Light trajectories and thermal shadows casted by black holes in a cavity

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Received June 4, 2022
Revised November 9, 2023
Accepted November 15, 2023
Published November 27, 2023

Abstract. We explore the shadows and the photon rings casted by black holes in cavity. Placing the observer inside such an isothermal background, we examine the influence of the cavity temperature $T_{\text{cav}}$ and the charge $Q$ on the involved optical features. After studying the effect of the horizon radius by varying $Q$, we investigate the thermal behaviors of the black hole shadows in a cavity. For fixed charge values, we find that the shadow radius $r_s$ increases by decreasing $T_{\text{cav}}$. Varying such a temperature, we discuss the associated energy emission rate. After that, we show that the curves in the $r_s - T_{\text{cav}}$ plane share similarities with the $G - T$ curves of the Anti de Sitter black holes. Then, we study the trajectories of the light rays casted by black holes in a cavity. We further observe that the light trajectory behaviors are different than the ones of the non rotating black holes due to the cavity effect. Finally, we provide an evidence for the existence of an universal ratio defined in terms of the photon sphere radius and the impact parameter. Concretely, we obtain an optical ratio $\frac{b_{\text{sp}}}{r_{\text{ap}}} \sim \sqrt{3}$.

Keywords: Exact solutions, black holes and black hole thermodynamics in GR and beyond, gravity

ArXiv ePrint: 2206.00615

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1 Introduction

Recently, many efforts have been devoted to the study of optical and thermodynamic properties of the black holes built from different classical and quantum gravity models. These findings have been supported by the detections and the observational investigations of such fascinating objects. Concretely, the Even Horizon Telescope (EHT) international collaboration has provided an image of an accretion flow through the vicinity of the supermassive black hole in M87\(^*\) [1–3]. The associated region called shadow is surrounded by a bright ring known as the photon sphere. Since the elaboration of such an image and the associated data by EHT, various works dealing with the geometric properties of the black hole shadows have been proposed by examining the involved sizes and shapes [4–20]. In four dimensions, it has been shown that the non-rotating black hole solutions exhibit a perfect circular geometry where the size can be controlled by parameters such as the charge [8, 21, 22]. For the rotating black holes, however, the circular configuration is deformed and distorted generating non trivial shapes including D and cardioid ones [23–26]. Other generic shadow degeneracy scenarios associated with the Kerr-Newman-Taub-NUT black hole solutions with observer extra discrete symmetries have been provided. It has been remarked that EHT data can be exploited to impose constraints on the black hole parameters. These constraints could be used to built models matching with the EHT observations considered as a reference for testing the obtained theoretical results of the black hole physics.

Beside the optical features, the black hole thermodynamics has received also a remarkable interest showing non trivial results corresponding to the criticality and the stability behaviors. A crucial focus has been devoted to the study of the black holes on the Anti de Sitter (AdS) geometries [27, 28, 31, 32]. The needed quantities have been nicely computed for various gravity theories. Considering the cosmological constant as a pressure, certain black holes show similarities with van der Waals fluid systems where some universal properties have been largely studied in different gravity theories. In particular, universal critical numbers predicted by four dimensional AdS black holes have been obtained either from the P-V diagram or the Hawking-Page (HP) transition [28–30]. For such a transition, two universal critical constants based on the HP and minimum black hole thermodynamical transition points have been found [30]. They are given by \( C_S = \frac{S_{\text{HP}} - S_{\text{min}}}{S_{\text{min}}} \) and \( C_T = \frac{T_{\text{HP}} - T_{\text{min}}}{T_{\text{min}}} \). It has been revealed that the AdS black holes can be in a stable thermal equilibrium with radiation [27, 31, 33, 34].
Hawking and Page (HP) have provided theoretically an evidence for the existence of transitions in the phase space of the (non-rotating uncharged) Schwarzschild-AdS black hole. A first order phase transition in the charged (non-rotating) Reissner Nordstrom-AdS (RN-AdS) black hole space-time has been studied in different backgrounds including Dark Energy (DE) and Dark Matter (DM) [21, 35, 36]. The effect of such a dark sector on the optical aspect has been also investigated for different backgrounds. In this way, the shadows and the deflection angle of the light rays by black holes in arbitrary dimensions have been discussed in [13, 21–23]. The effect of DE and the space-time dimension on the involved optical quantities has been inspected. Varying the space-time dimension with the presence the DE effect by means of the DE field intensity and the state parameter, it has been observed that the shadow size and the deflection angle get modified. For instance, it has been revealed that the shadow size increases by increasing the DE field intensity. However, it decreases by augmenting the space-time dimension [22]. The influence of DM on the shadows and the photon rings of a stringy black hole illuminated by accretions has been also studied. It has been found that the photon sphere radius increases with the DM parameter [37].

Motivated by the investigations on singular space-times, interplays between the black hole thermodynamics and the optical properties have been established. It has been shown that the shadow size can provide information on the HP phase transitions, the critical behaviors and the microstructure states of the black holes living in AdS geometries [38–40]. It can generate also data on the geothermodynamics. In particular, it has been shown that the black hole shadows could provide correct information concerning the small/large black hole phase structure and the involved microstructure [41]. It has been shown that the relation between the shadows and the thermodynamics of the black hole has been also developed for regular space-times [42]. Using the elliptic function analysis, it has been explored further a fundamental connection between the AdS black hole thermodynamics and the deflection angle of the light rays. Concretely, various thermodynamics behaviors of such black holes have been approached in terms of the deflection angle variations [43].

It has been shown that the black holes enclosed by a cavity could solve the stability problem [44–48]. This isothermal geometry has been considered as a tool to provide a stable solution for black hole physics. Precisely, it has been revealed that the Schwarzschild black holes in a cavity can be thermally stable. They involve the phase structures and the transition behaviors as the ones appearing in the Schwarzschild-AdS black holes. By considering canonical ensembles, a similar interplay has been observed in the Reissner-Nordstrom (RN) black holes. More recently, the phase structures of extended models in a cavity have been investigated where the Hawking-Page-like and van der Waals-like phase transitions have taken place [49]. It has been suggested that the black hole physics in non-flat spacetimes could provide relevant findings even if they do not necessarily describe the universe. In some cases, the latter could be illustrated by geometries with a small cosmological constant [50–53]. Moreover, the physics in cavities could unveil data on the vacuum state, the particle creation and the annihilation behaviors. The study of the cavity in such space-time geometries could support the correspondence between the conformal field theories in flat spaces and their duals in cosmological backgrounds. This could be exploited to extract features of black holes [54–56].

The aim of this work is to explore for the first time the optical behaviors of the black holes in a cavity using thermodynamical ones. Precisely, we investigate the shadows and the photon rings casted by black holes surrounded by such an isothermal generic space, by examining the effect of the cavity temperature $T_{\text{cav}}$ and the charge $Q$. After studying the effect of the horizon radius on the visualizing shadows of the proposed black holes by varying
the charge, we move to study the associated thermal behaviors. For fixed charge values, we observe that the shadow radius $r_s$ increases by decreasing the cavity temperature $T_{cav}$. Varying such a temperature, we discuss the energy emission rate. Moreover, we show that the curves in the $r_s - T_{cav}$ plane share similarities with the $G - T$ curves of the AdS black holes. Then, we investigate the trajectories of the light rays casted by the black holes in a cavity. We reveal that the light trajectories are different than the light ray behaviors of the non rotating black holes due to the cavity effect. Finally, we provide an evidence for the existence of an universal ratio associated with the photon sphere radius and the impact parameter. Precisely, we find the optical ratio $b_{sp} \sim \sqrt{3}$.

This paper is organized as follows. In section 2, we present a concise review on the black holes surrounded by a cavity. In section 3, we approach the black hole shadows in terms of the horizon radius. Section 4 is devoted to the thermal behaviors of such shadows. In section 5, we discuss the trajectories of the light rays casted by the black holes in a cavity. Conclusions and open questions are given in the last section.

2 Black holes in a cavity

In this section, we present a concise review on the black holes in a cavity background. In particular, we give the associated relevant concepts. It is recalled that in asymptotically AdS geometries, the black holes can be considered thermodynamically stable [44, 49]. In this way, the AdS boundary serves as a reflective wall. Alternatively, various works have reported that the black holes enclosed by a cavity can be also thermally stable. It has been suggested that the cavity presence could overcome the thermodynamics stability problem of the black holes in asymptotically flat spaces. Motivated by these activities, we would like to investigate other properties including the optical ones by examining the shadow and the light ray behaviors near such black holes. This could unveil certain interesting behaviors by considering static and spheric metric structures of the black holes in a cavity physical system. In this way, the 4-dimensional metric solution can be expressed as follows

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.1)$$

where $f(r)$ is the metric function of a black hole in a cavity. In terms of the involved parameters [49], this metric function takes the following form

$$f(r) = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{Q^2}{r r_+}\right). \quad (2.2)$$

In this relation, $r_+$ indicates the horizon radius given by

$$r_+ = m + \sqrt{m^2 - Q^2}, \quad (2.3)$$

where $m$ and $Q$ are the charge and the mass parameters of the black holes, respectively. For $Q = 0$, however, we recover the Schwarzschild black hole solution [57]. A close examination shows that many models could be approached depending on the observer positions. For simplicity reasons, however, we could place the observer inside the cavity surrounded region. This situation can be ensured by imposing the following constraints

$$\begin{cases} r_{cav} > r_+ \\ r_{cav} > r_{ob}, \end{cases} \quad (2.4)$$

$$\begin{cases} r_{cav} > r_+ \\ r_{cav} > r_{ob}, \end{cases} \quad (2.5)$$
being interpreted as thermodynamic and optical conditions, respectively. In figure 1, we illustrate the representation of a black hole in a cavity with the above requirements.

The red line represents the light transmitting to the observer placed at \( r_{\text{ob}} \) being smaller to the cavity radius \( r_{\text{cav}} \). However, the blue surface corresponds to the equatorial plane.

3 Shadow behaviors of the black holes in a cavity system

In this section, we investigate the optical properties of the black hole in a cavity. In particular, we study the shadow behaviors by varying the involved black hole parameters.

Roughly, we approach the shadow geometrical configurations in terms of the one-dimensional real closed curves obtained from the equations of motion [8, 22, 25]. It has been remarked that the shadow can be considered as the most relevant concept which could provide predictions associated with the black hole existence. This optical concept can be approached via two geometric deformations corresponding to the size and the shape generating data on the black hole parameters. Different geometries have been obtained going beyond the circle described by non-rotating spherical solutions. Roughly, we consider the Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial \tau} = -\frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, \tag{3.1}
\]

where \( S \) and \( \lambda \) are the Jacobi action and the affine parameter along the geodesics, respectively. \( p^{\alpha} \) represents the conjugate momentum of the black hole in a cavity system. In the spherically symmetric spacetime, the Hamiltonian which describes the photon motion can be expressed as follows

\[
H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta = 0. \tag{3.2}
\]

Considering a particular motion of the photons in the equatorial plane \( \theta = \frac{\pi}{2} \), the Hamiltonian equation reduces to the following form

\[
(r f(r) p_r)^2 - r^2 p_t^2 + f(r) p_t^2 = 0. \tag{3.3}
\]
Applying the Hamilton Jacobi formalism, the equations describing the motion of the photons can be formulated as

\[
\begin{align*}
\frac{dt}{d\lambda} &= \frac{E}{f(r)}, \\
\frac{dr}{d\lambda} &= \pm \sqrt{f(r) \left( \frac{E^2}{f(r)} - \frac{L^2}{r^2} \right)}, \\
\frac{d\phi}{d\lambda} &= -\frac{L}{r^2},
\end{align*}
\]

where \( E = -p_t \) and \( L = p_\phi \) are the conserved total energy and the conserved angular momentum of the photon, respectively. It turns out that the geometric shape of a black hole can be completely described by the limit of their shadows being the visible shape of the unstable closed geodesic curves of the photons. To approach such behaviors, one can exploit the radial equation of motion given by

\[
\left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}}(r) = 0,
\]

where \( V_{\text{eff}}(r) \) indicates the effective potential for a radial particle motion in the space-time. In particular, it reads as

\[
V_{\text{eff}} = f(r) \left( \frac{L^2}{r^2} - \frac{E^2}{f(r)} \right).
\]

An examination reveals that the maximal value of the effective potential corresponds to the radius of the circular orbits \( r_{sp} \). To get the radius of the photon unstable circles, one should consider the following constraint

\[
V_{\text{eff}} \bigg|_{r=r_{sp}} = 0.
\]

Using eq. (3.6) and eq. (3.7), one can obtain

\[
\begin{align*}
f(r_{sp}) \left( \frac{L^2}{r_{sp}^2} - \frac{E^2}{f(r_{sp})} \right) &= 0, \\
L^2 \left( \frac{r_{sp} f'(r_{sp}) - 2f(r_{sp})}{r_{sp}^3} \right) &= 0,
\end{align*}
\]

where the notation \( f'(r) = \frac{\partial f(r)}{\partial r} \) has been used. By the help of the eq. (3.8), we can find the equation constraint concerning the radius of the unstable photon spheres via the relation

\[
\begin{align*}
r_{sp} f'(r_{sp}) - 2f(r_{sp}) &= 0.
\end{align*}
\]

The real and positive solution of this equation is given by

\[
r_{sp} = \frac{3(Q^2 + r_+^2) + \sqrt{9Q^4 - 14Q^2 r_+^2 + 9r_+^4}}{4r_+}.
\]

It has been remarked that the expression of the photon sphere radius \( r_{sp} \) which depends on the charge \( Q \) and the horizon radius \( r_+ \) can recover the previous results. Taking \( Q = 0 \) and
\( r_+ = 2M \), we obtain the radius of the unstable photon sphere of the Schwarzschild black hole solutions \([8, 10, 22]\). By the help of the radial and the angular geodesic equations, the orbit equation for the photon reads as

\[
\frac{dr}{d\phi} = \pm \frac{r^2}{L} \sqrt{f(r) \left( \frac{E^2}{f(r)} - \frac{L^2}{r^2} \right)}.
\]  

(3.11)

The photon orbit is constrained by

\[
\frac{dr}{d\phi} \bigg|_{r = r_{sp}} = 0.
\]

(3.12)

Using eq. (3.12), the previous equation take the following form

\[
\frac{dr}{d\phi} = \pm r \sqrt{f(r) \left( \frac{r^2 f(r_{sp})}{r_{sp}^2 f(r)} - 1 \right)}.
\]

(3.13)

Considering the light ray sent from a static observer situated at \( r_{ob} \) inside the cavity and transmitted into the past with an angle \( \alpha_{ob} \), we have

\[
\cot \alpha_{ob} = \sqrt{g_{rr} \frac{dr}{d\phi} \bigg|_{r = r_{ob}}} = \frac{1}{r \sqrt{f(r)}} \frac{dr}{d\phi} \bigg|_{r = r_{ob}}.
\]

(3.14)

Exploiting eq. (3.14), one finds the angle of the observer as a function of the various parameters

\[
\sin^2 \alpha_{ob} = \frac{f(r_{ob}) r_{sp}^2}{r_{ob}^2 f(r_{sp})}.
\]

(3.15)

In this context, one can get the angular radius of the black hole shadows as a function of the circular orbit radius of the photon appearing in eq. (3.10). Precisely, the shadow radius of the black hole being observed by a static observer placed at \( r_{ob} \) is given by

\[
r_s = r_{ob} \sin \alpha_{ob} = R \sqrt{\frac{f(r_{ob})}{f(R)}} \bigg|_{R = r_{sp}}.
\]

(3.16)

The apparent shape of the black hole shadow in a cavity system can be obtained by using the celestial coordinates \( x \) and \( y \) which can be expressed as follows

\[
x = \lim_{r_0 \to -\infty} \left( -r_0^2 \sin \theta_0 \frac{d\phi}{dr} \bigg|_{(r_0, \theta_0)} \right),
\]

\[
y = \lim_{r_0 \to -\infty} \left( r_0^2 \frac{d\theta}{dr} \bigg|_{(r_0, \theta_0)} \right).
\]

(3.17)

In figure 2, we plot the shadow geometries for different values of the charge \( Q \). As expected, the geometry of the shadows is a perfect circular due to the absence of the rotating parameter. It has been remarked that the charge of the black hole in a cavity is a relevant parameter controlling the optical behaviors including the shadow size. Indeed, the latter increases with the horizon radius. Taking a fixed value of \( r_+ \), the shadow size increases by increasing the charge \( Q \). It is observed that for \( Q = 0 \), we recover the shadows of the Schwarzschild black hole \([8, 10, 22]\). To inspect the shadow radius behaviors, we consider its variation in terms of \( r_+ \) for different values of the charge. This optical aspect is illustrated in figure 3. This figure confirms the previous findings where the radius augments with the charge. This quantity has been interpreted as a geometric parameter controlling the shadow size.
4 Shadow thermal behaviors of black holes

Motivated by the results associated with the link between the black holes in a cavity and the thermodynamics of the AdS black holes, we study shadow behaviors by varying the cavity temperature. It is recalled that the temperature is a crucial quantity needed to approach the stability behaviors. Indeed, the Hawking temperature is given by

$$T_H = \frac{1}{4\pi} \left. \frac{df(r)}{dr} \right|_{r=r_+}.$$  \hspace{1cm} (4.1)

In the cavity system, however, the temperature is related to the cavity radius $r_{cav}$. Concretely, it is expressed as follows

$$T_{cav} = \frac{T_H}{\sqrt{f(r_{cav})}}$$  \hspace{1cm} (4.2)
which can be given in terms of the involved physical parameters [44, 45, 49]. According to such works, the calculations provide the following expressions

$$T_{cav} = \frac{r_+^2 - Q^2}{4\pi r_+ \sqrt{f(T_{cav})}}. \quad (4.3)$$

It has been remarked that the cavity temperature $T_{cav}$ and the shadow geometry in a cavity are controlled by the horizon radius. Inspired by such a link, we investigate the shadow behaviors by varying the temperature. It has been observed that, for $r_{cav} > r_+$, the temperature of the cavity system is similar to the one of the charged AdS black hole. The only difference is for $r_{cav} = r_+$ where the cavity temperature $T_{cav}$ diverges [49]. As proposed in figure 1, the observer is placed inside the cavity to visualize the transmitting lights in terms of the shadows described by real closed curves. By the help of the method reported in [40], we study the thermal shadow behaviors. In figure 4, we illustrate the shadows as functions of the cavity temperature $T_{cav}$ for fixed values of the charge $Q$. To get a concrete model, we consider a situation where one has $r_{cav} = 20$ and $r_{ob} = 15$. In this way, the shadow size increases by decreasing the temperature of the cavity system. It follows from this figure that the charge $Q$ decreases the shadow radius of the black hole in a cavity. Augmenting the charge $Q$, we observe that the temperature variation is contrary to the horizon radius one as illustrated in figure 3. These inverse behaviors come from the relationship between the cavity temperature and the horizon radius $r_+$. Indeed, when $T_{cav}$ decreases (increases), $r_+$ increases (decreases). Moreover, the shadow properties of the AdS black hole solutions have been conserved in the black hole inside the cavity. In fact, it is an increasing (decreasing) function of the mass (temperature) [10, 22]. Such thermal behaviors push one to inspect other black hole properties needed to support the present findings. The corresponding energy emission rate could be investigated. It is worth noting that near to the horizon of the black hole, the quantum fluctuations can create and annihilate pairs of particles. In this way, the particles with positive energies can escape through tunneling from the black hole associated with the Hawking radiation. In what follows, we discuss the involved energy emission rate in
Figure 5. Energy emission rate for different values of $T_{\text{cav}}$ and $Q$, by placing the observer in the equatorial plane and taking $r_{\text{ob}} = 15$ and $r_{\text{cav}} = 20$.

For a distant observer with the above optical and the thermodynamical conditions, the high energy absorption cross section could provide data on the black hole shadows. In this regard, it has been remarked that the absorption cross section of the black hole can oscillate to an approximated constant value $\sigma_{\text{lim}} = \pi r_s^2$. The energy emission rate can be written as

$$\frac{d^2 E(\omega)}{d\omega dt} = \frac{2\pi^3 r_s^3 \omega^3}{e^{\frac{T_H}{\omega}} - 1},$$

where $\omega$ indicates the emission frequency [22, 24, 25, 58], and where $T_H$ is the associated Hawking temperature. Using the link with the cavity temperature, this expression takes the following form

$$\frac{d^2 E(\omega)}{d\omega dt} = \frac{2\pi^3 r_s^3 \omega^3}{e^{\sqrt{f(r_{\text{cav})}T_{\text{cav}}}}} - 1.$$ (4.5)

The energy emission rate is illustrated in figure 5 as a function of $\omega$ by varying $T_{\text{cav}}$ and $Q$. For small values of the cavity temperature, it has been observed that the charge does not bring any relevant effect. Augmenting such a temperature, the energy emission rate maximum increases by decreasing the charge $Q$. Fixing the charge values, this maximum increases with $T_{\text{cav}}$. This shows that $T_{\text{cav}}$ and $Q$ involve opposite effects on the involved optical behaviors to ensure the stability. This finding confirms the previous obtained results dealing with the black holes in cavities.

Inspired by the similarities between the black hole thermodynamics in AdS geometries and in isothermal backgrounds, we approach the cavity temperature from an optical point of view. Indeed, we plot in figure 6 the cavity temperature as a function of the shadow radius by varying the charge $Q$. The shadow radius increases by decreasing $T_{\text{cav}}$ and decreases with the charge $Q$. This behavior is confirmed in figure 6, by using the optical and the thermodynamical conditions given by eq. (2.4) and eq. (2.5) for $r_{\text{ob}} = 3$ associated with an observer placed closed to the sphere photon radius of the black hole in a cavity. An examination shows that the curves in the $r_s - T_{\text{cav}}$ plane share similarities with the $G - T$ curves of the charged and rotating AdS black holes, where $G$ and $T$ are the Gibbs free energy and the temperature, respectively. To unveil such similar behaviors, we should exploit the thermodynamical quantities. According to [49], the Gibbs free energy expression of a black hole in a cavity takes the following form

$$G = r_{\text{cav}} \left[ \frac{r_s + \frac{Q^2}{r_s r_{\text{cav}}}}{12\sqrt{f(r_{\text{cav}})}} - 8 + \frac{2}{3} \right].$$ (4.6)
Figure 6. The behaviors of the shadow radius in terms of the cavity temperature. In particular, we take the observer in the equatorial plane and positioned in $r_{\text{ob}} = 3$ and the cavity radius $r_{\text{cav}} = 20$.

Solving the equation $G = 0$, we can get the value of the phase transition temperature $T_{\text{HP}}$. Varying the charge $Q$, this temperature is found to be constant

$$T_{\text{HP}} = 0.0284. \quad (4.7)$$

Moreover, it has been observed that the $r_s - T_{\text{cav}}$ curves involve the swallowtail behavior observed in the $G - T$ plane. Indeed, the $T_{\text{HP}}$ temperature in the $G - T$ plane coincides with the $T_{\text{HP}}$ temperature in the $r_s - T_{\text{cav}}$ plane. This result could be used to confirm the relation between the thermodynamics and the optical aspect of the black holes in a cavity. As expected, the thermal quantities of the black holes including the phase transition temperature $T_{\text{HP}}$ can be approached using the shadow configurations.

5 Trajectories of the light rays by black holes in a cavity

In this section, we study the trajectory of the light rays casted by the black holes in a cavity. In particular, we determine the light trajectory around the black hole in a cavity by varying the cavity temperature $T_{\text{cav}}$. The light trajectories casted by the black holes in cavities can be established by using numerical computations associated with the following orbit equation

$$\frac{du}{d\phi} = \sqrt{\frac{1}{b^2} - u^2 (1 - ur_+)} \left(1 - \frac{Q^2 u}{r_+} \right), \quad (5.1)$$

where one has used the change variable $u = \frac{1}{r}$ and where $b$ is the so-called the impact parameter given by $b = \frac{|L|}{E} \quad [59, 60]$. Using Eq. (5.1), we can solve $\phi$ with respect to $u$ in order to depict the trajectory of the light ray in a cavity background. To approach the trajectories of the light rays casted by the black holes in such a system, one should use the impact parameter to identify three possible regions. Indeed, these regions can be determined by the help of the effective potential given by eq. (3.6). In figure 7, we plot such an effective potential as a function of the radial coordinate $r$ for different values of the cavity temperature $T_{\text{cav}}$ and the charge $Q$. 
This potential increases and reaches a maximum at the photon sphere associated with \( b_{sp} \) representing the impact parameter of the spinning light rays. This quantity verifies the following constraint
\[
V_{\text{eff}}(r_{sp}) = \frac{1}{b_{sp}^2}.
\] (5.2)

Two values of \( T_{\text{cav}} \) will be dealt with, being \( T_{\text{cav}} = 0.04, 0.08 \). For \( Q = 0.3 \), they provide two impact parameter values \( b_{sp} = 5.39 \) and \( b_{sp} = 2.46 \), as shown in figure 7, corresponding to the photon sphere radius \( r_{sp} = 3.09 \) and \( r_{sp} = 1.38 \), respectively. For \( Q = 0.1 \), however, the corresponding impact parameter and the photon sphere radius increase. Fixing the charge value, the effective potential increases by decreasing the cavity temperature. We observe that the photon sphere radius \( r_{sp} \) also decreases by increasing the cavity temperature. It has been remarked that the impact parameter value \( b_{sp} \) provides the trajectories of the light rays in three different regions. These regions are denoted by region ①, region ② and region ③ corresponding to \( b < b_{sp} \), \( b = b_{sp} \) and \( b > b_{sp} \), respectively.

In the region ①, the light ray falls into the black hole in a cavity due to the values of the impact parameter lower to \( b_{sp} \). In region ③, however, the light rays near the black hole in a cavity system are reflected back. In the region ②, the light rays come into the photon sphere making an infinite turn number around the black hole due to the non vanishing value of the angular velocity. The associated orbit is circular and unstable. To examine these regions, we plot in figure 8 the trajectories of the light rays in the polar coordinates \((r, \phi)\) for different values of the cavity temperature \( T_{\text{cav}} \) and the charge \( Q \). To analyse the effect of the cavity temperature on the light ray trajectories, we vary the impact parameter \( b \) in the range \([0, 7]\). The step between two values of the impact parameter is 1/10 for all light
Figure 8. The trajectories of the light rays for different values of the cavity temperature. The horizon $r_+$ and photon sphere $r_{sp}$ are represented by the black disc and black dashed circle, respectively. The observer is situated in the equatorial plane and positioned in $r_{ob} = 15$. We take also the cavity radius $r_{cav} = 20$.

A close examination reveals that the horizon and the photon sphere radii decrease by increasing the cavity temperature. For fixed values of the charge, this confirms the previous results associated with the shadow and the photon sphere radii as functions of the cavity temperature. For $T_{cav} = 0.04$, it has been remarked that the region $\circled{3}$ is small than the region $\circled{1}$. An inverse behavior is observed for $T_{cav} = 0.08$. For such a value, the region $\circled{3}$ is large compared to the region $\circled{1}$. Moreover, the region of the reflected light rays enlarges with the cavity temperature. This shows that such a temperature can be considered as a relevant quantity modifying the light ray behaviors near a black hole. However, it has been remarked that the charge does not affect such trajectories. An examination shows that the light ray
behaviors of the black holes in a cavity system are different than the light trajectories around the non rotating black holes without cavities including the DE contributions [10, 22]. It has been observed that such a distinction comes from the cavity effect on the black hole object. By numerically examining the optical behavior plotted in the above figures, we collect optical results for various moduli space regions in table 1.

According to the reported data, we explore universal optical ratios. This could be understood as universal optical behaviors completing the ones obtained in the black hole thermodynamics [29, 30]. We could observe a relevant optical behavior for small values of the involved parameters. In particular, we anticipate the following universal optical ratio

\[
\frac{b_{sp}}{r_{sp}} = \sqrt{3} + \xi(T_{cav}, Q)
\]

where \(\xi\) can be considered as an extra contribution going to zero for small values of \(T_{cav}\) and \(Q\). This universal ratio has not been observed only in the black hole in a cavity but also in the ordinary Schwarzschild (Sch) black holes without a cavity [61]. After a consistence verification, we find

\[
\frac{b_{sp}(Sch)}{r_{sp}(Sch)} = \frac{3\sqrt{3}m}{3m} = \sqrt{3}.
\]

In the charged solution, this ratio is checked to be close to \(\sqrt{3}\).
6 Conclusion and open questions

In this work, we have explored the optical behaviors casted by the black holes in a cavity. Concretely, we have studied the shadows and the photon rings of the black holes surrounded by a cavity, where the effect of the temperature and the charge have been examined. After studying the impact of the horizon radius on the illustrating images of the black holes by varying the charge, we have investigated the thermal shadow behaviors in the cavity system imposing the geometric conditions. For fixed charge values, we have revealed that the shadow radius $r_s$ increases by decreasing the cavity temperature $T_{cav}$. Moreover, we have shown that the $r_s - T_{cav}$ curves involve swallowtail behaviors usually observed in the $G - T$ plane of the ordinary AdS black holes. Among others, we have found that the $T_{HP}$ temperature in the $G - T$ curves coincides with the $T_{HP}$ temperature in the $r_s - T_{cav}$ plane. The present findings could support the relation between the thermodynamics and the optical aspect of the black holes in a cavity.

In this way, the thermodynamical quantities of the black holes such as the phase transition temperature $T_{HP}$ could be approached using the shadow formalism in cavity systems. Then, we have established the trajectory of the light rays casted by the black holes in cavities. Precisely, we have observed that the light trajectories of the black holes in a cavity background are different than the light ray behaviors of the ordinary solutions. This distinction comes from the cavity effect on the black hole object. Finally, we have found the optical universal ratio associated with the photon sphere radius and the impact parameter. For small values of the charge $Q$ and the cavity temperature $T_{cav}$, we have found an optical ratio $b_{sp}/r_{sp} \sim \sqrt{3}$. This universal optical ratio has been verified for the Schwarzschild black hole without a cavity which equals to $\sqrt{3}$.

This work comes up with certain questions. This study associated with circular curves could be adopted to various models by enlarging the moduli space. Rotating black holes in a cavity could be considered as a possible extension even the associated computations will be complicated. The corresponding shadow curves could provide extra information derived from elliptic geometries. It should be interesting also to consider generic optical pictures by examining other optical properties including the deflection angle of the light rays. We hope address such questions in future works.

Acknowledgments

The authors would like to thank N. Askour, H. El Moumni and Y. Hassouni, for collaborations on related topics. This work is partially supported by the ICTP through AF. The authors would like to thank the anonymous referees for interesting comments and suggestions.

References

[1] Event Horizon Telescope collaboration, First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole, Astrophys. J. Lett. 875 (2019) L4 [arXiv:1906.11241] [inSPIRE].

[2] Event Horizon Telescope collaboration, First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring, Astrophys. J. Lett. 875 (2019) L5 [arXiv:1906.11242] [inSPIRE].

– 14 –
[23] A. Belhaj, H. Belmahi and M. Benali, Superentropic AdS black hole shadows, *Phys. Lett. B* **821** (2021) 136619 [arXiv:2110.06771] [INSPIRE].

[24] A. Belhaj et al., Shadows of 5D black holes from string theory, *Phys. Lett. B* **812** (2021) 136025 [arXiv:2008.13478] [INSPIRE].

[25] A. Belhaj et al., Cosmological constant effect on charged and rotating black hole shadows, *Int. J. Geom. Meth. Mod. Phys.* **18** (2021) 2150188 [arXiv:2007.09058] [INSPIRE].

[26] A. Belhaj et al., Black hole shadows in M-theory scenarios, *Int. J. Mod. Phys. D* **30** (2021) 2150026 [arXiv:2008.09908] [INSPIRE].

[27] G.A. Marks, F. Simovic and R.B. Mann, Phase transitions in 4D Gauss-Bonnet-de Sitter black holes, *Phys. Rev. D* **104** (2021) 104056 [arXiv:2107.11352] [INSPIRE].

[28] D. Kubiznak and R.B. Mann, P-V criticality of charged AdS black holes, *JHEP* **07** (2012) 033 [arXiv:1205.0559] [INSPIRE].

[29] A. Belhaj, M. Chabab, H. El Moumni and M.B. Sedra, On Thermodynamics of AdS Black Holes in Arbitrary Dimensions, *Chin. Phys. Lett.* **29** (2012) 100401 [arXiv:1210.4617] [INSPIRE].

[30] A. Belhaj et al., On universal constants of AdS black holes from Hawking-Page phase transition, *Phys. Lett. B* **811** (2020) 135871 [arXiv:2010.07837] [INSPIRE].

[31] N. Altamirano, D. Kubiznak and R.B. Mann, Reentrant phase transitions in rotating anti-de Sitter black holes, *Phys. Rev. D* **88** (2013) 101502 [arXiv:1306.5756] [INSPIRE].

[32] D. Kubiznak and F. Simovic, Thermodynamics of horizons: de Sitter black holes and reentrant phase transitions, *Class. Quant. Grav.* **33** (2016) 245001 [arXiv:1507.08630] [INSPIRE].

[33] A. Belhaj et al., On Thermodynamics of AdS Black Holes in M-Theory, *Eur. Phys. J. C* **76** (2016) 149 [arXiv:1503.07308] [INSPIRE].

[34] A. Belhaj et al., On Heat Properties of AdS Black Holes in Higher Dimensions, *JHEP* **05** (2015) 149 [arXiv:1509.02196] [INSPIRE].

[35] A. Belhaj et al., Kerr-AdS black hole behaviors from dark energy, *Int. J. Mod. Phys. D* **29** (2020) 2050069 [INSPIRE].

[36] A. Belhaj et al., Dark energy effects on charged and rotating black holes, *Eur. Phys. J. Plus* **134** (2019) 422 [arXiv:1912.08687] [INSPIRE].

[37] X.-X. Zeng, K.-J. He and G.-P. Li, Effects of dark matter on shadows and rings of Brane-World black holes illuminated by various accretions, *Sci. China Phys. Mech. Astron.* **65** (2022) 290411 [arXiv:2111.05090] [INSPIRE].

[38] M. Zhang and M. Guo, Can shadows reflect phase structures of black holes?, *Eur. Phys. J. C* **80** (2020) 790 [arXiv:1909.07033] [INSPIRE].

[39] A. Belhaj et al., Thermal Image and Phase Transitions of Charged AdS Black Holes using Shadow Analysis, *Int. J. Mod. Phys. A* **35** (2020) 2050170 [arXiv:2005.05893] [INSPIRE].

[40] X.-C. Cai and Y.-G. Miao, Can we know about black hole thermodynamics through shadows?, *arXiv:2107.08352* [INSPIRE].

[41] C. Wang, B. Wu, Z.-M. Xu and W.-L. Yang, Ruppeiner geometry of the RN-AdS black hole using shadow formalism, *Nucl. Phys. B* **976** (2022) 115698 [arXiv:2107.12615] [INSPIRE].

[42] S. Guo, G.-R. Li and G.-P. Li, Shadow thermodynamics of an AdS black hole in regular spacetime *, *Chin. Phys. C* **46** (2022) 095101 [arXiv:2205.04957] [INSPIRE].

[43] A. Belhaj, H. Belmahi, M. Benali and A. Segui, Thermodynamics of AdS black holes from deflection angle formalism, *Phys. Lett. B* **817** (2021) 136313 [INSPIRE].
[44] H.W. Braden, J.D. Brown, B.F. Whiting and J.W. York Jr., Charged black hole in a grand canonical ensemble, *Phys. Rev. D* 42 (1990) 3376 [inSPIRE].

[45] P. Wang, H. Wu and H. Yang, Thermodynamic Geometry of AdS Black Holes and Black Holes in a Cavity, *Eur. Phys. J. C* 80 (2020) 216 [arXiv:1910.07874] [inSPIRE].

[46] H. El Moumni and J. Khalloufi, Nonlinear-Maxwell-Yukawa de-Sitter black hole thermodynamics in a cavity: I — Canonical ensemble, *Nucl. Phys. B* 973 (2021) 115593 [inSPIRE].

[47] H. El Moumni and J. Khalloufi, Nonlinear-Maxwell-Yukawa de-Sitter black hole thermodynamics in a cavity: II — Grand canonical ensemble, *Nucl. Phys. B* 977 (2022) 115731 [inSPIRE].

[48] F. Simovic and R.B. Mann, Critical Phenomena of Charged de Sitter Black Holes in Cavities, *Class. Quant. Grav.* 36 (2019) 014002 [arXiv:1807.11875] [inSPIRE].

[49] W.-B. Zhao, G.-R. Liu and N. Li, Hawking-Page phase transitions of the black holes in a cavity, *Eur. Phys. J. Plus* 136 (2021) 981 [arXiv:2012.13921] [inSPIRE].

[50] J. Sola Peracaula, The cosmological constant problem and running vacuum in the expanding universe, *Phil. Trans. Roy. Soc. Lond. A* 380 (2022) 20210182 [arXiv:2203.13757] [inSPIRE].

[51] Supernova Search Team collaboration, Observational evidence from supernovae for an accelerating universe and a cosmological constant, *Astron. J.* 116 (1998) 1009 [astro-ph/9805201] [inSPIRE].

[52] S.M. Carroll, The Cosmological constant, *Living Rev. Rel.* 4 (2001) 1 [astro-ph/0004075] [inSPIRE].

[53] Planck collaboration, Planck 2015 results. XIII. Cosmological parameters, *Astron. Astrophys.* 594 (2016) A13 [arXiv:1502.01589] [inSPIRE].

[54] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* 2 (1998) 253 [hep-th/9802150] [inSPIRE].

[55] M. Beutter, A. Pargner, T. Schwetz and E. Todarello, Axion-electrodynamics: a quantum field calculation, *JCAP* 02 (2019) 026 [arXiv:1812.05487] [inSPIRE].

[56] H. Xu, Y.C. Ong and M.-H. Yung, Landauer’s principle in qubit-cavity quantum-field-theory interaction in vacuum and thermal states, *Phys. Rev. A* 105 (2022) 012430 [arXiv:2109.08391] [inSPIRE].

[57] R. André and J.P.S. Lemos, Thermodynamics of d-dimensional Schwarzschild black holes in the canonical ensemble, *Phys. Rev. D* 103 (2021) 064069 [arXiv:2101.11010] [inSPIRE].

[58] S.-W. Wei and Y.-X. Liu, Observing the shadow of Einstein-Maxwell-Dilaton-Axion black hole, *JCAP* 11 (2013) 063 [arXiv:1311.4251] [inSPIRE].

[59] X.-X. Zeng, H.-Q. Zhang and H. Zhang, Shadows and photon spheres with spherical accretions in the four-dimensional Gauss-Bonnet black hole, *Eur. Phys. J. C* 80 (2020) 872 [arXiv:2004.12074] [inSPIRE].

[60] X.-X. Zeng and H.-Q. Zhang, Influence of quintessence dark energy on the shadow of black hole, *Eur. Phys. J. C* 80 (2020) 1058 [arXiv:2007.06333] [inSPIRE].

[61] I. Bengtsson, S. Holst and E. Jakobsson, Classics Illustrated: Limits of Spacetimes, *Class. Quant. Grav.* 31 (2014) 205008 [arXiv:1406.4326] [inSPIRE].