Space-dependent diffusion with stochastic resetting: A first-passage study

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We explore the effect of stochastic resetting on the first-passage properties of space-dependent diffusion in presence of a constant bias. In our analytically tractable model system, a particle diffusing in a linear potential \( U(x) \propto \mu |x| \) with a spatially varying diffusion coefficient \( D(x) = D_0 |x| \) undergoes stochastic resetting, i.e., returns to its initial position \( x_0 \) at random intervals of time, with a constant rate \( r \). Considering an absorbing boundary placed at \( x_a < x_0 \), we first derive an exact expression of the survival probability of the diffusing particle in the Laplace space and then explore its first-passage to the origin as a limiting case of that general result. In the limit \( x_a \to 0 \), we derive an exact analytic expression for the first-passage time distribution of the underlying process. Once resetting is introduced, the system is observed to exhibit a series of dynamical transitions in terms of a sole parameter, \( \nu := (1 + \mu D_0^{-1}) \), that captures the interplay of the drift and the diffusion. Constructing a full phase diagram in terms of \( \nu \), we show that for \( \nu < 0 \), i.e., when the potential is strongly repulsive, the particle can never reach the origin. In contrast, for weakly repulsive or attractive potential (\( \nu > 0 \)), it eventually reaches the origin. Resetting accelerates such first-passage when \( \nu < 3 \), but hinders its completion for \( \nu > 3 \). A resetting transition is therefore observed at \( \nu = 3 \), and we provide a comprehensive analysis of the same. The present study paves the way for an array of theoretical and experimental works that combine stochastic resetting with inhomogeneous diffusion in a conservative force-field.

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I. INTRODUCTION

Stochastic resetting implies a situation, where an ongoing dynamical process is stopped at random time epochs to start anew. It has gained overwhelming attention in recent times because of its spontaneous ubiquity in numerous natural and man made systems. For example, stock market crashes may reset the asset prices by drastically reducing those to some prior value. Epidemics and natural disasters may have a similar effect on the population of a living species in a certain locality. Search processes may also reset, examples include foraging animals returning to their habitats due to fatigue or extreme weather. In computer science, it has long been known that resetting certain algorithms may significantly enhance their performance by reducing the effective run time. At the microscopic level, resetting is an indispensable part of the classical Michaelis–Menten reaction scheme and therefore, is crucial to the understanding of a variety of cellular processes. For all these reasons and others, resetting and its applications have created a central point of scientific interest in recent years.

Diffusion with stochastic resetting serves as a classic model to explore resetting phenomena, where the completion of a first-passage process is accelerated due to resetting. When the diffusion occurs in the presence of a bias, resetting either facilitates or hinders the resulting first-passage process. As system parameters are varied, resetting may invert its role, which leads to a resetting transition. In recent years, diffusion with resetting in various potential landscapes have thoroughly been explored. In all these studies, however, the diffusion is assumed to be independent of the position of the particle.

Space-dependent or inhomogeneous diffusion frequently arises in a number of soft-matter systems. For instance, diffusion of tRNA inside the ribosome is found to be position-dependent in a recent study. The diffusion coefficient of a Brownian particle in the vicinity of a wall or surface is greatly reduced due to hydrodynamic interactions and the mutual diffusion coefficient of two particles in a suspension depends on their separating distance. A particle in geometric confinement undergoes diffusion that depends on its position, e.g., colloidal particles in porous media, particles trapped in vesicles, or in between two nearly parallel walls. Brownian particles confined in a narrow channel with uneven boundaries or inside a helical tube experience an effective space-dependent diffusivity along its direction of transport. Diffusion of a colloidal particle in a reversible chemical polymer gel, micro-magnetic dynamics in ferromagnetic systems, and reaction-diffusion inside a narrow channel also generate space-dependent diffusivity. Other popular examples of heterogeneous diffusion include dynamics of fluid membranes and entangled polymer suspensions. In a separate context, the presence of a space-dependent (multiplicative) noise term in Brownian dynamics has been found to manifest noise induced transition and asymmetric localization of particles and many other interesting transport phenomena.

In the present work, we consider an exactly solvable model system, where a particle in a linear potential \( U(x) \propto \mu |x| \) diffusing inhomogeneously in space with a diffusion coefficient \( D(x) = D_0 |x| \) is subject to stochastic resetting. The space-dependent nature of the diffusion reduces the fluctuations of motion as the particle approaches the origin, while such fluctuations are enhanced when it moves away from the origin.

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We provide a detailed analysis of the possible effects of resetting on the first-passage properties of this system and construct a phase diagram, where dynamical transitions between different phases are observed by tuning a single parameter, $\nu \equiv (1 + \mu D_0^{-1})$, that captures the interplay of the drift and diffusion for the present system.

The rest of this paper is organized as follows. We start in Sec. II where we consider a particle undergoing space-dependent diffusion with resetting in presence of a constant bias and study its first-passage to an absorbing boundary. In particular, we derive an exact expression of the survival probability of the particle in the Laplace space and utilize the same to calculate the first-passage time. The results obtained in Sec. II hold for any arbitrary position of the absorbing boundary, provided it is placed between the origin and the initial position of the particle. In Sec. III, we explore the limiting case, where the absorbing boundary is placed at the origin. There, we first study the underlying process and derive an exact analytical expression for the first-passage time distribution. In the same Section, we investigate the effect of resetting on the system when it diffuses to the origin. The resetting transition is discussed in Sec. IV. In Sec. V, we construct a full phase diagram for the present problem and draw the final conclusions in Sec. VI.

II. SPACE-DEPENDENT DIFFUSION: FIRST-PASSAGE WITH Resetting

A. The model

Consider a particle diffusing in a linear potential $U(x) = U_0 |x|$ with a space-dependent diffusion coefficient that varies linearly with the distance from the origin as $D(x) = D_0 |x|$, where $D_0 > 0$ is the proportionality constant. Note that when $U_0 > 0$, the potential is attractive, whereas it is repulsive for $U_0 < 0$. Assume that the particle is stochastically reset to a position $x_r > 0$ [see Fig. 1(a)] with a constant rate $\zeta$, which implies that the random times between two consecutive resetting events are drawn from an exponential distribution with mean $\zeta^{-1}$. Letting $p_r(x,t|x_0)$ denote the conditional probability density of finding the particle at position $x$ at time $t$, provided that the initial position was $x_0$, the Fokker-Planck equation for the process with resetting [1] can be written as

$$\frac{\partial p_r(x,t|x_0)}{\partial t} = \mu \frac{\partial}{\partial x} p_r(x,t|x_0) + D_0 \frac{\partial^2}{\partial x^2} [xp_r(x,t|x_0)]$$

$$ - r p_r(x,t|x_0) + r \delta(x-x_r),$$

where $\delta(x-x_r)$ is a Dirac delta function. Here $\mu \equiv U_0 \xi^{-1}$ is the constant drift velocity. $\xi$ being the friction coefficient. Since $\xi$ is always positive, the drift acts towards the origin ($\mu > 0$) when the potential is attractive and away from the origin ($\mu < 0$) when the potential is repulsive [see Fig. 1(b),(c)]. Note that when $r = 0$, Eq. (1) boils down to the Fokker-Planck equation for the underlying process without resetting. Once resetting is introduced, i.e., for $r > 0$, there is a loss of probability from position $x$ and a subsequent gain of probability at position $x_r$. The last two terms on the right hand side of Eq. (1) account for this additional probability flow, which is proportional to $r$, the rate of resetting.

Consider an absorbing boundary at $x_a < x_0$ [see Fig. 1], which implies that when the particle, starting at $x_0 > 0$, hits that boundary for the first time, it is immediately removed from the system, leading to $p_r(x_a,t|x_0) = 0$. In terms of the survival probability $Q_r(t|x_0) := \int_0^\infty p_r(x,t|x_0) dx$, i.e. the probability that the particle exists in the interval $\Omega = [x_a, \infty]$, the backward Fokker Planck equation [12] for the above process is given by

$$\frac{\partial Q_r(t|x_0)}{\partial t} = -\mu \frac{\partial Q_r(t|x_0)}{\partial x_0} + D_0 \frac{\partial^2 Q_r(t|x_0)}{\partial x_0^2} - r Q_r(t|x_0) + r Q_r(t|x_r),$$

with the initial condition $Q_r(0|x_0) = 1$ and the boundary condition $Q_r(t|x_a) = 0$. Laplace transforming Eq. (2) we obtain

$$D_0 \frac{\partial^2 \tilde{Q}_r(s|x_0)}{\partial x_0^2} - s \frac{\partial \tilde{Q}_r(s|x_0)}{\partial x_0} - (s+r)\tilde{Q}_r(s|x_0) = -[1 + r \tilde{Q}_r(s|x_r)],$$

where $\tilde{Q}_r(s|x_0) := \int_0^\infty e^{-st} Q_r(t|x_0) dt$ denotes the Laplace transform of $Q_r(t|x_0)$. Letting $T_r$ denote the first-passage time (FPT) to the absorbing boundary placed at $x_a$, we recall that the probability density of $T_r$ is given by $-\partial Q_r(t|x_0)/\partial t$. This allows us to calculate any moment of $T_r$ from $\tilde{Q}_r(s|x_0)$ following the relation

$$\langle T^n_r \rangle = \int_0^\infty t^n \left[ -\frac{\partial \tilde{Q}_r(s|x_0)}{\partial t} \right] dt$$

$$\equiv n(-1)^{n-1} \left[ \frac{d^{n-1} \tilde{Q}_r(s|x_0)}{ds^{n-1}} \right]_{s=0}.$$
Since we are interested in the first-passage properties of the system, next we solve Eq. \((\mathbf{3})\) in order to find out the survival probability in the Laplace space.

### B. The survival probability

Eq. \((\mathbf{3})\) is a linear non-homogeneous differential equation. In order to convert it to a homogeneous one, we first consider a constant shift as

\[
\tilde{q}_r(s|x_0) := \frac{1}{s + r} \tilde{Q}_r(s|x_0) - \left[ \frac{1 + r \tilde{Q}_r(s|x_0)}{s + r} \right].
\]

\((5)\)

Eq. \((\mathbf{3})\) in terms of \(\tilde{q}_r(s|x_0)\) reads

\[
D_0 x_0 \frac{\partial^2 \tilde{q}_r(s|x_0)}{\partial x_0^2} - \mu \frac{\partial \tilde{q}_r(s|x_0)}{\partial x_0} - (s + r)\tilde{q}_r(s|x_0) = 0.
\]

\((6)\)

Performing a variable transformation as \(\rho(x_0) = \sqrt{x_0}\) and considering that \(\tilde{q}_r(s|\rho) = \rho^s \tilde{y}(s|\rho)\), where \(v\) is an arbitrary constant, we can rewrite Eq. \((\mathbf{6})\) in terms of \(\tilde{y}(s|\rho(x_0))\) as

\[
\frac{\partial^2 \tilde{y}(s|\rho)}{\partial \rho^2} + \left( \frac{c_1}{\rho} \right) \frac{\partial \tilde{y}(s|\rho)}{\partial \rho} + \left[ \frac{c_2}{\rho^2} - 4 \left( \frac{s + r}{D_0} \right) \right] \tilde{y}(s|\rho) = 0,
\]

\((7)\)

where \(c_1 = 2v - 1 - 2\mu D_0^{-1}\) and \(c_2 = v|v - 2(1 + \mu D_0^{-1})|\).

Assigning \(c_1 = 1\), we get \(v = (1 + \mu D_0^{-1})\), which in turn leads to \(c_2 = -v^2\). Therefore, Eq. \((\mathbf{7})\) reduces to

\[
\frac{\partial^2 \tilde{y}(s|\rho)}{\partial \rho^2} + \left( \frac{1}{\rho} \right) \frac{\partial \tilde{y}(s|\rho)}{\partial \rho} = \left[ \frac{v}{\rho} \right]^2 + 4 \left( \frac{s + r}{D_0} \right) \tilde{y}(s|\rho).
\]

\((8)\)

Eq. \((\mathbf{8})\) is a modified Bessel equation with general solution\(^{59}\)

\[
\tilde{y}(s|\rho) = \begin{cases} A_+ I_v \left( 2 \sqrt{\frac{x_0}{D_0}} \rho \right) + B_+ K_v \left( 2 \sqrt{\frac{x_0}{D_0}} \rho \right) & \text{if } v > 0, \\ A_- I_{-v} \left( 2 \sqrt{\frac{x_0}{D_0}} \rho \right) + B_- K_{-v} \left( 2 \sqrt{\frac{x_0}{D_0}} \rho \right) & \text{if } v < 0. \end{cases}
\]

\((9)\)

Here \(I_v(y) := \sum_{n=0}^\infty \frac{1}{\Gamma(n + v + 1)} \left( \frac{y}{2} \right)^{2n+v}\) is the modified Bessel function of the first kind\(^{30}\) and \(K_v(y) = \frac{\pi}{\sin(v \pi)} I_{-v}(y)\) is the modified Bessel function of the second kind\(^{30}\), defined in terms of \(I_v(\cdot)\).

Recalling that \(\rho(x_0) = \sqrt{x_0}\) and \(\tilde{q}_r(s|x_0) = x_0^{v/2} \tilde{y}(s|x_0)\), from Eq. \((\mathbf{5})\) and Eq. \((\mathbf{9})\) we obtain the general solution of Eq. \((\mathbf{3})\) as

\[
\tilde{Q}_r(s|x_0) = \begin{cases} A_+ x_0^{v/2} I_v \left( 2 \sqrt{\frac{x_0}{D_0}} \right) + B_+ x_0^{v/2} K_v \left( 2 \sqrt{\frac{x_0}{D_0}} \right) & \text{if } (1 + \frac{v}{D_0}) > 0, \\ A_- x_0^{v/2} I_{-v} \left( 2 \sqrt{\frac{x_0}{D_0}} \right) + B_- x_0^{v/2} K_{-v} \left( 2 \sqrt{\frac{x_0}{D_0}} \right) & \text{if } (1 + \frac{v}{D_0}) \leq 0. \end{cases}
\]

\((10)\)

In order to obtain the specific solution of Eq. \((\mathbf{3})\) from Eq. \((\mathbf{10})\), we need to find out the explicit expressions of \(A_\pm\) and \(B_\pm\) from the boundary conditions.

Since \(\tilde{Q}_r(s|x_0)\) should be finite even at \(x_0 \to \infty\), we set \(A_\pm = 0\). The absorbing boundary at \(x_a\) leads to \(\tilde{Q}_r(s|x_a) = 0\), which gives

\[
B_\pm = - \left[ 1 + r \tilde{Q}_r(s|x_a) \right] \frac{\frac{1}{2} \left( 1 + \frac{\nu}{D_0} \right) K_\pm \left( 2 \sqrt{\frac{x_a}{D_0}} \right)}{s + r}.
\]

\((11)\)

Note that \(K_1 + \frac{\nu}{D_0} (\cdot) = K_{-1} - \frac{\nu}{D_0} (\cdot)\), which leads to \(B_+ = B_-\).

Substituting \(A \equiv A_\pm\) and \(B \equiv B_\pm\) in Eq. \((\mathbf{10})\) we get

\[
\tilde{Q}_r(s|x_0) = \frac{1 + r \tilde{Q}_r(s|x_a)}{s + r} \left[ 1 - \frac{x_a^{v/2} K_v \left( 2 \sqrt{\frac{x_a}{D_0}} \right)}{x_0^{v/2} K_v \left( 2 \sqrt{\frac{x_0}{D_0}} \right)} \right].
\]

\((12)\)

where \(v := (1 + \mu D_0^{-1})\), as obtained earlier. Note that for attractive potential \(v > 1\), whereas for repulsive potential \(v < 1\).

### C. The first-passage time: Mean and standard deviation

Recalling Eq. \((\mathbf{1})\), we see that the mean FPT from \(x_0\) to \(x_a\) in presence of resetting can be obtained as \(\langle T_r \rangle = \left[ \tilde{Q}_r(s|x_0) \right]_{s=0} \).
In a similar spirit, the second moment of $T_r$ is obtained following Eq. (4) as $\langle T_r^2 \rangle = -2[\partial \hat{Q}_r(s|x_0)/\partial s]_{s=0}$. Utilizing that relation and setting $\alpha := 2\sqrt{r/D_0}$, we calculate the standard deviation of the FPT, $\sigma(T_r) := \sqrt{\langle T_r^2 \rangle - \langle T_r \rangle^2}$, that reads

$$\sigma(T_r) = \frac{1}{\alpha} \left[ \sqrt{\frac{\alpha \sqrt{x_a} K_{\nu-1}(\alpha \sqrt{x_a}) K_{\nu}(\alpha \sqrt{x_0}) + K_{\nu}(\alpha \sqrt{x_0}) \left[ \left( \frac{\alpha \sqrt{x_0}}{\alpha \sqrt{x_a}} \right)^{\nu} K_{\nu-1}(\alpha \sqrt{x_a}) - \alpha \sqrt{x_0} K_{\nu-1}(\alpha \sqrt{x_0}) \right]} \right] - 1, \quad (15)$$

Fig. 2 suggests that the effect of resetting should depend significantly on the placement of the absorbing boundary with respect to the origin, a special feature that can be attributed to the inhomogeneous nature of the diffusion. Recalling that the diffusion coefficient $D(x)$ varies linearly with $x$, we see that it vanishes at the origin. Hence, as the particle moves close to the origin, its dynamics gets drift-dominated. Therefore, it is not expected to ever reach the origin when the potential is repulsive. Introduction of resetting to the system might help the particle reach the origin in this case. In stark contrast, when the potential is attractive, the particle should always reach the origin; resetting can either accelerate or delay such first-passage. Note that once the particle reaches the origin, the attractive potential will not allow it to leave, hence in absence of the absorbing boundary it is expected to stay there forever. Motivated by the above possibilities, in the rest of this paper we perform a comprehensive analysis of the first-passage of the system to the origin.

III. REACHING THE ORIGIN

In this Section, we explore the first-passage of the particle to the origin as a limiting case ($x_a \to 0$) of the general results obtained in the previous Section. To start with, we consider the survival probability from Eq. (13) in the limit $x_a \to 0$. The limiting expression of the modified Bessel function $K_\nu(\cdot)$ for small arguments \cite{Lebedev} for $\nu \leq 0$ gives $\lim_{x_a \to 0} x_a^{-\nu/2}/K_\nu(\alpha \sqrt{x_a}) \simeq 0$, which leads to $\hat{Q}_r(s|x_0) \simeq s^{-1}$, i.e., $\hat{Q}_r(|x_0|) \simeq 1$. Therefore, the survival probability is always conserved to unity for $\nu := (1 + \mu D_0^{-1}) < 0$, which means that the particle can never reach the origin when the linear potential is strongly repulsive ($\mu < -D_0$), even when it is reset at $x_0$ with a rate $r > 0$. Resetting is expected to lead to a non-equilibrium steady state in this case; we will address that elsewhere.

As we are interested in the first-passage properties of the system in the present work, our discussion will henceforth be restricted to $\nu \geq 0$. In what follows, we start with the underlying process ($r \to 0$) to study its first-passage to the origin and then, introduce resetting to explore its effect on such first-passage.
A. First-passage without resetting

In the absence of resetting, i.e., for \( r \to 0 \), Eq. (13) boils down to

\[
\tilde{Q}_0(s|x_0) = \lim_{r \to 0} \tilde{Q}_r(s|x_0) = \frac{1}{s} \left[ 1 - \left( \frac{x_0}{D_0} \right)^2 \right] \tilde{K}_2 \left( 2 \sqrt{\frac{s x_0}{D_0}} \right),
\]

(16)

where \( \tilde{Q}_0(s|x_0) := \int_0^\infty e^{-s \tilde{t}} \tilde{Q}_0(t|x_0)dt \) is the survival probability of the underlying process in the Laplace space. For \( v > 0 \), the limiting expression for the modified Bessel function \( K_v(\cdot) \) for small arguments\(^{39}\) leads to \( \lim_{t \to 0} e^{-\nu/2} K_v(\alpha \sqrt{x_0}) \approx 2^{-1} \Gamma(v)/\nu \), where \( \Gamma(v) := \int_0^\infty t^{v-1} e^{-t} dt \) is the Gamma function. Plugging in this expression into Eq. (13), we obtain the survival probability in the limit \( x_0 \to 0 \), which reads

\[
\tilde{Q}_0(s|x_0) = \frac{1}{s} \left[ 1 - \frac{2}{\Gamma(v)} \left( \frac{x_0}{D_0} \right)^2 \right] \tilde{K}_2 \left( 2 \sqrt{\frac{s x_0}{D_0}} \right).
\]

(17)

Letting \( T_0 \) denote the FPT of the underlying process and recalling that the probability density of \( T_0 \) is given by\(^{32}\)

\[
f_{T_0}(t) = -dQ_0(t|x_0)/dt,
\]

we see that \( \tilde{Q}_0(s|x_0) = [1 - T_0(s)]/s \). Eq. (17) thus gives

\[
\tilde{T}_0(s) := \int_0^\infty e^{-s \tilde{t}} f_{T_0}(t)dt = \frac{2}{\Gamma(v)} \left( \frac{x_0}{D_0} \right)^2 \tilde{K}_2 \left( 2 \sqrt{\frac{s x_0}{D_0}} \right).
\]

(18)

Here \( \tilde{T}_0(s) \) is the Laplace transform of \( f_{T_0}(t) \). Next, we consider the following identity\(^{22}\)

\[
\int_0^\infty p^{-(\gamma+1)} \exp \left[ -p - \frac{\beta^2}{4p} \right] dp = 2 \left( \frac{\beta}{2} \right) \gamma K_\gamma(\beta),
\]

(19)

which holds for \( |\arg(\beta)| < \pi/2 \) and \( \Re(\beta^2) > 0 \). Comparing Eqs. (18) and (19), we identify \( \gamma = v \), \( \beta = 2\sqrt{s x_0/D_0} \) and \( p = \nu \), which allows us to write

\[
f_{T_0}(t) = t^{-(\nu+1)} / \Gamma(v) \left( \frac{x_0}{D_0} \right)^v \exp \left[ -\frac{x_0}{D_0} t \right].
\]

(20)

Eq. (20) thus presents the first-passage time distribution to the origin for a particle that undergoes space-dependent diffusion in a weakly repulsive or attractive linear potential \( (v > 0) \). Note that in the long time limit \( f_{T_0}(t) \to t^{-(\nu+1)} \). Since \( f_{T_0}(t) = -dQ_0(t|x_0)/dt \), the survival probability in the limit \( t \to \infty \) decays as \( Q_0(t|x_0) \to 0 \). Thus \( v \) governs the decay of the survival probability of the underlying process, and hence can be identified as the “persistence exponent”\(^{61}\) for the present problem. It is evident from Eq. (20) that the tail of the distribution \( f_{T_0} \) gets heavier as \( v \) decreases. Resetting the system with a suitable rate can then effectively shorten that heavy tail of the FPT distribution, thereby accelerating the resulting first-passage.

The mean FPT to the origin in the absence of resetting can be derived from Eq. (20) as

\[
\langle T_0 \rangle := \int_0^\infty t f_{T_0}(t)dt = \frac{x_0}{D_0(v-1)}.
\]

(21)

Eq. (21) clearly indicates that the mean FPT for the underlying process diverges when \( v \leq 1 \). In other words, when the potential is repulsive \( (\mu < 0) \) or when the particle freely diffuses \( (\mu = 0) \) with a diffusion coefficient \( D(x) = D_0|x| \), it takes infinite mean time to reach the origin in the limit \( r \to 0 \). It can reach the origin in finite mean time only when the potential is attractive \( (\mu > 0) \). Eq. (21) thus analytically validates our physical intuition.

The second moment of the FPT can be derived from Eq. (20) as

\[
\langle T_0^2 \rangle := \int_0^\infty t^2 f_{T_0}(t)dt = \frac{x_0^2}{D_0^2(v-1)(v-2)},
\]

which shows that \( \langle T_0^2 \rangle \) diverges for \( v < 2 \) and is finite only when \( v > 2 \). Note that the first and second moments of \( T_0 \) can be well calculated from Eq. (17) utilizing the relations \( \langle T_0 \rangle = [\tilde{Q}_0(s|x_0)]_{s=0} \) and \( \langle T_0^2 \rangle = -2[\partial^2 \tilde{Q}_0(s|x_0)/\partial s]^2 \), respectively. The standard deviation in \( T_0 \), \( \sigma(T_0) := \sqrt{\langle T_0^2 \rangle - \langle T_0 \rangle^2} \), thus reads

\[
\sigma(T_0) = \frac{x_0}{D_0(v-1)} \frac{1}{\sqrt{v-2}}.
\]

(22)

Comparing Eqs. (21) and (22), we see that for \( 1 < v \leq 2 \), the standard deviation diverges whereas the mean is finite. Eqs. (21) and (22) also indicate that \( \sigma(T_0) = \langle T_0 \rangle \) when \( v = 3 \). For \( 2 < v < 3 \), \( \sigma(T_0) \) and \( \langle T_0 \rangle \) are both finite, but \( \sigma(T_0) > \langle T_0 \rangle \), which means that the fluctuations in the first-passage time \( T_0 \) around its mean are high. In contrast, for \( v > 3 \), \( \sigma(T_0) < \langle T_0 \rangle \), i.e., the fluctuations in \( T_0 \) around \( \langle T_0 \rangle \) are less. Therefore, the persistent exponent \( \nu := (1 + \mu D_0^{-1}) \), which characterizes the nature (attractive or repulsive) and relative strength of the potential (manifested by the drift \( \mu \)) over diffusion (manifested by \( D_0 \)), marks the signature of different dynamical phases for the underlying process. Next, we investigate the effect of resetting on those phases.

B. First-passage with resetting

When the absorbing boundary is placed at the origin, the survival probability in presence of resetting can be obtained from Eq. (13) utilizing the liming expression \( \lim_{r \to 0} e^{-\nu/2} K_v(\alpha \sqrt{x_0}) \approx 2^{-1} \sqrt{\nu} / \alpha^\nu \) as before, and that gives

\[
\tilde{Q}_r(s|x_0) = \frac{1}{s + r} \left[ \frac{x_0}{D_0} \right]^2 \tilde{K}_2 \left( 2 \sqrt{\frac{x_0 + \nu r}{D_0}} \right).
\]

(23)

The associated mean FPT can either be obtained directly from Eq. (13) utilizing the above liming expression for \( x_0 \to 0 \) or
setting $s = 0$ in Eq. (23), and it reads

$$
\langle T_r \rangle = \frac{1}{r} \left[ \frac{\Gamma(v)}{2 \left[ \Gamma (\nu) + \frac{\Gamma (1/2)}{\sqrt{\nu}} \right]} - 1 \right].
$$

(24)

Recalling the definition of the Gamma function, we observe from Eq. (24) that $\langle T_r \rangle$ is finite for $v > 0$. Comparing with Eq. (21), we see that while the mean FPT to the origin diverges for $0 < v \leq 1$ (i.e., when the potential is either nonexistent or weakly repulsive) for the underlying process, it becomes finite in presence of resetting.

The standard deviation of the FPT, in a similar spirit as in case of $\langle T_r \rangle$, can be calculated directly from Eq. (15) using the limiting expression for $x_v \to 0$. Alternatively, it can be derived by first calculating the second moment from Eq. (23) and then utilizing Eq. (24). In either way it leads to

$$
\sigma(T_r) = \frac{1}{r} \sqrt{\left[ \frac{\Gamma(v)}{\Gamma(v) - 4 \left[ \frac{\Gamma (\nu)}{\Gamma (1/2)} \right]^{-1/2} K_\nu \left( \frac{2 \sqrt{\Gamma(1/2)}}{x_v} \right)} - 1 \right]}. 
$$

(25)

In Fig. 3 we plot the mean FPT and the standard deviation of the FPT as functions of $r$ from Eqs. (21) and (22), respectively, for different values of the persistent exponent $v$. It shows that for $r \to 0$, the mean $\langle T_r \rangle$ and the standard deviation $\sigma(T_r)$ of the FPT, both diverge for $0 < v < 1$ [panel (a)], while for $1 < v < 2$ the mean FPT is finite, but the standard deviation diverges [panel (b)]. For $v > 2$, both the mean and the standard deviation become finite for $r \to 0$, though $\sigma(T_{r\to0}) > \langle T_{r\to0} \rangle$ for $2 < v < 3$ [panel (c)], whereas $\sigma(T_{r\to0}) < \langle T_{r\to0} \rangle$ for $v > 3$ [panel (d)]. Therefore, all the results in the limit $r \to 0$ are in agreement with our previous derivations.

Panels (a)–(c) of Fig. 3 suggest that when $0 < v < 3$, $\langle T_r \rangle$ shows a non-monotonic variation with the resetting rate. This implies that when the potential is either weakly repulsive or weak to moderately attractive, resetting expedites the first-passage of the particle to the origin. In contrast, panel (d) of Fig. 3 shows that for $v > 3$, the mean FPT monotonically increases with $r$, thereby suggesting that resetting can only delay the first-passage to the origin when the potential is strongly attractive. This marks the signature of a resetting transition, which is expected as $v$ increases beyond a tipping point. In the following Section, we present a comprehensive analysis of the resetting transition for the present system.

IV. THE RESETTING TRANSITION

We observe from Fig. 3 that as $v$ increases, the variation of the mean FPT with the resetting rate changes from non-monotonic to monotonic. In other words, the optimal resetting rate, i.e., the rate of resetting that minimizes the mean FPT, is non-zero for smaller values of $v$ and it reduces to zero as $v$ grows. This indicates that the optimal resetting rate, denoted $r^\ast$, should serve as a suitable observable to study the resetting transition. Motivated by this idea, we proceed to explore the resetting transition in terms of the optimal resetting rate.

A. The optimal resetting rate

In order to study the optimal resetting rate, we define a new variable $z := \alpha \sqrt{x_0} = 2\sqrt{r\alpha_0}/D_0$, which leads to

$$
r = \frac{z^2 D_0}{4 x_0}.
$$

(26)

Eq. (24) in terms of $z$ reads

$$
\langle T_r \rangle = \left( \frac{4 x_0}{D_0 z^2} \right) \left[ \frac{2^{\nu-1} \Gamma(v)}{z^\nu K_\nu(z)} - 1 \right].
$$

(27)

Since $r^\ast$ minimizes the mean FPT, we have $\left. d \langle T_r \rangle / dr \right|_{r=r^\ast} = 0$. Eq. (26) suggests that $d \langle T_r \rangle / dz = (2 x_0 D_0^{-1} / z) d \langle T_r \rangle / dz$. 

FIG. 3. The mean FPT $\langle T_r \rangle$ [from Eq. (21)] and the standard deviation of the FPT $\sigma(T_r)$ [from Eq. (22)] vs. the resetting rate $r$, for four different phases of space-dependent diffusion in a linear potential. Solid lines indicate $\langle T_r \rangle$ and dashed lines show $\sigma(T_r)$. Curves of similar color denote the same value of $v$, while the colored circles mark the minimum value of $\langle T_r \rangle$ for each choice of $v$. Panel (a): For $0 < v < 1$, the mean $\langle T_r \rangle$ and the standard deviation $\sigma(T_r)$ of the FPT, both diverge in the limit $r \to 0$ (no resetting). Panel (b): For $1 < v < 2$, while $\langle T_r \rangle$ is finite in the limit $r \to 0$, $\sigma(T_r)$ diverges. Panel (c): For $2 < v < 3$, the mean and the standard deviation of FPT, both are finite for $r \to 0$, and $\sigma(T_{r\to0}) > \langle T_{r\to0} \rangle$. Panel (d): For $v > 3$, both the mean and standard deviation are finite in the limit $r \to 0$, and $\sigma(T_{r\to0}) < \langle T_{r\to0} \rangle$. Panels (a)–(c) show that for $0 < v < 3$, $\langle T_r \rangle$ exhibits non-monotonic variation with the resetting rate, whereas panel (d) shows that for $v > 3$, the mean FPT monotonically increases with $r$. In all panels we have taken $x_0 = 2.0$ and $D_0 = 1.0$. 

Differentiating Eq. (27) with respect to $z$ we obtain the following transcendental equation

$$F(z; \nu) := 4z^\nu K_\nu(z) + z^{2\nu} \Gamma(\nu) \left[ \frac{zK_{\nu-1}(z)}{K_\nu(z)} - 2 \right] = 0. \quad (28)$$

In Fig. 4(a) we graphically solve Eq. (28). The solutions, $z^\ast$, when plugged into Eq. (26), give the optimal resetting rate $r^\ast$. In Fig. 4(b) we plot $r^\ast$ vs. $\nu$ for different values of $x_0D_0^{-1}$ to find that $r^\ast$ is non-zero only when $\nu < 3$. This implies that resetting expedites the first-passage to the origin for weakly repulsive/attractive potential. In contrast, $r^\ast$ becomes zero for $\nu \geq 3$, which means that resetting can no longer assist the first-passage to the origin when the potential becomes strongly attractive. The optimal resetting rate thus marks the point of resetting transition at $\nu = 3$ (i.e., $\mu = 2D_0$).

Recalling Eqs. (21) and (22), we see that resetting accelerates the first-passage of the particle to the origin when the fluctuations in the FPT of the underlying process around its mean are high. This agrees with the general theory of first-passage with resetting\[13\]. Note that the condition of resetting transition is unaffected by the distance of the initial position $x_0$ from the origin. This is strikingly different from homogeneous diffusion in a linear potential, where the condition reads $\mu = 2D/x_0$, $D$ being the constant diffusion coefficient.

Fig. 4(b) also exhibits an interesting non-monotonic behavior of $r^\ast$ for $\nu < 1$, where the linear potential is repulsive in nature. Starting from $\nu \rightarrow 0^+$, $r^\ast$ increases to attain a maximum, and then gradually decreases until it vanishes at $\nu = 3$. This initial rise is apparently counter-intuitive, but it can be physically understood as follows. In the present context, the particle undergoes space-dependent diffusion; hence the fluctuations in its movement get much more prominent as its distance from the origin increases. It moves less erratically as it approaches $x = 0$, which makes that regime drift-dominated. As discussed earlier, resetting can expedite the first-passage to the origin by cutting short the trajectories that tend to move away from $x = 0$. In addition, it might rescue the particle, somewhat trapped in the interval $0 < x < x_0$ because of the low effective diffusion, especially when the drift velocity is weak. The latter role of resetting is somewhat analogous to the action of a repulsive potential ($\mu < 0$), as it drives the particle away from the origin. This explains the initial increase of $r^\ast$ as the repulsion weakens, i.e., $\nu$ grows, until a tipping point. After that, the role of resetting in minimizing the lifetime of the trajectories where the particle diffuses away from the origin ($x \gg x_0$) becomes predominant and the optimal resetting rate gradually decreases with $\nu$. The non-monotonic variation of $r^\ast$ for smaller values of $\nu$ can thus be attributed to the space-dependent nature of the diffusion.

B. The maximal speedup

The optimal resetting rate minimizes the mean FPT and thereby leads to the maximal speedup for the resulting first-passage process. This inspires us to quantify the maximal speedup as the ratio between the mean FPT for the underlying process and the process under optimal resetting. Utilizing Eq. (21) and Eq. (27), we can write

$$\frac{\langle T_0 \rangle}{\langle T_{r^\ast} \rangle} = \left\{ \begin{array}{ll} \frac{z^{2\nu}\nu K_\nu(z^\ast)}{4[2^{\nu-1}\Gamma(\nu)-z^\nu \nu K_\nu(z^\ast)]} & \text{for } \nu < 3, \\ 1 & \text{for } \nu \geq 3, \end{array} \right. \quad (29)$$

where we considered the fact that for $\nu \leq 3$, $z^\ast = 0$ and plugged in that into Eq. (27).

In Fig. 4(c), we plot the maximal speedup from Eq. (29) vs. $\nu$, which shows that the introduction of resetting renders the infinite mean FPT of the underlying process finite for $\nu \leq 1$, leading to infinite speedup. For $1 < \nu < 3$, the maximal speedup is finite but greater than unity, which suggests that resetting still expedites the first-passage in this case. In contrast, for $\nu \leq 3$ it becomes unity that implies that the underlying process ends faster compared to that with resetting in this regime. Next, to complete the analysis of the resetting transition, we investigate the stochastic fluctuations in the first-passage time at the optimal resetting rate.

---

**FIG. 4.** Panel (a): Graphical solution of Eq. (28) for different values of $\nu$. The solutions, $z^\ast$, are marked by colored circles. Panel (b): The optimal resetting rate $r^\ast$ vs. $\nu$ for different values of $x_0D_0^{-1}$, calculated by plugging in $z^\ast$ into Eq. (26). For $\nu < 1$, the potential is repulsive, whereas it is attractive for $\nu > 1$. The non-zero $r^\ast$ values for $\nu < 3$ (white regime) indicate that here resetting expedites the first-passage to the origin. In contrast, $r^\ast = 0$ values for $\nu \geq 3$ implies that resetting can no longer accelerate the first-passage process (gray regime). This leads to the resetting transition at $\nu = 3$. Panel (c): Main: The maximal speedup $\langle T_0 \rangle/\langle T_{r^\ast} \rangle$ vs. $\nu$ from Eq. (29). For $\nu \leq 1$, the maximal speedup is infinite, and it decays to unity at $\nu = 3$, the point of resetting transition. Inset: The relative fluctuations in FPT for optimal resetting, $\sigma(T_{r^\ast})/\langle T_{r^\ast} \rangle$, vs. $\nu$ from Eq. (21), indicating that the resetting transition occurs at $\nu = 3$.\[13\]
C. The fluctuations in FPT for optimal resetting

Recalling that $z := \sqrt{r_0/D_0}$, and plugging in the same into Eq. (25), we obtain an expression of $\sigma(T_r)$ as a function of $z$ as

$$
\sigma(T_r) = \left( \frac{4x_0}{D_0z^2} \right)^{1/2} \frac{\Gamma(\nu) \left[ \Gamma(\nu) - 4 \left( \frac{z}{2} \right)^{1+\nu} K_{\nu-1}(z) \right]}{4 \left( \frac{z}{2} \right)^{2\nu} [K_{\nu}(z)]^2} - 1.
$$

Combining Eq. (27) with Eq. (30), we can readily express the relative stochastic fluctuations, defined as $\sigma(T_r)/\langle T_r \rangle$, in terms of $z$. Setting $z = z^*$ [solutions of Eq. (28)] we obtain the identity $(z^*/2)^{1+\nu} \Gamma(\nu) K_{\nu-1}(z^*) = (z^*/2)^{\nu} \Gamma(\nu) K_{\nu}(z^*) - 2 (z^*/2)^{2\nu} [K_{\nu}(z^*)]^2$, and incorporating that in the expression of $\sigma(T_r)/\langle T_r \rangle$, we finally get

$$
\frac{\sigma(T_r)}{\langle T_r \rangle} = \begin{cases} 
1 & \text{for } \nu < 3 \\
\frac{1}{\sqrt{2}} & \text{for } \nu \geq 3.
\end{cases}
$$

Here we utilized the results for the underlying process from Eqs. (21) and (22) for $\nu \geq 3$. In the inset of Fig. 5, we plot the relative fluctuations to observe the signature of the resetting transition in terms of $\nu$.

Eq. (31) proves analytically that for $\nu < 3$, the stochastic fluctuations around the mean FPT is always unity for optimal resetting. This agrees with the results established by the general theory of first-passage with resetting. Looking back at panels (a)–(c) of Fig. 3, we see that the curves representing the mean FPT and the standard deviation of the FPT always intersect at the optimal resetting rate, which supports Eq. (31).

V. THE PHASE DIAGRAM

In the previous Sections, we have thoroughly explored the possible ways in which stochastic resetting can affect the first-passage of a particle to the origin, when the particle undergoes space-dependent diffusion in presence of a constant bias. In doing so, we observed that such effect of resetting is guided solely by the parameter $\nu := (1 + \mu D_0^{-1})$, the persistent exponent of the underlying process. This allows us to construct a full phase diagram for the present problem, as displayed in Fig. 5.

From Fig. 5, we observe that the entire range of $\nu \in (-\infty, \infty)$ can be divided into five distinct phases and dynamical transitions occur between these phases when $\nu$ is tuned. For $-\infty < \nu \leq 0$ the potential is strongly repulsive, and the particle never reaches the origin in that case, which leads to phase I in Fig. 5.

For weakly repulsive potential ($0 < \nu < 1$) and in absence of any bias ($\nu = 1$), resetting renders the infinite mean FPT for the underlying process finite and this marks phase II. When the potential is weakly attractive ($1 < \nu \leq 2$), the mean FPT of the underlying process is finite, but introduction of resetting decreases it further, which constructs phase III. Note that both in phase II and III, the infinite standard deviation of the FPT for the underlying process becomes finite due to resetting.

For weak to moderately attractive potential ($2 < \nu \leq 3$), for $r \to 0$ both the mean FPT and the standard deviation of the FPT are finite, but $\sigma(T_r) > \sigma(T_0)$. Resetting still expedites the first-passage to the origin in this regime, which is displayed as phase IV in Fig. 5. Summarizing the above results, we see that resetting accelerates the first-passage of the particle to the origin for phases II–IV [marked in different shades of blue in Fig. 5], which is confirmed by the non-monotonic variation of $\langle T_r \rangle$ with the resetting rate and the non-zero optimal resetting rates marked by the point of intersection of the curves representing $\langle T_r \rangle$ and $\sigma(T_r)$ in each phase.
In contrast, for $3 < v < \infty$, i.e., when the potential is strongly attractive, in absence of resetting $\sigma(T_0) < \langle T_r \rangle$. Introduction of resetting delays the first-passage to origin in this regime, which marks phase V of Fig. 5. This phase can be identified from the monotonic increase in $\langle T_r \rangle$ with $r$ and the fact that the curves representing $\langle T_r \rangle$ and $\sigma(T_r)$ do not intersect here (the optimal resetting rate is zero, as marked in Fig. 5). The resetting transition thus occurs at $v = 3$, where the mean FPT of the underlying process is exactly equal to the standard deviation of the FPT.

VI. CONCLUSIONS

In this article, we presented a comprehensive analysis of the effect of stochastic resetting on the first-passage properties of heterogeneous diffusion in presence of a constant bias. In our model, a particle that diffuses in a potential $U(x) \propto \mu|x|$ with a space-dependent diffusion coefficient $D(x) = D_0|x|$, is subject to stochastic resetting with a constant rate $r$. Assuming an absorbing boundary placed at a position $x_0 < x_0$, where $x_0 > 0$ is the initial position of the particle, we derived an exact expression of the survival probability in the Laplace space and is the initial position of the particle, we derived an exact expression of the survival probability in the Laplace space and is the initial position of the particle, we derived an exact expression of the survival probability in the Laplace space.

In that limit, i.e., when $x_0 \to 0$, we first presented an in depth analysis of the underlying process ($r \to 0$) that includes the derivation of an exact analytic expression of the first-passage time distribution. When subjected to resetting, the system is observed to undergo a series of dynamical transitions as a single parameter $v := (1 + \mu D_0^{-1})$, is tuned. For $v < 0$, when the potential is strongly repulsive, the particle never reaches the origin. Resetting is expected to generate a non-equilibrium steady state in that case, which we plan to study elsewhere. For $v > 0$ the potential is either weakly repulsive or attractive, and the particle can eventually reach the origin. Resetting accelerates the completion of the associated first-passage process for $v < 3$, but delays it when $v \geq 3$. We provided a detailed account of the resetting transition observed at $v = 3$.

Space-dependent diffusion naturally gives rise to a noise that is multiplicative in nature, and is crucial in modeling a large variety of diffusion processes. We are hopeful that the present study will inspire a series of theoretical and experimental works that bring together heterogeneous diffusion and stochastic resetting.

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DATA AVAILABILITY

The data that supports the findings of this work are available within the article.

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