Theory of Plasma-Cascade Instability
Vladimir N. Litvinenko*1,2 and Gang Wang2,1

1 Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY
2 Collider-Accelerator Department, Brookhaven National Laboratory, Upton, NY

Abstract. In this paper we present the theory of a novel micro-bunching instability occurring in charged particle beams propagating along a straight trajectory: a Plasma-Cascade Instability (PCI). This instability was confirmed by 3D numerical simulations [1] and observed experimentally [2]. It can be driven by variation of beam’s density and/or particle’s mobility. The PCI can strongly amplify noise in the beam and drastically reduce its quality. Conversely, such instability can drive novel high-power sources of radiation or can be used as a broadband amplifier.

PACS numbers: 52.59.Sa, 29.27.-a, 41.60.Cr, 41.75.Ak, 29.20.Ej, 52.20.-j, 52.35.Qz

I. Introduction

High brightness intense charged particle beams are central for high luminosity hadron colliders [3-7], X-ray free-electron-lasers (FEL) [8-22] as well as for future hadron beam coolers [23-24], X-ray FEL oscillators [26-29], and plasma-wake-field accelerators with TV/m accelerating gradients [30-38]. Preservation of the beam quality during generation, acceleration, transportation and compression is important for attaining the desirable properties of the beam. On the other hand, instabilities in the beams can be deliberately built-in to attain specific results. The most known application is the FEL instability used for generating coherent radiation from THz to X-rays [39-43]. Less known applications are Coherent electron Cooling (CeC) of hadron beams [44-49] or generation of broad-band high power radiation by micro-bunched beams [50-52].

The PCI is the micro-bunching instability occurs in a beam propagating along a straight line. It differs from the well-known and well-studied conventional micro-bunching instabilities [53-72] requiring bending of the beam trajectory.

Theory of the PCI is important both for understanding of the nature of this instability and for predicting condition when PCI can occur. We will start from discussions of a simple model for such instability and continue with a rigorous derivation of the integral equation describing the PCI in 3D case. We will continue with reducing the integral equation to a second order ordinary differential equation for two specific cases of momenta distribution. We conclude with deriving equation for longitudinal PCI and discussions of our PCI theory in the framework of other instabilities.

II. Simple model of Plasma-Cascade Instability

Let’s consider an evolution of perturbations in a charged homogeneous beam using the co-moving frame of reference, where motion of particles can be considered non-relativistic. It is well

1 For example, in a magnetic chicane or in an arc of an accelerator.
known that small density perturbations $|\delta n| / n_o << 1$ in a cold homogeneous beam (e.g., non-neutral plasma) will undergo oscillations with plasma frequency [73-74],

$$\ddot{n} + \omega_p^2 n = 0; \quad \omega_p = \frac{c}{4\pi n_r r_c}$$  (1)

where $n_r$ is the particles density in the rest frame of particles, $c$ is the speed of light, $r_c = e^2 / mc^2$ is the classical radius, $e$ is charge and $m$ is its rest mass of the particles. Equation (1) has a trivial oscillatory solution for an arbitrary infinitesimal $\delta n(\vec{r})$:

$$\ddot{n} = \delta n(\vec{r}) \cdot \cos(\omega_p t + \varphi),$$  (2)

with constants of motion $\delta n, \varphi$ determined by initial conditions. During this stable oscillation after each quarter of the plasma period the electrostatic energy (e.g., that of the density perturbation) is transferred to the kinetic energy (e.g., velocity perturbation), and vice versa. In plasma with finite temperature this stable oscillation will eventually decay [73-74].

Situation does change if the density of the beam, and the corresponding plasma frequency, vary in time. The oscillator equation with time-dependent frequency

$$\ddot{n} + \omega_p(t)n = 0,$$  (3)

may result in unstable oscillations with exponentially growing amplitude. In general, Eq. (3) does not have an analytical solution. But its solution can be represented by a linear symplectic transformation [75-78], which can be formally expressed in the form of ordered matrix exponent:

$$\begin{bmatrix}
\dot{n}(t_2) \\
\dot{n}(t_2)
\end{bmatrix} = M(t_1|t_2) \begin{bmatrix}
\dot{n}(t_1) \\
\dot{n}(t_1)
\end{bmatrix}; M(t_1|t_2) = \exp \left[ \int_{t_1}^{t_2} D(t) dt \right]; D(t) = \begin{bmatrix}
0 & 1 \\
-\omega_p(t) & 0
\end{bmatrix},$$  (4)

calculated in a following sequence:

$$\exp \left[ \int_{t_1}^{t_2} D(t) dt \right] = \lim_{N \to \infty} \prod_{n=1}^{N} M_n \equiv M_N \ldots M_2 M_1; \Delta t = \frac{t_2 - t_1}{N}; t_n^* \in \{t_1 + (n-1)\Delta t, t_1 + n\Delta t\}$$

$$M_n = \exp \left[ D(t_n^*) \Delta t \right] = \begin{bmatrix}
\cos \Delta \varphi_n & \sin \Delta \varphi_n / \omega_p(t_n^*) \\
-\omega_p(t_n^*) \sin \Delta \varphi_n & \cos \Delta \varphi_n
\end{bmatrix}; \Delta \varphi_n = \omega_p(t_n^*) \Delta t$$  (5)

with $\det M(t_1|t_2) = \exp \int_{t_1}^{t_2} \text{Trace}[D(t)] dt = 1$. In spite of the “focusing” appearance of equation (3), the resulting matrix can correspond to either stable or unstable oscillations.

---

2 In this paper we are using SGS units. While using $e$ and $m$ is typical for describing electron or proton beams, for charged ion beams one should replace them with $q=Ze$ and $M=Am$ using the charge state $Z$ and atomic number $A$ of the ion.
Periodic modulation of oscillator frequency could result in exponential growth of oscillation amplitude - the phenomena well known as the parametric resonance [79]. Both the width and growth rate of this parametric instability can be enormous when the span of the frequency modulation is large. The well-known example of such instability is a system of periodic focusing lenses with focal lengths \( F \) less than a quarter of distance between them \( L \). The instability is such system has growth rate per cell of [78]:

\[
\lambda = -\left( \frac{L}{2F} - 1 \right) \left( 1 + \sqrt{1 - \left( \frac{L}{2F} - 1 \right)^2} \right),
\]

which be arbitrary large for \( L > 4F \). Such exponential instabilities are also well known in accelerators, where a solution of \( s \)-dependent Hill’s equation [75-78]:

\[
\frac{d^2x}{ds^2} + K_1(s)x = 0
\]

can be unstable even for non-negative \( K_1(s) \geq 0 \).

These examples prove that modulation of the beam parameters can result in an instability of plasma oscillation and growth of the density modulation in the beam, e.g., in an instability. The beam density modulation can be the result from the changes in the transverse distribution (e.g., from transverse focusing or defocusing forces), from the bunch \(^3\) compression or decompression, or from changes in the beam energy. It is obvious that changes in transverse and longitudinal sizes of the bunch results in the change of its volume and in the corresponding changes in the particles density.

In contrast, changes in the beam energy do not affect particle’s density in the laboratory frame but affect mobility of the particles. Because of the Lorentz transformation, in the instantaneous comoving frame the beam acceleration leads to bunch elongation, increase of distance between particles, and corresponding reduction in their density. Naturally, the bunch deceleration has the opposite effect.

Using the comoving beam frame is very attractive because the traditional set of Vlasov-Maxwell equations is reduced to non-relativistic case of Vlasov-Poisson equations, which is easier to solve. But unfortunately, in the case of beam change its energy (or bending its trajectory the co-moving is no longer inertial - with well-known unpleasant consequences.

Hence, in this paper we will use laboratory frame for our general theory and will use the comoving frame only for specific cases.

III. 3D theory of Plasma-Cascade Instability

Let’s consider bunch of particles whose motion is described using a standard accelerator coordinate system:

\(^3\) Here we use traditional definition for bunch as an assemble of particles limited in all spatial dimensions. In contrast, a beam can be either a sequence of bunches or a continuous flow of particles.
\[
\ddot{r} = \ddot{r}_o(s) + q_1 \cdot \ddot{n}(s) + q_2 \cdot \ddot{b}(s); \quad \ddot{\tau} = \frac{d\ddot{r}_o}{ds}; \quad \ddot{n} = -\frac{d\ddot{r}_o}{ds} \left| \frac{d\ddot{r}_o}{ds} \right| \ddot{b} = [\ddot{n} \times \ddot{\tau}]; \\
\frac{d\ddot{\tau}}{ds} = -K_o(s) \cdot \ddot{n}; \quad \frac{d\ddot{n}}{ds} = K_o(s) \cdot \ddot{\tau} - K_o(s) \cdot \ddot{b}; \quad \frac{d\ddot{b}}{ds} = K_o(s) \cdot \ddot{n};
\]

where \( \ddot{r}_o(s) \) is trajectory of the reference particle, \( K_o(s) = 1/\rho(s) \) is the curvature of the trajectory, and \( K_o(s) \) is its torsion. In this case the length along (azimuth) the reference trajectory \( s = \int |d\ddot{r}_o| \) serves as independent variable [76,80-82] with accelerator Hamiltonian of:

\[
H^* = -\left(1 + K_o q_1 \right) \sqrt{\left(\frac{H - e\varphi}{c^2} - m^2 c^2 \right)^2 - \left( p_1 - \frac{e}{c} A_1 \right)^2 - \left( p_2 - \frac{e}{c} A_2 \right)^2} \]

\[
-\frac{e}{c} A_3 - K_o q_1 \left( p_2 - \frac{e}{c} A_2 \right) - K_o q_2 \left( p_1 - \frac{e}{c} A_1 \right) \\
A_3 = (1 + K_o q_1) A_s + K\left(q_2 A_1 - q_1 A_2\right);
\]

and the set of three Canonical pairs of variables 4

\[
\{q_1, p_1\}, \{q_2, p_2\}, \{-ct, p_\tau\}; \\
p_1 = p_{m1} - \frac{e}{c} A_1; p_2 = p_{m2} - \frac{e}{c} A_2; p_\tau = \frac{E + e\varphi}{c};
\]

where \( t \) is arrival time of particle to the azimuth \( s \), \( \ddot{p}_m = \ddot{n} \cdot \ddot{r}_m + \ddot{b} \cdot \ddot{p}_m + \ddot{\tau} \cdot \ddot{p}_m \) and \( E \) are particle’s mechanical momentum and energy, correspondingly, and \( \{\varphi, A\} \) is the 4-potential of EM field. Introduction for the “paraxial” canonical pair:

\[
\{q_3, p_3\}; q_3 = c(t_o(s) - t); p_3 = \frac{E - E_o(s)}{c} + e \frac{\varphi(\ddot{r}_o; t) - \varphi(\ddot{r}_o; t)}{c}
\]

reduces the Hamiltonian (7) by

\[
\frac{c}{v_o(s)} p_3 - q_3 \frac{d}{ds} \left( E_o(s) + e \frac{\varphi(\ddot{r}_o; t)}{c} \right),
\]

where we used obvious \( \frac{dt_o(s)}{ds} = \frac{1}{v_o(s)} \) and

\[
E_o(s) = \gamma_o(s) mc^2 = c \sqrt{p_o^2(s) + m^2 c^2}; \quad \beta_o(s) = \frac{v_o(s)}{c} = \sqrt{1 - \gamma_o^{-2}(s)}.
\]

4 It is important to note that this is curvilinear coordinate system with neither unit or diagonal metric tensor [76,80-82].
For compactness of our expressions, we will define set of coordinates, \( q \), and corresponding Canonical momenta, \( p \), as well as the phases space vector \( \xi \):

\[
q = \{ q_1, q_2, q_3 \}; \quad p = \{ p_1, p_2, p_3 \}; \quad \xi^T = \{ q, p \};
\]
\[
\bar{q} = \tilde{e}_1 q_1 + \tilde{e}_2 q_2 + \tilde{e}_3 q_3; \quad \bar{p} = \tilde{e}_1 p_1 + \tilde{e}_2 p_2 + \tilde{e}_3 p_3; \quad \tilde{e}_1 = n; \tilde{e}_2 = b; \tilde{e}_3 = \tau
\]

for 3D case. Number of components are proportionally reduced for 2D and 1D cases. Equations of motion can be rewritten in a symplectic form as \(^{[82]}\)

\[
\xi^T = [q, p]; \quad \frac{d\xi}{ds} = S_k \frac{\partial H}{\partial \xi_k} \iff \frac{d\xi}{ds} = S \frac{\partial H}{\partial \xi};
\]

\[
S \equiv \left[ S_{ik} \right] = \begin{bmatrix} 0 & I_{3 \times 3} \\ -I_{3 \times 3} & 0 \end{bmatrix}; \quad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad S^2 = -I_{6 \times 6};
\]

where \( 0 \) is 3x3 zero matrix (see Appendix A for further discussion) and index \( ^T \) indicates transposition of matrices, including transferring a column in a row and vice versa.

Motion of particles is determined by initial conditions\(^6\)

\[
Q \equiv q(s = 0); \quad P \equiv p(s = 0) \iff \bar{Q} \equiv \bar{q}(s = 0); \quad \bar{P} \equiv \bar{p}(s = 0).
\]

Solved equations of motion

\[
q = q(Q, P, s); \quad p = p(Q, P, t)s; \quad \xi^T(X, s) = [q, p]; \quad X^T = [Q, P]
\]

represent the Canonical transformation from \( \{ Q, P \} \) to \( \{ q, p \} \) and vice versa. It means that inverse transformation

\[
Q = Q(q, p, s); \quad P = P(q, p, s); \quad X = \{ Q, P \} = X(\xi, t)
\]

exists and also represents the Canonical transformation from \( \{ q, p \} \) to \( \{ Q, P \} \). The transformation (15) results in trivial Hamiltonian system with set of canonical variables \( \{ Q, P \} \) and a zero Hamiltonian:

\[
\bar{H}(Q, P, s) = 0. \quad ^7
\]

---

\(^5\) Further in the paper we will use Einstein’s convention of summation by repeated indices, e.g.,

\[
a_{i}b_{i} = \sum_{i} a_{i}b_{i}; \quad a_{i}b_{i}c_{nk} = \sum_{i} \sum_{k} a_{i}b_{i}c_{nk}.
\]

\(^6\) We will interchangeably use both vector and functional appearances of the coordinates and momenta for compactness of formulae.

\(^7\) Generally speaking, this transformation can leave \( \bar{H}(Q, P, s) = f(s) \), which easily can be removed by a trivial Canonical transformation \( F = Q_i P_i - \int f(z) dz \).
Since we are considering instability in charge particle beams, we assume that solution for an unperturbed distribution function \( F_o \) is known and it satisfies the self-consistent Vlasov equation \(^8\) [83]:

\[
\frac{\partial F_o(\xi, s)}{\partial s} + S_{ik} \frac{\partial F_o(\xi, s)}{\partial \xi_i} \frac{\partial H_o(\xi, s)}{\partial \xi_k} = 0;
\]

\[
\tilde{F}_o(X) \equiv F_o(X, s = 0) \Rightarrow F_o(\xi, s) = \tilde{F}_o(X(\xi, s)).
\]

Let’s now consider an infinitesimally small perturbation of the distribution function, \( \tilde{f} \),

\[
F(\xi, s) = F_o(\xi, s) + \tilde{f}(\xi, s); \quad \| \tilde{f}(\xi, s) \| \ll \| F_o(\xi, s) \|;
\]

e.g., \( \tilde{f}(\xi, s) = O(\varepsilon) \| F_o(\xi, s) \|; \quad \varepsilon \ll 1 \), and the corresponding weak perturbation in the Hamiltonian:

\[
H(\xi, s) = H_o(\xi, s) + \tilde{h}(\xi, s); \quad \tilde{h}(\xi, s) = O(\varepsilon) \| H_o(\xi, s) \|.
\]

Applying Canonical transformation (15) we reduce the Hamiltonian (19) to the perturbation term

\[
\tilde{H}(X, s) = \tilde{h}(X, s) \equiv \tilde{h}(\xi(X, s), s),
\]

with Vlasov equations for the corresponding variation of the initial distribution function \( \tilde{f} \):

\[
\tilde{F}(X, s) = \tilde{F}_o(X) + \tilde{f}(X, s); \quad \tilde{f}(\xi, s) \equiv \tilde{f}(X(\xi, s), s);
\]

\[
\frac{\partial \tilde{f}}{\partial s} + S_{ik} \frac{\partial F_o}{\partial X_i} \frac{\partial \tilde{h}}{\partial X_k} + S_{ik} \frac{\partial f}{\partial X_i} \frac{\partial h}{\partial X_k} = 0.
\]

This method, called “the variation on initial values” in analytical mechanics [84] or as “the method of trajectories” in plasma physics [74], is well known. By assuming that solutions for self-consistent trajectories in eqs. (13-15) are known, it allows us to remove (at least formally) dynamic terms and to reduce the Vlasov equations to one comprising only of the perturbation terms. Next standard step is the linearization of the Vlasov equation by recognizing that third term in eq. (21) has order of \( O(\varepsilon^2) \):

\[
\frac{\partial \tilde{f}}{\partial s} + S_{ik} \frac{\partial F_o}{\partial X_i} \frac{\partial \tilde{h}}{\partial X_k} = S_{ik} \frac{\partial f}{\partial X_i} \frac{\partial h}{\partial X_k} = O(\varepsilon^2) \rightarrow 0;
\]

\[
\frac{\partial \tilde{f}}{\partial s} + \frac{\partial F_o}{\partial Q_i} \frac{\partial \tilde{h}}{\partial P_i} - \frac{\partial F_o}{\partial P_i} \frac{\partial \tilde{h}}{\partial Q_i} = 0.
\]

\(^8\) Self-consistent distribution function, which we use as the known background, would include all macroscopic collective effects such as space-charge and wake-fields induced by the bunch. Generally speaking, the self-consistent Hamiltonian would have functional dependence on the initial beam distribution \( \tilde{F}_o(X) \), e.g., \( H = H_o(\xi, P, \tilde{F}_o(X)) \). This fact does not change validity and applicability of the Vlasov equation (17).
A number of further assumptions are needed to derive either analytically solvable equations or those reducible to a directly solvable set of ordinary differential or integral equations. It is well known that a generic 3D evolution of a finite size charged beam is analytically intractable. Rare exceptions, such as non-physical but self-consistent Kapchinsky-Vladimirsky (KV) distribution, only attest to the case.

One typical simplification used in the theory of beam instabilities is an assumption of an infinite homogenous plasma. While this approach is not applicable for all collective effects in a beam with finite sizes, it has limited applicability for analyzing evolution of perturbation with periods significantly smaller than typical scales of density uniformity.

It is intuitively understandable that scales of the beam uniformity define the scale of the perturbations when the infinite homogenous plasma methods can be used as a good approximation. Detailed studies of this approximation are presented in Appendix B and can be summarized as following: for 3D Fourier components, the \( k \)-vector must satisfy all of the following conditions:

\[
\tilde{k} = k_\perp + \hat{e}_3 a_3; \quad a_3 \cdot \sqrt{\frac{k_\perp^2 + k_3^2}{\gamma^2}} \gg 1; \quad \gamma \approx \frac{m c^2}{E},
\]

which include scaling of the longitudinal component of the wavevector \( k_3 \) and the bunch length \( a_3 \) with the relativistic factor \( \gamma \). Since we considering a generic beam transport system, which can include beam’s acceleration, compression or decompression, focusing and bending of its trajectory, we shall also assume that changes in the beam and the accelerator parameters at the scale of the density modulation are negligible:

\[
||\nabla g|| \ll |\tilde{k}||g|
\]

where \( g \) is an any generic parameter of the accelerator, including but not limited to the beam’s energy, velocity, sizes, the accelerator EM fields, the curvature and the torsion of the reference beam trajectory.

Next step in evaluating the instability requires linearization of the symplectic map \( \xi = \textbf{M} : \chi \) using \( 6 \times 6 \) symplectic transport matrix [76,80-82]:

\[
\xi = \textbf{M}(s) \chi; \quad \chi = \textbf{M}^{-1}(s) \xi; \quad \textbf{M}(0) = I_{6 \times 6};
\]

which should be evaluated self-consistently, including macroscopic collective effects. It is convenient to identify four \( 3 \times 3 \) block-matrices in the transport matrix:

9 Typically, the combination of Vlasov and Maxwell equations is not directly solvable because it contains partial derivatives.

10 Nonlinearity of the map would result in nonlinear, position dependent transformation of the \( k \)-vector canceling advantages offered by Fourier transformation.
It is easy to show that this is no longer a problem for a beam with finite sizes and finite emittances. When used one of eq. (26) which provide explicit connections between the local and initial coordinates and momenta:

\[ \begin{bmatrix} q \\ p \end{bmatrix} = M(s) \begin{bmatrix} Q \\ P \end{bmatrix}; \quad \begin{bmatrix} Q \\ P \end{bmatrix} = M^{-1}(s) \begin{bmatrix} q \\ p \end{bmatrix}; \]

\[ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \quad M^{-1} = -SM^T S = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}; \]

It worth noticing that in this notation three degrees of motion are decoupled when all four 3x3 matrices, A, B, C and D are diagonal (see Appendix A for more details).

Matrix A plays especial role for this instability since is determinant represent the degree of the three-dimensional bunch compression:

\[ \frac{d^2 \rho(\bar{q})}{ds^2} + k_p^2 \tilde{\rho}(\bar{q}) = 0; \tilde{\rho}(\bar{q}) = \tilde{\rho}_o(\bar{q}) e^{i(k_p - \omega_0) t}; \omega_0 = c \beta k_p. \]  

(28)

where used one of eq. (27) to connect local beam densities (at azimuth s) with their initial values at s=0:

\[ P = -C^T q + A^T p \Rightarrow p = (A^T)^{-1} (P + C^T q); \]

\[ dp^3 \bigg|_{q=const} = \frac{1}{\det A} d \left( P + C^T q \right)^3 \Rightarrow \frac{1}{\det A} \int_{q_{min}}^{q_{max}} \int_{p_{min}}^{p_{max}} d \left( P + C^T q \right)^3 = \frac{1}{\det A}. \]

\[ A^T B P = q = const \Rightarrow Q = A^{-1} (q - BP); \]

\[ F \left( q, \left( A^T \right)^{-1} \left( P + C^T q \right) \right) = F_o \left( A^{-1} (q - BP), P \right). \]

One of important consequences of using the assumption of the infinite homogeneous plasma results in requirement of \( \det A > 0 \). Otherwise, because of the infinite size of the plasma, beam density would become infinitely large, e.g. unphysical. This is extremely simple to show for the 1D case:

\[ M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}; \quad M^{-1} = -SM^T S = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}, \]

when \( m_{11} \) plays the role of the \( \det A \) and the change in the line density can be easily expressed as

\[ \rho(q,s) = \int_{-\infty}^{\infty} F_o (-m_{21} q + m_{11} p) dp = \frac{1}{m_{11}} \int_{-\infty}^{\infty} F_o (p) dp = \frac{n_o}{m_{11}}. \]

It is easy to show that this is no longer a problem for a beam with finite sizes and finite emittances.

As shown in Appendices C, D and E, density perturbation will generate additional potentials of the EM field resulting in perturbation of the accelerator Hamiltonian (see equations (E5)):
\[ \ddot{h} = \frac{4\pi e^2}{c} \int \frac{\bar{\rho}_k e^{i\mathbf{q} \cdot \mathbf{r}}}{\gamma_o^2 \beta_o^2 k_\perp^2 + k_3^2} \; dt = -\frac{\partial \ddot{h}}{\partial \mathbf{q}} = -\frac{4\pi e^2}{c} \int \frac{i\mathbf{k} \cdot \bar{\rho}_k e^{i\mathbf{q} \cdot \mathbf{r}}}{\gamma_o^2 \beta_o^2 k_\perp^2 + k_3^2} \; dt \]  

(30)

We can easily connect the \( \bar{\rho}_k \) at location \( s \) with Fourier harmonic of \( \bar{f} \). Taking into account conservation of the phases-space volume \( dq^3 dp^3 = det \mathbf{M} \cdot dQ^3 dP^3 = dQ^3 dP^3 dQ^3 dP^3 = dq^3 dp^3 \) and conservation of the phase space density \( \bar{f}(X, s) \equiv \bar{f}(\xi(X, s), s) \) we get:

\[ \bar{\rho}_k \equiv \bar{\rho}(s, \bar{k}(s)) = \frac{1}{(2\pi)^3} \int f(Q, P, s) e^{-i\mathbf{q} \cdot \mathbf{r}(X)} dQ^3 dP = \frac{1}{(2\pi)^3} \int \bar{f} e^{-i\mathbf{k} \cdot \mathbf{q} + A^{-1} B \cdot \mathbf{P}} dQ^3 dP^3, \]

(31)

where we used \( \mathbf{q} = \mathbf{A} \cdot \mathbf{Q} + \mathbf{B} \cdot \mathbf{P} \) as equivalent of \( q = A Q + B P \) in eq. (27), a compact notation for convolution of two vectors and a matrix:

\[ \bar{x} \cdot \mathbf{A} \cdot \bar{y} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} x_i y_j \]

and explicit indication that the \( k \)-vector \( \bar{k}(s) \) is function of \( s \):\(^{11}\)

\[ \bar{k}(s) = \bar{k}_o \cdot A^{-1}(s); \bar{k}_o = \bar{k}(s = 0); \bar{k}(s) \cdot \mathbf{q} = \bar{k}_o \cdot \mathbf{Q} + \bar{k}_o \cdot A^{-1} \mathbf{B} \cdot \mathbf{P}. \]

(32)

It means that matrix \( \mathbf{A} \), the spatial components of the transport matrix, also defines evolution of the \( k \)-vector with initial value of \( \bar{k}_o \):

\[ k^T(s) = [k_1(s), k_2(s), k_3(s)]; k_o = A^T k(s) \Leftrightarrow k(s) = (A^T)^{-1} k_o. \]

(33)

We can assume, without loss of generality, that initial distribution is an arbitrary integrable function of momenta \(^{12}\):

\[ F_o = n_o F_o(P); \int_{-\infty}^{\infty} F_o(P) dP^3 = 1; n_o = \frac{j_o}{e c}, \]

(34)

where \( j_o \) is the initial beam current density. It is important to note that in contrast with velocity-dependent spatial density of the beam, \( n_i = j_o / e v_o \), the \( n_o = \beta_o n_r \) has a well-defined finite value.

Now we can rewrite Vlasov equation (22) using relations from eq. (29):

\[ \text{For compactness, in places where it cannot cause confusion, we omit explicit indication } s\text{-dependence, for example using } A^{-1} \mathbf{B} \text{ instead of } A(s)^{-1} \mathbf{B}(s) \text{ in this equation.} \]

\[ \text{For plasma to remain uniform the distribution must have form of } f(p + M q) \text{. Initial linear} \]

\[ \text{correlations between } p \text{ and } q \text{ can be incorporated into the transport matrix (26).} \]
\( P = -C^\top q + A^\top p; \ dP_i = A_{ij} dp_j - C_{ji} dq_j; \)
\[
\frac{\partial \tilde{f}}{\partial s} = -n_o \frac{\partial F_o}{\partial P_i} \frac{\partial P_i}{\partial P_j} \delta \left( \frac{dp_j}{ds} \right) - n_o \frac{\partial F_o}{\partial P_i} \frac{\partial P_i}{\partial q_j} \delta \left( \frac{dq_j}{ds} \right) = n_o \frac{\partial F_o}{\partial P_i} A_{ji} \frac{\partial \tilde{h}}{\partial q_j};
\]
and taking into account that
\[
\delta \left( \frac{dq_j}{ds} \right) = \frac{\partial \tilde{h}(\xi, s)}{\partial p_i} = 0.
\]
Using (30) we arrive to self-consistent Vlasov equations:
\[
\frac{\partial \tilde{f}}{\partial s} = n_o \frac{\partial F_o}{\partial P_i} A_{ji}(s) F_j(q, s); \quad \tilde{F}(q, s) = \frac{\partial \tilde{h}}{\partial q} = \frac{4\pi e^2}{c} \int \frac{i\vec{k} \cdot \vec{\rho}_e e^{ikq} dk^3}{\gamma_o^2 \beta_o^2 k_\perp^2 + k_3^2},
\]
which is suitable for the Fourier transform
\[
\tilde{f}_{\vec{k}}(P, s) = \frac{1}{2\pi} \int f(Q, P, s) e^{-i\vec{k} \cdot \vec{Q}} dQ^3;
\]
resulting in
\[
\frac{\partial \tilde{f}_{\vec{k}}(P, s)}{\partial s} = n_o \frac{\partial F_o}{\partial P_i} A_{ji}(t) \int dQ^3 e^{-i\vec{k} \cdot \vec{Q}} F_j(q, t).
\]
The later has to be evaluated at \( \vec{P} = \text{const} \) using established relations between \( k \)-vectors (33):
\[
\tilde{F}_{\vec{k}} = \int e^{-i\vec{k} \cdot \vec{Q}} F(q, s) dQ^3 \bigg|_{\vec{P} = \text{const}} = \frac{4\pi e^2}{c} \int \frac{i\vec{k} \cdot \vec{\rho}_e dk^3}{\gamma_o^2 \beta_o^2 k_\perp^2 + k_3^2} \left( \frac{1}{2\pi} \right)^3 \int e^{i\vec{k} \cdot \vec{Q}} dQ^3 = \frac{e^{i\vec{k} \cdot \vec{A}}}{\text{det} \vec{A}} \delta(\vec{k} - \vec{k}_o \vec{A}^{-1}),
\]
resulting in
\[
\frac{\partial \tilde{f}_{\vec{k}}(P, s)}{\partial s} = \frac{4\pi n_o e^2}{c} \frac{\tilde{\rho}(s, \vec{k}(s))}{\gamma_o(s)^2 \beta_o(s)^2 k_\perp(s)^2 + k_3(s)^2} \frac{e^{i\vec{k} \cdot \vec{A}^{-1}(s) \vec{B}(s)}}{\text{det} \vec{A}(s)} \left( \frac{i k o \partial F_o}{\partial P_i} \right),
\]
where we took into account that \( k_j(s) A_{ij}(s) \frac{\partial F_o}{\partial P_i} = k_o \frac{\partial F_o}{\partial P_i} \). This equation can be integrated:
\[
\tilde{f}_{\vec{k}}(P, s) = \tilde{f}_{\vec{k}}(P, 0) + \frac{4\pi n_o e^2}{c} \int \frac{e^{i\vec{k} \cdot \vec{A}^{-1}(s) \vec{B}(s)}}{\text{det} \vec{A}(s)} \frac{\tilde{\rho}(s)(\vec{k}) d\vec{k}}{\gamma_o(s)^2 \beta_o(s)^2 k_\perp(s)^2 + k_3(s)^2}.
\]
Rewriting (31) as
\[
\tilde{\rho}(s, \vec{k}(s)) = \frac{1}{(2\pi)^3} \int \tilde{f}_{\vec{k}}(P, s) \cdot e^{-i\vec{k} \cdot \vec{A}(s) \vec{B}(s)} dP^3,
\]
turns eq. (38) into a directly solvable integral equation:

$$\hat{\rho}(s, \vec{k}(s)) = \hat{\rho}_{k_0}(s) + \frac{4\pi i e^2 n_c}{c} \int_o^s \hat{\rho}(\xi, \vec{k}(\xi)) d\xi \int \frac{e^{i\vec{k} \cdot (\vec{u}(\xi) - \vec{u}(s))}}{\det A(\vec{\xi})} \frac{\gamma_o(\vec{\xi})^2 \beta_o(\vec{\xi})^2 k_+^2(\vec{\xi}) + k_-^2(\vec{\xi})}{\partial F_0/\partial P_i} dP^3;$$

$$U(s) = A^{-1}(s)B(s); \hat{\rho}_{k_0}(s) = \int e^{-i\vec{k} \cdot (\vec{u}(s) - \vec{u})/\beta_o} f_{\vec{k}}(P, 0) dP^3.$$  

(40)

While this equation already can be used for evaluation of the instability, it can be further simplified by eliminating convolution $\sum_{i=1}^3 \frac{\partial F_0}{\partial P_i} k_i'$. Using integration by parts

$$\int \frac{\partial F_0}{\partial P_i} \phi dP_i = F_0 \phi|_{P_i=\pm\infty} - \int F_0 \frac{\partial \phi}{\partial P_i} dP_i$$

and $F_0(P_i = \pm\infty) = 0$ we get:

$$\sum_{i=1}^3 k_{i1} \frac{\partial}{\partial P_i} e^{i\vec{k} \cdot (\vec{u}(s) - \vec{u})/\beta_o} = -i k_0^2 (u(s) - u(\xi));$$

$$k_o = |\vec{k}^o|; \vec{k}^o = k_o \vec{\nu}_o; u(s) = \vec{\nu}_o \cdot \vec{U}(\xi) \cdot \vec{\nu}_o \equiv \sum_{i,j} U_{ij}(\xi) \cdot \nu_i \nu_j,$$

where we introduced unit vector $\vec{\nu}_o$ in the direction of the initial $k$-vector. If it is convenient to extent is definition of this dimensionless vector as

$$\vec{k}(s) = k_o \vec{\nu}(s); \vec{v}(s) = \vec{k} \cdot \vec{A}^{-1}(s).$$

(42)

We show in eq. (A12) of Appendix A that $AB^T = BA^T$, which also means that $U = A^{-1}B$ is also a symmetric matrix.

Combining eqs. (40) and (41) This brings us to the final form of integral equation for this instability:

$$\hat{\rho}(s, \vec{k}(s)) = \int_o^s \hat{\rho}(\xi, \vec{k}(\xi)) \cdot K(\xi)(u(s) - u(\xi)) L_o(s, \vec{k}(s)) d\xi + \hat{\rho}_{k_0}(s);$$

$$K(\xi) = \frac{4\pi n e^2}{c \det A(\vec{\xi}) \nu(\vec{\xi})} L_o(k_o, s, \vec{k}(s)) = \int e^{i(\vec{k} \cdot \vec{u}(s) - \vec{k} \cdot \vec{u})/\beta_o} F_0(P) dP^3;$$

$$\vec{k}(\xi) = \vec{k}_o \cdot \vec{U}(\xi) \equiv k_o \vec{\nu}_o \cdot \vec{U}(\xi); \nu(s) = \gamma_o(s) \beta_o(s) \nu_+ s^2 + \nu_3(s)^2;$$

(43)

which can be solved numerically for any accelerator.

It is important to note that in the kernel of the integral equation (43) there in only one term, the Landau damping [87], $L_o$, depend on the absolute value of the $k$-vector. The rest of the terms, $K$ and $u$, are defined by the geometry (e.g. direction of the initial $k$-vector), the components of accelerator transport matrix in form of matrix $U = A^{-1}B$ and $s$-dependent denominator.
The most non-trivial construction is actually \( \mathbf{v}(\zeta) \), which is the result of the asymmetry introduced by Lorentz transformation of the 4-potential:

\[
\mathbf{v}(s) = \mathbf{\tilde{v}} \cdot \mathbf{\tilde{G}}(s) \cdot \mathbf{\tilde{v}}; \mathbf{\tilde{G}} = \mathbf{A}^{-1} \begin{bmatrix} 
\gamma \beta^2 & 0 & 0 \\
0 & \gamma \beta^2 & 0 \\
0 & 0 & 1 
\end{bmatrix} (\mathbf{A}^{-1})^T. \tag{44}
\]

Furthermore, the convolution \( u(s) = \mathbf{\tilde{v}} \cdot \mathbf{\tilde{U}}(\zeta) \cdot \mathbf{\tilde{v}} \) (41) has important non-trivial properties that it is nonnegative monotonic function with positive derivative (see eq. (A16) in Appendix A):

\[
u(s) \geq 0; \quad \nu'(s) > 0. \tag{45}\]

Generally speaking, for a beam with arbitrary momentum spread the equation (43) cannot be either evaluated analytically or reduced in complexity. But physical nature of various terms can be identified by considering specific cases. For example, the integral over the momenta is known as Landau damping [87] and can be easily evaluated for Gaussian distribution:

\[
F_o(P) = \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left( -\frac{P_i^2}{2\sigma_i^2} \right) \tag{46}
\]

generating exponential term

\[
L_d = \int e^{i(i(s) - i(s))} F_o(P) dP^3 = \prod_{i=1}^{3} \exp \left( -\frac{k_i^2 \sigma_i^2 (\xi - \xi(s))^2}{2} \right); \tag{47}
\]

corresponding to the decay of the modulation during the interval \( (\xi, s) \).

Equation (43) is the most general equation that describes evolution of high-frequency modulation in beams driven by space charge effects. It can be also used to describe one dimensional or 2D instabilities. For example, it is easy to show that conventional longitudinal microwave instability [53-71] can be also described by this equation under simplified assumptions. Specifically, conventional theory of longitudinal microwave instability assumes that in straight sections the longitudinal motion is frozen and energy modulation resulted from accumulated space-charge forces is transferred into density by \( R_{56} \) of a magnetic system. Furthermore, space charge is frequently neglected in the bending magnetic system. Hence, eq. (43) is an universal equation for description of instabilities driven by the space charge.

### IV. Specific cases

In this section, we review some specific cases which are of interest for this paper. In some cases, we can simplify eq. (43) or reduce it to second-order ordinary differential equation (ODE).
Let’s consider cases when the Landau damping term allows separation of variables $s$ and $\zeta$\textsuperscript{13}:

$$L_d(s, \zeta) = \Lambda(\zeta) \Lambda^{-1}(s); \Lambda(s) = e^{-\phi(s)};$$

and the integral equation (33) becomes:

$$\tilde{q}(s) = -\int_o^s \tilde{q}(\zeta) K(\zeta)(u(s) - u(\zeta)) d\zeta + \tilde{q}_0(s); \tag{49}$$

for scaled density modulation $\tilde{q}(s) = e^{\phi(s)} \tilde{\rho}(s, \tilde{k}(s))$. Combination of first and second derivatives of eq. (49) transfers it into the second order ODE:

$$\tilde{q}'' - \alpha' \cdot \tilde{q}' + Ku' \cdot \tilde{q} = \tilde{q}_0'' - \alpha' \cdot \tilde{q}_0';$$

$$\tilde{q}_0(s) = e^{\phi(s)} \tilde{\rho}_0(s); \alpha = \ln \frac{u'}{u_0'}; u'_0 = u'(0),$$

where we used fact that $u'>0$ and standard notation for derivatives $f' = \frac{df}{ds}; f'' = \frac{d^2f}{ds^2}$. Finally, this equation can be also reduced to inhomogeneous Hill’s equation:

$$\tilde{q}'' + \tilde{K}(s) \tilde{q} = \zeta(s); \tilde{K}(s) = K(s)u'(s) - \frac{\alpha'(s)^2}{4} + \frac{\alpha''(s)}{2};$$

$$\tilde{q} = e^{-\frac{\alpha(s)}{2}} \tilde{q} \equiv \sqrt{\frac{u'}{u'(s)}} e^{\phi(s)} \tilde{\rho}(s, \tilde{k}s); \zeta(s) = e^{-\frac{\alpha(s)}{2}} \left( \tilde{q}_0''(s) - \tilde{q}_0'(s) \alpha'(s) \right).$$

It is well known that solution of homogeneous Hill’s equation is represented by symplectic matrix (see eqs. (3-5)):

$$\begin{bmatrix} \dot{\tilde{q}}(s) \\ \dot{\tilde{q}}'(s) \end{bmatrix} = \mathbf{R}(s) \begin{bmatrix} \tilde{q}(0) \\ \tilde{q}'(0) \end{bmatrix}; \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}; \mathbf{R}' = \begin{bmatrix} 0 & 1 \\ -\tilde{K}(s) & 0 \end{bmatrix} \mathbf{R}; \det \mathbf{R} = 1;$$

which also defines general solutions of inhomogeneous equation:

$$\dot{q}(s) = r_{11}(s) \dot{q}(0) + r_{12}(s) \dot{q}'(0) + \int_0^s \left( r_{11}(\zeta) r_{12}(s) - r_{11}(s) r_{12}(\zeta) \right) \zeta(\zeta) d\zeta. \tag{53}$$

Hence, solution of the homogenous Hill’s equation $\dot{q}'' + \tilde{K}(s) \dot{q} = 0$ is the key for investigation of this instability, when the separation (48) is possible. Cold beam with momenta distribution

$$F_0(P) = \delta(P_1) \delta(P_2) \delta(P_3)$$

\textsuperscript{13} Unfortunately, as can be seen from eq. (47), such separation is impossible for Gaussian momenta distribution.
definitely satisfy this requirement with $\phi = 1$. More general and interesting is the case of beam with $\kappa$-1 momentum distribution in all directions:

$$F_0(P) = F_{\kappa-1}(P) = \frac{1}{\pi^3} \prod_{i=1}^{3} \frac{\sigma_i}{\sigma_i^2 + P_i^2},$$

allowing to integrate over the momenta:

$$L_\sigma(\zeta, s) = \int e^{\frac{i\kappa_i(s-\kappa(\zeta))}{\sigma}} F_0(P) dP^3 = e^{-\sum_{i=1}^{3} \sigma_i |\kappa_i(s)-\kappa(\zeta)|}.$$  \hspace{1cm} (54)

If condition $|\kappa_i(s)| \geq |\kappa_i(\zeta)|$ for all three components $i = 1, 2, 3$, we can used $|\kappa_i(s) - \kappa_i(\zeta)|$ and separated variables:

$$L_\sigma(s, \zeta) = e^{\phi} e^{-\phi(s)}; \phi(\zeta) = \sum_{i=1}^{3} \sigma_i |\kappa_i(\zeta)|;$$  \hspace{1cm} (55)

resulting in a second order ODE (50) for $\hat{q}_\sigma(s) = \hat{\rho}(s, \kappa(s)) \cdot \exp(\phi(s))$, and in Hill’s equation (51) for $\hat{q}(s) = \hat{\rho}(s, \kappa(s)) \sqrt{u'/u'(s)} \cdot \exp(\phi(s))$.

This is a good place to discuss driving term, $\zeta(s)$, in the right-hand side (r.h.s.) of Hills equation

$$\zeta(s) = e^{\frac{q}{\sigma}} (\hat{q}_o'' - \hat{q}_o' \alpha'); \hat{q}_o(s) = e^{\phi(s)} \hat{\rho}_o(s) = e^{\phi(s)} \int e^{-i\kappa_i(s)} \hat{f}_{\kappa_i}(P, 0).$$  \hspace{1cm} (57)

Generally speaking, for an arbitrary initial perturbation $\hat{f}(P, Q)$, both $\hat{q}''$ and $\hat{q}_o''$ are not equal zero and Hill’s equation remains inhomogeneous. One case is an exception: when the initial perturbation is fully defined by the density perturbation:

$$\hat{f}(P, Q) = \hat{\rho}_o(Q) \cdot F_{\kappa-1}(P) \to \int e^{-i\kappa_i(s)} \hat{\rho}_o(s) \cdot \hat{f}_{\kappa_i}(P, 0) = \hat{\rho}_o e^{-\phi(s)};$$  \hspace{1cm} (58)

all derivatives of $\hat{q}_o$ are equal zero, and the Hill’s equation becomes homogenous:

$$\hat{f}(P, Q) = \hat{\rho}_o(Q) \cdot F_{\kappa-1}(P) \to \hat{q}'' + \hat{K}(s)\hat{q} = 0.$$  \hspace{1cm} (59)

While the conditions $|\kappa_i(s)| \geq |\kappa_i(\zeta)|$ for all $s \geq \zeta$ are frequently satisfied, they also can be violated in the case of an arbitrary coupling. In fact, it is possible to construct matrix $\mathbf{U}$ that one component of vector $\mathbf{\tilde{q}} = \{\kappa \mathbf{U}\}$ turns from non-zero value at $\zeta$ to zero at $s > \zeta$. Emittance exchange lattices can serve as an example. If even one of $|\kappa_i(s)| \geq |\kappa_i(\zeta)|$ conditions is violated, the separation becomes impossible.

As we shown in Appendix A (see eq.(A29)) that in the case of uncoupled motion the matrix $\mathbf{U}$ is diagonal with monotonically growing diagonal terms.
\[ U(s) = \left[ \delta_y \right] \mu_i(s); \mu_i(0) = 0; \mu_i(s) > \mu_i(\zeta) \quad \forall \zeta < s \; \delta_y \mu_i \]

which means that
\[
|\kappa_i(s)| = |k_o \mu_i(s)| \quad (60)
\]

are also monotonic functions satisfying conditions \( |\kappa_i(s)| \geq |\kappa_i(\zeta)|; s \geq \zeta \).

Hence, we proved that in accelerator with decoupled motion one can use second order ODE (50) or Hill’s equation (51) for beam with \( \kappa^{-1} \) momentum (energy) distributions. This also include linear accelerators using solenoids – the equations of motion are decoupled by using torsion \( \kappa_o(s) = -\frac{eB}{2p_o c} \) (see eq. (6) [86]).

For the beam with the constant density and constant energy propagating in a drift space we have
\[
A = D = I; \quad C = 0; \quad U(s) = B(s) = \frac{1}{\gamma_o \beta_o mc} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s / \gamma_o^2 \beta_o^2 \end{bmatrix};
\]
\[
\tilde{k} = \text{const}; \quad \tilde{p}_{0\tilde{k}} = \text{const}; \quad u = \frac{s}{\gamma_o^3 \beta_o^3 mc} \left( \gamma_o^2 \beta_o^2 \tilde{k}^2 + k_3^2 \right); \quad u'K = \frac{4\pi n_o e^2}{\gamma_o^3 \beta_o^3 mc} = \text{const}; \quad u'' = 0.
\]

For cold plasma oscillations it results in well known \( \tilde{k} \)-independent equation:
\[
\frac{d^2 \tilde{\rho}}{ds^2} + k_p^2 \tilde{\rho} = 0; \quad k_p^2 = \frac{4\pi n_o e^2}{\gamma_o^3 \beta_o^3 mc},
\]

which, after applying the inverse Fourier transformation, becomes the carbon copy of eq. (2) but in the laboratory frame:
\[
\frac{d^2 \rho(\bar{q})}{ds^2} + k_p^2 \rho(\bar{q}) = 0; \quad \rho(\bar{q}) = \tilde{p}_o(\bar{q}) e^{i(k_p - \omega_o t)} \quad \omega_o = c\beta_o k_p.
\]

For beams with \( \kappa^{-1} \) momentum distribution (54) propagating in a drift space with the constant density and constant energy, \( \tilde{k} \)-dependence occurs via Landau damping term and \( \bar{q}'' \) as the driving term:
\[
\bar{q}'' + k_p^2(s) \bar{q} = \bar{q}_o''; \quad \bar{q} = e^{i(s)} \tilde{p}_o q_o = e^{i(s)} \tilde{p}_o q_o;
\]
\[
\phi(s) = \frac{s}{\gamma_o \beta_o mc} \left( \sigma_1 |k_1| + \sigma_2 |k_2| + (\gamma_o \beta_o)^{-2} \sigma_3 |k_3| \right)
\]
\[
\tilde{p}_o q_o = \int \exp \left( i \frac{s}{\gamma_o \beta_o mc} \left( k_1 \sigma_1 P_1 + \sigma_2 k_2 P_2 + (\gamma_o \beta_o)^{-2} \sigma_3 k_3 P_3 \right) \right) f_{\tilde{k}o}(P, 0) dP^3.
\]
As one important consequence of eq. (64) is that for beam propagating in straight section Landau damping decrement for transverse modulation is boosted by factor \( \gamma_o^2 \beta_o^2 \), which is typically \( \gg 1 \). In other words, for \( k_{\perp 2} |\sigma| \gg k_0 |\sigma| \), the Landau damping term is significantly larger than for \( k_{\perp 2} = 0 \). This is one of the reasons why longitudinal PCI is of special interest in the paper.

Let’s consider 1D longitudinal instability in a beam propagating along straight trajectory, e.g. when the longitudinal and transverse motion are decoupled:

\[
A(s) = \begin{bmatrix} A_\perp(s) & 0 \\ 0 & a_\parallel(s) \end{bmatrix}; \quad B(s) = \begin{bmatrix} B_\perp(s) & 0 \\ 0 & b_\parallel(s) \end{bmatrix}; \quad \tilde{\kappa}(s) = \tilde{e}_3 k(s) = \tilde{e}_3 \frac{k_p}{a_\parallel(s)}. \tag{65}
\]

Evolution of this instability can be described either by integral equation:

\[
\tilde{\rho}(s, k(s)) = -\frac{4\pi n_c e^2}{c} \int \tilde{\rho}(\xi, k(\xi)) K_\parallel(\xi)(u(s) - u(\xi)) d\xi e^{-ik_p(u(s) - u(\xi))} F_\parallel(P) dP + \tilde{\rho}_{ok}(s); \tag{66}
\]

\[
K_\parallel(\xi) = \frac{4\pi n_c e^2}{c} \frac{a_\parallel(\xi)}{\det A_\perp(\xi)}; \quad u(\xi) = \frac{b_\parallel(\xi)}{a_\parallel(\xi)}; \quad \tilde{\rho}_{ok}(s) = \int e^{-ik_p(0-s)} \tilde{f}_{k_\parallel}(P, 0) dP.
\]

or for \( k-1 \) longitudinal momentum distribution by differential equation:

\[
\tilde{q}'' - \xi(s) \cdot \tilde{q}' + k_p^2(s) \tilde{q} = \tilde{q}'' - \xi(s) \cdot \tilde{q}'; \tag{67}
\]

\[
k_p^2(s) = \frac{4\pi}{(\gamma_o \beta_o)^3} \cdot \frac{n_c r_c}{a_\parallel \det A_\perp} \cdot \xi(s) = \frac{d}{ds} \left( \ln a_\perp^2 (\gamma_o \beta_o)^3 \right); \quad \tilde{q}'' = \tilde{\rho}(s, a_\parallel(s)) e \frac{k_h s \sigma_3}{a_\parallel(s)} ,
\]

where \( k_p(s) \) is \( s \)-depended frequency (is \( s \) domain), \( k / a_\parallel(s) \) is scaled wavenumber of the perturbation, and \( \xi(s) \) represents an addition term which, depending on tits sign, either damps or amplifies modulation. The corresponding Hill’s equation has the same driving term but slightly different \( s \)-depended frequency:

\[
\tilde{q}'' + k_p^2 \tilde{q} = a_\parallel(\gamma_o \beta_o)^3 \left( \tilde{q}'' - \tilde{q}' \frac{u''}{u'} \right); \quad k_p^2 = k_p^2 - \frac{\xi'^2(s)}{4} + \frac{\xi''(s)}{2}. \tag{68}
\]

For beam with the constant energy and no-compression (\( a_\parallel = 1 \)) equations (67) and (68) become identical and describe PCI driven only by transverse focusing:

\[
a_\parallel = 1; \quad \gamma_o \beta_o = \text{const}; \quad b_\parallel = \frac{S}{(\gamma_o \beta_o)^3 mc} \quad \tilde{q}'' + k_p^2(s) \tilde{q} = \tilde{q}'' ; \tag{69}
\]

\[
k_p^2(s) = \frac{4\pi}{(\gamma_o \beta_o)^3} \cdot \frac{n_c r_c}{\det A_\perp(s)}; \quad \rho_k(s) = \tilde{q}(s) e \frac{k_h s \sigma_3}{(\gamma_o \beta_o)^3 \gamma_o \beta_o}.
\]
V. Conclusions.

In this paper we developed general self-consistent 3D theory of high-frequency microbunching instability driven by space charge forces. We derived the directly solvable integral equation (43) fully describing any such instability within the well-defined range of assumption summarized by eqs. (23-24). We also derived conditions when this integral equation can be reduced to a second order ODE (51).

This theory is applicable to all accelerators and it accurately describes both the newly discovered PCI and conventional MBI within the range of assumption summarized in eq. (23). Furthermore, this theory goes beyond traditional model used to describe conventional MBI as alternating pairs “space-charge kick” – “R_{56} drift”. Our solution is self-consistent including continuous evolution in all elements of accelerators. As can be seen from eq. (32), bending of the particle trajectory couples longitudinal and transverse modulations resulting in non-trivial evolution in accelerator arcs or chicanes.

The authors would like to thank all of their colleagues from BNL who contributed to the CeC project, with special acknowledgements going to the CeC group and Dr. Thomas Roser for encouragement and unrelenting support of this research. The first author would also like to thank Prof. Pietro Musumeci (UCLA), who mentioned during our discussion that the modulation of the transverse beam size can violate energy conservation in longitudinal plasma oscillations. This notion sparked the initial impulse for us to investigate if the modulation of the transverse beam size could cause a broad-band longitudinal instability.

This research was supported by NSF grant PHY-1415252, by DOE NP office grant DE- FOA-0000632, and by Brookhaven Science Associates, LLC under Contract No. DE-SC0012704 with the U.S. Department of Energy.
Appendix A – System Hamiltonian and equations of motion

Traditionally in accelerator physics literature the vector $X$ in phase space is combined from canonical pairs of coordinates and momenta $X^T = [...(q_i,P_i)...] = [...(x_{2i-1},x_{2i})...]$. In this paper, following [82], we use equivalent but different structure of the $X$-vector, which clearly separate coordinates and momenta and simple form of the matrix of symplectic generator, $S$:

$$X^T = [x_1,...,x_{2n}] = [Q^T,P^T]; Q^T = [q_1,...,q_n]; P^T = [p_1,...,p_n];$$

$$\left\{ \begin{align*}
\frac{dq_i}{dt} &= \frac{\partial H}{\partial P_i} \\
\frac{dP_i}{dt} &= -\frac{\partial H}{\partial q_i}
\end{align*} \right\} \iff \frac{dX}{dt} = S \frac{dH}{dt} \iff \frac{dx_i}{dt} = S_{ij} \frac{dH}{dx_j};$$

$$S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}; \quad I_{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Use of these notations is especially convenient for linear maps in the form of $2nx2n$ symplectic transport matrices:

$$X(t_2) = M(t_2|t_1) X(t_1) \equiv M(t_2|t_1) \begin{bmatrix} Q(t_1) \\ P(t_1) \end{bmatrix};$$

$$M^T S M = S M S^T = S; \quad M^{-1} = -S M^T S;$$

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \quad M^{-1} = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix};$$

$$Q(t_2) = A Q(t_1) + B P(t_1); P(t_2) = C Q(t_1) + D P(t_1);$$

$$Q(t_i) = D^T Q(t_2) - B^T P(t_2); P(t_i) = -C^T Q(t_2) + A^T P(t_2);$$

providing explicit connection between coordinates and momenta with their initial values and vice versa. It also provides important properties of the block matrices which can be very useful for the evaluation of complex expression. Specifically, symplecticity of transport matrix requires that four $n xn$ matrices $AB^T, DC^T, A^T C, D^T B$ will be symmetric

$$\left( AB^T \right)^T = AB^T; \left( DC^T \right)^T = DC^T, \left( A^T C \right)^T = A^T C, \left( D^T B \right)^T = D^T B$$

(A3)

and that

$$AD^T - BC^T = I; \quad (A^T D - C^T B) = I.$$  

(A4)

In the case of uncoupled motion, all four matrices $A, B, C, D$ become diagonal making conditions (A3) trivial and turning (A4) into simple conditions for diagonal components:
\[ A_i D_i - B_i C_i = 1; \ i = 1, \ldots, n \]  
(A5)
equivalent to unity of determinants for individual 2x2 matrices in notations (A2):

\[
M_i = \begin{bmatrix}
  A_i & B_i \\
  C_i & D_i
\end{bmatrix}; \quad \text{det} \ M_i = 1. \quad (A6)
\]

Before ending this Appendix, we would like to point out that one can use time as independent variable with traditional set of Canonical pairs:

\[
\{ x, p_x \}, \{ y, p_y \}, \{ z, p_z \} X^T = [ \tilde{r}, \tilde{P} ]. \quad (A7)
\]

with traditional Hamiltonian [79]. The motion can be also expanded about the reference trajectory:

\[
\tilde{P} = \dot{x} p + \dot{y} p + \dot{z} (p_0 + p); \quad \tilde{q} = \ddot{r} - \dot{z} ; \quad \{ x, p_x \} = p_x + e A_x (\tilde{r}, t); \quad \{ y, p_y \} = p_y + e A_y (\tilde{r}, t); \\
\{ \zeta = z - z_0(t), p_z = p + e A_z (\tilde{r}, t) - p_0(t) \}; \quad \{ p_0(t) + e A_z (\tilde{r}_0(t), t) \}; \\
q \equiv \{ q_1, q_2, q_3 \} = \{ x, y, \zeta \}; \quad p \equiv \{ p_1, p_2, p_3 \} = \{ \pi, \pi, \pi \}; \\
H = H_o(q, p, t) = c \sqrt{1 + \left( \tilde{P}^2 - \frac{e}{c} A \right)^2} + e \varphi - \dot{z} p_z - \zeta p_0(t). \quad (A8)
\]

where \( \tilde{r}_0(t) = \dot{z} \cdot z_0(t) \) is the position of the reference particle (usually the center of the bunch) moving with designed momentum \( p_0(t) \) along z-axis, and \( \tilde{A}(\tilde{r}, t), q(\tilde{r}, t) \) are the vector and the scalar potential of the EM field. The reference particle has the design energy and velocity along the z-axis:

\[
E_o(t) \equiv \gamma_o(t) mc^2 = c \sqrt{p_0^2(t) + m^2 c^2}; \quad \beta_0(t) \equiv \frac{v_o(t)}{c} = \sqrt{1 - \gamma_o^{-2}(t)}. \quad (A9)
\]

Since both systems are Hamiltonian and pairs have the same dimensionally, the transformation from \( t \)-based to \( s \)-based description represents Canonical transformation [79] and, therefore, both direct and inverse transformation are symplectic:

\[
\tilde{X} = X(X, s) \iff X = X(\tilde{X}, t); \\
\tilde{J}(X, s) = \frac{DX}{DX} \equiv \left[ \frac{\partial \tilde{x}}{\partial x} \right]; \quad \tilde{J}^T S \tilde{J} = S; \quad J(\tilde{X}, t) = \frac{DX}{DX} \equiv \left[ \frac{\partial x}{\partial \tilde{x}} \right]; \quad J^T S J = S; \quad (A10)
\]

\[
S = \begin{bmatrix}
  \sigma & 0 & 0 \\
  0 & \sigma & 0 \\
  0 & 0 & \sigma
\end{bmatrix}; \quad \sigma = \begin{bmatrix}
  0 & 1 \\
  -1 & 0
\end{bmatrix}; \quad J(\tilde{X}(X, s), t(s)) = J(X, s)^{-1};
\]
In the case of linear matrix transformations used for infinitesimally small deviations from the reference there is direct connection between two transport matrices:

\[
\delta X(t) = T(0|t)\delta X(0); \delta \tilde{X}(s) = M(0|s)\delta \tilde{X}(0);
\]

\[
T(0|t, (s)) = J(s)^{-1} M(0|s) J(0);
\]

which makes these descriptions interchangeable.

Matrix \( A^{-1}B \) plays critical role in the instability integral equation (43). It’s properties can be studied for a Hamiltonian system describing a generic linear system:

\[
H = \frac{1}{2} \xi^T H(s) \xi; \quad H^T = H = \begin{bmatrix} H_q & H_m^T \\ H_m & H_p \end{bmatrix}; \quad H_{q,p}^T = H_{q,p}.
\]

Using equations of motion:

\[
M' = SH \cdot M; \quad A' = H_m A + H_p C; \quad B' = H_m B + H_p D.
\]

Taking into account that \((A^{-1})' = -A^{-1}A'A^{-1}\), we get

\[
(A^{-1}B)' = A^{-1}B' - A^{-1}A'A^{-1}B = A^{-1}H_p \left(D - CA^{-1}B\right),
\]

which can be turned into

\[
(A^{-1}B)' = A^{-1}H_p \left(A^T\right)^{-1}
\]

\[
D - CA^{-1}B = \left(DA^T - CB^T\right) \left(A^{-1}\right)^T = \left(A^{-1}\right)^T;
\]

\[
(A^{-1}B)' = A^{-1}H_p \left(A^T\right)^{-1}
\]

using symplecticity conditions \((A4) \ A^{-1}B = B^T \left(A^T\right)^{-1}\) and \(DA^T - CB^T = I\) to show that

\[
D - CA^{-1}B = D - CB^T \left(A^T\right)^{-1} = \left(DA^T - CB^T\right) \left(A^T\right)^{-1} = \left(A^T\right)^{-1}.
\]

It is possible to show for an arbitrary accelerator [76,80-82] that \( H_p \) is a diagonal with positive diagonal terms:

\[
H_p = \frac{1}{\gamma \beta \mu \nu c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(\frac{\gamma \beta \mu \nu c}{c}\right) \end{bmatrix}
\]

This allow us to prove that for an arbitrary accelerator with invertible matrix \( A \), the convolution \( u(s) \) in eq. (43) is nonnegative monotonically growing function with \( u'(s) > 0 \).
\[ u(s) = v_o^T (A(s)^{-1} B(s)) v_o \equiv \bar{v}_o^T (A^{-1} B) \bar{v}_o ; \quad B(0) = 0 \Rightarrow u(0) = 0 ; \quad v(s) = A^T (s)^{-1} v_o ; \]
\[ u'(s) = v_o^T (A^{-1} B)' v_o = v^T(s) H_p(s) v(s) = \sum_{i=1}^{3} H_{i}(s) v_i^2(s) > 0. \quad (A16) \]
\[ u(s) = \int_0^s \left( \sum_{i=1}^{3} H_{i}(s) v_i^2(s) \right) d\zeta \geq 0. \]

In other words, we proved that convolution of any constant vector with matrix \( A^{-1} B \) is nonnegative monotonically growing function.

In the chase of uncoupled motion, all 2x2 matrices are diagonal and

\[
\frac{d}{ds} (A^{-1} B)' = \begin{bmatrix} \alpha_1(s) & 0 & 0 \\ 0 & \alpha_2(s) & 0 \\ 0 & 0 & \alpha_3(s) \end{bmatrix} = \frac{1}{\gamma_s \beta_o mc} \begin{bmatrix} a_{11}^{-2} & 0 & 0 \\ 0 & a_{22}^{-2} & 0 \\ 0 & 0 & (a_{33} \gamma_o \beta_o)^{-2} \end{bmatrix} ; \alpha_i(s) > 0; \\
\mu_i(s) = \int_0^s (A^{-1} B)' d\zeta \geq 0; \quad A^{-1} B = \int_0^s (A^{-1} B) d\zeta = \begin{bmatrix} \mu_1(s) & 0 & 0 \\ 0 & \mu_2(s) & 0 \\ 0 & 0 & \mu_3(s) \end{bmatrix} = \delta \mu_i. \quad (A19) 
\]

i.e. diagonal term of matrix \( A^{-1} B \) are monotonously growing positive functions:
\[ \forall s_1 > s_2; \mu_i(s_1) > \mu_i(s_2). \quad (A20) \]

**Appendix B – Conditions for applicability of short period perturbations or, in other words, assumption of homogeneous infinite plasma.**

Fourier or Laplace transformations are frequently used to solve the linearized Vlasov equation. The main problem from inhomogeneous distribution (or finite size of the beam) that it results in coupling Fourier harmonics of the perturbation with those the background, e.g., applying Fourier transformation to eq. (17)

\[
\int dQ^3 e^{-iQ \cdot \tilde{\kappa}} \left( \frac{\partial \tilde{f}_K}{\partial t} + \frac{\partial F_{\alpha}}{\partial \vec{Q}} \frac{\partial \tilde{h}}{\partial \vec{P}} - \frac{\partial F_{\alpha}}{\partial \vec{P}} \frac{\partial \tilde{h}}{\partial \vec{Q}} \right) = \frac{\partial \tilde{f}_K}{\partial t} + \int d\kappa^3 \left\{ F_{\alpha} \left[ \tilde{\kappa} \cdot \frac{\partial \tilde{h}_{-\kappa}}{\partial \vec{P}} \right] - \tilde{h}_{-\kappa} \left[ \tilde{\kappa} - \tilde{\kappa} \right] \cdot \frac{\partial F_{\alpha}}{\partial \vec{P}} \right\}, 
\]

does not result in separation of the Fourier harmonics. In this sense, this equation is as complicated the original Vlasov equation.

Let’s consider a beam with typical scales of the inhomogeneity, \( a_{k,x,z} \), which are not necessarily of the same order of magnitude:
The conditions for separation of the Fourier harmonics in (B.1) are easiest to derive in the comoving frame of reference, where \( a_z \) is increased by relativistic factor \( \gamma = \frac{E_0}{mc^2} \). For simplicity we will consider that motion of particles in the comoving frame is non-relativistic and we can neglect effects of magnetic field, e.g., assume \( \vec{B} = 0 \). In this case the perturbed Hamiltonian is a simple linear function of the electric field potential. In this case Maxwell equations are reduced to two simple equations for electric field:

\[
\text{div} \vec{E} = 4 \pi \rho; \quad \text{curl} \vec{E} = 0. \tag{B1}
\]

First, let’s establish relations between parameters in laboratory and comoving frame. It is well known that that length of the bunch is scaled-up in the comoving frame by the relativistic factor \( \gamma \)

\[
a_{z, cm} = \gamma a_z \tag{B2}
\]

and the \( \vec{k} = (\omega / c, \vec{k}) \) transforms a 4-vector [87] proving well-known relations using the Lorentz transformation:

\[
\vec{k}_{cm} \equiv \vec{\kappa} = \hat{z} \kappa_z + \hat{\kappa}_\perp; \quad \omega_{cm} = 0; \quad \vec{k}_{lab} = \hat{z} k_z + \hat{\kappa}_\perp; \\
\vec{\kappa}_\perp = \vec{k}_\perp; \quad k_z = \gamma_o \left( \kappa_z + \beta_o \frac{\omega_{cm}}{c} \right) = \gamma_o \kappa_z; \quad \omega_{lab} = \gamma_o \left( \omega_{cm} + v_o \kappa_z \right) = v_o \kappa_z; \tag{B3}
\]

where we use indexed “cm” and “lab” for the comoving and laboratory frames, correspondingly.

Further in this Appendix we will use the comoving frame and will drop the index “c”. The natural conditions for neglecting the beam’s edges, the transitions and the reflection effects is that there must be a significant number of oscillations in each direction at the typical scales of the inhomogeneity, e.g.:

\[
\kappa_{x,y} a_{x,y} >> 2\pi; \quad \gamma \kappa_z a_z = k_z a_z >> 2\pi. \tag{B4}
\]

Second, and much more convoluted, condition is that Fourier harmonic of the induced electric field (and therofor of the perturbation in the Hamiltonian) are linear functions of the harmonic of the charge density perturbation,

\[
\rho_{\kappa} = e \int_{-\infty}^{\infty} d\vec{r} e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \bar{f}_{\kappa}(\vec{r},\vec{\nu},t) d\vec{\nu}. \tag{B5}
\]

where \( \bar{f}_{\kappa} \) is perturbation of the distribution function in the comoving frame. In the infinite charged plasma periodic density perturbation results is periodic electric field aligned with the \( \vec{k} \)-vector:
\[ E = \tilde{E}_k e^{i \kappa r}; \tilde{E}_k = \tilde{E}_{k||} + \tilde{E}_{k\perp}; \tilde{E}_{k||} = \frac{\kappa (\tilde{k} \cdot \tilde{E}_k)}{k^2} = \frac{\kappa}{k} E_k; \]

\[ \text{curl} \tilde{E} = 0 \rightarrow \kappa \times \tilde{E}_k = 0 \rightarrow \tilde{E}_{k\perp} = 0 \rightarrow \tilde{E} = \frac{\kappa}{k} E_k; \]

\[ \text{div} \tilde{E} = 4\pi \rho_e e^{i \kappa r} \rightarrow i\kappa \cdot \tilde{E}_k = i\kappa E_k = 4\pi \rho_e, \]

resulting in well known

\[ \tilde{E}_k \equiv 4\pi \rho_e \frac{\kappa}{i k^2}. \quad (B7) \]

For the non-uniform density resulting field will deviate from intuitive extensions of (B7) by \( \delta \tilde{E} \):

\[ \tilde{E} = 4\pi \rho_e (\tilde{r}) \frac{\kappa}{i k^2} e^{i \kappa r} + \delta \tilde{E}, \quad (B8) \]

and we need to find conditions when \( \kappa |\delta \tilde{E}| \ll 4\pi |\rho_e| \cdot \) We get the following using (B3):

\[ \text{div} \tilde{E} = 4\pi \rho_e (\tilde{r}) e^{i \kappa r} + 4\pi \left( \frac{\kappa \cdot \tilde{\nabla} \rho_e (\tilde{r})}{k^2} \right) e^{i \kappa r} + \text{div} \delta \tilde{E} = 4\pi \rho_e (\tilde{r}) e^{i \kappa r}; \]

\[ \text{curl} \tilde{E} = 4\pi \left( \frac{\kappa \times \tilde{\nabla} \rho_e (\tilde{r})}{k^2} \right) e^{i \kappa r} + \text{curl} \delta \tilde{E} = 0; \]

\[ \text{div} \delta \tilde{E} = -4\pi \left( \frac{\kappa \cdot \tilde{\nabla} \rho_e (\tilde{r})}{k^2} \right) e^{i \kappa r} \sim \kappa |\delta \tilde{E}| \]

\[ \text{curl} \delta \tilde{E} = -4\pi \left( \frac{\kappa \times \tilde{\nabla} \rho_e (\tilde{r})}{k^2} \right) e^{i \kappa r} \sim \kappa |\delta \tilde{E}| \]

While the error estimation resulting from \( \text{div} \tilde{E} = 4\pi \rho_e \) improves on the intuitive requirement (B4):

\[ \frac{\partial \delta E_x}{\partial x} + \frac{\partial \delta E_y}{\partial y} + \frac{\partial \delta E_z}{\partial z} = -4\pi e^{i \kappa r} \left( \kappa_x \frac{\partial \rho_e}{\partial x} + \kappa_y \frac{\partial \rho_e}{\partial y} + \kappa_z \frac{\partial \rho_e}{\partial z} \right); \frac{\partial \rho_e}{\partial x_i} \sim \frac{\rho_e}{\sigma_i}; \]

\[ -ie^{i \kappa r} \left( \frac{\partial \delta E_x}{\partial x} + \frac{\partial \delta E_y}{\partial y} + \frac{\partial \delta E_z}{\partial z} \right) \sim \left( \kappa_x \delta E_x + \kappa_y \delta E_y + \kappa_z \delta E_z \right) \sim 4\pi \frac{\rho_e}{k^2} \left( \frac{\kappa}{a_x} + \frac{\kappa}{a_y} + \frac{\kappa}{a_z} \right) \quad (B10) \]

\[ \frac{\text{div} \delta \tilde{E}}{\text{div} \tilde{E}} \sim \frac{\kappa_x}{a_x} + \frac{\kappa_y}{a_y} + \frac{\kappa_z}{a_z} < 1 \]

the error estimations resulting from \( \text{curl} \tilde{E} = 0 \)
is much more important because it links all three dimensions:

\[
\begin{align*}
\frac{\partial \delta E_y}{\partial y} + \frac{\partial \delta E_z}{\partial z} \sim |\kappa_y \delta E_z| + |\kappa_z \delta E_y| \sim \frac{4\pi}{\kappa^2} |\rho_x| \left( \frac{\kappa_y}{\kappa a_y} + \frac{\kappa_z}{\kappa a_z} \right); \\
\frac{\partial \delta E_x}{\partial x} + \frac{\partial \delta E_z}{\partial z} \sim |\kappa_x \delta E_z| + |\kappa_z \delta E_x| \sim \frac{4\pi}{\kappa^2} |\rho_x| \left( \frac{\kappa_x}{\kappa a_x} + \frac{\kappa_z}{\kappa a_z} \right); \\
\frac{\partial \delta E_x}{\partial x} + \frac{\partial \delta E_y}{\partial y} \sim |\kappa_x \delta E_y| + |\kappa_y \delta E_x| \sim \frac{4\pi}{\kappa^2} |\rho_x| \left( \frac{\kappa_x}{\kappa a_x} + \frac{\kappa_y}{\kappa a_y} \right). 
\end{align*}
\]

This allows us to estimate errors for each component of electric field:

\[
|\delta E_x| \sim |\bar{E}| \left( \frac{1}{\kappa a_x} \frac{\kappa_x}{\kappa_y} + \frac{1}{\gamma \kappa a_y} \frac{\kappa_y}{\kappa_x} \right); \\
|\delta E_y| \sim |\bar{E}| \left( \frac{1}{\kappa a_y} \frac{\kappa_y}{\kappa_x} + \frac{1}{\gamma \kappa a_x} \frac{\kappa_x}{\kappa_y} \right); \\
|\delta E_z| \sim |\bar{E}| \left( \frac{1}{\kappa a_z} \frac{\kappa_x}{\kappa_y} + \frac{1}{\gamma \kappa a_y} \frac{\kappa_y}{\kappa_x} \right).
\]

and

\[
|\delta E_x| \sim |\bar{E}| \left( \frac{1}{\gamma \kappa a_y} \frac{\kappa_z}{\kappa_x} \right); \\
|\delta E_y| \sim |\bar{E}| \left( \frac{1}{\kappa a_y} \frac{\kappa_z}{\kappa_x} \right); \\
|\delta E_z| \sim |\bar{E}| \left( \frac{1}{\kappa a_z} \frac{\kappa_z}{\kappa_x} \right).
\]

Now, let’s introduce the following definitions:
These conditions are most important for the case of the longitudinal density modulation. 

Electric field are sufficient conditions in the co-

Lorentz transformation. It means that

And finally, the combination gives of all estimations result in the following:

\[
\varepsilon_x = \frac{1}{\kappa a_x}; \varepsilon_y = \frac{1}{\kappa a_y}; \varepsilon_z = \frac{1}{\gamma \kappa a_z}; \\
\text{And} \quad r_{xy} = \frac{\kappa_x}{\kappa_y}; r_{xz} = \frac{\kappa_x}{\kappa_z}; r_{yz} = \frac{\kappa_y}{\kappa_z}.
\]

and rewrite (B13-14) as

\[
|\delta E_x| \sim |\bar{E}| \cdot \min \left( \varepsilon_x + \frac{\varepsilon_y}{r_{xy}}, \varepsilon_z + \frac{\varepsilon_y}{r_{yz}}, \varepsilon_x \right) \leq \varepsilon_x + \varepsilon_y + \varepsilon_z; \\
|\delta E_y| \sim |\bar{E}| \cdot \min \left( \varepsilon_y + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_y + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_y \right) \leq \varepsilon_x + \varepsilon_y + \varepsilon_z; \\
|\delta E_z| \sim |\bar{E}| \cdot \min \left( \varepsilon_z + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_y + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_z \right) \leq \varepsilon_x + \varepsilon_y + \varepsilon_z;
\]

And finally, the combination gives of all estimations result in the following:

\[
\min \left( r_x, \frac{1}{r_y} \right) = 1, r \geq 0
\]

\[
|\delta E_x| \sim |\bar{E}| \cdot \min \left( \varepsilon_x + \frac{\varepsilon_y}{r_{xy}}, \varepsilon_y + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_z + \frac{\varepsilon_y}{r_{xy}} \right) \leq \varepsilon_x + \varepsilon_y + \varepsilon_z;
\]

\[
|\delta E_y| \sim |\bar{E}| \cdot \min \left( \varepsilon_y + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_y + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_y \right) \leq \varepsilon_x + \varepsilon_y + \varepsilon_z;
\]

\[
|\delta E_z| \sim |\bar{E}| \cdot \min \left( \varepsilon_z + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_y + \frac{\varepsilon_x}{r_{xy}}, \varepsilon_z \right) \leq \varepsilon_x + \varepsilon_y + \varepsilon_z;
\]

It means that

\[
\kappa a_x \gg 1; \kappa a_y \gg 1; \gamma \kappa a_z \gg 1.
\]

are sufficient conditions in the co-moving frame for eq. (B7) to be a valid approximation for the electric field.

Lorentz transformation (B#) changes these conditions to the lab-frame as follows:

\[
a_{x,y} \cdot \sqrt{\frac{k_x^2}{\gamma^2} + \frac{k_y^2}{\gamma^2}} \gg 1; \quad a_z \cdot \sqrt{\gamma^2 k_z^2 + \frac{k_y^2}{\gamma^2}} \gg 1.
\]

These conditions are most important for the case of the longitudinal density modulation

\[
\hat{k}_{lab} = \hat{z}k_{i}; \quad k_{i} \gg \text{max} \left( \frac{\gamma}{a_{x,y}, a_z}; \frac{1}{a_z} \right),
\]

which we will use late in this paper.
Appendix C. Fields transfer from the comoving frame

An instantaneous comoving frame (with a fixed velocity \( \mathbf{v}_o \) along the direction of the reference trajectory) can be used for EM field evaluation and transferring it back to laboratory frame.

In this Appendix we will use metric of special relativity for 4-dimensional time and space:

\[
\begin{align*}
(a')_i &= \left(a_o, \vec{a}\right); \quad (b')_i = \left(b_o, -\vec{b}\right); \quad a'b_i = a_o b_o - \vec{a} \cdot \vec{b},
\end{align*}
\]

three 4-vectors: the time space, the k-vector and the 4-potential of EM field,

\[
\begin{align*}
(x')^i &= \left(ct, \vec{r}\right); \quad (k')^i = \left(\frac{\omega}{c}, \vec{k}\right); \quad (A')_i = \left(\phi, \vec{A}\right),
\end{align*}
\]

and fact that phase of oscillations is 4-scalar

\[
\phi = -k x^i = \vec{k} \cdot \vec{r} - \omega t = \text{inv},
\]

and is invariant of Lorentz transformation. Here we will use Lorentz transformation into and from inertial frame moving along z-axis with velocity \( \mathbf{v}_o \):

\[
\hat{\mathbf{z}} = \tilde{\mathbf{r}}(s_o); \quad \tilde{\mathbf{v}}_o = \mathbf{v}_o \cdot \hat{\mathbf{z}}(s_o) = \text{const}; \quad \beta_o = \frac{\mathbf{v}_o}{c}; \quad \gamma_o = \frac{1}{\sqrt{1 - \beta_o^2}};
\]

\[
\begin{align*}
a_{ol} &= \gamma_o (a_{oc} + \beta_o a_{zc}); \quad a_{zl} = \gamma_o (a_{zc} + \beta_o a_{oc});
\end{align*}
\]

\[
\begin{align*}
a_{oc} &= \gamma_o (a_{oc} - \beta_o a_{zl}); \quad a_{zc} = \gamma_o (a_{zl} - \beta_o a_{oc});
\end{align*}
\]

where we use subscripts \( c \) and \( i \) for variables in the comoving and the laboratory frame, correspondingly.

Using standard assumption that motion of particles in this frame is non-relativistic, \( |\tilde{\mathbf{v}}_o| \ll c \), allows us to neglect magnetic field and use zero vector potential \(^{14}\)

\[
\varphi_c^k = \left(\varphi_c, \vec{A}_c\right); \quad \vec{A}_c = 0; \quad \Delta \varphi_c = -4\pi \rho(\vec{r}_c).
\]

Applying Fourier transform \( f(\vec{k}_c) = \int f(\vec{r}_c) e^{-i\vec{k}_c \cdot \vec{r}_c} d\vec{r}_c \), we obtain

\[^{14}\text{Here we are using standard assumption that motion of particles in the comoving frame is non-relativistic and magnetic field can be neglected. Second standard assumption is that plasma frequency is slow when compared with } c|\vec{k}|: \omega_p \ll c|\vec{k}| \text{ allows to neglect second driving term in Maxwell equation from magnetic field: } \text{curl} \vec{B} = 4\pi \mathbf{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \text{.} \]
\[
\varphi_c(\vec{k}) = \frac{4\pi\rho_{\vec{k}}}{k_c^2}; \quad A_c(\vec{k}) = 0; \quad \tilde{\varphi}_c = \frac{4\pi\rho_{\vec{k}}}{k_c^2} e^{ik_c}; \quad \tilde{A}_c = \frac{4\pi\rho_{\vec{k}}}{k_c^2} e^{-ik_c}; \quad (C6)
\]

where we used the facts that \( \omega_c = 0 \). The 4-potential (C6) we can be easily transferred to the laboratory frame:

\[
\tilde{\varphi}_l = \gamma_\rho \tilde{\varphi}_c; \quad \tilde{A}_l = \tilde{\varphi}_l - \frac{4\pi\rho_{\vec{k}}}{k_i} e^{i(k\beta\beta_0)}; \quad \tilde{A}_l = \gamma_\rho \tilde{\varphi}_c (1, \beta_0, 0, 0).
\]

and expressed through the components in the laboratory frame:

\[
\tilde{\varphi}_l = -\frac{4\pi\rho_{\vec{k}}}{k_i - \beta_0 k_i} e^{i(k_i \beta - k_i \beta_0产学)}, \quad \tilde{A}_l = \tilde{\varphi}_l + \tilde{\alpha}_l = \beta_0 \tilde{\varphi}_l + \tilde{\alpha}_l = \tau(s_o)
\]

taking into account scaling of the density by the factor \( \gamma_\rho \):

\[
\rho_{\vec{k}} = \gamma_\rho \rho_{\vec{k}}, \quad (C9)
\]

and confection between k-vectors:

\[
k_{\parallel} = \gamma_\rho k_{\parallel}, \quad \omega_{\parallel} = c k_{\parallel} = \gamma_\rho \beta_0 c k_{\parallel} = \beta_0 c k_{\parallel}, \quad \tilde{k}_{\parallel} = \tilde{k}_{\parallel}, \quad \tilde{k}_\perp = \tilde{k}_\perp - \beta_0\rho_{k_c}^2.
\]

Equation (C8) is identical to the EM potentials in Cartesian coordinates, that we derived in the next Appendix.

**Appendix D. Expression for charge and current density modulation**

Solving Maxwell equations require knowledge of the charge and current densities as functions of coordinates and time:

\[
div B = 0; \quad div E = 4\pi \rho; \quad curl B = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B} + \frac{4\pi}{c} \vec{j}; \quad curl \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \frac{4\pi}{c} \vec{j}; \quad (D1)
\]

where \( f(\vec{r},\vec{v},t) \) is the particles distribution function in the \( (\vec{r},\vec{v}) \) configuration space. Using \( s \) as independent variable makes connection between \( \rho, \vec{j} \) and the phase space distribution function \( F(q,p,s) \) non-trivial, where \((q,p)\) is conjugate Canonical set of coordinates and momenta. This Appendix is dedicated for establishing such connection and finding corresponding 4-potentila of the EM field.

Let’s introduce instantaneous Cartesian coordinate system with z-axis along the reference trajectory at \( s = \vec{s}_o \) (see equation (6) in the main text):
\[ \hat{x} = \hat{n}(s_o); \hat{y} = \hat{b}(s_o); \hat{z} = \hat{t}(s_o) = \frac{d\hat{r}_o(s)}{ds} \left|_{s=s_o} \right.; \quad \frac{d\hat{n}}{ds} = K(s) \cdot \hat{r} - \kappa(s) \cdot \hat{b}; \quad \frac{d\hat{b}}{ds} = \kappa(s) \cdot \hat{n}; \]
\[ \ddot{r} = \ddot{r}_o(s_o) + \dot{x} \cdot x + \dot{y} \cdot y + \dot{z} \cdot z = \ddot{r}_o(s) + \ddot{b}(s)q_1 + \ddot{b}(s)q_2; \]
\[ q_1 = \ddot{n}(s) \left( \hat{x} \cdot x + \hat{y} \cdot y + \hat{z} \cdot z \right) - \ddot{r}_o(s); q_2 = \ddot{b}(s) \left( \hat{x} \cdot x + \hat{y} \cdot y + \hat{z} \cdot z \right) - \ddot{r}_o(s); \]
\[ q_3 = c \left( t_o(s) - t \right). \]

with following ratios in the vicinity of \( r_o(s) \):
\[ dx = dq_1 + q \kappa(s_o) ds; dy = dq_2 - q \kappa(s_o) ds; dz = \left( 1 + q \kappa(s_o) \right) ds; \]
\[ q_3 = c \left( t_o(s) - t \right); dq_3 = \frac{ds}{\beta_o(s_o)} - c dt. \]

where we used eq. (6) for reference trajectory. At fixed \( s \):
\[ ds = 0 \rightarrow dz = 0; dq_1 = dx; dq_2 = dy; dq_3 = -c dt. \]

Number of particles confined in infinitesimal volume \( dq^3 \) at fixed \( s=s_o \) is defined as:
\[ dn = \left| dq^3 \right| \int \tilde{f}(q,p,s) dp^3 = cdx dt \int \tilde{f}(q,p,s) dp^3 \]

is identical to number of particles passing through the elementary area \( dx dy \) locates at \( s=s_o \) in time interval \( dt \):
\[ \tilde{j}(\vec{r},t) = \int \tilde{v} f(\vec{r},\vec{v},t) d\vec{v}^3; \quad dn = d\vec{a} \cdot \tilde{j} \cdot dt; d\vec{a} = \hat{z} dx dy; \]
\[ dn = dx dy dt \int \tilde{v} f(\vec{r},\vec{v},t) d\vec{v}^3, \]
resulting in
\[ \int \tilde{f}(q,p,s) dp^3 = \frac{1}{c} \int \tilde{v} f(\vec{r},\vec{v},t) d\vec{v}^3 \]

Using paraxial approximation \( \tilde{v}_z = v_o + \delta v; |\delta v| \ll v_o \), and neglecting \( \delta v \) in the integral (D7), we can express \( \rho, \tilde{j} \) using phases space distribution function as:
\[ \rho(\vec{r},t) = \frac{\rho(q,s)}{\beta_o(s)}; \tilde{j}(\vec{r},t) = \hat{z} c \rho(\vec{r},t); \rho(q,s) = c \int \tilde{f}(q,p,s) dp^3. \]

Applying Fourier transformation\(^{15} \)
\[ g_k \equiv g(k_1,k_2,k_3,s) = \int g(q,s) e^{-i\vec{q}\cdot\vec{r}} dq^3 \]

\(^{15} \) In this Appendix, we use interchangeably both the compact, \( g_k \), and detailed, \( g(k_1,k_2,k_3,s) \), notation for the Fourier components defined in the accelerator coordinates.
where we used $\delta \left( \frac{k_z}{\beta_o} - \kappa_z \right) = \beta_o \delta \left( k_o - \beta_o \kappa_z \right)$ and singularity of Dirac’s $\delta$-function:
\[ g(x)\delta(x-y) = g(y)\delta(x-y). \]

To find 4-potential induced by such perturbation we can use Lorenz gauge \( \frac{\partial \varphi}{c \partial t} + \text{div} \vec{A} = 0 \) providing for separation of equations for each component of 4-potential [88]^{16}:

\[
\frac{\partial^2 \varphi}{c^2 \partial t^2} - \Delta \varphi = 4\pi \rho; \quad \frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \Delta \vec{A} = 4\pi \vec{j},
\]

which can be Fourier transformed to:

\[
\phi(\vec{k}, \omega) e^{i(\vec{k}\cdot \vec{r} - \omega t)} = \frac{4\pi \rho(\vec{k}, \omega)}{\vec{k}^2 - k_o^2} e^{i(\vec{k}\cdot \vec{r} - \omega t)} = \frac{8\pi^2 \epsilon}{c} \frac{f(\kappa, \kappa_y, \beta_o \kappa_z, \vec{s})}{\vec{k}^2 - \beta_o^2 \kappa_z^2} \delta(k_o - \beta_o \kappa_z) e^{i(\vec{k}\cdot \vec{r} - \omega t)},
\]

(D15)

\[ \vec{A}(\vec{k}, \omega) = \vec{k} \phi(\vec{k}, \omega) \frac{k_o}{\kappa_z} = \vec{z} \beta_o \phi(\vec{k}, \omega). \]

In the inverse Fourier transform

\[ \phi(\vec{r}, t) = \frac{1}{(2\pi)^4} \int \phi(\vec{k}, \omega) e^{i(\vec{k}\cdot \vec{r} - \omega t)} d\omega d\vec{k}^2, \]

\[ \delta \text{-function makes integral over } \omega \text{ is straight forward:} \]

\[
\frac{1}{2\pi} \int e^{i\omega(t, \tau - t)} g(\vec{k}, \omega) \delta \left( \frac{\omega}{c} - \beta_o \kappa_z \right) d\omega = \frac{c}{2\pi} g(\vec{k}, \rho \kappa_z) e^{i\beta_o \kappa_z (\tau - t)}; \quad \tau(\vec{s}) = c(t_o(\vec{s}) - t),
\]

(D16)

with the remaining integral of

\[
\phi(\vec{r}, t) = 4\pi \epsilon \int \frac{g(\kappa, \kappa_y, \beta_o \kappa_z, \vec{s})}{\vec{k}^2 - \beta_o^2 \kappa_z^2} e^{i(\vec{k}\cdot \vec{r} - \omega t)} d\vec{k}^3 = \frac{\vec{z} \beta_o \phi(\vec{r}, t)}{(2\pi)^4}; \quad \vec{A}(\vec{r}, t) = \vec{z} \beta_o \phi(\vec{r}, t).
\]

(D17)

Taking into account expansion (D13), the exponent in (D17) can be expressed using the accelerator coordinates:

\[ \vec{k} \vec{r} + \beta_o \kappa_z \tau(\vec{s}) = \vec{k} \vec{q}; \quad k_{1, 2} = \kappa_x; k_3 = \beta_o \kappa_z; \]

(D18)

and using ratio \( \beta_o d\vec{k}^3 = d\vec{s} \) we get expression connecting the 4-potential and density perturbation in the accelerator coordinates:

\[
\phi(\vec{q}, s) \equiv \phi(\vec{r}, t) = 4\pi \epsilon \beta_o \gamma \int \frac{\vec{f}}{\gamma (\kappa^2 + k_z^2)} e^{i\vec{k}\cdot \vec{q}} d\vec{k}^3 = \frac{\vec{z} \beta_o \phi(\vec{q}, s)}{(2\pi)^4}; \quad \vec{A}(\vec{q}, s) = \vec{z} \beta_o \phi(\vec{q}, s),
\]

(D19)

where we take into account

^{16} The Lorentz gauge can be used for time-dependent component of the EM field, which is of interest in this paper.
Appendix E. Perturbed Hamiltonian

As derived in Appendix D, density perturbation results in additional 4-potential

$$\delta \phi' = \left\{ \delta \phi, \delta \tilde{A} \right\}; \delta \tilde{A} = \dot{z} \beta_o \delta \phi.$$  \hspace{1cm} (E1)

which we will consider being infinitesimally small: \( \delta \phi \sim O(e), e \ll 1 \). Goal of this Appendix is to define additional term of the reduced accelerator Hamiltonian (7-9) resulting from the density perturbation:

$$h' = -\left(1 + Kq_1 \right) \sqrt{\left( \frac{E_o + cP_3 - e \phi_{\perp} - e \delta \phi}{c^2} \right)^2 - m^2 c^2 - \left( \frac{P_1 - e}{c} A_1 \right)^2 - \left( \frac{P_2 - e}{c} A_2 \right)^2}$$

$$- \frac{e}{c} \left(1 + Kq_1 \right) \left( A_z + \beta_o \delta \phi \right) + \kappa q_1 P_2 - \kappa q_2 P_1 - \frac{c}{v_o(s)} P_3 + q_s \frac{d}{ds} \left( E_o(s) + e \frac{\phi(\tilde{r}_o(s), t)}{c} \right)$$  \hspace{1cm} (E2)

where we used explicit expression for \( A_3 \) component of the vector potential (7). Perturbation of the Hamiltonian is coming only from first two terms in r.h.s of (E2).

$$\delta h' = -\left(1 + Kq_1 \right) \sqrt{\left( \frac{E - e \delta \phi}{c^2} \right)^2 - m^2 c^2 - \tilde{p}_{\perp}^2 + \frac{e}{c} \beta \delta \phi}$$

$$p_z = \sqrt{\frac{E^2}{c^2} - m^2 c^2 - \tilde{p}_{\perp}^2};$$

$$\frac{E}{c} = H - e \phi; \ \beta_o = \frac{v_o}{c}; \ \beta_z = \frac{v_z}{c} = \frac{cp_z}{E}; 1 - \beta_o^2 = \gamma_o^{-2};$$

$$\delta h' = \left(1 + Kq_1 \right) \frac{e}{c} \delta \phi \left( \frac{c}{v_z} - \frac{v_o}{c} \right) = \frac{1}{\beta_o \gamma_o} \frac{c}{e} \delta \phi \left[ 1 + \gamma_o^2 \left( \frac{\beta_o}{\beta_z} - 1 \right) \right] \left(1 + Kq_1 \right).$$  \hspace{1cm} (E3)

First, in paraxial approximation term \( |Kq_1| \ll 1 \) can be dropped. It is also easy to show that second term in the figure brackets is infinitivally small in the case of paraxial motion resulting in non-relativistic motion in the co-moving frame:

$$\gamma_o^2 \left( \frac{\beta_o}{\beta_z} - 1 \right) = \frac{\gamma_o^2 \beta_{\perp}^2}{\beta_z (\beta_o + \beta_z)} + \frac{\gamma - \gamma_o}{\gamma \beta_z (\beta_o + \beta_z)} \sim \frac{\beta_{\text{cm}}^2}{2} + \frac{\delta \gamma}{2 \gamma} \ll 1.$$  \hspace{1cm} (E4)

31
Specifically, \( c\gamma_o\beta_\perp \) is the transverse velocity in the co-moving frame and \( \frac{\delta y}{\gamma} \) is the relative energy deviation in the beam. Both of these values assumed to be infinitesimally small. As the result, the perturbation of the Hamiltonian is reduced to as simple:

\[
\delta h^* = \frac{1}{\beta_o^2 c} \frac{e}{\gamma_o} \frac{4\pi e^2}{c} \int \frac{\tilde{\rho}_k}{\gamma_o^2 \beta_o^2 k^2} e^{i\delta q} \frac{dk^3}{(2\pi)^3};
\]

\[
\tilde{\rho}_k = \int e^{-i\delta q} f(q, p, s) dp dq.
\] (E5)

References

[1] *Simulations of Coherent Electron Cooling with Two Types of Amplifiers*, Jun Ma, Gang Wang, Vladimir Litvinenko, International Journal of Modern Physics A (IJMPA), Vol. 34 (2019) 1942029

[2] *Plasma-Cascade Instability*, Vladimir N. Litvinenko, Yichao Jing, Dmitry Kayran, Patrick Inacker, Jun Ma, Toby Miller, Irina Petrushina, Igor Pinayev, Kai Shih, Gang Wang, Yuan H. Wu, accepted for publication in Physical Review Accelerators and Beams (2021)

[3] "Challenges and Goals for Accelerators in the XXIst Century", Edited by S. Myers and O. Bruning Published by World Scientific Publishing Co. Pte. Ltd., 2016. ISBN

[4] LHC design report, [http://documents.cern.ch/cgi-bin/](http://documents.cern.ch/cgi-bin/)

[5] "A Large Hadron Electron Collider at CERN", J. L. Abelleira Fernandez t, C. Adolphsen, A.N. Akay, H. Aksakal, J.L. Albacete et al., Journal of Physics G: Nuclear and Particle Physics, Volume 39, Number 7, July 2012, pp. 75001-75630 (630) [http://iopscience.iop.org/0954-3899/39/7/075001](http://iopscience.iop.org/0954-3899/39/7/075001)

[6] ATLAS collaboration (2018). "Observation of H→bb decays and VH production with the ATLAS detector". arXiv:1808.08238 [hep-ex]

[7] CMS collaboration (2018), "Observation of Higgs Boson Decay to Bottom Quarks", Physical Review Letters 121, 12180 (2018)

[8] P. Emma, et al., Nature Photonics 4 (2010) 641

[9] C. Pellegrini, X-ray free-electron lasers: from dreams to reality, Physica Scripta, Volume 2016, Number T169

[10] D. Pile, First light from SACLA, Nature Photonics 5, pages 456–457 (2011)

[11] T. Ishikawa et al., A compact X-ray free-electron laser emitting in the sub-angstrom region, Nat. Photon. 2012, 6, 540–544

[12] European XFEL. Available online: [https://www.xfel.eu](https://www.xfel.eu)

[13] Commissioning and First Lasing of the European XFEL, H. Weise and W. Decking, 38th International Free-Electron Laser Conference, FEL 2017, Santa Fe, USA, 20 Aug 2017 - 25 Aug 2017 5 pp. (2017)

[14] In Soo Ko et al., Construction and Commissioning of PAL-XFEL Facility, Applied Sciences 2017, 7(5), 479

[15] C.J. Milne et al., Swiss FEL: The Swiss X-ray Free Electron Laser, Applied Sciences 2017, 7(7), 720

[16] A. Doerr, The new XFELs, Nature Methods, volume 15, page 33 (2018)
[17] Pohang Accelerator Laboratory, http://pal.postech.ac.kr/paleng/.
[18] P. Ball, Nature 548 (2017) 507.
[19] Z. Huang, I. Lindau, Nature Photonics 6 (2012) 505–506.
[20] H.-S. Kang, et al., Nature Photonics 11 (2017) 708.
[21] Challenges of High Photon Energy, High-Repetition Rate XFELs, MaRIE Summer 2016 Workshop Series, August 9-10, 2016, https://www.lanl.gov/science-innovation/science-facilities/marie/_assets/docs/challenges-high-photon-energy.pdf
[22] B. Eliasson, and Chuan-Sheng Liu, An electromagnetic gamma-ray free electron laser, Journal of Plasma Physics, 79 (06). 2013, pp. 995-998. ISSN 0022-3778, http://www.slac.stanford.edu/cgi-wrap/getdoc/slac-pub-15106.pdf
[23] Electron-ion collider: The next QCD frontier, A. Accardi, J.L. Albacete, M. Anselmino, N.. Armesto, E.C. Aschenauer, et al., The European Physical Journal A 52 (9), 268
[24] A Large Hadron Electron Collider at CERN: Report on the Physics and Design Concepts for Machine and Detector, J.L.A Fernandez, C. Adolphsen, A.N. Akay, H. Aksakal, J.L. Albacete, et al., Journal of Physics G: Nuclear and Particle Physics 39 (7), 075001
[25] D. Boer, M Diehl, R. Milner, R. Venugopalan, W. Vogelsang, A. Accardi, et al., Gluons and the quark sea at high energies: Distributions, polarization, tomography, arXiv preprint, arXiv:1108.1713
[26] K.J. Kim, Y.V. Shvyd’ko, and S. Reiche, Phys. Rev. Lett. 100, 244802 (2008)
[27] K.J. Kim and Y.V. Shvyd’ko, Phys. Rev. ST Accel. Beams 12, 030703 (2009)
[28] V.N. Litvinenko, Optics-free FEL oscillator, Proceeding of FEL’2002 conference, August 2002, Argonne, IL, 2003 Elsevier Science B.V. (2003) II-23
[29] V.N. Litvinenko, Y. Hao, D. Kayran, D. Trbojevic, Optics-free X-ray FEL Oscillator, Proceedings of 2011 Particle Accelerator Conference, New York, NY, USA, March 25-April 1, 2011, p. 802
[30] J. Rosenzweig, et al, Nucl. Instr. Meth. Phys. Re. A, 410, 3 (1998) 532-543.
[31] C. B. Schroeder, et al, Phys. Rev. ST-AB, 13, 101301 (2010).
[32] J. Rosenzweig, Presentation at Towards a 5th Generation Light Sources Workshop, Avalon, Catalina Island, October, 2010.
[33] C. Jing, WF-Note-241, http://www.hep.anl.gov/awa/links/wfnotes.htm
[34] K. Nakajima, et al, Nucl. Instr. Meth. Phys. Re. A, 375 (1996) 593-596.
[35] I. Blumenfeld, et al, Nature 445 Feb. 2007 pp.741-744.
[36] S. Corde et al., Nature, 27 August 2015
[37] N. Jain, T.M. Antonsen, J.P. Palastro, Physical Review Letters 115, 195001 (2015)
[38] A. Zholents et al. A preliminary design of the collinear dielectric wakefield accelerator, Nucl. Inst. Meth. Phys. Res. A, v. 829, p.190, 2016.
[39] S. Antipov, C. Jing, M. Fedurin, W. Gai, et al., Phys. Rev. Lett. 108, 144801, 2012
[40] G.P. Williams, Filling the THz gap—high power sources and applications, Reports on Progress in Physics, Volume 69, Number 2, 2005
[41] C. Sung, S. Ya. Tochitsky, S. Reiche, J. B. Rosenzweig, C. Pellegrini, and C. Joshi, Phys. Rev. ST Accel. Beams 9, 120703, 2006
[42] D. Ratner, R. Abela, J. Amann, C. Behrens, D. Bohler et al., Physical Review Letters 114, 05480 (2015)
[43] E. Allaria, R. Appio, L. Badano, W.A. Barletta, S. Bassanese et al., Nature Photonics, 6, 699 (2012)
[44] V.N. Litvinenko, Y.S. Derbenev, Coherent Electron Cooling, Phys. Rev. Lett. 102, 114801 (2009)
[45] D. Ratner, Microbunched Electron Cooling for High-Energy Hadron Beams, Phys. Rev. Lett. 111 084802 (2013)
[46] V.N. Litvinenko, Y.S. Derbenev, Proc. of 29th International Free Electron Laser Conference, Novosibirsk, Russia, 2007, p. 268
[47] V.N. Litvinenko, Advances in Coherent Electron Cooling, In Proceedings of COOL 2013 workshop, June 2013, Mürren, Switzerland, p. 175, ISBN 978-3-95450-140-3
[48] V.N. Litvinenko, G. Wang, D. Kayran, Y. Jing, J. Ma, I. Pinayev, Plasma-Cascade micro-bunching Amplifier and Coherent electron Cooling of a Hadron Beams, arXiv:1802.08677, February 2018, https://arxiv.org/pdf/1802.08677.pdf
[49] Plasma-Cascade Instability- theory, simulations and experiment, Vladimir N. Litvinenko, Gang Wang, Yichao Jing, Dmitry Kayran, Jun Ma, Irina Petrushina, Igor Pinayev and Kai Shih, arXiv:1902.10846
[50] A. Chao, E. Granados, X. Huang, D. Ratner, H.-W. Luo, In Proceedings of IPAC2016, Busan, Korea, p. 1048
[51] D.F. Ratner, A.W. Chao, Phys. Rev. Lett. 105, 154801 (2010)
[52] Y. Jiao, D.F. Ratner, A.W. Chao, Physical Review ST Accelerators and Beams 14, 110702 (2011)
[53] R.A. Lacey, Phys. Rev. Lett. 114, 142301, 2015
[54] E. L. Saldin, E. A. Schneidmiller, and M.V. Yurkov, Nucl. Instrum. Methods Phys. Res., Sect. A 490, 1 (2002).
[55] S. Heifets, S. Krinsky, G. Stupakov, Physical Review ST Accelerators and Beams 5, 064401 (2002).
[56] E. A. Schneidmiller and M.V. Yurkov, Phys. Rev. ST Accel. Beams 13, 110701 (2010)
[57] T. Shaftan and Z. Huang. Phys. Rev. ST AB, 7:080702, 2004.
[58] M. Venturini, Physical Review ST Accelerators and Beams 10, 104401 (2007).
[59] M. Venturini, Models of longitudinal space-charge impedance for microbunching instability. Phys. Physical Review ST Accelerators and Beams 11, 034401 (2008)
[60] D. Ratner, Z. Huang, A. Chao, Three-Dimensional Analysis of Longitudinal Space Charge Microbunching Starting From Shot Noise Presented at FEL08, Gyeongju, Korea, 24-29 Aug (2008)
[61] M. Dohlus, E. A. Schneidmiller, and M.V. Yurkov, Phys. Rev. ST Accel. Beams 14, 090702 (2011).
[62] A. Gover and E. Dyunin, Phys. Rev. Lett. 102, 154801 (2009).
[63] R. Akre, D. Dowell, P. Emma, J. Frisch, S. Gilevich et al., Phys. Rev. ST Accel. Beams 11, 030703 (2008)
[64] A. Marinelli and J.B. Rosenzweig, Microscopic kinetic analysis of space-charge induced optical microbunching in a relativistic electron beam, Phys. Rev. ST Accel. Beams 13, 110703 (2010)
[65] A. Marinelli, E. Hemsing, M. Dunning, D. Xiang, S. Weathersby, F. O'Shea, I. Gadjev, C. Hast, and J. B. Rosenzweig, Phys. Rev. Lett. 110, 264802 (2013).
[66] Z. Huang and K.-J. Kim, Phys. Rev. ST Accel. Beams 5, 074401 (2002).
[67] Z. Huang, M. Borland, P. Emma, J. Wu, C. Limborg, G. Stupakov, and J. Welch, Phys. Rev. ST Accel. Beams 7, 074401 (2004).
[68] D. Ratner, *Much Ado About Microbunching: Coherent Bunching in High Brightness Electron Beams*, PhD thesis, Department of Applied Physics, Stanford University, 2011.

[69] A. Marinelli, E. Hemsing, and J.B. Rosenzweig, *Three dimensional analysis of longitudinal plasma oscillations in a thermal relativistic electron beam*, Physics of Plasmas, 18:103105, 2011.

[70] V.N. Litvinenko, G. Wang, *Relativistic Effects in Micro-Bunching*, Proceedings of FEL2014, Basel, Switzerland, August 25-29, 2014, THP035, http://www.fel2014.ch/prepress/FEL2014/papers/thp035.pdf

[71] D. Ratner, *Microbunching Instability: Review of Current Models, Observations, and Solutions*, ICFA Beam Dynamics Newsletter No. 49 p. 112 (2012)

[72] A. Fedotov et al., *Parametric collective resonances and space-charge limit in high-intensity rings*, PR STAB 6, 094201, 2003

[73] D.R. Nicholson, *Introduction in Plasma Theory*, John Wiley & Sons, 1983

[74] R. C. Davidson, *Physics of Nonneutral Plasmas*, Addison- Wesley, Reading, MA, 1990

[75] E.D. Courant, H.S. Snyder, Ann. Phys. 3 (1958) 1.

[76] A.A. Kolomensky and A.N. Lebedev, *Theory of Cyclic Accelerators*, North Holland, 1966

[77] S.Y. Lee, *Accelerator Physics*, World Scientific, 2011

[78] H. Wiedemann, *Particle Accelerator Physics*, Springer, 2007

[79] L.D. Landau, E.M. Lifshitz, *Classical Mechanics*, Elsevier Science, 1976, ISBN-10 0750628960

[80] V. N. Litvinenko, *Analytical Tools in Accelerator Physics*, C-A/AP/406 note, September 2010, https://www.bnl.gov/isd/documents/74289.pdf

[81] USPAS Winter 2008 Accelerator Physics Course, V.N. Litvinenko, September 1, 2010, pp. 1-116, http://uspas.fnal.gov/materials/08UCSC/Accelerator_Physics1.pdf

[82] Alex J. Dragt, *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics*, University of Maryland, Center for Theoretical Physics, Department of Physics, 1997

[83] A. A. Vlasov, *On Vibration Properties of Electron Gas*, J. Exp. Theor. Phys. (in Russian). 8 (3) 1938, 291.

[84] V. I Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag New York 1989

[85] I.M. Kapchinsky and V.V. Vladimirsky, in Proceedings of the International Conference on High Energy Accelerators and Instrumentation, CERN Scientific Information Service, Geneva, 1959, p. 274.

[86] *Compensating effect of solenoids with quadrupole lenses*, Vladimir N Litvinenko, Alexander A Zholents, arXiv:1809.11138, 28 Sep 2018, https://arxiv.org/abs/1809.11138

[87] L.D. Landau, *On the vibration of the electronic plasma*, JETP 16 (1946), 574. English translation in J. Phys. (USSR) 10 (1946), 25.

[88] L.D. Landau, and E.M. Lifshitz, *The Classical Theory of Fields*, Butterworth-Heinemann, 1975