Supersolidity of the $\alpha$ cluster structure in the nucleus $^{12}\text{C}$

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Abstract

For more than half a century, the structure of $^{12}\text{C}$, such as the ground band, has been understood to be well described by the three $\alpha$ cluster model based on a geometrical crystalline picture. On the contrary, recently it has been claimed that the ground state of $^{12}\text{C}$ is also well described by a nonlocalized cluster model without any geometrical configurations originally proposed to explain the dilute gas-like Hoyle state, which is now considered to be a Bose-Einstein condensate of $\alpha$ clusters. The challenging unsolved problem is how we can reconcile the two exclusive $\alpha$ cluster pictures of $^{12}\text{C}$, crystalline vs nonlocalized structure. We show that the crystalline cluster picture and the nonlocalized cluster picture can be reconciled by noticing that they are a manifestation of supersolidity with properties of both crystallinity and superfluidity. This is achieved through a superfluid $\alpha$ cluster model based on effective field theory, which treats the Nambu-Goldstone zero-mode rigorously. For several decades, scientists have been searching for a supersolid in nature. Nuclear $\alpha$ cluster structure is considered to be the first confirmed example of a stable supersolid.
A supersolid [1–11] is a solid that exhibits the property of superfluidity. Supersolids have been searched for in He II and recently in the Bose-Einstein condensate (BEC) of atomic gas. Superfluidity is caused by spontaneous symmetry breaking (SSB) of the global phase and has been known even for small systems with ten or fewer interacting and trapped particles such as parahydrogen [12–14] and He clusters [15, 16]. In nuclei, it has been shown that the interacting gas-like three $\alpha$ clusters in the $0^+_2$ (7.65 MeV) Hoyle state of $^{12}\text{C}$ is a BEC [17–25] and a superfluid cluster model reproduced the experimental data well. In the present paper, we show that crystalline $\alpha$ cluster structure, which has been described successfully by many cluster models, has the simultaneous properties of crystallinity and superfluidity. That is, the $\alpha$ cluster structure is a stable supersolid.

The geometrical $\alpha$ cluster model based on the crystalline picture [26–29], which was originally proposed following the intuitive geometrical classical picture [30–32], has witnessed much success in recent decades [33–38]. This model has explained the structure of light nuclei [33–35] (typically $\alpha+\alpha$ cluster structure in $^8\text{Be}$ and $\alpha+^{16}\text{O}$ cluster structure in $^{20}\text{Ne}$) and in medium-weight and heavy nuclei [36] (typically $\alpha+^{40}\text{Ca}$ structure in $^{44}\text{Ti}$ and $\alpha+^{208}\text{Pb}$ structure in $^{212}\text{Po}$). The emergence of cluster structure is found to be a consequence of the Pauli principle [37, 38].

The $\alpha$ cluster model picture has been successful not only in understanding the structure in the bound energy region but also anomalous large angle scattering, and Airy structure in nuclear prerainbows and rainbows over a wide range of scattering energies in a unified way, as evidenced typically by $\alpha+^{16}\text{O}$ and $\alpha+^{40}\text{Ca}$ systems [38, 40]. The $\alpha$ cluster picture based on crystallinity has been confirmed from negative energy to high positive energy.

The localized three $\alpha$ cluster structure of $^{12}\text{C}$, Fig. 1(a), has been thoroughly studied using cluster models based on geometrical three $\alpha$ configurations, including the generator coordinate method (GCM) with the Brink wave function [41, 42], the resonating group method (RGM) [43, 44], semi-microscopic three $\alpha$ boson models using the orthogonality condition model (OCM) [45] and the Faddeev equation [46]. They all support the geometrical three $\alpha$ cluster structure in $^{12}\text{C}$ such as the triangle geometry for the ground band, whose precise shape may be determined by experiment [47].

On the other hand, it has recently been claimed that typical $\alpha$ cluster structures such as the ground band of $^{20}\text{Ne}$ and $^{12}\text{C}$ can be understood by the completely opposing picture of a nonlocalized cluster model (NCM) without any geometrical configuration [48–51]. This was
originally proposed \cite{17, 18} to explain the dilute gas-like \( \alpha \) cluster structure of the Hoyle state in \( ^{12}\text{C} \), which is considered to be a BEC of \( \alpha \) clusters \( \cite{19, 25} \).

The two concepts of the *geometrical crystalline* cluster, Fig. 1(a), and the *nonlocalized* cluster, Fig. 1(b), are apparently incompatible with each other. The challenge is to solve this puzzle by reconciling the two exclusive pictures because both pictures explain the \( \alpha \) cluster structure in the bound and quasi-bound energies almost equally well.

The purpose of this paper is to show that the crystalline cluster picture of Fig. 1(a) and the nonlocalized cluster picture of Fig. 1(b) can be reconciled. We achieve this by hypothesizing that the \( \alpha \) cluster structure has the properties shown in Fig. 1(d) of both crystallinity, Fig. 1(a), and superfluidity, Fig. 1(c), simultaneously. We confirm this hypothesis for the historically most thoroughly studied \( N-\alpha \) cluster nucleus, \( ^{12}\text{C} \). It is essential to rigorously treat the Nambu-Goldstone (NG) zero mode due to the spontaneous symmetry breaking of the global phase in finite systems with a small number of particles. This has not been respected in traditional cluster models.

For this purpose, we use a field theoretical superfluid cluster model (SCM) in which the order parameter that satisfies the Gross-Pitaevskii equation is defined and the number fluctuation of \( \alpha \) clusters is taken into account. The formulation was originally developed
to study a gas-like BEC state \[23–25\]. The model Hamiltonian for a bosonic field \(\hat{\psi}(x)\) \((x = (x, t))\) representing the \(\alpha\) cluster is given as follows:

\[
\hat{H} = \int d^3 x \, \hat{\psi}^\dagger(x) \left( -\frac{\nabla^2}{2m} + V_{\text{ex}}(x) - \mu \right) \hat{\psi}(x) + \frac{1}{2} \int d^3 x \, d^3 x' \, \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') U(|x - x'|) \hat{\psi}(x') \hat{\psi}(x),
\]

with \(V_{\text{ex}}\) and \(U(|x - x'|)\) being a mean field potential in which the \(\alpha\) clusters are trapped and the residual \(\alpha-\alpha\) interaction, respectively. We set \(\hbar = c = 1\).

When superfluidity of \(\alpha\) clusters occurs, i.e. the global phase symmetry of \(\hat{\psi}\) is spontaneously broken, we decompose \(\hat{\psi}\) as \(\hat{\psi}(x) = \xi(r) + \hat{\varphi}(x)\) where the c-number \(\xi(r) = \langle 0 | \hat{\psi}(x) | 0 \rangle\) is an order parameter and is assumed to be real and isotropic. To obtain the excitation spectrum, we need to solve three coupled sets of equations, the Gross–Pitaevskii (GP) equation, the Bogoliubov-de Gennes (BdG) equations, and the zero-mode equation. The GP equation determines the order parameter, \(\xi\), and is given by

\[
\left\{-\frac{\nabla^2}{2m} + V_{\text{ex}}(r) - \mu + V_H(r)\right\} \xi(r) = 0,
\]

where \(V_H(r) = \int d^3 x' \, U(|x - x'|) \xi^2(r')\). \(\xi\) is normalized with the superfluid particle number \(N_0\) as \(\int d^3 x \, |\xi(r)|^2 = N_0\). The superfluid density is given by \(\rho_s = |\xi(r)|^2/N_0\). The BdG equations describe the collective oscillations of the superfluid and are given by

\[
\int d^3 x' \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ -\mathcal{M}^* & -\mathcal{L}^* \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \omega_n \begin{pmatrix} u_n \\ v_n \end{pmatrix},
\]

where

\[
\mathcal{M}(x, x') = U(|x - x'|) \xi(r) \xi(r'),
\]

\[
\mathcal{L}(x, x') = \delta(x - x') \left\{-\frac{\nabla^2}{2m} + V_{\text{ex}}(r) - \mu + V_H(r)\right\} + \mathcal{M}(x, x').
\]

The index \(n = (n, \ell, m)\) stands for the main, azimuthal, and magnetic quantum numbers. The eigenvalue \(\omega_n\) is the excitation energy of the BdG mode. For isotropic \(\xi\), the BdG eigenfunctions can be taken to have separable forms, \(u_n(x) = U_{n\ell}(r)Y_{\ell m}(\theta, \phi)\), \(v_n(x) = V_{n\ell}(r)Y_{\ell m}(\theta, \phi)\). We necessarily have an eigenfunction belonging to a zero eigenvalue, \((\xi(r), -\xi(r))^t\), and its adjoint function \((\eta(r), \eta(r))^t\) is obtained as \(\eta(r) = \partial \xi(r)/\partial N_0\). The field operator is expanded as \(\hat{\varphi}(x) = -i \dot{Q}(t) \xi(r) + \hat{P} \eta(r) + \sum_n \{\hat{a} u_n(x) + \hat{a}^\dagger v_n^*(x)\}\) with
the commutation relations $[\hat{Q}, \hat{P}] = i$ and $[\hat{a}_n, \hat{a}^\dagger_{n'}] = \delta_{nn'}$. The operator $\hat{a}_n$ is an annihilation operator of the BdG mode, and the pair of canonical operators $\hat{Q}$ and $\hat{P}$, which are called the NG or zero-mode operators, originate from the SSB of the global phase. The nonlinear Hamiltonian for $\hat{Q}$ and $\hat{P}$, $\hat{H}^{QP}$, whose explicit forms are given in Ref. [25], gives a discrete spectrum in the zero–mode equation $\hat{H}^{QP} |\Psi_\nu\rangle = E_\nu |\Psi_\nu\rangle (\nu = 0, 1, \cdots)$, just as a one-dimensional quantum mechanical Hamiltonian with a binding potential does. The total unperturbed Hamiltonian is $\hat{H}_u = \hat{H}^{QP} + \sum_n \omega_n \hat{a}^\dagger_n \hat{a}_n$. The states that we consider are $|\Psi_\nu\rangle |0\rangle_{ex}$ with energy $E_\nu$, called the zero-mode state, and $|\Psi_0\rangle \hat{a}_n^\dagger |0\rangle_{ex}$ with energy $\omega_n$, called the BdG state, where $\hat{a}_n |0\rangle_{ex} = 0$.

As in Refs. [23–25], we take $V_{ex}(r) = m\Omega^2 r^2 / 2$, and $U(|x - x'|) = V_r e^{-\mu^2 |x - x'|^2} - V_a e^{-\mu_a^2 |x - x'|^2}$, with $V_r$ and $V_a$ being the strength parameters of the short-range repulsive potential due to the Pauli principle [52] and long-range attractive potential, respectively. The chemical potential is fixed by the specification of the superfluid particle number $N_0$. We identify the ground state as the vacuum $|\Psi_0\rangle |0\rangle_{ex}$. The range parameters $\mu_a$ and $\mu_r$ are fixed to the values 0.475 fm$^{-1}$ and 0.7 fm$^{-1}$ in Ref. [52], respectively. The two potential parameters, $\Omega$, which controls the size of the system, and $V_r$, which prevents collapse of the condensate, are determined to be $\Omega = 4.093$ MeV/\(\hbar\) and $V_r = 610$ MeV. These reproduce the experimental root mean square (rms) radius, 2.45 fm, of the ground state, $|\Psi_0\rangle |0\rangle_{ex}$ and the energy level of the $0^+_2$ Hoyle state, identified as the first excited zero-mode state $|\Psi_1\rangle |0\rangle_{ex}$.

Before proceeding to the calculated results and discussions, we briefly mention what is newly developed from our previous papers [23–25], where we focused on gas-like dilute $\alpha$ cluster states like the Hoyle state with a considerably large condensation rate such as 70% and showed that they can be understood as a BEC of $\alpha$ clusters in the strict sense that the global phase of the system is spontaneously broken. BEC occurs whatever the condensation rate and the density distribution if the global phase is locked. What is new in the present paper is that SSB of the global phase of the system with non-gas-like $\alpha$ cluster structure occurs stably even under small condensation rate such as 5%, which has never been thought about before in studies of $\alpha$ cluster condensation in nuclei. Furthermore, the negative parity states of $^{12}$C are also treated, whereas the previous studies of gas-like BEC states [23–25] discussed only the positive parity states. We find that our theory reproduces well the observed negative parity states in $^{12}$C, $3^-$ at 9.64 MeV and $1^-$ at 10.85 MeV, for the first time from the viewpoint of superfluidity of the $\alpha$ cluster structure.
In Fig. 2 the energy levels calculated using our SCM without any geometrical crystallinity, Fig. 1(c), and assuming a small superfluid density (condensation rate) of 5%, i.e. \( N_0 = 0.05 \times 3 \), are compared with the experimental data and other \( \alpha \) cluster model calculations based on the geometrical crystalline picture, Fig. 1(a). In the SCM the \( J^\pi = 2^+, 4^+, 3^-, \) and \( 1^- \) states emerge as BdG mode excitations, and the \( 0^+_2 \) Hoyle state appears as an NG zero-mode excitation on the superfluid vacuum. The agreement of the \( 3^- \) and \( 1^- \) states with experiment is good. The agreement of the \( 2^+ \) and \( 4^+ \) states with experiment would be improved if the deformation of \( V_{ex} \) were taken into account, since it would shift the excitation energy of the \( 2^+ \) and \( 4^+ \) states downward and upward, respectively. The agreement of the SCM with experiment is comparable to the GCM and RGM calculations, both of which locate the rotational band \( 2^+ \) and \( 4^+ \) states with an equilateral triangular \( \alpha \) cluster configuration considerably lower than the experimental data. In Fig. 2(c) the \( 3^- \) and \( 1^- \) states have the equilateral and non-equilateral triangular configurations of the three \( \alpha \) clusters, respectively. We consider that the \( \alpha \) cluster structure involves both properties of geometrical crystallinity and superfluidity, which evokes Landau’s two-fluid model (normal fluid and superfluid) of \( \text{He II} \) and the duality (particle and wave) of light, in which the superfluidity and the wave nature are both caused by the formation of a coherent wave function (order parameter) due to the BEC of the bosons belonging to a zero eigenvalue.

In Fig. 3(a) the calculated eigenfunction \( \xi(r) \) and its adjoint eigenfunction \( \eta(r) \) are displayed. We see that the number fluctuation of the superfluid \( \alpha \) clusters in the ground state, \( \eta \), is highest near the surface region and decreases toward the inner and outer regions. In Fig. 3(b) \( \rho_s \) and \( \rho \) represent the probabilities of finding the superfluid \( \alpha \) clusters and nucleons, respectively. \( \rho_s \) is highest in the center of the nucleus and gradually decreases toward the surface region. The non-superfluid normal density may be defined as \( \rho_n \equiv \rho - \rho_s \). \( \rho_s \) is much smaller than \( \rho \). However, it is this small superfluid density component that causes the coherent wave nature of the system. The predisposition of the superfluid fraction component \( \rho_s \) in the ground state of \( \text{^{12}C} \) arises partly due to the orthogonality to the BEC Hoyle state.

In Fig. 4 the BdG wave functions \( \mathcal{U}_{nl}(r) \) and \( \mathcal{V}_{nl}(r) \) for the \( 2^+ \) and \( 3^- \) states are displayed. The peak of \( \mathcal{U}_{nl}(r) \) for \( \ell \neq 0 \) is located in the surface region because of the repulsive force between the \( \alpha \) clusters and moves outward with increasing \( \ell \) due to the centrifugal force. The magnitude of \( \mathcal{V}_{nl}(r) \) is negligible for the \( 2^+ \) and \( 3^- \) states, implying no Bogoliubov
mixing in these states due to the small condensation rate.

We proceed to understand why the apparently exclusive pictures of SCM, Fig. 1(c), and GCM, Fig. 1(a), give similar results. The geometrical structure in Fig. 1(a) has not previously been considered to be related to superfluidity of $\alpha$ clusters. Also, no attention has been paid to the treatment of the global phase of the wave function with a geometrical
configuration. However, we note that in Fig. 1(d) bosons, α clusters of the Brink model in the GCM, which are sitting in the 0s state of distinct (due to the Pauli principle) harmonic oscillator potentials and are arranged with the geometrical configuration of Fig. 1(a), can form a coherent wave. This is suggestive of the optical lattice [6, 55–57] in which trapped cold atom bosons form a coherent condensed (superfluid) wave function. In fact, the de Broglie wavelength of each 0s state α cluster with very low energy is far larger than the geometrical distance \( d \) between the α clusters. This means that the phases of the waves are locked to form a coherent wave function, i.e. superfluidity (condensation) of the system. This logic is general and independent of the geometrical configuration and number of α clusters involved, \( N \). Therefore in principle, whatever the geometrical configuration, triangle, linear chain \( N \)-α cluster (\( N = 2, 3, 4, \cdots \)) or tetrahedron (\( N = 4 \)), trigonal bipyramid (\( N = 5 \)), etc., the geometrical α cluster structures have the potential to form a coherent wave function (superfluidity). Whether the state is superfluid depends on \( \rho_s \), which encapsulates the structure and degree of clustering.

The present study finds that the superfluid ground state is stable with a condensation rate that is 5%, giving similar energy levels to the GCM, RGM, and experiment, as shown in Fig. 2. This strongly supports the view of a geometrical α cluster structure for the ground

![FIG. 4. Calculated BdG wave functions \( U_{n\ell}(r) \) (solid lines) and \( V_{n\ell}(r) \) (dotted lines) for the 2\(^+\) \((n = 0, \ell = 2)\) and 3\(^-\) \((n = 0, \ell = 3)\) states.](image-url)
FIG. 5. Energy levels of $^{12}$C calculated in the nonlocalized cluster model (NCM) \[59, 60\] with (a) Volkov force No.1 and (b) Volkov force No.2 are compared with (c) the experimental energy levels \[53\], (d) the superfluid cluster model (SCM) calculations and the three boson model calculations using (e) the OCM \[45\] and (f) the Faddeev model \[46\].

state with superfluid density that is sufficient to form a coherent wave. We note that the emergence of the coherent wave function due to condensation in nature is possible even if the condensation rate is not large. In fact, it was shown through systematic calculations \[25\] that BEC of $\alpha$ clusters like the Hoyle state occurs stably even under a small condensation rate such as 20%. We also note that the superfluidity of heavy nuclei occurs due to the Cooper pairs generated by a small number of nucleons near the Fermi surface \[54\] as well the BEC of He II with a condensation rate of approximately 10% \[58\]. It is useful to decompose the density distribution $\rho^{GCM}$ due to the GCM wave function $\Psi^{GCM}$ based on Fig. 1(a) as $\rho^{GCM} = \rho^{GCM}_s + \rho^{GCM}_n$ where $\rho^{GCM}_s$ is the superfluid density due to the coherent wave function (order parameter) and $\rho^{GCM}_n$ is the noncondensed component.

In Fig. 5 the energy levels of the NCM calculations \[59, 60\] with the Volkov force \[61\] are compared with the SCM calculations, the experimental data and the three $\alpha$ boson model calculations based on the geometrical crystalline picture using the OCM \[45\] and the Faddeev model \[46\]. In Fig. 5(a) and (b) $1^-$ is absent because it was not reported in Refs. \[59, 60\]. The NCM calculations reproduce the experimental energy level ordering, similar to the GCM, RGM, and SCM, although the $2^+$ and $4^+$ states are at considerably lower energy than the experimental data. Both the NCM and SCM locate the $2^+$ state
deviated from experiment, downward and upward, respectively. The SCM gives better agreement with the experimental $4^+$ state. While the ground band of the NCM forms a rotational band, whose moment of inertia is much larger (1.8 times for V1 and 1.5 times for V2) than the experimental value, the SCM seems not to form a rotational band. This is understood as follows. While in the NCM calculations the three $\alpha$ clusters are assumed to sit in the deformed harmonic potential with the oscillator parameters $B_x=B_y\neq B_z$ for $x$, $y$ and $z$ directions, namely the trapping (container) potential in Fig. 1(b) is deformed, the trapping potential of the SCM is assumed to be spherical in the present calculations. If the SSB of the rotational invariance is introduced in the external potential $V_{\text{ex}}$ in Eq. (1) of the SCM by using a deformed trapping harmonic oscillator potential with $\Omega_x=\Omega_y\neq\Omega_z$ corresponding to $B_x=B_y\neq B_z$ in the NCM, the rotational character of the ground band would be recovered. A reduction of the moment of inertia due to the superfluid component $\rho_s$ may be expected.

The NCM wave function $\Psi^{\text{NCM}}$ given by Eq. (3) of Ref.[17] is obtained by constraining the generating function of the GCM to the Gaussian form $f(R) = \exp(-R^2/\beta^2)$[18]. It spans a subspace of the whole GCM Hilbert space. Physically, $\Psi^{\text{NCM}}$ approximately extracts a condensate-like component from the whole GCM wave function $\Psi^{\text{GCM}}$. Although $\Psi^{\text{NCM}}$ with all the $\alpha$ clusters sitting in the 0s state as in Fig. 2(b) under the antisymmetrization operator between the clusters is a nonlocalized cluster wave function, it is not an exact condensate wave function in the sense that the number fluctuations of the $\alpha$ clusters are not taken into account. It may be called a pseudo-condensed model because it involves the condensate component, which can be dominant in the Hoyle state. The overlap of the GCM wave function $\Psi^{\text{GCM}}$ of Refs.[41][42] with $\Psi^{\text{NCM}}$ is about 0.93 for the ground state[48]. The ground state GCM (RGM) wave function, $\Psi^{\text{GCM}}(\Psi^{\text{RGM}})$ of Refs. [41][44], can be represented well by a single $\Psi^{\text{NCM}}(\beta)$ with a large overlap of almost 100%[51]. This means that the GCM (RGM) wave functions based on the geometrical picture, Fig. 1(a), almost equivalently involve the nonlocalized cluster of Fig. 1(b). In other words, $\Psi^{\text{GCM}}$ has duality involving both the crystalline, Fig. 1(a), and nonlocalized cluster, Fig. 1(b), nature simultaneously. Similar to $\rho_s^{\text{GCM}}$, $\rho_s^{\text{NCM}}$ may be decomposed as $\rho_s^{\text{NCM}}=\rho_s^{\text{SCM}}+\rho_s^{\text{NCM}}$. Physically $\rho_s^{\text{NCM}}$ in Fig. 1(b) corresponds to $\rho_s^{\text{SCM}}$ in Fig. 1(c). It is now clear that $\Psi^{\text{GCM}}$ should not be regarded simply as a crystalline wave function as it appears in Brink's wave function since it involves the dual nature of crystallinity and nonlocalized coherent wave
structure via $\rho_s$. A crystalline $\alpha$ cluster structure with superfluidity is called a supersolid.

Considering the duality of the $\alpha$ cluster structure, it is natural that the energy spectrum of $^{12}\text{C}$ is described by both the GCM and RGM based on the crystalline picture and by the SCM and NCM based on the nonlocalized cluster picture with wave nature. Since the coherent wave function is represented by $\Psi$ with a common phase $\Phi$, i.e. $\Psi = |\Psi| \exp(i\Phi)$ [54], it is natural that such a wave function can be represented by a single wave function if the parameter $\beta$, which determines the size parameter of the trapping harmonic oscillator potential, is properly chosen. Because of the duality, it is not surprising that not only in the 5-$\alpha$ nucleus $^{20}\text{Ne}$ [48, 49] but also in many $\alpha$ cluster nuclei [17, 18, 50, 51, 60, 62–67], which have been successfully described by the crystalline picture, the wave functions are well represented by a single nonlocalized cluster wave function. It is a manifestation of the wave nature of the duality of the $\alpha$ cluster structure.

We can see the duality of crystallinity and nonlocalized cluster in the most typical well-developed crystalline dumbbell $\alpha$ cluster of $^{8}\text{Be}$. The solved GCM wave function of the ground band, $\Psi^{\text{GCM}}$, is well represented with the overlaps, 0.96, 0.96, and 0.93 for the $0^+$, $2^+$, and $4^+$ states, respectively, by a single Brink wave function $\Psi^{\text{Brink}}(R)$ with the distance parameter $R=3.5$ fm [68]. At the same time, $\Psi^{\text{GCM}}$ is represented by a single $\Psi^{\text{NCM}}$ with an overlap of almost 100% [18]. This means that the $\alpha$ cluster of $^{8}\text{Be}$ has the duality of crystalline and nonlocalized cluster nature. $^{8}\text{Be}$, which is a starting nucleus with a two-$\alpha$ linear chain structure in the Ikeda diagram [27, 69] and in the extended Ikeda diagram [40, 70], can be considered a prototype example of such duality of the $\alpha$ cluster structure in nuclei.

To summarize, we have shown that the energy levels with the $\alpha$ cluster structure in $^{12}\text{C}$ as well as the rms radius of the ground state, which have been understood to have a crystalline structure described well by cluster models, can also be described by a superfluid $\alpha$ cluster model. The $\alpha$ cluster structure is found to have duality, simultaneously exhibiting properties that would intuitively be considered mutually exclusive: crystallinity and nonlocalized cluster structure. The $\alpha$ cluster wave function based on a crystalline picture involves a superfluid component, whose coherent wave represents the nonlocalization of the wave function described by a nonlocalized cluster model. Considering the $\alpha$ cluster structure as a supersolid having crystallinity and superfluidity, it is not at all surprising that it exhibits the crystalline and the nonlocalized pictures simultaneously. The emergence of the low-
lying collective Hoyle state as a Nambu-Goldstone mode is a manifestation of the SSB of
the global phase of the ground state with crystallinity. The supersolidity of the \( \alpha \) cluster
structure is considered to be the first confirmed example of a stable supersolid in nature.

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