Gravitational Lensing and the Variability of $G$

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Abstract

The four observables associated with gravitational lensing of distant quasars by intervening galaxies: image splittings, relative amplifications, time delays, and optical depths, provide separate measures of the strength of the gravitational constant $G$ at cosmological distances. These allow one, in principle, to factor out unknown lensing parameters to directly to probe the variation of $G$ over cosmological time. We estimate constraints on $\dot{G}$ which may be derivable by this method both now and in the future. The limits one may obtain can compete or exceed other direct limits on $\dot{G}$ today, but unfortunately extracting this information, is not independent of the effort to fix other cosmological parameters such as $H_0$ and $\Omega_0$ from lensing observations.

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1. Introduction

The gravitational constant, \( G \), is the poorest measured fundamental constant in nature. In fact, it may not even be a constant at all. The exceedingly small value of \( G \), coupled with the large value of the age of the universe encouraged speculation early on, first following Dirac and then spurred by the advent of Brans-Dicke cosmology, that the two quantities may be somehow tied together \cite{Dirac1937, Dyson}. Moreover, because classical general relativity cannot be quantized, there has been a recurring interest in the possibility that GR arises as the low energy limit of a more fundamental theory. In such a theory, the gravitational constant may arise dynamically, associated with the vacuum expectation value of some field (or dynamics of some internal space). Since this dynamical value may be time dependent, so may \( G \). Over the past year, largely as a result of considerations based on extensions of the original old inflationary models \cite{Guth1981, Linde1982, Albrecht1982, Steinhardt1990}, there has been a renewed interest \cite{La1989} in the possibility that the gravitational constant has varied on cosmological timescales.

There exist several sensitive direct probes of a monotonic change in the gravitational constant during the present epoch, including the use of pulsar timing measurements and radar experiments, all of which suggest that \( \dot{G}/GH \leq 0.4 \) today. \cite{Shapiro1964, Shapiro1971, Helling1987, Reasen1983, Damour1988} At the opposite extreme, calculations of primordial nucleosynthesis put indirect limits on \( \dot{G} \) during the first seconds of the big bang expansion from limits on the observed Helium abundance \cite{Accetta1990}. If the variation of \( G \) has followed a constant power law in time, the latter limit (\( \dot{G}/GH \leq 0.01 \)) is stronger than the
direct limits on the variation today. What has been lacking however is any way to
directly probe the value of $G$ at times between these two epochs. Since it has even
been proposed that $G$ may oscillate in time [Accetta and Steinhardt 1991], a direct
measure of $G$ at intermediate times would be of great interest. It is the purpose
of this paper to suggest that observations of gravitational lensing could, in princi-
ple, provide such a measure, and to investigate the realistic limits which it may be
possible to obtain.

On first thought it is not clear that lensing can constrain $G$. While the
bend angle which light rays are subject to is directly related to the strength of the
gravitational constant at the time light rays pass the lensing object, the quantity
which enters into all formulas is the product $GM$, where $M$ is the mass of the
lensing object. Unless $M$ can be determined independently a separate extraction
of $G$ seems impossible. However, it is not the actual bend angle which is directly
observed in gravitational lensing. All lensing observables depend also (in somewhat
different ways) upon the distance of the lensing galaxy and the quasar as inferred
from their redshifts. The distance redshift relation depends upon the time-averaged
value of $G$, which for redshifts of $O(1)$ can be a significant fraction of the lifetime
of the universe. Thus for any lensing system a prediction of one observable based
on a measurement of another can give a signal of the time variability of $G$. What
remains to be seen however is exactly how sensitive such a comparison is, and how
much it depends on our knowledge, or lack thereof, of cosmological parameters such
as the Hubble constant $H_0$, the density parameter $\Omega_0$, and even the cosmological
constant $\Lambda$.

The organization of the paper is as follows: in section 2 we outline our nota-
ations and conventions and introduce the models we will use. In section 3 we consider constraints from lensing statistics and in section 4 we discuss constraints which can be derived from individual lensing systems. Section 5 contains our conclusions.

2. Cosmology and Lens Models

The observables of interest in gravitational lensing depend upon the combination $GM$ (where $M$ is the mass of the lensing galaxy) and the distance to the galaxy and source. If it is assumed that the bending occurs predominantly as the light rays pass through the local region of the lensing galaxy and thus the time required is much shorter than the time scale over which $G$ varies significantly, the effect of the variation of $G$ will be to replace $GM$ by $G_l M$ (where $G_l$ is $G$ at the time of lensing) and also to alter the distance-red shift relation. Measuring the first effect is cleanest, in principle, because it is not dependent upon cosmological modelling. Unfortunately, unless there is an independent way to determine the mass of the lensing galaxy the first effect alone is unmeasurable. Since both velocity dispersion, and stellar luminosity will also depend upon $G$, there are no observables which seem to allow $M$ to be independently extracted.

Hence, to proceed, we must consider some specific cosmological model, incorporating a variable $G$. We will consider for definiteness a Brans-Dicke (BD) theory (this is perhaps the simplest viable extension of GR with a varying gravitational constant and is often used in connection with extended inflationary models). While our discussion will be in terms of BD cosmology, the general features should be characteristic of any model with varying $G$. In particular, these ideas could be applied to any theory based on the Friedman-Robertson-Walker metric (which has gained more ex-
perimental support recently from the isotropy of the cosmic microwave background) with an evolution equation for the scale factor determined by the equation of state of matter which also incorporates a varying value of $G$ consistently in the equations of motion (e.g. Dyson, Beckenstein 1977, Beckenstein and Meisels 1980).

In the BD cosmology the line element is the usual FRW metric

$$ds^2 = -dt^2 + R^2(t)[d\chi^2 + s_k(\chi)^2d\Omega]$$

where $s_k = \sinh(\chi), \chi, \sin(\chi)$ for $k = -1, 0, 1$. Einstein’s equations are modified and a new dynamical field $\phi$, with $G \sim \phi^{-1}$, is introduced. For large time the general solution of the Brans-Dicke field equations will be matter dominated and in many cases of interest (e.g. Weinberg 1972, La and Steinhardt 1989, Steinhardt and Accetta 1990) $\phi \sim R^\sigma$, where $\sigma$ will be a function of the Brans-Dicke scalar-tensor coupling constant, $\omega$, which tends to zero in the limit $\omega \to \infty$ (where Einstein’s theory is recovered).

The independent limits on the scalar-tensor coupling constant in the simplest Brans-Dicke theory are already far more stringent than we will place from the variation of $G$ (Reasonberg 1979), but our purpose here is to use this model merely as a testing ground to explore the sensitivity of lensing parameters to $G$.

In the ($k = 0$) examples cited above $\sigma = (1 + \omega)^{-1}$. The evolution equation for the scale factor in a matter dominated epoch is

$$\left(\frac{\dot{R}}{R}\right)^2 + k\frac{\dot{R}}{R^2} = \frac{8\pi G_0}{3} \rho_0 \left(\frac{2\omega + 3}{2\omega + 4}\right) \left(\frac{R_0}{R}\right)^{3+\sigma} + \left(\frac{\sigma^2 \omega}{6} - \sigma\right) \left(\frac{\dot{R}}{R}\right)^2$$

If we define $\eta = R/R_0$ and

$$\Omega_0 = \left(\frac{8\pi G_0 \rho_0}{3H_0^2}\right) \left(\frac{2\omega + 3}{2\omega + 4}\right) \left(1 + \sigma - \frac{\sigma^2 \omega}{6}\right)^{-1}$$
we can rewrite this as
\[ \dot{\eta}^2 + (\Omega_0 - 1)H_0^2 = \Omega_0 H_0^2 \eta^{-1(1+\sigma)} \] (4)
\[ \frac{k}{R_0^2} = (\Omega_0 - 1)H_0^2 \left( 1 + \sigma - \frac{\omega \sigma^2}{6} \right) \] (5)

Note that if one instead were simply to allow \( G \) to vary as a power law and were to use Einstein’s equations unchanged one would obtain the same result (4,5), except without the last factor on the r.h.s. of (5). For a fixed value of \( \Omega_0 \) and \( H_0 \), it is this factor which causes \( R_0 \) to vary with \( \omega \).

The measure of distance we will use is the angular diameter distance
\[ d_A = \frac{R_0}{1 + z} s_k(\chi) \] (6)
which assumes that the lensed rays traverse a mean filled “beam”. (Similar, but algebraically more complex constraints can be obtained in the case of an “empty” beam approximation, where affine angles and distances are used (i.e. see [Turner et al. 1983, Krauss and White 1991]). The mean filled beam approximation is probably closer to the actual situation, however, it has been shown that the uncertainty due to clumpiness of matter in the beam trajectory can be one of the main sources of uncertainty in the analysis of individual lensing systems [Alcock and Anderson 1985]. If the Hubble constant were independently measured it is possible that this uncertainty could be reduced since it also enters into the determinations of \( H_0 \) from time delays in lensing systems.

Once the evolution equation for the scale factor, \( R(t) \), is specified we can solve for the distance redshift relation in the usual way [Weinberg 1972]. For the special case \( k = 0 \) we obtain a simple expression for the distance as a function of
\[ x = 1 + z, \text{ (similar expressions for } \Omega_0 \neq 1 \text{ can also be obtained.}) \]

\[ d_A = \frac{1}{\beta H_0 x^\beta} (1 - x^{-\beta}) \quad (7) \]

where \( \beta = (1 + \sigma)/2. \) The distance is plotted in figure 1 for \( \beta = 0.4, 0.5, 0.6, \) and reduces to the usual expression [Turner et al. 1984] in the limit of constant \( G \) \( (\omega \to \infty, \beta \to 1/2). \) As a guide to the expected magnitude of \( \beta \) one would like to obtain sensitivity to, notice that an assumed variation-since-lensing of

\[ \frac{\Delta G}{G} = \frac{G_l - G_0}{G_0} = 20\% \Rightarrow \sigma = \frac{\log 1.2}{\log x_l} \sim 0.2 \quad (8) \]

for lenses \( z_l \sim 1.5. \) This corresponds to \( \beta \sim 0.6. \) If we take the age of the universe to be \( t_0 = 2/(3 + \sigma) H_0^{-1} = 10^{10} \text{yr}, \) this then gives

\[ \frac{\dot{G}}{G}|_0 = -\sigma H_0 \sim 10^{-11} \text{yr}^{-1} \quad (9) \]

which is comparable with other direct measures of \( \dot{G}/G. \)

We will consider two simplified lens models in what follows: the point mass and the isothermal sphere lenses. The point mass lens is chosen for its simplicity, the isothermal lens because the flatness of rotation curves of galaxies suggest \( \rho \sim r^{-2} \) is a reasonable approximation to galactic mass distributions (at least asymptotically). While for any actual lens system these models are overly simplistic they serve to illustrate the main points. The observables for these lens systems which we would want to examine for sensitivity to \( \beta \) are: time delays between images, angles between images and ratio in brightness of the images, as well as the (differential and total) optical depth for lensing.
3. Lensing statistics

The formalism appropriate to gravitational lensing statistics was first developed in [Turner et al. 1984] and latter generalized to arbitrary Robertson-Walker cosmologies in [Gott et al. 1989] (see also [Krauss and White 1991] for a recent presentation). The key quantity is the optical depth, or integrated probability of lensing, $\tau$, assuming a non-evolving population of galaxies, modelled as singular isothermal spheres. This depth is relatively free of matter clustering uncertainties [Alcock and Anderson 1985] but is sensitive to variations in the distance-redshift relation, which makes it a good probe of cosmology.

As an example consider the expression for $\tau$, for a $k = 0$ universe. Including the $\beta$ dependence of the bend angle ($\alpha \sim G$) we obtain

$$\tau(y = 1 + z) = \frac{F}{\beta^2} \int_1^y dx \left( x^{-3-\beta}(x^\beta - 1)^2 \right)^2$$

where $H_0R_0 = 1$ and $n_0$ is the comoving number density of galactic lenses, which we assume in this instance are all identical (non-evolving) isothermal spheres producing identical bend angles $\alpha_0$. The bend angle $\alpha_0$ is related to measured velocity dispersions of nearby galaxies today. For further details of these definitions see [Turner et al. 1984]. Equation (10) reduces to eqn (2.26c) of [Turner et al. 1984] in the limit $\beta \to 1/2$. The integral can be done analytically but the result is cumbersome and is not shown here. The optical depth vs redshift is shown in figure 2 for $\beta = 0.4, 0.5, 0.6$. As can be seen the variation with $\beta$ is slight making this a poor measure of $\dot{G}/G$. A similarly small dependence on $\beta$ is shown by the differential optical depth $d\tau/dz$. Since the distance-redshift relation becomes less $\beta$ dependent as $\Omega_0$ decreases we expect the variation in $\tau$ to be less than above when $\Omega_0 < 1$, al-
though this is somewhat offset by the $\beta$ dependence of $F$ coming through $R_0$. Thus variations of $G$ going as a power law in time have little effect on lensing statistics, at least at the level where these statistics are likely to be determined in the foreseeable future.

4. Individual systems

A better hope of constraining $\dot{G}$ comes from examining the observables associated with multiply imaged quasars (i.e. see Hewitt et al. 1988). Specifically we will be interested in the observables: time delay, image splitting and image magnification for our two model lenses. The strategy will be the following: each observable will depend both on $GM$, and on $d_A(G)$. If we have more than two observables for each system, then we hope to overly constrain the system so that we can check for consistency between the different determinations of these quantities from each observable.

1) Point mass lens.

For these lenses the time delay $\Delta t$ and ratio in magnitude of images $r$ are related through [Krauss and Small 1991]

\[ \Delta t = 2GM(1 + z)[(r - \frac{1}{\sqrt{r}}) + \log(r)] \] (11)

(the $(1 + z)$ factor is absent in microlensing [Krauss and Small 1991]) so these two parameters can be used to infer $GM$ at the time of lensing, independently of the cosmological distance-redshift relation. If it is not possible to measure $\Delta t$ in the lens system, or if the measure has a large uncertainty, $GM$ must be obtained some other way, e.g. from virial velocity measurements. Any limit on $\beta$ will depend on how
well this quantity is known.

Given $GM$ we can use the observed angular splitting of images and the relation

$$\Delta \theta = \frac{4GM}{c^2S} \frac{r - 1}{\sqrt{r^{1/2} - 1}}$$

(12)

to determine $S = D_S/D_{LS}$ where $D_S$ and $D_{LS}$ are the angular diameter distances from the observer to the source and from the lens to the source respectively. This is a function only of the (known) redshifts, $\Omega_0$ and $\beta$, e.g. in the $k = 0$ case

$$S \equiv \frac{s_0(\chi_S)}{s_0(\chi_S - \chi_L)} = \frac{1 - x_S^{-\beta}}{x_L^{-\beta} - x_S^{-\beta}}$$

(13)

so a knowledge of the redshifts allows a determination of $\beta$ (up to clumping uncertainties [Alcock and Anderson 1985]) if we assume a value for $\Omega_0$ (or conversely a determination of $\Omega_0$ if we know $\beta$). Notice that $S$ is a ratio of distances and so is independent of $H_0$. As an example if we take $z_L = 1, z_S = 3$ then for $k = 0$, mean filled beam, $S$ is a monotonically increasing function of $\beta$ varying from 2.3 to 2.5 as $\beta$ runs from 0.4 to 0.6 as can be seen in figure 3. Given the above and the fact that typical image splittings can be $\sim 3'' - 7''$ it is not impossible that a good measurement of the angular splitting could limit $\beta$ to be in the range competitive with other direct probes of $\dot{G}$.

2) Isothermal Sphere

For the somewhat more realistic, isothermal sphere model the situation is simpler (in principle). If the velocity dispersion of the lensing galaxy, $\sigma_{||}$, is known, say from measurements of the rotation curves, a measure of the angular splitting
allows us to immediately infer $S$:

$$\Delta \theta = \frac{2\alpha}{c^2 S} = \frac{8\pi \sigma_\parallel^2}{c^2 S} \Rightarrow S = \frac{8\pi}{\Delta \theta} \left( \frac{\sigma_\parallel}{c} \right)^2$$

(14)

In fact a simultaneous measurement of the time delay, which gives us $D = D_L/S$ (where $D_L$ is the angular diameter distance from the observer to the lens),

$$\Delta t = 32\pi^2 (1 + z) \left( \frac{\sigma_\parallel}{c} \right)^4 \frac{D}{c}$$

(15)

and the image splitting would in principle allow us to measure both $\beta$ and $\Omega_0$ (up to uncertainties in $H_0$, because $D_L$ has the dimensions of distance and hence is dependent upon $H_0$) because the dependence of $D$ and $S$ on $\beta$ and $\Omega_0$ is different\textsuperscript{4}. We expect the strongest constraint on $\beta$ for fixed $\Omega_0$ to come from $S$ however.

\textsuperscript{4} See figure 4.

The singular isothermal sphere (SIS) model is probably still too naive to apply to actual individual lens systems. One should at least include the effects of a finite galactic core \cite{Hinshaw and Krauss 1987, Krauss and White 1991}. Alternatively, a more complicated, but more general model, the ‘elliptical lens’ \cite{Narayan and Grossman 1988}, is available for use in extracting these quantities for individual galactic lenses. Nevertheless, the SIS model should give a general idea of the methodology to be used, and the possible sensitivity to $\beta$. Using these other models in an application of these ideas to actual lenses would merely require replacing the above equations for $S$ and $\Delta t$ with somewhat more complicated equations which would include the lensing parameters fit by the observations.

Because of the simplicity of the SIS model we only had to make due with 2 lensing observables to overconstrain the system. We note however that the ratio of
image amplifications itself is also dependent on $S$ and $\beta$, and so can also be used to probe for consistency when more complicated fits to galactic lenses are required.

5. Conclusions

While we have demonstrated here that gravitational lensing provides in principle a direct sensitivity to variations in $G$ over cosmological time, our results suggest that to be competitive with limits on $\dot{G}$ at the present time, lensing parameters must be extracted from observations at the level of 10% or better—a daunting but not impossible task. Statistical measures such as the optical depth do not seem sufficiently sensitive to $\dot{G}/G$, the effects of reasonable changes in $G$ being swamped by larger uncertainties from our present lack of knowledge of $\Omega_0$ and $H_0$. For individual lens systems a simplified model suggests that it may be possible to see variations of the order $\dot{G}/G \sim 10^{-11}/\text{yr}$ if accurate measures of the angular splitting, amplifications, and perhaps also time delays become available for a system with source and lens at relatively high redshifts ($z_S \sim 2, 3$ and $z_L \sim 1$). Pessimistically, it is worth noting that a possible variation in $G$ is yet one more uncertainty which could limit one’s ability to extract $H_0$ from measurements of time delays in individual systems. On the other hand, if $H_0$ and $\Omega_0$ are measured reliably by independent means, one’s ability to probe for variations in $G$ will improve.

Nevertheless, in spite of the limitations of this method, it is worth emphasizing that it does provide perhaps the only ‘direct’ probe of variations in $G$ during intermediate times between the present epoch, and the nucleosynthesis era in the very early universe. We have placed ‘direct’ in quotation marks because as we have demonstrated, one’s ability to extract information on $\dot{G}$ is intertwined with our
knowledge (or ignorance) of the proper cosmological model for the evolution of the universe during this time. In this regard, we also note that for our analysis, we used as an analytic tool to probe the sensitivity of lensing, a simple Brans-Dicke cosmological model, which in fact is already ruled out by other constraints for the parameter range which would produce the level of time variation probed here. In this case, $G$ would vary as a simple power law with time. We expect our results would be applicable for any similar, more viable, model. Of course, there are other possibilities, including an oscillatory behaviour of $G$ with a cosmologically interesting period (i.e. [Accetta and Steinhardt 1991]). One would need specific models to perform an analysis similar to that performed here, but it may be that for such scenarios, gravitational lensing could provide sensitive limits, by comparing results obtained from lensing systems at different redshifts.
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