On $W$-representations of $\beta$- and $q, t$-deformed matrix models

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ABSTRACT

$W$-representation realizes partition functions by an action of a cut-and-join operator on the vacuum state with a zero-mode background. We provide explicit formulas of this kind for $\beta$- and $q, t$-deformations of the simplest rectangular complex matrix model. In the latter case, instead of the complicated definition in terms of multiple Jackson integrals, we define partition functions as the weight-two series, made from Macdonald polynomials, which are evaluated at different loci in the space of time variables. Resulting expression for the $\hat{W}$ operator appears related to the problem of simple Hurwitz numbers (contributing are also the Young diagrams with all but one lines of length two and one). This problem is known to exhibit nice integrability properties. Still the answer for $\hat{W}$ can seem unexpectedly sophisticated and calls for improvements. Since matrix models lie at the very basis of all gauge- and string-theory constructions, our exercise provides a good illustration of the jump in complexity between $\beta$- and $q, t$-deformations – which is not always seen at the accidently simple level of Calogero-Ruijsenaars Hamiltonians (where both deformations are equally straightforward). This complexity is, however, quite familiar in the theories of network models, topological vertices and knots.

1 Introduction

Eigenvalue matrix models [1] play an increasingly important role in modern theory. Originally they appear as convenient representatives of universality classes of important statistical distributions (random matrix ensembles) [2] and as exactly solvable examples of quantum field and string theory models [3]. Today they are also considered as exhaustively describing the non-perturbative sector, where they appear through either the ADHM description of instanton calculus [4], or the free-field formalism in conformal theory [5] or as the outcome of localization methods [6]. A variety of important applications stimulates the study of matrix models per se, which revealed a lot of important structures, widely used in modern physics, and points to the hidden matrix-model structures far beyond their traditional areal. At early stages the central properties seemed to be integrability and exhaustive set of Ward identities (Virasoro constrains), reviewed in [7] and later promoted to a powerful method of the AMM/EO topological recursion [8]. Today they are considered as consequences of a more fundamental "superintegrability" property $\langle \text{character} \rangle = \text{character}$ [9], which survives also in tensor models and, perhaps, even knot theory. One more conceptually important description is in terms of evolution in the moduli space of coupling constants – known in matrix-model context as $W$-representations [10].

The goal of this paper is to make a step towards unification of these two approaches. We work out a $W$-representation of $q, t$-models, which are the first example, where the superintegrability-based definition from [11] is indisputably simpler than the conventional one [12] through a matrix-like Jackson integral. In this case the latter would be a network-model lifting [13, 14] of Dotsenko-Fateev matrix model [5], which is a free-field description of intertwiner (topological vertex) convolutions in DIM algebra [15], where integrals over the matrix eigenvalues (which are the arguments of the screening integrands) are actually substituted by sums over Young diagrams and, further, over plane partitions. The former definition, at least for the simplest representations of DIM, is just a relatively easy sum over Macdonald functions [11], and we will show that its $W$-representation is, as expected, a direct generalization of the formula for traditional matrix models. It is governed by a relative of Calogero-Ruijsenaars Hamiltonian [16] (which is the $\beta$ and $q, t$-deformation of the simplest of cut-and-join operators from [17]). However, for generic $q$ and $t$ this "relative" appear to be far more sophisticated and is not just a naive "neighbor harmonic" of the same operator (like it remains to be at the level of $\beta$-deformation). In this paper we provide just a raw formula, which deserves further rewriting and better understanding.

2 The definition of Gaussian $q, t$-model

For ordinary Gaussian matrix models one can explicitly calculate any particular correlator (and various generating functions of those) and observe a spectacular fact, that averaging preserves the shape of some functions, i.e. a function of original ("microscopical") variables (fields) after ("functional") integration becomes
the same function of the final ("macroscopical") variables. Moreover, these eigenfunctions of "renormalization group evolution" in matrix models are very simple: just the Schur functions, i.e. the characters of the underlying symmetry. Exact statement in the case of the two simplest, Gaussian Hermitian and complex matrix models, are [18]:

\[
Z_N(p) = c_N \int dM e^{-\frac{1}{2} \text{Tr} M^2 + \sum_k \frac{p_k M_{kk}^2}{k}} = \sum_R \frac{\text{Schur}_R(p_k = \delta_{k,2}) \cdot \text{Schur}_R(p_k = N)}{\text{Schur}_R(p_k = \delta_{k,1})} \cdot \text{Schur}_R(p)
\]

\[
Z_{N_1 \times N_2}(p) = c \int dM e^{-\frac{1}{2} \text{Tr}(MM^t) + \sum_k \frac{p_k (M_{kk}^t)^2}{k}} = \sum_R \frac{\text{Schur}_R(p_k = N_1) \cdot \text{Schur}_R(p_k = N_2)}{\text{Schur}_R(p_k = \delta_{k,1})} \cdot \text{Schur}_R(p)
\]

This fact does not follow directly neither from integrability nor from the Virasoro constraints – instead they both are its corollaries. Thus it can be considered as the long-hunted for formulation of the superintegrability property, of which the above two are the two complementary implications.

Perhaps, even more interesting is the simplicity of the formulas [11], which calls for all kinds of generalizations – and one of the most interesting for today’s applications is to q, t-models, which are now intensively studied by far more complicated techniques. Instead, according to [11], partition functions of the Gaussian q, t-models can be just postulated to be

\[
Z(p) = \sum_R \frac{\langle M_R(P) \rangle}{||M_R||^2} \cdot M_R(p) = \left\langle \exp \left( \sum_k \frac{t^k - t^{-k}}{q^k - q^{-k}} \cdot p_k P_k \right) \right\rangle
\]

where \( M_R \{ p \} \) are Macdonald polynomials [22], and average over the fields \( P_k \) is defined through

\[
\left\langle M_R(P) \right\rangle_{N_1 \times N_2} = \frac{M_R(\pi^{(N_1)}) M_R(\pi^{(N_2)})}{M_R(\delta_{k,1}^*)}
\]

and

\[
\left\langle M_R(P) \right\rangle_N = \frac{M_R(\delta_{k,2}^*)}{M_R(\delta_{k,1}^*)} \cdot M_R(\pi^{(N)})
\]

respectively for the complex and Hermitian models. Here the topological locus and the important loci in the space of time variables \( \{ p_k \} \) are

\[
\delta_{k,n}^{(N)} = \frac{t^{Nk} - t^{-Nk}}{t^k - t^{-k}} \quad t = q^{\beta} \rightarrow 1 \quad N
\]

and

\[
\delta_{k,n} = n \cdot \frac{(q - q^{-1})^{k/n}}{t^k - t^{-k}} \cdot \delta_{k,n} \quad t = q^{\beta} \rightarrow 1 \quad \frac{\delta_{k,n}}{\beta}
\]

where \( \delta_{k,n} = 1 \) when \( k \) is divisible by \( n \) and zero otherwise. We showed also what happens in the limit when both \( t \) and \( q \) tend to unity, but \( t = q^{\beta} \) – this is what is usually called \( \beta \)-deformation [23] (in Nekrasov calculus \( \beta = -\epsilon_1/\epsilon_2 \)). Alternatively q, t-model can be defined as a free-field correlator [12] in DIM-based network models [13], see also [14] – but exact comparison remains to be done. There the time-variables \( P_k \) are actually the fields, generalizing the standard matrix model expressions \( P_k = \text{Tr}(XX^t)^k \) and \( P_k = \text{Tr} X^k \) in complex and Hermitian cases. Note that Gaussian model is different from the logarithmic (Dotsenko-Fateev) one, directly relevant for the AGT relation [24], where the analogue of the basic relations [18] and [22] involves generalized Macdonald functions at the l.h.s. (with the deformation parameter \( \epsilon = \epsilon_1 + \epsilon_2 \), measuring the deviation from \( \beta = 1 \)) and Nekrasov functions at the r.h.s., see [12, 25, 27] for the step-by-step solution of that model and the resulting exhaustive proof of the AGT relation on the lines of [28] and 29.

In this paper we look for still another definition/reformulation of the Gaussian q, t-model: in terms of the W-representations [10], i.e. represent the partition functions [11] as

\[
Z(p) = e^{\dot{W}(p)} \cdot 1
\]

with operator \( \dot{W} \) depending on p-variables and p-derivatives \( \partial/\partial p_k \).
3 \( \beta \)-deformation of Gaussian Hermitian and complex models \( (t = q^\beta) \)

To avoid writing extra formulas, we present here the \( W \)-representation in the \( \beta \)-deformed case: it is easy to return to the original result of [10] and [30] by putting \( \beta = 1 \) and substituting Jack polynomials by the ordinary Schur functions. For examples of \( W \)-representations in other conventional models see [31, 32] and [10] (it deserves mentioning that the celebrated Rosso-Jones formula for torus knots [33, 34] is also an example).

The \( \beta \)-deformed Gaussian Hermitian model (also known as Gaussian \( \beta \)-ensemble) was described in detail in [35]. It can be defined as an eigenvalue integral

\[
\tilde{Z}_\beta^{(N)} \{ p \} \sim \int \frac{1}{\prod_{i<j}(x_i - x_j)^{2\beta}} \prod_{i=1}^{N} \exp \left( \sum_{k} \frac{p_kx_k}{k} \right) e^{-x_i^2/2} dx_i
\]

The averages, calculated in [35], are reproduced by the straightforward \( \beta \)-deformation of the standard \( W \)-representation of [10] \footnote{I am indebted to Tomas Prochazka and Piotr Sulkowski for raising the question about this old unpublished complement to [10] and [35] – and to Shamil Shakirov for confirming the formula.}

\[
\tilde{Z}_\beta^{(N)} \{ p \} = e^{-Nt_0} \cdot \exp \left\{ \frac{1}{2} \sum_{a,b=0} \left( abt_a t_b + \frac{\partial}{\partial t_{a+b-2}} + \beta \cdot (a + b + 2)t_{a+b+2} \frac{\partial^2}{\partial t_a \partial t_b} \right) + \frac{1 - \beta}{2} \sum_{a=0} (a + 1)(a + 2)t_a \frac{\partial}{\partial t_a} \right\} \cdot e^{Nt_0} = 
\]

\[
= \exp \left\{ \frac{\beta N - \beta + 1}{2} Np_2 + \frac{Np_2^2}{2} + \frac{1}{2} \sum_{a,b=1} \left( (a + b - 2)p_a p_b \frac{\partial}{\partial p_a + b - 2} + \beta \cdot a b p_{a+b+2} \frac{\partial^2}{\partial p_a \partial p_b} \right) + \beta N \sum_{a=1} a p_{a+2} \frac{\partial}{\partial p_a} + \frac{1 - \beta}{2} \left( \sum_{a=1} a(a + 1)p_{a+2} \frac{\partial}{\partial p_a} \right) \right\} \cdot 1 = e^{\tilde{W}_\beta^{(N)}} \cdot 1
\]

where \( W \)-operator in the exponent is the \(-2\)-th harmonic of the the Calogero-Ruijsenaars Hamiltonian [10], e.g. of (23) in [29], and we presented expressions in terms of both popular choices of time-variables: \( t_k \) and \( p_k = kt_k \).

The character expansion of [19] in this case is in terms of Jack polynomials \( \text{Jack}_R \{ p \} = \mathcal{M}_R \{ p \}_{t=q^\beta \rightarrow 1} : \)

\[
\tilde{Z}_\beta^{(N)} \{ p \} = \sum_R \beta^{R/2} \cdot \frac{\text{Jack}_R \{ \delta_{k,2} \} \cdot \text{Jack}_R \{ p_k = N \}}{\| \text{Jack}_R \|^2} \cdot \frac{\text{Jack}_R \{ \frac{p_k}{N} \}}{\text{Jack}_R \{ X \}}
\]

and contributing, as usual in Gaussian Hermitian models, are only Young diagrams \( R \) of even size \( |R| \). A typical example of the entries in this expression:

\[
\text{Jack}_{[3,2,1]} \{ p_k = N \} = \frac{(\beta N + 2)(\beta N + 1)(N - 1)(N - 2)(\beta N - \beta + 1)}{(2\beta + 1)^2(3\beta + 2)}
\]

\[
\text{Jack}_{[3,2,1]} \{ p_k = \delta_{k,1} \} = \frac{\beta^3}{(2\beta + 1)^2(3\beta + 2)}, \quad \text{Jack}_{[3,2,1]} \{ p_k = \delta_{k,2} \} = \frac{\beta(\beta - 1)}{(2\beta + 1)^2(3\beta + 2)}
\]

In particular, one can see that the vanishing Gaussian average \( \langle \text{Schur}_{[3,2,1]} \{ X \} \rangle = 0 \) becomes non-vanishing after a \( \beta \)-deformation to \( \langle \text{Jack}_{[3,2,1]} \{ X \} \rangle_\beta \sim (\beta - 1) \).

We wrote eq. [10] in conventional notation of [35]. To make it consistent with the general expression in the present text one should rescale \( p_k \rightarrow \beta p_k \), to get an expansion in \( \text{Jack}_p \) :

\[
\tilde{Z}_\beta^{(N)} \{ p \} := \tilde{Z}_\beta^{(N)} \{ \beta p \} = \sum_R \beta^{R/2} \cdot \frac{\text{Jack}_R \{ \delta_{k,2} \} \cdot \text{Jack}_R \{ p_k = N \}}{\| \text{Jack}_R \|^2} \cdot \text{Jack}_R \{ \frac{p_k}{N} \} = e^{\tilde{W}_\beta^{(N)}} \cdot 1 = 
\]

\[
= \exp \left\{ \frac{(\beta N - \beta + 1)\beta N p_2 + \beta^2 p_2^2}{2} + \frac{1}{2} \sum_{a,b=1} \left( \beta \cdot (a + b - 2)p_a p_b \frac{\partial}{\partial p_a + b - 2} + a b p_{a+b+2} \frac{\partial^2}{\partial p_a \partial p_b} \right) + \beta N \sum_{a=1} a p_{a+2} \frac{\partial}{\partial p_a} + \frac{1 - \beta}{2} \left( \sum_{a=1} a(a + 1)p_{a+2} \frac{\partial}{\partial p_a} \right) \right\} \cdot 1
\]
Likewise, for the \( \beta \)-deformation of the complex model we have

\[
\sum_{R} \beta^{\left|R\right|}, \text{Jack}_{R} \{ p_{k} = N_{1} \} \cdot \text{Jack}_{R} \{ p_{k} = N_{2} \}, \frac{\text{Jack}_{R} \{ p \}}{||\text{Jack}_{R}||^{2}} = e^{{\hat{W}}_{\beta}^{(N_{1} \times N_{2})}} \cdot 1 =
\]

\[
= \exp \left\{ \beta^{2} N_{1} N_{2} p_{1} + \sum_{a,b = 1} \left( \beta \cdot (a + b - 1) p_{a} p_{b} \frac{\partial}{\partial p_{a+b-1}} + a b p_{a+b+1} \frac{\partial^{2}}{\partial p_{a} \partial p_{b}} \right) + \beta (N_{1} + N_{2}) \sum_{a = 1} a p_{a+1} \frac{\partial}{\partial p_{a}} + (1 - \beta) \left( \sum_{a = 1} a^{2} p_{a+1} \frac{\partial}{\partial p_{a}} \right) \right\} \cdot 1
\]

These are the two formulas which we want to lift to the \( q, t \) models. Note that the weights \( \beta^{\left|R\right|/2} \) and \( \beta^{\left|R\right|} \) at the l.h.s. of these formulas automatically appear in the limit \( t = q^{2} \to 1 \) from the factor \( \frac{1}{2} \) in (6).

4 On ambiguity of W-representation

In fact, \( W \)-representation is ambiguous in the following sense. If we only demand that the l.h.s. of, say, (13) at \( \beta = 1 \), is reproduced term-by-term by the action of some evolution operator on unity, then the operator at the r.h.s. of (13) can be modified by adding to the exponent of any linear combination of infinitely-many new operators. The simplest of them is

\[
\sigma_{3} \{ p \} \cdot \left( 2(N_{1} N_{2} + 1) \cdot \frac{\partial}{\partial p_{2}} - (N_{1} + N_{2}) \cdot \frac{\partial^{2}}{\partial p_{1}^{2}} - \frac{4(N_{1}^{2} - 1)(N_{2}^{2} - 1)}{(N_{1} N_{2} + 4)(N_{1} + N_{2})^{2} + 2(N_{1} N_{2} + 1)} \cdot p_{2} \frac{\partial^{2}}{\partial p_{2}^{2}} + \ldots \right)
\]

with a coefficient, which arbitrarily depends on \( p_{k} \) – if one prefers to preserve grading, it should be restricted to \( \sigma_{3} \{ p \} = \sigma_{(3)} p_{3} + \sigma_{(2,1)} p_{2} p_{1} + \sigma_{(1,1,1)} p_{1}^{3} \). At the level 4 there are two new operators with arbitrary coefficients:

\[
\sigma_{4a} \{ p \} \cdot \left( 3(N_{1} N_{2} + 1)(N_{1} N_{2} + 2) \cdot \frac{\partial}{\partial p_{3}} - (N_{1}^{2} + 3 N_{1} N_{2} + N_{2}^{2} + 1) \cdot \frac{\partial^{2}}{\partial p_{1}^{2}} + \ldots \right) +
\]

\[
+ \sigma_{4b} \{ p \} \cdot \left( 2(N_{1} N_{2} + 1) \cdot \frac{\partial^{2}}{\partial p_{2} \partial p_{1}} - (N_{1} + N_{2}) \cdot \frac{\partial^{3}}{\partial p_{1}^{3}} + \ldots \right)
\]

and so on. Note that the second operator in (15) resembles a \( p_{1} \)-derivative of (14).

One can fix this freedom by restricting the \( N \)-dependence of \( W \) to contain just three structures: \( N_{1} N_{2}, N_{1} + N_{2} \) and 1 – this picks up the nice formula (13) and forbids all the ugly ambiguities.

5 Towards W-representation for q, t-models

While \( \beta \)-deformation of \( W \)-representation is very simple, this is not quite the case for the generic \( q \) and \( t \) and we construct it in five steps.

First will be specification of \( N \)-dependence, see eq. (18) below.

Second is finding the first contributions to the \( N \)-independent part \( \hat{W} \) of the evolution operator, see (19).

Third – the study of its ambiguities, see (20)

Forth – understanding the peculiar tri-linear structure (22) of \( \hat{W} \) and

Fifth – providing a generating function (24) for its generic term.

The most surprising in this story is that one expects the relevant \( W \) operator to be a close relative of Calogero-Ruijsenaars Hamiltonian for generic \( q \) and \( t \),

\[
\hat{H} = \oint \frac{dz}{z} \exp \left( \sum_{k=1}^{\infty} \frac{(1 - t^{-2k}) z^{k} p_{k}}{k} \right) \exp \left( \sum_{k=1}^{\infty} \frac{q^{2k} - 1}{z^{k}} \frac{\partial}{\partial p_{k}} \right) =
\]

\[
= \sum_{m=0}^{\infty} t^{-2m} \cdot \text{Schur}_{[m]} \left\{ (t^{2k} - 1) p_{k} \right\} \cdot \text{Schur}_{[m]} \left\{ (q^{2k} - 1) k \frac{\partial}{\partial p_{k}} \right\}
\]

– the simplest in the operator family, of which Macdonald polynomials are the eigenfunctions. However, to get the needed \( W \) operator, which provides the \( W \)-representation of a \( q, t \)-model, it is not enough just to change the power of \( z \) in \( \oint \frac{dz}{z} \). The answer is substantially more involved – what, of course, is not really a surprise for
experts in \(q,t\)-deformations. The structures, revealed in this basic exercise, can now be expected to emerge in more sophisticated applications.

To avoid repeating the lengthy story twice we concentrate in this paper on the \(q,t\)-deformation of the complex model, and our target is the operator \(\mathcal{W}\) in

\[
\mathcal{Z}_{N_1 \times N_2} = \sum_{m=0} \mathcal{Z}_m \equiv \sum_R \mathcal{M}_R \{ \pi^{(N_1)} \} \cdot \mathcal{M}_R \{ \pi^{(N_2)} \} \cdot \mathcal{M}_R \{ p \} \frac{\delta_{\epsilon_{k,1}}}{|\mathcal{M}_R|^2} = e^{\mathcal{W}} \cdot 1 \tag{17}
\]

which is naturally graded, and \(\mathcal{Z}_m\) denotes the contribution of the Young diagrams of the size (number of boxes) \(|R| = m\).

It turns out that from the point of view of \(N\)-dependence the \(\mathcal{W}\) operator contains just three different structures: \(t^{N_1+N_2} + t^{-N_1-N_2}, t^{N_1-N_2} + t^{N_2-N_1}\) and \(t^{-N_1-N_2}:

\[
\mathcal{W} = \frac{q^2}{q^{2(m-1)}} \left\{ \left( t^{N_1+N_2} + t^{-N_1-N_2} \right) \left( \frac{p_1}{q^{2(m-1)}} + \sum_{m=2} \frac{m \delta_{m-1}^{(m-1)}}{q^{2(m-1)}} \right) \right. + \\
+ \left. \left( t^{N_1-N_2} + t^{N_2-N_1} \right) \left( -\frac{p_1}{q^{2(m-1)}} + \sum_{m=2} \frac{m \delta_{m-1}^{(m-1)}}{q^{2(m-1)}} \right) \right\} + t^{-N_1-N_2} \cdot \hat{W} \tag{18}
\]

with the underlined \(p_1\)-terms combined into \(\mathcal{Z}_1 = \frac{(t^{N_1+t^{-N_1}})(t^{N_2+t^{-N_2}})}{(q^{2(m-1)})^2} p_1 = \frac{(t^{N_1})(t^{N_2})}{(q)^2} \cdot p_1\) and with a somewhat more sophisticated \(N\)-independent \(\hat{W}\):

\[
\hat{W} = \frac{1}{(q^{-1})(q^{-2})} \sum_{m=2} m S_{1m} \delta_{1m}^{(m)} + \\
+ \sum_{m=3} \frac{p_{m-1}}{q^{m-1}} \left( -\frac{2q^2 + t^2}{q} \delta_{m-1}^{(m-1)} + \frac{t^2}{q} \delta_{m-2}^{(m-2)} + \frac{t^4}{q^2} \delta_{m-3}^{(m-3)} + \cdots + \frac{(t^{2m-2} - 1)q^{2m-2}}{(q^2 - 1)(q^4 - 1)} \delta_{m-1}^{(m-1)} \right) + \\
+ \sum_{m=3} \frac{p_{m-1} p_1 (t^{2(m-1)} - t^{m-1})}{(q^{2(m-1)} - 1)} \left( -\delta_{m-1}^{(m-1)} + \frac{t^2}{q^2} \delta_{m-2}^{(m-2)} + \frac{t^4}{q^4} \delta_{m-3}^{(m-3)} + \cdots + \frac{(t^{2m-2} - 1)q^{2m-2}}{(q^2 - 1)(q^4 - 1)} \delta_{m-1}^{(m-1)} \right) + \cdots + \\
+ \frac{q^{2m-2}}{q^{2(m-1)}} \sum_{m=5} m S_{1m-3} \cdot \left( \frac{\delta_{m-3}^{(m-3)} - \delta_{m-2}^{(m-2)}}{q^2} + \frac{t^2 \delta_{m-3}^{(m-3)}}{q^4} - \frac{t^4}{(q^2 - 1)} \delta_{m-1}^{(m-1)} \right) + \\
+ \frac{q^{2m-2}}{q^{2(m-1)}} \sum_{m=4} m S_{1m-2} \cdot \left( \frac{\delta_{m-2}^{(m-2)} - \delta_{m-1}^{(m-1)}}{q^2} \right)
\]

In these formulas we use the condensed notation, which is natural already at the level of \([16]\):

\[
S_R = \text{Schur}_R \{ (t^{2k} - 1)p_k \}, \quad \hat{D}_k = (q^{2k} - 1)k \frac{\partial}{\partial p_k} \quad \text{and} \quad \hat{D}_R = \text{Schur}_R \{ \hat{D}_k \} \tag{20}
\]

Note that the first line in \([18]\) contains a Schur polynomial \(\hat{D}_{m-1}\), while the second line – just a single derivative \(\hat{D}_{m-1}\). It is clear that one more combination actually appears in these formulas,

\[
P_k = \frac{p_k}{q^{2k} - 1}, \quad \text{so that} \quad \hat{D}_k = k \frac{\partial}{\partial P_k} \tag{21}
\]

However, before making use of this observation, one should answer another obvious question.

## 6 Ambiguities at generic \(q, t\)

Comparing sophisticated \([19]\) with the simply-looking \([18]\) it is natural to ask, if one can use the ambiguity in \(\mathcal{W}\) to simplify \(\hat{W}\). This desire is only strengthened by the fact that the result \([18]\), though very nice from all other perspectives, makes fighting ambiguity after the full-fledged \(q,t\)-deformation more difficult. Indeed, in sect.\(4\) we fixed the ambiguity by restricting the \(N\)-dependence, but now this is impossible – all ambiguities preserve...
the fact that there are just three different structures: $t^{N_1+N_2} + t^{-N_1-N_2}$, $t^{N_1-N_2} + t^{N_2-N_1}$, and $t^{-N_1-N_2}$. In fact, and it is the $\hbar$-expansion which converts them into many independently-looking structures in the $\beta$-deformed formulas.

What saves the situation is that the ambiguities are still in one-to-one correspondence with those in sec.4, in particular, the three first three

$$
\sigma_3 \left\{ -q^2 \left( t^{N_1+N_2} + t^{-N_1-N_2} \right) \hat{D}_{[1,1]} - q^2 \left( t^{N_1-N_2} + t^{N_2-N_1} \right) \hat{D}_{[2]} + \frac{1}{t^{N_1+N_2}} \left( (q^2 + t^2) \hat{D}_{[1,1]} + (q^2 t^2 + 1) \hat{D}_{[2]} \right) + \frac{(t^2 - 1) \beta}{t^{N_1+N_2}} (q^2 \hat{D}_{[3]} + (q^2 t^2 + 1) \hat{D}_{[2,1]} + t^2 \hat{D}_{[1,1,1]}) + \cdots \right\} +
$$

$$
+ \sigma_{4a} \left\{ \left( t^{N_1+N_2} + t^{-N_1-N_2} \right) \hat{D}_{[2,1]} + \left( t^{N_1-N_2} + t^{N_2-N_1} \right) \hat{D}_{[3]} + \frac{t^{-N_1-N_2}}{q^4} \left( -q^2 (q^2 t^2 - t^2 + 1) \hat{D}_{[3]} - (q^2 t^2 + 1)^2 \hat{D}_{[2,1]} - (q^2 t^4 + q^4 + t^2) \hat{D}_{[1,1,1]} + \cdots \right) \right\} +
$$

$$
+ \sigma_{4b} \left\{ \left( t^{N_1+N_2} + t^{-N_1-N_2} \right) \hat{D}_{[1,1,1]} - \left( t^{N_1-N_2} + t^{N_2-N_1} \right) \hat{D}_{2,1} - \frac{t^{-N_1-N_2}}{q^4} \left( -q^2 (q^2 t^2 - q^2 + 1) \hat{D}_{[3]} - (q^2 t^2 + 1)^2 \hat{D}_{[2,1]} - (q^2 t^4 + q^4 + t^2) \hat{D}_{[1,1,1]} + \cdots \right) \right\}
$$

to be compared with (14) and (15) (with $\sigma$’s differing by rescalings and linear combinations). This means that one can eliminate the ambiguity for arbitrary $q,t$ by simply requiring that all $\sigma = 0$, i.e., with a certain abuse of terminology, that the $\beta$-deformed counterpart of the $W$-representation is (12). Exact specification of our choice (19) is that all $\sigma = 0$.

7 Condensed form of the W-representation

Coming back to (19) one can see that the operator $\hat{W}$ has a peculiar tri-linear structure:

$$
\hat{W} = \sum_{m=2} \sum_{Q} \sum_{m-1} \sum_{k} C^Q_k \cdot P_k \cdot S_{[1^{m-k}]} \cdot \hat{D}_Q
$$

with relatively simple $q,t$-dependent (but $p$-independent) coefficients $C^Q_k$. Indeed,

$$
\hat{W} = \sum_{m=2} \left\{ \left( t^2 - 2q^2 \right) P_m - P_{m-1} \cdot S_{[1]} \right\} \cdot \hat{D}_{[m-1]} + \cdots + \left( \sum_{k=1}^{m} \frac{\rho_k P_k}{q^{2k-1}} \cdot S_{[1^{m-k}]} \right) \cdot \hat{D}_{[m-1]} \right\} +
$$

$$
+ \left( (q^4 - t^4) P_m + t^2 P_{m-1} S_{[1]} - P_{m-2} S_{[1]} \right) \cdot \frac{\hat{D}_{[m-2,1]}}{q^2} +
$$

$$
+ \left( \left( t^2 - q^2 \right)^2 P_m - (t^2 - q^2) P_{m-1} S_{[1]} + P_{m-2} S_{[1]} \right) \cdot \frac{\hat{D}_{[m-3,2]}}{q^4} +
$$

$$
+ \left( (t^6 - q^6) P_m - t^4 P_{m-1} S_{[1]} + t^2 P_{m-2} S_{[1]} - P_{m-3} S_{[1]} \right) \cdot \frac{\hat{D}_{[m-2,1]}}{q^4} + \cdots
$$

(23)

where

$$
\rho_k = (-)^{k+1} \cdot \left( t^{2k-2} \cdot \frac{q^{2k-1} - 1}{q^2 - 1} + \frac{t^{2k-2} - 1}{t^2 - 1} \right) = (-)^{k+1} \cdot \left( q^{2k-2} \cdot \frac{q^{2k-2} - 1}{q^2 - 1} + \frac{t^{2k-2} - 1}{t^2 - 1} \right)
$$

(24)

This structure is seen even better, if one uses a further condensed notation:

$$
P_m^{(k)} := \sum_{l=0}^{k} (-)^l t^{2(k-l)} P_{m-1} S_{[1^l]}^{[1^l]}
$$

(25)
In these terms we can list many more items:

\[ \dot{W} = \sum_{m=2}^{\infty} \left( -q^2 P_m^{(0)} \cdot \dot{D}_{[m-1]} + \sum_{k=1}^{m} \rho_k P_k S_{[1m-k]} \cdot \frac{\dot{D}_{[m-1]}}{q^{2(m-2)}} \right) + \sum_{m=3} \left( P_m^{1} - q^2 P_m^{(0)} \right) \cdot \dot{D}_{[m-1]} - \sum_{m=4} \left( \frac{P_m^2 - q^{4} P_m^{(0)}}{q^2} \right) \frac{\dot{D}_{[m-2.1]}}{q^2} + \sum_{m=5} \left( \frac{P_m^3 - q^{2} P_m^{(1)}}{q^2} \right) \frac{\dot{D}_{[m-3.2]}}{q^2} + \sum_{m=6} \left( \frac{P_m^4 - q^{6} P_m^{(0)}}{q^4} \right) \frac{\dot{D}_{[m-3.1,1.1]}}{q^4} \right) + + \sum_{m=7} \left( \frac{P_m^5 - q^{8} P_m^{(1)}}{q^4} \right) \frac{\dot{D}_{[m-4.2.1]}}{q^4} - \left( \frac{P_m^6 - q^{8} P_m^{(0)}}{q^6} \right) \frac{\dot{D}_{[m-4.1,1,1,1.1]}}{q^6} \right) + + \sum_{m=8} \left( \frac{P_m^7 - q^{10} P_m^{(0)}}{q^6} \right) \frac{\dot{D}_{[m-5.2.1.1.1,1.1]}}{q^6} + \sum_{m=9} \left( \frac{P_m^9 - q^{12} P_m^{(0)}}{q^{10}} \right) \frac{\dot{D}_{[m-6.1,1,1,1,1,1,1,1]}}{q^{10}} \right) + + \sum_{m=10} \left( \frac{P_m^{10} - q^{14} P_m^{(0)}}{q^{12}} \right) \frac{\dot{D}_{[m-7.2.1.1,1,1,1,1,1,1,1]}}{q^{12}} + \ldots \]

and it gets clear, that contributing are only derivatives \(D_0\) with Young diagrams of the type \(Q = [s, 2n_2, 1^n_1]\), i.e. with at most one line of the length \(s > 2\) — exactly those which define the simple Hurwitz numbers, see \[17\][30][36]\ and references therein. The size of the diagram is fixed to be \(|Q| = s + 2n_2 + n_1 = m - 1\), i.e.

\[ \dot{W} = \sum_{m=2}^{\infty} \left( -q^2 P_m^{(0)} \cdot \dot{D}_{[m-1]} + \sum_{k=1}^{m} \rho_k P_k S_{[1m-k]} \cdot \frac{\dot{D}_{[m-1]}}{q^{2(m-2)}} \right) + \sum_{s+2n_2+n_1=m-1}^{m} \left( -n_1 \left( P_{m+n_1+2n_2}^{(n_1+n_1+2n_1+1) - q^{2(n_1+1)}} P_{m+n_1+2n_2}^{(n_1+n_1+2n_1+1)} \right) \dot{D}_{[m, 2n_2, 1^n_1]} \right) \]

Underlined are the items, which deviate from the general rule and need a separate treatment. We can attempt to improve the last formula by lifting restriction \(s \geq 2\) on the length of the first row in the diagram \(Q\) (when \(n_2 = 0\)). This requires switching from \(\rho_k\) to a slightly nicer \(\hat{\rho}_k\), with the help of the following manipulation:

\[ \sum_{k=1}^{m} \rho_k P_k S_{[1m-k]} = \sum_{k=1}^{m} (-)^{k+1} \left( \frac{t^{2k-2} - q^{2k-1}}{q^2-1} + \frac{t^{2k-2} - 1}{t^2-1} \right) P_k S_{[1m-k]} \] \[ \rho_k \Rightarrow \] \[ P_m^{(m-1)} \left( -t^{2(m-1)-l} P_{m-l} S_{[1]} \right) = \sum_{k=1}^{m} \left( -t^{m-k} t^{2k-2} P_{m-l} S_{[1]} \right) \] \[ \sum_{k=1}^{m} \rho_k P_k S_{[1m-k]} = (-)^{m} \rho_k P_{m-1}^{(m-1)} + \sum_{k=1}^{m} (-)^{k+1} \left( \frac{t^{2k-2} - q^{2k-1}}{q^2-1} + \frac{t^{2k-2} - 1}{t^2-1} \right) \hat{\rho}_k P_k S_{[1m-k]} \]

However, this is not sufficient to provide the second term in the would-be coefficient \(P_m^{(m-1)} - q^{2m} P_m^{(0)}\) of \(\dot{D}_{[m-1]}\).

8 From derivatives to shifts

So far we expanded \(\dot{W}\) operators in \(p\)-derivatives. However, after \(q\)-deformation one can instead consider difference operators (shifts of \(p_k\)). Transformation between derivatives to shifts is provided by the Cauchy formula \[22\], recently reviewed from related perspective in \[51\], which expresses shifts through Schur polynomials.
of derivatives:

$$\exp \left( \sum_k \bar{p}_k \frac{\partial}{\partial p_k} \right) = \sum_R \text{Schur}_R \{ \bar{p}_k \} \cdot \text{Schur}_R \left\{ k \frac{\partial}{\partial p_k} \right\} \quad (28)$$

In particular, for $\bar{p}_k = z^k$, we get a contribution from only single-line Young diagrams $R = [m]$ (symmetric representations):

$$\exp \left( \sum_k z^k \frac{\partial}{\partial p_k} \right) = \sum_m z^m \cdot \text{Schur}_{[m]} \left\{ k \frac{\partial}{\partial p_k} \right\} \quad (29)$$

Changing sign, we can restrict to single-row diagrams $R = [1^m]$ (antisymmetric representations):

$$\exp \left( - \sum_k z^k \frac{\partial}{\partial p_k} \right) = \sum_m (-)^m z^m \cdot \text{Schur}_{[1^m]} \left\{ k \frac{\partial}{\partial p_k} \right\} \quad (30)$$

Tri-linear structure (22) implies that in addition to two exponentials in (10), providing Schur functions of $(t^{2k} - 1)p_k$ and $\hat{D}_k$ one needs an additional insertion like $\sum_k p_k z^{-k} = \sum_k \frac{p_k z^{-k}}{q}$ in the integrand. However, more important is that in (20) contributing are diagrams $Q$ with arbitrary number of rows $l_Q$, thus the single-$z$ integrals like (10) should be substituted by multiple-$z$ integrals (because of the restriction on the line lengths, one can actually hope to manage with finitely many $z$-variables). Amusingly, such integrals are also expected in description of generalized Kerov functions [38] – and the corresponding theory still remains to be built.

9 Conclusion

In this paper we conjectured a prototype (26) of the $W$-representation of the basic $q, t$-model, generalizing the Gaussian complex-matrix models. As suggested in [11] we used as a definition of $q, t$-model a sum of Macdonald averages, defined through the character-preservation postulate of [9]. Exact relation to network formulation of [13] remains to be worked out, together with a possible derivation from the $W$-representation of the deep $R$-matrix structures and Knizhnik-Zamolodchikov equations, which so far are revealed (also, just partly) only in the network-model approach [39]. An important step in this direction can be further lifting to generalized Macdonald functions, depending on collections of Young diagrams, which are capable to describe many free fields and thus the full-fledged networks with many horizontal lines.

Our resulting formula (26) is hardly final – rather an intermediate one. It is still far from ideal and should be further improved and converted into something, probably more similar to the Calogero-Ruijsenaars Hamiltonian (10). In continuous models $W$-representations always involve just another harmonic of the same operator, whose zero-harmonic provides the Hamiltonian – of which the relevant characters were the eigenfunctions. Surprisingly or not, a similar relation in the $q, t$-deformed case is far from obvious. This should have something to do with the fact that the relevant integrals in network models are multiple Jackson integrals, which are actually sums over (collections of) Young diagrams – and the notion of harmonics in this case is not fully obvious. Further improvement (condensed notation) of (26) should be an important, and, perhaps, the most straightforward step towards building appropriate formalism. An important fact in this relations is that the asymmetry between $q$ and $t$, which is very strong in the starting definition (2) of the $q, t$ model, almost disappears at the level of the $N$-independent $W$-operator in (26) – as one expects in the context of network models. Of separate interest and significance is the detailed study of particular limits – not only to $\beta$-deformed ($t = q^\beta \rightarrow 1$), but also to the Hall-Littlewood ($t = 0$) case.

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for various extensions beyond torus knots and associated $W$-operators see [34].