Results from Bosonisation for Resonant Tunneling through a Quantum Dot in an Aharanov-Bohm Ring

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Abstract

We study coherent charge tunneling through a one-dimensional interacting ring with a one-dimensional quantum dot embedded in one of its arms through bosonisation. The symmetries of the effective action explain many of the features such as phase change between resonances, in-phase successive resonances and phase-locking, which have been observed in experiments of coherent transport in mesoscopic rings, with a quantum dot. We also predict changes in the behaviour of the tunneling conductance in the presence of an Aharanov-Bohm flux through the ring. We argue that these results hold true in general for any dot.

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Recent electron interferometry experiments\cite{1, 2, 3} on mesoscopic Aharanov-Bohm (AB) rings are of fundamental interest as these probe not only the total transmission through the resonant tunneling structure but also the phase associated with the electron transport. The first such experiment by Yacoby et al\cite{1} on an AB ring with a resonant tunneling structure in the form of a quantum dot showed that there exists a coherent component in the transport through the dot. Further, this coherent transport is characterized by unusual features - successive AB conductance peaks are in phase and there is an abrupt change in phase by $\pi$ when the conductance reaches a maximum. More recent experiments\cite{2, 3} confirm this picture and also observe a phase drop of $\pi$ between successive conductance peaks.

There have been many theoretical attempts\cite{4} to explain these features. The abrupt phase change by $\pi$ when the conductance peak reaches a maximum has been explained in terms of the phase locking imposed by the condition that the two terminal conductance is an even function of the magnetic field\cite{5}. Wu et al\cite{6} suggest that the ‘in phase’ behaviour of successive conductance peaks arises due to the fact that resonant tunneling through the whole system can be observed only when the phase shift introduced by the resonant state of the dot coincides with the transmission phase of the rest of the ring. Kang\cite{7}, in a recent work, uses the Friedel sum rule for the effective single particle levels in the quantum dot and a non interacting tight binding representation for the electrons on the ring to explain some of these features. However, a proper understanding of the various unusual features seen experimentally is still lacking.

Motivated by this, in this letter, we study the problem of coherent transport in a single channel electron ring connected to external leads at $X_L$ and $X_R$ with a 1-D resonant tunneling structure embedded in one of its arms (Fig.(1)). The single channel model is appropriate for a very narrow ring where one expects only a few 1-D channels to contribute to the transmission. We use the bosonization approach pioneered for transmission problems by Kane and Fisher\cite{8}, and successfully applied to study transport in 1-D wires in various contexts\cite{9}. We explain several of the distinct features characterizing coherent transport in the interferometer device geometry in terms of the symmetries of the theory. We also study the problem in the presence of an external magnetic flux. This approach also allows us to study the interacting problem as well.

We begin with the tight-binding Hamiltonian for spinless fermions on a ring with a hopping parameter $t$ and a short range repulsive Coulomb interaction $U$. If the ring is pierced by a magnetic flux $\Phi = \int_{0}^{L} A_\phi dx$ where $A_\phi$ is the component of the vector potential along the ring and $L$ is the length of the circumference of the ring, then the Hamiltonian can be written as

$$H = t \sum_{j=1}^{N} (e^{-\frac{i\delta}{\Phi_0}}\psi_j^\dagger\psi_{j+1} + h.c.) + U \sum_{j=1}^{N} (\psi_j^\dagger\psi_j)(\psi_{j+1}^\dagger\psi_{j+1})$$

where $\delta = 2\pi \Phi / \Phi_0$, $\Phi_0 = hc/e$ is the flux quantum and $N$ is the total number of sites on the ring. The fermions satisfy periodic boundary conditions on the ring - $\psi_{N+1} = \psi_1$. A gauge transformation on the fermions $\psi_j \rightarrow e^{\frac{i\delta}{\Phi_0}}\psi_j$ leads to the usual Hubbard form for the
Hamiltonian
\[ H = t \sum_{j=1}^{N} (\psi_j^\dagger \psi_{j+1} + h.c) + U \sum_{j=1}^{N} ((\psi_j^\dagger \psi_j)(\psi_{j+1}^\dagger \psi_{j+1})) \tag{2} \]

with the periodic boundary conditions on the fermions now changed to \( \psi_{N+1} = e^{i\delta} \psi_1 \). In the low-energy, long-wavelength limit, the fermion fields can be expanded about the right and left Fermi momentum points \( \pm k_F \): \( \psi(x) = e^{-ik_F x} \psi_L(x) + e^{ik_F x} \psi_R(x) \), where \( \psi_L(x) \) and \( \psi_R(x) \) are left and right moving Fermi fields. Linearizing the dispersion and using the standard bosonization technique, we can express, in the continuum limit, the fermion fields in terms of two bosonic fields \( \theta(x) \) and \( \phi(x) = \int_0^x \Pi(x') dx' \) (where \( \Pi(x) \) is the momentum of the \( \theta(x) \) field) satisfying the commutation relations \([\phi(x), \theta(x')] = i \Theta(x - x') \):

\[
\psi_L(x) = e^{-i\sqrt{\pi} (\theta(x) - \phi(x))} \\
\psi_R(x) = e^{i\sqrt{\pi} (\theta(x) + \phi(x))}.
\] (3)

The corresponding bosonic Hamiltonian on the ring is given by
\[
H_{ring} = v_F \int_0^L dx \left[ \frac{g}{2} (\nabla \phi)^2 + \frac{1}{2g} (\nabla \theta)^2 \right] \tag{4}
\]

where \( v_F \) is the Fermi velocity and \( g \) is related to \( U \) as \( g^{-2} = (1 + \frac{U}{\pi v_F}) \). We restrict ourselves to repulsive interactions for which \( g < 1 \). (\( g = 1 \) is the noninteracting limit and \( g > 1 \) for attractive interactions.) In the absence of any magnetic flux through the ring, the fermions satisfy periodic boundary conditions. Consistent with these boundary conditions, one can show that the bosonic Hamiltonian is symmetric under \( \theta(x) \rightarrow \theta(x) + \sqrt{\pi} \). This symmetry represents the discrete particle nature of the electrons\[^8\]. It can also be easily seen that a non-zero magnetic flux through the ring which couples to \( \partial_x \theta(x) = \partial_x \phi(x) \) leaves the \( \theta \) field unchanged, but transforms the \( \phi \) field as \( \phi(x) \rightarrow \phi(x) \pm \delta / \sqrt{\pi} \), where \( \delta \) is as defined earlier and \(+/−\) is because right and left moving fermion fields respond to the flux in opposite ways.

\* Fig 1. \* Schematic diagram of the Aharanov-Bohm ring coupled to leads on the left and right through tunneling junctions and having a dot in one of its arms.
The one-dimensional dot can be simply modelled by a symmetric double barrier potential, which, in turn, can be taken to be that of two δ-function barriers positioned at \( x = \frac{L}{4} - \frac{d}{2} \) and \( x = \frac{L}{4} + \frac{d}{2} \) as shown in Fig. (1). It is appropriate to consider the large barrier limit where Coulomb blockade effects can occur. The effective Hamiltonian for the dot can then be written as[8]

\[
H_{\text{dot}} = V\left[\cos(2\sqrt{\pi}\theta_1(\tau) - k_F d/2) + \cos(2\sqrt{\pi}\theta_2 + k_F d/2)\right] + V_G \frac{\theta_2(\tau) - \theta_1(\tau)}{\sqrt{\pi}} \tag{5}
\]

where \( \theta_1(\tau) = \theta(x = L/4 - d/2, \tau) \), \( \theta_2(\tau) = \theta(x = L/4 + d/2, \tau) \), \( V \) is the strength of the δ-function potentials and \( V_G \) is the gate voltage which couples to the electrons between the two barriers. Such a system is known to have resonant tunneling behaviour when the Luttinger parameter \( g > 1/4 \) [3].

The leads are taken to be non-interacting and are coupled to the ring through tunnel junctions at \( X_L \) and \( X_R \) as shown in Fig.(1). The lead-ring interaction can be described by a Hamiltonian of the form

\[
H_{\text{lead–ring}} = [t_L b_L^\dagger(X_L)\psi_L(X_L) + h.c] + [t_R b_R^\dagger(X_R)\psi_R(X_R) + h.c] \tag{6}
\]

where \( b_L \) and \( b_R \) are the non-interacting fermion operators on the left and right leads respectively, and \( \psi_L, \psi_R \) are the fermion operators on the ring defined earlier.

To analyse the transport properties of the system modelled by the action

\[
S = \int d\tau dx H = \int d\tau \int_0^L dx [S_{\text{ring}} + H_{\text{dot}} + H_{\text{lead–ring}}], \tag{7}
\]

we first consider the case where there is no flux through the ring. Following Kane and Fisher[3], we see that the quadratic degrees of freedom away from the barriers can be integrated out and the effective action can be expressed as

\[
S_{\text{eff}} = \frac{1}{g} \sum_{i\omega_n} |\omega_n| \left\{ |\chi(\omega_n)|^2 + \frac{\pi}{4} |n(\omega_n)|^2 \right\} + \int d\tau \left[ \frac{1}{2} \bar{U}(n - n_0)^2 + V \cos 2\sqrt{\pi}\chi \cos \pi n \right] \tag{8}
\]

with \( \chi/\sqrt{\pi} = (\theta_1 + \theta_2)/2\sqrt{\pi} \) interpreted as the number of particles transferred across the barriers and \( n = (\theta_2 - \theta_1)/\sqrt{\pi} + k_F d/2\pi \) as the number of particles between the two barriers. The first term in \( V_{\text{eff}} \) can be interpreted as the energy cost to put \( n \) particles in the quantum dot. The optimum value for the number of particles between the barriers \( n_0 \) is controlled by \( V_G \) and depends on \( \bar{U} = \pi \hbar v_F / g^2 d \). The effective action is invariant under \( \chi \rightarrow \chi + \sqrt{\pi} \), corresponding to the transfer of an electron across the island or the dot with no change in the charge state of the dot. However, at values of \( V_G \) tuned such that the optimum value is \( n_0 = 1/2 \) [8], \( (V_G = \frac{\pi \hbar v_F}{g^2 d} (-\rho_F d + n_0)) \), there is no extra cost to the energy to change the charge state of the dot by one. This corresponds to the fact that exactly at resonance, there exists an additional symmetry in the effective action, with \( \chi \rightarrow \chi + \sqrt{\pi}/2 \), along with a corresponding change in the charge state of the island \( n \rightarrow n + 1 \) - i.e. at each resonance, the symmetry is equivalent to changing the charge state of the island by 1 and transferring 1/2
an electron across the barrier. Now, let us study what happens to the fermion fields at gate voltages tuned such that the resonance condition is satisfied for the dot. As we have seen above, when the dot goes through a resonance, half an electron is transferred across the dot with a change in the charge state of the dot by unity. Transport of half an electron across the barrier corresponds to the following transformations for the boson fields on the upper arm of the ring - \( \theta(x) \rightarrow \theta(x) + \sqrt{\pi q} \) and \( \phi(x) \rightarrow \phi(x) \pm \sqrt{\pi q} \) with \( q \) half an integer and +/- for the \( R/L \) movers in the second equation. This leads to a phase shift of \( \pi \) at each resonance.

In terms of the gate voltage, the resonance occurs at \( V_G = \frac{\pi \hbar v_F}{g d} (-\rho_F d + q + 1/2) \). We shall call the resonances for which \( q = \text{even integer} \) as ‘odd’ resonance and \( q = \text{odd integer} \) as ‘even’ resonance. Note that spacing between resonances is given by \( \Delta V_G = \frac{\pi \hbar v_F}{g^2 d} \) for the one dimensional dot. (In general, the spacing depends on the capacitance of the dot and is given by \( \Delta V_G = \frac{Q}{C} \), where \( C \) is the capacitance of the dot. and is given by \( \Delta V_G = \frac{Q}{C} \).

In open geometries where the Luttinger liquid wire is connected to external voltage reservoirs, conductance experiments measure only the transmission amplitudes which depend on the energy but not on the phase and hence, maxima in the transmission amplitudes occur both for odd and even resonances. However, in an interferometry geometry like that of Fig.(1), the transmission characteristics depend crucially on the interference patterns between the electrons travelling through the two different paths. For the electron travelling through the upper arm with the embedded dot, even (odd) resonances lead to phase shifts of \( 2\pi(\pi) \) respectively. Constructive interference with the electron travelling through the lower arm can therefore occur only at ‘even resonances’, which can be detected at the leads as peaks in the conductance oscillations. So conductance maxima only occur at even resonances and the spacing of the gate voltages at which the maxima now occur is twice that observed in conductance measurements done in open geometries. This implies that at the maxima, the charge state of the island or the dot changes by even integers. Moreover, since the phase change between two successive even resonances is \( 2\pi \), this also explains why successive conductance maxima are always in phase. Odd resonances, on the other hand, lead to destructive interference with the electrons travelling through the lower arm and occur in between successive conductance maxima and are characterized by a phase change of \( \pi \). Thus, the phase drop of \( \pi \) between successive conductance maxima occurs because the gate voltage goes through the odd resonance of the dot. One would not expect conductance maxima at these values of the gate voltage. In fact, destructive interference at these resonances should make the tunneling conductance go to zero. However, this would be true only if we considered only the two direct path contributions from \( X_L \) to \( X_R \) through the upper and lower arms in the path integral. But when the direct path contributions are zero, we must include the effects of multipath contributions to the conductance amplitude which lead to small but nonzero values for the tunelling conductance.

The above analysis is valid for frequencies \( \omega_n < \frac{\pi v_F}{g^2 \hbar d} \), because at higher frequencies, the electron simply sees two independent barriers and there is no resonant tunneling. By the same reasoning, \( \omega_n > \frac{\pi v_F}{g^2 \hbar (L - d)} \) to ensure that the complementary distance is sufficiently large, so that the electron sees the two barriers sequentially and there is no
resonant tunneling. Also, to ensure that the one-dimensional physics of the ring is being probed, the temperatures have to be greater than $T_L = \hbar v_F / k_B L$.

The symmetry $\chi \rightarrow \chi + \sqrt{\pi}$ is a generic symmetry of the action, whereas $\chi \rightarrow \chi + \sqrt{\pi}/2$, $n \rightarrow n + 1$ is a symmetry only at resonance. Hence, there is a phase change of $\pi$ at each successive resonance, where the charge state of the island changes by unity. However, in between resonances, the symmetry is restored to $\chi \rightarrow \chi + \sqrt{\pi}$ which does not allow phase changes. *Phase rigidity is thus a consequence of this symmetry.* What about the scale over which the symmetry changes or the width of the resonances? The naive expectation for the width of the resonance is that it be of the same order of the energy scale in the problem, which is the dot energy scale or $\hbar v_F / g^2 d$. However, for mesoscopic systems, the system size offers another energy scale of the order of $1/L$, which is much smaller than $1/d$ in the limit where this analysis is valid and can lead to much narrower resonances. Moreover, for interacting systems in one dimension, it is well-known that resonances are extremely narrow, degenerating into $\delta$-function peaks as $T \rightarrow 0$. Hence, we suggest that *the extra-ordinary abruptness of the phase change on scales much smaller than $k_B T$ can be explained by a combination of two facts*. One is that the the resonance is related to the symmetry of the fermions on the ring, whose relevant energy scale is given by $1/L$. The second is that for interacting electrons, resonances are extremely narrow. *In fact, a study of the scale over which the phase change takes place is equivalent to the study of the resonance line shape for the resonance peak.* Thus, we predict that the width over which $\pi$ changes should be the same as the width of the resonance maxima. The measurement of this width should thus be a measurement of the Luttinger parameter $g$ in the one-dimensional wire [11].

When we introduce flux through the ring, we see that the symmetries on the bosonic fields at even and odd resonances through the upper arm are also changed to $\theta(x) \rightarrow \theta(x) + \sqrt{\pi} q$ and $\phi(x) \rightarrow \phi(x) \pm \sqrt{\pi} q \pm \delta / \sqrt{\pi}$, where $q$ is integer or half-integer for even and odd resonances and $+/-$ is for the $R/L$ movers respectively. For the lower arm, they are given by $\theta(x) \rightarrow \theta(x) + \sqrt{\pi} q$ and $\phi(x) \rightarrow \phi(x) \pm \sqrt{\pi} q \pm \delta / \sqrt{\pi}$. But here, $q$ is always an integer.

Now let us consider some particular cases.

- **$\delta = \pi$**
  At odd resonances, both $\psi_L(x)$ and $\psi_R(x)$ acquire a phase shift $\pi$ as they travel through the dot. But they also acquire a phase shift of $\pi$ due to the flux. Hence, in this case, there is a destructive interference with the corresponding fields from the lower arm when $q$ is an integer - i.e at even resonances. When $q$ is odd, the fermions through the upper and lower arms interfere constructively to give rise to conductance peaks. Thus, the position of the conductance maxima shift to the position of the minima when there was no flux.

- **$\delta = \pi/2$**
  When one-quarter fluxes are introduced, there is yet another twist which comes into play. Precisely at $\delta = \pi/2$, it becomes possible to have constructive interference between left-moving electrons through the upper arm with right moving holes through
the lower arm and vice-versa. The phases of the left and right movers at even resonance after one full circuit through the ring are given by

\[ \psi_L \rightarrow e^{-i\pi/2}\psi_L, \quad \psi_R \rightarrow e^{i\pi/2}\psi_R \quad \text{with dot} \]
\[ \psi^C_L \rightarrow e^{i\pi/2}\psi^C_L, \quad \psi^C_R \rightarrow e^{-i\pi/2}\psi^C_R \quad \text{without dot} \]  

(9)

and at odd resonance by

\[ \psi_L \rightarrow e^{i\pi/2}\psi_L, \quad \psi_R \rightarrow e^{-i\pi/2}\psi_R \quad \text{with dot} \]
\[ \psi^C_L \rightarrow e^{3i\pi/2}\psi^C_L, \quad \psi^C_R \rightarrow e^{-3i\pi/2}\psi^C_R \quad \text{without dot} \]  

(10)

In both cases, it is easy to see that \( \psi_L \) through the upper arm and \( \psi^C_R \) through the lower arm have the same phases and interfere constructively and so do \( \psi_R \) and \( \psi^C_L \). Also, this constructive interference happens both for the odd and even resonances and there should be maxima in the transmission conductance for both cases. Hence, the spacing between conductance peaks should be halved as compared to the spacing without any flux. This halving of the spacing of the conductance maxima at these values of \( \delta \) has also been noted by Kang[7], who computed the tunneling conductance explicitly using the Friedel sum rule for phase change through the dot. However, here we understand the reason why the odd resonances survive at these particular values of the external flux in terms of the symmetries of the theory.

- \( \delta = \) arbitrary

Here, as for the case when \( \delta = \pi \), we still expect the transmission at even resonances, where there is no phase shift through the dot, to interfere constructively and lead to conductance maxima. However, the position of the conductance maxima shift continuously as a function of the flux.

Note that the entire analysis has no dependence on the value of the Luttinger parameter \( g \) except that it be within the range where resonant tunneling behaviour is allowed. The only difference that one expects between \( g = 1 \) and \( g \neq 1 \) is in the widths of the regions of the phase change, and the widths of the transmission maxima. At \( T = 0 \), and in the thermodynamic limit, for \( g \neq 1 \), the resonance peaks are expected to be infinitely sharp. However, for finite \( T \) and for mesoscopic lengths \( L \), one expects the widths to have appropriate power law dependences on both these quantities. Whether the abrupt nature of the phase change is related to the well-known fact that interactions appear to narrow resonances, or whether it depends only on the fact that the scale over which the resonance occurs is related to the ring energy scales rather than the dot energy scales, is a more detailed question, which needs the explicit computation of the tunneling conductance and the line shapes[11].

In conclusion, in the above analysis, we have explicitly used the effective action for a 1-D resonant tunneling structure to show that the effect of going through the dot resonant state leads to a change in phase for the electron fields which can then be evaluated. But a similar analysis is also valid for a general dot, as long as the dot is embedded in a narrow wire
where the single channel approximation holds good. The dot can be thought of as providing effective single particle energy levels for resonance, \( Q^2/C \), where \( C \) is the capacitance of the dot, as well as a phase change whenever an odd number of fermions tunnel through it. Transmission through the dot can then be thought of in terms of hopping between a local impurity situated at the site of the dot and the electron fields \([10]\). The effect of such a local impurity on the fermions in the ring is to cause phase shifts which in the bosonic representation can be expressed in terms of transformations on the associated boson fields. A more detailed analysis in terms of boundary conformal field theory and an explicit computation of the tunneling conductance will be reported elsewhere \([11]\).

Thus, we have been able to explain many of the distinct features seen in experiments on coherent transport through a mesoscopic ring with a dot embedded in one of its arms, through the symmetries of the effective action for the coherent transport. Conductance maxima occur only at even resonances which allow for constructive interference between the two different paths and since the same symmetry exists at all even resonances, this also explains why successive maxima are always in phase. Between two successive maxima, there is an odd resonance which is characterized by a phase drop of \( \pi \). However, note that this means that the two transmission maxima are separated from one another by the addition of two electrons to the dot, whereas experiments seem to indicate that it is more likely that the maxima are separated by the addition of a single electron. To put it another way, in general, one would not expect the Coulomb blockade minima to coincide with the minima expected due to destructive interference.

The phase rigidity between phase changes is explained as a consequence of the symmetry of the action which corresponds to the discrete particle nature of the electrons. The abruptness of the phase change can be related both to the fact that the underlying symmetry change occurs over electron energy scales and not over dot energy scales and also to the narrowness of the resonances expected for interacting fermions. Within this picture, in fact, it is harder to understand why the width of the conductance maxima follows the standard non-interacting Breit-Wigner form. In other words, the conductance maxima behave as if single electrons are tunneling through, whereas the minima behave as if the system is interacting. This is still a puzzle that has to be understood better.

In the presence of an Aharanov-Bohm flux through the ring, the positions of the resonances are shifted. We also see a period doubling in the case when the flux through the ring is one-quarter of the flux quantum, in agreement with the result of Kang. At a more detailed level, the non-zero tunneling conductance amplitude at the 'odd resonances' require multi-path contributions or contributions of higher dimensional operators in the path integral. We expect our results to hold even for a general dot. At resonance, the effect of the dot is expected to be that of a local impurity on the fermions in the ring, leading to phase shifts.
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