Conserved Monodromy $R^T \tilde{T} \circ \tilde{T} \circ = \tilde{T} \circ \tilde{T} \circ R^T$ Algebra in the Quantum Self-Dual Yang-Mills System

LING-LIE CHAU AND ITARU YAMANAKA
Department of Physics, University of California, Davis, California 95616

ABSTRACT

We find a conserved monodromy matrix differential operator $\tilde{T} \circ$ in the quantum Self-Dual Yang-Mills (SDYM) system and show that it satisfies the exchange algebra $R^T \tilde{T} \circ \tilde{T} \circ = \tilde{T} \circ \tilde{T} \circ R^T$. From its two infinitesimal forms, we obtain the infinite conserved quantum nonlocal-charge algebras and the infinite conserved Yangian algebras. It is remarkable that such conserved algebras exist in a four-dimensional nontrivial quantum field theory with interactions.
In two-dimensional (2-d) quantum solvable systems, revealing algebraic structure of operators is a key step to solving the system. Among the algebras, the exchange algebra of quantum monodromy matrix \( \tilde{T} \), plays an important role in the quantum inverse scattering method. Monodromy matrix is the boundary value of the solution to the linear systems or the group-valued element constructed from the conserved Lie-algebra-valued local currents which satisfy the Kac-Moody affine algebra. It generates conserved charges that lead to the complete solutions of some 2-d systems.\(^{[1]}\)

In Ref.\(^{[2]}\), we had formulated the quantum self-dual Yang-Mills (SDYM) system in terms of the group-valued local field \( \tilde{J} \). In this paper, we construct a conserved quantum monodromy matrix operator \( \tilde{T} \circ \) in this nontrivial four dimensional quantum field theory with interactions. The monodromy matrix \( \tilde{T} \circ \), besides being a quantum field operator, contains differential operators. It obeys a quantum exchange algebra \( R^T \tilde{T} \circ \tilde{T} \circ = \tilde{T} \circ \tilde{T} \circ R^T \). (We use a tilde on top of a letter to indicate that it is a quantum field operator; and a circle after a letter to denote that it contains the differential operator.) Taylor-expanding \( \tilde{T} \circ \) in its spectral parameter, we obtain the conserved infinite quantum nonlocal charges and their algebras; Taylor-expanding first in its exponent in the spectral parameter we obtain the conserved infinite Yangian charges and their Yangian algebras. These algebras in the four-dimensional (4-d) SDYM quantum field theory have many more generalized features than those in two dimensions.

It is remarkable that such conserved algebras exist in a 4-d nontrivial quantum field theory with interactions. Because of their generalized features, the techniques developed in two dimensions to extract physical information are not directly applicable. It is important and interesting to eventually find out the physical implications of these algebras.

The Quantum SDYM System:

First, we briefly review the quantum Self-Dual Yang-Mills theory. As formulated in Ref.\(^{[2]}\), it is characterized by a quantum field Hamiltonian

\[
\tilde{H}_{\text{int}} = -\alpha \int \int \int d\bar{y}dzd\bar{z}\{Tr\{((\partial_z\tilde{J}^{-1})(\partial_z\tilde{J}))
\]

\[
+ \int_0^1 d\rho Tr\{(\tilde{J}^{-1}\partial_\rho\tilde{J})[((\partial_z\tilde{J}^{-1})(\partial_z\tilde{J}) - (\partial_z\tilde{J}^{-1})(\partial_z\tilde{J})]}}\},
\]

(1)

where \( \tilde{J} = \tilde{J}(y,\bar{y},z,\bar{z}) \) is a 2 \(\times\) 2 matrix operator field depending on the 4-d coordinates \( y, \bar{y}, z, \bar{z} \); and \( y \) is the time. (In this theory, the \( z \) and \( \bar{z} \) are discretized.

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To save writing, we use integration in $z$ and $\bar{z}$ to denote summation in these variables.\cite{2}

A group-valued local quantum field $\tilde{J}$ satisfies the fixed-time-$y$ exchange algebra

$$\tilde{J}_1 \tilde{J}_\Pi = 1_{\Pi,\Pi} \tilde{J}_1 R^J_{\Pi,\Pi},$$  \hspace{1cm} (2)

where

$$R^J_{\Pi,\Pi} \equiv P_{\Pi,\Pi} \{ q^{\Delta t \epsilon(\tilde{y} - \tilde{y}')} \delta_{zz'} t^q_{\Pi,\Pi} - q^{\Delta t \epsilon(\tilde{y} - \tilde{y}')} \delta_{zz'} s^q_{\Pi,\Pi} \},$$  \hspace{1cm} (3)

and

$$t^q_{\Pi,\Pi} = diag\{1, \frac{1}{q + q^{-1}} \begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix}, 1\}, \hspace{1cm} s^q_{\Pi,\Pi} = 1_{\Pi,\Pi} - t^q_{\Pi,\Pi};$$

$$q \equiv e^{-(i\hbar/2a^2) \delta_{zz'} \delta_{zz'}}; \hspace{1cm} P_{\Pi,\Pi} = diag\{1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1\}.$$ \hspace{1cm} (4)

Note that $P_{\Pi,\Pi}$ is the exchange matrix for tensor space $\Pi$ and $\Pi$, i.e. $P_{\Pi,\Pi} A_1 B_\Pi = A_\Pi B_1 P_{\Pi,\Pi}$. The constant $q \neq 1$ is from the quantum $\hbar \neq 0$ effect. It also depends in a nontrivial way upon the cutoff $a^2$ and the type of Hamiltonian characterized by the parameter $\alpha$. The field $\tilde{J}^{-1}$, which satisfies the relation $\tilde{J}^{-1}\tilde{J} = 1 = \tilde{J}\tilde{J}^{-1}$, obeys the fixed-time-$y$ exchange algebra

$$\tilde{J}^{-1}_\Pi \tilde{J}_1 = \tilde{J}_1 R^J_{\Pi,\Pi} \tilde{J}^{-1}_\Pi.$$  \hspace{1cm} (5)

The equation of motion obtained from $\partial_y (\tilde{J} \partial_y \tilde{J}^{-1}) = i/\hbar [\tilde{H}_{\text{int}}, \tilde{J} \partial_y \tilde{J}^{-1}]$ is

$$\partial_y (\tilde{J} \partial_y \tilde{J}^{-1}) = -\partial_z (\tilde{J} \partial_z \tilde{J}^{-1}).$$  \hspace{1cm} (6)

From the above exchange algebra, Eq.(2), we showed in Ref.[2] that the current $\tilde{j} \equiv \tilde{J} \partial_y \tilde{J}^{-1}$ satisfies a 4-d fixed-time-$y$ Kac-Moody affine algebra with a central charge $\delta'(\tilde{y}_1 - \tilde{y}_2)$ -term

$$[\tilde{j}_1, \tilde{j}_\Pi] = [P_{\Pi,\Pi}, \tilde{j}_\Pi] \delta(\tilde{y}_1 - \tilde{y}_2) \delta_{zz_1} \delta_{\bar{z}_1,\bar{z}_2} + P_{\Pi,\Pi} \delta'(\tilde{y}_1 - \tilde{y}_2) \delta_{zz_1} \delta_{\bar{z}_1,\bar{z}_2}.$$  \hspace{1cm} (7)

It is a relation true only at fixed time-$y$ since $\tilde{j}(y, z, \tilde{y}, \bar{z})$ is not conserved.

On the other hand, we can show\cite{2} that the $z$-integrated (summed) current
\( \tilde{J}(\tilde{y}, \tilde{z}) = \int_{-l}^{l} \tilde{j}(z, \tilde{y}, \tilde{z}) \, dz \) is conserved by checking \( \partial_y \tilde{J} = 1/h[H_{\text{int}}, \tilde{J}\partial_y \tilde{J}^{-1}] = 0 \). Summing Eq.(7) in \( z_1 \) and \( z_2 \), we obtain the conserved Kac-Moody affine algebra,

\[
[(\tilde{\nabla}_{\tilde{y}_1})_1 \circ, (\tilde{\nabla}_{\tilde{y}_2})_2 \circ] = [P_{\Pi} \delta(\tilde{y}_1 - \tilde{y}_2), (\tilde{\nabla}_{\tilde{y}_2})_1 \circ] \delta_{\tilde{z}_1, \tilde{z}_2},
\]  

where

\[
\tilde{\nabla}_{\tilde{y}} \circ \equiv 2l \partial_{\tilde{y}} \circ + \tilde{J}.
\]

Here the circle of \( \partial_{\tilde{y}} \circ \) signifies its differential-operator property, \( \partial_{\tilde{y}} \circ f(y) = \partial_y f(y) + f(y) \partial_{\tilde{y}} \circ \) or \([\partial_{\tilde{y}} \circ, f(y)] = \partial_y f(y)\). Notice that this affine algebra of \( \tilde{\nabla}_{\tilde{y}} \circ \) has no central-charge \( \delta'(\tilde{y}_1 - \tilde{y}_2) \)-term. This is crucial for deriving the exchange algebra of the monodromy matrix operator \( \tilde{T} \circ, R^{T} \tilde{T} \circ \tilde{T} \circ = \tilde{T} \circ \tilde{T} \circ R^{T} \), which we shall derive in the next section.

**The \( R^{T} \tilde{T} \circ \tilde{T} \circ = \tilde{T} \circ \tilde{T} \circ R^{T} \) Exchange Algebra:**

Since there is no central charge term in Eq.(8), the \( \tilde{\nabla}_{\tilde{y}} \circ \)'s at different points of \( \tilde{z} \) commute and we can always define a path-ordered exponential function of \( \tilde{\nabla}_{\tilde{y}} \circ \)

\[
\check{\psi}(\tilde{y}, \tilde{z}; \lambda) \circ \equiv \check{\mathcal{P}} \exp(\lambda \int_{-\tilde{l}}^{\tilde{z}} d\tilde{z} \tilde{\nabla}_{\tilde{y}} \circ),
\]

where \( \lambda \) is an arbitrary complex number. For \( \tilde{z} \) equal to the boundary value, it becomes the monodromy matrix operator

\[
\check{T} \circ = [\check{\psi}(\tilde{y}, \tilde{z}; \lambda) \circ]_{\tilde{z} = \tilde{l}}.
\]

The path-ordered integration is the path-ordered product of the following infinitesimal elements:

\[
\check{L}_n \circ \equiv (1 + \lambda \Delta \tilde{z} \tilde{\nabla}_{\tilde{y}} \circ), \text{ and } \tilde{z} = n\Delta \tilde{z};
\]

\[
\check{T} \circ = (1 + \lambda \Delta \tilde{z} \tilde{\nabla}_{\tilde{y}} \circ)_{\tilde{z} = \tilde{l}}(1 + \lambda \Delta \tilde{z} \tilde{\nabla}_{\tilde{y}} \circ)_{\tilde{z} = \tilde{l} - \Delta \tilde{z}} \cdots (1 + \lambda \Delta \tilde{z} \tilde{\nabla}_{\tilde{y}} \circ)_{\tilde{z} = -\tilde{l}}
\]

\[
= \check{L}_{\tilde{z} = \tilde{l}} \circ \check{L}_{\tilde{z} = \tilde{l} - \Delta \tilde{z}} \circ \cdots \check{L}_{\tilde{z} = -\tilde{l}} \circ .
\]

Notice we treat \( \tilde{y} \) and \( \tilde{y}' \) as independent variables, i.e., \( \partial_{\tilde{y}} f(\tilde{y}') = 0 \). Using the Kac-Moody affine algebra of \( \tilde{\nabla}_{\tilde{y}} \circ \), Eq.(8), we can straightforwardly show that the
\(\tilde{L}_n\)'s satisfy the following exchange algebras

\[
R^T_{I,\Pi}(\lambda, \mu) [\tilde{L}_n(\lambda)]_\circ [\tilde{L}_n(\mu)]_\Pi \circ = [\tilde{L}_n(\mu)]_\Pi \circ [\tilde{L}_n(\lambda)]_\circ R^T_{I,\Pi}(\lambda, \mu), \quad \text{for } n = m, \tag{13}
\]

where

\[
R^T_{I,\Pi}(\lambda, \mu) \equiv 1 - P_{I,\Pi} \frac{\lambda \mu}{\lambda - \mu} \delta(\bar{y} - \bar{y}'); \tag{14}
\]

and

\[
(\tilde{L}_n)_\circ (\tilde{L}_m)_\Pi = (\tilde{L}_m)_\Pi (\tilde{L}_n)_\circ, \quad \text{for } n \neq m. \tag{15}
\]

Using the fact that \(\tilde{T}\circ\) is the product of \(\tilde{L}_n\circ\), Eq.(12), we then derive the exchange algebra of the monodromy matrix,

\[
R^T_{I,\Pi}(\lambda, \mu) \tilde{T}_I(\lambda) \circ \tilde{T}_\Pi(\mu) \circ = \tilde{T}_\Pi(\mu) \circ \tilde{T}_I(\lambda) \circ R^T_{I,\Pi}(\lambda, \mu). \tag{16}
\]

Notice that it would not be possible to derive Eq.(16) from Eqs.(13)-(15), if there were a central-charge \(\delta'(\bar{y}_1 - \bar{y}_2)\)-term in the Kac-Moody affine algebra of \(\tilde{\nabla}_{\bar{y}}\circ\), Eq.(8).

Eq.(16) is the central result of this paper. This 4-d exchange algebra has two generalizations when compared to those in two dimensions\[1\]: the monodromy matrix contains differential operator \(\partial_{\bar{y}}\circ\) and it depends on coordinate \(\bar{y}\).

It is known that the conserved currents in the quantum WZNW theory\[3,4\] and the non-conserved local currents in the quantum principal chiral model\[5\] in two dimensions satisfy the affine algebra with central charge and no consistent exchange algebra for their monodromy matrix can be constructed. Therefore, it is interesting and remarkable that a conserved quantum algebra \(R^T \tilde{T} \circ \tilde{T} \circ = \tilde{T} \circ \tilde{T} \circ R^T\) exists for a nontrivial 4-d quantum field theory with interactions.

**Infinite Conserved Nonlocal-Charge Algebras and Infinite Conserved Yangian Algebras:**

Let’s first discuss the infinite conserved non-local charge algebras. We Taylor-expand \(\tilde{T}(\lambda)\circ\) in \(\lambda\) and \(\partial_{\bar{y}}\circ\)

\[
\tilde{T}_\circ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{t}^{(m,n)}(\bar{y}) \lambda^m (\partial_{\circ})^n, \tag{17}
\]

which defines the quantum nonlocal charges \(\bar{t}^{(m,n)}(\bar{y})\). They have the following expressions in terms of the field \(\tilde{J}\):

\[
\bar{t}^{(0,0)} = 1, \tag{18}
\]
and obtain the Yangian charges and their algebra:

\[ \tilde{t}^{(1,0)} = \int_{-\bar{l}}^\bar{l} dz \tilde{\nabla}_y \circ 1 = \int_{-\bar{l}}^\bar{l} dz \int_{-\bar{l}}^\bar{l} dz \tilde{J} \partial_y \tilde{J}^{-1}, \]  
(19)

\[ \tilde{t}^{(2,0)} = \int_{-\bar{l}}^\bar{l} d\bar{z}_2 \tilde{\nabla}_y \circ \int_{-\bar{l}}^\bar{l} d\bar{z}_1 \tilde{\nabla}_y \circ 1, \]  
(20)

and

\[ \tilde{t}^{(n,0)} = \int_{-\bar{l}}^\bar{l} dz_n \tilde{\nabla}_y \circ \int_{-\bar{l}}^\bar{l} d\bar{z}_{n-1} \tilde{\nabla}_y \circ \ldots \int_{-\bar{l}}^\bar{l} d\bar{z}_1 \tilde{\nabla}_y \circ 1. \]  
(21)

Substituting Eq.(17) into Eq.(16), we show that the nonlocal-charges satisfy the following algebras:

\[ [\tilde{t}_I^{(1,0)}, \tilde{t}_I^{(1,0)}] = [P_{1,\bar{I}}, \tilde{t}_I^{(1,0)}] \delta(\bar{y} - \bar{y}') - \tilde{t}_I^{(1,1)} \partial_{\bar{y}} \delta(\bar{y} - \bar{y}') P_{1,\bar{I}}, \]  
(22)

\[ [\tilde{t}_I^{(2,0)}, \tilde{t}_I^{(1,0)}] = [P_{1,\bar{I}}, \tilde{t}_I^{(2,0)}] \delta(\bar{y} - \bar{y}') - (\tilde{t}_I^{(2,1)} \partial_{\bar{y}} + \tilde{t}_I^{(2,2)} \partial_{\bar{y}}^2) \delta(\bar{y} - \bar{y}') P_{1,\bar{I}}, \]  
(23)

and

\[ [\tilde{t}_I^{(n,0)}, \tilde{t}_I^{(m,0)}] = P_{1,\bar{I}} (\sum_{i=0}^{\infty} \tilde{t}_I^{(n+i,0)} \tilde{t}_I^{(m-i-1,0)}) \delta(\bar{y} - \bar{y}') \]  
\[ - P_{1,\bar{I}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (\sum_{i=0}^{\infty} \tilde{t}_I^{(n+i,0)} \tilde{t}_I^{(m-i,0)p}) \delta_{\bar{y}}^p \delta(\bar{y} - \bar{y}'), \]  
(24)

with positive integers \( m, n, p \). In Eq.(24), we have defined \( \tilde{t}^{(n,p)} = 0 \) for \( n < p \). The first nonlocal charge algebra, Eq.(22), is nothing but the \( z \)- and \( \bar{z} \)-summed Kac-Moody affine algebra of Eq.(7).

Next, we make the \( \lambda \)-expansion of the monodromy matrix first in the exponent and obtain the Yangian charges and their algebra:

\[ \tilde{T}_c = \sum_{n=0}^{\infty} \left( \sum_{m=n}^{\infty} \tilde{\omega}^{(m,n)} \lambda^m \right) (\partial_{\bar{y}}^n), \]  
(25)

where the \( \tilde{\omega}^{(m,n)} \)'s are the Yangian charges and they satisfy the following algebras

\[ [\tilde{\omega}_I^{(1,0)}, \tilde{\omega}_I^{(1,0)}] = ([P_{1,\bar{I}}, \tilde{\omega}_I^{(1,0)}] \delta(\bar{y} - \bar{y}') - \tilde{\omega}_I^{(1,1)} \partial_{\bar{y}} \delta(\bar{y} - \bar{y}') P_{1,\bar{I}}), \]  
(26)

\[ [\tilde{\omega}_I^{(2,0)}, \tilde{\omega}_I^{(1,0)}] = [P_{1,\bar{I}}, \tilde{\omega}_I^{(2,0)}] \delta(\bar{y} - \bar{y}') - \]  
\[ \{ (\tilde{\omega}_I^{(2,1)} + \frac{1}{2} (\tilde{\omega}_I^{(1,1)})^2 - \frac{1}{2} \tilde{\omega}_I^{(1,1)} \tilde{\omega}_I^{(1,1)} - \frac{1}{2} \tilde{\omega}_I^{(1,1)} \tilde{\omega}_I^{(2,0)} ) \delta_{\bar{y}} + \tilde{\omega}_I^{(2,2)} \partial_{\bar{y}}^2 \} \delta(\bar{y} - \bar{y}') P_{1,\bar{I}}, \]  
(27)

and etc.
For these infinite algebras we have not found an explicit general expression for
the higher order terms, as we have done for the previous case, Eq.(24).

These algebras have many generalized new features compared to those in two
dimensions \([6,7,8]\): (1) the charges \(\tilde{t}\)'s and \(\tilde{\omega}\)'s depend on coordinates \(\bar{y}\); (2) there are
the central charge \(\partial_y^n \delta(\bar{y} - \bar{y}')\)-terms.

The \(\tilde{\omega}^{(m,n)}\)'s and the \(\tilde{t}^{(m,n)}\)'s are related
\[
\tilde{t}^{(1,n)} = \tilde{\omega}^{(1,n)}, \tag{28}
\]
\[
\tilde{t}^{(2,n)} = \tilde{\omega}^{(2,n)} + \frac{1}{2}(\tilde{\omega}^{(1,n)})^2, \tag{29}
\]

**Co-product**

Motivated by Drinfel’d’s work \([6]\), we define co-product for our algebras:
\[
\Delta(\tilde{T}_I(\lambda) \circ) \equiv \tilde{T}^A_I(\lambda) \circ \tilde{T}^B_I(\lambda) \circ, \tag{30}
\]
where super \(A,B\) stands for the different Fock spaces the operators \(\tilde{T} \circ\) acts on; thus, in the right hand side (r.h.s.) of Eq.(30), the product of two monodromy
matrices is a product merely in the matrix sense (not in the quantum-field-operator
sense since the two operators act on different Fock spaces and they commute).

We shall show that we can introduce the following distributive rule for the
coproduct
\[
\Delta(\tilde{T}_I \circ \tilde{T}_\Pi \circ) = \Delta(\tilde{T}_I) \circ \Delta(\tilde{T}_\Pi) \circ, \tag{31}
\]
and that it is consistent with the exchange relation, Eq.(16). Taking the co-product
of both sides of Eq.(16), we obtain
\[
R^T_{I,\Pi} \Delta(\tilde{T}^A_I(\lambda)) \circ \Delta(\tilde{T}^A_\Pi(\mu)) \circ = \Delta(\tilde{T}^A_\Pi(\mu)) \circ \Delta(\tilde{T}^A_I(\lambda)) \circ R^T_{I,\Pi}; \tag{32}
\]
on the other hand, we need to check that this equality is true directly from the
definition of co-products:

l.h.s. of Eq.(32)
\[
= R^T_{I,\Pi} \Delta(\tilde{T}^A_I(\lambda)) \circ \Delta(\tilde{T}^A_\Pi(\mu)) \circ; \quad \text{from the definition of co-products,}
\]
\[
= R^T_{I,\Pi} \tilde{T}^A_I(\lambda) \circ \tilde{T}^B_I(\lambda) \circ \tilde{T}^A_\Pi(\mu) \circ \tilde{T}^B_\Pi(\mu) \circ; \quad \text{from } \tilde{T}^B_I(\lambda) \circ \tilde{T}^A_\Pi(\mu) \circ = \tilde{T}^A_\Pi(\mu) \circ \tilde{T}^B_I(\lambda) \circ,
\]
\[
= R^T_{I,\Pi} \tilde{T}^A_I(\lambda) \circ \tilde{T}^A_\Pi(\mu) \circ \tilde{T}^B_I(\lambda) \circ \tilde{T}^B_\Pi(\mu) \circ; \quad \text{using the exchange algebra, Eq.(6) twice,}
\]
\[
= \tilde{T}^A_\Pi(\mu) \circ \tilde{T}^A_I(\lambda) \circ \tilde{T}^B_\Pi(\mu) \circ \tilde{T}^B_I(\lambda) \circ R^T_{I,\Pi}; \quad \text{using } \tilde{T}^A_I(\lambda) \circ \tilde{T}^B_\Pi(\mu) \circ = \tilde{T}^B_\Pi(\mu) \circ \tilde{T}^A_I(\lambda) \circ,
\]
= \tilde{T}_\Pi^A(\mu) \circ \tilde{T}_\Pi^B(\mu) \circ \tilde{T}_I^A(\lambda) \circ \tilde{T}_I^B(\lambda) \circ R_{I,\Pi}^T; \quad \text{from the definition of co-product,}
= \Delta(\tilde{T}_\Pi^A(\mu)) \circ \Delta(\tilde{T}_I^A(\lambda)) \circ R_{I,\Pi}^T = \text{r.h.s. of Eq.}(32); \quad (33)
therefore, we now have demonstrated the consistency of our co-product.

Expanding Eq.(30), we obtain the co-products of the charges.

The co-products of the nonlocal charges are

\[ \Delta(t^{(1,0)}) = (t^{(1,0)})^A + (t^{(1,0)})^B, \quad \text{(34)} \]
\[ \Delta(t^{(2,0)}) = (t^{(2,0)})^A + (t^{(2,0)})^B + (t^{(1,0)})^A(t^{(1,0)})^B + (t^{(1,1)})^A\partial_y(t^{(1,0)})^B, \quad \text{and etc.; (35)} \]
\[ \Delta(t^{(1,1)}) = (t^{(1,1)})^A + (t^{(1,1)})^B, \quad \text{(36)} \]
\[ \Delta(t^{(2,2)}) = (t^{(2,2)})^A + (t^{(2,2)})^B + (t^{(1,1)})^A(t^{(1,1)})^B, \quad \text{(37)} \]
and etc.

and etc. (In the last two co-products, we use the co-product rule for the parameter \( l \), \( \Delta(l) = 2l \).)

Similarly, we can also checked that these co-products for the Yangian charges, Eqs.(38)-(41), are consistent with the Yangian algebras, Eqs.(26)-(27).
**Automorphism:**

The exchange algebra of monodromy matrix, Eq. (16), has an automorphism, since the $R$-matrix, Eq.(14), is in invariant under the following transformation of the spectral parameters,

\[
\frac{1}{\lambda} \to \frac{1}{\lambda} - \nu \quad \text{and} \quad \frac{1}{\mu} \to \frac{1}{\mu} - \nu, \tag{42}
\]

where $\nu$ is an arbitrary complex number as are $\lambda$ and $\mu$. This implies that given Eq.(16), the following equation is also true

\[
R_{\text{II}}^T(\lambda, \mu) \tilde{T}_1(\frac{\lambda}{1-\nu\lambda}) \circ \tilde{T}_2(\frac{\mu}{1-\nu\mu}) = \tilde{T}_2(\frac{\lambda}{1-\nu\lambda}) \circ \tilde{T}_1(\frac{\mu}{1-\nu\mu}) \circ R_{\text{II}}^T(\lambda, \mu), \tag{43}
\]

with the same $R_{\text{II}}^T(\lambda, \mu)$ matrix as in Eq.(16). Expanding out in $\lambda$ and $\mu$, we obtain the automorphism in $\tilde{t}$'s and $\tilde{\omega}$'s

\[
\begin{align*}
\tilde{t}^{(0,n)} & \to \tilde{t}^{(0,n)}_\nu = \tilde{t}^{(0,n)}, \\
\tilde{t}^{(1,n)} & \to \tilde{t}^{(1,n)}_\nu = \tilde{t}^{(1,n)}, \\
\tilde{t}^{(2,n)} & \to \tilde{t}^{(2,n)}_\nu = \tilde{t}^{(2,n)} + \nu \tilde{t}^{(1,n)}, \\
\tilde{t}^{(3,n)} & \to \tilde{t}^{(3,n)}_\nu = \tilde{t}^{(3,n)} + 2\nu \tilde{t}^{(2,n)} + \nu^2 \tilde{t}^{(1,n)},
\end{align*}
\]

and etc;

\[
\begin{align*}
\tilde{\omega}^{(1,n)} & \to \tilde{\omega}^{(1,n)}_\nu = \tilde{\omega}^{(1,n)}, \\
\tilde{\omega}^{(2,n)} & \to \tilde{\omega}^{(2,n)}_\nu = \tilde{\omega}^{(2,n)} + \nu \tilde{\omega}^{(1,n)}, \\
\tilde{\omega}^{(3,n)} & \to \tilde{\omega}^{(3,n)}_\nu = \tilde{\omega}^{(3,n)} + 2\nu \tilde{\omega}^{(2,n)} + \nu^2 \tilde{\omega}^{(1,n)},
\end{align*}
\]

and etc.

The $\tilde{t}^{(m,n)}_\nu$'s and the $\tilde{\omega}^{(m,n)}_\nu$'s satisfy the same nonlocal-charge algebras, Eqs.(22)-(24), and the same Yangian algebras, Eqs.(26)-(27), respectively.

Our algebras also have the automorphism in $\tilde{y}$ and $\tilde{z}$ coordinates because the $R^T$-matrix depends only on

the difference of two $\tilde{y}$ coordinates and is independent of $\tilde{z}$.
Discussions and Conclusion:

The Yangian algebras in two dimensions have provided the powerful means in constructing S matrices for some integrable systems, e.g., the sine-Gordon model\cite{7} and Gross-Neveu model\cite{8,9}. We cannot directly transfer the techniques developed in two dimensions to extract physical implications, since the algebras we have obtained in four dimensions have many generalized features. What is surprising is that a nontrivial 4-d quantum field theory like the SDYM system can have these beautiful conserved algebras. It is a challenge to us to eventually decode the physical implications of these conserved algebras.
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