Classical particle exchange: a quantitative treatment

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Abstract

The “classic” analogy of classical repulsive interactions via exchange of particles is revisited with a quantitative model and analyzed. This simple model based solely upon the principle of momentum conservation yields a nontrivial, conservative approximation at low energies while also including a type of “relativistic” regime in which the conservative formulation breaks down. Simulations are presented which are accessible to undergraduate students at any level in the physics curriculum as well as analytic treatments of the various regimes which should be accessible to advanced undergraduate physics majors.
I. INTRODUCTION

Countless students in introductory physics learn that the “exchange of virtual particles” is responsible for the fundamental forces of nature. Several popular introductory textbooks contain diagrams which sketch how classical particle exchange could plausibly explain the qualitative nature of repulsive forces.\(^1\)\(^2\) Furthermore, some texts even attempt to construct analogies for how attractive forces could arise from complicated exchanges of classical objects.\(^3\)\(^4\) In this paper, we wish to address the gaping hole in the literature regarding how such pictures may be quantitatively useful in understanding the connection between fundamental interactions and momentum transfer through mediating particles.

Just as physical theories are only useful within certain domains of validity, analogies are only helpful until their meanings are stretched to a point at which the usefulness breaks down. To properly analyze fundamental interactions, the methods of quantum field theory provide the tools necessary for obtaining quantitatively accurate results. Ref. \(^5\) provides a particularly illuminating discussion of how gravitational, electrostatic and nuclear potentials arise as either attractive or repulsive interactions by using the path integral formulation of quantum field theory. Additionally, by casually invoking the energy-time version of the Heisenberg uncertainty principle, one may obtain surprisingly accurate information regarding the force laws resulting from electromagnetic and nuclear interactions.\(^6\) The focus of the present work is not to require an idealized analysis within classical mechanics to describe the nature of fundamental interactions, but to explore how effective forces between particles which are spatially separated can arise within classical dynamics.

A student needs only very basic tools to explore the implications of a particular particle exchange model. With easily acquired numerical results, an advanced student may apply the mathematical analysis required to obtain both exact and asymptotic results. The goal of the present work is to present a quantitative approach, accessible at both introductory and advanced levels, which thoroughly analyzes a particular model for interactions based on classical physics.

In particular, we consider a system of two massive particles, each of mass \(M\), which interact with each other via the exchange of two mediating particles, each of mass \(m \ll M\), which are taken to always move at speed \(c\) and interact with the heavier particles through inelastic collisions, always emerging with speed \(c\) relative to a stationary lab (or
“ground”) frame. Though this model is admittedly artificial compared to the quantum field theories describing the known fundamental interactions, the reasoning required for a careful, quantitative analysis are quite useful in understanding the realistic interactions that do occur in nature through mediating quantum fields.\footnote{5}

A notable shortcoming of the classical particle-exchange analogy is its inability to describe attractive forces.\footnote{7} While it is possible to invoke quantum fluctuations in energy to explain attractive nuclear forces in a qualitative manner,\footnote{8} we emphasize that attractive interactions emerge naturally from classical scalar field theory.\footnote{9} Such a rigorous discussion of the origin of attractive forces implicitly requires a discussion of quantum theory, as these interactions rely on the wave-like nature of matter. Consequently, such treatment is beyond the scope of the present work, as we wish to present a model which may be thoroughly analyzed classically.

This paper is arranged as follows: in Sec. II we present a model for classical particle exchange and explore some basic consequences through simulations and physical reasoning, both of which are appropriate for students in introductory physics courses. Sec. III contains a thorough analysis of the model employing advanced physical reasoning and special functions to verify the speculative results obtained through careful estimation in Sec. II. Finally, we summarize the results in Sec. IV.

II. MODEL

We wish to investigate the classical picture of particle exchange as a mechanism for interaction between two massive particles. We imagine two particles each of mass $M$ exchanging small particles, each of mass $m \ll M$ as shown in Fig. 1. The analogy is often made to a pair of ice skaters (or rollerbladers) tossing a ball back and forth.\footnote{1–4} Each time one skater catches the ball and throws it back, a small amount of momentum is imparted to the skater, resulting in an effective repulsive force between the skaters which is mediated by the ball being tossed. We construct a quantitative model for this type of interaction by taking the smaller particle’s velocity to be a constant, given speed $c$. We choose the label $c$ with no reference whatsoever to the speed of light, though we will see that our $c$ plays a role in our model which is rather similar to that of the actual speed of light in electromagnetism, allowing us to explore a sort of “non-relativistic” limit of the model for speeds $v \ll c$. In order to keep the system’s center of mass at rest, we shall consider a symmetric setup in
which two small particles are exchanged. When the smaller, mediating particles approach each other we assume that they pass through one another without interaction or collide elastically.\textsuperscript{10}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Two particles of mass $M$ experience a repulsive “force” which is mediated by the exchange of a smaller particle of mass $m \ll M$.}
\end{figure}

Since the mediating particles always move at speed $c$, the collisions involving the massive\textsuperscript{11} particles with the mediating particles would not result in momentum transfer if the collisions were elastic. To obtain nontrivial momentum transfer, we must consider inelastic collisions which result in an incremental increase in the system’s kinetic energy after each collision. We shall explore whether the work required for this change in kinetic energy may be associated with an effective potential energy for the system. Taking the large, right-moving particle to be moving at speed $v$, momentum conservation applied to a single collision gives

$$Mv_n + mc = Mv_{n+1} - mc,$$

or $\delta v \equiv v_{n+1} - v_n = 2mc/M$. With repeated collisions of this form, the two massive particles will accelerate away from their common center of mass in a manner qualitatively similar to the motion experienced by two like charges placed near each other and released. We employ two approaches to investigate the quantitative nature of this effective force law. First, we simulate the system as described, obtaining numerically an effective force law which decreases as $r^{-1}$ for small velocities $v \ll c$, where $r$ is the instantaneous separation between the two massive particles. Second, the discrete sequence of collisions leads to a recursion relation which allows us to obtain a closed-form expression for $r_n$, the separation
distance immediately preceding the $n^{th}$ collision. While exact, this closed-form expression for $r_n$ is less than transparent regarding the physics of the system. In the following section, we apply continuum approximation to uncover the effective dynamics analytically in various limits.

A. Full simulation

The full simulation consists of integrating the Newtonian equations of motion for free particles moving at constant speeds and monitoring for a “collision” at which point each massive particle is given a boost in speed $\delta v = 2mc/M$ and the mediating particles are reflected with equal momenta in the opposite directions. Letting $x^{(1)}$ ($x^{(2)}$) denote the position of the right-moving (left-moving) particle and $v^{(1)}$ ($v^{(2)}$) its velocity, we consider the following initial conditions:

$$x^{(1)}(0) = -x^{(2)}(0) = \frac{r_0}{2},$$

$$v^{(1)}(0) = v^{(2)}(0) = 0.$$  \hspace{1cm} (2)

The mediating particles are initially located at the origin and begin moving in opposite directions toward the massive particles at $t = 0$ with speed $c$. Letting the positions of the mediating particles be given by $X^{(i)}$ for $i = 1, 2$, it is an instructive exercise to numerically integrate the equations of motion

$$\frac{dx^{(i)}}{dt} = v^{(i)},$$

$$\frac{dX^{(i)}}{dt} = V^{(i)}.$$  \hspace{1cm} (4)

with $V^{(1)} = +c$ and $V^{(2)} = -c$ at $t = 0$. To monitor for collisions, at each time step $\Delta t$ we check for the following condition:

$$|x^{(i)} - X^{(j)}| < \epsilon,$$  \hspace{1cm} (6)

indicating that the mediating particle nearest the $i^{th}$ particle has come within a small distance $\epsilon$ of the massive particle’s location. When this occurs, we make the following adjustment to the equations of motion:

$$v^{(i)} \to v^{(i)} + \frac{2mc}{M} \text{sign} (V^{(j)}),$$

$$V^{(j)} \to -V^{(j)}.$$  \hspace{1cm} (7)
indicating that a collision has occurred, resulting in momentum transfer. Results are generally insensitive to the time-step size, provided $\epsilon \ll c\Delta t$. Fig. 2 depicts the numerically computed average acceleration as a function of separation distance for $m = 0.005M$. For the computation of acceleration, we only use the separation distance and corresponding time just after collision events, since each massive particle’s acceleration is formally zero between collisions. Note that for position measurements which are taken at unequal time increments, we require the following discrete representation of its second temporal derivative

$$\frac{d^2r}{dt^2}\bigg|_{r=r_n} \approx \frac{r_{n+1} - r_n}{t_{n+1} - t_n} - \frac{r_n - r_{n-1}}{t_n - t_{n-1}}.$$  \hspace{1cm} (9)

A strong linear trend on a log-log plot demonstrates the power-law nature of the force law,

$$\frac{d^2x}{dt^2} \propto r^{b_1},$$  \hspace{1cm} (10)

with $b_1 \approx -1$. This result is consistent with a rough estimation of the rate of momentum transfer for $v \ll c$. Each collision is associated with transfer of momentum

$$\delta p = 2mc.$$  \hspace{1cm} (11)
For $v \ll c$, the massive particles do not move appreciably during one collision cycle. Let $r$ denote the instantaneous separation distance between the massive particles. Beginning with the mediating particles at the origin, one cycle requires each particle to cover a distance $\frac{r}{2}$ to collide with the massive particles and then another distance of $\frac{r}{2}$ to return to the origin. Thus, a single collision cycle associated with a momentum transfer $\delta p$ requires a time

$$\delta t = \frac{r}{c}. \quad (12)$$

The average force experienced by each massive particle is then

$$F_{\text{ave}} \simeq \frac{\delta p}{\delta t} = \frac{2mc^2}{r}. \quad (13)$$

The validity of this crude estimate will be examined more carefully in the next section, but for now it serves to make the results in Fig. 2 appear rather plausible. One might worry about the implications of an inverse-linear force law, since this could potentially be associated with a logarithmic potential energy function, just as in the case of two uniformly charged wires of infinite length. In the case of point particles, the potential does not asymptotically approach a constant value at large distances and should result in ever-increasing speeds as the massive particles move farther away from each other. This does not appear consistent with the model, as the mechanism for momentum does not allow the mediating particles to travel faster than speed $c$, so the speeds of the massive particles should be bounded by this limit. The resolution of this apparent paradox will be addressed below where we must refine the simulation method in order to access much longer times.

### B. Calculation of collision times

The results so far suggest a disconnect between the low-energy behavior of the model and the high-energy “speed limit” of $c$, which should be enforced by the mediating particles. To obtain some resolution, we must explore extremely large timescales, thus allowing the massive particles to approach high speeds, $v^{(1,2)} \sim c$. Because the time between subsequent collisions grows at an accelerated rate as the massive particles spread apart and speed up, the basic scheme outlined above becomes impractical. In fact, most of the computation is entirely unnecessary since all particles move with constant velocities until a collision occurs. Starting from one collision event, the time for the next collision may be computed using the
instantaneous velocities of all particles, and this process may be repeated. Though the time between collisions grows rapidly, the computation time of this scheme grows linearly with number of collisions, not with the elapsed time as before.

To proceed, let us consider a single collision event shown in Fig. [1]. With both mediating particles at the origin and instantaneous separation $r_n$ between the outwardly moving massive particles, the next collision will occur after the mediating particles have reached the massive particles, requiring a time

$$\delta t_n = \frac{r_n/2}{c - v_n},$$

(14)

corresponding to traveling a distance of $\frac{r_n}{2}$ with speed $c - v_n$ relative to the outwardly moving, massive particles. After time $\delta t_n$ has elapsed, collisions occur resulting in the mediating particles reversing directions and

$$v_n \rightarrow v_{n+1} \equiv v_n + \frac{2mc}{M}.$$  

(15)

The cycle completes when the mediating particles return to the origin. By symmetry, this also requires time $\delta t_n$, so the entire elapsed time for a complete cycle is $2\delta t_n$, or

$$t_{n+1} = t_n + \frac{r_n}{c - v_n}.$$  

(16)

To update the positions of the massive particles, we note that before the collision, each particle was moving away with speed $v_n$ with respect to the ground for time $\delta t_n$. After the collision, each particle moves away from the system’s center of mass for time $\delta t_n$ with the updated speed, $v_{n+1}$. Thus, the separation distance increases by an amount $2v_n \delta t_n + 2v_{n+1} \delta t_n$, or

$$r_{n+1} = r_n + 2v_n \delta t_n + 2v_{n+1} \delta t_n.$$  

(17)

Eqs. (14)-(17) constitute a closed recursion relation which may be iteratively advanced to obtain the velocity, separation distance and time corresponding to the beginning of each collision cycle.

For a point of comparison, we may take the approximate force law in Eq. (13) and write Newton’s second law for the motion of the right-moving particle,

$$M \frac{d^2 x^{(1)}}{dt^2} = \frac{2mc^2}{r}.$$  

(18)

Applying the symmetry of the system, we have $r = 2x^{(1)}$ and may change variables,

$$\frac{d^2 x^{(1)}}{dt^2} = \frac{d^2 r}{dt^2} = \frac{1}{2} \frac{d}{dr} \left( r^2 \right).$$  

(19)
Writing $\dot{r} = 2v$, where $v$ represents the speed of each massive particle, we may integrate both sides to obtain

$$v^2(r) = v_0^2 + \frac{2mc^2}{M} \ln \frac{r}{r_0},$$

which represents a statement of conservation of energy with a potential energy given by

$$U(r) = \frac{2mc^2}{M} \ln a r,$$

for some arbitrary length scale $a$. We refer to Eq. (20) as the non-relativistic approximation, as its derivation relies on assuming $v \ll c$. The term “non-relativistic” (NR) as used here does not refer to speeds much less than the actual speed of light but those significantly smaller than the mediating particle speed $c$. The role played by $c$ in this model is similar to that of the actual speed of light in electrodynamics, but we stress that special relativity and the actual speed of light play no role in this model. Improvements to this low-energy approximation will be explored in the next section, but we are in a position to compare its predictions to the full simulation. Fig. 3 depicts the predictions of Eq. (20) compared to the actual simulation information contained in Eqs. (14)-(17). As expected, the non-relativistic approximation breaks down as the massive particles’ speeds approach $c$. For large separation distances, the massive particle speeds do not increase as sharply with increasing distance as...
the non-relativistic approximation predicts. Indeed, once the massive particles reach a speed of \( c \), the mediating particles, also traveling at speed \( c \), are unable to catch up to the massive particles. Correspondingly the recursion relations break down and no more collisions are found. Specifically, as \( v_n \to c \) from below, we have \( \delta t_n \to \infty \). If the massive particle speed becomes exactly \( c \), \( \delta t_n \) does not exist and no further collisions occur. Another possibility is that a single collision changes \( v_n \) from just below \( c \) to just above \( c \). In this case, \( \delta t_n \) formally becomes negative and we conclude similarly that no further collisions occur.

The behavior of the system explored thus far can be summarized as follows: for arbitrary initial separations, the massive particles are repelled from each other by the effective force provided by the mediating particles. At long times, the speeds (with respect to the ground) of the massive particles approach \( c \), the speed of the mediating particles. While an approximate statement of energy conservation has been derived (see Eq. (20)) for low speeds \( v \ll c \), the associated potential is problematic as it has no lower bound for \( r \to \infty \). An unlimited amount of potential may be converted into the massive particle’s kinetic energy resulting in the erroneous prediction that for any initial separation, both massive particles will continue to accelerate rather than asymptotically approach finite speeds. That the initial separation distance has no effect on the final speeds of the massive particles suggests that the system is not conservative. In the next section, we will carefully examine this system using analytic tools to quantitatively explore some of these issues.

III. ANALYTIC APPROACH

A. Exact solution to recursion relation

The discrete sequence of collisions described by Eqs. (14)-(17) can be analyzed exactly, yielding a closed-form expression for \( r_n \), the separation distance after \( n \) collisions. Eq. (15) simply states that the velocity increases by a constant amount after each collision, or

\[
v_n = \frac{2mnc}{M}.
\]
Inserting Eq. (22) into Eq. (17) and using Eq. (14), we have

\[ r_{n+1} = r_n + 2 \left[ v_n + \frac{mc}{M} \right] c - v_n, \]  
(23)

\[ = \left( 1 + \frac{2m(n+1)}{M} \right) r_n. \]  
(24)

Proceeding iteratively,

\[ r_1 = \left( 1 + \frac{2m}{M} \right) r_0, \]  
(25)

\[ r_2 = \left( 1 + \frac{4m}{M} \right) \left( 1 + \frac{2m}{M} \right) r_0, \]  
(26)

\[ \vdots \]  
(27)

\[ r_n = \left( 1 + \frac{2nm}{M} \right) \prod_{k=0}^{n-1} \left( 1 + \frac{2km}{M} \right) r_0. \]  
(28)

By employing the Gamma function, which satisfies

\[ \Gamma(x + 1) = x\Gamma(x), \]  
(29)

and reduces to the factorial for integer arguments, \( n! = \Gamma(n + 1) \), we may write this as

\[ r_n = \frac{\Gamma \left( \frac{M}{2m} + n \right) \Gamma \left( \frac{M}{2m} - n \right)}{\left[ \Gamma \left( \frac{M}{2m} \right) \right]^2} \left( 1 - \left( \frac{2mn}{M} \right) \right) r_0. \]  
(30)

The derivation of Eq. (30) from Eq. (28) requires use of Eq. (29), the property

\[ \Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}, \]  
(31)

and their mathematical offspring,

\[ \Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)}. \]  
(32)

**B. Limiting cases**

As an exact, closed-form solution, Eq. (30) contains all of the physics we have encountered up to this point. The low-energy force law in Eq. (13) was previously derived using physical reasoning, but we can demonstrate that it also follows from the exact solution rather than
appealing to comparisons such as Fig. (2). To this end, let us define $\alpha \equiv \frac{M}{2m}$ and take the natural logarithm of Eq. (30), obtaining

$$\ln \frac{r}{r_0} = \ln \Gamma (\alpha - n) + \ln \Gamma (\alpha - n) - 2 \ln \Gamma (\alpha)$$

$$+ \ln \left[ 1 - \left( \frac{n}{\alpha} \right)^2 \right].$$

(33)

To investigate the dynamics for $m \ll M$ and $v \ll c$, we examine the limit $\alpha \to \infty$ with $n \ll \alpha$. We first apply Stirling’s approximation to the Gamma functions,

$$\ln \Gamma (\alpha \pm n) \simeq (\alpha \pm n) \ln [\alpha \pm n],$$

(34)

$$\ln \Gamma (\alpha) \simeq \alpha \ln \alpha.$$  

(35)

Applying the limit $n \ll \alpha$ and expanding the logarithms according to

$$(1 \pm x) \ln [1 \pm x] \simeq x + \frac{x^2}{2},$$

(36)

we recover the result

$$\ln \frac{r}{r_0} \simeq \frac{n^2}{\alpha},$$

(37)

which is equivalent to Eq. (20) with $v_0 = 0$ upon the identification $n \to \frac{M}{2m} \frac{v}{c}$ (see Eq. (22)).

Alternatively, we may consider the limit $v \to c$. Note that Eq. (30) diverges as $n \to \alpha$, indicating that this only occurs as $r \to \infty$. Implicit in this relation is the upper limit on the number of collisions before the massive particles reach terminal velocity,

$$n_{\text{max}} = \frac{M}{2m}.$$ 

(38)

We may probe the system at long times by letting $n = \alpha - \epsilon$ for $\epsilon \ll 1$. Eq. (30) then becomes

$$\frac{r_n}{r_0} = \frac{\Gamma (2\alpha)}{[\Gamma (\alpha)]^2} \Gamma (\epsilon) \cdot \frac{2\epsilon}{\alpha}.$$ 

(39)

Employing the small-argument expansion

$$\Gamma (\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon),$$

(40)

where $\gamma \simeq 0.577$ is the Euler-Mascheroni constant, we may expand Eq. (39) to obtain

$$\frac{r}{r_0} \simeq \frac{2\Gamma (2\alpha)}{\alpha [\Gamma (\alpha)]^2} (1 - \gamma \epsilon).$$

(41)
Taking $\epsilon \to 0$ is equivalent to letting $v \to c$, and we obtain

$$r \to r_c \equiv \frac{4m\Gamma\left(\frac{M}{m}\right)}{M\left[\Gamma\left(\frac{M}{2m}\right)\right]^2} r_0$$

as $v \to c$. \hfill (42)

For $r > r_c$, each massive particle is moving at the same speed as the mediating particles and experiences no subsequent collisions with the mediating particles. Some mystery may be removed from Eq. (42) by taking the natural logarithm of both sides and applying Stirling’s approximation, this time keeping several terms

$$\ln \Gamma(x) \simeq x \ln x - x - \frac{1}{2} \ln \frac{x}{2\pi}.$$ \hfill (43)

When the smoke clears, we have the compact result

$$r_c = \sqrt{\frac{2m}{\pi M}} 2^{\frac{M}{m}} r_0.$$ \hfill (44)

That is, at a finite separation distance, the massive particles attain their maximum speeds $v = c$. We note that since the massive particles always evolve to this state regardless of initial separation (i.e., various amounts of supposed “potential energy” in the initial state with no kinetic energy) energy cannot be conserved in this system. States with different energies all evolving into a single high-energy state requires sources or sinks in energy. However, the low-energy, non-relativistic approximation is quite useful for describing the dynamics at low energies. Unfortunately, unlike the Coulomb repulsion, there exist no initial conditions for which the relativistic limit is avoided.

### C. Relativistic corrections

The discrete relations in Eqs. (15)-(16) may be formally interpreted as differential equations by applying the convention

$$v_{n+1} - v_n = \frac{\Delta v}{\Delta n} \to \frac{dv}{dn},$$ \hfill (45)

with a similar relation for $t_n \to t(n)$. One then obtains

$$\frac{dv}{dt} = \frac{dv}{dn} \frac{dn}{dt} = \frac{2mc}{Mr} (c - v).$$ \hfill (46)

Note that this corresponds to an acceleration given by the force in Eq. (13) with corrections which are first-order in $\beta \equiv \frac{v}{c}$. Unlike the non-relativistic limit, this acceleration explicitly
drops to zero as \( v \to c \). Furthermore the explicit appearance of \( v \) in the force indicates a non-conservative nature to this force. Employing the chain rule as for the non-relativistic limit, we obtain the following equation for \( \beta(r) \),

\[
r\beta \frac{d\beta}{dr} = \frac{m}{M} (1 - \beta). \tag{47}
\]

Eq. (47) is separable and admits the closed-form solution

\[
\left( \frac{r}{r_0} \right)^{\frac{m}{M}} = \frac{e^{-\beta}}{1 - \beta}. \tag{48}
\]

This may be inverted to yield a formula for \( v = \beta c \)

\[
v(r) = c \left[ 1 + W \left( -\left(\frac{r_0}{r}\right)^{m/M} \frac{e}{1} \right) \right], \tag{49}
\]

where \( W(z) \) is the Lambert-W function,\(^{18}\) defined as the principal value of

\[
z = W(z)e^{W(z)}. \tag{50}
\]

While the solution clearly satisfies \( v(r_0) = 0 \) and

\[
\lim_{r \to \infty} v(r) = c [1 + 0] = c, \tag{51}
\]

the time required for this to happen (rigorously, for \( |c - v| < 2mc/M \)) is quite large, and unfortunately for the theory, this does not appear to agree very well with the simulation or exact solution (see Fig. 4), breaking down even before the non-relativistic approximation breaks down. There is an equally curious situation that occurs in electromagnetism. The general solutions to Maxwell’s equations for known sources rely on fairly complex expressions involving evaluation of physical quantities at retarded times. However, by expanding these expressions the lowest-order term is the \textit{instantaneous} Coulomb term. This appears to be a rather deep result also showing up in quantum electrodynamics\(^{19}\) and quantum gravity\(^{20}\). The refined approximation in this section is only part of the required correction to the non-relativistic limit, and some potentially “fortuitous” cancellation between this modification and the rest of the terms being neglected is required to obtain a result more accurate than the NR approximation. An example of this sort of fortunate cancellation from classical physics may be observed by considering the electric field due to an arbitrary configuration of currents and charges, given by one of Jefimenko’s equations,\(^{15}\)

\[
E(r, t) = \frac{1}{4\pi\varepsilon_0} \int \left[ \frac{\rho(r', t_r)}{R^2} \hat{R} + \frac{\dot{\rho}(r', t_r)}{cR} \hat{R} + \frac{\dot{J}(r', t_r)}{c^2R} \hat{R} \right] d\tau', \tag{52}
\]
where $\rho$ is charge density, $\mathbf{J}$ is current density, $R = |\mathbf{r} - \mathbf{r}'|$, and the retarded time is given by $t_r \equiv t - R/c$. Following an exercise in a popular text on electrodynamics, one may consider constant currents for which $\dot{\mathbf{J}} = 0$ (i.e., the third term in Eq. (52) disappears). In this case, a miraculous cancellation occurs, yielding

$$ E(r, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r', t) d\tau'}{R^2} $$

(53)

where the correction to the instantaneous Coulomb potential and the second term in Eq. (52) cancel perfectly. That is, despite the explicit appearance of corrections of order $\beta$ and evaluation of functions at $t_r$ instead of instantaneous time $t$, the field turns out to be the instantaneous Coulomb-like contribution. The “relativistic correction” in the particle-exchange model appears to be analogous to evaluating the field at retarded times without including the additional corrections, resulting in a less-accurate result at short times.

IV. SUMMARY

In this paper we thoroughly examined a simple model for classical interactions through the exchange of mediating particles in which momentum conservation is enforced for each
collision. As demonstrated in simulations and analytic reasoning, the resulting interactions yield an effectively conservative theory at low energies with a $1/r$ force. The conservative approximation breaks down at high energies, and regardless of initial separation, the massive particles both eventually reach the maximum speed allowed by the physical mechanism of energy transfer within the system.

The classical particle exchange analogy of ice skaters throwing a ball back and forth has typically been used as an illustration in public outreach presentations and in teaching, from general education science courses to introductory and advanced physics courses. However, the analogy has value as a physical system for students to investigate quantitatively. The phenomenon can be used in various contexts including homework, an in-class activity, a computational physics exercise, or assessment. Furthermore, it can be used at both the introductory and advanced level in the undergraduate curriculum.

In introductory physics, students learning computational modeling\textsuperscript{[21]} can investigate the phenomenon numerically. Derivation of the change in speed of a massive particle, $\delta v = 2mc/M$, using Conservation of Momentum (Eq. 1) is a straightforward exercise in introductory physics. Students can also explore and describe the position-time and velocity-time graphs. Because position and velocity change abruptly, introductory students have the opportunity to fit a smooth function to values that change discretely. Furthermore, teachers can use this system to assess understanding of potential energy functions (Eq. 21) and conservation of energy. Having already studied systems of particles interacting via the inverse-square law, students can practice applying a similar analysis to the $1/r$ force, possibly preparing them for similar forces that arise in an E&M course. Finally, as shown in this paper, teachers can also use the system as an application in a junior/senior level course in mechanics\textsuperscript{[22]} or mathematical physics where students are expected to explore the model in its limit using more advanced computational and analytical techniques.

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10 The physics is unchanged if we interpret this lack of interaction as a perfectly elastic collision between the mediating particles. In fact, in one dimension energy conservation and momentum conservation only allow for exchange of incoming momenta in two-particle scattering events.\(^\text{[12]}\)

11 We use the term “massive” to describe the particles of mass \(M \gg m\), but this convenient terminology is not meant to suggest that the mediating particles are massless. While we take \(m \ll M\), the model requires a nonzero mediating particle mass \(m\).

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