Nontrivial critical crossover between directed percolation models: Effect of infinitely many absorbing states

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At non-equilibrium phase transitions into absorbing (trapped) states, it is well known that the directed percolation (DP) critical scaling is shared by two classes of models with a single (S) absorbing state and with infinitely many (IM) absorbing states. We study the crossover behavior in one dimension, arising from a considerable reduction of the number of absorbing states (typically from the IM-type to the S-type DP models), by following two different (excitatory or inhibitory) routes which make the auxiliary field density abruptly jump at the crossover. Along the excitatory route, the system becomes overly activated even for an infinitesimal perturbation and its crossover becomes discontinuous. Along the inhibitory route, we find continuous crossover with the universal crossover exponent $\phi \simeq 1.78(6)$, which is argued to be equal to $\nu_3$, the relaxation time exponent of the DP universality class on a general footing. This conjecture is also confirmed in the case of the directed Ising (parity-conserving) class. Finally, we discuss the effect of diffusion to the IM-type models and suggest an argument why diffusive models with some hybrid-type reactions should belong to the DP class.

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I. INTRODUCTION

The directed percolation (DP) has been studied extensively as one of typical dynamic critical phenomenon far from equilibrium [1]. Nonequilibrium phase transitions of systems with a unique absorbing state are found to belong to the DP class if no symmetry or conservation of the order parameter is present [2]. Even systems with infinitely many (IM) absorbing states such as the pair contact process (PCP) [3] are believed to share the same critical behavior with the DP, but the theoretical understanding of the IM-type DP (DP_{IM}) models is still lacking. For example, the phenomenological field theory introduced in Ref. [4] and elaborated in Ref. [5] shows inconsistency with the numerical studies. In fact, the field theory based on the phenomenological Langevin equation predicts that the PCP should belong to the dynamic percolation rather than the DP class [5]. Besides the universality issue, its spreading dynamics is also not fully understood [8-10].

One may reduce the number of absorbing states significantly by introducing particle diffusion (PCP with diffusion or PCPD) [5]. Surprisingly, the PCPD has brought up serious turmoil, which could not be settled down in spite of extensive numerical [5, 11, 12] and analytical [12] studies. The answer seems to be one of two possibilities: The PCPD belongs to the DP class with a long transient, or forms a new universality class distinct from known universality classes to date. As an attempt to resolve the issue, the present authors suggested two different approaches to the PCPD.

First, introducing the dynamic perturbation which is implemented by the biased hopping, we showed that the one-dimensional PCPD with biased diffusion (driven PCPD or DPCPD), exhibits critical scaling distinct from the unbiased PCPD, but instead shares the critical behavior with the two-dimensional PCPD without bias (dimensional reduction) [13, 14]. Since all known DP models are robust against the biased diffusion [13, 14], a DPCPD-type variant can serve as a litmus test for the PCPD scaling [15]. Second, we studied the crossover behavior from the PCPD to the DP by introducing single-particle annihilation/branching reactions and showed that there are diverging crossover scales with the universal nontrivial crossover exponent [16]. These results provided another evidence that the PCPD is distinct from the DP.

In this paper, we study the crossover behavior from the DP_{IM} to the DP in order to understand better the difference between these two “equivalent” (DP) universality classes. Actually, it would be absurd to talk about the crossover between two models belonging to the identical universality class (see Sec. [II]). However, the DP_{IM} models differ from the DP models in regard to the “non-order” parameter at the transitions, which shows a discontinuous singularity at the crossover. This singularity induces a well-defined and nontrivial crossover from the DP_{IM} to the DP, more generally a crossover arising from a considerable reduction of the number of absorbing states (between two different DP_{IM}).

We find two distinct crossover behaviors depending on the routes to reduce the number of absorbing states. Infinitesimal inclusion of an “excitatory” process (like single-particle branching) makes the system overly active, which gives rise to a discontinuous crossover (see Sec. [III]). While, the opposite “inhibitory” route (like single-particle annihilation) reveals a contin-
uous crossover with the nontrivial crossover exponent \( \phi \simeq 1.78(6) \). We find that this crossover exponent is universal for various kinds of models including the PCP and the triplet contact process (TCP). We argue that the crossover exponent is not independent but equal to \( \nu \), the relaxation time exponent of the DP universality class on a general footing. This conjecture is also confirmed in the case of the directed Ising (parity-conserving) class (see Sec. IV).

The PCPD per se can be considered one of crossover models from the DP to the DP by allowing diffusion, which reduces the number of absorbing states considerably (from exponentially many to linearly many absorbing states with respect to system size). However, the crossover study from the PCP to the PCPD does not give any useful information on the DP, because the particle diffusion makes the system more active (excitatory) and the crossover turns out to be discontinuous.

Finally, we study the crossover from the diffusive reaction models to the DP. Such an example is the crossover from the PCPD to the DP studied in Ref. [16]. In Sec. V, the hybrid-type models with diffusion (tp12: 2A → \( \emptyset \), 3A → 4A) [2] are perturbed by adding a single-particle annihilation process \( (A \rightarrow \emptyset) \) and its crossover to the DP is investigated. These hybrid-type models (where the branching process is of higher order than the annihilation process) are numerically known to belong to the DP class. As expected, the critical line emanates “linearly” from the hybrid-type model point. We suggest an argument why these models should belong to the DP rather than a PCPD-type nontrivial class.

II. CROSSOVER BETWEEN THE IDENTICAL UNIVERSALITY CLASS?

This section considers a \( d \)-dimensional stochastic system of hard core particles with dynamics summarized in Table I. In Ref. [17], it is shown that two different stochastic systems modeled by tilded and untilded rates are equivalent if transition rates satisfy the relations

\[
\tilde{\sigma} = \mu \sigma, \quad 2 \tilde{D} + \tilde{\xi} = 2D + \xi, \\
\tilde{D} + \frac{\tilde{\sigma}}{2} = D + \frac{\sigma}{2}, \\
\tilde{\lambda} + \frac{\tilde{\eta}}{2} + \frac{\tilde{\sigma}}{2} = \frac{1}{\mu} \left( \lambda + \frac{\eta}{2} + \frac{\sigma}{2} \right), \\
\tilde{\lambda} + \tilde{\eta} + \tilde{\sigma} - 2 \tilde{D} = \lambda + \eta + \sigma - 2D, 
\]

where any positive number \( \mu \) which renders all tilded and untilded transition rates nonnegative is physically meaningful. By equivalence is meant that all correlation functions of the model with tilded parameters can be deduced from those of the model with untilded parameters and vice versa. For instance, the particle density \( \rho \) at time \( t \) of two different stochastic many body systems be-

| TABLE I: \( d \)-dimensional reaction-diffusion processes of single species with hard core exclusion and their rates. |
|-----------------|-----------------|-----------------|
| Diffusion       | A\( \emptyset \) → \( \emptyset A \) with rate \( D/d \) |
| Pair annihilation| A\( A \) → \( \emptyset \) with rate \( \lambda/d \) |
| Coalescence     | A\( A \) → A\( \emptyset \) with rate \( \eta/(2d) \) |
| Death           | A → \( \emptyset \) with rate \( \xi \) |
| Branching       | \( \emptyset A \) → A\( A \) with rate \( \sigma/(2d) \) |
| Branching       | A\( \emptyset \) → A\( A \) with rate \( \sigma/(2d) \) |

comes [17]

\[
\rho(D, \lambda, \eta, \xi, \sigma; \rho_0, t) = \frac{1}{\mu} \tilde{\rho}(\tilde{D}, \tilde{\lambda}, \tilde{\eta}, \tilde{\xi}, \tilde{\sigma}; \tilde{\rho}_0, t),
\]

if the initial density of two systems has the relation \( \tilde{\rho}_0 = \mu \rho_0 \). Needless to say, both \( \rho_0 \) and \( \tilde{\rho}_0 \) should lie between 0 and 1. Although the equivalence is shown only for the one-dimensional systems in Ref. [17], Eq. (1) is generally true for any \( d \)-dimensional systems, which can be easily shown by the same technique developed in Ref. [17].

Now consider the branching annihilating random walks with one offspring (BAW1) [18] which corresponds to the model with \( \xi = 0 \) in Table I. The parameters used in Ref. [18] in one dimension are \( D = p/2, \lambda = \mu, \) and \( \eta = \sigma = 1 - p \) with the tuning parameter \( p \). If \( w \equiv \mu - 1 \) is very small and nonnegative, the solution of Eq. (1) up to the order of \( w \) is

\[
\tilde{D} - D = -\frac{w}{2} \sigma, \quad \tilde{\sigma} = \sigma, \quad \tilde{\xi} = w \sigma, \\
\tilde{\lambda} - \lambda \simeq -w(\eta + 2 \lambda), \quad \tilde{\eta} - \eta \simeq w(\eta + 2 \lambda - 2 \sigma).
\]

If \( w \) is sufficiently small, it is always possible to associate the BAW1 with a stochastic process with spontaneous death in an equivalent way with all nonnegative rates. Since the BAW1 in high dimensions is also known to have a nontrivial transition point [14], the following discussion is valid in any spatial dimension. The transition points for stochastic systems with small \( w \) can be always calculated exactly from Eq. (1) (approximately from Eq. (2)), if the transition point of the BAW1 is given.

The conclusions from the above analysis are two-fold. First, it is clear from Eq. (3) that the phase boundary (critical line) should meet the BAW1 transition point linearly with finite slope, as \( w \) vanishes. This implies that there is no additional singularity involved near \( w = 0 \), which is fully expected from the crossover between models with the identical universality class. If one defines the crossover exponent \( \phi \) from the shape of the critical line near \( w = 0 \) (see Sec. III), one can say that \( \phi = 1 \). Second, the critical decay of the density is given by \( \tilde{\rho}_c(t) = (1 + w)p_c(t) \) from Eq. (2), which implies that there is no diverging crossover time scale for small \( w \).

Since the introduction of the spontaneous death does not change the structure of the absorbing phase space
(single absorbing state) let alone the universality class, the above analysis is in good harmony with the naive expectation as to the “crossover” between two models belonging to the identical class. In the next section, however, we will show that the substantial change of the absorbing phase space without affecting the universality class will trigger a nontrivial crossover.

III. CROSSOVER FROM THE DP_{1M} TO THE DP

Unlike the BAW1, the pair contact process (PCP) is the prototype of the DP_{1M} models with exponentially many absorbing states. By introducing single-particle reactions to the PCP, the number of absorbing states changes drastically from infinity to one. This section shows that this qualitative change is reflected into the singular behavior of the phase boundary close to the PCP transition point in one dimension.

The dynamics of the model is summarized as

\[ AA \xrightarrow{p} \emptyset, \quad AA\emptyset \xrightarrow{(1-p)/2} A\emptyset, \quad \emptyset A \xrightarrow{(1-q)w^2/2} \emptyset, \quad A\emptyset \xrightarrow{qw^2/2} AA, \]

where \( 0 \leq q \leq 1 \). For the PCP at \( w = 0 \), any configuration without a pair of neighboring particles (a mixture of isolated particles and vacant sites) is absorbing and its number grows exponentially with system size. The order parameter of the PCP is the pair density (the number density of \( AA \) pairs) and the particle density field is auxiliary which is finite even in the absorbing phase. At nonzero \( w \), an isolated particle becomes active and only the vacuum becomes the true absorbing state. In this case, the particle density is usually adopted as the order parameter and the pair density scales in the same way.

Figure 1 locates the transition point of the PCP \( (w = 0) \) at \( p_0 = 0.077 0905(5) \) by exploiting the critical decay of the pair density as \( \rho_p(t) \sim t^{-\delta} \) with \( \delta \) to be the critical exponent of the DP class whose accurate value can be found in \[20\]. In numerical simulations, the system size is \( L = 2^{18} \) and the number of independent samples are 750, 1500, and 400 for the data in the active, critical, and absorbing phases, respectively. The flatness of \( \rho_p(t)t^\delta \) over four log decades in time confirms the solid DP critical scaling of the PCP.

At finite \( w \), the model still belongs to the DP class irrespective of \( q \). Unlike the PCPD to the DP crossover model in Ref. \[16\], however, the critical lines show two completely different singular behaviors, depending on the value of \( q \). For large \( q \), the activity of the system is enhanced by additional single-particle reaction processes (excitatory process) and the system becomes overly activated even with infinitesimal \( w \). The critical line does not converge to the PCP critical point as \( w \) decreases to zero \((w = 0^+)\) and shows a discontinuous jump. On the other hand, for small \( q \), the system activity is suppressed (inhibitory process) and the system becomes more inactive. The critical line nicely converges to the PCP critical point and shows a continuous crossover with a nontrivial crossover exponent.

First, we choose the \( q = 1 \) case as a typical excitatory route of the crossover from the DP_{1M} to the DP. As shown in Fig. 2, the critical line approaches \( p \approx 0.1448 \) as \( w \) approaches zero, which is by far above the criti-

![Figure 1](image1.png)

**FIG. 1:** (Color online) Semilogarithmic plot of \( \rho_p(t)t^\delta \) vs \( t \) of the PCP near criticality with \( \delta = 0.1595 \) to be the exponent of the DP. Since the upper (lower) curve veers up (down), we estimate the critical point as \( p_0 = 0.077 0905(5) \) with the error in the last digit by 5.

![Figure 2](image2.png)

**FIG. 2:** (Color online) Plots of \( \rho_p(t)t^\delta \) vs \( t \) for \( w = 10^{-3}, 10^{-4}, \) and \( w = 10^{-5} \) at \( q = 1 \) close to the critical points in semi-logarithmic scales. Again, \( \delta \) assumes the DP value. The curves in the middle correspond to \( p = 0.1451, 0.1448, \) and 0.14484 (from bottom to top), respectively and the value of \( p' \)'s of other two curves are \( \pm 0.0001 \) off from the middle value. As \( w \) becomes smaller, the relaxation time becomes larger though the critical point does not change much and approaches \( p \approx 0.1448 \).
The critical value of the PCP ($p_0 \simeq 0.077$). So there is a big jump of the critical line at $w = 0$. The discontinuity can be understood as follows: Consider a system with $p$ slightly above the PCP critical point $p_0$ and $0 < w \ll \tau^{-1}$ where $\tau$ is the relaxation time which is finite off criticality. Then the single-particle branching event ($A \rightarrow 2A$) with the characteristic time of $w^{-1}$ occurs effectively after the system falls into one of the PCP absorbing states in which the isolated particle density is finite. Since the branching event creates a new pair, the system is reactivated and performs the damage-spreading-type “defect dynamics” for some time proportional to $\tau$ and again falls into one of the PCP absorbing states. This defect dynamics continues forever with the period of time $w^{-1}$.

As the particle density is finite (and quite large) even in the PCP absorbing states, the time-averaged particle density in this iterated process should be finite in this region of the phase diagram. This implies that the continuous absorbing phase transition at infinitesimal $w$ into vacuum should occur way above $p_0$, which is consistent with our finding. Note that the discontinuity in the auxiliary field (particle) density is crucial in this crossover.

Actually, the same argument can be applied to the crossover from the PCP to the PCPD. We can introduce the diffusion rather than the single-particle reactions and again consider $p$ slightly above $p_0$. Let $\rho_0$ denote the isolated particle density at the PCP absorbing states, then the characteristic length scale between isolated particles is $1/\rho_0$. If $0 < Dp_0^2 \ll \tau^{-1}$ with the diffusion constant $D$, the “defect dynamics” will continue again indefinitely for small $D$. So the phase boundary in the $D - p$ plane should have a discontinuity at $D = 0$.

Let us turn to the crossover model with $q = 0$, which should represent a typical inhibitory route. Table II summarizes the critical points of the model for some $w$’s at $q = 0$ and the corresponding phase boundary is plotted in Fig. 3. Unlike the previous case, the reactive phase shrinks continuously with the rate of additional single-particle annihilation process and the phase boundary is continuous. The usual analysis method can be applied to this case [21]. If we define $\Delta = (p_0 - p)/p_0$ and $\Delta_c(w) = (p_0 - p_c(w))/p_0$, the phase boundary is well fitted by $\Delta_c \sim w^{1/\phi}$ with $\phi^{-1} = 0.56(2)$ or $\phi = 1.78(6)$; see the inset of Fig. 3. Let us assume the existence of the well-defined crossover scaling which is described by the scaling function [21]

$$\rho_p(\Delta, w; t) = t^{-\delta} F(\Delta^\nu t, w^\mu t), \quad (5)$$

where $\rho_p$ is the pair density and $\mu = \nu / \phi$ with $\phi$ estimated in the above. We examine whether the scaling function in Eq. (4) correctly describe the crossover near the PCP critical point.

First, we measure the pair density for various $w$’s at

![FIG. 3: (Color online) Phase boundary of the model of Eq. (4) at $q = 0$ in $w - p$ plane. Symbols locate the numerically estimated critical points. The error of the critical point is smaller than the symbol size. The curve shows the least-square-fit result of the phase boundary. The absorbing (active) phase is above (below) the curve. Inset: the same but the vertical axis is $\Delta_c(w)$ in log-log scale. The slope corresponds to the inverse of the crossover exponent which is estimated as $\phi = 1.78(6)$.](image)

![FIG. 4: (Color online) Log-log plot of the scaling function Eq. (5) for the PCP crossover model using $\delta = 0.1595$ and $\mu = 0.97$. All curves are collapsed into a single curve.](image)

| $w$ | $p_c(w)$|
|---|---|
| 0 | 0.077 0905(5) |
| $10^{-5}$ | 0.077 002(3) |
| $5 \times 10^{-5}$ | 0.076 885(3) |
| $10^{-4}$ | 0.076 784(4) |
| $2 \times 10^{-4}$ | 0.076 642(2) |
| $3 \times 10^{-4}$ | 0.076 530(5) |
| $4 \times 10^{-4}$ | 0.076 432(2) |
| $5 \times 10^{-4}$ | 0.076 345(2) |
| $6 \times 10^{-4}$ | 0.076 264(1) |
| $10^{-3}$ | 0.075 988(4) |
Scaling function $H$ at finite $w$ and $w = 0$. The asymptotic value of $w = 0$ is different from the $w \to 0$ limit.

$\Delta = 0$. From the scaling ansatz $[5]$, the pair density at $\Delta = 0$ should collapse as

$$t^\delta \rho_p(t) = G(w t^\parallel t).$$

With $\mu_\parallel \approx 0.97$, all curves for the pair density are collapsed into a single curve as Fig. 4 shows. Next, we take $\Delta = 0$ should collapse as $\Delta = 0$. From the scaling ansatz (5), the pair density at the DP Eq. (5) correctly describes the crossover behavior from all curves at different critical points collapse well into a single curve. Hence we conclude that the scaling function

$$t^\delta \rho_p(\Delta_c(w); t) = H(w t^\parallel t),$$

where $H(x)$ approaches a constant as $x \to \infty$. In Fig. 4 all curves at different critical points collapse well into a single curve. Hence we conclude that the scaling function Eq. (5) correctly describes the crossover behavior from the DP$_{1M}$ to the DP.

Since the models at $w = 0$ and at $w \neq 0$ belong to the same DP universality class, it is natural to ask what is the origin of such a nontrivial singularity near the PCP critical point. The inset of Fig. 5 gives a hint to this question, which shows that the scaling function $H$ (the amplitude of the critical decay) does not approach the PCP value as $w$ goes to zero, i.e., it is not continuous at $w = 0$. The discontinuity in this amplitude must originate again from the discontinuity in the auxiliary field (particle) density. One can see it directly from the behavior of the particle density ($\rho$). Unlike the pair density, $\rho$ cannot be described by the scaling function $[5]$. Consider again the case at $\Delta = 0$ and nonzero $w$. For any finite value of $w$ in the thermodynamic limit, $\rho(t)$ approaches to zero as $t \to \infty$; see Fig. 6. On the other hand, the model at $\Delta = w = 0$ (the critical PCP), has a nonzero density of $\rho(t)$ as $t \to \infty$. In other words, the $w \to 0$ limiting process is different from the $w = 0$ model itself in regard to the auxiliary field density.

To check the universality of the crossover exponent, we study the modified PCP with the replacement of $2A \to 3A$ with $3A \to 4A$ in Eq. (4) which is the model of Eq. (4) with no diffusion ($D = 0$). This model also has infinitely many absorbing states and belongs to the DP$_{1M}$ class. By introducing single-particle reactions ($w \neq 0$), the same crossover behavior is found as the above (data not shown).

We also study more general crossover behavior from one DP$_{1M}$ to another DP$_{1M}$ with the considerably reduced number of absorbing states. To be specific, we consider the triplet contact process (TCP) and its crossover model by introducing the $2A \to A$ process without spontaneous death. The TCP with pair dynamics is defined

\begin{align*}
\delta &= 0.1595 \quad \text{and} \quad \mu_\parallel = 0.97 \text{ at criticality in semi-log scales. Inset: Scaling function } H \\
&\Rightarrow \text{all curves at different critical points collapse well into a single curve. Hence we conclude that the scaling function}
\end{align*}
as

\[
\begin{align*}
\text{AAA} & \xrightarrow{p} \text{000}, \\
\text{AAA0} & \xrightarrow{0} \text{AAA}, \\
\text{0AA} & \xrightarrow{(1-p)/2} \text{AAA}, \\
\text{A0} & \xrightarrow{w/2} \text{A0}, \\
\text{0A} & \xrightarrow{w/2} \text{00}.
\end{align*}
\]

The above model has infinitely many absorbing states, but with nonzero \( w \) the number of absorbing states is greatly reduced. At \( w = 0 \), there is again a jump in the auxiliary field density (here, the pair density). We found that the critical point for the TCP at \( w = 0 \) is \( p_c = 0.036865(5) \), exploiting the DP critical scaling (data not shown). Figure 7 shows the scaling plot of the triplet density \( \rho_1 \) in the same way as in Fig. 4. We also measured the crossover exponent from the phase boundary and found the same exponent (data not shown).

Hence we conclude that there is the well-defined and universal crossover scaling from the DP\(_{\text{IM}}\) to the DP which is mediated by the significant reduction of the number of absorbing states. The discontinuity in the auxiliary field plays a crucial role in this nontrivial crossover.

IV. CONJECTURE ON THE CROSSOVER EXPONENT \( \phi \)

The crossover exponent from the DP\(_{\text{IM}}\) to the DP is estimated as \( \phi = 1.78(6) \). Since this crossover occurs between the same universality class, we are suspicious that \( \phi \) may not be independent but related to the well-known DP critical exponents. Actually, we argue that the crossover exponent is given by the DP relaxation time exponent: \( \phi = \nu || \simeq 1.733 \) (or \( \mu || = 1 \)) which is compatible with the numerical estimation within error. The reason is as follows: Take the model of Eq. \( \text{(4)} \) at \( q = 0 \). The critical line should be determined by the competition between the single-particle annihilation process (\( A \rightarrow \emptyset \)) parameterized by \( w \) and the multi-particle (pair) reaction process \( 2A \rightarrow \emptyset \) or \( 3A \) parameterized by \( \Delta \sim (p_0 - p) \). We expect both events should appear at the same time scale along the critical line to balance off each other. Since the single-particle (auxiliary field) density is finite at the PCP (DP\(_{\text{IM}}\)) point, the time scale for \( A \rightarrow \emptyset \) should be simply proportional to \( w^{-1} \). The time scale for the pair reaction process should be given by the relaxation time scale \( \tau \sim \Delta^{-\nu ||} \). Consequently, the critical line is determined as \( \Delta_c \sim (p_0 - p_c(w)) \sim w^{1/\nu ||} \), which yields \( \phi = \nu || \) and equivalently \( \mu || = 1 \).

Considering the crossover between the DP models with a unique absorbing state, the time scale for the process parameterized by \( w \) is proportional to \( w^{-\nu ||} \) like the other competing process because the auxiliary field (particle) density is also vanishing critically as \( w \) decreases to zero. In this case, we get \( \phi = 1 \) and equivalently \( \mu || = \nu || \), which is consistent with our result in Sec. \( \text{III} \).

Since our argument for the crossover exponent is generally applicable to any universality class, we can check its validity through studying the similar type crossover between the directed Ising (DI) class models \( \text{[22]} \). Consider a one-dimensional system with two species, say \( A \) and \( B \). Between the same species, hard core exclusion is applied, but different species can reside at the same site. The dynamic rules are as follows: The dynamics always starts with an \( A \) particle. A randomly chosen \( A \) particle can hop to one of nearest neighbors with probability \( p \). If two \( A \) particles meet at the same site by hopping, both particles are removed with probability \( \frac{w}{2} \). If this annihilation attempt fails, the particle goes back to the original site. With probability \( 1 - p \), a \( B \) particle is generated at the same site occupied by the chosen \( A \) particle. If that site is already occupied by another \( B \) particle, the two \( B \) particles transmute to two \( A \) particles which will be placed at two nearest neighbor sites. If any of transmuted \( A \) particles is placed at the site already occupied by another \( A \) particle, both particles are annihilated immediately. In summary, \( 2A \rightarrow \emptyset \), \( A \rightarrow A + B \), and \( 2B \rightarrow 2A \) processes are allowed with \( A \) particle diffusion.

Since \( B \) particles are not allowed to hop, the system is inactive without an \( A \) particle but only with \( B \) particles. The number of the absorbing states grows exponentially with system size and the auxiliary field \( (B \) particle) density is finite at the absorbing transition. Besides, the number of \( A \) particles is conserved modulo 2, which is the characteristic of the DI (or parity-conserving) class.

As in Sec. \( \text{III} \) we study the crossover by introducing spontaneous annihilation of \( B \) particles (inhibitory route) with rate \( w \). As expected, we find the DI critical scaling for both \( w = 0 \) and \( w \neq 0 \) cases. We locate the critical line by exploiting the known DI critical exponents \( \text{[22]} \), which is summarized in Table \( \text{III} \) for some \( w \)'s. Since the critical exponent \( \nu || \) of the DI class (\( \simeq 3.25 \)) is much larger than that of the DP (\( \simeq 1.733 \)), the accuracy of the critical points in Table \( \text{III} \) is worse than that for the DP cases. From these data, one can estimate the crossover exponent \( 1/\phi || \) as \( 0.34(4) \) which should be compared with \( 1/\nu || \) of the DI class (\( \simeq 0.31 \)). Hence we conclude that our argument for \( \phi = \nu || \) also applies to the crossover from the IM-type DI to the DI models.

| \( w \)       | \( p_c(w) \)        |
|-------------|-----------------|
| 0           | 0.4518(1)       |
| \( 10^{-4} \) | 0.4443(2)       |
| \( 3 \times 10^{-4} \) | 0.4405(2)  |
| \( 10^{-3} \) | 0.4353(1)       |
| \( 3 \times 10^{-3} \) | 0.4276(1)  |
V. DIFFUSION EFFECT

This section studies how the crossover scaling is affected if particles are allowed to hop in models considered in Sec. III. With single-particle diffusion, the PCP becomes the PCPD where the particle (auxiliary field) density as well as the pair density vanishes at criticality even without any single-particle reaction process. The crossover from the PCPD to the DP caused by including single-particle reactions has been studied previously by the present authors [16] where the value of the crossover exponent is reported as $1/\phi = 0.58(3)$. This value is quite close to that obtained for the crossover from the DP to the PCPD in Sec. III. This similarity may mislead one to jump to the wrong conclusion that the critical nature of the PCP and the PCPD is equivalent. However, one should remember that the origin of the nontrivial crossover from the PCP to the DP lies in the finiteness of the auxiliary field density at the PCP critical point, while it vanishes at the PCPD critical point. So, if the PCPD belongs to the DP class, then one should expect a trivial crossover with $\phi = 1$. Our finding of the nontrivial crossover in [16] implies that the PCPD class is distinct from the DP class. Hence it is likely that the similarity of two crossover exponents be a mere coincidence.

The distinction of the PCPD from the DP can also be evidenced by the study of the crossover model with the hybrid-type reaction dynamics $2A \rightarrow 0$ and $3A \rightarrow 4A$. Without diffusion, this model belongs to the DP class and its crossover to the DP was studied in Sec. III. Even if particles are allowed to diffuse, this model (called as tp12) is numerically known to belong to the DP class [9].

It will be illuminating to see how the diffusion in the tp12 can change the crossover behavior to the DP. The dynamics of the model is summarized as

\[
\begin{align*}
AA &\xrightarrow{p} 00, \\
apAAA &\xrightarrow{1-p/2} AAAA, \\
apA &\xrightarrow{w/2} 00, \\
A0 &\xrightarrow{D} 0A,
\end{align*}
\]

where $D = (1-w)/2$. The model at $w = 0$ is the tp12. Our numerical results in Fig. 8 show the typical “crossover” behavior between the identical universality class discussed in Sec. III. The critical line converges to the $w = 0$ point linearly ($\phi = 1$) and there is no diverging time scale as $w$ becomes smaller with almost perfect collapse of all critical density decay curves near $w \approx 0$. Our crossover study provides another strong numerical evidence that the tp12 belongs to the DP class.

If the PCPD does not belong to the DP class, why should the tp12 belong to the DP class? The PCPD and the tp12 seem quite similar in the sense of multi-particle nature in reaction dynamics and also the absorbing space structure (vacuum and a single-particle state). However, they are quite different in the role of diffusing isolated particles. See Fig. 9 for the space-time configurations for the tp12 and the PCPD at criticality, starting from the low density initial condition without pairs. The diffusing isolated particles of the tp12 cannot increase the number of particles in most cases, because $2A \rightarrow 0$ dynamics dominates over $3A \rightarrow 4A$ dynamics: Pairs generated by collisions of two isolated particles evaporate before greeting another isolated particle to become “active” triplets. Consequently there is effectively no feedback mechanism from isolated particles to increase the particle density or make the system more active. Therefore the region of isolated particles can be regarded as absorbing like in the PCP model. This case may correspond to the no-feedback point ($r = 0$) for the generalized PCPD (GPCPD) studied in [10], which is the DP fixed point. This argument can be generalized to systems with hybrid-type reaction dynamics $mA \rightarrow (m+k)A$ and $nA \rightarrow (n-t)A$ with

![FIG. 9: (Color online) Space-time configurations of (a) the tp12 and (b) the PCPD at criticality. The initial density is (a) $\frac{1}{2}$ for the tp12 and (b) $\frac{1}{12}$ for the PCPD with size $L = 2^{10}$. Isolated particles (particles in a cluster) are designated by the blue (red) color.](image-url)
$m > n$ and $k, l > 0$, which are numerically shown to belong to the DP class \cite{Henrichsen2000}.

The isolated particles of the PCPD, however, cannot be regarded as absorbing as Fig. 9 shows; the isolated particles may affect the critical spreading actively, because both dynamics of $2A \to \emptyset$ and $2A \to 3A$ compete each other and consequently there is an effective feedback mechanism from diffusing particles to make the system active. This corresponds to the GPCPD with long-term memory effects at $r \neq 0$ \cite{Noh2004}.

VI. SUMMARY AND CONCLUSION

In summary, we studied the crossover from the model belonging to the directed percolation (DP) class with infinitely many absorbing states (DPIM) to the DP model by reducing the number of absorbing states significantly. The crossover is found to be well described by the usual crossover scaling function for the order parameter. The crossover exponent $\phi$ is argued to be related to one of the DP critical exponent, i.e., $\phi = \nu_{\parallel}$, which is further evidenced by the similar crossover model belonging to the directed Ising class. The origin of the diverging scale and the nontrivial crossover comes from the discontinuity of the auxiliary field density at the DPIM critical point. Our study for the first time presents the existence of the nontrivial scaling in the DPIM, which is compared with the study on the spreading exponents.

We also studied how the crossover behavior from the DPIM to the DP is affected by particle diffusion. The crossover from the pair contact process with diffusion (PCPD) to the DP studied in Ref. \cite{Park2006} is well classified by the nontrivial crossover exponent. On the other hand, the tp12 which is known to belong to the DP class is characterized by the trivial “crossover” between the identical class detailed in Sec. \cite{Janssen2004}. This provides an additional evidence supporting that the PCPD is distinct from the DP. In addition, we suggest an argument based on the role of diffusing isolated particles why the tp12 should belong to the DP class, but the PCPD does not need to be.

It will be a challenging problem to see if the crossover scaling from the DPIM to the DP can be anticipated in the framework of the field theory \cite{Henrichsen2000}.

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