SOME REMARKS ON THE CORRESPONDENCE BETWEEN ELLIPTIC CURVES AND FOUR POINTS IN THE RIEMANN SPHERE

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ABSTRACT. In this paper we relate some classical normal forms for complex elliptic curves in terms of 4-point sets in the Riemann sphere. Our main result is an alternative proof that every elliptic curve is isomorphic as a Riemann surface to one in the Hesse normal form. In this setting, we give an alternative proof of the equivalence between the Edwards and the Jacobi normal forms. Also, we give a geometric construction of the cross ratios for 4-point sets in general position.

INTRODUCTION

A complex elliptic curve is by definition a compact Riemann surface of genus 1. By the uniformization theorem, every elliptic curve is conformally equivalent to an algebraic curve given by a cubic polynomial in the form

\begin{equation}
E: \quad y^2 = 4x^3 - g_2 x - g_3, \quad \text{with } \Delta = g_2^3 - 27g_3^2 \neq 0,
\end{equation}

this is called the Weierstrass normal form. For computational or geometric purposes it may be necessary to find a Weierstrass normal form for an elliptic curve, which could have been given by another equation. At best, we could predict the right changes of variables in order to transform such equation into a Weierstrass normal form, but in general this is a difficult process.

A different method to find the normal form (1) for a given elliptic curve, avoiding any change of variables, requires a degree 2 meromorphic function on the elliptic curve, which by a classical theorem always exists and in many cases it is not difficult to compute. By the Riemann-Hurwitz formula, this map has four branch points and, by composing with a Möbius transformation, we can assume that one point is at \(\infty\) and the other points are three complex numbers \(\{z_1, z_2, z_3\}\). Let \(C\) be the centroid of the points \(z_i\) and \(e_i = z_i - C\). With this points we are able to find the constants \(g_2\) and \(g_3\) of (1) as the coefficients of the polynomial

\begin{equation}
4(x - e_1)(x - e_2)(x - e_3).
\end{equation}

This motivates the study of another classical normal forms in this setting. We are interested in relating the branch points of degree 2 meromorphic functions on the elliptic curves with 4-point configurations derived from a given normal form.

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This relation naturally leads us to the study of the cross ratio of 4-point sets. See
Example 1 and Example 2 for illustrate the above discussion.

Our main result is an alternative proof that every elliptic curve is isomorphic
as a Riemann surface to one in the Hesse normal form (see Theorem 4). In this
setting, we give an alternative proof of the equivalence between the Edwards and
the Jacobi normal forms (see Theorem 3). We give a geometric construction of the
cross ratios for 4-point sets in general position in terms of the angles of certain
curvilinear triangles (see Figure 3 and Figure 4). We don’t know whether this
construction was known earlier.

This paper is organized as follows: in Section 1 we explain briefly the corre-
spondence between elliptic curves and 4-point sets in the Riemann sphere. Also
we give some remarks on the cross ratio of 4-point sets. In Section 2 we describe
the configuration associated to the Legendre and the Weierstrass normal forms. In
Section 3 we describe the configuration of points for the Jacobi and the Edwards
normal forms and how they are related. Finally, in Section 4 we prove the main
theorem.

1. CORRESPONDENCE BETWEEN $\mathcal{M}$ AND FOUR POINTS

Recall that an elliptic curve is a Riemann surface of genus one. We denote by $\mathcal{M}$
the moduli of elliptic curves, i.e., the set of classes of isomorphism of elliptic curves,
where by isomorphism we mean a biholomorphism between Riemann surfaces.

Every elliptic curve admits a meromorphic function of degree 2 and by the
Riemann-Hurwitz formula this map has four critical values. Reciprocally, given
a 4-point set in the Riemann sphere there is a degree 2 meromorphic function with
critical values on those points. The above correspondence is summed in following
theorem (see [2, §6.3.2]):

**Theorem 1.** There is a bijection between $\mathcal{M}$ and 4-point sets on the Riemann
sphere modulo M"obius transformations.

**Remark.** In this paper we assume that the complex plane $\mathbb{C}$ has the canonical
orientation, the positive orientation is the counterclockwise orientation.

Now we will discuss a geometric interpretation of the cross ratio of 4-point sets
when one of them is at $\infty$ and the other three complex numbers are not collinear.

1.1. THE CROSS RATIO AND THREE POINTS IN THE COMPLEX PLANE. Recall that the
cross ratio $\chi(z_1, z_2, z_3, z_4)$ of four distinct and ordered points $(z_1, z_2, z_3, z_4)$ in the
Riemann sphere, is the value $\mu(z_4)$ where $\mu$ is the unique M"obius transformation
which sends orderly the three points $(z_1, z_2, z_3)$ to $(0, 1, \infty)$. When the four points
are complex numbers the cross ratio becomes:

$$\chi(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}.$$  \hspace{1cm}(3)

We begin with this elementary remark:

**Lemma 1.** Suppose that $\{z_1, z_2, z_3, \infty\}$ and $\{w_1, w_2, w_3, \infty\}$ are two 4-point sets
in the Riemann sphere which are M"obius equivalent, then there exists an affine
transformation $az + b$ that maps one set to the other.
Proof. Suppose that $\mu(z)$ is the Möbius transformation which sends $\{z_1, z_2, z_3, \infty\}$ to $\{w_1, w_2, w_3, \infty\}$. If $\mu$ fixes $\infty$ there is nothing to do. If $\mu(\infty) = w_i$, we can take the Möbius transformation $T$ that interchanges $\infty$ with $w_i$ and interchange the other points. Such transformation exists because this kind of permutations do not change the cross ratio. Then the affine transformation is $T \circ \mu$. \hfill $\Box$

The Lemma 1 has interesting consequences. First, since we can send every point to $\infty$ by a Möbius transformation it follows directly from Lemma 1 and Theorem 1 the corollary:

Corollary 1. There is a bijection between $\mathcal{M}$ and the sets of three points on the complex plane $\mathbb{C}$ modulo complex affine transformations.

Remark. In what follows we assume that our three points are not collinear; also we suppose that the triangle formed by these points is positively oriented.

We say that two sets of three points in the complex plane are equivalent if there is an affine map $az + b$ which maps one set to the other. Recall that, by definition, two triangles are similar if they have the same angles. And two (oriented) triangles are similar if and only if there is an affine transformation $az + b$, with $a, b \in \mathbb{C}$ and $a \neq 0$, which maps one triangle to the other. It follows from the Corollary 1 that two sets of three points are equivalent if and only if their (oriented) triangles are similar.

From the above discussion follows the proposition:

Proposition 1. Let $z_1, z_2, z_3$ be three non collinear points in the complex plane. The number $\lambda$ is a cross ratios for the 4-point set $\{z_1, z_2, z_3, \infty\}$ if and only if the oriented triangle $\{0, 1, \lambda\}$ is similar to the oriented triangle $\{z_1, z_2, z_3\}$. Therefore there are generically six equivalent cross ratios of each 4-point set $\{z_1, z_2, z_3, \infty\}$ (see Figure 1).
We will see in Section 4 that for not collinear points the only exception is when the oriented triangle \( \{z_1, z_2, z_3\} \) is equilateral. If \( \lambda \) is a cross ratio for some 4-point set, the other values are given by (10).

The proof of Lemma 1 tells us how to construct a cross ratio for a particular order:

**Proposition 2.** Let \( z_1, z_2, z_3 \) be three non collinear points in the complex plane forming an oriented triangle with respective angles \( \alpha, \beta, \gamma \). The cross ratios \( \lambda_1 = \chi(z_1, z_2, z_3, \infty) \) and \( \lambda_2 = \chi(z_1, z_2, \infty, z_3) \) can be obtained by the geometric construction illustrated in Figure 2.

In order to find geometrically the cross ratio for a 4-point set in general position in the complex plane we need to find the angles of the curvilinear triangles of we call the shape of a 4-point set, which we describe bellow.

### 1.2. The 4-point shape.

In what follows we assume that our four points are in general position, i.e. they not lie in a same circle.

Let \( \{z_1, z_2, z_3, z_4\} \) be a 4-point set in the complex plane, for each three points there are exactly one circle passing through them, then there are four circles associated to these points. Consider the curvilinear triangles formed by arcs of these circles whose vertices are in each set of three points of \( \{z_1, z_2, z_3, z_4\} \) (see Figure 3), we suppose that they are oriented (with the positive orientation of \( \mathbb{C} \)).

These four curvilinear triangles have the same angles: to see this, observe, like in the Lemma 1 that the permutations of kind 2+2, i.e. product of two transpositions do not change the cross ratio, then there exist a Möbius transformation which permutes the four points interchanging a pair of points \( z_i \) with \( z_j \) and interchanging the remainder two. This kind of transformations form a group isomorphic to
the Klein group. As Möbius transformations maps circles to circles conformally, preserving the orientation, this group acts transitively in the four triangles.

From the last argument also follows that two equivalent 4-point sets in the complex plane have oriented curvilinear triangles with the same angles.

If we allow generalized circles the same is true when one point is $\infty$ and the other are non collinear complex numbers, in this case one curvilinear triangle is an euclidean triangle, the other triangles have an arc and two lines as its sides (see Figure 4). The euclidean triangle is positively oriented, which induce an orientation to the other triangles.

Reciprocally, if their oriented curvilinear triangles have the same angles they are equivalent, we can see it sending one point to $\infty$ and considering the previous case of euclidean triangles. So we can define the shape of a 4-point set in the Riemann sphere as the oriented curvilinear triangles obtained in the before construction.

Then we have the following theorem:

**Theorem 2.** Two 4-point set in general position in the Riemann sphere are Möbius equivalent if and only if their shapes have the same angles (see Figure 3 and Figure 4).

Then, the cross ratios of four points in the complex plane can be found geometrically analogously to 1.1, we exemplify the construction of two of them:

**Proposition 3.** Let $\{z_1, z_2, z_3, z_4\}$ four points in the complex plane in general position. Consider the oriented curvilinear triangle with vertices in $z_1$, $z_2$ and $z_3$ with respective angles $\alpha, \beta, \gamma$. The cross ratios $\lambda_1 = \chi(z_1, z_2, z_3, z_4)$ and $\lambda_2 = \chi(z_1, z_2, z_4, z_3)$ can be obtained by the geometric construction illustrated in Figure 5.
Figure 4. Curvilinear triangles of four points in the Riemann sphere with one point at $\infty$.

Figure 5. Geometric construction of $\lambda_1 = \chi(z_1, z_2, z_3, z_4)$ and $\lambda_2 = \chi(z_1, z_2, z_4, z_3)$.

2. Legendre and Weierstrass normal form.

Recall that if $p(x)$ is a cubic polynomial with complex coefficients and different roots, the Riemann surface $E$: $y^2 = p(x)$ has the projection to the first coordinate as a meromorphic function of degree two which ramifies exactly in the roots of $p(x)$ and $\infty$ (see [1, 8.10]), then, by the Corollary 1 every elliptic curve is isomorphic to one elliptic curve of that form. By the discussion in the previous section, two
such elliptic curves $E_1$: $y^2 = p(x)$, $E_2$: $y^2 = q(x)$ are isomorphic if and only if the roots of $p(x)$ and $q(x)$ are equivalent or equivalently if their oriented triangles have the same angles, for non-degenerated triangles.

Given four points we can also take some cross ratio of them, so we can take the elliptic curve as

$$E: \quad y^2 = x(x-1)(x-\lambda), \quad \lambda \in \mathbb{C} - \{0,1\}. \quad (4)$$

This is called a Legendre normal form of an elliptic curve. Two such elliptic curves $E_1$: $y^2 = x(x-1)(x-\lambda_1)$, $E_2$: $y^2 = x(x-1)(x-\lambda_2)$ are isomorphic if and only if the $\lambda_1$ and $\lambda_2$ are equivalent, we explained in 1.1 how to read geometrically into this equivalence: they are equivalent if and only if the oriented triangles with vertices at 0, 1, $\lambda_1$ and 0, 1, $\lambda_2$ are similar.

Another normal form is obtained considering the centroid $C$ of three points $\{z_1, z_2, z_3\}$ in the complex plane to obtain three points $e_i = z_i - C$ whose centroid is at 0, so the polynomial $4(x - e_1)(x - e_2)(x - e_3)$ has the form $4x^3 - g_2x - g_3$, then the elliptic curve is given by:

$$E: \quad y^2 = 4x^3 - g_2x - g_3, \quad \Delta = g_2^3 - 27g_3^2 \neq 0, \quad (5)$$

where $\Delta = g_2^3 - 27g_3^2$ is the discriminant of the cubic polynomial. Recall $\Delta \neq 0$ if and only if the cubic polynomial has distinct roots. This kind of elliptic curves are called a Weierstrass normal form. Thus we can interpret geometrically the Weierstrass normal form as all the sets of three points in the complex plane whose centroid is at 0. As we know the centroid can be found geometrically, as the intersection of the three medians. As affine transformations preserve centroids, we have that two elliptic curves in the Weierstrass normal form are isomorphic if and only if the roots of the polynomials differ by a complex factor.

2.1. Some examples. To illustrate the above discussion we give a couple of examples. First we compute a Weierstrass and a Legendre normal form of the Fermat cubic in this setting:

**Example 1.** The Fermat cubic is an elliptic curve given by $x^3 + y^3 = 1$. We can take for example the meromorphic function $x + y$, it is of degree 2 with critical values 0 and the cubic roots of 4, sending 0 to $\infty$ by $1/z$, we obtain that the Fermat cubic is the equilateral triangle with vertices in the cubic roots of 1/4. In this case the centroid is already in the origin, then the Weierstrass normal form is $y^2 = 4x^3 - 1$. Denote by $\rho$ the cubic root $\exp(2\pi i/3)$. To obtain the cross ratio hang the equilateral triangle on the interval $[0,1]$, the vertices are $\{0,1,-\rho\}$. So the Legendre normal form is $y^2 = x(x-1)(x+\rho)$.

For the next example we need elliptic curves given by quartic equations. Given a quartic polynomial $q(x)$ with complex coefficients and distinct roots, the Riemann surface $E$: $y^2 = q(x)$ has the projection to the first coordinate as a meromorphic function of degree 2 whose critical values are exactly the roots of $q(x)$ (see [1, 8.10]). Note that the curve $E$ has not smooth projective closure in $\mathbb{CP}^2$.

**Example 2.** In [7, §2.3.5] is considered the following family of algebraic curves: $F: \quad x^n + y^m = 1 \quad (n \geq m \geq 2)$. For the cases $(4,2)$, $(3,3)$ and $(3,2)$ they are the elliptic curves:
\[ x^4 + y^2 = 1, \quad x^3 + y^3 = 1, \quad x^3 + y^2 = 1. \]

With our method we can see that the last two are isomorphic between them and non-isomorphic to the first. For that, consider the projection to the first coordinate on the curve \((n, m) = (4, 2)\), it ramifies on the fourth roots of 1, as they are in the same circle, the cross ratio is real. The second case was calculated in the previous example, it corresponds to equilateral triangle. In the third case the projection to the first coordinate ramifies on the cubic roots of unity and \(\infty\). By the previous discussion it follows the affirmation.

3. The Jacobi and Edwards normal form

Recall that every 4-point set in the real line can be mapped by some Möbius function to four symmetric points respect to zero \(\{a, -1/a, -a, 1/a\}\). In fact, this can be done with every four points in the complex plane. Because, for \(a \in \mathbb{C} - \{0\}\) such that \(a^4 \neq 1\), the cross ratio

\[
\varphi(a) = \chi \left( a, -\frac{1}{a^2}, -a, \frac{1}{a^2} \right) = \left( \frac{1 - a^2}{1 + a^2} \right)^2
\]

is well defined, and as function of \(a\) is meromorphic in all Riemann sphere, therefore take all values in \(\hat{\mathbb{C}} - \{0, 1, \infty\}\) when \(a^4 \neq 1\) and \(a \neq 0\). Then all elliptic curves are isomorphic to one elliptic curve in the form

\[
y^2 = (x^2 - a^2) \left( a^2 x^2 - 1 \right), \quad \text{with } a^4 \neq 1 \text{ and } a \neq 0.
\]

The projection in the first coordinate is a meromorphic function of degree 2 whose critical values are \(\{a, -1/a, -a, 1/a\}\). Note that (7) can be reduced to the curve:

\[
y^2 = (x^2 - 1) \left( k^2 x^2 - 1 \right), \quad \text{with } k^2 \neq \pm 1 \text{ and } k \neq 0,
\]

where \(k = a^2\), this can be achieved applying the transformation \(\mu(z) = z/a\) to the points \(\{a, -1/a, -a, 1/a\}\). The equation (8) is called a Jacobi normal form of an elliptic curve. Therefore every elliptic curve is isomorphic to one elliptic curve in the Jacobi normal form. In fact this elliptic curve is isomorphic to the normal form defined by Edwards, see [3]. The Edwards normal form is by definition:

\[
x^2 + y^2 = a^2 + a^2 x^2 y^2, \quad \text{with } a^4 \neq 1 \text{ and } a \neq 0,
\]

in this case the projection to the first coordinate is of degree 2 and ramifies exactly on \(\{a, -1/a, -a, 1/a\}\) (see Figure 6).

We summarize the previous discussion in the following theorem:

**Theorem 3.** Every elliptic curve \(X\) is isomorphic to one elliptic curve in the form

\[
y^2 = (x^2 - a^2) \left( a^2 x^2 - 1 \right), \quad \text{with } a^4 \neq 1 \text{ and } a \neq 0,
\]

which is isomorphic to the elliptic curve in the Edwards normal form (9) and isomorphic to the elliptic curve in the Jacobi normal form (8) with \(k = a^2\).

For different approach of the above equivalence consult [3].
4. The $J$ invariant and the Hesse normal form

We say that $\lambda$ and $\lambda' \in \hat{\mathbb{C}} - \{0, 1, \infty\}$ are equivalent if $\{0, 1, \lambda, \infty\}$ and $\{0, 1, \lambda', \infty\}$ are Möbius equivalent, it occurs if and only if there is a Möbius transformation $\mu$ such that permutes $\{0, 1, \infty\}$ and satisfies $\mu(\lambda) = \lambda'$. Computing these cross ratios, we obtain that the points equivalent to $\lambda$ are:

$$\lambda, \frac{1}{\lambda}, \lambda - 1, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda}.$$  \hfill (10)

Define the rational function

$$J(\lambda) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$  \hfill (11)

this is a rational map of degree 6 called the $J$ invariant, it is an invariant because $\lambda$ is equivalent to $\lambda'$ if and only if $J(\lambda) = J(\lambda')$ (our definition of $J$ is according to [2, §6.3.2], it differs from the definition of other authors by some factor). Then, the $J$ invariant is a function well-defined in $\mathcal{M}$ (viewed as four points on the sphere modulo Möbius transformations), the value of $J$ in a set of four points $\{z_1, z_2, z_3, z_4\}$ is obtained by $J(\lambda)$, where $\{0, 1, \lambda, \infty\}$ is a 4-point set equivalent to $\{z_1, z_2, z_3, z_4\}$.

In the following diagram we represent the critical values and the critical points of $J$, the weight of the arrows represent the multiplicity of the critical point:

\begin{align*}
(12) & \quad 0 \quad 1 \quad \infty \quad 2 \quad \frac{1}{2} \quad -1 \quad -\rho \quad -\rho^2 \\
\end{align*}
where $\rho$ is the cubic root $\exp(2\pi i/3)$. Note that the Fermat cubic is a critical point of $J$, which corresponds to an equilateral triangle. The $J$ invariant can be factorized by the following rational functions (see [2, p. 93]):

\[
\hat{C} = \frac{3x + 1}{3 - x}, \quad \hat{C} = \frac{33}{3 - x}, \quad \hat{C},
\]

(the constant $-3^3$ in the last function in (13) differs from the constant that appear in [2, p. 93], which seems to contain a mistake).

Consider the elliptic curve in the form

\[
x^3 + y^3 + 1 = 3kxy, \quad k \in \mathbb{C} \text{ with } k^3 \neq 1,
\]
called the Hesse normal form. As in Example 1 we take the function $x+y$, this is also of degree two with critical values on $-k$ and on the roots of the cubic polynomial

\[
z^3 - 3kz^2 + 4
\]
this polynomial has discriminant $\Delta = 4^23^3(k^3 - 1)$, then by our hypothesis there are three distinct roots. So we can take the value of the $J$ invariant in the set of critical values, obtaining a function $\varphi(k)$ depending on $k$, and well defined in $\hat{C} - \{1, \rho, \rho^2, \infty\}$. Since the roots of a polynomial depends continuously of the coefficients, $\varphi(k)$ is a continuous function on $\hat{C} - \{1, \rho, \rho^2, \infty\}$. And it is also analytic in $\mathbb{C} - \{0, 1, \rho, \rho^2\}$, as we can see looking at the general formulas of cubic equations (taking locally suitable branches of the roots) and taking the formula (3) with $\nu = -k$:

\[
z_\nu = k - \rho^\nu \alpha - \frac{1}{\rho^\nu} \frac{k^2}{\alpha}, \quad \nu = 1, 2, 3, \quad \text{where } \alpha = \sqrt[3]{2 - k^3 + 2i\sqrt{k^3 - 1}}.
\]

Let us analyse what is happening in $\infty$ and in the cubic roots of 1. If $k_n$ is a sequence such that $k_n \to \rho^\nu$, again as the roots depend continuously of the coefficients, for each $n$, we can choose a labelling of the roots of $z^3 - 3k_nz^2 + 4$ such that the cross ratios tends to $\infty$, then $\varphi(k_n) \to \infty$.

On the other hand, if $k_n \to \infty$, we affirm that $\varphi(k_n) \to \infty$. Here is a sketch of the proof: considering the equation (16), we have that $\alpha/k$ tends to $-1, -\rho$ or $-\rho^2$, when $k \to \infty$, depending of the choice of the branch of the cubic root in $\alpha$. With this, it is straightforward to check that $\chi(z_1, z_2, z_3, -k)$ tends to $0, 1$ or $\infty$, when $k \to \infty$, according to the choice of the branch. Then, if we take for each $n$, $k = k_n$ in (16) we have that the cross ratios $\chi(z_1, z_2, z_3, -k_n)$ accumulate on an subset of $\{0, 1, \infty\}$ when $k_n \to \infty$. In any case $J$ tends to $\infty$.

Then $\varphi(k)$ is a continuous function on all the Riemann sphere. Hence a rational function, whose poles are $\infty$ and the cubic roots of 1.

Hence, for every $\lambda \in \hat{C} - \{0, 1, \infty\}$ there exists $k \in \mathbb{C}$ with $k^3 \neq 1$, such that $\varphi(k) = J(\lambda)$, then $\lambda$ is equivalent to one cross ratio of the set formed by $-k$ and the roots of $z^3 - 3kz^2 + 4$.

We summarize the previous discussion in the following theorem:

**Theorem 4.** Every elliptic curve $X$ is isomorphic to one elliptic curve given in the Hesse normal form:
Figure 7. A configuration of points for the Hesse normal form with \( k = -2.39882 + 1.27189i \).

\[
(17) \quad x^3 + y^3 + 1 = 3kxy, \quad k \in \mathbb{C} \text{ with } k^3 \neq 1,
\]

which is isomorphic to the elliptic curve in the form

\[
(18) \quad y^2 = (x + k)(x^3 - 3kx^2 + 4), \quad k \in \mathbb{C} \text{ with } k^3 \neq 1.
\]

For different approaches consult [4], [5] and [6]. According to the formula obtained in [6] or in [4] the rational function \( \varphi(k) \) should be:

\[
(19) \quad \varphi(k) = \frac{27}{4} \left( \frac{k(k^3 + 8)}{4(k^3 - 1)} \right)^3.
\]

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