Applying chiral perturbation to twisted mass lattice QCD

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We have explored twisted mass LQCD (tmLQCD) analytically using chiral perturbation theory, including discretization effects up to $O(a^2)$, and working at next-to-leading (NLO) order in the chiral expansion. In particular we have studied the vacuum structure, and calculated the dependence of pion masses and decay constants on the quark mass, twisting angle and lattice spacing. We give explicit examples for quantities that both are and are not automatically improved at maximal twisting.

1. Introduction

There has been much interest recently in the twisted mass formulation of lattice QCD [1,2] (see Ref. [3] for a recent review). We study here the importance of the symmetry breaking inherent in tmLQCD, which can be done analytically using the chiral effective theory including effects of discretization.

2. The effective chiral Lagrangian

To derive an effective continuum chiral theory for tmLQCD, we follow the two-step procedure of Ref. [4]. We first write down an effective continuum Lagrangian at the quark level that describes the long distance physics of the underlying lattice theory, and then match it onto an effective chiral Lagrangian. The details of this procedure are given in Ref. [5].

The effective continuum Lagrangian is constrained to be invariant under the symmetries of the lattice theory. It has form:

$$L_{eff} = L_g + \bar{\psi}(\gamma \cdot D + m + i \tau_3 \mu)\psi + b_1 \bar{\psi} \sigma_{\mu \nu} F_{\mu \nu} \psi + O(a^2),$$

where $L_g$ is the continuum gluon Lagrangian, $m$ is the physical quark mass, defined as usual by

$$m = Z_m (m_0 - \bar{m}_c) / a,$$

and $\mu$ is the physical twisted mass defined by

$$\mu = Z_{\mu} (\mu_0 - \bar{\mu}_c) / a,$$

$m_0$ and $\mu_0$ are bare normal and twisted mass parameters respectively. Note that the lattice symmetries forbid additive renormalization of $\mu_0$.

In writing down (1), we have dropped terms that vanish by equations of motion. We have also dropped terms of order higher than quadratic in $\{m, \mu, a\}$ (factors of $\Lambda_{QCD}$ are implicit here and henceforth) in anticipation of the power-counting in our chiral effective theory. The net result is that (1) differs from that for the standard Wilson theory given in [4] only by the addition of a twisted mass term.

Next we write down a generalization of the continuum chiral Lagrangian that includes the effects of the Pauli term. We use the power counting scheme:

$$1 \gg \{m, p^2, a\} \gg \{m^2, mp^2, p^4, am, ap^2, a^2\},$$

where $m$ is a generic mass parameter that can be either $m$ or $\mu$. The chiral Lagrangian is built from the standard $SU(2)$ matrix-valued field $\Sigma$. It can be obtained from the quark Lagrangian, (1), by a standard spurion analysis. We must introduce a spurion matrix $\hat{A}$ for the Pauli term, as well as the usual spurion $\chi$ for the mass terms. Both transform in the same way as the $\Sigma$ field. At the end of the analysis the spurions are set to their respective constant values:

$$\chi \rightarrow 2 B_0 (m + i \tau_3 \mu) = \hat{m} + i \tau_3 \hat{\mu},$$

$$\hat{A} \rightarrow 2 W_0 a = \hat{a},$$

where $B_0 = Z_{\mu} / Z_m$, $W_0 = Z_m / Z_{\mu}$, $Z_m$, $Z_{\mu}$ are renormalization factors.
where $B_0$ and $W_0$ are unknown dimensionful parameters. The difference from the standard construction comes only from the inclusion of the $\mu$ term in the constant value of $\chi$. Thus, we can read off the form of the chiral Lagrangian for tmLQCD from Ref. [6], where the Lagrangian for untwisted Wilson fermions was worked out to quadratic order in our expansion. The only extension we make is to include sources for currents and densities.

3. The phase diagram of tmLQCD

Using the chiral Lagrangian, we have extended the analysis of Ref. [4] into the twisted mass plane. The details of the analysis are given in Ref. [5]. We record here the main results.

In the region $a \gg m' \sim \mu \sim a^2$ where the Aoki phase occurs for untwisted Wilson fermions ($m' = m + aW_0/B_0$ is the shifted mass due to effect of the Pauli term, as noted in Ref. [4]), the potential energy is dominated by:

$$V_\chi = -\frac{c_1}{4} \text{Tr}(\Sigma + \Sigma^\dagger) + \frac{c_3}{4} \text{Tr}[i(\Sigma - \Sigma^\dagger)\tau_3]$$

$$+ \frac{c_2}{16} [\text{Tr}(\Sigma + \Sigma^\dagger)]^2,$$ (4)

where $c_1 \sim m'$, $c_3 \sim \mu$, and $c_2 \sim a^2$ are all of the same order. As in the untwisted case, it is the competition between the leading order mass terms and the $a^2$ term at NLO that can lead to interesting phase structure.

To determine the vacuum structure we find the condensate, $\Sigma_0$, which minimizes the potential. Parametrizing using $\Sigma_0 = A_m + iB_m \cdot \tau$, where $A_m^2 + B_m^2 = 1$, the functional forms of $A_m$ and $B_m$ are found by solving a quartic equation. The minimum is found to lie in the $\Sigma_3$ plane in flavor space.

The vacuum phase structure depends on the sign of the coefficient $c_2$, which is a linear combination of low-energy constants (LEC’s) in the chiral Lagrangian. Fig. 1 illustrates the two possibilities.

For $c_2 > 0$, the usual Aoki phase (first noted in Ref. [4]) occurs, but is confined to a short segment on the untwisted axis. In the Aoki phase segment, the charged pions are massless but the neutral pion is not. All three pions become massless only at the second-order endpoints.

For $c_2 < 0$, there is a first-order transition line extending a distance $\Delta \mu \sim a^2$ in the twisted direction. The charged pions are not massless anywhere on this first-order line (including the endpoints), while the neutral pion becomes massless at the two second-order endpoints. In fact the neutral pion mass for $c_2 < 0$ has the same dependence on $\mu$ along the first-order transition line as that on $m$ along the Aoki phase line for $c_2 > 0$. Evidence for the phase structure in the $c_2 < 0$ scenario has recently been seen in numerical simulations [10].
4. Pion masses and pion decay constant

In the region \( m \sim \mu \sim a \gg a^2 \sim am \sim m^2 \) away from the “Aoki phase” (where most simulations will be done), we find that the splitting of pion masses due to parity-flavor breaking first occurs at NLO:

\[
\Delta m^2_\pi \equiv m^2_{\pi_3} - m^2_{\pi_{1,2}} = \frac{C \hat{a}^2 \mu^2}{(\hat{m} + \hat{a})^2 + \mu^2},
\]

where \( C \) is a linear combination of the LEC’s. The mass splitting is \( \propto a^2 \) for any values of \( \hat{m} \) and \( \mu \) within the region we are considering, not just at maximal twisting. For \( \hat{m}, \mu \gg \hat{a} \), it becomes \( C \hat{a}^2 \sin^2 \omega \), the form expected from the symmetries of tmLQCD [2]. Note that at NLO, there are only tree level contributions to the mass splitting, whose full expression agrees with that of Ref. [11].

For pion decay constants, we find no splitting un-expectedly, and the pion decay constants will be given elsewhere [12].

5. Current matrix elements

Once the twisting angle is defined, the physical untwisted currents can be obtained simply from the appropriate functional derivative of the effective chiral Lagrangian with respect to the sources. We use the non-perturbative definition given by:

\[
\tan \omega = \langle \partial_\mu V^2_\mu P^1 \rangle / \langle \partial_\mu A^1_\mu P^1 \rangle,
\]

where the defining correlators are evaluated in the chiral theory using single pion contributions. We find the twisting angle is related to the parameters in the chiral theory by:

\[
\tan \omega = B_{3,m}/(A_m + 4W_{10}\hat{a}/f^2),
\]

where \( W_{10} \) is one of the unknown LEC’s.

As an example, consider the physical untwisted axial vector current:

\[
[A^a_\mu](\omega) = \left( \cos(\omega)(1 - \delta^{3a}) + \delta^{3a} \right) A^a_\mu + \sin(\omega) \varepsilon^{ab\mu} V^b_\mu.
\]

The effect of parity-flavor breaking can be seen when \([A^a_\mu] \) is expanded in terms of the pion fields. There are terms with even number of pion fields that are usually forbidden by parity, and the coefficients of the expansion depend on the flavor index.

The physical, single pion matrix element of the axial vector current, \([0][A^a_\mu][\pi_b] \), defines the pion decay constant, is an example of a quantity that is automatically \( O(a) \) improved. But the unphysical, two-pion matrix elements are not (and are not expected to be, according to the general theory of Ref. [2]). For instance, let \( q = p_2 - p_1 \ll m^2_\pi \), then for \( a = 1, 2 \), we find:

\[
\langle \pi_a(p_1)[A^a_\mu][\pi_3(p_2)] = -i(p_1 + p_2)B_{3,m}\hat{a}D_1
\]

\[
+ iq \left[ 4B_{3,m}\hat{a}D_2 - \frac{8B_{3,m}\hat{a}}{f^2}(\hat{a}D_3 + \hat{a}^3/m_\pi^2 D_4) \right],
\]

where the \( D_i \)’s are linear combinations of LEC’s. The \( 1/m_\pi^2 \) term comes from a diagram which contains a three-pion vertex that arises from parity-flavor breaking. Since the continuum limit must be taken before the chiral limit to avoid entering the Aoki phase, there is no divergence.

A potential use of the unphysical two-pion current matrix elements is to provide a way of calculating the new LEC’s introduced by discretization into the chiral Lagrangian.

REFERENCES

1. R. Frezzotti et al., JHEP 0108, 058 (2001).
2. R. Frezzotti and G. Rossi, hep-lat/0306014.
3. R. Frezzotti, “Twisted-mass QCD,” plenary talk at Lattice 2004.
4. S. Sharpe and R. Singleton, Jr., Phys. Rev. D 58, 074501 (1998).
5. S. Sharpe and J. Wu, hep-lat/0407025.
6. O. Bär, G. Rupak, and N. Shoresh, hep-lat/0306021.
7. G. Münster and C. Schmidt, Europhys. Lett. 66, 652 (2004).
8. G. Münster, hep-lat/0407006.
9. S. Aoki, Phys. Rev. D 30, 2653 (1984).
10. F. Farchioni et al., hep-lat/0406039.
11. L. Scorzato, hep-lat/0407023.
12. S. Sharpe and J. Wu, in preparation.