On S-Comultiplication Modules

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Abstract. Let $R$ be a commutative ring with $1 \neq 0$ and $M$ be an $R$-module. Suppose that $S \subseteq R$ is a multiplicatively closed set of $R$. Recently, Sevim et al. in ([19], Turk. J. Math. (2019)) introduced the notion of $S$-prime submodule which is a generalization of prime submodule and used them to characterize certain class of rings/modules such as prime submodules, simple modules, torsion free modules, $S$-Noetherian modules and etc. Afterwards, in ([2], Comm. Alg. (2020)), Anderson et al. defined the concept of $S$-multiplication modules and $S$-cyclic modules which are $S$-versions of multiplication and cyclic modules and extended many results on multiplication and cyclic modules to $S$-multiplication and $S$-cyclic modules. Here, in this article, we introduce and study $S$-comultiplication module which is the dual notion of $S$-multiplication module. We also characterize certain class of rings/modules such as comultiplication modules, $S$-second submodules, $S$-prime ideals, $S$-cyclic modules in terms of $S$-comultiplication modules.

1. Introduction

Throughout this article, we focus only on commutative rings with a unity and nonzero unital modules. Let $R$ will always denote such a ring and $M$ will denote such an $R$-module. This paper aims to introduce and study the concept of $S$-comultiplication module which is both the dual notion of $S$-multiplication modules and a generalization of comultiplication modules. Sevim et al. in their paper [19] gave the concept of $S$-prime submodules and used them to characterize certain classes of rings/modules such as prime submodules, simple modules, torsion-free modules and $S$-Noetherian rings. A nonempty subset $S$ of $R$ is said to be a multiplicatively closed set (briefly, m.c.s) of $R$ if $0 \notin S$, $1 \in S$ and $st \in S$ for each $s, t \in S$. From now on $S$ will always denote a m.c.s of $R$. Suppose that $P$ is a submodule of $M$, $K$ is a nonempty subset of $M$ and $J$ is an ideal of $R$. Then the residuals of $P$ by $K$ and $J$ are defined as follows:

$$ (P : K) = \{ x \in R : xK \subseteq P \} $$

$$ (P : M J) = \{ m \in M : Jm \subseteq P \}. $$

In particular, if $P = 0$, we sometimes use $\text{ann}(K)$ instead of $(0 : K)$. Recall from [19] that a submodule $P$ of $M$ is said to be an $S$-prime submodule if $(P : M) \cap S = \emptyset$.

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and there exists $s \in S$ such that $am \in P$ for some $a \in R$ and $m \in M$ implies either $sa \in (P : M)$ or $sm \in P$. Particularly, an ideal $I$ of $R$ is said to be an $S$-prime ideal if $I$ is an $S$-prime submodule of $M$. We here note that if $S \subseteq u(R)$, where $u(R)$ is the set of all units in $R$, the notion of $S$-prime submodule is in fact prime submodule.

Recall that an $R$-module $M$ is said to be a multiplication module if each submodule $N$ of $M$ has the form $N = IM$ for some ideal $I$ of $R$ [12]. It is easy to note that $M$ is a multiplication module if and only if $N = (N : M)M$ [16]. The author in [16] showed that for a multiplication module $M$, a submodule $N$ of $M$ is prime if and only if $(N : M)$ is a prime ideal of $R$ [16, Corollary 2.11].

The dual notion of prime submodule which is called second submodule was first introduced and studied by S. Yassemi in [20]. Recall from that a nonzero submodule $P$ of $M$ is said to be a second submodule if for each $a \in R$, the homothety $P \xrightarrow{a} P$ is either zero or surjective. Note that if $P$ is a second submodule of $M$, then $ann(P)$ is a prime ideal of $R$. For the last twenty years, the dual notion of prime submodule has attracted many researchers and it has been studied in many papers. See, for example, [5, 6, 7, 9, 13] and [14]. Also the notion of comultiplication module which is the dual notion of multiplication module was first introduced by Ansari-Toroghy and Farshadifar in [8] and has been widely studied by many authors. See, for instance, [1, 10, 11] and [15]. Recall from [8] that an $R$-module $M$ is said to be a comultiplication module if each submodule $N$ of $M$ has the form $N = (0 :_M I)$ for some ideal $I$ of $R$. Note that $M$ is a comultiplication module if and only if $N = (0 :_M ann(N))$.

Recently, Anderson et al. in [2], introduced the notions of $S$-multiplication modules and $S$-cyclic modules, and they extended many properties of multiplication and cyclic modules to these two new classes of modules. They also showed that for $S$-multiplication modules, any submodule $N$ of $M$ is $S$-prime submodule if and only if $(N : M)$ is an $S$-prime ideal of $R$ [2, Proposition 4]. An $R$-module $M$ is said to be an $S$-multiplication module if for each submodule $N$ of $M$, there exist $s \in S$ and an ideal $I$ of $R$ such that $sN \subseteq IM \subseteq N$. Also $M$ is said to be an $S$-cyclic module if there exists $s \in S$ such that $sM \subseteq Rm$ for some $m \in M$. They also showed that every $S$-cyclic module is an $S$-multiplication module and they characterized finitely generated multiplication modules in terms of $S$-cyclic modules (See, [2, Proposition 5] and [2, Proposition 8]).

Farshadifar, currently, in her paper [17] defined the dual notion of $S$-prime submodule which is called $S$-second submodule and investigate its many properties similar to second submodules. Recall that a submodule $N$ of $M$ is said to be an $S$-second if $ann(N) \cap S = \emptyset$ and there exists $s \in S$ such that either $saN = 0$ or $saN = sN$ for each $a \in R$. In particular, the author in [17] investigate the $S$-second submodules of comultiplication modules. Here, we introduce $S$-comultiplication module which is the dual notion of $S$-multiplication modules and investigate its many properties. Recall that an $R$-module $M$ is said to be an $S$-comultiplication module if for each submodule $N$ of $M$, there exist an $s \in S$ and an ideal $I$ of $R$ such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$.

Among other results in this paper, we characterize certain classes of rings/modules such as comultiplication modules, $S$-second submodules, $S$-prime ideals, $S$-cyclic
modules (See, Theorem 1, Theorem 2, Proposition 3, Theorem 4, Theorem 5, Theorem 6 and Theorem 7). Also, we prove the S-version of Dual Nakayama’s Lemma (See, Theorem 8).

2. S-comultiplication modules

**Definition 1.** Let $M$ be an $R$-module and $S \subseteq R$ be a m.c.s of $R$. $M$ is said to be an $S$-comultiplication module if for each submodule $N$ of $M$, there exist an $s \in S$ and an ideal $I$ of $R$ such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$. In particular, a ring $R$ is said to be an $S$-comultiplication ring if it is an $S$-comultiplication module over itself.

**Example 1.** Every $R$-module $M$ with $\text{ann}(M) \cap S \neq \emptyset$ is trivially an $S$-comultiplication module.

**Example 2.** *(An $S$-comultiplication module that is not $S$-multiplication)*

Let $p$ be a prime number and consider the $\mathbb{Z}$-module

$$E(p) = \{ \alpha = \frac{m}{p^n} + \mathbb{Z} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\} \}.$$ 

Then every submodule of $E(p)$ is of the form $G_t = \{ \alpha = \frac{m}{p^n} + \mathbb{Z} : m \in \mathbb{Z} \}$ for some fixed $t \geq 0$. Take the multiplicatively closed set $S = \{ 1 \}$. Note that $(G_t : E(p))E(p) = 0_{E(p)} \neq G_t$ for each $t \geq 1$. Then $E(p)$ is not an $S$-multiplication module. Now, we will show that $E(p)$ is an $S$-comultiplication module. Let $t \geq 0$. Then it is easy to see that $(0 :_{E(p)} \text{ann}(G_t)) = (0 :_{E(p)} p^t \mathbb{Z}) = G_t$. Therefore, $E(p)$ is an $S$-comultiplication module.

**Example 3.** Every comultiplication module is also an $S$-comultiplication module. Also the converse is true provided that $S \subseteq u(R)$.

**Example 4.** *(An $S$-comultiplication module that is not comultiplication)*

Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$ and $S = \text{reg}(\mathbb{Z}) = \mathbb{Z} - \{0\}$. Now, take the submodule $N = m\mathbb{Z}$, where $m \neq 0, \pm 1$. Then $(0 : \text{ann}(m\mathbb{Z})) = \mathbb{Z} \neq m\mathbb{Z}$ so that $M$ is not a comultiplication module. Now, take a submodule $K$ of $M$. Then $K = k\mathbb{Z}$ for some $k \in \mathbb{Z}$. If $k = 0$, then choose $s = 1$ and note that $s(0 : \text{ann}(K)) = (0 : k\mathbb{Z})$. If $k \neq 0$, then choose $s = k$ and note that $s(0 : \text{ann}(K)) \subseteq k\mathbb{Z} = K \subseteq (0 : \text{ann}(K))$. Therefore, $M$ is an $S$-comultiplication module.

**Lemma 1.** Let $M$ be an $R$-module. The following statements are equivalent.

(i) $M$ is an $S$-comultiplication module.

(ii) For each submodule $N$ of $M$, there exists $s \in S$ such that $s(0 :_M \text{ann}(N)) \subseteq N \subseteq (0 :_M \text{ann}(N))$.

(iii) For each submodule $K, N$ of $M$ with $\text{ann}(K) \subseteq \text{ann}(N)$, there exists $s \in S$ such that $sN \subseteq K$.

**Proof.** $(i) \Rightarrow (ii)$: Suppose that $M$ is an $S$-comultiplication module and take a submodule $N$ of $M$. Then by definition, there exist $s \in S$ and an ideal $I$ of $R$ such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$. Then note that $IN = (0)$ and so $I \subseteq \text{ann}(N)$. This gives that $s(0 :_M \text{ann}(N)) \subseteq s(0 :_M I) \subseteq N \subseteq (0 :_M \text{ann}(N))$ which completes the proof.
which is a generalization of Noetherian rings and they extended many properties if it is an $x$-Noetherian ring and $x$-Noetherian module. Then $\text{ann}_S M$ is an $x$-comultiplication module. Since $\text{ann}(K) \subseteq \text{ann}(N)$, we have $(0 :_M \text{ann}(N)) \subseteq (0 :_M \text{ann}(K))$ and so

$$s_1s_2(0 :_M \text{ann}(N)) \subseteq s_2N \subseteq s_2(0 :_M \text{ann}(N)) \subseteq s_2(0 :_M \text{ann}(K)) \subseteq K$$

which completes the proof.

(iii) $\Rightarrow$ (i): Suppose that (iii) holds. Let $N$ be a submodule of $M$. Then it is clear that $\text{ann}(N) = \text{ann}(0 :_M \text{ann}(N))$. Then by (iii), there exists $s \in S$ such that $s(0 :_M \text{ann}(N)) \subseteq N \subseteq (0 :_M \text{ann}(N))$.

(ii) $\Rightarrow$ (i): It is clear. $\square$

Let $S$ be a m.c.s of $R$. The saturation $S^*$ of $S$ is defined by $S^* = \{x \in R : x|s \text{ for some } s \in S\}$. Also $S$ is said to be a saturated m.c.s of $R$ if $S = S^*$. Note that $S^*$ is always a saturated m.c.s of $R$ containing $S$.

**Proposition 1.** Let $M$ be an $R$-module and $S$ be a m.c.s of $R$. The following assertions hold.

(i) Let $S_1$ and $S_2$ be two m.c.s of $R$ and $S_1 \subseteq S_2$. If $M$ is an $S_1$-comultiplication module, then $M$ is also an $S_2$-comultiplication module.

(ii) $M$ is an $S$-comultiplication module if and only if $M$ is an $S^*$-comultiplication module, where $S^*$ is the saturation of $S$.

**Proof.** (i): Clear.

(ii): Assume that $M$ is an $S^*$-comultiplication module. Since $S \subseteq S^*$, the result follows from the part (i).

Suppose $M$ is an $S^*$-comultiplication module. Take a submodule $N$ of $M$. Since $M$ is $S^*$-comultiplication module, there exists $x \in S^*$ such that $x((0 :_M \text{ann}(N)) \subseteq N \subseteq (0 :_M \text{ann}(N))$ by Lemma 1. Since $x \in S^*$, there exists $s \in S$ such that $x|s$, that is, $s = rx$ for some $r \in R$. This implies that $s(0 :_M \text{ann}(N)) \subseteq x((0 :_M \text{ann}(N)) \subseteq N \subseteq (0 :_M \text{ann}(N))$. Thus, $M$ is an $S$-comultiplication module. $\square$

Anderson and Dumitrescu, in 2002, defined the concept of $S$-Noetherian rings which is a generalization of Noetherian rings and they extended many properties of Noetherian rings to $S$-Noetherian rings. Recall from [4] that a submodule $N$ of $M$ is said to be an $S$-finite submodule if there exists a finitely generated submodule $K$ of $M$ such that $sN \subseteq K \subseteq N$. Also, $M$ is said to be an $S$-Noetherian module if its each submodule is $S$-finite. In particular, $R$ is said to be an $S$-Noetherian ring if it is an $S$-Noetherian $R$-module.

**Proposition 2.** Let $R$ be an $S$-Noetherian ring and $M$ be an $S$-comultiplication module. Then $S^{-1} M$ is a comultiplication module.

**Proof.** Let $W$ be a submodule of $S^{-1} M$. Then, $W = S^{-1} N$ for some submodule $N$ of $M$. Since $M$ is an $S$-comultiplication module, there exists $s \in S$ such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$ for some ideal $I$ of $R$. Then, we get $S^{-1}(s(0 :_M I)) = S^{-1}((0 :_M I)) \subseteq S^{-1}N \subseteq S^{-1}((0 :_M I))$ that is $S^{-1}N = S^{-1}((0 :_M I))$. Now, we will show that $S^{-1}((0 :_M I)) = (0 :_{S^{-1}M} S^{-1} I)$. Let $\overline{w} \in S^{-1}((0 :_M I))$
where \( m \in (0 :_M I) \) and \( s' \in S \). Then, we have \( Im = (0) \) and so \((S^{-1} I)(\frac{m}{s}) = (0)\). This implies that \( \frac{m}{s} \in (0 :_{S^{-1}M} S^{-1}I) \). For the converse, let \( \frac{m}{s} \in (0 :_{S^{-1}M} S^{-1}I) \). Then, we have \((S^{-1} I)(\frac{m}{s}) = (0)\). This implies that, for each \( x \in I \), there exists \( s'' \in S \) such that \( s''xm = 0 \). Since \( R \) is an \( S \)-Noetherian ring, \( I \) is \( S \)-finite. So, there exists \( s' \in S \) and \( a_1, a_2, \ldots, a_n \in I \) such that \( s' I \subseteq (a_1, a_2, \ldots, a_n) \subseteq I \). As \((S^{-1} I)(\frac{m}{s}) = (0)\) and \( a_i \in I \), there exists \( s_i \in S \) such that \( s_i a_i m = 0 \). Now, put \( t = s_1 s_2 \cdots s_n s' \in S \). Then we have \( ta_i m = 0 \) for all \( a_i \) and so \( tIm = 0 \). Then we deduce \( \frac{m}{s} = \frac{m}{s} \in S^{-1}(0 :_M I) \). Thus, \( S^{-1}(0 :_M I) = (0 :_{S^{-1}M} S^{-1}I) \) and so \( W = S^{-1} N = (0 :_{S^{-1}M} S^{-1}I) \). Therefore, \( S^{-1} M \) is a comultiplication module. □

Recall from [2] that a m.c.s \( S \) of \( R \) is said to satisfy maximal multiple condition if there exists \( s \in S \) such that \( t \) divides \( s \) for each \( t \in S \).

**Theorem 1.** Let \( M \) be an \( R \)-module and \( S \) be a m.c.s. of \( R \) satisfying maximal multiple condition. Then, \( M \) is an \( S \)-comultiplication module if and only if \( S^{-1} M \) is a comultiplication module.

**Proof.** \((\Rightarrow)\) Suppose that \( W \) is a submodule of \( S^{-1} M \). Then \( W = S^{-1} N \) for some submodule \( N \) of \( M \). Since \( M \) is an \( S \)-comultiplication module, there exist \( t' \in S \) and an ideal \( I \) of \( R \) such that \( t' (0 :_M I) \subseteq N \subseteq (0 :_M I) \). This implies that \( IN = (0) \) and so \( S^{-1} (IN) = (S^{-1} I)(S^{-1} N) = 0 \). Then we have \( S^{-1} N \subseteq (0 :_{S^{-1}M} S^{-1}I) \). Let \( \frac{m}{s} \in (0 :_{S^{-1}M} S^{-1}I) \). Then we get \( \frac{m}{s} = 0 \) for each \( a \in I \) and this yields that \( u a m' = 0 \) for some \( u \in S \). As \( S \) satisfies maximal multiple condition, there exists \( s \) such that \( u | s \) for each \( u \in S \). This implies that \( s = u x \) for some \( x \in R \). Then we have \( s a m' = x u a m' = 0 \). Then we conclude that \( I s m' = 0 \) and so \( s m' \in (0 :_M I) \). This yields that \( t' s m' \in t'(0 :_M I) \subseteq N \) and so \( \frac{m}{s} = \frac{t' s m'}{t's ss} \in \subseteq S^{-1} N \). Then we get \( S^{-1} N = (0 :_{S^{-1}M} S^{-1}I) \) and so \( S^{-1} M \) is a comultiplication module.

\((\Leftarrow)\) Suppose that \( S^{-1} M \) is a comultiplication module. Let \( N \) be a submodule of \( M \). Since \( S^{-1} M \) is comultiplication, \( S^{-1} N = (0 :_{S^{-1}M} S^{-1}I) \) for some ideal \( I \) of \( R \). Then we have \((S^{-1} I)(S^{-1} N) = S^{-1} (IN) = 0 \). Then for each \( a \in I, m \in N \), we have \( \frac{m}{s} = 0 \) and thus \( u a m = 0 \) for some \( u \in S \). By maximal multiple condition, there exists \( s \) such that \( s a m = 0 \) and so \( s IN = 0 \). This implies that \( N \subseteq (0 :_M sI) \). Now, let \( m \in (0 :_M sI) \). Then \( I s m = 0 \) so it is easily seen that \((S^{-1} I)(\frac{m}{s}) = 0 \). Then we conclude that \( \frac{m}{s} \in (0 :_{S^{-1}M} S^{-1}I) = S^{-1} N \). Then there exists \( x \in S \) such that \( x m \in N \). Again by maximal multiple condition, \( sm \in N \). Then we have \( s(0 :_M sI) \subseteq N \subseteq (0 :_M sI) \). Since \( sI \) is an ideal of \( R \), \( M \) is an \( S \)-comultiplication module.

**Theorem 2.** Let \( f : M \to M' \) be an \( R \)-homomorphism and \( t \text{Ker}(f) = (0) \) for some \( t \in S \).

\((i)\) If \( M' \) is an \( S \)-comultiplication module, then \( M \) is an \( S \)-comultiplication module.

\((ii)\) If \( f \) is an \( R \)-epimorphism and \( M \) is an \( S \)-comultiplication module, then \( M' \) is an \( S \)-comultiplication module.

**Proof.** \((i)\) Let \( N \) be a submodule of \( M \). Since \( M' \) is an \( S \)-comultiplication module, there exist \( s \in S \) and an ideal \( I \) of \( R \) such that \( s(0 :_M I) \subseteq f(N) \subseteq (0 :_{M'} I) \). Then, we have \( f(N) = (0) \) and so \( f(N) \subseteq \text{Ker} f \). Since \( t \text{Ker}(f) = (0) \), we have \( t(N) = (0) \) and so \( N \subseteq (0 :_M tI) \). Now, we will show that \( t^2 s(0 :_M tI) \subseteq N \subseteq (0 :_M tI) \). Let \( m \in (0 :_M tI) \). Then, we have \( tIm = 0 \) and so
$f(tIm) = tf(m) = If(tm) = 0$. This implies that $f(tm) \in (0 : M_1 I)$. Thus, we have $sf(tm) = f(stm) \in s(0 : M_1 I) \subseteq f(N)$ and so there exists $y \in N$ such that $f(stm) = f(y)$ and so $stm - y \in Ker(f)$. Thus, we have $t(stm - y) = 0$ and so $t^2 sm = tx$. Then we obtain

$$t^2 s(0 : M_1 I) \subseteq tN \subseteq N' \subseteq (0 : M_1 I).$$

Now, put $t^2 s = s' \in S$ and $J = tI$. Thus,

$$s'(0 : M_1 J) \subseteq N' \subseteq (0 : M_1 J).$$

Therefore, $M$ is an $S$-comultiplication module.

(ii) Let $N'$ be a submodule of $M'$. Since $M$ is an $S$-comultiplication module, there exist $s \in S$ and an ideal $I$ of $R$ such that

$$s(0 : M_1 I) \subseteq f^{-1}(N') \subseteq (0 : M_1 I).$$

This implies that $fN' = (0)$ and so $f(I^{-1}(N')) = fN' = (0)$ since $f$ is surjective. Then, we have $N' \subseteq (0 : M_1 I)$. On the other hand, we get $f(s(0 : M_1 I)) = sf((0 : M_1 I)) \subseteq f(N') = N'$. Now, let $m' \in (0 : M_1 I)$. Then, $Im' = 0$. Since, $f$ is epimorphism, there exists $m \in M$ such that $m' = f(m)$. Then, we have $Im' = If(m) = f(Im) = 0$ and so $Im \subseteq Ker f$. Since $tKer(f) = 0$, we have $tIm = 0$ and so $tm \in (0 : M_1 I)$. Then we get $f(tm) = tf(m) = tm' \in f((0 : M_1 I)).$

Thus, we have $t(0 : M_1 I) \subseteq f((0 : M_1 I))$ and hence $st(0 : M_1 I) \subseteq sf((0 : M_1 I)) \subseteq N' \subseteq (0 : M_1 I)$. Thus, $M'$ is an $S$-comultiplication module.

As an immediate consequences of previous theorem, we give the following explicit results.

**Corollary 1.** Let $M$ be an $R$-module, $N$ be a submodule of $M$ and $S$ be a m.c.s of $R$. Then we have the following.

(i) If $M$ is an $S$-comultiplication module, then $N$ is an $S$-comultiplication module.

(ii) If $M$ is an $S$-comultiplication module and $tM \subseteq N$ for some $t \in S$, then $M/N$ is an $S$-comultiplication $R$-module.

**Proposition 3.** Let $M_i$ be an $R_i$-module and $S_i$ be a m.c.s of $R_i$ for each $i = 1, 2$. Suppose that $M = M_1 \times M_2$, $R = R_1 \times R_2$ and $S = S_1 \times S_2$. The following assertions are equivalent.

(i) $M$ is an $S$-comultiplication $R$-module.

(ii) $M_1$ is an $S_1$-comultiplication $R_1$-module and $M_2$ is an $S_2$-comultiplication $R_2$-module.

**Proof.** (i) $\Rightarrow$ (ii) : Assume that $M$ is an $S$-comultiplication $R$-module. Take a submodule $N_1$ of $M_1$. Then, $N_1 \times \{0\}$ is a submodule of $M$. Since $M$ is an $S$-comultiplication module, there exist $s = (s_1, s_2) \in S_1 \times S_2$ and an ideal $J = I_1 \times I_2$ of $R$ such that $(s_1, s_2)(0 : M_1 I_1 \times I_2) \subseteq N_1 \times \{0\} \subseteq (0 : M_1 I_1 \times I_2)$, where $I_i$ is an ideal of $R_i$. Then we can easily get $s_1(0 : M_1 I_1) \subseteq N_1 \subseteq (0 : M_1 I_1)$ which shows that $M_1$ is an $S_1$-comultiplication module. Similarly, taking a submodule $N_2$ of $M_2$ and a submodule $\{0\} \times N_2$ of $M$, we can show that $M_2$ is an $S_2$-comultiplication module.

(ii) $\Rightarrow$ (i) : Now, assume that $M_1$ is an $S_1$-comultiplication module and $M_2$ is an $S_2$-comultiplication module. Let $N$ be a submodule of $M$. Then we can write $N = N_1 \times N_2$ for some submodule $N_i$ of $M_i$. Since $M_1$ is an $S_1$-comultiplication module,

$$s_1(0 : M_1 I_1) \subseteq N_1 \subseteq (0 : M_1 I_1)$$
for some ideal \( I_1 \) of \( R_1 \) and \( s_1 \in S_1 \). Since \( M_2 \) is an \( S_2 \)-comultiplication module,
\[
    s_2(0 :_{M_2} I_2) \subseteq N_2 \subseteq (0 :_{M_2} I_2)
\]
for some ideal \( I_2 \) of \( R_2 \) and \( s_2 \in S_2 \). Put \( s = (s_1, s_2) \in S \). Then,
\[
    s(0 :_{M} I_1 \times I_2) = s_1(0 :_{M_1} I_1) \times s_2(0 :_{M_2} I_2)
    \subseteq N_1 \times N_2 \subseteq (0 :_{M_1} I_1) \times (0 :_{M_2} I_2) = (0 :_{M} I_1 \times I_2)
\]
where \( I_1 \times I_2 \) is an ideal of \( R \) and \( (s_1, s_2) \in S \), as needed. \( \square \)

**Theorem 3.** Let \( M = M_1 \times M_2 \times \cdots \times M_n \) be an \( R = R_1 \times R_2 \times \cdots \times R_n \)
module and \( S = S_1 \times S_2 \times \cdots \times S_n \) be a \( \mathbb{m.c.s.} \) of \( R \) where \( M_i \) are \( R_i \)-modules and \( S_i \) are \( \mathbb{m.c.s.} \) of \( R_i \) for all \( i \in \{1, 2, \ldots, n\} \), respectively. The following statements are equivalent.

(i) \( M \) is an \( S \)-comultiplication \( R \)-module.

(ii) \( M_i \) is an \( S_i \)-comultiplication \( R_i \)-module for each \( i = 1, 2, \ldots, n \).

**Proof.** Here, induction can be applied on \( n \). The statement is true when \( n = 1 \). If \( n = 2 \), result follows from Proposition 3. Assume that statements are equivalent for each \( k < n \). We will show that it also holds for \( k = n \). Now, put \( M' = M_1 \times M_2 \times \cdots \times M_{n-1} \), \( R = R_1 \times R_2 \times \cdots \times R_{n-1} \) and \( S = S_1 \times S_2 \times \cdots \times S_{n-1} \). Note that \( M = M' \times M_n \), \( R = R' \times R_n \) and \( S = S' \times S_n \). Then by Proposition 3 \( M \) is an \( S \)-comultiplication \( R \)-module if and only if \( M' \) is an \( S' \)-comultiplication \( R' \)-module and \( M_n \) is an \( S_n \)-comultiplication \( R_n \)-module. The rest follows from induction hypothesis. \( \square \)

Let \( p \) be a prime ideal of \( R \). Then we know that \( S_p = (R - p) \) is a \( \mathbb{m.c.s.} \) of \( R \). If an \( R \)-module \( M \) is an \( S_p \)-comultiplication for a prime ideal \( p \) of \( R \), then we say that \( M \) is a \( p \)-comultiplication module. Now, we will characterize comultiplication modules in terms of \( S \)-comultiplication modules.

**Theorem 4.** Let \( M \) be an \( R \)-module. The following statements are equivalent.

(i) \( M \) is a comultiplication module.

(ii) \( M \) is a \( \mathbb{P} \)-comultiplication module for each prime ideal \( \mathbb{P} \) of \( R \).

(iii) \( M \) is an \( \mathbb{M} \)-comultiplication module for each maximal ideal \( \mathbb{M} \) of \( R \).

(iv) \( M \) is an \( \mathbb{M} \)-comultiplication module for each maximal ideal \( \mathbb{M} \) of \( R \) with \( M \mathbb{M} \neq 0 \).

**Proof.** (i) \( \Rightarrow \) (ii) : Follows from Example 3.

(ii) \( \Rightarrow \) (iii) : Follows from the fact that every maximal ideal is prime.

(iii) \( \Rightarrow \) (iv) : Clear.

(iv) \( \Rightarrow \) (i) : Suppose that \( M \) is an \( \mathbb{M} \)-comultiplication module for each maximal ideal \( \mathbb{M} \) of \( R \) with \( M \mathbb{M} \neq 0 \). Take a submodule \( N \) of \( M \) and a maximal ideal \( \mathbb{M} \) of \( R \). If \( M \mathbb{M} = 0 \), then clearly we have \( N \mathbb{M} = (0 :_{\mathbb{M}} \text{ann}(N)) \mathbb{M} \). So assume that \( M \mathbb{M} \neq 0 \). Since \( M \) is an \( \mathbb{M} \)-comultiplication module, there exists \( s_M \notin \mathbb{M} \) such that \( s_M(0 :_{\mathbb{M}} \text{ann}(N)) \subseteq N \). Then we have
\[
    (0 :_{\mathbb{M}} \text{ann}(N)) \mathbb{M} = (s_M(0 :_{\mathbb{M}} \text{ann}(N))) \mathbb{M} \subseteq N \mathbb{M} \subseteq (0 :_{\mathbb{M}} \text{ann}(N)) \mathbb{M}.
\]
Thus we have \( N \mathbb{M} = (0 :_{\mathbb{M}} \text{ann}(N)) \mathbb{M} \) for each maximal ideal \( \mathbb{M} \) of \( R \). Therefore, \( N = (0 :_{\mathbb{M}} \text{ann}(N)) \) so that \( M \) is a comultiplication module. \( \square \)

Now, we shall give the \( S \)-version of Dual Nakayama’s Lemma for \( S \)-comultiplication module. First, we need the following Proposition.
**Proposition 4.** Let $M$ be an $S$-comultiplication $R$-module. Then,
(i) If $I$ is an ideal of $R$ with $(0 :_M I) = 0$, then there exists $s \in S$ such that $sM \subseteq IM$.
(ii) If $I$ is an ideal of $R$ with $(0 :_M I) = 0$, then for every element $m \in M$, there exists $s \in S$ and $a \in I$ such that $sm = am$.
(iii) If $M$ is an $S$-finite $R$-module and $I$ is an ideal of $R$ with $(0 :_M I) = 0$, then there exist $s \in S$ and $a \in I$ such that $(s + a)M = 0$.

**Proof.** (i) : Suppose that $I$ is an ideal of $R$ with $(0 :_M I) = 0$. Then we have $((0 :_M I) : M) = (0 : IM) = (0 : M)$. Then by Lemma 3 (iii), there exists $s \in S$ such that $sM \subseteq IM$.

(ii) : Suppose that $I$ is an ideal of $R$ with $(0 :_M I) = 0$. Then for any $m \in M$, we have $(0 : Rm) = ((0 :_M I) : Rm) = (0 : Im)$. Again by Lemma 3 (iii), there exists $s \in S$ such that $sRm \subseteq Im$ and so $sm = am$ for some $a \in I$.

(iii) : Suppose that $M$ is an $S$-finite $R$-module and $I$ is an ideal of $R$ with $(0 :_M I) = 0$. Then there exists $t \in S$ such that $tM \subseteq Rm_1 + Rm_2 + \cdots + Rm_n$ for some $m_1, m_2, \ldots, m_n \in M$. Since $(0 :_M I) = 0$, by (i), there exists $s \in S$ such that $sM \subseteq IM$. This implies that $stM \subseteq tIM$. Then we obtain that $stM \subseteq tIM = tM \subseteq I(Rm_1 + Rm_2 + \cdots + Rm_n) = Im_1 + Im_2 + \cdots + Im_n$. Then for each $i = 1, 2, \ldots, n$, we have $stm_i = a_1m_1 + a_2m_2 + \cdots + a_im_n$ and so $-a_{1i}m_1 - a_{2i}m_2 - \cdots - (st-a_{ii})m_i + \cdots - a_{ni}m_n = 0$. Now, let $\Delta$ be the following matrix

$$\begin{bmatrix}
st-a_{11} & -a_{12} & \cdots & -a_{1n} 
-a_{21} & st-a_{22} & \cdots & -a_{2n} 
\vdots & \vdots & \ddots & \vdots 
-a_{ni} & -a_{ni} & \cdots & st-a_{nn}
\end{bmatrix}_{n \times n}$$

Then we have $|\Delta| m_i = 0$ for each $i = 1, 2, \ldots, n$. Thus we obtain that $t |\Delta| M = 0$. This implies that $t(s^nt^n + a)M = (s^nt^{n+1} + at)M = 0$ for some $a \in I$. Now, put $u = s^nt^{n+1} \in S$ and $b = at \in I$. Then we have $(u + b)M = 0$ which completes the proof. □

**Theorem 5.** *(S-Dual Nakayama’s Lemma)* Let $M$ be an $S$-comultiplication module, where $S$ is a m.c. of $R$ satisfying maximal multiple condition. Suppose that $I$ is an ideal of $R$ such that $tI \subseteq \text{Jac}(R)$ for some $t \in S$. If $(0 :_M tI) = 0$, then there exists $s \in S$ such that $sM = 0$.

**Proof.** Suppose that $S$ satisfies maximal multiple condition. Then there exists $s \in S$ such that $t|s$ for each $t \in S$. Let $I$ be an ideal of $R$ with $tI \subseteq \text{Jac}(R)$ for some $t \in S$ and $(0 :_M tI) = 0$. Then for each $m \in M$, by Proposition 2 (ii), there exists $t' \in S$ such that $t'Rm \subseteq tIm$ and so $s^2t'Rm \subseteq s^2tIm \subseteq s^2Im$. Now, put $u = s^2t'$. By maximal multiple condition, we have $sRm \subseteq uRm \subseteq s^2Rm$ and so $sm = s^2am$ for some $a \in R$. On the other hand, we note that $sI \subseteq tI \subseteq \text{Jac}(R)$. Thus we have $s(1 - sa)m = 0$. Since $sa \in \text{Jac}(R)$, we get $1 - sa$ is an unit and so $sm = 0$. Thus we have $sM = 0$. □

**Corollary 2.** *(Dual Nakayama’s Lemma)* Let $M$ be a comultiplication module and $I$ an ideal of $R$ such that $I \subseteq \text{Jac}(R)$. If $(0 :_M I) = 0$, then $M = 0$.

**Proof.** Take $S = \{1\}$ and apply Theorem 5. □
3. S-cyclic modules

In this section, we investigate the relations between S-comultiplication modules and S-cyclic modules.

**Proposition 5.** Let $M$ be an $S$-comultiplication $R$-module and $N$ be a minimal ideal of $R$ such that $(0:_MN) = 0$. Then, $M$ is an $S$-cyclic module.

**Proof.** Choose a nonzero element $m$ of $M$. Since $M$ is an $S$-comultiplication module, there exist $s \in S$ and an ideal $I$ of $R$ such that $s(0:_MI) \subseteq Rm \subseteq (0:_MI)$. By the assumption $(0:_MN) = 0$, we have

$$s((0:_MN):_MI) \subseteq Rm \subseteq ((0:_MN):_MI) \Rightarrow s(0:_MI) \subseteq Rm \subseteq (0:_MI).$$

Since $0 \subseteq NI \subseteq N$ and $N$ is minimal ideal of $R$, either $NI = N$ or $NI = 0$. If the former case holds, we have $s(0:_MN) \subseteq Rm \subseteq (0:_MN)$. This means that $Rm = 0$, a contradiction. The second case implies the equality $s(0:_M0) \subseteq Rm \subseteq (0:_M0)$. It means $sM \subseteq Rm \subseteq M$ proving that $M$ is S-cyclic.

**Proposition 6.** Let $M$ be an $S$-comultiplication module of $R$. Let $\{M_i\}$ be a collection of submodules of $M$ with $\bigcap_i M_i = 0$. Then, for every submodule $N$ of $M$, there exists an $s \in S$ such that

$$s \bigcap_i (N + M_i) \subseteq N \subseteq \bigcap_i (N + M_i).$$

**Proof.** Let $N$ be a submodule of $M$. Since $M$ is an $S$-comultiplication module, we have $s(0:_M\text{ann}(N)) \subseteq N \subseteq (0:_M\text{ann}(N))$ for some $s \in S$. This implies $s(\bigcap_i M_i :_M \text{ann}(N)) \subseteq N \subseteq (\bigcap_i M_i :_M \text{ann}(N))$ since $\bigcap_i M_i = 0$. Then, we obtain $s(\bigcap_i (M_i :_M \text{ann}(N))) \subseteq N \subseteq \bigcap_i (M_i :_M \text{ann}(N))$. Thus,

$$s \bigcap_i (N + M_i) \subseteq s \bigcap_i (M_i :_M \text{ann}(N)) \subseteq N \subseteq \bigcap_i (N + M_i).$$

**Proposition 7.** Let $M$ be an $S$-comultiplication module. Then, for each submodule $N$ of $M$ and each ideal $I$ of $R$ with $N \subseteq s(0:_MI)$ for some $s \in S$, there exists an ideal $J$ of $R$ such that $I \subseteq J$ and $s(0:_MJ) \subseteq N$.

**Proof.** Let $N$ be a submodule of $M$. Since $M$ is an $S$-comultiplication module, $s(0:_M\text{ann}(N)) \subseteq N \subseteq (0:_M\text{ann}(N))$ for some $s \in S$. So, we obtain $s(0:_M\text{ann}(N)) \subseteq N \subseteq s(0:_MI)$. Taking $J = I + \text{ann}(N)$,

$$s(0:_MJ) = s(0:_M I + \text{ann}(N)) \subseteq s(0:_MI) \cap s(0:_M\text{ann}(N)) \subseteq s(0:_M\text{ann}(N)) \subseteq N.$$

Recall that an $R$-module $M$ is said to be a torsion free if the set of torsion elements $T(M) = \{m \in M : rm = 0$ for some $0 \neq r \in R\}$ of $M$ is zero. Also $M$ is called a torsion module if $T(M) = M$. We refer the reader to [3] for more details on torsion subsets $T(M)$ of $M$.

**Theorem 6.** Every $S$-comultiplication module is either $S$-cyclic or torsion.

**Proof.** Let $M$ be an $S$-comultiplication module. Assume that $M$ is not an $S$-cyclic module and $\text{ann}_R(m) = 0$ for some $m \in M$. Since $Rm$ is a submodule of $M$ and $M$ is an $S$-comultiplication module, we have $s(0:_M\text{ann}(m)) \subseteq Rm \subseteq (0:_M$.
ann\((m)\). It gives \(sM \subseteq Rm \subseteq M\) for some \(s \in S\). This contradiction completes the proof. Hence, \(\text{ann}(m) \neq 0\) for all \(m \in M\) proving that \(M\) is torsion module. \(\square\)

**Theorem 7.** Let \(R\) be an integral domain and \(M\) be an \(S\)-finite and \(S\)-comultiplication module. If \(sM\) is faithful for each \(s \in S\), then \(M\) is an \(S\)-cyclic module.

**Proof.** Suppose that \(M\) is not an \(S\)-cyclic module. Then \(M\) is a torsion module from Theorem \(6\). Since \(M\) is an \(S\)-finite module, there exist \(s \in S\) and \(m_1, m_2, \ldots, m_n \in M\) such that \(sM \subseteq Rm_1 + Rm_2 + \cdots + Rm_n\). This implies that 
\[
\text{ann}(Rm_1 + Rm_2 + \cdots + Rm_n) = \bigcap_{i=1}^{n} \text{ann}(m_i) \subseteq \text{ann}(sM) = 0
\]
since \(sM\) is faithful. Hence, \(M\) is an \(S\)-cyclic module. \(\square\)

Recall from [19] that an \(R\)-module \(M\) is said to be an \(S\)-torsion free module if there exists \(s \in S\) and whenever \(am = 0\) for some \(a \in R\) and \(m \in M\), then either \(sa = 0\) or \(sm = 0\).

**Theorem 8.** Every \(S\)-comultiplication \(S\)-torsion free module is an \(S\)-cyclic module.

**Proof.** Let \(M\) be an \(S\)-comultiplication and \(S\)-torsion free module. If \(sM = 0\) for some \(s \in S\), then \(M\) is an \(S\)-cyclic module. So assume that \(sM \neq 0\) for each \(s \in S\). Since \(M\) is an \(S\)-torsion free module, there exists \(t' \in S\) and whenever \(am = 0\) for some \(a \in R\) and \(m \in M\), then either \(t'a = 0\) or \(t'm = 0\). Since \(t'M \neq 0\), there exists \(m \in M\) such that \(t'm \neq 0\). As \(M\) is an \(S\)-comultiplication module, there exists \(t \in S\) such that \(t(0 :_M \text{ann}(m)) \subseteq Rm\). Since \(\text{ann}(m)m = 0\) and \(M\) is \(S\)-torsion free module, we conclude either \(t'\text{ann}(m) = 0\) or \(t'm = 0\). The second case is impossible. So we have \(t'\text{ann}(m) = 0\) and so \(t'M \subseteq (0 :_M \text{ann}(m))\). This implies that \(t't'M \subseteq t(0 :_M \text{ann}(m)) \subseteq Rm\) where \(t't' \in S\), namely, \(M\) is an \(S\)-cyclic module. \(\square\)

Let \(K\) be a nonzero submodule of \(M\). \(K\) is said to be an \(S\)-minimal submodule if \(L \subseteq K\) for some submodule of \(M\), then there exists \(s \in S\) such that \(sK \subseteq L\).

**Theorem 9.** Every \(S\)-comultiplication prime \(R\)-module \(M\) is \(S\)-minimal.

**Proof.** Let \(M\) be an \(S\)-comultiplication prime \(R\)-module. Assume that \(N\) is a submodule of \(M\). Since \(M\) is prime, \(\text{ann}(N) = \text{ann}(M)\). Also, \((0 :_M \text{ann}(N)) = (0 :_M \text{ann}(M))\). Since \(M\) is an \(S\)-comultiplication module, \(s(0 :_M \text{ann}(N)) \subseteq N \subseteq (0 :_M \text{ann}(N))\) for some \(s \in S\). Hence, we get \(s(0 :_M \text{ann}(M)) \subseteq N \subseteq (0 :_M \text{ann}(M))\) and it shows that \(sM \subseteq N \subseteq M\). Therefore, \(M\) is \(S\)-minimal. \(\square\)

4. S-second submodules of S-comultiplication modules

This section is dedicated to the study of \(S\)-second submodules of \(S\)-comultiplication module. Now, we need the following definition.

**Definition 2.** Let \(M\) and \(M'\) be two \(R\)-modules and \(f : M \rightarrow M'\) be an \(R\)-homomorphism.

(i) If there exists \(s \in S\) such that \(f(m) = 0\) implies that \(sm = 0\), then \(f\) is said to be an \(S\)-injective (or, just \(S\)-monic).

(ii) If there exists \(s \in S\) such that \(sM' \subseteq \text{Im} f\), then \(f\) is said to be an \(S\)-epimorphism (or, just \(S\)-epic).
The following proposition is explicit. Let $M$ be an $R$-module. An element $x \in R$ is called a zero divisor on $M$ if there exists $0 \neq m \in M$ such that $xm = 0$, or equivalently, $\text{ann}_M(x) \neq \{0\}$. The set of all zero divisor elements of $R$ on $M$ is denoted by $z(M)$.

**Proposition 8.** Let $M$ and $M'$ be two $R$-modules and $f : M \to M'$ be an $R$-homomorphism.

(i) $f$ is $S$-monic if and only if there exists $s \in S$ such that $s \text{Ker}(f) = \{0\}$.

(ii) If $f$ is monic, then $f$ is $S$-monic for each $m.c.s$ $S$ of $R$. The converse holds in case $S \subseteq R - z(M)$.

(iii) If $f$ is epic, then $f$ is $S$-epic for each $m.c.s$ $S$ of $R$. The converse holds in case $S \subseteq u(R)$.

Recall from [19] that a submodule $P$ of $M$ with $(P : M) \cap S = \emptyset$ is said to be an $S$-prime submodule if there exists a fixed $s \in S$ and whenever $am \in P$ for some $a \in R, m \in M$, then either $sa \in (P : M)$ or $sm \in P$. In particular, an ideal $I$ of $R$ is said to be an $S$-prime ideal if $I$ is an $S$-prime submodule of $M$. We note here that Acrif and Hamed, in their paper [18], studied and investigated the further properties of $S$-prime ideals. Now, we give the following needed results which can be found in [19].

**Proposition 9.** (i) ([19] Proposition 2.9) If $P$ is an $S$-prime submodule of $M$, then $(P : M)$ is an $S$-prime ideal of $R$.

(ii) ([19] Lemma 2.16) If $P$ is an $S$-prime submodule of $M$, there exists a fixed $s \in S$ such that $(P : M s') \subseteq (P : M s)$ for each $s' \in S$.

(iii) ([19] Theorem 2.18) $P$ is an $S$-prime submodule of $M$ if and only if $(P : M s)$ is a prime submodule of $M$ for some $s \in S$.

By the previous proposition, we deduce that $P$ is an $S$-prime submodule if and only if there exists a fixed $s \in S$ such that $(P : M s)$ is a prime submodule and $(P : M s') \subseteq (P : M s)$ for each $s' \in S$.

Sevim et al. in [19] gave many characterizations of $S$-prime submodules. Now, we give a new characterization of $S$-prime submodules from another point of view.

Recall that a homomorphism $f : M \to M'$ is said to be an $S$-zero if there exists $s \in S$ such that $sf(m) = 0$ for each $m \in M$, that is, $s \text{Im} f = \{0\}$.

**Proposition 10.** Let $P$ be a submodule of $M$ with $(P : M) \cap S = \emptyset$. The following statements are equivalent.

(i) $P$ is an $S$-prime submodule of $M$.

(ii) There exists a fixed $s \in S$, for any $a \in R$ and the homothety $M/P \cong M/P$, either $S$-zero or $S$-injective with respect to $s \in S$.

**Proof.** $(i) \Rightarrow (ii)$: Suppose that $P$ is an $S$-prime submodule of $M$. Then there exists a fixed $s \in S$ such that $am \in P$ for some $a \in R, m \in M$ implies that $saM \subseteq P$ or $sm \in P$. Now, take $a \in R$ and assume that the homothety $M/P \cong M/P$ is not $S$-injective with respect to $s \in S$. Then there exists $m \in M$ such that $a(m + P) = am + P = 0_{M/P}$ but $s(m + P) \neq 0_{M/P}$. This gives that $am \in P$ and $sm \notin P$. Since $P$ is an $S$-prime submodule, we have $sa \in (P : M)$ and thus $sam' \in P$ for each $m' \in M$. Then we have $sa(m' + P) = 0_{M/P}$ for each $m' \in M$, that is, the homothety $M/P \cong M/P$ is $S$-zero with respect to $s$.

$(ii) \Rightarrow (i)$: Suppose that $(ii)$ holds. Let $am \in P$ for some $a \in R$ and $m \in M$. Assume that $sm \notin P$. Then we deduce the homothety $M/P \cong M/P$ is not
is an $S$-injective. Thus by (ii), $M/P \rightarrow M/P$ is $S$-zero with respect to $s \in S$, namely, $sa(m' + P) = 0_{M/P}$ for each $m' \in M$. This yields that $sa \in (P : M)$. Therefore, $P$ is an $S$-prime submodule of $M$. \qed

It is well known that a submodule $P$ of $M$ is a prime submodule if and only if every homothety $M/P \rightarrow M/P$ is either injective or zero. This fact can be obtained by Proposition 10 by taking $S \subseteq u(R)$.

Recall from [7] that a submodule $N$ of $M$ with $\text{ann}(N) \cap S = \emptyset$ is said to be an $S$-second submodule if there exists $s \in S$, $srN = 0$ or $srN = sN$ for each $r \in R$. Motivated by Proposition 10, we give a new characterization of $S$-second submodules from another point of view. Since the proof is similar to Proposition 10, we omit the proof.

**Theorem 10.** Let $N$ be a submodule of $M$ with $\text{ann}(N) \cap S = \emptyset$. The following assertions are equivalent.

(i) $N$ is an $S$-second submodule.

(ii) There exists $s \in S$ such that for each $a \in R$, the homothety $N \xrightarrow{a} N$ is either $S$-zero or $S$-surjective with respect to $s \in S$.

(iii) There exists a fixed $s \in S$, for each $a \in R$, either $saN = 0$ or $sN \subseteq aN$.

The author in [7] proved that if $N$ is an $S$-second submodule of $M$, then $\text{ann}(N)$ is an $S$-prime ideal of $R$ and the converse holds under the assumption that $M$ is comultiplication module [7] Proposition 2.9]. Now, we show that this fact is true even if $M$ is an $S$-comultiplication module.

**Theorem 11.** Let $M$ be an $S$-comultiplication module. The following statements are equivalent.

(i) $N$ is an $S$-second submodule of $M$.

(ii) $\text{ann}(N)$ is an $S$-prime ideal of $R$ and there exists $s \in S$ such that $sN \subseteq s'N$ for each $s' \in S$.

**Proof.** (i) $\Rightarrow$ (ii) : The claim follows from [7] Proposition 2.9] and [7] Lemma 2.13).

(ii) $\Rightarrow$ (i) : Suppose that $\text{ann}(N)$ is an $S$-prime ideal of $R$. Now, we will show that $N$ is an $S$-second submodule of $M$. To prove this, take $a \in R$. Since $\text{ann}(N)$ is an $S$-prime ideal, by Proposition 1 there exists $s \in S$ such that $\text{ann}(sN)$ is a prime ideal and $\text{ann}(s'N) \subseteq \text{ann}(sN)$ for each $s' \in S$. Assume that $saN \neq (0)$. Now, we shall show that $sN \subseteq aN$. Since $M$ is an $S$-comultiplication module, there exists $s' \in S$ and an ideal $I$ of $R$ such that $s'(0 :_M I) \subseteq aN \subseteq (0 :_M I)$. This implies that $aI \subseteq \text{ann}(N)$. Since $\text{ann}(N)$ is an $S$-prime ideal, there exists $s \in S$ such that either $sa \in \text{ann}(N)$ or $sI \subseteq \text{ann}(N)$ by Proposition 1. The first case impossible since $saN \neq (0)$. Thus we have $I \subseteq \text{ann}(sN)$. Then we have $s's(0 :_M \text{ann}(sN)) \subseteq s'(0 :_M I) \subseteq aN$. This implies that $s's^2N \subseteq s's(0 :_M \text{ann}(sN)) \subseteq aN$. Then by (ii), $sN \subseteq s's^2N \subseteq aN$. Then by Theorem 10(iii), $N$ is an $S$-second submodule of $M$. \qed

**Theorem 12.** Let $M$ be a comultiplication module. The following statements are equivalent.

(i) $N$ is a second submodule of $M$.

(ii) $\text{ann}(N)$ is a prime ideal of $R$.
Proof. Take $S \subseteq u(R)$ and note that $S$-comultiplication module and comultiplication modules are equal. On the other hand, second submodule and $S$-second submodules are equivalent. The rest follows from Theorem 11.

Theorem 13. Let $M$ be an $S$-comultiplication module and let $N$ be an $S$-second submodule of $M$. If $N \subseteq N_1 + N_2 + \cdots + N_m$ for some submodules $N_1, N_2, \ldots, N_m$ of $M$, then there exists $s \in S$ such that $sN \subseteq N_i$ for some $1 \leq i \leq m$.

Proof. Suppose that $N$ is an $S$-second submodule of an $S$-comultiplication module $M$. Suppose that $N \subseteq \sum_{i=1}^{m} N_i$ for some submodules $N_1, N_2, \ldots, N_m$ of $M$. Then we have $\text{ann} \left( \sum_{i=1}^{m} N_i \right) = \bigcap_{i=1}^{m} \text{ann} (N_i) \subseteq \text{ann} (N)$. Since $N$ is an $S$-second submodule, by Theorem 11, $\text{ann} (N)$ is an $S$-prime ideal of $R$. Then by [19], Corollary 2.6, there exists $s \in S$ such that $s \text{ann} (N_i) \subseteq \text{ann} (N)$ for some $1 \leq i \leq m$. This implies that $\text{ann} (N_i) \subseteq \text{ann} (sN)$. Then by Lemma 1 (iii), $stN \subseteq N_i$ for some $t \in S$ which completes the proof.

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