ENERGY STABILITY AND CONVERGENCE OF SAV BLOCK-CENTERED FINITE DIFFERENCE METHOD FOR GRADIENT FLOWS*

XIAOLI LI †, JIE SHEN ‡, AND HONGXING RUI §

Abstract. We present in this paper construction and analysis of a block-centered finite difference method for the spatial discretization of the scalar auxiliary variable Crank-Nicolson scheme (SAV/CN-BCFD) for gradient flows, and show rigorously that scheme is second-order in both time and space in various discrete norms. When equipped with an adaptive time strategy, the SAV/CN-BCFD scheme is accurate and extremely efficient. Numerical experiments on typical Allen-Cahn and Cahn-Hilliard equations are presented to verify our theoretical results and to show the robustness and accuracy of the SAV/CN-BCFD scheme.

Key words. scalar auxiliary variable (SAV), gradient flows, energy stability, block-centered finite difference, error estimates, adaptive time stepping

AMS subject classifications. 65M06, 65M12, 65M15, 35K20, 35K35, 65Z05

1. Introduction. Gradient flows are widely used in mathematical models for problems in many fields of science and engineering, particularly in materials science and fluid dynamics, cf. [1, 2, 31, 21] and the references therein. Therefore it is important to develop efficient and accurate numerical schemes for their simulation. There exists an extensive literature on the numerical analysis of gradient flows, see for instance [3, 14, 7, 10, 23, 9, 15] and the references therein.

In the algorithm design of gradient flows, an important goal is to guarantee the energy stability at the discrete level, in order to capture the correct long-time dynamics of the system and provide enough flexibility for dealing with the stiffness problem induced by the thin interface. Many schemes for gradient flows are based on the traditional fully-implicit or explicit discretization for the nonlinear term, which may suffer from harsh time step constraint due to the thin interfacial width [11, 22]. In order to deal with this problem, the convex splitting approach [18, 24, 16] and linear stabilization approach [17, 22, 28, 33] have been widely used to construct unconditionally energy stable schemes. However, the convex splitting approach usually leads to nonlinear schemes and linear stabilization approach is usually limited to first-order accuracy.

Recently, a novel numerical method, the so called invariant energy quadratization (IEQ), was proposed in [29, 32, 30]. This method is a generalization of the method of Lagrange multipliers or of auxiliary variable. The IEQ approach is remarkable as it permits us to construct linear, unconditionally stable, and second-order unconditionally energy stable schemes for a large class of gradient flows. However, it leads to coupled systems with variable coefficients that may be difficult or expensive to solve. The scalar auxiliary variable (SAV) approach [21, 20] was inspired by the IEQ approach, which inherits its main advantages but overcomes many of its shortcomings. In particular, in a recent paper [19], the authors established the first-order convergence and error estimates for the semi-discrete SAV scheme.

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†School of Mathematics, Shandong University, Jinan 250100, China. Email: xiaolisdu@163.com.
‡Corresponding Author. Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA. Email: shen7@purdue.edu.
§School of Mathematics, Shandong University, Jinan 250100, China. Email: hxrui@sdu.edu.cn.
In this paper, we construct a SAV/CN scheme with block-centered finite differences for gradient flows, carried out a rigorous stability and error analysis, and implemented an adaptive time stepping strategy so that the time step is only dictated by accuracy rather than by stability. The block-centered finite difference method can be thought as the lowest order Raviart-Thomas mixed element method with a suitable quadrature. Its main advantage over using a regular finite difference method is that it can approximate both the phase function and chemical potential with Neumann boundary conditions in the mixed formulation to second-order accuracy, and it guarantees local mass conservation. Our approach for error estimates here is very different from that in [19] which is based on deriving $H^2$ bounds for the numerical solution. However, this approach can not be used in the fully discrete case with finite-differences in space. The essential tools used in the proof are the summation-by-parts formulae both in space and time to derive energy stability, and an induction process to show that the discrete $L^\infty$ norm of the numerical solution is uniformly bounded, without assuming a uniform Lipschitz condition on the nonlinear potential. To the best of the authors’ knowledge, this is the first paper with rigorous proof of second-order convergence both in time and space for a linear scheme to a class of gradient flows without assuming a uniform Lipschitz condition for the nonlinear potential.

The paper is organized as follows. In Section 2, we describe our numerical scheme, including the temporal discretization and spatial discretization. In Section 3, we demonstrate the energy stability for our SAV/CN-BCFD scheme. In Section 4, we carry out error estimates for the SAV/CN-BCFD schemes. In Section 5, we present some numerical experiments to verify the energy stability and accuracy of the proposed schemes.

Throughout the paper we use $C$, with or without subscript, to denote a positive constant, which could have different values at different places.

2. The SAV/CN-BCFD scheme. Given a typical energy functional [19]:

$$E(\phi) = \int_{\Omega} \left( \frac{\lambda}{2} \phi^2 + \frac{1}{2} |\nabla \phi|^2 \right) dx + E_1(\phi), \quad (2.1)$$

where $\Omega$ is a rectangular domain in $\mathbb{R}^2$, $\lambda \geq 0$ and $E_1(\phi) = \int_{\Omega} F(\phi) dx \geq -c_0$ for some $c_0 > 0$, i.e., it is bounded from below. We consider the following gradient flow:

$$\begin{align*}
\frac{\partial \phi}{\partial t} &= MG \mu, \quad \text{in } \Omega \times J, \\
\mu &= -\Delta \phi + \lambda \phi + F'(\phi), \quad \text{in } \Omega \times J, 
\end{align*} \quad (2.2)$$

where $J = (0, T]$, and $T$ denotes the final time. $M$ is the mobility constant which is positive. The chemical potential $\mu = \frac{\partial E}{\partial \phi}$. $G = -1$ for the $L^2$ gradient flow and $G = \Delta$ for the $H^{-1}$ gradient flow. $F(\phi)$ is the nonlinear free energy density and we focus on as an example, when $E_1(\phi) = \int_{\Omega} \alpha (1 - \phi^2)^2 dx$, the $L^2$ and $H^{-1}$ gradient flows are the well-known Allen-Cahn and Cahn-Hilliard equations, respectively.

The boundary and initial conditions are as follows.

$$\begin{align*}
\partial_n \phi |_{\partial \Omega} &= 0, \quad \partial_n \mu |_{\partial \Omega} = 0, \\
\phi |_{t=0} &= \phi_0, 
\end{align*} \quad (2.3)$$

where $n$ is the unit outward normal vector of the domain $\Omega$. The equation satisfies the following energy dissipation law:

$$\frac{dE}{dt} = \int_{\Omega} \frac{\partial \phi}{\partial t} \mu dx = M \int_{\Omega} \mu G \mu dx \leq 0. \quad (2.4)$$
2.1. The semi discrete SAV/CN scheme. We recall the SAV/CN scheme introduced in [21] first.

Let $C_0 > c_0$ so that $E_1(\phi) + C_0 > 0$. Without loss of generality, we substitute $E_1$ with $E_1 + C_0$ without changing the gradient flow. Then $E_1$ has a positive lower bound $\tilde{C}_0 = C_0 - c_0$, which we still denote as $C_0$ for simplicity.

In the SAV approach, a scalar variable $r(t) = \sqrt{E_1(\phi)}$ is introduced, and the system (2.2) can be transformed into:

\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= M \mathcal{G} \mu, \\
\mu &= -\Delta \phi + \lambda \phi + \frac{r}{\sqrt{E_1(\phi)}} F'(\phi), \\
r_t &= \frac{1}{2} \int_{\Omega} F'(\phi) \phi_t d\mathbf{x},
\end{align*}
\]

Then, the SAV/CN scheme is given as follows:

\[
\begin{align*}
\frac{\phi^{n+1} - \phi^n}{\Delta t} &= M \mathcal{G} \mu^{n+1/2}, \\
\mu^{n+1/2} &= -\Delta \phi^{n+1/2} + \lambda \phi^{n+1/2} + \frac{r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} F'(\phi^{n+1/2}), \\
\frac{r^{n+1} - r^n}{\Delta t} &= \frac{1}{2} \int_{\Omega} F'(\phi^{n+1/2}) \frac{\phi^{n+1} - \phi^n}{\Delta t} d\mathbf{x},
\end{align*}
\]

where $\phi^{n+1/2} = \frac{1}{2} (\phi^n + \phi^{n+1})$, $r^{n+1/2} = \frac{1}{2} (r^n + r^{n+1})$, $\tilde{\phi}^{n+1/2}$ can be any explicit approximation of $\phi(t^{n+1/2})$ with an error of $O(\Delta t^2)$. For instance, we may let $\tilde{\phi}^{n+1/2}$ be the extrapolation by

\[
\tilde{\phi}^{n+1/2} = \frac{1}{2} (3 \phi^n - \phi^{n-1}).
\]

2.2. Spacial discretization. We apply the BCFD method on the staggered grids for the spacial discretization.

First we give some preliminaries. Let $L^m(\Omega)$ be the standard Banach space with norm

\[
\|v\|_{L^m(\Omega)} = \left( \int_{\Omega} |v|^m d\Omega \right)^{1/m}
\]

For simplicity, let

\[
(f, g) = (f, g)_{L^2(\Omega)} = \int_{\Omega} f g d\Omega
\]

denote the $L^2(\Omega)$ inner product, $\|v\|_{\infty} = \|v\|_{L^\infty(\Omega)}$. And $W^{k,p}(\Omega)$ be the standard Sobolev space

\[
W^{k,p}(\Omega) = \{ g : \|g\|_{W^{k,p}(\Omega)} < \infty \},
\]

where

\[
\|g\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha g\|_{L^p(\Omega)}^p \right)^{1/p}
\]

The grid points are denoted by

\[
(x_{i+1/2}, y_{j+1/2}), \quad i = 0, ..., N_x, \quad j = 0, ..., N_y,
\]
and the notations similar to those in [26] are used.

\[ x_i = (x_{i-1} + x_{i+1})/2, \quad i = 1, \ldots, N_x, \]
\[ h_x = x_{i+1} - x_{i-1}, \quad i = 1, \ldots, N_x, \]
\[ y_j = (y_{j-1} + y_{j+1})/2, \quad j = 1, \ldots, N_y, \]
\[ h_y = y_{j+1} - y_{j-1}, \quad j = 1, \ldots, N_y, \]

where \( h_x \) and \( h_y \) are grid spacings in \( x \) and \( y \) directions, and \( N_x \) and \( N_y \) are the number of grids along the \( x \) and \( y \) coordinates, respectively.

Let \( g_{i,j}, g_{i+rac{1}{2},j}, g_{i,j+rac{1}{2}} \) denote \( g(x_i,y_j), g(x_{i+rac{1}{2}},y_j), g(x_i,y_{j+rac{1}{2}}) \). Define the discrete inner products and norms as follows,

\[ (f,g)_m = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_x h_y f_{i,j} g_{i,j}, \]
\[ (f,g)_x = \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} h_x h_y f_{i+rac{1}{2},j} g_{i+rac{1}{2},j}, \]
\[ (f,g)_y = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y-1} h_x h_y f_{i,j+rac{1}{2}} g_{i,j+rac{1}{2}}, \]
\[ (v,r)_{TM} = (v_1, r_1)_x + (v_2, r_2)_y. \]

For simplicity, from now on we always omit the superscript \( n \) (the time level) if the omission does not cause conflicts. Define

\[ [d_x g]_{i+rac{1}{2},j} = (g_{i+1,j} - g_{i,j})/h_x, \]
\[ [d_y g]_{i,j+rac{1}{2}} = (g_{i,j+1} - g_{i,j})/h_y, \]
\[ [D_x g]_{i,j} = (g_{i+rac{1}{2},j} - g_{i-rac{1}{2},j})/h_x, \]
\[ [D_y g]_{i,j} = (g_{i,j+rac{1}{2}} - g_{i,j-rac{1}{2}})/h_y, \]
\[ [d_t g]_{i,j}^n = (g^n_{i,j} - g^{n-1}_{i,j})/\Delta t. \]

The following discrete-integration-by-part lemma [26] plays an important role in the analysis.

**Lemma 1.** Let \( q_{i,j}, w_{1,i,j+1/2,j} \) and \( w_{2,i,j+1/2,j} \) be any values such that \( w_{1,i,j+1/2,j} = w_{1,N_x+1/2,j} = w_{2,i,1/2} = w_{2,i,N_y+1/2} = 0 \), then

\[ (q, D_x w_1)_m = -(d_x q, w_1)_x, \]
\[ (q, D_y w_2)_m = -(d_y q, w_2)_y. \]

**2.2.1. SAV/CV-BCFD scheme for \( H^{-1} \) gradient flow.** Let us denote by \( \{ Z^n, W^n, R^n \}_{n=0}^N \) the BCFD approximations to \( \{ \phi^n, \mu^n, r^n \}_{n=0}^N \). The scheme for \( H^{-1} \)
gradient flow is as follows: for $1 \leq i \leq N_x$, $1 \leq j \leq N_y$,

\[
\begin{align*}
[d_t Z]^{n+1}_{i,j} &= M[D_x d_x Z + D_y d_y Z]^{n+1/2}_{i,j}, \\
W^{n+1/2}_{i,j} &= -[D_x d_x Z + D_y d_y Z]^{n+1/2}_{i,j} + \lambda Z_{i,j}^{n+1/2} \\
&\quad + \frac{R^{n+1/2}}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})}} F'(\tilde{Z}^{n+1/2}), \\
d_t R^{n+1} &= \frac{1}{2}\frac{2}{E_1^h(\tilde{Z}^{n+1/2})^2} (F'(\tilde{Z}^{n+1/2}), d_t Z^{n+1}_m),
\end{align*}
\tag{2.13}
\]

where $\hat{Z}^{n+1/2}$ is an approximation of $\hat{Z}^{n+1/2}$, and

\[
E_1^h(\hat{Z}^{n+1/2}) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_x h_y F(\hat{Z}^{n+1/2}_{i,j}).
\]

The boundary and initial approximations as follows.

\[
\begin{align*}
[d_x Z]^{n+1/2}_{i,j} &= [d_x Z]^{n+1/2}_{i,j} = 0, & 1 \leq j \leq N_y, \\
[d_y Z]^{n+1/2}_{i,j} &= [d_y Z]^{n+1/2}_{i,j} = 0, & 1 \leq i \leq N_x, \\
[d_x W]^{n+1/2}_{i,j} &= [d_x W]^{n+1+1/2}_{i,j} = 0, & 1 \leq j \leq N_y, \\
[d_y W]^{n+1/2}_{i,j} &= [d_y W]^{n+1+1/2}_{i,j} = 0, & 1 \leq i \leq N_x, \\
Z^{0}_{i,j} &= \phi_{0,i,j}, & 1 \leq i \leq N_x, 1 \leq j \leq N_y.
\end{align*}
\tag{2.16}
\]

**Remark.** The solution procedure of the above scheme is described in detail in \cite{21, 20}, and hence is omitted here.

### 2.2.2. SAV/CV-BCFD scheme for $L^2$ gradient flow.
Let us denote by \{Z^n, W^n, R^n\} the BCFD approximations to {φ^n, µ^n, r^n} \(_n=0^N\). The scheme for $L^2$ gradient flow is as follows: for $1 \leq i \leq N_x$, $1 \leq j \leq N_y$,

\[
\begin{align*}
[d_t Z]^{n+1}_{i,j} &= -M W^{n+1/2}_{i,j}, \\
W^{n+1/2}_{i,j} &= -[D_x d_x Z + D_y d_y Z]^{n+1/2}_{i,j} + \lambda Z^{n+1/2}_{i,j} \\
&\quad + \frac{R^{n+1/2}}{\sqrt{E_1^h(\tilde{Z}^{n+1/2})}} F'(\tilde{Z}^{n+1/2}), \\
d_t R^{n+1} &= \frac{1}{2}\frac{2}{E_1^h(\tilde{Z}^{n+1/2})^2} (F'(\tilde{Z}^{n+1/2}), d_t Z^{n+1}_m),
\end{align*}
\tag{2.19}
\]

where $\hat{Z}^{n+1/2}$ is an approximation of $\hat{Z}^{n+1/2}$. The boundary and initial conditions are given in (2.16).

### 3. Unconditional energy stability.
We demonstrate below that the full discrete SAV/CN-BCFD schemes are unconditionally energy stable with the discrete energy functional

\[
E_d(Z^n) = \frac{\lambda}{2} ||Z^n||^2_m + \frac{1}{2} ||dZ^n||^2_{TM} + ||R^n||^2,
\tag{3.1}
\]

where $dZ = (d_x Z, d_y Z)$. 

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\section{H^{-1} gradient flow.}

\textbf{Theorem 2.} The scheme (2.13)-(2.15) is unconditionally stable and the following discrete energy law holds for any $\Delta t$:

\begin{equation}
\frac{1}{\Delta t} [E_d(Z^{n+1}) - E_d(Z^n)] = -M \|dW^{n+1/2}\|_{T,M}^2, \quad \forall n \geq 0. \tag{3.2}
\end{equation}

\textit{Proof.} Multiplying equation (2.13) by $W_{i,j}^{n+1/2}h_xh_y$, and making summation on $i,j$ for $1 \leq i \leq N_x, 1 \leq j \leq N_y$, we have

\begin{equation}
(d_tZ^{n+1},W^{n+1/2})_m = M(D_xd_xW^{n+1/2} + D_yd_yW^{n+1/2},W^{n+1/2})_m. \tag{3.3}
\end{equation}

Using Lemma 1, equation (3.3) can be transformed into the following:

\begin{equation}
(d_tZ^{n+1},W^{n+1/2})_m = -M(\|d_xW^{n+1/2}\|_x^2 + \|d_yW^{n+1/2}\|_y^2) = -M\|dW^{n+1/2}\|_{T,M}^2. \tag{3.4}
\end{equation}

Multiplying equation (2.14) by $d_tZ^{n+1}h_xh_y$, and making summation on $i,j$ for $1 \leq i \leq N_x, 1 \leq j \leq N_y$, we have

\begin{equation}
(d_tZ^{n+1},W^{n+1/2})_m = -(D_xd_xZ^{n+1/2} + D_yd_yZ^{n+1/2},d_tZ^{n+1})_m
+ \frac{R^{n+1/2}}{\sqrt{E_1^b(Z^{n+1/2})}}(F'(\tilde{Z}^{n+1/2}),d_tZ^{n+1})_m
+ \lambda(Z^{n+1/2},d_tZ^{n+1})_m. \tag{3.5}
\end{equation}

Using Lemma 1 again, the first term on the right hand side of equation (3.5) can be written as:

\begin{equation}
-(D_xd_xZ^{n+1/2} + D_yd_yZ^{n+1/2},d_tZ^{n+1})_m = \|dZ^{n+1}\|_{T,M}^2 - \|dZ^n\|_{T,M}^2 \tag{3.6}
\end{equation}

Multiplying equation (2.15) by $R^{n+1} + R^n$ leads to

\begin{equation}
\frac{(R^{n+1})^2 - (R^n)^2}{\Delta t} = \frac{R^{n+1/2}}{\sqrt{E_1^b(Z^{n+1/2})}}(F'(\tilde{Z}^{n+1/2}),d_tZ^{n+1})_M. \tag{3.7}
\end{equation}

Combining equation (3.7) with equations (3.4) - (3.6) gives that

\begin{equation}
\frac{(R^{n+1})^2 - (R^n)^2}{\Delta t} + \lambda \|Z^{n+1}\|_{m}^2 - \|Z^n\|_{m}^2
\end{equation}

\begin{equation}
\frac{\|dZ^{n+1}\|_{T,M}^2 - \|dZ^n\|_{T,M}^2}{2\Delta t}
\end{equation}

\begin{equation}
= -M\|dW^{n+1/2}\|_{T,M}^2 \leq 0,
\end{equation}

which implies the desired results (3.2). \hfill \Box
3.2. \( L^2 \) gradient flow. For \( L^2 \) gradient flow, we shall only state the result, as its proof is essentially the same as for the \( H^{-1} \) gradient flow.

**Theorem 3.** The scheme (2.17)-(2.19) is unconditionally stable and the following discrete energy law holds for any \( \Delta t \):

\[
\frac{1}{\Delta t}[E_d(Z^{n+1}) - E_d(Z^n)] = -M\|W^{n+1/2}\|^2_m, \quad \forall n \geq 0. \tag{3.9}
\]

4. Error estimates. In this section, we derive our main results of this paper, i.e., error estimates for the fully discrete SAV/CN-BCFD schemes.

For simplicity, we set

\[
e^n_\phi = Z^n - \phi^n, \quad e^n_\mu = W^n - \mu^n, \quad e^n_r = R^n - r^n.
\]

4.1. \( H^{-1} \) gradient flow. We shall first derive error estimates for the case of \( H^{-1} \) gradient flow.

**Theorem 4.** We assume that \( F(\phi) \in C^3(\mathbb{R}) \) and \( \phi \in W^{1,\infty}(J;W^{4,\infty}(\Omega)) \cap W^{3,\infty}(J;W^{1,\infty}(\Omega)), \mu \in L^\infty(J;W^{4,\infty}(\Omega)). \) Let \( \Delta t \leq C(h_x + h_y), \) then for the discrete scheme (2.13)-(2.15), there exists a positive constant \( C \) independent of \( h_x, h_y \) and \( \Delta t \) such that

\[
\begin{align*}
\|Z^{k+1} - \phi^{k+1}\|_m &+ \|dZ^{k+1} - d\phi^{k+1}\|_{TM} + |R^{k+1} - r^{k+1}| \\
+ & \left( \sum_{n=0}^k \Delta t \|dW^{n+1/2} - d\mu^{n+1/2}\|_{TM}^2 \right)^{1/2} \\
+ & \left( \sum_{n=0}^k \Delta t \|W^{n+1/2} - \mu^{n+1/2}\|_m^2 \right)^{1/2} \\
\leq & C(\|\phi\|_{W^{1,\infty}(J;W^{4,\infty}(\Omega))} + \|\mu\|_{L^\infty(J;W^{4,\infty}(\Omega))})(h_x^2 + h_y^2) \\
+ & C\|\phi\|_{W^{3,\infty}(J;W^{1,\infty}(\Omega))}\Delta t^2.
\end{align*}
\]

We shall split the proof of the above results into three lemmas below.

**Lemma 5.** Under the condition of Theorem 4, there exists a positive constant \( C \) independent of \( h_x, h_y \) and \( \Delta t \) such that

\[
\begin{align*}
(e^{k+1})_r^2 + & \frac{1}{2} \|d_{\phi}^{k+1}\|^2_{TM} + \frac{1}{2} \|e^{k+1}\|^2_{e\phi} + \frac{M}{2} \sum_{n=0}^k \Delta t \|d_{\mu}^{n+1/2}\|^2_{TM} \\
\leq & C \sum_{n=0}^{k+1} \Delta t \|d_{\phi}^{n}\|^2_{TM} + \frac{M}{2} \sum_{n=0}^{k+1} \Delta t \|e^{n+1/2}\|^2_{e\phi} \\
+ & C \sum_{n=0}^{k+1} \Delta t \|e^{n}\|^2_{e\phi} + C \sum_{n=0}^{k+1} \Delta t (e^n_r)^2 \\
+ & C(\|\phi\|_{W^{1,\infty}(J;W^{4,\infty}(\Omega))} + \|\mu\|_{L^\infty(J;W^{4,\infty}(\Omega))})(h_x^4 + h_y^4) \\
+ & C\|\phi\|_{W^{3,\infty}(J;W^{1,\infty}(\Omega))}\Delta t^4.
\end{align*}
\]

**Proof.** Denote

\[
\begin{align*}
\delta_x(\phi) & = d_x \phi - \frac{\partial \phi}{\partial x}, \quad \delta_y(\phi) = d_y \phi - \frac{\partial \phi}{\partial y}, \\
\delta_x(\mu) & = d_x \mu - \frac{\partial \mu}{\partial x}, \quad \delta_y(\mu) = d_y \mu - \frac{\partial \mu}{\partial y}.
\end{align*}
\]
Subtracting equation (2.5) from equation (2.13), we obtain

\[ [d_t e_{\phi}]_{i,j}^{n+1} = M[D_x (d_x e_{\mu} + \delta_x (\mu)) + D_y (d_y e_{\mu} + \delta_y (\mu))]_{i,j}^{n+1/2} + T_{1,i,j}^{n+1/2} + T_{2,i,j}^{n+1}, \]

where

\[ T_{1,i,j}^{n+1/2} = \frac{\partial \phi}{\partial t} |_{i,j}^{n+1/2} - [d_t \phi]_{i,j}^{n+1} \leq C \| \phi \|_{W^{3,\infty}(J;L^\infty(\Omega))} \Delta t^2, \]

\[ T_{2,i,j}^{n+1/2} = M[D_x \frac{\partial \mu}{\partial x} + D_y \frac{\partial \mu}{\partial y}]_{i,j}^{n+1/2} - M \Delta \mu_{i,j}^{n+1/2} \leq CM (h_x^2 + h_y^2) \| \mu \|_{L^\infty(J;W^{4,\infty}(\Omega))}. \]

Subtracting equation (2.6) from equation (2.14) leads to

\[ e_{\mu,i,j}^{n+1/2} = - [D_x (d_x e_{\phi} + \delta_x (\phi)) + D_y (d_y e_{\phi} + \delta_y (\phi))]_{i,j}^{n+1/2} + \lambda e_{\phi,i,j}^{n+1/2} + \frac{R_{i,j}^{n+1/2} F' (\tilde{Z}_{i,j}^{n+1/2})}{\sqrt{E_1 (\tilde{Z}_{i,j}^{n+1/2})}} \]
\[ - \frac{r_{i,j}^{n+1/2}}{\sqrt{E_1 (\phi_{i,j}^{n+1/2})}} F' (\phi_{i,j}^{n+1/2}) + T_{3,i,j}^{n+1/2}, \]

where

\[ T_{3,i,j}^{n+1/2} = \Delta \phi_{i,j}^{n+1/2} - [D_x \frac{\partial \phi}{\partial x} + D_y \frac{\partial \phi}{\partial y}]_{i,j}^{n+1/2} \leq C (h_x^2 + h_y^2) \| \phi \|_{L^\infty(J;W^{4,\infty}(\Omega))}. \]

Subtracting equation (2.7) from equation (2.15) gives that

\[ d_t e_{r}^{n+1} = \frac{1}{2 \sqrt{E_1 (\tilde{Z}_{i,j}^{n+1/2})}} \left( F' (\tilde{Z}_{i,j}^{n+1/2}, d_t Z^{n+1})_m \right) \]
\[ - \frac{1}{2 \sqrt{E_1 (\phi_{i,j}^{n+1/2})}} \int_\Omega F' (\phi_{i,j}^{n+1/2}) \phi_{i,j}^{n+1/2} \, dx + T_{4,i,j}^{n+1/2}, \]

where

\[ T_{4,i,j}^{n+1/2} = r_{i,j}^{n+1/2} - d_t e_{r}^{n+1} \leq C \| r \|_{W^{3,\infty}(J)} \Delta t^2. \]

Multiplying equation (4.3) by \( e_{\mu,i,j}^{n+1/2} h_x h_y \), and making summation on \( i,j \) for \( 1 \leq i \leq N_x, 1 \leq j \leq N_y \), we have

\[ (d_t e_{\phi}^{n+1}, e_{\mu,i,j}^{n+1/2})_m = M \left( D_x (d_x e_{\mu} + \delta_x (\mu))^{n+1/2} + D_y (d_y e_{\mu} + \delta_y (\mu))^{n+1/2}, e_{\mu,i,j}^{n+1/2} \right)_m \]
\[ + (T_{1}^{n+1/2}, e_{\mu,i,j}^{n+1/2})_m + (T_{2}^{n+1/2}, e_{\mu,i,j}^{n+1/2})_m. \]

Using Lemma 1, we can write the first term on the right hand side of equation (4.10)
Thanks to Cauchy-Schwarz inequality, the last two terms on the right hand side of equation (4.11) can be transformed into:

\[
-M(\delta_x(\mu)^{n+1/2}, d_x e_\mu^{n+1/2})_x - M(\delta_y(\mu)^{n+1/2}, d_y e_\mu^{n+1/2})_y
\]

\[
\leq M \|d\mu^{n+1/2}\|_{L_M}^2 + C\|\mu\|_{L^{\infty}(\Omega)}^2 (h_x^4 + h_y^4).
\]  

(4.12)

Multiplying equation (4.6) by \(d_t e_{\phi,i,j}^{n+1} h_x h_y\), and making summation on \(i, j\) for \(1 \leq i \leq N_x, 1 \leq j \leq N_y\), we have

\[
(e_{\phi,i,j}^{n+1/2}, d_t e_{\phi,i,j}^{n+1})_m = -(D_x (d_x e_\phi + \delta_x(\phi))^{n+1/2} + D_y (d_y e_\phi + \delta_y(\phi))^{n+1/2}, d_t e_{\phi,i,j}^{n+1})_m
\]

\[
+ \left(\frac{R^{n+1/2}}{E_1(\tilde{Z}^{n+1/2})} F'((\tilde{Z})^{n+1/2}) - \frac{r^{n+1/2}}{E_1(\tilde{Z}^{n+1/2})} F'(\phi^{n+1/2}), d_t e_{\phi,i,j}^{n+1}\right)_m
\]

\[
+ \lambda e_{\phi,i,j}^{n+1/2}, d_t e_{\phi,i,j}^{n+1})_m + (T_3^{n+1/2}, d_t e_{\phi,i,j}^{n+1})_m.
\]  

(4.13)

Similar to the estimate of equation (3.6), the first term on the right hand side of equation (4.13) can be transformed into the following:

\[
-(D_x (d_x e_\phi + \delta_x(\phi))^{n+1/2} + D_y (d_y e_\phi + \delta_y(\phi))^{n+1/2}, d_t e_{\phi,i,j}^{n+1})_m
\]

\[
= (d_x e_{\phi,i,j}^{n+1/2}, d_t d_x e_{\phi,i,j}^{n+1})_x + (d_y e_{\phi,i,j}^{n+1/2}, d_t d_y e_{\phi,i,j}^{n+1})_y
\]

\[
+ \left(\delta_x(\phi)^{n+1/2}, d_t d_x e_{\phi,i,j}^{n+1/2}\right)_x + \left(\delta_y(\phi)^{n+1/2}, d_t d_y e_{\phi,i,j}^{n+1/2}\right)_y
\]

\[
\leq \frac{\|d\phi^{n+1/2}\|_{L_M}^2}{2\Delta t} + \frac{\|d\phi^{n+1/2}\|_{L_M}^2}{2\Delta t} + (\delta_x(\phi)^{n+1/2}, d_t d_x e_{\phi,i,j}^{n+1/2})_x
\]

\[
+ (\delta_y(\phi)^{n+1/2}, d_t d_y e_{\phi,i,j}^{n+1/2})_y.
\]  

(4.14)

The second term on the right hand side of equation (4.13) can be rewritten as follows:

\[
\left(\frac{R^{n+1/2}}{E_1(\tilde{Z}^{n+1/2})} F'((\tilde{Z})^{n+1/2}) - \frac{r^{n+1/2}}{E_1(\tilde{Z}^{n+1/2})} F'(\phi^{n+1/2}), d_t e_{\phi,i,j}^{n+1}\right)_m
\]

\[
= r^{n+1/2} \left(\frac{F'((\tilde{Z})^{n+1/2})}{E_1(\tilde{Z}^{n+1/2})} - \frac{F'(\phi^{n+1/2})}{E_1(\phi^{n+1/2})}ight)_m
\]

\[
+ r^{n+1/2} \left(\frac{F'(\phi^{n+1/2})}{E_1(\phi^{n+1/2})} - \frac{F'((\tilde{Z})^{n+1/2})}{E_1((\tilde{Z})^{n+1/2})}\right)_m
\]

\[
+ r^{n+1/2} \left(\frac{F'((\tilde{Z})^{n+1/2})}{E_1((\tilde{Z})^{n+1/2})}, d_t e_{\phi,i,j}^{n+1}\right)_m.
\]  

(4.15)
Recalling equation (4.3), the first term on the right hand side of equation (4.15) can be transformed into the following:

\[
M^{r+1/2}F'(\tilde{Z}^{r+1/2}) \sqrt{E_1^b(\tilde{Z}^{r+1/2})} \leq M^{r+1/2} \left( \frac{F'(\tilde{Z}^{r+1/2})}{\sqrt{E_1^b(\tilde{Z}^{r+1/2})}} - \frac{F'(\tilde{\phi}^{r+1/2})}{\sqrt{E_1^b(\tilde{\phi}^{r+1/2})}} \right),
\]

where \(d_x F'(\tilde{Z}^{r+1/2})\) is the derivative of \(F\) evaluated at \(\tilde{Z}\) and \(d_x F'(\tilde{\phi}^{r+1/2})\) is the derivative of \(F\) evaluated at \(\tilde{\phi}\). Note that \(d_x F'(\tilde{Z}^{r+1/2})\) and \(d_x F'(\tilde{\phi}^{r+1/2})\) are both positive.

Using the Cauchy-Schwarz inequality, we can deduce that

\[
\left\| Z^n \right\|_\infty \leq C_*.
\]

This hypothesis will be verified in Lemma 7 using a bootstrap argument.

Finally, we shall first make the hypothesis that there exists a positive constant \(C_*\) such that

\[
\| Z^n \|_\infty \leq C_*.\]

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Similarly we can obtain
\[
M r^{n+1/2} \left( \frac{F'(\hat{Z}^{n+1/2})}{E_1^{h}(\hat{Z}^{n+1/2})} - \frac{F'(\hat{\phi}^{n+1/2})}{E_1^{h}(\hat{\phi}^{n+1/2})} \right), d_t e_\phi^{n+1/2} m \\
\leq \frac{M}{6} \left\| d_\mu e^{n+1/2}_\mu \right\|_H^2 + C \| r \|_{L^\infty(J)} \left( \| e^{n}_\phi \|_m^2 + \| e^{n-1}_\phi \|_m^2 \right) \\
+ C \| r \|_{L^\infty(J)} \left( \| d_\gamma e^{n}_\gamma \|_2^2 + \| d_\gamma e^{n-1}_\gamma \|_2^2 \right) \\
+ C \| \| \|_{L^\infty(J;W^3,\infty(\Omega))} (\hat{h}_x^4 + \hat{h}_y^4). 
\]

Then equation (4.16) can be estimated by:
\[
r^{n+1/2} \left( \frac{F'(\hat{Z}^{n+1/2})}{E_1^{h}(\hat{Z}^{n+1/2})} - \frac{F'(\hat{\phi}^{n+1/2})}{E_1^{h}(\hat{\phi}^{n+1/2})} \right), d_t e^{n+1}_\phi m \\
\leq \frac{M}{6} \left\| d_\mu e^{n+1/2}_\mu \right\|_H^2 + C \| r \|_{L^\infty(J)} \left( \| e^{n}_\phi \|_m^2 + \| e^{n-1}_\phi \|_m^2 \right) \\
+ C \| r \|_{L^\infty(J)} \left( \| d_\gamma e^{n}_\gamma \|_H^2 + \| d_\gamma e^{n-1}_\gamma \|_H^2 \right) \\
+ C \| \| \|_{L^\infty(J;W^3,\infty(\Omega))} (\hat{h}_x^4 + \hat{h}_y^4) + C \| \|_{W^3,\infty(J;L^\infty(\Omega))} \Delta t^4. 
\]

Similar to (4.16), the second term on the right hand side of equation (4.15) can be controlled by:
\[
r^{n+1/2} \left( \frac{F'(\hat{Z}^{n+1/2})}{E_1^{h}(\hat{Z}^{n+1/2})} - \frac{F'(\hat{\phi}^{n+1/2})}{E_1^{h}(\hat{\phi}^{n+1/2})} \right), d_t e^{n+1}_\phi m \\
\leq \frac{M}{6} \left\| d_\mu e^{n+1/2}_\mu \right\|_H^2 + C \| r \|_{L^\infty(J)} \left( \| h_x^4 + \hat{h}_y^4 \right) \\
+ C \| \| \|_{L^\infty(J;W^3,\infty(\Omega))} (\hat{h}_x^4 + \hat{h}_y^4) \\
+ C \| \|_{W^3,\infty(J;W^1,\infty(\Omega))} \Delta t^4. 
\]

The third term on the right hand side of equation (4.13) can be estimated by:
\[
\lambda(e^{n+1/2}_\phi, d_t e^{n+1}_\phi)_m = \lambda \frac{\| e^{n+1}_\phi \|_m^2 - \| e^{n}_\phi \|_m^2}{2 \Delta t}. 
\]

Multiplying equation (4.8) by $e^{n+1}_r + e^n_r$ leads to
\[
\frac{(e^{n+1}_r)^2 - (e^n_r)^2}{\Delta t} = \frac{e^{n+1}_r}{E_1^{h}(\hat{Z}^{n+1/2})} (F'(\hat{Z}^{n+1/2}), d_t Z^{n+1})_m \\
- \frac{e^{n+1}_r}{E_1^{h}(\hat{\phi}^{n+1/2})} \int_\Omega F'(\hat{\phi}^{n+1/2}) \phi_t^{n+1/2} d\mathbf{x} \\
+ T^{n+1/2}_4 \cdot (e^{n+1}_r + e^n_r). 
\]
\[
\frac{e_r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} (F'(Z^{n+1/2}), d_t Z^{n+1})_m - \frac{e_r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} \int_{\Omega} F'(\phi^{n+1/2}) \phi_t^{n+1/2} \, dx
\]
\[
= \frac{e_r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} \left( (F'(\phi^{n+1/2}), d_t \phi^{n+1})_m - \int_{\Omega} F'(\phi^{n+1/2}) \phi_t^{n+1/2} \, dx \right)
\]
\[
+ \frac{e_r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} (F'(\tilde{Z}^{n+1/2}), d_t \phi^{n+1})_m
\]
\[
+ e_r^{n+1/2} \frac{F'(\tilde{Z}^{n+1/2})}{\sqrt{E_1(\tilde{Z}^{n+1/2})}} - \frac{F'(\phi^{n+1/2})}{\sqrt{E_1(\phi^{n+1/2})}}, d_t \phi^{n+1})_m.
\]

Since \( F(\phi) \in C^3(\mathbb{R}) \), we have that

\[
\frac{e_r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} (F'(\tilde{Z}^{n+1/2}), d_t \phi^{n+1})_m - \frac{e_r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} \int_{\Omega} F'(\phi^{n+1/2}) \phi_t^{n+1/2} \, dx
\]
\[
\leq C(e_r^{n+1/2})^2 + C\|\phi\|_{W^{1,\infty}(J; L^{\infty}(\Omega))}(\|e_\phi^n\|_m^2 + \|e_\phi^{n-1}\|_m^2).
\]

Recalling the midpoint approximation property of the rectangle quadrature formula, we can obtain that

\[
\frac{e_r^{n+1/2}}{\sqrt{E_1(\phi^{n+1/2})}} \left( (F'(\phi^{n+1/2}), d_t \phi^{n+1})_m - \int_{\Omega} F'(\phi^{n+1/2}) \phi_t^{n+1/2} \, dx \right)
\]
\[
\leq C(e_r^{n+1/2})^2 + C\|\phi\|_{W^{1,\infty}(J; L^{\infty}(\Omega))}(h_x^2 + h_y^2).
\]

Combining equation (4.24) with equations (4.10)-(4.27) and using Cauchy-Schwarz inequality result in

\[
\frac{(e_r^{n+1})^2 - (e_r^n)^2}{\Delta t} + \frac{\|d\phi^{n+1}\|_{T_M}^2 - \|d\phi^n\|_{T_M}^2}{2\Delta t}
\]
\[
+ \lambda \frac{\|e_\phi^{n+1}\|_m^2 - \|e_\phi^n\|_m^2}{2\Delta t} + M\|d\phi^{n+1/2}\|_{T_M}^2
\]
\[
\leq \frac{M}{2} \|d\phi^{n+1/2}\|_{T_M}^2 + C\|r\|_{L^\infty(J)}^2(\|e_\phi^n\|_m^2 + \|e_\phi^{n-1}\|_m^2)
\]
\[
+ C\|r\|_{L^\infty(J)}^2(\|d\phi^{n+1/2}\|_{T_M}^2 + \|d\phi^{n-1}\|_{T_M}^2)
\]
\[
- (\delta_x(\phi)^{n+1/2}, d_x e_\phi^{n+1/2})_x - (\delta_y(\phi)^{n+1/2}, d_y e_\phi^{n+1/2})_y
\]
\[
- (T_1^{n+1/2}, d_t e_\phi^{n+1})_m - (T_2^{n+1/2}, e_\mu^{n+1/2})_m
\]
\[
- (T_1^{n+1/2}, e_\mu^{n+1/2})_m - T_1^{n+1/2} \cdot (e_r^{n+1} + e_r^n)
\]
+ C(\epsilon_r n^{1/2})^2 + C||\phi||^2_{W^{1,\infty}(J; L^\infty(\Omega))} (||e_\phi^n||^2_m + ||e_\phi^{n-1}||^2_m) \\
+ C(||\phi||^2_{W^{1,\infty}(J; L^\infty(\Omega))} + ||\mu||^2_{L^\infty(J; W^{4,\infty}(\Omega))})(h_x^4 + h_y^4) \\
+ C||\phi||^2_{W^{3,\infty}(J; L^\infty(\Omega))}\Delta t^4. 
\tag{4.29}

From the discrete-integration-by-parts,

\[ \sum_{n=0}^{k} \Delta t(f^n, d_t g^{n+1}) = -\sum_{n=1}^{k} \Delta t(d_t f^n, g^n) \]

\[ + (f^k, g^{k+1}) + (f^0, g^0). \]

we find

\[ \sum_{n=0}^{k} \Delta t(T_3^{n+1/2}, d_t e_\phi^{n+1}) \]

\[ = -\sum_{n=1}^{k} \Delta t(d_t T_3^{n+1/2}, e_\phi^n) + (T_3^{k+1/2}, e_\phi^{k+1}) + (T_3^{1/2}, e_\phi^0) \]

\[ \leq C \sum_{n=1}^{k} \Delta t ||e_\phi^n||^2_m + \frac{\lambda}{4} ||e_\phi^{k+1}||^2_m + C||\phi||^2_{W^{1,\infty}(J; W^{4,\infty}(\Omega))}(h_x^4 + h_y^4). \]

Similarly we have

\[ -\sum_{n=0}^{k} \Delta t(\delta_x(\phi) n^{1/2}, d_t d_x e_\phi^{n+1/2})_x - \sum_{n=0}^{k} \Delta t(\delta_y(\phi) n^{1/2}, d_t d_y e_\phi^{n+1/2})_y \]

\[ \leq C \sum_{n=1}^{k} \Delta t ||d e_\phi^n||^2_{T_M} + \frac{\lambda}{4} ||e_\phi^{k+1}||^2_m + C||\phi||^2_{W^{1,\infty}(J; W^{3,\infty}(\Omega))}(h_x^4 + h_y^4). \]

Multiplying equation (4.28) by \(\Delta t\), summing over \(n\), \(n = 0, 1, \ldots, k\) and combining with equations (4.31) and (4.32), we can obtain (4.2).

\textbf{Lemma 6.} Under the condition of Theorem 4, there exists a positive constant \(C\) independent of \(h_x\), \(h_y\) and \(\Delta t\) such that

\[ ||e_\phi^{k+1}||^2_m + M \sum_{n=0}^{k} \Delta t ||e_\phi^{n+1/2}||^2_m \]

\[ \leq C \sum_{n=0}^{k} \Delta t (e_r^{n+1})^2 + C \sum_{n=0}^{k} \Delta t ||e_\phi^n||^2_m \]

\[ + \frac{M}{4} \sum_{n=0}^{k} \Delta t ||d e_\phi^{n+1/2}||^2_{T_M} + C \sum_{n=0}^{k} \Delta t ||d e_\phi^{n+1/2}||^2_{T_M} \]

\[ + C(||\mu||^2_{L^\infty(J; W^{4,\infty}(\Omega))} + ||\phi||^2_{L^\infty(J; W^{4,\infty}(\Omega))})(h_x^4 + h_y^4) \]

\[ + C||\phi||^2_{W^{3,\infty}(J; L^\infty(\Omega))}\Delta t^4. \]

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Proof. Multiplying equation (4.3) by $e^{n+1/2}_\phi i,j h_x h_y$, and making summation on $i,j$ for $1 \leq i \leq N_x$, $1 \leq j \leq N_y$, we have

$$(d_x e^{n+1/2}_\phi, e^{n+1/2}_\phi)_m$$

$$= M \left( D_x (d_x e_\mu + \delta_x(\mu))^{n+1/2} + D_y (d_y e_\mu + \delta_y(\mu))^{n+1/2}, e^{n+1/2}_\phi \right)_m$$

$$+ (T_1^{n+1/2}, e^{n+1/2}_\phi)_m + (T_2^{n+1/2}, e^{n+1/2}_\phi)_m.$$ \hfill (4.34)

Using Lemma 1, the first term on the right hand side of equation (4.34) can be transformed into the following:

$$M \left( D_x (d_x e_\mu + \delta_x(\mu))^{n+1/2} + D_y (d_y e_\mu + \delta_y(\mu))^{n+1/2}, e^{n+1/2}_\phi \right)_m$$

$$- M \left( (d_x e_\mu + \delta_x(\mu))^{n+1/2}, d_x e^{n+1/2}_\phi \right)_x$$

$$- M \left( (d_y e_\mu + \delta_y(\mu))^{n+1/2}, d_y e^{n+1/2}_\phi \right)_y.$$ \hfill (4.35)

The first term on the right hand side of equation (4.35) can be estimated as:

$$- M \left( (d_x e_\mu + \delta_x(\mu))^{n+1/2}, d_x e^{n+1/2}_\phi \right)_x$$

$$= -M \left( (d_x e_\mu + \delta_x(\mu))^{n+1/2}, (d_x e_\phi + \delta_x(\phi))^{n+1/2} \right)_x$$

$$+ M (d_x e^{n+1/2}_\mu, \delta_x(\phi))x - M (\delta_x(\mu)^{n+1/2}, d_x e^{n+1/2}_\phi)_x$$

$$\leq M \left( e^{n+1/2}_\mu, D_x (d_x e_\phi + \delta_x(\phi))^{n+1/2} \right)_m$$

$$+ \frac{M}{4} \|d_x e^{n+1/2}_\phi\|_x^2 + C \|d_x e^{n+1/2}_\phi\|_x^2$$

$$+ C(\|\mu\|_{L^\infty(J;W^{3,\infty}(\Omega))} + \|\phi\|_{L^\infty(J;W^{3,\infty}(\Omega))})(h_x^4 + h_y^4).$$ \hfill (4.36)

In the $y$ direction, we have the similar estimates. Then the left hand side in (4.35) can be bounded by:

$$M \left( D_x (d_x e_\mu + \delta_x(\mu))^{n+1/2} + D_y (d_y e_\mu + \delta_y(\mu))^{n+1/2}, e^{n+1/2}_\phi \right)_m$$

$$\leq M \left( e^{n+1/2}_\mu, D_x (d_x e_\phi + \delta_x(\phi))^{n+1/2} + D_y (d_y e_\phi + \delta_y(\phi))^{n+1/2} \right)_m$$

$$+ \frac{M}{4} \|d_y e^{n+1/2}_\phi\|_y^2 + C \|d_y e^{n+1/2}_\phi\|_y^2$$

$$+ C(\|\mu\|_{L^\infty(J;W^{3,\infty}(\Omega))} + \|\phi\|_{L^\infty(J;W^{3,\infty}(\Omega))})(h_x^4 + h_y^4).$$ \hfill (4.37)

Thanks to (4.6) and (4.15), the first term on the right hand side of (4.37) can be estimated as follows:

$$M \left( e^{n+1/2}_\mu, D_x (d_x e_\phi + \delta_x(\phi))^{n+1/2} + D_y (d_y e_\phi + \delta_y(\phi))^{n+1/2} \right)_m$$

$$= M \left( e^{n+1/2}_\mu, \frac{R^{n+1/2}}{\sqrt{E_1^0(\tilde{Z}^{n+1/2})}} F'(\tilde{Z}^{n+1/2}) + \frac{R^{n+1/2}}{\sqrt{E_1^0(\phi^{n+1/2})}} F'(\phi^{n+1/2}) \right)_m$$

$$+ M(e^{n+1/2}_\mu, \lambda e^{n+1/2}_\phi)_m + M(e^{n+1/2}_\mu, \Phi^{n+1/2})_m - M\|e^{n+1/2}_\mu\|_m^2$$

$$\leq \frac{M}{2} \|e^{n+1/2}_\mu\|_m^2 + C(e^{n+1/2}_\mu + e^{n+1/2}_\phi)^2 + C(\|\phi\|_{L^\infty(J;W^{4,\infty}(\Omega))}^2(h_x^4 + h_y^4).$$ \hfill (4.38)
Combining equation (4.34) with equations (4.37) and (4.38) and multiplying equation (4.28) by $2\Delta t$, summing over $n$, $n = 0, 1, \ldots, k$ lead to (4.33).

Lemma 7. Under the condition of Theorem 4, there exists a positive constant $C_*$ independent of $h_x$, $h_y$ and $\Delta t$ such that

$$\|Z^n\|_\infty \leq C_*$$ for all $n$.

Proof. We proceed in two steps.

**Step 1** (Definition of $C_*$): Using the scheme (2.13)-(2.15) for $n = 0$ and applying the inverse assumption, we can get the approximation $Z^1$ with the following property:

$$\|Z^1\|_\infty \leq \|Z^1 - \phi\|_\infty + \|\phi\|_\infty \leq \|Z^1 - \Pi_h \phi\|_\infty + \|\Pi_h \phi - \phi\|_\infty + \|\phi\|_\infty$$

$$\leq C h^{-1} (\|Z^1 - \phi\|_m + \|\phi - \Pi_h \phi\|_m) + \|\Pi_h \phi - \phi\|_\infty + \|\phi\|_\infty$$

$$\leq C (h + h^{-1} \Delta t^2) + \|\phi\|_\infty \leq C.$$

where $h = \max\{h_x, h_y\}$ and $\Pi_h$ is an bilinear interpolant operator with the following estimate [6]:

$$\|\Pi_h \phi - \phi\|_\infty \leq C h^2. \quad (4.39)$$

Thus we can choose the positive constant $C_*$ independent of $h$ and $\Delta t$ such that

$$C_* \geq \max\{\|Z^1\|_\infty, 2\|\phi\|_\infty\}.$$

**Step 2** (Induction): By the definition of $C_*$, it is trivial that hypothesis (4.17) holds true for $l = 1$. Supposing that $\|Z^{l-1}\|_\infty \leq C_*$ holds true for an integer $l = 1, \ldots, k + 1$, with the aid of the estimate (4.42), we have that

$$\|Z^l - \phi^l\|_m \leq C (\Delta t^2 + h^2).$$

Next we prove that $\|Z^l\|_\infty \leq C_*$ holds true. Since

$$\|Z^l\|_\infty \leq \|Z^l - \phi^l\|_\infty + \|\phi^l\|_\infty \leq \|Z^l - \Pi_h \phi^l\|_\infty + \|\Pi_h \phi^l - \phi^l\|_\infty + \|\phi^l\|_\infty$$

$$\leq C h^{-1} (\|Z^l - \phi^l\|_m + \|\phi^l - \Pi_h \phi^l\|_m) + \|\Pi_h \phi^l - \phi^l\|_\infty + \|\phi^l\|_\infty$$

$$\leq C_1 (h + h^{-1} \Delta t^2) + \|\phi^l\|_\infty. \quad (4.40)$$

Let $\Delta t \leq C_2 h$ and a positive constant $h_1$ be small enough to satisfy

$$C_1 (1 + C_2^2) h_1 \leq \frac{C_*}{2}.$$

Then for $h \in (0, h_1]$, we derive from (4.40) that

$$\|Z^l\|_\infty \leq C_1 (h + h^{-1} \Delta t^2) + \|\phi^l\|_\infty$$

$$\leq C_1 (h_1 + C_2^2 h_1) + \frac{C_*}{2} \leq C_*.$$

This completes the induction. □

We are now in position to prove our main results.
Proof of Theorem 4. Thanks to the above three lemmas, we can obtain
\[
(e_{r}^{k+1})^2 + \frac{1}{2} \| \frac{d e_{\phi}^{k+1}}{d t} \|_{M}^2 + \| e_{\phi}^{k+1} \|_{m}^2 \\
+ \frac{M}{4} \sum_{n=0}^{k} \Delta t \| d e_{\mu}^{n+1/2} \|_{T M}^2 + \frac{M}{2} \sum_{n=0}^{k} \Delta t \| e_{\mu}^{n+1/2} \|_{m}^2
\leq C \sum_{n=0}^{k+1} \Delta t \| d e_{\phi}^{n} \|_{T M}^2 + C \sum_{n=0}^{k+1} \Delta t (e_{r}^{n})^2
\leq C(\| \phi \|_{W^{1,\infty}(J; W^{4,\infty} (\Omega))) + \| \mu \|_{L^{\infty}(J; W^{4,\infty}(\Omega))) (h_x^4 + h_y^4))
+ C\| \phi \|_{W^{3,\infty}(J; W^{1,\infty}(\Omega))) \Delta t^4.
\]
Finally applying the discrete Gronwall’s inequality, we arrive at the desired result:
\[
(e_{r}^{k+1})^2 + \| d e_{\phi}^{k+1} \|_{T M}^2 + \| e_{\phi}^{k+1} \|_{m}^2 \\
+ \sum_{n=0}^{k} \Delta t \| d e_{\mu}^{n+1/2} \|_{T M}^2 + \sum_{n=0}^{k} \Delta t \| e_{\mu}^{n+1/2} \|_{m}^2
\leq C(\| \phi \|_{W^{1,\infty}(J; W^{4,\infty} (\Omega))) + \| \mu \|_{L^{\infty}(J; W^{4,\infty}(\Omega))) (h_x^4 + h_y^4))
+ C\| \phi \|_{W^{3,\infty}(J; W^{1,\infty}(\Omega))) \Delta t^4.
\]
Thus, the proof of Theorem 4 is complete.

4.2. $L^2$ gradient flow. For the $L^2$ gradient flow, we shall only state the error estimates below, as their proofs are essentially the same as for the $H^{-1}$ gradient flow.

Theorem 8. We assume that $F(\phi) \in C^3(\mathbb{R})$ and $\phi \in W^{1,\infty}(J; W^{4,\infty}(\Omega)) \cap W^{3,\infty}(J; W^{1,\infty}(\Omega))$ and $\Delta t \leq C(h_x + h_y)$. Then for the discrete scheme (2.17)-(2.19), there exists a positive constant $C$ independent of $h_x$, $h_y$ and $\Delta t$ such that
\[
\| Z^{k+1} - \phi^{k+1} \|_{m} + \| d Z^{k+1} - d \phi^{k+1} \|_{T M} + | R^{k+1} - r^{k+1} | \\
\leq C\| \phi \|_{W^{3,\infty}(J; W^{1,\infty}(\Omega))) \Delta t^2 + C\| \phi \|_{W^{3,\infty}(J; W^{1,\infty}(\Omega))) (h_x^2 + h_y^2).
\]

5. Numerical simulations. We present in this section various numerical experiments to verify the energy stability and accuracy of the proposed numerical schemes.

5.1. Accuracy test for Allen-Cahn and Cahn-Hilliard equations. We consider the free energy
\[
E(\phi) = \int_{\Omega} \left( \frac{1}{2} | \nabla \phi |^2 + \frac{1}{4\epsilon^2} (\phi^2 - 1)^2 \right) \, dx.
\]
and for better accuracy, rewrite it as
\[
E(\phi) = \int_{\Omega} \left( \frac{1}{2} | \nabla \phi |^2 + \frac{\beta}{2\epsilon^2} \phi^2 + \frac{1}{4\epsilon^2} (\phi^2 - 1 - \beta)^2 - \frac{\beta^2}{2\epsilon^2} \right) \, dx,
\]
where $\beta$ is a positive number to be chosen. To apply our schemes (2.13)-(2.15) or (2.17)-(2.19) to the system (2.2), we drop the constant in the free energy and specify the operator $G$, the energy $E_1(\phi)$ and $\lambda$ as follows:
\[
G = -(-\Delta)^s, \quad E_1(\phi) = \frac{1}{4\epsilon^2} \int_{\Omega} (\phi^2 - 1)^2 \, dx, \quad \lambda = \frac{\beta}{\epsilon^2}.
\]
The system (2.2) becomes the standard Allen-Cahn equation with \( s = 0 \), and the standard Cahn-Hilliard equation with \( s = 1 \).

We denote
\[
\begin{align*}
\| f - g \|_{\infty, 2} &= \max_{0 \leq n \leq k} \{ \| f^n + q - g^n + q \|_X \}, \\
\| f - g \|_{L^2} &= \left( \sum_{n=0}^{k} \Delta t \| f^n + q - g^n + q \|_X^2 \right)^{1/2}, \\
\| R - r \|_{\infty} &= \max_{0 \leq n \leq k} \{ R^{n+1} - r^{n+1} \},
\end{align*}
\]

where \( q = \frac{1}{2}, 1 \) and \( X = m, TM \).

In the following simulations, we choose \( \Omega = (0, 1) \times (0, 1) \) and \( C_0 = 0.5 \).

### 5.1.1. Convergence rates of the SAV/CN-BCFD scheme for Allen-Cahn equation.

**Example 1.** We take \( T = 0.5, \ G = -1, \beta = 0, \ M = 0.01, \epsilon = 0.08, \Delta t = 5E-4 \), and the initial solution \( \phi_0 = \cos(\pi x) \cos(\pi y) \). To get around the fact that we do not have possession of exact solution, we measure Cauchy error, which is similar to [5, 27, 7]. Specifically, the error between two different grid spacings \( h \) and \( h/2 \) is calculated by \( \| e_\zeta \| = \| e_{h}/2 \| \).

The numerical results are listed in Table 1. We observe the second-order convergence predicted by the error estimates in Theorem 8.

| \( h \) | \( \| e_Z \|_{\infty, 2} \) Rate | \( \| e_{\Delta Z} \|_{\infty, 2} \) Rate | \( \| e_W \|_\infty \) Rate |
|---|---|---|---|
| 1/10 | 6.36E-3 — | 5.96E-2 — | 5.93E-3 — |
| 1/20 | 1.59E-3 2.00 | 6.91E-3 2.01 | 1.20E-3 2.02 |
| 1/40 | 3.41E-4 2.00 | 1.73E-3 2.00 | 3.00E-4 2.00 |
| 1/80 | 8.51E-5 2.00 | 4.31E-4 2.00 | 7.49E-5 2.00 |

### 5.1.2. Convergence rates of SAV/CN-BCFD scheme for Cahn-Hilliard equation.

**Example 2.** We take \( T = 0.5, \ G = \Delta, \beta = 0, \ M = 0.01, \epsilon = 0.2, \Delta t = 5E-4 \), with the same initial solution as in Example 1. The numerical results are listed in Tables 2 and 3. Again, we observe the expected second-order convergence rate in various discrete norms.

| \( h \) | \( \| e_Z \|_{\infty, 2} \) Rate | \( \| e_{\Delta Z} \|_{\infty, 2} \) Rate | \( \| e_W \|_\infty \) Rate |
|---|---|---|---|
| 1/10 | 5.49E-3 — | 2.78E-2 — | 4.88E-3 — |
| 1/20 | 1.36E-3 2.01 | 6.91E-3 2.01 | 1.20E-3 2.02 |
| 1/40 | 3.41E-4 2.00 | 1.73E-3 2.00 | 3.00E-4 2.00 |
| 1/80 | 8.51E-5 2.00 | 4.31E-4 2.00 | 7.49E-5 2.00 |

### 5.2. Coarsening dynamics and adaptive time stepping.

In this example, we simulate the coarsening dynamics of the Cahn-Hilliard equation.

Since the scheme (2.13)-(2.15) is unconditionally energy stable, we can choose time steps according to accuracy only with an adaptive time stepping. Actually in
Table 3

Errors and convergence rates of example 2.

| $h$   | $\|e_W\|_{2,2}$ | Rate | $\|e_dW\|_{2,2}$ | Rate |
|-------|------------------|------|------------------|------|
| 1/10  | 2.50E-2          | —    | 2.18E-1          | —    |
| 1/20  | 6.11E-3          | 2.03 | 5.46E-2          | 2.00 |
| 1/40  | 1.52E-3          | 2.01 | 1.37E-2          | 2.00 |
| 1/80  | 3.79E-4          | 2.00 | 3.42E-3          | 2.00 |

many situations, the energy and solution of gradient flows can vary drastically in certain time intervals, but only slightly elsewhere. In order to maintain the desired accuracy, we adjust the time sizes based on an adaptive time-stepping strategy below (Ref. [13, 20]). We update the time step size by using the formula

**Algorithm 1** Adaptive time stepping procedure

Given: $Z^n$ and $\Delta t^n$.

1: Compute $Z^{n+1}_{Ref}$ using a first order SAV-BCFD scheme and $\Delta t^n$.
2: Compute $Z^{n+1}$ using the SAV/CN-BCFD scheme (2.13)-(2.15) and $\Delta t^n$.
3: Calculate $e^{n+1} = \|Z^{n+1}_{Ref} - Z^{n+1}\|/\|Z^{n+1}\|$.
4: If $e^{n+1} > tol$ then
   Recalculate time step $\Delta t^n \leftarrow \max\{\Delta t_{min}, \min\{A_{dp}(e^{n+1}, \Delta t^n), \Delta t_{max}\}\}$.
5: goto 1
6: else
   Update time step $\Delta t^{n+1} \leftarrow \max\{\Delta t_{min}, \min\{A_{dp}(e^{n+1}, \Delta t^n), \Delta t_{max}\}\}$.
7: endif

$$A_{dp}(e, \Delta t) = \rho \left(\frac{tol}{e}\right)^{1/2} \Delta t, \quad (5.4)$$

where $\rho$ is a default safety coefficient, $tol$ is a reference tolerance, and $e$ is the relative error at each time level. In this simulation, we take

$$\begin{cases}
\mathcal{G} = \Delta, \quad \Delta t_{max} = 10^{-2}, \quad \Delta t_{min} = 10^{-5}, \quad tol = 10^{-3}, \\
M = 0.002, \quad \epsilon = 0.01, \quad \beta = 6, \quad \rho = 0.9,
\end{cases}$$

with a random initial condition with values in $[-0.05, 0.05]$, and the initial time step is taken as $\Delta t_{min}$.

To demonstrate the effectiveness of the SAC/CN-BCFD scheme with adaptive time stepping, we compute the reference solutions with a small uniform time step $\Delta t = 10^{-5}$ and a large uniform time step $\Delta t = 10^{-3}$ respectively. Characteristic evolutions of the phase functions are presented in Fig. 1. We also present in Fig. 2 the energy evolutions and the roughness of interface, where the roughness measure function $R(t)$ is defined as follows:

$$R(t) = \sqrt{\frac{1}{|\Omega|} \int_{\Omega} (\phi - \bar{\phi})^2 d\Omega}, \quad (5.5)$$

with $\bar{\phi} = \frac{1}{|\Omega|} \int_{\Omega} \phi d\Omega$. One observes that the solution obtained with adaptive time steps is consistent with the reference solution obtained with a small time step, while
\[ \Delta t = 10^{-5} \]

(a) \( T = 0.02 \)  
(b) \( T = 0.10 \)  
(c) \( T = 1.0 \)

Adaptive

(d) \( T = 0.02000 \)  
(e) \( T = 0.10000 \)  
(f) \( T = 0.99990 \)

\[ \Delta t = 10^{-3} \]

(g) \( T = 0.02 \)  
(h) \( T = 0.1 \)  
(i) \( T = 1.0 \)

**Fig. 1.** Snapshots of the phase function among small time steps, adaptive time steps and large time steps in example 3.

the solution with large time step deviates from the reference solution. This is also verified by both the energy evolutions and roughness measure function \( R(t) \). We present in Fig. 3 the adaptive time steps for different \( \epsilon = 0.02, 0.01, 0.005 \). We observe that there are about two-orders of magnitude variation in the time steps with the adaptive time stepping, which indicates that the adaptive time stepping for the SAV/CN-BCFD scheme is very efficient.

**Fig. 2.** Numerical comparisons of discrete scaled surface energy and roughness for the simulation of spinodal decomposition in example 3.

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Fig. 3. Adaptive time steps for different $\epsilon$: (a) $\epsilon = 0.02$, (b) $\epsilon = 0.01$, (c) $\epsilon = 0.005$

REFERENCES

[1] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, The Journal of chemical physics, 28 (1958), pp. 258–267.

[2] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. III. nucleation in a two-component incompressible fluid, The Journal of chemical physics, 31 (1959), pp. 688–699.

[3] W. Chen, Y. Liu, C. Wang, and S. Wise, Convergence analysis of a fully discrete finite difference scheme for the Cahn-Hilliard-Hele-Shaw equation, Mathematics of Computation, 85 (2015), pp. 2231–2257.

[4] X. Chen, C. M. Elliott, A. Gardiner, and J. J. Zhao, Convergence of numerical solutions to the Allen-Cahn equation, Applicable Analysis, 69 (1998), pp. 47–56.

[5] Y. Chen and J. Shen, Efficient, adaptive energy stable schemes for the incompressible Cahn-Hilliard Navier-Stokes phase-field models, Journal of Computational Physics, 308 (2016), pp. 40–56.

[6] C. N. Dawson, M. F. Wheeler, and C. S. Woodward, A two-grid finite difference scheme for nonlinear parabolic equations, SIAM journal on numerical analysis, 35 (1998), pp. 435–452.

[7] A. E. Diegel, X. H. Feng, and S. M. Wise, Analysis of a mixed finite element method for a Cahn-Hilliard-Darcy-Stokes system, SIAM Journal on Numerical Analysis, 53 (2015), pp. 127–152.

[8] A. E. Diegel, C. Wang, and S. M. Wise, Stability and convergence of a second-order mixed finite element method for the Cahn-Hilliard equation, Ina Journal of Numerical Analysis, 36 (2015), pp. 1867–1891.

[9] C. M. Elliott, D. A. French, and F. A. Milner, A second order splitting method for the Cahn-Hilliard equation, Numerische Mathematik, 54 (1989), pp. 575–590.

[10] X. Feng, Fully discrete finite element approximations of the Navier-Stokes-Cahn-Hilliard diffuse interface model for two-phase fluid flows, Siam Journal on Numerical Analysis, 44 (2006), pp. 1049–1072.

[11] X. Feng and A. Prohl, Numerical analysis of the Allen-Cahn equation and approximation for mean curvature flows, Numerische Mathematik, 94 (2003), pp. 33–65.

[12] X. Feng and A. Prohl, Error analysis of a mixed finite element method for the Cahn-Hilliard equation, Numerische Mathematik, 99 (2004), pp. 47–84.

[13] H. Gomez and T. J. R. Hughes, Provably unconditionally stable, second-order time-accurate, mixed variational methods for phase-field models, Journal of Computational Physics, 230 (2011), pp. 5310–5327.

[14] G. Grün, On convergent schemes for diffuse interface models for two-phase flow of incompressible fluids with general mass densities, SIAM Journal on Numerical Analysis, 51 (2013), pp. 3036–3061.

[15] J. Guo, C. Wang, S. Wise, and X. Yue, An $h^2$ convergence of a second-order convex-splitting, finite difference scheme for the three-dimensional Cahn-Hilliard equation, Commun. Math. Sci., 14 (2016), pp. 489–515.

[16] Z. Hu, S. M. Wise, C. Wang, and J. S. Lowengrub, Stable and efficient finite-difference nonlinear-multigrid schemes for the phase field crystal equation, Journal of Computational Physics, 228 (2009), pp. 5323–5339.

[17] C. Liu, J. Shen, and X. Yang, Dynamics of defect motion in nematic liquid crystal flow: modeling and numerical simulation, Commun. Comput. Phys, 2 (2007), pp. 1184–1198.

[18] J. Shen, C. Wang, X. Wang, and S. M. Wise, Second-order convex splitting schemes for gradient flows with Ehrlich-Schwoebel type energy: application to thin film epitaxy, SIAM Journal on Numerical Analysis, 50 (2012), pp. 105–125.
[19] J. Shen and J. Xu, *Convergence and error analysis for the scalar auxiliary variable (SAV) schemes to gradient flows*, SIAM Journal on Numerical Analysis, 56 (2018), pp. 2895–2912.

[20] J. Shen, J. Xu, and J. Yang, *A new class of efficient and robust energy stable schemes for gradient flows*, arXiv preprint arXiv:1710.01331, (2017).

[21] J. Shen, J. Xu, and J. Yang, *The scalar auxiliary variable (SAV) approach for gradient flows*, Journal of Computational Physics, 353 (2018), pp. 407–416.

[22] J. Shen and X. Yang, *Numerical approximations of Allen-Cahn and Cahn-Hilliard equations*, Discrete Contin. Dyn. Syst., 28 (2010), pp. 1669–1691.

[23] J. Shen and X. Yang, *A phase-field model and its numerical approximation for two-phase incompressible flows with different densities and viscosities*, SIAM Journal on Scientific Computing, 32 (2010), pp. 1159–1179.

[24] C. Wang and S. M. Wise, *An energy stable and convergent finite-difference scheme for the modified phase field crystal equation*, SIAM Journal on Numerical Analysis, 49 (2011), pp. 945–969.

[25] L. Wang and H. Yu, *Convergence analysis of an unconditionally energy stable linear Crank-Nicolson scheme for the Cahn-Hilliard equation*, arXiv preprint arXiv:1710.09604, (2017).

[26] A. Weiser and M. F. Wheeler, *On convergence of block-centered finite differences for elliptic problems*, SIAM Journal on Numerical Analysis, 25 (1988), pp. 351–375.

[27] S. Wise, J. Kim, and J. Lowengrub, *Solving the regularized, strongly anisotropic Cahn-Hilliard equation by an adaptive nonlinear multigrid method*, Journal of Computational Physics, 226 (2007), pp. 414–446.

[28] C. Xu and T. Tang, *Stability analysis of large time-stepping methods for epitaxial growth models*, SIAM Journal on Numerical Analysis, 44 (2006), pp. 1759–1779.

[29] X. Yang, *Linear, first and second-order, unconditionally energy stable numerical schemes for the phase field model of homopolymer blends*, J. Comput. Phys., 327 (2016), pp. 294–316, http://dx.doi.org/10.1016/j.jcp.2016.09.029, http://dx.doi.org/10.1016/j.jcp.2016.09.029.

[30] X. Yang and G. Zhang, *Numerical approximations of the Cahn-Hilliard and Allen-Cahn equations with general nonlinear potential using the Invariant Energy Quadratization approach*, arXiv preprint arXiv:1712.02760, (2017).

[31] P. Yue, J. J. Feng, C. Liu, and J. Shen, *A diffuse-interface method for simulating two-phase flows of complex fluids*, Journal of Fluid Mechanics, 515 (2004), pp. 293–317.

[32] J. Zhao, X. Yang, Y. Gong, and Q. Wang, *A novel linear second order unconditionally energy stable scheme for a hydrodynamic-tensor model of liquid crystals*, Computer Methods in Applied Mechanics and Engineering, 318 (2017), pp. 803–825.

[33] J. Zhao, X. Yang, J. Li, and Q. Wang, *Energy stable numerical schemes for a hydrodynamic model of nematic liquid crystals*, SIAM Journal on Scientific Computing, 38 (2016), pp. A3264–A3290.

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