BOCHNER FORMULA, BERNSTEIN TYPE ESTIMATES, AND POROUS MEDIA EQUATION ON LOCALLY FINITE GRAPHS

LI MA

ABSTRACT. In this paper, we consider three typical problems on a locally finite connected graph. The first one is to study the Bochner formula for the Laplacian operator on a locally finite connected graph. We use the Bochner formula to derive the Bernstein type estimate of the heat equation. The second is to derive the Reilly type formula of the Laplacian operator. The last one is to obtain global positive solution to porous-media equation via the use of Aronson-Benilan argument. There is not much work in the direction of the study of nonlinear heat equations on locally finite connected graphs.

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1. INTRODUCTION

In this paper, we study some typical problems related to heat equations and porous-media equation on a locally finite connected graph. We do believe that the study of nonlinear heat equations on locally finite connected graphs is an important subject as like it is in the Riemannian geometry (see [3]). After some thinking, we immediately realize that the Sobolev type inequality on graphs [1] plays a key role in such a research. However, Sobolev type inequality on graphs is not a topic of this paper. We first study the Bochner formula for the Laplacian operator on a locally finite connected graph. Our Bochner formula is new and should be very useful in the study of eigenvalue estimate of the Laplacian operators on graphs. In fact, by invoking the trick of integration by part on locally finite connected graphs, one may also obtain the Reilly formula on locally finite connected graphs. Once we have the Bochner formula, we use it to study the global behavior of the bounded solution to the heat equation. It is quite nature to ask if we can get a Bernstein type estimate for the solutions to the heat equation on locally finite connected graphs. We can obtain this result. The last question under our consideration of this paper is to obtain global positive solution to porous-media equation via the use of Aronson-Benilan argument. This is a hard

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question since we may not have the Sobolev compactness imbedding theorem and it is not easy to obtain the global solution from the exhaustion domain method. We can overcome this difficulty by using the Aronson-Benilan type estimate of the bounded solutions to the Porous-media equation. Our main results are (5), theorem 3 and theorem 4 below.

In the previous posted in arxiv draft version of this paper, we studied the McKean type eigenvalue estimate. Prof. J.Dodzuik informed me that he and his coauthor had done it in [4] many year ago. Here we would like to express my hearty thanks to him for sending to us their paper [4]. We may call the locally finite graph with McKeen type inequality the McKean graph. Anyway, it is a interesting problem to study related heat equation on McKean graphs and we hope to study it in near future.

Here is the plan of the paper. In section 2 we study the maximum principle of heat equation and for porous media equation. In section 3 we obtain the Bochner formula and the Bernstein type estimate. In the last section 4 we consider the locally bounded global solution to the Porous-media equation.

2. the Maximum principles

We start from recalling some definitions and the maximum principle for bounded solution to heat equation. Let $G = (V(G), E(G))$ be an infinite, locally finite, connected graph without loops or multiple edges where $V = V(G)$ is the set of vertices of $G$ and $E = E(G)$ is the set of edges. We still write $x \in G$ when $x$ is a vertex of $G$. We use the notation $x \sim y$ to indicate the edge connects the vertex $x$ to its neighbor vertex $y$. We equip $V$ with the symmetric weight $\mu_{xy} \geq 0$ associated to the edge $x \sim y$ such that $\sum_{x \sim y} \mu_{xy} > 0$ for each $x \in V$ and we always assume that our edges are unoriented in the sense that $\mu_{xy} = \mu_{yx}$. We call such a graph the short name the weighted graph. Let $d_x = \sum_{x \sim y} \mu_{xy} > 0$.

We define the space of all square summable functions on $G$,

$l^2(V) = \{f : V \rightarrow \mathbb{R}; \sum_{x \in V} d_x f(x)^2 < \infty\}$

with the inner product

$$(f, g) = \sum_{x \in V} d_x f(x) g(x).$$

Define on $l^2(V)$ the Laplacian operator for the function $f$,

$$\Delta f(x) = \frac{1}{d_x} \sum_{x \sim y} \mu_{xy} (f(y) - f(x)).$$

and the norm of the gradient of the function $f$ by

$$|\nabla f|^2(x) = \frac{1}{d_x} \sum_{x \sim y} \mu_{xy} (f(y) - f(x))^2.$$
Note that
\[(\Delta f(x))^2 \leq |\nabla f|^2(x).\]

Fix \(x_0 \in G\) and let \(r(x) = d(x, x_0)\). Let
\[d_{\pm}(x) = \sum_{y \sim x : r(y) = r(x) \pm 1} \mu_{xy}.\]
be the number of vertices which are 1 step closer or further to \(x_0\) than \(x\).
Define the mean curvature \(H(x)\) of the sphere of radius \(r(x)\) to \(x_0\) by
\[H(x) = \Delta r(x) = \frac{1}{d_x} \sum_{x \sim y} \mu_{xy}(r(y) - r(x)).\]

It can be verified that
\[H(x) = \frac{d_+(x) - d_-(x)}{d_x}.\]

Then we have the following maximum principle ([2] [7] [5] [5]).

**Theorem 1.** Assume that there exists some \(x_0 \in G\) and a constant \(C \geq 0\) such that \(H(x) \leq C\) on \(V\). Let \(u_0(x)\) be any bounded function on \(G\). Then any bounded solution \(u(t, x)\) to the heat equation
\[u_t = \Delta u\]
with initial data \(u(0) = u_0\) satisfies
\[\sup_G |u(t, x)| \leq \sup_G |u_0(x)|\]
for every \(t \geq 0\).

The proof of the result above is standard. By the result above we can derive the uniqueness of bounded solution to the heat equation on \(V\). In fact the claim follows by considering differences of bounded solutions with same initial condition. Actually we can extend the maximum principle to positive solution to the porous media type equation
\[u_t = \Delta \log u, \quad (0, \infty) \times V\]
with bounded initial data. Again the proof is standard, for completeness, we give the proof.

**Theorem 2.** Assume that there exists some \(x_0 \in G\) and a constant \(C \geq 0\) such that \(H(x) \leq C\) on \(V\). Let \(v_0(x)\) be any bounded positive function on \(G\). Then any bounded positive solution \(v(t, x)\) to the heat equation (7) with initial data \(v(0) = v_0\) satisfies
\[\sup_G v(t, x) \leq \sup_G v_0(x)\]
for every \(t \geq 0\).
Proof. By considering $v(t,x)$ and rescaling the time variable, we may assume $\sup_{G} v_0(x) = 1$. Set $u(t,x) = \log v(t,x)$ and $u_0(x) = \log v_0(x)$. Then $u$ satisfies that

$$e^u u_t = \Delta u.$$  

Let $M_1 = \sup\{u(t,x); (t,x) \in (0,T) \times V\}$. Note that $M_2 := \sup\{u_0(x), x \in V\} = 0$. Clearly we may assume $M_1 \geq 0$; otherwise we are done. We consider for positive $C > 0$ and $R > 0$ the function

$$w(t,x) = u(t,x) - \frac{M_1}{R}(d(x,x_0) + Ct).$$

If we denote by $B_R = B_R(x_0)$ the ball with radius $R$ and the center $x_0$, we may conclude $w(t,x) \leq 0$ for $(t,x) \in \{0\} \times B_R \cup [0,T) \times \partial B_R$, which is the parabolic boundary of $[0,T) \times B_R(x_0)$. Assume that $w(t,x)$ attains its positive maximum at the interior point $(t_-, x_-)$ of $(0,T) \times B_R$, we may assume that

$$w_t > 0, \quad \Delta w \leq 0$$

at this point. This fact implies that at $(t_-, x_-)$, $u > 0$,

$$e^u u_t \geq u_t > \frac{M_1 C}{R},$$

and

$$\Delta u \leq \frac{M_1 C}{R}.$$  

However, this is impossible due to (3). Then we have $w(t,x) \leq 0$ in $[0,T) \times B_R(x_0)$, which is equivalent to $u(t,x) \leq \frac{M_1}{R}(d(x,x_0) + Ct)$. Letting $R \to \infty$ we obtain $u(t,x) \leq 0$ on $[0,T) \times V$, which gives us (2). \qed

The maximum principle above gives us a comparison lemma for the porous media equation (1). We shall use this fact in section five.

3. Bochner formula and Bernstein estimate for heat equation

Following the method of Bakry-Emery, we define

$$\Gamma(f,g) = \frac{1}{2}\{\Delta(fg)(x) - f(x)\Delta g(x) - g(x)\Delta f(x)\}$$

and

$$\Gamma_2(f,g) = \frac{1}{2}\{\Delta \Gamma(fg)(x) - \Gamma(f, \Delta g)(x) - \Gamma(g, \Delta f)(x)\}.$$  

Then, by direct computation,

$$\Delta f^2(x) = 2f(x)\Delta f(x) + |\nabla f|^2(x),$$

$$\Gamma(f,g)(x) = \frac{1}{2d_x} \sum_{y \sim x} \mu_{xy}(f(y) - f(x))(g(y) - g(x)), $$

$$\Gamma_2(f,f)(x) = \frac{1}{2}|\nabla f|^2(x),$$
\[ \Gamma_2(f, f)(x) = \frac{1}{4} |D^2 f|^2(x) - \frac{1}{2} |\nabla f|^2(x) + \frac{1}{2} (\Delta f)^2(x), \]

where
\[
|D^2 f|^2(x) := \frac{1}{d_x} \sum_{y \sim x} \frac{\mu_{xy}}{dy} \sum_{z \sim y} \mu_{yz} |f(x) - 2f(y) + f(z)|^2.
\]

We now compute the Bochner formula for the function \( f \).
\[ -\Delta |\nabla f|^2(x) = -|D^2 f|^2(x) + \frac{2}{d_x} \sum_{y \sim x} \frac{\mu_{xy}}{dy} \sum_{z \sim y} \mu_{yz} (f(x) - 2f(y) + f(z))(f(x) - f(y)). \]

Set
\[ I = \frac{2}{d_x} \sum_{y \sim x} \frac{\mu_{xy}}{dy} \sum_{z \sim y} \mu_{yz} (f(x) - 2f(y) + f(z))(f(x) - f(y)). \]

Note that
\[
I = 2|\nabla f|^2(x) + \frac{2}{d_x} \sum_{y \sim x} \mu_{xy} (f(x) - f(y)) \Delta f(y)
\]
\[
= 2|\nabla f|^2(x) + \frac{2}{d_x} \sum_{y \sim x} \mu_{xy} (f(x) - f(y)) (\Delta f(y) - \Delta f(x)) + \Delta f(x) \Delta f(y)
\]
\[
= 2|\nabla f|^2(x) + 2|\Delta f|^2(x) - 2(\nabla f, \nabla f)(x).
\]

Then we have the following Bochner formula:
\[ -\Delta |\nabla f|^2(x) = -|D^2 f|^2(x) + 2|\nabla f|^2(x) + 2|\Delta f|^2(x) - 2(\nabla f, \nabla f)(x). \]

We now use this formula to derive the Bernstein type estimate for the bounded solution \( f(t, x) \) to the heat equation
\[ f_t = \Delta f \]
with initial data \( f_0 \). Using (5) we get that
\[ (\partial_t - \Delta)(\frac{1}{2} |\nabla f|^2(t, x)) = -\frac{1}{2} |D^2 f|^2(t, x) + |\nabla f|^2(t, x) + |\Delta f|^2(t, x). \]

Assume on \( G \) the curvature condition
\[ \Gamma_2(f, f) \geq \frac{1}{m} (\Delta f)^2(x) + \frac{k}{2} |\nabla f|^2(x), \]
for some constants \( m > 0 \) and \( k \in \mathbb{R} \).

Using (5) we know that
\[
\frac{1}{2} |D^2 f|^2(x) \geq (k + 1)|\nabla f|^2(x) + \left( \frac{2}{m} - 1 \right)(\Delta f)^2(x).
\]

Inserting this back to (5) we get that
\[ (\partial_t - \Delta)(\frac{1}{2} |\nabla f|^2(t, x)) \leq -k |\nabla f|^2(t, x) + (2 - \frac{2}{m}) |\Delta f|^2(t, x). \]

Using
\[ |\Delta f|^2(x) \leq |\nabla f|^2(x) \]
we obtain that
\[ (\partial_t - \Delta)(\frac{1}{2} |\nabla f|^2(t, x)) \leq (-k(2 - \frac{2}{m})) |\nabla f|^2(t, x). \]
Recall that
\[(\partial_t - \Delta)f^2(t, x) = -|\nabla f|^2(t, x).\]
Then we can choose \(\alpha > 0\) such that
\[(\partial_t - \Delta)(t|\nabla f|^2(t, x) + \alpha f^2(t, x)) \leq 0.\]
Using the maximum principle we then have
\[t|\nabla f|^2(t, x) + \alpha f^2(t, x) \leq \alpha \sup_G f_0^2(x)\]
Since \(f(t, x)\) is uniformly bounded in \(t\), we know that there exists \(t_k \to \infty\) such that \(f(t_k, x) \to f_\infty(x)\) for each \(x \in G\), and \(\lim_{t_k \to \infty} |\nabla f(t_k, x)|^2 \to 0\), which implies that \(f_\infty(x) = \text{const.}\). In conclusion we have the result below.

**Theorem 3.** Assume that \(G\) is a locally finite connected graph with curvature condition (7). Assume that there exists some \(x_0 \in G\) and a constant \(C \geq 0\) such that \(H(x) \leq C\) on \(V\). Let \(f_0(x)\) be any bounded function on \(G\). Then any bounded solution \(f(t, x)\) to the heat equation
\[f_t = \Delta f\]
with initial data \(f(0, x) = f_0(x)\) exists globally and there exists \(t_k \to \infty\) such that \(f(t_k, x) \to f_\infty(x)\) where \(f_\infty(x)\) is a constant function.

**4. Global solution to the porous-media equation**

Given any bounded positive function \(u_0 : V \to \mathbb{R}_+\). We consider the global existence of the positive solution \(u(t, x)\) to the porous-media equation
\[u_t = \Delta \log u, \quad \text{in} \quad (0, \infty) \times V\]
with the initial data \(u(0) = u_0\). Just like in the Euclidean domain case, we may define the equation \((\ref{eq:porous})\) in the distribution sense (in time variable). Namely for any compact domain supported function \(\phi\) (which is smooth in the \(t\)-variable) defined on space \((0, \infty) \times V\), we have
\[-\int u \phi_t = \int \log u \Delta \phi,\]
where the integration is taken over the space \((0, \infty) \times V\).

Take any finite subgraph \(\Omega \subset V\). We may first consider \((\ref{eq:porous})\) in \((0, \infty) \times \Omega\) with initial data and boundary condition \(u_0\). Let \(f = \frac{1}{2} \log u\). Then \(u = e^{2f}\) and it satisfies the equivalent problem
\[e^f(e^f)_t = \Delta f, \quad \text{in} \quad (0, \infty) \times V.\]

Actually we can get the local in time solution \(u_\Omega\) to \((\ref{eq:porous})\) (respectively \(f_\Omega = \frac{1}{2} \log u_\Omega\) to \((\ref{eq:porous_2})\)) in \((0, T) \times \Omega\) (for some \(T > 0\)) by using the discrete Morse flow method \([6]\).
For $N > 1$ an integer and any $T > 0$, let
\[ h = T/N, \quad t_n = nh, \quad n = 0, 1, 2, \ldots, N. \]
Assume that we have constructed $f_j \in L^2(\Omega)$, $0 \leq j \leq n - 1$, and $f_{n-1}$ is a minimizer of the functional
\[
I_{n-1}(f) = \frac{1}{2h} \int_{\Omega} |e^f - e^{f_{n-2}}|^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla f|^2 \, dx
\]
on the space $H = \{ f \in L^2(\Omega) \mid f - f_0 = 0, \text{ on } \partial \Omega \}$. Note that $H$ is a closed convex subset of $L^2(\Omega)$. Define
\[
I_n(f) = \frac{1}{2h} \int_{\Omega} |e^f - e^{f_{n-1}}|^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla f|^2 \, dx
\]
on $H$. It is clear that the infimum is finite and, by applying the Poincaré inequality to $f - f_0$ (see [1]), that any minimizing sequence is bounded in $H$. By the direct method in the calculus of variations, one concludes that $I_n$ has a unique minimizer $f_n$ in $H$ which satisfies
\[
\frac{1}{h} \left( e^f - e^{f_{n-1}} \right) e^f = \Delta f
\]
along with the uniform energy bound
\[
\frac{1}{2h} \int_{\Omega} |e^{f_n} - e^{f_{n-1}}|^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla f_n|^2 \, dx \leq \frac{1}{2} \int_{\Omega} |\nabla f_{n-1}|^2 \, dx \leq C. \tag{10}
\]
We define $f_N(t) \in L^2$ for $t \in [0, T]$ such that, for $n = 1, \ldots, N$,
\[
f_N(t) = f_n, \quad t \in [t_{n-1}, t_n].
\]
We further define, for $n = 1, \ldots, N$,
\[
\partial_t e^{f_N}(t) = \frac{1}{h} (e^{f_n} - e^{f_{n-1}}), \quad t \in [t_{n-1}, t_n].
\]
Then $f_N$ satisfies
\[
e^{f_N} \partial_t e^{f_N}(t) = \Delta f_N
\]
in $\Omega \times (0, T)$. Note that the energy bound [10] implies that
\[
\int_0^T \int_{\Omega} e^{2f_N} + \sup_t \int_{\Omega} |\nabla f_N|^2 \, dx \leq 5C.
\]
We may use the Poincaré inequality to get the uniform $L^2(\Omega)$ bound of $\{f_N\}$. Taking a subsequence of $\{f_N\}$ that converges in $L^\infty_t H$, one obtains a limit $f \in L^\infty_t H$ that satisfies
\[
e^f \partial_t e^f = \Delta f
\]
in distribution sense in the domain $\Omega \times (0, T)$.
To get the globally defined solution, we need the linear upper bound for $u_\Omega = e^{2f}$ and we follow a well-known argument due to Aronson and Benilan. Let $\lambda > 1$. Define
\[
w_\lambda(t, x) = \lambda u_\Omega(\lambda^{-1} t, x).
\]
Then \( w_\lambda(t, x) \) satisfies \( S \) in \((0, T) \times \Omega\) with the initial data and boundary condition \( \lambda u_0(x) \), which is bigger than \( u_0(x) \). By using the comparison principle we know that

\[
w_\lambda(t, x) > u_\Omega(t, x) \quad \text{in} \quad (0, T) \times \Omega.
\]

Set

\[
v_\lambda(t, x) = w_\lambda(t, x) - u_\Omega(t, x).
\]

Then

\[
\frac{\partial}{\partial \lambda} v_\lambda(t, x) \geq 0, \quad \text{in} \quad (0, T) \times \Omega,
\]

or equivalently \( u \leq t^{-1} u \) for \( u = u_\Omega \), which by integration, implies that \( u_\Omega(t, x) \leq C(1 + t) \) where \( C > 0 \) is a constant depending only on \( u_0 \). Hence we can extend the solution \( u_\Omega(t, x) \) globally. Take \( \Omega = \Omega_j \) where \( V = \bigcup \Omega_j \), \( \Omega_j \subset \Omega_{j+1} \) are exhaustion finite subgraphs of \( V \). Then we get a sequence of solutions \( \{u_j\} \) defined on \( \Omega_j \times (0, \infty) \). By taking diagonal subsequence we can get a sub-convergence sequence on any finite subset of \( V \), still denoted by \( \{u_j\} \) and a global (locally bounded) solution \( u(t, x) \) of \( S \) with initial data \( u_0 \) such that

\[
u(t, x) = \lim_{j \to \infty} u_j(t, x),
\]

locally in \((0, \infty) \times V\). In summary, we then have

**Theorem 4.** For any bounded positive function \( u_0 : V \to \mathbb{R}_+ \), there exists a global solution to \( S \) with initial data \( u_0 \).

The uniqueness question to \( S \) is a interesting (may be very difficult) problem and it can be considered by using the maximum principle. We leave it open to interesting readers.

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L. Ma, Distinguished Professor, Department of mathematics, Henan Normal University, Xinxiang, 453007, China

E-mail address: lma@tsinghua.edu.cn