ON THE REGULARITY FOR 3D NAVIER-STOKES EQUATION

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Abstract. In this paper we will prove that the vorticity belongs to $L^\infty(0, T; L^2(\Omega))$ for 3D incompressible Navier-Stokes equation with periodic initial-boundary value conditions, then the existence of a global smooth solution is obtained. Our approach is to construct a set of auxiliary problems to approximate the original one of vorticity equation.

Keywords. Navier-Stokes equation; Regularity; Vorticity.

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1. Introduction

Let $\Omega = (0, 1)^3$, and $\mathcal{D}(\Omega)$ be the space of $C^\infty$ functions with compact support contained in $\Omega$. Some basic spaces will be used in this paper:

$\mathcal{V} = \{ u \in \mathcal{D}(\Omega), \div u = 0 \}$
$V = \text{the closure of } \mathcal{V} \text{ in } H^1(\Omega)$
$H = \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega)$

The velocity-pressure form for Navier-Stokes equation is

$$
\begin{align*}
\partial_t u_1 + u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 + u_3 \partial_{x_3} u_1 + \partial_{x_1} p &= \Delta u_1 \\
\partial_t u_2 + u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + u_3 \partial_{x_3} u_2 + \partial_{x_2} p &= \Delta u_2 \\
\partial_t u_3 + u_1 \partial_{x_1} u_3 + u_2 \partial_{x_2} u_3 + u_3 \partial_{x_3} u_3 + \partial_{x_3} p &= \Delta u_3
\end{align*}
$$

(1)

with the initial conditions $(u_1, u_2, u_3)|_{t=0} = (u_{10}, u_{20}, u_{30})(x)$, henceforth we always ignore the assumption of sufficient smoothness of the initial conditions. Moreover, the periodic boundary conditions are

$$
u_i(x + e_j, t) = u_i(x, t), \quad i, j = 1, 2, 3$$

and the incompressible condition is

$$
\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0
$$

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where $x = (x_1, x_2, x_3)$ is a point of $\mathbb{R}^3$, and $e_j$ is $j^{th}$ unit vector in $\mathbb{R}^3$.

$u = (u_1, u_2, u_3)$ is velocity, $p$ is pressure, and $\nu > 0$ is viscosity.

The vorticity-velocity form for Navier-Stokes equation is

\[ \begin{align*}
\partial_t \omega_1 + u_1 \partial_{x_1} \omega_1 + u_2 \partial_{x_2} \omega_1 + u_3 \partial_{x_3} \omega_1 - \omega_1 \partial_{x_1} u_1 - \omega_2 \partial_{x_2} u_1 - \omega_3 \partial_{x_3} u_1 &= \Delta \omega_1 \\
\partial_t \omega_2 + u_1 \partial_{x_1} \omega_2 + u_2 \partial_{x_2} \omega_2 + u_3 \partial_{x_3} \omega_2 - \omega_1 \partial_{x_1} u_2 - \omega_2 \partial_{x_2} u_2 - \omega_3 \partial_{x_3} u_2 &= \Delta \omega_2 \\
\partial_t \omega_3 + u_1 \partial_{x_1} \omega_3 + u_2 \partial_{x_2} \omega_3 + u_3 \partial_{x_3} \omega_3 - \omega_1 \partial_{x_1} u_3 - \omega_2 \partial_{x_2} u_3 - \omega_3 \partial_{x_3} u_3 &= \Delta \omega_3
\end{align*} \]

(2)

with the initial conditions $\omega_1(x_1, x_2, x_3)|_{t=0} = (\omega_{10}, \omega_{20}, \omega_{30}) = (\text{curl} \upsilon_{10}, \text{curl} \upsilon_{20}, \text{curl} \upsilon_{30})$, and the periodic boundary conditions:

$$\omega_i(x + e_j, t) = \omega_i(x, t), \quad i, j = 1, 2, 3$$

and the incompressible condition:

$$\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0$$
$$\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0$$

We here recall the global $L^2$-estimate from [4] for the Navier-Stokes equation of velocity-pressure form. Since

$$\int \Omega u_i (u_1 \partial_{x_1} u_i + u_2 \partial_{x_2} u_i + u_3 \partial_{x_3} u_i) = \frac{1}{2} \int \Omega (u_1 \partial_{x_1} u_i^2 + u_2 \partial_{x_2} u_i^2 + u_3 \partial_{x_3} u_i^2)$$

$$= -\frac{1}{2} \int \Omega u_i^2 \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0 \quad i = 1, 2, 3$$

$$\int \Omega (u_1 \partial_{x_1} p + u_2 \partial_{x_2} p + u_3 \partial_{x_3} p) = -\int \Omega p (\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3) = 0$$

and

$$\int \Omega u_i \Delta u_i = \int \Omega u_i (\partial_{x_1}^2 u_i + \partial_{x_2}^2 u_i + \partial_{x_3}^2 u_i)$$

$$= -\int \Omega ((\partial_{x_1} u_i)^2 + (\partial_{x_2} u_i)^2 + (\partial_{x_3} u_i)^2)$$

then

$$\int \Omega u_1 \partial_{x_1} u_1 + \int \Omega u_1 (u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 + u_3 \partial_{x_3} u_1) + \int \Omega u_1 \partial_{x_1} p = \int \Omega u_1 \Delta u_1$$
$$\int \Omega u_2 \partial_{x_2} u_2 + \int \Omega u_2 (u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + u_3 \partial_{x_3} u_2) + \int \Omega u_2 \partial_{x_2} p = \int \Omega u_2 \Delta u_2$$
$$\int \Omega u_3 \partial_{x_3} u_3 + \int \Omega u_3 (u_1 \partial_{x_1} u_3 + u_2 \partial_{x_2} u_3 + u_3 \partial_{x_3} u_3) + \int \Omega u_3 \partial_{x_3} p = \int \Omega u_3 \Delta u_3$$
so that
\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} \left( u_1^2 + u_2^2 + u_3^2 \right) + \int_{\Omega} \left( \left( \partial_{x_1} u_1 \right)^2 + \left( \partial_{x_2} u_1 \right)^2 + \left( \partial_{x_3} u_1 \right)^2 + \left( \partial_{x_2} u_2 \right)^2 + \left( \partial_{x_3} u_2 \right)^2 + \left( \partial_{x_3} u_3 \right)^2 \right) = 0
\]

it follows that
\[
\int_{\Omega} (u_1^2 + u_2^2 + u_3^2) + \int_{0}^{T} \left( \| \nabla u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_2 \|_{L^2(\Omega)}^2 + \| \nabla u_3 \|_{L^2(\Omega)}^2 \right) = \int_{\Omega} (u_{10}^2 + u_{20}^2 + u_{30}^2)
\]
Hence we have
\[
\sup_{t \in (0,T)} \int_{\Omega} (u_1^2 + u_2^2 + u_3^2) < +\infty \quad (3)
\]
\[
\int_{0}^{T} (\| \nabla u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_2 \|_{L^2(\Omega)}^2 + \| \nabla u_3 \|_{L^2(\Omega)}^2) < +\infty \quad (4)
\]
Above \( u \) can be interpreted as the Galerkin approximation of the solution, but (3) and (4) are also true for the solution of problem (1).

The rest of sections are arranged as follows: In section 2 and 3, we introduce a set of auxiliary problems and prove the uniform boundedness and the existence of their solutions in \( L^\infty(0,T;L^2(\Omega)) \). Then it is shown that the solutions of the auxiliary problems converge to that of Navier-Stokes equation with vorticity-velocity form, which also belongs to \( L^\infty(0,T;L^2(\Omega)) \). Final section will present the solution of Navier-Stokes equation with velocity-pressure form belongs to \( L^\infty(0,T;H^2(\Omega)) \).

### 2. Auxiliary problems

For the 3D regularity, we only need to prove that the vorticity in (2) belongs to \( L^\infty(0,T;L^2(\Omega)) \).

Given a partition with respect to \( t \) as follows:
\[
0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots < t_N = T
\]
On each \( t \in (t_{k-1}, t_k) \), we introduce an auxiliary problem:
\[
\begin{align*}
\partial_t \omega_1 + \nabla_1 \cdot \nabla_1 \omega_1 + \nabla_2 \cdot \nabla_2 \omega_1 + \nabla_3 \cdot \nabla_3 \omega_1 &= \Delta \omega_1 + \partial_{x_1} \omega_1 + \partial_{x_2} \omega_1 + \partial_{x_3} \omega_1 + \partial_{x_1} q = \Delta \tilde{\omega}_1 \\
\partial_t \omega_2 + \nabla_1 \cdot \nabla_2 \omega_2 + \nabla_2 \cdot \nabla_2 \omega_2 + \nabla_3 \cdot \nabla_3 \omega_2 &= \Delta \omega_2 + \partial_{x_1} \omega_2 + \partial_{x_2} \omega_2 + \partial_{x_3} \omega_2 + \partial_{x_2} q = \Delta \tilde{\omega}_2 \\
\partial_t \omega_3 + \nabla_1 \cdot \nabla_3 \omega_3 + \nabla_2 \cdot \nabla_3 \omega_3 + \nabla_3 \cdot \nabla_3 \omega_3 &= \Delta \omega_3 + \partial_{x_1} \omega_3 + \partial_{x_2} \omega_3 + \partial_{x_3} \omega_3 + \partial_{x_3} q = \Delta \tilde{\omega}_3
\end{align*}
\]
where the initial value is assumed to be \( \tilde{\omega}_i(x,t_{k-1}) = \tilde{\omega}_i^{k-1}(x), \quad \tilde{\omega}_i(x,0) = \omega_{i0}(x), \quad i = 1, 2, 3 \), and
\[
\tilde{w}_i^k(x) = \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \tilde{w}_i(x, t) dt
\]

and
\[
\tilde{u}_i^k(x) = \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} u_i(x, t) dt, \quad i = 1, 2, 3
\]

In addition, let \( \varepsilon > 0 \), we construct a mollifier \( J_\varepsilon \in C_0^\infty(\mathbb{R}^3) \) such that
i) \( J_\varepsilon(x) \geq 0, \quad x \in \mathbb{R}^3 \),
ii) \( J_\varepsilon(x) = 0 \) if \( |x| \geq \varepsilon \), and
iii) \( \int_{\mathbb{R}^3} J_\varepsilon(x) dx = 1 \).

Then a convolution is defined as
\[
\tilde{w}_i^k(x) = J_\varepsilon \ast \tilde{w}_i^k(x) = \int_{\mathbb{R}^3} J_\varepsilon(x - y) \tilde{w}_i^k(y) dy
\]

where we assume that a zero extension of \( \tilde{w}_i^k \) be made outside \( \Omega \).

Similarly we can set the periodic boundary conditions :
\[
\tilde{w}_i(x + e_j, t) = \tilde{w}_i(x, t), \quad i, j = 1, 2, 3
\]

and the incompressible condition :
\[
\partial_{x_1} \tilde{w}_1 + \partial_{x_2} \tilde{w}_2 + \partial_{x_3} \tilde{w}_3 = 0 \\
\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0
\]

It is easy to check that
\[
\partial_{x_1} \tilde{w}_1 + \partial_{x_2} \tilde{w}_2 + \partial_{x_3} \tilde{w}_3 = 0 \Rightarrow \partial_{x_1} \tilde{w}_1^k + \partial_{x_2} \tilde{w}_2^k + \partial_{x_3} \tilde{w}_3^k = 0 \\
\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0 \Rightarrow \partial_{x_1} u_1^k + \partial_{x_2} u_2^k + \partial_{x_3} u_3^k = 0
\]

In the section 3, by means of the Galerkin method and the compactness imbedding theorem, we can prove the local existence of the weak solutions of these systems for each \( (t_{k-1}, t_k) \) being small enough. Below we also interpret \( \tilde{w} \) as the Galerkin approximation of the solution of the problems (5), and first prove that \( \tilde{w}, t \in (0, T) \) belong to \( L^\infty(0, T; L^2(\Omega)) \). In section 4, an approach of approximation is used to assert that the solution of (2) also belongs to \( L^\infty(0, T; L^2(\Omega)) \).

Since
\[
\int_{\Omega} \left( \tilde{w}_1 (\tilde{u}_1^k \partial_{x_1} \tilde{w}_1^k + \tilde{u}_2^k \partial_{x_2} \tilde{w}_1^k + \tilde{u}_3^k \partial_{x_3} \tilde{w}_1^k) \\
+ \tilde{w}_2 (\tilde{u}_1^k \partial_{x_1} \tilde{w}_2^k + \tilde{u}_2^k \partial_{x_2} \tilde{w}_2^k + \tilde{u}_3^k \partial_{x_3} \tilde{w}_2^k) \\
+ \tilde{w}_3 (\tilde{u}_1^k \partial_{x_1} \tilde{w}_3^k + \tilde{u}_2^k \partial_{x_2} \tilde{w}_3^k + \tilde{u}_3^k \partial_{x_3} \tilde{w}_3^k) \right)
\]
\[
\begin{align*}
&= - \int_{\Omega} \left( \omega_1 \left( \partial_{x_1} (\omega_1 u_1^1) + \omega_1 \partial_{x_2} (\omega_1 u_2^1) + \omega_1 \partial_{x_3} (\omega_1 u_3^1) \right) \\
&\quad + \omega_2 \partial_{x_1} (\omega_2 u_1^2) + \omega_2 \partial_{x_2} (\omega_2 u_2^2) + \omega_2 \partial_{x_3} (\omega_2 u_3^2) \\
&\quad + \omega_3 \partial_{x_1} (\omega_3 u_1^3) + \omega_3 \partial_{x_2} (\omega_3 u_2^3) + \omega_3 \partial_{x_3} (\omega_3 u_3^3) \right) \\
&= - \int_{\Omega} \left( \omega_1 \left( \partial_{x_1} (\omega_1 u_1^1) + \omega_1 \partial_{x_2} (\omega_1 u_2^1) + \omega_1 \partial_{x_3} (\omega_1 u_3^1) \right) \\
&\quad + \omega_2 \partial_{x_1} (\omega_2 u_1^2) + \omega_2 \partial_{x_2} (\omega_2 u_2^2) + \omega_2 \partial_{x_3} (\omega_2 u_3^2) \\
&\quad + \omega_3 \partial_{x_1} (\omega_3 u_1^3) + \omega_3 \partial_{x_2} (\omega_3 u_2^3) + \omega_3 \partial_{x_3} (\omega_3 u_3^3) \right) \\
&\quad + \omega_2 \partial_{x_1} (\omega_2 u_1^2) + \omega_2 \partial_{x_2} (\omega_2 u_2^2) + \omega_2 \partial_{x_3} (\omega_2 u_3^2) \\
&\quad + \omega_3 \partial_{x_1} (\omega_3 u_1^3) + \omega_3 \partial_{x_2} (\omega_3 u_2^3) + \omega_3 \partial_{x_3} (\omega_3 u_3^3) \\
&\quad + \omega_2 \partial_{x_1} (\omega_2 u_1^2) + \omega_2 \partial_{x_2} (\omega_2 u_2^2) + \omega_2 \partial_{x_3} (\omega_2 u_3^2) \\
&\quad + \omega_3 \partial_{x_1} (\omega_3 u_1^3) + \omega_3 \partial_{x_2} (\omega_3 u_2^3) + \omega_3 \partial_{x_3} (\omega_3 u_3^3) \\
&\quad + \omega_2 \partial_{x_1} (\omega_2 u_1^2) + \omega_2 \partial_{x_2} (\omega_2 u_2^2) + \omega_2 \partial_{x_3} (\omega_2 u_3^2) \\
&\quad + \omega_3 \partial_{x_1} (\omega_3 u_1^3) + \omega_3 \partial_{x_2} (\omega_3 u_2^3) + \omega_3 \partial_{x_3} (\omega_3 u_3^3) \\
&= - \int_{\Omega} (\omega_1 \partial_{x_1} q + \omega_2 \partial_{x_2} q + \omega_3 \partial_{x_3} q) = - \int_{\Omega} q \left( \partial_{x_1} \omega_1 + \partial_{x_2} \omega_2 + \partial_{x_3} \omega_3 \right) = 0 \\
\text{furthermore} \\
\int_{\Omega} \omega_1 \Delta \omega_1 = \int_{\Omega} \omega_1 \left( \partial_{x_1}^2 \omega_1 + \partial_{x_2}^2 \omega_1 + \partial_{x_3}^2 \omega_1 \right) = - \int_{\Omega} \left( \left( \partial_{x_1} \omega_1 \right)^2 + \left( \partial_{x_2} \omega_1 \right)^2 + \left( \partial_{x_3} \omega_1 \right)^2 \right)
\end{align*}
\]
Then from (5) we have

\[
\int_\Omega \omega_1 \partial_t \omega_1 + \int_\Omega \omega_1 (\overrightarrow{\partial}_1 \partial_x \omega_1^1 + \overrightarrow{\partial}_2 \partial_x \omega_1^2 + \overrightarrow{\partial}_3 \partial_x \omega_1^3)
- \int_\Omega \omega_1 (\overrightarrow{\partial}_1 \partial_x \omega_1^1 + \overrightarrow{\partial}_2 \partial_x \omega_1^2 + \overrightarrow{\partial}_3 \partial_x \omega_1^3) + \int_\Omega \omega_1 \partial_x \omega_1 = \int_\Omega \omega_1 \Delta \omega_1
\]

\[
\int_\Omega \omega_2 \partial_t \omega_2 + \int_\Omega \omega_2 (\overrightarrow{\partial}_1 \partial_x \omega_2^1 + \overrightarrow{\partial}_2 \partial_x \omega_2^2 + \overrightarrow{\partial}_3 \partial_x \omega_2^3)
- \int_\Omega \omega_2 (\overrightarrow{\partial}_1 \partial_x \omega_2^1 + \overrightarrow{\partial}_2 \partial_x \omega_2^2 + \overrightarrow{\partial}_3 \partial_x \omega_2^3) + \int_\Omega \omega_2 \partial_x \omega_2 = \int_\Omega \omega_2 \Delta \omega_2
\]

\[
\int_\Omega \omega_3 \partial_t \omega_3 + \int_\Omega \omega_3 (\overrightarrow{\partial}_1 \partial_x \omega_3^1 + \overrightarrow{\partial}_2 \partial_x \omega_3^2 + \overrightarrow{\partial}_3 \partial_x \omega_3^3)
- \int_\Omega \omega_3 (\overrightarrow{\partial}_1 \partial_x \omega_3^1 + \overrightarrow{\partial}_2 \partial_x \omega_3^2 + \overrightarrow{\partial}_3 \partial_x \omega_3^3) + \int_\Omega \omega_3 \partial_x \omega_3 = \int_\Omega \omega_3 \Delta \omega_3
\]

so that

\[
\frac{1}{2} \partial_t \int_\Omega (\omega_1^2 + \omega_2^2 + \omega_3^2) + \int_\Omega ((\partial_x \omega_1)^2 + (\partial_x \omega_2)^2 + (\partial_x \omega_3)^2) + (\partial_x \omega_2)^2 + (\partial_x \omega_3)^2 + (\partial_x \omega_1)^2) + (\partial_x \omega_2)^2 + (\partial_x \omega_3)^2 + (\partial_x \omega_1)^2
\]

\[
- \int_\Omega (\overrightarrow{\partial}_1 \partial_x \omega_1^1 + \overrightarrow{\partial}_2 \partial_x \omega_1^2 + \overrightarrow{\partial}_3 \partial_x \omega_1^3)
+ \overrightarrow{\partial}_2 \partial_x \omega_1^2 + \overrightarrow{\partial}_3 \partial_x \omega_1^3)
+ \overrightarrow{\partial}_3 \partial_x \omega_1^3 + \overrightarrow{\partial}_2 \partial_x \omega_3^2 + \overrightarrow{\partial}_3 \partial_x \omega_3^3)
+ \int_\Omega (\overrightarrow{\partial}_1 \partial_x \omega_2^1 + \overrightarrow{\partial}_2 \partial_x \omega_2^2 + \overrightarrow{\partial}_3 \partial_x \omega_2^3)
+ \overrightarrow{\partial}_2 \partial_x \omega_2^2 + \overrightarrow{\partial}_3 \partial_x \omega_2^3)
+ \overrightarrow{\partial}_3 \partial_x \omega_2^3 + \overrightarrow{\partial}_2 \partial_x \omega_3^2 + \overrightarrow{\partial}_3 \partial_x \omega_3^3)
+ \overrightarrow{\partial}_3 \partial_x \omega_3^3)
= 0
\]

it follows that

\[
\partial_t \int_\Omega (\omega_1^2 + \omega_2^2 + \omega_3^2) + 2 \int_\Omega ((\partial_x \omega_1)^2 + (\partial_x \omega_2)^2 + (\partial_x \omega_3)^2) + (\partial_x \omega_2)^2 + (\partial_x \omega_3)^2 + (\partial_x \omega_1)^2
+ (\partial_x \omega_2)^2 + (\partial_x \omega_3)^2 + (\partial_x \omega_1)^2
\]

\[
\leq 2 \int_\Omega (\overrightarrow{\partial}_1 \partial_x \omega_1^2 + \overrightarrow{\partial}_2 \partial_x \omega_2^2 + \overrightarrow{\partial}_3 \partial_x \omega_3^2)
+ \overrightarrow{\partial}_2 \partial_x \omega_1^2 + \overrightarrow{\partial}_3 \partial_x \omega_2^2 + \overrightarrow{\partial}_3 \partial_x \omega_3^2)
+ \overrightarrow{\partial}_3 \partial_x \omega_1^2 + \overrightarrow{\partial}_2 \partial_x \omega_2^2 + \overrightarrow{\partial}_3 \partial_x \omega_3^2)
\]

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by using Young inequality: \( uv \leq \frac{1}{4}u^2 + v^2 \).

Thus,

\[
\begin{align*}
&\int_\Omega (\dot{\omega}^1_t + \dot{\omega}^2_t + \dot{\omega}^3_t) + \int_{t_{k-1}}^t \int_\Omega ((\partial_x \dot{\omega}^1_1)^2 + (\partial_x \dot{\omega}^1_2)^2 + (\partial_x \dot{\omega}^1_3)^2 \\
&\quad + (\partial_x \dot{\omega}^2_1)^2 + (\partial_x \dot{\omega}^2_2)^2 + (\partial_x \dot{\omega}^2_3)^2 \\
&\quad + (\partial_x \dot{\omega}^3_1)^2 + (\partial_x \dot{\omega}^3_2)^2 + (\partial_x \dot{\omega}^3_3)^2) \\
&\quad + \int_\Omega ((\partial_x \dot{\omega}^1_1)^2 + (\partial_x \dot{\omega}^1_2)^2 + (\partial_x \dot{\omega}^1_3)^2 \\
&\quad + (\partial_x \dot{\omega}^2_1)^2 + (\partial_x \dot{\omega}^2_2)^2 + (\partial_x \dot{\omega}^2_3)^2 \\
&\quad + (\partial_x \dot{\omega}^3_1)^2 + (\partial_x \dot{\omega}^3_2)^2 + (\partial_x \dot{\omega}^3_3)^2)
\leq \int_\Omega (\hat{\omega}_1^k \cdot \hat{\omega}_2^k \cdot \hat{\omega}_3^k) + \\
&\quad + 4 \int_{t_{k-1}}^t \left( \|\ddot{\mathbf{f}}_1\|_{L^\infty(\Omega)}^2 + \|\ddot{\mathbf{f}}_2\|_{L^\infty(\Omega)}^2 + \|\ddot{\mathbf{f}}_3\|_{L^\infty(\Omega)}^2 \right) \int_\Omega (\hat{\omega}_1^k \cdot \hat{\omega}_2^k \cdot \hat{\omega}_3^k)
\leq \int_\Omega (\hat{\omega}_1^k \cdot \hat{\omega}_2^k \cdot \hat{\omega}_3^k) + \\
&\quad + 4 \Delta t_k \left( \|\ddot{\mathbf{f}}_1\|_{L^\infty(\Omega)}^2 + \|\ddot{\mathbf{f}}_2\|_{L^\infty(\Omega)}^2 + \|\ddot{\mathbf{f}}_3\|_{L^\infty(\Omega)}^2 \right) \left( \|\ddot{\mathbf{f}}_1\|_{L^2(\Omega)}^2 + \|\ddot{\mathbf{f}}_2\|_{L^2(\Omega)}^2 + \|\ddot{\mathbf{f}}_3\|_{L^2(\Omega)}^2 \right)
\end{align*}
\]

(6)

Note that

\[
\|\dddot{\mathbf{f}}_i\|_{L^2(\Omega)}^2 = \int_\Omega \left( \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \dot{\omega}_i(x, t) dt \right)^2 \leq \frac{1}{\Delta t_k^2} \int_\Omega \Delta t_k \int_{t_{k-1}}^{t_k} \dot{\omega}_i(x, t) dt
\]

and similarly

\[
\|\dddot{\mathbf{f}}_i\|_{L^2(\Omega)}^2 \leq \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \|u_i\|_{L^2(\Omega)}^2, \quad i = 1, 2, 3
\]

In addition, a convolution inequality in [1] is applied to get

\[
\|\dddot{\mathbf{f}}_i\|_{L^\infty(\Omega)} = \|J \ast \dddot{\mathbf{f}}_i(x)\|_{L^\infty(\Omega)} \leq \|J\|_{L^1(B\varepsilon)} \|\dddot{\mathbf{f}}_i\|_{L^2(\Omega)} \leq \frac{1}{\mu \varepsilon} \sup_{(t_{k-1}, t_k)} \|u_i\|_{L^2(\Omega)}
\]

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where \( B_\varepsilon = \{ x : |x| < \varepsilon \} \) and \( \| \mathbf{u}_k \|_{L^\infty(\Omega)} = \| \mathbf{u}_k \|_{L^\infty(\mathbb{R}^2)} \), the quantity \( \mu_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). We still need further assuming that \( \varepsilon \to 0 \) and \( \frac{\Delta t_k}{\mu_k} \to 0 \) as \( k \to \infty \) or \( \Delta t_k \to 0 \).

From (3) we have

\[
\int_\Omega (\tilde{\omega}^2_1 + \tilde{\omega}^2_2 + \tilde{\omega}^2_3) + \int_{t_{k-1}}^t \left( \| \nabla \tilde{\omega}_1 \|^2_{L^2(\Omega)} + \| \nabla \tilde{\omega}_2 \|^2_{L^2(\Omega)} + \| \nabla \tilde{\omega}_3 \|^2_{L^2(\Omega)} \right) \\
\leq \int_\Omega (\tilde{\omega}^k_{1-1} + \tilde{\omega}^k_{2-1} + \tilde{\omega}^k_{3-1}) + \\
+ \frac{4 \Delta t_k}{\mu_\varepsilon} \sup_{(t_{k-1}, t_k)} \left\{ \| u_1 \|^2_{L^2(\Omega)} + \| u_2 \|^2_{L^2(\Omega)} + \| u_3 \|^2_{L^2(\Omega)} \right\} \cdot \sup_{(t_{k-1}, t)} \int_\Omega (\tilde{\omega}^2_1 + \tilde{\omega}^2_2 + \tilde{\omega}^2_3)
\]

By (3) we have

\[
\sup_{(t_{k-1}, t_k)} \left\{ \| u_1 \|^2_{L^2(\Omega)} + \| u_2 \|^2_{L^2(\Omega)} + \| u_3 \|^2_{L^2(\Omega)} \right\}
\]

\[
\leq K_0 = \sup_{t \in (0, T)} \int_\Omega (u_1^2 + u_2^2 + u_3^2) + \int_0^T \left( \| \nabla u_1 \|^2_{L^2(\Omega)} + \| \nabla u_2 \|^2_{L^2(\Omega)} + \| \nabla u_3 \|^2_{L^2(\Omega)} \right) < +\infty
\]

Thus,

\[
\left( 1 - 4K_0 \frac{\Delta t_k}{\mu_\varepsilon} \right) \sup_{t \in (t_{k-1}, t_k)} \int_\Omega (\tilde{\omega}^2_1 + \tilde{\omega}^2_2 + \tilde{\omega}^2_3) + \\
+ \int_{t_{k-1}}^{t_k} \left( \| \nabla \tilde{\omega}_1 \|^2_{L^2(\Omega)} + \| \nabla \tilde{\omega}_2 \|^2_{L^2(\Omega)} + \| \nabla \tilde{\omega}_3 \|^2_{L^2(\Omega)} \right) \leq \int_\Omega (\tilde{\omega}^k_{1-1} + \tilde{\omega}^k_{2-1} + \tilde{\omega}^k_{3-1})
\]

Now we set

\[
M_0 = \int_\Omega (\omega^2_{10} + \omega^2_{20} + \omega^2_{30}) \\
M_k = \sup_{t \in (t_{k-1}, t_k)} \int_\Omega (\tilde{\omega}^2_1 + \tilde{\omega}^2_2 + \tilde{\omega}^2_3) \\
\delta_k = \int_{t_{k-1}}^{t_k} \left( \| \nabla \tilde{\omega}_1 \|^2_{L^2(\Omega)} + \| \nabla \tilde{\omega}_2 \|^2_{L^2(\Omega)} + \| \nabla \tilde{\omega}_3 \|^2_{L^2(\Omega)} \right)
\]

\[
k = 1, \ldots, N
\]

then we have

\[
\left( 1 - 4K_0 \frac{\Delta t_k}{\mu_\varepsilon} \right) M_k + \delta_k \leq M_{k-1}
\]

The partition is assumed to be fine enough. Because of the local existence of Galerkin solution in section 3 and the absolute continuity of integration with
respect to $t$, it is valid that $\delta_k \to 0$ as $\Delta t_k \to 0$.

We may first consider the case that

$$M_{k-1} \frac{\Delta t_k}{\delta_k} \to 0, \text{ as } \Delta t_k \to 0$$

which may be a subsequence $k'$, still denoted $k$. At this time, we can choose $\varepsilon_k$ on each $(t_{k-1}, t_k)$ such that

$$\mu_{\varepsilon_k} = 4K_0 M_{k-1} \frac{\Delta t_k}{\delta_k} \text{ and } 1 - 4K_0 \frac{\Delta t_k}{\mu_{\varepsilon_k}} \geq \frac{1}{2}$$

$$\varepsilon = \max_k \{ \varepsilon_k \}.$$

Then we obtain

$$\left( 1 - 4K_0 \frac{\Delta t_k}{\mu_{\varepsilon_k}} \right) M_k + \delta_k = \left( 1 - \frac{\delta_k}{M_{k-1}} \right) M_k + \delta_k \leq M_{k-1}$$

it follows that $M_k \leq M_{k-1}$.

Otherwise, $\delta_k \leq O(\Delta t_k)M_{k-1}$. This leads to that

$$\left\{ \| \nabla \omega_1 \|^2_{L^2(\Omega)} + \| \nabla \omega_2 \|^2_{L^2(\Omega)} + \| \nabla \omega_3 \|^2_{L^2(\Omega)} \right\}$$

$$\leq \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \left( \| \nabla \omega_1 \|^2_{L^2(\Omega)} + \| \nabla \omega_2 \|^2_{L^2(\Omega)} + \| \nabla \omega_3 \|^2_{L^2(\Omega)} \right) \leq O(1)M_{k-1}$$

Since these $(t_{k-1}, t_k)$ are of finite length, the number of them is finite. According to Cauchy-Schwartz inequality, similar to (6), we have

$$\partial_t \int_{\Omega} (\hat{\omega}^2_1 + \hat{\omega}^2_2 + \hat{\omega}^2_3) + \int_{\Omega} \left[ (\partial_{x_1} \hat{\omega}_1)^2 + (\partial_{x_2} \hat{\omega}_1)^2 + (\partial_{x_3} \hat{\omega}_1)^2 + (\partial_{x_2} \hat{\omega}_2)^2 + (\partial_{x_3} \hat{\omega}_2)^2 + (\partial_{x_1} \hat{\omega}_3)^2 + (\partial_{x_2} \hat{\omega}_3)^2 + (\partial_{x_3} \hat{\omega}_3)^2 \right]$$

$$\leq 4 \left\{ \left( \int_{\Omega} \hat{\omega}_1^4 \right)^{\frac{1}{2}} \left( \int \omega_1^4 \right)^{\frac{1}{2}} + \left( \int_{\Omega} \hat{\omega}_2^4 \right)^{\frac{1}{2}} \left( \int \omega_2^4 \right)^{\frac{1}{2}} + \left( \int_{\Omega} \hat{\omega}_3^4 \right)^{\frac{1}{2}} \left( \int \omega_3^4 \right)^{\frac{1}{2}} \right\}$$

$$= 4 \left\{ \| \omega_1 \|^2_{L^4(\Omega)} \left( \| \omega_1 \|^2_{L^2(\Omega)} + \| \omega_2 \|^2_{L^2(\Omega)} + \| \omega_3 \|^2_{L^2(\Omega)} \right) \right\}$$

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Therefore, independent of \( \omega \) such that

\[
\| \mathbf{u}^i \|_{L^4(\Omega)}^2 \leq C_1 \left( \| \mathbf{u}^i \|_{L^2(\Omega)}^2 + \| \nabla \mathbf{u}^i \|_{L^2(\Omega)}^2 \right), \quad i = 1, 2, 3
\]

Therefore,

\[
\int_{\Omega} \left( \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2 \right) + \int_{t_{k-1}}^t \left( \| \nabla \tilde{\omega}_1 \|_{L^2(\Omega)}^2 + \| \nabla \tilde{\omega}_2 \|_{L^2(\Omega)}^2 + \| \nabla \tilde{\omega}_3 \|_{L^2(\Omega)}^2 \right) \\
\leq \int_{\Omega} \left( \tilde{\omega}_1^{k-12} + \tilde{\omega}_2^{k-12} + \tilde{\omega}_3^{k-12} \right) + \\
+ C_2 \left( \| \mathbf{n}^1 \|_{L^2(\Omega)}^2 + \| \mathbf{n}^2 \|_{L^2(\Omega)}^2 + \| \mathbf{n}^3 \|_{L^2(\Omega)}^2 \right) + \\
\times \left( \| \nabla \mathbf{n}^1 \|_{L^2(\Omega)}^2 + \| \nabla \mathbf{n}^2 \|_{L^2(\Omega)}^2 + \| \nabla \mathbf{n}^3 \|_{L^2(\Omega)}^2 \right) \\
\times \left( \| \nabla \mathbf{n}^1 \|_{L^2(\Omega)}^2 + \| \nabla \mathbf{n}^2 \|_{L^2(\Omega)}^2 + \| \nabla \mathbf{n}^3 \|_{L^2(\Omega)}^2 \right)
\]

A convolution inequality in [1] is applied to get

\[
\| \mathbf{u}_i \|_{L^2(\Omega)}^2 \leq \| J \cdot \mathbf{n}_i \|_{L^2(\mathbb{R}^3)} \leq \| \nabla \mathbf{u}_i \|_{L^2(\mathbb{R}^3)} \| \mathbf{n}_i \|_{L^2(\Omega)}^2 \\
= \| \mathbf{u}_i \|_{L^2(\Omega)}^2 \leq \sup_{(t_{k-1}, t_k)} \| \mathbf{n}_i \|_{L^2(\Omega)}^2
\]

Thus we have

\[
\int_{\Omega} \left( \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2 \right) + \int_{t_{k-1}}^t \left( \| \nabla \tilde{\omega}_1 \|_{L^2(\Omega)}^2 + \| \nabla \tilde{\omega}_2 \|_{L^2(\Omega)}^2 + \| \nabla \tilde{\omega}_3 \|_{L^2(\Omega)}^2 \right) \\
\leq \int_{\Omega} \left( \tilde{\omega}_1^{k-12} + \tilde{\omega}_2^{k-12} + \tilde{\omega}_3^{k-12} \right) + \\
+ C_2 \left( \sup_{(t_{k-1}, t_k)} \{ \| u_1 \|_{L^2(\Omega)}^2 + \| u_2 \|_{L^2(\Omega)}^2 + \| u_3 \|_{L^2(\Omega)}^2 \} \right) + \\
\times \left( \sup_{(t_{k-1}, t_k)} \{ \| u_1 \|_{L^2(\Omega)}^2 + \| u_2 \|_{L^2(\Omega)}^2 + \| u_3 \|_{L^2(\Omega)}^2 \} \right)
\]
\[ 
\times \int_{t_{k-1}}^{t} \int_{\Omega} \left( \dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2 \right) + \\
+ C_3 \left( \Delta t_k \sup_{(t_{k-1}, t_k)} \left\{ \| u_1 \|_{L^2(\Omega)}^2 + \| u_2 \|_{L^2(\Omega)}^2 + \| u_3 \|_{L^2(\Omega)}^2 \right\} + \\
+ \int_{t_{k-1}}^{t_k} \left\{ \| \nabla u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_2 \|_{L^2(\Omega)}^2 + \| \nabla u_3 \|_{L^2(\Omega)}^2 \right\} \right) M_{k-1}
\]

where \( C_2, C_3 > 0 \) are constants independent of \( k \). Set

\[ K_k^* = \Delta t_k \sup_{(t_{k-1}, t_k)} \left\{ \| u_1 \|_{L^2(\Omega)}^2 + \| u_2 \|_{L^2(\Omega)}^2 + \| u_3 \|_{L^2(\Omega)}^2 \right\} + \\
+ \int_{t_{k-1}}^{t_k} \left\{ \| \nabla u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_2 \|_{L^2(\Omega)}^2 + \| \nabla u_3 \|_{L^2(\Omega)}^2 \right\}
\]

and

\[ f_k(t) = \sup_{(t_{k-1}, t)} \int_{\Omega} \left( \dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2 \right) \]

Then we arrive at

\[ f_k(t) \leq M_{k-1} + C_2 \frac{1}{\Delta t_k} K_k^* \int_{t_{k-1}}^{t} f_k(t) + C_3 M_{k-1} \]

By using Gronwall inequality it follows that

\[ M_k \leq (1 + C_3 K_k^*) \exp \left( C_2 K_k^* \right) M_{k-1} \]

Note that

\[ \sum_{k=1}^{N} K_k^* \leq T \sup_{t \in (0, T)} \int_{\Omega} (u_1^2 + u_2^2 + u_3^2) + \int_{0}^{T} \left( \| \nabla u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_2 \|_{L^2(\Omega)}^2 + \| \nabla u_3 \|_{L^2(\Omega)}^2 \right) \]

\[ \leq (T + 1) K_0 < +\infty \]

Hence, combining above two cases, we obtain

\[ M_1 \leq (1 + C_3 K_1^*) \exp \left( C_2 K_1^* \right) M_0 \]

\[ M_2 \leq (1 + C_3 K_1^*) (1 + C_3 K_2^*) \exp \left( C_2 \sum_{k=1}^{2} K_k^* \right) M_0 \]

\[ \ldots \]

\[ M_N \leq \prod_{k=1}^{N} (1 + C_3 K_k^*) \exp \left( C_2 \sum_{k=1}^{N} K_k^* \right) M_0 \]
Note that
\[
\prod_{k=1}^{N} (1 + C_3 K_k^*) = \exp \left( \ln \prod_{k=1}^{N} (1 + C_3 K_k^*) \right)
\]
\[
= \exp \left( \sum_{k=1}^{N} \ln(1 + C_3 K_k^*) \right) \leq \exp \left( C_3 \sum_{k=1}^{N} K_k^* \right) = \exp(C_3(T + 1)K_0)
\]

These mean that
\[
M_k \leq M_0 \exp \left( (C_2 + C_3) (T + 1)K_0 \right) \quad k = 1, \cdots, N
\]

Finally we get
\[
\sup_{t \in (0,T)} \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) \leq \max_k \{M_k\}
\]
\[
\leq M_0 \exp \left( (C_2 + C_3) (T + 1)K_0 \right)
\]

This conclusion is also true for the weak solution of problem (5) by means of the result of section 3 and the lower limit of Galerkin sequence according to the page 196 of [4].

3. Existence

In this section we have to consider the existence of solutions of the auxiliary problems. We just need to consider the following system on \((0, \delta)\):

\[
\begin{align*}
\partial_t \tilde{\omega}_1 + \overline{\omega}_1 \partial_x, \overline{w}_1 + \overline{\omega}_2 \partial_x, \overline{w}_2 + \overline{\omega}_3 \partial_x, \overline{w}_3 - \overline{\omega}_1 \partial_x, \overline{u}_1 - \overline{\omega}_2 \partial_x, \overline{u}_2 - \overline{\omega}_3 \partial_x, \overline{u}_3 + \partial_x q &= \Delta \tilde{\omega}_1 \\
\partial_t \tilde{\omega}_2 + \overline{\omega}_1 \partial_x, \overline{w}_2 + \overline{\omega}_2 \partial_x, \overline{w}_2 + \overline{\omega}_3 \partial_x, \overline{w}_3 - \overline{\omega}_1 \partial_x, \overline{u}_2 - \overline{\omega}_2 \partial_x, \overline{u}_2 - \overline{\omega}_3 \partial_x, \overline{u}_3 + \partial_x q &= \Delta \tilde{\omega}_2 \\
\partial_t \tilde{\omega}_3 + \overline{\omega}_1 \partial_x, \overline{w}_3 + \overline{\omega}_2 \partial_x, \overline{w}_3 + \overline{\omega}_3 \partial_x, \overline{w}_3 - \overline{\omega}_1 \partial_x, \overline{u}_3 - \overline{\omega}_2 \partial_x, \overline{u}_3 - \overline{\omega}_3 \partial_x, \overline{u}_3 + \partial_x q &= \Delta \tilde{\omega}_3
\end{align*}
\]

with initial value \(\tilde{\omega}_i(x,0) = \omega_{i0}(i = 1, 2, 3)\) and
\[
\overline{w}_i(x) = \frac{1}{\delta} \int_0^{\delta} \tilde{\omega}_i(x,t)dt
\]
and
\[
\overline{u}_i(x) = \frac{1}{\delta} \int_0^{\delta} u_i(x,t)dt, \quad \overline{u}_i(x) = J_{x} * \overline{w}_i(x), \quad i = 1, 2, 3
\]
as well as the incompressible conditions:

\[
\begin{align*}
\partial_x \tilde{\omega}_1 + \partial_x \tilde{\omega}_2 + \partial_x \tilde{\omega}_3 &= 0 \quad \Rightarrow \quad \partial_x \overline{w}_1 + \partial_x \overline{w}_2 + \partial_x \overline{w}_3 = 0 \\
\partial_x u_1 + \partial_x u_2 + \partial_x u_3 &= 0 \quad \Rightarrow \quad \partial_x \overline{u}_1 + \partial_x \overline{u}_2 + \partial_x \overline{u}_3 = 0
\end{align*}
\]
(i) The Galerkin procedure is applied. For each $m$ and $i = 1, 2, 3$ we define an approximate solution $(\tilde{\omega}_{1m}, \tilde{\omega}_{2m}, \tilde{\omega}_{3m})$ as follows:

$$\tilde{\omega}_{im} = \sum_{j=1}^{m} g_{ij}(t)w_{ij}$$

where $\{w_{i1}, \cdots, w_{im}, \cdots\}$ is the basis of $W$, and $W$ = the closure of $\mathcal{Y}$ in the Sobolev space $W^{2,1}(\Omega)$, which is separable and is dense in $V$. Thus

$$(\partial_t \tilde{\omega}_{im}, w_{il}) + (\nabla \tilde{\omega}_{im}, \nabla w_{il}) + ((\mathbf{\Pi} \cdot \nabla) \tilde{\omega}_{im}, w_{il}) - ((\mathbf{\Psi}_m \cdot \nabla) \tilde{\mathbf{r}}, w_{il}) = 0 \quad (8)$$

$t \in (0, \delta), \quad l = 1, \cdots, m$

$$\tilde{\omega}_{im}(0) = \omega_{i0}^m$$

where $\omega_{i0}^m$ is the orthogonal projection in $H$ of $\omega_{i0}$ onto the space spanned by $w_{i1}, \cdots, w_{im}$. Therefore,

$$\sum_{j=1}^{m} (w_{ij}, w_{il}) g'_{ij}(t) + \sum_{j=1}^{m} (\nabla w_{ij}, \nabla w_{il}) g_{ij}(t) + \sum_{j=1}^{m} \{((\mathbf{\Pi}(t) \cdot \nabla) w_{ij}, w_{il}) - ((w_{ij} \cdot \nabla) w_{il}, \mathbf{\Pi}_i(t))\} \bar{g}_{ij}(t) = 0$$

where $\bar{g}_{ij}(t) = \frac{1}{\delta} \int_{t}^{\delta} g_{ij}(t) dt$, and $u_i \in L^\infty(0, T; H)$ from section 1 which are determined by equations (1). Inverting the nonsingular matrix with elements $(w_{ij}, w_{il}), \ 1 \leq j, l \leq m$, we can write above system in the following form

$$g'_{ij}(t) + \sum_{l=1}^{m} \alpha_{ijl} g_{il}(t) + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}(t) = 0 \quad (9)$$

where $\alpha_{ijl}, \beta_{ijl}$ are constants.

The initial conditions are equivalent to

$$g_{ij}(0) = g_{ij}^{0} = \text{the } j^{th} \text{ component of } \omega_{i0}^{m}$$

We construct a sequence $\{g_{ij}^k\}$ by using a successive approximation:

$$g_{ij}^{1} = -\sum_{l=1}^{m} \alpha_{ijl} g_{il}^{0} - \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{0} \Rightarrow g_{ij}^{1} = g_{ij}^{0} - \int_{0}^{t} \left( \sum_{l=1}^{m} \alpha_{ijl} g_{il}^{0} + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{0} \right)$$

$$g_{ij}^{2} = -\sum_{l=1}^{m} \alpha_{ijl} g_{il}^{1} - \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{1} \Rightarrow g_{ij}^{2} = g_{ij}^{0} - \int_{0}^{t} \left( \sum_{l=1}^{m} \alpha_{ijl} g_{il}^{1} + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{1} \right)$$

$$\cdots$$

$$g_{ij}^{k} = -\sum_{l=1}^{m} \alpha_{ijl} g_{il}^{k-1} - \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{k-1} \Rightarrow g_{ij}^{k} = g_{ij}^{0} - \int_{0}^{t} \left( \sum_{l=1}^{m} \alpha_{ijl} g_{il}^{k-1} + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{k-1} \right)$$

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so that

\[ |g_{ij}^k(t) - g_{ij}^{k-1}(t)| \leq \int_0^t \left( \sum_{l=1}^m |\alpha_{ijl}| |g_{il}^{k-1}(t) - g_{il}^{k-2}(t)| + \sum_{l=1}^m |\beta_{ijl}| |g_{il}^{k-1}(t) - g_{il}^{k-2}(t)| \right) \]

it follows that

\[
\max_{i,j} \sup_t |g_{ij}^k(t) - g_{ij}^{k-1}(t)| \leq \max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + |\beta_{ijl}|) \cdot t \cdot \max_{i,j} \sup_t |g_{ij}^{k-1}(t) - g_{ij}^{k-2}(t)|
\]

Taking \( \delta = \frac{1}{\max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + 2|\beta_{ijl}|)} \), as \( t \leq \delta \), then choosing \( \delta^* \):

\[
0 < \delta^* = \frac{\max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + |\beta_{ijl}|)}{\max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + 2|\beta_{ijl}|)} < 1
\]

we have

\[
\max_{i,j} \|g_{ij}^k - g_{ij}^{k-1}\|_\infty \leq \delta^* \cdot \max_{i,j} \|g_{ij}^{k-1} - g_{ij}^{k-2}\|_\infty \leq (\delta^*)^{k-1} \cdot \max_{i,j} \|g_{ij}^1 - g_{ij}^0\|_\infty
\]

For any \( n, k \) (we can set \( n > k \) without loss of generality), we get

\[
\max_{i,j} \|g_{ij}^n - g_{ij}^k\|_\infty \leq \max_{i,j} \|g_{ij}^n - g_{ij}^{n-1}\|_\infty + \cdots + \max_{i,j} \|g_{ij}^{k+1} - g_{ij}^k\|_\infty
\]

\[
\leq ((\delta^*)^{n-1} + \cdots + (\delta^*)^k) \cdot \max_{i,j} \|g_{ij}^1 - g_{ij}^0\|_\infty = (\delta^*)^{k-1} \cdot \max_{i,j} \|g_{ij}^1 - g_{ij}^0\|_\infty
\]

\[
\to 0 \quad (k \to \infty)
\]

Thus, for every \( i = 1, 2, 3; \quad j = 1, \cdots, m \), \( \{g_{ij}^k\} \) is a Cauchy sequence in \( L^\infty(0, \delta) \). Since \( L^\infty(0, \delta) \) is complete, then there exists a function \( g_{ij}^* \in L^\infty(0, \delta) \) such that

\[
\|g_{ij}^k - g_{ij}^*\|_\infty \to 0 \quad \text{as} \quad k \to \infty.
\]

From

\[
g_{ij}^k(t) = g_{ij}^0 - \int_0^t \left( \sum_{l=1}^m \alpha_{ijl}g_{il}^{k-1}(t) + \sum_{l=1}^m \beta_{ijl}g_{il}^{k-1}(t) \right)
\]

let \( k \to \infty \), it follows that

\[
g_{ij}^*(t) = g_{ij}^0 - \int_0^t \left( \sum_{l=1}^m \alpha_{ijl}g_{il}^0(t) + \sum_{l=1}^m \beta_{ijl}g_{il}^0(t) \right)
\]

i.e., \( g_{ij}^* \) is a solution of the system (9) on \( (0, \delta) \) for which \( g_{ij}^*(0) = g_{ij}^0, \quad i = 1, 2, 3; \quad j = 1, \cdots, m \).
(ii) 
\[
\sum_{i=1}^{3} (\partial_t \tilde{\omega}_{im}, \tilde{\omega}_{im}) + \sum_{i=1}^{3} (\nabla \tilde{\omega}_{im}, \nabla \tilde{\omega}_{im}) + \sum_{i=1}^{3} ((\overline{\mathbf{v}} \cdot \nabla) \overline{\mathbf{v}}, \tilde{\omega}_{im}) - \sum_{i=1}^{3} ((\overline{\mathbf{v}'} \cdot \nabla) \overline{\mathbf{v}'}, \tilde{\omega}_{im}) = 0
\]

Then we write
\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{i=1}^{3} \| \tilde{\omega}_{im} \|_{L^2(\Omega)}^2 \right) + \sum_{i=1}^{3} \| \nabla \tilde{\omega}_{im} \|_{L^2(\Omega)}^2 - \sum_{i=1}^{3} ((\overline{\mathbf{v}} \cdot \nabla) \overline{\mathbf{v}}, \tilde{\omega}_{im}) + \sum_{i=1}^{3} ((\overline{\mathbf{v}'} \cdot \nabla) \overline{\mathbf{v}'}, \tilde{\omega}_{im}) = 0
\]

Similar to those in the section 2, and \( \eta \) is chosen to be small enough, we have
\[
\sum_{i=1}^{3} \| \tilde{\omega}_{im} \|_{L^2(\Omega)}^2 + \int_0^\eta \left( \sum_{i=1}^{3} \| \nabla \tilde{\omega}_{im} \|_{L^2(\Omega)}^2 \right) \leq 2 \left( \sum_{i=1}^{3} \| \omega_{i0} \|_{L^2(\Omega)}^2 \right)
\]
as \( 1 - 4 K_0 \eta/\mu \geq 1/2 \). Hence
\[
\sup_{t \in (0, \eta)} \left( \sum_{i=1}^{3} \| \tilde{\omega}_{im} \|_{L^2(\Omega)}^2 \right) \leq 2 \left( \sum_{i=1}^{3} \| \omega_{i0} \|_{L^2(\Omega)}^2 \right)
\]
and
\[
\sum_{i=1}^{3} \| \tilde{\omega}_{im}(\eta) \|_{L^2(\Omega)}^2 + \int_0^\eta \sum_{i=1}^{3} \| \nabla \tilde{\omega}_{im} \|_{L^2(\Omega)}^2 \leq 2 \left( \sum_{i=1}^{3} \| \omega_{i0} \|_{L^2(\Omega)}^2 \right)
\]
The inequalities (10) and (11) are valid for any fixed \( \delta \leq \eta \).

(iii) Let \( \tilde{\omega}_m \) denote the function from \( \mathbb{R} \) into \( V \), which is equal to \( \tilde{\omega}_m \) on \((0, \delta)\) and to 0 on the complement of this interval. The Fourier transform of \( \tilde{\omega}_m \) is denoted by \( \tilde{\omega}_m^\wedge \). We want to show that
\[
\int_{-\infty}^{+\infty} |\tau|^2 \gamma \left( \sum_{i=1}^{3} \| \tilde{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 \right) d\tau < +\infty
\]
for some \( \gamma > 0 \). Along with (11) this will imply that
\[
\tilde{\omega}_m \text{ belongs to a bounded set of } H^\gamma(\mathbb{R}, V, H)
\]
and will enable us to apply the result of compactness.

We observe that \( \mathbf{K} \) can be written as
\[
\frac{d}{dt} \left( \sum_{i=1}^{3} (\tilde{\omega}_{im}, w_{ij}) \right) = \sum_{i=1}^{3} (f_{im}, w_{ij}) + \sum_{i=1}^{3} (\omega_{i0}^m, w_{ij}) \eta_0 - \sum_{i=1}^{3} (\tilde{\omega}_{im}(\delta), w_{ij}) \eta_0
\]
where $\eta_0$, $\eta_\delta$ are Dirac distributions at 0 and $\delta$, and
\[
f_{im} = -\Delta \tilde{\omega}_{im} + (\overline{\omega} \cdot \nabla)\omega_{im} - (\omega_m \cdot \nabla \overline{\omega}_i)
\]
\[
f^0_{im} = f_{im} \text{ on } (0, \delta), \quad 0 \text{ outside this interval}
\]

By the Fourier transform,
\[
2\pi \tau \sum_{i=1}^{3} (\hat{\omega}_{im}, w_{ij}) = \sum_{i=1}^{3} (\hat{f}_{im}, w_{ij}) + \sum_{i=1}^{3} (\omega_m^{\delta}, w_{ij}) - \sum_{i=1}^{3} (\hat{\omega}_{im}(\delta), w_{ij}) \exp(-2\pi \delta \tau)
\]
where $\hat{\omega}_{im}$ and $\hat{f}_{im}$ denoting the Fourier transforms of $\hat{\omega}_{im}$ and $f_{im}$ respectively.

We multiply above equality by $\hat{g}_{ij}(\tau) = \text{Fourier transform of } g_{ij}$ and add the resulting equation for $j = 1, \cdots, m$, we get
\[
2\pi \tau \sum_{i=1}^{3} \|\hat{\omega}_{im}(\tau)\|^2_{L^2(\Omega)} = \sum_{i=1}^{3} (\hat{f}_{im}(\tau), \hat{\omega}_{im}(\tau)) + \sum_{i=1}^{3} (\omega_m^{\delta}, \hat{\omega}_{im}(\tau)) - \sum_{i=1}^{3} (\hat{\omega}_{im}(\delta), \hat{\omega}_{im}(\tau)) \exp(-2\pi \delta \tau)
\]

For some $\varphi_i \in V$,
\[
\int_{0}^{\delta} \sum_{i=1}^{3} (f_{im}, \varphi_i) = \int_{0}^{\delta} \sum_{i=1}^{3} (-\Delta \tilde{\omega}_{im}, \varphi_i) + \int_{0}^{\delta} \sum_{i=1}^{3} ((\overline{\omega} \cdot \nabla)\omega_{im}, \varphi_i) - \int_{0}^{\delta} \sum_{i=1}^{3} ((\omega_m \cdot \nabla)\overline{\omega}_i, \varphi_i)
\]
\[
= \int_{0}^{\delta} \sum_{i=1}^{3} (\nabla \tilde{\omega}_{im}, \nabla \varphi_i) - \int_{0}^{\delta} \sum_{i=1}^{3} ((\overline{\omega} \cdot \nabla)\omega_{im}, \varphi_i) + \int_{0}^{\delta} \sum_{i=1}^{3} ((\omega_m \cdot \nabla)\overline{\omega}_i, \varphi_i)
\]
\[
\leq \int_{0}^{\delta} \sum_{i=1}^{3} \|\nabla \tilde{\omega}_{im}\|_{L^2(\Omega)} \|\nabla \varphi_i\|_{L^2(\Omega)} +
\]
\[
+ 2 \int_{0}^{\delta} \left( \sum_{i=1}^{3} \|\overline{\omega}_i\|_{L^2(\Omega)} \right)^{1/2} \left( \sum_{i=1}^{3} \|\omega_{im}\|_{L^2(\Omega)} \right)^{1/2} \left( \sum_{i=1}^{3} \|\nabla \varphi_i\|_{L^2(\Omega)} \right)^{1/2}
\]
\[
\leq \int_{0}^{\delta} \left( \sum_{i=1}^{3} \|\nabla \tilde{\omega}_{im}\|_{L^2(\Omega)} \right)^{1/2} \left( \sum_{i=1}^{3} \|\nabla \varphi_i\|_{L^2(\Omega)} \right)^{1/2} +
\]
\[
+ 2 \delta \left( \sum_{i=1}^{3} \|\overline{\omega}_i\|_{L^2(\Omega)} \right)^{1/2} \left( \sum_{i=1}^{3} \|\omega_{im}\|_{L^2(\Omega)} \right)^{1/2} \left( \sum_{i=1}^{3} \|\nabla \varphi_i\|_{L^2(\Omega)} \right)^{1/2}
\]
\[
\leq \int_{0}^{\delta} \left( \sum_{i=1}^{3} \|\nabla \tilde{\omega}_{im}\|_{L^2(\Omega)} \right)^{1/2} \|\nabla \varphi\|_{V} +
\]
\[
+ 2 \delta \left( \sup_{(0, \delta)} \sum_{i=1}^{3} \|u_i\|_{L^2(\Omega)} \right)^{1/2} \left( \sup_{(0, \delta)} \sum_{i=1}^{3} \|\tilde{\omega}_{im}\|_{L^2(\Omega)} \right)^{1/2} \|\nabla \varphi\|_{V}
\]
this remains bounded according to (3), (4), and (10), (11). Therefore
\[ \int_0^\delta \| \hat{f}_{im}(t) \|_V \, dt = \int_0^\delta \sup_{\| \varphi \|_V = 1} \sum_{i=1}^3 (f_{im}, \varphi_i) < +\infty \]
it follows that
\[ \sup_{\tau \in \mathbb{R}} \| \hat{f}_{im}(\tau) \|_V < +\infty, \quad \forall m \]
Due to (10), we have
\[ \| \hat{\omega}_{im}(0) \|_{L^2(\Omega)} < +\infty, \quad \| \hat{\omega}_{im}(\delta) \|_{L^2(\Omega)} < +\infty \]
then by Poincare inequality,
\[ |\tau| \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 \leq c_1 \sum_{i=1}^3 \| \hat{f}_{im}(\tau) \|_V \| \hat{\omega}_{im}(\tau) \|_V + c_2 \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)} \]
\[ \leq c_3 \left( \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 + \sum_{i=1}^3 \| \nabla \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)} \right) \]
(12)
For \( \gamma \) fixed, \( \gamma < 1/4 \), we observe that
\[ |\tau|^{2\gamma} \leq c_4(\gamma) \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}}, \quad \forall \tau \in \mathbb{R} \]
Thus by (12),
\[ \int_{-\infty}^{+\infty} |\tau|^{2\gamma} \left( \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 \right) \, d\tau \leq c_4(\gamma) \int_{-\infty}^{+\infty} \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}} \left( \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 \right) \, d\tau \]
\[ \leq c_5 \int_{-\infty}^{+\infty} \frac{1}{1 + |\tau|^{1-2\gamma}} \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)} \, d\tau \]
\[ + \ c_6 \int_{-\infty}^{+\infty} \frac{1}{1 + |\tau|^{1-2\gamma}} \sum_{i=1}^3 \| \nabla \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)} \, d\tau \]
\[ + \ c_7 \int_{-\infty}^{+\infty} \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 \, d\tau \]
Because of the Parseval equality,
\[ \int_{-\infty}^{+\infty} \sum_{i=1}^3 \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 \, d\tau = \int_0^\delta \sum_{i=1}^3 \| \hat{\omega}_{im}(t) \|_{L^2(\Omega)}^2 \, dt \]
\[ \leq \delta \sup_{(0,\delta)} \sum_{i=1}^3 \| \hat{\omega}_{im} \|_{L^2(\Omega)}^2 < +\infty \]
\[ \int_{-\infty}^{+\infty} \sum_{i=1}^3 \| \nabla \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)}^2 \, d\tau = \int_0^\delta \sum_{i=1}^3 \| \nabla \hat{\omega}_{im}(t) \|_{L^2(\Omega)}^2 \, dt < +\infty \]
as \( m \to \infty \). By Cauchy-Schwarz inequality and the Parseval equality,

\[
\int_{-\infty}^{+\infty} \frac{1}{1 + |\tau|^{-2\gamma}} \sum_{i=1}^{3} \| \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)} d\tau \\
\leq \left( \int_{-\infty}^{+\infty} \frac{1}{(1 + |\tau|^{-2\gamma})^2} \right)^{1/2} \left( \int_{0}^{\delta} \sum_{i=1}^{3} \| \hat{\omega}_{im}(t) \|_{L^2(\Omega)}^2 dt \right)^{1/2} < +\infty
\]

\[
\int_{-\infty}^{+\infty} \frac{1}{1 + |\tau|^{-2\gamma}} \sum_{i=1}^{3} \| \nabla \hat{\omega}_{im}(\tau) \|_{L^2(\Omega)} d\tau \\
\leq \left( \int_{-\infty}^{+\infty} \frac{1}{(1 + |\tau|^{-2\gamma})^2} \right)^{1/2} \left( \int_{0}^{\delta} \sum_{i=1}^{3} \| \nabla \hat{\omega}_{im}(t) \|_{L^2(\Omega)}^2 dt \right)^{1/2} < +\infty
\]

as \( m \to \infty \) by \( \gamma < 1/4 \) and (ii).

(iv) The estimate (10) and (ii) enable us to assert the existence of an element \( \tilde{\omega}^* \in L^2(0, \delta; V) \cap L^\infty(0, \delta; H) \) and a subsequence \( \tilde{\omega}_{m'} \) such that

\( \tilde{\omega}_{m'} \to \tilde{\omega}^* \) in \( L^2(0, \delta; V) \) weakly, and in \( L^\infty(0, \delta; H) \) weak-star, as \( m' \to \infty \).

Due to (iii) we also have

\( \tilde{\omega}_{m'} \to \tilde{\omega}^* \) in \( L^2(0, \delta; H) \) strongly as \( m' \to \infty \).

This convergence result enable us to pass to the limit.

Let \( \psi_i \) be a continuously differentiable function on \( (0, \delta) \) with \( \psi_i(\delta) = 0 \). We multiply (8) by \( \psi_i(t) \) then integrate by parts. This leads to the equation

\[
- \int_{0}^{\delta} \sum_{i=1}^{3} (\hat{\omega}_{im'}(t), \partial_t \psi_i(t) w_{ij}) dt + \int_{0}^{\delta} \sum_{i=1}^{3} (\nabla \hat{\omega}_{im'}, \psi_i(t) \nabla w_{ij}) dt \\
+ \int_{0}^{\delta} \sum_{i=1}^{3} ((\nabla \cdot \nabla) \omega_{im'}, w_{ij} \psi_i(t)) - \int_{0}^{\delta} \sum_{i=1}^{3} ((\nabla \omega_{im'}, \nabla) \omega_i, w_{ij} \psi_i(t)) = \sum_{i=1}^{3} (\omega_{m0}^i, w_{ij}) \psi_i(0)
\]

Since \( \hat{\omega}_{im'} \) converges to \( \tilde{\omega}^* \) in \( L^2(0, \delta; H) \) strongly as \( m' \to \infty \), then \( \overline{\omega}_{im'} \) also converges strongly to \( \overline{\omega}_i^* \), and

\[
\int_{0}^{\delta} \sum_{i=1}^{3} (\hat{\omega}_{im'}, \partial_t \psi_i(t) w_{ij}) dt \to \int_{0}^{\delta} \sum_{i=1}^{3} (\tilde{\omega}^*_i, \partial_t \psi_i(t) w_{ij}) dt \\
\int_{0}^{\delta} \sum_{i=1}^{3} (\nabla \hat{\omega}_{im'}, \psi_i(t) \nabla w_{ij}) dt = - \int_{0}^{\delta} \sum_{i=1}^{3} (\hat{\omega}_{im'}, \psi_i(t) \Delta w_{ij}) dt \\
\to - \int_{0}^{\delta} \sum_{i=1}^{3} (\tilde{\omega}^*_i, \psi_i(t) \Delta w_{ij}) = \int_{0}^{\delta} \sum_{i=1}^{3} (\nabla \tilde{\omega}_i^*, \psi_i(t) \nabla w_{ij}) dt
\]
\[
\int_0^\delta \sum_{i=1}^3 ((\overline{\nabla} \cdot \nabla) \varpi_{im'}, w_{ij} \psi_i(t)) = - \int_0^\delta \sum_{i=1}^3 ((\overline{\nabla} \cdot \nabla) w_{ij} \psi_i(t), \varpi_{im'})
\]

\[
\rightarrow - \int_0^\delta \sum_{i=1}^3 (\overline{\nabla} \cdot \nabla) w_{ij} \psi_i(t, \varpi_i(t)) = \int_0^\delta \sum_{i=1}^3 ((\overline{\nabla} \cdot \nabla) \varpi_i, w_{ij} \psi_i(t))
\]

Thus, in the limit we find

\[
- \int_0^\delta \sum_{i=1}^3 (\tilde{\omega}_i^*, \partial_t \psi_i(t)) + \int_0^\delta \sum_{i=1}^3 (\nabla \tilde{\omega}_i^*, \psi_i(t) \nabla v_i) dt
\]

\[
+ \int_0^\delta \sum_{i=1}^3 ((\overline{\nabla} \cdot \nabla) \varpi_i, v_i \psi_i(t)) - \int_0^\delta \sum_{i=1}^3 ((\overline{\nabla} \cdot \nabla) \varpi_i, v_i \psi_i(t)) = \sum_{i=1}^3 (\omega_{i0}, v_i) \psi_i(0)
\]

(13)

holds for \( v_i = w_{i1}, w_{i2}, \cdots \); by this equation holds for \( v_i = \) any finite linear combination of the \( w_{ij} \), and by a continuity argument above equation is still true for any \( v_i \in V \). Hence we find that \( \tilde{\omega}_i^* (i = 1, 2, 3) \) is a Leray-Hopf weak solution of the system (7).

Finally it remains to prove that \( \tilde{\omega}_i^* \) satisfies the initial conditions. For this we multiply (7) by \( v_i \psi_i(t) \), after integrating some terms by parts, we get

\[
- \int_0^\delta \sum_{i=1}^3 (\tilde{\omega}_i^*, \partial_t \psi_i(t)) + \int_0^\delta \sum_{i=1}^3 (\nabla \tilde{\omega}_i^*, \psi_i(t) \nabla v_i) dt
\]

\[
+ \int_0^\delta \sum_{i=1}^3 ((\overline{\nabla} \cdot \nabla) \varpi_i, v_i \psi_i(t)) - \int_0^\delta \sum_{i=1}^3 ((\overline{\nabla} \cdot \nabla) \varpi_i, v_i \psi_i(t)) = \sum_{i=1}^3 (\omega_{i0}, v_i) \psi_i(0)
\]

By comparison with (13),

\[
\sum_{i=1}^3 (\tilde{\omega}_i^*(0) - \omega_{i0}, v_i) \psi_i(0) = 0
\]

Therefore we can choose \( \psi_i \) particularly such that

\[
(\tilde{\omega}_i^*(0) - \omega_{i0}, v_i) = 0, \quad \forall v_i \in V
\]
4. Convergence

Now the partition is refined infinitely and \( \varepsilon \) becomes sufficiently small, we will prove that there exists some subsequence of the solutions of auxiliary problems which converges to a weak solution of (2).

Since
\[
\sup_{t \in (0, T)} \int_{\Omega} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) < +\infty
\]
the family \((\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)\) is uniformly bounded in \(L^2(0, T; H) \cap L^\infty(0, T; H)\), then we can choose \( \varepsilon' \to 0 \) \((\varepsilon' = O(\frac{1}{k}))\) and \( k' \to \infty \), or \( \Delta t' \to 0 \), such that there exists a subsequence \((\tilde{\omega}_1', \tilde{\omega}_2', \tilde{\omega}_3')\) converging weakly in \(L^2(0, T; H)\) and weak-star in \(L^\infty(0, T; H)\) to some element \((\omega_1^*, \omega_2^*, \omega_3^*)\). On the other hand, because \(\tilde{\omega}_i(i = 1, 2, 3)\) belong to \(L^2(0, T; H)\), we can verify that
\[
\omega_i(x, t) = \left\{ \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \tilde{\omega}_i(x, t) dt, \quad t \in (t_{k-1}, t_k) \subset (0, T) \right\}
\]
also belongs to \(L^2(0, T; H)\).

In the same way, we know from (3) that the function
\[
\mathbf{u}_i(x, t) = \left\{ \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} u_i(x, t) dt, \quad t \in (t_{k-1}, t_k) \subset (0, T) \right\}
\]
belongs to \(L^2(0, T; H)\).

Finally we will prove that \((\omega_1^*, \omega_2^*, \omega_3^*)\) is a solution of the vorticity-velocity form of Navier-Stokes equation (2).

Taking \(\varphi_i \in C^\infty((0, T) \times \mathbb{R}^3), \quad (i = 1, 2, 3)\) with a period on \(\Omega\), and
\[
\partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2 + \partial_{x_3}\varphi_3 = 0
\]
we have
\[
\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\Omega} \varphi_i (\partial_t \tilde{\omega}_i + \mathbf{u}_i \cdot \nabla \tilde{\omega}_i) + \mathbf{u}_1 \cdot \nabla \cdot \mathbf{u}_1 + \mathbf{u}_2 \cdot \nabla \cdot \mathbf{u}_2 + \mathbf{u}_3 \cdot \nabla \cdot \mathbf{u}_3 -
\]

Here $\tilde{\omega}_i$ ($i = 1, 2, 3$) denote the collection of those solutions of problem (5) defined on every $(t_{k-1}, t_k)$. Integrating by parts we get

$$\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\Omega} (\varphi_1 \partial_t \tilde{\omega}_1 + \omega_1^k (\varphi_1 \partial_x \tilde{\omega}_1 + \varphi_1 \partial_x \tilde{\omega}_1^k) + (\varphi_2 \partial_x \varphi_1 + \varphi_1 \partial_x \tilde{\omega}_2) + (\varphi_3 \partial_x \varphi_1 + \varphi_1 \partial_x \tilde{\omega}_3) \right) -$$

$$- \omega_2^k (\varphi_1 \partial_x \tilde{\omega}_1 + \varphi_1 \partial_x \tilde{\omega}_1^k) + (\varphi_2 \partial_x \varphi_1 + \varphi_1 \partial_x \tilde{\omega}_2) + (\varphi_3 \partial_x \varphi_1 + \varphi_1 \partial_x \tilde{\omega}_3) \right) +$$

$$+ q \partial_x \varphi_1 + \tilde{\omega}_1 \Delta \varphi_1 = \sum_{k=1}^{N} \int_{\Omega} (\varphi_1 (x,t_k) \tilde{\omega}_1 (x,t_k) - \varphi_1 (x,t_{k-1}) \tilde{\omega}_1 (x,t_{k-1}))$$

$$\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\Omega} (\varphi_2 \partial_x \varphi_2 + \omega_2^k (\varphi_2 \partial_x \tilde{\omega}_2 + \varphi_2 \partial_x \tilde{\omega}_2^k) + (\varphi_3 \partial_x \varphi_2 + \varphi_2 \partial_x \tilde{\omega}_3) \right) -$$

$$- \omega_3^k (\varphi_2 \partial_x \tilde{\omega}_2 + \varphi_2 \partial_x \tilde{\omega}_2^k) + (\varphi_3 \partial_x \varphi_2 + \varphi_2 \partial_x \tilde{\omega}_3) \right) +$$

$$+ q \partial_x \varphi_2 + \tilde{\omega}_2 \Delta \varphi_2 = \sum_{k=1}^{N} \int_{\Omega} (\varphi_2 (x,t_k) \tilde{\omega}_2 (x,t_k) - \varphi_2 (x,t_{k-1}) \tilde{\omega}_2 (x,t_{k-1}))$$

$$\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\Omega} (\varphi_3 \partial_x \varphi_3 + \omega_3^k (\varphi_3 \partial_x \tilde{\omega}_3 + \varphi_3 \partial_x \tilde{\omega}_3^k) + (\varphi_3 \partial_x \varphi_3 + \varphi_3 \partial_x \tilde{\omega}_3) \right) -$$

$$- \omega_3^k (\varphi_3 \partial_x \tilde{\omega}_3 + \varphi_3 \partial_x \tilde{\omega}_3^k) + (\varphi_3 \partial_x \varphi_3 + \varphi_3 \partial_x \tilde{\omega}_3) \right) +$$

$$+ q \partial_x \varphi_3 + \tilde{\omega}_3 \Delta \varphi_3 = \sum_{k=1}^{N} \int_{\Omega} (\varphi_3 (x,t_k) \tilde{\omega}_3 (x,t_k) - \varphi_3 (x,t_{k-1}) \tilde{\omega}_3 (x,t_{k-1}))$$

From section 2 we have the following conclusions:

$\tilde{\omega}_i \to \omega_i^* \text{ in } L^2(0,T;H) \text{ weakly, and in } L^\infty(0,T;H) \text{ weak-star}$

$\tilde{\omega}_i \to \omega_i^* \text{ in } L^2(0,T;H) \text{ weakly}$

as $k' \to \infty$, or $\Delta t'_k \to 0$.

In addition, for a certain solution $u$ of (1), we can prove due to (3) and (4) that
\( \overline{u}_i \rightarrow u_i \) in \( L^2(0,T;H) \) strongly as \( \varepsilon \rightarrow 0 \) and \( k \rightarrow \infty \), or \( \Delta t_k \rightarrow 0 \).

In fact, we have

\[
\| \overline{u}_i - \overline{u}_i \|_{L^2(\Omega)} = \int_{|y| \leq \varepsilon} J_\varepsilon(y) \| \overline{u}_i(x-y) - \overline{u}_i(x) \|_{L^2(\Omega)} dy
\]

\[
\leq \sup_{|y| \leq \varepsilon} \| \overline{u}_i(x-y) - \overline{u}_i(x) \|_{L^2(\Omega)} \rightarrow 0
\]

as \( \varepsilon \rightarrow 0 \). We can take \( \varepsilon = O(\frac{1}{k}) \).

Set \( Q = (0,T) \times \overline{\Omega} \), \( \Delta t = \max \{ \Delta t_k \} \). \( \forall \varepsilon > 0 \), and \( u_i \in L^2(0,T;L^2(\Omega)) \), there exists a \( v_i \in C^\infty(0,T;L^2(\Omega)) \) such that

\[
\| u_i - v_i \|_{L^2(Q)} < \varepsilon
\]

By means of the same partition as that for \( \overline{u}_i \) to construct \( v_i \), since there exists a constant \( C > 0 \) such that \( \| \partial_t v_i \|_{L^2(\Omega)} \leq C \), and \( \max_i \| \overline{u}_i - v_i \|_{L^2(\Omega)} \leq C \Delta t \), it follows that

\[
\| \overline{v}_i - v_i \|_{L^2(Q)} = \left( \int_0^T \| \overline{v}_i - v_i \|_{L^2(\Omega)}^2 \right)^{1/2} \leq CT^{1/2} \Delta t
\]

Thus

\( \overline{v}_i \rightarrow v_i \) \( (L^\infty(0,T;L^2(\Omega))) \), \( \text{as } \Delta t \rightarrow 0 \)

Take \( \Delta t \) such that \( \| \overline{v}_i - v_i \|_{L^2(Q)} < \varepsilon \). Moreover,

\[
\int_0^T \| \overline{v}_i - v_i \|_{L^2(\Omega)}^2 = \sum_{k=1}^N \left\| \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} (u_i - v_i) \|_{L^2(\Omega)}^2 \Delta t_k
\]

\[
\leq \sum_{k=1}^N \left\| \left( \int_{t_{k-1}}^{t_k} (u_i - v_i) \right)^2 \right\|_{L^2(\Omega)}^{1/2} \leq \int_0^T \| u_i - v_i \|_{L^2(\Omega)}^2
\]

so that \( \| \overline{v}_i - v_i \|_{L^2(Q)} \leq \| u_i - v_i \|_{L^2(Q)} < \varepsilon \). Therefore,

\[
\| \overline{v}_i - u_i \|_{L^2(Q)} \leq \| u_i - v_i \|_{L^2(Q)} + \| v_i - \overline{v}_i \|_{L^2(Q)} + \| \overline{v}_i - \overline{u}_i \|_{L^2(Q)} < 3\varepsilon
\]
Hence as $\Delta t \to 0$, we have $\| \overline{u}_i - u_i \|_{L^2(\Omega)} \to 0$.

Finally we obtain

$$\| \overline{u}_i - u_i \|_{L^2(\Omega)} \leq \left( \int_0^T \| \overline{u}_i - u_i \|_{L^2(\Omega)}^2 \right)^{1/2} + \| \overline{u}_i - u_i \|_{L^2(\Omega)} \to 0$$

as $k \to \infty$ or $\Delta t_k \to 0$.

These convergence results enable us to pass the limit. That is,

$$\sum_{k'} \int_{t_{k'-1}}^{t_k} \int_{\Omega} \left( \tilde{\omega}_1 \partial_t \varphi_1 + \overline{\omega}_1' \left( \overline{\omega}_1' \partial_{x_1} \varphi_1 + \overline{\omega}_2' \partial_{x_2} \varphi_1 + \overline{\omega}_3' \partial_{x_3} \varphi_1 \right) - \frac{\overline{\omega}_1'}{\omega_1'} \omega_1' \partial_{x_1} \varphi_1 \right) - \omega_1' \partial_{x_1} \varphi_1 + \overline{\omega}_2' \partial_{x_2} \varphi_1 + \overline{\omega}_3' \partial_{x_3} \varphi_1 + q \partial_{x_1} \varphi_1 + \tilde{\omega}_1 \Delta \varphi_1 \right)$$

$$= \int_\Omega (\varphi_1(x,T)\tilde{\omega}_1(x,T) - \varphi_1(x,0)\tilde{\omega}_1(x,0))$$

$$\sum_{k'} \int_{t_{k'-1}}^{t_k} \int_{\Omega} \left( \tilde{\omega}_2 \partial_t \varphi_2 + \overline{\omega}_2' \left( \overline{\omega}_1' \partial_{x_1} \varphi_2 + \overline{\omega}_2' \partial_{x_2} \varphi_2 + \overline{\omega}_3' \partial_{x_3} \varphi_2 \right) - \frac{\overline{\omega}_2'}{\omega_2'} \omega_2' \partial_{x_1} \varphi_2 \right) + \overline{\omega}_2' \partial_{x_2} \varphi_2 + \overline{\omega}_3' \partial_{x_3} \varphi_2 + q \partial_{x_2} \varphi_2 + \tilde{\omega}_2 \Delta \varphi_2$$

$$= \int_\Omega (\varphi_2(x,T)\tilde{\omega}_2(x,T) - \varphi_2(x,0)\tilde{\omega}_2(x,0))$$

$$\sum_{k'} \int_{t_{k'-1}}^{t_k} \int_{\Omega} \left( \tilde{\omega}_3 \partial_t \varphi_3 + \overline{\omega}_3' \left( \overline{\omega}_1' \partial_{x_1} \varphi_3 + \overline{\omega}_2' \partial_{x_2} \varphi_3 + \overline{\omega}_3' \partial_{x_3} \varphi_3 \right) - \frac{\overline{\omega}_3'}{\omega_3'} \omega_3' \partial_{x_1} \varphi_3 \right) + \overline{\omega}_3' \partial_{x_2} \varphi_3 + \overline{\omega}_3' \partial_{x_3} \varphi_3 + q \partial_{x_3} \varphi_3 + \tilde{\omega}_3 \Delta \varphi_3$$

$$= \int_\Omega (\varphi_3(x,T)\tilde{\omega}_3(x,T) - \varphi_3(x,0)\tilde{\omega}_3(x,0))$$

This is equivalent to

$$\int_0^T \int_{\Omega} \left\{ (\omega_1^2 \partial_t \varphi_1 + \omega_2^2 \partial_t \varphi_2 + \omega_3^2 \partial_t \varphi_3) + \omega_1^2 \Delta \varphi_1 + \omega_2^2 \Delta \varphi_2 + \omega_3^2 \Delta \varphi_3 + \omega_1^2 (u_1 \partial_{x_1} \varphi_1 + u_2 \partial_{x_2} \varphi_1 + u_3 \partial_{x_3} \varphi_1) + \omega_2^2 (u_1 \partial_{x_1} \varphi_2 + u_2 \partial_{x_2} \varphi_2 + u_3 \partial_{x_3} \varphi_2) + \omega_3^2 (u_1 \partial_{x_1} \varphi_3 + u_2 \partial_{x_2} \varphi_3 + u_3 \partial_{x_3} \varphi_3) \right\}$$

$$= \int_\Omega \{ (\varphi_1(x,T)\omega_1^2(x,T) + \varphi_2(x,T)\omega_2^2(x,T) + \varphi_3(x,T)\omega_3^2(x,T)) - (\varphi_{10}(x)\omega_{10}(x) + \varphi_{20}(x)\omega_{20}(x) + \varphi_{30}(x)\omega_{30}(x)) \}$$
Here we also have
\[
\omega^*_i(x,0) = \omega_{i0}(x), \quad \varphi_i(x,0) = \varphi_{i0}(x), \quad i = 1, 2, 3
\]
Hence we know that there exists some \(\omega^*_i\) which belongs to \(L^\infty(0,T;L^2(\Omega))\) and is a Leray-Hopf weak solution of (2).

Note that a weak formulation of the following equations:
\[
\omega = \text{curl} \ u \\
\int_0^T \int_\Omega \varphi \cdot [\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu \Delta \omega] = 0
\]
are equivalent to
\[
\int_0^T \int_\Omega \tilde{\varphi} \cdot [\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u] = 0
\]
for any \(\varphi \in C^\infty((0,T) \times \mathbb{R}^3)\) with a period on \(\Omega\), and \(\tilde{\varphi} = \text{curl} \varphi\), in some distribution sense.

5. Nonexistence
Consider a series
\[
\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}
\]
where \(s\) is a complex variable, \(s = \sigma + it\). Its analytic continuation is called Riemann zeta-function. The existence and infinity of its nontrivial zeros on the critical line \(\sigma = \frac{1}{2}\) are already known.

Let
\[
\xi(s) = \frac{1}{2} s(s - 1) \pi^{-\frac{s}{2}} \Gamma \left( \frac{1}{2} \right) \zeta(s)
\]
which is an entire function, and all zeros of \(\xi(s)\) coincide with the nontrivial zeros of \(\zeta(s)\).

Writing
\[
\Xi(z) = \xi \left( \frac{1}{2} + iz \right), \quad z = t - i\tilde{\sigma}, \quad \tilde{\sigma} = \sigma - \frac{1}{2}
\]
then, as well known, we have an analytic expression of \(\Xi(s)\) without those poles and trivial zeros of \(\zeta(s)\) as follows:
\[
\Xi(z) = 2 \int_0^\infty \Phi(t) \cos(zt)dt \quad (\ast)
\]

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where
\[ \Phi(t) = 2 \sum_{n=1}^{\infty} \left(2n^4 \pi^2 e^{\frac{2}{3}t} - 3n^2 \pi e^{\frac{2}{3}t}\right) e^{-n^2 \pi e^{\frac{2}{3}t}} \]

Let \( z = y - ix \), then
\[
\cos z = \frac{1}{2} \left(e^{i z} + e^{-i z}\right) = \frac{1}{2} (e^{x+iy} + e^{-x-iy}) \\
= \frac{1}{2} \left[ e^x (\cos y + i \sin y) + e^{-x} (\cos y - i \sin y) \right] \\
= \frac{e^x + e^{-x}}{2} \cos y + i \frac{e^x - e^{-x}}{2} \sin y \\
= \cosh x \cos y + i \sinh x \sin y
\]
where \( \cosh x \) and \( \sinh x \) are two hyperbolic functions.

From (*) we know that
\[
\Xi(z) = \xi \left(\frac{1}{2} + iz\right) = \xi \left(\frac{1}{2} + x + iy\right) \\
= 2 \int_{0}^{\infty} \Phi(t) \cos(zt) \, dt
\]
where
\[ \Phi(t) = 2 \sum_{n=1}^{\infty} \left(2n^4 \pi^2 e^{\frac{2}{3}t} - 3n^2 \pi e^{\frac{2}{3}t}\right) e^{-n^2 \pi e^{\frac{2}{3}t}} \]

It follows that
\[
\Xi(z) = 2 \int_{0}^{\infty} \Phi(t) \left\{ \cosh(zt) \cos(yt) + i \sinh(zt) \sin(yt) \right\} dt \\
= 2 \int_{0}^{\infty} \Phi(t) \cosh(zt) \cos(yt) \, dt + i 2 \int_{0}^{\infty} \Phi(t) \sinh(zt) \sin(yt) \, dt
\]
Thus \( \Xi(z) \) is divided into the real and imaginary parts as
\[
f(x, y) = 2 \int_{0}^{\infty} \Phi(t) \cosh(zt) \cos(yt) \, dt \\
g(x, y) = 2 \int_{0}^{\infty} \Phi(t) \sinh(zt) \sin(yt) \, dt
\]
then \( \Xi(z) = 0 \) is equivalent to the following system of equations:
\[
f(x, y) = 0, \quad g(x, y) = 0
\]
For any fixed \( x_0 > 0 \), we will consider a property of zeros of \( f(x_0, y) \) and \( g(x_0, y) \) in this section.
Lemma 1. For a fixed \( x_0 > 0 \), \( f(x_0, y) \) or \( g(x_0, y) \) with respect to \( y \) is not of any zero of second or higher even order.

In fact, let the arbitrary \( \delta > 0 \) be small enough, if \( f(x_0, y_0) = 0 \), then
\[
\begin{align*}
f(x_0, y_0 - \delta) & \cdot f(x_0, y_0 + \delta) \\
= & \int_0^\infty \Phi(t) \cosh(x_0 t) \cos(y_0 - \delta) t dt \cdot \int_0^\infty \Phi(t) \cosh(x_0 t) \cos(y_0 + \delta) t dt \\
= & \int_0^\infty \Phi(t) \cosh(x_0 t) \left[ \cos(y_0 t) \cos(\delta t) + \sin(y_0 t) \sin(\delta t) \right] dt \times \\
& \times \int_0^\infty \Phi(t) \cosh(x_0 t) \left[ \cos(y_0 t) \cos(\delta t) - \sin(y_0 t) \sin(\delta t) \right] dt \\
= & \left( \int_0^\infty \Phi(t) \cosh(x_0 t) \cos(y_0 t) \cos(\delta t) dt \right)^2 - \left( \int_0^\infty \Phi(t) \cosh(x_0 t) \sin(y_0 t) \sin(\delta t) dt \right)^2
\end{align*}
\]

An application of the following asymptotic expansion
\[
\sin x = x - \frac{1}{3!} \theta_1(x) x^3, \quad |\theta_1(x)| \leq 1
\]
\[
1 - \cos x = \frac{1}{2!} x^2 - \frac{1}{4!} \theta_2(x) x^4, \quad |\theta_2(x)| \leq 1
\]
leads to
\[
\begin{align*}
f(x_0, y_0 - \delta) & \cdot f(x_0, y_0 + \delta) \\
= & \left( \int_0^\infty \Phi(t) \cosh(x_0 t) \cos(y_0 t) (1 - \cos \delta t) dt \right)^2 - \left( \int_0^\infty \Phi(t) \cosh(x_0 t) \sin(y_0 t) \sin(\delta t) dt \right)^2
\end{align*}
\]
\[
\begin{align*}
= & \left( \int_0^\infty \Phi(t) \cosh(x_0 t) \cos(y_0 t) \left[ \frac{1}{2!} (\delta t)^2 - \frac{1}{4!} \theta_2(\delta t) (\delta t)^4 \right] dt \right)^2 - \\
& - \left( \int_0^\infty \Phi(t) \cosh(x_0 t) \sin(y_0 t) \left[ (\delta t) - \frac{1}{3!} \theta_1(\delta t) (\delta t)^3 \right] dt \right)^2 \\
= & \delta^4 \left( \int_0^\infty t^2 \Phi(t) \cosh(x_0 t) \cos(y_0 t) \left[ \frac{1}{2} - \frac{1}{24} \theta_2(\delta t) (\delta t)^2 \right] dt \right)^2 - \\
& - \delta^2 \left( \int_0^\infty t \Phi(t) \cosh(x_0 t) \sin(y_0 t) \left[ 1 - \frac{1}{6} \theta_1(\delta t) (\delta t)^2 \right] dt \right)^2
\end{align*}
\]
\[
< 0
\]
Thus, as \( f(x_0, y_0 - \delta) > 0 \), \( f(x_0, y_0 + \delta) < 0 \), or vice versa. The conclusion is obtained.

Similarly, using (**) we also have
\[
\begin{align*}
g(x_0, y_0 - \delta) & \cdot g(x_0, y_0 + \delta) \\
= & \int_0^\infty \Phi(t) \sinh(x_0 t) \sin(y_0 - \delta) t dt \cdot \int_0^\infty \Phi(t) \sinh(x_0 t) \sin(y_0 + \delta) t dt
\end{align*}
\]
= \int_0^\infty \Phi(t) \sinh(x_0 t) \left[ \sin(y_0 t) \cos(\delta t) - \cos(y_0 t) \sin(\delta t) \right] \, dt \times \\
\times \int_0^\infty \Phi(t) \sinh(x_0 t) \left[ \sin(y_0 t) \cos(\delta t) + \cos(y_0 t) \sin(\delta t) \right] \, dt \\
= \left( \int_0^\infty \Phi(t) \sinh(x_0 t) \sin(y_0 t) \cos(\delta t) \, dt \right)^2 - \left( \int_0^\infty \Phi(t) \sinh(x_0 t) \cos(y_0 t) \sin(\delta t) \, dt \right)^2 \\
= \left( \int_0^\infty \Phi(t) \sinh(x_0 t) \sin(y_0 t) \cos(\delta t) \, dt \right)^2 - \left( \int_0^\infty \Phi(t) \sinh(x_0 t) \cos(y_0 t) \sin(\delta t) \, dt \right)^2 \\
\delta^4 \left( \int_0^\infty t^2 \Phi(t) \sinh(x_0 t) \sin(y_0 t) \left[ \frac{1}{2} - \frac{1}{24} \theta_2(\delta t) (\delta t)^2 \right] \, dt \right)^2 - \\
- \delta^2 \left( \int_0^\infty t \Phi(t) \sinh(x_0 t) \cos(y_0 t) \left[ 1 - \frac{1}{6} \theta_1(\delta t) (\delta t)^2 \right] \, dt \right)^2 < 0 \\
Thus, the same conclusion is obtained.

**Lemma 2.** For a fixed \( x_0 > 0 \), \( f(x_0, y) \) or \( g(x_0, y) \) with respect to \( y \) is not of any zero of third or higher odd order.

We assume for sake of contradiction that this conclusion fails, then \( f'_y(x_0, y_0) = 0 \), and consider further that

\[
f'_y(x_0, y_0 - \delta) \cdot f'_y(x_0, y_0 + \delta) \]
\[
= \left\{ - \int_0^\infty t \Phi(t) \cosh(x_0 t) \sin(y_0 - \delta) t \, dt \right\} \cdot \left\{ - \int_0^\infty t \Phi(t) \cosh(x_0 t) \sin(y_0 + \delta) t \, dt \right\} \\
= \int_0^\infty t \Phi(t) \cosh(x_0 t) \left[ \sin(y_0 t) \cos(\delta t) - \cos(y_0 t) \sin(\delta t) \right] \, dt \times \\
\times \int_0^\infty t \Phi(t) \cosh(x_0 t) \left[ \sin(y_0 t) \cos(\delta t) + \cos(y_0 t) \sin(\delta t) \right] \, dt \\
= \left( \int_0^\infty t \Phi(t) \cosh(x_0 t) \sin(y_0 t) \cos(\delta t) \, dt \right)^2 - \left( \int_0^\infty t \Phi(t) \cosh(x_0 t) \cos(y_0 t) \sin(\delta t) \, dt \right)^2 \\
= \left( \int_0^\infty t \Phi(t) \cosh(x_0 t) \sin(y_0 t) \left( 1 - \cos \delta t \right) \, dt \right)^2 - \left( \int_0^\infty t \Phi(t) \cosh(x_0 t) \cos(y_0 t) \sin(\delta t) \, dt \right)^2 \\
= \left( \int_0^\infty t \Phi(t) \cosh(x_0 t) \sin(y_0 t) \left[ \frac{1}{2} \theta_2(\delta t)^2 - \frac{1}{4} \theta_2(\delta t)^4 \right] \, dt \right)^2 -
It follows that \( f'_y(x_0, y_0 - \delta) > 0 \), if \( f'_y(x_0, y_0 + \delta) < 0 \), or vice versa. This is inconsistent with the previous assumption that \( f'_y(x_0, y) \) is of a zero of second or higher even order.

Similarly, we take proof by contradiction for \( g(x_0, y) \) to assume that the conclusion fails, then \( g'_y(x_0, y_0) = 0 \)

\[
g'_y(x_0, y_0 - \delta) \cdot g'_y(x_0, y_0 + \delta) = \int_0^\infty t \Phi(t) \sinh(x_0 \delta t) \cos(y_0 - \delta) t \, dt \cdot \int_0^\infty t \Phi(t) \sinh(x_0 \delta t) \cos(y_0 + \delta) t \, dt
\]

\[
= \int_0^\infty t \Phi(t) \sinh(x_0 t) \cos(y_0 t) \cos(\delta t) \, dt \times \int_0^\infty t \Phi(t) \sinh(x_0 t) \cos(y_0 t) \cos(\delta t) \, dt
\]

\[
= \left( \int_0^\infty t \Phi(t) \sinh(x_0 t) \cos(y_0 t) \cos(\delta t) \, dt \right)^2 - \left( \int_0^\infty t \Phi(t) \sinh(x_0 t) \sin(y_0 t) \sin(\delta t) \, dt \right)^2
\]

\[
= \left( \int_0^\infty t \Phi(t) \sinh(x_0 t) \cos(y_0 t) \cos(\delta t) \, dt \right)^2 - \left( \int_0^\infty t \Phi(t) \sinh(x_0 t) \sin(y_0 t) \sin(\delta t) \, dt \right)^2
\]

\[
= \left( \int_0^\infty t \Phi(t) \sinh(x_0 t) \cos(y_0 t) \left[ \frac{1}{2} \sinh(\delta t) - \frac{1}{4!} \theta_2(\delta t) (\delta t)^3 \right] \, dt \right)^2
\]

\[
= \delta^4 \left( \int_0^\infty t^3 \Phi(t) \sinh(x_0 t) \cos(y_0 t) \left[ \frac{1}{2} - \frac{1}{24} \theta_2(\delta t) (\delta t)^2 \right] \, dt \right)^2
\]

\[
- \delta^2 \left( \int_0^\infty t^2 \Phi(t) \sinh(x_0 t) \cos(y_0 t) \left[ 1 - \frac{1}{6} \theta_1(\delta t) (\delta t)^2 \right] \, dt \right)^2
\]

\[< 0\]

It follows that \( g'_y(x_0, y) \) changes sign near the point \( y_0 \). This is also inconsistent with the assumption that \( g'_y(x_0, y) \) is of a zero of second or higher even order.

It is impossible for an analytic function to be of an zero of fractional order, otherwise a singularity of some its derivative will appear. Combining Lemma 1 and 2, we have
**Theorem 1.** For any fixed \(x_0 > 0\), all zeros of \(f(x_0, y)\) and \(g(x_0, y)\) with respect to \(y\) are simple.

In the following, we deal with the nonexistence of any zero of \(\Xi(z)\) standing outside the critical line \(x = 0\) or \(\sigma = \frac{1}{2}\). Since \(\Xi(z)\) is an even function, we only need considering the half-plane \(x > 0\).

We can prove that the system of equations \(f(x, y) = 0\), \(g(x, y) = 0\) has no real solution as \(x > 0\). That is, when \(x > 0\), \(f(x, y)\) and \(g(x, y)\) can not vanish at the same time.

In fact, given any a point \((x_0, y_0)\) on the half-plane \(x > 0\), if \(g(x_0, y_0) \neq 0\), then the system of equations has no solution. Otherwise, \(g(x_0, y_0) = 0\), that is, \((x_0, y_0)\) satisfies some equation \(g(x, y) = 0\). Since \(g(x, y)\) is continuous and differentiable, then there exists a neighborhood

\[
N_{\delta_1}(x_0, y_0) = \{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1\}
\]

such that \(g_x'(x, y) \neq 0\), according to Theorem 1. From existence theorem of implicit function it follows that the equation \(g(x, y) = 0\) decides a continuous and smooth curve \(y = \phi(x)\) in the neighborhood. If \(f(x_0, y_0) = 0\) at the same time, then \(f_y'(x_0, y_0) \neq 0\) according to Theorem 1. By means of the Cauchy-Riemann equation, we have

\[
g_x'(x_0, y_0) = -f_y'(x_0, y_0) \neq 0
\]

Since \(g_x'(x, y)\) is also continuous and differentiable, then there exists another neighborhood with the same center

\[
N_{\delta_2}(x_0, y_0) = \{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2 \leq \delta_1\}
\]

such that \(g_x'(x, y) \neq 0\). Let the point \((x_1, y_1) \in N_{\delta_2}(x_0, y_0)\) be on the curve \(y = \phi(x)\) and \(x_1 \neq x_0\), as \(\delta_2\) is small enough, we have

\[
g(x_1, y_1) = g_x'(x_0, y_1)(x_1 - x_0) + O((x_1 - x_0)^2) \neq 0
\]

This leads to a contradiction.

As a conclusion, we obtain

**Theorem 2.** For any \(z = y - ix, \ x > 0\), the system of equations \(f(x, y) = 0\) and \(g(x, y) = 0\) has no real solution. That is,

\[
\Xi(z) = \xi \left(\frac{1}{2} + x + iy\right) \neq 0
\]
6. Regularity

We can still use Galerkin procedure as in section 3. Since \( V \) is separable there exists a sequence of linearly independent elements \( w_1, \cdots, w_{im}, \cdots \) which is total in \( V \). For each \( m \) we define an approximate solution \( u_{im} \) of (1) as follows:

\[
 u_{im} = \sum_{j=1}^{m} g_j(t) w_j
\]

and

\[
 \int_{\Omega} \partial_t u_{1m} w_{1j} + \int_{\Omega} (u_{1m} \partial_{x_1} u_{1m} + u_{2m} \partial_{x_2} u_{1m} + u_{3m} \partial_{x_3} u_{1m}) w_{1j} + \int_{\Omega} \partial_{x_1} p w_{1j} = \int_{\Omega} \Delta u_{1m} w_{1j}
\]

\[
 \int_{\Omega} \partial_t u_{2m} w_{2j} + \int_{\Omega} (u_{1m} \partial_{x_1} u_{2m} + u_{2m} \partial_{x_2} u_{2m} + u_{3m} \partial_{x_3} u_{2m}) w_{2j} + \int_{\Omega} \partial_{x_2} p w_{2j} = \int_{\Omega} \Delta u_{2m} w_{2j}
\]

\[
 \int_{\Omega} \partial_t u_{3m} w_{3j} + \int_{\Omega} (u_{1m} \partial_{x_1} u_{3m} + u_{2m} \partial_{x_2} u_{3m} + u_{3m} \partial_{x_3} u_{3m}) w_{3j} + \int_{\Omega} \partial_{x_3} p w_{3j} = \int_{\Omega} \Delta u_{3m} w_{3j}
\]

\[
 u_{im}(0) = u_{im}^{m_0}, \quad j = 1, \cdots, m
\]

(14)

where \( u_{im}^{m_0} \) is the orthogonal projection in \( H \) of \( u_{i0} \) on the space spanned by \( w_1, \cdots, w_{im} \).

We now are allowed to differentiate (14) in the \( t \), we get

\[
 \int_{\Omega} \partial_{tt} u_{1m} w_{1j} + \int_{\Omega} (\partial_t u_{1m} \partial_{x_1} u_{1m} + \partial_t u_{2m} \partial_{x_2} u_{1m} + \partial_t u_{3m} \partial_{x_3} u_{1m}) w_{1j} + 
\]

\[
 + \int_{\Omega} (u_{1m} \partial_{x_1} \partial_t u_{1m} + u_{2m} \partial_{x_2} \partial_t u_{1m} + u_{3m} \partial_{x_3} \partial_t u_{1m}) w_{1j} + \int_{\Omega} \partial_{x_1} \partial_t p w_{1j} = \int_{\Omega} \Delta \partial_t u_{1m} w_{1j}
\]

\[
 \int_{\Omega} \partial_{tt} u_{2m} w_{2j} + \int_{\Omega} (\partial_t u_{1m} \partial_{x_1} u_{2m} + \partial_t u_{2m} \partial_{x_2} u_{2m} + \partial_t u_{3m} \partial_{x_3} u_{2m}) w_{2j} + 
\]

\[
 + \int_{\Omega} (u_{1m} \partial_{x_1} \partial_t u_{2m} + u_{2m} \partial_{x_2} \partial_t u_{2m} + u_{3m} \partial_{x_3} \partial_t u_{2m}) w_{2j} + \int_{\Omega} \partial_{x_2} \partial_t p w_{2j} = \int_{\Omega} \Delta \partial_t u_{2m} w_{2j}
\]

\[
 \int_{\Omega} \partial_{tt} u_{3m} w_{3j} + \int_{\Omega} (\partial_t u_{1m} \partial_{x_1} u_{3m} + \partial_t u_{2m} \partial_{x_2} u_{3m} + \partial_t u_{3m} \partial_{x_3} u_{3m}) w_{3j} + 
\]

\[
 + \int_{\Omega} (u_{1m} \partial_{x_1} \partial_t u_{3m} + u_{2m} \partial_{x_2} \partial_t u_{3m} + u_{3m} \partial_{x_3} \partial_t u_{3m}) w_{3j} + \int_{\Omega} \partial_{x_3} \partial_t p w_{3j} = \int_{\Omega} \Delta \partial_t u_{3m} w_{3j}
\]

\[
 j = 1, \cdots, m
\]

(15)

We multiply (15) by \( g_{ij}(t) \) and add the resulting equations for \( j = 1, \cdots, m \), we
find
\[
\frac{1}{2} \int_{\Omega} (\partial_t u_{1m})^2 + \int_{\Omega} \partial_t u_{1m}(\partial_t u_{1m} \partial_t x_1 u_{1m} + \partial_t u_{2m} \partial_t x_2 u_{1m} + \partial_t u_{3m} \partial_t x_3 u_{1m}) + \\
+ \int_{\Omega} \partial_t u_{1m}(u_{1m} \partial_t x_1 \partial_t u_{1m} + u_{2m} \partial_t x_2 \partial_t u_{1m} + u_{3m} \partial_t x_3 \partial_t u_{1m}) + \int_{\Omega} \partial_t \partial_t u_{1m} \partial_t x_1 \partial_t p = \int_{\Omega} \partial_t u_{1m} \Delta \partial_t u_{1m}
\]
\[
\frac{1}{2} \int_{\Omega} (\partial_t u_{2m})^2 + \int_{\Omega} \partial_t u_{2m}(\partial_t u_{1m} \partial_t x_1 u_{2m} + \partial_t u_{2m} \partial_t x_2 u_{2m} + \partial_t u_{3m} \partial_t x_3 u_{2m}) + \\
+ \int_{\Omega} \partial_t u_{2m}(u_{1m} \partial_t x_1 \partial_t u_{2m} + u_{2m} \partial_t x_2 \partial_t u_{2m} + u_{3m} \partial_t x_3 \partial_t u_{2m}) + \int_{\Omega} \partial_t \partial_t u_{2m} \partial_t x_2 \partial_t p = \int_{\Omega} \partial_t u_{2m} \Delta \partial_t u_{2m}
\]
\[
\frac{1}{2} \int_{\Omega} (\partial_t u_{3m})^2 + \int_{\Omega} \partial_t u_{3m}(\partial_t u_{1m} \partial_t x_1 u_{3m} + \partial_t u_{2m} \partial_t x_2 u_{3m} + \partial_t u_{3m} \partial_t x_3 u_{3m}) + \\
+ \int_{\Omega} \partial_t u_{3m}(u_{1m} \partial_t x_1 \partial_t u_{3m} + u_{2m} \partial_t x_2 \partial_t u_{3m} + u_{3m} \partial_t x_3 \partial_t u_{3m}) + \int_{\Omega} \partial_t \partial_t u_{3m} \partial_t x_3 \partial_t p = \int_{\Omega} \partial_t u_{3m} \Delta \partial_t u_{3m}
\]
and
\[
\int_{\Omega} (\partial_t u_{1m} \partial_t x_1 \partial_t p + \partial_t u_{2m} \partial_t x_2 \partial_t p + \partial_t u_{3m} \partial_t x_3 \partial_t p) = - \int_{\Omega} \partial_t \partial_t t(\partial_t x_1 u_{1m} + \partial_t x_2 u_{2m} + \partial_t x_3 u_{3m}) = 0
\]

Since
\[
\int_{\Omega} \partial_t u_{im}(u_{1m} \partial_t x_1 \partial_t u_{im} + u_{2m} \partial_t x_2 \partial_t u_{im} + u_{3m} \partial_t x_3 \partial_t u_{im}) = \\
= \frac{1}{2} \int_{\Omega} (u_{1m} \partial_t x_1 (\partial_t u_{im})^2 + u_{2m} \partial_t x_2 (\partial_t u_{im})^2 + u_{3m} \partial_t x_3 (\partial_t u_{im})^2) \\
= - \frac{1}{2} \int_{\Omega} (\partial_t u_{im})^2(\partial_t x_1 u_{1m} + \partial_t x_2 u_{2m} + \partial_t x_3 u_{3m}) = 0
\]

and
\[
\int_{\Omega} \partial_t u_{im} \Delta \partial_t u_{im} = \int_{\Omega} \partial_t u_{im}(\partial_{x_1}^2 \partial_t u_{im} + \partial_{x_2}^2 \partial_t u_{im} + \partial_{x_3}^2 \partial_t u_{im}) = \\
= - \int_{\Omega} ((\partial_x \partial_t u_{im})^2 + (\partial_{x_2} \partial_t u_{im})^2 + (\partial_{x_3} \partial_t u_{im})^2), \quad i = 1, 2, 3
\]

then
\[
\frac{1}{2} \int_{\Omega} ((\partial_t u_{1m})^2 + (\partial_t u_{2m})^2 + (\partial_t u_{3m})^2) + \\
+ \|\nabla \partial_t u_{1m}\|_{L^2(\Omega)}^2 + \|\nabla \partial_t u_{2m}\|_{L^2(\Omega)}^2 + \|\nabla \partial_t u_{3m}\|_{L^2(\Omega)}^2 \\
\leq \|\partial_t u_{1m}\|_{L^4(\Omega)} \left(\|\partial_t u_{1m}\|_{L^4(\Omega)} \|\partial_t x_1 u_{1m}\|_{L^2(\Omega)} + \|\partial_t u_{2m}\|_{L^4(\Omega)} \|\partial_t x_2 u_{1m}\|_{L^2(\Omega)} + \\
+ \|\partial_t u_{3m}\|_{L^4(\Omega)} \|\partial_t x_3 u_{1m}\|_{L^2(\Omega)} \right) \\
+ \|\partial_t u_{2m}\|_{L^4(\Omega)} \left(\|\partial_t u_{1m}\|_{L^4(\Omega)} \|\partial_t x_2 u_{2m}\|_{L^2(\Omega)} + \|\partial_t u_{2m}\|_{L^4(\Omega)} \|\partial_t x_2 u_{2m}\|_{L^2(\Omega)} + \\
+ \|\partial_t u_{3m}\|_{L^4(\Omega)} \|\partial_t x_3 u_{2m}\|_{L^2(\Omega)} \right) \\
+ \|\partial_t u_{3m}\|_{L^4(\Omega)} \left(\|\partial_t u_{1m}\|_{L^4(\Omega)} \|\partial_t x_3 u_{3m}\|_{L^2(\Omega)} + \|\partial_t u_{2m}\|_{L^4(\Omega)} \|\partial_t x_3 u_{3m}\|_{L^2(\Omega)} + \\
+ \|\partial_t u_{3m}\|_{L^4(\Omega)} \|\partial_t x_3 u_{3m}\|_{L^2(\Omega)} \right)
\]
\[ + \| \partial_t u_{3m} \|_{L^2(\Omega)} \left( \| \partial_{t} u_{1m} \|_{L^2(\Omega)} + \| \partial_{x_1} u_{3m} \|_{L^2(\Omega)} + \| \partial_{x_2} u_{3m} \|_{L^2(\Omega)} \right) + \]
\[ + \| \partial_t u_{3m} \|_{L^2(\Omega)} \| \partial_{x_2} u_{3m} \|_{L^2(\Omega)} \]
\[ \leq \left( \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right)^{1/2} \left( \sum_{j=1}^{3} \| \partial_t u_{jm} \|^2_{L^2(\Omega)} \right)^{1/2} \left( \sum_{i,j=1}^{3} \| \partial_{x_i} u_{jm} \|^2_{L^2(\Omega)} \right)^{1/2} \]
where
\[ \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \leq 2 \sum_{i=1}^{3} \left( \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right)^{1/2} \left( \| \nabla \partial_t u_{im} \|_{L^2(\Omega)} \right)^{3/2} \]
\[ \leq 2 \left( \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right)^{1/4} \left( \sum_{i=1}^{3} \| \nabla \partial_t u_{im} \|^2_{L^2(\Omega)} \right)^{3/4} \]
so that
\[ \partial_t \left( \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right) + 2 \left( \sum_{i=1}^{3} \| \nabla \partial_t u_{im} \|^2_{L^2(\Omega)} \right) \]
\[ \leq 2 \left( \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right)^{1/4} \left( \sum_{i=1}^{3} \| \nabla \partial_t u_{im} \|^2_{L^2(\Omega)} \right)^{3/4} \left( \sum_{i=1}^{3} \| \nabla u_{im} \|^2_{L^2(\Omega)} \right)^{1/2} \]
\[ \leq 3 \left( \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right)^{1/2} \left( \sum_{i=1}^{3} \| \nabla u_{im} \|^2_{L^2(\Omega)} \right)^{1/2} + \left( \sum_{i=1}^{3} \| \nabla \partial_t u_{im} \|^2_{L^2(\Omega)} \right) \]
it follows that
\[ \partial_t \left( \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right) + \left( \sum_{i=1}^{3} \| \nabla \partial_t u_{im} \|^2_{L^2(\Omega)} \right) \leq \phi_m(t) \left( \sum_{i=1}^{3} \| \partial_t u_{im} \|^2_{L^2(\Omega)} \right) \]
where
\[ \phi_m(t) = 1 + 3 \left( \sum_{i=1}^{3} \| \nabla u_{im} \|^2_{L^2(\Omega)} \right)^2 \]

Introducing a stream function: \( \psi = (\psi_2, \psi_2, \psi_3) \),
\[ \text{curl} \psi = (\partial_{x_2} \psi_3 - \partial_{x_3} \psi_2, \ \partial_{x_3} \psi_1 - \partial_{x_1} \psi_3, \ \partial_{x_1} \psi_2 - \partial_{x_2} \psi_1) \]
According to \( \omega = \text{curl} \ u \), \( u = \text{curl} \psi \) and \( \text{div} \ \psi = 0 \), we have
\[ \text{curl} \ \text{curl} \ \psi = -\Delta \psi = \omega, \ \ -\Delta \text{curl} \ \psi = \text{curl} \ \omega \]
That is, \(-\Delta u = \text{curl} \ \omega \). Then \((-\Delta u, \ u) = \langle \text{curl} \ \omega, \ u \rangle \), where
\[ (-\Delta u, \ u) = \sum_{i=1}^{3} (-\Delta u_i, \ u_i) = \sum_{i=1}^{3} (\nabla u_i, \ \nabla u_i) = \sum_{i=1}^{3} \| \nabla u_i \|^2_{L^2(\Omega)} \]

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\[ \text{curl} \omega, u = (\partial_{x_2} \omega_3 - \partial_{x_3} \omega_2, u_1) + (\partial_{x_3} \omega_1 - \partial_{x_1} \omega_3, u_2) + (\partial_{x_1} \omega_2 - \partial_{x_2} \omega_1, u_3) \]
\[ = -(\omega_3, \partial_{x_2} u_1) + (\omega_2, \partial_{x_3} u_1) - (\omega_1, \partial_{x_3} u_2) + (\omega_3, \partial_{x_1} u_2) - (\omega_2, \partial_{x_1} u_3) + (\omega_1, \partial_{x_2} u_3) \]
\[ = (\omega_1, \partial_{x_2} u_3 - \partial_{x_3} u_2) + (\omega_2, \partial_{x_3} u_1 - \partial_{x_1} u_3) + (\omega_3, \partial_{x_1} u_2 - \partial_{x_2} u_1) \]
\[ = (\omega, \text{curl} u) = (\omega, \omega) = \sum_{i=1}^{3} \|\omega_i\|_{L^2(\Omega)}^2 \]

Hence,
\[ \left( \sum_{i=1}^{3} \|\nabla u_i\|_{L^2(\Omega)}^2 \right)^{1/2} = \left( \sum_{i=1}^{3} \|\omega_i\|_{L^2(\Omega)}^2 \right)^{1/2} \]

it follows that
\[ \phi_m(t) = 1 + 3^3 \left( \sum_{i=1}^{3} \|\omega_{im}\|_{L^2(\Omega)}^2 \right)^2 < +\infty \]

By the Gronwall inequality,
\[ \frac{d}{dt} \left\{ \sum_{i=1}^{3} \|\partial_t u_{im}\|_{L^2(\Omega)}^2 \right\} \exp \left( - \int_0^t \phi_m(s) ds \right) \leq 0 \]

whence
\[ \sup_{t \in (0, T)} \left( \sum_{i=1}^{3} \|\partial_t u_{im}(t)\|_{L^2(\Omega)}^2 \right) \leq \left( \sum_{i=1}^{3} \|\partial_t u_{im}(0)\|_{L^2(\Omega)}^2 \right) \exp \left( \int_0^T \phi_m(s) ds \right) \]

Therefore
\[ \partial_t u_{im} \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad i = 1, 2, 3 \]

Similar to the Theorem 3.8 in Chapter 3 of [4], we obtain
\[ u_i \in L^\infty(0, T; H^2(\Omega)), \quad i = 1, 2, 3 \]

Noting that \((-\Delta u, v) = (-\partial_t u - (u \cdot \nabla) u, v)\). Since \(\partial_t u\) and \((u \cdot \nabla) u\) are of some degree of continuity, then \(u\) can reach a higher degree of continuity, based on the smoothing effect of inverse elliptic operator \(\Delta^{-1}\). By repeated application of this process one can prove that the solution \(u\) is in \(C^\infty(\Omega \times (0, T))\).
References

[1] R. A. Adams, and J. J. F. Fournier, *Sobolev Spaces*, Second ed., Pure and Applied Mathematics, Elsevier, Oxford, (2003);

[2] O.A.Ladyženskaya, V.A.Solonnikov, and N.N.Ural’ceva, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, (1988);

[3] Qun Lin, and Lung-an Ying, *Interval Vortex Methods*, Numerical Methods for PDEs, 30: pp.1368-1396, (2014);

[4] R. Temam, *Navier-Stokes equations Theory and numerical analysis*, Reprint of the 1984, AMS Chelsea Publishing, Providence, R.I., (2001).