Cayley properties of the line graphs induced by consecutive layers of the hypercube

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Abstract. Let \( n > 3 \) and \( 0 < k < \frac{n}{2} \) be integers. In this paper, we investigate some algebraic properties of the line graph of the graph \( Q_n(k, k+1) \) where \( Q_n(k, k+1) \) is the subgraph of the hypercube \( Q_n \) which is induced by the set of vertices of weights \( k \) and \( k+1 \). In the first step, we determine the automorphism groups of these graphs for all values of \( k \). In the second step, we study Cayley properties of the line graph of these graphs. In particular, we show that for \( k > 2 \), if \( 2k + 1 \neq n \), then the line graph of the graph \( Q_n(k, k+1) \) is a vertex-transitive non Cayley graph. Also, we show that the line graph of the graph \( Q_n(1, 2) \) is a Cayley graph if and only if \( n \) is a power of a prime \( p \).

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1. Introduction

In this paper, a graph \( \Gamma = (V, E) \) is considered as an undirected simple graph where \( V = V(\Gamma) \) is the vertex-set and \( E = E(\Gamma) \) is the edge-set. For all the terminology and notation not defined here, we follow [2,8,9].

Let \( n \geq 1 \) be an integer. The hypercube \( Q_n \) is the graph whose vertex set is \( \{0, 1\}^n \), where two \( n \)-tuples are adjacent if they differ in precisely one coordinates. The hypercube \( Q_n \), has been extensively studied. Nevertheless, many open questions remain. Harary, Hayes, and Wu [11] wrote a comprehensive survey on hypercube graphs. In the graph \( Q_n \), the layer \( L_k \) is the set of vertices which contain \( k \) 1s, namely, vertices of weight \( k \), \( 1 \leq k \leq n \). We denote by \( Q_n(k, k+1) \), the subgraph of \( Q_n \) induced by layers \( L_k \) and \( L_{k+1} \). If \( n = 2k + 1 \), then the graph \( Q_{2k+1}(k, k+1) \) has been investigated from various
aspects, by various authors and is called the middle layer cube \([7,11,13,30]\) or regular hyperstar graph \([17,18,19,23]\). It has been conjectured by Dejter, Erdős, and Havel \([13]\) among others, that \(Q_{2k+1}(k, k+1)\) is hamiltonian.

The following figure shows the graph \(H(5, 2)\) \((Q_5(2, 3))\) in plane. Note that in this figure the set \(\{i, j, k\}\) \(\{(i, j)\}\) is denoted by \(ijk\) \((ij)\).

![Fig 1. The regular hyperstar graph H(5,2)](image)

The study of vertex-transitive graphs has a long and rich history in discrete mathematics. Prominent examples of vertex-transitive graphs are Cayley graphs which are important in both theory as well as applications. Vertex-transitive graphs that are not Cayley graphs, for which we use the abbreviation VTNCG, have been an object of a systematic study since 1979 \([3,10]\). In trying to recognize whether or not a vertex-transitive graph is a Cayley graph, we are left with the problem of determining whether the automorphism group contains a regular subgroup \([2]\). The reference \([1]\) is an excellent source for studying graphs that are VTNCG. In particular, determining the automorphism group of a given graph can be very useful in determining whether this graph is a Cayley graph. In this area of research, in algebraic graph theory, there are various works and some of the recent papers in this scope are \([3,10,14,15,22,23,24,25,26,28,29]\). In this paper, we investigate some algebraic properties of the line graph of the graph \(Q_n(k, k+1)\), in particular, we study cayleyness of this graph.

We can consider the graph \(Q_n\) from another point of view. The Boolean lattice \(BL_n, n \geq 1\), is the graph whose vertex set is the set of all subsets of \([n] = \{1, 2, ..., n\}\), where two subsets \(x\) and \(y\) are adjacent if their symmetric difference has precisely one element. In the graph \(BL_n\), the layer \(L_k\) is the set of \(k\)-subsets of \([n]\). We denote by \(BL_n(k, k+1)\), the subgraph of \(BL_n\) induced by layers \(L_k\) and \(L_{k+1}\).

Note that if \(A\) is a subset of \([n]\), then the characteristic function of \(A\) is the function \(\chi_A: [n] \rightarrow \{0, 1\}\), with the rule \(\chi_A(x) = 1\), if and only if \(x \in A\). We now can show that the mapping \(\chi: V(BL_n) \rightarrow V(Q_n)\), defined by the
rule, $\chi(A) = \chi_A$, is a graph isomorphism. Now, it is clear that the graph $Q_n$ is isomorphic with the graph $BL_n$, by an isomorphism that induces an isomorphism from $BL_n(k, k+1)$ to $Q_n(k, k+1)$. For this reason, in the sequel, we work on the graph $BL_n(k, k+1)$ and for abbreviation, we denote it by $B(n, k)$. We know that $\binom{n}{k} = \binom{n}{n-k}$, hence $B(n, k) \cong B(n, n-k)$. Therefore, in the sequel we assume that $k < \frac{n}{2}$.

2. Preliminaries

The group of all permutations of a set $V$ is denoted by $\text{Sym}(V)$ or just $\text{Sym}(n)$ when $|V| = n$. A permutation group $G$ on $V$ is a subgroup of $\text{Sym}(V)$. In this case we say that $G$ act on $V$. If $\Gamma$ is a graph with vertex-set $V$, then we can view each automorphism of $\Gamma$ as a permutation of $V$, and so $\text{Aut}(\Gamma)$ is a permutation group. Let the group $G$ act on $V$, we say that $G$ is transitive (or $G$ acts transitively on $V$) if there is just one orbit. This means that given any two element $u$ and $v$ of $V$, there is an element $\beta$ of $G$ such that $\beta(u) = v$.

The graph $\Gamma$ is called vertex-transitive, if $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. The action of $\text{Aut}(\Gamma)$ on $V(\Gamma)$ induces an action on $E(\Gamma)$, by the rule $\beta\{x, y\} = \{\beta(x), \beta(y)\}$, $\beta \in \text{Aut}(\Gamma)$, and $\Gamma$ is called edge-transitive if this action is transitive. The graph $\Gamma$ is called symmetric, if for all vertices $u, v, x, y, \beta$, $\alpha$ of $\Gamma$ such that $u$ and $v$ are adjacent, and $x$ and $y$ are adjacent, there is an automorphism $\alpha$ such that $\alpha(u) = x$, and $\alpha(v) = y$. It is clear that a symmetric graph is vertex-transitive and edge-transitive.

For $v \in V(\Gamma)$ and $G = \text{Aut}(\Gamma)$, the stabilizer subgroup $G_v$ is the subgroup of $G$ containing all automorphisms which fix $v$. In the vertex-transitive case all stabilizer subgroups $G_v$ are conjugate in $G$, and consequently are isomorphic. In this case, the index of $G_v$ in $G$ is given by the equation, $|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|$. If each stabilizer $G_v$ is the identity group, then every element of $G$, except the identity, does not fix any vertex, and we say that $G$ acts semiregularly on $V$. We say that $G$ act regularly on $V$ if and only if $G$ acts transitively and semiregularly on $V$, and in this case we have $|V| = |G|$.

Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, ..., n\}$. The Johnson graph $J(n, k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices $v, w$, are adjacent if and only if $|v \cap w| = k - 1$. The Johnson graph $J(n, k)$ is a vertex-transitive graph [9]. It is an easy task to show that the set $H = \{f_\theta \mid \theta \in \text{Sym}([n])\}$, $f_\theta(x_1, ..., x_k) = \{\theta(x_1), ..., \theta(x_k)\}$, is a subgroup of $\text{Aut}(J(n, k))$ [9]. It has been shown that $\text{Aut}(J(n, k)) \cong \text{Sym}([n])$, if $n \neq 2k$, and $\text{Aut}(J(n, k)) \cong \text{Sym}([n]) \times \mathbb{Z}_2$, if $n = 2k$, where $\mathbb{Z}_2$ is the cyclic group of order 2 [5,15,25].

Let $G$ be any abstract finite group with identity 1, and suppose $\Omega$ is a set of $G$, with the properties:

(i) $x \in \Omega \implies x^{-1} \in \Omega$, (ii) $1 \notin \Omega$.

The Cayley graph $\Gamma = \Gamma(G; \Omega)$ is the (simple) graph whose vertex-set and edge-set defined as follows:
$$V(\Gamma) = G, E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}.$$ It can be shown that a connected graph \(\Gamma\) is a Cayley graph if and only if \(\text{Aut}(\Gamma)\) contains a subgroup \(H\), such that \(H\) acts regularly on \(V(\Gamma)\) [2, chapter 16].

3. Main results

**Definition 3.1.** Let \(n \geq 4\) be an integer and \([n] = \{1, 2, ..., n\}\). Let \(k\) be an integer such that \(1 \leq k < \frac{n}{2}\). The graph \(B(n, k)\) is a graph with the vertex set \(V = \{v \mid v \subset [n], |v| \in \{k, k+1\}\}\) and the edge set \(E = \{\{v, w\} \mid v, w \in V, v \subset w \text{ or } w \subset v\}\).

**Example.** According to the Definition 3.1. the following figure shows \(B(5, 1)\) in plane.

![Figure 2: B(5,1)](image)

Note that in the above figure \(i = \{i\}, ij = \{i, j\}\).

**Remark 3.2.** In the sequel, we denote every set \(\{x_1, x_2, ..., x_t\}\) by \(x_1x_2...x_t\).

We see that in \(\Gamma = B(n, k)\), if \(v = x_1...x_k \in P_1 = \{v \mid v \subset [n], |v| = k\}\), then

$$N(v) = \{x_1...x_ky_1, ..., x_1...x_ky_{n-k}\},$$

where \(\{x_1, ..., x_k, y_1, ..., y_{n-k}\} = [n] = \{1, ..., n\}\). Hence, \(\text{deg}(v) = |N(v)| = n - k\).

On the other hand, if \(w = x_1...x_kx_{k+1} \in P_2 = \{v \mid v \subset [n], |v| = k + 1\}\), then

$$N(w) = \{u \mid u \subset w, |u| = k\},$$

and hence \(|N(w)| = \text{deg}(w) = k + 1\). Therefore, if \(k \neq \frac{n-1}{2}\), then we have \(k + 1 \neq n - k\), and thus if \(k \neq \frac{n-1}{2}\) the graph \(B(n, k)\) is not a regular graph.

Since every vertex of \(B(n, k)\) which is in \(P_1\) is of degree \(n - k\) and \(|P_1| = \binom{n}{k}\) then the number of edges of \(B(n, k)\) is \((n - k)\binom{n}{k}\) and the number of vertices of \(B(n, k)\) is \(\binom{n}{k} + \binom{n}{k+1}\).
Proposition 3.3. The graph $B(n,k)$ is bipartite and connected.

Proof. Let $P_1 = \{v \mid v \in V(B(n,k)), |v| = k\}$ and $P_2 = \{v \mid v \in V(B(n,k)), |v| = k+1\}$. Then from the definition of the graph $B(n,k)$ it follows that $V = V(B(n,k)) = P_1 \cup P_2$, $P_1 \cap P_2 = \emptyset$ and every edge $e = \{x,y\}$ of $B(n,k)$ is such that only one of $x$ or $y$ is in $P_1$ and the other is in $P_2$.

We now show that $B(n,k)$ is a connected graph. Let $x, y$ be two vertices of $B(n,k)$. In the first step, let $x, y$ be in $P_1$. Let $x = x_1x_2...x_k$, $y = y_1y_2...y_k$ and $|x_1...x_k \cap y_1...y_k| = k - t, 0 \leq t \leq k - 1$. We can show by induction on $t$ that $d(x,y) \leq 2t$, where $d(x,y)$ is the distance of vertices $x$ and $y$ in $B(n,k)$. Let $|x \cap y| = |x_1...x_k \cap y_1...y_k| = k - 1$, then we have $x = x_1...x_{k-1}u, y = x_1...x_{k-1}v$, for some $u, v \in [n]$, $v \neq u$. Now if $z = x_1...x_{k-1}uv$ then $P : x, z, y$ is a path between $x$ and $y$ and we have $d(x,y) = 2 = 2t$, for $t = 1$.

Now, suppose that the assertion is true for $t = m$, where $1 \leq m < k - 1$. Let $|x \cap y| = k - (m+1)$. Let $x = x_1...x_{k-m-1}u_1...u_{m+1}$, $y = x_1...x_{k-m-1}v_1...v_{m+1}$. Then for the vertex, $z_1 = x_1...x_{k-m-1}u_1...u_mv_1$, we have $|z_1 \cap y| = k - m, |z_1 \cap x| = k - 1$, hence by the assumption of induction we have $d(x,z_1) = 2$, and $d(z_1,y) = 2m$, therefore $d(x,y) = d(x,z_1) + d(z_1,y) = 2 + 2m = 2(m+1)$ (in fact, we can show that in this case, $d(x,y) = 2(m+1)$).

In the second step, let $x \in P_1$ and $y \in P_2$. If $x = x_1x_2...x_k$, $y = y_1y_2...y_ky_{k+1}$, then $z = y_1...y_k \in P_1$ and $z$ is adjacent to $y$. Now, by what we have seen in the first step, there is a path between $x$ and $z$ in $B(n,k)$, and therefore there is a path between $x$ and $y$ in $B(n,k)$.

In the last step, let $x, y \in P_2$. If $x = x_1...x_kx_{k+1}$, $y = y_1...y_kx_{k+1}$. Then, for $z = x_1...x_k$ we have $z \in P_1$ and $z$ is adjacent to $x$, and so according to the second step, there is a path between $y$ and $z$, and therefore there is a path between $x$ and $y$. 

By the method which we used in the proof of Proposition 3.3. we can deduce the following result.

Corollary 3.4. Let $D$ be the diameter of the graph $B(n,k)$. If $n \neq 2k+1$, then $D = 2(k+1) = 2k+2$. If $n = 2k+1$, then $D = 2k+1$.

We know that every vertex-transitive graph is a regular graph, so if $\Gamma$ is not a regular graph, then $\Gamma$ is not a vertex-transitive graph. Thus, if $n \neq 2k+1$, then $B(n,k)$ is not a vertex-transitive graph.

Let $V = V(B(n,k))$ be the vertex set of $B(n,k)$. Then, for each $\sigma \in Sym([n])$, the mapping,

\[ f_\sigma : V \rightarrow V, \ f_\sigma(v) = \{\sigma(x) \mid x \in v\}, \ v \in V, \]

is a bijection of $V$ and $f_\sigma$ is an automorphism of the graph $B(n,k)$. In fact, for each edge $e = \{v,w\} = \{x_1...x_k, x_1...x_kx_{k+1}\}$, we have,

\[ f_\sigma(e) = \{f_\sigma(v), f_\sigma(w)\} = \{\sigma(x_1)...\sigma(x_k), \sigma(x_1)...\sigma(x_k)\sigma(x_{k+1})\}. \]
and consequently \( f_\sigma(e) \) is an edge of \( B(n, k) \). Similarly, if \( f = \{x, y\} \) is not an edge of \( B(n, k) \), then \( f_\sigma(f) = \{f_\sigma(x), f_\sigma(y)\} \) is not an edge of \( B(n, k) \). Therefore, if \( H = \{f_\sigma \mid \sigma \in \text{Sym}([n])\} \), then \( H \) is a subgroup of the group \( G = \text{Aut}(B(n, k)) \). In fact, we show that if \( n \neq 2k+1 \), then \( \text{Aut}(B(n, k)) = H \), and if \( n = 2k+1 \), then \( \text{Aut}(B(n, k)) = H \times \mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) is the cyclic group of order 2. It is an easy task to show that \( H \cong \text{Sym}([n]) \).

**Proposition 3.5.** If \( \Gamma = B(n, k) \), then \( \Gamma \) is edge-transitive. Moreover if \( n = 2k+1 \), then \( \Gamma \) is vertex-transitive.

*Proof.* If \( e_1 = \{x_1, \ldots, x_k, x_1, \ldots, x_k, x_{k+1}\} \) and \( e_2 = \{y_1, \ldots, y_k, y_1, \ldots, y_k, y_{k+1}\} \) are edges of \( \Gamma \), then we define the mapping,

\[
\theta = \left( \begin{array}{c} x_1, \ldots, x_k, x_{k+1}, u_1, \ldots, u_{n-k-1} \\ y_1, \ldots, y_k, y_{k+1}, v_1, \ldots, v_{n-k-1} \end{array} \right)
\]

where \( \{x_1, \ldots, x_{k+1}, u_1, \ldots, u_{n-k-1}\} = \{1, \ldots, n\} \) and \( \{y_1, \ldots, y_{k+1}, v_1, \ldots, v_{n-k-1}\} \).

It is an easy task to show that \( \theta \in \text{Sym}([n]) \). Therefore, \( f_\theta \in S = \{f_\sigma \mid \sigma \in \text{Sym}([n])\} \) is an edge of \( \Gamma \), and we have \( f_\theta(e_1) = e_2 \).

We now assume that \( n = 2k+1 \). For each vertex \( v \) in \( V = V(B(n, k)) \), let \( v^c \) be the complement of the set \( v \) in \([n]\). We define the mapping \( \alpha : V \to V \) by the rule, \( \alpha(v) = v^c \), for every \( v \) in \( V \). Since the complement of a \( k \)-subset of the set \([n]\) is a \( k+1 \)-subset of \([n]\), then \( \alpha \) is an automorphism of \( B(n, k) \). If \( v \) and \( w \) are \( k \)-subsets of \([n]\), then \( f_\theta(v) = w \). If \( v \) is a \( k \)-subset and \( w \) is a \( k+1 \)-subset of \([n]\), then \( \alpha(w) \) is a \( k \)-subset of \([n]\), hence there is some \( f_\theta \in \text{Aut}(B(n, k)) \) such that \( f_\theta(v) = \alpha(w) \), and thus \( (\alpha f_\theta)(v) = w \).

In the sequel, we determine \( \text{Aut}(B(n, k)) \), the automorphism group of the graph \( B(n, k) \).

**Lemma 3.6.** Let \( n \) and \( k \) be integers with \( n > k \geq 1 \), and let \( \Gamma = (V, E) = B(n, k) \), with \( V = P_1 \cup P_2 \), \( P_1 \cap P_2 = \emptyset \), where \( P_1 = \{v \mid v \in [n], |v| = k\} \) and \( P_2 = \{w \mid w \subset [n], |w| = k+1\} \). If \( f \) is an automorphism of \( \Gamma \) such that \( f(v) = v \) for every \( v \in P_1 \), then \( f \) is the identity automorphism of \( \Gamma \).

*Proof.* Note that since \( f \) is a permutation of the vertex set \( V \) and \( f(P_1) = P_1 \), then \( f(P_2) = P_2 \). Let \( w \in P_2 \) be an arbitrary vertex. Since \( f \) is an automorphism of the graph \( \Gamma \), then for the set \( N(w) = \{v \mid v \in P_1, v \leftrightarrow w\} \), we have \( f(N(w)) = \{f(v) \mid v \in P_1, v \leftrightarrow w\} = N(f(w)) \). On the other hand, since for every \( v \in P_1 \), \( f(v) = v \), then \( f(N(w)) = N(w) \), and therefore \( N(f(w)) = N(w) \). In other words, \( w \) and \( f(w) \) are \((k+1)\)-subsets of \([n]\) such that their family of \( k \)-subsets are the same. Now, it is an easy task to show that \( f(w) = w \). Therefore, for every vertex \( x \) in \( \Gamma \) we have \( f(x) = x \) and thus \( f \) is the identity automorphism of \( \Gamma \).
Remark 3.7. If in the assumptions of the above lemma, we replace with $f(v) = v$ for every $v \in P_2$, then we can show, by a similar discussion, that $f$ is the identity automorphism of $\Gamma$.

Lemma 3.8. Let $\Gamma = (V, E)$ be a connected bipartite graph with partition $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$. Let $f$ be an automorphism of $\Gamma$. If for a fixed vertex $v \in V_1$, we have $f(v) \in V_1$, then $f(V_1) = V_1$ and $f(V_2) = V_2$. Also, if for a fixed vertex $v \in V_1$, we have $f(v) \in V_2$, then $f(V_1) = V_2$ and $f(V_2) = V_1$.

Proof. In the first step, we show that if $w \in V_1$ then $f(w) \in V_1$. We know that if $w \in V_1$, then $d_\Gamma(v, w) = d(v, w)$, the distance between $v$ and $w$ in the graph $\Gamma$, is an even integer. Assume $d(v, w) = 2l$, $0 \leq 2l \leq D$, where $D$ is the diameter of $\Gamma$. We prove by induction on $l$, that $f(w) \in V_1$. If $l = 0$, then $d(v, w) = 0$, thus $w = v$, and hence $f(w) = f(v) \in V_1$. Suppose that if $w_1 \in V_1$ and $d(v, w_1) = 2(k - 1)$, then $f(w_1) \in V_1$.

Theorem 3.10. Let $n$ and $k$ be integers with $\frac{n}{2} > k \geq 1$, and let $\Gamma = (V, E) = B(n, k)$, with vertex set $V = P_1 \cup P_2$, $P_1 \cap P_2 = \emptyset$, where $P_1 = \{v \mid v \in [n], |v| = k\}$ and $P_2 = \{w \mid w \in [n], |w| = k + 1\}$. Let $f$ be an automorphism of the graph $\Gamma$. If $n \neq 2k + 1$, then $f(V_1) = V_1$ and $f(V_2) = V_2$. Also, if $n = 2k + 1$, then $f(V_1) = V_1$ and $f(V_2) = V_2$.

Proof. We know that if $v \in P_1$, then $\deg(v) = n - k$, and if $w \in P_2$, then $\deg(w) = k + 1$. If $n \neq 2k + 1$, then $n - k \neq k + 1$. Hence, if $v \in P_1$, then $f(v) \notin P_2$ (note that $\deg(f(v)) = \deg(v)$). Therefore, if $v \in P_1$, then $f(v) \in P_1$. Now, since by proposition 3.3. the graph $B(n, k)$ is connected and bipartite, thus by lemma 3.8. we conclude that $f(V_1) = V_1$ and $f(V_2) = V_2$.

We now are ready to prove one of the important results of this paper.

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a permutation of $V_1$ where $f|_{V_1}$ is the restriction of $f$ to $V_1$. Let $\Gamma_2 = J(n,k)$ be the Johnson graph with the vertex set $V_1$. Then, the vertices $v, w \in V_1$ are adjacent in $\Gamma_2$ if and only if $|v \cap w| = k - 1$.

We assert that the permutation $g = f|_{V_1}$ is an automorphism of the graph $\Gamma_2$.

For proving our assertion, it is sufficient to show that if $v, w \in V_1$ are such that $|v \cap w| = k - 1$, then we must have $|g(v) \cap g(w)| = k - 1$. Note that since $v, w$ are $k$-subsets of $[n]$, then if $u$ is a common neighbor of $v, w$ in the graph $\Gamma = B(n, k)$, then the set $u$ contains the sets $v$ and $w$. In particular $u$ contains the $(k + 1)$-subset $v \cup w$. now, since $u$ is a $(k + 1)$-subset of $[n]$, then we have $u = v \cup w$. In other words, vertices $v$ and $w$ have exactly one common neighbor, namely, the vertex $u = v \cup w$, in the graph $\Gamma = B(n, k)$.

We assert that $|g(v) \cap g(w)| = k - 1$. In fact, if $|g(v) \cap g(w)| = k - h < k - 1$, then $h > 1$ and hence $|g(v) \cup g(w)| = k + h \geq k + 2$. Hence, there is no $(k + 1)$-subsets $t$ in $[n]$, such that $g(v) \cup g(w) \subset t$. In other words, the vertices $g(v)$ and $g(w)$ have no common neighbor in the graph $B(n, k)$, which is impossible. Note that $f$ is an automorphism of the graph $\Gamma = B(n, k)$, hence the number of common neighbors of $v$ and $w$ in $\Gamma$ is equal to the number of the common neighbors of $f(v) = g(v)$ and $f(w) = g(w)$ in the graph $\Gamma$.

Our argument shows that the permutation $g = f|_{V_1}$ is an automorphism of the Johnson graph $\Gamma_2 = J(n, k)$ and therefore by [5 chapter 9, 25] there is a permutation $\theta \in Sym([n])$ such that $g = f_\theta$.

On the other hand, we know that $f_\theta$ by its natural action on the vertex set of the graph $\Gamma = B(n, k)$ is an automorphism of $\Gamma$. Therefore, $l = f_\theta^{-1}f$ is an automorphism of the graph $\Gamma = B(n, k)$ such that $l$ is the identity automorphism on the subset $P_1$. We now can conclude, by Lemma 3.6, that $l = f_\theta^{-1}f$, is the identity automorphism of $\Gamma$, and therefore $f = f_\theta$.

In other words, we have proved that if $f$ is an automorphism of $\Gamma = B(n, k)$, then $f = f_\theta$, for some $\theta \in Sym([n])$, and hence $f \in H \ (n \neq 2k + 1)$. We now deduce that $G = H$.

(b) In this step, we prove the theorem for the case $n = 2k + 1$. Firstly, note that the mapping $\alpha : V(\Gamma) \rightarrow V(\Gamma)$, defined by the rule, $\alpha(v) = v^c$, where $v^c$ is the complement of the subset $v$ in $[n] = [2k+1] = \{1,2,\ldots,2k+1\}$, is an automorphism of the graph $\Gamma = B(n, k)$. In fact, if $A, B$ are subsets of $[n]$ such that $A \subset B$, then $B^c \subset A^c$, and hence if $\{A,B\}$ is an edge of the graph $B(n, k)$, then $\{\alpha(A), \alpha(B)\}$ is an edge of the graph $B(n, k)$. Therefore we have, $\alpha \leq Aut(B(n, k))$. Note that the order of $\alpha$ in the group $G = Aut(B(n, k))$ is 2, and hence $\alpha \cong \mathbb{Z}_2$. Let $H = \{f_\theta \mid \theta \in Sym([n])\}$. We have seen already that $H \cong Sym([n])$ and $H \leq Aut(B(n, k))$. We can see that $\alpha \not\in H$, and for every $\theta \in Sym([n])$, we have, $f_\theta \alpha = \alpha f_\theta$ [25]. Therefore,

$$Sym([n]) \times \mathbb{Z}_2 \cong H \times < \alpha > \cong < H, \alpha >$$

$$= \{f_\gamma \alpha^i \mid \gamma \in Sym([n]), 0 \leq i \leq 1 \} = S$$
is a subgroup of $G = \text{Aut}(\Gamma)$. We show that $G = S$.

Let $f \in \text{Aut}(\Gamma) = G$. We show that $f \in S$. There are two cases.

(i) There is a vertex $v \in V_1$ such that $f(v) \in V_1$, and hence by Lemma 3.8. we have $f(V_1) = V_1$.

(ii) There is a vertex $v \in V_1$ such that $f(v) \in V_2$, and hence by Lemma 3.8. we have $f(V_1) = V_2$.

Let $f(V_1) = V_1$. Then, by a similar argument which we did in (a), we can conclude that $f = f_\theta$, where $\theta \in \text{Sym}(n)$. In other words, $f \in S$.

We now assume that $f(V_1) \neq V_1$. Then, $f(V_1) = V_2$. Since the mapping $\alpha$ is an automorphism of the graph $\Gamma$, then $f\alpha$ is an automorphism of $\Gamma$ such that $f\alpha(V_1) = f(\alpha(V_1)) = f(V_2) = V_1$. Therefore, by what is proved in (i), we have $f\alpha = f_\theta$, for some $\theta \in \text{Sym}(n)$. Now since $\alpha$ is of order 2, then $f = f_\theta \alpha \in S = \{f_\gamma \alpha^i \mid \gamma \in \text{Sym}(n), 0 \leq i \leq 1\}$.

Let $\Gamma$ be a graph. The line graph $L(\Gamma)$ of the graph $\Gamma$ is constructed by taking the edges of $\Gamma$ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in $\Gamma$ have a common vertex. It is an easy task to show that if $\theta \in \text{Aut}(\Gamma)$, then the mapping $f(\theta) : V(L(\Gamma)) \to V(L(\Gamma))$ defined by the rule,

$$f(\theta)(\{u, v\}) = \{\theta(u), \theta(v)\}, \{u, v\} \in E(\Gamma),$$

is an automorphism of the graph $L(\Gamma)$. Hence, it is clear that if a graph $\Gamma$ is edge-transitive, then its line graph is vertex transitive. There is an important relation between $\text{Aut}(\Gamma)$ and $\text{Aut}(L(\Gamma))$. In fact, we have the following result [2, chapter 15].

**Theorem 3.11.** The mapping $\theta : \text{Aut}(\Gamma) \to \text{Aut}(L(\Gamma))$ defined by the rule,

$$\theta(g)\{u, v\} = \{g(u), g(v)\}, g \in \text{Aut}(\Gamma), \{u, v\} \in E(\Gamma),$$

is a group homomorphism and in fact we have;

(i) $\theta$ is a monomorphism provided $\Gamma \neq K_2$;

(ii) $\theta$ is an epimorphism provided $\Gamma$ is not $K_4$, $K_4$ with one edge deleted, or $K_4$ with two adjacent edges deleted.

Let $[n] = \{1, ..., n\}, n \geq 4$ and $\Gamma = B(n, k)$. We let the graph $\Gamma_1 = L(\Gamma)$, the line graph of the graph $\Gamma$. Then, each vertex in $\Gamma_1$ is of the form $\{A, Ay\}$, where $A \subseteq [n], |A| = k, y \in [n] - A$ and $Ay = A \cup \{y\}$. Two vertices $\{A, Ay\}$ and $\{B, Bz\}$ are adjacent in $\Gamma_1$ if and only if $A = B$ and $y \neq z$, or $Ay = Bz$ and $A, B$ are distinct $k$-subset of $Ay$. In other words, if $A = x_1 ... x_k \subseteq [n]$, then for the vertex $v = \{A, Ay_1\}$ of $\Gamma_1 = L(B(n, k))$, we have;

$$N(v) = \{\{A, Ay_2\}, ..., \{A, Ay_{n-k}\}, \{Ay_1 - \{x_1\}, Ay_1\}, ..., \{Ay_1 - \{x_k\}, Ay_1\}\}.$$

Hence, the graph $L(B(n, k))$ is a regular graph of valency $n - k - 1 + k = n - 1$. In other words, the degree of each vertex in the graph $L(B(n, k))$ is independent of $k$. In fact, the graph $L(B(n, k))$ is a vertex-transitive graph, because by Proposition 3.5. the graph $B(n, k)$ is an edge-transitive graph.
The following figure shows the graph $L(B(4,1))$ in plane.

![Figure 3: L(B(4,1))](image)

Note that in the above figure $i, ij = \{\{i\}, \{i, j\}\}$.

We know by Theorem 3.3, the graph $B(n, k)$ is a connected graph, hence its line graph, namely, the graph $L(B(n, k))$ is a connected graph [4]. The graph $L(B(n, k))$ has some interesting properties, for example, if $n, k$ are odd integers, then $L(B(n, k))$ is a hamiltonian graph. In fact, if $v$ is a vertex in $B(n, k)$, then $deg(v) \in \{k + 1, n - k\}$, therefore, if $n, k$ are odd integers, then the degree of each vertex in the graph $B(n, k)$ is an even integer, and hence $B(n, k)$ is eulerian [4]. Consequently, in this case the graph $L(B(n, k))$ is a hamiltonian graph.

We know that the graph $L(B(n, k))$ is a vertex-transitive graph. There is a well known conjecture in graph theory which asserts that almost all vertex-transitive connected graphs are hamiltonian [21]. Depending on this conjecture and what is mentioned in above, it seems that the following conjecture has an affirmative answer.

**Conjecture** The line graph of the graph $B(n, k)$, namely, $L(B(n, k))$ is a hamiltonian graph.

A graph $\Gamma$ is called an integral graph, if all of its eigenvalues are integers. The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974 [12]. In general, the problem of characterizing integral graphs seems to be very difficult. There are good surveys in this area (for example [6]).

In the scope of the present paper, we have the following result.

**Fact** (Mirafzal [27, Theorem 3.4]) Let $n > 3$ be an integer. Then, the graph $L(B(n, 1))$ is a vertex-transitive integral graph with distinct eigenvalues $-2, -1, 0, n - 2, n - 1$. 

On the automorphism group of the line graph of the graph $B(n,k)$, namely, $L(B(n,k))$, by Proposition 3.3, Theorem 3.10, and Theorem 3.11, we have the following result.

**Theorem 3.12.** Let $n \geq 4$, $[n] = \{1, \ldots, n\}$, $1 \leq k < \frac{n}{2}$. If $\Gamma = B(n,k)$ and $n \neq 2k + 1$, then $\text{Aut}(L(\Gamma)) \cong \text{Sym}([n])$. If $n = 2k + 1$, then $\text{Aut}(L(\Gamma)) \cong \text{Sym}([n]) \times \mathbb{Z}_2$.

We now ready to determine the values of $n, k$, such that the graph $L(B(n,k))$ is a non Cayley graph.

A permutation group $G$, acting on a set $V$, is $k$-homogeneous if its induced action on $V^{(k)}$ is transitive, where $V^{(k)}$ is the set of all $k$-subsets of $V$. Also we say that $G$ is $k$-transitive if $G$ is transitive on $V^{(k)}$, where $V^{(k)}$ is the set of $k$-tuples of distinct elements of $V$. Note that if $G$ is $k$-homogeneous, then we have $\binom{n}{k} \equiv |G|$, and if $G$ is $k$-transitive, then we have $\frac{n!}{(n-k)!} \equiv |G|$. If the group $G$ acts regularly on $V^{(k)}$, then $G$ is said to be sharply $k$-transitive on $V$. This means that for given two $k$-tuples in $V^{(k)}$, there is a unique permutation in $G$ mapping one $k$-tuple to the other.

We need the following two results which can be find in [8].

**Theorem 3.13.** [20] Let $G$ be a group $k$-homogeneous on a finite set $\Omega$ of $n$ points, where $n \geq 2k$. Then, if $k \geq 2$,

(a) the permutation group $G$ is $(k-1)$-transitive, and
(b) if also $k \geq 5$, the given permutation group $G$ is $k$-transitive.

The following result is a very deep result in group theory.

**Theorem 3.14.** [16] Let $G$ be a group $k$-homogeneous but not $k$-transitive on a finite set $\Omega$ of $n$ points, where $n \geq 2k$. Then, up to permutation isomorphism, one of the following holds:

(i) $k = 2$ and $G \leq \text{AGL}(1,q)$ with $n \equiv 3 \pmod{4}$;
(ii) $k = 3$ and $\text{PSL}(2,q) \leq G \leq \text{PGL}(2,q)$, where $n - 1 \equiv 3 \pmod{4}$;
(iii) $k = 3$ and $G = \text{AGL}(1,8), \text{AGL}(1,8)$ or $\text{AGL}(1,32)$; or
(iv) $k = 4$ and $G = \text{PSL}(2,8), \text{PGL}(2,8)$ or $\text{PGL}(2,32)$.

Note that here $q$ is a power of a prime integer, and $\Gamma L(1,q)$ is the group of mappings $x \mapsto ax^\sigma + b$ on $GF(q)$, where $a \neq 0$ and $b$ are in $GF(q)$ and $\sigma \in \text{Aut}(GF(q))$. $\text{AGL}(1,q)$ consists of those mappings with $\sigma = 1$. All the groups listed in the theorem are assumed to act in their usual permutation representations.

In the sequel, we also need the following result [31, chapter 7].

**Theorem 3.15.** Let $G$ be a sharply $2$-transitive permutation group on a finite set $V$. Then the degree of $G$ is $p^m$ for some prime $p$. Moreover, $G$ is similar to a subgroup of $\text{Aff}(W)$ which contains the translation group where $W$ is a vector space of dimension $m$ over $GF(p)$.

We now proceed to prove one the most important results of our work.
**Theorem 3.16.** Let \( n, k \) be integers, \( 3 < n, 1 \leq k < \frac{n}{2} \) and \( n \neq 2k + 1 \). Then the graph \( L(B(n, k)) \) is a non Cayley graph if each of the following holds,

(i) \( k = 2 \), and \( n - 1 \neq q \equiv 3 \pmod{4} \); where \( q \) is a power of a prime integer.

(ii) \( k \geq 3 \).

(iii) \( k = 1 \) and \( n \) is not a power of a prime integer.

**Proof.** On the contrary, assume that the graph \( \Gamma = L(B(n, k)) \) is a Cayley graph, then the automorphism group \( Aut(\Gamma) \) contains a subgroup \( G \) such that \( G \) acts regularly on the vertex set of \( \Gamma \) [2, chap 16]. We know by Theorem 3.12. that \( Aut(\Gamma) = \{ f_\theta \mid \theta \in Sym([n]) \} \). We let \( G_1 = \{ \theta \mid f_\theta \in G \} \), then \( G_1 \) is a subgroup of \( Sym([n]) \) which is isomorphic with the group \( G \), and \( G_1 \) is \((k + 1)\)-homogeneous on the set \([n] = \{1, 2, ..., n\} \). Note that each vertex \( v = \{ A, Ax \} \) in the graph \( L(B(n, k)) \) consist of a \(k\)-subset and a \((k + 1)\)-subset of \([n] \) and the group \( G \) acts transitively on the vertex-set of the graph \( \Gamma \). In fact, if \( A, B \) are given \((k + 1)\)-subsets of the set \([n] \), then we choose \(k\)-subsets \( C, D \) of \( A, B \) respectively, and thus \( \{ A, C \}, \{ B, D \} \) are vertices of the graph \( \Gamma = L(B(n, k)) \), and hence there is some element \( f_\theta \in G \) such that \( f_\theta(\{ A, C \}) = \{ \theta(A), \theta(C) \} = \{ B, D \} \), which implies that \( \theta(A) = B \), where \( \theta \in G_1 \).

We assert that that the group \( G_1 \) is a \((k + 1)\)-transitive permutation group on the set \([n] \). In the first step, we assume that \( k \geq 2 \). If \( k \geq 4 \), then \( k + 1 \geq 5 \), now since \( G_1 \) is a \((k + 1)\)-homogeneous permutation group on the set \([n] = \{1, 2, ..., n\} \), then by Theorem 3.13. \( G_1 \) is a \((k + 1)\)-transitive permutation group on the set \([n] \). If \( k = 3 \), then \( G_1 \) is a 4-homogeneous permutation group on the set \([n] \), then by Theorem 3.13. \( G_1 \) is a 3-transitive permutation group on the set \([n] \). Note that \( |G_1| = |G| = |V(L(B(n, 3)))| = (n-3)(n-4)/3 \), then \( n(n-1)(n-2) \) divides \((n-3)(n-4)/3 \). Therefore, 6 divides \( n-3 \), and hence \( n = 6t + 3 \), for some \( t \geq 2 \)(note that if \( k = 3 \), then \( n \geq 10 \)).

Now, by comparing the order of the group \( G_1 \) with orders of the groups which appear in the cases (ii),(iii),(iv) in Theorem 3.14. we conclude by Theorem 3.14. that the group \( G_1 \) is a \((k + 1)\)-transitive permutation group on the set \([n] \)(note that if \( G_1 = AGL(1, 8) \), then \( n = 8 \) and \( |G_1| = 56 \), if \( G_1 = AGL(1, 8) \), then \( |G_1| = 168 \), if \( G_1 = PGL(2, 8) \), then \( n = 8 \) and \( |G_1| = 9 \times 8 \times 7 \), if \( G_1 = PGL(2, 8) \), then \( n = 8 \) and \( |G_1| = 18 \), if \( G_1 = PGL(2, 32) \), then \( n = 32 \) and \( |G_1| = 30 \).)

Hence, if \( u_1 = (x_1, x_2, ..., x_k, x_{k+1}) \), is a \((k + 1)\)-tuple of distinct elements of \([n] \), then for \( u_2 = (x_1, x_{k+1}, x_k, ..., x_3, x_2) \), there is an element \( \theta \in G_1 \) such that,

\[
\theta(u_1) = (\theta(x_1), \theta(x_2), ..., \theta(x_k), \theta(x_{k+1})) = u_2 = (x_1, x_{k+1}, x_k, ..., x_3, x_2)
\]

Now, for the \(k\)-subset \( A = \{x_2, ..., x_k, x_{k+1}\} \) we have;

\[
\theta(A) = \{\theta(x_2), \theta(x_3), ..., \theta(x_k), \theta(x_{k+1})\} = A
\]

Note that if \( k = 1 \), then \( \theta \) can be the identity element of the group \( G_1 \), but if \( k > 1 \), then \( \theta \neq 1 \), and hence in the first step we assume that \( k > 1 \).
Therefore, if we consider the $k$-subset $A = \{x_2, \ldots, x_k, x_{k+1}\}$ of $[n]$, then $v = \{A, Ax_1\}$ is a vertex of the graph $\Gamma = L(B(n, k))$, and thus, for the element $f_\theta$ in the group $G$ we have,

$$f_\theta(v) = \{\theta(A), \theta(Ax_1)\} = \{A, A\theta(x_1)\} = \{A, Ax_1\} = v$$

which is a contradiction, because $1 \neq f_\theta \in G_\theta$ and the group $G$ acts regularly on the vertex set of the graph $\Gamma$. Consequently, if $k > 1$, then the graph $L(B(n, k))$ is a vertex-transitive non-Cayley graph.

We now assume that $k = 1$. Before proceeding, we mention that we do not luckily need Theorem 3.14. in the sequel, because, this theorem do not work in this case. We know that $|V| = n(n - 1)$ and $G$ is a subgroup of $Aut(\Gamma)$ which is regular on the set $V$, thus $|G| = n(n - 1)$ and hence $|G_1| = n(n - 1)$. We assert that $G_1$ is 2-transitive on the set $[n]$. If $(i, j)$ and $(r, s)$ are 2-tuples of distinct elements of $[n]$, then $\{i, ij\}$ and $\{r, rs\}$ are vertices of the graph $L(B(n, 1))$, and thus there is some $f_\phi$ in $G$ such that

$$f_\phi(\{i, ij\}) = \{\phi(i), \phi(i)\phi(j)\} = \{r, rs\}$$

which implies that for $\phi \in G_1$ we have $\phi(i) = r, \phi(j) = s$, namely $\phi(i, j) = (r, s)$.

Therefore, $G_1$ is a 2-transitive group on $[n]$ of order $n(n - 1)$, and hence $G_1$ is sharply 2-transitive on $[n]$. Therefore, by Theorem 3.15. the integer $n$ is of the form $p^n$ for some prime $p$. In other words, if $n$ is not a power of a prime, then the graph $L(B(n, 1))$ has no subgroup $G$ in its automorphism group such that $G$ acts regularly on the vertex set of $L(B(n, 1))$, and hence this graph is not a Cayley graph.

The above theorem do not say anything if $k = 1$ and $n$ is a power of a prime, but we can show that the graph $L(B(4, 1))$ which is displayed in Figure 4. is a Cayley graph. In fact we have the following result.

**Proposition 3.17.** Let $G = A_4$, the alternating group of degree 4 on the set $[4] = \{1, 2, 3, 4\}$. Let $\rho = (1, 2, 3)$ and $a = (1, 2)(3, 4)$. If $\Gamma$ is the Cayley graph $Cay(G; S)$ where $S = \{\rho, \rho^2, a\}$, then $\Gamma$ is isomorphic with $B(L(4, 1))$. In other words, $L(B(4, 1))$ is a Cayley graph.

**Proof.** Let $b = (1, 3)(2, 4)$, $c = (1, 4)(2, 3)$. Now, since $H = \{1, \rho, \rho^2\}$ is a subgroup of $G$, then $G = H \cup aH \cup bH \cup cH$. Note that $K = \{1, a, b, c\}$, is an abelian subgroup of the group $G$, with the property that the order of each non identity element of $G$ is 2, and in $G$ we have $ab = c, bc = a, ca = b$.

We know that in the Cayley graph $Cay(G; S)$, each vertex $v$ is adjacent to every vertex $vs, s \in S$ and hence if $v \in \{a, b, c\}$, then $N(v) = \{va, v\rho, v\rho^2\}$, where $N(v)$ is the set of neighbors of $v$. Now a simple computation shows that Figure 3, displays the graph $Cay(A_4, S)$ in the plane, is isomorphic with the graph $L(B(4, 1))$. In fact, we have,

$$\rho^{-2}c\rho^2 = \rho^{-2}(1, 4)(2, 3)\rho^2 = (3, 4)(1, 2)a \in S,$$

which implies that $\rho^2$ is adjacent to $c\rho^2$.

$$(\alpha \rho)^{-1}c\rho = \rho^{-1}(ac)\rho = \rho^{-1}b\rho = \rho^{-1}(1, 3)(2, 4)\rho = a \in S,$$

which implies that $\alpha \rho
is adjacent to \(c\rho\).

\[(a\rho^2)^{-1}b\rho^2 = \rho^{-2}ab\rho^2 = \rho^{-2}c\rho^2 = \rho^{-2}(1,4)(2,3)\rho^2 = (3,4)(1,2) = a \in S,\]

which implies that \(a\rho^2\) is adjacent to \(b\rho^2\).

\(cb = a \in S\) which implies that \(c\) is adjacent to \(b\).

Also, note that \(H, aH, bH, cH\), are 3-cliques in \(Cay(G; S)\).

---

**Theorem 3.18.** Let \(\Gamma\) be a regular graph of order \(n(n-1)\) and valency \(n-1\).

Suppose that in \(\Gamma\) there are \(n\) disjoint \((n-1)\)-cliques \(D_1, \ldots, D_n\), such that \(V(\Gamma) = (\cup D_i)_{i \in [n]}\) and for each pair of distinct cliques \(D_i, D_j\) in \(\Gamma\) there is exactly one pair of vertices \(v_i, v_j\) such that \(v_i \in D_i, v_j \in D_j\) and \(v_i\) is adjacent to \(v_j\). Then, \(\Gamma\) is isomorphic with \(L(B(n,1))\).

**Proof.** In the first step, note that each vertex \(v\) in a \((n-1)\)-clique \(D\) is adjacent to exactly one vertex \(w\) which is not in \(D\), because \(v\) is of degree \(n-1\). We now, choose one of the cliques in \(\Gamma\) and label it by \(C_1\). We let vertices in \(C_1\) are \(\{v[1,12], \ldots, v[1,1n]\}\). If \(D\) is a clique in \(\Gamma\) different from \(C_1\), then there is exactly one integer \(j, 2 \leq j \leq n\), such that \(v[1,1j]\) is adjacent to exactly one vertex of \(D\), say, \(v_D\). Then, we label the clique \(D\) by \(C_j\). We now label the vertices in \(C_j\) as follows,

If \(v \in C_j\), then there is exactly one \(i, i \in \{1, 2, \ldots, n\}, i \neq j\) such that \(v\) is adjacent to exactly one vertices in \(C_i\), now in such a case, we label \(v\) by \(v[j, ij]\).

Now, it is an easy task to show that the mapping \(\phi : V(L(B(n,1)) \rightarrow V(\Gamma)\), defined by the rule, \(\phi([j, ij]) = v[j, ij]\) is a graph isomorphism. \(\Box\)
Let $H, K$ be groups, with $H$ acting on $K$ in such way that the group structure of $K$ is preserved (for example $H$ is a subgroup of automorphisms of the group $K$). So for each $u \in K$ and $x \in H$ the mapping $u \mapsto u^x$ is an automorphism of $K$ (Note that the action of $H$ on $K$ is not specified directly). The semi-direct product of $K$ by $H$ denoted by $K \rtimes H$ is the set,

$$K \rtimes H = \{(u, x) \mid u \in K, x \in H\}$$

with binary operation $(u, x)(v, y) = (uv^{x^{-1}}, xy)$

**Theorem 3.19.** if $n = p^m$, for some prime $p$, then the line graph of $B(n, 1)$, namely, the graph $L(B(n, 1))$ is a Cayley graph.

**Proof.** Consider the finite field $GF(n)$ and let $K$ be the group $K = (GF(n), +)$. For each $0 \neq a \in K$, we define the mapping $f_a : K \rightarrow K$, by the rule $f_a(x) = ax$, $x \in K$. Then, $f_a$ is an automorphism of the group $K$ and $H = \{f_a \mid 0 \neq a \in K\}$ is a group (with composition of functions) of order $n - 1$ which is isomorphic with the multiplicative group of the field $GF(p^n)$. If we let $G = K \rtimes H$, then $G$ is a well defined group of order $n(n - 1)$. Note that $G$ is not an abelian group. Let $T = \{(0, h) \mid h \in H\}$. Then, $T$ is a subgroup of order $n - 1$ in the group $G$ which is isomorphic with $H$ and $[G : T] = n$, where $[G : T]$ is the index of $T$ in $G$. Hence, there are elements $b_1, \ldots, b_n$ in $G$ such that $G = b_1T \cup \ldots \cup b_nT$, and if $i \neq j$, then $b_iT \cap b_jT = \varnothing$. Note that $f_{-1} \in H$ and $f_{-1}^2 = i$, where $i$ is the identity element of $H$. Then, for the element $\alpha = (1, f_{-1})$, we have,

$$\alpha^2 = (1, f_{-1})(1, f_{-1}) = (1 + (-1)1, (f_{-1})^2) = (0, i) = e,$$

where $e$ is the identity element of $G$. Note that $\alpha \notin T$.

We now let $\Gamma = Cay(G; S)$, where $S = (T - \{e\}) \cup \{\alpha\}$. Note that $S = S^{-1}$, because $T$ is a subgroup of $G$ and $\alpha^2 = 1$ (and consequently $\alpha^{-1} = \alpha \in S$). Since, in the graph $\Gamma$ vertices $x, y$ are adjacent if and only if $x^{-1}y \in S$, then for each $i, 1 \leq i \leq n$, the subgraph induced by the set $C_i = b_iT$, is a $(n - 1)$-clique in $\Gamma$. Since $\alpha \notin T$, then $C_i \cap C_i\alpha = \varnothing$. In fact, if $v = b_\alpha t = b_\beta t_\alpha$, $t_\alpha, t \in T$, then $t = t_\alpha$, and hence $\alpha \in T$ which is a contradiction. Therefore, if $v \in C_j$, then $v$ is adjacent to $\alpha v$ and $\alpha v \in C_i$, $i \neq j$. Since $\Gamma$ is a regular graph of valency $n - 1$, then for each vertex $v \in C_i$ there is exactly one $j, j \neq i$ such that $v$ is adjacent to exactly one vertex $w$ in $C_j$. Now, by Theorem 3.18, we conclude that $\Gamma$ is isomorphic with $L(B(n, 1))$, and therefore the graph $L(B(n, 1))$ is a Cayley graph.

\[ \square \]

4. **Conclusion**

In this paper, we determined the automorphism group of the graph $Q_n(k, k + 1)$, where $Q_n(k, k + 1)$ is the subgraph of the hypercube $Q_n$ which is induced
by the set of vertices of weights $k$ and $k + 1$, for all $n > 3$ and $0 < k < \frac{n}{2}$. Then, we studied some algebraic properties of the line graph of these graphs. In particular, we proved that for $k > 2$, if $2k + 1 \neq n$, then the line graph of the graph $Q_n(k, k + 1)$ is a vertex-transitive non-Cayley graph. Also, we showed that the line graph of the graph $Q_n(1, 2)$ is a Cayley graph if and only if $n$ is a power of a prime $p$.

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