The Umbral operator and the integration involving generalized Bessel-type functions

1 Ramanujan master theorem and its implication

The Ramanujan master theorem (RMT) \([7, 11]\) state that if the function \(f\) is defined through the series expansion

\[
f(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \varphi(r)
\]

with \(\varphi(0) \neq 0\), then the following identity holds

\[
\int_0^\infty x^{v-1} f(x) dx = \Gamma(v) \varphi(-v).
\]

where \(\Gamma(z)\) is the Euler’s gamma function. This identity has been proved rigorously in \([1, 2]\). A simpler, albeit heuristic, proof can be achieved by exploiting umbral method \([18]\) can be found in \([4, 9]\). By setting

\[
f(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \hat{c}^r \varphi(0)
\]

with the umbral operator \(\hat{c}\) defined by

\[
\hat{c}^r \varphi(0) = \varphi(r).
\]
the function $f$ can be expressed as an pseudo-exponential function $f(x) = e^{\hat{c}x} \varphi(0)$, then the integral (2) reduce to the form

$$\int_0^\infty x^{\nu-1} e^{\hat{c}x} \varphi(0) dx = \Gamma(\nu) \hat{c}^{-\nu} \varphi(0)$$  \hspace{1cm} (5)

The umbral formalism technique of RMT as in (5) provides a very flexible and powerful tool for new results including different aspect of the special functions theory. This technique is also useful to make an extension of RMT. In this aspect several articles are available in the literature. In [9], Górska have discussed a number of possible extension of RMT and also developed the implication of the procedure for the theory of special functions. In [6] the operational method has been exploited to evaluate integrals of various type. For example, integral

$$I_n(a, b, \alpha) = \int_{-\infty}^{\infty} dx (ax + b)^n e^{-\alpha x^2}$$  \hspace{1cm} (6)

can be worked out using a general procedure based on the generating function (GF) method and $I_n(a, b, \alpha)$ can be expressed in terms of two-variable the Gould-Hopper Hermite type polynomials [3, 22, 23]

$$H_n(x, y) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k} y^k}{(n-2k)! k!}$$

as

$$I_n(a, b, \alpha) = \sqrt{\frac{\pi}{\alpha}} H_n \left( b, \frac{a^2}{4\alpha} \right)$$  \hspace{1cm} (7)

More details about the GF method can be found in the book [19].

The present paper is organized as follows: In Section 2, class of integral involving generalized Bessel and generalized Struve functions using GF method are given. The technique is based on the formal reduction of functions in this family to Gaussians. As an application of the results, the evaluation of integrals involving trigonometric functions, are given in Section 3. The concluding remark and further aspect of this technique are mentioned in Section 4.

2 Integrals involving the generalized Bessel functions and generalized Struve functions

The purpose of this work is to evaluate integral involving the generalized Bessel function of the first kind $W_{p, \beta, \gamma}$ defined for complex $z \in \mathbb{C}$ and $\beta, \gamma, p \in \mathbb{C}$ by

$$W_{p, \beta, \gamma}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \gamma^k}{\Gamma(p + \frac{\beta+1}{2} + k)!} \left( \frac{z}{2} \right)^{2k+p}$$  \hspace{1cm} (8)

More properties including recurrence relation, differential equation, functional inequalities, geometric properties etc. of the function $W_{p, \beta, \gamma}$ can be found in [8, 12] and references therein. Various properties of generalized Bessel functions of two variable and truncated polynomials are found in [27, 28]. It is worth mentioning that, $W_{p, 1, 1} = J_p$ is Bessel function of order $p$ and $W_{p, 1, -1} = I_p$ is modified Bessel function of order $p$. Also, $W_{p, 2, 1} = 2j_p/\sqrt{\pi}$ is spherical Bessel function of order $p$ and $W_{p, 2, -1} = 2i_p/\sqrt{\pi}$ is modified spherical Bessel function of order $p$. Thus the study of the integral involving $W_{p, \beta, \gamma}$ will give far reaching results than the result in [6].

In this framework, first it is necessary to rewrite the generalized Bessel functions $W_{p, \beta, \gamma}$ using the operator $\hat{c}$. Let $p, \beta, \alpha, x \in \mathbb{R}, \alpha \neq 0$, then

$$W_{p, \beta, \alpha}(x) = \left( \frac{x\hat{c}}{2} \right)^p \exp \left\{ -\hat{c} \left( \frac{x}{2} \right)^2 \right\} \varphi(0)$$  \hspace{1cm} (9)
where
\[ \hat{c}^\mu \varphi(0) = \varphi(\mu), \quad \varphi(\mu) = \frac{1}{\Gamma\left(\mu + \frac{\beta + 1}{2}\right)} \] (10)

By treating \( \hat{c} \) as an ordinary constant, several important integrals are evaluated in sequel. The presence of exponential in the new representation make the evaluation of integration more convenient than the usual methods. Among the large number of works regarding the umbral calculus, for a remarkably clear, insightful, and systematic exposition of the investigations carried out by various authors in the field of umbral calculus and its applications, the interested reader should refer also to the greatest contribution of Rota’s school for the first set umbral calculus on a firm logical foundation by using operator methods [13, 14]. Lately numerous of results were given by Rota and his coworkers (see [15–17]). In the context of study the special functions by Umbral calculus method, we refer [25, 26].

A careful manipulation (also see [24]) from (10) yields that
\[ (11) \]

where \( B(.x; y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \) is the well-known beta function. The identity (11) is used to prove main result in sequel.

The polynomial \( B_n(x, y; p, \beta) \) which have the generating function \( e^{xt}M_{-1, p+\beta/2}(yt^2) \) and satisfy the differential
\[ \hat{c} \partial_y B_n(x, y; p, \beta) = \partial_y^2 B_n(x, y; p, \beta), \quad B_n(x, 0; p, \beta) = x^n \varphi \left( p + \frac{\beta}{2} - 1 \right) \]

has an important role in this article. Here, \( M_{\alpha_1, \beta_1}(x) \) is the Wright-Bessel functions [19] defined as
\[ M_{\alpha_1, \beta_1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(k\alpha_1 + \beta_1)} = \hat{c}(\beta_1 - 1) \exp(\hat{c}\alpha_1 x)\varphi(0). \]

Thus,
\[ B_n(x, y; p, \beta) = \exp(\hat{c}^{-1}y\partial_y^2)x^n \varphi \left( p + \frac{\beta}{2} - 1 \right). \]

By considering the operational definition of the two dimensional Hermite polynomial \( H_n(x, y) = \exp(y\partial_y^2)x^n \), it is immediate that
\[ B_n(x, y; p, \beta) = \xi^{(n+\frac{\beta}{2} - 1)}H_n(x, \hat{c}^{-1}y)\varphi(0) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}y^k}{(n-2k)!\Gamma(p-k+\beta-\frac{\beta}{2})} \]

Next we give sequence of result related to the integration involving generalized Bessel functions.

**Proposition 2.1.** Let \( y, \alpha > 0 \) and \( \beta \in \mathbb{R} \). Then
\[ \int_{-\infty}^{\infty} W_{0, \alpha, \beta}(\sqrt{\gamma} x) dx = \frac{1}{\Gamma(\frac{\beta}{2})} \sqrt{\frac{\pi}{\alpha\gamma}}. \]

**Proof.** For \( y, \alpha > 0 \), it is evident from (9) that
\[ W_{0, \alpha, \beta}(\sqrt{\gamma} x) = \exp\left\{ -\hat{c}\alpha y \left( \frac{x}{2} \right)^2 \right\} \varphi(0). \]

This together with RMT as defined in (5) imply that
\[ \int_{-\infty}^{\infty} W_{0, \alpha, \beta}(\sqrt{\gamma} x) dx = 2 \int_0^{\infty} \exp\left\{ -\hat{c}\alpha y x^2 \right\} \varphi(0) dx = \int_0^{\infty} x^{\frac{\beta}{2} - 1} \exp\left\{ -\hat{c}\alpha y x \right\} \varphi(0) dx \]
\[ = 2\Gamma(1/2)\hat{c}^{-1/2} \frac{\sqrt{\alpha\gamma}}{\Gamma(\frac{\beta}{2})} \varphi(0) = \frac{2\sqrt{\pi}}{\Gamma(\frac{\beta}{2})}. \]
The following result give the integration of generalized Bessel function with the product of algebraic expression \((ax + b)^n\).

**Proposition 2.2.** Let \(\gamma, \alpha > 0\) and \(\beta, a, b \in \mathbb{R}\). Suppose that \(p > n\). Then

\[
\int_{-\infty}^{\infty} \frac{(ax + b)^n}{(\sqrt{T} x)^p} W_{p, \beta, \alpha}(\sqrt{T} x) dx = \frac{\Gamma(p + \beta - 1/2)}{2^{p-1} \Gamma(p + \beta/2)} \sqrt{\frac{\pi}{a \gamma}} B_n \left( b, \frac{a^2}{a \gamma} ; p, \beta \right)
\]  

(14)

**Proof.** An application of (7) and (9) yields

\[
\int_{-\infty}^{\infty} \frac{(ax + b)^n}{(\sqrt{T} x)^p} W_{p, \beta, \alpha}(\sqrt{T} x) dx = \int_{-\infty}^{\infty} dx (ax + b)^n \exp \left\{ -\frac{\gamma}{4} x^2 \right\} \varphi(0) dx
\]

\[
= \frac{\gamma^{p-\frac{1}{2}}}{2^{p-1}} \sqrt{\frac{\pi}{a \gamma}} H_n \left( b, \frac{a^2}{a \gamma} \right) \varphi(0)
\]  

(15)

Now taking \(\mu_1 = p + \beta/2 - 1\) and \(\mu_2 = (1 - \beta)/2\) in (11), and then using identity (12) it follows that

\[
\gamma^{p-\frac{1}{2}} H_n \left( b, \frac{a^2}{a \gamma} \right) \varphi(0) = B \left( p + \beta - \frac{1}{2}, \frac{1 - \beta}{2} \right) \gamma^{p+\beta-\frac{1}{2}} H_n \left( b, \frac{a^2}{a \gamma} \right) \varphi(0) \gamma^{\frac{1-n}{2}} \varphi(0)
\]

\[
= \frac{B \left( p + \beta - \frac{1}{2}, \frac{1 - \beta}{2} \right)}{\Gamma \left( \frac{1-n}{2} \right)} B_n(x, y; p, \beta).
\]

This together with (15) leads to the identity (14). \(\square\)

**Remark 2.3.** For any function \(f\) which can be represented as

\[f(x, m) = \sum_{k=0}^{m} f_k x^k \quad (m < p),\]

the following identity holds

\[
\int_{-\infty}^{\infty} f(ax + b) \frac{W_{p, \alpha, \beta}(\sqrt{T} x)}{(\sqrt{T} x)^p} dx = \frac{\Gamma(p + \beta - 1/2)}{2^{p-1} \Gamma(p + \beta/2)} \sqrt{\frac{\pi}{a \gamma}} \sum_{k=0}^{m} f_k B_k \left( b, \frac{a^2}{a \gamma} ; p, \beta \right).
\]

The next result emphasis on the integration of linear combination of generalized Bessel functions.

**Proposition 2.4.** Let \(\alpha > 0\) and \(\beta, a, b \in \mathbb{R}\). Then

\[
\int_{-\infty}^{\infty} F_{n, \beta, \alpha}(x; a, b) dx = \frac{2\pi}{\sqrt{-\alpha}} n! \sum_{k=0}^{[n/2]} \frac{b^{n-2k} a^{2k}}{(4\alpha)^k (n - 2k)! k! \Gamma(k + \frac{\beta}{2})}
\]

(16)

where, \(F_{n, \beta, \alpha}\) is the linear combination of \(W_{p, \alpha, \beta}\), defined as

\[F_{n, \beta, \alpha}(x; a, b) := \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k W_{n-k, \alpha, \beta}(x).\]

**Proof.** By a computation, from (9), it follows that

\[F_{n, \beta, \alpha}(x; a, b) = \left( \frac{a \gamma}{2} x + b \right)^n \exp \left\{ -\frac{\gamma}{4} \left( \frac{x}{2} \right)^2 \right\} \varphi(0).\]
This along with (7) yields
\[
\int_{-\infty}^{\infty} dx F_{n,\beta,\alpha}(x; a, b) = \int_{-\infty}^{\infty} dx \left( \frac{aC}{2} x + b \right)^n \exp \left\{ -\alpha \left( \frac{x}{2} \right)^2 \right\} \varphi(0) = \frac{2\sqrt{\pi}}{\sqrt{\alpha}} \epsilon^{1/2} H_n \left( b, \frac{a^2 C}{4\alpha} \right) \varphi(0)
\]
\[
= \frac{2\sqrt{\pi}}{\sqrt{\alpha}} n! \sum_{k=0}^{[n/2]} \frac{b^{n-2k} a^{2k}}{(4\alpha)^k (n-2k)! k! \Gamma(k + \frac{\beta}{2})}.
\]

Next result provides the integration of generalized Bessel functions of integer order \( n \).

**Proposition 2.5.** Let \( \alpha > 0 \) and \( \beta \in \mathbb{R} \). Suppose that \( b_n = \int_{-\infty}^{\infty} W_{n,\alpha,\beta}(x) dx \) for \( n = 0, 1, 2, \cdots \). Then
\[
b_{2n} = \frac{(2n)! \sqrt{\pi}}{2^{2n-1} \alpha^{n+\frac{1}{2}} n! \Gamma(n + \frac{\beta}{2})}, \quad \text{and} \quad b_{2n+1} = 0.
\]

**Proof.** First note that \( H_{2n}(0, y) = \frac{(2n)! y^n}{n!} \) and \( H_{2n+1}(0, y) = 0 \). Now an application of (4) yields
\[
b_n = \int_{-\infty}^{\infty} W_{n,\alpha,\beta}(x) dx = \int_{-\infty}^{\infty} \left( \frac{xC}{2} \right)^n \exp \left\{ -\alpha \left( \frac{x}{2} \right)^2 \right\} \varphi(0) dx = \frac{\epsilon^{n-1/2} \sqrt{\pi}}{2^{n-1} \alpha^{3/2}} H_n \left( 0, \frac{1}{\alpha} \right) \varphi(0).
\]

Thus,
\[
b_{2n+1} = \frac{\epsilon^{2n+1-1/2} \sqrt{\pi}}{2^{2n-1} \alpha^{3/2}} H_{2n+1} \left( 0, \frac{1}{\alpha} \right) \varphi(0) = 0,
\]
\[
b_{2n} = \frac{\epsilon^{2n-1/2} \sqrt{\pi}}{2^{2n-1} \alpha^{3/2}} H_{2n} \left( 0, \frac{1}{\alpha} \right) \varphi(0) = \frac{(2n)! \sqrt{\pi}}{2^{2n-1} \alpha^{n+\frac{1}{2}} n! \Gamma(n + \frac{\beta}{2})}.
\]

Now consider the generalized Struve function of the first kind \( H_{p,\beta,\alpha} \) defined for complex \( z \in \mathbb{C} \) and \( p, \beta, \alpha \in \mathbb{C} \) (\( \text{Re}(p) > -1 \)) by
\[
H_{p,\beta,\alpha}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{\Gamma(p + 1 + \frac{\beta}{2} + k) \Gamma(k + \frac{\beta}{2})} \left( \frac{z}{2} \right)^{2k+p+1}
\]
(17)

More properties of generalized Struve function can be found in the work of Yagmur and Orhan [21]. For the purpose, we assume that \( x \in \mathbb{R} \) and \( p, \beta, \alpha \in \mathbb{R} \). Then the umbral formulation of \( H_{p,\beta,\alpha} \) is
\[
H_{p,\beta,\alpha}(x) = \hat{c}_1 \frac{a^p + \frac{1}{2}}{\hat{c}_2 \alpha} \left( \frac{x}{2} \right)^{p+1} \frac{1}{1 + \hat{c}_1 \hat{c}_2 \alpha} \varphi_1(0) \varphi_2(0).
\]
(18)

Here the operator \( \hat{c}_i \) \( (i = 1, 2) \) acts only on \( \varphi_i(0) \) as given in (10).

**Proposition 2.6.** Let \( \mu, p, \beta, \alpha(\neq 0) \in \mathbb{R} \). Suppose that \( \mu + p \) are not even integers. Then
\[
\int_{0}^{\infty} x^\mu H_{p,\beta,\alpha}(x) dx = \frac{2\mu \pi}{(a + \frac{\mu + p}{2}) \sin \left( (\mu + p) \frac{\pi}{2} \right)} \frac{1}{\Gamma(1 - \mu - p) \Gamma(\frac{\mu + p}{2})}.
\]
(19)

**Proof.** Using the concept of Laplace transform, (18) can be rewritten as
\[
H_{p,\beta,\alpha}(x) = \hat{c}_1 \frac{a^p + \frac{1}{2}}{\hat{c}_2 \alpha} \left( \frac{x}{2} \right)^{p+1} \int_{0}^{\infty} \exp \left( -s \left( 1 + \hat{c}_1 \hat{c}_2 \alpha \left( \frac{x}{2} \right)^2 \right) \right) \varphi_1(0) \varphi_2(0) ds.
\]
This along with \( \Gamma(x) = \int_{0}^{\infty} s^{x-1} \exp(-s) ds \) and Euler’s reflection formula for Gamma function imply that
\[
\int_{0}^{\infty} x^\mu H_{p,\beta,\alpha}(x) dx = \frac{\hat{c}_1 \frac{a^p + \frac{1}{2}}{\hat{c}_2 \alpha}}{2^{p+1}} \int_{0}^{\infty} x^{\mu + p + 1} \exp \left( -s \left( 1 + \hat{c}_1 \hat{c}_2 \alpha \left( \frac{x}{2} \right)^2 \right) \right) \varphi_1(0) \varphi_2(0) dx ds.
\]
\[
\frac{2^{\mu} t^{\frac{\mu+\nu}{2}} \exp(-s t) \exp(-t) \varphi_1(0) \varphi_2(0) dt ds}{(\alpha)^{1+\frac{\mu+\nu}{2}}} = \frac{2^{\mu} \Gamma(1+\frac{\mu+\nu}{2}) \Gamma(-\frac{\mu+\nu}{2})}{(\alpha)^{1+\frac{\mu+\nu}{2}}} \frac{1}{\Gamma(\frac{\nu}{2}) \Gamma(\frac{\mu+\nu}{2})}.
\]

Several consequences of Propositions 2.1–2.6 are discussed in Section 3. The interesting point is that the improper (with infinite limits) integrals of trigonometric functions which are mainly evaluated by using Cauchy residue theorem, can also be evaluated by judicious choice of parameters value from the results in this section.

## 3 Applications

This section emphasis on the evaluation of improper integration of trigonometric functions. For this purpose, the umbral method, introduced in earlier sections, is used. It is worth mentioning here that sine and cosine functions satisfy the following relation with generalized Bessel function. For \( \alpha \in \mathbb{R}, \alpha \neq 0 \), it follows from (9) that

\[
\cos(\alpha x) = \sqrt{\frac{\pi x}{2}} \mathcal{W}_{-1/2,1,\alpha^2}(x) = \sqrt{\pi e^{-1/4}} \exp\left\{-\frac{\hat{c} a^2}{4} x^2\right\} \varphi(0)
\]

(20)

\[
\sin(\alpha x) = \alpha \sqrt{\frac{\pi x}{2}} \mathcal{W}_{1/2,1,\alpha^2}(x) = \alpha \sqrt{\pi e^{-1/4}} \exp\left\{-\frac{\hat{c} a^2}{4} x^2\right\} \varphi(0)
\]

(21)

\[
\cosh(\alpha x) = \sqrt{\frac{\pi x}{2}} \mathcal{W}_{-1/2,1,-\alpha^2}(x) = \sqrt{\pi e^{-1/4}} \exp\left\{-\frac{\hat{c} a^2}{4} x^2\right\} \varphi(0)
\]

(22)

\[
\sinh(\alpha x) = \alpha \sqrt{\frac{\pi x}{2}} \mathcal{W}_{1/2,1,-\alpha^2}(x) = \alpha \sqrt{\pi e^{-1/4}} \exp\left\{-\frac{\hat{c} a^2}{4} x^2\right\} \varphi(0)
\]

(23)

### Proposition 3.1

For \( a > 0 \) and \( b > 0 \), the following two identities can be proved by using RMT involving umbral operator:

\[
\int_0^\infty \cos(ax^2) \cos(2b x) dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left( \cos\left(\frac{b^2}{a}\right) + \sin\left(\frac{b^2}{a}\right) \right)
\]

(24)

\[
\int_0^\infty \sin(ax^2) \cos(2b x) dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left( \cos\left(\frac{b^2}{a}\right) - \sin\left(\frac{b^2}{a}\right) \right)
\]

(25)

### Proof.

From (20), it is evident that

\[
\cos(ax^2) = \sqrt{\pi} e^{-1/4} \exp\left\{-\frac{\hat{c} a^2}{4} x^2\right\} \varphi(1)\]

(26)

\[
\cos(2b x) = \sqrt{\pi} e^{-1/4} \exp\left\{-\frac{\hat{c} b^2}{4} x^2\right\} \varphi(2)
\]

(27)

Similarly, (21) yields

\[
\sin(ax^2) = \frac{\alpha \sqrt{\pi x^2}}{2} \exp\left\{-\frac{\hat{c} a^2}{4} x^2\right\} \varphi_3(0)
\]

(28)

Now (26) and (27) together imply that the left hand side of (24) have umbral operator representation as

\[
I_1 = \int_0^\infty \cos(ax^2) \cos(2b x) dx = \pi (\hat{c}_1 \hat{c}_2)^{-1/2} \int_0^\infty \exp\left\{-\frac{\hat{c}_1 a^2}{4} x^2\right\} \exp\left\{-\frac{\hat{c}_2 b^2}{4} x^2\right\} dx \varphi_1(0) \varphi_2(0).
\]
Since \( \exp \{-c_2 b^2 x^2\} = \sum_{k=0}^{\infty} (-1)^k \frac{c_2^k b^{2k}}{k!} x^{2k} \), \( I_1 \) can be rewritten by interchanging the summation and integration, and thus have the new form

\[
I_1 = \pi (c_1 c_2)^{-1/2} \sum_{k=0}^{\infty} (-1)^k \frac{c_1^k b^{2k}}{k!} \int_0^{\infty} x^{2k} \exp \left\{ -\frac{c_1 a^2}{4} x^4 \right\} dx \varphi_1(0)\varphi_2(0).
\]

Further a substitution of \( x^4 = t \) and then some computation along with RMT lead to

\[
I_1 = \frac{\pi (c_1 c_2)^{-1/2}}{4} \sum_{k=0}^{\infty} (-1)^k \frac{c_1^k b^{2k}}{k!} \left( \frac{c_1 a^2}{4} \right)^{-\frac{k}{2} - \frac{1}{4}} \Gamma \left( \frac{k}{2} + \frac{3}{4} \right) \varphi_1(0)\varphi_2(0)
\]

\[
= -\frac{\pi}{2\sqrt{2a}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma \left( \frac{k}{2} + \frac{1}{4} \right)}{2^{k-1/2}} \left( \frac{2b^2}{a} \right)^k \left( \frac{b^2}{a} \right) c_2 \left( \frac{k}{2} - \frac{1}{4} \right) \varphi_1(0)\varphi_2(0)
\]

Now the Legendre duplication formula \( \sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + 1/2) \) yields

\[
\Gamma \left( \frac{k}{2} + \frac{1}{4} \right) = \frac{2^{k-1/2} \sqrt{\pi} \Gamma \left( k + \frac{1}{4} \right)}{\Gamma \left( \frac{k}{2} + \frac{3}{4} \right)}
\]

Also by an application of the identity \( \Gamma(x) \Gamma(1-x) = \pi / \sin(\pi x) \), it follows that

\[
\Gamma \left( \frac{1}{4} - \frac{k}{2} \right) \Gamma \left( \frac{k}{2} + \frac{3}{4} \right) = \frac{\pi}{\sin \left( \frac{k\pi}{2} + \frac{3\pi}{4} \right)}
\]

Finally

\[
I_1 = \frac{\sqrt{\pi}}{2\sqrt{2a}} \sum_{k=0}^{\infty} (-1)^k \frac{\sin \left( \frac{2k\pi}{4} + \frac{3\pi}{4} \right)}{k!} \left( \frac{b^2}{a} \right)^k
\]

\[
= \frac{\sqrt{\pi}}{2\sqrt{2a}} \left( \sum_{k=0}^{\infty} \frac{\sin \left( \frac{2k\pi}{2} + \frac{3\pi}{4} \right)}{(2k)!} \left( \frac{b^2}{a} \right)^{2k} - \sum_{k=0}^{\infty} \frac{\sin \left( \frac{2k+1\pi}{2} + \frac{3\pi}{4} \right)}{(2k+1)!} \left( \frac{b^2}{a} \right)^{2k+1} \right)
\]

\[
= \frac{\sqrt{\pi}}{2\sqrt{2a}} \left( \sum_{k=0}^{\infty} (-1)^k \frac{\sin \left( \frac{3\pi}{4} \right)}{(2k)!} \left( \frac{b^2}{a} \right)^{2k} - \sum_{k=0}^{\infty} (-1)^k \frac{\sin \left( \frac{5\pi}{4} \right)}{(2k+1)!} \left( \frac{b^2}{a} \right)^{2k+1} \right)
\]

\[
= \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left( \cos \left( \frac{b^2}{a} \right) + \sin \left( \frac{b^2}{a} \right) \right).
\]

Similarly, by using (27) and (28), it follows that

\[
I_2 = \int_0^{\infty} \sin(\pi x) \cos(2bx) dx = \pi \left[ \frac{c_3}{c_2} \int_0^{\infty} \exp \left\{ -\frac{c_3 a^2}{4} x^4 \right\} \exp \left\{ -c_2 b^2 x^2 \right\} dx \varphi_1(0)\varphi_2(0) \right]
\]

When proceeding similarly as in the proof of \( I_1 \), it can be shown (details for the proof are omitted due to identical computation) that \( I_2 = \sqrt{(\pi/8a)} \left( \cos \left( \frac{b^2}{a} \right) - \sin \left( \frac{b^2}{a} \right) \right) \)

Next result evaluates the integrations involving sine and cosine functions with arguments as power of algebraic terms.

**Proposition 3.2.** For \( a > 0 \) and \( r \geq 1 \) the following two identities can be verified by RMT involving umbral operator:

\[
\int_0^{\infty} \sin(ax^r) dx = \frac{\Gamma \left( \frac{1}{2r} \right) \sin \left( \frac{\pi}{2r} \right)}{rad^{1/r}}
\]
Proof. It is evident from (21) that the integral in (29) can be represented in term of umbral operator.

\[
\int_0^\infty \cos(ax^r)dx = \frac{\Gamma\left(\frac{1}{2r}\right) \cos\left(\frac{\pi}{2r}\right)}{ra^{1/r}}
\]  

(30)

Now the required integration can be evaluated by using RMT as follows:

\[
a \frac{\sqrt[4]{\pi} \varepsilon^{1/2}}{2} \int_0^\infty x^r \exp\left(-\frac{a^2}{4} x^2\right) dx \varepsilon(0) = a \frac{\sqrt[4]{\pi} \varepsilon^{1/2}}{4r} \int_0^\infty x^{n+1/2} \exp\left(-\frac{a^2}{4} x\right) dx \varepsilon(0)
\]

\[
= a \frac{\sqrt[4]{\pi} \varepsilon^{1/2}}{4r} \Gamma\left(\frac{1}{2r} + \frac{1}{2}\right)\left(\frac{a^2}{4}\right)^{-\left(\frac{1}{2r} + \frac{1}{2}\right)} \varepsilon(0)
\]

\[
= \sqrt[4]{\pi} 2^{1/2-1} r a^{1/2} \Gamma\left(\frac{1}{2r} + \frac{1}{2}\right)\left(\frac{1}{2}\right) \varepsilon(0)
\]

Finally, by the Legendre duplication formula \(\sqrt[4]{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z)\Gamma(z+1/2)\) and by an application of the identity \(\Gamma(x)\Gamma(1-x) = \pi / \sin(\pi x)\), it follows that

\[
\int_0^\infty \sin(ax^r)dx = \frac{\Gamma\left(\frac{1}{2}\right) \sin\left(\frac{\pi}{2r}\right)}{ra^{1/2}}.
\]

Similarly, the integration in (30) can also be evaluated by the same technique as above by writing \(\cos(ax^r)\) in terms of umbral operator by considering (20).

\[
\square
\]

4 Concluding Remark

It is evident from Section 2 that the umbral method can easily be used as a tool to evaluate complicated integrals involving generalized Bessel and generalized Struve functions. Though the results obtained in earlier sections have intensive computation and shrewd manipulation, the benefit of consideration of such general framework are that through the judicious choice of parameter \(p, \beta\) and \(\alpha\), they generate several interesting application, which include extending the result of previous work.

In this context it is worth comparing the result obtained in Section 2 with results derived in [4]-[9]. The appropriate choice of the parameters \(p, \beta\) and \(\alpha\) yields the following consequences.

1. The results obtained in Section 2 are the generalization of many results obtained by Babusci et al. [6] respectively. For example, if \(\beta = \alpha = 1\) Proposition 2.1 is equivalent to the identity (7) in [6] while the identity (8) of the same article is a special case of Proposition 2.2.

2. For \(a \neq 0\), the inequality \(p > n\) in Proposition 2.2 is sharp in the sense that the identity (14) is not valid for some \(a, b, n\) if \(p \leq n\). For example: let \(a > 0, b = 0\) and \(\alpha = \beta = \gamma = n = 1\). If \(p = 1/2\), it is easy to find that right-hand side of (14) is zero. Since \(W_{1/2,1,1}(x) = \sqrt{2/(\pi x)} \sin(x)\), the left-hand side reduces to the integration \(a^n \int_{-\infty}^{\infty} \sqrt{x} \sin(x)dx\) which even does not converge. On contrary, if \(a = 0\), the condition \(p > n\) could be relaxed and replaced by \(p > 0\). Thus

\[
\int_{-\infty}^{\infty} \frac{1}{(\sqrt[4]{\pi})^p} W_{\rho, \beta, \alpha}(\sqrt[4]{\pi}) dx = \frac{1}{2^{p-1} \Gamma\left(p + \frac{\beta}{2}\right) \sqrt{\alpha \gamma}}.
\]
which is a generalized form of the integration
\[
\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, dx = \pi.
\]

3. For \( \alpha = 1 \) and \( \beta = 2 \), the Proposition 2.5 leads to the integration involving spherical Bessel \( j_n = \sqrt{\pi} W_{n,2,1}/2 \) function and which is also solved in [6] (see equation (17) and (18), page-1). Now as \( j_1(x) = (\sin(x) - x\cos(x))/x \) and \( j_2(x) = (3\sin(x) - 3x\cos(x) - x^2\sin(x))/x^3 \),
\[
\int_{-\infty}^{\infty} \frac{\sin(x) - x\cos(x)}{x^2} \, dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{3\sin(x) - 3x\cos(x) - x^2\sin(x)}{x^3} \, dx = \frac{\pi}{2}.
\]

4. Taking \( \mu = -p - 1 \) in (19) it follows that
\[
\int_{0}^{\infty} x^{-p-1} H_{p,1,1}(x) \, dx = \frac{2^{-p-1}\pi}{\Gamma(1+p)}.
\]
This integration is listed in [10, 6.813.2]. Similarly the formula 6.811.1 and 6.813.1 listed in [10] can be obtained from (19) by choosing \( \mu = 0 \) and \( \mu = p - 1 \) respectively.

5. The integrations (24), (25), (29), (30) are listed in [10].

We conclude the article with the remark that many improper integral involving sine, cosine, hyperbolic sine and hyperbolic cosine functions which are listed in [10] can be evaluated easily by the adopted method in this article and the list of such evaluation will appear in future work.

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