G-capacity under degenerate case and its application

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Abstract. In this paper, we first find a type of viscosity solution of $G$-heat equation under degenerate case, and then obtain the related $G$-capacity $c(B_T \in A)$ for any Borel set $A$. Furthermore, we prove that $I_A(B_T)$ has no quasi-continuous version when it is not a constant function.

Key words. G-heat equation, G-expectation, G-capacity, Quasi-continuous, Viscosity solution

AMS subject classifications. 60H10

1 Introduction

Motivated by model uncertainty in finance, Peng [11, 14] introduced the notions of $G$-expectation $\hat{E}[\cdot]$ and $G$-Brownian motion $(B_t)_{t \geq 0}$ via the following $G$-heat equation:

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x),$$

(1.1)

where $G(a) = \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-)$ for $a \in \mathbb{R}$, $\sigma > 0$ and $a \in [0, \sigma]$. For any bounded and continuous function $\varphi$, we have $\hat{E}[\varphi(x + B_t)] = u(t, x)$, where $u$ is the viscosity solution of (1.1). Under the $G$-expectation framework, the corresponding stochastic calculus of Itô’s type was also established in Peng [11, 12].

The $G$-expectation can be also seen as a upper expectation. Indeed, Denis et al. [2] obtained a representation theorem of $G$-expectation $\hat{E}[\cdot]$ by stochastic control method:

$$\hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X] \text{ for each } X \in Lip(\Omega).$$

where $\mathcal{P}$ is a family of weakly compact probability measures on $(\Omega, B(\Omega))$. Moreover, they gave the characterization of $L^p_G(\Omega)$ for $p \geq 1$. The representation theorem was also obtained in [4] by a simple probabilistic method.

Denis et al. [2] (see also [4]) introduced the notion of $G$-capacity $c(\cdot)$ in $G$-expectation space and showed that each random variable in $L^p_G(\Omega)$ has a quasi-continuous version with respect to $c(\cdot)$. Under the non-degenerate case, i.e. $\sigma > 0$, Hu et al. [6] proved that $c(B_T = a) = 0$ for each $(T, a) \in (0, \infty) \times \mathbb{R}$ by finding a kind of viscosity supersolution of $G$-heat equation (1.1), and further obtained that $I_{[a,b]}(B_T)$, $a \leq b$, is in

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$L_G^1(\Omega)$, which has important application in stochastic recursive optimal control problem under $G$-expectation space (see [3]). As far as we know, there is no result about the above two points under the degenerate case, i.e. $\underline{\sigma} = 0$.

In this paper, we first study $c(\{B_T \in A\})$ for $A \in \mathcal{B}(\mathbb{R})$ under degenerate case. The known method for calculating $G$-capacity is to find the “similarity solution” of $G$-heat equation (1.1) (see [5, 10, 15]). But this method is no longer suitable for some cases such as $A = (\infty, a] \cup [b, \infty)$ with $a < b$. To overcome this difficult, we use stochastic control method to find a type of viscosity solution of $G$-heat equation (1.1), and then obtain $c(\{B_T \in A\})$, which can provide some examples for checking the convergence rate of Peng’s central limit theorem (see [7, 9]) under degenerate case. As an application, we prove that $I_A(B_T)$ is not in $L_G^1(\Omega)$ for any $A \in \mathcal{B}(\mathbb{R})$ with $A \neq \emptyset$ and $A \neq \mathbb{R}$, which is completely different from the case $\underline{\sigma} > 0$.

This paper is organized as follows. In Section 2, we present some basic notions and results of $G$-expectation. In Section 3, we obtain the $G$-capacity $c(\{B_T \in A\})$ for any Borel set $A$ under degenerate case. As an application, we prove that $I_A(B_T)$ is not in $L_G^1(\Omega)$ for any $A \in \mathcal{B}(\mathbb{R})$ with $A \neq \emptyset$ and $A \neq \mathbb{R}$ in Section 4.

## 2 Preliminaries

We recall some basic notions and results of $G$-expectation. The readers may refer to [11, 14] for more details.

Let $\Omega = C [0, \infty)$ be the space of real-valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$. Let $B_t(\omega) := \omega_t$, for $\omega \in \Omega$ and $t \geq 0$ be the canonical process. Set

$$Lip (\Omega) := \{ \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}) : n \in \mathbb{N}, \ 0 < t_1 < \cdots < t_n, \ \varphi \in C_{b,Lip}(\mathbb{R}^n) \},$$

where $C_{b,Lip}(\mathbb{R}^n)$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^n$. It is easy to verify that

$$Lip (\Omega) = \{ \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}) : n \in \mathbb{N}, \ 0 < t_1 < \cdots < t_n, \ \varphi \in C_{b,Lip}(\mathbb{R}^n) \}.$$

Let

$$G(a) := \frac{1}{2} (\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-) \text{ for } a \in \mathbb{R},$$

where $\bar{\sigma} > 0$ and $\underline{\sigma} \in [0, \bar{\sigma}]$. The $G$-expectation $\mathbb{E} : Lip (\Omega) \rightarrow \mathbb{R}$ is defined by the following two steps.

Step 1. For each $X = \varphi(B_s - B_t)$ with $0 \leq s \leq t$ and $\varphi \in C_{b,Lip}(\mathbb{R})$, we define

$$\mathbb{E} [X] = u(t - s, 0),$$

where $u$ is the viscosity solution of (1.1).

Step 2. For each $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ with $0 < t_1 < \cdots < t_n$ and $\varphi \in C_{b,Lip}(\mathbb{R}^n)$, we define

$$\mathbb{E} [X] = \varphi_0,$$
where \( \phi_0 \) is obtained via the following procedure:

\[
\phi_{n-1}(x_1, \ldots, x_{n-1}) = \hat{E} \left[ \phi \left( x_1, \ldots, x_{n-1}, B_{t_n} - B_{t_{n-1}} \right) \right],
\]

\[
\phi_{n-2}(x_1, \ldots, x_{n-2}) = \hat{E} \left[ \phi_{n-1} \left( x_1, \ldots, x_{n-2}, B_{t_{n-1}} - B_{t_{n-2}} \right) \right],
\]

\[
\vdots
\]

\[
\phi_1(x_1) = \hat{E} \left[ \phi_2 \left( x_1, B_{t_2} - B_{t_1} \right) \right],
\]

\[
\phi_0 = \hat{E} \left[ \phi_1 \left( B_{t_1} \right) \right].
\]

The following is the definition of the viscosity solution of \((1.1)\) (see [1]).

**Definition 2.1** A real-valued continuous function \( u \in C \left( [0, \infty) \times \mathbb{R} \right) \) is called a viscosity subsolution (resp. supersolution) of \((1.1)\) on \([0, \infty) \times \mathbb{R}\) if \( u(0, \cdot) \leq \phi(\cdot) \) (resp. \( u(0, \cdot) \geq \phi(\cdot) \)), and for all \( (t^*, x^*) \in (0, \infty) \times \mathbb{R} \), \( \phi \in C^2 \left( (0, \infty) \times \mathbb{R} \right) \) such that \( u(t^*, x^*) = \phi(t^*, x^*) \) and \( u < \phi \) (resp. \( u > \phi \)) on \((0, \infty) \times \mathbb{R} \setminus (t^*, x^*)\), we have

\[
\partial_t \phi(t^*, x^*) - G \left( \partial^2_{xx} \phi(t^*, x^*) \right) \leq 0 \quad \text{(resp.} \geq 0).\]

A real-valued continuous function \( u \in C \left( [0, \infty) \times \mathbb{R} \right) \) is called a viscosity subsolution (resp. supersolution) of \((1.1)\) if it is both a viscosity subsolution and a viscosity supersolution of \((1.1)\) on \([0, \infty) \times \mathbb{R}\).

The space \((\Omega, \text{Lip}(\Omega), \hat{E})\) is called a \(G\)-expectation space. The corresponding canonical process \((B_t)_{t \geq 0}\) is called a \(G\)-Brownian motion. The \(G\)-expectation \(\hat{E} : \text{Lip}(\Omega) \to \mathbb{R}\) satisfies the following properties: for each \( X, Y \in \text{Lip}(\Omega), \)

(i) Monotonicity: If \( X \geq Y \), then \( \hat{E}[X] \geq \hat{E}[Y] \).

(ii) Constant preservation: \( \hat{E}[c] = c \) for \( c \in \mathbb{R} \).

(iii) Subadditivity: \( \hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y] \).

(iv) Positive homogeneity: \( \hat{E}[\lambda X] = \lambda \hat{E}[X] \) for \( \lambda \geq 0 \).

For every \( p \geq 1 \), we denote by \( L^p_G(\Omega) \) the completion of \( \text{Lip}(\Omega) \) under the norm \( \| X \|_p := \left( \hat{E} \left[ |X|^p \right] \right)^{1/p} \). The \(G\)-expectation \( \hat{E}[X] \) can be extended continuously to \( L^1_G(\Omega) \) under the norm \( \| \cdot \|_1 \), and \( \hat{E} : L^1_G(\Omega) \to \mathbb{R} \) still satisfies (i)-(iv).

Denis et al. [2] (see also [3]) proved the following representation theorem.

**Theorem 2.2** There exists a weakly compact set of probability measures \( \mathcal{P} \) on \((\Omega, \mathcal{B}(\Omega))\) such that

\[
\hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X] \quad \text{for each} \quad X \in L^1_G(\Omega),
\]

where \( \mathcal{B}(\Omega) = \sigma(B_t : t \geq 0) \). \( \mathcal{P} \) is called a set that represents \( \hat{E} \).
Remark 2.3 Denis et al. [2] gave a concrete $\mathcal{P}$ that represents $\hat{\mathbb{E}}$ as follows. Let $(W_t)_{t \geq 0}$ be a 1-dimensional classical Brownian motion defined on a Wiener probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, P^W)$, and let $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ be the natural filtration generated by $W$. The set of probability measures $\mathcal{P}_1$ on $(\Omega, \mathcal{B}(\Omega))$ is defined by

$$\mathcal{P}_1 = \left\{ P = P^W \circ (X^v)^{-1} : X^v_t = \int_0^t v_s dW_s, (v_s)_{s \leq T} \in M^2(0, T; [\sigma, \sigma]) \text{ for any } T > 0 \right\},$$

where $M^2(0, T; [\sigma, \sigma])$ is the space of all $\tilde{\mathcal{F}}_t$-adapted processes $(v_s)_{s \leq T}$ with $v_s \in [\sigma, \sigma]$. Then $\mathcal{P} = \bar{\mathcal{P}}_1$ represents $\hat{\mathbb{E}}$, where $\mathcal{P}_1$ is the closure of $\mathcal{P}_1$ under the topology of weak convergence.

The $G$-capacity associated to $\mathcal{P}$ is defined as

$$c(D) = \sup_{P \in \mathcal{P}} P(D) \text{ for } D \in \mathcal{B}(\Omega). \quad (2.1)$$

An important property of this capacity is that $c(F_n) \downarrow c(F)$ for any closed sets $F_n \downarrow F$.

A set $A \subset \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. A property holds "quasi-surely" (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$ q.s. For this $\mathcal{P}$, set

$$L^p(\Omega) := \left\{ X \in \mathcal{B}(\Omega) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \right\} \text{ for } p \geq 1.$$ 

It is easy to check that $L^p_G(\Omega) \subset L^p(\Omega)$. For each $X \in L^1(\Omega)$,

$$\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X]$$

is still called $G$-expectation and satisfies (i)-(iv).

Now we review the characterization of $L^p_G(\Omega)$ for $p \geq 1$.

**Definition 2.4** A function $X : \Omega \to \mathbb{R}$ is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set $O \subset \Omega$ with $c(O) < \varepsilon$ such that $X|_{\Omega \setminus O}$ is continuous.

**Definition 2.5** We say that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ such that $X = Y$, q.s.

**Theorem 2.6** ([2, 4]) For each $p \geq 1$, we have

$$L^p_G(\Omega) = \left\{ X \in \mathcal{B}(\Omega) : \lim_{N \to \infty} \hat{\mathbb{E}} \left[ |X|^p I_{\{|X| \geq N\}} \right] = 0 \text{ and } X \text{ has a quasi-continuous version} \right\}.$$

The following Fatou’s property is important in $G$-expectation space.

**Theorem 2.7** ([2, 4]) Let $\{X_n\}_{n=1}^\infty \subset L^1_G(\Omega)$ satisfy $X_n \downarrow X$ q.s. Then

$$\hat{\mathbb{E}}[X_n] \downarrow \hat{\mathbb{E}}[X]. \quad (2.2)$$

Moreover, if $X \in L^1_G(\Omega)$, then

$$\hat{\mathbb{E}}[X_n - X] \downarrow 0.$$
3 Main results

In this section, we consider the case $\sigma = 0$, then the $G$-heat equation is

$$\partial_t u - \frac{1}{2} \sigma^2 (\partial_{xx} u)^+ = 0, \quad u(0, x) = \varphi(x). \quad (3.1)$$

The following theorem is our main result.

**Theorem 3.1** Let $\sigma = 0$ and $\sigma > 0$. For each given $T > 0$ and $A \in \mathcal{B}(\mathbb{R})$, we have

(i) If $\rho(A) := \inf \{|x| : x \in A\} = 0$, then $c\{B_T \in A\} = 1$;

(ii) If $A \subseteq [0, \infty)$ or $A \subseteq (-\infty, 0]$, then $c\{B_T \in A\} = \Phi \left( \frac{\rho(A)}{\sigma \sqrt{T}} \right)$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{t^2}{2} \right) dt$;

(iii) If $\rho(A) \neq 0$, $A \not\subseteq [0, \infty)$ and $A \not\subseteq (-\infty, 0]$, then

$$c\{B_T \in A\} = \sum_{i=-\infty}^{\infty} sgn(i) \left[ \Phi \left( \frac{2i(\rho(A^+) + \rho(A^-)) + \rho(A^-)}{\sigma \sqrt{T}} \right) + \Phi \left( \frac{2i(\rho(A^+) + \rho(A^-)) + \rho(A^+)}{\sigma \sqrt{T}} \right) \right],$$

where $sgn(x) := I_{[0,\infty)}(x) - I_{(-\infty,0)}(x)$, $\rho(A^+) := \inf\{x : x \in A, x \geq 0\}$, $\rho(A^-) := \inf\{-x : x \in A, x \leq 0\}$.

In order to prove this theorem, we need the following lemma. By Remark 2.3, we know that

$$\sup_{\omega \in M^2(0, T ; [0, \infty])} P^W \left( \int_0^T v_s dW_s \in A \right) \leq c\{B_T \in A\} \text{ for } A \in \mathcal{B}(\mathbb{R}), \quad (3.2)$$

where $(W_t)_{t \geq 0}$ is a 1-dimensional classical Brownian motion defined on a Wiener probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, P^W)$. The proof of the following lemma is accomplished by the proper construction of $u_n$ on the assumption that the equal sign in (3.2) holds and the optimal control in (3.2) is $(\pi I_{[0, \tau_n \wedge \tau]}(s))_{s \leq T}$ for $A = \{b, l\}$ with $b \leq 0 \leq l$, where

$$\tau_a := \inf\{t \geq 0 : \sigma W_t = a\} \text{ for } a \in \mathbb{R}.$$ 

In turn, the conclusion of the lemma also shows that the above assumption is true.

**Lemma 3.2** Let $\sigma = 0$ and $\sigma > 0$. Then, for each given $T > 0$ and $b < 0 < l$, we have

$$c\{B_T \in \{b, l\}\} = \sum_{i=-\infty}^{\infty} sgn(i) \left[ \Phi \left( \frac{2i(l-b) - b}{\sigma \sqrt{T}} \right) + \Phi \left( \frac{2i(l-b) + l}{\sigma \sqrt{T}} \right) \right].$$

**Proof.** For each fixed $n \geq 1$, we first prove that

$$u_n(t, x) := \sum_{i=-\infty}^{\infty} sgn(i) \left[ \Phi \left( \frac{2i(l-b) + x - b}{\sigma \sqrt{\frac{1}{n} + t}} \right) + \Phi \left( \frac{2i(l-b) + l - x}{\sigma \sqrt{\frac{1}{n} + t}} \right) \right] I_{(b,l)}(x)$$

$$+ \Phi \left( \frac{|b - x| \wedge |l-x|}{\sigma \sqrt{\frac{1}{n} + t}} \right) I_{(-\infty, b] \cup [l, \infty)}(x)$$

is a viscosity solution of (3.1) on $(0, \infty) \times \mathbb{R}$.

On the one hand, we need to prove that $u_n$ is a viscosity subsolution of (3.1). Indeed, it suffices to do three steps to show that $\partial_t \psi(t^*, x^*) - \frac{1}{2} \sigma^2 (\partial_{xx} \psi(t^*, x^*))^+ \leq 0$ for all $(t^*, x^*) \in (0, \infty) \times \mathbb{R}$, $\psi \in C^2((0, \infty) \times \mathbb{R})$ such that $\psi(t^*, x^*) = u_n(t^*, x^*)$ and $\psi \geq u_n$. 

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Step 1. If \( x^* \in (-\infty, b) \cup (l, \infty) \), it is easy to know from extreme value theory that

\[
\partial_{xx}^2 \psi(t^*, x^*) \geq \partial_{xx}^2 u_n(t^*, x^*), \quad \partial_t \psi(t^*, x^*) = \partial_t u_n(t^*, x^*)
\]

and it follows by simple calculation that

\[
\partial_t u_n(t^*, x^*) = -\frac{1}{2} \Phi'(\frac{|b - x^*| \land |l - x^*|}{\sigma \sqrt{\frac{1}{n} + t^*}}) \frac{|b - x^*| \land |l - x^*|}{\sigma \sqrt{\frac{1}{n} + t^*}}^3
\]

and

\[
\partial_{xx}^2 u_n(t^*, x^*) = \Phi''(\frac{|b - x^*| \land |l - x^*|}{\sigma \sqrt{\frac{1}{n} + t^*}}) \frac{1}{\sigma^2 \left( \frac{1}{n} + t^* \right)}.
\]

By the definition of \( \Phi(x) \), we know that \( \Phi''(x) > 0 \) holds if \( x > 0 \), and

\[
\Phi''(x) = -x \Phi'(x).
\]

Then we obtain

\[
\partial_{xx}^2 u_n(t^*, x^*) > 0 \quad \text{and} \quad \partial_t u_n(t^*, x^*) - \frac{1}{2} \sigma^2 \partial_{xx}^2 u_n(t^*, x^*) = 0,
\]

which implies \( \partial_t \psi(t^*, x^*) - \frac{1}{2} \sigma^2 \left( \partial_{xx}^2 \psi(t^*, x^*) \right)^\dagger \leq 0 \).

Step 2. If \( x^* \in (b, l) \), by simple calculation, we can still get

\[
\partial_t u_n(t^*, x^*) = -\frac{1}{2} \sum_{i=-\infty}^{\infty} \text{sgn}(i) \left[ \Phi' \left( \frac{|2i(l-b) + x^* - b|}{\sigma \sqrt{\frac{1}{n} + t^*}} \right) \frac{|2i(l-b) + x^* - b|}{\sigma \sqrt{\frac{1}{n} + t^*}^3} + \Phi' \left( \frac{|2i(l-b) + l - x^*|}{\sigma \sqrt{\frac{1}{n} + t^*}} \right) \frac{|2i(l-b) + l - x^*|}{\sigma \sqrt{\frac{1}{n} + t^*}^3} \right]
\]

and

\[
\partial_{xx}^2 u_n(t^*, x^*) = \sum_{i=-\infty}^{\infty} \text{sgn}(i) \left[ \Phi'' \left( \frac{|2i(l-b) + x^* - b|}{\sigma \sqrt{\frac{1}{n} + t^*}} \right) + \Phi'' \left( \frac{|2i(l-b) + l - x^*|}{\sigma \sqrt{\frac{1}{n} + t^*}} \right) \right] \frac{1}{\sigma^2 \left( \frac{1}{n} + t^* \right)^2}.
\]

It can be verified by the definition of \( \Phi(x) \) that \( \partial_t u_n(t^*, x^*) - \frac{1}{2} \sigma^2 \partial_{xx}^2 u_n(t^*, x^*) = 0 \). Since \( \partial_{xx}^2 \psi(t^*, x^*) \geq \partial_{xx}^2 u_n(t^*, x^*) \), and if we want to replicate the idea we had in step 1, we just have to prove \( \partial_t u_n(t^*, x^*) > 0 \).

It is well-known that the stopping time \( \tau_{b-x} \land \tau_{l-x} \) has the following density for \( x \in (b, l) \) (see \[8\])

\[
P^W \{ \tau_{b-x} \land \tau_{l-x} \in ds \} = \frac{1}{\sqrt{2\pi \sigma^2 s^3}} \sum_{i=-\infty}^{\infty} \left\{ (2i(l-b) - b + x) \exp \left( -\frac{(2i(l-b) - b + x)^2}{2\sigma^2 s} \right) + (2i(l-b) + l - x) \exp \left( -\frac{(2i(l-b) + l - x)^2}{2\sigma^2 s} \right) \right\} \, ds.
\]

By tedious calculation, we can get

\[
u_n(t, x) = \int_0^{t-l} P^W \{ \tau_{b-x} \land \tau_{l-x} \in ds \} \, dx.
\]
which implies $\partial_t u_n (t^*, x^*) > 0$, so we get $\partial_{xx}^2 u_n (t^*, x^*) > 0$. Moreover,

$$\partial_t \psi (t^*, x^*) - \frac{1}{2} \sigma^2 (\partial_{xx}^2 \psi (t^*, x^*))^+ \leq 0.$$  

**Step 3.** If $x^* \in \{b, l\}$, we know by the definition of $\psi$ that

$$\psi (t, x^*) \geq u_n (t, x^*) = 1 \text{ and } \psi (t^*, x^*) = u_n (t^*, x^*) = 1,$$

it then follows that $\partial_t \psi (t^*, x^*) = 0$. Hence, we obtain $\partial_t \psi (t^*, x^*) - \frac{1}{2} \sigma^2 (\partial_{xx}^2 \psi (t^*, x^*))^+ \leq 0$. Thus, from step 1 to step 3, we know that $u_n$ is a viscosity subsolution of (3.1).

On the other hand, we need to prove that $u_n$ is a viscosity supersolution of (3.1). So, let us show in two steps that $\partial_t \psi (t^*, x^*) - \frac{1}{2} \sigma^2 (\partial_{xx}^2 \psi (t^*, x^*))^+ \geq 0$ for all $(t^*, x^*) \in (0, \infty) \times \mathbb{R}$, $\psi \in C^2 ((0, \infty) \times \mathbb{R})$ such that $\psi (t^*, x^*) = u_n (t^*, x^*)$ and $\psi \leq u_n$.

**Step 4.** If $x^* \notin \{b, l\}$, similar to the proof idea of step 1 and step 2, we can also get the same results as follows:

$$\partial_{xx}^2 \psi (t^*, x^*) \leq \partial_{xx}^2 u_n (t^*, x^*), \quad \partial_{xx}^2 u_n (t^*, x^*) > 0, \quad \partial_t \psi (t^*, x^*) = \partial_t u_n (t^*, x^*).$$

and

$$\partial_t u_n (t^*, x^*) - \frac{1}{2} \sigma^2 \partial_{xx}^2 u_n (t^*, x^*) = 0$$

Thereby, we have $(\partial_{xx}^2 \psi (t^*, x^*))^+ \leq (\partial_{xx}^2 u_n (t^*, x^*))^+$. Moreover, we show that

$$\partial_t \psi (t^*, x^*) - \frac{1}{2} \sigma^2 (\partial_{xx}^2 \psi (t^*, x^*))^+$$

$$\geq \partial_t u_n (t^*, x^*) - \frac{1}{2} \sigma^2 (\partial_{xx}^2 u_n (t^*, x^*))^+$$

$$= \partial_t u_n (t^*, x^*) - \frac{1}{2} \sigma^2 \partial_{xx}^2 u_n (t^*, x^*)$$

$$= 0$$

**Step 5.** If $x^* \in \{b, l\}$, then we know that

$$\psi (t^*, x^*) = u_n (t^*, x^*) \text{ and } \psi (t^*, x) \leq u_n (t^*, x),$$

which implies that

$$\partial_{x^+} \psi (t^*, x^*) \leq \partial_{x^+} u_n (t^*, x^*) \text{ and } \partial_{x^-} \psi (t^*, x^*) \geq \partial_{x^-} u_n (t^*, x^*).$$

which is (3.3).

It is easy to check by the definition of $\Phi (x)$ that

$$\partial_{x^-} u_n (t^*, b) = \frac{2}{\sigma \sqrt{2 \pi (\frac{1}{t} + t^*)}} > 0,$$

$$\partial_{x^+} u_n (t^*, l) = -\frac{2}{\sigma \sqrt{2 \pi (\frac{1}{t} + t^*)}} < 0.$$  

Now we claim that $\partial_{x^+} u_n (t^*, b) < 0$. Otherwise, $\partial_{x^+} u_n (t^*, b) \geq 0$. Noting that $\partial_{xx}^2 u_n (t^*, x) > 0$ for $x \in (b, l)$ and $u_n (t^*, b) = 1$, we obtain $u_n (t^*, x) > 1$ for $x \in (b, l)$, which contradicts to $u_n \leq 1$. Hence,
we have $\partial_x u_n(t^*, b) < 0$. Similarly, we can get $\partial_x u_n(t^*, l) > 0$. So we can not find $\psi \in C^2((0, \infty) \times \mathbb{R})$ satisfying (3.3). Thus, from (H4)-(H5), we know that $u_n$ is also a viscosity supersolution of (3.1).

In conclusion, $u_n$ is a viscosity solution of (3.1). Then we obtain
\[
\hat{E} [u_n (0, B_T)] = u_n(T, 0).
\]
Noting that $u_n(0, x) \downarrow I_{(b,l)}(x)$, we can deduce by Theorem 2.7 that
\[
c(\{B_T \in \{b,l\}\}) = \hat{E} [I_{(b,l)}(B_T)] = \lim_{n \to \infty} \hat{E} [u_n (0, B_T)] = \lim_{n \to \infty} u_n(T, 0),
\]
which implies the desired result. □

**Remark 3.3** Under the condition of Lemma 3.2, by using the similar method, we can get
\[
c(\{B_T \in \{b,l\}\}) = c(\{B_T \in (-\infty, b] \cup [l, \infty)\}) \text{ for } b < 0 < l
\]
and
\[
c(\{B_T = a\}) = c(\{B_T \geq |a|\}) = c(\{B_T \leq -|a|\}) = \Phi \left( \frac{|a|}{\sigma \sqrt{T}} \right) \text{ for } a \in \mathbb{R}.
\]
The value $c(\{B_T \geq |a|\})$ was obtained in [10, 15] by using “similarity solution” method.

**Proof of Theorem 3.1** If $\rho(A) = 0$, we can find a sequence $\{a_n : n \geq 1\} \subset A$ such that $a_n \to 0$. By Remark 3.3, we get
\[
c(\{B_T \in A\}) \geq \lim_{n \to \infty} c(\{B_T = a_n\}) = \lim_{n \to \infty} \Phi \left( \frac{|a_n|}{\sigma \sqrt{T}} \right) = 1,
\]
which implies (i).

If $A \subset [0, \infty)$, we can find a sequence $\{x_n : n \geq 1\} \subset A$ such that $x_n \to \rho(A)$. By Remark 3.3, we know that
\[
c(\{B_T \in A\}) \geq \lim_{n \to \infty} c(\{B_T = x_n\}) = \Phi \left( \frac{\rho(A)}{\sigma \sqrt{T}} \right).
\]
Noting that $A \subset [\rho(A), \infty)$, by Remark 3.3, we get
\[
c(\{B_T \in A\}) \leq c(\{B_T \geq \rho(A)\}) = \Phi \left( \frac{\rho(A)}{\sigma \sqrt{T}} \right).
\]
Thus we have $c(\{B_T \in A\}) = \Phi \left( \frac{\rho(A)}{\sigma \sqrt{T}} \right)$. By the similar method for $A \subset (-\infty, 0]$, then we obtain (ii).

If $\rho(A) \neq 0$, $A \subset [0, \infty)$ and $A \subset (-\infty, 0]$, then we can find two sequences $\{b_n : n \geq 1\}$ and $\{c_n : n \geq 1\}$ in $A$ such that $b_n < 0 < c_n$, $-b_n \to \rho(A^-)$ and $c_n \to \rho(A^+)$. By Lemma 3.2, Remark 3.3 and $A \subset (-\infty, -\rho(A^-)] \cup [\rho(A^+), \infty)$, we get
\[
c(\{B_T \in A\}) \geq \lim_{n \to \infty} c(\{B_T \in \{b_n, c_n\}\}) = c(\{B_T \in (-\rho(A^-), \rho(A^+))\}),
\]
\[
c(\{B_T \in A\}) \leq c(\{B_T \in (-\infty, -\rho(A^-)] \cup [\rho(A^+), \infty)\}),
\]
which implies $c(\{B_T \in A\}) = c(\{B_T \in (-\rho(A^-), \rho(A^+))\})$. Thus we obtain (iii). □
4 Application to G-expectation

For each \( \phi_n \in C_b(\mathbb{R}) \) such that \( \phi_n \downarrow I_A \), we know by (2.2) that
\[
\hat{E}[\phi_n(B_T)] \downarrow c(\{B_T \in A\}).
\] (4.1)

The following theorem is the application of Theorem 3.1.

**Theorem 4.1** Let \( \sigma = 0 \) and \( \sigma > 0 \). Then, for each given \( T > 0 \), \( A \in \mathcal{B}(\mathbb{R}) \) with \( A \neq \emptyset \) and \( A \neq \mathbb{R} \), we have \( I_A(B_T) \notin L^1_G(\Omega) \).

**Proof.** Due to \( A \neq \emptyset \) and \( A \neq \mathbb{R} \), then one of the following two results must hold.

(i) There exist a point \( x_0 \in A \) and a sequence \( \{x_k : k \geq 1\} \subset A \) such that \( x_k \rightarrow x_0 \).

(ii) There exist a point \( x_0 \in A^c \) and a sequence \( \{x_k : k \geq 1\} \subset A \) such that \( x_k \rightarrow x_0 \).

If (i) holds and \( I_A(B_T) \in L^1_G(\Omega) \), then
\[
h_n(B_T) \vee I_A(B_T) \in L^1_G(\Omega) \text{ and } h_n(B_T) \vee I_A(B_T) - I_A(B_T) \downarrow 0,
\]
where
\[
h_n(x) = [1 + n(x - x_0)]I_{[x_0 - \frac{1}{n},x_0]}(x) + [1 - n(x - x_0)]I_{[x_0,x_0 + \frac{1}{n}]}(x).
\]
By (2.2), we have
\[
\hat{E}[h_n(B_T) \vee I_A(B_T) - I_A(B_T)] \downarrow 0 \text{ as } n \rightarrow \infty.
\] (4.2)

Moreover, we also know that \( h_n(B_T) \vee I_A(B_T) - I_A(B_T) \geq h_n(x_k)I_{\{x_k\}}(B_T) \) for any \( k \geq 1 \). Then we deduce by Theorem 3.1 that
\[
\hat{E}[h_n(B_T) \vee I_A(B_T) - I_A(B_T)] \geq \lim_{k \rightarrow \infty} h_n(x_k)c(\{B_T = x_k\}) = c(\{B_T = x_0\}) > 0,
\]
which contradicts to (4.2). Thus \( I_A(B_T) \notin L^1_G(\Omega) \).

If (ii) holds, then \( A^c \) satisfies (i). Thus we obtain \( I_{A^c}(B_T) \notin L^1_G(\Omega) \), which implies \( I_A(B_T) = 1 - I_{A^c}(B_T) \notin L^1_G(\Omega) \). \( \square \)

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