How To Fix Non-Perturbatively A Parameter Dependent Covariant Gauge On The Lattice

L. Giusti∗, M. L. Paciellob, S. Petrarca†, B. Taglientib

aBoston University - Department of Physics, 590 Commonwealth Avenue, Boston MA 02215 USA
bINFN, Sezione di Roma 1, P.le A. Moro 2, I-00185 Roma, Italy
cDipartimento di Fisica, Università di Roma "La Sapienza"

We describe how to overcome some problems that usually prevent from obtaining an efficient algorithm to fix a generic covariant gauge on the lattice. This gauge is the lattice equivalent of the generic gauge usually adopted in perturbative calculations. It depends on a parameter whose value can be varied in order to check the gauge dependence of measured matrix elements.

Before going into details, we want to make a preliminary remark concerning the definition of the gauge potential on the lattice. As it is well known the gauge potential, on the lattice, is a derived quantity being the links the natural fields. The link is the parallel transporter of the theory and is connected with the gauge potential only in the continuum limit. Of course, it is necessary to choose a definition of in terms of the links in order to fix the gauge; this is not a problem because any definition differing from another by irrelevant terms ( ) must go to the same continuum operator up to a constant. This trivial, though fundamental, consideration was checked non-perturbatively by numerical simulations, obtaining a spectacular coincidence (see Fig. between the rescaled values of the correlator:

\[ \langle A_i, A_i \rangle(t) = \frac{1}{3V^2} \sum_i \sum_{x,y} Tr\langle A_i(x,t)A_i(y,0) \rangle, \]

(2)

built up assuming the following different definitions:

\[ A_\mu(x) \equiv \frac{(U_\mu(x) - U_\mu^\dagger(x))_{\text{traceless}}}{2ia\alpha}, \]

(3)

\[ A'_\mu(x) \equiv \frac{(U_\mu(x))^2 - (U_\mu^\dagger(x))^2)_{\text{traceless}}}{4ia\alpha}. \]

(4)

We found that the expected relation \( A'_\mu(x) = C(g_0)A_\mu(x) + O(a^2) \) is fulfilled and the effect of order terms is negligible in the measured matrix elements. This result substantially supports the correctness of the continuum limit of matrix elements built up by the insertion of operators representing \( A_\mu \) on the lattice.

Actually the freedom of the definition on the lattice, instead of being considered problematic, has been rarely exploited in literature and it is interesting to quote two recent papers that take into account this possibility for different purposes. Ref. proposed to use, in the Landau gauge, the exponential form for the relation between \( U_\mu \) and \( A_\mu \) instead of the linear one (3).
by applying a steepest descent iterative algorithm which minimizes an ad hoc functional \( F[G] \) chosen in order to have minimum (extremum) corresponding to a gauge transformation satisfying the gauge condition.

The definition of \( \langle \mathcal{O} \rangle \) shows that before applying to each thermalized configuration the gauge fixing algorithm one has to generate a set of \( \Lambda \), one for each lattice site, distributed as a gaussian of width \( \alpha \). Instead of taking more different sets of \( \Lambda \) as it is suggested by (3) and then averaging, it is reasonable to take only one set of \( \Lambda \) for each configuration. In fact one generates for each thermalized configuration a different sets of \( \Lambda \) matrices with the same width \( \alpha \).

The functional \( F[G] \), that in the Landau gauge is given by \( F_L[G] \equiv ||A^G||^2 = \int \text{Tr} (A^G(x)A^G(x)) \, d^4x \), must be changed in order to reach the new gauge condition. This problem was discussed in [4] where the following more complicated functional form was suggested:

\[
F[G] = \int d^4x \text{Tr} \left[ \partial_\mu A_{\mu}^{G(\alpha)} - \Lambda \right]^2 .
\]

The stationary points of this functional correspond to the following gauge condition:

\[
D_\mu \partial_\nu (\partial_\mu A_{\nu}^{G(\alpha)} - \Lambda) = 0 .
\]

Of course, when \( \partial_\mu A_{\mu}^{G(\alpha)}(x) - \Lambda(x) = 0 \), the new functional is stationary but there can be other non trivial solutions. This fact is obviously a problem of the new functional, and simpler functional forms are difficult to find, as it has been discussed in [4]. We will assume the functional form (6) and we will ignore the problem of spurious solutions in the following. The choice of the form of \( F[G] \) is not sufficient to solve the numerical problem of fixing the generic covariant gauge because, due to the complicated structure of (6), it is likely that the usual gauge fixing algorithm either does not converge or takes too much computer time. This problematic convergence can be traced back to the following reason. In the Landau case, being \( F_L \) proportional to \( U_\mu \), when the gauge fixing algorithm visits the lattice point \( \tilde{x} \) it puts only the gauge transformation in the point \( \tilde{x} \) different to the identity and then locally \( F_L \) depends linearly on \( G(\tilde{x}) \). Of course this property

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Comparison of the matrix elements of \( \langle A_i^g(A) \rangle(t) \) (crosses) and the rescaled \( \langle A_i A_i \rangle \cdot C_2^2(g_0) \) (open circles) as function of time for a set of 50 thermalized \( SU(3) \) configurations at \( \beta = 6.0 \) with a volume \( V \cdot T = 8^3 \cdot 16 \), \( C_t = .729(2) \). The data have been slightly displaced in \( t \) to help eye, the errors are jackknife. This figure is taken from ref. [1].}
\end{figure}
is not valid anymore for a naïve transcription on the lattice of $F[G]$ because this shows a natural quadratic dependence on $G(\tilde{x})$.

At this point one can take advantage of the freedom to choose the discretisation of $A_\mu$. In fact it is possible to find a discretisation ("driven discretisation") for the functional $F[G]$ such that it takes only a local linear dependence on $G(\tilde{x})$.

This aim can be reached by allowing different terms of $F$ to be chosen in such a way to guarantee the local linear dependence on $G(\tilde{x})$, instead of being an algebraic consequence of a particular definition of $A_\mu$. In the following we show some examples of the discretisation for the terms contained in the functional $F[G]$:

\[ 2iagA_\mu(x) = U_\mu(x) - U_\mu^\dagger(x); \]
\[ a^2g^2A_\mu(x)A_\nu(x) = I + U_\mu^\dagger(x)U_\nu(x) - U_\nu(x) - U_\nu^\dagger(x); \]
\[ a^2g^2A_\mu(x - \mu)A_\nu(x - \nu) = I + U_\mu(x - \mu)U_\nu(x - \nu) - U_\mu(x - \mu) - U_\nu^\dagger(x - \nu). \]

Each term locally transforms linearly in $G(\tilde{x})$ and it is a possible discretisation of the continuum operator written on the left side of each expression. Using the driven discretisation the expression of $F[G]$ assumes the following compact form:

\[ F[G] = \frac{1}{V a^2 g^2} Tr \sum_x J^G(x) J^{G^\dagger}(x), \]

where

\[ J(x) = -8I + \sum_\nu \left[ U_\nu^\dagger(x - \nu) + U_\nu(x) \right] - iag \Lambda. \]

As it is shown in Fig. 2, the number of sweeps necessary in order to reach a prefixed quality of the gauge fixing increases when $\alpha$ decreases. Each sweep takes around 20% more computer time than in the case of the functional $F_L$. The quality of the gauge fixing is measured by adopting as quality factor the variation $\delta F$ of $F[G]$ between a sweep and the next.

Further details on the formalism, spurious solutions and checks will be discussed in a forthcoming paper (Ref. [5]).

**REFERENCES**

1. L. Giusti, M.L. Paciello, S. Petrarca, B. Taglienti, M. Testa, Phys. Lett. **B432** (1998) 196; and Nucl. Phys. Proc. Suppl. **73** (1999) 862.
2. H. Nakajima, S. Furui, Nucl. Phys. Proc. Suppl. **73** (1999) 865.
3. F. D. R. Bonnet, P. O. Bowman, D. B. Leinweber, A. G. Williams, D. G. Richards, Discretisation Errors in Landau Gauge on the Lattice, hep-lat/9905006; see also this Conference.
4. L. Giusti, Nucl. Phys. **B498** (1997) 331.
5. L. Giusti, M.L. Paciello, S. Petrarca, B. Taglienti, in preparation.