Introduction to the method of multiple scales

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1 Introduction

Perturbation methods are aimed at finding approximate analytic solutions to problems whose exact analytic solutions can not be found. The setting where perturbation methods are applicable is where there is a family of equations $\mathcal{P}(\varepsilon)$, depending on a parameter $\varepsilon \ll 1$, and where $\mathcal{P}(0)$ has a known solution. Perturbation methods are designed to construct solutions to $\mathcal{P}(\varepsilon)$ by adding small corrections to known solutions of $\mathcal{P}(0)$. The singular aim of perturbation methods is to calculate corrections to solutions of $\mathcal{P}(0)$. Perturbation methods do not seek to prove that a solution of $\mathcal{P}(0)$, with corrections added, is close to a solution of $\mathcal{P}(\varepsilon)$ for $\varepsilon$ in some finite range with respect to some measure of error. It’s sole aim is to compute corrections and to make sure that the first correction is small with respect to the chosen solution of $\mathcal{P}(0)$, that the second correction is small with respect to the first correction and so on, all in the limit when $\varepsilon$ approaches zero. This formal nature and limited aim of is why we prefer to call it perturbation methods rather than perturbation theory. A mathematical theory is a description of proven mathematical relations among well defined objects of human thought. Perturbation methods does not amount to a mathematical theory in this sense. It is more like a very large bag of tricks whose elements have a somewhat vague domain of applicability and where the logical relations between the tricks are not altogether clear, to put it nicely.

After all this negative press you might ask why we should bother with this subject at all and why we should not rather stay with real, honest to God, mathematics. The reason is simply this: If you want analytic solutions to complex problems, it is the only game in town. In fact for quantum theory, which is arguably our best description of reality so far, perturbation methods is almost always the first tool we reach for. For the quantum theory of fields, like quantum electrodynamics, perturbation methods are essentially the only tools available. These theories are typically only known in terms of perturbation expansions. You could say that we don’t actually know what the mathematical description of these very fundamental theories are. But at the same time, quantum theory of fields give some of the most accurate, experimentally verified, predictions in all of science.

So clearly, even if perturbation methods are somewhat lacking in mathematical justification, they work pretty well. And in the end that is the only thing that really counts.

These lecture notes are not meant to be a general introduction to the wide spectrum of perturbation methods that are used across science. Many textbooks exists whose aim is to give such a broad overview that include the most commonly used perturbation methods\cite{2,10,3,9}. Our aim is more limited, and we focus on one such method that is widely used in nonlinear optics and laser physics. This is the method of multiple scales. The method of multiple scales is described in all respectable books on perturbation methods and there are also more specialized books on singular perturbation methods where the method of multiple scales has a prominent place\cite{6,5}. There are however quite different views on how the method is to be applied and what its limitations are.
Therefore, the description of the method appears quite different in the various sources, depending on the views of the authors. In these lecture notes we describe the method in a way that is particularly well suited to the way it is used in nonlinear optics and laser physics. The source that is closest to our approach is [11].

We do not assume that the reader has had any previous exposure to perturbation methods. The lecture notes therefore starts off with three sections where the basic ideas of asymptotic expansions are introduced and illustrated using algebraic equations. The application to differential equations starts in section four where we use regular perturbation expansions to find approximate solutions to ODEs. In section five we introduce the method of multiple scale and apply it to weakly nonlinear ODEs. We move on to weakly nonlinear dispersive PDEs in section six and finally move on to the Maxwell equations in section seven. Several exercises involving multiple scales for ODEs and PDEs are included in the lecture notes.

2 Regular and singular problems, applications to algebraic equations.

In this section we will introduce perturbation methods in the context of algebraic equations. One of the main goals of this section is to introduce the all-important distinction between regular and singular perturbation problems, but we also use the opportunity to introduce the notion of a perturbation hierarchy and describe some of its general properties.

2.1 Example 1: A regularly perturbed quaderatic equation

Consider the polynomial equation

\[ x^2 - x + \varepsilon = 0 \] (1)

This is our perturbed problem \( P(\varepsilon) \). The unperturbed problem \( P(0) \), is

\[ x^2 - x = 0 \] (2)

The unperturbed problem is very easy to solve

\[ x^2 - x = 0 \]

\[ \therefore \]

\[ x_0 = 0 \] (3)

\[ x_1 = 1 \] (4)

Let us focus on \( x_1 \) and let us assume that the perturbed problem has a solution in the form of a perturbation expansion

\[ x(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + ... \] (5)
where \( a_0 = 1 \). Our goal is to find the unknown numbers \( a_1, a_2, \ldots \). These numbers should have a size of order 1. This will ensure that \( \varepsilon a_1 \) is a small correction to \( a_0 \), that \( \varepsilon^2 a_2 \) is a small correction to \( \varepsilon a_1 \) and so on, all in the limit of small \( \varepsilon \). As we have stressed before, maintaining the ordering of the perturbation expansion is the one and only unbreakable rule when we do perturbation calculations. The perturbation method now proceeds by inserting the expansion (5) into equation (1) and collecting terms containing the same order of \( \varepsilon \).
The perturbation hierarchy (8) is easy to solve and we find

\[ a_0 = 1 \]  \hspace{1cm} (9)
\[ a_1 = -1 \]  \hspace{1cm} (10)
\[ a_2 = -1 \]

and our perturbation expansion to second order in \( \varepsilon \) is

\[ x(\varepsilon) = 1 - \varepsilon - \varepsilon^2 + ... \]  \hspace{1cm} (11)

For this simple case we can solve the unperturbed problem directly using the solution formula for a quadratic equation. Here are some numbers

| \( \varepsilon \)   | Exact solution | Perturbation solution |
|-------------------|----------------|-----------------------|
| 0.001             | 0.998999      | 0.998999              |
| 0.01              | 0.989898      | 0.9989900             |
| 0.1               | 0.887298      | 0.890000              |

We see that our perturbation expansion is quite accurate even for \( \varepsilon \) as large as 0.1.

Let us see if we can do better and find an even more accurate approximation by extending the perturbation expansion to higher order in \( \varepsilon \). In fact let us take the perturbation expansion to infinite order in \( \varepsilon \).

\[ x(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + ... = a_0 + \sum_{n=1}^{\infty} \varepsilon^n a_n \]  \hspace{1cm} (12)

Inserting (12) into (1) and expanding we get

\[
\begin{align*}
(a_0 + \sum_{n=1}^{\infty} \varepsilon^n a_n)(a_0 + \sum_{m=1}^{\infty} \varepsilon^m a_m) - a_0 - \sum_{n=1}^{\infty} \varepsilon^n a_n + \varepsilon & = 0 \\
\downarrow \\
a_0^2 - a_0 + \sum_{p=1}^{\infty} \varepsilon^p (2a_0 - 1)a_p + \sum_{p=2}^{\infty} \varepsilon^p \left( \sum_{m=1}^{p-1} a_m a_{p-m} \right) + \varepsilon & = 0 \\
\downarrow \\
a_0^2 - a_0 + \varepsilon ((2a_0 - 1)a_1 + 1) + \sum_{p=2}^{\infty} \varepsilon^p \left( (2a_0 - 1)a_p + \sum_{m=1}^{p-1} a_m a_{p-m} \right) & = 0
\end{align*}
\]  \hspace{1cm} (13)

Therefore the complete perturbation hierarchy is

\[ a_0(a_0 - 1) = 0 \]  \hspace{1cm} (14)
\[ (2a_0 - 1)a_1 = -1 \]  \hspace{1cm} (15)
\[ (2a_0 - 1)a_p = - \sum_{m=1}^{p-1} a_m a_{p-m}, \quad p \geq 2 \]
The right-hand side of the equation for \( a_p \) only depends on \( a_j \) for \( j < p \). Thus the perturbation hierarchy is an infinite system of nonlinear equations that is coupled in such a special way that we can solve them one by one. The perturbation hierarchy truncated at order 4 is

\[
\begin{align*}
(2a_0 - 1)a_1 &= -1 \\
(2a_0 - 1)a_2 &= -a_1^2 \\
(2a_0 - 1)a_3 &= -2a_1a_2 \\
(2a_0 - 1)a_4 &= -2a_1a_3 - a_2^2
\end{align*}
\]

Using \( a_0 = 1 \), the solution of the hierarchy is trivially found to be

\[
\begin{align*}
a_1 &= -1 \\
a_2 &= -1 \\
a_3 &= -2 \\
a_4 &= -5
\end{align*}
\]

For \( \varepsilon = 0.1 \) the perturbation expansion gives

\[
x(0.1) = 0.8875...
\]

whereas the exact solution is

\[
x(0.1) = 0.8872...
\]

we are clearly getting closer. However we did not get all that much in return for our added effort.

Of course we did not actually have to use perturbation methods to find solutions to equation (1) since it is exactly solvable using the formula for the quadratic equation. The example however illustrate many general features of perturbation calculations that will appear again and again in different guises.

### 2.2 Example 2: A regularly perturbed quintic equation.

Let us consider the equation

\[
x^5 - 2x + \varepsilon = 0
\]

This is our perturbed problem \( \mathcal{P}(\varepsilon) \). For this case perturbation methods are neccessary since there is no solution formula for general polynomial equations of order higher than four. The unperturbed problem \( \mathcal{P}(0) \), is

\[
x^5 - 2x = 0
\]

It is easy to see that the unperturbed equation has a real solution

\[
x \equiv a_0 = \sqrt[5]{2}
\]
We will now construct a perturbation expansion for a solution to (20) starting with the solution \( x = a_0 \) of the unperturbed equation (21).

\[ x(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots \quad (23) \]

Inserting (23) into equation (20) and expanding we get

\[
\begin{align*}
(a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots)^5 & = 0 \\
-2(a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots) + \varepsilon & = 0 \\
\downarrow & \\
a_0^5 + 5a_0^4(\varepsilon a_1 + \varepsilon^2 a_2 + \ldots) + 10a_0^3(\varepsilon a_1 + \ldots)^2 & + \ldots \\
-2a_0 - 2\varepsilon a_1 - 2\varepsilon^2 a_2 - \ldots + \varepsilon & = 0 \\
\downarrow & \\
a_0^5 - 2a_0 + \varepsilon(1 + 5a_0^4 a_1 - 2a_1) + \varepsilon^2(5a_0^4 a_2 + 10a_0^3 a_1^2 - 2a_2) + \ldots & = 0 \\
\end{align*}
\quad (24) \]

Thus the perturbation hierarchy to order two in \( \varepsilon \) is

\[
\begin{align*}
a_0^5 - 2a_0 & = 0 \\
(5a_0^4 - 2)a_1 & = -1 \\
(5a_0^4 - 2)a_2 & = -10a_0^3 a_1^2 \\
\end{align*}
\quad (25) \]

Observe that the first equation in the hierarchy, for \( a_0 \), is nonlinear in whereas the equations for \( a_p \) are linear in \( a_p \) for \( p > 0 \). All the linear equations are defined in terms of the same linear operator \( L(\cdot) = (5a_0^4 - 2)(\cdot) \). This is the same structure that we saw in the previous example. If the unperturbed problem is linear the first equation in the hierarchy will also be linear. The perturbation hierarchy is easy to solve and we find

\[
\begin{align*}
a_1 & = -\frac{1}{5a_0^4 - 2} = -\frac{1}{8} \\
a_2 & = -\frac{10a_0^3 a_1^2}{2a_0^4 - 2} = -\frac{5\sqrt{8}}{256} \\
\end{align*}
\quad (28) \]

The perturbation expansion to second order is then

\[ x(\varepsilon) = \sqrt{2} - \frac{1}{8} \varepsilon - \frac{5\sqrt{8}}{256} \varepsilon^2 + \ldots \quad (29) \]

Here are some numbers

| \( \varepsilon \) | Exact solution | Perturbation solution |
|---|---|---|
| 0.001 | 1.18908 | 1.18908 |
| 0.01 | 1.19795 | 1.19795 |
| 0.1 | 1.17636 | 1.17638 |

Perturbation expansions for the other solutions to equation (20) can be found by starting with the other four solutions of the equation (21). In this way we
get perturbation expansions for all the solutions of (20), and the effort was not much larger than for the quadratic equation.

If we can find perturbation expansions for all the solutions of a problem \( P(\varepsilon) \), by starting with solutions of the unperturbed problem \( P(0) \), we say that \( P(\varepsilon) \) is a regular perturbation of \( P(0) \). If the perturbation is not regular it is said to be singular. This distinction applies to all kinds of perturbation problems whether we are looking at algebraic equations, ordinary differential equations or partial differential equations. Clearly, for polynomial equations a necessary condition for being a regular perturbation problem is that \( P(\varepsilon) \) and \( P(0) \) have the same algebraic order. This is not always the case as the next example shows.

2.3 Example 3: A singularly perturbed quadratic equation.

Let us consider the following equation

\[
\varepsilon x^2 + x - 1 = 0
\]  

This is our perturbed problem \( P(\varepsilon) \). The unperturbed problem \( P(0) \), is

\[
x - 1 = 0
\]

There is only one solution to the unperturbed problem

\[
x \equiv a_0 = 1
\]

Let us find a perturbation expansion for a solution to (30) starting with the solution (32) of the unperturbed problem.

\[
x(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + ...
\]

Inserting (33) into equation (30) and expanding we get

\[
\varepsilon(a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + ...)^2 + a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + ... - 1 = 0
\]

\[
\varepsilon(a_0^2 + 2a_0 a_1 + ...) + a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + ... - 1 = 0
\]

\[
a_0 - 1 + \varepsilon(a_1 + a_0^2) + \varepsilon^2(a_2 + 2a_0 a_1) + ... = 0
\]

The perturbation hierarchy up to second order in \( \varepsilon \) is thus

\[
a_0 = 1
\]

\[
a_1 = -a_0^2
\]

\[
a_2 = -2a_0 a_1
\]
The solution of the perturbation hierarchy is
\[ a_0 = 1 \]  
\[ a_1 = -1 \]  
\[ a_2 = 2 \]  

and the perturbation expansion for the solution to equation (30) starting from the solution \( x = 1 \) to the unperturbed problem (31) is
\[ x(\varepsilon) = 1 - \varepsilon + 2\varepsilon^2 + ... \]  

In order to find a perturbation expansion for the other solution to the quadratic equation (30), the unperturbed problem (31) is of no help.

However, looking at equation (30) we learn something important: In order for a solution different from \( x = 1 \) to appear in the limit when \( \varepsilon \) approaches zero, the first term in (30) can not approach zero. This is only possible if \( x \) approaches infinity as \( \varepsilon \) approaches zero.

Inspired by this, let us introduce a change of variables
\[ x = \varepsilon^{-p} y \]  
where \( p > 0 \). If \( y \) is of order one as \( \varepsilon \) approaches zero then \( x \) will approach infinity and will thus be the solution we lost in (31). Inserting (38) into (30) give us
\[ \varepsilon(\varepsilon^{-p}y)^2 + \varepsilon^{-p}y - 1 = 0 \]
\[ \Downarrow \]
\[ \varepsilon^{1-2p} y^2 + \varepsilon^{-p}y - 1 = 0 \]
\[ \Downarrow \]
\[ y^2 + \varepsilon^{p-1}y - \varepsilon^{2p-1} = 0 \]  

The idea is now to pick a value for \( p \), thereby defining a perturbed problem \( \mathcal{P}(\varepsilon) \), such that \( \mathcal{P}(0) \) has a solution of order one. For \( p > 1 \) we get in the limit when \( \varepsilon \) approaches zero the problem
\[ y^2 = 0 \]  

which does not have any solution of order one. One might be inspired to choose \( p = \frac{1}{2} \). We then get the equation
\[ \sqrt{\varepsilon}y^2 + y - \sqrt{\varepsilon} = 0 \]  

which in the limit when \( \varepsilon \) approaches zero turns into
\[ y = 0 \]  

This equation clearly has no solution of order one. Another possibility is to choose \( p = 1 \). Then we get the equation
\[ y^2 + y - \varepsilon = 0 \]  

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In the limit when \( \varepsilon \) approaches zero this equation turns into
\[
y^2 + y = 0 \quad (44)
\]
This equation has a solution \( y = -1 \) which is of order one. We therefore proceed with this choice for \( p \). We introduce a perturbation expansion for the solution to (43) that starts at the solution \( y \equiv a_0 = -1 \) to the unperturbed equation (44).
\[
x(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots \quad (45)
\]
Inserting the perturbation expansion (45) into equation (43) and expanding we get
\[
(a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots)^2 + a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots - \varepsilon = 0
\downarrow
a_0^2 + a_0 + \varepsilon((2a_0 + 1)a_1 - 1) + \varepsilon^2((2a_0 + 1)a_2 + a_1^2) + \ldots = 0 \quad (46)
\]
The perturbation hierarchy to second order in \( \varepsilon \) is then
\[
a_0^2 + a_0 = 0 \quad (47)
(2a_0 + 1)a_1 = 1
(2a_0 + 1)a_2 = -a_1^2
\]
We observe in passing that the perturbation hierarchy has the special structure we have seen earlier. The solution to the perturbation hierarchy is
\[
a_1 = -1 \quad (48)
a_2 = 1
\]
and the perturbation expansion to second order in \( \varepsilon \) is
\[
y(\varepsilon) = -1 - \varepsilon + \varepsilon^2 + \ldots \quad (49)
\]
Going back to the original coordinate \( x \) we finally get
\[
x(\varepsilon) = -\varepsilon^{-1} - 1 + \varepsilon + \ldots \quad (50)
\]
Even for \( \varepsilon \) as large as 0.1 the perturbation expansion and the exact solution, \( x_E(\varepsilon) \), are close
\[
x(\varepsilon) = -\varepsilon^{-1} - 1 + \varepsilon + \ldots \approx -10.900.. \quad (51)
x_E(\varepsilon) = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon} \approx -10.916..
\]
The perturbation problem we have discussed in this example is evidently a singular problem. For singular problems, a coordinate transformation, like the one defined by (38), must at some point be used to transform the singular perturbation problem into a regular one.
At this point I need to be honest with you: There is really no general rule for how to find the right transformations. Skill, experience, insight and sometimes even dumb luck is needed to succeed. This is one of the reasons why I prefer to call our subject perturbation methods and not perturbation theory. Certain classes of commonly occurring singular perturbation problems has however been studied extensively and rules for finding the correct transformations has been designed. In general what one can say is that some kind of scaling transformation, like in (38), is almost always part of the mix.

3 Asymptotic sequences and series.

When using perturbation methods our main task is to investigate the behaviour of unknown functions \( f(\varepsilon) \), in the limit when \( \varepsilon \) approaches zero. This is what we did in examples 1 to 3.

The way we approach this problem is to compare the unknown function \( f(\varepsilon) \) to one or several known functions when \( \varepsilon \) approaches zero. In example 1 and 2 we compared with the known functions \( \{1, \varepsilon, \varepsilon^2, \ldots\} \) whereas in example 3 we used the functions \( \{\varepsilon^{-1}, 1, \varepsilon, \ldots\} \). In order to facilitate such comparisons we introduce the "large-o" and "little-o" notation.

**Definition 1** Let \( f(\varepsilon) \) be a function of \( \varepsilon \). Then

i) \( f(\varepsilon) = O(g(\varepsilon)), \quad \varepsilon \to 0 \quad \iff \quad \lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| \neq 0 \)

ii) \( f(\varepsilon) = o(g(\varepsilon)), \quad \varepsilon \to 0 \quad \iff \quad \lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0 \)

Thus \( f(\varepsilon) = O(g(\varepsilon)) \) means that \( f(\varepsilon) \) and \( g(\varepsilon) \) are of roughly the same size when \( \varepsilon \) approaches zero and \( f(\varepsilon) = o(g(\varepsilon)) \) means that \( g(\varepsilon) \) is much larger than \( f(\varepsilon) \) when \( \varepsilon \) approaches zero.

We have for example that

1. \( \sin(\varepsilon) = O(\varepsilon), \quad \varepsilon \to 0, \) because
   
   \[
   \lim_{\varepsilon \to 0} \left| \frac{\sin(\varepsilon)}{\varepsilon} \right| = 1 \neq 0
   \]

2. \( \sin(\varepsilon^2) = o(\varepsilon), \quad \varepsilon \to 0, \) because
   
   \[
   \lim_{\varepsilon \to 0} \left| \frac{\sin(\varepsilon^2)}{\varepsilon} \right| = \lim_{\varepsilon \to 0} \left| \frac{2\varepsilon \cos(\varepsilon^2)}{1} \right| = 0
   \]

3. \( 1 - \cos(\varepsilon) = o(\varepsilon), \quad \varepsilon \to 0, \) because
   
   \[
   \lim_{\varepsilon \to 0} \left| \frac{1 - \cos(\varepsilon)}{\varepsilon} \right| = \lim_{\varepsilon \to 0} \left| \frac{\sin(\varepsilon)}{1} \right| = 0
   \]
4. \( \ln(\varepsilon) = o(\varepsilon^{-1}), \quad \varepsilon \to 0, \) because

\[
\lim_{\varepsilon \to 0} \left| \frac{\ln(\varepsilon)}{\varepsilon^{-1}} \right| = \lim_{\varepsilon \to 0} \left| \frac{\varepsilon^{-1}}{\varepsilon^{-2}} \right| = \lim_{\varepsilon \to 0} \varepsilon = 0
\]

When we apply perturbation methods we usually use a whole sequence of comparison functions. In examples 1 and 2 we used the sequence

\[ \{\delta_n(\varepsilon) = \varepsilon^n\}_{n=1}^\infty \]

and in example 3 we used the sequence

\[ \{\delta_n(\varepsilon) = \varepsilon^n\}_{n=-1}^\infty \]

What is characteristic about these sequences is that

\[ \delta_{n+1}(\varepsilon) = o(\delta_n(\varepsilon)), \quad \varepsilon \to 0 \quad (52) \]

for all \( n \) in the range defining the sequences. Sequences of functions that satisfy the condition \((52)\) are called asymptotic sequences.

Here are some asymptotic sequences

1. \( \delta_n(\varepsilon) = \sin(\varepsilon)^n \)
2. \( \delta_n(\varepsilon) = \ln(\varepsilon)^n \)
3. \( \delta_n(\varepsilon) = (\sqrt[3]{\varepsilon})^n \)

Using the notion of asymptotic sequence we can define asymptotic expansion analogous to the way infinite series are defined in elementary calculus.

**Definition 2** Let \( \{\delta_n(\varepsilon)\} \) be an asymptotic sequence. Then a formal series

\[ \sum_{n=1}^{\infty} a_n \delta_n(\varepsilon) \quad (53) \]

is an asymptotic expansion for a function \( f(\varepsilon) \) as \( \varepsilon \) approaches zero if

\[ f(\varepsilon) - \sum_{n=1}^{N} a_n \delta_n(\varepsilon) = o(\delta_N(\varepsilon)), \quad \varepsilon \to 0 \quad (54) \]
Observe that

\[ f(\varepsilon) - a_1 \delta_1(\varepsilon) = o(\delta_1(\varepsilon)), \quad \varepsilon \to 0 \]

\[ \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - a_1 \delta_1(\varepsilon)}{\delta_1(\varepsilon)} = 0 \]

\[ \lim_{\varepsilon \to 0} a_1 - \frac{f(\varepsilon)}{\delta_1(\varepsilon)} = 0 \]

\[ a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\delta_1(\varepsilon)} \quad (55) \]

in an entirely similar way we find that for all \( m \geq 1 \) that

\[ a_m = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{n=1}^{m-1} a_n \delta_n(\varepsilon)}{\delta_m(\varepsilon)} \quad (56) \]

This shows that for a fixed asymptotic sequence, the coefficients of the asymptotic expansion for a function \( f(\varepsilon) \) are determined by taking limits. Observe that the formula (55) does not require differentiability for \( f(\varepsilon) \) at \( \varepsilon = 0 \). This is very different from Taylor expansions which require that \( f(\varepsilon) \) is infinitely differentiable at \( \varepsilon = 0 \).

This is a hint that asymptotic expansions are much more general than the usual convergent expansions, like for example power series, that we known from elementary calculus. In fact asymptotic expansions may well diverge but this does not make them less useful! The following example was first discussed by Leonard Euler in 1754.

### 3.1 Euler’s example

Let a function \( f(\varepsilon) \) be defined by the formula

\[ f(\varepsilon) = \int_0^\infty dt \frac{e^{-t}}{1 + \varepsilon t} \quad (57) \]

The integral defining \( f(\varepsilon) \) converge very fast and because of this \( f(\varepsilon) \) is a very smooth function, in fact it is infinitely smooth and moreover analytic in the complex plane where the negative real axis has been removed.

Using the properties of telescoping series we observe that for all \( m \geq 0 \)

\[ \frac{1}{1 + \varepsilon t} = \sum_{n=0}^{m} (-\varepsilon t)^n + \frac{(-\varepsilon t)^{m+1}}{1 + \varepsilon t} \quad (58) \]
Inserting (58) into (57) we find that

\[ f(\varepsilon) = S_m(\varepsilon) + R_m(\varepsilon) \]  

(59)

where

\[ S_m(\varepsilon) = \sum_{n=0}^{m} (-1)^n n! \varepsilon^n \]  

(60)

\[ R_m(\varepsilon) = (-\varepsilon)^{m+1} \int_0^{\infty} \frac{t^{m+1} e^{-t}}{1 + \varepsilon t} \, dt \]

For the quantity \( R_m(\varepsilon) \) we have the estimate

\[ |R_m(\varepsilon)| \leq \varepsilon^{m+1} \int_0^{\infty} \frac{t^{m+1} e^{-t}}{1 + \varepsilon t} \leq \varepsilon^{m+1} \int_0^{\infty} t^{m+1} e^{-t} = (m+1)! \varepsilon^{m+1} \]  

(61)

from which it follows that

\[ \lim_{\varepsilon \to 0} \left| \frac{R_m(\varepsilon)}{\varepsilon^m} \right| \leq \lim_{\varepsilon \to 0} (m+1)! \varepsilon = 0 \]  

(62)

Thus we have proved that an asymptotic expansion for \( f(\varepsilon) \) is

\[ f(\varepsilon) = \sum_{n=0}^{\infty} (-1)^n n! \varepsilon^n \]  

(63)

It is on the other hand trivial to verify that the formal power series

\[ \sum_{n=0}^{\infty} (-1)^n n! \varepsilon^n \]  

(64)

diverge for all \( \varepsilon \neq 0 \! \). In figure 1 we compare the function \( f(\varepsilon) \) with what we get from the asymptotic expansion for a range of \( \varepsilon \) and several truncation levels for the expansion. From this example we make the following two observations that are quite generic with regards to the convergence or divergence of asymptotic expansions.

Firstly, the asymptotic expansion (63) is an accurate representation of \( f(\varepsilon) \) in the limit when \( \varepsilon \) approaches zero even if the expansions is divergent. Secondly, adding more terms to the expansion for a fixed value of \( \varepsilon \) makes the expansion into a worse approximation.

In reality we are most of the time, because of algebraic complexity, only able to calculate a few terms of an asymptotic expansion. Thus convergence properties of the expansion are most of the time unknown. As this example shows, convergence properties are also not relevant for what we are trying to achieve when we solve problems using perturbation methods.
Figure 1: Comparing the exact(blue) expression for $f(\varepsilon)$ with the asymptotic expansion (63) containing ten(red) and twenty(yellow) terms

4 Regular perturbation expansions for ordinary differential equations.

It is now finally time to start solving differential equations using asymptotic expansions. Let us start with a simple boundary value problem for a first order ordinary differential equation.

4.1 Example 1: A weakly nonlinear boundary value problem.

Consider the following boundary value problem

$$ y'(x) + y(x) + \varepsilon y^2(x) = x, \quad 0 < x < 1 $$

$$ y(1) = 1 $$

(65)

where $\varepsilon$ as usual is a small number. Since the differential equation is nonlinear and nonseparable, this is a nontrivial problem. The unperturbed problem is

$$ y'(x) + y(x) = x, \quad 0 < x < 1 $$

$$ y(1) = 1 $$

(66)

The unperturbed problem is easy to solve since the equation is a first order linear equation. The general solution to the equation is

$$ y(x) = x - 1 + Ae^{-x} $$

(67)
The arbitrary constant $A$ is determined from the boundary condition

$$y(1) = 1$$

$$\downarrow$$

$$1 - 1 + Ae^{-1} = 1$$

$$\downarrow$$

$$A = e$$  \hfill (68)

Thus the unique solution to the unperturbed problem is

$$y_0(x) = x - 1 + e^{1-x}$$  \hfill (69)

We now want to find an asymptotic expansion for the solution to the perturbed problem (65) starting from the solution $y_0(x)$. We thus postulate an expansion

$$y(x; \varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \ldots$$  \hfill (70)

Inserting (70) into (65) and expanding we get

$$\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots\right)' + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots + \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots)^2 = x$$

$$\downarrow$$

$$y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \ldots + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots + \varepsilon(y_0^2 + 2\varepsilon y_0 y_1 + \ldots) = x$$

$$\downarrow$$

$$y_0' + y_0 + \varepsilon(y_1' + y_1 + y_0^2) + \varepsilon^2(y_2' + y_2 + 2y_0 y_1) + \ldots = x$$  \hfill (72)

We must also expand the boundary condition

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \ldots = 1$$  \hfill (73)

From (72) and (73) we get the following perturbation hierarchy

$$y_0'(x) + y_0(x) = x$$

$$y_0(1) = 1$$  \hfill (74)

$$y_1'(x) + y_1(x) = -y_0^2(x)$$

$$y_1(1) = 0$$  \hfill (75)

$$y_2'(x) + y_2(x) = -2y_0(x)y_1(x)$$

$$y_2(1) = 0$$  \hfill (76)

We observe that the perturbation hierarchy has the special structure that we have noted earlier. All equations in the hierarchy are determined by the linear
operator $\mathcal{L} = \frac{d}{dx} + 1$. The first boundary value problem in the hierarchy has already been solved. The second equation in the hierarchy is

$$y_1'(x) + y_1(x) = -y_0^2(x) \quad (77)$$

Finding a special solution to this equation is simple:

$$y_1'(x) + y_1(x) = -y_0^2(x)$$

$\downarrow$

$$(y_1(x)e^x)' = -y_0^2(x)e^x$$

$\downarrow$

$$y_1(x) = -e^{-x} \int_0^x dx' e^{x'} y_0^2(x') \quad (78)$$

inserting the formula for $y_0(x)$ into (78), integrating and adding a general solution of the homogenous equation, we get the general solution to (77) in the form

$$y_1(x) = -x^2 + 4x - 5 + (A_1 + 5)e^{-x} + (2x - x^2)e^{1-x} - e^{2-x} + e^{2-2x} \quad (79)$$

Using the boundary condition for $y_1(x)$ we find that $A_1 = e^2 - 5$ and we can conclude that the perturbation expansion to first order in $\varepsilon$ is

$$y(x; \varepsilon) = x - 1 + e^{1-x} + \varepsilon (-x^2 + 4x - 5 + (2x - x^2)e^{1-x} + e^{2-2x}) + ... \quad (80)$$

The general solution to the third equation in the perturbation hierarchy is in a similar way found to be

$$y_2(x) = A_2 e^{-x} - 2e^{-x} \int_0^x dx' e^{x'} y_0(x') y_1(x') \quad (81)$$

As you can see the algebraic complexity increase rapidly. The function $y_2(x)$ will have a large number of terms, even for this very simple example.

Recall that we are only ensured that the correction $\varepsilon y_1(t)$ is small with respect to the unperturbed solution $y_0(t)$ in the limit when $\varepsilon$ approaches zero. The perturbation method does not say anything about the accuracy for any finite value of $\varepsilon$. The hope is of course that the perturbation expansion also gives a good approximation for some range of $\varepsilon > 0$.

For this problem we do not have an exact solution that can be used to investigate the accuracy of the perturbation expansion for finite values of $\varepsilon$. This is the normal situation when we apply perturbation methods. The only way to get at the accuracy of the perturbation expansion is to compare it to an approximate solution found by some other, independent, approximation scheme. Often this involve numerical methods, but it could also be another perturbation method.

In figure (2) we compare our perturbation expansion to a high precision numerical solution in the domain $0 < x < 1$. We observe that even for $\varepsilon$ as large as 0.05 our perturbation expansion give a very accurate representation of the solution over the whole domain.

However, as the next example show, things are not always this simple.
4.2 Example 2: A weakly damped linear oscillator.

Consider the following initial value problem

\[ y''(t) + \epsilon y'(t) + y(t) = 0, \quad t > 0 \quad (82) \]
\[ y(0) = 1 \]
\[ y'(0) = 0 \]

This is our perturbed problem \( \mathcal{P}(\epsilon) \). The unperturbed problem, \( \mathcal{P}(0) \), is

\[ y''(t) + y(t) = 0 \quad (83) \]
\[ y(0) = 1 \]
\[ y'(0) = 0 \]

The general solution to the unperturbed equation is evidently

\[ y_0(t) = A_0 e^{it} + A_0^* e^{-it} \quad (84) \]

The initial condition is satisfied if

\[ A_0 + A_0^* = 1 \quad (85) \]
\[ iA_0 - iA_0^* = 0 \]

which has the unique solution \( A_0 = \frac{1}{2} \). Thus the unique solution to the unperturbed problem is

\[ y_0(t) = \frac{1}{2} e^{it} + (*) \quad (86) \]
where $z + (*)$ means $z + z^\ast$. This is a very common notation.

We want to find a perturbation expansion for the solution of the perturbed problem, starting with the solution $y_0$ of the unperturbed problem. The simplest approach is to use an expansion of the form

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) \ldots$$  \hspace{1cm} (87)

We now as usual insert (87) into the perturbed equation (82) and expand

$$(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots)'' + \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots)' + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots = 0$$

$$\Downarrow$$

$$y''_0 + y_0 + \varepsilon(y''_1 + y_1 + y''_0) + \varepsilon^2(y''_2 + y_2 + y'_1) + \ldots = 0$$  \hspace{1cm} (88)

We must in a similar way expand the initial conditions

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \ldots = 1$$  \hspace{1cm} (89)

$$y'_0(0) + \varepsilon y'_1(0) + \varepsilon^2 y'_2(t) + \ldots = 0$$

From equations (88) and (89) we get the following perturbation hierarchy

$$y''_0 + y_0 = 0, \quad t > 0$$ \hspace{0.2cm} (90)

$$y'_0(0) = 1$$

$$y_0(0) = 0$$

$$y''_1 + y_1 = -y'_0, \quad t > 0$$

$$y'_1(0) = 0$$

$$y_1(0) = 0$$

$$y''_2 + y_2 = -y'_1, \quad t > 0$$

$$y'_2(0) = 0$$

$$y_2(0) = 0$$

We note that the perturbation hierarchy has the special form discussed earlier. Here the linear operator determining the hierarchy is $L = \frac{d^2}{dt^2} + 1$.

The first initial value problem in the hierarchy has already been solved. The solution is (86). Inserting $y_0(t)$ into the second equation in the hierarchy we get

$$y''_1 + y_1 = -\frac{i}{2}e^{it} + (*)$$  \hspace{1cm} (91)

Looking for particular solutions of the form

$$y''_1(t) = Ce^{it} + (*)$$

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will not work here because the right-hand side of (91) is a solution to the homogenous equation. In fact (91) is a harmonic oscillator driven on resonance. For such cases we must rather look for a special solution of the form

$$y_1(t) = Cte^{it} + (\ast)$$

By inserting (92) into (91) we find $C = -\frac{1}{4}$. The general solution to equation (91) is then

$$y_1(t) = A_1e^{it} - \frac{1}{4}te^{it} + (\ast)$$

Applying the initial condition for $y_1(t)$ we easily find that $A_1 = -\frac{i}{4}$. Thus the perturbation expansion to first order in $\varepsilon$ is

$$y(t) = \frac{1}{2}e^{it} + \varepsilon \frac{1}{4}(i-t)e^{it} + (\ast)$$

Let $y_E(t)$ be a high precision numerical solution to the perturbed problem (82). For $\varepsilon = 0.01$ we get for increasing time

| $t$  | $y_E$  | $y$  |
|-----|--------|------|
| 4   | -0.6444| -0.6367 |
| 40  | -0.5426| -0.5372 |
| 400 | -0.0722| 0.5295  |

The solution starts out by being quite accurate, but as $t$ increases the perturbation expansion eventually loose any relation to the exact solution. The true extent of the disaster is seen in figure (3).

So what is going on, why is the perturbation expansion such a bad approximation in this example?

Observe that $y_1$ contain a term that is proportional to $t$. Thus as $t$ grows the size of $y_1$ also grows and when

$$t \sim \frac{1}{\varepsilon}$$

the second term in the perturbation expansion become as large as the first term. The ordering of the expansion breaks down and the first correction, $\varepsilon y_1$, is of the same size as the solution to the unperturbed problem, $y_0$.

The reason why the growing term, $y_1$, is a problem here, but was not a problem in the previous example is that here the domain for the independent variable is unbounded.

Let us at this point introduce some standard terminology. The last two examples involved perturbation expansions where the coefficients depended on a parameter. In general such expansions takes the form

$$f(\varepsilon; x) \sim \sum_{n=1}^{\infty} a_n(x)\delta_n(\varepsilon), \quad \varepsilon \to 0$$

(96)
Figure 3: Comparing the direct perturbation expansion (red) and a high precision numerical solution (green)
where the parameter, \( x \), ranges over some domain \( V \subset \mathbb{R}^m \) for some \( m \). In example one, \( V \) is the interval \([0, 1]\) whereas in example two, \( V \) is the unbounded interval \((0, \infty)\).

With the introduction of a parameter dependence of the coefficients, a breakdown of order in the expansion for some region(s) in \( V \) becomes a possibility. We saw how this came about in example 2.

And let me be clear about this: Breakdown of order in parameter dependent perturbation expansions is not some weird, rarely occurring, event. On the contrary it is very common.

Thus methods have to be invented to handle this phenomenon, which is called nonuniformity of asymptotic expansions. The multiple scale method is designed to do exactly this.

5 The multiple scale method for weakly nonlinear ordinary differential equations.

In the previous section we saw that trying to represent the solution to the problem

\[
y''(t) + \varepsilon y'(t) + y(t) = 0, \quad t > 0 \tag{97}
\]

\[
y(0) = 1
\]

\[
y'(0) = 0
\]

using a regular perturbation expansion

\[
y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)...
\]

leads to a nonuniform expansion where ordering of the terms broke down for \( t \sim \frac{1}{\varepsilon} \). In order to understand how to fix this, let us have a look at the exact solution to (97). The exact solution can be found using characteristic polynomials. We get

\[
y(t) = Ce^{-\frac{1}{2}\varepsilon t}e^{i\sqrt{1 - \frac{1}{4}\varepsilon^2}t} + (*)
\]

where

\[
C = \frac{-\lambda^*}{\lambda - \lambda^*}, \quad \lambda = -\frac{1}{2}\varepsilon + i\sqrt{1 - \frac{1}{4}\varepsilon^2}
\]

If we expand the square root in the exponent with respect to \( \varepsilon \), we get

\[
y(t) \approx Ce^{-\frac{1}{2}\varepsilon t}e^{it}e^{-\frac{1}{4}\varepsilon^2 t} + (*)
\]

Observe that if \( f(\xi) \) is a function whose derivative is of order one, then the function

\[
g_n(t) = f(\varepsilon^n t)
\]

satisfy

\[
\triangle g_n(t) = g_n(t + T) - g_n(t) \approx \varepsilon^n f'(\varepsilon^n t)T = O(1) \iff T \sim \varepsilon^{-n}
\]
We express this by saying that the function $g_n(t)$ vary on the time scale $t_n = \varepsilon^{-n}t$. If we now look at equation (101) we see that the approximate solution vary on three separate time scales $t_0 = \varepsilon^0t, t_1 = \varepsilon^{-1}t$ and $t_2 = \varepsilon^{-2}t$. If we include more terms in the Taylor expansion for the square root in (99) the resulting solution will depend on even more time scales.

Inspired by this example we postulate the existence of a function

$$h = h(t_0, t_1, t_2, \ldots)$$ (104)

such that

$$y(t) = h(t_0, t_1, t_2, \ldots)|_{t_j = \varepsilon^{j}t}$$ (105)

is a solution to problem (97). Using the chain rule we evidently have

$$\frac{dy}{dt}(t) = \{(\partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2} + \ldots)h\}|_{t_j = \varepsilon^{j}t}$$

which we formally write as

$$\frac{d}{dt} = \partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2} + \ldots$$ (106)

The function $h$ is represented using a perturbation expansion of the form

$$h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots$$ (107)

The multiple scale method now proceed by substituting (106) and (107) into the differential equation

$$y''(t) + \varepsilon y'(t) + y(t) = 0$$ (108)

and expanding everything in sight.

\[
\begin{align*}
(\partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2} + \ldots)(\partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2} + \ldots) &
(\partial_{t_0} + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + \varepsilon(\partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2} + \ldots) \\
(\partial_{t_0} + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots & = 0 \\
\downarrow \\
(\partial_{t_0} + \varepsilon(\partial_{t_0 t_1} + \partial_{t_1 t_0}) + \varepsilon^2(\partial_{t_0 t_2} + \partial_{t_1 t_1} + \partial_{t_2 t_0}) + \ldots) &
(\partial_{t_0} + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + \varepsilon(\partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2} + \ldots) \\
(\partial_{t_0} + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots = 0 \\
\downarrow \\
\partial_{t_0}t_0h_0 + h_0 + \varepsilon(\partial_{t_0 t_0}h_1 + h_1 + \partial_{t_0 t_1}h_1 + \partial_{t_1 t_0}h_0 + \partial_{t_1}h_0) \\
+ \varepsilon^2(\partial_{t_0}t_1h_2 + h_2 + \partial_{t_0}t_1h_1 + \partial_{t_1}t_0h_0 + \partial_{t_1}h_1 + \partial_{t_2}h_0 + \partial_{t_1}h_0 + \partial_{t_1}h_1) + \ldots = 0
\end{align*}
\] (109)
which gives us the following perturbation hierarchy to second order in $\varepsilon$

$$
\partial_{t_0 t_0} h_0 + h_0 = 0 \tag{110}
$$

$$
\partial_{t_0 t_0} h_1 + h_1 = -\partial_{t_0 t_1} h_0 - \partial_{t_1 t_0} h_0 - \partial_{t_0} h_0
$$

$$
\partial_{t_0 t_0} h_2 + h_2 = -\partial_{t_0 t_1} h_1 - \partial_{t_1 t_0} h_1 - \partial_{t_0 t_2} h_0 - \partial_{t_1 t_1} h_0 - \partial_{t_2 t_0} h_0 - \partial_{t_1} h_0 - \partial_{t_0} h_1
$$

We observe in passing that the perturbation hierarchy has the special form we have seen several times before. Here the common differential operator is $L = \partial_{t_0 t_0} + 1$.

At this point a remark is in order. It is fair to say that there is not a full agreement among the practitioners of the method of multiple scales about how to perform these calculations. The question really hinges on whether to take the multiple variable function $h(t_0, t_1, \ldots)$ seriously or not. If you do, you will be lead to a certain way of doing these calculation. This is the point of view used in most textbooks on this subject. We will not follow this path here. We will not take $h$ seriously as a multiple variable function and never forget that what we actually want is not $h$ but rather $y$, which is defined in terms of $h$ through equation (105). This point of view will lead us to do multiple scale calculations in a different way from what you see in most textbooks. This way is very efficient and will make it possible to go order $\varepsilon^2$ and beyond without being overwhelmed by the amount of algebra that needs to be done. What I mean when I say that we will not take $h$ seriously as a multiple variable function will become clear as we proceed.

One immediate consequence is already evident from the way I write the perturbation hierarchy. Observe that I keep

$$
\partial_{t_0 t_0} h_k, \text{ and } \partial_{t_1 t_1} h_k \tag{111}
$$

as separate terms, I don’t use the equality of cross derivatives to simplify my expressions. This is the first rule we must follow when we do multiple scale calculations in the way I am teaching you in these lecture notes. If we took $h$ seriously as a multiple variable function we would put cross derivatives equal. The second consequence of our choise is to disregard the initial values for the time being. We will fit the initial values at the very end of our calculations rather than do it at each order in $\varepsilon$ like in example 1 and example 2.

Let us now proceed to solve the equations in the perturbation hierarchy. At order $\varepsilon^0$ we have the equation

$$
\partial_{t_0 t_0} h_0 + h_0 = 0 \tag{112}
$$

When we are applying multiple scales to ordinary differential equations we always use the general solution to the order $\varepsilon^0$ equation. For partial differential
equations this will not be so, as we will see later. The general solution to (112) is evidently
\[ h(t_0, t_1, ..) = A_0(t_1, t_2, ..)e^{it_0} + (*) \] (113)
Observe that the equation only determines how \( h_0 \) depends on the fastest time scale \( t_0 \), the dependence on the other time scales \( t_1, t_2, .. \) is arbitrary at this point and is reflected in the fact that integration ”constant” \( A_0 \) is actually a function depending on \( t_1, t_2, .. \).

We have now solved the order \( \varepsilon \) equation. Inserting the expression for \( h_0 \) into the order \( \varepsilon \) equation, we get after some simple algebra
\[ \partial_{t_0} h_1 + h_1 = -2i(\partial_{t_1} A_0 + \frac{1}{2} A_0)e^{it_0} + (*) \] (114)
We now need a particular solution to this equation. Observe that since \( A_0 \) only depends on the slow time scales \( t_1, t_2, .. \) equation (114) is in fact a harmonic oscillator driven on resonance. It is simple to verify that it has a particular solution of the form
\[ h_1(t_0, t_1, ..) = -t_0(\partial_{t_1} A_0 + \frac{1}{2} A_0)e^{it_0} \] (115)
But this term is growing and will lead to breakdown of ordering for the perturbation expansion (107) when \( t_0 \sim \varepsilon^{-1} \). This breakdown was exactly what we tried to avoid using a multiple scales approach!

But everything is not lost, we now have freedom to remove the growing term by postulating that
\[ \partial_{t_1} A_0 = -\frac{1}{2} A_0 \] (116)
With this choise, the order \( \varepsilon \) equation simplifies into
\[ \partial_{t_0} h_1 + h_1 = 0 \] (117)
Terms in equations leading to linear growt like in (115) are traditionally called secular terms. The name are derived from the Latin word soeculum that means century and are used here because this kind of nonuniformity was first observed on century time scales in planetary orbit calculations.

At this point we introduce the second rule for doing multiple scale calculations in the particular way that I advocate in these lecture notes. The rule is to disregard the general solution of the homogenous equation for all equations in the perturbation hierarchy except the first. We therefore choose \( h_1 = 0 \) and proceed to the order \( \varepsilon^2 \) equation using this choise. The equation for \( h_2 \) then simplifies into
\[ \partial_{t_0} h_2 + h_2 = -2i(\partial_{t_2} A_0 - \frac{i}{2} \partial_{t_1} A_0 - \frac{i}{2} \partial_{t_1} A_0)e^{it_0} + (*) \] (118)
We have a new secular term and in order to remove it we must postulate that
\[ \partial_{t_2} A_0 = \frac{i}{2} \partial_{t_1} A_0 + \frac{i}{2} \partial_{t_1} A_0 \] (119)

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Using this choice, our order $\varepsilon^2$ equation simplifies into
\[
\partial_{t_0} h_2 + h_2 = 0 \tag{120}
\]
For this equation we use, according to the rules of the game, the special solution $h_2 = 0$.

What we have found so far is then
\[
h(t_0, t_1, t_2, ..) = A_0(t_1, t_2, ..)e^{it_0} + (\star) + O(\varepsilon^3) \tag{121}
\]
where
\[
\begin{align*}
\partial_{t_1} A_0 &= -\frac{1}{2} A_0 \tag{122} \\
\partial_{t_2} A_0 &= \frac{i}{2} \partial_{t_1} t_1 A_0 + \frac{i}{2} \partial_{t_1} A_0 \tag{123}
\end{align*}
\]
At this point you might ask if we actually have done something useful. Instead of one ODE we have ended up with two coupled partial differential equations, and clearly if we want to go to higher order we will get even more partial differential equations.

Observe that if we use (122) we can simplify equation (123) by removing the derivatives on the right hand side. Doing this we get the system
\[
\begin{align*}
\partial_{t_1} A_0 &= -\frac{1}{2} A_0 \tag{124} \\
\partial_{t_2} A_0 &= -\frac{i}{8} A_0 \tag{125}
\end{align*}
\]
The first thing that should come to mind when we see a system like (124) and (125) is that the count is wrong. There is one unknown function, $A_0$, and two equations. The system is overdetermined and will get more so if we extend our calculations to higher order in $\varepsilon$. Under normal circumstances, overdetermined systems of equations have no solutions, which for our setting means that under normal circumstances the function $h(t_0, t_1, t_2, ..)$ does not exist! This is what I meant when I said that we will not take the functions $h$ seriously as a multiple variable function. For systems of first order partial differential equations like (124), (125) there is a simple test we can use to decide if a solution actually exist. This is the cross derivative test you know from elementary calculus. Taking $\partial_{t_2}$ of equation (124) and $\partial_{t_1}$ of equation (125) we get
\[
\begin{align*}
\partial_{t_2 t_1} A_0 &= \partial_{t_2} \partial_{t_1} A_0 = -\frac{1}{2} \partial_{t_2} A_0 = -\frac{i}{16} A_0 \tag{126} \\
\partial_{t_1 t_2} A_0 &= \partial_{t_1} \partial_{t_2} A_0 = -\frac{i}{8} \partial_{t_1} A_0 = -\frac{i}{16} A_0
\end{align*}
\]
According to the cross derivative test the overdetermined system (124), (125) is solvable. Thus in this case the function $h$ exists, at least as a two variable function. To make sure that it exists as a function of three variables we must
derive and solve the perturbation hierarchy to order $\varepsilon^3$, and then perform the cross derivative test. For the current example we will never get into trouble, the many variable function $h$ will exist as a function of however many variables we want. But I want you to reflect on how special this must be. We will at order $\varepsilon^n$ have a system of $n$ partial differential equations for only one unknown function! In general we will not be so lucky and this is the reason why we can not take $h$ seriously as a many variable function.

So should we be disturbed by the nonexistence of solutions in the general case? Actually no, and the reason is that we do not care about $h(t_0, t_1, \ldots)$. What we care about is $y(t)$.

Inspired by this let us define an amplitude, $A(t)$, by

\[ A(t) = A_0(t_1, t_2, \ldots)|_{t_j = \varepsilon t} \quad (127) \]

Using this and equations (105), (121) our perturbation expansion for $y(t)$ is

\[ y(t) = A(t)e^{it} + (*) + O(\varepsilon^3) \quad (128) \]

For the amplitude $A(t)$ we have, using equations (106), (124), (125) and (127)

\[
\frac{dA}{dt}(t) = \left\{ (\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \ldots)A_0(t_1, t_2, \ldots) \right\}|_{t_j = \varepsilon t} \\
\downarrow \\
\frac{dA}{dt}(t) = \left\{ -\frac{1}{2} A_0(t_1, t_2, \ldots) - \varepsilon^2 \frac{i}{8} A_0(t_1, t_2, \ldots) \right\}|_{t_j = \varepsilon t} \\
\downarrow \\
\frac{dA}{dt} = -\frac{1}{2} A - \varepsilon^2 \frac{i}{8} A \quad (129) \]

This equation is our first example of an amplitude equation. The amplitude equation determines through equation (127) the perturbation expansion for our solution to the original equation (97). The amplitude equation is of course easy to solve and we get

\[ y(t) = Ce^{-\frac{1}{2} \varepsilon t} e^{it} e^{-\frac{i}{8} \varepsilon^2 t} + (*) + O(\varepsilon^3) \quad (130) \]

The constant $C$ can be fitted to the initial conditions. What we get is equal to the exact solution up to second order in $\varepsilon$ as we see by comparing with (101).

Let us next apply the multiple scale method to some weakly nonlinear ordinary differential equations. For these cases no exact solution is known so the multiple scale method will actually be useful!
5.1 Example 1

Consider the initial value problem

\[ \frac{d^2 y}{dt^2} + y = \varepsilon y^3 \quad (131) \]
\[ y(0) = 1 \]
\[ \frac{dy}{dt}(0) = 0 \]

If we try to solve this problem using a regular perturbation expansion we will get secular terms that will lead to breakdown of ordering on a time scale \( t \sim \varepsilon^{-1} \). Let us therefore apply the multiple scale approach. We introduce a function \( h \)

\[ y(t) = h(t_0, t_1, t_2, \ldots) \quad |t_j = \varepsilon^{j} \quad (132) \]

and expansions

\[ \frac{d}{dt} = \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ... \quad (133) \]
\[ h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ... \quad (134) \]

Inserting these expansions into (131) and expanding we get

\[ (\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...) \]
\[ (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ... + h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ... \]
\[ = \varepsilon(h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ...)^3 \]
\[ \downarrow \]
\[ (\partial_{t_0} + \varepsilon (\partial_{t_0} + \partial_{t_1} + \varepsilon^2(\partial_{t_0} + \partial_{t_1} + \partial_{t_2} + ...)) \]
\[ (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ... + h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ... \]
\[ = \varepsilon h_0^3 + 3\varepsilon^2 h_0^2 h_1 + ... \]
\[ \downarrow \]
\[ \partial_{t_0} h_0 + h_0 + \varepsilon(\partial_{t_0} h_1 + h_0 + \partial_{t_0} h_0 + \partial_{t_2} h_0) \]
\[ + \varepsilon^2(\partial_{t_0} h_2 + h_2 + \partial_{t_0} h_1 + \partial_{t_2} h_1 + \partial_{t_2} h_0 + \partial_{t_4} h_0 + \partial_{t_2} h_0 \]
\[ + \partial_{t_2} h_0) + ... = \varepsilon h_0^3 + 3\varepsilon^2 h_0^2 h_1 + ... \quad (135) \]

Which gives us the following perturbation hierarchy to second order in \( \varepsilon \)

\[ \partial_{t_0} h_0 + h_0 = 0 \quad (136) \]

\[ \partial_{t_0} h_1 + h_1 = h_0^3 - \partial_{t_0} h_0 - \partial_{t_2} h_0 \]

\[ \partial_{t_0} h_2 + h_2 = 3h_0^2 h_1 - \partial_{t_0} h_1 - \partial_{t_2} h_1 - \partial_{t_0} h_0 - \partial_{t_2} h_0 \]

\[ - \partial_{t_1} h_0 - \partial_{t_3} h_0 \]
The general solution to the first equation in the perturbation hierarchy is

\[ h_0 = A_0(t_1, t_2, ...) e^{i \alpha_0} + (*) \]  

(137)

Inserting this into the right hand side of the second equation in the hierarchy and expanding we get

\[ \partial_{t_0} h_1 + h_1 = (3|A_0|^2 A_0 - 2i\partial_{t_1} A_0) e^{i \alpha} + A_0^3 e^{3i \alpha} + (*) \]  

(138)

In order to remove secular terms we must postulate that

\[ \partial_{t_1} A_0 = -\frac{3i}{2} |A_0|^2 A_0 \]  

(139)

This choice simplify the equation for \( h_1 \) into

\[ \partial_{t_0} h_1 + h_1 = A_0^3 e^{3i \alpha} + (*) \]  

(140)

According to the rules of the game we now need a particular solution to this equation. It is easy to verify that

\[ h_1 = -\frac{1}{8} A_0^3 e^{3i \alpha} + (*) \]  

(141)

is such a particular solution.

We now insert \( h_0 \) and \( h_1 \) into the right hand side of the third equation in the perturbation hierarchy and find

\[ \partial_{t_0} h_2 + h_2 = (-\frac{3}{8} |A_0|^4 A_0 - 2i\partial_{t_2} A_0 - \partial_{t_1} A_0) e^{i \alpha} + (*) + NST \]  

(142)

where \( NST \) is an acronym for "nonsecular terms". Since we are not here planning to go beyond second order in \( \epsilon \), we will at this order only need the secular terms and group the rest into \( NST \). In order to remove the secular terms we must postulate that

\[ \partial_{t_2} A_0 = \frac{3i}{16} |A_0|^4 A_0 + \frac{i}{2} \partial_{t_1} A_0 \]  

(143)

As before it make sense to simplify (143) using equation (139). This leads to the following overdetermined system of equations for \( A_0 \)

\[ \partial_{t_1} A_0 = -\frac{3i}{2} |A_0|^2 A_0 \]  

(144)

\[ \partial_{t_2} A_0 = -\frac{15i}{16} |A_0|^4 A_0 \]

Let us check solvability of this system using the cross derivative test

\[ \partial_{t_2 t_1} A_0 = -\frac{3i}{2} \partial_{t_2} (A_0^2 A_0^*) \]

\[ = -\frac{3i}{2} (2A_0 \partial_{t_2} A_0 A_0^* + A_0^2 \partial_{t_2} A_0^*) \]

\[ = -\frac{3i}{2} \left( 2A_0 \left( -\frac{15i}{16} |A_0|^4 A_0^* \right) A_0^* + A_0^2 \left( \frac{15i}{16} |A_0|^4 A_0^* \right) \right) \]

\[ = -\frac{45}{32} |A_0|^6 A_0 \]
\[ \partial_{t_1 t_2} A_0 = -\frac{15i}{16} \partial_{t_1} (A_0^3 A_0^* A_0^{-2}) \]  
\[ = -\frac{15i}{16} (3A_0^2 \partial_{t_1} A_0 A_0^2 + 2A_0^3 \partial_{t_1} A_0^*) \]  
\[ = -\frac{15i}{16} \left( 3A_0^2 \left( -\frac{3i}{2} |A_0|^2 A_0 \right) A_0^{-2} + 2A_0^3 A_0^* \left( \frac{3i}{2} |A_0|^2 A_0^* \right) \right) \]  
\[ = -\frac{45}{32} |A_0|^6 A_0 \]  
(145)

The system is compatible and thus the function \( h_0 \) exists as a function of two variables. Of course, whether or not \( h_0 \) exists is only of academic interest for us since our only aim is to find the solution of the original equation \( y(t) \). Defining an amplitude, \( A(t) \) by

\[ A(t) = A_0(t_1, t_2, ...) |_{t_j = \varepsilon j t} \]  
(149)

we find that the solution is

\[ y(t) = A(t)e^{it} - \varepsilon \frac{1}{8} A^3 e^{3it} + (\ast) + O(\varepsilon^2) \]  
(150)

where the amplitude satisfy the equation

\[ \frac{dA}{dt}(t) = \{ (\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...) A_0(t_1, t_2, ...) \} |_{t_j = \varepsilon j t} \]  
\[ \downarrow \]  
\[ \frac{dA}{dt}(t) = \{ -\varepsilon \frac{3i}{2} |A_0|^2 A_0(t_1, t_2, ...) - \varepsilon^2 \frac{15i}{16} |A_0|^4 A_0(t_1, t_2, ...) \} |_{t_j = \varepsilon j t} \]  
\[ \downarrow \]  
\[ \frac{dA}{dt} = -\varepsilon \frac{3i}{2} |A|^2 A - \varepsilon^2 \frac{15i}{16} |A|^4 A \]  
(151)

Observe that this equation has a unique solution for a given set of initial conditions regardless of whether the overdetermined system \((144)\) has a solution or not. Thus doing the cross derivative test was only motivated by intellectual curiosity, we did not have to do it.

In summary, \((150)\) and \((151)\), determines a perturbation expansion for \( y(t) \) that is uniform for \( t \lesssim \varepsilon^{-2} \).

At this point it is reasonable to ask in which sense we have made progress. We started with one second order nonlinear ODE for a real function \( y(t) \) and have ended up with one first order nonlinear ODE for a complex function \( A(t) \). This question actually has two different answers.

The first one is that it is possible to get an analytical solution for \((151)\) whereas this is not possible for the original equation \((131)\). This possibility might however easily get lost as we proceed to higher order in \( \varepsilon \), since this will add more terms to the amplitude equation. But even if we can not solve the
amplitude equation exactly, it is a fact that amplitude equations with the same mathematical structure will arise when we apply the multiple scale method to many different equations. Thus any insight into an amplitude equation derived by some mathematical analysis has relevance for many different situations. This is clearly very useful.

There is however a second, more robust, answer to the question of whether we have made progress or not. From a numerical point of view there is an important difference between (131) and (151). If we solve (131) numerically the time step is constrained by the oscillation period of the linearized system
\[ \frac{d^2 y}{dt^2} + y = 0 \] (152)

which is of order \( T \sim 1 \), whereas if we solve (151) numerically the timestep is constrained by the period \( T \sim \varepsilon^{-1} \). Therefore if we want to propagate out to a time \( t \sim \varepsilon^{-2} \) we must take on the order of \( \varepsilon^{-2} \) time steps if we use (152) whereas we only need on the order of \( \varepsilon^{-1} \) time steps using (151). If \( \varepsilon \) is very small the difference in the number of time steps can be highly significant. From this point of view, the multiple scale method is a reformulation that is the key element in a very fast numerical method for solving weakly nonlinear ordinary and partial differential equation.

Let us next turn to the problem of fitting the initial conditions. Using equation (150) and the initial conditions from (131) we get, truncating at order \( \varepsilon \), the following equations
\[ A(0) - \varepsilon \frac{1}{8} A^3(0) + (*) = 1 \] (153)
\[ iA(0) - \varepsilon \left( \frac{3i}{2} |A(0)|^2 A(0) + \frac{3i}{8} A^3(0) \right) + (*) = 0 \]
The solution for \( \varepsilon = 0 \) is
\[ A(0) = \frac{1}{2} \] (154)

For \( \varepsilon > 0 \) we solve the equation by Newton iteration starting with the solution for \( \varepsilon = 0 \). This will give us the initial condition for the amplitude equation correct to this order in \( \varepsilon \).

In figure (4) we compare the multiple scale solution, keeping only the first term in the amplitude equation, to a high precision numerical solution for \( \varepsilon = 0.1 \) for \( t \sim \varepsilon^{-2} \). We see that the perturbation solution is very accurate for this range of \( t \). In figure (5) we do the same comparison as in figure (4) but now for \( t \sim \varepsilon^{-3} \). As expected the multiple scale solution and the numerical solution starts to deviate for this range of \( t \). In figure (6) we make the same comparison as in figure (5), but now include both terms in the amplitude equation. We see that high accuracy is restored for the multiple scale solution for \( t \sim \varepsilon^{-3} \).
Figure 4: Comparing the multiple scale solution, while keeping only the first term in the amplitude equation (red), to a numerical solution (green) for $t \sim \varepsilon^{-2}$.

Figure 5: Comparing the multiple scale solution, while keeping only the first term in the amplitude equation (red), to a numerical solution (green) for $t \sim \varepsilon^{-3}$. 

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5.2 Example 2

Let us consider the weakly nonlinear equation

$$\frac{d^2 y}{dt^2} + \frac{dy}{dt} + \epsilon y^2 = 0, \quad t > 0 \quad (155)$$

We want to apply the multiple scale method and introduce a function $h(t_0, t_1, t_2, \ldots)$ such that

$$y(t) = h(t_0, t_1, t_2, \ldots)|_{t_j = \epsilon^j t} \quad (156)$$

is a solution to equation (155). As usual we have the formal expansions

$$\frac{d}{dt} = \partial_{t_0} + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2} + \ldots \quad (157)$$

$$h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \ldots \quad (158)$$
Inserting (156), (157) and (158) into equation (155) and expanding, we get

\[(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \ldots)(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \ldots)
\]

\[(h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + (\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \ldots)
\]

\[= -\varepsilon(h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots)^2
\]

\[
(\partial_{t_{0t_0}} + \varepsilon(\partial_{t_{0t_1}} + \partial_{t_{1t_0}}) + \varepsilon^2(\partial_{t_{0t_2}} + \partial_{t_{1t_1}} + \partial_{t_{2t_0}}) + \ldots)
\]

\[(h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + (\partial_{t_{0t_0}} + \varepsilon \partial_{t_{0t_1}} + \varepsilon^2 \partial_{t_{0t_2}} + \ldots)
\]

\[= -\varepsilon h_0^2 - \varepsilon^2 2h_0 h_1 + \ldots
\]

which gives us the perturbation hierarchy

\[
\partial_{t_{0t_0}} h_0 + \partial_{t_0} h_0 = 0
\]

\[
\partial_{t_{0t_0}} h_1 + \partial_{t_0} h_1 = -h_0^2 - \partial_{t_{0t_1}} h_0 - \partial_{t_{1t_0}} h_0 - \partial_{t_1} h_0
\]

\[
\partial_{t_{0t_0}} h_2 + \partial_{t_0} h_2 = -2h_0 h_1 - \partial_{t_{0t_1}} h_1 - \partial_{t_{1t_0}} h_1 - \partial_{t_{0t_2}} h_0 - \partial_{t_{1t_1}} h_0 - \partial_{t_1} h_1 - \partial_{t_2} h_0
\]

The general solution to the first equation in the perturbation hierarchy is

\[h_0(t_0, t_1, t_2, \ldots) = A_0(t_1, t_2, \ldots) + B_0(t_1, t_2, \ldots) e^{-t_0}
\]

where \(A_0\) and \(B_0\) are real functions of their arguments. Inserting \(h_0\) into the second equation in the hierarchy we get

\[
\partial_{t_{0t_0}} h_1 + \partial_{t_0} h_1 = -\partial_{t_1} A_0 - A_0^2 + (\partial_{t_1} B_0 - 2A_0 B_0) e^{-t_0} - B_0^2 e^{-2t_0}
\]

In order to remove secular terms we must postulate that

\[
\partial_{t_1} A_0 = -A_0^2
\]

\[
\partial_{t_1} B_0 = 2A_0 B_0
\]

equation (162) simplifies into

\[
\partial_{t_{0t_0}} h_1 + \partial_{t_0} h_1 = -B_0^2 e^{-2t_0}
\]

which has a special solution

\[h_1(t_0, t_1, \ldots) = -\frac{1}{2} B_0^2 e^{-2t_0}
\]
Inserting (161) and (165) into the third equation in the perturbation hierarchy we get
\[ \partial_{t_0} h_2 + \partial_{t_0} h_2 = -\partial_{t_1} A_0 - \partial_{t_1} A_0 + (\partial_{t_0} B_0 - \partial_{t_1} B_0)e^{-t_0} + NST \] (166)

In order to remove secular terms we must postulate that
\[ \partial_{t_2} A_0 = -\partial_{t_1} A_0 \] (167)
\[ \partial_{t_2} B_0 = \partial_{t_1} B_0 \]

We can as usual use (163) to simplify (167). We are thus lead to the following overdetermined system for \( A_0 \) and \( B_0 \).
\[ \partial_{t_1} A_0 = -A_0^2 \] (168)
\[ \partial_{t_1} B_0 = 2A_0 B_0 \]
\[ \partial_{t_2} A_0 = -2A_0^3 \]
\[ \partial_{t_2} B_0 = 2A_0^3 B_0 \]

In order to satisfy our academic curiosity, let us do the cross derivative test for solvability of (168).
\[ \partial_{t_1 t_2} A_0 = -2\partial_{t_1} A_0^2 = -6A_0^2 \partial_{t_1} A_0 = 6A_0^4 \] (169)
\[ \partial_{t_2 t_1} A_0 = -\partial_{t_2} A_0^2 = -2A_0 \partial_{t_2} A_0 = 4A_0^4 \] (170)
\[ \partial_{t_1 t_2} B_0 = 2\partial_{t_1} (A_0 B_0) = 4A_0 \partial_{t_1} A_0 B_0 + 2A_0^2 \partial_{t_1} B_0 = 0 \]
\[ \partial_{t_2 t_1} B_0 = 2\partial_{t_2} (A_0 B_0) = 2\partial_{t_2} A_0 B_0 + 2A_0 \partial_{t_2} B_0 = 0 \]

We see that the test fails, so the system (168) has no solutions. However the multiple scale method does not fail since we are not actually interested in the functions \( A_0 \) and \( B_0 \) that defines \( h_0 \), but is rather interested in the function \( y(t) \). Define two amplitudes \( A(t) \) and \( B(t) \) by
\[ A(t) = A_0(t_1, t_2, ...)|_{t_j = \varepsilon^j} \]
\[ B(t) = B_0(t_1, t_2, ...)|_{t_j = \varepsilon^j} \]

then the solution to (155) is
\[ y(t) = A(t) + B(t)e^{-t} - \varepsilon \frac{1}{2} B^2(t)e^{-2t} + O(\varepsilon^2) \] (172)

where the amplitudes \( A(t) \) and \( B(t) \) satisfy the equations
\[ \frac{dA}{dt}(t) = \{((\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)A_0(t_1, t_2, ...))|_{t_j = \varepsilon^j t} \}
\[ \downarrow \]
\[ \frac{dA}{dt}(t) = \{-\varepsilon A^2(t_1, t_2, ...) - 2\varepsilon^2 A(t_1, t_2, ...)^3\}|_{t_j = \varepsilon^j t} \]
\[ \downarrow \]
\[ \frac{dA}{dt} = -\varepsilon A^2 - 2\varepsilon^2 A^3 \] (173)
and
\[
\frac{dB}{dt}(t) = \{(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...) B_0(t_1, t_2, ...)\}_{t_j = \varepsilon/t}
\]
\[\downarrow\]
\[
\frac{dB}{dt}(t) = \{(2\varepsilon A_0(t_1, t_2, ...) B_0(t_1, t_2, ...) + 2\varepsilon^2 A_0^2(t_1, t_2, ...) B_0(t_1, t_2, ...))\}_{t_j = \varepsilon/t}
\]
\[\downarrow\]
\[
\frac{dB}{dt} = 2\varepsilon AB + 2\varepsilon^2 A^2 B
\]
(174)

Given the initial conditions for $A$ and $B$, equations (173) and (174) clearly has a unique solution and our multiple scale method will ensure that the perturbation expansion (172) will stay uniform for $t \lesssim \varepsilon^{-2}$. As for the previous example the initial conditions $A(0)$ and $B(0)$ are calculated from the initial conditions for (155) by a Newton iteration. Thus we see again that the existence or not of $h(t_0, ...)$ is irrelevant for constructing a uniform perturbation expansion.

The system (173) and (174) can be solved analytically in terms of implicit functions. However, as we have discussed before, analytical solvability is nice, but not robust. If we take the expansion to order $\varepsilon^3$, more terms are added to the amplitude equations and the property of analytic solvability can easily be lost. What is robust is that the presence of $\varepsilon$ in the amplitude equations makes (173) and (174) together with (172) into a fast numerical scheme for solving the ordinary differential equation (155). This property does not go away if we take the perturbation expansion to higher order in $\varepsilon$.

5.3 Exercises

For the following initial value problems, find asymptotic expansions that are uniform for $t \ll \varepsilon^{-2}$. You thus need to take the expansions to second order in $\varepsilon$. Compare your asymptotic solution to a high precision numerical solution of the exact problem. Do the comparison for several values of $\varepsilon$ and show that the asymptotic expansion and the numerical solution of the exact problem deviates when $t \sim \varepsilon^{-2}$.

Problem 1:

\[
\frac{d^2 y}{dt^2} + y = \varepsilon y^2
\]
\[y(0) = 1\]
\[\frac{dy}{dt}(0) = 0\]
Problem 2:

\[ \frac{d^2 y}{dt^2} + y = \varepsilon (1 - y^2) \frac{dy}{dt} \]
\[ y(0) = 1 \]
\[ \frac{dy}{dt}(0) = 0 \]

Problem 3:

\[ \frac{d^2 y}{dt^2} + y = \varepsilon (y^3 - 2 \frac{dy}{dt}) \]
\[ y(0) = 1 \]
\[ \frac{dy}{dt}(0) = 0 \]

Problem 4: Let the initial value problem

\[ \frac{d^2 y}{dt^2} + \frac{dy}{dt} + \varepsilon y^2 = 0, \quad t > 0 \]
\[ y(0) = 1 \]
\[ y'(0) = 1 \]

be given. Design a numerical solution to this problem based on the amplitude equations (173), (174) and (172). Compare this numerical solution to a high precision numerical solution of (175) for \( t \ll \varepsilon^{-2} \). Use several different values of \( \varepsilon \) and show that the multiple scale solution and the high precision solution deviate when \( t \sim \varepsilon^{-2} \).

6 The multiple scale method for weakly nonlinear dispersive wave equations.

It is now finally time to start applying the multiple scale method to partial differential equations. The partial differential equations that are of interest in optics are almost always hyperbolic, weakly dispersive and nonlinear. We will therefore focus all our attention on such equations. The simplest such equation is the following

6.1 Example 1

Let us consider the equation

\[ \partial_t u - \partial_{xx} u + u = \varepsilon u^2 \]

(176)

Inspired by our work on ordinary differential equations we introduce a function \( h(x_0, t_0, x_1, t_1, ...) \) such that

\[ u(x, t) = h(x_0, t_0, x_1, t_1, ...) |_{t_j = \varepsilon^j, x_j = \varepsilon^j x} \]

(177)
is a solution of (176). The derivatives turns into
\[ \partial_t = \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \ldots \]
\[ \partial_x = \partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2} + \ldots \]
and for \( h \) we use the expansion
\[ h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots \]
Inserting (177), (178) and (179) and expanding everything in sight, we get
\[ (\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \ldots) (\partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2} + \ldots) \]
\[ (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) - \]
\[ (\partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2} + \ldots) (\partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2} + \ldots) \]
\[ (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) \]
\[ = \varepsilon (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots)^2 \]
\[ \downarrow \]
\[ (\partial_{t_0 t_0} + \varepsilon (\partial_{t_0 t_1} + \partial_{t_1 t_0}) + \varepsilon^2 (\partial_{t_0 t_2} + \partial_{t_1 t_1} + \partial_{t_2 t_0}) + \ldots) \]
\[ (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) - \]
\[ (\partial_{x_0 x_0} + \varepsilon (\partial_{x_0 x_1} + \partial_{x_1 x_0}) + \varepsilon^2 (\partial_{x_0 x_2} + \partial_{x_1 x_1} + \partial_{x_2 x_0}) + \ldots) \]
\[ (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) + (h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots) \]
\[ = \varepsilon (h_0^2 + 2\varepsilon h_0 h_1 + \ldots) \]
\[ \downarrow \]
\[ \partial_{t_0 t_0} h_0 + \varepsilon (\partial_{t_0 t_1} h_1 + \partial_{t_1 t_0} h_0 + \partial_{t_0 t_0} h_0) + \]
\[ \varepsilon^2 (\partial_{t_0 t_0} h_2 + \partial_{t_0 t_1} h_1 + \partial_{t_1 t_0} h_0 + \partial_{t_0 t_2} h_0 + \partial_{t_1 t_1} h_0 + \partial_{t_2 t_0} h_0) - \]
\[ \partial_{x_0 x_0} h_0 - \varepsilon (\partial_{x_0 x_1} h_1 + \partial_{x_1 x_0} h_0 + \partial_{x_0 x_0} h_0) - \]
\[ \varepsilon^2 (\partial_{x_0 x_0} h_2 + \partial_{x_0 x_1} h_1 + \partial_{x_1 x_0} h_0 + \partial_{x_0 x_2} h_0 + \partial_{x_1 x_1} h_0 + \partial_{x_2 x_0} h_0) \]
\[ + h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots \]
\[ = \varepsilon h_0^2 + 2\varepsilon^2 h_0 h_1 + \ldots \]
which gives us the perturbation hierarchy
\[ \partial_{t_0 t_0} h_0 - \partial_{x_0 x_0} h_0 + h_0 = 0 \]
\[ \partial_{t_0 t_1} h_1 - \partial_{x_0 x_0} h_1 + h_1 = h_0^2 - \partial_{t_0 t_1} h_0 - \partial_{t_1 t_0} h_0 \]
\[ + \partial_{x_0 x_1} h_0 + \partial_{x_1 x_0} h_0 \]
\[ \partial_{t_0 t_2} h_2 - \partial_{x_0 x_0} h_2 + h_2 = 2h_0 h_1 - \partial_{t_0 t_1} h_1 - \partial_{h_1 t_0} h_1 \]
\[ - \partial_{t_1 t_0} h_0 - \partial_{t_2 t_0} h_0 + \partial_{x_0 x_1} h_1 + \partial_{x_1 x_0} h_1 \]
\[ + \partial_{x_0 x_2} h_0 + \partial_{x_1 x_1} h_0 + \partial_{x_2 x_0} h_0 \]
\[ = 38 \]
For ordinary differential equations we used the general solution to the order $\varepsilon^0$ equation. For partial differential equations we can not do this. We will rather use a finite sum of linear modes. The simplest possibility is a single linear mode which we use here

$$h_0(t_0, x_0, t_1, x_1, \ldots) = A_0(t_1, x_1, \ldots) e^{i(kx_0 - \omega t_0)} + (*)$$  \hspace{1cm} (184)

Since we are not using the general solution we will in not be able to satisfy arbitrary initial conditions. However, in optics this is perfectly alright since most of the time the relevant initial conditions are in fact finite sums of wave packets or even a single wave packet. Such initial conditions can be included in the multiple scale approach that we discuss in this section. For (184) to actually be a solution to (181) we must have

$$\omega = \omega(k) = \sqrt{1 + k^2}$$  \hspace{1cm} (185)

which we of course recognize as the dispersion relation for the linearized version of (176). With the choice of signs used here (184) will represent a right-moving disturbance.

Inserting (184) into (182) we get

$$\partial_{t_0} h_1 - \partial_{x_0, x_1} h_1 + h_1 = 2|A_0|^2$$  \hspace{1cm} (186)

$$+ A_0^2 e^{2i(kx_0 - \omega t_0)} + A_0^2 e^{-2i(kx_0 - \omega t_0)}$$  \hspace{1cm} (187)

$$+(2i\omega \partial_t A_0 + 2ik \partial_x A_0) e^{i(kx_0 - \omega t_0)}$$

$$- (2i\omega \partial_t A_0 + 2ik \partial_x A_0) e^{-i(kx_0 - \omega t_0)}$$  \hspace{1cm} (188)

In order to remove secular terms we must postulate that

$$2i\omega \partial_t A_0 + 2ik \partial_x A_0 = 0$$  \hspace{1cm} (189)

$$\uparrow$$

$$\partial_t A_0 = -\frac{k}{\omega} \partial_x A_0$$

Here we assume that the terms

$$e^{2i(kx_0 - \omega t_0)}, e^{-2i(kx_0 - \omega t_0)}$$

are not solutions to the homogenous equation

$$\partial_{t_0} h_1 - \partial_{x_0, x_1} h_1 + h_1 = 0$$

For this to be true we must have

$$\omega(2k) \neq 2\omega(k)$$  \hspace{1cm} (190)

and this is in fact true for all $k$. This is however not generally true for dispersive wave equations. Whether it is true or not will depend on the exact form of the
dispersion relation for the system of interest. In optics, equality in (190), is called *phase matching* and is of outmost importance.

The equation for $h$ now simplify into

\[
\frac{\partial h}{\partial t} + \frac{\partial x}{\partial t} h = 2 |A_0|^2 + A_0^2 e^{2i(kx_0 - \omega t_0)} + A_0^{*2} e^{-2i(kx_0 - \omega t_0)} \tag{191}
\]

According to the rules of the game we need a special solution to this equation. It is easy to verify that

\[
h_1 = 2 |A_0|^2 - \frac{1}{3} A_0^2 e^{2i(kx_0 - \omega t_0)} - \frac{1}{3} A_0^{*2} e^{-2i(kx_0 - \omega t_0)} \tag{192}
\]

is such a special solution. Inserting (184) and (192) into (183), we get

\[
\frac{\partial h}{\partial t} - \frac{\partial x}{\partial t} h + h = (2i \omega \frac{\partial h}{\partial t} A_0 + 2i k \frac{\partial x}{\partial t} A_0 - \frac{\partial x}{\partial t} A_0 + \frac{10}{3} |A_0|^2 A_0) e^{i(kx_0 - \omega t_0)} + NST + (*) \tag{193}
\]

In order to remove secular terms we must postulate that

\[
2i \omega \frac{\partial h}{\partial t} A_0 + 2i k \frac{\partial x}{\partial t} A_0 - \frac{\partial x}{\partial t} A_0 + \frac{10}{3} |A_0|^2 A_0 = 0 \tag{194}
\]

Equations (189) and (194) is, as usual, an overdetermined system. In general it is not an easy matter to verify that an overdetermined system of partial differential equations is solvable and the methods that do exist to adress such questions are mathematically quite sophisticated. For the particular case discussed here it is however easy to verify that the system is in fact solvable. But, as we have stressed several times in these lecture notes, we are not really concerned with the solvability of the system (189), (194) for the many variabable function $A_0$. We are rather interested in the function $u(x,t)$ which is a solution to (176).

With that in mind, we define an amplitude

\[
A(x,t) = A_0(t_1, x_1, \ldots)|_{t_j = \varepsilon^j, x_j = \varepsilon^j} \tag{195}
\]

The solution to (176) is then

\[
u(x,t) = A(x,t) e^{i(kx - \omega t)} + \varepsilon (2|A|^2(x,t) - \frac{1}{3} A^2(x,t) e^{2i(kx_0 - \omega t_0)} - \frac{1}{3} A^2(x,t) e^{-2i(kx_0 - \omega t_0)}) + O(\varepsilon^2) \tag{196}
\]

where $A(x,t)$ satisfy a certain amplitude equation that we will now derive.

Multiplying equation (189) by $\varepsilon$, equation (194) by $\varepsilon^2$ and adding the two
expressions, we get

\[
\varepsilon (2i\omega \partial_{t_1} A_0 + 2ik \partial_{x_1} A_0) \\
+ \varepsilon^2 (2i\omega \partial_{t_2} A_0 + 2ik \partial_{x_2} A_0 - \partial_{t_1 t_1} A_0 + \partial_{x_1 x_1} A_0 + \frac{10}{3} |A_0|^2 A_0) \\
= 0
\]

\[
\downarrow
\]

\[
2i\omega (\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2}) A_0 + 2ik (\partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2}) A_0 \\
-(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2})^2 A_0 + (\partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2})^2 A_0 + \varepsilon^2 \frac{10}{3} |A_0|^2 A_0 = 0 \quad (198)
\]

where we have used the fact that \(A_0\) does not depend on \(t_0\) and \(x_0\) and where the equation (198) is correct to second order in \(\varepsilon\). If we now evaluate (198) at \(x_j = \varepsilon^j x, t_j = \varepsilon^j t\) and, using (178) and (195), we get the amplitude equation

\[
2i\omega \partial_{t} A + 2ik \partial_{x} A - \partial_{tt} A + \partial_{xx} A + \varepsilon^2 \frac{10}{3} |A|^2 A = 0 \quad (199)
\]

\[
\downarrow
\]

\[
\partial_{t} A = -\frac{k}{\omega} \partial_{x} A - \frac{i}{2\omega} \partial_{t} A + \frac{i}{2\omega} \partial_{xx} A + \varepsilon^2 \frac{5i}{3\omega} |A|^2 A \quad (200)
\]

This equation appears to have a problem since it contains a second derivative with respect to time. The initial conditions for (176) is only sufficient to determine \(A(x,0)\). However, in order to be consistent with the multiple scale procedure leading up to (200) we can only consider solutions such that

\[
\partial_{t} A \sim -\frac{k}{\omega} \partial_{x} A \sim \varepsilon \quad (201)
\]

\[
\downarrow
\]

\[
\partial_{tt} A \sim \left( \frac{k}{\omega} \right)^2 \partial_{xx} A \sim \varepsilon^2
\]

Thus we can, to second order in \(\varepsilon\), rewrite the amplitude equation as

\[
\partial_{t} A = -\frac{k}{\omega} \partial_{x} A + \frac{i}{2\omega^3} \partial_{xx} A + \varepsilon^2 \frac{5i}{3\omega} |A|^2 A \quad (202)
\]

This is now first order in time and has a unique solution for a given initial condition \(A(x,0)\).

The multiple scale procedure demands that the amplitude \(A(x,t)\) vary slowly on scales \(L = \frac{2\pi}{k}, T = \frac{2\pi}{\omega}\). This means that (196) and (202) can be thought of as a fast numerical scheme for wavepacket solutions to (176). If these are the kind of solutions that we are interested in, and in optics this is often the case, it is much more efficient to use (196) and (202) rather than having to resolve the scales \(L\) and \(T\) by integrating the original equation (176).
Of course this particular amplitude equation is the famous Nonlinear Schrödinger equation and can be solved analytically using the Inverse Scattering Transform. But analytic solvability, however nice it is, is not robust. If we want to propagate our waves for \( t \lesssim \varepsilon^{-3} \), the multiple scale procedure must be extended to order \( \varepsilon^3 \), and additional terms will appear in the amplitude equation. These additional terms will destroy many of the wonderful mathematical properties of the Nonlinear Schrödinger equation but it will not destroy the fact that it is the key element in a fast numerical scheme for wave packet solutions to (170).

### 6.2 Example 2

Let us consider the equation

\[
        u_{tt} + u_{xx} + u_{xxxx} + u = \varepsilon u^3
\]  

Introducing the usual tools for the multiple scale method, we have

\[
u(x, t) = h(x_0, t_0, x_1, t_1, \ldots)|_{t_j = \varepsilon^j t, x_j = \varepsilon^j x}
\]  

\[
\partial_t = \partial_{t_0} + \varepsilon \partial_{t_1} + \ldots
\]

\[
\partial_x = \partial_{x_0} + \varepsilon \partial_{x_1} + \ldots
\]

\[
h = h_0 + \varepsilon h_1 + \ldots
\]

Inserting these expressions into (203) and expanding we get

\[
(\partial_{t_0} + \varepsilon \partial_{t_1} + \ldots)(\partial_{t_0} + \varepsilon \partial_{t_1} + \ldots)(h_0 + \varepsilon h_1 + \ldots) +
\]

\[
(\partial_{x_0} + \varepsilon \partial_{x_1} + \ldots)(\partial_{x_0} + \varepsilon \partial_{x_1} + \ldots)(h_0 + \varepsilon h_1 + \ldots) +
\]

\[
(\partial_{x_0} + \varepsilon \partial_{x_1} + \ldots)(\partial_{x_0} + \varepsilon \partial_{x_1} + \ldots)(h_0 + \varepsilon h_1 + \ldots)
\]

\[
= \varepsilon(h_0 + \ldots)^3
\]

\[
(\partial_{t_0} + \varepsilon(\partial_{t_0 t_1} + \partial_{t_1 t_0}) + \ldots)(h_0 + \varepsilon h_1 + \ldots) +
\]

\[
(\partial_{x_0} + \varepsilon(\partial_{x_0 x_1} + \partial_{x_1 x_0}) + \ldots)(h_0 + \varepsilon h_1 + \ldots) +
\]

\[
(\partial_{x_0} + \varepsilon(\partial_{x_0 x_1} + \partial_{x_1 x_0}) + \ldots)(\partial_{x_0 x_0} + \varepsilon(\partial_{x_0 x_1} + \partial_{x_1 x_0}) + \ldots)
\]

\[
(h_0 + \varepsilon h_1 + \ldots) = \varepsilon h_0^3 + \ldots
\]

\[
\partial_{t_0} t_0 h_0 + \varepsilon(\partial_{t_0 t_0} h_1 + \partial_{t_0 t_1} h_0 + \partial_{t_1 t_0} h_0) +
\]

\[
\partial_{x_0} x_0 h_0 + \varepsilon(\partial_{x_0 x_0} h_1 + \partial_{x_0 x_1} h_0 + \partial_{x_1 x_0} h_0) +
\]

\[
\partial_{x_0} x_0 x_0 x_0 h_0 + \varepsilon(\partial_{x_0 x_0 x_0} h_1 + \partial_{x_0 x_0 x_1} h_0 + \partial_{x_0 x_0 x_1} h_0 + \partial_{x_0 x_0 x_1} h_0 + \partial_{x_0 x_1 x_0 x_0} h_0 + \partial_{x_0 x_1 x_0 x_1} h_0 + \partial_{x_1 x_0 x_0 x_0} h_0 + \partial_{x_1 x_0 x_1 x_0} h_0 + \partial_{x_1 x_0 x_1 x_1} h_0)
\]

\[
= \varepsilon h_0^3 + \ldots
\]
which gives us the perturbation hierarchy

\[ \partial_{t_0} h_0 + \partial_{x_0} h_0 + \partial_{x_0 x_0 x_0} h_0 = 0 \] (205)

\[ \partial_{t_1} h_1 + \partial_{x_0} h_1 + \partial_{x_0 x_0 x_0} h_1 = h_0^3 \] (206)

\[ -\partial_{t_1} h_0 - \partial_{t_0} h_0 - \partial_{x_0} h_0 - \partial_{x_1} h_0 \]
\[ -\partial_{x_0 x_0 x_0} h_1 - \partial_{x_0 x_1} h_0 - \partial_{x_1 x_0 x_0} h_0 - \partial_{x_1 x_0} h_0 + \partial_{x_1 x_0 x_0 x_0} h_0 \]

For the order \( \varepsilon^0 \) equation we choose a wave packet solution

\[ h_0(x_0, t_0, x_1, t_1, \ldots) = A_0(x_1, t_1, \ldots) e^{i(kx_0 - \omega t_0)} + (\ast) \] (207)

where the dispersion relation is

\[ \omega = \sqrt{k^4 - k^2 + 1} \] (208)

Inserting (207) into (206) we get after a few algebraic manipulations

\[ \partial_{t_0} h_1 + \partial_{x_0} h_1 + \partial_{x_0 x_0 x_0} h_1 = \]
\[ (2i\omega \partial_{t_1} A_0 - 2ik \partial_{x_1} A_0 + 4ik^3 \partial_{x_1} A_0 + 3|A_0|^2 A_0) e^{i(kx_0 - \omega t_0)} \]
\[ + A_0^3 e^{3i(kx_0 - \omega t_0)} + (\ast) \] (210)

(211)

In order to remove secular terms we must postulate that

\[ 2i\omega \partial_{t_1} A_0 - 2ik \partial_{x_1} A_0 + 4ik^3 \partial_{x_1} A_0 + 3|A_0|^2 A_0 = 0 \] (212)

But using the dispersion relation (208) we have

\[ -2ik + 4ik^3 = 2i\omega \omega' \]

so that (212) simplifies into

\[ 2i\omega (\partial_{t_1} A_0 + \omega' \partial_{x_1} A_0) + 3|A_0|^2 A_0 = 0 \] (213)

Introducing an amplitude

\[ A(x, t) = A_0(x_1, t_1, \ldots)|_{x_j = \varepsilon x_j, t_j = \varepsilon t} \]

we get, following the approach from the previous example, the amplitude equation

\[ 2i\omega (\partial_t A + \omega' \partial_x A_0) = -3|A|^2 A \] (214)

This equation together with the expansion

\[ u(x, t) = A(t) e^{i(kx - \omega t)} + (\ast) + O(\varepsilon) \] (215)

constitute a fast numerical scheme for wave packet solutions to (203) for \( t \lesssim \varepsilon^{-1} \).

Of course this particular amplitude equation can be solved analytically, but as
stressed earlier, this property is not robust and can easily be lost if we take the expansion to higher order in \( \varepsilon \).

There is however one point in our derivation that we need to look more closely into. We assumed that the term

\[ A_0^3 e^{3i(kx_0-\omega t_0)} \]  

(216)

was not a secular term. The term is secular if

\[ \omega(3k) = 3\omega(k) \]  

(217)

Using the dispersion relation (208) we have

\[ \omega(3k) = 3\omega(k) \]

\[ \Downarrow \]

\[ \sqrt{81k^4 - 9k^2 + 1} = 3\sqrt{k^4 - k^2 + 1} \]

\[ \Downarrow \]

\[ 81k^4 - 9k^2 + 1 = 9k^4 - 9k^2 + 9 \]

\[ \Downarrow \]

\[ k = \pm \frac{1}{\sqrt{3}} \]  

(218)

Thus the term (216) can be secular if the wave number of the wave packet is given by (218). This is another example of the phenomenon that we in optics call phase matching. As long as we stay away from the two particular values of the wave numbers given in (218), our expansion (214) and (215) is uniform for \( t \ll \varepsilon^{-1} \). However if the wave number takes on one of the two values in (218), nonuniformities will make the ordering of the expansion break down for \( t \sim \varepsilon^{-1} \). However this does not mean that the multiple scale method breaks down. We only need to include a second amplitude at order \( \varepsilon^0 \) that we can use to remove the additional secular terms at order \( \varepsilon^1 \). We thus, instead of (207), use the solution

\[ h_0(x_0, t_0, x_1, t_1, ...) = A_0(x_1, t_1, ...)e^{i(kx_0-\omega t_0)} + B_0(x_1, t_1, ...)e^{3i(kx_0-\omega t_0)} + (\ast) \]  

(219)

(220)

where \( k \) now is given by (218). Inserting this expression for \( h_0 \) into the order \( \varepsilon \) equation (206) we get after a fair amount of algebra, the equation

\[ \partial_{t_0 t_0} h_1 + \partial_{x_0 x_0} h_1 + \partial_{x_0 x_0 x_0} h_1 = \]

\[ (2i\omega\partial_{t_1} A_0 - 2ik\partial_{x_1} A_0 + 4ik^3\partial_{x_1} A_0 \]

\[ + 3|A_0|^2 A_0 + 6|B_0|^2 A_0 + 3A_0^{*2} B_0)e^{i(kx_0-\omega t_0)} \]

\[ + (6i\omega\partial_{t_1} B_0 - 6ik\partial_{x_1} B_0 + 108ik^3\partial_{x_1} B_0 \]

\[ + 3|B_0|^2 B_0 + 6|A_0|^2 B_0 + A_0^{*3})e^{3i(kx_0-\omega t_0)} \]

\[ + NST + (\ast) \]
In order to remove secular terms we must postulate the two equations

\[
2i\omega \partial_t A_0 - 2ik \partial_x A_0 + 4ik^3 \partial_x A_0 + 3|A_0|^2 A_0 + 6|B_0|^2 A_0 + 3A_0^2 B_0 = 0
\]

\[
6i\omega \partial_t B_0 - 6ik \partial_x B_0 + 108ik^3 \partial_x B_0 + 3|B_0|^2 B_0 + 6|A_0|^2 B_0 + A_0^3 = 0
\]

Using the dispersion relation we have

\[
-6ik + 108ik^3 = 2i\omega (3k)
\]

Inserting this into (222) simplifies it into

\[
2i\omega (3k)(\partial_t A_0 + \omega'(3k)\partial_x A_0) = -3|A_0|^2 A_0 - 6|B_0|^2 A_0 - 3A_0^2 B_0
\]

Introducing amplitudes

\[
A(x,t) = A_0(x_1,t_1,...)|_{x_j = \epsilon x_j, t_j = \epsilon t}
\]

\[
B(x,t) = B_0(x_1,t_1,...)|_{x_j = \epsilon x_j, t_j = \epsilon t}
\]

the asymptotic expansion and corresponding amplitude equations are for this case

\[
u(x,t) = A(x,t)e^{i(kx - \omega t)}
\]

\[
+ B(x,t)e^{3i(kx - \omega t)} + (*) + O(\epsilon)
\]

\[
2i\omega (3k)(\partial_t A + \omega'(3k)\partial_x A) = -3|A|^2 A - 6|B|^2 A - 3A^2 B
\]

\[
2i\omega (3k)(\partial_t A + \omega'(3k)\partial_x A) = -3|B|^2 B + 6|A|^2 B + A^3
\]

The same approach must be used to treat the case when we do not have exact phase matching but we still have

\[
\omega(3k) \approx 3\omega(k)
\]

### 6.3 Exercises

In the following problems use the methods from this section to find asymptotic expansions that are uniform for \( t \ll \epsilon^2 \). Thus all expansions must be taken to second order in \( \epsilon \).

**Problem 1:**

\[
u_{tt} - u_{xx} + u = \epsilon^2 u^3
\]

**Problem 2:**

\[
u_{tt} - u_{xx} + u = \epsilon(u^2 + u_x^2)
\]
Problem 3: \[ u_{tt} - u_{xx} + u = \varepsilon (uu_{xx} - u^2) \]

Problem 4: \[ u_t + u_{xxx} = \varepsilon u^2 u_x \]

Problem 5: \[ u_{tt} - u_{xx} + u = \varepsilon (u_x^2 - uu_{xx}) \]

7 The multiple scale method for Maxwell’s equations

In optics the equations of interest are of course Maxwell’s equations. For a situation without free charges and currents they are

\begin{align*}
\partial_t B + \nabla \times E &= 0 \quad (225) \\
\partial_t D - \nabla \times H &= 0 \\
\nabla \cdot D &= 0 \\
\nabla \cdot B &= 0
\end{align*}

The material of interest is almost always nonmagnetic so that

\begin{align*}
H &= \frac{1}{\mu} B \\
D &= \varepsilon_0 E + P
\end{align*}

The polarization is in general a sum of a terms that is linear in \( E \) and one that is nonlinear in \( E \). We have

\[ P = P_L + P_{NL} \]

(227)

where the term linear in \( E \) has the general form

\[ P_L(x, t) = \varepsilon_0 \int_{-\infty}^{t} dt' \chi(t - t') E(x, t') \]

(228)

Thus the polarization at a time \( t \) depends on the electric field at all times previous to \( t \). This memory effect is what we in optics call \emph{temporal dispersion}. The presence of dispersion in Maxwell equations spells trouble for the integration of the equations in time, we can not solve them as a standard initial value problem. This is of course well known in optics and various more or less ingenious methods has been designed for getting around this problem. In optical pulse propagation one gets around the problem by solving Maxwell’s equations approximately as a boundary value problem rather than as an initial value problem. A very general version of this approach is the well known UPPE\[8,\,4,\,7\] propagation scheme. In these lecture notes we will, using the multiple scale method, derive
approximations to Maxwell’s equations that can be solved as an initial value problem.

In the explicit calculations that we do we will assume that the nonlinear polarization is generated by the Kerr effect. Thus we will assume that

$$P_{NL} = \varepsilon_0 \eta E \cdot E$$  \hfill (229)

where $\eta$ is the Kerr coefficient. This is a choice we make just to be specific, the applicability of the multiple scale method to Maxwell’s equations in no way depend on this particular choice for the nonlinear response.

Before we proceed with the multiple scale method we will introduce a more convenient representation of the dispersion. Observe that we have

$$P_L(x, t) = \varepsilon_0 \int_{-\infty}^{t} dt' \chi(t - t') E(x, t')$$  \hfill (230)

$$= \varepsilon_0 \int_{-\infty}^{\infty} d\omega \tilde{\chi}(\omega) \hat{E}(x, \omega) e^{-i\omega t}$$

$$= \varepsilon_0 \int_{-\infty}^{\infty} d\omega \left( \sum_{n=0}^{\infty} \frac{\tilde{\chi}^{(n)}(0)}{n!} \omega^n \right) \hat{E}(x, \omega) e^{-i\omega t}$$

$$= \varepsilon_0 \int_{-\infty}^{\infty} d\omega \hat{\chi}(\omega) \hat{E}(x, \omega) e^{-i\omega t}$$

$$= \varepsilon_0 \int_{-\infty}^{\infty} d\omega \hat{\chi}(\omega) \hat{E}(x, \omega) e^{-i\omega t}$$

$$= \varepsilon_0 \chi(i\partial_t) E(x, t)$$

where $\tilde{\chi}(\omega)$ is the Fourier transform of $\chi(t)$. These manipulations are of course purely formal, in order to make them into honest mathematics we must dive into the theory of pseudo differential operators. In these lecture notes we will not do this as our focus is on mathematical methods rather than mathematical theory.

Inserting (226), (227), (228) and (230) into (225) we get Maxwell’s equations in the form

$$\partial_t B + \nabla \times E = 0$$

$$\partial_t E - c^2 \nabla \times B + \partial_t \tilde{\chi}(i\partial_t) E = -c^2 \mu_0 \partial_t P_{NL}$$

$$\nabla \cdot (E + \tilde{\chi}(i\partial_t) E) = -\frac{1}{\varepsilon_0} \nabla \cdot P_{NL}$$

$$\nabla \cdot B = 0$$
7.1 TE scalar wave packets

Let us first simplify the problem by only considering solutions of the form

\[ E(x, y, z, t) = E(x, z, t) \text{e}^y(232) \]

\[ B(x, y, z, t) = B_1(x, z, t) \text{e}^x + B_2(x, z, t) \text{e}^z \]

For this simplified case Maxwell’s equations takes the form

\[ \partial_t B_1 - \partial_z E = 0 \] (233)
\[ \partial_t B_2 + \partial_x E = 0 \]
\[ \partial_t E - c^2(\partial_z B_1 - \partial_x B_2) + \partial_t \tilde{\chi}(i\partial_t) E = -\partial_t P_{NL} \]
\[ \partial_x B_1 + \partial_z B_2 = 0 \]

where

\[ P_{NL} = \eta E^3 \] (234)

It is well known that this vector system is fully equivalent to the following scalar equation

\[ \partial_{tt} E - c^2 \nabla^2 E + \partial_{tt} \tilde{\chi}(i\partial_t) E = -\partial_{tt} P_{NL} \] (235)

where we have introduced the operator

\[ \nabla^2 = \partial_{xx} + \partial_{zz} \] (236)

Equation (235) will be the starting point for our multiple scale approach, but before that I will introduce the notion of a formal perturbation parameter. For some particular application of equation (235) we will usually start by making the equation dimension-less by picking some scales for space, time, and \( E \) relevant for our particular application. Here we don’t want to tie our calculations to some particular choice of scales and introduce therefore a formal perturbation parameter in the equation multiplying the nonlinear polarization term. Thus we have

\[ \partial_{tt} E - c^2 \nabla^2 E + \partial_{tt} \tilde{\chi}(i\partial_t) E = -\varepsilon^2 \eta \partial_{tt} E^3 \] (237)

Starting with this equation we will proceed with our perturbation calculations assuming that \( \varepsilon \ll 1 \) and in the end we will remove \( \varepsilon \) by setting it equal to 1.

What is going on here is that \( \varepsilon \) is a "place holder" for the actual small parameter that will appear in front of the nonlinear term in the equation when we make a particular choice of scales. Using such formal perturbation parameters is very common.

You might ask why I use \( \varepsilon^2 \) instead of \( \varepsilon \) as formal perturbation parameter? I will not answer this question here but will say something about it at the very end of the lecture notes. We proceed with the multiple scale method by introducing the expansions

\[ \partial_t = \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ... \] (238)
\[ \nabla = \nabla_{0} + \varepsilon \nabla_{1} + \varepsilon^2 \nabla_{2} + ... \]
\[ e = e_{0} + \varepsilon e_{1} + \varepsilon^2 e_{2} + ... \]
\[ E(x, t) = e(x_0, t_0, x_1, t_1, ...)|_{t_j = \varepsilon^j t, x_j = \varepsilon^j x} \]
where
\[ \nabla_j = (\partial_{x_j}, \partial_{z_j}) \tag{239} \]
is the gradient with respect to \( x_j = (x_j, z_j) \). We now insert (238) into (237) and expand everything in sight
\[
(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)
\]
\[
(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)
\]
\[
\hat{\chi}(i \partial_{t_0} + i \varepsilon \partial_{t_1} + i \varepsilon^2 \partial_{t_2} + ...)(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)
\]
\[
(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)^3
\]
\[
\downarrow
\]
\[
(\partial_{t_0 t_0} + \varepsilon(\partial_{t_0 t_1} + \partial_{t_1 t_0}) + \varepsilon^2(\partial_{t_0 t_2} + \partial_{t_1 t_1} + \partial_{t_2 t_0}) + ...)
\]
\[
(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)^2
\]
\[
(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)^3
\]
\[
(\partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + ...)^4
\]
\[
\downarrow
\]
\[
49
\]
\[ \partial_{t_0}c_0 + \varepsilon (\partial_{t_0}c_1 + \partial_{t_0}c_1 + \partial_{t_0}c_0 + \partial_{t_0}c_0 + \partial_{t_0}c_0 + \partial_{t_0}c_0) + ... \\
+ \varepsilon^2 (\partial_{t_0}c_2 + \partial_{t_0}c_1 + \partial_{t_0}c_1 + \partial_{t_0}c_0 + \partial_{t_0}c_1 + \partial_{t_0}c_0 + \partial_{t_0}c_0) + ... \\
- \varepsilon^2 \nabla_0^2 c_0 - \varepsilon^2 (\nabla_0^2 c_1 + \nabla_1 \cdot \nabla_0 c_0 + \nabla_0 \cdot \nabla_1 c_0) \\
- \varepsilon^2 \nabla_0^2 c_2 + \nabla_1 \cdot \nabla_0 c_1 + \nabla_0 \cdot \nabla_1 c_1 \\
+ \nabla_0^2 c_0 + \nabla_1 \cdot \nabla_1 c_0 + \nabla_0 \cdot \nabla_2 c_0 + ... \\
+ \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 + \varepsilon (\partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 + \varepsilon^2 (\partial_{t_0} \hat{\chi}(i\partial_{t_0})c_2 \\
+ \partial_{t_0} \hat{\chi}(i\partial_{t_0})i\partial_{t_1}c_1 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_1 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_1 \\
+ \partial_{t_0} \hat{\chi}(i\partial_{t_0})i\partial_{t_1}c_0 - \frac{1}{2} \partial_{t_0} \hat{\chi}(i\partial_{t_0}) \partial_{t_1}c_0 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})i\partial_{t_1}c_0 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 \\
+ \partial_{t_0} \hat{\chi}(i\partial_{t_0}) \partial_{t_1}c_0 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 \\
+ \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 + ... \\
= -\varepsilon^2 \partial_{t_0}c_0^3 + ... \\
\]

which gives us the perturbation hierarchy

\[ \partial_{t_0} \hat{c}_0 - \varepsilon^2 \nabla_0^2 \hat{c}_0 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 = 0 \] (240)

\[ \partial_{t_0} \hat{c}_1 - \varepsilon^2 \nabla_0^2 \hat{c}_1 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_1 = \] (241)

\[ -\partial_{t_0} \hat{c}_0 - \partial_{t_1}c_0 - \varepsilon^2 \nabla_1 \cdot \nabla_0 c_0 - \varepsilon^2 \nabla_0 \cdot \nabla_1 c_0 \\
- \partial_{t_0} \hat{\chi}(i\partial_{t_0})i\partial_{t_2}c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 \\
\]

\[ \partial_{t_0} \hat{c}_2 - \varepsilon^2 \nabla_0^2 \hat{c}_2 + \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_2 = \] (242)

\[ -\partial_{t_0} \hat{c}_1 - \partial_{t_1}c_1 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_1 - \nabla_1 \cdot \nabla_0 \hat{\chi}(i\partial_{t_0})c_1 \\
- \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_1 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_1 - \partial_{t_1}c_0 - \partial_{t_2}c_0 \\
- \varepsilon^2 \nabla_1 \cdot \nabla_0 c_1 - \varepsilon^2 \nabla_0 \cdot \nabla_1 c_1 - \varepsilon^2 \nabla_2 \cdot \nabla_0 c_0 - \varepsilon^2 \nabla_0 \cdot \nabla_2 c_0 \\
- \nabla_0 \cdot \nabla_2 c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})i\partial_{t_2}c_0 + \frac{1}{2} \partial_{t_0} \hat{\chi}(i\partial_{t_0}) \partial_{t_1}c_0 \\
- \partial_{t_0} \hat{\chi}(i\partial_{t_0})i\partial_{t_1}c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 \\
- \partial_{t_1}c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 - \partial_{t_0} \hat{\chi}(i\partial_{t_0})c_0 \\
\]

For the order \( \varepsilon^0 \) equation we choose the wave packet solution

\[ c_0(x_0, t_0, x_1, t_1, ..) = A_0(x_1, t_1, ...) e^{i\theta_0} + (*) \] (243)

where

\[ \mathbf{x}_j = (x_j, z_j) \] (244)

\[ \theta_0 = \mathbf{k} \cdot \mathbf{x}_0 - \omega t_0 \]
and where \( \mathbf{k} \) is a plane vector with components \( \mathbf{k} = (\xi, \eta) \). In (244) \( \omega \) is a function of \( k = ||\mathbf{k}|| \) that satisfy the dispersion relation
\[
\omega^2 n^2(\omega) = c^2 k^2
\]  
where the refractive index is defined by
\[
n^2(\omega) = 1 + \tilde{\chi}(\omega)
\]
We now must calculate the right-hand side of the order \( \varepsilon \) equation. Observe that
\[
\partial_{t_1} e_0 = -i \omega \partial_{t_1} A_0 e^{i \theta_0} + (*) \\
\partial_{t_0} e_0 = -i \omega \partial_{t_0} A_0 e^{i \theta_0} + (*) \\
\nabla_1 \cdot \nabla_0 e_0 = ik \nabla_1 A_0 \cdot \mathbf{u} e^{i \theta_0} + (*) \\
\nabla_0 \cdot \nabla_1 e_0 = ik \nabla_0 A_0 \cdot \mathbf{u} e^{i \theta_0} + (*) \\
\partial_{t_0,t_0} \tilde{\chi}'(i \partial_{t_0}) i \partial_{t_1} e_0 = -i \omega \tilde{\chi}'(\omega) \partial_{t_1} A_0 e^{i \theta_0} + (*) \\
\partial_{t_0,t_0} \tilde{\chi}(i \partial_{t_0}) e_0 = -i \omega \tilde{\chi}(\omega) \partial_{t_1} A_0 e^{i \theta_0} + (*) \\
\partial_{t_0,t_0} \tilde{\chi}(i \partial_{t_0}) e_0 = -i \omega \tilde{\chi}(\omega) \partial_{t_1} A_0 e^{i \theta_0} + (*)
\]
where \( \mathbf{u} \) is a unit vector in the direction of \( \mathbf{k} \). Inserting (247) into (241) we get
\[
\partial_{t_1} e_1 = c^2 \nabla_0^2 e_1 + \partial_{t_0,t_0} \tilde{\chi}(i \partial_{t_0}) e_1 = \\
-\{ -2i \omega \partial_{t_1} A_0 - 2ic^2 k \mathbf{u} \cdot \nabla_1 A_0 \\
- i \omega^2 \tilde{\chi}'(\omega) \partial_{t_1} A_0 - 2i \omega \tilde{\chi}(\omega) \partial_{t_1} A_0 \} e^{i \theta_0} + (*)
\]
In order to remove secular terms we must postulate that
\[
-2i \omega \partial_{t_1} A_0 - 2ic^2 k \mathbf{u} \cdot \nabla_1 A_0 - i \omega^2 \tilde{\chi}'(\omega) \partial_{t_1} A_0 - 2i \omega \tilde{\chi}(\omega) \partial_{t_1} A_0 = 0
\]
\[
\Downarrow
\omega(2n^2 + \omega \tilde{\chi}'(\omega)) \partial_{t_1} A_0 - 2ic^2 k \mathbf{u} \cdot \nabla_1 A_0 = 0
\]
Observe that from the dispersion relation (245) we have
\[
\omega^2 n^2(\omega) = c^2 k^2 \\
\Downarrow
\omega^2 (1 + \tilde{\chi}(\omega)) = c^2 k^2 \\
\Downarrow
2 \omega \omega' n^2(\omega) + \omega^2 \tilde{\chi}'(\omega) \omega' = 2c^2 k \\
\Downarrow
\omega(2n^2 + \omega \tilde{\chi}'(\omega)) \omega' = 2c^2 k
\]
Thus (249) can be written in the form
\[
\partial_{t_1} A_0 + v_g \cdot \nabla_1 A_0 = 0
\]  
(250)
where $v_g$ is the group velocity

$$v_g = \omega'(k)u$$

(251)

The order $\varepsilon$ equation simplifies into

$$\partial_{t_0t_0} e_1 - c^2\nabla_0^2 e_1 + \partial_{t_0t_0} \tilde{\chi}(i\partial_{t_0}) e_1 = 0$$

(252)

According to the rules of the game we choose the special solution

$$e_1 = 0$$

(253)

for (252). We now must compute the right-hand side of the order $\varepsilon^2$ equation. Observe that

$$\partial_{t_2t_0} e_0 = -i\omega \partial_{t_2} A_0 e^{i\theta_0} + (*)$$

(254)

$$\partial_{t_1t_1} e_0 = \partial_{t_1t_1} A_0 e^{i\theta_0} + (*)$$

$$\partial_{t_0t_0} e_0 = -i\omega \partial_{t_0} A_0 e^{i\theta_0} + (*)$$

$$\nabla_2 \cdot \nabla_0 e_0 = iku \cdot \nabla_2 A_0 e^{i\theta_0} + (*)$$

$$\nabla_1 \cdot \nabla_1 e_0 = \nabla_1^2 A_0 e^{i\theta_0} + (*)$$

$$\nabla_0 \cdot \nabla_2 e_0 = iku \cdot \nabla_2 A_0 e^{i\theta_0} + (*)$$

$$\partial_{t_0t_0} \tilde{\chi}(i\partial_{t_0}) i\partial_{t_2} e_0 = -i\omega^2 \tilde{\chi}'(\omega) \partial_{t_2} A_0 e^{i\theta_0} + (*)$$

$$\frac{1}{2} \partial_{t_0t_0} \tilde{\chi}''(i\partial_{t_0}) \partial_{t_1t_1} e_0 = -\frac{1}{2} \omega^2 \tilde{\chi}''(\omega) \partial_{t_1t_1} A_0 e^{i\theta_0} + (*)$$

Inserting (253) and (254) into the right-hand side of the order $\varepsilon^2$ equation we get

$$\partial_{t_0t_0} e_2 - c^2\nabla_0^2 e_2 + \partial_{t_0t_0} \tilde{\chi}(i\partial_{t_0}) e_2 =$$

$$-\{ -2i\omega \partial_{t_2} A_0 + \partial_{t_1t_1} A_0 - 2ic^2 k u \cdot \nabla_2 A_0 - c^2 \nabla_1^2 A_0$$

$$-i\omega^2 \tilde{\chi}'(\omega) \partial_{t_2} A_0 + \frac{1}{2} \omega^2 \tilde{\chi}''(\omega) \partial_{t_1t_1} A_0 + 2\omega \tilde{\chi}'(\omega) \partial_{t_1t_1} A_0$$

$$-2i\omega \tilde{\chi}(\omega) \partial_{t_2} A_0 + \tilde{\chi}(\omega) \partial_{t_1t_1} A_0 - 3\omega^2 |A_0|^2 \} e^{i\theta_0} + NST + (*)$$

(255)
In order to remove secular terms we must postulate that
\[-2i\omega \partial_{t_2} A_0 + \partial_{t_1 t_1} A_0 - 2i c^2 k u \cdot \nabla_2 A_0 - c^2 \nabla^2_1 A_0 - i\omega^2 \tilde{\chi}'(\omega) \partial_{t_2} A_0 \]
\[+ \frac{1}{2} \omega^2 \tilde{\chi}''(\omega) \partial_{t_1 t_1} A_0 + 2\omega \tilde{\chi}'(\omega) \partial_{t_1 t_1} A_0 - 2i\omega \tilde{\chi}(\omega) \partial_{t_2} A_0 + \tilde{\chi}(\omega) \partial_{t_1 t_1} A_0 \]
\[-3\omega^2 \eta |A_0|^2 = 0 \] (257)

Using the dispersion relation (246), equation (256) can be simplified into
\[\partial_{t_2} A_0 + v_g \cdot \nabla_2 A_0 - i\beta \nabla^2_1 A_0 + i\alpha \partial_{t_1 t_1} A_0 - i\gamma |A_0|^2 A_0 = 0 \] (258)

where
\[\alpha = \omega' n^2 + 2\omega \tilde{\chi}'(\omega) + \frac{1}{2} \omega^2 \tilde{\chi}''(\omega) \]
\[\beta = \frac{\omega'}{2k} \]
\[\gamma = \frac{3\eta \omega^2 \omega'}{2c^2 k} \]

Defining an amplitude \(A(x, t)\) by
\[A(x, t) = A_0(x_1, t_1, ...) |_{t_j = e^{\epsilon t}, x_j = e^{\epsilon x}} \] (261)

and proceeding in the usual way, using (250) and (258), we get the following amplitude equation
\[\partial_t A + v_g \cdot \nabla A - i\beta \nabla^2 A + i\alpha \partial_{tt} A - i\gamma |A|^2 A = 0 \] (262)

where we have put the formal perturbation parameter equal to 1. From what we have done it is evident that for
\[E(x, t) = A(x, t)e^{i(k \cdot x - \omega t)} + (\ast) \]

to be an approximate solution to (237) we must have
\[\gamma |A|^2 \sim \beta \nabla^2 A \sim \alpha \partial_{tt} A \sim O(\epsilon^2) \]
\[\partial_t A \sim v_g \cdot \nabla A \sim O(\epsilon) \] (264)

where \(\epsilon\) is a number much smaller than 1. Under these circumstances (262), (263)
is the key elements in a fast numerical scheme for wave packet solutions to (237). Because of the presence of the second derivative with respect to time, equation (262) can not be solved as a standard initial value problem. However, because of (264) we can remove the second derivative term by iteration
\[\partial_t A = -v_g \cdot \nabla A \sim O(\epsilon) \]
\[\downarrow \]
\[\partial_{tt} A = (v_g \cdot \nabla)^2 A \sim O(\epsilon^2) \] (265)

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which leads to the equation
\[ \partial_t A + \mathbf{v}_g \cdot \nabla A - i\beta \nabla^2 A + i\alpha (\mathbf{v}_g \cdot \nabla)^2 A - i\gamma |A|^2 A = 0 \] (266)
which can be solved as a standard initial value problem. In deriving this equation we assumed that the terms proportional to
\[ e^{\pm 3\pi i (k \cdot x - \omega t)} \]
where nonsecular. For this to be true we must have
\[ \omega(3k) \neq 3\omega(k) \] (267)
where \( \omega(k) \) is a solution to \( (245) \). If an equality holds in \( (267) \) we have phase matching and the multiple scale calculation has to be redone, starting at \( (243) \), using a sum of two wave packets with the appropriate center wave numbers and frequencies instead of the single wavepacket we used in the calculation leading to \( (262) \). It could also be the case that we are modelling a situation where several wave packets are interacting in a Kerr medium. For such a case we would instead of \( (243) \) use a finite sum of wave packets
\[ e_0(x_0, t_0, x_1, t_1, \ldots) = \sum_{j=0}^{N} A_j(x_1, t_1, \ldots) e^{i\theta_j} + (*) \] (268)
Calculations analogous to the ones leading up to equation \( (262) \) will now give a separate equation of the type \( (262) \) for each wave packet, unless we have phase matching. These phase matching conditions appears from the nonlinear term in the order \( \varepsilon^2 \) equation and takes the familiar form
\[ k_j = s_1 k_{j_1} + s_2 k_{j_2} + s_3 k_{j_3} \] (269)
\[ \omega(k_j) = s_1 \omega(k_{j_1}) + s_2 \omega(k_{j_2}) + s_3 \omega(k_{j_3}) \]
where \( s = \pm 1 \). The existence of phase matching leads to coupling of the amplitude equations. If \( (269) \) holds, the amplitude equation for \( A_j \) will contain a coupling term proportional to
\[ A_{j_1}^{s_1} A_{j_2}^{s_2} A_{j_3}^{s_3} \] (270)
where by definition \( A_j^{-1} = A_j \) and \( A_j^{-1} = A_j^* \).
We have seen that assuming a scaling of \( \varepsilon \) for space and time variables and \( \varepsilon^2 \) for the nonlinear term leads to an amplitude equation where second derivatives of space and time appears at the same order as the cubic nonlinearity. This amplitude equation can thus describe a situation where diffraction, group velocity dispersion and nonlinearity are of the same size. Other choices of scaling for space, time and nonlinearity will lead to other amplitude equations where other physical effects are of the same size. Thus choice of scaling is determined by what kind of physics we want to describe.
7.2 Linearly polarized vector wave packets

Up til now all applications of the multiple scale method has involved scalar equations. The multiple scale method is not limited to scalar equations and is equally applicable to vector equations. However, for vector equations we need to be more careful than for the scalar case when it comes to eliminating secular terms. We will here use Maxwell’s equations (231) to illustrate how the method is applied to vector equations in general. Assuming, as usual, a polarization response induced by the Kerr effect, our basic equations are

\[
\begin{align*}
\partial_t B + \nabla \times E &= 0 \quad (271) \\
\partial_t E - c^2 \nabla \times B + \partial_t \tilde{\chi}(i\partial_t)E &= -\varepsilon^2 \eta \partial_t (E^2 E) \\
\nabla \cdot B &= 0 \\
\nabla \cdot E + \tilde{\chi}(i\partial_t) \nabla \cdot E &= -\varepsilon^2 \eta \nabla \cdot (E^2 E)
\end{align*}
\]

where we have introduced a formal perturbation parameter in front of the nonlinear terms. We now introduce the usual machinery of the multiple scale method. Let \(e(x_0, t_0, x_1, t_1, \ldots)\) and \(b(x_0, t_0, x_1, t_1, \ldots)\) be functions such that

\[
E(x, t) = e(x_0, t_0, x_1, t_1, \ldots)|_{x_j=\varepsilon^j x, t_j=\varepsilon^j t} \quad (272)
\]

\[
B(x, t) = b(x_0, t_0, x_1, t_1, \ldots)|_{x_j=\varepsilon^j x, t_j=\varepsilon^j t}
\]

and let

\[
\begin{align*}
\partial_t &= \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \ldots \quad (273) \\
\nabla \times &= \nabla_0 \times + \varepsilon \nabla_1 \times + \varepsilon^2 \nabla_2 \times + \ldots \\
\nabla \cdot &= \nabla_0 \cdot + \varepsilon \nabla_1 \cdot + \varepsilon^2 \nabla_2 \cdot + \ldots \\
e &= e_0 + \varepsilon e_1 + \varepsilon^2 e_2 + \ldots \\
b &= b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \ldots
\end{align*}
\]

We now insert (273) into (271) and expand everything in sight to second order in \(\varepsilon\). Putting each order of \(\varepsilon\) to zero separately gives us the perturbation hierarchy.

At this point you should be able to do this on your own so I will just write down the elements of the perturbation hierarchy when they are needed.

The order \(\varepsilon^0\) equations, which is the first element of the perturbation hierarchy, is of course

\[
\begin{align*}
\partial_{t_0} b_0 + \nabla_0 \times e_0 &= 0 \quad (274) \\
\partial_{t_0} e_0 - c^2 \nabla_0 \times b_0 + \partial_{t_0} \tilde{\chi}(i\partial_{t_0}) e_0 &= 0 \\
\nabla_0 \cdot b_0 &= 0 \\
\nabla_0 \cdot e_0 + \tilde{\chi}(i\partial_{t_0}) \nabla_0 \cdot e_0 &= 0
\end{align*}
\]

For the order \(\varepsilon^0\) equations we chose a linearly polarized wave packet solution. It must be of the form

\[
\begin{align*}
e_0(x_0, t_0, x_1, t_1, \ldots) &= \omega A_0(x_1, t_1, \ldots) q e^{i\theta_0} + (*) \quad (275) \\
b_0(x_0, t_0, x_1, t_1, \ldots) &= k A_0(x_1, t_1, \ldots) t e^{i\theta_0} + (*)
\end{align*}
\]
where
\[ \theta_0 = k \cdot x_0 - \omega t_0 \] (276)
and where
\[ \omega = \omega(k) \]
is a solution to the dispersion relation
\[ \omega^2 n^2(\omega) = c^2 k^2 \] (277)
The orthogonal unit vector \( q \) and \( t \) span the space transverse to \( k = ku \), and the unit vectors \( (q, t, u) \) define a positively oriented frame for \( \mathbb{R}^3 \).

The order \( \varepsilon \) equations are
\[
\begin{align*}
\partial_{t_0} b_1 + \nabla_0 \times e_1 &= -\partial_{t_0} b_0 - \nabla_1 \times e_0 \quad (278) \\
\partial_{t_0} e_0 - c^2 \nabla_0 \times b_0 + \partial_{t_0} \bar{\chi}(i\partial_{t_0})e_0 &= -(\omega n^2(\omega) + \omega^2 \bar{\chi}'(\omega))\partial_{t_1} A_0 q \\
\nabla_0 \cdot b_1 &= -\nabla_1 \cdot b_0 \\
\nabla_0 \cdot e_1 + \bar{\chi}(i\partial_{t_0})\nabla_0 \cdot e_1 &= -\nabla_1 \cdot e_0 - \bar{\chi}(i\partial_{t_0})\nabla_1 \cdot e_0 - i\bar{\chi}'(i\partial_{t_0})\partial_{t_1} \nabla_0 \cdot e_0
\end{align*}
\]
Inserting (275) into (278), we get
\[
\begin{align*}
\partial_{t_0} b_1 + \nabla_0 \times e_0 &= -\{k \partial_{t_1} A_0 t + \omega \nabla_1 A_0 \times q\} e^{i\theta_0} + (*) \quad (279) \\
\partial_{t_0} e_0 - c^2 \nabla_0 \times b_0 + \partial_{t_0} \bar{\chi}(i\partial_{t_0})e_0 &= -(\omega n^2(\omega) + \omega^2 \bar{\chi}'(\omega))\partial_{t_1} A_0 q \\
\nabla_0 \cdot b_0 &= -k \nabla_1 A_0 \cdot t e^{i\theta_0} + (*) \\
\nabla_0 \cdot e_0 + \bar{\chi}(i\partial_{t_0})\nabla_0 \cdot e_0 &= -\{\omega n^2(\omega) \nabla_1 A_0 \cdot q\} e^{i\theta_0} + (*)
\end{align*}
\]
If we can find a special solution to this system that is bounded we will get a perturbation expansion that is uniform for \( t \leq \varepsilon^{-1} \). We will look for solutions of the form
\[
\begin{align*}
e_1 &= ae^{i\theta_0} + (*) \quad (280) \\
b_1 &= be^{i\theta_0} + (*)
\end{align*}
\]
where \( a \) and \( b \) are constant vectors. Inserting (280) into (279), we get the following linear algebraic system of equations for the unknown vectors \( a \) and \( b \)
\[
\begin{align*}
-i\omega b + ik u \times a &= -\{k \partial_{t_1} A_0 t + \omega \nabla_1 A_0 \times q\} \\
-i\omega n^2(\omega) a - ic^2 k u \times b &= -\{\omega n^2(\omega) + \omega^2 \bar{\chi}'(\omega)\} \partial_{t_1} A_0 q \\
-c^2 k \nabla_1 A_0 \times t \\
\n\n\n\end{align*}
\]
(281) 
(282) 
(283) 
(284)

Introduce the longitudinal and transverse parts of \(a\) and \(b\) through
\[
a_{\|} = (u \cdot a)u, \quad a_{\perp} = a - a_{\|}
\]
\[
b_{\|} = (u \cdot b)u, \quad b_{\perp} = b - b_{\|}
\]

Then from (283) and (284) we get
\[
a_{\|} = (i\omega k\nabla A_0 \cdot q)u \tag{286}
\]
\[
b_{\|} = (i\nabla A_0 \cdot t)u \tag{287}
\]

However the longitudinal part of (281) and (282) will also determine \(a_{\|}\) and \(b_{\|}\). These values must be the same as the ones just found in (286), (287). These are solvability conditions. Taking the longitudinal part of (281) we get
\[
-i\omega u \cdot b = -\omega u \cdot (\nabla A_0 \times q) \tag{288}
\]
\[
\checkmark
\]
\[
u \cdot b = i\nabla A_0 \cdot t
\]

which is consistent with (287). Thus this solvability condition is automatically satisfied. Taking the longitudinal part of (282) we get
\[
-i\omega u \cdot a = c^2 k u \cdot (\nabla A_0 \times t) \tag{289}
\]
\[
\checkmark
\]
\[
u \cdot a = i\frac{\omega}{k} \nabla A_0 \cdot q
\]

which is consistent with (286). Thus this solvability condition is also automatically satisfied. The transversal part of (281) and (282) are
\[
-i\omega b_{\perp} + ik u \times a_{\perp} = -\{k\partial_t A_0 + \omega \nabla A_0 \cdot u\}t \tag{290}
\]
\[
-i\omega^2 a_{\perp} - ic^2 k u \times b_{\perp} = -\{\omega(n^2(\omega) + \omega \hat{\chi}'(\omega))\partial_t A_0 + c^2 k \nabla A_0 \cdot u\}q
\]

This linear system is singular, the determinant is zero because of the dispersion relation (277). It can therefore only be solved if the right-hand side satisfy a certain solvability condition. The most effective way to find this condition is to use the Fredholm Alternative. It say that a linear system
\[
Ax = c
\]

has a solution if and only if
\[
f \cdot c = 0
\]
for all vectors \(f\), such that
\[
A^\dagger f = 0
\]
where \(A^\dagger\) is the adjoint of \(A\).
The matrix for the system (290) is

\[
M = \begin{pmatrix}
  iku \times & -i\omega \\
  -i\omega n^2 & -ic^2 ku \times
\end{pmatrix}
\]

The adjoint of the matrix is clearly

\[
M^\dagger = \begin{pmatrix}
  -iku \times & -i\omega n^2 \\
  -i\omega & ic^2 ku \times
\end{pmatrix}
\]

(291)

The null space of the adjoint is thus determined by

\[
-iku \times \alpha - i\omega n^2 \beta = 0 \tag{292}
\]

\[
-i\omega \alpha + ic^2 ku \times \beta = 0
\]

A convenient basis for the null space is

\[
\left\{ \begin{pmatrix}
  -c^2 kq \\
  \omega t
\end{pmatrix}, \begin{pmatrix}
  c^2 kt \\
  \omega q
\end{pmatrix} \right\}
\]

(293)

The first basis vector gives a trivial solvability condition whereas the second one gives a nontrivial condition which is

\[
c^2 k \{ k \partial_t A_0 + \omega \nabla_1 A_0 \cdot u \} + \omega \{ \omega (n^2(\omega) + \omega \hat{\chi}'(\omega)) \partial_t A_0 + c^2 k \nabla_1 A_0 \cdot u \} = 0
\]

\[
\Downarrow
\]

\[
\omega^2 (2n^2 + \omega \hat{\chi}'(\omega)) \partial_t A_0 + 2c^2 ku \cdot \nabla_1 A_0 = 0 \tag{294}
\]

Observe that from the dispersion relation (277) we have

\[
\omega^2 n^2(\omega) = \omega^2 (1 + \hat{\chi}(\omega)) = c^2 k^2
\]

\[
\Downarrow
\]

\[
2\omega \omega' n^2 + \omega^2 \hat{\chi}'(\omega) \omega' = 2c^2 k
\]

\[
\Downarrow
\]

\[
\omega (2n^2 + \omega \hat{\chi}'(\omega)) \omega' = 2c^2 k
\]

Using (295) in (294) the solvability condition can be compactly written as

\[
\partial_t A_0 + v_g \cdot \nabla_1 A_0 = 0 \tag{296}
\]

where \(v_g\) is the group velocity

\[
v_g = \frac{d\omega}{dk} u
\]

(297)

The system (290) is singular but consistent. We can therefore disregard the second equation and look for a special solution of the form

\[
a_\perp = a q
\]

(298)

\[
b_\perp = 0
\]

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Inserting (298) into the first equation in (290) we get easily

$$a_\perp = i \left\{ \partial_t A_0 + \frac{\omega}{k} u \cdot \nabla_1 A_0 \right\} q$$  \hspace{1cm} (299)$$

From (286), (287), (298) and (299) we get the following bounded special solution to the order $\varepsilon$ equations

$$e_1 = \{ i(\partial_t A_0 + \frac{\omega}{k} u \cdot \nabla_1 A_0)q + i(\frac{\omega}{k} q \cdot \nabla_1 A_0)u \} e^{i\theta_0} + (*)$$ \hspace{1cm} (300)$$

$$b_1 = \{ i(t \cdot \nabla_1 A_0)u \} e^{i\theta_0} + (*)$$

The order $\varepsilon^2$ equations are

$$\partial_\varepsilon b_2 + \nabla_0 \times e_2 = -\{ \partial_t b_1 + \nabla_1 \times e_1 + \partial_\varepsilon b_0 + \nabla_2 \times e_0 \}$$  \hspace{1cm} (301)$$

$$\begin{align*}
\partial_\varepsilon e_2 - c^2 \nabla_0 \times b_2 + \varepsilon_0 \chi (i\partial_\varepsilon) e_2 &= -\{ \partial_t e_1 - c^2 \nabla_1 \times b_1 + \partial_\varepsilon e_0 \} \\
- c_2 \nabla_2 \times b_0 + \varepsilon_0 \chi (i\partial_\varepsilon) e_1 + i\partial_\varepsilon \chi (i\partial_\varepsilon) \partial_1 e_1 + i\partial_\varepsilon \chi (i\partial_\varepsilon) \partial_2 e_0 \\
\partial_\varepsilon \chi (i\partial_\varepsilon) e_0 + i\partial_\varepsilon \chi' (i\partial_\varepsilon) \partial_1 e_0 + i\partial_\varepsilon \chi'' (i\partial_\varepsilon) \partial_2 e_0 \\
- \frac{1}{2} \partial_\varepsilon \chi''' (i\partial_\varepsilon) \partial_1 \partial_2 e_0 + \eta \partial_\varepsilon e_0^2 e_0 \}
\end{align*}$$

$$\nabla_0 \cdot b_2 = -\{ \nabla_1 \cdot b_1 + \nabla_2 \cdot b_0 \}$$

$$\begin{align*}
\nabla_0 \cdot e_2 + \varepsilon_0 \chi (i\partial_\varepsilon) \nabla_0 \cdot e_2 &= -\{ \nabla_1 \cdot e_1 + \nabla_2 \cdot e_0 + \chi (i\partial_\varepsilon) \nabla_1 \cdot e_1 \\
+ i\chi' (i\partial_\varepsilon) \partial_1 \nabla_0 \cdot e_1 + \varepsilon_0 \chi (i\partial_\varepsilon) \nabla_2 \cdot e_0 + i\chi' (i\partial_\varepsilon) \partial_2 \nabla_0 \cdot e_0 \\
+ i\chi'' (i\partial_\varepsilon) \partial_1 \nabla_1 \cdot e_0 - \frac{1}{2} \chi''' (i\partial_\varepsilon) \partial_1 \partial_2 \nabla_0 \cdot e_0 + \eta \nabla_0 \cdot (e_0^2 e_0) \}
\end{align*}$$
We now insert (275) and (300) into (301). This gives us

\[
\begin{align*}
\partial_t b_2 + \nabla_0 \times e_2 &= -\{i(\partial_t, \nabla_1 A_0 \cdot t) u + i \nabla_1 \partial_t A_0 \times q \} \quad (302) \\
+ i \frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot u) \times q + i \frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot q) \times u + k \partial_{t_2} A_0 t \\
+ \omega \nabla_2 A_0 \times q \} e^{i\theta_0} + (*) \\
\end{align*}
\]

\[
\begin{align*}
\partial_t e_2 - c^2 \nabla_0 \times b_2 + \partial_{t_0} \tilde{\chi}(i \partial_{t_0}) \} e_2 &= -\{i F(\omega) \partial_{t_1 t_1} A_0 q \\
+ i G(\omega)(\partial_{t_1} \nabla_1 A_0 \cdot u) q + i G(\omega)(\partial_{t_1} \nabla_1 A_0 \cdot q) u - i c^2 \nabla_1 (\nabla_1 A_0 \cdot t) \times u \\
- c^2 k \nabla_2 A_0 \times t + H(\omega) \partial_{t_2} A_0 q - 3i c^2 |A_0|^2 A_0 \} e^{i\theta_0} + (*) \\
\end{align*}
\]

\[
\begin{align*}
\nabla_0 \cdot b_2 &= -\{i \nabla_1 (\nabla_1 A_0 \cdot t) \cdot u + k \nabla_2 A_0 \cdot t \} e^{i\theta_0} + (*) \\
\end{align*}
\]

\[
\begin{align*}
\nabla_0 \cdot e_2 + \tilde{\chi}(i \partial_{t_0}) \} \nabla_0 \cdot e_2 &= -\{i c^2 \nabla_1 \partial_{t_1} A_0 \cdot q + i c^2 \nabla_1 (\nabla_1 A_0 \cdot u) \cdot q \\
+ i c^2 \nabla_1 (\nabla_1 \cdot q) \cdot u + \omega n^2 \nabla_2 A_0 \cdot q \} e^{i\theta_0} + (*) \\
\end{align*}
\]

where we have defined

\[
\begin{align*}
F(\omega) &= n^2 + 2 \omega \tilde{\chi}'(\omega) + \frac{1}{2} \omega^2 \tilde{\chi}''(\omega) \\
G(\omega) &= \frac{\omega}{k} (n^2 + \omega \tilde{\chi}'(\omega)) \\
H(\omega) &= \omega(n^2 + \omega \tilde{\chi}'(\omega)) \\
\end{align*}
\]

Like for the order \( \varepsilon \) equations, we will look for bounded solutions of the form

\[
\begin{align*}
e_2 &= a e^{i\theta_0} + (*) \\
b_2 &= b e^{i\theta_0} + (*) \\
\end{align*}
\]

Inserting (304) into (302) we get the following linear system of equations for the
constant vectors $a$ and $b$.

$$-i\omega b + ik u \times a = -\{i(\partial_{t_1} \nabla_1 A_0 \cdot t)u + i\nabla_1 \partial_{t_1} A_0 \times q\} \quad (305)$$

$$+ i\frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot u) \times q + i\frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot q) \times u + k\partial_{t_2} A_0 t$$

$$+ \omega \nabla_2 A_0 \times q\}$$

$$-i\omega n^2(\omega)a - ic^2k u \times b = -\{iF(\omega)\partial_{t_1} A_0 q$$

$$+ iG(\omega)\partial_{t_1} A_0 \cdot u)q + iG(\omega)\partial_{t_1} A_0 \cdot q)u$$

$$-ic^2\nabla_1 (\nabla_1 A_0 \cdot t) \times u - c^2k\nabla_2 A_0 \times t$$

$$+ H(\omega)\partial_{t_2} A_0 q - 3i\eta \omega^4 |A_0|^2 A_0\}$$

$$iku \cdot b = -\{i\nabla_1 (\nabla_1 A_0 \cdot t) \cdot u + k\nabla_2 A_0 \cdot t\} \quad (307)$$

$$i\omega n^2u \cdot a = -\{in^2\nabla_1 \partial_{t_1} A_0 \cdot q + in^2\frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot u) \cdot q$$

$$+ in^2\frac{\omega}{k} \nabla_1 (\nabla_1 \cdot q) \cdot u + \omega n^2\nabla_2 A_0 \cdot q\} \quad (308)$$

We introduce longitudinal and transversal vector components for $a$ and $b$ like before and find from (307) and (308) that

$$a_{||} = (-\frac{1}{k} \nabla_1 \partial_{t_1} A_0 \cdot q - \frac{\omega}{k^2} \nabla_1 (\nabla_1 A_0 \cdot u) \cdot q$$

$$- \frac{\omega}{k^2} \nabla_1 (\nabla_1 A_0 \cdot q) \cdot u + i\frac{\omega}{k} \nabla_2 A_0 \cdot q)u \quad (309)$$

$$b_{||} = (i\nabla_2 A_0 \cdot t - \frac{1}{k} \nabla_1 (\nabla_1 \cdot t) \cdot u)u \quad (310)$$

The longitudinal part og (305) is

$$u \cdot b = \frac{1}{\omega} \{\partial_{t_1} \nabla_1 A_0 \cdot t - \nabla_1 \partial_{t_1} A_0 \cdot t - \frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot u) \cdot t + i\omega \nabla_2 A_0 \cdot t\} \quad (312)$$

in order for (312) to be consistent with (311) we find that the following solvability condition must hold

$$\partial_{t_1} \nabla_1 A_0 \cdot t = \nabla_1 \partial_{t_1} A_0 \cdot t \quad (313)$$

The longitudinal part of (306) is

$$u \cdot a = \frac{1}{\omega n^2} \{G(\omega)\partial_{t_1} \nabla_1 A_0 \cdot q + ic^2k\nabla_2 A_0 \cdot q\} \quad (314)$$

in order for (314) to be consistent with (309) we find, after a little algebra, that the solvability condition

$$\frac{\omega}{k} n^2(\omega) \nabla_1 \partial_{t_1} A_0 \cdot q + G(\omega)\partial_{t_1} \nabla_1 A_0 \cdot q =$$

$$-c^2\nabla_1 (\nabla_1 A_0 \cdot q) \cdot u - c^2\nabla_1 (\nabla_1 A_0 \cdot u) \cdot q \quad (315)$$
must hold. The transverse parts of \((305)\) and \((306)\) are

\[
-\omega b + iku \times a = -\{i \nabla_1 \partial_{t_1} A_0 \cdot u + i \frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot u) \cdot u \tag{316}
\]

\[
-\frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot q) \cdot q + k \delta t_2 A_0 + \omega \nabla_2 A_0 \cdot u \}
\]

\[
-\{i \frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot q) \cdot t \}
\]

The matrix for this linear system is the same as for the order \(\varepsilon\) case \((290)\), so that the two solvability conditions are determined, through the Fredholm Alternative, by the vectors \((293)\). The solvability condition corresponding to the first of the vectors in \((293)\) is

\[
(-c^2k)(-i \frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot q) \cdot t) + \omega(-ic^2 \nabla_1 (\nabla_1 A_0 \cdot t) \cdot q) = 0
\]

\[
\nabla_1 (\nabla_1 A_0 \cdot q) \cdot t = \nabla_1 (\nabla_1 \cdot t) \cdot q \tag{317}
\]

and the solvability condition corresponding to the second vector in \((293)\) is

\[
c^2k(-i \nabla_1 \partial_{t_1} A_0 \cdot u + i \frac{\omega}{k} \nabla_1 (\nabla_1 A_0 \cdot u) \cdot u
\]

\[
-k \nabla_1 (\nabla_1 A_0 \cdot q) \cdot q + k \partial_{t_2} A_0 + \omega \nabla_2 A_0 \cdot u \} + \omega(-\{i F(\omega) \partial_{t_1 t_1} A_0
\]

\[
+i G(\omega) \partial_{t_1} \nabla_1 A_0 \cdot u - ic^2 \nabla_1 (\nabla_1 A_0 \cdot t) \cdot t + c^2k \nabla_2 A_0 \cdot u
\]

\[
+H(\omega) \partial_{t_2} A_0 - 3\eta \omega^4 |A_0|^2 A_0 \} q = 0
\]

\[
\nabla_1 (\nabla_1 A_0 \cdot q) \cdot q + \nabla_1 (\nabla_1 A_0 \cdot t) \cdot t - \nabla_1 (\nabla_1 A_0 \cdot u) \cdot u
\]

\[
+i \alpha \partial_{t_1 t_1} A_0 - i \gamma |A_0|^2 A_0 = 0 \tag{318}
\]

where we have defined

\[
\alpha = \frac{\omega' F(\omega)}{2c^2 k} \tag{319}
\]

\[
\beta = \frac{\omega'}{2k} \tag{320}
\]

\[
\gamma = \frac{3\eta \omega^4}{2c^2 k} \tag{321}
\]

\[
\delta_1 = \frac{\omega'}{2\omega} \tag{322}
\]

\[
\delta_2 = \frac{\omega' G(\omega)}{2c^2 k} \tag{323}
\]

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We have now found all solvability conditions. These are (313),(315),(317) and (318). We define an amplitude $A(x,t)$ by

$$A(x,t) = A_0(x_1,t_1,\ldots)|_{x_j = \varepsilon^j x, t_j = \varepsilon^j t}$$

and derive the amplitude equations from the solvability conditions in the usual way. This gives us the following system

$$\partial_t \nabla A \cdot t = \nabla \partial_t A \cdot t$$  \hspace{1cm} (321)$$

$$\frac{\omega}{k} n^2(\omega) \nabla \partial_t A \cdot q + G(\omega) \nabla \nabla A \cdot q = -c^2 \nabla (\nabla A \cdot q) \cdot u - c^2 \nabla (\nabla A \cdot u) \cdot q$$ \hspace{1cm} (322)$$

$$\nabla (\nabla A_0 \cdot q) \cdot t = \nabla (\nabla A \cdot t) \cdot q$$ \hspace{1cm} (323)$$

$$\partial_t A + v_g \cdot \nabla A + i \delta_1 \nabla \partial_t A \cdot u + i \delta_2 \partial_t \nabla A \cdot u$$

$$-i \beta (\nabla (\nabla A \cdot q) \cdot q + \nabla (\nabla A \cdot t) \cdot t - \nabla (\nabla A \cdot u) \cdot u)$$

$$+ io \partial_{tt} A - i \gamma |A|^2 A = 0$$  \hspace{1cm} (324)$$

where we as usual have set the formal perturbation parameter equal to 1. Equations (321) and (323) are automatically satisfied since $A(x,t)$ is a smooth function of space and time. We know that only amplitudes such that

$$\partial_t A \sim -v_g \cdot \nabla A = \omega' \nabla u$$ \hspace{1cm} (325)$$

can be allowed as solutions. This is assumed by the multiple scale method. If we insert (325) into (322), assume smoothness and use the dispersion relation, we find that (322) is automatically satisfied. The only remaining equation is then (324) and if we insert the approximation (325) for the derivatives with respect to time in the second and third term of (324) we get, using the dispersion relation, that (324) simplify into

$$\partial_t A + v_g \cdot \nabla A - i \beta \nabla^2 A + io \partial_{tt} A - i \gamma |A|^2 A = 0$$ \hspace{1cm} (326)$$

Here we have also used the fact that

$$qq + tt + uu = I$$

The amplitude $A$ determines the electric and magnetic fields through the identities

$$\mathbf{E}(x,t) \approx \{(\omega A + i(\frac{\omega}{k} - \omega')\mathbf{u} \cdot \nabla A)\mathbf{q}$$

$$+ i(\frac{\omega}{k} \mathbf{q} \cdot \nabla A)\mathbf{u}\} e^{i(kx - \omega t)} + (*)$$

$$\mathbf{B}(x,t) \approx \{k \mathbf{A} t + i(\mathbf{t} \cdot \nabla A)\mathbf{u}\} e^{i(kx - \omega t)} + (*)$$
The equations (326) and (327) are the key elements in a fast numerical scheme for linearly polarized wave packet solutions to Maxwell’s equations. Wave packets of circular polarization or arbitrary polarization can be treated in an entirely similar manner as can sums of different polarized wave packets.

We of course recognize the amplitude equation (326) as the 3D nonlinear Schrödinger equation including group velocity dispersion. As we have seen before, an equation like this can be solved as an ordinary initial value problem if we first use (325) to make the term containing a second derivative with respect to time into one containing only space derivatives.

The derivation of the nonlinear Schrödinger equation for linearly polarized wave packets I have given in this section is certainly not the fastest and simplest way this can be done. The main aim in this section was to illustrate how to apply the multiple scale method for vector PDEs in general, not to do it in the most effective way possible for the particular case of linearly polarized electromagnetic wave packets.

All the essential elements we need in order to apply the method of multiple scales to problems in optics and laser physics, and other areas of science too, are at this point known. There are no new tricks to learn. Using the approach described in these lecture notes, amplitude equations can be derived for most situations of interest. Applying the method is mechanical, but for realistic systems there can easily be a large amount of algebra involved. This is unavoidable, solving nonlinear partial differential equations, even approximately, is hard.

In these lecture notes we have focused on applications of the multiple scale method for time-propagation problems. The method was originally developed for these kind of problems and the mechanics of the method is most transparent for such problems. However the method is by no means limited to time propagation problems.

Many pulse propagation schemes are most naturally formulated as a boundary value problem where the propagation variable is a space variable. A very general scheme of this type is the well known UPPE scheme. More details on how the multiple scale method is applied for these kind of schemes can be found in [4], [7] and [1].

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