Tiling and spectral properties of near-cubic domains

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1 Introduction

Let $E$ be a measurable set in $\mathbb{R}^n$ such that $0 < |E| < \infty$. We will say that $E$ tiles $\mathbb{R}^n$ by translations if there is a discrete set $T \subset \mathbb{R}^n$ such that, up to sets of measure 0, the sets $E + t : t \in T$ are mutually disjoint and $\bigcup_{t \in T} (E + t) = \mathbb{R}^n$. We call any such $T$ a translation set for $E$, and write $E + T = \mathbb{R}^n$. A tiling $E + T = \mathbb{R}^n$ is called periodic if it admits a period lattice of rank $n$; it is a lattice tiling if $T$ itself is a lattice. Here and below, a lattice in $\mathbb{R}^n$ will always be a set of the form $T\mathbb{Z}^n$, where $T$ is a linear transformation of rank $n$.

It is known [19], [18] that if a convex set $E$ tiles $\mathbb{R}^n$ by translations, it also admits a lattice tiling. A natural question is whether a similar result holds if $E$ is “sufficiently close” to being convex, e.g. if it is close enough (in an appropriate sense) to a $n$-dimensional cube. In this paper we prove that this is indeed so in dimensions 1 and 2; we also construct a counterexample in dimensions $n \geq 3$.

A major unresolved problem in the mathematical theory of tilings is the periodic tiling conjecture, which asserts that any $E$ which tiles $\mathbb{R}^n$ by translations must also admit a periodic tiling. (See [3] for an overview of this and other related questions.) The conjecture has been proved for all bounded measurable subsets of $\mathbb{R}$ [14], [12] and for topological discs in $\mathbb{R}^2$ [2], [8]. Our Theorem 2 and Corollary 1 prove the conjecture for near-square domains in $\mathbb{R}^2$. We emphasize that no assumptions on the topology of $E$ are needed; in particular, $E$ is not required to be connected and may have infinitely many connected components.

Our work was also motivated in part by a conjecture of Fuglede [1]. We call a set $E$ spectral if there is a discrete set $\Lambda \subset \mathbb{R}^n$, which we call a spectrum for $E$, such that $\{e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(E)$. Fuglede conjectured that $E$ is spectral if and only if it tiles $\mathbb{R}^n$ by translations, and proved it under the assumption that either the translation set $T$ or the spectrum $\Lambda$ is a lattice. This problem was addressed in many recent papers (see e.g. [4], [7], [10], [13], [14], [15], [16], [17]), and in particular the conjecture has been proved for convex regions in $\mathbb{R}^2$ [3], [5], [3].

It follows from our Theorem 1 and from Fuglede’s theorem that the conjecture is true for $E \subset \mathbb{R}$ such that $E$ is contained in an interval of length strictly less than $3|E|/2$. (This was proved in [15] in the special case when $E$ is a union of finitely many intervals of equal length.) In dimension 2, we obtain the “tiling $\Rightarrow$ spectrum” part of the conjecture for near-square domains. Namely, if $E \subset \mathbb{R}^2$ tiles $\mathbb{R}^2$ and satisfies the assumptions of Theorem 2 or Corollary 1, it also admits a lattice tiling, hence it is a spectral set by Fuglede’s theorem on the lattice case of his conjecture. We do not know how to prove the converse implication.

Our main results are the following.
Theorem 1 Suppose $E \subseteq [0, L]$ is measurable with measure 1 and $L = 3/2 - \epsilon$ for some $\epsilon > 0$. Let $\Lambda \subset \mathbb{R}$ be a discrete set containing 0. Then
(a) if $E + \Lambda = \mathbb{R}$ is a tiling, it follows that $\Lambda = \mathbb{Z}$.
(b) if $\Lambda$ is a spectrum of $E$, it follows that $\Lambda = \mathbb{Z}$.

The upper bound $L < 3/2$ in Theorem 1 is optimal: the set $[0, 1/2] \cup [1, 3/2]$ is contained in an interval of length $3/2$, tiles $\mathbb{Z}$ with the translation set $\{0, 1/2\} + 2\mathbb{Z}$, and has the spectrum $\{0, 1/2\} + 2\mathbb{Z}$, but does not have either a lattice translation set or a lattice spectrum. This example has been known to many authors; an explicit calculation of the spectrum is given e.g. in [14].

Theorem 2 Let $E \subset \mathbb{R}^2$ be a measurable set such that $[0, 1]^2 \subset E \subset [-\epsilon, 1 + \epsilon]^2$ for $\epsilon > 0$ small enough. Assume that $E$ tiles $\mathbb{R}^2$ by translations. Then $E$ also admits a tiling with a lattice $\Lambda \subset \mathbb{R}^2$ as the translation set.

Our proof works for $\epsilon < 1/33$; we do not know what is the optimal upper bound for $\epsilon$.

![Figure 1: Examples of near-square regions which tile $\mathbb{R}^2$. Note that the second region also admits aperiodic (hence non-lattice) tilings.](image)

Corollary 1 Let $E \subset \mathbb{R}^2$ be a measurable set such that $|E| = 1$ and $E$ is contained in a square of sidelength $1 + \epsilon$ for $\epsilon > 0$ small enough. If $E$ tiles $\mathbb{R}^2$ by translations, then it also admits a lattice tiling.

Theorem 3 Let $n \geq 3$. Then for any $\epsilon > 0$ there is a set $E \subset \mathbb{R}^n$ with $[0, 1]^n \subset E \subset [-\epsilon, 1 + \epsilon]^n$ such that $E$ tiles $\mathbb{R}^n$ by translations, but does not admit a lattice tiling.

2 The one-dimensional case

In this section we prove Theorem 1. We shall need the following crucial lemma.

Lemma 1 Suppose that $E \subseteq [0, L]$ is measurable with measure 1 and that $L = 3/2 - \epsilon$ for some $\epsilon > 0$. Then

$$|E \cap (E + x)| > 0 \quad \text{whenever } 0 \leq x < 1.$$  \hfill (1)
Proof of Lemma 1. We distinguish the cases (i) $0 \leq x \leq 1/2$, (ii) $1/2 < x \leq 3/4$ and (iii) $3/4 < x < 1$.

(i) $0 \leq x \leq 1/2$

This is the easy case as $E \cup (E + x) \subseteq [0, L + 1/2] = [0, 2 - \epsilon]$. Since this interval has length less than 2, the sets $E$ and $E + x$ must intersect in positive measure.

(ii) $1/2 < x \leq 3/4$

Let $x = 1/2 + \alpha$, $0 < \alpha \leq 1/4$. Suppose that $|E \cap (E + x)| = 0$. Then $1 + 2\alpha \leq 3/2$ and $|(E \cap [0, x]) \cup (E \cap [x, 2x])| \leq x$, as the second set does not intersect the first when shifted back by $x$. This implies that $|E| \leq x + (3/2 - \epsilon - 2x) = 3/2 - \epsilon - x = 1 - \epsilon - \alpha < 1$, a contradiction as $|E| = 1$.

(iii) $3/4 \leq x < 1$

Let $x = 3/4 + \alpha$, $0 < \alpha < 1/4$. Suppose that $|E \cap (E + x)| = 0$. Then $|(E \cap [0, 3/4 - \alpha - \epsilon]) \cup (E \cap [3/4 + \alpha, 3/2 - \epsilon])| \leq 3/4 - \alpha - \epsilon$, for the second set translated to the left by $x$ does not intersect the first. This implies that $|E| \leq (3/4 - \alpha - \epsilon) + 2\alpha + \epsilon = 3/4 + \alpha < 1$, a contradiction.

We need to introduce some terminology. If $f$ is a nonnegative integrable function on $\mathbb{R}^d$ and $\Lambda$ is a subset of $\mathbb{R}^d$, we say that $f + \Lambda$ is a packing if, almost everywhere,

$$\sum_{\lambda \in \Lambda} f(x - \lambda) \leq 1. \quad (2)$$

We say that $f + \Lambda$ is a tiling if equality holds almost everywhere. When $f = \chi_E$ is the indicator function of a measurable set, this definition coincides with the classical geometric notions of packing and tiling.

We shall need the following theorem from \[10\].

**Theorem 4** If $f, g \geq 0$, $\int f(x)dx = \int g(x)dx = 1$ and both $f + \Lambda$ and $g + \Lambda$ are packings of $\mathbb{R}^d$, then $f + \Lambda$ is a tiling if and only if $g + \Lambda$ is a tiling.

**Proof of Theorem 4.** (a) Suppose $E + \Lambda$ is a tiling. From Lemma 1 it follows that any two elements of $\Lambda$ differ by at least 1. This implies that $\chi_{[0,1]} + \Lambda$ is a packing, hence it is also a tiling by Theorem 1. Since $0 \in \Lambda$, we have $\Lambda = \mathbb{Z}$.

(b) Suppose that $\Lambda$ is a spectrum of $E$. Write

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$$

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for the measure of one unit mass at each point of \( \Lambda \). Our assumption that \( \Lambda \) is a spectrum for \( E \) implies that
\[
|\hat{\chi}_E|^2 + \Lambda = \mathbb{R}
\]
is a tiling (see, for example, [10]). This, in turn, implies that \( \text{dens } \Lambda = 1 \).

We now use the following result from [10]:

**Theorem 5** Suppose that \( f \geq 0 \) is not identically 0, that \( f \in L^1(\mathbb{R}^d) \), \( \hat{f} \geq 0 \) has compact support and \( \Lambda \subset \mathbb{R}^d \). If \( f + \Lambda \) is a tiling then
\[
\text{supp } \hat{\delta}_\Lambda \subseteq \{ \hat{f} = 0 \} \cup \{0\}. \tag{3}
\]

Let us emphasize here that the object \( \hat{\delta}_\Lambda \), the Fourier Transform of the tempered measure \( \delta_\Lambda \), is in general a tempered distribution and need not be a measure.

For \( f = |\chi_E|^2 \) Theorem 5 implies
\[
\text{supp } \hat{\delta}_\Lambda \subseteq \{0\} \cup \{\chi_E * \hat{\chi}_E = 0\}, \tag{4}
\]
since \( \chi_E * \hat{\chi}_E \) is the Fourier transform of \( |\chi_E|^2 \) (where \( \hat{g}(x) = \overline{g(-x)} \)). But
\[
\{\chi_E * \hat{\chi}_E = 0\} = \{x : |E \cap (E + x)| = 0\}.
\]
This and Lemma [1] imply that
\[
\text{supp } \hat{\delta}_\Lambda \cap (-1,1) = \{0\}. \tag{5}
\]

Let
\[
K_\delta(x) = \max \{0, 1 - (1 + \delta)|x|\} = (1 + \delta)\chi_{I_\delta} * \hat{\chi}_{I_\delta}(x),
\]
where \( I_\delta = [0, \frac{1}{1+\delta}] \), be a Fejér kernel (we will later take \( \delta \to 0 \)). Then \( \hat{K}_\delta = (1 + \delta)|\chi_{I_\delta}|^2 \) is a non-negative continuous function and, after calculating \( \hat{\chi}_{I_\delta} \), it follows that
\[
\hat{K}_\delta(0) = \frac{1}{1 + \delta}
\]
and
\[
\{x : \hat{K}_\delta(x) = 0\} = (1 + \delta)(\mathbb{Z} \setminus \{0\}). \tag{5}
\]

Next, we use the following result from [11]:

**Theorem 6** Suppose that \( \Lambda \in \mathbb{R}^d \) is a multiset with density \( \rho \), \( \delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda \), and that \( \hat{\delta}_\Lambda \) is a measure in a neighborhood of 0. Then \( \hat{\delta}_\Lambda(\{0\}) = \rho \).

**Remark.** The proof of Theorem [11] shows that the assumption of \( \hat{\delta}_\Lambda \) being a measure in a neighborhood of zero is superfluous, if one knows a priori that \( \hat{\delta}_\Lambda \) is supported only at zero, in a neighborhood of zero. Indeed, what is shown in that proof is that, as \( t \to \infty \), the quantity \( \hat{\delta}_\Lambda(\phi(tx)) \) remains bounded, for any \( C_c^\infty \) test function \( \phi \). If \( \hat{\delta}_\Lambda \) were not a measure near 0 but had support only at 0, locally, this quantity would grow like a polynomial in \( t \) of degree equal to the degree of the distribution at 0.
Applying Theorem 3 and the remark following it we obtain that \( \hat{\delta}_A \) is equal to \( \delta_0 \) in a neighborhood of 0, since \( \Lambda \) has density 1.

Next, we claim that

\[ \sum_{\lambda \in \Lambda} \hat{K}_\delta(x - \lambda) = 1, \quad \text{for a.e. } x. \]

Indeed, take \( \psi_\epsilon \) to be a smooth, positive-definite approximate identity, supported in \( (-\epsilon, \epsilon) \), and take \( \epsilon = \epsilon(\delta) \) to be small enough so that \( \text{supp} \psi_\epsilon \ast K_\delta \subset (-1,1) \). We have then

\[
\begin{align*}
\sum_{\lambda \in \Lambda} \hat{K}_\delta(x - \lambda) &= \lim_{\epsilon \to 0} \sum_{\lambda \in \Lambda} \hat{\psi}_\epsilon(x - \lambda) \hat{K}_\delta(x - \lambda) \\
&= \lim_{\epsilon \to 0} \delta_\Lambda \left( (\hat{\psi}_\epsilon \hat{K}_\delta)(x) \right) \\
&= \lim_{\epsilon \to 0} \hat{\delta}_\Lambda \left( (\psi_\epsilon \ast K_\delta)(x) \right) \\
&= \lim_{\epsilon \to 0} \delta_0 \left( (\psi_\epsilon \ast K_\delta)(x) \right) \quad \text{(for } \epsilon \text{ small enough)} \\
&= \lim_{\epsilon \to 0} \psi_\epsilon \ast K_\delta(0) \\
&= K_\delta(0) \\
&= 1,
\end{align*}
\]

which establishes the claim. Applying this for \( x \to 0 \) and isolating the term \( \lambda = 0 \) we get

\[
1 = \frac{1}{1 + \delta} + \sum_{0 \neq \lambda \in \Lambda} \hat{K}_\delta(-\lambda).
\]

Letting \( \delta \to 0 \) we obtain that \( \hat{K}_\delta(-\lambda) \to 0 \) for each \( \lambda \in \Lambda \setminus \{0\} \), which implies that each such \( \lambda \) is an integer, as \( \mathbb{Z} \setminus \{0\} \) is the limiting set of the zeros of \( \hat{K}_\delta \).

To get that \( \Lambda = \mathbb{Z} \) notice that \( \chi_{[0,1]} + \Lambda \) is a packing. By Theorem 3 again we get that \( \chi_{[0,1]} + \Lambda \) is in fact a tiling, hence \( \Lambda = \mathbb{Z} \).

\[ \square \]

3 Planar regions

**Proof of Theorem 2** We denote the coordinates in \( \mathbb{R}^2 \) by \( (x_1, x_2) \). For \( 0 \leq a \leq b \leq 1 \) we will denote

\[
\begin{align*}
E_1(a,b) &= (E \cap \{a \leq x_1 \leq b, \ x_2 \leq 0\}) \cup \{a \leq x_1 \leq b, \ x_2 \geq 0\}; \\
E_2(a,b) &= (E \cap \{a \leq x_1 \leq b, \ x_2 \geq 0\}) \cup \{a \leq x_1 \leq b, \ x_2 \leq 0\}; \\
F_1(a,b) &= (E \cap \{a \leq x_1 \leq b, \ x_1 \leq 0\}) \cup \{a \leq x_2 \leq b, \ x_1 \geq 0\}; \\
F_2(a,b) &= (E \cap \{a \leq x_2 \leq b, \ x_1 \geq 0\}) \cup \{a \leq x_2 \leq b, \ x_1 \leq 0\}.
\end{align*}
\]

We will also use \( S_{a,b} \) to denote the vertical strip \([a,b] \times \mathbb{R}\). Let \( v = (v_1, v_2) \in \mathbb{R}^2 \). We will say that \( E_2(a,b) \) complements \( E_1(a',b') + v \) if \( E_1(a',b') + v \) is positioned above \( E_2(a,b) \) so that (up to sets of measure 0) the two sets are disjoint and their union is \( S_{a,b} \). In particular, we must have \( a' + v_1 = a \) and \( b' + v_1 = b \). We will write \( \tilde{E}_1(a,b) = S_{a,b} \setminus E_1(a,b) \), and similarly for \( E_2 \). Finally, we write \( A \sim B \) if the sets \( A \) and \( B \) are equal up to sets of measure 0.
Lemma 2 Let $0 < s'' < s' < s < 2s''$. Suppose that $E_1(a, a + s) + v$, $E_1(a, a + s') + v'$, $E_1(a, a + s'') + v''$ complement $E_2(b - s, b)$, $E_2(b - s', b)$, $E_2(b - s'', b)$ respectively. Then the points $v, v', v''$ are collinear. Moreover, the absolute value of the slope of the line through $v, v''$ is bounded by $c(2s'' - s)^{-1}$.

Applying the lemma to the symmetric reflection of $E$ about the line $x_2 = 1/2$, we find that the conclusions of the lemma also hold if we assume that $E_2(a, a + s) + v$, $E_2(a, a + s') + v'$, $E_2(a, a + s'') + v''$ complement $E_1(b - s, b)$, $E_1(b - s', b)$, $E_1(b - s'', b)$ respectively. Furthermore, we may interchange the $x_1$ and $x_2$ coordinates and obtain the analogue of the lemma with $E_1, E_2$ replaced by $F_1, F_2$.

Proof of Lemma 2. Let $v = (v_1, v_2)$, $v' = (v'_1, v'_2)$, $v'' = (v''_1, v''_2)$. We first observe that if $v_1 = v''_1$, it follows from the assumptions that $v = v''$ and there is nothing to prove. We may therefore assume that $v_1 = v''_1$. We do, however, allow $v' = v$ or $v' = v''$.

It follows from the assumptions that $E_2(b - s', b)$ complements each of $E_1(a, a + s'') + v''$, $E_1(a + s' - s'', a + s') + v'$, $E_1(a + s - s'', a + s) + v$. Hence

$$E_1(a + s' - s'', a + s') \sim E_1(a, a + s'') + (v'' - v'),$$

$$E_1(a + s - s'', a + s) \sim E_1(a, a + s'') + (v'' - v).$$

Let $n$ be the unit vector perpendicular to $v - v''$ and such that $n_2 > 0$. For $t \in \mathbb{R}$, let $P_t = \{x : x \cdot n \leq t\}$. We define for $0 \leq c \leq c' \leq 1$:

$$\alpha_{c,c'} = \inf\{t \in \mathbb{R} : |E_1(c, c') \cap P_t| > 0\},$$

$$\beta_{c,c'} = \sup\{t \in \mathbb{R} : |\overline{E_1(c, c')} \setminus P_t| > 0\}.$$

We will say that $x$ is a low point of $E_1(c, c')$ if $x \in S_{c,c'}$, $x \cdot n = \alpha_{c,c'}$, and for any open disc $D$ centered at $x$ we have

$$|D \cap E_1(c, c')| > 0.$$

Similarly, we call $y$ a high point of $\overline{E_1(c, c')} \setminus P_t$ if $y \in S_{c,c'}$, $y \cdot n = \beta_{c,c'}$, and for any open disc $D$ centered at $y$ we have

$$|D \cap \overline{E_1(c, c')}| > 0.$$  

It is easy to see that such points $x, y$ actually exist. Indeed, by the definition of $\alpha_{c,c'}$ and an obvious covering argument, for any $\alpha > \alpha_{c,c'}$ there are points $x' \cdot n \leq \alpha$ and that holds for any disc $D$ centered at $x'$. Thus the set of such points $x'$ has at least one accumulation point $x$ on the line $x \cdot n = \alpha_{c,c'}$. It follows that any such $x$ is a low point of $E_1(c, c')$. The same argument works for $y$.

The low and high points need not be unique; however, all low points $x$ of $E_1(c, c')$ lie on the same line $x \cdot n = \alpha_{c,c'}$ parallel to the vector $v - v''$, and similarly for high points. Furthermore, the low and high points of $E_1(c, c')$ do not change if $E_1(c, c')$ is modified by a set of measure 0.

Let now $A = E_1(a, a + s'')$, and let $x$ be a low point of $A$. Since $s < 2s''$, we have

$$B := E_1(a, a + s) = E_1(a, a + s') \cup E_1(a + s - s'', a + s) \sim A \cup (A + v'' - v),$$

hence $x$ is also a low point of $B$ with respect to $v - v''$. Now note that

$$E_1(a + s' - s'', a + s') \sim A + (v'' - v')$$
intersects any open neighbourhood of \( x + (v'' - v') \) in positive measure. But on the other hand, \( E_1(a + s' - s'', a + s') \subset B \). By the extremality of \( x \) in \( B \), \( x + (v'' - v') \) lies on or above the line segment joining \( x \) and \( x + (v'' - v) \), hence \( v'' - v' \) lies on or above the line segment joining \( 0 \) and \( v'' - v \).

Repeating the argument in the last paragraph with \( x \) replaced by a high point \( y \) of \( \tilde{E}_1(a, a + s'') \), we obtain that \( v'' - v' \) lies on or below the line segment joining \( 0 \) and \( v'' - v \). Hence \( v, v', v'' \) are collinear.

Finally, we estimate the slope of the line through \( v, v'' \). We have to prove that
\[
\frac{2s'' - s}{s - s''} |v''_2 - v_2| \leq \epsilon
\]  
(recall that \( v''_1 - v_1 = s - s'' \)). Define \( x \) as above, and let \( k \in \mathbb{Z} \). Iterating translations by \( v - v'' \) (in both directions), we find that \( x + k(v - v'') \) is a low point of \( B \) as long as it belongs to \( B \), i.e.
as long as
\[
a \leq x_1 + k(s - s'') \leq a + s.
\]
The number of such \( k \)'s is at least \( \frac{s}{s - s''} - 1 \). On the other hand, all low points of \( B \) lie in the rectangle \( a \leq x_1 \leq a + s, -\epsilon \leq x_2 \leq 0 \). Hence
\[
\left( \frac{s}{s - s''} - 2 \right) |v''_2 - v_2| \leq \epsilon,
\]
which is (8). □

We return to the proof of Theorem 3. Since \( E \) is almost a square, we know roughly how the translates of \( E \) can fit together. Locally, any tiling by \( E \) is essentially a tiling by a “solid” \( 1 \times 1 \) square with “margins” of width between 0 and \( 2\epsilon \) (see Fig. 2).

We first locate a “corner”. Namely, we may assume that the tiling contains \( E \) and its translates \( E + u, E + v \), where
\[
1 \leq u_1 \leq 1 + 2\epsilon, \quad -2\epsilon \leq u_2 \leq 2\epsilon, \quad (9)
\]
\[
0 \leq v_1 \leq \frac{1}{2} + \epsilon, \quad 1 \leq v_2 \leq 1 + 2\epsilon. \quad (10)
\]
This can always be achieved by translating the tiled plane and taking symmetric reflections of it if necessary.

Let \( E + w \) be the translate of \( E \) which fits into this corner:
\[
v_1 + 1 \leq w_1 \leq v_1 + 1 + 2\epsilon, \quad u_2 + 1 \leq w_2 \leq u_2 + 1 + 2\epsilon. \quad (11)
\]
We will prove that \( w = u + v \) (without the \( \epsilon \)-errors).

From (11), (9), (10) we have
\[
1 \leq w_1 \leq \frac{3}{2} + 3\epsilon, \quad -4\epsilon \leq w_2 - v_2 \leq 4\epsilon.
\]
Hence \( w \) satisfies both of the following.
Figure 2: A “corner” and a fourth near-square.

(A) $E_2(0, 1 - (w_1 - u_1))$ complements $E_1(w_1 - u_1, 1) + (w - u)$, and

$$1 - (w_1 - u_1) = 1 - w_1 + u_1 \geq 1 + 1 - \left(\frac{3}{2} + 3\epsilon\right) = \frac{1}{2} - 3\epsilon,$$

$$|(w_1 - u_1) - v_1| = |(w_1 - v_1) - u_1| \leq 2\epsilon.$$

(B) $-4\epsilon \leq w_2 - v_2 \leq 4\epsilon$, and $F_2(r, t)$ complements $F_1(r', t') + (w - v)$, where

$$r = \max(0, w_2 - v_2), \quad r' = \max(0, v_2 - w_2),$$

$$t = 1 - \max(0, v_2 - w_2), \quad t' = 1 - \max(0, w_2 - v_2).$$

If $w = u + v$, we have $w - u = v$, $w - v = u$, hence by considering the “corner” $E, E + u, E + v$ we see that both (A) and (B) hold. Assuming that $\epsilon$ is small enough, we shall prove that:

1. All points $w$ satisfying (A) lie on a fixed straight line $l_1$ making an angle less than $\pi/4$ with the $x_1$ axis.

2. All points $w$ satisfying (B) lie on a fixed straight line $l_2$ making an angle at most $\pi/4$ with the $x_2$ axis.

It follows that there can be at most one $w$ which satisfies both (A) and (B), since $l_1$ and $l_2$ intersect only at one point. Consequently, if $E + w$ is the translate of $E$ chosen as above, we must have $w = u + v$. Now it is easy to see that $E + \Lambda$ is a tiling, where $\Lambda$ is the lattice $\{ku + mv : k, m \in \mathbb{Z}\}$.

We first prove 1. Suppose that $w, w', w'', \ldots$ (not necessarily all distinct) satisfy (A). By the assumptions in (A), we may apply Lemma 2 with $E_1$ and $E_2$ interchanged and with $a = 0, b = 1, s = 1 - (w_1 - u_1), s' = 1 - (w'_1 - u_1), \ldots \geq \frac{1}{2} - 3\epsilon$. From the second inequality in (A) and the triangle inequality we also have $|s - s''| \leq 4\epsilon$. We find that all $w$ satisfying (A) lie on a line $l_1$ with slope bounded by

$$\frac{\epsilon}{2s'' - s} \leq \frac{\epsilon}{s'' - |s'' - s|} \leq \frac{\epsilon}{1/2 - 7\epsilon},$$

which is less than 1 if $\epsilon < 1/16$. 


To prove 2., we let \( w, w', w'' \) be three (not necessarily distinct) points satisfying (B) and such that \( w_2 \leq w_2' \leq w_2'' \). We then apply the obvious analogue of Lemma \( \text{\ref{lemma:property}} \) with \( E_1, E_2 \) replaced by \( F_1, F_2 \) and with \( a = \max(v_2 - w_2, 0) \leq 4\epsilon, b = 1 - \max(v_2 - w_2) \geq 1 - 4\epsilon \). From the estimates in (B) we have \( 1 - 16\epsilon \leq s, s', s'' \leq 1, \) hence \( |2s'' - s| \geq 2 - 32\epsilon - 1 = 1 - 32\epsilon \). We conclude that all \( w \) satisfying (B) lie on a line \( l_2 \) such that the inverse of the absolute value of its slope is bounded by \( \frac{1}{1 - 32\epsilon} \). This is at most 1 if \( \epsilon \leq 1/33 \). 

\[ \Box \]

**Proof of Corollary** \( \text{\ref{corollary}} \). Let \( Q = [0,1] \times [0,1] \). By rescaling, it suffices to prove that for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( E \subset Q, E \) tiles \( \mathbb{R}^2 \) by translations, and \( |E| \geq 1 - \delta, \) then \( E \) contains the square \( Q_\epsilon = [\epsilon, 1 - \epsilon] \times [\epsilon, 1 - \epsilon] \) (up to sets of measure 0). The result then follows from Theorem \( \text{\ref{theorem:main}} \).

Let \( E \) be as above, and suppose that \( Q_\epsilon \setminus E \) has positive measure. Since \( E \) tiles \( \mathbb{R}^2 \), there is a \( v \in \mathbb{R}^2 \) such that \( |E \cap (E + v)| = 0 \) and \( |Q_\epsilon \cap (E + v)| > 0 \). We then have

\[ |E \cup (E + v)| = |E| + |E + v| \geq 2 - 2\delta, \]

but also

\[ |E \cup (E + v)| \leq |Q \cup (Q + v)| \leq 2 - \epsilon^2, \]

since \( E \subset Q, E + v \subset Q + v, \) and \( Q_\epsilon \cap (Q + v) \neq \emptyset \) so that \( |Q \cap (Q + v)| \geq \epsilon^2 \). This is a contradiction if \( \delta \) is small enough.

\[ \Box \]

**4 A counterexample in higher dimensions**

In this section we prove Theorem \( \text{\ref{theorem:counterexample}} \). It suffices to construct \( E \) for \( n = 3 \), since then \( E \times [0,1]^{n-3} \) is a subset of \( \mathbb{R}^n \) with the required properties.

Let \( (x_1, x_2, x_3) \) denote the Cartesian coordinates in \( \mathbb{R}^3 \). It will be convenient to rescale \( E \) so that \( [\epsilon, 1]^3 \subset E \subset [0,1 + \epsilon]^3 \).

\[
\begin{array}{c}
\text{A} \\
\text{B} \\
\text{P} \\
\text{Q}
\end{array}
\]

\[
\begin{array}{c}
\text{D} \\
\text{C} \\
\text{S} \\
\text{R}
\end{array}
\]

**Figure 3:** The construction of \( E \).

We construct \( E \) as follows. We let \( E \) be bounded from below and above by the planes \( x_3 = 0 \) and \( x_3 = 1 \) respectively. The planes \( x_1 = \epsilon, x_1 = 1, x_2 = \epsilon, x_2 = 1 \) divide the cube \([0,1 + \epsilon]^3\) into
9 parts (Figure 3). The middle part is entirely contained in \( E \). We label by \( A, B, C, D, P, Q, R, S \) the remaining 8 segments as shown in Figure 3. We then let

\[ E \cap P = P \cap \left\{ 0 \leq x_3 \leq \frac{1}{8} \text{ or } \frac{1}{2} \leq x_3 \leq \frac{5}{8} \right\}, \]

\[ E \cap R = R \cap \left\{ 0 \leq x_3 \leq \frac{1}{8} \text{ or } \frac{1}{2} \leq x_3 \leq \frac{5}{8} \right\}, \]

\[ E \cap Q = Q \cap \left\{ 0 \leq x_3 \leq \frac{1}{4} \text{ or } \frac{3}{8} \leq x_3 \leq \frac{3}{4} \text{ or } \frac{7}{8} \leq x_3 \leq 1 \right\}, \]

\[ E \cap S = S \cap \left\{ 0 \leq x_3 \leq \frac{1}{4} \text{ or } \frac{3}{8} \leq x_3 \leq \frac{3}{4} \text{ or } \frac{7}{8} \leq x_3 \leq 1 \right\}, \]

and

\[ E \cap A = A \cap \left\{ 0 \leq x_3 \leq \frac{1}{16} \right\}, \]

\[ E \cap C = A \cap \left\{ \frac{1}{2} \leq x_3 \leq \frac{9}{16} \right\}, \]

\[ E \cap B = B \cap \left\{ \frac{5}{16} \leq x_3 \leq \frac{3}{4} \right\}, \]

\[ E \cap D = D \cap \left\{ 0 \leq x_3 \leq \frac{1}{4} \text{ or } \frac{13}{16} \leq x_3 \leq 1 \right\}. \]

We also denote \( K = \bigcup_{j \in \mathbb{Z}} (E + (0, 0, j)) \).

Let \( E + T \) be a tiling of \( \mathbb{R}^3 \), and assume that \( 0 \in T \). Suppose that \( E + v \) and \( E + w \) are neighbours in this tiling so that the vertical sides of \((E \cap P) + v\) and \((E \cap Q) + w\) meet in a set of non-zero two-dimensional measure. Then we must have \( v - w = (0, 1, (v - w)_3) \), where \((v - w)_3 \in \{ \pm \frac{1}{4}, \pm \frac{3}{4} \}\). A similar statement holds with \( P, Q \) replaced by \( R, S \) and with the \( x_1, x_2 \) coordinates interchanged. We deduce that the tiling consists of copies of \( E \) stacked into identical vertical “columns” \( K_{ij} = K + (i, j, t_{ij}) \), arranged in a rectangular grid in the \( x_1x_2 \) plane and shifted vertically so that \( t_{i+1,j} - t_{ij} \) and \( t_{i,j+1} - t_{ij} \) are always \( \pm \frac{1}{4} \). We will use matrices \((t_{ij})\) to encode such a tiling or portions thereof.

It is easy to see that \((t_{ij})\), where \( t_{ij} = 0 \) if \( i + j \) is even and \( \frac{1}{4} \) if \( i + j \) is odd, is indeed a tiling. It remains to show that \( E \) does not admit a lattice tiling. Indeed, the four possible choices of the generating vectors in any lattice \((t_{ij})\) with \( t_{ij} = \pm \frac{1}{4} \) produce the configurations

\[
\begin{pmatrix}
0 & t \\
2t & 0
\end{pmatrix}, \quad
\begin{pmatrix}
2t & t \\
0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & t \\
-t & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & -t \\
t & 0
\end{pmatrix}.
\]

But it is easy to see that the corners \( A, B, C, D \) do not match if so translated.

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