Interesting dynamics at high mutual inclination in the framework of the Kozai problem with an eccentric perturber

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5 May 2014

Abstract
We study the dynamics of the 3-D three-body problem of a small body moving under the attractions of a star and a giant planet which orbits the star on a much wider and elliptic orbit. In particular, we focus on the influence of an eccentric orbit of the outer perturber on the dynamics of a small highly inclined inner body. Our analytical study of the secular perturbations relies on the classical octupole Hamiltonian expansion (third-order theory in the ratio of the semi-major axes), as third-order terms are needed to consider the secular variations of the outer perturber and potential secular resonances between the arguments of the pericenter and/or longitudes of the node of both bodies. Short-period averaging and node reduction (by adoption of the Laplace plane reference frame) reduce the problem to two degrees of freedom. The four-dimensional dynamics is analyzed through representative planes which identify the main equilibria of the problem. As in the circular problem (i.e. perturber on a circular orbit), the ”Kozai-bifurcated” equilibria play a major role in the dynamics of an inner body on quasi-circular orbit: its eccentricity variations are very limited for mutual inclination between the orbital planes smaller than $\sim 40^\circ$, while they become large and chaotic for higher mutual inclination. Particular attention is also given to a region around $35^\circ$ of mutual inclination, detected numerically by Funk et al. (2011) and consisting of long-time stable and particularly low eccentric orbits of the small body. Using a 12th-order Hamiltonian expansion in eccentricities and inclinations, in particular its action-angle formulation obtained by Lie transforms in Libert & Henrard (2008), we show that this region presents an equality of two fundamental frequencies and can be regarded as a secular resonance. Our results also apply to binary star systems where a planet is revolving around one of the two stars.

Key words: planetary systems, celestial mechanics, methods: analytical, planets and satellites: dynamical evolution and stability, binaries: general

1 Introduction
The (inner) Lidov-Kozai mechanism (Kozai 1962; Lidov 1962) is a well-known secular resonance of the restricted three-body problem, which can be reduced two degrees of freedom after short-period averaging and node reduction (see for instance Malige et al. 2002). Kozai (1962) showed that a highly inclined asteroid perturbed by Jupiter periodically exchanges its eccentricity and inclination. Its analytical theory relied on the assumption that Jupiter’s orbit is circular, so that the problem is integrable. Since its discovery, the Kozai resonance has found numerous applications in studies of planetary and stellar systems.

Recently, analytical studies (e.g. Michtchenko et al. 2006, Libert & Henrard 2007) have shown the possibility that extrasolar planetary systems can be in a long-term stable highly non-coplanar configuration, sometimes due to a secular Kozai-type phase-protection mechanism. For instance, Libert & Tsiganis (2009) found that $\nu$ Andromedae, HD 12661, HD 169830 and HD 74156 extrasolar two-planet systems have orbital parameters compatible with a Kozai-resonant state, if their (unknown) mutual inclination is at least $45^\circ$.

The Kozai dynamical phenomenon is also well-known in studies of binary systems (e.g. Innanen et al. 1997, Wu & Murray 2003, Fabrycky & Tremaine 2007), in particular in the S-type configuration (a planet revolves around one of the primaries) where the orbit of a highly inclined planet can undergo large amplitude oscillations of its eccentricity. A similar Kozai-resonant evolution can be observed in the C-type configuration, where the planet is in orbit around the binary (e.g. Migaszewski & Goździewski 2011).

Concerning the planetary three-body problem, the discovery of giant extrasolar planets on eccentric orbit raises the question of the influence of their eccentricity on potential asteroids and Earth-mass companions on inclined orbit. In a preliminary numerical study of Funk et al. (2011), the long-term stability of inclined fictitious Earth-mass planets in the habitable zone of extrasolar giant
planets discovered so far is analyzed. They have realized a parametric analysis, where several giant planet’s eccentricities are considered, while the Earth-like body is initially on a circular orbit, closer to the star than the giant gas. Their simulations show that, for distant orbits, test-planets below a critical inclination of approximately 40° are in a stable configuration with gas giants on either circular (i.e. well-known result associated with the Koziak mechanism) or elliptic orbit. Furthermore, for gas giant on eccentric orbit, the small companion exhibits non-negligible variations in eccentricity, except for a region around 35° of mutual inclination of the orbital planes, consisting of long-time stable and low eccentric orbits of the Earth-like body. So the influence of the eccentricity of the perturber on the dynamics of inclined Earth-mass planets seems to be significant and deserve to be studied in more detail with dynamical tools. This is the goal of the present contribution.

In this work, we focus on the 3-D three-body problem of a small body moving under the attractions of a star and a giant planet which orbits the star on a much wider and elliptic orbit. Our analytical study of the secular perturbations relies on the classical octupole Hamiltonian expansion (third-order theory in the ratio of the semi-major axes), widely used in planetary and stellar systems (e.g. Ford et al. 2000, Lee & Peale 2003, Migaszewski & Gozdziewski 2011). Actually, third-order terms are needed to introduce the secular variations of the eccentricity of the perturber. Indeed, the second-order quadrupole approximation does not depend on the argument of the pericenter of the perturber, whose eccentricity is thus an integral of motion (e.g. Harrington 1969, Lidov & Ziglin 1974, Ferrer & Osaci 1994 and Farago & Laskar 2010). The third-order terms introduce qualitative changes in the dynamics and can explain the aforementioned dynamical features observed by Funk et al. (2011), as we will show in this work. Let us note that, even if the octupole development is an analytical expansion of the three-body problem whatever their masses, we only focus on planetary systems with a small value of the inner body’s mass hereafter. This problem is sometimes called the reduced spatial three-body problem. Since we consider the (very limited) effect of the small mass on its companion, the secular evolution of the outer body is considered, and so are the potential secular resonances between the arguments of the pericenter and/or longitudes of the node of both bodies.

The paper is organized as follows. In Section 2, the octupole Hamiltonian formulation is recalled. Section 3 analyzes the four-dimensional secular dynamics of the elliptic spatial three-body problem, by means of 2-D geometric representations called representative planes. Section 4 focusses on the dynamical feature around 35° of mutual inclination of the orbital planes, described in Funk et al. (2011). Finally our results are summarized in Section 5.

2 OCTUPOLE HAMILTONIAN FORMULATION

Let us consider a system consisting of an inner small body ($m_1$) and an outer giant planet ($m_2$) orbiting a star ($m_3$) (also called inner three-body problem, see Farago & Laskar 2010). Due to their masses, the inner body will be named the perturbed body and the outer one the perturber in the following. We focus on the spatial (or 3-D) problem where both planetary orbits are mutually inclined. In the Solar System, this configuration corresponds for instance to an asteroid perturbed by Jupiter. Let us note that studies on the secular evolution of asteroids are mostly realized under the assumption that Jupiter’s orbit is circular (e.g. Koziak 1962). Since the discovery of extrasolar systems, inclined test particles, representing Earth-mass planets with weak gravitational effects on a system composed of a star and a gas giant, are another application of the spatial problem. However, as many giant extrasolar planets have eccentric orbit, one may wonder the influence of the eccentricity of such a Jupiter-like planet on its Earth-mass companion(s). To address this question, we consider in the following that perturber is on an eccentric orbit.

The spatial model of the three-body problem can be described using the canonical heliocentric formulation (see Poincare 1896, Laskar & Robutel 1995):

$$\mathcal{H} = \sum_{j=1}^{2} \left\{ \frac{\mathbf{p}_j^2}{2m_j} + \frac{G(m_0 + m_j)m_i}{\mathbf{r}_j} - \frac{Gm_im_j}{||\mathbf{r}_1 - \mathbf{r}_2||} + \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{m_0} \right\},$$

where $\mathbf{r}_j$ are the position vectors of $m_j$ relative to the star, $\mathbf{p}_j$ are their conjugate momenta relative to the barycenter of the three-body system, and $m_0' = (1/m_0 + 1/m_j)^{-1}$ are the reduced masses. Let us recall that the heliocentric velocities $\dot{\mathbf{r}}_j$ will not follow the direction given by $\mathbf{p}_j$, and thus the ellipses are not tangent to the real trajectory. The first term of the expansion is the sum of the keplerian velocities related to the canonical heliocentric velocities $\mathbf{q}_j = \mathbf{r}_j + \dot{\mathbf{r}}_j$, where $\dot{\mathbf{r}}_j = \mathbf{v}_j = \sqrt{\mu_j} \frac{\mathbf{r}_j}{||\mathbf{r}_j||}$, $\mu_j = Gm_0m_j/(r_j^2)$ are the planet’s reduced masses, $\mathbf{r}_j = (x_j, y_j, z_j)$ are their positions, $\dot{\mathbf{r}}_j = (\dot{x}_j, \dot{y}_j, \dot{z}_j)$ are their heliocentric velocities related to the canonical heliocentric velocities $\dot{\mathbf{q}}_j = \mathbf{v}_j - \mathbf{e}_j \mathbf{r}_j$, where $\mathbf{e}_j = (e_j, 0, 0)$ are their eccentricities, $e_j = \sqrt{1 - \frac{a_j^2}{c_j}}$ are the semi-major axes. The semi-major axes, widely used in planetary and stellar systems, are composed of a star and a gas giant, are another application of the spatial problem.

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the following way - see Prado (2005) for a similar 2-D calculation:

\[
\cos S = \left( R_1 \begin{pmatrix} \cos f_1 \\ \sin f_1 \end{pmatrix} \right)^T \cdot \left( R_2 \begin{pmatrix} \cos f_2 \\ \sin f_2 \end{pmatrix} \right)
\]

(4)

\[
r-R_1^T R_2 \equiv \begin{pmatrix} \cos f_1 & \sin f_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos f_2 \\ \sin f_2 \end{pmatrix}
\]

(5)

\[
= \tilde{a} \cos f_1 + \tilde{b} \sin f_1
\]

(6)

where \( R_1(i_1, \omega_1, \Omega_1) \) and \( R_2(i_2, \omega_2, \Omega_2) \) are Eulerian rotations of the orbital reference frames of the mass \( m_1 \) and \( m_2 \) respectively, and \( \tilde{a} \) and \( \tilde{b} \) have rather simple expressions:

\[
\tilde{a} = R_{11} \cos f_2 + R_{12} \sin f_2, \quad \tilde{b} = R_{21} \cos f_2 + R_{22} \sin f_2,
\]

(7)

\( R_{ij} \) being the element in the \( i \)th row and \( j \)th column of the matrix \( R \). The averaging of the Hamiltonian \( \mathcal{H} \) over the short-period terms,

\[
\langle H_{DP} \rangle_{M_1, M_2} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} H_{DP} dM_1 dM_2,
\]

(8)

is realized, in practice, over the eccentric anomaly \( E_1 \) of the perturbed body and over the true anomaly \( f_2 \) of the outer perturbing body. The intermediate results after the first averaging are the following:

\[
(H_{DP0})_{M_1} = \frac{G m_1 m_2}{r_2^4},
\]

(9)

\[
(H_{DP1})_{M_1} = 3 \frac{G m_1 m_2}{2a_2^3} \frac{dz_2}{r_2} a_1 e_1 \tilde{a},
\]

(10)

\[
(H_{DP2})_{M_1} = \frac{G m_1 m_2}{4 a_2^3} \left[ \frac{dz_2}{r_2} a_1^2 \right] \left[ (20\tilde{a}^2 - 15\tilde{b}^2 - 9) e_1^2 + 3(\tilde{a}^2 + \tilde{b}^2) - 2 \right],
\]

(11)

\[
(H_{DP3})_{M_1} = \frac{5G m_1 m_2}{16 a_2^4} \frac{dz_2}{r_2} a_1^4 e_1 \tilde{a} \left[ (20\tilde{a}^2 - 15\tilde{b}^2 - 9) e_1^2 + 15(\tilde{a}^2 + \tilde{b}^2) - 12 \right],
\]

(12)

where \( H_{DPi} \) means the term of the direct part \( \mathcal{K} \) associated to the \( i \)th Legendre polynomial. For the averaging over the true anomaly of the outer body, we first replace \( \tilde{a} \) and \( \tilde{b} \) by their values (see Eq. (7)), and obtain the following first terms of the secondly averaged Hamiltonian:

\[
(H_{DP0})_{M_1,M_2} = -\frac{G m_1 m_2}{a_2^2},
\]

(13)

\[
(H_{DP1})_{M_1,M_2} = 0,
\]

(14)

the terms \( (H_{DP2})_{M_1,M_2} \) and \( (H_{DP3})_{M_1,M_2} \) being too long to be exposed here. As we can see, the first two terms are constant in the secular problem and do not contribute to the averaged Hamiltonian \( \mathcal{K} \). Let us note that an alternative development of the secular expansion using Hansen coefficients can be found in Laskar & Boué (2010).

To simplify the formulation of the averaged Hamiltonian, the Jacobi’s reduction, also known as the elimination of the nodes (Jacobi 1842), allows us to reduce the expansion to a two degree of freedom function only. The reduction is based on the existence of additional integrals of motion, namely the invariance of the total angular momentum, \( \mathbf{C} \), both in norm and in direction. The constant direction of the vector \( \mathbf{C} \) defines an invariant plane perpendicular to it. This plane is known as the invariant Laplace plane. The choice of this plane as reference plane implies the following relations:

\[
\Omega_1 - \Omega_2 = \pm 180^\circ
\]

(15)

\[
G_1 \cos i_1 + G_2 \cos i_2 = C
\]

(16)

\[
G_1 \sin i_1 - G_2 \sin i_2 = 0
\]

(17)

with \( C \) being the norm of the total angular momentum. Let us note that, in our study, the invariant Laplace plane coincides almost with the Jupiter-like planet’s orbital plane, since its inclination relative to the invariant plane, \( i_2 \), is of the order of \( m_1/m_2 \) by relation (17).

Another quantity, related to the total angular momentum, is frequently used. This is known as the angular momentum deficit (Laskar 1997):

\[
\mathcal{A} = \sum_{j=1}^{2} L_j \sqrt{1 - e_j^2} \cos i_j = L_1 + L_2 - C.
\]

(18)

Finally, we present the octupole expansion of the Hamiltonian \( \mathcal{H} \) averaged over the short-period terms and expressed in the invariant Laplace plane, using the succinct formulation introduced by Ford et al. (2000):

\[
\mathcal{K} = -\alpha^3 \frac{G m_1 m_2 L_1^3}{16a_2^3} \left[ \left( 2 + 3\left( 1 - \frac{G_1}{L_1} \right)^2 \right) \left( 3 \cos^2 i - 1 \right) + 15 \left( 1 - \frac{G_1}{L_1} \right)^2 \left( 1 - \cos^2 i \right) \cos 2g_1 \right]
\]

(19)

\[
\times \left[ 1 - \frac{G_2}{L_2} \right] \left( 1 - \frac{G_2}{L_2} \right) \left[ A(\cos g_1 \cos g_2 - \cos i \sin g_1 \sin g_2) + 10 \left( \frac{G_2}{L_2} \right) \cos i (1 - \cos^2 i) \sin g_1 \sin g_2 \right] \left[ \frac{G_1}{L_1} \right] \left[ \frac{G_2}{L_2} \right] \left[ \frac{G_1}{L_1} \right] \left[ \frac{G_2}{L_2} \right]
\]

where

\[
\cos i = \frac{c_i^2 - c_i^2 \tilde{a}^2}{2a_2^2},
\]

\[
B = 7 - 5 \left( \frac{G_1}{L_1} \right)^2 - 7 \left( 1 - \frac{G_2}{L_2} \right)^2 \cos 2g_1
\]

\[
A = 7 - 3 \left( \frac{G_2}{L_2} \right)^2 - \frac{5}{2} (1 - \cos^2 i) B,
\]

\[
i = i_1 + i_2 \text{ being the mutual inclination. The equations of motion associated to Hamiltonian (19) are}
\]

\[
\dot{g}_1 = \frac{\partial \mathcal{K}}{\partial G_1}, \quad \dot{g}_2 = -\frac{\partial \mathcal{K}}{\partial G_2}
\]

(20)

One has to keep in mind that such an approach is limited to small values of the semi-major axes ratio, namely \( \alpha < 0.1 \). To consider larger values of the ratio, a development to higher order is needed, as done by Kozai (1962).

For a Jupiter-like planet on a circular orbit \( (G_2 = L_2) \), the formulation (19) simplifies to the quadrupole approximation (second-order terms in \( \alpha \)). Then the secular Hamiltonian does not depend on \( G_2 \), and the norm of the associated momentum \( G_2 \) is an integral of motion, which means that the eccentricity of the outer body is constant in this formulation. As a result, the problem is integrable and this approximation is studied in many papers (e.g. Harrington 1969, Laskar & Zahn 1977, Ferrer & Osácar 1994 and Farago & Laskar 2010, or in the artificial satellite context e.g. Laskar 1986, Russell & Brinckerhoff 2009 and Del塞te et al. 2010).

In the present work, no assumption on the eccentricity of the Jupiter-like planet is considered. The variation of the eccentricity of the perturber is introduced through the octupole terms (third-order terms in \( \alpha \)). As these terms depend on the variable \( g_2 \), the problem can not be reduced to one degree of freedom, and it induces qualitative changes on its dynamics, as shown in the next section.
3 GEOMETRIC REPRESENTATION OF THE DYNAMICS

In this section, we study the dynamics of the two degree of freedom Hamiltonian (19) by means of 2-D geometric representations, called representative planes (see Micchelenko et al. 2006, Libert & Henrard 2007). The idea is to choose a 2-D plane of initial conditions which is suitable for the analysis of the stationary solutions of the secular two degree of freedom problem. This plane should be representative in the sense that we aim to find a plane such that it contains the initial conditions of orbits representative of each class of orbits.

Such a plane can be obtained by fixing $g_1$ and $g_2$ to values that verify the conditions

$$G_1 = \frac{\partial K}{\partial g_1} = 0 \quad \text{and} \quad G_2 = \frac{\partial K}{\partial g_2} = 0, \quad (21)$$

i.e. according to the symmetries of the secular 3-D problem, $(2g_1, \Delta \omega = g_1 - g_2) = (0, 0), (0, \pi), (\pi, 0)$ and $(\pi, \pi)$. Indeed, the secular Hamiltonian function can be developed in Fourier series of cosinus terms, whose generic argument is

$$\phi = j_1 g_1 + j_2 g_2 + k \Delta \Omega, \quad (22)$$

where $j_1$ and $j_2$ are of the same parity ($j_1, j_2, k$ are integers), and $\Delta \Omega = \pi$ after node reduction. As a result, conditions (21) are verified when $\sin \phi = 0$, i.e. $(2g_1, \Delta \omega) = (0, 0), (0, \pi), (\pi, 0)$ and $(\pi, \pi)$. These four pairs of angles define four distinct quarters of the representative plane.

In the following, we choose the geometric representation introduced by Migaszewski & Goździewski (2011), and defined as $x = e_1 \cos g_1$ and $y = e_2 \cos g_2$ with $\sin 2g_1 = \sin \Delta \omega = 0$. On this representative plane, the level curves of constant Hamiltonian are plotted for given values of AMD, $\alpha = \alpha_1/\alpha_2$ and $\mu = m_1/m_2$. The boundary of permitted motion is defined as $i = 0$, 180° hereafter. Let us recall that the eccentricities and mutual inclination are related through the integral of AMD.

We insist on the fact that the representative plane is not a phase space or a surface of section. However, all orbits have to cross the representative plane (i.e. pass through the conditions $G_1 = 0$ and $G_2 = 0$). As a result, conditions (21) are verified when $\sin \phi = 0$, i.e. $(2g_1, \Delta \omega) = (0, 0), (0, \pi), (\pi, 0)$ and $(\pi, \pi)$. These four pairs of angles define four distinct quarters of the representative plane.

When the perturbing body is on a circular orbit, the octupole formulation coincides with the quadrupole approximation. Since the quadrupole approach $K_{quad}(e_1, e_2)$ is integrable, its dynamics can be represented on the phase space $(e_1 \cos \omega_1, e_1 \sin \omega_1)$ (see e.g. Thomas & Morbidelli 1996 for more detail). However, for nearly circular orbit of the perturber, the expansion (19) is four-dimensional and a first picture of the dynamics consists in the plot of the level curves of constant Hamiltonian in the aforementioned representative plane. This representation is given in Figure 1 for several values of AMD: 8.09 \times 10^{-8}, 4.79 \times 10^{-8}, 1.11 \times 10^{-8}, 1.80 \times 10^{-8}, 2.683 \times 10^{-8} and 2.7 \times 10^{-7}. For all these values the maximal mutual inclination between the two orbital planes, $i_{\text{max}}$, is reached at the origin $(e_1 = e_2 = 0)$, while the border enclosing the possible dynamics of the problem represents the coplanar case $(i = 0)$. The five first AMD values considered here correspond to $i_{\text{max}}$ of 20°, 50°, 80°, 110° and 180° respectively. In the bottom right panel of Figure 1 the region of permitted motion separates into two parts which are bordered by the curves $i = 0$ (higher absolute values of $e_2$) and $i = 180°$ (smaller absolute values of $e_2$).

As explained hereabove, the structure of the geometric representation reveals the equilibria of the problem. For small mutual inclination ($i_{\text{max}} = 20°$, see Figure 1 top left), circular orbit of the inner body corresponds to a stable equilibrium and no variation in eccentricity is possible. For larger inclinations ($i_{\text{max}} = 50°$ and $80°$, top right and middle left panels respectively), the point $e_1 = 0$ becomes an unstable equilibrium, and a separatrix divides the left panel of the representative plane (where $2g_1 = \pi$): the closed region is characterized by the libration of $g_1$ around 90° or 270° and the region outside the separatrix by the circulation of this angle. The two stable equilibria (at $g_1 = 90°$ and $g_1 = 270°$) created by bifurcation of the equilibrium at circular orbit are referred to as Kozai equilibria, by analogy to the restricted problem (Kozai 1962; Lidov 1962). This change of stability of the central equilibrium induces large variation in eccentricity for an inner body initially on a nearly circular orbit, since its real motion (short periods included) will stay close to the separatrix of the reduced problem. The maximal mutual inclination corresponding to the change of stability of the central equilibrium, called critical mutual inclination, has been calculated by Kozai (1962): it drops from the well-known value 39.23° to 32°, as the semi-major axes ratio increases from 0 to 0.5. For the parameters of Figure 1 the critical mutual inclination is 39.1°.

Additional bifurcations of these equilibria occur for higher values of mutual inclination (see middle right panel of Figure 1). For more detail, we refer to the complete study of these equilibria and their stability realized by Migaszewski & Goździewski (2009) for the three-body problem. For increasing values of AMD, the equilibrium at the origin vanishes and the region of permitted motion is divided into two islands. The dynamics is then governed by two families of equilibria: the equilibria related to the bifurcation of the Kozai equilibria and located at the border of permitted motion (called solutions IVA by Migaszewski & Goździewski 2009), and those related to the bifurcation of the central equilibrium and appearing close to the $e_1 = 0$ axis (called solution IIIa and...
Kozai problem with an eccentric perturber

Figure 1. Level curves of constant Hamiltonian (19) in the representative plane \((e_1 \cos 2g_1, e_2 \cos \Delta \omega)\) for different values of \(\text{AMD}\) such that the mutual inclination at the origin is 20° (top left), 50° (top right), 80° (middle left), 110° (middle right), 180° (bottom left). In the bottom right panel, the origin \(e_1 = e_2 = 0\) does not belong to the region of permitted motion. Other parameters are \(\alpha = 0.05\) and \(m_1/m_2 = 10^{-4}\).

by Migaszewski & Goździewski (2009). Let us note that these last ones are unstable.

In the Laplace plane reference frame, Libert & Henrard (2008) have shown that, when the orbit is outside the Kozai-resonant area, the global extrema of the eccentricities are reached when \(\sin \Delta \omega = 0\) (see also Michtchenko et al. 2006), while their local extrema are reached when \(\sin 2g_1 = 0\). Example of such a behavior is illustrated in Fig. 1 (top right panel) where the four points of intersection of a given orbit are symbolized by '*' signs. As they are located in the four quadrants, the dynamics of this orbit is characterized by the circulation of both angles 2\(g_1\) and \(\Delta \omega\). For a Kozai-resonant system considered in the Laplace plane reference frame, Libert & Tsiganis (2009) have shown that the eccentricities of both planets are not coupled, the eccentricity of the inner planet being extremal when \(\sin 2g_1 = 0\), and the one of the outer planet when \(\sin \Delta \omega = 0\). The '+' signs in Fig. 1 (top right panel) show a Kozai-resonant orbit: all the intersection points are located on the left part of the representation, indicating the libration of \(g_1\) resonant angle.

For the reasons given above, a particular interest of such a geometric view of the dynamics is to give an estimation of the
3.2 Elliptic orbit of the perturber

In this section we assume an elliptic orbit of the perturber and study the dynamical evolution of the system by means of the representative plane. While the dynamics of Figure 1 focus on nearly circular orbit of $m_2$, we keep increasing the AMD values to reach higher eccentricities of this body. First let us consider $e_2$ close to 0.1. Three different values of AMD are displayed in Figure 2. They are chosen such that the mutual inclinations of the orbits with initial eccentricities $e_1=0$ and $e_2=0.1$ is $20^\circ$ (left), $50^\circ$ (middle) and $80^\circ$ (right).

Our first observation is the similarity to the dynamics of the bottom right panel of Figure 1. So the 3-D elliptic three-body problem is affected by two kinds of equilibria only: the “Kozai-bifurcated” equilibria at very high value of $e_1$ and the equilibria at circular orbit of the inner body.

Although the dynamics is very similar for the three values of AMD displayed in Figure 2 the shifting on the y-axis is obvious and explains the different dynamics observed for a given system considered at various mutual inclinations. In order to explain analytically the results of Funk et al. (2011) (behavior of an Earth-like body initially on an inner circular orbit in a gas-giant system), let us consider the evolution of a two-planet system whose initial eccentricities are $e_1=10^{-6}$ and $e_2=0.1$. The intersection points of the evolution of the system with the representative plane are denoted by '*' signs in Figure 2.

For a small mutual inclination ($i=20^\circ$), $g_1$ and $\Delta \omega$ circulate and both variations in $e_1$ and $e_2$ are so limited that the four expected intersection points seem gathered on two points only (see left panel of Figure 2). For a mutual inclination of the orbits of $50^\circ$ (middle panel), the system is destabilized by the unstable equilibria: $g_1$ oscillates between libration and circulation and $e_1$ reaches values as high as 0.55. The same instability is present for the orbit of the third panel of Figure 2 ($i=80^\circ$), where $e_1$ reaches a value close to 1. If we had extended the integration time, the intersection points would not be regular anymore, showing the chaotic evolution of highly inclined systems due to the closeness to the ”separatrix”. These two different long-term evolutions ($i=20^\circ$ and $i=50^\circ$) are illustrated in Figure 3 by means of a numerical integration of the octupole Hamiltonian equations (20).

This change of dynamical behavior can be easily deduced from the shape of the Hamiltonian curves on the representative plane, as it can be seen on Figure 2. For small mutual inclinations (i.e. close to the borders of higher absolute values of $e_2$), the
Figure 3. Long-term evolution of a system consisting of a small body initially on nearly circular orbit ($e_1 = 10^{-6}$) and an outer body whose initial eccentricity is $e_2 = 0.1$. Arguments of pericenter are fixed to $g_1 = 0^\circ$. The initial mutual inclination between both orbital planes is $i = 20^\circ$ (left panel) and $i = 50^\circ$ (right panel). The change of dynamics is obvious.

Figure 4. Detailed views of Figure 3.
systems whose inner orbit is circular correspond to extrema of the Hamiltonian curve (left panel of Figure 5). On the other hand, for mutual inclinations higher than a value close to 40°, there exists another intersection point, belonging to the same curve of constant Hamiltonian, and of same eccentricity $e_2$ (right panel of Figure 5). The abcissa of these intersection points represent the secular variation of the eccentricity of $m_1$. For a higher eccentricity of the outer body, the dynamics is similar, as shown in Figure 5 ($e_2 = 0.5$).

So we conclude that an inner small body on a quasi-circular orbit attracted by a giant companion on an elliptic orbit behaves secularly in a similar way as in the circular three-body problem: small periodic variations of its eccentricity when the mutual inclination of the orbital planes is small, on the contrary to the large chaotic variation observed for mutual inclinations higher than a critical value of $\sim 40°$. These analytical results are consistent with the numerical study of Funk et al. (2011).

However, it is interesting to note that, even if the representative planes of Figures 1 and 5 precisely depict the dynamics around the central and Kozai families of equilibria, some additional dynamical features can be “hidden”. For instance, the $\phi'$ symbols in Figure 1 (top right panel) identify an orbit characterized by a libration of the angle $\Delta \omega$ around 180° (and simultaneous circulation of $\dot{\omega}_1$). This kind of behavior is classified as mode 2 by Michtchenko et al. (2006) and its existence can not be deduced from the analysis of our representative plane. In the same way, no particular dynamics associated to $\sim 35°$ of mutual inclination, as the one reported by Funk et al. (2011) and described in the next section, is visible on the representative planes of Figure 4.

### 4 INTERESTING DYNAMICS AROUND 35° OF MUTUAL INCLINATION

Funk et al. (2011) have realized a numerical study of the long-term stability of inclined fictitious Earth-mass planets moving under the attraction of an eccentric giant planet. The small body is initially on a quasi-circular orbit. Although the massless companion exhibits periodically and limited variation in eccentricity for a mutual inclination smaller than $\sim 40°$ (observation in agreement with our analytic study of the previous section), their simulations have identified a dynamical region around 35° of mutual inclination consisting of long-time stable and particularly low eccentric orbits.

This feature is well described by our octupole approximation, as it is shown in Figure 6 where numerical integrations of the Hamiltonian equations (20) are used to deduce the maximal eccentricity variation of the inner body on initially quasi-circular orbit ($e_1 = 10^{-6}$), and for mutual inclination of the orbital planes up to 50°. Left panel shows that, for small values of $i$, the variation of $e_1$ is negligible when the orbit of the perturber is quasi-circular (due to the presence of the central equilibrium of Figure 1), while, for eccentric orbit of the perturber, the higher the value of $e_2$, the wider the secular variation of $e_1$. This variation is even wider for high semi-major axes ratios, as shown in the right panel of Figure 6. For mutual inclinations higher than $\sim 38°$, the instability related to the Kozai bifurcations and described in the previous section produces the important increase of the secular variations of $e_1$ (see the right sides of the graphs). The main new feature of Figure 6 is the sudden decrease of the maximal $e_1$ variation around a value close to 35° for all semi-major axes ratios and eccentricities of the gas giant, as observed in the numerical study of Funk et al. (2011). Let us note that this region around 35° is not present for a circular orbit of the perturber and is much in evidence for high semi-major axes ratios, so that the change of dynamics is due to the third-order terms of the octupole expansion. The aforementioned behavior is also illustrated, in Figure 7 by means of long-term evolutions of systems with mutual inclination of 34°, 35.55° and 38° (numerical integration of the octupole Hamiltonian equations (20)). In the following, this feature is analyzed in more detail.
Kozai problem with an eccentric perturber

Figure 6. Maximal eccentricity variation of the inner small planet on quasi-circular orbit ($e_1 = 10^{-6}$) reached during its secular evolution, for mutual inclination of the orbital planes up to 50°. The integration time is fixed to $10^7$ years.

Figure 7. Long-term evolution of systems with mutual inclination of 34°, 35.55° and 38°. Other initial orbital elements are: $a_1 = 0.05$, $a_2 = 1$, $e_1 = 10^{-6}$, $e_2 = 0.3$, $g_1 = 0°$ and $g_2 = 0°$. The masses are $m_1 = 10^{-4}M_{Jup}$ and $m_2 = 1M_{Jup}$. 
To understand this particular behavior, we decide to realize an analytical study of the frequencies of the system, similar to the one of Libert & Henrard (2008). Using a 12th-order expansion of the perturbative potential in powers of the eccentricities and the inclinations, they have performed Lie transformations to introduce an action-angle formulation of the Hamiltonian and identify the analytical expressions of the four fundamental frequencies of the 3-D secular (non-resonant) three-body problem. This study has been realized in two reference frames: a general one and the Laplace plane reference frame. Our aim in the present section is to wonder whether the dynamics around 35° pointed out in Figures 6 and 7 corresponds to a commensurability between the fundamental frequencies. For the sake of completeness, their analytical study is briefly described here.

The Hamiltonian function expanded in powers of the eccentricities and the inclinations and averaged over the mean anomalies \( M_i \) reads:

\[
K = \frac{G m_1 m_2}{a_2} \sum_{k_1,j_1,k_2,j_2} B^{k_1,j_1}_{k_2,j_2} E_1^{k_1(1j_1)} E_2^{k_2(2j_2)} S_1^{k_1j_1} S_2^{k_2j_2} \cos \Phi,
\]

with \( \Phi = \sum_{i,j} p_i q_i - \sum_{k,j} \mathcal{L}_i \mathcal{L}_j \), \( E_i = \sqrt{2\mathcal{L}_i/\mathcal{H}} \), and \( S_i = \sqrt{2E_i/\mathcal{R}} \). The canonical variables in formula (24) are the classical modified Delaunay’s elements:

\[
\begin{align*}
\lambda_i &= \text{mean longitude} \\
p_i &= \text{longitude of the pericenter} \\
q_i &= \text{longitude of the node}
\end{align*}
\]

The indices \( (k,i,l \in \mathbb{N}) \) are positive integers. The coefficients \( B^{k_1,j_1}_{k_2,j_2} \) depend only on the ratio \( a_1/a_2 \) of the semi-major axes. The secular Hamiltonian is a four degree of freedom problem. Let us note that it only depends on three angles, as

\[
\Phi = \sum_{i,j} p_i q_i - \sum_{k,j} \mathcal{L}_i \mathcal{L}_j - k(q_1 - q_2).
\]

As shown in Libert & Henrard (2007), the numerical convergence of the secular series (24) is very good for a large set of parameters, even for moderate values of the eccentricities and the inclinations. The development is limited to order 12 in the eccentricities and the inclinations, which means that are kept in the Hamiltonian the terms for which the sum of the exponents of \( E_1, E_2, S_1 \) and \( S_2 \) is lower or equal to 12.

In order to obtain the analytic expressions of the four fundamental frequencies, they have used a Lie transform perturbation scheme (Deprit 1969) to average the Hamiltonian (24) over the secular variables \( p_i^{\prime}\) and \( q_i^{\prime} \) (i.e. the secular variables after a “reducing rotation” (Hénon 1988)). After a second Lie transform on the combination \( \dot{p}_i^{\prime} + \dot{p}_i^{\prime} + \dot{q}_i^{\prime} - 3\dot{q}_2^{\prime} \), they get the following action-angle formulation of the Hamiltonian – we refer to Libert & Henrard (2008) for more detail:

\[
\mathcal{K} = \sum_{i,j \leq j_1 \leq 2} C_{i,j_1,j_2} \dot{E}_1^{2j_1} \dot{E}_2^{2j_2} S_1^{2j_1} S_1^{2j_2}.
\]

The associated Hamiltonian equations lead to the expression of the four frequencies:

\[
\begin{align*}
\dot{p}_i^{\prime} &= -\frac{(x_{22})}{2} \sum_{l_1,j_1,j_2} 2l_1 C_{i,j_1,j_2} \dot{E}_1^{2l_1-j_1} \dot{E}_2^{2j_2} S_1^{2j_1} \\
\dot{p}_j^{\prime} &= \mu \sum_{l_1,j_1,j_2} 2l_1 C_{i,j_1,j_2} \dot{E}_1^{2l_1-j_1} \dot{E}_2^{2j_2} S_1^{2j_1} \\
\dot{q}_i^{\prime} &= -\frac{(x_{22})}{2} \sum_{l_1,j_1,j_2} 2l_1 C_{i,j_1,j_2} \dot{E}_1^{2l_1-j_1} \dot{E}_2^{2j_2} S_1^{2j_1} S_1^{2j_2} \\
\dot{q}_j^{\prime} &= 0,
\end{align*}
\]

\( \mu \) being the mass ratio \( m_1/(m_1 + m_2) \). The unit of frequency is the keplerian frequency \( \omega_0 = \sqrt{G m_0/a_2^3} \) of the mass \( m_0 \) multiplied by the mass ratio \( (m_1 + m_2) / m_0 \). Let us note that, in the Laplace plane reference frame, the long-term evolution of the orbital elements can be described by only two frequencies and their linear combinations:

\[
f_i = \dot{p}_i^{\prime} + \dot{q}_i^{\prime} \text{ and } f_2 = -\dot{p}_i^{\prime} + \dot{q}_i^{\prime}.
\]

In the following, we study the evolution of these frequencies with increasing values of the mutual inclination between the orbital planes. By resorting to a frequency analysis (Laskar 1993) on the data sets obtained with the octupole approximation, Table 1 identifies the main combinations of the fundamental frequencies common to the evolutions of the orbital elements for \( i = 30^\circ \). The frequencies are listed by decreasing amplitude of the trigonometric term and noted \( c_1 \) (highest amplitude) to \( c_5 \). Bold type \( c_1 \) corresponds to the precession rate of an angular variable in circulation. Last columns display the identifications of the different combinations in terms of the fundamental frequencies \( (\dot{p}_i^{\prime}, \dot{p}_j^{\prime} \text{ and } \dot{q}_i^{\prime}) \) and the two frequencies \( f_1 \) and \( f_2 \) respectively.

As can be observed in Table 1, the two frequencies, \( f_1 = 0.01184 \) and \( f_2 = 0.00837 \) (values calculated from equation (28) of the analytical 12th-order expansion, in their unit of frequency), correspond to the precession rates of the arguments of the pericenter \( \omega_0 \) and \( \omega_{12} \) respectively. Last column shows that all the frequencies of the orbital evolutions are linear combinations of these two frequencies. In particular, the main frequency of the eccentricities is the precession rate of \( \Delta\omega \) and corresponds to \( f_1 - f_2 \). Let us note that the analytical frequencies given by equation (28) are very close to the ones identified by the frequency analysis on the octupole approximation: \( f_{1oct} = 0.01192 \) and \( f_{2oct} = 0.00839 \). This small shift in the periods (less than \( 10^2 \) years) is due to the limitations of both approximations with respect to the semi-major axes ratio or the eccentricities and inclinations.

To analyze the dynamics around a mutual inclination of 35° in more detail, we examine the evolution of the two frequencies for increasing mutual inclination values. Figure 8 shows the evolution of the periods associated to \( f_1 \) and \( f_2 \) for mutual inclination up to 38°. The two curves intersect when \( i = 36^\circ \), namely 35.53° in the octupole formulation and 35.9° in the eccentricities and inclinations development.

As a result, the coupling between the frequencies of the orbital elements is different for this particular value of the mutual inclination, as shown in Table 2 (frequency analysis on the octupole approximation). Indeed, all the frequencies are combinations of \( f = f_1 = f_2 \) and a very small frequency \( g \). The change of dy-
dynamics induced by the commensurability $f_1 = f_2$, is obvious when looking at the $Δω$'s evolution in Figure 7, the angle is in libration for the very low eccentric orbit at $i = 35.55^\circ$. As it corresponds to a behavior modification of an angle, this particular dynamics can thus be regarded as a secular resonance.

All this study is realized in the Laplace plane reference frame. If we consider the orbital plane of the giant planet as reference plane ($i_2 = 0$), the evolution of the eccentricities is similar and the resonant angle becomes $ω_1 - Ω_1$, as illustrated in Figure 9.

5 CONCLUSION

In the present work, we focused on the study of the 3-D elliptic three-body problem of a small mass under the attraction of an outer giant body. The influence of an eccentric orbit of the perturber on the dynamics of a small inclined inner body has, to our knowledge, not yet been investigated in the literature. Particular attention has been given to a region around $35^\circ$ of mutual inclination consisting of long-time stable and particularly low eccentric orbits of the small body. Using a 12th-order Hamiltonian expansion in eccentricities and inclinations, in particular its action-angle formulation obtained by Lie transforms in Libert & Henrard (2008), we have shown that this region corresponds to a commensurability of the two frequencies that are the precession rates of the arguments of the pericenter $ω_1$ and $ω_2$. It explains the change of dynamics of the angle $Δω$ which starts to evolve in libration. This particular dynamics can thus be regarded as a secular resonance. The same analysis can be realized with the orbital plane of the giant planet as the reference plane (i.e. no adoption of the Laplace plane) to identify $ω_1 - Ω_1$ as the resonant angle of this reference frame.

This study also applies to binary star systems, where a planet is revolving around one of the two stars (inner problem), since the mass ratio between the bodies is of the same order ($µ ≈ 10^{-3}$).

The region around $35^\circ$ could belong to the habitable zone of the system and be of particular interest for the research of life in extrasolar systems, as it consists of stable orbits with limited eccentricity variation of the planet, which means a constant distance between the planet and the host star.
ACKNOWLEDGEMENT

The work of A.-S. Libert is supported by an F.R.S.-FNRS Postdoctoral Research Fellowship. A.-S. Libert warmly thanks the team of Vienna for fruitful discussions. Numerical simulations were made on the local computing resources (Cluster URBM-SYSDYN) at the University of Namur (FUNDP, Belgium).

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