Wall Crossing and Instantons in Compactified Gauge Theory

Heng-Yu Chen\(^1\), Nick Dorey\(^2\) and Kirill Petunin\(^2\)

\(^1\)Department of Physics, University of Wisconsin, Madison, WI 53706, USA.

and

\(^2\)DAMTP, Centre for Mathematical Sciences
University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, UK

Abstract
We calculate the leading weak-coupling instanton contribution to the moduli-space metric of \(\mathcal{N} = 2\) supersymmetric Yang-Mills theory with gauge group \(SU(2)\) compactified on \(\mathbb{R}^3 \times S^1\). The results are in precise agreement with the semiclassical expansion of the exact metric recently conjectured by Gaiotto, Moore and Neitzke based on considerations related to wall-crossing in the corresponding four-dimensional theory.
1 Introduction

Supersymmetric gauge theories provide a setting where interesting non-perturbative phenomena such as confinement and chiral symmetry breaking can be described with exact analytic formulae [1, 2]. Results of this kind rely on the powerful constraints of supersymmetry and also on conjectured duality properties of the theories in question. Often the resulting exact formulae can be expanded at weak coupling where they provide a precise prediction for instanton effects which can then be compared to direct semiclassical computations in weakly coupled gauge theory [3]. Such calculations then provide a strong check on conjectured dualities and other assumptions underlying the exact analysis.

For the case of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in four dimensions, an exact description of the low-energy effective action and of the BPS spectrum was provided by Seiberg and Witten (SW) in [1]. More precisely, for a given value of the electric and magnetic charges, the SW solution determines the mass of the corresponding BPS state if it is present in the theory. However, except in the simplest cases, the question of which states are present in the spectrum at a given point of the moduli space remained open. In the weak coupling regime the question can be answered by a semiclassical analysis of the spectrum of monopoles and dyons. However, the moduli space contains walls of real codimension one on which BPS states become marginally stable and can decay. Recently Kontsevich and Soibelman [7, 8] proposed a precise algorithm which determines how the spectrum changes across these curves of marginal stability in the moduli space. Given the weak coupling spectrum, their results effectively determine the spectrum at any point in the moduli space.

Another perspective on the wall-crossing conjecture of [7, 8] is obtained by considering the behaviour of the corresponding theory compactified down to three dimensions on a circle. After compactification, the BPS states of the four-dimensional theory yield distinct instanton corrections to the hyper-Kähler metric on the moduli space of the compactified theory. On general grounds this metric is expected to be smooth and at first sight this seems to be in contradiction with the discontinuous changes in the four-dimensional BPS spectrum across walls of marginal stability. A remarkable resolution of this puzzle has recently been suggested by Gaiotto, Neitzke and Moore (GMN) [9]. They showed that the a smooth metric can be obtained precisely when the spectrum obeys the Kontsevich-Soibelman wall crossing formula. Indeed, this condition leads to a set of integral equations which determine the moduli-space metric exactly for any value of $\mathcal{N} = 2$.

---

1Direct semiclassical tests of the Seiberg-Witten solution itself were initiated in [4, 5] and eventually completed in [6].

2Here the instantons refered to are really instantons of the low-energy effective theory including all quantum corrections. They should be distinguished from the instantons appearing in the semiclassical calculation presented below which correspond to finite-action solutions of the equations of motion of the underlying gauge theory.
the compactification radius! The main purpose of this paper is to investigate the weak-coupling contents of their results, and to test these against first-principles semiclassical calculations.

In the following we will focus on the simplest case, of \( \mathcal{N} = 2 \) SUSY Yang-Mills theory with gauge group \( SU(2) \) compactified to three dimensions on circle of radius \( R \). The moduli space is parameterised by a single scalar VEV, \( a \), and the dynamical scale of the corresponding four-dimensional theory is denoted \( \Lambda \). In the weak-coupling limit, \( |a| \gg |\Lambda| \) we will see that the GMN results give rise to a sum over semi-classical instanton contributions carrying magnetic charge. Each instanton contribution has a prefactor which is a complicated function of dimensionless parameter \( |a|R \). As found in previous investigations of instanton effects in compactified gauge theory \([13, 14]\), the leading semiclassical contribution can be expanded in two distinct ways. For \( |a|R \gg 1 \) the result can be expanded as a sum over the contributions of magnetic monopoles and Julia-Zee dyons regarded as classical solutions of finite Euclidean action on \( \mathbb{R}^3 \times S^1 \). We focus on the contributions of unit magnetic charge but arbitrary electric charge. We reproduce these terms in the GMN result by a direct semiclassical calculation. An important feature first noticed in \([15]\) is that, unlike similar calculations in four dimensions, the functional determinants corresponding to fluctuations of the bose and fermi fields do not cancel. We evaluate the ratio of fluctuation determinants from first principles and find that it precisely reproduces the prefactors appearing in the weak-coupling expansion of the GMN results. Given the relation between the GMN integral equations and the Kontsevich-Soibelman conjecture, this can also be regarded as a first-principles test of the latter.

As one goes towards smaller values of the dimensionless radius \( |a|R \), the sum over the electric charges of the semiclassical dyons diverges and requires Poisson resummation. As in \([13, 14]\), the resulting series also admits an interpretation in terms of classical configurations of finite action. The relevant configurations are an infinite tower of twisted monopoles obtained by applying large gauge transformations to the BPS monopole \([16]\). Finally, in the three-dimensional limit, \( |a|R \rightarrow 0 \), the twisted monopoles decouple and only the contribution of a single BPS monopole remains. In the context of \( \mathcal{N} = 4 \) SUSY Yang-Mills realised on the world volume of two parallel D3 branes in IIB string theory, the relation between the two corresponding expansions can be understood as T-duality to an equivalent configuration of D2 branes in the IIA theory. It would be interesting to find a similar stringy interpretation in the present context.

Another interesting question concerns the three-dimensional limit of the GMN results. Taking the limit \( |a|R \rightarrow 0 \) with the effective three dimensional coupling held fixed, we obtain \( \mathcal{N} = 4 \) SUSY Yang-Mills in three dimensions with gauge group \( SU(2) \). The exact metric on the moduli space of this theory was conjectured to coincide with the Atiyah-Hitchin metric in \([17]\). This

\[3\text{Strictly speaking the full spectrum is already known in this simple case and agrees with the conjecture of [7,8]. However the resulting formula of [9] for the hyper-Kähler metric on the moduli space of the compactified theory still yields non-trivial predictions at weak coupling. In addition, we expect our calculation to generalise straightforwardly to cases with larger gauge groups and/or additional matter where the wall-crossing formula remains conjectural.} \]
proposal was then tested against an explicit semiclassical calculation of the one-monopole contribution [15]. Taking the three-dimensional limit of the full GMN integral equations is difficult for reasons explained in [10], and it is not straightforward to recover the Atiyah-Hitchin metric in this approach. On the other hand, the leading semiclassical contribution to the GMN metric studied here can easily be continued to three-dimensions after the Poisson resummation described above. We show that it reproduces the one-monopole contribution to the Atiyah-Hitchin metric including the correct numerical prefactor. As explained in [15], this coefficient together with the constraints of supersymmetry and other global symmetries uniquely determines the Atiyah-Hitchin metric.

2 Moduli Space, Wall Crossing Formula and BPS Spectrum

To begin, let us review the relevant results in [9] and set the subsequent notations. We consider the pure $\mathcal{N} = 2$ supersymmetric gauge theory in $d = 4$ dimensions with gauge group $G = SU(2)$ and dynamical scale $\Lambda$. This theory has a Coulomb branch $\mathcal{B}$ where the complex adjoint scalar field in the vector multiplet, $\phi$, acquires a VEV and the $SU(2)$ gauge group is broken down to $U(1)$. The massless bosonic fields on the Coulomb branch consist of a $U(1)$ gauge field and a complex scalar $a$ whose VEV (also denoted as $a$) parametrises $\mathcal{B}$ as a complex manifold. It is also convenient to define a gauge-invariant order parameter $u = \langle \text{Tr} \phi^2 \rangle$ which provides a globally-defined coordinate on $\mathcal{B}$.

The spectrum of the theory on the Coulomb branch $\mathcal{B}$ contains BPS states $\gamma = (n_e, n_m)$ carrying electric and magnetic charges, $n_e$ and $n_m$, under the unbroken $U(1)$. Each BPS state carries central charge $Z_\gamma(u)$

$$Z_\gamma(u) = n_e a(u) + n_m a_D(u),$$

which lies on a lattice in the complex plane with periods $a$ and $a_D$. The magnetic period is determined by the prepotential $F(a)$ [1] via

$$a_D = \frac{\partial F(a)}{\partial a}. \quad (2)$$

The prepotential $F(a)$ also determines the low-energy effective gauge coupling:

$$\tau_{\text{eff}}(a) = \frac{4\pi i}{g_{\text{eff}}^2(a)} + \frac{\Theta_{\text{eff}}(a)}{2\pi} = \frac{\partial^2 F(a)}{\partial a^2}. \quad (3)$$

The exact low-energy action and BPS spectrum for the $SU(2)$ theory were determined in [1] by demanding a consistent realisation of electric-magnetic duality on the moduli space $\mathcal{B}$. In the exact formulae the central charges $a(u)$ and $a_D(u)$ are identified as periods of a meromorphic differential on a certain elliptic curve. In this paper we will mostly be interested in the weak coupling regime where $|a| \gg |\Lambda|$. In this case the effective coupling constant can be approximated by its one-loop value,

$$\tau_{\text{eff}}(a) \simeq \frac{2i}{\pi} \log \frac{a}{\Lambda}, \quad (4)$$
up to corrections in powers of \((\Lambda/a)^4\) coming from four-dimensional Yang-Mills instantons. To similar accuracy the magnetic central charge is given as \(a_D \simeq \tau_{\text{eff}} a\).

The exact mass formula for BPS states of charge \(\gamma\) is \(M_\gamma = |Z_\gamma|\) where the central charge \(Z_\gamma\) defined in (1). As mentioned in the introduction, the Seiberg-Witten solution does not immediately specify the set of values of \(\gamma\) which are present in the theory. Formally this corresponds to determining the values of the second helicity supertrace \(\Omega(\gamma, u) = -\frac{1}{2} \text{Tr}_{\text{BPS},\gamma}(-1)^{J_3}(2J_3)^2\) at each point on the Coulomb branch. Here \(J_3\) is any generator of the rotation subgroup of the massive little group, and this yields \(\Omega(\gamma, u) = -2\) for the vector multiplet and \(\Omega(\gamma, u) = +1\) for half-hypermultiplet.

In the weakly coupled region \(|a| \gg |\Lambda|\) the BPS spectrum can be determined by semiclassical analysis. It consists of the W-bosons of charges \(\pm(1,0)\) and an infinite tower of Julia-Zee dyons \(\pm(n,1)\) with unit magnetic charge and arbitrary integer electric charge \(4n\). The plus and minus signs correspond to charge conjugation. As we vary the modulus \(u\), the spectrum changes continuously except on the curves of real codimension one where one or more BPS states becomes marginally stable. The degeneracies \(\Omega(\gamma, u)\) can change discontinuously as we cross such a “wall of marginal stability”. Following [9], to describe this phenomenon, we associate to each BPS state of charge vector \(\gamma\) a ray \(l_\gamma\) in a complex \(\zeta\)-plane:

\[
l_\gamma := \left\{ \zeta : \frac{Z_\gamma(u)}{\zeta} \in \mathbb{R}_- \right\}.
\] (5)

These rays rotate in the \(\zeta\)-plane as we move in \(\mathcal{B}\). At a point on the wall of marginal stability, a set of BPS rays \(\{Z_\gamma(u)\}\) become aligned, and their charges can be parametrized as \(\{N_1\gamma_1 + N_2\gamma_2\}, N_1, N_2 > 0\), for some primitive vectors \(\gamma_{1,2}\) with \(Z_{\gamma_1}/Z_{\gamma_2} \in \mathbb{R}_+\) [9].

The \(\mathcal{N} = 2\) theory with gauge group \(SU(2)\) offers a concrete example of the wall-crossing phenomenon [1]. The Coulomb branch \(\mathcal{B}\) corresponds to the complex plane with two singular points at \(u = \pm \Lambda^2\). The weak-coupling region where \(|u| \gg |\Lambda|^2\) is disconnected from the region near the origin by a closed curve of marginal stability which passes through the singular points. As we enter the strongly coupled region near the origin in \(\mathcal{B}\), most of the BPS states present in the semiclassical spectrum decay, and we are left with only two BPS states: They are the magnetic monopoles \(\pm(0,1)\) and the dyons of unit electric charges denoted either as \(\pm(1,-1)\) or \(\pm(1,1)\), depending on whether we approach the curve of marginal stability from upper or lower half plane in \(\mathcal{B}\). Physically, taking the results in [18] as an example, as we approach curve of marginal stability from the upper half plane, the W-bosons \(\pm(1,0)\) decays into a (anti-) monopole \(\pm(0,1)\) and a dyon of charge \(\pm(1,-1)\); while a dyon \(\pm(n,1)\) decays into \(n + 1\) (anti-) monopoles \(\pm(0,1)\) and \(n\) dyons of charge \(\pm(1,1)\). Similar analysis can be done for the lower half plane.

---

\(^{4}\)In this paper we shall follow the same normalization convention for electric charges as in [1], which differs the convention used in [9] by a factor of 2.

\(^{5}\)The auxiliary complex variable \(\zeta\) is known as the spectral parameter. After adding the point at infinity, \(\zeta\) parametrises \(\mathbb{C}P^1\).
Although the $SU(2)$ theory is well understood, theories with additional matter and or larger gauge groups have a very complicated set of walls of marginal stability. Until recently, a systematic determination of the BPS spectrum for generic $N = 2$ SUSY gauge theories and all regions of the moduli space remained elusive. In recent work Kontsevich and Soibelman [7, 8] have proposed a wall-crossing formula which potentially solves this problem. Here we will be mostly interested in the consequences of their conjecture for the corresponding theory compactified to three dimensions. In the following we briefly review the main results of [9]. Focusing on four-dimensional $N = 2$ Euclidean SYM with gauge group $SU(2)$, we consider its compactification on the space of $\mathbb{R}^3 \times S^1$, where $S^1 : x^4 \sim x^4 + 2\pi R$ has radius $R$. On length-scales much larger than $R$, the effective action on the Coulomb branch becomes three-dimensional. In addition to the complex scalar $a$ of the four-dimensional theory, there are now two additional real periodic scalar fields. The first one is the electric Wilson line, which comes from the component $A_4$ of the $U(1)$ gauge along $S^1$ denoted as

$$\theta_e = \oint_{S^1} A_4 dx^4. \quad (6)$$

Large gauge transformations shift the value of $\theta_e$ by integer multiples of $2\pi$. Thus, we periodically identify $\theta_e \sim \theta_e + 2\pi$. In three dimensions, we can also dualize the $U(1)$ Abelian gauge field $A_i$, $i = 1, 2, 3$ in favor of another real scalar $\theta_m \in [0, 2\pi]$ known as the magnetic Wilson line. This appears in the classical action in the combination $n_m \theta_m$ where $n_m$ is the total magnetic charge. Dirac quantisation forces $n_m$ to take integer values and, correspondingly, the theory is invariant under shifts of $\theta_m$ by integer multiples of $2\pi$.

Taking into account the scalars $\{a, \bar{a}, \theta_e, \theta_m\}$, the Coulomb branch of the compactified theory is a manifold $\mathcal{M}$ with real dimension four. The low-energy effective field theory on the Coulomb branch is then given by a $d = 3$ sigma model with target $\mathcal{M}$. The condition for the effective theory to admit eight unbroken supercharges is for the moduli space $\mathcal{M}$ to be hyper-Kähler. A hyper-Kähler manifold is Kähler with respect to a triplet of complex structures $J_\alpha$, $\alpha = 1, 2, 3$, satisfying the $su(2)$ algebra $J_\alpha J_\beta = -\delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} J_\gamma$ and $J_\alpha^2 = -1$. For each $J_\alpha$, there is a corresponding symplectic form $\omega_\alpha$, and it is related to the metric $g$ on $\mathcal{M}$ by $\omega_\alpha = -g J_\alpha$. More generally we can form linear combinations $c^\alpha J_\alpha$, with $\sum_{\alpha=1}^3 c^2_\alpha = 1$, and the corresponding Kähler form is $c^\alpha \omega_\alpha$. The manifold is therefore Kähler with respect to a $\mathbb{C}P^1$ worth of complex structures. Let the complex variable $\zeta$ parametrise such $\mathbb{C}P^1$, the twistor space $\mathcal{T}$ of a hyper-Kähler manifold is constructed so as to incorporate all possible complex structures, topologically $\mathcal{T} \sim \mathcal{M} \times \mathbb{C}P^1$ [19]. In particular, choosing the complex coordinates holomorphic with respect to $J_3$, we can now organise the general Kähler form as

$$\omega(\zeta) = -i \frac{1}{2\zeta} \omega_+ + \omega_3 - i \frac{1}{2} \zeta \omega_-, \quad (7)$$

where we have defined $\omega_\pm = \omega_1 \pm i \omega_2$ and the metric $g$ is extracted from $\zeta$ independent component of $\omega(\zeta)$. The twistor space $\mathcal{T}$ plays an important role in the construction of metric on $\mathcal{M}$ in [9].

---

6See also [20] for a nice review on the twistor theory for hyper-Kähler manifolds.
In the limit of large $R$, the low-energy effective action can be obtained by dimension reduction of the four-dimensional low-energy theory. To describe the action we define the complex combination $z = \theta_m - \tau_{\text{eff}} \theta_e$ which parametrises a torus with complex structure $\tau_{\text{eff}}(a)$. In this limit the moduli space $M$ corresponds to a fibration of this torus over the Coulomb branch $B$ of the four-dimensional theory. The real bosonic part of the resulting action is given in terms of scalar fields $\{a, \bar{a}, \theta_e, \theta_m\}$ as

$$S_B = \frac{1}{4} \int d^3 x \left( \frac{4\pi R}{g_{\text{eff}}^2} \partial_\mu a \partial^\mu \bar{a} + \frac{g_{\text{eff}}^2}{16\pi^3 R} \partial_\mu z \partial^\mu \bar{z} \right).$$

(8)

In addition, surface terms give rise to pure imaginary terms in the action depending on the total electric and magnetic charge,

$$S_{\text{Im}} = i \left( n_e + \frac{\Theta_{\text{eff}}}{2\pi} n_m \right) \theta_e + i n_m \theta_m.$$

(9)

The term proportional to $\Theta_{\text{eff}}$ arises from dimensional reduction of the $F \wedge F$ term in the low-energy action of the four-dimensional theory after replacing $A_4$ by $\theta_e/2\pi R$. The corresponding fermionic terms in the action take the form

$$S_F = \frac{2\pi R}{g_{\text{eff}}^2} \int d^3 x \left( i \bar{\psi} \sigma^\mu \partial_\mu \psi + i \bar{\lambda} \sigma^\mu \partial_\mu \lambda \right),$$

(10)

where $\lambda$ and $\psi$ are the dimensional reduction along the $x_4$ direction of the four-dimensional Weyl fermions in the $U(1)$ vector multiplet whose lowest component is the scalar $a$.

Comparing with a general three-dimensional $\sigma$-model we can also extract the leading $R \rightarrow \infty$ behavior of the hyper-Kähler metric on $M$,

$$g^{sf} = R(\text{Im} \tau_{\text{eff}})|da|^2 + \frac{1}{4\pi^2 R}(\text{Im} \tau_{\text{eff}})^{-1}|dz|^2.$$

(11)

This was called the “semi-flat” metric in [9], as the two-torus spanned by $\{\theta_e, \theta_m\}$ is flat. The metric (11) also makes apparent that $g^{sf}$ is Kähler with respect to the complex structure where $\{a, z\}$ are holomorphic coordinates. Going to finite radius $R$, this semi-flat metric gets corrected quantum mechanically by instanton contributions which arise from the four-dimensional BPS states whose worldlines now wrap around $S^1$. Until recently, no analytic formulae were available for the exact metric except in the three-dimensional limit $R \rightarrow 0$ where it becomes the Atiyah-Hitchin metric [17]. The approach of [9] seeks to determine the exact metric by insisting that the total contribution of four-dimensional BPS states remains smooth despite the discontinuous changes of the spectrum dictated by the Kontsevich-Soibelman conjecture. We now review their main results.

The metric is effectively determined once the one-parameter family of Kähler forms $\omega(\zeta)$ introduced above is known. In particular the Kähler form $\omega$ and metric $g$ are both derived from a Kähler potential $K$, with $\omega = i \partial^2 K/(\partial z^a \partial \bar{z}^b) dz^a \wedge d\bar{z}^b$ and $g = 2 \partial^2 K/(\partial z^a \partial \bar{z}^b) dz^a d\bar{z}^b = 2g_{ab} dz^a d\bar{z}^b$, respectively. For any complex symplectic manifold one can always find Darboux coordinates in which
the symplectic form becomes canonical. In the present case we introduce complex coordinates $X_e(ζ)$ and $X_m(ζ)$, in terms of which

$$ω(ζ) = -\frac{1}{4π^2 R} \frac{dX_e}{X_e} \wedge \frac{dX_m}{X_m}. \tag{12}$$

More generally, we also introduce a corresponding Darboux coordinate $X_γ(ζ)$ associated with any vector $γ$ in the charge lattice determined by the relation $X_{γ_1 + γ_2} = X_{γ_1}X_{γ_2}$ where $X_γ = X_e$ for $γ = (1, 0)$ and $X_γ = X_m$ for $γ = (0, 1)$.

In the large-$R$ limit, the semi-flat metric (11) corresponds to the choice,

$$X^\text{sf}_γ(ζ) = \exp \left( πRζ + iθ_γ + πR\bar{Z}_γζ \right). \tag{13}$$

It turns out that this asymptotic behavior, along with the discontinuities of $X_γ(ζ)$ across the wall of marginal stability, as governed by Konsevich-Soibelman algebra is enough to determine $X_γ(ζ)$ (and hence, the metric on $M$) at any point on the complex $ζ$-plane. Explicitly, the coordinate $X_γ(a, θ, ζ)$ derived in [9] obeys the following integral equation:

$$X_γ(ζ) = X^\text{sf}_γ(ζ) \exp \left[ \frac{1}{2πi} \sum_{γ'∈Γ} Ω(γ'; u)⟨γ, γ'⟩ \int_{l_{γ'}} \frac{dζ'}{ζ' - ζ} \log \left( 1 - σ(γ')X_{γ'}(ζ') \right) \right], \tag{14}$$

Let us define various quantities appeared above: $⟨γ, γ'⟩$ is the symplectic product between two charge vectors, $γ = (n_e, n_m)$ and $γ' = (n'_e, n'_m)$, which we can take to be

$$⟨γ, γ'⟩ = ⟨(n_e, n_m), (n'_e, n'_m)⟩ = -n_e n'_m + n'_e n_m, \tag{15}$$

and $σ(γ') = (-1)^{2n'_e n'_m}$, known as a “quadratic refinement”. The summation in (14) is over the set of charges $Γ$ in the theory and the BPS ray $L_{γ'}$ associated with $γ'$ is defined analogously to (5).

### 3 Semiclassical Limit of the Wall-Crossing Formula

The integral equation (14) for the Darboux coordinate $X_γ(ζ)$ is of a standard type similar to those arising in the context of the Thermodynamic Bethe Ansatz [11] [8]. Taking the logarithm of this equation we see that the right hand side contains a source term corresponding to the semi-flat expression and an integral convolution. As usual we can solve the equation iteratively, order by order in appropriate expansion parameter, when the second term is smaller than the first. In [9] an expansion of this sort was obtained at large radius $R|a| ≫ 1$, for any point on the Coulomb branch. The contributions of a BPS state with charge $γ$ is exponentially supressed by a factor of $\exp(-2πR|Z_γ|)$ and, when $R|a| ≫ 1$, this factor is small for all $γ$. Here we are instead interested in a weak coupling expansion of the integral equation. Thus, we will restrict our attention to the semiclassical region of the moduli space, $|a| ≫ Λ$, where $g^2_\text{eff} ≪ 1$ while holding the dimensionless...
quantity $R|a|$ fixed. As we explain below, the quantity $\exp(-2\pi R|Z_\gamma|)$ is then suppressed for all states with non-zero magnetic charge. In the weak coupling spectrum described above, this is the case for all states except the massive gauge bosons $W^\pm$. This means we must effectively resum the contributions from these states. This is the main difference between the weak-coupling expansion described below and the large-$R$ expansion of [9].

We begin by decomposing the Darboux coordinate $X_\gamma(\zeta)$ as

$$X_\gamma(\zeta) = [X_e(\zeta)]^{n_e} [X_m(\zeta)]^{n_m}, \quad \gamma = (n_e, n_m).$$

The integral equation [14] for the electric and the magnetic Darboux coordinates is then given as

$$X_e(\zeta) = X_{e}^{\text{sf}}(\zeta) \exp \left[ -\frac{1}{2\pi i} \sum_{\gamma' \in \Gamma} c_e(\gamma') \mathcal{I}_{\gamma'}(\zeta) \right], \quad c_e(\gamma') = \Omega(\gamma'; u_0(1, 0), \gamma'),$$

$$X_m(\zeta) = X_{m}^{\text{sf}}(\zeta) \exp \left[ -\frac{1}{2\pi i} \sum_{\gamma' \in \Gamma} c_m(\gamma') \mathcal{I}_{\gamma'}(\zeta) \right], \quad c_m(\gamma') = \Omega(\gamma'; u_0(0, 1), \gamma'),$$

where $X_{e}^{\text{sf}}(\zeta)$ and $X_{m}^{\text{sf}}(\zeta)$ are given by [13] with $Z_\gamma$ replaced by $Z_e$ and $Z_m$, respectively, and $\mathcal{I}_{\gamma'}(\zeta)$ is defined to be

$$\mathcal{I}_{\gamma'}(\zeta) = \int_{I_{\gamma'}} \frac{d\zeta'}{\zeta' - \zeta} \log(1 - \sigma(\gamma')X_{\gamma'}(\zeta')).$$

Now, taking the weak coupling limit, to one loop order we find

$$\log X_{e}^{\text{sf}}(\zeta) = \pi R a \zeta^{-1} + i\theta_e + \pi R a \zeta,$$

$$\log X_{m}^{\text{sf}}(\zeta) = \pi R a_{\text{eff}}(a) \zeta^{-1} + i\theta_m + \pi R a_{\text{eff}}(a) \zeta.$$ (20)

We can see that in this limit $\log |X_{e}^{\text{sf}}| \gg \log |X_{m}^{\text{sf}}|$, this has interesting consequences for deriving an iterative solution to $X_\gamma(\zeta)$. Explicitly, let us expand $\log X_{e}(\zeta)$ and $\log X_{m}(\zeta)$ for the weak coupling spectrum of $N = 2 SU(2)$ gauge theory using [17], [18]:

$$\log X_{e}(\zeta) = \log X_{e}^{\text{sf}}(\zeta) - \frac{1}{2\pi i} \sum_{n'_e \in \mathbb{Z}} \sum_{n'_m = \pm 1} c_e(\gamma') \mathcal{I}_{(n'_e, n'_m)}(\zeta),$$

$$\log X_{m}(\zeta) = \log X_{m}^{\text{sf}}(\zeta) - \frac{c_m(W^+)}{2\pi i} \mathcal{I}_{(1, 0)}(\zeta) - \frac{c_m(W^-)}{2\pi i} \mathcal{I}_{(-1, 0)}(\zeta) - \sum_{n'_e \in \mathbb{Z}} \sum_{n'_m = \pm 1} \frac{c_m(\gamma')}{2\pi i} \mathcal{I}_{(n'_e, n'_m)}(\zeta).$$

The central charge has the form $Z_\gamma(a) = a(n_e + n_m r_{\text{eff}}(a))$. The mass of a BPS particle $\gamma = (n_e, n_m)$ at weak-coupling limit is given by

$$|Z_\gamma(a)| = |a| \sqrt{\left(n_m \frac{4\pi}{g_{\text{eff}}} \right)^2 + \left(n_e + n_m \frac{\Theta_{\text{eff}}}{2\pi} \right)^2},$$

where $g_{\text{eff}}$ and $\Theta_{\text{eff}}$ now denote the effective coupling constant and the effective vacuum angle. The BPS spectrum $\Gamma$ consists of the $W$-bosons of charges $\pm(1, 0)$ which we denoted as $W^\pm$ and only contribute to the middle two terms in (22), and the remaining summations in (21) and (22) are over the infinite tower of dyons with charges $\pm(n, 1)$, $n \in \mathbb{Z}$.
Let us further describe our weak coupling, iterative approach to solving \( \log X_c \). At the leading order, we substitute the semi-flat coordinates [20] into the right hand side of [21, 22] and ignore the components which vanish as \( g_{\text{eff}} \rightarrow 0 \). As the result, the magnetic coordinate \( X_m \) receives additional order one contribution due to the \( W \)-bosons, while the dyon contributions to \((X_e, X_m)\) are exponentially suppressed as \( \sim \exp(-c/\delta_{\text{eff}}^2) \) along the integration contours \( \{ l_i \} \). We shall denote the resultant coordinates at this order as \((X_e^{(0)}, X_m^{(0)})\). Thus we have

\[
\log X_e^{(0)}(\zeta) = \log X_e^{sf}(\zeta), \quad \log X_m^{(0)}(\zeta) = \log X_m^{sf}(\zeta) + \log D(\zeta),
\]

where we have defined the short-hand notation for the integral:

\[
\mathcal{I}^{(0)}_{(n'_e, n'_m)}(\zeta) = \int_{l_i'_{\zeta'}} \frac{d\zeta' \zeta' + \zeta}{\zeta' - \zeta} \log \left( 1 - \left[ X_e^{(0)}(\zeta') \right]^{n'_e} \left[ X_m^{(0)}(\zeta') \right]^{n'_m} \right).
\]

We shall denote \((X_e, X_m)\) into

\[
\log X_e(\zeta) = \log X_e^{(0)}(\zeta) + \delta \log X_e(\zeta), \quad \log X_m(\zeta) = \log X_m^{(0)}(\zeta) + \delta \log X_m(\zeta).
\]

By plugging \((X_e^{(0)}, X_m^{(0)})\) into [21] and [22], we can read off the subleading-order corrections to the coordinates:

\[
\delta \log X_e(\zeta) = -\frac{1}{2\pi} \sum_{n'_e \in \mathbb{Z}} \sum_{n'_m = \pm 1} \gamma_{\zeta} I_{(n'_e, n'_m)}^{(0)}(\zeta),
\]

where we have defined the short-hand notation for the integral:

\[
\mathcal{I}^{(0)}_{(n'_e, n'_m)}(\zeta) = \int_{l_i'_{\zeta'}} \frac{d\zeta' \zeta' + \zeta}{\zeta' - \zeta} \log \left( 1 - \left[ X_e^{(0)}(\zeta') \right]^{n'_e} \left[ X_m^{(0)}(\zeta') \right]^{n'_m} \right).
\]

We will soon see that these terms generate the exponentially suppressed dyon contributions.

To extract the corresponding correction to the metric on \( \mathcal{M} \), we compute the symplectic form \( \omega(\zeta) \) [12], including the corrections [27] and [28] to [21]. The result yields

\[
\omega(\zeta) \approx -\frac{1}{4\pi^2 R} d(\log X_e^{(0)}(\zeta) + \delta \log X_e(\zeta)) \wedge d(\log X_m^{(0)}(\zeta) + \delta \log X_m(\zeta))
\]

\[
= \omega^{sf}(\zeta) + \omega^{W}(\zeta) + \omega^{\text{dyon}}(\zeta) + \mathcal{O}(\delta^2),
\]

where in the first line we have re-written \( X_c(\zeta) \) in terms of \( X_e(\zeta) \) and \( X_m(\zeta) \) using [11]. The various terms in [30] are then given by

\[
\omega^{sf}(\zeta) = -\frac{1}{4\pi^2 R} d \log X_e^{sf}(\zeta) \wedge d \log X_m^{sf}(\zeta),
\]

\[
\omega^{W}(\zeta) = -\frac{1}{4\pi^2 R} d \log X_e^{sf}(\zeta) \wedge d \log D(\zeta),
\]

\[
\omega^{\text{dyon}}(\zeta) = -\frac{1}{4\pi^2 R} \left( d \delta \log X_e(\zeta) \wedge d \log X_m^{(0)}(\zeta) + d \log X_e^{(0)}(\zeta) \wedge d \delta \log X_m(\zeta) \right).
\]
functions. The resulting geometry corresponds to a finite shift of the coupling constant:

\[ \pi R \]

further allowed 2

As mentioned above, we are taking weak coupling limit to be naturally interpreted as summing up exponentially suppressed, instanton-like, contributions.

To go to small values of \( R \) arbitrary, we set \( \Omega(\gamma') = 2 \). In \cite{9}, the authors further considered the large radius limit \( R \gg |\Lambda| \) and further allowed \( 2\pi R|a| \gg 1 \), this sets \( K_{\nu}(2\pi R|ka|) \sim e^{-2\pi R|ka|} \), and the series in \cite{35} and \cite{36} can be naturally interpreted as summing up exponentially suppressed, instanton-like, contributions.

As mentioned above, we are taking weak coupling limit \( |a/\Lambda| \gg 1 \) while keeping \( 2\pi R|a| \) fixed and arbitrary. To go to small values of \( R|a| \), one instead needs to Poisson resum the series of Bessel functions \cite{21}. The resulting geometry corresponds to a finite shift of the coupling constant \cite{22}:

\[
\frac{2\pi R}{g_{\text{eff}}^2} - \frac{2\pi R}{g_{\text{eff}}} - \sum_{n \in \mathbb{Z}} \frac{1}{2\pi |M(n)|} \implies |M(n)| = \sqrt{|a|^2 + \left( \frac{g_e}{2\pi R} + \frac{n}{R} \right)^2},
\]

which determines the harmonic function \( V \) in the Gibbons-Hawking ansatz for the hyper-Kähler metric (see eq. (4.2) of \cite{9}). In the limit \( R \to 0 \), this reduces to the shift \( 1/e_{\text{eff}}^2 \to 1/e_{\text{eff}}^2 - 1/(2\pi M_W) \) where \( 1/e_{\text{eff}}^2 = 2\pi R/g_{\text{eff}}^2 \) and \( M(0) \) are the gauge coupling and the mass of \( W \)-boson in three dimensions \cite{17} \cite{18}.

Now for the more complicated \( \omega^{\text{dyon}}(\zeta) \), our strategy here is to evaluate the integrals involved by saddle point approximation, as they are dominated by the exponential terms at weak coupling. Explicitly we have the following series for \( \omega^{\text{dyon}}(\zeta) \) in \cite{33}:

\[
\omega^{\text{dyon}}(\zeta) = \sum_{\gamma'=\{n',\pm 1\}} \omega_{\gamma'}(\zeta),
\]

\[
\omega_{\gamma'}(\zeta) = -\frac{1}{4\pi^2 R} \frac{d\mathcal{X}_{\gamma'}^{(0)}(\zeta)}{X_{\gamma'}^{(f)}(\zeta)} \wedge \left( \frac{1}{2\pi i} \int_{l_{\gamma'}} \frac{dz'}{z'} \frac{\mathcal{X}_{\gamma'}^{(0)}(\zeta')}{\mathcal{X}_{\gamma'}^{(f)}(\zeta')} \right) \approx -\frac{1}{4\pi^2 R} \frac{d\mathcal{Y}_{\gamma'}^{(0)}(\zeta)}{X_{\gamma'}^{(f)}(\zeta)} \wedge \left( \frac{1}{2\pi i} \int_{l_{\gamma'}} \frac{dz'}{z'} \frac{\mathcal{X}_{\gamma'}^{(0)}(\zeta')}{\mathcal{X}_{\gamma'}^{(f)}(\zeta')} \right),
\]

where we have used the fact that \( \Omega(\gamma', a) = 1 \) for all the dyon states. Along each integration contour \( l_{\gamma'} : Z_{\gamma'}/z' \in \mathbb{R}_- \), the zeroth order Darboux coordinate \( \mathcal{X}_{\gamma'}^{(0)}(\zeta') = \mathcal{X}_{\gamma'}^{(0)}(\zeta')D(\zeta') \) is proportional to
exponential factor \( \exp \left[ -\pi R |Z_{\gamma'}| (|\zeta'| + 1/|\zeta'|) \right] \), which ensures the convergence of the integral. At the weak coupling \(|\tau_{\text{eff}}| \gg 1\) and \(|Z_{\gamma'}| \gg 1\), we can therefore Taylor-expand \( X_{\gamma'}^{(0)}(\zeta')/(1 - X_{\gamma'}^{(0)}(\zeta')) \) in the integrand above into

\[
\frac{X_{\gamma'}^{(0)}(\zeta')}{1 - X_{\gamma'}^{(0)}(\zeta')} = \sum_{k=1}^{\infty} \left[ X_{\gamma'}^{(0)}(\zeta') \right]^k, \quad \zeta' \in \mathcal{L}_{\gamma'}, \tag{40}
\]

and perform saddle point analysis for each term in the expansion. Here we have also further approximated \( dX^{(0)}(\zeta)/X^{(0)}(\zeta) \) by \( dX_{\gamma}^{(0)}(\zeta)/X_{\gamma}^{(0)}(\zeta) \), as the contribution proportional to \( d\mathcal{D}(\zeta)/\mathcal{D}(\zeta) \) is of higher order in \( g_{\text{eff}}^2 \) in our saddle point analysis. The saddle point analysis amounts to extremizing \( \exp[-k\pi R |Z_{\gamma'}| (|\zeta'| + 1/|\zeta'|)] \) with respect to \(|\zeta'| \) in each term of (40), we can then deduce that the saddle point sits at \( \zeta' = -Z_{\gamma'}/|Z_{\gamma'}| \). Upon substitution and performing the Gaussian fluctuation integral, the leading expression for \( \omega_{(\gamma',k)}^{\text{dyon}}(\zeta) \) is given by

\[
\omega^{\text{dyon}}(\zeta) = \sum_{\gamma' = (\eta', \pm 1)}^{\infty} \sum_{k=1}^{\infty} \omega_{(\gamma',k)}(\zeta), \tag{41}
\]

\[
\omega_{(\gamma',k)}(\zeta) = \mathcal{J}_{(\gamma',k)} \left. dX_{\gamma}^{(0)}(\zeta) / X_{\gamma}^{(0)}(\zeta) \right|_{\zeta} \left[ Z_{\gamma'} \right] \left[ \frac{dZ_{\gamma'}}{Z_{\gamma'}} - \frac{d\bar{Z}_{\gamma'}}{\bar{Z}_{\gamma'}} \right] - \left( \frac{dZ_{\gamma'}}{\zeta} - \zeta d\bar{Z}_{\gamma'} \right), \tag{42}
\]

\[
\mathcal{J}_{(\gamma',k)} = \frac{1}{8\pi^2 i} \left. D(-e^{i\delta_{\gamma'}}) n_m' \right|_{n_m'}^k \exp \left[ k(-2\pi R |Z_{\gamma'}| + i \theta_{\gamma'}) \right], \tag{43}
\]

\[
\log D(-e^{i\delta_{\gamma'}}) n_m' \approx \pm \log D(\mp i) = \frac{1}{\pi i} \int_0^\infty \frac{dy}{y} \left[ \frac{y + i}{y - i} \log \left( 1 - e^{-\pi R |a|(y+1/y)+i\theta_e} \right) \right.
\]

\[
\left. - \frac{y - i}{y + i} \log \left( 1 - e^{-\pi R |a|(y+1/y)-i\theta_e} \right) \right]. \tag{44}
\]

Here \( e^{i\delta_{\gamma'}} = \frac{n'_m + \tau_{\text{eff}} a_m' n_m}{n'_m + \tau_{\text{eff}} a_m n_m} \), and in the weak coupling limit we have further approximated \( e^{i\delta_{\gamma'}} \approx i n_m' \), this is consistent with the saddle point approximation and ensures the resultant one loop determinant \( D \) being real. By substituting \( y = e^t \) and using \( n_m' = \pm 1 \), the equation (44) can be re-expressed as

\[
\pm \log D(\mp i) = \frac{2}{\pi} \int_0^\infty \frac{dt}{\cosh t} \left[ \log \left( 1 - e^{-2\pi R |a| \cosh t+i\theta_e} \right) + \log \left( 1 - e^{-2\pi R |a| \cosh t-i\theta_e} \right) \right]. \tag{45}
\]

In the next section we will show that this expression precisely corresponds to the ratio of one-loop determinants corresponding to small fluctuations around a classical dyon contribution.

Now we can extract the correction to the symplectic form on the moduli space \( \mathcal{M} \) from the \( \zeta' \)-independent part of \( \omega^{\text{dyon}}(\zeta) \). As \( g_{\text{eff}}^2 \to 0 \), keeping only the leading terms with \( k = 1 \) and focusing on only instanton (i.e. \( n'_m = +1 \)) contributions, we obtain

\[
\omega^{\text{inst.}}_{3} = \sum_{\gamma' = (\eta', 1)} \mathcal{J}_{(\gamma',1)} \left( 2\pi R dZ_{\gamma'} \wedge d\bar{Z}_{\gamma'} + i |Z_{\gamma'}| d\theta_{\gamma'} \wedge \left( \frac{dZ_{\gamma'}}{Z_{\gamma'}} - \frac{d\bar{Z}_{\gamma'}}{\bar{Z}_{\gamma'}} \right) \right)
\]

\[
= \sum_{\gamma' = (\eta', 1)} \mathcal{J}_{(\gamma',1)} \left( 2\pi R |n'_e + \tau_{\text{eff}}(a)|^2 da \wedge d\bar{a} + i |n'_e + \tau_{\text{eff}}(a)|^2 d\theta_{\gamma'} \wedge |a| \left( \frac{da}{a} - \frac{d\bar{a}}{\bar{a}} \right) \right), \tag{46}
\]
where \( \omega_3^{\text{dyon}} = \omega_3^{\text{inst}} + \omega_3^{\text{inst}} \) (\( \omega_3^{\text{inst}} \) corresponds to \( n'_m = -1 \) contributions). Here in the second line of (46) we have used the weakly coupled expression for the central charge \( Z_{\gamma'} = a(n'_e + \tau_{\text{eff}}(a)) \). Explicitly, let us write out the \( g_{a\bar{a}} \) component from (46), which gives the dominant contribution in weak-coupling expansion:

\[
g_{a\bar{a}}^{\text{inst}} = \frac{\sqrt{R}}{4\pi} \sum_{\gamma'=(n'_e,1)} \frac{\mathcal{D}(-i)|Z_{\gamma'}|^{3/2}}{|a|^2} \exp\left(-2\pi R|Z_{\gamma'}| + i\theta_{\gamma'}\right). \tag{47}
\]

Other metric components \( g_{az}^{\text{inst}}, g_{a\bar{z}}^{\text{inst}} \) which are suppressed by \( g_{\text{eff}}^2 \) can also be readily extracted from (46). By including these additional metric components and using the complex coordinates \((z, \bar{z})\) introduced earlier we can calculate a Kähler potential, \( K^{\text{dyon}} \) corresponding to the Kähler form \( \omega_3^{\text{dyon}} \),

\[
\theta_m = \frac{1}{2} \left( z + \bar{z} \right) + \frac{i}{\text{Im} \tau_{\text{eff}}} \left( z - \bar{z} \right), \quad \theta_e = \frac{i}{2\text{Im} \tau_{\text{eff}}}(z - \bar{z}),
\]

we recover the Kähler potential for the Kähler form (46):

\[
K^{\text{dyon}} = \sum_{\gamma'=(n'_e,\pm 1)} \frac{\mathcal{D}(-i)}{4\pi^3 R^{3/2} \sqrt{|Z_{\gamma'}|}} \exp\left(-2\pi R|Z_{\gamma'}| + i\theta_{\gamma'}\right). \tag{49}
\]

Finally we further expand the metric (47) at weak coupling as

\[
g_{a\bar{a}}^{\text{inst}} \approx \frac{\sqrt{R}}{4\pi} \left( \frac{4\pi}{g_{\text{eff}}^2} \right)^{3/2} \sum_{\gamma'=(n'_e,1)} \frac{\mathcal{D}(-i)}{|a|^{1/2}} \exp\left(-S_{\text{Mon}} - S_{\phi}(n'_e)\right), \tag{50}
\]

\[
S_{\text{Mon}} = \frac{8\pi^2 R |a|}{g_{\text{eff}}^2} - i \theta_m, \tag{51}
\]

\[
S_{\phi}(n'_e) = \frac{g_{\text{eff}}^2 R |a|}{4} \left( n'_e + \frac{\Theta_{\text{eff}}}{2\pi}\right)^2 - in'_e \theta_e. \tag{52}
\]

The exponent is the Euclidean action of a magnetic monopole thought of as a static field configuration on \( \mathbb{R}^3 \times S^1 \). The remaining term \( S_{\phi}(n'_e) \) is the leading contribution from the electric charge of the dyon. In four dimensional theory, the corresponding contribution to the dyon mass to comes from the slow motion of the monopole in the \( S^1 \) factor of its moduli space [23]. The further shift of the electric charge \( n'_e \rightarrow n'_e + \Theta_{\text{eff}}/2\pi \) corresponds to the familiar Witten effect [24].

One additional subtlety described in Section 4.1 of [9] concerns the appropriate choice of coordinates on the torus fibre of the moduli space. When working near a singularity in the moduli space where a ratio of BPS particle masses vanishes it is appropriate to change variables to a coordinate which is single-valued in a neighbourhood of the singular point. In the present case we are interested in the semiclassical region of the moduli space near infinity and the singularity corresponds to the logarithm in the one-loop effective coupling [4]. Adapting eq. (4.13) of [9] to this case, the corresponding change of variable is

\[
\theta_m \rightarrow \theta'_m = \theta_m + \frac{\Theta_{\text{eff}}}{2\pi} \theta_e.
\]
Implementing this replacement, equation (52) becomes
\[
S(\phi) = \frac{g_{\text{eff}}^2 R|a|}{4} \left( n'_e + \Theta_{\text{eff}} \frac{2\pi}{2} \right)^2 - i \left( n'_e + \Theta_{\text{eff}} \frac{2\pi}{2} \right) \theta_e.
\] (53)
and we see that the imaginary part of the action \( S_{\text{Mon.}} + S(\phi) \) agrees with eqn. (10) obtained directly from dimensional reduction of the four-dimensional metric.

The hyper-Kähler metric on \( \mathcal{M} \) completely determines the terms in the low-energy effective action for the massless fields with at most two derivatives or four fermions. These terms correspond to a three-dimensional supersymmetric sigma model with target space \( \mathcal{M} \);
\[
S_{\text{eff}}^{(3D)} = \frac{1}{4} \int d^3x \left( g_{ij}(X) \left[ \partial_\mu X^i \partial^\mu X^j + i \bar{\Omega}^i \partial \Omega^j \right] + \frac{1}{6} R_{ijkl}(\bar{\Omega}^i \cdot \Omega^k)(\bar{\Omega}^j \cdot \Omega^l) \right). \] (54)
Here \( \{X^i\} \) are four bosonic scalar fields and \( \{\Omega_i^\alpha\} \) are their Majorana fermionic superpartners.

As usual, the bosonic scalar fields \( \{X^i\} \) define the coordinates on the quantum moduli space \( \mathcal{M} \). In the semiclassical limit, the metric goes to its semiflat value and, choosing complex coordinates \( \{a, \bar{a}, z, \bar{z}\} \), the bosonic action must coincide with (8). If we choose appropriately rescaled coordinates defined by
\[
a = \frac{g_{\text{eff}}}{2 \sqrt{\pi R}} X^1, \quad z = \frac{4 \pi \sqrt{\pi R}}{g_{\text{eff}}} X^2,
\] (55)
then the semi-flat metric \( g^{sf} \) (11) simply reduces to the flat metric \( \delta_{ij} \), \( i, j = 1, 2 \) in this limit.

By comparing the fermionic terms in the action with (10), we can rewrite the action in terms of the Weyl spinors \( \{\lambda_\alpha, \bar{\lambda}_\dot{\alpha}, \psi_\alpha, \bar{\psi}_\dot{\alpha}\} \) of four-dimensional \( U(1) \) vector multiplet. Following [15], we rewrite the latter in terms three-dimensional Majorana fermions \( \{\chi^a, \bar{\chi}^a\} \) via
\[
\lambda_\alpha = \chi^1_\alpha, \quad \epsilon_{\alpha\beta}\bar{\chi}^\dot{\beta} = \chi^2_\alpha, \quad \psi_\alpha = \bar{\chi}^1_\alpha, \quad \epsilon_{\alpha\beta}\bar{\psi}^\dot{\beta} = \bar{\chi}^2_\alpha.
\] (56)
The Majorana fermions \( \{\Omega_i^\alpha\} \) appearing in (54) can be then be expressed as
\[
\Omega_\alpha^i = M^{ic}(X)\bar{\chi}_\bar{c}^\alpha, \quad \Omega_\bar{\alpha}^\dot{i} = M^{\dot{i}c}(X)\chi^c_\alpha, \quad c = 1, 2,
\] (57)
where \( M^{ic}(X) \) and \( M^{\dot{i}c}(X) \) are undetermined matrices which can depend non-trivially on the bosonic scalars \( X^i \). Matching with the fermion kinetic terms in (10) imposes the normalisation condition,
\[
\delta_{ij}M^{ia}(X)M^{\dot{b}\dot{a}}(X) = \left( \frac{8\pi R}{g_{\text{eff}}^2} \right) \delta^{ab}.
\] (58)
In a vacuum where \( \theta_e = 0 \), the relation between the fermion bilinears appearing in (54) and the four dimensional fermions can be made explicit as,
\[
\bar{\Omega}^1 \cdot \Omega^1 = \left( \frac{8\pi R}{g_{\text{eff}}^2} \right) \lambda \cdot \bar{\lambda}, \quad \bar{\Omega}^2 \cdot \Omega^2 = \left( \frac{8\pi R}{g_{\text{eff}}^2} \right) \psi \cdot \bar{\psi}.
\] (59)
The four-fermion term in the action \((54)\) involves the Riemann tensor of the hyper-Kähler metric on \(\mathcal{M}\). The leading semiclassical computation is computed using our result \((50)\) for the metric. As the metric is Kähler in the complex coordinates \(\{a, \bar{a}, z, \bar{z}\}\), we conclude that at the leading order in \(g\) expansion, the only non-vanishing components, up to the symmetries of the Riemann tensor, are

\[
R_{a\bar{z}z\bar{a}} = R_{a\bar{a}z\bar{z}} = g_{a\bar{p}} \partial_z (g^{p\bar{q}} \partial_{\bar{z}} g_{a\bar{q}}), \quad p, q = \{a, z\},
\]

and we can reduce the expressions above to

\[
R_{a\bar{z}z\bar{a}} = R_{a\bar{a}z\bar{z}} = -\frac{1}{4} g_{a\bar{a}}^{\text{inst}}. \tag{61}
\]

In terms of \(\{X^i, X^\bar{j}\}\) defined in \((55)\), we can relate the Riemann tensor \((60)\) extracted from the integral formula in [9] with the one in new coordinates via

\[
R_{\bar{1}\bar{2}\bar{1}\bar{2}} = \left| \frac{da}{dX^1} \frac{dz}{dX^2} \right|^2 R_{a\bar{z}z\bar{a}} = (2\pi)^2 R_{a\bar{z}z\bar{a}}. \tag{62}
\]

The Riemann tensor captures the quantum corrections to the metric, both perturbatively and non-perturbatively. Now we can use the above conversion between \(\Omega^{1,2}\) and \(\lambda, \psi\) \((59)\) to extract the prediction for the four fermion in the low-energy effective Lagrangian from \((50)\). After taking into account the symmetries of the Riemann tensor, and restricting again to the leading \(k = n'_m = 1\) sector, we obtain\footnote{Strictly speaking the four-fermion vertex is only correct as written in a vacuum where \(\theta_e = 0\). For \(\theta_e \neq 0\), the matrices appearing in \((57)\) effect a rotation which changes the chirality of the vertex but preserves the overall normalisation which is subject to \((58)\). We will supress this subtlety in the following.}

\[
S_{4F} = \frac{2^{9/2} \pi}{R_{|a|^{1/2}}} \left( \frac{2\pi R}{g_{\text{eff}}^2 (a)} \right)^{7/2} D(-i) \exp [-S_{\text{Mon}}] \sum_{n'_e \in \mathbb{Z}} \exp \left[ -S^{(n'_e)}_\varphi \right] \int d^3 x (\psi \cdot \bar{\psi})(\lambda \cdot \bar{\lambda}). \tag{63}
\]

We shall next verify this term in the effective action via a direct semiclassical calculation.

## 4 Semiclassical Instanton Calculation

In this Section we will compute the monopole and dyon contributions to the action from first principles. We focus on the leading contribution of magnetic charge \(n_m = k = 1\) and arbitrary electric charges \(n_e \in \mathbb{Z}\) to an appropriate four fermionic correlator to be defined momentarily. A similar calculation in the three-dimensional limit was performed in [15] and for the corresponding theory with 16 supercharges on \(\mathbb{R}^3 \times S^1\) in [13, 14]. We will refer the reader to these references for some of the details.

We begin by considering a static BPS monopole of the \(\mathcal{N} = 2\) theory in four-dimensional Minkowski spacetime. The bosonic moduli of this soliton consist of three coordinates \(X^1, X^2, X^3\) specifying the position of its centre in \(\mathbb{R}^3\) and a global \(U(1)\) charge angle parametrized by \(\varphi \in [0, 2\pi]\). The moduli space is thus \(\mathbb{R}^3 \times S^1\). As the configuration is one-half BPS, there are four fermionic
zero modes in the monopole background, as generated by half of the eight supercharges. It is convenient to work in a formalism where monopoles are preserved by supercharges of the same four-dimensional chirality, to do so we embed the monopole as self-dual field configuration in an auxiliary four-dimensional $N = 2$ gauge theory where the scalar field in the BPS equation arises corresponds to a component of the four-dimensional gauge field. In general, this is not the same as the four-dimensional theory we compactified on $\mathbb{R}^3 \times S^1$ in the previous section where the corresponding field remains a scalar. However, as we discuss below, the fermions of the two theories are related to each other by an $R$-symmetry rotation (see [13]).

The left and right handed Weyl fermions of the auxiliary theory are denoted $\rho^{A}_\delta$ and $\bar{\rho}^{\dot{A}}_\delta$ respectively where $A = 1, 2$ and $\delta, \dot{\delta} = 1, 2$. In terms of these fermions the four zero modes of the instanton are all left-handed yielding a non-zero contribution to the correlator,

$$G_4(y_1, y_2, y_3, y_4) = \langle \prod_{A=1}^{2} \rho^{A}_1(y_{2A-1})\rho^{A}_2(y_{2A}) \rangle.$$ \hspace{1cm} (64)

corresponding to a vertex of the form $(\bar{\rho}^1 \cdot \bar{\rho}^1)(\bar{\rho}^2 \cdot \bar{\rho}^2)$ in the low-energy effective action. The fermions of the auxiliary theory are related to the original four-dimensional Weyl fermions by an $SO(3)$ $R$-symmetry rotation which mixes left and right-handed chiralities but preserves the normalisation of the four-fermion vertex in the effective Lagrangian. In a vacuum where $\theta_e = 0$ the zero modes of a monopole are chirally symmetric in the original four-dimensional theory, and the explicit relation takes the form \[10\]

$$(\bar{\rho}^1 \cdot \bar{\rho}^1)(\bar{\rho}^2 \cdot \bar{\rho}^2) = (\psi \cdot \bar{\psi})(\lambda \cdot \bar{\lambda}).$$ \hspace{1cm} (65)

In the weak-coupling approximation we can replace the fermions in the correlation function (64) with their zero mode values multiplied by corresponding Grassmann collective coordinates $\xi^{A}_\delta$. The explicit form of the zero modes is given in Appendix C of [15]. As we are interested in comparing with the low-energy effective action we focus on the large-distance limit of the correlation function and of the fermion zero modes. We can then express the large distance limit of $\rho^{A}_\delta$ in terms of $\xi^{A}_\delta$ and the three dimensional Dirac fermion propagator $S_F(x) = \gamma^\mu x^\mu/(4\pi |x|^2)$ as

$$\rho^{(LD)}_A(y) = 8\pi(S_F(y - X))^\beta_\alpha \xi^{A}_\beta.$$ \hspace{1cm} (66)

In the four-dimensional theory, the semiclassical dynamics of monopoles is described by supersymmetric quantum mechanics on the moduli space [29]. For a single monopole of mass $M = 4\pi |a|/g^2$, this corresponds to the dynamics of a free non-relativistic particle moving on $\mathbb{R}^3 \times S^1$. These bosonic degrees of freedom have four free fermionic superpartners. The collective coordinate Lagrangian takes the form

$$L_{QM} = L_X + L_\phi + L_\xi,$$ \hspace{1cm} (67)

\[10\]As in the previous section, when $\theta_e \neq 0$, the rotation leads to chirally asymmetric vertex when written in terms of $\lambda$ and $\psi$. \hfill \vspace{1cm}

15
where we have \( L_X = \frac{M}{a^2} \dot{\bar{X}}(t)^2 \), \( L_\varphi = \frac{1}{2} \frac{M}{a^2} (\varphi)^2 \) and \( L_\xi = \frac{M}{a^2} \xi^A \xi^A \), where the dot denotes a time derivative. The combination \( \frac{M}{a^2} \) is the moment of inertia of a monopole with respect to global gauge rotation, \( L_\varphi \) describes a free particle of mass \( \frac{M}{a^2} \) moving along \( S^1_\varphi \) with \( \varphi \in [0, 2\pi] \).

The quantity of interest here is the large distance behavior of the four fermion correlation function as defined above. To pass to the theory on \( \mathbb{R}^3 \times S^1 \), we Wick rotate the collective coordinate quantum mechanics described above to a periodic Euclidean time identified with the \( x_4 \) coordinate introduced earlier. There are periodic boundary conditions for both bosons and fermions to preserve supersymmetry. At leading semiclassical order the fermionic fields in the correlator are replaced by their values in the monopole background. The resulting large distance correlation function then takes the following form:

\[
\mathcal{G}_4(y_1, y_2, y_3, y_4) = \int [d\mu] \prod_{A=1}^2 \rho_1^{(LD)} A(y_{2A-1}) \rho_2^{(LD)} A(y_{2A}),
\]

\[
\int [d\mu] = \frac{1}{4\pi^2} \int [d^3X(x^4)] [d\varphi(x^4)] [d^4\xi(x^4)] |\mathcal{R}| \exp \left[ - \int_0^{2\pi R} dx^4 L_{QM} \right] \exp \left[ - \frac{8\pi^2 R |a|}{g^2} + i\theta_m \right],
\]

where superscript "LD" on the fermionic zero modes indicates their large distance behaviors as given in (66). The prefactor of \( 1/4\pi^2 \) arises from the Jacobian for the change of variables from bosonic fields to the four bosonic collective coordinates and can be traced to the same factor in the standard formula [25] given as eq. (114) in [15]. The integration measure \([d\mu]\) consists of bosonic \([d^3X][d\varphi]\) and fermionic \([d^4\xi]\) zero mode measures, and the one-loop determinant \( \mathcal{R} \) encoding the non-zero mode fluctuations, all weighted by the monopole effective action \( \exp[\int_0^{2\pi R} dx^4 L_{QM} - S_{Mon.}] \) given in (67). We shall now evaluate various contributions in turns following [14] and [28].

For the bosonic \( \int [d^3X(x^4)] \exp[\int_0^{2\pi R} dx^4 L_X] \) and the fermionic \( \int [d^4\xi(x^4)] \exp[\int_0^{2\pi R} dx^4 L_\xi] \) zero mode measures, first we note that \( \tilde{X}(x^4) \) and \( \xi^A_\alpha(x^4) \) now need to satisfy periodic boundary condition \( \tilde{X}(x^4) = \tilde{X}(x^4 + 2\pi R) \) and \( \xi^A_\alpha(x^4) = \xi^A_\alpha(x^4 + 2\pi R) \). This implies that for the free Lagrangians \( L_X \) and \( L_\xi \), the path integrals are dominated by the constant classical paths which, with slight abuse of notations, we again denote as \( \tilde{X} \) and \( \xi^A_\alpha \). We can then expand around the classical paths:

\[
\tilde{X}(x^4) = \tilde{X} + \delta \tilde{X}(x^4), \quad \xi^A_\alpha(x^4) = \xi^A_\alpha + \delta \xi^A_\alpha(x^4),
\]

and decompose the path integrals into

\[
\int [d^3X(x^4)] \exp \left[ - \int_0^{2\pi R} dx^4 L_X \right] = \int d^3X \int [d^3\delta \tilde{X}(x^4)] \exp \left[ - \int_0^{2\pi R} dx^4 \frac{M}{2} (\delta \tilde{X}(x^4))^2 \right],
\]

\[
\int [d^4\xi(x^4)] \exp \left[ - \int_0^{2\pi R} dx^4 L_\xi \right] = \int d^4\xi \int [d^4\delta \xi(x^4)] \exp \left[ - \int_0^{2\pi R} dx^4 \frac{M}{2} \delta \xi^A_\alpha(x^4) \delta \xi^A_\alpha(x^4) \right].
\]
The Gaussian integrals in (71) and (72) over $\delta X(x^4)$ and $\delta \xi(x^4)$ can be readily evaluated using standard textbook results, and we obtain:

$$
\int [d^3 X(x^4)] \exp \left[ - \int_0^{2\pi R_0} dx^4 L_X \right] = \int d^3 X \left[ \sqrt{\frac{M}{2\pi(2\pi R)}} \right]^3 ,
$$

$$
\int [d^4 \xi(x^4)] \exp \left[ - \int_0^{2\pi R_0} dx^4 L_\xi \right] = \int d^4 \xi \left[ \sqrt{\frac{M}{2\pi(2\pi R)}} \right]^{-4} .
$$

(73)

Next consider the path integral for $\varphi$, which encodes the motion of the monopole along $S^1$. The conjugate momentum $P_\varphi = \frac{M}{|a|^2} \dot{\varphi}$ to $\varphi$, is identified with the electric charge and is naturally quantised in integer units. The corresponding Hamiltonian is $H_\varphi = \frac{1}{2} |a|^2 P^2_\varphi$. The resulting states in four dimensions carry one unit of magnetic charge and $P_e = n_e$ units of electric charge and are naturally identified as the corresponding BPS dyons. We can then equate the path integral for $\int [d\varphi(x^4)] \exp \left[ - \int_0^{2\pi R_0} dx^4 L_\varphi \right]$ with the quantum mechanical partition function $\text{Tr}[e^{-\left(2\pi R_0\right) H_\varphi}]$, where the trace sums over the eigenstates $\sim e^{in_e\varphi}$, $n_e \in \mathbb{Z}$ of $H_\varphi$ and can be readily evaluated to give:

$$
\int [d\varphi(x^4)] \exp \left[ - \int_0^{2\pi R_0} dx^4 L_\varphi \right] = \sum_{n_e \in \mathbb{Z}} \exp \left[ - \frac{1}{2} \frac{|a|^2}{M} n_e^2 \right] .
$$

(74)

A further phase in the classical action arises from the surface terms coupling to electric and magnetic charge. Including a bare vacuum angle $\Theta$ and allowing for the Witten effect which shifts $n_e \rightarrow n_e + \frac{\Theta}{2\pi}$, the summation in (74) is replaced by

$$
\sum_{n_e \in \mathbb{Z}} \exp \left[ - \frac{1}{2} \frac{|a|^2}{M} \left( n_e + \frac{\Theta}{2\pi} \right)^2 + i \left( n_e + \frac{\Theta}{2\pi} \right) \theta_e \right] , \quad M = \frac{4\pi}{g^2 |a|} .
$$

(75)

We note that this matches the corresponding sum appearing in the GMN prediction [54] up to a replacement of the bare coupling and vacuum angle by their one-loop renormalised counterparts.

To complete the semiclassical integration measure, in addition to the zero modes discussed so far, it is necessary to include the non-zero mode fluctuations which lead to a non-cancelling ratio $R$ of functional determinants. Again, we start by reviewing the situation in the four-dimensional theory where similar fluctuations are taken into account in the calculation by Kaul [28] of the one-loop corrections to the monopole mass. In this case, the spatial fluctuations of the scalars, spinors and ghosts around the static monopole background are all described in terms of two operators $\Delta_{\pm}$ given explicitly as

$$
\Delta_+ = -D^2_j + |a|^2 , \quad \Delta_- = -D^2_j + |a|^2 + 2\epsilon_{ijk} \sigma_i F_{jk}^{\text{Mon.}} , \quad i, j = 1, 2, 3 .
$$

(76)

(77)

Here the three dimensional covariant derivative $D_j = \partial_j + i A_j^{\text{Mon.}}$ is with respect to background static monopole and, as above, $a$ is the VEV of the complex scalar in the massless $U(1)$ vector multiplet. The one-loop correction to the monopole mass in four dimensions then involves the
logarithm of the ratio $[\det(\Delta_+)/\det'(\Delta_-)]^{1/2}$, where the prime indicates that we have removed the zero mode contribution. As above, we are interested in the corresponding fluctuations around the monopole, thought of as a static configuration of finite Euclidean action on $\mathbb{R}^3 \times S^1$. In the absence of Wilson line (i.e. for $\theta_e = 0$), the corresponding fluctuation operators for our calculation are

$$D_\pm = \Delta_\pm + \left(\frac{\partial}{\partial x_4}\right)^2,$$

(78)

where the extra derivatives wrt to $x_4$ take account of the Fourier modes of each fluctuation field on $S^1$. We then identify the corresponding one-loop contribution to the path integral measure as

$$R = \left[\frac{\det(D_+)}{\det'(D_-)}\right]^{1/2}.$$

(79)

By translation invariance on $S^1$, we can decompose any eigenfunction of $D_\pm$ as $\Phi_\pm(\vec{x}, x^4) = \phi_\pm(\vec{x}) f_\pm(x^4)$, where $\phi_\pm(\vec{x})$ satisfy

$$\Delta_\pm \psi_\pm(\vec{x}) = \lambda^2_{\pm} \phi_\pm(\vec{x}),$$

(80)

while $f_\pm(x^4)$ along the compactified circle take the plane-wave form $f_\pm(x^4) \sim e^{i \omega_\pm x^4}$. In a supersymmetric theory, there are equal total number of non-zero eigenvalues for both bosonic and fermionic fields, this naively implies that their contributions cancel completely and $R = 1$. However the spectra of $D_\pm$ contain both normalizable bound states and continuous scattering states, as inherited from $\Delta_\pm$, the precise cancellation requires identical densities of bosonic and fermionic eigenvalues. As discovered by [28] this is not the case in the monopole background. The same effect leads to the non-cancelling one-loop determinant in the three-dimensional instanton calculation of [15] and we find a similar effect in the present case of $\mathbb{R}^3 \times S^1$.

Using the operator identity $\log \det(M) = \text{Tr} \log(M)$, we can rewrite $R$ as the following integral expression:

$$R = (2\pi R)^{-2} \exp \left[ \frac{1}{2} \text{Tr}_x \log[\det_4 D_+] - \frac{1}{2} \text{Tr}_x \log[\det_4 D_-] \right] = (2\pi R)^{-2} \exp \left[ \frac{1}{2} \int_{|a|}^\infty d\lambda \delta \rho(\lambda) \log |K(\lambda, 2\pi R)| \right],$$

(81)

$$K(\lambda, 2\pi R) = \det_{x^4} \left( \frac{\partial}{\partial x_4} \right)^2 + \lambda^2,$$

(82)

where the overall normalisation constant $(2\pi R)^{-2}$ was introduced so that $R$ goes over to the corresponding three-dimensional quantity calculated in [15]:

$$R^{(3D)} = \left[ \frac{\det(\Delta_+)}{\det'(\Delta_-)} \right]^{1/2} = 4M_W^2.$$  

(83)

\footnote{To properly take the three-dimensional limit, we need to first Poisson re-sum the explicit logarithmic expressions arising in (81), cf. (87), and (95) in the next section, before setting $R \to 0$.}
in the limit \( R \to 0, g^2 \to 0 \) with the three-dimensional gauge coupling \( e^2 = \frac{g^2}{2\pi R} \) held fixed. The quantity \( \delta \rho(\lambda) = \rho_+(\lambda) - \rho_-(\lambda) \) is the difference between densities of eigenvalues of the operators \( \Delta_+ \) and \( \Delta_- \). This quantity was determined using the Callias index theorem in [28]. In our notation the result of [28] is,

\[
d\lambda \delta \rho(\lambda) = -\frac{2|a|d\lambda^2}{\pi \lambda^2 \sqrt{\lambda^2 - |a|^2}}. \tag{84}
\]

The remaining kernel \( K(\lambda, 2\pi R) \) is precisely the partition function of harmonic oscillator with frequency \( \omega = \lambda \) at inverse temperature \( \beta = 2\pi R \).

\[
K(\lambda, 2\pi R)^{-1} = \frac{\exp[-\pi R\lambda]}{1 - \exp[-2\pi R\lambda]}. \tag{85}
\]

Introducing a non-vanishing Wilson line \( \theta_e \) corresponds to turning on the fourth component of the gauge field. This can be incorporated in the operators \( D_\pm \) given in (78) by the minimal coupling prescription,

\[
\frac{\partial}{\partial x_4} \rightarrow \frac{\partial}{\partial x_4} + n_e \frac{\theta_e}{2\pi R}
\]

which introduces a chemical potential which shifts the oscillator frequencies to the complex values \( \omega = \lambda + in_e \frac{\theta_e}{2\pi R} \). The fluctuation modes of each adjoint field include modes with \( n_e = \pm 1 \) filling out the supermultiplet of the \( W^{\pm} \) bosons. Summing over both contributions we find \( K = K_+ K_- \) where

\[
K_{\pm}(\lambda, \theta_e, 2\pi R)^{-1} = \frac{\exp[-\pi R\lambda \pm i\theta_e/2]}{1 - \exp[-2\pi R\lambda \pm i\theta_e]}. \tag{86}
\]

Substituting (86) and (84) into (81), and explicit change of variable \( \lambda = 2|a| \cosh t \), the one-loop determinant \( R \) is given by:

\[
\log R = -4R|a| \cosh^{-1} \frac{\Lambda_{UV}}{|a|} - 2\log(2\pi R)
+ \frac{2}{\pi} \int_0^\infty \frac{dt}{\cosh t} \log \left( 1 - e^{-2\pi R|a| \cosh t + i\theta_e} \right) + \frac{2}{\pi} \int_0^\infty \frac{dt}{\cosh t} \log \left( 1 - e^{-2\pi R|a| \cosh t - i\theta_e} \right) . \tag{87}
\]

where we have evaluated the integral over the eigenvalues with a UV cut-off \( \Lambda_{UV} \). The UV divergence in the first term is precisely that encountered in the four-dimensional calculation of [28]. The divergence is cancelled by the counter-term which is responsible for coupling constant renormalisation in the vacuum sector and the net effect is to replace the classical coupling \( g^2 \) appearing in the monopole mass by the one-loop effective coupling, \( g_{2 \text{eff}}^2(a) \). Similarly the chiral anomaly results in the replacement of the classical vacuum angle \( \Theta \) by its effective counterpart \( \Theta_{\text{eff}}(a) \) as defined above. The remaining finite terms yield a complicated function of the dimensionless parameter \( |a|/R \). However, we recognise the integral in the second line as precisely the same appearing in the definition (45) of the quantity \( \log D(-i) \) in the semiclassical expansion of the GMN result.
Collecting all the pieces and summing over electric charges \( n'_e \), we can extract the four-fermion vertex in the low-energy effective action from examining the large distance behavior of the four fermion correlation function \( G_4(y_1, y_2, y_3, y_4) \). Substituting (66), (73) and (87) into (68) and (69), the four-fermion correlation function is given by

\[
G_4(y_1, y_2, y_3, y_4) = 2^{13/2} \pi R |a| \left( \frac{2\pi R}{g_{\text{eff}}^2} \right)^{-1/2} \exp \left[ -S_{\text{Mon}} \right] \sum_{n'_e \in \mathbb{Z}} \exp \left[ -S_{\varphi}^{(n'_e)} \right] 
\times \int d^3 X \epsilon^{\alpha' \beta' \gamma'} S_F(y_1 - X)_{\alpha \alpha'} S_F(y_2 - X)_{\beta \beta'} S_F(y_3 - X)_{\gamma \gamma'} S_F(y_4 - X)_{\delta \delta'}.
\]

(88)

Here we have taken into account the one-loop renormalisation effect discussed earlier in the previous paragraph, so that the monopole and dyon actions \( S_{\text{Mon}} \) and \( S_{\varphi}^{(n'_e)} \) are given in terms of the effective parameters, as:

\[
S_{\text{Mon}} = \frac{8\pi^2 R}{g_{\text{eff}}^2} |a| - i \theta_m,
\]

(89)

\[
S_{\varphi}^{(n'_e)} = \frac{g_{\text{eff}}^2 R |a|}{4} \left( n'_e + \frac{\Theta_{\text{eff}}}{2\pi} \right)^2 - i \left( n'_e + \frac{\Theta_{\text{eff}}}{2\pi} \right) \theta_e.
\]

(90)

In (88), we have also used the relation between \( D(-i) \) and \( R \). Finally, for consistency, the same renormalisation of the classical coupling \( g^2 \), which leads to its replacement by the corresponding effective coupling \( g_{\text{eff}}^2(a) \) in the exponent, must also be implemented wherever the coupling appears \(^{12}\). In terms of the low-energy effective action, the resulting correlator corresponds to the appearance of a four-fermion interaction term of the form

\[
S_{4F} = 2^{9/2} \pi R |a|^{1/2} \left( \frac{2\pi R}{g_{\text{eff}}^2(a)} \right)^{7/2} D(-i) \exp \left[ -S_{\text{Mon}} \right] \sum_{n'_e \in \mathbb{Z}} \exp \left[ -S_{\varphi}^{(n'_e)} \right] \int d^3 \lambda (\lambda \cdot \bar{\lambda}) \gamma^\lambda \lambda_{\varphi}^\lambda,
\]

(91)

we see that this exactly matches the prediction coming from the integral equations of [9] given in (63)!

5 Interpolating to Three Dimensions

Having matched the predicted action (63) and the semiclassical result (91) for the dyon contributions, in this section we explain the relation to the semiclassical instanton result for the three-dimensional theory found in [15] which confirms that in the limit \( R \to 0 \), the hyper-Kähler metric on the Coulomb branch is given by Atiyah-Hitchin manifold [30]. To achieve this, we take the metric (17) which sums over all the electric charges of the dyons \( \{n'_e\} \) and Poisson resum it using the standard formula:

\[
\sum_{k=-\infty}^{+\infty} f(k) = \sum_{n=-\infty}^{+\infty} \hat{f}(n), \quad \hat{f}(n) = \int_{-\infty}^{+\infty} f(k) e^{-2\pi i n k} dk.
\]

(92)

\(^{12}\)Concretely, this renormalisation corresponds to a divergent contribution to the instanton measure arising from loop diagrams of perturbation theory in the monopole background.
This procedure exchanges the momentum modes along the compact direction which are identified with the electric charges of dyons with a corresponding set of winding modes \[^{13}\]. This resummation is necessary because the sum over electric charges appearing in \[^{21}\] and \[^{91}\] diverges in the \( R |a| \to 0 \) limit. We can in fact directly perform the Poisson resummation on the expression \[^{17}\] for the metric component \( g_{a\bar{a}} \). The relevant Fourier transform can be evaluated using eq. (6.726-4) in \[^{26}\] \[^{13}\], allowing us to rewrite the expression as,

\[
\tilde{g}_{a\bar{a}}^{\text{inst.}} = \frac{4\pi}{g_{\text{eff}2}} \sum_{n \in \mathbb{Z}} |a|^2 D(-i) \frac{|M(n)|^2}{|M(n)|^2} \exp \left( \frac{-8\pi^2 R}{g_{\text{eff}2}^2} |M(n)| + i\Psi(n) \right),
\]

where we have used the short-hand notation:

\[
|M(n)| = \sqrt{|a|^2 + \left( \frac{\theta_e}{2\pi R} + \frac{n}{R} \right)^2}, \quad \Psi(n) = \theta_m - n\Theta_{\text{eff}}.
\]

One can also Taylor expand and Poisson re-sum the prefactor \[^{15}\] to demonstrate that \( D(-i) \) satisfies the equation:

\[
\frac{d \log D(-i)}{d(2\pi R|a|)} = 2 \left( \sum_{n \in \mathbb{Z}} \frac{1}{(2\pi R)|M(n)|} - \frac{1}{\pi} \text{Arcsinh} \left( \frac{\Lambda_{\text{UV}}}{|a|} \right) \right).
\]

The quantity \( M(n) \) appearing in the exponent of \[^{93}\] corresponds to the Euclidean action of a “twisted monopole” \[^{16}\] \[^{27}\] \[^{13}\] \[^{14}\]. These are BPS field configuration in the compactified gauge theory on \( R^3 \times S^1 \) which are obtained by applying a large gauge transformation of the form \( A_4(x) \to A(x) + \partial \chi(x) \) with \( \chi(x_4 + 2\pi R) = \chi(x_4) + 2n\pi \). As \( \theta_e = \int dx^4 A_4 \), the Wilson line undergoes a periodic shift \( \theta_e \to \theta_e + 2n\pi \) under this transformation. These transformations are topologically non-trivial and are classified by an element of \( \pi_1(S^1) = \mathbb{Z} \). This leads to an infinite tower of field configurations for each value of of the magnetic charge labelled by the winding number \( n \). Summing over these configurations ensures that the metric retains the correct periodicity in \( \theta_e \).

To compare our prediction with the three-dimensional result, we take the limit \( R \to 0 \) while keeping the three-dimensional gauge coupling \[^{14}\] fixed: \( 1/e_{\text{eff}}^2 = 2\pi R/g_{\text{eff}2}^2 \). Note that \( |M(n)| \to \infty \) for \( n \neq 0 \), and thus only the \( n = 0 \) terms in \[^{93}\] and \[^{95}\] survive in this limit. In the strict three-dimensional limit, \( \{ \text{Re}(a), \text{Im}(a), \theta_e/2\pi R \} \) transform as a \( 3 \) under global \( SU(2)_N \) symmetry, hence we can rotate into \( \theta_e = 0 \) vacuum, and \[^{95}\] integrates into \( D(-i) = (4\pi R|a|)^2 \). Using this symmetry, \[^{91}\] , and the normalization factors given in \[^{58}\]-\[^{59}\], we can then deduce that the following four-fermions vertex is generated in the low-energy effective action:

\[
S_{4F} = \frac{2^7 \pi^3 M_W}{e_{\text{eff}}^8} \exp \left( \frac{-4\pi}{e_{\text{eff}}^2} M_W + i\theta_m \right) \int d^4x \left( \bar{\psi} \gamma^\mu (\lambda \cdot \partial) \right),
\]

\[^{13}\] The required result is obtained by approximating the Bessel function in the integrand of eq. (6.726-4) by its asymptotic form for large arguments.

\[^{14}\] Note that in \[^{15}\] the three and four-dimensional couplings are related via \( 1/e^2 = R/g^2 \), which differs from our convention by a factor of \( 2\pi \).
where $M_W = \sqrt{|a|^2 + (\theta_c/(2\pi R))^2}$. Comparing with eqs. (29, 34) in [15], we obtained a perfect match with the four-fermion vertex generated from the three-dimensional one instanton semiclassical computation, which was also in agreement with the prediction coming from the exact Atiyah-Hitchin metric given in eq. (54) of [15].

Acknowledgements

HYC is supported in part by NSF CAREER Award No. PHY-0348093, DOE grant DE-FG-02-95ER40896, a Research Innovation Award and a Cottrell Scholar Award from Research Corporation, and a Vilas Associate Award from the University of Wisconsin. KP is supported by a research studentship from Trinity College, Cambridge.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19 [Erratum-ibid. B 430 (1994) 485] [arXiv:hep-th/9407087].
[2] N. Seiberg and E. Witten, Nucl. Phys. B 431 (1994) 484 [arXiv:hep-th/9408099].
[3] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, Phys. Rept. 371 (2002) 231 [arXiv:hep-th/0206063].
[4] D. Finnell and P. Pouliot, Nucl. Phys. B 453, 225 (1995) [arXiv:hep-th/9503115].
[5] N. Dorey, V. V. Khoze and M. P. Mattis, Phys. Rev. D 54 (1996) 2921 [arXiv:hep-th/9603136].
[6] N. A. Nekrasov, Adv. Theor. Math. Phys. 7 (2004) 831 [arXiv:hep-th/0206161].
[7] M. Kontsevich and Y. Soibelman, arXiv:0811.2435 [math.AG].
[8] M. Kontsevich and Y. Soibelman, arXiv:0910.4315 [math.AG].
[9] D. Gaiotto, G. W. Moore and A. Neitzke, arXiv:0807.4723 [hep-th].
[10] D. Gaiotto, G. W. Moore and A. Neitzke, arXiv:0907.3987 [hep-th].
[11] A. B. Zamolodchikov, Nucl. Phys. B 342, 695 (1990).
[12] S. Alexandrov and P. Roche, arXiv:1003.3964 [hep-th].
[13] N. Dorey, JHEP 0104, 008 (2001) [arXiv:hep-th/0010115].
[14] N. Dorey and A. Parnachev, JHEP 0108, 059 (2001) [arXiv:hep-th/0011202].
[15] N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong and S. Vandoren, Nucl. Phys. B 502, 59 (1997) [arXiv:hep-th/9703228].
[16] K. M. Lee and P. Yi, Phys. Rev. D 56, 3711 (1997) [arXiv:hep-th/9702107].
[17] N. Seiberg and E. Witten, arXiv:hep-th/9607163.

[18] F. Ferrari and A. Bilal, Nucl. Phys. B 469, 387 (1996) arXiv:hep-th/9602082.

[19] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, Commun. Math. Phys. 108, 535 (1987).

[20] I. T. Ivanov and M. Rocek, Commun. Math. Phys. 182, 291 (1996) arXiv:hep-th/9512075.

[21] H. Ooguri and C. Vafa, Phys. Rev. Lett. 77 (1996) 3296 arXiv:hep-th/9608079.

[22] N. Seiberg and S. H. Shenker, Phys. Lett. B 388, 521 (1996) arXiv:hep-th/9608086.

[23] E. Tomboulis and G. Woo, Nucl. Phys. B 107, 221 (1976).

[24] E. Witten, Phys. Lett. B 86 (1979) 283.

[25] C. W. Bernard, Phys. Rev. D 19 (1979) 3013.

[26] I. S. Gradshteyn and I. M. Ryzhik, “Table of Integrals, Series, and Products”, Academic Press.

[27] N. Dorey, JHEP 9907, 021 (1999) arXiv:hep-th/9906011.

[28] R. K. Kaul, Phys. Lett. B 143, 427 (1984).

[29] J. P. Gauntlett, Nucl. Phys. B 411, 443 (1994) arXiv:hep-th/9305068.

[30] M. F. Atiyah and N. J. Hitchin, “The Geometry and Dynamics of Magnetic Monopoles. M.B. Porter Lectures”, Princeton, USA: Univ. Pr. (1988) 133p.