SOME RESULTS ON SECOND-ORDER ELLIPTIC OPERATORS WITH POLYNOMIALLY GROWING COEFFICIENTS IN $L^p$-SPACES

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Abstract. In this paper we study the second order elliptic operator of the form

$$A := (1 + |x|^{\alpha})\Delta + b|x|^{\alpha - 2}x \cdot \nabla - c|x|^{\alpha - 2} - |x|^\beta, \quad x \in \mathbb{R}^N,$$

where $\alpha \in [0, 2)$, $\beta > 0$, and $b, c \in \mathbb{R}$ are allowed to assume any value. We prove that, for any $1 < p < \frac{N - \alpha}{2 - \alpha}$, the closure of $(A, C^\infty_c(\mathbb{R}^N))$ generates an analytic $C_0$-semigroup in $L^p(\mathbb{R}^N)$.

In the limit case, that is, $p = \frac{N - \alpha}{2 - \alpha}$ we also require an additional condition

$$\left(\frac{N}{p} - 2 + \alpha\right)\left(\frac{N}{p'} - \alpha + b\right) + c > 0.$$

Moreover, we prove that the maximal realization of the operator $A$ generates a positive bounded analytic semigroup in $L^p(\mathbb{R}^N)$ for $p > \frac{N - \alpha}{2}$.

1. Introduction

Second order elliptic operators with bounded coefficients have been widely studied in the literature and nowadays they are well understood. In recent years there have been substantial developments in the theory of second order elliptic operators with unbounded coefficients. The latter arise as a model in many field of science, especially in stochastic analysis and mathematical finance, where stochastic models lead to equations with unbounded coefficients, e.g., the well known Black-Scholes equation. The developments regarding operators with unbounded coefficients are well documented see, for example, [4], [2], [5], [6], [7], [9], [10], [13], [14], [15], [16], [18], [19], [20], [21], [22], [3] and the references therein.

The present paper is devoted to the study of the following elliptic operator

$$Au(x) = (1 + |x|^{\alpha})\Delta u(x) + b|x|^{\alpha - 2}x \cdot \nabla u(x) - c|x|^{\alpha - 2}u(x) - |x|^\beta u(x), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in [0, 2)$, $\beta > 0$ and $b, c$ are real numbers.

The case $\alpha > 2$ has been treated in [1]. In particular, the authors studied the complete operator $A$ under the assumptions $\alpha > 2$, $\beta > \alpha - 2$, $b \in \mathbb{R}$ and $c = 0$. The main result is that the semigroup generated by the realization $A_p$ of the elliptic operator $A$ in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$ with domain

$$D_p(A) = \{u \in W^{2,p}(\mathbb{R}^N) : (1 + |x|^{\alpha})|D^2u|, (1 + |x|^{\alpha - 1})|\nabla u|, |x|^\beta u \in L^p(\mathbb{R}^N)\},$$

is strongly continuous and analytic. See also [4] for the case $b = c = 0$. 

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The operator $A$ without potential term $c|x|^{\alpha-2} + |x|^\beta$ is studied in [10] for $\alpha \in [0, 2]$ and in [21] for $\alpha > 2$. After a modification of the drift term $b|x|^{\alpha-2}x \cdot \nabla$ near the origin for $\alpha < 2$, it is proven in [10] that the $L^p$-realization of $A$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$. Whereas for $\alpha > 2$, in [21], the generation of an analytic semigroup in $L^p(\mathbb{R}^N)$ is stated for $p > \frac{N-2}{2}$.

The Schrödinger-type operators $(1 + |x|^\alpha)\Delta - |x|^\beta$ in $L^p(\mathbb{R}^N)$ have been considered in [15] and [4]. In [15] the generation of an analytic semigroup in $L^p(\mathbb{R}^N)$ for $\alpha \in [0, 2]$ and $\beta > 2$ is obtained. In [4] the same result is obtained in the case $\alpha > 2$ and $\beta > \alpha - 2$.

The operator with only diffusion term was studied in [18] and [22]. In [18] the authors proved that for $N \geq 3$, $p > \frac{N}{N-2}$ and $\alpha > 2$, the maximal realization of the operator $(1 + |x|^\alpha)\Delta$ in $L^p(\mathbb{R}^N)$ generates an analytic semigroup which is contractive if and only if $p \geq \frac{N+\alpha-2}{N-2}$. The case $\alpha \leq 2$ has been treated in [22] and in this case no constraints on $p$ appear.

In [16] the authors showed that the operator $|x|^\alpha \Delta + b|x|^{\alpha-2}x \cdot \nabla - c|x|^{\alpha-2} - |x|^\beta$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ if $s_1 + \min\{0, 2 - \alpha\} < \frac{N}{p} < s_2 + \max\{0, 2 - \alpha\}$, where $s_i$ are the roots of the equation $c + s(N - 2 + b - s) = 0$.

We note here that methods and results are different for $\alpha \leq 2$ or $\alpha > 2$. In particular, the generation results for $\alpha > 2$ depend upon the dimension $N$; except for the case when the potential term has the form $|x|^\beta$ with $\beta > \alpha - 2$.

Our aim in this paper is to complete the picture considering the complete operator

$$A = (1 + |x|^\alpha)\Delta + b|x|^{\alpha-2}x \cdot \nabla - c|x|^{\alpha-2} - |x|^\beta$$

in the case $\alpha < 2$. We note that with the assumptions on $\beta > 0$ and $\alpha \in [0, 2]$ the operator $A$ has unbounded coefficients at infinity and local singularities in the drift and potential term. We will first apply perturbation theorems for linear m-accretive operators in order to show that the operator $\tilde{A} = (1 + |x|^\alpha)\Delta - c|x|^{\alpha-2} - |x|^\beta$ with $b = 0$ and under the assumption $\beta > \alpha - 2$ is quasi maximal accretive on $L^p(\mathbb{R}^N)$ for $1 < p < \frac{N}{2-\alpha}$ and for every $c \in \mathbb{R}$, moreover $C^\infty_c(\mathbb{R}^N)$ is a core for it. On the other hand, if $p = \frac{N-\alpha}{2-\alpha}$, we require an additional condition, that is, $\left(\frac{N}{p} - 2 + \alpha\right)\left(\frac{N}{p} - \alpha\right) + c > 0$. We then use the results obtained for the Schrödinger operator $\tilde{A}$ to prove the maximality of the complete operator $A$ and hence, we deduce that the closure of $A$ generates an analytic $C_0$-semigroup.

The paper is organized as follows. In Section 2 we discuss the sectoriality of $A$ in the sense of [12, Definitions 1.5.8]. Section 3 is concerned to the quasi m-accretiveness of the sum of two linear m-accretive operators $A_0 = (1 + |x|^\alpha)\Delta - |x|^\beta$ and the multiplication operator $W = |x|^{\alpha-2}$ in $L^p$-spaces; the proof is based on perturbation argument established by N. Okazawa [23]. The main task in Section 4 is to prove the essential maximality of the operator $A$ in order to get generation results. Finally, Section 5 deals with the m-accretivity of the operator $A_{p, \text{max}}$, the maximal realization of $A$. The generation results for $A_{p, \text{max}}$ in $L^p(\mathbb{R}^N)$ for $p > \frac{N-\alpha}{N-2}$ are deduced by duality.

**Notations.** We fix some notations that we use throughout this paper. $\langle \cdot, \cdot \rangle$ denotes the pairing between $X$ and $X^*$ (the dual space of $X$). For a closed linear operator $A$ on a Banach space $X$ we denote by $D(A)$, $R(A)$, the domain and the range of $A$, respectively. We denote by $A^*$ the adjoint of an operator $A$. In general we use standard notations for function spaces,
we denote by $L^p(\mathbb{R}^N)$ and $W^{2,p}(\mathbb{R}^N)$ the standard $L^p$ and Sobolev spaces, respectively. We will denote the space of smooth and compactly supported functions on $\mathbb{R}^N$ by $C^k_c(\mathbb{R}^N)$, $k \in \mathbb{N}$.

If $f$ is smooth enough we set
\[
|\nabla f(x)|^2 = \sum_{i=1}^N |D_i f(x)|^2, \quad |D^2 f(x)|^2 = \sum_{i,j=1}^N |D_{ij} f(x)|^2.
\]
For any $x_0 \in \mathbb{R}^N$ and any $r > 0$ we denote by $B(x_0, r) \subset \mathbb{R}^N$ the open ball, centered at $x_0$ with radius $r$. We simply write $B(r)$ when $x_0 = 0$. Finally, by $x \cdot y$ we denote the Euclidean scalar product of the vectors $x, y \in \mathbb{R}^N$.

Other notations will be introduced according to necessity.

2. Sectoriality of $A$

In this section we discuss the sectoriality of the operator $A$. To this purpose a $L^p-$generalization of the classical Hardy inequality is needed. If $p \geq 2$ one has
\[
2 \left( \frac{N-2}{p} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} dx
\]
for every $u \in C^1_c(\mathbb{R}^N)$. As regards the case $1 < p < 2$ the following inequality holds
\[
2 \left( \frac{N-2}{p} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} dx - 2p^2 (N-2) \sigma_{N-1} R^{N-2} \delta_{p/2} \leq \int_{\mathbb{R}^N} |\nabla u|^2 (u^2 + \delta)^{p/2} dx
\]
for every $u \in C^1_c(B(R))$ (see [23, Lemma 2.2, Lemma 2.3]). If $u \in W^{2,p}(\mathbb{R}^N)$ then by [17, Corollary 2.2] we have that the function $|u|^{p-2} |\nabla u|^2$ is in $L^1(\mathbb{R}^N)$ (actually one should consider the function $|u|^{p-2} |\nabla u|^2 \chi_{\{u \neq 0\}}$); hence we can take the limit as $\delta \to 0$ in (2.2) and use (2.1) for functions in $W^{2,p}(\mathbb{R}^N)$.

The following result shows that the operator $A$ is quasi-sectorial.

Proposition 2.1. Let $p > 1$. Then for every $0 \leq \alpha < 2$, $\beta > 0$ and $b, c \in \mathbb{R}$, there exist $\omega > 0$ and $l_\alpha > 0$ such that for every $u \in C^\infty_c(\mathbb{R}^N)$ we have
\[
|\text{Im}(\hat{A} u, |u|^{p-2} u)| \leq l_\alpha^{-1} - |\text{Re}(\hat{A} u, |u|^{p-2} u)|
\]
where $\hat{A} = A - \omega$.

Proof. Let $\hat{A} = A - \omega$, $u \in C^\infty_c(\mathbb{R}^N)$ and set $u^* = |u|^{p-2}$. Multiplying $\hat{A} u$ by $u^*$ and integrating over $\mathbb{R}^N$, one obtains
\[
\int_{\mathbb{R}^N} \hat{A} u u^* dx = \int_{\mathbb{R}^N} a(x)|u|^{p-4} |\text{Re}(\bar{u}\nabla u)|^2 dx - \int_{\mathbb{R}^N} a(x)|u|^{p-4} |\text{Im}(\bar{u}\nabla u)|^2 dx
\]
\[
- \int_{\mathbb{R}^N} \hat{u}|u|^{p-2} \nabla a(x) \nabla u dx - \int_{\mathbb{R}^N} \hat{a}(x)|u|^{p-4} \bar{u}\nabla u|u|^{p-2} \nabla u dx
\]
\[
+ b \int_{\mathbb{R}^N} \hat{u}|u|^{p-2} |x|^{\beta N - 1} |x| \nabla u dx - \int_{\mathbb{R}^N} |x|^{\beta} + c |x|^{\alpha-2} + \omega |u|^p dx
\]
where $a(x) = 1 + |x|^\alpha$. We note here that the integration by part in the singular case $1 < p < 2$ is allowed thanks to [17].
By taking the real part of the left and the right hand side, we have

\[
\text{Re} \left( \int_{\mathbb{R}^N} \hat{A} u u^* \, dx \right) = -(p-1) \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx \\
- \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Im}(\bar{u} \nabla u)|^2 \, dx - \int_{\mathbb{R}^N} |u|^{p-2} \nabla a(x) \text{Re}(\bar{u} \nabla u) \, dx \\
+ b \int_{\mathbb{R}^N} |u|^{p-2} |x|^{\alpha-2} \text{Re}(\bar{u} \nabla u) \, dx - \int_{\mathbb{R}^N} \left( |x|^\beta + c |x|^{\alpha-2} + \omega \right) |u|^p \, dx \\
= -(p-1) \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx - \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Im}(\bar{u} \nabla u)|^2 \, dx \\
+ \int_{\mathbb{R}^N} \left( \frac{(\alpha-b)(N-2+\alpha)}{p} |x|^{\alpha-2} - |x|^\beta - \omega \right) |u|^p \, dx.
\]

Let \( p \geq 2 \), using the identity \( \text{Re}(u \nabla u) = |u| |\nabla u| \) and the generalized Hardy inequality (2.1) for \(|u|\), we have

\[
- (p-1) \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx \\
\leq -(p-1) \frac{1}{2} \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx - \frac{(p-1)}{2} \int_{\mathbb{R}^N} |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx \\
\leq - \frac{(p-1)}{2} \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx + \frac{(N-2)}{2} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} \, dx.
\]

Therefore,

\[
\text{Re} \left( \int_{\mathbb{R}^N} \hat{A} u u^* \, dx \right) \\
\leq - \frac{(p-1)}{2} \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx - \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Im}(\bar{u} \nabla u)|^2 \, dx \\
+ \int_{\mathbb{R}^N} \left( -c_1 \frac{1}{|x|^2} + \left( \frac{(\alpha-b)(N-2+\alpha)}{p} - c \right) |x|^{\alpha-2} - |x|^\beta - \omega \right) |u|^p \, dx
\]

where \( c_1 = \frac{p-1}{2} \left( \frac{N-2}{p} \right)^2 \). We can choose \( \omega > 0 \) such that

\[
- \frac{c_1}{|x|^2} + \left( \frac{(\alpha-b)(N-2+\alpha)}{p} - c \right) |x|^{\alpha-2} - |x|^\beta - \omega \leq - \left( \frac{(\alpha-b)(N-2+\alpha)}{p} - c \right) |x|^{\alpha-2}.
\]

Furthermore,

\[
- \text{Re} \left( \int_{\mathbb{R}^N} \hat{A} u u^* \, dx \right) \geq \frac{p-1}{2} \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx \\
+ \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Im}(\bar{u} \nabla u)|^2 \, dx + c_2 \int_{\mathbb{R}^N} |u|^p |x|^{\alpha-2} \, dx \\
= \frac{(p-1)}{2} B^2 + C^2 + c_2 D^2,
\]
where \( c_2 = \left| \frac{(\alpha - b)(N-2+\alpha)}{p} - c \right| \) is a positive constant and we have set
\[
B^2 = \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Re}(\bar{u} \nabla u)|^2 \, dx
\]
\[
C^2 = \int_{\mathbb{R}^N} a(x) |u|^{p-4} |\text{Im}(\bar{u} \nabla u)|^2 \, dx
\]
\[
D^2 = \int_{\mathbb{R}^N} |x|^{\alpha-2} |u|^p \, dx.
\]

Taking now the imaginary part of the left and the right hand side in (2.4), we obtain
\[
\text{Im} \left( \int_{\mathbb{R}^N} \hat{A} u u^* \, dx \right) = - (p-2) \int_{\mathbb{R}^N} a(x) |u|^{p-4} \text{Im}(\bar{u} \nabla u) \text{Re}(\bar{u} \nabla u) \, dx
\]
\[
- \int_{\mathbb{R}^N} |u|^{p-2} \nabla a(x) \text{Im}(\bar{u} \nabla u) \, dx + b \int_{\mathbb{R}^N} |u|^{p-2} |x|^{\alpha-1} \frac{x}{|x|} \text{Im}(\bar{u} \nabla u) \, dx.
\]

Moreover,
\[
\left| \text{Im} \left( \int_{\mathbb{R}^N} \hat{A} u u^* \, dx \right) \right| \leq |p-2| \left( \int_{\mathbb{R}^N} |u|^{p-4} a(x) |\text{Re}(\bar{u} \nabla u)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^{p-4} a(x) |\text{Im}(\bar{u} \nabla u)|^2 \, dx \right)^{\frac{1}{2}}
\]
\[
+ |\alpha - b| \left( \int_{\mathbb{R}^N} |u|^{p-4} |x|^\alpha |\text{Im}(\bar{u} \nabla u)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u|^p |x|^{\alpha-2} \, dx \right)^{\frac{1}{2}}
\]
\[
\leq |p-2| |BC| + |\alpha - b| |CD|.
\]

Hence, it is possible to determine a positive constant \( l_\alpha \), such that
\[
\frac{(p-1)}{2} B^2 + C^2 + c_2 D^2 \geq l_\alpha \{ |p-2| |BC| + |\alpha - b| |CD| \}.
\]

As a consequence of the above estimates, we conclude the sectorial estimate (2.3), that is
\[
\left| \text{Im} \left( \int_{\mathbb{R}^N} \hat{A} u u^* \, dx \right) \right| \leq l_\alpha^{-1} \left[ - \text{Re} \left( \int_{\mathbb{R}^N} \hat{A} u u^* \, dx \right) \right].
\]

\[\Box\]

3. The Operator \( A \) for \( b = 0 \)

In this section we consider the operator \( A_0 = (1 + |x|^{\alpha}) \Delta - |x|^{\beta} \) plus the potential \( W = |x|^{\alpha-2} \) and show that \( -A_0 + cW \) defined on a suitable domain \( D(A_0) \) is maximal accretive on \( L^p(\mathbb{R}^N) \) for \( 1 < p < \frac{\alpha}{\alpha - 2} \) and for every \( c \in \mathbb{R} \). Moreover \( C^\infty_c(\mathbb{R}^N) \) is a core for \( A_0 - cW \).

In [15] L. Lorenzi and A. Rhandi proved that for \( 0 \leq \alpha \leq 2 \) and \( \beta \geq 0 \) the realization of \( A_0 \) in \( L^p(\mathbb{R}^N) \) with domain \( D(A_0) = \{ u \in W^{2,p}(\mathbb{R}^N) : (1 + |x|^{\alpha}) |D^2 u|, (1 + |x|^{\alpha}) \nabla u, |x|^{\beta} \in L^p(\mathbb{R}^N) \} \) generates a strongly continuous analytic semigroup. Moreover \( D(A_0) \) coincides with the maximal domain of \( A_0 \) and \( C^\infty_c(\mathbb{R}^N) \) is a core for \( (A_0, D(A_0)) \).

In order to treat \( -A_0 + cW \) we use the following perturbation theorem due to N. Okazawa, see [23, Theorem 1.7].
Theorem 3.1. Let $A$ and $B$ be linear $m$-accretive operators in $L^p(\mathbb{R}^N)$. Let $D$ be a core for $A$ and let $\{B_\varepsilon\}$ be the Yosida approximation of $B$. Assume that

(i) there are constants $a_1, a_2 \geq 0$ and $k_1 > 0$ such that for all $u \in D$ and $\varepsilon > 0$

$$\text{Re}(Au, F(B_\varepsilon u)) \geq k_1 \|B_\varepsilon u\|_p^2 - a_2 \|u\|_p^2 - a_1 \|B_\varepsilon u\|_p \|u\|_p$$

where $F$ is the duality map on $L^p(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$. Then $B$ is $A$-bounded with $A$-bound $k_1^{-1}$:

$$\|Bu\|_p \leq k_1^{-1} \|Au\|_p + k_0 \|u\|, \quad u \in D(A) \subset D(B).$$

Assume further that

(ii) $\text{Re}(u, F(B_\varepsilon u)) \geq 0$, for all $u \in L^p(\mathbb{R}^N)$ and $\varepsilon > 0$;

(iii) there is $k_2 > 0$ such that $A - k_2 B$ is accretive.

Set $k = \min\{k_1, k_2\}$. If $t > -k$ then $A + tB$ with domain $D(A + tB) = D(A)$ is $m$-accretive and any core of $A$ is also a core for $A + tB$. Furthermore, $A - kB$ is essentially $m$-accretive on $D(A)$.

In our situation the Yosida approximation of $W$ is given by $W_\varepsilon(x) = \frac{W}{1 + \varepsilon W} = \frac{1}{|x|^{N + \alpha}}$ and the duality map is given by $F(W_\varepsilon) = W_\varepsilon^p u|u|^{p-2} W_\varepsilon u|_p^{2-p}$.

We will use a $L^p$-generalization of the classical Hardy inequality. If $p > 1$ the following inequality holds

$$\left(\frac{N - 2 + \delta}{p}\right)^2 \int_{\mathbb{R}^N} |v|^p |x|^2 dx \leq \int_{\mathbb{R}^N} |\nabla v|^2 |v|^{p-2} |x|^\delta dx \quad (3.1)$$

defined for every $v \in C^1_c(\mathbb{R}^N)$ and $\delta \in \mathbb{R}$.

Lemma 3.2. Let $p > 1$. Assume that $N > 2$, $0 \leq \alpha < 2$, $\beta > \alpha - 2$.

(i) If $p < \frac{N - \alpha}{2 - \alpha}$, then for every $u \in C^\infty_c(\mathbb{R}^N)$ and for every $k \in \mathbb{R}$ there exists $a_1 \geq 0$ such that

$$\text{Re}(-Au, W_\varepsilon^{p-1} u |u|^{p-2}) \geq k_1 \|W_\varepsilon u\|_p^p - a_1 \|W_\varepsilon u\|_p^{p-1} \|u\|_p.$$  \hspace{1cm} (3.2)

(ii) If $p = \frac{N - \alpha}{2 - \alpha}$, then for every $u \in C^\infty_c(\mathbb{R}^N)$ we have

$$\text{Re}(-Au, W_\varepsilon^{p-1} u |u|^{p-2}) \geq k_1 \|W_\varepsilon u\|_p^p - a_1 \|W_\varepsilon u\|_p^{p-1} \|u\|_p$$ \hspace{1cm} (3.3)

where $k_1 = k_1(b, c) = \left(\alpha - 2 + \frac{N}{p}\right) (-\alpha + b + \frac{N}{p}) + c$ and $a_1 \geq 0$.

**Proof.** Let $u \in C^\infty_c(\mathbb{R}^N)$. Observe that the terms $|x|^{\alpha-2}$ and $|x|^{\alpha-2} x \cdot \nabla u$ belong to $L^p(\mathbb{R}^N)$ thanks to the condition $p \leq \frac{N}{2 - \alpha}$. Multiplying $Au$ by $W_\varepsilon^{p-1} u |u|^{p-2}$ and integrating over $\mathbb{R}^N$ we obtain the following. We note here that the integration by part in the singular case $1 < p < 2$ is allowed thanks to [17].

$$- \int_{\mathbb{R}^N} Au W_\varepsilon^{p-1} u |u|^{p-2} dx = - \int_{\mathbb{R}^N} a(x) \Delta u W_\varepsilon^{p-1} u |u|^{p-2} + b |x|^{\alpha-2} x \cdot \nabla u W_\varepsilon^{p-1} u |u|^{p-2} dx$$

$$+ \int_{\mathbb{R}^N} c |x|^{\alpha-2} W_\varepsilon^{p-1} u |u|^p + |x|^3 W_\varepsilon^{p-1} u |u|^p dx,$$
where \(a(x) = (1 + |x|^\alpha)\). Then
\[
- \int_{\mathbb{R}^N} a(x) \Delta u W_{\epsilon}^{p-1}|u|^{p-2} dx = \int_{\mathbb{R}^N} \nabla u \cdot \nabla (a(x) W_{\epsilon}^{p-1}|u|^{p-2}) dx
\]
\[
= \int_{\mathbb{R}^N} \nabla u \cdot \nabla (\pi|u|^{p-2}) a(x) W_{\epsilon}^{p-1} + a(x) \nabla u \cdot \nabla W_{\epsilon}^{p-1}|u|^{p-2} + |x|^\alpha \nabla u W_{\epsilon}^{p-1}|u|^{p-2}.
\]

Recalling that
\[
\text{Re} \left( \nabla u \cdot \nabla (\pi|u|^{p-2}) \right) = (p - 1)|u|^{p-4} \text{Re} (\pi \nabla u)^2 + |u|^{p-4} \text{Im} (\pi \nabla u)^2,
\]
and
\[
\text{Re} (\pi \nabla u) = |u| \nabla |u|
\]
we have
\[
\text{Re} \left( - \int_{\mathbb{R}^N} a(x) \Delta u W_{\epsilon}^{p-1}|u|^{p-2} + b|x|^\alpha \nabla u W_{\epsilon}^{p-1}|u|^{p-2} dx \right)
\]
\[
= (p - 1) \int_{\mathbb{R}^N} a(x) W_{\epsilon}^{p-1}|u|^{p-4} \text{Re} (\pi \nabla u)^2 dx
\]
\[
+ \int_{\mathbb{R}^N} a(x) W_{\epsilon}^{p-1}|u|^{p-4} \text{Im} (\pi \nabla u)^2 dx
\]
\[
+ \frac{1}{p} \int_{\mathbb{R}^N} a(x) \nabla W_{\epsilon}^{p-1} \cdot \nabla |u|^p + (\alpha - b) W_{\epsilon}^{p-1}|x|^\alpha \nabla |u|^p dx.
\]

Then,
\[
\text{Re} \left( - \int_{\mathbb{R}^N} Au W_{\epsilon}^{p-1}|u|^{p-2} dx \right)
\]
\[
\geq (p - 1) \int_{\mathbb{R}^N} (1 + |x|^\alpha) W_{\epsilon}^{p-1}|u|^{p-2} |\nabla |u|^2 dx
\]
\[
+ \frac{1}{p} \text{div} \left[ a(x) \nabla W_{\epsilon}^{p-1} + (\alpha - b) W_{\epsilon}^{p-1}|x|^\alpha \nabla |u|^p + |x|^\beta W_{\epsilon}^{p-1} |u|^p \right] dx
\]
\[
= (p - 1) \int_{\mathbb{R}^N} (1 + |x|^\alpha) W_{\epsilon}^{p-1}|u|^{p-2} |\nabla |u|^2 dx
\]
\[
- \frac{1}{p} \int_{\mathbb{R}^N} \left[ \nabla a(x) \cdot \nabla W_{\epsilon}^{p-1} + a(x) \Delta W_{\epsilon}^{p-1} + (\alpha - b)|x|^\alpha (\alpha - 2 + N) W_{\epsilon}^{p-1} + x \cdot \nabla W_{\epsilon}^{p-1} \right] |u|^p dx
\]
\[
+ \int_{\mathbb{R}^N} c|x|^\alpha W_{\epsilon}^{p-1}|u|^p + |x|^\beta W_{\epsilon}^{p-1} |u|^p \right] dx
\]
\[
= (p - 1) \int_{\mathbb{R}^N} (1 + |x|^\alpha) W_{\epsilon}^{p-1}|u|^{p-2} |\nabla |u|^2 dx
\]
\[
- \frac{1}{p} \int_{\mathbb{R}^N} \left[ (2\alpha - b)|x|^\alpha \nabla W_{\epsilon}^{p-1} + (1 + |x|^\alpha) \Delta W_{\epsilon}^{p-1} + (\alpha - b)(\alpha - 2 + N) |x|^\alpha W_{\epsilon}^{p-1} \right] |u|^p dx
\]
\[
+ \int_{\mathbb{R}^N} c|x|^\alpha W_{\epsilon}^{p-1}|u|^p + |x|^\beta W_{\epsilon}^{p-1} |u|^p \right] dx.
\]
We first note that
\[ \nabla W_{p-1}^\varepsilon = \gamma W_{p-1}^\varepsilon |x|^{-\alpha} x, \quad \text{and} \]
\[ \Delta W_{p-1}^\varepsilon = |x|^{-\alpha} \gamma (N - \alpha + p(\alpha - 2)W_{p-1}^\varepsilon |x|^{2-\alpha}) W_{p-1}^\varepsilon , \]
with \( \gamma = (\alpha - 2)(p - 1) \), to infer that
\[
\begin{align*}
\text{Re} \left( - \int_{\mathbb{R}^N} A u W_{p-1}^\varepsilon |u|^{p-2} dx \right) \\
\geq (p - 1) \int_{\mathbb{R}^N} (1 + |x|^\alpha) W_{p-1}^\varepsilon |u|^{p-2} |\nabla u|^2 dx \\
- \frac{1}{p} \int_{\mathbb{R}^N} [(2\alpha - b)\gamma + (1 + |x|^\alpha)\gamma |x|^{-\alpha} (N - \alpha + p(\alpha - 2)|x|^{-\alpha + 2} W_{p-1}^\varepsilon ) \\
+ (\alpha - b)(\alpha - 2 + N)|x|^{\alpha - 2} W_{p-1}^\varepsilon |u|^p dx \\
+ \int_{\mathbb{R}^N} c|x|^{\alpha - 2} W_{p-1}^\varepsilon |u|^p + |x|^\beta W_{p-1}^\varepsilon |u|^p dx.
\end{align*}
\] (3.4)

Now, we begin estimating the term \((p - 1) \int_{\mathbb{R}^N} (1 + |x|^\alpha) W_{p-1}^\varepsilon |u|^{p-2} |\nabla u|^2 dx\) by mean of the Hardy inequality (3.1). First, we choose \( v = W_{p-1}^\varepsilon |u| \) and \( \delta = \alpha \).

One has
\[
\left( \frac{\alpha - 2 + N}{p} \right)^2 \int_{\mathbb{R}^N} |x|^{\alpha - 2} W_{p-1}^\varepsilon |u|^p dx \leq \int_{\mathbb{R}^N} |x|^{\alpha} W_{p-1}^\varepsilon |u|^{p-2} |\nabla u|^2 dx \\
+ \frac{\gamma^2}{p^2} \int_{\mathbb{R}^N} |x|^{-\alpha + 2} W_{p+1}^\varepsilon |u|^p dx - \frac{2}{p^2} \int_{\mathbb{R}^N} (|x|^{\alpha} \Delta W_{p-1}^\varepsilon + \alpha |x|^{\alpha - 2} x \cdot \nabla W_{p-1}^\varepsilon ) |u|^p dx
\]
that is
\[
\int_{\mathbb{R}^N} |x|^{\alpha} W_{p-1}^\varepsilon |u|^{p-2} |\nabla u|^2 dx \geq \\
\int_{\mathbb{R}^N} \left[ \left( \frac{\alpha + N - 2}{p} \right)^2 |x|^{\alpha - 2} W_{p-1}^\varepsilon - \frac{\gamma^2}{p^2} |x|^{-\alpha + 2} W_{p-1}^\varepsilon \\
+ \frac{2}{p^2} \gamma (N - \alpha + p(\alpha - 2)|x|^{-\alpha + 2} W_{p-1}^\varepsilon ) + \frac{2}{p^2} \gamma \alpha \right] W_{p-1}^\varepsilon |u|^p dx.
\]

In the same way we choose \( v = W_{p-1}^\varepsilon |u| \) and \( \delta = 0 \) to obtain
\[
\left( \frac{N - 2}{p} \right)^2 \int_{\mathbb{R}^N} |x|^{-2} W_{p-1}^\varepsilon |u|^p dx \leq \int_{\mathbb{R}^N} W_{p-1}^\varepsilon |u|^{p-2} |\nabla u|^2 dx \\
+ \frac{\gamma^2}{p^2} \int_{\mathbb{R}^N} |x|^{-2} W_{p+1}^\varepsilon |u|^p dx - \frac{2}{p^2} \int_{\mathbb{R}^N} \Delta W_{p-1}^\varepsilon |u|^p dx
\] (3.5)
that is
\[
\int_{\mathbb{R}^N} W_{\varepsilon}^{p-1}|u|^{p-2} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^N} |x|^{-\alpha} W_{\varepsilon}^p |u|^p
\]
\[
\left[ \left( \frac{2-N}{p} \right)^2 |x|^{\alpha-2} W_{\varepsilon}^{-1} - \frac{\gamma^2}{p^2} |x|^{-\alpha+2} W_{\varepsilon} + \frac{2}{p^2} \gamma (N-\alpha + p(\alpha - 2)|x|^{-\alpha+2} W_{\varepsilon}) \right] \right] \, dx.
\]

Therefore, putting together (3.4), (3.5) and (3.6), we get
\[
\text{Re} \left( -\int_{\mathbb{R}^N} Au W_{\varepsilon}^{p-1} |u|^{p-2} \, dx \right) \geq \int_{\mathbb{R}^N} W_{\varepsilon}^p |u|^p [\]
\[
- \frac{1}{p} (2\alpha - b) \gamma - \frac{1}{p} \gamma (N-\alpha) + (p-1) \left( \frac{2}{p^2} \gamma (N-\alpha) + \frac{2}{p^2} \gamma \alpha \right)
\]
\[
+ \left( -\frac{1}{p} \gamma p(\alpha - 2) + (p-1) \left( -\frac{\gamma^2}{p^2} + \frac{2}{p^2} \gamma p(\alpha - 2) \right) \right) |x|^{-\alpha+2} W_{\varepsilon}
\]
\[
+ \left( -\frac{1}{p} (\alpha - b)(\alpha + N - 2) + (p-1) \left( \frac{\alpha + N - 2}{p} \right)^2 + c \right) |x|^{-\alpha+2} W_{\varepsilon}^{-1}
\]
\[
+ |x|^{-\alpha} \left( -\frac{1}{p} \gamma (N - \alpha) + (p-1) \frac{2}{p^2} \gamma (N-\alpha) \right)
\]
\[
+ |x|^{-\alpha} \left( -\frac{1}{p} \gamma p(\alpha - 2) + (p-1) \left( -\frac{\gamma^2}{p^2} + \frac{2}{p^2} \gamma p(\alpha - 2) \right) \right) |x|^{-\alpha+2} W_{\varepsilon}
\]
\[
+ |x|^{-\alpha} (p-1) \left( \frac{N - 2}{p} \right)^2 |x|^{-\alpha-2} W_{\varepsilon}^{-1}
\]
\[
+ |x|^{2-\alpha+2} |x|^{-\alpha-2} W_{\varepsilon}^{-1} \right] \, dx
\]
\[
= \int_{\mathbb{R}^N} W_{\varepsilon}^p |u|^p [\]
\[
+ \frac{\gamma}{p^2} (-2\alpha + (N-\alpha)(p-2)) + \frac{\gamma}{p}
\]
\[
- \frac{\gamma}{p^2} (\alpha - 2) |x|^{-\alpha+2} W_{\varepsilon}
\]
\[
+ \left( \frac{\alpha - 2 + N}{p} \left( b + \frac{-\alpha + (p-1)(N-2)}{p} \right) + c \right) |x|^{-\alpha-2} W_{\varepsilon}^{-1}
\]
\[
+ |x|^{-\alpha} \left( \frac{\gamma}{p^2} (N-\alpha)(p-2) \right)
\]
\[
- |x|^{-\alpha} \left( \frac{\gamma^2}{p^2} (\alpha - 2) \right) |x|^{-\alpha+2} W_{\varepsilon}
\]
\[
+ |x|^{-\alpha} (p-1) \left( \frac{N - 2}{p} \right)^2 |x|^{-\alpha-2} W_{\varepsilon}^{-1}
\]
\[
+ |x|^{2-\alpha+2} |x|^{-\alpha-2} W_{\varepsilon}^{-1} \right] \, dx.
\]
Now, taking into account that $|x|^{-\alpha+2}W_\varepsilon \leq 1$, $|x|^\alpha W_\varepsilon^{-1} = 1 + \varepsilon |x|^\alpha$ and that $\gamma \leq 0$, we finally have
\[
\text{Re} \left( -\int_{\mathbb{R}^N} AuW_\varepsilon^{p-1}\overline{u}|u|^{p-2}dx \right) \geq \int_{\mathbb{R}^N} W_\varepsilon^p |u|^p \left[ \frac{\gamma}{p^2} (-2\alpha + (N - \alpha)(p - 2)) + b\gamma - \frac{\gamma}{p^2}(\alpha - 2) + \frac{\alpha - 2 + N}{p} \left( b + \frac{-\alpha + (p - 1)(N - 2)}{p} \right) + c \right. \\
+ |x|^{-\alpha} \left[ \frac{\gamma}{p^2} (N - \alpha)(p - 2) - \frac{\gamma}{p^2} (\alpha - 2) + (p - 1) \left( \frac{N - 2}{p} \right)^2 \right] \\
+ |x|^{\alpha-2} \left[ \frac{\alpha - 2 + N}{p} \left( b + \frac{-\alpha + (p - 1)(N - 2)}{p} \right) + c \right] + |x|^{-\alpha+2} dx \\
= \int_{\mathbb{R}^N} W_\varepsilon^p |u|^p \left[ (\alpha - b + \gamma) \frac{\gamma + \alpha + N - 2}{p} + (p - 1) \left( \frac{\gamma + \alpha + N - 2}{p} \right)^2 + c \right. \\
+ |x|^{-\alpha} \left[ \frac{-\gamma + N - 2}{p} + (p - 1) \left( \frac{\gamma + N - 2}{p} \right)^2 \right] \\
+ |x|^{\alpha-2} \left[ \frac{\alpha - 2 + N}{p} \left( b + \frac{-\alpha + (p - 1)(N - 2)}{p} \right) + c \right] + |x|^{-\alpha+2} dx \\
= \int_{\mathbb{R}^N} W_\varepsilon^p |u|^p \left[ k_1 + |x|^{-\alpha} k_2 + |x|^{\alpha-2} k_3 + |x|^{-\alpha+2} k_4 + |x|^{\alpha+2} \right] dx,
\]
where $k_1, k_2, k_3, k_4$ are explicit constants which are given by
\[
k_1 = -(\alpha - b + \gamma) \frac{\gamma + \alpha + N - 2}{p} + (p - 1) \left( \frac{\gamma + \alpha + N - 2}{p} \right)^2 + c = \left( \alpha - 2 + \frac{N}{p} \right) \left( -\alpha + b + \frac{N}{p} \right) + c,
\]
\[
k_2 = -\gamma \frac{\gamma + N - 2}{p} + (p - 1) \left( \frac{\gamma + N - 2}{p} \right)^2 = \left( \frac{\alpha}{p} - 2 + \frac{N}{p} \right) \left( -\alpha + \frac{N}{p} \right),
\]
\[
k_3 = \frac{\alpha - 2 + N}{p} \left( b + \frac{-\alpha + (p - 1)(N - 2)}{p} \right) + c,
\]
and
\[
k_4 = (p - 1) \left( \frac{N - 2}{p} \right)^2.
\]
If $\gamma + N - 2 > 0$ which in turn implies $p < \frac{N - \alpha}{2 - \alpha}$ (that is our natural condition) one has $k_2 \geq 0$.

Now, it suffices to add and subtract to the right hand side the term $a_1 \int_{\mathbb{R}^N} W_\varepsilon^{p-1}|u|^p dx$, where $a_1 \geq 0$ is a suitable constant to be chosen later. First, using Hölder inequality we obtain
\[
\int_{\mathbb{R}^N} W_\varepsilon^{p-1}|u|^p dx \leq \left( \int_{\mathbb{R}^N} (W_\varepsilon^{p-1}|u|^{p-1})^p dx \right)^{\frac{p}{p'}} \left( \int_{\mathbb{R}^N} |u|^p \right)^{\frac{1}{p}} = \|W_\varepsilon u\|_{p}^{p-1}\|u\|_{p}.
Hence, we get
\[
\text{Re}\left(-\int_{\mathbb{R}^N} A u W_{q}^{p-1} |u|^{p-2} dx\right) \geq \\
\int_{\mathbb{R}^N} \left[ k_1 + |x|^{-\alpha} k_2 + |x|^{\alpha-2} \epsilon k_3 + |x|^{-2} \epsilon k_4 + |x|^{\beta-\alpha+2} + a_1 (|x|^{2-\alpha} + \epsilon) \right] W_{q}^{p} |u|^{p} dx \\
- a_1 \|W_{q} u\|_{p}^{p-1} \|u\|_{p}.
\]

Now, in order to establish (3.2) and (3.3) we analyze the function
\[
q(x) = k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} + |x|^{2-\alpha} a_1 + \epsilon (|x|^{\alpha-2} k_3 + |x|^{2} k_4 + a_1).
\]

We first consider the case \( k_2 > 0 \) and claim that for every \( k > 0 \) there exists a suitable \( a_1 \geq 0 \) such that \( q(x) \geq k \). Indeed, observe that since \( k_4 > 0 \) and \( -2 < -\alpha \) we have
\[
\lim_{x \to 0} |x|^{\alpha-2} k_3 + |x|^{-2} k_4 = +\infty \quad \text{and} \quad \lim_{x \to \infty} |x|^{\alpha-2} k_3 + |x|^{-2} k_4 = 0;
\]
then the function \(|x|^{\alpha-2} k_3 + |x|^{-2} k_4\) attains a minimum \( v_1 \). Taking \( a_1 > |v_1| \) we have
\[
q(x) \geq k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} + |x|^{2-\alpha} a_1.
\]

Now set \( k \in \mathbb{R} \) and observe that \( \lim_{x \to 0} k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} - k = +\infty \) and \( \lim_{x \to \infty} k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} - k = +\infty; \) then the function \( k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} - k \) attains a minimum \( \nu_2 \) and, moreover, there exists \( r_1 \) such that for \( |x| \leq r_1 \) we have \( k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} - k > 0 \). Then we can choose \( a_1 \) such that \( r_1^{2-\alpha} a_1 \geq |\nu_2| \) and have that \( k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} + |x|^{2-\alpha} a_1 - k > 0 \) for every \( x \in \mathbb{R}^N \). Therefore taking \( a_1 \geq \max\{|v_1|, \frac{|\nu_2|}{r_1^{2-\alpha}}\} \) we obtain the desired result.

As regards the case \( k_2 = 0 \) we can argue in the same way taking into account that \( \lim_{x \to 0} k_1 + |x|^{-\alpha} k_2 + |x|^{\beta-\alpha+2} - k = k_1 \).

As an easy consequence of the above result one has that \( W \) is \( A_0 \)-bounded that is
\[
\|W u\|_{p} \leq k^{-1} \|A_0\| + C\|u\|, \quad u \in D(A_0) \subset D(W).
\]

Now, we are in position to prove the main result of this section.

**Theorem 3.3.** Assume that \( N > 2, 0 \leq \alpha < 2, \beta > \alpha - 2 \).

(i) If \( 1 < p < \frac{N-\alpha}{2-\alpha} \), then for every \( c \in \mathbb{R} \), \( A_0 - c W \) is quasi \( m \)-accretive on \( D(A_0) \) and \( C^\infty_c(\mathbb{R}^N) \) is a core.

(ii) If \( p = \frac{N-\alpha}{2-\alpha} \) and if \( c > -k_1 \), where \( k_1 = \left( \alpha - 2 + \frac{N}{p} \right) \left( -\alpha + \frac{N}{p} \right) \), then \( A_0 - c W \) is quasi \( m \)-accretive on \( D(A_0) \) and \( C^\infty_c(\mathbb{R}^N) \) is a core. Moreover \( A_0 - k_1 W \) is essentially \( m \)-accretive.

**Proof.** In order to apply Theorem 3.1, we observe that both operators \( A_0 \) and \( W \) are \( m \)-accretive in \( L^p(\mathbb{R}^N) \). Then Lemma 3.2, with \( b = c = 0 \), yields (i) in Theorem 3.1 with \( a_2 = 0 \), moreover if \( b = c = 0 \) we have that \( k_1 > 0 \). Observe that (i) holds also for \( -A_0 + \lambda \) for \( \lambda > 0 \) by changing the value of \( a_1 \).

The second assumption (ii) in Theorem 3.1 is obviously satisfied. The last one, (iii), is obtained by Proposition 2.1 setting \( b = 0 \).

□
Remark 3.4. In the case $p = \frac{N-\alpha}{2-\alpha}$ we observe that the condition

$$\left(\alpha - 2 + \frac{N}{p}\right) \left(-\alpha + \frac{N}{p'}\right) + c > 0$$

is the same condition obtained by G. Metafune et al. See [16, Theorem 4.5] for the completely homogeneous operator $|x|^\alpha \Delta + b |x|^{\alpha-2} \nabla \cdot x - c |x|^{\alpha-2}$.

4. Maximality of $A$

In the following section we will prove the essential maximality of $(A, C^\infty_c(\mathbb{R}^N))$. The first result is the density in $L^p(\mathbb{R}^N)$ of the range of $\lambda - A$ for some $\lambda > 0$.

Proposition 4.1. Let $0 \leq \alpha < 2$ and $1 < p < \frac{N-\alpha}{2-\alpha}$ or $p = \frac{N-\alpha}{2-\alpha}$ and

$$b \left(\frac{\alpha + N - 2}{2}\right) + \left(\frac{N}{p} - 2 + \alpha\right) \left(\frac{N}{p'} - \alpha\right) + c > 0.$$ 

Then the range $(\lambda - A)(C^\infty_c(\mathbb{R}^N))$ is dense in $L^p(\mathbb{R}^N)$ for some $\lambda > 0$.

Proof. Suppose that there exists $f \in L^{p'}(\mathbb{R}^N)$ such that

$$\int f(x)(\lambda - A)\varphi dx = 0, \text{ for all } \varphi \in C^\infty_c(\mathbb{R}^N).$$

We have to show that $f = 0$.

Let $\phi = (1 + |x|^\alpha)^{b/\alpha}$, where $b \in \mathbb{R}$ represents the coefficient of the drift term of $A$ and set $\varphi = \frac{f}{\sqrt{\varphi}} \in C^\infty_c(\mathbb{R}^N)$. We note that the function $\phi$ is the function for which the following holds

$$\frac{1}{\phi} \text{div} (\phi \nabla \varphi) = \Delta \varphi + b \frac{|x|^\alpha - 2}{1 + |x|^\alpha} x \cdot \nabla \varphi, \quad \varphi \in C^\infty_c(\mathbb{R}^N).$$

A simple computation gives

$$(-A) \varphi = \frac{1}{\sqrt{\phi}} \left[-(1 + |x|^\alpha) \Delta \varphi + U \varphi\right], \quad (4.1)$$

where

$$U = (1 + |x|^\alpha) \left[-\frac{1}{4} \left|\nabla \phi\right|^2 + \frac{1}{2} \frac{\Delta \phi}{\phi}\right] + |x|^{\beta} + c|x|^{\alpha-2},$$

that is

$$U = \tilde{c}|x|^{\alpha-2} - \frac{|x|^{2\alpha-2}}{1 + |x|^\alpha} \left(-\frac{b^2}{4} + \frac{b\alpha}{2}\right) + |x|^{\beta}$$

where $\tilde{c} = b \left(\frac{N-\alpha}{2}-\alpha\right) + c$.

Now by (4.1) we have

$$\int \overline{f}(\lambda - A)\varphi = \int \frac{\overline{f}}{\sqrt{\varphi}} \left[-(1 + |x|^\alpha) \Delta \varphi + U \varphi\right] = 0.$$
We define \( h = \sqrt{f} \in L^p_{\text{loc}}(\mathbb{R}^N) \subset L^1_{\text{loc}}(\mathbb{R}^N) \), then
\[
\int h \left[-(1 + |x|^\alpha)\Delta + U + \lambda\right] v = 0.
\]

Hence
\[
\int h(1 + |x|^\alpha)\Delta v = \int h v (\lambda + U)
\]
and since \( p(\alpha - 2) + N > \alpha > 0 \) we have that \( |x|^{\alpha - 2} \in L^p_{\text{loc}}(\mathbb{R}^N) \) and then \( \lambda + U \in L^p_{\text{loc}}(\mathbb{R}^N) \). Then we have
\[
\Delta [h(1 + |x|^{\alpha})] = h (\lambda + U).
\]
The following Kato’s inequality will be the key tool to prove the generation.

**Proposition 4.2.** (*Kato’s inequality*)\(^1\). Let \( u \in L^1_{\text{loc}}(\mathbb{R}^N) \) be such that its distributional laplacian, \( \Delta u \) is also in \( L^1_{\text{loc}}(\mathbb{R}^N) \). Define
\[
\text{sgn}(u)(x) = \begin{cases} 
u(x)/|u(x)|, & \text{if } u(x) \neq 0 \\ 0, & \text{if } u(x) = 0 \end{cases}
\]
here, \( u \text{ sgn}(u) = |u| \). Then, in the sense of distributions
\[
\Delta |u| \geq \text{Re}[\text{sgn}(u)\Delta u].
\]

Applying (4.2) to \( h(1 + |x|^{\alpha}) \) we obtain for every \( v \in C^\infty_c(\mathbb{R}^N) \)
\[
\int |h(1 + |x|^{\alpha})|\Delta v \geq \text{Re} \int (\text{sign } h)v \Delta [h(1 + |x|^{\alpha})].
\]
Then up to change \( \lambda \) for a fixed \( \varepsilon > 0 \) we have
\[
(\text{sign } h)\Delta [h(1 + |x|^{\alpha})] = |h| (\lambda + U) \geq |h| \left( \lambda + (\tilde{c} - \varepsilon)|x|^{\alpha - 2} + |x|^\beta \right).
\]
Then
\[
\int |h|(1 + |x|^{\alpha})\Delta v \geq \int \left| h \right| \left( \lambda + (\tilde{c} - \varepsilon)|x|^{\alpha - 2} + |x|^\beta \right) v
\]
and hence
\[
\int |h|(\lambda - A_1)v \leq 0
\]
for all \( v \in C^\infty_c(\mathbb{R}^N) \) where \( A_1 = (1 + |x|^{\alpha})\Delta - (\tilde{c} - \varepsilon)|x|^{\alpha - 2} - |x|^\beta \). By Theorem 3.3 we have that \( (\lambda - A_1)C^\infty_c(\mathbb{R}^N) = L^p(\mathbb{R}^N) \) for some \( \lambda > 0 \) if \( 1 < p < \frac{N - \alpha}{2 - \alpha} \). If \( p = \frac{N - \alpha}{2 - \alpha} \) we require
\[
b \left( \frac{\alpha + N - 2}{2} \right) + \left( \frac{N}{p} - 2 + \alpha \right) \left( \frac{N}{p'} - \alpha \right) + c > \varepsilon.
\]
Hence
\[
\int |h|g \leq 0
\]
\(^1\)What we call \( \text{sgn}(u) \), Kato calls \( \text{sgn}(\bar{u}) \).
for all $g \in L^p(\mathbb{R}^N)$. Take $g_n = \sqrt{\phi} |f|^{p-1} \vartheta_n \in L^p(\mathbb{R}^N)$ where $\vartheta_n$ is a cutoff function such that $0 \leq \vartheta \leq 1$, $\vartheta = 1$ in $B(n)$ and $\vartheta = 0$ in $B(2n)^c$. It holds

$$\int |f|^{p} \vartheta_n \leq 0$$

for all $n \in \mathbb{N}$. This implies that $f = 0$ and then maximality is proved.

Now, we are ready to prove the main result of this section.

**Theorem 4.3.** Let $0 \leq \alpha < 2$ and either $1 < p < \frac{N-\alpha}{2-\alpha}$ or else $p = \frac{N-\alpha}{2-\alpha}$ and

$$\left( \frac{N}{p} - 2 + \alpha \right) \left( \frac{N}{p} - \alpha + b \right) + c > 0 \quad (4.2)$$

Then the closure of $(A, C_c^\infty(\mathbb{R}^N))$ generates an analytic $C_0$–semigroup.

**Proof.** We observe that since $1 < p < \frac{N-\alpha}{2-\alpha}$ we have that $p(\alpha - 1) > -N$ and $p(\alpha - 2) > -N$, then the drift and the potential terms of $A$ belong to $L^p(\mathbb{R}^N)$. As a consequence $A$ maps $C_c^\infty(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ and $A$ is well defined in $C_c^\infty(\mathbb{R}^N)$ as an operator acting on $L^p(\mathbb{R}^N)$.

First we consider the case $1 < p < \frac{N-\alpha}{2-\alpha}$. By Proposition 2.1, for $\lambda$ greater then a suitable $\lambda_0 > 0$, the operator $-A + \lambda$ is accretive, then it is closable, by Proposition 4.1, the range of $-A + \lambda$ (we can take $\lambda > \lambda_0$) is dense in $L^p(\mathbb{R}^N)$. Hence the closure of $A$ is quasi $m$–accretive. By Lumer and Phillips Theorem, cf. e.g. [24, Ch. I, Theorem. 4.3] or [8, Theorem 1.1], the closure of $A$ is the generator of a $C_0$–semigroup on $L^p(\mathbb{R}^N)$. Since once again by Proposition 2.1 $A$ is quasi-sectorial, then the semigroup is analytic.

Now consider the case $p = \frac{N-\alpha}{2-\alpha}$. In order to emphasize the dependence by the coefficients $b$ and $c$ we denote the operator $A$ by

$$A_{b,c} := (1 + |x|^{\alpha})\Delta + b|x|^{\alpha-2}x \cdot \nabla - c|x|^{\alpha-2} - |x|^\beta, \quad x \in \mathbb{R}^N.$$ 

Fix $b \in \mathbb{R}$ and choose $c_1$ such that the hypotheses of Proposition 4.1 are satisfied and such that the constant $k_1(b, c_1)$ of Lemma 3.2 is positive. Arguing as before we have that the closure $-A_{b,c_1}$ of $-A_{b,c_1}$ is quasi $m$–accretive. We apply once again Okazawa’s perturbation Theorem 3.1 to the operator $-A_{b,c_1}$ perturbed by $W$. Since $k_1(b, c_1) > 0$, Lemma 3.2 gives condition (i) of Theorem 3.1. The second assumption (ii) is obviously satisfied. The last one, (iii), is obtained by Proposition 2.1. So we have that $-A_{b,c_1} + c_2 W$ is quasi $m$–accretive if $c_2 > -k_1(b, c_1)$ and quasi essentially $m$–accretive if $c_2 = -k_1(b, c_1)$, moreover $C_c^\infty(\mathbb{R}^N)$ is a core, then the closure of $-A + c_2 W$ coincides with the closure of $-A + c_2 W$.

Setting $c = c_1 + c_2$ we have that $c_2 \geq -k_1(b, c_1)$ is equivalent to $k_1(b, c) > 0$ which is the condition (4.2). Then the closure of $A_{b,c}$ is quasi $m$–accretive. Therefore, arguing as before, the closure of $A$ generate an analytic semigroup.

**Remark 4.4.** In the case $p = \frac{N-\alpha}{2-\alpha}$ we observe that the condition

$$\left( \frac{N}{p} - 2 + \alpha \right) \left( \frac{N}{p} - \alpha + b \right) + c > 0$$

is the same condition as G. Metafune et al. in [16, Theorem 4.5].
5. M-ACCRETIVITY OF $A_{p,\text{max}}$

In the previous section we obtained that the operator $A$ with domain $C^\infty_c(\mathbb{R}^N)$ is essentially quasi $m$-accretive in $L^p(\mathbb{R}^N)$ for $1 < p < \frac{N-\alpha}{2-\alpha}$. In the limit case, $p = \frac{N-\alpha}{2-\alpha}$, the additional condition $\left(\frac{N}{p} - 2 + \alpha\right)\left(\frac{N}{p} - \alpha + b\right) + c > 0$ is required.

Since $D_0 = C^\infty_c(\mathbb{R}^N \setminus \{0\}) \subset C^\infty_c(\mathbb{R}^N)$ one can also infer that $(A, D_0)$ is essentially quasi $m$-accretive for $1 < p \leq \frac{N-\alpha}{2-\alpha}$. In this section we study the $m$-accretivity of the maximal realization of $A$ defined as

$$
\begin{align*}
A_{p,\text{max}} &:= Au, \\
D(A_{p,\text{max}}) &:= \{ u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) : Au \in L^p(\mathbb{R}^N) \}.
\end{align*}
$$

$A_{p,\text{min}}$ denotes the minimal realization of $A$: the closure in $L^p(\mathbb{R}^N)$ of $(A, D_0)$.

We use a duality argument to prove that $A_{p,\text{max}}$ is $m$-accretive, i.e.

$$
\text{Re} \int_{\mathbb{R}^N} A_{p,\text{max}} u \bar{u} |u|^{p-2} dx \geq 0 \quad \forall u \in D(A_{p,\text{max}}), \quad R(1 + A) = L^p(\mathbb{R}^N).
$$

To this purpose we need the following proposition which is a consequence of local elliptic regularity.

**Proposition 5.1.** The operator $A_{p,\text{max}}$ with domain $D(A_{p,\text{max}})$ is closed.

Now take the formal adjoint of the differential operator $A$ that is given by

$$
\tilde{A} = (1 + |x|^\alpha)\Delta + \tilde{b}|x|^\alpha - \tilde{c}|x|^\beta,
$$

where the coefficients $\tilde{b}$ and $\tilde{c}$ are defined as

$$
\begin{align*}
\tilde{b} &= 2\alpha - b, \\
\tilde{c} &= (b - \alpha)(\alpha - 2 + N).
\end{align*}
$$

**Proposition 5.2.** $A^*_{p,\text{min}} = \tilde{A}_{p',\text{max}}$ and $A^*_{p,\text{max}} = \tilde{A}_{p',\text{min}}$.

**Proof.** We have that

$$
D(A^*_{p,\text{min}}) = \{ u \in L^{p'}(\mathbb{R}^N) | \exists f \in L^{p'}(\mathbb{R}^N) \text{ s.t. } \int u A\varphi = \int f \varphi, \forall \varphi \in D(A_{p,\text{min}}) \}.
$$

Since $D_0 \subset D(A_{p,\text{min}})$ and the coefficients of $A$ are bounded in every compact subset of $\mathbb{R}^N \setminus \{0\}$ by local elliptic regularity (see [11, Theorem 9.19]) we have that $D(A^*_{p,\text{min}}) \subset W^{2,p}_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ then we can integrate by parts and obtain $\int u A\varphi = \int \tilde{A} u \varphi$. Then $A^*u = Au \in L^p(\mathbb{R}^N)$, therefore we can conclude that $D(A^*_{p,\text{min}}) = D(\tilde{A}_{p',\text{max}})$ and $A^*_{p,\text{min}} = \tilde{A}_{p',\text{min}}$.

The second assertion is obtained by duality, taking into account the closedness of the operators.

By applying propositions 5.1 and 5.2 to the operator $\tilde{A}$ we obtain the main result of this section.

**Theorem 5.3.** Let $\frac{N-\alpha}{2-\alpha} < p < \infty$. Assume that $\alpha \in [0, 2)$, $\beta > 0$ and $b, c \in \mathbb{R}$. Then $A_{p,\text{max}}$ generates a positive bounded analytic semigroup in $L^p(\mathbb{R}^N)$. 

**Proof.** It is sufficient to write the conditions of Theorem 4.3 for the operator \( \tilde{A} \) in \( L^p \), and one has from the duality argument in Proposition 5.2 that \( A^*_{p, \text{min}} = \tilde{A}_{p', \text{max}} \) and \( (A_{p, \text{max}})^* = (\tilde{A})_{p', \text{min}} \). Hence, we deduce the generation results for \( A_{p, \text{max}} \).

We observe that in the case \( \frac{N-\alpha}{N-2} < p < \frac{N-\alpha}{2-\alpha} \) the minimal and the maximal domains coincide \( D(A_{p, \text{min}}) \equiv D(A_{p, \text{max}}) \). Finally we can state the following theorem.

**Theorem 5.4.** Let \( \alpha > 4 - N \) then for every \( b, c \in \mathbb{R} \), the closure of \( (A, D_0) \) generates an analytic semigroup.

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