Stability and Boundedness of Solutions to Some Nonautonomous Multidimensional Nonlinear Systems

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Abstract. Assessment of degree of boundedness/stability of multidimensional nonlinear systems with time-dependent and especially nonperiodic coefficients is an important applied problem which has no adequate resolution yet. Most of the known techniques mostly provide computationally intensive and conservative stability criteria in this area which frequently fail to gage the degrees of stability and especially boundedness of solutions to the corresponding systems. Recently, we outline a new approach to this task resting on analysis of solutions to a scalar auxiliary equation bounding from above time-histories of the norms of solutions to the original systems. This paper develops a new technique casting the auxiliary equation in a simplified form which, in turn, amplifies its application domain and reduces the computational hamper of our prior approach. Consequently, we developed novel boundedness/stability criteria and estimated the trapping/stability regions for some multidimensional nonlinear systems with time – dependent coefficients. This let us to assess in target simulations the degree of boundedness/stability of multidimensional, nonlinear and nonautonomous systems which were intractable to our prior methodology.

Keywords: Nonlinear systems with variable coefficients; Stability and boundedness of solutions; Estimation of trapping/stability regions.

1. Introduction

Analysis of boundedness and stability of nonlinear systems with variable and nonperiodic coefficients plays an important role in various engineering and natural science problems which, for instance, aids the design of robust controllers and observers. It appears that currently there are no confirmable necessary and sufficient conditions of local stability of the trivial solution to homogeneous systems of such kind, see, e.g. [1-7]. Sufficient conditions of asymptotic stability of the trivial solution to a system

\[ \dot{x} = \varphi(t, x), \varphi(t, 0) = 0, \ x: [t_0, \infty) \rightarrow \mathbb{R}^n \]  

(1.1)

were initiated by Lyapunov [8] in terms of widely used thereafter Lyapunov exponents, \( \chi(x(t)) \) that were defined as follows

\[ \chi(x(t)) = \limsup_{t \to \infty} \frac{1}{t} \ln \|x(t)\| \]  

(1.2)

where \( \| \| \) is a vector’s norm. It was shown in [8] that under some additional conditions on \( \varphi(t, x) \) the trivial solution to (1.1) is asymptotically stable if the linearization of this system at zero is regular and its maximal Lyapunov exponent is negative, see also see [1] for contemporary review of this subject. Yet, the confirmation of the former condition presents considerable problem in applications. Subsequently, it was demonstrated by Perron [9] that arbitrary small perturbations can reverse sign in the Lyapunov exponents of a linearized system and alter its stability. In turn, this attests that Lyapunov’s regularity condition is essential. Subsequent works were focused on recasting the Lyapunov stability conditions through analysis of separation of solutions to the linearized system. Finally, the necessary and sufficient conditions of stability of Lyapunov exponents of the linearized system were given in [10] and [11]. However, it turned out that these conditions are not confirmable since a sensible mechanism of their verification does not exist yet.

Application of the concept of the generalized exponents, that was introduced in [6], provides sufficient stability conditions of the trivial solution to (1.1) which, in principle, can be checked with the aid of numerical simulations. Still, the upper generalized exponent is larger or equal to the maximal Lyapunov exponent which heightens conservatism of this more robust approach.

To retrieve a simple sufficient stability condition from this last approach, we firstly write (1.1) as follows,

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\[
\dot{x} = B(t)x + f_s(t, x) + F_x(t), \quad \forall t \geq t_0, \quad x \in \mathbb{H} \subset \mathbb{R}^n, \quad f_s(t, 0) = 0
\]

\(x(t_0, t_0, x_0) = x_0\) \quad (1.3)

where continuous functions, \(f_s : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(F_x : [t_0, \infty) \rightarrow \mathbb{R}^n\), continuous matrix, \(A(t) \in \mathbb{R}^{n \times n}, \quad \forall t \geq t_0\), \(t_0 \in \mathbb{T} := [\zeta, \infty), \quad \zeta \in \mathbb{R}\), \(x_0 \in \mathcal{G} \subset \mathbb{R}^n\), \(\mathcal{G}\) is a bounded neighborhood about \(x \equiv 0\), \(F_x(t) = F_0\eta(t)\)

\[
\sup_{t \geq t_0} \|\eta(t)\| = 1, \quad F_0 \in \mathbb{R}_{\geq 0}, \quad \mathbb{R}_{\geq 0} \text{ is a set of nonnegative real numbers, } \|\| \text{ stands for induced 2-norm of a matrix or 2-norm of a vector and } x(t, t_0, x_0) := x(t_0, x_0) \text{ is the solution to (1.1). To shorten the notation, we will write below that } x(t, t_0, x_0) \equiv x(t, x_0) \text{ and assume that (1.1) possesses a unique solution } \forall t \geq t_0.
\]

We also will assess stability of the trivial solution to the following homogeneous equation,

\[
\dot{x} = B(t)x + f_s(t, x)
\]

\(x(t_0, x_0) = x_0\) \quad (1.4)

and its linearization,

\[
\dot{x} = B(t)x
\]

\(x(t_0, x_0) = x_0\) \quad (1.5)

Obviously, the solution to (1.5) can be written as \(x(t, x_0) = W(t, t_0)x_0\), where \(W(t, t_0) = w(t)w^{-1}(t_0)\) and \(w(t)\) are transition and fundamental solution matrices for (1.5), respectively.

Next, pretend that \(f_s(t, x)\) obeys Lipschitz continuity condition,

\[
\|f_s(t, x)\| \leq l_1(t, t_0)\|x\|, \quad \forall x \in \Omega_1 \subset \mathbb{R}^n, \quad \forall t \geq t_0.
\]

(1.6)

where \(\Omega_1\) is a bounded neighborhood of \(x \equiv 0\) and \(l_1(t, t_0) \leq \hat{l}_1 \in \mathbb{R}_{>0}, \quad \forall t \geq t_0\) is a continuous function and,

\[
\|W(t, t_0)\| \leq Ne^{-\nu(t-t_0)}, \quad \forall t \geq t_0, \quad N, \quad \nu > 0.
\]

(1.7)

and

\[
\hat{l}_1 - \nu < 0
\]

(1.8)

Then the trivial solution to (1.3) is asymptotically stable if (1.6), (1.7) and (1.8) hold [4, 6, 7].

A more general but less tractable condition on the norm of transition matrix was presented in [1, 2],

\[
\|W(t, t_0)\| \leq N \exp\left\{\int_{t_0}^{t} \gamma(s)ds\right\}, \quad \forall t > t_0 > 0, \quad \gamma : [t_0, \infty) \rightarrow \mathbb{R}, \quad N \in \mathbb{R}_{>0}
\]

(1.9)

where it was shown that (1.6), (1.9) and the following condition,

\[
\lim_{t \to \infty} \sup_{t \to t_0} \left\{1/\left(1+(t-t_0)\right)\right\} \int_{t_0}^{t} \gamma(s)ds + N\hat{l}_1 < 0
\]

(1.10)

embrace asymptotic stability of the trivial solution of (1.4). More compelling stability conditions of (1.4) were given in [12], yet verification of these conditions can be computationally challenging as well.

In control literature analysis of stability of nonautonomous nonlinear systems was aided by application of the Lyapunov functions method [15 – 21]. Nonetheless, application of this methodology is strenuous in that area.

The problem of estimating the stability regions of autonomous nonlinear systems has concerned numerous publications in the last few decades [22-38] but applications of these techniques fail for systems with time-varying coefficients. To our knowledge, the problem of estimating the trapping regions for nonautonomous nonlinear systems has not been addressed practically in the current literature.

In [13] we developed a technique for estimation of the upper bounds of the norms of solutions to (1.1) or (1.2) and use it to distinguish boundedness/stability conditions and estimate the trapping/stability regions for these systems. Consequently, under normalizing condition, \(w(t_0) = 1\), we derive the following inequality, \(\|x(t, x_0)\| \leq X(t, X_0), \quad \forall t \geq t_0\), where \(x(t, x_0)\) is a solution to (1.1) and \(X(t, X_0)\) is a solution to a scalar equation,
\[
\dot{X} = p(t) X + c(t)\|f_\epsilon(t, x(t, x_0)) + F_\epsilon(t)\|
\]

\[X(t_0, x_0) = \|w^{-1}(t_0) x_0\| = X_0\]

(1.11)

where

\[p(t) = d\left(\ln\|w(t)\|\right) / dt\]

(1.12)

and

\[c(t) = \|w(t)\| w^{-1}(t) = \sigma_{\text{max}}(w) / \sigma_{\text{max}}(w)\]

(1.13)

In turn, [13] has also introduced a nonlinear extension of Lipschitz continuity condition which was casted as follows,

\[\|f_\epsilon(t, x)\| \leq L(t, \|x\|), \quad \forall x \in \Omega_2 \subset \mathbb{R}^n, \quad \forall t \geq t_0\]

(1.14)

where \( L : [t_0, \infty) \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is continuous function in \( t \) and \( \|x\| \), \( L(t,0) = 0 \), and \( \Omega_2 \) is a bounded neighborhood of \( x = 0 \). Note that [13] defines (1.14) in a closed form if \( f_\epsilon \) is either a piece-wise polynomial function in \( x \) or can be approximated by such function with, e.g., bounded in \( \Omega_2 \) error term, which, for instance, can be written in Lagrange form. In the former cases, \( \Omega_2 = \mathbb{R}^n \) and in the latter case, \( \Omega_2 = \mathbb{R}^n \) if the error term is also globally bounded for \( \forall x \in \mathbb{R}^n \). Thus, condition, \( \Omega_2 = \mathbb{R}^n \) holds for various nonlinear systems emerging in science and engineering applications.

Consequently, using (1.14) we reviewed (1.11) in the following form

\[\dot{z} = p(t) z + c(t) (L(t, z) + \|f_\epsilon(t, z)\|)\]

\[z(t_0, z_0) = \|w^{-1}(t_0) z_0\| = z_0\]

(1.15)

It turned out that (1.15) provides sound estimates of trapping/stability regions under assumption that the bound of \( \|f_\epsilon(t, x)\| \) is only known – a frequent pronouncement in the area of robust control. But such estimates become more conservative if \( f_\epsilon(t, x) \) is defined explicitly. Consequently, we refined this methodology in [14], where (1.15) was used to estimate the error of successive approximations to solutions to (1.3) or (1.4) stemming from the trapping/stability regions of corresponding systems. This modified approach has enhanced our boundedness/stability criteria and estimated in successive approximations the corresponding trapping/stability regions with increased accuracy.

Nonetheless, the methodologies developed in [13] and [14] work under condition that \( c(t) < \infty, \quad \forall t \geq t_0 \), which considerably limits the scope of its applications. In fact, it follows from (1.13) that frequently, \( \lim_{t \rightarrow \infty} c(t) = \infty \) even if \( A \) is a Hurwitz and time invariant matrix.

The current paper lifts this limitation for a practically important class of nonlinear systems with variable and nonperiodic coefficients. It develops a new approach yielding a counterpart equation to (1.11) with \( c(t) = 1 \) and \( p(t) = \text{const} \) under some conditions that are frequent met in various applications. This new technique naturally voids elaborate simulations of \( w(t), p(t) \) and \( c(t) \) that were required previously. Consequently, we amend our boundedness and stability criteria and estimate the trapping/stability regions for systems that were intractable to our former methodology.

2. Modified Scalar Auxiliary Equation

Firstly, we average \( B(t) \) as follows, \( A(t_0) = \lim_{t_0 \rightarrow \infty} (t-t_0)^{-1} \int_{t_0}^t B(s) \, ds < \infty, \quad \forall t_0 \in T \), where a constant matrix, \( A(t_0) \in \mathbb{R}^{n \times n}, \ A(t_0) \neq 0, \|A(t_0)\| < \infty, \forall t_0 \in T \). In turn, let us assume that \( A \) is a constant and
diagonalizable matrix with eigenvalues, 

\[ \lambda_k = \alpha_k \pm i \beta_k, \quad i = \sqrt{-1}, \quad 1 \leq k \leq n, \quad \alpha_k, \beta_k \in \mathbb{R}. \]

We will also presume that some \( \lambda_k \) can be real, i.e., \( \beta_k = 0, \quad \forall k \in [2K - 1, n] \), where \( K \geq 1 \) is an appropriate integer. Let us surmise that \( \alpha_k \geq \alpha_{k+1}, \quad k \in [1, n-1] \) and define a square diagonal matric, \( \Lambda = A + i B \) with \( A = \text{diag}(\alpha_1, \ldots, \alpha_n) \) and \( B = \text{diag}(\beta_1, \ldots, \beta_n) \).

Next, we write (1.3) as follows,

\[ \dot{x} = Ax + G(t)x + f(t, x) + F_\alpha(t), \quad t \in [t_0, \infty) \]

\[ x(t_0) = x_0 \]

(2.1)

where \( G_\alpha(t) = B(t) - A \). In turn, we rewrite (2.1) into the eigenbasis of \( A \) as follows,

\[ \dot{y} = \lambda y + G(t)y + f(t, y) + F(t), \quad t \in [t_0, \infty), \quad y \in \mathbb{R}^n \]

\[ y(t_0, y_0) = y_0 = V^{-1}x_0 \]

(2.2)

where \( y = V^{-1}x, \quad V \in \mathbb{R}^{n \times n} \) is the eigenmatrix of \( A \), \( G = V^{-1}G_\alpha V \), \( f(t, y) = V^{-1}f_\alpha(t, Vy) \) and \( F(t) = V^{-1}F_\alpha \). Subsequently, we represent (2.2) in the following form,

\[ \dot{y} = (\lambda I + i B)y + (A - \lambda I)y + G(t)y + f(t, y) + F(t) \]

\[ y(t_0, y_0) = y_0 = V^{-1}x_0 \]

(2.3)

where \( \lambda \in \mathbb{R} \) is defined below. Then, using matrix exponential we write the fundamental matrix of solutions to equation \( \dot{y} = (\lambda I + i B)y \) as follows,

\[ w(t) = \exp(\lambda I + i B)(t-t_0) = e^{\lambda(t-t_0)} \exp(iB)(t-t_0) \]

(2.4)

where \( I \) is an identity matrix. Henceforth, we get that \( \|w(t)\| = e^{\lambda(t-t_0)} \), \( \|w^{-1}(t)\| = e^{-\lambda(t-t_0)} \) since \( \|\exp(iB)(t-t_0)\| = 1 \), which implies that \( c(w(t)) = 1 \) and \( p(w(t)) = \lambda \) if \( w(t) \) is defined by (2.4).

Next, adopting (2.4), we can write equation (1.15) for (2.3) as follows,

\[ \dot{z} = (\lambda + \|A - \lambda I\|)z + \|G(t)\|z + L(t, z) + \|F(t)\| \]

\[ z(t_0, z_0) = z_0 = \|V^{-1}x_0\| \]

(2.5)

where \( z(t, z_0) \geq \|y(t, y_0)\|, \quad \forall t \geq t_0 \) and \( y \in \Omega_2 \).

In sequel, we define \( \hat{\lambda} \) using the following condition,

\[ \min_{\hat{\lambda}} \left( \lambda + \|A - \hat{\lambda}I\| \right) \]

(2.6)

maximizing the degree of stability of a scalar linear equation, \( \dot{z} = (\lambda + \|A - \lambda I\|)z \). Since \( A - \hat{\lambda}I \) is a diagonal matrix, \( \|A - \hat{\lambda}I\| = \max_k |\hat{\lambda}_k - \hat{\lambda}| \) and (2.6) can be written as,

\[ \min_{\hat{\lambda}} \left( \lambda + \max_k |\hat{\lambda}_k - \hat{\lambda}| \right) \]

which, obviously, yields that \( \hat{\lambda} = \alpha_n \) and, consequently, \( \left( \lambda + \|A - \hat{\lambda}I\| \right)_{\hat{\lambda} = \alpha_n} = \alpha_1 \). This let us to write (2.5) as follows,

\[ \dot{z} = (\alpha_1 + \|G(t)\|)z + L(t, z) + \|F(t)\| \]

\[ z(t_0, z_0) = z_0 = \|V^{-1}x_0\| \]

(2.7)
To develop a less conservative version of (2.7) we set that matrix, $D(t) = \text{Im}\left(diag(G(t))\right)$, $D \in \mathbb{R}^{n_{sc} \times n_{sc}}$, $B_{+}(t) = B + D(t)$ and $G_{-} = G - D$, and rewrite (2.3) as follows,

$$
\begin{align*}
\dot{y} &= (\lambda I + iB_{+}(t))y + (A - \lambda I)y + G_{-}(t)y + f(t, y) + F(t) \\
y(t_0, y_0) &= V^{-1}x_0
\end{align*}
$$

Let us recall that a fundamental solution matrix for equation, $\dot{y} = (\lambda I + iB_{+}(t))y$ can be written as,

$$
\begin{align*}
w_{+}(t) = \exp(\lambda I + iB_{+}(t)(t - t_0))
\end{align*}
$$

as well since $B_{+}$ is a diagonal matrix which, as prior, implies that

$$
\begin{align*}
\|w_{+}(t)\| &= e^{\lambda(t - t_0)}, \|w^{-1}_{+}(t)\| = e^{-\lambda(t - t_0)}, c\left(w_{+}(t)\right) = 1 \text{ and } p\left(w_{+}(t)\right) = \alpha_i. \quad \text{Lastly, a less conservative counterpart of (2.7) can be written as follows,}
\end{align*}
$$

$$
\begin{align*}
z = (\alpha_i + \|G_{-}(t)\|)z + L(t, z) + \|F(t)\|
\end{align*}
$$

(2.8)

Now we present the pivot inequality,

$$
\begin{align*}
\|x(t, x_0)\| \leq \|V\| \|z(t, z_0)\|, \quad z_0 = \|V^{-1}x_0\|, \quad \forall x(t, x_0) \in \Omega_2, \forall t \geq t_0,
\end{align*}
$$

(2.9)

bounding the norm of solutions to (1.3) by matching solutions of the scalar equation (2.8), which expedites our inferences in the next two sections.

Note that selection of $\lambda$ through (2.6) naturally equates the degrees of stability of linearized autonomous version of equation (2.1) and its scalar match, i.e., linearized autonomous components in (2.8). Furthermore, (2.8) is defined explicitly, whereas definition of its counterpart given in [13] requires numerical evaluation of $W(t)$ and subsequently, $p\left(W(t)\right)$ and $c\left(W(t)\right)$. Albeit that the current version of auxiliary equation is gaining efficiency and widening the scope of the relevant applications, its primary counterpart can provide sharper estimates in common application domain.

To straight up further referencing, we write a homogeneous counterpart of (2.8) as follows,

$$
\begin{align*}
z = (\alpha_i + \|G_{-}(t)\|)z + L(t, z)
\end{align*}
$$

(2.10)

Subsequently, we estimate solutions to (1.3) and (1.4) via assessing the behavior of the corresponding solutions to the scalar equations (2.8) and (2.10). For this sake we assume as well that these last equations admit unique solutions $\forall t \geq t_0$.

Local analysis of boundedness of solutions to (2.8) and stability of the trivial solution to (2.10) is simplified under the following condition,

$$
\begin{align*}
L(t, z) \leq l_2(t)z, \quad \forall t \geq t_0, \quad z \in [0, \hat{z}], \quad \hat{z} > 0
\end{align*}
$$

(2.11)

which can be interpreted as application of Lipschitz continuity condition to a scalar nonnegative function $L(t, z)$ with $z \geq 0$. Note that definition of $l_2(t)$ is exceedingly simplified for scalar functions. Thus, the auxiliary equation can be locally linearized by application of (2.11) which straightforward the relevant inferences and provides the corresponding conditions in a closed form.

3. Linearized auxiliary equation

Application of (2.11) to (2.8) yields its linear counterpart that can be written as follows,
\[
\dot{z} = \mu(t) z + \|F(t)\|, \quad z \in [0, \hat{z}], \quad \forall t \geq t_0 \\
\begin{align*}
z(t_0, z_0) &= z_0 = \|V^{-1}x_0\| \\
\end{align*}
\]  
\( (3.1) \)

where \( \mu(t) = \alpha_t + \|G_\mu(t)\| + l_2(t) \) and \( \|F(t)\| = F \|V^{-1}\eta(t)\| \).

Obviously, \( (3.1) \) admits the following solution,
\[
z(t, z_0) = z_0 z_h(t) + \int_{t_0}^{t} \theta(t, \tau) \|F(\tau)\| \, d\tau
\]
\( (3.2) \)

where \( z_h(t) = \exp \left( \int_{t_0}^{t} \mu(\tau) \, d\tau \right) \) and \( \theta(t, \tau) = \exp \left( \int_{\tau}^{t} \mu(s) \, ds \right) \). In the reminder of this section, we will assume without repetition that \( G_\mu(t) \) and \( l(t) \) are continuous functions ensuring continuity of \( z(t, z_0), \forall t \geq t_0 \). To simplify subsequent references, we will write as well a homogeneous counterpart to \( (3.1) \) as follows,
\[
\dot{z} = \mu(t) z \\
\]
\( (3.3) \)

Clearly, \( (3.3) \) is stable if and only if,
\[
\int_{t_0}^{t} \mu(\tau) \, d\tau < \infty, \quad \forall t \geq t_0
\]
\( (3.4) \)

and asymptotically stable if and only if
\[
\lim_{t \to +\infty} \left( \int_{t_0}^{t} \mu(\tau) \, d\tau \right) = -\infty
\]
\( (3.5) \)

In turn, stability of the trivial solution to \( (1.2) \) can be inferred through analysis of behavior of solutions to \( (3.3) \) if \( \vec{z}(t, z_0) \in [0, \hat{z}] \) and \( x(t, x_0) \in \Omega_2, \forall t \geq t_0 \). This comprises the following

**Theorem 1.** Assume that \( (3.3) \) is stable. Then the trivial solution of \( (1.2) \) is stable as well. Assume, additionally, that \( \|x_0\| \) is a sufficiently small value. Then, the following inequality holds,
\[
\|x(t, x_0)\| \leq \|V\| \|z(t, z_0)\|, \quad z_0 = \|V^{-1}x_0\|, \quad \forall t \geq t_0, \quad x(t, x_0) \in \Omega_2
\]
\( (3.6) \)

where \( \vec{z}(t, z_0) \) is a solution to \( (3.3) \). Note that if \( \Omega_2 = \mathbb{R}^n \), then \( (3.6) \) holds \( \forall x_0 \in \mathbb{R}^n \).

**Proof.** In fact, since \( (3.3) \) is stable and \( z_h(t) \) is a continuous function, \( \sup_{t \geq t_0} z_h(t) = z_s \), where \( 1 \leq z_s < \infty \), which yields that \( \sup_{t \geq t_0} \vec{z}(t, z_0) = z_s x_0 \).

Next, let \( S \) be a ball with radius \( r = \|V\| z_m z_0 \) that is centered at \( x = 0, \vec{z} \geq z_s, z_0 \in [0, z_{max}] \). In turn, let us choose \( z_0 \) such that \( S \subset \Omega_2 \), but \( S \cap \partial \Omega_2 = \emptyset \), where \( \partial \) and \( \partial \Omega_2 \) stand for empty set and the boundary of \( \Omega_2 \), respectively. Next, assume in contradiction that for some small values of \( \|x_0\| \in S \), \( \exists t_1 > t_0 \) such that \( x(t_1, x_0) \notin S \), but \( x(t_1, x_0) \in \Omega \). This yields that \( \|x(t_1, x_0)\| > r \) and consequently,
\[
\|x(t_1, x_0)\| > \|V\| \sup_{t \geq t_0} \vec{z}(t, z_0).
\]

Yet, the last inequality fails since \( (3.6) \) holds for \( t = t_1 \) which yields that,
\[
\|x(t_1, x_0)\| \leq \|V\| \vec{z}(t_1, z_0) \leq \|V\| z_s z_0 = r \text{. Thus, } x(t_1, x_0) \in S \subset \Omega_2, \forall t \geq t_0 \text{ and, consequently, } (3.6) \text{ holds} \quad \square \
Note that (3.6) includes an additional condition that is missed in (2.9) since this inequality is consequence of application of both inequalities (1.14) and (2.11).

**Theorem 2.** Assume that (3.3) is asymptotically stable. Then the trivial solution of (1.2) is asymptotically stable as well. If additionally, $\|x_0\|$ is a small value, then (3.6) holds as well.

**Proof.** Literally, under this more conservative condition (3.6) holds due to prior statement which affirms the current one.

Next, we present some efficient stability conditions appending the above statements.

**Corollary 1.** Assume that either (3.4) or (3.5) holds. Then the trivial solution to (1.4) is either stable or asymptotically stable, respectively, and, in turn, inequality (3.6) holds for sufficiently small values of $\|x_0\|$.

**Proof.** Really, under the above conditions (3.3) is either stable or asymptotically stable, which, due to Theorems 1 and 2, assures this statement.

**Corollary 2.** Assume that $\mu(t) \leq -\nu$, $\forall t \geq t_0$, $\nu > 0$. Then, the trivial solution of (1.2) is asymptotically stable and inequality (3.6) holds for some small values of $\|x_0\|$.

**Proof.** In fact, under the above conditions (3.3) is asymptotically stable. Thus, this statement holds due to Theorem 2.

In sequel, let us define $\gamma = \lim_{t \to \infty} \sup_{t_0 \leq t < \infty}(t-t_0)^{-1} \int_{t_0}^{t} H(\tau) d\tau$, where $H(t) = \|G_e(t)\| + l(t)$, which steers us to

**Corollary 3.** Assume that $\phi = \alpha_1 + \gamma < 0$ and $\|x_0\|$ is sufficiently small. Then, the trivial solution of (1.2) is asymptotically stable and (3.6) holds for sufficiently small values of $\|x_0\|$.

**Proof.** In fact, condition $\phi < 0$ implies that the Lyapunov exponent of (3.3) is negative which yields that the trivial solution of (3.3) is asymptotically stable [1] and, due to theorem 2, (3.6) holds for sufficiently small values of $\|x_0\|$.

**Corollary 4.** Assume that $\alpha_1 < 0$ and either $\int_{t_0}^{\infty} H(\tau) d\tau < \infty$ or $\lim_{t \to \infty} H(t) = 0$. Then, the trivial solution of (1.2) is asymptotically stable and (3.4) holds for sufficiently small values of $\|x_0\|$.

**Proof.** In fact, either condition of this statement implies asymptotic stability of (3.3), see [6], chapter 4, sec. 3.

Note that the above statements grant stability conditions and bounds of solutions to (1.4) in closed form if $f(t, x)$ is a continuous piecewise polynomial function in $x$ since in this case (1.14) and (2.11) can be represented in analytical form. In contrast, utility of either (1.8) or (1.10) requires simulations of $w(t)$ and subsequent estimations of the corresponding parameters or function from data simulated on the chosen time interval. The solution of such inverse problems can be sensitive to variation of various parameters of this set up, e.g., the length of time interval, etc. Consequently, to our knowledge, this problem has not been attended yet for a practically feasible system.

In turn, let us develop some criteria of boundedness of solutions to (1.1). Clearly, a solution to (3.1), $z(t, z_0) < \infty$, $\forall t \geq t_0$ if (3.3) is either stable or asymptotically stable and

$$\rho(t_0) = \sup_{t \geq t_0} \int_{t_0}^{t} V^{-1} \eta(t) d\tau < \infty, \forall t_0 \in T,$$

which leads to the following,
Theorem 3. Assume that (3.3) is stable and (3.7) holds and both \( \|x_0\| \) and \( F_0 \) are sufficiently small. Then, \( \|x(t, x_0)\| = O(\|x_0\| + F_0) \), \( \forall t \geq t_0 \) and inequality (3.6) holds, where \( x(t, x_0) \) and \( z(t, z_0) \) are solutions to equations (1.1) and (3.1), respectively. Furthermore, if, additionally, (3.3) is asymptotically stable then \( \lim_{t \to \infty} \|x(t, x_0)\| = O(F_0) \).

Proof. In fact, the conditions of this statement imply that \( \sup_{t \geq t_0} z(t, z_0) = z, z_0 + \rho(t_0) F_0 = r < \infty \). Choosing \( \hat{z} \) suitably large and \( z_0 \) and \( F_0 \) appropriately small, we get that \( \|x_0\| \) is fitly small, which let us to repeat the steps used prior to prove theorem 1. This implies that under the above conditions, \( x(t, x_0) \in \Omega_2, \forall t \geq t_0 \) which affirms (3.6) and, consequently, the above statement \( \square \)

Corollary 5. Assume that \( \mu(t) \leq -\nu, \forall t \geq t_0, \nu > 0 \) and both \( \|x_0\| \) and \( F_0 \) are appropriately small and (3.7) holds. Then, inequality (3.6) holds and \( \lim_{t \to \infty} \|x(t, x_0)\| \leq F_0 / \nu \).

Proof. In fact, the conditions of Theorem 3 hold under assumptions of this statement which implies that (3.6) holds as well. Consequently, we get that \( \lim_{t \to \infty} z_h(t) = 0 \) and \( \lim_{\kappa \leq t \to \infty} t \int_{t_0}^{t} \theta(t, \tau) \|F(\tau)\| d\tau \leq F_0 / \nu \) \( \square \).

Next, let us assume that the Lyapunov exponent of (3.3), \( \phi = \alpha_i + \gamma < 0 \) which implies that \( z_h(t, z_0) \leq D_i (\epsilon_i) e^{-\rho(t-t_0)} \) and \( \theta(t, \tau) \leq D_i (\epsilon_i) e^{-\rho(t-t_0)} \), where, \( -\rho_i = \phi + \epsilon_i \cdot \rho_i > 0, k = 1, 2 \).

Corollary 6. Assume that \( \phi < 0 \) and both \( \|x_0\| \) and \( F_0 \) are appropriately small and (3.7) holds. Then, inequality (3.6) holds and \( \lim_{t \to \infty} \|x(t, x_0)\| \leq D_i (\epsilon_i) F_0 / \rho \).

Proof. Really, theorem 3 holds under conditions of this statement affirming that (3.6) and, in turn, this statement holds \( \square \).

Let us repeat lastly that if, in turn, \( f(t, x) \) is a continuous piecewise polynomial function in \( x \), then, \( \Omega_2 \equiv \mathbb{R}^n \), which implies that our boundedness conditions can be aliened in closed form and hold for sufficiently large and positive values of \( \hat{z}, \forall x_0 \in \mathbb{R}^n \) and \( F_0 \in \mathbb{R}^n \).

4. Nonlinear auxiliary equation

Analysis of solutions to a scalar nonlinear equation (2.8) conveys fewer conservative criteria of boundedness/stability of the initial nonlinear and multidimensional equations and enfolds estimation of their trapping/stability regions which are intractable in applications of linearized auxiliary equation. Below we present for convenience some standard definitions of trapping/stability regions, see also [14], where such definitions were also adopted.

Definition 1. A compact set of initial vectors, that includes zero-vector, is called a trapping region of equation (1.3) about zero if condition, \( x_0 \in \mathbb{N}_1 \) implies that \( x(t, x_0) \in \mathbb{N}_1, \forall t \geq t_0 \).

Clearly this definition acknowledges that \( \mathbb{N}_1 \) is an invariant set of (1.3) containing zero.

Definition 2. An open set of initial vectors, that includes zero-vector, is called a stability region of the trivial solution to (1.4) if condition, \( x_0 \in \mathbb{N}_2 \) implies that \( \lim_{t \to \infty} x(t, x_0) = 0 \).

In general, (2.8) is a nonintegrable, nonlinear but scalar equation with variable coefficients and behavior of its solutions can be readily assessed in simulations or by laying out some integrable autonomous equation bounding (2.8). The former approach is substantially abridged due to the following.
**Theorem 4.** Let initial values, \( z'_0 \leq z''_0 \), then \( z(t, z'_0) \leq z(t, z''_0), \forall t \geq t_0 \), where \( z(t, z_0) \) is a solution to (2.8).

**Proof.** In fact, the solutions to (2.8) do not intersect in \( t \times y \) plane due to uniqueness □

Application of the above statement grants some boundedness/stability criteria for equations (1.1) and (1.2), respectively. Firstly, let us define a set of centered at zero concentric ellipsoids, \( E(z) \subset \mathbb{R}^n \) as follows,

\[
E(z) : \| V^{-1}x \| = z \in \mathbb{R}_{\geq 0}
\]

(4.1)

Also, we assume that \( \partial E(z) \subset \mathbb{R}^{n-1} \) defines the boundaries of these ellipsoids and \( E_-(z) = E(z) - \partial E(z) \).

To streamline further derivations, we accept in this section without repetition that \( \Omega_2 = \mathbb{R}^n \) and \( \hat{z} \) is aptly large. This prompts the following.

**Theorem 5.** Assume that the trivial solution to (2.10) is either stable or asymptotically stable. Then the trivial solution to (1.4) is stable/asymptotically stable as well, respectively. Furthermore, let the interval \( [0, \bar{z}] \) defines the maximal stability region of asymptotically stable trivial solution to (2.10). Then the set \( E_-(\bar{z}) \) is contained in stability region of (1.4).

**Proof.** Really, the proof of this statement follows from inequality (2.9), where it is presumed that \( x(t, x_0) \) and \( z(t, z_0) \) are solutions to equations (1.4) and (2.10), respectively □

**Theorem 6.** Assume that interval \( [0, \bar{z}] \) defines a trapping region of (2.8) about \( z \equiv 0 \), i.e., \( \| z(t, z_0) \| \leq \sigma \in \mathbb{R}^n \), \( \forall z_0 \in [0, \bar{z}], \forall t \geq t_0 \). Then, the ellipsoid \( E(\bar{z}) \) is included into the trapping region of equation (1.1) about zero, i.e., \( \| x(t, x_0) \| < \infty, \forall x_0 \in E(\bar{z}), \forall t \geq t_0 \) and inequality (2.9) holds.

**Proof.** As formerly, the proof of this statement follows from inequality (2.9), where \( x(t, x_0) \) and \( z(t, z_0) \) are considered as solutions to equations (1.3) and (2.8), respectively □

Thus, the problem of estimating the trapping/stability regions about point \( x \equiv 0 \) of multidimensional equations (1.1) or (1.2) is reduced to reckoning the threshold value of \( z_0 \) splitting the interval of initial values of solutions to (2.8) in two parts associating with qualitatively distinct behavior of such solutions on long time-intervals. Then the boundary of trapping/stability region for either (1.3) or (1.4) is given by the following formula, \( z_0 = \| V^{-1}x_0 \| \). In turn, the task of simulating the threshold value is markedly simplified since \( z(t, z_0) \) is monotonically increases in \( z_0, \forall t \geq t_0 \).

Yet, the structure of solutions to (2.8) and, in turn, its multidimensional counterparts (1.1) and (1.2) can be further divulged through analysis of solutions to additionally simplified equations which either bound or approximate (2.8). In this context we acknowledge that under (2.11) solutions to linear equations (3.1) and (3.2) bound from above the corresponding solutions to (2.8) and (2.10) that are stemmed from the same initial values. In turn, a complementary stability condition can be recouped using the Massera, Chetaev and Malkin theorem, which let to relax (2.11) into a more general condition, see [1] for a contemporary reference and citations. In this regard, we assume that

\[
L(t, z) = L_1(t)z + L_2(t, z), \text{where } L_1 : [t_0, \infty) \to \mathbb{R}^r_{\geq 0}, L_2 : [t_0, \infty) \times \mathbb{R}^r_{\geq 0} \to \mathbb{R}^r_{\geq 0}
\]

and

\[
L_2 (t, z) \leq l_3 z', r > 1, \forall z \in [0, \bar{z}], l_3, \bar{z} \in \mathbb{R}^r_{\geq 0}
\]

(4.2)

Then (2.10) can be written as

\[
\hat{z} = \eta (t) z + L_2(t, z)
\]

(4.3)
where \( \eta(t) = \alpha_1 + \left\| G_\tau(t) \right\| + L_1(t) \). Next, let \( \varphi(t, \tau) = \int_\tau^t \eta(s) \, ds \).

\[
\varphi(t, \tau) \leq \varphi_0 - \varphi_1 (t - \tau) + \varphi_2 \tau, \quad \varphi_0, \varphi_1, \varphi_2 \in \mathbb{R}_{>0}, \varphi_2 \in \mathbb{R}_{\geq0},
\]

(4.4) and

\[
(r-1)\varphi_1 \geq \varphi_2
\]

(4.5)

This comprises the following.

**Theorem 7.** Assume that (4.2) is contended with sufficiently small \( l_3 \) and (4.4) and (4.5) hold as well. Then, the trivial solution of (4.3) is asymptotically stable which, in turn, implies that the trivial solution of the corresponding equation (1.4) is asymptotically stable as well.

**Proof.** Indeed, the proof of this statement directly follows from application of the mentioned above theorem to equation (1.4).\( \square \)

Some additional approaches to analysis of the structure of solutions to a scalar auxiliary equation were outlined in [13]. Below we complement and apply some of these techniques to a modified auxiliary equation (2.8) which is developed in this paper. With this intend, let us write two scalar and autonomous equations as follows,

\[
\begin{align*}
\dot{z}_1 &= \kappa_+ z_1 + L_+ (z_1) + F_0 \\
\dot{z}_2 &= \kappa_- z_2 + L_- (z_2) + F_-
\end{align*}
\]

(4.6)

\[
\begin{align*}
\dot{z}_{20} &= \left\| V^{-1} x_0 \right\| \\
\dot{z}_{21} &= \left\| V^{-1} x_0 \right\|
\end{align*}
\]

(4.7)

where \( \kappa_+ = \alpha_1 + \sup_{t \geq t_0} \left\| B_+ (t) \right\|, \quad \kappa_- = \alpha_1 + \inf_{t \geq t_0} \left\| B_- (t) \right\|, \quad L_+ = \sup_{t \geq t_0} L(t, z), \quad L_- = \inf_{t \geq t_0} L(t, z), \quad z \in \mathbb{R}^n, \)

\[
F_- = F_0 \inf_{t \geq t_0} \eta(t).
\]

Obviously, right – sides of the last two integrable and scalar equations bound from above and below the one in (2.8), which, due to comparison principle, implies that, \( z_2(t, z_0) \leq z(t, z_0) \leq z_1(t, z_0) \), \( \forall t \geq t_0 \), where \( z(t, z_0) \) is a solution to (2.8). In turn, the structures of solutions to (4.6) and (4.7) are determined by location and stability of their fixed solutions. Application of such reasoning to (4.6) yields sufficient conditions for boundedness of solutions to (2.8) and stability of the trivial solution to (2.10) which, consequently, embraces the corresponding statements for solutions to (1.3) and (1.4). Conversely, resolving behavior of solutions to (4.7), brings necessary conditions for boundedness of solutions to (2.8) and stability of the trivial solution to (2.10). Let us first recap the akin sufficient conditions in the following.

**Theorem 8.** Assume firstly that \( F_0 = 0 \) and (I) \( z_- \) is a unique unstable fix solution to (4.6). Then the trivial solution to (2.10) is asymptotically stable, \( \lim_{t \to \infty} z(t, z_0) = 0, \forall z_0 \in [0, z_-] \) and, consequently, \( z_- \) bounds from above the stability region of its trivial solution, which, in turn, implies that the trivial solution to (1.4) is asymptotically stable and \( \lim_{t \to \infty} \| x(t, x_0) \| = 0, \forall x_0 \in E_\tau (z_-) \), i.e. \( E_\tau (z_-) \) is enclosed into the stability region of the trivial solution to (1.4). (II) Let \( z_- \) is a stable and unique fix solution to (4.6). Then, \( z(t, z_0) \leq z_- \), \( \forall z_0 \leq z_- \) and, \( z_- \) bounds from above the trapping region of (2.10) about \( z \equiv 0 \) which, in turn, implies that \( \| x(t, x_0) \| \leq \| x(t, z_-) \|, \forall x_0 \in E_\tau (z_-) \) and \( E_\tau (z_-) \) is enclosed into the trapping region of the trivial solution to (1.4).

Next, let us assume that \( F_0 > 0 \), \( \kappa_- < 0 \) and (III) (4.6) has two fix solutions \( 0 < Z_2 < Z_1 \) corresponding to simple roots of its right – side. Then, \( Z_1 \) and \( Z_2 \) are unstable and stable fix solutions to (4.6), respectively, and
\( z(t, z_0) \leq Z_2, \forall z_0 \leq Z_2 \) and \( \lim_{t \to \infty} z(t, z_0) = Z_2, \forall z_0 < Z_1 \), which, consequently, yields that
\[
\|x(t, x_0)\| \leq \max\left\{ \|v\|Z_2, \forall x_0 \in E(Z_2) \right\}, \forall t \geq t_0,
\]
\[
\lim \sup_{t \to \infty} \|x(t, x_0)\| \leq \|v\|Z_2, \forall x_0 \in E_-(Z_2)
\]
Lastly, let us assume that \( F_0 > 0, \kappa < 0 \) and (VI) \( Z_1 = Z_2 = Z \), then, \( z(t, z_0) \leq Z, \forall z_0 < Z \) and
\[
\lim_{t \to \infty} z(t, z_0) = Z, \forall z_0 < Z \text{ which implies that } \|x(t, x_0)\| \leq \|v\|Z, \forall x_0 \in E(Z), \forall t \geq t_0.
\]
**Proof.** The proof of this statement immediately follows from qualitative analysis of behavior of solutions to (4.6) and application of inequality (2.9). Let, for instance, show that \( Z_1 \) and \( Z_2 \) are unstable and stable fixed solutions to (4.6). In fact, \( z_1(0) = F_0 > 0 \) which implies that continuous right-side of (4.6) is a positive function, \( \forall z < Z_2 \)
since \( Z_2 \) is a simple and smallest root of equation, \( \kappa z_2 + L_1(z_2) + F_0 = 0 \). Hence \( Z_2 \) is stable and, in turn, \( Z_1 \) is unstable fix solutions to (4.6). □

Application of similar reasoning to equation (4.7) leads to necessary conditions for boundedness and stability of the relevant solutions to (2.8) and (2.10), respectively, which can aid simulations of these equations. Nonetheless, these conditions do not directly endorse the consistent properties of solutions to (1.3) or (1.4). Yet, utility of the lower bound of solution to (2.8) can simplify approximate resembling of the trapping/stability regions for (1.3) or (1.4). In fact, assume that \( z_0 \) is a threshold initial value for (4.7) such that, for instance, \( \lim_{t \to \infty} z_2(t, z_0) < \infty, \forall z_0 \leq z_0 \)
and \( \lim_{t \to \infty} z_2(t, z_0) = \infty, \forall z_0 > z_0 \). Then, \( \lim_{t \to \infty} z_2(t, z_0) = \infty, \forall z_0 > z_0 \) as well. Thus, \( z_0 \) yields an upper estimate of the actual threshold initial value of solutions to (2.8) which can be accessed through analytical reasoning.

Next, we endorse an additional lower bound for solutions to (2.10) through utility of integrable Bernoulli equation bounding from below the right-side of (2.10). In fact, assume that \( L(t, z) \) is a polynomial in \( z \) that can be written as follows, \( L(t, z) = L_1(t) z + L_2(t) z^2 + P_3(t, z) \), where \( L_1(t), L_2(t) > 0, \forall t \geq t_0 \) and, \( P_3(t, z) \geq 0, \forall t \geq t_0 \) is a polynomial in \( z \) with left out constant, linear and quadratic terms. This enfolds the following Bernoulli equation,
\[
\dot{z}_3 = \left( \alpha + \|G_-(t) + L_1(t)\| \right) z_3 + L_2(t) z_3^2
\]
\[
z_3(t_0, z_0) = z_0 = \|v^{-1} x_0\|
\]
Clearly, \( z_3(t, z_0) \leq z(t, z_0), \forall t \geq t_0 \). Thus, the threshold initial value procuring in testing of solutions to an integrable equation (4.8) exceeds the one which can retrieve in simulations of (2.10). The next section assesses that these values turn out to be quite close to each other in simulations which are presented below.

5. Simulations

This section applies the developed methodology to decipher the evolutions of solutions’ norms and appraise the degree of stability/boundedness of two nonlinear systems with time-dependent nonperiodic coefficients which are common in various applications [40]. These systems comprise two coupled Van der Pol – like or Duffing - like oscillators with variable coefficients. In both cases direct utility of our prior technique, which is based on application of (1.15) [13], is compromised since in these cases, \( \lim_{t \to \infty} c(t) = \infty \).

5.1 Coupled Van der Pol – like system

Equation (2.1) can be turned in such system if we assume that
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-(\omega_1^2 + d) & -\alpha_1 & d & 0 \\
0 & 0 & 0 & 1 \\
d & 0 & -(\omega_2^2 + d) & -\alpha_2
\end{pmatrix},
G_x = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-g_{21}(t) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -g_{43}(t) & 0
\end{pmatrix},
\]

\[
F_x(t) = \begin{pmatrix}
0 \\
F_{x_1}(t) \\
F_{x_2}(t)
\end{pmatrix}^T.
\]

\[
f = \begin{pmatrix}
0 \\
-m_1x_2^3 \\
-m_2x_4^3
\end{pmatrix}^T
\]

where \(\alpha_1 = 0.4, \alpha_2 = 0.2, \omega_1^2 = 1, \omega_2^2 = 4, g_{21} = a_1 \sin r_1 t + a_2 \sin r_2 t, g_{43} = b_1 \sin s_1 t + b_2 \sin s_2 t, a_1 = a_2 = b_1 = b_2 = 0.1, r_1 = 3.14, r_2 = 6.15, s_1 = 3.1, s_2 = 6.28, F_{x_1} = F_{x_2} = F_0 \sin q t, i = 1, 2, q_1 = 5.43, q_2 = 10\). Note that in most but all our simulations we assumed that \(m_1 = m_2 = 60\).

Next, let \(V = \begin{bmatrix} v_{ij} \end{bmatrix}, i, j = 1, \ldots, 4\) is the eigenmatrix for \(A\) and in equation (2.2)
\[
A = \text{diag}(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*), \quad \lambda_1^* = \lambda_2^* = \alpha_k + i\beta_k, \quad \lambda_3^* = \lambda_4^* = -i\beta_k, k = 1, 2, \alpha_1 \geq \alpha_2, G(t) = V^{-1}G \dot{V}.
\]

\[
F(t) = V^{-1}F_x \quad \text{and} \quad f = V^{-1}\begin{pmatrix}
0 \\
-\mu_1 \left(\sum_{k=1}^{4} v_{2k}y_k \right)^3 \\
-\mu_2 \left(\sum_{k=1}^{4} v_{4k}y_k \right)^3
\end{pmatrix}^T.
\]

Finally, we set that in equation (2.8), \(G_\omega = G - \text{Im}\left(\text{diag}(G)\right)\) and \(L(t, z) = \|V^{-1}\delta\|z^3\), where \(\delta = [0 \quad \delta_1 \quad 0 \quad \delta_2]^T\).

\[
\delta_1 = -\mu_1 \begin{pmatrix}
\text{abs}(v_{21}^3) + 3\text{abs}(v_{21}v_{22})^2 + 3\text{abs}(v_{21}^2v_{22}) + \text{abs}(v_{22}^3) + \\
\frac{1}{3}\left(\text{abs}(v_{21}v_{23}^2) + 2\text{abs}(v_{21}v_{22}v_{23}) + \text{abs}(v_{22}v_{23}) + \text{abs}(v_{21}v_{24}) + \\
2\text{abs}(v_{21}v_{22}v_{24}) + \text{abs}(v_{22}v_{24}) + \\
2v_{22}v_{23}v_{24} + v_{23}v_{24}^2 + \\
\text{abs}(v_{23}^3) + 3\text{abs}(v_{23}v_{24})^2 + \text{abs}(v_{24}^3)
\end{pmatrix}
\]

\[
\delta_2 = -\mu_2 \begin{pmatrix}
\text{abs}(v_{41}^3) + 3\text{abs}(v_{41}v_{42})^2 + 3\text{abs}(v_{41}^2v_{42}) + \text{abs}(v_{42}^3) + \\
\frac{1}{3}\left(\text{abs}(v_{41}v_{43}^2) + 2\text{abs}(v_{41}v_{42}v_{43}) + \text{abs}(v_{42}v_{43}) + \text{abs}(v_{41}v_{44}) + \\
2\text{abs}(v_{41}v_{42}v_{44}) + \text{abs}(v_{42}v_{44}) + \\
2v_{42}v_{43}v_{44} + v_{42}v_{44}^2 + \\
\text{abs}(v_{43}^3) + 3\text{abs}(v_{43}v_{44})^2 + \text{abs}(v_{44}^3)
\end{pmatrix}
\]
The results of simulation of the Van der Pol-like system with indicated above parameters are partly shown in Fig. 1. Fig. 1a contrasts in solid, datched and dotted lines time-histories of the norms of solutions to equations (1.4), (2.10) and (2.7) with \( F_0 = 0 \), respectively. Note that in this case \( \mu_1 = \mu_2 = 0.1 \) and \( x_0 \) is located afar of the boundary of stability region. Clearly, utility of (2.10) notably enhances the estimates delivered through application of a more conservative equation (2.7) with \( F_0 = 0 \) and adequately estimate the norm of actual solutions to (1.4).

In subsequent simulations presented in this subsection we set that, \( \mu_1 = \mu_2 = 60 \). Figs. 1b - d display in datched and solid lines projections of the boundary of the stability region on some coordinate planes which are acquired in simulations of equations (1.4) and (2.10) corresponding to the Van der-Pol-like model. In fact, simulations of (2.10) estimate the threshold value of \( z_0 \) which, in turn, defines the boundary ellipsoid through application of the formula, \( z_0 = \| \nu^{-1} x_0 \| \). Clearly, the attained estimates turn out to be fairly considerate if the structure and parameters of the model are defined precisely. Yet, the practical merit of these estimates becomes more apparent for systems under uncertainty, where, for instance, it is presumed that at most the norms of some model components or parameters are defined presicelly [4, 5].

### 5.2 Coupled Duffing-like system

For this system (5.1) remains intact but (5.2) should be adjusted as follows,

\[
f = \begin{pmatrix} 0 & -\mu_1 x_1^3 & 0 \\ 0 & 0 & -\mu_2 x_3^3 \end{pmatrix} \]

which, in turn, requires some alteration in (5.3) and (5.4), i.e., \( v_{2k} \to v_{i_k} \) and \( v_{4k} \to v_{3_k} \), respectively. Fig. 2 shows in datched and solid lines projections on some coordinate -planes of the boundary of stability region of the trivial solution to (1.4) corresponding to the coupled Duffing model and equation (2.10), respectively. As in section 5.1, the current estimations turn out to be fairly conservative if for precisely defined systems but turn out to be more...
compelling for systems under uncertainties. Note that simulations on Fig. 2b are started at $t_0 = 1.43$, whereas other plots in this figure are simulated with $t_0 = 0$. Clearly, comparison of figures 2a and b shows that the shape of attractor’s boundary on these figures are noticeably affected by initial time moment. However, such dependence might become less pronounced for other values of $t_0$ conceivably due to almost periodic nature of time-dependent coefficients adopting in this model.

Note that simulations on Fig. 2b are started at $t_0 = 1.43$, where is other plots in this figure are simulated with $t_0 = 0$. Clearly, comparison of figures 2a and b shows that the shape of attractor’s boundary on these figures are noticeably affected by initial time moment. However, such dependence might become less pronounced for other values of $t_0$ conceivably due to almost periodic nature of time-dependent coefficients adopting in this model.

Next, we review some results on estimating of the trapping regions for equation (1.3). Fig. 3 displays in dashed and solid lines projections on the coordinate – planes of the boundary of the trapping region about zero for equation (1.3) corresponding to coupled Duffing – like model with variable coefficients and relevant equation (2.8), respectively. In these simulations we set that $\mu_1 = \mu_2 = 0.1$ and $F_{01} = 0.01$, $F_{02} = 0$. Our outcomes in these simulations are comparable to ones that are presented prior and show that our fairly conservative estimates squeeze and smooth the actual boundary of the trapping region and become more appealing for systems under uncertainty. Yet, utility of these estimates for precisely defined systems provides numerical affirmation of our boundedness and stability criteria. Nonetheless, combining these techniques with the appropriate successive approximations should considerably refine the accuracy of the corresponding estimates, see [14], where such strategy was developed and applied to the suitable systems.

After all, we would also mention that threshold values, which are obtained through simulations of (2.8) and analytical solutions to (4.8), turn out to be close in our simulations which practically superimpose the corresponding boundary-curves on our plots.
Fig. 3a displays in dashed and solid lines projection on $x_1 \times x_2$ -plane of the boundary of trapping region obtaining in simulations of nonhomogeneous equation (1.3) describing coupled Duffing model and corresponding equation (2.8), respectively.

Fig. 3a displays in dashed and solid lines projection on $x_1 \times x_3$ -plane of the boundary of trapping region obtaining in simulations of nonhomogeneous equation (1.3) describing coupled Duffing model and corresponding equation (2.8), respectively.

Fig. 3a displays in dashed and solid lines projection on $x_3 \times x_4$ -plane of the boundary of trapping region obtaining in simulations of nonhomogeneous equation (1.3) describing coupled Duffing model and corresponding equation (2.8), respectively.

Fig. 3a displays in dashed and solid lines projection on $x_2 \times x_4$ -plane of the boundary of trapping region obtaining in simulations of nonhomogeneous equation (1.3) describing coupled Duffing model and corresponding equation (2.8), respectively.

6. Conclusion and upcoming studies

In [13] we derived a scalar and nonlinear auxiliary equation with solutions bounding from above the norms of solutions to the original multidimensional, nonlinear and nonautonomous systems and show the application of this technique to assessment of boundedness and stability of nonlinear time-invariant systems. Yet, the application domain of this approach is constrained since the underlying auxiliary equation contains a function often approaching infinity with $t \to \infty$. The current paper presents a novel technique casting the auxiliary equation in a form escaping this limitation for substantial class of systems arising in applications. Furthermore, the new auxiliary equation is simpler than the prior one and more efficient in computations. Still in the common application domain the current estimates can be more conservative than their prior counterparts.

Next, we present various boundedness and stability criteria stemmed from analysis of novel linearized and nonlinear auxiliary equations. Lastly, we authenticate our study in inclusive simulations of the systems with typical dissipative and conservative nonlinear components that were intractable to our prior technique. The simulations show that our upper estimates of the norms of solutions to nonlinear and nonautonomous systems turn out to be adequate if their initial points are located afar from the boundaries of the corresponding trapping/stability regions. Yet, the boundaries of the trapping/stability regions are estimated rather conservatively by our current technique if the corresponding systems are defined precisely. But such estimates turn out to be more appealing for systems under uncertainty.

Note also that the precision of our estimates can be substantially improved through integration of our current technique with the methodology of successive approximations that was outlined in [14]. This should grant extension of such combine approach to applications that were prior constrained.

Our current results should aid further developments of techniques for enhanced gaging of degree of stability of a wider class of practically important nonautonomous nonlinear systems as well as to evoke some analytical techniques for more efficient analysis of the nonlinear auxiliary equation. These topics will be addressed in our subsequent studies.
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