Mixed $H_–/H_\infty$ Fault Detection Filtering for Itô-Type Affine Nonlinear Stochastic Systems

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Abstract- This paper studies the mixed $H_–/H_\infty$ fault detection filtering of Itô-type nonlinear stochastic systems. Mixed $H_–/H_\infty$ filtering combines the system robustness to the external disturbance and the sensitivity to the fault of the residual signal. Firstly, for Itô-type affine nonlinear stochastic systems, some sufficient criteria are obtained for the existence of $H_–/H_\infty$ filter in terms of Hamilton-Jacobi inequalities (HJIs). Secondly, for a class of quasi-linear Itô systems, a sufficient condition is given for the existence of $H_–/H_\infty$ filter by means of linear matrix inequalities (LMIs). Finally, a numerical example is presented to illustrate the effectiveness of the proposed results.

Keywords: Stochastic systems, fault detection, nonlinear systems, $H_–/H_\infty$ fault detection filtering.

1 Introduction

Along with the development of modern industrial production, higher requirements for safety and reliability have been put forward. In order to ensure safety and reliability in industrial process, various techniques for fault detection, fault isolation and fault estimation have appeared

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Fault detection filter (FDF) is to use the estimated value of the system state and the measurement output to generate the residual signal to detect the system fault. According to the real time comparison between the designed residual evaluation function and the corresponding threshold, we are in a position to determine whether there is a fault occurring. In real world, some unknown disturbance signals existing in dynamic systems will fluctuate the residual. Therefore, one needs to design robust FDF to exclude these unknown external effect. $H_\infty$ control is one of the most important robust control methods since G. Zames’ fundamental work \cite{41} published, which has been studied extensively \cite{5,12,14,28,29,12,42,43,45}, and has been applied to event triggers \cite{13,32}, fault diagnosis \cite{18,30,48}, sliding mode control method \cite{35} and adaptive control method \cite{16}. $H_\infty$ FDF is to design the filter such that the $L_2$-gain from the external interference to the residual signal is less than the given attenuation level $\gamma > 0$, which reflects the robust ability of the concerned systems. $H_-$ FDF is to design the filter such that the $L_2$-gain from the fault signal to the residual signal is larger than the given sensitivity level $\delta > 0$, which measures the sensitivity of the considered systems to the fault signal. Different from the sole $H_\infty$ or $H_-$ FDF, mixed $H_-/H_\infty$ FDF is a combination of $H_\infty$ FDF and $H_-$ FDF, which not only meets the robustness requirement, but also requires the residual signal to be sensitive enough to the fault information \cite{4,34}. So $H_-/H_\infty$ FDF is one of the most popular robust design methods.

Stochastic Itô-type systems are ideal mathematical models in finance mathematics \cite{37}, systems biology \cite{6,7}, benchmark mechanical systems \cite{36}, so the study for stochastic systems has attracted many researchers’ interest, and stochastic control has become one of the most important research fields in modern control theory \cite{10,39,40}. In 1998, $H_\infty$ control of linear Itô systems was first investigated in \cite{15,31}, and from then, nonlinear $H_\infty$ control \cite{2,42} and filtering \cite{43}, mixed $H_2/H_\infty$ control \cite{45} of Itô systems have been solved. $H_-$ FDF is to use the $H_-$ index to measure the minimum influence of the fault on the residual signal \cite{19,20,21,30,48}, where \cite{24} and \cite{19,20} were about FDFs of linear time-invariant and time-varying deterministic systems, respectively. Generally speaking, for linear time-invariant/time-varying deterministic systems, the corresponding FDF design can be turned into solving some LMIs \cite{24,38,48}/differential Riccati equations (DREs) \cite{19,20}. In \cite{30}, FDF of nonlinear switched stochastic systems was discussed based on T-S fuzzy model approach. While the fault isolation problem for discrete-time fuzzy interconnected systems with unknown interconnections was considered in \cite{46}. The reference \cite{17} investigated the adaptive fuzzy output feedback fault-tolerant optimal control problem for a class of single-input and single-output nonlinear systems in strict feedback form.
A robust fault detection $H_\infty$ observer was constructed for a Takagi-Sugeno fuzzy model with sensor faults and unknown bounded disturbances via an LMI formulation in [4]. The reference [34] studied $H_\infty$ fault detection observer design in the finite frequency domain for a class of linear parameter-varying descriptor systems. Recently, we analyze the $H_\infty$ index for a class of linear discrete-time stochastic Markov jump systems with multiplicative noise [21]. It can be found that, up to now, there are few works on mixed $H_\infty$ FDF for nonlinear Itô-type stochastic systems.

Motivated by the aforementioned reason, this paper studies $H_\infty$ FDF of affine nonlinear Itô systems, where a Luenberger type observer is considered in designing $H_\infty$ FDF, which can not only suppress the effect of external interference on the residual signal below a level $\gamma > 0$, but also guarantee the residual signal to be sensitive to the fault signal. The contributions of this paper can be summarized as follows:

• Applying Itô formula together with square completion technique, some sufficient conditions are obtained for the existence of $H_\infty$ FDF of affine stochastic Itô systems in terms of coupled HJIs. As corollaries, the existence conditions of $H_\infty$ FDF of linear stochastic Itô systems are also presented in terms of algebraic Riccati inequalities (ARIs), which can be transformed into solving LMIs by Matlab LMI Toolbox. How to solve the HJIs is a challenging problem, and a potential powerful technique to solve the coupled HJIs can refer to [1] by using the neural network approach.

• We also study $H_\infty$ FDF for a class of quasi-linear stochastic Itô systems. It is shown that for such a class of nonlinear stochastic systems, the corresponding $H_\infty$ FDF can be designed via solving some LMIs instead of HJIs, which is very convenient in practice.

The organization of this paper is as follows: In Section 2, we make some preliminaries to introduce useful definitions and lemmas. In Section 3, some sufficient conditions for the existence of $H_\infty$ fault detection filter are presented based on HJIs. Applying LMI-based technique, in Section 4, for a class of quasi-linear stochastic systems, the corresponding $H_\infty$ FDF design is converted into solving LMIs. Section 5 presents an example to illustrate the effectiveness of our given results. Section 6 concludes this paper with some remarks and perspectives.

For convenience, this paper adopts the following standard notations:

\( \mathcal{R}^+ := [0, +\infty) \); \( M' \): the transpose of the matrix \( M \) or vector \( M \); \( M > 0 \) (\( M < 0 \)): the matrix \( M \) is a positive definite (negative definite) real symmetric matrix; \( I_n \): \( n \times n \) identity matrix; \( \mathcal{R}^n \): the \( n \)-dimensional real Euclidean vector space with the norm \( \| x \| = \sqrt{\sum_{k=0}^{n} x_k^2} \); \( \mathcal{R}^{n \times m} \): the
the fault information to be detected. We assume that all functions with respect to an increasing σ-algebra $\{F_t\}_{t \geq 0}$ satisfying $\|v(t)\|_{L^2_{\mathbb{F}}} := E \int_0^\infty \|v(t)\|^2 dt < \infty$; A function $f(x)$ is called a positive function, if $f(x) > 0$ for any $x \neq 0$, and $f(0) = 0$; $C^2(U; X)$: the class of $X$-valued functions $V(x)$, which are twice continuously differential with respect to $x \in U$, except possibly at the point $x = 0$; $C^{2,1}(U \times \mathbb{R}^+; X)$: the class of $X$-valued functions $V(x,t)$, which are twice continuously differential with respect to $x \in U$, and once continuously differential with respect to $t \in \mathbb{R}^+$, except possibly at the point $x = 0$.

2 Preliminaries

Consider the following affine nonlinear Itô stochastic system (the time variable $t$ is suppressed):

$$
\begin{cases}
    dx = [f_1(x) + g_1(x)v + h(x)f] \, dt \\
    \quad + [f_2(x) + g_2(x)v] \, dw,
\end{cases}
$$

\hspace{1cm} (1)

where $x(t) \in \mathbb{R}^n$ is the $n$-dimensional state vector, $y \in \mathbb{R}^{n_y}$ is the $n_y$-dimensional measurement output, $v \in \mathbb{R}^{n_v}$ stands for the exogenous disturbance signal with $v \in L^2_{\mathbb{F}}(\mathbb{R}^+, \mathbb{R}^{n_v})$, and $w(t)$ is a 1-D standard Wiener process defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}, \mathcal{P})$ with the σ-field $\mathcal{F}_t$ generated by $w(\cdot)$ up to time $t$. $f(t) \in \mathbb{R}^{n_t}$ denotes the fault information to be detected. We assume that all functions $f_1(x)$, $g_1(x)$, $h(x)$, $f(t)$, $f_2(x)$, $g_2(x)$, $l(x)$, $m(x)$ and $n(x)$ are continuous, which satisfy certain conditions as linear growth condition and Lipschitz condition such that the state equation in (1) has a unique strong solution.

In this paper, we adopt the following Luenberger-type observer as FDF of system (1):

$$
\begin{cases}
    d\hat{x}(t) = \hat{f}(\hat{x}(t)) \, dt + \hat{h}(\hat{x}(t)) (y(t) - l(\hat{x}(t))) \, dt, \\
    \dot{r}(t) = \hat{s}(\hat{x}(t)) (y(t) - l(\hat{x}(t))),
\end{cases}
$$

\hspace{1cm} (2)

where $\hat{x}(t)$ is the estimated value of $x(t)$, $r(t)$ is viewed as the residual signal, $\hat{f}$, $\hat{h}$, and $\hat{s}$ are the filter functions to be designed.
Set $$\eta(t) = \left[ x(t) \quad \dot{x}(t) \right]'$$, then we get the following augmented system:

$$\left\{ \begin{array}{l}
  d\eta = \left( \tilde{f}_1(\eta) + \tilde{g}_1(\eta)v + \tilde{h}(\eta)f \right) dt \\
  \quad + \left( \tilde{f}_2(\eta) + \tilde{g}_2(\eta)v \right) dw, \\
  r(t) = \tilde{s}(\eta, v, f), \\
  \eta(0) = \begin{bmatrix} x_0 \\
  0 \end{bmatrix} \in \mathbb{R}^n, \\
\end{array} \right.$$  \hspace{1cm} (3)

where

$$\tilde{f}_1(\eta) = \begin{bmatrix} f_1(x) \\
  \hat{f}(\hat{x}) + \hat{h}(\hat{x})(l(x) - l(\hat{x})) \end{bmatrix}, \quad \tilde{g}_1(\eta) = \begin{bmatrix} g_1(x) \\
  \hat{h}(\hat{x})m(x(t)) \end{bmatrix},$$

$$\tilde{h}(\eta) = \begin{bmatrix} h(x) \\
  \hat{h}(\hat{x})n(x) \end{bmatrix}, \quad \tilde{f}_2(\eta) = \begin{bmatrix} f_2(x) \\
  0 \end{bmatrix},$$

$$\tilde{g}_2(\eta) = \begin{bmatrix} g_2(x) \\
  0 \end{bmatrix}, \quad \tilde{s}(\eta, v, f) = \tilde{s}(\hat{x}) [l(x) + m(x)v + n(x)f - l(\hat{x})].$$

2.1 Definitions and lemmas

For our needs, we introduce some definitions as follows.

**Definition 1** An Itô-type stochastic differential system

$$\left\{ \begin{array}{l}
  dx(t) = f(x(t)) dt + g(x(t)) dw(t), \\
  x(0) = x_0 \in \mathbb{R}^n \\
\end{array} \right.$$  \hspace{1cm} (4)

is said to be exponentially stable in mean square sense, if there exist constants $$\beta \geq 1$$ and $$\alpha > 0$$, such that the solution $$x(t)$$ of system (4) satisfies

$$E\|x(t)\|^2 \leq \beta e^{-\alpha t}\|x_0\|^2, \quad t \geq 0.$$  

The residual signal $$r(t)$$ needs to be measured and calculated in real time to judge whether there is a noticeable fault occurring. Therefore, it is worth emphasizing that the comprehensive ability to reflect external interferences and internal fault information are important for $$r(t)$$ in the designed filter. In order to reasonably analyze and study the capacity of $$r(t)$$, we introduce $$H_\infty$$ index and $$H_\infty$$ index to describe system (3).
Definition 2 For stochastic system (3), its sensitive operator $L_{f,r}$ and $H_-$ index are respectively defined as

$$L_{f,r} : f(t) \in L^2_\mathbb{F}(\mathbb{R}^+, \mathbb{R}^{n_f}) \mapsto r(t) \in L^2_\mathbb{F}(\mathbb{R}^+, \mathbb{R}^{n_r})$$

and

$$\|L_{f,r}\|_- = \inf_{v(t) \equiv 0, \eta(0) = 0, f(t) \not\equiv 0, f(t) \in L^2_\mathbb{F}(\mathbb{R}^+, \mathbb{R}^{n_f})} \frac{\|r(t)\|_{L^2_\mathbb{F}}}{\|f(t)\|_{L^2_\mathbb{F}}},$$

Meanwhile, we define the perturbation operator $L_{v,r}$ and $H_\infty$ index as

$$L_{v,r} : v(t) \in L^2_\mathbb{F}(\mathbb{R}^+, \mathbb{R}^{n_v}) \mapsto r(t) \in L^2_\mathbb{F}(\mathbb{R}^+, \mathbb{R}^{n_r})$$

and

$$\|L_{v,r}\|_\infty = \sup_{v(t) \equiv 0, \eta(0) = 0, v(t) \not\equiv 0, v(t) \in L^2_\mathbb{F}(\mathbb{R}^+, \mathbb{R}^{n_v})} \frac{\|r(t)\|_{L^2_\mathbb{F}}}{\|v(t)\|_{L^2_\mathbb{F}}},$$

respectively.

The purpose in this paper is to design a mixed $H_-/H_\infty$ FDF for system (1). Below, we give the definition of mixed $H_-/H_\infty$ FDF.

Definition 3 A FDF (2) is called the mixed $H_-/H_\infty$ FDF, if for any given scalars $\gamma > 0$ and $\delta > 0$, the following requirements are satisfied simultaneously.

- The augmented system (3) is internally stable, that is, when $f(t) \equiv 0$ and $v(t) \equiv 0$ in system (3), the following system

$$(5) \begin{cases} d\eta(t) = \tilde{f}_1(\eta(t))dt + \tilde{f}_2(\eta(t))dw(t), \\ \eta(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n_u}, \end{cases}$$

is exponentially stable in mean square sense.
- The augmented system (3) is externally stable, that is, when \( f(t) \equiv 0, \eta(0) = 0 \) in system (3), for any nonzero \( v(t) \in L^2_F(\mathbb{R}^+, \mathbb{R}^n v) \), the \( L_2 \)-gain from \( v(t) \) to \( r(t) \) of the following system

\[
\begin{aligned}
d\eta(t) &= \left( \tilde{f}_1(\eta(t)) + \tilde{g}_1(\eta(t))v(t) \right) dt + \left( \tilde{f}_2(\eta(t)) + \tilde{g}_2(\eta(t))v(t) \right) dw(t), \\
r(t) &= \hat{s}(\hat{x}(t))(l(x(t)) + m(x(t))v(t) - l(\hat{x}(t)))
\end{aligned}
\]

is less than or equal to \( \gamma > 0 \), i.e., the \( H_\infty \) index \( \|L_{v, r}\|_\infty \leq \gamma \).

- The residual signal is enough sensitive to the fault, that is, when \( v(t) \equiv 0, \eta(0) = 0 \) in system (3), for any nonzero \( f(t) \in L^2_F(\mathbb{R}^+, \mathbb{R}^n f) \), the \( L_2 \)-gain from \( f(t) \) to \( r(t) \) of the following system

\[
\begin{aligned}
d\eta(t) &= \left( \tilde{f}_1(\eta(t)) + \tilde{h}(\eta(t))f(t) \right) dt + \tilde{f}_2(\eta(t))dw(t), \\
r(t) &= \hat{s}(\hat{x}(t))(l(x(t)) + n(x(t))f(t) - l(\hat{x}(t)))
\end{aligned}
\]

is large than or equal to \( \delta > 0 \), i.e., the \( H_- \) index \( \|L_{f, r}\|_- \geq \delta \).

In Definition 3, \( \gamma > 0 \) is called as the disturbance attenuation level, and \( \delta > 0 \) as the fault sensitivity level.

The following lemmas will play important roles in this study.

**Lemma 1** \([45]\) For \( x, b \in \mathbb{R}^n \), if \( A \) is a real symmetric matrix with appropriate dimension, \( A^{-1} \) exists. Then we have

\[
x' Ax + x' b + b' x = (x + A^{-1} b)' A (x + A^{-1} b) - b' A^{-1} b.
\]

**Lemma 2** \([27, 45]\) Suppose there exists a function \( V(\eta, t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}) \). An infinitesimal generator \( LV(\eta, t) : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R} \) associated with (3) is given by

\[
\begin{aligned}
LV(\eta, t) &= \frac{\partial V'}{\partial t} + \frac{\partial V'}{\partial \eta} \left( \tilde{f}_1(\eta) + \tilde{g}_1(\eta)v + \tilde{h}(\eta)f(t) \right) \\
&\quad + \frac{1}{2} \left( \tilde{f}_2(\eta) + \tilde{g}_2(\eta)v \right) \frac{\partial^2 V}{\partial \eta^2} \left( \tilde{f}_2(\eta) + \tilde{g}_2(\eta)v \right),
\end{aligned}
\]

and

\[
EV(\eta(t), t) = EV(\eta(t_0), t_0) + E \int_{t_0}^{t} LV(\eta(s), s) ds, \ 0 \leq t_0 < t < +\infty.
\]
2.2 Residual evaluation

After obtaining the gain matrices of the filter, we are in the position to discuss the residual estimation. For the purpose of evaluating the residual signal, one tends to adopt a threshold \( J_{th} > 0 \), and the \( J_{th} \) conforms to the following decision logic:

\[
\begin{align*}
J_r(t) > J_{th} \Rightarrow \text{faults} \Rightarrow \text{alarm}, \\
J_r(t) \leq J_{th} \Rightarrow \text{fault-free},
\end{align*}
\]

where the residual evaluation function \( J_r(t) \) is defined by

\[
J_r(t) := \left( \frac{1}{t} \int_0^t r'(s)r(s)ds \right)^{\frac{1}{2}}, J_r(0) = 0,
\]

and the threshold \( J_{th} \) is determined by

\[
J_{th} := \sup_{f(t)\equiv 0, \forall t\in \mathbb{L}_2^2} \mathbb{E}J_r(T),
\]

where \( T \) is the evaluation window.

3 FDF for Affine Nonlinear Stochastic Systems

In this section, we will give our main results about the mixed \( H_-/H_\infty \) FDF for affine nonlinear stochastic systems.

**Theorem 1** For any given disturbance attenuation level \( \gamma > 0 \) and fault sensitivity level \( \delta > 0 \), if there exist positive constants \( c_1, c_2, c_3, \varepsilon_1, \varepsilon_2 \) and a positive Lyapunov function \( V(\eta) \in C^2(\mathbb{R}^n; \mathbb{R}^+) \), such that

\[
c_1\|\eta\|^2 \leq V(\eta) \leq c_2\|\eta\|^2,
\]

and \( V(\eta) \) solves the following two coupled HJIs

\[
\begin{align*}
&\begin{cases}
(1 + \varepsilon_1)\|\dot{s}(\hat{x}) (l(x) - l(\hat{x}))\|^2 + \frac{\partial V}{\partial \eta} \tilde{f}_1(\eta) \\
- \left( \frac{1}{2} \bar{g}_2(\eta) + \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta) + \frac{1}{2} \bar{g}_1'(\eta) \frac{\partial V}{\partial \eta} \right) \cdot \left( \frac{1}{2} \bar{g}_2(\eta) + \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta) + (1 + \varepsilon_1^{-1})\|\dot{s}(\hat{x})m(x)\|^2 I \\
- \gamma^2 I \right)^{-1} \left( \frac{1}{2} \bar{g}_2(\eta) + \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta) + \frac{1}{2} \bar{g}_1'(\eta) \frac{\partial V}{\partial \eta} \right) \\
+ \frac{1}{2} \bar{f}_2(\eta) + c_3\|\eta\|^2 \leq 0,
\end{cases} \\
&\begin{cases}
\frac{1}{2} \bar{g}_2(\eta) + \frac{\partial^2 V}{\partial \eta^2} \tilde{g}_2(\eta) + (1 + \varepsilon_1^{-1})\|\dot{s}(\hat{x})m(x)\|^2 I \\
- \gamma^2 I < 0,
\end{cases}
\end{align*}
\]
Lemma 2, for any $T > 0$, So, by Theorem 4.4 of [26] and (12), the system (5) is exponentially stable in mean square sense. Note that for any $\varepsilon > 0$, we have

\[
(1 - \varepsilon^{-1}) \|\hat{s}(\hat{x})n(x)\|^2 I - \delta^2 I > 0
\]

for some filter functions $\hat{f}(\cdot)$, $\hat{h}(\cdot)$, $\hat{s}(\cdot)$ with suitable dimensions. Then the desired $H_\infty$ FDF is obtained by [2].

**Proof:** Firstly, we show that system (3) is internally stable, or equivalently, the system (5) is exponentially stable in mean square sense. For system (5), apply Lemma 2 and consider (13), we have

\[
\mathcal{L}V(\eta)_{|f=0,v=0} = \frac{\partial V}{\partial \eta} \hat{f}_1(\eta) + \frac{1}{2} \hat{f}_2(\eta) \frac{\partial^2 V}{\partial \eta^2} \hat{f}_2(\eta) \leq -c_3 \|\eta\|^2.
\]

So, by Theorem 4.4 of [26] and (12), the system (5) is exponentially stable in mean square sense.

Next, we will show that the system (6) is also externally stable, i.e., $\|\mathcal{L}_{v,T}\|_{\infty} \leq \gamma$. From Lemma 2 for any $T > 0$, $v(t) \in L^2_T(\mathbb{R}^+, \mathbb{R}^n)$, and the initial value $\eta(0) = 0$, we have

\[
E \int_0^T \left( \|r(s)\|^2 - \gamma^2 \|v(s)\|^2 \right) ds
= E \int_0^T \left( \|r(s)\|^2 - \gamma^2 \|v(s)\|^2 + \mathcal{L}V(\eta(s))_{|f=0} \right) ds
= EV(\eta(T)) + V(\eta(0))
= E \int_0^T \left[ \|r(s)\|^2 - \gamma^2 \|v(s)\|^2 + \frac{\partial V}{\partial \eta} \left( \hat{f}_1(\eta(s)) + \hat{g}_1(\eta(s))v(s) \right) \right] ds - EV(\eta(T)).
\]

Note that for any $\varepsilon_1 > 0$, we have

\[
\|r(s)\|^2 \leq (1 + \varepsilon_1) \|\hat{s}(\hat{x})m(x(s)) - l(\hat{x}(s))\|^2
+ (1 + \varepsilon_1^{-1}) \|\hat{s}(\hat{x}(s))m(x(s))\|^2 \|v(s)\|^2.
\]
So

\[
E \int_0^T \left( \| r(s) \|^2 - \gamma^2 \| v(s) \|^2 \right) ds \\
\leq E \int_0^T \left\{ \left[ (1 + \varepsilon_1) \| \hat{s}(\hat{x}(s)) (l(x(s)) - l(\hat{x}(s))) \|^2 \right] \\
+ \left[ (1 + \varepsilon_1^{-1}) \| \hat{s}(\hat{x}(s)) m(x(s)) \|^2 - \gamma^2 \| v(s) \|^2 \right] \\
+ \frac{\partial V'}{\partial \eta} \left( \tilde{f}_1(\eta(s)) + \tilde{g}_1(\eta(s)) v(s) \right) + \frac{1}{2} \left( \tilde{g}_2(\eta(s)) v(s) \right)' \right\} ds - EV(\eta(T)),
\]

By Lemma \[1\]

\[
E \int_0^T \left( \| r(s) \|^2 - \gamma^2 \| v(s) \|^2 \right) ds \\
\leq E \int_0^T \left\{ \left[ (1 + \varepsilon_1) \| \hat{s}(\hat{x}(s)) (l(x(s)) - l(\hat{x}(s))) \|^2 \right] \\
+ \frac{\partial V'}{\partial \eta} \left( \tilde{f}_1(\eta(s)) \right) - \left( \frac{1}{2} \tilde{g}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(s)) \right) \\
+ \left( \frac{1}{2} \tilde{g}_1(\eta(s))' \frac{\partial V}{\partial \eta} \right)' \Lambda_2^{-1} \left( \frac{1}{2} \tilde{g}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(s)) \right) \\
+ \frac{1}{2} \tilde{g}_1(\eta(s))' \frac{\partial V}{\partial \eta} + [v + \Lambda_1]' \Lambda_2 [v + \Lambda_1] \\
+ \frac{1}{2} \tilde{f}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(s)) \right\} ds - EV(\eta(T)),
\]

where

\[
\Lambda_1 = \Lambda_2^{-1} \left( \frac{1}{2} \tilde{g}_2(\eta) \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta) + \frac{1}{2} \tilde{g}_1(\eta) \frac{\partial V}{\partial \eta} \right),
\]

\[
\Lambda_2 = \frac{1}{2} \tilde{g}_2(\eta) \frac{\partial^2 V}{\partial \eta^2} \tilde{g}_2(\eta) + (1 + \varepsilon_1^{-1}) \| \hat{s}(\hat{x}) m(x) \|^2 I - \gamma^2 I.
\]

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By (13), $A_2 < 0$, then inequality (16) yields that

\[
E \int_0^T (\|r(s)\|^2 - \gamma^2 \|v(s)\|^2) \, ds \\
\leq E \int_0^T \left\{ (1 + \varepsilon_1) \|\dot{s}(\dot{x}(s)) (l(x(s)) - l(\dot{x}(s)))\|^2 \\
+ \frac{\partial V'}{\partial \eta} \tilde{f}_1(\eta(s)) - \left( \frac{1}{2} \tilde{g}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(s)) \right) \\
+ \frac{1}{2} \tilde{g}_1'(\eta(s)) \frac{\partial V'}{\partial \eta} \Lambda_2^{-1} \left( \frac{1}{2} \tilde{g}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(s)) \right) \\
+ \frac{1}{2} \tilde{f}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(s)) \right\} \, ds - EV(\eta(T)),
\]

(17)

Now, the following inequality holds under the condition (13):

\[
E \int_0^T (\|r(s)\|^2 - \gamma^2 \|v(s)\|^2) \, ds \leq 0.
\]

Let $T \to \infty$ in the above inequality, we have

\[
E \int_0^\infty \|r(t)\|^2 dt \leq \gamma^2 E \int_0^\infty \|v(t)\|^2 dt, \forall v(t) \in \mathcal{L}_2^2(\mathcal{R}^+; \mathcal{R}^n).
\]

(18)

Thus, the external stability of the system (3) is shown.

Finally, we show the $H_-$ index to satisfy $\|\mathcal{L}_{f,r}\|_\infty \geq \delta$. By Itô formula, for any $T > 0$, $t \in [0, T]$, $f(t) \not\equiv 0$, $v(t) \equiv 0$, and the initial value $\eta(0) = 0$, we have

\[
E \int_0^T (\|r(s)\|^2 - \delta^2 \|f(s)\|^2) \, ds \\
= E \int_0^T (\|r(s)\|^2 - \delta^2 \|f(s)\|^2 - \mathcal{L}V(\eta(s))) \, ds \\
+ EV(\eta(T)) - V(\eta(0)) \\
= E \int_0^T \left[ \|r(s)\|^2 - \delta^2 \|f(s)\|^2 - \frac{\partial V'}{\partial \eta} \left( \tilde{f}_1(\eta(s)) \\
+ h(\eta(s)) f(s) \right) - \frac{1}{2} \tilde{f}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(s)) \right] \, ds \\
+ EV(\eta(T)) \\
\geq E \int_0^T \left[ (1 - \varepsilon_2) \|\dot{s}(\dot{x}(s)) (l(\dot{x}(s)) - l(x(s)))\|^2 \\
+ (1 - \varepsilon_2^{-1}) \|\dot{s}(\dot{x}(s)) n(x(s))\|^2 \|f(s)\|^2 - \delta^2 \|f(s)\|^2 \\
- \frac{\partial V'}{\partial \eta} \left( \tilde{f}_1(\eta(s)) + \tilde{h}(\eta(s)) f(s) \right) \\
- \frac{1}{2} \tilde{f}_2(\eta(s))' \frac{\partial^2 V}{\partial \eta^2} \tilde{f}_2(\eta(t)) \right] \, ds + EV(\eta(T)).
\]

(19)
The above inequality makes use of the following relation (\(\forall \varepsilon_2 > 0\))

\[
\|\hat{s}(\hat{x}) (l(x) + n(x)f - l(\hat{x}))\|^2 \\
\geq (1 - \varepsilon_2)\|\hat{s}(\hat{x}) (l(\hat{x}) - l(x))\|^2 + (1 - \varepsilon_2^{-1})\|\hat{s}(\hat{x})n(x)\|^2 \|f\|^2.
\]

By Lemma 1

\[
E \int_0^T (\|r(s)\|^2 - \delta^2 \|f(s)\|^2) \, ds \\
\geq E \int_0^T \left\{ \begin{array}{l}
- c_3\|\eta(s)\|^2 - \frac{\partial V'}{\partial \eta} \hat{f}_1(\eta(s)) - \frac{1}{2} \hat{f}_2(\eta(s))\frac{\partial^2 V}{\partial \eta^2}\hat{f}(\eta(s)) \\
+ \left[ f(s) - \frac{1}{2} (I - \delta^2 I)^{-1} \tilde{h}(\eta(s))\frac{\partial V}{\partial \eta} \right] \cdot (1 - \varepsilon_2^{-1})\|\hat{s}(\hat{x}(s))n(x(s))\|^2 (I - \delta^2 I)^{-1} \\
+ \left[ f(s) - \frac{1}{2} (I - \delta^2 I)^{-1} \tilde{h}(\eta(s))\frac{\partial V}{\partial \eta} \right] - \frac{1}{2} \frac{\partial V'}{\partial \eta} \tilde{h}(\eta(s)) \\
\cdot [(1 - \varepsilon_2^{-1})\|\hat{s}(\hat{x}(s))n(x(s))\|^2 (I - \delta^2 I)^{-1} \\
\hat{h}(\eta(s))\frac{\partial V}{\partial \eta} \right] \, ds.
\end{array} \right.
\]

By the similar technique used in proving (18), from inequality (14), we obtain that for any \(\forall \eta(t) \in L_2^x(\mathbb{R}^+, \mathbb{R}^n)\),

\[
E \int_0^\infty \|r(s)\|^2 \, ds \geq \delta^2 E \int_0^\infty \|f(s)\|^2 \, ds.
\]

The proof is completed. \(\square\)

**Remark 1** In fact, when we consider \(H_{\infty} \) index \(\|L_{v,r}\|_{\infty}\) and \(H_- \) index \(\|L_{f,r}\|_-\), the same Lyapunov function is not necessary. That is, we can choose two different Lyapunov functions \(V_1(\eta)\) and \(V_2(\eta) \in C^2(\mathbb{R}^n; \mathbb{R}^+)\) in calculating

\[
\|r(s)\|^2 - \gamma^2 \|v(s)\|^2 + \mathcal{L}V_1(s)|_{v=0} \leq 0
\]

and

\[
\|r(s)\|^2 - \delta^2 \|f(s)\|^2 + \mathcal{L}V_2(s)|_{v=0} \geq 0,
\]

which would reduce the conservatism of Theorem 7.

**Corollary 1** For given disturbance attenuation level \(\gamma > 0\) and fault sensitivity level \(\delta > 0\), if for any \(\eta \in \mathbb{R}^n\), there exist positive constants \(c_1, c_2, c_3, \varepsilon_1, \varepsilon_2\) and two positive Lyapunov
functions \( V_1 \in C^2(\mathcal{R}^{n_0}; \mathcal{R}^+) \) satisfying (12) and \( V_2 \in C^2(\mathcal{R}^{n_0}; \mathcal{R}^+) \), which solve the following coupled HJIs:

\[
\begin{aligned}
(1 + \varepsilon_1)\|\hat{s}(\hat{x}) (l(\hat{x}) - l(x))\|^2 + c_3\|\eta\|^2 + \frac{\partial V_1}{\partial \eta} \tilde{f}_1(\eta) \\
- \left( \frac{1}{2} \tilde{g}_2(\eta)' \frac{\partial^2 V_1}{\partial \eta^2} \tilde{f}_2(\eta) + \frac{1}{2} \tilde{g}'_1(\eta) \frac{\partial V_1}{\partial \eta} \right) \\
\cdot \left( \frac{1}{2} \tilde{g}_2(\eta)' \frac{\partial^2 V_1}{\partial \eta^2} \tilde{g}_2(\eta) + (1 + \varepsilon_1^{-1})\|\hat{s}(\hat{x})m(x)\|^2 I \\
- \gamma^2 I \right)^{-1} \frac{1}{2} \tilde{g}_2(\eta)' \frac{\partial^2 V_1}{\partial \eta^2} \tilde{g}_2(\eta) + (1 + \varepsilon_1^{-1})\|\hat{s}(\hat{x})m(x)\|^2 I \\
+ \frac{1}{2} \tilde{f}_2(\eta)' \frac{\partial^2 V_1}{\partial \eta^2} \tilde{f}_2(\eta) \leq 0,
\end{aligned}
\]

\[
\begin{aligned}
\frac{1}{2} \tilde{g}_2(\eta)' \frac{\partial^2 V_2}{\partial \eta^2} \tilde{g}_2(\eta) + (1 + \varepsilon_1^{-1})\|\hat{s}(\hat{x})m(x)\|^2 I \\
- \gamma^2 I < 0,
\end{aligned}
\]

and

\[
\begin{aligned}
(1 - \varepsilon_2)\|\hat{s}(\hat{x}) (l(\hat{x}) - l(x))\|^2 - \frac{\partial V_2}{\partial \eta} \tilde{f}_1(\eta) \\
- \frac{1}{2} \tilde{f}_2(\eta)' \frac{\partial^2 V_2}{\partial \eta^2} \tilde{f}_2(\eta) - \frac{1}{2} \tilde{g}'_1(\eta) \frac{\partial^2 V_2}{\partial \eta^2} \tilde{g}_2(\eta) (1 - \varepsilon_2^{-1}) \\
\cdot \|\hat{s}(\hat{x})n(x)\|^2 I - \delta^2 I \right)^{-1} \tilde{h}(\eta)' \frac{\partial V_2}{\partial \eta} \geq 0, \\
(1 - \varepsilon_2^{-1})\|\hat{s}(\hat{x})n(x)\|^2 I - \delta^2 I > 0
\end{aligned}
\]

for some filter functions \( \tilde{f}, \tilde{h}, \hat{s} \) with suitable dimensions. Then the desired \( H_- / H_\infty \) FDF is given by (12).

If we consider a special case that \( m(x) \equiv 0 \) and \( g_2(x) \equiv 0 \) in system (11), then, we can get the following result.

**Corollary 2** For given disturbance attenuation level \( \gamma > 0 \) and fault sensitivity level \( \delta > 0 \), if for any \( \eta \in \mathcal{R}^{n_0} \), there exist constants \( c_1, c_2, c_3, \varepsilon_1, \varepsilon_2 \) and two positive Lyapunov functions \( V_1 \in C^2(\mathcal{R}^{n_0}; \mathcal{R}^+) \) satisfying (12) and \( V_2 \in C^2(\mathcal{R}^{n_0}; \mathcal{R}^+) \) solving the following two coupled HJIs:

\[
\begin{aligned}
(1 + \varepsilon_1)\|\hat{s}(\hat{x}) (l(\hat{x}) - l(x))\|^2 + c_3\|\eta\|^2 + \frac{\partial V_1}{\partial \eta} \tilde{f}_1(\eta) \\
+ \frac{1}{4} \gamma^{-2} \left( \tilde{g}'_1(\eta) \frac{\partial V_1}{\partial \eta} \right)' \left( \tilde{g}'_1(\eta) \frac{\partial V_1}{\partial \eta} \right) \\
+ \frac{1}{2} \tilde{f}_2(\eta)' \frac{\partial^2 V_1}{\partial \eta^2} \tilde{f}_2(\eta) \leq 0
\end{aligned}
\]
and

\[
\begin{array}{l}
(1 - \varepsilon_2) \| \dot{s}(\dot{x}) (l(\dot{x}) - l(x)) \|^2 - \frac{\partial V_h}{\partial \eta} \tilde{f}_1(\eta) \\
- \frac{1}{2} \tilde{f}_2(\eta)^T \frac{\partial^2 V_h}{\partial \eta^2} \tilde{f}_2(\eta) - \frac{1}{2} \frac{\partial V_h}{\partial \eta} \tilde{h}(\eta) \left( (1 - \varepsilon_2^{-1})ight) \\
\cdot \| \dot{s}(\dot{x}) n(x) \|^2 I - \delta^2 I \right)^{-1} \tilde{h}(\eta)' \frac{\partial V_h}{\partial \eta} \geq 0, \\
(1 - \varepsilon_2^{-1}) \| \dot{s}(\dot{x}) m(x) \|^2 I - \delta^2 I > 0
\end{array}
\] (24)

for some filter functions \( \tilde{f}, \tilde{h}, \dot{s} \) with suitable dimensions. Then (2) is a desirable \( H_-/H_\infty \) FDF for system (1) when \( m(x) \equiv 0 \) and \( g_2(x) \equiv 0 \).

It is well known that for some practical models, not only the state, but also the external disturbance \[47\] and fault signal maybe corrupted by noise. For example, the nonlinear stochastic \( H_\infty \) control of Itô type differential systems with all the state, control input and external disturbance-dependent noise (\((x, u, v)\)-dependent noise for short) was studied in \[44\]. Therefore, a mixed \( H_-/H_\infty \) FDF for nonlinear stochastic systems with \((x, v, f)\)-dependent noise deserves further study, which motivates us to consider the following system

\[
\begin{align*}
\dot{x}(t) &= (f_1(x(t)) + g_1(x(t))v(t) + h_1(x(t))f(t)) dt \\
&\quad + (f_2(x(t)) + g_2(x(t))v(t) + h_2(x(t))f(t)) dw(t), \\
y(t) &= l(x(t)) + m(x(t))v(t) + n(x(t))f(t), \\
x(0) &= x_0 \in \mathcal{R}^n.
\end{align*}
\] (25)

As so, we can get the following augmented system:

\[
\begin{align*}
\dot{\eta}(t) &= \left( \tilde{f}_1(\eta(t)) + \tilde{g}_1(\eta(t))v(t) + \tilde{h}_1(\eta(t))f(t) \right) dt \\
&\quad + \left( \tilde{f}_2(\eta(t)) + \tilde{g}_2(\eta(t))v(t) + \tilde{h}_2(\eta(t))f(t) \right) dw(t), \\
\dot{s}(\eta(t)) &= r(t), \\
\eta(0) &= \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in \mathcal{R}^{n_\eta},
\end{align*}
\] (26)

where

\[
\begin{align*}
\tilde{h}_1(\eta(t)) &= \begin{bmatrix} h_1(x) \\ \tilde{h}(\dot{x}(t)) n(x(t)) \end{bmatrix}, \\
\tilde{h}_2(\eta(t)) &= \begin{bmatrix} h_2(x) \\ 0 \end{bmatrix}.
\end{align*}
\]
**Theorem 2** For any given disturbance attenuation level $\gamma > 0$ and fault sensitivity level $\delta > 0$, if for all $\eta \in \mathbb{R}^{n_\eta}$, the following two coupled HJIs

\[
\begin{align*}
&\begin{cases}
(1 + \varepsilon_1)\|\dot{s}(\hat{x})(l(x) - l(\hat{x}))\|^2 + c_3\|\eta\|^2 + \frac{\partial V_1}{\partial \eta} \tilde{f}_1(\eta) \\
- \left(\frac{1}{2} \tilde{g}_2(\eta) \tilde{V}_2 \tilde{f}_2(\eta) + \frac{1}{2} \hat{g}_1(\eta) \frac{\partial V_1}{\partial \eta}\right) \\
\cdot \left(\frac{1}{2} \tilde{g}_2(\eta) \tilde{V}_2 \tilde{f}_2(\eta) + \frac{1}{2} \hat{g}_1(\eta) \frac{\partial V_1}{\partial \eta}\right) \\
- \gamma^2 I^{-1} \cdot \left(\frac{1}{2} \eta \eta^T + \gamma^2 I\right) \tilde{f}_2(\eta) \leq 0, \\
\frac{1}{2} \tilde{g}_2(\eta) \tilde{V}_2 \tilde{f}_2(\eta) + (1 + \varepsilon_1^{-1})\|\dot{s}(\hat{x})m(x)\|^2 I \\
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
&\begin{cases}
(1 - \varepsilon_2)\|\dot{s}(\hat{x})(l(x) - l(\hat{x}))\|^2 - \frac{\partial V_1}{\partial \eta} \tilde{f}_1(\eta) \\
- \frac{1}{2} \tilde{f}_2(\eta) \tilde{V}_2 \tilde{f}_2(\eta) - \left(\frac{1}{2} \tilde{h}_2(\eta) \tilde{V}_2 \tilde{f}_2(\eta) \\
+ \frac{1}{2} \hat{h}_1(\eta) \frac{\partial V_1}{\partial \eta}\right) \\
\cdot \left(\frac{1}{2} \eta \eta^T + \delta^2 I\right) \tilde{f}_2(\eta) \leq 0, \\
\frac{1}{2} \tilde{h}_2(\eta) \tilde{V}_2 \tilde{f}_2(\eta) + (1 - \varepsilon_2^{-1})\|\dot{s}(\hat{x})n(x)\|^2 I \\
\end{cases}
\end{align*}
\]

admit a set of solutions $(V_1 > 0, V_2 > 0, \tilde{f}, \tilde{h}, \hat{f}, \hat{h}, c_1 > 0, c_2 > 0, c_3 > 0, c_1 > 0, \varepsilon_1 > 0, \varepsilon_2 > 0)$, where

$V_1 \in C^2(\mathbb{R}^{n_\eta}; \mathbb{R}^+) \text{ satisfying (23)}$, $V_2 \in C^2(\mathbb{R}^{n_\eta}; \mathbb{R}^+) \text{ and } c_1, c_2, c_3, \varepsilon_1, \varepsilon_2 \text{ are positive constants}$. Then the FDF for system (25) is given by (23).

**Proof:** Repeating the same procedure as in Theorem 1 and Corollary 1. The proof is completed. □

Below, we consider the following linear stochastic system

\[
\begin{align*}
&\begin{cases}
dx(t) = (A_0 x(t) + B_0 v(t) + C_0 f(t)) \, dt \\
+ (A_1 x(t) + B_1 v(t) + C_1 f(t)) \, dw(t),
y(t) = A_2 x(t) + B_2 v(t) + C_2 f(t)
\end{cases}
\end{align*}
\]
as well as the following linear FDF

\[
\begin{aligned}
\begin{cases}
d\hat{x}(t) = \left( \hat{A}\hat{x}(t) + \hat{B}(y(t) - A_2\hat{x}(t)) \right) dt, \\
r(t) = \hat{S}(y(t) - A_2\hat{x}(t)).
\end{cases}
\end{aligned}
\]  

(30)

Set \( \eta(t) = \begin{bmatrix} x(t)' & \hat{x}(t)' \end{bmatrix}' \), then we get the following augmented system

\[
\begin{aligned}
\begin{cases}
d\eta(t) = (\tilde{A}_0\eta(t) + \tilde{B}_0v(t) + \tilde{C}_0f(t))dt \\
\quad + (\tilde{A}_1\eta(t) + \tilde{B}_1v(t) + \tilde{C}_1f(t))dw(t), \\
r(t) = \tilde{A}_2\eta(t) + \tilde{B}_2v(t) + \tilde{C}_2f(t),
\end{cases}
\end{aligned}
\]

(31)

where

\[
\begin{aligned}
\tilde{A}_0 &= \begin{bmatrix} A_0 & 0 \\ \hat{B}A_2 & A - \hat{B}A_2 \end{bmatrix}, \\
\tilde{B}_0 &= \begin{bmatrix} B_0 \\ \hat{B}\hat{B}_2 \end{bmatrix}, \\
\tilde{C}_0 &= \begin{bmatrix} C_0 \\ \hat{B}C_2 \end{bmatrix}, \\
\tilde{A}_1 &= \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \\
\tilde{B}_1 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\
\tilde{C}_1 &= \begin{bmatrix} C_1 \\ 0 \end{bmatrix}, \\
\tilde{A}_2 &= \begin{bmatrix} \hat{S}A_2 & -\hat{S}A_2 \end{bmatrix}, \\
\tilde{B}_2 &= \hat{S}B_2, \\
\tilde{C}_2 &= \hat{S}C_2.
\end{aligned}
\]

Using Theorem 2, it is easy to show the following result.

**Theorem 3** For any given disturbance attenuation level \( \gamma > 0 \) and fault sensitivity level \( \delta > 0 \), if there exist two positive definite matrices \( P_1 \) and \( P_2 \) solving the following ARIs:

\[
\begin{aligned}
\mathbb{R}_1 := (1 + \varepsilon_1)\varpi + P_1\tilde{A}_0 + \tilde{A}_0'P_1 - (\tilde{B}_1'P_1\tilde{A}_1 + \tilde{B}_0'P_1)' \\
\quad - (\tilde{B}_1'P_1\tilde{B}_1 + (1 + \varepsilon_1^{-1})\||\hat{S}\hat{B}_2||^2I - \gamma^2I\|^{-1} \\
\quad - (\tilde{B}_1'P_1\tilde{B}_1 + (1 + \varepsilon_1^{-1})\||\hat{S}\hat{B}_2||^2I - \gamma^2I\|) < 0
\end{aligned}
\]

(32)

and

\[
\begin{aligned}
\mathbb{R}_2 := (1 - \varepsilon_2)\varpi - P_2\tilde{A}_0 - \tilde{A}_0'P_2 - (\tilde{C}_1'P_2\tilde{A}_1 + \tilde{C}_0'P_2)' \\
\quad - (\tilde{C}_1'P_2\tilde{C}_1 + (1 - \varepsilon_2^{-1})\||\hat{S}\hat{C}_2||^2I - \delta^2I\|^{-1} \\
\quad - (\tilde{C}_1'P_2\tilde{C}_1 + (1 - \varepsilon_2^{-1})\||\hat{S}\hat{C}_2||^2I - \delta^2I\|) \geq 0.
\end{aligned}
\]

(33)

where

\[
\begin{aligned}
\varpi &= \begin{bmatrix} A_1'\hat{S}'\hat{S}A_2 & -A_2'\hat{S}'\hat{S}A_2 \\ -A_2'\hat{S}'\hat{S}A_2 & A_2'\hat{S}'\hat{S}A_2 \end{bmatrix}
\end{aligned}
\]

and \( c, \varepsilon_1 \) and \( \varepsilon_2 \) are positive constants. Then the FDF for system (29) is given by (30).
Proof: For system (31), we choose two Lyapunov functions as $V_1(\eta) = \eta'P_1\eta$ and $V_2(\eta) = \eta'P_2\eta$. By Theorem 2, we obtain

\begin{align}
(1 + \varepsilon_1)\|\hat{S}(A_2x - A_2\hat{x})\|^2 + c\|\eta\|^2 - (\hat{B}_1'P_1\hat{A}_1\eta + \hat{B}_0'P_1\eta)' \\
(\hat{B}_1'P_1\hat{B}_1 + (1 + \varepsilon_1^{-1})\|\hat{S}\hat{B}_2\|^2I - \gamma^2I)^{-1}(\hat{B}_1'P_1\hat{A}_1\eta + \hat{B}_0'P_1\eta) + 2\eta'P_1\hat{A}_0\eta + \eta'\hat{A}_1'P_1\hat{A}_1\eta \leq 0,
\end{align}

and

\begin{align}
(1 - \varepsilon_2)\|\hat{S}(A_2x - A_2\hat{x})\|^2 - (\hat{C}_1'P_2\hat{A}_1\eta + \hat{C}_0'P_2\eta)' \\
(-\hat{C}_1'P_2\hat{C}_1 + (1 - \varepsilon_2^{-1})\|\hat{S}\hat{C}_2\|^2I - \delta^2I)^{-1}(\hat{C}_1'P_2\hat{A}_1\eta + \hat{C}_0'P_2\eta) - 2\eta'P_2\hat{A}_0\eta - \eta'\hat{A}_1'P_2\hat{A}_1\eta \geq 0,
\end{align}

(34) and (35) are equivalent to

\begin{align}
\eta'\mathcal{R}_1\eta \leq 0,
\end{align}

(36)

\begin{align}
(\hat{B}_1'P_1\hat{B}_1 + (1 + \varepsilon_1^{-1})\|\hat{S}\hat{B}_2\|^2I - \gamma^2I < 0
\end{align}

and

\begin{align}
\eta'\mathcal{R}_2\eta \geq 0,
\end{align}

(37)

\begin{align}
-\hat{C}_1'P_2\hat{C}_1 + (1 - \varepsilon_2^{-1})\|\hat{S}\hat{C}_2\|^2I - \delta^2I > 0,
\end{align}

respectively, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined respectively in (32) and (33). (36) and (37) are equivalent to (32) and (33), respectively. The proof is thus completed. □

Theorem 3 is regarding $H_\infty$ FDF for linear stochastic systems based on ARIs. (32) and (33) can be converted into LMIs by the technique in the next section, hence, $H_\infty$ FDF of the system (29) can be easily designed by Matlab LMI Toolbox.

4 LMI-Based Approach for Quasi-Linear Systems

Generally speaking, for general nonlinear stochastic systems, it is not easy to design the $H_\infty$ filter due to the difficulty in solving HJIs. However, for a class of special nonlinear stochastic systems called “quasi-linear stochastic systems”, the filtering design problem can be converted into solving LMIs as done in [43].
We consider the following quasi-linear stochastic system governed by Itô differential equation

\[
\begin{align*}
  dx(t) &= \left(A_0 x(t) + F_0(x(t)) + B_0 v(t) + C_0 f(t)\right) dt \\
  &\quad + \left(A_1 x(t) + F_1(x(t)) + B_1 v(t) + C_1 f(t)\right) dw(t), \\
  y(t) &= A_2 x(t) + B_2 v(t) + C_2 f(t),
\end{align*}
\]

where \( F_i(0) = 0, \ i = 1, 2 \). As a matter of fact, in (25), if one takes \( g_1(x) = B_0, \ h_1(x) = C_0, \ g_2(x) = B_1, \ h_2(x) = C_1, \ l(x) = A_2, \ m(x) = B_2, \ n(x) = C_2, \) and regards \( A_0 x(t) + F_0(x(t)) \) and \( A_1 x(t) + F_1(x(t)) \) as the Taylor’s series expansions of \( f_1(x) \) and \( f_2(x) \), respectively, then the state equation of (25) comes down to the first equation of (38).

For the quasi-linear stochastic system (38), we consider linear FDF (30). Set \( \eta(t) = \begin{bmatrix} x(t) \  \dot{x}(t) \end{bmatrix}' \), then we get the following augmented system:

\[
\begin{align*}
  d\eta(t) &= (\tilde{A}_0 \eta(t) + \tilde{F}_0(\eta(t)) + \tilde{B}_0 v(t) + \tilde{C}_0 f(t)) dt \\
  &\quad + (\tilde{A}_1 \eta(t) + \tilde{B}_1 v(t) + \tilde{F}_1(\eta(t)) + \tilde{C}_1 f(t)) dw(t), \\
  r(t) &= \tilde{A}_2 \eta(t) + \tilde{B}_2 v(t) + \tilde{C}_2 f(t),
\end{align*}
\]

where

\[
\tilde{F}_0(\eta(t)) = \begin{bmatrix} F_0(x(t)) \\
  0 \end{bmatrix}, \quad \tilde{F}_1(\eta(t)) = \begin{bmatrix} F_1(x(t)) \\
  0 \end{bmatrix}.
\]

For any given disturbance attenuation level \( \gamma > 0 \) and fault sensitivity level \( \delta > 0 \), we want to seek the filtering parameters \( \hat{A}, \hat{B} \) and \( \hat{S} \), such that \( \|L_{e,r}\|_\infty < \gamma \) and \( \|L_{f,r}\|_- > \delta \).

**Assumption 1** Suppose there exists a scalar \( \alpha > 0 \), such that

\[
\|\tilde{F}_i(\eta)\| \leq \alpha \|\eta\|, \ i = 1, 2.
\]

**Lemma 3** Given scalars \( \gamma > 0, \delta > 0 \), if the following three matrix inequalities

\[
0 < P \leq \beta I,
\]

\[
\begin{bmatrix}
  \hat{A}_2^T \hat{A}_2 + \hat{A}_0 P + P \hat{A}_0 + P + 4\alpha^2 \beta I & P \hat{B}_0 & \hat{A}_1^T P & \hat{A}_1^T P & 0 \\
  * & -\gamma^2 I + \hat{B}_2^T \hat{B}_2 & 0 & \hat{B}_1^T P & \hat{B}_1^T P \\
  * & * & -P & 0 & 0 \\
  * & * & * & -P & 0 \\
  * & * & * & * & -P
\end{bmatrix} < 0
\]

(42)
admit a pair of positive solutions \((P > 0, \beta > 0)\), then the augmented system \([39]\) is internally stable. Moreover, \(\|L_{e,r}\|_\infty < \gamma\) and \(\|L_{f,r}\|_\infty > \delta\).

**Proof:** Firstly, we prove that system \([39]\) is internally stable, i.e., the following system

\[
d\eta(t) = (\dot{A}_0\eta(t) + \tilde{F}_0(\eta(t)))\,dt + (\dot{A}_1\eta(t) + \tilde{F}_1(\eta(t)))\,dw(t)
\]

is exponentially stable in mean square sense. For convenience, in this section, we denote \(\mathcal{L}_1\) as the infinitesimal generator of system \([39]\). To prove the internal stability of \([39]\), we take the Lyapunov function as \(V(\eta) = \eta'P\eta\), where \(P > 0\) is the solution of \([41]-[43]\). By Itô formula, for system \([44]\),

\[
\mathcal{L}_1 V(\eta)|_{v=0,f=0} = \eta' (\dot{A}_0'P + P\dot{A}_0 + \dot{A}_1'P\dot{A}_1)\eta + 2\eta'P\tilde{F}_0 + \tilde{F}_1'P\tilde{F}_1 + 2\eta'\dot{A}_1'P\tilde{F}_1.
\]

By \([40]\), it follows that

\[
2\eta'P\tilde{F}_0 + \tilde{F}_1'P\tilde{F}_1 + 2\eta'\dot{A}_1'P\tilde{F}_1 \leq \eta'P\eta + \tilde{F}_0'P\tilde{F}_0 \\
+2\tilde{F}_1'P\tilde{F}_1 + \eta'\dot{A}_1'P\dot{A}_1\eta \leq \eta'(P + \dot{A}_1'P\dot{A}_1 + 3\alpha^2\beta I)\eta.
\]

Substituting the above inequality into \([45]\), it yields that

\[
\mathcal{L}_1 V(\eta)|_{v=0,f=0} \leq \eta'(\dot{A}_0'P + P\dot{A}_0 + 2\dot{A}_1'P\dot{A}_1 + P + 3\alpha^2\beta I)\eta.
\]

Obviously, under the condition \([42]\), we have

\[-\theta := \lambda_{max}(\dot{A}_0'P + P\dot{A}_0 + 2\dot{A}_1'P\dot{A}_1 + P + 3\alpha^2\beta I) < 0.\]

So

\[
\mathcal{L}_1 V(\eta)|_{v=0,f=0} \leq -\theta\|\eta\|^2.
\]

According to \([26]\), system \([39]\) is exponentially stable in mean square sense.
Secondly, we prove \( \| \mathcal{L}_{v,r} \|_\infty < \gamma \) with \( \eta(0) = 0 \) and \( f(t) \equiv 0 \) in (39). Note that when \( f(t) \equiv 0 \) and \( \eta(0) = 0 \), (39) becomes

\[
\begin{aligned}
\eta(0) &= 0, \\
\Pi &= \tilde{\mathbb{E}}_t \mathbb{E}_t \mathbb{E}_t \mathbb{E}_t \leq \mathbb{E}_t \mathbb{E}_t \mathbb{E}_t \mathbb{E}_t \equiv 0 \text{ in (39)}.
\end{aligned}
\]

For system (47), using Itô formula to \( V(\eta) = \eta' P \eta \), we have

\[
E \int_0^T (||r(t)||^2 - \gamma^2 ||v(t)||^2) dt
= E \int_0^T [\mathcal{L}_1 V(\eta(t))]_{f \equiv 0} + ||r(t)||^2 - \gamma^2 ||v(t)||^2] dt
- EV(\eta(T))
\leq E \int_0^T \left[ \eta'(t) \left( \tilde{A}_2 \tilde{A}_2 + \tilde{A}_0 P + P \tilde{A}_0 + 2 \tilde{A}_1 P \tilde{A}_1 + P + 3 \alpha^2 \beta I \right) \eta(t) + 2 \eta(t)' P \tilde{B}_0 v(t) + v(t)' \tilde{B}_1' P \tilde{B}_1 v(t) + 2 \eta(t)' \tilde{A}_1' P \tilde{B}_1 v(t) + 2 \tilde{F}_1' P \tilde{B}_1 v(t) + v(t)' (-\gamma^2 I + \tilde{B}_2 \tilde{B}_2) v(t) \right] dt
- EV(\eta(T))
\leq E \int_0^T \left[ \eta'(t) \left( \tilde{A}_2 \tilde{A}_2 + \tilde{A}_0 P + P \tilde{A}_0 + 2 \tilde{A}_1 P \tilde{A}_1 + P + 4 \alpha^2 \beta I \right) \eta(t) + 2 \eta(t)' P \tilde{B}_0 v(t) + v(t)' \tilde{B}_1' P \tilde{B}_1 v(t) + 2 \eta(t)' \tilde{A}_1' P \tilde{B}_1 v(t) + v(t)' (-\gamma^2 I + \tilde{B}_2 \tilde{B}_2) v(t) \right] dt
- EV(\eta(T))
= E \int_0^T \left[ \eta(t)' \begin{bmatrix} \eta(t) \\ v(t) \end{bmatrix} \right] \left. \begin{bmatrix} \Pi \end{bmatrix} \begin{bmatrix} \eta(t) \\ v(t) \end{bmatrix} \right. dt - EV(\eta(T))
\leq E \int_0^T \left[ \eta(t)' \begin{bmatrix} \eta(t) \\ v(t) \end{bmatrix} \right] \begin{bmatrix} \Pi \end{bmatrix} \begin{bmatrix} \eta(t) \\ v(t) \end{bmatrix} dt,
\]

where

\[
\Pi := \begin{bmatrix} \Pi_{11} & P \tilde{B}_0 + \tilde{A}_1' P \tilde{B}_1 \\ * & -\gamma^2 I + \tilde{B}_2' \tilde{B}_2 + 2 \tilde{B}_1' P \tilde{B}_1 \end{bmatrix}
\]

with

\[
\Pi_{11} = \tilde{A}_2 \tilde{A}_2 + \tilde{A}_0 P + P \tilde{A}_0 + 2 \tilde{A}_1 P \tilde{A}_1 + P + 4 \alpha^2 \beta I.
\]
By Schur’s complement and inequality (42), we get $\prod < 0$. Thus, for any $T > 0$,
\[
E \int_0^T \|r(t)\|^2 \, dt \\
\leq \gamma^2 E \int_0^T \|v(t)\|^2 \, dt + \lambda_{\text{max}}(\prod) E \int_0^T \|Z(t)\|^2 \, dt \\
\leq (\gamma^2 + \lambda_{\text{max}}(\prod)) E \int_0^T \|v(t)\|^2 \, dt,
\]
where $Z(t) = \begin{bmatrix} \eta(t)' & v(t)' \end{bmatrix}'$. Let $T \to \infty$ in above, then for any nonzero $v \in L_2^2(\mathbb{R}^+, \mathbb{R}^{nv})$, 
$\|L_{v,r}\|_\infty \leq (\gamma^2 + \lambda_{\text{min}}(\prod))^{1/2} < \gamma$, so the external stability of (39) is proved.

Finally, we show $\|L_{f,r}\|_\infty > \delta$ when $v \equiv 0$ and $\eta(0) = 0$ in (39). Consider system
\[
\begin{aligned}
d\eta(t) &= (\bar{A}_0 \eta(t) + \bar{F}_0(\eta(t)) + \bar{C}_0 f(t))dt \\
&\quad + (\bar{A}_1 \eta(t) + \bar{F}_1(\eta(t)) + \bar{C}_1 f(t))dw, \\
\eta(0) &= 0, \\
r(t) &= \bar{A}_2 \eta(t) + \bar{C}_2 f(t).
\end{aligned}
\]
(48)

According to Lemma 2 for system (48), we are in a position to show that
\[
E \int_0^T \left( \|r(t)\|^2 - \delta^2 \|f(t)\|^2 \right) \, dt \\
= E \int_0^T \left( \|r(t)\|^2 - \delta^2 \|f(t)\|^2 - L_1 V(\eta(t))|_{v \equiv 0} \right) \, dt \\
\quad + EV(\eta(T)) - EV(\eta(0)) \\
\geq E \int_0^T \left[ -\eta'(t) \left( -\bar{A}_2 \bar{A}_2 + \bar{A}_0 P + P \bar{A}_0 + 2 \bar{A}_1 P \bar{A}_1 \\
+ P + 4 \alpha^2 \beta I \right) \eta(t) - 2 \eta(t)' P \bar{C}_0 f(t) \\
- 2 f(t)' \bar{C}_1 f(t) - 2 \eta(t)' \bar{A}_1 P \bar{C}_1 f(t) \\
+ f(t)'(-\delta^2 I + \bar{C}_2 \bar{C}_2) f(t) \right] dt + EV(\eta(T)) \\
= E \int_0^T \left[ \eta(t)' f(t)' \right] \begin{bmatrix} \eta(t) \\ f(t) \end{bmatrix} dt + EV(\eta(T)),
\]
(49)
where, by (43),
\[
\begin{bmatrix} \Phi_{11} & -P \bar{C}_0 - \bar{A}_1' P \bar{C}_1 \\
* & -\delta^2 I + \bar{C}_2' \bar{C}_2 - 2 \bar{C}_1' P \bar{C}_1 \end{bmatrix} > 0
\]
with
\[
\Phi_{11} = \bar{A}_2' \bar{A}_2 - \bar{A}_0' P - P \bar{A}_0 - 2 \bar{A}_1' P \bar{A}_1 - P - 4 \alpha^2 \beta I.
\]
So
\[
E \int_0^T \| r(t) \|^2 dt \\
\geq \delta^2 E \int_0^T \| f(t) \|^2 dt + \lambda_{\min}(\mathcal{M}) E \int_0^T \| \mathcal{M}(t) \|^2 dt,
\]
(50)
where \( \mathcal{M}(t) = \begin{bmatrix} \eta(t)' & f(t)' \end{bmatrix} \). For \( f(t) \neq 0, f \in L^2_\infty(\mathbb{R}^+, \mathbb{R}^n) \), let \( T \to \infty \) in (50), we have
\[
E \int_0^\infty \| r(t) \|^2 dt \\
\geq \delta^2 E \int_0^\infty \| f(t) \|^2 dt + \lambda_{\min}(\mathcal{M}) E \int_0^\infty \| \mathcal{M}(t) \|^2 dt \\
\geq \delta^2 E \int_0^\infty \| f(t) \|^2 dt + \lambda_{\min}(\mathcal{M}) E \int_0^\infty \| f(t) \|^2 dt
\]
which yields that \( \| \mathcal{L}_{f,r} \| \geq (\delta^2 + \lambda_{\min}(\mathcal{M}))^{1/2} > \delta \) due to \( \lambda_{\min}(\mathcal{M}) > 0 \). The proof is completed.
\( \Box \)

Based on Lemma [11] an \( H_-/H_\infty \) FDF can be designed in terms of LMIs for system (38). For convenience, we set \( P \) a real symmetric diagonal matrix such as \( P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0 \), then

\[
\mathcal{A}_0 := P \tilde{A}_0 = \begin{bmatrix} P_1 A_0 & 0 \\ \tilde{B} A_2 & \tilde{A} - \tilde{B} A_2 \end{bmatrix},\\
\mathcal{B}_0 := P \tilde{B}_0 = \begin{bmatrix} P_1 B_0 \\ \tilde{B} B_2 \end{bmatrix},\\
\mathcal{C}_0 := P \tilde{C}_0 = \begin{bmatrix} P_1 C_0 \\ \tilde{B} C_2 \end{bmatrix},\\
\mathcal{A}_1 := P \tilde{A}_1 = \begin{bmatrix} P_1 A_1 & 0 \\ 0 & 0 \end{bmatrix},\\
\mathcal{B}_1 := P \tilde{B}_1 = \begin{bmatrix} P_1 B_1 \\ 0 \end{bmatrix},\\
\mathcal{C}_1 := P \tilde{C}_1 = \begin{bmatrix} P_1 C_1 \\ 0 \end{bmatrix},\\
\mathcal{A}_2 := \tilde{A}_2 \tilde{A}_2 = \begin{bmatrix} A_2 \tilde{S} A_2 & -A_2 \tilde{S} A_2 \\ -\tilde{A}_2 \tilde{S} A_2 & \tilde{A}_2 \tilde{S} A_2 \end{bmatrix},\\
\mathcal{B}_2 := \tilde{B}_2 \tilde{B}_2 = \tilde{B}_2 \tilde{S} B_2,\\
\mathcal{C}_2 := \tilde{C}_2 \tilde{C}_2 = \tilde{C}_2 \tilde{S} C_2.
\]

where \( \begin{bmatrix} \tilde{A} & \tilde{B} \end{bmatrix} = P \begin{bmatrix} \tilde{A} & \tilde{B} \end{bmatrix}, \tilde{S} = \tilde{S} \tilde{S}. \)

**Theorem 4** If there exists the solution \( (P_1 > 0, P_2 > 0, \beta > 0, \tilde{A}, \tilde{B}, \tilde{S}) \) solving the following LMIs:

\[
0 < \begin{bmatrix} P_1 & 0 \\ P_2 & \end{bmatrix} \leq \beta I,
\]
(51)
and

\[
\begin{bmatrix}
\ell_{11} & \mathcal{C}_0 & \mathcal{A}'_1 & \mathcal{A}'_1 & 0 \\
* & \delta^2 I - \mathcal{C}_2 & 0 & \mathcal{C}'_1 & \mathcal{C}'_1 \\
* & * & -P & 0 & 0 \\
* & * & * & -P & 0 \\
* & * & * & * & -P \\
\end{bmatrix} < 0,
\]

with

\[h_{11} = \mathcal{A}_2 + \mathcal{A}'_0 + \mathcal{A}_0 + P + 4\alpha^2 \beta I\]

and

\[\ell_{11} = -\mathcal{A}_2 + \mathcal{A}'_0 + \mathcal{A}_0 + P + 4\alpha^2 \beta I,\]

then \[(30)\] is a mixed \(H_-/H_\infty\) FDF of the system \[(38)\]. In this case, the admissible filter matrices can be given by

\[\hat{\mathcal{S}} = \hat{\mathcal{S}}^{\frac{1}{2}}, \quad \hat{\mathcal{A}} = P_2^{-1} \hat{\mathcal{A}}, \quad \hat{\mathcal{B}} = P_2^{-1} \hat{\mathcal{B}}.\]

5 Numerical Example

In this section, one numerical example is provided to illustrate the effectiveness of our main results.
Example 1 Consider the nonlinear stochastic system (38) with the following parameters:

\[
A_0 = \begin{bmatrix} -6.01 & -2.94 \\ -2.94 & -6.17 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.91 & -0.44 \\ -1.31 & -0.39 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0.43 & 0.15 \\ -0.09 & 0.07 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.37 \\ -0.41 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 0.35 \\ -0.6 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1.21 \\ -0.11 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -3.67 \\ 0.51 \end{bmatrix},
\]

\[
F_1(x(t)) = B_1 = C_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
F_0(x(t)) = 0.5 \begin{bmatrix} \sin(x_1(t)) \\ \sin(x_2(t)) \end{bmatrix}.
\]

In addition, we choose the $H_{\infty}$ performance level $\gamma = 1$ and $H_{-}$ performance level $\delta = 0.5$. For the above parameters, by using Matlab LMI Toolbox, the solutions of LMIs in Theorem 4 for $\{P_1 > 0, P_2 > 0, \beta > 0, \hat{A}, \hat{B}, \hat{S}\}$ are obtained as

\[
P_1 = \begin{bmatrix} 0.5248 & -0.0397 \\ -0.0397 & 0.3805 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.4799 & 0 \\ 0 & 0.4799 \end{bmatrix},
\]

\[
\hat{A} = \begin{bmatrix} -2.0747 & -0.8242 \\ 0.8209 & -2.0835 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0.0032 & -0.0529 \\ -0.0054 & -0.0441 \end{bmatrix},
\]

\[
\hat{S} = \begin{bmatrix} 0.4052 & 0.2562 \\ 0.2562 & 0.2745 \end{bmatrix}, \quad \beta = 6.
\]

Thus, the desired filter matrices of the $H_{-}/H_{\infty}$ FDF are as follows:

\[
\hat{A} = \begin{bmatrix} -4.3228 & -1.7172 \\ 1.7103 & -4.3411 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0.0067 & -0.1102 \\ -0.0113 & -0.0919 \end{bmatrix},
\]

\[
\hat{S} = \begin{bmatrix} 0.0067 & -0.1102 \\ -0.0113 & -0.0919 \end{bmatrix}.
\]

We use Matlab to simulate the state trajectories $x(t)$ and the filter trajectories $\hat{x}(t)$ of system (39) under $v(t) \equiv 0$, $f(t) \equiv 0$ and $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; see Figures 7 and 8. From Figures 7 and 8, it is easy to see that system (39) is internally stable.
To show the effectiveness of the designed $H_-$/$H_\infty$ filter, we assume $v(t) = 0.9^t \in L^2_2(\mathbb{R}^+, \mathbb{R})$ and $f(t) = f_0.4(t) = \begin{cases} 0.4, & t \in [10, 20] \\ 0, & \text{else} \end{cases} \in L^2_2(\mathbb{R}^+, \mathbb{R})$. The residual evaluation function is

$$J_r(t) = \left( E \left\{ \frac{1}{t} \int_0^t r'(s)r(s)ds \right\} \right)^{\frac{1}{2}}.$$

After 100 times Monte Carlo simulations without fault influence, $J_{th} = 0.4$ with evaluation window $T = 5$. The residual evaluation function $J_r(t)$ and fault signal $f(t)$ are depicted in Figure 3. However, if the fault signal is weaker, the FDF may fail to alarm, which can be seen from Figure 4 when we set $f(t) = f_{0.1}(t) = \begin{cases} 0.1, & t \in [10, 20] \\ 0, & \text{else} \end{cases} \in L^2_2(\mathbb{R}^+, \mathbb{R})$. In practice, one
needs to select a suitable combination of $\gamma$ and $\delta$ according to practical engineering requirements. In this example, if we set $\gamma = 0.78$, $\delta = 1.41$, $f(t) = f_{0.1}(t)$, the $H_{-}/H_{\infty}$ FDF parameters can be computed as

$$
\hat{A} = \begin{bmatrix} -4.8359 & -3.3771 \\ 3.3761 & -4.509 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} -0.0177 & -0.1615 \\ -0.0124 & -0.1198 \end{bmatrix},
$$

$$
\hat{S} = \begin{bmatrix} 0.6030 & 0.2787 \\ 0.2787 & 0.4084 \end{bmatrix}
$$

as well as $J_{th} = 0.0205$. From Figure 5, we can see that the fault sensitivity is improved.
6 Conclusion

In this paper, the $H_-/H_\infty$ FDF design for Itô-type affine nonlinear stochastic systems and quasi-linear systems have been discussed, and sufficient conditions for the existence of the desired FDF have been given via HJIs and LMIs, respectively. The key to the FDF design of affine nonlinear stochastic systems is how to solve the coupled HJIs, this is a very challenging problem, and some potential effective approaches to overcome this difficulty may refer to [8, 25] for global linearization method, [1] for neural network method, and [9] for fuzzy interpolation method. In addition, from our simulation example, in order to select a more suitable combination of $(\gamma, \delta)$, a co-design algorithm for $H_-$ index and $H_\infty$ index is necessary. Wherever possible, the smaller $\gamma > 0$ and the larger $\delta > 0$, the better the performance of FDF. However, from the second inequalities of HJIs (13) and (14), $\gamma > 0$ cannot be arbitrarily small and $\delta > 0$ cannot be arbitrarily large. In order to obtain the optimal selection $(\gamma^*, \delta^*)$, we may turn to Pareto optimization method [22, 23] together with convex optimization [3]. Pareto optimization is a co-operative game, its application to $H_-/H_\infty$ FDF design will be our future work.

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