ON TORSION SECTIONS OF ELLIPTIC FIBRATIONS

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ABSTRACT. Let $E$ be an elliptic curve over the function field $\mathbb{Q}(t)$. Suppose that for every number field $L \neq \mathbb{Q}$ and every element $\tau \in L$ such that the specialization $E_{\tau}$ is smooth, the curve $E_{\tau}$ has a non-trivial torsion point over $L$. We show that $E$ has a non-trivial torsion point over $\mathbb{Q}(t)$. This provides evidence in support of a question of Graber-Harris-Mazur-Starr on rational pseudo-sections of arithmetic surjective morphisms.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a variety of positive dimension defined over a field $K$. In [1], Graber-Harris-Mazur-Starr investigate various scenarios under which the geometry of $X$ and the algebra of $K$ would guarantee that $X$ has a $K$-rational point. One of the questions they raise is the following [1, Question 7, p. 540]:

Let $K$ be a number field, and let $C$ be a smooth curve of genus $\geq 1$ over the function field $K(t)$. Suppose that for every non-trivial algebraic extension $L/K$ and every element $t_0 \in L$ such that $C_{t_0}$ has a smooth specialization at $t_0$, the curve $C_{t_0}$ has a $L$-rational point. Does $C$ have a $K(t)$-rational point?

In this note we provide evidence in support of this question. Before we state our result we first set up some notation. For the rest of this note, $K$ is a number field, and $B$ is either $\mathbb{P}^1$ or an elliptic curve over $K$ of positive Mordell-Weil rank. Fix a $K$-rational, ample Cartier divisor $D$ on $B$. Then for any finite extension $L/K$, define a height function $H_{B,L}$ on $B$ as follows. If $B = \mathbb{P}^1$, take $H_{B,L}$ to be the multiplicative height on $\mathbb{P}^1(L)$ [2, p. 174]; otherwise, take $H_{B,L}$ to be $H_{B,D,L}$, the multiplicative height on $B(L)$ with respect to the divisor $D$. Then the theorems of Schanuel [2, Thm. B.6.2] and Néron [2, Thm. B.6.3] give the asymptotic estimate

$$N_{B,L}(x) := \#\{P \in B(L) : H_{B,L}(P) < x\} \sim \begin{cases} \alpha(B,L)x^2 & \text{if } B = \mathbb{P}^1, \\ \beta(B,D,L)(\log x)^{\text{rank}(B/L)/2} & \text{otherwise}, \end{cases}$$

where $\alpha(B,L)$ and $\beta(B,D,L)$ are explicit, positive constants.

Theorem. With the notation as above, let $E$ be an elliptic curve over the function field $K(B)$. For any point $P \in B(L)$, write $E_P$ for the specialization of $E$ at $P$. Suppose that for every non-trivial finite extension $L/K$, there exists a constant $\lambda_{E,L} > 1/2$ such that

$$\#\{P \in B(L) : H_{B,L}(P) < x, \ E_P \text{ is smooth and } E_P(L)_{\text{tor}} \neq 0\} \gg_{E,L} N_{B,L}(x)^{\lambda_{E,L}}.$$

Then $E$ has a non-trivial torsion point over $K(B)$.

1991 Mathematics Subject Classification. Primary 11G35; Secondary 11G05, 14G05.

Key words and phrases. Elliptic curve, elliptic surfaces, height, torsion points.

Supported in part by NSA.
Remark 1. We can simplify condition 11 by stipulating that every smooth specialization of \( \mathcal{E} \) at every rational point of \( B(L) \) have a non-trivial torsion point. On the other hand, for any integer \( D \neq 0, 1, -432 \), the torsion subgroup of \( y^2 = x^3 + D \) over \( \mathbb{Q} \) is \( \mathbb{Z}/3, \mathbb{Z}/2 \) or trivial, depending on whether \( D \) is a square, a cube, or neither [3, p. 34]. Thus the curve \( y^2 = x^3 + t \) has no non-trivial torsion point over \( \mathbb{Q}(t) \), while \( N_{\mathbb{P}^1, \mathbb{Q}}(x)^{1/2} \) of its specializations above rational points on \( \mathbb{P}^1(\mathbb{Q}) \) of height \( < x \) do. In particular, the bound \( \lambda_{E, L} > 1/2 \) is optimal.

Remark 2. Our argument makes crucial use of Merel’s theorem on torsion points of elliptic curves over number fields [4]. If we have the analogous result for Abelian varieties, we can readily extend the Theorem to families of Abelian varieties fibered over \( B \).

2. The absolute case

Lemma. With the notation as above, let \( L \) be a number field containing \( K \), and let \( E \) be an elliptic curve defined over the function field \( L(B) \). Then the condition 11 implies that \( E \) has a non-trivial torsion point defined over \( L(t) \).

Proof. Fix an elliptic fibration \( \pi : \mathcal{E} \to \mathbb{P}^1 \) over \( L \) with generic fiber \( E \). Then every torsion point \( T \) of \( E(L(B)) \) corresponds to an \( L \)-rational, \( L \)-irreducible torsion multisection \( \mathcal{T} \) of \( \pi \), such that \( T \) and \( T' \) correspond to the same \( \mathcal{T} \) if and only if \( T \) and \( T' \) fall into the same \( \text{Gal}(L(B)/L(B)) \)-orbit. In particular, the degree of \( \pi_{\mathcal{T}} \) is precisely the cardinality of this Galois orbit.

Denote by \( \Delta_{\pi} \subset B(\mathcal{T}) \) the discriminant locus of \( \pi \). Then for every \( P \in B(L) - \Delta_{\pi} \), every \( L \)-rational torsion point of the smooth fiber \( E_P \) belongs to a unique \( L \)-rational, \( L \)-irreducible torsion multisection \( \mathcal{T} \). Thanks to Merel’s theorem [4], only finitely many such \( \mathcal{T} \) can have \( L \)-rational points above \( B(L) - \Delta_{\pi} \). The condition 11 then implies that there exists at least one such \( \mathcal{T} \) with

\[
\# \{ P \in B(L) - \Delta_{\pi} : H_{B, L}(P) < x \text{ and } \mathcal{T}(L) \cap E_P(L)_{\text{tor}} \neq \emptyset \} \gg E \ N_{B, L}(x)^{\lambda_{E, L}}.
\]

In particular, \( \mathcal{T}(L) \) is infinite, so \( \mathcal{T} \) is absolutely irreducible, and hence a (possibly singular) curve over \( L \) with geometric genus \( \rho_\mathcal{T} \leq 1 \). Since \( \pi_{\mathcal{T}} \) is a morphism, properties of height functions imply that \( H_{\mathcal{T}, \pi^*(D), L} \), the height function on \( \mathcal{T} \) with respect to the Cartier divisor \( \pi^*(D) \), satisfies (cf. [2, Thm. B.2.5(b), Thm. B.3.2(b), Remark B.3.2.1(b)]

\[
\# \{ P \in \mathcal{T}(L) : H_{\mathcal{T}, \pi^*(D), L}(P) < x \} \gg_{\mathcal{T}, D, \pi} N_{B, L}(x)^{\deg(\pi_{\mathcal{T}})\lambda_{E, L}}.
\]

Suppose \( \rho_\mathcal{T} = 1 \), so \( \pi_{\mathcal{T}} \) is an \( L \)-isogeny between the two curves \( \mathcal{T}, B \) of geometric genus 1. That means the two curves have the same Mordell-Weil rank over \( L \). But since \( \lambda_{E, L} > 1/2 \) and \( B(L) \) is infinite, [3] would contradict Néron’s theorem [2, Thm. B.6.3] unless \( \pi_{\mathcal{T}} \) has degree one, i.e. unless \( \mathcal{T} \) is an actual, non-trivial, \( L \)-rational torsion section of \( \pi \), in which case we are done.

Next, suppose \( \rho_\mathcal{T} = 0 \), so \( \mathcal{T} \) is \( L \)-birational to \( \mathbb{P}^1 \) since \( \mathcal{T}(L) \) is not empty, and the existence of the non-constant morphism \( \pi_{\mathcal{T}} \) implies that \( B = \mathbb{P}^1 \). Denote by \( \pi' : \mathbb{P}^1 \to \mathcal{T} \) the
desingularization of $\mathcal{T}$. It is defined over $L$; so does $\psi = \pi \circ \pi' : \mathbb{P}^1 \to \mathbb{P}^1$. Then \textup{[2]} implies that
\begin{equation}
\# \{ P \in \mathcal{T}(L) : H_{\mathbb{P}^1,\psi^*(D),L}(P) < x \} \gg_E N_{\mathbb{P}^1,L}(x)^{\deg(\pi)\deg(\pi')\lambda_{E,L}}.
\end{equation}

The Picard group of $\mathbb{P}^1$ is trivial, so properties of height functions \textup{[2]} Thm. B.3.2(d)] imply that $H_{\mathbb{P}^1,\psi^*(D),L}$ and the standard multiplicative height $H_{\mathbb{P}^1,L}$ differs by a positive multiple bounded from above and from zero. Schanuel’s theorem then implies that the left side of \textup{(4)} is $\ll N_{\mathbb{P}^1,L}(x)$. Since $\lambda_{E,L} > 1/2$, this forces $\deg(\pi) = \deg(\pi') = 1$, whence $\mathcal{T}$ is an actual section, as desired.

\section{Proof of the Theorem}

For every non-trivial finite extension $L/K$, the Lemma furnishes a non-trivial torsion point $T_L$ of $E$ over $L(B)$. Suppose $E$ is not a constant elliptic curve over $\overline{K}$, i.e. there does not exist an elliptic curve $E_0$ defined over $K$ such that $E$ and $E_0$ are isomorphic over $\overline{K}(B)$. Then the Mordell-Weil group of $E$ over $\overline{K}(B)$ is finitely generated \textup{[3]} Thm. III.6.1], and hence $E(L(B))_{\text{tor}}$ is bounded independent of the finite extension $L/K$. In particular, we can find two finite extensions $L_1, L_2$ of coprime degree over $K$ such that $T_{L_1} = T_{L_2}$. Then this common, non-trivial torsion point is defined over $L_1(B) \cap L_2(B) = K(B)$, as desired.

Now, suppose $E$ is a constant elliptic curve over $\overline{K}$. Then $E(\overline{K}(B))$ is no longer finitely generated, and we need to proceed differently. We now give an arithmetic argument that is in fact applicable to all $E$.

Fix an elliptic fibration $\pi : \mathcal{E} \to B$ over $L$ with generic fiber $E$. Fix a $K$-rational non-empty affine open set $U \subset B$ over which $\pi$ is smooth, and denote by $R_U$ the corresponding affine coordinate ring. For any finite extension $L/K$ and any $P \in U(L)$, denote by $\hat{R}_{U,P}$ the completion of $R_U \otimes_K L$ at the maximal $m_P$ corresponding to the $L$-rational point $P$, and by $\hat{L}_P$ its field of fractions. Then the formal group argument in \textup{[5]} Prop. VII.3.1], which does not require that $\hat{R}_{U,P}$ have a finite residue field, implies that $E(\hat{L}_P)_{\text{tor}}$ injects into $\mathcal{E}_P(\hat{R}_{U,P}/m_P) \simeq \mathcal{E}_P(L)$ under the specialization map. Take $L/K$ to be a non-trivial extension of the form $L = K(\sqrt[q]{t})$ with $q$ a rational prime, apply Merel’s theorem and we see that $\#E(L(B))_{\text{tor}}$ is uniformly bounded for all such $L$. That means we can find two such $L_1 \neq L_2$ with $T_{L_1} = T_{L_2}$. Since $L_1 \cap L_2 = K$, we are done. \hfill \Box

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\begin{itemize}
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