LOCAL COCYCLES AND CENTRAL EXTENSIONS FOR
MULTI-POINT ALGEBRAS OF KRICEVER-NOVIKOV TYPE

MARTIN SCHLICHENMAIER

Abstract. Multi-point algebras of Krichever-Novikov type for higher genus Riemann surfaces are generalisations of the Virasoro algebra and its related algebras. Complete existence and uniqueness results for local 2-cocycles defining almost-graded central extensions of the functions algebra, the vector field algebra, and the differential operator algebra (of degree \( \leq 1 \)) are shown. This is applied to the higher genus, multi-point affine algebras to obtain uniqueness for almost-graded central extensions of the current algebra of a simple finite-dimensional Lie algebra. An earlier conjecture of the author concerning the central extension of the differential operator algebra induced by the semi-infinite wedge representations is proved.

1. Introduction

Algebras of Krichever-Novikov type are important examples of infinite-dimensional associative algebras or Lie algebras. They generalize the Witt algebra, its universal central extension (the Virasoro algebra) and related algebras like the untwisted affine (Kac-Moody) algebra. One way to describe the Witt algebra is to define it as the algebra of those meromorphic vector fields on the Riemann sphere \( S^2 = \mathbb{P}^1(\mathbb{C}) \) which have only poles at 0 and \( \infty \). It admits a standard basis \( \{ e_n = z^{n+1} \frac{dz}{dz}, \ n \in \mathbb{Z} \} \). The Lie structure is the Lie bracket of vector fields. One obtains immediately

\[
[e_n, e_m] = (m - n)e_{n+m}.
\]

By introducing the degree \( \deg(e_n) := n \) it becomes a graded Lie algebra. In such a way all related algebras to the Witt algebra can be given as meromorphic objects on \( S^2 \) which are holomorphic outside 0 and \( \infty \). Of special importance besides the vector field algebra are the function algebra, i.e. the algebra of Laurent polynomials \( \mathbb{C}[z, z^{-1}] \), the (Lie algebra) \( g \)-valued meromorphic functions, i.e. the loop or current algebra \( g \otimes \mathbb{C}[z, z^{-1}] \) with structure

\[
[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}, \quad x, y \in g, \quad n, m \in \mathbb{Z},
\]

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and their central extensions, the Virasoro algebra, the Heisenberg algebra and the untwisted affine (Kac-Moody) algebras.

If one replaces $S^2$ by higher genus compact Riemann surfaces (or equivalently by smooth projective curves over $\mathbb{C}$) and allows, instead at two points, poles at a set $A$ of finitely many points which is divided into two disjoint nonempty subsets $I$ and $O$, one obtains in a similar way the algebras of Krichever-Novikov type (see Section 2 for details). For higher genus and two points the vector field algebra and the function algebra were introduced by Krichever and Novikov [13, 14, 15], the corresponding affine algebras by Sheinman [36, 37]. Its multi-point generalization were given by the author [24, 25, 26, 27], including the function algebra, the vector field algebra, the differential operator algebra, and the current algebra [29, 30, 33] and their central extensions.

For the algebras related to the Virasoro algebra the fact that they are graded is of importance in many contexts. In particular, this plays a role in their representation theory (e.g. highest weight representations, Verma modules, etc.). It turns out that a weaker concept, an almost-grading, will be enough to guarantee the availability of certain methods in representation theory of infinite-dimensional algebras. Almost-grading means that for pairs of homogeneous elements of degree $n$ and $m$ the result is in a fixed range (not depending on $n$ and $m$) around the “ideal” value $n + m$ (see Definition 2.2).

In the works cited above it is shown that there exists for any splitting of $A$ into $I \cup O$ a grading, such that the algebras and their modules are almost-graded.

Krichever-Novikov type algebras appear e.g. in string theory, in conformal field theory and also in the theory of integrable models. In particular, in closed string theory in the interpretation of the Riemann surface $M$ as possible world sheet of the theory, the points in $I$ correspond to free incoming strings and the points in $O$ to free outgoing strings. The non-simply-connectedness of $M$ corresponds to string creation, annihilation and interaction. Furthermore these algebras have relations to moduli spaces, e.g. [34, 35].

In all the above-mentioned fields the passage to central extensions of the algebras are of fundamental importance. Typically, by some necessary regularization procedure one obtains only projective representations of the involved algebra which can be given as linear representations of a suitable central extension. Such a central extension is given by a 2-cocycle of the Lie algebra cohomology with values in the trivial module. For the representation theory it is fundamental to extend the almost-grading to the central extension. This requires that the defining 2-cocycle is local, where we understand by a local cocycle a cocycle which vanishes if calculated for pairs of homogeneous elements of degree $n$ and $m$ if the sum $n + m$ lies outside a certain fixed range (not depending on $n$ and $m$).

For the considered algebras there are certain cocycles geometrically defined. These cocycles are given in (3.10), (3.14), and (3.26). They are obtained by integration over cycles on the Riemann surface with the points in $A$ removed. If one chooses as integration cycle a cycle $C_S$ which separates the points in $I$ from the points in $O$ one obtains a local cocycle with respect to the almost-grading introduced by the splitting $A = I \cup O$.

In this article I show that (up to coboundary) all local cocycles are scalar multiple of the above mentioned geometric cocycles obtained by integration along a separating
cycle. The result is formulated in Theorems 4.3, 4.5, 4.6, and 4.8. In the function algebra case one obtains uniqueness only if one requires the cocycle to be a multiplicative or a $L$-invariant cocycle (see Definition 3.3). These properties are typically fulfilled in the applications under consideration. In particular, we obtain

$$\dim H^2_{loc,*}(A, C) = \dim H^2_{loc}(L, C) = 1, \quad \dim H^2_{loc}(D^1, C) = 3.$$  

Here $A$ denotes the function algebra, $L$ the vector field algebra, and $D^1$ the algebra of differential operators of degree $\leq 1$, $H_{loc}$ denotes the subspace of cohomology classes containing at least one local cocycle, and $H^2_{loc,*}(A, C)$ denotes local cocycles which are (equivalently) either multiplicative or $L$-invariant.

Clearly, the classical case ($g = 0$ and two points) is contained as a special case in the general results. In the classical case the result for the Witt algebra is the well-known fact, that the Virasoro algebra is the universal central extension of the Witt algebra. The statement for the differential operator algebra in the classical case was shown by Arbarello, De Concini, Kac and Procesi [1]. For the vector field algebra in higher genus with two points Krichever and Novikov supplied a proof of the uniqueness in a completely different manner than presented here. Assuming that every cocycle is of geometric origin they used “discrete Baker-Akhieser functions” to identify the integration cycle [13, 14].

The content of the article is as follows. In Section 2 the necessary basic informations about the geometric setup and the studied algebras and its modules are given. In Section 3 central extensions and cocycles are studied. In Section 4 local cocycles are introduced and the main results about uniqueness are formulated. Section 5 contains the proofs. The technique presented there involves the almost-grading and consists essentially in setting-up a suitable recursion between different levels. In the vector field and differential operator algebra case the explicit description of the basis elements via rational functions and theta function respectively is needed.

In [27, 29] the author formulated a conjecture about the cocycle of the differential operator algebra associated to a representation on the semi-infinite wedge forms of weight $\lambda$. In Section 6 it is shown that the conjecture follows from the results obtained in this article (Theorem 6.4). In particular, the identified cocycle extends to the whole differential operator algebra of arbitrary degree.

Section 7 deals with an application to central extensions of current algebras $g \otimes A$. In particular, if $g$ is a finite-dimensional simple Lie algebra any almost-graded central extension of $g \otimes A$ is obtained by a scalar multiple of a geometric cocycle for which the integration is over a separating cycle, see Theorem 7.3.

There are some articles addressing the different question of determining the full cohomology space (or at least its dimension) of some of the algebras considered here. For the vector field algebra $L$ see for example results by Wagemann [39, 40] based on work of Kawazumi [12]. From these it follows that $\dim H^2(L, C) = 2g + N - 1$, where $g$ is the genus of the Riemann surface $M$ and $N$ is the number of points in $A$. See also some earlier work of Millionshchikov [17] in which he proves finite-dimensionality. Further there is the work of Getzler [5], Wodzicki [21], and Li [16] on the differential operator algebra of all degrees, and Kassel and Loday [10, 11], Bremner [2, 3], and others for the current algebras. These results can not be used in the theory of highest weight...
representations of the algebras, because the almost-grading (via the locality) cannot be incorporated. In general, the full cohomology spaces are higher dimensional. Roughly speaking, the deRham cohomology of $M \setminus A$ is responsible for the Lie algebra cocycles. For the classical case the deRham cohomology space is one-dimensional. Hence, in this case (and only in this case) all 2-cocycle classes are local classes and we recover the classical results. But in general, to identify the local cocycle classes seems to be a difficult task. The approach presented here is completely different. We do not use the partial results on the general cohomology mentioned above, but use a direct approach. In addition, we deal systematically with even a broader class of algebras.

2. The multi-point algebras of Krichever-Novikov type

2.1. Geometric set-up and the algebra structure.

Let $M$ be a compact Riemann surface of genus $g$, or in terms of algebraic geometry, a smooth projective curve over $\mathbb{C}$. Let $N, K \in \mathbb{N}$ with $N \geq 2$ and $1 \leq K < N$. Fix $I = (P_1, \ldots, P_K)$, and $O = (Q_1, \ldots, Q_{N-K})$ disjoint ordered tuples of distinct points ("marked points" "punctures") on the curve. In particular, we assume $P_i \neq Q_j$ for every pair $(i, j)$. The points in $I$ are called the in-points the points in $O$ the out-points. Sometimes we consider $I$ and $O$ simply as sets and set $A = I \cup O$ as a set.

Let $\mathcal{K}$ be the canonical line bundle of $M$. Its associated sheaf of local sections is the sheaf of holomorphic differentials. Following the common practice I will usually not distinguish between a line bundle and its associated invertible sheaf of section. For every $\lambda \in \mathbb{Z}$ we consider the bundle $\mathcal{K}^\lambda := \mathcal{K}^\otimes \lambda$. Here we use the usual convention: $\mathcal{K}^0 = \mathcal{O}$ is the trivial bundle, and $\mathcal{K}^{-1} = \mathcal{K}^*$ is the holomorphic tangent line bundle (resp. the sheaf of holomorphic vector fields). After fixing a theta characteristic, i.e. a bundle $S$ with $S^{\otimes 2} = \mathcal{K}$, we can also consider $\lambda \in \frac{1}{2} \mathbb{Z}$ with respect to the chosen theta characteristics. In this article we will only need $\lambda \in \mathbb{Z}$. Denote by $\mathcal{F}^\lambda$ the (infinite-dimensional) vector space\(^1\) of global meromorphic sections of $\mathcal{K}^\lambda$ which are holomorphic on $M \setminus A$.

Special cases, which are of particular interest to us, are the quadratic differentials ($\lambda = 2$), the differentials ($\lambda = 1$), the functions ($\lambda = 0$), and the vector fields ($\lambda = -1$). The space of functions I will also denote by $\mathcal{A}$ and the space of vector fields by $\mathcal{L}$. By multiplying sections with functions we again obtain sections. In this way the space $\mathcal{A}$ becomes an associative algebra and the spaces $\mathcal{F}^\lambda$ become $\mathcal{A}$-modules.

The vector fields in $\mathcal{L}$ operate on $\mathcal{F}^\lambda$ by taking the Lie derivative. In local coordinates

\begin{equation}
L_{e}(g) := (e(z) \frac{d}{dz}) \cdot (g(z) \, dz^\lambda) := \left( e(z) \frac{dg}{dz}(z) + \lambda \, g(z) \frac{de}{dz}(z) \right) \, dz^\lambda.
\end{equation}

Here $e \in \mathcal{L}$ and $g \in \mathcal{F}^\lambda$. To avoid cumbersome notation I used the same symbol for the section and its representing function. If there is no danger of confusion I will do the same in the following.

The space $\mathcal{L}$ becomes a Lie algebra with respect to the Lie derivative (2.1) and the spaces $\mathcal{F}^\lambda$ become Lie modules over $\mathcal{L}$. As usual I write $[e, f]$ for the bracket of the

\(^1\)For $\lambda = \frac{1}{2} + \mathbb{Z}$ we should denote the vector space by $\mathcal{F}^\lambda_{\theta}$ and let $S$ go through all theta characteristics.
vector fields. Its local form is

\[(2.2) \quad [e(z) \frac{d}{dz}, f(z) \frac{d}{dz}] = \left( e(z) \frac{df}{dz}(z) - f(z) \frac{de}{dz}(z) \right) \frac{d}{dz}. \]

For the Riemann sphere \((g = 0)\) with quasi-global coordinate \(z\) and \(I = (0)\) and \(O = (\infty)\) the introduced function algebra is the algebra of Laurent polynomials \(\mathbb{C}[z, z^{-1}]\) and the vector field algebra is the Witt algebra, i.e. the algebra whose universal central extension is the Virasoro algebra. We denote for short this situation as the classical situation.

The vector field algebra \(\mathcal{L}\) operates on the algebra \(\mathcal{A}\) of functions as derivations. Hence it is possible to consider the semi-direct product \(\mathcal{D}^1 = \mathcal{A} \times \mathcal{L}\). This Lie algebra is the algebra of differential operators of degree \(\leq 1\) which are holomorphic on \(M \setminus A\).

As vector space \(\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{L}\) and the Lie product is given as

\[(2.3) \quad [(g, e), (h, f)] := (e \cdot h - f \cdot g, [e, f]). \]

There is the short exact sequence of Lie algebras

\[(2.4) \quad 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{D}^1 \longrightarrow \mathcal{L} \longrightarrow 0. \]

Obviously, \(\mathcal{L}\) is also a subalgebra of \(\mathcal{D}^1\). The vector spaces \(\mathcal{F}^\lambda\) become \(\mathcal{D}^1\)-modules by the canonical definition

\[(2.5) \quad (g, e) \cdot v = g \cdot v + e \cdot v, \quad v \in \mathcal{F}^\lambda. \]

By universal constructions algebras of differential operators of arbitrary degree can be considered \([27, 29, 32]\). There is another algebra of importance, the current algebra. It will be defined in Section 7.

Let \(\rho\) be a meromorphic differential which is holomorphic on \(M \setminus A\) with exact pole order 1 at the points in \(A\) and given positive residues at \(I\) and given negative residues at \(O\) (of course obeying the restriction \(\sum_{P \in I} \text{res}_P(\rho) + \sum_{Q \in O} \text{res}_Q(\rho) = 0\)) and purely imaginary periods. There exists exactly one such \(\rho\) (see \([23, \text{p.116}]\)). For \(R \in M \setminus A\) a fixed point, the function \(u(P) = \text{Re} \int_P^R \rho\) is a well-defined harmonic function. The family of level lines \(C_\tau := \{p \in M \mid u(P) = \tau\}, \ \tau \in \mathbb{R}\) defines a fibration of \(M \setminus A\). Each \(C_\tau\) separates the points in \(I\) from the points in \(O\). For \(\tau \ll 0\) \((\tau \gg 0)\) each level line \(C_\tau\) is a disjoint union of deformed circles \(C_i\) around the points \(P_i, i = 1, \ldots, K\) (of deformed circles \(C_i^*\) around the points \(Q_i, i = 1, \ldots, N - K\)).

For \(f \in \mathcal{F}^\lambda\) and \(g \in \mathcal{F}^\mu\) we have \(f \otimes g \in \mathcal{F}^{\lambda+\mu}\). In particular for \(\mu = 1 - \lambda\) we obtain a meromorphic differential.

**Definition 2.1.** The Krichever-Novikov pairing \((KN\ \text{pairing})\) is the pairing between \(\mathcal{F}^\lambda\) and \(\mathcal{F}^{1-\lambda}\) given by

\[(2.6) \quad \langle f, g \rangle := \frac{1}{2\pi i} \int_{C_\tau} f \otimes g = \sum_{P \in I} \text{res}_P(f \otimes g) = -\sum_{Q \in O} \text{res}_Q(f \otimes g), \]

where \(C_\tau\) is any non-singular level line.
The last equality follows from the residue theorem. Note that in (2.6) the integral does not depend on the level line chosen. We will call any such level line or any cycle cohomologous to such a level line a separating cycle. In particular, the KN pairing can be described as

\[
\langle f, g \rangle = \frac{1}{2\pi i} \sum_{i=1}^{K} \int_{C_i} f \otimes g = \frac{1}{2\pi i} \int_{C_S} f \otimes g = -\frac{1}{2\pi i} \sum_{i=1}^{N-K} \int_{C^*_i} f \otimes g.
\]

2.2. Almost-graded structure.

For infinite dimensional algebras and their representation theory a graded structure is usually of importance to obtain structure results. A typical example is given by the Witt algebra \( \mathcal{W} \). \( \mathcal{W} \) admits a preferred set of basis elements given by

\[
\{ e_n = z^{n+1} \frac{d}{dz} \mid n \in \mathbb{Z} \}.
\]

One calculates \([e_n, e_m] = (m - n)e_{n+m}\). Hence \( \deg(e_n) := n \) makes \( \mathcal{W} \) to a graded Lie algebra.

In our more general context the algebras will almost never be graded. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure (they call it a quasi-graded structure), will be enough to develop an interesting theory of representations (highest weight representations, Verma modules, etc.).

**Definition 2.2.** (a) Let \( \mathcal{L} \) be an (associative or Lie) algebra admitting a direct decomposition as vector space \( \mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n \). The algebra \( \mathcal{L} \) is called an almost-graded algebra if (1) \( \dim \mathcal{L}_n < \infty \) and (2) there are constants \( R \) and \( S \) with

\[
\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m+R} \mathcal{L}_h, \quad \forall n, m \in \mathbb{Z}.
\]

The elements of \( \mathcal{L}_n \) are called homogeneous elements of degree \( n \).

(b) Let \( \mathcal{L} \) be an almost-graded (associative or Lie) algebra and \( \mathcal{M} \) an \( \mathcal{L} \)-module with \( \mathcal{M} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n \) as vector space. The module \( \mathcal{M} \) is called an almost-graded module, if (1) \( \dim \mathcal{M}_n < \infty \), and (2) there are constants \( R' \) and \( S' \) with

\[
\mathcal{L}_m \cdot \mathcal{M}_n \subseteq \bigoplus_{h=n+m+R'} \mathcal{M}_h, \quad \forall n, m \in \mathbb{Z}.
\]

The elements of \( \mathcal{M}_n \) are called homogeneous elements of degree \( n \).

By a weak almost-grading we understand an almost-grading without requiring the finite-dimensionality of the homogeneous subspaces.

For the 2-point situation for \( M \) a higher genus Riemann surface and \( I = \{ P \}, O = \{ Q \} \) with \( P, Q \in M \), Krichever and Novikov introduced an almost-graded structure of the algebras and the modules by exhibiting special bases and defining their elements to be the homogeneous elements. In \( [26, 27] \) its multi-point generalization was given, again by exhibiting a special basis. (See also Sadov \( [22] \) for some results in similar directions.)

In more detail, for fixed \( \lambda \) and for every \( n \in \mathbb{Z} \), and \( i = 1, \ldots, K \) a certain element \( f_{n,p}^{\lambda} \in F^{\lambda} \) is exhibited. The \( f_{n,p}^{\lambda} \) for \( p = 1, \ldots, K \) are a basis of a subspace \( F_n^{\lambda} \) and it is
shown that
\[ \mathcal{F}^\lambda = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n^\lambda. \]

The subspace \( \mathcal{F}_n^\lambda \) is called the *homogeneous subspace of degree* \( n \).

The basis elements are chosen in such a way that they fulfill the duality relation with respect to the KN pairing \( (2.6) \)
\[ \langle f_{n,p}^\lambda, f_{m,r}^{1-\lambda} \rangle = \delta_{m-n}^m \cdot \delta_{r}^r. \]

This implies that the KN pairing is non-degenerate.

We will need as additional information about the elements \( f_{n,p}^\lambda \) that
\[ \text{ord}_P(f_{n,p}^\lambda) = (n + 1 - \lambda) - \delta_{r}^r, \quad i = 1, \ldots, K. \]

After choosing local coordinates \( z_p \) at the points \( P_p \) the scalar can be fixed by requiring
\[ f_{n,p}^\lambda(z_p) = z_p^{n-\lambda}(1 + O(z_p)) (dz_p)^\lambda. \]

To give an impression of the type of conditions at \( O \) let me consider two cases. For \( N = K + 1 \) and \( O = \{Q_1\} \) for \( g \geq 2, \lambda \neq 0,1 \) and a generic choice for the points in \( A \) (or \( g = 0 \) without any restriction) we set
\[ \text{ord}_{Q_i}(f_{n,p}^\lambda) = -K \cdot (n + 1 - \lambda) + (2\lambda - 1)(g - 1). \]

For \( N = 2K \) and \( O = \{Q_1, Q_2, \ldots, Q_K\} \) we set
\[ \begin{align*}
\text{ord}_{Q_i}(f_{n,p}^\lambda) &= -(n + 1 - \lambda), \quad i = 1, \ldots, K - 1 \\
\text{ord}_{Q_N}(f_{n,p}^\lambda) &= -(n + 1 - \lambda) + (2\lambda - 1)(g - 1).
\end{align*} \]

For \( \lambda = 0 \) or \( \lambda = 1 \) (and hence for all \( \lambda \) in the case of genus \( g = 1 \)) for small \( n \) some modifications are necessary. For \( g \geq 2 \) and for certain values of \( n \) and \( \lambda \) such modifications are also needed if the points are not in generic positions. See \[26\] for the general recipe. By Riemann-Roch type arguments it is shown in \[24\] that there is up to a scalar multiple only one such \( f_{n,p}^\lambda \).

For the basis elements \( f_{n,p}^\lambda \) in \[25\] explicit descriptions in terms of rational functions (for \( g = 0 \)), the Weierstraß \( \sigma \)-function (for \( g = 1 \)), and prime forms and theta functions (for \( g \geq 1 \)) are given. For a description using Weierstraß \( \wp \)-function, see \[21\], \[28\]. We will need such a description at a certain step in our proofs.

If \( f \in \mathcal{F}^\lambda \) is any element then it can be written as \( f = \sum_{m,r} \alpha_{m,r} f_{m,r}^{\lambda} \). To simplify notation I will sometimes use \( \sum'_{m,r} \) to denote the double sum \( \sum_{m \in \mathbb{Z}} \sum_{r=1}^K \). The symbol \( \sum' \) denotes that only finitely many terms will appear in the sum. Via \( (2.10) \) the coefficients can be calculated as
\[ \alpha_{m,r} = \langle f, f_{m,r}^{1-\lambda} \rangle = \frac{1}{2\pi i} \int_{C_S} f \otimes f_{m,r}^{1-\lambda}. \]

By considering the pole order at \( I \) and \( O \) a possible range for non-vanishing \( \alpha_{m,r} \) is given. A detailed analysis \[27\], \[26\] yields
Theorem 2.3. With respect to the above introduced grading the associative algebra $A$, and the Lie algebras $L$, and $D$ are almost-graded and the modules $F^\lambda$ are almost-graded modules over them. In all cases the lower shifts in the degree of the result (e.g. the numbers $R, R'$ in (2.8) and (2.9)) are zero.

The upper shifts can be explicitly calculated. We will not need them here. Let us abbreviate for terms of higher degrees as the one under consideration the symbol $h.d.t.$.

By calculating the exact residues in the case of the lower bound we obtain

Proposition 2.4.

$$A_{n,p} \cdot A_{m,r} = \delta_p^r \cdot A_{n+m,r} + h.d.t., \quad A_{n,p} \cdot f_{m,r} = \delta_p^r \cdot f_{n+m,r} + h.d.t.,$$

$$[e_{n,p}, e_{m,r}] = \delta_p^r \cdot (m-n) \cdot e_{n+m,r} + h.d.t., \quad e_{n,p} \cdot f_{m,r} = \delta_p^r \cdot (m+\lambda n) \cdot f_{n+m,r} + h.d.t..$$

Note that the grading does not depend on the numbering of the points in $I$. Also the filtration $F^\lambda_{(n)}$ introduced by the grading does not depend on renumbering the points in $O$ because

$$F^\lambda_{(n)} := \bigoplus_{m \geq n} F^\lambda_m = \{ f \in F^\lambda \mid \text{ord}_P(f) \geq n - \lambda, \forall P \in I \}. $$

But this is an invariant description. It also shows that a different recipe for the orders at $O$ will not change the filtration.

Remark 2.5. In the following we have also to consider the case when we interchange the role played by $I$ and $O$. We obtain a different grading $^* \in I^*$ introduced by $I^* = O$. This grading we call inverted grading. For $N > 2$ this not only a simple inversion and a translation. Homogeneous elements of the original grading in general will not be homogeneous anymore and vice versa. Denote the homogeneous objects and basis with respect to the new grading also by $^*$. By considering the orders at the points $P_i$ and $Q_j$ and using (2.10) we obtain

$$F^\lambda_n \subseteq \bigoplus_{h = -\alpha n - L_1}^{\alpha n + L_2} F^n_{h}, \quad F^\lambda_{n} \subseteq \bigoplus_{h = -\beta n - L_3}^{\beta n + L_4} F^n_{h},$$

with $\alpha, \beta > 0$ and $L_1, L_2, L_3, L_4$ numbers which do not depend on $n$ and $m$.

Let me introduce the following notation:

$$A_{n,p} := f_{n,p}^0, \quad e_{n,p} := f_{n,p}^{-1}, \quad \omega^{n,p} := f_{n,p}^1, \quad \Omega^{n,p} := f_{n,p}^2.$$ 

3. Cocycles and central extensions

In this section I consider central extensions of the above introduced algebras. In quantum theory one is typically forced (e.g. by regularization procedures) to consider projective representations of the algebras which correspond to linear representations of centrally extended algebras.

Let $G$ be any Lie algebra (over $\mathbb{C}$). A (one-dimensional) central extension $\hat{G}$ is the middle term of a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C} \longrightarrow \hat{G} \longrightarrow G \longrightarrow 0,$$
such that $C$ is central in $\hat{G}$. Two central extensions $\hat{G}_1$ and $\hat{G}_2$ are called equivalent if there is a Lie isomorphism $\varphi : \hat{G}_1 \to \hat{G}_2$ such that the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & C & \longrightarrow & \hat{G}_1 & \longrightarrow & G & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & C & \longrightarrow & \hat{G}_2 & \longrightarrow & G & \longrightarrow & 0 \\
\end{array}
$$

is commutative.

Central extensions are classified up to equivalence by the second Lie algebra cohomology space $H^2(G, C)$ (where $C$ is considered as the trivial module), i.e. by 2-cocycles up to coboundaries. An antisymmetric map

$$
\gamma : G \times G \to C
$$

is a 2-cocycle if

$$
\gamma([f,g], h) + \gamma([g,h], f) + \gamma([h,f], g) = 0, \quad \forall f, g, h \in G.
$$

A 2-cocycle is a coboundary if there is a linear map $\phi : G \to C$ such that

$$
\gamma(f, g) = \phi([f, g]), \quad \forall f, g \in G.
$$

In the following we will only deal with 2-cocycles which we will just call cocycles. Given a cocycle $\gamma$ the central extension can be explicitly given by the vector space direct sum $\hat{G} := C \oplus G$ with the Lie bracket given by the structure equations (with $\hat{e} := (0, e)$ and $t := (1, 0)$)

$$
[\hat{e}, \hat{f}] := [e, f] + \gamma(e, f) \cdot t, \quad [t, \hat{G}] = 0.
$$

In terms of short exact sequences we obtain

$$
\begin{array}{ccccccc}
0 & \longrightarrow & C & \longrightarrow & \hat{G} = C \oplus G & \longrightarrow & G & \longrightarrow & 0 \\
\end{array}
$$

Changing the cocycle by a coboundary corresponds to choosing a different linear lifting map of $p_2$ other than $i_2$.

In the following subsections we are considering cocycles for the algebras $A$ (considered as abelian Lie algebra), $L$ and $D^1$ and the by the cocycles defined central extensions. In Section 7 we consider cocycles of the current algebras (multi-point and higher genus). For the classical situation the cocycles are either given purely algebraic in terms of structure constants or as integrals (or residues) of objects expressed via the quasi-global coordinate $z$. Typically they are not invariantly defined. The classical expressions need some counter terms involving projective and affine connections.

**Definition 3.1.** Let $(U_\alpha, z_\alpha)_{\alpha \in J}$ be a covering of the Riemann surface by holomorphic coordinates, with transition functions $z_\beta = f_{\beta \alpha}(z_\alpha)$. A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ resp. $T = (T_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) projective (resp. affine) connection if it transforms as

$$
R_\beta(z_\beta) \cdot (f'_{\beta \alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta \alpha}), \quad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2} \left( \frac{h''}{h'} \right)^2
$$
the Schwartzian derivative, respectively

\[ T_\beta(z_\beta) \cdot f^\alpha_{\beta} = T_\alpha(z_\alpha) + \frac{f^{\prime\prime}_{\beta,\alpha}}{f^\prime_{\beta,\alpha}}. \]

Here \( \prime \) denotes differentiation with respect to the coordinate \( z_\alpha \).

It follows from (3.8) and (3.9) that the difference of two affine (projective) connections is always a usual (quadratic) differential.

**Proposition 3.2.** Let \( M \) be any compact Riemann surface.

(a) There exists always a holomorphic projective connection.

(b) Given a point \( P \) on \( M \) there exists always a meromorphic affine connection which is holomorphic outside \( P \) and has there at most a pole of order 1

**Proof.** (a) is a classical result, e.g. see [7, 8]. (b) is shown in [27, 32]. \( \square \)

For the following I will choose a fixed holomorphic projective connection \( R^{(0)} \) and a fixed meromorphic affine connection \( T^{(0)} \) with at most a pole of order 1 at the point \( Q_1 \). All other connections with poles only at the points in \( A \) can be obtained by adding elements of \( F_1 \), resp. \( F_2 \), to these reference connections.

### 3.1. Central extensions of the function algebra.

The function algebra considered as Lie algebra is abelian. Hence any antisymmetric bilinear form will define a 2-cocycle. For any \( f, g \in A \) and any linear form \( \phi \) we obtain \( \phi([f, g]) = 0 \). Hence, there will be no coboundary, i.e. \( H^2(A, \mathbb{C}) \cong \bigwedge^2 A \).

In the following we will consider cocycles which are of geometric origin. Let \( C \) be any differentiable cycle in \( M \setminus A \) then

\[ \gamma : A \times A \to \mathbb{C}, \quad \gamma_C(g, h) := \frac{1}{2\pi i} \int_C gdh \]

is antisymmetric because \( 0 = \int_C d(gh) = \int_C gdh + \int_C hdg \). Hence, this defines a cocycle. Note that \( C \) can be replaced by any homologous cycle (assuming that it is still a differentiable curve) in \( H_1(M \setminus A, \mathbb{Z}) \), because the differential \( fdg \) is holomorphic on \( M \setminus A \). Any cocycle obtained via choosing a cycle \( C \) in (3.10) is called a geometric cocycle.

**Definition 3.3.** (a) A cocycle \( \gamma \) for \( A \) is multiplicative if it fulfills the “cocycle condition” for the associative algebra \( A \), i.e.

\[ \gamma(f \cdot g, h) + \gamma(g \cdot h, f) + \gamma(h \cdot f, g) = 0, \quad \forall f, g, h \in A. \]

(b) A cocycle \( \gamma \) for \( A \) is \( \mathcal{L} \)-invariant if

\[ \gamma(e, g, h) = \gamma(e, h, g), \quad \forall e \in \mathcal{L}, \forall g, h \in A. \]

Both properties are of importance. Below we will show that a cocycle of the function algebra which is obtained via restriction from the differential operator algebra will be \( \mathcal{L} \)-invariant. In Section 6 we will show that cocycles obtained by pulling back the standard cocycle of \( \mathfrak{gl}(\infty) \) (see its definition there) via embeddings of \( A \) into \( \mathfrak{gl}(\infty) \) respecting the almost-grading will be multiplicative.
Proposition 3.4. The cocycle \( \gamma_C \) is multiplicative and \( \mathcal{L} \)-invariant.

**Proof.** That \( \gamma_C \) is multiplicative follows from \( \int_C d(fgh) = 0 \) and Leibniz rule. To see the \( \mathcal{L} \)-invariance, first note that we have \( e . dh = d(e . h) \) for \( e \in \mathcal{L} \) and \( h \in \mathcal{A} \), i.e. the differentiation and the Lie derivative commute. Second, we have \( e . \omega = d(\omega(e)) \) for \( e \in \mathcal{L} \) and \( \omega \in \mathcal{F}^1 \). Both claims can be directly verified in local coordinates. Now

\[
\int_C (e . f) dg = \int_C e . (f dg) - \int_C f . (e . dg) = - \int_C f . (d(e . g)) = \int_C (e . g) df.
\]

In the first step we used \( e . (a \otimes b) = (e . a) \otimes b + a \otimes (e . b) \) for \( a \in \mathcal{F}^\lambda \) and \( b \in \mathcal{F}^\mu \); in the second step that the first integral vanishes due to the fact that it is differential (using \( e . \omega = d(\omega(e)) \)), and in the last step the antisymmetry of the cocycle. \( \Box \)

3.2. Central extensions of the vector field algebra.

In the classical situation there is up to equivalence and rescaling only one nontrivial central extension of the Witt algebra, the Virasoro algebra. In terms of generators \( e_n \) the standard form of the cocycle is

\[
\gamma(e_n, e_m) = \frac{1}{12} (n^3 - n) \delta_{n,-m}.
\]

For the higher genus multi-point situation we consider for each cycle \( C \) (or cycle class) with respect to the chosen projective connection \( \mathcal{R}(0) \)

\[
\gamma_{C,R(0)}(e,f) := \frac{1}{24\pi i} \int_C \left( \frac{1}{2} (e'' f - f'' e) - R^{(0)} \cdot (e' f - e f') \right) dz.
\]

Recall that we use the same letter for the vector field and its local representing function. This cocycle was introduced for the \( N = 2 \) case by Krichever and Novikov [13, 14]. As shown in [27, 26] it can be extended to the multi-point situation. There it was also shown that the integrand is indeed a differential and that it defines a cocycle.

Next we consider coboundaries. A cocycle which is a coboundary can be given as \( D\phi(e,f) = \phi([e,f]) \) with a linear form \( \phi \). We have fixed a projective connection. If we choose another projective connection \( R \) which has only poles at \( A \), then \( R = R^{(0)} + \Omega \) with a meromorphic quadratic differential with poles only at \( A \). We calculate

\[
\gamma_{C,R}(e,f) - \gamma_{C,R(0)}(e,f) = \frac{1}{24\pi i} \int_C \Omega (e f' - f e') dz = \frac{1}{24\pi i} \int_C \Omega \otimes [e,f].
\]

This implies that the two cocycles are cohomologous.

The linear forms on \( \mathcal{L} \) can be given in terms of the dual elements of \( e_{n,p} \). We can employ the KN pairing (2.6) and can give \( \phi \) by

\[
\phi(e) = \langle W, e \rangle, \quad \text{with} \quad W = \sum_{n \in \mathbb{Z}} \sum_r \beta_{n,r} \Omega^{n,r},
\]

see (2.7). Here the outer sum can reach indeed from \(-\infty \) to \(+\infty \). Recall that

\[
\langle \Omega^{n,r}, e_{m,r} \rangle = \delta_{n,m} \cdot \delta_{r,0}.
\]
and that for a fixed $e$ only finitely many terms in (3.16) will be nonzero. In this way we can give any coboundary by choosing such an infinite sum $W$. Let us denote this coboundary by

\begin{equation}
W(e, f) = \langle W, [e, f] \rangle.
\end{equation}

We will call a cocycle a geometric cocycle if it can be represented as (3.14) with a suitable cycle $C$ where the reference connection might be replaced by a meromorphic projective connection $R$.

**Remark 3.5.** One part of the Feigin-Novikov conjecture says that every cocycle of the vector field algebra is cohomologous to a linear combination of geometric cocycles obtained by integration along the basis cycles in $H_1(M \setminus A, \mathbb{Z})$. This (and the more general conjecture) was proven by Wagemann [39, 40] based on work of Kawazumi [12]. We will not use this classification result in the following. Instead we will show directly that every local cocycle (see Definition 4.1) and more generally every cocycle which is bounded from above will be a geometric cocycle involving only the cycles $C_1, C_2, \ldots, C_K$.

### 3.3. Central extensions of the differential operator algebra.

Due to the exact sequence of Lie algebras

\begin{equation}
0 \to A \to D^1 \to \mathcal{L} \to 0
\end{equation}

every cocycle $\gamma^{(v)}$ of $\mathcal{L}$ will define via pull-back a cocycle $p^*_2(\gamma)$ on $D^1$. Restricted to the subspace $\mathcal{L}$ in $D^1$ it will be exactly the cocycle $\gamma^{(v)}$ and it will vanish if one of the arguments is from $A$. We will denote this cocycle on $D^1$ also by $\gamma^{(v)}$.

The situation is slightly more complicated for the function algebra $A$ in $D^1$.

**Proposition 3.6.** A cocycle $\gamma^{(f)}$ of $A$ can be extended to a cocycle of $D^1$ if and only if $\gamma^{(f)}$ is $\mathcal{L}$-invariant i.e.

\begin{equation}
\gamma^{(f)}(e, g, h) = \gamma^{(f)}(e, h, g), \quad \forall e \in \mathcal{L}, \forall g, h \in A.
\end{equation}

**Proof.** Let $\tilde{\gamma}$ be a cocycle for $D^1$ and $\gamma$ its restriction to $A$. If we write down the cocycle condition for the elements $e \in \mathcal{L}$ and $g, h \in A$ we obtain (3.20). Vice versa: We define the extended bilinear map

\begin{equation}
\tilde{\gamma} : D^1 \times D^1 \to \mathbb{R}, \quad \tilde{\gamma}((g, e), (h, f)) := \gamma^{(f)}(g, h).
\end{equation}

Clearly it is antisymmetric. We have to check the cocycle condition. By linearity it is enough to do this for “pure” elements $(e, f, g)$. If at least 2 of them are vector fields or all of them are functions then each of the terms in the cocycle relation vanishes separately. It remains $e \in \mathcal{L}$ and $f, g \in A$. Because $[f, g] = 0$ the cocycle condition is equivalent to (3.20). \qed

By Proposition 3.6 the geometric cocycles fulfill (3.20). Hence,

**Proposition 3.7.** The geometric cocycles $\gamma^{(f)}_C(f, g) = \frac{1}{2\pi i} \int_C f dg$ can be extended to $D^1$. 

Let $\gamma$ be an arbitrary cocycle of $D^1$, and let $\gamma^{(f)}$ be its restriction to $A$ and $\gamma^{(v)}$ its restriction to $L$ and both of them extended by zero to $D^1$ again. Then $\gamma^{(m)} = \gamma - \gamma^{(f)} - \gamma^{(v)}$ will again be a cocycle. It will only have nonzero values for $e \in L$ and $f \in A$ and fulfill $\gamma^{(m)}(f,e) = -\gamma^{(m)}(f,e)$. We call $\gamma^{(m)}$ a mixing cocycle. This decomposition of $\gamma = \gamma^{(f)} + \gamma^{(v)} + \gamma^{(m)}$ is unique.

Coboundaries for $D^1$ are given again by choosing linear forms on $D^1$. The dual spaces to the functions (vector fields) are given by the differentials (quadratic differentials) with the KN pairing as duality. Hence let

\begin{equation}
V = \sum_{n \in \mathbb{Z}} \sum_r \alpha_{n,r}\omega^{n,r}, \quad W = \sum_{n \in \mathbb{Z}} \sum_r \beta_{n,r}\Omega^{n,r}
\end{equation}

be possibly both-sided infinite sums then

\begin{equation}
\phi((f,e)) = \langle V, f \rangle + \langle W, e \rangle.
\end{equation}

The corresponding coboundary is given as

\begin{equation}
\phi([(g,e),(h,f)]) = \phi((e\cdot h - f \cdot g,[e,f])) = \langle V, e \cdot h - f \cdot g \rangle + \langle W, [e,f] \rangle.
\end{equation}

This implies that the splitting into the three types remains if we pass to cohomology. The coboundary for $\gamma^{(v)}$ will be given by $W$, the coboundary for $\gamma^{(m)}$ will be given by $V$, and there is of course no coboundary for $\gamma^{(f)}$.

We want to study the mixing cocycles in more detail.

**Proposition 3.8.** Every bilinear form $\gamma : L \times A \to \mathbb{C}$ fulfilling

\begin{equation}
\gamma([e,f],g) - \gamma(e,f \cdot g) + \gamma(f,e \cdot g) = 0, \forall e, f \in L, \forall g \in A
\end{equation}

defines by antisymmetric extension and by setting it zero on $A \times A$ and $L \times L$ a mixing cocycle for $D^1$.

**Proof.** Let $\gamma$ be a bilinear form extended as described. Per construction it is antisymmetric. The only cocycle condition which does not trivially vanish is the one involving two vector fields $e$ and $f$ and one function $g$. This cocycle condition is exactly (3.25). \qed

**Proposition 3.9.** Let $C$ be any cycle on the Riemann surface $M$. And let $T^{(0)}$ be the meromorphic affine reference connection which has at most a pole of order 1 at $Q_1$ and is holomorphic elsewhere. Then

\begin{equation}
\gamma_{C,T^{(0)}}(e,g) = -\gamma_{C,T^{(0)}}(g,e) = \frac{1}{2\pi i} \int_C (e \cdot g'' + T^{(0)} \cdot (e \cdot g')) \, dz
\end{equation}

defines a mixing cocycle.

This has been shown in [27] (see also [32]) The addition of an affine connection is necessary because otherwise the integrand would not be a differential. As in the vector field case two cocycles obtained by different meromorphic affine connections with poles only at $A$ will be cohomologous. Recall that $e \cdot g = e \cdot g'$, where the l.h.s. is the Lie derivative with the vector field and the r.h.s. is the multiplication with the local representing function.

As explained above the coboundaries can be given via $E_V(e,g) = \langle V, e,g \rangle$. Again cocycles obtained via (3.26) with suitable affine connections are called geometric cocycles.
In all three cases, of special importance are integration over the cycles $C_1, C_2, \ldots, C_K$ around the points $P_i, i = 1, \ldots, K$ and integration over the cycle $C_S = \sum_i C_i$. The corresponding cocycles we will denote also by

$$\gamma_i^{(f)}, \gamma_i^{(m)}, \gamma_i^{(v)}, \quad i = 1, \ldots, K,$$

$$\gamma_S^{(f)}, \gamma_{S,T}^{(m)}, \gamma_{S,R}^{(v)}.$$ 

The $S$ stands for the separating cycle $C_S$. If the connection is the reference connection we will sometime drop it in the notation. A cocycle obtained via integration over a separating cycle I will call a separating cocycle.

**Proposition 3.10.** In the following let $\gamma$ be either the function cocycle (3.10), the vector field cocycle (3.14), or the mixing cocycle (3.26).

(a) The cocycles $\gamma_i = \gamma C_i$ for $i = 1, \ldots, K$ are linearly independent cohomology classes.

(b) The separating cocycle $\gamma_S$ is not cohomologous to zero.

**Proof.** The claim (b) follows from (a) because $\gamma_S = \sum_i \gamma_i$. Now assume a linear relation

$$\sum_{i=1}^K \alpha_i \gamma_i = 0$$

in the cohomology space.

(i) We do first the function case. We evaluate this relation for the pairs $(A_{-1,r}, A_{1,r})$ with $r = 1, \ldots, K$ and obtain $\alpha_r = 0$ (there is no nontrivial coboundary). Hence, (a).

(ii) Mixing case: The relation says there is a $V$ as in (3.22) such that $\sum_{i=1}^K \alpha_i \gamma_i = E_V$ (a possible $D_W$ will not contribute). We evaluate this relation for pairs of elements $(e_{-n,r}, A_{n,r})$ with $r = 1, \ldots, K$ and obtain

$$\alpha_r \cdot n(n-1) - \langle V, e_{-n,r} \cdot A_{n,r} \rangle = \alpha_r \cdot n(n-1) - \langle V, \sum_{h=0}^{L_2} \sum_t b_{(-n,r)(n,r)}^{(h,t)} A_{h,t} \rangle$$

$$= \alpha_r \cdot n(n-1) - B(e_{-n,r}, A_{n,r}) = 0,$$

with $B(e_{-n,r}, A_{n,r}) := \sum_{h=0}^{L_2} \sum_t \alpha_k \cdot b_{(-n,r)(n,r)}^{(h,t)}$. Here we used the almost-graded structure (5.17) and the KN pairing (2.6).

**Claim 3.11.** $B(e_{-n,r}, A_{n,r}) = O(n)$.

Note that $L_2$ is a constant independent of $n$. Hence the summation range will stay the same. But the coefficients may change with $n$. We have to show that they are at most of order $n \to \infty$. This follows from the explicit description of the basis elements of $F^\lambda$ in terms of rational functions for $g = 0$ and theta-functions and prime forms for $g \geq 1$ given in [25]. The details of the proof of the claim can be found in the appendix. Hence, $\alpha_r \cdot n(n-1) = O(n)$ which implies necessarily $\alpha_r = 0$.

(iii) The vector field case is completely analogous with the modification that as “test pairs” we take $(e_{-n,r}, e_{n,r})$ and obtain

$$\alpha_r \cdot (n+1)n(n-1) - C(e_{-n,r}, e_{n,r}) = 0$$

with (3.25). Again

**Claim 3.12.** $C(e_{-n,r}, e_{n,r}) = O(n)$.

And we conclude as above.  \(\square\)
4. Uniqueness Results for Local Cocycles

**Definition 4.1.** (a) Let $\mathcal{G} = \bigoplus_{n \in \mathbb{Z}} \mathcal{G}_n$ be an almost-graded Lie algebra. A cocycle $\gamma$ for $\mathcal{G}$ is called local if there exist $M_1, M_2 \in \mathbb{Z}$ with

\[
\forall n, m \in \mathbb{Z} : \quad \gamma(\mathcal{G}_n, \mathcal{G}_m) \neq 0 \implies M_2 \leq n + m \leq M_1.
\]

(b) A cocycle $\gamma$ for $\mathcal{G}$ is called bounded from above if there exists $M_1 \in \mathbb{Z}$ with

\[
\forall n, m \in \mathbb{Z} : \quad \gamma(\mathcal{G}_n, \mathcal{G}_m) \neq 0 \implies n + m \leq M_1.
\]

If a cocycle is local the almost-grading of $\mathcal{G}$ can be extended to $\hat{\mathcal{G}} = \mathbb{C} \oplus \mathcal{G}$ by defining $\text{deg } \hat{x} = \text{deg } x$ and $\text{deg } t = 0$. Here $\hat{x} = (0, x)$ and $t = (1, 0)$. We call such an extension an almost-graded extension, or a local extension. Krichever and Novikov [13] introduced the notion of local cocycles in the two point case and coined the name. It might have been more suitable to use the name “almost-graded cocycle” instead of “local cocycle”. In any case, local cocycles are globally defined in contrast to their names.

**Theorem 4.2.** (a) The geometric cocycles $\gamma_{S, (f)}$, $\gamma_{S, (v)}$, and $\gamma_{S, (m)}$ are local cocycles which are bounded from above by zero.

(b) The geometric cocycles $\gamma_{C_i, (f)}$, $\gamma_{C_i, (v)}$, and $\gamma_{C_i, (m)}$ for $i = 1, \ldots, K$ are bounded from above by zero.

(c) For an arbitrary meromorphic projective connection $R$ and an arbitrary meromorphic affine connection $T$ which are holomorphic outside of $A$ the cocycles $\gamma_{S, R}^{(v)}$ and $\gamma_{S, T}^{(m)}$ are local.

**Proof.** Recall that the index $S$ means integration over a separating cocycles. The value of the above cocycles for homogeneous elements can be calculated by calculating residues at the points $P_1, \ldots, P_K$. Considering the order of the elements at these points we obtain that in case (b) the cocycles are bounded from above by zero. Now $\gamma_S = \sum_i \gamma_i$, hence $\gamma_S$ is bounded from above by zero. But equivalently the integration over a separating cycle can be done by calculation of residues at the points $Q_1, \ldots, Q_{N-K}$. This yields also a lower bound for them. (See [27] for explicit formulas for the lower bounds). Hence (a) follows. As long as we add meromorphic 1-differentials (resp. quadratic differentials) which have only poles at the points of $A$ to the affine (resp. projective) reference connection the bounds for the cocycles will change but they will stay local. The upper bound zero will not change if we add only 1-differentials (resp. quadratic differentials) with maximal pole order 1 (resp. pole order 2) at the points in $I$. □

We call a cohomology class a local cohomology class if it contains a cocycle which is local. This implies that by choosing a suitable lift of the elements of $\mathcal{G}$ to $\hat{\mathcal{G}}$ the almost-grading of $\mathcal{G}$ can be extended to $\hat{\mathcal{G}}$. If $\gamma_1$ and $\gamma_2$ are local then the sum $\gamma_1 + \gamma_2$ will also be local. Hence the local cohomology classes will be a subspace of $H^2(\mathcal{G}, \mathbb{C})$ which we denote by $H^2_{\text{loc}}(\mathcal{G}, \mathbb{C})$. Note that not necessarily every element in a local cohomology class will be a local cocycle.
Theorem 4.3. (a) A cocycle $\gamma$ for the function algebra $A$ which is either multiplicative or $L$-invariant is local if and only if it is a multiple of the separating cocycle, i.e. there exists $\alpha \in \mathbb{C}$ such that
\begin{equation}
\gamma(f, g) = \alpha \gamma_s(f, g) = \frac{\alpha}{2\pi i} \int_{C_s} f dg.
\end{equation}
(b) A local cocycle will be bounded by zero and for the values at the upper bound we have
\begin{equation}
\gamma(A_{-n, r}, A_{n, s}) = \alpha \cdot n \cdot \delta_r^s, \quad \text{with} \quad \alpha = \gamma(A_{-1, r}, A_{1, r})
\end{equation}
for any $r = 1, \ldots, K$.

As a consequence we obtain immediately

Theorem 4.4. (a) A local cocycle for the function algebra which is multiplicative is also $L$-invariant and vice versa.
(b) Denote by $H^2_{\text{loc}, \ast}(A, \mathbb{C})$ the subspace of cocycles which are local and multiplicative (or equivalently local and differential), then $\dim H^2_{\text{loc}, \ast}(A, \mathbb{C}) = 1$.

Note that for the algebra $A$ there are no nontrivial coboundaries.

Theorem 4.5. (a) A cocycle for the vector field algebra $\gamma$ is a local cocycle if and only if $\gamma$ is the sum of a multiple of the separating cocycle with projective connection $R^{(0)}$ and of a coboundary $D_W$, i.e. there exist $\alpha \in \mathbb{C}$ and $W = \sum_{n=M_2}^{M_1} \sum_{r} \beta_{n, r} \Omega^{n, r}$ such that
\begin{equation}
\gamma(e, f) = \alpha \gamma_{S, R^{(0)}}(e, f) + D_W(e, f), \quad \text{with} \quad D_W(e, f) = \langle W, [e, f] \rangle.
\end{equation}
(b) If $\alpha \neq 0$ then $\gamma(e, f) = \alpha \gamma_{S, R}(e, f)$, with a projective connection $R$ which has only poles at the points in $A$.
(c) If $\gamma$ is a local cocycle which is bounded from above by zero then at level zero the cocycle is given by
\begin{equation}
\gamma(e_n, e_{-n}) = \left( \frac{(n+1)n(n-1)}{12} \cdot \alpha + nb_r \right) \delta_s^r.
\end{equation}
with $\alpha := 2\gamma(e_2, e_{-2}) - 4\gamma(e_1, e_{-1})$, and $b_r := \gamma(e_1, e_{-1})$.

Here $\alpha$ can be calculated with respect to any $r$.
(d) $\dim H^2_{\text{loc}}(\mathcal{L}, \mathbb{C}) = 1$.

Theorem 4.6. (a) A mixing cocycle for the differential operator algebra $\gamma$ is a local cocycle if and only if $\gamma$ is the sum of a multiple of the separating cocycle with affine connection $T^{(0)}$ and of a coboundary $E_V$, i.e. there exist $\alpha \in \mathbb{C}$ and $V = \sum_{n=M_2}^{M_1} \sum_{r} \beta_{n, r} \omega^{n, r}$ such that
\begin{equation}
\gamma(e, g) = \alpha \gamma_{S, T^{(0)}}(e, g) + E_V(e, g), \quad \text{with} \quad E_V(e, g) = \langle V, e.g \rangle.
\end{equation}
(b) If $\alpha \neq 0$ then $\gamma(e, g) = \alpha \gamma_{S, T}(e, g)$ with an affine connection $T$ which has only poles at the points in $A$. 

(c) If $\gamma$ is a local cocycle which is bounded from above by zero then at level zero the cocycle is given by
\begin{equation}
\gamma(e_{n,r}, A_{n,s}) = (n(n-1)\alpha + n \cdot b_r) \cdot \delta_{rs}.
\end{equation}
Here $\alpha$ can be calculated with respect to any $r$.

(d) The subspace of local cohomology classes which are given by mixing cocycles is one-dimensional.

Theorem 4.3, Theorem 4.5, and Theorem 4.6 will be proved in the following section. In addition some more statements about cocycles which are bounded from above are given. It will turn out (Theorems 5.7, 5.12, 5.18) that they are geometric cocycles involving as integration paths only the cycles $C_i$ around the points $P_1, \ldots, P_K$. This implies that for $K = 1$ this path will be a separating cycle. Hence,

**Proposition 4.7.** Let $K = 1$ (e.g. $N = 2$).

(a) Every multiplicative or $\mathcal{L}$-invariant cocycle for the function algebra which is bounded from above will be local.

(b) Every cocycle for $\mathcal{L}$ or $\mathcal{D}^1$ which is bounded from above is cohomologous to a local cocycle.

By passing to the inverted grading (see Remark 2.5) the proposition is also true if we replace “bounded from above” by bounded from below.

An arbitrary cocycle for the differential operator algebra can uniquely be decomposed into 3 cocycles of fixed type. Hence, we obtain as a corollary of the above theorems

**Theorem 4.8.** A cocycle $\gamma$ for the differential operator algebra is local if and only if it is a linear combination of the cocycle obtained by extension of the separating cocycle for the function algebra, the cocycle obtained by pulling back the separating vector field cocycle and the separating mixing cocycle with meromorphic affine and projective connections $T$ and $R$ holomorphic outside $A$
\begin{equation}
\gamma = a_1 \gamma^{(f)}_S + a_2 \gamma^{(m)}_{S,T} + a_3 \gamma^{(e)}_{S,R} + E_V + D_W,
\end{equation}
and coboundary terms $E_V + D_W$. If $a_2, a_3 \neq 0$ then the coboundary terms can be included into the connections $R$ and $T$.

(b) The subspace of $\mathbb{H}^2_{loc}(\mathcal{D}^1, \mathbb{C})$ of cocycles cohomologous to local cocycles is three-dimensional and is generated by the cohomology classes of the separating cocycles of function, mixing and vector field type.

In the classical case, i.e. $M = \mathbb{P}^1(\mathbb{C})$ and $A = \{0, \infty\}$, the statement about the vector field algebra is the well-known fact that the Witt algebra possess a one-dimensional universal central extension, the Virasoro algebra. In the standard description the cocycle is local. In this case for the differential operator algebra the result was proved by Arbarello, De Concini, Kac and Procesi [1].

For the higher genus two-point situation the result for the vector field algebra was proved by Krichever and Novikov [13, 14] starting from the general assumption that the cocycle will be a geometric cocycle and determining the defining cycle using “discrete
Baker-Akhieser functions”. The method of the proof is completely different. I will not use in my proof their results. Indeed, I will obtain an independent proof of it.
5. **The proofs**

5.1. **Multiplicative cocycles for the function algebra.**

In the following \( \gamma \) denotes a multiplicative cocycle for the function algebra which is bounded from above. For a pair \((A_{n,p}, A_{m,r})\) we call the sum \( l = n + m \) the level of the pair. The pairs of level \( l \) can be written as \( \gamma(A_{n,p}, A_{-n+l,r}) \).

We make descending recursion on the level \( l \). First we will show that starting at a level \( l > 0 \) for which the values of the cocycle will be zero for \( l \) and all higher levels the values will also be zero for all levels between 1 and \( l \). Then we will show that for all levels less than zero the cocycle values are determined by its values at level 0. Finally, we analyse the level zero. In particular it will turn out that all possible values for level zero can be realized by suitable linear combinations of the geometric cocycles \( \gamma_r, r = 1, \ldots, K \). We conclude that \( \gamma \) itself is a linear combination. Boundedness from below will only allow a combination for which all coefficients are the same.

**Lemma 5.1.** The elements \( \gamma(A_{n,r}, A_{-n+l,s}) \) of level \( l \) for \( r \neq s \) are universal linear combinations of elements of level \( \geq (l + 1) \).

**Proof.** By the multiplicativity

\[
\gamma(A_{0,r} \cdot A_{n,r}, A_{-n+l,s}) + \gamma(A_{n,r} \cdot A_{-n+l,s}, A_{0,r}) + \gamma(A_{-n+l,s} \cdot A_{0,r}, A_{n,r}) = 0.
\]

We replace the products with the help of the almost-grading for \( A \), i.e by

\[
A_{n,r} \cdot A_{m,s} = \delta^{s} \cdot A_{n+m,r} + \sum_{h=n+m+1}^{n+m+L} \sum_{t} a^{(h,t)}_{(n,r),(m,s)} A_{h,t},
\]

where \( a^{(h,t)}_{(n,r),(m,s)} \in \mathbb{C} \), and \( L \) is the upper bound for the almost grading. As usual any summation range over the second index is \( \{1, \ldots, K\} \). Hence for \( r \neq s \)

\[
\gamma(A_{n,r} + h.d.t., A_{-n+l,s}) + \gamma(h.d.t., A_{0,r}) + \gamma(h.d.t., A_{n,r}) = 0.
\]

Here \( h.d.t. \) should denote linear combinations of elements of degree which do not contribute to the levels under considerations. This implies \( \gamma(A_{n,r}, A_{-n+l,s}) \) can be expressed as linear combinations of values of the cocycle of higher level than \( l \). The coefficients appearing in this linear combination only depend on the geometric situation, i.e. on the structure constants of the algebra and not on the the cocycle under consideration. This should be understood by the term “universal linear combination” in the theorem. \( \square \)

**Remark 5.2.** In the following we will use the phrase “can be expressed by elements of higher level”, “determined by higher level”, or simply “\( = h.l. \)” to denote that it is a universal linear combination of cocycle values for pairs of homogeneous elements of level higher than the level under consideration. By the level we understand the sum of the degree of the two arguments. In particular, if two cocycles are given by higher level and they coincide in higher levels, they will coincide also for the elements under consideration.

**Lemma 5.3.** The value \( \gamma(A_{0,r}, A_{l,r}) \) can be expressed by elements of level \( \geq l + 1 \).
Proof. By the multiplicativity
\[ \gamma(A_{0,r} \cdot A_{0,r}, A_{l,r}) + \gamma(A_{0,r} \cdot A_{l,r}, A_{0,r}) + \gamma(A_{l,r} \cdot A_{0,r}, A_{0,r}) = 0. \]
Using the almost-grading we obtain
\[ \gamma(A_{0,r}, A_{l,r}) + 2 \cdot \gamma(A_{l,r}, A_{0,r}) = \text{h.l.}. \]
By the antisymmetry of the cocycle the claim follows.

We do not need it in the following. But for completeness let me note

Lemma 5.4. \( \gamma(1, f) = 0, \quad \forall f \in A. \)

Proof. From \( (1 \cdot 1, f) + (1 \cdot f, 1) + (f \cdot 1, 1) = 0 \) we conclude \( 0 = \gamma(1, f) + 2\gamma(f, 1) = \gamma(f, 1). \)

By Lemma 5.1 only the case \( r = s \) is of importance at the level \( l \). Hence, to simplify notation we will suppress in the following the second index. Starting from
\[ \gamma(A_{k} \cdot A_{n}, A_{m}) + \gamma(A_{n} \cdot A_{m}, A_{k}) + \gamma(A_{m} \cdot A_{k}, A_{n}) = 0 \]
we obtain
\[ \gamma(A_{k+n}, A_{m}) + \gamma(A_{n+m}, A_{k}) + \gamma(A_{m+k}, A_{n}) = \text{h.l.}. \] (5.2)
We specialize this for \( m = -1 \) and \( m = 1 \):
\[ \gamma(A_{k+n}, A_{-1}) + \gamma(A_{n-1}, A_{k}) + \gamma(A_{k-1}, A_{n}) = \text{h.l.}, \] (5.3)
\[ \gamma(A_{k+n}, A_{1}) + \gamma(A_{n+1}, A_{k}) + \gamma(A_{k+1}, A_{n}) = \text{h.l.}, \] (5.4)
and set in (5.3) \( k = l - n + 1 \) and in (5.2) \( k = l - n - 1 \) (\( l \) denotes the level) to obtain
\[ \gamma(A_{l+1}, A_{-1}) + \gamma(A_{n-1}, A_{l-1}) + \gamma(A_{l-1}, A_{n}) = \text{h.l.}, \] (5.5)
\[ \gamma(A_{l-1}, A_{1}) + \gamma(A_{n+1}, A_{l+1}) + \gamma(A_{l+1}, A_{n}) = \text{h.l.}. \] (5.6)
Subtracting (5.5) from (5.6) we obtain the recursion formula
\[ \gamma(A_{n+1}, A_{l-(n+1)}) = \gamma(A_{n-1}, A_{l-(n-1)}) - \gamma(A_{-1}, A_{l+1}) + \gamma(A_{1}, A_{l-1}) + \text{h.l.}. \] (5.7)
If we set \( n = -m \) and \( k = l \) in (5.2) we obtain
\[ \gamma(A_{l-m}, A_{m}) + \gamma(A_{0}, A_{l}) + \gamma(A_{l+m}, A_{-m}) = \text{h.l.}. \]
From Lemma 5.3 it follows that \( \gamma(A_{0}, A_{l}) \) is of higher level, hence
\[ \gamma(A_{m}, A_{l-m}) = -\gamma(A_{-m}, A_{l+m}) + \text{h.l.}. \] (5.8)
For \( m = 1 \) we obtain \( \gamma(A_{1}, A_{l-1}) = -\gamma(A_{-1}, A_{l+1}) + \text{h.l.}, \) which we can plug into (5.7) to obtain
\[ \gamma(A_{n+1}, A_{l-(n+1)}) = \gamma(A_{n-1}, A_{l-(n-1)}) + 2\gamma(A_{1}, A_{l-1}) + \text{h.l.}. \] (5.9)
Hence, the knowledge of \( \gamma(A_{0}, A_{l}) \) and \( \gamma(A_{1}, A_{l-1}) \) will fix the complete cocycle at level \( l \) by the knowledge of the higher levels. But \( \gamma(A_{0}, A_{l}) \) itself is fixed by higher level (Lemma 5.3), hence \( \gamma(A_{1}, A_{l-1}), \) or equivalently \( \gamma(A_{-1}, A_{l+1}) \) will fix everything.
First we consider the level \( l = 0 \) and obtain the recursion
\[ \gamma(A_{n+1}, A_{l-(n+1)}) = \gamma(A_{n-1}, A_{l-(n-1)}) + 2\gamma(A_{1}, A_{l-1}) + \text{h.l.}. \]
This implies
\begin{equation}
\gamma(A_n, A_{-n}) = n \cdot \gamma(A_1, A_{-1}) + \text{h.l.}
\end{equation}

**Lemma 5.5.** The level \( l \) for \( l \neq 0 \) is completely determined by higher levels.

**Proof.** First consider \( l > 0 \). We have to show that \( \gamma(A_1, A_{l-1}) \) is determined by higher levels. For \( l = 1 \) we obtain \( \gamma(A_1, A_{l-1}) = \gamma(A_1, A_0) \) which is determined by higher level (see Lemma 5.3). For \( l = 2 \) we obtain \( \gamma(A_1, A_{l-1}) = \gamma(A_1, A_1) = 0 \) by the antisymmetry. Hence we can assume \( l > 2 \). We set in (5.12) \( k = l - r - 1, n = 1, m = r \) and obtain
\begin{equation}
\gamma(A_1, A_{l-1}) + \gamma(A_r, A_{l-r}) - \gamma(A_{r+1}, A_{l-r-1}) = \text{h.l.}.
\end{equation}
Set \( m := \frac{l-2}{2} \) for \( l \) even or \( m := \frac{l-1}{2} \) for \( l \) odd. We let \( r \) run trough 1, 2, \ldots, \( m \) and obtain from (5.11) \( m \) equations. The first equation will always be
\[ 2 \cdot \gamma(A_1, A_{l-1}) - \gamma(A_2, A_{l-2}) = \text{h.l.} \]
The last equation will depend on the parity of \( l \). For \( l \) even and \( r = m \) the last term on the l.h.s. of (5.11) will be \( \gamma(A_{\frac{l}{2}}, A_{\frac{l}{2}}) \), which vanishes. For \( l \) odd the last term of the last equation will coincide with the second term. Hence
\[ \gamma(A_1, A_{l-1}) + 2 \cdot \gamma(A_{\frac{l}{2}}, A_{\frac{l}{2}}) = \text{h.l.} \]
will be the last equation. In this case we divide it by 2. All these equations are added up. As result we obtain
\[(m + \epsilon) \cdot \gamma(A_1, A_{l-1}) = \text{h.l.},\]
where \( \epsilon = 1 \) for \( l \) even 1/2 for \( l \) odd. This shows the claim for \( l > 0 \).

For \( l < 0 \) note that we can equally determine \( \gamma(A_{-1}, A_{l+1}) \) to fix the cocycle. Now the arguments work completely in the same way as above. The claim for \( l = -1, -2 \) follows immediately. We plug \( k = l - r + 1, n = -1, m = r \) into (5.12) and obtain
\begin{equation}
\gamma(A_{-1}, A_{l+1}) + \gamma(A_r, A_{l-r}) - \gamma(A_{r-1}, A_{l-r+1}) = \text{h.l.}.
\end{equation}
We set \( m := \frac{-l-2}{2} \) for \( l \) even and \( m := \frac{-l-1}{2} \) for \( l \) odd and consider the equation (5.12) for \( r = -1, -2, \ldots, -m \). They have the similar structure as for \( l > 0 \) and we can add them up again to obtain the statement about \( \gamma(A_{-1}, A_{l+1}) \).

**Proposition 5.6.** Let \( \gamma \) be multiplicative cocycle which is bounded from above then:
(a) 0 is also an upper bound, i.e \( \gamma(A_n, A_m) = 0 \) for \( n + m > 0 \).
(b) It is determined by its value at level 0.
(c) The level zero is given as
\begin{equation}
\gamma(A_{n,r}, A_{-n,s}) = n \cdot \delta_s^r \cdot \alpha_r, \quad \text{with } \alpha_r := \gamma(A_{1,r}, A_{-1,r})
\end{equation}
for \( n \in \mathbb{Z}, r, s = 1, \ldots, K \).

**Proof.** Assume \( \gamma \) to be bounded. If \( M > 0 \) is an upper bound then by Lemma 5.3 and Lemma 5.4 its values at the level \( M \) are linear combinations of levels \( > M \). Hence they are also vanishing on level \( M \) and finally 0 is also an upper bound. This proves (a). Part (b) follows again from the above lemmas. For \( r \neq s \) the Equation (5.13) follows from Lemma 5.4. For \( r = s \) this is (5.10) which has to be applied for each \( r \) separately.
Theorem 5.7. The space of multiplicative cocycles for the function algebra which are bounded from above is \( k \)-dimensional. A basis is given by the cocycles

\[
\gamma_i(f,g) = \frac{1}{2\pi i} \int_{C_i} f dg, \quad i = 1, \ldots, K.
\]

Proof. From the Proposition 5.6 it follows that the space is at most \( K \)-dimensional. By Theorem 4.2 and Proposition 3.4 the geometric cocycles \( \gamma_i \) are of this type, see Proposition 3.6. Also the geometric cocycles have this property (Proposition 3.10) they are linearly independent, hence a basis.

Proof of Theorem 4.3 (multiplicative case). If \( \gamma \) is a multiple of the separating cocycle then it is local (see Theorem 4.2). Now assume that \( \gamma \) is multiplicative and local. Hence it is bounded from above and can be written as \( \gamma = \sum_i \alpha_i \gamma_i \). We have to show that \( \alpha_1 = \alpha_2 = \cdots = \alpha_K \). For \( K = 1 \) claim (a) is immediate. If we interchange the role of \( I \) and \( O \) we obtain the inverted grading (see Remark 2.5) which we denote by \( \ast \). A cocycle which is bounded from below with respect to the old grading is bounded from above with respect to the new grading. Denote by \( C_i^\ast \) circles around the points \( Q_i \) for \( i = 1, \ldots, N - K \), and by \( \gamma_i^\ast(f,g) = \frac{1}{2\pi i} \int_{C_i^\ast} f dg \) the corresponding geometric cocycle. Using Theorem 5.7 we obtain \( \gamma = \sum_{i=1}^{N-K} \alpha_i^\ast \gamma_i^\ast \) with certain \( \alpha_i^\ast \in \mathbb{C} \). Again if \( N - K = 1 \) then there is just one cocycle which is then a separating cocycle and (a) is proven. Hence we assume \( N - K > 1 \). By subtracting the two presentation of the same cocycle and regrouping the summation we obtain

\[
0 = (\alpha_1 + \alpha_1^\ast) \sum_{i=1}^K \gamma_i + \sum_{k=2}^K (\alpha_k - \alpha_1) \gamma_k - \sum_{k=2}^{N-K} (\alpha_k^\ast - \alpha_1^\ast) \gamma_k^\ast.
\]

Here we used \( \sum_{i=1}^K \gamma_i + \sum_{i=1}^{N-K} \gamma_i^\ast = 0 \).

For each \( k = 2, \ldots, K \) separately we take the pair of functions \( f_n \) and \( g_n \) which are uniquely defined for infinitely many \( n \) with \( n \gg 0 \) by the conditions \( \text{ord}_P(f_n) = -n \), \( \text{ord}_{P_1}(f_n) = n - g \) and \( \text{ord}_{P_n}(g_n) = n \), \( \text{ord}_{P_1}(f_n) = -n - g \), the requirement that they are holomorphic elsewhere and that with respect to the chosen local coordinate \( z_k \) at \( P_k \) the leading coefficient is 1. Then \( \sum_{k=1}^K \gamma_k(f_n, g_n) = 0 \) because the elements to not have poles at the points \( Q_j \). All terms in the sum (5.15) are zero with the exception of \( \gamma_k(f_n, g_n) = n \). This implies that \( \alpha_k = \alpha_1 \). In a completely analogous way \( \alpha_1^\ast = \alpha_1^\ast \). Hence it remains a multiple of the separating cocycle. This can only be zero if \( \alpha_1 = -\alpha_1^\ast \). This shows (a). The explicit form (4.4) of the level zero follows from Proposition 3.10 and the fact that for the geometric cocycle we have \( \gamma_S(A_{-1,r}, A_{1,r}) = 1 \). Hence (b).

5.2. \( \mathcal{L} \)-invariant cocycles for the function algebra.

In this subsection I consider cocycles of the function algebra which are \( \mathcal{L} \)-invariant, i.e.

\[
\gamma(e.g, h) - \gamma(e.h, g) = 0, \quad e \in \mathcal{L}, g, h \in \mathcal{A}.
\]

Cocycles which are obtained via restriction of cocycles of the algebra \( \mathcal{D}^1 \) of differential operators are of this type, see Proposition 3.10. Also the geometric cocycles have this property (Proposition 3.4).
By the almost-graded action of $\mathcal{L}$ on $\mathcal{A}$ we have

$$e_{n,r}A_{m,s} = \delta_{r}^{s} \cdot m \cdot A_{n+m,r} + \sum_{h=n+m+1}^{n+m+L_2} \sum_{t} b_{(n,r),(n,s)}^{(h,t)} A_{h,t}$$

with $b_{(n,r),(m,s)}^{(h,t)} \in \mathbb{C}$, and $L_2$ the upper bound for the almost-graded structure.

Using (5.16) we get

$$\gamma(e_{n,r}A_{m,s}, A_{p,t}) = \gamma(e_{n,r}A_{p,t}, A_{m,s}),$$

and with the almost-graded structure

$$\delta_{r}^{s} \cdot m \cdot \gamma(A_{n+m,r}, A_{p,t}) = \delta_{p}^{m} \cdot p \cdot \gamma(A_{p+n,r}, A_{m,s}) + h.l..$$

For $r = t \neq s$ we obtain $p \cdot \gamma(A_{p+n,r}, A_{m,s}) = h.l.$ for any $p$. This implies that Lemma 5.1 is also true for $\mathcal{L}$-invariant cocycles. Hence it is enough to consider $s = r = t$. We will drop again the second index and obtain

$$m \cdot \gamma(A_{m+n}, A_{p}) = p \cdot \gamma(A_{p+n}, A_{m}) + h.l..$$

If we set $n = 0$ we obtain

$$(m + p) \cdot \gamma(A_{m}, A_{p}) = h.l..$$

Note that $m + p$ is the level. Hence for level $l \neq 0$ everything is determined by higher levels. This is Lemma 5.6 now for $\mathcal{L}$-invariant cocycles.

Let us assume that $\gamma$ is bounded from above then (as above) it will also be bounded by zero. For level 0 we set $p = -(n + 1)$ and $m = 1$ in (5.19) and obtain with the antisymmetry of the cocycle

$$\gamma(A_{n+1}, A_{-(n+1)}) = (n + 1) \cdot \gamma(A_{1}, A_{-1}),$$

which corresponds to (5.10). The proofs of Proposition 5.6, of Theorem 5.7 and of Theorem 4.3 rely only on these lemmas and the relation (5.10). Hence we obtain that they are also valid if we replace “multiplicative” by “$\mathcal{L}$-invariant”. In particular, we obtain

**Proposition 5.8.** (a) The space of $\mathcal{L}$-invariant cocycles for the function algebra which are bounded from above is $K$-dimensional. A basis is given by the cocycles

$$\gamma_{i}(f, g) = \frac{1}{2\pi i} \int_{C_{i}} f dg, \quad i = 1, \ldots, K.$$ 

(b) A bounded cocycle for the function algebra is multiplicative if and only if it is $\mathcal{L}$-invariant.

5.3. Mixing local cocycles for the differential operator algebra.

In this subsection I consider those cocycles defined for the differential operator algebra $\mathcal{D}^{1}$ which vanish on the subalgebra $\mathcal{A}$ of functions and the subalgebra $\mathcal{L}$ of vector fields. We start from the cocycle relation for $e, g \in \mathcal{L}$ and $g \in \mathcal{A}$

$$\gamma([e, f], g) - \gamma(e, f.g) + \gamma(f, e.g) = 0,$$

where $[e, f]$ is the commutator of $e$ and $f$.
Adding these equations yields
\[(5.32)\]
\[\gamma([e_{k,r}, e_{n,s}], A_{m,t}) - \gamma(e_{k,r}, e_{n,s}, A_{m,t}) + \gamma(e_{n,s}, e_{k,r}, A_{m,t}) = 0.\]

We use the almost-graded structure (5.17) and
\[(5.25)\]
\[\gamma(\delta_{k,r} \cdot (n-k) \cdot e_{k+n,r} + \sum_{h=n+m+1}^{n+m+L_3} \sum_{l} c^{(h,t)}_{(n,r),(m,s)} e_{h,t} \]
with \(c^{(h,t)}_{(n,r),(m,s)} \in \mathbb{C}\), and \(L_3\) the upper bound for the almost-graded structure of \(L\).

Again we want to make induction on the level of the elements. If we plug (5.17) and (5.25) into (5.24) we obtain as relation on level \(l = (n + m + k)\)
\[(5.26)\]
\[\delta_{k,r} \cdot (n-k) \cdot \gamma(e_{k+n,r}, A_{m,t}) - \delta_{s,t} \cdot m \cdot \gamma(e_{k,r}, A_{m+n,s}) + \delta_{r,m} \cdot \gamma(e_{n,s}, A_{m+k,t}) = \text{h.l.} .\]

If all \(r, s, t\) are mutually distinct, this does not produce any relation on this level. For \(s = t \neq r\), \(m = -1\) and \(n = p + 1\) we obtain \(\gamma(e_{k,r}, A_{p,s}) = \text{h.l.} .\). Hence,

**Lemma 5.9.** The cocycle values \(\gamma(e_{k,r}, A_{p,s})\) for \(r \neq s\) can be expressed as universal linear combinations of cocycle values of higher level.

Again we use the phrase “universal linear combination” to denote the situation explained in Remark 5.2.

This shows that it is enough to consider elements with the same second index. We will drop it in the notation. The equation (5.26) can now be written as:
\[(5.27)\]
\[(n-k) \cdot \gamma(e_{k+n}, A_m) - m \cdot \gamma(e_k, A_{m+n}) + m \cdot \gamma(e_{n}, A_{m+k}) = \text{h.l.} .\]

Setting \(k = 0\) in (5.27) yields
\[(5.28)\]
\[(n+m)\gamma(e_{n}, A_m) = m\gamma(e_0, A_{m+n}) + \text{h.l.} .\]

**Lemma 5.10.** (a) If the level \(l = (n+m) \neq 0\) then
\[(5.29)\]
\[\gamma(e_n, A_m) = \frac{m}{n+m} \cdot \gamma(e_0, A_{n+m}) + \text{h.l.} .\]

(b) \(\gamma(e_n, A_0)\) for all \(n \in \mathbb{Z}\) is given by higher levels.

**Proof.** Part (a) is obtained by dividing (5.28) by \(n + m \neq 0\). For \(n \neq 0\) we obtain (b) by setting \(m = 0\). We set \(m = 1\) and \(n = -1\) in (5.28) and get (b) also for \(n = 0\).

Hence, as long as the level \(l \neq 0\), there is for each level only one parameter which can be adjusted, then everything is fixed by the higher levels.

It remains to deal with the level 0 case. We set \(k = -n - m\) in (5.27) and obtain
\[(5.30)\]
\[(2n + m) \cdot \gamma(e_{-m}, A_m) - m \cdot \gamma(e_{-(n+m)}, A_{n+m}) + m \cdot \gamma(e_n, A_{-n}) = \text{h.l.} .\]

We specialize further to \(m = 1\) and \(m = -1\)
\[(5.31)\]
\[(2n + 1)\gamma(e_{-1}, A_1) - \gamma(e_{-(n+1)}, A_{n+1}) + \gamma(e_n, A_{-n}) = \text{h.l.} .\]
\[(5.32)\]
\[(2n - 1)\gamma(e_1, A_{-1}) + \gamma(e_{-(n-1)}, A_{n-1}) - \gamma(e_n, A_{-n}) = \text{h.l.} .\]

Adding these equations yields
\[(5.33)\]
\[\gamma(e_{-(n+1)}, A_{n+1}) = \gamma(e_{-(n-1)}, A_{n-1}) + (2n - 1) \cdot \gamma(e_1, A_{-1}) + (2n + 1) \cdot \gamma(e_{-1}, A_1) + \text{h.l.}\]
Recall that \( \gamma(e_0, A_0) = \text{h.l.} \) hence the values on level zero are uniquely fixed by \( \gamma(e_1, A_{-1}) \) and \( \gamma(e_{-1}, A_1) \). We use
\[
(5.34) \quad \alpha := 1/2 \left( \gamma(e_1, A_{-1}) + \gamma(e_{-1}, A_1) \right), \quad \beta_0 := -\gamma(e_{-1}, A_1),
\]
and obtain
\[
(5.35) \quad \gamma(e_{-(n+1)}, A_{n+1}) = \gamma(e_{-(n-1)}, A_{n-1}) + 2(2n-1)\alpha - 2\beta_0.
\]
The starting elements of the recursion are
\[
(5.36) \quad \gamma(e_0, A_0) = \text{h.l.}, \quad \gamma(e_{-1}, A_1) = -\beta_0.
\]
By induction this implies
\[
(5.37) \quad \gamma(e_{-n}, A_n) = n(n-1)\alpha - n\beta_0 + \text{h.l.}.
\]

Proposition 5.11. (a) For a mixing cocycle the level 0 is fixed by the data
\[
(5.38) \quad \alpha_r := 1/2 \left( \gamma(e_{1,r}, A_{-1,r}) + \gamma(e_{-1,r}, A_{1,r}) \right), \quad \beta_{0,r} = -\gamma(e_{-1,r}, A_{1,r}),
\]
for \( r = 1, \ldots, K \) via
\[
(5.39) \quad \gamma(e_{-n,r}, A_{n,s}) = (n(n-1)\alpha_r - n\beta_{0,r}) \cdot \delta_s^r + \text{h.l.},
\]
where \( \text{h.l.} \) denotes a universal linear combination of cocycle values of level \( > 0 \).

(b) A mixing cocycle which is bounded from above is uniquely given by the collection of values
\[
(5.40) \quad \gamma(e_{1,r}, A_{-1,r}), \gamma(e_{-1,r}, A_{1,r}), \quad r = 1, \ldots, K, \quad \gamma(e_{0,r}, A_{n,r}), \quad n \in \mathbb{Z} \setminus \{0\}.
\]

Proof. Our analysis above works for every \( r \) separately. Lemma 5.9 gives the statement for \( r \neq s \). This shows (a).

Let \( \gamma_1 \) and \( \gamma_2 \) be two cocycles bounded from above, which have the same set of values \( 5.40 \). Let \( L \) be a common upper bound. Recall that at level \( l \neq 0 \) the elements of this level are fixed as certain universal linear combinations of elements of level higher than \( l \) and the element \( \gamma(e_0, A_l) \). Hence, the two cocycles coincide for every level \( l > 0 \). For level 0 we use part (a), hence they coincide on level 0 and further on on every level. \( \square \)

Theorem 5.12. (a) Let \( \gamma \) be a mixing cocycle which is bounded from above by \( M \) then there exist \( \alpha_r \in \mathbb{C}, \ r = 1, \ldots, K \) and a formal sum of 1-differentials
\[
(5.41) \quad V = \sum_{n=-M}^{n=M} \sum_s \beta_{n,s} \omega^{n,s}, \quad \beta_{n,s} \in \mathbb{C},
\]
such that
\[
(5.42) \quad \gamma = \sum_{r=1}^K \alpha_r \gamma_r + E_V, \quad E_V(e, g) = \langle V, e.g \rangle.
\]

(b) The cohomology space of mixing cocycles bounded from above is \( K \)-dimensional and generated by the classes \( \{ \gamma_r \}, \ r = 1, \ldots, K \).

Recall that \( \gamma_r \) denotes the cocycle obtained by \( 3.26 \) where we integrate over \( C_r \) using our fixed reference affine connection \( T^{(0)} \).
Proof. In view of Proposition 5.11 for proving (a) it is enough to show that we can realize by such a combination all values

\[ \gamma(e_{0,r}, A_{n,r}), n \in \mathbb{Z}, n \leq M, n \neq 0, \quad \gamma(e_{1,r}, A_{-1,r}), \quad \gamma(e_{-1,r}, A_{1,r}), \quad r = 1, \ldots, K. \]

Consider a cocycle given as such a linear combination. We will show that we can recursively adjust all parameter to realize all possible values. The affine connection \( T^{(0)} \) is fixed. It does not have poles at the points in \( I \). Recall the orders of the basis elements

\[ \ord_{P_{r}}(e_{n,r}) = n + 1, \quad \ord_{P_{r}}(A_{n,r}) = n, \quad \ord_{P_{r}}(w^{n,r}) = -n - 1. \]

The orders increase by 1 at the points \( P_{s} \) with \( s \neq r \). The highest level is \( M \). Assume \( M > 0 \). We set all \( \beta_{m,r} = 0 \) for \( m > M \). The first set of values we have to realize are \( \gamma(e_{0,r}, A_{M,r}) \). But in this case the first term in the expression does not contribute, only \( E_{V}(e_{0,r}, A_{M,r}) \) will contribute. But this term calculates to

\[ \sum_{n=\infty}^{M} \sum_{s} \beta_{n,s} \langle \omega^{n,s}, e_{0,r} \cdot A_{M,r} \rangle = \sum_{n=\infty}^{M} \sum_{s} \beta_{n,s} \langle \omega^{n,s}, M \cdot A_{M,r} + h.d.t. \rangle = \beta_{M,r} \cdot M. \]

As long as \( M > 0 \) we can divide by \( M \) and obtain the prescribed value \( \gamma(e_{0,r}, A_{M,r}) \) for any \( r \). For \( M - 1 > 0 \) we can do the same and obtain

\[ \gamma(e_{0,r}, A_{M-1,r}) = (M - 1) \beta_{M-1,r} + \sum_{s=1}^{K} \beta_{M,s} \cdot b^{(M,s)}_{(0,r),(M-1,r)}, \]

where the coefficients \( b^{(M,s)}_{(0,r),(M-1,r)} \) are the structure constants introduced in (5.17). This can be done recursively as long as the level is \( > 0 \).

For the level 0 we have also a contribution of the first term. We calculate

\[ \gamma(e_{1,r}, A_{-1,r}) = 2\alpha_{r} + \beta_{0,r} + h.d.t., \quad \gamma(e_{-1,r}, A_{1,r}) = -\beta_{0,r} + h.d.t., \]

where the higher degree terms are already determined. This implies, that by setting

\[ \alpha_{r} = 1/2(\gamma(e_{1,r}, A_{-1,r}) + \gamma(e_{-1,r}, A_{1,r})) + h.d.t., \quad \beta_{0,r} = -\gamma(e_{-1,r}, A_{1,r}) + h.d.t. \]

the level zero will have the prescribed values.

For \( l < 0 \) the argument to determine \( \beta_{l,r} \) will work as for \( l > 0 \) with the only modification, that we pick-up additional elements due to the first integral (which also involves the expansion of \( T^{(0)} \)).

(b) follows now from (a) by Proposition 3.10.

Proof of Theorem 4.6. First recall that a separating cocycle with finite sum in the coboundary is local (see Theorem 4.12). To prove the opposite we use the same technique as in the proof of Theorem 4.13 presented in Section 5.1. The arguments are completely similar up to the formulation of the equivalent of Equation (5.15). The corresponding equation is

\[ 0 = (\alpha_{1} + \alpha_{1}^{*}) \sum_{i=1}^{K} \gamma_{i} + \sum_{k=2}^{K} (\alpha_{k} - \alpha_{1}) \gamma_{k} - \sum_{k=2}^{N-K} (\alpha_{k}^{*} - \alpha_{1}^{*}) \gamma_{k}^{*} + E_{V} - E_{V^{*}}. \]
Here

\[(5.50)\quad V = \sum_{n=M_1}^{n=M_2} \sum_{r} \beta_{n,r} \omega^{n,r}, \quad V^* = \sum_{n=M_1}^{n=M_2} \sum_{r} \beta^*_{n,r} \omega^{n,r}, \]

and * denotes the opposite grading. Again if \( K = 1 \) (or \( N - K = 1 \)) the integration cycle will be the separating cycle.

Similar as there we consider for infinitely many \( n \) with \( n \gg 0 \) a function \( g_n \) and a vector field \( e_n \) defined by \( \text{ord}_{P_k}(g_n) = n \), \( \text{ord}_{P_k}(e_n) = n + 1 \), \( \text{ord}_{P_k}(e_n) = n - g \), and \( \text{ord}_{Q_k}(e_n) = 1 \) (this is due to the possible pole of \( T(0) \) at \( Q_1 \)).

\[\text{Proposition 5.14.}\]

\[\text{The lower bounds follows from ord}_P = \text{local. This implies that the sum } (5.51) \text{ (see Proposition 5.14 below). This shows (a). If } \alpha = \alpha_1 \neq 0 \text{ then we can write } T := T(0) + 1/(\alpha V) \text{ and obtain } \gamma = \alpha \cdot \gamma_{S,T}. \text{ This shows (b). Part (c) is only a specialization of Proposition 5.11 and the fact that for the separating cocycle } \gamma_S(e_{1,r}, A_{1,r}) = 0 \text{ and } \gamma_S(e_{1,r}, A_{-1,r}) = 2. \text{ Part (d) follows from (a) using Proposition 3.10.}\]

\[\text{Proposition 5.14.}\]

\[(a) \text{ Let } \gamma^{(m)}_S \text{ be the separating mixing cocycle. Then } \gamma = \gamma^{(m)}_S + E_V \text{ with } V = \sum_{m=M}^{m=M_2} \sum_r \alpha_{m,r} \Omega^{m,r} \text{ is local if and only if } V \text{ is a finite sum.}\]

\[(b) \text{ Let } \gamma^{(v)}_S \text{ be the separating vector field cocycle. Then } \gamma = \gamma^{(v)}_S + D_W \text{ with } W = \sum_{m=M}^{m=M_2} \sum_r \beta_{m,r} \Omega^{m,r} \text{ is local if and only if } W \text{ is a finite sum.}\]

**Proof.** I will only proof (a). The proof of (b) is completely analogous. That finiteness of the sum implies locality follows from Theorem 4.12. For the opposite direction assume locality of \( \gamma \). With \( \gamma \) also \( \gamma^{(m)}_S = E_V \) will be local. Hence it is enough to proof the claim for \( \gamma = E_V \) with \( V = \sum_{m=M}^{m=M_2} \sum_r \alpha_{m,r} \Omega^{m,r} \). We might even assume that \( M \) is suitable negative. Assume that \( V \) is not finite. Let \( m_0 \) be such that (1) \( E_V(e_{n,r}, A_{m,s}) = 0 \) for \( n + m_0 \leq m_0 \), (2) there exists an \( r \) with \( \alpha_{m_0,r} \neq 0 \) and (3) \( m_0 < 0 \). We use the
condition $E_V(e_{k,r}, A_{m_0-k,r}) = 0$ for all $k \geq 0$ and calculate with the almost graded structure (5.17)

\[ \alpha_{m_0,r} + \frac{1}{m_0-k} \sum_{h=m_0+1}^{m_0+L_2} \sum_t \alpha_{h,t} d_{(k,r)(m_0-k,r)}^{(h,t)} = 0, \quad k \in \mathbb{N}_0. \]

If $L_2 = 0$ (which is the case for the classical situation) this already implies that $\alpha_{m_0,r} = 0$ in contradiction to the assumption. Now assume $L_2 > 0$ then (5.53) gives a homogeneous system of infinitely many independent equations for the $\alpha_{h,t}$ ($m_0+1 \leq h \leq m_0+L$, $1 \leq t \leq K$) and $a_{m_0,r}$. We obtain only the trivial solution $\alpha_{m_0,r}$ in contradiction to the assumption. \hfill \Box

5.4. Cocycles for the vector field algebra.

For the cocycles of the vector field algebra the statements and the proofs are quite similar to the mixing cocycle case. Instead of affine connections projective connections will appear.

If we plug the almost-grading (5.25) into the cocycle condition (5.54)

\[ \gamma([e_{n,r}, e_{m,s}], e_{p,t}) + \gamma([e_{m,s}, e_{p,t}], e_{n,r}) + \gamma([e_{p,t}, e_{n,r}], e_{m,s}) = 0 \]

for triples of basis elements we obtain

\[ \delta^s_r \cdot (m-n) \gamma(e_{n+m,r}, e_{p,t}) + \delta^s_t \cdot (p-m) \gamma(e_{m+p,s}, e_{n,r}) + \delta^t_r \cdot (n-p) \gamma(e_{n+p,r}, e_{m,s}) = h.l. \]

Again, if $s, t, r$ are mutually distinct this does not produce any relation on level $n+m+p$.

For $s = r \neq t$ we obtain

\[ (m-n) \gamma(e_{n+m,r}, e_{p,t}) = h.l. \]

For $k \in \mathbb{Z}$ we set $m := \frac{k+1}{2}$, $n := \frac{k-1}{2}$ if $k$ is odd, and $m := \frac{k+2}{2}$, $n := \frac{k-2}{2}$ if $k$ is even. In both cases we obtain that $\gamma(e_{k,r}, e_{p,t}) = h.l.$

**Lemma 5.15.** For $r \neq t$ the value of the cocycle $\gamma(e_{k,r}, e_{p,t})$ is given as a universal linear combination of values of the cocycle at higher level.

Hence again only $r = s = t$ is of importance and we will drop the second index. We obtain

\[ (m-n) \cdot \gamma(e_{n+m}, e_p) + (p-m) \cdot \gamma(e_{m+p}, e_n) + (n-p) \cdot \gamma(e_{n+p}, e_m) = h.l. \]

If we set $n = 0$ and use the antisymmetry we obtain

\[ (m+p) \gamma(e_m, e_p) + (p-m) \gamma(e_{m+p}, e_0) = h.l. \]

Hence,

**Lemma 5.16.** If the level $l = m + p \neq 0$ then

\[ \gamma(e_m, e_p) = \frac{m-p}{m+p} \cdot \gamma(e_{m+p}, e_0) + h.l. \]
It remains to deal with the level zero case. Clearly \( \gamma(e_0, e_0) = 0 \) due to antisymmetry. Setting \( p = -(n + 1) \) and \( m = 1 \) in (5.57) we get
\[
(5.60) \quad (n - 1)\gamma(e_{n+1}, e_{-(n+1)}) = (n + 2)\gamma(e_n, e_{-n}) - (2n + 1)\gamma(e_1, e_{-1}) + \text{h.l.}
\]
This recursion fixes the level zero starting from higher level and the values of \( \gamma(e_1, e_{-1}) \) and \( \gamma(e_2, e_{-2}) \).

**Proposition 5.17.** For a cocycle for the vector field algebra the level zero is given by the data \( \alpha_r \) and \( \beta_r \) for \( r = 1, \ldots, K \) fixed by
\[
(5.61) \quad \alpha_r = \frac{1}{6} (\gamma(e_{2r}, e_{-2r}) - 2\gamma(e_{1r}, e_{-1r})) \quad \beta_r = \gamma(e_{1r}, e_{-1r})
\]
and higher level values via
\[
(5.62) \quad \gamma(e_k, e_{-k}) = ((k + 1)k(k - 1)\alpha_r + k\beta_r) \delta_r^s + \text{h.l.}
\]
Where h.l. denotes a universal linear combination of values of the cocycle evaluated at higher levels.

(b) A cocycle which is bounded from above is uniquely given by the collection of values
\[
\gamma(e_1, e_{-1}), \gamma(e_{2r}, e_{-2r}), \quad r = 1, \ldots, K, \quad \gamma(e_{n+1}, e_{-n}), \quad n \in \mathbb{Z} \setminus \{0\}, \quad r = 1, \ldots, K.
\]

**Proof.** (a) For \( r \neq s \) Lemma 5.15 gives the claim. For \( r = s \) Equation (5.60) gives the recursive relation. It remains to show the explicit formula. By antisymmetry it is enough to consider \( k > 0 \) and there it follows from induction starting with \( k = 1 \) and \( k = 2 \).

Part (b) follows with the same arguments as in the proof of Proposition 5.11. \( \square \)

**Theorem 5.18.** (a) Let \( \gamma \) be a cocycle for the vector field algebra which is bounded from above by \( M \), then there exists a sum \( W = \sum_{n=-\infty}^{n=M} \sum_r \beta_{n,r} \Omega^{n,r} \) of quadratic differentials and a collection \( \alpha_k \in \mathbb{C}, \quad k = 1, \ldots, K \) such that \( \gamma \) is the linear combination
\[
(5.64) \quad \gamma(e, f) = \sum_{i=1}^{K} \alpha_i \gamma_i(e, f) + D_W(e, f)
\]
with
\[
(5.65) \quad D_W(e, f) = \sum_{n=-\infty}^{n=M} \sum_r \beta_{n,r} \frac{1}{2\pi i} \int_{C_i} \Omega^{n,r}[e, f] = \langle W, [e, f] \rangle.
\]

(b) The cohomology space of mixing cocycles bounded from above is \( K \)-dimensional and generated by the classes \([\gamma_r], \quad r = 1, \ldots, K\). **Proof.** The proof of (a) is completely analogous the proof presented for the mixing cocycles. It allows to calculate \( \beta_{n,r} \) recursively from above to obtain any \( \gamma(e_{n,r}, e_{0,r}) \).

On level zero we calculate
\[
(5.66) \quad \gamma(e_{1r}, e_{-1r}) = -2\beta_{0,r} + \text{h.d.t.}, \quad \gamma(e_{2r}, e_{-2r}) = \frac{1}{2} \alpha_r - 4\beta_{0,r} + \text{h.d.t.}
\]
Hence,
\[
(5.67) \quad \alpha_r = 2\gamma(e_{2r}, e_{-2r}) - 4\gamma(e_{1r}, e_{-1r}) + \text{h.d.t.}, \quad \beta_{0,r} = -1/2 \gamma(e_{1r}, e_{-1r}) + \text{h.d.t.}
\]
will realize the given values. The argument for negative level is again the same as for
the mixing cocycle.
(b) follows again from (a) by Proposition 3.10.

Proof of Theorem 4.5. The proof has completely the same structure as the proof of
the function case and the mixing case respectively. For the testing vector fields \( f_n \),
and \( e_n \), for infinitely many \( n \) with \( n \gg 0 \) we require the orders
\[
\text{ord}_{P_k}(f_n) = n + 1, \quad \text{ord}_{P_k}(e_n) = -n + 1, \quad \text{ord}_{P_1}(e_n) = n - 3g + 1 \text{ and the condition that they should be regular elsewhere and normalized at } P_k.
\]
Then
\[
(\alpha_k - \alpha_1) \frac{1}{24}(n + 1)n(n-1) + A(e_n, f_n) = 0,
\]
\[
A(e_n, f_n) := DV(e_n, f_n) - DV^*(e_n, f_n).
\]

As above (see also the appendix)

Claim 5.19. \( A(e_n, f_n) = O(n) \).

By letting \( n \) go to \( \infty \) we conclude that \( \alpha_k = \alpha_1 \) and obtain all the other results in
the same way as for the mixing cocycle. In particular for \( \alpha \neq 0 \) we can suitable rescale \( W \)
and incorporate it into the projective connection. This shows (b) The behaviour of level zero follows from Proposition 5.17 and the fact that for the separating cocycle \( \gamma_S \) we have \( \gamma_S(e_{2r}, e_{-2r}) = 1/2 \) and \( \gamma_S(e_{1r}, e_{-1r}) = 0 \). Part (d) follows from (a) with Proposition 3.10.

6. An application: \( \overline{gl}(\infty) \) and wedge representations for the
differential operator algebra

6.1. The infinite matrix algebra \( \overline{gl}(\infty) \).

First let me recall the following facts about infinite-dimensional matrix algebras (see
[9] for details). Let \( \text{mat}(\infty) \) be the vector space of (both-sided) infinite complex matrices. An element \( A \in \text{mat}(\infty) \) can be given as
\[(6.1)\]
\[A = (a_{ij})_{i,j \in \mathbb{Z}}, \quad a_{ij} \in \mathbb{C}.
\]

Consider the subspaces
\[
\begin{align*}
\mathfrak{gl}(\infty) := & \{ A = (a_{ij}) \mid \exists r = r(A) \in \mathbb{N} : a_{ij} = 0, \text{ if } |i|, |j| > r \}, \\
\overline{\mathfrak{gl}}(\infty) := & \{ A = (a_{ij}) \mid \exists r = r(A) \in \mathbb{N} : a_{ij} = 0, \text{ if } |i-j| > r \}.
\end{align*}
\]

The matrices in \( \mathfrak{gl}(\infty) \) have “finite support”, the matrices in \( \overline{\mathfrak{gl}}(\infty) \) have “finitely many diagonals”. The elementary matrices \( E_{kl} \) are given as
\[
E_{kl} = (\delta_{k,l} \delta_{i,j})_{i,j \in \mathbb{Z}}.
\]
For \( \mu = (\ldots, \mu_{-1}, \mu_0, \mu_1, \ldots) \in \mathbb{C}^\mathbb{Z} \) and \( r \in \mathbb{Z} \) set
\[
(6.3)\]
\[A_r(\mu) := \sum_{i \in \mathbb{Z}} \mu_i E_{i,i+r}
\]
to denote a diagonal matrix where the diagonal is shifted by \( r \) positions to the right.
The elements \( \{ E_{kl} \} \) are a basis of \( \mathfrak{gl}(\infty) \), the elements \( \{ A_r(\mu) \} \) are a generating set.
for $\overline{gl}(\infty)$. The subspaces $gl(\infty)$ and $\overline{gl}(\infty)$ of $\text{mat}(\infty)$ become associative algebras with the usual matrix product. To see that the multiplication is well-defined for $\overline{gl}(\infty)$ the fact that every element has only finitely many diagonals is of importance. (Note that $\text{mat}(\infty)$ itself is not an algebra.) With the commutator they become infinite-dimensional Lie algebras. In the terminology of Kac and Raina the algebra $\overline{gl}(\infty)$ is $\overline{a}_\infty$.

The Lie algebra $\overline{gl}(\infty)$ admits a standard 2-cocycle. For $A = (a_{ij}) \in \overline{gl}(\infty)$ set $\pi(A) = (\pi(A)_{ij})$ the matrix defined by

$$\pi(A)_{ij} := \begin{cases} a_{ij}, & i \geq 0, j \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$  

The cocycle is defined by

$$\alpha(A, B) := \text{tr}(\pi([A, B]) - [\pi(A), \pi(B)]).$$

Note that the matrix expression under the trace has finite support, hence the trace is well-defined. Restricted to the subalgebra $gl(\infty)$ the cocycle vanishes. The following proposition is well-known. E. g. a proof can be found in [4].

**Proposition 6.1.** The bilinear form $\alpha$ defines a cocycle which is not cohomologous to zero. The continuous cohomology $H^2_{\text{cont}}(\overline{gl}(\infty), \mathbb{C})$ is one-dimensional and generated by $\alpha$.

Let $\tilde{gl}(\infty)$ be the central extension defined via the cocycle class $[\alpha]$.

**Proposition 6.2.** The cocycle $\alpha$ is a multiplicative cocycle, i.e.

$$\alpha(A \cdot B, C) + \alpha(B \cdot C, A) + \alpha(C \cdot A, B) = 0.$$  

**Proof.** Let us decompose the matrices $A$, $B$ and $C$ into the following four boxes

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \quad \text{with} \quad X_4 = \pi(X).$$

We will not distinguish between the boxes and the matrices in $\overline{gl}(\infty)$ obtained by filling them up again to elements of $\overline{gl}(\infty)$. In particular $X = X_1 + X_2 + X_3 + X_4$. The matrices $X_2$ and $X_3$ have finite support. A direct calculation shows

$$\alpha(X, Y) = \text{tr}(X_3Y_2) - \text{tr}(Y_3X_2).$$

Hence

$$\alpha(AB, C) = \text{tr}(A_3B_1C_2) + \text{tr}(A_4B_3C_2) - \text{tr}(C_3A_1B_2) - \text{tr}(C_3A_2B_4).$$

Because all products have finite support all the traces make sense. Permuting cyclically and adding the results gives (6.6).

For the generators of $\overline{gl}(\infty)$ we calculate

$$\alpha(A_r(\mu), A_{-s}(\mu')) = 0, \quad r \neq s,$$
and

\begin{equation}
\alpha(A_r(\mu), A_{-r}(\mu')) = \begin{cases} 
0, & r = 0 \\
\sum_{k=0}^{r-1} \mu_k \mu'_{k+r}, & r < 0 \\
-\sum_{k=0}^{r-1} \mu_k \mu'_{r-k}, & r > 0.
\end{cases}
\end{equation}

For the basis elements of \( F^\lambda \) we introduce a linear order in a lexicographical way, i.e. \((n, r) > (m, s)\) if \( n > m \) or \((n = m \text{ and } r > s)\). Set \( v_{K_{n+r}} := f^\lambda_{n,r} \). In this way we can assign by the \( A, L \) or \( D^1 \)-module structure of \( F^\lambda \) to every element of \( A \) and \( L \) an infinite matrix in the usual way if we use the basis elements \( v_j \) together with its numbering. The almost-grading of the module structure guarantees that the matrix will be in \( \mathfrak{gl}(\infty) \).

Denote the induced Lie homomorphism or the homomorphism of associative algebras by \( \Phi_\lambda \). By the almost-graded structure we can write

\begin{equation}
\Phi_\lambda(A_{n, r}) = \sum_{r=-K(n+L_1)}^{-Kn} A_r(\mu), \quad \Phi_\lambda(e_{n, r}) = \sum_{r=-K(n+L_2)}^{-Kn} A_r(\mu'),
\end{equation}

with elements \( \mu, \mu' \in \mathbb{C}_{\lambda} \) given by the structure constants. The numbers \( L_1 \) and \( L_2 \) are the upper bounds for the almost-graded structure. The cocycle \( \alpha \) can be pulled back to \( A, L \) and \( D^1 \) to obtain a cocycle \( \gamma_\lambda \) by

\begin{equation}
\gamma_\lambda(e, g) = \Phi_\lambda^*(\alpha)(e, g) = \alpha(\Phi_\lambda(e), \Phi_\lambda(g)).
\end{equation}

**Proposition 6.3.** The cocycle \( \gamma_\lambda \) obtained by pulling back \( \alpha \) is a local cocycle which is bounded from above by zero. As cocycle of \( A \) it is multiplicative.

**Proof.** By (6.12), (6.10) and (6.11) we see that it is indeed local and bounded by zero from above. Proposition 6.2 shows that it is multiplicative.

Let me remark that the multiplicativity follows also indirectly because \( \gamma_\lambda \) on \( A \) is obtained by restriction of a differential cocycle and is local, see Theorem 4.4. But the property expressed in Proposition 6.2 is also important in more general situation, [38].

**Theorem 6.4.** The cocycle \( \gamma_\lambda = \Phi_\lambda^*(\alpha) \) can be written as the following linear combination of the separating cocycles introduced above,

\begin{equation}
\gamma_\lambda = \Phi_\lambda^*(\alpha) = -\left( \gamma^{(g)}_S + \frac{1-2\lambda}{2} \gamma^{(m)}_{S,T_{\lambda}} + 2(6\lambda^2 - 6\lambda + 1) \gamma^{(v)}_{S,R_{\lambda}} \right),
\end{equation}

with a suitable meromorphic affine connection \( T_\lambda \) and a projective connection \( R_\lambda \) without poles outside of \( A \) and at most poles of order one at the points in \( I \) for \( T_\lambda \) and order two for \( R_\lambda \).

**Proof.** The existence of such a linear combination with possible coboundary terms follows from the uniqueness results of Section 4. It remains to calculate the scalar factors. But from the explicit expressions (6.11) of the cocycle \( \alpha \) we calculate immediately

\[ \gamma_\lambda(A_{1,r}, A_{-1,r}) = 1, \]
\[ \gamma_\lambda(e_{1,r}, e_{-1,r}) = -\lambda(\lambda - 1), \quad \gamma_\lambda(e_{2,r}, e_{-2,r}) = -(1-2\lambda)^2 + 2\lambda(2-2\lambda), \]
\[ \gamma_\lambda(e_{1,r}, A_{-1,r}) = \lambda - 1, \quad \gamma_\lambda(e_{-1,r}, A_{1,r}) = \lambda. \]
The only structure constants necessary for the above calculations are the values given in Proposition 2.4. We use (4.4), (4.6) and (4.8) to calculate the factors in the combination. All factors in front of the basic separating cocycles are non-zero and the coboundary terms can be incorporated into the connections $T_\lambda$ and $R_\lambda$. Note that the overall minus sign could be removed by rescaling the central element $t$ by $(-1)$.

**Remark 6.5.** Recall that the three separating cocycles in (6.14) are linearly independent. Hence, the central extensions $\hat{D}_\lambda$ of the differential operator algebra associated to different weights $\lambda$ are not even after rescaling the central element equivalent. If we consider only the centrally extended $A$ we see that the same central extension $\hat{A}$ will do. Clearly, the obtained central extensions $\hat{L}_\lambda$ to different $\lambda$ of $L$ will be after rescaling of the central element be equivalent. But the explicit element in the class will depend on the weight $\lambda$ via the projective connection $R_\lambda$.

**Remark 6.6.** In this article we considered only $\lambda \in \mathbb{Z}$. But for $\lambda \in \frac{1}{2}\mathbb{Z}$ the formula (6.14) will also be true with the only exception of $\lambda = 1/2$. Here the mixing cocycle will vanish, but a boundary term $E_{V_{1/2}}$ will remain. Hence

$$\gamma_{1/2} = -\gamma^{(f)} + \gamma^{(v)}_{S,R_{1/2}} + E_{V_{1/2}}.$$ (6.15)

Let me indicate the relevance of Theorem 6.4. In quantum field theory one is usually searching for highest weight representations of the symmetry algebra. The modules $F^\lambda$ are clearly not of this type. But there is procedure (which for the classical situation is well-known) how to construct from $F^\lambda$ the space of semi-infinite wedge forms $H^\lambda$ and to extend the action to it. The naively extended action will not be well-defined. It has to be regularized. See [27] and [32] for the details. As in the classical case the regularization is done by embedding the algebras via the action on an ordered basis of $F^\lambda$ into $\widehat{gl}(\infty)$ and by using the standard regularization procedure there. One obtains for $\widehat{gl}(\infty)$ only a projective action which can be described as a linear representation of the centrally extended algebra $\hat{gl}(\infty)$ defined via the cocycle $\alpha$. Pulling back the cocycle we obtain an action of $\hat{A}$, $\hat{L}_\lambda$ and $\hat{D}_\lambda$ on $H^\lambda$. We are exactly in the situation discussed above.

**Theorem 6.7.** The space of semi-infinite wedge forms $H^\lambda$ carries a representation of centrally extended algebras $\hat{A}$, $\hat{L}_\lambda$ and $\hat{D}_\lambda$. The defining cocycle $\gamma_\lambda$ for the central extension is given by (6.14). The cocycles for $\hat{A}$ and $\hat{L}_\lambda$ are obtained by restricting $\gamma$ to the subalgebras.

In [27,32] the algebra of differential operators $D_\lambda$ on $F^\lambda$ of arbitrary degree is introduced. The action can be extended to $H^\lambda$ if we pass to the central extension obtained from pulling back $\alpha$. Hence we obtain a central extension $\hat{D}_\lambda$ and a cocycle for this algebra.

**Proposition 6.8.** The algebra $D_\lambda$ of meromorphic differential operators holomorphic outside $A$ admits a central extension $\hat{D}_\lambda$. The restriction of the defining cocycle to the subalgebra $D^1$ of differential operators of degree less or equal one is given by (6.14). The algebra $\hat{D}_\lambda$ can be realized as operators on the space of semi-infinite wedge forms.
Further details will appear in [32]. In the classical situation this extension is the extension given by the Radul cocycle [20]. Note that Radul gave it only for $\lambda = 0$. Again for the classical situation and again only for $\lambda = 0$, Li [16] showed that this is the only linear combination of cocycles for $D_1$ which can be extended to $D_0$.

7. An application: Cocycles for the affine algebra

Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra with a fixed invariant, symmetric bilinear form $(\cdot, \cdot)$, i.e. a form obeying $([x, y], z) = (x, [y, z])$. Further down we will assume nondegeneracy. For the semi-simple case the Cartan-Killing form will do. The multi-point higher genus current algebra (or multi-point higher genus loop algebra) is defined as

$$(7.1) \tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathcal{A}, \quad \text{with Lie product} \quad [x \otimes f, y \otimes g] := [x, y] \otimes f \cdot g.$$

We introduce a grading in $\tilde{\mathfrak{g}}$ by defining

$$(7.2) \quad \deg(x \otimes A_{n, p}) := n.$$

This makes $\tilde{\mathfrak{g}}$ to an almost-graded Lie algebra.

Important classes of central extensions $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ are given by

$$(7.3) \quad [x \otimes f, y \otimes g] = [x, y] \otimes (fg) + (x, y) \cdot \gamma_C(f, g) \cdot t, \quad [t, \tilde{\mathfrak{g}}] = 0,$$

where $\gamma_C(f, g) = \frac{1}{2\pi i} \int_C fdg$ is a geometric cocycle for the function algebra obtained by integration along the cycle $C$. As usual I set $\widetilde{x \otimes f} := (0, x \otimes f)$. These algebras are called the higher genus (multi-point) affine Lie algebras (or Krichever-Novikov algebras of affine type).

In the classical situation these are nothing else then the usual affine Lie algebras (i.e. the untwisted affine Kac-Moody algebras). For higher genus such algebras were introduced by Sheinman [36, 37] for the two point situation and by the author for the multi-point situation [29, 30]. See also Bremner [2, 3] for related work. From the purely algebraic context, i.e $\mathcal{A}$ an arbitrary commutative algebra without a grading they were studied earlier by Kassel [10], Kassel and Loday [11], and others. For the $C^\infty$-case see also Pressley and Segal [19].

From Theorem 4.3 we immediately get

**Proposition 7.1.** Assume that $(\cdot, \cdot)$ is nondegenerate, then the cocycle $(x, y) \cdot \gamma_C(f, g)$ is local if and only if the integration cycle $C$ is a separating cycle $C_S$.

We might even assume a more general situation:

**Proposition 7.2.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra which fulfills the condition $[\mathfrak{g}, \mathfrak{g}] \neq 0$. Let $\gamma$ be a local cocycle for the current algebra $\tilde{\mathfrak{g}}$ of the $\mathfrak{g}$. Assume that there is a nondegenerate invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ and a bilinear form $\gamma^{(f)}$ on $\mathcal{A}$ such that $\gamma$ can be written as

$$(7.4) \quad \gamma(x \otimes f, y \otimes z) = (x, y) \cdot \gamma^{(f)}(f, g),$$

then $\gamma^{(f)}$ is a multiple of the separating cocycle for the function algebra.
Proof. First, $\gamma^{(f)}$ is obviously antisymmetric and hence a cocycle for $\mathcal{A}$. We calculate
\begin{equation}
\gamma([x \otimes f, y \otimes g], z \otimes h) = \gamma([x, y] \otimes f \cdot g, z \otimes h) = ([x, y], z) \gamma^{(f)}(f \cdot g, h).
\end{equation}
For the cocycle condition for the elements $x \otimes f, y \otimes g$ and $z \otimes h$ we have to permute this cyclically and add the result up. We obtain (using the invariance of $(.,.)$)
\begin{equation}
([x, y], z) \left( \gamma^{(f)}(f \cdot g, h) + \gamma^{(f)}(g \cdot h, f) + \gamma^{(f)}(h \cdot f, g) \right) = 0.
\end{equation}
By the condition $[g, g] \neq 0$ and by the nondegeneracy of $(.,.)$ it follows that $\gamma^{(f)}$ is a multiplicative cocycle. Applying Theorem 4.3 yields the claim.

**Theorem 7.3.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with Cartan-Killing form $(.,.)$, then every local cocycle for the current algebra $\mathfrak{g} = \mathfrak{g} \otimes \mathcal{A}$ is cohomologous to a cocycle given by
\begin{equation}
\gamma(x \otimes f, y \otimes g) = a \cdot \frac{(x, y)}{2\pi i} \int_{C_S} f dg,
\end{equation}
with $a \in \mathbb{C}$.
In particular, $H^2_{\text{loc}}(\mathfrak{g}, \mathbb{C})$ is one-dimensional and up to equivalence and rescaling there is only one nontrivial local central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$.

Proof. Kassel [10] proved that the algebra $\mathfrak{g} = \mathfrak{g} \otimes \mathcal{A}$ for any commutative algebra $\mathcal{A}$ over $\mathbb{C}$ and any $\mathfrak{g}$ a simple Lie algebra admits a universal central extension. It is given by
\begin{equation}
\widehat{\mathfrak{g}}^{\text{univ}} = \left( \Omega^1_{\mathcal{A}} / d\mathcal{A} \right) \oplus \mathfrak{g},
\end{equation}
with Lie structure
\begin{equation}
[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y)f dg, \quad [\Omega^1_{\mathcal{A}} / d\mathcal{A}, \widehat{\mathfrak{g}}^{\text{univ}}] = 0.
\end{equation}
Here $\Omega^1_{\mathcal{A}} / d\mathcal{A}$ denotes the vector space of Kähler differentials of the algebra $\mathcal{A}$. The elements in $\Omega^1_{\mathcal{A}}$ can be given as $f dg$ with $f, g \in \mathcal{A}$, and $f dg$ denotes its class modulo $d\mathcal{A}$. This universal extension is not necessarily one-dimensional. Let $\hat{\mathfrak{g}}$ be any one-dimensional central extension of $\mathfrak{g}$. It will be given as a quotient of $\widehat{\mathfrak{g}}^{\text{univ}}$. Up to equivalence it can be given by a Lie homomorphism $\Phi$
\begin{equation}
\widehat{\mathfrak{g}}^{\text{univ}} = \Omega^1_{\mathcal{A}} / d\mathcal{A} \oplus \mathfrak{g} \xrightarrow{\Phi=(\varphi, id)} \widehat{\mathfrak{g}} = \mathbb{C} \oplus \mathfrak{g}
\end{equation}
with a linear form $\varphi$ on $\Omega^1_{\mathcal{A}} / d\mathcal{A}$. The structure of $\hat{\mathfrak{g}}$ is then equal to
\begin{equation}
[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y)\varphi(f dg)t, \quad [t, \hat{\mathfrak{g}}] = 0.
\end{equation}
In our situation $M \setminus A$ is an affine curve and $\Omega^1_{\mathcal{A}} / d\mathcal{A}$ is the first cohomology group of the complex of meromorphic functions on $M$ which are holomorphic on $M \setminus A$ (similar arguments can be found in an article by Bremner [2]). By Grothendieck’s algebraic deRham theorem [3, p.453] the cohomology of the complex is isomorphic to the singular cohomology of $M \setminus A$. Hence such a linear form $\varphi$ can be given by choosing a linear combination of cycle classes in $M \setminus A$ and integrating the differential $f dg$ over this combination. By Theorem 4.3 the locality implies that the combination is a multiple of the separating cocycle. \qed
Remark 7.4. There is a warning in order. The claim of the above theorem is not true for \( g \) only reductive. As a nontrivial example take \( g = gl(n) \) and \( \psi \) any antisymmetric bilinear form on \( A \). Then

\[
\gamma(x \otimes f, y \otimes g) = \text{tr}(x)\text{tr}(y)\psi(f, g)
\]
defines a cocycle. But \( \psi \) can be chosen to be local without being a geometric cocycle.

Further details will appear in a forthcoming paper [31].

APPENDIX A. ASYMPTOTIC EXPANSIONS

In this appendix I show Claim 5.13. The proofs of Claims 5.19, 5.11, and 5.12 are completely analogous. Recall the claim: \( A(e_n, g_n) = O(n) \). First we deal with the genus \( g = 0 \) case. We might assume that \( P_1 \) corresponds to \( z = 0, P_k \) to \( z = 1 \), and \( Q_1 \) to \( z = \infty \).

For Claim 5.13 we have \( g_n = z^{-n}(z - 1)^n, e_n = z^n(z - 1)^{-n+1} \frac{d}{dz} \). This implies: \( e_n g'_n = z^{-1} \cdot n \). Hence the claim. (For Claim 5.19 the elements are given by \( f_n = z^{-n+1}(z - 1)^{n+1} \frac{d}{dz}, e_n = z^{n+1}(z - 1)^{-n+1} \frac{d}{dz} \), and we calculate \([e_n, f_n] = 2z(-n+nz) \frac{dz}{dz} \).)

For genus \( g \geq 1 \) we restrict the presentation to the case of generic positions of the points in \( A \). For a nongeneric position there might appear an additional factor. It will not depend on \( n \) hence it can be ignored in the analysis. Also we might assume that \( n \gg 0 \) to avoid special prescription necessary for small \( n \) in the case of weight 0 and weight 1.

In [25] for certain elements of \( \mathcal{F}^h \) explicit expressions were given. The formulas there are valid if the required orders at the points sum up to \( (2\lambda - 1)g - 2\lambda \). This is exactly the case for the elements considered here.

The building blocks are (see [25]):

1. the prime form \( E(P, Q) \), which is a multivalued holomorphic form on \( M \times M \) of weight \(-1/2\) in each argument. It will vanish only only along the diagonal; the zero will be of first order,
2. the \( \sigma \)-differential which is a multivalued holomorphic form of weight \( g/2 \) without zeros,
3. the well-known \( \partial \)-function on the Jacobian of \( M \),
4. the Jacobi map \( J \), which embeds \( M \) into its Jacobian,
5. the Riemann vector \( \Delta \in \mathbb{C}^g \) (see [18], I, p.149).

First we deal with the mixing cocycle situation. We abbreviate

\[
S(g_n, P) := J(P) - (n + g)J(P_1) + nJ(P_k) + \Delta,
S(e_n, P) := J(P) + (n - 3g)J(P_1) - (n - 1)nJ(P_k) + J(Q_1) + 3\Delta.
\]

For the case of a mixing cocycle the elements are given as (see [25], Equation (18))

\[
g_n := \beta_1^{-1}E(P, P_1)^{-n-g}E(P, P_k)^n \sigma(P)^{-1} \partial(S(g_n, P)),
\]

with \( \beta_1 := E(P_k, P_1)^{-n-g} \sigma(P_k)^{-1} \partial(S(g_n, P)) \in \mathbb{C} \),

\[
e_n := \beta_2^{-1}E(P, P_1)^{-n-3g}E(P, P_k)^{-n+1}E(P, Q_1) \sigma(P)^{-3} \partial(S(e_n, P)),
\]

with \( \beta_2 := E(P_k, P_1)^{-n-3g}E(P_k, Q_1) \sigma(P_k)^{-3} \partial(S(e_n, P)) \in \mathbb{C} \).
By the genericity $\vartheta(S(e_n, P_k))$ and $\vartheta(S(g_n, P_k))$ will not be zero. We calculate $e_n g'_n$, where we take the derivative with respect to the local variable at the point $P$. We obtain

$$
e_n g'_n = \frac{E(P, P_1)^{-4g-1} \sigma(P)^{-5} E(P, Q_1)}{E(P_k, P_1)^{-4g} \sigma(P_k)^{-4} E(P_k, Q_1)} \times$$

$$\times \left( (-n - g)E(P, P_k)\sigma(P) \frac{\vartheta(S(e_n, P))\vartheta(S(g_n, P))}{\vartheta(S(e_n, P_k))\vartheta(S(g_n, P_k))}E'(P, P_1) + (n - 1)E(P, P_1)\sigma(P) \frac{\vartheta(S(e_n, P))\vartheta(S(g_n, P))}{\vartheta(S(e_n, P_k))\vartheta(S(g_n, P_k))}E'(P, P_k) + E(P, P_1)\sigma'(P) \frac{\vartheta(S(e_n, P))\vartheta(S(g_n, P))}{\vartheta(S(e_n, P_k))\vartheta(S(g_n, P_k))}E(P, P_k) + E(P, P_1)\sigma(P) \frac{\vartheta(S(e_n, P))\vartheta(S(g_n, P))J'(P)}{\vartheta(S(e_n, P_k))\vartheta(S(g_n, P_k))E(P, P_k)} \right)$$

The $n$-dependence is only due to the obvious multiplicative factors of the first two terms and the quotients of the theta functions. But for the latter quotients we obtain

$$\left( \frac{\vartheta(S(e_n, P))\vartheta(S(g_n, P))}{\vartheta(S(e_n, P_k))\vartheta(S(g_n, P_k))} \right)^{(k)} = O(1), \quad k \in \mathbb{N}_0,$$

$$\left( \frac{\vartheta(S(e_n, P))\vartheta(S(g_n, P))J'(P)}{\vartheta(S(e_n, P_k))\vartheta(S(g_n, P_k))} \right)^{(k)} = O(n), \quad k \in \mathbb{N}_0.$$
References

[1] E. Arbarello, C. De Concini, V.G. Kac, and C. Procesi, *Moduli spaces of curves and representation theory*, Commun. Math. Phys. **117** (1988), no. 1, 1–36.

[2] M. Bremner, *Universal central extensions of elliptic affine Lie algebras*, J. Math. Phys. **35** (1994), no. 12, 6685–6692.

[3] M. Bremner, *Four-point affine Lie algebras*, Proc. Am. Math. Soc. **123** (1995), no. 7, 1981–1989.

[4] D.B. Fuks, *Cohomology of infinite-dimensional Lie algebras. Transl. from the Russian by A. B. Sosinskiy.*, Contemporary Soviet Mathematics. New York: Consultants Bureau. XII, 1986.

[5] E. Getzler, *Cyclic homology and the Beilinson-Manin-Schechtman central extension*, Proc. Amer. Math. Soc. **104** (1988), 729–734.

[6] Ph. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley, New York, 1978.

[7] R.C. Gunning, *Lectures on Riemann surfaces*, Princeton Mathematical Notes. Princeton. N. J., Princeton University Press, 1966.

[8] N.S. Hawley and M. Schiffer, *Half-order differentials on Riemann surfaces*, Acta Math. **115** (1966), 199–236.

[9] V.G. Kac and A.K. Raina, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, Advanced Series in Mathematical Physics, Vol. 2. Singapore-New Jersey-Hong Kong: World Scientific. IX, 1987.

[10] Ch. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra*, J. Pure Appl. Algebra **34** (1984), 265–275.

[11] Ch. Kassel and J.-L. Loday, *Extensions centrales d’algèbres de Lie*, Ann. Inst. Fourier **32** (1982), no. 4, 119–142.

[12] N. Kawazumi, *On the complex analytic Gel’fand-Fuks cohomology of open Riemann surfaces*, Ann. Inst. Fourier **43** (1993), no. 3, 655–712.

[13] I.M. Krichever and S.P. Novikov, *Algebras of Virasoro type, Riemann surfaces and structures of the theory of solitons*, Funktional Anal. i. Prilozhen. **21** (1987), 46–63.

[14] I.M. Krichever and S.P. Novikov, *Virasoro type algebras, Riemann surfaces and strings in Minkowski space*, Funktional Anal. i. Prilozhen. **21** (1987), 47–61.

[15] I.M. Krichever and S.P. Novikov, *Algebras of Virasoro type, energy-momentum tensors and decompositions of operators on Riemann surfaces*, Funktional Anal. i. Prilozhen. **23** (1989), 46–63.

[16] W.-L. Li, *2-cocycles on the algebra of differential operators on the circle*, Pis’ma Zh. Eksp. Teor. Fiz **50** (1989), no. 8, 341–343.

[17] A. Ruffing, Th. Deck, and M. Schlichenmaier, *String branchings on complex tori and algebraic representations of generalized Krichever-Novikov algebras*, Lett. Math. Phys. **26** (1992), 23–32.

[18] V.A. Sadov, *Bases on multipunctured Riemann surfaces and interacting strings amplitudes*, Commun. Math. Phys. **136** (1991), 585–597.

[19] M. Schlichenmaier, *Introduction to Riemann surfaces, algebraic curves and moduli spaces*, Lecture Notes in Physics, vol. 322, Springer, Berlin, Heidelberg, New York, 1990.

[20] M. Schlichenmaier, *Krichever-Novikov algebras for more than two points*, Lett. Math. Phys. **19** (1990), 151–165.

[21] M. Schlichenmaier, *Krichever-Novikov algebras for more than two points: explicit generators*, Lett. Math. Phys. **19** (1990), 327–336.

[22] M. Schlichenmaier, *Central extensions and semi-infinite wedge representations of Krichever-Novikov algebras for more than two points*, Lett. Math. Phys. **20** (1991), 33–46.

[23] M. Schlichenmaier, *Verallgemeinerte Krichever - Novikov Algebren und deren Darstellungen*, Ph.D. thesis, Universität Mannheim, 1990.
[28] M. Schlichenmaier, *Degenerations of generalized Krichever-Novikov algebras on tori*, Jour. Math. Phys. *34* (1993), 3809–3824.

[29] M. Schlichenmaier, *Differential operator algebras on compact Riemann surfaces*, Generalized Symmetries in Physics (Clausthal 1993, Germany) (H.-D. Doebner, V.K. Dobrev, and A.G. Ushveridze, eds.), World Scientific, 1994.

[30] M. Schlichenmaier, *Zwei Anwendungen algebraisch-geometrischer Methoden in der theoretischen Physik: Berezin-Toeplitz-Quantisierung und globale Algebren der zweidimensionalen konformen Feldtheorie*, Habilitation thesis, 1996.

[31] M. Schlichenmaier, *Higher genus affine algebras of Krichever-Novikov type*, preprint 2002.

[32] M. Schlichenmaier, *Algebras of meromorphic differential operators on higher genus Riemann surfaces and semi-infinite wedge representations*, in preparation.

[33] M. Schlichenmaier and O.K. Sheinman, *The Sugawara construction and Casimir operators for Krichever-Novikov algebras*, J. Math. Sci., New York *92* (1998), no. 2, 3807–3834.

[34] M. Schlichenmaier and O.K. Sheinman, *Wess-Zumino-Witten-Novikov theory, Knizhnik-Zamolodchikov equations, and Krichever-Novikov algebras, I.*, Russian Math. Surv. (Uspeki Math. Naukii) *54* (1999), 213–250, math.QA/9812083

[35] M. Schlichenmaier and O.K. Sheinman, *Wess-Zumino-Witten-Novikov theory, Knizhnik-Zamolodchikov equations, and Krichever-Novikov algebras, II.*, in preparation.

[36] O.K. Sheinman, *Elliptic affine Lie algebras*, Funct. Anal. Appl. *24* (1990), no. 3, 210–219.

[37] O.K. Sheinman, *Affine Lie algebras on Riemann surfaces*, Funct. Anal. Appl. *27* (1993), no. 4, 266–272.

[38] O.K. Sheinman, *The second order Casimirs for the affine Krichever-Novikov algebras \( \hat{gl}_{g,2} \) and \( \hat{sl}_{g,2} \)*, Moscow Mathematical Journal *1* (2001), no. 4, 605–628.

[39] F. Wagemann, *Some remarks on the cohomology of Krichever-Novikov algebras*, Lett. Math. Phys. *47* (1999), no. 2, 173–177, *Erratum*: Lett. Math. Phys. 52(2000), 349.

[40] F. Wagemann, *Density of meromorphic in holomorphic vector fields*, preprint 2001.

[41] M. Wodzicki, *Cyclic homology of differential operators*, Duke Math. J. *54* (1987), 641–647.

(Martin Schlichenmaier) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MANNHEIM, A5, D-68131 MANNHEIM, GERMANY

E-mail address: schlichenmaier@math.uni-mannheim.de