Examples of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry breaking

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ABSTRACT

In this paper we consider gauged $\mathcal{N} = 2$ supergravities which arise in the low-energy limit of type II string theories and study examples which exhibit spontaneous partial supersymmetry breaking. For the quantum $STU$ model we derive the scalar field space and the scalar potential of the $\mathcal{N} = 1$ supersymmetric low-energy effective action. We also study the properties of the Minkowskian $\mathcal{N} = 1$ supersymmetric ground states for a broader class of supergravities including the quantum $STU$ model.

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1 Introduction

After the initial no-go theorem of Refs. [1, 2] explicit \( \mathcal{N} = 2 \) supergravities with spontaneous partial supersymmetry breaking were first discussed in Refs. [3–5] following the global analysis of Refs. [6, 7]. Recently a systematic analysis was performed in Refs. [8–10] using the embedding tensor formalism [11]. It was shown that both electrically and magnetically charged hypermultiplets have to be in the spectrum in order to circumvent the no-go theorem of Refs. [1, 2]. The presence of two holomorphic commuting Abelian isometries in the hypermultiplet sector are an additional necessary ingredient of the supergravity to show partial supersymmetry breaking.

Below the scale of the supersymmetry breaking \( m_{3/2} \) one can construct a low-energy effective \( \mathcal{N} = 1 \) action in terms of the \( \mathcal{N} = 2 \) “input data” by integrating out all heavy modes with masses of order \( \mathcal{O}(m_{3/2}) \). The resulting \( \mathcal{N} = 1 \) field space is a quotient of the original \( \mathcal{N} = 2 \) quaternionic-Kähler manifold with respect to the two isometries. The properties of the two isometries ensure that the quotient is a Kähler manifold consistent with the \( \mathcal{N} = 1 \) of the effective theory [9, 10]. If only these two isometries are gauged, no further supersymmetry breaking is possible. However, if additional isometries, which do not participate in the \( \mathcal{N} = 2 \to \mathcal{N} = 1 \) breaking, are gauged, the superpotential \( \mathcal{W} \) and the \( \mathcal{D} \)-terms can be non-trivial and possibly induce the breaking of \( \mathcal{N} = 1 \) at a lower scale.

In this paper we consider \( \mathcal{N} = 2 \) supergravities which arise as the low energy effective theory of type II string compactifications, i.e., supergravities which are in the image of the c-map [12]. In this class of supergravities the quaternionic-Kähler manifold of the hypermultiplets is ‘special’ in that it has a specific fibration structure with a base which is determined by a holomorphic prepotential \( \mathcal{G} \). As a consequence of the fibration isometries exist which can induce partial supersymmetry breaking.

In the resulting \( \mathcal{N} = 1 \) backgrounds some of the original scalar fields of the \( \mathcal{N} = 2 \) theory acquire a vacuum expectation value and become massive. For special quaternionic-Kähler manifolds only two scalars of the fibre are fixed while in the base the number of massive scalars depends on the form of the prepotential \( \mathcal{G} \). For a quadratic \( \mathcal{G} \) no scalars are fixed while a generic \( \mathcal{G} \) fixes all scalars in the base. We focus on two specific examples where the base of the fibration is a complex three-dimensional manifold determined by the cubic \( STU \) and quantum \( STU \) prepotentials. In both models one complex scalar field of the base together with three fields in the fibre remain free and we explicitly determine \( \mathcal{W} \) and \( \mathcal{D} \) below the scale of partial supersymmetry breaking together with the \( \mathcal{N} = 1 \) scalar field space. For the \( STU \)-model we find the latter to be the symmetric space \( SO(4) \times SO(2) \).

We also discuss the conditions for \( \mathcal{N} = 1 \) supersymmetric Minkowskian vacua for generic supergravities in the image of the c-map. The superpotential and the \( \mathcal{D} \)-term depend on two holomorphic prepotentials, the \( \mathcal{G} \) of the hypermultiplet sector together with the \( \mathcal{F} \) which encodes the couplings of the vector multiplets. We show that the form of these prepotentials does not only determine the dimension of the \( \mathcal{N} = 1 \) scalar field space but also the existence of \( \mathcal{N} = 1 \) supersymmetric minima.

This paper is organized as follows. In Section 2 we recapitulate the relevant aspects of four-dimensional gauged \( \mathcal{N} = 2 \) supergravities, partial \( \mathcal{N} = 2 \to \mathcal{N} = 1 \) supersymmetry
breaking and the resulting $\mathcal{N} = 1$ low energy effective action. In Section 3 we focus on special quaternionic-Kähler manifolds which arise in the hypermultiplet sector of type II string compactifications and recall for this case the quotient construction leading to the $\mathcal{N} = 1$ low energy effective action. In Section 4 we calculate the $\mathcal{N} = 1$ Kähler potential explicitly for the (quantum) STU model. In Section 5 we compute the superpotential and the $\mathcal{D}$-terms for generic special quaternionic-Kähler manifolds and for the quantum STU-model. In Section 6 we determine the supersymmetric ground states of the $\mathcal{N} = 1$ theory. Appendix A discusses field redefinitions that are used to bring the quantum STU Kähler potential to its final form and Appendix B contains the calculation of Killing prepotentials that appear in the superpotential and $\mathcal{D}$-terms.

## 2 Partially broken $\mathcal{N} = 2$ supergravities

### 2.1 $\mathcal{N} = 2$ supergravity in four dimensions

In order to set the stage let us briefly recall some properties of four-dimensional $\mathcal{N} = 2$ supergravity with gauged Abelian isometries in the hypermultiplet sector (for a review, see e.g. [13]). The spectrum consists of a gravitational multiplet, $n_v$ vector multiplets and $n_h$ hypermultiplets. The gravitational multiplet $(g_{\mu\nu}, \Psi_{\mu}, \Lambda_0)$ contains the spacetime metric $g_{\mu\nu}$, $\mu, \nu = 0, \ldots, 3$, two gravitini $\Psi_{\mu}$, $A = 1, 2$, and the graviphoton $A_0$. A vector multiplet $(A_\mu, \lambda^A, t)$ contains a vector $A_\mu$, two gaugini $\lambda^A$ and a complex scalar $t$, while a hypermultiplet $(\zeta_\alpha, q^u)$ contains two hyperini $\zeta_\alpha$ and four real scalars $q^u$.

The Lagrangian of the scalar fields is given by

$$L = g_{ij}(t, \bar{t}) \partial_i t^i \partial_{\bar{j}} \bar{t}^j + h_{uv}(q) D_\mu q^u D^\mu q^v - V(t, q),$$

where the indices take the values $i, \bar{j} = 1, \ldots, n_v$, $u, v = 1, \ldots, 4n_h$. $g_{ij}(t, \bar{t})$ is the metric of a $2n_v$-dimensional special-Kähler manifold $M_v$ and $h_{uv}(q)$ is the metric of the $4n_h$-dimensional quaternionic-Kähler manifold $M_h$. From (2.1) we see that the total scalar field space locally is the direct product

$$M = M_v \times M_h.$$  

The metric of the special-Kähler manifold is given by

$$g_{ij} = \partial_i \partial_{\bar{j}} K^\nu, \quad \text{where} \quad K^\nu = -\ln i (\bar{X}^I \mathcal{F}_I - X^I \bar{\mathcal{F}}_I)$$

is the Kähler potential. $X^I(t)$ and $\mathcal{F}_I(t)$, $I = 1, \ldots, n_v + 1$, are holomorphic functions of the $t^i$ with $\mathcal{F}_I = \partial \mathcal{F}/\partial X^I$ being the derivatives of a holomorphic prepotential $\mathcal{F}(X)$ which is homogenous of degree two. There is a choice of coordinates, called “special coordinates”, where $X^I = (t^i, 1).$\footnote{We are choosing this convention in order to be consistent with Ref. [10] later on.}

$h_{uv}$ in (2.1) denotes the metric on the $4n_h$-dimensional quaternionic-Kähler manifold $M_h$. These manifolds admit three almost complex structures $J^x$, $x = 1, 2, 3$ that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} 1 + \epsilon^{xyz} J^z,$$
Spontaneous supersymmetry breaking can be analyzed in terms of the scalar parts of the supersymmetry transformations

\[ \nabla \text{derivative} \]

\[ V^\Lambda \text{symmetric transformations} \]

\[ \text{given by} \]

\[ t^\text{scalar partners} \]

\[ \text{charges imposes} \]

\[ \text{with the metric} \ h_{uv} \text{being hermitian with respect to all three of them. Supersymmetry requires the existence of a principal} \ SU(2) \text{-bundle over} \ M_h \text{with a curvature two-form} \]

\[ K^x = d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z , \]

\[ \text{where} \ \omega^x \text{denotes the one-form connection on the} \ SU(2) \text{-bundle.} \]

For simplicity we consider only Abelian gauge fields and hence their supersymmetric scalar partners \( t^i \) are neutral. The scalars \( q^u \) in the hypermultiplets on the other hand can be charged and their covariant derivatives in (2.1) are defined as

\[ D_\mu q^u = \partial_\mu q^u - A_\mu^i \Theta^i_\lambda k^u_\lambda + B_{\mu \ell} \Theta^{i \lambda}_\ell k^u_\lambda , \]

where \( \lambda \) labels the non-trivial Killing vectors \( k_\lambda(q) \) on \( M_h \) and \( A_\mu^i \) are electric vectors while \( B_{\mu \ell} \) are magnetic duals. The charges \( \Theta^i_\lambda \) and \( \Theta^{i \lambda}_\ell \) are the electric and magnetic parts of the embedding tensor which in the following we frequently combine into the symplectic object \( \Theta^x_\lambda = (\Theta^i_\lambda, -\Theta^{i \lambda}_\ell) \) [11]. Mutual locality of electric and magnetic charges imposes

\[ \Theta^{[i \lambda} \Theta^{\ell]} = 0 . \]

Finally, the scalar potential \( V \) appearing in (2.1) is given by

\[ V = -12 S_{AB} \bar{S}^{AB} + g_{ij} W^{iAB} W^{j}_{AB} + 2 N^A_{\lambda} N^\alpha_{A} , \]

where the couplings \( S_{AB}, W^{iAB} \) and \( N^A_{\alpha} \) denote the scalar part of the fermionic supersymmetric transformations given by

\[ S_{AB} = \frac{1}{2} e^{K^i/2} \nabla^A \Theta^\lambda_\Lambda P^u_\lambda \epsilon_{AC} (\sigma^x)^C , \]

\[ W^{iAB} = i e^{K^i/2} g^{ij} (\nabla^j V^\Lambda) \Theta^\lambda_\Lambda P^x_\lambda (\sigma^x)^C \epsilon^{CB} , \]

\[ N^A_{\alpha} = 2 e^{K^i/2} \nabla^A \Theta^\lambda_\Lambda \epsilon^{AC} U_{\alpha u} k^u_\lambda . \]

\( V^\Lambda \) is a holomorphic symplectic vector defined by \( V^\Lambda \equiv (X^I, F_I) \) with Kähler-covariant derivative \( \nabla_i V^\Lambda = \partial_i V^\Lambda + (\partial_i K^\nu) V^\nu . \) \( \epsilon_{AB} \) is the two-dimensional \( \epsilon \)-tensor, \( \epsilon^{AB} \) its inverse and \( (\sigma^x)^A_B \) are the standard Pauli matrices. The isometries on \( M_h \) generated by \( k^x_\lambda \) can be characterized by a triplet of Killing prepotentials (moment maps) \( P^x_\lambda \) defined by

\[ -2 k^u_\lambda K^x_{uv} = \nabla_v P^x_\lambda = \partial_v P^x_\lambda + \epsilon^{xyz} \omega^y_v P^z_\lambda , \]

where \( K^x_{uv} \) are the coefficients of the two-forms defined in (2.5) and \( \omega^y_v \) is the \( SU(2) \) connection. \( U_{\alpha u} \) is the vielbein of \( M_h \) which can be used to express the metric as

\[ h_{uv} = U_{\alpha u} \epsilon_{AB} C_{\alpha \beta} U^{\beta v} , \]

where \( C_{\alpha \beta} \) is the \( Sp(n_h) \) invariant metric.

### 2.2 Spontaneous \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 1 \) supersymmetry breaking

Spontaneous supersymmetry breaking can be analyzed in terms of the scalar parts of the supersymmetry transformations

\[ \delta_\epsilon \Psi_{\mu A} = D_\mu \epsilon^*_A - S_{AB} \gamma_\mu \epsilon^B + \ldots , \]

\[ \delta_\epsilon \lambda^i_A = W^{iAB} \epsilon^B + \ldots , \]

\[ \delta_\epsilon \zeta_\alpha = N^A_{\alpha} \epsilon_A + \ldots , \]
with $S_{AB}, W^{iAB}$ and $N_a^A$ given in (2.9) and $e^A$ being the $SU(2)$ doublet of supersymmetry parameters. Partial supersymmetry breaking occurs whenever the theory has a background where one linear combination of supersymmetry transformations is nonzero while the second vanishes \[1, 2\].\(^2\) In this case one of the two gravitinos gains a mass $m_{3/2}$ via the super-Higgs mechanism and the unbroken supersymmetry implies that this heavy gravitino is part of an entire $\mathcal{N} = 1$ massive spin-3/2 multiplet with spin content $s = (3/2, 1, 1, 1/2)$. The two massive vectors in this multiplet are the graviphoton together with a vector of a vector multiplet while the necessary Goldstone fields are recruited from two charged hypermultiplets \[3, 4, 8\]. This in turn implies that (at least) two isometries $k_1$ and $k_2$ on $M_h$ with particular properties have to be gauged. Concretely, these isometries must be such that for the choice $P_{1,2} = 0$, the other Killing prepotentials have to satisfy $P_{1}^1 = -P_{2}^2$, $P_{1}^2 = P_{2}^1$ \[9\]. If additional isometries are gauged, we insist that they do not contribute to the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking but may break $\mathcal{N} = 1$ at some lower scale.

The general solution of partial $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking is derived in Refs. \[8, 9\] and will not be repeated here in all generality. For a given supergravity it constrains the embedding tensor or in other words the structure of the gauge charges. In this paper we confine our interest to Minkowski backgrounds\(^3\) where the embedding tensors that partially break supersymmetry can be expressed in the form \[8\]

\[
\Theta_{1} = \text{Re} \left( \mathcal{F}_{IJ} C^{J} \right), \quad \Theta_{1}^{1} = \text{Re} C^{I},
\]

\[
\Theta_{2} = \text{Im} \left( \mathcal{F}_{IJ} C^{J} \right), \quad \Theta_{2}^{1} = \text{Im} C^{I},
\]

with $C^I$ being complex constants satisfying

\[
\mathcal{C}^I (\text{Im} \mathcal{F}_{IJ} C^{J}) = 0 .
\]

Since the embedding tensor has to be constant, (2.13) stabilizes $\text{rk}(\mathcal{F}_{IJ} C^{J})$ of the $n_v$ complex coordinates on $M_v$ by the condition \[10\]

\[
\mathcal{F}_{IJ} C^{J} = \text{const.} \quad \text{or} \quad \mathcal{F}_{IJ} C^{J} \delta X^K = 0 .
\]

This implies that only a submanifold $\bar{M}_v \subset M_v$ descends to the $\mathcal{N} = 1$ theory. A similar situation occurs in the hypermultiplet sector as we will see in Section 3.2.

Below the scale of supersymmetry breaking one can derive an effective low energy $\mathcal{N} = 1$ theory by integrating out all massive fields of $O(m_{3/2})$. This includes the massive spin-3/2 multiplet but as we just saw in (2.15) further multiplets can become massive and thus have to be integrated out. The light scalar fields (denoted by $M^U$ in the following) of the resulting effective $\mathcal{N} = 1$ theory have the standard sigma-model couplings \[15\]

\[
\mathcal{L} = - K_{UV} D_{\mu} M^U D^{\mu} \bar{M}^\mu - V ,
\]

where $K_{UV} = \partial_V \bar{\partial}_V K$ denotes the Kähler metric of the scalar field space and $K$ its Kähler potential. The scalar potential is given by

\[
V = e^K \left( K_{UV} D_{\mu} W D^{\mu} \bar{W} - 3 |W|^2 \right) + \frac{1}{2} \left( \text{Re} f \right)_{IJ} D^I D^J .
\]

\(^2\)This can only be achieved if magnetically charged hypermultiplets are in the spectrum \[3-8\].

\(^3\) For examples of $\mathcal{N} = 1$ AdS vacua in this class of theories see e.g. \[14\].
where $W$ is the superpotential and $D_U W = \partial_U W + (\partial_U K) W$ its Kähler-covariant derivative. $D^I$ are the $D$-terms for the light $\mathcal{N} = 1$ vector multiplets and the holomorphic gauge kinetic function is given by $f_{IJ} = i \mathcal{F}_{IJ}$.

It was shown in [8] that integrating out the two massive vector bosons amounts to taking the quotient of $\hat{M}_h$ with respect to the two gauged isometries $k_1$ and $k_2$ while integrating out the additional massive scalars simply yields a submanifold of the $\mathcal{N} = 2$ geometry. Therefore the scalar field space of the effective $\mathcal{N} = 1$ theory is given by

$$M^{\mathcal{N}=1} = \hat{M}_h \times \hat{M}_v,$$

where

$$\hat{M}_h \subset M_h / \langle k_1, k_2 \rangle \quad \hat{M}_v \subset M_v.$$  \hspace{1cm} (2.18)

In Refs. [9,10] it was shown that $M^{\mathcal{N}=1}$ is indeed a Kähler manifold. Its Kähler potential is $K = K^v + \hat{K}$ and we give the explicit form of $\hat{K}$ in the next section for the specific subclass of Kähler manifolds which descend from special quaternionic Kähler manifolds.

If only the two isometries required for partial supersymmetry breaking are gauged, the superpotential and the $D$-terms vanish in a Minkowski background. To get a nontrivial scalar potential, additional isometries have to be gauged at a scale $\tilde{m}$ below $m_3/2$. The corresponding superpotential and $D$-terms are then given by [9]

$$W = e^{-\hat{K}/2} V^A \Theta^\lambda \lambda P_{\lambda^-,} \hspace{1cm} (2.20)$$

$$D^I = -\Pi^I_J \Gamma^J_K (\text{Im} \mathcal{F})^{-1} K^L \left( \Theta^\lambda_L - \mathcal{F}_{LM} \Theta^{M\lambda} \right) P^{3\lambda},$$

with $P_{\lambda^-} \equiv P_{\lambda^1} - i P_{\lambda^2}$ and $\Pi^I_J$ and $\Gamma^J_K$ are projectors that arise when projecting out the heavy gauge bosons

$$\Pi^I_J = \delta^I_J - 2 e^{K^v} X^I X^K \text{Im} (\mathcal{F})_{KJ},$$

$$\Gamma^J_K = \delta^I_J - \frac{C^{(P)} I \bar{C}^{(P)}_K \text{Im} (\mathcal{F})_{KJ}}{C^{(P)} M \text{Im} (\mathcal{F})_{MN} C^{(P)} N}, \quad \text{with } C^{(P)} I = \Pi^I_J C^{J}.$$  \hspace{1cm} (2.22)

After this general discussion let us now turn to a specific class of $\mathcal{N} = 2$ theories which arise at the tree level of type II string theories compactified on Calabi-Yau threefolds.

3 Quotient construction for special Kähler manifolds

3.1 Special quaternionic-Kähler manifolds

In this paper we focus on the subclass of $\mathcal{N} = 2$ theories where $M_h$ is restricted to be a special quaternionic-Kähler manifold. Such manifolds are constructed by fibering a specific $(2n_h + 2)$-dimensional $G$-bundle over a $(2n_h - 2)$-dimensional special-Kähler submanifold $M_{sk}$. Let us denote the complex coordinates of $M_{sk}$ by $z^a, a = 1, \ldots, n_h - 1$ and $Z^A = (z^a, 1)$, $A = 1, \ldots, n_h$ and the holomorphic prepotential by $\mathcal{G}(Z)$. In this notation the Kähler potential $K^h$ of $M_{sk}$ is given in analogy with (2.3) by

$$K^h = -\ln i \left( \bar{Z}^A \mathcal{G}_A - Z^A \bar{\mathcal{G}}_A \right) = -\ln \left( -2 Z^A N_{AB} \bar{Z}^B \right),$$  \hspace{1cm} (3.1)
where we defined 

\[ N_{AB} = \text{Im} \mathcal{G}_{AB}. \]  

(3.2)

The (real) coordinates of the fibre are denoted by \( \phi, \bar{\phi}, \xi^A, \bar{\xi}_A \) and the metric is given by [16]

\[
h_{wq}(q) \partial_{\mu} q^\nu \partial^\mu q^\nu = -(\partial \phi)^2 - e^{4\phi}(\partial \bar{\phi} + \bar{\xi}_A \partial \xi^A - \xi^A \partial \bar{\xi}_A)^2 + g_{ab} \partial z^a \partial z^b + e^{2\phi} \text{Im} \mathcal{M}^{AB}(\partial \bar{\xi} - \mathcal{M} \partial \xi)_A(\partial \bar{\xi} - \mathcal{M} \partial \xi)_B,
\]

(3.3)

where \( g_{ab} \) is the metric on \( M_{sk} \) and

\[
\mathcal{M}_{AB} = \check{\mathcal{G}}_{AB} + 2i \frac{N_{AC}N_{BD}Z^CZ^D}{N_{DC}Z^CZ^D}.
\]

(3.4)

The metric (3.3) has \((2n_h + 2)\) isometries generated by the Killing vectors

\[
k_\phi = \frac{1}{2} \frac{\partial}{\partial \phi} - \frac{1}{2} \frac{\partial}{\partial \bar{\phi}} - \frac{\xi}{2} \frac{\partial}{\partial \xi^A} - \frac{\bar{\xi}_A}{2} \frac{\partial}{\partial \bar{\xi}_A}, \quad k_{\bar{\phi}} = -2 \frac{\partial}{\partial \bar{\phi}} ,
\]

(3.5)

\[
k_A = \frac{\partial}{\partial \xi_A} + \frac{\bar{\xi}}{2} \frac{\partial}{\partial \bar{\phi}}, \quad \tilde{k}^A = \frac{\partial}{\partial \xi_A} - \bar{\xi}^A \frac{\partial}{\partial \phi}.
\]

They act transitively on the \( G \)-fibre coordinates and the subset \( \{k_A, \tilde{k}^A, k_{\bar{\phi}}\} \) spans a Heisenberg algebra which is graded with respect to \( k_\phi \). The commutation relations are

\[
[k_{\phi}, k_{\bar{\phi}}] = k_{\bar{\phi}}, \quad [k_{\phi}, k_A] = \frac{1}{2} k_A, \quad [k_{\bar{\phi}}, \tilde{k}^A] = \frac{1}{2} \tilde{k}^A, \quad [k_A, \tilde{k}^B] = -\delta^B_A k_{\bar{\phi}},
\]

(3.6)

with all other commutators vanishing. Gauged supergravities in this class of theories have been discussed in [17, 18].

### 3.2 Quotient construction for generic base

Let us now review the explicit construction of \( \check{M}_h \subseteq M_h/(k_1, k_2) \) in a Minkowski background with \( M_h \) being special quaternionic, following [9, 10]. Instead of using \( \phi, \bar{\phi}, \xi^A, \bar{\xi}_A \) it is more convenient to use another set of coordinates \((z^a, w^0, w_A)\) which are holomorphic on \( M_h \) with respect to one of its complex structures, \( J^3 \). \( z^a \) are the holomorphic coordinates on \( M_{sk} \) as introduced in Section 3.1 while \( w^0 \) and \( w_A \) are given by

\[
w^0 = e^{-2\phi} + i(\bar{\phi} + \xi^A(\bar{\xi}_A - \mathcal{G}_{AB}\xi^B)) , \quad w_A = -i(\bar{\xi}_A - \mathcal{G}_{AB}\xi^B).
\]

(3.7)

Before we proceed, let us also record the inverse transformations which are given by

\[
\phi = -\frac{1}{2} \ln (\text{Re} w^0 - \text{Re} w_AN^{-1AB} \text{Re} w_B),
\]

\[
\bar{\xi}_A = -\text{Re}(\mathcal{G}_{AB}N^{-1BC}\bar{w}_C),
\]

\[
\xi^A = -\text{Re}(N^{-1AB}\bar{w}_B).
\]

(3.8)

The Kähler potential \( \check{K} \) of the quotient \( \check{M}_h \) is derived from

\[
\check{K} = K^h + 2\phi = -\ln i(\bar{Z}^A\mathcal{G}_A - Z^A\check{\mathcal{G}}_A) - \ln (\text{Re} w^0 - \text{Re} w_AN^{-1AB} \text{Re} w_B),
\]

(3.9)
where additional constraints have to be imposed. Moreover, since \( \hat{M}_h \) is a quotient of codimension 2, one of the complex coordinates in (3.9) is redundant. In the rest of this section we explicitly give the constraints and their solutions.

The two Killing vectors \( k_1, k_2 \) which are relevant for the quotient construction are not unique. They are linear combinations of the Killing vectors (3.5) given by

\[
\begin{align*}
k_1 &= \text{Re} \, D^A k_A + \text{Re}(D^A \mathcal{G}_{AB}) \tilde{k}^B + \text{Re} \left( i \, D^A w_A \right) k_{\phi} , \\
k_2 &= \text{Im} \, D^A k_A + \text{Im}(D^A \mathcal{G}_{AB}) \tilde{k}^B + \text{Im} \left( i \, D^A w_A \right) k_{\phi} ,
\end{align*}
\]

where the non-uniqueness is parameterized by \( n_h - 1 \) complex constants \( D^A \). The condition that \( k_1 \) and \( k_2 \) commute can now be expressed as

\[
\bar{D}^A N_{AB} D^B = 0 .
\]

In addition, all prefactors in (3.10) have to be constant giving the conditions

\[
\begin{align*}
D^A \mathcal{G}_{AB} &= \text{const.} , \\
D^A w_A &= \tilde{C} ,
\end{align*}
\]

where \( \tilde{C} \) are arbitrary complex constants. (Note the analogy of (3.11) and (3.12) with (2.14) and (2.15).) The condition (3.12) fixes a subset of scalar fields in the special-Kähler base \( M_{sk} \) while the condition (3.13) fixes one complex scalar field in the fiber. Note that due to (3.12) the constraint (3.11) does not impose any condition on the scalar fields, but merely constrains the vector \( D \) to lie on the boundary of the domain of the \( Z \) coordinates, which is given by \(-2Z^A N_{AB} Z^B = e^{-k^A} > 0\). This also implies that \( Z \) and \( D \) should not be proportional to each other \( Z \not\sim D \).

Let us elaborate on the implication of (3.12). As in (2.15) it is equivalent to

\[
\mathcal{G}_{ABC} D^B \delta Z^C = 0 ,
\]

and thus fixes \( \text{rk}(\mathcal{G}_{ABC} D^B) \) of the \( n_h - 1 \) complex coordinates on \( M_{sk} \) [10]. The degree-2 homogeneity of the prepotential implies \( \mathcal{G}_{ABC} Z^C = 0 \) and thus \( Z^C \) is a null eigenvector of the \( n_h \times n_h \) matrix \( \mathcal{G}_{ABC} D^B \). Therefore it cannot have full rank and we have

\[
\text{rk}(\mathcal{G}_{ABC} D^B) \leq n_h - 1 .
\]

Denoting by \( \hat{n}_h \) the number of unfixed complex coordinates on \( M_{sk} \) we have

\[
\hat{n}_h = n_h - 1 - \text{rk}(\mathcal{G}_{ABC} D^B) .
\]

Table 3.1 displays the corresponding dimensions for some typical prepotentials. We see that a generic prepotential of degree three or higher can fix all \( n_h - 1 \) fields \( z^a \), while for quadratic \( \mathcal{G} \) (3.12) is trivially satisfied and no \( z^a \) are fixed, since the second derivatives \( \mathcal{G}_{AB} \) are constant. Intermediate examples where some of the \( z^a \) are fixed are given by the cubic \( STU \)- and quantum-\( STU \) prepotentials. They were discussed in ref. [10] and will be recalled in more detail in Section 4.

Before we proceed, let us note that Table 3.1 applies to any special Kähler manifold that is restricted by the conditions (3.11) and (3.12). Thus the same argument can
Table 3.1: Number of base coordinates descending to $\hat{M}_h$ for some typical prepotentials $G$

| $G$             | rk($G_{ABC}D^B$) | $\hat{n}_h$ |
|-----------------|------------------|-------------|
| generic         | $n_h - 1$        | 0           |
| quadratic       | 0                | $n_h - 1$   |
| (quantum) STU   | 2                | 1           |

be applied to the vector multiplet sector as it is determined by the analogous equations (2.14) and (2.15). Let $\hat{n}_v \equiv \dim\mathcal{M}_v$, then Table 3.1 can be used for the vector multiplets with the substitutions

$$G \to \mathcal{F}, \quad D^B \to C^J, \quad n_h - 1 \to n_v, \quad \hat{n}_h \to \hat{n}_v. \quad (3.17)$$

Let us now describe the explicit construction of the quotient $\hat{M}_h$ following ref. [10]. One first combines the two Killing vectors given in (3.10) into one holomorphic vector

$$k \equiv k_1 - ik_2 = -4\bar{D}^A \text{Re} \ w_A \frac{\partial}{\partial w^0} - 2\bar{D}^B N_{BA} \frac{\partial}{\partial w_A} \quad (3.18)$$

where we inserted (3.5) and used (3.7) to change to holomorphic coordinates. The quotient is taken by identifying fiber coordinates on the integral curve $s$ generated by $k$

$$s \sim s' = e^{\lambda k} \left( w_0, w_A \right) = \left( w_0 + \lambda \delta_1 + \lambda^2 \delta_2, w_A + \lambda \delta_A \right), \quad (3.19)$$

with $\lambda \in \mathbb{C}$ and

$$\delta_1 = -4\bar{D}^A \text{Re} \ w_A, \quad \delta_2 = 2\bar{D}^B N_{BA} \bar{D}^A, \quad \delta_A = -2\bar{D}^B N_{BA}. \quad (3.20)$$

By inserting (3.19) into (3.9) (and using (3.11)), one can check that $k$ is indeed a Killing vector as $\hat{K}$ satisfies

$$\hat{K}(Z^A, w^0, w_A) = \hat{K}(Z^A, w^0, w_A). \quad (3.21)$$

To remove the coordinate that is redundant on the quotient, we use an ansatz similar to Eq. (3.13)

$$Z^A w_A = \bar{D}, \quad \bar{D} \in \mathbb{C}. \quad (3.22)$$

The conditions (3.13) and (3.22) determine two of the $w_A$ in terms of the other coordinates. Let us denote the two dependent coordinates by $w_i, w_j, i < j$ while the independent coordinates are $w_a, a = 1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n_h$ with $\hat{i}, \hat{j}$ omitted. $w_i, w_j$ are then given by

$$w_i = \frac{\left( (Z^a D^a - Z^a D^j)w_a - Z^i \bar{C} + D^j \bar{D} \right)}{D^i Z^j - D^j Z^i}, \quad (3.23)$$

$$w_j = -\frac{\left( (Z^a D^a - Z^a D^i)w_a - Z^i \bar{C} + D^i \bar{D} \right)}{D^i Z^j - D^j Z^i}.$$  

Due to (3.11) $\bar{C}$ is constant along the integral curves of $k$

$$\bar{C}' = e^{\lambda k} \bar{C} = \bar{C} + \lambda D^A \delta_A = \bar{C}, \quad (3.24)$$
so after the elimination of two fiber coordinates, one has to transform the remaining fiber coordinates and $\tilde{D}$ to stay on the quotient. The Kähler potential is of course still invariant under these transformations

$$\tilde{K}(Z^A, w^0, w'_a, \tilde{D}) = \hat{K}(Z^A, w^0, w_a, \tilde{D}) .$$

The ansatz (3.22) was chosen such that the transformation of $\tilde{D}$ is always non-zero

$$\tilde{D}' = e^{\lambda k} \tilde{D} = \tilde{D} + \lambda Z^A \delta_A ,$$

due to $Z^A \delta_A \neq 0$ [8]. This guarantees that each equivalence class $[w^0, w_A]$ contains for each $\tilde{D} \in \mathbb{C}$ exactly one representative, i.e. the quotient is isomorphic to the submanifold obtained by fixing $\tilde{D}$. One can now freely choose a $\tilde{D}$ to pick a representative $(w^0, w_a)$ for the quotient. In total, 2 of the $n_h + 1$ complex fiber coordinates are fixed in the $N = 1$ theory. Including the $n_h$ remaining base coordinates, $\tilde{M}_h$ thus has complex dimension $n_h + n_h - 1$.

### 4 Kähler potential of the quantum STU-model

As an explicit example let us study in detail the quantum STU-model with the prepotential

$$G = \frac{Z^1 Z^2 Z^3}{Z^1 Z^3} + \frac{\alpha (Z^2)^3}{3 Z^4} , \quad \alpha \in \mathbb{R} .$$

It defines a 16-dimensional quaternionic Kähler manifold via (3.3) in the original $N = 2$ supergravity with $n_h = 4$ hypermultiplets. For $\alpha = 0$ (called the STU-model) the manifold is the symmetric space

$$M_h = \frac{SO(4, 4)}{SO(4) \times SO(4)} ,$$

with the six-dimensional special Kähler base

$$M_{sk} = \left( \frac{SU(1, 1)}{U(1)} \right)^3 .$$

Using (3.1) and (3.2) one determines the Kähler potential on the base (before taking the quotient) to be

$$e^{-K^h} = -2 Z^A N_{AB} \tilde{Z}^B = -\frac{8}{3} \text{Im} Z^3 (3 \text{Im} Z^1 \text{Im} Z^3 + \alpha (\text{Im} Z^2)^2) .$$

In order to apply the quotient construction, let us display the condition (3.12) for the prepotential (4.1) [10]

$$(D^4 Z^2 - D^2)(D^4 Z^3 - D^3) = \text{const.} ,$$

$$(D^4 Z^1 - D^1)(D^4 Z^3 - D^3) + \alpha (D^4 Z^2 - D^2)^2 = \text{const.} ,$$

$$\tilde{(D^4 Z^1 - D^1)(D^4 Z^2 - D^2)} = \text{const.} ,$$

$$(2 D^4 Z^1 Z^3 - Z^2 (D^1 Z^3 + Z^1 D^3) + \alpha (Z^2)^2 (\frac{2}{3} D^4 Z^2 - D^2) = \text{const.} .$$
It is easy to see that these equations fix all $Z^A$ unless $Z^2 = \frac{D^2}{D^4}$. With $Z^2$ fixed in this way, the second and fourth equations in (4.5) both reduce to

\[(D^1 Z^1 - D^1)(D^4 Z^3 - D^3) = \text{const.} \ , \tag{4.6}\]

so there are two solutions that leave either $Z^1$ or $Z^3$ free. We pick

\[Z^1 = \frac{D^1}{D^4} , \quad Z^2 = \frac{D^2}{D^4} , \quad Z^3 \text{ arbitrary} \ . \tag{4.7}\]

For simplicity we set $D^4 = 1$ in the following which can always be achieved by rescaling Eqs. (3.11)-(3.13). The condition (3.11) that $D^4$ has to be null with respect to $N_{AB}$ then reads

\[\text{Im } D^2 \left[ \text{Im } D^1 \text{Im } D^3 + \frac{1}{3} \alpha \left( \text{Im } D^2 \right)^2 \right] = 0 \ . \tag{4.8}\]

$\text{Im } D^2 \neq 0$ has to hold in order to keep (4.4) finite. This implies that the bracket in (4.8) must be zero and we have

\[\text{Im } D^1 \neq 0 , \quad \text{Im } D^2 \neq 0 , \quad \text{Im } D^3 = -\frac{\alpha (\text{Im } D^2)^2}{3 \text{Im } D^1} . \tag{4.9}\]

Furthermore, the bracket in (4.4) must be non-zero which implies

\[\text{Im } Z^3 \neq -\frac{\alpha (\text{Im } D^2)^2}{3 \text{Im } D^1} . \tag{4.10}\]

In summary we have on the base

\[Z^1 = D^1 \in \mathbb{C} \setminus \mathbb{R} , \quad Z^2 = D^2 \in \mathbb{C} \setminus \mathbb{R} , \quad Z^4 = D^4 = 1 , \quad Z^3 + \frac{1}{3} \alpha (\text{Im } D^2)^2 \in \mathbb{C} \setminus \mathbb{R} , \quad D^3 + \frac{1}{3} \alpha (\text{Im } D^2)^2 \in \mathbb{R} . \tag{4.11}\]

Since (4.4) must be positive, the choice of $D^4$ determines in which half-plane $Z^3$ has to lie

\[\text{Im } D^1 \text{Im } D^2 \leq 0 \iff \text{Im } Z^3 \geq -\frac{\alpha (\text{Im } D^2)^2}{3 \text{Im } D^1} . \tag{4.12}\]

In the fiber two complex coordinates are fixed by (3.23). Since $Z$ and $D$ only differ in their third component, the denominator in (3.23) is only non-zero if either $i$ or $j$ is 3. One can show [19] that the final result does depend on the specific choice only up to holomorphic coordinate transformations and thus without loss of generality we can set $i = 3, j = 4$ in the following to eliminate $w_3$ and $w_4$ in the fiber. Furthermore, we also use the fact that we can freely choose a $\tilde{D}$ and set $\tilde{D} = \tilde{C}$ to obtain

\[w_3 = 0 , \quad w_4 = \tilde{C} - D^1 w_1 - D^2 w_2 . \tag{4.13}\]

In order to compute the Kähler potential $\hat{K}$ given in (3.9) on the quotient $\hat{M}_h$ we first compute

\[N^{-1AB}(\alpha) = g(Z) h(Z) \left( 6 (\text{Im } Z^1 \text{Im } Z^3)^2 N^{-1AB}(\alpha = 0) \right. \]

\[+ \alpha \text{Im } Z^2 \left( \text{Re}(Z^A Z^B) + 2 \text{Re } Z^A \text{Re } Z^B - 2 \delta^{AB} (\text{Im } Z^A)^2 \right. \]

\[- \left. (\delta^A_1 \delta^B_3 + \delta^A_3 \delta^B_1)(6 \text{Im } Z^1 \text{Im } Z^3 + 2 \alpha (\text{Im } Z^2)^2) \right) \ , \tag{4.14}\]

11
where we abbreviated
\[ g(Z) = (6 \text{Im} Z^1 \text{Im} Z^3 + 2\alpha (\text{Im} Z^2)^2)^{-1}, \quad h(Z) = (\text{Im} Z^1 \text{Im} Z^3 - \alpha (\text{Im} Z^2)^2)^{-1}, \]
and
\[
N^{-1AB}(\alpha = 0) = -\frac{1}{2 \text{Im} Z^1 \text{Im} Z^2 \text{Im} Z^3} \begin{pmatrix}
|Z^1|^2 & \text{Re}(Z^1 Z^2) & \text{Re}(Z^1 Z^3) & \text{Re}(Z^1) \\
\text{Re}(Z^1 Z^2) & |Z^2|^2 & \text{Re}(Z^2 Z^3) & \text{Re}(Z^2) \\
\text{Re}(Z^1 Z^3) & \text{Re}(Z^2 Z^3) & |Z^3|^2 & \text{Re}(Z^3) \\
\text{Re}(Z^1) & \text{Re}(Z^2) & \text{Re}(Z^3) & 1
\end{pmatrix}.
\]

Inserting (4.4), (4.14), (4.11) and (4.13) into (3.9), we obtain
\[
e^{-\hat{K}} = -\frac{4}{3} \text{Im} D^2 g(Z^3)^{-1} \text{Re} w^0 \\
- 4h(Z^3) \left( \text{Im} D^1 \text{Im} Z^3 | \text{Im} D^1 \bar{w}_1 - \text{Im} D^2 w_2 - i \text{Re} \tilde{C} |^2 \\
- \frac{2}{3} (\text{Im} D^2)^2 \left( 3 (\text{Re} \tilde{C} + \text{Im} D^a \text{Im} w_a)^2 - (\text{Im} D^1 \text{Re} w_1 - \text{Im} D^2 \text{Re} w_2)^2 \right) \right),
\]
where
\[
g(Z^3) \equiv g(Z^1 = D^1, Z^2 = D^2, Z^3) , \quad h(Z^3) \equiv h(Z^1 = D^1, Z^2 = D^2, Z^3) .
\]
By a set of field redefinitions – discussed in Appendix A – \( \hat{K} \) can be brought into the form
\[
e^{-\hat{K}} = (Z^3 + \bar{Z}^3)(w^0 + \bar{w}^0) - (w_1 + \bar{w}_2)(\bar{w}_1 + w_2) + \frac{\alpha [(w_1 + \bar{w}_2) + (\bar{w}_1 + w_2)]^2}{4\alpha - \frac{3 \text{Im} D^1}{2(\text{Im} D^2)^2} (Z^3 + \bar{Z}^3)}. 
\]
For \( \alpha = 0 \) (\(STU\)-model) we thus obtain
\[
e^{-\hat{K}} = (Z^3 + \bar{Z}^3)(w^0 + \bar{w}^0) - (w_1 + \bar{w}_2)(\bar{w}_1 + w_2) .
\]
This is the Kähler potential of the eight-dimensional homogeneous Kähler manifold \([20]\)
\[
\hat{M}_h = \frac{SO(4,2)}{SO(4) \times SO(2)}. 
\]

Let us briefly discuss the fact that the quantum \(STU\) Kähler potential has a singularity in \(Z^3\). In equation (4.17), the singularity is present in the form of \(h(Z^3)\) and located at \(\text{Im} Z^3 = \frac{\alpha (\text{Im} D^2)^2}{\text{Im} D^1}\). It is confined to only one of the two possible domains of \(Z^3\) given by (4.12).

5 The \(\mathcal{N} = 1\) scalar potential

5.1 Special quaternionic manifolds

To get a non-zero scalar potential, let us now consider the case where \(n > 2\) linearly independent isometries are gauged. The gauge bosons are recruited among the graviphoton
and the vectors of the vector multiplets, and the number of available gauge bosons limits the number of possible gaugings to obey

\[ n \leq n_v + 1 \quad \text{. (5.1)} \]

The \( n \) commuting Killing vectors are parametrized as in (3.10) by the linear combination

\[ k_\lambda = r^A_\lambda k^A + s_{\lambda B} \tilde{k}^B + t_\lambda k^\phi, \quad \lambda = 1, \ldots, n \quad \text{, (5.2)} \]

where \( r^A_\lambda, s_{\lambda B}, t_\lambda \) are constant parameters. \( k^\phi \) does not appear in (5.2) as there are no linear independent commuting Killing vectors involving \( k^\phi \) (this was shown in [19]). \( k_1 \) and \( k_2 \) are the two Killing vectors (3.10) that ensure partial supersymmetry breaking and thus we have

\[ D^A = r^A_1 + i r^A_2, \quad D^A g_{AB} = s_{1B} + i s_{2B} \quad \text{. (5.3)} \]

The additional Killing vectors \( k_{\lambda > 2} \) do not participate in the partial supersymmetry breaking, as already discussed in Section 2.2.

\( k^A, \tilde{k}^B, k^\phi \) form a \((2n_h + 1)\)-dimensional Heisenberg algebra, which has maximal Abelian dimension \((n_h + 1)\) [21]. This number is obviously the upper bound for the number of Abelian gaugings

\[ n \leq n_h + 1 \quad \text{. (5.4)} \]

Demanding that all \( k_\lambda \) commute, the commutation relations (3.6) imply \( \frac{1}{2} n(n - 1) \) conditions on the real parameters \( r^A_\lambda, s_{\lambda B}, t_\lambda \) which take a form analogous to the locality constraint (2.7) of the embedding tensor

\[ r^A_\lambda s_{\rho A} = 0 \quad \text{. (5.5)} \]

For \( \lambda \geq 3 \) and \( \rho = 1, 2 \) they can be brought into a more useful form by inserting (5.3)

\[ (s_{\lambda B} - r^A_\lambda g_{AB}) D^B = 0 \quad \text{. (5.6)} \]

Similarly, the embedding tensor constraint (2.7) with \( \lambda \geq 3 \) and \( \kappa = 1, 2 \) reads

\[ C^I (\Theta^J_\lambda - F_{IJ} \Theta^K_\lambda) = 0 \quad \text{, (5.7)} \]

after inserting the explicit \( \Theta^1_\lambda \) and \( \Theta^2_\lambda \) from (2.13).

With coordinates and a basis of Killing vectors on \( M_h \) at hand, the superpotential (2.20) and \( D \)-terms (2.21) can be calculated more explicitly for a special quaternionic hypermultiplet sector. The only parts of the superpotential and the \( D \)-terms that depend on the fields of the hypermultiplets are the Killing prepotentials \( P_\lambda^- \) and \( P_\lambda^3 \), which are calculated in Appendix B. When inserted, (2.20) and (2.21) take the form

\[ W = 2X^I (\Theta^J_\lambda - F_{IJ} \Theta^K_\lambda) (s_{\lambda B} - r^A_\lambda g_{AB}) Z^B \quad \text{, (5.8)} \]

\[ D^I = -i G^I (\text{Im} F)^{-1 KL} (\Theta^J_L - F_{LM} \Theta^K_M) \text{ Re} ((s_{\lambda B} - r^A_\lambda g_{AB}) N^{-1 BC} \bar{w}_C - \tilde{t}_\lambda) \quad \text{, (5.9)} \]
Since the constraints (5.6) and (5.7) take a similar form, the superpotential (5.8) is symmetric under the exchange of \((X^I, F_I)\) and \((Z^A, G_A)\). This can be made manifest by rewriting (5.8) in the symplectic form

\[
\mathcal{W} = 2V^A \Theta_\Lambda^\lambda s_{\Lambda \Sigma} U^\Sigma, \quad (5.10)
\]

using the symplectic vectors \(s_{\Lambda \Sigma} \equiv (s_{\Lambda A}, -r_\Lambda^A)\) and \(U^\Sigma \equiv (Z^A, G_A)\). If we define the constant matrix \(\Theta_{\Lambda \Sigma} \equiv \Theta_\Lambda^\lambda s_{\Lambda \Sigma}\) and insert it into (5.10), we get the superpotential in the same form as given in \([9]\). The lesson we take away from rederiving the result in the form (5.10) is that the rank of \(\Theta_{\Lambda \Sigma}\) is at most \(n - 2\). This can be seen by recalling that the first two gauged isometries do not contribute to the superpotential which implies \(\lambda = 3, \ldots, n\) in (5.10). Thus \(\Theta_{\Lambda \Sigma}\) is the product of a \((n - 2)\)-column matrix and a \((n - 2)\)-row matrix and its rank is bounded by \(n - 2\).

### 5.2 STU and quantum STU models

We now evaluate (5.8) and (5.9) for the quantum STU model with an arbitrary vector multiplet sector. Let us start with the superpotential. It can be simplified by using (5.6) and (4.11) which imply

\[
(s_{\Lambda B} - r_\lambda^A G_{AB}) Z^B = (s_{\Lambda B} - r_\lambda^A G_{AB})(Z^B - D^B) = B^\text{STU}_\lambda (Z^3 - D^3), \quad (5.11)
\]

where we defined the constants

\[
B^\text{STU}_\lambda = s_{\lambda 3} - r_\lambda^A G_{A3}. \quad (5.12)
\]

They are constants since \(G_{A3}\) does not depend on \(Z^3\)

\[
G_{A3} = \partial_A \frac{Z^1 Z^2}{Z^4} \bigg|_{Z^i = D^i, Z^2 = D^2, Z^4 = 1} = \text{const.} . \quad (5.13)
\]

Inserting into (5.10) yields

\[
\mathcal{W} = 2V^A \Theta_\Lambda^\lambda B^\text{STU}_\lambda (Z^3 - D^3). \quad (5.14)
\]

Note that the only term which contains a scalar from a hypermultiplet is the overall factor \((Z^3 - D^3)\). This term can never vanish since the domains of \(Z^3\) and \(D^3\) are disjunct due to (4.11). We will see in the next section that as a consequence, consistency with a Minkowski background requires \(\mathcal{W}\) to vanish for an appropriate choice of prepotentials \(F\) in the vector multiplet sector.

The \(D\)-terms for the quantum STU model were calculated in \([19]\) and are given by (5.9) with the substitution

\[
N^{-1BA} \tilde{w}_A = -\frac{3g(Z^3)}{\Im D^2} \Re Z^B \left(\tilde{C} + i \Im D^a \tilde{w}_a\right) + \frac{h(Z^3)}{2 \Im D^2} \left(\Im Z^B \Im D^a - 2\delta^B_a (\Im D^a)^2\right) \tilde{w}_a. \quad (5.15)
\]
6 Supersymmetric vacua of the $\mathcal{N} = 1$ theory

In this section we study the conditions for supersymmetric vacua of the scalar potential (2.17) which occur for $\langle D_U W \rangle = \langle D^I \rangle = 0$.\footnote{For the non-supersymmetric vacua the analysis is less systematic and strongly depends on the type of prepotential.} We do not confine our analysis to the $STU$-models but consider the prepotentials given in Table 3.1. Since throughout this paper we assumed a Minkowskian background, consistency demands that the supersymmetric vacua also have $\langle W \rangle = 0$.

Let us start by showing that this consistency condition is implied by the conditions we stated before. From its definition we have $D_U W = \partial_U W + (\partial_U K)W$ with $K = \dot{K} + K^v$. Since $W$ (defined in (2.20)) and $K^v$ (defined in (2.3)) are independent of $w^0$ and $w_A$, their partial derivatives with respect to the fiber coordinates vanish and thus we have

$$D_w W = (\partial_w \dot{K}) W. \quad (6.1)$$

$\dot{K}$ is given in (3.9) and we find

$$\partial_w \dot{K} = -(\text{Re} \, w^0 - \text{Re} \, w_A N^{-1AB} \text{Re} \, w_B)^{-1} = -e^{\dot{K} - K^h} \neq 0. \quad (6.2)$$

As a consequence $\langle D_w W \rangle = 0$ necessarily implies $\langle W \rangle = 0$.

Let us now turn to the covariant derivatives with respect to the base coordinates $X^I$ and $Z^\hat{A}$. The index $\hat{I}$ labels the $\hat{n}_v$ $\mathcal{N} = 1$ scalar fields descending from the vector multiplets, while $\hat{A}$ labels the $\hat{n}_h$ scalar fields descending from hypermultiplets. For a (quantum) $STU$ prepotential $G$, we only have $Z^3$ on the base, i.e., $\hat{A} = 3$. A quadratic $G$ does not fix any base coordinates and $\hat{A} = 1, \ldots, n_h - 1$, while a generic $G$ fixes all fields.

The two factors $X^I(\Theta^\lambda_I - F_{IJ} \Theta^J \lambda)$ and $(s_{\lambda B} - r_\lambda^A G_{AB})Z^B$ that appear in the superpotential (5.8) are both at most linear in the fields $X^I$ or $Z^\hat{A}$ for all prepotentials appearing in Table 3.1. For the quantum $STU$ prepotential this can be explicitly seen from (5.11), for quadratic prepotentials $G_{AB}, F_{IJ}$ are constant while generic prepotentials fix all fields in the base so that $X^I(\Theta^\lambda_I - F_{IJ} \Theta^J \lambda)$ or $(s_{\lambda B} - r_\lambda^A G_{AB})Z^B$ are constant altogether. To make this more explicit let us denote the constant parts by $A^\lambda$ and $B_\lambda$ respectively so that we have

$$X^I(\Theta^\lambda_I - F_{IJ} \Theta^J \lambda) \equiv \begin{cases} (X^3 - C^3)A^\lambda_{\text{STU}}, & F \text{ quantum } STU, \\ X^I A^\lambda_{\text{quad},I}, & F \text{ quadratic,} \\ A^\lambda_{\text{gen}}, & F \text{ generic,} \end{cases} \quad (6.3)$$

$$(s_{\lambda B} - r_\lambda^A G_{AB})Z^B \equiv \begin{cases} B^\lambda_{STU}(Z^3 - D^3), & G \text{ quantum } STU, \\ B^{\text{quad}}_{\lambda B} Z^B, & G \text{ quadratic,} \\ B^\lambda_{\text{gen}}, & G \text{ generic.} \end{cases}$$

Altogether we thus have nine possible combinations forming the superpotential. Any of the four combinations of quantum $STU$ and generic prepotentials necessarily have $W \equiv 0$. In these cases the superpotential consists of the constant factor $A^\lambda B_\lambda$ possibly multiplied by $(X^3 - C^3)$ or $(Z^3 - D^3)$ which are non-zero due to (4.11). Hence $\langle W \rangle = 0$.
implies $A^\lambda B_\lambda = 0$ and thus $\mathcal{W} \equiv 0$. In these cases supersymmetric vacua only exist if the superpotential is set to zero by choosing an appropriate embedding tensor and gauged Killing vectors.

The four combinations of a quadratic prepotential with a quantum $STU$ or generic prepotential also only have supersymmetric vacua for $\mathcal{W} \equiv 0$. This can be seen from $\langle D_X \mathcal{W} \rangle = 0$ which for $\mathcal{G}$ generic (and $\langle \mathcal{W} \rangle = 0$) implies

$$\langle \partial_X \mathcal{W} \rangle = 2A_{\text{quad},I}^\lambda B_{\text{gen}}^\lambda = 2A_{\text{quad},I}^\lambda B_{\text{gen}}^\lambda = 0 \, ,$$

and thus $\mathcal{W} \equiv 0$. For a quantum $STU$ prepotential $\mathcal{G}$ one has

$$\langle \partial_X \mathcal{W} \rangle = 2A_{\text{quad},I}^\lambda B_{\text{STU}}^\lambda (Z^3 - D^3) = 0 \, .$$

Since again $(Z^3 - D^3)$ cannot vanish $A_{\text{quad},I}^\lambda B_{\text{STU}}^\lambda = 0$ and thus $\mathcal{W} \equiv 0$ has to hold. We see that in general a quantum $STU$ or generic prepotential $\mathcal{G}$ leads to supersymmetric backgrounds with non-trivial superpotentials only if the “other” prepotential $\mathcal{F}$ is such that $X^I(\Theta_\lambda - \mathcal{F}_I \Theta_{J,\lambda})$ is not just linear in the fields $X^I$.

As the last case we need to discuss both prepotentials being quadratic. From (6.3) we learn

$$\langle \partial_X \mathcal{W} \rangle = 2A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}} Z^A = 0 \, , \quad \langle \partial_Z \mathcal{W} \rangle = 2X^I A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}} = 0 \, .$$

The conditions (6.6) fix $\text{rk}(A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}})$ of the fields $Z^A$ and $X^I$. $A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}}$ is the product of a $(n_v + 1) \times (n - 2)$ and a $(n - 2) \times n_h$ matrix, so its rank is at most $\min(n_v + 1, n_h, n - 2)$. (The reason why $\lambda$ takes only $n - 2$ values is because the first two gauged isometries do not contribute to the superpotential.) We know from (5.6) and (5.7) that both $A_{\text{quad},I}^\lambda$ and $B_{\lambda A}^{\text{quad}}$ have a non-trivial null eigenvector

$$C^I A_{\text{quad},I}^\lambda = 0 \, , \quad B_{\lambda A}^{\text{quad}} D^A = 0 \, ,$$

which restricts the possible rank of $A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}}$ to obey the stronger condition

$$\text{rk}(A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}}) \leq \min(n_v, n_h - 1, n - 2) = n - 2 \, ,$$

where the equality is due to (5.1) and (5.4). Now remember that $Z$ can not be proportional to $D$ (and analogously $X \not\sim C$). For $A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}}$ to have two more non-trivial null eigenvectors in addition to $C^I$ and $D^A$, we have to demand

$$\text{rk}(A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}}) < n_v \, ,$$

$$\text{rk}(A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}}) < n_h - 1 \, .$$

The first condition (6.9) is automatically satisfied due to (6.8) and (5.1), so (6.10) is the consistency condition that has to be imposed if there are two quadratic prepotentials. In this case, there are $n_h - 1 + n_v - 2\text{rk}(A_{\text{quad},I}^\lambda B_{\lambda A}^{\text{quad}})$ complex flat directions among the $X^I$ and $Z^A$.

Let us summarize the results so far in this section. For quadratic, quantum $STU$ or generic prepotentials one can have non-trivial superpotentials with supersymmetric vacua only when both $\mathcal{G}$ and $\mathcal{F}$ are quadratic and (6.10) holds.
We will now discuss how many fields have to be fixed to satisfy in addition the $D$-term condition $\langle D^I \rangle = 0$. We first recall that the projectors $\Pi^I_J$ and $\Gamma^J_K$ defined in (2.22) project out two directions in the field space. Applied to the $D$-terms in (5.9) one observes that also two of the initially $n_v + 1$ $D$-terms are projected out leaving at most $n_v - 1$ $D$-terms linearly independent. In our basis of fermions the $D$-terms are complex with their phase being a gauge freedom. So there can be only up to $n_v - 1$ real conditions implied by $\langle D^I \rangle = 0$. Since the index $\lambda$ in (5.9) only takes $n - 2$ values, this number is reduced further to $n - 2$, which is smaller than $n_v - 1$ due to (5.1).

For $n_h \geq 3$ there are fiber coordinates left after the quotient construction which can be used to solve the $D$-term condition independently of the prepotentials $F$ and $G$. The $n - 2$ real Killing prepotentials $P^3_\lambda$ (given in (B.7)) vanish for

$$\text{Re} \left( (s_{AB} - r^A \mathcal{G}_{AB}) N^{-1BC} \bar{w}_C - t_\lambda \right) = 0. \quad (6.11)$$

Two of the $w_C$ coordinates are already fixed leaving the $n_h - 2$ complex fields $w_a$ to solve (6.11). Thus, the $n - 2$ $P^3_\lambda$ can be set to zero if $2(n_h - 2) \geq n - 2$ which is always satisfied due to (5.4). This solution leaves at least $2n_h - n - 2$ real flat directions among the $w_a$ on top of the flat directions left in the base after solving the $F$-term condition. In addition $w^0$ is always a complex flat direction.

For $n_h = 2$, $\langle D^I \rangle = 0$ can only be achieved by fixing base coordinates $X^I, Z^A$ which may be possible depending on the form of $F$ and $G$. However, if there is a non-trivial superpotential some of the base coordinates might already be fixed by the $F$-term condition. For the quantum $STU$ prepotentials, $F_{IJ}$ and $G_{AB}$ only depend on $X^3$ or $Z^3$, respectively and these fields can be fixed to set some of the $D$-terms to zero. For $n_h = 2$, only one additional gauging is allowed, so all $D$-terms are proportional to just one Killing prepotential $P^3_3$, which can be set to zero by fixing the real or imaginary part of $Z^3$.

## 7 Conclusion

In this paper, we studied explicit examples of supergravities which exhibit spontaneous partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking in a Minkowskian background. In the hypermultiplet sector we confined our analysis to special quaternionic-Kähler manifolds which, as we reviewed, do have isometries that, when appropriately gauged, can induce the partial supersymmetry breaking. We considered the explicit example of a special quaternionic-Kähler manifold with the (quantum) $STU$ model as the base. In this case the base is complex three-dimensional while the fibre is complex five-dimensional. In the $\mathcal{N} = 1$ background one of the base and three of the fibre coordinates span the scalar field space. For the (classical) $STU$ model the quaternionic-Kähler manifold is given by $\frac{SO(4,4)}{SO(4) \times SO(4)}$ while the $\mathcal{N} = 1$ quotient is the Kähler manifold $\frac{SO(4,2)}{SO(4) \times SO(2)}$. For the quantum $STU$ model $K$ was explicitly computed but the corresponding Kähler manifold no longer is a simple symmetric space.

We also considered the situation where additional isometries are gauged to induce $D$-terms together with a non-trivial superpotential. Both depend on the choice of the prepotentials $F$ and $G$. We analyzed the conditions for supersymmetric minima of the scalar potential for the nine cases where $F$ and $G$ are quadratic, quantum $STU$ or generic.
We found that only for $\mathcal{F}$ and $\mathcal{G}$ both being quadratic a non-zero superpotential with supersymmetric minima can exist. However, there may exist supersymmetric minima for other classes of prepotentials. The $D$-terms pose no obstruction for supersymmetric vacua if one considers at least three hypermultiplets.

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**Appendix**

### A Field redefinitions for the Kähler potential

In this appendix, we display the holomorphic field redefinitions that bring the Kähler potential (4.17) into the form (4.19). $\tilde{C}$ in (4.17) can be eliminated by the field redefinition

\[
2 \text{Im } D^1 w_1 + i \text{Re } \tilde{C} \rightarrow w_1, \\
-2 \text{Im } D^2 w_2 - i \text{Re } \tilde{C} \rightarrow w_2,
\]

resulting in

\[
e^{-\hat{K}} = -g(Z^3)^{-1} \left[ \frac{4}{3} \text{Im } D^2 \text{Re } w^0 + \frac{1}{6} h(Z^3)(w_1 + \bar{w}_2)(\bar{w}_1 + w_2) \right]
- \frac{2}{3} (\text{Im } D^2)^2 h(Z^3) \left[(w_1 + \bar{w}_2) - (\bar{w}_1 + w_2)\right]^2.
\]

This expression can be further simplified by shifting $Z^3$ in such a way that its two domains given by (4.12) are separated by the real axis instead of the line $\text{Im } Z^3 = -\frac{\alpha (\text{Im } D^2)^2}{3 \text{Im } D^1}$

\[
Z^3 \rightarrow Z^3 - i \frac{\alpha (\text{Im } D^2)^2}{3 \text{Im } D^1}.
\]

In addition we exchange real and imaginary part of $Z^3$ and absorb some constants in $w^0$ by

\[
-\text{Im } Z^3 \rightarrow Z^3, \\
-2 \text{Im } D^1 \text{Im } D^2 w^0 \rightarrow w^0.
\]

Inserted into (4.15) one finds

\[
g(Z^3) \rightarrow (6 \text{Im } D^1 \text{Re } Z^3)^{-1}, \]

\[
h(Z^3) \rightarrow (\text{Im } D^1 \text{Re } Z^3 - \frac{4}{3} (\text{Im } D^2)^2)^{-1},
\]

which puts $\hat{K}$ into the final form (4.19).
B The Killing prepotentials $P_\lambda^-$ and $P_\lambda^3$

In this appendix we explicitly compute the Killing prepotentials $P_\lambda^-$ and $P_\lambda^3$ for special quaternionic manifolds. In this case the quaternionic vielbein $U_A^\alpha$ used in (2.11) reads

$$U_A^\alpha = U_u^\alpha dq^u = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{u} & \bar{e} & -v & -E \\ \bar{v} & E & u & e \end{pmatrix}, \quad (B.1)$$

with the one-forms

$$u = ie^{K/2+\phi}Z^A(d\tilde{\xi}_A - \mathcal{M}_{AB}d\xi^B),$$

$$v = \frac{1}{2}e^{2\phi}[de^{-2\phi} - i(d\tilde{\phi} + \tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A)],$$

$$E^k = -\frac{i}{2}e^{\phi-K/2}\Pi_A^b N^{-1AB}(d\tilde{\xi}_B - \mathcal{M}_{BC}d\xi^C),$$

$$e^b = \Pi_A^b dZ^A. \quad (B.2)$$

The SU(2) connections $\omega^x$ are given by

$$\omega^1 = i(\bar{u} - u), \quad \omega^2 = u + \bar{u},$$

$$\omega^3 = \frac{i}{2}(v - \bar{v}) - ie^{K/2}(Z^A N_{AB}d\bar{Z}^B - \bar{Z}^A N_{AB}dZ^B). \quad (B.3)$$

The Killing prepotentials $P_\lambda^x$ of a special quaternionic-Kähler manifold take the simple form

$$P_\lambda^x = \omega^x_k k^u_\lambda. \quad (B.4)$$

Inserting the Killing vectors (5.2) in terms of the basis vectors (3.5) we find

$$P_\lambda^- \equiv P_\lambda^1 - i P_\lambda^2 = (\omega^1 - i\omega^2) k^u_\lambda$$

$$= -2i u_\lambda \left( r_\lambda A k^u_A + s_{\lambda A} \tilde{k}_A^{Au} + t_\lambda k^u_\phi \right) \quad (B.5)$$

$$= 2e^{K/2}(s_{\lambda B} - r_\lambda A G_{AB}) Z^B,$$

where we used the identity $Z^A \mathcal{M}_{AB} = Z^A \mathcal{G}_{AB}$ that follows from the definition of $\mathcal{M}_{AB}$. For $P_\lambda^3$ we obtain analogously

$$P_\lambda^3 = \omega^3_k k^u_\lambda = e^{2\phi} \left( r_\lambda A \tilde{\xi}_A - s_{\lambda A} \xi^A - t_\lambda \right). \quad (B.6)$$

Using (3.8) one can switch to holomorphic coordinates

$$P_\lambda^3 = e^{2\phi} \text{Re} \left( (s_{\lambda B} - r_\lambda A G_{AB}) N^{-1BC} \bar{w}_C - t_\lambda \right). \quad (B.7)$$

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