma 7.5 and the observations made after the definition (7.2) of $X_n$, we obtain the following which concludes Step 1:

**Lemma 7.6:** The sequence $(X_n)_{n \geq 1}$ is a descending sequence of algebraic group subshifts of $V^H$.  

STEP 2: Since \( H \) is of algebraic Markov type, Lemma 7.6 and Theorem 4.3 imply that the descending sequence \((X_n)_{n \geq 1}\) must stabilize and consist of algebraic group subshifts of finite type of \(V^H\). In particular, there exists \( N \geq 1 \) such that \( X_n = X_N =: X \) for every \( n \geq N \).

For the notations, let \( \mathbb{Z}_{<n} = \{ x \in \mathbb{Z} : x < n \} \) for every \( n \in \mathbb{Z} \); then we deduce from (7.2) that

\[
X = \bigcap_{n \geq 1} X_n = \{ x|_H : x \in \Sigma, x|_{\Phi(H \times \mathbb{Z}_{<0})} = \varepsilon^{\Phi(H \times \mathbb{Z}_{<0})} \}.
\]

Note that \( X \) is nonempty since it clearly contains \( \varepsilon^H \). It follows immediately from (7.7) that

\[
L := \{ x \in \Sigma : x|_{\Phi(H \times \mathbb{Z}_{<0})} = \varepsilon^{\Phi(H \times \mathbb{Z}_{<0})}, x|_H \in X \}
\]

\[
= \{ x \in \Sigma : x|_{\Phi(H \times \mathbb{Z}_{<0})} = \varepsilon^{\Phi(H \times \mathbb{Z}_{<0})} \}.
\]

For every \( v \in X \), consider the following subset of \( L \),

\[
L(v) := \{ x \in \Sigma : x|_{\Phi(H \times \mathbb{Z}_{<0})} = \varepsilon^{\Phi(H \times \mathbb{Z}_{<0})}, x|_H = v \}.
\]

Then by definition of \( L(v) \) and the relation (7.7), it is clear that \( L(v) \) is nonempty for every \( v \in X \).

Let \( \Omega := \Phi(H \times I_N) \subset G \). Consider the subshift \( \Sigma' := \Sigma(V^G; \Omega, \Sigma_\Omega) \) of \( V^G \) (see Definition (1.1)). It is clear that \( \Sigma \subset \Sigma' \). We are going to prove the converse inclusion.

Let \( y \in \Sigma' \) be a configuration. We aim to find a corresponding \( v \in X \) with certain properties. Note that \( a = \Phi((0_H, 1)) \in G \) and let

\[
g = (N + 1)a = \Phi((0_H, N + 1)) \in G.
\]

Then by definition of \( \Sigma' \), there exist \( z_0, z_1 \in \Sigma \) such that \( (z_0)|_\Omega = y|_\Omega \) and \( (z_1)|_{a + \Omega} = y|_{a + \Omega} \). It follows that for \( z = z_0(z_1)^{-1} \in \Sigma \), we have

\[
z|_{\Phi(H \times \{1, \ldots, N\})} = \varepsilon^{\Phi(H \times \{1, \ldots, N\})}.
\]

Now define \( v := ((-g)z)|_H \).

Recall that \( G^*_N := \{-N, \ldots, -1\} \). We deduce from (7.2) and the relation

\[
((-g)z)|_{\Phi(H \times \{-N, \ldots, -1\})} = \varepsilon^{\Phi(H \times \{-N, \ldots, -1\})} = \varepsilon^{\Phi(H \times G^*_N)}
\]

that \( v \in X_N \). Since \( X_N = X \), it follows that \( v \in X \).
Given this \( v \), let \( c \in L(v) \). Since the subshift \( \Sigma \) is \( G \)-invariant, we have \( gc \in \Sigma \) and the configuration \( x := (gc)^{-1}z_0 \in \Sigma \) satisfies

\[
x|_{\Phi(H \times \{0, \ldots, N+1\})} = y|_{\Phi(H \times \{0, \ldots, N+1\})}.
\]

As \( X = X_k \) for all \( k \geq N \), an immediate induction by a similar argument shows that there exists a sequence \((x_m)_{m>N} \subset \Sigma \) such that for every \( m > N \),

\[
x_m|_{\Phi(H \times \{0, \ldots, m\})} = y|_{\Phi(H \times \{0, \ldots, m\})}.
\]

We remark that any given finite subset of \( G \) is contained in some translate of the sets \( \Phi(H \times \{0, \ldots, m\}) \) for some \( m \geq 1 \). Consequently, the above paragraph shows that \( y \) belongs to the closure of \( \Sigma \) in \( V^G \) with respect to the prodiscrete topology. As \( \Sigma \) is closed in \( V^G \), it follows that \( y \in \Sigma \).

Therefore, \( \Sigma' \subset \Sigma \) and we conclude that \( \Sigma = \Sigma' = \Sigma(V^G; \Omega, \Sigma_\Omega) \).

**STEP 3:** We regard \( \Sigma_\Omega \) as a subshift of \( U^H \) with respect to the shift action given by the group \( H \) with the alphabet \( U := V^{\Phi(D \times I_N)} \) which is clearly an algebraic group.

As \( \Sigma \) is closed in \( V^G \) with respect to the prodiscrete topology, it follows from Lemma 3.1 that \( \Sigma_\Omega \) is also closed in \( U^H \). Moreover, as \( \Sigma \) is an algebraic group subshift, we deduce that \((\Sigma_\Omega)_{\Phi(E \times I_N)} \subset U^E \) is an algebraic subgroup for every finite subset \( E \subset H \). Therefore, \( \Sigma_\Omega \) is an algebraic group subshift of \( U^H \).

Since \( H \) is a group of algebraic Markov type, \( \Sigma_\Omega \) is an algebraic group subshift of finite type of \( U^H \). Hence, there exists a finite subset \( D \subset H \) such that \( \Sigma_\Omega = \Sigma(U^H; D, P) \) where

\[
P := (\Sigma_\Omega)_{\Phi(D \times I_N)} = \Sigma_{\Phi(D \times I_N)}
\]

is an algebraic subgroup of \( U^D = V^{\Phi(D \times I_N)} \).

Finally, it is straightforward from the above that

\[
\Sigma = \Sigma' = \Sigma(V^G; \Omega, \Sigma_\Omega)
= \Sigma(V^G; \Phi(H \times I_N), \Sigma(U^H; D, P))
= \Sigma(V^G; \Phi(D \times I_N), P)
\quad \text{(by Lemma 2.2}).
\]

Since \( \Phi(D \times I_N) \) is finite and \( P \) is an algebraic subgroup of \( V^{\Phi(D \times I_N)} \), we conclude that \( \Sigma \) is an algebraic group subshift of finite type of \( V^G \). The proof of Theorem 7.3 is complete.  \( \blacksquare \)
7.3. Application on groups of algebraic Markov type. As an immediate application of Theorem 7.3 and Proposition 7.2, we obtain the following property on the class of groups of algebraic Markov type.

**Corollary 7.7:** Let $0 \to H \to G \to C \to 0$ be an extension of countable groups. Suppose that $C$ is cyclic and $H$ is a group of algebraic Markov type. Then $G$ is also a group of algebraic Markov type. ■

**Proof of Theorem 7.1.** The argument below is standard and similar to the proof of [31, Theorem 4.2]. Suppose that $G$ is a countable, polycyclic-by-finite group. Then $G$ admits a subnormal series $G = G_n \supset G_{n-1} \supset \cdots \supset G_0 = \{1_G\}$ whose factors are cyclic groups. Since a trivial group is of algebraic Markov type (see the remark following Definition 6.2), an immediate induction using Theorem 7.3 and Proposition 7.2 shows that $G_0, \ldots, G_n = G$ are all of algebraic Markov type. The proof of Theorem 7.1 is completed. ■

8. Inverse images of homomorphisms of algebraic group subshifts

For the proof of Theorem 1.9, we remark that by Theorem 1.6, $\Sigma_1$ and $\Sigma_2$ are algebraic group subshifts of finite type of $U^G$ and $V^G$ respectively. By Theorem 4.5, the algebraic group sofic subshift $\tau(\Sigma_1)$ is in fact an algebraic group subshift of $V^G$. Hence it is also an algebraic group subshift of finite type of $V^G$ by Theorem 1.6.

The rest of the proof of Theorem 1.9 is a direct consequence of Theorem 1.6 and the following general result.

**Proposition 8.1:** Let $G$ be a countable group and let $U, V$ be algebraic groups over an algebraically closed field. Let $\Sigma$ be an algebraic group subshift of $V^G$ and let $\tau \in \text{Hom}_{U,V,G\text{-algr}}(U^G,V^G)$. Then $\tau^{-1}(\Sigma)$ is an algebraic group subshift of $U^G$.

**Proof.** Since $\tau$ is $G$-equivariant and is continuous with respect to the prodiscrete topology, $\tau^{-1}(\Sigma)$ is clearly $G$-invariant and is closed in $U^G$ with respect to the prodiscrete topology.

Since $\tau \in \text{Hom}_{U,V,G\text{-algr}}(U^G,V^G)$, there exists a finite subset $M \subset G$ and a homomorphism of algebraic groups $\mu : U^M \to V$ such that

$$\tau(x)(g) = \mu((g^{-1}x)|_M)$$

for all $x \in U^G$ and $g \in G$. (8.1)
We can assume that $1_G \in M$ since otherwise, we can simply replace $M$ by $M \cup \{1_G\}$ and compose $\mu$ with the canonical projection $U^{M \cup \{1_G\}} \rightarrow U^M$.

Let $F \subset G$ be finite subset. Then $F \subset FM$ and $\Sigma_F$ is an algebraic subgroup of $U^{FM}$ and we have a map (cf. [7, Section 3]):

$\tau^+_F : U^{FM} \rightarrow V^F$

defined by $\tau^+_F (c) = \tau(u)|_F$ for every $c \in U^{FM}$ and every $u \in U^G$ such that $u|_{FM} = c$. Then $\tau^+_F$ is a certain fibered product over $g \in F$ of the homomorphisms of algebraic groups $U^gM \rightarrow V(g)$ naturally induced by $\mu$. Hence, $\tau^+_F$ is a homomorphism of algebraic groups for every finite subset $F \subset G$.

Let $E \subset G$ be a fixed finite subset. Since $G$ is countable, there exists an increasing sequence $(E_n)_{n \geq 0}$ of finite subsets of $G$ containing $E$ and such that $G = \bigcup_{n \geq 0} E_n$.

For every $n \geq 0$, let $\psi_n : U^{E_nM} \rightarrow U^E$ be the canonical homomorphism of algebraic groups induced by the inclusion $E \subset E_nM$ (cf. Lemma 4.7). For every $n \geq 0$, we define respectively algebraic subgroups of $U^{E_nM}$ and $U^E$ by

$W_n := (\tau^+_n)^{-1}(\Sigma_{E_n}), \quad Z_n := \psi_n(W_n).$  

We claim that $(\tau^{-1}(\Sigma))_E = \bigcap_{n \geq 0} Z_n$. Indeed, if $z \in (\tau^{-1}(\Sigma))_E$ then there exists $x \in U^G$ such that $z = x|_E$ and $\tau(x) \in \Sigma$ thus $x|_{E_nM} \in W_n$ and $z = \psi_n(x|_{E_nM}) \in Z_n$ for all $n \geq 0$. Hence, $(\tau^{-1}(\Sigma))_E \subset \bigcap_{n \geq 0} Z_n$.

For the converse implication, let $z \in \bigcap_{n \geq 0} Z_n$. For every $n \geq 0$, we define $S_n := \psi_n^{-1}(z) \cap W_n$. Then $S_n$ is a translate of the algebraic subgroup $\text{Ker}(\psi_n) \cap W_n$. Since $z \in Z_n$, the set $S_n$ is nonempty for every $n \geq 0$.

We thus obtain an inverse system $(S_n)_{n \geq 0}$ whose transition maps are the restrictions to $S_m$ of the canonical homomorphism of algebraic groups $U^{E_nM} \rightarrow U^{E_mM}$ (cf. Lemma 4.7) for all $m \geq n \geq 0$.

By Lemma 2.4, there exists $y \in \varprojlim_{n \geq 0} S_n \subset U^G$. Since $y|_{E_nM} \in \psi_n^{-1}(z)$ for all $n \geq 0$, we find that $y|_E = z$. As $y|_{E_nM} \in W_n$ for every $n \geq 0$, we infer from (8.2) that $\tau(y)|_{E_n} = \tau^+_n(y|_{E_nM}) \in \Sigma_{E_n}$. Hence, it follows Lemma 2.5 that $\tau(y) \in \varprojlim_{n \geq 0} \Sigma_{E_n} = \Sigma$. We deduce that $(\tau^{-1}(\Sigma))_E \supset \bigcap_{n \geq 0} Z_n$. This proves the claim that $(\tau^{-1}(\Sigma))_E = \bigcap_{n \geq 0} Z_n$.

By the Noetherianity of the Zariski topology, $\bigcap_{n \geq 0} Z_n$ is an algebraic subgroup of $U^E$ and thus so is $(\tau^{-1}(\Sigma))_E$. We can thus conclude that $\tau^{-1}(\Sigma)$ is an algebraic group subsequhift of $U^G$. \[\square\]
9. Generalizations

In this section, we obtain axiomatic generalizations of our main results where the alphabets can now be taken for example as Artinian groups or Artinian modules over a ring.

9.1. Admissible Artinian group structures.

Definition 9.1: Let $\Gamma$ be a group. An **admissible Artinian structure** on $\Gamma$ is a sequence $\mathcal{H} = (\mathcal{H}_n)_{n \geq 1}$ where every $\mathcal{H}_n$ is a collection of subgroups of $\Gamma^n$ with the following stability properties:

1. $\{1\}, \Gamma \in \mathcal{H}_1$;
2. for every $m \geq n \geq 1$ and for every projection $\pi: \Gamma^m \to \Gamma^n$ induced by an injection $\{1, \ldots, n\} \to \{1, \ldots, m\}$, we have $\pi(H_m) \in \mathcal{H}_n$ and $\pi^{-1}(H_n) \in \mathcal{H}_m$ for every $H_m \in \mathcal{H}_m$ and $H_n \in \mathcal{H}_n$;
3. for every $n \geq 1$ and $H, K \in \mathcal{H}_n$, we have $H \cap K \in \mathcal{H}_n$;
4. for every $n \geq 1$, every descending sequence $(H_k)_{k \geq 0}$ of subgroups of $\Gamma^n$, where $H_k \in \mathcal{H}_n$ for every $k \geq 0$, eventually stabilizes.

In this case, we say that $(\Gamma, \mathcal{H})$, or simply $\Gamma$ when there is no possible confusion, is an **admissible Artinian group structure**. For every $n \geq 1$, elements of $\mathcal{H}_n$ are called **admissible subgroups** of $\Gamma^n$.

We have the following simple observation:

**Lemma 9.2:** Let $\Gamma$ be a group with an admissible Artinian structure $(\mathcal{H}_n)_{n \geq 1}$ and let $H \subset \Gamma$ be an admissible subgroup. Let $\mathcal{H}'_n := \{H^n \cap H_n : H_n \in \mathcal{H}_n\}$. The following holds:

1. for every $m \geq 1$, the group $\Gamma^m$ admits an admissible Artinian group structure given by $(\mathcal{H}_{mn})_{n \geq 1}$;
2. for every $H_k \in \mathcal{H}_n$, $H^n \cap H_k \in \mathcal{H}'_n$ is an admissible subgroup of $\Gamma^n$;
3. the group $H$ admits an induced admissible Artinian structure $(\mathcal{H}'_n)_{n \geq 1}$ given by $\mathcal{H}'_n$.

**Proof.** The point (i) is straightforward. For (ii), we first show by induction that $H^n \in \mathcal{H}_n$ for every $n \geq 1$. Indeed, this is true for $n = 1$ by assumption on $H$. Suppose that $H^k \in \mathcal{H}_k$ for some $k \geq 1$. Using the condition (2) for $\Gamma$, it is not hard to see that $A \Gamma (1, \ldots, k) \times \Gamma^{k+1} \in \mathcal{H}_{k+1}$.
and $B = \Gamma^\{1\} \times H^{\{2,\ldots,k+1\}} \in \mathcal{H}_{k+1}$. It follows from the condition (3) for $\Gamma$ that

$$H^{k+1} = H^{\{1,\ldots,k+1\}} = A \cap B \in \mathcal{H}_{k+1}.$$

Therefore, (ii) follows immediately also by the condition (3) for $\Gamma$.

For (iii), let us check the conditions in Definition 9.1. The conditions (1) and (3) are trivial. For (2), let $m \geq n \geq 1$ and let $\{1,\ldots,n\} \rightarrow \{1,\ldots,m\}$ be an injection. Let $\pi: \Gamma^m \rightarrow \Gamma^n$ and let $\pi_H: H^m \rightarrow H^n$ be the induced projections. Let $H_n \in \mathcal{H}_n$ and $H_m \in \mathcal{H}_m$. By (ii), $H^m \cap H_m \in \mathcal{H}_m$ and $H^n \cap H_n \in \mathcal{H}_n$. It follows from the condition (2) for $\Gamma$ and (ii) that

$$\pi_H(H^m \cap H_m) = \pi(H^m \cap H_m) \in \mathcal{H}_n \quad \text{and} \quad \pi_H^{-1}(H^n \cap H_n) = H^m \cap \pi_H^{-1}(H_n) \in \mathcal{H}_m.$$

Thus (2) is verified.

Thanks to (ii), it is clear that the condition (4) for $H$ is a direct consequence of the condition (4) for $\Gamma$.

**Example 9.3:** Let $\Gamma$ be a group equipped with a collection $A$ of its subgroups which is stable by taking intersection and such that $\{1\}, \Gamma \in A$ and every descending sequence of subgroups taken in $A$ eventually stabilizes. Then $A$ induces an admissible Artinian structure $\mathcal{H} = (\mathcal{H}_n)_{n \geq 1}$ on $\Gamma$ defined by $\mathcal{H}_n := A \times \ldots \times A$ ($n$-times) for every $n \geq 1$.

Note that $\mathcal{H}$ is the smallest admissible Artinian structure of $\Gamma$ such that every element of $A$ is an admissible subgroup of $\Gamma$.

**Definition 9.4:** Let the notations be as above. We say that $\mathcal{H} = (\mathcal{H}_n)_{n \geq 1}$ is the **induced product admissible Artinian structure** on $\Gamma$ by $A$.

**Example 9.5:** Every algebraic group $V$ over an algebraically closed field, resp. every compact Lie group $W$, admits a canonical admissible Artinian structure given by all algebraic subgroups of $V^n$, resp. by all closed subgroups of $W^n$, for $n \geq 1$.

**Example 9.6:** Recall that a group $\Gamma$ is **Artinian** if every descending sequence of subgroups of $\Gamma$ eventually stabilizes.

For example, finite groups are Artinian and for every prime number $p$, the subgroup

$$\mu_{p^n} := \{z \in \mathbb{C}^*: \exists n \geq 0, z^{p^n} = 1\}$$

of the multiplicative group $(\mathbb{C}^*, \times)$ is Artinian.
It is well-known that finite direct products of Artinian groups are Artinian 
(see, e.g., the proof of [17, Lemma 4.3]).

Then, it is clear that every Artinian group $\Gamma$ admits a canonical admissible 
Artinian structure given by all subgroups of $\Gamma^n$ for every $n \geq 1$.

**Example 9.7:** Similarly, every Artinian (left or right) module $M$ over a ring $R$ 
is equipped with a canonical admissible Artinian structure given by all $R$- 
submodules of $M^n$ for all $n \geq 1$.

**Definition 9.8:** Let $\Gamma$ and $\Gamma'$ be admissible Artinian group structures. An ad-
missible Artinian group structure $H$ on $\Gamma \times \Gamma'$ is said to be **compatible** with $\Gamma$ 
and $\Gamma'$ if the following holds:

\((\ast)\) for every integer $m \geq 1$ and for the canonical projections 
$p: (\Gamma \times \Gamma')^m \to \Gamma^m$

and

$p': (\Gamma \times \Gamma')^m \to (\Gamma')^m,$

and for all admissible subgroups $P \subseteq (\Gamma \times \Gamma')^m$, $Q \subseteq \Gamma^m$ and $Q' \subseteq (\Gamma')^m$, 
the groups $p(P)$, $p'(P)$ are respectively admissible subgroups of $\Gamma^m$ 
and $(\Gamma')^m$ and the groups $p^{-1}(Q)$, $(p')^{-1}(Q')$ are admissible subgroups 
of $(\Gamma \times \Gamma')^m$.

A homomorphism of abstract groups $\varphi: \Gamma \to \Gamma'$ is said to be **admissible**
if there exists a compatible Artinian group structure on $\Gamma \times \Gamma'$ such that the 
graph $\{(x, \varphi(x)) : x \in \Gamma\}$ is an admissible subgroup of $\Gamma \times \Gamma'$.

Several useful remarks are in order.

**Remark 9.9:** Homomorphisms of algebraic groups, homomorphisms of compact Lie groups, homomorphisms of Artinian groups, and morphisms of $R$-modules 
are all admissible with the canonical admissible Artinian structures of algebraic 
groups, compact Lie groups, Artinian groups, and $R$-modules respectively that 
are described in the above examples.

**Remark 9.10:** Given an admissible Artinian group structure $(\Gamma, \mathcal{H})$ as in Def-
inition 9.1, it is clear that the projections $\pi: \Gamma^m \to \Gamma^n$ induced by injections 
$\{1, \ldots, n\} \to \{1, \ldots, m\}$ are admissible homomorphisms of admissible Artinian group structures on $\Gamma^m$ and $\Gamma^n$ for every $m \geq n \geq 1$. Moreover, by Lemma 9.2, 
the restrictions of the canonical projections $\pi: \Gamma^m \to \Gamma^n$ to admissible sub-
groups of $\Gamma^m$ are also admissible homomorphisms.
9.2. Finiteness properties of admissible group shifts. We first extend both the definition of group shifts with finite group alphabets in the literature (cf. [10]) and Definition 1.2 of algebraic group subshifts to the case where the alphabets are admissible Artinian group structures as follows.

**Definition 9.11:** Let $G$ be a group. Let $A$ be an admissible Artinian group structure. A subshift $\Sigma \subset A^G$ is called an **admissible group subshift** if it is closed in $A^G$ with respect to the prodiscrete topology and if the restriction $\Sigma_E \subset A^E$ is an admissible subgroup for any finite subset $E \subset G$.

In the above definition and in what follows, the admissible Artinian structure of $A^E$ is induced by that of $A^{\{1,\ldots,|E|\}}$ via any bijection $\{1,\ldots,|E|\} \to E$. The bijection can be arbitrary because of Definition 9.1.(2) with $m = n$.

**Example 9.12:** Let $G$ be a group and let $A$ be an Artinian group (resp. an Artinian module over a ring $R$). Then $A^G$ is naturally an abstract group with pointwise group operations (resp. an $R$-module with pointwise group operations and with pointwise $R$-action). Let $\Sigma$ be an abstract subgroup (resp. an $R$-submodule) of $A^G$ which is also a closed subshift. Then $\Sigma$ is an admissible subshift of $A^G$ with respect to the canonical admissible Artinian structure on $A$ (cf. Example 9.6, Example 9.7).

Likewise, algebraic group subshifts are also admissible subshifts with respect to the canonical admissible Artinian structure on algebraic groups.

As an application of the proofs of our main theorems, we obtain the following general result:

**Theorem 9.13:** Let $G$ be a polycyclic-by-finite group and let $A$ be an admissible Artinian group structure. Then every admissible group subshift of $A^G$ is a subshift of finite type. Moreover, there exists a finite subset $D \subset G$ and an admissible subgroup $W$ of $A^D$ such that $\Sigma = \Sigma(A^G; D, W)$.

Before giving the proof of Theorem 9.13, we establish some key lemmata on inverse systems of admissible Artinian group structures.

**Lemma 9.14:** Let $\Gamma$ be an admissible Artinian group structure. Let $(X_n)_{n \geq 0}$ be a descending sequence of left translates of admissible subgroups of $\Gamma$. Then the sequence $(X_n)_{n \geq 0}$ eventually stabilizes.
Proof. By hypothesis, we can find a sequence of elements \((g_n)_{n \geq 0}\) of \(\Gamma\) and a sequence of admissible subgroups \((H_n)_{n \geq 0}\) of \(\Gamma\) such that \(X_n = g_nH_n\) for every \(n \geq 0\). Fix an integer \(n \geq 0\), it follows that \(g_{n+1}H_{n+1} \subset g_nH_n\). Then \(g_n^{-1}g_{n+1}H_{n+1} \subset H_n\) thus \(g_n^{-1}g_{n+1} \in H_n\) and therefore \(g_n^{-1}g_{n+1} = h_n \in H_n\). Hence, \((g_n^{-1}g_{n+1}H_n)_{n \geq 0}\) is a descending sequence of admissible subgroups of \(\Gamma\). Since \(\Gamma\) is an admissible Artinian group structure, there exists an integer \(N \geq 0\) such that \(H_n = H_N\) for all \(n \geq N\).

Finally, remark that if \(g, h \in \Gamma\) and \(H \subset \Gamma\) is a subgroup such that \(gH \subset hH\) then necessarily \(gH = hH\). Applying this remark to the sequence \((g_nH_n)_{n \geq N}\), we can thus conclude that \(X_n = X_N\) for all \(n \geq N\) and the proof is completed.

Lemma 9.15: Let \((\Gamma_i, \varphi_{ij})_{i, j \in I}\) be an inverse system indexed by a countable directed set \(I\), where every \(\Gamma_i\) is an admissible Artinian group structure and the transition maps \(\varphi_{ij} : \Gamma_j \to \Gamma_i\) are admissible homomorphisms for all \(i < j\). Suppose that \(X_i\), for every \(i \in I\), is a left translate of an admissible subgroup of \(\Gamma_i\) and that \(\varphi_{ij}(X_j) \subset X_i\) for all \(i < j\) in \(I\). Then the induced inverse subsystem \((X_i)_{i \in I}\) satisfies

\[
\lim_{\leftarrow \in I} X_i \neq \emptyset.
\]

Proof. The proof below is a straightforward modification of [26, Proposition 4.2] using the Mittag-Leffler condition argument. We give the details here for the sake of completeness.

First, since \(I\) is countable, we can suppose without loss of generality that \((I, \prec) = (\{n \in \mathbb{Z} : n \geq 0\}, \leq)\) as directed sets (see [26, Proposition 4.2]). For every integer \(n \geq 0\), we define \(Y_n \subset \Gamma_n\) by

\[
Y_n := \bigcap_{m \geq n} \varphi_{nm}(X_m).
\]

By hypotheses, we deduce that \((\varphi_{nm}(X_m))_{m \geq n}\) is a descending sequence of left translates of admissible subgroups of \(\Gamma_n\). Since \(\Gamma_n\) is an admissible Artinian group structure for every \(n \geq 0\), it follows from Lemma 9.14 that we can define, by an immediate induction on \(n\), an increasing sequence of integers \((k_n)_{n \geq 0}\) such that \(k_n \geq n\) and for all \(k \geq k_n\), we have \(Y_n = \varphi_{nk}(X_k) \neq \emptyset\).
For all integers \( m \geq n \geq 0 \), consider the induced maps
\[
f_{nm} := \varphi_{nm}|Y_m : Y_m \to Y_n
\]
and note from the choice above of the integers \( k_m \geq k_n \) that
\[
f_{nm}(Y_m) = \varphi_{nm}(\varphi_{mk_m}(X_k)) = \varphi_{nk_m}(X_k) = Y_n.
\]
We thus obtain the universal inverse system \((Y_n, f_{nm})\) associated with \((X_n, \varphi_{nm})\) in which the transition maps are surjective maps between nonempty spaces. Since the directed set is countable, it follows that \( \lim_{n \geq 0} Y_n \) is nonempty. Therefore, we conclude that \( \lim_{n \geq 0} X_n = \lim_{n \geq 0} Y_n \) is nonempty. The proof is completed.

Proof of Theorem 9.13. The proof is exactly the same, mutatis mutandis, as the proof of Theorem 7.1. It suffices to replace algebraic groups, algebraic subgroups, and homomorphisms of algebraic groups respectively by admissible Artinian group structures, admissible subgroups, and admissible homomorphisms of admissible Artinian group structures. Applications of Lemma 2.4 in the proof should also be replaced everywhere by applying Lemma 9.15 instead. We remark also that the Noetherianity of the Zariski topology on algebraic groups is now replaced by the descending chain condition in the definition of admissible Artinian group structures.

9.3. Generalizations of other main results. In fact, by repeating the exact same definitions and proofs in the present paper to admissible Artinian group structures instead of algebraic groups, as we did for Definition 9.11 and Theorem 9.13, all the results of the paper remain valid. We omit the details of similar straightforward verifications of the following principal results.

THEOREM 9.16: Let \( G \) be a countable group and let \( A \) be an admissible Artinian group structure. Let \( D \subset G \) be a finite subset and let \( P \subset A^D \) be an admissible subgroup. Then the subshift of finite type \( \Sigma(A^G; D, P) \) is an admissible group subshift of \( A^G \).

Proof. Similar to the proof of [6, Corollary 6.3].

Let the notations and hypotheses be as in Theorem 9.16; such a subshift \( \Sigma(A^G; D, P) \) is called an admissible group subshift of finite type of \( A^G \).
Proposition 9.17: Let $G$ be a countable group and let $A$ be an admissible Artinian group structure. The following are equivalent:

(a) every admissible group subshift of $A^G$ is an admissible group subshift of finite type;

(b) every descending sequence of admissible group subshifts of $A^G$ eventually stabilizes.

Proof. Similar to the proof of Proposition 6.1. \[\blacksquare\]

Corollary 9.18: Let $G$ be a polycyclic-by-finite group and let $A$ be an admissible Artinian group structure. Then every descending sequence of admissible group subshifts of $A^G$ eventually stabilizes.

Proof. It is a consequence of Proposition 9.17 and Theorem 9.13. \[\blacksquare\]

Now let $G$ be a group and let $A, B$ be admissible Artinian group structures.

Definition 9.19: A cellular automaton $\tau: A^G \to B^G$ is an admissible group cellular automaton if $\tau$ admits a memory set $M$ whose associated local defining map $\mu: A^M \to B$ is an admissible homomorphism of admissible Artinian group structures (cf. Definition 9.8).

In this case, consider an arbitrary finite subset $F \subset G$ and the induced map (cf. [7, Section 3])

$$\tau^+_F: A^{FM} \to B^F$$

given by $\tau^+_F(c) = \tau(x)|_F$ for every $c \in A^{FM}$ and $x \in A^G$ such that $x|_{FM} = c$. Then we have the following:

Lemma 9.20: $\tau^+_F$ is an admissible homomorphism of Artinian group structures.

Proof. Indeed, for every $c \in A^{FM}$, we have $\tau^+_F(c)(g) = \mu(g^{-1}c|_{FM})$ for every $g \in F$ where we recall that $g^{-1}c \in A^{g^{-1}FM}$ is defined by

$$(g^{-1}c)(h) := c(gh)$$

for every $h \in g^{-1}FM$. For each $g \in F$, let $\Gamma(g) \subset A^{gM} \times B^g$ be the graph of the map $\mu_g: A^{gM} \to B^g$ defined by

$$\mu_g(x) := \mu(g^{-1}x)$$

and let $\pi_g: A^{FM} \times B^F \to A^{gM} \times B^g$ be the canonical projection.
By Definition 9.8, there is an Artinian group structure $H$ of $A^M \times B$ compatible with the Artinian group structures of $A^M$ and $B$ so that the graph $\Gamma(g)$ is an admissible subgroup of $A^gM \times B^\{g\}$ for every $g \in F$.

We show that $H$ induces a compatible Artinian group structure on $A^{FM} \times B^F$ as follows. Let $\iota: FM \to F \times M$ be any set-theoretic injection. Let 

$$\pi: A^{F \times M} \times B^F \to A^{F \times M}$$

be the canonical projection. Then 

$$A^{FM} \times B^F = \pi^{-1}(A^{\iota(FM)} \times \{1_A\}^{F \times M \setminus \iota(FM)})$$

which is an admissible subgroup of $A^{F \times M} \times B^F$ by Property $(\ast)$ in Definition 9.8. Here, if 

$$\iota(FM) = F \times M,$$

then by convention, we set 

$$A^{\iota(FM)} \times \{1_A\}^{F \times M \setminus \iota(FM)} = A^{F \times M}.$$  

Hence, $H$ induces a compatible Artinian group structure on $A^{FM} \times B^F$ (cf. Lemma 9.2(ii)).

Then it is not hard to see that the graph of $\tau^+_F$ is the following intersection:

$$\Gamma_{\tau^+_F} = \bigcap_{g \in F} \pi^{-1}_g(\Gamma(g)).$$

On the other hand, it follows from Definition 9.8 that for every $g \in F$, $\pi^{-1}_g(\Gamma(g))$ is an admissible subgroup of $A^{FM} \times B^F$ (with both regarded as admissible subgroups of $A^{F \times M} \times B^F$). As intersections of admissible subgroups are also admissible subgroups, we deduce that $\Gamma_{\tau^+_F}$ is an admissible subgroup of $A^{FM} \times B^F$. Hence, $\tau^+_F$ is an admissible homomorphism.

**Theorem 9.21:** Let $G$ be a polycyclic-by-finite group. Let $A, B$ be admissible Artinian group structures. Let $\Sigma_1$ and $\Sigma_2$ be respectively admissible group subshifts of $A^G$ and $B^G$. Let $\tau: A^G \to B^G$ be an admissible group cellular automaton. Then $\tau^{-1}(\Sigma_2)$ and $\tau(\Sigma_1)$ are respectively admissible group subshifts of finite type of $A^G$ and $B^G$.

**Proof.** Similar to Theorem 1.9.
**Corollary 9.22:** Let $G$ be a polycyclic-by-finite group and let $A$ be an admissible Artinian group structure. Let $\Sigma$ be an admissible group subshift of $A^G$. Let $\psi: A^G \to A^G$ be an admissible group cellular automaton. Suppose that $\psi(\Sigma) \subset \Sigma$ and let $\tau: \Sigma \to \Sigma$ be the restriction map. Then the limit set $\Omega(\tau)$ is an admissible group subshift of finite type of $A^G$ and $\tau$ is stable.

**Proof.** Similar to the proof of Corollary 1.10.

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10. **Density of periodic configurations in algebraic subshifts and admissible group subshifts**

In this section, we obtain some generalizations to the algebraic setting of a known density result in the classical setting with finite alphabets.

We then give some direct consequences of our main results on the density of periodic configurations in algebraic group subshifts and more generally in admissible group subshifts. A counter-example is given at the end of the section. See also [27] for another application of our main results on the pseudo-orbit tracing property.

10.1. **Languages, words and irreducibility of subshifts.** Let $A$ be a finite set. We recall some useful terminologies concerning languages and words of a given subshift $\Sigma \subset A^Z$. Let $A^*$ be the free monoid generated by $A$. Elements of $A^*$ are finite words $w = a_1 a_2 \cdots a_k$, where $n \geq 0$, with letters $a_i \in A$ for all $1 \leq i \leq k$. The concatenation of words defines the monoid structure on $A^*$ whose identity element is the empty word denoted by $\varepsilon \in A^*$. For every word $w = a_1 a_2 \cdots a_k$, we define its **length** by $|w| := k$.

Given a word $w = a_1 a_2 \cdots a_n \in A^*$ of length $n \geq 1$, we define a periodic configuration $w^\infty \in A^Z$ given by $w^\infty(i + kn) = a_i$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n$.

A word $w \in A^*$ is a **subword** of a configuration $x \in A^Z$ if either $w$ is the empty word or there exist integers $m \geq n$ such that $w = x(n)x(n+1)\cdots x(m)$.

Consider now a subshift $X \subset A^Z$. The **language** of $X$ is the subset $L(X) \subset A^*$ consisting of all words $w \in A^*$ such that $w$ is a subword of some configuration in $X$. 

Now let $G$ be a group and let $A$ be a set. Let $\Sigma \subset A^G$ be a subshift. Then $\Sigma$ is irreducible, if it for any $y, z \in \Sigma$ and any finite subsets $E, F \subset G$, there exist $x \in \Sigma$ and $g \in G$ such that $x|_E = y|_E$ and $(gx)|_F = z|_F$. If $G = \mathbb{Z}$, this means that for all $u, v \in L(\Sigma)$, there exists $w \in L(\Sigma)$ such that $uwv \in L(\Sigma)$.

We say that $\Sigma$ is strongly irreducible if there exists a finite subset $\Delta \subset G$ with $1_G \in \Delta$ with the following property. If $E, F$ are finite subsets of $G$ such that $E \cap F \Delta = \emptyset$, then, for any $y, z \in \Sigma$, there exists $x \in \Sigma$ such that $x|_E = y|_E$ and $x|_F = z|_F$.

10.2. Density of periodic configurations in algebraic $W$-subshifts. As a generalization of strongly irreducible subshifts and of irreducible subshifts of finite type, the following definition is introduced in [4] (cf. [4, Proposition 4.4, Proposition 4.5]).

Definition 10.1 (cf. [4]): Let $A$ be a set. An irreducible subshift $\Sigma \subset A^\mathbb{Z}$ is called a $W$-subshift if there exists an integer $N \geq 0$ such that for every $u \in L(\Sigma)$, there exists $c \in L(\Sigma)$ such that $|c| \leq N$ and $ucu \in L(\Sigma)$.

We obtain below a generalization to the algebraic setting of a criterion of the density of periodic configurations with finite alphabet (cf. [4, Theorem 5.1]).

Theorem 10.2: Let $V$ be a complete algebraic variety over an algebraically closed field. Let $\Sigma \subset V^\mathbb{Z}$ be an algebraic $W$-subshift. Then $\Sigma$ contains a dense set of periodic configurations.

Proof (Benjy Weiss + $\varepsilon$). We follow the steps in the proof of [4, Theorem 5.1] which, as stated by the authors in [4], was suggested by Benjy Weiss. We only need to prove that every word $w \in L(\Sigma)$ is a subword of some periodic configuration of $\Sigma$.

As $\Sigma$ is a $W$-subshift, we can find an integer $N \geq 0$ satisfying the condition described in Definition 10.1. Let

$$Z := \prod_{k=1}^{N} V^{\{1,\ldots,k\}}.$$  

Since $V$ is a complete algebraic variety by hypothesis, so is $Z$. For each word $u \in L(\Sigma)$, consider the subset $F(u) \subset Z$ which represents all nonempty words $c \in A^*$ of length at most $N$ such that $ucu \in L(\Sigma)$.
By the choice of $N$, the set $F(u)$ is nonempty for every word $u \in L(\Sigma)$. We claim that $F(u)$ is in fact an algebraic subvariety of $Z$. To see this, let $m = |u|$ and define for every $1 \leq k \leq N$ a subset of $V\{-m+1, \ldots, k+m\}$:

$$uV^{1,k} := \{ x \in V\{-m+1, \ldots, k+m\} : x(-m+1) \cdots x(0) = x(k+1) \cdots x(k+m) = u \in L(\Sigma) \}.$$ 

Let

$$Y_u := \Sigma\{-m+1, \ldots, k+m\} \cap uV^{1,k} \subset V\{-m+1, \ldots, k+m\}.$$ 

We consider also $\pi_k : V\{-m+1, \ldots, k+m\} \to V\{0, \ldots, k\}$ the canonical projection induced by the inclusion $\{0, \ldots, k\} \subset \{-m+1, \ldots, k+m\}$. Then we have for every $1 \leq k \leq N$ that

$$F(u) \cap V^{1,k} = \pi_k(Y_u).$$

Note that $\Sigma\{-m+1, \ldots, k+m\}$ is clearly a complete algebraic subvariety of $V\{-m+1, \ldots, k+m\}$. On the other hand, $uV^{1,k}$ is a complete algebraic subvariety of $V\{-m+1, \ldots, k+m\}$. It follows that $Y_u$ is also a complete algebraic subvariety of $V\{-m+1, \ldots, k+m\}$. Note that $\pi_k$ is clearly an algebraic morphism. Therefore, $F(u)$ is a finite union of algebraic subvarieties of $Z$ so that it is indeed an algebraic subvariety of $Z$.

Now, we claim that there exists a word $u_0 \in L(\Sigma)$ with the property that there does not exist $u \in L(\Sigma)$ such that $F(u) \subsetneq F(u_0)$. Indeed, suppose the contrary; then we can find a sequence of words $(w_n)_{n \geq 0}$ in $L(\Sigma)$ such that $F(w_{n+1}) \nsubset F(w_n)$ for all $n \geq 0$. Thus, we obtain an infinite descending sequence $(F(w_n))_{n \geq 0}$ of Zariski closed subsets of $Z$, which contradicts the Noetherianity of the Zariski topology.

Given such a word $u_0 \in L(\Sigma)$, let $c_0 \in F(u_0)$ be an arbitrary fixed word. Suppose that $v \in A^*$ is such that $u_0vu_0 \in L(\Sigma)$. Then it is immediate that $F(u_0vu_0) \subset F(u_0)$ and hence by the choice of $u_0$, we must have

$$F(u_0vu_0) = F(u_0).$$

In particular, $c_0 \in F(u_0vu_0)$ and thus $u_0vu_0c_0u_0vu_0 \in L(\Sigma)$. From this point, the rest of the proof is exactly the same as in the proof of [4, Theorem 5.1].

10.3. SOME CONSEQUENCES ON ADMISSIBLE GROUP SUBSHIFTS.

**Corollary 10.3:** Let $A$ be an admissible Artinian group structure. Let $\Sigma \subset A^Z$ be an irreducible admissible group subshift. Then $\Sigma$ contains a dense set of periodic configurations.
Proof. Theorem 9.13 implies that $\Sigma$ is an admissible group subshift of finite type of $A^\mathbb{Z}$. Hence, there exists an interval $M = \{-k, \ldots, k\} \subset \mathbb{Z}$ where $k \geq 0$ such that $\Sigma = \Sigma(A^\mathbb{Z}; M, \Sigma_M)$. The argument below is standard (see, for example, [10]).

Let $y \in \Sigma$ be a configuration. Let $\varepsilon \in A$ be the neutral element of the group $A$ and let $e := \varepsilon^M \in A^M$. We regard $e \in L(\Sigma)$ as a word of length $|M|$. For every $n \geq 0$, define

$$w_n := y|_{\{-n, \ldots, n\}} \in L(\Sigma).$$

Since $\Sigma$ is irreducible, there exists a word $u_n \in L(\Sigma)$ such that $e u_n w_n \in L(\Sigma)$. Again, by the irreducibility of $\Sigma$, there exists $v_n \in L(\Sigma)$ such that

$$e u_n w_n v_n e \in L(\Sigma).$$

Consider the periodic configuration $x_n \in A^\mathbb{Z}$ which is a suitable translate of $(e u_n w_n v_n)^\infty$ such that

$$(x_n)|_{\{-n, \ldots, n\}} = w_n = y|_{\{-n, \ldots, n\}} \in A^{\{-n, \ldots, n\}}.$$  

It is clear that $(g x_n)|_M \in \Sigma_M$ for all $g \in \mathbb{Z}$. Thus, $x_n \in \Sigma(A^\mathbb{Z}; M, \Sigma_M) = \Sigma$.

Therefore, we obtain a sequence of periodic configurations $(x_n)_{n \geq 0}$ of $\Sigma$ which converges to $y$. The proof is thus completed.

The following theorem is proved in [4, Theorem 1.1] where it was stated for a finite alphabet but the proof actually holds for an arbitrary alphabet.

**Theorem 10.4** (cf. [4]): Let $G$ be a residually finite group and let $A$ be a set. Suppose that $\Sigma \subset A^G$ is a strongly irreducible subshift of finite type and that there exists a periodic configuration in $\Sigma$. Then $\Sigma$ contains a dense set of periodic configurations.

Every polycyclic-by-finite group is residually finite and every group subshift contains the zero configuration which is obviously periodic. Hence, we obtain the following immediate consequence of Theorem 10.4 and Theorem 9.13:

**Corollary 10.5:** Let $G$ be a polycyclic-by-finite group and let $A$ be an admissible Artinian group structure. Let $\Sigma \subset A^G$ be a strongly irreducible admissible group subshift. Then $\Sigma$ contains a dense set of periodic configurations.
10.4. A COUNTER-EXAMPLE. Let $G$ be a finitely generated abelian group and let $A$ be a finite group. By [17, Corollary 7.4], we know that for every closed subshift $\Sigma \subset A^G$ which is also an abstract subgroup, the set of periodic configurations of $\Sigma$ is dense in $\Sigma$ with respect to the prodiscrete topology.

However, when the alphabet is not a compact space, the set of periodic configurations of an admissible group subshift may fail to be dense.

**Example 10.6:** Let $G = \mathbb{Z}$ and let $A = \mathbb{Q}$ be the field of rational numbers. Let $D = \{0, 1\} \subset G$ and let

$$P = \{(x_0, x_1) \in A^D : x_1 = ax_0\}$$

for some constant $a \in \mathbb{Q}$ such that $a > 1$. Consider the subshift of finite type $\Sigma := \Sigma(A^G; D, P)$ of $A^G$. Then it is clear that $\Sigma = \mathbb{Q}c$ where $c \in A^G$ is the configuration defined by $c(n) = a^n$ for every $n \in \mathbb{Z}$. Hence, $\Sigma$ is also a $\mathbb{Q}$-vector subspace of $A^G$. However, the only periodic configuration of $\Sigma$ is the zero-configuration $0^G$ since $a^n \to +\infty$ when $n \to +\infty$. It follows that $\Sigma$ does not contain a dense subset of periodic configurations.

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