On regular Stein neighborhoods of a union of two maximally totally real subspaces in $\mathbb{C}^n$

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Abstract We construct regular Stein neighborhoods of a union of two maximally totally real subspaces $M = (A + iI)\mathbb{R}^n$ and $N = \mathbb{R}^n$ in $\mathbb{C}^n$, provided that the eigenvalues of the real $n \times n$ matrix $A$ are sufficiently small. This result is applied to provide regular Stein neighborhoods of an immersed totally real $n$-manifold in a complex $n$-manifold, with only finitely many double points, and such that the union of the tangent spaces at each double point in some local coordinates coincides with $M \cup N$, described above.

Keywords Stein neighborhoods · Totally real subspaces · Strictly plurisubharmonic functions · Strong deformation retraction

1 Introduction

To solve certain problems in complex analysis it is a very useful property for a subset of a manifold to have open Stein neighborhoods together with some additional suitable properties. It is important to control the homotopy type or the shape of the neighborhoods, and hence having the so-called regular neighborhoods; these are neighborhoods which admit a strong deformation retraction to a given set.

It is well known that a Euclidean distance function to a totally real submanifold in $\mathbb{C}^n$ defines tubular strongly pseudoconvex neighborhoods (see Nirenberg and Wells [10]). Furthermore, every closed real surface which is smoothly immersed into a complex surface has a basis of regular Stein neighborhoods, provided that there are only finitely many double points and only hyperbolic complex points, and they are all of special type (see Forstnerič...
A provided that the eigenvalues of it with the distance function to $M$ topology of the sublevel sets of $\rho$ on $\mathbb{R}^n$ cases. Proposition 4.3] by the author further extend the above result, but still only for some special Levi forms. It enables us to prove the existence of regular Stein neighborhoods of much more streamlined computations, which are essential in examining the positivity of their $M$ squared Euclidean distance functions to $A$ is polynomially convex near the origin if $A$ has no purely imaginary eigenvalue of modulus greater than one. In this case one has a smooth non-negative plurisubharmonic function defining function for $M(A)\cup N$ near the origin (see e.g. [15, Theorem 1.3.8]), and patching it with the distance function to $M(A)\cup N$ yields a plurisubharmonic defining function $\rho$ for $M(A)\cup N$ (see [3, Lemma IV] or [12, Lemma 5.3]). However, we have no control on the topology of the sublevel sets of $\rho$, hence to find regular neighborhoods additional hypotheses on $\rho$ are needed. As in [16] we prefer to work with functions, depending polynomially in squared Euclidean distance functions to $M$ and $N$ respectively, but we are now able to give a much more streamlined computations, which are essential in examining the positivity of their Levi forms. It enables us to prove the existence of regular Stein neighborhoods of $M(A)\cup N$, provided that the eigenvalues of $A$ are sufficiently close to zero (see Theorem 4.2). We add here that by Weinstock’s result, small analytic anulli prevent $M(A)\cup N$ from having Stein neighborhoods if $A$ has purely imaginary eigenvalue of modulus greater than one.

At the end we also apply our result to get regular neighborhoods of immersed totally real $n$-manifolds in a complex $n$-manifold with only finitely many double points, and such that the union of the tangent spaces at each intersection in some local coordinates coincides with $M(A)\cup N$, described above.

2 The Euclidean distance function to a totally real subspace

Throughout this paper $z = (z_1, \ldots, z_n)$ will denote the standard holomorphic coordinates and $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ the corresponding real coordinates on $\mathbb{C}^n = (\mathbb{R} + i\mathbb{R})^n \approx \mathbb{R}^n + i\mathbb{R}^n \approx \mathbb{R}^{2n}$ with respect to $z_j = x_j + iy_j, j \in \{1, \ldots, n\}$. By $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively, we denote the Euclidean inner product and Euclidean distance on $\mathbb{C}^n$:

$$\langle \xi, \eta \rangle = \sum_{j=1}^{n} \overline{\xi}_j \eta_j, \quad |\xi| = \sqrt{\langle \xi, \xi \rangle}, \quad \xi = (\xi_1, \ldots, \xi_n), \quad \eta = (\eta_1, \ldots, \eta_n).$$

In real notation with $\xi_j = \sigma_j + i\tau_j$ and $\eta_j = \mu_j + i\nu_j$ for $j \in \{1, \ldots, n\}$ we have

$$\langle \xi, \eta \rangle = \sum_{j=1}^{n} (\sigma_j \mu_j + \tau_j \nu_j), \quad \xi = (\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n), \quad \eta = (\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n).$$

We also introduce the usual multiindex notation

$$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad \lambda = (\lambda_1, \ldots, \lambda_n), \quad |\lambda| = \sum_{j=1}^{n} \lambda_j.$$

Recall that a real linear subspace in $\mathbb{C}^n$ is called totally real if it contains no complex subspace. Let now $M$, $N$ be linear totally real subspaces of maximal dimension in $\mathbb{C}^n$, ...
intersecting transversally. It is not difficult to prove (see e.g. [17]) that there exists a non-singular complex linear transformation which maps $N$ onto $\mathbb{R}^n \cong (\mathbb{R} + i0)^n \subset \mathbb{C}^n$ and $M$ onto $M(A) = (A + iI)\mathbb{R}^n$, where $(A - iI)$ is invertible and $A$ is the real Jordan canonical form, i.e. $A$ is a block matrix, having zero-matrices as off-diagonal blocks, and each of the main diagonal blocks satisfies one of the two conditions listed below:

**Case 1** A matrix with $a \in \mathbb{R}$ on the main diagonal, possibly with $\delta \in \mathbb{R}\{0\}$ on the upper diagonal, and zeros otherwise, i.e.

$$
\begin{bmatrix}
a & \delta \\
a & \ddots \\
& \ddots & \ddots & \delta \\
& & a \\
\end{bmatrix},
$$

(2.1)

**Case 2** A block matrix, having $2 \times 2$ main diagonal blocks with complex eigenvalues, possibly with the $2 \times 2$ identity-matrix $I_2$ multiplied by $\delta \in \mathbb{R}\{0\}$ on the upper diagonal, and $2 \times 2$ zero-matrices otherwise, i.e.

$$
\begin{bmatrix}
C & \delta I_2 \\
C & \ddots \\
& \ddots & \ddots & \delta I_2 \\
& & C \\
\end{bmatrix},
\quad C = \begin{bmatrix} c & -b \\ b & c \end{bmatrix} \in \mathbb{R}^{2\times2}, \ c^2 + (1 - b^2)^2 \neq 0.
$$

(2.2)

Moreover, $A$ is uniquely determined up to the order of diagonal blocks, and a non-zero real number $\delta$ can be chosen arbitrarily. Also, the degenerate case (i.e. $1 \times 1$ matrix in case (2.1) and $2 \times 2$ matrix in case (2.2)) is considered here, though it lacks an upper diagonal or block-upper diagonal, respectively. If $A$ is diagonalizable, then all diagonal blocks are degenerate.

For any $\delta \in \mathbb{R}$ we set

$$
A_\delta = \text{diag}(A_{\delta,1}, \ldots, A_{\delta,\alpha})
$$

(2.3)

to be a $n \times n$ block diagonal matrix and such that for every $j \in \{1, \ldots, \alpha\}$ the diagonal block $A_{\delta,j} \in \mathbb{R}^{n_j \times n_j}$ is of the form (2.1) or (2.2), with $n_1 + \cdots + n_\alpha = n$.

**Lemma 2.1** Let $A_\delta$ for $\delta \in \mathbb{R}$ be defined as in (2.3), and let $d_{M(A_\delta)}(x, y)$ be the squared Euclidean distance function from $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ to $M(A_\delta) = (A_\delta + iI)\mathbb{R}^n$ in $\mathbb{R}^{2n}$. If $A_\delta$ is non-diagonalizable, then

$$
d_{M(A_\delta)}(x, y) = d_{M(A_0)}(x, y) + q_\delta(x, y),
$$

where $q_\delta$ is a homogeneous polynomial of degree 2 in $x, y$, and such that its coefficients are rational functions in $\delta$ that vanish at $\delta = 0$. Furthermore, when $A_0 = \text{diag}(C_1, \ldots, C_\beta, a_1, \ldots, a_\gamma) \in \mathbb{R}^{n \times n}$, $C_j = \begin{bmatrix} c_j & -b_j \\ b_j & c_j \end{bmatrix}$ for $1 \leq j \leq \beta$, then

$$
d_{M(A_0)}(x, y) = \sum_{j=1}^{\beta} \frac{(x_{2j-1} - c_jy_{2j-1} + b_jy_{2j})^2 + (x_{2j} - c_jy_{2j} - b_jy_{2j-1})^2}{1 + c_j^2 + b_j^2} + \sum_{j=2\beta+1}^{2\beta+\gamma} \frac{(x_j - a_jy_j)^2}{1 + a_j^2}.
$$

(2.4)
In particular, if \( A_0 = \text{diag}(C_1, \ldots, C_\beta) \) (respectively \( A_0 = \text{diag}(a_1, \ldots, a_n) \)), then \( d_{M(A_0)}(x, y) \) is equal to the first term (respectively second term for \( \beta = 0 \)) of \( (2.4) \).

**Proof** Let \( A_\delta = \text{diag}(A_{\delta,1}, \ldots, A_{\delta,\alpha}) \) with \( A_{\delta,\alpha} \in \mathbb{R}^{n_{\delta} \times n_{\delta}}, n_1 + \cdots + n_\alpha = n \) and let \( M(A_{\delta,j})^\perp \) be the orthogonal complement of \( M(A_{\delta,j}) \) in \( \mathbb{C}^{n_j} \) with respect to the standard inner product. This yields the direct sum decompositions of spaces \( M(A_\delta) = (A_\delta + iI)\mathbb{R}^n \) and its orthogonal complement \( M(A_\delta)^\perp \) in \( \mathbb{C}^n \), respectively, into pairwise orthogonal linear subspaces:

\[
M(A_\delta) = M_{\delta,1} \oplus \ldots \oplus M_{\delta,\alpha}, \quad M(A_\delta)^\perp = M_{\delta,1}^\perp \oplus \ldots \oplus M_{\delta,\alpha}^\perp,
\]

where \( M_{\delta,j} = \{0\}^{n_1+\cdots+n_{j-1}} \times M(A_{\delta,j}) \times \{0\}^{n_{j+1}+\cdots+n_\alpha} \) and its orthogonal complement is \( M_{\delta,j}^\perp = \{0\}^{n_1+\cdots+n_{j-1}} \times M(A_{\delta,j})^\perp \times \{0\}^{n_{j+1}+\cdots+n_\alpha} \). It follows that \( d_{M(A_\delta)} = \sum_{j=1}^\alpha d_{M(A_{\delta,j})} \)

where we denoted by \( d_{M(A_\delta)} \) (respectively \( d_{M(A_{\delta,j})} \)) the squared Euclidean distance to \( M(A_\delta) \) in \( \mathbb{C}^n \) (respectively \( M(A_{\delta,j}) \) in \( \mathbb{C}^{n_j} \)). It is thus sufficient to prove the lemma for the case when \( A_\delta \) is of the form \( (2.1) \) or \( (2.2) \).

First, we consider the case when \( A_\delta \in \mathbb{R}^{n \times n} \) is of the form \( (2.1) \). In this case \( M(A_\delta) \) is given as a span of \( n \) linearly independent vectors

\[
M(A_\delta) = \text{Span}\{f_j + ae_j + \delta e_{j-1}\}_{2 \leq j \leq n} \cup \{ae_1 + f_1\}, \quad (2.5)
\]

where \( \{e_1, f_1, \ldots, e_n, f_n\} \) is the standard ortho-normal basis of \( \mathbb{R}^n \) and \( a \in \mathbb{R} \). We observe further that the orthogonal complement of \( M(A_\delta) \) is then equal to

\[
M(A_\delta)^\perp = \text{Span}\{e_j - af_j - \delta f_{j+1}\}_{1 \leq j \leq n-1} \cup \{e_n - af_n\}. \quad (2.6)
\]

Indeed, since every \( e_j \) for \( j \in \{1, \ldots, n\} \) is orthogonal to all but one vector in the span \( (2.6) \), this span contains \( n \) linearly independent vectors. To simplify the computations we denote the vectors in \( (2.6) \) by

\[
g_j = e_j - af_j - \delta f_{j+1}, \quad 1 \leq j \leq n-1, \quad g_n = e_n - af_n.
\]

We proceed with the Gram-Schmidt process \( g'_1 = g_1, g'_m = g_m - \sum_{j=1}^{m-1} (g_m \cdot g'_j)/|g'_j|^2 g'_j \) for \( m \in \{2, \ldots, n\} \), to obtain the orthogonal basis of \( M(A_\delta)^\perp \). It can be shown by induction that

\[
g'_j = e_j - af_j - \delta v_j, \quad j \in \{1, \ldots, n\}, \quad (2.7)
\]

where the components of \( v_j \) are rational functions in \( \delta \) and without a pole at \( \delta = 0 \). Indeed, if \( g'_j \) is of the form \( (2.7) \), then \( |g'_j|^2 \) is a rational function in \( \delta \) and it has no zero at \( \delta = 0 \), and \( \langle g_m, g'_j \rangle \) is a rational function in \( \delta \) and without a pole at \( \delta = 0 \). The squared Euclidean distance of \( (x, y) \) to \( M(A_\delta) \) is thus equal to

\[
d_{M(A_\delta)}(x, y) = \sum_{j=1}^n \frac{|g'_j|^2}{|g_j|^2} = \sum_{j=1}^n \frac{\langle e_j - af_j, (x, y) \rangle^2}{|e_j - af_j - \delta v_j|^2} - \delta \sum_{j=1}^n \frac{\langle v_j, (x, y) \rangle^2 |2g'_j + \delta v_j, (x, y)|}{|g_j|^2} + \delta \sum_{j=1}^n \frac{\langle e_j - af_j, (x, y) \rangle^2 |2g'_j + \delta v_j, v_j|}{|e_j - af_j|^2 |g'_j|^2}. \quad (2.8)
\]
Observe that the last two terms are homogeneous polynomials of degree 2 in \(x, y\) and such that their coefficients are rational functions in \(\delta\) and neither of them has a pole at \(\delta = 0\). Further, for \(\delta = 0\) in (2.6) we get \(M(A_0)^\perp\) as a span of orthogonal vectors, and hence the first term in (2.8) is precisely

\[
d_{M(A_0)}(x, y) = \sum_{j=1}^{n} \frac{(x_j - ay_j)^2}{1 + a^2}.
\]

In a similar fashion we now deal with the case when \(A_\delta\) is of the form (2.2) and with \(n\) even. We have

\[
M(A_\delta) = \text{Span}\{ce_j + f_j + be_{j+1} + \delta e_{j-1}, -be_j + f_{j+1} + ce_{j+1} + \delta e_{j-1}\}_{j \in \{3, 5, \ldots, n-1\}} \\
\quad \quad \cup \{ce_1 + f_1 + be_2, -be_1 + f_2 + ce_2\},
\]

\[
M(A_\delta)^\perp = \text{Span}\{e_j - cf_j + bf_{j+1} - \delta f_{j+2}, e_{j+1} - bf_j - cf_{j+1} - \delta f_{j+3}\}_{j \in \{1, 3, \ldots, n-3\}} \\
\quad \quad \cup \{-cf_{n-1} + e_{n-1} + bf_n, -bf_{n-1} + e_n - cf_n\},
\]

where \(b, c \in \mathbb{R}\) and the vectors in the spans are again linearly independent. We set

\[
h_j = -cf_j + e_j + bf_{j+1} - \delta f_{j+2}, \quad j \in \{1, 3, \ldots, n - 3\},
\]

\[
h_j = -bf_{j-1} + e_j - cf_j - \delta f_{j+2}, \quad j \in \{2, 4, \ldots, n - 2\},
\]

\[
h_{n-1} = -cf_{n-1} + e_{n-1} + bf_n, \quad h_n = -bf_{n-1} + e_n - cf_n,
\]

and the Gram-Schmidt process yields the orthogonal basis \(\{h_1', \ldots, h_n'\}\) of \(M(A_\delta)^\perp\):

\[
\begin{align*}
\text{for } j &= 1, 3, \ldots, n - 1, \\
\quad h_j' &= -cf_j + e_j + bf_{j+1} - \delta w_j, \\
\quad h_j' &= -bf_{j-1} + e_j - cf_j - \delta w_j, \quad j \in \{2, 4, \ldots, n\},
\end{align*}
\]

where the components of \(w_j\) are rational functions in \(\delta\) having no pole at \(\delta = 0\). It follows that

\[
d_{M(A_\delta)}(x, y) = \sum_{j=1,3,\ldots,n-1} \frac{(-cf_j + e_j + bf_{j+1}, (x, y))^2}{|cf_j + e_j + bf_{j+1}|^2} + \sum_{j=2,4,\ldots,n} \frac{(-bf_{j-1} + e_j - cf_j, (x, y))^2}{|bf_{j-1} + e_j - cf_j|^2}
\]

\[
+ \delta \sum_{j=1,3,\ldots,n-1} \frac{(-cf_j + e_j + bf_{j+1}, (x, y))^2(2h_j' + \delta w_j, w_j)}{|cf_j + e_j + bf_{j+1}|^2|h_j'|^2}
\]

\[
+ \delta \sum_{j=2,4,\ldots,n} \frac{(-bf_{j-1} + e_j - cf_j, (x, y))^2(2h_j' + \delta w_j, w_j)}{|bf_{j-1} + e_j - cf_j|^2|h_j'|^2}
\]

\[- \delta \sum_{j=1}^{n} \frac{|w_j, (x, y)|^2 (2h_j' + \delta w_j, (x, y))}{|h_j'|^2}.
\]

Again, the sums in the last three terms are polynomials of degree 2 in \(x, y\) and such that their coefficients are rational functions in \(\delta\) and neither of them has a pole at \(\delta = 0\). Using (2.9) for \(\delta = 0\) we see that the first two terms in the above sum are precisely

\[
d_{M(A_0)}(x, y) = \sum_{j=1}^{n} \frac{(x_{2j-1} - cy_{2j-1} + by_{2j})^2}{1 + c^2 + b^2} + \frac{(x_{2j} - cy_{2j} - by_{2j})^2}{1 + c^2 + b^2}.
\]

This completes proof of the lemma. □
3 Local construction at the intersection

Given a $C^2$-function $f : \Omega \to \mathbb{R}$ on an open set $\Omega \subset \mathbb{C}^n$ we denote the holomorphic and antiholomorphic derivatives by $\frac{\partial f}{\partial z_j} = \frac{1}{2} \left( \frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right)$, $\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right)$, and the complex Hessian matrix is defined by

$$H^C(f)(z) = \left[ \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z) \right]_{j,k=1}^n, \quad z \in \Omega.$$ 

For $1 \leq r \leq n$ we also introduce the notation

$$\left( \frac{\partial f}{\partial z^r} \right)_r = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_r} \right), \quad \left( \frac{\partial f}{\partial \bar{z}^r} \right)_r = \left( \frac{\partial f}{\partial \bar{z}_1}, \ldots, \frac{\partial f}{\partial \bar{z}_r} \right), \quad H^C_r(f) = \left[ \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r.$$ 

A function $f$ is strictly plurisubharmonic if and only if $H^C(f)(z)$ is a positive definite Hermitian matrix at each point $z \in \Omega$, and this is the case precisely when all leading principal minors of $H^C(f)$ are positive on $\Omega$, i.e. $\det(H^C_r(f))$ is positive on $\Omega$ for all $r \in \{1, \ldots, n\}$. The notion of strictly plurisubharmonic functions can easily be extended to complex manifolds (see e.g. [7]).

To simplify the computations concerning the complex Hessian matrices or their determinants in the rest of this section, let us further introduce the notation

$$\xi^T \eta = \sum_{j=1}^r \xi_j \eta_j, \quad \xi^T \eta = \begin{bmatrix} \xi_j \eta_k \end{bmatrix}_{j,k=1}^r, \quad \xi = (\xi_1, \ldots, \xi_r), \quad \eta = (\eta_1, \ldots, \eta_r), \quad (3.1)$$

We state a simple fact about the determinant of a sum of rank-one matrices:

**Lemma 3.1** Let $u_1, \ldots, u_s, v_1, \ldots, v_s \in \mathbb{C}^2$, where $s \in \mathbb{N}$, and let $B \in \mathbb{C}^{2 \times 2}$. Then

$$\det \left( B + \sum_{l=1}^s u_l v_l^T \right) = \det(B) + \text{Tr}(B) \sum_{l=1}^s v_l^T u_l - \sum_{l=1}^s v_l^T B u_l + \det \left( \sum_{l=1}^s u_l v_l^T \right),$$

$$\det \left( \sum_{l=1}^s u_l v_l^T \right) = \sum_{1 \leq l < m \leq s} (u_l^T v_l u_m^T v_m - u_l^T v_m u_m^T v_l).$$

**Proof** We prove the lemma by induction on $s$. The case $s = 1$ is a simple identity

$$\det(B + u v^T) = \det(B) + \text{Tr}(B) u^T v - v^T B u, \quad B \in \mathbb{C}^{2 \times 2}, \quad u, v \in \mathbb{C}^2. \quad (3.2)$$

Next, assuming that the statement holds for some $s \in \mathbb{N}$ and applying (3.2), we get

$$\det \left( B + \sum_{l=1}^{s+1} u_l v_l^T \right) = \det \left( B + \sum_{l=1}^s u_l v_l^T \right) + \text{Tr} \left( B + \sum_{l=1}^s u_l v_l^T \right) u_{s+1}^T v_{s+1}$$

$$- v_{s+1}^T \left( B + \sum_{l=1}^s u_l v_l^T \right) u_{s+1} \quad = \quad \det(B) + \text{Tr}(B) \sum_{l=1}^s v_l^T u_l - \sum_{l=1}^s v_l^T B u_l$$

$$+ \sum_{1 \leq l < m \leq s} (u_l^T v_l u_m^T v_m - u_l^T v_m u_m^T v_l)$$

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Let $A$ be a real Jordan canonical form, and let $d_{M}, d_{N}$, respectively, be the distance functions with respect to two totally real subspaces intersecting transversally. Has the following properties:

\[
\frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}
\]

\[
\begin{align*}
\det (H^E_r (\rho)) &= \Delta^2 - \Gamma^2 + \frac{2b}{1+b^2+c^2} \frac{\partial}{\partial u} (d_{M}, d_{N}) d_{M} + \Delta \frac{\partial}{\partial v} (d_{M}, d_{N}) d_{N} \\
&\quad + 2 \Delta \Re \left( \sum_{j=1,2} \frac{\partial d_{M}}{\partial z_j} \frac{\partial d_{N}}{\partial z_j} \right) + 2 \Delta \Im \left( \sum_{j=1,2} \frac{\partial d_{M}}{\partial z_j} \frac{\partial d_{N}}{\partial \bar{z}_j} - \frac{\partial d_{N}}{\partial z_j} \frac{\partial d_{M}}{\partial \bar{z}_j} \right) \\
&\quad + \det (H^R_r (d_{M}, d_{N})) Z_{1,2}.
\end{align*}
\]

In addition, if $n = 2$, then $|\frac{\partial d_{M}}{\partial z}|^2 = d_{M}$, $|\frac{\partial d_{N}}{\partial z}|^2 = d_{N}$ and $0 \leq Z_{1,2} \leq d_{M} d_{N}$.

**Proof** By a straightforward computation we obtain

\[
\frac{\partial^2}{\partial z_j \partial \bar{z}_k} = \frac{\partial^2}{\partial u^2} (d_{M}, d_{N}) \frac{\partial d_{M}}{\partial z_j} \frac{\partial d_{M}}{\partial \bar{z}_k} + \frac{\partial^2}{\partial v^2} (d_{M}, d_{N}) \frac{\partial d_{N}}{\partial z_j} \frac{\partial d_{N}}{\partial \bar{z}_k} + \frac{\partial}{\partial u} (d_{M}, d_{N}) \left( \frac{\partial d_{M}}{\partial z_j} \frac{\partial d_{N}}{\partial \bar{z}_k} + \frac{\partial d_{N}}{\partial z_j} \frac{\partial d_{M}}{\partial \bar{z}_k} \right)
\]
\[
+ \frac{\partial P}{\partial u} (d_M, d_N) \frac{\partial d_M}{\partial z_j \partial \bar{z}_k} + \frac{\partial P}{\partial v} (d_M, d_N) \frac{\partial d_N}{\partial z_j \partial \bar{z}_k}.
\]

(3.3)

The leading principal submatrix of the complex Hessian matrix of \( \rho \), consisting of the first \( r \) rows and in the first \( r \) columns, \( r \in \{1, 2, \ldots, n\} \), can thus in matrix notation (see (3.1)) be written as

\[
H^C_r(\rho) = H^C_r(d_M) \frac{\partial P}{\partial u} (d_M, d_N) + H^C_r(d_N) \frac{\partial P}{\partial v} (d_M, d_N) + L_r,
\]

(3.4)

where

\[
L_r = \frac{\partial^2 P}{\partial u^2} (d_M, d_N) \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r^T + \frac{\partial^2 P}{\partial v^2} (d_M, d_N) \left( \frac{\partial d_N}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T + \frac{\partial^2 P}{\partial u \partial \bar{v}} (d_M, d_N) \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T + \frac{\partial^2 P}{\partial v \partial \bar{u}} (d_M, d_N) \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r \left( \frac{\partial d_N}{\partial z} \right)_r^T.
\]

(3.5)

Denoting by \( I_r \) the \( r \times r \) identity-matrix, we immediately get (\( N = \mathbb{R}^n \)):

\[
d_N(x, y) = \sum_{j=1}^n y_j^2, \quad \frac{\partial d_N}{\partial z_j}(x, y) = -iy_j, \quad j \in \{1, \ldots, n\},
\]

\[
H^C_r(d_N) = \left[ \frac{\partial^2 d_N}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r = \frac{1}{2} I_r, \quad r \in \{1, \ldots, n\}.
\]

(3.6)

To prove (2) we now assume that \( A = \text{diag}(a_1, \ldots, a_n) \in \mathbb{R}^{n \times n} \) is a diagonal matrix. A simple computation then yields (see also Lemma 2.1):

\[
d_M(x, y) = \sum_{j=1}^n \frac{(x_j - a_j y_j)^2}{1 + a_j}, \quad \frac{\partial d_M}{\partial z_j}(x, y) = \frac{(1 + ia_j)(x_j - a_j y_j)}{1 + a_j}, \quad j \in \{1, \ldots, n\},
\]

\[
H^C_r(d_M) = \left[ \frac{\partial^2 d_M}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r = \frac{1}{2} I_r, \quad r \in \{1, \ldots, n\}.
\]

(3.7)

Using (3.4), (3.6), (3.7) and since \( \Delta = \frac{1}{2} \left( \frac{\partial P}{\partial u} (d_M, d_N) + \frac{\partial P}{\partial v} (d_M, d_N) \right) \), we obtain

\[
H^C_r(\rho) = \Delta I_r + L_r.
\]

(3.8)

Observe that for \( 2 \leq r \leq n \) the matrix \( L_r \) is of rank two, and hence its possible minors of order greater than 2 vanish. Further, the following fact related to properties of the characteristic polynomial is well-known

\[
\det(\lambda I_r + B) = \lambda^r + c_{r-1} \lambda^{r-1} + \cdots + c_1 \lambda + c_0, \quad \lambda \in \mathbb{C}, \quad B \in \mathbb{C}^{r \times r},
\]

(3.9)

where the coefficient \( c_j \) for \( j \in \{1, \ldots, r-1\} \) is equal to the sum of all principal minors of \( B \) or order \( r - j \). Therefore, from (3.8) for \( \Delta = \Delta, \quad B = L_r \), we deduce

\[
\det \left( H^C_r(\rho) \right) = \Delta^r + \text{Tr}(L_r) \Delta^{r-1} + \Delta^{r-2} \sum_{1 \leq j < k \leq r} \text{Minor}_{j,k}(L_r), \quad 2 \leq r \leq n.
\]

(3.10)

Here \( \text{Minor}_{j,k}(L_r) \) is the principal minor of \( L_r \) with respect to the \( j \)-th, \( k \)-th rows and columns; this is the determinant of the principal submatrix of \( L_r \), formed by taking the elements in the \( j \)-th, \( k \)-th columns, and in the \( j \)-th, \( k \)-th rows. It follows immediately from (3.5) that

\[
\text{Tr}(L_r) = \frac{\partial^2 P}{\partial u^2} (d_M, d_N) \left| \left( \frac{\partial d_M}{\partial z} \right)_r \right|^2 + \frac{\partial^2 P}{\partial v^2} (d_M, d_N) \left| \left( \frac{\partial d_N}{\partial z} \right)_r \right|^2.
\]
Also, we apply Lemma 3.1 to the principal submatrix of \( L_r \) with respect to the \( j, k \)-th rows and columns. After regrouping the like terms, we get

\[
\text{Minor}_{j,k}(L_r) = \left( \frac{\partial^2 P}{\partial u \partial v}(d_M,d_N) \right) Z_{j,k},
\]

where

\[
Z_{jk} = \left( \frac{\partial d_M}{\partial z_j}, \frac{\partial d_M}{\partial z_k} \right)^T \left( \frac{\partial d_M}{\partial z_j}, \frac{\partial d_M}{\partial z_k} \right) - \left( \frac{\partial d_N}{\partial z_j}, \frac{\partial d_N}{\partial z_k} \right)^T \left( \frac{\partial d_N}{\partial z_j}, \frac{\partial d_N}{\partial z_k} \right).
\]

Finally, (3.9), (3.10), (3.11) imply the formula in (2).

Next, we prove (3); in this case we have (see Lemma 2.1):

\[
d_M(x_1, y_1, x_2, y_2) = \frac{(x_1 - cy_1 + by_2)^2}{1 + b^2 + c^2} + \frac{(x_2 - cy_2 - by_1)^2}{1 + b^2 + c^2}.
\]

A simple computation shows that

\[
\frac{\partial d_M}{\partial z_1}(x_1, y_1, x_2, y_2) = \frac{1+ic}{1+b^2+c^2}(x_1 - cy_1 + by_2) + \frac{ib}{1+b^2+c^2}(x_2 - cy_2 - by_1),
\]

\[
\frac{\partial d_M}{\partial z_2}(x_1, y_1, x_2, y_2) = \frac{1+ic}{1+b^2+c^2}(x_2 - cy_2 - by_1) - \frac{ib}{1+b^2+c^2}(x_1 - cy_1 + by_2),
\]

\[
\left[ \frac{\partial^2 d_M}{\partial z_j \partial z_k} \right]_{j,k=1} = \frac{1}{2} \left[ \begin{array}{cc} 1 & 2ib \\ -2ib & 1 + b^2 + c^2 \end{array} \right].
\]

We recall that \( \Delta = \frac{1}{2} \left( \frac{\partial P}{\partial u}(d_M,d_N) + \frac{\partial P}{\partial v}(d_M,d_N) \right) \) and set \( \Gamma = \frac{b}{1+b^2+c^2} \frac{\partial P}{\partial u}(d_M,d_N) \), \( K_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). From (3.6), (3.13) and (3.4) for \( r = 2 \) we thus obtain

\[
H^C_2(\rho) = \Delta I_2 + i \Gamma K_2 + L_2.
\]

Using (3.8) for \( r = 2 \), \( B = i \Gamma K_2 + L_2 \), \( \lambda = \Delta \), and since \( \text{Tr}(i \Gamma K_2 + L_2) = \text{Tr}(L_2) \), it further follows that

\[
\det(H^C_2(\rho)) = \Delta^2 + \Delta \text{Tr}(L_2) + \det(i \Gamma K_2 + L_2).
\]

To apply Lemma 3.1 for \( B = i \Gamma K_2 \), we first compute

\[
\left( \frac{\partial d_M}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_M}{\partial z} \right)_2 = - \frac{2ib}{1+b^2+c^2} d_M, \quad \left( \frac{\partial d_N}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_N}{\partial z} \right)_2 = 0,
\]

\[
\left( \frac{\partial d_N}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_M}{\partial z} \right)_2 + \left( \frac{\partial d_M}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_N}{\partial z} \right)_2 = 2i \text{Im} \left( \frac{\partial d_N}{\partial z_2}, \frac{\partial d_M}{\partial z_1}, -\frac{\partial d_N}{\partial z_1}, \frac{\partial d_M}{\partial z_2} \right),
\]

and it then yields

\[
\det(i \Gamma K_2 + L_2) = - \Gamma^2 - \frac{2b}{1+b^2+c^2} \Gamma \frac{\partial^2 P}{\partial u \partial v}(d_M,d_N) d_M + 2 \Gamma \text{Im} \left( \frac{\partial d_N}{\partial z_2}, \frac{\partial d_M}{\partial z_1}, -\frac{\partial d_N}{\partial z_1}, \frac{\partial d_M}{\partial z_2} \right) + \det(L_2).
\]

We now combine this with (3.10), (3.11) for \( r = 2 \), \( j = 1, k = 2 \) and (3.14). To conclude the proof of (3) (respectively (2)), we use (3.6), (3.7) for \( n = r = 2 \) (respectively (3.13)), to see that

\[
\left( \frac{\partial d_M}{\partial z} \right)_2^2 = d_M, \quad \left( \frac{\partial^2 d_N}{\partial z^2} \right)_2^2 = d_N.
\]

Since \( \frac{\partial^2 d_M}{\partial z_1 \partial z_2} = \frac{1}{2} \) by (3.7), (3.13), the property (1) follows from (3.4) (\( r = 1 \)).
The following lemma is essential for the construction of regular Stein neighborhoods in the proof of Theorem 4.2. It provides a suitable plurisubharmonic function near the intersection of the union of two totally real subspaces. It generalizes [16, Lemma 3.3] to the higher dimensional case, and slightly extends it in the two-dimensional case.

**Lemma 3.3** Let $A$ be a real diagonalizable $n \times n$ matrix, and let $d_M$ and $d_N$, respectively, be the squared Euclidean distance functions to $M = (A + iI)\mathbb{R}^n$ and $N = \mathbb{R}^n$ in $\mathbb{C}^n$. If $A$ satisfies one of the conditions

1. $A = \text{diag}(a, \ldots, a)$, $|a| \leq \frac{1}{15}$, $n > 2$,
2. $A = \text{diag}(a_1, a_2)$, $|a_1|, |a_2| \leq \frac{1}{15}$,
3. $A = \begin{bmatrix} c & -b \\ b & c \end{bmatrix}$, $|c|, |b| \leq \frac{1}{16}$,

then there exists a homogeneous polynomial $P \in \mathbb{R}[u, v]$ such that the function

$$\rho = P(d_M, d_N)$$

is strictly plurisubharmonic precisely on $\mathbb{C}^n \setminus \{0\}$, and such that

$$\{ P = 0 \} \cap (\mathbb{R}_+)^2 = (\mathbb{R}_+ \times \{0\}) \cup \{(0) \times \mathbb{R}_+\},$$

$$\nabla P(u, v) \neq 0, \ u, v \in \mathbb{R}^+, \ \frac{\partial P}{\partial u}(u, 0) = 0 = \frac{\partial P}{\partial v}(0, v), \ u, v \in \mathbb{R}_+. \quad (3.15)$$

Moreover, if $A$ satisfies (1) for $n > 2$ and $P(u, v) = u^2v + uv^2$, then it follows

$$H^C_r(\rho)(x, y) \geq \frac{1}{5} \sum_{j=1}^n \left( \frac{(x_j - ay_j)^2}{1 + a_j^2} \right)^r + y_j^r, \quad r \in [1, \ldots, n]. \quad (3.16)$$

**Proof** Recall first that $d_N(x, y) = \sum_{j=1}^n y_j^2$ and $\frac{\partial d_N}{\partial x_j}(x, y) = -iy_j, \ j \in \{1, \ldots, n\}$.

When $A = \text{diag}(a_1, \ldots, a_n)$ is a diagonal matrix, then $d_M(x, y) = \sum_{j=1}^n \frac{(x_j - ay_j)^2}{1 + a_j^2}$ (see Lemma 2.1) with $\frac{\partial d_M}{\partial x_j}(x, y) = \frac{(1 + ia_j)(x_j - ay_j)}{1 + a_j^2}$ and

$$\text{Re}\left( \frac{\partial d_M}{\partial x_j}(x, y) \right)_r, \text{Re}\left( \frac{\partial d_N}{\partial x_j}(x, y) \right)_r = \sum_{j=1}^n \frac{a_j(x_j - ay_j)y_j}{1 + a_j^2}, \quad r \in [1, \ldots, n]. \quad (3.17)$$

Next, for $r \in [1, \ldots, n]$ we introduce the following useful notation

$$d_{M,r}(x, y) = \left| \frac{\partial d_M}{\partial x_j}(x, y) \right|_r^2 = \sum_{j=1}^n \frac{(x_j - ay_j)^2}{1 + a_j^2},$$

$$d_{N,r}(x, y) = \left| \frac{\partial d_N}{\partial x_j}(x, y) \right|_r^2 = \sum_{j=1}^n y_j^2,$$

$$\vartheta_r = \max_{1 \leq j \leq r} \frac{|a_j|}{\sqrt{1 + a_j^2}}. \quad (3.18)$$

Using the triangle and the Cauchy–Schwarz inequality, respectively, we then get

$$\left| \text{Re}\left( \frac{\partial d_M}{\partial x_j} \right)_r, \text{Re}\left( \frac{\partial d_N}{\partial x_j} \right)_r \right| \leq \vartheta_r (d_{M,r}d_{N,r})^{\frac{1}{2}}.$$
We also compute the expression $Z_{jk}$ in Lemma 3.2 (2):

$$Z_{jk}(x, y) = \sum_{m=j,k} \left| \frac{\partial M}{\partial z_m}(x, y) \right|^2 \sum_{m=j,k} \left| \frac{\partial N}{\partial z_m}(x, y) \right|^2 - \left| \sum_{m=j,k} \frac{\partial M}{\partial z_m}(x, y) \frac{\partial N}{\partial z_m}(x, y) \right|^2$$

$$= \frac{(x_j - a_j y_j)^2 y_j^2}{1 + a_j^2} - 2 \frac{(x_j - a_j y_j)(x_k - a_k y_k) y_j y_k}{(1 + a_j^2)(1 + a_k^2)} + \frac{(x_k - a_k y_k)^2 y_k^2}{1 + a_k^2}.$$

Let us now consider the case $A = \text{diag}(a, \ldots, a) \in \mathbb{R}^{n \times n}$. We see that

$$\sum_{1 \leq j < k \leq r} Z_{jk}(x, y) = \sum_{1 \leq j < k \leq r} \frac{1}{1 + a^2} \left( (x_j - a y_j) y_j - (x_k - a y_k) y_k \right)^2$$

$$= \frac{r}{1 + a^2} \sum_{j=1}^r (x_j - a y_j)^2 y_j - \frac{1}{1 + a^2} \left( \sum_{j=1}^r (x_j - a y_j) y_j \right)^2,$$

where the last equality is obtained by Lagrange's identity (see [14, p. 38–39]). Thus

$$0 \leq \sum_{1 \leq j < k \leq r} Z_{jk} \leq d_{M,r} d_{N,r}, \quad 2 \leq r \leq n. \quad (3.20)$$

Next, we apply Lemma 3.2 (2) to $P(u, v) = u^2 v + v^2 u$ with $\det(H_{\mathbb{R}}(P)(u,v)) = -4(a^2 + uv + u^2 v)$. Together with (3.18), (3.19), (3.20), we get the lower bound for the determinant of the Hessian matrix of $\rho = P(d_M, d_N)$ on $\mathbb{C}^n \setminus \{0\}$:

$$\Delta^{2-r} \det(H_{\mathbb{C}}(\rho)) \geq \Delta^2 + \Delta \left( 2 d_N d_{M,r} + 2 d_M d_{N,r} - 4 \partial_r (d_M + d_N)(d_{M,r}, d_{N,r}) \frac{1}{2} \right)$$

$$- 4 \left( d_M^2 + d_M d_N + d_N^2 \right) d_{M,r} d_{N,r}, \quad (3.21)$$

where $\Delta = \frac{1}{2} \left( \frac{\partial P}{\partial (d_M, d_N)} + \frac{\partial P}{\partial (d_M, d_N)} \right) = \frac{1}{2} (d_M^2 + 4 d_M d_N + d_N^2)$ and $2 \leq r \leq n$. For every $r \in \{1, \ldots, n\}$ let the expression on the right-hand side of the inequality (3.21) be given as $\Psi_{r, \frac{1}{2}}$. Taking $\partial_r = \frac{1}{4}$ and regrouping the terms of $\Psi_{r, \frac{1}{4}}$ yields

$$\Psi_{r, \frac{1}{4}} = d_M d_N \left( 2 d_{M,r} d_N + 2 d_{N,r} d_M - 4 d_{M,r} d_{N,r} \right)$$

$$+ \left( d_M^2 + d_N^2 \right) \left( 2 d_M d_N + d_M d_{N,r} + d_{M,r} d_N - 4 d_{M,r} d_{N,r} \right)$$

$$+ d_M^2 d_N \left( 2 d_{N,r} - \frac{5}{2} d_{N,r} d_{M,r} \right)^{\frac{1}{2}} + \frac{25}{32} d_M$$

$$+ d_N^3 d_M \left( 2 d_{M,r} - \frac{5}{2} d_{M,r} d_{N,r} \right)^{\frac{1}{2}} + \frac{25}{32} d_N$$

$$+ \frac{1}{8} d_M^3 \left( d_M + 4 d_{M,r} d_{N,r} \right)^{\frac{1}{2}} + 4 d_N$$

$$+ \frac{1}{8} d_N^3 \left( d_N + 4 d_{M,r} d_{N,r} \right)^{\frac{1}{2}} + 4 d_M$$

$$+ \left( \frac{1}{10} d_M^4 - \frac{41}{32} d_M^2 d_N + \frac{9}{2} d_M^2 d_N^2 - \frac{41}{32} d_M d_N^3 + \frac{1}{16} d_N^4 \right) + \frac{1}{40} \left( d_M^4 + d_N^4 \right). \quad (3.22)$$
Since $d_M \geq d_{M,r}$ and $d_M \geq d_{M,r}$ for all $r$, the first six terms in the above sum are non-negative. The eighth term is a symmetric homogeneous polynomial in $d_M$ and $d_N$. It is positive on $(M \cup N) \setminus \{0\}$, while on $\mathbb{C}^n \setminus (M \cup N)$ we can respectively factor out $d_M^2 d_N^2$, setting $W = \frac{d_M}{d_N} + \frac{d_N}{d_M}$, and after regrouping the terms, we obtain $d_M^2 d_N^2 \left( \frac{1}{10} W^2 - \frac{41}{32} W + \frac{43}{16} \right)$, which is positive there. The last term is clearly positive everywhere except at the origin, thus it follows that $\Psi_{r, \vartheta}$ is positive on $\mathbb{C}^n \setminus \{0\}$. As $\Psi_{r, \vartheta}$, is a decreasing function with respect to $\vartheta_r$ and $\vartheta_r = \frac{|a|}{\sqrt{1+a^2}}$ (see (3.18) and remember that $A = \text{diag}(a, \ldots, a)$) increases with respect to $|a|$, we thus have $\Psi_{r, \vartheta} > 0$ on $\mathbb{C}^n \setminus \{0\}$ for $\vartheta_r \leq \frac{1}{4}$ (hence $|a| \leq \frac{1}{\sqrt{15}}$), $r \in \{1, \ldots, n\}$.

From (3.21), (3.22), it immediately follows that $\det(H_{r}^C(\rho)) \geq 0$, with equality precisely at the origin, provided that $2 \leq r \leq n$, $|a| \leq \frac{1}{\sqrt{15}}$, and in addition we have

$$\det(H_r^C(\rho))(x, y) \geq \frac{1}{40} \left( d_M^2 + d_N^4 \right)^2 \left( \frac{1}{2} (d_M^2 + d_M d_N + d_N^2) \right)^{r-2} (x, y)$$

$$\geq \frac{1}{5.2^r} \left( d_M^2 + d_N^2 \right)^2 (x, y) \geq \frac{1}{5.2^r} \sum_{j=1}^n \left( \frac{(r-ay_j)^2}{1+a^2} \right)^r + y_{2r}^r$$

Further, by combining Lemma 3.2 (1) and (3.18), (3.19), we obtain for $\vartheta_1 \leq \frac{1}{4}$ that

$$H_1^C(\rho) \geq \Delta + \left( 2d_M d_{M,1} + 2d_M d_{M,1} - 4 \vartheta_1 (d_M + d_N) (d_{M,1} d_{N,1})^\frac{1}{2} \right)$$

$$\geq d_M \left( \frac{1}{8} d_M - (d_{M,1} d_{N,1})^\frac{1}{2} + d_{N,1} \right) + d_N \left( \frac{1}{8} d_N - (d_{M,1} d_{N,1})^\frac{1}{2} + d_{M,1} \right)$$

$$\geq \frac{3}{8} (d_M^2 + d_N^2) + 2d_M d_N.$$  \hspace{1cm} (3.23)

and we deduce that $H_1^C(\rho)$ vanishes at the origin and is positive elsewhere for $|a| \leq \frac{1}{\sqrt{15}}$ (hence $\vartheta_r \leq \frac{1}{4}$). This concludes the proof that $\rho = d_M^2 d_N + d_M d_N^2$ is strictly plurisubharmonic precisely on $\mathbb{C}^n \setminus \{0\}$, if $M = M(\text{diag}(a, \ldots, a))$, $|a| \leq \frac{1}{\sqrt{15}}$.

We proceed with the case $A = \text{diag}(a_1, a_2)$. Using Lemma 3.2 (1),(2), again applied to $P(u, v) = u^2 v + v^2 u$ and $\rho = P(d_M, d_N)$, together with the estimates (3.18), (3.19), $0 \leq Z_{12} \leq d_M d_N$, we get exactly (3.21) for $r = 2$ and (3.23). Much as in the case $A = \text{diag}(a, \ldots, a)$, the condition $|a_1|, |a_2| \leq \frac{1}{\sqrt{15}}$ (and hence $\vartheta_1, \vartheta_2 \leq \frac{1}{4}$) now implies that $H_2^C(\rho)$ is positive definite everywhere, except at the origin.

In particular, if $A = \text{diag}(a, a)$ we obtain even better upper bound for $|a|$. Lemma 3.2 (2) applied to any polynomial $P$ (and $\rho = P(d_M, d_N)$), together with (3.17), (3.18), (3.19) for $r = 2$, and setting $Y = \sum_{j=1}^2 \frac{(r-ay_j)}{\sqrt{1+a^2}} \sqrt{1+a^2}$, then yields

$$\det(H_2^C(\rho)) = \Delta \left( \frac{\partial^2 P}{\partial u^2} (d_M, d_N) d_M + \frac{\partial^2 P}{\partial v^2} (d_M, d_N) d_N - 2 \frac{\partial^2 P}{\partial u \partial v} (d_M, d_N) \vartheta Y \right)$$

$$+ \Delta^2 + \det(H^R(\rho)(d_M, d_N)) (d_M d_N - Y^2).$$  \hspace{1cm} (3.24)

For $P(u, v) = u^3 v + 5u^2 v^2 + uv^3$, we get $\Delta = \frac{1}{2} (d_M^3 + 13d_M^2 d_N + 13d_M d_N^2 + d_N^3)$,

$$\det(H^R(\rho)(d_M, d_N)) = -3 (3d_M^4 + 20d_M^3 d_N + 94d_M^2 d_N^2 + 20d_M d_N^3 + 3d_N^4),$$

and we further have

$$\det(H_2^C(\rho)) = R(d_M, d_N) Y^2 + \vartheta S(d_M, d_N) Y + T(d_M, d_N).$$  \hspace{1cm} (3.25)
where
\[
R(u, v) = 3(3u^4 + 20u^3v + 94u^2v^2 + 20uv^3 + 3v^4),
\]
\[
S(u, v) = -(3u^5 + 59u^4v + 302u^3v^2 + 302u^2v^3 + 59uv^4 + 3v^5),
\]
\[
T(u, v) = \frac{1}{4}u^6 + 22u^5v + 403u^4v^2 + 44u^3v^3 + 403u^2v^4 + 22uv^5 + v^6).
\]

The discriminant of the expression (3.25) with respect to \(Y\) is for \(\theta = \frac{2}{3}\) equal to
\[
-\frac{1}{5}(45u^{10} + 906u^9v + 25889u^8v^2 + 127768u^7v^3 + 582402u^6v^4 - 207076u^5v^5
+ 582402u^4v^6 + 127768u^3v^7 + 25889u^2v^8 + 906uv^9 + 45v^{10}),
\]
which is negative everywhere, except for \(u = v = 0\). If \(|a| \leq \frac{2}{\sqrt{3}}\) (and hence \(\theta \leq \frac{2}{3}\)), we have \(\det(H_C^2(\rho)) \geq 0\) with equality precisely at the origin. Next, by combining Lemma 3.2 (1) and (3.19) for \(r = 1\), we deduce that
\[
H_C^1(\rho) \geq \frac{1}{2}(d_M^3 + 13d_M^2d_N + 13d_Md_N^2 + d_N^3) + 2d_N(3d_M + 5d_N)d_M,d_N,
+ 2d_M(3d_N + 5d_M)d_N, - \theta(6d_M^2 + 40dMd_N + 6d_N^2)(d_M,d_N,1)^\frac{1}{2}.
\]

By regrouping the terms for \(\theta \leq \frac{2}{3}\) \(|a| \leq \frac{2}{\sqrt{3}}\), we further get
\[
H_C^2(\rho) \geq \frac{2}{3}(d_Md_N,d_N - d_Nd_M)^2 + d_M^2(\frac{1}{2}d_M - 4(d_M,d_N,1)^{1/2} + 8d_N,1)
+ d_N^2(\frac{1}{2}d_N - 4(d_M,d_N,1)^{1/2} + 8d_M,1) + 6d_Nd_M(d_M - 2(d_M,d_N,1)^{1/2} + d_N)
+ 6d_Nd_M(d_M - 2(d_M,d_N,1)^{1/2} + d_N,1)^{1/2} + d_Md_N(d_M + d_N).
\]

Since \(d_M \geq d_M,1\) and \(d_N \geq d_N,1\), all the terms on the right hand-side of the equality are non-negative, and in addition the last term vanishes precisely at the origin. This proves that \(\rho = d_M^3d_N + 5d_M^2d_N^2 + d_Md_N^3\) for \(M = M(\text{diag}(a,a))\) with \(|a| \leq \frac{2}{\sqrt{3}}\) is strictly plurisubharmonic precisely on \(\mathbb{C}^n \setminus \{0\}\).

Finally, let \(n = 2\) and assume that \(A\) has complex eigenvalues. By (2.4) we have \(d_M(x_1, y_1, x_2, y_2) = \frac{(x_1-cy_1+by_2)^2}{1+b^2+c^2} + \frac{(x_2-cy_2-by_1)^2}{1+b^2+c^2}\) and the holomorphic derivatives of \(d_M\) are of the form (3.13). By regrouping the terms we easily see that
\[
\text{Re}\left(\frac{\partial d_M}{\partial z_1} \frac{\partial d_N}{\partial \overline{z}_1}\right) = \frac{1}{1+b^2+c^2} y_1(c(x_1 - cy_1 + by_2) + b(x_2 - cy_2 - by_1)),
\]
\[
\text{Re}\left(\sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \overline{z}_j}\right) = \frac{b}{1+b^2+c^2} y_2(x_1 - cy_1 + by_2) - y_1(x_2 - cy_2 - by_1)
+ \frac{c}{1+b^2+c^2} (y_1(x_1 - cy_1 + by_2) + y_2(x_2 - cy_2 - by_1)),
\]
\[
\text{Im}\left(\frac{\partial d_N}{\partial \overline{z}_2} \frac{\partial d_M}{\partial z_2} - \frac{\partial d_N}{\partial z_1} \frac{\partial d_M}{\partial \overline{z}_1}\right) = \frac{1}{1+b^2+c^2} (y_1(x_2 - cy_2 - by_1) - y_2(x_1 - cy_1 + by_2)).
\]

Applying the Cauchy–Schwarz inequality respectively to the first or third expression and to each term of the sum in the second expression, we get
\[
\left|\text{Re}\left(\frac{\partial d_M}{\partial z_1} \frac{\partial d_N}{\partial \overline{z}_1}\right)\right| \leq \frac{\sqrt{b^2+c^2}}{\sqrt{1+b^2+c^2}} (d_Md_N)^{1/2},
\]
and

\[
\begin{align*}
\text{Re}\left( \sum_{j=1,2} \frac{\partial^2 M}{\partial z_j^2} \frac{\partial^2 N}{\partial z_j^2} \right) & \leq \frac{|b|+|c|}{\sqrt{1+b^2+c^2}} (d_M d_N)^{\frac{1}{2}}, \\
\text{Im}\left( \frac{\partial^2 N}{\partial z_1^2} \frac{\partial^2 M}{\partial z_1^2} - \frac{\partial^2 M}{\partial z_1^2} \frac{\partial^2 N}{\partial z_1^2} \right) & \leq \frac{1}{\sqrt{1+b^2+c^2}} (d_M d_N)^{\frac{1}{2}}. \quad (3.27)
\end{align*}
\]

We now apply Lemma 3.2 (1), once more to \(\rho = d_M^2 d_N + d_N^2 d_M\). Using respectively (3.26) and the rough estimates \(|\frac{\partial^2 M}{\partial z_1^2}|^2, |\frac{\partial^2 N}{\partial z_1^2}|^2 \geq 0, \frac{\sqrt{b^2+c^2}}{\sqrt{1+b^2+c^2}} \leq \frac{1}{4}\) for \(|b|, |c| \leq \frac{1}{16}\), then after regrouping the terms we obtain (\(\Delta = \frac{1}{2}(d_M^2 + 4d_M d_N + d_N^2)\)):

\[
H_1^C(\rho) \geq \Delta + \left(2d_N |\frac{\partial^2 M}{\partial z_1^2}|^2 + 2d_M |\frac{\partial^2 N}{\partial z_1^2}|^2 - 4 \frac{\sqrt{b^2+c^2}}{\sqrt{1+b^2+c^2}} (d_M + d_N)(d_M d_N)^{\frac{1}{2}} \right)
\]

\[
\geq \frac{1}{2} \left( d_M^2 + 4d_M d_N + d_N^2 \right) - (d_M + d_N)(d_M d_N)^{\frac{1}{2}}
\]

\[
\geq \frac{1}{2} (d_M + d_N)(\sqrt{d_M} - \sqrt{d_N})^2 + d_M d_N.
\]

It is immediate that \(H_1^C(\rho)\) in non-negative and vanishes precisely at the origin. Furthermore, Lemma 3.2 (3) (with \(0 \leq Z_{12} \leq d_M d_N\)) and (3.27) yield

\[
\text{det}(H_2^C(\rho)) \geq \Delta^2 - \Gamma^2 + \left(\Delta - \frac{2b}{1+b^2+c^2}\right)(2d_N) d_M + 2\Delta d_M d_N
\]

\[
- 2\left(\Delta + \frac{|b|+|c|}{\sqrt{1+b^2+c^2}} (d_M d_N)^{\frac{1}{2}} + |\Gamma| \frac{1}{\sqrt{1+b^2+c^2}} (d_M d_N)^{\frac{1}{2}} \right)(2d_M + 2d_N)
\]

\[
- 4 \left(d_M^2 + d_M d_N + d_N^2\right)d_M d_N.
\]

where \(\Gamma = \frac{b}{1+b^2+c^2}(2d_M d_N + d_N^2)\). Using the estimates \(1 + b^2 + c^2 \geq 1\) and \(|c|, |b| \leq \frac{1}{16}\) respectively, and each time regrouping the like terms, we further get

\[
\text{det}(H_2^C(\rho)) \geq \frac{1}{4} d_M^4 + \frac{17}{2} d_M^2 d_N^2 + \frac{1}{4} d_M^4 - b^2 d_N^2 \left(12d_M^2 + 8d_M d_N + d_N^2\right)
\]

\[
- 2|b| \left(d_M^2 + 8d_M d_N + 3d_N^2\right)(d_M + d_N)(d_M d_N)^{\frac{1}{2}}
\]

\[
- 2|c| \left(d_M^2 + 4d_M d_N + d_N^2\right)(d_M + d_N)(d_M d_N)^{\frac{1}{2}}
\]

\[
\geq \frac{1}{2} d_M d_N^2 \left(2\sqrt{d_M} - \sqrt{d_N}\right)^2 + \frac{1}{128} d_M^2 d_N \left(7\sqrt{d_M} - 16\sqrt{d_N}\right)^2
\]

\[
+ \frac{1}{8} d_M^3 \left(\sqrt{d_M} - \sqrt{d_N}\right)^2 + \frac{1}{8} d_N^3 \left(2\sqrt{d_M} - \sqrt{d_N}\right)^2 + \frac{67}{128} d_M^3 d_N
\]

\[
+ \frac{1}{256} d_M^4 + \left(\frac{31}{256} d_M^4 - \frac{33}{32} d_M d_N^2 + \frac{285}{64} d_M^2 d_N^2 - \frac{33}{32} d_N^3 d_M + \frac{31}{256} d_N^4\right).
\]

The first six terms are non-negative, while the last one is positive everywhere, except at the origin. Indeed, on \(\mathbb{C}^2 \setminus (M \cup N)\) it can be seen as \(d_M^2 d_N^2 \left(\frac{31}{256} W^2 - \frac{33}{32} W + \frac{539}{128}\right)\), \(W = \frac{d_M}{d_N} + \frac{d_N}{d_M}\). Hence \(\rho\) is strictly plurisubharmonic on \(\mathbb{C}^2 \setminus \{0\}\).

To finish the proof of the lemma we observe that, if \(P\) is either \(P(u, v) = u^2 v + uv^2\) or \(P(u, v) = u^3 v + 5u^2 v^2 + uv^3\), the property (3.15) is clearly satisfied. \(\square\)
4 Regular Stein neighborhoods

A system of open Stein neighborhoods \( \{ \Omega_\epsilon \}_{\epsilon \in (0,1)} \) of a set \( S \) in a complex manifold \( X \) is called a regular, if for every \( \epsilon \in (0,1) \) we have

1. \( \Omega_\epsilon = \bigcup_{\epsilon \leq i < \epsilon} \Omega_i, \quad \overline{\Omega_\epsilon} = \bigcap_{\epsilon > i} \Omega_i, \)
2. \( S = \bigcap_{\epsilon \in (0,1)} \Omega_\epsilon \) is a strong deformation retract of every \( \Omega_\epsilon \) with \( \epsilon \in (0,1) \).

One way to construct such a system of neighborhoods is to find a non-negative function \( \rho \), which is strictly plurisubharmonic in a neighborhood of \( S \), and such that \( S = \{ \rho = 0 \} = \{ \nabla \rho = 0 \} \). Observe that in this case the sublevel sets \( \Omega_\epsilon = \{ \rho < \epsilon \} \) for \( \epsilon \) small enough are Stein, and the flow of the negative gradient vector field \( -\nabla \rho \) gives the strong deformation retraction of \( \Omega_\epsilon \) to \( S \). Note that the assumptions on \( \rho \) can be slightly weakened (see e.g. Theorem 4.2 or [16, Theorem 4.1]).

For the sake of completeness we also recall the following fact about homogeneous polynomials [16, Lemma 3.2], which will be used later on.

**Lemma 4.1** Let \( Q, R \) be real homogeneous polynomials of the same even degree. Assume further that \( Q \) is vanishing at the origin and is positive elsewhere. Then for any sufficiently small constant \( \epsilon_0 > 0 \), it follows that \( Q \geq \epsilon_0 \cdot |R| \), with equality precisely at the origin.

We are now ready to prove the main result.

**Theorem 4.2** Let \( A \) be a real \( n \times n \) matrix such that \( A - i I \) is invertible and let \( M(A) = (A + i I)\mathbb{R}^n \). Then the union \( M(A) \cup \mathbb{R}^n \) has a regular system of strongly pseudoconvex Stein neighborhoods if any of the following conditions is fulfilled:

1. The real part of every eigenvalue of \( A \) is equal to a constant \( a \) such that \( |a| \leq \frac{1}{\sqrt{n}} \), and the absolute value of the imaginary part of any eigenvalue is
   \[
   \leq \left( 10e(360n^n(n!)) \left( e^{1+\frac{23}{90}} - \frac{1}{3} \left( \frac{1}{13} \right)^n \right) \right)^{-1}.
   \]
2. The absolute values of the real and the imaginary parts of the eigenvalues of \( A \) are
   \[
   \leq \left( 10e \cdot (1080n^n(n!)) e^\frac{59}{270} (e - \frac{1}{3} (\frac{1}{13}))^n \right)^{-\frac{1}{2}}.
   \]
3. \( n = 2 \) and all eigenvalues of \( A \) are real, and such that either their absolute values are \( \leq \frac{1}{\sqrt{15}} \) or the eigenvalues are equal and their absolute values are \( \leq \frac{2}{3} \).
4. \( n = 2 \) and the absolute values of the real and the imaginary parts of the eigenvalues of \( A \) are \( \leq \frac{1}{\sqrt{16}} \).

Moreover, away from the origin the neighborhoods coincide with sublevel sets of the squared Euclidean distance functions to \( M(A) \) and \( N = \mathbb{R}^n \) respectively.

It is clear that non-singular linear transformations preserve regularity and Steinness of the system of neighborhoods. According to the notes in Sec. 2, the general case of the union of two totally real subspaces \( M, N \) of maximal dimension, intersecting at the origin, thus reduces to the situation described in Theorem 4.2, i.e. \( N = \mathbb{R}^n \) and \( M = (A + i I)\mathbb{R}^n \), where \( i \) is not the eigenvalue of \( A \in \mathbb{R}^{n \times n} \).

**Proof of Theorem 4.2** Our goal is to construct a function \( \rho \), which is strictly plurisubharmonic on \( \mathbb{C}^n \setminus \{0\} \), and such that \( M(A) \cup \mathbb{R}^n = \{ \rho = 0 \} = \{ \nabla \rho = 0 \} \). Since any real non-singular matrix \( V \) maps \( M(A) = (A + i I)\mathbb{R}^n \) onto \( M(VAV^{-1}) = (VAV^{-1} + i I)\mathbb{R}^n \), we see that \( \tilde{\rho} = \rho \circ V^{-1} \) with respect to \( M(VAV^{-1}) \) inherits all the above properties of \( \rho \). It is thus
sufficient to consider the case when \( A = A_\delta \) is in the Jordan canonical form (2.3), where the parameter \( \delta \) can be chosen arbitrarily.

If either of the conditions (3) or (4) of the theorem is satisfied, then Lemma 3.3 (2),(3) immediately implies the existence of the function \( \rho \) with the listed properties.

To find such \( \rho \) when any of the other two conditions is fulfilled, some additional analysis needs to be done. We first set the notation. For any pair of matrices \( B', B \in \mathbb{R}^{n \times n} \) and a homogeneous polynomial \( P \) of degree \( d \geq 2 \) in two variables (to be chosen later on), we define

\[
\rho_B = P(d_{M(B)}, d_N), \quad \rho_{B'} = P(d_{M(B')}, d_N),
\]

\[
\Phi_r(B', B) = \det \left( H_r^C(\rho_{B'}) \right) - \det \left( H_r^C(\rho_B) \right), \quad r \in \{1, \ldots, n\}.
\]

(4.1)

Observe that \( d_{M(B)}, \rho_B \) and \( \det(H_r^C(\rho_B)) \) with \( r \in \{1, \ldots, n\} \), respectively, are homogeneous polynomials of degree 2 in variables \( x, y \) and the coefficients of \( q_\delta \) are rational functions in \( \delta \) that vanish for \( \delta = 0 \). It follows that

\[
\Phi_r(A_\delta, A_0) = \det \left( H_r^C(\rho_{A_\delta}) \right) - \det \left( H_r^C(\rho_{A_0}) \right), \quad \text{for } r \in \{1, \ldots, n\},
\]

(4.2)

is a homogeneous polynomial of degree \((2d - 2)r\) in \( x, y \), and the coefficients of \( \Phi_r(A_\delta, A_0) \) tend to 0 as \( \delta \) approaches 0.

By Lemma 4.1 it suffices to prove that \( \rho_{A_0} \) is strictly plurisubharmonic everywhere except maybe at the origin, provided that the constant \( \delta \) is small enough. Since \( \rho_{A_0} = P(d_{M(A_0)}, d_N) \) for some homogeneous polynomial \( P \) we then have

\[
\nabla \rho_{A_\delta} = \frac{\partial P}{\partial x}(d_{M(A_\delta)}, d_N)\nabla d_{M(A_\delta)} + \frac{\partial P}{\partial y}(d_{M(A_\delta)}, d_N)\nabla d_{M(A_\delta)}. \quad \text{Hence, if we choose} \quad P \quad \text{so that it satisfies (3.15), it will immediately imply}
\]

\( M(A_\delta) \cup N = \{\rho_{A_\delta} = 0\} = \{\nabla \rho_{A_\delta} = 0\} \).

Let us now consider the case (1) of the theorem. Therefore, suppose that the real parts of all the eigenvalues of \( A_0 \) are equal to some constant \( a \) with \( |a| \leq \frac{1}{\sqrt{15}} \), i.e. \( A_0 \) is of the form as in Lemma 2.1 with \( a_j = c_k = a \) for \( j \in \{1, \ldots, n\}, k \in \{1, \ldots, \beta\} \). We set \( D = \text{diag}(a, \ldots, a) \) and compute

\[
d_{M(A_0)}(x, y) = \frac{1}{1+a^2} \sum_{j=1}^n (x_j - ay_j)^2 + R_D(x, y),
\]

(4.3)

where

\[
R_D(x, y) = \sum_{j=1}^{\beta} b_j \left( y_{j-1}^2 + y_{j-1}^2 \right) + 2y_{j-1}(x_{j-1} - ay_{j-1}) - 2y_{j-2}(x_{j-2} - ay_{j-2}) \left( 1 + a^2 + b_j \right) ^2
\]

\[
- \sum_{j=1}^{\beta} b_j^2 \left( x_{j-1} - ay_{j-1} \right)^2 + (x_{j-2} - ay_{j-2})^2 \left( 1 + a^2 + b_j \right) (1 + a^2) \right).
\]

(4.4)

Using the notation (4.1) we further have

\[
\det(H_r^C(\rho_{A_0})) = \det(H_r^C(\rho_D)) + \Phi_r(A_0, D), \quad r \in \{1, \ldots, n\},
\]

(4.5)

\( \Phi_r(A_0, D) \) is a homogeneous polynomial of degree \( 4r \) in \( \frac{x-ay}{\sqrt{1+a^2}} \), \( y \), and its coefficients are rational functions in \( b \), and such that they vanish as \( b \) approaches 0. By choosing \( P(u, v) = u^2v + v^2u \) in (4.1), we make the function \( \rho_D \) strictly plurisubharmonic, and we get the lower bound for \( \det(H_r^C(\rho_D)) \) (see Lemma 3.3 (1) and (3.16)). To estimate \( \Phi_r(A_0, D) \) we shall
write it as a sum of monomials in \( x - ay \), \( y, b \), then examine their coefficients and count the number of terms. Since this number is large and the terms are difficult to control, some less precise estimates will be done.

Taking \( P(u, v) = u^2 v + v^2 u \) in (3.3) and regrouping the last two terms yields

\[
(H^C(\rho_{A_0}))_{j,k} = 2\left( d_N \frac{\partial d_M(\rho_{A_0})}{\partial z_j} \frac{\partial d_M(\rho_{A_0})}{\partial \bar{z}_k} + d_M(\rho_{A_0}) \frac{\partial d_N}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_k} \right) \\
+ 2(d_M(\rho_{A_0}) + d_N) \left( \frac{\partial d_M(\rho_{A_0})}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_k} + \frac{\partial d_M(\rho_{A_0})}{\partial \bar{z}_j} \frac{\partial d_N}{\partial z_k} \right) \\
+ 2d_M(\rho_{A_0})d_N \left( \frac{\partial^2 d_M(\rho_{A_0})}{\partial z_j \partial \bar{z}_k} + \frac{\partial^2 d_N}{\partial z_j \partial \bar{z}_k} \right) + \left( d_N^2 \frac{\partial^2 d_M(\rho_{A_0})}{\partial z_j \partial \bar{z}_k} + d_M^2(\rho_{A_0}) \frac{\partial^2 d_N}{\partial z_j \partial \bar{z}_k} \right),
\]

while simple computations and regrouping the like-terms give

\[
\frac{\partial d_M(\rho_{A_0})}{\partial z_j} (x, y) = (1 + ia)(x_j - \bar{y}_j) \frac{1}{1 + a^2}, \quad j \in \{2 \beta + 1, \ldots, 2 \beta + \gamma \}
\]

\[
\frac{\partial d_M(\rho_{A_0})}{\partial \bar{z}_j} (x, y) = (1 + ia)(x_j - \bar{y}_j) - \frac{(1 + ia)(1 + a^2)}{(1 + a^2)(1 + a^2 + b_j^2)},
\]

\[
H^C(d_M(\rho_{A_0})) = \left[ \frac{\partial^2 d_M(\rho_{A_0})}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^{2\beta + \gamma} = \bigoplus_{j=1}^{\beta} \frac{1}{2} \left[ \begin{array}{c} 2ib_j \\ \frac{1}{1 + b_j^2 + a^2} \end{array} \right] \oplus \frac{1}{2} I_y.
\]

Next, we fix \( j, k \in \{1, \ldots, n\} \). To simplify the computation of \( (H^C(\rho_D))_{j,k} \) we introduce new variables \( X_j = \frac{x_j - ay_j}{\sqrt{1 + a^2}} \) for all \( j \in \{1, \ldots, n\} \) and \( X = (X_1, \ldots, X_n) \). By inserting the expressions computed in (3.6), (4.3), (4.4), (4.7) applied for \( b = 0 \) (i.e. \( A_0 = D = \text{diag}(a, \ldots, a) \)) into (4.6), we obtain

\[
(H^C(\rho_D))_{j,k} (x, y) = \left( \sum_{l=1}^{n} 2y_l^2 X_l X_k + \sum_{l=1}^{n} 2X_l^2 y_l y_k \right) \\
+ \left( \sum_{l=1}^{n} 2iX_l^2 X_k y_l - \sum_{l=1}^{n} 2X_l^2 y_l X_k + \sum_{l=1}^{n} 2i\bar{y}_l^2 X_l y_k - \sum_{l=1}^{n} 2\bar{y}_l^2 y_l X_k \right) \\
+ \sum_{l,m=1}^{n} 2\delta_{jk} y_l^2 X_m + \left( \sum_{l,m=1}^{n} \frac{1}{2} \delta_{jk} y_l^2 y_m^2 + \sum_{l,m=1}^{n} \frac{1}{2} \delta_{jk} X_l^2 X_m^2 \right),
\]

where we denoted \( \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \). Thus \( (H^C(\rho_D))_{j,k} \) is written as a sum of \( 4n^2 + 6n \) monomials in \( X, y \) of degree 4 and with the modulus of the coefficients equal to \( \frac{1}{2} \) or 2. Indeed, the first two terms in (4.6) give respectively \( 2n \) and \( 4n \) monomials, with the absolute value of the coefficients 2, and the third (respectively fourth) term contributes either no terms for \( j \neq k \) or \( n^2 \) (respectively \( 2n^2 \)) terms with coefficient 2 (respectively \( \frac{1}{2} \)) for \( j = k \). If
we further divide each monomial with coefficient \( \pm 2 \) (respectively \( \pm 2i \)) into a sum of four monomials with the coefficient \( \pm \frac{1}{2} \) (respectively \( \pm \frac{1}{2}i \)), it follows that

\[
(H^C(\rho_D))_{j,k}(x,y) = \frac{1}{2} \sum_{l=1}^{m} t_l X^{\alpha_l} Y^{\omega_l}, \quad |\alpha_l| + |\omega_l| = 4, \ |t_l| = 1
\]

(4.9)

with \( m = 6n^2 + 24n \) for \( j = k \), or \( m = 24n \) otherwise; \( \alpha_l, \omega_l \) are the multiindices. Furthermore, using only distributivity property (no regrouping of like-terms or adding them), then for a permutation \( \pi \) from the symmetric group \( S_r \) with \( r \in \{1, \ldots, n\} \), we get \((H^C(\rho_D))_{1,\pi(1)} \cdots (H^C(\rho_D))_{r,\pi(r)}(x,y)\) of the form

\[
(\frac{1}{2})^r \sum_{l=1}^{m} \tilde{t}_l X^{\tilde{\alpha}_l} Y^{\tilde{\omega}_l}, \quad |\tilde{\alpha}_l| + |\tilde{\omega}_l| = 4r, \ |\tilde{t}_l| = 1,
\]

(4.10)

where \( \tilde{m} \) is the product of the numbers of terms of \( r \) factors of the form (4.9).

In a similar fashion we now present \((H^C(\rho_{A_0}))_{j,k}\) in the form

\[
\sum_{l=1}^{m_0} \frac{\tau_l(a) b^{k}_{l} X^{\mu_l} Y^{\nu_l}}{(1 + a^2)^{\mu_l} (1 + a^2 + b_{l}^2)^{\nu_l} (1 + a^2 + b_{l}^2)^{\rho_l}},
\]

(4.11)

where \( \tau_l(a) \) depends only on \( a \) for any \( l \). We first halve the coefficients equal to 2 in (4.19), and hence write \( d_{M(\rho_{A_0})} \) (see (4.3)) as a sum of the form (4.11) with \(|\tau_l(a)| \in \{1, \sqrt{1 + a^2}\}, |\mu_l| + |\nu_l| = 2, \eta_l, \sigma_l, \theta_l \in \{0, 1\} \) and \( m = n + 8\beta \). Next, \( \frac{\partial d_{M(\rho_{A_0})}}{\partial z_j} \) already consists of one (respectively five) such term if \( 1 \leq j - 2\beta \leq \gamma \) (respectively \( 1 \leq j \leq 2\beta \)). By denoting

\[
\Lambda_r = \{(2j - 1, 2j)\}_{1 \leq j - 1 \leq r} \cup \{(2j, 2j - j)\}_{1 \leq j \leq r} \cup \{(j, j)\}_{1 \leq j \leq r},
\]

we see that \( \frac{\partial^2 d_{M(\rho_{A_0})}}{\partial z_j^2 \partial z_k} \) (respectively \( \frac{\partial^2 d_{M(\rho_{A_0})}}{\partial z_j^2 \partial \sigma_k} + \frac{\partial^2 d_{M(\rho_{A_0})}}{\partial z_j \partial \sigma_k} \)) is either equal to either \( \frac{1}{2} \) (respectively 1) for \( j = k \), or \( \pm \frac{ib_j}{1 + a^2 + b_j^2} \) for \( (j, k) \in \Lambda_r, j \neq k \), and vanishes otherwise (see (4.7)). We insert all these expressions along with (3.6) into (4.6) and we get \((H^C(\rho_{A_0}))_{j,k}(x,y)\) of the form (4.11) with \(|\mu_j| + |\nu_j| = 4\) and \( \eta_j, \sigma_j, \theta_j \in \{0, 2\} \). To control the terms better, we again use only distributivity property (no regrouping of like-terms, adding or canceling). The first two terms in (4.6) then contribute respectively at most \( 25n + (n + 8\beta) \) and \( 10(2n + 8\beta) \) summands with \(|\tau_l(a)| \in \{2, 2\sqrt{1 + a^2}, 2(1 + a^2)\} \). The last term in (4.6) for \( (j, k) \in \Lambda_r \) and \( j \neq k \) (respectively \( j = k \)) gives \( n^2 \) (respectively \( n^2 + (n + 8\beta) \)) summands with \( \tau_l(a) = \pm i \) (respectively \( |\tau_l(a)| \in \{\frac{1}{2}, \frac{1}{2}\sqrt{1 + a^2}, 2(1 + a^2)\} \)), while the third term in (4.6) adds \( n(n + 8\beta) \) summands with \(|\tau_l(a)| \in \{2, 2\sqrt{1 + a^2}, 2(1 + a^2)\} \) in both cases; otherwise (for \( (j, k) \notin \Lambda_r \)) these two terms contribute no summands. By quartering (respectively halving) the summands with \(|\tau_l(a)| \in \{2, 2\sqrt{1 + a^2}, 2(1 + a^2)\} \) (respectively \( \tau_l(a) = \pm i \)) we get that \((H^C(\rho_{A_0}))_{j,k}\) is of the form (4.11) with \(|\tau_l(a)| \leq \frac{1}{2}(1 + a^2)\) for any \( l \), and if in addition \((j, k) \in \Lambda_r \) (respectively \( j = k \)) it has at most

\[
4(46n + 88\beta) + 4n(n + 8\beta) + (n^2 + (n + 8\beta)^2) = 184n + 352\beta + 6n^2 + 48n\beta + 64\beta^2
\]

(respectively \( 184n + 352\beta \)) summands, hence \((2\beta \leq n)\):

\[
m_0 \leq \begin{cases} 360n + 46n^2, & (j, k) \in \Lambda_r \\ 360n, & \text{otherwise} \end{cases}
\]
Again, using only distributivity we get \((H^C_r(\rho_{A_0}))_{1,\pi(1)} \cdots (H^C_r(\rho_{A_0}))_{r,\pi(r)}(x, y)\) in the form
\[
\left(\frac{1}{2}\right)^r \sum_{l=1}^{m_0} \frac{\tilde{t}_l(a)}{(1+a^2)^{\bar{y}_{ij}}} \prod_{j=1}^{r} b_i X^\mu j y^\nu j, \quad |\mu| + |\nu| = 4r, \tilde{\eta}_l, \tilde{\theta}_j \in \{0, r\}
\] (4.12)
where \(|\tilde{t}_l(a)| \leq (1 + a^2)^r\) for all \(l\), and \(m_0\) is the product of the numbers ob terms of the corresponding factors.

We observe that \((H^C_r(\rho_D))_{j,k}\) and \((H^C_r(\rho_{A_0}))_{j,k}\) respectively contain substantialaly more terms for \((j, k) \in \Lambda_r\). Therefore we use a few facts from elementary combinatorial theory about permutations with forbidden positions to count the number of terms of \(\det(H^C_r(\rho_D))\) and \(\det(H^C_r(\rho_{A_0}))\). It is well known and follows easily by using the elementary inclusion-exclusion principle that the number of permutations in \(S_r\) with exactly \(s \in \{0, \ldots, r\}\) fixed points is \(\Delta_r,s = \frac{r!}{s!} \sum_{l=0}^{r-s} (-1)^l \binom{r}{l}.\) Combining this with (4.10) gives
\[
\det(H^C_r(\rho_D))(x, y) = \sum_{\pi \in S_r} \text{sgn}(\pi) \prod_{l=1}^{r} (H^C_r(\rho_D))_{l,\pi(l)} = \left(\frac{1}{2}\right)^r \sum_{s=0}^{m_{r,s}} \sum_{l=1}^{m_{r,s}} \tilde{t}_l(a) \prod_{j=1}^{r} \sum_{j=1}^{r} b_i X^\mu j y^\nu j,
\] (4.13)
where \(m_{r,s} = \Delta_r,s(24n + 6n^2)^t (24n)^{r-s} = \Delta_r,s(24n)^r (1 + \frac{1}{2}n)^s, |\mu_l| + |\nu_l| = 4r, |\eta_l| = 1.\)
Furthermore, let \(\Sigma_{r,s}\) be the set of all permutations \(\pi \in S_r\) with precisely \(s \in \{0, \ldots, r\}\) points \(l\) such that \((l, \pi(l)) \in \Lambda_r\) and we try to find the upper bound for \(|\Sigma_{r,s}|.\) Conveniently, the number of choices of exactly \(s\) points of the permutation such that \((l, \pi(l)) \in \Lambda_r\) is equal to the \(s\)-th coefficient of the so-called rook-polynomial corresponding to \(\Lambda_r\), and that is \((1 + x)^{2s} (1 + 4x + 2x^2)^{\beta_r}\) with \(r = 2\beta_r + \gamma_r\) (see e.g. [11, Chapter 7]). Trivially, the upper bound for the \(s\)-th coefficient of this polynomial is the \(s\)-th coefficients of the polynomial \((1 + 2x)^{2s} (1 + 4x + 2x^2)^{\beta_r}\) and that is \(2^s(r)\). Permuting the other \(n - r\) points in any way yields \(|\Sigma_{r,s}| \leq 2^s(r, r - s)!.\) From (4.12) we then deduce that
\[
\det(H^C_r(\rho_{A_0}))(x, y) = \left(\frac{1}{2}\right)^r \sum_{s=0}^{m_{r,s}} \sum_{l=1}^{m_{r,s}} \frac{\tilde{t}_l(a)}{(1+a^2)^{\bar{y}_{ij}}} \prod_{j=1}^{r} b_i X^\mu j y^\nu j,
\] (4.14)
where we have \(m_{r,s} = |\Sigma_{r,s}|(360n + 46n^2)^s(360n)^{r-s} = |\Sigma_{r,s}|(1 + \frac{23}{180}n)^s(360n)^r, |\mu_l| + |\nu_l| = 4r, |\eta_l|, \tilde{\theta}_j \in \{0, r\}, |\tilde{t}_l(a)| \leq (1 + a^2)^r.\)

It is clear that the terms of \(\det(H^C_r(\rho_{A_0}))(x, y)\) in (4.14) with \(|\eta_l| = 0\) are precisely the terms of \(\det(H^C_r(\rho_D))\), while for the terms with \(|b| \leq 1, |\tilde{\eta}_l| \geq 1\) we have \(\max_{1 \leq k \leq \beta} |b_k| \geq |b^{\tilde{\eta}_l}|\), Further, we have \((1 + a^2 + b^2_j) \geq 1\) for all \(j\) and
\[
|X^\alpha y^\omega| \leq \sum_{j=1}^{n} (X_j^{2r} + y_j^{2r}), \quad |\alpha| + |\omega| = 4r.
\] (4.15)
Combining all this with (4.1), (4.13), (4.14) and assuming \(|b| \leq 1\) yields
\[
|\Phi_r(A_0, D)(x, y)| \leq \left(\frac{1+a^2}{2}\right)^r \sum_{s=0}^{m_{r,s}} \left|\Sigma_{r,s}\right| (1 + \frac{23}{180}n)^s (360n)^r - \Delta_r,s(n + \frac{1}{2}n)^s (24n)^r
\]
\[
\max_{1 \leq j \leq \beta} |b_j| \sum_{j=1}^{n} (X_j^{2r} + y_j^{2r}).
\] (4.16)
Using $\Delta_{r,s} \geq \frac{1}{3} r_1^r$ and $|\Sigma_{r,s}| \leq 2^s r_1^s$ we easily get

$$
\sum_{s=0}^{r} |\Sigma_{r,s}|(1 + \frac{23}{180} n)^s - \sum_{s=0}^{r} \Delta_{r,s}(\frac{24}{360})^r (1 + \frac{1}{4} n)^s
$$

$$
\leq (r!) \sum_{s=0}^{r} \left( \frac{2^s}{3^s} (1 + \frac{23}{180} n)^s - \frac{1}{3} (\frac{1}{15})^r \frac{1}{4} n)^s \right) \leq (r!) \left( e^{2 + \frac{23}{90} n} - \frac{1}{3} (\frac{1}{15})^r e^{1 + \frac{n}{4}} \right).
$$

(4.17)

We now use (4.17) and replace $X_j$ back by $\frac{\sqrt{x_j-a_j}}{\sqrt{1+a_j^2}}$ for all $j$ in (4.16). Finally, applying (3.16) for $|a| \leq \frac{1}{\sqrt{r}}$ yields $|\Phi_r(A_0, D)| \leq H^C_r(\rho_D)$ (see (4.5)), with equality precisely at the origin, provided that

$$
\max_{1 \leq j \leq \beta} |b_j| \leq \left( 10(360n)^r (r!) \left( e^{1 + \frac{23}{90} n} - \frac{1}{3} (\frac{1}{15})^r e^{\frac{n}{4}} \right) \right)^{-1}.
$$

It concludes the proof of the existence of the appropriate function $\rho$ in case (1).

Next, we consider the case (2). Suppose $A_0$ is as in Lemma 2.1 and observe that

$$
d_{M(A_0)}(x, y) = \sum_{j=1}^{n} x_j^2 + R_0(x, y),
$$

$$
R_0(x, y) = \sum_{j=2\beta+1}^{2\beta+\gamma} a_j \frac{-2x_jy_j+a_j(y_j^2-x_j^2)}{1+a_j^2} + \sum_{j=1}^{\beta} \frac{(b_j^2+c_j^2)(y_{j+1}^2-y_{j-1}^2)}{1+b_j^2+c_j^2} + \sum_{j=1}^{\beta} \frac{2x_{j+1}y_j-c_jy_{j+1}-2x_j(c_jy_{j+1}+b_jy_{j-1})}{1+b_j^2+c_j^2}
$$

with $a = (a_{2\beta+1}, \ldots, a_{2\beta+\gamma})$, $c = (c_1, \ldots, c_2\beta)$, $b = (b_1, \ldots, b_2\beta)$, $2\beta + \gamma = n$. Straightforward computations and regrouping the like-terms further give

$$
\frac{\partial d_{M(A_0)}}{\partial \bar{z}_j}(x, y) = x_j + a_j \frac{(i-a_j)x_j-(1-i a_j)y_j}{1+a_j^2}, \quad 2\beta + 1 \leq j \leq 2\beta + \gamma,
$$

$$
\frac{\partial d_{M(A_0)}}{\partial \bar{z}_{j+1}}(x, y) = x_{j+1} - \frac{(c_j+b_j^2+y_j^2-y_{j+1}^2-y_{j-1}^2)}{1+b_j^2+c_j^2}, \quad 1 \leq j \leq \beta,
$$

$$
\frac{\partial d_{M(A_0)}}{\partial \bar{z}_{j+1}}(x, y) = x_{j+1} - \frac{(c_j+b_j^2+y_j^2-y_{j+1}^2-y_{j-1}^2)}{1+b_j^2+c_j^2}, \quad 1 \leq j \leq \beta.
$$

$$
H^C(d_{M(A_0)}) = \left[ \frac{\partial^2 d_{M(A_0)}}{\partial \bar{z}_j \partial \bar{z}_k} \right]_{j, k=1}^{2\beta+\gamma} = \frac{\beta}{2} \left[ \begin{array}{c}
1 & -\frac{2ib_j}{1+b_j^2+c_j^2} \\
-\frac{2ib_j}{1+b_j^2+c_j^2} & 1
\end{array} \right] \oplus \frac{1}{2} I_y.
$$

(4.20)

We have

$$
\det(D_{r,0}(A_0)) = \det(D_{r,0}(\rho_0)) + \Phi_r(A_0, 0), \quad r \in \{1, \ldots, n\},
$$

(4.21)

where $\Phi_r(A_0, 0)$ is a homogeneous polynomial of degree $4r$ in $x$, $y$, and its coefficients are rational functions in $a$, $b$, $c$, and such that they tend to 0 as $a$, $b$, $c$ all approach 0. We proceed mutatis-mutandis as in case (1), hence some details will be omitted.

To compute $(H^C)_j(x, y)$ we insert the expressions computed in (3.6), (4.18), (4.19), (4.20) applied for $a = b = c = 0$ (i.e. $A_0 = 0$) into (4.6) with $A_0 = 0$. We obtain precisely...
Using the estimates (4.24) and where (4.22) with

\[ \sum_{l=1}^{m_1} \frac{u_l \alpha_l \beta_l \gamma_l \delta_l \varepsilon_l \varphi_l}{\prod_{a=j,k}(1+a^2 \beta_a r^2 + b^2 \gamma_a s^2)} \cdot |\alpha_l| + |\beta_l| = 4, u_l \in \mathbb{C}, \eta_{a,l}, \theta_{a,l} \in [0, 1]. \]  

(4.22)

We first halve the coefficients equal to 2 in (4.19), and write \(d_{M(A_0)}\) (see (4.18)) in the form (4.22) with \(u_l \in \{\pm 1, \pm i\}\) and \(m_1 = n + 4 \gamma + 16 \beta\). Recall that \(\frac{\partial^2 d_{M(A_0)}}{\partial z_j \partial \bar{z}_k}\) (respectively \(\frac{\partial^2 d_{M(A_0)}}{\partial z_j \partial \bar{z}_k}\)) is either equal to either \(\frac{1}{2}\) (respectively \(1\)) for \(j = k, \) or \(\pm \frac{i b_j}{1+c_j+b_j} \) for \((j, k) \in \Lambda_r, j \neq k, \) and vanishes otherwise (see (4.20)). Again, we insert these expressions along with (3.6) into (4.6). By using only ditributivity, we obtain that the first two terms in (4.6) contribute respectively \((8n + (n + 4 \gamma + 16 \beta))\) and \((18n + 2n + 4 \gamma + 16 \beta)\) summands with \(u_l \in \{\pm 2, \pm 2i\}\). The last term adds \(n^2\) (respectively \(n^2 + (n + 4 \gamma + 16 \beta)^2\)) summands with \(u_l = \pm i\) (respectively \(u_l = \frac{1}{2}\)) when \(j = k\) (respectively \((j, k) \in \Lambda_r\), and the third one gives none or \(n(n + 4 \gamma + 16 \beta)\) summands with \(u_l \in \{\pm 2, \pm 2i\}\) in both cases; otherwise they vanish. Finally, by halving (respectively quartering) the summands with \(u_l = \pm i\) (respectively \(u_l \in \{\pm 2, \pm 2i\}\)) we get \(H^C(\rho_{A_0})_{j,k}\) in the form (4.22) with \(|u_l| = \frac{1}{2}\) for all \(l; \) if \((j, k) \in \Lambda_r\) (respectively \(j = k\), it has at most

\[ 4(118n + 76 \gamma + 304 \beta) + 4n(n + 4 \gamma + 16 \beta) + (n^2 + (n + 4 \gamma + 16 \beta)^2) \]

(respectively \(4(118n + 76 \gamma + 304 \beta)\)) summands, thus \(2(\beta + \gamma = n, 2 \beta = n)\):

\[
m_1 \leq \begin{cases} 
1080n + 118n^2, & (j, k) \in \Lambda_r \\
1080n, & \text{otherwise} 
\end{cases}
\]

Similar to (4.14) we now get

\[
\det(H^C_r(\rho_{A_0})) = \left(\frac{1}{2}\right)^r \sum_{s=0}^r \sum_{l=1}^{m_{r,s}} \frac{\mu_l}{\prod_{a=j,k}(1+a^2 \beta_a r^2 + b^2 \gamma_a s^2)} a^{\mu_l} b^{\nu_l} c^{\kappa_l} d^{\eta_l} e^{\theta_l} f^{\varphi_l},
\]

(4.23)

where \(m_{r,s} \equiv |\Sigma_{r,s}|(1080n + 118n^2)^s(1080n)^{r-s} = |\Sigma_{r,s}|(1 + \frac{69}{49}n)^s(1080n)^r, \) and \(|\mu_l| + |\nu_l| = 4r, |\theta_l|, \theta_{l,j} \in \{0, r\}, |\mu_l(a)| \leq (1 + a^2)^r.\)

The terms of \(\det(H^C_r(\rho_0))\) are precisely those terms of \(\det(H^C_r(\rho_{A_0}))\), which have \(|\mu_l| + |\nu_l| = 4r, |\theta_l|, \theta_{l,j} \in \{0, r\}, |\mu_l(a)| \leq (1 + a^2)^r.\)

Using the estimates (4.24) and \((1 + c_j^2 + b_j^2), (1 + a_j^2) \geq 1\) for all \(j,\) and applying (4.1), (4.13), (4.15) for \(X = x, \rho_D = \rho_0\) (remember that \(a = 0\)), we further obtain

\[
|\Phi_r(A_0, 0)| \leq \left(\frac{1}{2}\right)^r \sum_{s=0}^r \left|\Sigma_{r,s}\right|(1 + \frac{69}{49}n)^s(1080n)^r - \Delta_{r,s}(n + \frac{1}{4}n)^s(24n)^r \right) \times \psi_{abc} \sum_{j=1}^n \left(x_{j}^{2r} + y_{j}^{2r}\right)
\]

(4.24)
Remark 4.3 Due to the simplicity and clarity some crude calculations were made in the proof of Theorem 4.2 and Lemma 3.3, and hence the estimates on the eigenvalues of deformation retraction of its gradient vanishing precisely on $M$ of the Levi problem in $C^2$ be as in Lemma 2.1 and let

\[
\rho = \begin{pmatrix} a & \cdots & b \end{pmatrix}
\]

Remark 4.4 Observe also that our construction is stable under small perturbations. Let $A_0$ be as in Lemma 2.1 and let $\tilde{A}_0$ be its small perturbation:

\[
A_0 = \text{diag}(C_1, \ldots, C_\beta, a_1, \ldots, a_\gamma), \quad C_j = \begin{bmatrix} c_j & -b_j \\ b_j & c_j \end{bmatrix} \text{ for } 1 \leq j \leq \beta,
\]

\[
\tilde{A}_0 = \text{diag}(\tilde{C}_1, \ldots, \tilde{C}_\beta, a'_1, \ldots, a'_\gamma), \quad \tilde{C}_j = C_j + \epsilon_j I_2 - \epsilon'_j E_2 \text{ for } 1 \leq j \leq \beta,
\]

\[
\epsilon = (\epsilon_1, \ldots, \epsilon_{\beta+\gamma}, \epsilon', \ldots, \epsilon_{\beta}) \quad \tilde{a}_j = a_j + \epsilon_j + \beta \text{ for } 1 \leq j \leq \gamma,
\]

where $I_2$ is the $2 \times 2$ identity matrix and $E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We compute

\[
\frac{(x_j - (a_1 + \epsilon_1) y_j)^2}{1 + (a_j + \epsilon_j)^2} = \frac{(x_j - \tilde{a}_j y_j)^2}{1 + \tilde{a}_j^2} + \frac{\epsilon_j (1 + \tilde{a}_j^2) - 2 \tilde{a}_j y_j (1 + \tilde{a}_j^2)}{(1 + \tilde{a}_j^2)(1 + (a_1 + \epsilon_1)^2)},
\]

\[
= \frac{(x-cy+bu)^2 + (u-cy-bv)^2}{1 + (c+e)^2 + (b+e)^2} + \frac{(2c+e) + (2b+e')((x-cy+bu)^2 + (u-cy-bv)^2)}{(1 + (c+e)^2 + (b+e)^2)^2} + \frac{(e'v-cy)(2(x-cy+bu) + e'v-cy) + (\epsilon'v-cy)(2(u-cy-bv) - \epsilon'v-cy)}{(1 + (c+e)^2 + (b+e)^2)^2},
\]

\[
\leq (540n)^r (r!)^s \frac{2^r}{s!} \left(1 + \frac{59}{540} n\right)^s
\]

\[
- \frac{1}{3} \left( \frac{24}{1080} \right)^r \frac{1}{s!} \left(1 + \frac{59}{270} n\right)^s \right) \psi_{abc} \sum_{j=1}^n (x_j^{2r} + y_j^{2r})
\]

\[
\leq (540n)^r (r!)^s \left( e^{\frac{59}{270} n} - \frac{1}{3} \left( \frac{1}{45} \right)^r \right) \psi_{abc} \sum_{j=1}^n (x_j^{2r} + y_j^{2r})
\]

From (3.16) for $a = 0$ and by taking

\[
\psi_{abc} \leq \left( 10e \cdot (1080n)^r (r!)^s \left( e - \frac{1}{3} \left( \frac{1}{45} \right)^r \right) \right)^{-1}
\]

we deduce that $|\Phi_r(A_0, 0)| \leq H^C_r(\rho_{A_0})$, with equality precisely at the origin. It concludes the proof of the existence of the appropriate function $\rho$ also in case (2).

To complete the proof of the theorem we use a suitable partition of unity to glue $\rho$ away from the origin with the squared Euclidean distance functions to $M$ and $N$ respectively (see [16, Theorem 4.1] for details). We get a non-negative defining function $\rho_0$ for $M \cup N$ with its gradient vanishing precisely on $M \cup N$ and such that it is strictly plurisubharmonic on a neighborhood of $M \cup N$, but without the origin. By the classical result known as the solution of the Levi problem in $\mathbb{C}^n$ (see e.g. [8, Theorem 5.1.2]), the sublevel sets $\{\rho_0 < \epsilon\}$ are then Stein for any sufficiently small $\epsilon$. Finally, we observe that the flow of $-\nabla \rho_0$ yields a deformation retraction of $\{\rho_0 < \epsilon\}$ onto $M \cup N$.
and by replacing $x, y, u, v, c, b, \varepsilon, \varepsilon'$ respectively by $x_{j-1}, y_{j-1}, x_j, y_j, c_j, b_j, \varepsilon_j, \varepsilon'_j$ in the second expression, we deduce that $d_{M(A_0)} = d_{M(A_0)} + \tilde{q}$, where $\tilde{q}$ is a homogeneous polynomial of degree 2, and whose coefficients are rational functions in $\varepsilon$ that vanish as $\varepsilon$ approaches 0. Further, in the notation (4.1) we have $\rho_B = P(d_{M(B)}, d_N)$ for some polynomial $P$ of degree $d$, hence

$$\det \left( H^C_r(\rho_{A_0}) \right) = \det \left( H^C_r(\rho_A) \right) + \tilde{Q}_r, \quad r \in \{1, \ldots, n\},$$

where $\tilde{Q}_r$ is homogeneous polynomial of degrees $(2d - 2)r$ in $x, y$, and such that its coefficients tend to 0 as $\varepsilon$ approaches 0. If $\det \left( H^C_r(\rho_{A_0}) \right)$ vanishes at the origin and is positive everywhere else for all $r$, then by Lemma 4.1 the same holds for $\det \left( H^C_r(\rho_{A_0}) \right)$, provided that $\varepsilon$ is sufficiently close to 0.

Lemma 3.3 can be also applied to prove the existence of regular neighborhoods of certain totally real immersions of $n$-manifolds into complex $n$-manifold; for results on closed real surfaces immersed into complex surface see [2, Theorem 2.2], [13, Theorem 2], [16, Proposition 4.3]).

In connection to this we also note that Weinstock’s result has been recently generalised by Gorai [4] and Shafikov and Sukhov [12, Theorems 1.3 and 4.2], to the effect that a union of two maximal totally real submanifolds in $\mathbb{C}^n$, intersecting transversally at the origin, is polynomially convex near the origin, provided that the union of their tangent spaces at the origin is polynomially convex near the origin.

**Proposition 4.5** Let $\pi: Z \to X$ be a smooth totally real immersion of a closed $n$-manifold into a complex $n$-manifold $X$, and such that $\pi$ has only transverse double points (no multiple points) $q_1, \ldots, q_s \in \pi(Z)$ with $\pi^{-1}(q_j) = \{t_j, u_j\}$. For any $j \in \{1, \ldots, s\}$, let the image under the tangent map of the tangent spaces of $Z$ at $t_j$ and $u_j$ define a union of totally real subspaces in $T_{q_j} X \approx \mathbb{C}^n$, which is holomorphically-equivalent to $(A_j + iI)\mathbb{R}^n \cup \mathbb{R}^n$, where $A_j \in \mathbb{R}^{n \times n}$ with $A_j - iI$ invertible. If the entries of $A_j$ for all $j \in \{1, \ldots, s\}$ satisfy any of the conditions (1) or (3) in Lemma 3.3, then $\tilde{Z} = \pi(Z)$ has a regular Stein neighborhood basis.

**Proof** For every double point $q_j \in \tilde{Z}$ there exist local holomorphic coordinates $\psi_j: U_j \to V_j \subset \mathbb{C}^n$, such that $\psi_j(q_j) = 0$ and such that $\psi_j(\tilde{Z} \cap U_j) = S_j \cup T_j$, where $S_j, T_j$ are real $n$-submanifolds in $V_j$, intersecting only at the origin, and they are tangent to $M_j = (A_j + iI)\mathbb{R}^n$ and $N_j = \mathbb{R}^n$ there, respectively.

Next, by following the proof of the local tubular neighborhood theorem (see e.g. [9, p. 78-92]), we show that in a small neighborhood of a point $w_0$ on an $n$-submanifold $S \subset \mathbb{R}^m$ the Taylor expansions of the squared Euclidean distance functions to $S$ and to the affine tangent space to $S$ at $w_0$, respectively, agree to the terms of second order. Let $0 \in \mathbb{W} \subset \mathbb{R}^n$ and let $F: \mathbb{W}^r \to \mathbb{W}, F(0) = w_0$, be a parametrization of $S$ in a neighborhood $W \subset \mathbb{R}^m$ of $w_0$. Also, there exist orthonormal smooth vector fields $(v_1, \ldots, v_{m-n})$: $W \cap S \to \mathbb{R}^m$, spanning the normal space to $S$. We set $\Theta: \mathbb{W} \times \mathbb{R}^{m-n}, \Theta(\mu, \nu) = F(\mu) + \sum_{j=1}^{m-n} v_j \nu_j(\mu)$, and observe that the rank of the Jacobian of $\Theta$ at $(0,0)$ is maximal. Therefore, by the implicit mapping theorem, $\Theta$ is a diffeomorphism in a neighborhood of $(0,0)$, and let $\Phi(w) = (\mu(w), \nu(w))$ be its smooth inverse, defined in a small neighborhood $\tilde{W}$ of $w_0$. Since $F(\mu(w))$ is the nearest point on $S$ to a point $w \in \tilde{W}$, the squared Euclidean distance from $w$ to $S$ is equal to $d_S(w) = \sum_{j=1}^{m-n} v_j^2(w)$. Furthermore, $d_S$ vanishes on $S$, hence all derivatives of $v_j$ in the tangent directions to $S$ vanish at $w_0$, while the derivative of $v_j$ in the direction $v_j(w_0)$ is equal to 1 if $j = l$ and vanishes otherwise. The Taylor’s expansion $v_j(w) = v_j(w_0) + \mathcal{O}(w_0)$.
\begin{align*}
\{v_j(w_0), w - w_0\} + \sum_{|\alpha|=3} (w - w_0)^\alpha f_\alpha(w), \text{ where } f_\alpha \text{ is some smooth function for any multiindex } \alpha, \text{ further implies that } \\
d_S(w) = \sum_{j=1}^{m-n} (v_j(w_0), w-w_0)^2 + \sum_{|\alpha|=3} (w - w_0)^\alpha g_\alpha(w) = d_M(w) + \sum_{|\alpha|=3} (w - w_0)^\alpha g_\alpha(w),
\end{align*}
where \(d_M\) is the squared Euclidean distance to the affine tangent space to \(S\) at \(w_0\) and \(g_\alpha\) is a smooth function for every \(\alpha\).

Denoting by \(d_{S_j}, d_{T_j}, d_{M_j}, d_{N_j}\) respectively the squared Euclidean distances to \(S_j, T_j, M_j, N_j\), we see that for a homogeneous polynomial \(P\) of degree \(d\) we get

\[
\det H^C_r(P(d_{S_j}, d_{T_j})(x,y)) = \det H^C_r(P(d_{M_j}, d_{N_j})(x,y)) + \sum_{|\alpha|+|\beta|>(2r-2)d} x^\alpha y^\beta h_\alpha(x, y),
\]
where the determinants of the complex Hessians are homogeneous polynomials of degree \((2d-2)r\) in \(x, y\) and \(h_\alpha, \tilde{h}_\alpha\) are smooth for all \(\alpha\), with \(\tilde{h}_\alpha(0) = 0\). Lemma 3.3 now furnishes a homogeneous polynomial \(P\), satisfying (3.15), and such that \(P(d_{M_j}, d_{N_j})\) is strictly plurisubharmonic everywhere, except at the origin. Since \(\tilde{h}_\alpha(x, y)\) are sufficiently close to zero for all \(\alpha\), provided that \((x, y)\) is close enough to the origin, we use Lemma 4.1 to deduce that \(\rho_j = P(d_{S_j}, d_{T_j})\) is strictly plurisubharmonic in some punctured neighborhood of the origin. Moreover, property (3.15) yields \(S_j \cup T_j = \{\rho_j = 0\} = \{\nabla \rho_j = 0\}\). We set \(\varphi_j = \rho_j \circ \psi_j\), and note that \(\varphi_j\) inherits the properties from \(\rho_j\).

It is well known that the squared distance function to a totally real submanifold, relative to some Riemannian metric, is strictly plurisubharmonic in a neighborhood of the submanifold (see e.g. [13, Proposition 4.1]). Therefore there exists a strictly plurisubharmonic function \(\varphi_0\) in some open neighborhood of \(\tilde{Z} \setminus \{q_1, \ldots, q_s\}\). Much as in the proof of [13, Theorem 4.1] we now patch functions \(\varphi_j\) for \(j \in \{0, 1, \ldots, s\}\) to obtain a non-negative function \(\rho\), defined on a neighborhood \(\Omega\) of \(\tilde{Z}\), and such that \((\rho = 0) = \{\nabla \rho = 0\} = \tilde{Z}\) and such that \(\rho\) is strictly plurisubharmonic on \(\Omega \setminus \{q_1, \ldots, q_s\}\). Since the restriction of \(\rho\) to the compact analytic set would be plurisubharmonic (see [6, p. 180]) and non-constant, there can be no compact positive dimensional analytic subset in \(\Omega\). By Grauert’s result (see [5, Proposition 5]), the sublevel set \(\{\rho < \epsilon\}\) is then Stein for any sufficiently small \(\epsilon\). To conclude, the flow of \(-\nabla \rho\) yields a deformation retraction of \(\{\rho < \epsilon\}\) onto \(\tilde{Z}\).

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\square
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