SPECTRAL, STOCHASTIC AND CURVATURE ESTIMATES FOR
SUBMANIFOLDS OF HIGHLY NEGATIVE CURVED SPACES

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ABSTRACT. We prove spectral, stochastic and mean curvature estimates for complete \( m \)-submanifolds \( \varphi : M \to N \) of \( n \)-manifolds with a pole \( N \) in terms of the comparison isoperimetric ratio \( I_m \) and the extrinsic radius \( r_\varphi \leq \infty \). Our proof holds for the bounded case \( r_\varphi < \infty \), recovering the known results, as well as for the unbounded case \( r_\varphi = \infty \). In both cases, the fundamental ingredient in these estimates is the integrability over \((0, r_\varphi)\) of the inverse \( I_m^{-1} \) of the comparison isoperimetric radius. When \( r_\varphi = \infty \), this condition is guaranteed if \( N \) is highly negatively curved.

1. INTRODUCTION

Let \( N \) be a complete Riemannian \( n \)-manifold with a pole \( p \) and let \( \varphi : M \to N \) be an isometric immersion of a complete Riemannian \( m \)-manifold \( M \) into \( N \). The extrinsic radius \( r_\varphi \) is the (possibly extended) number

\[
 r_\varphi = \inf\{ r \in (0, \infty] : \varphi(M) \subset B_N(r) \},
\]

where \( B_N(r) \subseteq N \) is the geodesic ball centered at \( p \) of \( N \) with radius \( r \in (0, \infty] \). Suppose that the radial sectional curvatures of \( N \) at \( x \in B_N(r_\varphi) \) satisfies

\[
 K_N(x) \leq -G(\rho_N(x))
\]

where \( G : \mathbb{R} \to \mathbb{R} \) is a smooth even function and \( \rho_N(x) = \text{dist}_N(p, x) \).

Associate to \( N \), the \( m \)-dimensional model manifold \( M^m_\sigma \) with radial sectional curvature \(-G(r)\). Namely,

\[
 M^m_\sigma = ([0, r_\varphi] \times S^{m-1}, ds^2 = dr^2 + \sigma^2(r) d\theta^2),
\]

where \( \sigma \) denotes the unique solution of the Cauchy problem on \((0, r_\varphi] \)

\[
\left\{ \begin{array}{l}
 \sigma''(t) - G(t) \sigma(t) = 0, \\
 \sigma(0) = 0, \quad \sigma'(0) = 1,
\end{array} \right.
\]

which we assume to be positive and increasing on \([0, r_\varphi)\). Observe that if \( G \geq 0 \) on \([0, \infty) \) then \( \sigma'' \geq 0 \) everywhere and \( \sigma' \geq 1 > 0 \) on \([0, \infty) \). More generally, it follows
by a strengthened version of Kneser Theorem, see e.g. [10, Prop. 1.21], that if 
\( G_- = \max \{-G, 0\} \) satisfies

\[
t \int_{t}^{\infty} G_-(s) \, ds \leq \frac{1}{4},
\]

then \( \sigma' \geq 0 \).

Associate to the model \( M^m_\sigma \) the function \( I_m : [0, r_\varphi] \to \mathbb{R}_+ \) defined by

\[
I_m(r) = \frac{\sigma(r)^m}{\int_{0}^{r} \sigma(t)^{m-1} \, dt}.
\]

In geometric terms, \( I_m \) is the non-homogeneous isoperimetric ratio

\[
I_m(r) = \frac{\text{vol} \, \partial B_r(o)}{\text{vol} B_r(o)},
\]

where \( B_r(o) \) and \( \partial B_r(o) \) are the geodesic balls and spheres in \( M^m_\sigma \) of radius \( r > 0 \) and center at the pole \( o \) of the model. In particular, the Cheeger constant \( h(M^m_\sigma) \) of the model has the upper estimate

\[
\inf_{[0, +\infty)} I_m(r) \geq h(M^m_\sigma).
\]

Also associated to the model \( M^m_\sigma \), is the homogeneous isoperimetric ratio \( \mathcal{J}_m(r) \) which is defined by

\[
\mathcal{J}_m(r) = \frac{\sigma(r)^m}{\int_{0}^{r} \sigma(t)^{m-1} \, dt} = \frac{\text{vol} \, \partial B_r(o)}{\text{vol} B_r(o)}.
\]

We showed in [3] that if \( \varphi : M \to N \) is minimal and the following extrinsic conditions

\[
\mathcal{J}_m(t) \geq 0, \quad t \in (0, r_\varphi) \quad \text{and} \quad I_m^{-1} \in L^1(0, r_\varphi)
\]

hold, then the global mean exit time of \( M \) is finite, and, in fact,

\[
E_M \leq \int_{0}^{r_\varphi} I_m^{-1}(s) \, ds < \infty.
\]

In particular, \( M \) is not \( L^1 \)-Liouville. Three aspects of this result should be remarked.

First, the condition \( \mathcal{J}_m(t) \geq 0 \) is implied by \( -G \leq 0 \). Indeed,

\[
\mathcal{J}_m(t) = m \frac{\sigma' - \sigma^{m-1}}{\sigma} = \frac{1}{\int_{0}^{t} \sigma^{m-1} \, dt} \left( m \sigma' \int_{0}^{t} \sigma^{m-1} - \sigma^m \right),
\]

thus \( \mathcal{J}_m(t) \geq 0 \iff \left( m \sigma' \int_{0}^{t} \sigma^{m-1} - \sigma^m \right) \geq 0 \).

Now, \( m \sigma' \int_{0}^{t} \sigma^{m-1} - \sigma^m \to 0 \) as \( r \to 0+ \) and its derivative is

\[
m \sigma'' \int_{0}^{t} \sigma^{m-1}.
\]

Thus if \( \sigma'' \geq 0 \), that is, if the curvature is nonpositive, then \( m \sigma'' \int_{0}^{t} \sigma^{m-1} \geq 0 \) and therefore \( \mathcal{J}_m(t) \geq 0 \).

Note however that, while it is necessary that \( \sigma'' \geq 0 \) in a right neighborhood of 0, \( \mathcal{J}_m(t) \) could be nondecreasing even in the presence of some controlled negativity of \( \sigma'' \).
Second, the requirement \( I_m^{-1} \in L^1(0, r_{\phi}) \) is automatically satisfied if \( r_{\phi} < \infty \), and in this case, in [8 Thm.9] we recover S. Markvorsen’s result [15 Thm.1-item ii.] under the slightly weaker hypothesis \( J_m'(t) \geq 0 \).

Finally, in the case where the extrinsic diameter is infinite, the condition that \( I_m^{-1} \) is integrable is equivalent to the stochastic incompleteness and implies a great amount of negative curvature of the \( m \)-dimensional model \( M^m_\sigma \) (and thus of \( N \)).

The result [8 Thm.9] suggests that there should exist a correspondence between results valid for complete, bounded submanifolds of \( N \) and companion results for complete, unbounded immersions \( \varphi: M \to N \), into a manifold \( N \) with a pole, with radial sectional curvatures bounded above as in (1) and such that \( I_m^{-1} \in L^1(0, +\infty) \).

The purpose of this paper is to show that this correspondence does exist for a variety of results, including well known curvature, stochastic and spectral estimates for bounded submanifolds, of which we shall prove counterparts in the unbounded highly negatively curved setting.

2. Statement of the Results

Theorem 1 (Spectral estimates). Let \( \varphi: M \to N \) be an isometric immersion of a complete \( m \)-dimensional Riemannian manifold \( M \) into the complete \( n \)-dimensional Riemannian manifold \( N \) with a pole \( p \in N \). Let \( \rho_N(y) = \text{dist}_N(p, y) \) and assume that the radial sectional curvature of \( N \) satisfies

\[
\text{Sec}_{rad}^N(x) \leq -G(\rho_N(x))
\]

for some smooth, even function \( G \). Assume also that the solution \( \sigma \) of (2) satisfies \( \sigma' \geq 0 \) on \([0, \text{diam}_\varphi(M)]\), that \( J_m \) is nondecreasing in that interval, and

\[
A = \sup_M \frac{|H|}{I_m(\rho_N(\varphi(x)))} \leq 1,
\]

where \( H \) is the mean curvature vector field of \( \varphi \). Then

(a) The bottom of the spectrum of the Laplace-Beltrami operator of \( M \) satisfies the estimate

\[
\lambda^\ast(M) \geq \max \left\{ \frac{(1 - A) I_m^{-1}(r_{\varphi})}{\int_0^{r_{\varphi}} I_m(t)^{-1} dt}, \left(1 - A\right)^2 \inf_{[0, r_{\varphi}]} I_m(r)^2 \frac{4}{4} \right\}
\]

(b) If \( \varphi \) is proper in the geodesic ball \( B_N(r_{\varphi}) \), \( \int_0^{r_{\varphi}} I_m^{-1} dt < +\infty \) and \( A < 1 \) then the spectrum of \( -\Delta_M \) is discrete.

Observe that item (b) of Theorem 1 extends the main result of [6]. It is worth noticing that in the case where \( \sigma'/\sigma \) is nonincreasing, then \( J_m'(r) < 0 \). Indeed, it is easy to check that \( J_m \) satisfies the Riccati equation

\[
J_m' + I_m^2 = (m - 1) \frac{\sigma'}{\sigma} I_m,
\]

and that \( I_m(r) \sim m/r \) as \( r \to 0 \). It follows that \( I_m' < 0 \) and \( I_m > (m - 1)\sigma'/\sigma \) in a right neighborhood of 0, and an easy comparison argument shows that if the right
hand side of the Riccati equation is nonincreasing then \( I'_m < 0 \) where defined. In particular,
\[
\inf_{[\rho_\sigma]} I_m = \lim_{r \to r_\sigma} I_m(r) \geq \lim_{r \to r_\sigma} (m-1) \frac{\sigma'}{\sigma}.
\]
In the special case where \( K_N(x) \leq -k^2 < 0 \), so that \( M^m_{\sigma} = H^m_{-k^2} \) is hyperbolic space of curvature \(-k^2\), we have
\[
\inf_{[0, r_\sigma]} I_m = \begin{cases} 
(m-1)k \coth(kr_\sigma) & \text{if } r_\sigma < +\infty, \\
(m-1)k & \text{if } r_\sigma = +\infty
\end{cases}
\]
Thus, if \( r_\sigma = +\infty \) and \(|H| \leq H_\sigma < (m-1)k \), we have
\[
(1 - A)^2 \inf_{[0, +\infty]} \frac{I_m(r)^2}{4} = \left(1 - \frac{H_\sigma}{(m-1)k}\right)^2 \frac{(m-1)k^2}{4} = \frac{(m-1)k - H_\sigma}{4},
\]
and, in particular, we recover a result by L.-F. Cheung and P.-F. Leung, [11] Theorem 2], and Bessa and Montenegro, [4] Corollary 4.4]. Similarly, in the case where \( r_\sigma < +\infty \), if \( k > 0 \) \( H_\sigma < (m-1)k \coth(kr_\sigma) \), then
\[
(1 - A)^2 \inf_{[0, +\infty]} \frac{I_m(r)^2}{4} = \frac{(m-1)k \coth(kr_\sigma) - H_\sigma}{4},
\]
while if \( k = 0 \) and \( H_\sigma < (m-1)/r_\sigma \), then
\[
(1 - A)^2 \inf_{[0, +\infty]} \frac{I_m(r)^2}{4} = \frac{(m-1)/r_\sigma - H_\sigma}{4},
\]
and we recover results by K. Seo [17].

A closer inspection of the proof of Theorem [11] shows that if we let \( M = N \), then the conclusion holds without having to assume that \( \mathcal{S}_n \) be increasing. Thus we have

**Corollary 2.** Let \( N \) be a complete Riemannian \( n \)-manifold with a pole \( p \) and radial sectional curvature satisfying
\[
\text{Sec}_{\text{rad}}^N(x) \leq -G(\rho_N(x))
\]
where \( G: \mathbb{R} \to \mathbb{R} \) is a smooth even function and the solution \( \sigma \) of the initial value problem (2) satisfies \( \sigma' \geq 0 \) on \([0, +\infty)\). Then
\[
\lambda^*(N) \geq \lambda^*(M^m_{\sigma}) \geq \max \left\{ \frac{\inf_{[0, \sigma]} I_n(r)^2}{4}, \frac{1}{\int_0^\infty I_n^{-1}(r)dr} \right\}.
\]
Moreover, if \( \int_0^\infty I_n^{-1}(r)dr < +\infty \), then the spectrum of \( N \) is purely discrete.

Observe that both alternatives do occur:
\[
\frac{1}{\int_0^\sigma I_m(t)^{-1}dt} \geq \inf_{[0, \sigma]} \frac{I_m(r)^2}{4}
\]
or
\[
\frac{1}{\int_0^\sigma I_m(t)^{-1}dt} \leq \inf_{[0, \sigma]} \frac{I_m(r)^2}{4}
\]
as shown by the examples below. Indeed, consider the 2-dimensional model $M^2_{\sigma}$, with

$$\sigma(t) = \left(\frac{r^7}{2}\right) \exp \frac{r^6}{6}.$$ 

It is easy to show that

$$\int_0^{\infty} I_2(s)^{-1} ds = \frac{3 \cdot 2^{2/3} \sqrt{3}}{\pi} \approx 2.62 > \inf_{[0, \infty)} \frac{I_2(r)^2}{4} = \frac{36}{25} \cdot \left(\frac{2}{3}\right)^{-1/3} \approx 1.64.$$ 

On the other hand, if $M^m_{\sigma} = \mathbb{H}^m(-1)$ is a totally geodesic hyperbolic space in $N = \mathbb{H}^n(-1)$, then

$$\int_0^{\infty} I_m(s)^{-1} ds = 0$$ 

by stochastic completeness, while, as observed above,

$$\inf_{[0, \infty)} \frac{I_m(r)^2}{4} = \frac{(m-1)}{4}.$$ 

We next describe mean curvature estimates which extend previous results valid for bounded immersions obtained, in increasing generality, in [2], [12], [13], [14], and [16].

**Theorem 3** (Mean curvature estimates). Let $\varphi : M \to N$ be an isometric immersion of a stochastically complete, $m$-dimensional Riemannian manifold $M$ into a $n$-dimensional Riemannian manifold $N$ with a pole $o \in N$. Assume that the radial sectional curvature of $N$ satisfies

$$\text{Sec}_{\text{rad}}(x) \leq -G(\rho_N(x))$$

for some smooth, even function $G(t)$ satisfying (3). Assume also that

(i) $\mathcal{I}_m(r)$ is non-decreasing for $r \in (0, r_\varphi]$

(ii) $I_m(r)^{-1} \in L^1(0, r_\varphi).$

Then

$$\sup_M \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \geq 1$$

In particular, if $r_\varphi = +\infty$, ($\varphi$ is unbounded in $N$) then

$$\limsup_{x \to \infty} \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \geq 1.$$ 

If $r_\varphi = +\infty$ and $I_m(r)^{-1} \to 0$ as $r \to +\infty$ then

$$\sup_M |\mathbf{H}| = +\infty.$$ 

**Remark 4.** If the mean curvature $H$ of $M$ is bounded, then either $r_\varphi < +\infty$, and $\varphi$ has bounded image in $N$, or $I_m(r)^{-1} \to 0$ as $r \to +\infty$. An immediate consequence is that, under the above assumptions, an immersed submanifold with bounded mean curvature in $N$ is stochastically incomplete. This completes the picture initiated in
[8, Thm.10], where we proved that, regardless the condition (ii) on the isoperimetric ratio $I_m(\cdot)$, a properly immersed minimal submanifold of $N$ is not $L^1$-Liouville (hence stochastically incomplete).

We also note that, according to Theorem 3, Theorem 1 does not apply if $I_m^{-1}$ is integrable and $M$ is stochastically complete.

**Remark 5.** An application of de L’Hospital rule show that condition $I_m(\cdot)^{-1} \to 0$ as $r \to \infty$ holds provided $\frac{\sigma'}{\sigma} \to +\infty$, which in turn is typical of a super-exponential behavior of $\sigma(\cdot)$. We have already remarked that condition $I_m' \geq 0$ is satisfied provided $\sigma''(\cdot) \geq 0$. Finally, as already recalled, the integrability of $I_m(\cdot)^{-1}$ is equivalent to the stochastic incompleteness of the model $M_\sigma$ and is implied by a sufficiently fast growth of $\sigma$.

3. Preliminaries

Let $M$ be a smooth Riemannian manifold and $\Omega \subset M$ an arbitrary open subset. The fundamental tone $\lambda^*(\Omega)$ of $\Omega$, is defined by

$$\lambda^*(\Omega) = \inf \left\{ \int_{\Omega} | \nabla f |^2, f \in H^1_0(\Omega) \setminus \{0\} \right\},$$

where $H^1_0(\Omega)$ is the completion of $C_\infty^0(\Omega)$ with respect to the norm

$$\| \varphi \|^2_{\Omega} = \int_{\Omega} \varphi^2 + \int_{\Omega} | \nabla \varphi |^2.$$

When $\Omega = M$ is a complete non-compact Riemannian manifold, the fundamental tone $\lambda^*(M)$ coincides with the bottom $\inf\Sigma(-\Delta)$ of the $L^2$-spectrum $\Sigma(-\Delta) \subset [0, \infty)$ of the unique self-adjoint extension of the Laplacian $\Delta$ acting on $C_\infty^0(M)$ also denoted by $\Delta$. When $\Omega$ is compact with boundary $\partial \Omega$, then the fundamental tone is the bottom of the $L^2$-spectrum of the Friedrichs extension of $-\Delta$ initially defined on $C_\infty^0(\Omega)$. Moreover, there exists $u \in C_\infty^0(\Omega) \cap H^1_0(\Omega)$, positive in $\Omega$, satisfying $\Delta u + \lambda^*(\Omega) u = 0$, ($u_{|\partial \Omega} = 0$ if $\partial \Omega \neq \emptyset$ and piecewise smooth). The spectrum decomposes as $\Sigma(-\Delta) = \Sigma_p(-\Delta) \cup \Sigma_{ess}(-\Delta)$ where $\Sigma_p(-\Delta)$ is formed by eigenvalues with finite multiplicity and $\Sigma_{ess}(-\Delta)$ is formed by accumulation points of the spectrum and by the eigenvalues with infinite multiplicity. It is said that $M$ has discrete spectrum if $\Sigma_{ess}(-\Delta) = \emptyset$ and that $M$ has purely continuous spectrum if $\Sigma_p(-\Delta) = \emptyset$. It is well known that for every exhaustion of $M$ by relatively compact open sets $\{K_j\}$ with boundary, one has, $\inf \Sigma_{ess}(-\Delta) = \lim_{j \to +\infty} \lambda^*(M \setminus K_j)$. It follows that $-\Delta$ has pure discrete spectrum if and only if

$$\lim_{j \to +\infty} \lambda^*(M \setminus K_j) = \infty.$$

The following two lemmas are useful to obtain lower bounds for the fundamental tones of open sets of Riemannian manifolds.
Lemma 6 (4). Let \( \Omega \subset M \) be an open subset of a Riemannian manifold \( M \). Then the fundamental tone of \( \Omega \) is bounded below by

\[
\lambda^*(\Omega) \geq \frac{c(\Omega)^2}{4},
\]

where \( c(\Omega) = \sup \left\{ \inf_{\Omega} \text{div} X : X \in \mathcal{X}^\infty(\Omega), \text{div} X \geq 0 \right\} \) and \( \mathcal{X}^\infty(\Omega) \) is the set of all smooth vector fields in \( \Omega \).

Lemma 7 (Barta [3], [5]). Let \( \Omega \subset M \) be an open subset of a Riemannian manifold \( M \) and \( u : \Omega \to \mathbb{R} \) be a smooth positive function. Then

\[
\lambda^*(\Omega) \geq \inf_{\Omega} \left[ -\frac{\Delta u}{u} \right].
\]

4. PROOF OF THE RESULTS

Proof of Theorem 1. Recall that if \( \varphi : M \to N \) is an isometric immersion and \( g : N \to \mathbb{R} \) and \( F : \mathbb{R} \to \mathbb{R} \) are smooth functions, then for every \( X \in T_xM \) we have

\[
\text{Hess}(F \circ g \circ \varphi)(X, X) = F''(g(\varphi(x))) \langle \nabla^N g, d\varphi X \rangle^2 \\
+ F'(g(\varphi(x))) \left[ \text{Hess}^N g(d\varphi X, d\varphi X) + \langle \nabla^N g, II(X, X) \rangle \right].
\]

If \( g = \rho_N \) is the distance function, then the assumption on the sectional curvature of \( N \) implies

\[
\text{Hess}_N \rho_N(Y, Y) \geq \frac{\sigma' \left[ \langle Y, Y \rangle - \langle \nabla^N \rho_N, Y \rangle^2 \right]}{\sigma}
\]
on \( B_N(r_0) \).

Assuming that \( F' \geq 0 \), letting \( \{X_i\}_{i=1}^m \) be an orthonormal basis of \( T_xM \) and setting \( \rho_x = \rho_N(\varphi(x)) \), we obtain

\[
\Delta(F \circ \rho \circ \varphi)(x) \geq m \left( F' \frac{\sigma'}{\sigma}(\rho_x) + (F'' - F' \frac{\sigma'}{\sigma})(\rho_x) \sum_{i=1}^m \langle \nabla^N \rho_N, d\varphi X_i \rangle^2 \right)
\]

\[
+ F'(\rho_x) \langle \nabla^N \rho_N, H \rangle.
\]

Let

\[
F(r) = \int_0^r \frac{\int_0^s \sigma^{m-1}(s) \, ds}{\sigma^{m-1}(t)} \, dt,
\]

so that

\[
F'(r) = \int_0^r \frac{\sigma^{m-1}(s) \, ds}{\sigma^{m-1}(r)} = I_m^{-1}(r) > 0 \quad \text{and} \quad F''(r) = 1 - (m - 1) \frac{\sigma'}{\sigma} F'(r),
\]

which, inserted into the last inequality, yield

\[
\Delta(F \circ \rho \circ \varphi)(x) \geq m \left( F' \frac{\sigma'}{\sigma}(\rho_x) + \left[ (1 - m \frac{\sigma'}{\sigma} F'_o(\rho_x)) \sum_{i=1}^m \langle \nabla^N \rho, d\varphi X_i \rangle^2 \right] 
\]

\[
- F'(\rho_x) |H(\varphi(x))|.
\]
We complete \( \{ d\varphi X_i \}_{i=1}^m \) to an orthonormal basis \( \{ d\varphi X_i \}_{i=1}^m \cup \{ Y_j \}_{j=m+1}^n \) on \( T_{\varphi(x)} N \), and note that
\[
\sum_i \langle \nabla^N \rho, d\varphi X_i \rangle^2 + \sum_j \langle \nabla^N \rho, Y_j \rangle^2 = 1.
\]
Inserting this into the above inequality and using the assumption \( \mathcal{J}_m'(t) \geq 0 \) in the form
\[
m \frac{\sigma' \sigma}{\sigma} = m \frac{\sigma' \int_0^t \sigma^{-1}(s) ds}{\sigma^{-1}(t)} \geq 1
\]
we finally obtain
\[
\Delta (F \circ \rho \circ \varphi)(x) \geq 1 + \left[ m \left( F' \frac{\sigma'}{\sigma} \right)(\rho_x) - 1 \right] \sum_j \langle \nabla^N \rho, Y_j \rangle^2 - F'(\rho_x) |\mathbf{H}(\varphi(x))| \geq 1 - F'(\rho_x) |\mathbf{H}(\varphi(x))|.
\]
(7)
Thus, if \( X = \nabla (F \circ \rho \circ \varphi) \), we have
\[
div_M X \geq 1 - \sup_M F'(\rho_x) |\mathbf{H}(x)| = 1 - A
\]
and
\[
|X| \leq F'(\rho_x) = I_{m-1}(\rho_x) \leq \frac{1}{\inf_{[0,r\varphi]} I_m(r)},
\]
and we conclude that
\[
\lambda^*(M) \geq (1 - A)^2 \inf_{[0,r\varphi]} \left( I_m(r) \right)^2.
\]
The estimate
\[
\lambda^*(M) \geq \frac{1 - A}{\int_0^{r\varphi} I_m^{-1}(t) dt}
\]
in (a) is an application of Barta’s Theorem. We consider first the case where \( r\varphi = +\infty \), and assume that \( I_{m-1} \in L^1([0, +\infty)) \) for otherwise the estimate is trivial. Define
\[
\bar{F} = \int_r^{+\infty} I_{m-1}(t) dt,
\]
and let \( u = \bar{F} \circ \rho_N \circ \varphi \). Then \( u \) is positive on \( M \) and, since
\[
\bar{F}'(r) = - I_{m-1}^{-1}(r) = - \frac{\sigma' \sigma^{m-1} dt}{\sigma^{m-1}(r)} < 0 \quad \text{and} \quad \bar{F}''(r) = -1 - (m-1) \frac{\sigma'}{\sigma} \bar{F}'(r),
\]
a computation similar to that performed in the first part of the proof shows that
\[
-\Delta_M u \geq 1 + |\mathbf{H}| \bar{F}'(\rho_x) = 1 - \frac{|\mathbf{H}|}{I_{m}(r)} \geq 1 - A
\]
and it follows from Barta’s Theorem that
\[
\lambda^*(M) \geq \inf_M \left( \frac{-\Delta u}{u} \right) \geq \frac{1 - A}{\int_0^{r\varphi} I_m^{-1}(t) dt},
\]
as required.
The case where $r_\phi < +\infty$ is similar. Since to apply Barta’s theorem we need $u$ to be positive, we note that our assumptions imply that $I_m$ is well defined and positive in $[0, r_\phi + \varepsilon]$ for every $\varepsilon > 0$ sufficiently small. Next we let

$$\tilde{F}_\varepsilon(r) = \int_r^{r_\phi + \varepsilon} I_m^{-1} dt,$$

and define $u_\varepsilon$ accordingly. Arguing as above shows that

$$\lambda^+(M) \geq \inf_M \left( \frac{-\Delta u_\varepsilon}{u_\varepsilon} \right) \geq \frac{1 - A_\varepsilon}{\int_0^{r_\phi + \varepsilon} I_m^{-1} dt},$$

where

$$A_\varepsilon = \sup_M |\tilde{F}'_\varepsilon(\rho_x)H|.$$

The conclusion now follows letting $\varepsilon \to 0$.

Finally, if $\phi$ is proper, $I_m^{-1}$ is integrable on $[0, \infty)$ and $A < 1$, then the function $-u$ is bounded, proper, and satisfies

$$\Delta(-u) \geq 1 - A > 0$$

on $M$. Therefore, in the terminology of [9], it is a weak maximum principle violating exhaustion function, and the discreteness of the spectrum of $M$ follows from [9, Theorem 32].

**Proof of Theorem 3.** We maintain the notation of the first part of the proof of Theorem [11] and let $v = F \circ \rho \circ \phi$. Then $v$ is bounded above by the assumption that $I_m^{-1} \in L^1([0, r_\phi])$ and, by (7), it satisfies

$$\Delta_M v \geq 1 - |F'(\rho_x)| |H(x)| = 1 - \frac{|H(x)|}{I_m(\rho_x)}.$$

Since $M$ is assumed to be stochastically complete, by the weak maximum principle at infinity there exists a sequence $\{x_n\}$ in $M$ such that

$$\lim_{n} v(x_n) = \sup_{M} v \quad \text{and} \quad \liminf_{n} \Delta_M v(x_n) \leq 0.$$

Since $F$ is increasing, this implies that $\rho_{x_n} \to r_\phi$. In particular, if $\phi$ is unbounded, we conclude that

$$\liminf_{x \to +\infty} \left( 1 - \frac{|H(x)|}{I_m(\rho_x)} \right) \leq 0,$$

that is, $\limsup_{x \to \infty} \frac{|H(x)|}{I_m(\rho_x)} \geq 1$.

In the case where $\phi$ is bounded, we can still conclude that

$$\inf_{M} \left( 1 - \frac{|H(x)|}{I_m(\rho_x)} \right) \leq 0,$$

that is, $\sup_{M} \frac{|H(x)|}{I_m(\rho_x)} \geq 1$. 

\[\square\]
5. IMMERSIONS INTO PRODUCTS

The aim of this section is to prove versions of Theorems 1 and 3 for immersions into a product manifold $N \times L$, where the factor $N$ satisfies condition (1) in the Introduction. This clearly relaxes the curvature conditions imposed on the target manifold. As a counterpart, we need to strengthen the assumptions replacing conditions on $\mathcal{I}_m$ and $I_M$ with analogous conditions on $\mathcal{I}_{m-1}$ and $I_{m-1}$ respectively.

In some sense, we make up for the presence of the factor $L$ by imposing more negative curvature conditions on the factor $N$.

Theorem 8. Let $\varphi$ be an isometric immersion of a complete Riemannian manifold $M$ of dimension $m$ into the product $N \times \mathbb{R}^l$, where $L$ and $N$ are complete Riemannian manifolds of dimension $n$ and $l$, respectively. Assume that $N$ has a pole $p \in N$ and that its radial sectional curvature satisfies

$$ K_N(x) \leq -G(\rho_N(x)) $$

where $\rho_N(y) = \text{dist}_N(p,y)$ and $G$ is a smooth, even function on $\mathbb{R}$. Assume that $m \geq l + 1$ and that the solution $\sigma$ of (2) satisfies $\sigma' \geq 0$ on $[0, r_{\pi_N \varphi}]$, where $\pi_N$ is the projection onto $N$. Suppose further that $\mathcal{I}_{m-1}$ is nondecreasing in that interval, and that

$$ A = \sup_M \left\{ \frac{|H|_i(x)}{I_{m-1}(\rho_N(\pi_N \varphi(x)))} \right\} \leq 1, $$

where $H$ is the mean curvature vector field of $\varphi$. Then

(a) The bottom of the spectrum of the Laplace-Beltrami operator of $M$ satisfies the estimate

$$ \lambda_1^*(M) \geq \max \left\{ (1 - A)^2 \inf_{[0, r_{\pi_N \varphi}]} \frac{I_{m-1}(r)}{4}, (1 - A) \frac{1}{\int_{0}^{r_{\pi_N \varphi}} I_{m-1}(r) \, dr} \right\} $$

(b) If $\pi_N \circ \varphi$ is proper, $\int_0^{r_{\pi_N \varphi}} I_{m-1} \, dt < +\infty$ and $A < 1$ then the spectrum of $-\Delta_M$ is discrete.

Proof. We continue to keep the notation of the proof of Theorem 1. Assuming that $F$ is smooth and satisfies $F' \geq 0$, we consider the function $F \circ \rho_N \circ \pi_N \circ \varphi$, and argue as in Theorem 1 to obtain

$$ \Delta (F \circ \rho \circ \pi \circ \varphi)(x) \geq \left( F'' - F' \frac{\sigma'}{\sigma} \right)(\rho_i) \sum_{i=1}^m \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 $$

$$ + \left( F' \frac{\sigma'}{\sigma} \right)(\rho_i) \sum_{i=1}^m d\pi_N d\varphi(X_i)^2 + F' (\rho_i) \langle \nabla^N \rho_N, d\pi_N H \rangle. $$

Choosing

$$ F(r) = \int_0^r \frac{\int_0^s \sigma^{m-1-2} \, ds \, \sigma^{m-1-2}}{\sigma^{m-1-2} \, dt} \, dt, $$

and inserting the identities

$$ F'(r) = \frac{\int_0^r \sigma^{m-1-1}(s) \, ds}{\sigma^{m-1-1}(r)} = I_{m-1-}^{-1}(r) > 0 \quad \text{and} \quad F''(r) = 1 - (m - l - 1) \frac{\sigma'}{\sigma} F'(r), $$

we obtain...
into the last inequality yields
\[
\Delta_M(F \circ \rho \circ \pi \circ \varphi)(x) \geq \left[ 1 - (m - l - 1)F' \sigma' \right](\rho_x) \sum_{i=1}^{m} \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 + \left( F' \sigma' \right)(\rho_x) \sum_{i=1}^{m} |d\pi_N d\varphi X_i|^2 + F'(\rho_x) \langle \nabla^N \rho_N, d\pi_N H \rangle.
\]

Since \( d\pi_N \) is the orthogonal projection onto \( T_xN \) which is of codimension \( l \) in \( T_{\langle \cdot, \cdot \rangle}(N \times L) \), and \( d\varphi(X_i) \) are \( m \)-orthonormal vectors, it is easy to verify that
\[
\sum_{i=1}^{m} |d\pi_N d\varphi X_i|^2 \geq m - l = (m - l)|\nabla^N \rho_N|^2.
\]

Using this, the fact that \( |\nabla^N \rho|^2 = 1 \geq \sum_i \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 \), and the assumption that \( J_{m-l} \) is nondecreasing in the form
\[
(m - l) \frac{\sigma'}{\sigma} F'(r) \geq 1
\]
we conclude that the right hand side of the above inequality is bounded below by
\[
(m - l)F' \sigma' \left( \rho_x \right) \left[ 1 - \sum_{i=1}^{m} \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 \right] + \sum_{i=1}^{m} \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 - F'(\rho_x)|H| \geq 1 - F'(\rho_x)|H|
\]
Thus, if \( X = \nabla(F \circ \rho_N \circ \pi_N \circ \varphi) \), then
\[
\text{div}_M X \geq 1 - F'(\rho_x)|H|,
\]
and the estimate
\[
(1 - A)^2 \inf_{[0, r_{\pi_N \varphi}]} I_{m-l}(r)^2 \geq \frac{4}{4}
\]
follows arguing as in Theorem \[1\]

In a completely similar manner, the second estimate in a) follows applying Bart's theorem to the function \( u = \tilde{F} \circ \rho_N \circ \pi_N \circ \varphi \) with
\[
\tilde{F}(r) = \int_r^{\text{diam}(\pi_N \varphi(M))} I_{m-l}(t)^{-1} dt,
\]
and conclusion b) in the statement is obtained noticing that if \( \pi_N \circ \varphi \) is proper, and \( I_{m-l}^{-1} \) is integrable, then \( -u \) is a a weak maximum principle violating exhaustion function.

In a similar fashion we have the following analogue of Theorem \[3\] which complements previous results by L. J. Alias, G.P. Bessa and M. Dacjzer, \[1\].

**Theorem 9.** Let \( \varphi : M \to N \) be an isometric immersion of a stochastically complete, \( m \)-dimensional Riemannian manifold \( M \) into the product \( N \times L \), where \( N \) and \( L \) are complete Riemannian manifolds of dimension \( n \) and \( l \) respectively, with \( m \geq l + 1 \), and \( N \) satisfies the conditions listed in the statement of Theorem \[5\] Assume also that

(i) \( J_{m-l}(r) \) is non-decreasing on \([0, \pi_N(\varphi(M))]\),
(ii) \( I_{m-1}(r)^{-1} \in L^1(\mathbb{R}^+) \) if \( \pi_N \varphi \) is unbounded. Then
\[
\sup_M \frac{|H(x)|}{I_{m-1}(\rho_x)} \geq 1
\]
In particular, if \( r_{\pi_N \varphi} = +\infty \), \( (\pi_N \varphi \) is unbounded in \( N \)) then
\[
\limsup_{x \to \infty} \frac{|H(x)|}{I_{m-1}(\rho_x)} \geq 1.
\]
If \( r_{\pi_N \varphi} = +\infty \) and \( I_{m-1}(r)^{-1} \to 0 \) as \( r \to +\infty \) then
\[
\sup_M |H| = +\infty.
\]

We conclude this section noting that the above arguments can be used to give the following version for products of the already cited mean exit time comparison results obtained in [15].

**Theorem 10** (Stochastic estimates). Let \( M, N \) and \( L \) be complete Riemannian manifolds of dimensions \( m, n \) and \( l \), as in the statement of Theorem 9 with \( m \geq l + 1 \). Let \( \varphi : M \to N \times L \) be a minimal immersion, and assume that (i) and (ii) in the statement of Theorem 9 hold. Then \( M \) is not \( L^1 \)-Liouville.

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