Detection of genuine tripartite entanglement based on Bloch representation of density matrices

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Abstract We study the genuine multipartite entanglement in tripartite quantum systems. By using the Schmidt decomposition and local unitary transformation, we convert the general states to simpler forms and consider certain matrices from correlation tensors in the Bloch representation of the simplified density matrices. Using these special matrices we obtain new criteria for genuine multipartite entanglement. Detail examples show that our criteria are able to detect more tripartite entangled and genuine tripartite entangled states than some existing criteria.

Keywords Genuine multipartite entanglement · Separability · Correlation tensor

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1 Introduction

Quantum entanglement is a key resource in various fields of information processing such as quantum cryptography [1], teleportation [2] and dense coding [3] etc. Among various types of entanglement, genuine multipartite entanglement stands out to be advantageous in many applications [4]. Therefore, measuring and detection of genuine multipartite entanglement are important tasks.

In this paper, we first transform a general state into a simpler form by the Schmidt decomposition [17] and LU equivalence [18,19] to explain our approach. Suppose $|\varphi\rangle$ is a pure state of a composite system $H_1^{d_1} \otimes H_2^{d_2}$. Then there exist orthonormal states $|i_a\rangle$ for system $H_1^{d_1}$ and orthonormal states $|i_b\rangle$ for system $H_2^{d_2}$ such that $|\varphi\rangle$ has the Schmidt decomposition $|\varphi\rangle = \sum_i \tau_i |i_a\rangle |i_b\rangle$, where $\tau_i$ are non-negative real numbers satisfying $\sum_i \tau_i^2 = 1$, known as the Schmidt coefficients [17]. For a bipartite state on $H_1^{d_1} \otimes H_2^{d_2}$, $\rho$ can be expressed as

$$\rho = \rho_{12} = \frac{1}{d_1d_2} I_{d_1} \otimes I_{d_2} + \sum_i t_i \lambda_i^{(1)} \otimes I_{d_2} + \sum_j t_j \lambda_j^{(2)} \otimes I_{d_1} + \sum_{ij} t_{ij} \lambda_i^{(1)} \otimes \lambda_j^{(2)},$$

with two real orthogonal matrices $A = \{a_{ij}\} \in O(d_1^2 - 1)$ and $B = \{b_{ij}\} \in O(d_2^2 - 1)$. Therefore, $N(\rho') = A^t N(\rho) B$ [15]. Let $\| \cdot \|_{tr}$ stand for the trace norm defined by $\| N \|_{tr} = \sum_{i} \sigma_i = Tr \sqrt{N^t N}$, $N \in \mathbb{R}^{m \times n}$, where $\sigma_i (i = 1, 2, \ldots, \min (m, n))$ are the singular values of the matrix $N$. Then we have $\| N(\rho') \|_{tr} = \| N(\rho) \|_{tr}$ due to the fact that the singular values of a rectangular matrix $N$ are the same as those of $B^t N A$ when $A, B$ are orthogonal. Thus the study of $\rho$ can be translated into that of $\rho'$. Using this idea, we are able to construct some useful invariants of GME.

The paper is organized as follows: in Section 2, we present new separability criteria to detect GME for $(2 \times 2 \times 2)$-dimensional quantum states, a detailed example shows that our theorem is
more effective than previous available results. In Section 3, we generalize these criteria to \((d \times d \times d)\)-dimensional systems. Comments and conclusions are given in Section 4.

2 GME for \((2 \times 2 \times 2)\)-dimensional quantum states

We first consider the separability and GME of three qubit states. Denote by \(\lambda_i^{(j)}\) \((i = 1, 2, 3)\) the standard Pauli spin matrices \(\sigma_3, \sigma_1\) and \(\sigma_2\), associated with the \(j\)th qubit, respectively. A general three qubit state \(\rho\) can be written in the following Bloch representation,

\[
\rho = \frac{1}{8}(I \otimes I \otimes I + \sum_{i=1}^{3} t_i^{I} \lambda_i^{(1)} \otimes I \otimes I + \sum_{j=1}^{3} t_j^{2} I \otimes \lambda_j^{(2)} \otimes I + \sum_{k=1}^{3} t_k^{3} I \otimes I \otimes \lambda_k^{(3)} + \sum_{i,j=1, i \neq j}^{3} t_{ij}^{1} \lambda_i^{(1)} \otimes \lambda_j^{(2)} \otimes I + \sum_{i,k=1, i \neq k}^{3} t_{ik}^{12} \lambda_i^{(1)} \otimes I \otimes \lambda_k^{(3)} + \sum_{j,k=1, j \neq k}^{3} t_{jk}^{23} I \otimes \lambda_j^{(2)} \otimes \lambda_k^{(3)}) \tag{1}
\]

where \(t_i^{I} = Tr(\rho \lambda_i^{(1)} \otimes I \otimes I), t_j^{2} = Tr(\rho I \otimes \lambda_j^{(2)} \otimes I), t_k^{3} = Tr(\rho I \otimes I \otimes \lambda_k^{(3)}), t_{ij}^{12} = Tr(\rho \lambda_i^{(1)} \otimes \lambda_j^{(2)} \otimes I), t_{ik}^{12} = Tr(\rho \lambda_i^{(1)} \otimes I \otimes \lambda_k^{(3)}), t_{jk}^{23} = Tr(\rho I \otimes \lambda_j^{(2)} \otimes \lambda_k^{(3)})\) and \(t_{ijk} = Tr(\rho \lambda_i^{(1)} \otimes \lambda_j^{(2)} \otimes \lambda_k^{(3)})\).

Let \(T_{1}^{123}, T_{2}^{123}, T_{3}^{123}, T_{1}^{213}, T_{2}^{213}, T_{3}^{213}\) and \(T_{3}^{132}\) denote the matrices with entries \(t_{ijk}, t_{ij3}, t_{ki2}, t_{i1k}, t_{ij1}, t_{ij2}\) and \(t_{ij3}\) \((i, j, k = 1, 2, 3)\), respectively. For example,

\[
T_{1}^{123} = \begin{bmatrix} t_{111} & t_{121} & t_{131} \\ t_{112} & t_{122} & t_{132} \\ t_{113} & t_{123} & t_{133} \end{bmatrix}, \quad T_{2}^{123} = \begin{bmatrix} t_{111} & t_{211} & t_{311} \\ t_{112} & t_{212} & t_{312} \\ t_{113} & t_{213} & t_{313} \end{bmatrix}. \tag{2}
\]

Set \(N_{1}^{123} = 15T_{1}^{123} + T_{2}^{123} + T_{3}^{123}, N_{2}^{123} = 4T_{2}^{123}, N_{3}^{123} = 15T_{1}^{312} + T_{2}^{312} + T_{3}^{312}\) and \(T(\rho) = \frac{1}{3}(\|N_{1}^{123}\|_r + \|N_{2}^{123}\|_r + \|N_{3}^{123}\|_r)\).

**Remark 1.** There are two kinds of genuine three-qubit entangled pure states under stochastic local operations and communication (SLOCC), namely, the GHZ state and W state. Mixing the GHZ or W states with white noise, one can obtain different upper bounds of \(\|N^{fgh}\|_r\) to detect their entanglement. We find that \(T(\rho) = \frac{1}{3}(\|N_{1}^{123}\|_r + \|N_{2}^{123}\|_r + \|N_{3}^{123}\|_r)\) to detect more genuine tripartite entangled states in \(d \times d \times d\) dimensional systems.

Note that \(\|N_{1}^{123}\|_r\) is invariant under local unitary transformations. Suppose \(\rho' = (I \otimes U_2 \otimes U_3)\rho(I \otimes U_2^\dagger \otimes U_3^\dagger)\), where \(U_2, U_3 \in U(2), U_2^2 = \sum_{j=1}^{3} a_{ij}\lambda_j^{(2)}\) and \(U_3^3 = \sum_{j=1}^{3} b_{ij}\lambda_j^{(3)}\) for some coefficients \(a_{ij}\) and \(b_{ij}\). By [18 Lemma 2.1] one has that \(A = (a_{ij}), B = (b_{ij}) \in O(3)\) and

\[
T_{1}^{123}(\rho') = B^t T_{1}^{123}(\rho) A, \quad T_{2}^{123}(\rho') = B^t T_{2}^{123}(\rho) A, \quad T_{3}^{123}(\rho') = B^t T_{3}^{123}(\rho) A. \tag{3}
\]

Then we have

\[
N_{1}^{123}(\rho') = 15 B^t T_{1}^{123}(\rho) A + B^t T_{2}^{123}(\rho) A + B^t T_{3}^{123}(\rho) A = B^t N_{1}^{123}(\rho) A. \tag{4}
\]
The singular value decomposition of $N^{1|23}(\rho)$ is $N^{1|23}(\rho) = UDV$, where $U$ and $V$ are unitary matrices, $D$ is a diagonal matrix with singular value of $N^{1|23}(\rho)$. Then $N^{1|23}(\rho') = B'UDV A$ shares the same matrix $D$ with $N^{1|23}(\rho)$. Thus the singular values of $N^{1|23}(\rho)$ and $N^{1|23}(\rho')$ are the same. Therefore, $\|N^{1|23}(\rho')\|_{tr} = \|N^{1|23}(\rho)\|_{tr}$. Due to invariance of the trace norm under local unitary transformations, we can simplify form of the density matrix to study the entanglement of the tripartite quantum states.

We first consider biseparable pure states. If $\rho = |\phi\rangle\langle\phi|$ is $1|23$ separable under the bipartition of the first qubit and the last two qubits, i.e., $|\phi_{1|23}\rangle = |\phi_1\rangle \otimes |\phi_{23}\rangle \in H_1^2 \otimes H_{23}^1$. From Schmidt decomposition, we have

$$|\phi_{1|23}\rangle = \tau_0|0a\rangle + \tau_1|1b\rangle,$$

with $|\tau_0|^2 + |\tau_1|^2 = 1$. Taking into account local unitary equivalence in $H_1^2 \otimes H_{23}^3$, when $|\phi_{23}\rangle \in H_{23}^3$ is separable, we can transform $\{a, b\}$ into two orthonormal basis elements which constitute separable states: (i) $\{a, b\} = \{|00\rangle, |01\rangle\}$, i.e., $|\phi_{1|23}\rangle = |\phi_1\rangle \otimes |\phi_{23}\rangle \otimes |\phi_3\rangle$ is fully separable; when $|\phi_{23}\rangle \in H_{23}^3$ is entangled, we can transform $\{a, b\}$ into two orthonormal basis elements which constitute entangled states: (ii) $\{a, b\} = \{|00\rangle, |11\rangle\}$. The matrices $T_1^{1|23}$, $T_2^{1|23}$ and $T_3^{1|23}$ are given by, respectively,

$$(i): T_1^{1|23} = \begin{bmatrix} \tau_0^2 + \tau_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_2^{1|23} = \begin{bmatrix} 0 & 0 & 0 \\ 2\tau_0\tau_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_3^{1|23} = \begin{bmatrix} 0 & 0 & 0 \\ -2\tau_0\tau_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(ii): T_1^{1|23} = \begin{bmatrix} \tau_0^2 - \tau_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_2^{1|23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\tau_0\tau_1 & 0 \\ 0 & 0 & -2\tau_0\tau_1 \end{bmatrix}, \quad T_3^{1|23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\tau_0\tau_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find that after Schmidt decomposition and LU equivalence, the matrices (2) constructed by correlation tensors have been greatly simplified to (5) and (6), then we have the following separability criterion.

**Lemma 1** If the state $\rho \in H_1^2 \otimes H_2^2 \otimes H_3^2$ is a bipartite separable pure state, corresponding to the case (i) and (ii), we have

1. If $\rho$ is separable under bipartition $1|23$, then $\|N^{1|23}\|_{tr} \leq \sqrt{227}$ or $\sqrt{233}$.
2. If $\rho$ is separable under bipartition $2|13$, then $\|N^{2|13}\|_{tr} \leq 12$ or $4$.
3. If $\rho$ is separable under bipartition $3|12$, then $\|N^{3|12}\|_{tr} \leq \sqrt{227}$ or $\sqrt{233}$.

**Proof**

(1) If a pure tripartite qubit state $|\phi\rangle \in H_1^2 \otimes H_2^2 \otimes H_3^2$ is separable under the bipartition $1|23$, then for the case (i),

$$\|N^{1|23}\|_{tr} = \sqrt{15^2(\tau_0^2 + \tau_1^2)^2 + 8\tau_0^2\tau_1^2} \leq \sqrt{227}.$$  

And for the case (ii),

$$\|N^{1|23}\|_{tr} = 15\sqrt{(\tau_0^2 - \tau_1^2)^2 + 4\tau_0\tau_1}$$

$$= 15\sqrt{(1 - 2\tau_1^2)^2 + 4\tau_0\tau_1} \sqrt{1 - \tau_1^2} \leq \sqrt{233}.$$  

(9)
where the upper bound is obtained by taking the extreme value of the function with independent variable $\tau_1$.

(2) If $\rho$ is separable under bipartition 2|13, for the first case,

$$T^{2|13} = \begin{bmatrix} \tau_0^2 + \tau_1^2 & 0 & 0 \\ 0 & 2\tau_0\tau_1 & 0 \\ 0 & 0 & -2\tau_0\tau_1 \end{bmatrix},$$

we have

$$\|N^{2|13}\|_{tr} = 4(\tau_0^2 + \tau_1^2 + 4\tau_0\tau_1) \leq 12.$$  

And for the second case,

$$T^{2|13} = \begin{bmatrix} \tau_0^2 - \tau_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$\|N^{2|13}\|_{tr} = 4\sqrt{(\tau_0^2 - \tau_1^2)^2} \leq 4.$$  

(3) Using similar method, if $\rho$ is separable under bipartition 3|12, $\|N^{3|12}\|_{tr} \leq \sqrt{227}$ and $\sqrt{233}$ with respect to the case (i) and (ii), respectively. \(\square\)

**Remark 2.** Under Schmidt decomposition a general state can be written as $|\phi_{123}\rangle = |\phi_1\rangle \otimes |\phi_{23}\rangle = \tau_0|0a\rangle + \tau_1|1b\rangle$. Under LU equivalence we can choose $\{|a\rangle, |b\rangle\} = \{|00\rangle, |01\rangle\}$ in the case (i) to transform $\{|a\rangle, |b\rangle\}$ into two orthonormal bases which constitute separable states. Then $|\phi_{23}\rangle$ must be a separable state such that $|\phi_{23}\rangle = |\phi_2\rangle \otimes |\phi_3\rangle$, which implies that $|\phi\rangle$ is fully separable. Hence, we have $\|N^{1|23}\|_{tr} \leq \sqrt{227}$. Consequently, if $\|N^{1|23}\|_{tr} > \sqrt{227}$, $\rho$ is not fully separable. Therefore, Lemma 1 can also be used to detect the fully separability.

**Theorem 1** For a tripartite qubit mixed state $\rho$, if

$$T(\rho) = \frac{1}{3}(\|N^{1|23}\|_{tr} + \|N^{2|13}\|_{tr} + \|N^{3|12}\|_{tr}) > \sqrt{233},$$

then $\rho$ is genuine multipartite entangled.

**Proof** For a mixed state $\rho = \sum p_i \rho_i$, $\sum p_i = 1$, if $\rho$ is biseparable, by using Lemma 1, we have

$$T(\rho) = \frac{1}{3}(\|N^{1|23}(\rho)\|_{tr} + \|N^{2|13}(\rho)\|_{tr} + \|N^{3|12}(\rho)\|_{tr})$$

$$\leq \frac{1}{3} \sum p_i (\|N^{1|23}(\rho_i)\|_{tr} + \|N^{2|13}(\rho_i)\|_{tr} + \|N^{3|12}(\rho_i)\|_{tr})$$

$$\leq \frac{1}{3}(\sqrt{233} + \sqrt{233} + \sqrt{233})$$

$$= \sqrt{233}.$$  

Consequently, if $T(\rho) > \sqrt{233}$, $\rho$ is GME. \(\square\)

**Example 1** Consider the mixture of the W state with maximally mixed state,

$$\rho = \frac{1-x}{8} I_8 + x|W\rangle\langle W|,$$
Fig. 1 Detect GME for $\rho$ in Example 1, $f_1(x)$ from our Theorem 1 (solid straight line), $f_2(x)$ from Theorem 2 in [13] (dash-dot straight line) and $f_3(x)$ from Theorem 2 in [15] (dashed straight line).

where $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, $x \in [0,1]$, $I_8$ is $8 \times 8$ identity matrix. By calculation, we have that $\|N_1|_{23}\|_\text{tr} = \left(\sqrt{2941} - \frac{5\sqrt{233}}{6} + \sqrt{\frac{2941}{18}} + \frac{5\sqrt{3657}}{6} + 10\right)x$. Using Lemma 1, we have when $\|N_1|_{23}\|_\text{tr} > \sqrt{227}$, $\rho$ is not fully separable; when $\|N_1|_{23}\|_\text{tr} > \sqrt{233}$, $\rho$ is not separable under bipartition 1|23. Therefore, we have $\rho$ is not fully separable for $0 < x \leq 0.4296$ and not separable under bipartition 1|23 for $0 < x \leq 0.4352$. In [21], $\rho$ was detected as entangled for $0.619 < x \leq 1$. This shows that Lemma 1 detects more entangled states.

By using Theorem 1, we have $f_1(x) = T(\rho) - \sqrt{233} = \frac{1}{8}[2(\sqrt{\frac{2941}{18}} - \frac{5\sqrt{233}}{6} + \sqrt{\frac{2941}{18}} + \frac{5\sqrt{3657}}{6} + 10) + \frac{28}{3}]x - \sqrt{233}$, using Theorem 2 in [13], $f_2(x) > 0$ is used to detect GME for $0.791 < x \leq 1$. Set $f_3(x) = \frac{1}{\sqrt{6}}(\sqrt{66}x - 6)$, using Theorem 2 in [15], $\rho$ is GME if $f_3(x) > 0$, i.e., $0.7385 < x \leq 1$. The comparison is shown in Fig.1, where our result is able to detect more GME states.

3 GME for $(d \times d \times d)$-dimensional quantum states

Next we consider the separability and GME of $(d \times d \times d)$-dimensional quantum states. Let $\lambda_i^{(j)}$ ($j = 1, 2, 3, i = 1, \cdots, d^2 - 1$) denote the generators of the special unitary Lie group $SU(d)$ associated with the $j$th $d$-dimensional Hilbert space with orthonormal basis $\{ |a\rangle \}_{a=0}^{d-1}$ [20], for $i = 1, \cdots, d - 1$,

$$
\lambda_i^{(j)} = \sqrt{\frac{2}{i(i+1)}} \sum_{a=0}^{i-1} |a\rangle\langle a| - i|j\rangle\langle i|
$$

for $i = d, \cdots, \frac{(d+2)(d-1)}{2}$,

$$
\lambda_i^{(j)} = |j\rangle\langle k| + |k\rangle\langle j|
$$
when \( i = \frac{d(d+1)}{2}, \cdots, d^2 - 1, \)
\[
\lambda^{(j)} = -i(|j\rangle\langle k| - |k\rangle\langle j|),
\]
where \( 0 \leq j < k \leq d - 1. \)

A general tripartite quantum state \( \rho \in H_1^d \otimes H_2^d \otimes H_3^d \) can be written in the following Bloch representation,
\[
\rho = \frac{1}{d^3} I_d \otimes I_d \otimes I_d + \frac{1}{2d^3} \left( \sum_{i=1}^{d^2-1} t_i^1 \lambda^{(1)}_i \otimes I_d \otimes I_d + \sum_{j=1}^{d^2-1} t_j^2 \lambda^{(2)}_j \otimes I_d \otimes I_d + \sum_{k=1}^{d^2-1} t_k^3 \lambda^{(3)}_k \otimes I_d \otimes I_d \right)
\]
\[
\otimes I_d \otimes \lambda^{(3)}_k + \frac{1}{4d^3} \left( \sum_{i,j=1}^{d^2-1} t_{ij}^2 \lambda^{(1)}_i \otimes \lambda^{(2)}_j \otimes I_d \right)
\]
\[
+ \sum_{k=1}^{d^2-1} t_{ik}^3 \lambda^{(1)}_i \otimes \lambda^{(2)}_j \otimes \lambda^{(3)}_k + \frac{1}{8} \sum_{i,j,k=1}^{d^2-1} t_{ijk} \lambda^{(1)}_i \otimes \lambda^{(2)}_j \otimes \lambda^{(3)}_k,
\]
where \( I_d \) denotes the \( d \times d \) identity matrix, \( t_i^1 = \text{Tr}(\rho \lambda^{(1)}_i \otimes I_d \otimes I_d), t_j^2 = \text{Tr}(\rho \lambda^{(2)}_j \otimes I_d \otimes I_d), t_k^3 = \text{Tr}(\rho \lambda^{(3)}_k \otimes I_d \otimes I_d) \),
\( t_{ij}^2 = \text{Tr}(\rho \lambda^{(1)}_i \otimes \lambda^{(2)}_j \otimes I_d), t_{ik}^3 = \text{Tr}(\rho \lambda^{(1)}_i \otimes \lambda^{(3)}_k \otimes I_d), t_{jk}^3 = \text{Tr}(\rho \lambda^{(2)}_j \otimes \lambda^{(3)}_k \otimes I_d) \), and \( t_{ijk} \) denote the matrices with entries \( t_{1jk}, t_{djk}, t_{d(d+1)jk}, t_{11k}, t_{1jk}, t_{1d} \) and \( t_{ijk} \) when \( i, j, k = 1, 2, 3 \), respectively. Set \( N^{1|23} = 15T_1^{1|23} + T_2^{1|23} + T_3^{1|23}, N^{2|13} = 4T_2^{2|13}, N^{3|12} = 15T_1^{3|12} + T_2^{3|12} + T_3^{3|12} \) and \( T(\rho) = \frac{1}{3}(\|N^{1|23}\|_{tr} + \|N^{2|13}\|_{tr} + \|N^{3|12}\|_{tr}) \).

**Lemma 2** If the state \( \rho \in H_1^d \otimes H_2^d \otimes H_3^d \) is a bipartite separable pure state, corresponding to the case (i) and (ii), we have

1. If \( \rho \) is separable under bipartition \( 1|23 \), then \( \|N^{1|23}\|_{tr} \leq \sqrt{2} \sqrt{\frac{2}{d}}(\frac{4}{d} + 1) \) or \( \sqrt{\frac{233}{2} - \frac{2}{d} + 15 \sqrt{(1 - \frac{2}{d})(\frac{2}{d} - \frac{2}{d})}} \);
2. If \( \rho \) is separable under bipartition \( 2|13 \), then \( \|N^{2|13}\|_{tr} \leq 4 \left[ \sum_{k=1}^{d^2-1} \sqrt{\frac{1}{k(k+1)}} \sqrt{1 - \frac{1}{d} + \frac{2(d^2-1)}{d}} \right] \) or \( 4 \sqrt{\frac{2}{d}}(1 + \sqrt{1 - \frac{2}{d}}) \);
3. If \( \rho \) is separable under bipartition \( 3|12 \), then \( \|N^{3|12}\|_{tr} \leq \sqrt{2} \sqrt{\frac{2}{d}}(\frac{4}{d} + 1) \) or \( \sqrt{\frac{233}{2} - \frac{2}{d} + 15 \sqrt{(1 - \frac{2}{d})(\frac{2}{d} - \frac{2}{d})}} \);

Proof (1) If \( \rho = |\varphi\rangle\langle \varphi| \) is separable under the bipartition \( 1|23 \), i.e., \( |\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_{23}\rangle \in H_1^d \otimes H_2^d \), by Schmidt decomposition we have
\[
|\varphi\rangle = \tau_0 |a_0\rangle + \tau_1 |a_1\rangle + \cdots + \tau_{d-1} |d-1, a_{d-1}\rangle
\]
with \( \sum_i |\tau_i|^2 = 1 \). Taking into account the local unitary equivalence in \( H_2^d \otimes H_3^d \), if \( |\varphi_{23}\rangle \in H_2^d \otimes H_3^d \) is separable we can transform \( \{|a_0\rangle, |a_1\rangle, \cdots, |a_{d-1}\rangle\} \) into the orthonormal bases which constitute separable states: \( \{|a_0\rangle, |a_1\rangle, \cdots, |a_{d-1}\rangle\} \neq \{|00\rangle, |01\rangle, \cdots, |0, d-1\rangle\} \), i.e. \( |\varphi_{123}\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle \) is
fully separable. When $|\varphi_{23}\rangle \in H_1^{23}$ is entangled, we can transform $\{|a_0\rangle, |a_1\rangle, \ldots, |a_{d-1}\rangle\}$ into the orthonormal bases which constitute entangled states: (ii) $\{|a_0\rangle, |a_1\rangle, \ldots, |a_{d-1}\rangle\} = \{|00\rangle, |11\rangle, \ldots, |d-1, d-1\rangle\}$.

In the first case, $|\varphi\rangle = \tau_0|000\rangle + \tau_1|101\rangle + \cdots + \tau_{d-1}|d-1, 0, d-1\rangle$, the matrices $T_1^{1|23}$, $T_2^{1|23}$ and $T_3^{1|23}$ are given by

$$T_1^{1|23} = \begin{pmatrix}
\tau_0^2 + \tau_1^2 & \sqrt{2}(\tau_0^2 + \tau_1^2) & \cdots & \sqrt{2}(\tau_0^2 + \tau_1^2) \\
\sqrt{2}(\tau_0^2 - \tau_1^2) & \frac{1}{2}(\tau_0^2 - \tau_1^2) & \cdots & \sqrt{2}(\tau_0^2 - \tau_1^2) \\
\cdots & \cdots & \cdots & \cdots \\
\sqrt{2}(\tau_0^2 - \tau_1^2) & \frac{1}{2}(\tau_0^2 - \tau_1^2) & \cdots & \sqrt{2}(\tau_0^2 - \tau_1^2) \\
0 & 0 & \cdots & 0
\end{pmatrix},$$

$$T_2^{1|23} = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 2\tau_0\tau_1 & 2\sqrt{2\tau_0\tau_1} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix},$$

$$T_3^{1|23} = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}.$$

Thus

$$\|N^{1|23}\|_{tr} = \|\alpha\rangle\langle\beta\|_{tr} = \sqrt{15^2[(\tau_0^2 + \tau_1^2)^2 + (1 - \frac{2}{d})(\tau_0^2 - \tau_1^2)^2] + 8\tau_0^2\tau_1^2 \sqrt{2 - \frac{2}{d}}} \leq \sqrt{2 - \frac{2}{d}}[15^2(2 - \frac{2}{d}) + 2],$$

where

$$|\alpha\rangle = \left[ 15(\tau_0^2 + \tau_1^2) 15\sqrt{\frac{1}{3}}(\tau_0^2 - \tau_1^2) \cdots 15\sqrt{\frac{1}{(d-1)d}}(\tau_0^2 - \tau_1^2) 2\tau_0^2\tau_1^2 \cdots -2\tau_0^2\tau_1^2 \cdots 0 \right]^t,$$

$$|\beta\rangle = \left[ 1 \sqrt{\frac{1}{3}} \cdots \sqrt{\frac{2}{(d-1)d}} 0 \cdots 0 \right]^t,$$

we have used $\|\alpha\rangle\langle\beta\|_{tr} = \|\alpha\|\|\beta\|$ for vectors $|\alpha\rangle$ and $|\beta\rangle$.

In the second case, $|\varphi\rangle = \tau_0|000\rangle + \tau_1|111\rangle + \cdots + \tau_{d-1}|d-1, d-1, d-1\rangle$, the matrices $T_1^{1|23}$, $T_2^{1|23}$ and $T_3^{1|23}$ are given by
Detection of genuine tripartite entanglement based on Bloch representation of density matrices

Thus we have used

\[ \rho = \begin{bmatrix} \frac{1}{2}(\tau_0^2 + \tau_1^2) & \frac{1}{2}(\tau_0^2 - \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \cdots & 0 \\ \\ \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 + \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \cdots & 0 \\ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \\ \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 + \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \cdots & 0 \\ \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \]

where we have used \( \rho = \begin{bmatrix} \frac{1}{2}(\tau_0^2 + \tau_1^2) & \frac{1}{2}(\tau_0^2 - \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \cdots & 0 \\ \\ \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 + \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \cdots & 0 \\ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \\ \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 + \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \frac{2}{\sqrt{(d-1)d}}(\tau_0^2 - \tau_1^2) & \cdots & 0 \\ \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \)

Then

\[ N^{123} = 15(F + |\gamma\rangle\langle\zeta|) + T_2^{123} + T_3^{123}, \]

where \( F \) is a \((d^2 - 1)\)-dimensional square matrix which the first column is same as the first column of \( T_1^{123} \) and the other elements are zero, \( |\gamma\rangle = \begin{bmatrix} \tau_0^2 + \tau_1^2 & \sqrt{\frac{2}{(d-1)d}}(\tau_0^2 - \tau_1^2) \cdots \sqrt{\frac{2}{(d-1)d}}(\tau_0^2 - \tau_1^2) & 0 \end{bmatrix}^t \),

\[ |\zeta\rangle = \begin{bmatrix} 0 \\ \sqrt{\frac{1}{2}} \\ \cdots \\ \sqrt{\frac{2}{(d-1)d}} \end{bmatrix}. \]

Thus

\[
\|N^{123}\|_\text{tr} \leq 15(\|F\|_\text{tr} + \|\gamma\|_\text{tr}) + \|T_2^{123}\|_\text{tr} + \|T_3^{123}\|_\text{tr} \\
\leq 15\sqrt{(1 - 2\tau_1^2)^2 + (1 - \frac{2}{d})^2} + 15\sqrt{(1 - \frac{2}{d})(2 - \frac{2}{d})} + 4\sqrt{2\sqrt{\frac{1}{d}(1 - \tau_1^2)}} \]

where we have used \( \|A + B\|_\text{tr} \leq \|A\|_\text{tr} + \|B\|_\text{tr} \) for matrices \( A \) and \( B \) and the upper bound is obtained by taking the extreme value of the function with independent variable \( \tau_1 \).

(2) If \( \rho = |\phi\rangle\langle\phi| \) is separable under the bipartition \( 2|13 \), i.e., \( |\phi\rangle = |\phi_2\rangle \otimes |\phi_{13}\rangle \in H_2^d \otimes H_{13}^d \). We only need to consider two cases:
where 0 \leq k < j \leq d - 1. Then
\begin{align*}
\|N^{2|13}\|_{tr} &\leq 4\left[ \frac{1}{4} \sum_{k=1}^{d-2} \sqrt{\frac{1}{k(k+1)}} \sqrt{\frac{1}{k+1} - \frac{1}{d} \sum_{i=0}^{k-1} \tau_i^2 - k \tau_k^2} \right] + 4 \sum_{0 \leq i < j \leq d-1} |\tau_i \tau_j| \\
&\quad + \sum_{k=1}^{d-1} \frac{2}{k(k+1)} \left[ \sum_{i=0}^{k-1} \tau_i^2 + k \tau_k^2 \right] \\
&\leq 4\left[ \frac{1}{4} \sum_{k=1}^{d-2} \sqrt{\frac{1}{k(k+1)}} \sqrt{\frac{1}{k+1} - \frac{1}{d} \sum_{i=0}^{k-1} \tau_i^2} \right] + 4 \frac{(d-1) \sum_{i=0}^{d-1} \tau_i^2}{2} + 2(1 - \frac{1}{d}) \sum_{i=0}^{d-1} \tau_i^2 \\
&= 4\left[ \frac{1}{4} \sum_{k=1}^{d-2} \sqrt{\frac{1}{k(k+1)}} \sqrt{\frac{1}{k+1} - \frac{1}{d}} \right] + 2(d^2 - 1),
\end{align*}
(20)
where we have used \(\|A + B\|_{tr} \leq \|A\|_{tr} + \|B\|_{tr}\).

(ii) \(|\varphi\rangle = \tau_0|000\rangle + \tau_1|111\rangle + \cdots + \tau_{d-1}|d-1, d-1, d-1\rangle\).
We have \(T^{2|13} = T_1^{1|23} = F + |\gamma\rangle\langle \zeta|\), then
\begin{align*}
\|N^{2|13}\|_{tr} &= 4\sqrt{\frac{1}{2}} \left[ \left( \frac{\tau_0^2}{d} - \frac{\tau_1^2}{d} \right)^2 + \left( \frac{1 - \frac{2}{d}}{d} \right)^2 \right] \\
&\leq 4\sqrt{\frac{2}{d}} \left[ \left( 1 + \sqrt{\frac{1}{d}} \right) \right].
\end{align*}
(21)

(3) If \(\rho = |\varphi\rangle\langle \varphi|\) is separable under the bipartition 3|12, i.e., \(|\varphi\rangle = |\varphi_3\rangle \otimes |\varphi_{12}\rangle \in H^d_3 \otimes H^d_{12}\).
In the first case, we have \(T_1^{3|12} = (T_1^{1|23})^t, T_2^{3|12} = (T_2^{1|23})^t, T_3^{3|12} = (T_3^{1|23})^t\), therefore
\(\|N^{3|12}\|_{tr} = \|(N^{1|23})^t\|_{tr} \leq \sqrt{\left( 2 - \frac{2}{d} \right)[15^2(2 - \frac{2}{d}) + 2]}\).
In the second case, we have \(T_1^{3|12} = T_1^{1|23}, T_2^{3|12} = T_2^{1|23}, T_3^{3|12} = T_3^{1|23}\), therefore
\(\|N^{3|12}\|_{tr} = \|N^{1|23}\|_{tr} \leq \sqrt{233(2 - \frac{2}{d}) + 15(1 - \frac{2}{d})(2 - \frac{2}{d})}\).
Remark 3. After the Schmidt decomposition of a general state, we have $|\varphi_{23}\rangle = |\varphi_2\rangle \otimes |\varphi_3\rangle = \tau_0|a_0\rangle + \cdot \cdot \cdot + \tau_{d-1}|a_{d-1}\rangle$, under LU equivalence we choose $\{a_0, \cdot \cdot \cdot , a_{d-1}\} = \{|00\rangle, \cdot \cdot \cdot , |0, d-1\rangle\}$ in the case (i) to transform $\{a_0, \cdot \cdot \cdot , a_{d-1}\}$ into the orthonormal basis elements which constitute separable states, then $|\varphi_{23}\rangle$ must be a separable state such that $|\varphi_{23}\rangle = |\varphi_2\rangle \otimes |\varphi_3\rangle$, which implies that $|\varphi\rangle$ is fully separable, then we have $\|N^{123}\|_{tr} \leq \sqrt{(2 - \frac{2}{d})[15^2(2 - \frac{2}{d}) + 2]}$. Consequently, if $\|N^{123}\|_{tr} > \sqrt{(2 - \frac{2}{d})[15^2(2 - \frac{2}{d}) + 2]}$, $\rho$ is not fully separable. Therefore, Lemma 2 can also be used to detect not fully separable.

Theorem 2 For a mixed state $\rho \in H_1^d \otimes H_2^d \otimes H_3^d$, if it holds that

$$T(\rho) = \frac{1}{3}(\|N^{123}\|_{tr} + \|N^{213}\|_{tr} + \|N^{312}\|_{tr}) > M,$$  \hspace{1cm}  \text{(22)}$$

then $\rho$ is genuine multipartite entangled, where $M = \max\{\sqrt{2 - \frac{2}{d}}[15^2(2 - \frac{2}{d}) + 2]}, \sqrt{233(2 - \frac{2}{d})} + 15\sqrt{(1 - \frac{2}{d})(2 - \frac{2}{d})}, 4[4 \sum_{k=1}^{d-2} \frac{1}{k(k+1)} \sqrt{\frac{1}{k+1} - \frac{1}{d}} + \frac{2(d^2-1)}{d}]\}.$

Proof If a mixed state $\rho = \sum p_i \rho_i$, $\sum p_i = 1$, is biseparable, by using Lemma 2, we have

$$T(\rho) = \frac{1}{3}(\|N^{123}(\rho)\|_{tr} + \|N^{213}(\rho)\|_{tr} + \|N^{312}(\rho)\|_{tr})$$

$$\leq \frac{1}{3} \sum p_i(\|N^{123}(\rho_i)\|_{tr} + \|N^{213}(\rho_i)\|_{tr} + \|N^{312}(\rho_i)\|_{tr})$$

$$\leq \frac{1}{3}(M + M + M)$$

$$= M.$$

Consequently, if $T(\rho) > M$, $\rho$ is GME. $\square$

Remark 4. Using a different method and the matrix norm, the authors [13] considered the Ky Fan norm of the matricizations of tensors to derive GME conditions for tripartite qubits and four partite qubits. While in our current approach we have used the correlation tensors in the Bloch representation of density matrices coupled with a linear combination of the special matrices, our main structural matrices in the algorithm differ from other references both in structure and in form. We have not only employed the trace norm but also applied local unitary equivalence and the Schmidt decompositions to study the GME of arbitrary dimensional quantum systems. It turns out that our special matrices of correlation tensors constructed in this way greatly simplify the criteria operationally, as shown in Example 1 that our theorem is more effective than Theorem 2 in [13] for $d = 2$.

Example 2 Consider the mixed state in three-qutrit quantum system,

$$\rho = \frac{1-x}{27} I_{27} + x |W(3)\rangle \langle W(3)|, \hspace{0.5cm} 0 \leq x \leq 1,$$  \hspace{1cm}  \text{(23)}$$

where $|W(3)\rangle = \frac{1}{\sqrt{6}}(|001\rangle + |010\rangle + |100\rangle + |112\rangle + |121\rangle + |211\rangle)$ is the $3 \times 3 \times 3$ $W$ state, $I_{27}$ is $27 \times 27$ identity matrix. For $d = 3$, from Theorem 1 in [13] and our Theorem 2, we have $f_4(x) = 2.372684x - 2.177324$ and $f_5(x) = T(\rho) - M = 34.5797x - 27.6257$, respectively. When $f_4(x) > 0$, Theorem 1 in [13] detects the GME for $0.917663 < x \leq 1$, while our Theorem 2 detects the GME for $0.798899 < x \leq 1$, see Fig. 2, which shows that our result (Theorem 2) is able to detect more genuine tripartite entangled states.
4 Conclusions

We have studied the entanglement and genuine multipartite entanglement in tripartite quantum systems. The general state has been transformed into a simpler form by the Schmidt decomposition and local unitary transformation, then we have constructed some new operational matrices and considered the trace norm of the sum of these matrices, from which we have obtained new criteria in detecting tripartite entanglement for \((2 \times 2 \times 2)\) and general \((d \times d \times d)\) systems. Detailed examples show that our criteria are able to detect genuine tripartite entanglement more effective than some existing criteria. Our approach can be also applied to general multipartite systems.

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References

1. Ekert, A.K.: Quantum cryptography based on Bell’s theorem. Phys. Rev. Lett. 67, 661 (1991)
2. Bennett, C.H., Brassard, G., Jozsa, R.: Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett. 70, 1895 (1993)
3. Bennett, C.H., Wiesner, S. J.: Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. Phys. Rev. Lett. 69, 2881 (1992)
4. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865 (2009)
5. Ma, Z.H., Chen, Z.H., Chen, J.L., Spengler, C.: Measure of genuine multipartite entanglement with computable lower bounds. Phys. Rev. A 83, 062325 (2011)
6. Chen, Z.H., Ma, Z.H., Chen, J.L., Severini, S.: Improved lower bounds on genuine-multipartite-entanglement concurrence. Phys. Rev. A 85, 062320 (2012)
7. Hong, Y., Gao, T., Yan, F.: Measure of multipartite entanglement with computable lower bounds. Phys. Rev. A 86, 062323 (2012)
8. Bancal, J.D., Gisin, N., Liang, Y.C., Pironio, S.: Device-independent witnesses of genuine multipartite entanglement. Phys. Rev. Lett. 106, 250404 (2011)
9. Wu, J.Y., Kampermann, H., Bruß, D., Klöckl, C.: Determining lower bounds on a measure of multipartite entanglement from few local observables. Phys. Rev. A 86, 022319 (2012)
10. Chen, K., Wu, L.A.: A matrix realignment method for recognizing entanglement. Quantum Inf. Comput. 3, 193 (2003)
11. Li, M., Wang, J., Shen, S.Q., Chen, Z.H., Fei, S.M.: Detection and measure of genuine tripartite entanglement with partial transposition and realignment of density matrices. Sci. Rep. 7, 17274 (2018)
12. Huber, M., Sengupta, R.: Witnessing genuine multipartite entanglement with positive maps. Phys. Rev. Lett. 113, 100501 (2014)
13. de Vicente, J.I., Huber, M.: Multipartite entanglement detection from correlation tensors. Phys. Rev. A 84, 062306 (2011)
14. Markiewicz, M., Laskowski, W., Paterek, T.: Detecting genuine multipartite entanglement of pure states with bipartite correlations. Phys. Rev. A 87, 034301 (2013)
15. Li, M., Jia, L.X., Wang, J., Shen, S.Q., Fei, S.M.: Measure and detection of genuine multipartite entanglement for tripartite systems. Phys. Rev. A 96, 052314 (2017)
16. Zhao, J.Y., Zhao, H., Jing, N.H., Fei, S.M.: Detection of genuine multipartite entanglement in multipartite systems. Int. J. of Theor. Phys. 58, 3181 (2019)
17. Nielsen, M. A., Chuang, I. L.: Quantum Computation and Quantum Information. 109. Cambridge Univ. Press, Cambridge, (2000)
18. Jing, N., Yang, M., Zhao, H.: Local unitary equivalence of quantum states and simultaneous orthogonal equivalence. J. Math. Phys. 57, 062205 (2016)
19. Cui, M.Y., Chang, J.M., Zhao, M.J., Huang, X.F.; Zhang, T.G.: Local unitary invariants of quantum states. Int. J. Theor. Phys. 56, 3779 (2016)
20. de Vicente, J.I.: Separability criteria based on the Bloch representation of density matrices. Quantum Inf. Comput. 7, 624 (2007)
21. Weinstein, Y. S.: Tripartite entanglement witnesses and entanglement sudden death. Phys. Rev. A 79, 012318 (2009)