GLOBAL SOLUTION IN CRITICAL SPACES TO THE COMPRESSIBLE OLDROYD-B MODEL WITH NON-SMALL COUPLING PARAMETER

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Abstract. This paper is dedicated to the global well-posedness issue of the compressible Oldroyd-B model in the whole space \( \mathbb{R}^d \) with \( d \geq 2 \). By exploiting the intrinsic structure of the system, we prove that if the initial data is small enough (depending on the coupling parameter), this set of equations admits a unique global solution in a certain critical Besov space. This result partially improves the previous work by Fang and the author [J. Differential Equations, 256(2014), 2559–2602].

1. Introduction.

1.1. The modeling. The system of viscoelastic fluids of Oldroyd type on \((0, T^*) \times \mathbb{R}^d, d \geq 2\), obtained from the laws of conservation of mass, and of momentum, and from the constitutive equation of the fluid, takes the following form [36, 39]:

\[
\begin{align*}
\frac{\partial \rho^*}{\partial t^*} + \text{div}^*(\rho^* u^*) &= 0, \\
\rho^* \left( \frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla^*) u^* \right) &= \text{div}^* (\tau^* - p^* \text{Id}), \\
\tau^* + \lambda \frac{D_{t^*} \tau^*}{D_{t^*}} &= 2\eta \left( D^*(u^*) + \mu \frac{D_{t^*} D^*(u^*)}{D_{t^*}} \right).
\end{align*}
\]

The \( ^* \)-variables are the dimensional ones in \( \mathbb{R}^d \), and \( T^* > 0 \) is a dimensional time. The unknowns are the density \( \rho^* \), the velocity \( u^* \) and the symmetric tensor of constrains \( \tau^* \). \( \text{Id} \) is the identity tensor, and the smooth function \( p^* = p^*(\rho^*) \) is the pressure. Moreover, \( \eta \) is the total viscosity of the fluid, \( \lambda > 0 \) is the relaxation time, and \( \mu \) is the retardation time with \( 0 < \mu < \lambda \).

\( \frac{D_{t^*} \tau^*}{D_{t^*}} \) is an objective derivative of the tensor \( \tau^* \), defined by

\[
\frac{D_{t^*} \tau^*}{D_{t^*}} = \left( \frac{\partial}{\partial t^*} + (u^* \cdot \nabla^*) \right) \tau^* + \tau^* W^*(u^*) - W^*(u^*) \tau^* - \alpha (D^*(u^*) \tau^* + \tau^* D^*(u^*))
\]

where \( D^*(u^*) = 1/2 (\nabla^* u^* + (\nabla^* u^*)^\top) \), \( W^*(u^*) = 1/2 (\nabla^* u^* - (\nabla^* u^*)^\top) \) are the deformation tensor and the vorticity tensor respectively, and \( \alpha \) is a parameter in \([-1,1]\).

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System (1.1) is completed by the initial data
\[ \rho^*(0, \cdot) = \rho_0^*, \quad u^*(0, \cdot) = u_0^*, \quad \tau^*(0, \cdot) = \tau_0^*, \quad \text{in } \mathbb{R}^d. \]

The symmetric tensor of constraints \( \tau^* \) can be decomposed into the Newtonian part and the elastic part \( \tau_e^* \), i.e.,
\[ \tau^* = 2\eta_e D^*(u^*) + \tau_e^*, \quad (1.2) \]
where \( \eta_e := \eta \mu / \lambda \) is the solvent viscosity. Substituting (1.2) into (1.1), we find that \( \tau_e^* \) satisfies
\[ \tau_e^* + \lambda \frac{D_{\alpha} \tau_e^*}{Dt} = 2\eta_e D^*(u^*), \quad (1.3) \]
where \( \eta_e := \eta - \eta_s \) is the polymer viscosity.

For the sake of simplicity, we denote \( \tau_e^* \) by \( \tau^* \) from now on. Then it follows from (1.1)-(1.3) that \( (\rho^*, u^*, \tau^*) \) solves
\[
\begin{cases}
\frac{\partial \rho^*}{\partial t^*} + \text{div}^*(\rho^* u^*) = 0, \\
\rho^* \left( \frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla) u^* \right) + \nabla^* p^* = \eta_s (\Delta^* u^* + \nabla^* \text{div}^* u^*) + \text{div}^* \tau^*, \\
\tau^* + \lambda \frac{D_{\alpha} \tau^*}{Dt} = 2\eta_s D^*(u^*). 
\end{cases} \quad (1.4)
\]

Define the Mach number
\[ \epsilon := \frac{U_0}{(\frac{dp^*}{d\rho^*}(\bar{\rho}_0^*))^{\frac{3}{2}}}, \]
where \( U_0 \) is the typical velocity of the fluid, \( (\frac{dp^*}{d\rho^*}(\bar{\rho}_0^*))^{\frac{3}{2}} \) is the speed of sound in the same fluid at the same state, and \( \bar{\rho}_0^* \) is a positive constant. Next, as in [15], we rewrite the density \( \rho^* \) as
\[ \rho^* = \bar{\rho}_0^* + eb^*. \]

From this decomposition and the definition of Mach number \( \epsilon \), the pressure term in the momentum equation (1.4) reduces to
\[
\nabla^* p^*(\rho^*) = \epsilon \left( \frac{dp^*}{d\rho^*}(\bar{\rho}_0^* + eb^*) \right) \nabla^* b^*
\]
\[ = \epsilon \left( \frac{dp^*}{d\rho^*}(\bar{\rho}_0^* + eb^*) - \frac{dp^*}{d\rho^*}(\bar{\rho}_0^*) \right) \nabla^* b^* + \frac{(U_0)^2 \nabla^* b^*}{\epsilon}. \quad (1.5) \]

Then system (1.4) changes to be
\[
\begin{cases}
\frac{\partial b^*}{\partial t^*} + \frac{\bar{\rho}_0^* \text{div}^* u^*}{\epsilon} + \text{div}^* (b^* u^*) = 0, \\
(\bar{\rho}_0^* + eb^*) \left( \frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla) u^* \right) + \frac{(U_0)^2 \nabla^* b^*}{\epsilon} - \eta_s (\Delta^* + \nabla^* \text{div}^*) u^*
\quad = -\epsilon \left( \frac{dp^*}{d\rho^*}(\bar{\rho}_0^* + eb^*) - \frac{dp^*}{d\rho^*}(\bar{\rho}_0^*) \right) \nabla^* b^* + \text{div}^* \tau^*, \\
\tau^* + \lambda \frac{D_{\alpha} \tau^*}{Dt^*} = 2\eta_s D^*(u^*). 
\end{cases} \quad (1.6)
\]

Now let us introduce the dimensionless variables,
\[
x^* = L_0 x, \quad t^* = \frac{L_0}{U_0} t, \quad u^* = U_0 u, \quad \rho^* = \bar{\rho}_0 \rho, \quad b^* = \bar{\rho}_0 b, \quad \tau^* = \bar{\tau}_0 \tau, \quad p^*(\rho^*) = \bar{\rho}_0 p(\rho),
\]
where \( L_0, U_0 \) are the characteristic length and velocity, respectively.
where $L_0$ represents a typical length of the flow. The real number $\frac{\eta U_0 L_0}{\rho_0}$ and $T_0 = \frac{\mu U_0 L_0}{\rho_0}$ characterize the density and stress tensor of the fluid, respectively. Denote

$$\text{Re} := \frac{\rho_0^* \eta U_0 L_0}{\rho_0^*}, \quad \text{We} := \frac{\lambda U_0 L_0}{\eta}, \quad \omega := 1 - \frac{\mu}{\lambda},$$

which are Reynolds number, Weissenberg number and retardation number, respectively. We remark that $\omega \in (0, 1)$ is also called the coupling constant of the fluid in the literature.

In dimensionless variables, system (1.6) takes the following form:

$$\begin{aligned}
&\partial_t b + \frac{\text{Re} \, \text{div} u}{\epsilon} + \text{div}(bu) = 0, \\
&\text{Re} \left( \partial_t u + (u \cdot \nabla)u \right) + \nabla b - (1 - \omega)(\Delta + \nabla \text{div})u \\
&\quad = -\frac{(1 - \omega)e}{\text{Re} + \epsilon} (\Delta + \nabla \text{div})u + K^\epsilon(e) \nabla b - \frac{\text{Re} \, \text{div} \tau}{\epsilon} + \frac{\text{Re} + \epsilon}{\epsilon} \tau,
\end{aligned}$$

(1.7)

where

$$g_o(\tau, \nabla u) := \tau W(u) - W(u) \tau - \alpha (D(u) \tau + \tau D(u)),$$

with $D(u), W(u)$ and $\alpha$ defined as before. Moreover,

$$K^\epsilon(c) := \frac{c}{\text{Re} + c} - \epsilon^2 \frac{\text{Re} \left( \frac{d\rho}{d\rho} \left( \text{Re} + c \right) - \frac{d\rho}{d\rho} \left( \text{Re} \right) \right)}{\text{Re} + c}.$$

(1.8)

We would like to point out that, letting $\epsilon \to 0$ in (1.7), then we obtain (see [15]) the following incompressible Oldroyd-B model in dimensionless variables:

$$\begin{aligned}
&\text{Re} \left( u_t + (u \cdot \nabla)u \right) - (1 - \omega)\Delta u + \nabla \Pi = \text{div} \tau, \\
&\text{We}(\tau_t + (u \cdot \nabla)\tau + g_o(\tau, \nabla u)) + \tau = 2\omega D(u),
\end{aligned}$$

(1.9)

where $\Pi$ is the pressure which is the Lagrange multiplier for the divergence free condition.

In this paper, we are not interested in the low Mach number limit problem, instead, we just focus on the problem for which the Mach number is fixed. Up to a scaling (see [15]), it suffices to investigate the case $\epsilon = 1$. Setting $a := \frac{b}{\text{Re}}$, then $(a, u, \tau)$ solves

$$\begin{aligned}
&\partial_t a + \text{div} u + \text{div}(au) = 0, \\
&\partial_t u + (u \cdot \nabla)u - \frac{1}{\text{Re}} A u + \nabla a - \frac{1}{\text{Re}} \text{div} \tau \\
&\quad = -\frac{1}{\text{Re}} I(a) (A u + \text{div} \tau) + K(a) \nabla a, \\
&\partial_t \tau + (u \cdot \nabla)\tau + g_o(\tau, \nabla u) + \frac{1}{\text{We}} \tau = 2\omega \frac{\text{We}}{\text{We}} D(u),
\end{aligned}$$

(1.10)

where $A := (1 - \omega)(\Delta + \nabla \text{div})$, $I(a) := \frac{a}{1 + a}$ and

$$K(a) := \frac{a}{1 + a} - \frac{\frac{d\rho}{d\rho} \left( \text{Re} + 1 + a \right) - \frac{d\rho}{d\rho} \left( \text{Re} \right)}{1 + a}.$$
1.2. The main result. The theory of Oldroyd-B fluids recently gained quite some attention. Most of the results on Oldroyd-B fluids in the literature are about the incompressible model. The study of the incompressible Oldroyd-B model started by a pioneering paper given by Guillopé and Saut [18]. They proved that (i) system (1.9) admits a unique local strong solution in suitable Sobolev spaces $H^s(\Omega)$ for bounded domains $\Omega \subset \mathbb{R}^3$; and (ii) this solution is global provided the data as well as the coupling constant $\omega$ satisfy: $\omega$ between the velocity $u$ and the symmetric tensor of the constrains $\tau$ and the data are sufficiently small. For extensions to this results to the $L^p$-setting, see the work of Fernández-Cara, Guillén and Ortega [16]. Later on, Molinet and Talhouk [35] removed the smallness restriction on the coupling constant $\omega$ in [18].

The situation of exterior domains was considered first in [20], where the existence of a unique global strong solution defined in certain function spaces was proved provided the initial data and the coupling parameter $\omega$ are small enough. Recently, Fang, Hieber and the author [13] improved the main result given in [20] to the situation of non-small coupling constant.

For the scaling invariant approach, and $\Omega = \mathbb{R}^d, d \geq 2$, Chemin and Masmoudi in [7] proved the existence and uniqueness of the global solution to the Oldroyd-B model (1.9) with initial data $(u_0,\tau_0)$ belonging to the critical space $\left(B_{p,1}^{\frac{d}{p}-1}\right)^d \times \left(B_{p,1}^{\frac{d}{p}}\right)^{d \times d}$ for any $p \in [1, \infty)$. A smallness restriction on the coupling constant $\omega$ is needed in this result. Afterwards, for general $\omega \in (0,1)$, Chen and Miao [15] constructed global solutions to the incompressible Oldroyd-B model with small initial data in $B_{2,\infty}^s, s > \frac{d}{2}$. For the critical $L^p$ framework, Fang, Zhang and the author [42] removed the smallness restriction on $\omega$ in [7].

For the weak solutions of incompressible Oldroyd-B fluids, see the work of Lions and Masmoudi [34] for the case $\alpha = 0$. For the general case $\alpha \neq 0$, Hu and Lin [21] made some progress in the 2D case. As for the blow-up criterions of the incompressible Oldroyd-B model, there are works [7][25][30]. Besides, we would like to mention that Constantin and Kliegl [9] proved the global regularity of solutions in two dimensional case for the Oldroyd-B fluids with diffusive stress. An approach based on the deformation tensor can be found in [27][28][29][31][32][33][37][41].

On the other hand, the studies on compressible Oldroyd-B model have thrown up some interesting results. Lei [26] and Guillopé, Salloum and Talhouk [19] investigated the incompressible limit problem of the compressible Oldroyd-B model in a torus and bounded domain $\Omega \subset \mathbb{R}^3$, respectively. They showed that the compressible flows with well-prepared initial data converge to incompressible ones when the Mach number $\epsilon$ converges to zero. The case of ill prepared initial data was studied by Fang and the author [15] in the whole space $\mathbb{R}^d, d \geq 2$. In particular, if $\epsilon = 1$, we also obtained in [15] the existence and uniqueness of the global solution in critical spaces to system (1.7) with small coupling constant $\omega$. The unique local strong solution to (1.4) with initial density $\rho_0$ vanishing from below and a blow-up criterion for this solution were established in [14]. For the compressible Oldroyd type model based on the deformation tensor, see the results [12][22][38] and references therein.

The aim of this paper is to study the compressible Oldroyd-B model (1.10) in the critical framework. This approach goes back to the pioneering work by Fujita and Kato [17] for the classical incompressible Navier-Stokes equations,
It has been proved in [15] that the system (1.10) is local well-posed with initial data $a$, system (1.10) in [15]. More precisely, the space for (an extra condition on $a$ is needed to compensate the lack of scaling invariance of $\tau$ is a key point to get control on the term $\tilde{a}$.)

Similar to the case of isentropic compressible Navier-Stokes equations [10], the extra coupling term $\omega$ is needed to compensate the lack of scaling invariance of system (1.10) in [15]. More precisely, the space for $(a_0, u_0, \tau_0)$ that generates global solution is

\[
\begin{align*}
(\mathcal{B}^{\mathcal{I}}_2 & \times \mathcal{B}^{d-1}_2) \times (\mathcal{B}^{\mathcal{I}}_2), \\
& \text{is a critical space. Actually, it has been proved in [15] that the system (1.10) is local well-posed with initial data } (a_0, u_0, \tau_0) \in \mathcal{B}^{\mathcal{I}}_2 \times \mathcal{B}^{d-1}_2 \times (\mathcal{B}^{\mathcal{I}}_2). \\
& \text{However, to obtain the global solution, an extra condition on } a_0 \text{ is needed to compensate the lack of scaling invariance of system (1.10) in [15]. More precisely, the space for } (a_0, u_0, \tau_0) \text{ that generates global solution is } \mathcal{B}^{\mathcal{I}}_2 \times \mathcal{B}^{d-1}_2 \times (\mathcal{B}^{\mathcal{I}}_2). \\
\end{align*}
\]

(1.11)

Similar to the case of isentropic compressible Navier-Stokes equations [10], the extra condition $a_0 \in \mathcal{B}^{d-1}_2$ was used in [15] to obtain the estimate $a \in L^2_t(\mathcal{B}^{\mathcal{I}}_2)$, which is a key point to get control on the term $K(a)\nabla a$.

In this paper, we consider the global well-posedness issue of system (1.10) with initial data in the space given in (1.11). Different from our previous results in [15], the coupling constant $\omega$ we investigate here is not small any more and thus the problem is much more complicated. As a matter of fact, in [15] $(a, u)$ and $\tau$ are treated separately. To be more precise, we bound $(a, u)$ by using the estimates obtained by Danchin [10] for the linearized system of isentropic compressible Navier-Stokes equations, namely

\[
\begin{align*}
\begin{cases}
a_t + \Lambda d = 0, \\
d_t - \Delta d - \Lambda a = 0,
\end{cases}
\end{align*}
\]

(1.12)

where $\Lambda := (-\Delta)^{\frac{1}{2}}$. The linear coupling term $\nabla \rho$ in the momentum equation is regarded as a source term, and the symmetric tensor of constrains $\tau$ is bounded.
with the aid of the well known estimates for transport equation. This is an easy way to get the global estimates of \((a, u, \tau)\) since the coupling between \(a\) and \(\tau\) is neglected, nevertheless, in order to close the estimates, the price we have to pay is to impose some smallness restriction on the coupling constant \(\omega\). For general \(\omega \in (0, 1)\), we must consider fully the coupling between \(a\) and \(\tau\), and deal with \((a, u, \tau)\) as a whole.

We shall obtain the existence and uniqueness of a solution \((a, u, \tau)\) in \((\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)\) as a whole in the following space (we postpone the definition of Besov spaces and hybrid Besov spaces to Section 2).

For \(T > 0\) and \(s \in \mathbb{R}\), let us denote

\[
\mathcal{E}_T^s := \mathcal{C}_T(B_2^{s-1, s}) \cap L_1^1(B_2^{s+1, s}) \times \left( \mathcal{C}_T(B_2^{s-1, 1}) \cap L_1^1(B_2^{s+1, 1}) \right)^d.
\]

We use the notation \(\mathcal{E}^s\) if \(T = \infty\), changing \([0, T]\) into \([0, \infty)\) in the definition above.

Our main result reads as follows:

**Theorem 1.1.** Let \(d \geq 2\). Assume that \((a_0, u_0, \tau_0) \in B_2^{\frac{d}{2} - 1, \frac{d}{2}} \times \left( B_2^{\frac{d}{2} - 1} \right)^d \times \left( B_2^{\frac{d}{2} - 1} \right)^d\). There exist two positive constants \(c\) and \(M\), depending on \(d, \omega, \text{Re}\) and \(\text{We}\), such that if

\[
\|a_0\|_{B_2^{\frac{d}{2} - 1, \frac{d}{2}}} + \|u_0\|_{B_2^{\frac{d}{2} - 1}} + \|\tau_0\|_{B_2^{\frac{d}{2} - 1}} \leq c,
\]

then system \((1.10)\) admits a unique global solution \((a, u, \tau)\) in \(\mathcal{E}_T^{\frac{d}{2}}\) with

\[
\|(a, u, \tau)\|_{\mathcal{E}_T^{\frac{d}{2}}} \leq M \left( \|a_0\|_{B_2^{\frac{d}{2} - 1, \frac{d}{2}}} + \|u_0\|_{B_2^{\frac{d}{2} - 1}} + \|\tau_0\|_{B_2^{\frac{d}{2} - 1}} \right).
\]

**Remark 1.1.** The smallness restriction of the initial data in Theorem 1.1 depends on the coupling constant \(\omega\). It is very interesting to construct global strong solutions with initial data uniformly in \(\omega\), even for the incompressible Oldroyd-B model. We will consider this problem in the near future.

**Remark 1.2.** For incompressible Oldroyd-B model, using the estimates of the incompressible part in Section 3, we can give a new proof of the result in [42] for \(p = 2\) without resorting to the Green matrix of the corresponding linearized system.

**Remark 1.3.** We believe that our method is also applicable to the incompressible Oldroyd-B model with variable density.

### 1.3. The main ideas of the proof

Let us now explain the main ingredients of the proof. Motivated by our previous result for incompressible Oldroyd-B model [42], we first consider the system of \((a, u, \text{div}\tau)\). Indeed, in view of the scaling above, \(u\) possesses the same regularity with \(\text{div}\tau\) instead of \(\tau\), that is why it is more convenient to treat \((a, u, \text{div}\tau)\) as a whole in the process of energy estimates in Besov spaces. To do so, our proof relies heavily on the following decomposition on \(u\) and \(\text{div}\tau\):

\[
u = \mathbb{P} u + \mathbb{P}^\perp u, \quad \text{and} \quad \text{div}\tau = \mathbb{P} \text{div}\tau + \mathbb{P}^\perp \text{div}\tau,
\]

where \(\mathbb{P} := \text{Id} + \nabla(-\Delta)^{-1}\text{div}\) is the Leray operator, and \(\mathbb{P}^\perp := -\nabla(-\Delta)^{-1}\text{div}\). Applying \(\mathbb{P}^\perp\) and \(\mathbb{P}^\perp\text{div}\) to the second and third equation of \((1.10)\) respectively, we
obtain

\[
\begin{aligned}
\partial_t a + \operatorname{div} u + \operatorname{div}(au) &= 0, \\
\partial_t \mathbb{P}^1 u + \mathbb{P}^1 ((u \cdot \nabla) u) - \frac{2(1 - \omega)}{\Re} \Delta \mathbb{P}^1 u + \nabla a &= -\frac{1}{\Re} \mathbb{P}^1 \operatorname{div} \tau \\
\partial_t \mathbb{P}^1 \operatorname{div} \tau + \frac{1}{\We} \mathbb{P}^1 \operatorname{div} \tau &= -\mathbb{P}^1 \Delta \mathbb{P}^1 u = -\mathbb{P}^1 \operatorname{div} ((u \cdot \nabla) \tau + g_\alpha(\tau, \nabla u)),
\end{aligned}
\]  

(1.13)

and

\[
\begin{aligned}
\partial_t \mathbb{P} u + \mathbb{P} ((u \cdot \nabla) u) - \frac{1 - \omega}{\Re} \Delta \mathbb{P} u - \frac{1}{\Re} \mathbb{P} \operatorname{div} \tau &= -\mathbb{P} (I(a) (Au + \operatorname{div} \tau)), \\
\partial_t \mathbb{P} \operatorname{div} \tau + \frac{1}{\We} \mathbb{P} \operatorname{div} \tau &= -\mathbb{P} \Delta \mathbb{P} u = -\mathbb{P} \operatorname{div} ((u \cdot \nabla) \tau + g_\alpha(\tau, \nabla u)).
\end{aligned}
\]  

(1.14)

Obviously, the linear part of (1.14) is the same with that of the auxiliary system of \((u, \mathbb{P} \operatorname{div} \tau)\) for the incompressible Oldroyd-B model (see [12] (1.12)), so the key point of this paper is to deal with the so called compressible part, i.e., system (1.13).

Similar to the case of isentropic compressible Navier-Stokes equations [10], we study the high frequency and low frequency part of system (1.13) in different ways. Roughly speaking, it is necessary to bound

\[
k_q + \min(2^{2q}, 1) \int_0^t k_q dt',
\]  

(1.15)

where

\[
k_q = \begin{cases} 
\|a_q\|_{L^2} + \|\mathbb{P}^1 u_q\|_{L^2} + \|\mathbb{P}^1 \operatorname{div} \tau_q\|_{L^2}, & \text{if } q \leq q_0, \\
\|\nabla a_q\|_{L^2} + \|\mathbb{P}^1 u_q\|_{L^2} + \|\mathbb{P}^1 \operatorname{div} \tau_q\|_{L^2}, & \text{if } q > q_0,
\end{cases}
\]

for some \(q_0 \in \mathbb{Z}\), and

\[
(a_q, u_q, \tau_q) := (\hat{\Delta} q a, \hat{\Delta} q u, \hat{\Delta} q \tau).
\]

(1.16)

The definition of the operator \(\hat{\Delta} q\) can be found in Section 2. In order to get the decay of \(a\) and \(\operatorname{div} \tau\), we make full use of the linear coupling terms \(\nabla a\) and \(\operatorname{div} \tau\), and the estimates are very delicate both in low and high frequency cases. It is worth pointing out that in low frequency case, the linearized system of (1.13) behaves like the heat equation. This fact is far from obvious due to the lack of smoothing effect of \(a\) and \(\mathbb{P} \operatorname{div} \tau\). To exhibit this phenomenon, the estimates of the cross term \(\mathbb{P}^1 u_q | \Delta \mathbb{P}^1 \operatorname{div} \tau_q\) play an important role (see (3.12) and (3.27) for more details). Furthermore, different from the isentropic compressible Navier-Stokes equations, it is shown that there is a gap between the high frequency and low frequency estimates of \(k_q\). In other words, we can estimate (1.15) for \(q \geq q_0\) and \(q \leq q_1\) with \(q_1 < q_0\).

To overcome this difficulty, we introduce a new quantity

\[
\tilde{k}_q := \|a_q\|_{L^2} + \|\mathbb{P}^1 u_q\|_{L^2} + \|\Lambda^{-1} \mathbb{P}^1 \operatorname{div} \tau_q\|_{L^2}, \quad \text{if } q_1 < q \leq q_0.
\]

According to Bernstein’s inequality, \(\tilde{k}_q\) is equivalent to \(k_q\) if \(q_1 < q \leq q_0\). Therefore, it suffices to bound (1.15) with \(k_q\) replaced by \(\tilde{k}_q\) for \(q_1 < q \leq q_0\). This is the main novel part of this paper. Once the estimates of the compressible part \((a, \mathbb{P}^1 u, \mathbb{P} \operatorname{div} \tau)\) of \((u, \operatorname{div} \tau)\) is obtained, we can bound the incompressible part \((\mathbb{P} u, \mathbb{P} \operatorname{div} \tau)\) in a similar and easier way. Putting them together, we get the estimates of \((a, u, \operatorname{div} \tau)\). On this basis, we bound \(\tau\) directly via the third equation of (1.10).
and hence obtain the global estimates of \((a,u,\tau)\) in the end. More details can be found in Section 3.

The rest part of this paper is organized as follows. In Section 2, we introduce the tools (the Littlewood-Paley decomposition and paradifferential calculus) and give some nonlinear estimates in Besov space. Section 3 is devoted to the global estimates of the paralinearized system (3.1) of (1.10). The proof of Theorem 1.1 is given in Section 4.

**Notation.**

- For \(a, b \in L^2\), \((a|b)\) denotes the \(L^2\) inner product of \(a\) and \(b\).
- For \(f \in S'\), \(\hat{f} = \mathcal{F}(f)\) is the Fourier transform of \(f\); \(\mathcal{F}^{-1}(f)\) denotes the inverse Fourier transform of \(f\).
- Throughout the paper, \(C\) denotes various “harmless” positive constants, and we sometimes use the notation \(A \lesssim B\) as an equivalent to \(A \leq CB\). The notation \(A \approx B\) means that \(A \lesssim B\) and \(B \lesssim A\).

2. The functional tool box. The results of the present paper rely on the use of a dyadic partition of unity with respect to the Fourier variables, the so-called *Littlewood-Paley decomposition*. Let us briefly explain how it may be built in the case \(x \in \mathbb{R}^d\), and the readers may see more details in [1, 5]. Let \((\chi, \varphi)\) be a couple of \(C^\infty\) functions satisfying

\[
\text{Supp}\chi \subset \left\{ |\xi| \leq \frac{4}{3} \right\}, \quad \text{Supp}\varphi \subset \left\{ \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\},
\]

and

\[
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1,
\]

\[
\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \text{for } \xi \neq 0.
\]

Set \(\varphi_q(\xi) = \varphi(2^{-q}\xi), h_q = \mathcal{F}^{-1}(\varphi_q)\), and \(\tilde{h} = \mathcal{F}^{-1}(\chi)\). The dyadic blocks and the low-frequency cutoff operators are defined for all \(q \in \mathbb{Z}\) by

\[
\hat{\Delta}_q u = \varphi(2^{-q}D)u = \int_{\mathbb{R}^d} h_q(y)u(x-y)dy,
\]

\[
\hat{\mathcal{S}}_q u = \chi(2^{-q}D)u = \int_{\mathbb{R}^d} \tilde{h}_q(y)u(x-y)dy.
\]

Then

\[
u = \sum_{q \in \mathbb{Z}} \hat{\Delta}_q u, \quad (2.1)
\]

holds for tempered distributions *modulo polynomials*. As working modulo polynomials is not appropriate for nonlinear problems, we shall restrict our attention to the set \(S'_h\) of tempered distributions \(u\) such that

\[
\lim_{q \to -\infty} \|\hat{\mathcal{S}}_q u\|_{L^\infty} = 0.
\]

Note that (2.1) holds true whenever \(u\) is in \(S'_h\) and that one may write

\[
\hat{\mathcal{S}}_q u = \sum_{p \leq q-1} \hat{\Delta}_p u.
\]
Besides, we would like to mention that the Littlewood-Paley decomposition has a nice property of quasi-orthogonality:
\[ \Delta_p \Delta_q u \equiv 0 \text{ if } |p - q| \geq 2 \text{ and } \Delta_p (\delta_{q-1} u \Delta_q u) \equiv 0 \text{ if } |p - q| \geq 5. \tag{2.2} \]

One can now give the definition of homogeneous Besov spaces.

**Definition 2.1.** For \( s \in \mathbb{R}, (p, r) \in [1, \infty]^2 \), and \( u \in S'_h(\mathbb{R}^d) \), we set
\[ \|u\|_{\dot{B}^s_{p,r}} := \left\| 2^q \| \Delta_q u \|_{L^p} \right\|_{L^r}. \]

We then define the space \( \dot{B}^s_{p,r} := \{ u \in S'_h(\mathbb{R}^d), \|u\|_{\dot{B}^s_{p,r}} < \infty \} \).

Since homogeneous Besov spaces fail to have nice inclusion properties, it is wise to define hybrid Besov spaces where the growth conditions satisfied by the dyadic blocks are different for low and high frequencies. In fact, hybrid Besov spaces play a crucial role for proving global well-posedness of isentropic compressible Navier-Stokes equations in critical spaces [10]. Let us now define the hybrid Besov spaces that we need. Here our notations are somehow different from those in [10].

**Definition 2.2.** Let \( s, t \in \mathbb{R} \), and \( u \in S'_h(\mathbb{R}^d) \). For some fixed \( q_0 \in \mathbb{Z} \), we set
\[ \|u\|_{\dot{B}^s_{2,1}^{q_0}} := \sum_{q \leq q_0} 2^{qs} \| \Delta_q u \|_{L^2} + \sum_{q > q_0} 2^{qt} \| \Delta_q u \|_{L^2}. \]

We then define the space \( \dot{B}^{s,t}_{2,1} := \{ u \in S'_h(\mathbb{R}^d), \|u\|_{\dot{B}^{s,t}_{2,1}} < \infty \} \).

**Remark 2.1.** For all \( s, t \in \mathbb{R}, q_0 \in \mathbb{Z} \), and \( u \in S'_h(\mathbb{R}^d) \), setting
\[ \|u\|_{\dot{B}^{s,t}_{2,1}} := \sum_{q \leq q_0} 2^{qs} \| \Delta_q u \|_{L^2} + \sum_{q > q_0} 2^{qt} \| \Delta_q u \|_{L^2}, \]
then it is easy to verify that \( \|u\|_{\dot{B}^{s,t}_{2,1}} \approx \|u\|_{\dot{B}^{s,t}_{2,1}} \).

**Remark 2.2.** Our hybrid Besov spaces defined in Definition 2.2 is essentially the same as those in [10]. Of course, if \( s = t \), \( \dot{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} \).

- If \( s < t \),
\[ \|u\|_{\dot{B}^{s,t}_{2,1}} \approx \|u\|_{\dot{B}^{s}_{2,1}} + \|u\|_{\dot{B}^{t}_{2,1}}. \]

Hence, \( \dot{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} \cap \dot{B}^{t}_{2,1} \).

- If \( s > t \),
\[ \|u\|_{\dot{B}^{s}_{2,1} + \dot{B}^{t}_{2,1}} \approx \|u\|_{\dot{B}^{s,t}_{2,1}} \approx \min \left( \|u\|_{\dot{B}^{s}_{2,1}}, \|u\|_{\dot{B}^{t}_{2,1}} \right). \]

The following lemma describes the way derivatives act on spectrally localized functions.

**Lemma 2.1** (Bernstein’s inequalities [1]). Let \( k \in \mathbb{N} \) and \( 0 < r < R \). There exists a constant \( C \) depending on \( r, R \) and \( d \) such that for all \( (a, b) \in [1, \infty]^2 \), we have for all \( \lambda > 0 \) and multi-index \( \alpha \)
- If \( \text{Supp} f \subset B(0, \lambda R) \), then \( \sup_{|\alpha| = k} \| \partial^\alpha f \|_{L^r} \leq C^{k+1} \lambda^{k-d(\frac{1}{r} - \frac{1}{2})} \| f \|_{L^a} \).
- If \( \text{Supp} f \subset C(0, \lambda R, \lambda R) \), then \( C^{-k-1} \lambda^k \| f \|_{L^r} \leq \sup_{|\alpha| = k} \| \partial^\alpha f \|_{L^r} \leq C^{k+1} \lambda^k \| f \|_{L^r} \).

Let us now state some classical properties for the Besov spaces.

**Proposition 2.1** ([1]). For all \( s, s_1, s_2 \in \mathbb{R}, 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty \), the following properties hold true:
paraproduct between $f$ by means of paradifferential calculus. Firstly introduced by J. M. Bony in [2], the paraproduct is defined by

$$\tilde{T}_f g = \sum_{q \in \mathbb{Z}} \hat{\tilde{S}}_{q-1} f \hat{\Delta} q g,$$

and the remainder is given by

$$\tilde{R}(f, g) = \sum_{q \geq -1} \hat{\Delta} q f \hat{\tilde{\Delta}} q g$$

with

$$\hat{\tilde{\Delta}} q g = (\hat{\Delta} q-1 + \hat{\Delta} q + \hat{\Delta} q+1) g.$$

We have the following so-called Bony’s decomposition:

$$fg = \tilde{T}_f g + \tilde{T}_g f + \tilde{R}(f, g). \quad (2.3)$$

The paraproduct $\tilde{T}$ and the remainder $\tilde{R}$ operators satisfy the following continuous properties.

**Proposition 2.2** (II). For all $s \in \mathbb{R}$, $\sigma > 0$, and $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$, the paraproduct $\tilde{T}$ is a bilinear, continuous operator from $L^\infty \times B^s_{p,r} \to B^s_{p,r}$ and from $B^{-\sigma, r_1} \times B^s_{p,r_2}$ to $B^s_{p,r}$ with $\frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$. The remainder $\tilde{R}$ is bilinear continuous from $B^s_{p_1, r_1} \times B^s_{p_2, r_2}$ to $B^{s_1 + s_2}_{p_1 + p_2, r_2}$ with $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$.

In view of [23], Proposition 2.2 and Bernstein’s inequalities, one easily deduces the following product estimates:

**Corollary 2.1.** Let $p \in [1, \infty]$. If $s_1, s_2 \leq \frac{d}{p}$ and $s_1 + s_2 > d \max\{0, \frac{2}{p} - 1\}$, then there holds

$$\|uv\|_{B^s_{p,1} \times B^s_{p,1} \to B^{s_1 + s_2}_{p,1}} \leq C\|u\|_{B^{s_1}_{p,1}} \|v\|_{B^{s_2}_{p,1}}. \quad (2.4)$$

In the following, we shall give a commutator estimate, which will be used to deal with the convection terms.

**Lemma 2.2.** Let $\Delta := \sqrt{-\Delta}$, then

$$\|[(\Delta^{-1}, \hat{\tilde{S}}_{q-1} v \cdot \nabla) \hat{\Delta} q u]\|_{L^2} \leq C\|\nabla \hat{\tilde{S}}_{q-1} v\|_{L^\infty} \|\Delta^{-1} \hat{\Delta} q u\|_{L^2}. \quad (2.5)$$

**Proof.** Noting first that the support of $\hat{\tilde{S}}_{q-1} v \cdot \nabla \hat{\Delta} q u$ lies in the annulus $\bar{C} := B(0, \frac{s}{q}) + \{\frac{d}{2} \leq |\xi| \leq \frac{s}{q}\}$, we choose a smooth function $\hat{\phi}$ supported in an annulus and with value 1 on a neighborhood of $\bar{C}$. Denote $\hat{\Delta} q = \hat{\phi}(2^{-q} \xi)$, i.e., $(\hat{\Delta} q \phi)' = \hat{\phi}'(2^{-q} \xi) \hat{\phi}(\xi)$, for any $\phi \in \mathcal{S}$. Direct calculations yield $\Delta^{-1} \hat{\Delta} q = 2^{-q}(|\cdot|^{-1} \hat{\phi})(2^{-q} \xi)$.
Thus
\[ |\Lambda^{-1}, \check{S}_{q-1}v \cdot \nabla| \Delta_q u = |\Lambda^{-1} \check{\Delta}_q, \check{S}_{q-1}v \cdot \nabla| \Delta_q u \]
\[ = |2^{-q}(\cdot |^{-1}| \check{\varphi}(2^{-q}D), \check{S}_{q-1}v \cdot \nabla| \check{\Delta}_q u \]
\[ = 2^{-q} \sum_{1 \leq k \leq d} \int_{\mathbb{R}^d} 2^{qd} F^{-1}(\cdot |^{-1}| \check{\varphi})(2^q y) \]
\[ \times \left( \check{S}_{q-1}v^k(x-y) - \check{S}_{q-1}v^k(x) \right) \partial_k \Delta_q u(x-y) dy \]
\[ = -2^{-2q} \sum_{1 \leq k \leq d} \int_0^1 \int_{\mathbb{R}^d} 2^{qd} F^{-1}(\cdot |^{-1}| \check{\varphi})(2^q y) \]
\[ \times (2^q y) \cdot \nabla \check{S}_{q-1}v^k(x - ty) \partial_k \Delta_q u(x-y) dy dt. \]

Consequently, using convolution inequality, we infer that
\[ \| |\Lambda^{-1}, \check{S}_{q-1}v \cdot \nabla| \Delta_q u \|_{L^2} \leq C2^{-2q} \| \nabla \check{S}_{q-1}v \|_{L^\infty} \| \nabla \check{\Delta}_q u \|_{L^2} \| yF^{-1}(\cdot |^{-1}| \check{\varphi})\|_{L^1} \]
\[ \leq C \| \nabla \check{S}_{q-1}v \|_{L^\infty} \| \Lambda^{-1} \check{\Delta}_q u \|_{L^2}, \]
where we have used the fact that \( \check{\varphi} \) is a smooth function supported in an annulus, so is \(|\xi|^{-1}| \check{\varphi}(\xi)\), and hence \( yF^{-1}(\cdot |^{-1}| \check{\varphi}) \) is integrable. This completes the proof of Lemma 2.2.

The study of non-stationary PDEs requires spaces of the type \( L^p_T(X) = L^p(0, T; X) \) for appropriate Banach spaces \( X \). In our case, we expect \( X \) to be a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain the bounds in spaces which are not of the type \( L^p(0, T; B^s_{p, r}) \). That naturally leads to the following definition introduced by Chemin and Lerner in [6].

**Definition 2.3.** For \( \rho \in [1, +\infty] \), \( s \in \mathbb{R} \), and \( T \in (0, +\infty) \), we set
\[ \|u\|_{\dot{L}^p_T(B^s_{p, r})} = \left\| 2^{ns} \| \Delta_q u(t) \|_{L^p(B^r_{p, r})} \right\|_{L^r_T} \]
and denote by \( \dot{L}^p_T(B^s_{p, r}) \) the subset of distributions \( u \in S'(0, T) \times \mathbb{R}^N \) with finite \( \|u\|_{\dot{L}^p_T(B^s_{p, r})} \) norm. When \( T = +\infty \), the index \( T \) is omitted. We further denote \( \dot{C} = (0, \infty) \) and \( \dot{L}^p_T(B^s_{p, r}) \) is the \( \dot{C} \)-extension of \( \dot{L}^p_T(B^s_{p, r}) \).

**Remark 2.3.** All the properties of continuity for the paraproduct, remainder, and product remain true for the Chemin-Lerner spaces. The exponent \( \rho \) just has to be chosen according to Hölder’s inequality for the time variable.

**Remark 2.4.** The spaces \( \dot{L}^p_T(B^s_{p, r}) \) can be linked with the classical space \( L^p_T(B^s_{p, r}) \) via the Minkowski inequality:
\[ \|u\|_{\dot{L}^p_T(B^s_{p, r})} \leq \|u\|_{L^p_T(B^s_{p, r})} \quad \text{if} \quad r > \rho, \]
\[ \|u\|_{\dot{L}^p_T(B^s_{p, r})} \geq \|u\|_{L^p_T(B^s_{p, r})} \quad \text{if} \quad r \leq \rho. \]

3. **Linearized system.** We begin this section by giving the paralinearized version of system (1.10)
\[
\begin{cases}
\partial_t a + \text{div}(T_v a) + \text{div} u = F, \\
\partial_t u + \check{T}_v \cdot \nabla u - \frac{1}{\text{Re}} A u + \nabla a - \frac{1}{\text{Re}} \text{div} \tau = G, \\
\partial_t \tau + \check{T}_v \cdot \nabla \tau + \frac{1}{\text{We}} \tau - \frac{2\omega}{\text{We}} D(u) = L,
\end{cases}
\quad (3.1)
\]
where \( v, F, G \) and \( L \) are some known functions. The purpose of this section is to establish the following property of system (3.1).

**Proposition 3.1.** Let \((a, u, \tau)\) be the solution to (3.1). There exists a constant \( C \) depending on \( d, \text{Re}, \text{We} \) and \( \omega \), such that for all \( s \in \mathbb{R} \), we have

\[
\begin{aligned}
&\|a(t)\|_{\dot{B}^{-1,s}_{2,1}} + \|u(t)\|_{\dot{B}^{-1,s}_{2,1}} + \|\tau(t)\|_{\dot{B}^{-1}_{2,1}} \\
&+ \|a\|_{L^1_t(\dot{B}^{s+1}_{2,1})} + \|u\|_{L^1_t(\dot{B}^{s+1}_{2,1})} + \|\tau\|_{L^1_t(\dot{B}^{-1}_{2,1})} \\
&\leq C \exp\left(C\|\nabla v\|_{L^1_t(L^\infty)}\right) \left(\|a_0\|_{\dot{B}^{-1,s}_{2,1}} + \|u_0\|_{\dot{B}^{-1,s}_{2,1}} + \|\tau_0\|_{\dot{B}^{-1}_{2,1}}ight) \\
&+ \|F\|_{L^1_t(\dot{B}^{s+1}_{2,1})} + \|G\|_{L^1_t(\dot{B}^{s+1}_{2,1})} + \|L\|_{L^1_t(\dot{B}^{-1}_{2,1})}. 
\end{aligned}
\]

(3.2)

**Proof.** Before proceeding any further, let us localize the system (3.1). Similar to (1.13) and (1.14), using the notation in (1.16), we find that \((a_q, \mathbb{P}^\perp u_q, \mathbb{P}^\perp \text{div} \tau_q)\) and \((\mathbb{F} u_q, \mathbb{P} \text{div} \tau_q)\) satisfy respectively

\[
\begin{aligned}
&\partial_t a_q + \text{div}(v_q a_q) + \mathbb{P}^\perp \cdot \text{div}^\perp u_q = f_q, \\
&\partial_t \mathbb{P}^\perp u_q + v_q \cdot \nabla \mathbb{P}^\perp u_q - \frac{2(1 - \omega)}{\text{Re}} \Delta \mathbb{P}^\perp u_q + \nabla a_q - \frac{1}{\text{Re}} \mathbb{P}^\perp \text{div} \tau_q = g_q^a, \\
&\partial_t \mathbb{P}^\perp \text{div} \tau_q + v_q \cdot \nabla \mathbb{P}^\perp \text{div} \tau_q + \frac{1}{\text{Re}} \mathbb{P}^\perp \text{div} \tau_q - \frac{\omega}{\text{We}} \Delta \mathbb{P}^\perp u_q = h_q^a, 
\end{aligned}
\]

(3.3)

and

\[
\begin{aligned}
&\partial_t \mathbb{P} u_q + v_q \cdot \nabla \mathbb{P} u_q - \frac{\omega}{\text{Re}} \Delta \mathbb{P} u_q - \frac{1}{\text{Re}} \mathbb{P} \text{div} \tau_q = g_q^b, \\
&\partial_t \mathbb{P} \text{div} \tau_q + v_q \cdot \nabla \mathbb{P} \text{div} \tau_q + \frac{1}{\text{Re}} \mathbb{P} \text{div} \tau_q - \frac{\omega}{\text{We}} \Delta \mathbb{P} u_q = h_q^b, 
\end{aligned}
\]

(3.4)

where \( v_q := \dot{S}_{q-1} v, \)

\[
\begin{aligned}
f_q := \dot{\Delta}_q F + \text{div}(v_q a_q - \dot{\Delta}_q \dot{T}_v a), \\
g_q^a := \mathbb{P}^\perp \dot{\Delta}_q G + \left(v_q \cdot \nabla \Delta_q \mathbb{P}^\perp u - \mathbb{P}^\perp \dot{\Delta}_q \dot{T}_v \cdot \nabla u\right), \\
h_q^a := \mathbb{P}^\perp \Delta_q H + \left(v_q \cdot \nabla \Delta_q \mathbb{P}^\perp \text{div} \tau - \mathbb{P}^\perp \dot{\Delta}_q \dot{T}_v \cdot \nabla \text{div} \tau\right), \\
g_q^b := \dot{\Delta}_q G + \left(v_q \cdot \nabla \Delta_q \mathbb{P} u - \mathbb{P} \dot{\Delta}_q \dot{T}_v \cdot \nabla u\right), \\
h_q^b := \dot{\Delta}_q H + \left(v_q \cdot \nabla \Delta_q \mathbb{P} \text{div} \tau - \mathbb{P} \dot{\Delta}_q \dot{T}_v \cdot \nabla \text{div} \tau\right),
\end{aligned}
\]

with \( H^k := (\text{div} L)^k - \sum_{1 \leq i, l \leq d} \dot{T}_{\partial_i \tau_l} \partial_{\tau_i \tau_l} \).

Next, the proof will be divided into five parts. In the first two parts, we deal with the two sub-systems (3.3) and (3.4), and get the estimates for \((a, \mathbb{P}^\perp u, \mathbb{P}^\perp \text{div} \tau)\) and \((\mathbb{F} u, \mathbb{P} \text{div} \tau)\), respectively. They are the central part of the proof. In the third part, we put the results obtained in the first two parts together, and get the global estimate for \((a, u, \text{div} \tau)\). Unfortunately, the smoothing effect of the velocity \( u \) is not totally revealed there. Therefore, the fourth part is devoted to complete the smoothing effect of \( u \). Finally, we go to bond \( \tau \), and thus obtain the global estimates for \((a, u, \tau)\).

**I. The compressible part**

Since the parabolic-hyperbolic system (3.3) behaves differently in high and low frequency, we have to deal with the high and low frequency part of (3.3) in different ways. To simplify the presentation, in the following we give all the estimates which are needed both for high and low frequency cases. To begin with, taking the \( L^2 \)
inner product of \((3.3)_1\) with \(a_q\) and \(\Delta a_q\) respectively, integrating by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| a_q \|_{L^2}^2 + \langle \text{div} \mathbb{P}^\perp u_q | a_q \rangle = (f_q | a_q) - \frac{1}{2} \int \text{div} v_q | a_q \|^2, \tag{3.5}
\]
and
\[
\frac{1}{2} \frac{d}{dt} \| \nabla a_q \|_{L^2}^2 + \langle \Delta \mathbb{P}^\perp u_q | \nabla a_q \rangle = (\nabla f_q | \nabla a_q) + \int a_q \partial_i a_q \partial_i v_q^i - \frac{1}{2} \int \text{div} v_q | \nabla a_q \|^2. \tag{3.6}
\]
Next, taking the \(L^2\) inner product of \((3.3)_2\) with \(\mathbb{P}^\perp u_q\) and of \((3.3)_3\) with \(\mathbb{P}^\perp \text{div} \tau_q\) yields
\[
\frac{1}{2} \frac{d}{dt} \| \mathbb{P}^\perp u_q \|_{L^2}^2 + \frac{2(1 - \omega)}{\text{Re}} \| \nabla \mathbb{P}^\perp u_q \|_{L^2}^2 + \left( \mathbb{P}^\perp u_q | \nabla a_q \right) - \frac{1}{\text{Re}} \mathbb{P}^\perp u_q | \mathbb{P}^\perp \text{div} \tau_q = (g_q^\perp | \mathbb{P}^\perp u_q) + \frac{1}{2} \int \text{div} v_q | \mathbb{P}^\perp u_q \|^2, \tag{3.7}
\]
and
\[
\frac{1}{2} \frac{d}{dt} \| \mathbb{P}^\perp \text{div} \tau_q \|_{L^2}^2 + \frac{1}{\text{We}} \| \mathbb{P}^\perp \text{div} \tau_q \|_{L^2}^2 - \frac{2\omega}{\text{We}} \| \Delta \mathbb{P}^\perp u_q | \mathbb{P}^\perp \text{div} \tau_q \rangle = (h_q^\perp | \mathbb{P}^\perp \text{div} \tau_q) + \frac{1}{2} \int \text{div} v_q | \mathbb{P}^\perp \text{div} \tau_q \|^2. \tag{3.8}
\]
In order to deal with the low frequency part, we also need to give the \(L^2\) estimate of \(\Lambda^{-1} \mathbb{P}^\perp \text{div} \tau_q\). To do so, applying first the operator \(\Lambda^{-1}\) to third equation of \((3.3)_1\), and then taking the \(L^2\) inner product of the resulting equation with \(\Lambda^{-1} \mathbb{P}^\perp \text{div} \tau_q\), we let to
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^{-1} \mathbb{P}^\perp \text{div} \tau_q \|_{L^2}^2 + \frac{1}{\text{We}} \| \Lambda^{-1} \mathbb{P}^\perp \text{div} \tau_q \|_{L^2}^2 + \frac{2\omega}{\text{We}} \| \Delta \mathbb{P}^\perp u_q | \mathbb{P}^\perp \text{div} \tau_q \rangle = (\Lambda^{-1} h_q^\perp | \Lambda^{-1} \mathbb{P}^\perp \text{div} \tau_q) - (\Lambda^{-1} (v_q \cdot \nabla \mathbb{P}^\perp \text{div} \tau_q) | \Lambda^{-1} \mathbb{P}^\perp \text{div} \tau_q). \tag{3.9}
\]
Owing to the fact that the linear operator associated with \((3.3)_1\) cannot be diagonalized in a basis independent of \(\xi\), coercive estimates can not be achieved by means of a linear combination of \((3.5) - (3.9)\). We must make full use the linear coupling terms in \((3.3)\). Accordingly, the following three equalities of cross terms are given. More precisely, first of all, applying \(\nabla\) to the first equation of \((3.3)_1\), taking the \(L^2\) inner product of the resulting equation with \(\mathbb{P}^\perp u_q\). Secondly, taking the \(L^2\) inner product of the second equation of \((3.3)\) with \(\nabla a_q\). Summing up these two results, we find that
\[
\frac{d}{dt} (\nabla a_q | \mathbb{P}^\perp u_q) - \| \nabla \mathbb{P}^\perp u_q \|_{L^2}^2 + \| \nabla a_q \|_{L^2}^2 - \frac{2(1 - \omega)}{\text{Re}} \| \Delta \mathbb{P}^\perp u_q | \nabla a_q \rangle - \frac{1}{\text{Re}} \nabla v_q \partial_i \partial_i u_q^i = (\nabla f_q | \mathbb{P}^\perp u_q) + (g_q^\perp | \mathbb{P}^\perp u_q) + \int a_q \partial_i v_q^i \partial_i \mathbb{P}^\perp u_q^i. \tag{3.10}
\]
Similarly, for the cross terms \((\mathbb{P}^\perp u_q | \mathbb{P}^\perp \text{div} \tau_q)\) and \((\mathbb{P}^\perp u_q | \Delta \mathbb{P}^\perp \text{div} \tau_q)\), we have
\[
\frac{d}{dt} \| \mathbb{P}^\perp u_q \|_{L^2}^2 + \frac{2\omega}{\text{We}} \| \nabla \mathbb{P}^\perp u_q \|_{L^2}^2 - \| \mathbb{P}^\perp \text{div} \tau_q \|_{L^2}^2 + \frac{2(1 - \omega)}{\text{Re}} \| \Delta \mathbb{P}^\perp u_q | \mathbb{P}^\perp \text{div} \tau_q \rangle + (\nabla a_q | \mathbb{P}^\perp \text{div} \tau_q) + \frac{1}{\text{We}} \| \mathbb{P}^\perp u_q | \mathbb{P}^\perp \text{div} \tau_q \rangle = (g_q^\perp | \mathbb{P}^\perp \text{div} \tau_q) + (h_q^\perp | \mathbb{P}^\perp u_q) + \int \text{div} v_q \mathbb{P}^\perp u_q \cdot \mathbb{P}^\perp \text{div} \tau_q, \tag{3.11}
\]
and
\[
\frac{d}{dt} (P^l u_q | \Delta P^l \nabla \tau_q) + \frac{1}{\Re} \| \nabla P^l \nabla \tau_q \|_{L^2}^2 - 2 \frac{\omega}{\We} \| \Delta P^l u_q \|_{L^2}^2 \\
- 2 \frac{(1 - \omega)}{\Re} (\Delta P^l u_q | \Delta P^l \nabla \tau_q) + (\nabla a_q | \Delta P^l \nabla \tau_q) + \frac{1}{\We} (\Delta P^l u_q | P^l \nabla \tau_q) \\
= (\partial_P^l | \Delta P^l \nabla \tau_q) + (\partial_P^l | \Delta P^l u_q) - (v_q \cdot \nabla P^l u_q | \Delta P^l \nabla \tau_q) \\
- (v_q \cdot \nabla P^l \nabla \tau_q | \Delta P^l u_q).
\]
(3.12)

Now let us define
\[
Y_q := \left[ M \| P^l u_q \|_{L^2}^2 + 2 \Re \frac{1 - \omega}{\Re} \nabla a_q \right] + \left[ \frac{(1 - \omega)}{\omega \Re} \nabla P^l \nabla \tau_q \right]_{L^2}^2 \\
+ 2 \Re \frac{(1 - \omega)}{\Re} \nabla a_q | P^l u_q \right]_{L^2}^2 - 2 \left[ P^l u_q \left( \frac{1 - \omega}{\omega \Re} \nabla P^l \nabla \tau_q \right) \right]_{L^2}^2,
\]
for \( q > q_0 \),
\[
Y_q := \left[ \| a_q \|_{L^2}^2 + \| P^l u_q \|_{L^2}^2 + M' \| P^l \nabla \tau_q \|_{L^2}^2 \\
+ 2 \Re \frac{(1 - \omega)}{\Re} \nabla a_q | P^l u_q \right]_{L^2}^2 + 2 \left( P^l u_q \left( \frac{1 - \omega}{\omega \Re} \nabla P^l \nabla \tau_q \right) \right]_{L^2}^2,
\]
for \( q \leq q_1 \), and
\[
Y_q := \left[ \| a_q \|_{L^2}^2 + \| P^l u_q \|_{L^2}^2 + \frac{\We}{\Re} \| A^{-1} P^l \nabla \tau_q \|_{L^2}^2 \\
+ 2 \frac{\gamma(1 - \omega)}{\Re} \nabla a_q | P^l u_q \right]_{L^2}^2 - \frac{\beta(1 - \omega)}{\Re} \nabla a_q | P^l \nabla \tau_q \right]_{L^2}^2,
\]
for \( q_1 < q \leq q_0 \), where \( q_0, q_1, M, M', \gamma \) and \( \beta \) will be determined later. Next, we will estimate \( Y_q \) for all \( q \in \mathbb{Z} \) step by step.

**Step 1. high frequencies.**

Multiplying (3.6) and (3.10) by \( \left( \frac{1 - \omega}{\Re} \right)^2 \) and \( \frac{1 - \omega}{\Re} \) respectively, summing up the resulting equations yields
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \| \nabla a_q \|_{L^2}^2 + 2 \left( \frac{1 - \omega}{\Re} \nabla a_q \right) \right) \\
- \frac{1 - \omega}{\Re} \| \nabla P^l u_q \|_{L^2}^2 + \frac{1 - \omega}{\Re} \| \nabla a_q \|_{L^2}^2 - \frac{1}{\Re} \left( \frac{1 - \omega}{\Re} \nabla a_q | P^l \nabla \tau_q \right) \\
= 2 \left( \frac{1 - \omega}{\Re} \right)^2 \text{R.H.S. of (3.6)} + \frac{1 - \omega}{\Re} \text{R.H.S. of (3.10)}.
\end{align*}
\]
(3.13)

Multiplying (3.8) and (3.11) by \( \left( \frac{(1 - \omega)\We}{\Re} \right)^2 \) and \( \frac{(1 - \omega)\We}{\Re} \) respectively, summing up the resulting equations, we are led to
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \left( \frac{1 - \omega}{\Re} \We \nabla \tau_q \right) \right)_{L^2}^2 - 2 \left( P^l u_q \left( \frac{1 - \omega}{\Re} \We \nabla \tau_q \right) \right]_{L^2}^2,
\end{align*}
\]
By using Cauchy-Schwarz inequality, we have

\[
(1 - \omega)W e \frac{\omega}{\omega e} \mid \nabla a_q \mid \frac{(1 - \omega)We}{\omega e} \mid \nabla P^\perp u_q \mid \leq \frac{2(1 - \omega)}{\omega e} \mid \nabla P^\perp u_q \mid^2_{L^2} - \frac{1}{2(1 - \omega)} \mid \nabla P^\perp u_q \mid^2_{L^2} \]

we obtain

\[
\left( \frac{1 - \omega}{\omega e} \right)^2 \mid \nabla a_q \mid \frac{(1 - \omega)We}{\omega e} \mid \nabla P^\perp u_q \mid \leq \frac{1}{\omega e} \left( \frac{(1 - \omega)We}{\omega e} \right)^2 R.H.S. \text{ of } (3.8) - \frac{1}{\omega e} \left( \frac{1 - \omega)We}{\omega e} \right)^2 R.H.S. \text{ of } (3.11) \tag{3.14}
\]

Multiplying (3.7) and (3.13) by M and ReWe respectively, adding to (3.14), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ M \left\| \nabla a_q \right\|_{L^2}^2 + 2\text{ReWe} \left( \frac{1 - \omega}{\omega e} \right) \right] \leq \text{ReWe} \left( \left\| \nabla a_q \right\|_{L^2}^2 + \left\| \nabla P^\perp u_q \right\|_{L^2}^2 \right)
\]

and

\[
2 \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla P^\perp u_q \right\|_{L^2}^2 \leq 3 \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla P^\perp u_q \right\|_{L^2}^2 + \frac{2}{3} \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla P^\perp u_q \right\|_{L^2}^2 .
\]

It follows that

\[
M \left\| \nabla a_q \right\|_{L^2}^2 + 2\text{ReWe} \left( \frac{1 - \omega}{\omega e} \right) \leq \text{ReWe} \left( \left\| \nabla a_q \right\|_{L^2}^2 + \left\| \nabla P^\perp u_q \right\|_{L^2}^2 \right)
\]

\[
+ 2\text{ReWe} \left( \frac{1 - \omega}{\omega e} \right) \left\| \nabla a_q \right\|_{L^2}^2 + 2 \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla P^\perp u_q \right\|_{L^2}^2 \right)
\]

\[
\geq \left( M - \frac{3}{2} \text{ReWe} \right) \left\| \nabla a_q \right\|_{L^2}^2 + \frac{1}{2} \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla a_q \right\|_{L^2}^2
\]

\[
+ \frac{1}{3} \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla P^\perp u_q \right\|_{L^2}^2 \right) \tag{3.16}
\]

Moreover,

\[
(1 + \omega) \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla a_q \right\|_{L^2}^2 + \frac{1}{(1 - \omega)We} \left( \frac{(1 - \omega)We}{\omega e} \right)^2 \left\| \nabla P^\perp u_q \right\|_{L^2}^2 \right) .
\]
and hence
\[
(1 - \omega) \text{Re} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{1}{(1 - \omega) \text{Re}} \left\| (1 - \omega) \text{Re} \left[ \text{P}_1 \nabla \tau_q \right] \right\|_{L^2}^2
- (1 + \omega) \left( \nabla a_q \right) \left( \frac{(1 - \omega) \text{Re}}{\omega \text{Re}} \right) \left[ \text{P}_1 \nabla \tau_q \right]
\geq \frac{(1 - \omega)}{2} \left( 1 - \omega \right) \text{Re} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{1}{2 \text{Re}} \left\| (1 - \omega) \text{Re} \left[ \text{P}_1 \nabla \tau_q \right] \right\|_{L^2}^2
= \frac{(1 - \omega)^2 \text{Re}}{2} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{1}{2 \text{Re}} \left\| (1 - \omega) \text{Re} \left[ \text{P}_1 \nabla \tau_q \right] \right\|_{L^2}^2.
\]
(3.17)
The rest terms on the left hand side of (3.15) can be bounded as follows.
\[
\frac{M}{\text{Re}} \left\| (\text{P}_1 u_q | \nabla a_q) \right\| + \frac{1}{\text{Re}} \frac{M^2}{(1 - \omega)^2 \text{Re}} \left\| (\text{P}_1 u_q) \left( \frac{1 - \omega}{\omega \text{Re}} \frac{(1 - \omega) \text{Re}}{\omega \text{Re}} \right) \left[ \text{P}_1 \nabla \tau_q \right] \right\|_{L^2}
\leq \frac{1}{4} \frac{(1 - \omega)^2 \text{Re}}{\omega \text{Re}} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{M^2}{(1 - \omega)^2 \text{Re}} \left\| (\text{P}_1 u_q) \left( \frac{1 - \omega}{\omega \text{Re}} \frac{(1 - \omega) \text{Re}}{\omega \text{Re}} \right) \right\|^2_{L^2}
\leq \frac{1}{4} \frac{(1 - \omega)^2 \text{Re}}{\omega \text{Re}} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{M^2}{(1 - \omega)^2 \text{Re}} \left\| (\text{P}_1 u_q) \right\|^2_{L^2}
\leq \frac{1}{4} \frac{1}{\text{Re}} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{M^2}{(1 - \omega)^2 \text{Re}} \left\| (\text{P}_1 u_q) \right\|^2_{L^2}
\leq \frac{1}{4} \frac{1}{\text{Re}} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{M^2}{(1 - \omega)^2 \text{Re}} \left\| (\text{P}_1 u_q) \right\|^2_{L^2}.
\]
(3.19)
From (3.17)–(3.19), we arrive at
\[
(2M - 2 - \text{Re} \text{Re}) \frac{1 - \omega}{\text{Re}} \left\| \text{P}_1 u_q \right\|_{L^2}^2 + (1 - \omega) \text{Re} \left\| \nabla a_q \right\|_{L^2}^2
+ \frac{1}{(1 - \omega) \text{Re}} \left\| (1 - \omega) \text{Re} \left[ \text{P}_1 \nabla \tau_q \right] \right\|_{L^2}^2 - (1 + \omega) \left( \nabla a_q \right) \left( \frac{(1 - \omega) \text{Re}}{\omega \text{Re}} \right) \left[ \text{P}_1 \nabla \tau_q \right]
+ M (\text{P}_1 u_q | \nabla a_q) - \frac{M}{\text{Re}} (\text{P}_1 u_q | \text{P}_1 \nabla \tau_q) - \frac{1}{\text{Re}} \frac{M^2}{(1 - \omega)^2 \text{Re}} \left\| (\text{P}_1 u_q) \left( \frac{1 - \omega}{\omega \text{Re}} \frac{(1 - \omega) \text{Re}}{\omega \text{Re}} \right) \right\|_{L^2}^2
\geq \left( 2M - 2 - \text{Re} \text{Re} \right) \frac{1 - \omega}{\text{Re}} \left( \frac{4}{3} 2^{q_0} \right)^2 \frac{2M^2}{(1 - \omega)^2 \text{Re}} \left\| \text{P}_1 \nabla \tau_q \right\|_{L^2}^2
+ \frac{1}{4} (1 - \omega)^2 \text{Re} \left\| \nabla a_q \right\|_{L^2}^2 + \frac{1}{4 \text{Re}} \left\| (1 - \omega) \text{Re} \left[ \text{P}_1 \nabla \tau_q \right] \right\|_{L^2}^2.
\]
(3.20)
Taking $M = \text{Re} + 2$, and
\[
2^{q_0} \geq \left( \frac{4}{3} \right)^2 \frac{2 \text{Re} (\text{Re} + 2)}{(1 - \omega)^2 \text{Re}},
\]
(3.21)
then it is easy to verify that
\[
\begin{align*}
M - \text{Re}We - \frac{1}{2} & \geq \frac{1}{2}, \\
(2M - 2 - \text{Re}We)\frac{1}{1-\omega} - \left(\frac{1}{2} - q_0\right)^2 & \geq \frac{\text{Re}We + 2}{4} \cdot \frac{1}{1-\omega}.
\end{align*}
\]
(3.22)

Next, we estimate the right hand side of (3.15). Indeed, using Cauchy-Schwarz and
above inequality yields
\[
\begin{align*}
\text{Re} & \geq \frac{\text{Re}We \cdot \text{div} \tau_q}{\omega \text{Re}}.
\end{align*}
\]

In view of commutator estimates, cf. [1], we infer that
\[
\text{Re} \geq \frac{\text{Re}We \cdot \text{div} \tau_q}{\omega \text{Re}}.
\]

\[3.15\]

\[3.16\]

\[3.20\]

\[3.24\]

Now from (3.16), (3.20)–(3.24), we find that there exist two constants \(c_1\) and \(C\)
depending on \(d, \text{Re}, \text{We}\), and \(\omega\), such that if \(q > q_0\), then (3.15) implies that
\[
\begin{align*}
\frac{d}{dt} Y_q & + c_1 Y_q \\
& \leq C \left( \|\nabla \Delta_q F\|_{L^2} + \|\nabla \Delta_q G\|_{L^2} + \|\nabla \Delta_q H\|_{L^2} \\
& + \|\nabla v\|_{L^\infty} \sum_{|q' - q| \leq 4} (\|\nabla a_{q'}\|_{L^2} + \|u_{q'}\|_{L^2} + \|\text{div} \tau_{q'}\|_{L^2}) \right). 
\end{align*}
\]
\(3.25\)

**Step 2. low frequencies.**

Part (1). \(q \leq q_1\). Multiplying (3.10) and (3.11) by \(\frac{1}{\text{Re}^2}\) and \(\frac{\text{We}}{\text{Re}^3}\) respectively, and then adding them to (3.5) and (3.7), we get
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \|a_q\|_{L^2}^2 + \|\nabla a_q\|_{L^2}^2 + \frac{2}{\text{Re}} (\nabla a_q \cdot \nabla^\perp u_q) + \frac{2\text{We}}{\text{Re}} (\nabla a_q \cdot \nabla^\perp \text{div} \tau_q) \right] \\
+ \frac{1}{\text{Re}} \|\nabla^\perp u_q\|_{L^2} + \frac{1}{\text{Re}} \|\nabla a_q\|_{L^2}^2 - \frac{\text{We}}{(\text{Re})^2} \|\nabla^\perp \text{div} \tau_q\|_{L^2}^2 - \frac{2(1 - \omega)}{(\text{Re})^2} (\Delta \nabla^\perp u_q \cdot \nabla a_q) \\
- \frac{2(1 - \omega)\text{We}}{(\text{Re})^2} (\Delta \nabla^\perp u_q \cdot \nabla^\perp \text{div} \tau_q) - \frac{1}{(\text{Re})^2} (\nabla a_q \cdot \nabla^\perp \text{div} \tau_q) + \frac{\text{We}}{\text{Re}} (\nabla a_q \cdot \nabla^\perp \text{div} \tau_q) \\
= R.H.S. of (3.5) and (3.7) + \frac{1}{\text{Re}} R.H.S. of (3.10) + \frac{\text{We}}{\text{Re}} R.H.S. of (3.11).
\end{align*}
\]
\(3.26\)

Multiplying (3.8) and (3.12) by \(M' \left(\frac{\text{We}}{\text{Re}^2}\right)^2\) and \(\frac{\text{We}}{\text{Re}^3}\) respectively, adding them to the
above inequality yields
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \|a_q\|_{L^2}^2 + \|\nabla a_q\|_{L^2}^2 + M' \left(\frac{\text{We}}{\text{Re}^2}\right)^2 \|\nabla^\perp \text{div} \tau_q\|_{L^2}^2 + \frac{2}{\text{Re}} (\nabla a_q \cdot \nabla^\perp u_q) \right]
\end{align*}
\]
Now we estimate the cross terms on the left hand side of (3.27). From (3.28)–(3.32), it is easy to see that

\[
\frac{2\omega}{\Re} \|\Delta \mathbb{P}^\perp u_q\|_{L^2}^2 + \frac{1}{\Re} \left( \|\nabla a_q\|_{L^2}^2 + (1 - \frac{1}{\Re \mathcal{W}}) (\nabla a_q) \|\Delta \mathbb{P}^\perp u_q\|_{L^2} \right) + \frac{1}{\Re} \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2 - 2(1 - \omega) \left( \|\nabla a_q\|_{L^2} \|\Delta \mathbb{P}^\perp u_q\|_{L^2} \right) + \frac{M'}{\Re} \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2
\]

\[
\left( \frac{1}{\Re} - \frac{2(1 - \omega) + 2M'\omega}{\Re} \right) \|\Delta \mathbb{P}^\perp u_q\|_{L^2} \|\nabla a_q\|_{L^2} \Delta \mathbb{P}^\perp \nabla \tau_q\|_{L^2} - \frac{2(1 - \omega)}{\Re} \|\nabla a_q\|_{L^2} \|\Delta \mathbb{P}^\perp \nabla \tau_q\|_{L^2},
\]

provided

\[
\frac{2\omega}{\Re} \|\Delta \mathbb{P}^\perp u_q\|_{L^2}^2 + \frac{1}{\Re} \left( \|\nabla a_q\|_{L^2}^2 + \|\mathbb{P}^\perp u_q\|_{L^2}^2 \right) + \frac{1}{\Re} \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2 \leq \frac{1}{4} \left( \|\mathbb{P}^\perp u_q\|_{L^2}^2 + \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2 \right),
\]

(3.28)

provided

\[
2\left( \frac{\mathbb{P}^\perp u_q}{\Re} \|\Delta \mathbb{P}^\perp \nabla \tau_q\|_{L^2} \right) \leq \frac{1}{4} \left( \|\mathbb{P}^\perp u_q\|_{L^2}^2 + \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2 \right),
\]

(3.30)

provided

\[
2\left( \frac{\mathbb{P}^\perp u_q}{\Re} \|\Delta \mathbb{P}^\perp \nabla \tau_q\|_{L^2} \right) \leq \frac{1}{4} \left( \|\mathbb{P}^\perp u_q\|_{L^2}^2 + \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2 \right),
\]

(3.31)

From (3.28)–(3.32), it is easy to see that

\[
\|a_q\|_{L^2}^2 + \|\mathbb{P}^\perp u_q\|_{L^2}^2 + \frac{1}{4} \left( \|\mathbb{P}^\perp u_q\|_{L^2}^2 + \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2 \right) \geq \frac{3}{4} \|a_q\|_{L^2}^2 + \frac{1}{4} \|\mathbb{P}^\perp u_q\|_{L^2}^2 + (M' - \frac{17}{4}) \|\mathbb{P}^\perp \nabla \tau_q\|_{L^2}^2.
\]

(3.33)

Using Cauchy-Schwarz inequality and Lemma 2.1 over and over again, the rest cross terms on the left hand side of (3.27) can be bounded in a similar way. In fact,

\[
\frac{2(1 - \omega)}{(\Re)^2} \|\Delta \mathbb{P}^\perp u_q\|_{L^2} \|\nabla a_q\|_{L^2} \leq \frac{2(1 - \omega)}{(\Re)^2} \frac{8}{3} \|\mathbb{P}^\perp u_q\|_{L^2} \|\nabla a_q\|_{L^2}
\]
\[
\begin{align*}
&\leq \frac{8}{3\Re} 2^{q_1} \left( \frac{1}{\Re} \|\nabla P^\perp u_q\|_{L^2}^2 + \frac{1}{\Re} \|\nabla a_q\|^2_{L^2} \right) \\
&\leq \frac{1}{4} \left( \frac{1}{\Re} \|\nabla P^\perp u_q\|_{L^2}^2 + \frac{1}{\Re} \|\nabla a_q\|^2_{L^2} \right), \quad (3.34)
\end{align*}
\]

provided

\[
2^{q_1} \leq \frac{3}{32} \Re. \quad (3.35)
\]

\[
\left| \left( \frac{1}{\Re} - \frac{1}{\Re \We} \right) \left( \nabla a_q \cdot \frac{\We}{\Re} \nabla \div q \right) \right| \leq \frac{1}{4} \Re \|\nabla a_q\|^2_{L^2} + \left( \frac{\Re \We - 1}{\Re \We} \right)^2 \frac{1}{\Re \We} \left\| \nabla \div q \right\|^2_{L^2}. \quad (3.36)
\]

\[
\left| \left( \frac{1}{\Re} - \frac{2(1 - \omega) + 2M' \omega}{\Re} \right) \left( \Delta P^\perp u_q \cdot \frac{\We}{\Re} \Delta P^\perp \div q \right) \right| \leq \frac{1}{4} \Re \left( \frac{M'}{\We} \right) \left\| \Delta P^\perp u_q \right\|^2_{L^2} + \frac{8}{3} \left( 3 \cdot 2^{q_1} \right)^2 \left( \frac{4M' \We}{(M' - 1) \Re} + \frac{1}{\Re \We} \right) \frac{1}{\Re} \|\nabla P^\perp u_q\|^2_{L^2}. \quad (3.37)
\]

\[
\left| \left( \frac{1}{\Re} - \frac{2}{\Re} \left( 3 \cdot 2^{q_1} \right)^3 \|\nabla P^\perp u_q\|^2_{L^2} \right) \right| \leq \frac{2}{\Re} \left( 3 \cdot 2^{q_1} \right)^3 \left\| \nabla P^\perp u_q \right\|^2_{L^2} + \frac{4}{\Re} \left( 3 \cdot 2^{q_1} \right) \frac{1}{\Re} \|\nabla P^\perp u_q\|^2_{L^2}. \quad (3.38)
\]

Moreover,

\[
2^{q_1} \Re \|\nabla P^\perp u_q\|^2_{L^2} \leq 2 \left( \frac{8}{3} \cdot 2^{q_1} \right) \frac{1}{\Re} \|\nabla P^\perp u_q\|^2_{L^2}. \quad (3.39)
\]

It follows from (3.34)–(3.40) that

\[
\begin{align*}
\frac{1}{\Re} \|\nabla P^\perp u_q\|^2_{L^2} - \frac{2q}{\Re} \|\Delta P^\perp u_q\|^2_{L^2} + \frac{1}{\Re} \|\nabla a_q\|^2_{L^2} &+ \frac{M' - 1}{\We} \left\| \nabla \div q \right\|^2_{L^2} \\
+ \frac{1}{\We} \left\| \nabla \div q \right\|^2_{L^2} &- \frac{2(1 - \omega)}{(\Re)^2} \left( \Delta P^\perp u_q \nabla a_q \right) \\
+ \left( 1 - \frac{1}{\Re \We} \right) \left( \nabla a_q \cdot \frac{\We}{\Re} \nabla \div q \right) + \left( \nabla a_q \cdot \frac{\We}{\Re} \Delta P^\perp \div q \right)
\end{align*}
\]
Taking \( M' := 4 \left( \frac{3}{2} \cdot \frac{2}{(\text{ReWe})^2} \right)^{\frac{1}{2}} \), \( 2_{q1} \leq \left( \frac{8}{3} \cdot 2_{q1} \right)^{4} \leq \frac{1}{16} \), \( \left( \frac{8}{3} \cdot 2_{q1} \right)^{2} \leq \frac{1}{16} \), \( \frac{4}{2} \cdot \frac{2}{(\text{ReWe})^2} \leq \frac{1}{16} \), \( \left( \frac{8}{3} \cdot 2_{q1} \right)^{6} \) then (3.29), (3.32) and (3.35) hold, and

\[
\frac{3}{4} - \left( \frac{8}{3} \cdot 2_{q1} \right)^{2} \geq \frac{7}{16}, \quad \text{and} \quad \frac{3}{4} - \frac{3}{2} - \frac{(\text{ReWe} - 1)^2}{\text{ReWe}} \geq 2.
\]

Next, we estimate the right hand side of (3.27). To this end, notice first that integrating by parts yields

\[
- (v_q \cdot \nabla \mathbf{P}^u u_q | \Delta \mathbf{P}^u \text{div} \tau_q) - (v_q \cdot \nabla \mathbf{P}^u \text{div} \tau_q | \Delta \mathbf{P}^u u_q) = (\nabla v_q \nabla \mathbf{P}^u u_q | \nabla \mathbf{P}^u \text{div} \tau_q) + (\nabla v_q \nabla \text{div} \tau_q | \nabla \mathbf{P}^u u_q) - (\text{div} v_q \nabla \mathbf{P}^u u_q : \nabla \mathbf{P}^u \text{div} \tau_q).
\]
Multiplying (3.10) and (3.11) by $\gamma$

Similar to (3.24), we have

Substituting this equality to (3.27), using the fact $q \leq q_1$, it is not difficult to verify that

$$R.H.S. of \ (3.27)$$

$$\leq C \left( \|a_q\|_{L^2} + \|P^\perp u_q\|_{L^2} + \left\| \frac{We}{Re} P^\perp \text{div} \tau_q \right\|_{L^2} \right)$$

$$\cdot \left( \|f_q\|_{L^2} + \|g_q^2\|_{L^2} + \|h_q^2\|_{L^2} \right)$$

$$+ \|\nabla v_q\|_{L^\infty} \left( \|a_q\|_{L^2} + \|P^\perp u_q\|_{L^2} + \left\| \frac{We}{Re} P^\perp \text{div} \tau_q \right\|_{L^2} \right). \quad (3.48)$$

Now we infer from (3.33), (3.41)–(3.46), (3.48) and (3.49) that there exist two constants $c_2$ and $C$ depending on $d$, $\text{Re}$, $\text{We}$ and $\omega$, such that if $q \leq q_1$, then (3.27) implies

$$\frac{d}{dt} Y_q + c_2 2^{2q} Y_q$$

$$\leq C \left( \|\Delta_q F\|_{L^2} + \|P^\perp \Delta_q G\|_{L^2} + \|P^\perp \Delta_q H\|_{L^2} \right)$$

$$+ \|\nabla v\|_{L^\infty} \sum_{|q'-q|\leq 4} \left( \|a_{q'}\|_{L^2} + \|u_{q'}\|_{L^2} + \|\text{div} \tau_{q'}\|_{L^2} \right). \quad (3.50)$$

Part (ii). $q_1 < q \leq q_0$. Multiplying (3.9) by $\frac{We}{2\omega \text{Re}}$, adding the resulting equation to (3.5) and (3.7), we arrive at

$$\frac{1}{2} \frac{d}{dt} \left( \|a_q\|^2_{L^2} + \|P^\perp u_q\|^2_{L^2} + \frac{We}{2\omega \text{Re}} \|\Lambda^{-1} P^\perp \text{div} \tau_q\|^2_{L^2} \right)$$

$$+ \frac{2(1-\omega)}{\text{Re}} \|\nabla P^\perp u_q\|^2_{L^2} + \frac{1}{2\omega \text{Re}} \|\Lambda^{-1} P^\perp \text{div} \tau_q\|^2_{L^2}$$

$$\leq R.H.S. \ of \ (3.5) \ and \ (3.7) + \frac{We}{2\omega \text{Re}} \ R.H.S. \ of \ (3.9). \quad (3.51)$$

Multiplying (3.10) and (3.11) by $\gamma \frac{1-\omega}{\text{Re}}$ and $-\beta \frac{1-\omega}{\text{Re}}$ respectively, adding them to (3.51), we are led to

$$\frac{1}{2} \frac{d}{dt} \left( \|a_q\|^2_{L^2} + \|P^\perp u_q\|^2_{L^2} + \frac{We}{2\omega \text{Re}} \|\Lambda^{-1} P^\perp \text{div} \tau_q\|^2_{L^2} \right)$$

$$+ \frac{2(1-\omega)}{\text{Re}} \|\nabla a_q\|_{L^2}^2 + \frac{\beta(1-\omega)\text{We}}{\omega \text{Re}} \|P^\perp u_q\|_{L^2}^2$$

$$+ (2 - \gamma - \beta) \frac{1-\omega}{\text{Re}} \|\nabla P^\perp u_q\|^2_{L^2} + \frac{\beta(1-\omega)\text{We}}{2\omega (\text{Re})^2} \|\text{div} \tau_q\|^2_{L^2}$$
More precisely, all of them can be bounded by using Cauchy-Schwarz inequality and Lemma 2.1.

Now we estimate the cross terms in the left hand side of (3.52) one by one. Indeed, provided
\[
\beta \leq \frac{1}{16} 2^{-g_0} \sqrt{\frac{\omega \text{Re}}{\text{We}}},
\]
(3.56)
Consequently, if (3.54) and (3.56) hold, we have
\[
\begin{align*}
2\gamma \frac{1-\omega}{\text{Re}} |(\nabla \mathbf{a}_q | \mathbf{P}^\perp \mathbf{u}_q)| & \leq 8 \frac{2^{g_0}}{3} \frac{\gamma}{\text{Re}} \left( \| \mathbf{a}_q \|_{L^2}^2 + \| \mathbf{P}^\perp \mathbf{u}_q \|_{L^2}^2 \right) \\
\beta \left( \frac{1-\omega}{\omega \text{Re}} \right) \mathbf{P}^\perp \mathbf{u}_q | \mathbf{P}^\perp \mathbf{d} \mathbf{v} \mathbf{\tau}_q | & \leq \frac{1}{4} \| \mathbf{P}^\perp \mathbf{u}_q \|_{L^2}^2 + \left( \frac{\beta \text{We} 8}{\omega \text{Re} 3} 2^{g_0} \right)^2 \| \mathbf{V}^{-1} \mathbf{P}^\perp \mathbf{d} \mathbf{v} \mathbf{\tau}_q \|_{L^2}^2 \\
& \leq \frac{1}{4} \| \mathbf{P}^\perp \mathbf{u}_q \|_{L^2}^2 + \frac{1}{4} \frac{\text{We}}{\omega \text{Re}} \| \mathbf{A}^{-1} \mathbf{P}^\perp \mathbf{d} \mathbf{v} \mathbf{\tau}_q \|_{L^2}^2.
\end{align*}
\]
(3.55)
Moreover,
\[
2\gamma \frac{(1-\omega)^2}{(\text{Re})^2} |(\Delta \mathbf{P}^\perp \mathbf{u}_q | \nabla \mathbf{a}_q)| & \leq \left( \frac{8}{3} \frac{1 - \omega}{\text{Re}} \right) \left[ \frac{1}{4} \frac{1 - \omega}{\text{Re}} \left( || \nabla \mathbf{P}^\perp \mathbf{u}_q ||_{L^2}^2 + \gamma || \nabla \mathbf{a}_q ||_{L^2}^2 \right) \right] \\
& \leq \frac{1}{4} \left[ \frac{1 - \omega}{\text{Re}} \left( || \nabla \mathbf{P}^\perp \mathbf{u}_q ||_{L^2}^2 + \gamma || \nabla \mathbf{a}_q ||_{L^2}^2 \right) \right].
\]
(3.58)
provided
\[
\gamma \leq \left( 2^{-g_0} \frac{3}{32} \text{Re} \right)^2.
\]
(3.59)
\[
\frac{\gamma(1 - \omega)}{(\text{Re})^2} |(\nabla a_q|\mathbb{P}^\perp \text{div} \tau_q)|
\leq \left( \sqrt{\frac{7}{3}} \frac{2q_0}{\text{Re}} \right) \left( \frac{\gamma(1 - \omega)}{\text{Re}} \|\nabla a_q\|_{L^2}^2 + \frac{1}{\omega \text{Re}} \|\Lambda^{-1}\mathbb{P}^\perp \text{div} \tau_q\|_{L^2}^2 \right)
\leq \frac{1}{8} \left( \frac{\gamma(1 - \omega)}{\text{Re}} \|\nabla a_q\|_{L^2}^2 + \frac{1}{\omega \text{Re}} \|\Lambda^{-1}\mathbb{P}^\perp \text{div} \tau_q\|_{L^2}^2 \right),
\] (3.60)

provided
\[
\gamma \leq \left( \frac{2^{-q_0} 3}{32 \text{Re}} \right)^2 .
\] (3.61)

\[
\frac{\beta(1 - \omega)^2 \text{We}}{\omega (\text{Re})^2} |(\Delta \mathbb{P}^\perp u_q|\mathbb{P}^\perp \text{div} \tau_q)|
\leq \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{\omega (\text{Re})^2} \|\mathbb{P}^\perp \text{div} \tau_q\|_{L^2}^2 + \frac{\beta(1 - \omega) \text{We}}{\omega (\text{Re})^2} \left( \frac{8}{3} 2q_0 \right)^2 \|\nabla \mathbb{P}^\perp u_q\|_{L^2}^2
\leq \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{\omega (\text{Re})^2} \|\mathbb{P}^\perp \text{div} \tau_q\|_{L^2}^2 + \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{\omega \text{Re}} \|\nabla \mathbb{P}^\perp u_q\|_{L^2}^2,
\] (3.62)

provided
\[
\beta \leq \left( \frac{8}{3} 2q_0 \right)^{-2} \frac{\omega \text{Re}}{4 \text{We}} .
\] (3.63)

\[
\frac{\beta(1 - \omega) \text{We}}{2 \omega \text{Re}} |(\nabla a_q|\mathbb{P}^\perp \text{div} \tau_q)|
\leq \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{2 \omega (\text{Re})^2} \|\mathbb{P}^\perp \text{div} \tau_q\|_{L^2}^2 + \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{\omega \text{Re}} \|\nabla a_q\|_{L^2}^2,
\] (3.64)

provided
\[
\beta \leq \frac{\omega \gamma}{2 \text{Re} \text{We}} .
\] (3.65)

\[
\frac{\beta(1 - \omega)}{2 \omega \text{Re}} |(\mathbb{P}^\perp u_q|\mathbb{P}^\perp \text{div} \tau_q)|
= \frac{\beta(1 - \omega)}{2 \omega \text{Re}} |(\Delta \mathbb{P}^\perp u_q|\Lambda^{-1}\mathbb{P}^\perp \text{div} \tau_q)|
\leq \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{2 \omega \text{Re}} \|\Lambda^{-1}\mathbb{P}^\perp \text{div} \tau_q\|_{L^2}^2 + \frac{\beta^2 1 - \omega}{2 \omega \text{Re}} \|\nabla \mathbb{P}^\perp u_q\|_{L^2}^2
\leq \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{2 \omega \text{Re}} \|\Lambda^{-1}\mathbb{P}^\perp \text{div} \tau_q\|_{L^2}^2 + \frac{1}{4} \frac{\beta(1 - \omega) \text{We}}{\omega \text{Re}} \|\nabla \mathbb{P}^\perp u_q\|_{L^2}^2,
\] (3.66)

provided
\[
\beta \leq \sqrt{\frac{\omega}{2}} .
\] (3.67)

Collecting (3.54), (3.56), (3.59), (3.61), (3.63), (3.65) and (3.67), we choose
\[
\gamma \leq \min \left( \frac{1}{4} \frac{2^{-q_0} 3}{32 \text{Re}}, \left( \frac{2^{-q_0} 3}{32 \text{Re}} \right)^2 \right)
\] (3.68)
and

\[ \beta \leq \min \left( \frac{1}{4} 2^{-q_0} \frac{3}{16} \sqrt{\frac{\omega \text{Re}}{\text{We}}} \left( \frac{8}{3} \frac{2^{q_0}}{2} \right)^{-2} \frac{\omega \text{Re}}{4 \text{We} \cdot \sqrt{2}} \right). \]  

(3.69)

Then \(3.57\) and the following inequality hold,

\[ \frac{\gamma(1 - \omega)}{\text{Re}} \| \nabla a_q \|_{L^2}^2 + (2 - \gamma - \beta) \frac{1 - \omega}{\text{Re}} \| \nabla \mathbb{P}^\perp u_q \|_{L^2}^2 + \frac{\beta(1 - \omega) \text{We}}{2 \omega (\text{Re})^2} \| \nabla \mathbb{P}^\perp \nabla \tau_q \|_{L^2}^2 \]

\[ + \frac{1}{2 \omega \text{Re}} \| \mathbb{P}^\perp \nabla \tau_q \|_{L^2}^2 = \frac{2 \gamma (1 - \omega)^2}{(\text{Re})^2} (\mathbb{P}^\perp u_q) (\nabla a_q) \]

\[ + \frac{\gamma(1 - \omega)}{\text{Re}^2} (\nabla a_q) (\nabla \mathbb{P}^\perp \nabla \tau_q) + \frac{\beta(1 - \omega)^2 \text{We}}{\omega (\text{Re})^2} (\mathbb{P}^\perp u_q) (\nabla \mathbb{P}^\perp \nabla \tau_q) \]

\[ - \frac{\beta(1 - \omega) \text{We}}{2 \omega \text{Re}} (\nabla a_q) (\nabla \mathbb{P}^\perp \nabla \tau_q) + \frac{\beta(1 - \omega)}{2 \omega \text{Re}} (\nabla \mathbb{P}^\perp u_q) (\nabla \mathbb{P}^\perp \nabla \tau_q) \]

\[ \geq \frac{3 \gamma(1 - \omega)}{8} \| \nabla a_q \|_{L^2}^2 + \frac{3}{4} (1 - \omega) \| \nabla \mathbb{P}^\perp u_q \|_{L^2}^2 \]

\[ + \frac{\beta(1 - \omega) \text{We}}{8 \omega (\text{Re})^2} \| \nabla \mathbb{P}^\perp \nabla \tau_q \|_{L^2}^2 + \frac{1}{4 \omega \text{Re}} \| \mathbb{P}^\perp \nabla \tau_q \|_{L^2}^2. \]  

(3.70)

Now we are left to bound the right hand side of \(3.52\). To do so, noting first that

\[ (\mathbb{P}^\perp (v_q \cdot \nabla \mathbb{P}^\perp \nabla \tau_q) | \mathbb{P}^\perp \nabla \tau_q) = (\mathbb{P}^\perp (v_q \cdot \nabla \mathbb{P}^\perp \nabla \tau_q) | \mathbb{P}^\perp \nabla \tau_q) - \frac{1}{2} \int \nabla v_q | \mathbb{P}^\perp \nabla \tau_q |^2, \]

then Lemma \(2.2\) implies

\[ |(\mathbb{P}^\perp (v_q \cdot \nabla \mathbb{P}^\perp \nabla \tau_q) | \mathbb{P}^\perp \nabla \tau_q)| \leq C \| \nabla v_q \|_{L^\infty} \| \mathbb{P}^\perp \nabla \tau_q \|_{L^2}. \]  

(3.71)

As a result, by virtue of Cauchy-Schwarz inequality and the fact \( q \leq q_0 \), we arrive at

\[ R.H.S.\ of \ (3.52) \]

\[ \leq C \left( \| f_q \|_{L^2} + \| g_q^\prime \|_{L^2} + \| \mathbb{P}^\perp \nabla \tau_q \|_{L^2} \right) \]

\[ \times \left( \| a_q \|_{L^2} + \| \mathbb{P}^\perp u_q \|_{L^2} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \| \mathbb{P}^\perp \nabla \tau_q \|_{L^2} \right) \]

\[ + C \| \nabla v_q \|_{L^\infty} \left( \| a_q \|_{L^2} + \| \mathbb{P}^\perp u_q \|_{L^2} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \| \mathbb{P}^\perp \nabla \tau_q \|_{L^2} \right)^2 \]  

(3.72)

Using the commutator estimates in \(1\) once more, one easily deduces that

\[ \| f_q \|_{L^2} + \| g_q^\prime \|_{L^2} + \| \mathbb{P}^\perp \nabla \tau_q \|_{L^2} \]

\[ \leq C \left( \| \mathbb{P}^\perp \Delta_q F \|_{L^2} + \| \mathbb{P}^\perp \Delta_q G \|_{L^2} + \| \mathbb{P}^\perp \Delta_q H \|_{L^2} \right) \]

\[ + \| \nabla v \|_{L^\infty} \sum_{|q - q'| \leq 4} \left( \| a_{q'} \|_{L^2} + \| u_{q'} \|_{L^2} + \| \mathbb{P}^\perp \nabla \tau_{q'} \|_{L^2} \right). \]  

(3.73)

From \(3.57\), \(3.70\) and \(3.72\), we conclude that there exist two constants \(c_3\) and \(C\) depending on \(d, \text{Re}, \text{We}\) and \(\omega\), such that if \(q_1 < q \leq q_0\) and \(\gamma, \beta\) satisfy
(3.68) and (3.69), then (3.52) implies
\[
\frac{d}{dt} Y_q + c_3 2^{2q} Y_q \leq C \left( \| \Delta_q F \|_{L^2} + \| P_{\pm} \Delta_q G \|_{L^2} + \| \Lambda^{-1} P_{\pm} \Delta_q H \|_{L^2} \\
+ \| \nabla v \|_{L^\infty} \sum_{|q' - q| \leq 4} (\| a_q' \|_{L^2} + \| u_q' \|_{L^2} + \| \Lambda^{-1} \text{div} \tau_q' \|_{L^2}) \right). \tag{3.74}
\]

(II). The incompressible part

We begin this part by giving the following five equalities in which the $L^2$ estimates and cross terms of the corresponding incompressible part ($P_{u_q}$, $P_{\text{div} \tau_q}$) of $(u_q, \text{div} \tau_q)$ are involved. Since they are obtained in the same way with those in the compressible part, we give a list of the results directly.

\[
\frac{1}{2} \frac{d}{dt} \| P_{u_q} \|_{L^2}^2 + \frac{1 - \omega}{\text{Re}} \| \nabla P_{u_q} \|_{L^2}^2 = \frac{1}{\text{Re}} (P_{u_q} P_{\text{div} \tau_q}) = (g^P_{u_q} P_{u_q}) + \frac{1}{2} \int \text{div} u_q |P_{u_q}|^2. \tag{3.75}
\]

\[
\frac{1}{2} \frac{d}{dt} \| P_{\text{div} \tau_q} \|_{L^2}^2 + \frac{1}{\text{We}} \| P_{\text{div} \tau_q} \|_{L^2}^2 - \frac{\omega}{\text{We}} (|P_{u_q} P_{\text{div} \tau_q}) = (h^P_{u_q} P_{u_q}) + \frac{1}{2} \int \text{div} u_q |P_{\text{div} \tau_q}|^2. \tag{3.76}
\]

\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^{-1} P_{\text{div} \tau_q} \|_{L^2}^2 + \frac{1}{\text{We}} \| \Lambda^{-1} P_{\text{div} \tau_q} \|_{L^2}^2 + \frac{\omega}{\text{We}} (P_{u_q} P_{\text{div} \tau_q}) = (\Lambda^{-1} h^P_{u_q} |\Lambda^{-1} P_{\text{div} \tau_q}) - (\Lambda^{-1} (u_q \cdot \nabla P_{\text{div} \tau_q}) |\Lambda^{-1} P_{\text{div} \tau_q}). \tag{3.77}
\]

\[
\frac{d}{dt} (P_{u_q} |P_{\text{div} \tau_q}) + \frac{\omega}{\text{We}} \| \nabla P_{u_q} \|_{L^2}^2 - \frac{1}{\text{Re}} \| P_{\text{div} \tau_q} \|_{L^2}^2 - \frac{1 - \omega}{\text{Re}} (P_{u_q} |P_{\text{div} \tau_q}) + \frac{1}{\text{We}} (P_{u_q} |P_{\text{div} \tau_q}) = (g^{P_{u_q}} |P_{\text{div} \tau_q}) + (h^{P_{u_q}} P_{u_q}) + \int \text{div} u_q (P_{u_q} \cdot P_{\text{div} \tau_q}). \tag{3.78}
\]

\[
\frac{d}{dt} (P_{u_q} |\Delta P_{\text{div} \tau_q}) + \frac{1}{\text{Re}} \| \nabla P_{\text{div} \tau_q} \|_{L^2}^2 - \frac{\omega}{\text{We}} \| \Delta P_{u_q} \|_{L^2}^2 - \frac{1 - \omega}{\text{Re}} (\Delta P_{u_q} |\Delta P_{\text{div} \tau_q}) + \frac{1}{\text{We}} (\Delta P_{u_q} |P_{\text{div} \tau_q}) = (g^{P_{u_q}} |\Delta P_{\text{div} \tau_q}) + (h^{P_{u_q}} P_{u_q}) + (v_q \cdot \nabla P_{u_q} |\Delta P_{\text{div} \tau_q}) - (v_q \cdot \nabla P_{\text{div} \tau_q} |\Delta P_{u_q}). \tag{3.79}
\]

Denote
\[
\bar{Y}_q := \sqrt{2 \| P_{u_q} \|_{L^2}^2 + \left\| \frac{(1 - \omega)\text{We}}{\omega \text{Re}} P_{\text{div} \tau_q} \right\|_{L^2}^2 - 2 \left( \frac{(1 - \omega)\text{We}}{\omega \text{Re}} P_{\text{div} \tau_q} \right)^2},
\]
for $q > q_0$,
\[
\bar{Y}_q := \sqrt{\| P_{u_q} \|_{L^2}^2 + M' \| P_{\text{div} \tau_q} \|_{L^2}^2 + 2 \left( \frac{\text{We}}{\text{Re}} P_{\text{div} \tau_q} \right)^2 + 2 \left( \frac{\text{We}}{\text{Re}} \Delta P_{\text{div} \tau_q} \right)^2},
\]
for $q \leq q_1$, and
\[
\bar{Y}_q := \left[ \| \mathcal{P} u_q \|_{L^2}^2 + \frac{\text{We}}{Re} \| \Lambda^{-1} \mathcal{P} \text{div} \tau_q \|_{L^2}^2 - \frac{2\beta(1 - \omega) \text{We}}{Re} \left( \mathcal{P} u_q \tau \mathcal{P} \text{div} \tau_q \right) \right]^{\frac{1}{2}},
\]
for $q_1 < q \leq q_0$, where $q_0, q_1, M', \beta$ are given as in the compressible part. Next, we shall bound $\bar{Y}_q$ for all $q \in \mathbb{Z}$.

**Step 1. high frequencies.**

Similar to (3.14), from (3.76), (3.78), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2 - 2 \left( \mathcal{P} u_q \left( \frac{1 - \omega}{\text{We}} \mathcal{P} \text{div} \tau_q \right) \right) \right)
+ \frac{1}{(1 - \omega) \text{We}} \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2 - \frac{1}{\text{Re}} \| \nabla \mathcal{P} u_q \|_{L^2}^2
- \frac{1}{\text{We}} \left( \mathcal{P} u_q \left( \frac{1 - \omega}{\text{We}} \mathcal{P} \text{div} \tau_q \right) \right)
= \left( \frac{1 - \omega}{\text{We}} \right)^2 R.H.S. of (3.76) - \frac{(1 - \omega) \text{We}}{\text{Re}} R.H.S. of (3.78), (3.80)
\]

Multiplying (3.75) by 2, and then adding the resulting inequality to (3.80) yields
\[
\frac{1}{2} \frac{d}{dt} \left( 2 \| \mathcal{P} u_q \|_{L^2}^2 + \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2 - 2 \left( \mathcal{P} u_q \left( \frac{1 - \omega}{\text{We}} \mathcal{P} \text{div} \tau_q \right) \right) \right)
+ \frac{1}{(1 - \omega) \text{We}} \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2 - \frac{2}{\text{Re}} (\mathcal{P} \text{div} \tau_q \mathcal{P} u_q) - \frac{1}{\text{We}} \left( \mathcal{P} u_q \left( \frac{1 - \omega}{\text{We}} \mathcal{P} \text{div} \tau_q \right) \right)
= 2 R.H.S. of (3.75) + \left( \frac{1 - \omega}{\text{We}} \right)^2 R.H.S. of (3.76)
- \frac{(1 - \omega) \text{We}}{\text{Re}} R.H.S. of (3.78).
\]

Obviously,
\[
2 \| \mathcal{P} u_q \|_{L^2}^2 + \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2 - 2 \left( \mathcal{P} u_q \left( \frac{1 - \omega}{\text{We}} \mathcal{P} \text{div} \tau_q \right) \right)
\approx \| \mathcal{P} u_q \|_{L^2}^2 + \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2,
\]
and
\[
\frac{2}{\text{Re}} (\mathcal{P} \text{div} \tau_q \mathcal{P} u_q) + \frac{1}{\text{We}} \left( \mathcal{P} u_q \left( \frac{1 - \omega}{\text{We}} \mathcal{P} \text{div} \tau_q \right) \right)
= \frac{1 + \omega}{1 - \omega} \text{We} \left( \mathcal{P} u_q \left( \frac{1 - \omega}{\text{We}} \mathcal{P} \text{div} \tau_q \right) \right)
\leq \frac{1}{2} \frac{1}{1 - \omega} \text{We} \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2 + \left( \frac{4}{3} q_u^2 \right)^2 \left( \frac{2}{\text{Re}} \| \nabla \mathcal{P} u_q \|_{L^2}^2 \right)
\leq \frac{1}{2} \frac{1}{1 - \omega} \text{We} \left\| (1 - \omega) \text{We} \mathcal{P} \text{div} \tau_q \right\|_{L^2}^2 + \frac{1}{2} \frac{1}{\text{Re}} \| \nabla \mathcal{P} u_q \|_{L^2}^2,
\]
(3.83)
provided

\[
\left( \frac{4}{3} \right)^2 \frac{2}{(1-\omega)\text{We}} \leq \frac{1}{2} \frac{1-\omega}{\text{Re}}, \quad \text{i.e.} \quad \frac{2\gamma_0}{3(1-\omega)} \geq \frac{8}{3(1-\omega)} \sqrt{\frac{\text{Re}}{\text{We}}}
\]

(3.84)

It is easy to see that

\[
\left( \frac{4}{3} \right)^3 \sqrt{\frac{2\text{Re}(\text{ReWe}+2)}{(1-\omega)^3\text{We}}} \geq \frac{8}{3(1-\omega)} \sqrt{\text{Re}}\text{We}.
\]

Therefore, (3.84) is a consequence of (3.21). It follows from (3.83) and (3.84) that

\[
1 - \omega \frac{\text{Re}}{(1-\omega)\text{Re}} \left\| \nabla P u q \right\|_{L^2}^2 + \frac{1}{2} \frac{1}{(1-\omega)\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} P \text{div} q \right\|_{L^2}^2 \geq \frac{8}{3(1-\omega)} \sqrt{\text{Re}}\text{We}.
\]

(3.85)

Using Cauchy-Schwarz inequality and commutator estimates, we can bound the right hand side of (3.81) as follows,

\[
\text{R.H.S. of (3.81)} \leq C \left( \left\| \nabla v \right\|_{L^\infty} \left( \left\| P u q \right\|_{L^2} + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} P \text{div} q \right\|_{L^2} \right) \right)
\]

\[
\leq C \left( \left\| P u q \right\|_{L^2} + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} P \text{div} q \right\|_{L^2} \right)
\]

\[
\times \left( \left\| P \Delta q G \right\|_{L^2} + \left\| P \Delta q H \right\|_{L^2} + \left\| \nabla v \right\|_{L^\infty} \sum_{|q-q'| \leq 4} (\left\| u_{q'} \right\|_{L^2} + \left\| \text{div} q' \right\|_{L^2}) \right).
\]

(3.86)

Substituting (3.86) into (3.81), using (3.82) and (3.85), we find that there exist constants \( \tilde{c}_1 \) and \( C \) depending on \( d, \text{Re}, \text{We}, \) and \( \omega \), such that if \( q > q_0 \), there holds

\[
\frac{d}{dt} \tilde{Y}_q + \tilde{c}_1 \tilde{Y}_q \leq C \left( \left\| P \Delta q G \right\|_{L^2} + \left\| P \Delta q H \right\|_{L^2} + \left\| \nabla v \right\|_{L^\infty} \sum_{|q-q'| \leq 4} (\left\| u_{q'} \right\|_{L^2} + \left\| \text{div} q' \right\|_{L^2}) \right).
\]

(3.87)

**Step 2. low frequencies.**

Part (i). \( q \leq q_1 \). Similar to (3.27), a linear combination of (3.75), (3.76), (3.78) and (3.79) yields

\[
\frac{1}{2} \left\| P u q \right\|_{L^2}^2 + \frac{\text{We}}{\text{Re}} \left\| P \text{div} q \right\|_{L^2}^2 + 2 \left\| P u q \right\|_{L^2} \frac{\text{We}}{\text{Re}} P \text{div} q \]

(3.88)
where $M'$ is the same as in (3.27). The cross terms in (3.88) and the right hand side can be estimated in a similar manner as those in (3.27), accordingly, we infer from (3.88) that there exist constants $\tilde{c}_2$ and $C$ depending on $d, \Re, W, \omega$, such that if $q \leq q_1$, there holds

$$\frac{d}{dt} \tilde{Y}_q + \tilde{c}_2 2^{2q} \tilde{Y}_q \leq C \left( \|\mathfrak{P}\tilde{\Delta}_q G\|_{L^2} + \|\mathfrak{P}\tilde{\Delta}_q H\|_{L^2} + \|\nabla v\|_{L^\infty} \sum_{|q-q'| \leq 4} (\|u_q\|_{L^2} + \|\text{div} \tau_q\|_{L^2}) \right).$$

(3.89)

Part (ii). $q_1 < q \leq q_0$. Similar to (3.32), by a linear combination of (3.75), (3.77) and (3.78), we get

$$\frac{1}{2} \frac{d}{dt} \left( \|u_q\|_{L^2}^2 + \frac{\Re}{\omega \Re} \|\mathfrak{A}^{-1} \mathfrak{P}\text{div} \tau_q\|_{L^2}^2 - \frac{2\beta(1-\omega)}{\omega \Re} (\mathfrak{P}u_q | \mathfrak{P}\text{div} \tau_q) \right) + (1-\beta) \frac{1-\omega}{\omega \Re} \|\nabla \mathfrak{P}u_q\|_{L^2}^2 + \frac{\beta(1-\omega)\Re}{\omega (\Re)^2} \|\mathfrak{P}\text{div} \tau_q\|_{L^2}^2 + \frac{1}{\omega \Re} \|\mathfrak{A}^{-1} \mathfrak{P}\text{div} \tau_q\|_{L^2}^2$$

$$+ \frac{\beta(1-\omega)^2 \Re}{\omega (\Re)^2} (\mathfrak{P}u_q | \mathfrak{P}\text{div} \tau_q) - \frac{\beta(1-\omega)}{\omega \Re} (\mathfrak{P}u_q | \mathfrak{P}\text{div} \tau_q)$$

$$\leq R.H.S. \text{ of (3.75)} + \frac{\Re}{\omega \Re} R.H.S. \text{ of (3.77)} - \frac{\beta(1-\omega)}{\omega \Re} R.H.S. \text{ of (3.78)},$$

where $\beta$ is the same as in (3.52). Arguing as in the corresponding compressible case, we find that there exist constants $\tilde{c}_3$ and $C$ depending on $d, \Re, W, \omega$, such that if $q_1 < q \leq q_0$, (3.90) implies

$$\frac{d}{dt} \tilde{Y}_q + \tilde{c}_3 2^{2q} \tilde{Y}_q \leq C \left( \|\mathfrak{P}\tilde{\Delta}_q G\|_{L^2} + \|\mathfrak{P}\tilde{\Delta}_q H\|_{L^2} + \|\nabla v\|_{L^\infty} \sum_{|q-q'| \leq 4} (\|u_q\|_{L^2} + \|\text{div} \tau_q\|_{L^2}) \right).$$

(3.91)

(III). Global estimates of $(a, u, \text{div} \tau)$

Let $X_q := Y_q + \tilde{Y}_q$. Moreover, we set

$$s(q) := \begin{cases} 2^q, & \text{if } q > q_0, \\ 1, & \text{if } q \leq q_0, \end{cases} \quad \text{and} \quad \tilde{s}(q) := \begin{cases} 1, & \text{if } q > q_0, \\ 2^{2q}, & \text{if } q \leq q_0. \end{cases}$$
Recalling the definition of \( Y_q \) and \( \hat{Y}_q \), using Bernstein’s inequalities, we infer from (3.16), (3.33), (3.57) and the corresponding estimates for incompressible part that there exist constants \( \bar{c} \) and \( C \) depending on \( d, \Re, \We, \end{equation}

Now collecting (3.25), (3.50), (3.74), (3.87), (3.89) and (3.91), we conclude that there exist constants \( \bar{c} \) and \( C \) depending on \( d, \Re, \We, \omega \), such that for all \( q \in \mathbb{Z} \)

\[
\frac{d}{dt} X_q + \bar{c} \hat{s}(q) X_q \leq C \left( s(q) \| \hat{\Delta}_q F \|_{L^2} + \| \hat{\Delta}_q G \|_{L^2} + \| \hat{\Delta}_q H \|_{L^2} \right)
\]

+ \| \nabla v \|_{L^2} \sum_{|q-q'| \leq 4} X_{q'} .
\]

Performing a time integration in (3.93), multiplying the resulting inequality by 2(q(s−1)), and taking sum w. r. t. \( q \) over \( \mathbb{Z} \) yields

\[
\sum_{q \in \mathbb{Z}} 2^{q(s-1)} X_q(t) + \bar{c} \sum_{q \in \mathbb{Z}} \hat{s}(q) 2^{q(s-1)} \int_0^t X_q dt' \leq \sum_{q \in \mathbb{Z}} 2^{q(s-1)} X_q(0) + C \left( \| F \|_{L^1(\hat{B}^{s-1}_{2,1})} + \| G \|_{L^1_t(\hat{B}^{s-1}_{2,1})} + \| H \|_{L^1_t(\hat{B}^{s-1}_{2,1})} \right)
\]

\[
+ C \int_0^t \| \nabla v \|_{L^2} \sum_{q \in \mathbb{Z}} 2^{q(s-1)} X_q dt' .
\]

Using the fact (3.92), this inequality is nothing but

\[
\| a(t) \|_{\hat{B}^{s-1,1}_{2,1}} + \| u(t) \|_{\hat{B}^{s-1}_{2,1}} + \| \text{div} \tau(t) \|_{\hat{B}^{s-1}_{2,1}}
\]

\[
+ \| a \|_{L^1_t(\hat{B}^{s+1,1}_{2,1})} + \| u \|_{L^1_t(\hat{B}^{s+1,1}_{2,1})} + \| \text{div} \tau \|_{L^1_t(\hat{B}^{s+1,1}_{2,1})}
\]

\[
\leq C \left( \| a_0 \|_{\hat{B}^{s-1,1}_{2,1}} + \| u_0 \|_{\hat{B}^{s-1}_{2,1}} + \| \text{div} \tau_0 \|_{\hat{B}^{s-1}_{2,1}}
\]

\[
+ \| F \|_{L^1_t(\hat{B}^{s+1,1}_{2,1})} + \| G \|_{L^1_t(\hat{B}^{s+1,1}_{2,1})} + \| H \|_{L^1_t(\hat{B}^{s+1,1}_{2,1})}
\]

\[
+ \int_0^t \| \nabla v \|_{L^\infty} \left( \| a(t') \|_{\hat{B}^{s-1,1}_{2,1}} + \| u(t') \|_{\hat{B}^{s-1}_{2,1}} + \| \text{div} \tau(t') \|_{\hat{B}^{s-1}_{2,1}} \right) dt' .
\]

(IV). The smoothing effect of \( u \)

Applying \( \hat{\Delta}_q \) to the equation of (3.1) yields

\[
\partial_t u_q + v_q \cdot \nabla u_q - \frac{1}{\Re} A u_q + \nabla a_q - \frac{1}{\Re} \text{div} \tau_q = \hat{\Delta}_q G + (v_q \cdot \nabla u_q - \hat{\Delta}_q \hat{T}_v \cdot \nabla u).
\]

Taking the inner product of the above equation with \( u_q \), integrating by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \| u_q \|_{L^2}^2 + \frac{1 - \omega}{\Re} \left( \| \nabla u_q \|_{L^2}^2 + \| \text{div} u_q \|_{L^2}^2 \right)
\]

\[
= \frac{1}{2} \int \text{div} \tau_q |u_q|^2 - (\nabla a_q |u_q|) + \frac{1}{\Re} (\text{div} \tau_q |u_q|)
\]

\[
+ (\hat{\Delta}_q G |u_q|) + (v_q \cdot \nabla u_q - \hat{\Delta}_q \hat{T}_v \cdot \nabla u |u_q|).
\]
By virtue of Bernstein’s inequality and the commutator estimates, we easily get
\[ \|u_q(t)\|_{L^2} + \kappa \frac{1 - \omega}{\text{Re}} 2q^2\|u_q\|_{L^2(L^2)} \leq \|u_q(0)\|_{L^2} + C 2q\|a_q\|_{L^2_L(L^2)} + \frac{1}{\text{Re}} \|\text{div}\tau_q\|_{L^1_t(L^2)} + \|\dot{\Delta}_q G\|_{L^1_t(L^2)} + C \int_0^t \|\nabla v\| L^\infty \sum_{|q' - q| \leq 4} \|u_{q'}\|_{L^2} dt', \]
with positive constant \( \kappa \) depending on \( d \). Multiplying this inequality by \( 2^{q(s-1)} \) and adding up them over \( q > q_0 \), we arrive at
\[ \sum_{q > q_0} 2^{q(s-1)}\|u_q(t)\|_{L^2} + \sum_{q > q_0} 2^{q(s+1)}\|u_q\|_{L^2_t(L^2)} \leq C \left( \|u_0\|_{\dot{B}^{s-1}_{2,1}} + \sum_{q > q_0} 2^{qs}\|a_q\|_{L^1_t(L^2)} + \sum_{q > q_0} 2^{q(s-1)}\|\text{div}\tau_q\|_{L^1_t(L^2)} \right) + C\|G\|_{L^1_t(L^2)} + C \int_0^t \|\nabla v\| L^\infty \|u\|_{\dot{B}^{s-1}_{2,1}} dt'. \tag{3.95} \]
It follows from (3.94) and (3.95) that
\[ \|a(t)\|_{\dot{B}^{s-1}_{2,1}} + \|u(t)\|_{\dot{B}^{s-1}_{2,1}} + \|\text{div}\tau(t)\|_{\dot{B}^{s-1}_{2,1}} + \|F\|_{L^1_t(\dot{B}^{s-1}_{2,1})} + \|G\|_{L^1_t(\dot{B}^{s-1}_{2,1})} + \|H\|_{L^1_t(\dot{B}^{s-1}_{2,1})} + \int_0^t \|\nabla v\|_{L^\infty} (\|a(t')\|_{\dot{B}^{s-1}_{2,1}} + \|u(t')\|_{\dot{B}^{s-1}_{2,1}} + \|\text{div}\tau(t')\|_{\dot{B}^{s-1}_{2,1}}) dt'. \]
Consequently, using Gronwall’s inequality, we are led to
\[ \|a(t)\|_{\dot{B}^{s-1}_{2,1}} + \|u(t)\|_{\dot{B}^{s-1}_{2,1}} + \|\text{div}\tau(t)\|_{\dot{B}^{s-1}_{2,1}} + \|F\|_{L^1_t(\dot{B}^{s-1}_{2,1})} + \|G\|_{L^1_t(\dot{B}^{s-1}_{2,1})} + \|H\|_{L^1_t(\dot{B}^{s-1}_{2,1})} \leq C \exp \left( C \|\nabla v\|_{L^1_t(L^\infty)} \right) \left( \|a_0\|_{\dot{B}^{s-1}_{2,1}} + \|u_0\|_{\dot{B}^{s-1}_{2,1}} + \|\text{div}\tau_0\|_{\dot{B}^{s-1}_{2,1}} + \|F\|_{L^1_t(\dot{B}^{s-1}_{2,1})} + \|G\|_{L^1_t(\dot{B}^{s-1}_{2,1})} + \|H\|_{L^1_t(\dot{B}^{s-1}_{2,1})} \right). \tag{3.96} \]

(V). The damping effect of \( \tau \)
Applying \( \Delta_q \) to the equation of (3.13) yields
\[ \partial_t \tau_q + v_q \cdot \nabla \tau_q + \frac{\tau_q}{\text{We}} - \frac{2\omega}{\text{We}} D(u_q) = \Delta_q L + (v_q \cdot \nabla \tau_q - \Delta_q \dot{T}_v \cdot \nabla \tau). \]
Taking the inner product of the above equation with \( \tau_q \), integrating by parts, we have
\[ \frac{1}{2} \frac{d}{dt} \|\tau_q\|_{L^2}^2 + \frac{1}{\text{We}} \|\tau_q\|_{L^2}^2 = \frac{1}{2} \int \text{div}v_q |\tau_q|^2 + \frac{2\omega}{\text{We}} (D(u_q)|\tau_q|) + (\Delta_q L|u_q|) + (v_q \cdot \nabla \tau_q - \Delta_q \dot{T}_v \cdot \nabla \tau |\tau_q|). \]
Similar to the estimates of high frequency part of \( u \), one easily deduces that
\[
\|\tau_q(t)\|_{L^2} + \frac{1}{We} \|\tau_q\|_{L^1_t(L^2)} \leq C \left( \|\tau_q(0)\|_{L^2} + 2^q \|u_q\|_{L^2} + \|\Delta_q L\|_{L^2} + \int_0^t \|\nabla v\|_{L^\infty} \sum_{|q-q'| \leq 4} \|\tau_{q'}\|_{L^2} \, dt' \right).
\]
Multiplying the above equation by \( 2^{q_s} \), and summing over \( q \in \mathbb{Z} \), we find that
\[
\|\tau(t)\|_{\dot{B}^{2,1}_{2,1}} + \frac{1}{We} \|\tau\|_{L^1_t(\dot{B}^{2,1}_{2,1})} \leq C \left( \|\tau_0\|_{\dot{B}^{2,1}_{2,1}} + \|u\|_{L^1_t(\dot{B}^{2,1}_{2,1})} + \|L\|_{L^1_t(\dot{B}^{2,1}_{2,1})} + \int_0^t \|\nabla v\|_{L^\infty} \|\tau\|_{\dot{B}^{2,1}_{2,1}} \, dt' \right).
\]
Gronwall’s inequality then implies that
\[
\|\tau(t)\|_{\dot{B}^{2,1}_{2,1}} + \frac{1}{We} \|\tau\|_{L^1_t(\dot{B}^{2,1}_{2,1})} \leq C \exp \left( C\|\nabla v\|_{L^1_t(L^\infty)} \right) \left( \|\tau_0\|_{\dot{B}^{2,1}_{2,1}} + \|u\|_{L^1_t(\dot{B}^{2,1}_{2,1})} + \|L\|_{L^1_t(\dot{B}^{2,1}_{2,1})} \right). \tag{3.97}
\]
To close the estimate, substituting (3.96) into (3.97), we conclude that
\[
\|a(t)\|_{\dot{B}^{-1,\infty}_{2,1}} + \|u(t)\|_{\dot{B}^{-1,1}_{2,1}} + \|\tau(t)\|_{\dot{B}^{2,1}_{2,1}} + \|\Delta_q L\|_{\dot{B}^{-1,1}_{2,1}} + \|\Delta_q L\|_{\dot{B}^{2,1}_{2,1}} + \|\Delta_{q'} L\|_{\dot{B}^{2,1}_{2,1}} + \|\Delta_q L\|_{\dot{B}^{2,1}_{2,1}} + \|\Delta_{q'} L\|_{\dot{B}^{2,1}_{2,1}} \leq C \exp \left( C\|\nabla v\|_{L^1_t(L^\infty)} \right) \left( \|a_0\|_{\dot{B}^{-1,\infty}_{2,1}} + \|u_0\|_{\dot{B}^{-1,1}_{2,1}} + \|\tau_0\|_{\dot{B}^{2,1}_{2,1}} + \|\Delta_q L\|_{\dot{B}^{-1,1}_{2,1}} + \|\Delta_{q'} L\|_{\dot{B}^{2,1}_{2,1}} + \|\Delta_q L\|_{\dot{B}^{2,1}_{2,1}} + \|\Delta_{q'} L\|_{\dot{B}^{2,1}_{2,1}} \right). \tag{3.98}
\]
Recalling that \( H^k := (\text{div} L)^k - \sum_{1 \leq i,j \leq d} \partial_i v_j \partial_j \tau^{i,k} \) by virtue of Lemma 2.1 and Proposition 2.2, it is easy to see that
\[
\|H\|_{L^1_t(\dot{B}^{-1,1}_{2,1})} \leq C\|\nabla v\|_{L^1_t(L^\infty)} \|\tau\|_{\dot{B}^{2,1}_{2,1}} \int_0^t dt'. \tag{3.99}
\]
Combining (3.98) with (3.99), then (3.2) follows immediately. This completes the proof of Proposition 3.1.

4. Proof of Theorem 1.1. Since the local existence and uniqueness of the solution \((a, u, \tau)\) to (1.10) have been proved in [13], it suffices to show \( T^* = \infty \), where \( T^* \) is the maximal existence time of \((a, u, \tau)\). To this end, let us now denote \( V(t) := \int_0^t \|\nabla u(t')\|_{L^\infty} \, dt' \),

\[
X(t) := \|a\|_{L^\infty_t(\dot{B}^{-1,\frac{2}{d}}_{2,1}) \cap L^1_t(\dot{B}^{-1,1}_{2,1})} + \|u\|_{L^\infty_t(\dot{B}^{\frac{2}{d}-1,\frac{2}{d}}_{2,1})} + \|\Delta_q L\|_{L^\infty_t(\dot{B}^{\frac{2}{d}-1,\frac{2}{d}}_{2,1})} + \|\Delta_{q'} L\|_{L^\infty_t(\dot{B}^{\frac{2}{d}-1,\frac{2}{d}}_{2,1})},
\]

\[
U(t) := \|a\|_{L^1_t(\dot{B}^{\frac{2}{d}+1,\frac{2}{d}}_{2,1})} + \|u\|_{L^1_t(\dot{B}^{\frac{2}{d}+1,\frac{2}{d}}_{2,1})} + \|\Delta_q L\|_{L^1_t(\dot{B}^{\frac{2}{d}+1,\frac{2}{d}}_{2,1})},
\]

and

\[
X_0 := \|a_0\|_{\dot{B}^{-1,\frac{2}{d}}_{2,1}} + \|u_0\|_{\dot{B}^{-1,1}_{2,1}} + \|\tau_0\|_{\dot{B}^{2,1}_{2,1}}.
\]
Collecting all these estimates, we infer from (3.2) that

\[ F = -\text{div}(\tilde{T}_a^j u), \]

\[ G = \frac{1}{\text{Re}} I(a) (Au + \text{div} \nu) + \tilde{K}(a) \nabla a - \sum_{1 \leq j \leq d} \tilde{T}_j^j u^j, \]

\[ L = -g_\alpha(\tau, \nabla u) - \sum_{1 \leq j \leq d} \tilde{T}_j^\alpha u^j, \]

and employing the product estimates in Besov spaces, it is not difficult to verify that

\[ \|F\|_{L^1_t(B^{d-1}_2, 1)} \leq C\|\alpha\|_{L^\infty_t(B^{d-1}_2, 1)} \|u\|_{L^1_t(B^{d+1}_2, 1)}, \]

\[ \|G\|_{L^1_t(B^{d-1}_2, 1)} \leq C\|\alpha\|_{L^\infty_t(B^{d}_2, 1)} \left( \|u\|_{L^1_t(B^{d+1}_2, 1)} + \|\tau\|_{L^1_t(B^{d-1}_2, 1)} \right), \]

\[ + C\|\alpha\|_{L^\infty_t(B^{d}_2, 1)} \|u\|_{L^1_t(B^{d+1}_2, 1)} \]

\[ \leq C\|\alpha\|_{L^\infty_t(B^{d-1}_2, 1)} \left( \|u\|_{L^1_t(B^{d+1}_2, 1)} + \|\tau\|_{L^1_t(B^{d-1}_2, 1)} \right), \]

\[ + C\|\alpha\|_{L^\infty_t(B^{d}_2, 1)} \|u\|_{L^1_t(B^{d+1}_2, 1)} \]

\[ \|L\|_{L^1_t(B^{d-1}_2, 1)} \leq C\|\tau\|_{L^\infty_t(B^{d}_2, 1)} \|\nabla u\|_{L^1_t(B^{d}_2, 1)} + \sum_{1 \leq j \leq d} \|\tilde{T}_j^j u^j\|_{L^1_t(B^{d}_2, 1)}, \]

\[ \leq C\|\tau\|_{L^\infty_t(B^{d}_2, 1)} \|u\|_{L^1_t(B^{d+1}_2, 1)} + C\|\nabla \tau\|_{L^\infty_t(B^{d+1}_2, 1)} \|u\|_{L^1_t(B^{d+1}_2, 1)}, \]

\[ \leq C\|\tau\|_{L^\infty_t(B^{d}_2, 1)} \|u\|_{L^1_t(B^{d+1}_2, 1)}. \]

Collecting all these estimates, we infer from (5.2) that

\[ \sum_{\tau \in [0, T^*]} (X(t) - X_0) \leq C C^0 \sum_{\tau \in [0, T^*]} (X(t) - X_0), \]

\[ \text{for all } t \in [0, T^*]. \]

Let us denote by \( C_0 \) the embedding constant of \( B^{d-1}_2 \hookrightarrow L^\infty. \) Then it is easy to see that \( V(t) \leq C_0 U(t). \) Choosing a positive constant \( c_* \) satisfying

\[ e^{C C_0 c_*} \leq 2, \quad \text{and} \quad 2C_* \leq \frac{1}{2}, \]

Define \( T_1 \) be the supremum of all time \( T^* \in [0, T^*] \) such that

\[ U(t) \leq c_*, \quad \text{for all } t \in [0, T^*]. \]

Combining (4.2) with (4.3), (4.1) reduces to

\[ X(t) \leq 4 CX_0, \quad \text{for all } t \in [0, T_1]. \]

Noting that \( U(t) \leq X(t), \) for all \( t \in [0, T^*]. \) Taking \( X_0 \) so small that \( 4CX_0 \leq \frac{c_0}{2}, \)

we then have

\[ U(t) \leq \frac{c_*}{2}, \quad \text{for all } t \in [0, T_1]. \]

This implies that \( T_1 = T^* \), and (4.3) holds on the interval \([0, T^*]\) provided \( X_0 \leq \frac{c_0}{8C}. \)

Accordingly, (4.4) holds with \( T_1 \) replaced by \( T^* \), and hence \( T^* = \infty. \) This completes the proof of Theorem 1.1. \( \square \)
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