Meson Supercurrent State in High Density QCD

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The ground state of three flavor quark matter at asymptotically large density is believed to be the color-flavor-locked (CFL) phase. At non-asymptotic density the effect of the non-zero strange quark mass cannot be neglected. If the strange quark mass exceeds \( m_s < m_s^{\text{crit}} \sim m_u^{1/3} \Delta^{2/3} \), the CFL state becomes unstable toward the formation of a neutral kaon condensate. Recently, several authors discovered that for \( m_s \sim (2 \Delta p_F)^{1/2} \) the CFL state contains gapless fermions, and that the gapless modes lead to an instability in current-current correlation functions. Using an effective theory of the CFL state we demonstrate that this instability can be resolved by the formation of a meson supercurrent, analogous to Migdal’s \( p \)-wave pion condensate. This state has a non-zero meson current which is canceled by a backflow of gapless fermions.

Calculations based on weak coupling QCD show that the ground state of three flavor QCD at very high baryon density is the color-flavor-locked (CFL) phase. During the past several years a lot of effort has been devoted to the question how the CFL state is modified in the presence of a non-zero strange quark mass. Using a BCS-type gap equation it is easy to see that the most important parameter is \( m_s^2/(2p_F \Delta) \), where \( p_F \) is the Fermi momentum and \( \Delta \) is the gap in the chiral limit. At non-asymptotic density the CFL phase undergoes a phase transition to a kaon condensed phase. This transition is shifted to larger values of \( m_s \) if instanton effects are important, as is the case at moderate density. For \( m_s^2/(2p_F \Delta) \sim 2\Delta \) the effective field theory description breaks down and we expect a phase transition to a less symmetric phase to occur. More recently, several groups observed that gapless fermion modes can appear in the spectrum of the CFL phase in the vicinity of the point \( m_s^2/(2p_F \Delta) \sim \Delta \). Gapless modes also appear in the kaon condensed CFL phase, but the critical strange quark mass is shifted to somewhat larger values.

The problem is that gapless fermion modes in a weakly coupled (BCS) superfluid tend to cause instabilities in current-current correlation functions. These instabilities are quite generic and appear in a wide range of systems, including the \( 2SC \) phase, the CFL phase, and cold atomic gases. The presence of an instability implies that the homogeneous superfluid is not the correct ground state. One possibility is that the true ground state is an inhomogeneous superconductor of the type first considered by Larkin, Ovchinnikov, Fulde and Ferrell and generalized to QCD in effective theories of cold atoms.

Our starting point is the effective theory of the CFL phase derived in the \( \text{SU}(3) \) symmetry breaking in the CFL phase and the baryon fields originate from quark-hadron complementarity. The chiral field is given by \( \Sigma = \exp(i\phi^a \lambda^a/f_\pi) \) where \( f_\pi \) is the pion decay constant. \( M \) is the quark mass matrix. The chiral field and the mass matrix transform as \( \Sigma \rightarrow L \Sigma R^\dagger \) and \( M \rightarrow LM R^\dagger \) under chiral transformations \( (L, R) \in SU(3)_L \times SU(3)_R \). The baryon field \( N \) transforms according to the adjoint representation of flavor \( SU(3) \). We will expand \( N \) in terms of the baryon nonet \( (p, n, \Sigma^\pm, \Xi^\pm, \Xi^0, \Lambda^{8/3}) \). \( v^\mu = (1, \vec{v}) \) is the Fermi velocity, and \( \Delta \) is the superfluid gap. We have suppressed the singlet fields associated with the breaking of the exact \( U(1)_V \) and approximate \( U(1)_A \) symmetries.

The covariant derivative of the nucleon field is given by \( D_\mu N = \partial_\mu N + i[V_\mu, N] \). The vector and axial-vector currents are

\[
Y_\mu = -\frac{i}{2} \left\{ \xi \partial_\mu \xi^\dagger + \xi^\dagger \partial_\mu \xi \right\}, \quad A_\mu = -\frac{i}{2} \xi \left( \partial_\mu \Sigma^\dagger \right) \xi.
\]

where \( \xi \) is defined by \( \xi^2 = \Sigma \). It follows that \( \xi \) transforms as \( \xi \rightarrow L \xi U(x) = U(x) \xi \xi^R \) with \( U(x) \in SU(3)_V \).

Symmetry arguments can be used to determine the leading mass terms in the effective lagrangian. Bedaque and Schäfer observed that \( X_L = MM^\dagger/(2p_F) \) and \( X_R = M^\dagger M/(2p_F) \) enter the microscopic theory like the temporal components of left and right handed flavor gauge fields. We can make the effective lagrangian...
invariant under this symmetry by introducing the covariant derivatives
\[ D_0 N = \partial_0 N + i [\Gamma_0, N], \]
\[ \nabla_0 \Sigma = \partial_0 \Sigma + i X_L \Sigma - i \Sigma X_R \]
with \( \Gamma_0 = -i \{ \{ \partial_0 + i X_R \} \xi^1 + \xi^1 \{ \partial_0 + i X_L \} \xi \}. \)
If the density is very large and the QCD coupling is weak then the low energy constants can be calculated in perturbation theory. The leading terms in the meson and baryon sector were calculated in \[23\] and \[9\]. The results are
\[ f_\pi^2 = \frac{21 - 8 \log(2)}{18} \left( \frac{\mu^2}{2\pi^2} \right), \quad v_\pi^2 = \frac{1}{3}, \quad a_3 = \frac{3\Delta^2}{4\pi^2}. \]
and \( D = F = 1/2 \). The gap \( \Delta \) can also be computed in perturbative QCD, but we will not make use of these results and treat \( \Delta \) as a parameter.

The parameter that controls the effect of the strange quark mass is the effective chemical potential \( \mu_s = m_s^2/(2\rho) \). When \( \mu_s \) exceeds the mass of the kaon the CFL ground state becomes unstable and a Bose condensate of kaons is formed. If isospin was an exact symmetry the energy the \( \alpha \) is a maximal kaon condensate.\(^{13}\) The meson part of the effective lagrangian gives a positive contribution
\[ \mathcal{E} = \frac{1}{2} \left( \vec{\nabla} \phi_K \right) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -ie^{i\phi_K} \\ 0 & ie^{-i\phi_K} & 0 \end{array} \right). \]

In the following we compute the vacuum energy as a function of the kaon current \( j_K = \vec{\nabla} \phi_K \). The meson part of the effective lagrangian is
\[ \mathcal{E} = \frac{1}{2} \frac{\rho^2}{2\pi^2} \frac{2\rho}{\pi^2} \mathcal{J} j_K. \]

A negative contribution can arise from gapless fermions. In order to determine this contribution we have to calculate the fermion spectrum in the presence of a non-zero current. The relevant part of the effective lagrangian is
\[ \mathcal{L} = \text{Tr} \left( N^\dagger i\eta^\mu D_\mu N \right) + \text{Tr} \left( N^\dagger \gamma_5 \left( \rho_A + \vec{\nu} \cdot \vec{A} \right) N \right) + \frac{\Delta}{2} \left( \text{Tr} (NN) - \text{Tr} (N) \text{Tr} (N) + h.c. \right) \]
where we have used \( D = F = 1/2 \). The covariant derivative is \( D_0 N = \partial_0 N + i [\rho_V, N] \) and \( D_1 N = \partial_1 N + i \vec{v} \cdot \vec{V} [\rho_V, N] \) with \( \vec{V}, \vec{A} \) given in eqn. \[8\] and
\[ \rho_{V,A} = \frac{1}{2} \left( \xi M \xi^\dagger \pm \xi M^\dagger M \xi \right). \]

The vector potential \( \rho_V \) and the vector current \( \vec{V} \) are diagonal in flavor space while the axial potential \( \rho_A \) and the axial current \( \vec{A} \) lead to mixing. In order to exhibit the mechanism for the instability we will first discuss the simple case of vanishing axial vector couplings \( F = 0 \). We will then study the realistic case \( D = F = 1/2 \).

We can determine the spectrum by decomposing \( \rho_V \) and \( \vec{V} \) into isospin and hypercharge components. We find
\[ \rho_V = -\frac{\mu_s}{4} \hat{f}_3 - \frac{\mu_s}{4} \hat{Y}, \quad \vec{V} = \frac{j_K}{2} \hat{f}_3 + \frac{3j_K}{4} \hat{Y}, \]
where \( \hat{f}_3 = \lambda_3/2 \) and \( \hat{Y} = \lambda_8/\sqrt{3} \) are the isospin and hypercharge generators. The dispersion relations can now be obtained from the \( (I_3, Y) \) quantum numbers of the baryon nonet. The result is
\[ \omega_n, z_\bar{\nu} = \sqrt{\Delta^2 + l^2} \pm \hat{v} \cdot j_K, \]
\[ \omega_{\Sigma^+}, \Sigma^- = \sqrt{\Delta^2 + l^2} \pm \frac{1}{2} (\mu_s + \hat{v} \cdot j_K), \]
\[ \omega_p, z_{\bar{\nu}} = \sqrt{\Delta^2 + l^2} \pm \frac{1}{2} (\mu_s - \hat{v} \cdot j_K). \]

where \( l = \hat{v} \cdot p - p_F \) is the residual momentum. The neutral modes \( (\Sigma_0, \Lambda_0, \Lambda_8) \) are not affected by the current or the vector potential. Eqn. \[10\] shows that there are gapless fermion modes as \( \mu_s \) approaches \( 2\Delta \), and that gapless modes occur earlier if there is a non-zero current. The contribution to the vacuum energy from gapless modes is
\[ \mathcal{E} = 4 \frac{\mu^2}{2\pi^2} \int dl \int d\Omega \frac{4\pi}{\omega} \omega \theta(-\omega), \]

Translated to the effective QCD Lagrangian, the contribution of the kaon condensate against the formation of a non-zero current.

Consider a spatially varying \( U(1)_Y \) rotation of the maximal kaon condensate
\[ U(x) \xi_K^0 U^\dagger(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & ie^{i\phi_K(x)}/\sqrt{2} \\ 0 & ie^{-i\phi_K(x)}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \]
This state is characterized by non-zero currents
\[ \vec{V} = \frac{1}{2} \left( \vec{\nabla} \phi_K \right) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right), \]
where the factor 4 is a degeneracy factor and $d\Omega$ is an integral over the Fermi surface. Near the Fermi surface we can approximate

$$\omega_l = \Delta + \frac{i^2}{2\Delta} - \frac{1}{2} (\mu_s + \vec{v} \cdot \vec{j}_K).$$

The integral in eqn. (15) receives contributions from one of the pole caps on the Fermi surface. The result has exactly the same structure as the energy functional of a non-relativistic two-component Fermi liquid with non-zero polarization which was recently analyzed by Son and Stephanov [21]. Introducing dimensionless variables

$$x = \frac{j_K}{a\Delta}, \quad h = \frac{\mu_s - 2\Delta}{a\Delta},$$

we can write $E = cn f_h(x)$ with

$$f_h(x) = x^2 - \frac{1}{x} \left( (h + x)^{5/2} \Theta(h + x) - (h - x)^{5/2} \Theta(h - x) \right).$$

We have defined the constants

$$c = \frac{8^4}{2 \cdot 15^4 c_s^2 v_s^2}, \quad N = \frac{\mu_s^2 \Delta^2}{\pi^2}, \quad a = \frac{8^2}{15^2 c_s^2 v_s^2},$$

where $c_s = (21 - 8 \log(2)) / 36$ is the numerical coefficient that appears in the weak coupling result for $f_h$. According to the analysis in [20] the function $f_h(x)$ develops a non-trivial minimum if $h_1 < h < h_2$ with $h_1 \approx -0.067$ and $h_2 \approx 0.502$. In perturbation theory we find $a = 13.9$ and the kaon condensed ground state becomes unstable for $(\Delta - \mu_s/2) < 0.46\Delta$.

We will see that the instability is weaker if the axial coupling is taken into account. The axial vector potential and the axial vector current can be written as

$$\rho_A + \vec{v} \cdot \vec{A} = \frac{i}{2} (\mu_s + \vec{v} \cdot \vec{j}_K) \left( e^{i\phi_K} \hat{u}^+ - e^{-i\phi_K} \hat{u}^- \right),$$

where $\hat{u}^\pm = (\lambda_\theta \pm i \lambda_\gamma)/\sqrt{2}$ are the u-spin raising and lowering operators. The axial vector coupling leads to mixing among the charged $(p, \Xi^0, \Sigma^0, \Lambda^0, \Lambda^{0.8})$ baryons. Gapless fermions occur in the charged sector. The mixing in the charged sector is

$$\mathcal{L} = \frac{i}{2} (\mu_s + \vec{v} \cdot \vec{j}_K) \left\{ e^{i\phi_K} (\Sigma^0 - \Sigma^-) - e^{-i\phi_K} (\Sigma^0 - \Sigma^-) \right\}.$$  

The dispersion relation of the lowest mode is given by

$$\omega_l = \Delta + \frac{(l_0 - l)^2}{2\Delta} - \frac{3}{4} \mu_s - \frac{1}{4} \vec{v} \cdot \vec{j}_K,$$

where $l_0 = (\mu_s + \vec{v} \cdot \vec{j}_K)/4$. The calculation of the energy density is completely analogous to the one we performed in the previous section. The only difference is that the number of gapless modes is reduced, and that the effect of the current on the energy of the fermion is smaller. The energy functional is $E = cN f_h(x)$ with $x$ defined in eqn. (15) and $h = (3\mu_s - 4\Delta)/(a\Delta)$. The numerical coefficients are $c = 2/(15^4 c_s^2 v_s^2)$ and $a = 2/(15^2 c_s^2 v_s^2)$.

Using the weak coupling values for $v_s$ and $c_s$ we find $a = 0.43$ and the kaon condensed ground state becomes unstable for $(\Delta - 3\mu_s/4) < 0.007\Delta$. In Fig. 1 we show the energy density as a function of the current $j_K$ for several values of $\mu_s$ near the transition. We note that the mechanism for the instability is quite robust, but the value of the onset chemical potential depends sensitively on the values of the low energy constants. We also note that the homogeneous gapless kaon condensed phase becomes stable at even larger values of the effective chemical potential, $(\Delta - 3\mu_s/4) < -0.05$. This result may not be reliable, however, because the contribution from higher fermion modes has to be included.

We note that the ground state has no net current. This is clear from the fact that the ground state satisfies $\delta E / \delta (\nabla \phi_K) = 0$. As a consequence the meson current is canceled by an equal but opposite contribution from gapless fermions. This is analogous to what happens in p-wave pion condensate in nuclear matter [18] or in the LOFF phase [17]. We also expect that the ground state has no chromo-magnetic instabilities. The effective lagrangian given in eqn. (10) is formulated in terms of gauge invariant fields. Quark-hadron continuity implies that we should study the screening length of a $SU(3)_F$ gauge field. For diagonal gauge fields we find

$$m^2_{\nabla} = \frac{\partial^2 E}{\partial j_K^2} \bigg|_{j_K=0} = v_s^2 f_s^2 \left( 1 - \frac{5}{8\sqrt{h}} \theta(h) \right),$$

which shows the magnetic instability for $h > 0$ and has the characteristic square root singularity observed in microscopic calculations [11]. In this work we have calculated the energy functional for arbitrary values of $j_K$, not just the second derivative at the origin. We find that the phase transition to a supercurrent state is first order and that the instability sets in for $h > h_c$ with $h_c < 0$. We also observe that the second derivative at the new minimum is positive and of the same order of magnitude as the screening mass at $h = 0$.

Summary: We have that shown that for $\mu_s \sim 4\Delta/3$, which is the threshold for the appearance of gapless
fermion modes, the CFL phase becomes unstable toward the formation of a meson supercurrent. This state is analogous to the p-wave pion condensate studied in low density nuclear matter. In that context the name p-wave was motivated by the fact that the interaction leads to the formation of a condensate of the form \( \psi \cdot \sigma \psi \) for showing me a draft version of his closely related work. Supercurrent formation is driven by a \( \psi \cdot \sigma \psi \) interaction between baryons and mesons.

Our result suggests that the series of phases encountered in three flavor QCD as the density is lowered starting from an asymptotically large value consists of the CFL phase, an s-wave kaon condensate, and a p-wave kaon condensate. There are many questions that remain to be addressed. In this work we established the presence of an instability but did not determine the exact nature of the new ground state. We have to consider additional currents, the role of other light fermions, and the effects of electric charge neutrality. Once the currents become large the effective field theory description breaks down and more microscopic approaches along the lines of \([25, 26]\) are necessary. These methods might also be useful in order to determine the fate of the meson supercurrent state as the density is lowered even further and \( \mu_s \) approaches \( 2\Delta \). One possibility is that the supercurrent phase evolves into a LOFF-type state.

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