Cusp bifurcation in the eigenvalue spectrum of $\mathcal{PT}$-symmetric Bose-Einstein condensates

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A Bose-Einstein condensate in a double-well potential features stationary solutions even for attractive contact interaction as long as the particle number and therefore the interaction strength do not exceed a certain limit. Introducing balanced gain and loss into such a system drastically changes the bifurcation scenario at which these states are created. Instead of two tangent bifurcations at which the symmetric and antisymmetric states emerge, one tangent bifurcation between two formerly independent branches arises [D. Haag et al., Phys. Rev. A 89, 023601 (2014)]. We study this transition in detail using a bicomplex formulation of the time-dependent variational principle and find that in fact there are three tangent bifurcations for very small gain-loss contributions which coalesce in a cusp bifurcation.

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I. INTRODUCTION

Bose-Einstein condensates with attractive contact interactions become unstable if the number of particles exceeds a certain limit [1–3]. In mean-field approximation where the condensate is described by the Gross-Pitaevskii equation this effect manifests itself in a vanishing ground state. Above this limit the mean interaction caused by the particles is strong enough to constrict and ultimately collapse the condensate. Below this limit the stationary solutions of the Gross-Pitaevskii equation are observable even though for attractive interactions the ground state is not the global minimum of the mean-field energy [4].

Characteristic properties of Bose-Einstein condensates with attractive interaction could be used for an atomic soliton laser [1]. However, this requires the realization of a particle flow into and out of the condensate. Modeling such effects in mean-field approximation is done via imaginary potentials, thus rendering the Hamiltonian non-Hermitian [5, 6]. Such non-Hermitian systems have been widely studied [6–12] and are supported by comparison with many-particle calculations [13, 14]. Both in- and outcoupling of particles have been experimentally realized [15, 16].

Despite the non-Hermiticity real eigenvalues and thus true stationary states can be found if the Hamiltonian is $\mathcal{PT}$-symmetric [17–19] or pseudo-Hermitian [20–22]. This unique property motivated a variety of theoretical studies [23–31] and the experimental realization in optical wave-guide systems [23, 32–34].

In [35] we studied the eigenvalue spectrum and dynamical properties of a Bose-Einstein condensate, described by the dimensionless Gross-Pitaevskii equation

$$ (-\Delta + V_{\text{ext}} + 8\pi Na|\psi|^2)\psi = \mu \psi, \quad (1) $$

in a three-dimensional $\mathcal{PT}$-symmetric double-well potential

$$ V_{\text{ext}}(x) = \frac{1}{4} [x^2 - \omega^2(y^2 + z^2)] + v_0 e^{-\sigma x^2} + i\gamma e^{-\rho x^2}, \quad (2) $$

with $\omega = 2$, $v_0 = 4$, $\sigma = 0.5$ and $\rho \approx 0.12$. The symmetric real part of the potential is a harmonic trap which is separated into two wells by a Gaussian barrier. The antisymmetric imaginary part of the potential induces particle loss in the left well and particle gain in the right well whose strength is given by the gain-loss parameter $\gamma$.

It was found that four stationary states of the real double-well potential are created in the two lowest-lying tangent bifurcations at strong attractive interaction strengths $Na$. If gain and loss is introduced into the system, instead of two bifurcations there is only one bifurcation between two previously independent states, while the other two states vanish. However, the underlying bifurcation mechanism remained unclear. It is the purpose of this paper to clarify this mechanism and study the bifurcation scenario in detail classifying it as a cusp bifurcation [36].

The number of solutions is not conserved at bifurcations since the Gross-Pitaevskii equation is non-analytic due to its nonlinear part $|\psi|^2$. This problem is addressed by applying an analytic continuation, where the complex wave functions are replaced by bicomplex ones. The numerical results are then obtained via the time-dependent variational principle (TDVP).

We start with a short introduction to bicomplex numbers in Sec. II before discussing their application to the Gross-Pitaevskii equation and the TDVP in Sec. III. A detailed analysis of the eigenvalue spectrum and the bifurcation scenario is carried out in Sec. IV. Conclusions are drawn in Sec. V.

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II. BICOMPLEX NUMBERS

Consider a usual complex number \( z \in \mathbb{C} \), \( z = x + iy \) with \( x, y \in \mathbb{R} \) and the imaginary unit \( i^2 = -1 \). A bicomplex number is constructed by replacing the real and imaginary part of \( z \) by complex numbers with an additional imaginary unit \( j \) which also fulfills \( j^2 = -1 \),

\[
z = x + iy = (x_1 + jx_2) + i(y_1 + jy_2) \equiv z_1 + iz_2 + jz_3 + k z_4.
\]  

(3)

In the last step \( k = ij \) with \( k^2 = 1 \) was introduced. For complex wave functions the relation \( \psi_+ = \psi_- \) holds, as can be seen in Eqs. (5), and the Eqs. (9) are reduced to the Gross-Pitaevskii equation (1). Thus, the coupled Eqs. (9) still support all solutions of the Gross-Pitaevskii equation (1).

To solve the Eqs. (9) the TDVP is generalized for bicomplex differential equations. Our ansatz consists of coupled Gaussian wave packets which yields accurate results even for a small number of wave packets [39–41]. For the double-well potential studied in this work it was shown that two coupled Gaussian wave packets suffice and additional wave packets only lead to small corrections [35, 42]. For all calculations the following bicomplex ansatz is used

\[
\psi(x, z(t)) = \sum_{n=1}^{2} \exp(-x^T A^n x + b^n \cdot x + c^n),
\]  

(10)

with the bicomplex three-dimensional diagonal matrix \( A^n = \text{diag}(A^n_1, A^n_2, A^n_3) \), the bicomplex three-dimensional vector \( b^n = (b^n_1, b^n_2, b^n_3)^T \), and the bicomplex number \( c^n \). The vector \( z(t) \) contains all parameters \( A^n, b^n, c^n \) of the wave function. The ansatz is chosen such that all solutions are invariant under rotations around the \( x \) axis, i.e. we search only for wave functions that possess the symmetry of the potential. The complex components of the wave function can be written as

\[
\psi_\pm(x, z\pm(t)) = \sum_{n=1}^{2} \exp(-x^T A^n_{\pm} x + b^n_{\pm} \cdot x + c^n_{\pm})
\]  

(11)

using Eq. (8) since the exponential function is defined as a power series.

The TDVP makes use of the McLachlan variational principle [43] which demands that the variation of the functional

\[
I = \| i \dot{\varphi} - \mathcal{H} \psi \|^2
\]  

(12)

with respect to \( \varphi \) vanishes, then after the variation \( \varphi = \dot{\psi} \) is set. Using the representation with the idempotent basis (4) for \( \varphi, \psi \) and \( \mathcal{H} \) leads to

\[
I = (i \varphi_- - \mathcal{H}_- \psi_- | i \varphi_+ - \mathcal{H}_+ \psi_+ ) e_+ + (i \varphi_+ - \mathcal{H}_+ \psi_+ | i \varphi_- - \mathcal{H}_- \psi_- ) e_-
\]

\[
\equiv I_+ e_+ + I_- e_-
\]  

(13)

The two coefficients are complex conjugate, thus, the functional \( \dot{I} = \dot{I}_+ = \dot{I}_- \) already contains the full information.
The variation of the functional $\tilde{I}$ reads,
\[
\delta \tilde{I} = \langle i \delta \varphi_- | \dot{\varphi}_+ - \mathcal{H}_+ \psi_+ \rangle + \langle i \varphi_- - \mathcal{H}_- \psi_- | i \delta \varphi_+ \rangle \\
= \langle i \delta \varphi_- | \dot{\psi}_+ - \mathcal{H}_+ \psi_+ \rangle \delta \tilde{z}_+^* \\
+ \langle \dot{\psi}_- - \mathcal{H}_- \psi_- | i \partial_{x_+} \psi_+ \rangle \delta \tilde{z}_+ \\
= 0.
\] (14)

Since the variations $\delta \tilde{z}_+$ and $\delta \tilde{z}_+^*$ are independent both coefficients have to vanish, which can be written in the compact form
\[
\langle i \dot{\psi}_+ - \mathcal{H}_+ \psi_+ | \partial_{x_+} \psi_+ \rangle = 0.
\] (15)

Using the idempotent basis we can reduce the bicomplex equation to a pair of coupled complex equations.

The next step is to insert the ansatz (11) of the wave function into Eq. (15). The calculations are in full analogy with the TDVP for complex Gaussian wave packets [44] and therefore we only list the results.

The equations of motion for the parameters of the Gaussian wave packets read
\[
\begin{align*}
\dot{A}_\pm^m &= 4(A_\pm^m)^2 - V_\pm^{2m}, \quad (16a) \\
\dot{b}_\pm^m &= 4A_\mp^m b_\pm^m + v_1^m, \quad (16b) \\
\dot{c}_\pm^m &= 2 \text{tr} A_\pm^m - b_\pm^m - b_\mp^m + v_0^m. \quad (16c)
\end{align*}
\]

The coefficients of the effective potential $V_\pm^{2m}$, $v_1^{2m}$ and $v_0^{2m}$ are obtained by numerically solving
\[
\sum_{n=1}^{N} \langle x^T | V_\pm^{2m} | x + v_1^{2m} | x + v_0^{2m} \rangle g_\pm^{2m}
\]
\[
= \sum_{n=1}^{N} \langle x^T | (V_{\text{ext}} + 8\pi N a \psi_+^m | x + v_1^m) g_\pm^{2m} \rangle,
\] (17)

with $\alpha, \beta = 1, \ldots, d$; $m = 1, \ldots, N$ and $k + l = 0, 1, 2$, where $d$ is the dimension and $N$ the number of coupled Gaussians. For the ansatz (11) used in this work we have $d = 3$ and $N = 2$.

Stationary solutions are obtained by varying the parameters $A_\pm^m, b_\pm^m, (c_\pm^m - c_\mp^m)$, such that they fulfill the equations $\dot{A}_\pm^m = 0, b_\pm^m = 0, (c_\pm^m - c_\mp^m) = 0$ for $n = 1, \ldots, N$. Note that the differences $(c_\pm^m - c_\mp^m)$ are used since not all parameters $c_\pm^m$ are free due to the norm constraint and the choice of a global phase. Standard numerical methods for complex numbers can be used to achieve this since all parameters in Eqs. (16) and (17) are complex. This again shows the benefit of the idempotent basis.

IV. RESULTS

Before discussing the analytic continuation the real spectrum is investigated. In [35] the real eigenvalue spectrum of this system has already been studied, however, the bifurcation scenario at strong attractive interactions, on which we concentrate in this paper, was only discussed briefly. In particular the bifurcation scenario was only shown for the gain-loss parameters $\gamma = 0$ and $\gamma = 0.001$, which, as we will see, does not suffice to understand the bifurcation process in detail.

Figure 1 shows the relevant part of the eigenvalue spectrum as a function of the nonlinearity parameter $Na$ for different gain-loss parameters $\gamma$. All states are $\mathcal{PT}$-symmetric, therefore, their chemical potentials $\mu$ are real. For a vanishing gain-loss contribution, $\gamma = 0$, the two states $|A\rangle$ and $|B\rangle$, which originate from the ground and excited state of the double-well potential, emerge from independent tangent bifurcations $T1$ and $T2$. The bifurcation $T1$ gives birth to the states $|A\rangle$ and $|D\rangle$ whereas the states $|B\rangle$ and $|C\rangle$ emerge from the bifurcation $T2$. At $\gamma = 0.001$ the situation has changed fundamentally. The two branches $|A\rangle$ and $|B\rangle$ are no longer independent but emerge from the same bifurcation $T1$. In addition the branches $|C\rangle$ and $|D\rangle$ have vanished.

This behavior can be understood by studying the system for very small parameters $\gamma$. For $\gamma = 0.0004$ both tangent bifurcations $T1$ and $T2$, and thus, also the branches $|C\rangle$ and $|D\rangle$ are still present. However, the two pairs emerging from $T1$ and $T2$ are now connected by a new tangent bifurcation $T3$ at which the lower lying states vanish. If the gain-loss parameter is slightly increased the bifurcation $T3$ is shifted to lower values of $Na$ and approaches $T2$ ($\gamma = 0.0006$). For $\gamma = 0.0008$ both bifurcations $T2$ and $T3$ have united and vanished.

To discuss the propagation of the two tangent bifurcations in the parameter space in more detail, a phase
Their trajectories have the characteristic form of a cusp. Equation is analytically continued and the two coupled
bifurcations are present while in the other areas only one of these states exists. For increasing values of the gain-loss parameter γ, the
bifurcation T2 is strongly shifted to lower nonlinearity parameters Na. At γc ≈ 0.000766 the two bifurcations coincide and the state |C⟩ vanishes while the states |B⟩ and |D⟩ merge.

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FIGURE 2. (Color online) Trajectories of the two tangent bifurcations T2 and T3 in the γ-Na-parameter space. In the
shaded area all three states involved in the two bifurcations are present while in the other areas only one of these states
exists. For increasing values of the gain-loss parameter γ, the bifurcation T2 is strongly shifted to lower nonlinearity param-
eters Na. At γc ≈ 0.000766 the two bifurcations coincide and the state |C⟩ vanishes while the states |B⟩ and |D⟩ merge.

The bicomplex chemical potential µ of the stationary solutions is shown in Fig. 3. For the discussion of the results we again use the components of bicomplex numbers as introduced in Eq. (3) instead of the representation in the idempotent basis since this renders the interpretation of the results much clearer. The first thing to note is that all states have vanishing µi and µj components. Therefore µr = µ holds and all states considered are PT symmetric. All states discussed so far are still present in the spectrum and only have a non-vanishing µk component (solid lines). In addition a pair of bicomplex states with µ = µ1 ± jµ1 emerges at the tangent bifurcations T1, T2, and T3 (dotted lines). For γ > γc, the bifurcations T2 and T3 have vanished leading to a bicomplex branch that is completely independent from the remaining scenario. Independent of the parameters γ and Na there are four stationary solutions.
at the bifurcation T1 has not changed, however, the bifurcations T2 and T3 have vanished and the states [B] and [D] have merged. The bicomplex states that emerged from T2 and T3 have also merged and form independent branches that do not bifurcate with any real branch.

At the tangent bifurcations two eigenvectors and the corresponding eigenvalues become equal qualifying them as exceptional points of second order [46, 47]. The bicomplex analysis reveals that at the cusp point, where the bifurcations T2 and T3 merge, a total of three states, one with real eigenvalue and two with bicomplex eigenvalues, coalesce. This is characteristic of an exceptional point of third order which has already been observed in spectra with similar cusp-like behavior [48, 49].

The coalescence of two tangent bifurcations is the characteristic property of a cusp bifurcation [36] which is described by the normal form

\[
\dot{x} = x^3 + \rho x - \sigma,
\]

whose stationary solutions \(\dot{x} = 0\) are found by Cardano’s method. The spectrum of the normal form has two tangent bifurcations which vanish at the critical values \(\rho_c = \sigma_c = 0\). In Fig. 4 the normal form is compared with the eigenvalue spectrum of the double-well potential. The parameters \(\rho\) and \(\sigma\) play the role of the gain-loss parameter \(\gamma\) and the nonlinearity parameter \(N_a\), respectively. At the tangent bifurcations in the eigenvalue spectrum real values of the chemical potential \(\mu\) turn into bicomplex values with a \(j\) component, \(\mu = \mu_1 + j\mu_2\), whereas in the spectrum of the normal form real values become ordinary complex numbers. Thus, we have to compare \(\mu_1\) with \(\text{Re} x\) and \(\mu_2\) with \(\text{Im} x\). Again the real branches are drawn as solid lines whereas the complex and bicomplex branches are drawn as dotted lines.

In the regime \(\gamma < \gamma_c, \rho < \rho_c\) the two tangent bifurcations are still separated, at \(\gamma = \gamma_c, \rho = \rho_c\) the tangent bifurcations coalesce and finally for \(\gamma > \gamma_c, \rho > \rho_c\) the bifurcations have vanished. The normal form captures the qualitative behavior of the bifurcation scenario found in the eigenvalue spectrum, thus justifying its classification as a cusp bifurcation.

V. CONCLUSION AND OUTLOOK

This answers the question raised in [35] regarding the bifurcation scenario at strong attractive interactions and, thus, completes the discussion of the eigenvalue spectrum of the three-dimensional \(\mathcal{PT}\)-symmetric double-well potential. Formulating the TDVP with bicomplex numbers in the idempotent basis has proved to be useful to analytically continue the non-analytic Gross-Pitaevskii equation and analyze the bifurcation scenarios in the eigenvalue spectrum. Using this formalism will help tackling problems with more complicated \(\mathcal{PT}\)-symmetric potentials and additional interactions such as the dipolar interaction which might show an even richer variety of bifurcation scenarios.

[1] P. A. Ruprecht, M. J. Holland, K. Burnett, and M. Edwards, Phys. Rev. A 51, 4704 (1995).
[2] R. J. Dodd, M. Edwards, C. J. Williams, C. W. Clark, M. J. Holland, P. A. Ruprecht, and K. Burnett, Phys. Rev. A 54, 661 (1996).
[3] M. Houbiers and H. T. C. Stoof, Phys. Rev. A 54, 5055 (1996).
[4] C. Sackett, C. Bradley, M. Welling, and R. Hulet, Appl. Phys. B 65, 433 (1997).
[5] N. Moiseyev, Non-Hermitian Quantum Mechanics (Cambridge University Press, Cambridge, 2011).
[6] Y. Kagan, A. E. Muryshkev, and G. V. Shlyapnikov, Phys. Rev. Lett. 81, 933 (1998).
[7] P. Schlagheck and T. Paul, Phys. Rev. A 73, 023619 (2006).
[8] K. Rapedius and H. J. Korsch, J. Phys. B 42, 044005 (2009).
[9] K. Rapedius, C. Elsen, D. Wittbaut, S. Wimberger, and H. J. Korsch, Phys. Rev. A 82, 063601 (2010).
[10] F. K. Abdullaev, V. V. Konotop, M. Salerno, and A. V. Yulin, Phys. Rev. E 82, 056606 (2010).
[11] Y. V. Bludov and V. V. Konotop, Phys. Rev. A 81, 013625 (2010).
[12] D. Witthaut, F. Trimborn, H. Hennig, G. Kordas, T. Geisel, and S. Wimberger, Phys. Rev. A 83, 063608 (2011).
[13] K. Rapedius, J. Phys. B 46, 125301 (2013).
[14] D. Dast, D. Haag, H. Cartarius, and G. Wunner, Phys. Rev. A 90, 052120 (2014).
[15] T. Gericke, P. Wurtz, D. Reitz, T. Langen, and H. Ott, Nat. Phys. 4, 949 (2008).
[16] N. P. Robins, C. Figl, M. Jeppesen, G. R. Dennis, and J. D. Close, J. Phys. B 46, 125301 (2013).
[17] D. Dast, D. Haag, H. Cartarius, and G. Wunner, Phys. Rev. A 90, 052120 (2014).
[18] T. Gericke, P. Wurtz, D. Reitz, T. Langen, and H. Ott, Nat. Phys. 4, 949 (2008).
[19] C. M. Bender, J. Math. Phys. 40, 2201 (1999).
[20] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[21] A. Mostafazadeh, J. Math. Phys. 43, 205 (2002).
[22] A. Mostafazadeh, J. Math. Phys. 43, 2814 (2002).
[23] S. Klaiman, U. Günther, and N. Moiseyev, Phys. Rev. Lett. 101, 080402 (2008).
[24] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, Phys. Rev. A 84, 040101 (2011).
[25] S. Bittner, B. Dietz, U. Günther, H. L. Harney, M. Miski-Oglu, A. Richter, and F. Schäfer, Phys. Rev. Lett. 108, 024101 (2012).
[26] H. Cartarius, D. Haag, D. Dast, and G. Wunner, J. Phys. A 45, 444008 (2012).
[27] H. Cartarius and G. Wunner, Phys. Rev. A 86, 013612 (2012).
[28] T. Mayteevarunyoo, B. A. Malomed, and A. Reoksabutr, Phys. Rev. E 88, 022919 (2013).
[29] E. M. Graefe, H. J. Korsch, and A. E. Niederle, Phys. Rev. Lett. 101, 150408 (2008).
[30] E. M. Graefe, U. Günther, H. J. Korsch, and A. E. Niederle, J. Phys. A 41, 255206 (2008).
[31] E.-M. Graefe, J. Phys. A 45, 444015 (2012).
[32] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. 6, 192 (2010).
[33] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009).
[34] B. Peng, S. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang, Nat. Phys. 10, 394 (2014).
[35] D. Haag, D. Dast, A. Löhle, H. Cartarius, J. Main, and G. Wunner, Phys. Rev. A 89, 023601 (2014).
[36] T. Poston and I. Stewart, Catastrophe theory and its applications (Pitman, London, 1978).
[37] R. Gervais Lavoie, L. Marchildon, and D. Rochon, Ann. Funct. Anal. 1, 75 (2010).
[38] R. Gervais Lavoie, L. Marchildon, and D. Rochon, Adv. Appl. Clifford Algebras 21, 561 (2011).
[39] S. Rau, J. Main, and G. Wunner, Phys. Rev. A 82, 023610 (2010).
[40] S. Rau, J. Main, H. Cartarius, P. Köberle, and G. Wunner, Phys. Rev. A 82, 023611 (2010).
[41] S. Rau, J. Main, P. Köberle, and G. Wunner, Phys. Rev. A 81, 031605(R) (2010).
[42] D. Dast, D. Haag, H. Cartarius, G. Wunner, R. Eichler, and J. Main, Fortschr. Physik 61, 124 (2013).
[43] A. D. McLachlan, Mol. Phys. 8, 39 (1964).
[44] R. Eichler, D. Zajec, P. Köberle, J. Main, and G. Wunner, Phys. Rev. A 86, 053611 (2012).
[45] D. Dast, D. Haag, H. Cartarius, J. Main, and G. Wunner, J. Phys. A 46, 375301 (2013).
[46] W. D. Heiss, J. Phys. A 45, 444016 (2012).
[47] G. Demange and E. M. Graefe, J. Phys. A 45, 025303 (2012).
[48] R. Gutöhrlein, J. Main, H. Cartarius, and G. Wunner, J. Phys. A 46, 305001 (2013).
[49] M. Am-Shallem, R. Kosloff, and N. Moiseyev, “Exceptional points for parameter estimation in open quantum systems: Analysis of the Bloch equations,” (2014), arXiv:1411.6364 [quant-ph].