Convergence of the Stochastic Euler Scheme for Locally Lipschitz Coefficients

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Abstract Stochastic differential equations are often simulated with the Monte Carlo Euler method. Convergence of this method is well understood in the case of globally Lipschitz continuous coefficients of the stochastic differential equation. However, the important case of superlinearly growing coefficients has remained an open question. The main difficulty is that numerically weak convergence fails to hold in many cases of superlinearly growing coefficients. In this paper we overcome this difficulty and establish convergence of the Monte Carlo Euler method for a large class of one-dimensional stochastic differential equations whose drift functions have at most polynomial growth.

Keywords Euler scheme · Stochastic differential equations · Monte Carlo Euler method · Local Lipschitz condition

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1 Introduction

Many applications require the numerical approximation of moments or expectations of other functionals of the solution of a stochastic differential equation (SDE) whose coefficients are superlinearly growing. Moments are often approximated by discretizing time using the stochastic Euler scheme (see, e.g., [11, 15, 21]), a.k.a. the Euler–Maruyama scheme, and by approximating expectations with the Monte Carlo method. This Monte Carlo Euler method has been shown to converge in the case of globally Lipschitz continuous coefficients of the SDE (see, e.g., Sect. 14.1 in [11] and Sect. 12 in [15]). However, the important case of superlinearly growing coefficients has remained an open problem. The main difficulty is that numerically weak convergence fails to hold in many cases of superlinearly growing coefficients; see [7]. In this paper we overcome this difficulty and establish convergence of the Monte Carlo Euler method for a large class of one-dimensional SDEs with at most polynomial growing drift functions and with globally Lipschitz continuous diffusion functions; see Sect. 2 for the exact statement.

For clarity of exposition, we concentrate in this introductory section on the following prominent example. Let \( T \in (0, \infty) \) be fixed, and let \( (X_t)_{t \in [0,T]} \) be the unique strong solution of the one-dimensional SDE

\[
dX_t = -X_t^3 \, dt + \bar{\sigma} \, dW_t, \quad X_0 = x_0
\]

(1)

for all \( t \in [0, T] \), where \( (W_t)_{t \in [0,T]} \) is a one-dimensional standard Brownian motion with continuous sample paths and where \( \bar{\sigma} \in (0, \infty) \) and \( x_0 \in \mathbb{R} \) are given constants. Our goal is then to solve the cubature approximation problem of the SDE (1). More formally, we want to compute moments and, more generally, the deterministic real number

\[
\mathbb{E}\left[ f(X_T) \right]
\]

(2)

for a given smooth function \( f : \mathbb{R} \to \mathbb{R} \) whose derivatives have at most polynomial growth.

A frequently used scheme for solving this problem is the Monte Carlo Euler method. In this method, time is discretized through the stochastic Euler scheme, and expectations are approximated by the Monte Carlo method. More formally, the Euler approximation \( (Y_N^n)_{n \in \{0,1,\ldots,N\}} \) of the solution \( (X_t)_{t \in [0,T]} \) of the SDE (1) is defined recursively through \( Y_0^N = x_0 \) and

\[
Y_{n+1}^N = Y_n^N - \frac{T}{N} \left( Y_n^N \right)^3 + \bar{\sigma} \cdot \left( \frac{W_{(n+1)T}}{N} - \frac{W_nT}{N} \right)
\]

(3)

for every \( n \in \{0, 1, \ldots, N - 1\} \) and every \( N \in \mathbb{N} := \{1, 2, \ldots\} \). Moreover, let \( Y_{n,m}^N, n \in \{0, 1, \ldots, N\}, N \in \mathbb{N} \), for \( m \in \mathbb{N} \) be independent copies of the Euler approximation defined in (3). The Monte Carlo Euler approximation of (2) with \( N \in \mathbb{N} \) time