Beyond Logarithmic Corrections to Cardy Formula

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Abstract

As shown by Cardy \cite{1}, modular invariance of the partition function of a given unitary non-singular 2\textit{d} CFT with left and right central charges \( c_L \) and \( c_R \), implies that the density of states in a microcanonical ensemble, at excitations \( \Delta \) and \( \bar{\Delta} \) and in the saddle point approximation, is

\[ \rho_0(\Delta, \bar{\Delta}; c_L, c_R) = c_L \exp(2\pi \sqrt{c_L \Delta / 6}) \cdot c_R \exp(2\pi \sqrt{c_R \bar{\Delta} / 6}). \]

In this paper, we extend Cardy’s analysis and show that in the saddle point approximation and up to contributions which are \textit{exponentially suppressed} compared to the leading Cardy’s result, the density of states takes the form

\[ \rho(\Delta, \bar{\Delta}; c_L, c_R) = f(c_L \Delta) f(c_R \bar{\Delta}) \rho_0(\Delta, \bar{\Delta}; c_L, c_R), \]

for a function \( f(x) \) which we specify. In particular, we show that (i) \( \rho(\Delta, \bar{\Delta}; c_L, c_R) \) is the product of contributions of left and right movers and hence, to this approximation, the partition function of any modular invariant, non-singular unitary 2\textit{d} CFT is holomorphically factorizable and (ii) \( \rho(\Delta, \bar{\Delta}; c_L, c_R)/(c_L c_R) \) is only a function of \( c_L \Delta \) and \( c_R \bar{\Delta} \). In addition, treating \( \rho(\Delta, \bar{\Delta}; c_L, c_R) \) as the density of states of a microcanonical ensemble, we compute the entropy of the system in the canonical counterpart and show that the function \( f(x) \) is such that the canonical entropy, up to exponentially suppressed contributions, is simply given by the Cardy’s result \( \ln \rho_0(\Delta, \bar{\Delta}; c_L, c_R) \).

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1 Introduction

Conformal symmetry as a natural extension of the Poincaré symmetry has long been noted and extensively studied in the formulation of quantum field theories. In two dimensions, the conformal algebra becomes infinite dimensional and admits a central extension \( c \), the Virasoro algebra

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} n(n^2 - 1) \delta_{m+n}.
\]

As put forward and stressed in the classic work of Belavin-Polyakov-Zamolodchikov [2], conformal invariance in two dimensions can be powerful enough to render the conformal field theory (CFT) solvable. Being infinite dimensional, unitary irreducible representations (irreps) of two-dimensional conformal algebra are also infinite dimensional. Each irrep is composed of a highest weight state (corresponding to a primary operator) and its descendants, obtained by the action of \( L_n \), with \( n < 0 \), on the highest weight state. A CFT is then specified by determining the set of \( L_n \)’s and its primary fields together with their operator product expansions (OPEs). A detailed analysis of 2d CFTs and references to the original papers may be found in [3, 4].
In a unitary CFT all the highest weight states and hence all the states of the theory, have a non-negative norm. This implies that both the conformal weights $\Delta$ (eigenvalues of primary states under $L_0$) of all primary operators and the central charge $c$ are non-negative. In this work we will only be concerned with unitary 2d CFTs.

A classification of 2d CFTs may be arranged based on the value of the central charge $c$ and the spectrum of the conformal weights of primary fields. Except for some specific cases, such as CFTs with a finite number of primary operators (the minimal models) and WZNW models, 2d CFTs are not proven to be solvable [3]. In such cases one would like to know if conformal invariance teaches us anything about the spectrum and degeneracy of states of the CFT.

A crude though useful notion is whether the spectrum of the primary operators of the CFT is continuous (for singular CFTs) or discrete (for non-singular CFTs) as in the case of rational conformal field theories or their subclass of minimal models. String worldsheet theory on a $D$-dimensional flat target space, i.e. a system of $D$ free noncompact bosons, is a trivial example of singular CFT. This is due to the fact that as a result of translation symmetry of the target space, the states and their conformal weights are also labeled by the center of mass momentum which is a continuous parameter. Nonetheless, one may compactify the target space on $T^D$ and make the spectrum discrete, thus effectively turning it into a “non-singular CFT” with a discrete spectrum for primary operators [5]. In the pursuit of our investigation, we shall only consider non-singular unitary CFTs or CFTs which can be made non-singular by a “regularization” like the example of compactified string theory mentioned above. In section 2 we will give a more precise definition of what we mean by “non-singular CFT”.

2d CFTs may be defined on two-dimensional surfaces of various topology, in particular a torus $T^2$. Although not necessary, in many physically relevant cases one needs to make sure that the theory is modular invariant.\footnote{Modular invariance becomes a necessity if the 2d CFT in question is going to be viewed as worldsheet description of a string theory [5].} For the case at hand, where the torus $T^2$ is specified by the complex structure (or modular parameter) $\tau$, modular transformations are elements of $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z}) / \mathbb{Z}_2$ which act on $\tau$ (and simultaneously on $\bar{\tau}$) as

$$
\tau \rightarrow \tau' = \frac{a \tau + b}{c \tau + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSL}(2, \mathbb{Z}).
$$

(1.2)

Modular invariance then demands that $Z(\tau, \bar{\tau}) = Z(\tau', \bar{\tau'})$, where $Z$ is the partition function of the CFT on the torus. In a seminal work [1], Cardy realized and emphasized on the role of modular transformations and the restrictions modular invariance imposes on the content of
primary operators of the CFT. In the case of minimal models, modular invariance restricts the central charge to be a rational number less than one, explicitly \( c = 1 - \frac{6(p-q)^2}{pq} \), where \( p \) and \( q \) are co-prime integers \([3]\). Moreover, if we also demand unitarity, the central charge is further restricted to \( c = 1 - \frac{6}{(q+2)(q+3)} \), where \( q \) is a positive integer. The theory of free fermions on a \( T^2 \) is another example of modular invariant theory, if both periodic and anti-periodic boundary conditions are allowed.

Invariance under modular transformations has been employed in unitary non-singular CFTs to specify the density of states at a given conformal weight \( \Delta \). It was shown by Cardy \([1]\) that in the saddle point approximation and for \( \Delta \gg c \), the density of states \( \rho(\Delta) \) grows exponentially as \( \sqrt{\Delta} \)

\[
\rho(\Delta)_{\text{Cardy}} \sim \exp \left( 2\pi \sqrt{\frac{c\Delta}{6}} \right).
\] (1.3)

Cardy’s analysis may be repeated for a general 2d CFT with left and right central charges \( c_L \) and \( c_R \), at left and right excitations \( \Delta \) and \( \bar{\Delta} \) respectively, yielding

\[
\rho(\Delta, \bar{\Delta})_{\text{Cardy}} \sim e^{2\pi \sqrt{\frac{c_L\Delta}{6}}} \cdot e^{2\pi \sqrt{\frac{c_R\bar{\Delta}}{6}}}.
\] (1.4)

One can make two observations from the above Cardy formula: 1) The density of states in the saddle point approximation is the product of those in the left and right sectors or, equivalently, the partition function is the product of a holomorphic and an anti-holomorphic function which may be thought of as partition functions of the left and right sectors, respectively. In other words, in the Cardy’s saddle point approximation, modular invariance implies that any unitary non-singular 2d CFT is holomorphically factorizable. 2) The density of states is only a function of \( c \cdot \Delta \) and not of their arbitrary combination.

In what follows we expand upon Cardy’s considerations and by exploiting modular invariance further we explore the validity of the above two observations beyond Cardy’s first-order saddle point approximation. In particular, we show that for any modular invariant, unitary and non-singular 2d CFT the above two observations remain valid up to exponentially suppressed contributions, within saddle point approximation. We will discuss the precise meaning of these two expressions in sections 2 and 3. We should also mention that the expectation of

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\[ ^2 \text{PSL}(2, \mathbb{Z}) \text{ can be generated by the action of two independent elements, usually known as } S- \text{ and } T- \text{transformations, corresponding to } (a, b, c, d) = (0, -1, 1, 0) \text{ and } (1, 1, 0, 1), \text{ respectively. Our analysis here, following that of Cardy’s, is based on exploiting } S\text{-transformation. In an interesting paper, implications of } T\text{-transformation for the spectrum of modular invariant, unitary CFTs have also been discussed [6].} \]

\[ ^3 \text{As we discuss in section 5, holomorphic factorizability of a 2d CFT may also be relevant in the context of AdS/CFT and 3d quantum gravity theories [7].} \]
validity of the previous observations is compatible with the results of Carlip [8], where first order in perturbation theory beyond the Cardy formula was studied. Our analysis here is then an all-orders extension of Carlip’s results.

The rest of this paper is organized as follows. In section 2, we revisit the analysis of [8] more closely and extend it to all orders in perturbation theory. In this section, we have used the more formal language of path integral. Although a clean method in the sense of producing an all-orders result in the saddle point approximation, the path integral approach may obscure the physical intuition and some of the assumptions we have made in the course of the computation. Therefore, in section 3 we present another method based on perturbative, order by order expansion in the saddle point approximation, reproducing the results of section 2 in a Taylor series expansion. In section 4, using the expression for the microcanonical density of states obtained in sections 2 and 3, we compute the canonical partition function of the theory as well as its canonical entropy and compare the latter to the microcanonical results. In section 5, besides reviewing our results, we draw our conclusions and discuss the implications of our findings if the 2d CFT in consideration is regarded as the CFT dual to gravity on an AdS$_3$ background. In two appendices we have collected the details of some computations, the results of which have been used in sections 2 and 3.

2 Extension of Carlip’s Formulation

For a generic (Euclidean) unitary 2d CFT with left and right central charges $c_L$ and $c_R$ on a torus with complex structure $\tau = \tau_1 + i\tau_2$, the partition function is defined as

$$Z(\tau, \bar{\tau}) = \text{Tr} \left( e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0} \right) = \sum_{\Delta, \bar{\Delta}} \rho(\Delta, \bar{\Delta}) e^{2\pi i \tau \Delta} e^{-2\pi i \bar{\tau} \bar{\Delta}},$$

(2.1)

where $\bar{\tau} = \tau_1 - i\tau_2$, $\Delta$ and $\bar{\Delta}$, which we assume to take non-negative values, are respectively the spectrum of $L_0$ and $\bar{L}_0$, and $\rho(\Delta, \bar{\Delta})$ is the density of states with left and right energies $\Delta$ and $\bar{\Delta}$. Unitarity of the CFT implies that $\rho$ is positive definite (negative norm states would have contributed to the partition function with negative $\rho$). In general, $Z(\tau, \bar{\tau})$ is neither a holomorphic function of $\tau$ nor holomorphically factorizable, where $Z(\tau, \bar{\tau})$ factorizes to holomorphic and antiholomorphic parts, and the modular invariant partition function $\mathcal{Z}$ is related to $Z$ as

$$\mathcal{Z}(\tau, \bar{\tau}) = e^{-\frac{2\pi i}{24}(\tau c_L - \bar{\tau} c_R)} Z(\tau, \bar{\tau}).$$

(2.2)

If one dealt with a chiral CFT, or the trace in the partition function were only over the chiral sector of the Hilbert space of the theory (i.e. over the $\bar{L}_0 = 0$ sector), one could make use of powerful analytic (more precisely meromorphic) functional properties in
combination with modular invariance and express the partition function only in terms of its \textit{polar part} and hence arrive at the Rademacher expansion which completely specifies the partition function \cite{9,10}. We note that the Rademacher expansion can also be applied to holomorphically factorizable CFTs \cite{11}. However, these techniques are not available in general and one may wonder how far one can go relying only on modular invariance, the question we explore below.

In his influential paper \cite{1}, Cardy took the first steps into this direction, showing that for any matching “high-and-low-temperature” expansion of the partition function one may compute the density of states for any given $2d$ CFT. Cardy formula, among other things, implies that around the saddle point the partition function is the product of its holomorphic and anti-holomorphic parts. This last statement is equivalent to the fact that the total entropy of the $2d$ CFT is the sum of the entropy of left and right sectors. In \cite{8}, it was shown how one may extend Cardy’s saddle point analysis and compute the leading logarithmic correction to the Cardy formula. Remarkably, it was observed how the addition of logarithmic corrections to the partition function would not spoil its holomorphic factorizability. In this section, we wish to broaden Carlip’s method and show that indeed this \textit{approximate holomorphic factorizability} holds true to any perturbative order around the saddle point in the $1/\tau$ expansion, up to exponentially suppressed corrections. We should stress that in order to argue for our case we only resort to (i) modular invariance (ii) unitarity of the theory and (iii) that the CFT be non-singular, a term which will be defined momentarily. These are indeed very mild assumptions in the sense that they are satisfied by many CFTs.

We begin our analysis by recalling that the density of states at energies $\Delta$ and $\bar{\Delta}$ can be computed using contour integrals in the two corresponding complex planes of $q = e^{2\pi i \tau}$ and $\bar{q} = e^{-2\pi i \bar{\tau}}$

$$
\rho(\Delta, \bar{\Delta}) = \frac{1}{(2\pi i)^2} \int dq d\bar{q} \frac{1}{q^{\Delta+1}} \frac{1}{\bar{q}^{\bar{\Delta}+1}} Z(q, \bar{q}).
$$

(2.3)

Note that the variables $q$ and $\bar{q}$ are to be treated as independent variables and not as complex conjugate of each other. Equation (2.3) is the usual Laplace transform to move from canonical to microcanonical ensemble, and thus we regard $\rho(\Delta, \bar{\Delta})$ as the \textit{microcanonical} density of states. Let us also remark that for this Laplace transform to be well-defined, or stated otherwise, for (2.3) to follow from (2.1), one should assume that the spectrum of the CFT is labeled by a non-negative integer $n$ such that $\Delta_n - \Delta_0$ is a non-negative integer, where $\Delta_0$ is the ground state energy, and similarly for the anti-holomorphic sector. This requirement, explicitly having discrete and integer-valued $\Delta_n - \Delta_0$, is what defines a non-singular CFT. \footnote{Note that with this definition any rational CFT \cite{3} is non-singular while the converse is not necessarily true. See footnote 6 for further comments on this point.}
Next, we use modular invariance to relate $Z(\tau, \bar{\tau})$ to $Z(-1/\tau, -1/\bar{\tau})$. Doing so yields

$$Z(\tau, \bar{\tau}) = e^{\frac{2\pi i c L}{24}} e^{\frac{-2\pi i c R}{24}} \left( \sum_{n, \bar{n}=0} \rho(\Delta_n, \bar{\Delta}_\bar{n}) e^{-\frac{2\pi i (\Delta_n - \frac{c L}{24})}{\tau}} e^{\frac{2\pi i (\bar{\Delta}_\bar{n} - \frac{c R}{24})}{\bar{\tau}}} \right). \quad (2.4)$$

Inserting (2.4) into (2.3) we obtain

$$\rho(\Delta, \bar{\Delta}) = \sum_{n, \bar{n}=0} \rho(\Delta_n, \bar{\Delta}_\bar{n}) I(a, b_n) I(\bar{a}, \bar{b}_n), \quad (2.5)$$

where

$$I(a, b_n) = -\int_{0}^{i\infty(+)} d\tau e^{-2\pi i a \tau + 2\pi i b_n \tau} = (-i) \int_{-\infty}^{0(+)} \frac{d\tau}{\tau^2} e^{-2\pi b_n \tau - 2\pi \frac{\Delta}{\tau}}, \quad (2.6)$$

with

$$a = \Delta - \frac{c L}{24}, \quad \bar{a} = \bar{\Delta} - \frac{c R}{24},$$

$$b_n = -\Delta_n + \frac{c L}{24}, \quad \bar{b}_n = -\bar{\Delta}_n + \frac{c R}{24}. $$

In our conventions, $\Delta_n > \Delta_m$ if $n > m$ (and similarly for $\bar{\Delta}$’s). In particular, the ground state energy $\Delta_0$ is the smallest eigenvalue of $L_0$. We stress that having a non-singular CFT with a mass gap and discrete spectrum, besides in (2.3), has also been used in arriving at (2.5).

One may perform the integral (2.6) using contour integrals and the result will be independent of the details of the contour by virtue of Cauchy’s theorem [9]. It can be shown that for $a \leq 0$ ($\bar{a} \leq 0$) the integral (2.6) is zero and thus we restrict ourselves to positive $a$ ($\bar{a}$) only. Depending on the sign of $b_n$, the integral $I(a, b_n)$ is either of the form of a Bessel function of the first kind $J_n(z)$ or its modified version $I_n(z)$, for negative or for positive $b_n$, respectively. Thus, using (8.412.2) of [12], we recast (2.6) as

$$I(a, b_n) = \begin{cases} \frac{-2\pi}{\sqrt{a}} \frac{b_n}{b} I_1(4\pi \sqrt{|a| b_n}), & b_n > 0, \\ \frac{2\pi |b_n|}{\sqrt{a}} I_1(4\pi \sqrt{|a| |b_n|}), & b_n < 0. \end{cases} \quad (2.7)$$
Plugging the integrals (2.7) into (2.5), one obtains the recursion formula for \( \rho \)

\[
\rho(\Delta, \bar{\Delta}) = (2\pi)^2 \sum_{\Delta_n < 24, \Delta_n < \frac{cL}{24}} \rho(\Delta_n, \bar{\Delta}_n) \frac{|c_L - \Delta_n/24|}{u_n} I_1(4\pi u_n) \cdot \frac{|c_R - \bar{\Delta}_n/24|}{v_n} I_1(4\pi v_n) \\
- (2\pi)^2 \sum_{\Delta_n > 24, \Delta_n < \frac{cL}{24}} \rho(\Delta_n, \bar{\Delta}_n) \frac{|c_L - \Delta_n/24|}{u_n} J_1(4\pi u_n) \cdot \frac{|c_R - \bar{\Delta}_n/24|}{v_n} I_1(4\pi v_n) \\
- (2\pi)^2 \sum_{\Delta_n < 24, \Delta_n > \frac{cL}{24}} \rho(\Delta_n, \bar{\Delta}_n) \frac{|c_L - \Delta_n/24|}{u_n} I_1(4\pi u_n) \cdot \frac{|c_R - \bar{\Delta}_n/24|}{v_n} J_1(4\pi v_n) \\
+ (2\pi)^2 \sum_{\Delta_n > 24, \Delta_n > \frac{cL}{24}} \rho(\Delta_n, \bar{\Delta}_n) \frac{|c_L - \Delta_n/24|}{u_n} J_1(4\pi u_n) \cdot \frac{|c_R - \bar{\Delta}_n/24|}{v_n} J_1(4\pi v_n),
\]

(2.8)

where

\[
u_n = \sqrt{|c_L - \Delta_n/24|} \sqrt{|c_R - \bar{\Delta}_n/24|}, \quad \nu_n = \sqrt{|c_L - \Delta_n/24|} \sqrt{|c_R - \bar{\Delta}_n/24|}.
\]

(2.9)

Equation (2.8), one of our main results, provides a recursive exact formula for the density of states and in this respect it may be viewed as analog of the Rademacher expansion for a generic unitary, modular invariant and non-singular 2d CFT which is not necessarily holomorphic or holomorphically factorizable.

To obtain the expression for the density of states we should solve the above recursive equation for \( \rho \). Recalling the behavior of \( J_1(z) \) and \( I_1(z) \) for large arguments,

\[
J_1(z) \sim \sqrt{\frac{2}{\pi z}} \sin \left( \frac{\pi}{4} - z \right), \quad I_1(z) \sim \frac{1}{\sqrt{2\pi z}} e^z, \quad z \gg 1,
\]

(2.10)

one can show that, despite the fact that the last three lines of (2.8) contain \( J_1(z) \) and are sums over infinitely many states with (presumably) exponentially growing weights \( \rho(\Delta_n, \bar{\Delta}_n) \), as a result of the oscillatory behavior of \( J_1(z) \), they are nevertheless exponentially suppressed compared to the first line for large \( u_n \) and \( v_n \). The details of this analysis have been gathered in Appendix A. Therefore, up to exponentially suppressed contributions, the right-hand side of (2.8) is given by its first line, namely

\[
\rho(\Delta, \bar{\Delta}) = (2\pi)^2 \sum_{\Delta_n < \frac{cL}{24}, \Delta_n < \frac{cL}{24}} \rho(\Delta_n, \bar{\Delta}_n) \frac{|c_L - \Delta_n/24|}{u_n} I_1(4\pi u_n) \cdot \frac{|c_R - \bar{\Delta}_n/24|}{v_n} I_1(4\pi v_n).
\]

(2.11)

Due to the exponential behavior of \( I_n(z) \), we can safely deduce that in the saddle point approximation \( \Delta \gg c, \bar{\Delta} \gg \bar{c} \), only the \( n, \bar{n} = 0 \) terms, corresponding to the maximum
value of \( u_n \) and \( v_n \), dominate in the sum (2.11), again up to exponentially suppressed contributions.\(^5\) Hence in the saddle point approximation we can write the density of states \( \rho \) as

\[
\rho(\Delta, \bar{\Delta}) \simeq \left(\frac{\pi^2}{3}\right)^2 \rho_0 \tilde{c}_L \frac{I_1(S_L^0)}{S_L^0} \cdot \tilde{c}_R \frac{I_1(S_R^0)}{S_R^0},
\]

where \( \rho_0 = \rho(\Delta_0, \bar{\Delta}_0) \) is the degeneracy of the ground state which is taken to be equal to one, and

\[
S_L^0 = 2\pi \sqrt{\frac{c_L}{6} \left( \Delta - \frac{c_L}{24} \right)}, \quad S_R^0 = 2\pi \sqrt{\frac{c_R}{6} \left( \Delta - \frac{c_R}{24} \right)},
\]

with \(^6\)

\[
\tilde{c}_L = c_L - 24\Delta_0, \quad \tilde{c}_R = c_R - 24\bar{\Delta}_0.
\]

We emphasize that (2.12) captures all polynomial corrections to the Cardy formula, including its logarithmic corrections. It also establishes what we set to prove in this section, namely that any 2\(d\) CFT is holomorphically factorizable in the saddle point approximation up to exponentially suppressed contributions, \( i.e. \) the density of states is the product of the density of states of the left and right sectors, respectively. Moreover, as explicitly manifest in (2.12), \( \rho(\Delta, \bar{\Delta}) \) depends on the combination of \( S_L^0 \) and \( S_R^0 \) or, rigorously, it is of the form

\[
\rho(\Delta, \bar{\Delta}) = \tilde{c}_L \cdot \tilde{c}_R \cdot h(S_L^0) \cdot h(S_R^0).
\]

The prefactor \( \tilde{c}_L \cdot \tilde{c}_R \) may be understood recalling that \( \rho \) is the density of states whilst the number of states \( dN = \rho(\Delta, \bar{\Delta})d\Delta d\bar{\Delta} \) in the ranges \( (\Delta, \Delta + d\Delta) \) and \( (\Delta, \Delta + d\Delta) \) only depends on combinations of \( S_L^0 \) and \( S_R^0 \).

We conclude this section by computing the density of states at energy \( E \). For given states labeled by \( \Delta_n \) and \( \bar{\Delta}_n \) (\( \Delta_n, \bar{\Delta}_n \geq 0 \)), the energy \( E \) and angular momentum \( J \) are defined as

\[
\Delta - \frac{c_L}{24} = \frac{1}{2}(E + J), \quad \bar{\Delta} - \frac{c_R}{24} = \frac{1}{2}(E - J),
\]

\(^5\) Recalling (2.10), the ratio of \( n \)th and zeroth terms in the sum (2.11) for large \( u_n \) or \( v_n \) is proportional to \( e^{-4\pi(u_n - u) - 4\pi(v_n - v_n)} \). In the saddle point approximation \( u_n - u \gg 1 \) and \( v_n - v \gg 1 \).

\(^6\) It is usually assumed that the ground state energy is zero, \( i.e. \) \( \Delta_0 = \bar{\Delta}_0 = 0 \), for which case \( \tilde{c}_L = c_L, \tilde{c}_R = c_R \). However, there are interesting and important examples with non-zero \( \Delta_0 \). One famous example discussed in [8, 13] is the Liouville theory, for which \( \Delta_0 \) is such that \( \tilde{c} = 1 \). Another case with non-zero \( \Delta_0 \), occurs in the context of \( AdS_3 \) hairy black holes [14]. The linear dilaton CFT (see chapter 2 of [5]) provides another example of such theories. Consider \( D \)-dimensional bosonic string on the linear dilaton \( \Phi = V_\mu X^\mu \) background. The central charge of the corresponding worldsheet CFT is \( c = D + 6V^\mu V_\mu \) [5]. A straightforward and explicit calculation shows that the spectrum of this theory is the same as the one of a free bosonic string at \( V_\mu = 0 \), except for the fact that the zero-point energy is shifted by \( \Delta_0 = V^\mu V_\mu /4 \). Nonetheless, as is also stressed in footnote 1 of section 7 of [5], the density of states of this theory is the same as the one of a string theory at \( V_\mu = 0 \). This may be understood through our equations noting that in this case \( \tilde{c} = D + 6V^\mu V_\mu - 24\Delta_0 = D \) and \( \Delta - \frac{c_L}{24} = \Delta_{V = 0} + \frac{V_\mu V_\nu}{4} - \frac{D + 6V^\mu V_\mu}{24} = \Delta_{V = 0} - \frac{D}{24} \), which is \( V_\mu \) independent. Linear dilaton theory is an example of a non-singular CFT which is not necessarily rational. We would like to thank the anonymous referee for a comment on this point.
where
\[-(E + \frac{c_L}{12}) \leq J \leq E + \frac{c_R}{12}, \quad E + \frac{c_L + c_R}{24} \geq 0.\]

Next, we note that
\[\rho(\Delta, \bar{\Delta}) = \frac{1}{2} d^2N \int_{E}^{J} \rho(\Delta, \bar{\Delta}) = \frac{d^2N}{dEdJ},\]
where \(N\) is the total number of states. One may then compute \(dN/dE\) by integrating (2.12) over \(J\). We are only interested in high energy contributions, \(E \gg \frac{c_L}{24}, \frac{c_R}{24}\), for which \(-E \leq J \leq E\) and hence up to exponentially suppressed contributions we have
\[
\rho(E) \equiv \frac{dN}{dE} = \frac{1}{2} \int_{-E}^{E} dJ \rho(\Delta, \bar{\Delta}) = \frac{\pi^2}{3} \sqrt{c_L \cdot c_R} \int_{0}^{\frac{\pi}{2}} d\theta I_1(2\pi \sqrt{\frac{c_LE}{6}} \sin \theta) I_1(2\pi \sqrt{\frac{c_RE}{6}} \cos \theta)
= \frac{\pi^2}{3} c_{tot} \cdot \frac{I_1(S_{Cardy})}{S_{Cardy}} + \text{exponentially suppressed contributions},
\]
where
\[S_{Cardy} = 2\pi \sqrt{\frac{c_{tot}E}{6}}, \quad c_{tot} = \tilde{c}_L + \tilde{c}_R.\]

The details of the computation of the integral (2.17) are given in Appendix B. Thus, it is evident from (2.17) that \(\rho(E)\), which is the extension of the standard Cardy formula to all orders in perturbation\(^7\) in power of \(1/E\), has the same functional form as the generic density of states (2.12). In particular, we note that its dependence on \(E\) only appears through the \(c_{tot}E\) combination.

3 Cardy Formula to All Orders: Extension of the Saddle Point Analysis

In the previous section, under natural but at the same time strong assumptions of modular invariance, unitarity and non-singularity, we computed the density of states for any \(2d\) CFT to all orders in perturbation theory around the saddle point, up to exponentially suppressed contributions. In what follows, we employ the saddle point method as used by Carlip \([8]\) to go beyond his first-order analysis and reproduce our result of the previous section. In subsection 3.1, using the modular \(S\)-transformation \(\beta \rightarrow 4\pi^2/\beta\), we first generate Carlip's result \([8]\) on the logarithmic correction to the Cardy formula. Then, in subsection 3.2, we extend Carlip’s analysis to extract the leading-order correction as well as all the subleading

\[^{7}\text{It is an all-orders result in the sense that we are dropping terms of the type } \exp(-\alpha\sqrt{E}) \text{ and } \exp(-\beta/E), \text{ with } \alpha, \beta > 0.\]
contributions to the Cardy formula in the asymptotic regime where $\beta E$ is large. Finally, in subsection 3.3, we consider the full partition function, including both energy $E$ and angular momentum $J$. Under the same assumptions as before, we retrieve (2.12). Although the calculation is somewhat more involved, we find it helpful for two reasons: It shows more explicitly the notion of “up to exponentially suppressed contributions around the saddle point” and it provides us with a cross-check of the result derived in section 2. We remark that in this section $E$ is the energy of a state of a 2d CFT on a cylinder whereas $\Delta$ and $\bar{\Delta}$ of the previous section are the conformal weights of states of the CFT on $\mathbb{R}^2$ and hence they differ by $\frac{cL+cr}{24}$.

3.1 Logarithmic Corrections

We begin by writing the partition function in the usual fashion as

$$Z(\beta) = \sum_n \exp(-\beta E_n) = \int dE \rho(E) \exp(-\beta E), \quad (3.1)$$

where the spectral density $\rho(E)$ is a sum of delta functions with positive integer coefficients. In spite of the fact that $\rho(E)$ is a discrete sum of delta functions, in the asymptotic regime where $E$ is very large, the levels are very dense and hence is physically meaningful to approximate $\rho(E)$ with a smooth distribution of eigenvalues such that

$$\rho(E) = \exp(\sigma(E)). \quad (3.2)$$

We shall further assume that the error we may incur due to this approximation is significantly negligible compared to the error coming from truncating our expansion in corrections to the saddle point approximation at any order of our choice. In other words, we are suggesting to consider (3.2) as an exact equality for some smooth function $\sigma(E)$. This last remark allows us to rewrite (3.1) as (recall the second equality in (2.1))

$$Z(\beta) = \int dE \exp(\sigma(E) - \beta E), \quad (3.3)$$

and consider (3.3) as an equality at any order in the large-$\beta$ expansion.

In order to proceed, we draw on modular invariance to match the low-temperature expansion of $Z(\beta)$

$$Z(\beta) = \exp(-\beta E_0) + a_1 \exp(-\beta E_1) + \mathcal{O}(\exp(-\beta E_2)), \quad (3.4)$$

---

8The analysis in this subsection was communicated to the authors by Simeon Hellerman through earlier email exchanges.
as \( \beta \to \infty \) to its high-temperature expansion

\[
Z(\beta) = \exp(-4\pi^2 E_0/\beta) + a_1 \exp(-4\pi^2 E_1/\beta) + \mathcal{O}(\exp(-4\pi^2 E_2/\beta)),
\] (3.5)

as \( \beta \to 0^+ \). Furthermore, let us make an ansatz for the form of \( \sigma(E) \). In reality, since we want to reverse-engineer \( \sigma(E) \) to obtain the correct saddle point expansion, we ought to make an ansatz for \( \sigma'(E) \) rather than for \( \sigma(E) \)

\[
\sigma'(E) = E^{q-1} \cdot (b_0 + b_1 E^{-p} + b_2 E^{-2p} \ldots),
\] (3.6)

where \( q \) is the leading power of \( E \) in \( \sigma(E) \) and \( p \) is some power greater than zero. Equation (3.6) gives either the following high energy expansion for \( \sigma(E) \)

\[
\sigma(E) = \ln(K) + \sum_{m \geq 0} \frac{b_m}{q - mp} E^{q-mp},
\] (3.7)

if \( q \) is not a positive integer multiple of \( p \), or

\[
\sigma(E) = \ln(K) + b_{q/p} \ln(E) + \sum_{m \geq 0, m \neq q/p} \frac{b_m}{q - mp} E^{q-mp},
\] (3.8)

if \( q \) is a positive integer multiple of \( p \). In the last two equations, \( K \) is meant to be taken as a constant independent of \( E \).

With the stage now set, we are ready to attempt the evaluation of the integral (3.3) in the high-temperature expansion \( \beta \to 0^+ \) in terms of the unknown numbers \( p, q \) and \( b_m \). The saddle point for the integral occurs at the energy \( E_* \) satisfying

\[
\sigma'(E_*) = \beta.
\] (3.9)

From (3.6) and (3.9) we then obtain the leading-order saddle point equation

\[
b_0 E_*^{q-1} = \beta,
\] (3.10)

which yields the following leading-order solution\(^9\)

\[
E_* = (\beta/b_0)^{-\frac{1}{1-q}}.
\] (3.11)

From (3.11), we learn that \( q \) must be less than one since \( E_* \) must increase as \( \beta \to 0^+ \). Since at leading order the value of \( \sigma(E_*) - \beta E_* \) ought to be equal to \( \ln(Z(\beta)) \), we finally have

\[
\frac{1 - q}{q} b_0^{-\frac{1}{1-q}} \beta^{-\frac{q}{1-q}} + \mathcal{O} \left( \beta^{-\frac{q}{1-q}} \right) = -\frac{4\pi^2 E_0}{\beta} + \mathcal{O} \left( e^{-\frac{4\pi^2 (E_1 - E_0)}{\beta}} \right).
\] (3.12)

\(^9\)Since both the right-hand side of (3.10) and \( E_* \) are positive, so must \( b_0 \).
We draw the reader’s attention to the absence in the right-hand side of (3.12) of power-law (or logarithmic) contributions in $\beta$. This fact entails that the terms subleading to $\beta^{-1}$ at small $\beta$ vanish to all orders in perturbation theory in fluctuations around the saddle point. We shall use this observation to bootstrap the higher-order terms in $\sigma(E)$.

As it may readily be seen, the limit $\beta \to 0^+$ in (3.12) predicts that $q = \frac{1}{2}$ and

$$b_0 = 2\pi \sqrt{-E_0} = 2\pi \sqrt{|E_0|},$$

as $E_0 = -\frac{c_L + c_R}{24}$ in any unitary CFT. Thus, the leading high energy expression for $\sigma(E)$ is

$$\sigma(E) = 4\pi \sqrt{E|E_0|} = 4\pi \sqrt{\frac{(c_L + c_R)E}{24}} = 2\pi \sqrt{\frac{(c_L + c_R)E}{6}} + \mathcal{O}(E^{\frac{7}{2}-p}),$$

which gives the usual Cardy formula for $\rho(E)$

$$\rho(E) \simeq \exp \left\{ 2\pi \sqrt{\frac{(c_L + c_R)E}{6}} \right\}.$$

In fact, substituting in (3.15) the leading-order saddle point value, we do indeed find

$$\sigma(E_*) - \beta E_* = -\frac{4\pi^2 E_0}{\beta},$$

which shows we have made no mistakes at this order.

We now move on and compute the first-order correction. We have to be cautious about two possible sources of error in (3.12). First, an error might come from including either the term $b(q-p) E^{q-p} = \frac{b_1}{\frac{3}{2}-p} E^{\frac{7}{2}-p}$ in the expansion for $\sigma(E)$, if $p \neq \frac{1}{2}$, or the term $b_1 \ln(E)$, if $p = \frac{1}{2}$. The second possibly relevant source of error is the inclusion of the first correction to the saddle point approximation in the expression (3.3) for the partition function $Z(\beta)$. Let us look at the latter contribution first, which is nothing other than a Gaussian integral over fluctuations of $E$ about $E_*$. Defining $\epsilon \equiv E - E_*$, and expanding around the saddle point $\epsilon = 0$, we find

$$\sigma_{\text{leading}}(E) - \beta E = \frac{4\pi^2 |E_0|}{\beta} - \frac{\beta^3}{16\pi^2 |E_0|} \epsilon^2 + \mathcal{O}\left(\frac{\beta^5}{|E_0|^2 \epsilon^3}\right),$$

where

$$\sigma_{\text{leading}}(E) \equiv 2 b_0 E^{1/2} = 4\pi \sqrt{|E_0| E},$$

is the leading term in the logarithm of the density of states. Integrating over energies yields

$$\int dE \exp (\sigma_{\text{leading}}(E) - \beta E) = \exp\left(\frac{4\pi^2 |E_0|}{\beta}\right) \cdot \int d\epsilon \exp\left\{-\frac{\beta^3}{16\pi^2 |E_0|} \epsilon^2 + \mathcal{O}\left(\frac{\beta^5}{|E_0|^2 \epsilon^3}\right)\right\}$$

$$= \exp\left(\frac{4\pi^2 |E_0|}{\beta}\right) \cdot (4\pi^{3/2} |E_0|^{1/2} \beta^{-3/2}) \cdot (1 + \mathcal{O}(\beta/\epsilon)).$$

(3.19)
Hence,

\[ F(\beta) = F_{\text{Cardy}} + F_{\text{leading-fluctuation}} + F_{\text{leading-correction-to-\sigma}} + O(\beta/\epsilon), \]

where \( F(\beta) \equiv \ln Z(\beta) \) and

\[ F_{\text{Cardy}} \equiv \frac{4\pi^2|E_0|}{\beta}, \]

\[ F_{\text{leading-fluctuation}} \equiv \ln \left( \frac{4\pi^{3/2}\beta|E_0|^{1/2}}{\beta^{3/2}} \right), \]

and \( F_{\text{leading-correction-to-\sigma}} \) is the leading correction to \( F(\beta) \) due to the inclusion of the \( b_1 \) term in \( \sigma(E) \). In principle, we also ought to have included the contribution to \( F(\beta) \) coming from a shift of \( E_* \) as a function of \( \beta \). However, we shall see that this contribution vanishes at leading order. By modular invariance, \( F(\beta) \) must equal \( F_{\text{Cardy}} + O(\exp(-4\pi^2(E_1 - E_0)/\beta)) \) as \( \beta \to 0^+ \). From this last observation, we learn that the leading term in \( F_{\text{leading-correction-to-\sigma}} \) must precisely cancel \( F_{\text{leading-fluctuation}} \) in the limit \( \beta \to 0^+ \), since the latter goes as \( \ln \beta \), which is much larger than the correction of size \( O(\exp(-4\pi^2(E_1 - E_0)/\beta)) \) appearing in the right-hand side of the equation for modular invariance (3.12).

Having established that first-order corrections to the saddle point approximation vanish and thus do not act as possible sources of error, we turn to computing the leading correction to the value of \( E_* \) when the \( b_1 \) term is included. The saddle point equation (3.10) is then modified to

\[ E_*^{-1/2} + \frac{b_1}{b_0} E_*^{-1/2-p} = \frac{\beta}{b_0}, \]

where \( b_0 = 2\pi \sqrt{|E_0|} \). In order to find the first-order shift in \( E_* \), we separate \( E_* \) into a zeroth-order piece and a correction piece as

\[ E_* = E_*^{(0)} + E_*^{(1)}, \]

and expand the \( b_1 \)-corrected saddle point equation, treating \( E_*^{(1)} \) as a small quantity to be included only at first order.\(^{10}\)

Let us rewrite the saddle point equation (3.23) in a more appealing form as

\[ E_* \cdot \left( 1 + \frac{b_1}{b_0} E_*^{-p} \right)^{-2} = \frac{b_0^2}{\beta^2}, \]

more suitable to be expanded as

\[ (E_*^{(0)} + E_*^{(1)}) \left( 1 - 2\frac{b_1}{b_0} (E_*^{(0)})^{-p} \right) = \frac{b_0^2}{\beta^2}. \]

\(^{10}\)This treatment will be justified \textit{a posteriori} at large \( \beta \).
With a little algebra on (3.26), we find that the solution for $E_\ast^{(1)}$ is

$$E_\ast^{(1)} = 2 \frac{b_1}{b_0} (E_\ast^{(0)})^{1-p} + \mathcal{O} \left( (E_\ast^{(0)})^{1-2p} \right). \quad (3.27)$$

At this point, we investigate the two possible values $p$ can take which are either $p \neq 1/2$ or $p = 1/2$. There are two corrections to the saddle point value of the exponent $\sigma(E_\ast) - \beta E_\ast$. The first correction

$$\frac{b_1}{\frac{1}{2} - p} (E_\ast^{(0)})^{\frac{1}{2} - p} + \mathcal{O} \left( (E_\ast^{(0)})^{\frac{1}{2} - 2p} \right), \quad (3.28)$$

originates from the inclusion of $\frac{b_1}{\frac{1}{2} - p} E_\ast^{\frac{1}{2} - p}$ in the expression for $\sigma(E_\ast)$, whilst the second correction

$$E_\ast^{(1)} \cdot [\sigma' (E_\ast^{(0)}) - \beta + \mathcal{O} (\sigma'' (E_\ast^{(0)}))], \quad (3.29)$$

is a consequence of the shift in value of $E_\ast$. However, by virtue of the leading-order saddle point equation for $E_\ast^{(0)}$, the second correction vanishes to the order of interest so that the total leading contribution to $F(\beta)$ from the $b_1$ term is

$$F_{\text{leading-corr-to-}} = \frac{b_1}{\frac{1}{2} - p} (E_\ast^{(0)})^{\frac{1}{2} - p} + \mathcal{O} \left( (E_\ast^{(0)})^{\frac{1}{2} - 2p} \right)$$

$$= \frac{b_1}{\frac{1}{2} - p} b_0^{1 - 2p} \beta^{2p - 1} + \mathcal{O} \left( \beta^{4p - 1} \right), \quad (3.30)$$

for $p \neq \frac{1}{2}$. Thus, we gather that the value of $p$ cannot be smaller than $1/2$, otherwise there would be a nonvanishing term of order $\beta^{2p - 1}$ in $F(\beta)$, which must be absent. Conversely, $p$ cannot be greater than $1/2$: If that were the case, the $\ln(\beta)$ term in $F(\beta)$ coming from the fluctuation integral could not be canceled by the $b_1$ term in $\sigma(E)$. Hence, it follows that $p$ must be exactly equal to $1/2$, and that the leading large-$E$ behavior of $\sigma(E)$ must be

$$\sigma(E) = 4\pi \sqrt{|E_0|E} + b_1 \ln(E) + \ln(K) + \mathcal{O}(E^{-1/2}). \quad (3.31)$$

The partition function is then

$$Z(\beta) = K \int_{\Lambda_*} dE \exp \left[ -\beta \left( \sqrt{E} - b_0/\beta \right)^2 + \frac{b_0^2}{\beta} \right] E^{b_1}, \quad (3.32)$$

where $\Lambda_*$ is an IR cut-off. It is clear that $Z(\beta)$ is independent of $\Lambda_*$ as we are in the high $T$ (low $\beta$) regime: A different choice of $\Lambda_*$ would only affect the lower part of the spectrum of
the theory in consideration. Introducing \( x = \sqrt{E - b_0/\beta} \), (3.32) reduces to

\[
Z(\beta) = K \int_{-\infty}^{\infty} dx \left( x + \frac{b_0}{\beta} \right)^{2b_1+1} \exp \left[ \frac{b_0^2}{\beta} \right]
\]

\[
= 2K \exp \left[ \frac{b_0^2}{\beta} \right] \left( \frac{b_0}{\beta} \right)^{2b_1+1} \int_{-\infty}^{\infty} dx \ e^{-\beta x^2} \left( 1 + \mathcal{O}(\beta) \right)
\]

\[
= 2K \exp \left[ \frac{b_0^2}{\beta} \right] \left( \frac{b_0}{\beta} \right)^{2b_1+1} \left( \sqrt{\frac{\pi}{\beta}} + \mathcal{O}(1) \right). \quad (3.33)
\]

The first term in (3.33), namely \( \exp \left( \frac{b_0^2}{\beta} \right) \), is just the expression of the entropy as given by the Cardy formula. Hence, by matching (3.33) to the \( \beta \to 0^+ \) limit, we are led to solve

\[
2K \left( \frac{b_0}{\beta} \right)^{2b_1} \left( \frac{b_0}{\beta} \right) \sqrt{\frac{\pi}{\beta}} = 1,
\]

from which we find that

\[
b_1 = -\frac{3}{4}, \quad (3.35)
\]

\[
K = 2^{-1/2} |E_0|^{1/4}, \quad (3.36)
\]
in agreement with the values found in [8].

### 3.2 Beyond Logarithmic Corrections

Thus far in this section we have reproduced the logarithmic correction to the Cardy formula. However, we wish to go beyond the first-order correction and reproduce the expression (2.17) which is valid to all orders in the perturbative expansion in \( 1/\tau \). We do so by introducing yet a new variable \( y = \sqrt{\beta} x \) which transforms the partition function (3.32) into

\[
Z(\beta) = 2K \frac{e^{\frac{y^2}{2b_0}}}{\sqrt{b_0}} \int_{-\Lambda_{UV}}^{\Lambda_{UV}} dy \ e^{-y^2} P(y; \beta), \quad (3.37)
\]

where, recalling (3.8) with \( p = q = 1/2 \),

\[
P(y; \beta) = \left( 1 + \frac{y \sqrt{\beta}}{b_0} \right)^{-1/2} \prod_{m=2}^{\infty} \exp \left\{ \frac{2b_m}{1 - m} \left( \frac{\beta}{b_0} \right)^{m-1} \left( 1 + \frac{y \sqrt{\beta}}{b_0} \right)^{1-m} \right\}, \quad (3.38)
\]

is a polynomial which can be expanded in \( \frac{y}{b_0} \) if \( \Lambda_{UV} \lesssim \frac{b_0}{\sqrt{\beta}} \) as \( \beta \to 0^+ \).
Because of the Gaussian weight, the energy levels above the cut-off have a vanishing contribution to the partition function as $\Lambda_{UV} \to \infty$. Thus, we can rewrite (3.37) as

$$Z(\beta) = 2K \frac{e^{\frac{b_0^2}{2}}}{\sqrt{b_0}} \int_{-\infty}^{\infty} dy \ e^{-y^2} P(y; \beta) = 2K \sqrt{\frac{\pi b_0}{e^{\frac{b_0^2}{2}}}} \left(1 + \sum_{k=1}^{\infty} d_k \beta^k\right),$$

(3.39)

where $d_k$ is a function of $b_0$ and $b_m$ for $m = 2, \ldots, k - 1$. Modular invariance, as discussed in the previous subsection, implies that

$$d_k = 0, \quad k \in \mathbb{N}.$$  

(3.40)

One can then use (3.40) to determine the coefficients $b_m$ for all $m$, leading to an all-orders result. It can be shown that

$$b_m = \frac{c_m}{b_0^{m-1}},$$

(3.41)

where the constants $c_m$ are the coefficients of the asymptotic expansion of the logarithm of the Bessel function $I_1(z)$

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{3}{2} + k\right)}{(2z)^k k! \Gamma\left(\frac{3}{2} - k\right)} \approx \exp\left[z - \ln\left(\sqrt{2\pi z}\right) - \frac{3}{8z} - \frac{3}{16z^2} - \frac{21}{128z^3} - \frac{27}{128z^4} + O\left(\frac{1}{z^5}\right)\right].$$

(3.42)

In order to put our findings in a suggestive form and make contact with (2.17), we rewrite

$$\sigma(E) = \ln(K) - \frac{3}{4} \ln(E) + \sum_{m \geq 2} \frac{2b_m}{1 - m} E^{\frac{1}{2}(1-m)},$$

(3.43)

as

$$\sigma(S_{\text{Cardy}}) = S_{\text{Cardy}} + \ln\left(\frac{\pi^2}{3} c_{\text{tot}}\right) - \frac{3}{2} \ln(S_{\text{Cardy}}) + \sum_{m=2}^{\infty} \frac{2^m c_m}{m - 1} \left(\frac{1}{S_{\text{Cardy}}}\right)^{m-1}$$

or

$$\rho(E) = e^\sigma = \frac{\pi^2}{3} \frac{c_{\text{tot}}}{S_{\text{Cardy}}} I_1(S_{\text{Cardy}}),$$

(3.45)

where $S_{\text{Cardy}}$ and $c_{\text{tot}}$ are defined in (2.18). Equation (3.45) is obviously the same as the result (2.17) obtained in the previous section.

11We recall that $|E_0| = \frac{\pi^2}{2}, K = \frac{1}{2} \sqrt{\frac{b_0}{\pi}}$ and $b_0 = 2\pi \sqrt{|E_0|}$. 

### 3.3 Generic Case with $E$ and $J$

Ideally, we would like to extend our previous analysis and generalize it to the case in which $J \neq 0$ and $c_L \neq c_R$. On general grounds,

$$\sigma(\Delta, \bar{\Delta}) = \sum_{m=-M}^{\infty} \frac{a_m}{mr} \Delta^{p-mr} + \sum_{n=-N}^{\infty} \frac{\bar{a}_n}{ns} \bar{\Delta}^{q-ns} + \alpha \ln \Delta + \beta \ln \bar{\Delta} + \eta(\Delta, \bar{\Delta}),$$

where

$$\eta(\Delta, \bar{\Delta}) = \sum_{n,m \neq 0} c_{m,n} \Delta^{m,rs} \bar{\Delta}^{n,rs},$$

with $\Delta$ and $\bar{\Delta}$ related to $E$ and $J$ as in (2.15).

According to the lore of modular invariance, we have\(^{12}\)

$$Z(\tau, \bar{\tau}) = \int d\Delta \, d\bar{\Delta} \exp \left( \sigma(\Delta, \bar{\Delta}) - \tau \Delta - \bar{\tau} \bar{\Delta} \right) = \exp \left\{ -2\pi^2 \left( \frac{E_0}{\tau} + \frac{\bar{E}_0}{\bar{\tau}} \right) \right\},$$

where we have assumed $\Delta_{\text{min}} = E_0/2$ and $\bar{\Delta}_{\text{min}} = \bar{E}_0/2$. From our saddle point analysis, we are led to set $N = M = 1$, $r = s = \frac{1}{2}$, $p = q = 0$; also, $a_{-1}^2 = -2\pi^2 E_0$ and $\bar{a}_{-1}^2 = -2\pi^2 \bar{E}_0$.

Equation (3.48) then reads

$$Z(\tau, \bar{\tau}) = 4 \int_{x_0}^{\infty} dx \int_{\tilde{x}_0}^{\infty} d\tilde{x} \left( x + \frac{a_{-1}}{\tau} \right)^\alpha \left( \tilde{x} + \frac{\bar{a}_{-1}}{\bar{\tau}} \right)^\beta e^{(-\tau x^2 - \bar{\tau} \tilde{x}^2)}$$

$$= 4 \int_{x_0}^{\infty} dx \int_{\tilde{x}_0}^{\infty} d\tilde{x} \left( \frac{a_{-1}}{\tau} \right)^\alpha \left( \frac{\bar{a}_{-1}}{\bar{\tau}} \right)^\beta \left( 1 + \frac{\tau x}{a_{-1}} \right)^\alpha \left( 1 + \frac{\bar{\tau} \tilde{x}}{\bar{a}_{-1}} \right)^\beta e^{(-\tau x^2 - \bar{\tau} \tilde{x}^2)} \approx 1,$$

where $x_0 = -\frac{a_{-1}}{\tau}$ and $\tilde{x}_0 = -\frac{\bar{a}_{-1}}{\bar{\tau}}$. Since we are working with a Gaussian integral, we can take both $x_0, \tilde{x}_0 \to -\infty$. Thus, (3.49) implies that $\alpha = \beta = -\frac{3}{4}$ since $\frac{1}{\tau}$ should cancel on the left hand side.

To go beyond saddle point approximation, we need to show that $\eta(\Delta, \bar{\Delta}) = 0$. Let us consider a term in $\eta$ like the following

$$c_{m,n} \Delta^{m/2} \bar{\Delta}^{n/2}, \quad m \cdot n \neq 0.$$ (3.50)

This last term can be written as a function of $x$ and $\tilde{x}$ as

$$c_{m,n} \left( x + \frac{a_{-1}}{\tau} \right)^m \left( \tilde{x} + \frac{\bar{a}_{-1}}{\bar{\tau}} \right)^n \sim c_{m,n} \left( \frac{a_{-1}}{\tau} \right)^m \left( \frac{\bar{a}_{-1}}{\bar{\tau}} \right)^n P \left( \frac{\tau x}{a_{-1}}, \frac{\bar{\tau} \tilde{x}}{\bar{a}_{-1}} \right), \quad m \cdot n \neq 0,$$ (3.51)

\(^{12}\)Note that in this subsection $\tau (\bar{\tau})$ is equal to $-2\pi i \tau (2\pi i \bar{\tau})$ of the previous section.
where $P$ is a polynomial and $P(0,0) = 1$. After taking the expansion
\[
\exp \left\{ c_{m,n} \left( \frac{a-1}{\tau} \right)^m \left( \frac{\bar{a}-1}{\bar{\tau}} \right)^n \right\} \simeq 1 + c_{m,n} \left( \frac{a-1}{\tau} \right)^m \left( \frac{\bar{a}-1}{\bar{\tau}} \right)^n ,
\]
we deduce from (3.48) that
\[
c_{m,n} = 0 ,
\]
which implies that the theory is holomorphically factorized around the saddle point.

One can then readily continue the analysis along the lines worked out in subsection 3.2 and reproduce (2.12). Since the computation is basically the same as that shown in the previous subsection we do not repeat it here.

### 4 Canonical vs. Microcanonical Entropy

In the previous two sections, starting from the canonical partition function we derived the expression for the microcanonical density of states through a Laplace transform. Then, employing modular invariance of the partition function we fixed the form of the microcanonical density of states, up to exponentially suppressed contributions in the saddle point approximation. By means of the usual thermodynamical equations, one can then read the microcanonical entropy $S_{\text{m.c.}}$ upon taking the logarithm of the density of states. Let us first consider the pure imaginary $\tau$ case, corresponding to the $L_0 = \bar{L}_0 = E$ sector, for which the microcanonical entropy is
\[
S_{\text{m.c.}} = \ln (\rho(E)) = \ln \left( \frac{\pi^2}{3} c_{\text{tot}} \right) + \ln \left( \frac{I_1(S_{\text{Cardy}})}{S_{\text{Cardy}}} \right) \quad (4.1a)
\]
\[
= \ln \left( \frac{\pi^{3/2}}{3\sqrt{2}} c_{\text{tot}} \right) + S_{\text{Cardy}} - \frac{3}{2} \ln S_{\text{Cardy}} + \mathcal{O} \left( \frac{1}{S_{\text{Cardy}}} \right) , \quad (4.1b)
\]
where $S_{\text{Cardy}}$ is given in (2.18). We note that (4.1a) gives the microcanonical entropy up to exponentially suppressed terms whilst (4.1b) captures only the leading log-correction to the Cardy formula. The logarithmic correction was also discussed in [8].

Given the density of states $\rho(E)$ (2.17), one may insert it back into the expression for the partition function $Z$
\[
Z(\beta) = \int dE \ \rho(E) \ \exp(-\beta E) \quad ,
\]
and compute the canonical entropy $S_c$
\[
S_c = \ln Z - \beta \frac{\partial \ln Z}{\partial \beta} .
\]
The integral (4.2) can be performed using formula (6.620.4) of [12] to obtain
\[ Z(\beta) = \frac{1}{2} \left( e^{\frac{\pi^2}{6}c_{\text{tot}}T} - 1 \right), \]
where \( T = \beta^{-1} \) is the temperature. One then arrives at \(^{13}\)
\[ S_c = \ln 2 + \frac{\pi^2}{3} c_{\text{tot}} T + \mathcal{O}(e^{-\frac{\pi^2}{6}c_{\text{tot}}T}). \]
That is, the canonical entropy, in the saddle point approximation and up to exponentially suppressed contributions, is completely given by the Cardy formula and in particular there are no logarithm or other polynomially suppressed terms, which appeared in the microcanonical entropy (4.1a). In other words, the functional dependence of the microcanonical density of states (2.17), namely \( \frac{1}{8} I_1(S) \), is such that the Cardy formula is the exact expression for the canonical entropy, up to exponentially suppressed contributions.

The above argument can be readily generalized to the case with non-zero \( J \), for generic \( \Delta \) and \( \bar{\Delta} \). As we discussed, the microcanonical density of states (2.12) is the product of density of states in the left and right sectors. Inserting (2.12) into (2.1) the canonical partition function takes the form
\[ Z(\tau, \bar{\tau}) = Z_L(\tau) \cdot Z_R(\bar{\tau}), \]
where
\[ Z_L(\tau) = \frac{1}{2} \left( e^{\frac{\pi^2}{6} T_L} - 1 \right), \quad Z_R(\bar{\tau}) = \frac{1}{2} \left( e^{-\frac{\pi^2}{6} T_R} - 1 \right). \]
As the details of the computations exactly parallel those of the previous case, we do not repeat them again. The canonical entropy is then given by
\[ S_c = S_c^L + S_c^R, \]
where
\[ S_c^L = \ln Z_L - \tau \frac{\partial \ln Z_L}{\partial \tau}, \quad S_c^R = \ln Z_R - \bar{\tau} \frac{\partial \ln Z_R}{\partial \bar{\tau}}. \]
In terms of left and right temperatures \( T_L = -\frac{1}{2\pi^2} \) and \( T_R = \frac{1}{2\pi^2} \), the canonical entropy (4.8) is obtained to be \(^{14}\)
\[ S_c = 2 \ln 2 + \frac{\pi^2}{3} (\tilde{c}_L T_L + \tilde{c}_R T_R) + \mathcal{O}(e^{-\frac{\pi^2}{6}T_L}, e^{-\frac{\pi^2}{6}T_R}). \]

\(^{13}\)One may compute the microcanonical temperature \( T_{\text{m.c.}} \) given the microcanonical entropy: \( T_{\text{m.c.}}^{-1} = \frac{\partial S}{\partial E} \). Using (2.17) we obtain
\[ \frac{\pi^2}{3} c_{\text{tot}} T_{\text{m.c.}} = \frac{S_{\text{Cardy}} I_1(S_{\text{Cardy}})}{I_2(S_{\text{Cardy}})}, \]
where \( I_2(x) \) is the modified Bessel function and \( S_{\text{Cardy}} \) is given in (2.18).

\(^{14}\)Recall that for a generic \( T_L \) and \( T_R \), the partition function (2.1) may be written as \( \text{Tr} \left( e^{-\beta_L L_0} e^{-\beta_R \bar{L}_0} \right) \)
Again, we see that the Cardy formula gives the exact canonical entropy up to exponentially suppressed terms.

5 Discussion and Outlook

In this work, we have exploited $S$-transformation of the $\text{PSL}(2,\mathbb{Z})$ modular group to learn more about the density of states of a generic unitary, modular invariant and non-singular 2d CFT. Interestingly, we have found that the density of states is the product of those of the left and right sectors and, furthermore, it only depends on $S_0 \equiv \tilde{c}(\Delta - \frac{c}{24})$ with $\tilde{c} = c - 24\Delta_0$ in each sector, up to exponentially suppressed contributions in the saddle point approximation. Our main result, (2.12), may have diverse interesting physical implications. We discuss some of them below.

1. Noticing that the density of states only depends on $S_0$, one can construct a class of 2d CFTs with different central charges and different spectra, but with the same $S_0$. This class of CFTs, for which the densities of states are the same, is relevant for “orbifolded CFTs” or CFTs on an $\mathbb{R} \times S^1/\mathbb{Z}_k$ orbifold. As an evocative and related argument, we observe that any Virasoro algebra at central charge $c$ has infinitely many Virasoro subalgebras labeled by integer $k$ at central charge $ck$. This point has also been noted in [16]. To appreciate this last comment, let us consider the algebra (1.1) and the subset of its generators $L_{nk}$

$$\hat{L}_n = \frac{1}{k} L_{nk}, \quad n \neq 0,$$

$$\hat{L}_0 - \frac{\hat{c}}{24} = \frac{1}{k} \left( L_0 - \frac{c}{24} \right),$$

where $\hat{c} = ck$. It is then straightforward to see that the set of $\hat{L}_n$ also forms a Virasoro algebra at central charge $\hat{c}$. The relation between $\hat{L}_n$ and $L_n$ could be understood through a (non-single valued) conformal map $w = z^k$ on the complex plane. This map in turn, can be viewed as orbifolding the $z$-plane by $\mathbb{Z}_k$. According to this line of reasoning, it can therefore be stated that $\hat{c}(\Delta - \frac{\hat{c}}{24}) = c(\Delta - \frac{c}{24})$. This is suggesting that two conformal field theories with central charge and spectrum $(c, \Delta)$ and $(\hat{c}, \hat{\Delta})$ or equivalently $\text{Tr} \left( e^{-\beta H - \beta \mu J} \right)$, where $\beta$ is the inverse of canonical temperature, $\mu$ is the chemical potential for the angular momentum $J$, $L_0 + \bar{L}_0 = 2H$, $L_0 - \bar{L}_0 = 2J$ and $\beta_L = 1/T_L$, $\beta_R = 1/T_R$. Therefore,

$$\frac{2}{T} = \frac{1}{T_L} + \frac{1}{T_R}, \quad \frac{2\mu}{T} = \frac{1}{T_L} - \frac{1}{T_R}. $$
respectively, should have the same density of states up to exponentially suppressed contributions.

In light of the above arguments, it might be possible to relax the requirement of having integer-valued spacing of the conformal weights $\Delta_n$ (cf. discussions below (2.3)) while keeping the discreteness of the spectrum: For theories with $(kc, \Delta/k)$ one might hope to find a “dual” CFT with $(c, \Delta)$ (see the example below). In this “dual” picture the spacing of the spectrum and the mass gap $\delta M$ is then $1/k$, or in a more suggestive form $\delta M \sim 1/c$.

2. The observation in item 1. may be employed in the black hole microstate identifications within string theory. In particular, it may be used to relate black holes within the family of $(D0, D6)$-branes with different number of $D6$-branes. A discussion on the latter subject may be found in [17]. In this context families of $(kc, \Delta/k)$ CFTs are mapped to each other by T-duality. See also [18] for a related analysis.

3. Although we mainly focused on the analysis of 2d CFT, one of our main motivations to study this problem was the desire to shed new light on the AdS$_3$/CFT$_2$ correspondence (see [15] for a review). As first pointed out by Strominger [19], the Bekenstein-Hawking entropy of BTZ black holes [20] is correctly reproduced by the Cardy formula (1.4), which is nothing but the thermodynamical entropy of a 2d CFT with Brown-Henneaux central charge [21] at temperature equal to the Hawking temperature of the BTZ black hole.

As shown in details earlier (see also [8]), Cardy formula should be viewed as the first term in the saddle point expansion. The natural question is then: What do the corrections to the Cardy formula correspond to in the gravity picture?

Of course, before employing our results to address questions in AdS$_3$ gravity, one should make sure that the assumptions of modular invariance and non-singularity are expected to hold for the CFT proposed to be dual to quantum gravity in AdS$_3$. Naturally, modular invariance is needed if we want the dual gravity to be compatible with toroidal boundary conditions on the Euclidean AdS$_3$, see [7, 11]. The non-singularity and existence of a mass gap are harder to argue for. For example, for the $D1/D5$ system as discussed in [25], there are some regions in the parameter space where the dual theory is expected to have a continuous spectrum above a gap. This brings about a pathology in the dual 2d CFT. Nevertheless, in generic points of the parameter space of the $D1/D5$ system the dual CFT is non-singular. Then, it seems plausible to assume that the consequences of our findings may be of significance for quantum gravity in AdS$_3$. This is compatible with the arguments of [7, 11].
An idea to tackle the question posed above, as motivated by stringy $\alpha'$-corrections to supergravity (e.g. see [22] for a review), is that the Bekenstein-Hawking area law receives corrections from the higher derivative corrections to the Einstein-Hilbert action [23]. This idea is shown to work for certain four- and five-dimensional black holes in higher derivative gravity theories [22]. Accordingly, the corrections to the Cardy formula may then play the role of corrections to that action whilst the exponentially suppressed contributions may be associated with “stringy or non-perturbative corrections” which do not admit a semiclassical description. Stated differently, the existence of corrections to the Cardy formula implies that the effective semiclassical description of a quantum gravity in AdS$_3$ admitting a dual CFT description cannot simply be AdS$_3$ Einstein gravity without higher derivative corrections.\footnote{It has been argued that the log-corrections [8] and in general the whole Rademacher expansion [24], may be associated with quantum gravity effects.} However, 3$d$ gravity has specific features: It was shown by Saida and Soda [26] that any higher derivative corrections to Einstein gravity in AdS$_3$, irrespective of the details of such corrections, only result in a shift in the AdS$_3$ radius and hence a shift in the Brown-Henneaux central charge. It is obvious that our corrections to the Cardy formula cannot be captured by only a shift in the central charge. Therefore, this particular idea does not work and our corrections should be understood in a different way.

In view of the results of [26], the answer to the above question lies within the lines of section 4. The Bekenstein-Hawking entropy of a BTZ black hole should be viewed as entropy of a system in a \textit{canonical ensemble}. This is due to the fact that a black hole as perceived by an observer at infinity is a system at a given temperature, the Hawking temperature. This fact becomes more apparent recalling the Wald entropy formula [23], in which entropy is associated with a Noether charge while the temperature is fixed to be the Hawking temperature. As we showed in section 4, the Cardy formula (4.5) does not receive logarithmic or polynomial corrections in powers of $c\cdot T$, it is exact up to exponentially suppressed contributions.

4. An interesting and important question put forth in [7], and discussed further in [11], is whether there is a well-defined AdS$_3$ pure Einstein quantum gravity, defined by a path integral over the AdS$_3$ Einstein-Hilbert action, where the metric is the only dynamical field with prescribed boundary conditions [21]. In [11] a careful analysis of the path integral for pure AdS$_3$ Einstein gravity is carried out, taking into account contributions of all “Brown-Henneaux states”. (Brown-Henneaux states are the boundary excitations of AdS$_3$ background which respect the Brown-Henneaux boundary conditions [21]. These excitations are localized around the conformal boundary of AdS$_3$ [11].) It was
then argued that this path integral does not have the expected form of the partition function of a “physically sensible theory”.

Our findings may have implications on this question/puzzle. If there exists such an $\text{AdS}_3$ quantum theory (minimal, pure Einstein theory, or otherwise) of gravity and it admits a modular invariant, non-singular and unitary dual $2d$ CFT, then according to (2.12) this CFT is holomorphically factorizable and one may relax demanding this last feature as a requirement. Note also that (2.12) is obtained from (2.8) by discarding the exponentially suppressed contributions in the saddle point approximation and that in the derivation of (2.8) no assumption about holomorphic factorizability has been made. (Equation (2.8) is an exact result which is true for generic $2d$ CFTs whose partition functions are neither necessarily holomorphic nor holomorphically factorizable.) In other words, our equation (2.8) should be compared to equations (5.13) and (5.4) of [11], or more precisely to the modular invariant partition function given in equation (3.7) therein, which gives the entropy associated with a BTZ black hole computed from the partition function of $\text{AdS}_3$ gravity (calculated as described above). As another outcome of this comparison, we observe that equation (5.13) (or equation (3.7)) turns into the partition function of a “physically sensible theory” (cf. discussions of [11]) if the coefficients $C_\Delta$ are replaced with $\rho(\Delta_n, \bar{\Delta}_n)$ of our analysis. The physical meaning of this observation and its implications will be analyzed elsewhere.

5. In this work we mainly focused on the results “up to exponentially suppressed contributions in the saddle point approximation”. The interpretation of these exponentially suppressed terms, especially with regard to the question of $\text{AdS}_3$ quantum gravity and the (BTZ) black holes entropy for the cases involving holomorphic partition function, corresponding to BPS black holes, has been discussed in e.g. [9, 27]. Given our analysis here, it is another interesting question to study these contributions for the case of generic non-BPS black holes.

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A Suppression of the Oscillatory Terms in $\rho(\Delta, \tilde{\Delta})$

In this Appendix we show that the last three terms in (2.8) are indeed suppressed compared to the first term. To this end, we estimate these three terms and compare them against the first term. We do so by assuming that our claim holds, that is

$$\rho(\Delta, \tilde{\Delta}) \simeq (2\pi)^2 \tilde{c}_L \tilde{c}_R \frac{I_1(4\pi u_0)}{u_0} \cdot \frac{I_1(4\pi v_0)}{v_0},$$

where

$$u_0 = \sqrt{\tilde{c}_L (\Delta - \tilde{c}_L \frac{24}{24})}, \quad v_0 = \sqrt{\tilde{c}_R (\tilde{\Delta} - \tilde{c}_R \frac{24}{24})},$$

and then replace the above expression for $\rho$ into the sums in the last three terms in (2.8). Noting the exponential growth of the states by energy, the main contribution to the sums comes from the large $\Delta$ states, a region of the spectrum in which the sums could be approximated by integrals over $\Delta$. The integrals one needs to compute then take the form

$$I(a, b) = \int_0^\infty dE I_1(a\sqrt{E})J_1(b\sqrt{E})$$

$$= \int_0^\infty dx xI_1(ax)J_1(bx).$$

To compute the above integral, we make use of (8.447.2) and (6.511.1) of [12]

$$I_1(az) = \sum_{k=0}^\infty \frac{1}{k!(k+1)!} \left(\frac{az}{2}\right)^{2k+1},$$

$$1 = \int_0^\infty dx J_1(bx).$$

The integral (A.3) then becomes

$$I(a, b) = \sum_{k=0}^\infty \frac{1}{k!(k+1)!} \left(\frac{a}{2}\right)^{2k+1} c_{k+1}(b),$$

where

$$c_k(b) \equiv \int_0^\infty dx x^{2k} J_1(bx).$$

If we have the expression for $c_k$ we can then compute $I(a, b)$.

From (A.6), one can then show that

$$\frac{d^2}{db^2} c_k(b) = -c_{k+1} - \frac{d}{db} \left( \frac{1}{b} c_k(b) \right).$$
To obtain (A.7), we have used the following identities for $J_n(z)$

$$z \frac{d}{dz} J_1(z) = z J_0(z) - J_1(z), \quad J_1(z) = - \frac{d}{dz} J_0(z).$$  \hspace{1cm} (A.8)

The solutions to (A.7) are of the generic form

$$c_k(b) = \frac{1}{b^{2k+1+2\epsilon}} d_k,$$  \hspace{1cm} (A.9)

where $d_k$ is a $b$-independent parameter to be determined and $\epsilon$ is an arbitrary number. Inserting (A.9) into (A.7) we obtain

$$d_{k+1} = -4(k + \epsilon)(k + 1 + \epsilon)d_k, \quad d_0 = 1.$$  \hspace{1cm} (A.10)

We next note that from (A.6) one can read $c_k(b) = b^{-2k-1-2\epsilon} c_k(b = 1)$ if we regulate the integral by replacing $dx$ with $d^{1+2\epsilon}x$. In other words, $\epsilon$ should be viewed as a dimensional regularization parameter which will be taken to zero at the end of the computation. Therefore, to leading order in $\epsilon$ we have

$$d_{k+1} = (-1)^k 2^{2k} k!(k + 1)! d_1, \quad k \geq 0, \quad d_1 = -4\epsilon,$$  \hspace{1cm} (A.11)

and hence

$$\mathcal{I}(a, b) = \frac{d_1}{2b^{2(1+\epsilon)}} \sum_{k=0}^{\infty} (-1)^k \left( \frac{a}{b} \right)^{2k+1} = \frac{d_1}{2b^{2\epsilon}} \frac{a}{b} \frac{1}{a^2 + b^2},$$  \hspace{1cm} (A.12)

where we have assumed $b > a$ (for $a > b$ the sum is not convergent). Thus, $\mathcal{I}(a, b) = 0$ in the $\epsilon \to 0$ limit. We would like to comment that from (A.3) one can show that $\mathcal{I}(a, b) = \frac{1}{2\pi} \mathcal{I}(\frac{a}{b}, 1)$ and, as clearly seen from (A.12), our regularization respects this property.

We wish to conclude this Appendix by emphasizing that a similar result on the vanishing of contributions from the Bessel function $J_1(z)$, was established in Appendix B of [9] using a different regularization scheme.

### B Computation of the Bessel Function Integral

Here, we present a detailed computation of the integral (2.17). Noting that $I_n(z) = i^{-n} J_n(iz)$, instead of (2.17) one may compute

$$J(x, y) \equiv \int_0^{\pi} d\theta J_1(x \sin \theta) J_1(y \cos \theta).$$  \hspace{1cm} (B.1)

Let us sketch out the steps for doing this computation:
1. By means of formula (8.535) of [12], we express the $J_n(z)$ in (B.1) as

\[
J_1(x \sin \theta) = \sin \theta \sum_{k=0}^{\infty} \frac{1}{k!} J_{k+1}(x) \left( \frac{x}{2} \right)^k \cos^{2k} \theta, 
\]

\[
J_1(y \cos \theta) = \cos \theta \sum_{l=0}^{\infty} \frac{1}{l!} J_{l+1}(y) \left( \frac{y}{2} \right)^l \sin^{2l} \theta. 
\]

(B.2)

2. Using formula (3.621.5) of [12], we can perform the theta integral

\[
\int_{0}^{\pi/2} d\theta \sin^{2l+1} \theta \cos^{2k+1} \theta = \frac{k!l!}{2(k+l+1)!},
\]

to derive

\[
J(x, y) = \sum_{k,l=0}^{\infty} \frac{1}{2(k+l+1)!} J_{k+1}(x) J_{l+1}(y) \left( \frac{x}{2} \right)^k \left( \frac{y}{2} \right)^l. 
\]

(B.3)

3. With the help of formula (8.440) of [12]

\[
J_{k+1}(x) = (-1)^{k+1} \left( \frac{x}{2} \right)^{-(k+1)} \sum_{p=k+1}^{\infty} \frac{(-1)^p}{p!\Gamma(p-k)} \left( \frac{x}{2} \right)^{2p}, 
\]

we obtain

\[
J(x, y) = \frac{2}{xy} \sum_{p,q=1}^{\infty} \frac{(-1)^{p+q}}{p!q!} \left( \frac{x}{2} \right)^{2p} \left( \frac{y}{2} \right)^{2q} \cdot C_{p,q}, 
\]

(B.4)

where

\[
C_{p,q} = \sum_{k=0}^{p-1} \sum_{l=0}^{q-1} \frac{(-1)^{k+l}}{(p-k-1)!(q-l-1)!(k+l+1)!}, 
\]

\[
= (-1)^{p+q} \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} \frac{(-1)^{r+s}}{r!s!(p+q-r-s-1)!}. 
\]

(B.5)

4. One can show that

\[
C_{p,q} = \frac{1}{(p+q-1)!}. 
\]

(B.6)

5. Thus, we are able to rewrite (B.6) as

\[
J(x, y) = \frac{2}{xy} \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(p+1)!(q+1)!(p+q+1)!} \left( \frac{x}{2} \right)^{2(p+1)} \left( \frac{y}{2} \right)^{2(q+1)}. 
\]

(B.7)
6. Next, let us consider the expansion of \( J_1(\sqrt{x^2 + y^2}) \)

\[
Q(x, y) \equiv \sqrt{x^2 + y^2} J_1(\sqrt{x^2 + y^2}) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left( \frac{x^2 + y^2}{4} \right)^{n+1}
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{m=0}^{n+1} \frac{1}{m!(n+1-m)!} \left( \frac{x}{2} \right)^{2m} \left( \frac{y}{2} \right)^{2(n-m+1)}
\]

\[
= -2 \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(p+1)!(q+1)!(p+q+1)!} \left( \frac{x}{2} \right)^{2(p+1)} \left( \frac{y}{2} \right)^{2(q+1)}
\]

\[
+ 2 \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+1)!} \left( \frac{x}{2} \right)^{2(p+1)} + 2 \sum_{q=0}^{\infty} \frac{(-1)^q}{q!(q+1)!} \left( \frac{y}{2} \right)^{2(q+1)}.
\]

7. To compare (B.10) with (B.9), we decompose the above sum into three different regions, \( p, q = 0, \cdots, \infty, p = -1, q = 0, \cdots, \infty \) and \( p = 0, \cdots, \infty, q = -1 \) such that

\[
Q(x, y) = -2 \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(p+1)!(q+1)!(p+q+1)!} \left( \frac{x}{2} \right)^{2(p+1)} \left( \frac{y}{2} \right)^{2(q+1)}
\]

\[
+ 2 \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+1)!} \left( \frac{x}{2} \right)^{2(p+1)} + 2 \sum_{q=0}^{\infty} \frac{(-1)^q}{q!(q+1)!} \left( \frac{y}{2} \right)^{2(q+1)}.
\]

8. Using once more (8.440) of [12], we deduce

\[
J(x, y) = -\frac{\sqrt{x^2 + y^2}}{xy} J_1(\sqrt{x^2 + y^2}) + \frac{1}{y} J_1(x) + \frac{1}{x} J_1(y).
\]

We can now take \( x, y \) to be imaginary-valued and finally arrive at

\[
I(x, y) \equiv \int_{0}^{\pi} d\theta I_1(x \sin \theta) I_1(y \cos \theta) = -J(ix, iy)
\]

\[
= \frac{\sqrt{x^2 + y^2}}{xy} I_1(\sqrt{x^2 + y^2}) - \frac{1}{y} I_1(x) - \frac{1}{x} I_1(y).
\]

In our analysis, we are interested in the large \( x, y \) limit whereby the last two terms in (B.13) are exponentially suppressed compared to the first term and may thus be dropped.

References

[1] J. L. Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. B 270, 186 (1986).
[2] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Infinite Conformal Symmetry in Two-dimensional Quantum Field Theory*, Nucl. Phys. B 241, 333 (1984).

[3] P. di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer, 1997.

[4] S. V. Ketov, *Conformal Field Theory*, World Scientific, 1995.

P. H. Ginsparg, *Applied Conformal Field Theory*, hep-th/9108028.

M.R. Gaberdiel, *An Introduction to Conformal Field Theory*, Rept. Prog. Phys. 63, 607 (2000), hep-th/9910156.

G. B. Segal, *The Definition of Conformal Field Theory*, in: Topology, Geometry and Quantum Field Theory, volume 308 of London Math. Soc. Lecture Note Ser., p. 421ff. Cambridge University Press, 2004.

[5] J. Polchinski, *String Theory. Vol. 1: An Introduction to the Bosonic String*, Cambridge University Press, 1998.

[6] S. Hellerman, *A Universal Inequality for CFT and Quantum Gravity*, arXiv:0902.2790.

[7] E. Witten, *Three-Dimensional Gravity Revisited*, arXiv:0706.3359.

[8] S. Carlip, *What we don’t know about BTZ black hole entropy*, Class. Quant. Grav. 15, 3609 (1998), hep-th/9806026.

—, *Logarithmic Corrections to Black Hole Entropy from the Cardy Formula*, Class. Quant. Grav. 17, 4175 (2000), gr-qc/0005017.

[9] R. Dijkgraaf, J. M. Maldacena, G. W. Moore and E. P. Verlinde, *A Black Hole Farey Tail*, hep-th/0005003.

[10] J. Manschot and G. W. Moore, *A Modern Fareytail*, Commun. Num. Theor. Phys. 4, 103 (2010), arXiv:0712.0573.

[11] A. Maloney and E. Witten, *Quantum Gravity Partition Functions in Three Dimensions*, JHEP 1002, 029 (2010), arXiv:0712.0155.

[12] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition (February 2007), Academic Press.

[13] D. Kutasov and N. Seiberg, *Number of Degrees of Freedom, Density of States and Tachyons in String Theory and CFT*, Nucl. Phys. B 358, 600 (1991).

N. Seiberg, *Notes on Quantum Liouville Theory and Quantum Gravity*, Prog. Theor. Phys. Suppl. 102, 319 (1990).
[14] F. Correa, C. Martinez, R. Troncoso, Scalar Solitons and the Microscopic Entropy of Hairy Black Holes in Three Dimensions, arXiv:1010.1259.

[15] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N Field Theories, String Theory and Gravity, Phys. Rept. 323, 183 (2000), hep-th/9905111.

[16] M. Bañados, Embeddings of the Virasoro Algebra and Black Hole Entropy, Phys. Rev. Lett. 82, 2030 (1999), hep-th/9811162.

L. Borisov, M. B. Halpern and C. Schweigert, Systematic Approach to Cyclic Orbifolds, Int. J. Mod. Phys. A 13 125 (1998), hep-th/9701061, and references therein.

[17] V. Jejjala and S. Nampuri, Cardy and Kerr, JHEP 1002, 088 (2010), arXiv:0909.1110.

[18] F. Loran and H. Soltanpanahi, 5D Extremal Rotating Black Holes and CFT Duals, Class. Quant. Grav. 26, 155019 (2009), arXiv:0901.1595.

[19] A. Strominger, Black Hole Entropy from Near Horizon Microstates, JHEP 9802, 009 (1998), hep-th/9712251.

[20] M. Bañados, C. Teitelboim and J. Zanelli, The Black Hole in Three-dimensional Spacetime, Phys. Rev. Lett. 69, 1849 (1992), hep-th/9204099.

M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Geometry of the (2 + 1) Black Hole, Phys. Rev. D 48, 1506 (1993), gr-qc/9302012.

[21] J. D. Brown and M. Henneaux, Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity, Commun. Math. Phys. 104, 207 (1986).

[22] A. Sen, Black Hole Entropy Function, Attractors and Precision Counting of Microstates, Gen. Rel. Grav. 40, 2249 (2008), arXiv:0708.1270.

—, Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity, JHEP 0509, 038 (2005), hep-th/0506177.

[23] V. Iyer and R. M. Wald, Some Properties of Noether Charge and a Proposal for Dynamical Black Hole Entropy, Phys. Rev. D 50, 846 (1994), gr-qc/9403028.

[24] D. Birmingham and S. Sen, An Exact Black Hole Entropy Bound, Phys. Rev. D 63, 047501 (2001), hep-th/0008051.

D. Birmingham, I. Sachs and S. Sen, Exact Results for the BTZ Black Hole, Int. J. Mod. Phys. D 10, 833 (2001), hep-th/0102155.
[25] N. Seiberg and E. Witten, *The D1/D5 System and Singular CFT*, JHEP **9904**, 017 (1999), hep-th/9903224.

[26] H. Saida and J. Soda, *Statistical Entropy of BTZ Black Hole in Higher Curvature Gravity*, Phys. Lett. B **471**, 358 (2000), gr-qc/9909061.

[27] N. Banerjee, D. P. Jatkar and A. Sen, *Asymptotic Expansion of the $\mathcal{N}=4$ Dyon Degeneracy*, JHEP **0905**, 121 (2009), arXiv:0810.3472.

S. Murthy and B. Pioline, *A Farey Tale for $\mathcal{N}=4$ Dyons*, JHEP **0909**, 022 (2009), arXiv:0904.4253.