1. Introduction

The term Free Boundary Problem (FBP) refers, in the modern applied mathematical literature, to a problem in which one or several variables must be determined in different domains of the space, or space-time, for which each variable is governed in its domain by a set of state laws. If the domains were known, the problem reduces to solving the equations, usually ordinary differential equations (ODEs) or partial differential equations (PDEs). Now, the novelty of FBPs lies in the fact that the domains are a priori unknown and have to be determined as a part of the problem, thanks to a number of free boundary conditions that are derived from certain physical laws or other constraints governing the phase transition.

The interplay of the state laws for the single phases and the special phase transition conditions leads to a mathematical discipline that combines analysis and geometry in sophisticated ways, together with mathematical modeling based on physics and engineering. Let us point out a basic division of the set of free boundary problems: In the processes that evolve in time, one of the main difficulties of the theory is to track the movement of the free boundary or boundaries; these problems are also called moving free boundary problems. The processes where time does not appear have the common name, FBP. Another basic distinction is between several-phase problems and the simpler one-phase problems, where the free boundary is the bounding hypersurface of the phase that is governed by the equations. Normally, a one-phase problem can be considered as a two-phase problem with a trivial second phase, usually the vacuum.

It is probably very hard, if not impossible, to track the origins of free boundary problems. The origin of the modern discipline owes much the famous Stefan problem that describes the joint evolution of a liquid and a solid phase, a question considered in 1831 by G. Lamé and B. Clapeyron in relation to the problems of ice formation in the polar seas. The problem is named after J. Stefan [36], who introduced the general class of such problems around 1890, and performed both numerical and theoretical studies of the phase-transition problems related to heat transfer. The rigorous analytical treatment exceeded the tools of the time, and the difficulty may be understood from the fact that the basic existence and uniqueness result was only proved by S. Kamenomostskaya [28] and O. Oleinik [33], once the notion of weak solutions and nonlinear analysis were ready.

In the twentieth century, new directions were developed, and problems concerning free surfaces flows, shock waves, and water waves became of central importance. During early 50’s until early 70’s, several mathematicians (such as P. Lax, H. Lewy, J. L. Lions, G. Stampacchia, among others) contributed to the growth of the field, and eventually the weak and variational formulation and techniques for these problems were fully developed. The obstacle problem became a basic study case in the mathematical theory. There are many names here unmentioned, but not forgotten. The book by G. Duvaut and J. L. Lions was probably one of the first to make an extensive list of problems available to mathematical community. At that stage of the
development of the topic, the mathematical treatment was anchored in the new tools of non-linear functional analysis and PDEs, and several deep contributions were made to obtain the regularity of solutions and to develop the theory towards other directions.

The 70’s and 80’s saw a continuous progress in the treatment of an increasing number of models, some of them became classical in the analysis of FBPs: the dam problem, the plasma problem, the porous medium equation, the $p$-Laplacian equation, the Hele-Shaw problem, the Muskat problem, among many others. At the same time, the difficult task of understanding the finer points of the geometry of the free boundaries become central. The work of L. Caffarelli allowed to make a remarkable progress in the questions of regularity of FBPs after his seminal paper \cite{Caf80}. It is also the time when a mathematical FBP community was formed, a regular series of international FBP meetings was scheduled, and all this served to maintain the impulse and activate the innovation. With time, different books (\cite{GilbargTrudinger, CFL, Spruck, CaffarelliCvitanić, CaffarelliVazquez} to mention a few) were written to consolidate the progresses that had been done in different areas. See also the survey paper \cite{CaffarelliSurvey} reporting on the situation around year 2000.

Recent decades have witnessed a rapid widening of the subject area by incorporation of important free boundary topics coming from different areas: In Finance, FBPs appear to determine the optimal exercise value in Black-Scholes models; in Mathematical Biology, they indicate the moving fronts of populations or tumors; many new and emerging FBPs have arisen from Fluid Mechanics; free boundaries appear in aggregation/swarming processes; in Geometry, much attention is now given to geometrical flows, including the curvature flows where different types of curvatures involved.

2. **Current and Future Developments**

Today, the study of FBPs is intensely pursued from various aspects (experimental, numerical, and theoretical), the subject is continuously finding new grounds for applications, and new fundamental theoretical questions continue to emerge. These developments, in particular, ask for new analytical and numerical methods, and improvements of existing algorithms and tools to handle extremely complex problems.

This theme issue contains several articles that represent the mainstream in today’s and future directions of FBPs (even though not exhaustive).

2.1. **Theoretical developments.** From the current status of theoretical directions, we have selected a number of topics that will be covered in this issue.

**Nonlocal Phenomena:** Many recent developments involve phenomena for which the diffusion process is of a nonlocal nature. Thus, in continuum mechanics and fluid dynamics, owing to the presence of many scales such as in polymers, there exists a global interaction through force fields, or induced by spatial phenomena in cases of surface diffusivity. Indeed, if heat flow is blocked by an insulated wall, heat will propagate along the wall by the influence of the internal heat, through the Dirichlet to Neumann nonlocal diffusion kernel. This is the case of the quasi-geostrophic equation or models of planar crack propagation in viscoelastic solids. Nonlocal diffusion behaviour also takes place in probability, when the random kicks affecting the particles cease to be infinitesimal and become jump processes (Lévy processes). Many phenomena involving phase transitions and free boundaries re-appear in this context. Nonlocal behaviour has many aspects and scales (thin membrane, thick membrane, different scales for interior and boundary diffusivity, etc.) and presents many new mathematical difficulties. It plays a crucial role in applications such as elasticity (the Signorini problem), optimal insulation, and mathematical finance (American option with Lévy process).

It is also noteworthy that, in the last decade, there has been a very definite progress in the mathematical understanding of nonlocal problems of stationary or evolution type, with or
without free boundaries. Some important papers in the latter direction are [7, 8, 9, 38]. More references will be mentioned below.

**Two- and Multi-Phase Problems:** Many phenomena modeled by PDEs give rise to several phases of behaviour, in which two- or multi-phases of free boundaries are natural. They may arise as limiting cases of solutions to systems of equations. Several phases in flow problems also appear in applied problems such as in oil related industry (where several phases of materials meet or mix), e.g., oil, gas, air, saltwater, sand, etc. The well-known Muskat problem is an example of such phenomena, see [11]. Another problem is the so-called segregation in reaction-diffusion problems which models a competition between several species [30, 19]. The theory of these problems are still at an embryonic stage of development; see [16] for latest developments concerning segregation problems. We refer also to [34] and the references therein, for other types of two-phase problems related to obstacle problem.

**Combinatorial Aspects and Random Walks:** Many applied problems result in particle interactions in various forms. Such problems, driven by exterior or interior forces, chemical reactions, or population growth, have been studied to some extent within applied sciences. When particles move (e.g. cell growth, crystallisation, discrete flows) randomly and occupy regions, they interact with other particles in their environment. Such interaction can be of various forms: They can be annihilated, or annihilate other particles; they can also freeze or vaporise and behave in many other ways. Such movements usually are described by random walks, and the density function representing the population, or the amount of chemical substances, are given by discrete harmonic functions.

Models for one-phase version of these discrete problems have been developed by several mathematicians. We refer to one of the latest results, [27], and the references cited therein. The problem has some history in combinatorial problems such as chip firing, random algebraic sums (Smash-sum), and colouring problems. The two- and multi-phase versions of such problems have yet not been developed, even though there are huge amount of corresponding models.

**Shock Waves:** The study of solutions with shocks in various models of compressible fluid dynamics leads to well-known free boundary problems. One class of such problems involves steady and self-similar solutions of the multidimensional Euler equations. Shocks correspond to one kind of discontinuities in the fluid velocity, density, and pressure, which are discontinuities mathematically in the solutions or in their derivatives depending on the model equations (cf. [13, 12]). Important physical applications involving shocks include self-similar shock reflection-diffraction and steady shocks in nozzles and around wedges/cones or airfoils. Such shocks are important in the mathematical theory of multidimensional conservation laws since steady/self-similar solutions with shocks are building blocks and asymptotic attractors of general solutions (cf. [12, 13]). In recent years, mathematical progress has been achieved in several long-standing problems, especially shock reflection-diffraction problems, for potential flow in which many different flow regimes are possible. Future challenges concern:

- Complicated patterns of regular shock reflection-diffraction and other configurations for potential flow;
- Extension of the results of shock reflection-diffraction, including regular/Mach configurations, to the full Euler equations;
- Construction of steady solutions in a nozzle (e.g. the de Laval nozzle), or around a non-straight wedge/cone or an airfoil;
- Analysis of free boundaries in solutions to the multidimensional Riemann problem.

Other related discontinuities (as free boundaries) in compressible fluid mechanics include vortex sheets and entropy waves (cf. [13, 14]).
2.2. Numerical developments. In the early days, numerical approaches to FBPs were frequently ad hoc based, say, on trial free boundary methods or coordinate transformations. Systematic approaches based on variational methods, variational inequalities, and weak formulations of degenerate nonlinear parabolic equations were developed as the mathematical theory became available. In recent years, important developments include the following:

- Diffuse interface models (such as the phase field and Cahn-Hilliard models) with applications to curvature flows, solidification and phase transformations in material science;
- Level set methods for evolving fronts including applications to fluid flow and image processing;
- Variational front tracking methods for geometric PDEs; for instance, interfaces involving curvature effects (such as surface tension and bending);
- Extensive mathematical contributions to the stability, well-posedness and rigorous error analysis of discrete approximations to free boundary problems and degenerate nonlinear elliptic and parabolic equations.
- Adaptive methods appropriate for free boundary and interface problems.

The need to simulate ever-more complicated large systems, stemming from the increase in computing power and the development of computational tools, stimulates new questions and problems for analysts. Therefore, there is a natural symbiosis between analysts, modelers, and computational mathematicians. Increased computing power together with the demand of applications have led to the study of systems of PDEs in domains with complex morphology. These complex multi-physics models frequently involve interfaces and free boundaries. Although one may obtain detailed information about sub-problems (say, the obstacle problem) or local-in-time existence results by using analytical techniques, the full complex system is required to be simulated in scientific and engineering applications. Nonlinear degenerate PDEs with interfaces and free boundaries are notoriously difficult to solve numerically. The best numerical methods depend on good analytical approaches, and sometimes they promote new advances in the PDE theory, such as the formulation of mean curvature flows beyond the onset of singularities. Future trends in numerics may include the following:

- Surface finite elements for geometric PDEs, surface processes on interfaces;
- Diffuse interface and phase field methods for two phase flow with surfactants, phase transformations in materials;
- Numerical optimization of free boundaries for control and inverse problems;
- Burgeoning applications in biology and medicine;
- Numerical methods for fully nonlinear equations;
- Numerical methods in homogenization, random media and random surfaces including stochastic equations;
- Computational methods for free boundary problems for shock waves, vortex sheets, entropy waves, and related compressible flows;
- Adaptivity (mesh refinement, coarsening and smoothing) for surfaces including topological change.

2.3. Applications and the present volume. As indicated above, the study of FBPs is an extremely broad field due to the abundance of applications in various sciences and real world problems, including physics, chemistry, engineering, industry, finance, biology, and other areas. This theme issue is also a showcase of applications of FBPs with a wide coverage of different areas from fluid mechanics to biology/medicine to finance.
In [1], A. Alphonse and C. Elliott formulate a Stefan problem on an evolving hypersurface and study the well-posedness of weak solutions given $L^1$ data. As mentioned earlier, the Stefan problem is the prototypical time-dependent free boundary problem. It arises in various forms in many models in the physical and biological sciences. In order to do their analysis, they have to first develop function spaces and results to handle the equations on evolving surfaces in order to give a natural treatment of the problem. The applications to numerical treatment of applied problems are natural and interesting.

D. Apushkinskaya and N. Uraltseva in [2] study the regularity of free boundaries in problems with hysteresis that arise in modeling biological and chemical processes "with memory". Such models lead to two-phase free boundary problems.

In [3], D. Bucur and Velichkov show how the analysis of a general shape optimization problem of spectral type can be reduced to the analysis of particular free boundary problems. The paper presents an overview of recent developments.

L. Caffarelli and H. Shahgholian in [6] give a survey on the approaches to regularity theory for four different free boundary problems, which are obstacle, thin obstacle, minimal surfaces and cavitation problems.

The paper by J. A. Carrillo & J. L. Vázquez [10] is devoted to problems of diffusion or aggregation. Of concern is the interaction between nonlocal diffusion with long-range interactions and nonlinearities of the degenerate type that may slow the movement. The question is then whether there is a free boundary. The answer depends on the models in a delicate way. Once this settled, the regularity and asymptotic behaviour take the scene.

The paper by G.-Q. Chen & M. Feldman [13] provides a survey of recent activities in deriving and analyzing free boundary problems in shock reflection/diffraction and related transonic flow problems. Shock waves are steep fronts that propagate in the compressible fluids which are fundamental in nature, especially in high-speed fluid flows. It is shown that several longstanding shock reflection/diffraction problems can be formulated as free boundary problems, some recent progress in developing mathematical ideas, approaches, and techniques for solving these problems is discussed.

P. Constantin investigates in [15] the dynamics of vortex patches in the Yudovich phase space. He derives an approximation for the evolution of the vorticity in the case of nested vortex patches with distant boundaries, and studies its long time behaviour.

In [17], D. Córdoba, J. Gómez-Serrano, and A. Zlatos deal with a famous problem in the theory of flows in porous media, the Muskat Problem. They show that there exist solutions of the problem that shift stability regimes: They start unstable, then become stable, and finally return to the unstable regime.

A. Figalli and H. Shahgholian in [21] present a survey on unconstrained free boundary problems involving elliptic operators. The main objective is to discuss a unifying approach to the optimal regularity of solutions to these matching problems.

Biological applications have attracted enormous attention lately, because of their importance to society. A. Friedman in [24] reviews several free boundary problems that arise in the mathematical modeling of biological processes. The biological topics are quite diverse: cancer, wound healing, biofilms, granulomas, and atherosclerosis. For each of these topics there is a description of the biological background and the mathematical model.

J. Glimm et al. in [25] review the existence and non-uniqueness for the Euler equations of fluid flow. The non-uniqueness of solutions is a fundamental obstacle to scientific predictions, preventing the success of such goals as predictive science, validation of solutions and quantification of the uncertainties on which engineers base their design conclusions. Some of the non-uniqueness of the Euler equations appears to be mathematical in nature, and it may be
assumed to disappear after some future, deeper level of mathematical results have been obtained. However, some of the nonuniqueness is physical, or at least numerical in origin, due to the underspecification of the Euler equations on physical grounds. A mitigating strategy is proposed in the paper.

M. Hadzic and S. Shkoller in [26] study the stability of steady states in the Stefan problem for general boundary shapes, which is a contribution to the theory of one of the most classical examples of the free boundary field. They prove the global-in-time stability of the steady states of the classical one-phase Stefan problem without surface tension, assuming a sufficient degree of smoothness on the initial domain.

R. Nochetto and R. Otarola in [32] review the finite element approximation of the classical obstacle problem in energy and max-norms and derive error estimates for both the solution and the free boundary. They present an optimal error analysis for the thin obstacle problem. Finally, the authors discuss the localization of the obstacle problem for the fractional Laplacian and prove quasi-optimal convergence rates.

In the same area, B. Perthame and N. Vauchelet in [34] deal with the incompressible limit of a mechanical model of tumor growth with viscosity. In particular, various mathematical models of tumor growth are available in the literature. A first class of models describes the evolution of the cell number density when considered as a continuous visco-elastic material with growth. A second class of models describes the tumor, as a set and rules for the free boundary are given related to the classical Hele-Shaw model of fluid dynamics.

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References
[1] Alphonse, A. & Elliott, Ch. 2015 A Stefan problem on an evolving surface. Phil. Trans. R. Soc. A, 373, 20140279. (doi:10.1098/rsta.2014.0279). [In this issue]
[2] Apushkinskaya, D. & Uraltseva, N. 2015 Free boundaries in problems with hysteresis. Phil. Trans. R. Soc. A, 373, 20140271. (doi:10.1098/rsta.2014.0271). [In this issue]
[3] Bucur, D. & Velichkov B. 2015 A free boundary approach to shape optimization problems. Phil. Trans. R. Soc. A, 373, 20140273. (doi:10.1098/rsta.2014.0273). [In this issue]
[4] Caffarelli, L. A. 1977 The regularity of free boundaries in higher dimensions, Acta Math. 139, 155–184.
[5] Caffarelli, L. & Silvestre, S. 2005 A Geometric Approach to Free Boundary Problems, American Mathematical Society: Providence, RI.
[6] Caffarelli, L. & Shahgholian, H. 2015 Regularity of free boundaries: A heuristic retro. This issue.
[7] Caffarelli, L. & Silvestre, L. 2007 An extension problem related to the fractional Laplacian. Comm. Partial Diff. Eq. 32, 1245–1260.
[8] Caffarelli, L. A. & Vasseur, A. 2010 Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. Ann. of Math. (2), 171, 1903–1930.
[9] Caffarelli, L. & Vázquez, J. L. 2011 Nonlinear porous medium flow with fractional potential pressure. Arch. Ration. Mech. Anal. 202, 537–565.
[10] Carrillo, J. A. & Vázquez, J. L. 2015 Some free boundary problems involving nonlocal diffusion and aggregation. Phil. Trans. R. Soc. A, 373, 20140275. (doi:10.1098/rsta.2014.0275). [In this issue]
[11] Castro, A., Cordoba, D., Fefferman, C., Gancedo, F. & Lopez-Fernandez, M. 2012 Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves. Ann. of Math. (2), 175, 909–948.
[12] Chen, G.-Q. & Feldman, M. 2010 Global solutions of shock reflection by large-angle wedges for potential flow. Ann. of Math. (2), 171, 1067–1182.
[13] Chen, G.-Q. & Feldman, M. 2015 Free boundary problems in shock reflection/diffraction and related transonic flow problems. Phil. Trans. R. Soc. A, 373, 20140276. (doi:10.1098/rsta.2014.0276). [In this issue]
[14] Chen, G.-Q. & Wang, Y.-G. 2012 Characteristic Discontinuities and Free Boundary Problems for Hyperbolic Conservation Laws. In: Nonlinear Partial Differential Equations, The Abel Symposium 2010, Chapter 5, pp. 53–82, H. Holden and K. H. Karlsen (Eds.), Springer.
[15] Constantin, P. 2015 Far field perturbations of vortex patches. Phil. Trans. R. Soc. A, 373, 20140277. (doi:10.1098/rsta.2014.0277). [In this issue] This issue.
[16] Conti, M., Terracini, S. & Verzini, G. 2005 A variational problem for the spatial segregation of reaction-diffusion systems. Indiana Univ. Math. J. 54(3), 779–815.
[17] Cordoba, D., Gomez-Serrano, J. & Zlatos, A. 2015 A note on stability shifting for the Muskat problem. Phil. Trans. R. Soc. A, 373, 20140278. (doi:10.1098/rsta.2014.0278). [In this issue]
[18] Dafermos, C. M. 2010 Hyperbolic Conservation Laws in Continuum Physics. 3rd Ed., Heidelberg, Germany: Springer-Verlag.
[19] Dancer, E., Hilhorst, D., Mimura, M., & Peletier, L. A. 1999 Spatial segregation limit of a competition-diffusion system. European J. Appl. Math. 10, 97–115.
[20] Elliott, C. M. & Ockendon, J. R. 1982 Weak and Variational Methods for Moving Boundary Problems. Research Notes in Mathematics, 59. Pitman (Advanced Publishing Program): Boston, Mass.-London.
[21] Figalli, A. & Shahgholian, H. 2015 An overview of unconstrained free boundary problems. Phil. Trans. R. Soc. A, 373, 20140281. (doi:10.1098/rsta.2014.0281). [In this issue]
[22] Friedman, A. 1982 Variational Principles and Free Boundary Problems. John Wiley & Sons, Inc.: New York, 1982.
[23] Friedman, A., 2000 Free boundary problems in science and technology. Notices of AMS, 47, 854–861.
[24] Friedman, A. 2015 Free boundary problems in biology. Phil. Trans. R. Soc. A, 373, 20140368. (doi:10.1098/rsta.2014.0368). [In this issue]
[25] Glimm, J., Sharp, D. H., Lim, H., Kaufman, K., & Hu, W. 2015 Euler equations: Existence, nonuniqueness and mesh converged statistics. Phil. Trans. R. Soc. A, 373, 20140282. (doi:10.1098/rsta.2014.0282). [In this issue]
[26] Hadzić, M., Shkoller, S. 2015 Global stability of steady states in the Stefan problem for general boundary shapes. Phil. Trans. R. Soc. A, 373, 20140284. (doi:10.1098/rsta.2014.0284). [In this issue]
[27] Jerison, D., Levine, L. & Sheffield, S. 2013 Internal DLA in higher dimensions. Electron. J. Probab. 18, no. 98, 14 pp.
[28] Kamenomostskaya, S. L. 1958 On Stefan problem, Nauchnye Doklady Vysshey Shkoly, Fiziko-Matematicheskie Nauki 1, 60–62. [In Russian]
[29] Kinderlehrer, D. & Stampacchia, G. 1980 An Introduction to Variational Inequalities and Their Applications. Pure and Applied Mathematics, 88. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers]: New York-London.
[30] Matano, H. & Mimura, M. 1983 Pattern formation in competition-diffusion systems in nonconvex domains. Publ. Res. Inst. Math. Sci. 19, 1049–1079
[31] Meirmanov, A. M. 1992 The Stefan Problem. Translated from the Russian by Marek Niezgoda and Anna Crowley. With an appendix by the author and I. G. Götz. de Gruyter Expositions in Mathematics, 3. Walter de Gruyter & Co.: Berlin.
[32] Nochetto, R. & Otárola, E. 2015 Convergence rates for the classical, thin and fractional elliptic obstacle problems. Phil. Trans. R. Soc. A, 373, 20140449. (doi:10.1098/rsta.2014.0449). [In this issue]
[33] Oleinik, O. A. 1960 A method of solution of the general Stefan problem, Doklady Akademii Nauk SSSR (in Russian), 135, 1050–1057. MR 0125341, Zbl 0131.09202.
[34] Otarola, E. 2015 A method of solution of the general Stefan problem, Doklady Akademii Nauk SSSR (in Russian), 135, 1050–1057. MR 0125341, Zbl 0131.09202.
[35] Patrosyan, A., Shahgholian, H. & Ural’tseva, N. 2012 Regularity of Free Boundaries in Obstacle-Type Problems. Graduate Studies in Mathematics, 136. American Mathematical Society: Providence, RI.
[36] Stefan J. 1889 Über die Theorie der Eisschmelze, insbesondere über die Eisschmelze im Polarameere, Sitzungsberichte der Österreichischen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, Abteilung 2, Mathematik, Astronomie, Physik, Meteorologie und Technik, 99, 965–983.
[37] Vázquez, J. L. 2007 The Porous Medium Equation. Mathematical Theory. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press: Oxford.
[38] Vázquez, J. L. 2012 Nonlinear Diffusion with Fractional Laplacian Operators, In: Nonlinear Partial Differential Equations: the Abel Symposium 2010, pp. 271–298, H. Holden and K. H. Karlsen (Eds.), Springer.
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