1. Introduction

This is the opposite of a survey paper. Here we are interested in one example, usually known as the irrational rotation $C^*$-algebra and written $A_\lambda$ where $\lambda$ is some real (usually irrational) number between 0 and 1. There are a lot of choices for $\lambda$, so saying the irrational rotation $C^*$-algebra is already misleading. We will exaggerate and simplify a lot in this paper. So don’t rely on this paper for theorems. Think of it instead as your friend sitting across the table at Starbucks, giving you the right idea but .... leaving out the details.

We will show that the irrational rotation $C^*$-algebra arises in three quite different contexts (there are more as well):
Then we will show how to get some information out of the irrational rotation $C^*$-algebra - in fact we will sketch (mostly) how to retrieve $\lambda$ using $K$-theory for $C^*$-algebras, which means we are almost preserving $A_\lambda$ up to isomorphism. Thus it turns out that as $\lambda$ varies among the irrationals between 0 and $1/2$ there are uncountably many isomorphism classes of the algebras $A_\lambda$.

This paper, then, is an advertisement for $K$-theory by showing exactly one application of it.

2. Quantum Mechanics

In 1926-7 the quantum-mechanical revolution in physics changed our understanding of the world. As has been the pattern since, the physicists knew what they wanted, and the mathematicians struggled to keep up, to keep the physics honest (as a mathematician would put it).

The simplest model of the hydrogen atom revolved about two operators $P$ and $Q$ that were to measure position and momentum of the electron. Heisenberg and Max Born showed that if $Q$ is the position operator and $P$ the momentum operator, then we have the canonical commutation relation

\[ PQ - QP = -i\hbar \]

where $\hbar$ is Planck’s constant.

That is, not only is $PQ$ unequal to $QP$ (the order that you measure things makes a difference) but the difference was governed by a precise formula. (Note, by the way, that asking a physicist whether Planck’s constant is rational or irrational will get you a look of incredulity to this day. And this doesn’t even take into account the folks who like to “let Planck’s constant go to zero”.

Since in those days matrices were far more popular than linear operators, a lot of people tried to find matrices $P$ and $Q$ that satisfied the Heisenberg equation. If you try your hand with $2 \times 2$ matrices, for instance, you will see that this is not such an easy problem. This came to an abrupt halt in 1927 when Weyl and von Neumann observed (cf. [4], [8]) that it was impossible to find finite-dimensional matrices that do the job. The argument is very simple. Suppose that $P$ and $Q$ are $n \times n$ matrices with $PQ - QP = \lambda I$ (where $I$ is the identity matrix). Take the trace of both sides (simply add up the elements on the main diagonal of the matrices) to obtain

\[ \text{Trace}(PQ) - \text{Trace}(QP) = \text{Trace}(\lambda I) = n\lambda. \]

However, $\text{Trace}(PQ) = \text{Trace}(QP)$ for any finite-dimensional matrices, and so $n\lambda = 0$, implying that $\lambda = 0$ and $PQ = QP$.

We conclude that one must use infinite-dimensional matrices. So the better thing to do is to take $P$ and $Q$ to be self-adjoint unbounded operators on infinite-dimensional complex Hilbert space. Following Weyl [8], we set

\[ U_s = \exp(isP) \quad \text{and} \quad V_t = \exp(itQ). \]

\[ ^2 \text{But physicists do the latter all the time. It is Bohr's "correspondence principle" or passage to the semi-classical limit, "semi" because one keeps the Poisson bracket that is the shadow of the operator commutant, so physicists are quite comfortable with all this!} \]
The Stone-von Neumann theorem (cf. [7]) tells us that all such pairs of one-
parameter unitary groups are unique up to unitary equivalence. On setting
$s = t = 1$ and $\hbar = \lambda$ we obtain

$$UV = e^{2\pi i \lambda} V U$$

This is called the Weyl form of the canonical commutation relation. These operators
are bounded unitary operators on the same Hilbert space, $U, V \in B(\mathcal{H})$.
So we may take the (non-commuting) polynomial algebra generated by $U, V,$
and their adjoints. We then close up this algebra with respect to the operator
norm and reach our goal, the $C^*$-algebra $A_\lambda$, constructed visibly as a norm-closed,
*-closed subalgebra of $B(\mathcal{H})$.
This is the first construction of the $A_\lambda$. We may restrict attention to $\lambda \in [0, 1)$
and ask an elementary question: as $\lambda$ changes, how is $A_\lambda$ affected? It turns out
that the case of greatest interest is when $\lambda$ is irrational, and so we will restrict to
that case as needed.

3. HOMEOMORPHISMS OF THE CIRCLE

Let $\phi : S^1 \to S^1$ be rotation of the circle by $2\pi \lambda$ radians counterclockwise. Any
rotation is a homeomorphism, and thus determines an action of the integers on
the circle by sending $n$ to $\phi^n$. This defines an action of the integers on $C(S^1)$,
continuous complex-valued functions on the circle, and from this we will construct
a $C^*$-algebra

$$C(S^1) \rtimes \mathbb{Z}$$
as follows.
Take $\mathcal{H}$ to be the Hilbert space $L^2(S^1)$ and let $T \in B(\mathcal{H})$ be the bounded
invertible operator corresponding to rotation by $\phi$. Any $f \in C(S^1)$ gives a pointwise
multiplication operator $M_f \in B(\mathcal{H})$. Then $C(S^1) \rtimes \mathbb{Z}$ is the norm-closed $\ast$-algebra
generated by $T$ and by all of the $M_f$. Note that finite sums of the form

$$\sum_{n=-k}^{k} M_{f_n} T^n$$

are dense in $C(S^1) \rtimes \mathbb{Z}$. There is a unique normalized trace $\tau$ on $C(S^1) \rtimes \mathbb{Z}$ given
on finite sums by

$$\tau(\sum_{n} M_{f_n} T^n) = \int_{S^1} f_0(t) dt \in \mathbb{R}$$

where $dt$ is normalized Lebesgue measure on the circle. It is not at all hard to prove
that (for $\lambda$ irrational, for which the action of $\mathbb{Z}$ on the circle is free)

$$A_\lambda \cong C(S^1) \rtimes \mathbb{Z}$$

and in fact they have the same universal property.

4. FOLIATED SPACES

The local picture of a foliated space is $\mathbb{R}^p \times N$, where $N$ is some topological space.
A subset of the form $\mathbb{R}^p \times n$ is called a plaque and a measurable subset $T$ which
meets each plaque at most countably often (the simplest being $\{x\} \times N$ is called
a transversal. The global picture is more complicated. We say that a (typically
compact) space $X$ is a foliated space if each point in $X$ has an open neighborhood
homeomorphic to the local picture and if locally the plaques fit together smoothly.

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3A trace is an linear functional on the positive elements of the $C^*$-algebra taking values in
$[0, +\infty]$ and satisfying $\tau(x^*x) = \tau(xx^*)$ for all $x$ in the $C^*$-algebra.
A leaf is a maximal union of overlapping plaques and it by construction is a smooth $p$-dimensional manifold. See [3] for details and lots of examples.

NOTE: I AM ATTACHING PICTURES OF THE STARBUCKS MUGS THAT I GOT FROM THEIR WEBSITE THAT I WOULD LIKE TO USE FOR AUTHENTICITY!!! IF THE EDITORS OR STARBUCKS VETOES THIS THEN WE WILL HAVE TO GET SOMEONE TO MAKE REAL PICTURES OF FOLIATIONS FOR US - I DON'T KNOW HOW TO DO THIS.

Here is an example. Take a cylinder, which we shall visualize as the outside of a coffee mug with the bottom and handle removed!! We can foliate this in several ways.

(1) Take this tall mug (INSERT PICTURE OF TALL MUG)) and cut out the bottom and throw away the top. That leaves a cylinder. We can think of the cylinder being made up of circles- the circle that you see at each end and the infinite number of circles you would get by cutting through with a saw parallel to the circular end. In this case each leaf is diffeomorphic to a circle.

(2) We can think of the cylinder as being made up of straight lines- the lines that start at one end of the cylinder and extend perpendicularly to the other end. (as you would see on what’s left of this Starbucks mug (INSERT PICTURE OF RED MUG) after removing the handle and the bottom.)

More precisely, we can start with the space $[0, 2\pi] \times [0, 1]$ sitting in the first quadrant, and then define $X$ to be the quotient space obtained by identifying the point $(x, 0)$ to the point $(x, 1)$. Then the first foliation corresponds to taking the plaques to be of the form $(x, t)$ where $t$ varies from 0 to 1 and the second foliation corresponds to taking the plaques to be of the form $(t, y)$ where $t$ varies from 0 to $2\pi$.

Next step. We want to take the cylinder and glue the left and right ends together. Precisely, glue the point $(0, t)$ to the point $(1, t)$. This gives us a torus (aka the crust of a doughnut) and if you think carefully you will see that in both examples (1) and (2) we wind up with a foliation by circles. In fact the two foliations are mutually perpendicular.

(3) Now the critical step. Instead of gluing $(0, t)$ to the point $(1, t)$ we will glue $(0, t)$ to the point $(1, t + \lambda)$ (where " + " means addition mod $2\pi$. Now something quite unusual happens and here we must specify whether $\lambda$ is rational or not. If $\lambda$ is rational then each leaf of the foliation is actually a circle, though you will have to go around the torus several times to show this. On the other hand, if $\lambda$ is irrational, then you do not get circles: every leaf is a line $\mathbb{R}$. Furthermore, each line is wrapped about the torus infinitely often, so that the line is actually dense in the torus. This construction is called the Kronecker flow on the torus (though we don’t know if this is in Kronecker’s honor or if he actually invented it) and may also be described in terms of the differential equation $dy = \lambda dx$.

Every foliated space satisfying very minimal technical assumptions has a $C^*$-algebra associated to it. This is due to H. E. Winkelnkemper and to A Connes. The procedure has two steps. The first step is to associate a topological groupoid to the foliation. Then a general construction assigns a $C^*$-algebra to the groupoid.
In our context these are always stable C*-algebras; they have the form $A \otimes K$ where $K$ is the C*-algebra of the compact operators.\footnote{If you are an analyst you can think of $K$ as the smallest C*-algebra inside $B(H)$ that contains all of the operators with finite-dimensional range. If you are an algebraist at heart then you will be pleased to hear that $B(H)$ is a local ring, and $K$ is its unique maximal ideal.} Here are some examples:

1. If $X = F \times B$ for some smooth manifold $F$ and compact space $B$ with leaves of the form $F \times \{b\}$, then the foliation algebra is $C(B) \otimes K$.
2. More generally, if $X$ is the total space of a compact fibre bundle $F \rightarrow X \xrightarrow{\pi} B$ then it is a foliation, where the leaves of the foliation are the subsets of $X$ of the form $\pi^{-1}(b)$. The foliation algebra is simply $C(B) \otimes K$.
3. (The punch line) If $X$ is the torus foliated by lines as constructed above at irrational angle $\lambda$ then the foliation algebra is $A_\lambda \otimes K$.

There is a natural trace that arises in this construction as well. What is needed is an invariant transverse measure. A transversal is a measurable set that meets each leaf of the foliation at most countably many times. A transverse measure measures transversals, naturally enough. If it has enough nice properties then it is an invariant transverse measure. Not all foliations have them, but the ones we are looking at do. In the case of the fibre bundle above, the foliation algebra is simply $C(B) \otimes K$ and invariant transverse measures correspond to certain measures on $B$. Invariant transverse measures correspond to Ruelle-Sullivan currents in foliation theory (cf. [3] Ch. IV.)

In the case of the Kronecker flow on the torus, the invariant transverse measure may be constructed from Lebesgue measure on a transverse circle to the foliation. This passes to a trace on the foliation algebra which corresponds to the trace constructed above.

Suppose that $p \in A$ is a projection and we have normalized the trace $\tau$ so that $\tau(1) = 1$. Then $0 \leq \tau(p) \leq 1$ by elementary considerations. But what is the range of the map? In the case $A = M_n(\mathbb{C})$ the range of $\tau$ would be $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$. What happens for $A_\lambda$? Stay tuned.

Lest the reader feel cheated that we are not obtaining $A_\lambda$ on the nose, let us hasten to point out that if $A \otimes K \cong B \otimes K$ (this is called stably isomorphic) then $A$ and $B$ are strongly Morita equivalent, and conversely ($A$ and $B$ being separable) by deep results of Brown, Green, and Rieffel \cite{1}. So $A_\lambda$ and $A_\lambda \otimes K$ have the same representation theory, essentially, and, as we shall see, the same $K$-theory.

To summarize, we have shown that the C*-algebra $A_\lambda$ arises in three disparate arenas of mathematics. (There are others as well, but this should be enough to convince you that it happens a lot!) At this point, though, it is not at all clear to what extent the algebra is dependent upon $\lambda$. Let’s find out.

5. The World’s Fastest Intro to $K$-theory

Suppose first that $A$ is a unital C*-algebra. A projection $p$ is an element of $A$ that satisfies $p^2 = p = p^*$. There are always projections, namely 0 and 1. If $X$ is a connected space then these are the only projections in $C(X)$. On the other hand, $M_n(\mathbb{C})$ has lots of projections: for instance, take a diagonal matrix that has only...
ones and zero’s on the diagonal. It turns out that $C(X) \otimes M_n(\mathbb{C})$ can have very interesting projections - these correspond to vector bundles over $X$.

Let $P_n(A)$ denote the set of projections in $A \otimes M_n(\mathbb{C})$, and define $P_{\infty}(A)$ to be the union of the $P_n(A)$ (where we put $P_n$ inside of $P_{n+1}$ by sticking it in the upper left corner and adding zeros to the right and below.) Unitary equivalence and saying that $p$ is equivalent to $p \oplus 0$ puts a natural equivalence relation $\sim$ on $P_{\infty}(A)$. Then $P_{\infty}(A)/\sim$ has a natural direct sum operation, and we can turn it into an abelian group by doing the so-called Grothendieck construction (taking formal differences of projections). If you don’t like that, take the free abelian group on the equivalence classes of projections, and then divide out by the subgroup generated by all elements of the form $[P + Q] - [P] - [Q]$. This gives an abelian group denoted $K_0(A)^\mathbb{Z}$

We may regard $K_0$ as a functor on unital C*-algebras and maps, since if $f : A \to A'$ is unital then $f$ takes projections to projections, unitaries to unitaries, and preserves direct sum. If $A$ is not unital then we may form its unitization $A^+$ (for example, $C_0(X)^+ \cong C(X^+)$ where $X$ is locally compact and $X^+$ is its one-point compactification), and then define $K_0(A)$ to be the kernel of the map

$$K_0(A^+) \longrightarrow K_0(A^+/A) \cong \mathbb{Z}.$$  

Note that if $A$ is separable then there are at most countably many equivalence classes of projections, and hence $K_0(A)$ is a countable abelian group.

For example, take $A = \mathbb{C}$. Then $P_n(A)$ consists of all of the projections in $M_n(\mathbb{C})$. We learned in the second semester of Linear Algebra that every projection is unitarily equivalent to a diagonal matrix of the form $\text{diag}(1, 1, \ldots, 1, 0, 0, 0)$ and hence we may regard the equivalence classes of $P_n(\mathbb{C})$ to be the integers $\{0, 1, 2, \ldots, n\}$. Then the equivalence classes of $P_{\infty}(\mathbb{C})$ are classified by the ranks of the matrices which correspond to all of the natural numbers $\{0, 1, 2, \ldots\}$ and taking formal inverses we obtain $K_0(\mathbb{C}) \cong \mathbb{Z}$. Note that the same answer emerges if we take $A = M_j(\mathbb{C})$ for any $j$, since “matrices of matrices are matrices.” Similarly (but this requires a little work) we have the same answer if $A = K$ the compact operators. Actually we need something stronger that is based upon this idea, namely this fact:

$$K_0(A) \cong K_0(A \otimes M_n(\mathbb{C})) \cong K_0(A \otimes K),$$

which we will use without further comment. Next, we note that for commutative unital C*-algebras $A = C(X)$ with associated maximal ideal space the compact space $X$, then

$$K_0(C(X)) \cong K^0(X)$$

where $K^0(X)$ is the Grothendieck group generated by complex vector bundles over $X$.

Today we will need only $K_0$ but there is a $K_1$ as well.

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5 A projection $p$ in $A \otimes M_n(\mathbb{C})$ corresponds to a finitely generated projective $A$-module in a natural way (think of endomorphisms of $A \oplus \ldots \oplus A$) and hence we are really using the classical definition of $K_0(A)$ for arbitrary unital rings, except for the fact that there one would use all idempotents, not just projections. The two definitions are equivalent, though, by a version of Gram-Schmitt orthogonalization. Similarly one can use invertibles rather than unitaries in the homotopy definition of $K_*$.

6 If $A$ is unital then define $U_n(A)$ to be the group of unitaries in $A \otimes M_n(\mathbb{C})$ and for $n > 0$ let

$$K_j(A) = \lim_n \pi_{j-1}(U_n(A))$$
6. The $K$-theory of the irrational rotation $C^*$-algebra: the bad news

Now, what happens to the irrational rotation $C^*$-algebra? A seemingly elementary question arises first: does $A_\lambda$ have any non-trivial projections? This was open for several years, and it led to decisive work by the second author whose results, together with those of Pimsner-Voiculescu, we now describe. We are altering the historical order a bit in what follows - see Rieffel [6] for the truth.

If $\lambda$ is irrational then Pimsner and Voiculescu showed [5] that

$$K_0(A_\lambda) \cong \mathbb{Z} \oplus \mathbb{Z}$$

independent of $\lambda$. So using $K_0$ by itself we cannot distinguish the various $A_\lambda$.

7. Traces to the rescue

If $A$ is any $C^*$-algebra with a nice trace and $p$ and $q$ are orthogonal projections in $A$ then

$$\tau(p \oplus q) = \tau(p) + \tau(q)$$

and so the trace gives us a homomorphism

$$K_0(A) \xrightarrow{\tau} \mathbb{R}$$

of abelian groups. We have remarked previously that if $A$ is separable (which we assume henceforth) then $K_0(A)$ is a countable abelian group, and hence $\tau(K_0(A))$, the image of

$$\tau : K_0(A) \longrightarrow \mathbb{R}$$

is a countable subgroup of $\mathbb{R}$. There are a lot of countable subgroups of $\mathbb{R}$ (cf. [2])!

However, there is good news. Pimsner and Voiculescu [5] showed that the image of the trace

$$K_0(A_\lambda) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\tau} \mathbb{R}$$

takes values $\mathbb{Z} + \lambda\mathbb{Z}$, the subgroup of $\mathbb{R}$ generated by 1 and by $\lambda$. Now Rieffel had previously shown that every element of $(\mathbb{Z} + \lambda\mathbb{Z}) \cap [0,1]$ is in the range of a projection in $A_\lambda$. Combining these results gives us this omnibus isomorphism theorem:

**Theorem 7.1.** (Rieffel [6], Pimsner-Voiculescu [5])

1. If $\lambda$ is irrational then the image of

$$K_0(A_\lambda) \xrightarrow{\tau} \mathbb{R}$$

is exactly $\mathbb{Z} + \lambda\mathbb{Z}$.

2. There are uncountably many isomorphism classes of algebras $A_\lambda$ as $\lambda$ ranges among the irrational numbers in the interval $[0,1/2]$.

3. If $\lambda$ and $\mu$ are irrational numbers in the interval $[0,1/2]$ and $A_\lambda \cong A_\mu$ then $\lambda = \mu$. [7]

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For other irrational numbers $\nu$, take the fractional part $\{\nu\}$ and then use either $\{\nu\}$ or $1 - \{\nu\}$ to get into the $[0,\frac{1}{2}]$ range.

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[6]: Rieffel
[5]: Pimsner-Voiculescu
[7]: Bott
(4) If \( \lambda \) and \( \mu \) are irrational numbers in the interval \([0, \frac{1}{2}]\) and \( m \) and \( n \) are positive integers with

\[
A_\lambda \otimes M_m(\mathbb{C}) \cong A_\mu \otimes M_n(\mathbb{C})
\]

then \( \lambda = \mu \) and \( m = n \).

(5) The algebras \( A_\lambda \) and \( A_\mu \) are strongly Morita equivalent (that is, \( A_\lambda \otimes K \cong A_\mu \otimes K \)) if and only if \( \lambda \) and \( \mu \) are in the same orbit of the action of \( GL(2, \mathbb{Z}) \) on irrational numbers by linear fractional transformations.

So we see that the \( A_\lambda \) retain all of the sensitive information about the angle. If we think back to the origins of \( A_\lambda \) this seems really astonishing:

- The exact value of Planck’s constant (and whether or not it is rational) really does seem to make something of a difference!
- One irrational rotation of a circle is really not like another irrational rotation of a circle.
- The angle of the Kronecker flow deeply effects the geometry of the foliation.

Case studies are supposed to suggest questions for further study. We hope that this note has done so.

REFERENCES

[1] L. G. Brown, R. Green, and M.A. Rieffel, *Stable isomorphism and strong Morita equivalence of C*-algebras*, Pacific J. Math. 71 (1977), 349-363.

[2] L. Fuchs, *Infinite Abelian Groups*, Vol. II, Pure and Applied Mathematics. Vol 36-II, Academic Press, New York, 1973.

[3] C. C. Moore and C. Schochet, *Global Analysis on Foliated Spaces*. Second edition. Mathematical Sciences Research Institute Publications, 9. Cambridge University Press, New York, 2006.

[4] John von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955. (translation of the 1932 German original.)

[5] M. Pimsner and D. Voiculescu, Exact sequences for \( K \)-groups and \( Ext \)-groups of certain cross-product C*-algebras, J. Operator Theory 4 (1980), 93-118.

[6] M. A. Rieffel, *C*-algebras associated with irrational rotations, Pacific J. Math 93 (1981), 415-429.

[7] J. Rosenberg, *A Selective History of the Stone-von Neumann Theorem*, Contemp. Math 365, AMS, 2004.

[8] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, 1950.

Department of Mathematics, Technion, Haifa 32000, Israel
E-mail address: clsmath@gmail.com