On a distribution function of a probability measure involving a permutation

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Abstract

In [3], we have introduced a probability measure to study the power and exponential sums for a certain coding system. The distribution function of the probability measure gives explicit formulas for the power and exponential sums.

[3, Theorem 4] states that the higher order derivatives of the distribution function with respect to a certain parameter are expressed by a generalization of the Takagi function. In [3], we only gave the sketch of the proof of Theorem 4, because the complete proof is very long. The purpose of this paper is to give the complete proof of [3, Theorem 4].

1 Introduction

Let $q \geq 2$ be an integer and $\sigma$ be a permutation

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & q-1 \\ \sigma(0) & \sigma(1) & \cdots & \sigma(q-1) \end{pmatrix}.$$

Throughout the paper, we assume that $\sigma^q = \text{id}$. A probability measure involving $\sigma$ on the unit interval has been introduced in [3]. Let us recall the definition of the measure.
Let $I = I_0(0) = [0, 1]$, and for each positive integer $k$, let
\[
I_k(n) = \left[\frac{n}{q^k}, \frac{n + 1}{q^k}\right), \quad 0 \leq n \leq q^k - 2,
\]
\[
I_k(q^k - 1) = \left[\frac{q^k - 1}{q^k}, 1\right].
\]
We denote the $\sigma$-field $\sigma\{I_k(n); 0 \leq n \leq q^k - 1\}$ by $F_k$ and the $\sigma$-field $\bigvee_{k=0}^{\infty} F_k$ by $\mathcal{F}$.

**Definition 1.** Let $d = (d_0, \ldots, d_{q-2})$ be a vector with $0 < d_j < 1$ ($0 \leq j \leq q - 2$) and $0 < \sum_{j=0}^{q-2} d_j < 1$, and set $d_{q-1} = 1 - \sum_{j=0}^{q-2} d_j$. Let $r = (r_0, \ldots, r_{q-2})$ be a vector whose components satisfy the same conditions as those of $d$, and set $r_{q-1} = 1 - \sum_{j=0}^{q-2} r_j$. Then the probability measure $\mu_{d,r}$ involving a permutation $\sigma$ on $(I, \mathcal{F})$ is defined as follows.

(i) $\mu_{d,r}(I) = 1$,
(ii) $\mu_{d,r}(I_1(n)) = d_n, \quad 0 \leq n \leq q - 1$,
(iii) for $k \geq 2$,
\[
\mu_{d,r}(I_k(n)) = \mu_{d,r}(I_{k-1}(j)) \times r_{\sigma^l(j)}, \quad 0 \leq n \leq q^k - 1,
\]
where $j$ and $l$ are integers with $n = qj + l$ ($0 \leq l \leq q - 1$). The distribution function $L_{d,r}$ of $\mu_{d,r}$ is defined by
\[
L_{d,r}(x) = \mu_{d,r}([0, x]), \quad x \in I.
\]
For simplicity, we use the abbreviation $L_r$ for $L_{r,r}$.

The measure $\mu_{d,r}$ is a generalization of the multinomial measure (see [4]) and the Gray measure (see [2]).

There is an interesting relation between $L_r(x)$ and the exponential sum for a certain coding system related to paperfolding sequences (see [3, Theorem 1]). Moreover, since $L_r(x)$ is an analytic function of $r$ (see [3, Theorem 2]), the power sums for the coding system are related to the higher order derivatives of $L_r(x)$ with respect to $r$ (see [3, Theorem 3]).

[3, Theorem 4] states that the higher order derivatives of $L_r(x)$ with respect to $r$ are expressed by a generalization of the Takagi function. To describe [3, Theorem 4], we prepare several notations. Let $q, e_l$, and $u$ be vectors with
\[
q = \left(\frac{1}{q}, \ldots, \frac{1}{q}\right),
\]
\[
e_l = \left(\frac{0}{q}, \ldots, \frac{l}{q}, 0, \ldots, 0\right), \quad 0 \leq l \leq q - 2,
\]
\[
u = (u_0, \ldots, u_{q-2}), \quad u_l \in \mathbb{N} \cup \{0\},
\]
\[2\]
and define
\[ |u| = u_0 + u_1 + \cdots + u_{q-2}, \quad u^! = \prod_{l=0}^{q-2} u_l! . \]

For \( n \in \mathbb{N} \cup \{ 0 \} \), let \( r_{\sigma^n} \) be the vector with
\[ r_{\sigma^n} = (r_{\sigma^n(0)}, \ldots, r_{\sigma^n(q-2)}). \]

For a set \( S \), let \( 1_S \) be the indicator function of \( S \). Define the function \( \Phi_l \) on \( I \) by
\[ \Phi_l = \sum_{j=0}^{q-1} 1_{I_l(qj+\sigma^{-j})}, \quad 0 \leq l \leq q-1. \]

Let \( \phi(x) \) be the function on \( I \) such that \( \phi(x) = qx \pmod{1} \) with \( 0 \leq \phi(x) < 1 \) for \( x \in [0,1) \) and \( \phi(1) = 1 \). We use the notation
\[ f \circ \phi^j(x) = f(\phi(\phi(\cdots \phi(x)))) \]
for any function \( f \). We denote the Lebesgue measure on \( I \) by \( \mu \).

**Definition 2.** The generalized Takagi function \( T_{d,r,u}(x) \) is defined as follows.

(i) If \( u = e_l \), then
\[
T_{d,r,e_l}(x) = \frac{1}{q} \sum_{j=0}^{\infty} \sum_{n=0}^{q^{j-1}} \mu_d,r(I_j(n))1_{I_j(n)}(x) \int_0^{\phi_j(x)} \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) d\mu_{r_{\sigma^n},r}. 
\]

(ii) If \( |u| \geq 2 \), then
\[
T_{d,r,u}(x) = \sum_{j=0}^{\infty} \sum_{\alpha=0}^{q^{j-1}-1} \left( \frac{\Phi_{\alpha}}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) 
\times \sum_{n=0}^{q^{j+1}-1} \mu_d,r(I_j(n))1_{I_{j+1}(n)}(x) \left( T_{r_{\sigma^n},r,u-e_\alpha} \circ \phi^{j+1}(x) \right). 
\]

Then the higher order derivatives of \( L_r(x) \) with respect to \( r \) are expressed as the following.

**Theorem 1.** ([3, Theorem 4]) (i) If \( u = e_l \), then
\[
\frac{1}{q} \frac{\partial}{\partial r_l} L_r(x) = (1_{I_l(1)}(x) - 1_{I_l(q-1)}(x))(L_{q,r}(x) - x) 
\]

3
\[ + \left( \sum_{n=0}^{q-1} r_n I_{I_1(n)}(x) \right) q \mathcal{T}_{q,r,e_i}(x) + \int_0^x (1_{I_1(t)} - 1_{I_1(q-1)}) d\mu. \]

(ii) If \(|u| \geq 2\), then

\[ \frac{1}{qu!} \frac{\partial^{a_0 + \cdots + u_{q-2}}}{\partial r_0^{a_0} \cdots \partial r_{q-2}^{u_{q-2}}} L_r(x) = \sum_{j=0}^{q-2} (1_{I_1(j)}(x) - 1_{I_1(q-1)}(x)) q \mathcal{T}_{q,r,u-e}(x) \]

\[ + \left( \sum_{n=0}^{q-1} r_n I_{I_1(n)}(x) \right) q \mathcal{T}_{q,r,u}(x). \]

In [3], we only gave the sketch of the proof of the above Theorem 1, because the complete proof is very long. The purpose of this paper is to give the complete proof of Theorem 1.

Finally, we mention the previous works on studying relations between higher order derivatives of distribution functions and Takagi functions. Hata–Yamaguti [1] is the first work clarifying a relation between the first order derivative of \( L_r(x) \) with respect to \( r \) and the usual Takagi function in the dyadic case. In [5], Hata–Yamaguti’s result is studied from a viewpoint of the binomial measure, and, in [4], it is generalized in the \( q \)-adic case, in which the multinomial measure and its distribution function play essential roles. In Kobayashi [2], the Gray measure and its distribution function are studied from a viewpoint of [5] and [4]. Since the measure \( \mu_{d,r} \) is a generalization of the multinomial measure and Gray measure, Theorem 1 is a natural generalization of the results obtained in [1, 5, 4], and [2].

2 Preliminary lemmas

For a fixed \( k \in \mathbb{N} \), any integer \( n \) with \( 0 \leq n \leq q^k - 1 \) is expressed as \( n = \sum_{i=0}^{k-1} n_i q^i \), where \( n_i \in \{0, 1, \ldots, q-1\} \). We use the abbreviation \( n = n_{k-1} \cdots n_0 \) for \( n = \sum_{i=0}^{k-1} n_i q^i \), in which the length of the word is always equal to \( k \), and identify \( I_k(n) \) with \( I_k(n_{k-1} \cdots n_0) \).

Firstly, we study a relation between \( \phi \) and \( \mu_{d,r} \). We note a simple fact

\[ I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0) \subset I_i(c_{i-1} \cdots c_0) \iff b_{i-1} \cdots b_0 = c_{i-1} \cdots c_0. \]

Lemma 1. We have

\[ \phi^i \left( \bigcup_{0 \leq b_0, \ldots, b_{i-1} \leq q-1} I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0) \right) = I_k(a_{k-1} \cdots a_0). \]

Proof. By the definition of \( \phi \), we have

\[ \phi \left( \bigcup_{0 \leq b_0, \ldots, b_{i-1} \leq q-1} I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0) \right) \]
Hence we obtain the assertion. □

Lemma 1 is equivalent to the following.

**Lemma 2.** We have

$$1_{I_k(a_{k-1}\cdots a_0)} \circ \phi^i = 1_{\bigcup_{0 \leq b_0, \ldots, b_{i-2} \leq q-1} I_{i-1+k}(b_{i-2} \cdots b_0 a_{k-1} \cdots a_0)}.$$ 

**Lemma 3.** We have

$$\mu_{d,r}(I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0)) = \mu_{d,r}(I_{i}(b_{i-1} \cdots b_0)) \mu_{r_{a_{k-1}}, r}(I_k(a_{k-1} \cdots a_0)).$$

**Proof.** It follows from Definition 1 and the assumption $\sigma^q = id$ that

$$\mu_{d,r}(I_{i+k}(b_{i-1} \cdots b_0 a_{k-1} \cdots a_0)) = \mu_{d,r}(I_{i}(b_{i-1} \cdots b_0)) r_{\sigma_0}(a_{k-1}) r_{\sigma^1(a_k-1)} \cdots r_{\sigma^{a_1}(a_k)}$$

and

$$\mu_{r_{a_{k-1}}, r}(I_k(a_{k-1} \cdots a_0)) = r_{\sigma_0}(a_{k-1}) r_{\sigma^1(a_k-1)} \cdots r_{\sigma^{a_1}(a_k)}.$$

Hence we obtain the assertion. □

**Lemma 4.** For any $i \in \mathbb{N}$, $a \in \mathbb{N} \cup \{0\}$ with $0 \leq a \leq q^i - 1$, and $x \in I_i(a_{i-1} \cdots a_0)$, we have

$$1_{[0,\phi(x)]}(\phi^i(y)) \times 1_{I_i(a_{i-1} \cdots a_0)}(y) = 1_{[0,x]}(y) \times 1_{I_i(a_{i-1} \cdots a_0)}(y).$$  \hfill (1)

**Proof.** We prove this by induction on $i$. When $i = 1$, we have for $x \in I_1(a_0)$

$$\phi(y) \in [0, \phi(x)] \iff y \in \bigcup_{m=0}^{q-1} \left[ \frac{m}{q}, \frac{m}{q} + \left|\frac{a_0}{q}\right| \right],$$

and hence

$$\phi(y) \in [0, \phi(x)] \text{ and } y \in I_1(a_0) \iff y \in [0, x] \cap I_1(a_0),$$

from which we get

$$1_{[0,\phi(x)]}(\phi(y)) \times 1_{I_1(a_0)}(y) = 1_{[0,x]}(y) \times 1_{I_1(a_0)}(y).$$  \hfill (2)

By Lemma 2, we have

$$1_{I_1(a_0)}(\phi^i(y)) \times 1_{I_i(a_{i-1} \cdots a_1)}(y) = 1_{I_{i+1}(a_{i-1} \cdots a_0)}(y).$$  \hfill (3)
Therefore, if \( x \in I_{i+1}(a_i \cdots a_0) \) and (1) holds for \( i \), by (3), (2), and Lemma 2, we have
\[
\begin{align*}
1_{[0,n^i(x)]}(f^{i+1}(x)) & \times 1_{I_{i+1}(a_i \cdots a_0)}(y) \\
= 1_{[0,n^i(x)]}( \phi^i(y) ) & \times 1_{I_{i}(a_0)}( \phi^i(y) ) \times 1_{I_{i}(a_i \cdots a_1)}(y) \\
= 1_{[0,n^i(x)]}( \phi^i(y) ) & \times 1_{I_{i}(a_0)}( \phi^i(y) ) \times 1_{I_{i}(a_i \cdots a_1)}(y) \\
= 1_{[0,x]}(y) & \times 1_{I_{i}(a_i \cdots a_1)}(y) \times 1_{I_{i}(a_0)}( \phi^i(y) ) \\
= 1_{[0,x]}(y) & \times 1_{I_{i+1}(a_i \cdots a_0)}(y).
\end{align*}
\]
This completes the proof. \( \square \)

For any bounded \( \mathcal{F} \)-measurable function \( f \), let
\[
E_{\mu_{d,r}}(f) = \int f d\mu_{d,r},
\]
\[
E_{\mu_{d,r}}(f; I_k(n)) = \int_{I_k(n)} f d\mu_{d,r}.
\]

Lemmas 5 and 6 show that a kind of integration by substitution is valid.

**Lemma 5.** For any \( i \in \mathbb{N} \), \( a \in \mathbb{N} \cup \{0\} \) with \( 0 \leq a \leq q^n - 1 \), and a bounded \( \mathcal{F} \)-measurable function \( f \), we have
\[
E_{\mu_{d,r}}(f \circ \phi_i; I_i(a_{i-1} \cdots a_0)) = \mu_{d,r}(I_i(a_{i-1} \cdots a_0))E_{\mu_{a_0,r}}(f).
\]

**Proof.** Since a bounded \( \mathcal{F} \)-measurable function can be approximated by step functions, it suffices to show the equality for \( f = 1_{I_j(c_{j-1} \cdots c_0)} \). By Lemmas 2 and 3, we have
\[
\begin{align*}
E_{\mu_{d,r}}(1_{I_j(c_{j-1} \cdots c_0)} \circ \phi_i; I_i(a_{i-1} \cdots a_0)) \\
= \int_{I} 1_{\bigcup_{b_0 \cdots b_j} I_{i+j}(b_0 \cdots b_j c_{j-1} \cdots c_0)}(y) \times 1_{I_i(a_{i-1} \cdots a_0)}(y) \ d\mu_{d,r} \\
= \mu_{d,r}(I_{i+j}(a_{i-1} \cdots a_0 c_{i-1} \cdots c_0)) \\
= \mu_{d,r}(I_i(a_{i-1} \cdots a_0))E_{\mu_{a_0,r}}(1_{I_j(c_{j-1} \cdots c_0)}).
\end{align*}
\]

**Lemma 6.** For any \( i \in \mathbb{N} \), \( a \in \mathbb{N} \cup \{0\} \) with \( 0 \leq a \leq q^n - 1 \), a bounded \( \mathcal{F} \)-measurable function \( f \), and \( x \in I_i(a_{i-1} \cdots a_0) \), we have
\[
E_{\mu_{d,r}}(f \circ \phi_i; I_i(a_{i-1} \cdots a_0) \cap [0, x]) = \mu_{d,r}(I_i(a_{i-1} \cdots a_0))E_{\mu_{a_0,r}}(f; [0, \phi_i(x)]).
\]
Proof. By Lemmas 4 and 5 we obtain

\[
E_{\mu_d,r} (f \circ \phi^i; I_i(a_{i-1} \cdots a_0) \cap [0,x]) \\
= E_{\mu_d,r} ((f \circ \phi^i) \times 1_{[0,x]}; I_i(a_{i-1} \cdots a_0)) \\
= E_{\mu_d,r} ((f \circ \phi^i) \times (1_{[0,\phi^i(x)]} \circ \phi^i); I_i(a_{i-1} \cdots a_0)) \\
= \mu_d, r(I_i(a_{i-1} \cdots a_0))E_{\mu_d, r, a_0} (f \times 1_{[0,\phi^i(x)]}).
\]

Next, we discuss the conditional expectation \(E_{\mu_d,r} (\cdot | \mathcal{F}_k)\). For a bounded \(\mathcal{F}\)-measurable function \(g\), \(E_{\mu_d,r} (g| \mathcal{F}_k)\) is defined to be the \(\mathcal{F}_k\)-measurable function such that

\[
\int_G E_{\mu_d,r} (g| \mathcal{F}_k) d\mu_d,r = \int_G g d\mu_d,r, \quad \text{for all } G \in \mathcal{F}_k.
\]

Since \(\mathcal{F}_k\) is the finite set and \(E_{\mu_d,r} (g| \mathcal{F}_k)\) is \(\mathcal{F}_k\)-measurable, \(E_{\mu_d,r} (g| \mathcal{F}_k)\) is a step function with constant values on \(I_k(n)\)’s. In fact, it is written explicitly as

\[
E_{\mu_d,r} (g| \mathcal{F}_k) = \sum_{n=0}^{\phi^k-1} \frac{E_{\mu_d,r} (g; I_k(n))}{\mu_d, r(I_k(n))} 1_{I_k(n)}.
\]  

Lemma 7. Let \(g\) be a bounded \(\mathcal{F}\)-measurable function. If \(h\) is a \(\mathcal{F}_k\)-measurable function, and \(g\) satisfies \(E_{\mu_d,r} (g| \mathcal{F}_k) = 0\), then

\[
\int_0^x h g d\mu_d,r = h(x) \int_0^x g d\mu_d,r.
\]

Proof. By (4), \(E_{\mu_d,r} (g| \mathcal{F}_k) = 0\) is equivalent to \(E_{\mu_d,r} (g; I_k(n)) = 0\) for every \(n\). Since \(h\) is \(\mathcal{F}_k\)-measurable, it takes a constant value \(C_n\) on \(I_k(n)\). Hence

\[
E_{\mu_d,r} (h g; I_k(n)) = C_n E_{\mu_d,r} (g; I_k(n)) = 0.
\]

Thus we obtain for \(x \in I_k(m)\)

\[
\int_0^x h g d\mu_d,r = E_{\mu_d,r} (h g; I_k(m) \cap [0,x]) \\
= h(x) E_{\mu_d,r} (g; I_k(m) \cap [0,x]) = h(x) \int_0^x g d\mu_d,r.
\]

Since the equality is independent of \(m\), it is valid for \(x \in I\). \(\square\)
3 The Radon-Nikodym derivative on the finite set

Let \( e = (e_0, \ldots, e_{q-2}) \) and \( s = (s_0, \ldots, s_{q-2}) \) be vectors whose components satisfy the same conditions as those of \( d \) in Definition 1, and set \( e_{q-1} = 1 - \sum_{j=0}^{q-3} e_j \) and \( s_{q-1} = 1 - \sum_{j=0}^{q-2} s_j \).

**Definition 3.** The function \( Z(e, s; k) : I \to \mathbb{R} \) is defined by

\[
Z(e, s; k) = \sum_{n=0}^{q-1} \frac{\mu_{e, s}(I_k(n))}{\mu_{d, r}(I_k(n))} I_{k}(n), \quad k \in \mathbb{N} \cup \{0\}.
\]

**Remark 1.** \( Z(e, s; k) \) is the so-called Radon-Nikodym derivative \( d\mu_{e, s}/d\mu_{d, r} \) on \( F_k \).

We identify \( I_k(n) \) with \( I_k(n_{k-1} \cdots n_0) \) as in the previous section.

**Definition 4.** The function \( W(s ; r) : I \to \mathbb{R} \) is defined by

\[
W(s ; r) = \sum_{0 \leq b_0, b_1 \leq q-1} s_{\sigma b_1}(b_0) 1_{I_{2}(b_1b_0)}.
\]

The following propositions have been proved in [3].

**Proposition 1.** We have

\[
L_{e, s}(x) = \lim_{k \to \infty} \int_0^x Z(e, s; k) d\mu_{d, r},
\]

where the convergence is uniform for \( e = (e_0, \ldots, e_{q-2}) \) and \( s = (s_0, \ldots, s_{q-2}) \).

**Proposition 2.** For \( k \geq 1 \), we have

\[
Z(e, s; k+1) = (\prod_{i=0}^{k-1} W(s ; r) \circ \phi^i) Z(e, s; 1).
\]

4 Higher order derivatives of distribution functions

Firstly, we study a relation between \( L_s(x) = L_{s, s}(x) \) and \( L_{q,s}(x) \).

**Lemma 8.** We have

\[
L_s(x) = (q \sum_{n=0}^{q-1} s_n 1_{I_1(n)}(x))(L_{q,s}(x) - x) + q \int_0^x \sum_{n=0}^{q-1} s_n 1_{I_1(n)}d\mu,
\]

where \( \mu \) is the Lebesgue measure on \( I \).
Proof. Let \( x \in I_1(m) \) \((0 \leq m \leq q - 1)\). Then it follows that
\[
L_s(x) = L_s\left(\frac{m}{q}\right) + \mu_{s,s}\left(\frac{m}{q}, x\right)
\]
\[
= \sum_{n=0}^{m-1} s_n + \frac{s_m}{1/q} \mu_{s,s}\left(\frac{m}{q}, x\right)
\]
\[
= q s_m (L_{q,s}(x) - x) + \sum_{n=0}^{m-1} s_n + q s_m \left(x - \frac{m}{q}\right).
\] (5)

Noting that, for \( x \in I_1(m) \),
\[
\int_0^x 1_{I_1(n)} d\mu = \begin{cases} 
0, & n > m, \\
x - \frac{n}{q}, & n = m, \\
\frac{1}{q}, & n < m,
\end{cases}
\]
we have
\[
\sum_{n=0}^{m-1} s_n + q s_m \left(x - \frac{m}{q}\right) = \sum_{n=0}^{q-1} q s_n \int_0^x 1_{I_1(n)} d\mu.
\] (6)

Substituting (6) into (5) and replacing the range of the variable \( x \) to \( I \), we obtain the assertion. \( \square \)

By Lemma 8, we have easily the following relation between the higher order derivative of \( L_s(x) \) and that of \( L_{q,s}(x) \).

**Lemma 9.** (i) If \( u = e_1 \), then
\[
\frac{1}{q} \frac{\partial}{\partial s_l} L_s(x) = (1_{I_1(l)}(x) - 1_{I_1(q-1)}(x))(L_{q,s}(x) - x)
\]
\[
+ \left(\sum_{n=0}^{q-1} s_n 1_{I_1(n)}(x)\right) \frac{\partial}{\partial s_l} L_{q,s}(x) + \int_0^x (1_{I_1(l)} - 1_{I_1(q-1)}) d\mu.
\]

(ii) If \(|u| \geq 2\), then
\[
\frac{1}{q} \frac{\partial^{u_0 + \ldots + u_{q-2}}}{\partial s_0^{u_0} \ldots \partial s_{q-2}^{u_{q-2}}} L_s(x) = \sum_{j=0}^{q-2} u_j (1_{I_1(j)}(x) - 1_{I_1(q-1)}(x))
\]
\[
	imes \frac{\partial^{u_0 + \ldots + u_{j-1} + (u_j - 1) + u_{j+1} + \ldots + u_{q-2}}}{\partial s_0^{u_0} \ldots \partial s_{j-1}^{u_{j-1}} \partial s_j^{u_j-1} \partial s_{j+1}^{u_{j+1}} \ldots \partial s_{q-2}^{u_{q-2}}} L_{q,s}(x)
\]
\[
+ \left(\sum_{n=0}^{q-1} s_n 1_{I_1(n)}(x)\right) \frac{\partial^{u_0 + \ldots + u_{q-2}}}{\partial s_0^{u_0} \ldots \partial s_{q-2}^{u_{q-2}}} L_{q,s}(x).
\]
Next, we study the higher order derivative of $L_{q,s}(x)$. Let $\psi_u : \{1, 2, \ldots, |u|\} \to \{0, 1, \ldots, q - 2\}$ be a mapping such that $\#\{m; \psi_u(m) = j\} = u_j$, which is the same one as that of [4]. For example, if $q = 4$, $u = (u_0, u_1, u_2) = (1, 2, 0)$, then $\psi_u : \{1, 2, 3\} \to \{0, 1, 2\}$ is a mapping satisfying $\#\{m; \psi_u(m) = 0\} = 1$, $\#\{m; \psi_u(m) = 1\} = 2$, and $\#\{m; \psi_u(m) = 2\} = 0$. In fact, $\psi_u$ is one of three mappings

$$
\begin{cases}
\psi_u(1) = 0, \\
\psi_u(2) = 1, \\
\psi_u(3) = 1,
\end{cases}
\quad
\begin{cases}
\psi_u(1) = 1, \\
\psi_u(2) = 0, \\
\psi_u(3) = 1,
\end{cases}
\quad
\begin{cases}
\psi_u(1) = 1, \\
\psi_u(2) = 1, \\
\psi_u(3) = 0.
\end{cases}
$$

**Lemma 10.** We have

$$
\frac{\partial^{u_0 + \cdots + u_{q-2}}}{\partial s_0^{u_0} \cdots \partial s_{q-2}^{u_{q-2}}} L_{q,s}(x) \bigg|_{s=r} = u! \lim_{k \to \infty} \sum_{0 \leq i_1 < \cdots < i_{|u|} \leq k-2} \sum_{\psi_u} \int_0^x \prod_{m=1}^{|u|} \left( \frac{\Phi_{\psi_u(m)}}{r_{\psi_u(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^i d\mu_{q,r},
$$

where the sum $\sum_{\psi_u}$ is taken over all $\psi_u$'s.

**Proof.** By Propositions [1] and [2] with $e = d = q$, we have

$$
L_{q,s}(x) = \lim_{k \to \infty} \int_0^x \prod_{i=0}^{k-2} W\left[ \frac{s}{r} \right] \circ \phi^i d\mu_{q,r}.
$$

From the definitions of $W\left[ \frac{s}{r} \right]$ and $\Phi_I$, it follows that

$$
W\left[ \frac{s}{r} \right] \circ \phi^i = \sum_{l=0}^{q-1} \frac{s_l}{r_l} (\Phi_I \circ \phi^i).
$$

Let $K_{s,i} = \sum_{l=0}^{q-1} s_l (\Phi_I \circ \phi^i)$. We show the equality

$$
\sum_{l=0}^{q-1} \frac{s_l}{r_l} (\Phi_I \circ \phi^i) = \frac{K_{s,i}}{K_{r,i}}.
$$

Since $\sum_{l=0}^{q-1} \Phi_I = 1$, it holds that $\sum_{l=0}^{q-1} \Phi_I \circ \phi^i = 1$. For any $x \in I$, there exists a unique $m$ such that $\Phi_m \circ \phi^i(x) = 1$, $\Phi_I \circ \phi^i(x) = 0$ ($l \neq m$), and hence, both of $\sum_{l=0}^{q-1} \frac{s_l}{r_l} (\Phi_I \circ \phi^i(x))$ and $\frac{K_{s,i}}{K_{r,i}}(x)$ are $\frac{\sum_{l=0}^{q-1} s_l}{r_m}$. Combining (7), (8), and (9), we have

$$
L_{q,s}(x) = \lim_{k \to \infty} \int_0^x \prod_{i=0}^{k-2} \frac{K_{s,i}}{K_{r,i}} d\mu_{q,r}.
$$
For $a$ with $0 \leq a \leq q - 2$, 
\[
\frac{\partial}{\partial s_a} K_{s,i} = \frac{(\Phi_a - \Phi_{q-1}) \circ \phi^i}{K_{r,i}}.
\]

By the same argument as in \cite{[4], pp.459-460},
\[
\frac{\partial^{u_0 + \cdots + u_{q-2}}}{\partial s_0^{u_0} \cdots \partial s_{q-2}^{u_{q-2}}} \left[ \prod_{i=0}^{x} K_{s,i} \right]_{s=r} 
= u! \sum_{0 \leq i_1 < \cdots < i_{|u|} \leq k-2} \varphi_u \sum_{0 \leq i_1 < \cdots < i_{|u|} \leq k-2} \varphi_u \sum_{i=0}^{x} \left( \prod_{m=1}^{x} \frac{(\Phi_{|u|}(m) - \Phi_{q-1}) \circ \phi^m}{K_{s,i,m}} \right) \times \left( \prod_{i=0}^{k-2} \frac{K_{s,i}}{K_{r,i}} \right) d\mu_{q,r}.
\]

Hence
\[
\frac{\partial^{u_0 + \cdots + u_{q-2}}}{\partial s_0^{u_0} \cdots \partial s_{q-2}^{u_{q-2}}} \left[ \prod_{i=0}^{x} K_{s,i} \right]_{s=r} 
= u! \sum_{0 \leq i_1 < \cdots < i_{|u|} \leq k-2} \varphi_u \sum_{i=0}^{x} \left( \prod_{m=1}^{x} \frac{(\Phi_{|u|}(m) - \Phi_{q-1}) \circ \phi^m}{r_{|u|}(m) - r_{q-1}} \right) \circ \phi^m d\mu_{q,r}.
\]

From (10) and (11), the assertion follows. 

By Lemmas \cite{[4]} and \cite{[11]}, we obtain the following.

**Proposition 3.** (i) If $u = e_1$, then
\[
\left. \frac{1}{q} \frac{\partial}{\partial s_1} L_s(x) \right|_{s=r} = (1_{1_1}(t) - 1_{1_{q-1}}(t))(L_{q,r}(x) - x) 
+ \left( \sum_{n=0}^{q-1} r_n 1_{1_1}(n) \right) \lim_{k \to \infty} \sum_{0 \leq j \leq k-2} \int_{0}^{x} \left( \frac{\Phi_j}{r_{j}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j d\mu_{q,r} 
+ \int_{0}^{x} (1_{1_1}(t) - 1_{1_{q-1}}(t)) d\mu.
\]

(ii) If $|u| \geq 2$, then
\[
\left. \frac{1}{q u!} \frac{\partial^{u_0 + \cdots + u_{q-2}}}{\partial s_0^{u_0} \cdots \partial s_{q-2}^{u_{q-2}}} L_s(x) \right|_{s=r} = \sum_{j=0}^{q-2} (1_{1_j}(t) - 1_{1_{q-1}}(t)) 
\times \lim_{k \to \infty} \sum_{0 \leq i_1 < \cdots < i_{|u|} \leq k-2} \varphi_u \sum_{j=0}^{q-2} (1_{1_j}(t) - 1_{1_{q-1}}(t)) 
\times \lim_{k \to \infty} \sum_{0 \leq i_1 < \cdots < i_{|u|} \leq k-2} \varphi_u \sum_{i=0}^{x} \left( \prod_{m=1}^{x} \frac{(\Phi_{|u|}(m) - \Phi_{q-1}) \circ \phi^m}{r_{|u|}(m) - r_{q-1}} \right) \circ \phi^m d\mu_{q,r}.
\]
5 A recursive relation for $D_{d,r,u,k}(x)$

Based on the expression of Proposition 3, we introduce the function $D_{d,r,u,k}$.

**Definition 5.** The function $D_{d,r,u,k} : I \rightarrow \mathbb{R}$ is defined by

$$D_{d,r,u,k}(x) = \frac{1}{q} \sum_{0 \leq i_1 < \cdots < i_k \leq k} \sum_{\psi_u} \int_0^x \prod_{m=1}^{|u|} \left( \frac{\Phi_{\psi_u(m)}}{r_{\psi_u(m)}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^m d\mu_{d,r}.$$

We will give a recursive relation for $D_{d,r,u,k}(x)$ (see Proposition 4 below), which gives the definition of generalized Takagi functions.

**Lemma 11.** For any $k, \beta \in \mathbb{N}$ with $\beta + 2 > k$, and integers $l, k$ with $0 \leq l \leq q - 1$, $0 \leq n \leq q^k - 1$, we have

$$E_{\mu_{d,r}} \left( \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^\beta ; I_k(n) \right) = 0.$$

**Proof.** For the $q$-adic representations $n = n_{k-1} \cdots n_0$ and $qj + \sigma^{-j}(l) = j\sigma^{-j}(l)$, we have

$$E_{\mu_{d,r}} \left( \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^\beta ; I_k(n_{k-1} \cdots n_0) \right)$$

$$= \frac{1}{r_l} \sum_{j=0}^{q-1} E_{\mu_{d,r}} \left( 1_{I_{2(j\sigma^{-j}(l))}} \circ \phi^\beta ; I_k(n_{k-1} \cdots n_0) \right)$$

$$= \frac{1}{r_{q-1}} \sum_{j=0}^{q-1} E_{\mu_{d,r}} \left( 1_{I_{2(j\sigma^{-j}(q-1))}} \circ \phi^\beta ; I_k(n_{k-1} \cdots n_0) \right). \quad (12)$$

By Lemma 2

$$\sum_{j=0}^{q-1} E_{\mu_{d,r}} \left( 1_{I_{2(j\sigma^{-j}(l))}} \circ \phi^\beta ; I_k(n_{k-1} \cdots n_0) \right)$$

$$= E_{\mu_{d,r}} \left( 1_{\bigcup_{b_2,b_3,\ldots \leq q-1} \bigcup_{b_{k+1}=0}^{b_{k+1}} I_{\beta+2(b_{\beta+1} \cdots b_{\beta+k+1} \cdots b_1 \sigma^{-b_1}(l))} ; I_k(n_{k-1} \cdots n_0) \right)$$

$$= E_{\mu_{d,r}} \left( 1_{\bigcup_{b_2,b_3,\ldots \leq q-1} \bigcup_{b_{k+1}=0}^{b_{k+1}} I_{\beta+2(n_{k-1} \cdots n_0 b_{\beta-k+1} \cdots b_1 \sigma^{-b_1}(l))} \right)$$

$$= \mu_{d,r} \left( I_{\beta+2(n_{k-1} \cdots n_0 b_{\beta-k+1} \cdots b_1 \sigma^{-b_1}(l))} \right).$$

Here, by Lemma 3

$$\mu_{d,r} \left( I_{\beta+2(n_{k-1} \cdots n_0 b_{\beta-k+1} \cdots b_1 \sigma^{-b_1}(l))} \right)$$

$$= \mu_{d,r} \left( I_{\beta+1(n_{k-1} \cdots n_0 b_{\beta-k+1} \cdots b_1)} \right) \mu_{r_{\sigma^{-b_1} \cdot R}} \left( I_1(\sigma^{-b_1}(l)) \right)$$

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= r_l \mu_{d,r}(I_{\beta} + 1(n_{k-1} \cdots n_0 b_{\beta-k+1} \cdots b_1)),

and hence

\[
\sum_{j=0}^{q-1} E_{\mu_{d,r}}(I_{I_2(j+1)} \circ \phi^{j} ; I_k(n_{k-1} \cdots n_0)) = r_l \mu_{d,r}(I_k(n_{k-1} \cdots n_0)).
\]  \hspace{1cm} (13)

Substituting (13) into (12), we obtain the assertion. \hfill \Box

**Lemma 12.** For any \( u \) with \(|u| \geq 2 \) and \( k \in \mathbb{N} \), let \( \{\beta_m\}_{m=1}^{[u]} \) be a strictly increasing sequence with \( \beta_1 + 2 > k \). Then we have

\[
E_{\mu_{d,r}}\left( \prod_{m=1}^{[u]} \left( \frac{\Phi_{\psi_u(m)}}{\psi_u(m)} - \frac{\Phi_{q-1}}{\psi_u(m)} \right) \circ \phi^{\beta_m} ; I_k(n) \right) = 0
\]

for every \( 0 \leq n \leq q^k - 1 \).

*Proof.* Set \( \alpha = \psi_u(1) \). Then \( u_\alpha > 0 \) by the definition of \( \psi_u \). We classify the set of \( \psi_u \)'s by \( \alpha \). By the definition of \( \psi_u \),

\[
\psi_u : \{1, 2, \ldots, [u]\} \rightarrow \{0, 1, \ldots, q - 2\},
\]

\[
\# \{2 \leq m \leq [u] ; \psi_u(m) = j\} = \begin{cases} u_j - 1, & \text{if } j = \alpha, \\ u_j, & \text{if } j \neq \alpha, \end{cases}
\]

and

\[
\psi_{u-e_\alpha} : \{1, 2, \ldots, [u] - 1\} \rightarrow \{0, 1, \ldots, q - 2\},
\]

\[
\# \{1 \leq m \leq [u] - 1 ; \psi_{u-e_\alpha}(m) = j\} = \begin{cases} u_j - 1, & \text{if } j = \alpha, \\ u_j, & \text{if } j \neq \alpha. \end{cases}
\]

Hence, for any \( \psi_u \) there exists a unique \( \psi_{u-e_\alpha} \) such that

\[
\psi_u(m) = \psi_{u-e_\alpha}(m - 1), \quad 2 \leq m \leq [u].
\]  \hspace{1cm} (14)

It follows from (14) that

\[
E_{\mu_{d,r}}\left( \prod_{m=1}^{[u]} \left( \frac{\Phi_{\psi_u(m)}}{\psi_u(m)} - \frac{\Phi_{q-1}}{\psi_u(m)} \right) \circ \phi^{\beta_m} ; I_k(n) \right)
\]

\[= E_{\mu_{d,r}}\left( \left( \frac{\Phi_{\psi_u(1)}}{\psi_u(1)} - \frac{\Phi_{q-1}}{\psi_u(1)} \right) \circ \phi^{\beta_1} \times \left( \prod_{m=2}^{[u]} \left( \frac{\Phi_{\psi_u(m)}}{\psi_u(m)} - \frac{\Phi_{q-1}}{\psi_u(m)} \right) \circ \phi^{\beta_m} \right) ; I_k(n) \right) \]
= E_{μ_d,r} \left( \left( \frac{\Phi_α}{r_α} - \frac{\Phi_q-1}{r_q-1} \right) \circ φ^β_1 \right) \times \left( \prod_{m=1}^{[u]-1} \left( \frac{\Phi_{ψ_u-e_α(m)}}{r_{ψ_u-e_α(m)}} - \frac{\Phi_q-1}{r_q-1} \right) \circ φ^{β_{m+1}}; I_k(n) \right). \quad (15)

Here we express $I_k(n_{k-1} \cdots n_0)$, $n = n_{k-1} \cdots n_0$, as

$$I_k(n_{k-1} \cdots n_0) = \bigcup_{0 \leq b_0, \ldots, b_{β_1-1} \leq q-1} I_{β_1+2}(n_{k-1} \cdots n_0b_{β_1-1} \cdots b_0). \quad (16)$$

Since $\left( \frac{\Phi_α}{r_α} - \frac{\Phi_q-1}{r_q-1} \right) \circ φ^β_1$ in (15) is $F_{β_1+2}$-measurable (see Lemma 2), it takes a constant value $C_{β_{β_1-1} \cdots b_0}$ on $I_{β_1+2}(n_{k-1} \cdots n_0b_{β_1-1} \cdots b_0)$. Hence, by (15) and (16),

$$E_{μ_d,r} \left( \prod_{m=1}^{[u]-1} \left( \frac{\Phi_{ψ_u(m)}}{r_{ψ_u(m)}} - \frac{\Phi_q-1}{r_q-1} \right) \circ φ^{β_m}; I_k(n) \right) = \sum_{0 \leq b_0, \ldots, b_{β_1-1} \leq q-1} C_{β_{β_1-1} \cdots b_0} \times E_{μ_d,r} \left( \prod_{m=1}^{[u]-1} \left( \frac{\Phi_{ψ_u-e_α(m)}}{r_{ψ_u-e_α(m)}} - \frac{\Phi_q-1}{r_q-1} \right) \circ φ^{β_{m+1}}; I_{β_1+2}(n_{k-1} \cdots n_0b_{β_1-1} \cdots b_0) \right).$$

By repeating this $[u] - 1$ times, there exists an integer $l$ with $0 \leq l \leq q - 2$ such that $E_{μ_d,r} \left( \prod_{m=1}^{[u]} \left( \frac{\Phi_{ψ_u(m)}}{r_{ψ_u(m)}} - \frac{\Phi_q-1}{r_q-1} \right) \circ φ^{β_m}; I_k(n) \right)$ is a linear combination of $E_{μ_d,r} \left( \left( \frac{Φ_{l}}{r_{l}} - \frac{Φ_q-1}{r_q-1} \right) \circ φ^{β_{[u]+l}}; I_{β_{[u]+l}+2}(n') \right)$ over $n'$s. Therefore the assertion follows from Lemma 11.

By Lemmas 11 and 12 with $k = 1$, we have easily the following.

**Lemma 13.** For any $u$ with $[u] \geq 1$ and $\{β_m\}_{m=1}^{[u]}$ with $0 \leq β_1 < β_2 < \cdots < β_{[u]}$, we have

$$E_{μ_d,r} \left( \prod_{m=1}^{[u]} \left( \frac{Φ_{ψ_u(m)}}{r_{ψ_u(m)}} - \frac{Φ_q-1}{r_q-1} \right) \circ φ^{β_m} \right) = 0.$$  \hspace{1cm} \square

**Proposition 4.** (i) If $u = e_l$, then

$$D_{d,r,e_l,k}(x) = \frac{1}{q} \sum_{j=0}^{k} \sum_{n=0}^{q^j-1} μ_{d,r}(I_j(n)) \int_{0}^{φ^j(x)} \frac{Φ_{l}}{r_{l}} - \frac{Φ_q-1}{r_q-1} dμ_{r,α,n,r}.$$

(ii) If $[u] \geq 2$, then

$$D_{d,r,u,k}(x) = \sum_{j=0}^{k-[u]+1} \sum_{α=0}^{q-2} \sum_{r_{α} > 0} \left( \frac{Φ_{α}}{r_{α}} - \frac{Φ_q-1}{r_q-1} \right) \circ φ^j(x).$$
\[ \sum_{n=0}^{q^j+1-1} \mu_{d,r}(I_{j+1}(n)) \mathbf{1}_{I_{j+1}(n)}(x) \left( D_{r_o \cap r, u-e_{\alpha}, k-j-1} \circ \phi^{j+1}(x) \right). \]

**Proof.** Taking \( u = e_t \) in Definition 5 we have

\[ D_{d,r,e_t,k}(x) = \frac{1}{q} \sum_{j=0}^{k} \int_0^x \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j d\mu_{d,r}. \]

If \( x \in I_j(n) \), then, by Lemmas 11 and 6,

\[ \int_0^x \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j d\mu_{d,r} = E_{\mu_{d,r}} \left( \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j ; I_j(n) \cap [0,x] \right) \]

\[ = \mu_{d,r}(I_j(n)) \int_0^{\phi^j(x)} \left( \frac{\Phi_l}{r_l} - \frac{\Phi_{q-1}}{r_{q-1}} \right) d\mu_{r_o \cap r, n} \]

which gives (i).

We express the sum \( \sum_{0 \leq i_1 < \ldots < i_{|u|} \leq k} \) in Definition 5 as \( \sum_{j=0}^{k-|u|+1} \sum_{j+1 \leq i_2 < \ldots < i_{|u|} \leq k} \) then, set \( i_{m-1}' = i_m - j - 1 \). Then we have, by (14),

\[ D_{d,r,u,k}(x) = \frac{1}{q} \sum_{j=0}^{k-|u|+1} \sum_{0 \leq i_1' < \ldots < i_{|u|-1}' \leq k-j-1} \sum_{\psi u}
\]

\[ \int_0^x \left( \frac{\Phi_{\psi u}(1)}{r_{\psi u}(1)} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j \times \left( \sum_{m=2}^{\alpha} \left( \frac{\Phi_{\psi u}(m)}{r_{\psi u}(m)} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{j_{m-1}+j+1} \right) d\mu_{d,r} \]

\[ = \frac{1}{q} \sum_{j=0}^{k-|u|+1} \sum_{0 \leq i_1' < \ldots < i_{|u|-1}' \leq k-j-1} \sum_{\alpha=0}^{\alpha-2} \sum_{u_{\alpha} > 0}
\]

\[ \int_0^x \left( \frac{\Phi_{\alpha}}{r_{\alpha}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j \times \left( \prod_{m=1}^{\alpha-1} \left( \frac{\Phi_{\psi u-e_{\alpha}}(m)}{r_{\psi u-e_{\alpha}}(m)} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{j_{m-1}+j+1} \right) d\mu_{d,r}. \]

By Lemma 2 \( \left( \frac{\Phi_{\alpha}}{r_{\alpha}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j \) is \( F_{j+2} \)-measurable. From (11) and Lemmas 11 and 12 it follows that

\[ E_{\mu_{d,r}} \left( \prod_{m=1}^{\alpha-1} \left( \frac{\Phi_{\psi u-e_{\alpha}}(m)}{r_{\psi u-e_{\alpha}}(m)} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^{j_{m-1}+j+1} \bigg| F_{j+2} \right) = 0. \]

Hence we have, by Lemma 7

\[ D_{d,r,u,k}(x) = \frac{1}{q} \sum_{j=0}^{k-|u|+1} \sum_{0 \leq i_1' < \ldots < i_{|u|-1}' \leq k-j-1} \sum_{\alpha=0}^{\alpha-2} \sum_{u_{\alpha} > 0}
\]

\[ \left( \frac{\Phi_{\alpha}}{r_{\alpha}} - \frac{\Phi_{q-1}}{r_{q-1}} \right) \circ \phi^j(x) \]
where $\| \cdot \|$.

Lemmas 5 and 13 give, for every $x$

Lemma 6 gives, for $x$

Combining (17), (18), (19), and Definition 5, we obtain for

Lemma 14.

6 Completion of the proof of Theorem 1

which gives (ii).

Lemma 6 gives, for $x \in I_{j+1}(n)$,

Combining (17), (18), (19), and Definition 5 we obtain for $x \in I_{j+1}(n)$

Combining (17), (18), (19), and Definition 5 we obtain for $x \in I_{j+1}(n)$

which gives (ii).

6 Completion of the proof of Theorem 1

Lemma 14. For any $u$ with $|u| \geq 1$, we have

max

where $\| \cdot \|_\infty$ means the supremum norm for $x \in I$. 16
Proof. Define $\mathcal{M}_{d,r,u,j}(x)$ by

$$
\mathcal{M}_{d,r,u,j}(x) = \left\{ \begin{array}{ll}
\frac{1}{q} \sum_{n=0}^{q^j-1} \mu_d r(I_j(n)) I_j(n)(x) \int_0^{\phi_j(x)} \frac{\Phi_l}{r_l} - \frac{\Phi_{q,l-1}}{r_{q,l-1}} |d\mu_{r,v}, r, u = e_l, u_j| \\
\sum_{a=0}^{q^j-1} \sum_{\alpha \geq 0} \left( \frac{\Phi_{r,a}}{r_{q,l-1}} \right) \circ \phi_j(x) \\
\times \sum_{n=0}^{q^j-1} \mu_d r(I_j+1(n)) I_j+1(n)(x) |T_{r,v}, r, u-e_a, r \circ \phi_j^j(x)|, |u| \geq 2.
\end{array} \right.
$$

Since $|\Phi_l(x)| \leq 1$, we have for $j \in \mathbf{N} \cup \{0\}$

$$
\left| \frac{\Phi_l}{r_l} - \frac{\Phi_{q,l-1}}{r_{q,l-1}} \circ \phi_j(x) \right| \leq \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}}.
$$

Fix $x \in I$. For every $j$, there exits an $m_j$ such that $x \in I_j(m_j)$. Then, by (20),

$$
\mathcal{M}_{d,r.e_l,j}(x) \leq \frac{1}{q} \mu_d r(I_j(m_j)) \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}} \mu_{r,v}, r([0, \phi_j(x)])
$$

$$
\leq \frac{2}{q} \min_{0 \leq a \leq q-1} \{r_a\} \left( \max_{0 \leq a \leq q-1} \{a\}\right) \left( \max_{0 \leq a \leq q-1} \{r_a\}\right)^{j-1}
$$

$$
\leq \frac{2}{q} \min_{0 \leq a \leq q-1} \{r_a\} \left( \max_{0 \leq a \leq q-1} \{r_a\}\right)^{j-1}.
$$

Hence

$$
\sum_{j=0}^{\infty} \|\mathcal{M}_{d,r.e_l,j}\|_\infty \leq \frac{2}{q} \min_{0 \leq a \leq q-1} \{r_a\} \left( \max_{0 \leq a \leq q-1} \{r_a\}\right)^{-1}.
$$

From $\|T_{d,r,e_l}\|_\infty \leq \sum_{j=0}^{\infty} \|\mathcal{M}_{d,r.e_l,j}\|_\infty$ and (21), it follows that

$$
\sup_{d'} \|T_{d',r,e_l}\|_\infty \leq \frac{2}{q} \min_{0 \leq a \leq q-1} \{r_a\} \left( \max_{0 \leq a \leq q-1} \{r_a\}\right)^{-1}.
$$

Fix $x \in I$. For every $j$, there exits an $m_j$ such that $x \in I_j+1(m_j)$. Then, by (20), we have for $|u| \geq 2$

$$
\mathcal{M}_{d,r,u,j}(x) \leq \frac{2}{\min_{0 \leq a \leq q-1} \{r_a\}} \mu_d r(I_j+1(m_j)) \sum_{a=0}^{q-2} |T_{r,m_j,r,u-e_a, r} \circ \phi_j^j(x)|
$$

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We prove this by induction on $|u|$. 

Proof.

where 

$$|T_{r,\gamma_j,\gamma,\theta^j+1}(x)|$$

$$2(q - 1) \left( \min_{0 \leq a \leq q-1} \{r_a\} \right) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \max_{\gamma',\gamma'|=|u|-1} |T_{r,\gamma_j,\gamma,\theta^j}(x)|$$

Hence

$$\sum_{j=0}^{\infty} \|M_{d,r,u,j}\|_\infty \leq \frac{2(q - 1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \max_{\gamma',\gamma'|=|u|-1} \sup_{d'} \|T_{d',r,\gamma'}\|_\infty. \tag{23}$$

From $\|T_{d,r,u}\|_\infty \leq \sum_{j=0}^{\infty} \|M_{d,r,u,j}\|_\infty$ and (23), it follows that

$$\max_{\gamma',\gamma'|=|u|} \sup_{d'} \|T_{d',r,\gamma'}\|_\infty \leq \frac{2(q - 1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \max_{\gamma',\gamma'|=|u|-1} \sup_{d'} \|T_{d',r,\gamma'}\|_\infty. \tag{24}$$

By repeating this $|u| - 1$ times, there exists an integer $l$ with $0 \leq l \leq q - 1$ such that

$$\max_{\gamma',\gamma'|=|u|} \sup_{d'} \|T_{d',r,\gamma'}\|_\infty \leq \left( \frac{2(q - 1)}{\min_{0 \leq a \leq q-1} \{r_a\}} \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^j \right)^{|u|-1} \sup_{d'} \|T_{d',r,e_1}\|_\infty. \tag{24}$$

Combining (24) with (22), we obtain the assertion. 

Lemma 15. For any $u$ with $|u| \geq 1$, we have

$$\max_{\gamma',\gamma'|=|u|} \sup_{d'} \|T_{d',r,\gamma'} - D_{d',r,\gamma'}\|_\infty \leq P_{|u|-1}(k) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^k,$$

where $P_{|u|-1}(k)$ is a polynomial of $k$ with degree $|u| - 1$.

Proof. We prove this by induction on $|u|$. By the same argument as in the proof of Lemma 14

$$|T_{d,\gamma_j,e_1}(x) - D_{d,\gamma_j,k}(x)| \leq \sum_{j=k+1}^{\infty} |M_{d,\gamma_j,e_1}(x)| \leq \frac{2}{q} \left( \min_{0 \leq a \leq q-1} \{r_a\} \right) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^k,$$

Hence

$$\sup_{d'} \|T_{d',\gamma_1,e_1} - D_{d',\gamma_1,k}\|_\infty \leq \frac{2}{q} \left( \min_{0 \leq a \leq q-1} \{r_a\} \right) \left( \max_{0 \leq a \leq q-1} \{r_a\} \right)^k.$$
Fix \( x \in I \). For every \( j \), there exists an \( m_j \) such that \( x \in I_{j+1}(m_j) \). Then, by (20), we have for \( |u| \geq 2 \)

\[
|T_{d,r,u}(x) - D_{d,r,u,k}(x)| \\
\leq \sum_{j=k-|u|+1}^{\infty} \sum_{\alpha=0}^{q-2} \frac{2}{\min \{ r_a \}} \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{j} \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right)\phi^j(x) \\
\times \sum_{n=0}^{q^j-1} \mu_{d,r}(I_{j+1}(n)) \left| T_{r_{a^n},r,u-e_a} \circ \phi^{j+1}(x) \right| \\
+ \sum_{j=0}^{k-|u|+1} \sum_{\alpha=0}^{q-2} \frac{2}{\min \{ r_a \}} \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{j} \left( \frac{\Phi_\alpha}{r_\alpha} - \frac{\Phi_{q-1}}{r_{q-1}} \right)\phi^j(x) \\
\times \sum_{n=0}^{q^j-1} \mu_{d,r}(I_{j+1}(n)) \left| T_{r_{a^n},r,u-e_a} \circ \phi^{j+1}(x) - D_{r_{a^n},r,u-e_a,k-j-1} \circ \phi^{j+1}(x) \right| \\
\leq \frac{2(q-1)}{\min \{ r_a \}} \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{|u|+2} \min_{0 \leq a \leq q-1} \{ r_a \} \max_{u',|u'|=|u|-1} \sup_{d'} \left| T_{d',r,u'} \right| \\
+ \frac{2(q-1)}{\min \{ r_a \}} \sum_{j=0}^{k-|u|+1} \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{j} \max_{u',|u'|=|u|-1} \sup_{d'} \left| T_{d',r,u'} - D_{d',r,u',k-j-1} \right|.
\]

Hence, by Lemma [14] and the assumption of induction,

\[
\max_{u',|u'|=|u|-1} \sup_{d'} \left| T_{d',r,u'} - D_{d',r,u',k} \right| 
\]

\[
\ll \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{|u|+2} \sum_{j=0}^{k-|u|+1} \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{j} P_{|u|-2}(k - j - 1) \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{k-j-1} \\
= \left( \max_{0 \leq a \leq q-1} \{ r_a \} \right)^{|u|+2} \sum_{j=0}^{k-|u|+1} P_{|u|-2}(k - j - 1),
\]

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where the implied constant depends only on $q$, $r$, and $|u|$. Since $\sum_{j=0}^{k-|u|+1} P_{|u|-2}(k-j-1)$ is a polynomial of $k$ with degree $|u| - 1$, we obtain the assertion.

By Lemma 15 we see that

$$\lim_{k \to \infty} D_{d,r,u,k}(x) = T_{d,r,u}(x)$$

holds uniformly for $x \in I$. Thus we obtain Theorem 1 by Propositions 3 and 4.

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