$H_D$-Quantum Vertex Algebras

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Abstract

We discuss a class of quantum vertex algebras where not only the commutativity of vertex algebra is broken by a braiding map $S^{(\tau)}$, but also the translation covariance is broken by a translation map $S^{(\gamma)}$. The new class of quantum vertex operators satisfy a Braided Jacobi Identity containing both the braiding and the translation maps.

1 What is a Vertex Algebra?

According to Borcherds [Bor98], [Bor01] a vertex algebra can be thought of as a commutative, associative, unital singular algebra with infinitesimal translation symmetry. Usually this made precise as follows. We start with a State Space $V$ (a vector space) with a distinguished element $1$ (called the vacuum, playing the role of unit in $V$). $V$ has an action of the Hopf algebra $H_D = \mathbb{C}[D]$, where $D$ is the infinitesimal translation operator. Then for each $a \in V$ we have a (singular) operator $Y(a, z)$ of left multiplication by $a$:

$$Y(a, z): V \rightarrow V((z)).$$

We have the following axioms:

- **Vacuum:** $Y(1, z) = 1_V$, and $Y(a, z)1 = e^{zD}a$.
- **Translation Covariance:** $[D, Y(a, z)] = Y(Da, z) = \partial_z Y(a, z)$.
- **Commutativity:** For all $a, b \in V$ there is an $N$ such that

$$ (z_1 - z_2)^N[Y(a, z_1), Y(b, z_2)] = 0. $$

In this formulation in the multiplication $Y(a, z)b$ the left factor $a$ is treated differently than the right factor $b$; for instance $a$ has the variable $z$ associated to it, but $b$ has no variable attached. We find it useful to reformulate the theory so that
both factors are treated symmetrically. We attach to the two factors $a$ and $b$ two
variables, $z_1, z_2$, and define a singular multiplication

$$X_{z_1,z_2}: V \otimes V \to V[[z_1, z_2]]((z_1 - z_2)^{-1}),$$

$$a \otimes b \to X_{z_1,z_2}(a \otimes b) = e^{z_2 D} Y(a, z_1 - z_2)b.$$

Then one derives from the axioms for $Y$ the following properties of $X_{z_1,z_2}$:

- **Vacuum:** $X_{z_1,z_2}(a \otimes 1) = e^{z_1 D} a$, $X_{z_1,z_2}(1 \otimes a) = e^{z_2 D} a$.
- **Translation Covariance:** $e^\gamma D X_{z_1,z_2}(a \otimes b) = X_{z_1+\gamma,z_2+\gamma}(a \otimes b)$.
- **Commutativity:** $X_{z_1,z_2}(a \otimes b) = X_{z_2,z_1}(b \otimes a)$.

Relation of $X_{z_1,z_2}$ to $Y(a, z)$ is given by expansions. Let $i_{z_1,z_2}$ be the expansion
of a rational function in $z_1 - z_2$ in the region $|z_1| > |z_2|$. Then

$$i_{z_1,z_2} X_{z_1,z_2}(a \otimes b) = Y(a, z_1) Y(b, z_2) 1,$$

$$i_{z_2,z_1} X_{z_1,z_2}(a \otimes b) = Y(b, z_2) Y(a, z_1) 1,$$

$$i_{z_2,z_3} X_{z_2,z_3}(a \otimes b) = Y(Y(a, z_3) b, z_2) 1. \tag{1}$$

From this one easily derives the axioms for vertex algebras in terms of $Y$, starting
with the properties of $X_{z_1,z_2}$.

2 Quantum Vertex Algebras via Deformation.

As a motivation for our definition of a quantum vertex algebra consider a com-
mutative algebra $M$, with a multiplication

$$m: M \otimes M \to M,$$

so that we have in particular $m(a \otimes b) = m(b \otimes a)$. Now a quantization of
$(M, m)$ could be defined by introducing a formal variable $t$, and a deformed
multiplication

$$m_t: M_t \otimes M_t \to M_t, \quad M_t = M[[t]],$$

where $m = m_t \mod t$. In general $m_t$ will not longer be commutative. We
could require that there is a *Braiding Map*:

$$S: M_t \otimes M_t \to M_t \otimes M_t,$$

such that $m_t \circ S(a \otimes b) = m_t (b \otimes a)$. So the braiding describes the failing of
$m_t$ to be commutative. Now if the commutative algebra $M$ has some symmetry,
say via a group $G$ acting on $M$ (so that $m(ga \otimes gb) = gm(a \otimes b)$), then we
can expect that after quantisation the deformed multiplication is no longer $G$-
symmetric. Instead, we can require that there is for each $g \in G$ a map

$$S^g: M_t \otimes M_t \to M_t \otimes M_t,$$
such that
\[ gm_t \circ S^g(a \otimes b) = m_t(ga \otimes gb). \]
So a deformation of a commutative algebra with symmetry \((M, m, G)\) would be a quintuple \((M_t, m_t, S, S^g, G)\), satisfying a complicated system of axioms we don’t want to write down here, see [ABa].

Now a vertex algebra is a singular analog of a commutative algebra with as symmetry the Hopf algebra \(H_D\) of infinitesimal translations. So if we quantize we can expect that the commutativity and translation covariance are no longer exact, and that we need extra structures to describe the broken symmetries.

Introduce a quantum variable \(t\) and deformed singular multiplication
\[
X_{z_1, z_2} : V^{\otimes 2} \to V[[z_1, z_2]][z_1^{-1}, (z_1 - z_2)^{-1}][[t]].
\]
The commutativity and translation covariance are supposed to be not longer exact. The extra structure we need is

- a **Braiding** map:
  \[
  S^{(\tau)}_{z_1, z_2} : V^{\otimes 2} \to V^{\otimes 2}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}][[t]].
  \]

- a **Translation** map:
  \[
  S^{(\gamma)}_{z_1, z_2} : V^{\otimes 2} \to V^{\otimes 2}[z_1^{\pm 1}, z_2, (z_1 - z_2)^{-1}, (z_1 + \gamma)^{\pm 1}, z_2 + \gamma][[t]].
  \]
The deformed multiplication and the braiding and translation maps are supposed to have the following properties:

- **Braided Commutativity:**
  \[
  X_{z_1, z_2}(a \otimes b) = X_{z_2, z_1} \circ S^{(\tau)}_{z_2, z_1}(b \otimes a).
  \]

- (Broken) Translation covariance:
  \[
  e^{\gamma D} X_{z_1, z_2} \circ S^{(\gamma)}_{z_1, z_2}(a \otimes b) = X_{z_1 + \gamma, z_2 + \gamma}(a \otimes b).
  \]

- Plus a bunch of other axioms (Yang-Baxter, hexagon, ...), see [ABa])

This defines a \(H_D\)-quantum vertex algebra.

3 Vertex Operators and the Braided Jacobi Identity.

We define a 1-variable vertex operator as usual:
\[
Y(a, z)b = X_{z, 0}(a \otimes b),
\]
i.e., by evaluating the second variable of \(X\) at zero. (Note that in general in an \(H_D\)-quantum vertex algebra we can not evaluate the first variable at 0.) Then the relation between \(Y\) and \(X\) is given by a variant of (1), where we need to
insert braiding and translation matrices in appropriate places. More generally,
one shows that there exists

\[ X_{z_1, z_2, z_3} : V^{\otimes 3} \to V[[z_k]](z_i - z_j)^{-1}[[t]], \quad 1 \leq i < j \leq 3, i \leq k \leq 3 \]

such that

\[ i_{z_1; z_2} X_{z_1, z_2, z_3} = X_{z_2, 0}(1 \otimes X_{z_2, z_3}). \]

Then we have the following expansions: if \( A = a \otimes b \otimes c \), then

\[
\begin{align*}
    i_{z_1; z_2} X_{z_1, z_2, 0}(A) &= Y(a, z_1)Y(b, z_2)c, \\
    i_{z_2; z_1} X_{z_1, z_2, z_3}(A) &= Y_{z_2}(1 \otimes Y_{z_1})i_{z_2; z_1}S^{(\gamma)(12)}(b \otimes a \otimes c), \\
    i_{z_2; z_3} X_{z_2, z_3, z_0}(A) &= Y_{z_2}(Y_{z_3} \otimes 1)i_{z_2; z_3}S^{(12)}(b \otimes a \otimes c).
\end{align*}
\]

Here we write \( Y_{z}(a \otimes b) \) for \( Y(a, z)b \). From these expansions one derives the Braided Jacobi Identity for \( H_D \)-quantum vertex algebras:

\[
\begin{align*}
    i_{z_1; z_2} \delta(z_1 - z_2, z_3) & a(z_1) b(z_2) c - i_{z_2; z_1} \delta(z_1 - z_2, z_3) Y_{z_2}(1 \otimes Y_{z_1}) S^{(\gamma)(12)}(b \otimes a \otimes c) \\
    &= i_{z_2; z_3} \delta(z_1 + z_2 + z_3) Y_{z_2}(Y_{z_3} \otimes 1) S^{(12)}(b \otimes a \otimes c).
\end{align*}
\]

Here we write \( a(z) \) for \( Y(a, z) \).

### 4 Example: Hall-Littlewood polynomials.

The main inspiration for our construction came from the theory of quantum vertex operators for Hall-Littlewood polynomials introduced by Jing, [Jin91]. These quantum vertex operators occur naturally in a \( H_D \)-quantum vertex algebra \( V_{L, t} \) which is a deformation of the lattice vertex algebra based on the lattice \( L = Z \alpha \), with pairing \( L \otimes L \to Z, a \alpha \otimes b \alpha \rightarrow ab \). \( V_{L, t} \) is in a sense generated by \( e^\alpha \) and the braiding and translation maps in this case are given by

\[
S^{(\gamma)}_{z_1, z_2}(e^\alpha \otimes e^\alpha) = \frac{1 - t z_2 / z_1}{1 - t z_1 / z_2} e^\alpha \otimes e^\alpha, \quad S^{(\gamma)}_{z_1, z_2}(e^\alpha \otimes e^\alpha) = \frac{1 - t z_2 / z_1}{1 - t z_1 \gamma / z_1 + \gamma} e^\alpha \otimes e^\alpha.
\]

An effective method to do calculations in \( V_{L, t} \) and similar \( H_D \)-quantum vertex algebras is given by the theory of bicharacters, see [Ang06].

### 5 Conclusion and Outlook.

The \( H_D \)-quantum vertex algebras introduced above are generalizations of the quantum vertex operators of Etingof-Kazhdan [EK00]. In their theory the vertex operators are translation covariant, so that the translation maps \( S^{(\gamma)} \) are the identity.

Vertex algebras are commutative algebras with translation covariance and singularities in the product of vertex operators of the form \( (z_1 - z_2)^{-N} \). In our \( H_D \)-quantum vertex algebras the commutativity and translation covariance
is broken via nontrivial braiding and translation maps $S^{(\tau)}$ and $S^{(\gamma)}$, but the singularities are essentially still of the same type $(z_1 - z_2)^{-N}$. Now in examples of quantum vertex operators, see e.g., [FR97], one sees that in practice the product can have singularities of the form

$$\frac{1}{z_1 - p^kq^l z_2}.$$

This means that in such quantum vertex algebras one needs to extend the symmetry algebra $H_D = \mathbb{C}[D]$ by adding (group like) operators $T_p, T_q$ that act like

$$T_p f(z) = f(pz), \quad T_q f(z) = f(qz).$$

Note that $H_{p,q} = \mathbb{C}[T_p^{\pm 1}, T_q^{\pm 1}, d]$ is non commutative.

Now the basic formalism of vertex algebras (and $H_D$-quantum vertex algebras) is very much based on $H_D$. For instance, the delta distribution $\delta(z_1, z_2)$, which is ubiquitous in the theory, is the difference of two expansions of the basic singularity $\frac{1}{z_1 - z_2}$, and expansions are given by the exponential operators canonically associated to $H_D$.

Replacing the commutative Hopf algebras $H_D$ by the non commutative $H_{p,q}$ changes the basic framework of vertex algebras drastically: for instance, one needs a new theory of Dirac delta distributions adapted to $H_{p,q}$, [ABb].

Acknowledgments

The results in this paper are joint work with Iana Anguelova, see [ABa].

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