Uniform and $L^q$-Ensemble Reachability of Parameter-dependent Linear Systems

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Abstract

We consider families of linear systems that are defined by matrix pairs $(A(\theta), B(\theta))$ which depending on a parameter $\theta$ that is varying over a compact set in the plane. The focus of this paper is on the task of steering a family of initial states $x_0(\theta)$ in finite time arbitrarily close to a given family of desired terminal states $x^*(\theta)$ via a parameter-independent open-loop control input. In this case the pair $(A(\theta), B(\theta))$ is called ensemble reachable. Using well-known characterizations of approximate controllability for systems in Banach spaces, ensemble reachability of $(A(\theta), B(\theta))$ is equivalent to an infinite-dimensional extension of the Kalman rank condition. In this paper we investigate structural properties and prove a decomposition theorem according to the spectra of the matrices $A(\theta)$. Based on this results together with results from complex approximation and functional analysis we show necessary and sufficient conditions in terms of $(A(\theta), B(\theta))$ for ensemble reachability for families of linear systems $(A(\theta), B(\theta))$ defined on the Banach spaces of continuous functions and $L^q$-functions. The paper also presents results on output ensemble reachability for families $(A(\theta), B(\theta), C(\theta))$ of parameter-dependent linear systems.

Keywords: parameter-dependent systems, ensemble controllability, ensemble reachability, infinite-dimensional systems

MSC: 30E10, 47A16, 93B05

1 Introduction

In recent years the task of controlling a large, potentially infinite, number of states, or systems at once using a single open-loop input function or single feedback controller comprised an emerging field in mathematical systems and control theory. The term ensemble control is commonly used to refer to this problem area, cf. [6, Section 2.4]. We note that the term blending problem was also used in [33]. For instance, a typical situation is that the states or the observation of a system are not known exactly but only in terms of a probability distribution defined on the state space or in terms of snapshots of the outputs. In this case the ensemble control problem is to morph a given initial probability density function of the states into a desired one by transporting it along a linear system. This yields controllability problems for the Liouville equation and the Fokker-Plank equation. We refer to [6, 7, 12] for recent works on this topic. Similarly, the ensemble observation problem is to reconstruct the initial probability density function from output snapshots, cf. [38, 39].

In this paper we examine the ensemble control problem for systems that are subject to uncertainties, i.e. the systems depend on parameters and the goal is to achieve a control task by using a single open-loop input function which is independent from the parameter. More precisely, we consider families of parameter-dependent linear control systems

$$\frac{dx(t, \theta)}{dt} = A(\theta)x(t, \theta) + B(\theta)u(t)$$

$$x(0, \theta) = x_0(\theta) \in \mathbb{R}^n,$$ (1)

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where the matrices $A(\theta) \in \mathbb{C}^{n \times n}$ are assumed to depend continuously on the parameter $\theta$ which is varying over a nonempty compact set $\mathcal{P} \subset \mathbb{C}$. The regularity of the input matrix $B(\theta) \in \mathbb{C}^{n \times m}$ depends on the particular case under consideration and will then be specified. Note that, in the case $\mathcal{P} = \{\theta_1, ..., \theta_K\}$ the problem reduces to the classical parallel connection of finitely many linear systems. Since in this case the problem remains finite-dimensional the situation is well-understood, cf. [14]. We emphasize that the input operator $\|M\| = M_{\infty}$ is bounded linear with $M_{\infty} > 0$ where $M_{\infty}$ is any matrix norm. Note that the input operator $M_{\infty}$ is bounded linear. Throughout this paper we consider only separable $M$-Banach spaces.

For $u \in L^1([0, T], \mathbb{C}^m)$ or $u = (u_0, ..., u_{T-1})$, $u_i \in \mathbb{C}^m$, let $\varphi(T, \theta, u)$ denote the solution to (1) or (2), respectively.

The ensemble control problem for $M$ is the following. Given a family of initial states $x_0(\theta)$ and a family of terminal states $x^*(\theta)$ find an open-loop input function $u$ such that

$$\varphi(T, \theta, u) = x^*(\theta) \quad \text{for all } \theta \in \mathcal{P}.$$ 

The key point here is that the input function $u$ has to be independent of the parameters of the systems. We emphasize that without this crucial requirement the controllability analysis of systems (1) would be much simpler, cf. [9]. For continuous-time systems it is shown in [24, Theorem 3.1.1] that the ensemble control problem is never (exactly) solvable. The same arguments can be applied to discrete-time systems and show that the ensemble control problem is never exactly solvable for discrete-time systems. Therefore, only weaker notions of controllability are reasonable and we will focus the problem of steering a family of initial states $x_0(\theta)$ simultaneously in finite time $T > 0$ into a prescribed $\varepsilon$-neighborhood of a family of desired terminal states $x^*(\theta)$.

Before we define the reachability notion considered in this paper we present the notation that will be used in the following. Let $X_n(\mathcal{P})$ and $X_{n,m}(\mathcal{P})$ denote separable Banach spaces of functions defined on $\mathcal{P}$ with values in $\mathbb{C}^n$ and $\mathbb{C}^{n \times m}$, respectively. As usual we denote the space of continuous functions $A: \mathcal{P} \rightarrow \mathbb{C}^n$ by $C_n(\mathcal{P})$. We say that $X_n(\mathcal{P})$ is a separable $M$-Banach space if it is a separable Banach space and for any matrices $A \in C_{n,n}(\mathcal{P})$ the induced multiplication operator $M_A : X_n(\mathcal{P}) \rightarrow X_n(\mathcal{P})$,

$$M_A f(\theta) = A(\theta) f(\theta)$$

is bounded linear with

$$\|M_A\| = c \max_{\theta \in \mathcal{P}} \|A(\theta)\|_{n \times n},$$

where $c > 0$ and $\|\cdot\|_{n \times n}$ is any matrix norm. Note that the input operator $M_B : \mathbb{C}^m \rightarrow X_n(\mathcal{P})$,

$$M_B v(\theta) = B(\theta) v$$

is always bounded linear. Throughout this paper we consider only separable $M$-Banach spaces.

Then, we say that a pair $(A, B) \in C_{n,n}(\mathcal{P}) \times X_{n,m}(\mathcal{P})$ is ensemble reachable on $X_n(\mathcal{P})$ in time $T > 0$, if for all $f \in X_n(\mathcal{P})$ and $\varepsilon > 0$ there is an input $u \in L^1([0, T], \mathbb{C}^m)$ ($u = (u_0, ..., u_{T-1}), u_i \in \mathbb{C}^m$) such that

$$\|\varphi(T, u) - f\|_{X_n(\mathcal{P})} < \varepsilon.$$ 

If $T > 0$ is not fixed in advance, i.e. if $T > 0$ may additionally depend on the initial and terminal states $f_0$ and $f$, then the pair $(A, B)$ is called ensemble reachable on $X_n(\mathcal{P})$. For $X_n(\mathcal{P}) = C_n(\mathcal{P})$ we say that $(A, B)$ is uniformly ensemble reachable.

The ensemble control problem for parameter-dependent linear systems was introduced by [22] and we refer to [23] for a recent contribution. For nonlinear parameter-dependent systems results have been obtained by [1], where also the controllability of the moments of a parameter-dependent system is introduced. For linear parameter-dependent systems the moment controllability problem is also considered in [24] and [37]. Another aspect of ensemble control is steer the average of the ensemble states towards, or arbitrarily close to, a predefined value, cf. [25][10]. This is related to the problem of output ensemble reachability, which is discussed in Section 5 in this paper. We also
mention the work \cite{8} that considers the problem of asymptotically stabilizing a bilinear ensemble defined by a parameter-dependent controlled Bloch equation. Also we note that the previous contributions mentioned above consider only continuous-time systems and, hence, they use the notion ensemble controllability. However, since we are considering discrete-time systems as well and controllability and reachability are not equivalent for discrete-time systems we use the term ensemble reachability.

In this paper we consider structural properties of parameter-dependent linear systems of the form (I), which are equivalent to the linear control system

\[ \dot{x}(t) = M_A x(t) + M_B u(t) \]

on the separable \( \mathcal{M} \)-Banach space \( X_n(P) \). With this identification at hand, ensemble reachability of a parameter-dependent linear system \( (A, B) \in C_{n,n}(P) \times X_{n,m}(P) \) on the Banach space \( X_n(P) \) is equivalent to approximate controllability of the infinite-dimensional linear system \( (M_A, M_B) \) on the Banach space \( X_n(P) \). In the following we stick to the notion ensemble reachability to emphasize the special structure of the controllability problem, namely that the input space is finite-dimensional. Moreover, the problem is not covered by standard textbooks on infinite-dimensional linear system such as \cite{13} \cite{13}. We emphasize that the work of Triggiani \cite{34} provides a comprehensive treatment of controllability and observability of infinite-dimensional continuous-time linear systems on Banach spaces defined by bounded linear operators. In \cite{34}, Remark 3.1.2 it is shown that for arbitrary bounded linear operators \( A \) and \( B \) one has the following equivalences:

(a) System (6) is approximately controllable.

(b) There exists \( T > 0 \) such that system (6) is approximately controllable on \([0, T]\).

(c) For all \( T > 0 \), system (6) is approximately controllable on \([0, T]\).

Hence, the parameter-dependent ensemble \( (A, B) \in C_{n,n}(P) \times X_{n,m}(P) \) is \( X_n(P) \)-ensemble reachable if and only if it is on \( X_n(P) \)-ensemble reachable on some \([0, T]\).

The rest of the paper is organized as follows. Section 2 addresses structural properties of parameter-dependent linear systems defined on general separable \( \mathcal{M} \)-Banach spaces. These result are then used as building blocks in the upcoming sections to derive necessary and sufficient conditions for uniform ensemble reachability in Section 3 and \( L^2 \)-ensemble reachability in Section 4. Afterwards, Section 5 discusses the controllability properties of parameter-dependent systems with parameter independent outputs.

Notation and Definitions

We denote for a matrix \( A \in \mathbb{C}^{n \times m} \) the complex conjugate by \( A^* := \overline{A}^T \) and the its kernel by \( \ker A \). For \( \Omega \subset \mathbb{C} \) we say that \( \Omega \) does not separate the plane if \( \mathbb{C} \setminus \Omega \) is connected. Similarly we say that \( \Omega_1, \ldots, \Omega_n \) do not separate the plane (are non-separating) if \( \mathbb{C} \setminus (\cup \Omega_1 \cup \cdots \cup \Omega_n) \) is connected. Further, let \( \text{int}(\Omega) \) denote the interior of the set \( \Omega \) and let \( \overline{\Omega} \) denote the closure of \( \Omega \). We say that a set \( C \) is properly contained in \( \Omega \) if \( C \subset \text{int}(\Omega) \). A compact connected set in the complex plane containing more than one point is called a continuum. A set \( \Omega \) is locally connected if for every \( \omega \in \Omega \) and each neighborhood \( U \) of \( \omega \) there is a connected neighborhood \( V \) of \( \omega \) that is contained in \( U \). A set \( \Omega \) is called contractible if the identity map on \( P \) is homotopic to a constant mapping, i.e. for some \( p \in \Omega \) there is a continuous map \( F: [0,1] \times \Omega \rightarrow \Omega \) such that \( F(0,\omega) = \omega \) and \( F(1,\omega) = p \) for all \( \omega \in \Omega \). If \( \Omega \) is a finite set we denote by \( \text{card} \{\Omega\} \) the number of elements. If \( \Omega \subset \mathbb{R}^2 \) is compact we denote to volume by \( \text{vol}(\Omega) \). A path is a continuous image a compact interval and a Jordan curve is a homeomorphic image of the unit circle \( \partial \mathbb{D} \).

2 Decomposition of parameter-dependent linear systems

At the beginning of this section we shortly recap relevant known results on approximate controllability. In \cite{34} Theorem 3.1.1 it is shown that approximate controllability of (5) on \([0, T]\) is
equivalent to the density condition
\[ \sum_{k \in \mathbb{N}_0} \text{im} \mathcal{M}^k_A \mathcal{M}_B = X. \]

Taking the special structure of (3) into account, namely that \( \mathcal{M}_B \) has finite dimensional range, the latter density condition can be written as follows. Let \( b_1(\theta), \ldots, b_m(\theta) \) denote the columns of \( B(\theta) \) and let \( A^k b_j \) denote the continuous functions \( \theta \mapsto A(\theta)^k b_j(\theta) \) for \( j = 1, \ldots, m \) and \( k = 0, 1, 2, 3, \ldots \). Then, a pair \( (A, B) \in C_{n,n}(\mathcal{P}) \times X_{n,m}(\mathcal{P}) \) is ensemble reachable on \( X_n(\mathcal{P}) \) if and only if the set
\[ \mathcal{L}_{(A,B)} := \text{span}\{A^k b_j \mid 1 \leq j \leq m, \ k \in \mathbb{N}_0\} \]
is dense in \( X_n(\mathcal{P}) \) with respect to \( \| \cdot \|_{X_n(\mathcal{P})} \). Note that discrete-time systems are not considered in [34], but the latter equivalence also holds for discrete-time parameter-dependent systems, cf. [30] Theorem 1. Moreover, we emphasize that \( \mathcal{L}_{(A,B)} \) is dense in \( X_n(\mathcal{P}) \) if and only if for each \( \varepsilon > 0 \) and each \( f \in X_n(\mathcal{P}) \) there exist scalar polynomials \( p_1, \ldots, p_m \) such that
\[ \left\| \sum_{j=1}^m p_j(\theta) b_j - f \right\|_{X_n(\mathcal{P})} < \varepsilon. \quad (7) \]
The latter links on one hand ensemble reachability to polynomial approximation. On other hand, ensemble reachability of a pair \( (A,B) \) can also be expressed in terms of multiplicity of the matrix multiplication operator \( \mathcal{M}_A \), cf. for instance [17]. A bounded linear operator \( T \) defined on a separable Banach space \( X \) is called \( n \)-multicyclic if there is an \( n \)-tuple \( (x_1, \ldots, x_n) \in X \times \cdots \times X \) such that \( X \) equals the closure of
\[ \left\{ \sum_{k=1}^n p_k(T) x_k \mid p_k \text{ runs over all polynomials } , k = 1, \ldots, n \right\} \]
and \( n \) is minimal number such that this is possible. That is, \( (A,B) \) is ensemble reachability if and only if the matrix multiplication operator \( \mathcal{M}_A \) is \( n \)-multicyclic with \( n \)-tuple \( (b_1, \ldots, b_m) \).

Before stating and proofing structural results for ensemble reachable pairs \( (A,B) \) we present two auxiliary results. The first is devoted to the fact that previous works on ensemble reachability are limited to real pairs of matrix families \( (A(\theta), B(\theta)) \). Let \( X_n^C(\mathcal{P}) \) be a separable \( \mathcal{M} \)-Banach space of real-valued functions. Then \( X_n^C(\mathcal{P}) := \{g + ih \mid g, h \in X_n^C(\mathcal{P})\} \) denotes its complexification equipped with the norm \( \|g + ih\| := \max_{t \in [0,2\pi]} \|g \cos(t) - h \sin(t)\| \). For details we refer to [29].

**Proposition 1.** Let \( (A,B) \) be a real pair. Then \( (A,B) \) is ensemble reachable on \( X_n^C(\mathcal{P}) \) if and only if \( (A,B) \) ensemble reachable on \( X_n^C(\mathcal{P}) \), i.e. if for each \( \varepsilon > 0 \) and for each \( f \in X_n^C(\mathcal{P}) \) there are real polynomials \( p_1, \ldots, p_m \) such that
\[ \left\| \sum_{j=1}^m p_j(\theta) b_j - f \right\|_{X_n^C(\mathcal{P})} < \varepsilon. \]

**Proof.** For simplicity, let \( m = 1 \). Suppose that \( (A,B) \) is ensemble reachable on \( X_n^C(\mathcal{P}) \). Let \( \varepsilon > 0 \) and \( f \in X_n^C(\mathcal{P}) \). Then, there is a complex polynomial \( p(z) = c_0 + c_1 z + \cdots + c_k z^k \) such that
\[ \|p(A) b - f\|_{X_n^C(\mathcal{P})} < \varepsilon \]
In particular, for \( r(z) = \text{Re}(c_0) + \text{Re}(c_1) z + \cdots + \text{Re}(c_k) z^k \) we have
\[ \|r(A) b - f\|_{X_n^C(\mathcal{P})} < \varepsilon, \]
which shows the claim.

Conversely, let \( \varepsilon > 0 \) and \( f = g + ih \in X_n^C(\mathcal{P}) \). By assumption, there are real polynomials \( r \) and \( q \) such that
\[ \|r(A) b - g\|_{X_n^C(\mathcal{P})} < \frac{\varepsilon}{2} \quad \text{and} \quad \|q(A) b - h\|_{X_n^C(\mathcal{P})} < \frac{\varepsilon}{2}. \]
Thus, defining \( p(z) := r(z) + iq(z) \) we have

\[
\|p(A) b - f \|_{X_N^2(P)} < \varepsilon.
\]

This shows the assertion. \( \square \)

Second, we consider parameter-dependent systems and their restriction to subsets of the parameter space. Suppose \( P_1 \subset P_2 \) we say that a pair \( (X(P_2), X(P_1)) \) has the restriction property if the restriction operator \( \mathcal{R} : X(P_2) \to X(P_1) \) is well-defined, bounded and onto. Then we have the following

**Lemma 1.** (a) If \((A, B)\) is ensemble reachable on \( X_n(P_2) \) and if the pair \( (X_n(P_1), X_n(P_2)) \) has the restriction property then \((A, B)\) is ensemble reachable on \( X_n(P_1) \).

(b) Let \( P_1, P_2 \) be compact with \( P_1 \subset P_2 \). Then the pair \( (C_n(P_2), C_n(P_1)) \) has the restriction property.

(c) Let \( P_1, P_2 \) be measurable with \( P_1 \subset P_2 \). Then for all \( 1 \leq q < \infty \) the pair \( (L^q_n(P_2), L^q_n(P_1)) \) has the restriction property.

**Proof.** (a): Let \( \varepsilon > 0 \) and \( f \in X_n(P_1) \). By the restriction property, the restriction operator \( \mathcal{R} \) is onto, i.e., \( \sum_{j=1}^{m} p_j(A)b_j - f \in X(P_1) \) has a preimage, say \( \sum_{j=1}^{m} \tilde{p}_j(A)b_j - \tilde{f} \in X_n(P_2) \). Hence, it holds

\[
\| \sum_{j=1}^{m} p_j(A)b_j - f \|_{X_n(P_1)} = \| \mathcal{R}(\sum_{j=1}^{m} \tilde{p}_j(A)b_j - \tilde{f}) \|_{X_n(P_1)} \leq C\| \sum_{j=1}^{m} \tilde{p}_j(A)b_j - \tilde{f} \|_{X_n(P_2)} < C\varepsilon.
\]

(b): Follows from Tietze’s extensions theorem [28] 20.4.

(c): Obvious. \( \square \)

Next, we consider an array of linear parameter-dependent systems \( (A_{ij}, B_{ij}) \in C_{n_i, n_j}(P) \times X_{n_i, n_j}(P) \) with \( 1 \leq i \leq j \leq N \). Let \( \overline{n} = n_1 + \cdots + n_N \) and \( \overline{m} = m_1 + \cdots + m_N \). Define the associated upper triangular parameter-dependent systems by

\[
A = \begin{pmatrix}
A_{11} & \cdots & A_{1N} \\
& \ddots & \\
0 & & A_{NN}
\end{pmatrix} \in C_{\overline{n}, \overline{m}}(P), \quad B = \begin{pmatrix}
B_{11} & \cdots & B_{1N} \\
& \ddots & \\
0 & & B_{NN}
\end{pmatrix} \in X_{\overline{n}, \overline{m}}(P). \tag{8}
\]

Then, we obtain

**Proposition 2.** The upper triangular pair \((A, B)\) is ensemble reachable on \( X_{\overline{n}}(P) \) if the diagonal pairs \((A_{ii}, B_{ii})\) are ensemble reachable on \( X_{n_i}(P) \) for all \( i = 1, \ldots, N \).

**Proof.** Suppose the diagonal pairs \((A_{ii}, B_{ii})\) are ensemble reachable on \( X_{n_i}(P) \). We consider continuous-time systems. For simplicity we focus on \( N = 2 \), i.e., on

\[
\begin{align*}
\dot{x}_1(t) &= A_{11}(\theta)x_1(t) + A_{12}(\theta)x_2(t) + B_{11}(\theta)u_1(t) + B_{12}(\theta)u_2(t) \\
\dot{x}_2(t) &= A_{22}(\theta)x_2(t) + B_{22}(\theta)u_2(t).
\end{align*}
\]

The general case is treated, proceeding by induction. Let \( \varphi_i(t, \theta, u) \) denote the solution of the \( i \)-th component.

Let \( f = \begin{pmatrix} f_1 \  f_2 \end{pmatrix} \in X_{n_1 + n_2}(P) \) and \( \varepsilon > 0 \). By ensemble rachability of \((A_{22}, B_{22})\) there exist \( T_2 > 0 \) and \( u_2^* \in L^1([0, T_2], C^{m_2}) \) such that

\[
\| \varphi_2(T_2, \cdot, u_2^*) - f_2 \|_{X_{n_2}(P)} < \varepsilon.
\]
Let \( u = (u_t^1) \in L^1([0, T], \mathbb{C}^m) \) with \( u_t \in L^1([0, T], \mathbb{C}^m) \), where \( u_t^2(t) = 0 \) for all \( T < t \leq T \).

Applying the variations of constant formula we have

\[
\varphi(T, \theta, u) = \int_0^T e^{(t-s)A_{11}(\theta)} \left( A_{12}(\theta) x_2(s) + B_{11}(\theta) u_1(s) + B_{12}(\theta) u_2^* (s) \right) ds
\]

and thus

\[
\varphi(T, \theta, u) = \int_0^T e^{(T-s)A_{11}(\theta)} B_{11}(\theta) u_1(s) ds + \Psi(T, \theta, u_2^*),
\]

where

\[
\Psi(T, \theta, u_2^*) = \int_0^T e^{(T-s)A_{11}(\theta)} \left( A_{12}(\theta) \int_0^s e^{(s-\tau)A_{22}(\theta)} B_{22}(\theta) u_2^*(\tau) d\tau + B_{12}(\theta) u_2^*(s) \right) ds.
\]

By ensemble reachability of \((A_{11}, B_{11})\) on \( X_n(P) \), there exist \( T_1 > 0 \) and \( u_t^1 \in L^1([0, T_1], \mathbb{R}^m) \) with

\[
\left\| \int_0^{T_1} e^{(T-s)A_{11}(\cdot)} B_{11}(\cdot) u_1(s) ds - (f_1(\cdot) - \Psi(T_1, u_2^*)) \right\|_{X_n(P)} < \varepsilon.
\]

This in turn implies

\[
\| \varphi(T_1, \cdot, u) - f_1(\cdot) \|_{X_{n_1+n_2}(P)} < \varepsilon.
\]

Let \( T := \max(T_1, T_2) \) and \( u^* = (u_t^1) \in L^1([0, T], \mathbb{C}^m) \) with \( u_t^* \in L^1([0, T], \mathbb{R}^m) \), where \( u_t^*(t) = 0 \) for all \( T < t \leq T \). Then, we have

\[
\| \varphi(T, \cdot, u^*) - f \|_{X_{n_1+n_2}(P)} < \varepsilon.
\]

This shows the assertion. \( \Box \)

The latter statement is an extension and correction of \([30, \text{Proposition 2}]\), where it was claimed that the reverse implication also holds. This, however, is false in general, see Section 3.

Before we provide an infinite-dimensional analog for the reachability of the parallel connection of linear systems, we recall some notation and results from the literature that will be used in the latter. Let \( P \subseteq \mathbb{C} \) be compact and \( A \in C_{n,n}(P) \). Obviously, the spectrum of the bounded linear matrix multiplication operator \( \mathcal{M}_A : x(\theta) \mapsto A(\theta)x(\theta) \) acting on \( X_n(P) \) is contained in the union of all the eigenvalues of the matrices \( A(\theta) \), i.e.

\[
\sigma_{X_n(P)}(\mathcal{M}_A) \subseteq \bigcup_{\theta \in P} \sigma (A(\theta)),
\]

where \( \sigma(A(\theta)) \) denotes the spectrum of the matrix \( A(\theta) \) for a fixed \( \theta \in P \). This follows simply from the fact that for any \( \rho \in \mathbb{C} \setminus \bigcup_{\theta \in P} \sigma (A(\theta)) \) the multiplication operator induced by \((A(\theta) - \rho I_n)^{-1}\) is well-defined and bounded. In many standard cases, e.g. for \( X_n(P) = C_n(P) \) or \( X_n(P) = L^2(P) \), one has even equality in (9), cf. \([18]\). But there are also simple examples which demonstrate that equality in general false, e.g. for \( X := \{ x \in C([-1, 1], \mathbb{C}^n) \mid x(\theta) = 0 \ \text{for all} \ \theta \in [-1, 0] \} \). To avoid later confusions, it will be convenient to denote the above union by \( \text{spec} A \), i.e.

\[
\text{spec} A := \bigcup_{\theta \in P} \sigma (A(\theta)).
\]

**Theorem 1.** Let \( P \subseteq \mathbb{C} \) be compact. Suppose the pairs \((A_i, B_i) \in C_{n_i,n_i}(P) \times X_{n_i,m}(P) \), \( i = 1, 2, \ldots, N \) are ensemble reachable on \( X_n(P) \) and satisfy \( \text{spec} A_i \cap \text{spec} A_j = \emptyset \) for all \( i \neq j \) and \( \text{spec} A_i \) does not separate the plane for all \( i = 1, \ldots, N \). Then the pair

\[
\begin{pmatrix}
A_1 \\
\vdots \\
A_N
\end{pmatrix}, \begin{pmatrix}
B_1 \\
\vdots \\
B_N
\end{pmatrix} \in C_{\bar{n},n}(P) \times X_{\bar{n},m}(P), \quad \bar{n} = n_1 + \cdots + n_N
\]

is ensemble reachable on \( X_{\bar{n}}(P) \).
Proof. The proof will be given for the case \( \varepsilon > 0 \) to the general case. Let \( \varepsilon > 0 \) and 
where \( \gamma \) is a closed curve enclosing the spectrum of \( A \). That is, for any polynomial \( p \) there are polynomials \( q_1 \) and \( q_2 \) such that 

\[ \|p_1(A_1)b_1 - f_1\|_{X_{n_1}(P)} < \varepsilon \quad \text{and} \quad \|p_2(A_2)b_2 - f_2\|_{X_{n_2}(P)} < \varepsilon. \]

By assumption the compact sets spec \( A_1 \) and spec \( A_2 \) are disjoint and do not separate the plane, thus, the application of Lemma 6 (Appendix) yields disjoint compact sets \( K_1 \) and \( K_2 \) which do not separate the plane and properly contain spec \( A_1 \) and spec \( A_2 \), respectively. Then, we consider the functions 

\[
\begin{align*}
    &h_1: K_1 \cup K_2 \to \mathbb{C}, \quad h_1(z) = \begin{cases} 1 & \text{if } z \in K_1 \\ 0 & \text{if } z \in K_2 \end{cases} \\
    &h_2: K_1 \cup K_2 \to \mathbb{C}, \quad h_2(z) = \begin{cases} 0 & \text{if } z \in K_1 \\ 1 & \text{if } z \in K_2. \end{cases}
\end{align*}
\]

By Lemma 7 (Appendix) there are polynomials \( q_1 \) and \( q_2 \) such that 

\[ |h_1(z) - q_1(z)| < \varepsilon \quad \text{and} \quad |h_2(z) - q_2(z)| < \varepsilon \quad \text{for all } z \in K_1 \cup K_2. \]

Thus, one has 

\[ |q_1(z)| < (1 + \varepsilon) \quad \text{and} \quad |q_2(z)| < \varepsilon \quad \text{for all } z \in K_1. \]

Defining the polynomial 

\[ p(z) := q_1(z)p_1(z) + q_2(z)p_2(z), \]

it suffices to verify that 

\[ \left\| \begin{pmatrix} p(A_1)b_1 - f_1 \\ p(A_2)b_2 - f_2 \end{pmatrix} \right\|_{X_{n_1+n_2}(P)} \leq k \varepsilon, \]

for some \( k > 0 \). W.l.o.g. we consider the case \( i = 1 \) and show 

\[ \|p(A_1)b_1 - f_1\|_{X_{n_1}(P)} \leq k \varepsilon. \]

It holds 

\[
\begin{align*}
    \|p(A_1)b_1 - f_1\|_{X_{n_1}(P)} &= \|q_1(A_1)p_1(A_1)b_1 + q_2(A_1)p_2(A_1)b_1 - f_1\|_{X_{n_1}(P)} \\
    &\leq \|q_1(A_1)p_1(A_1)b_1 - p_1(A_1)b_1\|_{X_{n_1}(P)} + \|q_2(A_1)p_2(A_1)b_1\|_{X_{n_1}(P)} \\
    &\leq \|q_1(A_1) - I\| \|p_1(A_1)\| \|b_1\|_{X_{n_1}(P)} + \|p_1(A_1)b_1 - f_1\|_{X_{n_1}(P)} \\
    &\quad + \|q_2(A_1)p_2(A_1)\| \|b_1\|_{X_{n_1}(P)},
\end{align*}
\]

where \( \|q_1(A_1) - I\|, \|p_1(A_1)\| \) and \( \|q_2(A_1)p_2(A_1)\| \) are operator norms, cf. [11]. To show the claim we consider the latter terms separately and use the Dunford-Taylor formula, cf. [21] Chapter 1, § 5, Section 6]. That is, for any polynomial \( p \) and matrix \( M \) we have 

\[ p(M) = \frac{1}{2\pi i} \int_{\gamma} p(z)(zI - M)^{-1} \, dz, \]

where \( \gamma \) is a closed curve enclosing the spectrum of \( M \). Note that the polynomial \( \tilde{q}_1(z) := q_1(z) - 1 \) satisfies \( |\tilde{q}_1(z)| < \varepsilon \) for all \( z \in K_1 \) and, thus, we have 

\[
\begin{align*}
    &\|q_1(A_1) - I\| \leq \varepsilon \max_{\theta \in P} \|\tilde{q}_1(A_1(\theta))\|_{n_1 \times n_1} \leq \frac{1}{2\pi} \int_{\gamma} |\tilde{q}_1(z)| \|zI - A_1(\theta)\|^{-1}\|_{n_1 \times n_1} \, dz \\
    &\leq \frac{\varepsilon L^*}{2\pi} \max_{\theta \in P} \max_{z \in \gamma} \|zI - A_1(\theta)\|^{-1}\|_{n_1 \times n_1},
\end{align*}
\]
Similarly, it holds
\[
\left\| q_2(A_1) p_2(A_1) \right\| \leq \varepsilon \max_{\theta \in \mathbb{P}} \frac{1}{2\pi} \int_{\gamma} |q_2(z)| |p_2(z)| \left\| (zI - A_1(\theta))^{-1} \right\|_{n \times n} \, dz
\]
\[
\leq \varepsilon \frac{L}{2\pi} \max_{z \in \gamma} |p_2(z)| \max_{\theta \in \mathbb{P}} \max_{z \in \gamma} \left\| (zI - A_1(\theta))^{-1} \right\|_{n \times n}.
\]
Defining \( \beta_1 := \|b_1\|_{X_{n_1}(\mathbb{P})}, \alpha_1 := \max_{\theta \in \mathbb{P}} \max_{z \in \gamma} \left\| (zI - A_1(\theta))^{-1} \right\|_{n_1 \times n_1}, \alpha_2 := \max_{z \in \gamma} |p_2(z)|, \) and
\[
k := \frac{c^2 L}{2\pi} \alpha_1 \alpha_3 \beta_1 + 1 + \frac{L}{2\pi} \alpha_1 \alpha_2 \beta_1
\]
we obtain
\[
\left\| p(A_1) b_1 - f_1 \right\|_{X_{n_1}(\mathbb{P})} \leq k \varepsilon.
\]
This shows the assertion.

We note that a similar technique for the construction of the polynomials \( p_n \) has been used in \([2]\), where also interpolation properties are considered. Therein, the functions \( h_1 \) and \( h_2 \) are approximated by polynomials due to Walsh’s theorem \([35, \text{pp. 77/78}]\) that also provides an error bound.

In order to consider non-structured matrix pairs \((A, B)\) and the corresponding multiplication operators, we consider the following decompositions of spec \( A \). A multi-valued map \( \Gamma : \mathbb{P} \to \text{spec} A \) with \( \Gamma(\theta) \subset \sigma(A(\theta)) \) for all \( \theta \in \mathbb{P} \) is termed spectral selection. for \( A \). If \( \Gamma \) is continuous with respect to the Hausdorff metric, then it is called a continuous spectral selection. The selection is called single valued if \( \Gamma \) is single valued and will be denoted by \( \lambda : \mathbb{P} \to \mathbb{C} \). Moreover, two spectral selections \( \Gamma_1 \) and \( \Gamma_2 \) are pointwise disjoint if \( \Gamma_1(\theta) \cap \Gamma_2(\theta) = \emptyset \) for all \( \theta \in \mathbb{P} \). They are strictly disjoint if \( \Gamma_1(\mathbb{P}) \cap \Gamma_2(\mathbb{P}) = \emptyset \). Obviously, strict disjointness implies pointwise disjointness. Finitely many (continuous) spectral selections \( \Gamma_1, \ldots, \Gamma_k \) are called a (continuous) spectral family for \( A \) if
\[
\bigcup_{i=1}^k \Gamma_i(\theta) = \sigma(A(\theta))
\]
for all \( \theta \in \mathbb{P} \). Note, that \( \Gamma_1, \ldots, \Gamma_k \) are not required to be disjoint in any sense. Certainly, there exists always a continuous spectral family of \( A \), for instance the trivial one \( \Gamma(\theta) := \sigma(A(\theta)) \), and sometimes this is even the only one which is continuous as in the case
\[
A(\theta) := \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}, \quad \theta \in \mathbb{P} := [0,1].
\]
However, locally or if \( \mathbb{P} \) is homeomorphic to \([0,1]\) one has the following.

**Lemma 2.** Let \( \mathbb{P} \subset \mathbb{C} \) be compact and \( A \in C_{n,n}(\mathbb{P}) \).

(a) For every open subset \( U \subset \mathbb{P} \) there exists an open subset \( V \subset U \) such that the restriction \( A|_V \) allows a single valued continuous spectral family.

(b) If \( \mathbb{P} \) is homeomorphic to \([0,1]\) then there exists a global single valued continuous spectral family for \( A \).

**Proof.** (a): Assume w.l.o.g. \( U = \mathbb{P} \) and define \( k := \max_{\theta \in \mathbb{P}} |\sigma(A(\theta))| \). Choose \( \theta_0 \in \mathbb{P} \) with \( |\sigma(A(\theta_0))| = k \). Then there exist small disjoint neighbourhoods \( U_i \) with \( \lambda_i \in U_i \) for \( i = 1, \ldots, k \), where \( \sigma(A(\theta_0)) = \{\lambda_1, \ldots, \lambda_k\} \). Then Rouche’s Theorem \([23, \text{Theorem 10.43 (b)}]\) and the maximality of \( k \) guarantees the existence of an open neighbourhood \( V \) of \( \theta_0 \) with \( |\sigma(A(\theta)) \cap U_i| = 1 \) for \( i = 1, \ldots, k \) and all \( \theta \in V \). This allows to define a single-valued continuous spectral family on \( V \).

(b): See \([21, \text{§ II.5.2}].\)
The next step is to obtain a block-diagonal decomposition of the matrices $A(\theta)$. We will assume that the parameter set $P \subset \mathbb{C}$ is additionally contractible. Note that if $P$ is compact interval it is contractible.

**Lemma 3.** Let $P \subset \mathbb{C}$ be compact and contractible and let $A \in C_{n,n}(P)$. Assume that $\Gamma_1, \ldots, \Gamma_k$ is a pairwise pointwise disjoint continuous spectral family. Then there exists a continuous family of invertible matrices $T(\theta)$ such that

$$T(\theta)^{-1}A(\theta)T(\theta) = \begin{pmatrix} A_1(\theta) & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & A_k(\theta) \end{pmatrix}$$

(10)

and the spectra of $A_i(\theta)$ are given by $\Gamma_i(\theta)$ for all $\theta \in P$ and $i = 1, \ldots, k$.

**Proof.** Since $\Gamma_1(\theta), \ldots, \Gamma_k(\theta)$ are pairwise pointwise disjoint we can construct cycles $\Sigma_1(\theta), \ldots, \Sigma_k(\theta)$ in the complex plane for all $\theta \in P$ such that

$$\nu_{\Sigma_i(\theta)}(z) = \begin{cases} 1 & \text{for all } z \in \Gamma_i(\theta), \\ 0 & \text{for all } z \in \sigma(A(\theta)) \setminus \Gamma_i(\theta) \end{cases}$$

where $\nu_{\Sigma_i(\theta)}(z)$ denotes the winding number of $z \in \mathbb{C}$ with respect to $\Sigma_i(\theta)$. Hence, we can define the following spectral projections

$$P_i(\theta) = \frac{1}{2\pi i} \int_{\Sigma_i(\theta)} (zI - A(\theta))^{-1} \, dz.$$

We claim that the map $\theta \mapsto P_i(\theta)$ is continuous and that the rank of $P_i(\theta)$ is constant with respect to $\theta$. To see that $\theta \mapsto P_i(\theta)$ is continuous we first note that $\Gamma_i$ is continuous, i.e. the Hausdorff distance between $\Gamma_i(\theta)$ and $\Gamma_i(\theta')$ tends to zero as $\theta'$ tends to $\theta$. Therefore, $\Gamma_i(\theta')$ is contained in $\Omega_i(\theta') := \{ z \in \mathbb{C} \mid \nu_{\Sigma_i(\theta)}(z) = 1 \}$ for $\theta'$ sufficiently close to $\theta$ and thus, by Cauchy’s Theorem [28, Theorem 10.35], one has

$$P_i(\theta') = \frac{1}{2\pi i} \int_{\Sigma_i(\theta')} (zI - A(\theta'))^{-1} \, dz.$$

for $\theta'$ sufficiently close to $\theta$. From the above representation of $P_i(\theta')$, it follows that $\theta \mapsto P_i(\theta)$ is continuous. Next, note that $\sum_{i=1}^k \text{rank } P_i(\theta) = n$ holds for all $\theta \in P$. This follows from the fact that the $\Sigma_1 + \cdots + \Sigma_k$ circles around the spectrum of $A(\theta)$. Moreover, by continuity with respect to $\theta$ one knows that the rank of $P_k(\theta')$ is greater or equal to the rank of $P_k(\theta)$ in a neighborhood of $\theta$. Since this holds for all $i = 1, \ldots, k$ we conclude that the rank of $P_i(\theta)$ is locally constant with respect to $\theta$ and because of the connectedness of $P$ globally constant. Finally, we can apply a generalization of Doležal’s result [11] by Grasse [15] which guarantees for all $i = 1, \ldots, k$ the existence of a continuous family of matrices $T_i(\theta) \in \mathbb{C}^{n \times n_i}$ with $n_i := \text{rank } P_i(\theta)$ such that the columns of $T_i(\theta)$ span the image of $P_i(\theta)$. Hence, for $i = 1, \ldots, k$, we obtain

$$A(\theta)T_i(\theta) = T_i(\theta)A_i(\theta)$$

with $A_i(\theta) = (T_i(\theta)^*T_i(\theta))^{-1}T_i(\theta)^*A(\theta)T_i(\theta) \in \mathbb{C}^{n_k \times n_i}$. Stacking all $T_i(\theta)$ together, i.e. setting

$$T(\theta) := (T_1(\theta) \mid \ldots \mid T_k(\theta)) \in \mathbb{C}^{n \times n},$$

yields the desired result

$$T(\theta)^{-1}A(\theta)T(\theta) = \begin{pmatrix} A_1(\theta) & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & A_k(\theta) \end{pmatrix}.$$
Due to Lemma 11 and 13, it suffices in many cases to focus on contractible subset. Henceforth, we assume that \( P \subset \mathbb{C} \) is contractible. This includes the cases where \( P \) is compact interval. Let \((A, B) \in C_{n,n}(P) \times X_{n,m}(P)\) and let \( \Gamma_1, \ldots, \Gamma_k \) be a pairwise pointwise disjoint spectral family. Then the subsystems \((A_i, B_i) \in C_{n,n}(P) \times X_{n,m}(P)\) given by

\[
A_i(\theta) := \Pi_i T^{-1}(\theta) A(\theta) T(\theta) \Pi_i^*, \quad B_i(\theta) := \Pi_i T^{-1}(\theta) B(\theta)
\]

and \( \Pi_i := [0 \ldots 0 \ I_{n_i} \ 0 \ldots 0] \in \mathbb{C}^{n \times n} \) result from the decomposition of Lemma 13 are called associated subpairs. Moreover, the multiplication operator \( M_T : X(P) \to X(P), \ M_T x(\theta) := T(\theta) x(\theta) \) is referred to as associated transformation map.

**Theorem 2.** Let \( P \subset \mathbb{C} \) be compact and contractible and let \((A, B) \in C_{n,n}(P) \times X_{n,m}(P)\). Moreover, let \( \Gamma_1, \ldots, \Gamma_k \) be pairwise strictly disjoint continuous spectral family.

(a) If \((A, B)\) is ensemble reachable on \( X_n(P) \) then the associated subpairs \((A_i, B_i)\) are ensemble reachable on \( X_i(P) := \Pi_i M^{-1}_T X(P) \) for all \( i = 1, \ldots, k \).

(b) Conversely, if for all \( i = 1, \ldots, k \) the associated subpairs \((A_i, B_i)\) are ensemble reachable on \( X_i(P) := \Pi_i M^{-1}_T X(P) \) and if additionally the spectral sets \( \Gamma_i(P) \), \( i = 1, \ldots, k \) do not separate the plane, then \((A, B)\) is ensemble reachable on \( X_n(P) \).

Proof. (a): Assume that \((A, B)\) is ensemble reachable on \( X \). Then obviously \((T^{-1} \ A T^{-1} B)\) is also ensemble reachable on \( X \) and hence \((A_i, B_i) = (\Pi_i T^{-1} A T^{-1} \Pi_i, \Pi_i T^{-1} B)\) are ensemble reachable on \( X_i(P) := \Pi_i M^{-1}_T X(P) \) for \( i = 1, \ldots, k \).

(b): Conversely, assume that the pairs \((A_i, B_i)\) are ensemble reachable on \( X_i(P) := \Pi_i M^{-1}_T X(P) \) for \( i = 1, \ldots, k \). Then, according to Theorem 1 and the fact that spec \( A_i = \Gamma_i(P) \), the parameter-dependent family \((T^{-1} A T^{-1} B)\) is ensemble reachable on \( X \) and hence likewise \((A, B)\).

The significance of Theorem 2 is that it allows to decompose the ensemble reachability problem into several smaller problems according to the spectral decomposition of the matrix multiplication operator defined by \( A(\theta) \).

**Remark 1.** (a) In Theorem 2(b) the additional assumption that the spectral sets are non-separating is necessary, cf. Example 1 below.

(b) Lemma 13 is false without the assumption that \( P \) is contractible; even simply connectedness of \( P \) is not sufficient, cf. Example 2 below.

**Example 1.** The following example shows that the additional non-separating condition in Theorem 2(b) is necessary. Consider

\[
A(\theta) := \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & \theta \end{pmatrix} \quad \text{and} \quad b(\theta) := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for} \quad \theta \in P := [-\frac{1}{2}, \frac{1}{2}].
\]

Let \( \mathbb{D} \) denote the unit disc and let \( A(\mathbb{D}) \) denote the disc algebra, that is \( f \in A(\mathbb{D}) \) if and only if \( f : \mathbb{D} \to \mathbb{C} \) is holomorphic and \( f \) extends continuously to \( \overline{\mathbb{D}} \). Moreover, let \( X(P) := X_1(P) \times X_2(P) \) with

\[
X_1 := \{ x \in X(P) \mid x(-\frac{1}{2}) = x(\frac{1}{2}) \} \quad \text{and} \quad \exists f \in A(\mathbb{D}) : f(e^{2\pi i \theta}) = x(\theta) \quad \forall \theta \in P \}
\]

and \( X_2 := C(P) \). Then the subsystems \((A_1, b_1)\) and \((A_2, b_2)\) are ensemble reachable on \( X_1(P) \) and \( X_2(P) \) but \((A, b)\) is not ensemble reachable on \( X(P) := X_1(P) \times X_2(P) \).

This can be seen as follows. Choose \( f \in A(\mathbb{D}) \) and define \( f^*(\theta) := f(e^{2\pi i \theta}) \) for all \( \theta \in P \). Let \( p_n \) be a sequence of complex polynomials such that

\[
\max_{\theta \in \mathbb{D}} \| p_n(e^{2\pi i \theta}) - f^*(\theta) \| = \max_{\theta \in \mathbb{D}} \| p_n(e^{2\pi i \theta}) - f(e^{2\pi i \theta}) \| \to 0 \quad \text{for} \quad n \to \infty.
\]

Hence, due to the maximum principle, \( p_n(\theta) \) converges uniformly on \( \overline{\mathbb{D}} \) to the holomorphic function \( f \) and thus \( p_n(\theta) \) has to converge also to \( f(\theta) \) for all \( \theta \in [-\frac{1}{2}, \frac{1}{2}] \), i.e. there is no degree of freedom for choosing \( x_2(\theta) \).
Example 2. Let $P$ be a polish circle defined as follows. Let $P := P_1 \cup P_1 \cup P_3$ with

$$P_1 := \{0\} \times [-1, 1], \quad P_2 := \left\{ (t, \sin \left(\frac{1}{t}\right)) \mid t \in (0, \frac{1}{\pi}] \right\},$$
and

$$P_3 := \{ \gamma(t) \mid t \in [0, \frac{1}{\pi}] \},$$

where $\gamma(t)$ is any continuous one-to-one path which connects the points $(0, -1) \in P_1$ and $(\frac{1}{\pi}, 0) \in P_2$ and does not intersect $P_1 \cup P_2$ in any other point. Moreover, let $A(\theta)$ be given by

$$A(\theta) := \begin{cases} 
    \begin{pmatrix}
    0 & 0 \\
    0 & 0
    \end{pmatrix}, & \text{for } \theta \in P_1, \\
    \begin{pmatrix}
    \cos \frac{1}{t} & -\sin \frac{1}{t} \\
    \sin \frac{1}{t} & \cos \frac{1}{t}
    \end{pmatrix} \begin{pmatrix}
    t & 0 \\
    0 & -t
    \end{pmatrix} \begin{pmatrix}
    \cos \frac{1}{t} & \sin \frac{1}{t} \\
    -\sin \frac{1}{t} & \cos \frac{1}{t}
    \end{pmatrix}, & \text{for } \theta = (t, \sin \frac{1}{t}) \in P_2, \\
    \begin{pmatrix}
    t & 0 \\
    0 & -t
    \end{pmatrix}, & \text{for } \theta = \gamma(t) \in P_3.
\end{cases}$$

Then $P$ is simply connected but not contractible and $A(\theta)$ is continuous. However, $A(\theta)$ is not continuously diagonalizable.

3 Uniform ensemble reachability

In this section we focus on necessary and sufficient conditions for ensemble reachability on the separable Banach space of continuous functions, i.e. we consider $X_n(P) = C_n(P)$. To ease notation we shortly write $(A, B) \in C_{n,m}(P)$. In doing so, we will put special emphasis on conditions which are testable only in terms of the matrix pair $(A, B)$. We will first treat parameter-dependent systems with a single input and multi-input systems afterwards. Note that the results of this section extend previous results in [16, 23, 29, 30], where the parameter set is assumed to be a compact real interval.

Single-input parameter-dependent systems

We begin this section with conditions on the matrix pair $(A, B)$ that are necessary for uniform ensemble reachability. The following statement provides an extension of the necessary conditions for single-input parameter-dependent linear systems given in [16, Lemma 1].

Theorem 3. Let $P \subset \mathbb{C}$ be compact. Suppose $(A, b) \in C_{n,1}(P)$ is uniformly ensemble reachable. Then, the following necessary conditions hold:

(a) The pairs $(A(\theta), b(\theta))$ are reachable for all $\theta \in P$.

(b) The eigenvalues of $A(\theta)$ have geometric multiplicity one for all $\theta \in P$.

(c) $\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset$ for all $\theta, \theta' \in P$ with $\theta \neq \theta'$.

(d) If $P$ is additionally contractible and locally connected then one has $\text{card} \{ \sigma(A(\theta)) \} = n$ for generic $\theta \in P$, more precisely, then the set $\{ \theta \in P \mid \text{card} \{ \sigma(A(\theta)) \} = n \}$ is open and dense in $P$.

(e) The set $P \subset \mathbb{C}$ has no interior points (relative to $\mathbb{C}$).

Proof. (a): Let $\theta \in P$. Applying the restriction property, Lemma 11 to $P_1 := \{ \theta \}$ it follows that for every $\varepsilon > 0$ and $\xi \in \mathbb{C}^n$ there are $T > 0$ and an input $u$ such that the solution $(A(\theta), b(\theta))$, denoted by $\varphi(T, \theta, u)$ satisfies $\|\varphi(T, \theta, u) - \xi\| < \varepsilon$. The assertion then follows from the fact that reachable set are closed.

(b): Follows immediately from (a) together with the Hautus-Lemma [32, Lemma 3.3.7].
(c): Let \( \theta \neq \theta' \in \mathcal{P} \) and suppose that \( \lambda \in \sigma(A(\theta)) \cap \sigma(A(\theta')) \). Then, applying the restriction property to \( \mathcal{P}_1 := \{\theta, \theta'\} \) and using by part (a), the parallel connection
\[
\left( \begin{array}{cc}
A(\theta) & 0 \\
0 & A(\theta')
\end{array} \right) = \left( \begin{array}{cc}
b(\theta) & b(\theta')
0 & b(\theta')
\end{array} \right)
\]
is reachable. Then, we have
\[
\text{rank}\left( \begin{array}{cc}
\lambda I - A(\theta) & 0 \\
0 & \lambda I - A(\theta')
\end{array} \right) \leq 2n - 1 < 2n,
\]
a contradiction to the Hautus Lemma [22 Lemma 3.3.7].

(d): Let \( \mathcal{P}_n := \{ \theta \in \mathcal{P} \mid \text{card} \{\sigma(A(\theta))\} = n \} \). Obviously, due to Rouche’s Theorem \( \mathcal{P}_n \) is open. Therefore, it remains to show that \( \mathcal{P}_n \) is dense in \( \mathcal{P} \). Assume that \( \mathcal{P}_n \) is not dense. Then there exists a non-empty open subset \( U \subset \mathcal{P} \setminus \mathcal{P}_n \). Define \( m := \max_{\theta \in U} \text{card} \{\sigma(A(\theta))\} \). By assumption one has \( m < n \). Again, by Rouche’s Theorem, one can show that the non-empty set \( U_m := \{ \theta \in U \mid \text{card} \{\sigma(A(\theta))\} = m \} \) is open and that the algebraic multiplicities of the eigenvalues are locally constant in \( U_m \). Therefore, possibly by passing to a open subset, we can assume that the algebraic multiplicities of the eigenvalues \( \lambda_1(\theta), \ldots, \lambda_m(\theta) \) are locally constant in \( U_m \). Moreover, by part (b) we already know that the geometric multiplicities of the eigenvalues are constant, too. Hence, for all \( \theta_0 \in U_m \) there exists \( r_0 > 0 \) such that one can simultaneously transform \( A(\theta) \) into Jordan canonical from for all \( \theta \in K_{r_0}(\theta_0) := \{ \theta \in \mathcal{P} \mid ||\theta - \theta_0|| \leq r_0 \} \) (Note, here we use only continuity and not Doležal!). By Theorem 2 (a) and Lemma 1 (b) it suffices to consider a single Jordan block \( (J(\theta), b(\theta)) \) on \( \mathcal{P}_0 = K_{r_0}(\theta_0) \). The claim then follows as from arguments used in the proof Proposition 5.

(e): Assume that \( \theta_0 \in \mathcal{P} \) is an interior point of \( \mathcal{P} \). Moreover, according to part (d) (note that the restriction of \( (A, B) \) to a closure of an open ball satisfies the assumption of having no isolated points) we can assume w.l.o.g. that there exists \( r > 0 \) such that \( K_r(\theta_0) := \mathcal{P}_0 \subset \mathcal{P}_n \subset \mathcal{P} \). Now, applying Lemma 3 to the restriction of \( (A, B) \) to \( \mathcal{P}_0 \) and since \( (A(\theta), b(\theta)) \) is pointwise reachable there is continuous change of coordinates such that
\[
T(\theta)^{-1}A(\theta)T(\theta) = \begin{pmatrix}
\lambda_1(\theta) & 0 \\
0 & \lambda_2(\theta) & \ddots & \\
& & \ddots & 0 \\
& & & \lambda_k(\theta)
\end{pmatrix} 
\]
and
\[
T(\theta)^{-1}b(\theta) = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} \quad (15)
\]
where \( \lambda_i(\theta) \) for \( i = 1, \ldots, k \) are the disjoint eigenvalue curves of \( A(\theta) \) on \( K_r(\theta_0) = \mathcal{P}_0 \). By part (c) the curves \( \lambda_i \) are continuous and one-to-one on \( \mathcal{P}_0 \) for all \( i = 1, ..., n \). Thus, for e.g. \( i = 1 \) and for every \( \varepsilon > 0 \) and \( f_1 \in C(\mathcal{P}_0) \) by ensemble reachability there is a polynomial \( p \) such that \( |p(\lambda_1(\theta)) - f_1(\theta)| < \varepsilon \) for all \( \theta \in \mathcal{P}_0 \). In particular, for all \( z \in \lambda_1(\mathcal{P}_0) \) we have \( |p(z) - f_1(z)| < \varepsilon \). Thus, the continuous and non-analytic function \( f_1 \circ \lambda_1 \) can be uniformly approximated by polynomials on \( \lambda_1(\mathcal{P}_0) \) and since \( \lambda_1(\mathcal{P}_0) \) has interior points we obtain a contradiction cf. [28, 20.1] or Example 4 below.

In the following we consider sufficient conditions in terms of the pair \( (A, B) \) for uniform ensemble reachability. In the simplest case \( A(\theta) \) is one by one and a pair \( (a(\theta), b(\theta)) \) is reachable if and only if \( b(\theta) \neq 0 \). Next, we show that for scalar pairs the necessary conditions are also sufficient. Later, in Theorem 3 we will show that for non-scalar pairs uniform ensemble reachability can in general only be concluded under suitable extra assumptions.

**Proposition 3.** Let \( \mathcal{P} \subset \mathbb{C} \) be a compact and contractible. Then, the scalar pair \( (a, b) \in C_{1,1}(\mathcal{P}) \) is uniformly ensemble reachable if and only if \( a \) is one-to-one and \( b(\theta) \neq 0 \) for all \( \theta \in \mathcal{P} \).

**Proof.** The necessity part follows from Theorem 3 (a) and (c). To show sufficiency let w.l.o.g. \( b \equiv 1 \). Let \( f \in C(\mathcal{P}, \mathbb{C}) \) and \( \varepsilon > 0 \). It suffices to conclude that there is a polynomial \( p \) such that
\[
\sup_{\theta \in \mathcal{P}} |p(a(\theta)) - f(\theta)| < \varepsilon.
\]
Since $a: \mathbb{P} \to a(\mathbb{P}) \subset \mathbb{C}$ is continuous, one-to-one and onto $a(\mathbb{P})$ is compact, has empty interior and does not separate the complex plane. Thus, $f \circ a^{-1}: a(\mathbb{P}) \to \mathbb{C}$ is continuous and by Mergelyan’s Theorem [28, Theorem 20.5] there is a polynomial $p$ such that

$$\sup_{z \in a(\mathbb{P})} |f(a^{-1}(z)) - p(z)| < \varepsilon.$$ 

This shows the assertion. 

Recall that for matrices depending continuously on a parameter a continuous transformation to the Jordan canonical form is not available in general, cf. [21, Lemma 4 for ensemble reachability. A, $b$ is invertible for all $\theta$ uniformly ensemble reachable. Therefore from the same reasoning. 

**Lemma 4** (Controllability form). Let $\mathbb{P} \subset \mathbb{C}$ be compact and suppose that $(A(\theta), b(\theta))$ is reachable for all $\theta \in \mathbb{P}$. Then

$$T(\theta) := (b(\theta), A(\theta)b(\theta), \ldots, A^{n-1}(\theta)b(\theta))$$

is invertible for all $\theta$ and one has

$$A_c(\theta) = T(\theta)^{-1}A(\theta)T(\theta) = \begin{pmatrix} 0 & \cdots & 0 & a_0(\theta) \\ 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{n-1}(\theta) \end{pmatrix},$$

where $a_k(\theta)$ are the coefficients of the characteristic polynomial, i.e. $\chi_{A(\theta)}(\lambda) = z^n - a_{n-1}(\theta)z^{n-1} - \cdots - a_0(\theta)$. Moreover, the pair $(A, b)$ is uniformly ensemble reachable if and only if $(A_c, e_1)$ is uniformly ensemble reachable.

**Proof.** Let $\theta \in \mathbb{P}$. By the Kalman rank condition [32, Section 3.2, Theorem 3] the pair $(A(\theta), b(\theta))$ is reachable if and only if the matrix $T(\theta)$ has rank $n$, i.e. $T(\theta)$ is invertible. Further, by the Cayley-Hamilton Theorem we have $A(\theta)^n = a_0(\theta)I + \cdots + a_{n-1}(\theta)A(\theta)^{n-1}$ for all $\theta \in \mathbb{P}$ and therefore

$$T(\theta)A_c(\theta) = (b(\theta), A(\theta)b(\theta), \ldots, A^{n-1}(\theta)b(\theta))$$

$$= (A(\theta)b(\theta), A(\theta)^2b(\theta), \ldots, a_0(\theta)b(\theta) + \cdots + a_{n-1}(\theta)A(\theta)^{n-1}b(\theta))$$

$$= (A(\theta)b(\theta), A(\theta)^2b(\theta), \ldots, A^n(\theta)b(\theta)) = A(\theta)T(\theta).$$

Since $T(\theta)$ is invertible, $b(\theta), A(\theta)b(\theta), \ldots, A^{n-1}b(\theta)$ is a basis and therefore $T(\theta)^{-1}b(\theta) = e_1$.

To see the second claim, suppose that $(A_c, e_1)$ is uniformly ensemble reachable. Since $T(\cdot)$ is continuous and $\mathbb{P}$ is a compact we define $d := \sup_{\theta \in \mathbb{P}} \|T(\theta)\| < \infty$. So for any $p \in \mathbb{C}[z]$ and $f \in C_n(\mathbb{P})$ one has

$$\sup_{\theta \in \mathbb{P}} \|p(A(\theta)) \cdot b(\theta) - f(\theta)\| \leq \sup_{\theta \in \mathbb{P}} \|T(\theta)\| \sup_{\theta \in \mathbb{P}} \|p(A_c(\theta))e_1 - T(\theta)^{-1}f(\theta)\|$$

$$= d \sup_{\theta \in \mathbb{P}} \|p(A_c(\theta))e_1 - T(\theta)^{-1}f(\theta)\|.$$ 

As $T$ is invertible and continuous, we have $T^{-1}C_n(\mathbb{P}) = C_n(\mathbb{P})$ and thus, $(A, b)$ is uniformly ensemble reachable if $(A_c, e_1)$ is uniformly ensemble reachable. The reverse implication follows from the same reasoning.
The following statement extends Proposition \[3\] to non-scalar single input pairs. This requires a condition on the characteristic polynomials. As a consequence, the subsequent result, which is a generalization of \[24\] Theorem 2.1] provides a sufficient condition for uniform ensemble reachability. In contrast to the scalar case it is no longer necessary.

**Theorem 4.** Let $\mathbf{P} \subset \mathbb{C}$ be compact and contractible and let the pair $(A, b) \in C_{n,1}(\mathbf{P})$ satisfy the necessary conditions in Theorem 3. Then, $(A, b)$ is uniformly ensemble reachable if the characteristic polynomials of $A(\theta)$ take the form $z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0(\theta)$, for some $a_{n-1}, \ldots, a_1 \in \mathbb{C}$ and $a_0 \in C_1(\mathbf{P})$.

**Proof.** By Lemma 1\[ we can assume that $(A(\theta), b(\theta))$ is in controllability form, i.e.

$$A(\theta) = \begin{pmatrix} 0 & \cdots & 0 & a_0(\theta) \\ 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & a_{n-1} \end{pmatrix} \quad b(\theta) = e_1.$$  

To show the claim, we verify that for $\varepsilon > 0$ and $f \in C_n(\mathbf{P})$ there is a polynomial $p$ so that

$$\sup_{\theta \in \mathbf{P}} \|p(A(\theta))e_1 - f(\theta)\| < \varepsilon.$$  

To this end, let $g(z) := z^n - a_{n-1}z^{n-1} - \cdots - a_1 z$ and define

$$p(z) := \sum_{k=1}^{n} p_k(g(z))z^{k-1},$$  

with $p_k \in \mathbb{C}[z]$, $k = 1, \ldots, n$. As $A(\theta)^k e_1 = e_{k+1}$ and $g(A(\theta)) = a_0(\theta)I$ we have

$$p(A(\theta))e_1 = \sum_{k=1}^{n} p_k(g(A(\theta)))A(\theta)^{k-1} e_1 = \sum_{k=1}^{n} p_k(g(A(\theta))e_k = \begin{pmatrix} p_1(a_0(\theta)) \\ \vdots \\ p_n(a_0(\theta)) \end{pmatrix}.$$  

Consequently, it remains to conclude that for appropriate choices of $p_k$, $k = 1, \ldots, n$ one has

$$\sup_{\theta \in \mathbf{P}} |p_k(a_0(\theta)) - f_k(\theta)| < \varepsilon \quad \text{for all } k = 1, \ldots, n.$$  

The necessary conditions together with $\chi_A(\theta)(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0(\theta)$ imply that $a_0 : \mathbf{P} \to a_0(\mathbf{P})$ is bijective and has a continuous inverse $a_0^{-1} : a_0(\mathbf{P}) \to \mathbf{P}$. Therefore, as $\mathbf{P}$ is contractible it is simply connected. Moreover, by Theorem 3\[(e) the set $\mathbf{P}$ has no interior points and it follows that the interior of $a_0(\mathbf{P})$ is empty and $\mathbb{C} \setminus a_0(\mathbf{P})$ is connected. Thus, by Mergelyan’s Theorem \[28\] Theorem 20.5] there are polynomials $p_k$ such that

$$\sup_{z \in a_0(\mathbf{P})} |p_k(z) - f_k(a_0^{-1}(z))| < \varepsilon \quad \text{for all } k = 1, \ldots, n.$$  

This shows the assertion. \[\square\]

**Remark 2.** If $\mathbf{P}$ is compact real interval, the conclusions in the proofs of Proposition 3\[ and Theorem 4\[ can also be drawn by replacing Mergelyan’s theorem by Walsh’s extension \[35\] Theorem 8, § II.2.5] of the Weierstrass approximation theorem.

An essential impact of Theorem 1\[ is that it allows for a combination of existing sufficient conditions. For instance, the pair

$$A(\theta) = \begin{pmatrix} 0 & -\theta^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \theta^2 + 1 \end{pmatrix}, b(\theta) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{P} = [0, 1]$$
is uniformly ensemble reachable which follows from Theorem 1 together with Theorem 4. We
note that this conclusion cannot be drawn by solely applying Theorem 4. More generally, the
Theorems 1 and 2 allow for the following approach to investigate uniform ensemble reachability.
Determine the strictly disjoint continuous spectral family of $A(\theta)$, or equivalently of the matrix
multiplication operator and, by Theorem 1, it suffices to consider the resulting parts individually.
This leads to the following sufficient conditions.

**Theorem 5.** Let $P \subset \mathbb{C}$ be compact and contractible with empty interior (relative to $\mathbb{C}$). Then
$(A, b) \in C_{n,1}(P)$ is uniformly ensemble reachable if the following conditions are satisfied.

(a) The pairs $(A(\theta), b(\theta))$ are controllable for all $\theta \in P$.

(b) There exists a pairwise strictly disjoint continuous spectral family $\Gamma_i(\theta)$ with non-separating
spectral sets $\bigcup_{\theta \in P} \Gamma_i(\theta)$.

(c) The first derivative of the corresponding characteristic polynomials $\chi_i$ are $\theta$-independent, i.e.
$\chi_{A_i(\theta)}(z) = z^n_i + a_{i,n-1}z^{n_i-1} + \cdots + a_{i,1}z + a_{i,0}(\theta)$.

(d) The maps $\theta \mapsto a_{i,0}(\theta)$ are one-to-one.

**Proof.** By condition (b) we can apply Lemma 3 and conclude the existence of a continuous family
$T(\theta)$ of invertible transformations such that $A(\theta)$ becomes block-diagonal, cf. [10]. By Theorem 2 (b) it is sufficient to verify that each associated subsystem is uniformly ensemble reachable.
Since we have single-input pairs, the assumptions (a), (b) and (d) imply the necessity conditions in
Theorem 3 for each associated block system $(A_k, b_k)$. The assertion then follows from condition (c)
in turn with Theorem 4.

The subsequent statement provides a mild extension of [16, Theorem 1], where $P$ was assumed
to be a compact real and interval and follows from Theorem 5 together with Lemma 2 (b).

**Corollary 1.** Let $P \subset \mathbb{C}$ be compact and homeomorphic to $[0, 1]$. Then, the pair $(A, b) \in C_{n,1}(P)$
is uniformly ensemble reachable if the following conditions are satisfied.

(a) The pairs $(A, b)$ are controllable for all $\theta \in P$.

(b) $\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset$ for all $\theta, \theta' \in P$ with $\theta \neq \theta'$.

(c) The eigenvalues of $A(\theta)$ are simple for all $\theta \in P$.

**Multi-input parameter-dependent systems**

Now we investigate parameter-dependent systems with more than one input. As in the single-
input case we begin with necessary conditions for uniform ensemble reachability for pairs $(A, B) \in C_{n,m}(P)$.

**Theorem 6.** Let $P \subset \mathbb{C}$ be compact. If the pair $(A, B) \in C_{n,m}(P)$ is uniformly ensemble reachability the following necessary conditions hold:

(a) For each $\theta \in P$ the pair $(A(\theta), B(\theta))$ is reachable.

(b) For any $s \geq m + 1$ and pairwise distinct $\theta_i \neq \theta_j$, $i \neq j \in \{1, \ldots, s\}$ one has
$\sigma(A(\theta_1)) \cap \cdots \cap \sigma(A(\theta_s)) = \emptyset$.

**Proof.** Part (a) follows as in Theorem 3 and part (b) is a consequence of the Hautus-Lemma [32]
Lemma 3.3.7].
Before extending the known results in \[16\] Lemma 1 we recall the Hermite canonical form for a reachable system \((A,B) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m}\), cf. [32] pp. 194-196. Let \(b_i\) denote the \(i\)th column of \(B\). Select from left to right in the permutated Kalman matrix
\[
\begin{pmatrix}
(b_1 \quad A b_1 \quad \cdots \quad A^{n-1} b_1) \\
\vdots \\
(b_m \quad A b_m \quad \cdots \quad A^{n-1} b_m)
\end{pmatrix}
\]
the first linear independent columns. Then, one obtains a list of basis vectors
\[
b_1, \ldots, A b_1, \ldots, b_m, \ldots, A^{h_m-1} b_m
\]
of the reachability subspace. The integers \(h_1, \ldots, h_m\) are called the Hermite indices, where \(h_i := 0\) if the column \(b_i\) has not been selected. One has \(h_1 + \cdots + h_m = n\) if and only if \((A,B)\) is reachable. Similar to Lemma 4 (see also [20, Section 6.4.6, Scheme I]), the invertible transformation
\[
T = (b_1, \ldots, A b_1, \ldots, b_m, \ldots, A^{h_m-1} b_m)
\]
yields the Hermite canonical form
\[
\begin{pmatrix}
A_{11} & \cdots & A_{1m} \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{mm}
\end{pmatrix}, \quad
\begin{pmatrix}
b_1 & 0 \\
0 & b_m
\end{pmatrix},
\]
where the \(m\) single-input subsystems \((A_{kk}, b_k) \in \mathbb{C}^{n_k \times n_k} \times \mathbb{C}^{m_k}\) are reachable and in control canonical form. Note that \(b_1\) denotes the first standard basis vector.

**Theorem 7.** Let \(P \subset \mathbb{C}\) be compact and contractible with empty interior (relative to \(\mathbb{C}\)). Then \((A,B) \in C_{n,m}(P)\) is uniformly ensemble reachable if the following conditions are satisfied.

(a) The pairs \((A(\theta), B(\theta))\) are reachable for all \(\theta \in P\).

(b) The input Hermite indices of \((A(\theta), B(\theta))\) do not depend on \(\theta \in P\).

(c) The corresponding subpairs \((A_{ii}, b_i)\) are uniformly ensemble reachable.

**Proof.** By condition (b) the continuous family \(T(\theta)\) of invertible changes of coordinates transform the pair \((A,B)\) into the \([16]\). Since each associated subsystem \((A_{ii}, b_i)\) is uniformly ensemble reachable the claim follows from applying Proposition 4. \(\square\)

Using the derived sufficient conditions for single input pairs, the latter statement immediately provides an extension of [16] Theorem 1, where \(P\) was assumed to be a compact real interval.

**Corollary 2.** Let \(P \subset \mathbb{C}\) be compact and homomorphic to \([0,1]\) with empty interior (relative to \(\mathbb{C}\)). Then, the pair \((A,B) \in C_{n,m}(P)\) is uniformly ensemble reachable if the following conditions are satisfied.

(a) The pairs \((A(\theta), B(\theta))\) are reachable for all \(\theta \in P\).

(b) The input Hermite indices of \((A(\theta), B(\theta))\) do not depend on \(\theta \in P\).

(c) \(\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset\) for all \(\theta, \theta' \in P\) with \(\theta \neq \theta'\).

(d) The eigenvalues of \(A(\theta)\) are simple for all \(\theta \in P\).

In case the pair \((A,B) \in C_{n,m}(P)\) admits the special form \((\theta A, B)\), where \(A \in \mathbb{C}^{n \times n}\) and \(B \in \mathbb{C}^{n \times m}\), the above necessary and sufficient conditions can be stated more precisely, cf. [23, 53]. Next, we consider upper triangular pairs of continuous matrices \(A\) and \(B\), i.e.
\[
A(\theta) = \begin{pmatrix}
a_{11}(\theta) & \cdots & a_{1n}(\theta) \\
\vdots & \ddots & \vdots \\
a_{nn}(\theta)
\end{pmatrix} \quad \text{and} \quad B(\theta) = \begin{pmatrix}
b_{11}(\theta) & \cdots & b_{1n}(\theta) \\
\vdots & \ddots & \vdots \\
b_{nn}(\theta)
\end{pmatrix}.
\]

In this case we obtain the following sufficient condition for uniform ensemble reachability.
Proposition 4. Let \( P \subset \mathbb{C} \) be a compact and contractible. The pair \((A, B) \in C_{n,n}(P)\) defined in (20) is uniformly ensemble reachable if \( \text{rank } B(\theta) = n \) for all \( \theta \in P \) and \( a_{ii} \) is one-to-one for all \( i = 1, \ldots, n \).

Proof. By Proposition 2 it is sufficient to consider the diagonal pairs \((a_{ii}, b_{ii}) \in C_{1,1}(P)\). As \( \text{rank } B(\theta) = n \) for all \( \theta \in P \) it follows that \( b_{ii}(\theta) \neq 0 \) for all \( \theta \in P \) and for all \( i = 1, \ldots, n \). Since the functions \( a_{ii} \) are one-to-one for all \( i = 1, \ldots, n \) we can apply Proposition 3 and the claim follows. \( \square \)

Note that the converse of the latter statement is false in general. This can be seen using the sufficient conditions for single-input parameter-dependent systems, e.g. Corollary 1. Next we consider a case providing necessary and sufficient conditions. Suppose the eigenvalues of \( A \) have algebraic multiplicities greater then one and geometric multiplicity one, i.e. let \( \lambda \) be a compact and contractible. The pair \((J, B)\) which extends \( J, B \) to a diagonal pair \((a_{ii}, b_{ii}) \in C_{1,1}(P)\).

Proposition 5. Let \( P \subset \mathbb{C} \) be compact, contractible and locally connected. Then, the pair \((J, B) \in C_{n,n}(P)\) defined in \( J, B \) is uniformly ensemble reachable if and only if \( \text{rank } B = n \) and \( \lambda \) is one-to-one.

Proof. We begin with the sufficiency part. Suppose that \( \lambda \) is one-to-one and \( \text{rank } B = n \). Then, w.l.o.g. let \( B = I \) and by Proposition 2 it is sufficient to show that the pair \((\lambda, 1)\) is uniformly ensemble reachable for every \( i = 1, \ldots, n \). Since \( \lambda \) is one-to-one the claim follows from Proposition 3.

Conversely, suppose the pair \((J, B)\) is uniformly ensemble reachable. First, suppose that \( \lambda \) is not one-to-one, i.e. there are \( \theta_1 \neq \theta_2 \in P \) such that \( \lambda(\theta_1) = \lambda(\theta_2) =: \lambda \). From the proof of Lemma 1 in \( \lambda \) it follows that the finite-dimensional pair

\[
\begin{pmatrix}
J(\theta_1) & 0 \\
0 & J(\theta_2)
\end{pmatrix}
\begin{pmatrix}
B \\
B
\end{pmatrix}
\]

is reachable. Then, from the Hautus Lemma \( B \) it follows

\[
2n = \text{rank} \begin{pmatrix}
\lambda I - J(\theta_1) & 0 \\
0 & \lambda I - J(\theta_2)
\end{pmatrix} \leq 2n - 1,
\]

a contradiction.

To see the second claim, w.l.o.g. we treat the case \( n = 2 \). Suppose that \( \text{rank } B < 2 \), i.e. w.l.o.g. \( B = \begin{pmatrix} b_1 & 0 \\ 0 & 1 \end{pmatrix} \). Note that, since \((J(\theta), B)\) is reachable for every \( \theta \in P \) one has \( |\lambda_{12}(\theta)| \neq 0 \) for all \( \theta \in P \). Since \( \lambda \) is continuous and \( P \) is compact and locally connected, by Lemma 5 (in the Appendix), the set \( \lambda(P) \) contains a path \( \gamma \). We first discuss the cases where the length of \( \gamma \) is finite, i.e. \( L_\gamma < \infty \).

Let \( \varepsilon > 0 \) and \( f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C_2(P) \) with

\[
c = \varepsilon \left(2L_\gamma^{-1} + \frac{1 + |b_1|}{\min_{\theta \in P} |\lambda_{12}(\theta)|} \right) + 1.
\]

As the pair \((J, B)\) is uniformly ensemble reachable there is a polynomial \( p \) such that

\[
\left\| p(J) \begin{pmatrix} b_1 \\ 1 \end{pmatrix} - \begin{pmatrix} c \lambda_{12} \\ 0 \end{pmatrix} \right\|_\infty < \varepsilon.
\]
Thus, for some $\Delta = (\Delta_1)$ $\in C_2(\mathbb{P})$ with $\|\Delta\|_\infty < \varepsilon$ we have

$$p(J) \left( \begin{array}{c} b_1 \\ 1 \end{array} \right) - \left( \begin{array}{c} c \lambda_{12} \\ 0 \end{array} \right) = \Delta.$$ 

That is, cf. [19] Ch. 6.1, p. 386 we have

$$\left( \begin{array}{c} p(\lambda(\theta)) b_1 + p'(\lambda(\theta)) \lambda_{12}(\theta) \\ p(\lambda(\theta)) \end{array} \right) = \left( \begin{array}{c} c \lambda_{12}(\theta) + \Delta_1(\theta) \\ \Delta_2(\theta) \end{array} \right)$$

and, in particular, for all $\theta \in \mathbb{P}$ one has

$$|p(\lambda(\theta))| < \varepsilon \quad \text{and} \quad |p'(\lambda(\theta)) - c| \leq \frac{|\Delta_1(\theta) - b_1 \Delta_2(\theta)|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} \leq \varepsilon \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|}.$$ 

That is, for all $z \in \lambda(\mathbb{P})$ we have

$$|p(z)| < \varepsilon \quad \text{and} \quad |p'(z) - c| \leq \varepsilon \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|}.$$ 

Let $U := \gamma([0, 1]) \cup_{z \in \gamma([0, 1])} B_{\varepsilon}(z)$ and since $p'$ is uniformly continuous on $U$ there is a $\delta > 0$ such that for all $z, w \in U$ with $|z - w| < \delta$ one has $|p'(z) - p'(w)| < 1$. By Lemma 9 (Appendix) there is an $\tilde{N} \in \mathbb{N}$ such that for every $N \geq \tilde{N}$ there is a rectifiable polygon $\gamma_N$ such that

$$\|\gamma_N - \gamma\|_\infty < \min \{\varepsilon, \delta\}.$$ 

In particular, for all $N \geq \tilde{N}$ there is a rectifiable polygons $\gamma_N$ such that $|p'(\gamma_N(t)) - p'(\gamma(t))| < 1$ for all $t \in [0, 1]$. Let $z_1 \neq z_2$ denote the end points of the path $\gamma$. Then, for all $N \geq \tilde{N}$ one has

$$2\varepsilon > |p(z_2) - p(z_1)| = \left| \int_{\gamma_N} p'(\xi) \, d\xi \right| = \left| \int_0^1 \left( p'(\gamma_N(t)) - p'(\gamma(t)) + p'(\gamma(t)) - c + c \right) \cdot \dot{\gamma}_N(t) \, dt \right|$$

$$\geq c L_{\gamma_N} - \int_0^1 |p'(\gamma(t)) - c| \cdot |\dot{\gamma}_N(t)| \, dt - \int_0^1 |p'(\gamma_N(t)) - p'(\gamma(t))| \cdot |\dot{\gamma}_N(t)| \, dt$$

$$\geq \left( c - \varepsilon \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} - 1 \right) L_{\gamma_N}$$

and, thus taking the limit $N \to \infty$, we have $2\varepsilon \geq \left( c - \varepsilon \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} - 1 \right) L_{\gamma}$. In particular, it follows

$$c \leq \varepsilon \left( 2 L_{\gamma}^{-1} + \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} \right) + 1,$$

a contradiction.

Now, assume that $L_{\gamma} = \infty$ and let $\varepsilon > 0$ and $f(\theta) = (f_1(\theta))^1 \in C_2(\mathbb{P})$ with

$$c > \varepsilon \left( \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} \right) + 1.$$ 

Then, following the above arguments we have

$$2\varepsilon \geq \left( c - \varepsilon \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} - 1 \right) L_{\gamma_N}$$

By the choice of $c$ we have $c - \varepsilon \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} - 1 > 0$ and therefore,

$$\frac{2\varepsilon}{c - \varepsilon \frac{1 + |b_1|}{\min_{\theta \in \mathbb{P}} |\lambda_{12}(\theta)|} - 1} > L_{\gamma_N}$$

As $L_{\gamma_N} \to \infty$ as $N \to \infty$, we obtain a contradiction for $N$ sufficiently large. \(\square\)
Remark 3. Note that for a finite parameter set $P := \{\theta_1, \ldots, \theta_N\}$ the uniform ensemble reachability problem of the pair $(J, B)$ boils down to a standard interpolation problem which can be solved exactly even for single input systems (if $b_n \neq 0$).

We close this section with two examples. The first one illustrates that, in contrast to linear finite-dimensional systems, it is reasonable to consider parameter-dependent systems with more inputs than state variables, i.e. where $m > n$.

Example 3. Let $P = [-1, 1]$ and consider the pair $(a, B) \in C_{1,2}(P)$ defined by

$$a(\theta) = \theta^2 \quad \text{and} \quad B(\theta) = (1 \theta)$$

The pair satisfies the necessary conditions in Theorem A.5. To see that the pair is uniformly ensemble reachable, let $f \in C_1(P)$ and $\varepsilon > 0$ be given. Then, we have to verify the existence of two polynomials $p_1$ and $p_2$ such that $|f(\theta) - p_1(\theta^2) - \theta p_2(\theta^2)| < \varepsilon$ for all $\theta \in [-1, 1]$. By construction, we have

$$p_1(\theta^2) + \theta p_2(\theta^2) = c_0 + c_1\theta + c_2\theta^2 + \cdots + c_k\theta^k$$

where $c_0, \ldots, c_k$ denote the coefficients of $p_1$ and $c_2, \ldots, c_k$ denote the coefficients of $p_2$. Then, the claim follows by applying Walsh’s result [35, Theorem 8, II.2.5]. Also we note that the conclusion here cannot be drawn by applying the sufficient conditions in Theorem 7 as condition (c) is not satisfied.

The second example concerns the contractibility of the parameter space $P$ and shows that for the space of continuous functions $C(P)$ this assumption cannot be weakened.

Example 4. Let $P = \partial D$ and consider the pair $(a, b) \in C_{1,1}(P)$ defined by

$$a(\theta) = \theta^2 \quad \text{and} \quad b(\theta) = 1$$

The pair is not uniformly ensemble reachable. This can easily be seen using a continuous function $f: \partial D \to \mathbb{C}$ that has no analytic extension to $D$, e.g. $f(z) = \frac{1}{z}$. Suppose $(a, b)$ is uniformly ensemble reachable, then for $\varepsilon = 1$ there is a polynomial $p$ such that $|p(z) - f(z)| < 1$ for all $z \in \partial D$. Hence, it holds

$$|zp(z) - 1| < |z| = 1 \quad \text{for all } z \in \partial D.$$ 

Thus, for the non-constant function $q: \overline{D} \to \mathbb{C}, q(z) := zp(z) - 1$ we have that the maximum of $|q|$ is attained at $z = 0$, a contradiction to the maximum modulus theorem [28, 12.1].

4 \quad L^q$-ensemble reachability

In this section we focus on necessary and sufficient conditions for ensemble reachability on the separable Banach space $L^q(P)$. In a first step we derive necessary conditions for $L^q$-ensemble reachability. We start with an auxiliary Selection Lemma which might be of separate interest.

Lemma 5. Let $P \subset \mathbb{C}$ be compact and suppose the matrix-valued function $R: \theta \in P \to \mathbb{C}^{m \times m}$ is Lebesgue measurable. Then there exists a $L^\infty$-selection $\alpha: P \to \mathbb{C}^m$ of the set-valued map $\theta \mapsto \ker R(\theta)$ such that $||\alpha(\theta)|| = 1$ whenever $\ker R(\theta) \neq \{0\}$.

Proof. By [5, A5.8] there are compact subsets $J_k$ of $P$ such that $\mu(P \setminus \bigcup_{k=1}^\infty J_k) = 0$ and $\theta \mapsto R(\theta)^*$ is continuous on $J_k$ for every $k = 1, 2, 3, \ldots$. Consider the set-valued map $F: P \to \mathbb{C}^m$,

$$F(\theta) = \begin{cases} 
\{0\} & \text{if } \ker R(\theta)^* = \{0\} \\
\ker R(\theta)^* \cap \overline{B}_1(0) & \text{else.}
\end{cases}$$

Then, as $R^*$ is continuous on $J_k$ for every $k \in \mathbb{N}$ we have that the graph of $F|_{J_k}$ is closed and $F|_{J_k}$ is bounded. Then, applying [3, Theorem A.7.3] for every $k \in \mathbb{N}$ the lexicographical selection, denoted by $\theta \mapsto \xi_k(\theta)$, is measurable on $J_k$. Then, $\xi: \bigcup_{k \in \mathbb{N}} J_k \to \mathbb{C}^m$, $\xi|_{J_k}(\theta) = \xi_k(\theta)$ is measurable and can be extended to a measurable function $\alpha: P \to \mathbb{C}^m$. \qed

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We note that, even if the map \( \theta \mapsto R(\theta) \) is continuous, the set-valued map \( F \) does in general not have a continuous selection. This can be seen, for instance, using an example which is due to Rellich, cf. [21, §5.3].

**Theorem 8.** Let \( P \subset \mathbb{C} \) be compact and suppose \((A,B) \in C_{n,n}(P) \times L^q_{n,m}(P)\) is \( L^q \)-ensemble reachable. Then \((A,B) \in C_{n,n}(P) \times L^q_{n,m}(P)\) has to satisfy the following necessary conditions:

(a) The pairs \((A(\theta),B(\theta))\) are reachable for almost all \( \theta \in P \).

(b) The eigenvalues of \((A(\theta))\) have geometric multiplicity one for almost all \( \theta \).

**Proof.** (a) Suppose contrary that there is a set \( \Omega \subset P \) with positive (Lebesgue-) measure \( \lambda(\Omega) > 0 \) such that for all \( \theta \in \Omega \) the pair \((A(\theta),B(\theta))\) is not reachable. That is, for each \( \theta \in \Omega \) the rank of the Kalman matrix \( R(A(\theta),B(\theta)) = (B(\theta) A(\theta) B(\theta) \cdots A(\theta)^{n-1} B(\theta)) \) is at most \( n-1 \). Hence, for each \( \theta \in \Omega \) the dimension of the kernel of \( R(A(\theta),B(\theta))^* \) is greater or equal to one. Obviously, the map \( \theta \mapsto R(A(\theta),B(\theta))^* \) is measurable. By Lemma 3 there exists a \( L^\infty \)-function \( \xi : P \to \mathbb{C}^n \) such that \( \xi \neq 0 \) and \( \xi(\theta)^* R(A(\theta),B(\theta)) = 0 \) for almost all \( \theta \in P \). Consequently, the nonzero functional \( L^q_{n}(P) \ni f \mapsto \int_{\Omega} \xi(\theta)^* f(\theta) \, d\theta \) vanishes on the span of \( \{ A(\cdot) b(\cdot) \mid k = 0,1,2,\ldots \} \), which contradicts the density of \( \{ p(A(\cdot) b(p) \mid p \text{ polynomial} \} \) in \( L^q_{n}(P) \).

(b) This is an immediate consequence of (a).

**Example 5.** Let \( P := [0,1] \) and consider

\[
A(\theta) := \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

The pair satisfies the necessary conditions in Theorem 3. Note that, Theorem 8 shows that the pair is uniformly ensemble reachable over the parameter space \([c,1] \) for any \( c \in (0,1) \). Furthermore, by Theorem 8 the pair is not uniformly ensemble reachable over \([0,1]\), as the pair \((A(0),b)\) is not reachable.

However, we shall show that \((A,b)\) is \( L^q \)-ensemble reachable over \([0,1] \) for \( q \in [1,\infty) \). To see this, we fix \( f = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) \in L^2(P) \) and \( \varepsilon > 0 \) and show that there is a polynomial \( p \) such that for all \( \theta \in [0,1] \) one has

\[
|p(\theta) - f_1(\theta)| < \varepsilon \quad \text{and} \quad |p(\theta) - f_1(\theta)| < \varepsilon.
\]

To this end, we pick continuous functions \( g_1 \) and \( g_2 \) on \( P \) such that \( \|f_i - g_i\|_q < \frac{\varepsilon}{2} \) and \( g_i(0) = 0 \) for \( i = 1,2 \). Then, the continuous function \( h : [-1,1] \to \mathbb{C} \) defined by

\[
h(\theta) := \begin{cases} g_1(\theta) & \theta \in [0,1] \\ g_2(-\theta) & \theta \in [-1,0] \end{cases}
\]

can approximated uniformly by a polynomial \( p : [-1,1] \to \mathbb{C} \) such that \( \|p - h\|_{\infty} < \frac{\varepsilon}{2} \). Consequently, for every \( \theta \in P \) we have

\[
|p(\theta) - f_1(\theta)| < p(\theta) - h(\theta)| + |g_1(\theta) - f_1(\theta)| < \varepsilon \quad \text{and} \quad |p(\theta) - f_2(\theta)| < \varepsilon.
\]

**Single input parameter-dependent systems**

In order to obtain sufficient conditions we will make use of the observation that a single-input pair \((A,B) \in C_{n,n}(P) \times L^q_{n}(P)\) is \( L^q \)-ensemble reachable if and only if the multiplication operator

\[
\mathcal{M}_A : L^q_{n}(P) \to L^q_{n}(P) \quad \mathcal{M}_A f(\theta) = A(\theta) f(\theta)
\]

is cyclic and \( b \) is a cyclic vector for \( \mathcal{M}_A \). Similar to the uniform ensemble reachability case, we consider scalar ensembles first.

**Theorem 9.** Let \( P \subset \mathbb{C} \) be compact and \((a,b) \in C(P) \times L^q(P)\).
(a) If \((a, b)\) is \(L^q\)-ensemble reachable for \(q \in [1, \infty)\), then \(b(\theta) \neq 0\) for almost all \(\theta \in P\) and \(a\) is essentially univalent, i.e. \(a\) is one-to-one on a set of full measure.

(b) If \(a\) is essentially univalent, \(b(\theta) \neq 0\) for almost all \(\theta \in P\), and

\[
\inf_p \int_P |p(a) b - \pi b|^q \, d\theta = 0
\]

then \((a, b)\) is \(L^q\)-ensemble reachable for \(q \in [1, \infty)\).

\[\text{Proof.}\ (a): \text{Suppose The pair } (a, b) \text{ is } L^q\text{-ensemble reachable. Then } \{p(a) b \mid p \text{ polynomial } \} \text{ is dense in } L^q, \text{i.e. the multiplication operator } \mathcal{M}_a \text{ is cyclic and } b \text{ is a cyclic vector. Then, by [31, Lemma 3.1] the function } a \text{ is essentially univalent. Moreover, by Theorem 5(a) the pair } (a(\theta), b(\theta)) \text{ is reachable for almost all } \theta \in P. \text{ Using the Kalman rank condition we conclude that } b(\theta) \neq 0 \text{ for almost all } \theta \in P.\]

(b): This follows from similar arguments used in the proof of Proposition 2.5 in [27]. \[\square\]

5 Output ensemble reachability

In applications to, e.g. cell biology or quantum systems, a frequently met task is to extract information of the system from average measurements. This motivates the study of families of parameter-dependent systems where the measurements are given by an average output functional, i.e. the ensemble is of the form

\[
\frac{\partial x}{\partial t}(t, \theta) = A(\theta) x(t, \theta) + B(\theta) u(t)
\]

\[
y(t) = \int_P C(\theta) x(t, \theta) \, d\theta
\]

with initial condition \(x(0, \theta) = x_0(\theta) \in C^n\) and \(x_0 \in X\). We assume that \(X_n(P)\) is separable Banach space such that the operators \(\mathcal{M}_A\) and \(\mathcal{M}_B\), defined in [30, and \(C: X_n(P) \to \mathbb{C}^p\),

\[
C f = \int_P C(\theta) f(\theta) \, d\theta
\]

are bounded linear. A triple \((A, B, C) \in C_{n,n}(P) \times X_{n,m}(P) \times C_{p,n}(P)\) is called output ensemble reachable, if for any \(x_0 \in X_n(P)\) and any \(y^* \in \mathbb{C}^p\) there exist a \(T > 0\) and an input function \(u \in L^1([0, T], C^n)\) such that

\[
\int_P C(\theta) \varphi(T, \theta, u) \, d\theta = y^*. \quad (23)
\]

As the output space \(Y = \mathbb{C}^p\) is finite dimensional the latter is equivalent to approximate output reachability, i.e. for every \(y \in \mathbb{C}^p\) and every \(\varepsilon > 0\) there exist a \(T > 0\) and an input function \(u \in L^1([0, T], C^n)\) such that

\[
\left\| \int_P C(\theta) \varphi(T, \theta, u) \, d\theta - y^* \right\| < \varepsilon. \quad (24)
\]

In [34, Theorem 7.1.1] Triggiani has shown that an infinite dimensional systems of the form

\[
\dot{x}(t) = \mathcal{M}_A x(t) + \mathcal{M}_B u(t)
\]

\[
y(t) = C x(t)
\]

is output controllable on \([0, T]\) if and only if

\[
\text{span}\{\text{im } (\mathcal{C} \mathcal{M}_A^k \mathcal{M}_B), k = 0, 1, 2, \ldots\} = \mathbb{C}^p. \quad (26)
\]

Applying this to the present setting we obtain the following characterization of output ensemble reachability.
Theorem 10. The triple \((A,B,C) \in C_{n,n}(P) \times X_{n,m}(P) \times C_{p,n}(P)\) is output ensemble reachable if and only if
\[
\text{rank} \left\{ \int_{P} C(\theta)A(\theta)^{k}b_{j}(\theta) \, d\theta, \quad j = 1, \ldots, m, \, k = 0, 1, 2, \ldots \right\} = p.
\]

Note that this characterization contains a characterization of averaged controllability in [30] Theorem 3], which is obtained in the case \(C(\theta) = I \in \mathbb{R}^{n \times n}\). We note that in [30] the matrix \(A(\theta)\) is only assumed to be measurable. As a consequence the corresponding multiplication operator \(A\) defined in \([3]\) is unbounded in general and therefore the infinite-dimensional techniques of Triggiani [34] cannot be applied to the case of measurable system matrices \(A(\theta)\). We also refer to [24] for comments that compare the different approaches.

Theorem 11. \((A,B,C) \in C_{n,n}(P) \times C_{n,m}(P) \times C_{p,n}(P)\) is output ensemble reachable if

(a) \((A,B)\) is uniformly ensemble reachable.

(b) For some \(\theta \in P\) it holds \(\text{rank} \, C(\theta) = p\).

Proof. As the output space is finite dimensional it suffices to verify approximate output controllability. Let \(\varepsilon > 0\) and \(y \in C^{n}\) be given. Then, by condition (b) there is a \(\theta^* \in P\) such that \(\text{rank} \, C(\theta^*) = p\). Thus, there is a \(f^* \in C^{m}\) such that \(y = C(\theta^*)f^*\). For \(r > 0\) let \(B_{\theta}(\theta^*)\) denote the closed ball around \(\theta^*\) with radius \(r > 0\) and let \(P_{r} := B_{\theta}(\theta^*) \cap P\). Further, we choose a continuous function \(g_{r} : P \to [0,1]\) satisfying \(g_{r}(\theta^*) = 1\) and \(g_{r}(\theta) = 0\) for all \(\theta \in P \setminus P_{r}\) (e.g. a standard mollifier) and consider the nonnegative continuous function \(h_{r}(\theta) = \|g_{r}(\theta)C(\theta)f^* - y\|\). Since \(h_{r}(\theta^*) = 0\) for all \(r > 0\) and by continuity and nonnegativity, we get for \(r^* > 0\) sufficiently small
\[
\max_{\theta \in P_{r^*}} h_{r^*}(\theta) \leq \max_{\theta \in P_{r^*}} \|g_{r^*}(\theta)C(\theta)f^* - y\| < \frac{\varepsilon}{2}.
\]

Let \(c = \max_{\theta \in P} \sup_{\|x\| = 1} \|C(\theta)x\|\). As the pair \((A,B)\) is uniformly ensemble reachable, for \(\frac{\varepsilon}{2c \, \text{vol}(P)} > 0\) and \(f^*(\theta) = \frac{1}{\text{vol}(P_{r})} g_{r^*}(\theta)f^* \in C_{n}(P)\) there are \(T > 0\) and \(u : [0,T] \to C^{m}\) such that
\[
\|\varphi(T,\cdot,u) - \frac{1}{\text{vol}(P_{r})} g_{r^*}(\cdot)f^*\|_{C_{n}(P)} < \frac{\varepsilon}{2c \, \text{vol}(P)}.
\]

Consequently, we have
\[
\left\| \int_{P} C(\theta)\varphi(T,\theta,u) \, d\theta - y \right\| \leq \left\| \int_{P} C(\theta) \left( \varphi(T,\theta,u) - f^*(\theta) \right) \, d\theta \right\| + \left\| \int_{P} C(\theta)f^*(\theta) \, d\theta - y \right\|
\leq T \sup_{\|x\| = 1} \|C(\theta)x\| \|g_{r^*}(\theta)f^* - y\| \, d\theta
+ \frac{1}{\text{vol}(P_{r})} \int_{P_{r}} \|g_{r^*}(\theta)C(\theta)f^* - y\| \, d\theta < \varepsilon.
\]

This shows the assertion. 

We note that the latter result also holds for discrete-time parameter-dependent systems and we obtain

Corollary 3. Under the assumptions of Theorem\textsuperscript{11} the parameter-dependent system
\[
x_{t+1}(\theta) = A(\theta)x_{t}(\theta) + B(\theta)u_{t},
\]
\[
y_{t} = \int_{P} C(\theta)x_{t}(\theta) \, d\theta
\]
is output ensemble reachable.
Similarly to the case of ensemble reachability we are interested in conditions for output ensemble reachability that are testable just in terms the triple \((A, B, C)\). In the case \((A, B, C) \in C_{n,m}(P) \times C_{n,m}(P) \times C_{n,m}(P) =: C_{n,m,m}(P)\) Theorem 11 together with Corollary 2 yields the following sufficient conditions.

**Corollary 4.** \((A, B, C) \in C_{n,m,p}(P)\) is output ensemble reachable if

(i) \((A(\theta), B(\theta))\) is reachable for all \(\theta \in P\).

(ii) The input Hermite indices of \((A(\theta), B(\theta))\) do not depend on \(\theta \in P\).

(iii) For any pair of distinct parameters \(\theta, \theta' \in P, \theta \neq \theta'\), the spectra of \(A(\theta)\) and \(A(\theta')\) are disjoint:

\[
\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset.
\]

(iv) For each \(\theta \in P\), the eigenvalues of \(A(\theta)\) have algebraic multiplicity one.

(v) For some \(\theta \in P\) it holds \(\text{rank}(C(\theta)) = p\).

Let \(c_k(\theta), k = 1, ..., p\) denote the rows of \(C(\theta)\) and consider the functionals

\[
h_k: C(P, \mathbb{R}^n) \to \mathbb{R}, \quad h_k f = \int_P c_k(\theta)f(\theta) \, d\theta, \quad k = 1, ..., p
\]

If the triple \((A, B, C) \in C_{n,m,p}(P)\) is output ensemble reachable, then by 34 Corollary 6.2 the functionals \(h_1, ..., h_p\) are linearly independent. Thus, the proof of the latter shows that condition (b) implies that the functionals \(h_1, ..., h_p\) are linearly independent. In addition, by 34 Corollary 6.2 a triple \((A, B, C) \in C_{n,m,p}(P)\) is output ensemble reachable if the pair \((A, B) \in C_{n,m}(P)\) is uniformly \((L^\infty)\)-ensemble reachable and the functionals \(h_1, ..., h_p\) are linearly independent.

Thus, for the treatment of concrete triples the condition (b) in Theorem 11 can be weakened to demanding that the functionals \(h_1, ..., h_p\) are linearly independent, which has then to be checked in the specific case.

### 6 Appendix

**Lemma 6.** Let \(C_1, ..., C_n\) be disjoint compact sets that do not separate the plane. Then, there are disjoint continua \(K_1, ..., K_n\) that do not separate the complex plane such that \(C_l\) is properly contained in \(K_l\) for each \(l = 1, ..., n\).

**Proof.** We treat the case \(n = 2\), the general case follows then by induction. By assumption the compact sets \(C_1\) and \(C_2\) are disjoint and let \(d > 0\) denote their distance. Then, the open neighborhoods \(U = \{w \in \mathbb{C} | \|w - z\| < \frac{d}{2}\} \) for some \(z \in C_1\) and \(V = \{w \in \mathbb{C} | \|w - z\| < \frac{d}{2}\} \) for some \(z \in C_2\) are disjoint.

We consider the cases that one compact set is contained in the convex hull of the other, w.l.o.g. let \(C_2\) be contained in the convex hull of \(C_1\). In a first step we construct the continuum \(K_2\) containing \(C_2\) properly. Following the construction in the proof of Theorem 13.5 in 28 (grid size \(h \in (0, \frac{d}{\sqrt{2}})\) is sufficient) the neighborhoods \(V\) contains a positive oriented Jordan curve \(\Gamma_2\) enclosing \(C_2\). Then, \(K_2 := \Gamma_2 \cup \text{int}(\Gamma_2)\) defines the claimed continuum. Secondly, as \(C_1\) is compact there is a square \(Q\) that properly contains the convex hull of \(C_1\) and \(C_2\). Since \(\mathbb{C} \setminus (C_1 \cup C_2)\) is connected for every \(x \in \partial Q\) and \(y \in \partial C_2\) there is a path \(\gamma_{x,y}\) connecting \(x \in C_2\) and \(y \in Q\) such that \(\gamma_{x,y} \cap C_1 = \emptyset\). Let \(\gamma\) be such a path with shortest length and let \(d' > 0\) denote the Hausdorff distance between \(\gamma\) and \(C_1\). Then, \(\gamma \cup C_2\) is also a continuum with Hausdorff distance \(d'' = \min\{d, d'\}\) to \(C_1\). Then, repeating the construction above with grid size \(h \in (0, \frac{d''}{\sqrt{2}})\) the open neighborhood \(U = \{w \in \mathbb{C} | \|w - z\| < \frac{d''}{2}\} \) for some \(z \in C_1\) contains a positive oriented Jordan curve \(\Gamma_1\) enclosing \(C_1\). Then, \(K_1 := \Gamma_1 \cup \text{int}(\Gamma_1)\) defines the claimed continuum. For the other cases, the above construction can be applied directly to the compact sets \(C_1\) and \(C_2\). This shows the assertion. \(\square\)
The next result is trivial, but given that this defines a building block in the construction methods for ensemble reachability we state it separately for future reference.

Lemma 7. Let $K_1, K_2$ be disjoint continua that do not separate the complex plane. Then, the function $h: K_1 \cup K_2 \to \mathbb{C}$ defined by $h(z) = 1$ for all $z \in K_1$ and $h(z) = 0$ for all $z \in K_2$ can be uniformly approximated by polynomials.

Proof. The proof of Lemma 6 implies the existence of open neighborhoods $U$ and $V$ of $K_1$ and $K_2$, respectively, such that the function $h$ is analytic on $U \cup V$. Then, the assertion follows from Runge’s Approximation Theorem, cf. [23, Theorem 13.7].

Lemma 8. Let $P \subset \mathbb{C}$ be compact, contractible and locally connected. Suppose $\lambda: P \to \mathbb{C}$ is continuous. Then $\lambda(P)$ contains a path.

Proof. Let $p \in P$ and, as $P$ is locally connected, there is a connected neighborhood $V(p)$. As $P$ is compact, by [36, Theorem (14.3)] the set $V(p)$ is compact, connected and locally connected. Further, by the Hahn-Mazurkiewicz Theorem [36, Theorem (4.1)] there is a continuous mapping $f$ such that $f([0, 1]) = V(p)$. Since $\lambda$ is continuous, $\lambda(P)$ contains the path $\gamma: [0, 1] \to \lambda(P)$, $t \mapsto \gamma(t) := \lambda(f(t))$. This shows the assertion.

The next statement is well-known, e.g., Euler method for ordinary differential equations. For completeness, we provide a short proof.

Lemma 9. Let $\gamma$ be path in the plane. Then, for every $\varepsilon > 0$ there is a polygon $\gamma_N$ consisting of $N$ line segments such that $\|\gamma - \gamma_N\|_{\infty} < \varepsilon$.

Proof. Let $\varepsilon > 0$. Since $\gamma$ is uniformly continuous there is a $\delta > 0$ such that $\|\gamma(t) - \gamma(s)\| < \frac{\varepsilon}{2}$ for all $|t - s| < \delta$. Choose $N \in \mathbb{N}$ and 0 = $t_0 < t_1 < \cdots < t_N = 1$ such that $|t_{k+1} - t_k| < \delta$ for all $k = 0, \ldots, N$. Let $\gamma_N$ denote the polygon connecting the points $\gamma(t_0), \ldots, \gamma(t_N)$ by $N$ line segments. Then, for every $t \in [0, 1]$ there is a $t_k \in \{t_0, \ldots, t_N\}$ such that $|t - t_k| < \delta$ and $t_k \geq t$. Hence,

$$\|\gamma(t) - \gamma(t_k)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\gamma_N(t) - \gamma_N(t_k)\| \leq \|\gamma_N(t_{k+1}) - \gamma_N(t_k)\| = \|\gamma(t_{k+1}) - \gamma(t_k)\| < \frac{\varepsilon}{2}.$$  

Consequently, for every $t \in [0, 1]$ the latter implies

$$\|\gamma(t) - \gamma_N(t)\| \leq \|\gamma(t) - \gamma(t_k)\| + \|\gamma(t_k) - \gamma_N(t)\| < \frac{\varepsilon}{2} + \|\gamma_N(t) - \gamma_N(t)\| < \varepsilon.$$  

This shows the assertion.

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