WEAK TYPE BOUNDS FOR ROUGH MAXIMAL SINGULAR INTEGRALS NEAR $L^1$

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Abstract. In this paper it is shown that for $\Omega \in L \log L(S^{d-1})$, the rough maximal singular integral operator $T^*_\Omega$ is of weak type $L \log L(R^d)$. Furthermore, for $w \in A_1$ and $\Omega \in L^\infty(S^{d-1})$, it is shown that $T^*_\Omega$ is of weak type $L \log L(w)$ with weight dependence $[w]_{A_1} [w]_{A_\infty} \log([w]_{A_\infty} + 1)$, which is same as the best known constant for the singular integral $T_\Omega$.

1. Introduction

The rough maximal operator $M_\Omega$ for $\Omega \in L^1(S^{d-1})$ is defined as

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^d} \int_{|x-y| \leq r} \Omega\left(\frac{x-y}{|x-y|}\right) |f(y)| \, dy.$$ 

Let $\int_{S^{d-1}} \Omega(\theta) \, d\theta = 0$, where $d\theta$ is the surface measure on $S^{d-1}$. The rough singular integral is given by,

$$T_\Omega f(x) = \text{p.v.} \int \frac{1}{|x-y|^d} \Omega\left(\frac{x-y}{|x-y|}\right) f(y) \, dy,$$

and the truncated rough singular operator is defined as

$$T^*_\Omega f(x) = \int_{|x-y| > \epsilon} \frac{1}{|x-y|^d} \Omega\left(\frac{x-y}{|x-y|}\right) f(y) \, dy.$$

Consider the rough maximal singular operator,

$$T^*_\Omega f(x) = \sup_{\epsilon>0} |T^*_\Omega f(x)|.$$

For $\Omega \in L \log L(S^{d-1})$ the rough singular integrals $T_\Omega$ as well as rough maximal singular integral operator $T^*_\Omega$, were shown to be bounded in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ by Calderón and Zygmund [1]. The same result was also established by Duoandikoetxea and Rubio de Francia [7] for $\Omega \in L^q(S^{d-1})$, $q > 1$, using Fourier transform estimates and a double dyadic decomposition of the kernel $K(x) = \text{p.v.} |x|^{-d} \Omega\left(\frac{x}{|x|}\right)$.

The case $p = 1$ was more elusive. For $\Omega \in \text{Lip}(S^{d-1})$, $T_\Omega$ is a standard Calderón-Zygmund operator hence bounded from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. In [2] and [3], Christ and Rubio de Francia showed that $M_\Omega$ is weak $(1,1)$ for $\Omega \in L \log L(S^{d-1})$. It was shown that $T_\Omega$ is weak $(1,1)$ in dimension two independently in [3] and [14]. Finally, the case for all dimensions for $T_\Omega$ was resolved by Seeger [26] using microlocal analysis. However, the weak $(1,1)$ boundedness for $T^*_\Omega$
The aim of this paper is to study the end point boundedness properties of the operator $T_\Omega^\ast$. Using an extrapolation argument one can derive that $T_\Omega^\ast f$ is in $L^{1,\infty}$ if $f \in L \log L$ and supported in a cube. By a slight abuse of notation, we define an operator $T$ is of weak type $L \log L(\mu)$ for a measure $\mu$ if there exists a constant $C_{\mu,T} > 0$ such that
\[
\mu(x \in \mathbb{R}^d : |Tf(x)| > \alpha) \leq \frac{C_{\mu,T}}{\alpha} \int |f(x)| \log \log \left( e^2 + \frac{|f(x)|}{\alpha} \right) d\mu(x).
\]
holds for all $\alpha > 0$. We denote $\|T\|_{L \log L(\mu) \to L^{1,\infty}(\mu)}$ to be the smallest constant $C_{\mu,T} > 0$ satisfying the above inequality. We also write $T$ is weak type $L \log L$ if the underlying measure is the Lebesgue measure on $\mathbb{R}^d$.

Our main theorem is as follows:

**Theorem 1.1.** Let $\Omega \in L \log L(S^{d-1})$ and $\int_{S^{d-1}} \Omega(\theta)d\theta = 0$. Then $T_\Omega^\ast$ is of weak type $L \log L$.

Our result improves the following result of Honzík [15], proved very recently.

**Theorem 1.2 ([15]).** Let $\Omega \in L^{\infty}(S^{d-1})$ and $\epsilon > 0$. Then
\[
\|T_\Omega^\ast f\|_{L^{1,\infty}} \leq C_{\epsilon} \|f\|_{L(\log L)^{2+\epsilon}}
\]
for functions $f$ supported in an unit cube and
\[
\|f\|_{L(\log L)^{2+\epsilon}} = \inf \left\{ \alpha > 0 : \int |f| \left( \log \log \left( e^2 + \frac{|f|}{\alpha} \right) \right)^{2+\epsilon} \leq \alpha \right\}.
\]

We would like to indicate the following improvements of Theorem 1.2. First it encompasses larger domain $L \log L(L^d)$, secondly it extends the class of $\Omega \in L^{\infty}(S^{d-1})$ to $L \log L(S^{d-1})$. Lastly, Theorem 1.1 also implies a local Orlicz space estimate like Theorem 1.2. Namely for $f$ supported in a unit cube $Q$, we have
\[
\|T_\Omega^\ast f\|_{L^{1,\infty}} \lesssim \|f\|_{L \log L(\Omega)}.
\]

To see this, assume $\|f\|_{L \log L(\Omega)} \leq 1$. For $\alpha \leq 100$, observe that $T_\Omega^\ast f(x) \lesssim M_\Omega f(x)$ holds for $x \notin ((100d)Q)$ and hence the above inequality follows from weak $(1,1)$ boundedness of $M_\Omega$. The case $\alpha > 100$ follows from Theorem 1.1.

One can observe that applying the method given in [15] to $T_\Omega$ only yields $T_\Omega : L(\log L)^{1+\epsilon}(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ boundedly. In the aforementioned method, the function $f$ is decomposed based on the $L(\log L)^{\gamma}$ averages for $\gamma > 0$ instead of the more natural $L^1$ averages. Also, the approach in [15] relies on the double dyadic decomposition of the kernel $K$ at the beginning, as in [20], which forces extra assumption on the size of the function. In contrast to the decomposition considered in [15], our method is similar in the spirit of [20], in the sense that if we apply this method for $T_\Omega$ we will recover the well known weak $(1,1)$ estimate. We will employ a finer decomposition of the function based on its size as in [28].

In this paper we have also studied the operator with rough radial part. Let $h \in L^{\infty}(\mathbb{R}^d)$ be a radial function. Define
\[
T_{\Omega,h} f(x) = p.v. \int_{\mathbb{R}^d} \frac{h(x-y)}{|x-y|^{d-\alpha}} \Omega\left(\frac{x-y}{|x-y|}\right) f(y)dy.
\]
This operator was shown to be bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, $1 < p < \infty$, for $\Omega \in \text{Lip}(\mathbb{S}^{d-1})$ by R. Fefferman [11]. Later Duoandikoetxea and Rubio de Francia [7] improved this result for $\Omega \in L^q(\mathbb{S}^{d-1})$ for some $1 < q \leq \infty$ and further to $\Omega \in H^1(\mathbb{S}^{d-1})$ in [8]. We would like to mention that Bochner-Riesz operator at the critical index $\frac{d-1}{2}$ is a prime example of this class of operators. In [3] weak $(1,1)$ estimate for this operator was shown, assuming the $L^2$ boundedness of $T_{\Omega,h}$ for $\Omega \in L^\infty(\mathbb{S}^{d-1})$ and $\partial_\theta \Omega \in L^\infty(\mathbb{S}^{d-1})$. By standard argument the size condition $\Omega \in L^\infty(\mathbb{S}^{d-1})$ can be replaced by $\Omega \in L\log L(\mathbb{S}^{d-1})$. To the best of our knowledge weak $(1,1)$ boundedness of $T_{\Omega,h}$ for $h \in L^\infty$ and for $\Omega \in L\log L(\mathbb{S}^{d-1})$, without assuming any smoothness condition, is not known.

Define the maximal operator corresponding to $T_{\Omega,h}$ as

$$T_{\Omega,h}^* f(x) = \sup_{\epsilon > 0} \left| \int_{|x-y| > \epsilon} \frac{h(x-y)}{|x-y|^{d-1}} \Omega \left( \frac{x-y}{|x-y|} \right) f(y) dy \right|.$$ 

In [7] it was proved that $T_{\Omega,h}^*$ is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. Our result is the following.

**Theorem 1.3.** Let $\Omega \in L\log L(\mathbb{S}^{d-1})$ with $[\mathbb{S}^{d-1}] \Omega(\theta d\theta) = 0$, $\partial_\theta \Omega \in L^\infty(\mathbb{S}^{d-1})$, and $h \in L^\infty(\mathbb{R}^d)$ be a radial function. Then $T_{\Omega,h}^*$ is of weak type $L\log L$. 

In the last decade quantitative weighted boundedness of singular integral operator has been one of the main theme of research in harmonic analysis. The celebrated result of Hytönen [16] finally showed that the dependence of the $L^2(w)$ boundedness of the Calderón-Zygmund operator $T_{cz}$ on the $A_2$ characteristic of the weight $w$ is linear. The sparse techniques, which evolved, while addressing this problem, proved to be very significant. It is still unknown whether the quadratic dependence of $T_{\Omega} : L^2(w) \to L^2(w)$ on the $A_2$ characteristic of $w$ is sharp [20, 5, 22]. For the end point $p = 1$, the operator $T_{\Omega}$ is known to be bounded from $L^1(w) \to L^{1,\infty}(w)$ for $w \in A_1$ [29, 10]. To state a quantitative version of this result we will need the following definitions.

**Definition 1.4.** Let $w : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function. We say $w \in A_1$ if

$$[w]_{A_1} = \sup_{Q \subset \mathbb{R}^d} \left( \frac{1}{|Q|} \int_Q w(t) \, dt \right)
\left\| w^{-1} \right\|_{L^\infty(Q)}$$

is finite. For $1 < p < \infty$, we say $w \in A_p$ if

$$[w]_{A_p} = \sup_{Q \subset \mathbb{R}^d} \left( \frac{1}{|Q|} \int_Q w(t) \, dt \right)
\left( \frac{1}{|Q|} \int_Q w(t)^{-\frac{1}{p-1}} \, dt \right)^{p-1}$$

is finite, and $w \in A_\infty$ if

$$[w]_{A_\infty} = \sup_{Q \subset \mathbb{R}^d} \left( \int_Q w(t) \, dt \right)^{-1}
\left( \int_Q M(wxQ)(t) \, dt \right)$$

is finite, where $M$ is the Hardy-Littlewood maximal function.

In [24, 17] it was shown that

$$(1.1) \quad \| T_{cz} \|_{L^1(w) \to L^{1,\infty}(w)} \lesssim [w]_{A_1} \log([w]_{A_\infty} + 1).$$
Moreover, this dependence on weight characteristics is optimal [23]. In [21, 18] it was shown that the above dependence also holds for the maximal Calderón-Zygmund operator $T^*_{CZ}$, i.e.
\[
\|T^*_{CZ}\|_{L^1(w)\rightarrow L^{1,\infty}(w)} \lesssim [w]_{A_1} \log([w]_{A_{\infty}} + 1).
\]

For $\Omega \in L^\infty(S^{d-1})$ the following bound for $T_\Omega$ was obtained in [25],
\[
\|T_\Omega\|_{L^1(w)\rightarrow L^{1,\infty}(w)} \lesssim [w]_{A_1} [w]_{A_{\infty}} \log([w]_{A_{\infty}} + 1).
\]

It is not known whether the extra constant $[w]_{A_{\infty}}$, in comparison to $T_{CZ}$, can be removed. The following result asserts a similar weighted dependence for $T^*_{\Omega}$.

**Theorem 1.5.** Let $\Omega \in L^\infty(S^{d-1})$ with $\int_{S^{d-1}} \Omega(\theta) \, d\theta = 0$. Then for $w \in A_1$ and $\alpha > 0$, we have,
\[
\|T^*_{\Omega}\|_{L^1(w)\rightarrow L^{1,\infty}(w)} \lesssim [w]_{A_1} [w]_{A_{\infty}} \log([w]_{A_{\infty}} + 1).
\]

We would like to remark that if one follows the method given in [29] one gets that $T^*_{\Omega}$ is of weak type $L \log \log L(w)$. However our proof is inspired by the method given in [25] which yields better weight constant. Following the arguments of Theorem 1.5, a similar result can be obtained for the operator $T^*_{\Omega, h}$.

In section §2 we will give the proof of Theorem 1.1. §3 will contain the sketch of the proof of Theorem 1.3 and §4 is devoted to the proof of Theorem 1.5. Throughout this paper $A \lesssim B$ means there is a constant $C$ depending only on the dimension $d$ such that $A \leq CB$. For any cube $Q$ we denote $l(Q)$ to be the length of the cube. For us, a dyadic cube $Q$ means $Q = 2^j \left( (0,1]^d + l \right)$ for some $j \in \mathbb{Z}$ and $l \in \mathbb{Z}^d$. For $C > 0$, we define $CQ$ to be the cube with the same centre as $Q$ and length $Cl(Q)$. Also, for a non-negative weight $w$ and a set $E \subset \mathbb{R}^d$, we denote $|E|$ to be the Lebesgue measure of the set $E$ and $w(E) = \int_E w(t) \, dt$.

## 2. Proof of Theorem 1.1

We begin by discretising the supremum in $T^*_{\Omega}$. Let $\beta \in C^\infty_c(\mathbb{R}^d)$ be a function supported on the annulus $\{ \frac{1}{2} \leq |x| \leq 2 \}$ and $\sum_{i \in \mathbb{Z}} \beta_i(x) = 1$ for $x \neq 0$, where $\beta_i(x) = \beta(2^{-i}x)$. We have
\[
T^*_{\Omega} f \leq M_\Omega f + \sup_{k \in \mathbb{Z}} \left| \sum_{i > k} K^i * f \right|,
\]
where $K^i(x) = K(x) \beta_i(x)$. To see the last inequality, for an $\epsilon > 0$, let $k_\epsilon = \lfloor \log_2 \epsilon \rfloor$. Then we have $T^*_{\Omega} f \leq |K^{k_\epsilon} * f| + \left| \sum_{i > k_\epsilon} K^i * f \right| \lesssim M_\Omega f + \sup_{k \in \mathbb{Z}} \sum_{i > k} K^i * f$ and hence (2.1) follows.

$M_\Omega$ is known to be of weak type $(1,1)$, see [3]. So it remains to estimate $\sup_{k \in \mathbb{Z}} \left| \sum_{i > k} K^i * f \right|$.

### 2.1. Decomposition of the function

Let $f \in L (\log \log L(\mathbb{R}^d))$ and $\alpha > 0$. We choose a maximal collection $\{Q\}_{Q \in F}$ of dyadic cubes such that,
\[
\alpha < \frac{1}{|Q|} \int_Q |f| \leq 2^d \alpha.
\]
We define the exceptional set $E = \bigcup_{Q \in \mathcal{F}} (100d)Q$. Therefore $|E| \lesssim \frac{1}{\alpha} \|f\|_1$. We set

$$f = g + \sum_{Q \in \mathcal{F}} f_Q,$$

where

$$g = f \chi_{\mathbb{R}^d \setminus \bigcup Q} + f \chi_{(\bigcup Q) \cap \{ |f| \leq 2^{\alpha} \}},$$

$$f_Q = f \chi_{Q \cap \{ |f| > 2^{\alpha} \}},$$

and $0 < c_1 < \frac{1}{4}$ is independent of $\alpha$ and is to be chosen later.

By Lebesgue differentiation theorem, we have $|g| \lesssim \alpha$ and hence $\|g\|^2_2 \lesssim \alpha \|f\|_1$. We further decompose

$$f_Q = \sum_{n=1}^{\infty} f^n_Q,$$

where

$$f^n_Q = f_Q \chi_{\{ 2^{\alpha} 2^{(n-1)} < |f_Q| \leq 2^{\alpha} 2^n \}},$$

Clearly, we have $\sum_{n=1}^{\infty} \frac{1}{|Q|} \int |f^n_Q| \lesssim \alpha$. We write $f^n_Q = g^n_Q + b^n_Q$, where

$$g^n_Q(x) = \left( \frac{1}{|Q|} \int f^n_Q \right) \chi_Q(x), \text{ and } b^n_Q = f^n_Q - g^n_Q.$$

For $n \in \mathbb{N}$, we set $g^n = \sum_{Q \in \mathcal{F}} g^n_Q$, $b^n = \sum_{Q \in \mathcal{F}} b^n_Q$, and $f^n = \sum_{Q \in \mathcal{F}} f^n_Q$.

We state some basic properties of these functions which can be verified easily.

\begin{align*}
(2.2) & \quad \|f^n\|^2_2 \lesssim 2^{\alpha} 2^n \alpha \|f\|_1.
(2.3) & \quad \int b^n_Q = 0 \quad \forall n \in \mathbb{N}, \, Q \in \mathcal{F}.
(2.4) & \quad \left\| \sum_n \sum_{Q \in \mathcal{F}} g^n_Q \right\|_{\infty} \lesssim \alpha.
(2.5) & \quad \sum_n \sum_{Q \in \mathcal{F}} \left( \|g^n_Q\|_1 + \|b^n_Q\|_1 \right) \lesssim \sum_n \sum_{Q \in \mathcal{F}} \|f^n_Q\|_1 \lesssim \|f\|_1.
\end{align*}

By (2.4) and (2.5), we have $\|\sum_n g^n\|^2_2 \lesssim \alpha \|f\|_1$. Let $\phi \in C^\infty_c(\mathbb{R}^d)$ be a function supported in the ball $B(0, \frac{1}{2})$ satisfying $\int \phi = 1$. We smoothen the kernel $K^i$ by writing,

$$K^i_0 = K^i \ast \phi_{-i},$$

$$K^i_n = K^i \ast \phi_{2^n-i} \text{ for } n \in \mathbb{N},$$
where \( \phi_j(x) = 2^j \phi(2^j x) \).

To estimate the level set \( \{ x \in E^c : \sup_{k \in \mathbb{Z}} |\sum_{i > k} K^i * f| > \alpha \} \), We write

\[
\sum_{i > k} K^i * f = \sum_{i > k} K^i * g + \sum_{i > k} (K^i - K_n^i) * f^n + \sum_{i > k} \sum_{n \geq 1} (K_n^i - K_0^i) * g^n
\]

\[
+ \sum_{i > k} \sum_{n \geq 1} K_0^i * f^n + \sum_{i > k} \sum_{n \geq 1} (K_n^i - K_0^i) * b^n.
\]

We define

\[
\mathcal{H}_{k,1} = \sum_{i > k} K^i * g, \quad \mathcal{H}_{k,2} = \sum_{i > k} \sum_{n \geq 1} (K^i - K_n^i) * f^n, \quad \mathcal{H}_{k,3} = \sum_{i > k} \sum_{n \geq 1} (K_n^i - K_0^i) * g^n
\]

\[
\mathcal{H}_{k,4} = \sum_{i > k} \sum_{n \geq 1} K_0^i * f^n, \quad \mathcal{H}_{k,b} = \sum_{i > k} \sum_{n \geq 1} (K_n^i - K_0^i) * b^n.
\]

2.2. Estimate for Good parts. One can observe that

\[
|\widehat{K^i}(\xi)| \lesssim \min\{2^i |\xi|^a, 2^i |\xi|^{-a}\}, \quad 0 < a < 1
\]

also holds for \( \Omega \in L \log L(S^{d-1}) \) instead of more restrictive \( \Omega \in L^\infty(S^{d-1}) \). Taking into account of above observation and imitating the proof as in [7] (Theorem E) we have the following.

**Lemma 2.1.** Let \( \Omega \in L \log L(S^{d-1}) \) with \( \int_{S^{d-1}} \Omega(\theta)d\theta = 0 \) and \( g \in L^2(\mathbb{R}^d) \). Then

\[
\left\| \sup_{k \in \mathbb{Z}} \left| \sum_{i > k} K^i * g \right| \right\|_2 \lesssim \|g\|_2.
\]

We will show that \( \left\| \sup_{k \in \mathbb{Z}} |\mathcal{H}_{k,j}| \right\|_2 \lesssim \alpha \|f\|_1 \), for \( j = 1, 2, 3 \). Then by Chebyshev’s inequality, we have \( \left|\{ x \in E^c : \sup_{k \in \mathbb{Z}} |\mathcal{H}_{k,j}(x)| > \alpha/5 \} \right| \lesssim \frac{1}{\alpha} \|f\|_1 \) for \( j = 1, 2, 3 \).

First, we observe that \( \left\| \sup_{k \in \mathbb{Z}} |\mathcal{H}_{k,1}| \right\|_2 \lesssim \|g\|_2 \lesssim \alpha \|f\|_1 \), where the first inequality follows from Lemma 2.1.

To deal with \( \mathcal{H}_{k,j}, j = 2, 3 \), we need \( L^2 \) estimates for the following intermediary operators defined as

\[
T_{-1}^* h = \sup_k \left| \sum_{i > k} K_0^i * h \right|, \quad T_m^* h = \sup_{k \in \mathbb{Z}} \left| \sum_{i > k} (K_m^i - K_{m-1}^i) * h \right|, \quad m \in \mathbb{N}.
\]

The maximal operator \( T_{-1}^* \) behaves as maximally truncated Calderón-Zygmund operator. Precisely, we have the following.

**Lemma 2.2.** For \( \Omega \in L \log L(S^{d-1}) \), the operator \( T_{-1}^* \) is bounded in \( L^2 \) and weak type \((1,1)\).

**Proof.** We define the corresponding singular integral operator \( T_{-1} \) as

\[
T_{-1} f = \sum_{i \in \mathbb{Z}} K_0^i * f = S * f.
\]
We first show that the kernel $S$ satisfies the growth and Hörmander conditions:

$$|S(x)| \lesssim \frac{\|\Omega\|_1}{|x|}, \quad |S(x) - S(y)| \lesssim \|\Omega\|_1 \frac{|y|}{|x|^{d+1}} \quad \text{for } 2|y| < |x|.$$  

Indeed we have,

$$|K^i_0(x)| = \left| \int |y|^{-d} \Omega \left( \frac{y}{|y|} \right) \beta_i(y) \phi_{-i}(x - y) \, dy \right|$$

$$\lesssim \chi_{2^{-i-2} \leq |x| \leq 2^{i+2}}(x) \frac{1}{|x|} \int_{2^{i-1} \leq |y| \leq 2^{i+1}} |\Omega\left( \frac{y}{|y|} \right) \phi(2^{-i}(x - y))| 2^{-id} \, dy$$

$$\lesssim \chi_{2^{-i-2} \leq |x| \leq 2^{i+2}}(x) \frac{1}{|x|} \int_{S^{d-1}} \Omega(\theta) \left( \int_{r=0}^2 |\phi(2^{-i}x - r\theta)| r^{d-1} \, dr \, d\theta \right)$$

$$\lesssim \frac{\|\Omega\|_1}{|x|} \chi_{2^{-i-2} \leq |x| \leq 2^{i+2}}(x).$$

For Hörmander’s condition, we observe that

$$|\nabla K^i_0(x)| = \left| \int |y|^{-d} \Omega \left( \frac{y}{|y|} \right) \beta_i(y) \nabla \phi_{-i}(x - y) \, dy \right|$$

$$\lesssim \chi_{2^{-i-2} \leq |x| \leq 2^{i+2}}(x) \frac{1}{|x|} \int_{2^{i-1} \leq |y| \leq 2^{i+1}} |\Omega\left( \frac{y}{|y|} \right) \nabla \phi(2^{-i}(x - y))| 2^{-i(d+1)} \, dy$$

$$\lesssim \frac{\|\Omega\|_1}{|x|} \chi_{2^{-i-2} \leq |x| \leq 2^{i+2}}(x).$$

Thus we get $|\nabla S(x)| \leq \sum_{i \in \mathbb{Z}} |\nabla K^i_0(x)| \lesssim \frac{\|\Omega\|_1}{|x|^{d+1}}$. Therefore by mean value theorem, we have

$$|S(x) - S(y)| \lesssim \frac{\|\Omega\|_1 |y|}{|x|^{d+1}} \quad \text{for } 2|y| < |x|. $$

We also note that,

$$|\tilde{S}(\xi)| \leq \sum_{i \in \mathbb{Z}} \hat{K}^i(\xi) \hat{\phi}_{-i}(\xi) \lesssim \sum_{i:|2^i \xi| \leq 1} |2^i \xi|^{\alpha} + \sum_{i:|2^i \xi| > 1} |2^i \xi|^{-\alpha} \lesssim 1.$$  

Hence $T_{-1}$ is a $L^2$ bounded Calderón-Zygmund operator. And by standard Calderón-Zygmund theory [13], the maximally truncated singular integral operator $T^\#_{-1}$ is weak $(1,1)$, where $T^\#_{-1}f = \sup_{\epsilon > 0} |S_\epsilon * f|$ and $S_\epsilon = S\chi_{|.| > \epsilon}$.

Now $T^\#_{-1}f(x) \leq \sup_{k \in \mathbb{Z}} \left( \left| \int_{|x-y| > 2^{k+1}} \sum_{i > k} K^i_0(x - y) f(y) \, dy \right| + \left| \int_{|x-y| \leq 2^{k+1}} \sum_{i > k} K^i_0(x - y) f(y) \, dy \right| \right)$

$$\leq \sup_{k \in \mathbb{Z}} \left( \left| \int_{|x-y| > 2^{k+1}} \sum_{i > k} K^i_0(x - y) f(y) \, dy \right| + \left| \int_{|x-y| \leq 2^{k+1}} K^{k+1}_0(x - y) f(y) \, dy \right| \right)$$

(2.9) \quad \lesssim T^\#_{-1}f + Mf.

Hence $T^\#_{-1}$ is also $L^2$-bounded and weak type $(1,1)$. \qed
Now we provide an $L^2$ estimate for the operator $T_m^*$. The following lemma is a $L \log L$ counterpart for Lemma 8 in [15]. The estimate was used in [6] to provide a sparse domination for $T_m^*$. The proof follows in verbatim.

**Lemma 2.3.** There exists constant $c_2 > 0$ such that,

$$\|T_m^* h\|_2 \lesssim 2^{-c_2 m} \|h\|_2, \ m \in \mathbb{N}.$$

We now complete the estimates for $H_{k,j}$, $j = 2, 3$. Using Lemma 2.3 and choosing $c_1$ such that $c_1 < c_2$, we have

$$\|\sup_{k \in \mathbb{Z}} |H_{k,2}|\|_2 \lesssim \sum_{n \geq 1} \left( \sup_k \left\| \sum_{i>k}^{\infty} (K_m^K_n - K_{m-1}^i) * f^n \right\|_2 \right) \leq \sum_{n \geq 1} \sum_{m=n+1}^{\infty} \|T_m^* (f^n)\|_2 \lesssim \sum_{n \geq 1} \sum_{m=n+1}^{\infty} 2^{-c_2 m} \|f^n\|_2 \lesssim \sum_{n \geq 1} 2^{-c_2 n} (2^{c_1 n} \alpha \|f\|_1)^{\frac{1}{2}} \lesssim (\alpha \|f\|_1)^{\frac{1}{2}}.$$

For $j=3$, we write $K_m^K_n - K_m^K_n = \sum_{m=1}^{n} (K_m^K_n - K_{m-1}^i)$ and use Lemma 2.3 to get

$$\|\sup_{k \in \mathbb{Z}} |H_{k,3}|\|_2 \lesssim \|\sup_k \left\| \sum_{i>k}^{\infty} \sum_{n=m+1}^{\infty} (K_m^K_n - K_{m-1}^i) * g^n \right\|_2 \leq \sum_{m=1}^{\infty} \|T_m^* (\sum_{n=m}^{\infty} g^n)\|_2 \lesssim \sum_{m=1}^{\infty} 2^{-c_2 m} \|\sum_{n=m}^{\infty} g^n\|_2 \lesssim (\alpha \|f\|_1)^{\frac{1}{2}}.$$

The estimate for $|\{x \in E^c : \sup_k |H_{k,4}(x)| > \alpha/5\}|$ follows from weak $(1,1)$ boundedness of $T_m^*$ (Lemma 2.2) and $\|\sum_{n \in \mathbb{N}} f^n\|_1 \lesssim \|f\|_1$.

### 2.3. Estimates of Seeger.

Before estimating the bad part $H_{k,b}$, we state the estimates in [26], that were vital in proving weak type endpoint boundedness of $T_{\Omega}$.

Let $\{H^i\}_{i \in \mathbb{Z}}$ be a sequence of kernels satisfying,

$$\text{supp } (H^i) \subset \{2^{i-2} \leq |x| \leq 2^{i+2}\},$$

(2.10)
And for each $N \in \mathbb{N}$,
\begin{equation}
(2.11) \quad \sup_{0 \leq t \leq N} \sup_{i \in \mathbb{Z}} \left| \frac{\partial^t}{\partial r^t} H^i(r \theta) \right| \lesssim C_N,
\end{equation}
uniformly in $\theta \in S^{d-1}$ and $r \in \mathbb{R}^+$. Using the decomposition of $H^i$ in Section 2 of [26], we write
\begin{equation}
H^i = \Gamma^i_s + (H^i - \Gamma^i_s),
\end{equation}
and we have the following decay estimates.

**Lemma 2.4** ([26]). Let $I \subseteq \mathbb{Z}$ be an index set and $Q$ be a collection of disjoint dyadic cubes. Let $F_s = \sum_{Q \in \mathcal{Q}, l(Q) = 2^s} b_Q$, where $b_Q$ is supported in $Q$ and $\|b_Q\|_1 \lesssim \alpha|Q|$. Then there exists $\delta_1 > 0$ such that for $s > 3$, we have
\begin{equation}
(2.12) \quad \left\| \sum_{i \in I} \Gamma^i_s * F_{i-s} \right\|_2 \lesssim C_0 2^{-\delta_1 s} \alpha^\frac{1}{2} \left( \sum_{Q \in \mathcal{Q}} \|b_Q\|_1 \right)^{\frac{1}{2}}.
\end{equation}
If we also assume $\int b_Q = 0$, then
\begin{equation}
(2.13) \quad \|(H^i - \Gamma^i_s) * b_Q\|_1 \lesssim (C_0 + C_5d) 2^{-\delta_1 s} \alpha \|b_Q\|_1.
\end{equation}

For the proof of Lemma 2.4, we refer to the proofs of Lemma 2.1 and Lemma 2.2 of [26]. We note that the aforementioned proofs also work for any index $I \subseteq \mathbb{Z}$.

2.4. **Estimate for Bad part.** We now diverge from the proof of [28] in the sense that instead of conducting our analysis on the kernel side, we focus on the scales $s$ of the bad part. In fact, we have a much stronger decay in $s$ for the rough singular integrals than the more general Radon transforms considered in [28].

We begin by collecting cubes of same scales, namely
\begin{equation}
B^n_s = \sum_{Q \in \mathcal{F}, l(Q) = 2^s} b^n_Q.
\end{equation}
Hence $b^n = \sum_{s \in \mathbb{Z}} B^n_s$. Since $\text{supp}(K^n_s) \subset \{2^{i-2} \leq |x| \leq 2^{i+2}\}, \forall n \geq 0$, we have
\begin{equation}
(2.14) \quad \sum_{i > k} \sum_{n \in \mathbb{N}} \sum_{s < 3} (K^n_i - K^n_0) * B^n_{i-s}(x) = 0, \text{ for } x \in E^c.
\end{equation}
Therefore it remains to estimate $\left| \left\{ x \in \mathbb{R}^d : \sup_k C_n \sum_{i > k} \sum_{n \in \mathbb{N}} \sum_{s \geq 3} (K^n_i - K^n_0) * B^n_{i-s}(x) \right\|_1 > \frac{\alpha}{10} \right\|_1$.

Let $C > 0$ be a constant to be chosen later. By Chebyshev’s inequality and $\|K^n_i\|_1 \lesssim \|\Omega\|_{L^1(S^{d-1})}$, we have
\begin{align*}
\left| \left\{ x \in \mathbb{R}^d : \sup_k C_n \sum_{i > k} \sum_{n \in \mathbb{N}} \sum_{s \geq 3} (K^n_i - K^n_0) * B^n_{i-s}(x) \right\|_1 > \frac{\alpha}{10} \right\|_1 \\
\lesssim \frac{1}{\alpha} \sum_{i \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \sum_{s \geq 3} (\|K^n_i\|_1 + \|K^n_0\|_1) \|B^n_{i-s}\|_1
\end{align*}
\[
\begin{align*}
\lambda & \leq \frac{1}{\alpha} \sum_{n \in \mathbb{N}} \sum_{s=3}^{C_n} \sum_{i \in \mathbb{Z}} \sum_{Q, d(Q) = 2^i \cdot s} \| b_Q^n \|_1 \\
\lambda & \leq \frac{1}{\alpha} \sum_{n \in \mathbb{N}} \sum_{s=3}^{n} \| f_Q^n \|_1 \\
\lambda & \leq \frac{1}{\alpha} \sum_{n \in \mathbb{N}} \sum_{Q \in F} \int |f_Q^n| \log \left( e^2 + \frac{|f_Q^n|}{\alpha} \right) \\
\lambda & \leq \frac{1}{\alpha} \sum_{Q \in F} \int |f(x)| \log \left( e^2 + \frac{|f(x)|}{\alpha} \right) dx.
\end{align*}
\]

To deal with \( \Omega \in L \log L(\mathbb{S}^{d-1}) \), we decompose the kernel into bounded and integrable parts. In this regard, we define the set \( D_\delta = \{ \theta \in \mathbb{S}^{d-1} : |\Omega(\theta)| \leq 2^{\delta} \| \Omega \|_{L_1(\mathbb{S}^{d-1})} \} \), where the constant \( \delta > 0 \) is to be chosen later. We write

\[
\Omega = \Omega \chi_{D_\delta} + \Omega \chi_{D_\delta^c} = \Omega^+ + \Omega^-
\]

And thus \( K_n^i = K_n^{i+} + K_n^{i-} \), where \( K_n^{i+} = K_n^{i+} \ast \phi_{2^n-i} \) and \( K_n^{i\pm}(x) = |x|^{-d}|\Omega^{\pm}(\frac{x}{|x|})\beta_i(x) \). Therefore we have

\[
\begin{align*}
\left\{ x \in \mathbb{R}^d : \sup_k \sum_{i > k} \sum_{n \in \mathbb{N}} \sum_{s=3}^{C_n} \left( K_i^n - K_0^n \right) * B_{i-s}^n(x) > \frac{\alpha}{10} \right\} \\
\leq \left\{ x \in \mathbb{R}^d : \sup_k \sum_{i > k} \sum_{n \in \mathbb{N}} \sum_{s=3}^{C_n} \left( K_i^n - K_0^n \right) * B_{i-s}^n(x) > \frac{\alpha}{20} \right\} \\
+ \left\{ x \in \mathbb{R}^d : \sup_k \sum_{i > k} \sum_{n \in \mathbb{N}} \sum_{s=3}^{C_n} \left( K_i^n - K_0^n \right) * B_{i-s}^n(x) > \frac{\alpha}{20} \right\} \\
= B^+ + B^-
\end{align*}
\]

To bound \( B^+ \), by Chebyshev’s inequality we have

\[
B^+ \lesssim \frac{1}{\alpha} \sum_{i \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \sum_{s=3}^{C_n} \| (|K_i^n| + |K_0^n|) * B_{i-s}^n \|_1
\]

\[
\lesssim \frac{1}{\alpha} \sum_{n \in \mathbb{N}} \sum_{s=3}^{C_n} \| B_{i-s}^n \|_1 \| K_i^n \|_1
\]

\[
\lesssim \frac{1}{\alpha} \sum_{n \in \mathbb{N}} \sum_{Q \in F} \| b_Q^n \|_1 \sum_{s=3}^{\infty} \int_{|\Omega(\theta)| > 2^{\delta} \| \Omega \|_{L_1(\mathbb{S}^{d-1})}} |\Omega(\theta)| d\theta
\]

\[
\lesssim \frac{1}{\alpha} \| f \|_1 \int_{\mathbb{S}^{d-1}} |\Omega(\theta)| \cdot \text{card}\{ s \in \mathbb{N} : 2^{\delta s} < \frac{|\Omega(\theta)|}{\| \Omega \|_1} \} \ d\theta
\]
Lemma 2.6. By a change of variable and differentiating in radial variable one can verify that the kernel $B$ to estimate $C$ satisfies the estimates (2.10) and (2.11) with $K$.

By a change of variable and differentiating in radial variable one can verify that the kernel $H_i^n$ satisfies the estimates (2.10) and (2.11) with $C_N = 2^δs$ (or see [15]). Now, to deal with the maximal operator in $B^-$ we need a Cotlar type inequality that was first proved by Honzík [15]. But in contrast with [15], we group the indices $φ$ follows from the support condition of $φ$.

Lemma 2.5. Let $a_1, a_2, a_3 \in \mathbb{Z}$ with $a_1 < a_2 < a_3$. Then there exists $δ_2 > 0$ such that

$$\int |φ_{a_2} * φ_{a_1} - φ_{a_1}| \lesssim 2^δ(a_1 - a_2), \quad \text{supp}(φ_{a_2} * φ_{a_1} - φ_{a_1}) \subseteq B(0, 2^{-da_1}) \quad \text{and}$$

$$\int |φ_{a_1} * (φ_{a_3} - φ_{a_2})| \lesssim 2^δ(a_1 - a_2), \quad \text{supp}(φ_{a_1} * (φ_{a_3} - φ_{a_2})) \subseteq B(0, 2^{-da_1}).$$

Lemma 2.6. There exists a sequence of non-negative functions $\{v_i\}_{i \in \mathbb{Z}}$ with $\sup_{i \in \mathbb{Z}} \|v_i\|_1 \lesssim 1$ such that the following pointwise inequality holds:

$$\sup_{k \in \mathbb{Z}} \left| \sum_{i > k} H_i^m * B_i^n \right| \lesssim \sum_{r = 0}^{s 2^n + 2 - 1} M \left( \sum_{i \equiv r (mod \ s 2^n + 2)} H_i^m * B_i^n \right) + C_n, s \sum_{i \in \mathbb{Z}} \sum_{Q \in F, l(Q) = 2^i} v_i |b_Q^n|,$n $

(2.16)$

where $M$ is the Hardy-Littlewood maximal function, $C_{n, s} = s 2^n 2^{-δ_2s}$ and $δ_2$ is as in Lemma 2.5.

Proof. By Euclid’s algorithm, we write $i = ps 2^n + r, k = qs 2^n + r'$ with $0 \leq r, r' \leq s 2^n + 2 - 1$ and $i > k$ implies $p \geq q$. Hence we have

$$\sup_{k \in \mathbb{Z}} \left| \sum_{i > k} H_i^m * B_i^n \right| \leq \sup_{q \in \mathbb{Z}} \sup_{r' = 0, 1, \ldots, s 2^n + 2 - 1} \left| \sum_{p, k} \sum_{Q \in F, l(Q) = 2^p 2^n + 2 + r} H_m^{ps 2^n + 2 + r} * b_Q^n \right| \leq \sum_{r = 0}^{s 2^n + 2 - 1} \sup_{q \in \mathbb{Z}} \sum_{p \geq q} \sum_{Q \in F, l(Q) = 2^p 2^n + 2 + r} H_m^{ps 2^n + 2 + r} * b_Q^n.$$

Now we fix $r$ and write

$$\left| \sum_{p \geq q} \sum_{Q \in F, l(Q) = 2^p 2^n + 2 + r} H_m^{ps 2^n + 2 + r} * b_Q^n \right|$$
\[ \begin{aligned}
&\leq \sum_{p \geq q} \sum_{Q \in F: l(Q) = 2^{p\alpha n+2}+r-s} (H_{m}^{p\alpha n+2+r} - \phi_{t_q} \ast H_{m}^{p\alpha n+2+r}) \ast b_{Q}^{n} \\
&\quad + \left| \sum_{p \leq q} \sum_{Q \in F: l(Q) = 2^{p\alpha n+2}+r-s} H_{m}^{p\alpha n+2+r} \ast \phi_{t_q} \ast b_{Q}^{n} \right| \\
&\quad + \left| \sum_{p < q} \sum_{Q \in F: l(Q) = 2^{p\alpha n+2}+r-s} H_{m}^{p\alpha n+2+r} \ast \phi_{t_q} \ast b_{Q}^{n} \right| \\
&:= \mathcal{I}_{1,q} + \mathcal{I}_{2,q} + \mathcal{I}_{3,q},
\end{aligned} \]

where \( t_q = -qs^{n+2} - r + s^{n+1} \).

Clearly \( \mathcal{I}_{2,q} \lesssim M(\sum_{p \in \mathbb{Z}} \sum_{Q \in F: l(Q) = 2^{p\alpha n+2}+r-s} H_{m}^{p\alpha n+2+r} \ast b_{Q}^{n}) \).

To estimate \( \mathcal{I}_{1,q} \), we expand the kernel to have:

\[ |H_{m}^{p\alpha n+2+r} - \phi_{t_q} \ast H_{m}^{p\alpha n+2+r}| \]

\[ \leq \left| (\phi_{2m - ps^{n+2} - r} - \phi_{t_q} \ast \phi_{2m - ps^{n+2} - r}) \ast K^{(p\alpha n+2+r)} - \right| \\
\[ + \left| (\phi_{2m - 1 - ps^{n+2} - r} - \phi_{t_q} \ast \phi_{2m - 1 - ps^{n+2} - r}) \ast K^{(p\alpha n+2+r)} - \right|. \]

We estimate the first term, the second follows similarly. For \( p \geq q \), we have \( t_q > 2^{m} - ps^{n+2} - r \) and Lemma 2.5 implies

\[ |(\phi_{2m - ps^{n+2} - r} - \phi_{t_q} \ast \phi_{2m - ps^{n+2} - r}) \ast K^{(p\alpha n+2+r)} - | \]

\[ \lesssim 2^{\delta s} v_{ps^{n+2}+r} \int |(\phi_{2m - ps^{n+2} - r} - \phi_{t_q} \ast \phi_{2m - ps^{n+2} - r})| \]

\[ \lesssim 2^{\delta s} 2^{-\delta s} 2^{-\delta s (p-q)s^{n+2}} v_{ps^{n+2}+r}, \]

where \( v_{i} := 2^{-in} \chi_{2^{-i-2} \leq |x| \leq 2^{i+2}} \). Therefore we have

\[ \mathcal{I}_{1,q} \lesssim 2^{\delta s} 2^{-\delta s} \sum_{p \geq q} 2^{-\delta s (p-q)s^{n+2}} \sum_{Q \in F: l(Q) = 2^{p\alpha n+2}+r-s} v_{ps^{n+2}+r} \ast |b_{Q}^{n}| \]

\[ \leq 2^{-\delta s} \sum_{i \in \mathbb{Z}} \sum_{Q \in F: l(Q) = 2^{i-s}} v_{i} \ast |b_{Q}^{n}|. \]

Now we estimate \( \mathcal{I}_{3,q} \). \( p < q \), \( t_q < 2^{m-1} - ps^{n+2} - r \) and Lemma 2.5 implies

\[ |H_{m}^{p\alpha n+2+r} \ast \phi_{t_q}| \leq \left| (\phi_{2m - ps^{n+2} - r} - \phi_{2m - 1 - ps^{n+2} - r}) \ast \phi_{t_q} \ast K^{(p\alpha n+2+r)} - \right| \]

\[ \lesssim 2^{\delta s} v_{ps^{n+2}+r} \int |(\phi_{2m - ps^{n+2} - r} - \phi_{2m - 1 - ps^{n+2} - r}) \ast \phi_{t_q}| \]

\[ \lesssim 2^{\delta s} 2^{-\delta s} 2^{-\delta s (q-p-1)s^{n+2}} v_{ps^{n+2}+r}. \]
Hence \( T_{3,q} \leq 2^{-((\delta_2 - \delta_3)s)} \sum_{i \in \mathbb{Z}} \sum_{Q \in F} \sum_{l(Q) = 2^{i-s}} v_i \ast |b_Q^s| \).

Finally, taking supremum over \( q \in \mathbb{Z} \) and summing in \( r \) produces the desired inequality. \( \square \)

Now we conclude the proof of Theorem 1.1. Choose \( \delta < \frac{1}{2} \min \{ \delta_1, \delta_2 \} \) and \( C \delta > 100 \). By Lemma 2.6 we have,

\[
\mathcal{B}^- \lesssim \left\{ x \in \mathbb{R}^d : \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} \sup_{k > k} \left| \sum_{i > k} H_m^i \ast B_i^n(x) > \frac{\alpha}{20} \right| \right\}
\]

\[
\lesssim \left\{ x \in \mathbb{R}^d : \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} \sum_{i \in \mathbb{Z}} \sum_{Q \in F; l(Q) = 2^{i-s}} v_i \ast |b_Q^s(x)| > \frac{\alpha}{40} \right\}
\]

\[
+ \left\{ x \in \mathbb{R}^d : \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} \sum_{i \equiv r \ (mod \ s^{n+2})} M \left( \sum_{i \equiv r \ (mod \ s^{n+2})} H_m^i \ast B_i^n(x) > \frac{\alpha}{40} \right) \right\}
\]

\[= \mathcal{B}_{1}^- + \mathcal{B}_{2}^-\]

By Chebyshev’s inequality we have,

\[
\mathcal{B}_{1}^- \lesssim \frac{1}{\alpha} \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} \sum_{i \in \mathbb{Z}} \sum_{Q \in F; l(Q) = 2^{i-s}} \|v_i\| \|b_Q^s\|_1
\]

\[
\lesssim \frac{1}{\alpha} \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} \sum_{i \in \mathbb{Z}} \sum_{Q \in F; l(Q) = 2^{i-s}} \|b_Q^s\|_1
\]

\[
\lesssim \frac{1}{\alpha} \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} s^{2n+2-\delta} \|b_Q^s\|_1
\]

\[
\lesssim \frac{1}{\alpha} \sum_{m=1}^{\infty} 2^{98m} \sum_{n=m}^{\infty} \sum_{Q \in F} \|b_Q^s\|_1
\]

\[
\lesssim \frac{1}{\alpha} \|f\|_1
\]

To deal with \( \mathcal{B}_{2}^- \), we simply break the kernel into \( L^1 \) and \( L^2 \) parts. Indeed we have,

\[
\mathcal{B}_{2}^- \lesssim \left\{ x \in \mathbb{R}^d : \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} \sum_{i \equiv r \ (mod \ s^{n+2})} M \left( \sum_{i \equiv r \ (mod \ s^{n+2})} (\Gamma_{m,s}^i - H_{m}^i) \ast B_i^n(x) > \frac{\alpha}{80} \right) \right\}
\]

\[
+ \left\{ x \in \mathbb{R}^d : \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{C_{n+1}} \sum_{i \equiv r \ (mod \ s^{n+2})} M \left( \sum_{i \equiv r \ (mod \ s^{n+2})} \Gamma_{m,s}^i \ast B_i^n(x) > \frac{\alpha}{80} \right) \right\}
\]

\[= \mathcal{B}_{2,1}^- + \mathcal{B}_{2,2}^-\]
The estimate for $B_{2,1}$ follows by positivity and weak type $(1,1)$ boundedness of $M$ and estimate (2.13).

$$B_{2,1} \lesssim \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{\infty} s^{2n^2+2-1} \sum_{r=0}^{\infty} \left\| \sum_{i \equiv r \pmod{s2n^2}} (\Gamma^i_{m,s} - H^i_{m}) * B^n_{i-s} \right\|_1 \right)$$

$$\lesssim \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{\infty} s^{2n^2+2-1} \sum_{r=0}^{\infty} 2^{-(\delta_1-\delta)s} \|b^n_Q\|_1 \right)$$

$$\lesssim \frac{1}{\alpha} \left( \sum_{n=1}^{\infty} 2^{-98n} \sum_{Q \in F} \|b^n_Q\|_1 \right)$$

$$\lesssim \frac{1}{\alpha} \|f\|_1.$$ 

To estimate $B_{2,2}$, we use $L^2$ boundedness of $M$ and estimate (2.12) to get,

$$B_{2,2} \lesssim \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{\infty} s^{2n^2+2-1} \sum_{r=0}^{\infty} \left\| \sum_{i \equiv r \pmod{s2n^2}} \Gamma^i_{m,s} * B^n_{i-s} \right\|_2 \right)^2$$

$$\lesssim \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{\infty} s^{2n^2+2-1} \sum_{r=0}^{\infty} \left\| \sum_{i \equiv r \pmod{s2n^2}} \Gamma^i_{m,s} * B^n_{i-s} \right\|_2 \right)^2$$

$$\lesssim \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{\infty} s^{2n^2+2-1} \sum_{r=0}^{\infty} \left( \sum_{Q \in F} \|b^n_Q\|_1 \right) \left( \sum_{Q \in F} \|b^n_Q\|_1 \right) \right)^2$$

$$\lesssim \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_n}^{\infty} s^{2n^2+2-1} \sum_{r=0}^{\infty} \left( \sum_{Q \in F} \|b^n_Q\|_1 \right) \left( \sum_{Q \in F} \|b^n_Q\|_1 \right) \right)^2$$

$$\lesssim \frac{1}{\alpha} \|f\|_1.$$ 

where we have used Cauchy-Schwartz inequality in the second to last step.

3. Proof of Theorem 1.3

The proof of Theorem 1.3 follows the same line of arguments as of Theorem 1.1. In this section we will provide the necessary modifications required. Here we denote $K(x) = h(|x|)/\Omega \left( \frac{x}{|x|} \right) |x|^{-d}$. Eq. (2.1) will be replaced by

$$T_{\Omega,h}^* f \leq \|h\|_{\infty} M_\Omega f + \sup_{k \in \mathbb{Z}} \left| \sum_{i>k} K^i \ast f \right|.$$ 

We apply the same decomposition of the function $f$ as in Section 2.1. In [8], it was shown that $K^i$ satisfies the inequality (2.7). Hence, estimation for $\mathcal{H}_{k,1}$ will follow immediately.
It is enough to handle $\| \sum K_{i}^{n} * B_{1}^{n} \|_{2}$. Now,

$$\| \sum K_{i}^{n} * B_{1}^{n} \|_{2} = \sum_{i} \| K_{i}^{n} * B_{1}^{n} \|_{2} + 2 \sum_{i, i' \in I} \langle K_{i}^{n} * B_{1}^{n}, K_{i'}^{n} * B_{1}^{n} \rangle + \sum_{i, i' \in I} \langle K_{i}^{n} * B_{1}^{n}, K_{i'}^{n} * B_{1}^{n} \rangle$$

By Cauchy Schwartz inequality the third term is dominated by the first. Recall $\| \phi_{1} \| = 1$. Therefore,

$$\| K_{i}^{n} * B_{1}^{n} \|_{2} \leq 2^{-2(\delta_{1} - \delta)} \alpha \| h \|_{\infty} \| B_{1}^{n} \|_{1},$$

where the last inequality follows from the argument of inequality 2.3 of [3]. For the cross terms

$$\left| \langle K_{i}^{n} * B_{1}^{n}, K_{i'}^{n} * B_{1}^{n} \rangle \right| = \left| \langle K_{i}^{n} * \tilde{B}_{1}^{n}, \tilde{K}_{i'}^{n} * \tilde{B}_{1}^{n} \rangle \right| \leq \| \tilde{B}_{1}^{n} \|_{1} \| K_{i}^{n} * \tilde{K}_{i'}^{n} \|_{\infty}$$

By using Lemma 6.1 of [3], summing in $i, i'$ and Equation (3.1) we get the following.

**Lemma 3.1.**

$$\| \sum_{i} H_{i}^{n} * B_{1}^{n} \|_{2} \leq 2^{-(\delta_{1} - \delta) \alpha \frac{\hat{a}}{2}} \| h \|_{\infty} \left( \sum_{Q} \| b_{Q}^{n} \|_{1} \right)^{\frac{1}{2}}$$

In view of the above lemma we conclude the proof of Theorem 1.3

$$B_{2}^{-} \leq \frac{1}{\alpha^{2}} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_{n}}^{\infty} s^{2n+2} \sum_{r=0}^{s} \sum_{i \equiv r (mod 2n+2)} \| H_{i}^{n} * B_{1}^{n} \|_{2} \right)^{2}$$

$$\leq \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{s=C_{n}}^{\infty} s^{2n} 2^{-\delta_{1}} \left( \sum_{Q \in F} \| b_{Q}^{n} \|_{1} \right)^{\frac{1}{2}} \right)^{2} \| f \|_{1},$$

where $\alpha = 2^{-(\delta_{1} - \delta) s \frac{\hat{a}}{2}}$, $\beta = 2^{-\delta_{1} \alpha \frac{\hat{a}}{2}}$, $\gamma = 2^{-(\delta_{1} - \delta) \alpha \frac{\hat{a}}{2}}$, and $\delta = \delta_{1} - \delta$.
4. Proof of Theorem 1.5

Let \( f \in L \log \log L(\mathbb{R}^d) \) and \( w \in A_1 \). It is enough to show the appropriate boundedness for \( \sup |\sum_{i \geq k} K^i \ast f| \), taking into account Equation (2.1), \( M_{\Omega}f(x) \leq \|\Omega\|_{L^\infty} Mf(x) \) a.e. and \( \|M\|_{L^1(w) \to L^{1,\infty}(w)} \lesssim [w]_{A_1} \) (see [13]).

We follow the exact decomposition of the function \( f \) as in Section 2.1. In this weighted setting, we need a modification of the decomposition of the kernel. Let \( \epsilon_w > 0 \) be defined as

\[
\epsilon_w = \frac{\log([w]_{A_\infty} + 1)}{2(1 + c_d[w]_{A_\infty}(1 + \log([w]_{A_\infty} + 1)))}
\]

and \( \Delta \) be the greatest integer less than or equal to \( \epsilon_w^{-1} \). For this section, we denote \( K_n^i \) as

\[
K_n^i = K^i * \phi_{\Delta n^{-1}}, \ n \in \mathbb{N},
\]

and \( K_n^j \) is defined as earlier. We decompose our operator as in (2.6) with the modified \( K_n^i \).

The exceptional set \( E = \bigcup_{Q \in F}(100d)Q \) is estimated as follows

\[
w(E) \lesssim \sum_{Q \in F} \frac{w((100d)Q)}{|(100d)Q|}|Q| \leq \frac{[w]_{A_1}}{\alpha} \sum_{Q \in F} \inf_{y \in (100d)Q} w(y) \int_Q f \leq \frac{[w]_{A_1}}{\alpha} \|f\|_{L^1(w)}.
\]

To estimate \( \mathcal{H}_{k,j} = 1, 2, 3 \), the good parts, we need appropriate Fefferman-Stein type inequalities. The following inequality was proved in [6] (Theorem 1.3).

**Lemma 4.1 ([6])**. Let \( 1 < r < p \), then the following is true:

\[
\left\| \sup_i \sum_{i > k} K^i \ast g \right\|_{L^p(w)} \lesssim p^2(p')^{\frac{1}{p}}(r')^{\frac{1}{p'}} \|g\|_{L^p(M^r_w)},
\]

where \( M^r_w = M(w^r)^{\frac{1}{r}} \).

We need a Fefferman-Stein inequality for \( T_m^* \). By applying the abstract result, Theorem 3.3 of [6] (taking into consideration of their calculation of sparse constants in page 1890) to \( T_m^* \) we get \( (1 + \epsilon_w, 1 + \epsilon_w) \) sparse domination, namely we have

\[
|\langle T_m^* f, g \rangle| \lesssim \frac{2 - c_2 2^m - 1}{\epsilon_w} \sum_{Q \in S} |Q|^{-\frac{1}{1 + \epsilon_w}} \|f\|_{L^{1+\epsilon_w}(Q)} \|g\|_{L^{1+\epsilon_w}(Q)},
\]

where \( S \) is a \( \frac{1}{2} \)-sparse family of cubes i.e. for each \( Q \in S \) there exists \( E_Q \subset Q \) such that \( |E_Q| \geq \frac{1}{2}|Q| \) and \( \{E_Q : Q \in S\} \) are pairwise disjoint. Now, following exactly the proof of Theorem 1.3 of [6] and keeping track of the constant depending on \( m \), we obtain the required inequality.

**Lemma 4.2**. Let \( 1 < r < p \) be such that \( (r - 1) = 2\epsilon_w(pr - r + 1) \). Then we have

\[
\|T_m^* g\|_{L^p(w)} \lesssim 2^{-c_2 2^m - 1} p^2(p')^{\frac{1}{p'}}(r')^{\frac{1}{p'}} \|g\|_{L^p(M^r_w)},
\]

where \( c_2 > 0 \) as in Lemma 2.3.

We also state the optimal reverse Hölder inequality obtained in [19].
Lemma 4.3 (\cite{19}). Let $w \in A_{\infty}$. Then there exists an absolute constant $c_d > 0$ such that for any $r \in [1, 1 + \frac{1}{c_d|w|_{A_{\infty}}}]$ and cube $Q$, we have

\[
\left(\frac{1}{|Q|} \int_Q w^r\right)^{\frac{1}{r}} \leq \frac{2}{|Q|} \int_Q w.
\]

Consequently, the following pointwise inequality is true:

\[
M_{r_w} w \lesssim |w|_{A_1} w,
\]

where $r_w = 1 + \frac{1}{c_d|w|_{A_{\infty}}}$.

To estimate the bad part, for $\lambda > 0$, $s \geq 3$, we require some estimates of the measure of set $E_{\lambda}^s$ which is defined as

\[
E_{\lambda}^s = \left\{ x \in \mathbb{R}^d : \sum_{n=1}^{s/Csw} \sup_{k} \sum_{i > k} |(K_n^i - K_0^i) * B_i^n(x)| > \lambda \right\},
\]

where $C > 0$ to be chosen later and $s_w = |w|_{A_\infty} \log(|w|_{A_\infty} + 1)$. The first is a unweighted measure estimate with a decay in $s$ and the other is a simple weighted estimate.

Lemma 4.4. Let $0 < \lambda < 1$ and $0 < \delta < \min(\delta_1, \delta_2)/3$, where $\delta_1, \delta_2$ as in Lemma 2.4 and Lemma 2.5 respectively. Then for any non-negative weight $v$, we have

\[
|E_{\lambda}^s| \lesssim \frac{1}{\lambda^2} 2^{-\delta s} \sum_{Q \in \mathcal{F}} |Q|,
\]

\[
v(E_{\lambda}^s) \lesssim \frac{1}{\lambda} \sum_{Q \in \mathcal{F}} |Q| \inf_{Q} M v.
\]

Proof. We first observe that for the modified kernel $K_n^i - K_0^i = \sum_{m=1}^n (K_m^i - K_{m-1}^i) = \sum_{m=1}^n H_m^i$, the corresponding Cotlar-type inequality (2.16) is given by

\[
\sup_{k \in \mathbb{Z}} \left\{ \sum_{i > k} H_m^i * B_i^n \right\} \lesssim \sum_{r=0}^{\Delta s 2^{n+2}} M \left( \sum_{i \equiv r (\text{mod } s 2^{n+2})} H_m^i * B_i^n \right) + \Delta s 2^{n+2} \sum_{i \in \mathbb{Z}} \sum_{Q \in \mathcal{F} : l(Q) = 2^{n-s}} v_i |b_Q^n|,
\]

Therefore in regard to the above inequality and the decomposition in Section 2.3, we get

\[
|E_{\lambda}^s| \lesssim \left| \left\{ x \in \mathbb{R}^d : \sum_{n=1}^{s/Csw} \sum_{m=1}^n \Delta s 2^{n+2-\delta s} \sum_{i \in \mathbb{Z}} \sum_{Q \in \mathcal{F} : l(Q) = 2^{n-s}} v_i |b_Q^n|(x) > \frac{\lambda \alpha}{3} \right\} \right|
\]

\[
+ \left| \left\{ x \in \mathbb{R}^d : \sum_{n=1}^{s/Csw} \sum_{m=1}^n \sum_{r=0}^{\Delta s 2^{n+2-1}} M \left( \sum_{i \equiv r (\text{mod } s 2^{n+2})} \left( \Gamma_{m,s}^i - H_m^i \right) * B_i^n \right)(x) > \frac{\lambda \alpha}{3} \right\} \right|
\]

\[
+ \left| \left\{ x \in \mathbb{R}^d : \sum_{n=1}^{s/Csw} \sum_{m=1}^n \sum_{r=0}^{\Delta s 2^{n+2-1}} M \left( \sum_{i \equiv r (\text{mod } s 2^{n+2})} \Gamma_{m,s}^i * B_i^n \right)(x) > \frac{\lambda \alpha}{3} \right\} \right|
\]

\[
+ \left| \left\{ x \in \mathbb{R}^d : \sum_{n=1}^{s/Csw} \sum_{m=1}^n \sum_{r=0}^{\Delta s 2^{n+2-1}} M \left( \sum_{i \equiv r (\text{mod } s 2^{n+2})} \Gamma_{m,s}^i * B_i^n \right)(x) > \frac{\lambda \alpha}{3} \right\} \right|
\]

\[
+ \left| \left\{ x \in \mathbb{R}^d : \sum_{n=1}^{s/Csw} \sum_{m=1}^n \sum_{r=0}^{\Delta s 2^{n+2-1}} M \left( \sum_{i \equiv r (\text{mod } s 2^{n+2})} \Gamma_{m,s}^i * B_i^n \right)(x) > \frac{\lambda \alpha}{3} \right\} \right|
\]
\(= \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3\).

We now choose \(C\) to satisfy \(C\delta > 100\). By Chebyshev’s inequality and \(\|v_i\|_1 \lesssim 1\), we get

\[
\mathcal{E}_1 \lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} n \Delta s 2^{n+2 - \delta s} \sum_{Q \in F} \|b^n_Q\|_1
\]

\[
\lesssim \frac{1}{\lambda^s} 2^{-\delta} \sum_{Q \in F} |Q|
\]

The estimate for \(\mathcal{E}_2\) follows from positivity and weak \((1, 1)\) boundedness of \(M\) and inequality (2.13). Indeed we have

\[
\mathcal{E}_2 \lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \Delta s 2^{n+2 - \delta s} \sum_{r=0}^{\infty} \sum_{i \equiv r (\text{mod } \Delta s 2^{n+2})} \left\| \Gamma_{m,s}^i - H_m^i \right\|_1
\]

\[
\lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \Delta s 2^{n+2 - \delta s} \sum_{Q \in F} \|b^n_Q\|_1
\]

To estimate \(\mathcal{E}_3\), we repeatedly use Cauchy-Schwartz inequality, \(L^2\) boundedness of \(M\) and (2.12) to get,

\[
\mathcal{E}_3 \lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \Delta s 2^{n+2 - \delta s} \sum_{r=0}^{\infty} \sum_{i \equiv r (\text{mod } \Delta s 2^{n+2})} \left\| \Gamma_{m,s}^i - H_m^i \right\|_2
\]

\[
\lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} \Delta s 2^{n+2 - \delta s} \sum_{Q \in F} \|b^n_Q\|_1
\]

Thus the first inequality follows. For the weighted inequality, we apply Chebyshev’s inequality to get,

\[
v(E^n_\lambda) \lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{Q \in F : l_i(Q) = 2^{i-s}} \int \int |(K_n^i - K_0^i)(x - y)||b^n_Q(y)| \, dy \, v(x, dx)
\]

\[
\lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{Q \in F : l_i(Q) = 2^{i-s}} \int |b^n_Q(y)| \frac{1}{2^d} \int_{|x-y| \leq 2^{i+2}} \, v(x) \, dx \, dy
\]

\[
\lesssim \frac{1}{\lambda^s} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{Q \in F : l_i(Q) = 2^{i-s}} \int |b^n_Q(y)| \frac{1}{2^d} \inf_{z \in Q} \int_{|x-z| \leq 2^{i}} \, v(x) \, dx \, dy
\]
\[
\lesssim \frac{1}{\lambda \alpha} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{Q \in \mathcal{F}}: l(Q) = 2^{i-s} \|b^n_Q\|_{L^1} \inf_{z \in Q} M v(z)
\]
\[
\lesssim \frac{1}{\lambda \alpha} \sum_{n=1}^{\infty} \sum_{Q \in \mathcal{F}} \|f^n_Q\|_{L^1} \inf_{Q} M v
\]
\[
\lesssim \frac{1}{\lambda} \sum_{Q \in \mathcal{F}} |Q| \inf_{Q} M v.
\]

We now combine the estimates in Lemma 4.4 into a single weighted estimate using an interpolation scheme given in [9]. The proof follows similarly to that of estimate (5.9) in [9]. We include a sketch of the proof for the convenience of the reader.

**Lemma 4.5.** Let \( \theta_w = r_w^{-1} \), where \( r_w \) as in Lemma 4.3. Then we have the following

\[
w(E^k_{\lambda}) \lesssim \frac{1}{\alpha \lambda^2} 2^{-s\delta(1-\theta_w)} \|f\|_{L^1(M_{r_w} w)}.
\]

**Proof.** We first note that using Lemma 4.4 and Lemma 6 of [9], we get

\[
\int_{E^k_{\lambda}} \min(v(x), t) \, dx \lesssim \frac{1}{\lambda^2} \sum_{Q \in \mathcal{F}} |Q| \min(t 2^{-\delta s}, \inf_{Q} M v), \quad t \in \mathbb{R}^+.
\]

As \( v(x)^{\theta_w} = \theta_w(1 - \theta_w) \int_0^\infty \min(v(x), t) t^{-2+\theta_w} \, dt \), we have

\[
\int_{E^k_{\lambda}} v(x)^{\theta_w} \, dx = \theta_w(1 - \theta_w) \int_{E^k_{\lambda}} \int_0^\infty \min(v(x), t) t^{-2+\theta_w} \, dt \, dx
\]
\[
\lesssim \frac{1}{\lambda^2} \theta_w(1 - \theta_w) \sum_{Q \in \mathcal{F}} |Q| \int_0^\infty \min(t 2^{-\delta s}, \inf_{Q} M v) t^{-2+\theta_w} \, dt
\]
\[
\lesssim \frac{1}{\alpha \lambda^2} 2^{-s\delta(1-\theta_w)}\|f\|_{L^1(M_{r_w} w)}.
\]

Choosing \( v = w^r_w \) produces the desired inequality. \( \square \)

Now that we have all the ingredients, we complete the proof of Theorem 1.5. We choose

\[
p_w = 1 + \frac{1}{\log([w]_{A_{\infty}} + 1)}
\]

and \( r_w \) as in Lemma 4.3. To estimate \( H_{k,1} \), we use Chebyshev’s inequality, Lemma 4.1, and Lemma 4.3 to get,

\[
w\left\{ x \in E^c : \sup_k |H_{k,1}(x)| > \frac{\alpha}{5} \right\} \lesssim \frac{1}{\alpha \lambda^2} p_w^{2p-w_p} p'_w(r'_w)^{p-w_p} \|g\|_{L^w_{\lambda}(M_{r_w} w)}
\]
\[
\lesssim \frac{1}{\alpha} p_w^{2p-w_p} p'_w(r'_w)^{2p-w_p} \|f\|_{L^1(M_{r_w} w)}
\]
\[
\lesssim \frac{1}{\alpha} \log([w]_{A_{\infty}} + 1) [w]_{A_{\infty}} [w]_{A_1} \|f\|_{L^1(w)}.
\]
The estimate for \( \mathcal{H}_{k,2} \) follows from Lemma 4.2 and Lemma 4.3. Indeed

\[
\begin{align*}
&\quad \left\{ x \in E^c : \sup_k |\mathcal{H}_{k,2}(x)| > \frac{\alpha}{5} \right\} \\
&\leq \frac{1}{\alpha^{p_w}} \left( \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \| T_m^n(f^n) \|_{L^p_w(w)} \right)^{p_w} \\
&\leq \frac{1}{\alpha^{p_w}} \left( \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} 2^{-c^2 2^{m-1}} \left( \int |f^n|^{p_w} M_{r_w}(w) \right)^{\frac{1}{p_w}} \right)^{p_w} \\
&\leq \frac{1}{\alpha} \left( \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} 2^{-c^2 2^{m-1}} \left( \int |f^n|^{p_w} M_{r_w}(w) \right)^{\frac{1}{p_w}} \right)^{p_w} \\
&\leq \frac{1}{\alpha} \left( \sum_{n=1}^{\infty} 2^{-c^2 2^{m-1}} \left( \int |f^n|^{p_w} M_{r_w}(w) \right)^{\frac{1}{p_w}} \right)^{\frac{p_w}{n}} \\
&\leq \frac{1}{\alpha} \log([w]_{A_\infty} + 1)[w]_{A_\infty} [w]_{A_1} \| f \|_{L^1(w)},
\end{align*}
\]

where we used the fact that \( c_1 < c_2 \) and \( 1 < p_w < 10 \). The estimate for \( \mathcal{H}_{k,3} \) follows from Lemma 4.2 and Lemma 4.3. Indeed we have

\[
\begin{align*}
&\quad \left\{ x \in E^c : \sup_k |\mathcal{H}_{k,3}(x)| > \frac{\alpha}{5} \right\} \\
&\leq \frac{1}{\alpha^{p_w}} \left( \sum_{m=1}^{\infty} \left\| T_m^n \left( \sum_{n=m}^{\infty} g^n \right) \right\|_{L^p_w(w,E^c)} \right)^{p_w} \\
&\leq \frac{1}{\alpha^{p_w}} \left( \sum_{m=1}^{\infty} 2^{-c^2 2^{m-1}} \left( \int \left| \sum_{n=m}^{\infty} g^n(x) \right|^{p_w} M_{r_w}(w \chi_{E^c})(x) \, dx \right)^{\frac{1}{p_w}} \right)^{p_w} \\
&\leq \frac{1}{\alpha} \left( \sum_{m=1}^{\infty} 2^{-c^2 2^{m-1}} \left( \int \left| \sum_{Q \in F} \sum_{n=1}^{\infty} g^n(x) \right|^{p_w} M_{r_w}(w \chi_{E^c})(x) \, dx \right)^{\frac{1}{p_w}} \right)^{p_w} \\
&\leq \frac{1}{\alpha} \left( \int \sum_{Q \in F} \sum_{n=1}^{\infty} g^n(x) \, M_{r_w}(w \chi_{E^c})(x) \, dx \right) \\
&\leq \frac{1}{\alpha} \left( \int \sum_{Q \in F} \sum_{n=1}^{\infty} g^n(x) \, |M_{r_w}(w \chi_{E^c})(x) dQ\} \, dx \right) \\
&\leq \frac{1}{\alpha} \left( \int \sum_{Q \in F} \sum_{n=1}^{\infty} g^n(x) \, |M_{r_w}(w \chi_{E^c})(x) dQ\} \, dx \right) \\
&\leq \frac{1}{\alpha} \left( \int \sum_{Q \in F} \sum_{n=1}^{\infty} g^n(x) \, |M_{r_w}(w \chi_{E^c})(x) dQ\} \, dx \right) \\
&\leq \frac{1}{\alpha} \log([w]_{A_\infty} + 1)[w]_{A_\infty} [w]_{A_1} \| f \|_{L^1(w)},
\end{align*}
\]

where we used the fact that \( M_{r_w}(w \chi_{E^c})(x) \leq M_{r_w}(w \chi_{E^c})(y) \) for \( x, y \in Q \) in the third to last step. The proof of this fact follows exactly as that of \( M \) (see page 159 of [12]).
The estimate for $\mathcal{H}_{k,A}$ follows from inequalities (2.9), (1.2) and $\|\sum_{n=1}^{\infty} f^n\|_{L^1(w)} \lesssim \|f\|_{L^1(w)}$. We now turn to the estimate of $\mathcal{H}_{k,b}$. Using the support properties of the kernel (2.14), we get

$$w\{x \in E^c : \sup_k |\mathcal{H}_{k,b}(x)| > \frac{\alpha}{5}\} \leq w\{x \in E^c : \sup_k \left| \sum_{i>k} \sum_{n=1}^{\infty} \sum_{s=3}^{C_{nsw}-1} (K_n^i - K_0^i) * B_{i-s}^n(x) \right| > \frac{\alpha}{5}\}$$

$$\leq w\{x \in E^c : \sup_k \left| \sum_{i>k} \sum_{n=1}^{\infty} \sum_{s=3}^{C_{nsw}-1} (K_n^i - K_0^i) * B_{i-s}^n(x) \right| > \frac{\alpha}{10}\}$$

$$+ w\{x \in E^c : \sup_k \left| \sum_{i>k} \sum_{n=1}^{\infty} \sum_{s=3}^{C_{nsw}-1} (K_n^i - K_0^i) * B_{i-s}^n(x) \right| > \frac{\alpha}{10}\},$$

Using Chebyshev’s inequality, the first term is dominated by

$$\frac{1}{\alpha} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{s=3}^{C_{nsw}-1} \mathbb{E}_{F,i(Q)} \int |(K_n^i - K_0^i)(x-y)||b_Q^n(y)| \, w(x) \, dx$$

$$\lesssim \frac{1}{\alpha} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{s=3}^{C_{nsw}-1} \mathbb{E}_{F,i(Q)} \int |b_Q^n(y)| \frac{1}{2^d} \int |x-y| \leq 2^{i+2} w(x) \, dx \, dy$$

$$\lesssim \frac{1}{\alpha} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{s=3}^{C_{nsw}-1} \mathbb{E}_{F,i(Q)} \int |b_Q^n(y)| \frac{1}{2^d} \inf_{s \in Q} \int |x-z| \leq 2^i w(x) \, dx \, dy$$

$$\lesssim \frac{1}{\alpha} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{s=3}^{C_{nsw}-1} \mathbb{E}_{F,i(Q)} \|b_Q^n\|_{L^1} \inf_M Mw$$

$$\lesssim \frac{s_w}{\alpha} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{s=3}^{C_{nsw}-1} \mathbb{E}_{F,i(Q)} \|b_Q^n\|_{L^1} \inf_M Mw$$

$$\lesssim \frac{s_w}{\alpha} \sum_{Q \in F} \int |f_Q(x)| \log \left( e^2 + \frac{|f_Q(x)|}{\alpha} \right) \, dx \inf_M Mw$$

$$\lesssim \frac{1}{\alpha} \|w\|_A \|w\|_{A_{\infty}} \log([w]_{A_{\infty}} + 1) \int |f(x)| \log \left( e^2 + \frac{|f(x)|}{\alpha} \right) \, w(x) \, dx.$$

Let $c > 0$ be a constant such that $c\delta(1-\theta_w) \sum_{s=1}^{\infty} 2^{-\delta s(1-\theta_w)/3} = 1$ and

$$\lambda_s = \frac{c\delta(1-\theta_w)}{10} 2^{-\delta s(1-\theta_w)(1-\theta_w)/3}.$$

Applying Lemma 4.5 to the weight $w$ with $\lambda = \lambda_s$, we get

$$w\{x \in \mathbb{R}^d : \sup_k \left| \sum_{i>k} \sum_{n=1}^{\infty} \sum_{s=3}^{C_{nsw}-1} (K_n^i - K_0^i) * B_{i-s}^n(x) \right| > \frac{\alpha}{10}\}$$

$$\leq w\{x \in \mathbb{R}^d : \sum_{s=3}^{C_{nsw}-1} \sup_k \left| \sum_{i>k} (K_n^i - K_0^i) * B_{i-s}^n(x) \right| > \frac{\alpha}{10} \sum_{s=1}^{\infty} \left( \frac{c\delta(1-\theta_w)}{10} 2^{-\delta s(1-\theta_w)(1-\theta_w)/3} \right) s_w \}$$
\[
\sum_{s = C\lambda w}^{\infty} w(L^s_{\lambda w}) \\
\leq \sum_{s = C\lambda w}^{\infty} \frac{1}{\alpha} 2^{s}(s-C\lambda w)(1-\theta_{w})/3 2^{-s\delta(1-\theta_{w})} \|f\|_{L^1(M_{\lambda w})} \\
\leq \frac{2^{-s_w C\delta(1-\theta_{w})}}{\alpha(1-\theta_{w})^2} \|f\|_{L^1(M_{\lambda w})} \sum_{s = C\lambda w}^{\infty} 2^{-s(1-\theta_{w})}/3 \\
\leq \frac{2^{-s_w C\delta(1-\theta_{w})}}{\alpha(1-\theta_{w})^3} \|f\|_{L^1(M_{\lambda w})} \\
\leq \frac{1}{\alpha} \|f\|_{L^1(w)}.
\]

Hence the proof of Theorem 1.5 concludes.

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