ON REGULAR SEQUENCES IN THE FORM MODULE WITH APPLICATIONS TO LOCAL BÉZOUT INEQUALITIES

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Abstract. Let \( q \) denote an ideal in a Noetherian local ring \((A, m)\). Let \( \mathfrak{a} = a_1, \ldots, a_d \subseteq q \) denote a system of parameters in a finitely generated \( A \)-module \( M \). This note investigates an improvement of the inequality \( c_1 \cdot \ldots \cdot c_d \cdot e_0(q; M) \leq \ell_A(M/qM) \), where \( c_i \) denote the initial degrees of \( a_i \) in the form ring \( G_A(q) \). To this end, there is an investigation of regular sequences in the form module \( G_M(q) \) by homology of a factor complex of the Koszul complex. In a particular case, there is a discussion of classical local Bézout inequality in the affine \( d \)-space \( \mathbb{A}^d_\mathbb{k} \).

1. Introduction

The importance of an improvement of the inequality \( \ell_A(M/qM) \geq c_1 \cdot \ldots \cdot c_d \cdot e_0(q; M) \) has to do with Bézout’s Theorem in the projective plane. Let \( C = V(F), D = V(G) \subseteq \mathbb{P}_\mathbb{k}^2, \mathbb{k} = \mathbb{k} \), be two curves in the projective plane without a common component. Then

\[
\sum_{P \in C \cap D} \mu(P; C, D) = \deg C \cdot \deg D,
\]

where \( \mu(P; C, D) \) denotes the local intersection multiplicity of \( P \) in \( C \cap D \). In a particular case when \( P \) is the origin, it follows that \( \mu(P; C, D) = \ell_A(A/(f, g)A) \), where \( A = \mathbb{k}[x, y]_{(x, y)} \) and \( f, g \) denote the equations in \( A \). Note that \( \ell_A(A/(f, g)A) = e_0(f, g; A) \) as \( A \) is a regular local ring. Since \( C, D \) have no component in common, \( \{ f, g \} \) forms a system of parameters in \( A \). Then

\[
e_0(f, g; A) \geq c \cdot d \cdot e_0(m; A) = c \cdot d,
\]

since \( e_0(m; A) = 1 \), called the local Bézout inequality in the affine plane \( \mathbb{A}^2_\mathbb{k} \). Here \( c, d \) denote the initial degree of \( f, g \) respectively. This estimate is well-known (see for instance [3] or [6]) and proved by resultants or Puiseux expansions. Moreover, equality holds if and only if \( C, D \) intersect transversally at the origin. In other words \( f^*, g^* \), the initial forms of \( f, g \) in the form ring \( G_A(m) \cong \mathbb{k}[X, Y] \), is a homogeneous system of parameters.

First Bydžovský [5] and most recently Boda-Schenzel [2] presented an improvement of the local Bézout inequality. More precisely,

\[
e_0(f, g; A) \geq c \cdot d + t,
\]

where \( t \) is the number of common tangents of \( f, g \) at origin when counted with multiplicities.

We generalized their result to an arbitrary situation. To this end, let \( q \) denote an ideal in a Noetherian local ring \((A, m, \mathbb{k})\) such that \( \ell_A(M/qM) \) is finite for a finitely generated \( A \)-module \( M \). Let \( \mathfrak{a} = a_1, \ldots, a_d \subseteq q \) denote a system of parameters of \( M \) such that \( a_i \in q^{c_i} \setminus q^{c_i+1}, c_i > 0, \) for \( i = 1, \ldots, d \). Then we have the following result.

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Theorem 1.1. (Cor. 5.3) With the previous notations, if $\alpha^* G_A(q)$ contains a $G_M(q)$-regular sequence $b^* = b^*_1, \ldots, b^*_{d-1}$ and we choose $b_i$ for $i = 1, \ldots, d - 1$ as in Lemma 4.2. Then

$$\ell_A(M/qM) \geq c \cdot e_0(q; M) + r$$

where $c = c_1 \cdot \cdots \cdot c_d$ and $r = \ell_A([\text{Ext}^{d-1}_{G_A(q)}(G_A(q)/\alpha^* G_A(q), G_M(q))]_{-\tau-1})$ is a constant for all $n \gg 0$ and $\tau = c_1 + \ldots + c_d$.

There are also few applications of the previous theorem. We refer Section 5 and 6.

Another motivation for the author was a recent preprint [8]. In this preprint, the authors define a generalized Koszul complex $L_\bullet(G, q, M; n)$ which is factor complex of Koszul complex (see definition 2.3). There are criteria concerning regular sequences in a finitely generated $A$-module $M$, which deal the vanishing and rigidity of the Koszul homology (see [4] and [9]). We present the similar criteria concerning regular sequences in the form module $G_M(q)$ in terms of the homology modules $L_i(G, q, M; n)$ of the complex $L_\bullet(G, q, M; n)$. More precisely, let $\underline{a} = a_1, \ldots, a_d$ and $\underline{b} = b_1, \ldots, b_t$ denote two systems of elements of $A$. There is a following theorem.

Theorem 1.2. (1)(Theorem 3.1) With the previous notations, the following are equivalent:

(a) $\alpha^* = a^*_1, \ldots, a^*_d$ is $G_M(q)$-regular sequence.
(b) $L_1(\underline{a}, q, M; n) = 0$ for all $n$.
(c) $L_i(\underline{a}, q, M; n) = 0$ for all $i > 0$, for all $n$.

(2)(Theorem 4.3, Prop. 4.4) With the previous notations, if $\alpha^* G_A(q)$ contains a $G_M(q)$-regular sequence $b^* = b^*_1, \ldots, b^*_t$, then

$$L_i(\underline{a}, q, M; n) = 0$$

for all $i > t$, for all $n$.

The converse is also true. Moreover, given $b^* = b^*_1, \ldots, b^*_t$ we choose $b_i$ for $i = 1, \ldots, t$ as in Lemma 4.2, there is an isomorphism

$L_d-1(\underline{a}, q, M; n) \cong \bigcap_{i=1}^{d} (\underline{a} \cdot q^{n+b_i-\tau})M : qA_i/(\underline{b} \cdot q^{n+b_i-\tau})M$.

We refer Section 3 and 4 for the detail discussion about above theorem. In [2] and [8], authors posed the problem to study the Euler characteristic $\chi_A(\underline{a}, q, M)$ of the complex $K_\bullet(\underline{a}, q, M; n)$, see def. 2.3, independently of its value which is equal to $e_0(\underline{a}; M) - c_1 \cdot \cdots \cdot c_d \cdot e_0(q; M) \geq 0$ for $n \gg 0$, cf. [2]. In section 6, we discuss a few properties of this Euler characteristic. A further investigation of the geometric meanings of the length involved in Theorem 1.1 in affine space $A^n$ when $d \geq 3$ is in progress.

As a source for basic notions in Commutative Algebra, we refer to [1] and [9]. For results on Homological Algebra, we refer to [10] and [14].

2. Preliminaries

In this section, we present the basic notations, which we are going to use in upcoming sections. For more detail, we refer the text book [9] and lecture notes [11].

Notation 2.1. (1) Let $(A, m)$ be a local Noetherian ring, $q$ be an ideal in $A$ and $M$ be a finitely generated $A$-module. Then $q$ is said to be an ideal of definition with respect to $M$ if the length $\ell_A(M/qM)$ of $A$-module $M/qM$ is finite. Now, it is easily seen that the length of $A$-modules $M/q^nM$ is also finite for all $n \in \mathbb{N}$.

For $n$ large enough, $\ell_A(M/q^nM)$ becomes a polynomial, which is written as

$$\ell_A(M/q^nM) = \sum_{i=0}^{d} e_i(q; M) \left( \binom{n+d-i-1}{d-i} \right),$$
where degree $d$ is equal to $\dim M$ (see [9]). Here, $e_i(q; M)$ are called the Hilbert-Samuel local multiplicities of $M$ with respect to $q$. The first $e_0(q; M)$ of them is our main ingredient for the rest of the note, and we call it just the multiplicity of $M$ with respect to $q$.

(2) The Rees and form rings of $A$ with respect to $q$ are defined by

$$R_A(q) = \oplus_{n \geq 0} q^n T^n \subseteq A[T] \quad \text{and} \quad G_A(q) = \oplus_{n \geq 0} q^n/q^{n+1},$$

where $T$ denotes an indeterminate over $A$. The Rees and form modules are defined in the corresponding way by

$$R_M(q) = \oplus_{n \geq 0} q^n M T^n \subseteq M[T] \quad \text{and} \quad G_M(q) = \oplus_{n \geq 0} q^n M/q^{n+1} M.$$

(3) Assume that $m \in M$ such that $m \in q^i M \setminus q^{i+1} M$. We define $m^* := m + q^{i+1} M \in [G_M(q)]_c$. If $m \in \cap_{n \geq 1} q^n M$, then we write $m^* = 0$. Here $m^*$ and $c$ are called the initial form and initial degree of $m$ in $G_M(q)$ respectively. We refer [12] for more detail.

For basic results about multiplicities, we refer [4] and [9]. Another tool for the investigation is use of Koszul complex.

**Remark 2.2.** (Koszul Complex) Let $\underline{a} = a_1, \ldots, a_d$ denote a system of elements of the ring $A$. The Koszul complex $K_*(\underline{a}; A)$ is defined as follows: Assume that $F$ is a free $A$-module with basis $e_1, \ldots, e_d$. Then $K_i(\underline{a}; A) = \bigwedge^i F$ for $i = 1, \ldots, d$. A basis of $K_i(\underline{a}; A)$ is given by the wedge products $e_{j_1} \wedge \ldots \wedge e_{j_i}$ for $1 \leq j_1 < \ldots < j_i \leq d$. The boundary homomorphism $K_i(\underline{a}; A) \rightarrow K_{i-1}(\underline{a}; A)$ is defined by

$$d_{j_1 \ldots j_i} : e_{j_1} \wedge \ldots \wedge e_{j_i} \mapsto \sum_{k=1}^i (-1)^{k+1} a_{j_k} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_k} \wedge \ldots \wedge e_{j_i},$$

on the free generators $e_{j_1} \wedge \ldots \wedge e_{j_i}$. Also $K_*(\underline{a}; M) \cong K_*(\underline{a}; A) \otimes_A M$. We write $H_i(\underline{a}; M), i \in \mathbb{Z}$, for the $i$-th homology of $K_*(\underline{a}; M)$.

For more detail about Koszul homology, we refer [4] and [9]. The following are main ingredients for the investigation, see [8] for reference.

**Notation 2.3.** (Khadam-Schenzel [8]) (1) Assume that $a_i \in q^i$ for $i = 1, \ldots, d$ and $n$ is a non-negative integer. For $n < 0$, we assume that $q^n M = 0$. We define a complex $K_*(\underline{a}, \underline{q}; M; n)$ in the following way:

(i) The $i$-th term $K_i(\underline{a}, \underline{q}; M; n) := \oplus_{1 \leq j_1 < \ldots < j_i \leq d} q^{n-c_{j_1} \ldots - c_{j_i}} M$ for $0 \leq i \leq d$ and $K_i(\underline{a}, \underline{q}; M; n) = 0$ otherwise.

(ii) The boundary homomorphism $K_i(\underline{a}, \underline{q}; M; n) \rightarrow K_{i-1}(\underline{a}, \underline{q}; M; n)$ is defined by homomorphisms on each of the direct summands $q^{n-c_{j_1} \ldots - c_{j_i}} M$. On $q^{n-c_{j_1} \ldots - c_{j_i}} M$, it is the map given by $d_{j_1 \ldots j_i} \otimes 1_M$ restricted to $q^{n-c_{j_1} \ldots - c_{j_i}} M$, where $d_{j_1 \ldots j_i}$ denotes the homomorphism as defined in 2.2.

It is clear that $K_*(\underline{a}, \underline{q}; M; n)$ is a complex. Moreover, by construction, $K_*(\underline{a}, \underline{q}; M; n)$ is a sub complex of the Koszul complex $K_*(\underline{a}; M)$. We write $H_i(\underline{a}, \underline{q}; M; n), i \in \mathbb{Z}$, for the $i$-th homology of the complex $K_*(\underline{a}, \underline{q}; M; n)$. Note that $[K_*(\underline{a}, q T^n; R_M(q))]_n \cong K_*(\underline{a}, \underline{q}; M; n)$ for $n \in \mathbb{N}$.

(2) We define $L_*(\underline{a}, \underline{q}; M; n)$ as the quotient of the embedding $K_*(\underline{a}, \underline{q}; M; n) \rightarrow K_*(\underline{a}; M)$. That is, there is a short exact sequence of complexes

$$0 \rightarrow K_*(\underline{a}, \underline{q}; M; n) \rightarrow K_*(\underline{a}; M) \rightarrow L_*(\underline{a}, \underline{q}; M; n) \rightarrow 0.$$

Note that $L_i(\underline{a}, \underline{q}; M; n) \cong \oplus_{1 \leq j_1 < \ldots < j_i \leq d} M/q^{n-c_{j_1} \ldots - c_{j_i}} M$. The boundary homomorphisms are those induced by the Koszul complex. We write $L_i(\underline{a}, \underline{q}; M; n), i \in \mathbb{Z}$, for the $i$-th homology of the complex $L_*(\underline{a}, \underline{q}; M; n)$.

For more detail about complexes $K_*(\underline{a}, \underline{q}; M; n)$ and $L_*(\underline{a}, \underline{q}; M; n)$, and their relationship with the local cohomology module, we refer Khadam-Schenzel [8].
3. Regular Sequences in the Form and Rees Modules

There is a criterion in terms of Koszul homology which ensures whether a sequence of elements in \( A \) is \( M \)-regular or not. More precisely, the sequence \( \underline{a} = a_1, \ldots, a_d \) is \( M \)-regular if and only if \( H_i(\underline{a}; M) = 0 \) for all \( i > 0 \) (see [9]). In this section, we present a similar criterion in the form module \( G_M(q) \) in terms of the homology modules \( L_i(\underline{a}, q; M; n) \).

Let \( \underline{a}^* = a_1^*, \ldots, a_d^* \) denote a sequence of initial forms in the form ring \( G_A(q) \) with \( \deg a_i^* = c_i \) for \( i = 1, \ldots, d \). We start with the main result of the section.

**Theorem 3.1.** With the previous notations, the following are equivalent:

1. \( \underline{a}^* = a_1^*, \ldots, a_d^* \) is \( G_M(q) \)-regular sequence.
2. \( L_1(\underline{a}, q; M; n) = 0 \) for all \( n \).
3. \( L_i(\underline{a}, q; M; n) = 0 \) for all \( i > 0 \), for all \( n \).

**Proof.** For \( (1) \Rightarrow (3) \), we use induction on \( d \). If \( d = 1 \), then

\[
L_1(a_1, q; M; n) = q^nM : A a_1/q^{n-c_1}M,
\]

which is equal to zero for all \( n \) if and only if \( a_1^* = G_M(q) \)-regular. Now, by virtue of long exact homology sequence coming from the mapping cone construction of the complex \( L_*(\underline{a}, q; M; n) \) (see [8]) and by inductive step, \( L_i(\underline{a}, q; M; n) = 0 \) for all \( i > 1 \), for all \( n \), and

\[
L_1(\underline{a}, q; M; n) \cong 0 : L_0(\underline{a}, q; M; n-c_d) a_d.
\]

The latter is isomorphic to \( (\underline{a}', q^n)M : A a_d/(\underline{a}', q^{n-c_d})M \), which is equal to zero for all \( n \), see [13]. It is obvious that \( (3) \Rightarrow (2) \).

For \( (2) \Rightarrow (1) \), we apply induction on \( d \) once again. The case \( d = 1 \) is clear, see above. Again, from the mapping cone construction and by assumption,

\[
L_1(\underline{a}', q; M; n) = a_d L_1(\underline{a}', q; M; n-c_d),
\]

and therefore, by virtue of Nakayama lemma \( L_1(\underline{a}', q; M; n) = 0 \) for all \( n \). Hence \( \underline{a}^* = a_1^*, \ldots, a_{d-1}^* \) is a \( G_M(q) \)-regular sequence by induction. Moreover,

\[
0 = L_1(\underline{a}, q; M; n) \cong (\underline{a}', q^n)M : A a_d/(\underline{a}', q^{n-c_d})M
\]

for all \( n \), and hence \( a_d^* = G_M(q)/(\underline{a}^*)G_M(q) \)-regular. This finishes the argument. \( \square \)

The following is a consequence of the previous theorem.

**Corollary 3.2.** With the previous notations, \( \underline{a}^* = a_1^*, \ldots, a_d^* \) is a \( G_M(q) \)-regular sequence implies that \( aT^c = a_1T^{c_1}, \ldots, a_dT^{c_d} \) is an \( R_M(q) \)-regular sequence.

**Proof.** If \( \underline{a}^* = a_1^*, \ldots, a_d^* \) is a \( G_M(q) \)-regular sequence, then \( \underline{a} = a_1, \ldots, a_d \) is an \( M \)-regular sequence, see [13]. Hence \( H_i(\underline{a}; M) = 0 \) for all \( i > 0 \). Therefore, from the long exact sequence of homology coming from the short exact sequence of \( 2.3 \), \( H_i(\underline{a}, q; M; n) = 0 \) for all \( i > 0 \) and for all \( n \) (see 3.1). That is \( H_i(aT^{\infty}; R_M(q)) = 0 \) for all \( i > 0 \). Hence, by virtue of Koszul criterion, \( aT^c = a_1T^{c_1}, \ldots, a_dT^{c_d} \) is an \( R_M(q) \)-regular sequence. \( \square \)

4. A Formula for Homology

There is a classical result concerning the length of an \( M \)-sequence inside the ideal \( (a_1, \ldots, a_d) \) and vanishing of the Koszul homology. More precisely, if \( b_1, \ldots, b_t \) is an \( M \)-sequence contained in the ideal \( (a_1, \ldots, a_d) \), then \( H_i(\underline{a}; M) = 0 \) for all \( i > d - t \), and there is a formula

\[
H_{d-t}(\underline{a}; M) \cong (a_1, \ldots, a_d) M : A (b_1, \ldots, b_t) M,
\]

see [4]. In this section, we present the similar result for the homology modules \( L_i(\underline{a}, q; M; n) \).

We begin with a lemma.
Lemma 4.1. With the previous notations, let $b^*$ be a $G_M(q)$-regular element of degree $\beta$, then there is a short exact sequence of complexes

$$0 \to \mathcal{L}_s(a, q, M; n - \beta) \xrightarrow{b} \mathcal{L}_s(a, q, M; n) \to \mathcal{L}_s(a, q, M/bM; n) \to 0.$$  

In particular, there is the long exact homology sequence

$$\ldots \to L_i(a, q, M; n - \beta) \xrightarrow{b} L_i(a, q, M; n) \to L_i(a, q, M/bM; n) \to \ldots$$

Proof. The kernel of the map $\mathcal{L}_s(a, q, M; n - \beta) \xrightarrow{b} \mathcal{L}_s(a, q, M; n)$ is zero since $b^*$ is $G_M(q)$-regular. Also, it is easy to see that $\text{Coker}(m_k) = \mathcal{L}_s(a, q, M/bM; n)$. This provides the short exact sequence of complexes. By taking homology it yields the long exact sequence. \hfill \square

Let $b = b_1, \ldots, b_t$ denote a sequence of elements in $A$, and $b^* = b_1^*, \ldots, b_t^*$ denote a sequence of initial forms in $G_A(q)$ with $\deg b_i^* = \beta_i$. There is another technical lemma.

Lemma 4.2. With the previous notations, assume that $b^*G_A(q) \subseteq a^*G_A(q)$. Then there are elements $b_1^*, \ldots, b_t^* \in A$ such that

(i) $b_i^* = b_i^*$ for $i = 1, \ldots, t$.

(ii) $(b_1^*, \ldots, b_t^*)A \subseteq (a_1, \ldots, a_d)A$.

Proof. The containment relation of the assumption restricted to degree $n \in \mathbb{Z}$ provides

$$\left( \sum_{i=1}^t b_i q^{n-\beta_i} + q^{n+1} \right)/q^{n+1} \subseteq \left( \sum_{j=1}^d a_j q^{n-c_j} + q^{n+1} \right)/q^{n+1}$$

for all $n$, and hence $\sum_{i=1}^t b_i q^{n-\beta_i} \subseteq \sum_{j=1}^d a_j q^{n-c_j} + q^{n+1}$ for all $n$. Now choose $n = \beta_k$ and therefore $b_k \in \sum_{j=1}^d a_j q^{\beta_k-c_j} + q^{\beta_k+1}$. Whence there exist $r_{jk} \in q^{\beta_k-c_j}$ for $j = 1, \ldots, d$, such that $b_k - \sum_{j=1}^d a_j r_{jk} \in q^{\beta_k+1}$. Note that $\sum_{j=1}^d a_j r_{jk} \in q^{\beta_k} \setminus q^{\beta_k+1}$. We choose $b_k = \sum_{j=1}^d a_j r_{jk}$ for $k = 1, \ldots, t$, and this finishes the proof. \hfill \square

Now, we present the main result of the section.

Theorem 4.3. With the previous notations, if $a^*G_A(q)$ contains a $G_M(q)$-regular sequence $b^* = b_1^*, \ldots, b_t^*$, then

$$L_i(a, q, M; n) = 0 \text{ for all } i > d - t, \text{ for all } n.$$

Moreover, given $b^* = b_1^*, \ldots, b_t^*$ we choose $b_i$ for $i = 1, \ldots, t$ as in Lemma 4.2, there is an isomorphism

$$L_{d-t}(a, q, M; n) \cong \bigcap_{i=1}^d (b_i^* q^{n+\beta_i-\overline{c}}) M : M a_i/(b_i q^{n+\beta_i-\overline{c}}) M,$$

where $c := c_1 + \cdots + c_d, \overline{\beta} := c_1 + \cdots + c_{i-1} + c_{i+1} + \cdots + c_d$, and $\beta := \sum_{j=1}^t \beta_j$.

Proof. We proceed by induction on $t$. The vanishing $L_i(a, q, M; n) = 0$ for $i > d$ is trivial, and it is easily seen that $L_d(a, q, M; n) \cong \bigcap_{i=1}^d (b_i^* q^{n+\overline{c}}) M : M a_i/(b_i q^{n+\overline{c}}) M$. Now assume that $t > 0$ and $bA \subseteq aA$ by lemma 4.2. As by virtue of Valla-Valabrega [13], $b_1^*, \ldots, b_t^*$ is a $G_{M/bh,M}(q)$-regular sequence, therefore by induction $L_i(a, q, M/b_1 M; n) = 0$ for all $i > d - t + 1$ and for all $n$. Hence by lemma 4.1, $L_i(a, q, M; n) = 0$ for all $i > d - t + 1$ and for all $n$, and $L_{d-t+1}(a, q, M; n - \beta_1) = 0$ for all $n$. Note that $b_1 L_i(a, q, M; n) = 0$ for all $i$, for all $n$, see [8, Theorem 3.5(b)].

Again by induction

$$L_{d-t+1}(a, q, M/b_1 M; n) \cong \bigcap_{i=1}^d (b_i^* q^{n+\sum_{j=2}^d \beta_j-\overline{c}}) M : M a_i/(b_i q^{n+\sum_{j=2}^d \beta_j-\overline{c}}) M$$

and hence by using 4.1 and $b_1 L_i(a, q, M; n) = 0$ for all $i, n$, we get

$$L_{d-t+1}(a, q, M/b_1 M; n) \cong L_{d-t}(a, q, M; n - \beta_1).$$

This finishes the inductive argument. \hfill \square
There is a converse of the previous theorem.

**Proposition 4.4.** With the previous notations, assume that \(L_i(\underline{a}, q; M; n) = 0\) for all \(i > d - t\), for all \(n\), then \(\underline{a}^*G_A(q)\) contains a \(G_M(q)\)-regular sequence \(b^* = b_1^*, \ldots, b_t^*\).

**Proof.** Note that from the short exact sequence

\[
0 \to q^n M/q^{n+1} M \to M/q^{n+1} M \to M/q^n M \to 0,
\]

there is the following short exact sequence of complexes

\[
0 \to K_\bullet(\underline{a}^*; G_M(q))_n \to L_\bullet(\underline{a}, q; M; n + 1) \to L_\bullet(\underline{a}, q; M; n) \to 0
\]

for \(n \in \mathbb{N}\), where \(K_\bullet(\underline{a}^*; G_M(q))_n\) denotes the \(n\)th component of the Koszul complex of \(G_M(q)\) w.r.t. \(\underline{a}^* = a_1^*, \ldots, a_d^*\). From here, by view of long homology exact sequence

\[
\cdots \to H_i(\underline{a}^*; G_M(q))_n \to H_i(\underline{a}^*; G_M(q))_n \to 0
\]

for all \(i > d - t\), for all \(n\), and hence \(H_i(\underline{a}^*; G_M(q)) = 0\) for all \(i > d - t\). Now the result follows by virtue of Koszul homology, see [9]. \(\square\)

5. Applications

Let \((A, m)\) denote a local Noetherian ring and \(M\) be a finitely generated \(A\)-module with \(\dim M = d\). Let \(\underline{a} = a_1, \ldots, a_d\) denote a system of parameters of \(M\) such that \(\underline{a} \subset q\). We present the main result of the section.

**Theorem 5.1.** With the previous notations, if \(\underline{a}^*G_A(q)\) contains a \(G_M(q)\)-regular sequence of length \(d - 1\), then

1. \(\ell_A(L_1(\underline{a}, q; M; n))\) is a constant for all \(n \gg 0\).
2. \(\ell_A(M/\underline{a}M) = c \cdot e_0(q; M) + \ell_A(L_1(\underline{a}, q; M; n))\) for all \(n \gg 0\), where \(c = c_1 \cdots c_d\).

**Proof.** Note that the alternating sum of the lengths of modules in the complex \(L_\bullet(\underline{a}, q; M; n)\) is

\[
\sum_{i=0}^{d} (-1)^i \sum_{1 \leq j_1 < \cdots < j_i \leq d} \ell_A(M/q^{n-c_{j_1} \cdots - c_{j_i}} M),
\]

which is a weighted \(d\)-fold difference operator of Hilbert-Samuel polynomial for all \(n \gg 0\) and hence is a constant \(c \cdot e_0(q; M)\). Also, it coincides with the Euler characteristic

\[
\chi_A(L_\bullet(\underline{a}, q; M; n)) = \sum_{i \geq 0} (-1)^i \ell_A(L_i(\underline{a}, q; M; n))\]

for all \(n \gg 0\), see [8] for more detail. As \(L_0(\underline{a}, q; M; n) \cong M/(\underline{a}M, q^n M) = M/\underline{a}M\) for all \(n \gg 0\) since \(q^n M \subseteq \underline{a}M\), therefore (2) follows from theorem 4.3. Also, (1) follows from (2). This completes the argument. \(\square\)

Now, we describe the length \(\ell_A(L_1(\underline{a}, q; M; n))\).

**Proposition 5.2.** With the previous notations, if \(\underline{a}^*G_A(q)\) contains a \(G_M(q)\)-regular sequence \(b^* = b_1^*, \ldots, b_{d-1}^*\) such that \(\deg b_i^* = \beta_i\) and we choose \(b_i\) for \(i = 1, \ldots, d - 1\) as in Lemma 4.2. Then \(\ell_A(L_1(\underline{a}, q; M; n))\) might be broken into two pieces. That is,

\[
\ell_A(L_1(\underline{a}, q; M; n)) = \ell_A([b^*G_M(q) : \underline{a}^*/(b^*G_M(q))]_{n+\beta - \tau - 1}) + \ell_n,
\]

where

\[
\ell_n = \ell_A(\cap_{i=1}^{d}(\underline{b}, q^{n+\beta - \tau})M \cap_{i=1}^{d}(\underline{b}, q^{n+\beta - \tau})M : M a_i)/(\cap_{i=1}^{d}(\underline{b}, q^{n+\beta - \tau})M : M a_i) \cap (\underline{b}, q^{n+\beta - \tau - 1})M,
\]

with \(c = c_1 \cdots c_d\), \(\tau = \sum_{i=1}^{d} c_i \tau_i = c_1 + \cdots + c_{i-1} + c_{i+1} + \cdots + c_d\), and \(\beta = \sum_{j=1}^{d-1} \beta_j\).

Moreover, for \(n \gg 0\), all of the lengths involved here are constants and independent of the choice of \(b^*\). We write \(r = \ell_A([b^*G_M(q) : \underline{a}^*/(b^*G_M(q))]_{n})\) and \(\ell = \ell_n\) for \(n \gg 0\).
Proof. As $b^*$ is a $G_M(q)$-regular sequence, hence $G_M(q)/(b^*)G_M(q) \cong G_M/\hat{b}M(q)$, see [13]. Therefore, it is easily seen that
\[
[b^*G_M(q) : a^*/b^*G_M(q)]_n \cong (\cap_{i=1}^d (b, q^{a_i+1})M :M a_i) \cap (b, q^n)M/(b, q^{n+1})M.
\]
Now, we have the following short exact sequence
\[
0 \to [b^*G_M(q) : a^*/b^*G_M(q)]_{n+\beta-1} \to \mathcal{L}_1(a, q, M; n) \to \to (\cap_{i=1}^d (b, q^{a_i+\beta-1})M :M a_i)/(\cap_{i=1}^d (b, q^{a_i+\beta-1})M :M a_i) \cap (b, q^n+1)M \to 0,
\]
see theorem 4.3. By counting the lengths, it provides the first equality of the statement. The length of the module in the middle is constant for $n \gg 0$, see 5.1. Also, the length of the module in the left is constant for $n \gg 0$ since it is of dimension 1. By comparing the Hilbert polynomials, this proves that all the lengths are constants for all $n \gg 0$.

Note that
\[
\frac{b^*G_M(q) : a^*/b^*G_M(q)}{\cong \text{Ext}^{d-1}_{G_A(q)}(G_A(q)/a^*G_A(q), G_M(q))[-\beta].}
\]
Therefore, we conclude that $\ell_A([b^*G_M(q) : a^*/b^*G_M(q)]_n)$ is independent of the choice of $b^*$, and consequently, $\ell$ is also independent of the choice of $b^*$. This completes the proof. \hfill \Box

Now, we have the main result of the section, which is also the consequence of previous two results.

**Corollary 5.3.** With the previous notations, if $a^*G_A(q)$ contains a $G_M(q)$-regular sequence $b^* = b_1^*, \ldots, b_{d-1}^*$ and we choose $b_i$ for $i = 1, \ldots, d - 1$ as in Lemma 4.2. Then
\[
\ell_A(M/\hat{a}M) \geq c \cdot e_0(q; M) + \rho
\]
where $c = c_1 \cdot \ldots \cdot c_d$ and $\rho = \ell_A([\text{Ext}^{d-1}_{G_A(q)}(G_A(q)/a^*G_A(q), G_M(q))]_{n-\beta-1})$ is a constant for all $n \gg 0$ and $\rho = c_1 + \ldots + c_d$.

We mention a geometric application to local Bézout inequality in the affine plane $A^2_k$.

**Remark 5.4.** Let $k$ be an algebraically close field and $A = k[x, y]_{(x, y)}$ be a local ring. Also, let $f, g$ denote a system of parameters in $A$ and $\mathfrak{m}$ denote the maximal ideal of $A$. Then $B := k[X,Y] \cong G_A(\mathfrak{m})$ and $1 = e_0(\mathfrak{m}; A)$. Then the above two results imply that
\[
e_0(f, g; A) \geq c \cdot d + t,
\]
where $t$ denotes the number of common tangents to $f, g$ at origin when counted with multiplicities. Note that $\ell_A([f^*B : B g/f^*B]_n) = t$ for all $n \gg 0$ (see [2]).

**Problem 5.5.** Let $M = A = k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$ be the local ring and $q = \mathfrak{m} = (x_1, \ldots, x_d)A$, where $d \geq 3$. Let $a^* = a_1^*, \ldots, a_{d-1}^*$, then the author does not know the geometric interpretation of
\[
\rho = \ell_A([\text{Ext}^{d-1}_{G_A(q)}(G_A(q)/a^*G_A(q), G_M(q))]_n) = \ell_A([a^*G_A(\mathfrak{m})/a^*G_A(\mathfrak{m})]_{n+c_1+\ldots+c_{d-1}})
\]
for all $n \gg 0$. This problem can be related to the homological terms as in case of $d = 2$.

In the next, we present another consequence. More precisely, there is an upper bound to $\ell_A(M/\hat{a}M) - e_0(a; M) \geq 0$.

**Corollary 5.6.** With the previous notations, if $a^*G_A(q)$ contains a $G_M(q)$-regular sequence $b^* = b_1^*, \ldots, b_{d-1}^*$, then
1. $\ell_A(M/\hat{a}M) - e_0(a; M) \leq \ell_A(L_1(a, q, M; n))$ for all $n \gg 0$.
2. Equality occurs when $a^*$ is a system of parameters of $G_M(q)$. The converse is not true in general.
Proof. Since $c \cdot e_0(q; M) \leq e_0(q; M)$ (see [2]). Therefore claim in (1) follows from previous theorem 5.1. Note that
\[
\ell_A(M/aM) - e_0(q; M) = \ell_A(L_1(a, q, M; n)) \quad \text{for all } n \gg 0 \iff e_0(q; M) = c \cdot e_0(q; M),
\]
(see theorem 5.1). Now, the claim in (2) follows by [2, Theorem 5.1].

6. An Euler Characteristic

With the notations of the previous section, we have the following lemma.

Lemma 6.1. With the previous notations, if $a^n = a_1^*, \ldots, a_{d-1}^*$ is a $G_M(q)$-regular sequence, then
\[
e_0(q; M) = c \cdot e_0(q; M) + \ell_A(q^nM/\sum_{i=1}^{d} a_iq^{n-c_i}M)
\]
\[-\ell_A(\sum_{i=1}^{d-1} a_iq^{n+c_i-c_i}M : M a_d \cap q^nM/\sum_{i=1}^{d-1} a_iq^{n-c_i}M)
\]
for all $n \gg 0$, where $c = c_1 \cdot \ldots \cdot c_d$.

Proof. Note $H_i(a; M) = 0$ for all $i > 1$ since $a_1, \ldots, a_{d-1}$ is $M$-regular sequence, cf. [13]. Also, $L_i(a, q, M; n) = 0$ for all $i > 1$ and for all $n$, see 4.3. Moreover,
\[
L_0(a, q, M; n) = M/(a, q^nM)M \equiv M/aM = H_0(a; M)
\]
since $q^nM \subseteq aM$ for $n \gg 0$. Therefore, from the long exact homology sequence coming from the short exact sequence in 2.3, we get the following exact sequence
\[
0 \to H_1(a, q, M; n) \to H_1(a; M) \to L_1(a, q, M; n) \to H_0(a, q, M; n) \to 0,
\]
for all $n \gg 0$. Note that $H_1(a; M) \cong a'M : M a_d/a'M$, where $a' = a_1, \ldots, a_{d-1}$. By Theorem 5.1, we get
\[
\ell_A(M/aM) - \ell_A(a'M : M a_d/a'M) = c \cdot e_0(q; M) + \ell_A(H_0(a, q, M; n)) - \ell_A(H_1(a, q, M; n))
\]
for all $n \gg 0$. Finally, note that
\[
\ell_A(M/aM) - \ell_A(a'M : M a_d/a'M) = e_0(a; M), H_0(a, q, M; n) = q^nM/\sum_{i=1}^{d} a_iq^{n-c_i}M
\]
and
\[
H_1(a, q, M; n) = [H_1(aT^{c_1}; R_M(q))]_n \cong \sum_{i=1}^{d-1} a_iq^{n+c_i-c_i}M : M a_d \cap q^nM/\sum_{i=1}^{d-1} a_iq^{n-c_i}M
\]
since $a_1T^{c_1}, \ldots, a_{d-1}T^{c_{d-1}}$ is an $R_M(q)$-regular sequence, see 3.2. This finishes the proof.

Let $\chi_A(a, q, M; n)$ denote the Euler characteristic of the complex $K_n(a, q, M; n)$. With the assumption of previous lemma,
\[
\chi_A(a, q, M; n) = \ell_A(q^nM/\sum_{i=1}^{d} a_iq^{n-c_i}M) - \ell_A(\sum_{i=1}^{d-1} a_iq^{n+c_i-c_i}M : M a_d \cap q^nM/\sum_{i=1}^{d-1} a_iq^{n-c_i}M)
\]
which is a constant. Even in a more general situation, we have
\[
\chi_A(a, q, M; n) = e_0(a; M) - c \cdot e_0(q; M) \quad \text{for all } n \gg 0,
\]
see [2] or [8]. Moreover, the authors mentioned a problem of giving an interpretation to $\chi_A(a, q, M; n)$ for $n \gg 0$. In case of $M = A$, Bôda-Schenzel [2] proved that $\chi_A(a, q, M; n) \geq 0$.
Note that we have the following complex finishes (2).

**Corollary 6.2.** With the previous notations, if \( \mathfrak{a}^* = a_1^*, \ldots, a_{d-1}^* \) is a \( G_M(q) \)-regular sequence, then

1. \( \chi_A(\mathfrak{a}, q, M; n) \leq \ell_A(L_1(\mathfrak{a}, q, M; n)) \) for all \( n \gg 0 \).
2. Equality occurs if and only if \( M \) is Cohen-Macaulay.

**Proof.** Since \( \ell_A(M/\mathfrak{a}M) \geq e_0(\mathfrak{a}; M) \), therefore claim in (1) follows from previous theorem 5.1. Note that

\[
\chi_A(\mathfrak{a}, q, M; n) = \ell_A(L_1(\mathfrak{a}, q, M; n)) \quad \text{for all} \quad n \gg 0 \iff \ell_A(M/\mathfrak{a}M) = e_0(\mathfrak{a}; M),
\]

see theorem 5.1. The latter is equivalent to the fact that \( M \) is Cohen-Macaulay, see [4]. This finishes (2). \( \square \)

In the following, we discuss a few more properties of Euler characteristic \( \chi_A(\mathfrak{a}, q, M; n) \). We need the following lemma.

**Lemma 6.3.** With the previous notations, assume that \( a \in q^c \setminus q^{c+1} \) such that \( \dim M/aM = d - 1 \). Then the following holds.

1. If \( \dim 0 : M a \leq d - 2 \), then \( c \cdot e_0(q; M) \leq e_0(q; M/aM) \). Moreover, equality occurs if and only if \( \deg \ell_A(q^nM : a/(q^{n-c}M + 0 : M a)) \leq d - 2 \) for all \( n \gg 0 \).
2. If \( \dim 0 : M a = d - 1 \), then \( c \cdot e_0(q; M) + e_0(0 : M a) \leq e_0(q; M/aM) \). Moreover, equality occurs if and only if \( \deg \ell_A(q^nM : a/(q^{n-c}M + 0 : M a)) \leq d - 2 \) for all \( n \gg 0 \).

**Proof.** Note that we have the following complex

\[
\mathcal{L}_*(a, q, M; n) : 0 \rightarrow M/q^{n-c}M \xrightarrow{a} M/q^nM \rightarrow 0,
\]

and hence

\[
\ell_A(M/q^nM) - \ell_A(M/q^{n-c}M) = \ell_A(L_0(a, q, M; n)) - \ell_A(L_1(a, q, M; n)),
\]

where \( L_0(a, q, M; n) \cong M/(a, q^n)M \) and \( L_1(a, q, M; n) \cong q^nM : a/q^{n-c}M \). We break \( \ell_A(q^nM : a/q^{n-c}M) \) by using the following short exact sequence:

\[
0 \rightarrow (q^{n-c}M + 0 : M a)/q^{n-c}M \rightarrow q^nM : a/q^{n-c}M \rightarrow q^nM : a/(q^{n-c}M + 0 : M a) \rightarrow 0.
\]

Also, by Artin-Rees we have

\[
(q^{n-c}M + 0 : M a)/q^{n-c}M = 0 : M a/q^{n-c}M \cap 0 : M a = 0 : M a/q^{n-c-l}(q^lM \cap 0 : M a)
\]

for some \( l \in \mathbb{N} \) and for all \( n \geq l \). That is,

\[
\ell_A(q^nM : a/q^{n-c}M) = \ell_A(q^lM \cap 0 : M a/q^{n-c-l}(q^lM \cap 0 : M a)) + \ell_A(q^lM : a/(q^{n-c}M + 0 : M a)) + \ell_A(0 : M a/0 : M a \cap q^lM).
\]

By using last equation into \((*)\), we get

\[
\ell_A(M/q^nM) - \ell_A(M/q^{n-c}M) = \ell_A(M/(a, q^n)M) - \ell_A(q^lM \cap 0 : M a/q^{n-c-l}(q^lM \cap 0 : M a)) - \ell_A(q^nM : a/(q^{n-c}M + 0 : M a)) - \ell_A(0 : M a/0 : M a \cap q^lM),
\]

where all lengths involved are polynomials for \( n \gg 0 \) with \( \deg(\ell_A(M/q^nM) - \ell_A(M/q^{n-c}M)) = \deg(\ell_A(M/(a, q^n)M) - \deg(\ell_A(q^lM \cap 0 : M a/q^{n-c-l}(q^lM \cap 0 : M a)) = \dim 0 : M a \leq d - 1, \deg(\ell_A(q^nM : a/(q^{n-c}M + 0 : M a)) \leq d - 1 \) and \( \ell_A(0 : M a/0 : M a \cap q^lM) \) is a constant.
Also, leading terms of $\ell_A(M/q^nM) - \ell_A(M/q^{n-c}M)$ and $\ell_A(M/(a, q^nM)$ and $e_0(q; M/aM)$ respectively for all $n \gg 0$. Now, in case of (1), we get
\[ c \cdot e_0(q; M) \leq e_0(q; M/aM), \]
and in case of (2), we get
\[ c \cdot e_0(q; M) + e_0(q; 0 :_M a) \leq e_0(q; M/aM). \]
Note that leading term of $\ell_A(q^1M \cap 0 :_M a/q^{n-c-1}(q^1M \cap 0 :_M a))$ is $e_0(q; q^1M \cap 0 :_M a)$, which is equal to $e_0(q; 0 :_M a)$. Indeed, we have the following short exact sequence
\[ 0 \to q^1M \cap 0 :_M a \to 0 :_M a \to 0 :_M a/(q^1M \cap 0 :_M a) \to 0, \]
and $\dim 0 :_M a/(q^1M \cap 0 :_M a) = 0$ whereas $\dim 0 :_M a = \dim(q^1M \cap 0 :_M a)$. Therefore $e_0(q; 0 :_M a) = e_0(q; q^1M \cap 0 :_M a)$, cf. [9, Theorem 13.3]. Finally, equality in both cases occur if and only if $\deg \ell_A(q^nM : a/(q^{n-c}M + 0 :_M a)) \leq d - 2$ for all $n \gg 0$. \hfill \Box

There is the following consequence of previous lemma.

**Proposition 6.4.** With the previous notations, let $\underline{a} = a_1, \ldots, a_d$ be a system of parameters of $M$, then the following holds.

1. If $\dim 0 :_M a \leq d - 2$, then $\chi_A(\underline{a}, q, M) \geq \chi_A(\underline{a}', q, M/a_1M)$, where $\underline{a}' = a_2, \ldots, a_d$. Moreover, equality occurs if and only if
   \[ \deg \ell_A(q^nM : a_1/(q^{n-c}M + 0 :_M a_1)) \leq d - 2 \]
   for all $n \gg 0$.

2. If $\dim 0 :_M a = d - 1$, then $\chi_A(\underline{a}, q, M) + \chi_A(\underline{a}', q, 0 :_M a_1) \geq \chi_A(\underline{a}', q, M/a_1M)$, where $\underline{a}' = a_2, \ldots, a_d$. Moreover, equality occurs if and only if
   \[ \deg \ell_A(q^nM : a_1/(q^{n-c}M + 0 :_M a_1)) \leq d - 2 \]
   for all $n \gg 0$.

**Proof.** Note that $\underline{a}' = a_2, \ldots, a_d$ is a system of parameters for both $A$-modules $M/a_1M$ and $0 :_M a_1$. Also, it is a well known fact that
\[ e_0(q; \underline{a}, M) + e_0(q; \underline{a}' :_M a_1) = e_0(q; \underline{a}' :_M a_1), \]
see for example [4]. For (1), we use Lemma 6.3(1) and get $c \cdot e_0(q; M) \leq e_2 \cdot \ldots \cdot c_d \cdot e_0(q; M/a_1M)$, where $c = c_1 \cdot \ldots \cdot c_d$. Hence by definition
\[ \chi_A(\underline{a}, q, M) \geq \chi_A(\underline{a}', q, M/a_1M), \]
where the equality occurs if and only if $\deg \ell_A(q^nM : a/(q^{n-c}M + 0 :_M a_1)) \leq d - 2$ for all $n \gg 0$.

For (2), we use Lemma 6.3(2) and get
\[ c \cdot e_0(q; M) + e_2 \cdot \ldots \cdot c_d \cdot e_0(q; 0 :_M a_1) \leq e_2 \cdot \ldots \cdot c_d \cdot e_0(q; M/a_1M). \]
Hence by definition
\[ \chi_A(\underline{a}, q, M) + \chi_A(\underline{a}', q, 0 :_M a_1) \geq \chi_A(\underline{a}', q, M/a_1M), \]
where the equality occurs if and only if $\deg \ell_A(q^nM : a/(q^{n-c}M + 0 :_M a_1)) \leq d - 2$ for all $n \gg 0$. \hfill \Box

We finish with the following remark.
Remark 6.5. Lemma 6.3, with slight modification, originally proved by Flenner-Vogel [7]. More precisely, they proved the equality in lemma if and only if \(a^*\) is a parameter for \(G_M(q)\). The author of present note tried to prove directly that \(\deg \ell_A(q^nM : a / (q^{n-c_1} M + 0 : M a)) \leq d - 2\) for all \(n \gg 0\) if and only if \(a^*\) is a parameter for \(G_M(q)\). The "if" part is easy. Indeed, \(\deg \ell_A(q^nM : a / (q^{n-c_1} M + 0 : M a)) \leq d - 2\) for all \(n \gg 0\) implies that
\[
\deg \ell_A(q^nM : a / q^{n-c_1} M + (q^{n+1} M : a)) \leq d - 2 \quad \text{for all } n \gg 0,
\]
where
\[
q^nM : a / q^{n-c_1} M + (q^{n+1} M : a) \cong \ker(G_M(q)/a^*G_M(q) \to G_M/aM(q)).
\]
But, this is equivalent to \(\dim(\ker(G_M(q)/a^*G_M(q) \to G_M/aM(q))) \leq d - 1\) which is equivalent to the fact that \(a^*\) is a parameter for \(G_M(q)\).

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