Non-Abelian Tensor Gauge Fields  
and  
New Topological Invariants

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Abstract

In this article we shall consider the tensor gauge fields which are possible to embed into the existing framework of generalized YM theory and therefore allows to construct the gauge invariant and metric independent forms in 2n+4 and 2n+2 dimensions. These new forms are analogous to the Pontryagin-Chern-Simons densities in YM gauge theory and to the corresponding series of densities in 2n+3 dimensions constructed recently in arXiv:1205.0027.

Keywords: Gauge Fields; Tensor Gauge Fields; Chern-Simons secondary characteristics
1 Introduction

The Abelian and non-Abelian chiral anomalies can be determine by a differential geometric method without having to evaluate the Feynman diagrams [1 2 3 4 5 6 7 8 9 10 11 12 13 14 16 17]. The non-Abelian anomaly in $2n-2$-dimensional space-time may be obtained from the Abelian anomaly in $2n$ dimensions by a series of reduction (transgression) steps and can be represented in a compact integral form [6 7 8 9 11 12 13 14 17].

In $D = 2n$ dimensions, the $U_A(1)$ anomaly is given by a $2n$-form:

$$\mathcal{P}_{2n} = Tr(G^n) = d \omega_{2n-1}, \quad (1.1)$$

where $\omega_{2n-1}$ is the Chern-Simons form to $2n - 1$ dimensions [6 13]:

$$\omega_{2n-1}(A) = n \int_0^1 dt \; Tr(AG_t^{n-1}), \quad (1.2)$$

where $G = dA + A^2$ is the 2-form YM field-strength tensor of the 1-form vector field $A = -igA^a_{\mu}L_a dx^\mu$ and $G_t = tG + (t^2 - t)A^2$.

In the recent articles [18 19] the authors have found the similar invariants in non-Abelian tensor gauge field theory [20 21 22]. These forms are defined in dimensions $D = 2n + 3$:

$$\Gamma_{2n+3}(A, A_2) = Tr(G^nG_3) = d \sigma_{2n+2}, \quad (1.3)$$

where $G_3 = dA_2 + [A, A_2]$ is the 3-form field-strength tensor for the rank-2 gauge field $A_2 = -igA^a_{\mu\lambda}L_a dx^\mu \wedge dx^\lambda$ and $G_{3t} = tG_3 + (t^2 - t)[A, A_2]$. The $(2n+2)$-form $\sigma_{2n+2}$ is [19]:

$$\sigma_{2n+2}(A, A_2) = \int_0^1 dt \; Tr(AG_t^{n-1}G_3t + ... + G_t^{n-1}AG_3t + G_t^nA_2). \quad (1.4)$$

The very fact that the tensor gauge fields introduced in [20 21 22] are symmetric over their last indices (see equation (2.1) ) prevents the construction of the invariant forms involving higher rank tensor gauge fields, that is the fields of the rank higher than two. Our intension in this article is to demonstrate that a class of the non-Abelian tensor gauge fields (2.1) can and should be extended to include new fields and therefore allows to construct the invariant forms in $D = 2n + 4$ and $D = 2n + 2$ dimensions. These new forms $\Phi_{2n+4} = d\psi_{2n+3}$ and $\Omega_{2n+2} = d\chi_{2n+1}$ are analogous to the Pontryagin-Chern-Simons densities $\mathcal{P}_{2n} = d\omega_{2n-1}$ in YM gauge theory and to the corresponding series of densities $\Gamma_{2n+3} = d\sigma_{2n+2}$ found in [19].

In the next section we shall introduce the tensor gauge fields which are possible to embed into the existing framework of generalized YM theory, their gauge transformations and the corresponding field strength tensor. In the third and fourth sections the invariant
forms $\Phi_{2n+4}$ and $\Omega_{2n+2}$ will be constructed. In the fifth section we shall consider the most general tensor gauge fields from a geometrical point of view [20, 21, 22].

2 Non-Abelian Tensor Gauge Fields

In the model of massless tensor gauge fields suggested in [20, 21, 22] the gauge fields are defined as rank-$(s+1)$ tensors

$$A^a_{\mu\lambda_1...\lambda_s}(x),$$

which are totally symmetric with respect to the indices $\lambda_1...\lambda_s$. The number of symmetric indices $s$ runs from zero to infinity. A priori the tensor fields have no symmetries with respect to the first index $\mu$. The index $a$ corresponds to the generators $L_a$ of an appropriate Lie algebra. The extended non-Abelian gauge transformation $\delta_\xi$ of the tensor gauge fields is defined as

$$\delta_\xi A^a_\mu = \partial_\mu \xi^a - ig[A^a_\mu, \xi],$$

$$\delta_\xi A^a_{\mu\lambda} = \partial_\mu \xi^a_{\lambda} - ig[A^a_\mu, \xi^a_\lambda] - ig[A^a_{\mu\lambda}, \xi],$$

$$\delta_\xi A^a_{\mu\lambda_1...\lambda_2} = \partial_\mu \xi^a_{\lambda_1,\lambda_2} - ig[A^a_\mu, \xi^a_{\lambda_1,\lambda_2}] - ig[A^a_{\mu\lambda_1}, \xi^a_{\lambda_2}] - ig[A^a_{\mu\lambda_2}, \lambda_1] - ig[A^a_{\mu\lambda_1,\lambda_2}, \xi],$$

where $\xi^a_{\lambda_1...\lambda_s}(x)$ are totally symmetric gauge parameters and comprises a closed algebraic structure. The generalized field-strength tensors are defined as [20, 21, 22]:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ig[A^a_\mu, A^a_\nu],$$

$$G^a_{\mu\nu,\lambda} = \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda} - ig( [A^a_{\mu\lambda}, A^a_\nu] ),$$

$$G^a_{\mu\nu,\lambda\rho} = \partial_\mu A^a_{\nu\lambda\rho} - \partial_\nu A^a_{\mu\lambda\rho} - ig( [A^a_{\mu\lambda}, A^a_{\nu\rho}] + [A^a_{\mu\rho}, A^a_{\nu\lambda}] ),$$

and transform homogeneously with respect to the extended gauge transformations $\delta_\xi$. The tensor gauge fields are in the matrix representation $A^a_{\mu\lambda_1...\lambda_s} = (L_c)^{ab} A^c_{\mu\lambda_1...\lambda_s} = if^{abc} A^c_{\mu\lambda_1...\lambda_s}$ with $f^{abc}$ - the structure constants of the Lie algebra.

Using field-strength tensors one can construct infinite series of forms $\mathcal{L}_s$ invariant under the transformations $\delta_\xi$. They are quadratic in field-strength tensors. The first terms are given by the formula [20, 21, 22]:

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_2 + ... = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - \frac{1}{4}G^a_{\mu\nu,\lambda}G^a_{\mu\nu,\lambda} - \frac{1}{4}G^a_{\mu\nu,\lambda\rho}G^a_{\mu\nu,\lambda\rho} + \frac{1}{4}G^a_{\mu\nu,\lambda}G^a_{\mu\nu,\lambda} + \frac{1}{4}G^a_{\mu\nu,\rho}G^a_{\mu\nu,\rho} + \frac{1}{2}G^a_{\mu\nu}G^a_{\mu\lambda,\nu} + ...$$

(2.4)
The Lagrangian contains quadratic in gauge fields kinetic terms, as well as cubic and quartic terms describing non-linear interactions of gauge fields with dimensionless coupling constant $g$.

Here we shall consider the set of fields which can be embedded into the existing framework of generalized YM theory and should be unified with the previous system of fields (2.1). First let us consider the rank-3 field which is now antisymmetric over its last two indices

$$A^a_{\mu\sigma_1\sigma_2}(x) = -A^a_{\mu\sigma_2\sigma_1}(x),$$

for this field the gauge transformation should be defined in following way

$$\delta A_\mu = \partial_\mu \xi - ig[A_\mu, \xi],$$
$$\delta A_{\mu\sigma_1\sigma_2} = \partial_\mu - ig[A_\mu, \zeta_{\sigma_1\sigma_2}] - ig[A_{\mu\sigma_1\sigma_2}, \xi],$$

where the tensor gauge parameter is antisymmetric $\zeta^a_{\sigma_1\sigma_2} = -\zeta^a_{\sigma_2\sigma_1}$. As one can verify these transformations form a closed algebra because

$$[\delta \zeta, \delta \varphi] A_{\mu\nu\lambda} = \delta \chi A_{\mu\nu\lambda},$$

where

$$\chi = [\zeta, \varphi], \quad \chi_{\sigma_1\sigma_2} = [\zeta, \varphi_{\sigma_1\sigma_2}] + [\zeta_{\sigma_1\sigma_2}, \varphi].$$

It is useful to compare the gauge transformations of the $A^a_{\mu\lambda_1\lambda_2}$ in (2.2) and of the $A^a_{\mu\sigma_1\sigma_2}$ in (2.6). As one can see they formally coincide, except the term $[A_{\mu\lambda_1}, \xi_{\lambda_2}] + [A_{\mu\lambda_2}, \xi_{\lambda_1}]$, which is explicitly symmetric under $\lambda_1 \leftrightarrow \lambda_2$ permutations. Therefore these fields can and should be unified into a general rank-3 gauge field $A^a_{\mu\nu\lambda}$ of which the symmetric and antisymmetric parts, with respect to the last two indices, reproduce the original fields.

In general we shall define the infinite set of fields

$$A^a_{\mu\sigma_1\sigma_2\rho_1\rho_2...\kappa_1\kappa_2}(x)$$

which are antisymmetric with respect to the permutations of pairs of indices

$$\sigma_1 \leftrightarrow \sigma_2, \quad \rho_1 \leftrightarrow \rho_2, \quad ..., \quad \kappa_1 \leftrightarrow \kappa_2,$$

but are symmetric under any permutations of these pairs

$$\sigma_1\sigma_2 \leftrightarrow \rho_1\rho_2, \quad ..., \quad \sigma_1\sigma_2 \leftrightarrow \kappa_1\kappa_2, ...$$
Thus for these fields take place the following relations

\[
A^a_{\mu \sigma_1 \sigma_2 \rho_1 \rho_2 \ldots \kappa_1 \kappa_2} = -A^a_{\mu \sigma_2 \sigma_1 \rho_1 \rho_2 \ldots \kappa_1 \kappa_2} = -A^a_{\mu \sigma_1 \sigma_2 \rho_2 \rho_1 \ldots \kappa_1 \kappa_2} = \ldots
\]
\[
-\delta_{\rho_1 \rho_2 \ldots \kappa_1 \kappa_2}^a = A^a_{\mu \rho_1 \rho_2 \sigma_1 \sigma_2 \ldots \kappa_1 \kappa_2} = \ldots = A^a_{\mu \kappa_1 \kappa_2 \rho_1 \rho_2 \sigma_1 \sigma_2} = \ldots
\]
\[
= A^a_{\mu \sigma_1 \sigma_2 \kappa_1 \kappa_2 \rho_1 \rho_2} = \ldots
\]

To simplify the description of these fields one can say that we have the gauge fields of the same type as in (2.1), but now the indices \(\{\lambda\}\) are replaced by the symbols \(\{\hat{\sigma}\}\) which are now the multi-indices

\[
\lambda \rightarrow \hat{\sigma} \equiv (\sigma', \sigma'').
\]

The new class of fields (2.8) can be represented in the form

\[
A^a_{\mu \hat{\sigma}_1 \ldots \hat{\sigma}_s}(x),
\] (2.9)

these fields are totally symmetric under permutations of the symbols \(\hat{\sigma}_i\) and are antisymmetric under permutation of the indices within the each symbol \(\hat{\sigma}_i\). We shall define the gauge transformations of these fields as

\[
\delta A^a_\mu = \partial_\mu \zeta - ig[A_\mu, \zeta],
\] (2.10)
\[
\delta A^a_{\mu \hat{\sigma}_1} = \partial_\mu \zeta_{\hat{\sigma}_1} - ig[A_\mu, \zeta_{\hat{\sigma}_1}] - ig[A_{\mu \hat{\sigma}_1}, \zeta],
\]
\[
\delta A^a_{\mu \hat{\sigma}_1 \hat{\sigma}_2} = \partial_\mu \zeta_{\hat{\sigma}_1 \hat{\sigma}_2} - ig[A_\mu, \zeta_{\hat{\sigma}_1 \hat{\sigma}_2}] - ig[A_{\mu \hat{\sigma}_1}, \zeta_{\hat{\sigma}_2}] - ig[A_{\mu \hat{\sigma}_2}, \zeta_{\hat{\sigma}_1}] - ig[A_{\mu \hat{\sigma}_1 \hat{\sigma}_2}, \zeta],
\]

where the gauge parameters

\[
\zeta^a_{\hat{\sigma}_1 \ldots \hat{\sigma}_s}
\]

are totally symmetric under permutations of the symbols \(\hat{\sigma}_i\) and are antisymmetric under permutation of the indices within the each symbol \(\hat{\sigma}_i\). The field strength tensors are defined as follow

\[
G_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],
\] (2.11)
\[
G_{\mu \nu, \hat{\sigma}_1} = \partial_\mu A_{\nu \hat{\sigma}_1} - \partial_\nu A_{\mu \hat{\sigma}_1} - ig([A_\mu, A_{\nu \hat{\sigma}_1}] + [A_{\mu \hat{\sigma}_1}, A_\nu]),
\]
\[
G_{\mu \nu, \hat{\sigma}_1 \hat{\sigma}_2} = \partial_\mu A_{\nu \hat{\sigma}_1 \hat{\sigma}_2} - \partial_\nu A_{\mu \hat{\sigma}_1 \hat{\sigma}_2} - ig([A_\mu, A_{\nu \hat{\sigma}_1 \hat{\sigma}_2}] + [A_{\mu \hat{\sigma}_1}, A_{\nu \hat{\sigma}_2}]
+ [A_{\mu \hat{\sigma}_2}, A_{\nu \hat{\sigma}_1}] + [A_{\mu \hat{\sigma}_1 \hat{\sigma}_2}, A_\nu]),
\]

and are transforming homogeneously with respect to the gauge transformations \(\delta \zeta\) (2.10). In these multi-indices notation the above transformations are identical with the original...
transformations (2.2) therefore one can define the invariant Lagrangian, which formally coincides with the one defined in [20, 21, 22], by interchanging the \( \{ \lambda \} \)’s by \( \{ \sigma \} \)’s

\[
L' = -\frac{1}{4} G_{\mu \nu, \lambda \sigma} G_{\mu \nu, \lambda \sigma} - \frac{1}{4} C_{\mu \nu, \lambda \sigma} G_{\mu \nu, \lambda \sigma}
+ \frac{1}{4} C_{\mu \nu, \lambda \sigma} G_{\mu \nu, \lambda \sigma} + \frac{1}{4} C_{\mu \nu, \lambda \sigma} G_{\mu \lambda, \lambda \sigma} + \frac{1}{2} C_{\mu \nu, \lambda \sigma} G_{\mu \nu, \lambda \sigma} + \ldots
\]  

(2.12)

The total Lagrangian is a sum of the (2.4) and (2.12). Our next intention is to show that in addition to the above gauge invariant Lagrangian one can construct also the new metric independent densities which can be added to the Lagrangian, as it was in the lower dimensional case [18], and are relevant for the description of the potential anomalies.

3 New Invariant Densities in \( 2n + 4 \) Dimensions

In order to introduce higher-dimensional metric-independent densities it is convenient to use the language of forms [6, 7, 14]. At the same time we have to stress that this language can not be used in general description of these fields dynamics, because in the invariant Lagrangian considered in the previous section all components of the fields are present. While in the metric-independent densities only antisymmetric parts of the fields are participating. The one- and two-form gauge potentials are defined as

\[
A = -igA^a_{\mu} L_a dx^{\mu} \\
A_2 = -igA^a_{\mu \nu} L_a dx^{\mu} \wedge dx^{\nu}
\]

with the corresponding field-strength tensors (2.11)

\[
G = dA + A^2, \quad G_3 = dA_2 + [A, A_2].
\]

(3.1)

The Bianchi identities are of the form

\[
DG = 0, \quad DG_3 + [A_2, G] = 0,
\]

(3.2)

where \( DG = dG + [A, G] \) and \( DG_3 = dG_3 + \{ A, G_3 \} \). With the new gauge fields in hands we shall introduce the three-form gauge potential as

\[
A_3 = -igA^a_{\mu \nu, \lambda} L_a dx^{\mu} \wedge dx^{\nu} \wedge dx^{\lambda}
\]

and the corresponding field strength tensor (2.11)

\[
G_4 = dA_3 + \{ A, A_3 \}
\]

(3.3)

---

1 The tensor gauge fields considered in the proposed extension of the Yang-Mills theory (2.1) and (2.9) are neither totally symmetric nor they are totally antisymmetric, they are of the mixed-symmetry type. The investigation of the tensor fields of mixed-symmetry type were made in relation with the string field theory, where the expansion of the string field on the oscillators naturally introduces tensor fields of mixed-symmetry [27, 28, 30, 30, 31, 32]. There is also extended literature on the field-theoretical description of mixed-symmetry free fields and cubic interaction vertices between mixed-symmetry AdS fields within various approaches [33, 34, 35, 36, 37, 38, 39, 40, 41].

2 One should keep in mind that the fields and field-strength tensors, like \( A_{\mu \sigma_1, \sigma_2}^a \) and \( G_{\mu \nu, \sigma_1, \sigma_2} \), can not be expressed in terms of forms, but only their antisymmetric parts.
with the Bianchi identity
\[ DG_4 + [A_3, G] = 0, \]  
where \( DG_4 = dG_4 + [A, G_4] \).

Let us consider a higher-dimensional invariant density in \( 2n+4 \) space-time dimensions:
\[ \Phi_{2n+4} = Tr(G^n G_4), \]  
which is a natural generalization of the Chern-Pontryagin form \( \mathcal{P}_{2n} = Tr(G^n) \). By direct computation of the derivative one can prove that \( \Phi_{2n+4} \) is an exact form:
\[
d\Phi_{2n+4} = Tr(dGG^{n-1}G_4 + \ldots + G^{n-1}dGG_4 + G^n dG_4) \\
= Tr((dG + [A, G])G^{n-1}G_4 + \ldots + G^{n-1}(dG + [A, G])G_4 + G^n dG_4 - \\
- [A, G]G^{n-1}G_4 - \ldots - G^{n-1}[A, G]G_4) = \\
= Tr(DGG^{n-1}G_4 + \ldots + G^{n-1}DGG_4 + G^n (dG_4 + [A, G_4])) \\
= Tr(G^n (DG_4 + [A_3, G])) = 0.
\]

In this calculation one must change sign when transmitting the differential \( d \) through an odd form or commuting odd forms using the cyclic property of the trace, and use Bianchi identities as well. According to Poincaré’s lemma, this equation implies that \( \Phi_{2n+4} \) can be locally written as an exterior differential of a certain \( (2n+3) \)-form. In order to find the form of which \( \Phi_{2n+4} \) is the derivative we have to find its variation, induced by the variation of the fields \( \delta A \) and \( \delta A_3 \):
\[ \delta G = D(\delta A), \quad \delta G_4 = D(\delta A_3) + \{A_3, \delta A\} \]
yielding a variation of \( \Phi_{2n+4} \) which is a total derivative:
\[
\delta \Phi_{2n+4} = \delta Tr(G^n G_4) = Tr(D\delta AG^{n-1}G_4 + \ldots + G^{n-1}D\delta AG_4 + G^n D\delta A_3 + G^n \{A_3, \delta A\}) \\
= Tr(D\delta AG^{n-1}G_4 + \ldots + G^{n-1}D\delta AG_4 + G^n D\delta A_3 + \\
+ \delta A([A_3, G]G^{n-1} + G[A_3, G]G^{n-2} + \ldots + G^{n-1}[A_3, G])) = \\
= Tr(D\delta AG^{n-1}G_4 + \ldots + G^{n-1}D\delta AG_4 + G^n D\delta A_3 - \\
- \delta A DG_4 G^{n-1} - \ldots - \delta A G^{n-1} DG_4) = \\
= Tr(D\delta AG^{n-1}G_4 + \ldots + G^{n-1}\delta AG_4 + G^n \delta A_3) = \\
= d Tr(\delta AG^{n-1}G_4 + \ldots + G^{n-1}\delta AG_4 + G^n \delta A_3). \quad (3.6)
\]

Following [6], we introduce a one-parameter family of potentials and strengths through the parameter \( t (0 \leq t \leq 1) \):
\[ A_t = tA, \quad G_t = tG + (t^2 - t)A^2, \quad A_{3t} = tA_3, \quad G_{4t} = tG_4 + (t^2 - t)\{A, A_3\}. \quad (3.7) \]
so that the equation (3.6) can be rewritten as

$$\delta Tr(G^n t G_{4t}) = d Tr(\delta A_t G^{n-1}_{4t} + \ldots + G^{n-1}_{t} \delta A_t G_{4t} + G^n_t \delta A_{3t}).$$

With $\delta = \delta t(\partial/\partial t)$ and $\delta A_t = A_t \delta t$, $\delta A_{3t} = A_{3t} \delta t$ we shall get by integration the desired result:

$$Tr(G^n t G_4) = d \psi_{2n+3}, \quad (3.8)$$

where the corresponding secondary $(2n + 3)$-form is

$$\psi_{2n+3}(A, A_3) = \int_0^1 dt \ Tr(AG^{n-1}_{4t} + \ldots + G^{n-1}_{t} AG_{4t} + G^n_t A_3). \quad (3.9)$$

The expression (3.9) can easily be evaluated for $n=1$ in five dimensions:

$$\psi_5 = \int_0^1 dt \ Tr(AG_{4t} + G_t A_3) = Tr(GA_3) \quad (3.10)$$

In seven dimensions, $n = 2$, we have

$$\psi_7 = \int_0^1 dt \ Tr(AG_t G_{4t} + G_t AG_{4t} + G^2_t A_3),$$

and after integration over $t$ we get a secondary 7-form:

$$\psi_7(A, G, G_4) = \frac{1}{3} Tr(AGG + AG_4 G + A_3 G^2 - \frac{1}{2} A^3 G_4 - \frac{1}{2} (A^2 A_3 + AA_3 A + A_3 A^2)G + \frac{1}{2} A^4 A_3). \quad (3.11)$$

The new property of the last functional compared with $\psi_5$ is that when the field-strength tensors tend to zero, $G = G_4 = 0$, the above form does not vanish and is equal to

$$\frac{1}{6} Tr(A^4 A_3). \quad (3.12)$$

The subsequent forms $\psi_{2n+3}$ are in $D = 2n + 3 = 5, 7, 9, 11, \ldots$ dimensions.

4 \quad Invariant Forms in Eight and Ten Dimensions

In this section we focus on a series of invariant forms that can be constructed in $2n + 2$ dimensions.

We start with a special invariant form that can be constructed in eight dimensions

$$\Omega_8 = Tr(GG_6 + G_4 G_4), \quad (4.1)$$

where the 6-form strength tensor (2.11) is

$$G_6 = dA_5 + \{A, A_5\} + 2A_5^2, \quad DG_6 + 2[A_3, G_4] + [A_5, G] = 0 \quad (4.2)$$
and their gauge transformations \((A.1)\) are
\[
\delta G_6 = [G_6, \xi] + 2[G_4, \zeta_2] + [G, \zeta_4], \quad \delta G_4 = [G_4, \xi] + [G, \zeta_2]. \tag{4.3}
\]

Using the above formulas one can get convince that the 8-form \(\Omega_8\) is gauge invariant and is exact \(d\Omega = 0\). Thus we have
\[
\Omega_8 = d\chi_7, \quad \chi_7 = Tr(GA_5 + G_4 A_3). \tag{4.4}
\]

The next invariant form of a similar structure can be constructed in ten dimensions
\[
\Omega_{10} = Tr(GG_8 + 3G_4 G_6), \tag{4.5}
\]
where the 8-form strength tensor is
\[
G_8 = dA_7 + \{A, A_7\} + 3\{A_3, A_5\}, \quad DG_8 + 3[A_3, G_6] + 3[A_5, G_4] + [A_7, G] = 0 \tag{4.6}
\]
and the gauge transforms are
\[
\delta G_8 = [G_8, \xi] + 3[G_6, \zeta_2] + 3[G_4, \zeta_4] + [G, \zeta_6]. \tag{4.7}
\]

It is straightforward to show that the form \(\Omega_{10}\) is gauge invariant and is exact \(d\Omega_{10} = 0\). Thus
\[
\Omega_{10} = d\chi_9, \quad \chi_9 = Tr(GA_7 + 3G_4 A_5 + \frac{3}{2} G_6 A_3) \tag{4.8}
\]

The general form of these invariants can be written as
\[
\Omega_{2n+2} = Tr(GG_{2n} + \alpha_1 G_4 G_{2n-2} + \alpha_2 G_6 G_{2n-4} + \ldots + ) = d\chi_{2n+1} \tag{4.9}
\]
where
\[
\chi_{2n+1} = Tr(GA_{2n-1} + \beta_1 G_4 A_{2n-3} + \beta_2 G_6 A_{2n-5} + \ldots ). \tag{4.10}
\]
and \(\alpha_i, \beta_i\) are certain numerical coefficients. The forms \(\chi_{2n+1}\) are defined in \(D = 2n + 1 = 5, 7, 9, 11, \ldots\) dimensions. It is also true that \(\Omega_6 \equiv \Phi_6\), see \((3.8)\) and \((3.10)\).

5 Most General Tensor Gauge Fields

The tensor gauge fields \((2.1)\) introduced in \([20, 21, 22]\) do have a geometrical interpretation if one introduce a unite tangent vector \(e^\mu\) and consider the gauge field \(A_\mu(x, e)\) depending on this variable and then define the extended gauge transformation as in \([22]\)
\[
A'_\mu(x, e) = U(\xi) A_\mu(x, e) U^{-1}(\xi) - \frac{i}{g} \partial_\mu U(\xi) U^{-1}(\xi), \tag{5.1}
\]

where the unitary transformation matrix is given by the expression \( U(\xi) = \exp\{ig\xi(x, e)\} \)
and the gauge parameter \( \xi(x, e) \) has the following expansion
\[
\xi(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} \xi_{\lambda_1...\lambda_s}^a(x) L_a e^{\lambda_1} ... e^{\lambda_s}.
\] (5.2)

Using this language one can consider the fields of mixed symmetry (2.9) introduced in this article as the gauge field depending also on antisymmetric wedge product
\[
\omega^\hat{\sigma} = e^{\sigma_1} \wedge e^{\sigma_2}
\] (5.3)
so that
\[
A_{\mu}(x, e, \omega).
\] (5.4)

It’s expansion in \( \omega \) will generate all fields considered in the previous chapters, but in addition it will generate tenors fields which have both - vector \( \{\lambda\} \) and multi-indices \( \{\hat{\sigma}\} \)
\[
A_{\mu,\lambda_1,\lambda_2,...,\hat{\sigma}_1,\hat{\sigma}_2,...}
\] (5.5)
These fields are symmetric under any permutations of all these indices and antisymmetric under permutations within each multi-index. Because in four dimensions one can construct even higher rank independent wedge products of the vector \( e^\mu \), such as \( e^{\sigma_1} \wedge e^{\sigma_2} \wedge e^{\sigma_3} \) and \( e^{\sigma_1} \wedge e^{\sigma_2} \wedge e^{\sigma_3} \wedge e^{\sigma_4} \), one can consider tensor fields depending on these antisymmetric variables. If one use the multi-index variable \( \hat{\sigma} \) to denote double \( \hat{\sigma} \equiv (\sigma', \sigma'') \), triple \( (\sigma', \sigma'', \sigma''') \) and quadric \( (\sigma', \sigma'', \sigma''', \sigma''''\) multi-indices then the tensor gauge fields will be of the same form as in (5.5), but with double, triple and quadric multi-indices. It is not difficult to find out their gauge transformations and corresponding field strength tensors.

6 Conclusions

In this article we demonstrated that a class of the non-Abelian tensor gauge fields (2.1) considered in [20, 21, 22] can and should be extended to include new fields (2.8) and therefore allows to construct a metric independent and gauge invariant forms in \( D = 2n + 4 \) and \( D = 2n + 2 \) dimensions. These new forms \( \Phi_{2n+4} = d\psi_{2n+3} \) and \( \Omega_{2n+2} = d\chi_{2n+1} \) are analogous to the Pontryagin-Chern-Simons densities \( P_{2n} = d\omega_{2n-1} \) in YM gauge theory and to the corresponding series of densities \( \Gamma_{2n+3} = d\sigma_{2n+2} \) found in [19], yielding the potential anomalies in gauge field theory. The above general considerations should be supplemented by an explicit calculation of loop diagrams involving chiral fermions. The argument in favor of the existence of these potential anomalies is based on the
fact that they fulfill Wess-Zumino consistency conditions \[4, 6, 7, 8, 11, 12, 13, 14, 17, 23, 24, 25, 26\]. At the same time, these invariant densities constructed on the space-time manifold have their own independent value since they suggest the existence of new invariants characterizing topological properties of a manifold and can be added to the invariant Lagrangian.

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A Field Strengths Transformations

The gauge transformations \(\delta \xi\) of non-Abelian tensor gauge fields were defined in \[20, 21, 22\] as \(2.2\). The generalized field-strength tensors \(2.11\) transform homogeneously under these gauge transformations \(\delta \xi\):

\[
\delta G^a_{\mu\nu} = -ig[G_{\mu\nu} \xi], \\
\delta G^a_{\mu\nu,\lambda} = -ig( [G_{\mu\nu,\lambda} \xi] + [G_{\mu\nu} \xi_\lambda] ), \\
\delta G^a_{\mu\nu,\lambda\rho} = -ig( [G^b_{\mu\nu,\lambda\rho} \xi] + [G_{\mu\nu,\lambda} \xi_\rho] + [G_{\mu\nu,\rho} \xi_\lambda] + [G_{\mu\nu} \xi_{\lambda\rho}] ).
\]

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