Factoring out Free Fields

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Abstract

For a generic \(\mathcal{W}\) algebra, we give an algorithmic procedure for factoring out all fields of dimension 1/2, both bosonic and fermionic, and some fields of dimension 1. This generalizes and makes more explicit the Goddard-Schwimmer theorem for free fermions. We also show how the induced gravity theory for the original \(\mathcal{W}\) algebra containing the free fields relates to the theory where the fields are factored out.

hepth@xxx/9306129

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1 Introduction

Some years ago, Goddard and Schwimmer [1] proved that every meromorphic conformal theory can be factorized into free fermions (of spin $1/2$) and a part containing no free fermions. A consequence of this is that in the classification of $W$ algebras, spin $1/2$ fermions need never be considered. This is very fortunate, since the main method of constructing a large number of $W$ algebras, hamiltonian reduction (see e.g. [4] for a recent account of the classical case, and [5, 6, 7] for the quantum case), does not generally yield spin $1/2$ fields. (Supersymmetric reduction, see [6], does give weight $1/2$ fields.) However, [1] does not treat bosonic fields of weight $1/2$ (symplectic bosons). Mostly, it is assumed that these fields, too, can always be factorized, but up to now this was not yet explicitly proven. In this letter, we present an algorithmic procedure for factoring out of both fermionic and bosonic fields of weight $1/2$.

It was already noticed in [1] that in some cases (e.g. the $N = 4$ superconformal algebra) spin $1$ bosons can also be decoupled from a conformal theory. This is certainly not a general property. The factorization-algorithm presented here gives a criterion to decide when free bosons can be decoupled.

In the second part of this letter, we extend the results of [8]. There, it was shown how factoring out the fermions from the $N = 3, 4$ superconformal algebras, links linear and non-linear $N = 3, 4$ induced supergravity. We will show here how all the factorizable fields can in general be integrated out. Also, the criterion for factorizable $U(1)$ fields is rederived from the Ward identities.

2 Algorithms for factorisation

In the following subsections, we will show how various free fields can be decoupled by adding extra composite terms. This will be done recursively by removing the highest order pole in the OPE of the free field with the other fields in the algebra. For the energy-momentum tensor, the extra terms amount exactly to substracting the usual e.m tensor of the free field, so the new e.m. tensor is again a good Virasoro tensor (with a central charge $c$ shifted by minus the central charge of the free fields). The other fields of the theory may become non-primary after redefinition, but it should always be possible to find a new primary basis. We will not go further into this.

We will use the following convention for the OPE of two fields:

$$A(z)B(w) = \sum_{i \leq h(A,B)} [AB]_i (w) (z - w)^{-i}, \quad (1)$$

where $h(A, B)$ is usually the sum of the conformal weights of the fields. We define the modes $\hat{A}_m$ by

$$\hat{A}_m B \equiv [AB]_m. \quad (2)$$

2.1 Free fermions

For completeness, we first rederive the result of [1] in our formalism and give an explicit algorithm for the decoupling. Consider a theory containing a free

\footnote{This is actually a shift in the index $m$ with respect to the usual definition. This will of course be reflected in the commutators.}
fermion \( \psi \) (we will only consider the Neveu-Schwarz sector):

\[
[\psi \psi]_1 = \lambda .
\]  

(3)

We have the commutation relation

\[
\hat{\psi}_m \hat{\psi}_n = -\hat{\psi}_n \hat{\psi}_m + \lambda \delta_{m+n-1}.
\]  

(4)

Now suppose we have already \textit{partly} decoupled \( \psi \) from a certain field \( A \), more exactly

\[
\hat{\psi}_k A = 0 , \forall k \geq n + 1.
\]  

(5)

We can then add a composite field to \( A \):

\[
\Delta_n A \equiv -\frac{1}{\lambda} \hat{\psi}_1 \hat{\psi}_n A,
\]  

(6)

such that condition (5) is valid for all \( k \geq n \):

\[
\hat{\psi}_k (1 + \Delta_n) A = 0 , \forall k \geq n.
\]  

(8)

So \( P_n \equiv (1 + \Delta_n) \) is the projection operator on the kernel of \( \hat{\psi}_n \). The procedure can clearly be iterated, and at the end all the redefined fields will have a regular OPE with \( \psi \), which is the desired result. The complete projection operator becomes

\[
P \equiv \prod_n P_n.
\]  

(9)

For example, it is very easy to check that the e.m. tensor gets the expected correction

\[
\Delta_2 T = -\frac{1}{2\lambda} \psi' \psi.
\]  

(10)

Finally, note that through use of Jacobi identities, one sees that the redefined fields form a closed algebra, where all fields commute with \( \psi \).

2.2 Symplectic bosons

Suppose we have a couple of symplectic bosons

\[
[\xi^\pm \xi^-]_1 = \lambda = -[\xi^- \xi^+]_1,
\]

\[
[\xi^+ \xi^+]_1 = [\xi^- \xi^-]_1 = 0.
\]  

(11)

The operators \( \hat{\xi}^\pm_m \), as in (2), have commutation relations

\[
\hat{\xi}^\pm_m \hat{\xi}^\pm_n = \hat{\xi}^\pm_n \hat{\xi}^\pm_m ,
\]

\[
\hat{\xi}^\pm_m \hat{\xi}^-n = \hat{\xi}^-m \hat{\xi}^+n + \lambda \delta_{m+n-1}.
\]  

(12)

We can define the operators

\[
\Delta^\pm_n \equiv \sum_{i \geq 1} \frac{(-1)^i}{i! \lambda^i} (\hat{\xi}^\pm_{1-n})^i (\hat{\xi}^\pm_n)^i ,
\]  

(13)

\footnote{Note that the regular part of an OPE can be found from the general relation \( [AB]_{-n} = \frac{1}{n!} [A^{(n)} B]_0 \).}

\[
[AB]_{-n} = \frac{1}{n!} [A^{(n)} B]_0.
\]  

(7)
or
\[
P_n^\pm \equiv 1 + \Delta_n^\pm \equiv \exp \left[ \pm \frac{1}{\lambda} \xi_{1-n} \xi_n^\pm \right].
\] (14)

On any field \( A \), the sum (13) will stop after a finite number of terms, since for every \( n \geq 1 \) the conformal weight of \( \xi_n^\pm A \) is strictly smaller than that of \( A \) itself, and there is a lower bound on the dimension of the fields in the algebra. If
\[
\xi_k^\pm A = 0, \ \forall k \geq n + 1,
\] (15)
these operators satisfy the commutation relations
\[
\begin{align*}
\Delta_n^\pm \Delta_n^\mp A &= \Delta_n^\pm \Delta_n^\mp A, \\
\xi_n^\pm \Delta_n^\mp A &= -\xi_n^\pm A, \\
\xi_n^\pm \Delta_n^\mp A &= \Delta_n^\pm \xi_n^\mp A.
\end{align*}
\] (16)

So the complete projection operator is
\[
\begin{align*}
P_n \equiv P^+_n \circ P^-_n.
\end{align*}
\] (17)

Again, this procedure can be iterated up to \( n = 1 \), proving that the symplectic bosons can all be decoupled. Note that since the algorithm is only based on the commutation relations (12) it will work just as well for a generic \((\beta, \gamma)\) system. For a fermionic \((b, c)\) system the result is even simpler, since then the sum (13) reduces to its first term, with an additional minus sign:
\[
\Delta_n^\pm \equiv -\frac{1}{\lambda} \xi_{1-n} \xi_n^\pm.
\] (18)

### 2.3 \( U(1) \) currents

When trying to decouple a \( U(1) \) field, we will see that this is not always possible. In fact, there will be no way to set the first order pole to zero by adding correction terms. We now have
\[
[JJ]_2 = \lambda
\] (19)
and
\[
\dot{J}_m \dot{J}_n = \dot{J}_n \dot{J}_m + \lambda (m - 1) \delta_{m+n-2}.
\] (20)

Suppose \( \dot{J}_k A = 0, \ \forall k \geq n + 1 \), then it is easy to check that
\[
P_n \equiv \exp \left[ \frac{1}{(1-n)\lambda} \dot{J}_{2-n} \dot{J}_n \right]:
\] (21)
is the projection operator on the kernel of \( \dot{J}_n \). However, the whole scheme breaks down at \( n = 1 \), since there is a factor \((1-n)^{-1}\) that diverges. Also, it is important to notice that the conformal dimension of \( \dot{J}_1 A \) is equal to that of \( A \) itself, so \((\dot{J}_1)^i A \) need not be zero even for very large \( i \). From this we conclude that a sufficient condition for the decoupling of a \( U(1) \) current is
\[
\dot{J}_1 A = [JA]_1 = 0
\] (22)
for \textit{all} fields \( A \) of the theory. In fact, it is quite easy to show that (22) is also a necessary condition. For the fields of conformal dimension 1 this is obvious.
Now consider the lowest dimension $d$ where (22) is not satisfied. From Jacobi identities it follows that

$$[J[AB]_n]_1 = [[JA]_1B]_n + [A[JB]_1]_n.$$  \hspace{1cm} (23)

Thus $\hat{J}_1$ working on any \textit{composite} field of this dimension vanishes. So correction terms can only be non-composite. Clearly, if $\hat{J}_1$ does not vanish on all $N_d$ (non-composite) primary fields of dimension $d$, we can not make $N_d$ independent linear combinations on which it does. Note that (22) simply tells us that all fields should have zero $U(1)$ charge with respect to $J$.

Of course, we may try to decouple a larger part of the $\mathcal{W}$ algebra, e.g. a Kac-Moody algebra, containing $J$. In this case, condition (22) need not hold. In this letter, however, we will restrict ourselves to the case of $U(1)$ fields.

Finally, note that due to \cite{7}

$$[AJ]_1 = [JA]_1 + \sum_{i \geq 2} \frac{(-1)^i}{(i-1)!} \partial^{(i-1)}{[AJ]}_i,$$  \hspace{1cm} (24)

the condition (22) is not equivalent to $[AJ]_1 = 0$. Here, the criterion becomes that $[AJ]_1$ may not contain any primary fields. Indeed, if $A$ is a primary with respect to $T$ (and $J$ a primary of dimension 1), we see from eq. (23) with $B = T$ that $[JA]_1$ is primary. Because the primary at $[JA]_1$ is the same as the one in $[AJ]_1$, eq. (22) translates in the requirement that there is no primary in $[AJ]_1$.

3 Induced and effective $\mathcal{W}$ gravities

Suppose we have a $\mathcal{W}$ algebra with generators $\Phi^i$, then the induced action $\Gamma$ of the $\mathcal{W}$ gravity is defined by

$$Z[h] = e^{-\Gamma[h]} = \left\langle e^{-\frac{1}{\pi} \int h \Phi} \right\rangle_{OPE}.$$  \hspace{1cm} (25)

See \cite{9} for an extensive account of induced and effective $\mathcal{W}$ gravity theories.

Suppose the $\mathcal{W}$ algebra contains a free field $F$ that can be factored out. We will denote by $\tilde{\Phi}^i$ the generators (anti-) commuting with $F$. It can easily be shown that one can invert the algorithms of the previous section. More specifically, we can write

$$\Phi^i = \tilde{\Phi}^i + P^i[\tilde{\Phi}, F]$$  \hspace{1cm} (26)

where the $P^i[\tilde{\Phi}, F]$ are some differential polynomials with all terms at least of order 1 in $F$. For the fields $\tilde{\Phi}$ we then define the induced action $\tilde{\Gamma}$ and $\tilde{Z}$ as in (23). The main result of \cite{8}, in the case of the $N = 3, 4$ superconformal algebras, was that the induced action $\tilde{\Gamma}[h]$ of the non-linear supergravity could be obtained from the linear one by integrating out the free fields:

$$\tilde{Z}[h] = \int [dh_F] \ Z[h_{\Phi}, h_F].$$  \hspace{1cm} (27)

We will now give a heuristic argument that this should be the case for a generic $\mathcal{W}$ algebra.

We can compute $Z$ as follows

$$Z[h, h_F] = \left\langle \exp \left[ -\frac{1}{\pi} \int h_k(\tilde{\Phi}^k + P^k[\tilde{\Phi}, F]) + h_ff \right] \right\rangle_{OPE}. \hspace{1cm} (28)$$
We assume that there exists a path integral formulation for this expression. This could change the form of the polynomials $P_k$, due to normal ordering problems. Now we integrate (28) over $h_F$, and change the order of integration. The last term in the exponential gives us $\delta(F)$, so all terms containing $F$ can be dropped. The remaining expression is exactly $\tilde{Z}$.

Going to the effective theory, we define

$$e^{-W[t]} = \int [dh] Z[h] \exp \left[ \frac{1}{\pi} \int h \cdot t \right].$$

(29)

From relation (27), we get

$$\tilde{W}[\Phi] = W[t, t_F = 0].$$

(30)

Finally, if a free field $F$ can be integrated out this should reflect itself in the Ward identities

$$\bar{\partial} \frac{\delta Z}{\delta h_i(x, \bar{x})} = \frac{1}{\pi} \sum_{j,n} \frac{(-1)^{n-1}}{(n-1)!} \partial^{n-1} \left( h_j \left[ \Phi^i \Phi^j \right]_n e^{-\frac{1}{\pi} \int h \cdot \Phi} \right).$$

(31)

Indeed, when we factor out a fermion, the Ward identity corresponding to $h_i = h_\psi$ in (31) is

$$\bar{\partial} u^\psi = -\frac{\lambda}{\pi} h_\psi + F \left[ h, u, u^\psi \right],$$

(32)

where

$$u^i \equiv \frac{\delta \Gamma}{\delta h_i}. $$

(33)

Setting $u^\psi = 0$, we can now fill in the solution for $h_\psi$ in the other Ward identities. In this way, the fermion $\psi$ completely disappears from our theory. It is pretty obvious that the same can be done for a couple $(\xi^+, \xi^-)$ of symplectic bosons, by looking at the equations with $h_i = \xi^\pm$.

Now suppose we want to solve from these equations a particular source $h_J$ corresponding to a $U(1)$ field $J$. The Ward identity (31) of $h_J$ has an anomalous term proportional to $\partial h_J$. This means that we will only be able to remove $J$ if $h_J$ never appears underived. So our criterion for factoring out of a $U(1)$ field $J$ should be that in all Ward identities of the theory, the coefficient of $h_J$ vanishes.

If we look at (31) for some source $h_i$, this term is given by

$$- \frac{1}{\pi} \sum_n \frac{(-1)^{n-1}}{(n-1)!} h_J \partial^{n-1} \left[ \Phi_i J \right]_n (x) e^{-\frac{1}{\pi} \int h \cdot \Phi}. $$

(34)

Now we can use (24) to simplify this to

$$- \frac{1}{\pi} h_J \left[ J \Phi_i \right]_1 (x) e^{-\frac{1}{\pi} \int h \cdot \Phi}. $$

(35)

So we see that requiring this term to vanish yields exactly the condition (22)!

\hspace{1cm} \textsuperscript{3}Here we use the fact that the $\hat{\Phi}^i$ commute with $F$. 

\hspace{1cm}
4 Conclusion

We have given explicit algorithms for factoring out free fields, including a simple criterion for the factorisation of $U(1)$ fields. These algorithms are ideally suited for computer implementation, e.g using the Mathematica\textsuperscript{TM} package OPEdefs \cite{10}. We have also shown that this decoupling is equivalent to integrating out fields from an induced $W$ gravity theory. We have worked purely at the quantum mechanical level, but it is to be expected that analogous algorithms will exist in the classical case. In fact, recently \cite{11} a number of classical $W$ algebras were constructed by hamiltonian reduction, containing bosons of dimensions 1 and 1/2 that could be decoupled. In \cite{12} classical $W$ algebras obtained by supersymmetric hamiltonian reduction (based on $OSp(1|2)$ embeddings in affine Lie superalgebras) were shown to be equivalent to non-supersymmetric ones (based on $SL(2)$ embeddings), after factorization of bosonic and fermionic dimension 1/2 fields.

Acknowledgements
We would like to thank J. M. Figueroa-O’Farrill and W. Troost for helpful suggestions and comments.

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