Dirac Equation in the Magueijo-Smolin Approach of Double Special Relativity

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We reconsider in details the Dirac equation in the context of the Magueijo-Smolin approach to the Doubly Special Relativity. Starting from the deformed dispersion relation we obtain the Dirac equation in momentum space, allowing us to achieve a more in-depth study of its semiclassical approach. Finally by means of a deformed correspondence principle we gain access to an equation in the position space.

I. INTRODUCTION

It is commonly assumed that Heisenberg, intending to remove the electron self-energy divergence, considered as early as 1930 \[1\] the possibility of introducing a space with discrete structure into the formalism of the quantum mechanics. The symmetry breakdown of this space however with the non-conservation of the energy-momentum tensor on one hand and the success of the method based on the renormalization group on the other, explain why such an approach was abandoned. In more recent years the hypothetical existence of ultra-light energy cosmic rays particles that could violate the Greisen-Zatsepin-Kuzmin limit forced physicists to reconsider the fundamental concepts of the structure of our space-time \[2\]. This ideas was also intensely discussed, in particular it was well known that the Planck length \(l_P\) plays an important role in quantum gravity, more specifically in string theory and loops quantum gravity. This fundamental length is missing in special relativity where two different observers don’t measure the same lengths at the same moment. The fact that \(l_P\) cannot be reached and must be have the same value in all inertial frames is obviously in contradiction with special relativity, thus justifying the introduction of a modification of transformations and laws of this theory. Based on this idea, Amelino-Camelia \[3\] followed by Magueijo and Smolin \[4\], pioneered an extended form (Double Special Relativity or DSR) by introducing, in addition to the speed of light, a second parameter in the form of an energy scale \(\kappa = 1/ l_P\) (or Planck energy \(E_P = \sqrt{\frac{\kappa^2 c^5}{\hbar}} = 1.956 \times 10^9\) Joules = \(10^{19}\) GeV), that implies a noncommutative space-time structure, the \(\kappa\)– Minkowski space-time. At this level it is important to remark that, as stated by Amelino Camilia, this short distance scale \(l_P\) is relativistically fundamental in the same sense as the scale \(c\) in Special Relativity. These two fundamental scales are for example different of \(\hbar\) which is the scale of the non zero value of the angular momentum, because they affect the structure of the transformation rules between observers which is not the case of \(\hbar\). In fact these ideas had already been discussed for several years, though we shall not concern ourselves with the historical background, which can be found in the recent and very exhaustive paper of Amelino-Camelia \[5\]. Finally one of the last generalizations of the DSR is the Deformed General Relativity proposed in \[6\] where the geometric framework of an internal De Sitter space is associated with the non-commutative curved space.

Our purpose in this work is to reconsider in details the Dirac equation in the context of the Magueijo-Smolin approach to the DSR. In a recent paper \[8\] we already made a study of this subject and we found an equation which is a special case of this one in the context of the Amelino-Camilia DSR \[7\]. It is important to state that interesting studies with this equation have also been conducted, in particular in \[9\] where it was applied to a calculation of the hydrogen atom spectrum was carried out, or in \[10\] where the discreteness of the generalized uncertainty principle was used.

We begin in the first part by recalling some consequences of the deformed Heisenberg algebra, then starting from the deformed dispersion relation we focus on the obtaining of a Dirac equation in momentum space, allowing us to achieve a more in-depth study of its semiclassical approach. Finally by means of a deformed correspondence principle we gain access to an equation in the position space and by consequently to a deformed current.
II. MAGUEIJO-SMOLIN MODEL

We have choose the Magueijo-Smolin approach mainly because it is one of the simplest to handle mathematically but also because its results are profound and far from trivial.

A. Deformed Heisenberg algebra

To define an energy scale, these workers used a nonlinear action of the Lorentz group on the momentum space \[4\], they then determined how a classical particle in a Minkowski space with a diagonal metric tensor \(g_{\mu\nu} = (1, -1, -1, -1)\), has its position and momentum which obey the following deformed Heisenberg algebra defined by means of Poisson brackets

\[
\{x^\mu, x^\nu\} = \lambda \left(x^\mu \delta^0{}_{\nu} - x^\nu \delta^0{}_{\mu}\right); \{x^\mu, p^\nu\} = -g^{\mu\nu} + \lambda p^\nu \delta^0{}_{\mu}; \{p^\mu, p^\nu\} = 0 \quad (1)
\]

where it is usual to introduce the new parameter \(\lambda = \kappa^{-1}\).

Important results are that all the Jacobi identities are conserved and the invariance of an expression with the quadriposition and the quadri-momentum under the DSR are given by

\[
\mathbf{x}^2 = g^{\mu\nu} x_\mu x_\nu \left(1 - \frac{E}{E_p}\right)^2 = \text{Cte}
\]

and

\[
\mathbf{p}^2 = \frac{g^{\mu\nu} p_\mu p_\nu}{\left(1 - \frac{E}{E_p}\right)^2} = \text{Cte},
\]

but, although the Heisenberg algebra is deformed, the Lorentz algebra is conserved

\[
\{J^{\mu\nu}, J^{\rho\sigma}\} = -g^{\mu\rho} J^{\nu\sigma} + g^{\nu\rho} J^{\mu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}. \quad (2)
\]

The position and momentum however are transformed by these Lorentz transformations under the deformed laws

\[
\begin{align*}
\{x^\mu, J^{\nu\rho}\} &= g^{\mu\rho} x^\nu - g^{\nu\rho} x^\mu + \lambda \left(p^\rho \delta^0{}_{\nu} - p^\nu \delta^0{}_{\rho}\right) x^\mu \\
\{p^\mu, J^{\nu\rho}\} &= g^{\mu\rho} p^\nu - g^{\nu\rho} p^\mu - \lambda \left(p^\rho \delta^0{}_{\nu} - p^\nu \delta^0{}_{\rho}\right) p^\mu.
\end{align*} \quad (3)
\]

We can note that this deformation acting only on the temporal part of the \(\kappa-\)Minkowski space, corresponds only to the action of the Lorentz boosts and is a particular case of a non-commutative deformation. The general case acts also on the spacial component by introducing a quadri-dimensional skew tensor \(\theta^{\mu\nu}\) (\(\theta\)-Minkowski space).

B. Magueijo-Smolin Energy dispersion relation

Starting from the above algebra laws (equations 1, 2 and 3) they deduced the following invariant energy dispersion relation \[4\]

\[
\frac{E^2 - p^2 c^2}{\left(1 - \frac{E}{E_p}\right)^2} = \frac{E_0^2}{\left(1 - \frac{E}{E_p}\right)^2}.
\]

Which with the help of the relation

\[
E_0 = \frac{mc^2}{1 + \frac{mc^2}{E_p}}
\]

becomes

\[
\frac{E^2 - p^2 c^2}{\left(1 - \frac{E}{E_p}\right)^2} = m^2 c^4, \quad (4)
\]
being called the Casimir mass.

We can observe that this relation is not invariant by the transformation which reverses the sign of the energy. We shall consider this specific point later. This relation can be transformed in the form

\[ E^2 = p^2 c^2 + m^2 c^4 \left( 1 - \frac{E}{E_p} \right)^2, \]  

(5)

from we obviously find

\[ E_{\pm} = \frac{E_p M^2}{1 - M^2} \pm \sqrt{p^2 c^2 + \frac{E_p^2 M^2}{1 - M^2}}, \]

where the energies of particles and anti-particles are not symmetrical due to the term \(-\frac{E_p M^2}{1 - M^2}\). This remark will be important for the interpretation of the Dirac equation later on, which is a special form of the more general relation already studied in many papers. Indeed it should be noted that this equation is a special case of a more general expression for the dispersion relation that can be written

\[ E^2 = p^2 c^2 + m^2 c^4 + m^2 c^4 \left( \frac{E}{E_p} \right)^n, \]

but here the very interesting thing is that the energy dispersion relation can be separated between energy and momentum. Another important example is the ultra-high energy case given by

\[ E^2 = p^2 c^2 + m^2 c^4 + p^2 c^2 \left( \frac{E}{E_p} \right)^n, \]

with the special case of the toy model of Amelino Camelia (n = 1) which is unfortunately not separable

\[ E^2 = p^2 c^2 + m^2 c^4 + p^2 c^2 \left( \frac{E}{E_p} \right). \]

Returning now to equation (5) we very easily find

\[ (1 - M^2) \left( E + \frac{E_p M^2}{1 - M^2} \right)^2 = p^2 c^2 + m^2 c^4 + \frac{E_p^2 M^4}{(1 - M^2)}, \]

where \( M = \frac{mc^2}{E_p} \) so that we can recast this expression in the usual relation expression of non-deformed special relativity by making the following change of variables

\[
\begin{align*}
\mathcal{M} &= \frac{E_p M}{c^2 \sqrt{1 - M^2}} = \frac{m}{\sqrt{1 - M^2}}, \\
\mathcal{E} &= \sqrt{1 - M^2} \left( E + \frac{E_p M^2}{1 - M^2} \right) = \left( E + \frac{\mathcal{M}^2 c^n}{E_p} \right) \frac{m}{\mathcal{M}},
\end{align*}
\]

so that now

\[ \mathcal{E}^2 = p^2 c^2 + \mathcal{M}^2 c^4, \]  

(6)

with a new momentum quadrivector \( \mathcal{P} = (\vec{p}, \mathcal{E}) \) and for solutions

\[
\begin{align*}
\mathcal{E}_+ &= \sqrt{p^2 c^2 + \frac{E_p^2 M^2}{1 - M^2}}, \\
\mathcal{E}_- &= -\sqrt{p^2 c^2 + \frac{E_p^2 M^2}{1 - M^2}}.
\end{align*}
\]

This result although very simple is not trivial and has an interesting physical significance as it explains that a relativistic particle in the Magueijo-Smolin context can be mathematically considered as a particle with non-deformed energy dispersion relation, but with a deformed energy \( \mathcal{E} \) and a deformed mass \( \mathcal{M} \). Starting from this assertion we will easily be able to obtain or confirm a certain number of results.

Note that we can easily deduce the non-relativistic limits of the Magueijo-Smolin energy, already discussed in

\[ E_{M-S} = \frac{p^2}{2m} + \frac{mc^2}{1 + \frac{mc^2}{E_p}} + ..., \]
and of the new energy
\[ E = \frac{p^2}{2M} + \mathcal{M}c^2 + \ldots = \frac{p^2}{2m} + \frac{mc^2}{\sqrt{1 - \frac{m^2c^4}{E^2}}} + \ldots \]

We quickly see that its limits are very different
\[ \lim_{mc^2 \to E_p} E_M - S \to \frac{p^2}{2m} + \frac{1}{2}mc^2 + \ldots \quad \text{and} \quad \lim_{mc^2 \to E_p} \mathcal{E} \to \infty. \]

In this paragraph our purpose was obviously not to affirm a little as in \[1\] that, since one can pass via a map from the deformed energy dispersion relation to a non-deformed energy dispersion relation, DSR is simply equivalent to the special relativity. We are here in the situation where the sentence "finding a map between theories establishes their equivalence", as Amelino-Camelia wrote in \[5\], is not sufficient enough and proves nothing. It is important to point out that it is only in this "toy" Magueijo-Smolin context that we can easily find a non-deformed energy dispersion relation, this "kind of trick" is undoubtedly generally impossible. This opportunity will allow us to go further in our work and to say some new things about DSR.

C. Non-deformed Lorentz group

We have seen that the Lorentz group is not deformed, what then happens to Heisenberg algebra?

The relation are now given by
\[ \{x^\mu, x^\nu\} = \lambda \left( x^\rho \delta^\mu_\nu - x^\nu \delta^\mu_\rho \right); \{x^\mu, P^\nu\} = -G^\mu^\nu + \lambda P^\rho \delta^\mu_\rho; \{P^\mu, P^\nu\} = 0, \]

where we introduced a new mass and Planck energy dependent metric tensor
\[ G^{\mu \nu} = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

which becomes \( g^{\mu \nu} \) in the limit of \( \mathcal{M} \to m \) \((mc^2 \ll E_p)\).

The Jacobi identities are non-deformed and if we put \( J^{\mu \nu} = x^\mu P^\nu - x^\nu P^\mu \), we have the deformed Lorentz Lie algebra by the "new metric" \( G^{\mu \rho} \)
\[ \{J^{\mu \nu}, J^{\rho \sigma}\} = -G^{\mu \rho} J^{\nu \sigma} + G^{\nu \rho} J^{\mu \sigma} - G^{\nu \sigma} J^{\mu \rho} + G^{\mu \sigma} J^{\nu \rho}, \]

and the position and momentum are transformed by these Lorentz transformations under the deformed laws
\[ \begin{cases} \{x^\mu, J^{\rho \nu}\} = G^{\mu \rho} x^\nu + G^{\nu \rho} x^\mu + \lambda \left( P^\rho \delta^\mu_\nu - P^\nu \delta^\mu_\rho \right) x^\mu \\
\{P^\mu, J^{\rho \nu}\} = G^{\mu \rho} P^\nu + G^{\nu \rho} P^\mu + \lambda \left( P^\rho \delta^\mu_\nu - P^\nu \delta^\mu_\rho \right) P^\mu. \end{cases} \]

We thus obtain the same relations than J. Magueijo and L. Smolin (equations\[1\] and \[3\], but now with new momentum \( P^\mu \) and energy dependent metric tensor \( G^{\mu \nu} \).

III. DIRAC EQUATION IN MOMENTUM SPACE

Based on equation \[6\] one deduces the non-deformed Dirac equation from a direct standard manner
\[ \mathcal{E} U(P) = \left( \overrightarrow{c \alpha \cdot P} + \mathcal{M} c^2 \beta \right) U(P), \]

which becomes with the starting variables
\[ \sqrt{1 - M^2} \left( E + \frac{E_p M^2}{1 - M^2} \right) U(p) = \left( c \overrightarrow{c \alpha \cdot \mathbf{P}} + \frac{E_p M}{\sqrt{1 - M^2}} \beta \right) U(p), \]
that to say the same equation obtained by a different technique in our last publication on this subject \[8\]

\[ EU(p) = \left( \frac{-E_p M^2}{1-M^2} + \frac{c \vec{\alpha} \cdot \vec{p}}{\sqrt{1-M^2}} + \frac{E_p M}{1-M^2} \beta \right) U(p), \]

and we thus deduce the deformed Dirac Hamiltonian which is hermitian

\[ H_{Dirac}^{Deformed} = \frac{-E_p M^2}{1-M^2} + \frac{c \vec{\alpha} \cdot \vec{p}}{\sqrt{1-M^2}} + \frac{E_p M}{1-M^2} \beta \]

\[ = \frac{c \vec{\alpha} \cdot \vec{p}}{\sqrt{1-(mc^2/E_p)^2}} + \frac{mc^2}{1-(mc^2/E_p)^2} \left( \beta - mc^2 \right). \]

Interestingly particle anti-particle symmetry breakdown is again reflected at the Dirac Hamiltonian level by the presence of the term \(-E_p M^2\). We have thus retrieve in a very simple manner the Dirac Hamiltonian found in \[8\] by a different procedure. In addition, it is interesting to note that contrary to the approach developed previously in \[8\], the obtained Hamiltonian is directly Hermitian. We shall now consider some direct applications of this equation.

A. Solving the Dirac equation

The solution of the non-deformed Dirac equation for a free particle propagating in the z direction is given in the momentum space by

\[ U^\pm_a(p) = \sqrt{\frac{M c^2 + E_\lambda}{2E}} \left( \frac{\chi_{a\sigma \gamma p}}{Mc^2+E_\lambda} \chi_a \right) \exp \left( \frac{i((pz - E_\lambda t)/\hbar)\cdot p}{p} \right), \]

where \( \lambda = \pm 1 \) according to the case of particle or antiparticle and the spinor field is obviously for \( a = 1 \) : \( \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)

and for \( a = 2 \) : \( \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

B. Pauli equation

The consequences of this change in variables is also valid for the Pauli equation which is the non-relativistic limit of the Dirac equation for an electron in a magnetic field. Recall that from the standard Dirac equation we arrive at the relation with an electromagnetics field

\[ \left( \begin{array}{c} \pi^0 - mc^2 \\ (\vec{\sigma} \cdot \vec{\pi}) c \\ (\vec{\sigma} \cdot \vec{\pi}) c - (\pi^0 + mc^2) \end{array} \right) \left( \begin{array}{c} \phi(p) \\ \chi(p) \end{array} \right) = 0, \]

where

\( \pi^\mu = p^\mu + eA^\mu, \)

particularly

\[ \pi^0 = p^0 + \frac{V}{c} = mc^2 + \frac{V}{c}, \]

which gives in the non-relativistic limit

\[ \pi^0 = mc^2 + E_{NR} + \frac{V}{c}, \]

and

\[ E_{NR}\phi(p) = \left( \frac{(\vec{\sigma} \cdot \vec{\pi})^2}{2m} - eV \right) \phi(p). \]
In the deformed case we only make the change
\[ \pi^0 = M c^2 + E_{NR} + e \frac{V}{c}, \]
then
\[ E_{NR} \phi(p) = \left( \frac{(\vec{p}, \vec{\sigma})^2}{2M} - eV \right) \phi(p), \]
or
\[ E_{NR} \phi(p) = \left( \frac{(\vec{p}, \vec{\sigma})^2}{2m} \sqrt{1 - \left( \frac{mc^2}{E_p} \right)^2} - eV \right) \phi(p), \]
which becomes for the hamiltonian
\[ H_{Deformed}^{Pauli} = \frac{\vec{p}^2}{2M} + \frac{\hbar}{4M^2 c^2} \vec{\sigma} \left( \vec{\nabla} V(R) \wedge \vec{p} \right) + V(R), \]
which is now deformed by a \( M \) factor.

C. Deformed Berry phase

If we consider the adiabatic evolution of the Dirac equation in order to compute its Berry curvature in momentum space, we found \[13\] the following value of the Berry phase associated with a relativistic particle submitted to a potential \( V(\vec{r}) \) is
\[ \vec{\alpha}(\vec{p}, \vec{\sigma}) = \frac{c^2 \hbar (\vec{p} \wedge \vec{\sigma})}{2 E (E + mc^2)}, \]
which in the DRS now becomes
\[ \vec{\alpha}_D(\vec{p}, \vec{\sigma}) = \frac{c^2 \hbar (\vec{p} \wedge \vec{\sigma})}{2E (E + Mc^2)}, \]
and, if we have made the "standard" change
\[ \vec{\alpha}_D(\vec{p}, \vec{\sigma}) = \frac{\sqrt{1 - \lambda^2 m^2} c^2 \hbar (\vec{p} \wedge \vec{\sigma})}{2 \sqrt{(1 - \lambda^2 m^2) p^2 c^2 + m^2 c^4}} \left( \frac{1}{\sqrt{1 - \lambda^2 m^2}} \right) \],
which is the exact result of the publication \[8\]. The position operator is then also very easily to determine
\[ \vec{r}_D = \vec{r} + \vec{\alpha}_D. \]
We now have three kinds of coordinates: \( \vec{p}_D , \vec{p} \) and the canonical position \( \vec{R} \)

D. Deformed Berry curvature

For the Berry curvature found in \[13\] we have
\[ \left[ x^i_D, x^j_D \right] = i \hbar \delta^{ij} (\vec{p}, \vec{\sigma}) = -i \hbar^2 \epsilon^{ijk} \frac{c^4}{2E^3} \left( m \sigma^k + \frac{p_k (\vec{p}, \vec{\sigma})}{E + mc^2} \right), \]
then in DSR
\[ \vec{\theta}_D(\vec{p}, \vec{\sigma}) = -\hbar \frac{c^4}{2E^3} \left( M \vec{\sigma} + \frac{\vec{p} (\vec{p}, \vec{\sigma})}{E + Mc^2} \right) \]
\[ = -\frac{\hbar (1 - \lambda^2 m^2) c^4}{2 ((1 - \lambda^2 m^2) p^2 c^2 + m^2 c^4)} \left( mc^2 \vec{\sigma} + \frac{\vec{p} (\vec{p}, \vec{\sigma})}{\sqrt{1 - \lambda^2 m^2) p^2 c^2 + m^2 c^4 + mc^2}} \right), \]
which is also the same result.
E. Deformed dynamic equations with a Berry phase

From [13] the dynamic equations are now then written in the following form

\[
\begin{align*}
\frac{\text{d}\vec{p}}{\text{d}t} &= \frac{\sqrt{1-\lambda^2 m^2}}{\sqrt{(1-\lambda^2 m^2) p^2 + m^2 c^4}} \frac{\text{d}\vec{r}}{\text{d}t} \wedge \vec{\theta} \\
\frac{\text{d}\vec{r}}{\text{d}t} &= \frac{\text{d}\vec{p}}{\text{d}t} = -\nabla V(\vec{r}),
\end{align*}
\]

which gives as direct application the non-relativistic Dirac particle in an electric potential

\[
\begin{align*}
\frac{\text{d}X^i}{\text{d}t} &= \frac{p^i}{M} + \frac{e\hbar}{2M^2c^2} \varepsilon^{ijk} \sigma_j \partial_k V \\
\frac{\text{d}r^i}{\text{d}t} &= \frac{p^i}{M} + \frac{e\hbar}{2M^2c^2} \varepsilon^{ijk} \sigma_j \partial_k V.
\end{align*}
\]

IV. DIRAC EQUATION IN POSITION SPACE

We know that position space is fundamental to work in the lagrangian approach and particularly to study the different symmetries of the physical problem. But the fact that DSR in momentum space must be extended to position space is not trivial considering that in this context the Planck length acts as a fundamental finite length. The interpretation of DSR in position space then becomes problematic and we do not intend to cover that topic here. A deeper discussion can be found in several papers and for example a consistent manner is explained in [14] or in [15] where it is shown that this kind of problems occurs only in the classical picture (with $\hbar \to 0$) and not in the quantum picture. Finally we must also announce the two interesting methods for obtaining the position space of non-linear relativity developed in [16].

It should be recalled that it is only in the Magueijo-Smolin context where we can find a non-deformed energy dispersion relation that we can try to work in position space so easily. Indeed as we obtained a non-deformed Dirac equation in momentum space, the temptation was thus strong to switch to position space with the standard method.

A. Non-deformed Klein-Gordon and Dirac equations

If we start with the non-deformed energy dispersion relation

\[
E^2 = p^2 c^2 + M^2 c^4,
\]

which gives by the standard correspondence principle the following Klein-Gordon equation

\[
\left(-\frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} - \left(\frac{M c}{\hbar}\right)^2\right) \phi(\vec{r}, t) = -\Delta \phi(\vec{r}, t).
\]

1. Hermitian hamiltonian

This last equation implies that the non-deformed Dirac equation in position space can be written in the following form

\[
\left(i \frac{\partial}{\partial \tau} - i \vec{\alpha} \cdot \vec{\nabla} - M c^2 \beta\right) \psi(\vec{r}, t) = 0,
\]

where $\tau$, the time linked to the energy $E$, is by means of the definition of $E$

\[
E \to \frac{\partial}{\partial \tau} = \sqrt{1 - M^2} \left( \frac{\partial}{\partial t} + \frac{E PM^2}{1 - M^2} \right).
\]
We study now the equation

\[
\left\{ i \left( \sqrt{1-M^2} \left( \frac{\partial}{\partial t} + \frac{E_PM^2}{1-M^2} \right) \right) - i \vec{\alpha} \cdot \vec{\nabla} - Mc^2 \beta \right\} \psi(\vec{r},t) = 0,
\]

or

\[
i \frac{\partial}{\partial t} \psi(\vec{r},t) = \left( \frac{-E_PM^2}{1-M^2} + \frac{ic\vec{\alpha} \cdot \vec{\nabla}}{\sqrt{1-M^2}} + \frac{E_PM}{1-M^2} \beta \right) \psi(\vec{r},t),
\]

which gives the hermitian hamiltonian in momentum representation

\[
H_{\text{Deformed}}^{\text{Dirac}} = \frac{-E_PM^2}{1-M^2} + \frac{c \vec{\alpha} \cdot \vec{p}}{\sqrt{1-M^2}} + \frac{E_PM}{1-M^2} \beta,
\]

already find above in the equation (8). We found again retrieve the problem of the difference between particle or antiparticle as the reverse time symmetry is now broken.

2. Continuity equation

From the non-deformed Dirac equation in positions space (11) we can directly obtain its adjoint equation

\[
\bar{\psi}^+(\vec{r},t) \gamma^0 \left( i\hbar \gamma^0 \frac{\partial}{\partial \tau} - i\hbar \vec{\gamma} \cdot \vec{\nabla} + Mc^2 \right) = 0,
\]

where we now use the \( \gamma^\mu \) Dirac matrix. The continuity equation is obviously

\[
\frac{\partial \rho(x)}{\partial \tau} + \vec{\nabla} \cdot \vec{j}(\vec{r},t) = 0,
\]

with the usual notations

\[
\left\{ \begin{array}{l}
\rho(\vec{r},t) = \bar{\psi}(\vec{r},t) \gamma^0 \psi(\vec{r},t) \\
\vec{j}(\vec{r},t) = \bar{\psi}(\vec{r},t) \vec{\gamma} \psi(\vec{r},t),
\end{array} \right.
\]

and the standard notation of Dirac adjoint field

\[
\bar{\psi}(\vec{r},t) = \psi^+(\vec{r},t) \gamma^0.
\]

If we now wish to introduce the "physics" time \( t \) we have

\[
\sqrt{1-M^2} \left( \frac{\partial}{\partial t} + \frac{E_PM^2}{1-M^2} \right) (\psi^+(\vec{r},t)\psi(\vec{r},t)) - \vec{\nabla} \psi^+(\vec{r},t) \vec{\alpha} \psi(\vec{r},t) = 0,
\]

then

\[
\frac{\partial}{\partial t} (\psi^+(\vec{r},t)\psi(\vec{r},t)) - \vec{\nabla} \left( \frac{\psi^+(\vec{r},t) \vec{\alpha} \psi(\vec{r},t)}{\sqrt{1-M^2}} \right) = - \frac{E_PM^2}{1-M^2} (\psi(\vec{r},t) \gamma^0 \psi(\vec{r},t)).
\]

Puting

\[
\left\{ \begin{array}{l}
\rho(\vec{r},t) = \psi^+(\vec{r},t)\psi(\vec{r},t) = |\psi_0(\vec{r},t)|^2 + |\psi_1(\vec{r},t)|^2 + |\psi_2(\vec{r},t)|^2 + |\psi_3(\vec{r},t)|^2 \\
\vec{j}(x) = \frac{\psi^+(x) \vec{\alpha} \psi(x)}{\sqrt{1-M^2}},
\end{array} \right.
\]
we arrive at an equation which tells us that the current is non-conserved

$$\frac{\partial}{\partial t} \rho(\vec{r}, t) + \text{div} \vec{j}(\vec{r}, t) = -\frac{E_p M^2}{\sqrt{1 - M^2}} \rho(\vec{r}, t).$$

From this it is clear that current is not conserved due to the presence of the term $-\frac{E_p M^2}{\sqrt{1 - M^2}}$ which breaks the particle anti-particle energy symmetry. Note that, by shifting the energy levels by the constant term $-\frac{E_p M^2}{\sqrt{1 - M^2}}$, we can restate this symmetry as now $E = \pm \sqrt{p^2 c^2 + E^2 + \frac{E_p M^2}{\sqrt{1 - M^2}} \rho(\vec{r}, t)}$. Nevertheless the 3-D $\vec{j}(\vec{r}, t) = \psi(\vec{r}, t) \vec{\alpha} \psi(\vec{r}, t) \sqrt{1 - M^2}$ is still deformed. Introducing the velocity as $\vec{c} \sqrt{1 - M^2} - \vec{\alpha}$, and its eigenvalues by $\vec{c} = \pm C \vec{u}$, where $C = \frac{c}{\sqrt{1 - M^2}}$ and $\vec{u}$ is an unitary vector, the 3-D current as the usual expression $\vec{j}(x) = \rho(x) \vec{c}$.

B. Correspondence principle in $\kappa$–Minkowski space

The question is now to determine whether it is allowed to use a "correspondence principle" in this context? The Minkowski phase space obeys the following Heisenberg algebra with Poisson brackets

$$\{x^\mu, x'^\nu\} = 0; \{x^\mu, p^\nu\} = -g^{\mu\nu}; \{p^\mu, p'^\nu\} = 0.$$ 

It is obviously well known that in quantum mechanics we build operators $\hat{X}^\mu$ and $\hat{P}^\mu$ acting on a Hilbert space such as

$$\begin{align*}
\hat{X}^\mu \psi(x) &= x^\mu \psi(x) \\
\hat{P}^\mu \psi(x) &= \frac{\hbar}{i} \frac{\partial \psi(x)}{\partial x^\mu},
\end{align*}$$

and which obey the following Heisenberg algebra with commutators

$$\left[\hat{X}^\mu, \hat{X}^\nu\right] = 0; \left[\hat{X}^\mu, \hat{P}^\nu\right] = -i\hbar g^{\mu\nu}; \left[\hat{P}^\mu, \hat{P}^\nu\right] = 0.$$ 

We have thus achieved the correspondence principle of the quantum mechanics. In the case of the $\kappa$– Minkowski space the Heisenberg algebra with Poisson brackets is

$$\{x^\mu, x'^\nu\} = \lambda \left(x^\mu g^{0\nu} - x'^\nu g^{0\mu}\right); \{x^\mu, p^\nu\} = -g^{\mu\nu} + \lambda p^\nu g^{0\mu}; \{p^\mu, p'^\nu\} = 0.$$ 

and then the operators $\hat{X}^\mu$ and $\hat{P}^\mu$ are defined by the generators of the $\kappa$– Minkowski

$$\begin{align*}
\hat{X}^\mu &= x^\mu - \lambda \hbar g^{0\mu} x^\nu \frac{\partial}{\partial x^\nu} \\
\hat{P}^\mu &= \frac{\hbar}{i} \frac{\partial}{\partial x^\mu}. 
\end{align*}$$
We then arrive after a standard calculus at the following Heisenberg algebra with commutators

\[
\left[ \hat{X}^\mu, \hat{X}^\nu \right] = i\hbar \left( \hat{X}^\mu g^\nu_\mu - \hat{X}^\nu g^\mu_\mu \right); \left[ \hat{X}^\mu, \hat{P}^\nu \right] = -i\hbar \left( -g^\mu_\nu + \lambda \hat{P}^\nu \right); \left[ \hat{P}^\mu, \hat{P}^\nu \right] = 0.
\]

In conclusion we can see that the standard correspondence principle (case of the "non deformed theory") is associated with a "deformed correspondence principle" which is true in this context of \(\kappa-Minkowski\) space.

C. Deformed Klein-Gordon and Dirac equations

If we start with the energy dispersion relation of Magueijo-Smolin

\[
E^2 = p^2 c^2 + m^2 c^4 \left( 1 - \frac{E}{E_p} \right)^2,
\]

we can directly deduce, by means of this "deformed correspondence principle", the following Klein-Gordon equation

\[
\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \left( \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \right)^2 \right) \tilde{\phi}(\vec{r}, t) = -\Delta \tilde{\phi}(\vec{r}, t),
\]

which becomes

\[
\left( \frac{i}{c} \partial_t - \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \right) \left( \frac{i}{c} \partial_t + \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \right) \tilde{\phi}(\vec{r}, t) = \left( i \vec{\sigma} \cdot \vec{\nabla} \right) \tilde{\phi}(\vec{r}, t),
\]

then

\[
\left( \frac{i}{c} \partial_t - \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \right) \left( \frac{i}{c} \partial_t + \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \right) \tilde{\phi}(\vec{r}, t) = \left( i \sigma^1 \vec{\nabla} \right) \tilde{\phi}(\vec{r}, t),
\]

where we have posed

\[
\left( \frac{i}{c} \partial_t \right) \left( \frac{i}{c} \partial_t + \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \right) \tilde{\phi}(\vec{r}, t) = \left( i \sigma^1 \vec{\nabla} \right) \tilde{\phi}(\vec{r}, t).
\]

We can now transform these equations in the following 4D-spinorial equation

\[
\begin{pmatrix}
  i\gamma^0 \partial_t & -i \vec{\gamma} \cdot \vec{\nabla} \\
  i \vec{\gamma} \cdot \vec{\nabla} & -i \partial_t
\end{pmatrix}
\begin{pmatrix}
  \tilde{\phi}(\vec{r}, t) \\
  \tilde{\eta}(\vec{r}, t)
\end{pmatrix}
= \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \begin{pmatrix}
  \tilde{\phi}(\vec{r}, t) \\
  \tilde{\eta}(\vec{r}, t)
\end{pmatrix},
\]

which is written by means of the Dirac matrix \(\gamma^\mu\)

\[
\begin{pmatrix}
  i\gamma^0 \partial_t & -i \vec{\gamma} \cdot \vec{\nabla} \\
  i \vec{\gamma} \cdot \vec{\nabla} & -i \partial_t
\end{pmatrix}
\begin{pmatrix}
  \tilde{\phi}(\vec{r}, t) \\
  \tilde{\eta}(\vec{r}, t)
\end{pmatrix}
= \frac{mc}{\hbar} \left( 1 - \frac{i\hbar}{E_p} \partial_t \right) \begin{pmatrix}
  \tilde{\phi}(\vec{r}, t) \\
  \tilde{\eta}(\vec{r}, t)
\end{pmatrix},
\]

which gives this non-hermitian hamiltonian in momentum representation

\[
\tilde{H}^{Deformed}_{Dirac} = -E_p M^2 \left( 1 - \frac{E}{E_p} \right) + M \beta \frac{\vec{\alpha} \cdot \vec{P}}{1 - \frac{E}{E_p}} + \frac{E_p M}{1 - \frac{E}{E_p}} \beta.
\]

which is evidently different of the hermitian one given by the equation (8). Finally like we easily check that

\[
\left[ \tilde{H}^{Deformed}_{Dirac}, J^i \right] = 0,
\]

with \(J^i = L^i + \frac{\hbar}{2} \Sigma^i\), this hamiltonian describes well a particle with spin 1/2.
1. Transition from hermitian to non-hermitian hamiltonian

We have already found in a different context the manner of passing from this hermitian to the non-hermitian hamiltonian

\[ \tilde{H}^{Deformed}_{Dirac} = D^{-1} H^{Deformed}_{Dirac} D, \]

this hermitic but non-unitary matrix being defined by

\[ \begin{cases} D &= a + b\beta \\ D^{-1} &= \frac{a - bM}{1 - M^2} + \frac{b - aM}{1 - M^2}\beta, \end{cases} \]

where

\[ \begin{cases} a &= \sqrt{\frac{1 + \sqrt{1 - M^2}}{2}} \\ b &= \sqrt{\frac{M^2}{2(1 + \sqrt{1 - M^2})}}. \end{cases} \]

Naturally we can easily see that the eigenvalues of this two hamiltonians are real.

2. Continuity equation

From equation (??) we have now in the position space the following adjoint Dirac equation

\[ \tilde{\psi}^+ (\vec{r}, t) \gamma^0 \left( i \left( \gamma^0 + M \right) \frac{\partial}{\partial t} - i \vec{\gamma}. \vec{\nabla} + mc^2 \right) = 0, \]

we pose then

\[ \tilde{\psi}^+ (\vec{r}, t) = \tilde{\psi}^+ (\vec{r}, t) \left( \gamma^0 + M \right), \]

and by a simple combination of these equations we obtain

\[ \frac{\partial}{\partial t} \left( \tilde{\psi}^+ (\vec{r}, t) \gamma^0 \tilde{\psi} (\vec{r}, t) \right) - \vec{\nabla}. \left( \tilde{\psi} (\vec{r}, t) \gamma^0 \frac{\gamma^0 - M}{1 - M^2} \gamma^0 \gamma^0 \tilde{\psi} (\vec{r}, t) \right) = 0, \]

which is now a conserved current relation

\[ \partial_{\mu} j^\mu (\vec{r}, t) = 0, \]

where the charge and current densities are defined by its components

\[ \begin{cases} \rho(x) &= \frac{1}{e} \tilde{\psi}^+ (\vec{r}, t) \gamma^0 \tilde{\psi} (\vec{r}, t) = \frac{1}{e} \tilde{\psi}^+ (\vec{r}, t) (1 + \beta M)^0 \tilde{\psi} (\vec{r}, t) \\ j^\mu (x) &= \tilde{\psi} (\vec{r}, t) \gamma^\mu \gamma^0 \gamma^0 \tilde{\psi} (\vec{r}, t) = \tilde{\psi}^+ (\vec{r}, t) \gamma^\mu \tilde{\psi} (\vec{r}, t). \end{cases} \]

As the hamiltonian is not hermitian we have already announced that the density of charge probability is

\[ \rho(\vec{r}, t) = \frac{1}{e} \left\{ \left( 1 + M \right) \left( |\tilde{\psi}_1(\vec{r}, t)|^2 + |\tilde{\psi}_2(\vec{r}, t)|^2 \right) + \left( 1 - M \right) \left( |\tilde{\psi}_3(\vec{r}, t)|^2 + |\tilde{\psi}_4(\vec{r}, t)|^2 \right) \right\}, \]

which is only positive always if: \( m < \frac{E_p}{c^2} \sim 10^{-8} \text{ Kg.} \)
We have
\[ \vec{j}(\vec{r},t) = \rho(\vec{r},t) \vec{v}, \]
we then deduce the matrix velocity
\[ \vec{v} = c \gamma^0 - M \frac{\gamma^0}{1-M^2} = c \begin{pmatrix}
\sqrt{\frac{1-M}{1+M}} & 0 & 0 & 0 \\
0 & \sqrt{\frac{1-M}{1+M}} & 0 & 0 \\
0 & 0 & \sqrt{1+M} & 0 \\
0 & 0 & 0 & \sqrt{1+M} \\
\end{pmatrix} \gamma, \]
which has as the same eigenvalues than in the first case
\[ v_i = \pm Cu_i. \]
The consequences of the zitterbewegun problem are thus the same as in the preceding case.

To summarize this paragraph, we have seen two manners of attempting to introduce a continuity equation, one with a hermitian hamiltonian which leads to a non-conserved current and the other with a non-hermitian hamiltonian leading to a conserved current. These two possibilities result in the same deformed zitterbewegun problem.

V. CONCLUSION

The main aim of this paper was to demonstrate how in the Magueijo-Smolin DSR context, we can introduce a non-deformation energy dispersion relation, deduce a Dirac equation in momentum space and then discuss in the position space the possibility of a current conservation equation. The other significant result is the fact that we can work with a non-deformed energy dispersion relation and the same algebra relations but defined with new momentum and metric tensor. Finally we easily discovered again the presence of Berry phase in the adiabatic approximation of the Dirac equation in momentum space.

The prospects of this approach could be of at least be of two types, firstly toward a direct study of another equations as Kemmer-Duffin-Petiau or Feschbach-Villars, secondly toward a generalisation of the type ”Gravity’s rainbow” of J. Magueijo and L. Smolin [17]. To finish we wish to insist one last time on the simplicity of the deformed energy dispersion relation which has enabled us to continue the calculus so easily. It is obviously that the generalisation of this kind of results in the case of a general non deformed energy dispersion is certainly not trivial.

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