Moving embedded lattice solitons

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It was recently proved that solitons embedded in the spectrum of linear waves may exist in discrete systems, and explicit solutions for isolated unstable embedded lattice solitons (ELS) of a differential-difference version of a higher-order NLS equation were found [Physica D 197 (2004) 86]. The discovery of these ELS gives rise to relevant questions such as the following: are there continuous families of ELS?, can ELS be stable?, is it possible for ELS to move along the lattice?, how do ELS interact?. The present work addresses these questions by showing that a novel equation (a discrete version of a complex modified KdV equation which includes next-nearest-neighbor couplings) has a two-parameter continuous family of exact ELS. These solitons can move with arbitrary velocities across the lattice, and the numerical simulations demonstrate that these ELS are completely stable. Moreover, the numerical tests show that these ELS are robust enough to withstand collisions, and the result of a collision is only a shift in the positions of the solitons. The model may apply to the description of a Bose-Einstein condensate with dipole-dipole interactions between the atoms, trapped in a deep optical-lattice potential.

PACS numbers: 05.45.Yv, 63.20.Ry, 45.05.+x, 05.45.-a

In nonlinear systems where solitons can exist, the propagation of small-amplitude linear waves, which obey the linearized version of the nonlinear equations, is possible too. However, for a soliton to exist, it is absolutely necessary that no resonances occur between the soliton and these linear waves. Otherwise, the soliton would decay due to an energy transfer towards the linear waves. Based on this no-resonance argument, it was frequently assumed that the solitons’ internal frequencies could not be contained in the linear spectrum of the system, i.e., they could not lie within the band of frequencies permitted to linear waves. However, at the end of the nineties exceptions to this rule were found, and a special type of solitons were discovered, which do not resonate with linear waves, in spite of having frequencies immersed in the spectrum of these waves. In 1999 these peculiar solitary waves were given the name of embedded solitons (ES), and in the following years a number of models supporting ES were identified. Most of these models describe continuous systems. However, some examples of discrete ES were recently found too. These embedded lattice solitons (ELS) are isolated solutions which are stable against small perturbations in the linear approximation, but are nonlinearly unstable. The discovery of these isolated unstable ELS triggered the search for models admitting continuous families of stable ELS. In this article we present a novel differential-difference equation which has a two-parameter continuous family of exact ELS. These solitons can move with arbitrary velocities across the lattice, and the numerical simulations show that they are completely stable solutions.

I. INTRODUCTION

In the late nineties, a new type of solitary waves was identified and given the name of embedded solitons (ES). At first, ES were found in nonlinear-optical models including quadratic (\(\chi^{(2)}\)) nonlinearities \textsuperscript{[9,10]}, and later they were discovered in hydrodynamic \textsuperscript{[7]} and liquid-crystal dynamical models \textsuperscript{[8]}. Quite recently, ES were found in discrete systems too, \textit{viz.}, in a finite-difference version of a higher-order NLS (nonlinear Schrödinger) equation \textsuperscript{[9]} and in a model of an array of linearly-coupled waveguides with the \(\chi^{(2)}\) and \(\chi^{(3)}\) (cubic) nonlinearities \textsuperscript{[10]}. The distinctive feature of the ES is that their internal frequencies and, in some cases, also their velocities, fall in a spectral region occupied by linear wave modes, where regular solitons cannot exist, as they would decay into radiation. That is, ES are exceptional states that do not emit linear waves, despite being in resonance with them. A qualitative explanation to the existence of ES was proposed in Ref. \textsuperscript{[3]} (see also \textsuperscript{[11]}), and more quantitative arguments were provided in Ref. \textsuperscript{[12]}.

A number of generic features of the ES found in continuous systems have been identified. In particular, ES are usually isolated solutions, although continuous families of ES may exist too, under special symmetry conditions \textsuperscript{[5,13,14]}. The ES may be stable against small perturbations in the linear approximation, but general arguments and direct simulations demonstrate that they are usually unstable at the second order, which is why they are often called “semi-stable” solutions. Nevertheless, examples of completely stable ES have been also found \textsuperscript{[8,13,15]}, and
their stability has been demonstrated in Refs. 13, 15.

Much less is known about embedded lattice solitons (ELS) in discrete systems. Only a few examples of models admitting isolated nonlinearly unstable ELS have been found 8, 11. Some issues which remain to be analyzed are, for example, the possibility of the existence of continuous families of ELS, and to determine if they may be completely stable. Furthermore, a challenging question is whether ELS may move across the lattice with arbitrary constant velocities. We address these issues in the present work, introducing a new nonlinear lattice model which gives rise to exact solutions for ELS which feature all these properties.

The paper is organized as follows. In Sec. 2 we present the new model and its continuous family of moving lattice solitons. In Sec. 3 we show that these moving solitons are indeed ELS. In Sec. 4 the behavior of perturbed and colliding ELS is investigated by means of direct simulations which, in particular, demonstrate that more general solitons with different amplitudes exist too. Section 5 concludes the paper.

II. THE MODEL AND ITS FAMILY OF MOVING LATTICE-SOLITON SOLUTIONS

Motivated by the recent discoveries of continuous families of ES in the complex modified Korteweg-de Vries (cmKdV) equation 8 and in a generalized third-order nonlinear Schrödinger equation 13, 15, in the present paper we will consider a differential-difference equation of the following form:

\[
\dot{r}_n = -2\varepsilon_0 (r_{n+1} - r_{n-1}) + \varepsilon_0 (r_{n+2} - r_{n-2}) + \rho |r_n|^2 (r_{n+1} - r_{n-1}) - \delta |r_n|^2 (r_{n+2} - r_{n-2})
\]

where \( r_n(t) \) is a complex variable, the overdot stands for \( d/dt \), and \( \varepsilon_0, \rho \) and \( \delta \) are real constants. At fixed \( t \), the discrete values \( r_n(t) \) can be considered as values of a continuous function \( u(x,t) \), evaluated at equally spaced lattice points. In other words, we can consider that \( r_n(t) \equiv u(x = n\Delta x, t) \), where \( \Delta x \) is the lattice spacing. If we now consider the limit when \( \Delta x \to 0 \), a standard limiting procedure (described in detail in Ref. 8) shows that the Eq. 11 goes over into the equation:

\[
\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^3 u}{\partial x^3} + \frac{1}{6} (\rho - 2\delta) \gamma |u|^2 \frac{\partial u}{\partial x} \tag{2}
\]

where \( \varepsilon = 2\varepsilon_0 (\Delta x)^3 \) and \( \gamma = 12\Delta x \). If \( \rho - 2\delta = 6 \) Eq. 2 is precisely the cmKdV equation studied in 8. It is worth observing that the coefficient in front of the nonlinear term in the cmKdV equation may be positive or negative, depending on the physical application. For example, in the description of light propagation in liquidcrystal waveguides, this coefficient is positive 8. On the other hand, the derivation of the cmKdV equation for strongly dispersive waves in a weakly nonlinear medium by means of a multiple-scale expansion, presented in Ref. 16, leads to a negative nonlinear coefficient.

If we neglect the nonlinear terms in Eq. 11 we arrive at a linear equation which describes a dispersive lattice (i.e., a discrete version of the linear dispersive equation \( u_t = \varepsilon u_{xxxx} \)). However, for the existence of ELS solutions, both nonlinear terms in Eq. 11 are needed.

By means of the staggering transformation,

\[
r_n(t) \equiv (-i)^n R_n(t)
\]

Eq. 11 can be cast in an alternative form,

\[
i\dot{R}_n + 2\varepsilon_0 (R_{n+1} - R_{n-1}) - \rho |R_n|^2 (R_{n+1} + R_{n-1})
\]

\[
+ i\varepsilon_0 (R_{n+2} - R_{n-2}) - i\delta |R_n|^2 (R_{n+2} - R_{n-2}) = 0.
\]

In this form, it is an extension of the well-known integrable Ablowitz-Ladik (AL) model 17, that corresponds to the terms in the first row in Eq. 4. The first (linear) additional term accounts for the higher-order lattice diffraction 18. As shown in Ref. 13, the third-order lattice diffraction can be induced by biasing a lattice (with the coupling between next-nearest neighbors) by a phase gradient applied along it (in particular, by means of the diffraction-management technique in an array of nonlinear optical waveguides 19). Then, the nonlinear terms in Eq. 11 may be regarded as generic nonlinear corrections to the second- and third-order lattice diffraction. In physical systems, nonlinear terms of these types should be induced by long-range nonlinear interactions between sites of the lattice. A realization of this is known in Bose-Einstein condensates (BECs) with dipole-dipole interactions between atoms, trapped in a deep periodic potential (which is created by means of an optical lattice) 20. Note that Eq. 11 does not include any on-site nonlinearity, which can be eliminated in the BEC by means of the known technique of the Feshbach resonance 21.

A basic finding of this work is that Eq. 11 has a family of exact soliton solutions, provided that \( \varepsilon_0 \rho < 0 \) and \( \delta > 2\rho \), of the following form:

\[
r_n(t) = A_{\text{sol}} \text{sech} \left( \frac{n\Delta x - at}{w} \right) e^{i(qt + nQ\Delta x)},
\]

where \( Q \) and \( \Delta x \) are free parameters, and the values of \( A_{\text{sol}}, w, a \) and \( q \) are determined by the following equations:

\[
A_{\text{sol}}^2 = \frac{\varepsilon_0}{\rho} \left( 2 - \frac{\delta}{\rho} \right),
\]

\[
\cos^2 \left( \frac{\Delta x}{w} \right) = \frac{\delta}{2\rho}.
\]

\[
\frac{a}{\Delta x} = -2\varepsilon_0 \cos (2Q\Delta x) \frac{w}{\Delta x} \sinh \left( \frac{2\Delta x}{w} \right)
\]

\[
+ 4\varepsilon_0 \cos (Q\Delta x) \frac{w}{\Delta x} \sinh \left( \frac{\Delta x}{w} \right),
\]

...
\( q = 2\varepsilon_0 \sin (2Q\Delta x) \cosh \left( \frac{2\Delta x}{w} \right) - 4\varepsilon_0 \sin (Q\Delta x) \cosh \left( \frac{\Delta x}{w} \right). \)  

(9)

For fixed values of the coefficients \( \varepsilon_0, \rho \) and \( \delta \), the Eqs. \[ \text{5-9} \] define a continuous family of moving lattice solitons, all with the same amplitude, due to Eq. \[ \text{11} \], but different widths \( w \) and velocities \( V = a/\Delta x \) (the latter being defined with respect to the discrete coordinate \( n \)). Because the value of \( \Delta x/w \) is fixed by Eq. \[ \text{7} \], and \( |\cos (2Q\Delta x)| \leq 1 \), Eq. \[ \text{8} \] shows that the velocity always takes values within a limited interval,

\[
\left| \frac{a}{\Delta x} \right| \leq 2 |\varepsilon_0| \frac{w}{\Delta x} \sinh \left( \frac{2\Delta x}{w} \right). \tag{10}
\]

It is relevant to stress that, contrary to a frequently made assumption, explicit counter-examples \[ \text{24, 25} \] demonstrate that the existence of exact solutions for moving solitons does not imply the integrability of the corresponding nonlinear lattice system. Nevertheless, the availability of exact soliton solutions capable of moving with arbitrary velocities across the lattice is a highly nontrivial feature, which permits soliton collisions to occur. This is an interesting process which shall be studied in Sec. 4.

To close this section we would like to notice that in the particular case when \( \rho = 2\delta \) the Eq. \[ \text{11} \] can be derived from a Hamiltonian function (which is a dynamical invariant of the equation), provided that suitable noncanonical Poisson brackets are introduced. As in this case \( (\rho = 2\delta) \) no soliton solutions for Eq. \[ \text{11} \] have been found, we will not analyze this case here. However, the existence of a Hamiltonian structure for Eq. \[ \text{11} \] is an interesting issue which merit further studies.

**III. SOLITON EMBEDDING**

We have seen that the Eq. \[ \text{11} \] has a continuous family of moving lattice solitons. Now we aim to determine whether these solitons are embedded in the spectrum of the radiation modes. To this end it is necessary to check if the intrinsic frequency of the soliton \[ \text{6} \] is contained (embedded) in the band of frequencies permitted to the small-amplitude linear waves (phonon modes) which are able to propagate in the lattice. If the soliton’s frequency falls into this band, it will be important to determine the wavenumbers corresponding to all the phonon modes which have the same frequency as the soliton, since these modes will be resonantly excited if the soliton is perturbed.

In order to determine the correct resonant wavenumbers it is necessary to take into account that the solitons defined by Eqs. \[ \text{6-9} \] are moving solitons. Due to this movement, the soliton’s intrinsic frequency must be compared to the linear spectrum of the phonon waves in the reference frame which moves along with the soliton (the transition to the moving reference frame, which is generally impossible in a nonlinear lattice model, is always possible in the linearized equation for the phonon modes). In this moving reference frame the coordinate \( n\Delta x \) is replaced by \( n\Delta x - at \), thus implying that the accordingly defined frequency includes the Doppler shift. Additional arguments (based on the Fourier expansion of the fields) showing that the resonance between moving solitons and linear waves must be studied in the moving reference frame were given in Refs. \[ \text{12, 13} \] and, in a more mathematically rigorous form, in Ref. \[ \text{26} \].

According to the previous paragraph, we look for phonon modes in the form: \( r_n(t) = \varepsilon \exp [i\Omega t - ik(n\Delta x - at)] \), with \( \varepsilon \) being an arbitrary small amplitude. The substitution of this expression in the linearized form of Eq. \[ \text{11} \] leads to the dispersion relation:

\[
\Omega (k) = 4\varepsilon_0 \sin (k\Delta x) \left[ 1 - \cos (k\Delta x) \right] - ka, \tag{11}
\]

the last term being the above-mentioned Doppler shift. The moving soliton will be embedded if its intrinsic frequency, which is \( - (q + aQ) \) according to Eq. \[ \text{14} \], is contained in the range of values of the function \( -\Omega (k) \) with the real argument \( k \) varying within \( -\infty < k < +\infty \). Since this range is the entire real axis (if \( a \neq 0 \)), all the moving lattice solitons given by Eq. \[ \text{5} \] are indeed embedded solitons.

In the particular case of \( a = 0 \), the corresponding quiescent soliton will be embedded if there is a real solution of the corresponding resonance condition, which follows from equating expressions \[ \text{9} \] and \[ \text{14} \]:

\[
q = 4\varepsilon_0 \sin (k\Delta x) \left[ 1 - \cos (k\Delta x) \right]. \tag{12}
\]

It is easy to see that real solutions to Eq. \[ \text{12} \] exist for \( |q| \leq 3\sqrt{3} |\varepsilon_0| \), otherwise the static soliton will not be embedded.

We would like to observe that some of the embedded solitons of the family defined by the Eqs. \[ \text{6-9} \] move with velocities which are contained within the range of phase velocities permitted to the phonon modes. From Eq. \[ \text{14} \] it follows that the phase velocities of these modes (defined in the laboratory reference frame) are defined by:

\[
v (k) = \frac{4\varepsilon_0}{k\Delta x} \sin (k\Delta x) \left[ 1 - \cos (k\Delta x) \right]. \tag{13}
\]

Consequently, the soliton’s velocity \( (V = a/\Delta x) \) will be within the range of this function if the equation:

\[
a = \frac{4\varepsilon_0}{k} \sin (k\Delta x) \left[ 1 - \cos (k\Delta x) \right], \tag{14}
\]

has real solution for \( k \). Depending on the values of \( \varepsilon_0, \rho, \delta, Q \) and \( \Delta x \), real solutions to Eq. \[ \text{14} \] may or may not exist.
From the previous paragraph it follows that the solitons of Eq. (11) can be classified in two groups. The first one contains those solitons whose internal frequencies lie within the range of the function $-\Omega (k)$, and whose velocities are not contained in the range of the function $\Omega (k)$. The second group includes those solitons which, in addition of having wavenumbers within the range of $-\Omega (k)$, have velocities which are contained in the range of the function $\Omega (k)$. In the following, the solitons of the first group will be called single-embedded solitons, and the solitons of the second group will be referred to as double-embedded solitons, to emphasize that in this case both, the wavenumbers and the velocities of the solitons, are contained in the corresponding linear spectra.

IV. PERTURBED AND COLLIDING LATTICE SOLITONS

A. Spectrum of emission from perturbed embedded solitons

Since the intrinsic frequencies of the soliton solutions of Eq. (11) fall within the phonon spectrum of the system, one might expect that resonances should occur between these solitons and the phonon modes, giving rise to the emission of resonant radiation. However, the distinctive feature of the ES is precisely the absence of this radiation. In the case of Eq. (2), the mechanism that suppresses the resonance between the ES and the linear waves can be understood if one calculates the Fourier transform $\tilde{u}(k, \omega)$ of the field $u(x, t)$. As demonstrated in Ref. [3], the calculation shows that $\tilde{u}(k, \omega)$ is proportional to a quotient, $P_2(\omega)/P_1(\omega)$, of two polynomials. $P_1(\omega)$ is an image of the linear part of the equation, and resonances occur at frequencies for which $P_1(\omega)$ vanishes, while $P_2(\omega)$ depends on the nonlinear part of the equation. In order to calculate the Fourier transform of the nonlinear term it is considered that $u(x, t)$ is a single-humped function (such as a hyperbolic secant times a complex exponential). In this way an explicit form for $P_2(\omega)$ can be obtained (see Eq. (52) in Ref. [3]). The crucial point which explains why the ES do not radiate is that $P_2(\omega)$ vanishes at the same points where $P_1(\omega) = 0$, provided that $u(x, t)$ is exactly an ES, and consequently the singularities in the expression $P_2(\omega)/P_1(\omega)$, which lead to the resonances, cancel out. However, if $u(x, t)$ is an altered (perturbed) ES, the exact cancelation does not take place, and hence transient emission from the perturbed ES will take place at the frequencies for which $P_1(\omega) = 0$. Generally, if the soliton’s intrinsic frequency is $\omega = -(q + aQ)$ [as in Eq. (3)], and $\Omega (k)$ is the linear dispersion relation for the phonon modes in the reference frame moving along with the soliton, such as in Eq. (11), then $P_1(\omega) = q + aQ - \Omega (k)$. Consequently, a perturbed ES is expected to emit radiation at wavenumbers $-k$ determined by the resonance condition $\Omega (k) = q + aQ$. In other words, the resonant wavenumbers are defined by the equation:

$$q + aQ = 4\varepsilon_0 \sin (k\Delta x) [1 - \cos (k\Delta x)] - ka.$$  (15)

To confirm that the perturbed solitons emit radiation at these wavenumbers, let us analyze two examples. In the first case let us consider the following parameters: $\varepsilon_0 = 1$, $\rho = -0.4$, $\delta = -0.85$, $Q = 0.61$ and $\Delta x = 3$. In this case, the soliton $A_1$ is a double-embedded one, with $A_{solv} = 0.56$, $w = 12.12$, $a = 7.75$ and $q = -5.1$. To verify that this soliton is double-embedded, in Fig. 1 we can see the graph of the function $v(k)\Delta x$ [with $v(k)$ given by Eq. (13)]. Since the maximum of this function is 7.967, the soliton’s parameter $a = 7.75$ lies within the range of $v(k)\Delta x$, thus implying that the soliton is indeed double-embedded. To perturb this soliton, we solved Eq. (11) numerically with the initial condition:

$$r_n(t = 0) = A_0 \text{sech} \left( \frac{n\Delta x}{w} \right) e^{i nQ\Delta x},$$  (16)

where the initial amplitude, $A_0 = 0.64$, is larger than the amplitude of the exact soliton, $A_{solv} = 0.56$. According to Eq. (15), the perturbed soliton will emit radiation at the wavenumbers:

$$-\frac{k}{2\pi} = -0.114, -0.073, -0.008, +0.103, +0.129.$$  (17)

The numerical solution of the Eq. (11) with the initial condition (16) is shown in Fig. 2 (for $t = 20$), and the corresponding power spectrum, $|r(k)|^2$, is displayed in Fig. 3 within the Brillouin zone, $-\pi/\Delta x < k < +\pi/\Delta x$, where the Fourier transform of the lattice field is defined as:

$$r(k) = \sum_n r_n\exp (i k n \Delta x).$$  (18)

The spectrum shown in Fig. 3 contains a central component, which is essentially the Fourier transform of the unperturbed soliton, and two symmetric radiation bands, approximately located at $0.07 \leq |k/2\pi| \leq 0.13$. These radiation bands contain all the expected resonant wavenumbers, except the value $-0.008$, which falls within the central band. The symmetry of the radiation bands is related to the fact that the oscillating behavior (in time) of the soliton solutions of Eq. (11) is described by a complex exponential exp ($i\omega_0 t$), whose real part oscillates as $\cos (\pm \omega_0 t)$. Due to this fact, a perturbed soliton is able to excite the phonon modes with frequencies $\pm \omega_0$, thus accounting for the occurrence of symmetrically reflected resonance numbers.

As a second example, let us perturb the single-embedded soliton corresponding to the parameters: $\varepsilon_0 = -4.8$, $\rho = 4.8$, $\delta = 10.3$, $Q = 7.9$ and $\Delta x = 0.4$. In this case the parameters of the soliton are: $A_{solv} = 0.381$, $w = 1.503$, $a = 15.812$ and $q = -0.771$. To confirm that this soliton is single-embedded, in Fig. 4 we display the
phase velocity (times $\Delta x$) of the radiation waves as given by Eq. (13). From this figure it is evident that the soliton’s parameter $\alpha = 15.812$ lies outside the range of the function $v(k) \Delta x$.

The resonance condition (14) defines in this case only one resonant wavenumber located at $-k/2\pi = -1.249$, which is very close to one of the edges of the Brillouin zone $|k/2\pi| \leq 1/(2\Delta x) = 1.25$. As in the previous example, we also expect a symmetrically reflected radiation peak located at $-k/2\pi = +1.249$, due to a complex factor describing the oscillating behavior of the perturbed soliton.

The numerical solution of Eq. (1) corresponding to an initial condition of the form (16) with $A_0 = 0.35$ (which is smaller than the amplitude of the exact soliton) shows that the perturbed pulse emits a radiation wavetrain to the left, as seen in Fig. 5. The spectrum of the numerical solution, shown in Fig. 6, exhibits two symmetric radiation peaks at $-k/2\pi = \pm 1.23$, which are very close to the expected values.

B. Robustness of the embedded solitons

The emission of radiation from a perturbed linearly stable ELS may result in its full decay, and consequently its stability (or, better said, robustness) must be tested in long simulations. It is relevant to mention that in the continuum limit, the ES solutions of Eq. (2) are stable because this equation gives rise to a continuous family of ES with arbitrary energy, and hence a perturbed soliton can shed off a part of its energy and settle down to an ES solution with a smaller energy $\delta$. In continuum models supporting isolated ES, the perturbed pulse may not be able to find a stationary soliton with a lower energy toward which it could relax, and hence it may go on radiating till complete decay. This scenario, referred to as the “semi-stability” (alias nonlinear instability) of the ES, really occurs in the continuum model with the combined $\chi^{(2)} \cdot \chi^{(3)}$ nonlinearity $\delta$.

In the present case, the stability of the ELS may be secured if a perturbed soliton can find a way to relax, after shedding off some radiation, to a new stationary soliton shape. Within the bounds of the exact solution family given by Eqs. (5)–(9), this process does not seem plausible, as all the exact solutions have a unique value of the amplitude. Nevertheless, it may happen that the exact family is only a subset of a more general manifold of soliton solutions. The possible existence of such an extended soliton family would not only provide for the stability as explained above, but it is a fundamentally important issue by itself.

To clarify the eventual fate of the perturbed solitons of Eq. (1), we now display some typical examples. In the first place, let us consider the double-embedded soliton corresponding to $\epsilon_0 = 1$, $\rho = -0.4$, $\delta = -0.85$, $Q = 0.61$ and $\Delta x = 3$. As stated before, the shape of this soliton is defined by the parameters $A_{\text{col}} = 0.56$, $w = 12.12$ and $a = 7.75$. If this soliton is perturbed by increasing its amplitude [for instance, taking the initial condition (16) with $A_0 = 0.64$], the perturbed pulse starts to oscillate. In the course of the evolution, the oscillations gradually fade away due to the radiation loss, as shown by the upper curve in Fig. 7, and the perturbed soliton tends to settle down into a stationary shape, which is quite possible, as the initial perturbation increased its norm.

On the other hand, if the height of the initial pulse is smaller than the amplitude of the exact soliton (for example, $A_0 = 0.48$), the norm of the perturbed pulse becomes smaller than the exact soliton’s norm, and therefore the pulse will not be able to relax back to the exact soliton. As explained above, such a perturbation might lead to the destruction of the soliton, unless it can find another stationary shape to which it may relax to. The evolution of this perturbed soliton was numerically simulated, and the lower curve in Fig. 7 shows the behavior of the soliton’s amplitude. As we can see in this figure, the perturbed soliton does not decay indefinitely. It again exhibits a damped oscillatory behavior (although the oscillation period is much longer than in the previous case, when the perturbation increased the norm), and it finally approaches a new equilibrium state. The behavior exhibited in Fig. 7 indicates that the double-embedded lattice solitons are completely stable pulses and also, as conjectured above, that the exact solitons seem to be just a subset of a broader family of discrete solitons. The complete characterization of this broader family lies outside the scope of the present work and it will not be addressed here.

The stability of the single-embedded lattice solitons of Eq. (1) is even clearer. Let us consider, for instance, the soliton corresponding to $\epsilon_0 = -4.8$, $\rho = 4.8$, $\delta = 10.3$, $Q = 7.9$, and $\Delta x = 0.4$. In this case, $A_{\text{col}} = 0.381$, $w = 1.503$, $q = -0.771$ and $a = 15.812$. As the value of the parameter $\alpha$ lies outside the range of the function $v(k) \Delta x$ (see Fig. 4), the soliton is indeed single-embedded (i.e., only in terms of its frequency, but not according to its velocity). To perturb this soliton, we first took the initial condition (16) with an increased amplitude, $A_0 = 0.415$. This perturbed pulse stabilizes itself quite rapidly, as we can see from the time evolution of its height, which is presented in the upper curve of Fig. 8. In a similar way, if the initial amplitude is decreased to $A_0 = 0.345$ (recall that the amplitude of the exact soliton is $A_{\text{col}} = 0.381$), the pulse also relaxes quickly, its height evolving as shown by the lower curve in Fig. 8. This figure indicates that the single-embedded soliton of Eq. (1) tested in these simulations is a stable solution. Figure 8 also shows that when the initial pulse is higher (lower) than the exact soliton, the amplitude of the established steady-state pulse will be still larger (smaller) than the amplitude $A_{\text{col}}$ of the exact soliton solution. A similar behavior was observed in the case of the ES of Eq. (2), and an explanation of this behavior (by means of a variational analysis) was provided in Ref. 8.

A comparison of Figs. 7 and 8 reveals an interesting
difference: while the amplitude of the perturbed double-embedded soliton shown in Fig. 7 exhibits well-defined oscillations, the amplitude of the single-embedded soliton seen in Fig. 8 evolves almost monotonically. This qualitative difference may be related to the fact that the resonance condition \( \delta = 4 \) determines five resonant wavenumbers in the case studied in Fig. 7, and only one in the case corresponding to Fig. 8. Actually, a precise explanation of the different behaviors exhibited in Figs. 7 and 8 remains a challenge for future work.

We again stress that the perturbed lattice solitons of Eq. (4) stabilize as new stable pulses whose heights are different from the unique value of the amplitude of the exact soliton solutions given by Eq. (6). As said above, this observation strongly suggests that the exact solutions given by Eqs. (5)-(9) constitute just a particular subset of a wider soliton family of the Eq. (1), which is a subject for a separate work.

Our simulations, performed at many values of the parameters, have never revealed an unstable soliton, even if the initial perturbation was actually large. An additional example is displayed in Fig. 9, for \( \varepsilon_0 = -1, \rho = 0.4, \delta = 0.85, \Delta x = 3 \) and \( Q = 0.61 \). The exact soliton corresponding to these values has \( A_{\text{sol}} = 0.56, w = 12.12, a = 7.75, \) and \( q = -5.103, \) while the initial condition was taken in the form of Eq. (16) with \( A_0 = 0.48, \) which is essentially lower than the amplitude of the exact solution. In this case again, the amplitude of this perturbed pulse exhibits damped oscillations and a very clear trend to the stabilization into a new stationary soliton, which does not belong to the family of the exact solutions.

\[ \rho = 4 \]

\section{C. Collisions between embedded solitons}

Once knowing that the ELS of Eq. (4) are stable against perturbations, it is natural to consider their robustness against collisions. To this end, we simulated collisions between two ELS moving in opposite directions. A typical example can be displayed for \( \varepsilon_0 = -4.8, \rho = 4.8, \delta = 10.3 \) and \( \Delta x = 0.4 \). In this case, all the solitons belonging to family \( (a, b) \) have the same amplitude and width, \( A_{\text{sol}} = 0.38 \) and \( w = 1.5 \), but may differ in the velocity, which is controlled by the parameter \( Q \). We can take a soliton moving to right, with \( Q_1 = 3.315 \) and \( a_1 = 15.7, \) and a second one travelling to left, with \( Q_2 = 7.9 \) and \( a_2 = -9.0. \) Initially, these solitons are placed at \( n\Delta x = -20 \) and \( n\Delta x = 12 \), respectively.

The collision between these solitons is shown in Fig. 10, which demonstrates that they recover their shapes and velocities after the collision. In Fig. 11, we specially display the shape of the solitons after the collision (at \( t = 11 \)). This figure shows that no radiation is emitted by the solitons after the collision, and it confirms that the solitons keep the same value of the amplitude, \( A_{\text{sol}} = 0.38 \), which they had before the collision. At the moment shown \( (t = 11) \), the centers of the pulses are located at \( n\Delta x = -78.6 \) and \( n\Delta x = 139.2 \). Without the collision, they would be found (at the same moment of time) at the points \( n\Delta x = -87 \) and \( n\Delta x = 152.7 \), respectively. We thus conclude that the only effect of the collision is a shift in the position of the solitons (as occurs usually in integrable systems). The same result was observed in simulations performed at other values of the parameters.

\section{V. Summary and Conclusions}

In this work we have introduced a new one-dimensional dynamical lattice model. The underlying equation includes next-nearest-neighbor couplings, and it may be regarded as a discretization of the complex modified KdV equation, or as an extension of the Ablowitz-Ladik model (in the staggered form). The model may apply to the description of a Bose-Einstein condensate with dipole interactions between atoms, trapped in a deep optical lattice.

A family of exact lattice-soliton solutions of the discrete equation was found, with the following characteristics:

(i) These solitons can move with arbitrary constant velocities (within a certain interval), without being hindered by the lattice discreteness. On the other hand, all the solitons have the same amplitude.
(ii) All the soliton solutions are indeed embedded (with the exception of a part of the subfamily of zero-velocity solitons) because their internal frequency lies within the phonon band (calculated in a moving reference frame which moves along with the soliton, and contains the corresponding Doppler shift).
(iii) The embedded solitons of Eq. (11) are not isolated solutions (unlike most other models admitting embedded solitons), but form a continuous two-parameter family.
(iv) Numerical simulations show that all the solitons are stable.
(v) The exact soliton solutions with the fixed amplitude constitute a subset a larger family of stable discrete moving solitons, which can be found in a numerical form. In particular, under a perturbation that reduces the soliton’s norm, a perturbed soliton relaxes to a new one which belongs to the extended family.
(vi) Collisions between the moving solitons known in the exact form are completely elastic (according to the results of systematic simulations), leading solely to a shift of the solitons’ positions.

Some of these features, such as the existence of a continuous family of moving lattice solitons and the elasticity of the collisions between them (as confirmed up to the numerical accuracy), suggest that the model may have a chance to be integrable. Investigation of this possibility poses a challenge for further studies of the model.

\section{Acknowledgements}

We sincerely thank A. Minzoni for his interest in this project and J. Yang for his useful comments and for pro-
Providing us with a preprint of Ref. [15]. DGSCA-UNAM (Dirección General de Servicios de Cómputo Académico of Universidad Nacional Autónoma de México) is acknowledged for granting us access to their computers Bakliz and Berenice. This work was financially supported from the grant DGAPA-UNAM IN112503. One of the authors (BAM) appreciates financial support from FENOMEC during his stay in Mexico.

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Figure captions

Fig. 1. The phase velocity of the linear waves (times ∆x) as a function of k [i.e., the right-hand side of Eq. (14)] for ε₀ = 1 and ∆x = 3.

Fig. 2. The perturbed double-embedded lattice soliton, at t = 20, which evolved from the initial condition (16), with A₀ = 0.64, w = 12.12, Q = 0.61, and ∆x = 3. The parameters of Eq. (1) are ε₀ = 1, ρ = −0.4, and δ = −0.85. In this figure and below, the shape of the soliton is shown as |rₙ| versus n.

Fig. 3. The power spectrum of the numerical solution corresponding to the perturbed double-embedded lattice soliton shown in Fig. 2.

Fig. 4. The phase velocity of the linear waves (times ∆x) as a function of k, for ε₀ = −4.8 and ∆x = 0.4.

Fig. 5. The perturbed single-embedded lattice soliton is shown at t = 10. It has evolved from the initial condition (16), with A₀ = 0.35, w = 1.503, Q = 7.9, and ∆x = 0.4. The parameters of Eq. (1) are ε₀ = −4.8, ρ = 4.8, and δ = 10.3. The velocity of the soliton is defined by the parameter a = 15.812.

Fig. 6. The power spectrum of the numerical solution corresponding to the perturbed single-embedded lattice soliton shown in Fig. 5.

Fig. 7. Evolution of the amplitudes of two perturbed double-embedded lattice solitons. In both cases, the initial conditions are given by Eq. (16), with w = 12.12, Q = 0.61, and ∆x = 3. The parameters of Eq. (1) are ε₀ = 1, ρ = −0.4, and δ = −0.85. The upper and lower curves correspond, respectively, to the initial amplitudes A₀ = 0.64 and A₀ = 0.48, which are larger and smaller than the value A_{sol} = 0.56 for the exact soliton solution in this case.

Fig. 8. Evolution of amplitudes of two perturbed single-embedded lattice solitons. In both cases, the initial conditions are taken as per Eq. (16), with w = 1.5, Q = 7.9, and ∆x = 0.4. Parameters of Eq. (1) are ε₀ = −4.8, ρ = 4.8, and δ = 10.3. The upper and lower curves correspond, respectively, to the initial amplitudes A₀ = 0.415 and A₀ = 0.345, which are larger and smaller than the value A_{sol} = 0.38 corresponding to the exact soliton in this case.

Fig. 9. Evolution of the amplitude of a perturbed soliton corresponding to the parameters ε₀ = −1, ρ = 0.4, δ = 0.85 and ∆x = 3. The initial pulse is take in the form of Eq. (16) with A₀ = 0.48, r = 0.61 and w = 12.12.

Fig. 10. Collision of two single-embedded lattice solitons, at ε₀ = −4.8, ρ = 4.8, δ = 10.3 and ∆x = 0.4. The solitons moving to the right and left are determined by parameters, respectively, Q₁ = 3.315, a₁ = 15.7, and Q₂ = 7.9, a₂ = −9.0.

Fig. 11. The shape of the lattice field after the collision shown in Fig. 10, at t = 11.
FIG. 3:

FIG. 4:
FIG. 7:

FIG. 8:
FIG. 9:

FIG. 10:
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