Computing Equivariant Homology with a Splitting Method

Yutao Liu
University of Chicago
Email: yutao492@math.uchicago.edu

September 19, 2022

Abstract: We develop a new method in the computation of equivariant homology, which is based on the splitting of cofiber sequences associated to universal spaces in the category of equivariant spectra. In particular, we will compute the equivariant homology of a point when $G = D_{2p}$ and $A_5$, with coefficients in $\mathbb{Z}$ and $A_G$.

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1 Introduction

Let $G$ be a compact Lie group. In [LMM81], Lewis, May and McClure defined Eilenberg-Maclane $RO(G)$-graded cohomology with Mackey functor coefficients. The purpose of this paper is to study a fundamental but hard question in equivariant stable homotopy: computing $RO(G)$-graded cohomology (or homology) of a point, which is equivalent to the equivariant homotopy of Eilenberg-Maclane spectra. We make this question more precise:

**Question 1.1.** Let $M$ be an arbitrary $G$-Mackey functor. Compute

$$\pi^G_V(HM) := [S^V, HM]^G$$

for any virtual $G$-representation $V$. Use $\star$ to denote the $RO(G)$-grading. Then we want to compute $\pi^G_{\star}(HM)$ as an $RO(G)$-graded abelian group.

Moreover, we have the Mackey functor valued homotopy

$$\pi^G_V(HM)(G/H) := [G/H + S^V, HM]^G \cong [S^V, HM]^H.$$

A more general question is to compute $\pi^G_{\star}(HM)$ as an $RO(G)$-graded Mackey functor.

In addition, when $M$ is chosen as a Green functor, which is a monoid in the category of Mackey functors, there will be extra multiplicative structures on the homotopy groups of $HM$. A final question is to compute $\pi^G_{\star}(HM)$ or $\pi^G_{\star}(HM)$ as an $RO(G)$-graded ring or Green functor.

Common choices of $M$ include constant Mackey functors $\mathbb{Z}, \mathbb{F}_p, \mathbb{Q}$ and the Burnside ring Mackey functor $A_G$, which is the unit object in the category of Mackey functors. All these choices admit natural Green functor structures.

When $G$ is a cyclic group with prime order, the question is completely solved in multiple ways. Partial computations have been done for larger groups like $C_{2^n}, C_{2^m},$ dihedral group $D_{2p}$, and quaternion group $Q_8$, for which we refer to [Ell20], [Geo21], [HK17], [KL20], [Lu21], [Zen18], and [Zou18]. We will develop a new method which uses these known computations as “building blocks” and enables the computation for a much larger collection of groups and Mackey functors.

The idea of our method is motivated by the study of rational $G$-spectra for finite $G$, which is first discussed by Greenlees and May:

**Theorem 1.2.** [GM95] There is an orthogonal basis $\{e_H : H \subset G\}$ of the rational Burnside ring, which only contains idempotent elements. For any rational $G$-spectrum $X$, define $e_H X$ as

$$e_H X := \text{colim}(X \xrightarrow{e_H} X \xrightarrow{e_H} X \xrightarrow{e_H} \ldots)$$

Then we have

$$X \cong \bigvee_H e_H X$$

and

$$[X, Y]^G \cong \prod_H [e_H X, e_H Y]^G$$

with one $H$ chosen from each conjugacy class of subgroups.

This theorem is further studied by Barnes in [Bar08], which reproves the splitting above by some topological constructions and sets up an algebraic model which completely describes the behavior of rational $G$-spectra.

One important idea in Barnes’ proof is to use the universal $G$-space $E\mathcal{F}_H$, which is characterized by the fixed point subspaces:

$$(E\mathcal{F}_H)^K \simeq \begin{cases} S^0, & \text{if } K \text{ is conjugate to } H, \\ \ast, & \text{otherwise.} \end{cases}$$
Theorem 1.3. [Bar08] For rational $G$-spectra $X, Y$, we have an isomorphism

$$[X, Y]^G \cong \prod_H [E\mathcal{F}_H \wedge X, E\mathcal{F}_H \wedge Y]^G$$

with one $H$ chosen from each conjugacy class. Moreover, the functor

$$X \mapsto \bigvee_H E\mathcal{F}_H \wedge X$$

is symmetric monoidal.

The computation in the rational world is not hard since the category of rational $G$-spectra splits into small and simple pieces (with one piece corresponding to each conjugacy class of subgroups). In fact, in order to obtain this full splitting, it suffices to invert all prime factors of $|G|$ instead of applying rationalization.

Our idea comes from a weaker splitting. If we just invert some prime factors of $|G|$, sometimes the category of $G$-spectra will still split, but into some less simple pieces. If we choose the inverted primes properly, some of the pieces will become computable, while the computation on other pieces can be decomposed into computations involving a single subgroup (usually a $p$-subgroup) of $G$. Thus if we know the equivariant homology for the pieces, we can glue them together to recover the equivariant homology for $G$.

Each localization loses torsion information at the inverted primes, but that information is retained at other localizations and no torsion information is lost when the pieces are finally glued together. That is, when $|G|$ contains multiple prime factors, we can apply the idea above multiple times with different prime factors inverted each time. The unlocalized homology can be recovered by collecting different localized data.

For example, let $G = D_{2p}$. When inverting 2 or $p$, we get two different splittings in the category of $G$-spectra. Both make the equivariant homology computable. Thus we can compute $\pi^G_*(HM)[1/2]$ and $\pi^G_*(HM)[1/p]$. The unlocalized $\pi^G_*(HM)$ can be computed by the following pullback diagram:

\[
\begin{array}{ccc}
\pi^G_*(HM) & \rightarrow & \pi^G_*(HM)[1/p] \\
\downarrow & & \downarrow \\
\pi^G_*(HM)[1/2] & \rightarrow & \pi^G_*(HM)[1/2, 1/p]
\end{array}
\]

For the other example $G = A_5$ in this paper, we apply three different splittings when elements in $\{2, 3\}$, $\{2, 5\}$, or $\{3, 5\}$ are inverted. The unlocalized homology still appears as the limit of a larger diagram.

The key element in our splitting method is the universal space $E\mathcal{F}$, where $\mathcal{F}$ is a family of subgroups of $G$. There is a natural cofiber sequence

$$E\mathcal{F}_+ \rightarrow S^0 \rightarrow E\mathcal{F},$$

where $E\mathcal{F}$ is the unreduced suspension of $E\mathcal{F}$. When certain prime factors of $|G|$ are inverted, the cofiber sequence above will split after taking $\Sigma^\infty$. Thus we have

$$X \simeq (E\mathcal{F}_+ \wedge X) \vee (E\mathcal{F} \wedge X)$$

for any $G$-spectrum $X$.

The inverted prime factors depend on the choice of $\mathcal{F}$. When we invert all prime factors of $|G|$, we can choose all families of subgroups together, which will result in a full splitting of $G$-spectra and recover Theorem 1.3. In our computation, we invert all but one prime factor and only use special choices of families. Details will be provided in sections 3 and 4.
**Structure of the paper:** In section 2, we will introduce the universal space $E\mathbb{F}$. We are especially interested in the equivariant homology of $E\mathbb{F}$ with coefficients in the Burnside ring Mackey functor $A_G$, which is the starting point of our splitting method. To be precise, we will prove:

**Proposition 1.4.** When $|G|$ is inverted, the Mackey functor valued homology

$$H^G_\ast(E\mathbb{F}; A_G)$$

concentrates in degree 0. Moreover, the trivial map $E\mathbb{F} \to \ast$ induces an inclusion

$$H^G_0(E\mathbb{F}; A_G) \hookrightarrow H^G_\ast(\ast; A_G) \cong A_G$$

which makes $H^G_\ast(E\mathbb{F}; A_G)$ into a direct summand of $A_G$.

In section 3, we will give a criterion about when the cofiber sequence

$$E\mathbb{F}_+ \to S^0 \to E\mathbb{F}$$

splits after applying $\Sigma^\infty$. We will also explain the choices of families in later computations and prove an important splitting theorem on the inverted prime factors.

Section 4 is the core of the paper, in which our splitting method is fully explained. We will describe the pieces of $G$-spectra after splitting. The decomposition into smaller subgroups of $G$ can be applied for most pieces, while the remaining ones can be computed by a cellular argument. The splitting method itself is not a computation, but a machine which uses the computations on smaller groups to obtain the data on larger groups. The input of the machine consists of equivariant homology on $X$. The splitting method itself splits after applying $\Sigma$.

In sections 5-8, we will apply our splitting method to compute

$$\pi_\ast^G(HM)$$

when $G = D_{2p}$ or $A_5$ and $M = \mathbb{Z}$ or $A_G$. The Mackey functors $\mathbb{Z}$ and $A_G$ are the “universal choices” of constant and general Mackey functors. In fact, the computational methods for $\mathbb{Z}$ or $A_G$ also work for other constant Mackey functors or general Mackey functors.

Explicit expressions of the homology with coefficients in $\mathbb{Z}$ will be provided. For coefficients in $A_G$, we will only show the decomposition into smaller groups without complete algebraic expressions because of the complexity.

In section 9, we will explain how to compute the Mackey functor valued homology for the case $M = \mathbb{Z}$. When we compute the homotopy of $H^G_\ast$ as Mackey functors instead of abelian groups, the input of our machine becomes the Mackey functor valued $\pi^H_\ast(H\mathbb{Z})$ for some $p$-subgroups $H$ of $G$. The structure maps in $\pi^G_\ast(H\mathbb{Z})$ can be expressed by the structure maps in $\pi^P_\ast(H\mathbb{Z})$, where $P \subset G$ is a Sylow subgroup. We will explicitly compute the Mackey functor valued $\pi^A_\ast^G(H\mathbb{Z})$ for some special choices of $V$.

**List of computations:**

- $\pi^{D_{2p}}_\ast(H\mathbb{Z})$: Theorem 5.7
- $\pi^{A_5}_\ast(H\mathbb{Z})$: Theorems 7.4, 7.5, 7.6, B.7, and B.15
- $\pi^{A_5}_\ast(HA_G)$: Theorems 8.2, 8.3, and 8.4

**Notations:** In this paper, we use $\ast$ when the homotopy or homology is graded over $\mathbb{Z}$, and use $\star$ when graded over $RO(G)$. We add an underline to express the Mackey functor valued homotopy or homology: $H^G_\ast\mathbb{F}$.

To be more precise, let $X$ be an unbased $G$-space and $M$ be a Mackey functor. Define the Mackey functor valued equivariant homology of $X$ with coefficients in $M$ as

$$H^G_\ast(X; M) := \mathbb{Z}^G(X_+ \wedge HM).$$

So we have

$$H^G_\ast(X; M)(G/H) = [\Sigma^\ast G/H_+, HM \wedge X_+]^G,$$

$$H^G_\ast(X; M) = H^G_\ast(X; M)(G/G).$$

Here $HM$ is the equivariant Eilenberg-Maclane spectrum corresponding to $M$.

In this paper, $G$ will always be a finite group.
Acknowledgment: The author would like to thank his PhD advisor, Peter May, for numerous suggestions on the organization and editing of this paper.

2 Universal spaces and homological properties

We introduce the universal space $E\mathcal{F}$ and study its equivariant homology in this section. The most important properties are given in Lemma 2.12 and Theorem 2.14.

2.1 Universal $G$-spaces

Definition 2.1. A family $\mathcal{F}$ is a collection of subgroups of $G$ which is closed under conjugation and taking subgroups. The corresponding universal space $E\mathcal{F}$ is an unbased $G$-space such that $(E\mathcal{F})^H$ is contractible for all $H \in \mathcal{F}$ and empty otherwise.

Remark 2.2. When $\mathcal{F}$ only contains the trivial subgroup, $E\mathcal{F}$ becomes $EG$, which is a contractible space with free $G$-action.

One construction of $E\mathcal{F}$ is given in [Dieck72] as a join of spaces. We will explain this idea later and use it to prove some important properties. An alternative construction is given by [Elm83] as a categorical bar construction.

We can fully characterize $E\mathcal{F}$ by fixed point subspaces:

Lemma 2.3. The universal space $E\mathcal{F}$ is unique up to weak equivalence.

Proof: Assume that $X,Y$ are $G$-spaces with the same conditions on fixed point subspaces as $E\mathcal{F}$.

If we have a $G$-map $X \rightarrow Y$, its restriction on each fixed point subspace is a weak equivalence since $X,Y$ have the same types of fixed point subspaces as either $\ast$ or $\emptyset$. Thus this map is a weak $G$-equivalence.

If we do not have a map between $X,Y$, consider $X \times Y$, which is another $G$-space with the same conditions on fixed point subspaces. The above argument shows that the projections $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$ are $G$-weak equivalences. Therefore, $E\mathcal{F}$ is unique up to weak equivalence. □

We assume $E\mathcal{F}$ to be a $G$-CW complex by applying the $G$-CW approximation.

Lemma 2.4. Any $G$-self map of $E\mathcal{F}$ is $G$-homotopic to the identity.

This lemma is implied by the following theorem:

Theorem 2.5. Let $O_G$ be the category of $G$-orbits. Define an $O_G$-space to be a contravariant functor from $O_G$ to topological spaces.

The following pair of functors

$$
\Phi : G - \text{spaces} \rightleftarrows O_G - \text{spaces} : \Psi
$$

defined by $\Phi(X)(G/H) := X^H$ and $\Psi(T) := T(G/{\{e\}})$ form a Quillen equivalence.

We refer to [May96, VI.6] for more details.

Proof of Lemma 2.4. Construct an $O_G$-space $T$ by setting

$$T(G/H) = \begin{cases} 
\ast, & \text{if } H \in \mathcal{F} \\
\emptyset, & \text{otherwise}
\end{cases}$$

Let $CT$ be a cofibrant approximation of $T$. Then

$$
\Psi(CT)^H = \Phi \Psi(CT)(G/H) \cong CT(G/H) \cong \begin{cases} 
\ast, & \text{if } H \in \mathcal{F} \\
\emptyset, & \text{otherwise}
\end{cases}
$$

Recall that

$$
\Phi E\mathcal{F}(G/H) = (E\mathcal{F})^H \cong \begin{cases} 
\ast, & \text{if } H \in \mathcal{F} \\
\emptyset, & \text{otherwise}
\end{cases}
$$
The unique map \( \Phi E \mathcal{F} \to T \) induces \( \Phi E \mathcal{F} \to CT \), whose adjoint \( E \mathcal{F} \to \Psi(CT) \) becomes a \( G \)-equivalence.

So we have
\[
[E \mathcal{F}, E \mathcal{F}]^G \cong [E \mathcal{F}, \Psi(CT)]^G \cong [\Phi E \mathcal{F}, CT]^G \cong [\Phi E \mathcal{F}, T]^G,
\]
which contains a single element.

Thus the identity map is the only self-map of \( E \mathcal{F} \) up to \( G \)-homotopy. \( \square \)

Notice that \( E \mathcal{F} \times E \mathcal{F} \) also appears as a universal space for \( \mathcal{F} \). Thus the above two lemmas imply:

**Lemma 2.6.** The universal space \( E \mathcal{F} \) is a \( G \)-topological semigroup up to homotopy.

Usually \( E \mathcal{F} \) is not a Hopf space since there is no unit. But with extra conditions, \( \Sigma^\infty E \mathcal{F}_+ \) will appear as a homotopy ring spectrum. We will explain this idea in section 3.

Notice that the based spaces \( E \mathcal{F}_+ \) and \( \widetilde{E} \mathcal{F} \) are also characterized by their fixed point subspaces:

\[
(E \mathcal{F}_+)^H \cong \begin{cases} S^0, & \text{if } H \in \mathcal{F}, \\ \ast, & \text{otherwise}. \end{cases}
\]

\[
(\widetilde{E} \mathcal{F})^H \cong \begin{cases} \ast, & \text{if } H \in \mathcal{F}, \\ S^0, & \text{otherwise}. \end{cases}
\]

Thus we can apply the same argument as above, but in a based version:

**Proposition 2.7.** Any self \( G \)-map of \( E \mathcal{F}_+ \) or \( \widetilde{E} \mathcal{F} \) which is a \( G \)-equivalence is homotopic to the identity.

The based spaces \( E \mathcal{F}_+ \) and \( \widetilde{E} \mathcal{F} \) are based \( G \)-topological semigroups up to \( G \)-homotopy. In fact, \( E \mathcal{F} \) is a based \( G \)-Hopf space since the natural inclusion \( S^0 \to E \mathcal{F} \) provides a unit.

### 2.2 \( \mathbb{Z} \)-graded homology with coefficients in \( A_G \)

We recall the definition of the Burnside ring and the corresponding Mackey functor:

**Definition 2.8.** For any finite group \( G \), the collection of isomorphism classes of finite \( G \)-sets forms a commutative monoid, with addition induced by disjoint unions. The **Burnside ring** \( A(G) \) is defined to be the group completion of this monoid. We can express \( A(G) \) as a free \( \mathbb{Z} \)-module, whose basis corresponds to isomorphism classes of \( G \)-orbits. We use \( \{G/H\} \) to denote the basis element of the orbit \( G/H \).

The **Burnside ring Mackey functor** \( A_G \) is defined by \( A_G(G/H) := A(H) \). The Mackey functor structure is given by the following maps:

For any \( L \subset H \), the transfer map
\[
T_L^H : A(L) \to A(H)
\]
sends \( \{L/K\} \) to \( \{H/K\} \) for any \( K \subset L \).

The restriction map
\[
R_L^H : A(H) \to A(L)
\]
sends \( \{H/N\} \) to itself, but viewed as an \( L \)-space.

When \( L = g^{-1}Hg \), we also have an isomorphism
\[
C_L^H : A(L) \to A(H)
\]
sending \( \{L/K\} \) to \( \{H/gKg^{-1}\} \) for any \( K \subset L \).

The Cartesian product of finite \( G \)-sets makes \( A(G) \) into a ring. This multiplicative structure also makes \( A_G \) into a Green functor, which is a monoid under the box product.

**Proposition 2.9.** For a \( G \)-CW complex \( X \), the integer graded homology of \( X \) with coefficients in \( A_G \) can be computed as
\[
H_*^G(X; A_G) \cong \bigoplus_H H_*(X^H/W_GH; \mathbb{Z})
\]
with one \( H \) chosen from each conjugacy class of subgroups of \( G \). Here \( W_GH \) is the Weyl group of \( H \) in \( G \).

As a Mackey functor,
\[
H_*^G(X; A_G)(G/L) \cong H_*^L(X; A_L) \cong \bigoplus K H_*(X^K/W_LK; \mathbb{Z})
\]
with one \( K \) chosen from each conjugacy class of subgroups of \( L \).
Proof: It suffices to prove the first equation. The second one can be proved by the same argument.

When we view $A_G$ as a covariant coefficient system, the restriction maps are removed. For any $G$-map $G/L_1 \to G/L_2$ induced by the multiplication of $g \in G$ (where we require $g^{-1} L_1 g \subset L_2$), we have an induced map

$$A(L_1) \to A(L_2)$$

sending each $\{L_1/H\}$ to $\{L_2/g^{-1} H g\}$. Therefore, as a coefficient system, $A_G$ can be decomposed as

$$A_G = \bigoplus_H A^H_G$$

with one $H \subset G$ chosen from each conjugacy class. Here $A^H_G$ is the sub-coefficient system of $A_G$ such that each $A^H_G(G/L) \subset A(L)$ is generated by all $\{L/g^{-1} H g\}$ with $g^{-1} H g \subset L$.

Now we have

$$H^*_G(X; A_G) \cong \bigoplus_H H^*_G(X; A^H_G).$$

It suffices to prove

$$H^*_G(X; A^H_G) \cong H_*(X^H/W_G H; \mathbb{Z})$$

for any $H \subset G$.

As described in [Wi75], the equivariant homology can be computed in a cellular way:

Let $\underline{C}_*(X)$ be the chain complex of contravariant coefficient systems such that

$$\underline{C}_*(X)(G/H) := C_*(X^H),$$

where $C_*(X^H)$ is the cellular chain complex of $X^H$ and the orbit maps are sent to conjugacies and inclusions of subcomplexes. For any covariant coefficient system $M$, $H^*_G(X; M)$ is the homology of the chain complex:

$$C^*_G(X; M) := \underline{C}_*(X) \otimes \mathcal{O}_G M,$$

where the tensor product is taken over the category $\mathcal{O}_G$ of $G$-orbits. To be more precise, we have

$$C^*_G(X; M) := \left( \bigoplus_{L \subset G} C_*(X^L) \otimes M(G/L) \right) / \sim.$$

For any $G$-map $f : G/L_1 \to G/L_2$, the equivalence relation identifies $f^* a \otimes b$ and $a \otimes f_* b$ for all $a \in C_*(X^L_1)$ and $b \in M(G(L_1))$.

Now we choose $M = A^H_G$. Since each $M(G/L)$ is freely generated by $\{L/g^{-1} H g\}$ and the structure maps send these basis elements to each other, $C^*_G(X; M)$ is the free $\mathbb{Z}$-module generated by the equivalence classes of $e \otimes \{L/g^{-1} H g\}$, for all cells $e$ in $X^L$ and $g^{-1} H g \subset L$.

We can eliminate the equivalence relation by the following four facts:

1. The free $\mathbb{Z}$-module $A^H_G(G/H)$ is generated by a single element $\{H/H\}$.
2. For any $e \otimes \{L/g^{-1} H g\}$, it is identified with $ge \otimes \{H/H\}$ by the equivalence relation.
3. If there exists another $g' \in G$ such that $ge \otimes \{H/H\} \sim e \otimes \{L/g^{-1} H g\} \sim g'e \otimes \{H/H\}$, then $\{L/g^{-1} H g\} = \{L/(g')^{-1} H g'\}$. Thus
   $$g^{-1} H g = l^{-1}((g')^{-1} H g')l$$
   for some $l \in L$. So $g' l g^{-1} \in W_G H$. Since $e$ is $L$-fixed, we have $(g' l g^{-1})(ge) = g'e$. Thus $ge$ and $g'e$ are in the same $W_G H$-orbit.
4. The converse of (3) is also true: $e_1 \otimes \{H/H\}$ and $e_2 \otimes \{H/H\}$ are identified if $e_1, e_2$ are in the same $W_G H$-orbit.

Now we have a 1-1 correspondence between the basis of $C^*_G(X; A^H_G)$ and the cells in $X^H/W_G H$. Therefore, we get

$$C^*_G(X; A^H_G) \cong C_*(X^H/W_G H).$$

Taking the homology on both sides gives us the required equation. □
Remark 2.10. Proposition 2.9 follows from the fact that the underlying coefficient system of $A_G$ splits into $A_H^G$. However, such splitting cannot be lifted to the Mackey functor level. Thus the decomposition does not work for the cohomology with coefficients in $A_G$.

In fact, since we cannot define orbit spectra on the complete $G$-universe, 

$$X \mapsto H_*(X^H/W_GH)$$

is a homology theory only for $G$-spaces, and hence cannot be represented by any $G$-spectrum.

2.3 Homology of $E_F$ in degree 0
Since $E_F$ only has empty and contractible fixed point subspaces, Proposition 2.9 helps us to compute the 0th degree equivariant homology of $E_F$ explicitly:

Proposition 2.11. The trivial map $E_F \to \{\ast\}$ implies an inclusion

$$H_G^0(E_F; A_G) \hookrightarrow H_G^0(\ast; A_G) \cong A_G.$$ 

The image of $H_G^0(E_F, A_G)(G/L)$ in $A(L)$ is generated by all $\{L/K\} \in A(L)$ with $K \in F$.

We already have a small splitting here:

Lemma 2.12. When $|G|$ is inverted, $M_F := H_G^0(E_F; A_G)$ is a direct summand of $A_G$ as Mackey functors.

Proof: Assume that $|G|$ is inverted everywhere. For any $H \subset G$, define a linear map

$$\chi_H : A(G) \to \mathbb{Z}$$

which sends each $G$-set $S$ to $|S^H|$.

Let $s_{(K,H)}$ denote $\chi_K(\{G/H\}) = |G/H|^K$, which can be computed as the product between $|W_GH|$ and the number of subgroups of $G$ containing $K$ and in the conjugacy class of $H$. Thus $s_{(K,H)} \neq 0$ if and only if $K$ is sub-conjugate to $H$.

When $|G|$ is inverted, all $|W_GH|$ and non-zero $s_{(K,H)}$ become invertible. We choose elements $e_H \in A(G)$ inductively by defining

$$e_H = |W_GH|^{-1}\left(|G/H| - \sum_K s_{(K,H)}^{-1} e_K\right)$$

with one $K$ chosen from each conjugacy class that contains a proper subgroup of $H$.

By induction, we have

$$\chi_K(e_H) = \begin{cases} 1, & \text{if } K \text{ is conjugate to } H \\ 0, & \text{otherwise} \end{cases}$$

Since the dimension of $A(G)$ agrees with the number of conjugacy classes, the collection of $e_H$, with one $H$ chosen from each conjugacy class, forms a basis of $A(G)$.

In general, for any $H \subset L \subset G$, define a linear map

$$\chi_H^L : A(L) \to \mathbb{Z}$$

which sends each $L$-set to the size of its $H$-fixed subset. We can get a similar basis $\{e_H^L\}$, with one $H$ chosen from each conjugacy class of subgroups of $L$, such that

$$\chi_K^L(e_H^L) = \begin{cases} 1, & \text{if } K \text{ is conjugate to } H \text{ in } L \\ 0, & \text{otherwise} \end{cases}$$

Now we discuss how these maps interact with transfer and restriction maps:

For any $K \subset L_1 \subset L_2 \subset G$, the restriction map $R_{L_1}^{L_2}$ keeps each $L_2$-set but views it as an $L_1$-set, hence does not change its $K$-fixed subset. So we have

$$\chi_K^{L_2} = \chi_K^{L_1} \circ R_{L_2}^{L_1}.$$
On the other hand, for any $K \subset L_1$, the elements in $\{L_1/K\}$ have isotropy groups conjugate to $K$ inside $L_1$. The elements in $T^{L_2}_{L_1}(\{L_1/K\}) = \{L_2/K\}$ have isotropy groups conjugate to $K$ inside $L_2$. There may be more isotropy groups. But these additional isotropy groups are still chosen from the conjugacy class of $K$ inside $G$. According to the construction of the basis element $e^L_H$, we have

$$\chi_H^{L_2}(T^{L_2}_{L_1}e^L_H) \neq 0 \text{ only if } H \text{ is conjugate to a subgroup of } K \text{ in } G$$

Let

$$N(L) := \bigcap_{H \subseteq L} \ker \chi_H^L$$

In other words, $N(L)$ is generated by $e^L_H$ for all $H \subset L$ and $H \notin \mathcal{F}$. The above discussion tells us that $N(L)$ is closed under transfer and restriction maps. Thus we get a sub-Mackey functor whose value at $G/L$ agrees with $N(L)$. Denote that as $N_\mathcal{F}$.

Notice that for each $L \subset G$, $e^L_H \in N(L) = N_\mathcal{F}(G/L)$ if $H \notin \mathcal{F}$, $e^L_H \in M_\mathcal{F}(G/L)$ if $H \in \mathcal{F}$, according to (Proposition 2.11). Moreover, consider any nontrivial element

$$a_1\{L/K_1\} + a_2\{L/K_2\} + \ldots + a_n\{L/K_n\} \in M_\mathcal{F}(G/L)$$

with $a_1, a_2, \ldots, a_n \neq 0$, $K_1, K_2, \ldots, K_n$ in different conjugacy classes in $\mathcal{F}$, and $|K_1| \leq |K_2| \leq \ldots \leq |K_n|$. The map $\chi_{K_n}^L$ sends \{I/K_1\}, \ldots, \{L/K_{n-1}\} to zero but \{L/K_n\} to a positive value. Thus this element is not in $\ker \chi_{K_n}^L$, and hence $M_\mathcal{F}(G/L) \cap N(L) = \emptyset$.

In conclusion, we have $A_G = M_\mathcal{F} \oplus N_\mathcal{F}$ and $M_\mathcal{F}$ appears as a direct summand.

(Proposition 2.11)

**Remark 2.13.** When the multiplicative structure is added into consideration, the maps $\chi_{K_n}^L$ appear as the components of the ring isomorphism from $A(L)$ to several copies of $\mathbb{Z}[[G]]^\times$, with one $H$ chosen from each conjugacy class inside $L$. The generators $e_{H}^{L}$ become idempotent elements. So we can view $M_{\mathcal{F}}$ as a direct summand of $A_{G}$ as Green functors.

### 2.4 Homology of $E\mathcal{F}$ in positive degrees

In positive degrees, we have:

**Theorem 2.14.** For any family $\mathcal{F}$, the $\mathbb{Z}$-graded, Mackey functor valued homology

$$H^G_\mathcal{F}(E\mathcal{F}; A_G)$$

contains only torsion when the degree is positive. Moreover, the torsion only has prime factors which divide $|G|$.

This is the most important property we want about $E\mathcal{F}$. We will use the rest of this section to prove it.

The main idea is an induction on the size of $\mathcal{F}$. For the base case, we have

**Lemma 2.15.** Let $BG = EG/G$ be the classifying space of principal $G$-bundles. Then $H_\ast(BG; \mathbb{Z})$ contains only torsion when the degree is positive. Moreover, the torsion only has prime factors which divide $|G|$.

This is a standard result about $EG$. We give one possible proof below.

**Proof:** Give $EG$ the standard $G$-CW structure which only contains $G$-free cells. Consider the map between cellular chain complexes induced by the projection $EG \to EG/G = BG$:

$$C_\ast(EG) \to C_\ast(BG)$$

Define another map between chain complexes

$$C_\ast(BG) \to C_\ast(EG)$$

which sends each cell $e$ in $BG$ to the sum of all cells in $EG$ which are sent to $e$ under $EG \to BG$. Then the composition

$$C_\ast(BG) \to C_\ast(EG) \to C_\ast(BG)$$

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is the multiplication by $|G|$ since all cells in $EG$ are $G$-free. The same $|G|$-multiplication passes to homology:

$$H_*(BG) \to H_*(EG) \to H_*(BG).$$

Since $EG$ is contractible, $H_*(EG) = \mathbb{Z}$, which is concentrated in degree 0. Thus when the degree is positive, $H_*(BG)$ contains only torsion which divides $|G|$. □

In the case when $\mathcal{F}$ only contains the trivial subgroup, $E\mathcal{F}$ becomes $EG$ and only has orbit spaces homotopic to $BH$ for $H \subset G$. Lemma 2.15 and Proposition 2.9 imply that it satisfies Theorem 2.14.

For general $\mathcal{F}$, we can construct $E\mathcal{F}$ as a join. More details about joins and their properties are given in Appendix A.

**Proposition 2.16.** The join of $G \times_{N_G H} EW_G H$, with one $H$ chosen from each conjugacy class in $\mathcal{F}$, is a valid construction for $E\mathcal{F}$. Here $N_G H$ is the normalizer of $H$ in $G$. We view $EW_G H$ as a $N_G H$-space. Define the product with $N_G H$ left-acting on $EW_G H$ and right-acting on $G$.

**Proof:** Notice that $E\mathcal{F}$ is characterized by its fixed point subspaces:

$$(E\mathcal{F})^K \simeq \begin{cases} \ast, & \text{if } K \in \mathcal{F}, \\ 0, & \text{if } K \notin \mathcal{F} \end{cases}$$

According to Lemma A.3. it suffices to show:

(a) If $K \in \mathcal{F}$, $(G \times_{N_G H} EW_G H)^K$ is contractible for some $H \in \mathcal{F}$.

(b) If $K \notin \mathcal{F}$, $(G \times_{N_G H} EW_G H)^K$ is empty for all $H \in \mathcal{F}$.

Consider any point in $G \times_{N_G H} EW_G H$ expressed by $a \times x$ for $a \in G$ and $x \in EW_G H$. For any $g \in G$, if the action of $g$ fixes the point $a \times x$:

$$g(a \times x) = ga \times x = a \times x,$$

then there is $h \in N_G H$, such that $hx = x$, $gah^{-1} = a$. Since all points in $EW_G H$ have isotropy group $H$, we have $h \in H$. Then $g = aha^{-1} \in aH a^{-1}$.

Therefore, the isotropy group of $a \times x$ is $aH a^{-1}$. Thus the $K$-fixed point subspace of $G \times_{N_G H} EW_G H$ is non-empty if and only if $K$ is sub-conjugate to $H$. Moreover, when $K = gHg^{-1}$, we have

$$(G \times_{N_G H} EW_G H)^K = \{g\} \times EW_G H,$$

which is contractible. The two conditions (a), (b) follows naturally. □

**Remark 2.17.** Since all points in $G \times_{N_G H} EW_G H$ have isotropy groups conjugate to $H$, according to Definition A.1, the collection of isotropy groups of all points in $E\mathcal{F}$ agrees with $\mathcal{F}$.

**Proof of Theorem 2.14** Assume that Theorem 2.14 is true for all families smaller than $\mathcal{F}$. Let $\mathcal{F}'$ be a smaller family which is obtained by removing the conjugacy class of one largest subgroup $H \in \mathcal{F}$. According to Proposition 2.16 we have

$$E\mathcal{F} \simeq E\mathcal{F}' \ast (G \times_{N_G H} EW_G H).$$

For any $K < L \subset G$, the discussion in the proof of Proposition 2.16 shows that the space $(G \times_{N_G H} EW_G H)^K$ is either empty or several copies of $EW_G H$. Since all points in each copy of $EW_G H$ have the same isotropy group, $(G \times_{N_G H} EW_G H)^K/L$ is either empty or the disjoint union of $BN$ for some subgroups $N \subset N_G H \subset G$. Lemma 2.15 shows that its positive degree homology only contains torsion dividing $|G|$. Thus Proposition 2.9 shows that $G \times_{N_G H} EW_G H$ satisfies Theorem 2.13.

Since $E\mathcal{F}'$ satisfies Theorem 2.14 according to Lemma A.3, now it suffices to prove the theorem for $E\mathcal{F}' \ast (G \times_{N_G H} EW_G H)$. Using $X(H)$ to denote $G \times_{N_G H} EW_G H$, the homology of $E\mathcal{F}' \times X(H)$ can be computed by the equivariant Künneth spectral sequence [LMD]:

$$E_{p,q}^2 = Tor_{p,q}^A (H_\ast^G (E\mathcal{F}'; A_G), H_\ast^G (X(H); A_G)) \Rightarrow H_\ast^G (E\mathcal{F}' \times X(H); A_G).$$
When we invert $|G|$, the homology of both $E\mathcal{F}'$ and $X(H)$ is concentrated in degree 0. Moreover, $H^G_{G}(E\mathcal{F}'; A_G) = M_{\mathcal{F}}$ appears as a direct summand of $A_G$ by Lemma 2.12 Thus the $E_2$-page collapses into a single box product:

$$E^2 = E_{0,0}^2 = M_{\mathcal{F}} \boxpl H^G_{G}(X(H); A_G).$$

Therefore, $E\mathcal{F}' \times X(H)$ has trivial homology in positive degrees when $|G|$ is inverted. □

For special choices of $\mathcal{F}$, the torsion in $H^G_{G}(E\mathcal{F}, A_G)$ may not see all prime factors of $|G|$. We emphasize the following criterion which is implied by Proposition 2.9:

**Proposition 2.18.** The torsion types in $H^G_{G}(E\mathcal{F}, A_G)$ come from $H^*(E\mathcal{F} K/L)$ for all $K \triangleleft L \subset G$.

### 3 Splitting cofiber sequences

Let $\mathcal{F}$ be a family of subgroups of $G$. Consider the cofiber sequence

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}$$

where the map $E\mathcal{F}_+ \to S^0$ sends $E\mathcal{F}$ into the non-basepoint of $S^0$.

#### 3.1 A criterion on splitting

**Theorem 3.1.** When certain prime factors of $|G|$ are inverted, the cofiber sequence above splits the category of $G$-spectra, in the sense that

$$X \simeq (E\mathcal{F}_+ \wedge X) \vee (\widetilde{E}\mathcal{F} \wedge X),$$

$$[E\mathcal{F}_+ \wedge X, \widetilde{E}\mathcal{F} \wedge Y]^G = [E\mathcal{F} \wedge Y, E\mathcal{F}_+ \wedge X]^G = 0$$

for any $G$-spectra $X, Y$. Moreover, the functors

$$X \mapsto E\mathcal{F}_+ \wedge X, \quad X \mapsto \widetilde{E}\mathcal{F} \wedge X$$

are symmetric monoidal.

The proof will give information about which primes must be inverted; see Remark 3.4.

**Remark 3.2.** In our later computations, we will focus on the case when $\mathcal{F}$ consists of all subgroups whose orders only contain prime factors in $T$. Here $T$ is an arbitrary set of prime factors of $|G|$. **Theorem 3.6** will show that it suffices to invert all prime factors not in $T$.

Consider the suspension of the cofiber sequence:

$$\Sigma^\infty E\mathcal{F}_+ \to S \to \Sigma^\infty \widetilde{E}\mathcal{F}.$$

**Theorem 3.3.** When $|G|$ is inverted, there exists a left inverse of $\Sigma^\infty E\mathcal{F}_+ \to S$.

**Proof:** According to Proposition 2.11, the map $\Sigma^\infty E\mathcal{F}_+ \to S$ induces an inclusion in homology:

$$\text{HA}_G(\Sigma^\infty E\mathcal{F}_+) =: M_{\mathcal{F}} \hookrightarrow \text{HA}_G S = A_G.$$

When $|G|$ is inverted, **Theorem 2.14** shows that the $\text{HA}_G$-homology is concentrated in degree 0. Thus it suffices to find a map $S \to \Sigma^\infty E\mathcal{F}_+$ which induces a projection from $A_G$ to $M_{\mathcal{F}}$ on their $\text{HA}_G$-homology. After composition with $\Sigma^\infty E\mathcal{F}_+ \to S$, we get a self-map of $\Sigma^\infty E\mathcal{F}_+$ that induces an isomorphism on $\text{HA}_G$-homology. Since $\Sigma^\infty E\mathcal{F}_+$ is a connective spectrum, this self map must be a weak equivalence.

For each $H \in \mathcal{F}$, the transfer and restriction maps in Mackey functors between images of $G/G$ and $G/H$ are induced by stable maps

$$T : S = \Sigma^\infty S^0 = \Sigma^\infty G/G_+ \to \Sigma^\infty G/H_+$$

$$R : \Sigma^\infty G/H_+ \to \Sigma^\infty G/G_+ = S.$$
$R$ is induced by the space-level map $G/H_+ \to S^0$, which sends $G/H$ into the non-basepoint of $S^0$. According to Remark 2.17, $G/H$ can be mapped into $E\mathcal{F}$. Thus $R$ is factorized as

$$\Sigma^\infty G/H_+ \to \Sigma^\infty E\mathcal{F}_+ \to S.$$  

Let $\iota_H$ be the composition of the first map above and $T$:

$$\iota_H : S \xrightarrow{T} \Sigma^\infty G/H_+ \to \Sigma^\infty E\mathcal{F}_+.$$  

The composition

$$S \xrightarrow{\iota_H} \Sigma^\infty E\mathcal{F}_+ \to S$$  

agrees with $R \circ T$, which is the equivariant Euler characteristic of $G/H$. We can fully describe the self-map on

$$HA_G S \cong A_G \cong \mathbb{Z}[S]$$  

induced by the Euler characteristic as follows:

For each $L \subset G$, the induced self-map on $A_G(G/L) = A(L)$ is the multiplication by the underlying $L$-set of $\{G/H\}$ (which is defined in Definition 2.8). We refer to [LMS86], V.1 and V.2 for more details.

Now we choose one $H$ from each conjugacy class in $\mathcal{F}$. Assign a number $c_H \in \mathbb{Z}[[G^{-1}]]$ for each such $H$. Consider the map

$$\sum_H c_H \iota_H : S \to \Sigma^\infty E\mathcal{F}_+.$$  

The composition

$$S \xrightarrow{\sum_H c_H \iota_H} \Sigma^\infty E\mathcal{F}_+ \to S$$  

induces a self-map on $A_G(G/L) = A(L)$ as the multiplication by $\sum_H c_H \{G/H\}$.

Recall that $HA_G(\Sigma^\infty E\mathcal{F}_+)(G/L) = M_{\mathcal{F}}(G/L) \subset A(L)$ is generated by all $\{L/J\}$ with $J \in \mathcal{F}$. The statement that $\sum_H c_H \iota_H$ induces the projection $A_G \to M_{\mathcal{F}}$ is equivalent to

$$\{L/J\} \cdot \sum_H c_H \{G/H\} = \{L/J\}$$

for any $J \in \mathcal{F}$, $J \subset L$.

Recall the ring map $\chi_K^L : A(L) \to \mathbb{Z}$ for any $K \subset L$ which sends each $L$-set to the number of $K$-fixed points. According to Remark 2.13, $\prod_{K \subset L} \chi_K^L$ is an injective ring homomorphism from $A(L)$ to copies of $\mathbb{Z}$. Thus it suffices to check the equation above on the images under each $\chi_K^L$. The equation above under $\chi_K^L$ becomes

$$|(L/J)^K| \cdot \sum_H c_H |(G/H)^K| = |(L/J)^K|.$$  

Notice that $|(L/J)^K| = 0$ for any $K \notin \mathcal{F}$ (since $K$ is not sub-conjugate to $J \in \mathcal{F}$). Thus it suffices to choose $c_H$ such that

$$\sum_H c_H |(G/H)^K| = \sum_H c_H s_{(K,H)} = 1, \forall K \in \mathcal{F}$$

with one $H$ chosen from each conjugacy class in $\mathcal{F}$.

Recall that $s_{(K,H)}$ is non-zero if and only if $K$ is sub-conjugate to $H$. Denote this relation by $[K] \leq [H]$. We have

$$\sum_{[K] \leq [H] \subset \mathcal{F}} c_H s_{(K,H)} = 1.$$  

Thus

$$c_K = s_{(K,K)}^{-1} \left( 1 - \sum_{[K] \leq [H] \subset \mathcal{F}} c_H s_{(K,H)} \right).$$  

Since $s_{(K,K)} = |W_{G,K}|$ is invertible when $|G|$ is inverted, $c_K$ can be chosen inductively from larger subgroups to smaller ones. Moreover, it’s clear that the denominator of any $c_K$ only contains prime factors dividing $|G|$.

Now the map $\sum_H c_H \iota_H$ gives us the required projection on the 0th homology, hence becomes the left inverse of $\Sigma^\infty E\mathcal{F}_+ \to S$.  \[\square\]
Proof of Theorem 3.1: Assume that $|G|$ is inverted. The left inverse in Theorem 3.3 and the multiplicative structure on space $E\mathcal{F}$ make $\Sigma\infty E\mathcal{F}+$ into a ring spectrum. So we have a splitting cofiber sequence of ring spectra:

$$\Sigma\infty E\mathcal{F}+ \to S \to \Sigma\infty \tilde{E}\mathcal{F}.$$ 

Thus we have

$$X \simeq (E\mathcal{F}+ \wedge X) \cup (\tilde{E}\mathcal{F} \wedge X).$$

Since

$$E\mathcal{F}+ \wedge E\mathcal{F}+ \simeq E\mathcal{F}+$$

and

$$\tilde{E}\mathcal{F} \wedge \tilde{E}\mathcal{F} \simeq \tilde{E}\mathcal{F},$$

the functors

$$X \mapsto E\mathcal{F}+ \wedge X, \ X \mapsto \tilde{E}\mathcal{F} \wedge X$$

are symmetric monoidal.

Moreover, $E\mathcal{F}+ \wedge \tilde{E}\mathcal{F} \simeq *$ since all its fixed point subspaces are contractible. Use $A,B$ to denote the suspensions of $E\mathcal{F}+$ and $\tilde{E}\mathcal{F}$ (in either order). The universal coefficient spectral sequence \cite{LM04} tells us that for any $G$-spectra $X,Y$, we have

$$\text{Ext}^*_G(B^*(A \wedge X), (B \wedge Y)) \Rightarrow (B \wedge Y)^*(A \wedge X) = [A \wedge X, \Sigma^* B \wedge Y]^G.$$ 

Since

$$B_.(A \wedge X) = \underline{Z}.(B \wedge A \wedge X) = 0,$$

the spectral sequence has trivial $E_2$-page. Thus we have

$$[E\mathcal{F}+ \wedge X, \tilde{E}\mathcal{F} \wedge Y]^G = [\tilde{E}\mathcal{F} \wedge Y, E\mathcal{F}+ \wedge X]^G = 0.$$

\hfill \Box

Now we can get a criterion about which prime factors of $|G|$ to be inverted according to the proofs above:

Remark 3.4. It suffices to invert the prime factors of the denominators of all $c_H$, and the torsion in $H^G(E\mathcal{F}; A_G)$.

According to Proposition 2.18, the torsion in $H^G(E\mathcal{F}; A_G)$ comes from the torsion in $H_* (E\mathcal{F}^K / L)$ for all $K \trianglelefteq L \subset G$.

3.2 A splitting theorem

In our later computations, we will use the following type of families:

Definition 3.5. Let $T$ be a set of prime factors of $|G|$. Define $\mathcal{F}_T$ to be the family of all subgroups whose orders only contain prime factors in $T$.

We will prove the following splitting theorem in the rest of this section:

Theorem 3.6. Theorem 3.1 holds when $\mathcal{F} = \mathcal{F}_T$ and all prime factors of $|G|$ not in $T$ are inverted.

According to Remark 3.4, it suffices to prove:

Proposition 3.7. For any $H \trianglelefteq L \subset G$ and $m > 0$, $H_m (E\mathcal{F}^H / L) = 0$.

Proposition 3.8. Consider numbers $c_H$ for all $H \in \mathcal{F}$ determined by the following equations: For any $K \in \mathcal{F}$, we have

$$\sum_H c_H \cdot |(G/H)^K| = 1.$$

Here we choose one $H$ from each conjugacy class inside $\mathcal{F}$. Then the denominators of all $c_H$’s become invertible after we invert all prime factors of $|G|$ not in $T$.

Proof of Proposition 3.7. We divide the proof into three steps:
Step 1: We begin with a simple case: $T = \{p\}$, $L = G$, and $H$ is the trivial subgroup. We want to show that $H_m(E\mathcal{F}/G) = 0$ for all $m > 0$.

Let $P$ be any Sylow $p$-subgroup of $G$. Fix any $G$-CW structure on $E\mathcal{F}$, which induces CW structures on both $E\mathcal{F}/P$ and $E\mathcal{F}/G$. The natural map $E\mathcal{F}/P \to E\mathcal{F}/G$ induces a map between cellular chain complexes:

$$
\phi : C_*(E\mathcal{F}/P) \to C_*(E\mathcal{F}/G).
$$

Define a map between abelian groups in the opposite direction

$$
\psi : C_*(E\mathcal{F}/G) \to C_*(E\mathcal{F}/P)
$$

as follows:

For any $G$-cell $G/H \land e$ of $E\mathcal{F}$, use $e_P$ and $e_G$ to denote the corresponding cells in $E\mathcal{F}/P$ and $E\mathcal{F}/G$. Let $Pg_1, Pg_2, \ldots, Pg_k$ be all right cosets of $P$ in $G$. Define $\psi$ by sending each $e_G \in C_*(E\mathcal{F}/G)$ to $\sum_i (g_ie)_P \in C_*(E\mathcal{F}/P)$.

We can show that $\psi$ is well-defined by the following two facts:

1. If $g, g'$ are in the same right coset of $P$, then $(ge)_P = (g'e)_P$. Thus $\psi$ does not depend on the choice of $g_1, g_2, \ldots, g_k$.
2. For any $e, e'$, $e_G = e'_G$ if and only if $e' = ge$ for some $g \in G$. Since $g_1g, g_2g, \ldots, g_kg$ also cover all right cosets of $P$, we have $\sum_i (g_i(g)e)_P = \sum_i ((g_ig)e)_P = \sum_i (g_ie)_P$. Thus $\psi$ is uniquely defined on each element of $C_*(E\mathcal{F}/G)$.

Moreover, $\psi$ commutes with the boundary maps:

We use $d, d_P, d_G$ to denote the cellular boundary maps for $E\mathcal{F}, E\mathcal{F}/P, E\mathcal{F}/G$ respectively. For any $G$-cell $G/H \land e$ of $E\mathcal{F}$, write

$$
d e = \sum_t \left( \sum_{g \in G} n_g \right) e_t = \sum_t \left[ \sum_i \left( \sum_{a \in P} n_{ag,a} \right) \right] e_t
$$

where the cells $e_t$ are in different $G$-orbits and $n_g \in \mathbb{Z}$. Then we have

$$
d_G e_G = \sum_t \left( \sum_{g \in G} n_g \right) \left( e_t \right)_G
$$

$$
d_P e_P = \sum_{t,i} \left( \sum_{a \in P} n_{ag,a} \right) \left( g_i e_t \right)_P
$$

Notice that

$$
d(\sum g e) = \sum_{g \in G} g(de) = \sum_t \left( \sum_{g \in G} n_g \right) \left( \sum_{g \in G} g \right) e_t = \sum_t \left[ \sum_i \left( \sum_{a \in P} n_{ag,a} \right) \right] e_t.
$$

Thus we have

$$
d_P(\sum_{g \in G} g e)_P = |P| \sum_{t,i} \sum_{g \in G} n_g(g_i e_t)_P.
$$

Since $\sum_{g \in G} (g e)_P = |P| \sum_{t,i} (g_i e)_P$, we have

$$
d_P(\sum_{i} (g_i e)_P) = \frac{1}{|P|} d_P(\sum_{g \in G} g e)_P = \sum_{t,i} \sum_{g \in G} n_g(g_i e_t)_P.
$$

Therefore, we get

$$
d_P(\psi e_G) = d_P(\sum_{i} (g_i e)_P) = \sum_{t,i} \sum_{g \in G} n_g(g_i e_t)_P
$$

$$
= \sum_{t} \sum_{g \in G} n_g(\sum_{i} (g_i e_t)_P) = \sum_{t} \sum_{g \in G} n_g \psi((e_t)_G) = \psi(d_G e_G).
$$

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Now $\psi$ becomes a morphism between chain complexes. Notice that $\phi \circ \psi$ is multiplication by $|G/P|$. Thus the induced map on homology

$$H_*(E\mathcal{F}/G) \xrightarrow{\psi_*} H_*(E\mathcal{F}/P) \xrightarrow{\phi_*} H_*(E\mathcal{F}/G)$$

is also multiplication by $|G/P|$, which is an isomorphism since $|G/P|$ is invertible.

Notice that $E\mathcal{F}^K \simeq *$ for any $p$-subgroup $K$. Thus $E\mathcal{F}$ is contractible as a $P$-space, and hence $H_m(E\mathcal{F}/P) = 0$ for all $m > 0$. The maps above imply that $H_m(E\mathcal{F}/G) = 0$ for all $m > 0$.

**Step 2:** We still assume $T = \{p\}$ but put no restrictions on $H, L$.

View $E\mathcal{F}^H$ as an $L/H$-space. Then for any $K \subset L/H$, $(E\mathcal{F}^H)^K = E\mathcal{F}^H \rtimes K$. It is contractible when both $H,K$ are $p$-groups, or empty otherwise. Thus $E\mathcal{F}^H$ is either empty or agrees with the $L/H$-universal space corresponding to the family of all $p$-subgroups of $L/H$. By induction on the size of the group, we can show that

$$H_m(E\mathcal{F}^H/L) = H_m(E\mathcal{F}^H/(L/H)) = 0$$

for all $m > 0$.

**Step 3:** Now we consider the general case $T = \{p_1,p_2,...,p_n\}$.

For any $i \in \{1,2,...,n\}$, let $P_i$ be a Sylow-$p_i$ subgroup of $G$. Then with the same argument as in step 1, there exist maps between chain complexes

$$C_*(E\mathcal{F}/G) \rightarrow C_*(E\mathcal{F}/P_i) \rightarrow C_*(E\mathcal{F}/G)$$

such that the induced map on homology

$$H_*(E\mathcal{F}/G) \rightarrow H_*(E\mathcal{F}/P_i) \rightarrow H_*(E\mathcal{F}/G)$$

is multiplication by $|G/P_i|$. Since $F$ contains all $p_i$-subgroups, $E\mathcal{F} \simeq *$ as a $P_i$-space. Thus $H_m(E\mathcal{F}/P_i) = 0$ for $m > 0$ and hence $|G/P_i| \cdot H_m(E\mathcal{F}/G) = 0$ for all $m > 0$.

Letting $i$ vary among $1,2,...,n$, we get

$$([G/P_1]|G/P_2|,...,|G/P_n|) \cdot H_m(E\mathcal{F}/G) = 0$$

for all $m > 0$. Since all prime factors of $G$ except $p_1,p_2,...,p_n$ are inverted, $([G/P_1]|G/P_2|,...,|G/P_n|)$ is inverted. Thus $H_m(E\mathcal{F}/G) = 0$ for any $m > 0$.

Using the same method as in step 2, we can prove $H_m(E\mathcal{F}^H/L) = 0$ for any $H \triangleleft L \subset G$ and $m > 0$. □

We now prove Proposition 3.8. Write $T = \{p_1,...,p_n\}$.

For any $H \subset G$, recall the ring map $\chi_H : A(G) \rightarrow \mathbb{Z}$ defined in the proof of Lemma 2.12 which sends each $G$-set $S$ to $|S|^H$. Let $C(G) := \prod_{[H]} \mathbb{Z}$. Then we have a ring map

$$\chi = \prod_{[H]} \chi_H : A(G) \rightarrow C(G).$$

The proof of Lemma 2.12 implies that $\chi$ is a ring isomorphism when $|G|$ is inverted. If we only invert the prime factors not in $T$, $\chi$ becomes a monomorphism and $C(G)/Im(\chi)$ only contains torsion of $p_1,...,p_n$.

We want to show that the denominators of all $c_H$’s are invertible, which is equivalent to the existence of the element $\sum_H c_H(G/H)$ in $A(G)$, when all prime factors of $|G|$ not in $T$ are inverted. Since $\chi$ is a monomorphism, this existence can be detected by the image of $\chi$:

$$\chi_K(\sum_H c_H(G/H)) = \sum_H c_H|(G/H)^K| = \begin{cases} 1, & \text{if } K \in \mathcal{F} \\ 0, & \text{otherwise} \end{cases}$$
In other words, it suffices to show that the element \( \alpha = (\alpha_H)_{|H|} \in C(G) \), which is defined as

\[
\alpha_H = \begin{cases} 
1, & \text{if } K \in \mathcal{F}, \\
0, & \text{otherwise}, 
\end{cases}
\]

is inside the image of \( \chi \).

We will need the following lemma:

**Lemma 3.9.** For any \( H \in \mathcal{F} \) and \( p \in T = \{p_1, p_2, \ldots, p_n\} \), there exists \( L \in \mathcal{F} \), such that \( \chi_H(G/L) \) is not divided by \( p \).

**Proof:** Without loss of generality, we assume that \( p = p_1 \).

Let \( M \) be the subgroup of \( H \) generated by all elements whose order is a power of one of \( p_2, p_3, \ldots, p_n \).

We observe that \( M \) is a normal subgroup of \( H \) since conjugations do not change the order of an element. Thus \( H \subseteq N_{G,M} \).

Write \( H = Q \ltimes M \) and \( N_{G,M} = W_{G,M} \ltimes M \) with \( Q \subset W_{G,M} \).

For any \( h \in H \), since \( H \in \mathcal{F} \), the order of \( h \) must have the form \( |h| = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \). For \( i = 2, 3, \ldots, n \), \( h^{|h|/p_1^{a_1}} \) has a \( p_1 \)-power order, hence is contained in \( M \). Since \( |h|/p_2^{a_2}, \ldots, |h|/p_n^{a_n} \) have the greatest common divisor \( p_1^{a_1} \), we have \( h^{p_1} \in M \). Therefore, \( H/M = Q \) is a \( p_1 \)-group.

Choose \( P \in W_{G,M} \) to be a Sylow-\( p_1 \) group containing \( Q \). Let \( L = P \ltimes M \supset H \). Then \( L \in \mathcal{F} \) since both \( |P| \) and \( |M| \) only contain prime factors inside \( \{p_1, \ldots, p_n\} \). We will show that \( \chi_H(G/L) \) is not divided by \( p_1 \).

Recall that \( \chi_H(G/L) = |(G/L)^H| \). For any \( g \in G \), \( gL \in G/L \) is fixed by \( H \) if and only if \( g^{-1}Hg \subset L \). If this happens, then \( g^{-1}Mg \subset L = P \ltimes M \).

Notice that \( M \) is generated by elements of prime power order for primes inside \( \{p_2, \ldots, p_n\} \). In other words, \( M \) is generated by its Sylow-\( p_i \)-subgroups, for \( i = 2, 3, \ldots, n \).

Let \( T \) be any Sylow \( p_i \)-subgroup of \( M \), with \( i \in \{2, 3, \ldots, n\} \). Then \( g^{-1}Tg \subset g^{-1}Mg \subset L \). Since \( |L| = |P| \cdot |M| \) and \( |P| \) is a \( p_1 \)-power, both \( T \) and \( g^{-1}Tg \) are Sylow-\( p_i \) groups of \( L \), hence are conjugate to each other inside \( L \). Since \( M \) is a normal subgroup of \( L \) and \( T \subset M \), we get \( g^{-1}Tg \subset M \). Since \( M \) is generated by its Sylow-\( p_2, \ldots, p_n \) groups, we have \( g^{-1}Mg = M \), and hence \( g \in N_{G,M} \).

When \( gL \in (G/L)^H \), we have \( g \in N_{G,M} \) and \( gL \in (N_{G,M}/L)^H \). Thus

\[
\chi_H(G/L) = |(G/L)^H| = |(N_{G,M}/L)^H| = |(W_{G,M}/P)^Q|.
\]

Recall that \( P \) is a Sylow-\( p_1 \) group of \( W_{G,M} \) containing \( Q \). Thus \( |W_{G,M}/P| \) is not divisible by \( p_1 \). Since the sizes of all \( Q \)-orbits except \( Q/Q \) are divided by \( p_1 \), the number of \( Q/Q \) orbits in \( W_{G,M}/P \) is not divided by \( p_1 \). Thus \( |(W_{G,M}/P)^Q| \) is not divided by \( p_1 \).

In conclusion, \( \chi_H(G/L) \) is not divided by \( p_1 \). \( \square \)

**Proof of Proposition 3.8** Since \( C(G) \) has finite rank and \( C(G)/\text{Im}(\chi) \) only contains torsion of \( p_1, \ldots, p_n \), there exist \( k_1, k_2, \ldots, k_n \in \mathbb{Z}^+ \), such that \( p_1^{k_1} \cdots p_n^{k_n}C(G) \subset \text{Im}(\chi) \).

We divide our proof into three steps:

**Step 1:** For any \( H \in \mathcal{F} \) and \( i \in \{1, \ldots, n\} \), there exists an element \( \beta(H,i) \in \text{Im}(\chi) \), such that, modulo \( p_i^{k_i} \), we have

\[
\beta(H,i)K \equiv \begin{cases} 
1, & \text{if } K \in [H] \\
1 \text{ or } 0, & \text{if } K \in \mathcal{F} \\
0, & \text{if } K \notin \mathcal{F}
\end{cases}
\]

According to Lemma 3.9, there exists \( L \in \mathcal{F} \) such that \( \chi_H(G/L) \) is not divided by \( p_i \). Notice that \( \chi_K(G/L) = 0 \) for any \( K \notin \mathcal{F} \). Since \( |(\mathbb{Z}/p_i^{k_i})^\times| = p_i^{k_i-1}(p_i - 1) \), the image of \( \{G/L\}^{p_i^{k_i-1}(p_i - 1)} \) is a valid choice for \( \beta(H,i) \).
Step 2: For any $H \in \mathcal{F}$, there exists an element $\beta(H) \in \text{Im}(\chi)$, such that, modulo $p_1^{k_1} \cdots p_n^{k_n}$, we have

$$\beta(H)_K \equiv \begin{cases} 1, & \text{if } K \in [H] \\ 1 \text{ or } 0, & \text{if } K \notin \mathcal{F} \\ 0, & \text{if } K \notin \mathcal{F} \end{cases}$$

For $i = 1, 2, \ldots, n$, choose $c_i \in \mathbb{Z}$ such that $c_1 \equiv 1 \pmod{p_1^{k_1}}$ and $c_1 \equiv 0 \pmod{p_j^{k_j}}$ for all $j \neq i$. We only need to choose

$$\beta(H) = c_1 \beta(H, 1) + \ldots + c_n \beta(H, n).$$

Step 3: Define $a + \ast b = a + b - ab$ for any $a, b$ in the same ring. Notice that for any $a, b, c \in \mathbb{Z}$ such that $a, b \equiv 0 \text{ or } 1 \pmod{c}$, we have

$$a + \ast b \equiv \begin{cases} 0, & \text{if } a \equiv b \equiv 0 \\ 1, & \text{otherwise} \end{cases}$$

mod $c$.

Choose one subgroup from each conjugacy class inside $\mathcal{F}$ and list them as $H_1, H_2, \ldots, H_m$. Let

$$\beta = \beta(H_1) + \ast \beta(H_2) + \ast \ldots \ast \beta(H_m)$$

Then modulo $p_1^{k_1} \cdots p_n^{k_n}$, we have

$$\beta_K \equiv \begin{cases} 1, & \text{if } K \in \mathcal{F} \\ 0, & \text{if } K \notin \mathcal{F} \end{cases}$$

Since $\beta \in \text{Im}(\chi)$ and $p_1^{k_1} \cdots p_n^{k_n} \cdot C(G) \subset \text{Im}(\chi)$, the special element $a = (\alpha_H|_{[H]})$, which is defined as

$$\alpha_H = \begin{cases} 1, & \text{if } H \in \mathcal{F}, \\ 0, & \text{if } H \notin \mathcal{F}, \end{cases}$$

must be contained in $\text{Im}(\chi)$. $\square$

4 Splitting methods

In this section, we will set up our splitting method in the computation of $\pi_\bullet^G(HM)$.

The procedure of the splitting method can be explained as follows:

(I) Fix a prime factor $p$ of $|G|$. Invert all other prime factors.

(II) Choose proper families $\mathcal{F}$ such that Theorem 3.1 holds. We will always choose $\mathcal{F} = \mathcal{F}_T$ for some set $T$ of prime factors of $|G|$. According to Theorem 3.6, Theorem 3.1 holds if $p \in T$.

(III) Apply Theorem 3.1 to split $HM$ into multiple pieces. Usually we need to choose multiple families and apply the splittings at the same time.

(IV) Compute the homotopy of each piece and glue them together. Now we have computed the localized $\pi_\bullet^G(HM)$ when all prime factors of $|G|$ except $p$ are inverted.

(V) List all prime factors of $|G|$ as $p_1, p_2, \ldots, p_n$. For $i = 1, 2, \ldots, n$, we can compute

$$R_i := \pi_\bullet^G(HM)[p_1^{-1}, \ldots, p_i^{-1}, \ldots, p_n^{-1}]$$

by the four steps above with $p = p_i$.

Consider the diagram with objects $R_1, R_2, \ldots, R_n, \pi_\bullet^G(HM)[[G]^{-1}]$ and morphisms

$$R_i \to \pi_\bullet^G(HM)[[G]^{-1}], \quad i = 1, 2, \ldots, n$$

as localizations. This is a diagram inside the category of $RO(G)$-graded rings. The limit of the diagram is exactly $\pi_\bullet^G(HM)$.

The hardest part is the computation of each piece in step (IV). In fact, most of the computation can be decomposed into smaller subgroups of $|G|$. In order to make this idea clear, first we consider a special case $M = \mathbb{Z}$. 17
4.1 Splitting method for $H\mathbb{Z}$

All arguments in this subsection still work if $\mathbb{Z}$ is replaced by other constant Mackey functors.

When $M = \mathbb{Z}$, we have no actual splittings in step (III) no matter which $F$ is chosen:

**Proposition 4.1.** If [Theorem 3.3] holds, we have $E\mathcal{F} \wedge H\mathbb{Z} \simeq \ast$. Thus

$$H\mathbb{Z} \simeq E\mathcal{F}_+ \wedge H\mathbb{Z}.$$

**Proof:** Consider the equivariant universal coefficient spectral sequence:

$$E^2_{*,*} = \text{Tor}_{\mathcal{A}_G}^*(HA_G, E\mathcal{F}_+; \mathbb{Z}) \Rightarrow H\mathbb{Z} \wedge E\mathcal{F}.$$

According to the proof of Lemma 2.12 $HA_G E\mathcal{F}$ is concentrated in degree 0 and appears as the direct summand $N\mathcal{F}$ of $A_G$. Thus the $E^2$-page is trivial except

$$E^2_{0,0} \cong \mathbb{Z} \boxtimes N\mathcal{F}.$$

Since

$$\mathbb{Z} = \mathbb{Z} \boxtimes A_G = \mathbb{Z} \boxtimes M\mathcal{F} \oplus \mathbb{Z} \boxtimes N\mathcal{F},$$

there must be one $\mathbb{Z}$ and one 0 in $\{\mathbb{Z} \boxtimes M\mathcal{F}, \mathbb{Z} \boxtimes N\mathcal{F}\}$.

Notice that $M\mathcal{F}(G/e) = H_0(E\mathcal{F}; \mathbb{Z}) = \mathbb{Z}$. Thus

$$\mathbb{Z} \boxtimes M\mathcal{F}(G/e) = \mathbb{Z}(G/e) \otimes M\mathcal{F}(G/e) = \mathbb{Z}.$$

Now we get

$$\mathbb{Z} \boxtimes M\mathcal{F} = \mathbb{Z}, \mathbb{Z} \boxtimes N\mathcal{F} = 0.$$

Therefore, the $E^2$-page of the spectral sequence is trivial, hence we have $\widetilde{E}\mathcal{F} \wedge H\mathbb{Z} \simeq \ast$. □

The product with $E\mathcal{F}_+$ can be computed in a more general way:

**Proposition 4.2.** Let $\mathcal{F}$ be an arbitrary family. Invert proper prime factors of $|G|$ such that [Theorem 3.3] holds. Then for any $G$-spectrum $X$, we have

$$\pi^G_V(E\mathcal{F}_+ \wedge X) \cong \text{Map}(M\mathcal{F}, \pi^G_V(X)).$$

Here $\pi^G_V(X)$ is the Mackey functor valued homotopy which sends each $G/H$ to $\pi^H_{V|H}(X)$.

We need a small lemma on the homotopy of $E\mathcal{F}$ first:

**Lemma 4.3.** Assume that proper primes are inverted such that [Theorem 3.3] holds. Then

$$\pi_*^G(\Sigma^\infty E\mathcal{F}_+) \cong \pi_*^G(S) \boxtimes M\mathcal{F},$$

$$\pi_*^G(\Sigma^\infty \widetilde{E}\mathcal{F}) \cong \pi_*^G(S) \boxtimes N\mathcal{F}.$$

Here $S = \Sigma^\infty S^0$ is the sphere spectrum.

**Proof:** According to Theorem 3.1 we have an equivalence of ring spectra:

$$S \simeq \Sigma^\infty E\mathcal{F}_+ \vee \Sigma^\infty \widetilde{E}\mathcal{F}.$$

Thus the product in $\pi_*^G(S)$ is induced by the products in $\pi_*^G(\Sigma^\infty E\mathcal{F}_+)$ and $\pi_*^G(\Sigma^\infty \widetilde{E}\mathcal{F})$.

In particular, the map

$$\pi_*^G(S) \boxtimes \pi_*^G(S) \to \pi_*^G(S)$$

is given by

$$\pi_*^G(S) \boxtimes \pi_*^G(S) \cong (\pi_*^G(\Sigma^\infty E\mathcal{F}_+) \boxtimes \pi_*^G(\Sigma^\infty E\mathcal{F}_+)) \oplus (\pi_*^G(\Sigma^\infty \widetilde{E}\mathcal{F}) \boxtimes \pi_*^G(\Sigma^\infty \widetilde{E}\mathcal{F})).$$
Here the map $f$ is the projection onto the first two summands. On the other hand, the map

$$\mathbb{S}(S) \otimes \mathbb{S}(S) \rightarrow \mathbb{S}(S)$$

is an isomorphism since $\mathbb{S}(S) \cong AG$ is the unit Mackey functor. Thus the projection $f$ must be an isomorphism and hence

$$\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) \otimes \mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) = \mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) \otimes \mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) = 0.$$

Since $\Sigma^\infty_{E,\mathcal{F}_+}$ and $\Sigma^\infty_{E,\mathcal{F}_-}$ are connective, we have

$$\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) \cong \mu_0(\mathcal{F}_+; AG) = M_{\mathcal{F}},$$

$$\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_-}) \cong \mu_0(\mathcal{F}_-; AG) = N_{\mathcal{F}}.$$

Thus

$$\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) \otimes N_{\mathcal{F}} = \mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) \otimes M_{\mathcal{F}} = 0$$

and hence

$$\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}) \cong \mathbb{S}(S) \otimes M_{\mathcal{F}},$$

$$\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_-}) \cong \mathbb{S}(S) \otimes N_{\mathcal{F}}.$$

\[\square\]

### Proof of Proposition 4.2

According to Theorem 3.1, we have

$$\pi^G_0(E,\mathcal{F}_+ \wedge X) = [S^V, E, \mathcal{F}_+ \wedge X]^G \cong [E, \mathcal{F}_+, \Sigma^{-V}X]^G = X^{-V}E,\mathcal{F}_+.$$  

View $X$ as an $S$-module and apply the equivariant universal coefficient spectral sequence:

$$\text{Ext}^*_{\mathbb{S}(S)}(\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}), X^{*-V}) \Rightarrow X^{*-V}E,\mathcal{F}_+.$$  

According to Lemma 4.3 and Lemma 2.12, we have

$$\text{Ext}^*_{\mathbb{S}(S)}(\mathbb{S}(\Sigma^\infty_{E,\mathcal{F}_+}), X^{*-V}) \cong \text{Ext}^*_{\mathbb{S}(S)}(\mathbb{S}(S) \otimes M_{\mathcal{F}}, X^{*-V})$$

$$\cong \text{Ext}^*_{\mu_0(\mathcal{F}_+; AG)}(M_{\mathcal{F}}, X^{*-V}) \cong \text{Map}(M_{\mathcal{F}}, \mathbb{S}(V))$$

and the $E_2$-page collapses to the bottom line. Thus we have

$$\pi^G_0(E,\mathcal{F}_+ \wedge X) = [S^V, E, \mathcal{F}_+ \wedge X]^G \cong \text{Map}(M_{\mathcal{F}}, \mathbb{S}(X)).$$

\[\square\]

Now we can decompose the $G$-homotopy of $E,\mathcal{F}_+ \wedge X$ onto the equivariant homotopy on $p$-subgroups of $G$.

### Theorem 4.4

Assume that all prime factors of $|G|$ except $p$ are inverted. Let $\mathcal{F} = \mathcal{F}_{p}$ and $P$ be a Sylow $p$-subgroup of $G$. For any virtual $G$-representation $V$, The composition

$$\pi^G_0(E,\mathcal{F}_+ \wedge X) \rightarrow \pi^G_0(X) \xrightarrow{\text{res}^G_P} \pi^G_{V|_p}(X)$$

is injective. Its image consists of all elements $x \in \pi^G_{V|_p}(X)$ such that

$$\text{res}^P_H(x) = c_g(\text{res}^P_{g^{-1}Hg}(x))$$
for any $H \subset P$ and $g \in G$ such that $g^{-1}Hg \subset P$. Here $c_g$ is the conjugation map induced by $g \in G$.

In addition, if $X$ is a ring spectrum, the composition

$$\pi^G_\bullet(E,\mathcal{F}_+ \wedge X) \to \pi^G_\bullet(X) \xrightarrow{\text{res}^G_p} \pi^P_\bullet(X)$$

preserves multiplicative structure since both composing maps do.

**Remark 4.5.** The image of $\text{res}^G_p$ in $\pi^P_{V|\mu}(X)$ does not only depend on $V|\mu$ since the choice of $g$ such that $g^{-1}Hg \subset P$ may not be contained in $P$, not even in $N_GP$. Thus it’s possible that for two different $G$-representations $V,W$, we have $V|\mu \cong W|\mu$ and yet $\pi^G_p(E,\mathcal{F}_+ \wedge X)$ and $\pi^G_W(E,\mathcal{F}_+ \wedge X)$ are different.

When $X = H\mathbb{Z}$ using Proposition 4.1 and Theorem 4.4, we can compute the $G$-homotopy of $H\mathbb{Z}$ from the $P$-homotopy of $H\mathbb{Z}$ with additional actions. Thus it suffices to choose $\mathcal{F}_{\{\mu\}}$ in step (III) to finish the splitting method.

**Proof of Theorem 4.4.** According to Proposition 4.2 we have

$$\pi^G_p(E,\mathcal{F}_+ \wedge X) \cong \text{Map}(M_{\mathcal{F}},\mathfrak{a}_G^G(X)).$$

As a sub-Mackey functor of $AG$, the proof of Lemma 2.12 also provides a description of $M_{\mathcal{F}}$. For any $H \subset G$, $M_{\mathcal{F}}(G/H)$ is generated by $[H/K] \in A(H)$ for all $p$-subgroups $K \subset H$. Let $P$ be a Sylow $p$-subgroup of $G$. For any $K \subset H$ and $g^{-1}Kg \subset P$, we have

$$[H/K] = \text{tr}^H_K \circ c_g \circ \text{res}^P_{g^{-1}Kg}([P/P]).$$

Thus each map from $M_{\mathcal{F}}$ to $\mathfrak{a}_G^G(X)$ is determined by the image of $[P/P]$ in $\pi^G_p(X)(G/P) = \pi^P_{V|\mu}(X)$. This gives us an injective map

$$\pi^G_p(E,\mathcal{F}_+ \wedge X) \hookrightarrow \pi^P_{V|\mu}(X).$$

Assume that $x \in \pi^P_{V|\mu}(X)$ is chosen in the image of $[P/P]$. Then $[H/K]$ must be mapped to $\text{tr}^H_K \circ c_g \circ \text{res}^P_{g^{-1}Kg}(x)$. However, there may be different choices of $g \in G$ to make $g^{-1}Kg$ contained in $P$. Assume that for $g_1, g_2 \in G$, we have $g_1^{-1}Kg_1, g_2^{-1}Kg_2 \subset P$. Then $[H/K]$ will be mapped to both

$$\text{tr}^H_K \circ c_{g_1} \circ \text{res}^P_{g_1^{-1}Kg_1}(x) \text{ and } \text{tr}^H_K \circ c_{g_2} \circ \text{res}^P_{g_2^{-1}Kg_2}(x).$$

These two elements must agree with each other if $x$ is a valid image of $[P/P]$. Since we can choose $H = K$, the restriction on $x$ can be rewritten as

$$\text{res}^P_H(x) = c_g(\text{res}^P_{g^{-1}Hg}(x))$$

for any $H \subset P$ and $g \in G$ such that $g^{-1}Hg \subset P$.

Finally, notice that

$$\pi^G_p(X) \cong \text{Map}(AG,\mathfrak{a}_G^G(X))$$

where each map from $AG$ to $\mathfrak{a}_G^G(X)$ corresponds to the image of $[G/G] \in AG(G/G) = A(G)$. Since $[P/P] = \text{res}^G_p[G/G]$. The map

$$\pi^G_p(E,\mathcal{F}_+ \wedge X) \hookrightarrow \pi^P_{V|\mu}(X)$$

we got before can be expressed as the composition

$$\pi^G_p(E,\mathcal{F}_+ \wedge X) \to \pi^G_p(X) \xrightarrow{\text{res}^G_p} \pi^P_{V|\mu}(X).$$

$\square$
4.2 General splitting method

Now we consider arbitrary $HM$. First we generalize [Theorem 4.4].

**Theorem 4.6.** Let $\mathcal{F}$ be an arbitrary family of subgroups of $G$. Invert proper prime factors such that [Theorem 3.1] holds.

For any virtual $G$-representation $V$ and $G$-spectrum $X$, $\pi^G_V(E\mathcal{F}+ \wedge X)$ is isomorphic to the limit of the diagram with objects

$$\pi^H_{V|H}(X), \ H \in \mathcal{F}$$

and morphisms

$$\text{res}^{H_2}_{H_1} : \pi^H_{V|H_2}(X) \to \pi^H_{V|H_1}(X), \ H_1 \subset H_2 \in \mathcal{F},$$

$$c_g : \pi^H_{V|H}(X) \to \pi^{gHg^{-1}}_{V|H}(X), \ H \in \mathcal{F}, \ g \in G.$$

The map from $\pi^G_V(E\mathcal{F}+ \wedge X)$ to each object $\pi^H_{V|H}(X)$ can be factorized as

$$\pi^G_V(E\mathcal{F}+ \wedge X) \to \pi^G_V(X) \xrightarrow{\text{res}^G_{\mathcal{F}+}} \pi^H_{V|H}(X).$$

Moreover, when $X$ is a ring spectrum, since taking limits preserves multiplicative structures, this decomposition can also be used to compute $\pi^G_{\mathcal{F}+}(E\mathcal{F}+ \wedge X)$ as a graded ring.

**Proof:** According to [Proposition 4.2] we have

$$\pi^G_V(E\mathcal{F}+ \wedge X) \cong \text{Map}(M_\mathcal{F}, \bigotimes^G_V X).$$

According to [Lemma 2.12] as a sub-Mackey functor of $A_G$, $M_\mathcal{F}$ is generated by $\{H/H\}$ for all $H \in \mathcal{F}$, with relations generated by

$$\text{res}^{H_2}_{H_1}(\{H_2/H_2\}) = \{H_1/H_1\}, \ H_1 \subset H_2 \in \mathcal{F},$$

$$c_g(\{H/H\}) = \{gHg^{-1}/gHg^{-1}\}, \ H \in \mathcal{F}, \ g \in G.$$

Each map from $M_\mathcal{F}$ to $\pi^G_V(E\mathcal{F}+ \wedge X)$ is determined by the images of $\{H/H\}$ in

$$\pi^G_V(X)(G/H) = \pi^H_{V|H}(X)$$

for all $H \in \mathcal{F}$ which are compatible with maps $\text{res}^{H_2}_{H_1}$ and $c_g$. Thus $\pi^G_V(E\mathcal{F}+ \wedge X)$ is expressed as the limit described in the theorem. \qed

Fix a prime factor $p$ of $|G|$ and invert all other prime factors. In step (III), we choose all $\mathcal{F}_T$ such that $p \notin T$ and $T$ does not contain all prime factors.

List all such families as $\mathcal{F}_1, ..., \mathcal{F}_t$. In step (IV), we apply [Theorem 3.1] and split $HM$ as follows:

$$HM \simeq ((E\mathcal{F}_1)_+ \wedge HM) \lor (E\mathcal{F}_1 \wedge HM)$$

$$\simeq ((E\mathcal{F}_1)_+ \wedge (E\mathcal{F}_2)_+ \wedge HM) \lor ((E\mathcal{F}_1)_+ \wedge E\mathcal{F}_2 \wedge HM) \lor (E\mathcal{F}_1 \wedge (E\mathcal{F}_2)_+ \wedge HM) \lor (E\mathcal{F}_1 \wedge E\mathcal{F}_2 \wedge HM) \simeq ...$$

Finally we have

$$HM \simeq \bigvee_{\lambda} E\lambda \wedge HM$$

for all maps $\lambda$ defined on $\{1, 2, ..., t\}$ sending each $i$ to either $(E\mathcal{F}_i)_+$ or $E\mathcal{F}_i$. Here $E\lambda$ is the smash product of all images of $\lambda$.

In practice, we can simplify the expression by the following properties on universal spaces:
Lemma 4.7. Let $\mathcal{F}$ and $\mathcal{F}'$ be families of subgroups of $G$. By checking the homotopy types of fixed point subspaces, we have
\[
E\mathcal{F} \wedge E\mathcal{F}' \simeq E(\mathcal{F} \cap \mathcal{F}'),
\]
\[
E\mathcal{F} \wedge \tilde{E}\mathcal{F}' \simeq E(\mathcal{F} \cup \mathcal{F}'),
\]
\[
E\mathcal{F} \wedge \tilde{E}\mathcal{F}' \simeq * \text{ if } \mathcal{F} \subset \mathcal{F}'.
\]

For any $\lambda$, if $\lambda(i) = (E\mathcal{F})_+$ for some $i$, we can apply Theorem 4.6 on
\[
E\lambda \wedge HM = (E\mathcal{F}_i)_+ \wedge \left( HM \wedge \bigwedge_{j \neq i} \lambda(j) \right)
\]
to decompose the $G$-homotopy of $E\lambda \wedge HM$ to the homotopy for smaller subgroups contained in $\mathcal{F}_i$.

The decomposition works for all pieces except one:
\[
\bigwedge_i \tilde{E}\mathcal{F}_i \wedge HM.
\]
According to Lemma 4.7, this piece can be expressed as $\tilde{E}\mathcal{F}_{\hat{p}} \wedge HM$, where $E\mathcal{F}_{\hat{p}}$ is the family of all subgroups $H$ such that not all prime factors of $|G|$ other than $p$ divide $|H|$.

We can use a cellular argument on this last piece: For any family $\mathcal{F}$, by checking the fixed point subspaces, we have
\[
G/H \wedge \tilde{E}\mathcal{F} \simeq *
\]
for all $H \in \mathcal{F}$. In other words, after smashing with $\tilde{E}\mathcal{F}_{\hat{p}}$, only a few $G$-cells will survive.

We make this idea more precise:

Proposition 4.8. Let $\mathcal{F}$ be a family of subgroups of $G$. Invert certain prime factors such that Theorem 3.1 holds. Then for any $G$-CW complex $X$ and $G$-spectrum $Y$, we have
\[
[X, \tilde{E}\mathcal{F} \wedge Y]^G \cong \bigcup_{H \notin \mathcal{F}} X^H, \tilde{E}\mathcal{F} \wedge Y]^G.
\]
\[
[Y, \tilde{E}\mathcal{F} \wedge X]^G \cong [Y, \tilde{E}\mathcal{F} \wedge \bigcup_{H \notin \mathcal{F}} X^H]^G.
\]

The reason is that $X$ and $\bigcup_{H \notin \mathcal{F}} X^H$ are $\tilde{E}\mathcal{F}$-equivalent.

When $G$ is not too complicated, the number of representation spheres will be greatly reduced, while we can compute the surviving ones directly as Bredon homology or cohomology. This is exactly how our computation works in sections 6 and 8.

Remark 4.9. Theoretically, our splitting methods work for all $G$-spectra besides $HM$. But the computation of $\pi^G_\star(\tilde{E}\mathcal{F}_{\hat{p}} \wedge X)$ depends on the structure of $X$ and could be quite hard when $G$ becomes larger.

However, this is not a problem when $G$ is small. For example, when $G = D_6$, the product with $\tilde{E}\mathcal{F}_{\hat{p}}$ ($p = 2$ or $3$) can be expressed as some geometric fixed point spectra. Thus it’s reasonable to say that all information of $D_6$-spectra can be encoded by $C_2$ and $C_3$-spectra. More details will be provided in section 6.

4.3 Computability remarks

When we apply the decomposition into smaller subgroups by Theorem 4.6 if the subgroups still contain multiple prime factors, we can apply the splitting method again. Finally everything will be decomposed into equivariant homotopy for subgroups with prime power orders.

Moreover, recall that the relation maps in Theorem 4.6 are generated by restriction maps $\text{res}_{H_2}^{H_1}$ and conjugation maps $c_g$. The restriction maps are usually not hard to compute in practice. Since the conjugacy maps are always isomorphisms, it suffices to consider the action of the Weyl group $W_G H$ on the $H$-homotopy.
In conclusion, in the computation of $\pi^G_\bullet(HM)$, our splitting method machine requires the following input:

1. $\pi^H_\bullet(HM)$ for all $H \subset G$ with prime power order;
2. The action of $W_G H$ over $\pi^H_\bullet|_{W_G H}(HM)$ for all $V \in RO(G)$;
3. The homotopy of $E\mathbb{F}_p \wedge HM$ for any prime factor $p$ of $|G|$.

Remark for (1): Currently, people do not know much about the equivariant homology of a point when the group has prime power order. The related results are listed below:

- (i) $\pi^G_\bullet(HM)$ for arbitrary $M$, which is computed by multiple people with different methods.
- (ii) $\pi^G_\bullet(H\mathbb{F}_2)$ by Zeng [Zen18] for odd $p$ and Georgakopoulos [Geo21] for $p = 2$. It’s likely that their methods can be generalized to arbitrary Mackey functors, although the computations would be very complicated.
- (iii) $\pi^G_\bullet(H\mathbb{F}_2)$ and partial information about $\pi^G_\bullet(H\mathbb{Z})$ by Ellis-Bloor [Ell20].
- (iv) $\pi^G_\bullet(H\mathbb{Z})$ by Lu [Lu21], only with additive structure.
- (v) $\pi^G_\bullet(H\mathbb{Z})$ by Holler and Kriz [HK17], only with additive structure.

The computation which works for $H\mathbb{Z}$ usually works for other constant Mackey functors as well.

Remark for (2): We need to further analyze the computation of $\pi^H_\bullet(HM)$ in order to compute the $W_G H$-action. The action is easier to obtained if we know the ring structure of $\pi^H_\bullet(HM)$ since the multiplication is compatible with the action.

If $\pi^H_\bullet(HM)$ is computed in a cellular way, we may also compute the action by its behavior on the representation spheres. An example is given in section 5.

Remark for (3): The computability depends on the types of representation spheres and subgroups of $G$. We will give a criterion on this at the end of section 8. (3) may be an obstruction after people know more equivariant computations in the future. At current time, however, all choices of $G$ and $M$ which pass (1) and (2) also pass (3).

In conclusion, now we have a list of groups for which the equivariant homology of a point is computable with our splitting method:

**Theorem 4.10.** For a finite group $G$, $\pi^G_\bullet(HM)$ is computable if one of the following conditions is satisfied:

1. For any Sylow subgroup $P$ of $G$, we have either $P = C_p$, or $P = C_{p^2}$ and $W_G P$ is cyclic;
2. $G = A_4$ or $A_5$ (only when $M$ is a constant Mackey functor);
3. $G = G_1 \times G_2$, such that $(|G_1|, |G_2|) = 1$ and both $\pi^G_\bullet(HM)$ and $\pi^{G_1}_\bullet(HM)$ are computable.

### 4.4 An algebraic point of view

There is a strong relation between Theorem 4.4 with a recent paper [Ang22], in which the same problem is considered in an algebraic way:

**Definition 4.11.** A Mackey functor $M$ is called cohomological if $tr^K_H \circ res^K_H$ is the multiplication by $|H/K|$ for any $K \subset H \subset G$.

**Theorem 4.12.** [Ang22, Theorem 3.1] Let $P_1, \ldots, P_n$ be Sylow subgroups of $G$, with one for each prime factor. There is an isomorphism

$$\oplus_i tr^G_{P_i} : \bigoplus_i M(G/P_i)/ \sim \rightarrow M(G/G),$$

where $\sim$ is generated by

$$tr^G_{P_i}(y) \sim tr^G_{P_j}(g^{-1}(y))$$

for any $y \in M(G/H)$ and $g \in G$ such that $H \subset P_i$, $g^{-1}Hg \subset P_j$. 

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This theorem can be used to compute equivariant homology with coefficients in a cohomological Mackey functor (like $\mathbb{Z}$) since

**Proposition 4.13.** [Ang22, Proposition 2.5] Let $X$ be a $G$-CW spectrum. If $M$ is a cohomological Mackey functor, then $\Sigma^G(X \wedge HM)$ is also cohomological.

Notice that, in *Theorem 4.12*, the equivalence relation is generated by the following two types:

1. The equivalence relation on one single summand $M(G/P_i)$:
   
   \[
   tr_{P_i}^G(y) \sim tr_{g^{-1}Hg}^P(y) \quad \text{for } y \in M(G/H) \text{ such that } H, g^{-1}Hg \subset P_i;
   \]

2. For $i \neq j$, we have
   
   \[
   tr_{P_i}^G(y) \sim tr_{P_j}^G(y) \quad \text{for any } y \in M(G/e).
   \]

This is equivalent to $tr_{P_i}^G \circ res_{P_i} = tr_{P_j}^G \circ res_{P_i}$ as part of the Mackey functor structure.

Since $tr_{P_i}^G \circ res_{P_i}$ and $tr_{P_j}^G \circ res_{P_i}$ are multiplications by $|G/P_i|$ and $|P_i|$, it’s not hard to check that *Theorem 4.12* is implied by (in fact equivalent to) its localized version:

**Proposition 4.14.** Assume that all prime factors of $|G|$ except $p$ are inverted. Let $P$ be a Sylow $P$-subgroup of $G$. For any cohomological Mackey functor $M$, we have an isomorphism

\[
tr_{P_i}^G : M(G/P)/\sim \rightarrow M(G/G),
\]

where $\sim$ is generated by

\[
tr_{P_i}^G(y) \sim tr_{g^{-1}Hg}^P(y) \quad \text{for any } y \in M(G/H) \text{ and } g \in G \text{ such that } H, g^{-1}Hg \subset P.
\]

This proposition is related to *Theorem 4.4* by the following two lemmas:

**Lemma 4.15.** With the same assumptions as in *Proposition 4.14*, let $R$ be the subgroup of $M(G/P)$ which consists of all elements $x$ such that

\[
res_H^P(x) = c_g(res_{g^{-1}Hg}^P(x))
\]

for any $H \subset P$ and $g \in G$ such that $g^{-1}Hg \subset P$. Then the composition

\[
R \hookrightarrow M(G/P) \xrightarrow{tr_{P_i}^G} M(G/G)
\]

is an isomorphism.

**Lemma 4.16.** Assume that all prime factors of $|G|$ except $p$ are inverted. Let $\mathcal{F} = \mathcal{F}_p$. Then for any cohomological Mackey functor $M$, we have

\[
\widetilde{E\mathcal{F}} \wedge HM \simeq *.
\]

**Remark 4.17.** When we are computing equivariant homology with coefficients in a cohomological Mackey functor, *Lemma 4.16* shows that smashing with $E\mathcal{F}_+$ makes no changes. Then *Lemma 4.15* shows that the methods in [Ang22] and our paper provide the same result.

In fact, our method also works for some non-cohomological $M$. We only need to guarantee

\[
\widetilde{E\mathcal{F}} \wedge HM \simeq *.
\]

This condition holds if and only if $M$ is an $M_\mathcal{F}$-module.

**Remark 4.18.** From an algebraic point of view, both our method and the method in [Ang22] share the same idea: Expressing the top level $M(G/G)$ of the Mackey functor $M$ by the lower levels.

The only difference is whether we use transfer maps or restriction maps, each of which has its own advantage: The expression with transfer maps does not require any localizations, while the expression with restriction maps preserves the multiplicative structure.
Thus \( M \) is an isomorphism. Thus \( \star = M \) is an isomorphism.

Fix \( z \in M(G/G) \). For any \( H \subset P \) and \( g \in G \) such that \( g^{-1}Hg \subset P \), we have

\[
c_g(\text{res}_{g^{-1}Hg}^P(\text{res}_P^G(z))) = c_g(\text{res}_{g^{-1}Hg}^G(c_g(z))) = \text{res}_H^G(c_g(z)) = \text{res}_H^G(\text{res}_P^G(z)).
\]

Since \( H, g \) are arbitrary, \( \text{res}_P^G(z) \) is contained in \( R \). Thus \( \text{res}_P^G \) can be viewed as an injective map from \( M(G/G) \) to \( R \), which is the right inverse of the composition

\[
R \hookrightarrow M(G/P) \xrightarrow{\text{tr}_P^G} M(G/G).
\]

On the other hand, for any \( x \in R \), choose \( H = P \) and \( g \in N_GP \). We get \( x = c_g(x) \). Thus

\[
\text{res}_P^G(\text{tr}_P^G(x)) = \sum_{g \in N_GP/P} c_g(x) = \text{tr}_{G/P}^G(x).
\]

Since \( \text{tr}_{G/P}^G \) is inverted, \( \text{res}_P^G \circ \text{tr}_P^G \) becomes a self-isomorphism of \( R \). Therefore, the composition

\[
R \hookrightarrow M(G/P) \xrightarrow{\text{tr}_P^G} M(G/G)
\]

is an isomorphism. \( \Box \)

**Proof of Lemma 4.15.** With the same argument as in the proof of Proposition 4.1, it suffices to prove \( M = M \square M \square \) for any cohomological Mackey functor \( M \).

For any \( H \subset G \), let \( P \) be a Sylow \( p \)-subgroup of \( H \). Then \( \text{tr}_P^H \circ \text{res}_P^H \) is multiplication by \( |H/P| \), hence is an isomorphism. Thus \( \text{res}_P^H \) is an injection.

Notice that for any \( p \)-subgroup \( K \) of \( G \), according to the definition of \( M \), we have \( M \subset A(K) \). Thus \( M \) is an isomorphism.

Consider the following commutative diagram.

\[
\begin{array}{ccc}
M(G/H) & \xrightarrow{\text{res}_P^H} & M \square M \square (G/H) \\
| & & |
\text{res}_P^G & \downarrow & \text{res}_P^G \\
M(G/P) & \xrightarrow{\text{res}_P^H} & M \square M \square (G/P)
\end{array}
\]

We have shown that the left vertical map and the bottom horizontal map are isomorphisms. Moreover, the top horizontal map is surjective, since

\[
M = (M \square M \square) \oplus (M \square N \square).
\]

Thus all maps in the diagram above are isomorphisms. Since \( H \subset G \) is arbitrary, we have \( M = M \square M \). \( \Box \)

## 5 Computation of \( \pi^{D_2p}_* (H \mathbb{Z}) \)

Starting from this section, we will apply our splitting method to compute \( \pi^{D_2p}_* (H \mathbb{Z}) \), \( \pi^{D_2p}_* (HA_\mathbb{Z}) \), \( \pi^{A_2}_* (H \mathbb{Z}) \), and \( \pi^{A_2}_* (HA_\mathbb{Z}) \) as \( RO(G) \)-graded rings.

We will use the same way to define generators as in [HHRY17] Definition 3.4:

**Definition 5.1.** For any actual \( G \)-representation \( V \) with \( V^G = 0 \), let \( v_V \in \pi_0(S^0) \) be the map \( S^0 \to S^V \) embedding \( S^0 \) to 0 and \( \infty \). We also use \( v_V \) to denote the Hurewicz image of \( v_V \in \pi_0(S^0) \) in \( \pi_0(H \mathbb{Z}) \).

For any actual orientable representation \( V \) of dimension \( n \), let \( w_V \) be the generator of \( \pi_{n-1}(H \mathbb{Z}) = H_{/n}^G(S^{V}; \mathbb{Z}) \) which restricts to the choice of orientation in

\[
H_{/n}^G(S^{V}; \mathbb{Z})(G/e) \cong H_n(S^n; \mathbb{Z}).
\]

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Some important relations on these generators are given below:

**Proposition 5.2.** (a) For any $V_1, V_2$,
\[ a_{V_1} a_{V_2} = a_{V_1 + V_2}, \quad u_{V_1 + V_2} = u_{V_1} u_{V_2}. \]

(b) Let $G_V$ be the isotropy subgroup of $V$. Then $|G/G_V|a_V = 0$.

(c) For $V, W$ both oriented with dimension 2, with $G_V \subset G_W$, we have
\[ a_W u_V = |G_W/G_V|a_V u_W. \]

**Convention:** For any $H \subset G$ and $H$-representations $V_1, V_2$, we will identify them in our computation if
\[ S^{V_1} \wedge HA_H \simeq S^{V_2} \wedge HA_H. \]

We may not have $V_1 \simeq V_2$. But in the computation of homology, it’s not necessary to distinguish these two representations.

For the rest of this section, let $G = D_{2p}$.

Up to conjugacy, the nontrivial subgroups of $G$ consist of $C_2$, $C_p$ and $G$ itself. According to the splitting method in section 4.1, the computation of $D_{2p}$-homotopy can be decomposed into computations of $C_2$ and $C_p$-homotopy.

We first list all irreducible representations for these subgroups except the trivial representation:

For $C_2$, we have the 1-dimensional sign representation $\sigma$.

For $C_p$, we have $p - 1$ different 2-dimensional rotation representations, which are identified according to the convention above. We denote them according to $\lambda$.

For $D_{2p}$, we have the sign representation $\sigma$ for $D_{2p}/C_p$, and the 2-dimensional dihedral representation $\gamma$, where all elements with order $p$ act as rotations and all elements with order 2 act as reflections.

### 5.1 Decomposition into $C_2$ and $C_p$

We invert either 2 or $p$ and apply [Theorem 4.4]. Notice that $C_2$ and $C_p$ have no non-trivial proper subgroups. Thus the relations in [Theorem 4.4] only contain the Weyl group action. Together with [Proposition 4.1] we have

\[ \pi^{D_{2p}}_{V/H}(H\mathbb{Z})[1/p] \cong \pi^{C_2}_{V/C_2}(H\mathbb{Z})[1/p], \]
\[ \pi^{D_{2p}}_{V/H}(H\mathbb{Z})[1/2] \cong \pi^{C_p}_{V/C_p}(H\mathbb{Z})^{D_{2p}/C_p}[1/2]. \]

It suffices to compute the action of $D_{2p}/C_p$ on $\pi^{C_p}_{V/C_p}(H\mathbb{Z})$. Express the generators of $D_{2p}$ by $\zeta$ and $\tau$ such that $\zeta^p = \tau^2 = 1$, $\zeta\tau = \tau\zeta^{-1}$.

First we consider the case that $V$ has no copies of $\sigma$:

**Proposition 5.3.** Let $V = k + n\gamma$ for some integers $k, n$. Then the action of $\tau$ on $[S^V, \Sigma^t H\mathbb{Z}]^{C_p}$ is multiplication by $-1$ if $|k - t| \equiv 2$ or $3$ (mod 4). Otherwise, the action is multiplication by 1.

**Proof:** Notice that $[S^V, \Sigma^t H\mathbb{Z}]^{C_p}$ can be viewed as the equivariant cohomology or homology of $S^{[n]\gamma}$ with coefficients in $\mathbb{Z}$. The computation can be done in a cellular way. We first give $S^{[n]\gamma}$ a $C_p$-CW structure.

We view $S^n$ as $\gamma$ compactified at $\infty$. The action of $\zeta$ is the counter-clockwise rotation by $2\pi/p$ and the action of $\tau$ is the reflection by the $x$-axis. The standard CW structure of $S^n$ can be described as follows:

As a based non-equivariant space, $S^n \cong S^2$ can be constructed by one single 2-cell and the base point at the origin. Denote that 2-cell by $a$.

As a based $C_p$-space, $S^n \cong S^3$ can be constructed by:
2-cells $b_1, b_2, \ldots, b_p$, such that each $b_i$ has image

$$\left\{ \left( r \cos \theta, r \sin \theta \right) : 0 \leq r \leq \infty, \frac{2\pi(i-1)}{p} \leq \theta \leq \frac{2\pi i}{p} \right\};$$

1-cells $c_1, c_2, \ldots, c_p$, such that each $c_i$ has image

$$\left\{ \left( r \cos \frac{2\pi i}{p}, r \sin \frac{2\pi i}{p} \right) : 0 \leq r \leq \infty \right\};$$

One 0-cell $d$ at $\infty$ and the base point at 0. Here the subscripts of $b_i$ and $c_i$ are defined modulo $p$.

For both $CW$ structures, the action of $\tau$ on $S^7$ can be made into a cellular map:

- The action of $\tau$ on the whole $S^7$ is a reflection. Thus $\tau a = -a$.
- The action of $\tau$ gives a permutation among the images of $b_1, b_2, \ldots, b_p$ and reverses the orientations of these 2-cells. Thus $\tau b_i = -b_{p+1-i}$.
- The action of $\tau$ gives a permutation among the images of $c_1, c_2, \ldots, c_p$ and keeps the orientations of these 1-cells. Thus $\tau c_i = c_{p+1-i}$.
- Finally, $\tau d = d$ since the point $\infty$ is fixed.

Since $S^{[n]} = S^7 \wedge S^7 \wedge \ldots \wedge S^7$, we can construct a $CW$ structure as follows:

For $i \in \mathbb{Z}$ and $0 \leq j \leq |n| - 1$, let

$$e_{2j+2,i} := (\zeta^i a, \ldots, \zeta^i a, b_i, d, \ldots, d)$$

$$e_{2j+1,i} := (\zeta^i a, \ldots, \zeta^i a, c_i, d, \ldots, d)$$

with $j$ copies of $\zeta^i a$ and $|n| - 1 - j$ copies of $d$.

We also have one 0-cell $e_0 := (d, \ldots, d)$ and the base point.

Notice that $\zeta(b_i = b_{i+1}, \zeta c_i = c_{i+1}$. This construction makes $S^{[n]}$ into a $C_p$-$CW$ complex with one $C_p$-cell (except the base point) in each degree between 0 and $2|n|$. The action of $\tau$ on $S^{[n]}$ is induced by the action on each copy of $S^7$, hence is also made cellular:

$$\tau e_{2j+2,i} = (-1)^{j+1} e_{2j+2,p+1-i}, \tau e_{2j+1,i} = (-1)^j e_{2j+1,p+1-i}, \tau e_0 = e_0.$$ 

Recall that $H^C_p(S^{[n]}; \mathbb{Z})$ and $H_C^*(S^{[n]}; \mathbb{Z})$ can be computed by the following chain and cochain complexes:

$$C^C_p(S^{[n]}; \mathbb{Z}) := C_p(S^{[n]}) \otimes_{C_p} \mathbb{Z}$$

$$C_C^*(S^{[n]}; \mathbb{Z}) := \text{Hom}_{C_p}(C_p(S^{[n]}); \mathbb{Z}),$$

where we view $\mathbb{Z}$ as a covariant coefficient system in the first equation and a contravariant one in the second equation.

The actions of $\tau$ on the homology and cohomology are obtained by applying $\tau$ on both $C^*_p(S^{[n]})$ and $\mathbb{Z}$.

Since $S^{[n]}$ has only $C_p$-free cells in positive degrees and only $C_p$-fixed cells in degree 0, and $\mathbb{Z}$ is the constant coefficient system, we have

$$C^C_p(S^{[n]}; \mathbb{Z}) := C_p(S^{[n]}) \otimes_{C_p} \mathbb{Z} \equiv C_p(S^{[n]})/C_p;$$

$$C_C^*(S^{[n]}; \mathbb{Z}) := \text{Hom}_{C_p}(C_p(S^{[n]}); \mathbb{Z}) \equiv \text{Hom}_\mathbb{Z}(C_p(S^{[n]})/C_p; \mathbb{Z}).$$

Let $e_{2j}$ and $e_{2j-1}$ be the orbits of cells $e_{2j,i}$ and $e_{2j-1,i}$. The induced $\tau$-action on $C^*_p(S^{[n]}; \mathbb{Z})/C_p$ is expressed as

$$\tau e_{2j} = (-1)^j e_{2j}, \quad \tau e_{2j-1} = (-1)^{j-1} e_{2j-1}.$$ 

Therefore, the $\tau$-action on the homology and cohomology is multiplication by $-1$ when the degree is congruent to 2 or 3 mod 4. Otherwise the $\tau$-action is multiplication by 1. □
Now we consider the case that \( V \) also contains copies of \( \sigma \). For any \( D_{2p} \)-spectra \( X, Y \), let \( \tau \) act on \( [X,Y]^{C_p} \) by conjugation. Since \( S^\sigma \cong S^1 \) as \( C_p \)-spaces, we have
\[
[X,Y]^{C_p} \cong [X, S^{m(\sigma-1)} \wedge Y]^{C_p}.
\]
Since \( \tau \) acts on \( S^\sigma \) as a reflection, the composition
\[
[X,Y]^{C_p} \cong [X, S^{m(\sigma-1)} \wedge Y]^{C_p} \cong [X, S^{m(\sigma-1)} \wedge Y]^{C_p} \cong [X,Y]^{C_p}
\]
agrees with the \( \tau \)-action on \( [X,Y]^{C_p} \) multiplied by \((-1)^m\).

Together with Proposition 5.3, we get

**Proposition 5.4.** Let \( V = k + m\sigma + n\gamma \). The action of \( \tau \) on \( [S^V, \Sigma^1 HZ]^{C_p} \) is multiplication by \(-1\) if \([k+m]/2 + m\) is odd. Otherwise, the action is multiplication by 1.

In conclusion, we have

**Proposition 5.5.** For any integers \( k, m, n \),
\[
[S^{k+m\sigma+n\gamma}, HZ]^G[1/p] \cong [S^{(k+n)+(m+n)\sigma}, HZ]^{C_p}[1/p],
\]
\[
[S^{k+m\sigma+n\gamma}, HZ]^G[1/2] \cong \begin{cases} 
0, & \text{if } [k+m]/2 + m \text{ is odd,} \\
[S^{(k+m)+n\lambda}, HZ]^{C_p}[1/2], & \text{if } [k+m]/2 + m \text{ is even.}
\end{cases}
\]

### 5.2 A complete expression

Now we combine Proposition 5.5 with the computations of \( \pi_*^{C_p}(HZ) \) and \( \pi_*^{C_p}(HZ) \) to get an explicit expression for \( \pi_*^{D_{2p}}(HZ) \). We refer to [Zen18], which provides a modern method to compute \( C_2 \) and \( C_p \)-homotopy of \( HZ \) and gives the following expression:

**Theorem 5.6.** The \( C_2 \) and \( C_p \)-homology of a point are given below:

\[
\pi_*^{C_2}(HZ) = \mathbb{Z}[u_{2\tau}, a_\sigma]/(2a_\sigma) \oplus \left( \bigoplus_{i>0} \mathbb{Z}(u_{2\tau}^{-i}) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/2(\Sigma^{-1}u_{2\sigma}^{-j}a_\sigma^{-k}) \right).
\]

\[
\pi_*^{C_p}(HZ) = \mathbb{Z}[u_{\lambda}, a_\lambda]/(pa_\lambda) \oplus \left( \bigoplus_{i>0} \mathbb{Z}(u_{\lambda}^{-i}) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/p(\Sigma^{-1}u_{\lambda}^{-j}a_\lambda^{-k}) \right).
\]

Here the generators \( u_{2\tau}, a_\sigma, u_{\lambda}, a_\lambda \) are defined in Definition 5.1.

Notice that Proposition 5.5 provides maps from \( \pi_*^{D_{2p}}(HZ) \) to \( \pi_*^{C_2}(HZ) \) and \( \pi_*^{C_p}(HZ) \). We can obtain generating elements in \( \pi_*^{D_{2p}}(HZ) \) by tracing the pre-image of generators in \( \pi_*^{C_2}(HZ) \) and \( \pi_*^{C_p}(HZ) \).

First we consider the case when \( p \) is inverted. According to the first equation in Proposition 5.5, we have

\[
\pi_0^{C_2}(HZ) \cong \pi_{c(1+\sigma-\gamma)}^{D_{2p}}(HZ), \quad \forall c \in \mathbb{Z}.
\]

We use \( u_{1-\sigma} \) to denote the generator of \( \pi_{1+\sigma-\gamma}^{D_{2p}}(HZ) \). Then \( u_{1-\sigma} \) is invertible.

\[
\pi_1^{C_2}(HZ) \cong \pi_{k+m\sigma+c(1+\sigma-\gamma)}^{D_{2p}}(HZ), \quad \forall c \in \mathbb{Z}.
\]

Thus the pre-images of \( u_{2\tau}, a_\sigma \in \pi_*^{C_2}(HZ) \) are \( u_{2\tau} u_{1-\sigma}^{-1}, a_\sigma u_{1-\sigma}^{-1} \in \pi_*^{D_{2p}}(HZ) \), for any \( c \in \mathbb{Z} \).

Therefore, \( \pi_*^{D_{2p}}(HZ) \) can be obtained by adding the invertible \( u_{1-\sigma} \) to \( \pi_*^{C_2}(HZ) \):

\[
\pi_*^{D_{2p}}(HZ)[1/p] = \mathbb{Z}[1/p][u_{2\tau}, a_\sigma, u_{1-\sigma}^{-1}]/(2a_\sigma).
\]
Thus the pre-image of

Thus we have

We still let

The main result of this section is given below:

Finally, we just need to glue

π_{\pi'}^C(H\mathbb{Z}) \cong \pi_{2(1-\sigma)}^{D_p}(H\mathbb{Z}), \forall c \in \mathbb{Z}.

Thus the pre-image of \( u_\lambda \in \pi_{\pi'}^C(H\mathbb{Z}) \) is \( u_{\gamma-\sigma}u_{2\sigma} \), for any \( c \in \mathbb{Z} \).

Thus the pre-image of \( a_\lambda \in \pi_{\pi'}^C(H\mathbb{Z}) \) is \( a_\gamma u_{2\sigma} \), for any \( c \in \mathbb{Z} \).

Therefore, we have

Finally, we just need to glue \( \pi_{\pi'}^{D_p}(H\mathbb{Z})[1/p] \) and \( \pi_{\pi'}^{D_p}(H\mathbb{Z})[1/2] \) together.

Theorem 5.7. Recall that we obtain \( a_\sigma, a_\gamma \) and \( u_{2\sigma} \) from [Definition 5.1] and choose \( u_{\gamma-\sigma} \) to be the generator of

Then we have

\begin{align*}
\pi_{\pi'}^{D_p}(H\mathbb{Z}) &= \mathbb{Z}[u_{\gamma-\sigma}, u_{2\sigma}, a_\sigma, a_\gamma]/(2a_\sigma, pa_\gamma) \\
&\oplus \left( \bigoplus_{i>0} \mathbb{Z}[u_{\gamma-\sigma}]/(u_{2\sigma}) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/[u_{2\sigma}]/(\sigma^{-1}u_{2\sigma}^j a_\sigma^k) \right) \\
&\oplus \left( \bigoplus_{j>0} \mathbb{Z}/[u_{\gamma-\sigma}]/(u_{2\sigma}) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/[u_{2\sigma}]/(\sigma^{-1}u_{2\sigma}^j a_\sigma^k) \right) \\
&\oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/[u_{\gamma-\sigma}]/(\sigma^{-1}u_{2\sigma}^j a_\sigma^k) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/[u_{2\sigma}]/(\sigma^{-1}u_{2\sigma}^j a_\sigma^k) \right) \\
&\oplus \left( \bigoplus_{j>0} \mathbb{Z}/[u_{\gamma-\sigma}]/(u_{2\sigma}) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/[u_{2\sigma}]/(\sigma^{-1}u_{2\sigma}^j a_\sigma^k) \right).
\end{align*}

6 Computation of \( \pi_{\pi'}^{D_p}(HA_G) \)

In this section, we will use the general splitting method mentioned in section 4.2 to compute \( \pi_{\pi'}^{D_p}(HA_G) \).

We still let \( G = D_{2p} \).

The main result of this section is given below:

Theorem 6.1. For any integers \( k, m, n \),

\[
\pi_{k+m+\sigma+n\gamma}(HA_G)[1/p] \cong \pi_{(k+m)+n\sigma}(HA_{C_2})[1/p] \oplus \pi_{k+m}(HA_{C_2})[1/p],
\]

\[
\pi_{k+m+\sigma+n\gamma}(HA_G)[1/2] \cong \begin{cases} 
0, & \text{if } [k + m]/2 + n \text{ is odd}, \\
\pi_{(k+m)+n\lambda}(HA_{C_2})[1/2], & \text{otherwise}. 
\end{cases}
\]

\[ \oplus \mathbb{Z}^2[1/2], \text{ if } k + n = 0, \]

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First we assume that $p$ is inverted.
According to the splitting method in section 4, we choose the family $\mathcal{F}_{(2)}$, which contains $C_2$ and the trivial subgroup. We have

$$HA_G \cong (E_{\mathcal{F}_{(2)}+} \land HA_G) \lor (\overline{E_{\mathcal{F}_{(2)}}} \land HA_G).$$

Since $A_G(G/H)$ has trivial $W_GH$-action for any $H \subset G$, the arguments towards the first equation in Proposition 5.5 still work for $E_{\mathcal{F}_{(2)}+} \land HA_G$:

**Proposition 6.2.** For any integers $k,m,n$,

$$[S^{k+m\sigma+n\gamma}, E_{\mathcal{F}_{(2)}+} \land HA_G]^G[1/p] \cong [S^{(k+n)+(m+n)\sigma}, HA_{C_2}]C_2[1/p].$$

For $\overline{E_{\mathcal{F}_{(2)}}} \land HA_G$, notice that $\gamma$ has trivial (only the origin) $C_p$ and $D_{2p}$ fixed point subspaces. According to Proposition 4.8 we have

**Proposition 6.3.** For any integers $k,m,n$,

$$[S^{k+m\sigma+n\gamma}, E_{\mathcal{F}_{(2)}+} \land HA_G]^G[1/p] \cong [S^{k+m\sigma}, E_{\mathcal{F}_{(2)}} \land HA_G]^G[1/p]$$

$$\cong [S^{k+m\sigma}, (E_{\mathcal{F}_{(2)}} \land HA_G)^G]_{C_p}[1/p] \cong [S^{k+m\sigma}, HA_{C_2}]C_2[1/p].$$

The last isomorphism is the computation of geometric fixed point spectra.

Next, we assume that 2 is inverted. Now we choose the family $\mathcal{F}_{(p)}$ instead, which contains $C_p$ and the trivial subgroup.

Similarly, the arguments toward the second equation of Proposition 5.5 still work for $E_{\mathcal{F}_{(p)}+} \land HA_G$:

**Proposition 6.4.** For any integers $k,m,n$,

$$[S^{k+m\sigma+n\gamma}, E_{\mathcal{F}_{(p)}+} \land HA_G]^G[1/2] \cong \begin{cases} 0, & \text{if } [k+m/2]+m \text{ is odd,} \\ [S^{(k+n)+(m+n)\sigma}, HA_{C_p}]C_p[1/2], & \text{if } [k+m/2]+m \text{ is even.} \end{cases}$$

By checking fixed point subspaces, the following maps

$$S^0 \leftrightarrow S^\sigma,$$

$$S^\gamma \rightarrow S^\gamma/C_p = S^{1+\sigma} \leftrightarrow S^1$$

become equivalences after smashing with $E_{\mathcal{F}_{(p)}}$. So we have

$$[S^{k+m\sigma+n\gamma}, E_{\mathcal{F}_{(p)}} \land HA_G]^G \cong [S^{k+n}, \overline{E_{\mathcal{F}_{(p)}}} \land HA_G]^G.$$
7 Computation of $\pi_A^G(H\mathbb{Z})$

In this section, we will use the splitting method to compute $\pi_A^G(H\mathbb{Z})$ for $G = A_5$. First we list all subgroups of $A_5$.

Subgroups of $A_5$ up to conjugacy:
- Sylow 2-subgroup $K_4 = ((12)(34), (13)(24))$, which is isomorphic to $C_2 \times C_2$;
- Sylow 3-subgroup $C_3 = ((123))$;
- Sylow 5-subgroup $C_5 = ((12345))$;
- Normalizers of Sylow subgroups:
  - $A_4 = N_GK_4 = ((12)(34), (123))$;
  - $D_6 = N_GC_3 = ((123), (12)(45))$;
  - $D_{10} = N_GC_5 = ((12345), (15)(24))$;
- Other subgroups: $C_2 = ((12)(34))$ and the trivial subgroup.

We also list all irreducible real representations of $A_5$ and its subgroups. Since there are lots of groups, we express the group and dimension of a representation in the subscript. For example, $V_{H,n}$ is an $n$-dimensional $H$-representation. The trivial representations of all groups will be written as 1.

Irreducible representations of $A_5$:
- $V_{A_5,5}^+$: This is obtained by viewing $A_5$ as the group of rotations of the regular icosahedron in the 3-dimensional space.
  - $V_{A_5,3}^+$: This is the same as $V_{A_5,5}^+$ except applying the conjugation by (12) (inside $S_5$) first.
  - For the two representations $V_{A_5,3}^+$ and $V_{A_5,3}^-$, we will see that all our computations which work for one of them also work for the other. This further implies that the two representation spheres are $HAG$-equivalent.
  - We use $V_{A_5,3}$ to denote both representations.
  - $V_{A_5,4}$: Consider the space of functions from $\{1, 2, 3, 4, 5\}$ to $\mathbb{R}$ whose images have sum 0. The action of $A_5$ is the natural permutation.
  - $V_{A_5,5}$: Consider the space of functions from antipodal pairs of vertices of the regular icosahedron, such that the sum of all images of the function is 0. The action of $A_5$ is induced by the rotations of the icosahedron.

Irreducible representations of $A_4$:
- $V_{A_4,2}$: All elements in $A_4$ with order 3 become rotations of either $2\pi/3$ or $\pi/3$. All elements with order 2 act trivially.
- $V_{A_4,3}$: Consider the space of functions from $\{1, 2, 3, 4\}$ to $\mathbb{R}$ whose images have sum 0. The action of $A_4$ is the natural permutation.

Irreducible representations of $D_{2p}$ ($p = 3$ or 5):
- $V_{D_{2p},1}$: The sign representation of $D_{2p}/C_p$, with $C_p$ acting trivially.
- $V_{D_{2p},2}$: Dihedral representations, where all elements with odd orders act as rotations, and all elements with order 2 act as reflections.

Irreducible representations of $K_4$:
- Three different sign representations $V_{K_4,1,1}, V_{K_4,1,2}, V_{K_4,1,3}$ which correspond to the three proper subgroups of $K_4$.
  - In addition, we use $V_{K_4,3}$ to denote $V_{K_4,1,1} + V_{K_4,1,2} + V_{K_4,1,3}$ (which is not irreducible).

Irreducible representations of $C_p$: ($p = 3$ or 5)
- $V_{C_p,2}$: All elements act as rotations.

Irreducible representations of $C_2$: $V_{C_2,1}$: Sign representation.

The representations of these groups are related by restrictions:
- The restrictions of $V_{A_5,3}$ in $A_4$, $D_6$, $D_{10}$ are $V_{A_4,3}$, $V_{D_6,1} + V_{D_6,2}$ and $V_{D_{10},1} + V_{D_{10},2}$.
- The restrictions of $V_{A_5,4}$ in $A_4$, $D_6$, $D_{10}$ are $1 + V_{A_4,3}$, $1 + V_{D_6,1} + V_{D_6,2}$ and $2V_{D_{10},2}$.
- The restrictions of $V_{A_5,5}$ in $A_4$, $D_6$, $D_{10}$ are $V_{A_4,2} + V_{A_4,3}$, $1 + 2V_{D_6,2}$ and $1 + 2V_{D_{10},2}$.
The restrictions of $V_{A_4,2}, V_{A_4,3}$ in $K_4$ are 2 and $V_{K_4,3}$.
The restrictions of $V_{D_{2p},1}, V_{D_{2p},2}$ in $C_p$ are 1 and $V_{C_p,2}$ for $p = 3, 5$.

Now we invert all prime factors of $|A_5|$ except $p \in \{2, 3, 5\}$. Applying Proposition 4.1 and Theorem 4.4, we know that $\pi_V^G(H\mathbb{Z})$ can be viewed as the subgroup of $\pi_{V|_{F}}^P(H\mathbb{Z})$, which consists of all element $x \in \pi_{V|_{F}}^P(H\mathbb{Z})$ such that

$$\text{res}^P_H(x) = c_g(\text{res}^P_{g^{-1}Hg}(x))$$

for any $H \subset P$ and $g \in G$ such that $g^{-1}Hg \subset P$. Here $P$ is a Sylow $p$-subgroup of $G$. The choices of $H, P, g$ in the condition above can be classified into three cases:

(1) $H = P$. Then $g^{-1}Hg \subset P$ is equivalent to $g \in N_G P$. The condition itself is equivalent to that $x$ is fixed under the action of $N_G P$ (or $W_G P$).

(2) $H$ is the trivial subgroup $e$. Notice that $\pi_A(H\mathbb{Z}) = \mathbb{Z}$. Thus the action of any $g \in G$ is the multiplication of 1 or $-1$. The action is trivial if $g$ has an odd order. Moreover, since $(12)(34) = (213)(324)$, which is the product of odd order elements, the action of any even order element is also trivial. Thus $c_g$ is the trivial action, and hence

$$\text{res}^P_H(x) = c_g(\text{res}^P_{g^{-1}Hg}(x))$$

is true for all $x$.

(3) $P = K_4$ and $H = C_2$. Then $g^{-1}Hg \subset P$ is equivalent to $g \in A_4 = N_G K_4$. The condition

$$\text{res}^P_H(x) = c_g(\text{res}^P_{g^{-1}Hg}(x)) = \text{res}^P_H(c_g(x))$$

must be true if $x$ is fixed under the action of $N_G P$.

In conclusion, (1) is the only case we need to consider. Theorem 4.4 can be rewritten as:

**Proposition 7.1.** Let $P$ be a Sylow $p$-subgroup of $G = A_5$. Assume that all prime factors of $|G|$ except $p$ are inverted. Then

$$\text{res}^P_G : \pi_V^G(H\mathbb{Z}) \rightarrow \pi_{V|_{F}}^P(H\mathbb{Z})$$

is injective with image $\pi_{V|_{F}}^P(H\mathbb{Z})^{W_G P}$.

In general, let $X$ be any $H A_G$-module. Then the composed map

$$\pi_V^G(E_{\mathbb{F}_p}^+ \wedge X) \rightarrow \pi_V^G(X) \xrightarrow{\text{res}^P_G} \pi_{V|_{F}}^P(X)$$

is injective with image $\pi_{V|_{F}}^P(X)^{W_G P}$.

**Remark 7.2.** Notice that in case (1), we only have $c_g$ with $g \in N_G P$. Thus the action of $W_G P$ on $\pi_{V|_{F}}^P(H\mathbb{Z})$ is determined by the restriction of $V$ in $N_G P$.

Now it suffices to compute $\pi_{A_5}^G(H\mathbb{Z})$ and the action of $W_G P$ for all Sylow subgroups $P$.

When $p = 3$ or 5, the computation is already done in section 5:

**Theorem 7.3.** For $p \in \{3, 5\}$, we have

$$\pi_{A_5}^C_p(H\mathbb{Z}) = \mathbb{Z}[u, a]/(pa) \oplus \left( \bigoplus_{i > 0} p\mathbb{Z}\langle u^{-i} \rangle \right) \oplus \left( \bigoplus_{j, k > 0} \mathbb{Z}/p(\Sigma^{-1}u^{-j}a^{-k}) \right).$$

Here $|a| = 2 - V_{C_p,2}$, $|a| = -V_{C_p,2}$.

For any $A_5$-representation $V$, write $V|_{D_{2p}} = l + mV_{D_{2p},1} + nV_{D_{2p},2}$. Let $\tau$ be the generator of $W_G C_p = D_{2p}/C_p$. The action of $\tau$ on $\pi_{A_5}^C_p(H\mathbb{Z})$ is the multiplication by $-1$ if $|l + m|/2 + m$ is odd. Otherwise, the action is trivial.

According to Proposition 7.1, we can compute $\pi_{A_5}^C_p(H\mathbb{Z})$ when 2, 3 or 5 are inverted:
Theorem 7.4. When 2, 5 are inverted, we have
\[ \pi^A_\star(H\Z) = \mathbb{Z}[u_{V_4-V_5}, u_{2V_5-V_5}] \otimes \]
\[ \mathbb{Z}[u_{V_5}, a_{V_5-1}]/(3a_{V_5-1}) \oplus \left( \bigoplus_{i>0} 3\mathbb{Z}(u_{V_5}^{-i}) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/3(\Sigma^{-1}u_{V_5}^{-j}a_{V_5-1}^{-k}) \right). \]
Here \( V_i \) denotes \( V_{A_5,i} \), \( i = 3, 4, 5 \). For any virtual representation \( V \), we have \( |a_V| = -V \), \( |u_V| = |V| - V \).

Theorem 7.5. When 2, 3 are inverted, we have
\[ \pi^A_\star(H\Z) = \mathbb{Z}[u_{V_5-V_4}, u_{2V_5-V_4}] \otimes \]
\[ \mathbb{Z}[u_{V_5}, a_{V_4}]/(5a_{V_4}) \oplus \left( \bigoplus_{i>0} 5\mathbb{Z}(u_{V_5}^{-i}) \right) \oplus \left( \bigoplus_{j,k>0} \mathbb{Z}/5(\Sigma^{-1}u_{V_5}^{-j}a_{V_4}^{-k}) \right). \]

For \( p = 2 \), we have a similar result:

Theorem 7.6. Let \( n_1, n_3, n_4, n_5 \) be arbitrary integers. Assume that 3, 5 are inverted. Then
\[ \pi^A_{n_1+n_3V_{A_5,3}+n_4V_{A_5,4}+n_5V_{A_5,5}}(H\Z) \cong \pi^{K_4}_{(n_1+n_4+2n_5)+(n_3+n_4+n_5)V_{K_4,4}}(H\Z) A_4/K_4. \]

The computation of \( \pi^{K_4}_{n+V_{K_4,4}}(H\Z) \) is given in Appendix B. Unlike Theorems 7.4 and 7.5, it is quite hard to give an explicit expression for this ring. We will provide a less explicit, but still computable expression in Theorems B.7 and B.15.

The unlocalized \( \pi^A_\star(H\Z) \) can be recovered from Theorems 7.4, 7.5, 7.6, B.7 and B.15.

8 Computation of \( \pi^A_\star(HA_G) \)

In fact, \( \pi^A_\star(HA_G) \) is not completely computable since part of the input, the \( K_4 \)-homology with coefficients in \( A_{K_4} \), is still unknown. Most of our result will be decompositions instead of actual computations.

Notation: In later cellular arguments, we will not distinguish \( G \)-CW complexes and \( G \)-CW spectra. Let \( X \) be a \( G \)-CW complex/spectrum and \( H \subset G \). Use \( \Phi^H X \) to denote the CW complex/spectrum consisting of \( H \)-fixed non-equivariant cells in \( X \).

- \( \Phi^H X \) only admits a \( W_G H \)-action, hence is not a \( G \)-space/spectrum in general.
- If \( X \) is a \( G \)-CW complex, \( \Phi^H X \) agrees with \( X^H \).
- For any \( G \)-CW complexes/spectra \( X, Y \), we have \( \Phi^H (X \wedge Y) \cong \Phi^H (X) \wedge \Phi^H (Y) \).

We will invert \( \{3, 5\}, \{2, 5\} \) and \( \{2, 3\} \) separately and apply three different splittings. The unlocalized \( \pi^A_\star(HA_G) \) can be recovered from Theorems 8.2, 8.3 and 8.4.

8.1 Statements of the results

According to the splitting method in section 4.2 and Lemma 4.7, we have the following splittings of \( HA_G \) when different primes are inverted.
Theorem 8.3. Assume that

\( HA_G \simeq (E \mathcal{F}_2 \cap HA_G) \lor (E \mathcal{F}_2 \lor E \mathcal{F}_5 \land HA_G) \lor (E \mathcal{F}_2 \land HA_G). \)

(b) When 2,5 are inverted, we have

\( HA_G \simeq (E \mathcal{F}_2 \cap HA_G) \lor (E \mathcal{F}_2 \lor E \mathcal{F}_5 \land HA_G) \lor (E \mathcal{F}_2 \land HA_G). \)

(c) When 2,3 are inverted, we have

\( HA_G \simeq (E \mathcal{F}_2 \cap HA_G) \lor (E \mathcal{F}_2 \lor E \mathcal{F}_5 \land HA_G) \lor (E \mathcal{F}_2 \land HA_G). \)

The computations of all pieces above are listed below:

Theorem 8.2. Assume that 3,5 are inverted. Let \( n_1, n_3, n_4, n_5 \) be arbitrary integers and

\[ V = n_1 + n_3 V_{A_5,3} + n_4 V_{A_5,4} + n_5 V_{A_5,5}. \]

(a) \( \pi^A_\chi (E \mathcal{F}_2 \cap HA_G) \equiv \pi^K_{k_4} (n_1 + n_3 + 2n_4 + n_5) V_{K_{A_4,3}} (HA_{A_4,3})^{A_4/K_4}. \)

(b) There is a pullback square

\[
\begin{array}{c}
\pi^A_\chi (E \mathcal{F}_2 \cap HA_G) \\
\downarrow \\
\pi^D_{\chi} (E \mathcal{F}_2 \cap HA_G) \\
\downarrow \\
\pi^C_{\chi} (E \mathcal{F}_2 \cap HA_G)
\end{array}
\]

(c) \( \pi^A_\chi (E \mathcal{F}_2 \cap HA_G) \equiv \pi^K_{10} (E \mathcal{F}_2 \cap HA_G). \)

(d) \( \pi^A_\chi (E \mathcal{F}_2 \cap HA_G) \equiv \pi_{n_1} (HZ) = \begin{cases} Z, & \text{if } n_1 = 0, \\ 0, & \text{otherwise}. \end{cases} \)

Theorem 8.3. Assume that 2,5 are inverted. Let \( n_1, n_3, n_4, n_5 \) be arbitrary integers and

\[ V = n_1 + n_3 V_{A_5,3} + n_4 V_{A_5,4} + n_5 V_{A_5,5}. \]

(a) \( \pi^A_\chi (E \mathcal{F}_2 \cap HA_G) \equiv \pi^K_{n_1 + n_3 + 2n_4 + n_5} (HA_{A_3,3})^{D_4/C_5}. \)

(b) There is a pullback square

\[
\begin{array}{c}
\pi^A_\chi (E \mathcal{F}_2 \cap HA_G) \\
\downarrow \\
\pi^D_{\chi} (E \mathcal{F}_2 \cap HA_G) \\
\downarrow \\
\pi^C_{\chi} (E \mathcal{F}_2 \cap HA_G)
\end{array}
\]

(c) \( \pi^A_\chi (E \mathcal{F}_2 \cap HA_G) \equiv \pi^K_{10} (E \mathcal{F}_2 \cap HA_G). \)

(d) \( \pi^A_\chi (E \mathcal{F}_3 \cap HA_G) = \begin{cases} Z, & \text{if } n_1 + n_5 = 0, \\ 0, & \text{otherwise}. \end{cases} \)
Theorem 8.4. Assume that 2, 3 are inverted. Let \( n_1, n_3, n_4, n_5 \) be arbitrary integers and
\[
V = n_1 + n_3 V_{A_3} + n_4 V_{A_4} + n_5 V_{A_5}.
\]
(a) \[
\pi^A_\nu(E, \mathcal{F}_{\{5\}} \wedge H A_G) \cong \pi^C_{n_1 + n_3 + n_4 + n_5 + 2n_4 + 2n_5} (H A_G)^{D_{10}/C_5}.
\]
(b) \[
\pi^A_\nu(E, \mathcal{F}_{\{2,5\}} \wedge H A_G) \cong \pi^D_{n_1} (E, \mathcal{F}_{\{3,5\}} \wedge H A_G).
\]
(c) \[
\pi^A_\nu(E, \mathcal{F}_{\{3,5\}} \wedge H A_G) \cong \pi^C_{n_1} (E, \mathcal{F}_{\{2,5\}} \wedge H A_G)^{D_{10}/C_3}.
\]
(d) \[
\pi^A_\nu(E, \mathcal{F}_{\{3,5\}} \wedge H A_G) = \begin{cases} \mathbb{Z}, & \text{if } n_1 + n_4 = 0, \\ 0, & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{Z}, & \text{if } n_1 + n_4 + n_5 = 0, \\ 0, & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{Z}, & \text{if } n_1 = 0, \\ 0, & \text{otherwise} \end{cases}
\]

Remark 8.5. Let \( \mathcal{F} \) be an arbitrary family of subgroups. Assume that Theorem 3.7 holds after inverting certain prime factors. Since \( E, \mathcal{F} \) is a direct summand of the sphere spectrum, \( H A_G \) appears as a direct summand of \( A_G \). Thus \( E, \mathcal{F} \wedge H A_G = H N_\mathcal{F} \) is still an Eilenberg-MacLane spectrum.

Therefore, when using Theorems 8.2, 8.3 and 8.4, we decompose different pieces into equivariant homology of a point with smaller subgroups and different coefficients. This guarantees that we can compute equivariant homology of a point by induction on the group size.

The proofs of parts (a), (b), (c) in Theorems 8.2, 8.3, 8.4 are similar. We will just prove these three parts in Theorem 8.2. The proofs of part (d) depend on the subgroups outside \( \mathcal{F} \), which are not symmetric when different prime factors are inverted. We will explain details for part (d) of Theorems 8.2, 8.3, 8.4 separately.

### 8.2 Proofs of Theorems 8.2, 8.3, 8.4

**Proof of Theorem 8.2** part (a): This is a direct consequence of Proposition 7.1. □

**Proof of Theorem 8.2** part (b): Apply Theorem 4.6 with \( \mathcal{F} = \mathcal{F}_{\{2,3\}} \) and \( X = E, \mathcal{F}_{\{2,5\}} \wedge H A_G \). We can express
\[
\pi^A_\nu(E, \mathcal{F}_{\{2,3\}} \wedge E, \mathcal{F}_{\{2,5\}} \wedge H A_G)
\]
as the limit of the diagram with objects
\[
\pi^H_{\nu|n}(E, \mathcal{F}_{\{2,5\}} \wedge H A_G), \; H \in \mathcal{F}_{\{2,3\}}.
\]
Notice that \( (E, \mathcal{F}_{\{2,5\}})^H = * \) unless \( H \in \{C_3, D_6, A_4, G\} \). Thus
\[
\pi^H_{\nu|n}(E, \mathcal{F}_{\{2,5\}} \wedge H A_G) = 0
\]
for all \( H \in \mathcal{F}_{\{2,3\}} \) except \( C_3, D_6, A_4 \).

The morphisms in the diagram can be simplified into Weyl group actions and restrictions. Notice that \( W_G D_6 \) and \( W_G A_4 \) are trivial, while \( W_G C_3 = D_6 \). Thus the Weyl group action on \( \pi^C_{n_1} \) is determined by the restriction map \( res_{C_3} \) when computing the limit. Therefore, the diagram is simplified as the pullback square given in the theorem. □

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Proof of Theorem 8.2 part (c): Apply Theorem 4.6 with $\mathcal{F} = \mathcal{F}_{(2,5)}$ and $X = E\mathcal{F}_{(2,3)} \wedge HA_G$. We can express
\[ \pi_V^A(E\mathcal{F}_{(2,5)} \wedge E\mathcal{F}_{(2,3)} \wedge HA_G) \]
as the limit of the diagram with objects
\[ \pi^H_{V|_{\mathcal{F}_{(2,3)}}}(E\mathcal{F}_{(2,3)} \wedge HA_G), \quad H \in \mathcal{F}_{(2,5)}. \]
Notice that $(E\mathcal{F}_{(2,3)})^H \cong *$ unless $H \in \{C_5, D_{10}, G\}$. Thus
\[ \pi^H_{V|_{\mathcal{F}_{(2,3)}}}(E\mathcal{F}_{(2,3)} \wedge HA_G) = 0 \]
for all $H \in \mathcal{F}_{(2,5)}$ except $C_5, D_{10}$.
Since $W_GC_5 = D_{10}$, the Weyl group action on $\pi^C_5$ is determined by the restriction $\text{res}_{C_5}$ when computing the limit. In addition, $W_GD_{10}$ is trivial. Thus the diagram is simplified to a single object $\pi^D_{V|_{\mathcal{F}_{(2,3)}}}(E\mathcal{F}_{(2,3)} \wedge HA_G)$. \(\square\)

Parts (a), (b), (c) of Theorems 8.3 and 8.4 can be proved with similar arguments. Now we focus on part (d).

Proof of Theorem 8.2 part (d): This is exactly the computation of geometric fixed point spectra since $\mathcal{F}_2$ consists of all proper subgroups of $G = A_5$. \(\square\)

Part (d) of Theorem 8.3 is more complicated since $\mathcal{F}_3$ does not contain all proper subgroups ($D_{10}$ is missing). We will have to apply a cellular argument with Proposition 4.8.

Notice that $E\mathcal{F}_3 \wedge HA_G \cong HN_{\mathcal{F}_3}$. Thus $\pi^A_3(E\mathcal{F}_3 \wedge HA_G)$ can be viewed as the integer-graded equivariant homology of the virtual representation spheres with coefficients in $N_{\mathcal{F}_3}$.

Since $\mathcal{F}_3$ contains all proper subgroups except the conjugacy class of $D_{10}$, Proposition 4.8 shows that any $G$-CW complex (or spectrum) $X$ is $E\mathcal{F}_3$-equivalent to
\[ \bigcup_{H \in [D_{10}]} \Phi^H X. \]

Now we only need to study the homology with coefficients in $N_{\mathcal{F}_3}$ for a special type of complexes:

Lemma 8.6. Assume that 2, 5 are inverted. Let $X$ be any $G$-CW complex such that
\[ X = \bigcup_{H \in [D_{10}]} \Phi^H X. \]
Then the natural maps $\Phi^{D_{10}} X \to X$ and $\Phi^G X \to X$ induce an isomorphism
\[ \tilde{H}^G(X; N_{\mathcal{F}_3}) \cong \tilde{H}_*(\Phi^{D_{10}} X) \oplus \tilde{H}_*(\Phi^G X). \]

Proof: The proof of Lemma 2.12 provides us a method to compute $N_{\mathcal{F}_3}$:
\[ N_{\mathcal{F}_3} \cong A_G/M_{\mathcal{F}_3} \]
where $M_{\mathcal{F}_3}(G/H) \subset A(H)$ is generated by all $\{H/K\}$ such that $K \in \mathcal{F}_3$.
We use $\mathcal{S}$ to denote the image of $S \in A_G(G/H)$ in $A_G/M_{\mathcal{F}_3}(G/H)$. Since $\mathcal{F}_3$ contains everything except $D_{10}$ and $G$, $N_{\mathcal{F}_3}(G/H) = 0$ for all $H \subset G$ except $N_{\mathcal{F}_3}(G/D_{10}) \cong \mathbb{Z}$
generated by $\{D_{10}/D_{10}\}$, and $N_{\mathcal{F}_3}(G/G) \cong \mathbb{Z}^2$.
generated by \([G/D_{10}] \) and \([G/G] \).

The transfer and restriction maps are inherited from \(A_G \):

\[
\text{tr}\left([D_{10}/D_{10}]\right) = [G/D_{10}],
\]

\[
\text{res}\left([G/D_{10}]\right) = \text{res}\left([G/G]\right) = [D_{10}/D_{10}].
\]

According to \cite{Wil75}, \(\tilde{H}^G(X; N_{\mathfrak{I}_3})\) is the homology of the complex

\[
\tilde{C}_*(X; N_{\mathfrak{I}_3}) := \left(\bigoplus_{H \subseteq G} \tilde{C}_*(\Phi H X) \oplus N_{\mathfrak{I}_3}(G/H)\right) / \sim.
\]

Since the fixed point subspaces of \(X\) are quite restricted, together with the expression of \(N_{\mathfrak{I}_3}\), this complex is simplified as

\[
\tilde{C}_*(X; N_{\mathfrak{I}_3}) \cong \tilde{C}_*(\Phi^{D_{10}} X) \oplus \tilde{C}_*(\Phi^G X).
\]

We get the required result by taking the homology. \(\square\)

**Proof of Theorem 8.2 part (d):** Recall that \(V = n_1 + n_3 V_{A_{3,3}} + n_4 V_{A_{3,4}} + n_5 V_{A_{3,5}}\). Let \(W = n_1 - V\). According to Proposition 4.8, \(S^W\) is \(E_{\mathfrak{I}_3}\)-equivalent to

\[
\bigcup_{H \in [D_{10}]} \Phi^H S^W.
\]

Thus

\[
\pi^{A_5}_V(E_{\mathfrak{I}_3} \wedge H A_G) = [S^V, E_{\mathfrak{I}_3} \wedge H A_G]^{A_5} \cong \left[S^{n_1}, \left(\bigcup_{H \in [D_{10}]} \Phi^H S^W \right) \wedge E_{\mathfrak{I}_3} \wedge H A_G \right]^{A_5}
\]

\[
\cong \tilde{H}_{n_1}^{A_5} \left(\bigcup_{H \in [D_{10}]} \Phi^H S^W; N_{\mathfrak{I}_3}\right).
\]

According to Lemma 8.6, this homology group is isomorphic to

\[
\tilde{H}_{n_1}(\Phi^{D_{10}} S^W) \oplus \tilde{H}_{n_1}(\Phi^G S^W).
\]

According to the definition of \(V_{A_{3,3}}, V_{A_{3,4}}, V_{A_{3,5}}\), it’s not hard to check:

\[
\Phi^G S^{V_{A_{3,3}}} \cong \Phi^G S^{V_{A_{3,4}}} \cong \Phi^G S^{V_{A_{3,5}}} \cong S^0,
\]

\[
\Phi^{D_{10}} S^{V_{A_{3,3}}} \cong \Phi^{D_{10}} S^{V_{A_{3,4}}} \cong S^0, \quad \Phi^{D_{10}} S^{V_{A_{3,5}}} \cong S^1.
\]

Since \(\Phi\) commutes with products and \(W = -n_3 V_{A_{3,3}} - n_4 V_{A_{3,4}} - n_5 V_{A_{3,5}}\), we have

\[
\Phi^{D_{10}} S^W \cong S^{-n_5}, \quad \Phi^G S^W \cong S^0.
\]

Therefore, we get

\[
\pi^{A_5}_V(E_{\mathfrak{I}_3} \wedge H A_G) = \begin{cases} \mathbb{Z}, & \text{if } n_1 + n_5 = 0, \\ \mathbb{Z}, & \text{if } n_1 = 0, \\ 0, & \text{otherwise} \end{cases}
\]

\(\square\)

**Remark 8.7.** When \(A_G\) is replaced by an arbitrary Mackey functor, the homotopy of \(E_{\mathfrak{I}_3} \wedge H A_G\) may not split into two pieces. In general, there may be non-trivial relations between \(\tilde{C}_*(\Phi^{D_{10}} X)\) and \(\tilde{C}_*(\Phi^G X)\) in the proof of Lemma 8.6.

However, the computation is still available. Since all irreducible representations have trivial \(G\)-fixed subspaces (only contain the origin), \(\tilde{C}_*(\Phi^G X) = \mathbb{Z}\) concentrated in degree 0 when \(X\) is a virtual representation sphere with no constant components. Thus we can still apply the same method when \(n_1 \neq 0\) and compute the other case directly.
The proof of Theorem 8.4 part (d) shares the same idea as Theorem 8.3 part (d) but is more complicated since $\mathcal{F}_\beta$ has two missing conjugacy classes $[D_6]$ and $[A_4]$.

Since $\mathcal{F}_\beta$ contains all proper subgroups except $D_6$, $A_4$, any $G$-CW complex $X$ is $\mathcal{F}_\beta$-equivalent to

$$\bigcup_{H \in [D_6] \cup [A_4]} \Phi^H X.$$

Similarly, an explicit expression for $N_{\mathcal{F}_\beta}(G/H)$ can be obtained from Lemma 2.12

$$N_{\mathcal{F}_\beta}(G/H) = 0 \text{ for all } H \subset G \text{ except }$$

- $N_{\mathcal{F}_\beta}(G/D_6) \cong \mathbb{Z}$ generated by $\{D_6/D_6\}$,
- $N_{\mathcal{F}_\beta}(G/A_4) \cong \mathbb{Z}$ generated by $\{A_4/A_4\}$, and
- $N_{\mathcal{F}_\beta}(G/G) \cong \mathbb{Z}^3$ generated by $\{G/D_6\}$, $\{G/A_4\}$, and $\{G/G\}$.

The structure maps are given by

- $\text{tr}^{G}_{D_6}(\{D_6/D_6\}) = \{G/D_6\}$,
- $\text{tr}^{G}_{A_4}(\{A_4/A_4\}) = \{G/A_4\}$,
- $\text{res}^{G}_{D_6}(\{G/D_6\}) = \text{res}^{G}_{D_6}(\{G/G\}) = \{D_6/D_6\}$,
- $\text{res}^{G}_{A_4}(\{G/A_4\}) = \text{res}^{G}_{A_4}(\{G/G\}) = \{A_4/A_4\}$,
- $\text{res}^{G}_{D_6}(\{G/G\}) = \{G/D_6\} = 0$.

By considering the cellular complex computing homology with coefficients in $N_{\mathcal{F}_\beta}$, we get an alternative version of Lemma 8.6

**Lemma 8.8.** Assume that 2, 3 are inverted. Let $X$ be any $G$-CW complex such that

$$X = \bigcup_{H \in [D_6] \cup [A_4]} \Phi^H X.$$

Then the natural maps $\Phi^{D_6} X \to X$, $\Phi^{A_4} X \to X$, and $\Phi^G X \to X$ induce an isomorphism

$$\tilde{H}^*_s(X; N_{\mathcal{F}_\beta}) \cong \tilde{H}_s(\Phi^{D_6} X) \oplus \tilde{H}_s(\Phi^{A_4} X) \oplus \tilde{H}_s(\Phi^G X).$$

**Proof of Theorem 8.4 part (d):** It follows from Lemma 8.8 and the fact that

- $\Phi^{D_6}(S^{V_5, 4}) \cong \Phi^{D_6}(S^{V_5, 5}) \cong \Phi^{A_4}(S^{V_5, 4}) \cong S^1$,
- $\Phi^{D_6}(S^{V_5, 3}) \cong \Phi^{A_4}(S^{V_5, 3}) \cong \Phi^{A_4}(S^{V_5, 5}) \cong S^0$.

\[ \square \]

**Remark 8.9.** The computation of $\pi_\beta^G(\mathcal{E} \mathcal{F}_\beta \wedge HM)$ can be divided into two parts: Find an expression on homology with coefficients in $N_{\mathcal{F}_\beta}$ and apply the expression on virtual representation spheres after removing most types of cells. This gives an intuitive criterion about how available such computation is.

The computability decreases if any of the following phenomena appears:

1. There exists $H \notin \mathcal{F}_\beta$ such that $N_G H \neq H$.
2. There exist proper subgroups $H_1, H_2 \notin \mathcal{F}_\beta$ such that $H_1 \subset H_2$. 

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9 Mackey functor valued homology

In this section, we will discuss the computation of the Mackey functor valued homology \( π^G_*(H \mathbb{Z}) \). All arguments and theorems in this section still work when \( \mathbb{Z} \) is replaced by other constant Mackey functors.

**Notation:** We use \( \Pi \) to denote the Mackey functor \( \mathbb{Z}^G(H \mathbb{Z}) \) when the virtual \( G \)-representation \( V \) is fixed. Then we have

\[
\Pi(G/H) = [G/H \ltimes S^V, H \mathbb{Z}]^G \cong [S^V, H \mathbb{Z}]^H = π^H_*(H \mathbb{Z}),
\]

\[
\Pi|_H = π^H_*(H \mathbb{Z}).
\]

When all prime factors of \( |G| \) except \( p \) are inverted, since \( H \mathbb{Z} = H \mathbb{Z} \ltimes E \mathbb{F}_p \), we can use Theorem 4.4 to compute \( [S^V, H \mathbb{Z}]^H \) as a subgroup of \( [S^V, H \mathbb{Z}]^{H_P} \) where \( P \) is a Sylow \( p \)-subgroup of \( H \). In order to get the Mackey functor structure on \( \Pi \), we still need to compute the restriction and transfer maps.

Consider any \( H_1 \subset H_2 \subset G \). Choose Sylow \( p \)-subgroups \( P_1, P_2 \) of \( H_1, H_2 \) such that \( P_1 \subset P_2 \). We will compute \( tr^{H_2}_{H_1} \) and \( res^{H_2}_{H_1} \) with the Mackey functor structure of \( \Pi|_{P_2} \) and conjugation maps.

The restriction map is simple:

**Proposition 9.1.** When computing \( \Pi(G/H_1) \) and \( \Pi(G/H_2) \) as subgroups of \( \Pi(G/P_1) \) and \( \Pi(G/P_2) \), \( res^{H_2}_{H_1} \) is determined by \( res^{P_2}_{P_1} \).

**Proof:** The inclusion \( \Pi(G/H_i) \hookrightarrow \Pi(G/P_i) \) is given by \( res^{H_i}_{P_i} \), \( i = 1, 2 \). The proposition follows from the fact that

\[
res^{P_2}_{P_1} \circ res^{H_2}_{H_1} = res^{H_1}_{P_1} \circ res^{H_2}_{H_1}.
\]

\( \square \)

The transfer map is more complicated:

Write the underlying \( P_2 \)-set of \( H_2/H_1 \) as the disjoint union of \( P_2 \)-orbits:

\[
H_2/H_1 = \bigsqcup_{i=1}^t P_2/Q_i.
\]

Choose \( g_1, g_2, ..., g_t \in H_2 \) such that each \( Q_i \) corresponds to \( g_i H_1 \) inside \( H_2/H_1 \). Then we have \( g_i^{-1} Q_i g_i \subset H_1 \), \( i = 1, 2, ..., t \).

**Proposition 9.2.** When computing \( \Pi(G/H_1) \) and \( \Pi(G/H_2) \) as subgroups of \( \Pi(G/P_1) \) and \( \Pi(G/P_2) \), \( tr^{H_2}_{H_1} \) is determined by

\[
\bigoplus_i tr^{P_2}_{Q_i} \circ c_{g_i} \circ res^{P_1}_{g_i^{-1} Q_i g_i}.
\]

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Proof: Notice that

$$\text{tr}_{H_2}^{H_1} : [(H_2/H_1)_+ \wedge S^V, H^2_2] \rightarrow [S^V, H^2_2]$$

is induced by the $H_2$-stable map

$$S^0 = \Sigma^{-W} S^W \rightarrow \Sigma^{-W} (H_2/H_1)_+ \wedge S^W = (H_2/H_1)_+.$$

Here $W$ is an $H_2$-representation with an embedding $H_2/H_1 \hookrightarrow W$. Taking a tubular neighborhood of $H_2/H_1$ inside $S^W$ and collapsing the complement to a point, we get the map $S^W \rightarrow (H_2/H_1)_+ \wedge S^W$ which defines the middle map above.

Now consider the commutative diagram

$$[(H_2/H_1)_+ \wedge S^V, H^2_2] \xrightarrow{\text{tr}_{H_1}^{H_2}} [S^V, H^2_2]$$

$$\downarrow$$

$$[(H_2/H_1)_+ \wedge S^V, H^2_2] \xrightarrow{\text{tr}_{H_2}^{H_1}} [S^V, H^2_2]$$

where the vertical maps come from taking the underlying $P_2$-maps, hence are inclusions by [Theorem 4.4](#).

The bottom horizontal map is induced by the underlying $P_2$-stable map of $S^0 \rightarrow (H_2/H_1)_+.$

For each $i = 1, 2, \ldots, t$, compose the $P_2$-stable map $S^0 \rightarrow (H_2/H_1)_+$ with the projection $H_2/H_1 \rightarrow P_2/Q_i$. The composition can be expressed by

$$S^0 = \Sigma^{-W} S^W \rightarrow \Sigma^{-W} (P_2/Q_i)_+ \wedge S^W = (P_2/Q_i)_+,$$

which exactly induces $\text{tr}_{Q_i}^{P_2}$.

Thus the bottom map

$$[(H_2/H_1)_+ \wedge S^V, H^2_2] \rightarrow [S^V, H^2_2]$$

is given by

$$[(H_2/H_1)_+ \wedge S^V, H^2_2] \equiv \bigoplus_{i=1}^t [(P_2/Q_i)_+ \wedge S^V, H^2_2] \xrightarrow{\oplus \text{tr}_{Q_i}^{P_2}} [S^V, H^2_2].$$

Moreover, the map

$$[(H_2/H_1)_+ \wedge S^V, H^2_2] \rightarrow [(P_2/Q_i)_+ \wedge S^V, H^2_2]$$

is obtained by taking projection and the underlying $Q_i$-map. Thus we can write it as

$$[(H_2/H_1)_+ \wedge S^V, H^2_2] \equiv [S^V, H^2_2] \xrightarrow{\text{res}_{H_1}^{H_2} g_i^{-1}} [S^V, H^2_2] \xrightarrow{\text{res}_{P_2}^{P_1} g_i^{-1}} [S^V, H^2_2].$$

Commute the conjugation by $g_i$ with the restriction map. The composition above becomes

$$c_{g_i} \circ \text{res}_{P_1}^{P_2} \circ \text{res}_{H_1}^{H_2} g_i^{-1}.$$

In conclusion, the left vertical map of the commutative diagram above can be factored as

$$\left(c_{g_i} \circ \text{res}_{P_1}^{P_2} \circ \text{res}_{H_1}^{H_2}\right)_i \circ \text{res}_{H_1}^{H_2}_i,$$

while the bottom horizontal map is given by $\oplus_i \text{tr}_{Q_i}^{P_2}$. Notice that $\text{res}_{P_1}^{P_2}$ is the map we used to compute $\Pi(G/H_1)$ by [Theorem 4.4](#). Thus $\text{tr}_{H_2}^{H_1}$ is determined by

$$\bigoplus_i \text{tr}_{Q_i}^{P_2} \circ c_{g_i} \circ \text{res}_{P_1}^{P_2} \circ \text{res}_{P_1}^{H_1}_i g_i^{-1}.$$

□
**Remark 9.3.** In practice, usually we do not need to compute both restriction and transfer maps. It’s possible that one of them is induced by the other together with the Mackey functor structure, especially when the Mackey functor is cohomological. Recall that the definition and properties of a cohomological Mackey functor are given in section 4.4.

Theoretically, whenever our splitting method in section 4 applies, we can always compute the homotopy of $H\mathbb{Z}$ as an $RO(G)$-graded Green functor instead of an $RO(G)$-graded ring since Theorem 4.4 also preserves the multiplicative structure. But it would be too complicated to finish the complete computation by hand.

For the rest of this section, we will give two examples to show how Proposition 9.1 and Proposition 9.2 work in practice.

**9.1 One example**

We will focus on $G = A_5$ since it is the most complicated group for which $\mathbb{Z}^G_*(H\mathbb{Z})$ is computable by our splitting methods. The computation for other smaller groups (like $D_{2p}$) will be similar and more simple.

When $H$ is a subgroup of $A_5$, a similar argument as in Proposition 7.1 gives us

**Proposition 9.4.** Assume that all prime factors of $|A_5|$ except $p$ are inverted. For any $H \subset A_5$, let $P$ be a Sylow $p$-subgroup of $H$. Then we have

$$\text{res}_P^H : \Pi(A_5/H) \to \Pi(A_5/P)^{WH}.$$  

We choose a specific virtual representation $V = V_{A_5,3} + V_{A_5,4} - V_{A_5,5} - 2$ and compute $\mathbb{Z}^G_*(H\mathbb{Z})$. We will provide details in the application of Proposition 9.1 and how to use the Mackey functor structure to simplify our computation.

We have $V|_{C_3} = 0, V|_{K_4} = V_{K_4,3} - 3, V|_{C_5} = V_{C_5,2} - 2$. The restrictions of $V$ as Mackey functors over $C_3, K_4, C_5$ are given below:

\[
\begin{array}{ccc}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
3 & 1 & 1 \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
1 & 2 & 5 \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
1 & 2 & 2
\end{array}
\]

The Weyl group actions are all trivial in this case.

Using Proposition 9.4 and Proposition 9.1 we can obtain the values of all levels in $\Pi$ and the restriction maps when two of the three prime factors 2, 3, 5 are inverted.

The first diagram below is the standard expression of an $A_5$-Mackey functor. The other three are the expressions of $\Pi$ (with only restriction maps) when 6, 10, or 15 is inverted respectively.
Now we can recover $\Pi$ (without transfer maps) when no prime factors are inverted, which is given in the left graph below. According to the fact that $\Pi$ is cohomological, the transfer maps are determined. The
9.2 Another example

Sometimes we may not be able to obtain all transfer maps from the restriction maps and Mackey functor structure, in which case we need to apply Proposition 9.2. Here is an example on its application.

We will still focus on $G = A_5$. We choose another specific virtual representation $V = 3 - V_{A_5,3} - V_{A_5,4}$ and compute $\pi_G^G(HZ)$. We have $V|_{C_3} = -2V_{C_3,2}$, $V|_{K_4} = 2 - 2V_{K_4,3}$, $V|_{C_5} = 2 - 3V_{C_5,2}$. The restrictions of $V$ as Mackey functors over $C_3, K_4, C_5$ are given below:

Here the Weyl group action is trivial, except that $A_4/K_4$ acts on $F^3$ by permuting the 3 copies of $F_2$. The restriction map $res_{C_2} : F^3_2 \to F_2$ is the projection onto one copy of $F_2$, depending on which $C_2 \subset K_4$ we choose. We will assume the restriction map to be $1 \oplus 0 \oplus 0$.

Now we can still combine Proposition 9.4 and Proposition 9.1 to compute $\Pi$ without transfer maps when different prime factors are inverted, and then recover the unlocalized version of $\Pi$. The answer is given in the left graph below. Notice that the values of $\Pi(A_5/H)$ and restriction maps are more complicated this time. We use $P$ to denote the natural projection.
The complete expression of $\Pi$ is given in the right graph above, where $I$ denotes the natural inclusion. Most of the transfer maps can be obtained from the fact that $\Pi$ is cohomological, except $tr_{A_4}^{-1}$, for which we have to use Proposition 9.2.

Assume that 3, 5 are inverted. We have $P_1 = P_2 = H_1 = K_4$, $H_2 = A_4$. The underlying $K_4$-space of $A_4/K_4$ is three copies of $K_4/K_4$ corresponding to the three $K_4$-cosets in $A_4/K_4$. Write $A_4/K_4 = \{g_1K_4, g_2K_4, g_3K_4\}$. Then we have $Q_1 = Q_2 = Q_3 = K_4$ corresponding to $g_1, g_2, g_3$.

Applying Proposition 9.2 we have

$$tr_{A_4}^{-1} = \bigoplus_{i=1}^3 tr_{A_4}^{-1} \circ c_{g_i} \circ res_{K_4} = c_{g_1} + c_{g_2} + c_{g_3}.$$

Thus the map

$$tr_{K_4} : \mathbb{F}_3^3 = \Pi(A_5/K_4) \rightarrow \Pi(A_5/A_4) \cong \Pi(A_5/K_4)^{A_4/K_4} = \mathbb{F}_2$$

sends each copy of $\mathbb{F}_2$ identically into $\mathbb{F}_2$. Adding the 3-torsion into consideration, we get

$$tr_{K_4} : \mathbb{F}_2 \cong (0,1)^3 \rightarrow \mathbb{F}_3 \times \mathbb{F}_2.$$
Appendix A  Homology of a join

In section 2.4, we construct the universal space $E \mathcal{F}$ as a join. We will provide more details and properties of a join in this appendix.

Definition A.1. The join $X \ast Y$ of two $G$-spaces $X, Y$ is defined as

$$X \ast Y := (X \times Y \times [0, 1] \sqcup X \sqcup Y)/\sim,$$

with equivalence relation given by projections

$$X \times Y \times \{0\} \to X, \quad X \times Y \times \{1\} \to Y.$$

The $G$-action on $X \ast Y$ is induced by the $G$-actions on $X, Y, X \times Y$.

It’s not hard to check:

Lemma A.2. For $G$-CW complexes $X, Y$, $X \ast Y$ has a natural $G$-CW structure. For any subgroup $H$,

(a) contractible if and only if at least one of $X^H, Y^H$ is contractible.

(b) empty if and only if both $X^H, Y^H$ are empty.

We can control the torsion types of $H^G_\ast(X \ast Y; A_G)$ as follows:

Lemma A.3. Assume that the Mackey functor valued homology of $X, Y, X \times Y$ with coefficients in $A_G$ only contains torsion in positive degrees. Then so does the homology of $X \ast Y$. Moreover, for any prime $p$, if $H^G_\ast(X \ast Y; A_G)$ contains $p$-torsion, then at least one of $X, Y, X \times Y$ has $p$-torsion in its homology with coefficients in $A_G$.

Proof: The homology of the join can be computed by the Mayer-Vietoris sequence:

$$X \ast Y = (X \times Y \times [0, 1] \sqcup X \sqcup Y)/\sim = (X \times Y \times \left[0, \frac{1}{2}\right] \sqcup X/\sim) \cup (X \times Y \times \left[\frac{1}{2}, 1\right] \sqcup Y/\sim).$$

We write $X \ast Y$ as the union of two mapping cylinders, which are homotopic to $X, Y$ respectively. The intersection of these two cylinders is $X \times Y \times \{1/2\} \simeq X \times Y$.

Thus we have a short exact sequence

$$0 \to C_\ast(X \times Y) \to C_\ast(X) \oplus C_\ast(Y) \to C_\ast(X \ast Y) \to 0$$

which passes to fixed point subspaces. So we get a long exact sequence on equivariant homology with coefficients in $A_G$:

$$\ldots \to H^G_{\ast+1}(X \ast Y; A_G) \to H^G_{\ast}(X \times Y; A_G) \to H^G_{\ast}(X; A_G) \oplus H^G_{\ast}(Y; A_G) \to H^G_{\ast}(X \ast Y; A_G) \to \ldots$$

On degree 0, the map

$$H^G_0(X \times Y; A_G) \to H^G_0(X; A_G) \oplus H^G_0(Y; A_G)$$

is always an inclusion since the 0th homology is determined by the number of connected components for each fixed point subspace. Thus the long exact sequence above implies that the homology of $X \ast Y$ in positive degrees only contains torsion in the homology of $X, Y, X \times Y$. □
Appendix B  Computation on $K_4$-homology

The computation of $\pi^K_*(H\mathbb{Z})$ in section 7 is not complete since $\pi^K_{*+nV_{K_4,3}}(H\mathbb{Z})$ and the $A_4/K_4$-action on it are required in the case when 3, 5 are inverted. We will compute this last missing part in this appendix.

For any $m, n \in \mathbb{Z}$, notice that $\pi^K_{m+nV_{K_4,3}}(H\mathbb{Z})$ can be expressed as the equivariant homology or cohomology of $S^{(n)}V_{K_4,3}$ with coefficients in $\mathbb{Z}$, which can be computed by considering an explicit CW structure on $S^{(n)}V_{K_4,3}$. In order to simplify our computation, especially for the multiplicative structure, we compare the homotopy of $H\mathbb{Z}$ with the homotopy of $H\mathbb{F}_2$, which is already computed in Ellis-Bloor’s thesis [Ell20].

Let $G = K_4$ throughout the section.

We use $V_1, V_2, V_3, V$ to denote the representations $V_{K_4,1,1}, V_{K_4,1,2}, V_{K_4,1,3}, V_{K_4,3}$. Let $H_1, H_2, H_3$ be the three proper subgroups of $K_4$ such that $V_i$ is the sign representation of $G/H_i, i = 1, 2, 3$.

Partial computation of $\pi^K_*(H\mathbb{F}_2)$ is given below.

**Definition B.1.** The positive cone $\star +$ of $RO(K_4)$ consists of all grades with the form $a + bV_2 + cV_3$ such that $b, c, d \leq 0$.

The negative cone $\star -$ of $RO(K_4)$ consists of all grades with the form $a + bV_2 + cV_3$ such that $b, c, d > 0$.

**Remark B.2.** Any element in $\star+$ can be written as $a - W$ for some $a \in \mathbb{Z}$ and actual representation $W$. The homology of a point in degree $a - W$ is exactly the homology of the representation sphere $S^W$ in degree $a$. This is why we call $\star+$ the positive cone.

Similarly, any element in $\star-$ can be written as $a + W$ for some actual representation $W$. The homology of a point in degree $a + W$ can be expressed as the cohomology of $S^W$ in degree $-a$. Thus we call $\star-$ the negative cone.

**Theorem B.3.** [Ell20, Theorem 4.14] Over the positive cone, we have

$$\pi^K_*(H\mathbb{F}_2) = \frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1y_2y_3 + y_2x_2y_3 + y_1y_2x_3)}.$$  

Here $|x_i| = -V_i$, $|y_i| = 1 - V_i$, $i = 1, 2, 3$.

**Theorem B.4.** [Ell20, Proposition 4.27 and Theorem 4.30] For the negative cone, consider the $\mathbb{F}_2$-linear span

$$\langle \frac{1}{x_1^i y_1^j x_2^k y_2^l x_3^m y_3^n} : i_1, j_1, i_2, j_2, i_3, j_3 \geq 0 \rangle.$$  

Define a self-map $f$ on the graded $\mathbb{F}_2$-module above by multiplication with $x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$. Then

$$\pi^K_{\star-}(H\mathbb{F}_2) = \Theta \cdot \ker(f)$$  

with $|\Theta| = V - 3$.

The ring structure is implied by the generators and $\Theta^2 = 0$. The action of $A_4/K_4 \cong C_3$ is the cyclic permutation on $x_1, x_2, x_3$ and $y_1, y_2, y_3$.

The relation between $H\mathbb{Z}$ and $H\mathbb{F}_2$ is given by

**Theorem B.5.** [Ell20, Theorem 4.40] The Bockstein spectral sequence computing the $RO(K_4)$-graded homology of a point with constant coefficients corresponding to

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{F}_2$$  

collapses to the $E^2$-page.
In other words, \( \pi_{a+bV}^{K_4} (H\mathbb{Z}) \) only contains \( \mathbb{F}_2 \) components unless \( a + 3b = 0 \). When \( a + 3b = 0 \), our computations will show that \( \pi_{-3b+bV}^{K_4} (H\mathbb{Z}) = \mathbb{Z} \).

Applying the Bockstein long exact sequence, we know that the map

\[
\pi_{a+bV}^{K_4} (H\mathbb{Z}) \to \pi_{a+bV}^{K_4} (HF_2)
\]

is an inclusion when \( a + 3b \neq 0 \), and becomes the projection \( \mathbb{Z} \to F_2 \) when \( a + 3b = 0 \). Therefore, we have

**Proposition B.6.** Both the ring structure and the action of \( A_4 / K_4 \) on \( \pi_{a+bV}^{K_4} (H\mathbb{Z}) \) are determined by their images in \( \pi_{a+bV}^{K_4} (HF_2) \).

Notice that \( \pi_{a+bV}^{K_4} (H\mathbb{Z}) \) is the equivariant homology or cohomology of \( S^{nV} \) with coefficients in \( \mathbb{Z} \). We can compute it by explicitly assigning a G-CW structure on \( S^{nV} \):

For any \( n > 0 \), \( S^{nV} = S^{nV_1} \wedge S^{nV_2} \wedge S^{nV_2} \). Each \( S^{nV_i} \) has the G-CW structure with one fixed 0-cell \( e_0 \), and one cell \( (G/H)_t \wedge e_i,j \) for each positive degree \( 0 < j < n \).

Write \( (G/H)_t \wedge e_i,j = e_i,j \vee e_i,j' \). Then the boundary map is expressed as

\[
\partial e_{i,j} = e_{i,j-1} + (-1)^{i-1} e_{i,j-1}, \quad \partial e_{i,j}' = e_{i,j-1}' + (-1)^{i-1} e_{i,j-1}, \quad \text{if } j > 1,
\]

\[
\partial e_{i,1} = \partial e_{i,1}' = e_0.
\]

We also formally define \( e_0' = e_0 \).

The G-CW structure on \( S^{nV} \) can be obtained by smashing the structures on \( S^{nV_i} \), \( i = 1, 2, 3 \). For any \( k, l, m \geq 0 \), the elements in

\[
\{e_{1,k}, e_{1,k}'\} \wedge \{e_{2,l}, e_{2,l}'\} \wedge \{e_{3,m}, e_{3,m}'\}
\]

consist of two \( G \)-cells if \( k, l, m \geq 0 \), or one \( G \)-cell otherwise.

In particular, if exactly one of \( k, l, m \) is zero, we obtain a \( G \)-free cell. If two of \( k, l, m \) are zero, the \( G \)-cell has isotropy group as one of \( H_1, H_2, H_3 \). When \( k = l = m = 0 \), we get the fixed cell \( e_{1,0} \wedge e_{2,0} \wedge e_{3,0} \).

### B.1 The positive cone

We give the expression of the positive cone of \( \pi_{a+bV}^{K_4} (H\mathbb{Z}) \) first:

**Theorem B.7.** The homotopy \( \pi_{a+bV}^{K_4} (H\mathbb{Z}) \), with \( b \leq 0 \), is the subring of

\[
\mathbb{Z}[x_1, y_1, x_2, y_2, x_3, y_3] / (2x_1, 2x_2, 2x_3, x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)
\]

generated by

1. \( x_1^1 y_1^1 x_2^2 y_2^1 x_3^1 y_3^3 \) such that
   \( i_1 + j_1 = i_2 + j_2 = i_3 + j_3 \),
   \( j_1 \equiv j_2 \equiv j_3 \mod 2 \),
   \( j_1j_2j_3 = 0 \).
2. \( x_1^1 y_1^1 x_2^1 y_2^1 x_3^3 y_3^1 + x_1^1 y_1^1 x_2^1 y_2^1 x_3^1 y_3^1 \) such that
   \( i_1 + j_1 = i_2 + j_2 = j_3 + 1 \),
   \( j_1 \equiv j_2 \equiv j_3 \mod 2 \).
3. \( x_1^1 y_1^1 y_2^1 x_3^1 y_3^1 + x_1^1 y_1^1 x_2^1 x_3^1 y_3^1 \) such that
   \( i_1 + j_1 = i_3 + j_3 = i_2 - 1 \),
   \( j_1 \equiv j_2 \equiv 0 \mod 2 \).
4. \( x_1^1 x_2^1 x_3^1 y_3^1 + x_1^1 x_2^1 x_3^1 y_3^1 \) such that
   \( i_2 + j_2 = i_3 + j_3 = i_1 - 1 \),
   \( j_2 \equiv j_3 \equiv 0 \mod 2 \).

The action of \( A_4 / K_4 = C_3 \) is the cyclic permutation on \( x_1, x_2, x_3 \) and \( y_1, y_2, y_3 \).
Remark B.8. The generators we give in the theorem above are not symmetric on $x_1, x_2, x_3$ and $y_1, y_2, y_3$. However, the subring itself will be symmetric after $x_1y_2y_3 + y_2x_2y_3 + y_1y_2x_3$ is quotiented out.

We will prove Theorem B.7 in the rest of this section.

For any $n \geq 0$, $\pi_n^{K^1}(HF)$ is the equivariant homology of $S^nV$ with coefficients in $\mathbb{Z}$, which is computed as the homology of the chain complex:

$$C_*^{K^1}(S^nV; \mathbb{Z}) := C_*(S^nV) \otimes \mathbb{Z}.$$ 

We can express the generators and the boundary map with the following notations:

**Definition B.9.** Let $\lambda$ be the linear self-map of $C_*^{K^1}(S^nV; \mathbb{Z})$ which exchanges $(k, l, m)$ and $(k, l, m)'$ if $k\ell m \neq 0$ and fixes $(k, l, m)$ if $k\ell m = 0$.

**Remark B.10.** For further convenience, we will use $\lambda(k, l, m)$ instead of $(k, l, m)'$. Moreover, we treat $\lambda$ as part of the coefficient of $(k, l, m)$. To be precise, for $a, b \in \mathbb{Z}$, $(a + b\lambda)(k, l, m)$ is the image of $(k, l, m)$ under the map $a \cdot \text{id} + b \cdot \lambda$, which contains $a$ copies of cell $(k, l, m)$ and $b$ copies of cell $(k, l, m)'$. We will call $a + b\lambda$ the coefficient of $(k, l, m)$ in this element.

For any degree $t$, $C_t^{K^1}(S^nV; \mathbb{Z})$ is generated by all $(k, l, m)$ and $\lambda(k, l, m)$ with $0 \leq k, l, m \leq n, k+l+m = t$. The boundary map is given below:

$$\partial(k, l, m) = (1 + (-1)^{k+1}\lambda)(k - 1, l, m) + (-1)^k(1 + (-1)^{l+1}\lambda)(k, l - 1, m) + (-1)^{k+l}(1 + (-1)^{m+1}\lambda)(k, l, m - 1)$$

if $k, l, m > 1$. When $k = 1$, we replace $1 + (-1)^{k+1}\lambda$ by 1. The cases $l = 1$ and $m = 1$ are similar.

$$\partial(k, l, 0) = (1 + (-1)^{k+1})(k - 1, l, 0) + (-1)^k(1 + (-1)^{l+1})(k, l - 1, 0)$$

if $k, l > 0$. The boundaries of $(k, 0, m)$ and $(0, l, m)$ are defined similarly.

$$\partial(k, 0, 0) = (1 + (-1)^{k+1})(k - 1, 0, 0)$$

if $k > 0$. The boundaries of $(0, l, 0)$ and $(0, 0, m)$ are defined similarly.

Moreover, $\partial$ commutes with $\lambda$.

The top degree can be computed directly:

**Lemma B.11.** For degree $3n$, $\ker \partial$ is generated by $(1 + (-1)^n\lambda)(n, n, n)$. Thus

$$H_{3n}^{K^1}(S^nV; \mathbb{Z}) = \mathbb{Z}, \quad H_{3n}^{K^1}(S^nV; \mathbb{F}_2) = \mathbb{F}_2.$$
It is quite hard to compute homology directly by \( \ker \partial / \im \partial \). However, we can use the Bockstein long exact sequence

\[
H^{K_4}_i(S^{nV}; \mathbb{Z}) \xrightarrow{2} H^{K_4}_{3i}(S^{nV}; \mathbb{Z}) \to H^{K_4}_{3i}(S^{nV}; \mathbb{F}_2) \to H^{K_4}_{3i-1}(S^{nV}; \mathbb{Z}) \xrightarrow{2} \cdots
\]

... \( \to H^{K_4}_1(S^{nV}; \mathbb{F}_2) \to \tilde{H}^{K_4}_0(S^{nV}; \mathbb{Z}) \xrightarrow{2} \tilde{H}^{K_4}_0(S^{nV}; \mathbb{Z}) \to \tilde{H}^{K_4}_0(S^{nV}; \mathbb{F}_2) \).

Since \( H^{K_4}_i(S^{nV}; \mathbb{Z}) \) only contains \( \mathbb{F}_2 \)-components except in the top degree, the long exact sequence is broken into several pieces:

\[
0 \to H^{K_4}_{3i}(S^{nV}; \mathbb{Z}) \xrightarrow{2} H^{K_4}_{3i}(S^{nV}; \mathbb{Z}) \to H^{K_4}_{3i}(S^{nV}; \mathbb{F}_2) \to 0,
\]

\[
H^{K_4}_{3i-1}(S^{nV}; \mathbb{Z}) = 0,
\]

\[
H^{K_4}_i(S^{nV}; \mathbb{F}_2) = H^{K_4}_i(S^{nV}; \mathbb{Z}) \oplus \tilde{H}^{K_4}_{i-1}(S^{nV}; \mathbb{Z}), \quad i = 1, 2, \ldots, 3n-1,
\]

\[
\tilde{H}^{K_4}_i(S^{nV}; \mathbb{Z}) = \tilde{H}^{K_4}_0(S^{nV}; \mathbb{F}_2).
\]

The first short exact sequence is \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_2 \to 0 \).

We can compute the homology with coefficients in \( \mathbb{Z} \) by the following strategy:

(1) Compute the dimension of \( H^{K_4}_i(S^{nV}; \mathbb{F}_2) \) as an \( \mathbb{F}_2 \)-vector space. This is not hard since the dimensions of \( \ker \partial \) and \( \im \partial \) with \( \mathbb{F}_2 \)-coefficients can be computed explicitly.

(2) Compute the dimension of \( H^{K_4}_i(S^{nV}; \mathbb{Z}) \) by the pieces of the Bockstein long exact sequence above.

(3) Guess elements in \( \ker \partial \) (with \( \mathbb{Z} \)-coefficients) whose images are \( \mathbb{F}_2 \)-independent modulo \( \im \partial \) (with \( \mathbb{F}_2 \)-coefficients). If we can find enough such elements to match the dimension of the homology, they must generate the whole homology group.

The boundary map with \( \mathbb{F}_2 \)-coefficients is simple:

\[
\partial(k, l, m) = (1 + \lambda) ((k - 1, l, m) + (k, l - 1, m) + (k, l, m - 1))
\]

if \( k, l, m > 0 \). If \( klm = 0 \), \( \partial(k, l, m) = 0 \).

**Lemma B.12.** With \( \mathbb{F}_2 \)-coefficients, \( \ker \partial \) is generated by:

\[
(k, l, m), \quad klm = 0,
\]

\[
(1 + \lambda)(k, l, m), \quad k, l, m > 0.
\]

**Proof:** All elements listed above are inside \( \ker \partial \). It suffices to show that those elements generate the whole kernel.

Assume that the coefficient of some \( (k, l, m) \) \((k, l, m > 0)\) is \( 1 \) or \( \lambda \) in some element \( Z \in \ker \partial \). Further assume that \( k \) is the smallest number among all such cells. Consider the coefficient of \( (k - 1, l, m) \) in \( \partial Z \), which should be 0.

If \( k = 1 \), the coefficient of \( (0, l, m) \) in \( \partial(1, l, m) \) is 1. Since \( (0, l, m) \) does not appear in the boundary of any other cells, its coefficient in \( \partial Z \) is 1 \( \neq 0 \), which is a contradiction.

If \( k > 1 \), the coefficient of \( (k - 1, l, m) \) in \( \partial(k, l, m) \) is \( 1 + \lambda \). The cell \( (k - 1, l, m) \) appears in the boundaries of \( (k, l, m), (k - 1, l + 1, m), (k - 1, l, m + 1) \). Since we already assume \( k \) to be the minimal such number, the other two cells cannot have 1 or \( \lambda \) as coefficients in \( Z \). Thus the coefficient of \( (k - 1, l, m) \) in \( \partial Z \) is \( 1 + \lambda \neq 0 \), which is a contradiction.

Therefore, the coefficient of any \( (k, l, m) \) \((k, l, m > 0)\) is either \( 1 + \lambda \) or 0 in any element in \( \ker \partial \). □

For each dimension \( t \), with \( \mathbb{F}_2 \)-coefficients, the dimension of \( \ker \partial \) agrees with the number of \( (k, l, m) \) such that \( 0 \leq k, l, m \leq n \) and \( k + l + m = t \). The dimension of \( \im \partial \) agrees with the number of \( (k, l, m) \) such that \( 0 < k, l, m \leq n \) and \( k + l + m = t + 1 \). After some combinatorial arguments, we get:
Proposition B.13. The dimensions of $H_*^{K_S}(S^{nV}; \mathbb{F}_2)$ in degrees $0, 1, ..., 3n$ are
$$1, 3, 5, ..., 2n - 1, 2n + 1, 2n, 2n - 1, 2n - 2, ..., 2, 1.$$  
This sequence is obtained by gluing two arithmetic sequences $1, 3, 5, ..., 2n + 1$ and $2n + 1, 2n, ..., 1$.
Thus the dimension of $H_*^{K_S}(S^{nV}; \mathbb{Z})$ in degrees $0, 1, ..., 3n - 1$ are
$$1, 2, 3, ..., n, n + 1, n - 1, n, n - 2, n - 1, n - 3, ..., 1, 2, 0$$  
To be precise, the sequence comes from three arithmetic sequences:
1. $1, 2, ..., n + 1$ for the first $n + 1$ terms;
2. $n + 1, n, n - 1, ..., 2$ for the $(n + 1)$th, $(n + 3)$th, $(n + 5)$th, ..., $(3n - 1)$th terms;
3. $n, n - 1, n - 2, ..., 1, 0$ for the $(n)$th, $(n + 2)$th, $(n + 4)$th, ..., $(3n)$th terms.
In addition, $H_*^{K_S}(S^{nV}; \mathbb{Z}) = \mathbb{Z}$.

Now we just need to guess enough elements in ker $\partial$ with $\mathbb{Z}$-coefficients:

Proposition B.14. If $n$ is odd, the generators of $H_*^{K_S}(S^{nV}; \mathbb{Z})$ can be represented by
\[
(1 - \lambda)(2i + 1, 2j + 1, n), \ 0 \leq i, j \leq \frac{n - 1}{2}; \\
(1 - \lambda)[(2i - 1, 2j, n) + (2i, 2j - 1, n)], \ 0 < i, j \leq \frac{n - 1}{2}; \\
(2i, 0, 2j), (0, 2i, 2j), \ 0 \leq i, j \leq \frac{n - 1}{2}; \\
(2i + 1, 0, 2j) - (2i, 0, 2j + 1), (0, 2i + 1, 2j) - (0, 2i, 2j + 1), \ 0 \leq i, j \leq \frac{n - 1}{2}.
\]
If $n$ is even, the generators of $H_*^{K_S}(S^{nV}; \mathbb{Z})$ can be represented by
\[
(1 + \lambda)(2i, 2j, n), \ 0 \leq i, j \leq \frac{n}{2}; \\
(1 + \lambda)[(2i + 1, 2j, n) - (2i, 2j + 1, n)], \ 0 \leq i, j \leq \frac{n - 2}{2}; \\
(2i, 0, 2j), (2j, 0, 2i), \ 0 \leq i, j \leq \frac{n}{2}; \\
(2i + 1, 0, 2j) - (2i, 0, 2j + 1), (0, 2i + 1, 2j) - (0, 2i, 2j + 1), \ 0 \leq i, j \leq \frac{n - 2}{2}.
\]
Some cells should appear multiple times in different classes, in which case we will only count once.

Proof: By combinatorial arguments, we can show that the number of such elements matches the dimension of the homology (although not simple). It suffices to show that, any non-zero $\mathbb{F}_2$-linear combination of the elements above is not in Im $\partial$ with $\mathbb{F}_2$-coefficients.

For any $Z = \sum_{\alpha \in J}(k_{\alpha}, l_{\alpha}, m_\alpha)$, assume that $\partial Z$ is a linear combination of the elements in the list above with $\mathbb{F}_2$-coefficients. Further assume that $|J|$ reaches its minimum.

For any $\alpha \in J$, we must have $k_{\alpha}, l_{\alpha}, m_\alpha \neq 0$. Otherwise $\partial(k_{\alpha}, l_{\alpha}, m_\alpha) = 0$ and we can remove this cell to make $J$ smaller.

Choose $(k, l, m) \in \{(k_{\alpha}, l_{\alpha}, m_\alpha)\}$ such that $m$ reaches its minimum. Notice that $(k, l, m - 1)$ does not appear in the boundary of any other cells inside $Z$. Thus the coefficient of $(k, l, m - 1)$ in $\partial Z$ is the same as the coefficient in $\partial(k, l, m)$, which is either 1 or $1 + \lambda$, hence non-zero. But $(k, l, m - 1)$ does not appear in the list above, which is a contradiction. □

Finally, we only need to check the image of $H_*^{K_S}(S^{nV}; \mathbb{Z})$ in $\pi_*^{K_S}(H\mathbb{F}_2)$ in order to obtain the multiplicative structure and the action of $A_4/K_4 = C_3$. This is not hard since the generators $x_i, y_i$ in Theorem B.3 come from the homology of $S^{nV}$, which can be decomposed into a $G/H_4 = C_2$ computation. To be precise, in the homology of $S^{nV}$, the cell $(k, l, m)$ or $(1 \pm \lambda)(k, l, m)$, when representing an element in homology, corresponds to $x_1^{n-k}x_2^{n-k-1}y_2x_3^{n-m}y_3^m$,

By transferring the generators in Proposition B.14 to elements in homology, we get Theorem B.7.
B.2 The negative cone

The expression of the negative cone of \( \pi_{*+V}^K(H\mathbb{Z}) \) is given below:

**Theorem B.15.** Define \( U \) as the \( \mathbb{F}_2 \)-span of all \( x_1^{-i_1}y_1^{-j_1}x_2^{-i_2}y_2^{-j_2}x_3^{-i_3}y_3^{-j_3} \) such that

\[
\begin{align*}
i_1 + j_1 &= i_2 + j_2 = i_3 + j_3, \\
j_1, j_2, j_3 &> 0, \quad i_1, i_2, i_3 \geq 0, \\
(i_1, i_2, i_3) &\neq (0, 0, 0).
\end{align*}
\]

Let \( f \) be the self-map on \( U \) given by multiplying \( x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 \).

Let \( T \) be the sub-module of \( U \) generated by all

\[
x_1^{-i_1}y_1^{-j_1} - x_2^{-i_2}y_2^{-j_2}x_3^{-i_3}y_3^{-j_3} + x_1^{-i_1-1}y_1^{-j_1}x_2^{-i_2}y_2^{-j_2}x_3^{-i_3-1}y_3^{-j_3} + x_1^{-i_1-1}y_1^{-j_1}x_2^{-i_2-1}y_2^{-j_2}x_3^{-i_3}y_3^{-j_3-1}
\]

such that \( j_1 \equiv j_2 \equiv j_3 \) (mod 2).

The homotopy \( \pi_{*+V}^K(H\mathbb{Z}) \), with \( b > 0 \), is given by

\[
4\mathbb{Z}\left(\bigoplus_{n>0}(y_1y_2y_3)^{-n}\right) \oplus (\ker(f) \cap T).
\]

We will use the rest of this section to prove **Theorem B.15**.

For any \( n > 0 \), \( \pi_{*+V}^K(H\mathbb{Z}) \) is the equivariant cohomology of \( S^nV \) with coefficients in \( \mathbb{Z} \), which is computed as the cohomology of the cochain complex

\[
C^*_K(S^nV; \mathbb{Z}) := \text{Hom}_{\mathcal{O}_G}(C_*(S^nV), \mathbb{Z}).
\]

Since all restriction maps in \( \mathbb{Z} \) are the identity, the cochain complex above agrees with

\[
\text{Hom}_{\mathbb{Z}}(C_*(S^nV/G), \mathbb{Z}).
\]

Thus we are computing the non-equivariant cohomology of the orbit space \( S^nV/G \).

We can express the cochain complex and the coboundary map with the following notations:

For any \( k, l, m \), let \( [k, l, m] \) be the function sending the orbit of

\[
e_1 \wedge e_2 \wedge e_3, m
\]
to 1 and all other cells to 0.

When \( klm \neq 0 \), let \( [k, l, m]' \) be the function sending the orbit of

\[
e_1' \wedge e_2 \wedge e_3, m
\]
to 1 and all other cells to 0.

Moreover, when some of \( k, l, m \) are greater than \( n \) or less than \( 0 \), we write \( [k, l, m] = 0 \).

**Definition B.16.** Let \( \lambda \) be the linear self-map of \( C^*_K(S^nV; \mathbb{Z}) \) which exchanges \( [k, l, m] \) and \( [k, l, m]' \) if \( klm \neq 0 \) and fixes \( [k, l, m] \) if \( klm = 0 \). We treat \( \lambda \) as part of the coefficient of \( [k, l, m] \) in the same way as in **Remark B.10**.

For any degree \( t \), \( C^t(S^nV/G; \mathbb{Z}) \) is generated by all \( [k, l, m] \) and \( \lambda[k, l, m] \) with \( 0 \leq k, l, m \leq n, k+l+m = t \).

The coboundary map is given below:

\[
\delta[k, l, m] = (1 + (-1)^k)\lambda[k + 1, l, m]
\]

\[
+(1)^k(1 + (-1)^l)\lambda[k, l + 1, m] + (-1)^{k+l}(1 + (-1)^m)\lambda[k, l, m + 1]
\]

if there is at most one 0 in \( [k, l, m] \). In addition, we have

\[
\delta[k, 0, 0] = (1 + (-1)^k)[k+1, 0, 0] + (-1)^k[k, 1, 0] + (-1)^k[k, 0, 1]
\]

The coboundaries of \( [0, l, 0] \) and \( [0, 0, m] \) are defined similarly.

Again, we can compute the top degree directly:
Lemma B.17. For degree $3n$, $1m\delta$ is generated by $(1 + (-1)^{n-1})\lambda[n,n]$. Thus $H_{K_i}^{3n}(S^{nV};\mathbb{Z}) = \mathbb{Z}$, $H_{K_i}^{3n}(S^{nV};\mathbb{F}_2) = \mathbb{F}_2$.

For the remaining degrees, first we describe $1m\delta$:

Lemma B.18. With $\mathbb{F}_2$-coefficients, $1m\delta$ is generated by:

$$(1 + \lambda)[k,l,m], \ k,l,m > 0,$$
\[\delta[0,0,m], \delta[0,l,0], \delta[k,0,0].\]

Proof: It’s not hard to check that, except $\delta[0,0,m], \delta[0,l,0], \delta[k,0,0]$, all other coboundaries are sums of $(1 + \lambda)[k,l,m]$ with $k,l,m > 0$. It suffices to check that any $(1 + \lambda)[k,l,m]$ can be expressed as a coboundary.

Consider an induction on $\min\{k,l,m\}$. Without loss of generality, assume that $k = \min\{k,l,m\}$. The base case $k = 1$ is given by $\delta[0,l,m] = (1 + \lambda)[1,l,m]$.

From case $i$ to $i + 1$: When $l,m \geq k = i + 1$, we have
\[\delta[i,l,m] = (1 + \lambda)[i + 1, l,m] + [i,l + 1,m] + [i, l + 1,m].\]

By induction, $(1 + \lambda)[i,l + 1,m]$ and $(1 + \lambda)[i,l,m + 1]$ are in $1m\delta$. Thus $(1 + \lambda)[i + 1,l,m]$ is also in $1m\delta$. □

Unlike the computation of the positive cone in section 7.1, we cannot explicitly guess the generators, since they are quite complicated. Instead, we will point out the types of cocycles with $\mathbb{F}_2$-coefficients which can be lifted to cocycles with $\mathbb{Z}$-coefficients.

Definition B.19. For $k,l,m \geq 0$ define
\[\langle k,l,m \rangle^+ := [2k + 1,2l,2m] + [2k,2l + 1,2m] + [2k,2l,2m + 1],\]
\[\langle k,l,m \rangle^- := [2k,2l - 1,2m - 1] - [2k - 1,2l,2m - 1] + [2k - 1,2l - 1,2m].\]

Notice that when $k = 0$, $\langle k,l,m \rangle^+$ contains one single term $[0,2l - 1,2m - 1]$. The cases when $l = 0$ or $m = 0$ are similar.

Proposition B.20. (a) Except in the top degree, any cocycle $Z$ with $\mathbb{Z}$-coefficients is a linear combination of $\langle k,l,m \rangle^+$ and $\langle k,l,m \rangle^-$ modulo $(2,1 + \lambda)$.

(b) On the other hand, if the sum of some $\langle k,l,m \rangle^+$ and $\langle k,l,m \rangle^-$ is a cocycle with $\mathbb{F}_2$-coefficients, it can be lifted to a cocycle with $\mathbb{Z}$-coefficients.

Since $(1 + \lambda)C^*(S^{nV},\mathbb{F}_2)$ is contained in $1m\delta$ according to Lemma B.18, this proposition gives us a complete description of the image of $H^*(S^{nV};\mathbb{Z})$ inside $H^*(S^{nV};\mathbb{F}_2)$.

Proof of Proposition B.20 part (a): First, we show that any generators of the cochain complex with forms $[2k,2l,2m]$ and $[2k + 1,2l + 1,2m + 1]$ do not appear in a cocycle modulo $(2,1 + \lambda)$.

Without loss of generality, assume that $k < n$. Consider the element $[2k + 1,2l,2m]$, which appears in the coboundary of $[2k,2l,2m], [2k + 1,2l - 1,2m], [2k + 1,2l,2m - 1]$ (or multiplied by $\lambda$), with coefficients $1 + \lambda, \pm(1 - \lambda), \pm(1 - \lambda)$. The coefficients of $[2k + 1,2l,2m]$ in $\delta\mathbb{Z}$ can never be zero unless the coefficient of $[2k,2l,2m]$ in $Z$ is a multiple of $1 - \lambda \in (2,1 + \lambda)$.

The case of $[2k + 1,2l - 1,2m + 1]$ can be proved in a similar way.

Next, we show that if $[2k + 1,2l,2m]$ appears in some cocycle $Z$ modulo $(2,1 + \lambda)$, then $[2k,2l + 1,2m]$ must also appear.

Consider the element $[2k + 1,2l + 1,2m]$, which appears in the coboundaries of $[2k + 1,2l,2m], [2k,2l + 1,2m], [2k + 1,2l + 1,2m - 1]$ (or multiplied by $\lambda$), with coefficients $\pm(1 - \lambda), \pm(1 - \lambda), 1 + \lambda$. Since the coefficient of $[2k + 1,2l + 1,2m]$ is 0 in $\delta\mathbb{Z}$, the only possibility is that $[2k + 1,2l + 1,2m - 1]$ does not appear in $Z$, while $[2k + 1,2l,2m]$ and $[2k,2l + 1,2m]$ have the same coefficient modulo $(2,1 + \lambda)$.

The same argument can be applied to any other pair of components in $\langle k,l,m \rangle^+$ or $\langle k,l,m \rangle^-$. □

In conclusion, we proved that any cocycle must be a linear combination of $\langle k,l,m \rangle^+$ and $\langle k,l,m \rangle^-$ modulo $(2,1 + \lambda)$. □
Proof of Proposition B.20 part (b): We only consider odd degrees, for which we only have \((k, l, m)^+\).

The case of even degrees can be proved in a similar way.

Notice that
\[
\delta(k, l, m)^+ = (1 - \lambda)([2k + 2, 2l, 2m] + [2k, 2l + 2, 2m] + [2k, 2l, 2m + 2]).
\]

Moreover, each \([2k, 2l, 2m]\) only appears in the coboundaries of \((k - 1, l, m)^+, (k, l - 1, m)^+, (k, l, m - 1)^+,\) with the same coefficient \(1 - \lambda\).

Assume that \(\sum_{\alpha \in J}(k_{\alpha}, l_{\alpha}, m_{\alpha})^+\) is a cocycle modulo 2. Then with \(\mathbb{Z}\)-coefficients, the coboundary consists of 2\((1 - \lambda)[2k, 2l, 2m]\) for some \(k, l, m\).

Consider any such \([2k, 2l, 2m]\). Find one \((k_{\alpha}, l_{\alpha}, m_{\alpha})^+\) whose coboundary contains \((1 - \lambda)[2k, 2l, 2m]\).

Without loss of generality, assume that we have \((k - 1, m, n)^+\). We change the component of \([2k - 1, 2m, 2n]\) to \(\lambda[2k - 1, 2m, 2n]\). The image of \((k - 1, m, n)^+\) is unchanged modulo \((2, 1 + \lambda)\). However, \(\delta(k - 1, m, n)^+\) is changed from
\[
(1 - \lambda)([2k, 2l, 2m] + [2k - 2, 2l + 2, 2m] + [2k - 2, 2l, 2m + 2])
\]

to
\[
(1 - \lambda)(-2k, 2l, 2m) + [2k - 2, 2l + 2, 2m] + [2k - 2, 2l, 2m + 2]).
\]

Thus we eliminate \(2(1 - \lambda)[2k, 2l, 2m]\) inside \(\delta \sum_{\alpha \in J}(k_{\alpha}, l_{\alpha}, m_{\alpha})^+\).

Applying the same procedure for each \([2k, 2l, 2m]\). Finally we can make \(\sum_{\alpha \in J}(k_{\alpha}, l_{\alpha}, m_{\alpha})^+\) into a cocycle with \(\mathbb{Z}\)-coefficients. \(\square\)

Now we still have the last piece in order to finish the ring structure of the negative cone:

Lemma B.21. Consider \(\Theta\) and \(y_{1}y_{2}y_{3}\) as generators of \(\pi_{-3}^{K_{4}}(HZ)\) and \(\pi_{-3}^{K_{4}}(HZ)\). We have \(\Theta y_{1}y_{2}y_{3} = 4\).

Proof: Consider the Mackey functor valued homotopy:
\[
\tilde{\pi}_{-3}^{K_{4}}(HZ) \quad \text{and} \quad \pi_{-3}^{K_{4}}(HZ).
\]

For any \(H \subset G = K_{4}\), we have
\[
\tilde{\pi}_{-3}^{K_{4}}(HZ)(G/H) = H_{3}^{H}(S_{V}; \mathbb{Z}) = \mathbb{Z},
\]
\[
\pi_{-3}^{K_{4}}(HZ)(G/H) = H_{3}^{H}(S_{V}; \mathbb{Z}) = \mathbb{Z}.
\]

The restriction and transfer maps are computed in [Ang22, Section 8]:
\[
\tilde{\pi}_{-3}^{K_{4}}(HZ) \rightarrow \tilde{\mathbb{Z}}, \quad \pi_{-3}^{K_{4}}(HZ) \rightarrow \mathbb{Z}.
\]

Here \(\tilde{\mathbb{Z}}\) is the \(K_{4}\)-Mackey functor with \(\mathbb{Z}\)-values and identity transfer maps.

The product between \(\Theta\) and \(y_{1}y_{2}y_{3}\) can be computed from the \(G/G\)-value of
\[
\tilde{\pi}_{-3}^{K_{4}}(HZ) \square \tilde{\pi}_{-3}^{K_{4}}(HZ) \rightarrow \tilde{\pi}_{0}^{K_{4}}(HZ),
\]
which is in fact \(\tilde{\mathbb{Z}} \square \tilde{\mathbb{Z}} \rightarrow \tilde{\mathbb{Z}}\). On the \(G/e\)-value, we have the common multiplication \(\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}\). Thus on the \(G/G\)-value, \((1, 1) \in \mathbb{Z} \times \mathbb{Z}\) is sent to 4 \(\in \mathbb{Z}\). \(\square\)

Theorem B.15 can be proved by combining Theorem B.4, Proposition B.20 and Lemma B.21.

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