FROM REPRESENTATIONS OF THE RATIONAL CHEREDNIK
ALGEBRA TO PARABOLIC HILBERT SCHEMES VIA THE
DUNKL-OPDAM SUBALGEBRA

E. GORSKY, J. SIMENTAL, AND M. VAZIRANI

Abstract. In this note we explicitly construct an action of the rational Cherednik algebra $H_{1,m/n}(S_n, \mathbb{C}^n)$ corresponding to the permutation representation of $S_n$ on the $\mathbb{C}^*$-equivariant homology of parabolic Hilbert schemes of points on the plane curve singularity $\{x^m = y^n\}$ for coprime $m$ and $n$. We use this to construct actions of quantized Gieseker algebras on parabolic Hilbert schemes on the same plane curve singularity, and actions of the Cherednik algebra at $t = 0$ on the equivariant homology of parabolic Hilbert schemes on the non-reduced curve $\{y^n = 0\}$. Our main tool is the study of the combinatorial representation theory of the rational Cherednik algebra via the subalgebra generated by Dunkl-Opdam elements.

Contents

1. Introduction 2
   1.1. Hilbert schemes on singular curves 2
   1.2. Quantized Gieseker varieties 4
   1.3. Coulomb branches and generalized affine Springer fibers 5
   1.4. Rational Cherednik algebras 6
   1.5. Relation to other work 8
   1.6. Structure of the paper 8
   Acknowledgments 9
2. Affine permutations 9
   2.1. The extended affine symmetric group 9
   2.2. $m$-stable and $m$-restricted permutations 12
   2.3. Lexicographic ordering and combinatorics of integer sequences 13
3. Rational Cherednik algebras 18
   3.1. Definition and basic properties 18
   3.2. An alternative presentation of $H_{t,c}$ 20
   4. A Mackey formula for $H_{t,c}$ 22
   4.1. Basis in $H_{t,c}$ 22
   4.2. Decomposition into $H_{t,c}(\mathfrak{u})$-modules 24
   5. Representation theory of $H_{t,c}$ 28
   5.1. Generalized eigenspaces and intertwining operators 28
   5.2. The standard module 29

Date: January 17, 2024.
E.G. was supported by the NSF grants DMS-1700814, DMS-1760329.
M.V. was supported by the Simons Foundation Collaboration Grant for Mathematicians, award number 319233.
1. Introduction

1.1. Hilbert schemes on singular curves. It is well-known and classical that, given a smooth algebraic curve, its Hilbert scheme of $k$ points is smooth and, in fact, isomorphic to the $k$-th symmetric product of the curve. On the contrary, much less is known in the case of a singular curve. In particular, Maulik [45] proved a conjecture of Oblomkov and Shende [49] relating the Euler characteristics of Hilbert schemes of points on a plane curve singularity to the HOMFLY-PT polynomial of its link. A more general conjecture of Oblomkov, Rasmussen and Shende [50, 27] relates the homology of these Hilbert schemes to the HOMFLY-PT homology of the link.

One possible approach to understanding Hilbert schemes of curves is by constructing an action of interesting algebras on their homology. Rennemo [55] constructed an action of the two-dimensional Weyl algebra on the homology of the Hilbert scheme of an integral locally planar curve (see also [47, 46]), and Kivinen [36] generalized this action to reduced locally planar curves with several components.

In this paper, we relate the geometry of (parabolic) Hilbert schemes on singular curves to the representation theory of the type $A$ rational Cherednik algebra and other related algebras.
More precisely, consider coprime positive integers $m$ and $n$, and let $C := \{x^m = y^n\}$ be a plane curve singularity in $\mathbb{C}^2$. We will always work locally near the origin, to simplify the notations we will always use $O_C$ for the completion of $\mathbb{C}[C]$ at $(0,0)$.

Note that for every ideal $I \subseteq O_C$ we have that $\dim(I/xI) = n$. We consider the parabolic Hilbert scheme $PHilb_{k,n+k}(C)$ that is the following moduli space of flags

$$(1) \quad PHilb_{k,n+k}(C) := \{O_C \supset I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k\}$$

where $I_k$ is an ideal in the ring of functions $O_C$ of codimension $s$. Moreover, we set $PHilb^e(C) := \sqcup_k PHilb_{k,n+k}(C)$. The natural $\mathbb{C}^*$ action on $C$ naturally lifts to $PHilb^e(C)$. Since $m$ and $n$ are coprime, the fixed points are precisely the flags of monomial ideals. In particular, the classes of these fixed points form a basis for the localized equivariant cohomology. The first main result of this paper is the following.

**Theorem 1.1.** There is a geometric action of the rational Cherednik algebra $H_{1,m/n}(S_n, \mathbb{C}^n)$ on the localized $\mathbb{C}^*$-equivariant homology of $PHilb^e(C)$. Moreover, with this action $H_*^{\mathbb{C}^*}(PHilb^e(C))$ gets identified with the simple highest weight module $L_{m/n}(\text{triv})$.

Recall that the rational Cherednik algebra $H_{t,c} := H_{t,c}(S_n, \mathbb{C}^n)$ contains the trivial idempotent $e := \frac{1}{n!} \sum_{P \in S_n} P$, and we can form the spherical subalgebra $eH_{t,c}e$. As a consequence of Theorem [1.1], we get that the spherical subalgebra acts on the equivariant homology of the Hilbert scheme $Hilb(C) := \sqcup_k Hilb_k(C)$.

**Corollary 1.2.** There is an action of the spherical rational Cherednik algebra $eH_{1,m/n}(S_n, \mathbb{C}^n)e$ on the localized $\mathbb{C}^*$-equivariant homology of $Hilb(C)$. Moreover, with this action $H_*^{\mathbb{C}^*}(Hilb(C))$ gets identified with $eL_{m/n}(\text{triv})$.

**Remark 1.3.** By [46, 47] the homology of the Hilbert schemes of singular curves is closely related to the homology of the corresponding compactified Jacobian, equipped with a certain “perverse” filtration. By [27, 51, 52, 58] the latter homology carries an action of the spherical trigonometric Cherednik algebra. [11]. Furthermore by [51, 52] the associated graded space admits a natural action of the spherical rational Cherednik algebra corresponding to the reflection representation of $S_n$ (also known as spherical rational Cherednik algebra of $\mathfrak{sl}_n$). The construction of this action uses global Springer theory developed by Yun [52].

The main advantage of our proof of Theorem [1.1] is that it does not use compactified Jacobians or perverse filtration at all. The generators of $H_{1,m/n}(S_n, \mathbb{C}^n)$ are identified with certain explicit operators in the homology of $PHilb^e(C)$.

We explore some ramifications of this result. In the theory of rational Cherednik algebras there is a “$t = 0$” and “$t = 1$” dichotomy, see Section [3.1] and in the statement of Theorem [1.1] we assume that $t = 1$. While the representation theory of the Cherednik algebra is very sensitive to this dichotomy, we have a version of Theorem [1.1] in the $t = 0$ case.

To this end, consider the non-reduced curve $C_0 := \{y^n = 0\}$. The punctual Hilbert scheme on $C_0$ is the moduli space of finite-codimensional ideals in the local ring $O_{C_0,0} = \mathbb{C}[x, y]/(y^n)$, and we may define the parabolic (punctual) Hilbert scheme $PHilb_{k,n+k}(C_0)$ analogously to (1). Again we set $PHilb^e(C_0) := \sqcup_k PHilb_{k,n+k}(C_0)$. We show the following “$t = 0$” (or “$m = \infty$”) analogue of Theorem [1.1].
Theorem 1.4. There is a geometric action of $H_{0,1}(S_n, \mathbb{C}^n)$ on the localized $\mathbb{C}^*$-equivariant cohomology of $\text{PHilb}^r(C_0)$, where $C_0$ is the non-reduced curve $\{y^n = 0\}$. Moreover, with this action $H^*_{\text{et}}(\text{PHilb}^r(C_0))$ gets identified with the polynomial representation $\Delta_{0,1}(\text{triv})$.

Similarly to Corollary 1.2 we get an action of the spherical subalgebra $eH_{0,1}e$ on the equivariant homology of $\text{Hilb}(C_0)$, and under this action $H^*_{\text{et}}(\text{Hilb}(C_0))$ gets identified with the polynomial representation of $eH_{0,1}e$.

1.2. Quantized Gieseker varieties. Another ramification of Theorem 1.1 connects parabolic Hilbert schemes to the representation theory of quantized Gieseker varieties. These are quantizations of the moduli space $\mathcal{M}(n,r)$ of rank $r$ torsion-free sheaves on $\mathbb{P}^2$ with fixed trivialization at infinity and second Chern class $c_2 = n$. The quantization, denoted $\mathcal{A}_c(n,r)$, depends on a parameter $c \in \mathbb{C}$, see Section 8 for a precise definition. For example, when $r = 1$, $\mathcal{M}(n,1)$ is simply the Hilbert scheme of $n$ points in $\mathbb{C}^2$ and $\mathcal{A}_c(n,1)$ is the spherical rational Cherednik algebra, see [22].

There is currently no presentation of the algebra $\mathcal{A}_c(n,r)$ by generators and relations. Nevertheless, Losev [41] managed to classify all finite-dimensional representations for a slightly smaller algebra $\mathcal{A}_c(n,r)$ such that $\mathcal{A}_c(n,r) = D(\mathbb{C}) \otimes \mathcal{A}_c(n,r)$. Namely, if $c = m/n$, $\gcd(m,n) = 1$ and $c$ is not in the interval $(-r,0)$ then $\mathcal{A}_c(n,r)$ has a unique irreducible finite-dimensional representation $\mathcal{E}_\pm(n,r)$, otherwise there are none. Furthermore, the action of $GL(r)$ on $\mathcal{M}(n,r)$ induces a quantum commutant map $\mathfrak{gl}(r) \to \mathcal{A}_c(n,r)$ and hence defines an action of $\mathfrak{gl}(r)$ on $\mathcal{E}_\pm(n,r)$. In [18] Etingof, Krylov, Losev and the second author computed the dimension and graded $\mathfrak{gl}(r)$ character of $\mathcal{E}_\pm(n,r)$.

In this paper we give a geometric construction of this representation for $m,n > 0$. Fix an integer $r > 0$, and denote by $\mathcal{C}_r(n) \subseteq \mathbb{Z}_{\geq 0}$ the set of $r$-tuples of non-negative integers that add up to $n$. For $\gamma = (\gamma_1, \ldots, \gamma_r) \in \mathcal{C}_r(n)$ we consider the following parabolic Hilbert scheme

$$
\text{PHilb}_{x}^{\gamma, x}(C) := \left\{ O_C \supseteq J^0 \supseteq J^1 \supseteq \cdots \supseteq J^r = xJ^0 : \dim(O_C/J^0) < \infty \text{ and } \dim(J^{i-1}/J^i) = \gamma_i \right\}.
$$

For example, $\text{PHilb}_{x}^{\gamma, x}(C) = \text{PHilb}_{1, \ldots, 1, x}^{(1<\cdots<x)}(C)$, where $(1,1,\ldots,1) \in \mathcal{C}_n(n)$ and $\text{Hilb}(C) := \sqcup_{k \geq 0} \text{Hilb}_k(C)$ is $\text{PHilb}_{0, 1, \ldots, 1}^{(n, \ldots, 1)}(C)$, where $(n) \in \mathcal{C}_1(n)$. We define the compositional parabolic Hilbert scheme of $C$ to be

$$
\text{CPHilb}_{x}^{\gamma, x}(C) := \bigcup_{\gamma \in \mathcal{C}_r(n)} \text{PHilb}_{x}^{\gamma, x}(C).
$$

Remark 1.5. Note that if $\gamma_i \leq 1$ for every $i$ then we have a natural isomorphism $\text{PHilb}_{x}^{\gamma, x}(C) = \text{PHilb}_{x}^{\gamma, x}(C)$. In particular, $\text{PHilb}_{x}^{\gamma, x}(C)^{i, \ldots, i} \subseteq \text{CPHilb}_{x}^{\gamma, x}(C)$. Similarly, if there exists $i$ such that $\gamma_i = n$ and $\gamma_j = 0$ for $j \neq i$ then we have a natural isomorphism $\text{PHilb}_{x}^{\gamma, x}(C) = \text{Hilb}(C)$, so that $\text{Hilb}(C)^{i, \ldots, i} \subseteq \text{CPHilb}_{x}^{\gamma, x}(C)$.

Remark 1.6. Note that we have chosen one projection to define our parabolic Hilbert schemes. We could have instead chosen the other projection so that, for $\gamma \in \mathcal{C}_r(m)$ we have the parabolic Hilbert scheme $\text{PHilb}_{x}^{\gamma, x}(C)$, where the condition
Therefore the homology of \( \text{CPHilb}^{r,y}(C) \) are disjoint unions of

Example 1.10. case of Theorem 1.7.

On the other hand, Remark 1.8. The appearance of the scheme \( \text{CPHilb}^{r,y}(C) \) opposed to \( \text{CPHilb}^{r,x}(C) \). The algebra \( C \) action of a reductive group

The BFN quantized Coulomb branch algebra corresponds to the choice

appears in \([18, \text{Corollary } 2.18]\). See Section 8 for more details.

Example 1.9. When \( r = 1 \), \( C_{c}(m) = \{(m)\} \) and \( \text{CPHilb}^{1,y}(C) = \text{Hilb}(C) \). Since in this case \( A_{m/n}(n,1) = e H_{1,m/n}(S_{m}, \mathbb{C}^{n})e \), we see that Corollary 1.2 is a special case of Theorem 1.7. (as opposed to \( \text{CPHilb}^{r,x}(C) \)) is explained as follows. Note that by Theorem 1.8, the algebra \( H_{1,n/m}(S_{m}, \mathbb{C}^{m}) \) acts on the equivariant cohomology of \( \text{PHilb}^{y}(C) \). The \( x,y \)-switch in Theorem 1.7 is then a geometric incarnation of the \( n,m \)-switch that appears in \([18, \text{Corollary } 2.18]\). See Section 8 for more details.

Example 1.10. When \( n = 1 \), the curve \( C \) is smooth, and all the spaces \( \text{PHilb}^{Y}(C) \) are disjoint unions of \( \mathbb{Z}_{\geq 0} \) copies of contractible spaces (labeled by \( \dim O_{C}/J^{0} \)). Therefore the homology of \( \text{CPHilb}^{r,y}(C) \) can be naturally identified with

On the other hand, \( \overline{C}_{c}(1,r) \) is isomorphic to a certain quotient of \( \mathcal{U}(\mathfrak{sl}(r)) \), and \( \overline{L}_{m,n}(n,r) \) is identified with \( S^{m}(C^{r}) \otimes \mathbb{C}[X] \).

1.3. Coulomb branches and generalized affine Springer fibers. From the action of a reductive group \( G \) on a vector space \( N \), Braverman, Finkelberg and Nakajima [4] construct an associative algebra called the Coulomb branch algebra, which is modeled after the equivariant homology of the affine grassmannian of \( G \), where multiplication is given by convolution. This algebra admits a natural quantization that appears when we take the loop rotation into account for the equivariance. Webster in [60] generalized their construction by introducing a category of line defects, where the BFN quantized Coulomb branch algebra appears as the endomorphism algebra of an object. Roughly speaking, a line defect consists of the choice of a parahoric subgroup \( P \subseteq G_{k} \) and a subspace \( L \subseteq N_{k} \) preserved by \( P \). The BFN quantized Coulomb branch algebra corresponds to the choice \( P = G_{k} \) and \( L = N_{k} \). It turns out that all of the algebras we work with in this paper appear as BFN quantized Coulomb branches or their generalizations:

- The spherical Cherednik algebra \( e H_{1,c}(S_{n}, \mathbb{C}^{n})e \) is the BFN quantized Coulomb branch for \( G = \text{GL}_{n} \) and \( N = \mathbb{C}^{n} \oplus \mathfrak{gl}_{n} \), [28, 61].
- The full Cherednik algebra \( H_{1,c}(S_{n}, \mathbb{C}^{n}) \) appears in the same setting as above, but choosing a nontrivial line defect associated to \( P = I \), the standard Iwahori subgroup, and \( L = \mathbb{O}^{n} \oplus i \), where \( i \) is the Lie algebra of the standard Iwahori, [61, 40].
• The quantized Gieseker variety $\mathcal{A}(n, r)$ is the BFN quantized Coulomb branch for $G = GL_n^r$ and $N = \mathbb{C}^n \oplus gl_n^{r}$. This follows from results of [48] and [42]. This is an example of symplectic duality [60] since $\mathcal{A}(n, r)$ appears both as the quantized Higgs branch for the Jordan quiver and the quantized Coulomb branch for the cyclic quiver with $r$ nodes.

The recent paper [34] constructs an action of the quantized Coulomb branch in the cohomology of generalized affine Springer fibers in the sense of [24], again by certain convolution diagrams. This has been extended to the parahoric setting in [23]. We identify the different parabolic Hilbert schemes we consider with generalized affine Springer fibers.

• For $\text{Hilb}(C)$, this is [23, Theorem 3.5].
• For $\text{PHilb}^p(C)$, see Proposition 7.19.
• For $\text{CPhilb}^r,y(C)$, see Proposition 8.9.

While we take this as a motivation for Theorems 1.1 and 1.7, our proofs do not use any of these technologies, in particular we do not obtain the action via convolution diagrams. The proofs of Theorems 1.1 and 1.4 are based on the study of the combinatorics of the various Hilbert schemes we consider, as well as the combinatorial representation theory of the rational Cherednik algebra. The development of this depends on a suitable presentation of this algebra, and we use work of Webster [61], and more recent work of LePage-Webster [40] to verify, in the case of the scheme $\text{PHilb}^p(C)$, that our action coincides with the one constructed in [23] via convolution diagrams, see Section 7.5.

On the contrary, there is no known set of generators and relations for the algebra $\mathcal{A}_{c}(n, r)$. However, we use Theorem 1.1 together with [18, Theorem 2.17] that constructs representations of $\mathcal{A}_{m/n}(n, r)$ starting from representations of $H_{n/m}(m)$ to prove Theorem 1.7.

1.4. Rational Cherednik algebras. The main idea behind the proof of Theorem 1.1 is to identify a basis in $L_{m/n}(\text{triv})$ that corresponds to the fixed-point basis in $H_{C}^{\tau}(\sqcup_k \text{PHilb}_{k,n+k}(C))$. Our main tool to construct this basis is a presentation of the rational Cherednik algebra $H_{t,c}(S_n, \mathbb{C}^n)$ that is better-suited for this purpose than the usual presentation. To lighten notation, we write $H_{C} = H_{t,c}(S_n, \mathbb{C}^n)$ below. Recall that, in its usual presentation, the algebra $H_{C}$ has generators $x_i, y_i$ $(i = 1, \ldots, n)$ and $S_n$. It is naturally graded, with $x_i$ of degree 1, $y_i$ of degree $-1$ and $S_n$ in degree zero. Dunkl and Opdam [15] constructed a family of commuting operators $u_1, \ldots, u_n$ of degree 0 in $H_{C}$. The algebra $H_{C}$ is, in fact, generated by $u_i$, the group algebra of $S_n$ and two additional generators

$$\tau := x_1(12\cdots n), \lambda := (12\cdots n)^{-1}y_1.$$ 

It is clear that $\tau, \lambda$ and $S_n$ already generate the algebra since one can obtain $x_1$ and $y_1$ (and hence all $x_i$ and $y_i$) using them. In Theorem 3.3 we give a complete list of relations between $\tau, \lambda, u_i$ and the generators of $S_n$. This presentation of the algebra $H_{C}$ has already appeared in the more complicated cyclotomic setting in the work of Griffeth [31] and Webster [61]. Since some relations become more transparent in the type $A$ setting, we present it in detail. The generators $u_i$ can be, in principle, eliminated, and the remaining relations are listed in Proposition 3.6.

We use the presentation of the algebra $H_{C}$ via the Dunkl-Opdam operators to, in the case where $c$ is a rational number with denominator precisely $n$, simultaneously
diagonalize the operators $u_i$ on the polynomial representation $\Delta_c(\text{triv})$ and give an explicit combinatorial description of the eigenvalues. We prove that the action of the operators $\tau$ and $\lambda$ sends an eigenvector to a multiple of another eigenvector, and describe the action of $S_n$ on an eigenbasis explicitly. We remark that this has already appeared in work of Griffeth, see [29, 30, 31] and Remark 1.13, but we reprove these results with combinatorics that are more amenable to our geometric goal.

**Theorem 1.11.** Let $c = m/n$, where $m, n \in \mathbb{Z}_{>0}; \gcd(m, n) = 1$. Then the following holds:

(a) $\Delta_c(\text{triv})$ has a basis $v_a$ labeled by sequences $a = (a_1, \ldots, a_n)$ of nonnegative integers. The action of $u_i, \tau$ and $\lambda$ in this basis is given by

$$u_i v_a = (a_i - (g_a(i) - 1)c)v_{\pi \cdot a}, \quad \tau v_a = v_{\pi \cdot a}, \quad \lambda v_a = (a_1 - (g_a(1) - 1)c)v_{\pi^{-1} \cdot a}$$

where $\pi(a_1, \ldots, a_n) = (a_n + 1, a_1, \ldots, a_{n-1})$ and $g_a$ is the minimal length permutation sorting the sequence $a$ to be non-decreasing.

(b) $L_c(\text{triv})$ has a basis $v_a$ labeled by sequences $(a_1, \ldots, a_n)$ such that $|a_i - a_j| \leq m$ for all $i, j$ and if $a_i - a_j = m$ then $i < j$.

The action of $S_n$ in the basis $v_a$ is given in Theorem 6.5.

**Remark 1.12.** Note that $\pi^{-1} \cdot a$ is well defined unless $a_1 = 0$. In this case $a_1 - (g_a(1) - 1)c = 0$, so $\lambda \cdot v_a$ is well defined.

The proof of Theorem 1.11 is based, roughly speaking, on the comparison of the basis of fixed points in $H^*(\sqcup_k \text{PHilb}_{k,n} + k(C))$ with the basis given by Theorem 1.11 b).

The proof of Theorem 1.11 uses a Mackey-type result for the algebra $H_c$. The algebra $H_n(u)$ generated by $u_1, \ldots, u_n$ and $S_n$ is isomorphic to the degenerate affine Hecke algebra of rank $n$. In Theorem 4.10 we construct a filtration of $\text{Res}_{H_n(u)} \Delta_c(\mu)$ by $H_n(u)$-modules and explicitly describe the subquotients. As a consequence, we are able to give a combinatorial basis of all standard modules $\Delta_c(\mu)$.

**Remark 1.13.** In [30, Theorem 5.1] Griffeth constructs, for generic values of the parameter $(t, c)$ an eigenbasis of every standard module, and in [29] he considers the case of the polynomial representation. Both Theorem 1.11 and the construction of a combinatorial basis for standard modules are a consequence of this and [30, Theorem 7.5] after specializing parameters. Our proof and construction of eigenbasis, using a Mackey-type formalism, is more conceptual and its combinatorics seem better-suited for geometric applications.

As a further application of the combinatorics of the Dunkl-Opdam presentation of the algebra $H_c$ we are able to give an explicit combinatorial construction of all the maps appearing in the BGG resolution of the module $L_c(\text{triv})$ for $c = m/n$, and we show that the complex formed by these maps is indeed exact. In particular, we give a new construction of this resolution that avoids appealing to the representation theory of finite Hecke algebras at roots of unity via the Knizhnik-Zamolodchikov functor, which uses techniques of complex analysis. Moreover, we are able to give a combinatorial basis in the spirit of that of Theorem 1.11 for every simple module $L_c(\mu)$, see Corollary 5.26.
Remark 1.14. More concretely, the standard modules $\Delta_c(n-\ell,1^\ell)$ and $\Delta_c(n-\ell+1,1^{\ell-1})$ have bases labeled by pairs $(a,T)$ and $(a',T')$ where $T$ and $T'$ are standard tableaux of the corresponding hook shapes. We explicitly compute matrix elements of the map between standard modules in this basis in the case $c=m/n$. As a consequence, we give two labelings of the basis in $L_c(n-\ell,1^\ell)$ presented either as a simple quotient of $\Delta_c(1^\ell,n-\ell)$, or as the radical of $\Delta_c(n-\ell-1,1^{\ell+1})$, and an explicit bijection between them.

1.5. Relation to other work. Finally, we would like to comment on the relations of our work to the existing literature. As we have mentioned above, the Dunkl-Opdam presentation of the Cherednik algebra has already appeared in work of Griffeth and Webster, [30, 31, 61], where it has been used for different purposes. In particular, Griffeth [30, 31] uses the fact that the operators $u_i$ are self-adjoint with respect to the Shapovalov form to compute the norm of elements in standard modules, see also [14], while Webster [61] uses the Dunkl-Opdam subalgebra to give a concrete equivalence between the category $O$ and modules over the quiver Hecke algebra.

By [46, 47] the homology of the Hilbert schemes of singular curves is closely related to the homology of the corresponding compactified Jacobian which is isomorphic to the affine Springer fiber in the affine Grassmannian. One would expect a similar connection between our parabolic Hilbert schemes and affine Springer fibers in the affine flag variety. These affine Springer fibers do admit affine pavings, and the combinatorics of the affine cells was studied in detail in [43, 50, 26].

The (co)homology of the affine Springer fibers in affine flag variety was studied in [27, 51, 52, 58] where it was proved that it carries an action of the trigonometric Cherednik algebra. Furthermore, this (co)homology has certain “perverse” filtration, and the associated graded space admits a natural action of the rational Cherednik algebra corresponding to the reflection representation of $S_n$ (also known as rational Cherednik algebra of $\mathfrak{sl}_n$). The construction of this action uses global Springer theory developed by Yun [62]. The combinatorics of finite dimensional representations of the rational Cherednik algebra for $\mathfrak{sl}_n$ was studied by Shin [53].

On the contrary, we find our construction to be more elementary than [51, 52]. Indeed, in our construction of geometric operators $\tau$ and $\lambda$ we use neither perverse filtration nor global Springer theory. The combinatorial presentation of the algebra is easier in the $\mathfrak{gl}_n$ setup. Still, we make an explicit comparison with the results of [26] in Section 6.3, see Remark 6.12.

1.6. Structure of the paper. The main body of the paper follows a reverse structure from the introduction. First we study the representation theory of rational Cherednik algebras and then we move on to Hilbert schemes. In Section 2 we look at the combinatorics of the affine symmetric group that will appear in the representation theory of the rational Cherednik algebra. We study the structure of the Cherednik algebra in Section 3.1 and, in particular, we give its alternative presentation that is better suited for geometry. Sections 4, 5 and 6 are devoted to the study of modules in category $O$ of the Cherednik algebra. In Section 4 we prove a Mackey-type formula for induced representations of the Cherednik algebra and apply it to study standard modules in Section 5. In particular we construct the BGG resolution of [2] in a purely combinatorial manner. Since the polynomial representation occupies a special place in geometric applications, we specialize the results of Section 5 to the polynomial representation in Section 6.
We turn to Hilbert schemes in Section 7. First, we examine the case of the reduced curve \( C = \{ x^m = y^n \} \) and prove Theorem 1.1, see Theorem 7.14. In this section, we also compare the parabolic Hilbert scheme to generalized affine Springer fibers, in particular proving that they admit a paving by affine cells, see Section 7.4. Section 8 is devoted to the compositional parabolic Hilbert scheme \( \text{CP Hilb}_{r,y}(C) \). We prove Theorem 1.7 as Theorem 8.6 and also realize this Hilbert scheme as a generalized affine Springer fiber. Finally, we study the case of the non-reduced curve \( C = \{ y^n = 0 \} \) in Section 9 where we prove Theorem 1.4, see Theorem 9.8.

Acknowledgments

We would like to thank Tudor Dimofte, Niklas Garner, Joel Kamnitzer, Oscar Kivinen, Ivan Losev, Alexei Oblomkov and Ben Webster for useful discussions. We would also like to thank Stephen Griffeth for comments on the relationship of this paper with some of his previous work. We are grateful to Sean Griffin for spotting a gap in a previous proof of Proposition 7.19. We would also like to thank the anonymous referees for their helpful suggestions on improving the organization and exposition of this paper.

2. Affine permutations

2.1. The extended affine symmetric group. In this section, we study the combinatorics of the extended affine symmetric group \( \tilde{S}_n \). For more details, see [28, 44].

**Definition 2.1.** The extended affine symmetric group is defined by the relations

\[
\tilde{S}_n = \langle \pi, s_i, 1 \leq i < n \mid s_is_{i+1}s_i = s_{i+1}s_is_i+1 \quad \text{for} \ 1 \leq i < n - 1, \\
\pi s_j = s_j \pi \quad \text{for} \ j \neq i \pm 1, \\
\pi s_i = s_{i+1}\pi \quad \text{for} \ 1 \leq i < n - 1, \\
s_i^2 = 1 \quad \text{for} \ i \in \mathbb{Z}/n\mathbb{Z} \rangle.
\]

Letting \( s_0 = \pi^{-1}s_1\pi \), we could consider generators \( s_i \) for \( i \in \mathbb{Z}/n\mathbb{Z} \). In this case \( \tilde{S}_n \) has as a subgroup the affine symmetric group \( \hat{S}_n = \langle s_i \mid i \in \mathbb{Z}/n\mathbb{Z} \rangle \). However for the purposes of this paper, we rarely take this point of view. Further, we will refer to elements \( p \in \tilde{S}_n \) as affine permutations, dropping the adjective “extended.”

We recall that \( \tilde{S}_n \) acts faithfully on \( \mathbb{Z} \) by \( n \)-periodic permutations, i.e. bijections \( p : \mathbb{Z} \to \mathbb{Z} \) such that \( p(i + n) = p(i) + n \). For this action \( \pi(i) = i + 1 \). It also acts on the set \( \mathbb{C}^n \) via:

\[
s_i \cdot (w_1, \ldots, w_i, w_{i+1}, \ldots, w_n) = (w_1, \ldots, w_{i+1}, w_i, \ldots, w_n) \\
\pi \cdot (w_1, \ldots, w_n) = (w_n + t, w_1, w_2, \ldots, w_{n-1})
\]

for a fixed parameter \( t \in \mathbb{C} \). It is convenient to extend an \( n \)-tuple to having coordinates indexed by all of \( \mathbb{Z} \) via \( w_{i+kn} = w_i - kt \). Then we may align the two actions, writing \( p \cdot (w_1, \ldots, w_n) = (w_{p-1(1)}, w_{p-1(2)}, \ldots, w_{p-1(n)}) \). This is consistent with our conventions taken in Remark 3.4 below.

Just as with the finite symmetric group, it is convenient to use window notation for affine permutations. The window notation of \( p \) is given by \([p(1), p(2), \ldots, p(n)]\), which determines \( p \) by periodicity.

\footnote{We drop the first relation when \( n = 2 \).}
We will follow the usual convention to extend Coxeter length $\ell$ from $S_n$ to $\widetilde{S}_n$ by setting $\ell(\pi) = 0$. Then length still counts the number of affine inversions, that is $\ell(p) = \#\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq n, i < j, p(i) > p(j)\}$.

**Definition 2.2.** Let us define the degree of $p \in \widetilde{S}_n$ to be $\frac{1}{n}\sum_{i=1}^{n}(p(i) - i)$.

Let $\widetilde{S}_n$ denote the submonoid of affine permutations $p$ such that $i > 0 \implies p(i) > 0$, i.e., the entries of $p$ in window notation are all positive.

Note $p \in \widetilde{S}_n$ iff it has degree 0. The only permutations in $\widetilde{S}_n^+$ of degree 0 are those in the finite symmetric group $S_n$.

Let $t_a \in \widetilde{S}_n$ denote translation by $a \in \mathbb{Z}^n$. In other words, its window notation is $t_a = [1+na_1, 2+na_2, \ldots, n+na_n]$. It is well-known that the subgroup $\{t_a \mid a \in \mathbb{Z}^n\}$ generated by translations is normal in $\widetilde{S}_n$ and it is isomorphic to $\mathbb{Z}^n$. Moreover, there is a semi-direct product decomposition $\widetilde{S}_n \simeq \mathbb{Z}^n \rtimes S_n$. This gives us the first part of the following Lemma, while the second assertion is obvious from the definitions.

**Lemma 2.3.** Any permutation $\omega \in \widetilde{S}_n$ can be uniquely written as $\omega = t_a g$ for $g \in S_n$, $a \in \mathbb{Z}^n$. Furthermore, $\omega \in \widetilde{S}_n^+$ if and only if $\omega = t_a g$ and $a_i \geq 0$ for all $i$.

Let sort($a$) denote the non-decreasing ordering of $a$, and $g_a \in S_n$ the shortest element such that $g_a \cdot a = \text{sort}(a)$. Note that the element $g_a$ is given by

$$g_a(i) = \sharp\{j : a_j < a_i\} + \sharp\{j : j \leq i \text{ and } a_i = a_j\}. \tag{5}$$

We denote $\omega_a := t_a g_a^{-1}$.

**Remark 2.4.** Note that the element $g_a \in S_n$ is uniquely specified by the requirement that the window notation of $t_a g_a^{-1}$ is increasing.

The following proposition follows easily from the semi-direct product decomposition of $\widetilde{S}_n$. In other words, we have a projection $\widetilde{S}_n \to \mathbb{Z}^n$ and the assignment $a \to \omega_a$ is a right inverse to this map.

**Proposition 2.5.** $\omega_a = \omega_b$ if and only if $a = b$.

In fact, the following stronger statement holds.

**Lemma 2.6.** Assume $\omega_a(i) = \omega_b(i)$ for some $i \in \{1, \ldots, n\}$. Then $g_a^{-1}(i) = g_b^{-1}(i)$ and $a_{g_a^{-1}(i)}(i) = b_{g_b^{-1}(i)}(i)$.

**Proof.** If $\omega_a(i) = \omega_b(i)$ then $g_a^{-1}(i) - g_b^{-1}(i) = n(b_{g_b^{-1}(i)} - a_{g_a^{-1}(i)})$. But $g_a^{-1}(i)$ and $g_b^{-1}(i)$ are in $\{1, \ldots, n\}$ so their difference is only divisible by $n$ if it is in fact 0. The result follows. \square

Let $L_{\min}^+(n)$ denote the set $L_{\min}^+(n) = \{\omega_a \in \widetilde{S}_n^- \mid a \in \mathbb{Z}_{\geq 0}^n\}$. Then $\omega = \omega_a \in \widetilde{S}_n^-$ is a minimal length (left) coset representative of $\widetilde{S}_n^-/S_n$, i.e., we have $0 < \omega(1) < \omega(2) < \cdots < \omega(n)$. Further note that the degree of $\omega_a$ as well as that of $t_a$ agrees with $||a|| := \sum a_i$.

It is easy to see the following holds.


Lemma 2.7. Let \( \omega \in \mathbf{L}^+_\text{min}(n) \) be of degree \( r > 0 \). Then there is a unique expression of the form

\[
\omega = (s_{\nu_r} \cdots s_2 s_1)\pi \cdots (s_{\nu_2} \cdots s_2 s_1)\pi(s_{\nu_1} \cdots s_2 s_1)\pi,
\]

where \( 0 \leq \nu_{i+1} \leq \nu_i \). In other words \( \nu \) is a partition with \( \nu_1 < n \) and \( r \) parts, and we allow parts to be zero.

Proof. Let us induct on \( r \), noting we exclude the case \( r = 0 \) from the hypotheses. This corresponds to \( \omega = \text{id} \). For \( r = 1 \) consider the window notation \( \omega = [\omega(1), \cdots, \omega(n)] \). Recall \( 0 < \omega(1) < \omega(2) < \cdots < \omega(n) \). In particular \( 0 \leq \omega(i) - i \) but \( n < \omega(n) \) since \( \omega \neq \text{id} \). Since the degree of \( \omega \) is \( 1 \), \( n = \sum_{i=1}^n (\omega(i) - i) \) which forces \( \omega(n) \leq 2n \) and hence \( 0 < \omega(n) - n \leq n \). Then \( \omega^{-1} = [\omega(n) - n, \omega(1), \cdots, \omega(n-1)] \in \sim_n^+ \) has degree 0 and so \( \omega^{-1} \in S_n \). Let \( k \) be maximal such that \( \omega(k) < \omega(n) - n \) and 0 otherwise, in which case we have \( \omega = \pi \). Then \( \omega^{-1} s_1 s_2 \cdots s_k \in S_n \cap \mathbf{L}^+_\text{min}(n) = \{\text{id}\} \) which implies \( \omega = s_k \cdots s_2 s_1 \pi \). This proves the base case.

Next assume the claim holds for all affine permutations in \( \mathbf{L}^+_\text{min}(n) \) of degree \( r < r \). Suppose \( \omega \) has degree \( r \). Choose \( k \) exactly as above, and note \( p = \omega^{-1} s_1 s_2 \cdots s_k \in \mathbf{L}^+_\text{min}(n) \) is of degree \( r - 1 \). By the inductive hypothesis, the claim holds for \( p \) with respect to \( r - 1 \) parts. We renumber as \( n > \nu_2 \geq \nu_3 \geq \cdots \geq \nu_r \geq 0 \).

Thus

\[
\omega = (s_{\nu_r} \cdots s_2 s_1)\pi \cdots (s_{\nu_2} \cdots s_2 s_1)\pi(s_{\nu_1} \cdots s_2 s_1)\pi.
\]

We need only show \( k > \nu_2 \) and then set \( \nu_1 = k \). Recall \( \nu_2 \) is maximal such that \( p(\nu_2) < p(n) - n \) and recall \( k \) is maximal such that \( \omega(k) < \omega(n) - n \). By choice of \( k \) we have \( p(k) = \omega(k) < \omega(n) - n \) and \( p(k+1) = \omega(n) - n \). If \( \nu_2 = k + 1 \) then \( p(\nu_2) = p(k+1) = \omega(n) - n \geq p(n) - n \) and if \( \nu_2 > k + 1 \) then \( p(\nu_2) = \omega(\nu_2 - 1) > \omega(n) - n \geq p_n - n \), both of which are contradictions.

Remark 2.8. Given \( \omega \in \mathbf{L}^+_\text{min}(n) \), the partition \( \nu \) can easily be obtained from the inversions of \( \omega \) as follows. For the transposed partition \( \nu^T \) which has \( n - 1 \) parts, \( \nu^T_i = \#\{k < i \mid \omega(k) > i\} \). Observe the length \( \ell(\omega) = |\nu| \).

Example 2.9. Let \( n = 5 \), \( a = (0, 2, 0, 0, 1) \). Thus \( g_a = [1, 5, 2, 3, 4] \), \( g_a^{-1} = [1, 3, 4, 5, 2] \), \( t_a = [1, 12, 3, 4, 10] \) and \( \omega_a = t_at_a^{-1} = [1, 3, 4, 10, 12] \) which has reduced word \( s_3 s_3 s_5 s_2 s_1 \pi s_3 s_2 s_1 \pi \) and so \( \nu = (3, 3, 1) \), \( \nu^T = (3, 2, 2) \). Note \( \omega(0) = 7, \omega(-1) = 5, \omega(-5) = 2 \) and

\[
\begin{align*}
\{k < 1 \mid \omega(k) > 1\} & = \{0, -1, -5\} & \nu^T_1 & = 3 \\
\{k < 1 \mid \omega(k) > 2\} & = \{0, -1\} & \nu^T_2 & = 2 \\
\{k < 1 \mid \omega(k) > 3\} & = \{0, -1\} & \nu^T_3 & = 2 \\
\{k < 1 \mid \omega(k) > 4\} & = \emptyset & \nu^T_4 & = 0 \\
\{k < 1 \mid \omega(k) > 5 = n\} & = \emptyset & \nu^T_5 & = 0.
\end{align*}
\]

There are other ways to obtain \( \nu \) from \( a \), but discussing them is beyond the scope of this paper. We will merely mention without proof one such way. Given \( a \) construct its \( n \)-abacus (with beads at heights determined by \( a \)) and then its corresponding \( n \)-core partition. Next, following Lapointe-Morse [39], remove all boxes from the \( n \)-core with hooklength \( > n \) and left-justify the remaining boxes. For the \( a \) given
above its 5-core is \((4, 3, 1)\), from which we remove its box in the upper left corner with hooklength 6 leaving us with \(\nu = (3, 3, 1)\).

2.2. \textit{m-stable and m-restricted permutations.} Here we recall some facts on \textit{m-stable} and \textit{m-restricted} affine permutations from \cite{20}.

\textbf{Definition 2.10.} (\cite{20}) We call an affine permutation \(\omega\) \textit{m-stable} if the inequality \(\omega(x + m) > \omega(x)\) holds for all \(x\). We call an affine permutation \(\omega\) \textit{m-restricted} if for all \(j < i\) one has \(\omega(j) - \omega(i) \neq m\).

It is clear that \(\omega\) is \textit{m-stable} if and only if \(\omega^{-1}\) is \textit{m-restricted}. Also, \(\omega\) is \textit{m-stable} if and only if
\[
\omega(\omega^{-1}(i) + m) > i \text{ for } i = 1, \ldots, n.
\]

\textbf{Definition 2.11.} We call a subset \(M \subset \mathbb{Z}^n\) \((m, n)\)-\textit{invariant} if \(M + n \subset M\) and \(M + m \subset M\).

If \(\omega \in \tilde{S}_n\) is an \textit{m-stable} permutation then for all \(i\) the set
\[
M^i_\omega = \{x \in \mathbb{Z}^n : \omega(x) \geq i\} = \omega^{-1}[i, +\infty).
\]
is \((m, n)\)-invariant. Indeed, if \(\omega(x) \geq i\) then \(\omega(x + n) = \omega(x) + n > i\) by definition of affine permutation and \(\omega(x + m) > \omega(x) \geq i\) because \(\omega\) is \textit{m-stable}.

Clearly, \(M^i_\omega + n = M^{i+1}_\omega + n\) and \(\omega\) is \textit{m-stable} if and only if \(M^i_\omega\) is \((m, n)\)-invariant for all \(i\). This implies the following useful proposition.

\textbf{Proposition 2.12.} An affine permutation \(\omega\) is \textit{m-stable} if and only if the sets \(M^i_\omega\) are \((m, n)\)-invariant for \(i = 1, \ldots, n\).

Next, we would like to characterize \textit{m-stable} and \textit{m-restricted} permutations using window notation, assuming \(\gcd(m, n) = 1\). As in \cite{20}, we use the affine permutation
\[
p_m := [0, m, \ldots, (n - 1)m].
\]

\textbf{Lemma 2.13.} Let \(\omega p_m = [x_1, \ldots, x_n]\). Then \(\omega\) is \textit{m-stable} if and only if
\[
x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1 + mn.
\]

\textbf{Proof.} It is sufficient to check the condition \(\omega(x + m) > \omega(x)\) for a single choice of \(x\) in each remainder modulo \(n\), in particular, for \(x = 0, m, 2m, \ldots, (n - 1)m\). Now for \(1 \leq i \leq n\) we have \(x_i = \omega(p_m(i)) = \omega((i - 1)m)\), so \(\omega\) is \textit{m-stable} if \(x_1 < \ldots < x_n\) and
\[
x_n = \omega((n - 1)m) < \omega(nm) = \omega(0) + nm = x_1 + mn.
\]

The condition \(x_1 < \ldots < x_n\) implies that we can write
\[
\omega p_m = t_a g_a^{-1}, \quad \omega = t_a g_a^{-1} p_m^{-1}, \quad \omega^{-1} = p_m g_a t_a
\]
for some vector \(a \in \mathbb{Z}^n\), and \(g_a\) as above. We can write
\[
\omega^{-1}(g_a^{-1}(i)) = p_m(-na g_a^{-1}(i) + i) = -na g_a^{-1}(i) + m(i - 1),
\]
so
\[
\omega^{-1}(i) = -na_i + m(g_a(i) - 1), \quad i = 1, \ldots, n.
\]

Hence, in window notation \(\omega^{-1} = [-na_1 + m(g_a(1) - 1), \ldots, -na_n + m(g_a(n) - 1)]\).

We get the following result:
Lemma 2.14. Let \( \gcd(m,n) = 1 \). A permutation \( \omega \) is \( m \)-stable if and only if \( \omega^{-1} \) can be written in the form \((\mathbf{a})\) for some vector \( \mathbf{a} \in \mathbb{Z}^n \) such that:
- \( a_i - a_j \leq m \) for all \( i, j \)
- \( a_i - a_j = m \) then \( i < j \).

Proof. Since \( \omega p_m = t_a g_a^{-1} = [x_1, \ldots, x_n] \), we get \( x_1 < \ldots < x_n \). We need to check the last condition \( x_n < x_1 + mn \) in terms of the vector \( a \).

Observe \( x_i = na_{g_a^{-1}(i)} + g_a^{-1}(i) \), so \( x_n < x_1 + mn \) if and only if either \( a_{g_a^{-1}(1)} + m > g_a^{-1}(n) \) or \( a_{g_a^{-1}(1)} + m = g_a^{-1}(n) \) and \( g_a^{-1}(1) > g_a^{-1}(n) \).

Now \( a_{g_a^{-1}(1)} = \min(a), a_{g_a^{-1}(n)} = \max(a) \), so either \( \max(a) - \min(a) < m \) or \( \max(a) - \min(a) = m \) and all appearances of \( \max(a) \) are to the left of all appearances of \( \min(a) \) in \( a \).

\( \square \)

Remark 2.15. The above results were stated in [26] for the affine symmetric group \( \tilde{S}_n \) (as opposed to extended affine \( \tilde{S}_n \)), but are equivalent to them after imposing the balancing condition for all affine permutations. In particular, \( p_m \) should be replaced by the degree 0 affine permutation \( \tilde{p}_m = [0, m, \ldots, (n-1)m - \kappa] \) where \( \kappa = \frac{1}{2}(mn-m-n-1) \). In particular, Lemma 2.13 can be rephrased by saying that \( \tilde{p}_m \) establishes a bijection between the set of \( m \)-stable affine permutations and the dilated fundamental alcove.

Example 2.16. Let \( n = 5, m = 3, a = (0,1,0,0,2) \). Thus \( g_a^{-1} = [1,3,4,2,5], \omega_a = [1,3,4,7,15] \), with inverses \( \omega^{-1} = [1,-1,2,3,-5] \) and \( g_a = [1,4,2,3,5] \).

Note \( \omega^{-1} = p_m \omega_a^{-1} = [0,3,6,9,12] \circ [1,-1,2,3,-5] = [0,4,3,6,2] \) is 3-restricted. Using \((\mathbf{a})\) we can also check \( \omega^{-1}(i) = -5a_i + 3g_a(i-1) \) as

\[
(0,4,3,6,2) = -5(0,1,0,0,2) + 3(0,3,1,2,4) = -5(0,1,0,0,2) + 3((1,4,2,3,5) - (1,1,1,1,1)).
\]

2.3. Lexicographic ordering and combinatorics of integer sequences. Recall that for \( a \in \mathbb{Z}^n \), we denote \(|a| := \sum_i a_i \). As in Section 2.1, we denote by \( g_a \in S_n \) the shortest element such that \( g_a \cdot a = \text{sort}(a) \).

Lemma 2.17. For every \( a \in \mathbb{Z}^n \), we have \( g_{\pi \cdot a} = g_a(12 \cdots n)^{-1} \). If \( a_i \neq a_{i+1} \), then we have \( g_{s_i \cdot a} = g_a s_i \).

Proof. We use the explicit equation \((\mathbf{a})\) for \( g_a \). Assume \( i \neq 1 \). Denote \( X_{\pi} := \{ j : (\pi \cdot a)_j < (\pi \cdot a)_i \} \) and \( Y_{\pi} := \{ j : (\pi \cdot a)_j = (\pi \cdot a)_i \} \) and \( j \leq i \). Similarly, denote \( X := \{ j : a_j < a_{i-1} \} \) and \( Y := \{ j : a_j = a_{i-1} \} \) and \( j \leq i - 1 \). So that \( g_{\pi \cdot a}(i) = g_{a}(i) X_{\pi} + g_{a}(i) Y_{\pi} \) and \( g_{\pi \cdot a}(i-1) = g_{a}(i) X + g_{a}(i) Y \). Note that, if \( j \neq 1 \), then \( j \in X_{\pi} \) (resp. \( j \in Y_{\pi} \)) if and only if \( j - 1 \in X \) (resp. \( j - 1 \in Y \)). Note also that we cannot have \( n \in Y \) because \( i - 1 < n \). Moreover, we have that \( 1 \in X_{\pi} \cup Y_{\pi} \) if and only if \( n \in X \) and, by the previous sentence, this happens if and only if \( n \in X \cup Y \). This shows that \( g_{\pi \cdot a}(i) = g_{a}(i) \). Note that this forces \( g_{\pi \cdot a}(1) = g_a(n) \). So \( g_{\pi \cdot a} = g_a(12 \cdots n)^{-1} \), as needed. The other equality is clear.

\( \square \)

It is easy to see that the assignment \( a \mapsto \omega_a \) gives a bijection between \( \mathbb{Z}^{n \geq 0} \) and the set \( \mathbb{L}^+_{\min}(n) \). More precisely, let us denote by \( P_k(n) \) the set \( \{ a \in \mathbb{Z}^n : |a| = k \} \). Inside, we have the sets
Moreover, the following properties are satisfied.

Lemma 2.18. The assignment $a \mapsto \omega_a$ gives a bijection between $\mathbb{Z}_{\geq 0}^n$ and the set $L_{\min}^+(n)$. Moreover,

(a) the set $\mathcal{P}_k(n)$ gets identified with

$$L_{\min}^+(n)_k := \{ \omega \in L_{\min}^+(n) : \deg \omega = k \}.$$  

(b) $\mathcal{P}_k'$ gets identified with $\{ \omega \in L_{\min}^+(n)_k : \omega(1) = 1 \}$.

(c) $\mathcal{P}_k^\circ$ with $\{ \omega \in L_{\min}^+(n)_k : \omega(1) > 1 \}$.

For affine permutations $\omega, \omega' \in \widetilde{S}_n$, we say that $\omega \succ_{\text{lex}} \omega'$ if the window notation of $\omega$ is greater than that of $\omega'$ in lexicographic ordering. More explicitly, $\omega \succ_{\text{lex}} \omega'$ if there exists $i \in \{1, \ldots, n\}$ such that $\omega(j) = \omega'(j)$ for $j = 1, \ldots, i - 1$ and $\omega(i) > \omega'(i)$.

We would like to study this partial order in more detail. In particular, we will see how it translates to $\mathbb{Z}_{\geq 0}^n$ under the bijection $a \mapsto \omega_a$. In order to do this, let us define a partial order on $\mathcal{P}_k(n)$ inductively. For $n = 2$ and even $k = 2\ell$, we have

$$(\ell, \ell) \prec (\ell + 1, \ell - 1) \prec (\ell - 1, \ell + 1) \prec \cdots \prec (2\ell, 0) \prec (0, 2\ell)$$

and for $k = 2\ell + 1$ odd we have

$$(\ell + 1, \ell) \prec (\ell, \ell + 1) \prec (\ell + 2, \ell - 1) \prec (\ell - 1, \ell + 2) \prec \cdots \prec (2\ell + 1, 0) \prec (0, 2\ell + 1).$$

Now assume we have defined partial orders on $\mathcal{P}_{k'}(n)$ for every $k'$. Let us define partial orders on $\mathcal{P}_{k}(n)$. The set $\mathcal{P}_0(n + 1)$ is a singleton so there is nothing to do. On $\mathcal{P}_1(n + 1)$ we have $(1, 0, \ldots, 0) \prec (0, 1, \ldots, 0) \prec \cdots \prec (0, 0, \ldots, 1)$. Assume that we have defined a partial order on $\mathcal{P}_k(n + 1)$. To define a partial order on $\mathcal{P}_{k+1}(n + 1)$, recall that we have a decomposition

$$\mathcal{P}_{k+1}(n + 1) := \mathcal{P}_{k+1}^\circ(n + 1) \cup \mathcal{P}_{k+1}^\prime(n + 1).$$

The map $\pi$ gives a bijection $\pi : \mathcal{P}_k(n + 1) \to \mathcal{P}_{k+1}^\circ(n + 1)$, and this gives a partial order on the set $\mathcal{P}_{k+1}^\circ(n + 1)$. Forgetting $a_1 = 0$, $\mathcal{P}_{k+1}^\prime(n + 1)$ is identified with $\mathcal{P}_{k+1}(n)$, and this gives a partial order on the set $\mathcal{P}_{k+1}^\prime(n + 1)$. Finally, we declare every element in $\mathcal{P}_{k+1}^\circ(n + 1)$ to be smaller than every element of $\mathcal{P}_{k+1}^\prime(n + 1)$. This gives a partial order on $\mathcal{P}_{k+1}(n + 1)$. Example 1 below gives some examples on how these partial orders look when $n = 3$. For each $a \in \mathbb{Z}_{\geq 0}^3$, we also include $\omega_a$, both in window notation and in its decomposition given by Lemma 2.7 for reference.

For another example, when $n = 4, k = 2$ we have $(1, 1, 0, 0) \prec (1, 0, 1, 0) \prec (1, 0, 0, 1) \prec (2, 0, 0, 0) \prec (0, 1, 1, 0) \prec (0, 1, 0, 1) \prec (0, 2, 0, 0) \prec (0, 0, 1, 1) \prec (0, 0, 2, 0) \prec (0, 0, 0, 2)$.

The following lemma gives properties of this partial order that will be important for us.

Lemma 2.19. With the partial order defined above, $\mathcal{P}_k(n)$ is linearly ordered. Moreover, the following properties are satisfied.

1. If $a \prec b$ in $\mathcal{P}_k(n)$, then $\pi \cdot a \prec \pi \cdot b$ in $\mathcal{P}_{k+1}(n)$.
2. If $a_n \geq a_1 > 0$, then $(a_1, a_2, \ldots, a_{n-1}, a_n) \prec (a_n + 1, a_2, \ldots, a_{n-1}, a_1 - 1)$.
3. If $a_i > a_{i+1}$, then $a \prec s_i \cdot a$. 


id

\[ 1, 2, 3 \]

\[ k = 0 \quad (0, 0, 0) \]

\[ \pi \quad s_1 \pi \quad s_2 s_1 \pi \]

\[ [2, 3, 4] \quad [1, 3, 5] \quad [1, 2, 6] \]

\[ k = 1 \quad (1, 0, 0) \prec (0, 1, 0) \prec (0, 0, 1) \]

\[ \pi^2 \quad \pi s_1 \pi \quad \pi s_2 s_1 \pi \quad s_1 \pi s_1 \pi \quad s_1 \pi s_2 s_1 \pi \quad s_2 s_1 \pi s_2 s_1 \pi \]

\[ [3, 4, 5] \quad [2, 4, 6] \quad [2, 3, 7] \quad [1, 5, 6] \quad [1, 3, 8] \quad [1, 2, 9] \]

\[ k = 2 \quad (1, 1, 0) \prec (1, 0, 1) \prec (2, 0, 0) \prec (0, 1, 0) \prec (0, 0, 1) \prec (0, 0, 2) \]

\[ \pi^3 \quad \pi^2 s_1 \pi \quad \pi^2 s_2 s_1 \pi \quad \pi s_1 \pi s_1 \pi \quad \pi s_1 \pi s_2 s_1 \pi \quad \pi s_2 s_1 \pi s_2 s_1 \pi \]

\[ [4, 5, 6] \quad [3, 5, 7] \quad [3, 4, 8] \quad [2, 6, 7] \quad [2, 4, 9] \quad [2, 3, 10] \]

\[ k = 3 \quad (1, 1, 1) \prec (2, 1, 0) \prec (1, 2, 0) \prec (2, 0, 1) \prec (0, 1, 2) \prec (3, 0, 0) \]

\[ s_1 \pi s_1 \pi s_1 \pi \quad s_1 \pi s_1 \pi s_2 s_1 \pi \quad s_1 \pi s_1 \pi s_2 s_1 \pi \quad s_2 s_1 \pi s_2 s_1 \pi \quad s_2 s_1 \pi s_2 s_1 \pi \quad s_2 s_1 \pi s_2 s_1 \pi \]

\[ [1, 6, 8] \quad [1, 5, 9] \quad [1, 3, 11] \quad [1, 2, 12] \]

\[ \prec (0, 2, 1) \quad \prec (0, 1, 2) \quad \prec (0, 3, 0) \quad \prec (0, 0, 3) \]

Figure 1. The partial order on \( a \in \mathcal{P}_k(3) \) for degrees \( k \leq 3 \). So that one may compare \( \prec \) to \( \geq_{lex} \) and to Bruhat order, above each \( a \) is the corresponding \( \omega_s \) both in window notation and it expression from Lemma 2.7

(4) If \( a_i > a_{i+1} \) and \( b \prec a \), then \( s_i \cdot b \prec s_i \cdot a \).

Proof. By induction, it follows easily that \( \mathcal{P}_k(n) \) is linearly ordered, as it is defined to be the concatenation of two linearly ordered sets. Property (1) is obvious from the definition. It remains to show (2), (3) and (4). Note that when \( n = 2 \) or \( k = 0, 1, 2, 3 \) and (4) are easy to check from the explicit definition of the partial order on \( \mathcal{P}_k(2) \) or on \( \mathcal{P}_k(1) \). So we may use an inductive procedure. We assume that (2), (3) and (4) are valid for \( \mathcal{P}_{k'}(n) \) for every \( k' \) and for \( \mathcal{P}_0(n + 1), \ldots, \mathcal{P}_k(n + 1) \), and we show that they are valid for \( \mathcal{P}_{k+1}(n+1) \). Recall that \( \mathcal{P}_{k+1}^0(n+1) = \pi(\mathcal{P}_k(n + 1)) \) and \( \mathcal{P}_{k+1}^0(n+1) = \mathcal{P}_{k+1}(n+1) \setminus \mathcal{P}_{k+1}^0(n + 1) \).

We start with (2). Note that if \( a \) is as in (2), then \( a \in \mathcal{P}_{k+1}^0(n+1) \). Then (2) happens if and only if in \( \mathcal{P}_k(n + 1) \) we have \((a_2, \ldots, a_n, a_{n+1}, a_1 - 1) < (a_2, \ldots, a_n, a_1 - 1, a_{n+1}) \). But this is clear because \( \mathcal{P}_k(n + 1) \) satisfies (3).

Now we move on to (3). Thanks to (1) and our inductive assumption, the only problem can arise with \( s_1 \); indeed, for \( i > 1 \) we can either go to \( \mathcal{P}_k(n) \), if \( a_1 > 0 \); or to \( \mathcal{P}_{k+1}(n - 1) \) if \( a_1 = 0 \) and the result follows by induction. So we assume \( i = 1 \). If \( a \in \mathcal{P}_{k+1}^0(n+1) \), then we can never have \( a_1 > a_2 \), so we may assume
that \( a \in \mathcal{P}_k^{n+1} \). If \( a_2 = 0 \), then \( s_1 \cdot a \in \mathcal{P}_k^{n+1} \), and we have \( a < s_1 \cdot a \) by definition. Otherwise, we may assume that \( a_1 > a_2 \geq 1 \). Then \( s_1 \cdot a > a \) is equivalent to, in \( \mathcal{P}_k^{n+1} \), having \((a_2, \ldots, a_{n+1}, a_1-1) < (a_1, a_3, \ldots, a_{n+1}, a_2-1)\). This is clear because \( \mathcal{P}_k(n+1) \) satisfies (2).

Finally, we check (4). Note that we cannot have \( i = 1 \) and \( a \in \mathcal{P}_k^{n+1}(n+1) \) simultaneously. We also cannot have \( a \in \mathcal{P}_k^{n+1}(n+1) \) and \( b \in \mathcal{P}_k^{n+1}(n+1) \) simultaneously. If both \( a \) and \( b \) belong to \( \mathcal{P}_k^{n+1}(n+1) \), the result follows by forgetting the initial 0 and using induction. If \( a \in \mathcal{P}_k^{n+1} \) and \( b \in \mathcal{P}_k^{n+1} \), then, since \( i \neq 1 \), \( s_i \) preserves both \( \mathcal{P}_k^{n+1}(n+1) \) and \( \mathcal{P}_k^{n+1} \) so the result is also clear. The result is also clear if both \( a, b \in \mathcal{P}_k^{n+1}(n+1) \) and \( i \neq 1 \). So it remains to check the case \( a, b \in \mathcal{P}_i, i = 1 \). If \( a_2 = 0, b_2 \neq 0 \), the result is clear.

If \( a_2, b_2 = 0 \), then we have that \( a \succ b \) if and only if \((0, a_3, \ldots, a_{n+1}, a_1 - 1) \succ (0, b_3, \ldots, b_{n+1}, b_1 - 1)\). This happens if and only if in \( \mathcal{P}_k(n) \) we have \((a_3, \ldots, a_{n+1}, a_1 - 1) \succ (b_3, \ldots, b_{n+1}, b_1 - 1)\). By (1), this implies \((a_1, a_3, \ldots, a_n) \succ (b_1, b_3, \ldots, b_n)\) in \( \mathcal{P}_k(n) \). But then \((0, a_1, a_3, \ldots, a_n) \succ (0, b_1, b_3, \ldots, b_n)\), which is what we wanted to show.

If \( a_2, b_2 \neq 0 \), we need to show that \((a_1, \ldots, a_{n+1}, a_2 - 1) \succ (b_1, \ldots, b_{n+1}, b_2 - 1)\). This happens if and only if \((a_3, \ldots, a_{n+1}, a_2 - 1, a_1 - 1) \succ (b_3, \ldots, b_{n+1}, b_2 - 1, b_1 - 1)\). But by assumption, \((a_3, \ldots, a_{n+1}, a_1 - 1, a_2 - 1) \succ (b_3, \ldots, b_1 - 1, b_2 - 1)\) and \(a_1 - 1 > a_2 - 1\). Since \( \mathcal{P}_k-1(n+1) \) satisfies (4), the result follows.

We now compare the order \( < \) to the lexicographic ordering via the bijection \( \omega : \mathbb{Z}_0^n \to L_{\min}^+ \). First, we have the following result, compare with Lemma 2.19(1).

**Lemma 2.20.** Let \( a, b \in \mathbb{Z}_0^n \) and assume \( \omega_a > 1_{\text{lex}} \omega_b \). Then, \( \omega_{\pi a} > 1_{\text{lex}} \omega_{\pi b} \).

**Proof.** If the window notation of \( \omega \) is \([\omega(1), \omega(2), \ldots, \omega(n)]\) then the window notation of \( \pi \omega \) is \([\omega(1) + 1, \omega(2) + 1, \ldots, \omega(n) + 1]\). Hence it is obvious that \( \omega > 1_{\text{lex}} \omega' \iff \pi \omega > 1_{\text{lex}} \pi \omega' \). Next, observe \( \omega_{\pi a} = \pi \omega_a \) from which the lemma follows. Indeed the entries of the window notation for \( \omega_a \) sort those of \( t_a \), and these have the form \( i + na_i \). On the other hand, the entries of \( t_{\pi a} \) are \( i + na_{\pi^-1(i)} = i + na_i - 1 \) which may we reindex as \( i + 1 + na_i \) as well as \( 1 + n(a_n + 1) = n + 1 + 1 + na_a \).

The next result tells us that, even though the partial order \( < \) looks complicated, it is in fact very natural when transported via the map \( a \mapsto \omega_a \).

**Lemma 2.21.** Let \( a, b \in \mathbb{Z}_0^n \) be such that \( \|a\| = \|b\| \). Then, \( a < b \) if and only if \( \omega_a > 1_{\text{lex}} \omega_b \).

**Proof.** Since for fixed degree we are dealing with linear orderings, by Lemma 2.18 we only need to check \( a \prec b \) implies \( \omega_a > 1_{\text{lex}} \omega_b \). Let us denote \( k := \|a\| = \|b\| \). The case when \( a \in \mathcal{P}_k \) and \( b \in \mathcal{P}_k \) follows from Lemma 2.18(b) and (c). The case when \( a, b \in \mathcal{P}_k \) follows from Lemma 2.20 and an inductive argument on \( k \).

Finally, assume \( a, b \in \mathcal{P}_k \) so that \( a_1 = b_1 = 0 \) and therefore \( \omega_a(1) = \omega_b(1) = 1 \). We have to consider \( \pi = (a_2, \ldots, a_n) \), \( b = (b_2, \ldots, b_n) \). By induction on \( n \) we may assume \( \omega_{\pi} > 1_{\text{lex}} \omega_{\pi} \). Note that we have \( g_a(i + 1) = g_{\pi}(i + 1) + 1 \) and that \( \pi_i = a_{i+1} \) for \( i = 1, \ldots, n - 1 \), and

\[
\omega_a = \left[1, \omega_{\pi}(1) + 1 + a_{g_{\pi}^{-1}(2)}, \ldots, \omega_{\pi}(n - 1) + 1 + a_{g_{\pi}^{-1}(n)}\right]
\]
and similarly for \( g_b, \omega_b \). By assumption, \( \omega_{\gamma} >_{1ex} \omega \). So there exists \( i_0 \in \{1, \ldots, n-1\} \) such that \( \omega_{\gamma}(i) = \omega(i) \) for \( i < i_0 \) and \( \omega_{\gamma}(i_0) > \omega(i_0) \). If \( i < i_0 \) then by Lemma 2.20 we get \( \omega_n(i + 1) = \omega_b(i + 1) \). Now, \( \omega_{\gamma}(i_0) > \omega(i_0) \) implies that \((n - 1)(a_{g_{\gamma}^{-1}(i_0)} - b_{g_{\gamma}^{-1}(i_0)} + g_{\gamma}^{-1}(i_0) - g_{\gamma}^{-1}(i_0)) > 0\), from where we deduce that \( a_{g_{\gamma}^{-1}(i_0)} \geq b_{g_{\gamma}^{-1}(i_0)} \). Finally,

\[
\omega_n(i_0 + 1) = \omega_{\gamma}(i_0) + 1 + a_{g_{\gamma}^{-1}(i_0)} > \omega_b(i_0) + 1 + b_{g_{\gamma}^{-1}(i_0)} = \omega_b(i_0 + 1)
\]

and we conclude that \( \omega_n >_{1ex} \omega_b \). \( \square \)

One can check Figure 11 to see examples of the structure described in both Lemmas 2.20 and 2.21, as well as Lemma 2.22 below.

Now let \( \omega \in L_1^+(n) \). Recall that by Lemma 2.7 that we may express \( \omega \) in the form

\[
\omega = (s_{r_0} \cdots s_2 s_1)_{\pi} \cdots (s_{r_\ell} \cdots s_2 s_1)_{\pi} (s_{\nu_1} \cdots s_2 s_1)_{\pi}
\]

where \( 0 \leq \nu_\ell \leq \cdots \leq \nu_1 < n \). We select \( \ell \leq r \) and \( j \leq \nu_\ell \) and consider the affine permutation

\[
\omega := (s_{r_\ell} \cdots s_2 s_1)_{\pi} \cdots (s_{\nu_1} \cdots s_j s_{j-1} \cdots s_1)_{\pi} \cdots (s_{\nu_1} \cdots s_2 s_1)_{\pi}
\]

where a hat over \( s_j \) means that we omit that transposition. Clearly, \( \omega \) belongs to the monoid \( \bar{S}_n^+ \), and \( \deg(\omega) = \deg(\bar{\omega}) \). Let us denote by \( \omega' \in L_1^+(n) \) the permutation whose window notation is the increasing arrangement of the window notation of \( \bar{\omega} \).

**Lemma 2.22.** We have \( \omega' >_{1ex} \omega \).

**Proof.** Let us start with the easy observation that, in window notation:

\[
(7) \quad s_k \cdots s_1 = [1, 2, \ldots, k, k + 2, \ldots, n, n + k + 1].
\]

Now let us split \( \omega \) as follows:

\[
\omega = \underbrace{(s_{\nu_\ell} \cdots s_1)_{\pi}}_{\alpha} \cdots \underbrace{(s_{\nu_+1} \cdots s_1)_{\pi}}_{\beta} \underbrace{(s_{\nu_1} \cdots s_1)_{\pi}}_{\gamma}
\]

Note that \( \alpha, \beta, \gamma \in L_1^+(n) \). Moreover, letting \( k := \nu_\ell \) since \( \nu_\ell \leq \nu_{\ell-1} \leq \cdots \leq \nu_1 \) it follows from (7) that \( \gamma = [1, 2, \ldots, k, \gamma(k + 1), \ldots, \gamma(n)] \), where \( k < \gamma(k + 1) < \cdots < \gamma(n) \). If we write \( \alpha = \alpha(1), \ldots, \alpha(n) \) we then have that

\[
\omega = [\alpha(1), \ldots, \alpha(j), \ldots, \alpha(k), \omega(k + 1), \ldots, \omega(p), \ldots, \omega(n)],
\]

where we let \( 1 \leq p \leq n \) be such that \( \gamma(p) = zn \). Observe \( z > 0 \).

Let us now compute \( \bar{\omega} \). Let \( \tilde{\beta} := s_{\nu_0} \cdots s_{j+1} s_{j} s_{j-1} \cdots s_1 \pi \), so that \( \bar{\omega} = \alpha \tilde{\beta} \gamma \). A straightforward computation shows that, in window notation, \( \tilde{\beta} = [1, 2, \ldots, j - 1, k, j + 1, \ldots, k, k + 2, \ldots, n, n + j] \). Then, \( \bar{\omega} = \omega \)

\[
[\alpha(1) \cdots \alpha(j - 1), \alpha(k + 1), \alpha(j + 1), \ldots, \alpha(k), \omega(k + 1), \ldots, \alpha(j + zn) \cdots \omega(n)],
\]

and in particular the window notation \( \omega \) and \( \bar{\omega} \) agree except for in the \( j \)th and \( p \)th entries. Since \( \omega \) is already sorted so its entries increase, to show \( \omega' >_{1ex} \omega \) it suffices to show \( \bar{\omega}(j) > \omega(j) \) and \( \bar{\omega}(p) > \omega(j) \). This will ensure that the first \( j - 1 \) entries of \( \omega \) and the sorted \( \omega' \) agree, but the \( j \)th entry of \( \omega' \) will be strictly larger.
than \( \omega(j) \). We compute \( \tilde{\omega}(j) = \alpha(k + 1) > \alpha(j) = \omega(j) \) since \( \alpha \in L_{\min}^+(n) \), and \( \tilde{\omega}(p) = \alpha(j) + zn > \alpha(j) = \omega(j) \).

\[ \square \]

Remark 2.23. From Lemma 2.22 we see that for \( \omega, p \in L_{\min}^+(n) \) of the same degree that

\( p <_{\omega} \omega \) implies \( p >_{1ex} \omega \)

in window notation. We could also deduce this from the characterization of Bruhat order for \( \tilde{S}_n \) given in Björner-Brenti [3, Theorem 8.3.7] (which one must extend appropriately to \( \tilde{S}_n \); this is easy if only comparing permutations of the same degree).

In particular, they characterize \( p \leq_{\omega} \omega \) for \( p, \omega \in \tilde{S}_n \) if \( p[i, j] \leq \omega[i, j] \) for all \( i, j \in \mathbb{Z} \) where

\[ p[i, j] = \# \{ a \leq i \mid p(a) \geq j \}. \]

We can prove (8) as follows.

Let \( p \neq \omega \in L_{\min}^+(n) \) be of the same degree, which means \( \sum_{i=1}^{n} p(i) = \sum_{i=1}^{n} \omega(i) \).

(This sum is \( n(n+1)/2 \) in the case \( p, \omega \in \tilde{S}_n \) ) Suppose that \( p >_{1ex} \omega \) which means \( p <_{1ex} \omega \). Hence for some \( 1 \leq \ell \leq n \) we have \( p(1) = \omega(1), p(2) = \omega(2), \ldots, p(\ell - 1) = \omega(\ell - 1) \) but \( p(\ell) < \omega(\ell) \). Since \( \sum_{i=1}^{\ell} p(i) = \sum_{i=1}^{n} \omega(i) \), there must be some \( \ell \leq i \leq n \) such that \( p(i) > \omega(i) \). Let \( i \leq n \) be the largest such \( i \). In other words for \( n \geq r > i \) we have \( p(r) \leq \omega(r) \). Let \( j := p(i) \). Let us compare \( p[i, j] \) and \( \omega[i, j] \).

Since \( \omega \in L_{\min}^+(n) \) we have \( \omega(1) < \omega(2) < \cdots < \omega(n) \) and so given \( a \) such that \( \omega(a) \geq j = p(i) > \omega(i) \) and \( a \leq i \) then \( i + 1 < a + kn < n \) for some \( k > 0 \). In particular \( p(a) = p(a + kn) - kn = \omega(a + kn) - kn = \omega(a) \geq j \). Hence \( \{ a \leq i \mid \omega(a) \geq j \} \subseteq \{ a \leq i \mid p(a) \geq j \} \). Further as \( \omega(i) < p(i) = j \) the element \( i \) does not belong to the first set but does to the second. This shows \( \omega[i, j] < \omega[i, j] \) and so \( p \not<_{\omega} \omega \). This proves (8).

We remark that even though \( >_{1ex} \) is a total order on \( \tilde{S}_n \), we only relate it to Bruhat order for two affine permutations of the same degree and that are both in \( L_{\min}^+(n) \).

3. Rational Cherednik algebras

Throughout the rest of the paper we take \( m, n \in \mathbb{Z}_{>0} \) and \( \gcd(m, n) = 1 \) unless otherwise stated.

3.1. Definition and basic properties. We work with the rational Cherednik algebra \( H_{t,c} := H_{t,c}(S_n, \mathbb{C}^n) \) of \( S_n \) acting on \( \mathbb{C}^n \) by permuting the coordinates. Let us recall that this is the quotient of the semidirect product algebra \( \mathbb{C}(x_1, \ldots, x_n, y_1, \ldots, y_n) \rtimes S_n \) by the relations

\[ [x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [y_i, x_j] = c(ij), \quad [y_i, x_i] = t - c \sum_{j \neq i} (ij) \]

Here \( t \) and \( c \) are complex parameters. Clearly, for a nonzero complex number \( a \in \mathbb{C}^* \), \( H_{at,ac} \cong H_{t,c} \). So we have the dichotomy \( t = 0 \) or \( t = 1 \). For most of the paper, we will assume that \( t = 1 \) and write \( H_c := H_{1,c} \).
We recall some basic facts about $H_{t,c}$ following [2]. The algebra $H_{t,c}$ is graded, with $x_i$ of degree 1, $y_i$ of degree $(-1)$, and $S_n$ in degree zero. When $t = 1$, the grading on $H_t$ is internal and defined by the Euler element

$$h = \frac{1}{2} \sum_i (x_i y_i + y_i x_i) = \sum_i x_i y_i + \frac{n}{2} - c \sum_{i < j} (ij).$$

Let us emphasize that the grading on $H_{t,c}$ is not internal. The algebra $H_{t,c}$ is also filtered, with $x_i$ and $y_i$ of filtration level 1 and $S_n$ of filtration level 0. An important PBW theorem states that

$$\text{gr } H_{t,c} = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes S_n,$$

where gr denotes associated graded with respect to this filtration. This implies that a basis of $H_{t,c}$ as a $\mathbb{C}$-vector space is given by $x^a y^b$, where $a \in S_n$, $x^a := x_1^{a_1} \cdots x_n^{a_n}$, $a \in \mathbb{Z}_{\geq 0}^n$, and similarly for $y^b$, $b \in \mathbb{Z}_{\geq 0}^n$. We will refer to this basis as the PBW basis of $H_{t,c}$. Another easy consequence of the PBW theorem is that $H_{t,c}$ contains the following subalgebras:

$$H_n(x) := \mathbb{C}[x_1, \ldots, x_n] \rtimes S_n, \quad H_n(y) := \mathbb{C}[y_1, \ldots, y_n] \rtimes S_n.$$

Next we need to consider some modules for $H_{t,c}$. We have the standard modules

$$\Delta_{t,c}(\mu) = \text{Ind}_{H_n(y)}^{H_{t,c}} V_\mu \simeq V_\mu \otimes \mathbb{C}[x_1, \ldots, x_n],$$

where $V_\mu$ is an irreducible representation of $S_n$ corresponding to the Young diagram $\mu$ of size $n$, and $y_i$ annihilate $V_\mu$. In particular, for $\mu = (n)$ the representation $V_\mu$ is trivial, and we get the polynomial representation $\Delta_{t,c}(\text{triv}) \simeq \mathbb{C}[x_1, \ldots, x_n]$.

When $t = 1$, it is not hard to see using the Euler element $h$ that $\Delta_c(\mu) := \Delta_{1,c}(\mu)$ has a unique irreducible quotient that we denote by $L_c(\mu)$. In fact, these are the simple objects of the category $O_c$, which is defined as the category of $H_c$-modules which are finitely generated over $\mathbb{C}[x_1, \ldots, x_n]$ and where $y_i$ act locally nilpotently. For example, the standard modules $\Delta_c(\mu)$ belong to $O_c$. We have the following facts about the category $O_c$.

**Theorem 3.1** ([2]). \(a\) If $c \in \mathbb{C} \setminus \mathbb{Q}$ then the category $O_c$ is semisimple, and all standard modules $\Delta_c(\mu)$ are irreducible. The same is true if $c$ is rational but its denominator is greater than $n$.

\(b\) Suppose that $c = m/n$ where $m, n \in \mathbb{Z}_{>0}, \gcd(m, n) = 1$. Then $\Delta_c(\mu)$ is irreducible, unless $\mu$ is a hook.

\(c\) Suppose that $c = m/n$ where $m, n \in \mathbb{Z}_{>0}, \gcd(m, n) = 1$, and let $\mu_l = (n - \ell, 1^\ell)$ be a hook partition. Then the morphisms between standard modules have the following form:

$$\text{Hom}_{H_t}(\Delta_c(\mu), \Delta_c(\mu')) = \begin{cases} \mathbb{C} & \mu = \mu', \\ \mathbb{C} & \mu = \mu_l, \mu' = \mu_{l-1} \text{ for some } \ell, \\ 0 & \text{otherwise.} \end{cases}$$

This theorem is proved in [2] using the Knizhnik-Zamolodchikov functor which compares the representation theory of $H_t$ to that of the type $A$ finite Hecke algebra. In this paper we give an alternative combinatorial proof. The “otherwise” case of (b) is proved in Lemma 5.11 and Lemma 5.14 while the interesting morphism $\Delta_c(\mu_\ell) \to \Delta_c(\mu_{\ell-1})$ is constructed in Proposition 5.16 Part (b) easily follows from (c), see Corollary 5.13.
The representation theory of the algebra $H_{0,1}$ is very different from that of $H_t$. For example, it is no longer true that the standard module $\Delta_{0,1}(\mu)$ has a unique irreducible quotient, moreover, the algebra $H_{0,1}$ is finite over its center so every irreducible $H_{0,1}$-module is finite-dimensional, \cite{10}. Still, in Section 9 we will use our results in the $t = 1$ case and a limiting procedure to give a combinatorial basis of $\Delta_{0,1}(\mu)$.

We will also consider the spherical subalgebra of $H_{t,c}$. Let $e := \frac{1}{n!} \sum_{p \in S_n} p \in \mathbb{C}S_n \subseteq H_{t,c}$ be the trivial idempotent for $S_n$. The spherical rational Cherednik algebra is the corner algebra $eH_{t,c}e$. We have an obvious functor $H_{t,c}$-mod $\rightarrow eH_{t,c}e$-mod, given by $M \mapsto eM = M^{S_n}$. When $t = 0$ or $t = 1$ and $c$ is not a negative real number, this functor is known to be an equivalence.

3.2. An alternative presentation of $H_{t,c}$. It will be convenient to work with a trigonometric presentation of the algebra $H_{t,c}$ that has already appeared in the work of Griffeth and Webster in the cyclotomic setting, \cite{31,61}. Since some relations become more transparent in the type $A$ setting, we recall this presentation in detail.

First, we have the Dunkl-Opdam elements in $H_{t,c}$:

$$u_i := x_i y_i - c \sum_{j < i} (ij).$$

Lemma 3.2. The Dunkl-Opdam elements generate a polynomial subalgebra of $H_{t,c}$.

Proof. It is straightforward to see that $u_i u_j = u_j u_i$. Since the leading term of $u_i$ is $x_i y_i$, the leading terms of $u_i$ in $\text{gr } H_{t,c}$ are algebraically independent, and hence $u_i$ are algebraically independent in $H_{t,c}$. \qed

We will denote this polynomial subalgebra by $A$.

We remark that we have the following relations where, as usual, $s_i = (i, i+1) \in S_n$ is a simple transposition.

\begin{align*}
\tag{9} s_i u_i &= u_{i+1} s_i + c \\
\tag{10} s_j u_i &= u_i s_j & \text{if } j \neq i, i-1
\end{align*}

Remark 3.3. These equations imply that $u_i$ and $s_j$ form a subalgebra in $H_{t,c}$ isomorphic to the degenerate affine Hecke algebra. We will denote this algebra by $H_n(u)$.

We will also need the following shift operators. Let $\tau := x_1 (12 \cdots n)$, $\lambda := (12 \cdots n)^{-1} y_1$. Note that, for every $i$, we have that $\tau = (1 \cdots i) x_i (i \cdots n)$ and
Remark 3.4. Given relations (14) and (12), it is convenient to define \( u_i \) by setting

\[
\begin{align*}
\tau u_i &= u_{i+1} \tau, \ i \neq n \\
\tau u_n &= (u_1 - t) \tau
\end{align*}
\]

It is clear that the elements \( s_1, \ldots, s_{n-1}, \tau \) and \( \lambda \) generate the algebra \( H_{t,c}. \) It turns out that, together with the \( u_i \), they give a presentation of this algebra.

Lemma 3.7. Let \( \lambda = (n \cdots i)y_i(i \cdots n) \). The following relations are straightforward to check.

\[
\begin{align*}
(11) & \quad \tau u_i = u_{i+1} \tau, \ i \neq n \\
(12) & \quad \tau u_n = (u_1 - t) \tau \\
(13) & \quad \lambda u_i = u_{i-1} \lambda, \ i \neq 1 \\
(14) & \quad \lambda u_1 = (u_n + t) \lambda \\
(15) & \quad s_i \tau = \tau s_{i-1}, \ i \neq 1 \\
(16) & \quad s_1 \tau^2 = \tau^2 s_{n-1} \\
(17) & \quad s_i \lambda = \lambda s_{i+1}, \ i \neq n - 1 \\
(18) & \quad s_{n-1} \lambda^2 = \lambda^2 s_1 \\
(19) & \quad \tau \lambda = u_1 \\
(20) & \quad \lambda \tau = u_n + t \\
(21) & \quad \lambda s_1 \tau = \tau s_{n-1} \lambda + c
\end{align*}
\]

It is clear that the elements \( s_1, \ldots, s_{n-1}, \tau \) and \( \lambda \) generate the algebra \( H_{t,c}. \) It turns out that, together with the \( u_i \), they give a presentation of this algebra.

The following theorem appears in work of Griffeth [31, Theorem 3.1] and Webster [61, Theorem 2.3] in the more complicated cyclotomic setting.

Theorem 3.5. Let \( H_{t,c} \) be the algebra generated by elements \( u_1, \ldots, u_n, \tau, \lambda \) and the symmetric group \( S_n, \) subject to the relations that \( [u_i, u_j] = 0 \) and \( (15) - (21). \) Then, \( H_{t,c} \cong H_{t,c}. \)

We refer to the aforementioned [31, 61] for a proof of this result. Here, we just remark that to recover \( x_i \) and \( y_i \) from \( \tau, \lambda \) and \( S_n \) we set

\[
x_i = s_{i-1} \cdots s_1 \tau s_{n-1} \cdots s_{i-1}, \quad y_i = s_i \cdots s_{n-1} \lambda s_1 \cdots s_{i-1}.
\]

Note that we can also eliminate the Dunkl-Opdam elements \( u_i \) from this presentation:

Proposition 3.6. The algebra \( H_{t,c} \) is generated by \( s_1, \lambda \) and \( \tau \) subject to the equations \( (15), (16), (17), (18), (21) \) and one more equation

\[
(22) \quad \lambda \tau = t + s_1 \cdots s_{n-1} \lambda s_{n-1} \cdots s_1 - c \sum_{i=1}^{n-1} s_1 \cdots s_i \cdots s_1
\]

The following lemma relates the nonnegative part of \( H_{t,c} \) to affine permutations.

Lemma 3.7. Let \( X \) denote the monoid of monomials in \( s_i \) and \( \tau \) (or, equivalently, in \( s_i \) and \( x_j \)). Then there is an isomorphism of monoids \( F_X : X \to \widetilde{S}_n^+ \) such that

\[
F_X(s_i) = s_i, \quad F_X(\pi) = \pi, \quad F_X(x_1^{a_1} \cdots x_n^{a_n}) = t^c
\]

for \( a_i \geq 0. \)

Proof. We can define \( F_X \) by \( F_X(s_i) = s_i, \ F_X(\pi) = \pi. \) Since the relations \( s_i \pi = \pi s_{i-1} \) and \( s_1 \pi^2 = \pi^2 s_{n-1} \) hold in \( \widetilde{S}_n, \ F_X \) is a homomorphism. Considering the window notation for \( s_i \) and \( \pi, \) it is easy to see the image is \( \widetilde{S}_n^+. \)
Now
\[ F_X(x_i) = s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i = [1, \ldots, i-1, i+1, \ldots, n] = \tau(0, \ldots, 0, 1, 0, \ldots, 0) \]
so \( F_X(x_1^{a_1}, \ldots, x_n^{a_n}) = \tau_a \) for all \( a \in \mathbb{Z}_{\geq 0}^n \).

Finally, by Lemma 2.3 any element of \( \tilde{S}_n^+ \) can be uniquely written as \( \omega = \tau_a g \)
for \( g \in S_n, a \in \mathbb{Z}_{\geq 0}^n \), while any element of \( X \) can be uniquely written as \( x_1^{a_1} \cdots x_n^{a_n} g \)
for \( g \in S_n \). Therefore \( F_X \) is a bijection.

**Corollary 3.8.** The monoid \( \tilde{S}_n^+ \) is generated by \( s_i, \pi \) modulo relations in \( S_n \) and
\[ s_i \pi = \pi s_{i-1}, \ s_1^2 = \pi^2 s_{n-1}. \]

Similarly, we have the following.

**Lemma 3.9.** Let \( Y \) denote the monoid of monomials in \( s_i \) and \( \lambda \) (or, equivalently, in \( s_i \) and \( y_j \)). Let \( \tilde{S}_n^- \) be the monoid generated by inverses of elements in \( \tilde{S}_n^+ \).

Then there is an isomorphism of monoids \( F_Y : Y \to \tilde{S}_n^- \) such that
\[ F_Y(s_i) = s_i, \ F_Y(\lambda) = \pi^{-1}, \ F_Y(y_1^{a_1} \cdots y_n^{a_n}) = \tau_{-a}. \]

**Remark 3.10.** The two isomorphisms \( F_X \) and \( F_Y \) are not compatible in the group \( \tilde{S}_n \) in the sense that relations between elements in the two monoids may not hold for their preimages in \( H_{l,c} \).

For instance \( \pi^2 \pi^{-1} = \pi^{-1} \pi \) in \( S_n \), whereas \( \tau \lambda \neq \lambda \tau \).

See equations 19 and 20.

### 4. A Mackey Formula for \( H_{l,c} \)

#### 4.1. Basis in \( H_{l,c} \)

In this section we present a basis in the algebra \( H_{l,c} \) using the generators from Section 3.2. It is an analogue of the PBW basis from Section 3.1.

Recall that \( H_n(y) \) is the subalgebra generated by \( S_n \) and \( y_i \) (or, equivalently, by \( S_n \) and \( \lambda \)), and that \( H_n(u) \) denotes the subalgebra generated by \( S_n \) and \( u_i \).

**Lemma 4.1.** (a) The algebra \( H_n(y) \) has a basis
\[ g\lambda(s_1 s_2 \cdots s_{\mu_1}) \cdot \lambda(s_1 s_2 \cdots s_{\mu_2}) \cdots \lambda(s_1 s_2 \cdots s_{\mu_r}) \]
for \( g \in S_n \) and \( 0 \leq \mu_{r'} \leq \cdots \leq \mu_1 \).

(b) The algebra \( H_{l,c} \) has a basis
\[ (s_{\nu_r} \cdots s_2 s_1) \tau \cdots (s_{\nu_1} \cdots s_2 s_1) \tau \cdot \lambda(s_1 s_2 \cdots s_{\mu_1}) \cdot \lambda(s_1 s_2 \cdots s_{\mu_2}) \cdots \lambda(s_1 s_2 \cdots s_{\mu_r}), \]
for \( g \in S_n \) and \( 0 \leq \nu_{r'} \leq \cdots \leq \nu_1, 0 \leq \nu_r \leq \cdots \leq \nu_1 \).

(c) The algebra \( H_{l,c} \) is free as a right \( H_n(y) \)-module with the basis
\[ (s_{\nu_r} \cdots s_2 s_1) \tau \cdots (s_{\nu_1} \cdots s_2 s_1) \tau. \]

**Proof.** By Lemma 3.7 any monomial in \( x_i \) of degree \( r \) can be written as \( F_X^{-1}(\omega) \) for \( \omega \in \tilde{S}_n^+ \) of also of degree \( r \). By Lemma 2.7 we can write
\[ \omega = (s_{\nu_r} \cdots s_2 s_1) \pi \cdots (s_{\nu_1} \cdots s_2 s_1) \pi \cdot g, \ 0 \leq \nu_r \leq \cdots \leq \nu_1. \]

so that
\[ F_X^{-1}(\omega) = (s_{\nu_r} \cdots s_2 s_1) \tau \cdots (s_{\nu_1} \cdots s_2 s_1) \tau \cdot g. \]

Similarly we can write
\[ F_Y^{-1}(\omega^{-1}) = g^{-1} \lambda(s_1 s_2 \cdots s_{\nu_1}) \cdot \lambda(s_1 s_2 \cdots s_{\nu_2}) \cdots \lambda(s_1 s_2 \cdots s_{\nu_r}). \]
The algebra $H_{t,c}$ has PBW basis $x_1^{a_1} \cdots x_n^{a_n} g y_1^{b_1} \cdots y_n^{b_n}$, and we can rewrite the monomials $x_1^{a_1} \cdots x_n^{a_n}$ and $y_1^{b_1} \cdots y_n^{b_n}$ as above independently. Finally, (c) is obvious from (a) and (b).

**Remark 4.2.** We can write this basis in more compact form $F_X^{-1}(\omega_1)g F_Y^{-1}(\omega_2)$, where $\omega_1$ and $\omega_2$ are minimal length coset representatives in $L^+_{\min}(n)$.

**Corollary 4.3.** Let $V$ be a finite dimensional representation of $S_n$ with basis $v_T$, $T \in \mathcal{T}$. We can regard it as a representation of $H_n(y)$ where $y_i$ act by 0. Then the induced representation $\Delta_{t,c}(V) := \text{Ind}^{H_n(y)}_{H_n(y)}(V)$ has the basis

$$v_T(\omega) := F_X^{-1}(\omega)v_T,$$

where $\omega$ is a minimal length coset representative in $L^+_{\min}(n)$ and $T \in \mathcal{T}$.

We define a partial order on the basis elements of $\Delta_{t,c}(V)$ in Corollary 4.3 as follows: $v_T(\omega) < v_T(\omega')$ if $\deg \omega = \deg \omega'$ and $\omega \geq_{\text{lex}} \omega'$.

**Example 4.4.** If $V$ is the trivial representation of $S_n$ then $\Delta_{t,c}(V)$ is just the polynomial representation. By Lemma 3.7 the basis $v_T(\omega)$ matches the monomial basis in $\mathbb{C}[x_1, \ldots, x_n]$ and by Lemma 2.21 the partial order we have defined coincides with the partial order $\prec$ defined in Section 2.2.

Next, we want to understand the action of the degenerate affine Hecke algebra $H_n(u)$ in this basis. Via the homomorphism $\text{ev}_0 : H_n(u) \to S_n$ that sends $u_1 \mapsto 0$ and $s_i \mapsto s_i$, the $u_i$ act on $V$ as Jucys-Murphy elements, and they can be simultaneously diagonalized.

**Lemma 4.5.** Suppose that $v_T \in V$ has weight $\omega$, i.e., it is a common eigenvector for the $u_i$ with eigenvalues $\omega_i$. Then

$$u_i(v_T(\omega)) = (\omega \cdot \omega_i)v_T(\omega) + \ell.o.t$$

$$= \omega_{\omega^{-1}(i)}v_T(\omega) + \ell.o.t$$

where $\omega$ is a minimal length coset representative in $L^+_{\min}(n)$, $\omega \cdot \omega_i$ is defined using the action (1) and $\ell.o.t$ denotes lower order terms.

**Proof.** As in Remark 3.3, to simplify notation, we define $u_i$ for all $i \in \mathbb{Z}$ by $u_{i+n} = u_i - t$, and likewise for $\omega_i$. Now the relations between $u_i, s_j$ and $\tau$ get the following form:

$$s_j u_i = u_{s_j(i)}s_j + \begin{cases} c & i \equiv j \mod n, \\ -c & i \equiv j + 1 \mod n, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\tau u_i = u_{\pi(i)} \tau.$$

Overall, we can write

$$F_X^{-1}(\omega)u_i = u_{\omega(i)}F_X^{-1}(\omega) + \ldots$$

so

$$u_i F_X^{-1}(\omega)v_T = F_X^{-1}(\omega)u_{\omega^{-1}(i)}v_T + \ldots = (\omega \cdot \omega_i) F_X^{-1}(\omega)v_T + \ldots,$$

and by Lemma 2.22 all extra terms are less than $F_X^{-1}(\omega)v_T$ in our order.

□
Corollary 4.6. The generalized eigenvalues of \( u_i \) on \( \Delta_{L_c}(V) \) are expressed as \( w = \omega \cdot \kappa_T \) where \( \kappa_T \) are eigenvalues of Jucys-Murphy operators for the basis \( v_T \) and \( \omega \in L^+_{\text{min}}(n) \).

Remark 4.7. Note that Lemma 4.5 relates the standard modules to the integrable modules of the trigonometric (aka degenerate) double affine Hecke algebra ([11]) via Suzuki’s localization, cf. [56, 57, 59].

4.2. Decomposition into \( H_n(u) \)-modules. In this section we give a more precise presentation of induced modules. First, we recall a useful construction of \( H_n(u) \)-modules which are induced from parabolic subgroups.

Let
\[
\mathcal{P}^+(n) = \{ a \in \mathbb{Z}^n_{\geq 0} \mid a_1 \leq a_2 \leq \cdots \leq a_n \} \\
\mathcal{P}^-(n) = \{ d \in \mathbb{Z}^n_{\geq 0} \mid d_1 \geq d_2 \geq \cdots \geq d_n \}.
\]

Let \( a \in \mathbb{Z}^n \) and let \( S(a) \) be its stabilizer in \( S_n \). In the special case \( d \in \mathcal{P}^-(n) \), the stabilizer \( S(d) \) is a standard parabolic subgroup. If \( (k_1, \ldots, k_n) \) is the composition of \( n \) that gives the multiplicities of the entries of \( d \), then \( S(d) = \langle s_i \mid d_i = d_{i+1} \rangle \simeq S_{k_1} \times \cdots \times S_{k_n} \). Note that this subgroup is conjugate to any such parabolic with the \( k_i \) reordered. Recall \( \omega_0 = [n, \ldots, 2, 1] \) is the longest element of \( S_n \). Let
\[
d^{\text{rev}} = \omega_0 \cdot d = (n, \ldots, d_2, d_1)
\]
so in particular \( S(d) \) and \( S(d^{\text{rev}}) \) are conjugate; and \( d \in \mathcal{P}^-(n) \iff d^{\text{rev}} \in \mathcal{P}^+(n) \). Let \( \omega_0^d \) be the longest element of \( S(d) \). Conjugation by \( \omega_0^d \) induces an isomorphism \( S(d) \to S(d^{\text{rev}}) \) we will denote \( \text{rev}_d \). Observe \( \omega_0^d \omega_0^d = g_d \) as it sorts \( d \) to \( d^{\text{rev}} = \text{sort}(d) \). In fact, we would have produced the same isomorphism conjugating by \( \omega_d^{-1} \), where we recall \( \omega_d \in L^+_{\text{min}}(n) \subseteq S_n \). Similarly, we have a corresponding parabolic subalgebra of \( H_n(u) \) we will denote
\[
H(d, u) := \langle s_i, u_j \mid d_i = d_{i+1}, 1 \leq j \leq n \rangle = \mathbb{C}[u_1, \ldots, u_n] \rtimes S(d).
\]

Just as we have an algebra automorphism \( \text{shift} : H_n(u) \to H_n(u) \) that sends \( s_i \mapsto s_i \) but does a constant shift \( u_i \mapsto u_i + t \), \( H(d, u) \) has a “finer” automorphism that is the identity on \( S(d) \) and shifts \( u_i \mapsto u_i + d_i t \). Using this shift map, we can extend
\[
\text{rev}_d : H(d, u) \to H(d^{\text{rev}}, u) \\
\quad s_i \mapsto s_{g_d(i)} = g_d s_i g_d^{-1} \\
\quad u_j \mapsto u_{g_d(j)} + d_j t.
\]

Given an \( H(d^{\text{rev}}, u) \)-module \( M \), via the above algebra isomorphism we can turn it into the “twisted” \( H(d, u) \)-module we denote \( M^{\text{rev}_d} \). Note that \( \text{rev}_d(s_i) = \omega_d^{-1} s_i \omega_d \) and \( \text{rev}_d(u_j) = u_{\omega_d^{-1}(j)} \), when we extend the notion of the \( u_j = u_{j+kn} + kt \) to be indexed by \( j \in \mathbb{Z} \) as in Remark 4.4. However, the map \( \text{rev}_d \) does not agree with conjugation by \( \omega_d^{-1} \), but under some lens it does up to “lower order terms” in a sense that will be made more precise in Remark 4.11 below. This is consistent with the observation that in \( H_n(u) \) we have \( u_j F_\chi^{-1}(\omega) = F_\chi^{-1}(\omega) u_{\omega^{-1}(j)} + \text{L.o.t.} \), where the latter are \( F_\chi^{-1} \) applied to terms lower than \( \omega \in S_n \) in Bruhat order.

Example 4.8. Let \( n = 5, d = (2, 2, 0, 0, 0) \), so \( d^{\text{rev}} = (0, 0, 0, 2, 2) \). Note \( \omega_d = [3, 4, 5, 11, 12], \omega_d^{-1} = [-6, -5, 1, 2, 3], S(d) \simeq S_2 \times S_3 \) and \( S(d^{\text{rev}}) \simeq S_3 \times S_2 \).
The permutations $\omega_0 = [5, 4, 3, 2, 1]$, $\omega_0^d = [2, 1, 5, 4, 3]$, $\omega_0 \omega_0^d = [4, 5, 1, 2, 3] = g_d$. The restricted module $\text{Res}_{H(d^{-\text{rev}}, u)} \text{triv} = M \otimes N$ where $M, N$ are one-dimensional spanned by weight vectors with $u$-weight $(0, -c, -2c)$ and $(-3c, -4c)$ respectively. The twisted $H(d, u)$-module $[\text{Res}_{H(d^{-\text{rev}}, u)} \text{triv}]_{\text{rev}, d}$ is one-dimensional, now spanned by weight vectors with $u$-weight $(-3c + 2t, -4c + 2t)$ and $(0, -c, -2c)$ respectively. It has the form $N^{\text{shift} \times 2} \otimes M$. The map $\text{rev}_d$ sends

$$
\begin{align*}
    s_1 &\mapsto s_4 & u_1 &\mapsto u_{-6} = u_4 + 2t \\
    s_3 &\mapsto s_1 & u_2 &\mapsto u_{-5} = u_5 + 2t \\
    s_4 &\mapsto s_2 & u_3 &\mapsto u_1 \\
    & & u_4 &\mapsto u_2 \\
    & & u_5 &\mapsto u_3
\end{align*}
$$

Just as the minimal length left coset representatives $\{\omega_a \mid a \in \mathcal{P}(n)\} = L^+_{\text{min}}(n) \subseteq \tilde{S}_n^+$ are those affine permutations whose window notation have positive increasing entries, the minimal length double coset representatives with respect to $S_n$ are those $\omega_a$ whose inverses’ window notation have increasing entries. These are exactly the $\omega_d$ for $d \in \mathcal{P}^-(n)$. See Example 4.8.

**Example 4.9.** Let $n = 3$, $d = (4, 1, 1)$, so $\omega_d = [5, 6, 13]$ is a minimal length double coset representative as $d \in \mathcal{P}^-(n)$. Note the double coset decomposes into left cosets as

$$
S_3[5, 6, 13]S_3 = [5, 6, 13]S_3 \sqcup [4, 6, 14]S_3 \sqcup [4, 5, 15]S_3
$$

i.e., $S_3 \omega_d S_3 = \omega_d S_3 \sqcup \omega_{s_1} d S_3 \sqcup \omega_{s_2 s_1} d S_3 = \omega_{(4, 1, 1)} S_3 \sqcup \omega_{(1, 4, 1)} S_3 \sqcup \omega_{(1, 1, 4)} S_3$.

$S_3/S(d)$ has minimal length left coset representatives \{id, $s_1, s_2 s_1$\}.

It is well known that $\mathbb{C}S_n$ is a free right module over $\mathbb{C}S(d)$ of rank $n!$. Given a representation $M$ of $S(d)$, we can consider the induced representation $\text{Ind}_{S(d)}^{S_n} M$ which has dimension $\frac{n!}{k_1! \cdots k_l!} \dim M$. Note that if $M$ is a $H(d, u)$-module, then $\text{Ind}_{S(d)}^{S_n} M$ naturally has a structure of $H_n(u)$-module, which agrees with $\text{Ind}_{H(d, u)}^{H_n(u)} M$.

**Theorem 4.10.** Let $V$ be a representation of $S_n$, inflated to a representation of $H_n(y)$ by setting $y_i$ to act as $0$. The induced module $\Delta_{d, c}(V) = \text{Ind}_{H_n(y)}^{H_n(u)}(V)$ has a filtration such that subquotients are isomorphic as $H_n(u)$-modules to the induced representations

$$
V_d := \text{Ind}_{H(d, u)}^{H_n(u)} \left[ \text{Res}_{H(d^{-\text{rev}}, u)}^{H_n(u)} V \right]_{\text{rev}, d}, \ d \in \mathcal{P}^-(n),
$$

where here we inflate $V$ along the homomorphism $e_{0}$.
Proof. Let $d \in \mathcal{P}^-(n)$. By Lemma 4.1 and Lemma 4.7 $H_{t,c}$ has filtrations

$$B_{\leq d} = \bigoplus_{a \in \mathcal{P}^-(n)} \mathbb{C}S_n F_{X}^{-1}(\omega_a) H_n(y)$$

$$B_{< d} = \bigoplus_{a \in \mathcal{P}^-(n)} \mathbb{C}S_n F_{X}^{-1}(\omega_a) H_n(y)$$

clearly preserved by $\mathbb{C}S_n$. By Lemma 4.11 the filtrations are also preserved by $H_n(u)$. These induce filtrations on $\Delta(V)$ with subquotients

$$V_d = B_{\leq d} \Delta_{t,c}(V) / B_{< d} \Delta_{t,c}(V).$$

In the following argument we lighten notation, writing $\hat{p}$ for $F_{X}^{-1}(p)$, so for instance the above expressions would become $\mathbb{C}S_n \overline{\omega}_H H_n(y)$.

Because $S_n \omega_d S_n = \bigcup_{g \in S_n / S(d)} g \omega_d S_n$, the following spaces are isomorphic not just as vector spaces, but as $\mathbb{C}S_n$-modules, $V_d \simeq \mathbb{C}S_n \overline{\omega}_d \otimes \mathbb{C}S_n V$. In particular, as a $\mathbb{C}S_n$-module, $V_d$ is generated by $\overline{\omega}_d \otimes V$, and is spanned by the independent spaces $\overline{\omega}_{H(d)} \otimes V$, for $g \in S_n / S(d)$. Note that if $s_i \in S(d)$ then

$$\overline{s_i \omega_d} \otimes V = \overline{\omega_d(s^{-1}_i \omega_d)} \otimes V = \overline{\omega_d \cdot \text{rev}_d(s_i)} \otimes V = \overline{\omega_d} \otimes \text{rev}_d(s_i)V.$$ 

Further as $u_j \overline{\omega_d} = (\overline{\omega_d \cdot \text{rev}_d(u_j)} + t.o.t.) \otimes V$, we have $u_j \overline{\omega_d} \otimes V = (\overline{\omega_d \cdot \text{rev}_d(u_j)} + t.o.t.) \otimes V$. On the other hand, these are killed in $V_d$. Thus we see that as an $H_n(u)$-module $V_d \simeq \text{Ind}_{H(d,u)}^{H_n(u)} \left[ \text{Res}_{H(d,u)}^{H_n(u)} V \right]_{\text{rev}_d}$. 

\[ \square \]

Remark 4.11. One can regard this theorem as a version of the classical Mackey formula:

$$\text{Res}_K \text{Ind}_H^G(\rho) = \bigoplus_{\omega \in K \backslash G / H} \text{Ind}_H^{K \cap \omega H \omega^{-1} \cap K}(\rho^\omega) = \bigoplus_{\omega \in K \backslash G / H} \text{Ind}_H^{K \cap \omega H \omega^{-1} \cap K}(\text{Res}_H^{K \cap \omega H \omega^{-1} \cap K}(\rho^\omega)),$$

where $G$ is a finite group, $H, K$ are its subgroups, $\rho$ is a representation of $H$ and $\rho^\omega(x) = \rho(\omega x \omega^{-1})$.

Our setting shares many features with classical Mackey for the case $G = \overline{S_n}$, $H = K = S_n$, where the minimal length double coset representatives are $\{ \omega_d | d \in \mathbb{Z}^n, d_1 \geq \cdots \geq d_n \}$. In that case $S(d) = \omega_d S_n \omega_d^{-1} \cap S_n$, $S(d_{rev}) = S_n \cap \omega_d^{-1} S_n \omega_d$ and one computes the action on an induced module via $p(\omega_d \otimes V) = \omega_d \otimes (\omega_d^{-1} \rho_d) V = \omega_d \otimes \rho_{d_{rev}}$, which is also equal to $\omega_d \otimes \rho_{d_{rev}}$. On the other hand, $\omega_d^{-1} H(d,u)$ is problematic on many levels. This is in part why we must work with the isomorphism $\text{rev}_d$ above.
While conjugation by $\omega_d^{-1}$ or by $g_d$ gives us in isomorphism from $S(d)$ to $S(d^{\text{rev}})$ when working inside of $S_n$, the most natural way to extend the notion of conjugation by $\omega_d^{-1}$ to $H_{t,c}$ does not send $H(d, u)$ to $H(d^{\text{rev}}, u)$. While $F_X^{-1}(\omega_d) =: \widetilde{\omega_d}$ is not invertible, this is not the main obstruction; one can localize and invert the $x_i$ (as one does with the trigonometric Cherednik algebra [50]). This essentially replaces $\pi$ with $\pi$ and adjoins $\pi^{-1}$, so would enlarge our algebra and embed a copy of $S_n$. We can define $\pi u_t u^{-1} = u_{\pi(i)} = u_{i+1}$ and $\pi^{-1} u_t \pi = u_{i-1}$ using the convention in Remark 3.4 and this is compatible with relation (11). This allows us to define conjugation by $\omega_d^{-1}$. It will still send $S(d) \to S(d^{\text{rev}})$ but will not send $H(d, u) \to H(d^{\text{rev}}, u)$ as conjugation by $S_n$ does not preserve $A$ (even though conjugation by $\pi$ does preserve $A$). For $g \in S_n$ recall that $H_n(u) \ni g^{-1} u g = u_{g^{-1}(i)} + \ell . o . t$, where here lower is with respect to $u$ degree. More specifically, $u_t g = g u_{g^{-1}(i)} + \text{terms} < 0$ in Bruhat order. These are the lower order terms we throw away when considering $V_d$ or $B_{<d}/B_{<d}$. Throwing away these lower order terms agrees with replacing conjugation by $\omega_d^{-1}$ with the isomorphism $\text{rev}_d : H(d, u) \to H(d^{\text{rev}}, u)$ when describing the Mackey filtration.

As a corollary to Theorem 4.10 we have the following.

**Corollary 4.12.** Let $V$ be a $\mathbb{C}S_n$-module such that when inflated along $\text{ev}_0$ to be an $H_n(u)$-module it has $u$-weight basis $\{ v_T \mid T \in T \}$. Let $w_T$ denote the weight of $v_T$. If we assume $t \neq 0$, then the $H_{t,c}$-module $\text{Ind}_{H_n(u)}^{H_{t,c}} V$ has finite dimensional generalized $u$-weight spaces and a generalized $u$-weight basis indexed by $P(n) \times T$.

Its generalized weights are

$$\{ \omega_n \cdot w_T \mid a \in P(n), T \in T \} = \{ g \omega_d \cdot w_T \mid d \in P^{-}(n), g \in S_n/S(d), T \in T \}.$$

When $t = 0$, the weights above are still given by the formula above but the $u$-weight spaces are no longer finite-dimensional. We study this case in detail in Section 9.

It is worth noting that given fixed $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, $d = (d_1, d_2, \ldots, d_n) \in P^{-}(n)$ the set

$$\{ g \omega_d \cdot w \mid g \in S_n/S(d) \} = \{ f \cdot (w_1 + d_n t, w_2 + d_{n-1} t, \ldots, w_n + d_1 t) \mid f \in S_n/S(d^{\text{rev}}) \}.$$

**Example 4.13.** Let us consider the Mackey formula for $M = \Delta_c(\text{triv})$ in the case $n = 2, c = 2, t = 1$. As we shall see, it is not $A$-semisimple. For weights of the form $w = (d, d)$, $\text{dim} M_{\text{gen}}^{(d,d)} = 2 > 1 = \text{dim} M_{(d,d)}$. For all other weights $w = (w_1, w_2)$ with $w_1 \neq w_2$ that occur, its generalized weight spaces $M_{\text{gen}}^w = M_w$ are 1-dimensional.

We have a single $T = \begin{bmatrix} 1 & 2 \\ \end{bmatrix}$ and $v_T$ has weight $w = (0, -c) = (0, -2)$. For $d \in P^{-}(n)$ we split into two cases according to $S(d)$.

**Case 1.** $d = (d, d) = d^{\text{rev}}$, $S(d) = S_2$. Thus our induction and restriction functors are trivial and $\text{Ind}_{H_{(d,d)}}^{H_{d^{\text{rev}}, u}} \text{Res}_{H_{(d^{\text{rev}}, u)}}^{H_{(d,d)}} = \text{triv}^{\text{rev}}$. The module $\text{triv}^{\text{rev}}$ still carries the trivial action of $S_2$, but $u_1 = 0 + d, u_2 = -c + d$. In other words it corresponds to a weight vector $v_d = v_{(d,d)} \in \Delta_c(\text{triv})$ of weight $w = (d, d - 2)$. Recall we require $d \geq 0$.

**Case 2.** $d = (d_1 > d_2)$, $d^{\text{rev}} = (d_2, d_1)$, $g_d = s_1$, and $S(d) = \{ \text{id} \} = S(d^{\text{rev}})$. Now $\text{Res}_{H_{d^{\text{rev}}, u}}^{H_{(d^{\text{rev}}, u)}} \text{triv} = \text{Res}_{A}^{H_{d^{\text{rev}}, u}} \text{triv} = (0) \boxtimes (-c)$ where we write the one-dimensional $A = \mathbb{C}[u_1] \otimes \mathbb{C}[u_2]$-module on which $u_1 - \alpha$ and $u_2 - \beta$ vanish as $(\alpha) \boxtimes (\beta)$. The twisted module is

$$[0] \boxtimes (-c) \rtimes = (-c + d_1) \boxtimes (0 + d_2) = (d_1 - 2) \boxtimes (d_2).$$
Finally
\[
\text{Ind}_{H_c}^{H_2}([\text{Res}_{H_d}^{H_2} \text{triv}])^{\text{rev}} = \text{Ind}_{h_c}^{H_2}(d_1 - 2) \boxtimes (d_2).
\]
This is an irreducible 2-dimensional \( H_2 \)-module.

In the special case \( d_2 = d_1 - 2 \) it is not \( A \)-semisimple. In other words the \( u_i \) act with Jordan blocks of size 2. The generalized \( \mathcal{w} = (d_1 - 2, d_2) = (d_2, d_2) \)-weight space is 2-dimensional and corresponds to the basis vectors in \( \Delta \) which by abuse of notation we can still call \( v_d = v(d_2 + 2, d_2) \) and \( v_{s_i} = v(d_2, d_2 + 2) \).

When \( d_2 \neq d_1 - 2 \) we get one-dimensional weight spaces spanned by \( v_d = v(d_1, d_2) \) of weight \( \mathcal{w} = (d_1 - 2, d_2) \) and \( v_{s_i} = v(d_2, d_1) \) of weight \( s_1 \cdot \mathcal{w} = (d_2, d_1 - 2) \). Because these (generalized) weights occur with multiplicity one, the Mackey filtration tells us these generalized weight spaces are true weight spaces.

5. Representation theory of \( H_{t,c} \)

5.1. Generalized eigenspaces and intertwining operators. As above, we will denote by \( \mathcal{A} \subseteq H_{t,c} \) the polynomial subalgebra generated by the Dunkl-Opdam elements \( u_1, . . . , u_n \).

For an \( H_{t,c} \)-module \( M \) and \( \mathcal{w} \in \mathbb{C}^n \), let \( M_{\mathcal{w}}^{\text{gen}} \) denote the generalized eigenspace with weight \( \mathcal{w} \), that is, \( (u_i - \mathcal{w}_i) \) acts locally nilpotently on \( M_{\mathcal{w}}^{\text{gen}} \) for every \( i \). We also denote by \( M_{\mathcal{w}} \subseteq M_{\mathcal{w}}^{\text{gen}} \) the subspace of honest simultaneous eigenvectors. At \( t = 1 \) the Euler element is \( h = \sum u_i + n/2 \) and it is therefore easy to see that every module in category \( \mathcal{O} \) is locally finite for the \( \mathcal{A} \)-action, so that it decomposes as the direct sum of its generalized weight spaces, and each such space is finite-dimensional. Note that this also follows easily from Theorem [1,10].

We are interested in the spectrum of \( \mathcal{A} \) on standard modules. To study it, we will make use of the following intertwining operators, cf. [12,30,37]

\[
\sigma_i := s_i - \frac{c}{u_i - u_{i+1}}, \quad i = 1, \ldots , n-1, \quad \tau = x_1(12\cdots n).
\]

Note that \( \tau \in H_{t,c} \), while the \( \sigma_i \) are elements of the localization \( H_{t,c}[(u_i - u_j)^{-1} : i \neq j] \). Alternatively, given a representation \( M \), we may think of \( \tau \) as an operator which is defined globally on \( M \), while \( \sigma_i \) is only defined on those generalized eigenspaces \( M_{\mathcal{w}}^{\text{gen}} \) for which \( \mathcal{w}_i - \mathcal{w}_{i+1} \neq 0 \), i.e., \( s_i \cdot \mathcal{w} \neq \mathcal{w} \).

Lemma 5.1. [30, (4.13)] We have \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) and, if \( i < n \), \( \tau = \sigma_{i+1} \tau \). Furthermore, \( \sigma_i \)

\[
\sigma_i^2 = \frac{(u_i - u_{i+1} - c)(u_i - u_{i+1} + c)}{(u_i - u_{i+1})^2}.
\]

and \( \lambda \sigma_i = \sigma_{i-1} \lambda \) if \( i > 1 \), also \( \lambda \sigma_i \tau = \tau \sigma_{n-1} \lambda - \frac{u_0}{u_0 - u_1} c = \tau \sigma_{n-1} \lambda - \frac{u_{n-1} + t}{u_n + t - u_1} c \).

It is not hard to see that we have

\[
\sigma_i : M_{\mathcal{w}}^{\text{gen}} \to M_{\mathcal{w}}^{\text{gen}}, \quad \tau : M_{\mathcal{w}}^{\text{gen}} \to M_{\mathcal{w}}^{\text{gen}}
\]

where the symmetric group \( S_n \) acts on \( \mathbb{C}^n \) by permuting the coordinates, and \( x \cdot (u_1, \ldots , u_n) = (u_n + t, w_1, \ldots , u_{n-1}) \) as in Equation [1]. In the first case we assume \( s_i \cdot \mathcal{w} \neq \mathcal{w} \) so \( \sigma_i \) is well-defined.

Remark 5.2. Note that, if \( \sigma_i | M_{\mathcal{w}}^{\text{gen}} = 0 \), then \( \mathcal{w}_i - \mathcal{w}_{i+1} = \pm c \). If \( M \) is free as a \( \mathbb{C}[x_1, \ldots , x_n] \)-module (for example, a standard module) then \( \tau | M_{\mathcal{w}}^{\text{gen}} \neq 0 \) provided \( M_{\mathcal{w}}^{\text{gen}} \neq 0 \).
**Remark 5.3.** Note that $\sigma_i = (s_i u_i - u_i s_i)/(u_i - u_{i+1})$. These operators are well-defined on any simple $H_{t,c}$-module on which $A$ acts semisimply. It is sometimes convenient to instead consider

$$\tilde{\sigma}_i := (s_i u_i - u_i s_i)/(u_i - u_{i+1} + c).$$

These also satisfy the braid relations and their quadratic relation becomes $\tilde{\sigma}_i^2 = 1$.

We will see below (see Section 6.2) that these operators are well-defined on $L_c(\text{triv})$.

Because the intertwiners satisfy the braid relations, $\sigma_\omega$ and $\tilde{\sigma}_\omega$ are well-defined for $\omega \in \tilde{S}_n$, if one takes the convention $\sigma_\pi = \tilde{\sigma}_\pi = \pi$.

Using intertwiners to construct and parameterize an $A$-weight basis for an $A$-semisimple (or calibrated) module, as well as giving the action of generators on that basis, follows ideas developed by Ram in [54] or Cherednik in [10]. In [54] the role of $A$ was instead played by an appropriate commutative subalgebra of the affine Hecke algebra, but the constructions apply in our context as well.

### 5.2. The standard modules.

From now on, unless otherwise explicitly stated, we will assume that the parameter $c$ has the form $c = m/n > 0$ with $\gcd(m, n) = 1$ and $t = 1$. In this section, we will analyze the action of the Dunkl-Opdam subalgebra on a standard module $\Delta_c(\mu)$. We will denote by $\text{SYT}(\mu)$ the set of standard tableaux on $\mu$. For $T \in \text{SYT}(\mu)$, $T_i$ denotes the box of $\mu$ labeled by $i$ under $T$, and $ct_T(i)$ is the content of this box.

**Definition 5.4.** Let $(a, T) \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu)$. Denote by $\omega(a, T) \in \mathbb{C}^n$ the weight whose $i$-th component is $w_i(a, T) = a_i - ct_T(g_a(i))c$ where, recall, $g_a \in \mathcal{S}_n$ is the minimal permutation that sorts $a$ increasingly.

From Lemma 5.17 it is clear that we have that the intertwining operators send $\tau: \Delta_c(\mu)_{\omega(a, T)} \rightarrow \Delta_c(\mu)_{\omega(\tau(a, T))}$ and, if $a_i \neq a_{i+1}$, $\sigma_i: \Delta_c(\mu)_{\omega(a, T)} \rightarrow \Delta_c(\mu)_{\omega(\sigma_i(a, T))}$.

The following result was proved in [30, Theorem 5.1].

**Theorem 5.5.** Let $c = m/n > 0$ with $\gcd(m, n) = 1$ and $M = \Delta_c(\mu)$. Then, for any $(a, T) \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu)$ we have $M_{\omega(a, T)} = M_{\omega(a, T)}^{\text{gen}}$ is 1-dimensional. Moreover, if $a_i > a_{i+1}$, then $\sigma_i |_{M_{\omega(a, T)}} \neq 0$, and the action of the Dunkl-Opdam subalgebra on $M$ is diagonalizable with eigenvalues given by $\omega(a, T)$.

**Proof.** The operators $u_i$ act on $V_{\mu}$ as classical Jucys-Murphy operators, and have spectrum $-ct_T(i)c$ for $T \in \text{SYT}(\mu)$. In other words the vector $\nu_T$ has weight $\omega_T = (-ct_T(1)c, \ldots, -ct_T(n)c)$. By Corollary 4.12 the generalized $u$-weights of $u_i$ on $\Delta_c(\mu)$ are given by $\omega_{a_i} \omega_T = (\omega(a, T)$, as now theorem follows from Lemma 5.6 below.

**Lemma 5.6.** Let $(a, T), (b, S) \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu)$. If $\omega(a, T) = \omega(b, S)$ then $a = b$ and $T = S$.

**Proof.** Assume $\omega(a, T) = \omega(b, S)$. Then, for every $i = 1, \ldots, n$,

$$a_i - b_i = c(\omega_{g_a(i)} - \omega_{g_b(i)}).$$

But $T$ and $S$ have the same shape $\mu$, so

$$ct_T(g_a(i)) - ct_S(g_b(i)) \in \{-n + 1, -n + 1, \ldots, 0, n - 2, n - 1\}$$

so, by our assumption on $c = m/n$, we must have $a_i - b_i = 0$. From here, we have $a = b$ and $ct_S(i) = ct_T(i)$ for every $i = 1, \ldots, n$, which implies $S = T$.  

\[ \Box \]
Remark 5.7. For arbitrary $t, c$, Corollary 4.12 still applies and the same proof shows that the generalized eigenvalues of $u_i$ on $\Delta_{c, \mu}$ are given by $u(a, T)$.

5.3. Recovering the action of $H_c$. Let us now see that for $c = m/n$, $\gcd(m, n) = 1$ the action of $H_c$ on the standard module $\Delta_c(\mu)$ is completely determined by the spectrum of the Dunkl-Opdam subalgebra. Fix an eigenbasis $\{v_T : T \in \text{SYT}(\mu)\}$ of $V_\mu$ for the Jucys-Murphy operators. Note that we have the basis $v_T(\omega)$, $\omega \in \mathbb{L}_{\min}^+(n)$ of $\Delta_c(\mu)$, cf. Corollary 4.3. For each $a \in \mathbb{Z}_{\geq 0}$ and every standard Young tableau $T$ on $\mu$, denote by $v(a, T) \in \Delta_c(\mu)v(a, T)$ a nonzero vector, normalized so that $v_T(\omega_a)$ appears with coefficient 1 in $v(a, T)$. The action of $H_c$ on $\Delta_c(\mu)$ is given by the following formulas:

Theorem 5.8. The module $\Delta_c(\mu)$ has a basis given by $\{v(a, T) : a \in \mathbb{Z}_{\geq 0}, T \in \text{SYT}(\mu)\}$, and the action of the algebra $H_c$ on $\Delta_c(\mu)$ is given by the following operators:

$$
\begin{align*}
&u_i v(a, T) = w_i v(a, T) \\
&\tau v(a, T) = v(\pi \cdot a, T) \\
&\lambda v(a, T) = w_1 v(\pi^{-1} \cdot a, T)
\end{align*}
$$

where $w = w(a, T)$. The action of $s_i$ with respect to this basis falls into the following five cases. Let $j = g_a(i)$. Then $s_i v(a, T) = 0$ if $a_i > a_{i+1}$, $s_i v(a, T) = v(s_i a, T)$ if $a_i = a_{i+1}$ and $j, j+1$ in same row of $T$, $s_i v(a, T) = -v(a, T)$ if $a_i = a_{i+1}$ and $j, j+1$ in same column of $T$.

Proof. Thanks to our normalization, we have that if $a_i > a_{i+1}$, then $\sigma_i v(a, T) = v(s_i a, T)$. From here and (23) one can deduce the first two cases.

Finding $s_i v(a, T)$ when $a_i = a_{i+1}$ is subtler. The weight of $s_i v(a, T)$ is $s_i \cdot w$. Note that, in this case, $g_a(i+1) = g_a(i) + 1 = j + 1$. Let us denote by $s_j(T)$ the tableau that is obtained from $T$ by permuting the entries $j$ and $j + 1$. Note that $s_j(T)$ may not be standard, and this is the case precisely when in the tableau $T$ we have

1. $T_j = (R, C)$ and $T_{j+1} = (R, C + 1)$ for some box $(R, C)$ in $\mu$, i.e., we see $j \not\in T$ or $j+1 \not\in T$,

2. $T_j = (R, C)$ and $T_{j+1} = (R + 1, C)$ for some box $(R, C)$ in $\mu$, i.e., we see $j \not\in T$ or $j+1 \not\in T$.

In these cases, $s_i \cdot w(a, T)$ is not of the form $w(a', T')$ for a standard tableau $T'$, so we must have $\sigma_i v(a, T) = 0$ and therefore $s_i v(a, T) = \pm v(a, T)$. Using the explicit
Corollary 5.13. torial proof. This is known via the KZ functor, cf. [2] and we have obtained a purely combina-

$\Delta$ Proof. $\mu$ submodule of $\Delta(\mu)$. We have recovered the action of $H_c$ on $\Delta_c(\mu)$. □

5.4. Maps between standards. In this section and the next one, we study maps between standard modules. Note that similar results were obtained in [1] for the case of trigonometric Cherednik algebras and in [54, 32] for the case of affine Hecke algebras.

Lemma 5.9. Suppose that $c = m/n, \gcd(m, n) = 1$. Let $d_i(\mu)$ be the number of boxes in $\mu$ with content $i \mod n$. Then for all $(a, T) \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu)$ one has

$\sharp\{j : nu_j(a, T) \equiv -mi \mod n\} = d_i(\mu)$.

Proof. We have

$nu_j(a, T) = na_j - m \text{ct}_T(g_a(j)) \equiv -m \text{ct}_T(g_a(j)) \mod n$.

Since $g_a$ is a permutation, $T_{g_a(j)}$ runs over all boxes in $\mu$ and the vector $\text{ct}_T(g_a(j))$ has exactly $d_i(\mu)$ entries equal to $i \mod n$. □

Remark 5.10. A similar argument and Remark 5.7 show that for $c = m/\ell$, $\gcd(m, \ell) = 1$ one has

$\sharp\{j : \ell u_j(a, T) \equiv -mi \mod \ell\} = d_i^{(\ell)}(\mu)$,

where $d_i^{(\ell)}(\mu)$ is the number of boxes in $\mu$ with content $i \mod \ell$.

Lemma 5.11. Suppose that $c = m/n, \gcd(m, n) = 1$. Let $\mu \neq \mu'$ be two partitions of $n$. Then $\text{Hom}_{H_c}(\Delta_c(\mu), \Delta_c(\mu')) = 0$ unless both $\mu$ and $\mu'$ are hook partitions.

Proof. Suppose that $\text{Hom}_{H_c}(\Delta_c(\mu), \Delta_c(\mu')) \neq 0$, then $w(a, T) = w(a', T')$ for some $(a, T) \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu)$ and $(a', T') \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu')$. By Lemma 5.9 we get $d_i(\mu) = d_i(\mu')$ for all $i$, which implies that $\mu$ and $\mu'$ have the same $n$-core [35].

Since $\mu$ has size $n$, either its $n$-core is empty and $\mu$ is a hook, or $\mu$ is an $n$-core itself. The same applies to $\mu'$, so they can share an $n$-core only if both partitions are hooks. □

Remark 5.12. A similar argument shows that for $c = m/\ell$, $\gcd(m, \ell) = 1$ one could possibly have $\text{Hom}_{H_c}(\Delta_c(\mu), \Delta_c(\mu')) \neq 0$ only if $\mu, \mu'$ have the same $\ell$-core. This is known via the KZ functor, cf. [2] and we have obtained a purely combinatorial proof.

Corollary 5.13. Let $c = m/n, \gcd(m, n) = 1$. If $\mu$ is not a hook partition then $\Delta_c(\mu)$ is irreducible.

Proof. The proof is standard but we include it here for completeness. If $R$ is a submodule of $\Delta(\mu)$, then (since the action of $y_1, \ldots, y_n$ is locally nilpotent) there is a vector $v \in R$ such that $y_1v = \cdots = y_nv = 0$. It spans a finite-dimensional subspace $U$ under the action of $\mathcal{S}_n$, and $\lambda(U) = 0$, it contains an irreducible representation of $\mathcal{S}_n$ isomorphic to $V_{\mu'}$. Then there is a nontrivial morphism $H_c$-modules $\Delta(\mu') \to \Delta(\mu)$ which sends $V_{\mu'}$ to this subspace. □
5.5. The BGG resolution. Throughout this section we assume $c = m/n, \gcd(m,n) = 1$.

Let us denote by $V_{\mu_{\ell}} := \bigwedge^\ell \mathbb{C}^{n-1}$ the hook representation of $S_n$, so that $\mu_{\ell}$ is the partition $(n-\ell, 1^\ell), \ell = 0, \ldots, n-1$. In particular, $V_{\mu_0}$ is the trivial representation and $V_{\mu_{n-1}}$ the sign representation. It is known \cite{2} that the representation $L_{m/n} := L_{m/n}^{(\text{triv})}$ admits a resolution
\begin{equation}
0 \to \Delta_c(\mu_{n-1}) \to \cdots \to \Delta_c(\mu_1) \to \Delta_c(\mu_0) \to 0
\end{equation}
that in fact coincides with the Koszul resolution of $L_{m/n}$ when considering a standard module as a $\mathbb{C}[x_1, \ldots, x_n]$-module. In this section, we will construct the resolution \cite{20} in a purely combinatorial manner. We remark that this has been recently generalized in \cite{20} to some other BGG resolutions.

Let us set up some notation. For each collection $1 < i_1 < \cdots < i_\ell \leq n$, let $T_{i_{i_1}<i_2<\cdots<i_\ell}$ be the tableau on $\mu_{\ell}$ that has the numbers $1, i_1, \ldots, i_\ell$ on its leg. Clearly, every tableau on $\mu_{\ell}$ is of this form.

Recall that for each element $(a, T) \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu)$ we have a nonzero vector $v(a, T) \in \Delta_c(\mu)_{v(a, T)}$. Clearly, every map on $\Delta_c(\mu)$ is completely determined by the image of the vectors $v_T = v(0, T), T \in \text{SYT}(\mu)$.

**Lemma 5.14.** Suppose that $\ell \neq j + 1$. Then $\text{Hom}_{\mu_{\ell}}(\Delta_c(\mu_\ell), \Delta_c(\mu_j)) = 0$.

**Proof.** Let $T = T_{i_{i_1}<i_2<\cdots<i_\ell}$ be a standard tableau of shape $\mu_{\ell}$, we have
\[n\omega_{i_1}(0, T) = m, \ldots, n\omega_{i_\ell}(0, T) = m\ell, n\omega_i(0, T) < 0 \text{ for } i \neq 1, i_1, \ldots, i_\ell.\]
Now suppose that $u(0, T) = u(a, T')$ for some $(a, T') \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu_j)$. One has
\[0 > n\omega_i(a, T') = na_i - m \cdot \text{ct}_T(g_a(i)) \geq -m \cdot \text{ct}_{T'}(g_a(i)) \text{ for } i \neq 1, i_1, \ldots, i_\ell,\]
so $\mu_j$ has at least $n - \ell - 1$ boxes with positive contents, and $\ell \geq j$.

Suppose that $\ell \geq j + 2$. It is easy to see that the equation
\[-m \text{ct}_T(i) = na_i - m \text{ct}_{T'}(g_a(i)),\]
implies
\[
\begin{cases}
    a_i = m \text{ and } \text{ct}_T(i) + n = \text{ct}_{T'}(g_a(i)) & i = i_{j+1}, \ldots, i_\ell, \\
    a_i = 0 \text{ and } \text{ct}_T(i) = \text{ct}_{T'}(g_a(i)) & \text{otherwise}.
\end{cases}
\]
By definition, this implies $g_a(i_{j+1}) = n - \ell + 1, \ldots, g_a(i_\ell) = n$, so
\[\text{ct}_{T'}(n - \ell + 1) = n - (j+1), \ldots, \text{ct}_{T'}(n) = n - \ell.\]
But this means that the first row of $T'$ contains the numbers $n, n-1, \ldots, n - \ell + 1$ in decreasing order, contradiction. \hfill \Box

**Remark 5.15.** Note that if a simple $L_c(\mu)$ appears as a composition factor inside a standard module $\Delta_c(\mu')$ then all weights $w(0, T)$ have to appear as weights of $\Delta_c(\mu')$, where $T$ is a standard Young tableau on $\mu$. Thus, the proof of Lemma 5.14 shows that the only composition factors of $\Delta_c(\mu_j)$ can be $L_c(\mu_j)$ and $L_c(\mu_{j+1})$.

Moreover, the multiplicity of $L_c(\mu)$ as a composition factor of $\Delta_c(\mu')$ is bounded above by the dimension of the (generalized) weight space $\Delta_c(\mu')_{w(0, T)}$ where $T$ is any standard Young tableau on $\mu$. Thus, $[\Delta_c(\mu_j) : L_c(\mu_{j+1})] \leq 1$. We will see in the next proposition that this multiplicity is always equal to 1.
Proposition 5.16. For $\ell = 1, \ldots, n - 1$, the space $\text{Hom}_F(\Delta_c(\mu_\ell), \Delta_c(\mu_{\ell-1}))$ is 1-dimensional. Up to a nonzero scalar, the unique homomorphism $\phi_\ell : \Delta_c(\mu_\ell) \to \Delta_c(\mu_{\ell-1})$ is determined by $\phi_\ell(v(0, T_{i_1, < \cdots < i_\ell})) = v(m_{e_{i_1}, T_{i_1, < \cdots < i_{\ell-1}}})$.

To prove Proposition 5.16 we will construct the maps $\phi_\ell$ using the results in Section 3.

Lemma 5.17. Let $D = \tau(\sigma_{n-1} \cdots \sigma_2 \sigma_1 \tau)^{m-1}$. Then for $1 < i < n$ $\sigma_i D = D \sigma_{i-1}$ and $u_i D = D u_{i-1}$, but $u_1 D = D(u_n + mt)$.

The proof is an easy computation we leave to the reader. Recall for $e_1 = (1,0,\ldots,0)$ that $H(e_1,u) = H_1(u) \otimes H_{n-1}(u)$.

Lemma 5.18. Let $U \subseteq V_{(n-\ell+1,1^{\ell-1})}$ be the $S_{n-1} \times S_1$-submodule spanned by all $v_T$ where $T = T_{i_1, i_2, < \cdots < i_{\ell-1}}$ with $i_{\ell-1} \neq n$. In particular $U \cong V_{(n-\ell+1,1^{\ell-1})}$ as an $S_{n-1}$-module.

(1) Let $t, c$ be such that $\Delta_{t,c}(n - \ell + 1, 1^{\ell-1})$ is $A$-semisimple. Then we have $H(e_1,u)DU \subseteq \Delta_{t,c}(n - \ell + 1, 1^{\ell-1})$ is an $H(e_1,u)$-submodule which is isomorphic to $V_{(n-\ell,1^{\ell-1})}$ as an $H_{n-1}(u)$-module on which $u_1$ acts identically as $c(\ell - n) + mt$.

(2) In the case $t = 1, c = \frac{m}{n}, \gcd(m,n) = 1$, then $u_1$ acts as $c\ell$ and $H_n(u)DU \cong \text{Ind}_{H(e_1,u)}^{H_n(u)}(c\ell) \boxtimes V_{(n-\ell,1^{\ell-1})}$.

Proof. The first statement follows from Lemma 5.17: Since $\Delta_{t,c}(n - \ell + 1, 1^{\ell-1})$ is $A$-semisimple the $\sigma_i$ act triangularly with respect to the $s_i$. So the action of the $\sigma_i$ on the inflation via $e_{v_0}$ of an $S_{n-1}$-module completely determines the $S_{n-1}$ structure. Recall that via $e_{v_0}$ the $u_i$ will act as Jucys-Murphy operators.

For the second statement, we use Lemma 5.17 to determine the action of $u_1$. Because $F_{\chi}^{-1}(\omega_{m_{e_{n}}}) = \tau(s_{n-1} \cdots s_2 s_1 \tau)^{m-1}$ is a minimal length double coset representative we get the second statement.

Lemma 5.19. $\text{Ind}_{H(e_1,u)}^{H_n(u)}(c\ell) \boxtimes V_{(n-\ell,1^{\ell-1})}$ has an $H_n(u)$-submodule isomorphic to $V_{(n-\ell,1^{\ell-1})}$ (inflated along $e_{v_0}$). In particular $u_1$ is identically zero on this submodule.

The proof is a standard result for the degenerate affine Hecke algebra.

Lemma 5.20. Let $M = \Delta_{t,c}(V)$ be an $H_{t,c}$-module which has a $H_n(u)$-submodule $N$ on which $u_1$ acts identically as zero. Then for $1 \leq i \leq n$ the $y_i$ act as zero on $N$.

Proof. Recall $u_1 = x_1 y_1$. Since $\Delta_{t,c}(V)$ is free as a $\mathbb{C}[x_1, \ldots, x_n]$-module, $x_1$ has no torsion so in particular $y_1$ is zero on $N$. As $N$ is $S_n$-invariant and $y_i = (1, 2, \ldots, i) y_1(i, \ldots, 2, 1)$, all the $y_i$ must act as zero.

Proof of Proposition 5.16. As a consequence of Lemma 5.20 we get that $\Delta(n - \ell + 1, 1^{\ell-1})$ has a $S_{n-1}$-submodule isomorphic to $V_{(n-\ell,1^{\ell-1})}$ on which all $y_i$ vanish. Thus Frobenius Reciprocity gives us a nonzero $H_{t,c}$ homomorphism

$$\Delta_c(n - \ell + 1, 1^\ell) \xrightarrow{\phi_\ell} \Delta_c(n - \ell + 1, 1^{\ell-1}).$$

This yields a proof of Proposition 5.16.

More concretely, we can normalize the basis $\{v_T \mid T \in \text{SYT}(\mu_t)\}$ of $V_{(n-\ell+1,1^{\ell-1})}$ so that we fix $v_T$ for $T = T_{2,3, \cdots, \ell+1}$ and take the other basis vectors to be
σ₀νₜ := νₜ for id ≤ 0 ≤ [1, n − ℓ + 1, . . . , n − 1, n, 2, 3, . . . , n − ℓ] in weak Bruhat order. (Recall as the σᵢ satisfy the braid relations, σ₀ makes sense.) Then φₜ is determined by

\[ νₜ \mapsto σ_ℓ \cdots σ_2σ₁DνT₂<\cdots<ₗ \]

where all tableau on the left of \( σ_ℓ \) have shape \((n − ℓ, 1^ℓ)\) but all tableau on the right have shape \((n − ℓ + 1, 1^{ℓ−1})\). More generally (noting \( i_{ℓ−1} ≥ ℓ \)) we have

\[ νT₁<₁₂<≤iₗ \mapsto σ_{i_{ℓ−1}} \cdots σ_2σ₁DνT₁<₁₂<≤i_{ℓ−1}. \]

In particular when \( i_ℓ = n \) we get

\[ νT₁<₁₂<≤iₙ \mapsto (σ_{n−1} \cdots σ_2σ₁σ)νT₁<₁₂<≤i_{n−1}. \]

Recall \((s_{n−1} \cdots s₁σ)ν = t_m e_n \geq 0\). One can easily check the above vectors’ u-weights are preserved by φₜ. It is only slightly more work to check with the above assignment that φₜ intertwines the σᵢ acting on the \( νₜ, T \in SYT(n − ℓ, 1^ℓ) \).

**Corollary 5.21.** For any \( ℓ = 0, 1, \ldots, n−1 \), the standard module \( Δ_c(µₜ) \) has a unique composition series \( 0 ⊆ I_ℓ ⊆ Δ_c(µₜ) \). Moreover, \( I_ℓ \cong L_c(µ⁺₁) \) and \( Δ_c(µₜ)/I_ℓ = L_c(µₜ) \).

**Proof.** From Remark 5.15 and Proposition 5.16 it follows that

\[ [Δ_c(µₜ) : L_c(µ)] = \begin{cases} 1 & \text{if } µ = µ⁺₁, µ⁺₁+1 \\ 0 & \text{otherwise} \end{cases} \]

moreover, \( L_c(µ₊₁) \) cannot appear as a quotient of \( Δ_c(µₜ) \). So defining \( I_ℓ := φₜ⁺₁(Δ_c(µ₊₁)) \) the result follows. \( □ \)

**Corollary 5.22.** We have \( \text{im}(φₜ⁺₁) = \text{ker}(φₜ) \). In other words, the complex \( Δ_c(µ₊₁) \stackrel{φₜ⁺₁}{\to} Δ_c(µₜ) \) is exact outside of degree 0 and coincides with (24).

**Proof.** It is enough to see that \( \text{ker}(φₜ) = I_ℓ \). For this, it is enough to see that \( φₜ \) is neither zero nor injective. That it is nonzero is obvious. Thanks to Lemma 5.14 we must have \( φₜ⁺₁ ∘ φₜ = 0 \). So \( φₜ⁺₁(I_ℓ) = 0 \) and \( φₜ \) is not injective. \( □ \)

5.6. Weight basis of simples. We continue assuming \( c = m/n \) with \( m \) and \( n \) coprime positive integers. In this section, we describe weights belonging to the maximal proper submodule of every standard module \( Δ_c(µ) \). Thanks to Corollary 5.13 this question is only interesting when \( µ = µₜ \) is a hook partition. Moreover, since \( Δ_c(µ₊₁) \) is simple, we may and will assume throughout this section that \( 0 ≤ ℓ < n−1 \).

**Lemma 5.23.** Let \((a, T) \in Z^n_{≥0} × SYT(µₜ)\). Then, there exists \((b, T') \in Z^n_{≥0} × SYT(µ₊₁)\) such that \( u(a, T) = u(b, T') \) if and only if either

- \( aₙ⁺₁(n) − m ≥ aₙ⁺₁(iₜ) \) or
- \( aₙ⁺₁(n) − m = aₙ⁺₁(iₜ) \) and \( gₙ⁻¹(n) > gₙ⁻¹(iₜ) \)

where \( iₜ \) is the number labeling the box with smallest content of \( µₜ \) on the tableau \( T \). Moreover, if this is the case, then \((b, T')\) is uniquely determined.

**Proof.** Following the notation of Section 5.5 let us denote \( T = T₁<≤<≤iₗ \). We will, first, see that there is a unique \( b \in Z^n \) (possibly with negative entries) and \( T' \) a
tableau on $\mu_{\ell+1}$ (possibly non-standard) such that $w(a, T) = w(b, T')$. Indeed, if such pair $(b, T')$ exists we must have

$$n(a_i - b_i) = m(cT(g_a(i)) - cT'(g_b(i)))$$

for every $i = 1, \ldots, n$. Since $m$ and $n$ are coprime and $\mu_{\ell}, \mu_{\ell+1}$ are adjacent hooks, we must have that either

(i) $a_i = b_i$ and $T_{g_a(i)} = T'_{g_a(i)}$ (meaning that this box is in $\mu_{\ell} \cap \mu_{\ell+1}$) or

(ii) $a_i - b_i = m$, $T_{g_a(i)}$ is the box of highest content in $\mu_{\ell}$, and $T'_{g_a(i)}$ is the box of lowest content in $\mu_{\ell+1}$.

Let $k \in \{1, \ldots, n\}$ be such that $T_k$ is the box of highest content of $\mu_{\ell}$. From (i) and (ii), the vector $b$ is uniquely specified: $b_i = a_i$ if $i \neq g_a^{-1}(k)$, and $b_{g_a^{-1}(k)} = a_{g_a^{-1}(k)} - m$. Moreover, the tableau $T'$ is also uniquely specified: $T'_{g_a(i)} = T_{g_a(i)}$ if $i \neq g_a^{-1}(k)$, and $T'_{g_a(g_a^{-1}(k))}$ is the box with lowest content in $\mu_{\ell+1}$. Our job now is to check that all coordinates of $b$ are non-negative and $T'$ is standard if and only if the conditions of the lemma are satisfied. Clearly, $b$ is non-negative if and only if $a_{g_a^{-1}(k)} \geq m$, so we will focus on the condition that $T'$ is standard.

Let us first verify that $T'$ is standard on $\mu_{\ell} \cap \mu_{\ell+1}$. Indeed, consider two consecutive boxes in $\mu_{\ell} \cap \mu_{\ell+1}$ and let $j_1 < j_2$ be their labels under $T$. Note that $j_1, j_2 \neq k$. By definition of $g_a$ and $b$ we have

$$b_{g_a^{-1}(j_1)} = a_{g_a^{-1}(j_1)} \leq a_{g_a^{-1}(j_2)} = b_{g_a^{-1}(j_2)}$$

and, if we have an equality, $g_a^{-1}(j_1) < g_a^{-1}(j_2)$. From the definition of $b_a$ it follows that $g_b g_a^{-1}(j_1) < g_b g_a^{-1}(j_2)$, as wanted.

So $T'$ is standard if and only if $g_b g_a^{-1}(i_\ell) < g_b g_a^{-1}(k)$. If $i_\ell = n$, we have $b_{g_a^{-1}(i_\ell)} = a_{g_a^{-1}(n)} > a_{g_a^{-1}(k)} - m = b_{g_a^{-1}(k)}$ and therefore $g_b g_a^{-1}(i_\ell) > g_b g_a^{-1}(k)$. Thus, we must have $k = n$ and $i_\ell < n$. It follows now that the tableau $T'$ is standard if and only if either $b_{g_a^{-1}(n)} > b_{g_a^{-1}(i_\ell)}$ or $b_{g_a^{-1}(n)} = b_{g_a^{-1}(i_\ell)}$ and $g_a^{-1}(n) > g_a^{-1}(i_\ell)$, which translates precisely into the conditions of the statement of the lemma. Finally, note that $a_{g_a^{-1}(n)} - m \geq a_{g_a^{-1}(i_\ell)}$ automatically implies $a_{g_a^{-1}(n)} - m \geq 0$. We are done. 

\begin{remark}
Note that for $\ell = 0$ there is a unique tableau $T$ on $\mu_0$ and $i_\ell = 1$.
\end{remark}

In this case, $a_{g_a^{-1}(1)} = \min a$ and $a_{g_a^{-1}(n)} = \max a$, so we recover the conditions defining the set $S$ in Section 6.3.

\begin{corollary}
Let $(a, T) \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu_{\ell})$. Then, $v(a, T) \in I_{\ell}$ if and only if there exists $(b, T') \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu_{\ell+1})$ such that $w(a, T) = w(b, T')$.
\end{corollary}

\begin{proof}
Since $I_{\ell} = \phi_{\ell+1}(\Delta_{\ell}(\mu_{\ell+1}))$, the necessity is clear. For sufficiency, assume that such $(b, T')$ exists. It is enough to see that $w(b, T') \notin I_{\ell+1}$ and to see this we can check that there does not exist $(d, T'') \in \mathbb{Z}_{\geq 0}^n \times \text{SYT}(\mu_{\ell+2})$ such that $w(b, T') = w(d, T'')$. So we have to check that $(b, T')$ does not satisfy the conditions of Lemma 6.23. Let $i_{\ell+1} := g_b g_a^{-1}(n)$, and note that $T'_{i_{\ell+1}}$ is precisely the box with lowest content in $\mu_{\ell+1}$. Now,

$$b_{g_b^{-1}(n)} - m = a_{g_a^{-1}(n)} - m \leq a_{g_a^{-1}(n)} - m = b_{g_a^{-1}(n)} = b_{g_b^{-1}(i_{\ell+1})}$$

If the inequality is strict, we are done. Else, we need to show that $g_b^{-1}(n) < g_b^{-1}(i_{\ell+1}) = g_a^{-1}(n)$. But in this case we have $a_{g_b^{-1}(n)} = a_{g_a^{-1}(n)}$ and the result now follows by the definition of $g_a$. 
\end{proof}
Corollary 5.26. Assume $0 \leq \ell < n - 1$ and let $(a, T) \in \mathbb{Z}_{>0}^n \times \text{SYT}(\mu_\ell)$. Let us denote by $i_\ell$ the label of the box with smallest content of $\mu_\ell$ under $T$. Then, $v(a, T) \in I_\ell$ if and only if either

- $a_{g_a^{-1}(n)} - m > a_{g_a^{-1}(i)}$ or
- $a_{g_a^{-1}(n)} - m = a_{g_a^{-1}(i)}$ and $g_a^{-1}(n) > g_a^{-1}(i_\ell)$.

It follows that $L_c(\mu_\ell) = \Delta_c(\mu_\ell) / I_\ell$ has a weight basis indexed by pairs $(a, T) \in \mathbb{Z}_{>0}^n \times \text{SYT}(\mu_\ell)$ such that

- $a_{g_a^{-1}(n)} - m < a_{g_a^{-1}(i)}$ or
- $a_{g_a^{-1}(n)} - m = a_{g_a^{-1}(i)}$ and $g_a^{-1}(n) < g_a^{-1}(i_\ell)$.

Remark 5.27. Note that, if $0 < \ell < n - 1$, then $L_c(\mu_\ell) \cong I_{\ell - 1}$. Thus, there is a weight-preserving bijection between pairs $(a, T) \in \mathbb{Z}_{>0}^n \times \text{SYT}(\mu_{\ell - 1})$ satisfying the condition marked with $\circ$ in Corollary 5.26 and those pairs $(b, T') \in \mathbb{Z}_{>0}^n \times \text{SYT}(\mu_\ell)$ satisfying the conditions marked with $\bullet$. This bijection is described in the proof of Lemma 5.23.

6. The polynomial representation of $H_c$

All of our geometric applications deal only with the representation $\Delta_c(\text{triv})$ where, as in Section 5, we have $c = m/n > 0$ with $\gcd(m, n)$ and triv is the trivial representation of $S_n$. Thus, we specialize many of the results of Section 5 to this special case. Note that as a $H_n(x)$-module we have $\Delta_c(\text{triv}) = \mathbb{C}[x_1, \ldots, x_n]$, where the action of $x_i \in H_n(x)$ is simply given by multiplication and the action of $S_n \subseteq H_n(x)$ is given by permuting the variables. For this reason, $\Delta_c(\text{triv})$ is commonly known as the polynomial representation of $H_c$. For the rest of this section, we assume $t = 1$. The case $t = 0$ will be treated in Section 9.

6.1. The action of the Dunkl-Opdam subalgebra. Following Definition 5.3 in the case where $\mu$ is the trivial partition, for $a \in \mathbb{Z}_{>0}^n$ we let $w := w(a) \in \mathbb{C}^n$ denote the weight whose $i$th component is $w_i = a_i - (g_a(i) - 1)c$. In other words, $w(a) = \omega_a(0, -c, -2c, \ldots, (1-n)c)$ where, as mentioned above, we specialize $t = 1$. Now we are ready to describe the spectrum of $\mathcal{A}$ on $\Delta_c(\text{triv})$, in the case where $c$ is generic. The proof of the following result is similar to that of Theorem 5.5, so we omit it.

Proposition 6.1. Assume that either $c \in \mathbb{C} \setminus \mathbb{Q}$ or that $c$ is a rational number with denominator greater than $n$. Let $M = \Delta_c(\text{triv})$. Then, $M_{w(a)}^{\text{gen}} \neq 0$ for every $a \in \mathbb{Z}_{>0}^n$, these are all the weight spaces of $\mathcal{A}$ on $M$, and each one of them is 1-dimensional (so that $M_{w(a)}^{\text{gen}} = M_{w(a)}$).

The following result is obtained by specializing Theorem 5.5 to the case where $\mu$ is the trivial partition of $n$.

Theorem 6.2. Let $c = m/n > 0$ with $\gcd(m, n) = 1$ and $M = \Delta(\text{triv})$. For any $a \in \mathbb{Z}_{>0}^n$, $M_{w(a)} = M_{w(a)}^{\text{gen}} \neq 0$. Moreover, if $a_i > a_{i+1}$, then $\sigma_i | M_{w(a)} \neq 0$.

Remark 6.3. Moreover, it is easy to show that for every element $a \in \mathbb{Z}_{>0}^n$, there exists a unique element $v_a \in M_{w(a)}$ of the form

$$v_a = x^a + \sum_{a' < a} k_{a, a'} x^{a'}$$
where \( \prec \) is the partial order on monomials defined in Section 2.3.

**Remark 6.4.** We can get an analogous result for the \( \mathfrak{sl}_n \) RCA, with the operators defined as in [53]. In particular, we can find a basis \( \{ v_a : a \in \mathbb{Z}_{\geq 0}^{n-1} \} \) of simultaneous eigenvectors for the \( \mathfrak{sl}_n \)-Dunkl-Opdam operators.

Note that the action of \( \tau \) is injective on \( \Delta_c(\text{triv}) \). Combinatorially, the action of \( \pi \) on the set of nonnegative sequences \( (a_1, \ldots, a_n) \) is injective, and any such sequence can be uniquely written as

\[
(a_1, \ldots, a_n) = \pi^k \cdot (0, b_2, \ldots, b_{n-1}),
\]

where \( (0, b_1, \ldots, b_{n-1}) \in \mathbb{Z}_{\geq 0}^n \). That is, the generating set for this action consists of sequences with \( a_1 = 0 \), and it is in bijection with the basis in the polynomial representation of \( \mathfrak{sl}_n \) RCA.

### 6.2. Recovering the action of \( H_c \)

We keep assuming that \( c = m/n > 0 \) with \( \gcd(m, n) = 1 \) and \( t = 1 \). The following theorem is a consequence of Theorem 5.8.

**Theorem 6.5.** The module \( \Delta_c(\text{triv}) \) has a basis given by \( \{ v_a : a \in \mathbb{Z}_{\geq 0}^n \} \), and the action of the algebra \( H_c \) on \( \Delta_c(\text{triv}) \) is given by the following operators:

\[
\begin{align*}
    u_i v_a &= w_i v_a \\
    \tau v_a &= \tau v_a \\
    \lambda v_a &= w_1 v_{\pi^{-1} a} \\
    s_i v_a &= \begin{cases} \\
        v_{s_i a} + \frac{c}{w_{i+1}} v_a & a_i > a_{i+1}, \\
        \frac{(w_i - w_{i+1}) (v_{s_i a} - v_i a)}{w_i - w_{i+1}} v_{s_i a} + \frac{c}{w_i - w_{i+1}} v_a & a_i < a_{i+1}, \\
        a_i = a_{i+1} & a_i = a_{i+1}
    \end{cases}
\end{align*}
\]

where we denote \( w_i = a_i - (g_n(i) - 1)c \).

For geometric applications, we will need a different basis of \( \Delta_c(\text{triv}) \) that gives nicer formulas for the action of the operators \( s_i \). This basis is a renormalization of the basis \( v_a \), but we have to be careful with the renormalization factor. The main result of this section is the following.

**Proposition 6.6.** There exists a function \( \varphi : \mathbb{Z}_{\geq 0}^n \to \mathbb{C}^\times \) such that, defining \( \tilde{v}_a := \varphi(a) v_a \), we have

\[
(1 - s_i) \tilde{v}_a = \frac{w_i - w_{i+1} - c}{w_i - w_{i+1}} (\tilde{v}_a - \tilde{v}_{s_i a})
\]

for every \( a \in \mathbb{Z}_{\geq 0}^n \) and \( i = 1, \ldots, n - 1 \).

**Proof.** We can define the \( \tilde{v}_a \) using the renormalized intertwiners \( \tilde{\sigma}_i \) of Remark 6.5. In particular, the vectors \( \tilde{v}_a = \tilde{\sigma}_{\omega_n} \tilde{v}_o \) for \( \omega_n \in L_{\min}(n) \) are uniquely determined after specifying \( \tilde{v}_o \). Then the existence of the function \( \varphi \) is obvious and \( \varphi(a) \) can be given as a product formula with terms indexed by the inversions of \( \omega_n \). In particular

\[
\varphi(a) = \frac{v_{s_i a}}{\varphi(s_i a)} = \begin{cases} \\
        v_{s_i a} & a_i > a_{i+1}, \\
        \frac{w_i - w_{i+1} - c}{w_i - w_{i+1}} v_{s_i a} & a_i < a_{i+1}, \\
        v_{s_i a} & a_i = a_{i+1}
    \end{cases}
\]

The formula for the action of \( 1 - s_i \) follows from Theorem 6.5. \( \square \)
Corollary 6.7. The action of the operators $u_i, \tau$ and $\lambda$ in the renormalized basis $\tilde{v}_a$ is given by the same equations as in Theorem 6.5

$$u_i \tilde{v}_a = u_i \tilde{v}_a, \quad \tau \tilde{v}_a = \tilde{v}_{\tau(a)}, \quad \lambda \tilde{v}_a = \tilde{v}_{\tau^{-1} \cdot a}.$$  

(26)

6.3. The radical of $\Delta_c(\text{triv})$. Finally, we will need an explicit weight basis of the simple quotient $L_c(\text{triv})$ of $\Delta_c(\text{triv})$. We define the set

$$S := \{(a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n : \max(a_i - a_j) > m, \text{ or } \max(a_i - a_j) = m \text{ and } a_i - a_j = m \text{ for some } j < i\}$$

The next result then follows from Corollary 5.26 applied to $\ell = 0$.

Proposition 6.8. The space

$$S := \bigoplus_{a \in S} \mathbb{C}v_a$$

is an $H_c$-submodule of $\Delta_c(\text{triv})$, and it coincides with the unique proper submodule of $\Delta_c(\text{triv})$ [19].

Now let $T := \mathbb{Z}_{\geq 0}^n \setminus S$. More explicitly,

$$T = \{a \in \mathbb{Z}_{\geq 0}^n : a_i - a_j \leq m \text{ for every } i, j; \text{ moreover, if } a_i - a_j = m \text{ then } j > i\}.$$  

(27)

Corollary 6.9. The module $L_c(\text{triv})$ has a basis $\{v_a : a \in T\}$. The action of $H_c$ on $L_c(\text{triv})$ is given by the same formulas as in Theorem 6.6 with the understanding that we set $v_a = 0$ if $a \notin T$.

Remark 6.10. Proposition 6.8 can be easily adapted to the $\mathfrak{sl}_n$-setting, cf. Remark 6.4. In that case, we have $S^{\mathfrak{sl}_n} := \{(a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} : \max(a_i) \geq m\}$. In particular, $T^{\mathfrak{sl}_n} = \{(a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} : a_i < m \text{ for every } i\}$ and we recover the formula $\dim(L_{\text{triv}}^{\mathfrak{sl}_n}) = m^{n-1}$.

As in Remark 6.4 we can prove that the action of $\pi$ is injective on the basis in $T$. The generating set for this action consists of sequences $(0, a_2, \ldots, a_n)$, and it is easy to see that such sequence is in $T$ if and only if $a_i < m$ for all $i$.

The following lemma provides an interpretation of the indexing set $T$ in terms of affine permutations.

Lemma 6.11. Let $w_i(a) := w(a)_i = a_i - \frac{a_i}{n}(g_n(i) - 1)$ be the weights of $u_i$ as above. Consider the affine permutation

$$\omega := [-n w_1(a), \ldots, -n w_n(a)]^{-1}.$$ 

Then the following statements hold:

(a) $(a_1, \ldots, a_n) \in T$ if and only if $\omega$ is $m$-stable.

(b) $a_i \geq 0$ for all $i$ if and only if $\omega p_m \in L_{\min}^+(n)$, where $p_m = [0, m, \ldots, (n-1)m]$.

Proof. We have

$$\omega^{-1}(i) = -n w_i(a) = -na_i + m(g_n(i) - 1),$$

so by (3) we have $\omega p_m = \tau_a g_a^{-1}$. By Lemma 2.14 $\omega$ is $m$-stable if and only if $a \in T$. Finally, $a_i \geq 0$ for all $i$ if and only if $\tau_a g_a^{-1} \in L_{\min}^+(n)$.  \(\Box\)
Remark 6.12. The action of \( \pi \) on \((a_1, \ldots, a_n)\) corresponds to the conjugation of \( \omega_n \) by \( \pi \in S_n \) which effectively slides the window in \( \omega_n \). Remark 6.10 gives a choice of a representative in each \( \pi \)-orbit with \( a_1 = 0 \) and \( m > a_i \geq 0 \) for \( i > 1 \).

From the viewpoint of affine permutations, a more natural choice of a representative is given by the balancing condition \( \sum_{i=1}^{n} \omega(i) = \frac{n(n+1)}{2} \). The corresponding permutations will be still \( m \)-stable, and by Remark 2.15 they are in bijection with the alcoves insider the \( m \)-dilated fundamental alcove.

Therefore we get an explicit bijection between the alcoves insider the \( m \)-dilated fundamental alcove, \( m \)-stable balanced affine permutations, and vectors of the form \((0, a_2, \ldots, a_n)\) with \( 0 \leq a_i < m \).

7. Singular curves

For coprime \( m, n \geq 1 \) we consider the plane curve singularity \( C = \{x^m = y^n\} \) at the origin. It has an action of \( \mathbb{C}^* \) given by \((x, y) \mapsto (s^n x, s^m y)\). This action extends to the local ring of functions on \( C \) which is isomorphic to \( \mathcal{O}_C = \mathbb{C}[x, y]/(x^m - y^n) \).

A homogeneous basis in \( \mathcal{O}_C \) can be described as follows:

\[
\mathcal{O}_C = \mathbb{C}[x](1, \ldots, y^{n-1})
\]

This presentation shows that \( \mathcal{O}_C \) is a free module over \( \mathbb{C}[x] \) of rank \( n \), and the multiplication by \( y \) is given by the matrix

\[
Y = \begin{pmatrix}
0 & 0 & \cdots & 0 & x^m \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]

7.1. Hilbert schemes on singular curves. By definition, the Hilbert scheme of \( k \) points on \( C \) is the moduli space of co-dimension \( k \) ideals

\[
\text{Hilb}_k(C) = \{ I \subset \mathcal{O}_C : I \text{ ideal, } \dim \mathcal{O}_C/I = k \}.
\]

The action of \( \mathbb{C}^* \) on \( C \) extends to an action on \( \text{Hilb}_k(C) \) for all \( k \). The fixed points of this action are monomial ideals. In terms of the identification (28) such an ideal is generated over \( \mathbb{C}[x] \) by monomials of the form \( \langle x^{c_1}, y x^{c_2}, \ldots, y^{n-1} x^{c_n} \rangle \). Since it is invariant under the multiplication of \( y \) (or the matrix \( Y \) above), we get a system of inequalities

\[
c_1 \geq c_2 \geq \ldots \geq c_n \geq c_1 - m.
\]

Note that \( \dim \mathcal{O}_C/I = k = \sum c_i \). In the notation of [19], such ideals can be represented by staircases of height \( n \) and width at most \( m \). See Figure 2.

Lemma 7.1. Suppose that \( I \subset \mathcal{O}_C \) is spanned over \( \mathbb{C}[x] \) by \( y^{\alpha_1} x^{c_1}, \ldots, y^{\alpha_n} x^{c_n} \) where \( \{\alpha_1, \ldots, \alpha_n\} = \{0, \ldots, n-1\} \). Then the following holds:

(a) If \( \max(c_i) - \min(c_i) > m \) then \( I \) is not an ideal in \( \mathcal{O}_C \) for any choice of \( \alpha_i \).

(b) If \( \max(c_i) - \min(c_i) \leq m \) then there exists a unique ideal \( I \) of this form.

Proof. Assume that \( I = \mathbb{C}[x]/(y^{\alpha_1} x^{c_1}, \ldots, y^{\alpha_n} x^{c_n}) \) is an ideal in \( \mathcal{O}_C \). Let \( \tilde{g} \in S_n \) be the permutation in \( S_n \) which sorts the \( \alpha_i \) in increasing order. Then by (30) \( c_{\tilde{g}^{-1}(1)} \geq c_{\tilde{g}^{-1}(2)} \geq \ldots \geq c_{\tilde{g}^{-1}(n)} \). Observe that \( c_{\tilde{g}^{-1}(1)} = \max(c_i) \) and \( c_{\tilde{g}^{-1}(n)} = \min(c_i) \).
The operator \( Y \) acting on the space \( I/xI \) has Jordan blocks of sizes \( \lambda_i \).

**Proof.** The space \( I/xI \) is spanned (over \( \mathbb{C} \)) by \( (v_1 = x^c_1, v_2 = yx^{c_2}, \ldots, v_n = y^{n-1}x^{c_n}) \). Clearly, if \( c_1 = c_2 \) then \( Y(v_1) = v_2 \), otherwise \( Y(v_1) \) vanishes in \( I/xI \).

Similarly, we see chains of vectors
\[
v_1 \xrightarrow{Y} \ldots \xrightarrow{Y} v_{\lambda_i} \xrightarrow{Y} 0, \quad v_{\lambda_i+1}^{-} \xrightarrow{Y} \ldots \xrightarrow{Y} v_{\lambda_i+\lambda_2}^{-} \xrightarrow{Y} 0, \quad \ldots, \quad v_{n-\lambda_{i+1}}^{-} \xrightarrow{Y} \ldots \xrightarrow{Y} v_n.
\]

Finally, \( Y(v_n) = x^{c_n+m} \), so if \( c_n + m > c_1 \) then \( Y(v_n) = 0 \), otherwise \( c_n + m = c_1 \) and \( Y(v_n) = v_1 \).

### 7.2. Parabolic Hilbert schemes on singular curves

Observe that for any ideal \( I \) we have \( \dim I/xI = n \). So we can define the parabolic Hilbert scheme as the moduli space of flags of ideals

\[
\text{PHilb}_{k,n+k}(C) := \{ \mathcal{O}_C \supset I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k : I_s \text{ ideal, } \dim \mathcal{O}_C/I_s = s \}
\]

and we define

\[
\text{PHilb}^x(C) := \bigsqcup_{k \geq 0} \text{PHilb}_{k,n+k}(C).
\]
Again, we would like to describe the fixed points of the $\mathbb{C}^*$ action on this variety explicitly, similarly to [33]. These are described by flags where all $I_n$ are monomial ideals. As above, for $i = 1, \ldots, n$ we can assume that the one-dimensional space $I_{k+i-1}/I_{k+i}$ is spanned by the monomial $y^\alpha x^\beta$ where $0 \leq \alpha_i \leq n-1$. In particular,

$$I_k = \mathbb{C}[x](y^\alpha x^\beta, \ldots, y^{\alpha_n} x^n).$$

Note that if $c_i = c_j$ and $i < j$ then $\alpha_i < \alpha_j$, so by Lemma 7.1 $\alpha_i$ are uniquely determined by $c_i$.

Furthermore, we can extend this construction by defining $I_{i+n} = x I_i$ for all integers $i \geq k$. Note that it follows that $\alpha_i = \alpha_i$ and $c_{i+n} = c_m + 1$ for all $i \geq 1$.

**Lemma 7.3.** The vector $c = (c_1, \ldots, c_n)$ determines a fixed point in $\text{P} \text{H} \text{i} \text{l} \text{b}_{k,n+k}$ if and only if either of the two equivalent conditions hold:

(a) For all $t > 0$ one has

$$\max_{i=t}^{i+n-1}(c_i) - \min_{i=t}^{i+n-1}(c_i) \leq m$$

(b) One has $\max_{i=1}^{n}(c_i) - \min_{i=1}^{n}(c_i) \leq m$ and whenever $c_j + m = c_i$ then $j < i$.

**Proof.** By construction, for all $t > 0$ the subspace $I_{k+i-1,t}$ is spanned over $\mathbb{C}[x]$ by the monomials $(y^\alpha x^\beta, \ldots, y^{\alpha_n} x^n)$, so by Lemma 7.1 it is an ideal if and only if (31) holds. This proves (a).

Now let us prove that (a) and (b) are equivalent. Indeed, the left hand side of (31) is $n$-periodic, so it is sufficient to consider $t \leq n$. Assume that $\max_{i=1}^{n}(c_i) - \min_{i=1}^{n}(c_i) \leq m$, then (31) does not hold if and only if there exists $i < t$ and $j \geq t$ such that $c_i = c_j + m$ and $c_{i+n} = c_i + 1 = c_j + m + 1$. 

We picture a fixed point in $\text{P} \text{H} \text{i} \text{l} \text{b}_{k,n+k}(C)$ as the staircase representing the ideal $I_k$, together with an enumeration of the boxes bordering it to its right, in such a way that the quotient $\mathcal{O}_C/I_{k+j}$ is spanned by the boxes under the staircase together with the boxes numbered 1, 2, ..., $j$. See Figure 3.

**Remark 7.4.** The proof of Lemma 7.3 is very similar to the proof of Lemma 2.12. Indeed, this is not a coincidence: let us parametrize the curve $C$ by $(x, y) = (z^a, z^b)$, then any monomial in $x$ and $y$ corresponds to a monomial in $z$. A monomial ideal in $\mathcal{O}_C$ then corresponds to an $(m, n)$-invariant subset in $\mathbb{Z}_{>0}$, and a flag of monomial ideals corresponds to a flag of $(m, n)$-invariant subsets. By Proposition 2.12 such flag determines an $m$-stable affine permutation. We conclude that fixed points on parabolic Hilbert scheme are in bijection with $m$-stable affine permutations $\omega$ such that $\omega p_m \in L^+_{\min}(n)$.

We define the line bundles $\mathcal{L}_i, 1 \leq i \leq n$ on the parabolic flag Hilbert scheme as follows. The fiber of $\mathcal{L}_i$ over the flag $I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+n} = xI_k$ is $I_{k+i-1}/I_{k+i}$. Then we have the following:

**Lemma 7.5.** There is a bijection between the eigenbasis $v_{\alpha}$ in $L_{m/n}(\text{triv})$ (defined in Corollary 6.9) and the set of $\mathbb{C}^*$ fixed points in $\text{P} \text{H} \text{i} \text{l} \text{b}^*(C)$. Under this bijection, the weight of $\mathcal{L}_i$ at a fixed point corresponds to the eigenvalue $n\omega_{n+i-1}(\alpha) + m(n-1)$ of the operator $n\omega_{n+i-1} + m(n-1)$ on $v_{\alpha}$.

**Proof.** Recall that by Corollary 6.9 the basis $v_{\alpha}$ in $L_{m/n}(\text{triv})$ is parametrized by sequences of nonnegative integers $\alpha = (a_1, \ldots, a_n)$ such that $a_i - a_j \leq m$ for every $i, j$, and if $a_i - a_j = m$ then $j > i$. The eigenvalues of $\omega_i$ are given by
that when a PHilb

\[ I_{15} = \langle x^3 y^2, x^5 y \rangle, I_{16} = \langle x^4 y^2, x^5 y, x^7 \rangle, I_{17} = \langle x^4 y^2, x^5 y \rangle. \]

Here \( y^{a_1} x^{c_1} = x^3 y^2, y^{a_2} x^{c_2} = x^7, y^{a_3} x^{c_3} = x^5 y. \)

\[ w_i = a_{i} - (g_{a}(i) - 1) \frac{m}{n} \]

where \( g_{a} \) is the permutation which sorts \( a \) in non-decreasing order (here we substituted \( c = \frac{m}{n} \)).

On the other hand, the fixed points in \( \text{PHilb}_{k,n+k} \) are determined by sequences of monomials \( (y^{a_i} x^{c_i}) \) where \( \max_{i=1}^{n} (c_i) - \min_{i=1}^{n} (c_i) \leq m \) and whenever \( c_j + m = c_i \) then \( j < i \). We remark that since \( y^{a_i} x^{c_i} \) spans the quotient \( I_{k+i-1}/I_{k+i} \) it follows that when \( c_i = c_j \) with \( i < j \) we have \( \alpha_i < \alpha_j \). Clearly, the assignment \( c_i = a_{n+1-i} \) is a bijection intertwining the restrictions on \( a_i \) and on \( c_i \). Note \( k = \sum c_i = \sum a_i = ||a|| \).

Finally, the line bundle \( \mathcal{L}_i \) has the equivariant weight \( ma_i + nc_i \). We have \( \alpha_i = \tilde{g}(i) - 1 \), where \( \tilde{g} \) is the permutation defined in the proof of Lemma 7.1 which sorts the \( \alpha_i \) in increasing order. Clearly, \( \tilde{g}(i) = n + 1 - g_{a}(n + 1 - i) \), hence

\[ ma_i + nc_i = m(n + 1 - g_{a}(n + 1 - i) - 1) + na_{n+1-i} = m(n-1) + m(1 - g_{a}(n + 1 - i)) + na_{n+1-i} = nw_{n+1-i} + m(n-1). \]

\[ \square \]

**Example 7.6.** For \( a = (0, \ldots, 0) \) we get \( w_i(a) = -(i-1) \frac{m}{n} \) while the corresponding fixed point in \( \text{PHilb}_{b,n} \) corresponds to the flag \( O_C \supseteq yO_C \supseteq \cdots \supseteq y^{n-1}O_C \). The section of \( \mathcal{L}_i \) is given by monomial \( y^{i-1} \) which has weight \( m-i \). Now

\[ nw_{n+1-i}(a) + m(n-1) = -m(n+1-i-1) + m(n-1) = m(i-1). \]

7.3. **Geometric operators.** There is a natural projection \( \pi : \text{PHilb}_{k,n+k} \rightarrow \text{Hilb}_k \) which sends a flag \( I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k \) to \( I_k \). The fibers of this projection are just the classical Springer fibers consisting of complete flags in \( I_k/xI_k \) invariant under the action of \( Y \). In particular, Lemma 7.2 immediately implies the following.

**Lemma 7.7.** Given an ideal \( I = \mathbb{C}[[x]] \langle y^{a_1} x^{c_1}, \ldots, y^{a_n} x^{c_n} \rangle \) in \( \text{Hilb}_k \) there are \((\lambda_1, \ldots, \lambda_\ell)\) fixed points in \( \text{PHilb}_{k,n+k} \) projecting to \( I \). There is a Springer action of \( S_\ell \times \cdots \times S_\ell \) on these fixed points, in which they span the induced representation from \( S_\lambda \times \cdots \times S_\lambda \) to \( S_n \).

Here \( \lambda \) is determined by \( c_i \) as in Lemma 7.2, and \( \ell \) is the length of \( \lambda \).

In what follows we will need a more explicit description of this action of \( Y \) on the fixed point basis. For this, we can also give a more explicit geometric description. Let \( \text{PHilb}^{(i)}_{k,n+k} \) denote the moduli space of flags of ideals

\[ \text{PHilb}^{(i)}_{k,n+k} = \{ I_k \supset I_{k+1} \supset \cdots \supset I_{k+i} \supset I_{k+i+2} \supset \cdots \supset I_{k+n} = xI_k \}. \]
There is a natural projection \( \pi_i : \text{PHilb}_{k,n+k} \to \text{PHilb}^{(i)}_{k,n+k} \). Let \( Z_i \subset \text{PHilb}^{(i)}_{k,n+k} \) denote the locus where \( yI_{k+i} \subset I_{k+i+2} \). The key properties of \( \pi_i \) are captured by the following lemma:

**Lemma 7.8.** (a) The map \( \pi_i \) is an isomorphism outside \( Z_i \) and a \( \mathbb{P}^1 \)-fibration over \( Z_i \).

(b) The preimage \( \pi^{-1}_i(Z_i) \) is cut out by a section of the line bundle \( L_i^{-1}L_{i+1} \).

(c) A fixed point corresponding to \( v_a \) is not in \( \pi^{-1}_i(Z_i) \) if and only if \( w_{n+1-i}(a) = w_{n-i}(a) - m \).

(d) The tangent bundle to the fiber of \( \pi_i \) over \( Z_i \) is isomorphic to \( L_iL_{i+1}^{-1} \).

**Proof.** (a) The fiber of \( \pi_i \) naturally corresponds to the space of \( y \)-invariant lines in two-dimensional space \( I_{k+i}/I_{k+i+2} \). Since \( y \) is nilpotent on \( I_{k+i}/I_{k+i+2} \), it is either identically zero and every line is \( y \)-invariant, or it is a Jordan block and has unique \( y \)-invariant line.

(b) A flag \( I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k \) is in \( \pi^{-1}_i(Z_i) \) if and only if \( yI_{k+i} \subset I_{k+i+2} \). Since \( yI_{k+i} \subset I_{k+i+1} \), we have a map \( s_y : \mathcal{L}_i \to \mathcal{L}_{i+1} \) which is equivalent to a section of \( L_i^{-1}L_{i+1} \).

(c) A fixed point is not in \( \pi^{-1}_i(Z_i) \) if and only if the weight of \( L_{i+1} \) differs from the weight of \( \mathcal{L}_i \) by \( m \). By Lemma 7.3 we get

\[
(n w_{n-i} + m(n - 1) = n w_{n+1-i} + m(n - 1) + m, \quad w_{n-i} = w_{n+1-i} + \frac{m}{n}.
\]

(d) Recall that the tangent space to \( \mathbb{P}^1 = \mathbb{P}(V) \) at a line \( \ell \) is canonically isomorphic to \( \text{Hom}(\ell, V/\ell) \). In our case \( \ell \simeq I_{k+i+1}/I_{k+i+2} = L_{i+1} \) and \( V/\ell \simeq I_{k+i}/I_{k+i+1} = \mathcal{L}_i \). So the tangent bundle to the fiber is isomorphic to the space \( \text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_i) \simeq \mathcal{L}_i L_{i+1}^{-1} \).

We can use the maps \( \pi_i \) to define the Springer action of \( S_n \) on the homology of \( \text{PHilb}_{k,n+k} \). Let \( \gamma_i : \pi^{-1}_i(Z_i) \to \text{PHilb}_{k,n+k} \) denote the natural inclusion map. By Lemma 7.8 we have well-defined Gysin maps \( \gamma^*_i : H_*(\text{PHilb}_{k,n+k}) \to H_*(\pi^{-1}_i(Z_i)) \) and \( \pi^*_i : H_*(Z_i) \to H_*(\pi^{-1}_i(Z_i)) \). Consider the composition

\[
B_i : H_*(\text{PHilb}_{k,n+k}) \xrightarrow{\gamma^*_i} H_*(\pi^{-1}_i(Z_i)) \xrightarrow{\pi^*_i} H_*(Z_i) \xrightarrow{\gamma^*_i} H_*(\pi^{-1}_i(Z_i)) \xrightarrow{\gamma^*_i} H_*(\text{PHilb}_{k,n+k}).
\]

By Lemma 7.5 we can identify the fixed point basis in the equivariant cohomology of \( \bigcup_k \text{PHilb}_{k,n+k} \) with the basis \( v_a \) in the representation \( L_{m/n} = L_{m/n}^{(\text{triv})} \). In fact, it is more natural to identify it with the renormalized basis \( \tilde{v}_a \).

**Lemma 7.9.** The action of \( B_i \) in the equivariant cohomology of \( \bigcup_k \text{PHilb}_{k,n+k} \) agrees with the action of \( 1 - s_{n-i} \) on \( L_{m/n} \), if we identify the fixed point basis in the former with \( \tilde{v}_a \).

**Proof.** We just need to compute the matrix elements of all the operators involved in the definition of \( B_i \). By Lemma 7.5 (b) the subvariety \( \pi^{-1}_i(Z_i) \) is cut out by a section of \( L_i^{-1}L_{i+1} \) corresponding to the map \( s_y : \mathcal{L}_i \to \mathcal{L}_{i+1} \). This map has weight \( m \), and so the Gysin map \( \gamma^*_i \) correspond to the multiplication by \( c_1(\mathcal{L}_{i+1}) - c_1(\mathcal{L}_i) - m \) which at a fixed point corresponds to the multiplication by \( (n w_{n-i} - n w_{n+1-i} - m) \). Note that by Lemma 7.3 (c) this annihilates the classes of all fixed points outside \( \pi^{-1}_i(Z_i) \).
The map $\pi_{i+}$ just maps the class of the fixed point in $\text{PHilb}_{k,n+k}$ to the class of the corresponding class in $\text{PHilb}_{k,n+k}$. The map $\pi_{i}'$, however, amounts to dividing by the cotangent weight of the fiber computed in Lemma 7.8 (d).

By combining these factors, it is now easy to compare the matrix elements of $B_i$ with the ones in Proposition 6.6 and observing that for $c = m/n$ one gets:

$$(1 - s_i)v_a = \frac{n\omega_a - n\omega_{i+1} - m}{n\omega_a} \cdot \omega_a - \omega_{s_i,a}$$

where $\omega = \omega(a)$.

We also have a geometric analogue of the shift operator $\tau$. Given a flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n+1} = xI_k$, we can consider the flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n+1} = xI_k \supset I_{k+n+1} = xI_{k+1}$. This defines a map $T : \text{PHilb}_{k,n+k} \rightarrow \text{PHilb}_{k,n+k+1}$.

**Definition 7.10.** We define $W_{k,n+k} \subset \text{PHilb}_{k,n+k}$ as the set of flags $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n+1} = xI_k$ such that $I_{k+n+1} \subset x\mathcal{O}_C$.

It is easy to see that $W_{k,n+k}$ is a closed subvariety in $\text{PHilb}_{k,n+k}$.

**Lemma 7.11.** The map $T : \text{PHilb}_{k,n+k} \rightarrow \text{PHilb}_{k,n+k+1}$ is injective and its image coincides with $W_{k+1,n+k+1}$. In particular, $\text{PHilb}_{k,n+k}$ and $W_{k+1,n+k+1}$ are isomorphic.

**Proof.** The image of $T$ is contained in $W_{k+1,n+k+1}$ by construction. Given a flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n+1} = xI_k$ in $W_{k+1,n+k+1}$, we have $I_{k+n} \subset x\mathcal{O}_C$, so we can define an ideal $I_k := x^{-1}I_k$. Since $I_{k+n} \subset xI_{k+1}$, we have $I_k \supset I_{k+1}$. Therefore $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n+1} = xI_k$ is a well defined point in $\text{PHilb}_{k,n+k}$ sent to the original flag by $T$.

Recall that the $\mathcal{L}_n$ has fibers $I_{k+n-1}/I_{k+n} = I_{k+n-1}/xI_n$. The inclusion $I_{k+n-1} \hookrightarrow \mathcal{O}_C$ induces a map $i : \mathcal{L}_n \hookrightarrow \mathcal{O}_C/x\mathcal{O}_C$.

**Lemma 7.12.** Define the covector $\eta : \mathcal{O}_C/x\mathcal{O}_C \rightarrow \mathbb{C}$ by the equation $\eta(y^{n-1}) = 1, \eta(y^k) = 0$ for $0 \leq k < n - 1$. Then $W_{k,n+k}$ is the zero locus of the composition

\[
s : \mathcal{L}_n \xrightarrow{i} \mathcal{O}_C/x\mathcal{O}_C \xrightarrow{\eta} \mathbb{C}
\]

or, equivalently, the zero locus of the section $s : \mathcal{C} \rightarrow \mathcal{L}_n^{-1}$.

**Proof.** Recall that $W_{k,n+k}$ is cut out by condition $I_{k+n-1} \subset x\mathcal{O}_C$ which is equivalent to vanishing of $i(\mathcal{L}_n)$. Since $i(\mathcal{L}_n)$ is a $y$-invariant subspace of $\mathcal{O}_C/x\mathcal{O}_C$ of dimension at most 1, either $i(\mathcal{L}_n) = 0$ or $i(\mathcal{L}_n) = \langle y^{n-1} \rangle$. Therefore $i(\mathcal{L}_n) = 0$ if and only if $\eta(i(\mathcal{L}_n)) = 0$.

Note that $\text{PHilb}_{k,n+k}$ is in general very singular and has several irreducible components. The section $s$ might vanish on some of these components identically. Still, by Lemma 7.12 we can define Gysin map

\[
J^* : H_*(\text{PHilb}_{k,n+k}) \rightarrow H_{*-2}(W_{k,n+k}),
\]

where $j = j_k$ is the inclusion $j : W_{k,n+k} \hookrightarrow \text{PHilb}_{k,n+k}$. We define $\Lambda$ as the composition

\[
\Lambda : H_*(\text{PHilb}_{k+1,n+k+1}) \xrightarrow{J^*} H_{*-2}(W_{k+1,n+k+1}) \xrightarrow{\cong} H_{*-2}(\text{PHilb}_{k,n+k}).
\]

**Lemma 7.13.** We have $T_\ast \circ \Lambda(-) = c_1(\mathcal{L}_n) \cap (-)$. 

Proof. Indeed, if $j : W_{k,n+k} \hookrightarrow \text{PHilb}_{k,n+k}$ is the inclusion, then
\[ T_s \circ \Lambda(-) = j_s j^*(-) = c_1(L_n) \cap (-) \]
by Lemma 7.12. \qed

**Theorem 7.14.** (a) The total localized equivariant homology
\[ U = \bigoplus_{k=0}^{\infty} H^*_\text{c}(\text{PHilb}_{k,n+k}) \]
has an action of the rational Cherednik algebra $H_{n,m}$. The action of $S_n$ is the Springer action described above, $u_{n+1-i} + m(n-1)$ correspond to capping with $c_1(L_1)$ and the operators $T$ and $\Lambda$ on $U$ correspond to the action of $\tau$ and $\lambda$.

(b) The representation $U$ is irreducible and isomorphic to $L_{n,m}(\text{triv})$. Under this isomorphism, fixed points of $\text{c}^*$ action correspond to the eigenbasis $v_\alpha$.

Proof. Let $s_i, u_i, \tau$ and $\lambda$ be the generators of $H_{1,c}$ where $c = m/n$. Recall that $H_{n,m}$ is isomorphic to $H_{1,m,n}$, under this isomorphism the generators $s_i, u_i, \tau$ and $\lambda$ of $H_{n,m}$ are mapped to $s_i, n u_i, \tau$ and $n \lambda$ respectively. Below, we will use this isomorphism to identify $L_{n,m}(\text{triv})$ with $L_{m/n}$.

By localization theorem [5 [29] $U$ is spanned by classes of fixed points. By Lemma 7.5 these are in bijection with the basis $v_\alpha$ (or, equivalently, $\bar{v}_\alpha$) in $L_{m/n}$.

This defines an isomorphism between $U$ and $L_{m/n}$ as vector spaces.

Next, we prove that the geometrically defined actions of $u_i, s_i, T$ and $\Lambda$ agree with the corresponding actions on $L_{m/n}$. This is done by explicitly comparing their matrix elements. For $u_i$ this follows from Lemma 7.9. For $s_i$ this follows from Lemma 7.13. For $T$ and $\tau$ it is easy to see from equation (20).

The action of $\Lambda$ is uniquely determined by Lemma 7.14. More precisely, the map $\eta$ in Lemma 7.12 has equivariant weight $-m(n-1)$ (since the weight of $y^{n-1}$ equals $m(n-1)$), while by Lemma 7.3 $L_n$ has weight $n u_1 + m(n-1)$. Therefore section $s$ in Lemma 7.12 has weight $n u_1$. By Lemma 7.13 we conclude that $T_s \circ \Lambda = n u_1 = u_1$.

Finally, the operators $u_i, s_i, T$ and $\Lambda$ satisfy the relations in $H_{n,m}$ since their counterparts on $L_{n,m}(\text{triv})$ do. Therefore there is indeed an action of $H_{n,m}$ on $U$ and it is an irreducible representation. \qed

**Remark 7.15.** In principle, one can check all the relations between the geometric operators directly (similarly to the computations in [9]), but the above proof seems to be more transparent.

**Remark 7.16.** Note the grading of $L_{m/n}$ by eigenvalues of the Euler operator, where $\deg(v_\alpha) = ||a|| = \sum a_i$ corresponds to grading by $k$ in $\bigoplus_k H^*_\text{c}(\text{PHilb}_{k,n+k})$.

Consider now the Hilbert scheme $\text{Hilb}(C) := \sqcup_k \text{Hilb}_k(C)$, and recall that we have defined $\text{PHilb}^*(C) := \sqcup_k \text{PHilb}_{k,n+k}(C)$. We have a $\text{c}^*$-equivariant projection $\Pi : \text{PHilb}^*(C) \to \text{Hilb}(C)$, $[I_k] \supset \cdots \supset [I_{k+n}] = [I_k]$ $\mapsto I_k$, that induces an $S_n$-invariant map on (localized) equivariant homology
\[ \Pi_* : H^*_\text{c}(\text{PHilb}^*(C)) \to H^*_\text{c}(\text{Hilb}(C)). \]

Now let $I_k \in \text{Hilb}_k(C)$ be a monomial ideal. Thanks to Lemma 7.7 and using the notation there, the span of the elements in $H^*_\text{c}(\text{PHilb}_{k,n+k}(C))$ mapping to $[I_k]$ is the induced representation $\text{Ind}_{S_1 \times \cdots \times S_\ell}^{S_n} \text{triv}$. Now by adjunction
Hom_{S_n}(\text{triv}, \text{Ind}_{S_n}^{S_{\lambda_1} \times \cdots \times S_{\lambda_k}} \text{triv}) = \text{Hom}_{S_{\lambda_1} \times \cdots \times S_{\lambda_k}}(\text{Res}_{S_n}^{S_{\lambda_1} \times \cdots \times S_{\lambda_k}} \text{triv}, \text{triv}) = \mathbb{C}

so up to scalars there is a unique \(S_n\)-equivariant section to the projection \(\Pi_* : \Pi_*^{-1}(\mathbb{C}[I_k]) \cap H^\ast_{SC}^{PC}(\text{PHilb}_{k+n+k}(\mathbb{C})) \to \mathbb{C}[I_k]\). As a consequence we get the following result.

**Proposition 7.17.** We may identify \(H^\ast_{SC}^{PC}(\text{Hilb}(\mathbb{C})) = H^\ast_{SC}^{PC}(\text{PHilb}^\circ(C))^{S_n}\) naturally. In particular, we obtain a geometric action of the spherical rational Cherednik algebra \(eH_{1,m/n}^e\) on \(H^\ast_{SC}^{PC}(\text{Hilb}(\mathbb{C}))\), that makes it an irreducible module isomorphic to \(eL_{m/n} = L_{m/n}^{S_n}\), where \(e := \frac{1}{n^m} \sum_{p \in S_n} p\) is the trivial idempotent in \(\mathbb{C}[S_n]\).

**Remark 7.18.** In [23], Garner and Kivinen study an action of the spherical rational Cherednik algebra on the homology of \(\text{Hilb}(\mathbb{C})\) using the Coulomb branch perspective. They identify \(\text{Hilb}(\mathbb{C})\) with a generalized affine Springer fiber and use the realization of \(eH_{1,m/n}^e\) as a quantized Coulomb branch algebra [38 61] to define an action via convolution diagrams. We will compare their construction to ours, in the parabolic setting, in Section 7.5.4.

7.4. Parabolic Hilbert schemes as generalized affine Springer fibers. The goal of this section is to show that \(\text{PHilb}^\circ(C) = \bigsqcup_k \text{PHilb}_{k,n+k}\) can be realized as a generalized affine Springer fiber. Thanks to [24], a consequence of this is that \(\text{PHilb}_{k,n+k}\) admits a paving by affine cells and therefore its cohomology is equivariantly formal.

Let us set \(G := \text{GL}_n\), acting on the vector space \(N := \mathbb{C}^n \oplus \mathfrak{gl}_n\), so that \(N\) is the representation space of the framed Jordan quiver:

\[
\begin{array}{c}
\circ \\
\downarrow \\
\bullet
\end{array}
\]

We will denote \(K := \mathbb{C}((\epsilon))\) and \(O := \mathbb{C}[[\epsilon]]\). We consider the groups \(G_0 \subseteq G_K\) of invertible \(O\)-linear (resp. \(K\)-linear) transformations on \(O^n\) (resp. \(K^n\)).

We choose an \(O\)-basis \(\{b_1, \ldots, b_n\}\) of \(O^n\). We define \(b_i\) for \(i \in \mathbb{Z}\) by setting \(b_{i+n} := cb_i\). The standard flag is the flag of \(O\)-lattices in \(K^n\)

\[
\cdots \supseteq J_{j-1} \supseteq J_j \supseteq J_{j+1} \supseteq \cdots
\]

where \(J_j\) is the \(O\)-span of \(\{b_{j+1}, b_{j+2}, \ldots, b_{j+n-1}\}\). We denote by \(I \subseteq G_K\) the standard Iwahori subgroup, that is, the stabilizer of the standard flag. The quotient space \(F \ell := G_K/I\) is known as the affine flag variety. This is an ind-scheme parametrizing flags of \(O\)-lattices \(\cdots \supseteq J_{j-1} \supseteq J_j \supseteq J_{j+1} \supseteq \cdots\) in \(K^n\) subject to the condition \(J_{j+n} = cJ_j\) for every integer \(j \in \mathbb{Z}\).

We will need a dual realization of \(F \ell\). Let us consider the \(K\)-dual \((K^n)^*\) of \(K^n\). This comes equipped with a dual basis \(b_1^*, \ldots, b_n^*\). As above, we set \(b_{i+n}^* := cb_i^*\), and the standard flag in \((K^n)^*\) is the flag of \(O\)-lattices in \(K^n\)

\[
\cdots \supseteq J_{j-1}^* \supseteq J_j^* \supseteq J_{j+1}^* \supseteq \cdots
\]

where \(J_j^*\) is the \(O\)-span of \(\{b_{j}^*, \ldots, b_{j+n-1}^*\}\). We remark that \(J_j^*\) is the standard lattice

\[
(O^n)^* = \{ f \in (K^n)^* \mid f(O^n) \subseteq O \}.
\]
Note that $G_K$ acts naturally on the space $(K^n)^*$, and we can identify $\mathcal{F}l$ with the set of $\mathcal{O}$-lattices $\cdots \supset J_{j-1} \supset J_j \supset J_{j+1} \supset \cdots$ in $(K^n)^*$ subject to the condition $J_{j+n} = c J_j$ for every integer $j \in \mathbb{Z}$, via the identification

$$[g] \mapsto \cdots \supset I_{j-1}^* g^{-1} \supset I_j^* g^{-1} \supset I_{j+1}^* g^{-1} \supset \cdots$$

The group $G_K$ acts on the module $N_K := \mathbb{K} \otimes N = K^n \oplus \mathfrak{gl}_n(K)$ in the natural way, and the subgroup $G_0 \subseteq G_K$ preserves the $\mathcal{O}$-submodule $N_0 := \mathcal{O} \otimes N \subseteq N_K$. Now we consider the element $Y \in \mathfrak{gl}_n(\mathcal{O})$ that in the basis $\{b_1, \ldots, b_n\}$ is represented by the matrix $[29]$, with $x$ replaced by $\epsilon$, and the element $(b_1, Y) \in N_0$. We will consider the generalized affine Springer fiber, cf. [4] [23] [24]

$$\text{Spr}(b_1, Y) := \{[g] \in \mathcal{F}l \mid (g^{-1} b_1, g^{-1} Y g) \in \mathcal{O}^n \oplus \mathfrak{i}) \subseteq \mathcal{F}l$$

where $i$ is the Lie algebra of the Iwahori subgroup $I$. More concretely, $i := \{X \in \mathfrak{gl}_n(\mathcal{O}) \mid X_{|_{x=0}} \text{ is lower triangular}\}$.

**Proposition 7.19.** We have an isomorphism

$$\text{Spr}(b_1, Y) \cong \bigsqcup_k \text{PHilb}_{k, n+k}$$

**Proof.** We use the presentation of $\mathcal{O}_C$ at the beginning of this section as a free $\mathbb{C}[x]$-module of rank $n$. In this presentation, an ideal of $\mathcal{O}_C$ corresponds to a $\mathbb{C}[x]$-submodule $I \subseteq \mathbb{C}[x]^n$ closed under the action of $Y$ in [24]. Similarly, an element of $\bigsqcup_k \text{PHilb}_{k, n+k}$ corresponds to a flag of $\mathbb{C}[x]$-submodules $\mathbb{C}[x]^n \supset I_k \supset \cdots \supset I_{k+n-1} \supset \mathfrak{i} x f_k$ such that dim $\mathbb{C}[x]^n/I_j = j < \infty$, each ideal $I_j$ is stable under the action of $Y$ and dim $I_j/I_{j+1} = 1$. Now, we identify $\mathcal{O}_C$ with the dual $(K^n)^*$ and, as above, we identify $[g] \in \mathcal{F}l$ with the flag $I_j^* g^{-1} \supset \cdots \supset I_{n-1}^* g^{-1} \supset I_n^* g^{-1} = \epsilon I_1^* g^{-1}$ where, as above, $I_1^* \supset I_2^* \supset \cdots$ is the standard flag. Identifying $x = \epsilon$, we see that to prove the proposition we have to check that the following conditions are equivalent:

1. $I_j^* g^{-1} \subseteq (\mathcal{O}^n)^*$ and $I_j^* g^{-1}$ is closed under the action of $Y$ for every $j \geq 1$.
2. $g^{-1} b_1 \in \mathcal{O}^n$ and $g^{-1} Y g \in i$.

Since $I_j^* g^{-1}$ is the $\mathcal{O}$-span of $\{b_j^* g^{-1}, \ldots, b_{j+n-1}^* g^{-1}\}$ and $b_{j+n}^* = eb_j^*$ for every $j$, it is easy to see that (1) $\Rightarrow$ (2). Let us check that (2) $\Rightarrow$ (1). First, we need to check that $b_j^* g^{-1}, \ldots, b_{n}^* g^{-1} \in (\mathcal{O}^n)^*$. That is, we need to show that $b_j^* g^{-1} \in (\mathcal{O}^n) \subseteq \mathcal{O}$ or, equivalently, that $b_j^* g^{-1}(b_j) \in \mathcal{O}$ for every $i, j = 1, \ldots, n$. Since $b_j^* \in (\mathcal{O}^n)^*$ for $i = 1, \ldots, n$, it is enough to show that $g^{-1} b_j \in \mathcal{O}^n$ for $j = 1, \ldots, n$. For $j = 1$, this is one of the conditions in (2). For $j > 1$, observe that $g^{-1} b_j = g^{-1} Y b_{j-1} = g^{-1} Y g^{-1} b_j$ since $g^{-1} Y g \in i \subseteq \mathfrak{gl}_n(\mathcal{O})$, the result follows by induction.

Now we need to show that $I_j^* g^{-1}$ is closed under the action of $Y$ for every $j \geq 1$, that is, we need to show that

$$I_j^* g^{-1} Y \subseteq I_j^* g^{-1}$$

It is clearly enough to do this for $j = 1, \ldots, n - 1$. The condition $g^{-1} Y g \in i$ is equivalent to $b_j^* g^{-1} Y \in \mathcal{O}$-span $\{b_i^* g^{-1}, \ldots, b_{j+n-1}^* g^{-1}\}$ for every $i = 1, \ldots, n$. This clearly implies that $I_j^* g^{-1}$ is closed under $Y$. \qed

**Remark 7.20.** Proposition 7.19 is a special case of a flag version of the main result of [23], which the authors kindly provided a preliminary version of.
Now we would like to verify that the generalized affine Springer fiber $\text{Spr}(b_1, Y)$ satisfies the conditions of [24, (3.2)]. Following that paper, let us denote by $\mathfrak{a} := X_*(A) \otimes \mathbb{Z} \mathbb{R}$, where $A \subseteq G = \text{GL}_n$ is a maximal torus, that we identify with the set of diagonal invertible matrices. For each weight $\xi \in \mathfrak{a}^*$, let us denote by $N_{\xi} \subseteq N$ the corresponding weight space. For $a \in \mathfrak{a}$ and $t \in \mathbb{R}$, we denote

$$N_{K, a, t} := \prod_{\xi \in \mathfrak{a}^*, d \in \mathbb{Z}} N_{\xi} \epsilon^d \subseteq N_K.$$

For $a \in \mathfrak{a}$, let $\mathfrak{g}_a := \mathfrak{g}_{K, a, 0} \cap \mathfrak{g}_0$. This is a Lie subalgebra of $\mathfrak{g}_0$ and we let $G_a \subseteq G_0$ be the corresponding subgroup, which is an Iwahori subgroup.

**Lemma 7.21.** There exist $a \in \mathfrak{a}$ and $t \in \mathbb{R}$ such that $G_a = I$ is the Iwahori subgroup, and $N_{K, a, t} = \mathbb{O}^n \oplus \mathbb{I}$.

**Proof.** Take any $a = \text{diag}(a_1, \ldots, a_n) \in \mathfrak{a}$ with $0 < a_1 < \cdots < a_n < 1$ and $t = 0$. It is straightforward to verify the result. \qed

Note that Lemma 7.21 tells us that $\text{Spr}(b_1, Y)$ is one of the varieties considered in [24, Section 3]. In the notation of that paper, we have

$$\text{Spr}(b_1, Y) = \mathcal{F}_a(t, (b_1, Y)).$$

In [24, Section 3.2] it was proved that $\mathcal{F}_a(t, (b_1, Y))$ has an affine paving provided that there exist $b \in \mathfrak{a}$ and $c \in \mathbb{R}$ such that the following conditions are satisfied:

- $c \geq t$
- $(b_1, Y) \in N_{K, b, c}$
- The projection $(b_1, Y)$ is $G$-good (in the sense of [24]), that is, no nonzero $G$-unstable covector in $N^*$ vanishes on the $\mathfrak{gl}_n$-orbit of $(b_1, Y)$

To verify these conditions, we consider $b = \text{diag}(c, 2c, \ldots, nc) \in \mathfrak{a}$, where $c := m/n$. Obviously $c > t = 0$, and is easy to check that $(b_1, Y) \in N_{K, b, c}$. For the last condition, we need to verify that the element

$$(b_1, Y) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in N$$

is $G$-good. This is a consequence of the following result.

**Proposition 7.22.** Let $X$ be a regular semisimple matrix and $v$ a cyclic vector for $X$. Then $(v, X) \in N$ is $G$-good.

To prove Proposition 7.22, we first give a necessary condition for a vector in the adjoint representation $\mathfrak{gl}_n$ to be unstable.

**Lemma 7.23.** Assume $B \in \mathfrak{gl}_n$ is $G$-unstable. Then, $B$ is nilpotent.

**Proof.** By definition, cf. [24], $B$ is unstable if and only if there exists a semisimple matrix $y$ and $t_1, \ldots, t_k > 0$ such that $B = B_1 + \cdots + B_k$, with $[y, B_k] = t_k B_k$. Since the $t_i$ are strictly positive and the filtration given by $y$ is bounded above, the result follows. \qed
Returning to the setting of Proposition 7.22 we may assume that $X$ is already in diagonal form. So $X = \text{diag}(x_i)$ with $x_i \neq x_j$ for $i \neq j$ and $v = (v_i)$, the cyclicity condition is equivalent to $v_i \neq 0$ for every $i$.

**Lemma 7.24.** Let $(w, B) \in N$ be such that $\text{tr}(B[\xi, X]) + w \cdot \xi v = 0$ for every $\xi \in \mathfrak{g}$. Then

1. $w_i = 0$ for every $i = 1, \ldots, n$.
2. $b_{ij} = 0$ for $i \neq j$.

**Proof.** The proof is straightforward, but let us give it for the sake of completeness. We have $[\xi, X] = (\xi_{ij}(x_i - x_j))_{ij}$ and $\xi v = (\sum_j \xi_{ij} v_j)_i$. Thus,

$$\text{tr}(B[\xi, X]) + w \cdot \xi v = \sum_{i,j=1}^n b_{ij} \xi_{ji}(x_j - x_i) + \xi_{ij} v_i w_j = 0$$

for every matrix $\xi \in \mathfrak{g}$. Taking the matrix $\xi$ with $\xi_{ii} = 1$ and all other coordinates 0 we see, using $v_i \neq 0$, that $w_i = 0$. Now take $i \neq j$. Taking the matrix $\xi$ with $\xi_{ij} \neq 0$ and all other coordinates 0 we see, using $x_i - x_j \neq 0$, that $b_{ij} = 0$. The result follows. \hfill \Box

**Proof of Proposition 7.22.** Let $(w, B) \in N^* \cong N$ be an unstable covector vanishing on $\mathfrak{g}_v \cdot (v, X)$, where we use the trace form to identify $N^* \cong N$. Thanks to Lemma 7.24 (1) we have that $w = 0$. It follows now from Lemma 7.23 that $B$ is nilpotent. But Lemma 7.24 (2) implies that $B$ is semisimple as well. So $B = 0$, and it follows that $(v, X)$ is $G$-good. \hfill \Box

From [24], we obtain the following result.

**Corollary 7.25.** The generalized affine Springer fiber $\text{Spr}(b_i, Y) = \bigsqcup_k \text{P} \text{Hilb}_{k,n+k}$ is paved by affine spaces. Thus, its cohomology is equivariantly formal.

**Remark 7.26.** The usual (as opposed to generalized) affine Springer fiber $\text{Spr}(Y)$ can be obtained by a similar construction for $N = \mathfrak{gl}_n$. Similarly to Proposition 7.19 it can be defined as the space of $Y$-invariant flags

$$\mathcal{I}_1^* g^{-1} \supseteq \cdots \supseteq \mathcal{I}_{n-1}^* g^{-1} \supseteq \mathcal{I}_n^* g^{-1} = \mathcal{I}_1^* g^{-1}$$

in $(\mathbb{K}^n)^*$, but these flags are no longer required to be contained in $(\mathbb{D}^n)^*$. It was proved in [43, 24] that for the same matrix $Y$ given by [29] the usual affine Springer fiber $\text{Spr}(Y)$ is paved by affine spaces, and the combinatorics of this paving was studied e.g. in [43, 26].

The Springer action of $S_n$ and the operator $T$ in cohomology of $\text{Spr}(Y)$ were considered in [62 51 52 58]. They were shown to generate the extended affine symmetric group, in particular, $T$ is invertible. Indeed,

$$T [\mathcal{I}_1^* g^{-1} \supseteq \cdots \supseteq \mathcal{I}_{n-1}^* g^{-1} \supseteq \mathcal{I}_n^* g^{-1} = \mathcal{I}_1^* g^{-1}] = [\mathcal{I}_1^* g^{-1} \supseteq \cdots \supseteq \mathcal{I}_{n-1}^* g^{-1} \supseteq \mathcal{I}_n^* g^{-1}]$$

while

$$T^{-1} [\mathcal{I}_1^* g^{-1} \supseteq \cdots \supseteq \mathcal{I}_{n-1}^* g^{-1} \supseteq \mathcal{I}_n^* g^{-1} = \mathcal{I}_1^* g^{-1}] = [\mathcal{I}_1^* g^{-1} \supseteq \cdots \supseteq \mathcal{I}_{n-1}^* g^{-1} \supseteq \mathcal{I}_n^* g^{-1}]$$

Furthermore, $S_n$, $T$ and line bundles $\mathcal{L}_i$ were used in [51, 52] to construct the action of the **trigonometric** Cherednik algebra on the equivariant homology of the affine Springer fiber.
In our setting, the failure of $T$ to be invertible gives rise to a new operator $\Lambda$ and together they generate the rational Cherednik algebra. This shows both the similarity and a subtle distinction between the trigonometric and rational setups.

### 7.5. Comparison to action by convolution diagrams.

The main result of [34] constructs an action of the Coulomb branch algebra for $(G, N)$ in the equivariant homology of any generalized affine Springer fiber for $(G, N)$ satisfying some mild assumptions. If the generalized affine Springer fiber is invariant under the loop rotation, then the action extends to the equivariant homology. The main result of [23] identifies the Hilbert schemes of points on arbitrary plane curve singularities with the generalized affine Springer fibers for $(G, N) = (GL_n, C^n \oplus gl_n)$, as in Section 7.5. By combining these results, [23] defines an action of the rational Cherednik algebra in the (equivariant) homology of Hilbert schemes of points on arbitrary plane curve singularities. The goal of this section is to compare their action with ours for the singularity $\{x^m = y^n\}$, see also [23, Section 4.3.2].

Let $t, c$ be formal variables and consider the $C[t, c]$-algebra $H_{t, c}(S_n, C^n)$ defined by the same relations as the usual Cherednik algebra but with the parameters $t, c$ replaced by the variables $t, c$. Thanks to work of Webster, see [40, 61], $H_{t, c} := H_{t, \tau}(S_n, C^n)$ is a generalized BFN Coulomb branch algebra.

Recall that if we have a reductive group $G$ acting on a vector space $N$, the BFN Coulomb branch algebra is defined as the equivariant Borel-Moore homology $H^*_\kappa \times C^*_\text{rot} (R_{G, N})$ where $R_{G, N}$ is a space modeled after the affine Grassmannian and $C^*_{\text{rot}}$ is the torus acting by loop rotations, see [4] for details. When $G = GL_n$ and $N = C^n \oplus gl_n$ we get precisely the spherical rational Cherednik algebra. To get the full Cherednik algebra, we need to replace $R_{G, N}$ with a larger space $R'_{G, N}$ that is rather modeled after the affine flag variety, so we have $H_{t, c} \cong H^*_I (C^*_{\text{rot}}) \times C^*_{\text{rot}} (R'_{G, N})$ where $I \subseteq G_K$ is the standard Iwahori and the action of $C^*_{\text{rot}}$ comes from the framing vector. The parameter $t$ is the $C^*_{\text{rot}}$-equivariant parameter, and the parameter $c$ is the $C^*_{\text{rot}}$-equivariant parameter. See [30, 61] for details.

To compare the actions we look at the isomorphism $H_{t, c} \cong H^*_I (C^*_{\text{rot}}) \times C^*_{\text{rot}} (R'_{G, N})$ constructed by Webster in [61, Lemma 4.2]. First, we have both algebras acting on a polynomial algebra $C[t, c][U_1, \ldots, U_n]$. On the Cherednik algebra side, this comes from identifying $U_i$ with the Dunkl-Opdam elements $u_i$, and we remark that this is not the usual polynomial representation of $H_{t, c}$, see [61, (2.17)–(2.22)]. On the Coulomb side, this comes from identifying $C[t, c][U_1, \ldots, U_n] \cong H^*_I (C^*_{\text{rot}}) \times C^*_{\text{rot}} (pt)$, where the $U_i$ are the Chern classes of the tautological line bundles on the affine flag variety. Both representations are faithful, and we need to identify the operators on $C[t, c][U_1, \ldots, U_n]$ corresponding to $\tau, \lambda$ and $S_n$.

According to [61, Lemma 4.2], the action of $\tau$ corresponds to the action of the correspondence:

$$T := \{(F_*, F'_*) \in \mathcal{F}l \times \mathcal{F}l : F_i = F'_{i-1}\},$$

while the action of $\lambda$ corresponds to the action of the correspondence:

$$L := \{(F_*, F'_*) \in \mathcal{F}l \times \mathcal{F}l : F_i = F'_{i+1}\}.$$

---

*Note that our $\tau$ is Webster’s $\sigma$, while our $\lambda$ is denoted $\tau$ by Webster.*
Remark 7.27. Note that the rational Cherednik algebra $H_{t,c}$ admits a Fourier transform, that is, a $\mathbb{C}[t,c]$-involution sending $y_i \mapsto x_i$, $x_i \mapsto -y_i$ and $s_i \mapsto s_i$. On the Coulomb branch setting, this automorphism interchanges the correspondences $T$ and $L$. So there is a choice of isomorphism $H_{t,c} \to H^+_s(\mathbb{C}^* \times \mathbb{C}^*_n) \times \mathbb{C}^*_0(G,N)$. To resolve this, we note that according to [23, Proposition 1.4] the action of $H_{t,c}$ on $H^+_0(\text{PHilb}^x(C))$ coming from a $\mathbb{C}[t,c]$-isomorphism $H_{t,c} \to H^+_s(\mathbb{C}^* \times \mathbb{C}^*_n) \times \mathbb{C}^*_0(G,N)$ factors through $H_{m/n} = H_{t,c}/(t-1,c-m/n)$ and we choose the isomorphism that sends the module constructed in [23, Theorem 4.9] to the category $\mathcal{O}_{m/n}$.

It follows from the comparison of convolution diagrams to correspondences in [23 Section 4.2.1] that the actions of $\tau, \lambda$ that we defined coincide with those defined by [23 Theorem 4.9 and Corollary 4.16]. The action of $S_n$ that we defined comes from projections to partial flag varieties, cf. Section 7.3 while that in [23] comes from the usual Springer action of $S_n$ on the homology of Springer fibers. The coincidence of these is well-known. Since the algebra $H_{t,c}$ is generated by $\tau, \lambda$ and $S_n$, Proposition 3.6 we obtain the following result, see also [23 Theorem 4.29].

Proposition 7.28. The action of $H_{m/n}$ on $H^+_0(\text{PHilb}^x(C))$ defined in Theorem 7.14 coincides with that constructed by Garner and Kivinen in [23 Proposition 1.4].

Corollary 7.29. There is an action of $H_{m/n}$ on the non-localized equivariant homology $H^+_0(\text{PHilb}^x(C))$ lifting the action from Theorem 7.14.

Remark 7.30. Let $C$ be a plane curve singularity and assume the $x$-projection $C \to \mathbb{C}$ has degree $n$. In [23], Garner and Kivinen construct an action of the algebra $H_{0,0} = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes S_n = H_{t,c}/(t,c)$ on the non-equivariant homology $H_*(\text{PHilb}^x(C))$; see also [34].

Remark 7.31. If $C = \{x^m = y^n\}$ and $\gcd(m, n) = d > 1$ then the curve $C$ has $d$ irreducible components. There is a $\mathbb{C}^*$ action on $C$ and on Hilbert schemes on $C$, and the results of [23] still apply, so one gets an interesting representation of the rational Cherednik algebra $H_{m/n}$ in the equivariant homology of $\sqcup_k \text{PHilb}^{k,n+k}(C)$. It would be very interesting to study this representation.

Note that the $\mathbb{C}^*$ action on the Hilbert schemes no longer has isolated fixed points, so even computing the character of this representation is a nontrivial problem. Nevertheless, we expect the representation to have minimal support in the sense of [17]. Indeed, the conjectures of [50] relate the homology of Hilb($C$) to the HOMFLY-PT invariant of the $(m,n)$ torus link. On the other hand, by [17] Theorem 4.11] the same invariant can be obtained as a character of a certain explicit minimally supported representation of the spherical rational Cherednik algebra with parameter $m/n$.

8. PARABOLIC HILBERT SCHEMES AND QUANTIZED GIESEKER VARIETIES

In this section, we use Theorem 7.14 together with 18 to study the geometric representation theory of quantized Gieseker varieties.

8.1. Quantized Gieseker varieties. Fix positive integers $n, r > 0$ and consider the vector space

$$ R := \mathfrak{g}l_n \oplus \text{Hom}(C^r, C^n). $$
We have a natural action of the group $GL_n$ on $R$, so every element $\xi \in gl_n$ induces a vector field on $R$, that we denote by $\xi_R$. In particular, $\xi_R \in D(R)$, the algebra of polynomial differential operators on $R$. Note that $GL_n$ acts on $D(R)$. Let $c \in \mathbb{C}$. It is straightforward to see that the following space is in fact an associative algebra,

$$A_c(n, r) := \left[ \frac{D(R)}{D(R)\{\xi_R - c\text{tr}(\xi) : \xi \in gl_n\}} \right]_{GL_n}$$

we call $A_c(n, r)$ a quantized Gieseker variety.

**Example 8.1.** When $r = 1$ then $A_c(n, r) = eH_{c,e}$, the spherical subalgebra in the type $\mathfrak{gl}_n$-Cherednik algebra. This follows from the main result of [22].

Let us now deal with the representation theory of $A_c(n, r)$. We follow [18] Section 3. Let $T_0 \subseteq GL_r$ be a maximal torus, and $T := \mathbb{C}^* \times T_0$. For each co-character $\nu : \mathbb{C}^* \to T$ we can define a category $O_\nu(A_c(n, r))$ of highest-weight $A_c(n, r)$-modules. The co-character $\nu$ has the form $t \mapsto (t^{\nu_0}, \nu'(t))$ for some co-character $\nu'$ of $GL_r$. If $\nu_0 \neq 0$, then $O_\nu(A_c(n, r))$ admits a module of Gelfand-Kirillov (GK)-dimension 1 if and only if $c = m/n$, where $\gcd(m, n) = 1$ and $c \notin (-r, 0)$. In this case, $O_\nu(A_c(n, r))$ admits a unique irreducible representation of GK-dimension 1, that we denote $L_{m/n}^+(n, r)$. Moreover, $L_{m/n}^+(n, r)$ depends only on the sign of $\nu_0$.

The next proposition follows from [18].

**Proposition 8.2.** Assume $m, n > 0$. We have a vector space isomorphism

$$L_{m/n}(n, r) = (L_{n/m}(\text{triv}) \otimes (\mathbb{C}^*)^{\otimes m})^S_m$$

where $L_{n/m}(\text{triv})$ is the simple highest weight representation of $H_{n/m}(S_m, \mathbb{C}^m)$ and the action of $S_m$ on $L_{n/m}(\text{triv}) \otimes (\mathbb{C}^*)^{\otimes m}$ is diagonal.

**Proof.** The $\mathfrak{sl}_m$-version of this result is [18] Corollary 2.18]. The $\mathfrak{gl}_n$-version is proved identically. Alternatively, it follows from the $\mathfrak{sl}_m$-version by multiplying both sides of [18] Corollary 2.18] by a polynomial algebra in one variable. \hfill \square

We would like to emphasize that in the statement of Proposition 8.2 there is a swap in the parameters $n, m$.

Let us elaborate on the statement of Proposition 8.2. A priori, it is only a vector space identification. However, we can recover the action of $A_{m/n}(n, r)$ on the space $(L_{n/m}(\text{triv}) \otimes (\mathbb{C}^*)^{\otimes m})^{S_m}$ as follows. First, we construct a matrix version of the rational Cherednik algebra.

**Definition 8.3.** Let $t, c \in \mathbb{C}$ and $m, r \in \mathbb{Z}_{>0}$. We define the algebra $H_{t,c}(m, r)$ as the quotient of the semidirect product $(\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m] \otimes (\text{End}(\mathbb{C}^r))^{\otimes m}) \rtimes S_m$ by the relations

- $[y_\ell, y_N] = 0 = [x_\ell, x_N]$ for any $\ell, N = 1, \ldots, m$.
- $[y_\ell, x_N] = c \left( \sum_{j=1}^r (E_{ij})_\ell (E_{ji})_N \right)$ if $\ell \neq N$.
- $[y_\ell, x_\ell] = t - c \sum_{N \neq \ell} \left( \sum_{j=1}^r (E_{ij})_\ell (E_{ji})_N \right)$

where $E_{ij}$ is the $r \times r$ matrix that has a 1 in the $(i, j)$-th position and zeroes everywhere else, and $(E_{ij})_\ell \in \text{End}(\mathbb{C}^r)^{\otimes m}$ is $\text{Id} \otimes \cdots \otimes \text{Id} \otimes E_{ij} \otimes \text{Id} \otimes \cdots \otimes \text{Id}$, where $E_{ij}$ is in the $\ell$-th position.
For example, when $r = 1$ we simply recover the rational Cherednik algebra $H_{t,c}(S_m, \mathbb{C}^m)$. To lighten notation but still emphasize the role of $m$ over $n$, we will write $H_{t,c}(m)$ in place of $H_{t,c}(S_m, \mathbb{C}^m)$ or $H_{t,c}$ below.

It is clear from the relations that if $M$ is an $H_{t,c}(m)$-module, then $M \otimes (\mathbb{C}^e)^\otimes m$ becomes an $H_{t,c}(m, r)$-module, where the elements $x_1, \ldots, x_m, y_1, \ldots, y_m$ act only on the $M$ tensor factor, the elements from $\text{End}(\mathbb{C}^e)^\otimes m$ act only on the $(\mathbb{C}^e)^\otimes m$ tensor factor, and $S_m$ acts diagonally. In fact, this defines a category equivalence $H_{t,c}(m)$-mod $\to H_{t,c}(m, r)$-mod, see [18]. Thus, the algebra $H_{1,n/m}(m, r)$ acts on $L_{n/m}(\text{triv}) \otimes (\mathbb{C}^e)^\otimes m$.

Now we can form the spherical subalgebra $eH_{1,n/m}(m, r)e$, with respect to the trivial idempotent $e = \frac{1}{m} \sum_{p \in S_m} p$, that acts on the space $(L_{n/m}(\text{triv}) \otimes (\mathbb{C}^e)^\otimes m)S_m$. Upon the identification $L_{m/n}(n, r) = (L_{n/m}(\text{triv}) \otimes (\mathbb{C}^e)^\otimes m)S_m$ of Proposition 8.2, the actions of $A_{m/n}(n, r)$ and $eH_{1,n/m}(m, r)e$ on their respective spaces get identified. This follows from [18] Section 2 after minor modifications.

8.2. Compositional parabolic Hilbert schemes, combinatorially. We consider the curve $C = \{x^m = y^n\}$. Let us consider the scheme

$$\text{CPHilb}^{r,y} := \{O_C \supseteq J^0 \supseteq \cdots \supseteq J^{r-1} \supseteq J^r = yJ^0\}$$

where $J^k$ are ideals in $O_C$ of finite codimension (not necessarily $k$). We have an action of $\mathbb{C}^*$ on $\text{CPHilb}^{r,y}$, and the fixed points can be identified with chains of monomial ideals. We can encode these as follows. Start with the monomial ideal $J^0 = \mathbb{C}[[y]](y^{c_1}, y^{c_2}, \ldots, x^{m-1}y^{c_m}) \subseteq O_C$. For $k = 1, \ldots, r$ let $\gamma_k := \dim(J^{k-1}/J^k) \geq 0$. Note that $\sum_{k=1}^r \gamma_k = m$. The space $J^{k-1}/J^k$ is spanned by the monomials $x^{\alpha_{k,1}}y^{\gamma_{k,1}}, \ldots, x^{\alpha_{k,k}}y^{\gamma_{k,k}}$ where $\alpha_{k,1} < \cdots < \alpha_{k,k}$. Note that if $c_{k,i} = c_{k',j}$ for some $k < k'$ then $\alpha_{k,i} < \alpha_{k',j}$. Moreover, if $c_{k,i} - c_{k',j} = n$ then $k' \leq k$.

Pictorially, we consider the staircase diagram defined by the ideal $J^0$ and we fill in the box corresponding to the monomial $x^{\alpha_{k,i}}y^{\gamma_{k,i}}$ with the number $k$. In particular, the number of boxes labeled by $k$ is precisely $\gamma_k$. See Figure 4. Note that the labels of these boxes are weakly increasing along each vertical run of the staircase diagram, where we read bottom-to-top. Moreover, if two labeled boxes are $n$ horizontal steps apart, then the label of the top box is no greater than that of the bottom box.

| $x^2$ | $x^2y$ | $x^2y^2$ | 4 |
|-------|--------|----------|---|
| $x$   | $xy$   | $xy^2$   | $xy^3$ | 2 |
| 1     | $y$    | $y^2$    | $y^3$  | $y^4$ | 4 |

Figure 4. An element of $\text{CPHilb}^{6,4}(\{x^3 = y^4\})$. Here, $J^0 = J^1 = (x^2y^3, xy), J^2 = J^3 = (x^2y^3, xy^6)$ and $J^4 = J^5 = J^6 = yJ^0 = (x^2y^3, xy^6)$. Also $\gamma = (0, 1, 0, 2, 0, 0) \in C_6(3)$ which corresponds to $2^14^2$. Note that the roles of $m$ and $n$, as well as those of $x$ and $y$ are different from those in Figures 2 and 3.
The localized equivariant homology \( H^*_e (\CPHilb^{\gamma,y}(C)) \) then admits a basis indexed by classes of fixed points. As in Section 7.1, see in particular Lemma 7.2 for a monomial ideal \( J^0 = \mathbb{C}[[y]][y_1^\ell, x y_2^2, \ldots, x^{m-1} y^m] \) we can define a composition \((\lambda_1, \ldots, \lambda_t)\) of \( m \). Thanks to the discussion above, the flags of monomial ideals that start with \( J^0 \) can be labeled by \( \ell \)-tuples of monomials \((m_1, \ldots, m_\ell)\), where \( m_i \) is a monomial of degree \( \lambda_i \) in \( r \) variables.

On the other hand, it follows from Lemma 7.7 that, as a \( S_m \)-module we have

\[
L_{n/m}(\text{triv}) = \bigoplus_{J^0 \subseteq O_C} \text{Ind}^S_{\lambda_1 \times \cdots \times \lambda_t} \text{triv}
\]

So that

\[
\mathcal{L}_{m/n}(n, r) = (L_{n/m}(\text{triv}) \otimes (\mathbb{C}^r)^{\otimes m})S_m
\]

which admits a basis in-
**Proof.** First, we verify that $\Pi^r$ is $S_{r, \text{rev}}$-invariant, that is, it is constant on $S_{r, \text{rev}}$-orbits. The group $S_{r, \text{rev}}$ is generated by simple reflections $s_i$, $i \notin \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$. It is enough to verify that $\Pi^r$ is invariant under each of these simple reflections.

By definition, $\Pi^r$ sends an element $(I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+m} = yI_k)$ to a flag involving only the ideals $I_k, I_{k+r}, I_{k+r+1}, \ldots, I_{k+r+1+y}$, and each one of these ideals has a multiplicity determined by the zeroes in $\gamma$. The invariance now follows from the explicit form of the action of $s_i$ obtained in Lemma 7.9.

Now we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{PHilb}^y(C) & \xrightarrow{\Pi^r} & \text{PHilb}^{\gamma, y}(C) \\
\downarrow \quad \Pi & & \downarrow \quad \tilde{\Pi} \\
\text{Hilb}(C) & & \text{Hilb}(C)
\end{array}
$$

The fiber of an ideal $I$ over $\Pi$ is precisely the Springer fiber $\text{Spr}(x) \subseteq \mathcal{F}(I/yI)$ consisting of full flags of subspaces in $I/yI \cong \mathbb{C}^m$ that are stable under the action of the nilpotent operator $x$. Likewise, the fiber of $I$ over $\Pi$ is the Spaltenstein variety $\text{Spr}^\gamma(x) \subseteq \mathcal{F}(I/yI)$, consisting of partial flags of subspaces in $I/yI$ that are stable under the action of $x$. It is a standard result from Springer theory, see e.g. [7] or [62, Section 2.6] that

$$H_*(\text{Spr}(x))^{S_r} = H_*(\text{Spr}^\gamma(x))$$

from which the result follows.

Thanks to the previous lemma and observing that $\gamma \mapsto \gamma^{\text{rev}}$ is an involution on $C_r(m)$ we get

$$H_*(\text{CPHilb}^r(C)) = \bigoplus_{\gamma \in C_r(m)} H_*(\text{PHilb}^{\gamma, y}(C)) = \bigoplus_{\gamma \in C_r(m)} H_*(\text{PHilb}^y(C))^{S_r},$$

on the other hand, we have the following well-known result.

**Lemma 8.5.** Let $V$ be a representation of $S_m$ and $r > 0$. Then

$$(V \otimes (\mathbb{C}^r)^{\otimes m})^{S_m} = \bigoplus_{\gamma \in C_r(m)} V^{S_r}.$$ 

Moreover, $V^{S_r}$ is the $\gamma$-weight space for the $\mathfrak{sl}(r)$ action on the left hand side.

**Proof.** Fix a basis $e_1, \ldots, e_r$ of $\mathbb{C}^r$. For $\gamma \in C_r(m)$, let $(\mathbb{C}^r)^{\otimes m}$ be the span of those tensors $e_{i_1} \otimes \cdots \otimes e_{i_m}$ such that $\gamma_j = z\{k : i_k = j\}$ (this is $\gamma$-weight subspace in $(\mathbb{C}^r)^{\otimes m}$). It follows by definition that $(\mathbb{C}^r)^{\otimes m}$ is stable under the action of $S_m$ and moreover that $(\mathbb{C}^r)^{\otimes m} = \text{Ind}_{S_r}^{S_m} \text{triv}$. Thus, we get $(\mathbb{C}^r)^{\otimes m} = \bigoplus_{\gamma \in C_r(m)} \text{Ind}_{S_r}^{S_m} \text{triv}$ and the result now follows by adjunction.

**Theorem 8.6.** Let $m$ and $n$ be coprime positive integers, and $r > 0$. There is an action of the algebra $A_{m/n}(n, r)$ on the (localized) equivariant homology $H_*^{C_r}(\text{CPHilb}^r(C))$, where $C$ is the singular curve $\{x^m = y^n\}$, and with this action we have $H_*^{C_r}(\text{CPHilb}^r(C)) \cong \mathcal{L}_{m/n}(n, r)$.
**Proof.** We have a natural action of the spherical subalgebra $eH_{1,n/m}(m,r)e$ on 
$$(L_{n/m}(\text{triv}) \otimes (C^r)^{\otimes m})^{S_m}.$$ 
Thanks to Theorem 4.4 the latter space can be identified with $(H_{C^r}^{\text{CPHilb}}(\gamma,y)(C)) \otimes (C^r)^{\otimes m})^{S_m}$ which in turn, by Lemmas 8.3 and 8.4, is naturally identified with $H_{C^r}^{\text{CPHilb}}(\gamma,y)(C)$. The result now follows from Proposition 8.2. □

**Example 8.7.** When $r = 1$, we have $\text{CPHilb}^{1,y}(C) = \text{Hilb}(C)$ and, up to [8 Proposition 9.5], we recover Proposition 7.17.

**Remark 8.8.** We can realize the generators $E_i, F_i$ of $\mathfrak{gl}(r)$ by explicit correspondences between $\text{PHilb}^{r,y}$ and $\text{PHilb}^{r,y}$ similar to [6 Theorem 3.4].

### 8.4. Compositional Hilbert schemes as generalized affine Springer fibers.

Just as with parabolic Hilbert schemes, the compositional parabolic Hilbert scheme $\text{CPHilb}^{r,y}(C)$ admits an interpretation as a generalized affine Springer fiber. In this setting, we let the group $G := \text{GL}_n^{x,r}$ act on the vector space $N := \mathbb{C}^n \oplus \mathfrak{gl}_n^{x,r}$ in the following way:

$$(g_0, g_1, \ldots, g_{r-1})(v, X_0, \ldots, X_{r-1}) = (g_0 v, g_1 X_0 g_0^{-1}, \ldots, g_0 X_{r-1} g_0^{-1}).$$

We can visualize $N$ in terms of representations of the following cyclic quiver:

![Diagram](attachment:image.png)

As in Section 7.4 we consider the groups $G_0 \subseteq G_K$. We will consider the affine Grassmannian

$$\mathcal{G}r_G := G_K/G_0 = (\text{GL}_{n,K}/\text{GL}_{n,0})^{x,r}$$

that parameterizes $r$-tuples of $0$-lattices inside $(K^n)^*$ via

$$[g_0, \ldots, g_{r-1}] \mapsto ((\mathbb{C}^n)^*g_0^{-1}, \ldots, (\mathbb{C}^n)^*g_{r-1}^{-1})$$

The group $G_K$ acts on $N_K := N \otimes K$, and $G_0$ preserves $N_0$. Recall the definition of $b_i \in \mathbb{C}^n$ and $Y \in \mathfrak{gl}_n(\mathcal{O})$ from Section 7.4. Here, we will consider the following generalized affine Springer fiber

$$\text{Spr}(b_1, \text{Id}, \ldots, Y) := \{[g] \in \mathcal{G}r_G \mid (g_0^{-1}b_1, g_0^{-1}g_1, \ldots, g_0^{-1}g_{r-1}g_{r-2}, g_0^{-1}Y g_{r-1}) \in N_0 \} \subseteq \mathcal{G}r_G.$$

### Proposition 8.9. We have an isomorphism

$$\text{Spr}(b_1, \text{Id}, \ldots, Y) \cong \text{CPHilb}^{r,y}(C).$$

**Proof.** By definition, an element $[g] = [g_0, \ldots, g_{r-1}] \in \mathcal{G}r_G$ belongs to the space $\text{Spr}(b_1, \text{Id}, \ldots, Y)$ if and only if $g_0^{-1}b_1 \in \mathbb{C}^n$, $g_0^{-1}g_1 \in \mathfrak{gl}_n(\mathcal{O})$ for $i = 0, \ldots, r-2$ and $g_0^{-1}Y g_{r-1} \in \mathfrak{gl}_n(\mathcal{O})$. It easily follows from here that $g_0^{-1}b_1 \in \mathbb{C}^n$ and $g_0^{-1}Y g_i \in \mathbb{C}^n$ for every $i = 0, \ldots, r-1$. Thanks to [23 Theorem 3.3] this implies that $\text{Spr}(b_1, \text{Id}, \ldots, Y) \subseteq \text{Hilb}(C)^{x,r}$. Let $(J^0, \ldots, J^{r-1}) \in \text{Hilb}(C)^{x,r}$ be the point corresponding to $[g_0, \ldots, g_{r-1}]$. The condition $g_{i+1}^{-1}g_i \in \mathfrak{gl}_n(\mathcal{O})$ for $i = 0, \ldots, r-2$
translates to \( J^0 \supseteq J^1 \supseteq \cdots \supseteq J^{r-1} \), while the condition \( g_0^{-1} Y g_{r-1} \in \mathfrak{gl}_n(\mathcal{O}) \) translates to \( J^{r-1} \supseteq y J^0 \). The result follows.

\[ \Box \]

**Remark 8.10.** Similar to the work of [23], the same proof shows that for an arbitrary plane curve singularity \( C \) such that the \( x \)-projection has degree \( n \), the scheme \( \mathrm{CPHilb}^{r,y}(C) \) can be presented as a generalized affine Springer fiber for \( G = \text{GL}^{x,r}_n \) and \( N := \mathbb{C}^n \oplus \mathfrak{gl}_n^{\oplus r} \).

**Remark 8.11.** Note that, just as the generalized affine Springer fibers considered in Section 7.4 are the intersection of an affine Springer fiber with the positive affine Grassmannian (cf. Remark 7.26), the generalized affine Springer fibers we consider here are (disjoint union of) affine Spaltenstein varieties with positive partial affine flag varieties.

Just as in Section 7.4 \( \text{Spr}(b_1, \text{Id}, \ldots, Y) \) can be realized as one of the varieties considered by [24]. Indeed, it is straightforward to verify that

\[ \text{Spr}(b_1, \text{Id}, \ldots, Y) = \mathcal{F}_a(t, (b_1, \text{Id}, \ldots, Y)) \]

where \( t = 0 \) and \( a \in \mathfrak{a} := X_*(A) \otimes \mathbb{R} \) is also 0 where, recall, \( A \subseteq G \) is a maximal torus. We can verify that \( \text{Spr}(b_1, \text{Id}, \ldots, Y) \) admits an affine paving as follows. Recall that we need to find \( b \in \mathfrak{a} \) and \( c \in \mathbb{R} \) satisfying the three conditions of Section 7.4. We can take \( c = m/n > t = 0 \) and \( b = (b^0, b^1, \ldots, b^{r-1}) \), where \( b^0 = b^1 = \cdots = b^{r-2} = \text{diag}(0,0,\ldots,0) \) and \( b^{r-1} = \text{diag}(c,c,\ldots,c) \). We need to verify that the element

\[ (b_1, \text{Id}, \ldots, Y) = (b_1, \text{Id}, \ldots, Y |_{\epsilon = 1}) \in N \]

is \( G \)-good. This follows because the element

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & \cdots & 0 & Y |_{\epsilon = 1} \\
\text{Id} & 0 & \cdots & 0 & 0 \\
0 & \text{Id} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & \cdots & \text{Id} & 0
\end{pmatrix}
\in \mathbb{C}^{rn} \oplus \mathfrak{gl}_{nr}
\]

is \( \text{GL}_{nr} \)-good, which in turn is a consequence of Proposition 7.22. Thus, thanks to [24] we get the following.

**Proposition 8.12.** The Hilbert scheme \( \text{CPHilb}^{r,y}(C) \) is paved by affine spaces. Thus, its cohomology is equivariantly formal.

**Remark 8.13.** Similarly to what is done in Section 7.4 one can show that for a composition \( \gamma \in \mathcal{C}_r(m) \) the variety \( \text{PHilb}^{\gamma,y}(C) \) admits a paving by affine spaces. This gives another proof of Proposition 8.12.

**Remark 8.14.** The algebra of functions on the Gieseker variety \( \mathcal{M}(n, r) \) is known, thanks to the results of Nakajima-Takayama [48], see also [13], to be the (non-quantized) Coulomb branch algebra for the gauge theory with gauge group \( G = \text{GL}^{x,r}_n \) and matter representation \( N = \mathbb{C}^n \oplus \mathfrak{gl}_n^{\oplus r} \) as defined in this section. Uniqueness of quantizations proved by Losev [42, Theorem 3.4] then shows that the algebra \( \mathcal{A}_r(n, r) \) is the corresponding quantized Coulomb branch algebra. It would be interesting to compare the action of \( \mathcal{A}_r(n, r) \) on \( H^*_C(\text{CPHilb}^{r,y}(C)) \) we have constructed here with an action by convolution diagrams as in [23, 34].
Theorem 9.2. The action of the Dunkl-Opdam subalgebra on \( \Delta(\text{triv}) = \mathbb{C}[x_1, \ldots, x_n] \) is still diagonalizable, and we will provide an explicit basis of \( \Delta(\text{triv}) \) completely analogous to that of Theorem 6.5. Since \( H_c \equiv H_{1,c} \), having \( c \to \infty \) will yield an action of the algebra \( H_{0,1} \).

9. Limit \( m \to \infty \)

Remark 8.15. Let \( C \) be a plane curve singularity such that the \( x \)-projection \( C \to \mathbb{C} \) has degree \( n \). One can use the techniques developed by Hilburn-Kamnitzer-Weeke in [34] and Garner-Kivinen in [23] to show that there is an action of the algebra of functions \( \mathbb{C}[M(n, r)] \) on the non-equivariant homology \( H_s(\text{CPHilb}^q(C)) \), cf. Section 7.5 and Remark 8.10.

Remark 8.16. As in Remark 7.31, we can consider the case \( C = \{ x^m = y^n \} \) for \( \gcd(m, n) = d > 1 \). In this case by [23] there is an action of the quantum Gieseker algebra \( \mathcal{A}_m(n, r) \) on \( H^*_C(\text{CPHilb}^q(C)) \). We expect this representation to have minimal support in the sense of [18]. Note that by [18] Theorem 2.17, Lemma 4.1] minimally supported representations of \( \mathcal{A}_m(n, r) \) are related to the minimally supported representations of \( H_m \) in a way similar to Proposition 8.2.

Proposition 9.1. For any \( c \in \mathbb{C} \), the action of the Dunkl-Opdam subalgebra on \( \Delta_c(\text{triv}) = \mathbb{C}[x_1, \ldots, x_n] \) is diagonalizable up to degree \( |c(n-1)| \). Moreover, up to this degree, the action of the algebra \( H_c \) is given by the same operators as in Theorem 6.5.

Proof. Following the strategy of the proof of Theorem 6.2, we need to construct the eigenvectors \( v_a \) for \( |a| < \sqrt{c(n-1)} \). The only obstruction to constructing these eigenvectors is that the intertwining operator \( \sigma_i \) may not be well-defined on the eigenspace \( M_{\mathcal{W}(a)} \). But this is only the case when \( \mathcal{W}(a)_i = \mathcal{W}(a)_{i+1} \). Recall that \( \mathcal{W}(a)_i - \mathcal{W}(a)_{i+1} = a_i - a_{i+1} - (g_n(i) - g_n(i+1))c \) where \( g_n \) is the shortest permutation that sorts \( a \). Since \( g_n(i) - g_n(i+1) \in \{ \pm 1, \ldots, \pm(n-1) \} \), the result follows.

Thanks to the previous proposition, letting \( c \to \infty \) and appropriately rescaling, we get the following \( "t = 0" \) analogue of Theorem 6.5.

Theorem 9.2. The \( H_{0,1} \)-module \( \Delta_{0,1}(\text{triv}) := H_{0,1} \otimes_{\mathbb{C}[y_1, \ldots, y_n]} S_{y_n} \text{triv} \) has a basis given by \( \{ v_a : a \in \mathbb{Z}^n \} \), and the action of the algebra \( H_{0,1} \) on \( \Delta_{0,1}(\text{triv}) \) is given by the following operators.

\[
\begin{align*}
   u_i v_a &= u_i v_a \\
   \tau v_a &= v_{\tau a} \\
   \lambda v_a &= \mathcal{W}_1 v_{\pi^{-1} a} \\
   s_i v_a &= \begin{cases} 
   v_{s_i a} + \frac{1}{(g_n(i)-g_n(i+1))} v_a \\
   (g_n(i)-g_n(i+1))v_{s_i a} + \frac{1}{(g_n(i)-g_n(i+1))} v_a 
   \end{cases} \\
&= a_i > a_{i+1}, \quad a_i < a_{i+1}, \quad a_i = a_{i+1}
\end{align*}
\]
where $w_i := w_i(a) = (1 - g_a(i))$ and, as before, $g_a$ is the minimal-length permutation that sorts $a$.

**Remark 9.3.** As above, one can also define the renormalized basis $\tilde{v}_a$ such that

$$(1 + s_i)\tilde{v}_a = \frac{g_a(i + 1) - g_a(i)}{g_a(i + 1) - g_a(i)} - \frac{1}{g_a(i + 1) - g_a(i)} \tilde{v}_{s_i a} + \frac{g_a(i + 1) - g_a(i)}{g_a(i + 1) - g_a(i)} \tilde{v}_a.$$  

**Remark 9.4.** We remark that, unlike the $t = 1$ case, the module $\Delta_{0,1}(\text{triv})$ is never irreducible. It has a unique irreducible graded quotient $L_{0,1}(\text{triv}) = \mathbb{C}[x_1, \ldots, x_n]/(\mathbb{C}[x_1, \ldots, x_n][x_1, x_2, \ldots, x_n])$.

Note that the proof of Theorem 9.2 can be extended to any Verma module $\Delta_{0,1}(\mu) := H_{0,1} \otimes_{\mathbb{C}[y]} \mathfrak{S}_n V_{\mu}$. In particular, we get that $\Delta_{0,1}(\mu)$ has a basis given by $v(a, T)$, where $a \in \mathbb{Z}_{\geq 0}^n$ and $T \in \text{SYT}(\mu)$. The action of $H_{0,1}$ on $\Delta_{0,1}(\mu)$ is given by

$$u_i v(a, T) = u_i(a, T) v(a, T)$$
$$\tau v(a, T) = v(\pi \cdot a, T)$$
$$\lambda v(a, T) = u_1(a, T) v(\pi^{-1} \cdot a, T)$$

$$s_i v(a, T) = \begin{cases} 
  v(s_i \cdot a, T) - A_2 v(a, T) & a_i > a_{i+1}, \\
  A_1 v(s_i \cdot a, T) + A_2 v(a, T) & a_i < a_{i+1}, \\
  (c T(j + 1) - c T(j)) v(a, T) & a_i = a_{i+1} \text{ and } s_j(T) \notin \text{SYT}(\mu), \\
  v(a, s_i(T)) - A_2 v(a, T) & a_i = a_{i+1} \text{ and } s_j(T) \in \text{SYT}(\mu)
\end{cases}$$

where $u_i(a, T) = - c T(g_a(i))$, we write $j = g_a(i)$ in the cases $a_i = a_{i+1}$, and where

$$A_1 = \frac{(c T(g_a(i)) - c T(g_a(i + 1))) - 1}{(c T(g_a(i)) - c T(g_a(i + 1)))^2}$$

and

$$A_2 = \frac{1}{c T(g_a(i)) - c T(g_a(i + 1))} = \frac{1}{c T(j) - c T(j + 1)}.$$

9.2. Hilbert scheme of the non-reduced line. On the geometric side, the curve $\{x^n = y^n\}$ has a natural limit at $m \to \infty$, namely, the non-reduced line $\{y^n = 0\}$. The ring of functions on $C_0 = \{y^n = 0\}$ has a basis $x^i y^j$ for $i \geq 0, n - 1 \geq j \geq 0$, as above.

The Hilbert scheme of points on $\{y^n = 0\}$ is the moduli space of ideals in the local ring

$$\mathcal{O}_{C_0,0} = \mathbb{C}[x, y]/y^n = \mathbb{C}[[x]]/\langle 1, \ldots, y^{n-1} \rangle.$$  

The multiplication by $y$ is given by the matrix similar to (29):

$$Y = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.$$  

We consider the $\mathbb{C}^*$ action on $C_0$ and on $\mathcal{O}_{C_0,0}$ such that $y$ has weight 1 and $x$ has weight 0. It naturally extends to the action on the punctual Hilbert scheme $\text{Hilb}_k(C_0, 0)$.  

Lemma 9.5. The fixed points of this action are isolated and correspond to monomial ideals.

Proof. An ideal \( I \) in \( \mathcal{O}_{C_0} \) is fixed under this \( \mathbb{C}^* \) action if and only if it is generated by functions \( y^{a_i}p_i(x) \) which are homogeneous in \( y \) but not necessary in \( x \). On the other hand, in the ring of formal power series \( p \mathbb{C} \), Remark 9.7.

Remark 9.6. It is important for the above proof that we work with the punctual scheme.

Remark 9.7. Unlike the curve \( \{x^m = y^n\} \), the curve \( C_0 \) has an action of another \( \mathbb{C}^* \) such that \( y \) has weight 0 and \( x \) has weight 1. The weight of this action on a monomial ideal \( I \) generated by \( y^{a_i}x^{c_i} \) equals \( \sum c_i = \dim \mathcal{O}_{C_0}/I = k \).

Similarly, one can define the parabolic Hilbert scheme \( \text{PHilb}_{k,n+k}(C_0) \) as the space of flags of ideals \( I_k \supset I_{k+1} \supset \cdots \supset I_0 = I \) in \( \mathcal{O}_{C_0} \), and \( \text{PHilb}^p(C_0) := \sqcup_k \text{PHilb}_{k,n+k}(C_0) \). The fixed points in \( \text{PHilb}^p(C_0) \) are determined by sequences of monomials \( (y^{a_i}x^{c_i}) \) with no restrictions on \( c_i \). As in Lemma 7.5, we have \( \alpha_i = g(i)-1 \), where \( g \) is the permutation which sorts \( c_i \) in non-increasing order (recall that when \( c_i = c_j \) with \( i < j \) we have \( \alpha_i < \alpha_j \)). We can write \( c_i = a_{n+1-i} \) and \( g(i) = n+1-g(n+1-i) \).

The construction of geometric operators corresponding to \( u_1, s_1, \tau \) and \( \lambda \) extends verbatim to this case, however, one needs to be careful with the equivariant weights. Now \( \mathcal{L}_i \) has the weight of the monomial \( (y^{a_i}x^{c_i}) \), that is

\[
c_1(\mathcal{L}_i) = \alpha_i = g(c(i)) - 1 = n - g_a(n+1-i) = (n-1) + w_{n+1-i}.
\]

The operators \( T \) and \( A \) can be defined as in Section 7.3 and their matrix elements can be computed similarly. Observe that \( \mathcal{O}_{C_0}/x\mathcal{O}_{C_0} \) still has a unique \( Y \)-invariant one dimensional subspace generated by \( y^{n-1} \) which has weight \( (n-1) \). The computation in Theorem 7.14 then implies \( T \circ A = u_1 \). We conclude the following:

Theorem 9.8. Consider the non-reduced curve \( C_0 = \{y^n = 0\} \) with the \( \mathbb{C}^* \) action \( (x, y) \mapsto (x, sy) \). Then the \( \mathbb{C}^* \) equivariant cohomology

\[
U_\infty = \bigoplus_{k=0}^{\infty} H^*_{{\mathbb{C}^*}}(\text{PHilb}_{k,n+k}(C_0))
\]

has an action of the rational Cherednik algebra \( H_{0,1} \) defined by the same operators in Theorem 7.14. This representation is isomorphic to the polynomial representation of \( H_{0,1} \).

Remark 9.9. Note that the work of Garner and Kivinen \cite{23} also accounts for the case of the non-reduced curve \( C_0 \). However, \cite{23} obtains an action of the Cherednik algebra \( H_{0,1} \) on the equivariant cohomology of \( \text{PHilb}^p(C_0) \) with respect to the group \( \mathbb{C}^* \times \mathbb{C}^* \), see Remark 7.7. In particular, we can obtain Theorem 9.8 from \cite{23} Proposition 1.5] after specializing one of the equivariant parameters to 0.

Finally, we would like to mention that the constructions of Section 5 can be extended to this setup, and we get the following result.
**Theorem 9.10.** With the same notation as in Theorem 9.8, the $C^*$-equivariant cohomology

$$H^*_C(\text{CPHilb}^{r,x}(C_0))$$

has an action of the spherical algebra $eH_{0,1}(n,r)e$, where $H_{0,1}(n,r)$ is the matrix version of the Cherednik algebra defined in Definition 8.3. This representation is isomorphic to the representation $(\mathbb{C}[x_1, \ldots, x_n] \otimes (\mathbb{C}^r)^{\otimes n})S_n$ defined in a natural way.

**Remark 9.11.** From its interpretation as a generalized affine Springer fiber, see Section 8.4, it follows that the homology $H^*_C(\text{CPHilb}^{r,x}(C_0))$ admits an action of a flavor deformation of the algebra of functions on the Gieseker variety $M(n,r)$. When $r = 1$, this flavor deformation is precisely $eH_{0,1}(n,1)e$, which is known to be commutative and it is in fact the algebra of functions on the Calogero-Moser space, [16]. It is unclear the relationship that the flavor deformation bears to $eH_{0,1}(n,r)e$ when $r > 1$.

**Conflict of interest.** The authors declare that they have no conflict of interest.

**References**

[1] M. Balagović. Irreducible modules for the degenerate double affine Hecke algebra of type $A$ as submodules of Verma modules. J. Combin. Theory Ser. A 133 (2015), 97–138.

[2] Yu. Berest, P. Etingof, V. Ginzburg, Finite-dimensional representations of rational Cherednik algebras, Int. Math. Res. Not. IMRN 2003, no. 19, 1053–1088.

[3] A. Björner, F. Brenti. Combinatorics of Coxeter groups, Graduate Texts in Mathematics, 231. Springer, New York, 2005.

[4] A. Braverman, M. Finkelberg, H. Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. Adv. Theor. Math. Phys. 22 (2018), no. 5, 1071–1147.

[5] M. Brion. Poincaré duality and equivariant (co)homology. The Michigan Mathematical Journal 48 (2000), no. 1, 77–92.

[6] J. Brundan. Symmetric functions, parabolic category $\mathcal{O}$ and the Springer fiber, Duke Math. J. 143 (2008), 41–79.

[7] J. Brundan, V. Ostrik. Cohomology of Spaltenstein varieties, Transf. Groups 16 619 (2011)

[8] D. Calaque, B. Enriquez, P. Etingof. Universal KZB equations: The elliptic case, In Algebra, arithmetic, and geometry: in honor of Yu. I. Manin, Vol. I, 165–266, Progr. Math. 269, Birkhäuser Boston, Inc., Boston, MA, 2009

[9] E. Carlsson, E. Gorsky, A. Mellit. The $h_{q,t}$ algebra and parabolic flag Hilbert schemes, Mathematische Annalen 376 (2020) 1303–1336

[10] I.V. Cherednik. Special bases of irreducible representations of a degenerate affine Hecke algebra, (Russian) Funktsional. Anal. i Prilozhen. 20 (1986), no. 1, 87–88.

[11] I. Cherednik. Double affine Hecke algebras. London Mathematical Society Lecture Note Series, 319. Cambridge University Press, Cambridge, 2005.

[12] I. Cherednik. Intertwining operators of double affine Hecke algebras. Selecta Math. (N.S.) 3 (1997), no. 4, 459–495.

[13] J. de Boer, K. Hori, H. Ooguri, Y. Oz. Mirror symmetry in three-dimensional gauge theories, quivers, and D-branes, Nuclear Phys. B 493 (1997), no. 1-2, 101–147

[14] C. F. Dunkl, S. Griffeth. Generalized Jack polynomials and the representation theory of rational Cherednik algebras, Selecta Math. (N.S.) 16 (2010) 791–818

[15] C. F. Dunkl, E. M. Opdam, Dunkl operators for complex reflection groups, Proc. London Math. Soc. (3) 86 (2003), no. 1, 70–108.

[16] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Invent. Mat. 147 (2002), no. 2, 243–348

[17] P. Etingof, E. Gorsky, I. Losev, Representations of rational Cherednik algebras with minimal support and torus knots, Adv. Math. 277 (2015), 124–180.
[18] P. Etingof, V. Krylov, I. Losev, J. Simental, *Representations with minimal support for quantized Gieseker varieties*, Math. Z. **298** (2021), no. 3-4, 1593–1621.

[19] P. Etingof, E. Stoica, *Unitary representations of rational Cherednik algebras*, Represent. Theory **13** (2009), 349–370.

[20] S. Fishel, S. Griffeth, E. Manosalva, *Unitary representations of the Cherednik algebra: $V^*$-homology*, Math. Z. **299** (2021), no. 3-4, 1593–1621.

[21] W. Fulton, *Intersection theory*, Second edition. Springer-Verlag, Berlin, 1998.

[22] W. L. Gan, V. Ginzburg, *Almost commuting variety, D-modules and Cherednik algebras*, Int. Mat. Res. Pap. **2006** (2006) 2649.

[23] N. Garber, O. Kivinen, *Generalized affine Springer theory and Hilbert schemes on planar curves*, Int. Math. Res. Not. IMRN **2022**, rnac038.

[24] M. Goresky, R. Kottwitz, R. Macpherson, *Purity of equivaled affine Springer fibers*, Represent. Theory **10** (2006), 130–146.

[25] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[26] W. Fulton, *Intersection theory*, Second edition. Springer-Verlag, Berlin, 1998.

[27] N. Garner, O. Kivinen, *Generalized affine Springer theory and Hilbert schemes on planar curves*, Int. Math. Res. Not. IMRN **2022**, rnac038.

[28] M. Goresky, R. Kottwitz, R. MacPherson, *Purity of equivaled affine Springer fibers*, Represent. Theory **10** (2006), 130–146.

[29] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[30] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[31] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[32] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[33] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[34] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[35] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[36] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[37] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[38] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[39] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[40] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[41] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[42] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[43] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[44] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[45] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.

[46] M. Goresky, R. Kottwitz, R. MacPherson, *Koszul duality, equivariant cohomology, and the localization theorem*, Invent. Math. **131** (1998), 25–83.
[47] L. Migliorini, V. Shende, A support theorem for Hilbert schemes of planar curves, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 6, 2353–2367.

[48] H. Nakajima, Y. Takayama, Cherkis bow varieties and Coulomb branches of quiver gauge theories of affine type A, Selecta Math, 23 (2017), 2553–2633.

[49] A. Oblomkov, V. Shende, The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link, Duke Math. J. 161 (2012), no. 7, 1277–1303.

[50] A. Oblomkov, J. Rasmussen, V. Shende, The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link, With an appendix by Eugene Gorsky, Geom. Topol. 22 (2018), no. 2, 645–691.

[51] A. Oblomkov, Z. Yun, Geometric representations of graded and rational Cherednik algebras, Adv. Math. 292 (2016), 601–706.

[52] A. Oblomkov, Z. Yun, The cohomology ring of certain compactified Jacobians, arXiv:1710.05391

[53] G. Shin, Useful operators in representations of the rational Cherednik algebra of type $sl_n$, Honam Math. J. 41 (2019), issue 2, 421–433

[54] A. Ram, Affine Hecke algebras and generalized standard Young tableaux, Special issue celebrating the 80th birthday of Robert Steinberg, J. Algebra 260 (2003), no. 1, 367–415.

[55] J. Rennemo, Homology of Hilbert schemes of points on a locally planar curve, J. Eur. Math. Soc. (JEMS) 20 (2018), no. 7, 1629–1654.

[56] T. Suzuki, Rational and trigonometric degeneration of the double affine Hecke algebra of type $A$, Int. Math. Res. Not. IMRN 2005, no. 37, 2249–2262.

[57] T. Suzuki, Classification of simple modules over degenerate double affine Hecke algebras of type $A$, Int. Math. Res. Not. (2003), no. 43, 2313–2339.

[58] M. Varagnolo, E. Vasserot, Finite-dimensional representations of DAHA and affine Springer fibers: the spherical case, Duke Math. J. 147 (2009), no. 3, 439–540.

[59] E. Vasserot. Induced and simple modules of double affine Hecke algebras. Duke Math. J. 126 (2005), no. 2, 251–324.

[60] B. Webster, Koszul duality between Higgs and Coulomb categories $O$, Preprint, arXiv 1611.06541

[61] B. Webster, Representation theory of the cyclotomic Cherednik algebra via the Dunkl-Opdam subalgebra, New York J. Math. 25 (2019), 1017–1047.

[62] Z. Yun, The spherical part of the local and global Springer actions, Math. Ann. 359 (2014), no. 3–4, 557–594.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, ONE SHIELDS AVENUE, DAVIS CA 95616

Email address: egorskiy@math.ucdavis.edu

MAX PLANCK INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY

Email address: simental@im.unam.mx

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, ONE SHIELDS AVENUE, DAVIS CA 95616

Email address: vazirani@math.ucdavis.edu