A QUICK AND DIRTY IRREDUCIBILITY TEST FOR
MULTIVARIATE POLYNOMIALS OVER $\mathbb{F}_q$

H.-C. GRAF V. BOTHMER AND F.-O. SCHREYER

Abstract. We provide some statistics about an irreducibility/reducibility test for multivariate polynomials over finite fields based on counting points. The test works best for polynomials in a large number of variables and can also be applied to black box polynomials.

1. Introduction

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$ be a polynomial. Since $f(x)$ can take only $q$ possible values for every point in $x \in \mathbb{A}^n(\mathbb{F}_q)$ we expect that $f(x) = 0$ for about $\frac{1}{q}$ of the points $\mathbb{A}^n(\mathbb{F}_q)$. If on the other hand $f = gh$ is a product of two polynomials $g, h \in \mathbb{F}_q[x_1, \ldots, x_n]$, we have $f(x) = 0$ if $g(x) = 0$ or $h(x) = 0$. So one might expect that products of polynomials satisfy $f(x) = g(x)h(x) = 0$ for approximately $\frac{2}{q} - \frac{1}{q^2}$ of the points $x \in \mathbb{A}^n(\mathbb{F}_q)$. This phenomenon is well explained by the Weil formulas [Mil80].

In this article we investigate the following irreducibility test for multivariate polynomials $f$ over $\mathbb{F}_q$:

Evaluate $f$ at $N$ random points. We reject the hypothesis that $f$ is reducible, if the fraction of zeros $\gamma_q(f)$ found is significantly smaller than $\frac{2}{q} - \frac{1}{q^2}$. Note that $99.5\%$ of all polynomial functions satisfy

$$\gamma_q(f) \leq \frac{1}{q} + 2.58 \sqrt{\frac{\frac{1}{q}(1 - \frac{1}{q})}{q^n}}.$$ 

This irreducibility test is quick, since the number of evaluations needed to detect a given percentage $1 - \epsilon$ of all products of polynomial functions or all general polynomial functions do not depend on the degree of the polynomials considered respectively, i.e.

$$N \sim O(-q \ln \epsilon).$$

On the other hand it is dirty, since it does not give a definite answer. Moreover we cannot make $\epsilon$ arbitrarily small, because $N$ is bounded by $q^n$, the number of $\mathbb{F}_q$ rational points in $\mathbb{A}^n(\mathbb{F}_q)$. There will always be a few polynomials that cannot be correctly classified by our method at all. For example the product of an irreducible, absolutely reducible polynomial with a further absolutely irreducible polynomial.
The test works for implicitly given (black box) polynomials as well. We give examples of such polynomials below.

The expected fraction of zeros for special classes of polynomials is also larger than \( \frac{1}{q} \). For example, the expected fraction of zeros for \( n \times n \) determinants is

\[
E(\gamma_{q,\text{det}}) = 1/q + 1/q^2 - 1/q^5 - 1/q^7 + O(1/q^{12})
\]

for \( n \geq 12 \).

**Notation 1.1.**

- \( \mathbb{F}_q \) \quad the finite field with \( q \) elements
- \( X \subset \mathbb{A}^n \) \quad an affine algebraic set
- \( X(\mathbb{F}_q) \) \quad the \( \mathbb{F}_q \)-rational points of \( X \)
- \( |X| = |X(\mathbb{F}_q)| \) \quad the number of \( \mathbb{F}_q \)-rational points of \( X \)
- \( \gamma_q(X) \) \quad the fraction of \( \mathbb{F}_q \)-rational points in \( \mathbb{A}^n \) that are contained in \( X \)
- \( B(N, p, k) = \binom{N}{k} p^k (1-p)^{N-k} \) \quad the binomial distribution
- \( N \) \quad the number of trials
- \( p \) \quad the success probability
- \( k \) \quad the number of successes
- \( N(\mu, \sigma) \) \quad the normal distribution with mean \( \mu \) and variance \( \sigma^2 \)

\( B(N, p) \) can be approximated by \( N(p, \sqrt{p(1-p)/N}) \).

2. Fractions of Zeros

**Example 2.1.** We choose fixed polynomials \( f_1, f_2 \) of degree 5 and \( f_3 \) of degree 10 in \( \mathbb{Z}[x_1, \ldots, x_4] \) with coefficients in \([-9, 9] \) using the random number generator of the computer algebra system MACAULAY 2 [GS] and consider \( f = f_1 f_2 + 7 f_3 \). Let \( X \) be the vanishing set \( V(f) \).

A black-box polynomial is a polynomial for which it is easy to check \( f(x) = 0 \), but the explicit formula for \( f \) in terms of the unknowns \( x_1 \ldots x_n \) is hard or impossible to write down.

**Example 2.2.** Let \( S_d \subset H^0(\mathbb{P}^2, \mathcal{O}(d)) \) be the hypersurface of singular homogeneous polynomials \( f \) of degree \( d \) in 3 variables. For each point \( f \in H^0(\mathbb{P}^2, \mathcal{O}(d)) \) it is easy to decide whether \( f \in S_d \) via the Jacobi criterion. On the other hand the equation of \( S_d \) in the \( \binom{d+2}{2} \) variables is not obvious.

**Example 2.3.** Let \( C \subset \mathbb{P}^4 \) be the determinantal curve of degree 10 and genus 6 defined by the maximal minors of the following \( 5 \times 3 \) matrix

\[
\begin{pmatrix}
  x_0 + x_1 - x_3 - x_4 & x_0 - x_1 - x_2 - x_4 & -x_0 + x_3 + x_4 \\
  -x_0 - x_2 + x_3 + x_4 & x_0 - x_1 - x_2 - x_3 + x_4 & -x_0 + x_1 - x_2 + x_3 + x_4 \\
  -x_0 - x_2 - x_3 - x_4 & -x_0 - x_1 - x_3 - x_4 & -x_1 + x_4 \\
  -x_1 - x_2 - x_3 + x_4 & -x_1 - x_2 & -x_1 + x_2 \\
  -x_0 + x_1 - x_2 - x_3 - x_4 & -x_0 + x_2 - x_3 + x_4 & x_0 - x_1 + x_2 + x_3 + x_4
\end{pmatrix}
\]

Let \( D = \{ H \in \mathbb{P}^4 \mid H \cap C \text{ is singular} \} \) be the dual variety of \( C \).
Definition 2.4. Let $X \subset \mathbb{A}^n$ an algebraic set. We denote by
\[ \gamma_q(X) := \frac{|X(\mathbb{F}_q)|}{|\mathbb{A}^n(\mathbb{F}_q)|} \]
the fraction of $\mathbb{F}_q$-rational points on $X$. In particular for a hypersurface $X = V(f)$ we have $\gamma_q(f) = \gamma_q(V(f))$. We call $\gamma_q(f)$ the fraction of $\mathbb{F}_q$-rational zeros of $f$.

Example 2.5. We estimate $\gamma_q$ in three of our examples by evaluating in $N = 1000$ random points over all primes up to 17. The following table gives the 99% confidence interval for $\gamma_q$:

| $q$ | $X$       | $S_8$        | $D$        |
|-----|-----------|--------------|------------|
| 2   | 56.7% ± 4.0% | 68.4% ± 2.9% | 55.3% ± 4.1% |
| 3   | 33.8% ± 3.9% | 42.3% ± 3.1% | 49.2% ± 4.1% |
| 5   | 17.9% ± 3.1% | 24.0% ± 2.6% | 24.9% ± 3.5% |
| 7   | 26.2% ± 3.6% | 16.8% ± 2.3% | 35.3% ± 3.9% |
| 11  | 9.3% ± 2.4%  | 8.9% ± 1.8%  | 8.0% ± 2.2%  |
| 13  | 8.6% ± 2.3%  | 9.6% ± 1.8%  | 8.4% ± 2.3%  |
| 17  | 5.2% ± 1.8%  | 8.1% ± 1.7%  | 5.9% ± 1.9%  |

In this article we will explain these numbers.

Remark 2.6. We can compute the true values $\gamma_2(X) = 56.3\%$, $\gamma_3(X) = 34.6\%$, $\gamma_5(X) = 18.7\%$ and $\gamma_7(X) = 27.6\%$ with the same effort, since there are less than 1000 rational points in $\mathbb{A}^4(\mathbb{F}_q)$ for $q \leq 7$.

To study the map $\gamma_q : \mathbb{F}_q[x_1 \ldots x_n] \to [0, 1], f \mapsto \gamma_q(f)$ we note that $\gamma_q(f)$ factors over the ring $R := \text{map}(\mathbb{A}^n(\mathbb{F}_q), \mathbb{F}_q)$:

\[
\begin{array}{ccc}
\mathbb{F}_q[x_1 \ldots x_n] & \xrightarrow{\gamma_q} & [0, 1] \\
\psi & \downarrow & \\
R & \leftarrow & \\
\end{array}
\]

Lemma 2.7. $\psi$ is surjective.

Proof. Since $|\mathbb{A}^n(\mathbb{F}_q)| = q^n < \infty$ we can find a polynomial with prescribed values at these points via interpolation. \hfill \square

We study the distribution of $\gamma_q$ on $R$ by regarding it as a random variable on the finite probability space $(R, \Omega, P)$ with $\Omega$ the sigma algebra of all subsets of $R$ and $P$ the constant probability measure.

Proposition 2.8. The distribution of $\gamma_q$ on $R$ is binomial
\[ P \left( \gamma_q = \frac{k}{q^n} \right) = B \left( q^n, \frac{1}{q}, k \right). \]
In particular the expectation value of $\gamma_q$ is $E(\gamma_q) = \frac{1}{q}$. 
Proof. We have to count the maps \( f \in \mathbb{R} \) that map precisely \( k \) different points to 0. Since the values at different points are independent, this number is
\[
\binom{q^n}{k} 1^k \cdot (q - 1)^{n-k}
\]
The probability that \( \gamma_q = \frac{k}{q^n} \) is therefore
\[
P\left( \gamma_q = \frac{k}{q^n} \right) = \binom{q^n}{k} \left( \frac{1}{q} \right)^k \cdot \left( \frac{q-1}{q} \right)^{n-k} = B\left( q^n, \frac{1}{q}, k \right)
\]
\[\Box\]

Example 2.9. Consider maps \( f \in R = \text{map}(\mathbb{A}^4(\mathbb{F}_{11}), \mathbb{F}_{11}) \). The distribution of fractions of zeros is
\[
P(\gamma_{11} = k/11^4) = B(11^4, 1/11, k).
\]
From its approximation by the normal distribution \( \mathcal{N}(0.0909, 0.0024) \) we obtain
\[
P(0.0847 \leq \gamma_{11} \leq 0.0971) \geq 99%.
\]
We now consider products. The random variable
\[
\gamma_{q,\cup}: \mathbb{R} \times \mathbb{R} \to [0,1], \gamma_{q,\cup}(f,g) = \gamma_q(fg) = |V(f) \cup V(g)|/q^n
\]
is assigned to each pair of functions the fraction of zeros of their product.

Proposition 2.10. On \( R \times R \) the distribution of \( \gamma_{q,\cup} \) is
\[
P(\gamma_{q,\cup} = k/q^n) = B\left( q^n, (2q-1)/q^2, k \right).
\]
In particular the expectation value of \( \gamma_{q,\cup} \) is
\[
E(\gamma_{q,\cup}) = \frac{2q-1}{q^2} = 1 - \left( \frac{q-1}{q} \right)^2.
\]
Proof. The value of \( f \cdot g \) in a point \( x \) depends on the values of \( f \) and \( g \) at \( x \). There are \( q^2 \) ways of choosing these values of which \( (q-1)^2 \) give \( (f \cdot g)(x) \neq 0 \). \[\Box\]

Example 2.11. Consider pairs \((f,g)\) of functions in \( R \) as in Example 2.9.
The distribution of \( \gamma_{11,\cup} \) is now
\[
P(\gamma_{11,\cup} = k/11^4) = B(11^4, 21/11^2, k).
\]
From its approximation by the normal distribution \( \mathcal{N}(0.1736, 0.0031) \), we obtain
\[
P(0.1655 \leq \gamma_{11,\cup} \leq 0.1816) \geq 99%
\]
Note that this range does not intersect
\[
P(0.0847 \leq \gamma_{11,\cup} \leq 0.0971) \geq 99%.
\]

Geometrically products of functions correspond to the union of their zero-sets. \( \gamma_q \) also behaves well under other geometric operations:
Points on a hypersurface of degree 10 in $\mathbb{A}^4$

Figure 1. 99% of polynomial functions on $\mathbb{A}^4$ have $\gamma_q$ between the continuous lines. 99% of products have $\gamma_q$ between the dashed lines.

**Proposition 2.12 (Intersection).** Let $X \subset \mathbb{A}^n$ be a subvariety. We consider the random variable

$$\gamma_{q, \cap X}: R \to [0, 1], \quad \gamma_{q, \cap X}(f) = |V(f) \cap X|/q^n.$$  

The distribution of $\gamma_{q, \cap X}$ is

$$P(\gamma_{q, \cap X} = k/q^n) = \binom{|X|}{1/q,k}.$$  

In particular, the expectation value of $\gamma_{q, \cap X}$ is $E(\gamma_{q, \cap X}) = \gamma_q(X)/q$, where $\gamma_q(X) = |X|/q^n$ is the fraction of points on $X$ in $\mathbb{A}^n(\mathbb{F}_q)$.

**Proof.** Clearly, $x \in X \cap V(f)$ if and only if $x \in X$ and $f(x) = 0$. Since the values of $f$ can be chosen independently on the points of $X$, we have

$$P(x \in \ker f \cap X | x \in X) = \frac{1}{q}.$$

□

**Corollary 2.13.** Consider the random variable

$$\gamma_{q, \cap}: R^c \to [0, 1], \quad \gamma_{q, \cap}(f_1, \ldots, f_c) = |V(f_1) \cap \cdots \cap V(f_c)|/q^n.$$  

Then the expected fraction of points is $E(\gamma_{q, \cap}) = \frac{1}{q^c}$.

**Proof.** Use Proposition 2.12 inductively. □

Notice that for polynomials $f_1, \ldots, f_c$ the expected codimension of $V(f_1, \ldots, f_c) \subset \mathbb{A}^n$ is also $c$.  

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*Figure 1.*
Singular curves in $\mathbb{P}^2$

**Figure 2.** The diagram shows the expectation values for various classes of polynomials in a large number of variables, and the measurement for $S_8$, the hypersurface of singular plane curves of degree 8. Note that the diagram tells that about 70% of all plane curves over $\mathbb{F}_2$ are singular.

**Proposition 2.14** (Substitution). Let $R^m = \text{map}(\mathbb{A}^n(\mathbb{F}_q), \mathbb{A}^m(\mathbb{F}_q))$ and $X \subset \mathbb{A}^m(\mathbb{F}_q)$ a subset. Consider the random variable

$$
\gamma_{q, \text{subst}} : R^m \to [0, 1], \quad \gamma_{q, \text{subst}}(\phi) = |\phi^{-1}X|/q^n
$$

The distribution of $\gamma_{q, \text{subst}}$ is

$$
P(\gamma_{q, \text{subst}} = k/q^n) = \mathcal{B}(q^n, \gamma_q(X), k).
$$

In particular the expectation value of $\gamma_{q, \text{subst}}$ is $E(\gamma_{q, \text{subst}}) = \gamma_q(X) = |X|/q^n$.

**Proof.** Choosing functions $f_1, \ldots, f_n$ is equivalent to independent choice of the image points. Therefore the probability of $\phi^{-1}(X)$ containing exactly $k$ points is the same as the probability of hitting $k$ points of $X$ while choosing $q^n$ points in $\mathbb{F}_q^n$. This gives the desired binomial distribution. □

3. Determinantal Varieties

Even though we have shown, that $E(\gamma_q) = \frac{1}{q}$ with a small variance on the set of all functions from $A$ to $\mathbb{F}_q$, there are special classes of functions that have larger expected $\gamma_q$.

It turns out that this behavior is common for determinants:
Proposition 3.1. Let $X \subset \mathbb{A}^{nm}$ be the determinantal variety of $n \times m$ matrices with $n \leq m$ of rank less than $n$. Then the fraction of points on $X$ is
\[
\gamma_q(X) = 1 - \prod_{i=0}^{n-1} \left(1 - \frac{1}{q^{m-i}}\right),
\]
i.e. $X$ contains $\gamma_q(X) \cdot q^{nm}$ points.

Proof. We prove that the number of matrices that have maximal rank is
\[
\prod_{i=0}^{n-1} (q^m - q^i)
\]
by induction. $M$ is a matrix of full rank if and only if the first $n - 1$ rows form a matrix of full rank and the last row is linearly independent of the first $n - 1$ rows. Since there are $q^{n-1}$ linear combinations of the first $n - 1$ rows we obtain a further factor $(q^m - q^{n-1})$.

\[\square\]

Corollary 3.2. On the space of matrices $R^{nm}$, consider the random variable
\[
\gamma_q,\det: R^{nm} \to [0, 1], \quad \gamma_q,\det(M) = \frac{|\{x \in \mathbb{A}^n \mid \text{rank } M(x) < n\}|}{q^n}.
\]
Then the fraction of zeros has expectation value
\[
E(\gamma_q,\det) = 1 - \prod_{i=0}^{n-1} \left(1 - \frac{1}{q^{m-i}}\right) = \frac{1}{q^{m-n+1}} + \ldots
\]
The distribution of $\gamma_q,\det$ is
\[
P(\gamma_q,\det = k/q^n) = \text{B}(q^n, E(\gamma_q,\det), k)
\]
Proof. Substitute functions for the variables in the generic $n \times m$ matrix and use Proposition 2.14.\[\square\]

In the special case of $n \times n$ square matrices we have
\[
E(\gamma_q,\det) = \frac{1}{q} + \frac{1}{q^2} - \frac{1}{q^5} - \frac{1}{q^7} + O(1/q^{12})
\]
for $n \geq 12$.

Example 3.3 (Example 2.2 continued). For small primes the divisor $S_d$ has more points than expected for irreducible polynomials, but not enough to seem reducible, see Figure 2. Our measurements are consistent with the well known fact that $S_d$ is an irreducible determinantal hypersurface [GKZ94, Chapter 13, Prop. 1.6 and 1.7].
Points on the dual variety of a curve in $C \subset \mathbb{P}^4$

![Graph showing fraction of zeros for determinantal and general estimates for Example 2.3.

\textbf{Figure 3.} $C$ has a simple node over $\mathbb{F}_7$ and is smooth over $\mathbb{F}_p$ for $p = 5, 11, 13, 17.$

4. Testing

To decide between two binomial distributions with success probabilities $p_1 < p_2$ and $N$ experiments, we compute empirical probability $\bar{p} = \frac{k}{N}$ and decide for $p_1$ if

$$\bar{p} \leq \bar{p}_{\text{middle}} = \sqrt{p_1 p_2 \frac{\sqrt{p_1(1-p_2)} + \sqrt{p_2(1-p_1)}}{\sqrt{p_1(1-p_1)} + \sqrt{p_2(1-p_2)}}} \approx \sqrt{p_1 p_2}.$$\

To achieve a confidence level of $1 - \epsilon$ we choose $s = s(\epsilon)$ such that

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-\frac{x^2}{2}} dx = \epsilon$$

and $N$ such that

$$\sqrt{N} \geq s(\epsilon) \frac{\sqrt{p_1(1-p_1)} + \sqrt{p_2(1-p_2)}}{p_2 - p_1}.$$\

In our case we have

$$p_1 \leq \frac{1}{q} + s(\epsilon) \sqrt{\frac{\frac{1}{q}(1 - \frac{1}{q})}{q^n}}$$

for $1 - \epsilon$ of all polynomials and

$$p_2 \geq \frac{2q - 1}{q^2} - s(\epsilon) \sqrt{\frac{\frac{2q-1}{q^2}(1 - \frac{2q-1}{q^2})}{q^n}}.$$\

for $1 - \epsilon$ of all products of polynomials. The decision based on the empirical probability $\bar{p} = \frac{k}{N}$, is then correct in $1 - \epsilon$ cases of the experiments. Note
however, that for fixed $n$ and $q$ we cannot make $\epsilon$ arbitrarily small, since we need $p_1 \leq p_2$.

An easy calculation gives the following estimate
\[
\sqrt{N} \geq s(\epsilon)\frac{(2q)^3}{q - 1 - 2sq^{n-2}}
\]
for $q \geq 3$, which approaches $s(\epsilon)\sqrt{2q}$ for large $n$ or $q$. Since $s(\epsilon) = O(\sqrt{-\ln(\epsilon)})$, we conclude that $N$ grows like $O(-q\ln(\epsilon))$.

For $\epsilon = 0.5\%$, $s = 2.58$, the number of trials needed is

\[
\begin{array}{cccccccc}
   & 2 & 3 & 5 & 7 & 11 & 13 & 17 \\
\hline
  n = 1 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
  n = 2 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
  n = 3 & \infty & \infty & \infty & 28373 & 2355 & 1908 & 1669 \\
  n = 4 & \infty & \infty & 1103 & 647 & 634 & 682 & 803 \\
  n = 5 & \infty & 1705 & 367 & 369 & 482 & 551 & 695 \\
  n = 6 & \infty & 384 & 259 & 308 & 447 & 521 & 673 \\
  n = 7 & 4457 & 224 & 225 & 289 & 437 & 513 & 667 \\
  n = 8 & 619 & 173 & 212 & 283 & 434 & 511 & 666 \\
  n = 9 & 295 & 151 & 206 & 280 & 433 & 511 & 666 \\
  n = 10 & 197 & 140 & 204 & 279 & 433 & 511 & 665 \\
\end{array}
\]

$\infty$ indicates that there are not enough points in $\mathbb{A}^n(\mathbb{P}_q)$ to perform the test for the required $\epsilon = 0.5\%$. In case we can perform the test, the deciding number of successes $N_{p_{middle}}$ is

\[
\begin{array}{cccccccc}
   & 2 & 3 & 5 & 7 & 11 & 13 & 17 \\
\hline
  n = 1 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
  n = 2 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
  n = 3 & \infty & \infty & 5607 & 301 & 207 & 139 \\
  n = 4 & \infty & \infty & 303 & 128 & 81 & 74 & 66 \\
  n = 5 & \infty & 754 & 101 & 73 & 61 & 59 & 57 \\
  n = 6 & \infty & 170 & 71 & 61 & 57 & 56 & 55 \\
  n = 7 & 2821 & 99 & 61 & 57 & 55 & 55 & 55 \\
  n = 8 & 391 & 76 & 58 & 56 & 55 & 55 & 55 \\
  n = 9 & 186 & 67 & 56 & 55 & 55 & 55 & 55 \\
  n = 10 & 125 & 62 & 56 & 55 & 55 & 55 & 55 \\
\end{array}
\]

5. Higher codimension

In principle this method can be applied to algebraic sets of higher codimension.

Consider two surfaces in $\mathbb{P}^4$ and their union. We would like to distinguish their union form the irreducible examples. One possibility is to consider the Chow form which is a determinantal hypersurface on $G(2,5)$ in this case. In Figure 4 we indicate the 5% and the 95% quantiles of $\gamma_q$ for the Chow forms of 100 Bordiga surfaces, elliptic scrolls and their unions. A second possibility is to count points and apply Corollary 3.2. As Figure 4 shows there is no difference between the two methods. The formula for the error
term underestimates the number of points on a elliptic scroll, because the scroll is irregular.

The method of searching points at random in higher codimensional subsets of rational varieties helped us in proving the existence of several interesting components of Hilbert schemes. [Sch96], [ST02], [vBEL04]

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