Efficiency estimation for an equilibrium version of Maxwell refrigerator

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Maxwell refrigerator as a device that can transfer heat from a cold to hot temperature reservoir making use of information reservoir was introduced by Mandal et al. [1]. The model has a two state demon and a bit stream interacting with two thermal reservoirs simultaneously. We work out a simpler version of the refrigerator where the demon and bit system interact with the reservoirs separately and for a duration long enough to establish equilibrium. The efficiency, η, of the device when working as an engine as well as the coefficient of performance (COP) when working as a refrigerator are calculated. It is shown that the maximum efficiency matches that of a Carnot engine/refrigerator working between the same temperatures, as expected. The COP at maximum power decreases as $\frac{1}{\beta_c}$ when $T_h > T_c \gg \Delta E$ ($k_B = 1$), where $T_h$ and $T_c$ are the temperatures of the hot and cold reservoirs respectively and $\Delta E$ is the level spacing of the demon. η at maximum power of the device, when working as a heat engine, is found to be $\frac{T_h}{T_h + T_c}$ when $T_c \ll \Delta E$ and

$\frac{1}{\beta_h}$ when $T_c \gg \Delta E$.

Maxwell [2] introduced an ‘intelligent’ demon that could discern the speed of molecules in two chambers connected by a trap door. Using this information and operating a trapdoor connecting the two chambers, the demon can transfer heat from cold to hot body, thus apparently violating the second law of thermodynamics. The discussions on this thought experiment raged through most of the twentieth century [3-7] and found a full resolution with the works of Landauer and Bennett [8, 10]. Crucial to the understanding of the fact is the problem that the demon needs to record the outcome of the measurement experiment and this involves increasing the information entropy of the memory registers. Resetting these memory elements leads to generation of entropy, which if accounted for explains the validity of second law [11].

Recently there has been a renewed interest in physics related to Maxwell demon to better understand the interrelation between information and thermodynamics in varied systems and devices. These include experimental work where Maxwell Demon like set up has been implemented and studied [12, 15], exploring the relevance of these ideas in biomolecular systems [16, 17] and molecular machines [18]. Theoretical studies on models of Maxwell’s demon like systems, both autonomous and with feedback loops [1, 19, 27], has been of carried out recently. The analysis of these models in the light of generalized fluctuation theorems in non-equilibrium thermodynamics has been of recent focus [24, 27].

One of the earliest exactly solvable model of Maxwell refrigerator (MR) was proposed by Mandal et al. [1]. It is an autonomous device consisting of a two level system (the demon) and a bit stream, that serves as a memory element, interacting simultaneously with two thermal reservoirs (Refer Fig. (1) in [1]). By solving for steady state of the demon using the master equation for the demon and bit system, it was shown that in certain regime of initial probability distribution of bits states the device acts as a refrigerator. In this work we present a related but simpler model, where the demon (two level system) first interacts with the hot reservoir and subsequently the demon and current bit in the bit stream interacts with the cold reservoir. After proving the validity of the second law for this system and working out various modes of operation of the device, we estimate the efficiency and COP of the device in these modes. Specifically we study the efficiency/COP in the reversible case and at maximum power.

The model consists of, as in the case of MR, a moving bit stream, a two level demon and two thermal reservoirs at temperatures $T_h$ and $T_c$. The primary difference between the new model and the MR model is in that we assume the bit plus demon system interacts with the two thermal reservoirs separately (see Fig. (1) given in the supplementary). Further, the interaction duration is taken to be large enough such that the system comes to equilibrium with the two reservoirs alternatively. We shall assume that the incoming bits are uncorrelated and the probability of a bit being in zero state is $p_0$. The rules for the transition when in contact with the two thermal reservoirs are such that while in contact with the hot reservoir, the bit states are left unaltered and while in contact with the cold reservoir transitions are allowed only between states 0d to 1u. The ratio of rates given by $\frac{p_{1u}}{p_{0d}} = e^{\beta_h \Delta E}$, where $\Delta E \equiv E_u - E_d$ being the energy difference between the two states of the demon and $\beta_h = \frac{1}{T_h}$ is the inverse temperature (we are using the convention that the Boltzmann constant, $k_B = 1$).

We will now solve for the probability of the outgoing bit to be 1. Consider the demon and the current bit in contact with the cold reservoir. Since the demon state would be determined by its contact with the hot reservoir with which it was in equilibrium, the probability that the state of the demon is down will be $\frac{1+e^{\sigma}}{2}$ and for it do be in the up state will be $\frac{1-e^{\sigma}}{2}$, where $\sigma = \tanh(\frac{\beta_c \Delta E}{2})$ with $\beta_c = \frac{1}{T_c}$. After the interaction of the demon and bit
combined system with the cold reservoir there are three possible ways for the current bit to end up in state 1: (i) The current bit at the beginning was 1 and the demon state at the beginning was down. The probability for this event is $p_1 (1+\sigma)/2$, where $p_1 = 1 - p_0$ is the probability for the incoming bit to be one. (ii) The current bit at the beginning was 0 and the demon state at the beginning was up. And the end state after interaction with the cold reservoir is $1u$. The probability for this event is $p_0 (1+\sigma)/(1-\omega)$, where $\omega = \tanh(\beta_T \Delta E)$ and $\beta_T = \frac{1}{T_T}$. (iii) The current bit at the beginning was 1 and the demon state at the beginning was up. And the end state after interaction with the cold reservoir is $1u$. The probability for this event is $p_1 (1-\sigma)/(1-\omega)^2$. Adding all the probabilities above we find the probability, $p_1'$, for the outgoing bit to be 1 is

$$p_1' = p_1 \frac{(1+\sigma)}{2} + p_0 \frac{(1+\sigma)}{2} \frac{(1-\omega)}{2} + p_1 \frac{(1-\sigma)}{2} \frac{(1-\omega)}{2}$$  

(1)

Since any absorption of heat from the cold reservoir comes with a flip of the current bit from 0 state to 1 state, the average heat transferred per cycle from the cold reservoir to the hot reservoir will be, $Q = \Phi \Delta E$, where $\Phi \equiv p_1' - p_1$ is the increase in the occurrence of 1s in the outgoing bit stream compared to the incoming stream. In the current model, the outgoing bits are not correlated as the demon equilibrates with the hot reservoir before it comes in contact with the cold reservoir again. This ensures that any memory of its state after the interaction with the cold reservoir (along with the bit state) in the previous cycle is lost. Putting in the value of $p_1'$ from Eq. (1) (defining $\delta \equiv p_0 - p_1$, $\epsilon \equiv -\frac{\omega - \sigma}{1-\omega\sigma}$), and simplifying one gets

$$\Phi = \frac{\delta - \epsilon}{4} (1 - \sigma \omega)$$  

(2)

As in the original MR model [1], one sees that there is a transfer of heat from cold to hot reservoir provided $\delta > \epsilon$. Parameter $\delta$ measures the capacity of the information reservoir (the incoming bit stream) to transfer heat from cold to hot reservoir and the parameter $\epsilon$ is a measure of the temperature difference between the reservoirs. It is to be noted that the expression for $\Phi$ in the current model is different from the one obtained for the MR model [1], both for the general case as well as for the special case where the interaction rate with the hot reservoir goes to infinity.

The seeming violation of the second law, in that one is able to transfer heat from cold body to hot body, is corrected for provided one considers the increase in information entropy of the bit stream. The change in information content of the bits is given by

$$\Delta S_B = p_1 \ln(p_1) + p_0 \ln(p_0) - p_1' \ln(p_1') - p_0' \ln(p_0')$$  

(3)

where the primed quantities refer to the probabilities associated with the outgoing bit stream. By Landauer’s principle this is the minimum entropy that will be generated as one resets the outgoing memory bits to the same probability distribution as the incoming ones. The entropy change in the thermal reservoirs is

$$\Delta S_{res} = \Phi (\beta_h - \beta_c)$$  

(4)

where we have put $\Delta E = 1$, defining the energy scales with respect to the level spacing. The sum of the above two entropies (given in Eq. (3) and Eq. (4)) is the average change in entropy of the system per cycle and can be written as

$$\Delta S_T = \frac{1 - \delta}{2} \ln \frac{1 - \delta}{1 - \delta - 2\Phi} + \frac{1 + \delta}{2} \ln \frac{1 + \delta}{1 - \delta - 2\Phi} + \Phi \ln \frac{(1 + \delta - 2\Phi)(1 - \epsilon)}{(1 - \delta - 2\Phi)(1 + \epsilon)}$$  

(5)

where we have used $p_0' = 1 - p_1'$, $p_1' = \Phi + p_1$, $p_1 = 1 - p_0$ and $p_0 = \frac{\delta + 1}{2}$.

It can be shown that $\Delta S_T$ is larger than or equal to zero for all parameter values, thus validating the second law. The first two terms in Eq. (5) is the Kullback-Leibler divergence between probability distribution of the incoming and the outgoing bits, $D_{KL}(p||p')$. The positivity of this quantity (by Gibb’s identity) ensures that the sum of first two term is always larger than or equal to zero. The third term (T3) in the RHS of the equation above can be shown to be always larger than or equal to zero by looking at three separate cases: $\delta = \epsilon, \epsilon < \delta \leq 1$ and $-1 \leq \delta < \epsilon$. When $\delta = \epsilon$, T3 is zero since $\Phi = 0$. When $\epsilon < \delta \leq 1, \Phi > 0$ and it can be shown that the argument inside the logarithm is larger than 1 (see supplementary). And when $-1 \leq \delta < \epsilon, \Phi < 0$ and it can be shown that

![FIG. 1. Various entropy terms as a function of $\delta$ for $T_c = 0.5$ and $T_h = 10$. This plot gives the generic behavior of these terms as function of $\delta$. Above $\delta > \epsilon$, $-\Delta S_{res}$ is positive indicating that heat is transferred from the cold reservoir to the hot reservoir. For $\delta_c < \delta < \epsilon$, heat is transferred from hot to cold reservoir, but the entropy change of bits is negative. This implies that in this regime the demon and bits system is working as an eraser.](image)
the argument inside the logarithm is between 0 and 1. These observations ensure that $T_3$ is always larger than or equal to zero thus proving that $\Delta S_T \geq 0$. Note that the process is reversible for the case when $\delta = \epsilon$ and there is no heat exchange between the thermal reservoirs for this case.

In Fig. 1 the variation of $\Delta S_B$ and $-\Delta S_{res}$ are plotted as a function of $\delta$ for $T_c = 0.5$ (in units of $\frac{\Delta S_B}{k_B}$) and $T_h = 10$. Note that $\Delta S_B$ is zero at $\delta = \epsilon$ and it has another zero for a lower value of $\delta$, which we refer to as $\delta_0(\omega, \epsilon)$. Since $\Delta S_T = \Delta S_B + \Delta S_{res} \geq 0$ and $\Delta S_T = 0$ at $\delta = \epsilon$, the straight line curve corresponding to $-\Delta S_{res}(\delta)$ is tangential to the curve $\Delta S_B(\delta)$ at the point $\delta = \epsilon$. The regime $\epsilon < \delta \leq 1$ is where the device works as a refrigerator, since in this region $Q$ is positive. In the regime $\delta_0 < \delta < \epsilon$, $\Delta S_B < 0$ and hence there is an erasure of the incoming bits on the average. Assuming a Szilard type reversible mechanism to convert the information entropy gained into work (and thus setting the probability distribution of bits to its incoming value), one can consider the $\Delta S_B < 0$ region as one where the system is working as a heat engine. When $-1 \leq \delta < \delta_0$, $\Delta S_B > 0$ and $Q < 0$ implying that the system is neither acting as a refrigerator nor a heat engine.

If $T_h \gg 1$, $\sigma$ can be put approximately to 0 and we have, $\epsilon \approx \omega$ and $\Phi \approx \frac{\epsilon}{1-\delta}$. Using these values in Eq. (3) for the change in bit entropy, one can solve for $\Delta S_B = 0$ giving the nontrivial root, $\delta_a = -\frac{\omega}{\epsilon}$ (see supplementary for details) in this limit. $\delta = \epsilon$ is in any case a root even without the above approximation. It is difficult to analytically solve $\Delta S_B = 0$ for arbitrary values of $T_c$ and $T_h$ but the phase diagram for the model can be found numerically. The phase diagram indicating three modes of operation for the demon and bits system as a function of $\delta$ and $\epsilon$ values for various fixed values of $T_c$ is shown in Fig. 2.

We now focus on the COP/efficiency of the device in its various regimes of operation. As discussed above when $\epsilon < \delta \leq 1$, the device acts as a refrigerator transferring heat $\Phi$ from the cold to the hot reservoir. For this device to work in a cyclic manner, one needs to reset the outgoing bits, such that the probability distribution for 1s and 0s are the same as that of the incoming bits. Since $\Delta S_B$ is positive in this regime, the resetting process will need a minimum work to be carried out by an amount equal to $T_h \Delta S_B$ by Landauer’s principle. Note that the resetting has to be done with the bits in contact with the hot bath as dissipating heat into the cold bath will be at cross purpose with the intention of cooling that bath. Thus the coefficient of performance (COP) of the device when working as a refrigerator is given by,

$$\text{COP} = \frac{\Phi}{T_h \Delta S_B}. \quad (6)$$

The variation of COP for the case when $T_c = 0.5$ and $T_h = 1$ is given in the inset of Fig. 3 where we have used the derived forms of $\Phi$ and $\Delta S_B$ (Eqs. (2) and (3)) to compute the curve. Note that the smallest value of delta in this plot is $\epsilon = 0.46$.

There are two important limits to look at in the refrigerator operation: (i) $\delta = \epsilon$, where the process is reversible and (ii) $\delta = 1$, where $\Phi$ is maximum and hence one gets maximum power. In the limit $\delta$ approaches $\epsilon$ from above, one can show that the COP tends to the limit $\frac{1}{2\pi-1}$ (see supplementary). This follows from the observation that $-\Delta S_{res}$ is tangential to the $\Delta S_B$ curve at $\delta = \epsilon$ in Fig. 1. The result is not surprising since in the limit $\delta$ approaches $\epsilon$, the refrigerator cycle becomes reversible and one should expect the COP to match with that of the Carnot refrigerator. In fact, it is an indication of the fact that our definition of COP is consistent with its traditional definition. Operating a refrigerator or engine reversibly is not practically very interesting as it takes infinite time to complete the cycle [28]. Finite time thermodynamics looks for optimization also in terms of power or rate at which heat is transferred. In the current device working as a refrigerator, maximum power can be attained by maximizing the value of $Q$. This happens at $\delta = 1$ where $\Phi$ has the largest positive value. With $\delta = 1$, the change in entropy of bits is given by

$$(\Delta S_B)_m = -\left(1 - \Phi_m \right) \ln \left(1 - \Phi_m \right) - \Phi_m \ln(\Phi_m) \quad (7)$$

where $\Phi_m = \frac{1-\epsilon}{1-\sigma \omega}$ is $\Phi$ evaluated for $\delta = 1$. Fig. 3 shows the COP of the refrigerator at maximum power (that is when $\delta = 1$) as a function of $T_h$ for different values of $T_c$: $T_c = 0.5$ and $T_c = 3$. As expected, the COP at maximum power lies below the Carnot refrigerator COP. It can be shown that in the limit of $T_h$
and $T_c$ both much larger than 1 (see supplementary),

$$\text{COP}_{\text{max}} = \frac{1}{(4 \ln 4 - 3 \ln 3)T_h}$$

and is independent of $T_c$.

We now look at the erasure/heat engine regime of the device. We shall assume extraction of work equal $T_h \Delta S_B$ via a reversible process of the kind carried out in a Szilard engine. The work is extracted in contact with the hot reservoir as that would give us maximum possible work output for a given $\Delta S_B$. The efficiency of the heat engine can be computed as,

$$\eta = \frac{-T_h \Delta S_B}{-\Phi - T_h \Delta S_B},$$

where the numerator on the right is the work extracted and the denominator is the net heat absorbed from the hot reservoir. $\Phi$ and $\Delta S_B$ are negative quantities when $\delta_a < \delta < \epsilon$.

As in the case of refrigerator COP, one can show that $\eta$ approaches the Carnot efficiency as $\delta$ approaches $\epsilon$ from below. This is not surprising as the engine is reversible in that limit. The variation of $\eta$ for fixed values of $T_c$ and $T_h$ ($T_c = 0.5$ and $T_h = 10$) are shown in the inset of Fig. 4. The efficiency at maximum power for fixed $T_h$ and $T_c$ can be computed by finding the value for $\delta$ for which $\Delta S_B$ is a minimum and evaluating $\eta$ at that value. Fig. 4 shows the variation of efficiency at maximum power as a function of $T_h$ for two fixed values of $T_c$. This curve has been computed numerically by finding the root of the equation $\frac{d}{d\delta} \Delta S_B(\delta) = 0$ and evaluating the efficiency at that value of $\delta$. As expected, the efficiency lies below the corresponding Carnot efficiency. It is clear from Eq. 9 that for large values of $T_h$, the efficiency will approach 1, just as it is for the Carnot engine.

One can analytically find $\eta_{\text{max}}$ for the case when $T_h \gg 1$. The value of $\delta$ leading to the minima of $\Delta S_B$ in this limit can be found to be the solution of the cubic equation $\delta^3 + \omega \delta^2 - 3 \delta + \omega = 0$ (see supplementary). For this case, if $T_c$ is also much larger than one, the region $\delta_a < \delta < \epsilon$ narrows down to zero as both $\delta_a$ and $\epsilon$ are proportional to $\omega$ when $T_h$ tends to large values. In the limit $T_c \ll 1$, $\omega$ can be approximated to 1 giving the root as $\delta = \sqrt{2} - 1$. Evaluating the value of $\eta$ for this limit one gets,

$$\eta_{\text{max}} = \frac{T_h}{0.779 + T_h}$$

that depends only on $T_h$ and is valid in the limit $T_h \gg 1$ and $T_c \ll 1$.

Efficiencies of engines and refrigerators at maximum power has been evaluated for a number of systems both macroscopic [28–30] and microscopic [31, 32]. The efficiency at maximum power depends on the particular model as well as the way optimization is carried out [24]. In the linear regime, with strong coupling and left-right symmetry, it has been shown that Curzon-Ahlborn result is universal up to second order in $\eta_c$, the Carnot efficiency [33]. Note that the results presented in this work does not depend on the $T_h \approx T_c$ limit but is an exact analysis of the particular model system studied. The presence of the energy scale corresponding to working substance (the level spacing of the demon) of the
microscopic device is what allows for the various approximations we have carried out.

It is to be kept in mind that the power optimization was done without taking into consideration the power maximization related to work done on resetting the bit (for refrigerator mode) or work extraction from erased bits (for the engine mode). As far as the COP of the refrigerator is concerned this will lead to a multiplication by a constant factor less than one because at maximum power the resetting will take more work than $T_b \Delta S_B$ to be performed. For the engine efficiency, power maximization in the work extraction process using the low entropy outgoing bit stream will lead to a value of work obtained that is less in magnitude than $T_b \Delta S_B$ in Eq. [9]. This would imply that the $\eta_{max}$ we found will just be an upper bound for the maximum power for the full device.

To summarize, we have studied a modified version of Maxwell refrigerator in which the bit demon system equilibrate with the thermal reservoirs alternately. The analysis of the current version is simple and allows one to study the efficiency of the device in detail. The second law of thermodynamics has been shown to hold in its generalized version which includes the Shannon entropy of the bit stream. The phase diagram of the device shows that the device can work as a refrigerator as well as a heat engine much like the original Maxwell refrigerator [1]. The power of the device has been maximized using the probability distribution of the incoming bits as the variational parameter. The efficiency studies lead to the interesting results that both the COP of the refrigerator and efficiency of the heat engine at maximum power can be independent of the temperature of the low temperature bath in the appropriate limits. These results could be of relevance to cellular level biological process, design of artificial molecular machines and thermodynamics of computation. The work could be generalized by making the interaction time of the device with the baths to be finite rather than letting it equilibrate. Another avenue to be explored will be quantum version of the device along the lines of recent work on a heat engine with two level system as the working substance [34].

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