Global entropy solutions to multi-dimensional isentropic gas dynamics with spherical symmetry

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Abstract

We are concerned with spherically symmetric solutions to Euler equations for multi-dimensional compressible fluids which have many applications in diverse real physical situations. The system can be reduced to one-dimensional isentropic gas dynamics with geometric source terms. Due to the presence of the singularity at the origin, there are few papers devoted to this problem. The present paper proves two existence theorems of global entropy solutions. The first one focuses on a case excluding the origin in which negative velocity is allowed, and the second one corresponds to a case which includes the origin with non-negative velocity. The $L^\infty$ compensated compactness framework and vanishing viscosity method are applied to prove the convergence of approximate solutions. In the second case, we show that if the blast wave initially moves outwards and the initial densities and velocities decay to zero with certain rates near the origin, then the densities and velocities tend to zero with the same rates near the origin for any positive time. In particular, the entropy solutions in two existence theorems are uniformly bounded with respect to time.

Keywords: isentropic gas, compensated compactness, uniform estimate

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1. Introduction

In this paper, we consider the Euler equations for compressible isentropic fluids with spherical symmetry which read
\[
\begin{align*}
\rho_t + \nabla \cdot \vec{m} &= 0, \quad \vec{x} \in \mathbb{R}^N, \\
\vec{m}_t + \nabla \cdot \left( \frac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p &= 0, \quad \vec{x} \in \mathbb{R}^N, 
\end{align*}
\]  
(1.1)

where $\rho, m$ and $p(\rho)$ denote the density, momentum and pressure of the gas respectively. The pressure takes the form of $p(\rho) = p_0 \rho^\gamma$, with $p_0 = \frac{\theta^2}{\gamma}, \theta = \frac{\gamma - 1}{\gamma}$ and $\gamma > 1$ being the adiabatic exponent.

We are interested in spherically symmetric solutions to system (1.1) with the form
\[
(\rho, \vec{m})(\vec{x}, t) = (\rho(x, t), m(x, t)\frac{x}{|x|}), x = |\vec{x}|. 
\]  
(1.2)

Then $(\rho(x, t), m(x, t))$ in (1.2) is governed by the one-dimensional Euler equations with geometric source terms:
\[
\begin{align*}
\rho_t + m_x &= -\frac{N-1}{x}m, \quad x \geq 0, \quad t > 0, \\
m_t + \left( \frac{m^2}{\rho} + p(\rho) \right)_x &= -\frac{N-1}{x}m^2, \quad x \geq 0, \quad t > 0. 
\end{align*}
\]  
(1.3)

For system (1.3), the study of spherically symmetric motion originates from several important applications, such as the theory of explosion waves in medium, and stellar dynamics including gaseous star formation and supernova formation. Note that the geometric source terms of (1.3) are singular at the origin, i.e. $x = 0$. In the present paper, we first study system (1.3) for the case where the origin is excluded. For simplicity, we consider (1.3) in the region outside the unit ball, that is,
\[
\begin{align*}
\nu_t + F(v)_x &= G(x, v), \quad x \in (1, +\infty), \quad t \in [0, +\infty), \\
\nu|_{t=0} &= \nu_0(x), \quad x \in [1, +\infty), \\
m|_{t=0} &= 0, \quad t \in [0, +\infty), 
\end{align*}
\]  
(1.4)

with initial data $\nu_0(x) \in L^\infty([1, +\infty))$, $v = (\rho, m)^T, F(v) = (m, \frac{m^2}{\rho} + p(\rho))^T, G(x, v) = (a(x)m, a(x)\frac{m^2}{\rho})^T$, where $a(x) = -\frac{N-1}{x}$. Then we consider the case where the origin is included, i.e. $x \geq 0$. Note that in this case, the initial boundary value problem is equivalent to the Cauchy problem of the compressible Euler equation (1.1) with spherically initial data
\[
(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)) \in L^\infty([0, +\infty)), \quad x \geq 0. 
\]  
(1.5)

The boundary condition $m = 0$ is based on the following: for classical solutions without vacuum to (1.3), $\frac{dx(t)}{dt} = u(x(t), t)$ defines a particle path $x(t)$, where $u = \frac{m}{\rho}$ is the velocity. Any two particle paths $x_1(t)$ and $x_2(t)$ preserve the mass within $[x_1(0), x_2(0)]$. Therefore the particle path starting from the boundary should stay on the boundary, which implies $u = 0$, i.e. $m = 0$ on the boundary. It does not matter whether the boundary is $x = 1$ or $x = 0$. 


There has been considerable progress on the existence of global entropy solutions for one dimension. Nishida [23] first proved the existence of large BV solutions for isothermal gas (i.e. \( \gamma = 1 \)) by the Glimm scheme. Nishida and Smoller [24] further studied the isentropic case (i.e. \( \gamma > 1 \)) under some restrictions on the initial data. Note that both works mentioned above consider the case of excluding vacuum.

If the initial values contain vacuum, Diperna [13] first proved the global existence of \( L^\infty \) entropy solutions with large initial data by the theory of compensated compactness for \( \gamma = 1 + \frac{2}{n+1} \), where \( n \geq 2 \) is any integer. Subsequently, Ding et al [11] and Chen et al [1] successfully extended the result to \( \gamma \in (1, \frac{5}{3}) \). Lions et al [16] and [17] treated the case \( \gamma > \frac{5}{3} \). Finally, Huang et al [14] solved the existence problem of \( L^\infty \) entropy solutions for \( \gamma = 1 \) through the compensated compactness and analytic extension methods.

For the inhomogeneous case, that is, where the right-hand side of (1.1) is not zero, Ding et al [12] established a general framework to investigate the global existence of \( L^\infty \) entropy solutions through the fractional step Lax–Friedrichs scheme. It should be noted that in the framework of [12], the approximate solutions are only required to be uniformly bounded in space \( x \), but not in time \( t \). The \( L^\infty \) norm of approximate solutions may increase with respect to time \( t \). Subsequently, extensive work has been produced on the inhomogeneous case—see [5, 8, 20, 21] and the references therein. To study the large time behavior of the entropy solution, it is important to show the uniform bound of solutions independent of time \( t \).

As shown in (1.3), the multi-dimensional compressible Euler system with spherical symmetry can be reduced to one-dimensional isentropic gas dynamics with geometric source terms, which may have singularity at \( x = 0 \). One of the main features is the resonance interaction among the characteristic mode and the geometrical source mode. The local existence of spherically symmetric solutions outside a solid ball centered at the origin was discussed by Makino et al [19] by the fractional step Lax–Friedrichs scheme. The global existence was first studied by Chen and Glimm [5], then by Tsuge for global existence with uniform estimates [27]. For the domain including the origin, Chen et al [2] proved a global existence theorem with large \( L^\infty \) data having only non-negative initial velocity. More interesting works can be found in Chen [2], Chen and Li [6], Li and Wang [15], Tsuge [27, 28], Yang [32, 33] and references therein. See also Wang and Wang [29, 30]. For further background information regarding the physical motivation for studying spherically symmetric solutions, please refer to [9, 10]. Recently, Chen et al [7] established a \( L^p \) global finite-energy entropy solution of isentropic Euler equations with spherical symmetry and large initial data.

Note that all of the above works are either based on numerical schemes, which need laborious estimates, or related with special solutions. In this paper, we apply the vanishing viscosity method together with the invariant region of a parabolic system with nonlinear source terms to obtain *a priori* uniform estimates of viscosity solutions. The revised version of the theory of the invariant region (lemma 2.1) is quite powerful and easy to use when dealing with source terms. This approach is valid for both the Cauchy problem and initial boundary value problem. In the first part of the present paper, we consider an initial boundary value problem outside a unit ball in which the origin is excluded. In the second part of the paper, we study the Cauchy problem so that the origin is included. In both cases, we obtain the uniform bound of viscosity solutions independent of time \( t \), while the \( L^\infty \) bound depends on time \( t \) in almost all previous works. This marks an important step in investigating the large time behavior of entropy solutions. It is worth pointing out that a set of new delicate control functions is designed to obtain the uniform bound of approximate solutions. Moreover, a new approach is proposed in the proof of the lower bound of density by using the appropriate decomposition of source terms of the heat equation and corresponding solutions successively. For the case excluding the origin,
we allow negative velocity when gas initially moves inwards. The major difficulties arise from handling the solid ball \( x = 1 \) and far field \( x = \infty \) at the same time. For the case including the origin, we also observe a new phenomena whereby if the blast wave initially moves outwards and the initial densities and velocities decay to zero with certain rates near the origin, then the densities and velocities tend to zero with the same rates near the origin for any positive time. It is remarked that in this case, the cavity phenomena occurs in the origin, but the gases move radially outwards. The method in our paper can be applied to solve the Euler equations with source terms, especially with geometric effect, such as two-dimensional radial gas flow, in gas flow through a general nozzle, etc. All of these results will be discussed in a forthcoming paper. Before formulating the main results, we define the entropy solutions of the initial boundary value problem and Cauchy problem respectively as follows.

**Definition 1.1.** A measurable function \( v(x, t) \in L^{\infty}((1, +\infty) \times \mathbb{R}^+) \) is called a global entropy solution of the initial boundary value problem (1.4) provided that

\[
\int_0^{+\infty} \int_1^{+\infty} (v \Phi_t + F(v) \Phi_x + G(x, v) \Phi) \, dx \, dt + \int_1^{+\infty} \Phi_0(x) \Phi(x, 0) \, dx = 0
\]

holds for any function \( \Phi \in C_1^0((1, +\infty) \times [0, +\infty)) \), and for any weak convex entropy pair \((\eta, q)(v)\), the inequality

\[
\eta(v)_t + q(v)_x - \nabla \eta(v) \cdot G(x, v) \leq 0
\]

holds in the sense of distributions.

The weak convex entropy-flux pair will be defined in the next section. The precise statement of the first result is given below.

**Theorem 1.1 (Excluding the origin).** Let \( 1 < \gamma \leq 3 \). Given any positive constant \( M_2 \), there exists a constant \( M_1 \), which is larger than \( M_2 \), such that if the initial and boundary data satisfy

\[
\rho_0(x) \geq 0, \quad \frac{m_0}{\rho_0} + \rho_0^0 \leq M_1 - M_2 x^{-\alpha}, \quad \frac{m_0}{\rho_0} - \rho_0^0 \geq -M_2 x^{-\alpha}, \text{ a.e.,}
\]

(1.8)

then there exists a global entropy solution of (1.4) satisfying

\[
\rho(x, t) \geq 0, \quad (\frac{m}{\rho} + \rho^0)(x, t) \leq M_1 - M_2 x^{-\alpha}, \quad (\frac{m}{\rho} - \rho^0)(x, t) \geq -M_2 x^{-\alpha}, \text{ a.e.,}
\]

(1.9)

in the sense of definition 1.1, where \( \alpha \) is any constant satisfying

\[
0 \leq \alpha \leq \frac{(N-1)\theta}{(1+\sqrt{\theta})^2}.
\]

**Remark 1.1.** For any \( M > M_1 \), theorem 1.1 still holds if \( M_1 \) is replaced by \( M \).

**Remark 1.2.** From (1.9), it implies that \( 0 \leq \rho^0 \leq \frac{M_1}{2}, -M_2 x^{-\alpha} \leq \frac{m}{\rho} \leq M_1 - M_2 x^{-\alpha} \).

**Remark 1.3.** From (1.8) and (1.9), the negative velocity is allowed.

**Remark 1.4.** The lower bound of \( \alpha \) is much more important than its upper bound. When \( \alpha \) becomes smaller, the increasing rate of \( -x^{-\alpha} \) to 0 decreases, which means that the range
of negative velocity is larger. In Tsuge [26], the initial data satisfies \( \frac{m_0}{\rho_0} - \rho_0^\theta \geq -C_\theta \frac{(N-1)\theta}{1+\sqrt{\theta}^2} \). More initial values with negative velocities are allowed here.

**Definition 1.2.** A measurable function \( \psi(x,t) \) is called a global entropy solution of the Cauchy problem (1.3) and (1.5) provided that

\[
\int_0^{+\infty} \int_0^{+\infty} (\psi_\Phi + F(\psi)\Phi_x + G(x,t,\psi)\Phi) \, dx \, dt + \int_0^{+\infty} \psi_0(x)\Phi(x,0) \, dx = 0
\]

holds for any function \( \Phi \in C^2_0((0, +\infty) \times [0, +\infty) \), and for any weak convex entropy pair \((\eta, q)\), the inequality

\[
\eta(\psi)_t + q(\psi)_x - \nabla \eta(\psi) \cdot G(x,t,\psi) \leq 0
\]

holds in the sense of distributions.

The second result of this paper is stated as follows.

**Theorem 1.2 (Including the origin).** Let \( \gamma > 1 \). Assume that for any nonnegative constants \( c \) and \( M \), there hold

\[
\rho_0(x) \geq 0, \quad \frac{m_0}{\rho_0} + \rho_0^\theta \leq M_3 x^\theta, \quad \frac{m_0}{\rho_0} - \rho_0^\theta \geq 0, \text{ a.e. } x \in [0, +\infty),
\]

then there exists a global entropy solution of (1.3) and (1.5) satisfying

\[
0 \leq \rho(x,t) \leq (\frac{M_3}{2})^\frac{1}{\gamma} x^\gamma, \quad 0 \leq m(x,t) \leq M_3 \rho(x,t) x^\theta \text{ a.e. } (x,t) \in [0, +\infty) \times \mathbb{R}^+.
\]

**Remark 1.5.** Theorem 1.2 means that if the blast wave initially moves outwards and \( \rho^\theta \) and \( u = \frac{m}{\rho} \) initially decay to zero with certain rates near the origin, then they tend to zero with the same rates near the origin for any positive time.

**Remark 1.6.** In theorem 1.2, the initial data can be allowed to tend to infinity at the far field.

**Remark 1.7.** The invariant region \( \frac{m}{\rho} + \rho^\theta \leq M, \frac{m}{\rho} - \rho^\theta \geq 0 \) was first observed in [2]. This corresponds to the special case \( c = 0 \) in theorem 1.2.

The main ingredient in proving theorem 1.1 is how to get the uniform estimates of viscosity solutions independent of viscosity \( \varepsilon \) and time \( t \). In fact, the viscosity solutions are uniformly bounded through a maximum principle for a parabolic system—see lemma 2.1 below. Roughly speaking, we first add viscous perturbation to the system (1.3) and get a viscous system (3.1), which can be reduced into a decoupled system (3.3) of Riemann invariants. Unfortunately, lemma 2.1 cannot be directly applied since the coefficients \( a_{12} \) and \( a_{21} \) of lemma 2.1 may be negative in the system (3.3), while they have to be negative for the application of the lemma. The key point is to introduce modified Riemann invariants to derive a new system (3.6) in which the coefficients have the desired sign. To prove theorem 1.2, we first introduce a space scaling transformation for both variables \( \rho = \tilde{\rho}_x, m = \tilde{m}_x \) and space coordinate \( \xi = \frac{1}{c-d+1} x^{c-d+1} \) (if \( c - d + 1 \neq 0 \)) or \( \xi = \ln x \) (if \( c - d + 1 = 0 \))—see section 4 below. Then we add viscous perturbations \((\varepsilon \tilde{\rho}_x, \varepsilon \tilde{m}_x)\) to the new system for
(\rho, m). Again using lemma 2.1, we obtain desired uniform estimates of viscosity solutions being independent of viscosity \( \varepsilon \) and time \( t \).

The present paper is organized as follows: in section 2, we construct approximate solutions by adding viscosity and some preliminaries are given. In section 3, we first obtain the uniform upper bounds independent of viscosity \( \varepsilon \) and time \( t \) for the viscosity solutions and then prove the \( H^{-1}_{\text{loc}} \) compactness for entropy–entropy flux pairs, and finally theorem 1.1. Section 4 is devoted to the proof of theorem 1.2, i.e. the existence of entropy solutions with spherical symmetry.

2. Preliminaries and formulations

We first introduce some basic facts for the system (1.3). The eigenvalues are

\[ \lambda_1 = \frac{m}{\rho} - \theta \rho^\theta, \quad \lambda_2 = \frac{m}{\rho} + \theta \rho^\theta, \]

(2.1)

where \( \theta = \frac{\gamma - 1}{2} \) and the corresponding right eigenvectors are

\[ r_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}. \]

(2.2)

The Riemann invariants \((w, z)\) are given by

\[ w = \frac{m}{\rho} + \rho^\theta, \quad z = \frac{m}{\rho} - \rho^\theta, \]

(2.3)

satisfying \( \nabla w \cdot r_1 = 0 \) and \( \nabla z \cdot r_2 = 0 \), where \( \nabla = (\partial_\rho, \partial_m) \) is the gradient with respect to \( U = (\rho, m) \). A pair of functions \((\eta, q) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2\) is defined to be an entropy–entropy flux pair of system (1.3) if it satisfies

\[ \nabla q(U) = \nabla \eta(U) \nabla \left[ \frac{m}{\rho} + p(\rho) \right]. \]

(2.4)

When

\[ \eta \bigg|_{\frac{m}{\rho} \text{ fixed}} \to 0, \quad \text{as } \rho \to 0, \]

\( \eta(\rho, m) \) is called weak entropy. Moreover, an entropy \( \eta(\rho, m) \) is convex (strictly convex) if the Hessian matrix \( \nabla^2 \eta(\rho, m) \) is nonnegative (positive). For example,

\[ \eta^*(\rho, m) = \frac{m^2}{2\rho} + \frac{p_0 \rho^{\gamma}}{\gamma - 1}, \quad q^*(\rho, m) = \frac{m^2}{2\rho^{\gamma}} + \frac{\gamma p_0^{\gamma-1} m}{\gamma - 1}, \]

(2.5)

is a strictly convex entropy pair. As shown in [16] and [17], any weak entropy for the system (1.3) is

\[ \eta = \rho \int_{-1}^{1} g \left( \frac{m}{\rho} + \rho^\theta s \right) (1 - s^2)^\lambda ds, \quad q = \rho \int_{-1}^{1} \left( \frac{m}{\rho} + \rho^\theta s \right) g \left( \frac{m}{\rho} + \rho^\theta s \right) (1 - s^2)^\lambda ds, \]

(2.6)

with \( \lambda = \frac{3 - \gamma}{4(\gamma - 1)} \) and \( g(\cdot) \in C^2(\mathbb{R}) \) is any function.

Now, we will introduce a revised version of the theory of the invariant region which is essentially based on the maximum principle for a parabolic equation—see [25].

**Lemma 2.1 (Maximum principle on bounded domain).** Let \( p(x, t), q(x, t), (x, t) \in [a, b] \times [0, T] \) be any bounded classical solution of the following quasilinear para-
bolic system
\[
\begin{align*}
    p_t + \mu_1 p_x &= p_{cx} + a_{11} p + a_{12} q + R_1, \\
    q_t + \mu_2 q_x &= q_{cx} + a_{21} p + a_{22} q + R_2,
\end{align*}
\]
with
\[
\begin{align*}
    p(x,0) &\leq 0, \quad q(x,0) \geq 0, \quad \text{for} \quad x \in [a,b], \\
    p(a,t) &\leq 0, \quad q(a,t) \geq 0, \quad \text{for} \quad t \in [0,T], \\
    p(b,t) &\leq 0, \quad q(b,t) \geq 0, \quad \text{for} \quad t \in [0,T],
\end{align*}
\]
where
\[
\mu_i = \mu_i(x,t,p(x,t),q(x,t)), a_{ij} = a_{ij}(x,t,p(x,t),q(x,t))
\]
and the source terms
\[
R_i = R_i(x,t,p(x,t),q(x,t),p_x(x,t),q_x(x,t)), i,j = 1,2, \forall (x,t) \in [a,b] \times [0,T].
\]
Here \(\mu, a_{ij}\) are bounded with respect to \((x,t,p,q) \in [a,b] \times [0,T] \times K\), where \(K\) is an arbitrary compact subset in \(\mathbb{R}^2\). \(a_{12}, a_{21}, R_1, R_2\) are continuously differentiable with respect to \(p,q\).

Assume that
\[
\begin{align*}
\text{(C1)} \quad & a_{12} \leq 0 \quad \text{holds for} \quad p = 0 \quad \text{and} \quad q \geq 0; \quad a_{21} \leq 0 \quad \text{holds for} \quad q = 0 \quad \text{and} \quad p \leq 0; \\
\text{(C2)} \quad & R_1 \leq 0 \quad \text{holds for} \quad p = 0 \quad \text{and} \quad q \geq 0; \quad R_2 \geq 0 \quad \text{holds for} \quad q = 0 \quad \text{and} \quad p \leq 0.
\end{align*}
\]
Then for any \((x,t) \in [a,b] \times [0,T], \quad p(x,t) \leq 0, \quad q(x,t) \geq 0.\]

3. Proof of theorem 1.1

3.1. Uniform upper bound estimate

We approximate (1.3) by adding artificial viscosity as follows:
\[
\begin{align*}
    \rho_t + m_x &= -\frac{N - 1}{x} m + \varepsilon \rho_{xx}, \\
    m_t + \left( \frac{m^2}{\rho} + p(\rho) \right)_x &= -\frac{N - 1}{x} m^2 + \varepsilon m_{xx} - \frac{2 \varepsilon \alpha M_2}{x^{\alpha+1}} \rho_{xx}.
\end{align*}
\]

We consider (3.1) on a cylinder \((1,b) \times \mathbb{R}^+, \text{with} \quad \mathbb{R}^+ = [0, +\infty), b := b(\varepsilon) \text{ satisfying} \lim_{\varepsilon \to 0} b(\varepsilon) = \infty\). The initial-boundary values are given by
\[
\begin{align*}
    (\rho,m)|_{t=0} &= (\rho_0(x), m_0(x)) = (\rho_0(x) + \varepsilon^\frac{1}{2}, \frac{m_0(x)}{\rho_0(x)} (\rho_0(x) + \varepsilon^\frac{1}{2})) \ast j^\varepsilon, 1 \leq x \leq b, \\
    (\rho,m)|_{x=1} &= (\rho_0(1),0), (\rho,m)|_{x=b} = (\rho_0(b),m_0(b)), t > 0,
\end{align*}
\]
where \(j^\varepsilon\) is a standard mollifier with small parameter \(\varepsilon > 0\). Next we will derive the uniform bound of the viscosity solutions by the maximum principle, i.e. lemma 2.1. Based on the Riemann invariants \(w\) and \(z\), we transform (3.1) into the following form:
From the initial condition (1.8), we set the control functions \((\phi, \psi)\):

\[
\phi = M_1 - M_2 x^{-\alpha} + \varepsilon e^{Ct}, \quad \psi = M_2 x^{-\alpha} + \varepsilon e^{Ct},
\]

where \(C\) and \(M_1\) are positive constants which will be determined later. Then a simple calculation shows that

\[
\phi_t = \varepsilon e^{Ct}, \quad \phi_x = \alpha M_2 x^{-\alpha - 1}, \quad \phi_{xx} = -\alpha (\alpha + 1) M_2 x^{-\alpha - 2};
\]

\[
\psi_t = \varepsilon e^{Ct}, \quad \psi_x = -\alpha M_2 x^{-\alpha - 1}, \quad \psi_{xx} = \alpha (\alpha + 1) M_2 x^{-\alpha - 2}.
\]

Define a modified Riemann invariant \((\bar{w}, \bar{z})\) as

\[
\bar{w} = w - \phi, \quad \bar{z} = z + \psi.
\]

We shall use lemma 2.1 to show \(\bar{w} \leq 0\) and \(\bar{z} \geq 0\) for any time. Inserting (4.4) into (3.3) yields the equations for \(\bar{w}\) and \(\bar{z}\):

\[
\begin{aligned}
\bar{w}_t + \lambda_2 \bar{w}_x &= \varepsilon \bar{w}_{xx} + 2 \varepsilon (w_x - \alpha M_2 x^{-\alpha - 1}) \frac{\rho_x}{\rho} - \theta \frac{N-1}{x} \rho^0 \bar{z} - \theta \frac{N-1}{x} \rho^2

\quad - \varepsilon \theta (\theta + 1) \rho^0 \rho_x^2, \\
\bar{z}_t + \lambda_1 \bar{z}_x &= \varepsilon \bar{z}_{xx} + 2 \varepsilon (z_x - \alpha M_2 x^{-\alpha - 1}) \frac{\rho_x}{\rho} + \theta \frac{N-1}{x} \rho^0 \bar{z} + \theta \frac{N-1}{x} \rho^2

\quad + \varepsilon \theta (\theta + 1) \rho^0 \rho_x^2.
\end{aligned}
\]

Note that

\[
\lambda_1 = \bar{z} - \psi + (1 - \theta) \rho^0, \quad \lambda_2 = \bar{w} + \phi + (\theta - 1) \rho^0.
\]

System (3.5) becomes

\[
\begin{aligned}
\bar{w}_t + (\lambda_2 - 2 \varepsilon \frac{\rho_x}{\rho}) \bar{w}_x &= \varepsilon \bar{w}_{xx} + a_{11} \bar{w} + a_{12} \bar{z} + R_1, \\
\bar{z}_t + (\lambda_1 - 2 \varepsilon \frac{\rho_x}{\rho}) \bar{z}_x &= \varepsilon \bar{z}_{xx} + a_{21} \bar{w} + a_{22} \bar{z} + R_2,
\end{aligned}
\]

where

\[
\begin{aligned}
a_{11} &= -\phi_x, \quad a_{12} = -\theta \frac{N-1}{x} \rho^0 \leq 0, \quad a_{21} = 0, \quad a_{22} = \psi_x + \theta \frac{N-1}{x} \rho^0, \\
R_1 &= -\phi_t - [\phi + (\theta - 1) \rho^0] \phi_x + \varepsilon \phi_{xx} - \varepsilon \theta (\theta + 1) \rho^0 \rho_x^2 + \theta \frac{N-1}{x} \rho^0 \psi + \theta \frac{N-1}{x} \rho^2, \\
R_2 &= \psi_t + [\psi + (1 - \theta) \rho^0] \psi_x - \varepsilon \psi_{xx} + \varepsilon \theta (\theta + 1) \rho^0 \rho_x^2 - \theta \frac{N-1}{x} \rho^0 \psi + \theta \frac{N-1}{x} \rho^2.
\end{aligned}
\]

A direct computation gives

\[
R_1 \leq (M_1 - M_2 x^{-\alpha} + 2 \varepsilon e^{Ct} + (\theta - 1) \rho^0) \psi_x + \theta \frac{N-1}{x} \rho^0 \psi + \theta \frac{N-1}{x} \rho^2 - \varepsilon e^{Ct}
\]

\[
= -\alpha M_1 x^{-\alpha - 1} - \alpha M_2 x^{-2 \alpha - 1} + \alpha M_2 (1 - \theta) \rho^0 x^{-\alpha - 1} + \theta (N - 1) M_2 \rho^0 x^{-\alpha - 1} - \theta \frac{N-1}{x} \rho^2 - 2 \varepsilon e^{Ct} M_2 x^{-\alpha - 1} - \varepsilon e^{Ct} + \theta \frac{N-1}{x} \rho^0 \varepsilon e^{Ct}.
\]
To ensure \( R_1 \leq 0 \), it is sufficient to show that
\[
\alpha M_2 x^{-\alpha} + M_2 [\alpha(1 - \theta) + \theta(N - 1)] \rho^\theta + \theta(N - 1) \rho^\theta \varepsilon e^{C t} x^\alpha
\]
\[
\leq \alpha M_2 + \theta(N - 1) \rho^\theta x^\alpha + 2 \varepsilon e^{C t} M_2 + \varepsilon C e^{C t} x^{\alpha + 1}.
\]
(3.7)

For the second term on the left-hand side of (3.7), by Cauchy–Schwartz inequality, we have
\[
M_2 [\alpha(1 - \theta) + \theta(N - 1)] \rho^\theta \leq \frac{1}{2} \theta(N - 1)^2 \rho^\theta + \frac{M_2^2 [\alpha(1 - \theta) + \theta(N - 1)]^2}{2\theta(N - 1)}.
\]

Since \( x \geq 1 \), the first and second terms in (3.7) can be controlled by choosing \( M_1 \geq M_2 \) and large enough.

For the third term, since \( \rho^\theta \leq \frac{1}{2}(1 + \rho^2) \), we choose \( C \) being at least larger than \( \frac{\theta(N - 1)}{2} \) and we will see that the choice of \( C \) also depends on \( R_2 \) later. Then \( \theta(N - 1) \rho^\theta \varepsilon e^{C t} x^\alpha \) can be controlled by \( \varepsilon C e^{C t} x^{\alpha + 1} \) and \( \theta(N - 1) \rho^2 x^\alpha \), as long as for fixed \( T \), choosing \( \varepsilon \) small enough such that \( \sqrt{\varepsilon} e^{C t} \leq \sqrt{\varepsilon} e^{C T} < 1 \) for \( t \leq T \). So it can be seen that \( \varepsilon \) relies on \( C \) and \( T \).

Thus \( R_1 \leq 0 \).

We now turn to the term \( R_2 \). Denote \( \beta = \frac{\rho}{\theta(N - 1)} \). By a direct calculation, we have
\[
R_2 \geq [-\psi + (1 - \theta) \rho^\theta] \psi_x - \varepsilon \psi_{xx} + \frac{\theta(N - 1)}{\theta(N - 1)^2} \rho^\theta \psi + \frac{\theta(N - 1)^2}{\theta(N - 1)} \rho^2 \psi + \psi_t
\]
\[
= \theta(N - 1) \left[ \rho^\theta - \frac{\alpha(1 - \theta) + \theta(N - 1)}{\theta(N - 1)} \rho^\theta \psi + \frac{\alpha \psi^2}{\theta(N - 1)} \right] - \varepsilon \alpha \psi e^{C t} + \frac{\varepsilon(1 - \theta) \rho^\theta \alpha e^{C t}}{x} + \varepsilon C e^{C t} - \varepsilon \alpha(1 + 1) M_2 x^{-\alpha - 2}
\]
\[
= \frac{\theta(N - 1)^2}{\theta(N - 1)} \left[ \left( \rho^\theta - \frac{\beta(1 - \theta) + 1}{2} \psi \right)^2 - \frac{(\beta(1 - \theta) + 1)^2}{4} \psi^2 + \beta \psi^2 \right]
\]
\[
+ \varepsilon \left[ -\alpha M_2 x^{-\alpha} + \varepsilon C e^{C t} \frac{\psi}{x} + C e^{C t} - \alpha(1 + 1) M_2 x^{-\alpha - 2} + (1 - \theta) \rho^\theta \alpha e^{C t} \frac{\psi}{x} \right].
\]
(3.8)

For the second term of the right-hand side of (3.8), choosing \( C \) also sufficiently large, and using \( \sqrt{\varepsilon} e^{C t} < 1 \) from the estimate of \( R_1 \) above, then its positivity is obtained. To ensure \( R_2 \geq 0 \), it remains to guarantee that
\[
\beta \psi^2 - \frac{(\beta(1 - \theta) + 1)^2}{4} \psi^2 \geq 0,
\]
i.e.,
\[
g(\beta) = \beta^2 (1 - \theta)^2 - 2 \beta (1 + \theta) + 1 \leq 0,
\]
which holds for any
\[
0 < \beta(N - 1) \beta_1 \leq \alpha \leq \beta(N - 1) \beta_2,
\]
where
\[
\beta_1 = \frac{1}{(1 + \sqrt{\theta})^2}, \quad \beta_2 = \frac{1}{(1 - \sqrt{\theta})^2},
\]
are the roots of the equation \( g(\beta) = 0 \). By the initial and boundary data (3.2), we obtain
\[
\bar{w}(x, 0) = w(x, 0) - \phi \leq 0, \quad \bar{z}(x, 0) = z(x, 0) + \psi \geq 0, \quad \text{for } x \geq 1;
\]
\( \dot{w}(1, t) \leq 0, \dot{z}(1, t) \geq 0, \text{ for } t > 0; \)

\( \dot{w}(b, t) \leq 0, \dot{z}(b, t) \geq 0, \text{ for } t > 0. \)

By lemma 2.1,

\[
\begin{align*}
 w(x, t) &\leq M_1 - M_2 x^{-\alpha} + \varepsilon e^{\Theta}, z(x, t) \geq -M_2 x^{-\alpha} - \varepsilon, \quad \text{i.e.} \\
 w(x, t) &\leq M_1 - M_2 x^{-\alpha} + \sqrt{\varepsilon}, z(x, t) \geq -M_2 x^{-\alpha} - \sqrt{\varepsilon},
\end{align*}
\]

where we have used the fact that \( \sqrt{\varepsilon}e^{\Theta} < 1. \) This gives the following theorem 3.1 for approximate solutions.

**Theorem 3.1 (\( L^\infty \) estimate: excluding the origin).** Let \( 1 < \gamma \leq 3. \) Given any positive constant \( M_1, \) which is larger than \( M_2, \) such that if

\[
\rho_0(x) \geq \varepsilon \hat{\approx}, \quad \frac{m_0}{\rho_0} + \rho_0^\theta \leq M_1 - M_2 x^{-\alpha} + \varepsilon, \quad \frac{m_0}{\rho_0} - \rho_0^\theta \geq -M_2 x^{-\alpha} - \varepsilon, \quad x \in [1, b],
\]

\[
\left( \frac{m}{\rho^\theta} + \rho_0^\theta \right)_{|x=1} \leq M_1 - M_2, \quad \left( \frac{m}{\rho^\theta} - \rho_0^\theta \right)_{|x=1} \geq -M_2,
\]

\[
\left( \frac{m}{\rho^\theta} + \rho_0^\theta \right)_{|x=b} \leq M_1 - M_2 b^{-\alpha} + \varepsilon, \quad \left( \frac{m}{\rho^\theta} - \rho_0^\theta \right)_{|x=b} \geq -M_2 b^{-\alpha} - \varepsilon, \quad t > 0,
\]

(3.9)

where \( \frac{\theta}{x} \leq \alpha \leq \frac{\theta}{(1 + \sqrt{\varepsilon})}, \) then the solutions of (3.1) and (3.2) satisfy

\[
\left( \frac{m}{\rho^\theta} + \rho_0^\theta \right)(x, t) \leq M_1 - M_2 x^{-\alpha} + \sqrt{\varepsilon}, \quad \left( \frac{m}{\rho^\theta} - \rho_0^\theta \right)(x, t) \geq -M_2 x^{-\alpha} - \sqrt{\varepsilon}.
\]

(3.10)

The only thing left to check in theorem 3.1 is condition (3.9) for initial data. It is easy to see that the function \( f(r) = (\varepsilon + r^\theta)^{\frac{\theta}{x}} - r - \varepsilon \hat{\approx} \) increases on \((0, \infty)\) when \( \theta \in (0, 1] \) and \( \varepsilon \) is small. By \( f(0) > 0, \) it follows that \( f(r) > 0 \) for \( r \geq 0. \) Hence the condition (3.9) holds.

### 3.2. Lower bound estimate of density

From the above argument, we know that the velocity \( u = \frac{m}{\rho} \) is uniformly bounded, i.e. \( |u| \leq M_1. \) The lower bound of density can be derived as in [18], but we introduce a different treatment for estimating an \( a \) \( priori \) lower bound for solutions of heat equations with general source terms. Set \( \varepsilon = \ln \rho, \) and we can get a scalar equation for \( \varepsilon, \) that is,

\[
\varepsilon_t + \varepsilon_x u + u_x = \varepsilon \varepsilon_{xx} + \varepsilon e^\frac{N-1}{x} u.
\]

(3.11)

From the initial-boundary value of (3.2), we have \( \varepsilon_{|x=0} = \ln \rho_0(x), \varepsilon_{|x=1} = \ln \rho_0(1), \varepsilon_{|x=b} = \ln \rho_0(b). \) Then it follows from (3.11) that

\[
\varepsilon_t - \varepsilon \varepsilon_{xx} = \varepsilon (e_x - \frac{u}{2\varepsilon})^2 - \frac{u^2}{4\varepsilon} - \frac{N-1}{x} u - u_x.
\]

To the best of our knowledge, we have not found any literature that ever stated the following lemma to obtain the lower bound of density.
Lemma 3.1. Assume that \( w \) is a classical solution of the heat equation:
\[
\begin{align*}
  w_t - \varepsilon w_{xx} &= f_1(x,t) + f_2(x,t) + h_t(x,t), & a < x < b, t > 0, \\
  w|_{t=0} &= \varphi(x), & a < x < b, \\
  w|_{x=a} &= \varphi(a), & w|_{x=b} = \varphi(b),
\end{align*}
\]
(3.12)
where \( f_1(x,t) \), \( f_2(x,t) \) and \( h_t(x,t) \) are bounded smooth functions and \( \varphi(x) \) is a bounded function. Then, there exists a positive constant \( C(a,b,\varepsilon,t) \), s.t.,
\[
w \geq -C(a,b,\varepsilon,t).
\]

Proof. The proof is given in the appendix.

By applying lemma 3.1, for
\[
w = e, f_1(x,t) = \varepsilon(e_x - \frac{u}{2\varepsilon})^2, f_2(x,t) = -\frac{u^2}{4\varepsilon} - \frac{N-1}{x} u, h_t(x,t) = u,
\]
it follows that
\[
\rho \geq e^{-C(a,b,\varepsilon,t)} > 0.
\]

From the local existence of approximate solutions, the upper and lower bound of density, we can conclude the following theorem for the global existence of approximate solutions.

Theorem 3.2. Under the assumption of the previous theorem, for any time \( T > 0 \), there exists \( \varepsilon_0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \), the initial-boundary value problem (3.1) and (3.2) admits a unique classical solution on \([1,b] \times [0,T]\) satisfying
\[
\rho(x,t) \geq e^{-C(\varepsilon,T)}, \left(\frac{m}{\rho} + \rho^\theta\right)(x,t) \leq M_1 - M_2x^{-\alpha} + \sqrt{\varepsilon}, \left(\frac{m}{\rho} - \rho^\theta\right)(x,t) \geq - M_2x^{-\alpha} - \sqrt{\varepsilon}.
\]

3.3. \( H_{\text{loc}}^{-1} \) compactness of the entropy pair

For any \( T \in (0,\infty) \), let \( \Pi_T = (1, +\infty) \times (0,T) \). We consider the entropy dissipation measures
\[
\eta(\rho,m) + q(\rho,m)_x,
\]
(3.13)
where \((\eta, q)\) is any weak entropy–entropy flux pair whose formula is given in (2.6). We will apply the Murat lemma to conclude that the entropy dissipation measures in (3.13) lie in a compact set of \( H_{\text{loc}}^{-1}(\Pi_T) \).

Lemma 3.2 (Murat [22]). Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, then
\[
(\text{compact set of } W_{\text{loc}}^{-1,1}(\Omega)) \cap (\text{bounded set of } W_{\text{loc}}^{-1,1}(\Omega)) \subset (\text{compact set of } H_{\text{loc}}^{-1}(\Omega)),
\]
where \( 1 < q \leq 2 < r \).

We write \( U = -\frac{N-1}{r}m, V = -\frac{N-1}{r} \frac{w^2}{\rho} \) for simplicity. Let \( K \subset \Pi_T \) be any compact set and choose \( \varphi \in C_0^\infty(\Pi_T) \) such that \( \rho|_K = 1 \) and \( 0 \leq \varphi \leq 1 \). When \( \varepsilon \) is small, \( K \subset (1,b(\varepsilon)) \times (0,T) \). Multiplying (3.1) by \( \nabla \eta^* \varphi \) with \( \eta^* \) being the mechanical entropy in (2.5), we obtain
\[
\varepsilon \int \int_{\Omega_t} (\rho, m) \nabla^2 \eta^* (\rho, m)^T \, d\mathbf{x} \, dt
\]
\[
= \int \int_{\Omega_t} \left[ (V \phi - 2 \varepsilon \phi \rho) \eta_\alpha^* + U \eta_\rho^* \phi + \eta^* q + q^* \varphi_x + \varepsilon \eta^* \varphi_{xx} \right] \, d\mathbf{x} \, dt. \tag{3.14}
\]

Note that
\[
(\rho, m) \nabla^2 \eta^* (\rho, m)^T = p_0 \gamma \rho^{\gamma-1} \rho^2 + \rho \mu_\alpha^2,
\]
and
\[
| (V \phi - 2 \varepsilon \phi \rho) \eta_\alpha^* | \leq \frac{\varepsilon p_0 \gamma}{2} \rho^{\gamma-1} \rho^2 + \varepsilon C \rho^2 m^2 \rho^{\gamma-1} + C \|V\|_{L^\infty},
\]
we get
\[
\left\{ \begin{array}{l}
\frac{\varepsilon}{2} \int \int_{\Omega_t} \varphi (\rho, m) \nabla^2 \eta^* (\rho, m)^T \, d\mathbf{x} \, dt \\
\leq \int \int_{\Omega_t} C (\varepsilon p_0 \gamma) \rho^{\gamma-1} \rho^2 \, d\mathbf{x} \, dt \\
+ \int \int_{\Omega_t} \left[ \eta^* \varphi_x + q^* \varphi_x + \varepsilon \eta^* \varphi_{xx} + (m_1^2 + \frac{\gamma}{\gamma - 1} m^2 \rho^{\gamma-1} p_0) \|U\|_{L^\infty} \right] \, d\mathbf{x} \, dt \\
\leq C (\varphi).
\end{array} \right.
\]

Thus we have arrived at
\[
\varepsilon \rho^{\gamma-2} \rho^2 + \varepsilon \mu_\alpha^2 \in L^2_{\text{loc}} (\Omega_T). \tag{3.15}
\]
Note that when \( \gamma > 2, \rho^{\gamma-2} \rho^2 \) is degenerate near \( \rho = 0 \). For simplicity, we assume that \( 1 < \gamma \leq 2 \). For any weak entropy–entropy flux pairs given in (2.6), as in (3.14), we have
\[
\eta + q = \varepsilon \eta \rho - \varepsilon (\rho, m) \nabla^2 \eta (\rho, m)^T + (\eta U + \eta_\rho V) - 2 \varepsilon \eta \rho \phi_x = \sum_{i=1}^{4} J_i. \tag{3.16}
\]
By (3.15) and the boundedness of \( \rho \) and \( \frac{\eta}{\rho} \) from theorem 3.1, we have
\[
\varepsilon \rho^2 + \varepsilon m^2 \in L^2_{\text{loc}} (\Omega_T). \tag{3.17}
\]
For \( J_1 \),
\[
\left| \int \int_{\Omega_t} \varepsilon \eta \rho \varphi_x \, d\mathbf{x} \, dt \right| \leq \left| \int \int_{\Omega_t} \varepsilon \eta \rho \varphi_x \, d\mathbf{x} \, dt \right| \leq \left| \int \int_{\Omega_t} (\varepsilon \eta \rho) \rho \varphi_x \, d\mathbf{x} \, dt \right| \leq C \left| \int \int_{\Omega_t} (\varepsilon \rho^2 + m^2) \varphi_x \, d\mathbf{x} \, dt \right| \leq C \int \int_{\Omega_t} (\varepsilon \rho^2 + m^2) \varphi_x \, d\mathbf{x} \, dt,
\]
then by (3.17), it is obvious that \( J_1 \) is compact in \( H^{-1}_{\text{loc}} (\Omega_T) \). Note that for any weak entropy, the Hessian matrix \( \nabla^2 \eta \) is controlled by \( \nabla^2 \eta^* \) (see [16]), i.e.
\[
(\rho, m) \nabla^2 \eta (\rho, m)^T \leq C (\rho, m) \nabla^2 \eta^* (\rho, m)^T, \tag{3.18}
\]
then \( J_2 \) is bounded in \( L^2_{\text{loc}} (\Omega_T) \) and then compact in \( W^{-1, \nu}_{\text{loc}} (\Omega_T) \) by the embedding theorem, for some \( 1 < \nu < 2 \). Similarly, \( J_3 \) and \( J_4 \) are bounded in \( L^2_{\text{loc}} (\Omega_T) \). Thus
\[
\eta + q \text{ is compact in } W^{-1, \nu}_{\text{loc}} (\Omega_T) \text{ for some } 1 < \nu < 2.
\]
On the other hand, $\eta_t + q_x$ is bounded in $W_{\text{loc}}^{-1,\infty}(\Omega_T)$. With the help of lemma 3.2, we conclude that

$$\eta_t + q_x, \text{is compact in } H_{\text{loc}}^{-1}(\Omega_T)$$

(3.19)

for all weak entropy–entropy flux pairs.

**Remark 3.1.** We focus on the uniform bound of $\rho$ and $m$. In the above argument, (3.19) still holds for the case where $\gamma > 2$ by a similar argument of [31]. See also [18]. We assume $1 < \gamma \leq 2$ for simplicity.

### 3.4. Entropy solution

By (3.19) and the compactness framework established in [11, 13, 16], we can prove that there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$ (still denoted by $(\rho^\varepsilon, m^\varepsilon)$) such that

$$(\rho^\varepsilon, m^\varepsilon) \to (\rho, m) \text{ in } L_p^\varepsilon(\Omega_T), \ p \geq 1.$$ (3.20)

As in [5, 6, 26, 27], we can prove that $(\rho, m)$ is an entropy solution to the initial-boundary value problem (1.4) and $m_{\text{loc}} = 0$ in the sense of the divergence-measure fields introduced in [3, 4]. Therefore, theorem 1.1 is complete.

**Remark 3.2.** The approach can also be applied to the Euler–Poisson system with spherical symmetry.

### 4. Proof of theorem 1.2

#### 4.1. Uniform upper bound estimate

Consider the system (1.3) on a cylinder $(a, b) \times \mathbb{R}^+$, with $\mathbb{R}^+ = [0, +\infty)$, $a := a(\varepsilon), b := b(\varepsilon) > 1$, and $\lim_{\varepsilon \to 0} a(\varepsilon) = 0$ for any $\delta \in \mathbb{R}$ and $\lim_{\varepsilon \to 0} b(\varepsilon) = \infty$. For example, $a(\varepsilon)$ can be taken as $-\frac{1}{\ln \varepsilon}$. We make a scaling transformation

$$\rho = \tilde{\rho}\tilde{x}, m = \tilde{m}\tilde{x}.$$ 

Taking $d = (\theta + 1)c > 0$, the system (1.3) can be rewritten as

$$\begin{cases}
\tilde{\rho}_t + x^{d-c}\tilde{m}_x = -(N-1+d)x^{d-c-1}\tilde{m}, \\
\tilde{m}_t + x^{d-c}\left(\frac{\tilde{\xi}^2}{\tilde{\rho}} + p(\tilde{\rho})\right)_x = [-2(d-c+N-1)\tilde{\xi}^2 \tilde{\rho} - (2d-c)p(\tilde{\rho})]x^{d-c-1}.
\end{cases}$$ (4.1)

If $c - d + 1 \neq 0$, let $\xi = \frac{1}{x^{c-d+1}}$. If $c - d + 1 = 0$, let $\xi = \ln x$. Then (4.1) becomes

$$\begin{cases}
\tilde{\rho}_t + \tilde{m}_\xi = -(N-1+d)x^{d-c-1}\tilde{m}, \\
\tilde{m}_t + \left(\frac{\tilde{\xi}^2}{\tilde{\rho}} + p(\tilde{\rho})\right)_\xi = [-2(d-c+N-1)\tilde{\xi}^2 \tilde{\rho} - (2d-c)p(\tilde{\rho})]x^{d-c-1}.
\end{cases}$$ (4.2)

We approximate (4.2) by adding artificial viscosity as follows:

$$\begin{cases}
\tilde{\rho}_t + \tilde{m}_\xi = -(N-1+d)x^{d-c-1}\tilde{m} + \varepsilon\tilde{\rho}_{\xi\xi}, \\
\tilde{m}_t + \left(\frac{\tilde{\xi}^2}{\tilde{\rho}} + p(\tilde{\rho})\right)_\xi = [-2(d-c+N-1)\tilde{\xi}^2 \tilde{\rho} - (2d-c)p(\tilde{\rho})]x^{d-c-1} + \varepsilon\tilde{m}_{\xi\xi}.
\end{cases}$$ (4.3)
The initial-boundary value conditions are given as follows:

\[(\tilde{\rho}, \tilde{m})|_{t=0} = (\tilde{\rho}_0(x), \tilde{m}_0(x)) \]

\[= (\tilde{\rho}_0(x) + \varepsilon \tilde{\xi}, (\tilde{m}_0(x) + \varepsilon \tilde{\xi})/\tilde{\rho}_0(x) + \varepsilon \tilde{\xi}) \chi_{[a(\varepsilon), b(\varepsilon)]}, x \in [a(\varepsilon), b(\varepsilon)], \]

\[(\tilde{\rho}, \tilde{m})|_{x=a(\varepsilon)} = (\tilde{\rho}_0(a(\varepsilon)), \tilde{m}_0(a(\varepsilon))) = (\tilde{\rho}_0(a(\varepsilon)), 0), \]

\[(\tilde{\rho}, \tilde{m})|_{x=b(\varepsilon)} = (\tilde{\rho}_0(b(\varepsilon)), \tilde{m}_0(b(\varepsilon))), t > 0, \tag{4.4} \]

where \(\tilde{\xi}\) is the standard mollifier and \(\chi\) is the characteristic function. As in the proof of theorem 1.1, the key point is to derive the uniform upper bound of the approximate solution \(\tilde{\rho}\) and \(\tilde{m}\).

By the definition of Riemann invariants, we have

\[w = \frac{m}{\rho} + \rho^\theta = \left(\frac{\tilde{m}}{\tilde{\rho}} - \rho^\theta\right) x^\theta := \tilde{w} x^\theta, \quad z = \frac{m}{\rho} - \rho^\theta = \left(\frac{\tilde{m}}{\tilde{\rho}} - \rho^\theta\right) x^\theta := \tilde{z} x^\theta. \]

Similarly, we have

\[\lambda_1 = \frac{m}{\rho} - \theta \rho^\theta = \left(\frac{\tilde{m}}{\tilde{\rho}} - \theta \rho^\theta\right) x^\theta := \tilde{\lambda}_1 x^\theta, \quad \lambda_2 = \frac{m}{\rho} + \theta \rho^\theta = \left(\frac{\tilde{m}}{\tilde{\rho}} + \theta \rho^\theta\right) x^\theta := \tilde{\lambda}_2 x^\theta. \]

It is obvious that the system (4.3) is equivalent to the following system

\[
\begin{cases}
\rho_t + m_x = -\frac{N-1}{x} m + \varepsilon \left[\rho_{xx} x^{2(d-c)} + (d - 3c) \rho_x x^{2(d-c)-1} + c(2c + 1 - d) \rho x^{2(d-c)-2}\right], \\
m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = -\frac{N-1}{x} \frac{m^2}{\rho} + \varepsilon \left[m_{xx} x^{2(d-c)} - (c + d) m_x x^{2(d-c)-1} + d(c + 1) m x^{2(d-c)-2}\right], \quad x \in [a(\varepsilon), b(\varepsilon)], t > 0. \tag{4.5}
\end{cases}
\]

By the rescaled Riemann invariants \(\tilde{w}\) and \(\tilde{z}\), we have

\[
\begin{cases}
\tilde{w}_t + \tilde{\lambda}_2 \tilde{w}_\xi = \varepsilon \tilde{w}_{\xi\xi} + 2\varepsilon \frac{\tilde{m}}{\tilde{\rho}} \tilde{w}_\xi + \varepsilon \theta(\theta + 1) \tilde{\rho}^{-1} \tilde{w}_\xi^2 \\
\tilde{z}_t + \tilde{\lambda}_1 \tilde{z}_\xi = \varepsilon \tilde{z}_{\xi\xi} + 2\varepsilon \frac{\tilde{m}}{\tilde{\rho}} \tilde{z}_\xi + \varepsilon \theta(\theta + 1) \tilde{\rho}^{-1} \tilde{z}_\xi^2 \tag{4.6}
\end{cases}
\]

Setting the control functions \((\phi, \psi) = (M_3 + 2\varepsilon, 0)\) and by the initial and boundary data (4.4), we have that

\[
\frac{m_0}{\rho_0} + \rho_0^\theta \leq (M_3 + 2\varepsilon) x^\theta, \quad \frac{m_0}{\rho_0} - \rho_0^\theta \geq 0, \quad \text{a.e.} \ x \in [a(\varepsilon), b(\varepsilon)],
\]

\[
\left(\frac{m}{\rho} + \rho^\theta\right)_{|x=a(\varepsilon)} \leq (M_3 + 2\varepsilon) a(\varepsilon)^{\theta}, \quad \left(\frac{m}{\rho} - \rho^\theta\right)_{|x=a(\varepsilon)} \geq 0,
\]

\[
\left(\frac{m}{\rho} + \rho^\theta\right)_{|x=b(\varepsilon)} \leq (M_3 + 2\varepsilon) b(\varepsilon)^{\theta}, \quad \left(\frac{m}{\rho} - \rho^\theta\right)_{|x=b(\varepsilon)} \geq 0. \tag{4.7}
\]

Define the modified Riemann invariants \((\tilde{w}, \tilde{z})\) as

\[
\tilde{w} = \tilde{w} - \phi, \quad \tilde{z} = \tilde{z} + \psi. \tag{4.8}
\]
The system (4.6) becomes

\[
\begin{align*}
\dot{w}(\xi,t) + \left( \lambda_2 - 2 \varepsilon \frac{\partial}{\partial \xi} \right) \dot{w}(\xi) &= \varepsilon \dot{w}(\xi) + a_{11} \dot{w}(\xi) + a_{12} \ddot{w}(\xi) + R_1, \\
\dot{z}(\xi,t) + \left( \lambda_1 - 2 \varepsilon \frac{\partial}{\partial \xi} \right) \dot{z}(\xi) &= \varepsilon \dot{z}(\xi) + a_{21} \dot{w}(\xi) + a_{22} \ddot{w}(\xi) + R_2,
\end{align*}
\]

(4.9)

where

\[a_{11} = 0, a_{12} = -\theta(N - 1 + d)\rho^2 \xi^{d-c-1} \leq 0,\]

\[a_{21} = 0, a_{22} = \left[ \frac{1}{4} \left( (c - d) - \theta(N - 1 + d) - \theta^2 c \right) \dot{z} + \frac{1}{2} \left( (c - d) + \theta^2 c \right) \xi \right] \xi^{d-c-1},\]

\[R_1 = \left[ \frac{1}{4}(c - d) \frac{\partial^2}{\partial^2 \xi} - \theta(N - 1 + d)\rho^2 \xi^{d-c-1} - \varepsilon \theta(\theta + 1)\rho^2 \xi \right] \xi^{d-c-1} \leq 0,\]

\[R_2 = \left[ \frac{1}{4}(c - d) + \theta(N - 1 + d) - \theta^2 c \right] \dot{w}(\xi)^2 \xi^{d-c-1} + \varepsilon \theta(\theta + 1)\rho^2 \xi \geq 0.\]

By the maximum principle lemma 2.1, we have

\[\dot{w}(\xi,t) \leq (M_3 + 2\varepsilon), \dot{z}(\xi,t) \geq 0,\]

which implies that

\[0 \leq \dot{\rho}(x,t) \leq \frac{M_3}{2} + \varepsilon, 0 \leq m(x,t) \leq (M_3 + 2\varepsilon)\dot{\rho}(x,t),\]

i.e.,

\[0 \leq \rho^2(x,t) \leq \frac{M_3}{2} + \varepsilon x^c, 0 \leq m(x,t) \leq (M_3 + 2\varepsilon)\rho(x,t)x^c. \]

(4.10)

Thus we have:

**Theorem 4.1 (L^\infty estimate: including the origin).** Let \( \gamma > 1 \). Assume that for any positive constants \( c \) and \( M_3 \), the initial and boundary data satisfy

\[\rho_0(x) \geq \dot{\varepsilon} x^c, \quad \frac{m_0}{\rho_0} + \dot{\rho}_0 \leq (M_3 + 2\varepsilon)x^c, \quad \frac{m_0}{\rho_0} - \dot{\rho}_0 \geq 0, \quad \text{a.e.} \ x \in [a(\varepsilon), b(\varepsilon)],\]

\[\left( \frac{m}{\rho} + \rho^2 \right) \big|_{x=a(\varepsilon)} \leq (M_3 + 2\varepsilon)a(\varepsilon)x^c, \quad \left( \frac{m}{\rho} - \rho^2 \right) \big|_{x=a(\varepsilon)} \geq 0,\]

\[\left( \frac{m}{\rho} + \rho^2 \right) \big|_{x=b(\varepsilon)} \leq (M_3 + 2\varepsilon)b(\varepsilon)x^c, \quad \left( \frac{m}{\rho} - \rho^2 \right) \big|_{x=b(\varepsilon)} \geq 0, \]

(4.11)

then the solution of (4.5) and (4.7) satisfies

\[0 \leq \rho(x,t) \leq \left( \frac{M_3}{2} + \varepsilon \right)^{\frac{1}{c-d+1}} x^c, 0 \leq m(x,t) \leq (M_3 + 2\varepsilon)\rho(x,t)x^c, \]

(4.12)

for \( x \in [a(\varepsilon), b(\varepsilon)] \).

4.2. Lower bound estimate

When \( c - d + 1 \neq 0 \), set \( \tilde{v} = ln \tilde{\rho} \), then we get a scalar equation for \( \tilde{v} \),

\[\tilde{v}_t + \tilde{v}_\xi u + \tilde{u}_\eta = \varepsilon \tilde{v}_\xi \xi + \varepsilon \tilde{v}_\xi^2 - \frac{N - 1 + b}{(c - d + 1)\xi} \tilde{u}. \]

(4.13)
Note that the velocity $\tilde{u} = \frac{\partial}{\partial t}$ is uniformly bounded, i.e., $|\tilde{u}| \leq C$, and
\[
\tilde{v}(\xi, 0) = \tilde{v}_0(\xi), \quad \frac{1}{c - d + 1} (a(\varepsilon))^{c - d + 1} \leq \xi \leq \frac{1}{c - d + 1} (b(\varepsilon))^{c - d + 1}.
\]
Following the same method as in section 3.2, we can show that $\tilde{\rho} \geq e^{-C(\varepsilon)}$. A similar argument can be applied to the case where $c - d + 1 = 0$. Consequently, we conclude the following theorem for the global existence of approximate solutions.

**Theorem 4.2.** Under the assumption of the previous theorem, for any time $T > 0$, there exists a positive constant $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$, the initial-boundary value problem (4.3) and (4.4) admits a unique classical solution on $[\rho(\varepsilon), b(\varepsilon)] \times [0, T]$ satisfying
\[
e^{-C(\varepsilon,T)} \varepsilon^c \leq \rho(x, t) \leq \left( \frac{M_1}{2} + \varepsilon \right)^{\frac{1}{2}} x^c, \quad 0 \leq m(x, t) \leq (M_3 + 2\varepsilon) \rho(x, t)x^c.
\]

**4.3. $H^\perp_{\varepsilon}$ compactness of an entropy pair**

For any $T \in (0, \infty)$, let $\Pi_T = (0, +\infty) \times (0, T)$. Let $K \subset \Pi_T$ be any compact set, and choose $\varphi \in C_c^\infty(\Pi_T)$ such that $\varphi|_K = 1$, and $0 \leq \varphi \leq 1$. When $\varepsilon$ is small, $K \subset (\rho(\varepsilon), b(\varepsilon)) \times (0, T)$. Similarly as in section 3.3, multiplying (4.5) by $\nabla \eta^* \varphi$ with $\eta^*$ the mechanical entropy, we obtain
\[
(\rho, m) \nabla^2 \eta^* (\rho, m)^T x^{2(\gamma - 1)} = (\rho_0^\gamma \rho^\gamma - 2 \rho_x^2 + \rho m_x^2) x^{2(\gamma - 1)},
\]
and
\[
(\varepsilon \rho^\gamma - 2 \rho_x^2 + \varepsilon \rho m_x^2) x^{2(\gamma - 1)} \in L^1_{\varepsilon}(\Pi_T), \quad \eta_x + q_x \text{ is compact in } H^\perp_{\varepsilon}(\Pi_T).
\]
which implies that for any weak entropy–entropy flux pairs $(\eta, q)$ given in (2.6), it holds
\[
\eta_t + q_x \text{ is compact in } H^\perp_{\varepsilon}(\Pi_T).
\]
Since the proof is almost the same as in section 3.3, we omit it here.

**4.4. Entropy solution**

By (4.15) and the compactness framework established in [11, 13, 16], we can prove that there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$ (still denoted by $(\rho^\varepsilon, m^\varepsilon)$) such that
\[
(\rho^\varepsilon, m^\varepsilon) \to (\rho, m) \text{ in } L^p_{\varepsilon}(\Pi_T), \quad p \geq 1.
\]

Note that $\rho = O(1)x^c$ and $m = O(1)x^c$ so that the right-hand sides of (1.3), that is, $\frac{1}{\rho} \frac{dm}{dx}$ and $\frac{m^c}{p}$, are integrable near the origin with respect to $x$. As in [5, 6, 26, 27], we can prove that $(\rho, m)$ is an entropy solution to the problem (1.3). Therefore, the proof of theorem 1.2 is complete.

We note that the entropy solution obtained above is exactly the entropy solution to the Cauchy problem of an isentropic gas dynamics system with spherical symmetry.

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Appendix

Proof of lemma 3.1:

**Proof.** We decompose $w$ into $w = \sum_{i=0}^{3} w_i$, where $w_2$ and $w_3$ are

$$w_2(x,t) = \int_a^b \int_0^t \Gamma(x-\xi, t-\tau) f_2(\xi, \tau) d\xi d\tau,$$

$$w_3(x,t) = \int_a^b \int_0^t \Gamma_3(x-\xi, t-\tau) h(\xi, \tau) d\xi d\tau,$$  \hspace{1cm} (A.1)

and $\Gamma$ is the heat kernel:

$$\Gamma(x-\xi, t-\tau) = \begin{cases} \frac{1}{(4\pi(t-\tau))^\frac{1}{2}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}}, & t > \tau; \\ 0, & t \leq \tau, \end{cases}$$  \hspace{1cm} (A.2)

and $w_0$ and $w_1$ are the solutions of the following problems respectively:

$$(P_0) : \begin{cases} w_0 - \varepsilon w_{0xx} = 0, a < x < b, \\
\left. w_0 \right|_{x=a} = \varphi(a), \left. w_0 \right|_{x=b} = \varphi(b) - w_2(a,t) - w_3(a,t), \\
\left. w_0 \right|_{x=a} = \varphi(a) - w_2(a,t) - w_3(a,t), \left. w_0 \right|_{x=b} = \varphi(b) - w_2(b,t) - w_3(b,t), \\
\left. w_0 \right|_{x=a} = \varphi(a) - w_2(a,t) - w_3(a,t), \left. w_0 \right|_{x=b} = \varphi(b) - w_2(b,t) - w_3(b,t), \end{cases}$$  \hspace{1cm} (A.3)

$$(P_1) : \begin{cases} w_1 - \varepsilon w_{1xx} = f_1(x,t), a < x < b, \\
\left. w_1 \right|_{x=a} = 0, \left. w_1 \right|_{x=b} = 0. \end{cases}$$  \hspace{1cm} (A.4)

It is obvious that $w_2 - \varepsilon w_{2xx} = f_2(x,t)$ and $w_3 - \varepsilon w_{3xx} = h(x,t)$. Thus $w = \sum_{i=0}^{3} w_i$ is the unique solution of equation (3.12). Note that

$$0 \leq \Gamma(x-\xi, t-\tau) \leq \frac{C}{(t-\tau)^{\frac{1}{2}}},$$

$$|\Gamma_3(x-\xi, t-\tau)| \leq \frac{C}{(t-\tau)^{\frac{1}{2}-\alpha}|x-\xi|^{2\alpha-1}},$$  \hspace{1cm} (A.5)

for any $\frac{1}{2} < \alpha < 1$. For example, $\alpha = \frac{2}{3}$, then $\Gamma_3$ is integrable with respect to $\xi$ and $\tau$ up to time $T > 0$. From (A.5), we obtain that $w_2$ and $w_3$ are bounded. Moreover, they are also bounded on the boundary $x = a, b$. We then turn to the problem $(P_0)$ and find that its solution $w_0$ is bounded. For the problem $(P_1)$, the solution $w_1 \geq 0$ due to $f_1(x,t) \geq 0$. Therefore, the proof of lemma 3.1 is complete. □
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