Stochastic Recursive Gradient Descent Ascent for Stochastic Nonconvex-Strongly-Concave Minimax Problems

Luo Luo ∗ Haishan Ye † Tong Zhang ‡

Abstract

We consider nonconvex-concave minimax problems of the form \( \min_x \max_y f(x, y) \), where \( f \) is strongly-concave in \( y \) but possibly nonconvex in \( x \). We focus on the stochastic setting, where we can only access an unbiased stochastic gradient estimate of \( f \) at each iteration. This formulation includes many machine learning applications as special cases such as adversary training and certifying robustness in deep learning. We are interested in finding an \( O(\varepsilon) \)-stationary point of the function \( \Phi(\cdot) = \max_y f(\cdot, y) \). The most popular algorithm to solve this problem is stochastic gradient descent ascent, which requires \( O(\kappa^3 \varepsilon^{-4}) \) stochastic gradient evaluations, where \( \kappa \) is the condition number. In this paper, we propose a novel method called Stochastic Recursive gradient Descent Ascent (SREDA), which estimates gradients more efficiently using variance reduction. This method achieves the best known stochastic gradient complexity of \( O(\kappa^3 \varepsilon^{-3}) \), and its dependency on \( \varepsilon \) is optimal for this problem.

1 Introduction

This paper considers the following minimax optimization problem

\[
\min_{x \in \mathbb{R}^d_1} \max_{y \in \mathbb{R}^d_2} f(x, y) \triangleq \mathbb{E} [F(x, y; \xi)],
\]

where the stochastic component \( F(x, y; \xi) \), indexed by some random vector \( \xi \), is \( \ell \)-gradient Lipschitz on average. This minimax optimization formulation includes many machine learning applications such as regularized empirical risk minimization \([33, 40]\), AUC maximization \([33, 37]\), adversary training \([13, 14]\) and certifying robustness in machine learning \([34]\). Many existing work \([6, 7, 10, 11, 21, 26, 27, 35, 37, 38, 40]\) focused on the convex-concave case of problem (1), where \( f \) is convex in \( x \) and concave in \( y \). For such problems, one can establish strong theoretical guarantees.

In this paper, we focus on a more general case of (1), where \( f(x, y) \) is \( \mu \)-strongly-concave in \( y \) but possibly nonconvex in \( x \). This case is referred to as stochastic nonconvex-strongly-concave minimax problems, and it is equivalent to the following problem

\[
\min_{x \in \mathbb{R}^d_1} \left\{ \Phi(x) \triangleq \max_{y \in \mathbb{R}^d_2} f(x, y) \right\}.
\]

Formulation \([2]\) contains several interesting examples in machine learning such as adversary training \([14]\) and certifying robustness in deep learning \([34]\).

∗ Shenzhen Research Institute of Big Data; rickyluoluo@gmail.com
† Shenzhen Research Institute of Big Data, The Chinese University of Hong Kong, Shenzhen; hsye.cs@outlook.com
‡ Hong Kong University of Science and Technology; tongzhang@ust.hk
Since $\Phi$ is possibly nonconvex, it is infeasible to find the global minimum in general. One important task of the minimax problem is finding an approximate stationary point of $\Phi$. A simple way to solve this problem is stochastic gradient descent with max-oracle (SGDmax) \cite{15, 19}. The algorithm includes a nested loop to solve $\max_{y \in \mathbb{R}^d} f(x, y)$ and use the solution to run approximate stochastic gradient descent (SGD) on $x$. Lin et al. showed that we can solve problem \cite{2} by directly extending SGD to stochastic gradient descent ascent (SGDA). The iteration of SGDA is just using gradient descent on $x$ and gradient descent on $y$ alternately. The complexity of SGDA to find $O(\varepsilon)$-stationary point of $\Phi$ in expectation is $O((\kappa^3 \varepsilon^{-4}) \log(1/\varepsilon))$. SGDA is more efficient than SGDmax because its complexity is $O((\kappa^3 \varepsilon^{-4}) \log(1/\varepsilon))$.

One insight of SGDA is that the algorithm selects an appropriate ratio of learning rates for $x$ and $y$. Concretely, the learning rate for updating $y$ is $O(\kappa^2)$ times that of $x$. Using this idea, it can be shown that the nested loop of SGDmax is unnecessary, and SGDA eliminates the logarithmic term in the complexity result. In addition, Rafique et al. \cite{30} presented some nested-loop algorithms that also achieved $O(\kappa^3 \varepsilon^{-4})$ complexity.

Recently, Thekumparampil et al. \cite{36} proposed a deterministic algorithm called proximal dual implicit accelerated gradient (ProxDIAG) to solve non-convex-concave minimax problem. The algorithm has a complexity with square root dependence on $\kappa$ for the strongly-concave case. However, the method does not cover the stochastic setting in this paper, and it only works for a special case of problem \cite{2} when the stochastic variable $\xi$ is finitely sampled from $\{\xi_1, \ldots, \xi_n\}$ (a.k.a. finite-sum case). That is,

$$f(x, y) = \frac{1}{n} \sum_{i=1}^{n} F(x, y; \xi_i). \quad (3)$$

In this paper, we propose a novel algorithm called Stochastic Recursive gradiEnt Descent Ascent (SREDA) for stochastic nonconvex-strongly-concave minimax problems. Unlike SGDmax and SGDA, which only iterate with current stochastic gradients, our SREDA updates the estimate recursively and it reduces the variance of the estimator.

The variance reduction techniques have been widely used in convex and nonconvex minimization problems \cite{1, 4, 12, 16, 17, 23, 24, 28, 31, 32, 39} and convex-concave saddle point problems \cite{7, 10, 11, 21, 27}. However, the nonconvex-strongly-concave minimax problems have two variables $x$ and $y$ and their roles in the objective function are quite different. To apply the technique of variance reduction, SREDA employs a concave maximizer with multi-step iteration on $y$ to simultaneously balance the learning rates, gradient batch sizes and iteration numbers of the two variables. We prove SREDA reduces the number of stochastic gradient evaluations to $O(\kappa^3 \varepsilon^{-3})$, which is the best known upper bound complexity. The result gives optimal dependency on $\varepsilon$ since the lower bound of stochastic first order algorithms for general nonconvex optimization is $O(\varepsilon^{-3})$ \cite{5, 12}. For finite-sum cases, the gradient cost of SREDA is $O\left(n \log(\kappa/\varepsilon) + \kappa^2 n^{1/2} \varepsilon^{-2}\right)$ when $n \geq \kappa^2$, and $O\left((\kappa^2 + \kappa n) \varepsilon^{-2}\right)$ when $n \leq \kappa^2$. This result is sharper than ProxDIAG \cite{36} in the case of $n$ is larger than $\sqrt{\kappa}$. We summarize the comparison of all algorithms in Table \ref{table:1}.

The paper is organized as follows. In Section \ref{section:2} we present notations and preliminaries. In Section \ref{section:3} we review the existing work for stochastic nonconvex-strongly-concave optimization and related techniques. In Section \ref{section:4} we present the SREDA algorithm and the main theoretical result. In Section \ref{section:5} we provide the proof of our main result of Theorem \ref{theorem:1}. Detailed lemmas and theorem are deferred in appendix. We conclude this work in Section \ref{section:6}. 

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Algorithm & Gradient Cost & Complexity \\
\hline
SGDmax & $O\left(n \log(\kappa/\varepsilon) + \kappa^2 n^{1/2} \varepsilon^{-2}\right)$ & $O(\kappa^3 \varepsilon^{-3})$ \\
SGDA & $O((\kappa^3 \varepsilon^{-4}) \log(1/\varepsilon))$ & $O((\kappa^3 \varepsilon^{-4}) \log(1/\varepsilon))$ \\
ProxDIAG & $O((\kappa^3 \varepsilon^{-4}) \log(1/\varepsilon))$ & $O((\kappa^3 \varepsilon^{-4}) \log(1/\varepsilon))$ \\
SREDA & $O(\kappa^3 \varepsilon^{-3})$ & $O(\kappa^3 \varepsilon^{-3})$ \\
\hline
\end{tabular}
\caption{Comparison of Algorithms}
\end{table}
The component function

Assumption 2. \( \ell > 0 \)

Assumption 3. \( \xi \) and random vector \( S \) is finite set \( F \) Lipschitz-gradient and strongly-concavity on \( G \) we have itly, but Lin et al. \cite{19} point out it is easily derived by standard arguments.

\[ \text{Table 1: We present the comparison on complexities of algorithms to solve problem (2) and (3).} \]

| Algorithm       | Stochastic                  | Finite-sum                  | Reference     |
|-----------------|-----------------------------|-----------------------------|---------------|
| SGDmax          | \( O(\kappa^3 \varepsilon^{-4} \log(1/\varepsilon)) \) | \( O(\kappa^2 n \varepsilon^{-2} \log(1/\varepsilon)) \) | \cite{15, 19} |
| PGSMD/PGSVRG    | \( O(\kappa^3 \varepsilon^{-4}) \)                      | \( O(\kappa^2 n \varepsilon^{-2}) \)                      | \cite{30}     |
| MGDA/HiBSA     | –                           | \( O(\kappa^4 n \varepsilon^{-2}) \)                      | \cite{20, 25} |
| ProxDIAG       | –                           | \( O(\kappa^{1/2} n \varepsilon^{-2}) \)                   | \cite{36}     |
| SGDA (GDA)     | \( O(\kappa^3 \varepsilon^{-4}) \)                      | \( O(\kappa^2 n \varepsilon^{-2}) \)                      | \cite{19}     |
| SREDA           | \( O(\kappa^3 \varepsilon^{-3}) \)                      | \( \begin{cases} O\left(n \log(\kappa/\varepsilon) + \kappa^2 n \varepsilon^{-2}\right), & n \geq \kappa^2 \\ O\left((\kappa^2 + \kappa n) \varepsilon^{-2}\right), & n \leq \kappa^2 \end{cases} \) | this paper    |

2 Notation and Preliminaries

We first introduce the notations and preliminaries used in this paper. For a differentiable function \( f(x, y) \) from \( \mathbb{R}^{d_1 \times d_2} \) to \( \mathbb{R} \), we denote the partial gradient of \( f \) with respect to \( x \) and \( y \) at \((x, y)\) as \( \nabla_x f(x, y) \) and \( \nabla_y f(x, y) \) respectively. We use \( \| \cdot \|_2 \) to denote the Euclidean norm of vectors. For a finite set \( S \), we denote its cardinality as \( |S| \). We assume that the minimax problem \( \Phi(x) \) satisfies the following assumptions.

\textbf{Assumption 1.} The function \( \Phi(\cdot) \) is lower bounded, i.e., we have \( \Phi^* = \inf_{x \in \mathbb{R}^{d_1}} \Phi(x) < +\infty \).

\textbf{Assumption 2.} The component function \( F \) has an average \( \ell \)-Lipschitz gradient, i.e., there exists a constant \( \ell > 0 \) such that for any \((x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, (x', y') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \) and random vector \( \xi \), we have

\[ \mathbb{E} \| \nabla F(x, y; \xi) - \nabla F(x', y'; \xi) \|^2_2 \leq \ell^2 \left( \| x - x' \|^2_2 + \| y - y' \|^2_2 \right). \]

\textbf{Assumption 3.} The component function \( F \) is concave in \( y \). That is, for any \( x \in \mathbb{R}^{d_1}, y, y' \in \mathbb{R}^{d_2} \) and random vector \( \xi \), we have

\[ F(x, y; \xi) \leq F(x, y'; \xi) + \langle \nabla_y F(x, y'; \xi), y - y' \rangle. \]

\textbf{Assumption 4.} The function \( f(x, y) \) is \( \mu \)-strongly-concave in \( y \). That is, there exists a constant \( \mu > 0 \) such that for any \( x \in \mathbb{R}^{d_1} \) and \( y, y' \in \mathbb{R}^{d_2} \), we have

\[ f(x, y) \leq f(x, y') + \langle \nabla_y f(x, y'), y - y' \rangle - \frac{\mu}{2} \| y - y' \|^2_2. \]

\textbf{Assumption 5.} The gradient of each component function \( F(x, y; \xi) \) has bounded variance. That is, there exists a constant \( \sigma > 0 \) such that for any random vector \( \xi \) and \((x, y) \in \mathbb{R}^{d_1 \times d_2} \), we have

\[ \mathbb{E} \| \nabla F(x, y; \xi) - \nabla f(x, y) \|^2_2 \leq \sigma^2 < \infty. \]

For the ease of presentation, we denote

\[ G(x, y; \xi) \triangleq (G_x(x, y; \xi), G_y(x, y; \xi)), \]

where \( G_x(x, y; \xi) = \nabla_y F(x, y; \xi) \) and \( G_y(x, y; \xi) = \nabla_y F(x, y; \xi) \). Under the assumptions of Lipschitz-gradient and strongly-concavity on \( f \), we can show that \( \Phi(\cdot) \) also has Lipschitz-gradient.
Lemma 1 (Lemma 3.3 of [19]). Under Assumptions 3 and 4, the function $\Phi(\cdot) = \max_{y \in \mathbb{R}^d} f(\cdot, y)$ is $(\ell + \kappa \ell)$-gradient Lipschitz. Additionally, the function $y^*(\cdot) = \arg \max_y f(\cdot, y)$ is unique defined.

Since $\Phi$ is differentiable, we may define $\varepsilon$-stationary point based on its gradient. The goal of this paper is to establish a stochastic gradient algorithm that output an $O(\varepsilon)$-stationary point in expectation.

Definition 1. We call $x$ an $O(\varepsilon)$-stationary point of the differentiable function $\Phi$ if $\|\nabla \Phi(x)\|_2 \leq O(\varepsilon)$.

3 Related Work

In this section, we review recent works for solving stochastic nonconvex-strongly-convex minimax problem [2] and introduce variance reduction techniques in stochastic optimization.

3.1 Nonconvex-Strongly-Concave Minimax

We present SGDmax in Algorithm 1. We can realize the max-oracle by stochastic gradient ascent (SGA) with $O(\kappa^3 \varepsilon^{-4} \log(1/\varepsilon))$ stochastic gradient evaluations to achieve sufficient accuracy. Using $S = \Theta(\kappa \varepsilon^{-2})$ guarantees that the variance of the stochastic gradients is less than $O(\kappa^{-1} \varepsilon^2)$. It requires $O(\kappa \varepsilon^{-2})$ iterations with step size $\eta = \Theta(1/(\kappa \ell))$ to obtain an $O(\varepsilon)$-stationary point of $\Phi$. The total stochastic gradient evaluation complexity is $O(\kappa^3 \varepsilon^{-4} \log(1/\varepsilon))$. The procedure of SGDA is shown in Algorithm 2.

Since variables $x$ and $y$ are not symmetric, we need to select different step sizes for them. In our case, we choose $\eta = \Theta(1/(\kappa \ell))$ and $\lambda = \Theta(1/\ell)$. This leads to an $O(\kappa^3 \varepsilon^{-4})$ complexity to obtain an $O(\varepsilon)$-stationary point with $S = O(\kappa \varepsilon^{-2})$ and $O(\kappa^3 \varepsilon^{-2})$ iterations [19]. Rafique et al. proposed proximally guided stochastic mirror descent and variance reduction (PGSMD/PGSVRG) whose complexity is also $O(\kappa^3 \varepsilon^{-4})$. Both of the above algorithms reveal that the key of solving problem efficiently is to update $y$ much more frequently than $x$. The natural intuition is that finding stationary point of a nonconvex function is typically more difficult than finding that of a concave or convex function. SGDmax implements it by updating $y$ more frequently (SGA in max-oracle) while SGDA iterates $y$ with a larger step size such that $\lambda/\eta = O(\kappa^2)$.

3.2 Variance Reduction Techniques

Variance reduction techniques has been widely used in stochastic optimization [2, 4, 12, 17, 23, 24, 28, 31]. One scheme of this type of methods is Stochastic Recursive grAdient algoritHm (SARAH) [23, 24]. Nguyen et al. first proposed it for convex minimization and established a convergence result. For nonconvex optimization, a closely related method is Stochastic Path-Integrated Differential EstimatoR (SPIDER) [12]. The algorithm estimates the gradient recursively together with a normalization rule, which guarantees the approximation error of the gradient is $O(\varepsilon^2)$ at each step. As a result, it can find $O(\varepsilon)$-stationary point of the nonconvex objective in $O(\varepsilon^{-3})$ complexity, which matches the lower bound [5]. This idea can also be extended to nonsmooth cases [28].

It is also possible to employ variance reduction to solve minimax problems. Most of the existing works focused on the convex-concave case. For example, [27] extend SVRG [16, 39] and SAGA [9] to solving strongly-convex-strongly-concave minimax problem in the finite-sum case, and established a linear convergence. One may also use the Catalyst framework [18, 27] and proximal point iteration [3, 21] to further accelerate when the problem is ill-conditioned. Recently, Chavdarova et al. [3] proposed an variance reduced extragradient algorithm that achieves a better upper bound.
Du and Hu [10], Du et al. [11] pointed out that for some special cases, the strongly-convex and strongly-concave assumptions of linear convergence for minimax problem may not be necessary.

Algorithm 1 SGDmax

1: Input $x_0$, learning rate $\eta > 0$, batch size $S > 0$, max-oracle accuracy $\zeta$
2: for $k = 0, \ldots, K$ do
3: draw $S$ samples $\{\xi_1, \ldots, \xi_S\}$
4: find $y_k$ so that $\mathbb{E}[f(x_k, y_k)] \geq \max_y f(x_k, y) - \zeta$
5: $x_{k+1} = x_k - \eta \cdot \frac{1}{S} \sum_{i=1}^{S} G_x(x_k, y_k; \xi_i)$
6: end for
7: Output: $\hat{x}$ chosen uniformly at random from $\{x_i\}_{i=0}^{K}$

Algorithm 2 SGDA

1: Input $(x_0, y_0)$, learning rate $\eta > 0$ and $\lambda > 0$, batch size $S > 0$
2: for $k = 0, \ldots, K$ do
3: draw $M$ samples $\{\xi_1, \ldots, \xi_S\}$
4: $x_{k+1} = x_k - \eta \cdot \frac{1}{S} \sum_{i=1}^{S} G_x(x_k, y_k; \xi_i)$
5: $y_{k+1} = y_k + \lambda \cdot \frac{1}{S} \sum_{i=1}^{S} G_y(x_k, y_k; \xi_i)$
6: end for
7: Output: $\hat{x}$ chosen uniformly at random from $\{x_i\}_{i=0}^{K}$

4 Algorithms and Main Results

In this section, we propose a novel algorithm for solving problem (2), which we call Stochastic Recursive gradiEnt Descent Ascent (SREDA). We show that the algorithm finds an $O(\varepsilon)$-stationary point with a complex of $O(\kappa^3 \varepsilon^{-3})$ stochastic gradient evaluations, and this result may be extended to the finite-sum case.

4.1 Stochastic Recursive Gradient Descent Ascent

SREDA uses variance reduction to track the gradient estimator recursively. Because there are two variables $x$ and $y$ in our problem (2), it is not efficient to combine SGDA with SPIDER [12] or (inexact) SARAH [23, 24] directly. The algorithm should approximate the gradient of $f(x_k, y_k)$ with small error, and keep the value of $f(x_k, y_k)$ sufficiently close to $\Phi(x_k)$. To achieve this, in the proposed method SREDA, we employ a concave maximizer with $O(\kappa)$ inner gradient ascent iterations on $y$. The details of SREDA and the concave maximizer are presented in Algorithm 3 and Algorithm 4 respectively.

In the initialization of SREDA, we can use inexact SARAH (iSARAH) [24] to maximize the strongly-concave function $f(x_0, \cdot)$ (or minimize the strongly-convex function $-f(x_0, \cdot)$). It achieves an approximate solution $y_0$ such that $\mathbb{E} \|\nabla_y f(x_0, y_0)\|_2^2 \leq \zeta = O(\kappa^{-2} \varepsilon^2)$ with a complex of $O(\kappa^2 \varepsilon^{-2} \log(\kappa/\varepsilon))$ [24].
Algorithm 3 SREDA

1: Input $x_0$, initial accuracy $\zeta$, learning rate $\eta_k, \lambda > 0$, batch size $S_1, S_2$ and periods $q, m > 0$.
2: $y_0 = \text{iSARAH}(-f(x_k, \cdot), \zeta)$
3: for $k = 0, \ldots, K - 1$ do
4:   if mod $(k, q) = 0$
5:     draw $S_1$ samples $\{\xi_1, \ldots, \xi_{S_1}\}$
6:     $v_k = \frac{1}{S_1} \sum_{i=1}^{S_1} G_x(x_k, y_k; \xi_i)$
7:     $u_k = \frac{1}{S_1} \sum_{i=1}^{S_1} G_y(x_k, y_k; \xi_i)$
8:   else
9:     // $\tilde{v}_\cdot$ and $\tilde{u}_\cdot$ are defined in Algorithm 4
10:    $v_k = \tilde{v}_{k-1, s_k-1} + 1$
11:    $u_k = \tilde{u}_{k-1, s_k-1} + 1$
12: end if
13: $x_{k+1} = x_k - \eta_k v_k$
14: $y_{k+1} = \text{ConcaveMaximizer} (k, m, S_2)$
15: end for
16: Output: $\hat{x}$ chosen uniformly at random from $\{x_i\}_{i=0}^{K-1}$

In the rest of this section, we show SREDA can keep the gradient with respect to $y$ less than $O(\kappa^{-2}\varepsilon^2)$ in expectation, and obtain an $O(\varepsilon)$-stationary point in $O(\kappa^3\varepsilon^{-3})$ stochastic gradient evaluations.

SREDA estimates the gradient of $f(x_k, y_k)$ by

$$(v_k, u_k) \approx (\nabla_x f(x_k, y_k), \nabla_y f(x, y)).$$

As illustrated in Algorithm 4, we evaluate the gradient of $f$ with a large batch size $S_1 = O(\kappa^2\varepsilon^{-2})$ at the beginning of each period, and update the gradient estimate recursively in concave maximizer with a smaller batch size $S_2 = O(\kappa\varepsilon^{-1})$.

For variable $x_k$, we adopt a normalized stochastic gradient descent rule with a learning rate

$$\eta_k = \min \left( \varepsilon \|v_k\|_2, \frac{1}{2\ell} \right) \cdot O(\kappa^{-1}).$$

With this step size, the change of $x_k$ is not dramatic at each iteration, which leads to accurate gradient estimates. When $v_k$ is large, we have $\eta_k = O(1/(\kappa\ell))$ which is larger than the stepsize $\eta_k = O(1/(\kappa^2\ell))$ of SGDA [19].

For variable $y_k$, we additionally expect $f(x_k, y_k)$ is a good approximation of $\Phi(x_k)$, which implies the gradient with respect to $y_k$ should be small enough. We hope to maintain the inequality $E \|\nabla_y f(x_k, y_k)\|_2^2 \leq O(\kappa^{-2}\varepsilon^2)$. Hence, we include a multi-step concave maximizer at line 14 with details given in Algorithm 4. This procedure can be regarded as one epoch of iSARAH [24]. We choose the step size $\lambda = O(1/\ell)$ and inner iteration number $m = O(\kappa)$, which simultaneously ensure that the gradient with respect to $y$ is small enough and the change of $y$ is not dramatic.
Algorithm 4 ConcaveMaximizer \((k, m, S_2)\)

1: Initialize \(\tilde{x}_{k,-1} = x_{k-1}, \tilde{y}_{k,-1} = y_{k-1}, \tilde{x}_{k,0} = x_{k+1}, \tilde{y}_{k,0} = y_k\).
2: draw \(S_2\) samples \(\{\xi_1, \ldots, \xi_{S_2}\}\)
3: \(\tilde{v}_{k,0} = v_k + \frac{1}{S_2} \sum_{i=1}^{S_2} G_x(\tilde{x}_{k,0}, \tilde{y}_{k,0}; \xi_i) - \frac{1}{S_2} \sum_{i=1}^{S_2} G_x(\tilde{x}_{k,-1}, \tilde{y}_{k,-1}; \xi_i)\)
4: \(\tilde{u}_{k,0} = u_k + \frac{1}{S_2} \sum_{i=1}^{S_2} G_y(\tilde{x}_{k,0}, \tilde{y}_{k,0}; \xi_i) - \frac{1}{S_2} \sum_{i=1}^{S_2} G_y(\tilde{x}_{k,-1}, \tilde{y}_{k,-1}; \xi_i)\)
5: \(x_{k,1} = \tilde{x}_{k,0}\)
6: \(\tilde{y}_{k,1} = \tilde{y}_{k,0} + \lambda \tilde{u}_{k,0}\)
7: for \(t = 1, \ldots, m + 1\) do
8: draw \(S_2\) samples \(\{\xi_{t,1}, \ldots, \xi_{t, S_2}\}\)
9: \(\tilde{v}_{k,t} = \tilde{v}_{k,t-1} + \frac{1}{S_2} \sum_{i=1}^{S_2} G_x(\tilde{x}_{k,t}, \tilde{y}_{k,t}; \xi_{t,i}) - \frac{1}{S_2} \sum_{i=1}^{S_2} G_x(\tilde{x}_{k,t-1}, \tilde{y}_{k,t-1}; \xi_{t,i})\)
10: \(\tilde{u}_{k,t} = \tilde{u}_{k,t-1} + \frac{1}{S_2} \sum_{i=1}^{S_2} G_y(\tilde{x}_{k,t}, \tilde{y}_{k,t}; \xi_{t,i}) - \frac{1}{S_2} \sum_{i=1}^{S_2} G_y(\tilde{x}_{k,t-1}, \tilde{y}_{k,t-1}; \xi_{t,i})\)
11: \(\tilde{x}_{k,t+1} = \tilde{x}_{k,t}\)
12: \(\tilde{y}_{k,t+1} = \tilde{y}_{k,t} + \lambda \tilde{u}_{k,t}\)
13: end for
14: Output: \(y_{k+1} = \tilde{y}_{k,s_k+1}\) and \(s_k\) is sampled from \(\{0, 1, \ldots, m\}\)

4.2 Complexity Analysis

As shown in Algorithm 3, SREDA updates variables with a large batch size per \(q\) iterations. We choose \(q = O(\varepsilon^{-1})\) because it is a good balance between the number of large batch evaluations with \(S_1 = O(\kappa^2 \varepsilon^{-2})\) samples and the concave maximizer with \(O(\kappa)\) iterations and \(S_2 = O(\kappa \varepsilon^{-1})\) samples.

Based on above parameter setting, we can obtain an approximate stationary point \(\check{x}\) in expectation such that \(\mathbb{E} \|\nabla \Phi(\check{x})\|_2 \leq O(\varepsilon)\) with \(K = O(\varepsilon^{-2})\) outer iterations. The total number of stochastic gradient evaluations of SREDA comes from the initial run of iSARAH, large batch gradient evaluation \((S_1\) samples\) and concave maximizer. That is,

\[
\text{StocGrad (Total)} = O(\kappa^3 \varepsilon^{-2} \log(\kappa/\varepsilon)) + O\left(\frac{K}{q} \cdot S_1\right) + O\left(K \cdot S_2 \cdot m\right)
\]

\[
= O(\kappa^3 \varepsilon^{-2} \log(\kappa/\varepsilon)) + O\left(\frac{\kappa \varepsilon^{-2}}{\varepsilon^{-1}} \cdot \kappa^2 \varepsilon^{-2}\right)
\]

\[
+ O\left(\kappa \varepsilon^{-2} \cdot \kappa \varepsilon^{-1} \cdot \kappa\right)
\]

\[
= O(\kappa^3 \varepsilon^{-3}).
\]

Assumption 1 means we can define

\[
\Delta_f = f(x_0, y_0) + \frac{\kappa^{-2} \varepsilon^2}{2\mu} - \Phi^* < \infty,
\]

and we can formally present the main result in Theorem 1.

**Theorem 1.** Under Assumptions 1-5 with the following parameter choices:

\[
\zeta = \kappa^{-2} \varepsilon^2, \eta_k = \min\left(\frac{\varepsilon}{5\kappa \ell \|v_k\|_2}, \frac{1}{10\kappa \ell}\right), \lambda = \frac{2}{7\ell},
\]

\[ q = \lceil \varepsilon^{-1} \rceil, \quad S_1 = \lceil 24\sigma^2 \kappa^2 \varepsilon^{-2} \rceil, \quad S_2 = \lceil \frac{7368}{175} \kappa q \rceil, \quad K = \lceil \frac{100\kappa\ell \varepsilon^{-2} \Delta_f}{9} \rceil \text{ and } m = \lceil 28\kappa - 1 \rceil, \]

Algorithm 4 outputs \( \hat{x} \) such that

\[ \mathbb{E} \| \nabla \Phi(\hat{x}) \|_2 \leq \frac{1073}{108} \varepsilon \]

with \( O(\kappa^3 \varepsilon^{-3}) \) stochastic gradient evaluations.

We should point out the complexity shown in Theorem 1 gives optimal dependency on \( \varepsilon \). We can prove it by construct the separate function as follows

\[ f(x, y) = g(x) + h(y), \]

where \( g \) is the nonconvex function in the lower bound analysis of 4, and \( h \) is an arbitrary smooth, \( \mu \)-strongly concave function. It is obviously that the lower bound complexity of finding an \( \varepsilon \)-stationary point of \( f \) is no smaller than that of finding an \( \varepsilon \)-stationary point of \( g \), which requires at least \( O(\varepsilon^{-3}) \) stochastic gradient evaluations.

### 4.3 Extension to Finite-sum Case

SREDA also works for nonconvex-strongly-concave minimax optimization in the finite-sum case with little modification of Algorithm 3. We just need to replace line 5-7 of Algorithm 3 with the full gradients, and use SARAH as initialization. We present the details in Algorithm 5. The algorithm is more efficient than ProxDIAG when \( n \geq \kappa^2 \). We state the result in Theorem 2 and provide the proof in appendix.

**Algorithm 5** SREDA (Finite-sum Case)

1. **Input** \( x_0 \), initial accuracy \( \zeta \), learning rate \( \eta_k, \lambda > 0 \), batch size \( S_1, S_2 \) and periods \( q, m > 0 \).
2. \( y_0 = \text{SARAH}( -f(x_k, \cdot), \zeta) \)
3. **for** \( k = 0, \ldots, K - 1 \) **do**
4. **if** \( \text{mod}(k, q) = 0 \)
5. \( v_k = \nabla_x f(x_k, y_k) \)
6. \( u_k = \nabla_y f(x_k, y_k) \)
7. **else**
8. \(/ / \check{v}_\cdot \text{ and } \check{u}_\cdot \text{ are defined in Algorithm 4} \)
9. \( v_k = \check{v}_{k-1,s_{k-1}+1} \)
10. \( u_k = \check{u}_{k-1,s_{k-1}+1} \)
11. **end if**
12. \( x_{k+1} = x_k - \eta_k v_k \)
13. \( y_{k+1} = \text{ConcaveMaximizer}(k, m, S_2) \)
14. **end for**
15. **Output**: \( \hat{x} \) chosen uniformly at random from \( \{x_i\}_{i=0}^{K-1} \)
Theorem 2. Suppose Assumption 1-5 hold. In the finite-sum case, we set the parameters

\[ \zeta = \kappa^{-2} \varepsilon^2, \eta_k = \min \left( \frac{\varepsilon}{5\kappa \ell \|v_k\|_2}, \frac{1}{10\kappa \ell} \right), \lambda = \frac{2}{7\ell}, q = \lceil \kappa^{-1} n^{1/2} \rceil, \]

\[ S_2 = \left\lceil \frac{7368}{175} \kappa q \right\rceil, K = \left\lceil \frac{100\kappa \varepsilon^{-2} \Delta f}{9} \right\rceil, m = \lceil 28\kappa - 1 \rceil. \]

Algorithm 5 outputs \( \hat{x} \) such that

\[ E \|\nabla \Phi(\hat{x})\|_2 \leq \frac{1073}{108} \varepsilon \]

with \( O(n \log(\kappa/\varepsilon) + \kappa^2 n^{1/2} \varepsilon^{-2}) \) stochastic gradient evaluations.

In the case of \( n \leq \kappa^2 \), we set the parameters

\[ \zeta = \kappa^{-2} \varepsilon^2, \eta_k = \min \left( \frac{\varepsilon}{5\kappa \ell \|v_k\|_2}, \frac{1}{10\kappa \ell} \right), \lambda = \frac{2}{7\ell}, q = 1, S_2 = 1, K = \left\lceil \frac{100\kappa \varepsilon^{-2} \Delta f}{9} \right\rceil, m = \lceil 28\kappa - 1 \rceil. \]

Algorithm 5 outputs \( \hat{x} \) such that

\[ E \|\nabla \Phi(\hat{x})\|_2 \leq \frac{1073}{108} \varepsilon \]

with \( O((\kappa^2 + \kappa n) \varepsilon^{-2}) \) stochastic gradient evaluations.

5 Convergence Analysis

We first present the outline of the analysis of Theorem 1. The detailed proofs of some lemmas and theorems shown in this section are deferred to appendix.

Different from Lin et al.’s analysis in [19], which directly considered the value of \( \Phi(x_k) \) and the distance \( \|y_k - y^*(x_k)\|_2 \), our proof mainly depends on \( f(x_k, y_k) \) and its gradient. We split the change of objective functions after one iteration on \( (x_k, y_k) \) into \( A_k \) and \( B_k \) as follows

\[ f(x_{k+1}, y_{k+1}) - f(x_k, y_k) = \frac{A_k}{A_k} + \frac{B_k}{B_k}, \]

where the quantity \( A_k \) provides the decrease of function value \( f \), and we want to show

\[ E[A_k] \leq -O(\kappa^{-1} \varepsilon); \] (4)

the quantity \( B_k \) can characterize the difference between \( f(x_{k+1}, y_{k+1}) \) and \( \Phi(x_{k+1}) \) and we want to ensure

\[ E[B_k] \leq O(\kappa^{-1} \varepsilon^2). \] (5)

By combining (4), (5) and the choice of \( K \) in Theorem 1, we can bound the average of \( \|v_k\|_2^2 \) in expectation

\[ \frac{1}{K} \sum_{k=0}^{K-1} E \|v_k\|_2 \leq O(\varepsilon). \]

We can also approximate \( E \|\Phi(x_k)\|_2 \) by \( E \|v_k\|_2 \) based on the following lemma.
Lemma 2. Under assumptions of Theorem 1, we have

$$E \| \nabla \Phi(x_k) \|_2 \leq E \| v_k \|_2 + \frac{4}{3} \varepsilon.$$ 

The output $\hat{x}$ of Algorithm 3 satisfies $E \| \nabla \Phi(x_k) \|_2 \leq O(\varepsilon)$. Based on the discussion of complexity in Section 4.2, we obtain the result of Theorem 1, which says that SREDA can obtain an $O(\kappa^3 \varepsilon^{-3})$ stationary point of $\Phi$ in expectation with $O(\kappa^3 \varepsilon^{-3})$ stochastic gradient evaluations.

We introduce two auxiliary quantities for our analysis

$$\delta_k = E \| \nabla y f(x_k, y_k) \|_2^2$$

and

$$\Delta_k = E \left[ \| v_k - \nabla x f(x_k, y_k) \|_2^2 + \| u_k - \nabla y f(x_k, y_k) \|_2^2 \right].$$

The rest of this section is organized as follows. We first present the initialization step by iSARAH formally. Then, we upper bound the auxiliary quantities $\delta_k$ and $\Delta_k$ by $O(\kappa^3 \varepsilon^{-3})$. Finally, we use $\delta_k$ and $\Delta_k$ to control $A_k$ and $B_k$ and prove the desired result of Theorem 1.

5.1 Initialization by iSARAH

We present the detailed procedure of iSARAH in Algorithm 6, which is used to initialize $y_0$ in SREDA (line 2 of Algorithm 3). We consider the following convex optimization problem

$$\min_w h(w) \triangleq E \left[ H(w; \xi) \right],$$

where $H$ is average $\ell$-gradient Lipschitz and convex, $h$ is $\mu$-strongly convex, and $\xi$ is a random vector. We have the following convergence result by using iSARAH to solve problem (6).

Lemma 3 (Corollary 4 of [24]). If we apply Algorithm 6 on problem (6) by setting $\gamma = O(1/\ell)$, $m' = O(\kappa)$, $b = O(\max(1/\zeta, \kappa))$ and $T = O(\log(1/\zeta))$, then, the total stochastic gradient complexity to achieve $E \| \nabla h(\hat{w}_T) \|_2^2 \leq \zeta$ is $O(\max(1/\zeta, \kappa) \log(1/\zeta))$.

According to Nguyen et al.'s proof of Lemma 3 with $h(\cdot) = -f(x_0, \cdot)$, we can archive $\hat{w}_T$ as $y_0$ such that

$$\delta_0 = E \| \nabla y f(x_0, y_0) \|_2^2 \leq \kappa^{-2} \varepsilon^2$$

by taking

$$\gamma = \frac{2}{\delta'}, \quad m' = [20\kappa - 1], \quad b = \max \left( 20\kappa - 10, 20\kappa^2 \varepsilon^{-2} \| \nabla y f(x_0, y^*(x_0)) \|_2^2 \right)$$

and

$$T = \left\lceil \log \left( \frac{4}{3} \kappa^2 \varepsilon^{-2} \| \nabla y f(x_0, y_0) \|_2^2 \right) \right\rceil.$$ 

The total number of stochastic gradient evaluations can be bounded as $(b + m')T = O(\kappa^2 \varepsilon^{-2} \log(\kappa/\varepsilon))$. We will provide the upper bound of $\delta_k$ and $\Delta_k$ by induction and the inequality (7) provides the induction base.
Algorithm 6 Inexact SARAH (iSARAH)

1: **Input** $\tilde{w}_0$, learning rate $\gamma > 0$, inner loop size $m'$, batch size $b > 0$
2: **for** $s = 1, \ldots, T$ **do**
3: draw $b$ samples $\{\xi_1, \ldots, \xi_b\}$
4: $w_0 = \tilde{w}_{s-1}$
5: $v_0 = \frac{1}{b} \sum_{i=1}^{b} \nabla H(w_0; \xi_i)$
6: $w_1 = w_0 - \gamma v_0$
7: **for** $t = 1, \ldots, m' - 1$ **do**
8: draw a samples $\xi_t$
9: $v_t = \nabla H(w_t; \xi_t) - \nabla H(w_{t-1}; \xi_t) + v_{t-1}$
10: $w_{t+1} = w_t - \gamma v_t$
11: **end for**
12: $\tilde{w}_s = w_{t'}, t'$ is uniformly sampled from $\{0, \ldots, m'\}$
13: **end for**
14: **Output**: $\tilde{w}_s$

5.2 Upper bound of $\Delta_k$ and $\delta_k$

Recall that we use the step size

$$\eta_k = \min \left( \frac{\varepsilon}{\delta \kappa \ell \|v_k\|_2}, \frac{1}{10 \kappa \ell} \right),$$

for SREDA in Theorem 1. We define $\varepsilon^2_{\delta} = \frac{\varepsilon^2}{25 \kappa^2 \ell^2}$ and the update rule of $x_k$ means for any $k \geq 0$, we have

$$\|x_{k+1} - x_k\|_2^2 \leq \varepsilon^2_{\delta}. \quad (8)$$

The following lemma says $E \|\tilde{u}_{k,t}\|_2^2$ in the concave maximizer decays along the iterations for appropriate step size $\lambda$, which implies that we can control $E \|y_{k+1} - y_k\|_2^2$ by martingale property as in SPIDER [12].

**Lemma 4.** For Algorithm 4 we have

$$E \|\tilde{u}_{k,t}\|_2^2 \leq \left( 1 - \frac{2 \mu \ell \lambda}{\mu + \ell} \right) \|\tilde{u}_{k,t-1}\|_2^2$$

for any $t \geq 1$ and $\lambda \leq 2/(\mu + \ell)$.

The concave minimization procedure in Algorithm 4 means we can establish the recursive relationship of $\delta_k$ and $\Delta_k$ in the following lemma.

**Lemma 5.** In Algorithm 2 and 4, for any $k = k_0 + 1, k_0 + 2, \ldots, k_0 + q - 1$, we have

$$\Delta_k \leq \Delta_{k_0} + \frac{3 \ell^2 \lambda^2}{S^2 (1 - \alpha)} \sum_{i=k_0}^{k-1} (\Delta_i + \delta_i + 2 \ell^2 \varepsilon^2_{\delta}) + \frac{(k - k_0) \ell^2 \varepsilon^2_{\delta}}{S^2},$$

11
\[ \delta_{k+1} \leq \left( \frac{2}{\mu \lambda(m+1)} + \frac{3\ell \lambda}{2 - \ell \lambda} \right) \delta_k + \left( 1 + \frac{3\ell \lambda}{2 - \ell \lambda} \right) \Delta_k + \left( 1 + \frac{2}{\mu \lambda(m+1)} + \frac{6\ell \lambda}{2 - \ell \lambda} \right) \ell^2 \varepsilon_x^2, \quad (9) \]

where \( \alpha = 1 - \frac{2\mu \ell \lambda}{\lambda + \ell} \).

Based on Lemma 4 and Lemma 5, we can upper bound \( \Delta_k \) and \( \delta_k \) by \( \mathcal{O}(\kappa^{-2} \varepsilon^2) \) as follows.

**Lemma 6.** In Algorithm 3 and 4, with parameters in Theorem 1, we have

\[ \Delta_k \leq \frac{1}{12} \kappa^{-2} \varepsilon^2 \quad \text{and} \quad \delta_k \leq \kappa^{-2} \varepsilon^2. \]

**Proof.** The setting of the parameters means

\[ \alpha = 1 - \frac{2\mu \ell \lambda}{\lambda + \ell} \leq 1 - \frac{2}{7\kappa}. \quad (10) \]

We can then prove the upper bound of \( \Delta_k \) and \( \delta_k \) by induction.

**Induction base:** The batch size \( S_1 \) and Assumption 5 means \( \Delta_0 \leq \frac{1}{24} \kappa^{-2} \varepsilon^2 < \frac{1}{12} \kappa^{-2} \varepsilon^2 \) and the output of iSARAH leads to \( \delta_0 \leq \kappa^{-2} \varepsilon^2 \).

**Induction step:** Assume that for any \( k' \leq k \), we have \( \Delta_{k'} \leq \frac{1}{12} \kappa^{-2} \varepsilon^2 \) and \( \delta_{k'} \leq \kappa^{-2} \varepsilon^2 \). If \( k = k_0 \) such that \( \text{mod}(k_0, q) = 0 \), we have \( \Delta_{k_0} \leq \frac{1}{24} \kappa^{-2} \varepsilon^2 \) based on the batch size \( S_1 \) and Assumption 5.

Let \( k'_0 = \lfloor k/q \rfloor \cdot q \). According Lemma 5, inequality (10) and the parameters setting in this theorem, we have

\[ \delta_{k+1} \leq \frac{3}{4} \delta_k + \frac{3}{2} \Delta_k + \frac{9}{4} \ell^2 \varepsilon_x^2 \leq \kappa^{-2} \varepsilon^2, \]

and

\[ \Delta_k \leq \Delta_{k'_0} + \frac{6\kappa}{7 S_2} \sum_{i=k'_0}^{k-1} (\Delta_i + \delta_i + 2\ell^2 \varepsilon_x^2) + \frac{(k - k'_0 + 1) \ell^2 \varepsilon_x^2}{S_2} \leq \frac{1}{12} \kappa^{-2} \varepsilon^2, \]

where the second inequalities in above two results are based on the induction hypothesis.

**5.3 The Proof of Theorem 1**

Based on the update of \( x_k \), we have

\[ A_k \leq -\eta_k \langle \nabla_x f(x_k, y_k), v_k \rangle + \frac{\ell \eta_k^2}{2} \| v_k \|_2^2 \]

\[ \leq \frac{\eta_k}{2} \| \nabla_x f(x_k, y_k) - v_k \|_2^2 - \left( \frac{\eta_k}{2} - \frac{\ell \eta_k^2}{2} \right) \| v_k \|_2^2, \quad (11) \]

where the first inequality is due to the average-Lipschitz of \( f \), and second comes from the Cauchy-Schwartz inequality.
The choice of step size $\eta_k$ implies that
\[
\left( \frac{\eta_k}{2} - \frac{\ell \eta_k^2}{2} \right) \|v_k\|_2^2 \geq \frac{9 \varepsilon^2}{100 \kappa \ell} \min \left( \frac{\|v_k\|_2}{\varepsilon}, \frac{\|v_k\|_2}{2 \varepsilon^2} \right)
\]
\[
\geq \frac{9 \varepsilon^2}{100 \kappa \ell} \left( \frac{\|v_k\|_2}{\varepsilon} - 2 \right) = \frac{9}{100 \kappa \ell} \left( \varepsilon \|v_k\|_2 - 2 \varepsilon^2 \right),
\]
where the first inequality is based on $\kappa \geq 1$ and the definition of $\eta_k$. The other inequality uses $\min(x, \frac{x^2}{2}) \geq |x| - 2$ for all $x$. By combining inequalities (11), (12) and taking expectation, we achieve
\[
\mathbb{E}[A_k] \leq \frac{1}{20 \kappa \ell} \mathbb{E} \|\nabla_x f(x_k, y_k) - v_k\|_2^2 - \frac{9}{100 \kappa \ell} \left( \varepsilon \mathbb{E} \|v_k\|_2 - 2 \varepsilon^2 \right)
\]
\[
\leq \frac{1}{20 \kappa \ell} \Delta_k - \frac{9}{100 \kappa \ell} \left( \varepsilon \mathbb{E} \|v_k\|_2 - 2 \varepsilon^2 \right).
\]

We can bound $\mathbb{E}[B_k]$ as follows:
\[
\mathbb{E}[B_k] \leq \mathbb{E}[f(x_{k+1}, y^*(x_{k+1})) - f(x_{k+1}, y_k)]
\]
\[
\leq \frac{1}{2\mu} \mathbb{E} \|\nabla_y f(x_{k+1}, y_{k+1})\|_2^2 = \frac{1}{2\mu} \delta_{k+1},
\]
where we use the definition of $y^*(\cdot)$ and inequality (19) in Lemma 7. Assumption 1 and the definition of $\Delta_f$ imply
\[
\Phi^* - f(x_K, y_K)
\]
\[
\leq f(x_K, y^*(x_K)) - f(x_K, y_K)
\]
\[
\leq \|\nabla_y f(x_K, y_K)\|_2^2 \leq \frac{\kappa^{-2} \varepsilon^2}{2 \mu}.
\]

By combining (13), (14), (15), Lemma 6 and summing over $k = 0, \ldots K - 1$, we obtain
\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \|v_k\|_2^2 \leq \frac{100 \kappa \ell}{9 \varepsilon} \left( \frac{1}{240 \ell} \kappa^{-1} + \frac{9}{50 \ell} \kappa^{-1} + \frac{1}{2 \mu} \kappa^{-2} \right) \varepsilon^2 + \frac{100 \kappa \ell \Delta_f}{9K \varepsilon}.
\]

Using $K = \lceil 100 \kappa \ell \varepsilon^{-2} \Delta_f / 9 \rceil$, we obtain
\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \|v_k\|_2^2 \leq \frac{100}{9} \left( \frac{1}{240} + \frac{9}{50} + \frac{1}{2} \right) \varepsilon + \varepsilon = \frac{929}{108} \varepsilon.
\]

This result and Lemma 2 implies that
\[
\mathbb{E} \|\nabla \Phi(\hat{x})\|_2 \leq \frac{1073}{108} \varepsilon.
\]

13
6 Conclusion

In this paper, we studied stochastic nonconvex-strongly-concave minimax problems. We proposed a novel algorithm called Stochastic Recursive gradiEnt Descent Ascent (SREDA). The algorithm employs variance reduction to solve minimax problems. Based on the appropriate choice of the parameters, we prove SREDA finds an $O(\varepsilon)$-stationary point of $\Phi$ with a stochastic gradient complex of $O(\kappa^3 \varepsilon^{-3})$. This result is better than state-of-the-art algorithms and optimal in its dependency on $\varepsilon$. We can also apply SREDA to the finite-sum case, and show that it performs well when $n$ is larger than $\kappa^2$.

There are still some open problems left. The complexity of SREDA is optimal with respect to $\varepsilon$, but whether it is optimal with respect to $\kappa$ is unknown. It is also possible to employ SREDA to reduce the complexity of stochastic nonconvex-concave minimax problems without the strongly-concave assumption.

References

[1] Zeyuan Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. *Journal of Machine Learning Research*, 18(1):8194–8244, 2017.

[2] Zeyuan Allen-Zhu. Natasha: Faster non-convex stochastic optimization via strongly non-convex parameter. In ICML, 2017.

[3] Zeyuan Allen-Zhu. Katyusha X: Practical momentum method for stochastic sum-of-nonconvex optimization. In ICML, 2018.

[4] Zeyuan Allen-Zhu and Elad Hazan. Variance reduction for faster non-convex optimization. In ICML, 2016.

[5] Yossi Arjevani, Yair Carmon, John C. Duchi, Dylan J. Foster, Nathan Srebro, and Blake Woodworth. Lower bounds for non-convex stochastic optimization. *arXiv preprint:1912.02365*, 2019.

[6] Antonin Chambolle and Thomas Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *Journal of mathematical imaging and vision*, 40(1):120–145, 2011.

[7] Tatjana Chavdarova, Gauthier Gidel, François Fleuret, and Simon Lacoste-Julien. Reducing noise in GAN training with variance reduced extragradient. In NIPS, 2019.

[8] Aaron Defazio. A simple practical accelerated method for finite sums. In NIPS, 2016.

[9] Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In NIPS, 2014.

[10] Simon S. Du and Wei Hu. Linear convergence of the primal-dual gradient method for convex-concave saddle point problems without strong convexity. In AISTATS, 2018.

[11] Simon S. Du, Jianshu Chen, Lihong Li, Lin Xiao, and Dengyong Zhou. Stochastic variance reduction methods for policy evaluation. In ICML, 2017.

[12] Cong Fang, Chris Junchi Li, Zhouchen Lin, and Tong Zhang. SPIDER: Near-optimal nonconvex optimization via stochastic path-integrated differential estimator. In NIPS, 2018.
[13] Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In NIPS, 2014.

[14] Ian Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and harnessing adversarial examples. arXiv preprint:1412.6572, 2014.

[15] Chi Jin, Praneeth Netrapalli, and Michael I. Jordan. What is local optimality in nonconvex-nonconcave minimax optimization? arXiv preprint:1902.00618, 2019.

[16] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In NIPS, 2013.

[17] Lihua Lei, Cheng Ju, Jianbo Chen, and Michael I. Jordan. Non-convex finite-sum optimization via SCSG methods. In NIPS, 2017.

[18] Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. Catalyst acceleration for first-order convex optimization: from theory to practice. Journal of Machine Learning Research, 18(212):1–54, 2018.

[19] Tianyi Lin, Chi Jin, and Michael I. Jordan. On gradient descent ascent for nonconvex-concave minimax problems. arXiv preprint:1906.00331, 2019.

[20] Songtao Lu, Ioannis Tsaknakis, Mingyi Hong, and Yongxin Chen. Hybrid block successive approximation for one-sided non-convex min-max problems: algorithms and applications. arXiv preprint:1902.08294, 2019.

[21] Luo Luo, Cheng Chen, Yujun Li, Guangzeng Xie, and Zhihua Zhang. A stochastic proximal point algorithm for saddle-point problems. arXiv preprint:1909.06946, 2019.

[22] Yurii Nesterov. Introductory lectures on convex optimization: A basic course, volume 87. Springer Science & Business Media, 2013.

[23] Lam M. Nguyen, Jie Liu, Katya Scheinberg, and Martin Takáč. SARAH: A novel method for machine learning problems using stochastic recursive gradient. In ICML, 2017.

[24] Lam M. Nguyen, Katya Scheinberg, and Martin Takáč. Inexact SARAH algorithm for stochastic optimization. arXiv preprint:1811.10105, 2018.

[25] Maher Nouiehed, Maziar Sanjabi, Tianjian Huang, Jason D. Lee, and Meisam Razaviyayn. Solving a class of non-convex min-max games using iterative first order methods. In NIPS, 2019.

[26] Yuyuan Ouyang and Yangyang Xu. Lower complexity bounds of first-order methods for convex-concave bilinear saddle-point problems. arXiv preprint:1808.02901, 2018.

[27] Balamurugan Palaniappan and Francis Bach. Stochastic variance reduction methods for saddle-point problems. In NIPS, 2016.

[28] Nhan H. Pham, Lam M. Nguyen, Dzung T. Phan, and Quoc Tran-Dinh. ProxSARAH: An efficient algorithmic framework for stochastic composite nonconvex optimization. arXiv preprint:1902.05679, 2019.

[29] Boris Teodorovich Polyak. Gradient methods for minimizing functionals. Zhurnal Vychislitel’noi Matematiki i Matematicheskoii Fiziki, 3(4):643–653, 1963.
[30] Hassan Rafique, Mingrui Liu, Qihang Lin, and Tianbao Yang. Non-convex min-max optimization: Provable algorithms and applications in machine learning. *arXiv preprint:1810.02060*, 2018.

[31] Sashank J. Reddi, Ahmed Hefny, Suvrit Sra, Barnabas Poczos, and Alex Smola. Stochastic variance reduction for nonconvex optimization. In *International conference on machine learning*, pages 314–323, 2016.

[32] Mark Schmidt, Nicolas Le Roux, and Francis Bach. Minimizing finite sums with the stochastic average gradient. *Mathematical Programming*, 162(1-2):83–112, 2017.

[33] Zebang Shen, Aryan Mokhtari, Tengfei Zhou, Peilin Zhao, and Hui Qian. Towards more efficient stochastic decentralized learning: Faster convergence and sparse communication. In *ICML*, 2018.

[34] Aman Sinha, Hongseok Namkoong, and John Duchi. Certifying some distributional robustness with principled adversarial training. *arXiv preprint:1710.10571*, 2017.

[35] Conghui Tan, Tong Zhang, Shiqian Ma, and Ji Liu. Stochastic primal-dual method for empirical risk minimization with O(1) per-iteration complexity. In *NIPS*, 2018.

[36] Kiran K. Thekumparampil, Prateek Jain, Praneeth Netrapalli, and Sewoong Oh. Efficient algorithms for smooth minimax optimization. In *NIPS*, 2019.

[37] Yiming Ying, Longyin Wen, and Siwei Lyu. Stochastic online AUC maximization. In *NIPS*, 2016.

[38] Junyu Zhang, Mingyi Hong, and Shuzhong Zhang. On lower iteration complexity bounds for the saddle point problems. *arXiv preprint:1912.07481*, 2019.

[39] Lijun Zhang, Mehrdad Mahdavi, and Rong Jin. Linear convergence with condition number independent access of full gradients. In *NIPS*, 2013.

[40] Yuchen Zhang and Lin Xiao. Stochastic primal-dual coordinate method for regularized empirical risk minimization. *The Journal of Machine Learning Research*, 18(1):2939–2980, 2017.
A Technical Tools

We first present some useful inequalities in convex optimization \cite{22,29} and martingale variance bound \cite{12}.

**Lemma 7** (\cite{22,29}). Suppose \( g(\cdot) \) is \( \mu \)-strongly convex and has \( \ell \)-Lipschitz gradient. Let \( w^* \) be the minimizer of \( g \). Then for any \( w \) and \( w' \), we have the following inequalities

\[
\langle \nabla g(w) - \nabla g(w'), w - w' \rangle \geq \frac{1}{\ell} \| \nabla g(w) - \nabla g(w') \|_2^2, \\
\| \nabla g(w) - \nabla g(w') \|_2 \geq \frac{\mu \ell}{\mu + \ell} \| w - w' \|_2 + \frac{1}{\mu + \ell} \| \nabla g(w) - \nabla g(w') \|_2, \\
2\mu (g(w) - g(w^*)) \leq \| \nabla g(w) \|_2^2.
\]

The inequalities (17) and (18) come from Theorem 2.1.5 and Theorem 2.1.12 of “Introductory lectures on convex optimization” \cite{22}, and the PL-inequality (19) can be found in Polyak \cite{29}.

**Lemma 8** (Lemma 1 of \cite{12}). Let \( V_k \) be estimator of \( B(z_k) \) as

\[
V_k = B_{S_*}(z_k) - B_{S_*}(z_{k-1}) + V_{k-1},
\]

where \( B_{S_*} = \sum_{B \in S_*} B_i \) satisfies

\[
\mathbb{E} [B_i(z_k) - B_i(z_{k-1}) \mid z_0, \ldots, z_{k-1}] = \mathbb{E} [V_k - V_{k-1} \mid z_0, \ldots, z_{k-1}],
\]

de \( B_i \) is \( L \)-Lipschitz continuous for any \( B_i \in S_* \). Then for all \( k = 1, \ldots, K \), we have

\[
\mathbb{E} \| V_k - B(z_k) \mid z_0, \ldots, z_{k-1} \|_2^2 \leq \| V_{k-1} - B(z_{k-1}) \|_2^2 + \frac{L^2}{|S_*|} \mathbb{E} \| z_k - z_{k-1} \|_2^2 \mid z_0, \ldots, z_{k-1} \].

B Some Results of Concave Maximizer

In this section, we show some results of concave maximizer Algorithm 4. They follow the proof of (inexact) SARAH \cite{23,24} and we present the details for completeness. We denote \( g(\cdot) = -f(x_{k+1}, \cdot) \) for fixed \( k \). We let \( \tilde{y}_k^* = \arg \min_y g(y) \) and \( \tilde{u}_{k,t} = -\tilde{u}_{k,t} \). It is obvious that \( g(\cdot) \) is \( \mu \)-strongly convex and has \( \ell \)-Lipschitz gradient. We first present some lemmas as follows.

**Lemma 9.** For Algorithm 4, we have

\[
\sum_{t=0}^m \mathbb{E} \| \nabla g(\tilde{y}_{k,t}) \|_2^2 \leq \frac{2}{\lambda} \mathbb{E} [g(\tilde{y}_{k,0}) - g(\tilde{y}_k^*)] + \sum_{t=0}^m \mathbb{E} \| \nabla g(\tilde{y}_{k,t}) - \tilde{u}_{k,t} \|_2^2 - (1 - \ell \lambda) \sum_{t=0}^m \mathbb{E} \| \tilde{u}_{k,t} \|_2^2.
\]

**Proof.** The update of Algorithm 4 implies that

\[
\mathbb{E} [g(\tilde{y}_{k,t+1})] \\
\leq \mathbb{E} [g(\tilde{y}_{k,t}) - \ell \langle \nabla g(\tilde{y}_{k,t}), \tilde{u}_{k,t} \rangle + \ell \lambda^2 2 \| \tilde{u}_{k,t} \|_2^2] \\
= \mathbb{E} [g(\tilde{y}_{t})] - \frac{\lambda}{2} \mathbb{E} \| \nabla g(\tilde{y}_t) \|_2^2 + \frac{\lambda}{2} \mathbb{E} \| \nabla g(\tilde{y}_t) - \tilde{u}_t \|_2^2 - \left( \frac{\lambda}{2} - \frac{\ell \lambda^2}{2} \right) \mathbb{E} \| \tilde{u}_t \|_2^2,
\]

\[
\frac{\lambda}{2} \mathbb{E} \| \nabla g(\tilde{y}_t) \|_2^2 + \frac{\lambda}{2} \mathbb{E} \| \nabla g(\tilde{y}_t) - \tilde{u}_t \|_2^2 - \left( \frac{\lambda}{2} - \frac{\ell \lambda^2}{2} \right) \mathbb{E} \| \tilde{u}_t \|_2^2,
\]
where the first inequality is based on the average $\ell$-Lipschitz gradient of $g$ and the equality follows from the the fact $\langle a, b \rangle = \frac{1}{2} \left( \|a\|_2^2 + \|b\|_2^2 - \|a - b\|_2^2 \right)$. By summing over $t = 0, \ldots, m$, we obtain

$$
\mathbb{E} [g(\hat{y}_{k,m+1})] \leq \mathbb{E} [g(\hat{y}_{k,0})] - \frac{\lambda}{2} \sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t})\|_2^2 + \frac{\lambda}{2} \sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_{k,t}\|_2^2 - \left( \frac{\lambda}{2} - \frac{\ell^2}{2} \right) \sum_{t=0}^{m} \mathbb{E} \|\hat{u}_{k,t}\|_2^2.
$$

This inequality is equivalent to

$$
\sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t})\|_2^2 \leq \frac{2}{\lambda} \mathbb{E} [g(\hat{y}_{k,0}) - g(\hat{y}_{k,m+1})] + \sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_{k,t}\|_2^2 - (1 - \ell \lambda) \sum_{t=0}^{m} \mathbb{E} \|\hat{u}_{k,t}\|_2^2
$$

where the second inequality follows from the fact that $\hat{y}^*_k$ is the minimizer of $g(\cdot)$. \qed

**Lemma 10.** For Algorithm 4, we have

$$
\mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_{k,t}\|_2^2 = \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|_2^2 - \sum_{j=1}^{t} \mathbb{E} \|\nabla g(\hat{y}_{k,j}) - \nabla g(\hat{y}_{k,j-1})\|_2^2 + \sum_{j=1}^{t} \mathbb{E} \|\hat{u}_{k,j} - \hat{u}_{k,j-1}\|_2^2
$$

for any $t \geq 1$.

**Proof.** Let $\mathcal{F}_j$ contains all information of $\hat{y}_{k,0}, \ldots, \hat{y}_{k,j}$ and $\hat{u}_{k,0}, \ldots, \hat{u}_{k,j-1}$. For $j \leq 1$, we have

$$
\mathbb{E} [\|\nabla g(\hat{y}_{k,j}) - \hat{u}_{k,j}\|_2^2 | \mathcal{F}_j]
$$

$$
= \|\nabla g(\hat{y}_{k,j-1}) - \hat{u}_{k,j-1}\|_2^2 + \|\nabla g(\hat{y}_{k,j}) - \nabla g(\hat{y}_{k,j-1})\|_2^2 + \mathbb{E} [\|\hat{u}_{k,j} - \hat{u}_{k,j-1}\|_2^2 | \mathcal{F}_j]
$$

$$
+ 2 \langle \nabla g(\hat{y}_{k,j-1}) - \hat{u}_{k,j-1}, \nabla g(\hat{y}_{k,j}) - \nabla g(\hat{y}_{k,j-1}) \rangle - 2 \langle \nabla g(\hat{y}_{k,j-1}) - \hat{u}_{k,j-1}, \mathbb{E} [\hat{u}_{k,j} - \hat{u}_{k,j-1} | \mathcal{F}_j] \rangle
$$

By taking expectation in the above equation, we have

$$
\mathbb{E} \|\nabla g(\hat{y}_{k,j}) - \hat{u}_{k,j}\|_2^2 = \mathbb{E} \|\nabla g(\hat{y}_{k,j-1}) - \hat{u}_{k,j-1}\|_2^2 - \mathbb{E} \|\nabla g(\hat{y}_{k,j}) - \nabla g(\hat{y}_{k,j-1})\|_2^2 + \mathbb{E} [\|\hat{u}_{k,j} - \hat{u}_{k,j-1}\|_2^2 | \mathcal{F}_j].
$$

By summing over $j = 1, \ldots, t$ for any $t \geq 1$, we have

$$
\mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_{k,t}\|_2^2 = \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|_2^2 - \sum_{j=1}^{t} \mathbb{E} \|\nabla g(\hat{y}_{k,j}) - \nabla g(\hat{y}_{k,j-1})\|_2^2 + \sum_{j=1}^{t} \mathbb{E} [\|\hat{u}_{k,j} - \hat{u}_{k,j-1}\|_2^2 | \mathcal{F}_j].
$$

\qed

**Lemma 11.** In Algorithm 4 with any $\lambda < 2/\ell$ and $t \geq 1$, we have

$$
\mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_{k,t}\|_2^2 = \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|_2^2 + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} [\|\hat{u}_{k,0}\|_2^2).
$$
Proof. Let $F_j$ be the definition in the proof of Lemma 10. For any $j \geq 0$, we let

$$
\nabla g_j(y) = -\frac{1}{S_2} \sum_{i=1}^{S_2} G_y(x_{k+1}, y; \xi_{j,i}),
$$

where $\{\xi_{j,i}\}_{i=1}^{S_2}$ comes from line 8 of Algorithm 4 with $t = j$. We can then write $\hat{u}_{k,j} = \hat{u}_{k,j-1} + \nabla g_j(\hat{y}_{k,j}) - \nabla g_j(\hat{y}_{k,j-1})$ and the update in Algorithm 4 implies

$$
\begin{align*}
E[\|\hat{u}_{k,j}\|^2_2 | F_j] &= \|\hat{u}_{k,j-1}\|^2_2 + \mathbb{E} \left[ \|\nabla g_j(\hat{y}_{k,j}) - \nabla g_j(\hat{y}_{k,j-1})\|_2^2 - \frac{2}{\lambda} (\nabla g_j(\hat{y}_{k,j-1}) - \nabla g_j(\hat{y}_{k,j}), \hat{y}_{t-1} - \hat{y}_t) | F_j \right] \\
&\leq \|\hat{u}_{k,j-1}\|^2_2 + \mathbb{E} \left[ \|\nabla g_j(\hat{y}_{k,j}) - \nabla g_j(\hat{y}_{k,j-1})\|_2^2 - \frac{2}{\ell \lambda} \|\nabla g_j(\hat{y}_{k,j-1}) - \nabla g_j(\hat{y}_{k,j})\|_2^2 | F_j \right] \\
&= \|\hat{u}_{k,j-1}\|^2_2 + \left( 1 - \frac{2}{\ell \lambda} \right) \mathbb{E} \left[ \|\nabla g_j(\hat{y}_{k,j-1}) - \nabla g_j(\hat{y}_{k,j})\|_2^2 | F_j \right],
\end{align*}
$$

where the inequality comes from (17) of Lemma 7.

By taking expectation for the above result and summing over $j = 1, \ldots, t$, we have

$$
\begin{align*}
\sum_{j=1}^{t} \mathbb{E} \|\hat{u}_{k,j} - \hat{u}_{k,j-1}\|^2_2 &\leq \frac{\ell \lambda}{2 - \ell \lambda} \left( \mathbb{E} \|\hat{u}_{k,0}\|^2_2 - \mathbb{E} \|\hat{u}_{k,t}\|^2_2 \right),
\end{align*}
$$

(20)

where we use the assumption $\lambda \leq 2/\ell$.

We can now achieve the desired result as follows

$$
\begin{align*}
\mathbb{E} &\|\nabla g(\hat{y}_{k,j}) - \hat{u}_{k,j}\|^2_2 \\
&\leq \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|^2_2 + \sum_{j=1}^{t} \mathbb{E} \|\hat{u}_{k,j} - \hat{u}_{k,j-1}\|^2_2 \\
&\leq \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|^2_2 + \frac{\ell \lambda}{2 - \ell \lambda} \left( \mathbb{E} \|\hat{u}_{k,0}\|^2_2 - \mathbb{E} \|\hat{u}_{k,t}\|^2_2 \right) \\
&\leq \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|^2_2 + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} \|\hat{u}_{k,0}\|^2_2,
\end{align*}
$$

where the first inequality follows from Lemma 10 and the second one comes from equation (20). \qed

We can now present the main result for the concave maximizer, which describes how the magnitude of gradient with respect to $y$ changes after executing Algorithm 4. This result will be used later to prove Lemma 5.

Lemma 12. In Algorithm 4, for any $k \geq 1$ and $\lambda \leq 1/\ell$, we have

$$
\mathbb{E} \|\nabla g(\hat{y}_{k,s_k})\|^2_2 \leq \frac{2}{\mu \lambda (m + 1)} \|\nabla g(\hat{y}_{k,0})\|^2_2 + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} \|\hat{u}_{k,0}\|^2_2 + \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|^2.
$$

Proof. By summing over the result of Lemma 11 for $t = 0, \ldots, m$, we obtain

$$
\sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_{k,t}\|^2_2 = (m + 1) \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|^2 + \frac{(m + 1) \ell \lambda}{2 - \ell \lambda} \mathbb{E} \|\hat{u}_{k,0}\|^2_2.
$$
Now this can be combined with Lemma 9 to obtain
\[
\sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t})\|_2^2 \leq \frac{2}{\lambda} \mathbb{E} [g(\hat{y}_{k,0}) - g(\hat{y}_k^*)] + \sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_t\|_2^2 - (1 - \ell \lambda) \sum_{t=0}^{m} \mathbb{E} \|\hat{u}_{k,t}\|_2^2 \tag{21}
\]
\[
\leq \frac{2}{\lambda} \mathbb{E} [g(\hat{y}_{k,0}) - g(\hat{y}_k^*)] + \sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t}) - \hat{u}_{k,t}\|_2^2
\]
\[
\leq \frac{2}{\lambda} \mathbb{E} [g(\hat{y}_{k,0}) - g(\hat{y}_k^*)] + \frac{(m + 1) \ell \lambda}{2 - \ell \lambda} \mathbb{E} \|\hat{u}_{k,0}\|_2^2 + (m + 1) \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|_2^2.
\]
We can now prove this theorem as follows:
\[
\mathbb{E} \|\nabla g(\hat{y}_{k,s_{k+1}})\|_2^2 = \frac{1}{m + 1} \sum_{t=0}^{m} \mathbb{E} \|\nabla g(\hat{y}_{k,t})\|_2^2 \leq \frac{2}{\lambda(m + 1)} \mathbb{E} [g(\hat{y}_{k,0}) - g(\hat{y}_k^*)] + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} \|\hat{u}_{k,0}\|_2^2 + \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|_2^2 \leq \frac{2}{\mu \lambda (m + 1)} \mathbb{E} \|\nabla g(\hat{y}_{k,0})\|_2^2 + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} \|\hat{u}_{k,0}\|_2^2 + \mathbb{E} \|\nabla g(\hat{y}_{k,0}) - \hat{u}_{k,0}\|_2^2,
\]
where the equality is due to the output of Algorithm 4. The first inequality is based on inequality (21), and the second one comes from inequality (19) of Lemma 7.

Now we present the detailed proof of Lemma 4 which is used in Section 5.

**B.1 The Proof of Lemma 4**

**Proof.** We use the same definition of $\mathcal{F}_t$ and $\nabla g_j(\cdot)$ as in Lemma 10 and 11. We have
\[
\mathbb{E} \left[ \|\hat{u}_{k,t}\|_2^2 \mid \mathcal{F}_t \right] = \|\hat{u}_{k,t-1}\|_2^2 + \mathbb{E} \left[ \|\nabla g_t(\hat{y}_{k,t-1}) - \nabla g_t(\hat{y}_{k,t-1})\|_2^2 - \frac{2}{\lambda} (\hat{y}_{k,t-1} - \hat{y}_{k,t}, \nabla g_t(\hat{y}_{k,t-1}) - \nabla g_t(\hat{y}_{k,t}))(\hat{u}_{k,t-1}) \mid \mathcal{F}_t \right]
\]
\[
\leq \|\hat{u}_{k,t-1}\|_2^2 + \mathbb{E} \left[ \|\nabla g_t(\hat{y}_{k,t-1}) - \nabla g_t(\hat{y}_{k,t-1})\|_2^2 - \frac{2 \mu \ell \lambda}{\mu + \ell} \|\hat{u}_{k,t-1}\|_2^2 - \frac{2}{\lambda(\mu + \ell)} \|\nabla g_t(\hat{y}_{k,t-1}) - \nabla g_t(\hat{y}_{k,t})\|_2 \mid \mathcal{F}_t \right]
\]
\[
= \left( 1 - \frac{2 \mu \ell \lambda}{\mu + \ell} \right) \|\hat{u}_{k,t-1}\|_2^2 + \left( 1 - \frac{2}{\lambda(\mu + \ell)} \right) \mathbb{E} \left[ \|\nabla g_t(\hat{y}_{k,t-1}) - \nabla g_t(\hat{y}_{k,t})\|_2 \mid \mathcal{F}_t \right]
\]
where the second inequality is based on result (18) of Lemma 7 and the last inequality is due to $\lambda \leq 2/(\mu + \ell)$. The definition of $\hat{u}_{k,t}$ means that we have proved the theorem.

**C The proof of Lemma 5**

**Proof.** Let
\[
\bar{\Delta}_{k,t} = \mathbb{E} \left( \|\hat{v}_{k,t} - \nabla_x f(\hat{x}_{k,t}, \hat{y}_{k,t})\|_2^2 + \|\hat{u}_{k,t} - \nabla_y f(\hat{x}_{k,t}, \hat{y}_{k,t})\|_2^2 \right).
\]
Then we have

\[
\bar{\Delta}_{k,0} = \mathbb{E} \left( \| \hat{v}_{k,0} - \nabla_x f(\hat{x}_{k,0}, \hat{y}_{k,0}) \|^2 + \| \hat{u}_{k,0} - \nabla_y f(\hat{x}_{k,0}, \hat{y}_{k,0}) \|^2 \right)
\leq \mathbb{E} \left( \| v_k - \nabla_x f(x_k, y_k) \|^2 + \| u_k - \nabla_y f(x_k, y_k) \|^2 \right) + \frac{\ell^2}{S_2} \mathbb{E} \left( \| x_{k,0} - x_k \|^2 + \| y_{k,0} - y_k \|^2 \right)
= \Delta_k + \frac{\ell^2}{S_2} \mathbb{E} \left( \| x_{k+1} - x_k \|^2 + \| y_{k+1} - y_k \|^2 \right)
\leq \Delta_k + \frac{\ell^2 \varepsilon_k^2}{S_2},
\]

where the first inequality comes from Lemma 8 by letting \( B(\cdot) = \nabla f(\cdot) \) and \( V_k = (v_k, u_k) \). Now for any \( k \geq 1 \), we have

\[
\Delta_k = \mathbb{E} \left( \| v_k - \nabla_x f(x_k, y_k) \|^2 + \| u_k - \nabla_y f(x_k, y_k) \|^2 \right)
= \bar{\Delta}_{k-1, s_{k-1}+1}
\leq \mathbb{E} \left( \| \hat{v}_{k-1,0} - \nabla_y f(\hat{x}_{k-1,0}, \hat{y}_{k-1,0}) \|^2 + \| \hat{u}_{k-1,0} - \nabla_y f(\hat{x}_{k-1,0}, \hat{y}_{k-1,0}) \|^2 \right)
+ \frac{\ell^2 s_{k-1}}{S_2} \sum_{t=0}^{s_{k-1}} \left( \| \hat{x}_{k-1,t+1} - \hat{x}_{k-1,t} \|^2 + \| \hat{y}_{k-1,t+1} - \hat{y}_{k-1,t} \|^2 \right)
= \bar{\Delta}_{k-1,0} + \frac{\ell^2 s_{k-1}}{S_2} \sum_{t=0}^{s_{k-1}} \lambda^2 \| \hat{u}_{k-1,t} \|^2
\leq \bar{\Delta}_{k-1,0} + \frac{\ell^2 \lambda^2 s_{k-1}}{S_2} \sum_{t=0}^{s_{k-1}} \alpha' \| \hat{u}_{k-1,t} \|^2
\leq \bar{\Delta}_{k-1,0} + \frac{\ell^2 \lambda^2}{S_2 (1 - \alpha)} \| \hat{u}_{k-1,0} \|^2
\]

where the first inequality is due to Lemma 8 and the second inequality comes from Lemma 9.

Now we can bound the quantity \( \| \hat{u}_{k,0} \|^2 \) as follows

\[
\mathbb{E} \| \hat{u}_{k,0} \|^2
= \mathbb{E} \| \hat{u}_{k,0} - \nabla_y f(x_{k+1}, y_k) + \nabla_y f(x_{k+1}, y_k) - \nabla_y f(x_k, y_k) + \nabla_y f(x_k, y_k) \|^2
\leq 3 \left( \mathbb{E} \| \hat{u}_{k,0} - \nabla_y f(x_{k+1}, y_k) \|^2 + \mathbb{E} \| \nabla_y f(x_{k+1}, y_k) - \nabla_y f(x_k, y_k) \|^2 + \mathbb{E} \| \nabla_y f(x_k, y_k) \|^2 \right)
\leq 3 \left( \| \hat{u}_{k,0} - \nabla_y f(x_{k,0}, \hat{y}_{k,0}) \|^2 + \ell^2 \varepsilon_k^2 + \delta_k \right)
= 3 \left( \bar{\Delta}_{k,0} + \delta_k + \ell^2 \varepsilon_k^2 \right)
\leq 3 \left( \Delta_k + \delta_k + 2\ell^2 \varepsilon_k^2 \right)
\]

where the first inequality comes from the fact \( \| a + b + c \|^2 \leq 3 \left( \| a \|^2 + \| b \|^2 + \| c \|^2 \right) \), the second inequality is based on the average Lipschitz gradient of \( f \), and the third inequality is due to inequality (22).
Combining the results of (23) and (24), we have

$$\Delta_k \leq \Delta_{k-1,0} + \frac{\ell^2 \varepsilon_x^2}{S_2(1-\alpha)} \| \tilde{u}_{k-1,0} \|^2$$

$$\leq \Delta_{k-1} + \frac{\ell^2 \varepsilon_x^2}{S_2} + \frac{3\ell^2 \varepsilon_x^2}{S_2} (\Delta_{k-1} + \delta_{k-1} + 2\ell^2 \varepsilon_x^2)$$

$$= \Delta_{k-1} + \frac{3\ell^2 \varepsilon_x^2}{S_2(1-\alpha)} (\Delta_{k-1} + \delta_{k-1} + 2\ell^2 \varepsilon_x^2) + \frac{\ell^2 \varepsilon_x^2}{S_2}.$$ 

Summing over the above inequality from $k_0$ recursively, we can prove the first part of this theorem

$$\Delta_k \leq \Delta_{k-1} + \frac{3\ell^2 \varepsilon_x^2}{S_2(1-\alpha)} (\Delta_{k-1} + \delta_{k-1} + 2\ell^2 \varepsilon_x^2) + \frac{\ell^2 \varepsilon_x^2}{S_2}$$

$$\leq \Delta_{k_0} + \frac{3\ell^2 \varepsilon_x^2}{S_2(1-\alpha)} \sum_{i=0}^{k-1} (\Delta_i + g_i + 2\ell^2 \varepsilon_x^2) + \frac{(k-k_0)\ell^2 \varepsilon_x^2}{S_2}.$$ 

We bound the quantity $\delta_{k+1}$ as follows:

$$\delta_{k+1} = \mathbb{E} \| \nabla f(x_{k+1}, y_{k+1}) \|^2$$

$$\leq \frac{1}{\mu \lambda (m+1)} \mathbb{E} \| \nabla f(x_{k+1}, y_k) \|^2 + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} \| \tilde{u}_{k,0} \|^2 + \mathbb{E} \| \nabla f(x_{k,0}, y_{k,0}) - \tilde{u}_{k,0} \|^2$$

$$= \frac{1}{\mu \lambda (m+1)} \left( \| \nabla f_y(x_{k+1}, y_k) - \nabla f_y(x_k, y_k) + \nabla f_y(x_k, y_k) \|^2 \right) + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} \| \tilde{u}_{k,0} \|^2 + \Delta_{k,0}$$

$$\leq \frac{2}{\mu \lambda (m+1)} \left( \| \nabla f_y(x_{k+1}, y_k) - \nabla f_y(x_k, y_k) \|^2 + \| \nabla f_y(x_k, y_k) \|^2 \right) + \frac{\ell \lambda}{2 - \ell \lambda} \mathbb{E} \| \tilde{u}_{k,0} \|^2 + \Delta_{k,0}$$

$$\leq \frac{2}{\mu \lambda (m+1)} \left( \ell^2 \varepsilon_x^2 + \delta_k \right) + \frac{3\ell \lambda}{2 - \ell \lambda} \left( \Delta_k + \delta_k + 2\ell^2 \varepsilon_x^2 \right) + \Delta_k + \frac{\ell^2 \varepsilon_x^2}{S_2}$$

$$\leq \frac{2}{\mu \lambda (m+1)} \left( \ell^2 \varepsilon_x^2 + \delta_k \right) + \frac{3\ell \lambda}{2 - \ell \lambda} \left( \Delta_k + \delta_k + 2\ell^2 \varepsilon_x^2 \right) + \Delta_k + \frac{\ell^2 \varepsilon_x^2}{S_2}$$

which is equivalent to the second part of this theorem. In the above derivation, the first inequality is according to Lemma [12], the second inequality is based on Young’s inequality, and the fourth inequality comes form the result of (24). The other steps are based on definitions.

\[ \square \]

**D The proof of Lemma [2]**

**Proof.** Consider $\nabla \Phi(x) = \nabla_x f(x, y^*(x))$. We have

$$\mathbb{E} \| \nabla \Phi(x_k) - \nabla_x f(x_k, y_k) \|^2$$

$$= \mathbb{E} \| \nabla_x f(x_k, y^*(x_k)) - \nabla_x f(x_k, y_k) \|^2$$

$$\leq \ell^2 \mathbb{E} \| y^*(x_k) - y_k \|^2$$

$$\leq \frac{\ell^2}{\mu^2} \mathbb{E} \| \nabla_y f(x_k, y^*(x_k)) - \nabla_y f(x_k, y_k) \|^2.$$
By Jensen’s inequality, we have

\[
\left( \mathbb{E} \| \nabla \Phi(x_k) - \nabla_x f(x_k, y_k) \|_2 \right)^2 \leq \mathbb{E} \| \nabla \Phi(x_k) - \nabla_x f(x_k, y_k) \|_2^2 \leq \varepsilon^2,
\]

which means

\[
\mathbb{E} \| \nabla \Phi(x_k) \|_2 = \mathbb{E} \| \nabla_x f(x_k, y_k) - (\nabla_x f(x_k, y_k) - \nabla \Phi(x_k)) \|_2
\leq \mathbb{E} \| \nabla_x f(x_k, y_k) \|_2 + \mathbb{E} \| \nabla_x f(x_k, y_k) - \nabla \Phi(x_k) \|_2
\leq \mathbb{E} \| \nabla_x f(x_k, y_k) \|_2 + \varepsilon.
\]

Similarly, we can prove

\[
\left( \mathbb{E} \| v_k - \nabla_x f(x_k, y_k) \|_2 \right)^2 \leq \mathbb{E} \| v_k - \nabla_x f(x_k, y_k) \|_2^2 \leq \Delta_k \leq \frac{1}{12} \kappa^{-2} \varepsilon^2,
\]

and

\[
\mathbb{E} \| \nabla_x f(x_k, y_k) \|_2 = \mathbb{E} \| v_k - (v_k - \nabla_x f(x_k, y_k)) \|_2
\leq \mathbb{E} \| v_k \|_2 + \mathbb{E} \| v_k - \nabla_x f(x_k, y_k) \|_2
\leq \mathbb{E} \| v_k \|_2 + \frac{1}{\sqrt{12}} \kappa^{-1} \varepsilon
\leq \mathbb{E} \| v_k \|_2 + \frac{3}{\sqrt{12}} \varepsilon.
\]

By combining the above results, we obtain

\[
\mathbb{E} \| \nabla \Phi(x_k) \|_2 \leq \mathbb{E} \| \nabla_x f(x_k, y_k) \|_2 + \varepsilon
\leq \mathbb{E} \| v_k \|_2 + \frac{1}{\sqrt{12}} \varepsilon + \varepsilon
\leq \mathbb{E} \| v_k \|_2 + \frac{4}{3} \varepsilon.
\]

\[\Box\]

E. The proof of Theorem 2

In the finite-sum case, we use the full gradient to replace the large batch sample size in stochastic case. The initialization of \( y_0 \) in this case is based on SARAH [23]. We can prove Theorem 2 with minor modifications on the analysis of Theorem 1.

E.1 Initialization by SARAH

We present the detailed procedure of SARAH [23] in Algorithm 7, which is used to initialize \( y_0 \) in SREDA for problem (3) (line 2 of Algorithm 5). We consider the following convex optimization problem

\[
\min_w h(w) \triangleq \frac{1}{n} \sum_{i=1}^{n} H(w; \xi_i),
\]

where \( H \) is average \( \ell \)-Lipschitz gradient and convex, \( h \) is \( \mu \)-strongly convex, and \( \xi_i \) is a random vector. We have the following convergence result by using SARAH to solve problem (25).
Algorithm 7 SARAH

1: Input \( \bar{w}_0 \), learning rate \( \gamma > 0 \), inner loop size \( m' \)
2: for \( s = 1, \ldots, T \) do
3: \( w_0 = \bar{w}_{s-1} \)
4: \( v_0 = \frac{1}{n} \sum_{i=1}^{n} \nabla H(w_0; \xi_i) \)
5: \( w_1 = w_0 - \gamma v_0 \)
6: for \( t = 1, \ldots, m' - 1 \) do
7: draw a samples \( \xi_t \)
8: \( v_t = \nabla H(w_t; \xi_t) - \nabla H(w_{t-1}; \xi_t) + v_{t-1} \)
9: \( w_{t+1} = w_t - \gamma v_t \)
10: end for
11: \( \tilde{w}_s = w_{t'}, \) where \( t' \) is uniformly sampled from \( \{0, 1, \ldots, m'\} \)
12: end for
13: Output: \( \tilde{w}_s \)

Lemma 13 (Corollary 3 of [23]). Run Algorithm 7 on problem (25) by setting

\[ \gamma = \frac{1}{2\ell}, \quad m' = \lceil 4.5\kappa \rceil, \quad \text{and} \quad T = \left\lceil \log \left( \frac{\|\nabla h(\tilde{w}_0)\|_2 / \zeta}{\log(9/7)} \right) \right\rceil. \]

Then, we obtain \( \tilde{w}_T \) such that \( \mathbb{E} \| \nabla \Phi(\tilde{x}) \|_2^2 \leq \zeta \) with \( (m' + n)T = \mathcal{O} \left( (n + \kappa) \log(1/\zeta) \right) \) stochastic gradient evaluations.

According to Lemma 13 [23] with \( h(\cdot) = -f(x_0, \cdot) \) and \( \zeta = \kappa^{-2}\varepsilon^2 \), we can set \( \tilde{w}_T \) as \( y_0 \) so that \( \mathbb{E} \| \nabla f(x_0, y_0) \|_2 \leq \kappa^{-2}\varepsilon^2 \) by taking

\[ \gamma = \frac{1}{2\ell}, \quad m' = \lceil 4.5\kappa \rceil, \quad \text{and} \quad T = \left\lceil \log \left( \frac{\|\kappa^2 \nabla h(\tilde{w}_0)\|_2^2 / \varepsilon^2}{\log(9/7)} \right) \right\rceil. \]

The total number of stochastic gradient evaluations is \( (b + m')T = \mathcal{O} \left( (n + \kappa) \log(\kappa/\varepsilon) \right) \).

E.2 The case of \( n \geq \kappa^2 \)

We set the parameters

\[ \zeta = \kappa^{-2}\varepsilon^2, \quad \eta_k = \min \left( \frac{\varepsilon}{5\kappa \ell \|v_k\|_2}, \frac{1}{10\kappa \ell} \right), \quad \lambda = \frac{2}{7\ell}, \quad q = \lceil \kappa^{-1} n^{1/2} \rceil, \]

\[ S_2 = \left\lceil \frac{7368}{175} q \right\rceil, \quad K = \left\lceil \frac{100\kappa \ell \varepsilon^{-2} \Delta_f}{9} \right\rceil, \quad \text{and} \quad m = \lceil 28\kappa - 1 \rceil. \]

Then the quantity \( \Delta_{k_0} \) is zero for any \( k_0 \) with \( \text{mod} \ (k_0, q) = 0 \). All analysis follows from that of Section 5 until (16) is satisfied. This because these different choices of \( q, \Delta_{k_0}, \) and the initialization of \( y_0 \) do not affect the proof of Lemma 6. Therefore we obtain

\[ \mathbb{E} \| \nabla \Phi(\tilde{x}) \|_2 \leq \frac{1073}{108} \varepsilon \]
by the parameters setting above. The total complexity is

\[
\text{StocGrad (Total)} = \text{StocGrad (SARAH)} + \text{StocGrad (FullGradient)} + \text{StocGrad (ConcaveMaximizer)}
\]

\[
= \mathcal{O}((n + \kappa) \log(\kappa/\varepsilon)) + \mathcal{O}\left(\frac{K}{q} \cdot n\right) + \mathcal{O}(K \cdot S_2 \cdot m)
\]

\[
= \mathcal{O}((n + \kappa) \log(\kappa/\varepsilon)) + \mathcal{O}\left(\frac{\kappa \varepsilon^{-2}}{\kappa^{-1} n^{1/2}} \cdot n\right) + \mathcal{O}\left(\kappa \varepsilon^{-2} \cdot n^{1/2} \cdot \kappa\right)
\]

\[
\leq \mathcal{O}\left(n \log(\kappa/\varepsilon) + \kappa^2 n^{1/2} \varepsilon^{-2}\right).
\]

### E.3 The case of \( n \leq \kappa^2 \)

We set the parameters

\[
\zeta = \kappa^{-2} \varepsilon^2, \; \eta_k = \min\left(\frac{\varepsilon}{5\kappa \ell \|v_k\|_2}, \frac{1}{10\kappa \ell}\right), \; \lambda = \frac{2}{7\ell}, \; q = 1,
\]

\[
S_2 = 1, \; K = \left\lceil\frac{100\kappa \ell \varepsilon^{-2} \Delta_f}{9}\right\rceil, \; \text{and} \; m = [28\kappa - 1].
\]

Then, we have \( \Delta_k = 0 \) for all \( k \) since \( q = 1 \). The analysis until Appendix B still holds but we need to refine some proofs of Section 5. By substituting \( \Delta_k = 0 \) in (9), we have

\[
\delta_{k+1} \leq \left(\frac{2}{\mu \lambda(m + 1)} + \frac{3\ell \lambda}{2 - \ell \lambda}\right) \delta_k + \left(1 + \frac{2}{\mu \lambda(m + 1)} + \frac{6\ell \lambda}{2 - \ell \lambda}\right) \ell^2 \varepsilon_x^2
\]

\[
\leq \frac{3}{4} \delta_k + \frac{9}{4} \ell^2 \varepsilon_x^2.
\]

This result combined with the initialization of \( \delta_0 \leq \kappa^{-2} \varepsilon^2 \) imply that we have \( \delta_k \leq \kappa^{-2} \varepsilon^2 \) for any \( k \geq 0 \). Consequently, the upper bound of \( B_k \) in inequality (11) still holds. The fact that \( \Delta_k = 0 \) also indicates the upper bound of \( A_k \) in inequality (13) will be tighter. Hence, all results from (11) to (16) still hold, and we can obtain

\[
\mathbb{E} \|\nabla \Phi(\hat{x})\|_2 \leq \frac{1073}{108} \varepsilon.
\]

The total complexity of the algorithm is

\[
\text{StocGrad (Total)} = \text{StocGrad (SARAH)} + \text{StocGrad (FullGradient)} + \text{StocGrad (ConcaveMaximizer)}
\]

\[
= \mathcal{O}((n + \kappa) \log(\kappa/\varepsilon)) + \mathcal{O}\left(\frac{K}{q} \cdot n\right) + \mathcal{O}(K \cdot S_2 \cdot m)
\]

\[
= \mathcal{O}((n + \kappa) \log(\kappa/\varepsilon)) + \mathcal{O}\left(\frac{\kappa \varepsilon^{-2}}{1} \cdot n\right) + \mathcal{O}(\kappa \varepsilon^{-2} \cdot 1 \cdot \kappa)
\]

\[
= \mathcal{O}\left((\kappa^2 + \kappa n) \varepsilon^{-2}\right).
\]