Dual curvature tensors and dynamics of gravitomagnetic matter

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Gravitomagnetic charge that can also be referred to as the dual mass or magnetic mass is the topological charge in gravity theory. A gravitomagnetic monopole at rest can produce a stationary gravitomagnetic field. Due to the topological nature of gravitomagnetic charge, the metric of spacetime where the gravitomagnetic matter is present will be nonanalytic. In this paper both the dual curvature tensors (which can characterize the dynamics of gravitational charge/monopoles) and the antisymmetric gravitational field equation of gravitomagnetic matter are presented. We consider and discuss the mathematical formulation and physical properties of the dual curvature tensors and scalar, antisymmetric source tensors, dual spin connection (including the low-motion weak-field approximation), dual vierbein field as well as dual current densities of gravitomagnetic charge. It is shown that the dynamics of gravitomagnetic charge can be founded within the framework of the above dual quantities. In addition, the dual relationship between the dynamical theories of gravitomagnetic charge (dual mass) and gravitoelectric charge (mass) is also taken into account in the present paper.

Keywords: gravitomagnetic charge, dual curvature tensors, gravitational field equation, antisymmetric source tensor
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I. INTRODUCTION

The gravitomagnetic monopole possesses the dual charge to the mass. In this sense, the gravitomagnetic charge can also be referred to as the dual mass [1,2]. Some authors referred to it as “magnetic mass” or magnetic-type mass (magnetic-like mass) [3]. Historically, Newman et al. discovered a stationary and spherically symmetric solution (now known as the NUT solution) that contains a second parameter \( \iota \) besides the mass \( m \), and regarded it as the empty-space generalization of the Schwarzschild solution [4]. Demiansky and Newman found that the NUT space is in fact the spacetime produced by a mass which has a gravitomagnetic charge [5]. Dowker and Roche independently rediscovered this interpretation of the NUT parameter \( \iota \) [6]. The cylindrically symmetric solution of gravitomagnetic charge (i.e., the cylindrical analogue of NUT space, which is the spacetime of a line gravitomagnetic monopole) [7] was considered by Nouri-Zonoz in 1997. The various properties of NUT space attracts attention of some authors [3,8,9]. Lynden-Bell and Nouri-Zonoz gave a new and more elementary derivation of NUT space based on the spherical symmetry of gravitoelectric and gravitomagnetic fields of NUT space and the spherical symmetry of its spatial metric \( \gamma_{ab} \) [8]. They proved that all geodesics of NUT space lie on spatial cones, and used this interesting theorem to determine the gravitational lensing properties of NUT space [8]. Miller studied the global properties of the Kerr-Taub-NUT metric [9]. Zimmerman and Shahir considered the geodesics in the NUT metric and found that the NUT geodesics are similar to the properties of trajectories for charged particles orbiting about a magnetic monopole [3]. Recently, the quantal field and differential geometric properties and related topics associated with the NUT space receive attention of several researchers. Bini et al. investigated a single master equation for the gauge- and tetrad- invariant first-order massless perturbation describing spin \( \leq 2 \) fields in the Kerr-Taub-NUT spacetime [10]. In their another paper, Bini et al. analyzed the parallel transport around closed circular orbits in the equatorial plane of the Taub-NUT spacetime to reveal the effect of the gravitomagnetic monopole parameter on circular holonomy transformations [11]. As far as the quantization of gravitomagnetic monopole is concerned, historically, many authors considered this problem by analogy with Dirac’s argument for the quantization of magnetic charge [6,12,13]. In history, it was clear to see that

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such an idea (as Dirac’s theory) might not lead to a consistent quantum field theory for the gravitomagnetic charge [8] if the gravitomagnetic charge is truly present in the universe. This topic will be further taken into consideration in our other work regarding the quantum theory of gravitomagnetic monopole.

Lynden-Bell and Nouri-Zonoz thought that it is natural to ask how general relativity must be modified to allow for gravitomagnetic monopole densities and currents [8]. They conjectured the generalization will be to spaces with unsymmetrical affine connections which have nonzero torsion [8]. However, by analogy with the electrodynamics which is only changed to include unsymmetrical affine connections which have nonzero torsion [8]. However, by analogy with the electrodynamics which is only changed to include

\[ \partial_{\mu} F_{\mu\nu} = j^\nu \]

but does not come with an additional vector potential [8,14] if the magnetic charge truly exists, here we will suggest another alternative to this problem, i.e., constructing an antisymmetric dual Einstein tensor to describe the gravitational field produced by gravitomagnetic matter. Note that here we will not introduce extra unsymmetrical affine connections.

This paper has a threefold purpose: to establish the theoretical basis (such as dual curvature tensors) on which the field equation of gravitomagnetic matter is suggested; to apply the gravitational field equation to some problems (such as the NUT metric); to consider other related topics (such as the dual metric, dual spin connection and dual current density). The physical reasons for the nature of the mathematical results obtained will be interpreted and discussed wherever possible. The paper presents a detailed and self-contained treatment of the fundamental problems of classical gravitomagnetic charge.

The paper is organized as follows: in Sec. II, by considering the connection among the NUT solution, Einstein’s equation and the dynamics of gravitomagnetic matter, it is shown that gravitomagnetic matter requires a gravitational field equation of its own; in Sec. III, we study the dual Riemann curvature tensors and dual Ricci curvature tensors. The relationship between these dual curvature tensors and the Einstein (and Ricci) tensor is also demonstrated. In Sec. IV, we take into consideration the comparison of the gravity theory (involving the gravitomagnetic charge) with the electrodynamics, which will provide clue to us on how to construct the gravitational field equation of dual mass (and hence its dynamics). Several fundamental subjects regarding the dynamics of gravitomagnetic matter are discussed in Sec. V, which are as follows: (i) by using the variational principle, the antisymmetric dual Einstein tensor is obtained with the dual curvature scalar; (ii) the antisymmetric source tensor of dual matter is constructed in terms of the four-dimensional velocity and the corresponding covariant derivatives; (iii) the dual spin connection which will arise in the field equation of dual mass is briefly suggested. The dual vierbein field, dual metric and dual current density of gravitomagnetic matter are taken into account in Sec. VI. The importance of these quantities derives from the fact that many dynamical variables (such as the source tensors) of physical interest regarding the gravitomagnetic matter can be expressed in terms of them. To consider the connection between the gravitomagnetic charge and the magnetic charge, in Sec. VII, we study the five-dimensional field equation of gravitational charge. In Sec. VIII, we discuss briefly the duality relationship between the dynamical theories of gravitomagnetic charge (dual mass) and gravitoelectric charge (mass). In Sec. IX, we conclude with some remarks.

II. WANTING A GRAVITATIONAL FIELD EQUATION NECESSARY

Is gravitomagnetic monopoles required of its own gravitational field equation for dealing with the gravitational effects (properties, phenomena)? Before proceeding with the treatment of dynamics of gravitomagnetic matter, we shall here consider a problem associated with the relationship among the field equation, the solutions and the Bianchi identities. In an attempt to investigate the metric resulting from the presence of the gravitomagnetic charge currents, we think that one may meet with problems concerning the choice of the “true” metric produced by gravitomagnetic charge from the nonanalytic solutions of Einstein’s field equation. We argue that in history the gravitational field equation of gravitomagnetic charge might get less attention than it deserves.

As is known to us all, in general relativity, the NUT metric (named after Newman, Tamborino and Unti) \( g_{\mu\nu}(m,\iota) \) is the one describing the gravitational distribution produced by both the gravitoelectric charge (with mass \( m \)) and the gravitomagnetic monopole (with dual mass\(^1\) \( \iota \)) fixed at the origin of the coordinate system [5]. Since it is well known that the NUT solution \( g_{\mu\nu}(m = 0, \iota \neq 0) \) satisfies Einstein’s vacuum equation \( \mathcal{R}_{\mu\nu} = 0 \) (and hence Einstein’s equation \( G_{\mu\nu} = 0 \) without matter), someone may hold that the dynamics of gravitomagnetic matter has already been embodied in Einstein’gravitational field equation, or that the gravitational field equation of gravitomagnetic charge has been involved in Einstein’s equation, or that the gravitational distribution of both mass and dual mass is governed together by Einstein’s equation. Thus from the point of view of these peoples it is concluded that there exists no

\(^1\)Dual mass is viewed as the gravitomagnetic monopole strength.
other gravitational field equation of gravitomagnetic matter than Einstein’s equation, or that the potential suggestion of its gravitational field equation is not essential in investigating the dynamics of gravitomagnetic monopole (should such exist). However, even though it sounds reasonable, we think that these viewpoints may be not the true case. We believe that, in fact, the gravitomagnetic matter needs its own gravitational field equation rather than Einstein’s equation, since the latter governs the spacetime of gravitoelectric matter (mass) only. In what follows it will just be concentrated on the field equation problem of gravitomagnetic charge by comparing it with the electromagnetic field equation of magnetic monopole.

The NUT solution \( g_{\mu\nu}(m = 0, \nu \neq 0) \), which are the nonanalytic functions, truly agrees with Einstein’s equation (empty-spacetime field equation), and it can truly describe the gravitational field near a gravitomagnetic monopole. This, however, does not means that all the nonanalytic solutions to Einstein’s equation belong to the contribution of gravitomagnetic charge. Namely, Einstein’s equation is overdetermined for determining the gravitational-potential solutions of gravitomagnetic charge. Moreover, it can be further concluded that the NUT solution satisfying Einstein’s equation is not necessarily the one only to Einstein’s equation, just like the case that a solution satisfying a Bianchi identity is not the one only to this Bianchi identity.

In order to clarify the above viewpoint further, let us consider an illustrative example. Since there are some analogies between general relativity and electromagnetism, and most of the essential features of dynamics of electric and magnetic charges share similarity to the gravity theories of gravitoelectric and gravitomagnetic charges, one can prove the previous interpretation to be sound by analogy with the Maxwellian equation in electrodynamics. It is well known that the electromagnetic field equation of electric charge and magnetic charge are written in the form \( \partial_\mu F^{\mu\nu} = j_\nu \) and \( \partial_\mu F^{\mu\nu} = j_\nu^E \) respectively. Here, the dual electromagnetic field tensor \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \) with \( \epsilon^{\mu\nu\alpha\beta} \) being the Levi-Civita tensor\(^2\). It is readily verified that the stationary monopole solution (say, the Wu-Yang solution \([15]^3\) also agrees with the field equation of electric charge (\( \partial_\mu F^{\mu\nu} = 0 \) without the electric current densities). But this does not means that the Wu-Yang solution should necessarily the one only to \( \partial_\mu F^{\mu\nu} = 0 \), since the equation (\( \partial_\mu F^{\mu\nu} = 0 \)) is over determined for determining the monopole solution (for instance, it cannot set the magnetic charge \( g \) in the Wu-Yang solution). Likewise, it is apparently seen that the electric charge solution of \( \partial_\mu F^{\mu\nu} = j_\nu^E \) automatically agrees with the electromagnetic field equation of magnetic charge (\( i.e., \partial_\mu \tilde{F}_{\mu\nu} = 0 \) without magnetic charge densities and currents). But it is clearly not suitable to say that the equation \( \partial_\mu F^{\mu\nu} = 0 \) is also the electromagnetic field equation of electric charge current \( j_\nu^E \). In fact, not all the analytical solutions which agree with \( \partial_\mu F^{\mu\nu} = 0 \) satisfy the equation \( \partial_\mu \tilde{F}_{\mu\nu} = j_\nu^E \). The reason for this may be as follows: for the latter field equation, the former equation is merely a Bianchi identity. Similarly, for the magnetic charge, \( \partial_\mu \tilde{F}_{\mu\nu} = 0 \) should also be viewed as a Bianchi identity, which may also be seen as follows: if the dual electromagnetic field tensor is defined to be \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \), and the electromagnetic field tensor \( \tilde{F}^{\mu\nu} \) expressed in terms of the dual electromagnetic field tensor takes the form \( \tilde{F}^{\mu\nu} = -\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \tilde{F}_{\alpha\beta} \), then the electromagnetic field equation of electric charge (\( i.e., \partial_\mu \tilde{F}_{\mu\nu} = 0 \) without the electric current densities) can just be rewritten as the form of a Bianchi identity. Thus, we show that the equation \( \partial_\mu F^{\mu\nu} = 0 \) in the framework of dual 4-D potential vector \( \tilde{A}^\nu \) is truly a Bianchi identity. Since \( \partial_\mu F^{\mu\nu} = 0 \) is only a Bianchi identity for the magnetic charge, it cannot be the electromagnetic field equation of monopole magnetic (similarly, Einstein’s equation in empty spacetime is not the gravitational field equation of gravitomagnetic matter, either). This, therefore, means that not all of the solutions satisfying this Bianchi identity may also satisfy \( \partial_\mu \tilde{F}_{\mu\nu} = j_\nu^E \), \( i.e., \) not all these solutions may surely belong to the effect of magnetic monopole (similarly, not all the empty-space generalizations of the solutions of Einstein’s equation belong to gravitomagnetic charge, \( i.e., \) Einstein’s field equation cannot govern independently the gravitational distribution produced by gravitomagnetic matter). In a word, gravitomagnetic matter should have its own gravitational field equation.

In view of the importance of gravitational field equation of gravitomagnetic monopoles, it is certainly desirable to construct a theoretical framework, within which the field equation (and hence its dynamics) can be established. First and foremost, we should study the dual curvature tensors in Riemann space.

\(^2\)Note that, in this paper, for the electrodynamics, the Levi-Civita tensor refers to that in Minkowski flat spacetime, while for the gravity theory, \( \epsilon^{\mu\nu\alpha\beta} \) stands for the Levi-Civita tensor in the curved spacetime.

\(^3\)The Wu-Yang solution is of the form (in spherical coordinate system) \( A_\phi = 0, A_\theta = 0, A_r = \frac{2(1 + \cos \theta)}{r \sin \theta} \), which is a nonanalytic function.
III. DUAL CURVATURE TENSORS IN RIEMANN SPACE

In Sec. II, we showed that the investigation of gravitational distribution in the vicinity of a gravitomagnetic charge/monopole should start with the needs of a gravitational field equation. In order to discover the gravitational field equation of gravitomagnetic matter, we will proceed in a manner entirely analogous to what has been done for that of magnetic monopole (i.e., first we define the dual electromagnetic field tensor $\mathcal{F}^{\mu\nu}$, and then we obtain the quantity $\partial_\nu \mathcal{F}^{\mu\nu}$ from the variational principle with the dual Lagrangian density).

To lay a mathematical foundation for the field equation of dual mass that we will develop in the next section, first we consider the properties of dual curvature tensors in Riemann space. The right- and left-dual Riemann curvature tensors may be defined as follows

$$\mathcal{R}^{\mu\tau\omega\nu} = \frac{1}{2}\epsilon^{\tau\omega\nu}_{\mu\rho} \mathcal{R}_{\mu\tau\rho\sigma}$$ (right dual), 

$$\epsilon^{\mu\tau\omega\nu} = \frac{1}{2}\epsilon^{\mu\tau\rho\sigma} \mathcal{R}_{\rho\sigma\omega\nu}$$ (left dual),

(3.1)

and consequently, the corresponding dual Ricci tensors take the form

$$\mathcal{R}^{\mu\nu} = g^{\tau\omega} \mathcal{R}^{\mu\tau\omega\nu} = -\frac{1}{2}\epsilon^{\tau\lambda\sigma}_{\mu\rho} \mathcal{R}_{\mu\tau\rho\sigma}$$ (right dual), 

$$\epsilon^{\mu\nu} = g^{\tau\omega} \mathcal{R}^{\mu\tau\omega\nu} = \frac{1}{2}\epsilon^{\mu\tau\rho\sigma} \mathcal{R}_{\rho\sigma\omega\nu}$$ (left dual),

(3.2)

respectively. Note that $\epsilon^{\mu\nu}_{\tau\lambda\sigma} \mathcal{R}_{\mu\tau\rho\sigma} = 0$, i.e., the right-dual Ricci tensor $\mathcal{R}^{\mu\nu}$ is identically zero. So, we think that $\mathcal{R}^{\mu\nu}$ may have no important value in dealing with the dynamics of dual mass and therefore we abandon it and only adopt the left-dual curvature tensors in the following discussion. In particular, in what follows, the left-dual Riemann tensor and its corresponding dual Ricci tensor will be denoted respectively by $\mathcal{R}^{\mu\tau\omega\nu}$ and $\tilde{\mathcal{R}}^{\mu\nu}$, i.e.,

$$\tilde{\mathcal{R}}^{\mu\tau\omega\nu} = \epsilon^{\mu\tau\omega\nu} = \frac{1}{2}\epsilon^{\mu\tau\rho\sigma} \mathcal{R}_{\rho\sigma\omega\nu}, \quad \tilde{\mathcal{R}}^{\mu\nu} = \epsilon^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\tau\rho\sigma} \mathcal{R}_{\rho\sigma\omega\nu}. \quad (3.3)$$

It should be pointed out that in this section, without loss of generality, we shall restrict ourselves to the left- and right-dual curvature tensors only. For other dual curvature tensors (such as mid-dual and two-side dual tensors) one may be referred to the appendix to this paper.

To consider further the properties of dual curvature tensor, we will present the following three interesting and useful identities regarding the above-mentioned dual curvature tensors:

$$\frac{1}{2}\epsilon^{\theta\tau\omega\nu} \tilde{\mathcal{R}}^{\mu}_{\tau\omega\nu} = -\frac{1}{2} \left( G^{\rho\mu} + G^{\sigma\mu} \right),$$

$$\frac{1}{2}\epsilon^{\theta\tau\omega\nu} \tilde{\mathcal{R}}^{\mu\tau\omega\nu} = -\frac{1}{2} \left( R^{\rho\mu} + R^{\sigma\mu} \right),$$

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} \left( \mathcal{R}^{\rho\mu} - \mathcal{R}^{\nu\mu} \right) = \frac{1}{2} \left[ \left( R^{\alpha\beta} - R^{\beta\alpha} \right) + \left( R^{\alpha\beta} - R^{\beta\alpha} \right) \right],$$

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} (\mathcal{R}^{\rho\mu} - \mathcal{R}^{\nu\mu} + \mathcal{R}'^{\rho\mu} - \mathcal{R}'^{\nu\mu}) = \mathcal{R}^{\alpha\beta} - \mathcal{R}^{\beta\alpha}. \quad (3.4)$$

Here $\mathcal{R}'^{\alpha\beta} = R^{\alpha\lambda} \lambda^{\beta}$, $\mathcal{R}'^{\mu\nu} = R^{\mu}_{\lambda} \lambda^{\nu}$. The proof of the first identity in Eq.(3.4) is given in Appendix 2 to this paper. Now let us look at the second identity in Eq.(3.4), which can be easily verified as follows:

$$\frac{1}{4}\epsilon^{\theta\tau\omega\nu} \epsilon_{\tau\omega\nu}^{\mu} \mathcal{R}^{\rho\mu} = -\frac{1}{2} \left( g^{\lambda\delta} \gamma^{\sigma\nu} - g^{\lambda\nu} g^{\sigma\theta} \right) \mathcal{R}^{\rho\mu}_{\lambda\sigma} = \frac{1}{2} \left( R^{\sigma\nu} \mu - R^{\nu\sigma} \mu \right), \quad (3.5)$$

where for the calculation of $\epsilon^{\theta\tau\omega\nu} \epsilon_{\tau\omega\nu}^{\mu}$, readers may be referred to Appendix 1. In Appendix 1 the Ricci tensor is so defined that $\mathcal{R}^{\delta\nu}_{\mu} = R^{\delta\mu}$. By using $R^{\delta\nu}_{\mu} = R^{\delta\mu}$, the second identity in Eq.(3.4) is proven correct. With the help of the formulae in Appendix 1, one can arrive at

\footnote{It should be pointed out that in the published paper [Gen. Relativ. Gravit. 34, 1423-1435 (2002)], however, we adopted the right-dual form rather than the left-dual one. Such a situation was caused by the fact that the form of the Riemann curvature tensor $\mathcal{R}^{\mu\nu\alpha\beta}$ applied in the GRG paper (see, for example, Eq.(7) in the above-mentioned GRG paper) refers to the form $\mathcal{R}^{\alpha\beta\mu\nu}$ expressed in the present paper.}

\footnote{It follows from the formula presented in Appendix 1 to this paper that the inner product of two Levi-Civita tensors/symbols (where the summation is to be carried out over the repeated index $\mu = 0, 1, 2, 3$) reads $\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu}^{\lambda\sigma} = -\left( g^{\lambda\nu} \gamma^{\sigma\omega} + g^{\lambda\beta} \gamma^{\nu\omega} + g^{\lambda\sigma} \gamma^{\nu\beta} \right)$.

\[ g^{\lambda\nu} \gamma^{\sigma\omega} + g^{\lambda\beta} \gamma^{\nu\omega} + g^{\lambda\sigma} \gamma^{\nu\beta} - g^{\lambda\nu} \gamma^{\beta\sigma} - g^{\lambda\sigma} \gamma^{\nu\beta} - g^{\lambda\beta} \gamma^{\nu\sigma} \]
Thus we obtain
\[
\frac{1}{2} \epsilon^{\mu \nu \alpha \beta} \bar{R}_{\mu \nu} = - \frac{1}{4} \left[ (R^{\alpha \beta} - R^{\beta \alpha}) + (R'_{\alpha \beta} - R'_{\beta \alpha}) \right], \quad - \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} \bar{R}_{\nu \mu} = - \frac{1}{4} \left[ (R^{\alpha \beta} - R^{\beta \alpha}) + (R'_{\alpha \beta} - R'_{\beta \alpha}) \right].
\]
(3.7)

It follows that
\[
\frac{1}{2} \epsilon^{\mu \nu \alpha \beta} (\bar{R}_{\mu \nu} - \bar{R}_{\nu \mu}) = - \frac{1}{2} \left[ (R^{\alpha \beta} - R^{\beta \alpha}) + (R'_{\alpha \beta} - R'_{\beta \alpha}) \right],
\]
(3.8)
which is the third identity in Eq.(3.4). Furthermore, by the aid of the formula in Appendix 1, one can show that the fourth identity in Eq.(3.4) holds also.

In the following we will briefly discuss the difference between \( R_{\mu \nu} \) and \( R'_{\mu \nu} \). For the Riemann tensor, even though \( R_{\alpha \mu \nu \beta} = - R_{\alpha \mu \beta \nu} \) holds, the relation \( R_{\alpha \mu \nu \beta} = - R_{\mu \alpha \nu \beta} \) is no longer valid due to the nonanalytic property of the metric tensors, which can be seen from the following calculation:
\[
R_{\alpha \mu \nu \beta} + R_{\mu \alpha \nu \beta} = \partial_\nu (\Gamma_{\alpha \mu \beta} + \Gamma_{\mu \alpha \beta}) - \partial_\beta (\Gamma_{\alpha \mu \nu} + \Gamma_{\mu \alpha \nu}) = (\partial_\nu \partial_\beta - \partial_\beta \partial_\nu) g_{\alpha \mu} \neq 0.
\]
(3.9)

Thus we have \( R_{\alpha \mu \nu \beta} = R_{\mu \alpha \beta \nu} + (\partial_\nu \partial_\beta - \partial_\beta \partial_\nu) g_{\alpha \mu} \). With the help of \( R_{\mu \nu} = g^{\alpha \beta} R_{\alpha \mu \nu \beta} \) and \( R'_{\mu \nu} = g^{\alpha \beta} R'_{\alpha \mu \nu \beta} \), one can arrive at
\[
R_{\mu \nu} = R'_{\mu \nu} + g^{\alpha \beta} (\partial_\nu \partial_\beta - \partial_\beta \partial_\nu) g_{\alpha \mu},
\]
(3.10)
which is the relation between \( R_{\mu \nu} \) and \( R'_{\mu \nu} \). Apparently, if \( g_{\alpha \mu} \) is an analytic function, then we have \( R_{\mu \nu} = R'_{\mu \nu} \). It should be noted that in the dynamics of gravitomagnetic matter, the metric is often nonanalytic. For this reason, one should distinguish \( R'_{\mu \nu} \) from \( R_{\mu \nu} \) in the treatment of the problems where both mass and dual mass are involved.

**IV. THE CONNECTION BETWEEN THE GRAVITY THEORY AND THE ELECTRODYNAMICS**

It is useful to consider the analogies between the gravity theory and the electrodynamics in developing the gravitational field equation of dual matter. This deep connection between them will make the establishment of gravity theory of gravitomagnetic matter more enlightening and possible.

The first analogy between them is that the Lagrangian density of electromagnetic fields can be constructed in terms of the dual field tensor \( \tilde{F}_{\mu \nu} \), i.e.,
\[
- \frac{1}{4} \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} = - \left( - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right) \quad \text{with} \quad \tilde{F}_{\mu \nu} = \frac{1}{2} \epsilon^{\alpha \beta} F_{\alpha \beta},
\]
(4.1)
the gravitational analogy to which arises also. By making use of the identities of Eq.(3.4), it is shown that
\[
\frac{1}{2} \epsilon^{\mu \tau \nu \omega} \tilde{R}_{\tau \omega \nu \mu} = - \bar{R}
\]
(4.2)
with \( \bar{R} \) being the curvature scalar.

The second analogy is that one can obtain the term \( \partial_\nu \tilde{F}_{\mu \nu} \) that appears on the left-handed side of the electromagnetic field equation of magnetic charge currents from the variational principle with the dual Lagrangian density \( L = - \frac{1}{4} \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} \). Note that here the variational principle is applied to \( L \) with respect to \( A^\mu \). In the similar manner, for the gravity theory, the dual Einstein tensor \( \tilde{G}_{\mu \nu} \) can also be derived via the variational principle with the dual curvature scalar \( \bar{R} \), i.e.,
\[
\delta \int_\Omega \sqrt{-g} \bar{R} d\Omega = 0 \Rightarrow \tilde{G}_{\mu \nu} = \bar{R}_{\mu \nu} - \bar{R}_{\nu \mu},
\]
(4.3)
where $\Omega$ denotes the spacetime region of volume integral and $d\Omega$ is the volume element. The dual curvature scalar $\tilde{R}$ is defined to be $\tilde{R} = g^{\mu\nu} R_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\lambda\sigma\tau} R_{\lambda\sigma\tau\mu}$ with $R_{\mu\nu} = \frac{1}{2} \epsilon^{\lambda\sigma\tau\mu} R_{\lambda\sigma\tau\nu}$. The relation (4.3) will be developed in more details in Sec. V.

The third analogy between the gravity theory and the electrodynamics may be seen from the following three set of expressions. The relation between the electromagnetic field tensor $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$ is

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta} F_{\alpha\beta}, \quad F_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta} \tilde{F}_{\alpha\beta}. \quad (4.4)$$

In the meanwhile, there exist the similar relations between curvature tensors and their dual. For example,

$$\tilde{R}_{\mu\nu\tau\omega} = \frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta} R_{\alpha\beta\tau\omega}, \quad R_{\mu\nu\tau\omega} = -\frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta} \tilde{R}_{\alpha\beta\tau\omega} \quad (4.5)$$

and

$$\tilde{R}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu}^{\lambda\sigma\tau} R_{\lambda\sigma\tau\nu}, \quad \frac{1}{2} (R_{\mu\nu} + R_{\nu\mu}) = -\frac{1}{2} \epsilon_{\mu}^{\lambda\sigma\tau} \tilde{R}_{\lambda\sigma\tau\nu}. \quad (4.6)$$

Note that the first expression of Eqs.(4.6) is due to the definition of the dual Ricci tensor $\tilde{R}_{\mu\nu}$ in (3.3), and the second expression of Eqs.(4.6) can be derived from the second identity in Eqs.(3.4).

We have, therefore, investigated in detail the properties of dual curvature tensors in Riemann space. Thus, the foundation of the dynamics of gravitomagnetic matter is laid in the preceding sections. Moreover, we have also discussed the problem of the requirement for a gravitational field equation of gravitomagnetic matter of its own in studying both the gravitational field distribution produced by the gravitomagnetic charge and the equation of motion of gravitomagnetic monopoles. In the following section, we will derive the gravitational field equation of gravitomagnetic matter from the variational principle with the dual curvature scalar.

V. GRAVITATIONAL FIELD EQUATION OF GRAVITOMAGNETIC MATTER

What in gravity theory may be caused by the existence of the gravitomagnetic monopoles? Will it lead to unsymmetrical metric and/or to modify Einstein’s field equation? As the topological charge in spacetime, gravitomagnetic charge will inevitably result in the nonanalytic parts in the metric functions. In this sense, gravitomagnetic monopole does not introduce the so-called extra gravitational potentials. So, here we need not employ the unsymmetrical metric in the treatment of the dynamics of gravitomagnetic matter. In order to determine the nonanalytic parts of the metric functions caused by the gravitomagnetic monopoles, we should first suggest a gravitational field equation of its own. A systematic procedure for the gravitational field equation of gravitomagnetic matter will be developed in this section. It will lay a physical foundation for the mathematical formalism in Sec. IV. If the gravitational charge is not present, then the contribution of this gravitational field equation without source term in general relativity is easily shown to vanish, on account of its status of Bianchi identity. But, it will play an essential role in treating the gravitational distributon problem of gravitomagnetic matter (should such exist). For examples, we will illustrate the application of this field equation to the NUT solution and the equation of motion of gravitomagnetic monopoles.

A. Dual curvature scalar and variational principle

With the introduction and motivation in the previous sections, we are now in a position to find a gravitational field equation of gravitomagnetic matter. For this aim, one should first construct a Lagrangian density that can describe the gravitational field with nonanalytic property. Such a Lagrangian density should be a one that will be vanishing if the metric function is analytic. The only one we can choose is just the dual curvature scalar $\tilde{R}$. So, the dual action $I'$ of the gravitational field produced by the gravitomagnetic matter may be of the form [1,2]

$$I' = \int_{\Omega} \sqrt{-g} \tilde{R} d\Omega = \int_{\Omega} \sqrt{-g} g^{\mu\nu} \left[ (1 - \zeta) \tilde{R}_{\mu\nu} + \zeta \tilde{R}_{\sigma\mu} \right] d\Omega, \quad (5.1)$$

which is a four-dimensional volume integral of the dual curvature scalar $\tilde{R}$. Note that the dual Ricci tensor $\tilde{R}_{\mu\nu}$ is asymmetric in $\mu, \nu$. So in Eq.(5.1), we introduce a certain parameter $\zeta$ to rewrite the dual action $I'$. The variation of the action $I'$ is
\[ \delta I' = \int_{\Omega} \delta \left( \sqrt{-g} g^{\mu\nu} \left[ (1 - \zeta) \hat{\mathcal{R}}_{\mu\nu} + \zeta \tilde{\mathcal{R}}_{\mu\nu} \right] \right) d\Omega + \int_{\Omega} \sqrt{-g} g^{\mu\nu} \left[ (1 - \zeta) \delta \hat{\mathcal{R}}_{\mu\nu} + \zeta \delta \tilde{\mathcal{R}}_{\mu\nu} \right] d\Omega, \quad (5.2) \]

The first term on the right-handed side of Eq.(5.2) can be rewritten as

\[ \int_{\Omega} \delta \left( \sqrt{-g} g^{\mu\nu} \left[ (1 - \zeta) \hat{\mathcal{R}}_{\mu\nu} + \zeta \tilde{\mathcal{R}}_{\mu\nu} \right] \right) d\Omega = \int_{\Omega} \sqrt{-g} \left[ (1 - \zeta) \left( \hat{\mathcal{R}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{\mathcal{R}} \right) + \zeta \left( \tilde{\mathcal{R}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{\mathcal{R}} \right) \right] \delta g^{\mu\nu} d\Omega = \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{\mathcal{R}} \right) - \zeta \left( \tilde{\mathcal{R}}_{\mu\nu} - \tilde{\mathcal{R}}_{\nu\mu} \right) \right] \delta g^{\mu\nu} d\Omega. \quad (5.3) \]

Now in the following we will calculate the second term on the right-handed side of Eq.(5.2). For convenience, we set \( A = \int_{\Omega} \sqrt{-g} g^{\mu\nu} \left[ (1 - \zeta) \delta \hat{\mathcal{R}}_{\mu\nu} + \zeta \delta \tilde{\mathcal{R}}_{\mu\nu} \right] d\Omega. \) The infinitesimal variation of the dual Ricci tensor \( \delta \tilde{\mathcal{R}}_{\mu\nu} \) is given as

\[ \delta \tilde{\mathcal{R}}_{\mu\nu} = \frac{1}{2} \delta \epsilon_{\mu}^{\lambda_\sigma \tau} \mathcal{R}_{\lambda_\sigma \tau \nu} + \frac{1}{2} \epsilon_{\mu}^{\lambda_\sigma \tau} \delta \mathcal{R}_{\lambda_\sigma \tau \nu}. \quad (5.4) \]

In Appendix 3, we show that the variation \( \int_{\Omega} \sqrt{-g} g^{\mu\nu} \frac{1}{2} \epsilon_{\mu}^{\lambda_\sigma \tau} \delta \mathcal{R}_{\lambda_\sigma \tau \nu} d\Omega \) is actually a surface integral over the boundary of the volume, namely, it can be transformed into a four-dimensional volume integral of a divergence. Since the variational principle assumes that the variation \( \delta g^{\mu\nu} \) vanishes on the boundary, it follows that \( \int_{\Omega} \sqrt{-g} g^{\mu\nu} \frac{1}{2} \epsilon_{\mu}^{\lambda_\sigma \tau} \delta \mathcal{R}_{\lambda_\sigma \tau \nu} d\Omega \) has no effect on the variation of the action and therefore gives no contribution to the derivation of the dual Einstein tensor \( \tilde{G}_{\mu\nu} \). We can, therefore, ignore this term\(^6\). Thus, the term \( \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \tilde{\mathcal{R}}_{\mu\nu} d\Omega \) in \( A \) can be reduced to

\[ \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \tilde{\mathcal{R}}_{\mu\nu} d\Omega = \frac{1}{2} \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \epsilon_{\mu}^{\lambda_\sigma \tau} \mathcal{R}_{\lambda_\sigma \tau \nu} d\Omega. \quad (5.5) \]

By using the formula in Appendix 4, Eq.(5.5) can be rewritten as

\[ \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \tilde{\mathcal{R}}_{\mu\nu} d\Omega = \frac{1}{2} \int_{\Omega} \sqrt{-g} g^{\mu\nu} \left( -g_{\mu\alpha} g_{\beta\gamma} \delta \epsilon^{\lambda_\sigma \tau} + \frac{1}{2} g_{\mu\beta} g_{\alpha\gamma} \delta \epsilon^{\lambda_\sigma \tau} \right) \mathcal{R}_{\lambda_\sigma \tau \nu} \delta g^{\alpha\beta} d\Omega = - \int_{\Omega} \sqrt{-g} \frac{1}{2} \left( -g^{\nu\beta} \epsilon_{\mu}^{\lambda_\sigma \tau} \mathcal{R}_{\lambda_\sigma \tau \nu} + \frac{1}{2} g_{\alpha\beta} \epsilon^{\nu \lambda_\sigma \tau} \mathcal{R}_{\lambda_\sigma \tau \nu} \right) \delta g^{\alpha\beta} d\Omega = - \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \hat{\mathcal{R}} \right) \delta g^{\mu\nu} d\Omega. \quad (5.6) \]

In the same fashion, the term \( \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \hat{\mathcal{R}}_{\nu\mu} d\Omega \) in \( A \) can be rewritten in the form

\[ \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \hat{\mathcal{R}}_{\nu\mu} d\Omega = - \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \hat{\mathcal{R}} \right) \delta g^{\mu\nu} d\Omega = - \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \hat{\mathcal{R}} \right) \delta g^{\mu\nu} d\Omega. \quad (5.7) \]

By introducing two parameters \( \xi \) and \( \zeta \), Eq.(5.6) and (5.7) can be rewritten as

\[ \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \tilde{\mathcal{R}}_{\mu\nu} d\Omega = -(1 - \xi) \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{\mathcal{R}} \right) \delta g^{\mu\nu} d\Omega - \xi \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \hat{\mathcal{R}} \right) \delta g^{\mu\nu} d\Omega. \quad (5.8) \]

and

\[ \int_{\Omega} \sqrt{-g} g^{\mu\nu} \delta \hat{\mathcal{R}}_{\nu\mu} d\Omega = -(1 - \zeta) \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \hat{\mathcal{R}} \right) \delta g^{\mu\nu} d\Omega - \zeta \int_{\Omega} \sqrt{-g} \left( \hat{\mathcal{R}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{\mathcal{R}} \right) \delta g^{\mu\nu} d\Omega. \quad (5.9) \]

\(^6\)Someone may argue that because of the symmetric property of the Christoffel symbol \( \Gamma^{\lambda}_{\mu\nu} \) in the indices \( \mu, \nu \), it is apparent to see that the integrand \( \sqrt{-g} g^{\mu\nu} \epsilon_{\mu}^{\lambda_\sigma \tau} \delta \mathcal{R}_{\lambda_\sigma \tau \nu} \) is strictly identical to zero, and it may drop out of the variation \( \delta I' \) immediately. For the dual formalism, however, this viewpoint is not appropriate for determining the dual Einstein tensor. To derive the dual Einstein tensor in a stringent way, if \( \int_{\Omega} \sqrt{-g} g^{\mu\nu} \epsilon_{\mu}^{\lambda_\sigma \tau} \delta \mathcal{R}_{\lambda_\sigma \tau \nu} d\Omega \) can be truly ignored, we should first show that \( \sqrt{-g} g^{\mu\nu} \epsilon_{\mu}^{\lambda_\sigma \tau} \delta \mathcal{R}_{\lambda_\sigma \tau \nu} \) can be rewritten as a covariant divergence of a certain vector indeed.
Hence, by using Eq.(5.8) and (5.9), one can express the second term on the right-handed side of Eq.(5.2) (i.e.,
\[ A = \int \sqrt{-g} g^{\mu \nu} \left[ (1 - \zeta) \delta R_{\mu \nu} + \zeta \delta \tilde{R}_{\nu \mu} \right] d\Omega \]
) as
\[ \int_{\Omega} \sqrt{-g} g^{\mu \nu} \left[ (1 - \zeta) \delta R_{\mu \nu} + \zeta \delta \tilde{R}_{\nu \mu} \right] d\Omega = [-(1 - \zeta)(1 - \xi) - \zeta \xi] \int_{\Omega} \sqrt{-g} \left( \tilde{R}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \tilde{R} \right) \delta g^{\mu \nu} d\Omega + [-(1 - \zeta)(1 - \zeta(1 - \zeta))] \int_{\Omega} \sqrt{-g} \left( \tilde{R}_{\mu \nu} - \tilde{R}_{\nu \mu} \right) \delta g^{\mu \nu} d\Omega. \] (5.10)

Namely, we have
\[ A = - \int_{\Omega} \sqrt{-g} \left( \tilde{R}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \tilde{R} \right) \delta g^{\mu \nu} d\Omega + [\zeta + \xi - \zeta(\xi + \zeta)] \int_{\Omega} \sqrt{-g} \left( \tilde{R}_{\mu \nu} - \tilde{R}_{\nu \mu} \right) \delta g^{\mu \nu} d\Omega. \] (5.11)

It follows from Eq.(5.2), (5.3) and (5.11) that the variation of \( I' \) is finally given
\[ \delta I' = [\xi - \zeta(\xi + \zeta)] \int_{\Omega} \sqrt{-g} \left( \tilde{R}_{\mu \nu} - \tilde{R}_{\nu \mu} \right) \delta g^{\mu \nu} d\Omega. \] (5.12)

If we set \( I = \frac{1}{\zeta - \zeta(\xi + \zeta)} I' \), the expression for the variation of \( I \) reads
\[ \delta I = \int_{\Omega} \sqrt{-g} \tilde{G}_{\mu \nu} \delta g^{\mu \nu} d\Omega \quad \text{with} \quad \tilde{G}_{\mu \nu} = \tilde{R}_{\mu \nu} - \tilde{R}_{\nu \mu}, \] (5.13)

where \( \tilde{G}_{\mu \nu} \) may be referred to as the dual Einstein tensor, which is the resulting one for the given dual action \( I \).

In what follows we will simplify the form of the dual Ricci tensor \( \tilde{R}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu}^{\lambda \sigma \tau} R_{\lambda \sigma \tau \nu} \), which yields
\[ \tilde{R}_{\mu \nu} = \frac{1}{4} \epsilon_{\mu}^{\lambda \sigma \tau} \frac{\partial}{\partial x^{\tau}} \left( \frac{\partial g_{\nu \lambda}}{\partial x^{\sigma}} - \frac{\partial g_{\nu \sigma}}{\partial x^{\lambda}} \right). \] (5.14)

Here any symmetric parts in \( R_{\lambda \sigma \tau \nu} \) have simply dropped out of \( \tilde{R}_{\mu \nu} \) because \( \epsilon_{\mu}^{\lambda \sigma \tau} \) is completely antisymmetric in \( \lambda, \sigma, \tau \). It is physically interesting that the \((00)\) component of the dual Ricci tensor is \( \tilde{R}_{00} = \frac{1}{4} \epsilon_{\mu}^{\lambda \sigma \tau} \frac{\partial}{\partial x^{\tau}} \left( \frac{\partial g_{0 \lambda}}{\partial x^{\sigma}} - \frac{\partial g_{0 \sigma}}{\partial x^{\lambda}} \right) \), which is somewhat analogous to its electromagnetic counterpart \( \frac{\partial}{\partial x^{\sigma}} F_{\mu \tau} = \frac{1}{4} \epsilon_{\mu}^{\lambda \sigma \tau} \frac{\partial}{\partial x^{\sigma}} \left( \partial_{\lambda} A_{\tau} - \partial_{\tau} A_{\lambda} \right) \). This, therefore, means that \( \tilde{G}_{\mu \nu} \) may truly serve as a dual Einstein tensor and would appear on the left-handed side of the gravitational field equation of gravitomagnetic charge. Clearly, the formalism for the dual Einstein tensor which we have developed here is rigorous. It may contribute to a better understanding of the dynamics of gravitomagnetic matter.

**B. Antisymmetrical source tensor**

The preceding subsection shows that the dual Einstein tensor \( \tilde{G}_{\mu \nu} \) is a second-rank antisymmetric one. So, the source tensor of gravitomagnetic matter, which appears on the right-handed side of the gravitational field equation of gravitomagnetic charge, should also be an antisymmetric one [1,2]. Such antisymmetrical tensors which are constructed in terms of the four-dimensional velocity vector \( U_{\nu} \), and its covariant derivatives with respect to the spacetime coordinates may be only the following two forms
\[ K_{\mu \nu} = D_{\mu} U_{\nu} - D_{\nu} U_{\mu} \equiv \partial_{\mu} U_{\nu} - \partial_{\nu} U_{\mu}, \quad H_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu}^{\lambda \rho} K_{\lambda \rho}. \] (5.15)

So, the second-rank antisymmetric source tensor of gravitomagnetic matter can take the following form (i.e., the linear combination of \( K_{\mu \nu} \) and \( H_{\mu \nu} \))
\[ S_{\mu \nu} = \kappa_{1} K_{\mu \nu} + \kappa_{2} H_{\mu \nu}, \] (5.16)

where \( \kappa_{1} \) and \( \kappa_{2} \) denote the two combination coefficients.

For the Fermionic field \( \psi \), its four-vector velocity (or current density) is \( U_{\mu} = e_{\mu}^{a} i \bar{\psi} \gamma_{a} \psi \), where \( e_{\mu}^{a} \) denotes the vierbein field with \( \mu, a = 0, 1, 2, 3 \) and \( \gamma_{a} \)'s stand for the Dirac matrices in the flat spacetime. The second-rank
antisymmetric source tensor of the gravitomagnetic Fermionic field $\psi$ can be constructed according to the above-mentioned definition\(^7\). The four-dimensional velocity (or momentum) of the Bosonic field is $U_\mu = \frac{1}{2i} (\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$, and the antisymmetric tensor $K_{\mu\nu}$ is therefore $K_{\mu\nu} = \frac{1}{2} (\partial_\mu \phi^* \partial_\nu \phi - \partial_\nu \phi^* \partial_\mu \phi)$, which can be used to express the antisymmetric source tensor $S_{\mu\nu}$ of the gravitomagnetic Bosonic field.

It should be noted that the source tensor of gravitomagnetic matter can also be expressed in terms of the dual current density (velocity) and the dual vierbein field $\tilde{e}_\mu \tilde{a}$, which will be discussed in Sec. VI.

C. Gravitational field equation of gravitomagnetic matter

In accordance with the above discussion, we discover the gravitational field equation of gravitomagnetic matter as follows (covariant form)

\[ \mathring{G}_{\mu\nu} = S_{\mu\nu}. \]  
(5.17)

Note that Einstein’ field equation is of symmetry in indices. In contrast, here the gravitational field equation is an antisymmetric one [1,2].

So far we have obtained the gravitational field equation of gravitomagnetic charge of its own. Now let us discuss the equation of motion of a gravitomagnetic monopole as a test particle (or a particle system consisting of gravitomagnetic monopoles). According to the gravitational field equation (contravariant form)

\[ \mathring{G}^{\mu\nu} = S^{\mu\nu}, \]  
(5.18)

we set a second-rank antisymmetric tensor $Z^{\mu\nu} = S^{\mu\nu} - \mathring{G}^{\mu\nu}$, the covariant divergence of which is expressed by

\[ Z^{\mu\nu}_{\;;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} Z^{\mu\nu}) = 0, \]  
(5.19)

which means that $Z^{\mu\nu} = \frac{1}{\sqrt{-g}} C^{\mu\nu}$ with $C^{\mu\nu}$ being a certain constant tensor, or a tensor whose divergence vanishes. Because of Eq.(5.18), $C^{\mu\nu} = 0$, i.e., $Z^{\mu\nu} = 0$. Note that even though the equation $Z^{\mu\nu} = 0$ can be derived straightforwardly from Eq.(5.18), here we would not think of it as the result of Eq.(5.18). On the contrary, we prefer to consider it as the result of the covariant divergence of the field equation Eq.(5.18). This, therefore, implies that the equation $Z^{\mu\nu} = 0$ is inevitably in connection with the kinematic equation of a system of particles consisting of gravitomagnetic charges.

The following calculation will confirm this interpretation: $S^{\mu\nu} U_\nu$ in $Z^{\mu\nu} U_\nu$ is expressed by

\[ S^{\mu\nu} U_\nu = \kappa_1 U_\nu (D^\mu U^\nu - D^\nu U^\mu) + \frac{1}{2} \kappa_2 \epsilon^{\mu\nu\alpha\beta} U_\nu (\partial_\alpha U_\beta - \partial_\beta U_\alpha) \]
\[ = -\kappa_1 \frac{D U_\mu}{ds} + \frac{1}{2} \kappa_2 \epsilon^{\mu\nu\alpha\beta} U_\nu (\partial_\alpha (g_{\beta\lambda} U^\lambda) - \partial_\beta (g_{\alpha\lambda} U^\lambda)) \]
\[ = -\kappa_1 \frac{D U_\mu}{ds} + \frac{1}{2} \kappa_2 \epsilon^{\mu\nu\alpha\beta} U_\nu (\partial_\alpha g_{\beta\lambda} - \partial_\beta g_{\alpha\lambda}) U^\lambda + \frac{1}{2} \kappa_2 \epsilon^{\mu\nu\alpha\beta} U_\nu (g_{\beta\lambda} \partial_\alpha - g_{\alpha\lambda} \partial_\beta) U^\lambda. \]  
(5.20)

It is noted that the second term $\frac{1}{2} \kappa_2 \epsilon^{\mu\nu\alpha\beta} U_\nu (\partial_\alpha g_{\beta\lambda} - \partial_\beta g_{\alpha\lambda}) U^\lambda$ on the right-handed side of (5.20) contains an expression $\sim \frac{1}{2} \kappa_2 \epsilon^{\mu\nu\alpha\beta} (\partial_\alpha g_{\beta\lambda} - \partial_\beta g_{\alpha\lambda}) U^\lambda$, which is just a gravitational (gravitomagnetic) Lorentz force, the electromagnetic counterpart of which is $g\epsilon^{\mu\nu\alpha\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) U_\nu$, with $g$ representing the magnetic charge. The fact that $\sim \frac{1}{2} \kappa_2 \epsilon^{\mu\nu\alpha\beta} U_\nu (\partial_\alpha g_{\beta\lambda} - \partial_\beta g_{\alpha\lambda}) U^\lambda$ resembles the expression for the Lorentz force in electrodynamics means that the equation $Z^{\mu\nu} = 0$ contains the dynamical equation of motion of a particle system composed of gravitomagnetic charges.

The antisymmetric field equation (5.18) of gravitomagnetic matter governs the gravitational field near the gravitomagnetic monopoles. It does not introduce new extra gravitational potentials. The role of this equation presented here is to determine the nonanalytic parts of the metric which is caused by the existence of the gravitomagnetic monopoles. If the gravitomagnetic charge does not exist in the universe, the vacuum case of Eq.(5.18) is just the Bianchi identity of Einstein’s field equation.

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\(^7\) The tensor $K_{\mu\nu}$ of Fermionic field is given as $i \left[ (\partial_\mu \tilde{\psi}) \gamma_\nu \psi - (\partial_\mu \psi) \gamma_\nu \tilde{\psi} \right] + i \tilde{\psi} (\gamma_\nu \partial_\mu - \gamma_\mu \partial_\nu) \psi + i \tilde{\psi} (\partial_\mu \gamma_\nu - \partial_\nu \gamma_\mu) \psi$. In the papers [1,2], the third term, $i \tilde{\psi} (\partial_\mu \gamma_\nu - \partial_\nu \gamma_\mu) \psi$, was unfortunately neglected.
D. About NUT metric

The NUT metric is the one produced by the stationary gravitomagnetic monopole at the origin of the coordinate system. Here we will discuss the relation between the NUT parameter $\iota$ and the gravitomagnetic charge. The contravariant-form field equation (5.18) of dual matter can be rewritten as

$$\tilde{R}^{\mu\nu} = \frac{1}{\sqrt{-g}} \tilde{\Gamma}^{\mu\nu}, \quad \tilde{S}^{\mu\nu} = \frac{1}{\sqrt{-g}} (\tilde{\Gamma}^{\mu\nu} - \tilde{\Gamma}^{\nu\mu}).$$

(5.21)

For a single gravitomagnetic monopole, the strength of which is $\iota'$, one may assume that $\tilde{\Gamma}^{0}_{0} = \iota' \delta(x)$. The four-dimensional volume integral of the term on the right-handed side of the field equation $\tilde{R}^{0}_{0} = \frac{1}{\sqrt{-g}} \tilde{\Gamma}^{0}_{0}$ is

$$\int_{\Omega} \frac{1}{\sqrt{-g}} \tilde{\Gamma}^{0}_{0} \, d\Omega = \iota'.$$

(5.22)

In what follows, we will consider the equation $\tilde{R}^{0}_{0} = \frac{1}{\sqrt{-g}} \iota' \delta(x)$, where $\tilde{R}^{0}_{0} = \frac{1}{\sqrt{-g}} \iota^{0\lambda\sigma\tau} \partial_\tau (\partial_\sigma g_{0\lambda} - \partial_\lambda g_{0\sigma})$ and $\iota^{0\lambda\sigma\tau} = \frac{1}{\sqrt{-g}} \epsilon^{0\lambda\sigma\tau}$. Here $\epsilon^{0\lambda\sigma\tau}$ is the Levi-Civita tensor in the flat Minkowski spacetime. It is readily verified that this equation can be rewritten as

$$\nabla \cdot \mathbf{B}_g = 4\iota'' \delta(x),$$

(5.23)

where $\iota'' = \iota' \delta(x^0)$, $\delta(x) = \delta(x^0) \delta(x)$. Here the gravitomagnetic field strength is defined as $\mathbf{B}_g = \nabla \times \mathbf{g}$ with the gravitomagnetic vector potentials being $\mathbf{g} = (g_{01}, g_{02}, g_{03})$.

It is well known that the gravitomagnetic vector potentials of NUT metric (without mass) can be expressed as

$$\mathbf{g} = \left(0, 0, -2f^2(r) \frac{\iota(1 - \cos \theta)}{r \sin \theta}\right)$$

(5.24)

in the spherical polar coordinate system, namely, $g_{0\phi} = -2f^2(r) \frac{\iota(1 - \cos \theta)}{r \sin \theta}$, where the function $f^2(r)$ is $f^2(r) = 1 - 2 \left(\frac{m r}{r^2 + z^2}\right)$. Thus it is easy to obtain

$$\nabla \times \mathbf{g} = \frac{-2\iota f^2(r)}{r^2} e_p + \frac{2}{r} \frac{\partial f^2(r)}{\partial r} \frac{\iota(1 - \cos \theta)}{r \sin \theta} e_\theta.$$

(5.25)

It should be noted that the divergence $\nabla \cdot (\nabla \times \mathbf{g})$ no longer vanishes at some spatial points (singularities) since here the gravitomagnetic vector potential is nonanalytic, which results from the presence of the gravitomagnetic monopoles. Hence the three-dimensional volume integral of $\nabla \cdot (\nabla \times \mathbf{g})$ is

$$\int_V \nabla \cdot (\nabla \times \mathbf{g}) \, dV = \oint_S (\nabla \times \mathbf{g}) \cdot dS = -8\pi \iota.$$

(5.26)

It follows from Eq.(5.23) and (5.26) that the relation between $\iota''$ and the NUT parameter $\iota$ is given

$$\iota'' = -2\pi \iota.$$

(5.27)

It should be emphasized that the Schwarzschild solution also satisfies the field equation (5.18) without source term, i.e., $\tilde{G}^{\mu\nu} = 0$. Someone may thus argue that the Schwarzschild solution is just the empty-space generalization of the solutions of Eq.(5.18). This situation is similar to the case of the NUT solution $g_{\mu\nu}(m = 0, \iota \neq 0)$, which was also thought of as the empty-space generalization of the solutions of Einstein’s equation [4].

How can we look upon the two solutions $g_{\mu\nu}(m = 0, \iota \neq 0)$ and $g_{\mu\nu}(m \neq 0, \iota = 0)$ in the NUT spacetime? It is believed that the vacuum equation ($\tilde{G}^{\mu\nu} = 0$) of dual matter may be viewed as the Bianchi identity of Einstein’s field equation, and Einstein’s vacuum equation is in turn the Bianchi identity of the field equation (5.18) of gravitomagnetic matter. For this reason, even though the solution $g_{\mu\nu}(m \neq 0, \iota = 0)$ is truly the one to Einstein’s field equation, it can also agree with the equation $\tilde{G}^{\mu\nu} = 0$ (Bianchi identity). So, it may be argued that the solution $g_{\mu\nu}(m \neq 0, \iota = 0)$ is the empty-space generalization of the solution of vacuum equation $\tilde{G}^{\mu\nu} = 0$. Likewise, the reason for Newman et al. to think of $g_{\mu\nu}(m = 0, \iota \neq 0)$ as the solution of Einstein’s vacuum equation [4] lies in that Einstein’s vacuum equation is just the Bianchi identity of Eq.(5.18) of gravitomagnetic matter. In fact, $g_{\mu\nu}(m = 0, \iota \neq 0)$ is only the one of the
solutions of Eq.(5.18), and furthermore, all the solutions satisfying Eq.(5.18) will also agree with Einstein’s vacuum equation (Bianchi identity). In this sense, the gravitomagnetic charge is truly required of its own gravitational field equation for us to treat its dynamical problems.

In addition, we consider briefly the stationary cylindrically symmetric exact solution of field equation (5.23). Suppose that the form of linear element describing the stationary cylindrically symmetric gravitomagnetic field is given by
\[ ds^2 = (dx^0)^2 - dx^2 - dy^2 - dz^2 + 2g_{0x}(y)dx^0 dx + 2g_{0y}(x)dx^0 dy, \]
where we assume that the gravitomagnetic potentials \( g_{0x} \) and \( g_{0y} \) are the functions with respect to \( y \) and \( x \), respectively. For the cylindrically symmetric nonvanishing uniform gravitomagnetic field produced by the gravitomagnetic charges (with a nonvanishing surface density of \( \sigma_M \)) present only in the \( x-y \) plane of \( z = 0 \), the gravitomagnetic field, \( \mathbf{B}_g \), which is defined as \( \nabla \times \mathbf{g} \), may be written
\[ (\mathbf{B}_g)_z = \frac{\sigma_M}{2}, \]
It follows from Eq.(5.30) that the direction of \( \mathbf{B}_g \) is parallel to the \( z \)-axis. The metric components of the uniform gravitomagnetic field, \( g_{0x}, g_{0y} \), are therefore readily obtained as follows
\[ g_{0x} = \frac{B_x}{2} y, \quad g_{0y} = -\frac{B_y}{2} x, \]
with \( B_y = \frac{\sigma_M}{2} \).

In order to obtain the contravariant metric \( g^{\mu \nu} \), we calculate the inverse matrix of the metric \((g_{\mu \nu})\), and the result is given as follows
\[ (g^{\mu \nu}) = \frac{1}{1 + g^0_{0x} + g^0_{0y}} \begin{pmatrix} 1 & g_{0x} & g_{0y} & 0 \\ g_{0x} & -1 + g_{0y}^2 & 0 & 0 \\ g_{0y} & 0 & -1 + g_{0x}^2 & 0 \\ 0 & 0 & 0 & -1 + g^2_{0x} + g^2_{0y} \end{pmatrix}. \]

E. Dual spin connection

Here we will consider the covariant derivative of Dirac field composed of gravitomagnetic charge in the local frame. Assume that an observer moves with proper three-acceleration \( \mathbf{a} \) and proper three-rotation \( \mathcal{J} \) inside an inertial frame of reference. Note that here the quantities and symbols just refer to those in the reference [16]. It is believed that the covariant derivative of gravitomagnetic-charge Dirac field is given as follows
\[ D_\mathcal{a} = \partial_\mathcal{a} - \frac{i}{16} \sigma_{\beta \gamma} \epsilon_{\beta \gamma} \lambda^\tau \Gamma^\mu_\lambda \tau \mathcal{a}. \]
Now we discuss the dual spin connection \(-\frac{i}{16} \sigma_{\beta \gamma} \epsilon_{\beta \gamma} \lambda^\tau \Gamma^\mu_\lambda \tau \mathcal{a}\). By using the following relations [16]
\[ \Gamma^i_{j0} = -\frac{\varepsilon_{ijk} \omega^k}{c^2}, \quad \Gamma^0_{00} = -\frac{a^0}{c^2}, \quad \Gamma^0_0 = \Gamma_{00} = 0, \]
\[ \frac{1}{2} \sigma^{ij} = -\frac{\mathbf{S}}{h}, \quad i \sigma^0 = \hat{\mathcal{a}}, \]
one can verify that
\[ -\frac{i}{16} \left( \sigma_{bc} \epsilon_{bc} ij \Gamma^i_{j0} + 2 \sigma_{bc} \epsilon_{bc} \partial_0 \Gamma_0^0 \right) = \frac{1}{1 + \frac{a^0}{c^2}} \left( \frac{i}{16} \sigma_{bc} \epsilon_{bc} ij \epsilon_{ijk} \frac{\omega^k}{c} + \frac{i}{8} \sigma_{bc} \epsilon_{bc} \frac{u^l a^l}{c^2} \right) \]
\[ = \frac{1}{1 + \frac{a^0}{c^2}} \left( \frac{1}{4c} \mathcal{J} \cdot \hat{\mathcal{a}} - \frac{i}{2he^2} \mathbf{a} \cdot \mathbf{S} \right), \]
where the Latin indices run over 1, 2, 3. Here \( \hat{\mathcal{a}} \) and \( \mathbf{S} \) denote the velocity operator and spin operator of Dirac field, respectively.

It is well known that a Dirac particle in a noninertial frame of reference will undergo the spin-rotation coupling \((\mathcal{J} \cdot \mathbf{S})\) [17,18] and velocity-acceleration coupling \((\mathbf{a} \cdot \hat{\mathcal{a}})\) [16]. However, it follows from (5.34) that for a gravitomagnetic-charge Dirac field, it experiences velocity-rotation coupling \((\mathcal{J} \cdot \hat{\mathcal{a}})\) and spin-acceleration coupling \((\mathbf{a} \cdot \mathbf{S})\), which are just the dual interactions of the above two couplings.
VI. DUAL METRIC AND DUAL CURRENT DENSITIES

A. Dual vierbein field and dual metric

It can be shown that in the weak gravitational field, Dirac matrices $\gamma^a$’s is rewritten as $\gamma^0 = \left(1 + \frac{\Phi}{2}\right)\beta$, $\gamma^i = g^{0i}\beta + \gamma^i_M$ \((i = 1, 2, 3)\) [1]. Here $\beta$ and $\gamma^i_M$ are the Dirac matrices in the flat Minkowski spacetime. $\phi$ and $g^{0i}$ can be viewed as the gravitational scalar potential and gravitomagnetic vector potentials. By using $\gamma^a = e^{\mu a}\gamma_\mu$ \((a\) denotes the indices of coordinate in Minkowski spacetime), one can obtain the relationship between the vierbein field $e^{\mu a}$ and the gravitational potential and gravitomagnetic vector potentials: $\epsilon^{\mu 0} \simeq 1 + \frac{\Phi}{2}$ and $\epsilon^{0\mu} \simeq g^{0\mu}$ \((\mu = 1, 2, 3)\). Here the dual vierbein field $\tilde{e}_{\mu a}$ is so defined that $\tilde{e}_{\mu a}$ and $\tilde{e}_{\mu a}$ satisfy the following relation

$$\partial_\mu \tilde{e}_{\nu a} - \partial_\nu \tilde{e}_{\mu a} = \frac{1}{2} \epsilon^{\mu \nu}_{\alpha \beta} \left( \partial_\alpha e_{\beta a} - \partial_\beta e_{\alpha a} \right).$$  \hspace{1cm} (6.1)

Moreover, the dual Dirac matrices can be defined as $\tilde{\gamma}_\mu = \tilde{e}_{\mu a}\gamma^a$. Multiplying the two sides of the above equation by $\gamma^a$, one can arrive at

$$\partial_\mu \tilde{\gamma}_\nu - \partial_\nu \tilde{\gamma}_\mu = \frac{1}{2} \epsilon^{\mu \nu}_{\alpha \beta} \left( \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha \right),$$  \hspace{1cm} (6.2)

which is the relation between the dual Dirac matrices $\tilde{\gamma}_\mu$ and the regular Dirac matrices $\gamma_\mu$.

The connection between the metric $g_{\mu \nu}$ and the flat metric $\eta_{ab}$ is $g_{\mu \nu} = e_{\mu}^a e_{\nu}^b \eta_{ab}$. Correspondingly, the relation between the dual metric $\tilde{g}_{\mu \nu}$ and the flat metric $\eta_{ab}$ may be $\tilde{g}_{\mu \nu} = e_{\mu a} \tilde{e}_{\nu b} \eta_{ab}$. It should be noted that even though we introduce the dual metric $\tilde{g}_{\mu \nu}$, in general, indices of tensors are raised and lowered still with the metric tensor $g_{\mu \nu}$ rather than with the dual metric $\tilde{g}_{\mu \nu}$.

In this subsection, we will discuss the dual covariant derivatives, $\tilde{\nabla}_\mu \psi$, of Fermionic field $\psi$ [19]. The dual spin connection of $\tilde{\nabla}_\mu \psi = \left( \partial_\mu + i\tilde{B}_\mu \right) \psi$ is defined by

$$\tilde{B}_\mu = \frac{1}{16} \epsilon_{\beta \gamma} e^{a \beta} e^{\alpha}_{\\mu} e^{c a}_{\\nu} \sigma^{b d} \tilde{\epsilon}_{\nu a} \tilde{e}_{\mu a},$$  \hspace{1cm} (6.3)

where $\sigma^{bd} = \frac{i}{2} \left[ \gamma^b, \gamma^d \right]$. It is essentially significant to construct the Lagrangian density of Fermionic-field gravitomagnetic charge, which may be given

$$\mathcal{L}_F = \sqrt{-g} \left\{ \frac{i}{2} \tilde{e}^{\mu a} \left[ \bar{\psi} \tilde{\gamma}_a \tilde{\nabla}_\mu \psi - \left( \tilde{\nabla}_\mu \bar{\psi} \right) \tilde{\gamma}_a \psi \right] - \tilde{m} \bar{\psi} \psi \right\},$$  \hspace{1cm} (6.4)

where $\tilde{m}$ denotes the dual mass of gravitomagnetic monopole, and $\tilde{\nabla}_\mu \bar{\psi} = \left( \partial_\mu - i\tilde{B}_\mu \right) \bar{\psi}$. The corresponding classical field equations of $\psi$ and $\bar{\psi}$ are as follows

$$\begin{align*}
\frac{i}{2} \tilde{e}^{\mu a} \tilde{\gamma}_a \tilde{\nabla}_\mu \psi + \frac{i}{2} \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \tilde{e}^{\mu a} \tilde{\gamma}_a \psi \right) - \tilde{m} \bar{\psi} \psi &= 0, \\
\frac{i}{2} \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \tilde{e}^{\mu a} \bar{\psi} \tilde{\gamma}_a \phi \right) - \frac{i}{2} \left( \tilde{\nabla}_\mu \bar{\psi} \right) \tilde{e}^{\mu a} \tilde{\gamma}_a \bar{\psi} - \tilde{m} \bar{\psi} &= 0.
\end{align*}$$  \hspace{1cm} (6.5)

It follows from Eqs.(6.5) that the conservation law of the dual current density is

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \tilde{e}^{\mu a} \tilde{\gamma}_a \psi \right) = 0,$$  \hspace{1cm} (6.6)

where the dual current density of dual mass is

$$\tilde{j}^\mu = i \tilde{\psi} \tilde{e}^{\mu a} \tilde{\gamma}_a \psi = \tilde{e}^{\mu a} j_a,$$  \hspace{1cm} (6.7)

which will be considered further in the following.

In addition, the Lagrangian density of the Bosonic-field gravitomagnetic matter may be of the form

$$\mathcal{L}_B = \sqrt{-g} \left[ \tilde{e}^{\mu a} \left( \partial_\mu \varphi^* \partial_a \varphi + \partial_a \varphi^* \partial_\mu \varphi \right) - \tilde{m}^2 \varphi^* \varphi \right].$$  \hspace{1cm} (6.8)
The corresponding classical field equations are

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \dot{e}^{\mu a} \partial_\mu \varphi \right) + \tilde{m}^2 \varphi = 0, \\
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \dot{e}_a^{\mu} \partial_\mu \varphi^* \right) + \tilde{m}^2 \varphi^* = 0.
\]

(6.9)

It follows that the conservation law of the dual current density takes the form

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} \dot{e}^{\mu a} \left( \varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^* \right) \right] = 0.
\]

(6.10)

Thus the dual current density of Bosonic-field gravitomagnetic matter can be written as

\[
\tilde{j}^\mu = \frac{1}{2i} \dot{\epsilon}^{\mu a} \left( \varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^* \right) = \dot{\epsilon}^{\mu a} j_a.
\]

(6.11)

Note that here the index \( a \) refers to the one in the flat Minkowski spacetime.

It is shown that in the above classical field equations of Fermionic and Bosonic field, gravitomagnetic monopole possesses the dual mass rather than the mass, namely, the concept of the mass is of no significance for the gravitomagnetic matter. Dual mass has its own gravitational nature and features different much from that of the mass. In this sense, matter may be classified into two categories, \( \text{i.e.} \), the regular matter and dual matter. The former has \textit{mass} while the latter has \textit{dual mass}.

### B. Dual current densities

In accordance with the expression (6.7), the dual velocity (or current density) of dual mass can be defined as \( \dot{U}_\mu = \dot{e}_\mu^a U_a \). To compare \( \partial_\mu \dot{U}_\nu - \partial_\nu \dot{U}_\mu \) with \( \partial_\mu U_\nu - \partial_\nu U_\mu \) is physically interesting, which is valuable for us to define a new kind of the dual vierbein field. The calculation of \( \partial_\mu \dot{U}_\nu - \partial_\nu \dot{U}_\mu \) and \( \partial_\mu U_\nu - \partial_\nu U_\mu \) yields

\[
\partial_\mu \dot{U}_\nu - \partial_\nu \dot{U}_\mu = \left[ \partial_\mu \dot{e}^{\nu a} - \partial_\nu \dot{e}^{\mu a} \right] (\dot{e}^{\mu a} \partial_\mu - \dot{e}^{\nu a} \partial_\mu) U^a, \\
\partial_\mu U_\nu - \partial_\nu U_\mu = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} (\partial_\alpha \dot{e}_{\beta a} - \partial_\beta \dot{e}_{\alpha a} - (\epsilon^{\alpha a} \partial_\beta - \epsilon^{\beta a} \partial_\alpha) \partial_\mu) U^a.
\]

(6.12)

Here we will adopt an alternative definition of the dual vierbein field \( \dot{e}_{\mu a} \), \( \text{i.e.} \),

\[
\partial_\mu \dot{e}^{\nu a} - \partial_\nu \dot{e}^{\mu a} = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} \left[ \partial_\alpha \dot{e}_{\beta a} - \partial_\beta \dot{e}_{\alpha a} - (\epsilon^{\alpha a} \partial_\beta - \epsilon^{\beta a} \partial_\alpha) \partial_\mu \right],
\]

(6.13)

which is different from (6.1). Consequently, this definition will lead to the following duality relation

\[
\partial_\mu \dot{e}^{\nu a} - \partial_\nu \dot{e}^{\mu a} = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} \left[ \partial_\alpha \dot{e}_{\beta a} - \partial_\beta \dot{e}_{\alpha a} - (\epsilon^{\alpha a} \partial_\beta - \epsilon^{\beta a} \partial_\alpha) \partial_\mu \right].
\]

(6.14)

Thus, it follows from Eq.(6.13) and (6.14) that Eq.(6.12) can be rewritten as \( \partial_\mu \dot{U}_\nu - \partial_\nu \dot{U}_\mu = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} (\partial_\alpha \dot{U}_{\beta a} - \partial_\beta \dot{U}_{\alpha a}) \) and \( \partial_\mu U_\nu - \partial_\nu U_\mu = -\frac{1}{2} \epsilon^{\mu \nu \alpha \beta} (\partial_\alpha \dot{U}_{\beta a} - \partial_\beta \dot{U}_{\alpha a}) \). So, the source tensors \( \mathcal{K}_{\mu \nu} \) and \( \mathcal{H}_{\mu \nu} \) defined in Sec. V can also be expressed in terms of the dual vierbein field (or dual current density), \( \text{i.e.} \), there is a duality relation between \( \mathcal{K}_{\mu \nu} \), \( \mathcal{H}_{\mu \nu} \) and \( \tilde{K}_{\mu \nu}, \tilde{H}_{\mu \nu} \).

\[
\mathcal{K}_{\mu \nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu = -\frac{1}{2} \epsilon^{\mu \nu \alpha \beta} (\partial_\alpha \dot{U}_{\beta a} - \partial_\beta \dot{U}_{\alpha a}) \equiv -\tilde{H}_{\mu \nu}, \\
\mathcal{H}_{\mu \nu} \equiv \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} (\partial_\alpha U_{\beta a} - \partial_\beta U_{\alpha a}) = \partial_\mu \dot{U}_\nu - \partial_\nu \dot{U}_\mu \equiv \tilde{K}_{\mu \nu}.
\]

(6.15)

The above discussion shows that if the vierbein field \( \dot{e}_{\mu a} \) is defined via Eq.(6.13), then there is no difference between the source tensor \( S_{\mu \nu} \) and the dual source tensor \( \tilde{S}_{\mu \nu} \). Here the dual source tensor \( \tilde{S}_{\mu \nu} \) is the linear combination of \( \tilde{K}_{\mu \nu} \) and \( \tilde{H}_{\mu \nu} \). In a word, both the current density and the dual current density can be used to define the source tensor of gravitomagnetic matter in the gravitational field equation.
VII. THE FIVE-DIMENSIONAL CASE

In order to find the potential connection between the gravitomagnetic charge and the magnetic charge, we will investigate the five-dimensional dynamics of gravitomagnetic matter. In the five-dimensional case, the dual Ricci tensor is

\[ \tilde{\mathcal{R}}_{AB} = \frac{1}{4} \epsilon_A^{CDE} \frac{\partial}{\partial x^E} \left( \frac{\partial \tilde{g}_{BC}}{\partial x^D} - \frac{\partial \tilde{g}_{DB}}{\partial x^C} \right), \]  

(7.1)

where \( A, B, C, D, E \) run over \( 0 - 4 \), instead of \( 0 - 3 \). In what follows Greek indices run over \( 0 - 3 \). By using the cylinder condition (i.e., the derivative of the metric with respect to \( x^4 \) is vanishing), we can obtain

\[ \tilde{\mathcal{R}}_{\mu\nu} = \frac{1}{4} \epsilon_{\mu}^{\lambda\sigma\tau} \partial_{\tau} (\partial_{\sigma} \tilde{g}_{\lambda\nu} - \partial_{\lambda} \tilde{g}_{\sigma\nu}) + \frac{1}{2} \epsilon_{\mu}^{\lambda\sigma\tau} \partial_{\tau} \partial_{\sigma} \tilde{g}_{\lambda\nu}, \]

\[ \tilde{\mathcal{R}}_{\mu4} = \frac{1}{4} \epsilon_{\mu}^{\lambda\sigma\tau} \partial_{\tau} (\partial_{\sigma} \tilde{g}_{4\nu} - \partial_{\lambda} \tilde{g}_{4\nu}), \]  

(7.2)

By using the Kaluza mechanism, the five-dimensional metric \( \tilde{g}_{AB} \) is adopted as follows [20]

\[ (\tilde{g}_{AB}) = \left( g_{\alpha\beta} + \kappa^2 \phi^2 A_\alpha A_\beta \right), \]  

(7.3)

where \( \phi \) and \( A_\alpha \) denote a certain (constant) scalar field and the electromagnetic field, respectively. Thus Eqs.(7.2) can be rewritten as

\[ \tilde{\mathcal{R}}_{\mu\nu} = \mathcal{R}_{\mu\nu} + \kappa^2 \phi^2 \epsilon_{\mu}^{\lambda\sigma\tau} \partial_{\tau} \partial_{\sigma} A_\nu, \]

\[ \tilde{\mathcal{R}}_{\mu4} = \frac{\kappa^2 \phi^2}{4} \epsilon_{\mu}^{\lambda\sigma\tau} \partial_{\tau} F_{\sigma\lambda}. \]  

(7.4)

So according to the field equation (5.21), i.e., \( \tilde{\mathcal{R}}_{\mu4} = \frac{1}{\sqrt{-g}} \tilde{\mathcal{Y}}_{\mu4} \), the second equation in Eqs.(7.4) can be rewritten as

\[ \frac{\kappa^2 \phi^2}{4} \epsilon_{\mu}^{\lambda\sigma\tau} \partial_{\tau} F_{\sigma\lambda} = \frac{1}{\sqrt{-g}} \tilde{\mathcal{Y}}_{\mu4}, \]  

(7.5)

which is actually the electromagnetic field equation of magnetic charge. This, therefore, means that the (\( \mu4 \)) component of the five-dimensional gravitational field equation of gravitomagnetic matter can be reduced to the electromagnetic field equation of magnetic monopole via the Kaluza mechanism. Thus the five-dimensional dynamics of gravitomagnetic monopole provides us with a potentially unified way to treat the magnetic monopole and gravitomagnetic monopole.

VIII. DUAL THEORY

In electrodynamics, it is shown that the dynamics of magnetic charge is just the dual theory to that of electric charge (for a concise formalism, see Appendix 6 to this paper). This phenomenon arises also in gravity theory. Now we are therefore led to consider in detail the duality relationship between the dynamical theories of gravitomagnetic charge (dual mass) and gravitoelectric charge (mass). First we define a second-rank antisymmetric gravitational potential tensor \( h^{\mu\nu} \) as an alternative to the description of the gravitational field. Such antisymmetric gravitational potential tensor \( h^{\mu\nu} \) may be so defined that the following expression

\[ \epsilon^{\mu\nu\tau\omega} = \sigma(x) \left( h^{\mu\omega} h^{\tau\nu} + h^{\tau\mu} h^{\omega\nu} + h^{\nu\mu} h^{\tau\omega} \right) \]  

(8.1)

is satisfied, where \( \sigma(x) \) is a scalar function to be determined.

Define a functional of \( h^{\lambda\sigma} \), say, \( \mathcal{N}_{\tau\omega\rho\mu} (h^{\lambda\sigma}) \), which is assumed to contain the derivatives of \( h^{\lambda\sigma} \) up to second order (i.e., it contains \( \partial_\rho h^{\lambda\sigma}, \partial_\eta h^{\lambda\sigma} \))\(^8\). Based on this, we assume that the gravitational Lagrangian density is chosen to be

\[^8\text{It is assumed that the field equations associated with the gravitational field is often of second differential order.}\]
By using the Euler-Lagrange equation, if the following equation is satisfied

$$\frac{\partial \mathcal{L}}{\partial h^{\lambda\sigma}} - \left[ \frac{\partial \mathcal{L}}{\partial h^{\lambda\sigma}_{\;\;\;\theta}} \right]_{,\theta} + \left[ \frac{\partial \mathcal{L}}{\partial h^{\lambda\sigma}_{\;\;\;\theta\eta}} \right]_{,\theta\eta} = \sqrt{-g} g^{\rho\nu} \left( N_{\lambda\omega\nu\sigma} - N_{\sigma\omega\nu\lambda} \right),$$

then it is believed that the functional $N_{\tau\omega\nu\mu}(h^{\lambda\sigma})$ is related closely to the dual Riemann tensor $\tilde{R}_{\tau\omega\nu\mu}$, namely, keeping in mind Eq.(8.2), we can set

$$\tilde{R}_{\tau\omega\nu\mu} = N_{\tau\omega\nu\mu}(h^{\lambda\sigma}),$$

which results from the expression (4.2), $\frac{1}{2} \varepsilon^{\mu\tau\omega\nu} \tilde{R}_{\tau\omega\nu\mu} = -\mathcal{R}$, in Sec. IV.

The roles of Eq.(8.1), (8.3) and (8.4) is as follows: Eq.(8.1) can be used to determine the scalar function $\sigma(x)$; Eq.(8.3) is applied to the determination of the form of the functional $N_{\tau\omega\nu\mu}(h^{\lambda\sigma})$; Eq.(8.4) can be employed to find the relationship between the antisymmetric gravitational potential tensor $h^{\lambda\sigma}$ and the symmetric metric tensor $g^{\lambda\sigma}$.

Because of the identity, $\frac{1}{2} \varepsilon^{\mu\tau\omega\nu} \tilde{R}_{\tau\omega\nu\mu} = -\mathcal{R}$, suggested in Sec. III, the substitution of Eq.(8.4) into this identity leads to $\frac{1}{2} \left( G^{\lambda\mu}_{\nu\sigma} + G^{\lambda\sigma}_{\nu\mu} \right) = -\frac{1}{2} \varepsilon^{\mu\tau\omega\nu} N_{\tau\omega\nu\mu}(h^{\lambda\sigma})$. Thus Einstein’s field equation is of the following form (rewritten in terms of the antisymmetric gravitational potential tensor $h^{\lambda\sigma}$ rather than of the symmetric metric tensor $g^{\lambda\sigma}$)

$$\frac{1}{2} \varepsilon^{\mu\tau\omega\nu} N_{\tau\omega\nu\mu}(h^{\lambda\sigma}) = -\kappa T^{\mu\nu},$$

with $T^{\mu\nu}$ being the energy-momentum tensor of matter. In this sense, we have found the duality relationship between the theories of mass and dual mass, and therefore developed a dual formalism (in terms of the antisymmetric gravitational potential tensor $h^{\lambda\sigma}$) which can express Einstein’s equation within its framework.

**IX. CONCLUDING REMARKS**

Not all of the nonanalytic solutions of Einstein’s vacuum equation belong to the contribution of gravitomagnetic charge. In an attempt to investigate the gravitational field caused by gravitomagnetic charge currents, we meet, however, with difficulties in selecting the “true” metric produced by gravitomagnetic charge from the solutions of the Bianchi identity (i.e., Einstein’s vacuum equation). For this reason, one should have a dynamics of gravitomagnetic charge. To achieve this aim, this paper gives a concise presentation of the mathematical formalism of the field equation of gravitomagnetic matter. We derive the antisymmetric dual Einstein tensor from the variational principle with the dual curvature scalar, and construct an antisymmetric source tensor, which appears on the right-handed side of the gravitational field equation of gravitomagnetic matter. It is also demonstrated that the topological dual charge needs its own gravitational dual spin connection and dual current density (and hence dual source tensor) constitute the dynamical framework of gravitomagnetic matter.

It follows from the point of view of the classical field equation that matter may be classified into two categories, i.e., the gravitoelectric and gravitomagnetic matter. It is clear that the concept of mass is of no significance for the gravitomagnetic matter. In this sense, we can say that the gravitomagnetic matter possesses the topological dual mass [1,2].

Lynden-Bell and Nouri-Zonoz considered briefly the problem as to how general relativity must be modified if the gravitomagnetic monopole truly exists in the universe [8]. They thought that the space where the gravitomagnetic
The monopole is present may possess the unsymmetrical affine connections [8]. Namely, their consideration means that the extra gravitational potentials should be introduced to allow for the gravitomagnetic charge. Indeed, the topological dual mass will introduce some additional new things in gravity theory, e.g., it will give rise to the topological singularities in spacetime, which is described by the nonanalytic metric functions. But there is no essential reason for cluing us on introducing the extra gravitational potentials. To deal with the dynamics of gravitomagnetic charge, the only thing we should do first is to introduce a new field equation (e.g., (5.17) and (5.18)), the vacuum case (without the sources) of which is just the Bianchi identity of Einstein’s equation.

Gravitomagnetic charge has some interesting relativistic quantum gravitational effects, e.g., the gravitational anti-Meissner effect, which may serve as an interpretation of the smallness of the observed cosmological constant. In accordance with quantum field theory, vacuum possesses infinite zero-point energy density due to the vacuum quantum fluctuations; whereas according to Einstein’s theory of General Relativity, infinite vacuum energy density yields the divergent curvature of space-time, namely, the space-time of vacuum is extremely curved. Apparently it is in contradiction with the practical fact [21]. In the context of quantum field theory a cosmological constant corresponds to the energy density associated with the vacuum and then the divergent cosmological constant may result from the infinite energy density of vacuum quantum fluctuations. However, a diverse set of observations suggests that the universe possesses a nonzero but very small cosmological constant [21–24]. How can we give a natural interpretation for the above paradox? Here, provided that vacuum matter is perfect fluid, which leads to the formal similarities between the weak-gravity equation in perfect fluid and the London’s electrodynamics of superconductivity, we suggest a potential explanation by using the cancelling mechanism via gravitational anti-Meissner effect: the gravitoelectric field (Newtonian field of gravity) produced by the gravitoelectric charge (mass) of the vacuum quantum fluctuations is exactly cancelled by the gravitoelectric field due to the induced current of the gravitomagnetic charge of the vacuum quantum fluctuations; the gravitomagnetic field produced by the gravitomagnetic charge (dual mass) of the vacuum quantum fluctuations is exactly cancelled by the gravitomagnetic field due to the induced current of the gravitoelectric charge (mass current) of the vacuum quantum fluctuations. Thus, at least in the framework of weak-field approximation, the extreme space-time curvature of vacuum caused by the large amount of the vacuum energy does not arise, and the gravitational effects of cosmological constant is eliminated by the contributions of the gravitomagnetic charge (dual mass). If gravitational anti-Meissner effect is of really physical significance, then it is necessary to apply this effect to the early universe where quantum and inflationary cosmologies dominate the evolution of the universe. Study of the geometric property in quantum regimes is an interesting and valuable direction.

We think that there might exist the formation/creation mechanism of gravitomagnetic charge in the gravitational interaction, just as some prevalent theories provide the theoretical mechanism of the existence of magnetic monopole in various gauge interactions [25]. But so far it has to yield any definite theories of how the gravitomagnetic matter forms (we think that it may arise from the interactions associated with the Chern-Simons gauge field). The magnetic monopole in the grand unified theory plays an important role in the early universe. The same may be said of gravitomagnetic monopole. If the gravitomagnetic monopoles truly exist in the universe, it will inevitably give rise to significant consequences and may also play an essential role in astrophysics and cosmology. In this paper we investigated the possibility of constructing a field equation that governs the gravitational field in the vicinity of this topological dual matter. We may conclude that both the dynamics of gravitomagnetic matter and other related topics associated with, for example, the creation mechanism of gravitomagnetic charge deserve further consideration.

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APPENDICES

Appendix 1. On the Levi-Civita tensor

The Levi-Civita tensor $\epsilon_{\mu\nu\alpha\beta}$ in the flat Minkowski spacetime is so defined that it changes sign when any pair of indices are interchanged and $\epsilon_{0123} = +1, \epsilon^{0123} = -1$. The Levi-Civita tensor $\epsilon_{0123}$ and $\epsilon^{0123}$ in the curved spacetime are defined to be $\epsilon_{0123} = \sqrt{-g} \epsilon_{0123}$ and $\epsilon^{0123} = \sqrt{-g} \epsilon^{0123}$, respectively, where $g$ is the determinant of the matrix $g_{\mu\nu}$.

One of the most important properties of the Levi-Civita tensor is that its covariant derivative is vanishing, which may be proved as follows:

The covariant derivative of $\epsilon_{\mu\nu\sigma\lambda}$ is given

$$
\epsilon_{\mu\nu\sigma\lambda} = \frac{\partial}{\partial x^\tau} \epsilon_{\mu\nu\sigma\lambda} - \Gamma^\lambda_{\nu\sigma} \epsilon_{\lambda\mu\rho\tau} - \Gamma^\lambda_{\nu\sigma} \epsilon_{\mu\lambda\rho\tau} - \Gamma^\lambda_{\nu\sigma} \epsilon_{\mu\nu\lambda\tau} - \Gamma^\lambda_{\tau\sigma} \epsilon_{\mu\nu\lambda\tau}.
$$

(A1)
As an illustrative example, we will calculate $\epsilon_{0123,\sigma}$ only. Here $\epsilon_{0123} = \sqrt{-g} \epsilon_{0123} = \sqrt{-g}$. It is easily obtained that

$$\frac{\partial}{\partial x^\tau} \epsilon_{0123} = \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \frac{\partial}{\partial x^\tau} g_{\alpha\beta}.$$ 

If $\mu, \nu, \tau, \sigma = 0, 1, 2, 3$, respectively, then using the antisymmetric property of $\epsilon_{\mu\nu\sigma\tau}$, one can arrive at

$$\Gamma^\lambda_{\mu\sigma} \epsilon_{\lambda\nu\sigma\tau} \rightarrow \Gamma^\lambda_{\mu\sigma} \epsilon_{\lambda\nu\sigma\tau} = \sqrt{-g} \Gamma^1_{1\sigma},$$

$$\Gamma^\lambda_{\theta\sigma} \epsilon_{\lambda\nu\theta\tau} \rightarrow \Gamma^\lambda_{\theta\sigma} \epsilon_{\lambda\nu\theta\tau} = \sqrt{-g} \Gamma^3_{3\sigma},$$

and

$$\Gamma^\lambda_{\tau\sigma} \epsilon_{\lambda\nu\tau\theta} \rightarrow \Gamma^\lambda_{\tau\sigma} \epsilon_{\lambda\nu\tau\theta} = \sqrt{-g} \Gamma^1_{1\sigma}.$$

So, it follows from (A1) and (A2) that the covariant derivative of the Levi-Civita tensor $\epsilon_{0123}$ is

$$\epsilon_{0123,\sigma} = \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \frac{\partial}{\partial x^\sigma} g_{\alpha\beta} - \sqrt{-g} \Gamma^\lambda_{\lambda\sigma} = 0.$$  \hspace{1cm} (A3)

An alternative derivation of the above result is given as follows:

The transformation relation between the Levi-Civita tensors in the noninertial and local inertial frames of reference satisfies $\epsilon_{\mu\nu\sigma\tau} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \frac{\partial \xi^\gamma}{\partial x^\sigma} \frac{\partial \xi^\delta}{\partial x^\tau} \epsilon_{\alpha\beta\gamma\delta}$. So, the derivative of $\epsilon_{\mu\nu\sigma\tau}$ reads

$$\frac{\partial \epsilon_{\mu\nu\sigma\tau}}{\partial x^\sigma} = \Gamma^\lambda_{\mu\sigma} \epsilon_{\lambda\nu\sigma\tau} + \Gamma^\lambda_{\nu\sigma} \epsilon_{\mu\lambda\sigma\tau} + \Gamma^\lambda_{\sigma\tau} \epsilon_{\mu\nu\lambda\sigma} + \Gamma^\lambda_{\tau\sigma} \epsilon_{\mu\nu\lambda\tau}.$$

The insertion of $\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \Gamma^\lambda_{\mu
u} \epsilon_{\lambda\alpha} \sigma\tau$ into (A4) yields

$$\frac{\partial \epsilon_{\mu\nu\sigma\tau}}{\partial x^\sigma} = \Gamma^\lambda_{\mu\sigma} \epsilon_{\lambda\nu\sigma\tau} + \Gamma^\lambda_{\nu\sigma} \epsilon_{\mu\lambda\sigma\tau} + \Gamma^\lambda_{\sigma\tau} \epsilon_{\mu\nu\lambda\sigma} + \Gamma^\lambda_{\tau\sigma} \epsilon_{\mu\nu\lambda\tau}.$$

It follows from Eq. (A1) and (A5) that $\epsilon_{\mu\nu\sigma\tau,\sigma} = 0$. Thus we have proven that the covariant derivative of the Levi-Civita tensor truly vanishes.

In the four-dimensional spacetime, the following four identities (which are the inner products of two Levi-Civita tensors) are useful for investigating the dynamics of gravitomagnetic matter:

$$\epsilon_{\mu\nu\alpha\beta} \epsilon^{\mu\nu\alpha\beta} = -4!,$$

$$\epsilon_{\mu\nu\alpha\beta} \epsilon^{\mu\nu\alpha\beta} = -3! g_{\mu\nu},$$

$$\epsilon_{\mu\nu\alpha\beta} \epsilon^{\lambda\nu\alpha\beta} = -2! \left(g_{\mu\lambda} g_{\nu\alpha} - g_{\mu\alpha} g_{\nu\lambda}\right),$$

$$\epsilon_{\mu\nu\alpha\beta} \epsilon^{\lambda\tau\alpha\beta} = - \left( g_{\mu\lambda} g_{\nu\tau} g_{\alpha\sigma} + g_{\mu\tau} g_{\nu\lambda} g_{\alpha\sigma} + g_{\mu\sigma} g_{\nu\lambda} g_{\alpha\tau} + g_{\mu\tau} g_{\nu\sigma} g_{\lambda\alpha} - g_{\mu\sigma} g_{\nu\tau} g_{\lambda\alpha} - g_{\mu\tau} g_{\nu\sigma} g_{\lambda\alpha} - g_{\mu\tau} g_{\nu\tau} g_{\alpha\sigma}\right).$$  \hspace{1cm} (A6)

**Appendix 2.** Proving the first identity of Eq.(3.4)

According to the fourth identity of Eq.(A6), one can arrive at

$$\epsilon^{\theta\tau\omega\nu} \epsilon^\mu_{\tau\lambda\sigma} = \epsilon^{\theta\tau\omega\nu} \epsilon^\mu_{\tau\lambda\sigma} = \epsilon^{\theta\tau\omega\nu} \epsilon_{\mu\lambda\sigma} = - \left(g^{\mu\nu} g^{\beta\lambda} g^{\sigma\nu} + g^{\mu\nu} g^{\theta\lambda} g^{\omega\nu} + g^{\mu\omega} g^{\nu\lambda} g^{\theta\sigma} - g^{\mu\nu} g^{\omega\lambda} g^{\theta\sigma} - g^{\mu\nu} g^{\lambda\nu} g^{\omega\sigma} - g^{\mu\omega} g^{\nu\lambda} g^{\sigma\nu}\right).$$  \hspace{1cm} (A7)

So, the further calculation yields

$$\epsilon^{\theta\tau\omega\nu} \Gamma^{\mu}_{\tau\omega\nu} = \frac{1}{2} \epsilon^{\theta\tau\omega\nu} \epsilon^{\mu\lambda\sigma} R_{\lambda\sigma\omega\nu} = - \left(R^{\theta\mu} - \frac{1}{2} g^{\mu\theta} R\right) = - \left(R^{\theta\mu} - \frac{1}{2} g^{\mu\theta} R\right).$$  \hspace{1cm} (A8)

If the Ricci tensor $R^{\mu\nu}$ and $\Gamma^{\mu\nu}$ are respectively defined to be $R^{\mu\nu} = R^{\mu\nu} - R^{\beta\nu} g_{\beta\mu} - R^{\beta\mu} g_{\beta\nu}$ and the curvature scalar $R$ is obtained via $R_{\omega\nu} = -g_{\alpha\beta} R^{\alpha\beta} = -R$, then we can readily show that this identity

$$\frac{1}{2} \epsilon^{\theta\tau\omega\nu} \Gamma^{\mu}_{\tau\omega\nu} = \frac{1}{2} (G^{\mu\theta} + G^{\theta\mu})$$  \hspace{1cm} (A9)
truly holds.

**Appendix 3.** Showing that \( \frac{1}{2} \int_{\Omega} \sqrt{-g} \epsilon_{\mu \tau}^{\nu \omega} \delta \mathcal{R}_{\tau \omega \mu \nu} d\Omega \) is a surface integral

For the first we obtain the variation of \( \mathcal{R}_{\tau \omega \mu \nu} \)

\[
\delta \mathcal{R}_{\tau \omega \mu \nu} = \frac{\partial}{\partial x^\nu} \delta \Gamma_{\tau, \omega \mu} - \frac{\partial}{\partial x^\mu} \delta \Gamma_{\tau, \omega \nu} - \delta \Gamma_{\omega, \mu} \Gamma_{\lambda, \tau \nu} - \Gamma_{\omega, \nu} \delta \Gamma_{\lambda, \tau \mu} + \delta \Gamma_{\omega, \mu} \Gamma_{\lambda, \tau \nu} + \Gamma_{\omega, \nu} \delta \Gamma_{\lambda, \tau \mu} \\
= \left( \frac{\partial}{\partial x^\nu} \delta \Gamma_{\tau, \omega \mu} - \Gamma_{\omega, \nu} \Gamma_{\lambda, \tau \mu} + \Gamma_{\omega, \mu} \delta \Gamma_{\lambda, \tau \nu} \right) - \left( \frac{\partial}{\partial x^\mu} \delta \Gamma_{\tau, \omega \nu} - \Gamma_{\omega, \mu} \delta \Gamma_{\lambda, \tau \nu} - \Gamma_{\omega, \nu} \delta \Gamma_{\lambda, \tau \mu} \right) \\
+ \Gamma_{\omega, \nu} \delta \Gamma_{\lambda, \tau \mu} - \Gamma_{\omega, \mu} \delta \Gamma_{\lambda, \tau \nu}. \\
\tag{A10}
\]

For the second, we analyze the third term \( \Gamma_{\omega, \nu} \delta \Gamma_{\lambda, \tau \mu} \) on the right-hand side of Eq. (A10). The following calculation is trivial

\[
\epsilon_{\mu \tau}^{\nu \omega} \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \tau \mu} = \frac{1}{2} \epsilon_{\mu \tau}^{\nu \omega} \Gamma_{\lambda, \omega \mu} \delta \eta_{\lambda \nu, \tau \mu} \\
= -\frac{1}{2} \epsilon_{\mu \tau}^{\nu \omega} \Gamma_{\lambda, \omega \mu} \left( \frac{\partial g_{\mu \lambda}}{\partial x^\tau} - \frac{\partial g_{\tau \mu}}{\partial x^\lambda} \right) \\
= -\epsilon_{\mu \tau}^{\nu \omega} \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \tau \mu}. \\
\tag{A11}
\]

In the same manner, we can rewrite \( -\epsilon_{\mu \tau}^{\nu \omega} \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \tau \nu} \) as follows

\[
-\epsilon_{\mu \tau}^{\nu \omega} \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \tau \nu} = \epsilon_{\mu \tau}^{\nu \omega} \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \nu \mu}, \\
\tag{A12}
\]

So, it follows from both Eq. (A11) and (A12) that one can obtain

\[
\epsilon_{\mu \tau}^{\nu \omega} \left( \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \tau \nu} - \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \nu \mu} \right) = \epsilon_{\mu \tau}^{\nu \omega} \left( -\Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \nu \mu} + \Gamma_{\lambda, \omega \mu} \delta \Gamma_{\lambda, \nu \mu} \right), \\
\tag{A13}
\]

which will lead to some changes in the expression for \( \epsilon_{\mu \tau}^{\nu \omega} \delta \mathcal{R}_{\tau \omega \mu \nu} \). Thus the integrand \( \epsilon_{\mu \tau}^{\nu \omega} \delta \mathcal{R}_{\tau \omega \mu \nu} \) may be rewritten as

\[
\epsilon_{\mu \tau}^{\nu \omega} \delta \mathcal{R}_{\tau \omega \mu \nu} = \epsilon_{\mu \tau}^{\nu \omega} \left( \frac{\partial}{\partial x^\nu} \delta \Gamma_{\tau, \omega \mu} - \Gamma_{\omega, \nu} \delta \Gamma_{\lambda, \tau \mu} + \Gamma_{\omega, \mu} \delta \Gamma_{\lambda, \tau \nu} \right) \\
- \epsilon_{\mu \tau}^{\nu \omega} \left( \frac{\partial}{\partial x^\mu} \delta \Gamma_{\tau, \omega \nu} - \Gamma_{\omega, \mu} \delta \Gamma_{\lambda, \tau \nu} - \Gamma_{\omega, \nu} \delta \Gamma_{\lambda, \tau \mu} \right) \\
= \epsilon_{\mu \tau}^{\nu \omega} \left[ (\delta \Gamma_{\tau, \omega \mu})_{\tau \nu} - (\delta \Gamma_{\tau, \omega \nu})_{\tau \mu} \right]. \\
\tag{A14}
\]

By using \( \epsilon_{\alpha \beta}^{\mu \nu} = \epsilon_{\alpha \beta}^{\mu \nu} = 0 \), one can therefore arrive at

\[
\frac{1}{2} \epsilon_{\mu \tau}^{\nu \omega} \delta \mathcal{R}_{\tau \omega \mu \nu} = B^\mu_{\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} B^\mu \right) \quad \text{with} \quad B^\mu = -\epsilon_{\mu \tau}^{\nu \omega} \delta \Gamma_{\tau, \omega \nu}. \\
\tag{A15}
\]

Hence, we have shown that \( \frac{1}{2} \int_{\Omega} \sqrt{-g} \epsilon_{\mu \tau}^{\nu \omega} \delta \mathcal{R}_{\tau \omega \mu \nu} d\Omega \) is truly a surface integral, i.e.,

\[
\frac{1}{2} \int_{\Omega} \sqrt{-g} \epsilon_{\mu \tau}^{\nu \omega} \delta \mathcal{R}_{\tau \omega \mu \nu} d\Omega = \int_{\Omega} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} B^\mu \right) d\Omega, \\
\tag{A16}
\]

which is a four-dimensional volume integral of the divergence \( \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} B^\mu \right) \), and hence has no contribution to the derivation of the dual Einstein tensor.

**Appendix 4.** The variation of the Levi-Civita tensor \( \epsilon_\mu^{\lambda \sigma \tau} \)

With the help of the two expressions \( \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\tau} = \frac{1}{2} \frac{\partial}{\partial x^\tau} - g_{\alpha \beta} \delta g^{\alpha \beta} \) and \( \delta g_{\mu \theta} = -g_{\mu \alpha} g_{\theta \beta} \delta g^{\alpha \beta} \), one can reach

\[
\delta \epsilon_\mu^{\lambda \sigma \tau} = \delta (g_{\mu \delta} \epsilon^{\delta \lambda \sigma \tau}) = \delta g_{\mu \theta} \epsilon^{\theta \lambda \sigma \tau} + g_{\mu \delta} \delta \epsilon^{\theta \lambda \sigma \tau} \\
= \left( -g_{\mu \alpha} g_{\theta \beta} \epsilon^{\theta \lambda \sigma \tau} + \frac{1}{2} g_{\mu \theta} g_{\alpha \beta} \epsilon^{\theta \lambda \sigma \tau} \right) \delta g^{\alpha \beta}. \\
\tag{A17}
\]
Thus we obtain the variation of the Levi-Civita tensor $\epsilon_{\mu}^{\lambda\sigma\tau}$.

Appendix 5. Mid-dual and two-side dual curvature tensors

In Sec. III, we only considered the left-dual and right-dual Riemann (and hence Ricci) curvature tensors, and showed that $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} R_{\mu\tau\lambda\sigma} \equiv 0$, $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} R_{\mu\tau\lambda\sigma} = \tilde{R}_{\nu\tau}$. In fact, there exist other kinds of dual Riemann tensors. See, for example, $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} R_{\mu\tau\lambda\sigma}$. It follows from the expression $R_{\mu\tau\lambda\sigma} = \frac{\partial}{\partial x^\rho} \Gamma^\rho_{\mu\tau\lambda\sigma} - \frac{\partial}{\partial x^\tau} \Gamma^\rho_{\mu\tau\lambda\sigma} = \Gamma^\rho_{\mu\tau\lambda\sigma} + \Gamma^\rho_{\tau\lambda\sigma\mu}$ that

$$R_{\mu\tau\lambda\sigma} = -R_{\tau\mu\lambda\sigma} + (\partial_\lambda \partial_\sigma - \partial_\sigma \partial_\lambda) g_{\mu\tau}. \quad (A18)$$

Thus one can obtain

$$\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} R_{\mu\tau\lambda\sigma} = \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} (\partial_\lambda \partial_\sigma - \partial_\sigma \partial_\lambda) g_{\mu\tau}. \quad (A19)$$

Note that here $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} (\partial_\lambda \partial_\sigma - \partial_\sigma \partial_\lambda) g_{\mu\tau} = \epsilon_{\nu}^{\mu\lambda\sigma} \partial_\lambda \partial_\sigma g_{\mu\tau} = \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \partial_\lambda (\partial_\sigma g_{\mu\tau} - \partial_\mu g_{\sigma\tau}) = -2\tilde{R}_{\nu\tau}$. So, a useful fact that no essential difference exists between $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} R_{\mu\tau\lambda\sigma}$ and $\tilde{R}_{\nu\tau}$ is thus demonstrated. This, therefore, means that we need not take account of $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} R_{\mu\tau\lambda\sigma}$ in considering the dynamics of gravitomagnetic charge.

In what follows, we will discuss the mid-dual and two-side dual curvature tensors. It will be shown that these two kinds of dual tensors are either trivial or equivalent to the left-dual tensor.

The mid-dual Riemann tensor under consideration is $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\nu}$, and the corresponding dual Ricci tensors can be written as

$$g^{\tau\omega} \left( \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\nu} \right) = 0, \quad \tilde{g}^{\mu\nu} \left( \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\nu} \right) = 0, \quad g^{\omega\mu} \left( \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\nu} \right) = \tilde{R}_{\tau\nu}, \ldots \quad (A20)$$

It is readily verified that the mid-dual tensors in (A20) are either trivial or equivalent to the left-dual tensor.

The two-side dual Riemann curvature tensor is defined to be $\frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\omega\nu}$, the corresponding Ricci tensors of which are given as follows

$$g^{\mu\nu} \left( \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\omega\nu} \right) = 0, \quad g^{\nu\tau} \left( \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\omega\nu} \right) = -\tilde{R}_{\nu\mu}, \quad g^{\omega\mu} \left( \frac{1}{2}\epsilon_{\nu}^{\mu\lambda\sigma} \tilde{R}_{\mu\tau\lambda\sigma\omega\nu} \right) = -\frac{1}{2}\tilde{R}_{\tau\mu\nu}, \ldots \quad (A21)$$

which are also trivial or equivalent to the left-dual tensor.

For this reason, in this paper we will not consider further other kinds of dual tensors such as mid-dual and two-side dual curvature tensors.

Appendix 6. The duality relation between the electromagnetic field equations of magnetic and electric charges

The electromagnetic field equations of magnetic and electric charges are of the form (where $j^\nu_e$ and $j^\nu_m$ are the electric and magnetic charge current densities, respectively)

$$\partial_\mu F^{\mu\nu} = j^\nu_e, \quad \partial_\mu \tilde{F}^{\mu\nu} = j^\nu_m \quad (A22)$$

with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $\tilde{F}^{\alpha\beta} = \frac{1}{2}E^{\alpha\beta\gamma} F_{\alpha\beta\gamma}$, $F^{\mu\nu} = -\frac{1}{2}E^{\mu\nu\alpha\beta} \tilde{F}_{\alpha\beta}$. The above electrodynamics is developed in the framework of the electromagnetic field $A_\mu$. In contrast, we can also propose a dual theory to the above one, where the formalism is constructed in the framework of the so-called dual electromagnetic field $\tilde{A}_\mu$, i.e.,

$$\tilde{F}^{\mu\nu} = \partial_\mu \tilde{A}^{\nu} - \partial_\nu \tilde{A}^\mu, \quad F^{\mu\nu} = \frac{1}{2}E^{\mu\nu\alpha\beta} \tilde{F}_{\alpha\beta}. \quad (A23)$$

The above theories are dual to each other and, furthermore, equivalent to one another. This, therefore, implies that one can study the electrodynamics not only in the framework of $A_\mu$ but also in the one of $\tilde{A}_\mu$. In Sec. VIII, the similar dual phenomenon in gravity theory is discussed, where a second-rank antisymmetric tensor $h_{\mu\nu}$ is defined to describe the gravitational field. So Einstein’s field equation can also be expressed inside the framework of $h_{\mu\nu}$, just as the fact that the electrodynamics can be described within the framework of $\tilde{A}_\mu$. 

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