Comprehensive proofs of localization in Anderson models with interaction. I. 
Two-particle localization estimates

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Abstract

We discuss the techniques and results of the multi-particle Anderson localization theory for disordered quantum systems with nontrivial interaction. After a detailed presentation of the approach developed earlier by Aizenman and Warzel, we extend their results to the models with exponentially decaying, infinite-range interaction.

1 Introduction. The motivation and the model

This manuscript is designed as the first issue in a mini-series aiming to present a survey of results and techniques of the rigorous localization theory of disordered quantum systems with nontrivial interaction. The first results in this direction, establishing the stability of Anderson localization in a two-particle system in $\mathbb{Z}^d$ with respect to a short-range interaction [16], have been immediately followed by the proofs of exponential spectral localization (cf. [6, 17]) and exponential strong dynamical localization (cf. [6]) in $N$-particle systems, for any fixed $N \geq 2$.

As it often happens, the first proofs are not necessarily the shortest and simplest ones. In particular, the method of [16, 17], a multi-particle adaptation of the variable-energy Multi-Scale Analysis (VEMSA) was later replaced by a significantly simpler one – multi-particle fixed-energy MSA (MP FEMSA). Such a simplification had a price: one had to design a spectral reduction (MP)FEMSA $\Rightarrow$ (MP)VEMSA; such a reduction was known for the systems without interaction, and its extension to interactive systems required a special form of the eigenvalue concentration (EVC) estimates, technically more involved than the conventional Wegner-type bounds. Another important ingredient of the spectral reduction came from the toolbox developed by Germinet and Klein [27], originally for the single-particle Anderson models.

The MPMSA, in its variable-energy version, was also extended to the continuous Anderson models [15, 37].

On the other hand, there has been a considerable time interval between the pioneering work by Aizenman and Warzel [6] on the MPFMM and the next bold step in that direction, made recently by Fauser and Warzel [26] who treated interactive Anderson models in $\mathbb{R}^d$ with infinite-range interaction. Quite naturally, the spectral analysis of unbounded random Schrodinger operators required various techniques which would be considerably simpler in the case of lattice systems, where "hard" functional analysis is often reduced to elementary linear algebra.

Taking into account the complexity of the original works, it seems reasonable to present their key techniques and ideas in a "nutshell", and in the simplest possible situation. This is how emerged the idea of the above mentioned the mini-series.

We start the first issue with a thorough presentation of the MPFMM, essentially for the reason that certain steps in popularization of the MPMSA, in its variable- and fixed-energy variants, have been already made in a recent monograph [18]. This manuscript, however, is not limited
to a mere review, for we show that the new ideas, developed by Aizenman and Warzel [6], can be easily adapted to the models with exponentially decaying interaction of infinite range. The lattice models with infinite-range interaction were studied in our paper [13] and, more recently, in our joint work with Yuri Suhov [19]. While a detailed presentation of the MPMSA is scheduled for subsequent issues, here we discuss some similarities and particularities of the two approaches, MPFMM and MPMSA.

In the multi-particle models with a finite-range interaction, the MPFMM, when applicable, provides the strongest decay bounds upon the eigenfunction correlators (EFC), as does its original, single-particle variant. In particular, such bounds are stronger than those proved with the help of the (single- or multi-particle) MSA, provided both methods apply to the same model. However, the relations between the two approaches are more complex (at least, for now) in the realm of multi-particle, interactive models than for the systems with no interaction.

Traditionally, the FMM is an unchallenged champion when it comes to the trees and other graphs with exponential growth of balls, while the MSA demonstrates its unparalleled flexibility in the situations where the "local" probability distribution of the random potential is not Hölder-continuous. As is well-known, for the lattice systems, it suffices to require the local distribution to be at least log-Hölder continuous, and for the systems in $\mathbb{R}^d$, with an alloy-type potential, the MSA establishes localization for any nontrivial probability distribution of the scatterers' amplitudes (cf. [10, 28]).

In the world of the interactive Anderson models, an additional parameter – the decay rate of the interaction – appears and, for the moment, sets apart the results that can be (or rather, have been) proved by the MPFMM and the MPMSA; we discuss this issue below.

We focus on the discrete systems for the obvious reason that this requires a minimum of analytical tools, making the proofs more comprehensive.

The choice of the two-particle systems for this, first issue in the planned mini-series, is motivated by several reasons.

- A number of geometrical arguments become most simple in the configuration space of $N = 2$ particles, so the 2-particle configuration space is $(\mathbb{Z}^d)^2 \cong \mathbb{Z}^{2d}$; in the illustrations provided in the present paper, we refer to the case where $d = 1$.

- It so happens that the existing decay bounds on the eigenfunctions (EFs) and on the EF correlators (EFCs) have been first proven with respect not to a norm-distance in $(\mathbb{Z}^d)^N$ (or, respectively, $(\mathbb{R}^d)^N$), but to a pseudo-distance, used explicitly in [6] and implicitly in [17]. Specifically, given two configurations of particles, $x$ with the particle positions $x_1, \ldots, x_N$ and $y$ with particles at $y_1, \ldots, y_N$, this is the Hausdorff distance $d_H(x, y)$ between the sets of the respective particle positions. While this is in general a complicated quantity, and a source of various unpleasant technical problems in the $N$-particle localization analysis, in the particular case $N = 2$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in (\mathbb{Z}^d)^2$, one has the identity $d_H(x, y) = d_S(x, y)$ where $d_S(\cdot, \cdot)$ is the symmetrized max-distance in $(\mathbb{Z}^d)^2$:

$$d_S(x, y) = \min \left[ |x - y|_\infty, |\pi(x) - y|_\infty \right]$$

with $|z|_\infty := \max \{|z_i|, 1 \leq i \leq d\}$ and (the only nontrivial) permutation $\pi \in S_2$ exchanges the particle positions: $\pi(x_1, x_2) = (x_2, x_1)$.

More to the point, the symmetrized max-distance is the natural max-distance in the configuration space of a system of two indistinguishable particles in $\mathbb{Z}^1$. The latter can be implemented$^2$

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$^1$We use this informal term, since, analytically, the description of disorder is not exactly the same for the discrete and continuous Anderson models. In the case of the lattice systems with, say, an IID potential, the suitable term here would be "single-site marginal" distribution.

$^2$For $d \geq 1$, the construction is slightly more complicated.
as the "half-space" \( \{ \mathbf{x} \in \mathbb{Z}^2 : (x_1 < x_2) \} \) (Fermi-particles) or, respectively, \( \{ \mathbf{x} \in \mathbb{Z}^2 : (x_1 \leq x_2) \} \) (Bose-particles).

• The most significant reason for choosing \( N = 2 \) comes from the regrettable fact that, although the above mentioned difficulty, appearing for \( N \geq 3 \) particles, had been partially overcome in [11, 12, 14], the solution proposed there applies so far to a limited class of random potentials. On the bright side, this class contains in particular the two most popular models of disorder used in physics (Gaussian distribution and uniform distribution in a finite interval), yet one is still far from the wealth of rigorous mathematical results of the 1-particle localization theory, where in the lattice systems of dimension \( d > 1 \) it suffices to require the probability distribution function (PDF) of the random potential to be log-Hölder continuous.

Summarizing, the 2-particle systems have been selected for this first issue in order to present the original methods from [6] and [16] in their best possible light, and with a minimum of technicalities that can easily start obscuring the key ideas in more general models.

► The mathematical theory of Anderson models with interaction is actually full of surprises. One of them is that, in contrast to the conventional, 1-particle theory, where since 1993 one has had two alternative, and mutually complementing, methods – one based on a multi-scale geometrical induction (Multi-Scale Analysis = MSA, going back to the pioneering works [23, 24]), and the other using, in a manner of speaking, a “mono-scale” strategy (the Aizenman–Molchanov =AM method, further developed in a series of subsequent works bearing a distinctive mark of Michael Aizenman’s enthusiasm, cf., e.g., [3–5]), these two approaches in the multi-particle localization theory finally settle on the common ground of the multi-scale induction, although they keep their distinctive features: the multi-particle MSA (MPMSA) makes use of bounds in probability, while its counterpart (MPFMM) developed by Aizenman and Warzel continues to successfully employ bounds in expectation.

► Another package of surprises (at least, that’s the way it looks so far) awaits one when it comes to the analysis of localization in presence of interactions of infinite range, decaying slower than exponentially. First, the MPFMM, applied recently by Fauser and Warzel [26] to the interacting systems in \( \mathbb{R}^d \), no longer provides exponential bounds on the EFCs and, as a result, on the decay of the eigenfunctions. Secondly, the MPMSA is no longer at a disadvantage (compared to the MPFMM) when it comes to the decay rate of the EFCs, i.e., the rate of the Strong Dynamical Localization (=SDL), and even provides the strongest possible result – an exponential decay – relative to the eigenfunctions.

► Finally, recall that we have already mentioned yet another surprise of the multi-particle Anderson theory: the physically most sound decay bounds (on EFs and on EFCs) for \( N \geq 3 \) particles, viz. the bounds in a (symmetrized) norm-distance and not the Hausdorff distance, have been established so far only with the help of the MPMSA.

It is to be stressed that the present manuscript is most certainly not intended as a replacement for (but only a complement to) the exposition in [6]. For instance, Aizenman and Warzel address in [6] the problem of perturbative parametric stability of the Anderson localization under a sufficiently weak interaction, in the situation where the 1-particle system exhibits strong localization in terms of the fractional moments. The most notable example of such a situation is localization of weakly (and locally) interacting one-dimensional quantum particles in a random environment.
1.1 The multi-particle Hamiltonian

For the reasons explained above, we consider the random Hamiltonian $H(\omega)$ acting as a bounded self-adjoint operator in $\ell^2((\mathbb{Z}^d)^2)$, of the form

$$H(\omega) = \sum_{j=1}^{2} \left( H_0^{(j)} + gV(x_j; \omega) \right) + U(x)$$

where $V : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}$ is a random field on $\mathbb{Z}^d$ (the configuration space of single particles), relative to a probability space $(\Omega, \mathcal{F}, P)$, $U$ is the operator of multiplication by the interaction potential $(x_1, x_2) \mapsto U(|x_1 - x_2|)$, and $H_0^{(j)}$, $j = 1, 2$, are replicas of the standard, nearest-neighbor lattice Laplacian on $\mathbb{Z}^d$, acting on the respective variables (particle positions) $x_j$.

1.2 Assumptions

In Theorems 1, 2 and 3 we consider the interaction potentials satisfying one the following conditions.

(U1) $\text{supp } U \subset [0, r_0]$, $r_0 < +\infty$.

(U2) $U(r) \leq e^{-m'r}$, $m' > 0$.

(U3) $U(r) \leq e^{-m'r\zeta}$, $m', \zeta > 0$.

Our main assumption on the external (random) potential is as follows.

(V1) The random field $V : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}$ is IID, a.s. bounded, with

$$P\{V(x; \omega) \in [0, 1]\} = 1,$$

and admits a bounded (common) marginal probability density $p_V$, with $\|p_V\|_\infty = p_V < +\infty$.

The non-negativity of $V$ is, of course, inessential, for the transformation $V \mapsto V + E$ results only in a spectral shift for $H$, leaving its eigenfunctions (of whatever nature) invariant. Making larger the amplitude of $V$ amounts to taking larger $|g|$.

The situation with the hypotheses required for the proof of Theorem 3 is a bit more complicated.

Its two-particle version, stated in this paper, does not actually require the existence of a bounded density; neither does the technique from [6]. In both cases, one can relax the condition of Lipschitz continuity of the marginal distribution to that of Hölder continuity.

On the other hand, in the general case where $N > 2$, the conditions depend at the moment upon the desired result: decay estimates in the Hausdorff distance still can be proved for $V$ with bounded density (or even Hölder-continuous PDF), but a more suitable decay in a norm-distance in the $N$-particle configuration space $(\mathbb{Z}^d)^N$ requires the following, stronger condition upon the random field $V$. As was already said, norm-distance bounds have not yet been proved with the help of the MPFMM.

Ref. [19], where localization bounds in a norm-distance were proved, relies on the following assumption (required only for $N > 2$):

(V2) The random field $V : \mathbb{Z} \times \Omega \rightarrow \mathbb{R}$ is IID, a.s. bounded, with

$$P\{V(x; \omega) \in [0, 1]\} = 1,$$

and admits a bounded marginal probability density $p_V$, with

$$p_*(1_{(0,1)}) \leq p_V(1_{(0,1)}) \leq p(1_{(0,1)}),$$
0 < p_1 < p < +\infty$, and bounded derivative $p'_{\nu}$ on $(0, 1)$.

The last condition certainly looks strange to a reader familiar with the eigenvalue concentration bounds (starting with the celebrated Wegner bound) for random Anderson Hamiltonians. We will explain the reasons for this condition in a forthcoming manuscript; here we simply mention that such a hypothesis appears in Ref. [14] used in [19].

In the case $N = 2$, the restrictive hypothesis (V2) can be substantially relaxed; the main assumption in Theorem 3 is as follows:

(V3) The random field $V : \mathbb{Z} \times \Omega \rightarrow \mathbb{R}$ is IID, with uniformly Hölder continuous marginal probability distribution function (PDF) $F_V(t) := \mathbb{P}\{V(x; \omega) \leq t\}$: for all $s \in [0, 1]$ and some $b \in (0, 1]$, $C < \infty$

$$\sup_{t \in \mathbb{R}} (F_V(t + s) - F_V(t)) \leq Cs^b.$$ 

The disorder amplitude $g > 0$, which we assume in this paper to be large enough, can be introduced, e.g., by putting a small factor $g^{-1}$ in front of the kinetic operator or a large factor $g$ in $V$, or else in a more subtle way, by assuming the marginal probability density of the IID random field $V$ to be small, viz. of order of $O(g^{-1})$. For definiteness, and also in order to follow more closely [2] and [6], we consider the potential of the form $gV(x; \omega)$ with $g \gg 1$. On one occasion (see Sect. 6), it will be convenient to change the attitude and work with operators $V + g^{-1}A$, where $A$ is the extended kinetic operator $H_0 + U$.

As was pointed out in Ref. [6], the assumption of existence and boundedness of the single-site marginal density of the random potential can be relaxed to Hölder continuity. However, it seems that this would require a modification of the proof of the uniform a priori bound on the fractional moments in Sect. 6. Specifically, the standard argument employed in the proof of the so-called weak $L^1$-bound, based on the linear transformation of the two-dimensional space supporting the reduced probability measure, requires a slightly more elaborate approach, developed in Ref. [2].

### 2 Basic notation and preliminary remarks

In the context of multi-particle Hamiltonians, we usually employ boldface notation for the objects referring to a multi-particle system, to visually distinguish them from their single-particle counterparts.

Recall the definition of the Hausdorff distance.

Given two subsets $X, Y$ of an abstract metric space $(\mathcal{M}, d)$, the Hausdorff distance between these subsets is given by

$$d_H(X, Y) := \max \left[ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right].$$

Associating with a configuration of $N$ distinguishable particles in $\mathbb{Z}^d$, $z = (z_1, \ldots, z_N)$, its "projection" to the 1-particle configuration space $\mathbb{Z}^d$, $\Pi z = \{z_1\} \cup \cdots \{z_N\}$, we extend $d_H$, defined for the subsets of $\mathbb{Z}^d$, to the particle configurations: $d_H(x, y) := d_H(\Pi x, \Pi y)$.

The following elementary statement explains what makes the 2-particle systems so special in the framework of our analysis.

**Lemma 2.1** (Hausdorff distance equals max-distance in $(\mathbb{Z}^d)^2$).

$$\forall x, y \in \mathbb{Z}^2 \quad d_H(x, y) = d_2(x, y). \quad (2.1)$$
Proof. Let \( R := d_S(x, y) = \min_{x \in S_2} |\pi(x) - y| \). Then for at least one choice of the vectors \( a = (a, a') \in [(x_1, x_2), (x_2, x_1)] \) and \( b = (b, b') \in [(y_1, y_2), (y_2, y_1)] \), with \( \{a, a'\} = \Pi x, \{b, b'\} = \Pi y \), one has

\[
\begin{align*}
|a - b| &= R \\
|a' - b'| &\leq R \\
\max |a - b', |a' - b| &\geq R, \\
i.e., |a - b| = R, \quad i.e., |\pi(a) - b| \geq R \quad \text{for Id} \neq \pi \in S_2.
\end{align*}
\]

If \( |a - b'| \geq R \), then \( d(a, \Pi b) = R \), while \( d(a', \Pi b) \leq d(a', b') \leq R \). In this case, we conclude that \( d_M(x, y) \equiv d_S(a, b) = R = d_S(x, y) \).

In the remaining case where \( |a' - b| \geq R \), the argument is similar and the final conclusion is the same. \( \square \)

From this point on, we use only \( d_S \), even in the arguments where \( d_M \) would be required in the case \( N \geq 3 \).

For a proper subset \( \emptyset \neq \Lambda \subset \mathbb{Z}^d \), we denote \( \partial^- \Lambda = \{y \in \Lambda : \text{dist}(x, \Lambda^c) = 1\}, \partial^+ \Lambda = \partial^- \Lambda^c \), with \( \Lambda^c := \mathbb{Z}^d \setminus \Lambda \).

We will systematically make use of the elementary inequality which is one of the cornerstones of the FMM technique: \( \forall s \in (0, 1) \mid \sum_n |a_n|^s \leq \sum_n |a_n|^\frac{s}{2} \).

Given finite lattice subsets \( \mathbb{A}_1 \subset (\mathbb{Z}^d)^2 \), with the edge boundary of \( \mathbb{A}_1 \) relative to \( \Lambda \),

\[ \partial \mathbb{A}_1 = \partial^{(\Lambda)} \mathbb{A}_1 := \{ (x, y) \in \mathbb{A}_1 \times \mathbb{A} : |x - y| = 1 \} \]

one can easily infer from the second resolvent identity the so-called Geometric Resolvent Equation (GRE) and the Geometric Resolvent Inequality (GRI) (cf., e.g., [31]). The latter will be used in its FMM-flavoured form (which we will call the FGRI): for any \( s \in (0, 1), x \in \mathbb{A}_1, y \in \Lambda \setminus \mathbb{A}_1 \), and for any \( E \) such that \( (H_\Lambda - E) \) and \( (H_{\Lambda_1} - E) \) are invertible,

\[
|G_\mathbb{A}_1(x, y; E)|^s \leq \sum_{(w, w') \in \partial \mathbb{A}_1} |G_{\mathbb{A}_1}(x, w; E)|^s |G_\mathbb{A}(w', y; E)|^s. \tag{2.2}
\]

A detailed discussion of various boundary conditions for the LSO (and of the related forms of the GRI) can be found in the review by Kirsch [31].

For brevity and following [31], we introduce the energy-disorder expectation \( \tilde{E}^I[\cdot] \): given a measurable function \( f : \mathbb{R} \times \Omega \) and an interval \( I \subset \mathbb{R} \), we set

\[
\tilde{E}^I[f(E, \omega)] := |I|^{-1} \int_I E[f(E, \omega)] \ dE,
\]

where \( E[\cdot] \) is the conventional expectation relative to \( (\Omega, P) \). For brevity, we keep the superscript "\( I \)" only where necessary or instructive and often write \( \tilde{E}[\cdot] \) instead of \( \tilde{E}^I[\cdot] \). Since the measure \( |I|^{-1} \ dE \) on \( I \subset \mathbb{R} \) is normalized, the inequality (2.2) implies a similar bound for the expectation \( \tilde{E}[\cdot] \) relative to the augmented probability space \( I \times \Omega \).

### 2.1 Structure of the paper

The central result presented in this paper is Theorem 1, proved by Aizenman and Warzel [6], while a more general Theorem 2 follows by a relatively simple adaptation of just one important ingredient of the proof (cf. Lemma 8.1 in Sect. 8).

We keep the main flow of argument in the proof of Theorem 1 as “linear” as possible, yet two exceptions seem appropriate.

- The first one concerns a key component of the FMM – an a priori bound by \( O(1) \) (indeed, by \( O(|y|^{-s}) \)) on the fractional moments (cf. Sect. 6).
The second one (cf. Sect. 7) is a fairly simple proof of decay bounds away from the support of the interaction $U$; see Fig. 1. The proofs of these two results can be read independently of the main body of the proof; the latter is essentially a multi-scale induction.

Appendix C could have been replaced with a mere reference to the work by Boole [7], but we break here with this common practice for several reasons. Firstly, Ref. [7] is not quite easy to find in libraries, although, with a bit of effort, it can be found in a downloadable form on Internet. Secondly, it would require at least an adaptation, both stylistic (recall that [7] was written in 1857) and notational. Thirdly, it seems that the shortest proof was given by Loomis [32], whose argument we reproduce almost verbatim in Appendix C.

## 3 Main results on the decay of the GFs

**Theorem 1** (Following Aizenman and Warzel [6]). Assume that the interaction potential $U$ is compactly supported (cf. Assumption (U1)) and the external random potential satisfies Assumption (V1). There exists $g_0 \in (0, +\infty)$ with the following properties.

For any finite connected subgraph $\Lambda \subset \mathbb{Z}^d$ and for all $g$ with $|g| \geq g_0$, the two-particle Hamiltonian $H_{\Lambda}^{(2)}(\omega)$ exhibits exponential decay of the fractional moments of the Green functions. Specifically, for some finite interval $I \subset \mathbb{R}$ containing the a.s. spectrum of $H_{\Lambda}^{(2)}(\omega)$, one has, with $m = m(g) \to +\infty$ as $g \to +\infty$,

$$\mathbb{E}^I \left| G_{\Lambda}^{(2)}(x, y; E) \right|^s \, dE \equiv \int_I \mathbb{E} \left[ \left| G_{\Lambda}^{(2)}(x, y; E) \right|^s \right] dE \leq e^{-md_{S}(x, y)}.$$  \hfill (3.1)

The same upper bound holds true for the two-particle Hamiltonian in the entire lattice $\mathbb{Z}^d$, i.e., for $H_{\{\mathbb{Z}^d\}^{(2)}(\omega)}$.

The result of Theorem 1 implies in fact strong dynamical localization (hence, spectral localization with probability one). Such an implication is neither new (this was done in [6]) nor difficult to prove, with the help of technical tools developed by Elgart et al. [22] (see also [18]) and by Germinet and Klein [27]. We plan to address it in a forthcoming issue. Recall that Ref. [6] provides a complete proof of exponential strong dynamical and spectral localization in the situation more general than the one covered by Theorem 1.

Using the augmented, energy-disorder measure space $I \times \Omega$ is mainly motivated by Lemma B.1 (cf. [6, Theorem 4.2]), allowing one to transform the decay properties of the EFCs into those of the [integrated] Green functions. On the other hand, it seems appropriate to stress here that the energy-disorder space $I \times \Omega$ was used already at an early stage of the development of the MSA by Martinelli and Scoppola [33] who proved that fast decay of the Green functions implied absence of a.c. spectrum; Refs. [22] and [18] improve this result.

We will focus on the finite-domain Hamiltonians $H_{\Lambda}^{(2)}(\omega)$, since the extension to an infinitely extended domain is obtained by a simple application of the Fatou lemma on converging measures. This argument can be found in Refs. [4, 5]; it is not specific to the localization analysis carried out with the help of the FMM.

Once the proof of Theorem 1 (for finite $\Lambda \subset \mathbb{Z}^d$) is completed, we will show that the main approach extends with no difficulty to infinite-range, exponentially decaying interactions, thus generalizing the original results by Aizenman and Warzel [6].

**Theorem 2.** Assume that the interaction potential $U$ in the two-particle Hamiltonian $H(\omega)$ decays exponentially fast at infinity (cf. Assumption (U2)) and the external random potential satisfies Assumption (V1). For $g_0$ large enough and for all $g$ with $|g| \geq g_0$, $H(\omega)$ exhibits
exponential decay of the fractional moments of the Green functions:

\[
\int_I \mathbb{E}[|G(x,y;E)|^s] \, dE \leq e^{-m d_S(x,y)} \tag{3.2}
\]

for some finite interval \(I \subset \mathbb{R}\) containing the a.s. spectrum of \(H(\omega)\).

Here a similar remark can be made, concerning the derivation of spectral and dynamical localization from the decay of the Green functions. Our proof is closer to the original technique from [6] than to a more advanced method developed by Fauser and Warzel [26] for differential operators.

Note also that Theorem 2 naturally and easily extends to more general locally finite graphs \(Z\), replacing \(\mathbb{Z}\), satisfying a condition of polynomially bounded growth of balls. We plan to discuss such an extension in a forthcoming issue.

On the other hand, the approach of Ref. [26] gives rise only to a sub-exponential decay of eigenfunction correlators, when \(U\) decays slower than exponentially, and this results to a sub-exponential decay bound on the eigenfunctions. As was said in the Introduction, here the MPFMM and the MPMSA are on essentially equal footage, and the latter even provides stronger (exponential) decay bounds on the eigenfunctions (albeit not on the EFCs). For this reason, we postpone to a forthcoming issue a detailed presentation of the MPFMM techniques in the case of sub-exponentially decaying interactions.

### 3.1 Statement of results on exponential decay of eigenfunctions

Here we briefly describe the kind of decay estimates that can be proved for the 2-particle systems with sub-exponentially decaying interaction, using the multi-particle multi-scale analysis.

**Theorem 3** (Cf. [19]). Assume that the interaction potential \(U\) in the two-particle Hamiltonian \(H(\omega)\) satisfies Assumption (U3), and the external potential satisfies (V3). Then for \(g_0 > 0\) large enough and for \(|g| \geq g_0\), with probability one, the two-particle Hamiltonian has pure point spectrum, and all its eigenfunctions decay exponentially fast at infinity. The eigenfunction correlators decay sub-exponentially fast at infinity.

As was already said, in the more general case where \(N \geq 3\), Assumption (V3) is to be replaced by a more restrictive one, (V2); otherwise, the existing techniques give rise only to Hausdorff-distance (and not norm-distance) decay of the EFCs.

### 4 Proof of Theorem 1

We follow closely the arguments from [6] (without repeating it every time again), taking shortcuts thanks to the simplifying assumption \(N = 2\), and adapting notation and calculations on an as-needed basis.

We will use the sequence of length scales \(\{L_k, k \geq 0\}\) defined by the recursion

\[
L_{k+1} := 2(L_k + 1), \quad k = 0, 1, \ldots, \tag{4.1}
\]

or, explicitly,

\[
L_k = 2^k(L_0 + 2) - 2. \tag{4.2}
\]

It suffices to treat the case where \(x, y\) with \(d_S(x, y) =: R > L_0\), for otherwise the decay bound can be absorbed in the constant factor. Given \(x, y\) with \(d_S(x, y) =: R > L_0\), there is a unique integer \(k \geq 0\) such that

\[
L_k < R = d_S(x, y) \leq L_{k+1}. \tag{4.3}
\]
This value \( k \) will be fixed until the end of the proof. Since \( k \) can be arbitrarily large, the proof will require a scale induction.

Further, assume w.l.o.g. that \( \text{diam} \Pi x \geq \text{diam} \Pi y \).

Next, by definition of the max-distance, there are the points \( x \in \Pi x, \ y \in \Pi y \), such that
\[
d_S(x, y) = R = |x - y|.
\] (4.4)

By the stochastic translation invariance of the random field \( V \), and the resulting diagonal shift-invariance of the EF correlators, we can assume w.l.o.g. that \( x = 0 \); this reduction is made mainly for making simpler the representation on Fig. 1, where it is instructive to indicate the positions of both coordinate axes (the replicas of the physical, 1-particle configuration space, assumed to be \( \mathbb{Z}^1 \)). With this reduction, for the rest of the proof, we have that

(i) \( \Pi x \ni 0 \);

(ii) there is a point \( y \in \Pi y \) such that
\[
|y| \equiv |y - 0| = d_S(x, y) = R > L_k.
\] (4.5)

We fix an arbitrarily large but finite connected subset \( \Lambda \subset \mathbb{Z}^d \); the connectedness is understood in the sense of graphs: \( \mathbb{Z}^d \) is endowed with the graph structure where a pair \((x, y)\) is an edge iff \( |x - y| := \sum_{i}|x_i - y_i| = 1 \), and \( \Lambda \) is considered as a subgraph of \( \mathbb{Z}^d \). For a given finite \( \Lambda \), the scale induction described below is to be stopped, once the scale \( L_k \) achieved at the \( k \)-th induction step is bigger or equal to \( \text{diam} \Lambda \).

The finiteness of \( \Lambda \) allows us to work with well-defined, finite-dimensional self-adjoint operators and their resolvents; the corresponding spectra are finite subsets of \( \mathbb{R} \).

Figure 1. Example for the proof of Theorem 1, with \( d = 1, N = 2 \).
The fractional moments $\mathbb{E}[|\cdot|^s]$ figuring in our formulae are computed for $s \in (0,1)$ small enough to guarantee the finiteness of the expectation; this is particularly important when the application of the Cauchy–Schwarz inequality gives rise to the exponent $2s$. The final bound on the Green functions can be extended to larger values of $s$ with the help of the "one-for-all" principle (cf. Lemma D.1). In any case, the exponentially small bounds on the fractional moments with any given $s > 0$ imply exponential strong dynamical (and spectral) localization.

**Step 1. Distant pairs of split configurations: no scale induction.** If at least one of the configurations $x$, $y$ has a large diameter, the required decay bound follows – without any scale induction – from Lemma 7.1, which takes now a simpler form, with $\text{diam } \Sigma x \geq \text{diam } \Sigma y$: for some $A, m \in (0, +\infty)$

$$
\mathbb{E}[|G(x,y)|^s] \leq A \exp \left( -m \min[|d_S(x,y)|, \text{diam } \Sigma x] \right).
$$

(4.6)

Specifically, if $\text{diam } \Sigma x > L_k/2$, (4.6) and (4.3) imply that

$$
\mathbb{E}[|G(x,y)|^s] \leq A e^{-\frac{m}{L_k}}.
$$

(4.7)

Hence we can focus in the rest of the proof on the pairs of configurations of diameter $\leq L_k/2$.

**Step 2. Decoupling inequality.** We start the analysis of the case where $\text{diam } \Sigma x, \text{diam } \Sigma y \leq L_k/2$, by establishing a decoupling inequality, essentially of the same nature as in the conventional, single-particle AM/FMM approach (cf. [2, 4]). However, the decoupling achieved here is not as "total" as usual (i.e., for $N = 1$), so one losess the valuable mono-scale structure of the AM method and has to resort to a multi-scale procedure (see Step 4).

It is convenient to introduce the point $0 := (0,0) \in (\mathbb{Z}^d)^2$. Since $\Sigma x \ni 0$ and $\text{diam } \Sigma x \leq L_k/2$, we have $x \in B_{L_k/2}(0)$; the latter cube is depicted as the dashed square on Fig. 1.

At least one coordinate of $y$ equals $y$; in the case $d = 1$, $N = 2$, possible positions of $y$ are indicated on Fig. 1 by black dots, outside the strip where $\text{diam } (\cdot) \leq L_k/2$, and by gray dots, inside the strip. In our argument, we are only concerned with the latter case (gray dots on Fig. 1). Since we only know that $|y - 0| > L_k$, the distance from $y$ to $B_{L_k}(0)$ is not necessarily large (it can be $= 1$). Still, $y$ is separated from $x \in B_{L_k/2}(0)$ by the belt of width $\geq L_k/2$, and it is this belt that will provide the desired decay bound for the EFC.

By the GRE applied to the cube $B_{L_k}(0)$, with $x$ inside and $y$ outside it,

$$
G(x,y) = \sum_{(w,w') \in \partial B_{L_k}(0)} G_{B_{L_k}}(0,x,w) \Delta(w,w') G(w',y)
$$

= \sum_{(w,w') \in \partial B_{L_k}(0)} G_{B_{L_k}}(0,x,w) G(w',y),
$$

(4.8)

yielding

$$
\mathbb{E}[|G(x,y)|^s] \leq |\partial B_{L_k}(0)| \max_{(w,w') \in \partial B_{L_k}(0)} \mathbb{E}[|G_{B_{L_k}}(0,x,w)|^s |G(w',y)|^s].
$$

Here $\Sigma y' = \{w_i, w_2'\}$ with $|0 - w_i'| = L_k + 1$ for at least one value $i \in \{1,2\}$; we fix such $i$ and set $u_1 = w_i', u_2 = y$. Then $u_1, u_2 \not\in B_{L_k}(0)$.

Fix $(w,w') \in \partial B_{L_k}(0)$. Introduce the sigma-algebra $\mathcal{F}_{\Lambda \setminus \{u_1, u_2\}}$ generated by $\{V(z); z \in \Lambda \setminus \{u_1, u_2\}\}$ and note that $G_{B_{L_k}}(0,x,w)$ is $\mathcal{F}_{\Lambda \setminus \{u_1, u_2\}}$-measurable, since $u_1, u_2 \not\in B_{L_k}(0)$, hence

$$
\mathbb{E}[|G(x,y)|^s] \leq \mathbb{E}[|G_{B_{L_k/2}}(0,x,w)|^s \mathbb{E}[|G(w',y)|^s |\mathcal{F}_{\Lambda \setminus \{u_1, u_2\}}]].
$$

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The conditional expectation in the above RHS is uniformly bounded, by virtue of Lemma 6.1 (cf. Eqn. (6.1)):
\[
\mathbb{E} \left[ |G(w', y)|^s \mid \mathcal{A}_{\{u_1, u_2\}} \right] \leq C_s |g|^{-s},
\]
thus
\[
\mathbb{E} \left[ |G(x, y)|^s \right] \leq C_s |g|^{-s} \mathbb{E} \left[ |G_{B_{Lk/2}(0)}(x, w)|^s \right].
\]

Assessing the above expectation is the most difficult task, and this hard work will be entrusted to the scale induction (we carefully avoid using the words which would infringe the MSA’s historical trademark).

**Step 3. Scaling step. I. Split configurations w.** We have $\Pi x \ni 0$, $\text{diam} x \leq L_k/2$, so the configuration $x$ is "clustered" – in the terminology of Ref. [6].

Consider first the case where $w$ is “split”: $\Pi w > L_k/2$. Further, $|w-0|=L_k$, $0$ is invariant w.r.t. the symmetry $(z_1, z_2) \mapsto (z_2, z_1)$, so $d_s(w, 0) = L_k$, while $x \in B_{L_k/2}(0)$, thus we also have
\[
d_s(x, w) \geq L_k - \frac{1}{2} L_k = \frac{1}{2} L_k,
\]
and this situation is covered by Lemma 7.1: with $\min[d_s(x, w), \text{diam} x] > L_k/2$, we have
\[
\mathbb{E} \left[ |G_{B_{L_k}(0)}(x, w)|^s \right] \leq \text{Const} |g|^{-s} e^{-m \frac{d}{L_k}}.
\]

**Step 4. Scaling step. II. Configurations w of restricted diameter.** To single out the kind of EF correlators we will be working with, introduce the following notation:
\[
\Upsilon(L) := \sup_{|I| \geq 1} \sum_{x \in B_{L_k/2}(0)} \sum_{w \in \partial B_{L_k/2}(0)} \mathbb{E}^I \left[ |G_A(x, w)|^s \right],
\]
In the scaling procedure described below, it will be used with $L = L_{k+1}$. We also need a slightly modified\(^3\) quantity, defined for $L = L_{k+1}$, $k \geq 0$:
\[
\tilde{\Upsilon}(L_{k+1}) := \sup_{|I| \geq 1} \sum_{x \in B_{L_{k+1}/2}(0)} \sum_{w \in \partial B_{L_{k+1}/2}(0)} \mathbb{E}^I \left[ |G_A(x, w)|^s \right].
\]

$\Upsilon(L_k)$, $\tilde{\Upsilon}(L_{k+1})$ are required to carry out the scale induction, but $\tilde{\Upsilon}(L_{k+1})$ is simpler to assess while performing the induction step.

We are going to show first that the quantities $\Upsilon(L_k)$, $k \geq 0$, satisfy the recursion
\[
\Upsilon(L_{k+1}) \leq \frac{a}{|g|^s} \Upsilon^2(L_k) + AL_{k+1}^{2q} e^{-2\nu L_k},
\]
and then infer from (4.12) (cf. Lemma A.1) that $\Upsilon(L_k)$ decay exponentially.

(4.i) It is convenient to approximate $\Upsilon(L_{k+1})$ by $\tilde{\Upsilon}(L_{k+1})$. Let us show that, for some $C$,
\[
0 \leq \Upsilon(L_{k+1}) - \tilde{\Upsilon}(L_{k+1}) \leq C L_{k+1}^{2Nd} e^{-m \frac{d}{L_{k+1}}}.
\]
The LHS inequality is obvious, since all the terms from $\tilde{\Upsilon}(L_{k+1})$ are present in $\Upsilon(L_{k+1})$. Consider any term $\mathbb{E} \left[ |G_A(x, w)|^s \right]$ figuring in the sum for $\Upsilon(L_{k+1})$ but absent in $\tilde{\Upsilon}(L_{k+1})$ (cf. (4.10)–(4.11)). The exclusion from $\tilde{\Upsilon}(L_{k+1})$ implies the LHS inequality in
\[
\frac{1}{2} L_k < \text{diam} \Pi w \leq \frac{1}{2} L_{k+1},
\]

\(^3\)Observe that, in the definition of $\Upsilon(L)$ with $L = L_{k+1}$, the diameter $L_k/2$ figuring in (4.11) would have to be replaced by a larger one: $L_{k+1}/2$. 

while the RHS inequality is due to the constraint figuring in the definition of $\Upsilon(L_{k+1})$. Further, $w \in \partial^* B_{L_{k+1}}(0)$ implies that $\Pi w \ni w$ with $|w - 0| = L_{k+1}$. Since $\text{diam } x \leq L_{k+1}/2$, it follows that
\[
d_S(x, w) = \min_{\pi \in \mathcal{G}_N} \max_i |x_{\pi(i)} - w_i| \geq \min_j |x_j - w|
\]
\[
\geq |0 - w| - |x_j - 0| \geq L_{k+1} - \text{diam } x \geq L_{k+1} - \frac{1}{2} L_{k+1} > \frac{1}{2} L_k.
\]

Now the lower bound $\text{diam } w > L_k/2$ allows us to apply Lemma 7.1 on $R$-distant configurations at least one of which is $R$-split (here we have $R = L_k/2$):
\[
\hat{E} \left[ |G_A(x, w)^*| \right] \leq A e^{-m x - k},
\]
so it remains only to assess the number of relevant terms, i.e., pairs $(x, w)$ figuring in (4.10). We have $x \in B_{L_{k+1}}(0)$ and $w \in \partial^* B_{L_{k+1}}(0)$ with $\text{diam } w \leq L_k/2$, thus $x, w \in B_{\frac{3}{2} L_k}(0)$. We conclude that the number of terms which constitute the difference $\Upsilon(L_{k+1}) - \hat{\Upsilon}(L_{k+1})$ is bounded by $\hat{C} L_{\frac{k+1}{2}}^N$. This completes the proof of (4.13).

(4.ii) Now we rescale the correlators $\Upsilon$, using in the process the reduced correlators $\hat{\Upsilon}$. This will be done with the help of a two-fold application of the FGRI.

Given the configurations $x$ with diam $x \leq L_k/2$ and $w$ with diam $w \leq L_k/2$, we can assume w.l.o.g. that $\Pi x \ni x$ (otherwise we perform a diagonal shift $(a, b) \mapsto (a + c, b + c)$ moving one of the particles in $x$ to $0 \in \mathbb{Z}^d$. Let $\Pi w \ni w, \hat{w} := (w, w)$. By a two-fold application of the FGRI, setting for brevity $B' = B_{\frac{1}{2} L_k}(0), B'' = B_{L_k}(\hat{w})$ (observe that $L_k \approx L_{k+1}/2$),
\[
\hat{E} \left[ |G(x, w)^*| \right] \leq \sum_{(u, u') \in \partial B'} \hat{E} \left[ |G_B'(x, u)^*| |G_B(u', v')^*| |G_B''(v, w)^*| \right].
\]
\[\tag{4.14}\]

Now we assess the middle factor in the RHS expectation. Here $|u' - 0| = L_k + 1$, so $\Pi u' \ni u'$ with $|u' - 0| = L_k + 1$, and $u' \notin B_{L_k}(0)$. Similarly, $\Pi v' \ni v'$ with $v' \notin B_{L_k}(w)$. (Cf. Fig. 2.) Therefore both $G_B'(x, u)$ and $G_B''(v, w)$ are measurable with respect to the sigma-algebra $\mathcal{F}_{\partial u', v'}$ generated by all $V(z; \cdot)$ with $z \neq u', v'$. At the same time, by Lemma 6.1,
\[
\hat{E} \left[ |G(u', v')^*| \right] \leq C|g|^{-s} < +\infty,
\]
\[\tag{4.15}\]
thus
\[
\hat{E} \left[ |G(x, w)^*| \right] \leq \frac{C}{g^s} |\partial B'| |\partial B''| \sum_{(u, u') \in \partial B'} \hat{E} \left[ |G_B'(x, u)^*| \right] \hat{E} \left[ |G_B''(v, w)^*| \right].
\]

By the Hermitian symmetry of the Green functions, both of the fractional moments in the RHS can be assessed in the same way: by the appropriate diagonal shift, leaving the expectation invariant, one of the particles in $w$ can be moved to 0, making the expectations $\hat{E} \left[ |G_B(x, u)^*| \right]$ and $\hat{E} \left[ |G_B''(w, u)^*| \right]$ similar. Thus it suffices to assess one of them, e.g., $\hat{E} \left[ |G_B(x, u)^*| \right]$. This is done in two steps:

(a) For $u$ with $\text{diam } u \leq L_k/2$ (see the green dots on Fig.2) we use the scale induction:
\[
\sum_{(u, u') \in \partial A_{L_k}(0)} \hat{E} \left[ |G_B'(x, u)^*| \right] \leq \Upsilon(L_k).
\]
Figure 1: Example for the Step (4ii). The blue dots represent the configurations \( x \) and \( w \) figuring in the reduced correlators \( \bar{\Upsilon} \) (cf. (4.11)). The points \( u, u' \) and \( v, v' \) are used in the FGRI. The green dots \( u, u' \) are the "split" configurations at the boundary, and the gray ones are "clustered". Each of the configurations \( u' \) and \( v' \) contains at least one point (\( u' \) and \( v' \), respectively) outside the area \( B_{L_k}(0) \cup B_{L_k}(w) \subset \mathbb{Z}^d \). This allows us to eliminate the Green function \( G(u', v'; E) \) in the RHS of Eqn. (4.14) with the help of Lemma 6.1, thus decoupling the remaining factors. The dashed areas are the cubes of radius \( L_k/2 \) in \((\mathbb{Z}^d)^2\).
(b) If $\text{diam } u > L_k/2$ (see the gray dots on Fig.2), then we use again Lemma 7.1: with $N = 2$, \[
\hat{E}[|G_{\mathbf{W}}(x, u)|^s] \leq Ae^{-m\frac{L_k}{4N}} = Ae^{-\frac{L_k}{4}}.
\]

Collecting all possible vertices $u$, falling into one of the above two categories (a) and (b), and upper-bounding the number of vertices from each category by $|\partial^* A(x)|$, we conclude that \[
\sum_u \hat{E}[|G_{\mathbf{W}}(x, u)|^s] \leq C|g|^{-s} (\Upsilon(L_k) + C' L^3_k e^{-mL_k}), \quad q = 2Nd - 2 = 2.
\]

Similarly, \[
\sum_v \hat{E}[|G_{\mathbf{V}}(v, w)|^s] \leq C|g|^{-s} (\Upsilon(L_k) + C' L^3_k e^{-\frac{L_k}{4}}).
\]

Therefore, \[
\hat{\Upsilon}(L_{k+1}) \leq C|g|^{-s} (\Upsilon(L_k) + C' e^{-\frac{L_k}{4}})^2.
\]

Finally, on account of the approximation formula (4.13), we obtain \[
\Upsilon(L_{k+1}) \leq C|g|^{-s} (\Upsilon(L_k) + C' e^{-\frac{L_k}{4}})^2 + AL_{k+1}^p e^{-\frac{L_k}{4}}.
\]

and with $\hat{\Upsilon}_k := \Upsilon(L_k) + C' e^{-\frac{L_k}{4}}$, $A' = 4\max(A, C')$, $M_k := 2C|g|^{-s}$, we come to the recursive inequality \[
\hat{\Upsilon}_{k+1} \leq \frac{1}{2} M_k \hat{\Upsilon}_k^2 + \frac{1}{2} A' L_{k+1}^p e^{-\frac{L_k}{4}}. \quad (4.16)
\]

An elementary calculation\footnote{A reader familiar with the work [27] by Germinet and Klein can recognize some elements of the proof of [27, Theorem 5.1] (cf. in particular [27, Eqn. (5.30)] in the argument used in Appendix A.)} (cf. Lemma A.1) shows that the recursion (4.16) implies \[
\forall k \geq 0 \quad \Upsilon(L_k) \leq \text{Const}(|g|, s, m) e^{-\mu L_k},
\]

with $\mu > 0$. For brevity, we omit here the explicit expression for the lower bound on $\mu$; see the details in Appendix A.

**Step 5. Conclusion.**

Given two configurations $x, y \in \Lambda^2$ with $d_S(x, y) = D$, there exist $\mu > 0$ such that \[
\hat{E}[|G(x, y; E)|^s] \leq \begin{cases} 
\text{Const } e^{-\frac{D}{2}}, & \text{if diam } (x) \vee \text{diam } (y) > D/2;
\text{Const } e^{-\mu D}, & \text{if diam } (x), \text{diam } (y) \leq D/2.
\end{cases}
\]

Thus the assertion of Theorem 1 is proved. \qed

Recall that the proof makes use of two important estimates, Lemma 6.1 on the finiteness of fractional moments, and Lemma 7.1 addressing the particle transfer processes to/from split configurations. Their proofs do not use the scale induction, and this is one of the reasons we prove them separately.

## 5 Proof of Theorem 2

A direct inspection of the proof of Theorem 1 shows that the finiteness of the range of interaction is used only in the proof of Lemma 7.1, providing an exponential upper bound on the fractional moment of the GFs $G(x, y; E)$ where at least one of the configurations $x, y$ is "split!", and the two are "distant". Further, Lemma 7.1 is proved without scale induction, as a separate statement
logically independent of the rest of the proof of Theorem 1. Therefore, it suffices to prove an analog of Lemma 7.1 for the exponentially decaying interactions of infinite range, and such a statement (Lemma 8.1) is proved in Section 8.

Replacing the statement of Lemma 7.1 with that of Lemma 8.1 in the arguments given in Section 4, the assertion of Theorem 2 follows. □

6 Finiteness of the fractional moments

A finite a priori bound on the fractional moments of the resolvents is one of the inescapable ingredients of the FMM, going back to the pioneering paper [2]. Its adaptation to the multiparticle models (cf. [6]) is, however, more involved than in the 1-particle theory. Both Ref. [6] and a more recent work [26], as well as Ref. [5] dedicated to the FMM for differential random operators, refer to some general results on maximally dissipative operators and related topics; cf., e.g., [1, 5], [8], [9], [20], [29], [34], and the monograph [38]. This list (certainly incomplete) constitutes a highly recommended introductory reading (300+ pages). In the present manuscript, however, we restrict our analysis to the lattice models, and actually work only with finite-dimensional operators, so it is possible to give an upshot of the required theory "in a nutshell", with a bare minimum of elementary technical tools. In fact, it all boils down to a straightforward application of the Boole formula. The reader can see, e.g. in [5], that a more general situation requires far-going generalizations of this simple identity.

Lemma 6.1. For any $s \in (0, 1)$ there exists $C_s < \infty$ such that for any finite connected subset $\Lambda \subset \mathbb{Z}^d$, any two sites $u_1, u_2 \in \mathbb{Z}^d$, with $n := \text{card } \{u_1, u_2\} \in \{1, 2\}$, and any pair of configurations $x, y \in \Lambda^2$ with $\Pi x \ni u_1, H y \ni u_2$, the following bound holds: some $C = C(s, F_V, n) < \infty$,\n
$$\mathbb{E} \left[ |G(x, y; E)|^s \left| \mathfrak{F}_{\neq u_1, u_2} \right| \right] \leq C|g|^{-s},$$

(6.1)

where $\mathfrak{F}_{\neq u_1, u_2}$ is the sigma-algebra generated by $\{V(u; \cdot), u \in \mathbb{Z}^d \setminus \{u_1, u_2\}\}$.

Proof. I. We consider first the case where $u_1 \neq u_2$.

As we shall see, the relevant representation of the random operator at hand is $gV(\omega) + A$, with the nonrandom component $A = H_0 + U$, and we work with the resolvent $G_g(E) = (gV(\omega) + A - E)^{-1}$. The random field $V$ is assumed bounded, $\|V(x; \cdot)\|_\infty < +\infty$, and it suffices to assume that $\|V(x; \cdot)\|_\infty \leq 1$, for larger values are simply obtained by taking $|g|$ larger. In fact, even the particular model where $V \sim \text{Unif}([0, 1])$ is of great interest, and it is one of the most popular models of disorder in physics. Then we can extract the factor $g$ and note that, with $B_g := g^{-1}A$, $\lambda = g^{-1}E$,

$$\mathbb{E} \left[ |G_g(x, y; E)|^s \left| \mathfrak{F}_{\neq u_1, u_2} \right| \right] = |g|^{-s} \mathbb{E} \left[ (1_y, (V(\omega) + B_g - \lambda)^{-1}1_x) \right].$$

In the rest of the proof, we work with the resolvent of the operator $V(\omega) + B_g$, at a rescaled energy $\lambda$ (which is fixed in the proof, anyway).

- Reduced probability space. Now the r.v. $V(x; \omega)$ vary inside $I = [0, 1]$ and admit a bounded probability density $p_V$, $\|p_V\|_\infty = \overline{p} < \infty$. The conditional distribution of $V$ given $\mathfrak{F}_{\neq u_1, u_2}$ gives rise to the reduced probability space $(A, \mathbb{P})$, where $A = I^2, \mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure $\text{mes}_1 \otimes \text{mes}_1$ on $A$, with density $(v_1, v_2) \mapsto \overline{p}(v_1, v_2) = p_V(v_1)p_V(v_2) \leq \overline{p}^2$. Then for any non-negative random variable $\tilde{\xi}$ on $(\Omega, \mathbb{F})$ and any $s \in (0, 1)$, the conditional expectation of $\tilde{\xi}$ given $\mathfrak{F}_{\neq u_1, u_2}$ has the form (below we allow the expectation to be $+\infty$)

$$0 \leq \mathbb{E} \left[ \tilde{\xi}^s(\omega) \left| \mathfrak{F}_{\neq u_1, u_2} \right| \right] = \int_A \int dv_1 \int dv_2 p_V(v_1)p_V(v_2) \xi^s(v_1, v_2; \cdot) \leq \overline{p}^2 \int_A \int dv_1 dv_2 \xi^s(v_1, v_2; \cdot),$$

(6.2)
where \( \zeta(v_1, v_2; \bullet) \) is obtained from \( \tilde{\zeta}(\omega) \) by identifying \( v_j \equiv V(u_j; \omega) \), with the remaining degrees of freedom fixed by conditioning (they are symbolically represented by \( \bullet \)). Now \( \zeta \) can be considered as a random variable on the square \( I^2 \) with the normalized Lebesgue measure. For the rest of the argument, \( P \{ \} \) and \( \bar{E} \{ \} \) refer to this new probability space. Let \( F_\zeta(t) = P \{ \zeta \leq t \} \), then

\[
\hat{E}[^*] = \int_A dv_1 dv_2 \zeta^*(v_1, v_2; \cdot) = \int_0^\infty t^s df_\zeta(t) = s \int_0^\infty t^{s-1}(1 - F_\zeta(t)) \, dt. \tag{6.3}
\]

We shall return to (6.3), once we obtain a suitable upper bound of the tail probability distribution function (below \( \text{mes} \) is the Lebesgue measure on \( A \))

\[
1 - F_\zeta(t) = \text{mes} \{ (v_1, v_2) \in A : \zeta(v_1, v_2) > t \}.
\]

**The Birman–Schwinger relation.** Introduce the sets

\[ S_j = \left( \{ u_j \} \times \mathbb{Z} \right) \cup (\mathbb{Z} \times \{ u_j \}) \cap \Lambda, \quad j = 1, 2, \]

the (multiplication) operators

\[
0 \neq C = 1_{S_1} + 1_{S_2} \geq 0, \quad D = 1_{S_1} - 1_{S_2},
\]

and the random variables

\[
\xi = \frac{1}{2}(V(x_1; \omega) + V(x_2; \omega)), \quad \eta = \frac{1}{2}(V(x_1; \omega) - V(x_2; \omega)).
\]

Then

\[
V(u_1; \omega)1_{S_1} + V(u_2; \omega)1_{S_2} = \xi C + \eta D
\]

and

\[
H(\omega) = \tilde{K}(\omega) + gV(u_1; \omega)1_{S_1} + gV(u_2; \omega)1_{S_2},
\]

\[
= \tilde{K} + g\xi C + g\eta D = K + g\xi C,
\]

where \( \tilde{K}(\omega) \) is \( \tilde{\mathfrak{g}}_{u_1, u_2} \)-measurable and \( K(\omega) = \tilde{K}(\omega) + \eta(\omega)D \).

The operator \( C \) is non-negative and not identically zero, so we can use the Birman–Schwinger identity for \( K_E = K - E \):

\[
C^{1/2}(K_E + g\xi C)^{-1}C^{1/2} = \left( C^{1/2}K_E^{-1}C^{1/2} + g\xi 1 \right)^{-1}.
\]

The operator

\[
K_{E, C} := C^{1/2}(K_E + g\xi C)^{-1}C^{1/2}
\]

is considered acting in the subspace of \( \mathcal{H} \),

\[
\mathcal{H}_{(u_1, u_2)} = (\text{Ker } C)^\perp = \text{Span } \{ 1_w : w \in \Lambda^2, Iw \cap \{ u_1, u_2 \} \neq \emptyset \},
\]

containing in particular \( 1_x \) and \( 1_y \). Its relevance is explained by the fact that both \( 1_x \) and \( 1_y \) are eigenvectors of \( C \) with positive eigenvalues,

\[
C1_x = 1_{S_1}1_x + 1_{S_2}1_x = (N_{u_1}(x) + N_{u_2}(x))1_x = \alpha_x 1_x,
\]

with

\[
N_w(u) := \text{card } \{ j \in \{1, 2\} : u_j = w \}\]

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(the number of particles in \( u \) at the position \( x \)), hence
\[
\begin{align*}
C^{1/2}1_x &= \alpha_x^{1/2}1_x, \\
C^{1/2}1_y &= \alpha_y^{1/2}1_y,
\end{align*}
\]
with \( 1 \leq \alpha_x, \alpha_y \leq 2 \). Therefore, with \( \alpha := (\alpha_x\alpha_y)^{-1/2} \in \left[ 1, 1/2 \right] \),
\[
G(x, y; E) = (1_y, (K_E + g\xi C)^{-1}1_x)
= \alpha (1_y, C^{1/2}(K_E + g\xi C)^{-1}C^{1/2}1_x)
= \alpha \left( 1_y, \left( C^{1/2}K_E^{-1}C^{1/2} + g\xi 1 \right)^{-1}1_x \right)
\]
Since \( \alpha \leq 1 \), one has an implication: for any \( t > 0 \),
\[
|G(x, y; E)| > t \implies \left| \left( 1_y, (K_{E,C} + g\xi 1)^{-1}1_x \right) \right| > t,
\]
where the RHS refers to the (finite-dimensional) space \( \mathcal{H}_{(u_1, u_2)} \).

- **The tail tale and the Boole formula.** Consider the linear change of variables \( \Phi : (v_1, v_2) \mapsto (\xi, \eta) = ((v_1 + v_2)/2, (v_1 - v_2)/2) \) with Jacobian = 2. Note that with \( (v_1, v_2) \in A \), \( \xi \) varies in \( [0, 1] \) and \( \eta \) in \( [-1/2, 1/2] \).

Let \( A' = \Phi(A) \subset [0, 1] \times [-1/2, 1/2] \), then by the Fubini theorem,
\[
\begin{align*}
\int_{\mathbb{R}^2} dv_1 dv_2 1_A 1_{M_t} &= 2 \int_{-1/2}^{1/2} d\eta \int_{\mathbb{R}} d\xi 1_{A'}(\xi, \eta) 1_{M_t} \circ \Phi^{-1}(\xi, \eta) \\
&\leq 2 \int_{-1/2}^{1/2} d\eta \int_{\mathbb{R}} d\xi 1_{M_t} \circ \Phi^{-1}(\xi, \eta) \\
&\leq 2 \sup_{\eta \in \mathbb{R}} \text{mes}(M_t(\eta))
\end{align*}
\]
where
\[
M_t(\eta) = \left\{ \xi \in \mathbb{R} : \left| (1_y, (K_{E,C,\eta} + g\xi 1)^{-1}1_x) \right| > t \right\}.
\]
The function
\[
\mathcal{R} : \xi \mapsto \left( 1_y, (K_{E,C,\eta} + g\xi 1)^{-1}1_x \right) \equiv g^{-1} \left( 1_y, (g^{-1}K_{E,C,\eta} + \xi 1)^{-1}1_x \right)
\]
is rational, with real simple poles,
\[
\mathcal{R}(\xi) = \sum_j \frac{g^{-1}c_j}{\lambda_j - \xi}, \quad \sum_j \left| g^{-1}c_j \right| \leq |g|^{-1} \quad \text{(by Bessel's inequality)},
\]
so we can again apply the Boole formula,
\[
\text{mes} \left\{ \xi : |\mathcal{R}(\xi)| > t \right\} = \frac{2 \sum c_j}{t} \leq \frac{2}{gt}.
\]
Therefore,
\[
1 = \text{mes} A \geq \int_{\mathbb{R}^2} dv_1 dv_2 1_A 1_{M_t} \leq 2 \cdot \frac{2}{gt} = \frac{4}{gt},
\]
yielding for the tail probability distribution function
\[
1 - F_\xi(t) \leq \min \left[ 1, 4g^{-1}t^{-1} \right].
\]
• The fractional moment. Return to the fractional moment in (6.3):

\[
\hat{E}[\zeta^s] = s \int_0^\infty t^{s-1}(1 - F(t)) \, dt \\
\leq s \int_0^\infty t^{s-1} \min(1, 4g^{-1}t^{-1}) \, dt \\
= s \int_0^{4/g} t^{s-1} \, dt + s \int_{4/g}^\infty t^{s-2} \, dt \\
= \frac{4^s}{g^s(1-s)}.
\]

Finally, for the conditional fractional moment in (6.1), we obtain, as asserted,

\[
\hat{E}[|G(x, y; E)|^s \mid \mathcal{F}_{u_1, u_2}] \leq \mathcal{F}^2 \hat{E}[\zeta^s] \leq \frac{C}{|y|^s(1-s)}.
\]

II. Now we turn to the simpler case where \( u_1 = u_2 = u \). The two-parameter operator families introduced above are controlled now with a single parameter. Consider the set

\[
S = (\{u\} \times \mathbb{Z}) \cup (\mathbb{Z} \times \{u\}) \cap \Lambda,
\]

and let \( C = 1_S \); then \( H = g\xi C + K(\omega) \), where \( K \) is \( \mathcal{F}_{\neq u} \)-measurable.

The operator \( C \) is non-negative and nonzero, so we can use the Birman–Schwinger identity for \( K_E = K - E \):

\[
C^{1/2}(\mathcal{F}_E + g\xi C)^{-1}C^{1/2} = \left( C^{1/2}\mathcal{F}_E^{-1}C^{1/2} - g\xi 1 \right)^{-1}.
\]

The functions \( 1_x, 1_y \) are eigenvectors of \( C \) with positive eigenvalues, viz.

\[
C 1_x = 1_S 1_x = N_u(x)1_x, C 1_y = 1_S 1_y = N_u(y)1_y,
\]

with \( N_w(u) := \text{card} \{ j \in \{1, 2\} : u_j = w \} \), hence

\[
C^{1/2}1_x = \alpha_x^{1/2}1_x, \quad C^{1/2}1_y = \alpha_y^{1/2}1_y,
\]

with \( 1 \leq \alpha_x, \alpha_y \leq 2 \). Therefore, as in (6.4), we obtain

\[
G(x, y; E) = (1_y, (\mathcal{F}_E + g\xi C)^{-1}1_x) = \alpha \left( 1_y, (C^{1/2}\mathcal{F}_E^{-1}C^{1/2} + g\xi 1)^{-1}1_x \right).
\]

The operator \( K_{E, C} := C^{1/2}(\mathcal{F}_E + g\xi C)^{-1}C^{1/2} \) acts in the subspace of \( \mathcal{H} \),

\[
\mathcal{H}_{(u)} = (\text{Ker} C) = \text{Span} \{ 1_w : w \in \Lambda^2, \Pi w \cap \{u\} \neq \emptyset \},
\]

containing \( 1_x \) and \( 1_y \). Since \( \alpha \geq 1 \), for any \( t > 0 \),

\[
|G(x, y; E)| > t \implies \left| (1_y, (\mathcal{F}_{E, C} + g\xi 1)^{-1}1_x) \right| > t.
\]

Introduce the rational function \( R : \xi \leftrightarrow (1_y, (\mathcal{F}_{E, \eta} + g\xi 1)^{-1}1_x) \),

\[
R(\xi) = \sum_j \frac{g^{-1}c_j}{\lambda_j - \xi} \sum_j |g^{-1}c_j| \leq g^{-1} \quad \text{(by Bessel's inequality)},
\]

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and the set $\mathcal{M}_t = \{\xi \in I : |R(\xi)| > t\}$ with $\operatorname{mes} \mathcal{M}_t \leq 1$. By the Boole inequality combined with the LHS inequality in (6.5),

$$\operatorname{mes} \{\xi : |R(\xi)| > t\} \leq \min \left[1, 2g^{-1}t^{-1}\right].$$

Hence

$$\hat{E}[\zeta^*] \leq \int_0^\infty t^s dF_\zeta(t) = s \int_0^\infty t^{s-1}(1 - F_\zeta(t)) dt \leq s \int_0^\infty t^{s-1} \min \left[1, 2g^{-1}t^{-1}\right] dt = \frac{2s}{g^s(1-s)}.$$

so the original (conditional) fractional moment in (6.1) is bounded:

$$\hat{E} \left[ |G(x, y; E)|^s \big| \mathfrak{F} \neq u, u_2 \right] \leq \frac{C}{g^s(1-s)}. \quad \square$$

7 Tunneling from split configurations. Finite-range interaction

**Lemma 7.1.** Assume that the interaction potential has finite range $r_0$. There exist some $A = A(d, r_0), m' > 0$ such that if the 1-particle system is $m$-localized with $m \geq 1$, then

$$E \left[ |G(x, y)|^s \right] \leq A \exp \left( -m' \min [d_S(x, y), \operatorname{diam}(x) \lor \operatorname{diam}(y)] \right). \quad (7.1)$$

**Proof.** The main argument benefits from the assumption $N = 2$ which simplifies the combinatorial analysis, as well as notation, since there is only one way to split a 2-particle configuration into two distant "clusters" - single-particle sub-configurations, in this case. Here $\Pi x = \{x_1, x_2\}$, $x_1, x_2 \in \mathbb{Z}^d$, $\operatorname{diam} x = |x_1 - x_2|$, and we assume that

$$d_S(x, y), |x_1 - x_2| \geq R > 0.$$ 

Thus either $\operatorname{dist}(x_i, \Pi y) \geq R$ for some $i \in \{1, 2\}$, or $\operatorname{dist}(y_j, \Pi x) \geq R$ for some $j \in \{1, 2\}$. In either case,

$$|x_1 - y_1| \lor |x_2 - y_2| \geq d_S(x, y) \geq R. \quad (7.2)$$

In the proof, we treat $U$ as a perturbation of the Hamiltonian $H^m$. With $H = H^m + U$, the FGRI gives

$$|G(x, y)|^s \leq |G^m(x, y)|^s + |(G^m U G)(x, y)|^s. \quad (7.3)$$

**Step 1.** Let us show that

$$\forall u, v \in \mathbb{Z}^d \quad \hat{E} \left[ |G^m(u, v)|^{2s} \right] \leq A e^{-m d_S(u, v)}. \quad (7.4)$$

This general bound will be first applied to the term $G^m(x, y)$ in (7.3), and later, at **Step 2**, to the expectations $G^m(x, w)$ in (7.3).

By assumption on of the 1-particle systems,

$$Q^{(1)}(x, y; \mathcal{R}) \leq A e^{-m |x-y|}.$$
For any interval \( I \subset \mathbb{R} \) we have a deterministic relation between the 2-particle and 1-particle EFC:

\[
Q^u_{\text{ni}}(u, v; I) = \sum_{\lambda_i \in \Sigma_1, \mu_i \in \Sigma_2} |\varphi_i(x_1)\overline{\varphi}_i(x_1)\psi_j(x_2)\overline{\psi}_j(y_2)|
\leq \sum_{\lambda_i \in \Sigma_1} |\varphi_i(x_1)\overline{\varphi}_i(y_1)| \sum_{\mu_j \in \Sigma_2} |\psi_j(x_2)\overline{\psi}_j(y_2)|
= Q^{(1)}(x_1, y_1)Q^{(2)}(x_2, y_2).
\]

Thanks to the a priori deterministic upper bound on the EFC, \( Q(u, v; E) \leq 1 \), we obtain, on account of (7.2),

\[
\hat{\mathbb{E}}\left[ |Q^u_{\text{ni}}(u, v; E)|^s \right] \leq \min \left\{ \hat{\mathbb{E}}\left[ Q^{(1)}(x_1, y_1) \right], \hat{\mathbb{E}}\left[ Q^{(2)}(x_2, y_2) \right] \right\}
\leq A \exp \left( -m \max [|x_1 - y_1|, |x_2 - y_2|] \right)
\leq Ae^{-m d_0(u, v)}.
\] (7.5)

(7.6)

(7.7)

We conclude this stage of analysis by applying Lemma B.1:

\[
\hat{\mathbb{E}}\left[ |Q^u_{\text{ni}}(u, v)|^s \right] \leq \frac{C}{1 - s} \hat{\mathbb{E}}\left[ |Q^u_{\text{ni}}(u, v)|^s \right] \leq \frac{C}{1 - s} Ae^{-m d_0(u, v)}.
\]

**Step 2.** Now consider the second, perturbative term in (7.3). Since \( U \) is diagonal in the delta-basis, and the support of the interaction energy function, \( w \mapsto U(w) \), is contained in the strip \( \mathbb{D}_{\mathcal{L}} := \{ z \in (\mathbb{Z}^d)^2 : \text{diam} z \leq r_0 \} \), we have

\[
\epsilon_R = \epsilon_R(\omega) := \|U\|^{-1} |(G^u_{\text{ni}} U G)(x, y)|^s
\leq \|U\|^{-1} \sum_{w \in \mathbb{D}_{\mathcal{L}}} |G^u_{\text{ni}}(x, w)|^s |U(w)|^s |G(w, y)|^s
\leq \sum_{w \in \mathbb{D}_{\mathcal{L}}} |G^u_{\text{ni}}(x, w)|^s |G(w, y)|^s.
\]

By the Cauchy–Schwarz inequality,

\[
\hat{\mathbb{E}}\left[ \epsilon_R \right] \leq \sum_{w \in \mathbb{D}_{\mathcal{L}}} \left( \hat{\mathbb{E}}\left[ |G^u_{\text{ni}}(x, w)|^{2s} \right] \right)^{1/2} \left( \hat{\mathbb{E}}\left[ |G(w, y)|^{2s} \right] \right)^{1/2}.
\] (7.8)

The last fractional moment in the RHS of (7.8) does not provide any significant contribution to the decay bound, for there is little we assumed about \( y \) (except that \( d_0(x, y) = R > L_k \)). Indeed, we merely claim that it is "harmless", making use of the results of Sect. 6: with \( 0 < 2s < 1 \),

\[
\hat{\mathbb{E}}\left[ |G(w, y)|^{2s} \right] \leq \frac{\text{Const}}{(1 - 2s)|g|^{2s}}.
\] (7.9)

Therefore,

\[
\hat{\mathbb{E}}\left[ \epsilon_R \right] \leq \frac{\text{Const}}{(1 - 2s)|g|^{2s}} \sum_{w \in \mathbb{D}_{\mathcal{L}}} \left( \hat{\mathbb{E}}\left[ |G^u_{\text{ni}}(x, w)|^{2s} \right] \right)^{1/2}.
\]

Applying again the general bound (7.4) with \((u, v) = (x, w)\), we can write

\[
\sum_{w \in \text{supp} U} \hat{\mathbb{E}}\left[ |G^u_{\text{ni}}(x, w)|^{2s} \right] \leq A \sum_{w \in \text{supp} U} e^{-m d(x, w)},
\] (7.10)
For \( w \in \text{supp } U \), the distance \( d(x, w) \) is essentially the distance from \( x \) to the diagonal \( \mathbb{D}_0 = \{(u, u), u \in \mathbb{Z}^d\} \). To be more precise, consider a point \( w \in \text{supp } U \), thus with \( \text{diam } w \leq r_0 \), and let

\[
d_S(x, w) = \max \left[ |x_1 - w'|, |x_2 - w''| \right], \quad \{w', w''\} = \Pi w.
\]

Then by the triangle inequality,

\[
r_0 \geq \text{diam } w = |w' - w''| \geq |x_1 - x_2| - |x_1 - w'| - |x_2 - w''|,
\]

hence

\[
d_S(x, w) \geq \frac{\text{diam } x - r_0}{2} \geq \frac{\text{diam } x - r_0}{2} =: R_x \geq \frac{1}{2}(R - r_0).
\]

(7.11)

Now we can apply crude upper bounds

\[
\text{card } \{w \in (\mathbb{Z}^d)^2 : d_S(x, w) = r\} \leq C r^{2d},
\]

(7.12)

and

\[
\sum_{w \in \text{supp } U} e^{-m d(x, w)} \leq \sum_{w \in \text{supp } U} e^{-m d(x, w)} \\
\leq C \sum_{r \geq R_x} r^{2d} e^{-m r} \leq C(R_x)^{2d} e^{-m R_x}
\]

(7.13)

\[
\leq C' R^{2d} e^{-m R}.
\]

Concluding, we obtain

\[
\hat{E}[\epsilon_R] \leq \text{Const } |g|^{s} e^{-\frac{m}{2} R}.
\]

(7.14)

Collecting (7.3), (7.4) with \((u, v) = (x, y)\), and (7.14), the claim follows.

8 Tunneling from split configurations. Infinite-range interaction

Lemma 8.1. Suppose that

\[
\min \left[ d_S(x, y), \text{diam } (x) \lor \text{diam } (y) \right] \geq R > 0.
\]

Then for some \( m' > 0 \), one has

\[
\hat{E}[|G(x, y)|^s] \leq Ae^{-m'R}.
\]

(8.1)

Proof. We assume that

\[
d_S(x, y), |x_1 - x_2| \geq R > 0.
\]

Thus either \( \text{dist}(x, \Pi y) \geq R \) for some \( i \in \{1, 2\} \), or \( \text{dist}(y_j, \Pi x) \geq R \) for some \( j \in \{1, 2\} \). In either case,

\[
|x_1 - y_1| \lor |x_2 - y_2| \geq d_S(x, y) \geq R.
\]

(8.2)

In the proof, we treat \( U \) as a perturbation of the Hamiltonian \( H_{ni} \), more precisely, we assess the effect of \( U \) on the fractional moment of the Green function \( G(x, y) \), with \( x \) far away from the support of the interaction potential, and with \( y \) distant from \( \Pi x \). The geometric analysis would be more involved for \( N \geq 3 \).

With \( H = H_{ni} + U \), the GRI gives

\[
|G(x, y)|^s \leq |G_{ni}(x, y)|^s + |(G_{ni} U G)(x, y)|^s.
\]

(8.3)
Step 1. As shown in the proof of Lemma 7.1, the Green functions of $H^0(\omega)$ satisfy

$$\forall u, v \in (\mathbb{Z}^d)^2 \quad \mathbb{E} \left[ |G^0(u, v)|^{2s} \right] \leq Ae^{-m d_\varepsilon(u, v)}. \quad (8.4)$$

Step 2. Consider the perturbation term in (8.3) We have

$$\epsilon_R = \epsilon_R(\omega) := |(G^0 U G)(x, y)|^s$$

$$\leq \left( \sum_{\text{diam } w \leq R/4} + \sum_{\text{diam } w > R/4} \right) \mathbb{E} \left[ |G^0(x, w)|^s |U(w^*)|G(w, y)|^s \right] \quad (8.5)$$

$$\leq \|U\|^s S_1 + e^{-\frac{4\pi R}{s}} S_2,$$

where

$$S_1 = \sum_{\text{diam } w \leq R/4} \mathbb{E} \left[ |G^0(x, w)|^s |G(w, y)|^s \right]$$

$$S_2 = \sum_{\text{diam } w > R/4} \mathbb{E} \left[ |G^0(x, w)|^s |G(w, y)|^s \right]$$

and the factor $e^{-\frac{4\pi R}{s}}$ is of course an upper bound on the interaction over the set $\{ w : \text{diam } w > R/4 \}$. Using the Cauchy–Schwarz inequality and the estimate (6.1), we obtain

$$S_2 \leq \sum_{\text{diam } w > R/4} \left( \mathbb{E} \left[ |G^0(x, w)|^{2s} \right] \right)^{1/2} \left( \mathbb{E} \left[ |G(w, y)|^{2s} \right] \right)^{1/2} \quad (8.6)$$

$$\leq \sum_{w \in (\mathbb{Z}^d)^2} \left( \mathbb{E} \left[ |G^0(x, w)|^{2s} \right] \right)^{1/2} \frac{\text{Const}}{(1 - 2s)|g|^{2s}},$$

provided, for $2s < 1$, so the above expectations are finite.

By (7.4) with $(u, v) = (x, w)$,

$$\sum_{w \in (\mathbb{Z}^d)^2} \mathbb{E} \left[ |G^0(x, w)|^{2s} \right] \leq A \sum_{w \in (\mathbb{Z}^d)^2} e^{-m d(x, w)} =: A' < +\infty, \quad (8.7)$$

since the function $w \mapsto e^{-m d(x, w)}$ is summable in $(\mathbb{Z}^d)^2$. Thus

$$e^{-\frac{4\pi R}{s}} S_2 \leq \frac{\text{Const}}{(1 - 2s)|g|^{2s}} e^{-\frac{4\pi R}{s}}. \quad (8.8)$$

Introduce the truncated interaction $U_R$ of range $R/4$, with the 2-body potential $U_R(r) = 1_{[r \leq R/4]} U(r)$, coinciding with $U(w)$ on the set $\{ w : \text{diam } w > R/4 \}$.

The same geometrical argument as in the proof of (7.11), with $r_0$ replaced now by $R/4$, gives

$$\min_{w \in \text{supp } U_{R/4}} \text{dist}(x, w) \geq \frac{R - \frac{R}{2}}{2} = \frac{3R}{8}. \quad (8.9)$$

Arguing as in the proof of Lemma 7.1, we obtain by (8.7) and (8.9),

$$\sum_{w \in \text{supp } U_{R/4}} \mathbb{E} \left[ |G^0(x, w)|^{2s} \right] \leq C \sum_{r \geq 3R/8} r^{2d} e^{-m r} \quad (8.10)$$

$$\leq C'e^{-c m R}.$$ 

Concluding, we obtain, for $0 < s < 1/3$,

$$\mathbb{E} [\epsilon_R] \leq \text{Const} |g|^{-s} \left( e^{-c m R} + e^{-\frac{4\pi R}{s}} \right). \quad (8.11)$$

Collecting (8.3), (8.4) and (8.11), the assertion follows.
Appendix A  Perturbed quadratic recursion

Fix any \(0 < \nu < m/4\). Let \(\beta_k := e^{2\nu}M_s^{1/2}k\), then the recursion (4.16)
\[
\hat{T}_{k+1} \leq \frac{1}{2}M_s \hat{T}_k^2 + \frac{1}{2}A'L_{k+1} P e^{-2\nu L_k}
\]
implies
\[
e^{-2\nu}M_s^{-1/2} \beta_{k+1} \leq \frac{1}{2}e^{-4\nu}M_s^{-1} \beta_k^2 + \frac{1}{2}A''L_{k+1} P e^{-2\nu L_k},
\]
for all \(k \geq 1\), with \(A'' = A'M_s^{1/2}e^{2\nu}\). If \(L_0\) (hence, every \(L_k\), \(k \geq 0\)) is large enough, depending on \(\frac{m}{2} - 2\nu > 0\), then
\[
\beta_{k+1} \leq \frac{1}{2}e^{-2\nu} \beta_k^2 + \frac{1}{2}e^{-2\nu} e^{-2\nu L_k}.
\]

To justify this statement, recall that it follows from the results of the single-particle variants of the MSA and FMM that the decay exponent \(m > 0\) (or, more precisely, a rigorous lower bound thereupon) can be chosen in the form \(m := c\ln |g|\), for \(|g|\) large enough. For example, it suffices to make use of the techniques from [2] or the finite-volume condition from [4], applied to single-site subsets of the lattice \(Z^1\). For arbitrarily small \(\delta > 0\) and \(L \gg 1\),
\[
a|g|^s L^b e^{-\nu L} \leq a|g|^s L^b e^{-c \ln |g| L} \leq a|g|^s e^{-(c-\delta) \ln |g| L} = a|g|^s \hat{L}^{(c-\delta) \ln |g| L} \leq e^{-2\nu L},
\]
with \(\nu\) arbitrarily close to \(m/4\).

Lemma A.1. Let the sequence \(\{L_k, k \geq 0\}\) satisfy the recursion \(L_{k+1} = 2L_k + 2\). Consider a sequence of positive numbers \(\{\beta_k\}\) with \(\beta_0 < e^{-\nu}\), for some \(\nu > 0\), and satisfying
\[
\forall k \in \mathbb{N} \quad \beta_{k+1} = \frac{1}{2}e^{-2\nu} \beta_k^2 + \frac{1}{2}e^{-2\nu} e^{-2\nu L_k}.
\]
Then for all \(k \geq 1\)
\[
\beta_k \leq \max \{e^{-\nu L_k}, e^{-\mu L_k}\},
\]
\[
\mu := \frac{\nu + \ln \beta_0^{-1}}{1 + L_0^{-\mu}}.
\]

Proof. • First, note that if there exists \(j \in \mathbb{N}\) such that
\[
\beta_j \leq e^{-\nu L_j},
\]
then by induction, for all \(k \geq j\) we have, using \(L_k = \frac{1}{2}L_{k+1} - 1\),
\[
\beta_{k+1} \leq \frac{1}{2}e^{-2\nu} e^{-2\nu L_k} + \frac{1}{2}e^{-2\nu} e^{-2\nu L_k} = e^{-2\nu} e^{-2\nu L_k} = e^{-\nu L_{k+1}}.
\]

• Next, suppose that (A.4) never occurs, so for all \(k\) we have
\[
\beta_{k+1} > e^{-\nu L_k}.
\]
Then $\frac{1}{2} e^{-2\nu \beta^2_k} + \frac{1}{2} e^{-2\nu e^{-\nu L_k}} < e^{-2\nu \beta^2_k}$, and by (A.1),
\[
\forall k \geq 0 \quad \beta_{k+1} < (e^{-\nu \beta_k})^2.
\]
By induction, with $\mu$ given by (A.3) and $2^k = L_k/(L_0 + 2)$, for all $k \geq 0$,
\[
\beta_k \leq (e^{-\nu \beta_0})^{2^k} = (e^{-\nu \beta_0})^{\frac{L_k}{L_0+2}} = e^{-\mu L_k},
\]  
(A.7)
with $\mu = -\ln(a^2 \beta_0^2)/(L_0 + 2) = \ln(a^{-1} \beta_0^{-1})/(1 + L_0/2)$.

Finally, if (A.6) holds on a finite integer interval $[0, j-1]$, and then one has (A.4), the inequality (A.7) is still valid for $k \in [0, j-1]$, while the bounds (A.4), (A.5) take over for the remaining values $k \geq j$.

Consequently,
\[
\hat{\Upsilon}_k \leq e^{2\nu M_s^{1/2}} e^{-\tilde{\mu} L_k}, \quad \tilde{\mu} := \min(\nu, \mu).
\]

**Appendix B  From the EF correlators to the Green functions**

In this section we provide a detailed proof of a fairly general relation between the resolvents and the EF correlators, which had been used in numerous works on the FMM. The single- or multiparticle nature of the Hamiltonian at hand is irrelevant, as long as the fractional moments of the EF correlators, which had been used in numerous works on the FMM. The single- or multi-

In this deterministic statement, the EF s are fixed, and the only relevant variable is $E$.

The GF is a rational function, and we divide it into the sum of two terms, according to the signs of the numerators:
\[
G = \sum_{E_i, c_i \geq 0} \frac{c_i}{E_i - E} + \sum_{E_i, c_i < 0} \frac{c_i}{E_i - E} =: G^{(+)} + G^{(-)}.
\]

We have
\[
\int_I |G(E)|^s \, dE \leq \int_I |G^{(+)}(E)|^s \, dE + \int_I |G^{(-)}(E)|^s \, dE
\]
Both integrals are assessed in the same way, so we focus on the first one.

It is convenient at this point to introduce probabilistic language, for we are going to apply a standard technique for the probability distribution functions (PDF), and consider the probability space $(I, \mathcal{B}_I, \text{mes}_I)$, where $\mathcal{B}_I$ is the Borel sigma-algebra and $\text{mes}_I := |I|^{-1}$ the normalized Lebesgue measure in $I \subset \mathbb{R}$. Further, consider the measurable function $G^{(\pm)} : E \mapsto G^{(\pm)}(E)$ (i.e., a "random variable" on $(I, \text{mes}_I)$), and let $F_{\pm}(t)$ be its PDF:
\[
F_{\pm}(t) = \text{mes}_I \{ E : |G^{\pm}| \leq t \}.
\]
Thus, denoting for brevity $Q$ where $Q$ is to reduce the estimate to that of the "tail distribution" (in $(I, \operatorname{mes} I)$) function $t \mapsto 1 - F(t) = \operatorname{mes} I \{G^+ > t\}$, and it is motivated by the Boole identity (cf. Proposition 4), applicable to any rational function with simple, real poles and positive expansion coefficients,

$$f : t \mapsto \sum_{i=1}^{n} \frac{c_i}{t_i - t}.$$ 

and stating that

$$\operatorname{mes} \{ \lambda : |f(\lambda)| > t \} = \frac{2 \sum_{i=1}^{n} c_i}{t},$$

hence

$$\operatorname{mes} I \{ \lambda : |f(\lambda)| > t \} \leq \frac{2 \sum_{i=1}^{n} c_i}{|I| t}.$$ 

Recall that, in fact, $c_i = \psi_i(x)\psi_i(y)$ (we choose the EFs real), so that

$$\sum_{\nu: c_{\nu} > 0} c_{\nu} \leq Q_+ (x,y), \quad \sum_{\nu: c_{\nu} < 0} (-c_{\nu}) \leq Q_-(x,y),$$

where $Q_\pm (x,y)$ are components of the EF correlator:

$$Q_+(x,y) + Q_-(x,y) = \sum_{\nu: \nu \in I} |c_{\nu}| \leq Q(x,y).$$

Thus, denoting for brevity $Q_\pm \equiv Q_\pm (x,y)$, we have

$$1 - F_\pm (t) \leq \min (|I|, 2Q_\pm t^{-1}) = |I| 1_{[0, 2Q_\pm / |I|]}(t) + \frac{2Q_\pm}{|I|} 1_{[2Q_\pm / |I|, +\infty]}(t)$$

and

$$\int_0^\infty t^s dF_\pm (t) \leq s \int_0^\infty t^{s-1} (1 - F_\pm (t)) dt.$$

$$\leq s |I| \int_0^{2Q_\pm / |I|} t^{s-1} dt + 2s Q_\pm \int_{2Q_\pm / |I|}^\infty t^{s-2} dt$$

$$= |I| \frac{(2Q_\pm)^s}{|I|^s} + 2s Q_\pm \frac{(2Q_\pm)^{s-1}}{|I|^{s-1}(1 - s)}$$

$$= \frac{(2Q_\pm)^s |I|^{1-s}}{1 - s}.$$ 

Therefore,

$$\int_I |G(x,y; E)|^s dE \leq \frac{2s |I|^{1-s}}{1 - s} (Q_+(x,y))^s + Q_-(x,y))^s$$

$$\leq \frac{2s (Q(x,y))^s |I|^{1-s}}{1 - s},$$

where the last inequality follows from $\frac{s^2 + s^2}{2} \leq \left(\frac{s + s}{2}\right)^s$, $s < 1$. \qed
Appendix C  Boole’s identity

While there seems to be a consensus that the result stated below was first discovered and proved by George Boole in 1857, we hesitate to refer to the original work [7] as the source of the most comprehensive proof. Instead, we provide a very short (10 lines) and elementary proof given almost a century later by Loomis [32]. Boole’s identity was rediscovered more than once and extended in various ways in the theory of the Hilbert transform and gave rise to a number of interesting applications; cf., e.g., [21, 30, 35, 36, 38] and references therein.

**Proposition 4.** Let be given real numbers \( \lambda_1 < \cdots < \lambda_n \) and positive real numbers \( c_1, \ldots, c_n \). Then

\[
\forall t > 0 \quad \text{mes} \left\{ x \in \mathbb{R} : \left| \sum_i \frac{c_i}{x - \lambda_i} \right| > t \right\} = \frac{2}{t} \sum_i c_i.
\]

**Proof.** [Cf. [32, Proof of Lemma 1]]. We assess first the Lebesgue measure of the set \( S_t \), where \( f(x) := \sum_i \frac{c_i}{x - \lambda_i} > t \). Since for all \( x \notin \{\lambda_1, \ldots, \lambda_n\} \)

\[
f'(x) = \sum_i \frac{-c_i}{(x - \lambda_i)^2} < 0,
\]

there are exactly \( n \) roots \( \kappa_i \) of the equation \( f(x) = t \), and one has \( \lambda_1 < \kappa_i < \lambda_{i+1} \), \( \kappa_n > \lambda_n \), thus \( S_+ = \cup_{i=1}^n I_i, \ I_i = (\lambda_i, \kappa_i) \), and \( \text{mes} S_+ = \sum_i (\kappa_i - \lambda_i) \).

Next, multiplying the equation \( f(x) := \sum_i \frac{c_i}{x - \lambda_i} = t \) by \( \prod_i (x - \lambda_i) \), we see that \( \kappa_i \) are the roots of the polynomial admitting two equivalent representations

\[
t \prod_i (x - \lambda_i) - \sum_{i=1}^n c_i \prod_{j \neq i} (x - \lambda_j) \equiv t \prod_i (x - \kappa_i).
\]

The identity for the sub-principal coefficients gives \( t \sum_i \lambda_i + \sum_i c_i = t \sum_i \kappa_i \), yielding \( \text{mes} S_+ = \sum_i (\kappa_i - \lambda_i) = t^{-1} \sum_i c_i \). Similarly, \( \text{mes} \{ x : f(x) < -t \} = t^{-1} \sum_i c_i \), so the assertion follows. \( \square \)

Appendix D  The "one-for-all" principle for the fractional moments

**Lemma D.1.** Let be given real numbers \( 0 < s_2 < t < s_1 < 1 \) and a complex-valued random variable \( X \) with finite absolute moments \( \hat{E} |X|^u \) of all orders \( u \in (0, s_1] \). Then

\[
\hat{E} |X|^t \leq (\hat{E} |X|^{s_1})^{\frac{t}{s_1}} \cdot (\hat{E} |X|^{s_2})^{\frac{t}{s_2}}.
\]  

(D.1)

**Proof.** First, represent \( t \) as a barycenter of \( s_1 \) and \( s_2 \),

\[
t = \alpha s_1 + (1 - \alpha)s_2 = \frac{s_1(t - s_2)}{s_1 - s_2} + \frac{s_2(s_1 - t)}{s_1 - s_2},
\]

and define two Hölder-conjugate exponents,

\[
q_1 = \frac{s_1 - s_2}{t - s_2}, \quad q_2 = \frac{s_1 - s_2}{s_1 - t}.
\]

Now the claim follows directly from the Hölder inequality:

\[
\begin{align*}
\hat{E} |X|^t &= \hat{E} |X|^s_1 |X|^{(1-\alpha)s_2} \\
&\leq \hat{E}^{1/q_1} \left[ |X|^{q_1 s_1} \right] \cdot \hat{E}^{1/q_2} \left[ |X|^{q_2 s_2} \right] \\
&= \hat{E}^{1/q_1} \left[ |X|^{s_1} \right] \cdot \hat{E}^{1/q_2} \left[ |X|^{s_2} \right]
\end{align*}
\]
or, more explicitly,
\[
\mathbb{E} \left[ |X|^t \right] \leq \left( \mathbb{E} \left[ |X|^{s_1} \right] \right)^{\frac{t-s_2}{s_1-s_2}} \cdot \left( \mathbb{E} \left[ |X|^{s_2} \right] \right)^{\frac{s_1-t}{s_1-s_2}}
\] (D.2)

\textbf{Remark D.1.} A sufficient condition for the convergence of the (absolute) moment of order \( u \in (0, 1) \) is the upper bound on the tail probabilities for the r.v. \(|X|\): with \( F_{|X|}(t) := P \{ |X| \leq t \} \) and some \( 0 < A < +\infty \),
\[
\int_{A}^{+\infty} \frac{1}{t^{1-u}} (1 - F_{|X|}(t)) \, dt < +\infty.
\]
In turn, the above condition is follows from a more explicit bound, often available in applications: for all sufficiently large \( t > 0 \),
\[
1 - F_{|X|}(t) \leq \frac{\text{Const}}{t^{u}}
\]
(the latter means, as usual, an upper bound by \( Ct^{-(u+\delta)} \) for some \( \delta > 0 \).)

\textbf{Acknowledgements}

It is a pleasure to thank Michael Aizenman, Simone Warzel, Jeffrey Schenker, Günter Stolz, Ivan Veselić and Misha Sodin for a number of fruitful discussions, related directly or indirectly to the Fractional Moment Method in its various forms.

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