Rotating black hole entropy from two different viewpoints

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Abstract
Using the brick-wall method, we study the entropy of Kerr–Newman black hole from two different viewpoints, a rest observer at infinity and a zero angular momentum observer near the horizon. We investigate this with a scalar field in the canonical quantization approach. An observer at infinity can take one of the two possible frequency ranges; one is with positive frequencies only and the other is with the whole range including negative frequencies. On the other hand, a zero angular momentum observer near horizon can take positive frequencies only. For the observer at infinity, the superradiant modes appear in either choice of the frequency ranges and the two results coincide. For the zero angular momentum observer superradiant modes do not appear due to the absence of the ergoregion. The resulting entropies from the two viewpoints turn out to be the same.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Since Bekenstein suggested that a black hole has an intrinsic entropy proportional to the surface area of its event horizon [1], there have been many works to explain its statistical origin [2]. Along this line, the ‘brick-wall model’ proposed by ’t Hooft [3] is to calculate the entropy of a black hole by counting the degrees of freedom near its horizon. By introducing a cutoff, the divergence due to the infinite blueshift near the horizon is removed [4–6]. Note that a global thermal equilibrium between the black hole and its surrounding is assumed in this model. Therefore, the original brick-wall model cannot be applied to a non-equilibrium system. However, the degrees of freedom are mostly concentrated near the horizon, it is good enough to consider only the narrow region near the horizon which is locally in thermal
equilibrium with the black hole. In this context, the ‘thin-layer method’ as an improved brick-wall method has also been introduced [7–9]. In the thin-layer method, the local thermal equilibrium is assumed. Assuming this kind of local thermal equilibrium near the horizon, one can calculate the entropies of various black holes.

When one applies this method to the rotating black hole case, one encounters the so-called superradiant modes. The superradiant modes are the special mode solutions of scalar fields that satisfy the Klein–Gordon equation in a given background spacetime of a rotating black hole, and their appearance is due to the presence of the ergoregion in which a particle cannot remain at rest as viewed from an observer at infinity. Counting these superradiant modes in the rotating case has caused some confusion in the evaluation of the entropy. Considering only non-superradiant modes, the entropies of various rotating black holes were evaluated in [10, 11]. Because of the divergence of the free energy from large azimuthal quantum number, they did not consider the entropy contribution from the superradiant modes. In [12, 13], an extra cutoff parameter was introduced in order to overcome the above divergence from the superradiant modes, and it yielded incorrect answers.

In [14], the superradiant modes were dealt with the correct quantization from the viewpoint of an observer at infinity in the rotating BTZ black hole case, and it was shown that the leading order divergence from the superradiant modes cancels the leading order divergence from the non-superradiant modes. Recently, Kenmoku et al. [15] evaluated the scalar field contribution to the rotating black hole entropy in an arbitrary D-dimensional spacetime. Although their result contains the contribution from the superradiant modes, in their calculation they did not separate the contributions from the superradiant and the non-superradiant modes. In the rotating BTZ black hole case, their result coincides with the one obtained in [14]. In the four-dimensional Kerr–Newman black hole case, besides the result of [15], so far there has been no correct calculation of the entropy from the superradiant part from the conventional viewpoint as in [14] in which the superradiant modes are considered to have positive frequencies.

In this paper, we reanalyze these results critically from a consistent setting and calculate the entropy of the Kerr–Newman black hole from the viewpoint of a rest observer at infinity (ROI) following the canonical quantization approach given in [16] as it was done in [14]. This result coincides with the result of [15]. We then calculate the entropy of the Kerr–Newman black hole from the viewpoint of a zero angular momentum observer (ZAMO) near the horizon. The result coincides with that from the ROI’s viewpoint.

The organization of the paper is as follows. In section 2, we calculate the entropy of a rotating black hole with a scalar field in the canonical quantization approach from the ROI’s viewpoint. In section 3, we calculate the entropy of a rotating black hole from ZAMO’s viewpoint. In section 4, we conclude with discussion.

2. Entropy from the viewpoint of ROI

In this section, we calculate the entropy of the Kerr–Newman (KN) black hole from the viewpoint of ROI using the brick-wall method with a massless real scalar field. The Kerr–Newman black hole solution is given by [13, 15, 17]

\[
\text{d}s^2 = g_{tt} \, \text{d}t^2 + 2g_{t\phi} \, \text{d}t \, \text{d}\phi + g_{\phi\phi} \, \text{d}\phi^2 + g_{rr} \, \text{d}r^2 + g_{\theta\theta} \, \text{d}\theta^2,
\]

where

\[
g_{tt} = \frac{\Delta - a^2 \sin^2 \theta}{\Sigma}, \quad g_{t\phi} = -\frac{a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma}, \quad g_{\phi\phi} = \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta, \quad g_{rr} = \frac{\Sigma}{\Delta}, \quad g_{\theta\theta} = \Sigma
\]
and
\[ \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2 + Q^2. \] (3)

Here \( M, \alpha, Q \) are mass, angular momentum per unit mass and charge of the black hole, respectively.

The Kerr–Newman black hole has two coordinate singularities, the outer and inner horizons, \( r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \) subject to a condition \( M^2 \geq a^2 + Q^2 \). The outer horizon is defined as the event horizon. The Kerr–Newman metric has a stationary limit surface which is the boundary of the ergoregion defined by \( r_{\text{erg}} = M + \sqrt{M^2 - a^2 \cos^2 \theta - Q^2} \). In the ergoregion, a particle cannot remain at rest as viewed from ROI.

Now we consider a quantum gas of scalar particles confined in a box near the horizon. With the metric (1), the matter action for a massless real scalar field \( \Phi_1 \) is given by
\[ I_{\text{matt}} = \int d^4x \sqrt{-g} \left( -\frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi \right). \] (4)
The resulting equation of motion is given by
\[ \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} \partial^{\mu} \Phi) = 0 \] (5)
with the boundary conditions
\[ \Phi(x) = 0 \quad \text{for} \quad r \leq r_+ + \epsilon \quad \text{and} \quad r \geq L. \] (6)
Here, \( r_+ \) is the outer horizon and \( r_+ + \epsilon \) and \( L \) are the boundaries of the spherical box assumed in the brick-wall model. We assume that the quantum gas is in thermal equilibrium with a black hole at temperature \( T \) and constant angular velocity \( \Omega \) with respect to a ROI inside the spherical box.

In the WKB approximation, assuming the wavefunction to be \( \Phi(x) = e^{-i\omega t + i\phi + iK(r, \theta)} \), the field equation (5) yields the following constraint condition:
\[ \frac{p_r^2}{g_{r r}} + \frac{p_\theta^2}{g_{\theta \theta}} = \frac{1}{\Gamma} \left( \omega^2 g_{\phi \phi} + 2\omega m g_{\phi \theta} + m^2 g_{tt} \right), \] (7)
where \( p_r = \partial K / \partial r, p_\theta = \partial K / \partial \theta, -\Gamma \equiv g_{t \phi}^2 - g_{tt} g_{\phi \phi} = \Delta \sin^2 \theta \geq 0 \) for \( r \geq r_+ \), and \( p_\phi = m \) is the azimuthal quantum number. The constraint condition (7) can be rewritten as
\[ \frac{p_r^2}{g_{r r}} + \frac{p_\theta^2}{g_{\theta \theta}} = \frac{g_{\phi \phi}}{-\Gamma} (\omega - m \Omega_+) (\omega - m \Omega_-), \] (8)
where
\[ \Omega_{\pm}(r, \theta) = -\frac{g_{\phi \phi}}{g_{t \phi}} \pm \sqrt{\left( \frac{g_{t \phi}}{g_{\phi \phi}} \right)^2 - \frac{g_{tt}}{g_{\phi \phi}}} \] (9)
are the maximum and minimum angular velocities that a particle can have. The limited range of the angular velocity is due to the restriction that no particle can move faster than light.

According to the semiclassical quantization rule, the number of modes is given by [3]
\[ \pi n(\omega, m) = \int_{r_+ + \epsilon}^{L} dr k(r; \omega, m). \] (10)
In order to get the expression for \( k(r; \omega, m) \) we first define the number of modes \( n(\omega) \) with energy not exceeding \( \omega \) as the sum over the phase space divided by the unit quantum volume \( (2\pi)^3 \) (with \( \hbar = 1 \)):
\[ (2\pi)^3 n(\omega) = \int dr dp_r d\theta dp_\theta d\phi dp_\phi. \] (11)
After integrating over the momenta $p_r$ and $p_\theta$ and setting $p_\phi = m$, the number of modes $n(\omega)$ can be expressed as follows:

$$n(\omega) = \frac{1}{8\pi^2} \int dr \, d\theta \, d\phi \, dm \sqrt{g_{rr}g_{\theta\theta}} \left( g_{\phi\phi}^{-1} \right) (\omega - m\Omega_+) (\omega - m\Omega_-). \tag{12}$$

Note that in performing the above integration a condition that the right-hand side of equation (8) be positive should be satisfied. This restriction comes from the way we evaluate the $p_\theta$ and $p_r$ integrals by calculating the area of the $p_\theta$–$p_r$ ellipse satisfying condition (8).

Therefore the radial wave number $k(r; \omega, m)$ defined in (10) is given as follows with the restriction that the right-hand side of equation (8) be positive:

$$k(r; \omega, m) = \int d\theta \, d\phi (N^{-2} (\omega - m\Omega_+) (\omega - m\Omega_-)). \tag{13}$$

where

$$N^{-2} = (-g^{zz})(g_{rr}g_{\theta\theta})^{1/2}/8\pi \quad \text{with} \quad g^{zz} = g_{\phi\phi}/\Gamma.$$ 

Since $k(r; \omega, m) \propto N^{-2}$ and $N \rightarrow 0$ near the horizon, the dominant contribution in equation (10) comes from integrating near the horizon region as the inner boundary of the brick wall approaches to it. We can see this by evaluating the number of modes per invariant volume $G(r, \theta; \omega, m)$ for given $\omega$ and $m$, which is given by the following relation:

$$n(\omega, m) = \int d\mu_{\text{inv}} G(r, \theta; \omega, m), \tag{14}$$

where

$$G(r, \theta; \omega, m) = \frac{1}{8\pi^2} \sqrt{g_{\phi\phi}} (\omega - m\Omega_+) (\omega - m\Omega_-)/(-\Gamma) \quad \text{and} \quad d\mu_{\text{inv}} = dr \, d\theta \, d\phi \sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}.$$ 

The number of modes per invariant volume is divergent at the horizon and decreases very rapidly as $r$ increases. Thus we introduce an ultraviolet cutoff at $r_+ + \epsilon$ near the horizon. The term $\sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}$ in the invariant volume diverges at a large distance. Therefore we also introduce an infrared cutoff at a large distance $L$.

The degrees of freedom are mostly concentrated near the horizon, and the main contribution to the entropy of the system comes from this region. This behavior is shown in figure 1, where $r_0$ is a chosen position at which the angular velocity of ZAMO at a given $(r, \theta)$ position ($\Omega_0 := \Omega_0(r, \theta) = -g_{\phi\phi}/g_{\phi\phi}$) differs by 1% from the angular velocity at the horizon ($\Omega_H = \Omega|_{r=r_m}$) to indicate how the degrees of freedom change as the distance from the horizon changes.

Both $\Omega_0$ converge to $\Omega_H$ near the horizon and vanish at infinity, so the angular velocity of particles near the horizon can be always thought of as $\Omega_H$. In particular, the maximum value $\Omega_+$ becomes the value $\Omega_H$ for a certain radius $r_m$. Since the angular velocity of a particle is less than $\Omega_H$ outside $r_m$, thermal equilibrium cannot be achieved for $r > r_m$. Thus the outer brick wall should be located inside the radius $r_m$, namely $L < r_m$. This is shown in figure 2.

In this paper, we assume that thermal equilibrium is maintained in the near horizon region which we are dealing with. Therefore, we can always regard the angular velocity of ZAMO at $(r, \theta)$ inside the brick wall, $\Omega_0(r, \theta)$, to be roughly equal to the horizon angular velocity $\Omega_H$, namely $\Omega_0(r, \theta) \simeq \Omega_H$ near the horizon, i.e., the range of integration giving dominant contributions.

We now consider the partition function for the scalar field following the work of [14]

$$Z(\beta, \Omega) = \text{Tr} \, e^{-\beta(\hat{H} - \Omega J)}, \tag{15}$$
The number of modes per invariant volume is shown at various polar angles. It is not changed by $\theta \rightarrow \pi - \theta$. Here, $r_0$ is a position at which $\Omega_0$ differs by 1% from $\Omega_H$ (see figure 2).

Possible angular velocities of particles are shown at $\theta = \pi/2$.

where $\hat{H}$ and $\hat{J}$ are the normal-ordered Hamiltonian and angular momentum operators of the quantized field and $\beta$ is the inverse temperature $T^{-1}$ with $k = 1$. By using the one-particle spectrum, we obtain the free energy from the partition function as

$$\beta F = -\ln Z = \sum_x \ln \sum_k [1 - e^{-\beta(\epsilon_k - \Omega_H)}]^k,$$

(16)
where $\lambda$ denotes the one-particle states for the free gas in the system and the occupation number $k$ takes the values $0, 1, 2, \ldots$, while $\varepsilon_k$ and $j_k$ are expectation values of the normal ordered $\hat{H}$ and $\hat{J}$ in the one-particle state $|1_\lambda\rangle$.

When we quantize matter fields in a stationary rotating black hole geometry, we have to take extra care to properly include the contribution from the superradiant modes which arise due to the presence of the ergoregion. Previously, the works of [10–13] failed to include this contribution properly. Following the guideline given in [4, 16], we now carry the canonical quantization for a scalar field in a rotating black hole system from the viewpoint of ROI. Here, we assume that the set $\varphi_{\omega,m}$ is the mode solutions satisfying the Klein–Gordon equation (5) and the constraint condition (8). We also exclude the mode solutions with the complex frequency which appeared in the context of vacuum instability related to the ergoregion [17–23]. We define the inner product of the scalar field for its norm following [24]:

$$\langle \Phi_1, \Phi_2 \rangle = \frac{i}{2} \int_{\mathcal{M}} \Phi_1^* \partial_\mu \Phi_2 d\Sigma^\mu.$$  \hfill (17)

Using the above definition, we obtain

$$\langle \varphi_{\omega,m}, \varphi_{\omega',m'} \rangle = \frac{i}{2} \int \varphi_{\omega,m}^* (\partial_t + \Omega_0 \partial_\phi) \varphi_{\omega',m'} N^{-1} d\Sigma,$$  \hfill (18)

where $d\Sigma^\mu = n^\mu d\Sigma, n^\mu = N^{-1}(\partial_t + \Omega_0 \partial_\phi)^\mu, N = (\omega'')^{-1/2}$. Then in the near horizon approximation $\Omega_0 \simeq \Omega_H$, the norm of a mode solution is given by

$$\langle \varphi_{\omega,m}, \varphi_{\omega,m} \rangle \simeq \int (\omega - m\Omega_H) |\varphi_{\omega,m}|^2 N^{-1} d\Sigma,$$  \hfill (19)

where $\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$. In order to make the norm positive $\omega$ and $m$ should satisfy the following condition:

$$\omega - m\Omega_H > 0.$$  \hfill (20)

There are two ways of imposing this condition: one way is to choose the frequency to be positive following the conventional interpretation as in [12–14], and the other way is to leave the frequency to be real as it is given in [15]. Now, we explain how the superradiant modes appear in these two approaches. (1) When $\omega > 0$, condition (20) becomes $\omega > m\Omega_H$ for any $m \in \mathbb{Z}$. (2) When $\omega < 0$, this condition becomes $0 > \omega > m\Omega_H$ for $m \in \mathbb{Z}_-$ only. In the conventional approach, condition (1) corresponds to the non-superradiant modes, and condition (2) corresponds to the superradiant modes. This can be seen by redefining $(\omega, m) := (-\tilde{\omega}, -\tilde{m})$ such that new $(\tilde{\omega}, \tilde{m})$ satisfies the conventional condition for superradiant modes $0 < \tilde{\omega} < \tilde{m}\Omega_H$ [14]. In the second approach [15], $\omega$ is allowed to have any real value and thus no need for the above separation. However, the superradiant modes should exist in this case too, and they correspond to the negative values of $\omega$. In the conventional canonical quantization procedure, one adopts the first approach [14], where one sets $(\varepsilon_\lambda, j_\lambda) = (\omega, m)$ for the non-superradiant (NS) modes and $(\bar{\varepsilon}_\lambda, \bar{j}_\lambda) = (-\omega, -m)$ for the superradiant (SR) modes with $\tilde{\omega} > 0, \tilde{m} \in \mathbb{Z}_+$. We can understand the above assignment as follows. In equation (16), if we naively assign $(\varepsilon_\lambda, j_\lambda)$ with positive $(\tilde{\omega}, \tilde{m})$, then the exponent $-\beta(\varepsilon_\lambda - \Omega j_\lambda)$ becomes positive since $0 < \tilde{\omega} < \tilde{m}\Omega_H$ for the superradiant modes. Consequently, the free energy in equation (16) cannot be well defined. On the other hand, if we assign $(\varepsilon_\lambda, j_\lambda)$ with negative $(-\tilde{\omega}, -\tilde{m})$, then the exponent $-\beta(\varepsilon_\lambda - \Omega j_\lambda)$ becomes negative for the superradiant modes and the free energy is well defined. Thus, the full set of solutions forms a complete basis with positive frequency whose norms are positive definite as follows:

$$\langle \varphi_{\omega,m}, \varphi_{\omega',m'} \rangle = \delta(\omega - \omega') \delta_{mm'} \quad \text{with} \quad \omega > 0, m \in \mathbb{Z}, \omega > m\Omega_H \quad \text{for NS},$$

$$\langle \varphi_{-\tilde{\omega},-\tilde{m}}, \varphi_{-\tilde{\omega}',-\tilde{m}'} \rangle = \delta(\tilde{\omega} - \tilde{\omega}') \delta_{\tilde{m}\tilde{m}'} \quad \text{with} \quad \tilde{\omega} > 0, \tilde{m} \in \mathbb{Z}_+, 0 < \tilde{\omega} < \tilde{m}\Omega_H \quad \text{for SR}.$$  \hfill (21)
The quantized scalar field can be expanded in terms of normal mode solutions as

$$\phi(x) = \sum_{\lambda \in \text{NS}} (a_{\omega, m} \phi_{\omega, m} + a_{\omega, m}^\dagger \phi_{\omega, m}^*) + \sum_{\lambda \in \text{SR}} (a_{-\tilde{\omega}, -\tilde{m}} \phi_{-\tilde{\omega}, -\tilde{m}} + a_{-\tilde{\omega}, -\tilde{m}}^\dagger \phi_{-\tilde{\omega}, -\tilde{m}}^*),$$

(22)

where $\lambda$ denotes the mode set $(\omega, m)$ and $(-\tilde{\omega}, -\tilde{m})$ for NS and SR modes, respectively. This can also be seen from the Hamiltonian operator in the ROI viewpoint

$$H = \sum_{\lambda \in \text{NS}} \omega \left( N_{\omega, m} + \frac{1}{2} \right) + \sum_{\lambda \in \text{SR}} (-\tilde{\omega}) \left( N_{-\tilde{\omega}, -\tilde{m}} + \frac{1}{2} \right),$$

(23)

where $N_{\omega, m} = a_{\omega, m}^\dagger a_{\omega, m}$ and $N_{-\tilde{\omega}, -\tilde{m}} = a_{-\tilde{\omega}, -\tilde{m}}^\dagger a_{-\tilde{\omega}, -\tilde{m}}$ are number operators. The angular momentum operator $J$ can be quantized by the same procedure too. If we define the vacuum state and one-particle states by the standard procedure, one can set $(\epsilon_{\lambda, j\lambda}) = (\omega, m)$ for the non-superradiant modes and $(\tilde{\epsilon}_{\lambda, j\lambda}) = (-\tilde{\omega}, -\tilde{m})$ for the superradiant modes in (16).

In the second approach [15], all the modes $(\omega, m)$ satisfy $\omega - m \Omega_H > 0$ giving the positive norm in equation (19). The frequency $\omega$ becomes negative for negative $m$. From the viewpoint adopted above, this can be understood as follows. We understood the conventional approach of [14] by introducing $\tilde{\omega}, \tilde{m}$ which have positive values satisfying the condition for the superradiant modes, $\tilde{\omega} < \tilde{m} \Omega_H$, and assigned them with negative of $(\epsilon_{\lambda, j\lambda})$. Thus the values of the original indices for the states $(\epsilon_{\lambda, j\lambda})$ run as follows:

$$(\epsilon_{\lambda, j\lambda}) = (\omega, m) \quad \text{for} \quad \text{NR modes with} \quad \omega > m \Omega_H,$$

$$(\tilde{\epsilon}_{\lambda, j\lambda}) = (-\tilde{\omega}, -\tilde{m}) \quad \text{for} \quad \text{SR modes with} \quad \tilde{m} \Omega_H > \tilde{\omega} > 0.$$  

Now, if we denote the indices $\epsilon_{\lambda, j\lambda}$ back into the original indices $\omega, m$, then $\omega$ and $m$ satisfy the original condition (20). This viewpoint was taken in [15]. Namely, in the SR modes the frequency $\omega$ becomes negative for negative $m$ [15]. Thus, the scalar field can be expanded as

$$\varphi(x) = \sum_{\lambda} (a_{\omega, m} \phi_{\omega, m} + a_{\omega, m}^\dagger \phi_{\omega, m}^*),$$

(24)

where the modes $\lambda = (\omega, m)$ satisfy the condition $\omega - m \Omega_H > 0$ for $\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$. Given the above analysis, the two approaches should yield the same result.

In this paper, we follow the first approach to evaluate the entropy of the scalar field in the rotating black hole case which has not been done before and will compare it with the result obtained in [15]. In the ensuing calculations that we perform in the conventional approach, we remove the tilde from $\tilde{\omega}, \tilde{m}$ in the SR modes for briefness.

In the conventional approach the free energy can be expressed with the sum of the NS and SR modes,

$$F = F_{\text{NS}} + F_{\text{SR}},$$

(25)

where the free energies of the NS and SR modes are given by

$$\beta F_{\text{NS}} = \sum_{\lambda \in \text{NS}} \int d\omega g(\omega, m) \ln[1 - e^{-\beta(\omega - m \Omega_H)}],$$

$$\beta F_{\text{SR}} = \sum_{\lambda \in \text{SR}} \int d\omega g(\omega, m) \ln[1 - e^{\beta(\omega - m \Omega_H)}].$$

(26)

(27)
Note that the density functions are given by $g(\omega, m) = \partial n(\omega, m)/\partial \omega$ and $g(\omega, m) = -\partial m(\omega, m)/\partial \omega$ for the NS and SR modes, respectively [9, 14].

Using the number of modes given by (10) and (13), we now calculate the free energy of the total system. The free energy for NS modes, equation (26), can be rewritten as

$$\beta F_{NS} = \frac{1}{\pi} \int_{m> m_{H}} \frac{d\omega}{\omega} \left[ \frac{1}{\pi} \int_{r_{+}}^{L} dk(r; m_{+}, m_{-}) \ln[1 - e^{-\beta(\omega - m_{-})}] \right]$$

where we integrated by parts with respect to $\omega$. For a computational convenience, we divide $F_{NS}$ into two parts

$$F_{NS} = F_{NS}^{(m>0)} + F_{NS}^{(m<0)},$$

which are given by

$$F_{NS}^{(m>0)} = -\frac{1}{\pi} \int \int d\theta d\phi \int_{r_{+}}^{Lr_{-}} dr \int_{m_{+}}^{m_{-}} dm \int_{m_{H}}^{\infty} d\omega \omega^{N-2} \frac{(\omega - m_{+})(\omega - m_{-})}{e^{\beta(\omega - m_{-})} - 1},$$

$$F_{NS}^{(m<0)} = -\frac{1}{\pi} \int \int d\theta d\phi \int_{r_{-}}^{r_{+}} dr \int_{m_{H}}^{m_{-}} dm \int_{m_{+}}^{\infty} d\omega \omega^{N-2} \frac{(\omega - m_{+})(\omega - m_{-})}{e^{\beta(\omega - m_{+})} - 1} - \frac{1}{\pi \beta} \int \int d\theta d\phi \int_{r_{-}}^{r_{+}} dr \int_{m_{H}}^{m_{-}} dm \omega^{N-2} \omega_{-} \ln(1 - e^{-\beta m_{-}}),$$

where we regarded the quantum number $m$ as a continuous variable. Here, we divided the range of $r$ integration into two parts for the negative $m$ case, since the minimum angular velocity $\Omega_{-}$ has positive value before $r_{eq}$ and has negative value beyond $r_{eq}$ as shown in figure 2.

On the other hand, the free energy of SR modes, equation (27), becomes

$$F_{SR} = -\frac{1}{\pi} \int \int d\theta d\phi \int_{r_{-}}^{r_{+}} dr \int_{m_{+}}^{m_{-}} dm \int_{m_{H}}^{\infty} d\omega \omega^{N-2} \frac{(\omega - m_{+})(\omega - m_{-})}{e^{\beta(\omega - m_{+})} - 1} + \frac{1}{\pi \beta} \int \int d\theta d\phi \int_{r_{-}}^{r_{+}} dr \int_{m_{H}}^{m_{-}} dm \omega^{N-2} \Omega_{-} \ln(1 - e^{-\beta m_{+}}).$$

We note that the second terms in (31) and (32) exactly cancel each other in the total free energy as in the BTZ case [14]. In evaluating the remaining terms, we adopt the near horizon approximation in which we assume that the dominant contribution comes only from the radial integration around the near horizon region, since in the remaining regions the degrees of freedom are negligible as shown in figure 1. Then equations (30)–(32) now yield the following:

$$F_{NS}^{(m>0)} = -\frac{\xi(4)}{\beta^4} \left[ \frac{1}{\pi} \int d\theta \frac{(r_{+}^2 + a^2)^{3/2}}{(r_{+} - r_{-})^{2}} \Sigma_{+} \left( \frac{1}{\epsilon} + F_{2} \ln \left( \frac{r_{+}}{\epsilon} \right) \right) - \frac{3(r_{+}^2 + a^2)^{3/2}}{a(r_{+} - r_{-})^{3/2}} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}) \right],$$

(33)
\[
F_{\text{MS}}^{(m<0)} = \frac{\xi(4)}{\beta^4} \left[ \frac{2(r_*^2 + a^2)^3}{\alpha(r_* - r_-)^{3/2}} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}) \right],
\]

\[
F_{\text{SR}} = -\frac{\xi(4)}{\beta^4} \left[ \frac{1}{\pi} \int d\theta \frac{(r_*^2 + a^2)^4 \sin \theta}{(r_* - r_-)^3 \Sigma_*} \left( \frac{1}{\epsilon} + F_2 \ln \left( \frac{r_*}{\epsilon} \right) \right) + \frac{(r_*^2 + a^2)^3}{\alpha(r_* - r_-)^{3/2}} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}) \right],
\]

where
\[
F_2 = 2 \left[ \frac{2(\Sigma_* r_+ a^2 \sin^2 \theta + r_+ [2r_*^4 + r_*^2 a^2 (5 \cos^2 \theta - 1) + a^4 (\cos^4 \theta + 3 \cos^2 \theta - 2)])}{(r_*^2 + a^2)^2 \Sigma_*} \right.
- \frac{1}{r_* - r_-}. \]

Using the thermodynamic relation, \( S = \beta^2 \theta F / \partial \beta |_{\beta=\beta_H} \), we obtain the entropy of the system from (33)–(35) as
\[
S_{\text{MS}} = \frac{\xi(4)}{16\pi^4} \left( \frac{r_* - r_-}{r_*^2 + a^2} \right)^3 \left[ \int d\theta \frac{(r_*^2 + a^2)^4 \sin \theta}{(r_* - r_-)^3 \Sigma_*} \left( \frac{1}{\epsilon} + F_2 \ln \left( \frac{r_*}{\epsilon} \right) \right) - \frac{\pi (r_*^2 + a^2)^{3/2}}{a} \frac{1}{\sqrt{\epsilon}} \right],
\]

\[
S_{\text{SR}} = \frac{\xi(4)}{16\pi^4} \left( \frac{r_* - r_-}{r_*^2 + a^2} \right)^3 \left[ \int d\theta \frac{(r_*^2 + a^2)^4 \sin \theta}{(r_* - r_-)^3 \Sigma_*} \left( \frac{1}{\epsilon} + F_2 \ln \left( \frac{r_*}{\epsilon} \right) \right) + \frac{\pi (r_*^2 + a^2)^{3/2}}{a} \frac{1}{\sqrt{\epsilon}} \right],
\]

where we used the relation between the Hawking temperature \( \beta_H^{-1} \) and the surface gravity \( \kappa_H \) of the Kerr–Newman black hole, \( \beta_H^{-1} = \kappa_H / 2\pi = (r_* - r_-) / 4\pi (r_*^2 + a^2) \). Note that the \( 1/\sqrt{\epsilon} \) order terms in (37) and (38) cancel each other even when \( \epsilon = \theta \) dependent. The total entropy is now given by
\[
S = S_{\text{MS}} + S_{\text{SR}} = \frac{\xi(4)}{8\pi^4} \int d\theta \frac{(r_*^2 + a^2)(r_* - r_-) \sin \theta}{\Sigma_*} \left( \frac{1}{\epsilon} + F_2 \ln \left( \frac{r_*}{\epsilon} \right) \right) + O(\sqrt{\epsilon}).
\]

where \( \Sigma_* = r_*^2 + a^2 \cos^2 \theta \). The above obtained total entropy exactly matches with the result of [15] except for the following: the logarithmically divergent subleading term is absent in [15]. This is because in [15] they restricted their concern to black hole singularities only up to simple zeros at the horizon. Thus, if they expanded the radial integral to the next order, they would get the same logarithmically divergent subleading term as in (39).

Finally, by introducing a new cutoff parameter \( \bar{\epsilon} \) instead of the original cutoff parameter \( \epsilon \) defined by the following relation,
\[
\frac{1}{\bar{\epsilon}^2} = \frac{1}{2880\pi} \left( \int_0^\pi \frac{d\theta}{\Sigma_*} \sin \theta \right) \frac{1}{\epsilon} = \frac{(r_* - r_-) \tan^{-1}(a/r_+)}{1440\pi r_+ a} \frac{1}{\bar{\epsilon}},
\]

we can express the total entropy in terms of the surface area of the event horizon,
\[
S = \left[ A_H / \bar{\epsilon}^2 + \frac{A_H (r_* - r_-)}{720\pi} \right] \left( \int_0^\pi \frac{d\theta}{\Sigma_*} \sin \theta \right) \ln \left( \frac{r_*}{\bar{\epsilon}} \right) + O(\sqrt{\bar{\epsilon}}).
\]

where \( A_H = 4\pi (r_*^2 + a^2) \) is the surface area of the horizon and the coefficient of the logarithmically divergent second term is finite. Choosing the invariant cutoff parameter \( \bar{\epsilon} \) to be twice the Planck length \( \bar{\epsilon} = 2l_p \) \( (l_p^2 = G_N) \), we retrieve the Bekenstein–Hawking relation in the leading order.
3. Entropy from the viewpoint of ZAMO near the horizon

In this section, we calculate the entropy from the viewpoint of ZAMO near the horizon. We first consider the near horizon line element of the Kerr–Newman black hole in the coordinates rotating with angular velocity of the black hole. Now we rewrite the Kerr–Newman metric (1) as

\[ ds^2 = \frac{1}{g_{tt}} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} (d\phi + \Omega_0 dt)^2, \]  

(42)

where \( \Omega_0(r, \theta) = -g_{t\phi}/g_{\phi\phi} \). Using the coordinate transformation

\[ \phi = \phi - \frac{\Omega_1}{\Omega_1 H} t, \]  

(43)

we change the metric (42) into the coordinates rotating with the angular velocity of the horizon, that is given by

\[ ds^2 = \frac{1}{g_{tt}} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} (d\phi - (\Omega_0 - \Omega_H) dt)^2. \]  

(44)

The above metric is diagonal only in the vicinity of the horizon due to the coordinate dependence of \( \Omega_0(r, \theta) \) which becomes \( \Omega_1 H \) at the horizon.

In the WKB approximation with \( \Psi(x) = e^{-iE t - i m \phi} \), the field equation (5) with (44) gives the following constraint condition near the horizon as

\[ \frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} = (-g^\theta\theta)([E - m(\Omega_0 - \Omega_H)]^2 - m^2 \tilde{\Omega}^2), \]  

(45)

\[ \approx (-g^\theta\theta)(E^2 - m^2 \tilde{\Omega}^2), \]  

(46)

where \( \tilde{\Omega} = (g^\theta\theta g_{\phi\phi})^{-1/2} \). Here we note that the constraint condition (46) for the momenta is the same as that in (7) obtained near the horizon from the ROI’s viewpoint if we set \( E := \omega - m \Omega_H \).

On the other hand, a ZAMO needs to measure the physical quantities locally in evaluating the entropy of the system. This observer measures local inverse temperature as \( \beta = \beta_H \sqrt{-g_{tt}} \) and energy as \( \epsilon = E/\sqrt{-g_{tt}} \), where \( \beta_H \) is the inverse Hawking temperature of the black hole measured by the ROI and \( \tilde{g}_{tt} := g_{tt} + 2 g_{t\phi} \Omega_0 + g_{\phi\phi} \Omega_0^2 = 1/g_{tt} \) [26]. Here, the combined product \( \beta \epsilon \) from the ZAMO’s viewpoint remains the same as \( \beta_H E \) from the ROI’s viewpoint as it was discussed in [26]. To assume thermal equilibrium near the horizon, we regard the local inverse temperature \( \beta \) to be approximately constant near the horizon as we did in the ROI case. Now we change the temperature and energy from the ROI viewpoint to the local temperature and energy measured by the ZAMO near the horizon and they can be done by approximating to \( \Omega_0 \approx \Omega_H \) from the above discussion.

Then the constraint condition (46) can be written in the near horizon approximation as

\[ \frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} = \left( \epsilon^2 - \frac{m^2}{g_{\phi\phi}} \right). \]  

(47)

The number of modes \( n(\epsilon, m) \) with energy less than \( \epsilon \) and with a fixed \( m \) can be calculated by integrating over the phase space. In the present case, the number of modes is given by

\[ \pi n(\epsilon, m) = \int_{r, r}^{L} dr k(r; \epsilon, m), \]  

(48)

where the radial wave number \( k(r; \epsilon, m) \) can be evaluated as in the previous section and is given by as follows with the restriction that the right-hand side of equation (47) be positive:

\[ k(r; \epsilon, m) = \frac{1}{8\pi^2} \int d\theta d\phi dp_{\phi} dp_r = \frac{1}{8\pi} \int d\theta d\phi (g_{rr} g_{\theta\theta})^{1/2} \left( \epsilon^2 - \frac{m^2}{g_{\phi\phi}} \right). \]  

(49)
Here we can expect that the entropy calculated by the ZAMO will be the same with one by the ROI as follows. We note that the number of modes in (48) is the same as that in (10) when \( \omega \) is replaced by \( E \). Then the term \( \beta F \) will be unchanged by using the invariance \( \beta \epsilon = \beta_H E \). Furthermore, the entropy \( S \) can also be written as

\[
S = \beta^2 \frac{\partial F}{\partial \beta} = \left( \beta \frac{\partial}{\partial \beta} - 1 \right) \beta F.
\]  

Thus we expect that the entropies are the same in two viewpoints; ROI and ZAMO.

The free energy is then given by

\[
\beta F = \sum_m \int d\epsilon \frac{\partial}{\partial \epsilon} \left[ \frac{1}{\pi} \int_{r_{r+}}^L dr k(r; \epsilon, m) \right] \ln[1 - e^{-\beta \epsilon}]
\]

\[
= -\frac{\beta}{\pi} \sum_m \int_{r_{r+}}^L dr \int d\epsilon \left[ \frac{k(r; \epsilon, m)}{e^{\beta \epsilon} - 1} \right]
\]

\[
+ \frac{1}{\pi} \sum_m \int_{r_{r+}}^L dr k(r; \epsilon, m) \ln[1 - e^{-\beta \epsilon}] \bigg|_{\epsilon_{\text{min}}(m)}^{\epsilon_{\text{max}}(m)}.
\]  

Note that there is no contribution from superradiant modes since there is no rotation of the frame from ZAMO’s viewpoint. For convenience, we also divide the free energy into \( m > 0 \) and \( m < 0 \) parts due to the restriction that the right-hand side of equation (45) be positive,

\[
F = F^{(m>0)} + F^{(m<0)},
\]  

where the two parts are given by

\[
F^{(m>0)} = -\frac{1}{\pi} \int d\theta \int d\phi \int_{r_{r+}}^L dr \int_0^\infty dm \int_{m/\sqrt{g_{\phi\phi}}}^\infty d\epsilon \frac{k(r; \epsilon, m)}{e^{\beta \epsilon} - 1},
\]  

\[
F^{(m<0)} = -\frac{1}{\pi} \int d\theta \int d\phi \int_{r_{r+}}^L dr \int_{-\infty}^0 dm \int_{-m/\sqrt{g_{\phi\phi}}}^{\infty} d\epsilon \frac{k(r; \epsilon, m)}{e^{\beta \epsilon} - 1}.
\]  

Now, the total free energy can be written as

\[
F = -\frac{2}{\pi} \int d\theta \int d\phi \int_{r_{r+}}^L dr \int_0^\infty dm \int_{m/\sqrt{g_{\phi\phi}}}^\infty d\epsilon \frac{k(r; \epsilon, m)}{e^{\beta \epsilon} - 1},
\]

\[
= -\frac{1}{4\pi^2} \int_{r_{r+}}^L dr \int d\theta \int d\phi (g_{rr} g_{\theta\theta})^{1/2} \int_0^\infty dm G_m|_{m/\sqrt{g_{\phi\phi}}},
\]  

where

\[
G_m(\epsilon) = \int d\epsilon \left( \frac{\epsilon^2 - m^2}{\epsilon^2 - \frac{m^2}{g_{\phi\phi}}} \right).
\]  

The integration (56) can be done straightforwardly, and we get

\[
\int_0^\infty dm G_m|_{m/\sqrt{g_{\phi\phi}}} = \frac{\zeta(4)\Gamma(4)}{3\beta^4} (g_{\phi\phi})^{1/2}.
\]  

Thus, the total free energy is given by

\[
F = -\frac{\zeta(4)}{\pi^2} \int_{r_{r+}}^L dr \int d\theta \int d\phi \frac{(g_{rr} g_{\theta\theta})^{1/2}}{\beta^4}.
\]
Using the thermodynamic relation \( S = \beta^2 \frac{\partial F}{\partial \beta} \bigg|_{\beta=\tilde{\beta}} \), the entropy of the system is given by

\[
S = \frac{4 \zeta(4)}{\pi^2} \int_{\epsilon}^{L} dr \int_{r_s}^{r_c} d\theta \int_{\phi_s}^{\phi_c} d\phi \sqrt{g_{rr} g_{\theta\theta} g_{\phi\phi}} \frac{1}{\beta^3},
\]

where we used the relation \( \tilde{\beta} = \beta_H / \sqrt{-g_{tt}} \). Evaluating the entropy with the near horizon approximation as we did in the ROI case, (59) becomes

\[
S = \frac{\zeta(4)}{8\pi^2} \int d\theta \left[ \left( \frac{r_s^2 + a^2}{r_s - r_c} \right) \sin \theta \left( \frac{1}{\epsilon} + F_2 \ln \left( \frac{L}{\epsilon} \right) \right) + O(\sqrt{\epsilon}) \right],
\]

where \( F_2 \) is given by (36) in the previous section. This result exactly coincides with the total entropy (39) obtained from the ROI’s viewpoint. Again introducing the invariant cutoff parameter \( \bar{\epsilon} \) defined by (40) in the previous section and setting \( \bar{\epsilon} = 2l_p \), we also obtain the Bekenstein–Hawking relation in the leading order.

4. Discussion

In this paper, using the brick-wall model with a scalar field in the canonical quantization approach we investigated the entropy of the Kerr–Newman black hole from two different viewpoints, a rest observer at infinity and a zero angular momentum observer near the horizon. The results from the two viewpoints coincide exactly. This is what we expected since the total entropy of the system must be the same physical quantity regardless of an observer.

As it is well known, the superradiant modes occur due to the presence of the ergoregion in a rotating black hole system. Incorporating the superradiant modes and evaluating the entropy of rotating black hole in the brick-wall model were a bit complicated and thus raised some confusion. This issue was not settled down until the work of Ho and Kang [14]. They evaluated the entropy of three-dimensional rotating BTZ black hole correctly first time in the brick-wall model. Recently in the work of [15], the result of [14] in the three-dimensional rotating BTZ black hole case was reproduced using a rather different approach in the brick-wall model and using the same method they also obtained the entropy of four-dimensional Kerr black hole. However, the approach of [15] was not quite conventional in such a way that negative frequencies are also allowed as superradiant modes. In the conventional approach of [14] only positive frequencies are allowed, and evaluating the contribution of the superradiant modes in terms of positive frequencies was rather tricky. In the conventional treatment for the superradiant modes, the frequencies satisfy the condition \( 0 < \omega < m\Omega_H \), and in the calculation of the free energy \( (\omega, m) \) should be regarded as \( (-\omega, -m) \) as first pointed out in [14]. Before the work of [14] people simply used \( (\omega, m) \) instead of \( (-\omega, -m) \) in the evaluation of the free energy, thus obtained a divergent contribution from the superradiant modes [12, 13]. While in the approach of [15] by allowing negative frequencies the superradiant modes were not separated from the non-superradiant modes in evaluating the free energy, thus the calculation becomes simplified quite a bit. Still it was not certain whether the approach of [14] would yield the same result as that in [15] in the four-dimensional Kerr black hole case, since the relation between the two approaches was not understood clearly so far. In this paper, we explain how the approaches of [14] and [15] can be understood on the same footing and show that the approach of [14] applied to the four-dimensional Kerr black hole case actually yields the same result as in [15].
It is also well known that the superradiant modes do not occur in a rotation-free co-moving system. Thus if we evaluate the entropy in this rotation free co-moving system (ZAMO) we are free from considering the troublesome superradiant modes. Therefore, we naturally expect that the evaluation of rotating black hole entropy would be quite simpler from ZAMO’s viewpoint. The entropy calculation from ZAMO’s viewpoint has not been performed so far, thus leaving the question that whether the entropy from ZAMO’s viewpoint actually agrees with the known result from ROI’s viewpoint unanswered. We confirm this in the latter part of this work.

We expect that if we apply this equivalence between the ROI’s and ZAMO’s in evaluating the entropy in the case of Kerr–Newman–de Sitter black hole we would get the result in a quite simpler fashion without complication of the superradiant modes [27].

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