COSMOLOGICAL MODELS BASED ON A STATISTICAL SYSTEM OF SCALAR CHARGED DEGENERATE FERMIONS AND AN ASYMMETRIC HIGGS SCALAR DOUBLET

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On the basis of the general relativistic statistical and kinetic theory, a consistent closed cosmological model is formulated. It is based on a statistical system of scalar-charged fermions interacting by means of classical and phantom scalar fields. Based on the study of the microscopic dynamics of scalar-charged particles, within the framework of the Lagrangian an Hamiltonian formalisms, a function of the dynamical mass of scalar-charged particles is constructed and it is shown that the nonnegativity condition for this function has to be removed for the consistency of the theory. On the basis of the Lagrangian formalism, equations of gravitational and scalar fields with singular sources are formulated and microscopic conservation laws are obtained. Within the framework of the general relativistic kinetic theory, macroscopic equations of gravitational and scalar fields are formulated and macroscopic conservation laws are obtained. The full correspondence of these equations to microscopic equations with singular sources is demonstrated. On the basis of the obtained equations, a cosmological model for a degenerate system of scalar-charged fermions is formulated. An exact solution of the constitutive equations for a degenerate scalar-charged plasma in the cosmological model is obtained, which allows significantly simplifying the original system of equations. On the basis of the obtained solution of the constitutive equations, two fundamentally different cosmological models are formulated, one of which has two types of single-scalar-charged fermions and the other has one kind of fermions charged with two charges of various nature. A qualitative analysis of the obtained 6-dimensional dynamical system for a two-component model is carried out. It is shown that in such models, acceleration deceleration modes become possible at the late stages of the evolution of the Universe.

Keywords: scalar-charged plasma, cosmological model, scalar field, asymmetric scalar doublet, qualitative analysis, macroscopic and microscopic equations

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1. Introduction

From a formal standpoint, phantom fields appear to have been introduced into gravity as one of the possible scalar field models in 1983 in [1]. In that study, as well as in later ones (see, e.g., [2], [3]), phantom fields were classified as scalar fields with the attraction of like-charged particles and were identified by the

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factor $\epsilon = -1$ in the energy–momentum tensor of the scalar field. We note that phantom fields in relation to wormholes and black universes were considered in [4], [5]. However, the introduction of phantom fields into the structure of quantum field theory encounters serious problems associated either with the probabilistic interpretation of quantum theory or with the problem of the stability of the vacuum state, due to the unboundedness of negative energy [6]. The negative kinetic term and the violation of the isotropic energy condition imply that energy is not bounded from below at the classical level, and hence negative norms appear at the quantum level. In turn, negative norms of quantum states generate negative probabilities, which contradicts the standard interpretation of quantum field theory [7], [8]. The requirement that the theory be unitary leads to instability when describing the interaction of quantized phantom fields with other quantized fields [9].

In [10], however, it was noted that the terms leading to instability can be regarded as corrections that are significant only at low energies below the physical cutoff. This approach allows considering the phantom field theories as some effective, physically acceptable theories, assuming that an effective theory allows embedding into some fundamental theory, for example, string field theory, which is consistent with the well-known idea of Carroll, Hoffman, and Trodden on effective field theory in which the phantom model can be viewed as a sector of a more fundamental theory [11] (also see [12]). In particular, it was shown in [11], [12] that there is a low-energy boundary such that the phantom field is stable during the lifetime of the Universe.

On the other hand, an analysis of the observations of the cosmological acceleration, as well as the associated barotropic coefficient $w = p/\epsilon$, carried out by various groups of researchers in recent years, shows that, apparently, it would be very difficult to do without phantom fields in cosmology. For example, SN-Ia data show a significant preference for “phantom” models and exclude the cosmological constant [13]. Strong restrictions can be obtained in combination with other observational data, including measurements of the Hubble parameter $H(z)$ at different redshifts. When combining standard rulers and standard clocks, the best match is observed when $w_0 = -1.01(+0.56 - 0.31)$ [14]. For the flat wCDM model, the constant parameter of the dark energy equation of state $w = -1.013(+0.068 - 0.073)$ [15] was measured (see also [16], [17]).

In a number of works by Aref'eva, Vernov, Koshelev, et al. [18]–[20], two-field cosmological models based on a pair of scalar fields, classical and phantom, were investigated, in which a negative kinetic term corresponds to the phantom field. In this case, under the assumption of a 6th-degree polynomial potential of the scalar fields, the classes of one-parameter and two-parameter exact solutions were found. As noted in [19], phantom fields [21] are involved in cosmology to provide the value of the barotropic coefficient $w < -1 (w: p = w\rho)$ in order to prevent “the big rip.” We note that in these papers, two-field cosmological models are substantiated by string theory, in which the tachyon describing brane decay is regarded as a phantom field.

In the works of Ignatiev, Agafonov, and Koch, a comprehensive study of incomplete cosmological models was carried out under the assumption that the Hubble constant is nonnegative for the cases of the classical Higgs vacuum field [22], the Higgs phantom field [23], [24], and the asymmetric scalar RR doublet (quintom) [25]–[29]. In such models, transitions of the cosmological evolution from the stage of expansion to the stage of contraction become possible (and conversely for a phantom field). If we discard a number of incorrect results, related just to the assumption of the nonnegativity of the Hubble constant, then one of the results of these studies can be summarized as follows: at the late stages of evolution, the cosmological model based on vacuum scalar fields always reaches inflation. The same result was confirmed by studies of the full model [30], in which the assumption of the nonnegativity of the Hubble constant was removed. Finally, in [31], a complete mathematical model of the cosmological evolution of an asymmetric vacuum scalar Higgs doublet (quintom) was investigated and it was shown that transitions from the cosmological contraction regime to the expansion regime and vice versa can occur in that model, and oscillatory modes with a change...
in the phases of contraction and expansion are also possible. We note that in [32], the prospects of quintom cosmology were linked precisely to the possibility of obtaining a cosmological scenario that would allow avoiding the Big Bang singularity. In addition, we note that contrary to a popular belief, the phantom field plays the role of a stabilizer for the stable accelerated expansion [31], thus being a necessary additional component of the cosmological model. Although the oscillatory regime discovered in [31] has not been sufficiently investigated, in all other cases vacuum quintom models exhibit the asymptotic behavior

\[ H(\pm \infty) \to \pm |H_0|, \quad \Omega(\pm \infty) \to 1, \]

where \( H(t) \) is the expansion rate (Hubble parameter), \( \Omega(t) \) is the invariant cosmological acceleration,

\[ H = \frac{\dot{a}}{a}, \quad \Omega = \frac{\ddot{a}a}{a^2}, \]

and \( H_0 \) is a constant that depends on the parameters of the model. We note a similar independent study [34], in which a detailed qualitative analysis of the quintom cosmological model with the exponential potential energy of the classical and phantom scalar fields was carried out.

On the other hand, in a number of earlier works based on the theory of statistical systems of scalar-charged particles, [35]–[38], in which the cosmological evolution of such systems was investigated, the possibility of four types of behavior of the corresponding cosmological models was shown, among which there were models with an intermediate ultrarelativistic stage and a final nonrelativistic stage [39], [40]. However, these studies were based, first, on an incomplete mathematical model, second, on the quadratic potential of scalar fields, and, third, on a scalar singlet. In [41], a cosmological model of the cosmological evolution of the statistical system of degenerate scalar-charged fermions interacting by means of a single scalar Higgs field, classical or phantom, is formulated. This paper also provides examples of numerical models of such systems, which radically differ in their behavior from models based on vacuum scalar fields. Finally, in the studies of one of the authors [42], [43], a mathematical model of gravitational perturbations of the cosmological system of scalar-charged fermions in the case of a scalar Higgs singlet (classical or phantom) was constructed and the stability of this cosmological model with respect to short-wave and scalar perturbations of the gravitational field was investigated. In these works, it was shown that the fermionic system is unstable at the early stages of cosmological expansion in the case of the classical Higgs interaction and is stable in the case of the phantom Higgs interaction.

These and other unique features of statistical systems of scalar-charged fermions indicate the need for a more detailed theoretical analysis of such systems: first, a rigorous microscopic and macroscopic substantiation of the theoretical model of the interaction of particles with scalar fields and, second, revealing the global properties of cosmological models based on statistical systems of scalar-charged particles. We note that the construction of a mathematical model for systems of scalar-charged particles encounters a number of serious theoretical problems associated, for example, with the determination of the total mass of particles in a scalar field, which requires careful analysis. In this regard, the problem arises of formulating a complete mathematical model of cosmological systems of scalar-charged particles with Higgs scalar fields, including an asymmetric scalar doublet.

Since the late 1990s, many researchers were studying cosmological models based on scalar and nonlinear fermionic (spinor) fields (see, e.g., [44]–[55] and the references therein). Interest in nonlinear spinor fields in cosmology is motivated by the fact that they can be used to solve a number of problems in cosmology: 1) accelerate the process of isotropization of an initially anisotropic Universe; 2) ensure the absence of the cosmological singularity, 3) generate a late-time cosmological acceleration. In addition, the nonlinear spinor field is a fairly good model of both an ideal fluid and various types of dark energy. In particular, the cosmological model based on the spinor quintom was investigated in [52].

\[ ^1 \text{Of course, this conclusion does not apply to cosmological models that include other material components in addition to scalar fields; see, e.g., [33].} \]
In this paper, we formulate a mathematical model of the cosmological statistical system of classical scalar-charged particles with Higgs scalar fields, investigate its main properties, and carry out its qualitative analysis for simple models of the scalar charge of particles in the Higgs doublet field. We consider a statistical system of scalar-charged degenerate fermions as a plasma model. The interest in this model is due to two circumstances. First, in the case of complete degeneracy, the expressions for macroscopic plasma scalars can be calculated in elementary functions, which allows a detailed analysis of such a system. The second, and more important circumstance is as follows. According to the well-known scenario, a degenerate Fermi system with interparticle interaction can transform into a Bose condensate with the properties similar to those of superfluidity and (scalar) superconductivity by the formation of Cooper pairs of like scalar-charged fermions. This condensate, as a system with minimum energy, can in principle be added to the scalar vacuum in cosmological models and can realize the observed component of dark bosonic matter. This is indicated by the fact that under conditions of sufficiently strong phantom scalar fields, the degree of degeneracy of scalar-charged fermions grows in the process of cosmological expansion [40].

2. The motion of a scalar-charged particle in scalar and gravitational fields

2.1. Lagrangian formalism for the description of the motion of a scalar-charged particle.
We refer to studies [37] and [39], which contain the correct generalization of the relativistic theory to both the case of phantom scalar fields and the sector of negative dynamical masses of scalar-charged particles. We consider the motion of a scalar-charged particle with scalar charges $q^r$ in scalar fields described by the potentials $\Phi^r(x^j), r = 1, N$. The only way to generalize the action of a massive free particle in a gravitational field to the case of a scalar-charged particle is\(^2\) [39]

$$S = -\int F(\Phi^1, \ldots, \Phi^N) \, ds,$$

where $F(\Phi^1, \ldots, \Phi^N)$ is a certain given function of scalar potentials. By varying action (3) along the trajectory of the particle, we obtain the equations of motion [39]

$$\frac{\delta u^i}{\delta s} = \partial_k \ln |F| \pi^{ik}(u),$$

where $u^i(s) = dx^i/ds$ is the velocity vector of the particle, $\delta/\delta s$ are absolute derivatives along the trajectory $x^i = x^i(s)$ (see, e.g., [56]),

$$\frac{\delta u^i}{\delta s} = \frac{du^i}{ds} + \Gamma^i_{kl} u^k \frac{du^l}{ds} = u^k \nabla_k u^i,$$

$\nabla_k$ is an operator of covariant differentiation with respect to the metric $g_{ik}\(^3\)$. Next, $\pi^{ik}(u)$ in (4) is a symmetric operator of orthogonal projection on the unit vector $u^i$,

$$\pi^{ik}(u) = u^i u^k - g^{ik}$$

such that

$$\pi_{ik}(u) u^k = 0, \quad \pi^{ik}(u) g_{ik} = -n + 1 = -3.$$

The first integral of the equations of motion for any functions $F(\Phi^1, \ldots, \Phi^N)$ is the normalization relation $(u, u) = \text{const}$. From the comparison with the action of a free massive particle of mass $m_0$ in the gravitational field (see, e.g., [57]), it follows that

$$F(0, \ldots, 0) = m_0,$$

where $m_0$ is a bare mass of the particle in absence of scalar fields.

\(^2\)Here and hereafter, the Planck system of units is chosen, $G = c = \hbar = 1$.

\(^3\)In what follows, wherever convenient, we use a comma to denote the covariant differentiation operator: $\nabla_i T = T_{,i}, \nabla^i T = T^{,i}$. 

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The fundamental principle of additivity of the Lagrange function taking limit relation (7) into account leads to the only possible form of the action, Eq. (3), and at the same time, the function $F(\Phi^1, \ldots)$:

$$F \equiv m_\star = m_0 + \sum_{r=1}^{N} q^r \Phi_r,$$

where $m_\star$ is a dynamical rest mass of the particle. We note that in the extended theory $[37]$, dynamical mass (8) can also be negative. However, it is this form of the dynamical (inert) mass of a particle that is a direct consequence of one of the fundamental principles, the principle of additivity of the Lagrange function. An attempt to place the right-hand side of equality (8) inside the modulus sign leads to a violation of the additivity principle and, at the same time, to a number of additional problems. We also note that the equations of motion of a scalar-charged particle, Eq. (4), are invariant under the substitution $m_\star \to -m_\star$.

We note that with this notation, the equations of motion of the particle (4) take the form

$$\frac{\delta u^i}{\delta s} = \partial_k \ln |m_\star|^\pi^{ik}(u).$$

The kinematic momentum of the particle $p^i$ lies on the dynamical mass shell $[37]$: 

$$(p, p) = m^2 \quad \Rightarrow \quad wp^i = \sqrt{m^2_\star + p^2},$$

where $\bar{p}^{(i)}$ are reference projections of the momentum vector, and $p^2$ is the physical momentum squared. We note that the dynamical mass of particles is not a heavy mass, which, in contrast to the dynamical mass, is determined by the total energy, i.e., $m = \sqrt{m^2_\star + p^2}$, whence $m(p = 0) = |m_\star|$ (see [37]). The symmetry requirement between particles and antiparticles ($q \to -q$) leads to the condition that the bare mass be equal to zero in formula (8) $[37]$. Accordingly, for the dynamical mass of a particle of the “a” type (to avoid confusion, we highlight the particle type index $a$ with parentheses) we assume in what follows that

$$m_\star \equiv m_{(a)} = \sum_{r} q^r_{(a)} \Phi_r.$$  

Thus, we do not exclude the possibility of a negative dynamical mass of fermions, in particular, the dynamical masses of particles and antiparticles differ in sign in this case while the total rest masses coincide. Hence, the total rest mass of a particle $m$ is equal to

$$m = |m_{(a)}|.$$  

2.2. Invariant canonical formalism for describing the motion of a scalar-charged particle.

The preceding consideration reveals the problem of the correct determination of the mass of a scalar-charged particle, which requires a deeper study and which is necessary for building a correct mathematical model of scalar-charged particles. In this regard, we turn to the invariant canonical (Hamiltonian) formalism of the equations of motion, which allows revealing some details that are not obvious in the Lagrangian formalism and which is required for a rigorous statistical description of a system of particles. General relativistic canonical equations of motion of scalar-charged particles were formulated in the studies of one of the authors $[58], [59]$ (see also $[37]$). Here, we briefly list the main points of this formalism. A normalized invariant volume element of the 8-dimensional phase space of a relativistic particle $\Gamma$, which is a vector
bundle $\Gamma = P(X) \times X$ with a Riemannian base $X(g)$ and a vector fiber $P(X)$ with respect to a pair of canonically conjugate dynamical variables $x^i$ (configuration coordinates) and $P_i$ (generalized momentum coordinates), is \cite{60}

$$d\Gamma = \frac{\varrho}{(2\pi)^3} dX dP = \frac{\varrho}{(2\pi)^3} dx^1 dx^2 dx^3 dx^4 dP_1 dP_2 dP_3 dP_4,$$

(13)

where

$$dX = \sqrt{-g} dx^1 dx^2 dx^3 dx^4, \quad dP = \frac{1}{\sqrt{-g}} dP_1 dP_2 dP_3 dP_4,$$

(14)

are the invariant volume elements of the configuration and momentum spaces, and $\varrho$ is the degeneration factor (for particles with spin $S$, $\varrho = 2S + 1$). Further, to shorten the notation, we also use the phase coordinates of the same name $\eta_a$, $a = 1, 8$:

$$\eta_i \equiv x^i, \quad \eta_{i+4} = P_i, \quad i = 1, 4,$$

(15)

in which the expression for the volume element of the phase space in (13) takes the simplest form

$$d\Gamma = \frac{\varrho}{(2\pi)^3} \prod_{a=1}^8 d\eta_a.$$  

(16)

The canonical equations of motion of a relativistic particle in the phase space $\Gamma$ have the form (see e.g., \cite{59} 

$$\frac{dx^i}{ds} = \frac{\partial H}{\partial P_i}, \quad \frac{dP_i}{ds} = -\frac{\partial H}{\partial x^i},$$

(17)

where $H(x, P)$ is a relativistically invariant Hamilton function and $u^i = dx^i/ds$ is a particle velocity vector.

Due to the antisymmetry of the canonical equations of motion (17) and the symmetry of phase volume (13) with respect to the canonical variables $\{x^i, P_i\}$, a differential relation \cite{60} that is known in classical dynamics as the Liouville theorem holds:

$$\frac{d\Gamma}{ds} = 0,$$

(18)

According to this relation, the phase volume of the world tube of particles is constant.

Calculating the total derivative of a function of dynamical variables $\Psi(x^i, P_k)$ and taking (17) into account, we find

$$\frac{d\Psi}{ds} = [H, \Psi],$$

(19)

where the invariant Poisson brackets are introduced as

$$[H, \Psi] = \frac{\partial H}{\partial P_i} \frac{\partial \Psi}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial \Psi}{\partial P_i}, \quad [H, \Psi] = -[\Psi, H].$$

(20)

We note that Poisson bracket (20) can be rewritten in a manifestly covariant form using an operator of covariant differentiation according to Cartan$^4$, $\nabla_i$ (covariant derivative in the bundle $\Gamma$ \cite{62}; see, e.g., \cite{60}),

$$\nabla_i = \nabla_i + \Gamma^k_{ij} P_k \frac{\partial}{\partial P_j},$$

(21)

$^4$Covariant derivatives according to Cartan were first introduced into relativistic statistics by Vlasov \cite{61}.
where $\nabla_i$ is an operator of covariant Ricci derivative and $\Gamma^k_{ij}$ are Christoffel symbols of the second kind with respect to the metric $g_{ij}$ of the base $X$. The operator $\tilde{\nabla}$ is defined such that

$$\tilde{\nabla}_i P_k \equiv 0 \quad (22)$$

and the following *symbolic* rule of differentiation of functions holds [60]:

$$\tilde{\nabla}_i \Psi(x, P) = \nabla_i [\Psi(x, P)], \quad (23)$$

This rule means that in order to calculate the Cartan derivative of a function $\Psi(x, P)$, it suffices to calculate its usual covariant derivative as if the momentum vector were covariantly constant. Due to this equality, the introduced operator is quite convenient for performing differential and integral operations in the phase space $\Gamma$. Thus, we write Poisson bracket (20) in a manifestly covariant form

$$[H, \Psi] \equiv \frac{\partial H}{\partial P_i} \tilde{\nabla}_i \Psi - \frac{\partial \Psi}{\partial P_i} \tilde{\nabla}_i H. \quad (24)$$

Further, due to (20), the Hamilton function is an integral of motion of the particle:

$$\frac{dH}{ds} = [H, H] = 0 \quad \Rightarrow \quad H = \text{const.} \quad (25)$$

Relation (25) can be called the normalization relation. Due to the linearity of the Poisson bracket, any continuously differentiable function $f(H)$ is also a Hamiltonian function. The only way to introduce an invariant Hamilton function that would be quadratic in the generalized momentum of a particle in the presence of only gravitational and scalar fields is

$$H(x, P) = \frac{1}{2} [\psi(x)(P, P) - \varphi(x)], \quad (26)$$

where $(a, b)$ is the scalar product of the vectors $a$ and $b$ with respect to the base metric,

$$(a, b) = g_{ik} a^i b^k,$$

and $\psi(x)$ and $\varphi(x)$ are certain scalar functions of scalar potentials. We choose the zero normalization of the Hamilton function in relation (25),

$$H(x, P) = \frac{1}{2} [\psi(x)(P, P) - \varphi(x)] = 0, \quad (27)$$

whence we find

$$(P, P) = \frac{\varphi}{\psi}. \quad (28)$$

From the first group of canonical equations of motion (17), we obtain a relation between the generalized momentum and the particle velocity vector

$$u^i \equiv \frac{dx^i}{ds} = \psi P^i \quad \Rightarrow \quad P^i = \psi^{-1} u^i. \quad (29)$$

Substituting (29) in normalization relation (28), we find

$$(u, u) = \psi \varphi.$$
Therefore, to satisfy the normalization relation for the particle velocity vector
\[(u, u) = 1, \quad (30)\]
we must have
\[\psi \varphi = 1 \implies \psi = \varphi^{-1}.\]
Thus, the invariant Hamilton function of a particle can be determined by just a single scalar function \(\varphi(x)\). Taking the last relation into account, we write the Hamilton function in final form
\[H(x, P) = \frac{1}{2} [\varphi^{-1}(x)(P, P) - \varphi(x)] = 0, \quad (31)\]
and from canonical equations (17) we obtain the connection between the generalized momentum and the particle velocity vector:
\[P^i = \varphi \frac{d x^i}{ds} \equiv \varphi u^i. \quad (32)\]

It follows from relation (28) that the generalized momentum vector and the velocity vector are timelike:
\[(P, P) = \varphi^2 \geq 0. \quad (33)\]

We note a relation that is important in what follows and is a consequence of (24), (26), and (33),
\[[H, P^k] = \nabla^k \varphi \equiv g^{ik} \partial_i \varphi, \quad (34)\]
where \(\nabla^i \equiv g^{ik} \nabla_k\) is a symbol of a covariant derivative.

Thus, the following statement is true.

**Statement 1.** Within the framework of the canonical formalism, the motion of a scalar-charged particle in a gravitational field is described by the phase trajectory \(x^i(s)\) in an 8-dimensional phase space, which is a vector bundle with a base \(X\) and a fiber \(P(X)\) with an invariant volume element \(d \mathcal{T} = dX dP\), Eq. (13), which is conserved along phase trajectories (17). The pair of phase coordinates of the particle \(\{x^i, P_i\}\), configuration coordinates and generalized momenta, are canonically conjugate, which manifests itself, first, in the conjugate character of the equations of motion (17), second, in the conservation of phase volume (18) and, third, in the conservation of the invariant Hamilton function along the phase trajectories, Eq. (25). In this case, the invariant Hamilton function of a scalar-charged particle is determined by just a single scalar function \(\varphi(x)\), Eq. (31), which depends on scalar potentials and is equal to zero along phase trajectories (27). The relation between the generalized momentum \(P_i\) and the particle velocity vector \(u^i = dx^i / ds\) is established by relation (32).

From the second group of canonical equations (17), we obtain the equations of motion in the Lagrangian formulation, which coincide with Eqs. (4),
\[\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = \partial_k \ln |\varphi| \pi^{ik}, \quad (35)\]
where \(\pi^{ik}\) is the tensor of orthogonal projection onto the direction \(u\), Eq. (5). The properties of tensor (6) and Euler equations (35) imply a rigorous consequence of orthogonality of the velocity and acceleration vectors:
\[g_{ik} u^i \frac{du^k}{ds} \equiv 0. \quad (36)\]
We note that Lagrangian equations of motion (35) are invariant with respect to the sign of the scalar function \( \varphi(x) \):

\[
\varphi(x) \to -\varphi(x).
\]

(37)

Hamilton function (37) is also invariant under the transformation (31) at its zero normalization. Therefore, relations (33) and (32) and Euler equations (35) imply that the square of \( \varphi \) has the meaning of the square of the dynamical mass of the particle, \( m_* \), in a scalar field:

\[
\varphi^2 = m_*^2.
\]

(38)

Action (9) corresponds to the specified choice of the Hamilton function. This function formally coincides with the Lagrange function of a relativistic particle with a rest mass \( m_* \) in a gravitational field (see, e.g., [57]).

We return to the question of choosing the function of the dynamical mass of the particle. Let the system have \( n \) different scalar fields, \( \Phi_r \), and each particle have \( n \) corresponding fundamental scalar charges, \( q_r \), \( (r = 1, n) \), which may include zero charges. The question about the choice of the function \( m_*(\Phi_r) \) arises.

Without specifying this function as yet, we note the following important circumstance. For canonical equations of motion (17) to admit a linear integral of motion \( \Psi = (\xi, P) = \text{const} \), it is necessary and sufficient according to (19) that \( [H, \Psi] = 0 \), which, in turn, is possible if and only if \( \xi \)

\[
(\xi, P) = \text{const} \iff L_{\xi} g_{ik} = 0,
\]

(39)

where \( L_{\xi} \) is the Lie derivative in the direction \( \xi \) (see, e.g., [63]).

We note that linear integrals of motion (39) have a transparent physical meaning: if \( \xi^i(x) \) is a spacelike vector \( (\xi^i, \xi^\alpha) < 0 \), then \( C_{\alpha} = (\xi^\alpha, P) \) is a projection of the particle total momentum vector, which is conserved along the direction \( \xi^i(x) \), while if \( \xi^i(x) \) is a timelike vector \( (\xi^i, \xi^\alpha) > 0 \), then \( C_4 = (\xi^4, P) \) is the conserved total energy of the particle.

We consider static fields \( g_{ik} \) and \( \Phi_r \) admitting a timelike Killing vector \( \xi^i = \delta^i_4 \) and \( \Phi(x^1, x^2, x^3) \), when the total energy of the charged particle is conserved, \( P_4 = E_0 = \text{const} > 0 \). We choose a frame of reference in which \( g_{\alpha 4} = 0, \alpha, \beta = 1, 3, \) such that the coordinate \( x^4 \) coincides with the world time \( t \). From the relation between the kinematic velocity vector \( u^i \) and the particle total momentum vector \( P_i \) (32), it then follows that

\[
P^i = \varphi u^i \Rightarrow u^4 = \frac{dt}{ds} = \frac{E_0}{\varphi}.
\]

(40)

Therefore, if we want to keep the same orientation of the world time \( t \) and the proper time \( s \) (i.e., \( u^4 = \frac{dt}{ds} > 0 \)), it is necessary to choose a mass function that would always remain nonnegative: \( m_* = |\varphi| > 0 \). However, such a conservative and, at first glance, correct approach used in [58], [59], [64], [1], contradicts the more fundamental principle of additivity of the Lagrange function. As shown in [35], the negativity of the particle dynamical mass function does not lead to any contradictions at the level of microscopic dynamics, because the observed kinematic momentum of the particle, which in our case exactly coincides with the total momentum,

\[
p^i = m_* \frac{dx^i}{ds} = P^i,
\]

just like the observed 3-dimensional velocity \( v^\alpha = dx^\alpha/dt = u^\alpha / u^4 \), in contrast to the unobservable kinematic 4-velocity of a particle, \( u^i \), preserves its orientation. The choice of a linear mass function in (31) that coincides with (12) corresponds to the principle of additivity of the action function. This choice also meets esthetic criteria, because in this case Hamilton function (31) is independent of the rest mass.
When choosing the dynamical mass in form (12), Hamilton function (31) and normalization relation (33) for the generalized momentum take the form

\[
H(x, P) = \frac{1}{2} \left[ m_*^{-1}(x)(P, P) - m_* \right] = 0, \\
(P, P) = m_*^2.
\] (42) (43)

We note the identities that are useful in what follows and are valid for Hamilton function (42):

\[
\vec{\nabla}_i H = -\nabla_i m_*,
\] (44)

\[
[H, \Psi] = \frac{1}{m_*} P^i \vec{\nabla}_i \Psi + \partial_i m_* \frac{\partial \Psi}{\partial P_i},
\] (45)

where \( \Psi(x, P) \) is an arbitrary function.

Thus, the following statement is holds.

**Statement 2.** From the standpoint of the canonical formalism, the choice of a dynamical mass in form (12) corresponds to the additivity principle of the Lagrange function of a scalar-charged particle. In the case \( m_* < 0 \), the relative orientation of the world time and the proper time of the particles \( u^4 = dt/ds < 0 \) changes, but the orientation of the observed kinematic momentum of the particle, Eq. (41), and the orientation of the observed 3-dimensional velocity vector \( \mathbf{v} \) of the particle are preserved.

3. Lagrangian formalism for a system of scalar-charged particles

3.1. Invariant and tensor functions of singular sources. Because the questions related to the dynamical mass are rather subtle, we provide a microscopic substantiation of the mathematical model of a plasma with scalar-charged particles that is considered in this paper. When formulating microscopic equations, it is necessary to correctly construct microscopic densities of singular field sources.

We say that a function \( D(x_1|x_2) \) is a two-point \( \delta \)-Dirac function invariant under nondegenerate transformations \( x^i = \phi^i(x^1, \ldots, x^n) \) if it defined on an \( n \)-dimensional Riemann manifold \( R_n \) and

\[
\int_{X_2} D(x_1|x_2)F(x_2) dX_2 = \begin{cases} F(x_1), & x_1 \in X_2, \\ 0, & x_1 \notin X_2, \end{cases}
\] (46)

where \( X_2 \subset R_n \), \( F(x) \) is an arbitrary tensor field on \( R_n \), \( dX = \sqrt{-g} dx^1 \ldots dx^n \) is the invariant volume element of \( R_n \), and

\[
D(x_1|x_2) = D(x_2|x_1),
\] (47)

\[
D(x'_1|x'_2) = D(x_1|x_2).
\] (48)

Along with the invariant Dirac \( \delta \)-function defined by properties (46)–(48), we can also consider a scalar density \( \Delta(x_1|x_2) \), which is usually called the Dirac \( \delta \)-function (see [65]),

\[
\Delta(x_1|x_2) = \frac{1}{\sqrt{-g}} D(x_1|x_2), \quad \Delta(x'_1|x'_2) = |J^{-1}(x_2)| \Delta(x_1|x_2),
\] (49)

where \( J \) is the Jacobian of a transformation.
It can be shown (see [65]) that the invariant Dirac δ-function obeys the following symbolic rule of covariant differentiation:

$$\frac{\partial}{\partial x^1}D(x_1|x_2) = D(x_1|x_2) \frac{\partial}{\partial x^2} \Rightarrow \frac{1}{\mathbf{\nabla}_i}D(x_1|x_2) = D(x_1|x_2)\mathbf{\nabla}_i,$$

where $\mathbf{\nabla}_i$ is an operator of covariant differentiation at the point $x_a$.

The geometric image of a classical particle is a time-like worldline $\Gamma_a: x^i = x^i(s_a) \equiv x^i_a$ along which a certain geometric object $\omega_a(x_a) \equiv \omega_a(s_a)$ is defined that characterizes its physical properties. We call this object a source. A classical point particle in scalar fields is associated with only one tensor object, the velocity vector $u^i_a = dx^i_a/ds_a$, and scalars, the scalar charges $q^r_{(a)}$ and the scalar dynamical mass $m_{(a)}$. Thus, the source of a classical particle can only have the structure $\omega_a(s_a) = \{q^r_{(a)}, m_{(a)}\} \times u^{i_1} \cdots u^{i_m}$. We define the density field of the source [65]

$$\Omega_a(x) = \int_{\Gamma_a} \omega_a D(x|x_a(s_a)) ds_a,$$

where the integration is carried out along the entire trajectory of the particle. Due to the definition of the invariant δ-function, the integration in (51) transfers the tensor properties of the object $\omega_a$ from the particle trajectory to the entire manifold $R_4$, by specifying the tensor field $\Omega_a(x)$.

If $R_4$ can be represented in some coordinate system as a direct product of a three-dimensional non-isotropic hypersurface $V_k$ and the congruence of coordinate lines $x_k$ normal to it at each point, then in this coordinate system the invariant Dirac-δ-function can be represented as the product of the 3-dimensional δ-function $D(\bar{x}_1|\bar{x}_2)$ invariant on the hypersurface $V_k$ and a one-dimensional δ-function $\delta(x|x)$

$$D(x_1|x_2) = D(\bar{x}_1|\bar{x}_2)\delta(x|x_{k_1k_2}),$$

where $\bar{x}$ are coordinates on $V_k$. In this coordinate system, the volume element $R_4$ is also represented as the product $dX = dV_k dx_k$, where $dV_k = \sqrt{-g(x)} |d^{n-1}\bar{x}$ is the area element of the hypersurface $V_k$. In what follows, we often perform such an operation in a synchronous frame when the normal vector $k_1$ is timelike. The metric $R_4$ has the following form in this frame:

$$ds^2 = d\tau^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha, \beta = 1, 3.$$

In this case, the 3-dimensional hypersurface is spacelike, and its area element is denoted by $dV$.

We consider the following source densities that have a simple physical meaning:

$$n^i(x) = \sum_a n^i_a(x) = \sum_a \int_{\Gamma_a} u^i_a(s_a) D(x|x_a) ds_a,$$

which is the particle number density vector (numeric vector), and

$$j^i_r(x) = \sum_a q^r_{(a)} n^i_a(x) = \sum_a q^r_{(a)} \int_{\Gamma_a} u^i_a(s_a) D(x|x_a) ds_a,$$

which is a current density vector of scalar charges.

We calculate the covariant divergences of these quantities. First, we consider an expression of the form

$$\nabla_i n^i_a(x) = \int_{\Gamma_a} u^i_a(s_a) \frac{\partial D(x|x_a)}{\partial x^i} ds_a.$$
We represent the integral in this expression as a curvilinear integral of the second kind taken along the entire particle trajectory,
\[ \nabla_i n_a^i(x) = \int_{\Gamma_a} \frac{\partial D(x|x_a)}{\partial x^i} \, dx_a, \]
and take symbolic rule (50) into account, according to which the operator of differentiation must act on unity in our case. Thus, the following relation always holds:
\[ \nabla_i n_a^i(x) = \int_{\Gamma_a} u_a^i(s_a) \frac{\partial D(x|x_a)}{\partial x^i} \, ds_a = 0. \] (55)
This relation has the form of a conservation law and establishes the obvious fact of the existence of a particle on its own trajectory. As a consequence of (55), the microscopic conservation laws are satisfied:
\[ \nabla_i n_a^i(x) = 0, \] (56)
\[ \nabla_j j_a^j(x) = 0. \] (57)

3.2. Microscopic field equations with singular sources. Taking the definition of the invariant \( \delta \)-function, Eq. (46), into account, the action for a type-“a” particle (9) can be rewritten as an integral over the volume of the Riemannian space \( R_4 \equiv X \):
\[ S = -\int_X dX \int_{\Gamma_a} D(x|x_a)m(a)(s_a) \, ds_a. \]
This allows us to write the total action integral for the system “scalar-charged particles (p) + scalar fields (s) + gravitational field (g)” in the form of an integral over \( X \),
\[ S = S_p + S_s + S_g = -\int_X dX \sum_a \int_{\Gamma_a} D(x|x_a)m(a)(s_a) \, ds_a - \int_X L_s dX - \frac{1}{16\pi} \int_X (R + 2\Lambda) dX, \] (58)
where \( \Lambda \) is the cosmological constant,
\[ L_s = \sum_r L(r) = \frac{1}{16\pi} \sum_r (\epsilon_r g^{ik}\Phi_{r,i}\Phi_{r,k} - 2V_r(\Phi_r)) \] (59)
is the Lagrange function of noninteracting scalar fields,
\[ V_r(\Phi_r) = -\frac{\alpha_r}{4} \left( \frac{\Phi_r^2 - \frac{m_r^2}{\alpha_m}}{\alpha_m} \right)^2 \] (60)
is the potential energy of the corresponding scalar fields, \( \alpha \) and \( \beta \) are their self-coupling constants, \( m_r \) are the masses of their quanta, and \( \epsilon_r = \pm 1 \) are indicators (the “+” sign corresponds to classical scalar fields, and “−” to phantom ones).

We calculate the variation of action (58) with respect to the dynamical variables of the particles, \( \delta_a S = \delta_a S_p \). In this case, it is necessary to take the differentiation property of the invariant \( \delta \)-function (50) into account, due to which the variation operation is also carried over to functions on the right,
\[ \delta D(x|y)F(y) = D(x|y) \delta F(y). \] As a result, we obtain
\[ \delta_a S = -\int_X dX \sum_a \int_{\Gamma_a} D(x|x_a)\delta_a(m(a)(s_a) \, ds_a). \]
Equating the variation $\delta_p S$ to zero, we obtain the equations of motion of scalar-charged particles in the form (10)

$$\frac{\delta u^i}{\delta s_a} = \partial_k \ln |m_{(a)}| \sigma^{ik}(u_a).$$  \hspace{1cm} (61)

Calculating the variation of the action of particles with respect to the scalar fields $\delta_r S_p$ and taking the definition of the dynamical mass of particles (12) into account, we similarly obtain

$$\delta_r S_p = - \int_X dX \sum_a q_{(a)}^r \int_{\Gamma_a} D(x|x_a) q_{(a)}^r \delta \Phi_r \ ds_a.$$  \hspace{1cm} (62)

Varying the actions of scalar fields with respect to scalar fields, we find the total variations $\delta_r S = \delta_r S_p + \delta_r S_e$:

$$\delta_r S = - \int_X dX \delta \Phi_r \left( \sum_a q_{(a)}^r \int_{\Gamma_a} D(x|x_a) ds_a + \frac{1}{8\pi} \left( e_r \Box \Phi_r + V_{\Phi_r} \right) \right),$$

where $\Box$ is the d’Alembert operator on the $g_{ik}$. Thus, we obtain microscopic equations for scalar fields:

$$e_r \Box \Phi + V_{\Phi_r} = -8\pi \sum_a q_{(a)}^r \int D(x|x_a) ds_a, \quad r = 1, N.$$  \hspace{1cm} (63)

We write scalar fields equations (62) in a more compact form:

$$e_r \Box \Phi + V_{\Phi_r} = -8\pi \sigma^r, \quad r = 1, N,$$

where the microscopic densities of the scalar charge of a system of particles with respect to the scalar field $\Phi_r$ are introduced:

$$\sigma^r = \sum_a \sigma^r_{(a)} = \sum_a q_{(a)}^r \int D(x|x_a) ds_a.$$  \hspace{1cm} (64)

They have a symmetry property with respect to scalar charges:

$$\sigma^r_{(a)} (-q_{(a)}^r) = -\sigma^r_{(a)} (q_{(a)}^r), \quad r = 1, N.$$  \hspace{1cm} (65)

Finally, we calculate the variation of action (58) with respect to the gravitational field. When calculating the variation of the action of particles, we obtain

$$\delta_g S_p = - \int_X dX \sum_a \int_{\Gamma_a} D(x|x_a) m_{(a)}(s_a) \delta_g ds_a.$$  \hspace{1cm} (66)

Thus, recalling the normalization of the velocity vector $(u_a, u_a) = 1$, we find

$$\delta_g ds_a = \frac{\partial}{\partial g_{ik}} \sqrt{g m_{(a)} u_a^i u_a^k} \delta g_{ik} = \frac{1}{2} u_a^i u_a^k \delta g_{ik} ds_a.$$  \hspace{1cm} (67)

The result of calculating this variation of the action functions of scalar fields and the gravitational field is known, and as a result we obtain the microscopic Einstein equations

$$G^i_k = R^i_k - \frac{1}{2} R \delta^i_k = 8\pi (T^i_{(p)} + T^i_{(s)}) + \Lambda \delta^i_k,$$  \hspace{1cm} (68)

where the microscopic energy–momentum tensor of scalar charged particles is introduced as

$$T^i_{(p)}(x) = \sum_a T^{i(k)}_{a} (x) = \sum_a \int_{\Gamma_a} m_{(a)}(s_a) u^i_a(s_a) u^k_a(s_a) D(x|x_a) ds_a$$  \hspace{1cm} (69)

and the energy–momentum tensor of scalar fields is

$$T^i_{(s)} = \frac{1}{16\pi} \sum_r (2\Phi_r \delta^i_k - \Phi_r \delta^i_k + 2V_r(\Phi_r) \delta^i_k).$$  \hspace{1cm} (70)
Remark 1. We note a very important direct consequence of the fundamental principle of additivity of the action function: the Einstein equations are determined by the energy–momentum tensor of particles with the dynamical masses of particles \( m_{(a)} \) in (12). This consequence is not quite usual because, at first glance, it can lead to negative values of the energy density of matter if \( m_{(a)} < 0 \). In this regard, taking (41) into account, we transform the expression for the energy–momentum tensor of scalar-charged particles, Eq. (67), as

\[
T_{ik}^{(p)}(x) = \sum_a \int_{\Gamma_a} P_a^i(s_a) u_a^k(s_a) D(x|x_a) \, ds_a = \sum_a \int_{\Gamma_a} P_a^i(s_a) D(x|x_a) \, dx_a^k.
\] (69)

We see that the apparently radical decision regarding the choice of the sign of the dynamical mass in (12) leads to the correct sign for the energy–momentum tensor of scalar-charged particles, due to (41). We note that if we discard the additivity principle of the Lagrange function as erroneous in our case and in the Lagrange function of a scalar-charged particle (9) replace its dynamical mass with the total mass \( m_* \rightarrow |m_*| \), then the particle equations of motion (4) do not change. In this case, however, the energy–momentum tensor of scalar-charged particles (67) and the density of scalar charges (64) change: the substitution \( m_* \rightarrow |m_*| \) is required in the energy–momentum tensor and the sign function \( \text{sgn}(m_{(a)}) \) has to be introduced as a multiplier in the integrand of the scalar charge density. As a result, the sign function \( \text{sgn}(m_{(a)}) \) has to be introduced as a multiplier in the integrand in (69), which exactly leads to negative values for the particle energy density.

Repeating similar calculations for the divergence of the energy–momentum tensor of particles (67) and taking equations of motion (61) into account, we obtain

\[
\nabla_k T_{ik}^{(a)}(x) = \int_{\Gamma_a} D(x|x_a) u_a^k u_a^i \nabla k m_{(a)}(s_a) \, ds_a \Rightarrow \nabla_k T_{ik}^{(p)} = \sum_a \int_{\Gamma_a} D(x|x_a) \nabla^a m_{(a)}(s_a) \, ds_a.
\] (70)

Using the symmetry of invariant \( \delta \)-function (47) and the definition of the particle dynamical mass in (12) and (64), we rewrite (70) in a more convenient form

\[
\nabla_k T_{ik}^{(p)} = \sum_a \sum_r \tilde{q}_a^i(r) \nabla^a \Phi_r \int_{\Gamma_a} D(x|x_a) \, ds_a \Rightarrow \nabla_k T_{ik}^{(p)} = \sum_r \sigma^r \nabla^i \Phi_r.
\] (71)

It is easy to verify that due to (71), (62), and (68) the conservation law for the total energy–momentum tensor of the system “scalar-charged particles + scalar fields” is automatically satisfied,

\[
\nabla_k T^{ik} = 0, \quad T^{ik} = T^{ik}_{(p)} + T^{ik}_{(s)}.
\] (72)

where the energy–momentum tensor of scalar-charged particles \( T^{ik} \) and the energy–momentum tensor of scalar fields \( T^{ik}_{(s)} \) are described by formulas (67) and (68).

4. Invariant distribution function, macroscopic averages, and transport equations in the general relativistic kinetic theory of scalar-charged plasma

In this paper, we consider a macroscopic model of scalar-charged plasma based on the general relativistic kinetic and statistical theory. The foundations of the general relativistic kinetic and statistical theory were laid in the 1960s in the works by Tauberg and Weinberg [66], Chernikov (see, e.g., [67]), Vlasov [61], and
others. Scalar fields were introduced into general relativistic statistics and kinetics in the early 1980s in the works by one of the authors [58], [59], [64], [1]. Further, in [35]–[37], a mathematical model of the statistical system of scalar-charged particles based on the microscopic description and the subsequent procedure of transition to the kinetic and hydrodynamical models was formulated. The already cited studies [37] and [39] contain the correct generalization of the general relativistic kinetic theory to the sector of phantom fields and negative dynamical masses.

4.1. Invariant relativistic distribution function of identical particles. The general formalism of invariant distribution functions was developed in [60], [68]. To take the possibility of a negative sign of the dynamical mass of particles into account, it is necessary to carefully apply this formalism to the considered case. To determine the macroscopic averages in a relativistic phase space, it is necessary to determine a unit timelike field of macroscopic observers, \( U_i(x) \): \((U, U) = 1\), according to whose clock the synchronization of the acts of measurement of individual particles are carried out. This timelike field, in turn, defines a spacelike three-dimensional surface, \( V_3 \), translations \( \delta x^i \) along which are orthogonal to the given field:

\[
V_3: \delta x^i U_i = 0, \tag{73}
\]

and translations \( dx^i \) along this field determine the synchronized proper time \( \tau \) of the observers:

\[
\frac{dx^i}{d\tau} = U^i \iff \frac{dx^i}{d\tau} U_i = 1 \Rightarrow d\tau = dx^i U_i. \tag{74}
\]

Thus, in the macroscopic reference frame of observers,

\[
X = V \times T \Rightarrow dX = dV \, d\tau. \tag{75}
\]

In this case, the connection of the particle proper time \( s \) with the synchronized proper time \( \tau \) of observers at each point of the configuration space is established by the relation

\[
\frac{d\tau}{ds} = U_i \frac{dx^i}{ds}. \tag{76}
\]

Using relation (41), which is a consequence of canonical equations (17), we finally obtain

\[
\frac{d\tau}{ds} = m^{-1}_\ast(U, P) \Rightarrow \frac{1}{m_\ast(s)} \frac{ds}{d\tau} = \frac{1}{(U(\tau), P(s))}. \tag{77}
\]

Relation (77) can be considered a differential equation for the function \( s(\tau) \), solving which we can determine the relation between the proper time of the particle and the time measured by the clocks of synchronized observers:

\[
s = s(\tau). \tag{78}
\]

Further, in the general relativistic kinetic theory, a system of particles is described by the distribution function \( F_a(x, P) \) of canonically conjugate dynamical variables \( \{x^i, P_i\} \) that is invariant in the 8-dimensional phase space \( \Gamma = P(X) \times X \). The invariant 8-dimensional distribution function of identical particles \( F(x, P) \) is introduced as follows [60]. Let the phase trajectory of a particle, determined by canonical equations (24) in the phase space \( \Gamma \), be

\[
x^i = x^i(s), \quad P_i = P_i(s) \quad \Rightarrow \quad \eta_a = \eta_a(s), \quad a = 1, 8. \tag{79}
\]

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Then the number of particles registered by observers in the region $d\Gamma$ of the phase space can be defined as [60]

$$dN(\tau) = F(x, P)\delta(s - s(\tau))d\Gamma.$$  \hspace{1cm} (80)

We note that the number of particles is a scalar, which, however, depends on the choice of the field of observers $U^i$, i.e., on the choice of the frame of reference in the Riemannian space $X$, while the 8-dimensional distribution function itself $F(x, P)$, introduced by relation (80), is invariant in the phase space $\Gamma$. We also note that it is impossible to give another definition of the invariant distribution function in the 8-dimensional phase space. All the previously introduced definitions of this function are special cases of (80), applicable in some special reference frames.

4.2. Macroscopic averages of the dynamical functions. The definition of the invariant distribution function (80) and other dynamical functions is the first key point of the relativistic kinetic theory, which depends on the sign of the dynamical mass of particles. Therefore, the determination of the appropriate operations requires special care. Let $\psi(x, P) \equiv \psi(\eta)$ be some scalar function of dynamical variables.

Then, according to (80), its macroscopic average $\Psi(\tau)$ in the region $\Omega \subset \Gamma$ is

$$\Psi(\tau) = \int_\Omega \psi(\eta(s)) dN = \int_\Omega F(\eta(s))\psi(\eta(s))\delta(s - s(\tau))d\Gamma.$$  \hspace{1cm} (81)

Assuming further, in accordance with (73) and (74), that $X = V \times T \Rightarrow dX = dV dt$, we write expression (81) in explicit form:

$$\Psi(\tau) = \frac{2S + 1}{(2\pi)^3} \int_V dV \int_0^\infty dt \int_{p(X)} dP \psi(\eta)F(\eta)\psi(\eta)\delta(s - s(\tau)).$$  \hspace{1cm} (82)

To integrate (82) over $t$ we use the relation between $t(s)$ (41) and $\tau(s)$, Eq. (78), and the properties of the Dirac $\delta$-function:

$$\delta(s - s(\tau))dt = \left| \frac{dt}{ds} \right| \delta(t - \tau)ds \equiv |m_+|^{-1}(U, P)\delta(t - \tau)dt.$$  \hspace{1cm} (83)

In deriving (83), we used the fact that the orientation of the generalized momentum, in contrast to the orientation of the kinematic velocity vector, is independent of the sign of the effective mass. Taking (83) into account in (82) and integrating over the time coordinate, we finally obtain

$$\Psi(\tau) = \frac{2S + 1}{(2\pi)^3} \int_V \frac{U_i}{|m_+|} \int_{p(X)} P^i\psi(\eta)F(\eta)\psi(\eta)dP.$$  \hspace{1cm} (84)

In particular, assuming $\psi = 1$, we obtain the total number of particles in the region $V$,

$$N(V) = \frac{2S + 1}{(2\pi)^3} \int_V \frac{U_i}{|m_+|} \int_{p(X)} P^iF(\eta)dP \equiv \int_V (U, n) dV,$$  \hspace{1cm} (85)

where the particle flux density vector is introduced as

$$n^i(x) = \frac{2S + 1}{(2\pi)^3|m_+|} \int_{p(X)} P^iF(\eta)dP.$$  \hspace{1cm} (86)

Further, due to the generalized momentum normalization relation (43), the invariant 8-dimensional distribution function $F(x, P)$ is singular on the mass shell. Therefore, we introduce a distribution function $f(x, P)$ that is nonsingular there with the help of the relation

$$F(x, P) = \delta(H(x, P))f(x, P).$$  \hspace{1cm} (87)
Evaluating the expression
\[ F(x, P) dP \equiv \frac{1}{\sqrt{-g}} dP_1 dP_2 dP_3 dP_4 \delta(H(x, P)) f(x, P) \]
using the Dirac \( \delta \)-function properties and the explicit form of Hamilton function (42), we find
\[ F(x, P) dP = \frac{1}{\sqrt{-g}} dP_1 dP_2 dP_3 \frac{|m_\ast|}{P_4^2} \delta(P_4 - P_4^+) f(x, P) \equiv |m_\ast| \delta(P_4 - P_4^+) f(x, P) dP_0, \tag{88} \]
where \( P_4^+ \equiv P_4 \) is a positive root of normalization equation (43) and
\[ dP_0 = \frac{1}{\sqrt{-g}} dP_1 dP_2 dP_3 \frac{P_4}{P_4^2} \tag{89} \]
is the volume element of the three-dimensional momentum space.

We note that due to the antisymmetry of Poisson bracket (20) and (25), the following relation holds for any function \( G(H) \):
\[ [H, G(H)F(x, P)] = G(H)[H, F(x, P)], \tag{90} \]
in particular,
\[ [H, \delta(H)f(x, P)] = \delta(H)[H, f(x, P)]. \tag{91} \]
As a result, formulas (84) for macroscopic averages can be written in terms of an \textit{even-dimensional nonsingular distribution function} \( f(x, P) \) as
\[ \Psi(\tau) = \frac{2S + 1}{(2\pi)^3} \int_V U_i dV \int_{P_+(X)} P_i \psi(\eta) f(\eta) dP_0, \tag{92} \]
where it is necessary to substitute the positive root of the mass-shell equation instead of \( P_4 \), and \( P_+ \) is the upper part of of the mass-shell pseudosphere. Thus, when passing to the 7-dimensional distribution function, the explicit dependence on the dynamical mass in these formulas disappears.

As a result, the following symbolic rule holds, understood in the sense of integration over the corresponding phase volumes:
\[ \psi(\eta) F(\eta) \delta(s - s(\tau)) d\Gamma \rightarrow \psi(\tilde{\eta}) f(\tilde{\eta})(U, P) dV dP_0, \tag{93} \]
where \( \tilde{\eta} \) are dynamical variables on the 6-dimensional subspace \( \Gamma_0(\tau) = V \times P_0 \subset \Gamma \) with the volume element
\[ d\Gamma_0 = dV dP_0. \tag{94} \]
In particular, for the \textit{flux density vector of the number of particles} (86) we obtain (in accordance with [56]) from (92) that
\[ n^i(x) = \frac{2S + 1}{(2\pi)^3} \int_{P_0(X)} P_i f(\eta) dP_0. \tag{95} \]
4.3. The macroscopic conservation laws. As was noted above, in the studies of one of the authors [58], [59], [64], [1], [35]–[37], the kinetic theory of scalar-charged plasma was formulated. In particular, in [37], a generalization to the sector of phantom scalar fields and negative dynamical masses was given. The transport equations\(^5\) are rigorous macroscopic consequences of kinetic theory, including the conservation law of a vector current corresponding to the microscopic conservation law in reactions of a certain fundamental charge \(G\) (if there exists such a conservation law),

\[ \nabla_i \sum_a q^r_{(a)} n^i_{(a)} = 0, \]

as well as the energy–momentum conservation laws for the statistical system,

\[ \nabla_k T_{ik}^p - \sum_r \sigma^r \nabla^i \Phi^r = 0, \]

where \(n^i_a\) is a numerical vector, \(T_{ik}^p\) is the particle energy–momentum tensor, and \(\sigma^r\) is the density of scalar charges with respect to the field \(\Phi^r\) [36], such that

\[ T_{ik}^p = \sum_a T_{ik}^{(a)}, \quad \sigma^r = \sum_a \sigma^r_{(a)}, \]

\[ T_{ik}^{(a)} = \frac{2S + 1}{(2\pi)^3} \int_{P_0} f_a(x, P) P^i P^k dP_0, \]

\[ \sigma^r_{(a)} = \frac{2S + 1}{(2\pi)^3} m(a) q^r_{(a)} \int_{P_0} f_a(x, P) dP_0. \]

Further, calculating the divergence of the total energy–momentum tensor of the “plasma + scalar fields” system according to (68), (97) and equating it to zero, we obtain

\[ \sum_r \nabla^i \Phi^r (e^r \Box \Phi^r + V^r_{\Phi^r} + 8\pi \sigma^r) = 0, \]

whence, under the condition of the functional independence of scalar fields, we obtain macroscopic equations for scalar fields with sources:

\[ e^r \Box \Phi^r + V^r_{\Phi^r} = -8\pi \sigma^r, \quad r = 1, N. \]

**Statement 3.** The comparison of formulas for microscopic tensor densities (53), (64), (67) and the corresponding macroscopic tensor densities (86), (99), (100) using relation (41) reveals their full correspondence up to a nonsingular operation of averaging, Eq. (92). The microscopic and macroscopic conservation laws for the corresponding quantities, (55) ↔ (96) and (71) ↔ (97), as well as the equations for scalar fields with sources, (63) ↔ (101), are identical. Thus, the introduced operations of calculating the macroscopic averages of dynamical functions are rigorously justified at the microscopic dynamical level.

4.4. Local thermodynamic equilibrium. At a sufficiently strong interparticle interaction, when the mean free path of particles becomes much smaller than the characteristic size of the statistical system, or the mean free path is much less than the characteristic evolution time of the statistical system, the local thermodynamic equilibrium (LTE) is established in the statistical system. Under the LTE conditions, the statistical system is isotropic and is described by locally equilibrium distribution functions (see [68] for the details)

\[ f^0_u = \left[ e^{(-\mu_{(a)} + (u, p))/\theta} \pm 1 \right]^{-1}, \]

\(^5\)The transport equations are integro-differential consequences of kinetic equations, the description of which would take too much space. For the details, including the generalization to \(T\)-noninvariant interactions, see [37].
where $\mu_{(a)}$ is the chemical potential, $\theta$ is the local temperature, and $u^i$ is the unit timelike vector of the macroscopic kinematic velocity of the statistical system; the sign “+” corresponds to fermions and “−” to bosons. Macroscopic moments with respect to isotropic distribution function (102) take the form of the corresponding moments of an ideal fluid for each components [35]:

$$n^{i}_{(a)} = n_a u^i, \quad T^{ik}_{(a)} = (\varepsilon_{(a)} + p_{(a)}) u^i u^k - p_{(a)} g^{ik},$$  \quad (103, 104)

where

$$(u, u) = 1. \quad (105)$$

Because the vector $u^i$ is the eigenvector of $T^{ik}_{(a)}$ corresponding to the eigenvalue $\varepsilon_{a}$, it is also the vector of the macroscopic dynamical velocity of the statistical system [56]. Normalization relation (105) implies the well-known identity

$$u^k_{;i} u_k \equiv 0, \quad (106)$$

which allows reducing energy–momentum conservation laws (97) to the form of the ideal hydrodynamics equations

$$(\varepsilon_p + p_p) u^i_{;k} u^k = (g^{ik} - u^i u^k) \left( p_{p,k} + \sum_r \sigma^{r} \Phi_{r,k} \right), \quad (107)$$

$$\nabla_k [(\varepsilon_p + p_p) u^k] = u^k \left( p_{p,k} + \sum_r \sigma^{r} \Phi_{r,k} \right), \quad (108)$$

and the fundamental charge conservation law (96) to the form

$$\nabla_k \rho^r u^k = 0, \quad (109)$$

where

$$\rho^r \equiv \sum_{a} q^r_{(a)} n_{(a)} \quad (110)$$

is the kinematic density of the scalar charge of the statistical system with respect to the scalar field $\Phi_r$.

Next, we calculate the macroscopic scalar densities $n_{(a)}$, $\varepsilon_{(a)}$, $p_{(a)}$ and $\sigma^{r}_{(a)}$ using definitions (86), (99), (100), (103), and (104). Contracting the relations (103) with the velocity vector $u_i$, and the relations (104) with $u_i u_k$ and $\pi_{ik}(u)$ (see (5)), and recalling the velocity vector normalization in (105), we express the macroscopic scalars as

$$n_{(a)} = u_i n^{i}_{(a)}, \quad \varepsilon_{(a)} = u_i u_k T^{ik}_{(a)}, \quad p_{(a)} = \frac{1}{3} \pi_{ik}(u) T^{ik}_{(a)}. \quad (111)$$

Thus, proceeding to the integration over the three-dimensional space of momenta with (88) and (89) in formulas (86), (99), and (100), and using the normalization relation for the generalized momentum (43), we find the macroscopic scalars

$$n_{(a)} = \frac{2S + 1}{(2\pi)^3} \int_{p_0} (u, P) f^{0}_{(a)}(x, P) \frac{dP_1 dP_2 dP_3}{\sqrt{-g P^4}},$$

$$\varepsilon_{(a)} = \frac{2S + 1}{(2\pi)^3} \int_{p_0} (u, P)^2 f^{0}_{(a)}(x, P) \frac{dP_1 dP_2 dP_3}{\sqrt{-g P^4}},$$

$$p_{(a)} = \frac{2S + 1}{3(2\pi)^3} \int_{p_0} |m_{(a)}^2 - (u, P)^2| f^{0}_{(a)}(x, P) \frac{dP_1 dP_2 dP_3}{\sqrt{-g P^4}},$$

$$\sigma^{r}_{(a)} = \frac{2S + 1}{(2\pi)^3} m_{(a)} q^{r}_{(a)} \int_{p_0} f^{0}_{(a)}(x, P) \frac{dP_1 dP_2 dP_3}{\sqrt{-g P^4}}.$$
We pass in the integrals here to a spherical coordinate system of the momentum space in a local frame \( u^i = \delta^i_1 \),

\[
P_{(1)} = |m_{(a)}| z \cos \phi \cos \theta, \quad P_{(2)} = |m_{(a)}| z \sin \phi \cos \theta, \quad P_{(3)} = |m_{(a)}| z \sin \theta, \quad P_{(4)} = |m_{(a)}| \sqrt{1 + z^2},
\]

\[
z \in [0, +\infty), \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \phi \in [0, 2\pi], \quad |J_p| = |m_{(a)}|^2 z^2 \cos \theta,
\]

\[
dP = \frac{dP_1 dP_2 dP_3}{\sqrt{-gP^4}} = \frac{m^2_{(a)}}{\sqrt{1 + z^2}} z^2 \cos \theta \, dz \, d\theta \, d\phi,
\]

where \( P_{(k)} \) are orthonormal components of the generalized momentum and \( J_p \) is the Jacobian of the transformation to the spherical coordinate system. We note that \( m_{(a)} \) is included in all formulas for microscopic scalars without the modulus sign, which is a consequence of the correct definition of scalar densities. Thus, integrating over angular momentum variables (112) in the formulas for macroscopic scalars, we finally obtain

\[
n_{(a)} = |m_{(a)}|^3 \frac{2S + 1}{2\pi^2} \int_0^\infty \frac{z^2 \, dz}{e^{-\gamma_{(a)} + \lambda_{(a)} \sqrt{1 + z^2}} + 1},
\]

\[
\varepsilon_{(a)} = m_{(a)}^4 \frac{2S + 1}{2\pi^2} \int_0^\infty \frac{\sqrt{1 + z^2} \, dz}{e^{-\gamma_{(a)} + \lambda_{(a)} \sqrt{1 + z^2}} + 1},
\]

\[
p_{(a)} = m_{(a)}^4 \frac{2S + 1}{6\pi^2} \int_0^\infty \frac{z^4 \, dz}{e^{-\gamma_{(a)} + \lambda_{(a)} \sqrt{1 + z^2}} + 1},
\]

\[
\sigma^r_{(a)} = m_{(a)}^3 \rho^r_{(a)} \frac{2S + 1}{(2\pi)^3} \int_0^\infty \frac{z^2 \, dz}{\sqrt{1 + z^2} e^{-\gamma_{(a)} + \lambda_{(a)} \sqrt{1 + z^2}} + 1},
\]

where \( \varepsilon_p = \sum \varepsilon_a, \quad p_p = \sum p_a, \quad \sigma^r = \sum \sigma^r_{(a)}, \quad \lambda_{(a)} = |m_{(a)}| / \theta, \quad \gamma_{(a)} = \mu_a / \theta, \quad S \) is the spin of the particles. Thus, the absolute magnitude of the dynamical mass is included in the final formulas for the equilibrium macroscopic scalars only through the equilibrium distribution function, via the energy \( P_A \).

We note that according to definitions (110), (113), and (116), the kinematic density of a scalar charge \( \rho^r \), Eq. (110), determining its conservation, does not coincide with the scalar density of the scalar charge \( \sigma^r \), which is a source of the scalar field \( \Phi_r \). These densities coincide in absolute value only for nonrelativistic plasmas, where average values of the momentum are small, \( \bar{z}^2 \to 0 \). In addition, for \( m_{(a)} < 0 \), these densities may differ in sign. We consider the transformation laws of macroscopic scalars in more detail.

Let an \( N \)-plet of scalar fields be given,

\[
\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_N\}.
\]

We first find how macroscopic scalar densities (113)–(116) transform under the reflection of scalar fields:

\[
\Phi : \Phi \to -\Phi.
\]

Under (118), the dynamical mass of particles transforms according to the law

\[
m_{(a)}(-\Phi) = -m_{(a)}(\Phi).
\]

It follows from formulas (113)–(116) that all macroscopic scalars can explicitly depend on scalar fields only through the dynamical mass:

\[
n_{(a)} = n_{(a)}(m_{(a)}(\Phi), \mu_{(a)}(\Phi), \theta), \quad \varepsilon_{(a)} = \varepsilon_{(a)}(m_{(a)}(\Phi), \mu_{(a)}(\Phi), \theta),
\]

\[
p_{(a)} = p_{(a)}(m_{(a)}(\Phi), \mu_{(a)}(\Phi), \theta), \quad \sigma^r_{(a)} = \sigma^r_{(a)}(\mu_{(a)}(\Phi), \theta).
\]
Thus, we obtain the transformation laws of macroscopic scalar densities (113)–(116) under the transformation $\Phi$ in (118):

\[ n_{(a)}(-\Phi) = \bar{n}_{(a)}(\Phi), \quad \varepsilon_{(a)}(-\Phi) = \bar{\varepsilon}_{(a)}(\Phi), \quad (121) \]
\[ p_{(a)}(-\Phi) = \bar{p}_{(a)}(\Phi), \quad \sigma_{(a)}^r(-\Phi) = -\bar{\sigma}_{(a)}^r(\Phi), \quad (122) \]

which means that the macroscopic scalar densities, except the scalar charge density, are invariant under transformation (118). In this case, under the condition of the continuity of the function $\sigma_{(a)}^r(\Phi_r)$ due to (122), the following relation holds (see Sec. 4.5):

\[ \sigma_{(a)}^r(0) = 0. \quad (123) \]

This is an important symmetry property due to which the macroscopic equations for scalar fields with sources (101) always have the trivial solution:

\[ (123) \quad \Rightarrow \quad \varepsilon_r \Box \Phi_0 + V_{\Phi_0}' \equiv -8\pi \sigma(\Phi_0), \quad \Phi_0 \equiv 0. \quad (124) \]

We note that the presence of a term with a constant rest mass in the formula for the dynamical mass, Eq. (12), would lead to the absence of the trivial solution for the macroscopic equations for scalar field with sources.

We consider now the scalar charge conjugation transformation:

\[ Q: q_{(a)}^r \leftrightarrow -q_{(a)}^r \Rightarrow q \leftrightarrow -q, \quad (125) \]

where $q$ is an $n \times N$ charge matrix:

\[ q = \begin{pmatrix} q_1^{(1)} & q_1^{(2)} & \cdots & q_1^{(n)} \\ q_2^{(1)} & q_2^{(2)} & \cdots & q_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ q_N^{(1)} & q_N^{(2)} & \cdots & q_N^{(n)} \end{pmatrix}. \]

Transformation (125) replaces scalar-charged particles with antiparticles. It is easy to see that the formulas for all macroscopic scalars (113)–(116) are charge-symmetric, including $\sigma_{(a)}^r$, in which the specificity of the scalar interaction is revealed (see, e.g., [39]):

\[ n_{(a)}(-q, -\gamma) = \bar{n}_{(a)}(q, \gamma), \quad \varepsilon_{(a)}(-q, -\gamma) = \varepsilon_{(a)}(q, \gamma), \quad (126) \]
\[ p_{(a)}(-q, -\gamma) = \bar{p}_{(a)}(q, \gamma), \quad \sigma_{(a)}^r(-q, -\gamma) = \bar{\sigma}_{(a)}^r(q, \gamma). \quad (127) \]

Thus, the following statement is true.

**Statement 4.** The choice of the dynamical mass in form (12), which allows negative values of scalar fields, leads to the correct sign of the macroscopic energy density and pressure of scalar-charged particles under local thermodynamic equilibrium conditions, as in the case of the microscopic density (see Remark 1).
We see that under the LTE conditions, the macroscopic conservation laws formally give $4 + N$ independent equations\(^6\) (107)–(109) for $5 + N$ macroscopic scalar functions $\varepsilon_p$, $p_p$, $n_r$, and three independent components of the velocity vector $u^i$. However, not all of the indicated macroscopic scalars are functionally independent, because they are all determined by locally equilibrium distribution functions (102). With the resolved chemical equilibrium conditions, when only one chemical potential remains independent, resolved mass-shell equation (see [35], [36] for the details), and given scalar potentials and the $4 + 2N$ scale factor, the macroscopic scalars $\varepsilon_p$, $p_p$, $n_r$, and $\sigma^r$ are determined by two scalars: some chemical potential $\mu$ and the local temperature $\theta$. Thus, system of equations (107)–(109) becomes fully determined.

We further investigate the question of how the equations for scalar fields (101) transform under the reflection of scalar fields (118) and under charge conjugation (125). Because Higgs potentials (151) are even functions of the scalar fields $\Phi_r$, we have $V'_r(\Phi) = -V'_r(-\Phi)$ under transformation (118). Recalling the transformation law of the scalar charge density (122), we conclude that the equations of scalar fields with a source, Eqs. (101), are invariant under reflection transformation (118). Further, because the scalar charges $q^r(\alpha)$ are not contained in the left-hand sides of scalar field equations (101) and the scalar charge densities $\sigma^r$, according to (127), are invariant under charge conjugation transformation (125), it follows that the equations of scalar fields with a source (101) are invariant under charge conjugation transformation (125). Thus, the following statement is true.

**Statement 5.** Macroscopic equations of scalar fields with sources (101) are invariant both under the reflection of scalar fields, Eq. (118), and under charge conjugation transformation (125), and in this case they always have trivial solution (124). This means, first, that if $\Phi = \Phi_0(x^i)$ is a solution of field equations (101), then $\Phi = -\Phi_0(x^i)$ with equivalent initial conditions (with a simultaneous change of the signs of the initial scalar potentials in the Cauchy problem) and equal parameters is also a solution of these field equations. Second, if $\Phi = \Phi_0(x^i)$ is a solution of field equations (101) at given scalar charges $Q$, then it is a solution of these equations for the charge-conjugate problem $Q \rightarrow -Q$. In particular, this implies that the signs of scalar potentials are independent of the signs of the charges of their sources and are determined only by the initial conditions of the Cauchy problem.

These features of the interaction of scalar fields with scalar charges distinguish this interaction from the interaction of vector and tensor fields with the corresponding charges. We can say that in the presence of scalar fields, a system of particles is charged with a scalar charge. The peculiarity of the situation is revealed in the fact that scalar charges are practically conserved according to charge conservation law (109), but their effective densities, which are sources of scalar fields, $\sigma^r$, fully depend on the value of scalar fields $\Phi_r$ through the dynamical mass.

**4.5. Locally equilibrium statistical system of scalar-charged fermions under conditions of complete degeneracy.** Further in this paper, we consider the equilibrium statistical system of fermions under conditions of complete degeneracy. In the case of the complete degeneracy of the statistical system of fermions,

$$\theta \rightarrow 0$$

(128)

and locally equilibrium distribution function of fermions (102) takes the form of a step function [36],

$$f^0_{(a)}(x, P) = \chi_+(\mu_{(a)} - \sqrt{m^2_{(a)} + P^2}),$$

(129)

where $\chi_+(z)$ is the Heaviside step function, $\mu_{(a)} \equiv \sqrt{m^2_{(a)} + f^2_{(a)}}$ and $f_{(a)}$ is the Fermi momentum for the type-“a” fermions.

\(^6\)One of the equations, Eq. (107), is dependent on the rest due to identity (106).
The result of integration of macroscopic densities (113)–(116) with respect to distribution (129) can be expressed in elementary functions [36]. For a multicomponent Fermi system with a multi-particle charge, we obtain the following expressions for macroscopic scalars:

\[
\begin{align*}
(a) &= \frac{1}{\pi^2} \pi^3, \\
\varepsilon_p &= \sum \frac{m^4}{8\pi^2} F_2(\psi(a)), \\
p_p &= \sum \frac{m^4}{24\pi^2} (F_2(\psi(a)) - 4F_1(\psi(a))), \\
\sigma^r &= \sum q^r(\psi(a)) m^3 \frac{2}{2\pi^2} F_1(\psi(a)),
\end{align*}
\]

where \(m(a)\) is described by formula (12) and dimensionless functions \(\psi(a)\) are introduced as

\[
\psi(a) = \frac{\pi(a)}{|m(a)|},
\]

which are equal to ratio of the Fermi momentum \(\pi(a)\) to the total energy of a fermion, and functions \(F_1(\psi)\) and \(F_2(\psi)\) are introduced to simplify the notation:

\[
\begin{align*}
F_1(\psi) &= \psi \sqrt{1 + \psi^2} - \ln(\psi + \sqrt{1 + \psi^2}), \\
F_2(\psi) &= \psi \sqrt{1 + \psi^2} (1 + 2\psi^2) - \ln(\psi + \sqrt{1 + \psi^2}).
\end{align*}
\]

The functions \(F_1(x)\) and \(F_2(x)\) are even,

\[
F_1(-x) = -F_1(x), \quad F_2(-x) = -F_2(x),
\]

and have the asymptotic forms

\[
\begin{align*}
F_1(x)|_{x \to 0} &\sim \frac{2}{3} x^3, \quad F_2(x)|_{x \to 0} \sim \frac{8}{3} x^3, \quad (F_2(x) - 4F_1(x))|_{x \to 0} \sim \frac{8}{5} x^5, \\
F_1(x)|_{x \to \pm \infty} &\sim x|x|, \quad F_2(x)|_{x \to \pm \infty} \sim 2x^3|x|.
\end{align*}
\]

We also write expressions for the derivatives of \(F_1(x)\) and \(F_2(x)\), which are useful in what follows:

\[
\begin{align*}
F_1'(x) &= \frac{2x^2}{\sqrt{1 + x^2}}, \quad F_2'(x) = 8x^2 \sqrt{1 + x^2}.
\end{align*}
\]

Adding both parts of relations (131) and (132), we can easily verify the identity

\[
\varepsilon_p + p_p = \frac{1}{3\pi^2} \sum m^4(\psi^3(a)) \sqrt{1 + \psi^2(a)}.
\]

To conclude this section, we find the asymptotics of the macroscopic scalars (130)–(133) as \(m(a) \to 0\), i.e., as \(\Phi \to 0\). It is necessary to use formula (134) and asymptotic formulas (139) for \(F_1(x)\) and \(F_2(x)\). We then find

\[
\begin{align*}
n_{(a)}|_{\Phi \to 0} &= \frac{1}{\pi^2} \pi^3, \quad \sigma^r_{(a)}|_{\Phi \to 0} \sim q^r(\pi(a)) \frac{m^2}{2\pi^2} \to 0, \\
\varepsilon_{(a)}|_{\Phi \to 0} &\sim 3p_{(a)}|_{\Phi \to 0} \sim \frac{1}{4\pi^2} \sum \pi^4(a) = \text{const}.
\end{align*}
\]

Thus, as we indicated above, \(\sigma^r_{(a)}(\Phi)\) is a continuous function as \(\Phi \to 0\), and in accordance with (123), \(\sigma^r_{(a)}(0) = 0\).
5. The cosmological model

We consider a homogeneous isotropic distribution of matter, assuming that space–time is described by the spatially flat Friedmann metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$ (143)

and all thermodynamic functions and scalar fields depend only on time. It is easy to verify that \( u_i = \delta^i_4 \) turns equations (107) into identities, and system of equations (108), (109) reduces to \( 1 + N \) material equations\(^7\)

\[
\dot{\varepsilon}_p + 3\frac{\dot{a}}{a}(\varepsilon_p + p_p) = \sum_r \sigma^r \dot{\Phi}_r, \tag{144}
\]

\[
\dot{\rho}^r + 3\frac{\dot{a}}{a}\rho^r = 0, \quad r = 1, N, \tag{145}
\]

where \( \dot{\phi} = d\phi/dt \).

5.1. Solution of the material equations for a degenerate plasma. Thus, there remain \( N + 1 \) differential equations for \( n + 1 \) thermodynamic functions \( \mu(a) \) and \( \theta \). Taking the limit \( \mu(a) \to \infty \) (or \( \theta \to 0 \)), we obtain a system of \( N + 1 \) equations for \( n \) functions, and the problem of the compatibility of these equations arises; this problem is independent of the presence of a scalar field. However, Eq. [36] shows that this is an apparent problem, in fact, no contradictions in system of equations (107)–(109) arise even in the case of a degenerate Fermi system. As it turns out, in this case, charge conservation laws (145) are in this case a direct consequence of the energy conservation law (144).

We consider energy conservation law (144) for the degenerate Fermi system. Using expressions (131)–(133) and (141) for macroscopic scalars, the definitions of the dynamical particle masses \( m(a) \) in (12), the functions \( \psi(a) \) (134), \( F_1(x) \) (135), and \( F_2(x) \) (136), as well as expressions (140) for their derivatives, after rather cumbersome calculations, we bring energy conservation law (144) to the explicit form

\[
\sum_{a=1}^n m(a)\pi_3(a)\sqrt{1 + \psi(a)} \frac{d}{dt} \ln(|a\pi(a)|) = 0, \tag{146}
\]

which, in view of the definition of the total mass in (12), can be represented explicitly as

\[
\sum_{a=1}^n \pi_3(a)\sqrt{1 + \psi(a)} \frac{d}{dt} \ln(|a\pi(a)|) \sum_{r=1}^N q_r(a)\Phi_r = 0. \tag{147}
\]

Further, using the definition of the kinematic charge density in (110) and expressions (130) for the density of the number of particles, we reduce the conservation laws of scalar charges (145) to the form

\[
\sum_{a=1}^n \pi_3(a) \frac{d}{dt} \ln(|a\pi(a)|)q_r(a) = 0, \quad r = 1, N. \tag{148}
\]

Equations (147), (148) are a system of \( N + 1 \) homogeneous algebraic equations for \( n \) unknowns \( X(a) = d/dt (a\pi(a)), a = 1, n \). This system always has a trivial solution \( X(a) = 0 \), and for \( n \leq N + 1 \) in the general case, it has only the trivial solution. We choose this solution as a solution of material equations (144), (145). We note that for a one-component or two-component system of fermions, the solution\(^7\) Here, we write the material equations in the case of \( N \) scalar fields.
of the material equations for \( X_{(a)} \) can only be trivial. However, even for any numbers \( n \) and \( N \), system of equations (147), (148) has only the trivial solution due to the arbitrariness in the parameters of scalar charges \( q_{(a)}^r \). Thus, we choose the solution

\[
\frac{d}{dt} \ln(a\pi_{(a)}) = 0 \quad \iff \quad a\pi_{(a)} = \text{const},
\]

(149)

We note that on one hand, this solution corresponds to the conservation law for the absolute value of the momentum of free particles in a homogeneous isotropic metric, and on the other hand, it was obtained in [40] for a one-component Fermi system.

**Statement 6.** The solution of material equations (144), (145) is the Fermi momentum conservation law (149) for each component of the Fermi system.

5.2. **Field equations.** As a field model, we next consider the asymmetric scalar Higgs doublet\(^8\) \( \{\Phi, \varphi\} \) with the Lagrange function \([28], [31]\]

\[
L_s = \frac{1}{8\pi} \left( \frac{1}{2} g^{ik}\Phi_i \Phi_k - V(\Phi) \right) + \frac{1}{8\pi} \left( -\frac{1}{2} g^{ik}\varphi_i \varphi_k - \mathcal{V}(\varphi) \right),
\]

(150)

where \( V(\Phi) \) is the potential energy of the classical scalar field and \( \mathcal{V}(\varphi) \) is the potential energy of the phantom field,

\[
V = -\frac{\alpha}{4} \left( \dot{\Phi}^2 - \frac{m^2}{\alpha} \right)^2, \quad \mathcal{V} = -\frac{\beta}{4} \left( \dot{\varphi}^2 - \frac{m^2}{\beta} \right)^2,
\]

(151)

\( \alpha \) and \( \beta \) are self-coupling constants, and \( m \) and \( \tilde{m} \) are masses of scalar bosons.

Next,

\[
T_s^{ik} = \frac{1}{8\pi} \left( \Phi^i \Phi^k - \frac{1}{2} g^{ik}\Phi^j \Phi^j + g^{ik}V(\Phi) \right) + \frac{1}{8\pi} \left( -\varphi^i \varphi^k + \frac{1}{2} g^{ik}\varphi^j \varphi^j + g^{ik}\mathcal{V}(\varphi) \right)
\]

(152)

is the energy–momentum tensor of the scalar doublet. We can omit the constant terms in Higgs potentials (151) by redefining the cosmological constant \( \Lambda_0 \):

\[
\Lambda \rightarrow \Lambda_0 - \frac{m^4}{4\alpha} - \frac{\tilde{m}^4}{4\beta}.
\]

(153)

Scalar fields are defined by equations for charged scalar fields with a source \([37]\),

\[
\Box \Phi + V'_\Phi = -8\pi\sigma_c, \quad -\Box \varphi + V'_\varphi = -8\pi\sigma_f, \quad (154)
\]

where \( \sigma_c \) and \( \sigma_f \) are scalar charge densities (133) for the respective classical and phantom fields. It can be shown that due to (97) and (154), the conservation law of the total energy–momentum tensor of the "plasma + charged scalar fields" system is identically satisfied:

\[
\nabla_i T^{ik} = \nabla_i (T_p^{ik} + T_s^{ik}) \equiv 0.
\]

(155)

The energy–momentum tensor of an asymmetric scalar doublet in an isotropic homogeneous metric also takes the form of an energy–momentum tensor of an ideal isotropic fluid,

\[
T_s^{ik} = (\varepsilon_s + p_s)u^iu^k - p_sg^{ik},
\]

(156)

\(^8\)The term “quintom” is also used in the literature.
where
\[ \varepsilon_s = \frac{1}{8\pi} \left( \frac{1}{2} \dot{\Phi}^2 + V(\Phi) - \frac{1}{2} \dot{\varphi}^2 + \mathcal{V}(\varphi) \right), \] (157)
\[ p_s = \frac{1}{8\pi} \left( \frac{1}{2} \dot{\Phi}^2 - V(\Phi) - \frac{1}{2} \dot{\varphi}^2 - \mathcal{V}(\varphi) \right), \] (158)
whence
\[ \varepsilon_s + p_s = \frac{1}{4\pi} (\dot{\Phi}^2 - \dot{\varphi}^2). \] (159)

Finally, equations of scalar fields (154) in the Friedmann metric take the form
\[ \ddot{\Phi} + \frac{3}{2} \dot{a} \dot{\Phi} + m^2 \Phi - \alpha \Phi^3 = -8\pi \sigma_c(t), \] (160)
\[ \ddot{\varphi} + \frac{3}{2} \dot{a} \dot{\varphi} - m^2 \varphi + \beta \varphi^3 = 8\pi \sigma_f(t). \] (161)

We further consider the standard Einstein equations with an unperturbed \( \Lambda \)-term, \( \Lambda_0 \):
\[ G^i_k = R^i_k - \frac{1}{2} R \delta^i_k = 8\pi T^i_k + \Lambda_0 \delta^i_k. \] (162)

We write the independent Einstein equations for Friedmann metric (143):
\[ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \sum_r \left( \frac{r \dot{\Phi}_r^2}{2} - \frac{m_r^2 \Phi_r^2}{2} + \frac{\alpha_r \Phi_r^4}{4} \right) + 8\pi p_p = \Lambda, \] (163)
\[ 3\frac{\ddot{a}^2}{a^2} - \sum_r \left( \frac{r \dot{\Phi}_r^2}{2} + \frac{m_r^2 \Phi_r^2}{2} - \frac{\alpha_r \Phi_r^4}{4} \right) - 8\pi \varepsilon_p = \Lambda, \] (164)
where \( \Lambda \) is renormalized cosmological constant (153).

We can obtain an independent Einstein equation [30] from (163) and (164),
\[ \dot{H} + 4\pi (\varepsilon + p) = 0, \] (165)
where \( \varepsilon = \varepsilon_p + \varepsilon_s \) and \( p = p_p + p_s \). Next, following [30], we introduce the total energy \( E \) of cosmological matter as
\[ E = \frac{1}{8\pi} (3H^2 - \Lambda_0) - \varepsilon, \] (166)
using which Einstein equation (164) can be written in the simple form
\[ E = 0, \] (167)
which reflects the fact that the total energy of a spatially flat Friedmann Universe is equal to zero.

Differentiating total energy (166) with respect to time and taking field equations (160) and (161) and relations (144), (157), (159), and (165) into account, we obtain the energy conservation law
\[ \frac{d}{dt} E = 0 \quad \Rightarrow \quad E = E_0. \] (168)
Thus, a consequence of the considered system of dynamical equations (144), (160), (161), (165) is the conservation law for the total energy of the cosmological system (168), \( E = E_0 \). Here, Einstein equation (167)
is the partial integral of this system, $E_0 = 0$. A similar situation arises for vacuum scalar fields [30]. This means that the first integral (167) can be regarded as an initial condition in the Cauchy problem for the cosmological model.

In particular, for a degenerate Fermi system, taking (141) into account, we find the following equation from (165):

$$\dot{H} + \frac{\dot{\Phi}^2}{2} - \frac{\dot{\phi}^2}{2} + \frac{4}{3\pi} \sum_a m_a^2 \psi_a^3 \sqrt{1 + \psi_a^2} = 0. \quad (169)$$

**Statement 7.** System of equations (144) with its solution given by (134), (160), (161), and (169) together with definitions (131)–(133) and integral condition (164) describes a closed mathematical model of the cosmological evolution of a completely degenerate Fermi system coupled to an asymmetric Higgs scalar doublet.

Following [30], we also introduce the effective energy of the cosmological system, which is nonnegative according to (166) and (167),

$$E_{\text{eff}} = \varepsilon + \frac{\Lambda_0}{8\pi} \geq 0 \Rightarrow E_{\text{eff}} = \frac{\dot{\Phi}^2}{2} - \frac{\dot{\phi}^2}{2} + V(\Phi) + V(\varphi) + \frac{\Lambda_0}{8\pi} \geq 0, \quad (170)$$

and an explicit expression for invariant cosmological acceleration (2) (see, e.g., [30])

$$\Omega = 1 + \frac{\dot{H}}{H^2}. \quad (171)$$

**5.3. Autonomous normal system of equations.** Choosing the variable $\xi(t)$ instead of the non-negative dynamical variable $a(t) \geq 0$,

$$\xi = \ln a, \quad \xi \in (-\infty, +\infty), \quad (172)$$

$$\dot{\xi} = H, \quad (173)$$

we use integral (149) of the conservation law for plasma (146) to find the function $\psi_{(a)}(t)$ in (134):

$$\psi_{(a)} = \frac{\pi_0^a e^{-\xi}}{|m(a)|}, \quad \pi_0^a = \pi_{(a)}(0). \quad (174)$$

With (12), we then obtain the function $\psi(t)$ in the form

$$\psi_{(a)} = \frac{\pi_0^a}{|\sum_r q_{(a)}^r \Phi_r|} e^{-\xi} \quad (175)$$

and scalar charge density (133) in the form

$$\sigma^r = \frac{1}{2\pi^2} \sum_a q_{(a)}^r F_1(\psi_{(a)}) \left( \sum_{r'} q_{(a)}^{r'} \Phi_{r'} \right)^3. \quad (176)$$

Further, setting

$$\Phi = Z, \quad \dot{\phi} = z \quad (177)$$
in the model of an asymmetric scalar doublet, we use this notation to write the dynamical equations that comprise a normal autonomous system of six ordinary differential equations (173), (177)–(180) for the dynamical variables $\xi, H, \Phi, Z, \varphi,$ and $z$:

\begin{align*}
\dot{H} &= -\frac{Z^2}{2} + \frac{z^2}{2} - \frac{4}{3\pi} \sum_a m^4(a) \psi^2(\phi(a)) \sqrt{1 + \psi^2(\phi(a))} = 0, \\
\dot{Z} &= -3\frac{\dot{a}}{a} Z - m^2 \Phi + \alpha \Phi^4 - 8\pi \sigma e, \\
\dot{z} &= -3\frac{\dot{a}}{a} z + m^2 \varphi - \beta \varphi^3 + 8\pi \sigma f,
\end{align*}

where the expression for $\sigma e$ is obtained from (176) by summation in the outer sum over the classical charges $q^c(a),$ and the expression for $\sigma f$ is obtained by summation over phantom charges $q^f(a).$

In these variables, the first integral (164) takes the form

\[ \Sigma_E: 3H^2 - \Lambda - \frac{Z^2}{2} + \frac{z^2}{2} + \frac{m^2}{2} \Phi^2 - \frac{\alpha \Phi^4}{4} + \frac{m^2}{2} \varphi^2 - \frac{\beta \varphi^4}{4} - \frac{1}{\pi} \sum_a m^4(a) F_2(\psi(a)) = 0. \]

Equation (181) is an algebraic equation for the dynamical variables $\xi, H, \Phi, Z, \varphi,$ and $z$ and describes a certain 5-dimensional hypersurface $\Sigma_E$ in a 6-dimensional arithmetic phase space of the dynamical system $\mathbb{R}_6 = \{\xi, H, \Phi, Z, \varphi, z\}.$ Following [30], this hypersurface is called the Einstein hypersurface. All phase trajectories of dynamical system (173), (177)–(180), like the initial points, must lie on the Einstein hypersurface. Because (181) is a first integral of the dynamical system, to solve the Cauchy problem it suffices to require that the initial point of the dynamical trajectory belong to the Einstein hypersurface (see [30], [31]). Thus, the following statement is true.

**Statement 8.** The dynamical system of the cosmological model based on a locally equilibrium system of scalar-charged degenerate fermions is completely described by the normal autonomous system of six ordinary differential equations (173), (177)–(180) with a single algebraic condition (181), which is the partial value of the first integral of this system. All trajectories of the dynamical system in the phase space $\mathbb{R}_6 = \{\xi, H, \Phi, Z, \varphi, z\}$ lie on the Einstein hypersurface $\Sigma_E$ (181). Due to the autonomy of the dynamical system, it is invariant under time translations

\[ t \to t + t_0, \]

which, in particular, always allows choosing $t_0$ such that

\[ a(0) = 1 \Rightarrow \xi(0) = 0. \]

**5.4. Two simple models of a fermion system.** We now discuss two fundamentally different simplest models of the statistical system of scalar-charged fermions with an asymmetric scalar doublet.

**Model 1.** A two-component statistical system of degenerate fermions where the particles of one component have the classical charge $e$ and the particles of the other component have a phantom scalar charge $\epsilon.$ In this case, two terms remain in the right-hand sides of formulas (113)–(116). We let $m_e$ and $m_\epsilon$ be the masses of the fermions (carriers of the corresponding charges):

\[ m_e = e\Phi, \quad m_\epsilon = \epsilon \varphi. \]
Model 2. One-component statistical system of fermions, scalar-charged with two charges \( \{e, \epsilon\} \) under conditions of complete degeneration. In this case, only one term remains in the right-hand sides of formulas (113)–(116). For the mass of the charge in this case, we have the formula

\[
m = e\Phi + e\varphi.
\]

(185)

The difference between models 1 and 2 consists, first, in the difference between formulas for dynamical masses: the first model has two dynamical masses \( m_c = e\Phi \) and \( m_l = e\varphi \), while in the second, there is only one mass \( m = e\Phi + e\varphi \). Second, in the second model there are two Fermi momenta \( \pi_c \) and \( \pi_l \) and hence two \( \psi \) functions: \( \psi_c = \pi_c/m_c(\Phi) \) and \( \psi_l = \pi_l/m_l(\varphi) \). Finally, in the corresponding formulas for macroscopic scalars, summation must be performed over the two components of the statistical system.

Further, in the case of model 1, the two fermion components in Eq. (169) must be summed, and in the case of model 2, a single term must be kept in the summand.

6. The analysis of model 1 with a two-component Fermi system of single-scalar-charged fermions

6.1. The dynamical system for model 1. In this case, formulas (175) and (176) take the form

\[
\psi_c = \frac{\pi_c^0}{|e\Phi|} e^{-\xi}, \quad \psi_l = \frac{\pi_l^0}{|e\varphi|} e^{-\xi},
\]

(186)

\[
\sigma_c = \frac{e^4}{2\pi^2} \Phi^3 F_1(\psi_c), \quad \sigma_l = \frac{e^4}{2\pi^2} \varphi^3 F_1(\psi_l).
\]

(187)

Further, Eqs. (160), (161) for the scalar doublet fields, can be written with the help of (177), (187) in the form

\[
\dot{Z} = -3HZ - m^2 \Phi + \Phi^3 \left( \alpha - \frac{4e^4}{\pi} F_1(\psi_c) \right),
\]

(188)

\[
\dot{z} = -3Hz + m^2 \varphi - \varphi^3 \left( \beta - \frac{4e^4}{\pi} F_1(\psi_l) \right).
\]

(189)

Einstein equation (169) and the first integral of the system of equations (181) for model 1 takes the form

\[
\dot{H} = -\frac{Z^2}{2} + \frac{z^2}{2} - \frac{4}{3\pi} e^4 \Phi^4 \psi_c^3 \sqrt{1 + \psi_c^2} - \frac{4}{3\pi} e^4 \varphi^4 \psi_l^3 \sqrt{1 + \psi_l^2},
\]

(190)

\[
3H^2 - \Lambda = \frac{Z^2}{2} + \frac{z^2}{2} - \frac{m^2 \Phi^2}{2} + \frac{\alpha \Phi^4}{4} - \frac{m^2 \varphi^2}{2} + \frac{\beta \varphi^4}{4} - \frac{e^4 \Phi^4}{\pi} F_2(\psi_c) - \frac{e^4 \varphi^4}{\pi} F_2(\psi_l) = 0.
\]

(191)

Further, the points of the phase space \( \mathbb{R}_4 \) where effective energy (170) is negative are inaccessible to the dynamical system. The inaccessible region is separated from the accessible region of the phase space by the zero effective energy hypersurface \( S_E \subset \mathbb{R}_6 \), which is a cylinder with the \( OH \) axis:

\[
S_E: \Lambda + \frac{e^4 \Phi^4}{\pi} F_2(\psi_c) + \frac{e^4 \varphi^4}{\pi} F_2(\psi_l) + \frac{Z^2}{2} - \frac{z^2}{2} - \frac{m^2 \Phi^2}{2} - \frac{\alpha \Phi^4}{4} + \frac{m^2 \varphi^2}{2} - \frac{\beta \varphi^4}{4} = 0,
\]

(192)

and the hypersurface of null effective energy (192) touches Einstein hypersurface (191) on the hyperplane \( H = 0 \):

\[
\Sigma_E \cap S_E = H = 0.
\]

(193)

As can be seen from Eq. (190), in the case of a scalar-neutral statistical system \( (e = \epsilon \equiv 0) \), the sign of the derivative of the Hubble constant is completely determined by the difference \( z^2 - Z^2 \): we always have
\( \dot{H} < 0 \) in the case of the dominance of the classical scalar field \( (Z^2 > z^2) \), and \( \dot{H} > 0 \) in the case of the dominance of the phantom scalar field \( (Z^2 < z^2) \). A variation of these factors in the course of cosmological evolution can lead to a wide variety of types of behavior of the cosmological model [31]. In the presence of charged matter, its contribution to this variation, as seen from (190), decreases the Hubble constant.

We note that instead of Eq. (190), we can consider an equivalent equation (see [30]), by substituting the expression for \( z^2/2 - Z^2/2 \) from (191) in (190):

\[
\dot{H} = -3H^2 + \Lambda + \frac{e^4 \Phi^4}{\pi} F_2(\psi_e) + \frac{e^4 \varphi^4}{\pi} F_2(\psi_i) - \frac{m^2 \Phi^2}{2} - \frac{m^2 \varphi^2}{2} + \frac{\alpha \Phi^4}{4} + \frac{\beta \varphi^4}{4} + \frac{4}{3\pi} e^4 \Phi^4 \psi_1^3 \sqrt{1 + \psi_e^2} + \frac{4}{3\pi} e^4 \varphi^4 \psi_1^3 \sqrt{1 + \psi_i^2}. \tag{194}
\]

Singular points of the dynamical system represented by a normal autonomous system of differential equations are determined by algebraic equations obtained by equating the derivatives of all dynamical variables to zero. Thus, we obtain the system of algebraic equations for the coordinates of singular points from (173), (177), (188)–(190):

\[
Z = 0, \quad z = 0, \tag{195}
\]

\[
H = 0, \tag{196}
\]

\[
\Phi^3 \left( \alpha - \frac{4e^4}{\pi} F_1(\psi_e) \right) - m^2 \Phi = 0, \tag{197}
\]

\[
\varphi^3 \left( \beta - \frac{4e^4}{\pi} F_1(\psi_i) \right) - m^2 \varphi = 0, \tag{198}
\]

\[
e^4 \Phi^4 \psi_1^3 \sqrt{1 + \psi_e^2} + e^4 \varphi^4 \psi_1^3 \sqrt{1 + \psi_i^2} = 0. \tag{199}
\]

In addition, we recall the total energy integral in Eq. (191): the singular point coordinates must satisfy this equation, which, in view of (195), takes the form

\[
\Lambda + \frac{e^4 \Phi^4}{\pi} F_2(\psi_e) + \frac{m^2 \Phi^2}{2} - \frac{\alpha \Phi^4}{4} + \frac{e^4 \varphi^4}{\pi} F_2(\psi_i) + \frac{m^2 \varphi^2}{2} - \frac{\beta \varphi^4}{4} = 0. \tag{200}
\]

We first consider Eq. (199). Using the definition of the functions \( \psi_{(a)} \) in (186), we rewrite this equation in the equivalent form

\[
e^{-3\xi} \left( |\epsilon \Phi| (\pi_e^0)^3 \sqrt{1 + \psi_e^2} + |\epsilon \varphi| (\pi_i^0)^3 \sqrt{1 + \psi_i^2} \right) = 0. \tag{201}
\]

It follows from (201) that either \( \Phi = \varphi = 0 \) or \( \xi = +\infty \) (where \( \psi_{(a)} = 0 \)), \( F_1(\psi_{(a)}) = 0 \), and \( F_2(\psi_{(a)}) = 0 \).

The first of these possibilities, with \( \Lambda = 0 \), gives a line of singular points

\[
M_\xi = (\xi, 0, 0, 0, 0, 0), \tag{202}
\]

because the zero solution for the potentials of scalar fields converts Eqs. (197) and (198) into identities, while the condition \( \Lambda = 0 \) is becomes an identity by Eq. (200).

In the case of the second possibility, Eq. (199) becomes an identity and Eqs. (197) and (198), except in the case of the trivial solution \( \Phi = \varphi = 0 \) studied above, leading to a straight line of singular points and the condition \( \Lambda = 0 \) have 8 more solutions at special values of the cosmological constant \( \Lambda_0 \):

\[
M_{\infty, \pm, \pm} = \left( +\infty, 0, 0, \pm \frac{m}{\sqrt{\alpha}}, \pm \frac{m}{\sqrt{\beta}}, 0 \right), \quad \Lambda_0 = 0,
\]

\[
M_{\infty, \pm, 0} = \left( +\infty, 0, 0, \pm \frac{m}{\sqrt{\alpha}}, 0, 0 \right), \quad \Lambda_0 = \frac{m^4}{4\beta}, \tag{204}
\]

\[
M_{\infty, 0, \pm} = \left( +\infty, 0, 0, 0, 0, \pm \frac{m}{\sqrt{\beta}} \right), \quad \Lambda_0 = \frac{m^4}{4\alpha}. \tag{205}
\]
with the ± signs in two terms in (203) taking independent values. We note that the value \( \xi = +\infty \)
corresponds to \( a = \infty \), i.e., the infinite future, and the value \( \xi = -\infty \) corresponds to \( a = 0 \), i.e., the
cosmological singularity.

**Statement 9.** Dynamical system (173), (177), (188)–(191) has singular points only for special values of the
cosmological constant \( \Lambda_0 \):

1) for \( \Lambda_0 = m^4/(4\alpha) + m^4/(4\beta) \) (\( \Lambda = 0 \)), straight line of singular points \( M_\xi \) (202) corresponding to zero
scalar potentials;

2) for \( \Lambda_0 = 0 \) (\( \Lambda = -m^4/(4\alpha) - m^4/(4\beta) \)), four singular points in the infinite future \( M_{\pm,\pm} \) (203) with
nonzero scalar potentials;

3) for \( \Lambda_0 = m^4/(4\beta) \) (\( \Lambda = -m^4/(4\alpha) \)), two singular points in the infinite future \( M_{\pm,0} \) (204) with a nonzero
potential of the phantom field;

4) for \( \Lambda_0 = m^4/(4\alpha) \) (\( \Lambda = -m^4/(4\beta) \)), two singular points in the infinite future \( M_{0,\pm} \) (205) with
a nonzero potential of the classical field.

Thus, for an arbitrary value of the cosmological constant, dynamical system (173), (177), (188)–(190)
has no singular points in general; for special values of the cosmological constant, it can have a straight line
of singular points, either 4 or 2 singular points in the infinite future.

We turn to the study of the nature of the singular points in those cases where they exist. Carrying
out standard calculations of the qualitative theory of dynamical systems (see, e.g., [69]), we arrive at the
following conclusion.

**Statement 10.** The eigenvalues of the matrix of the dynamical system on the line of singular points \( M_\xi \)
can only be sign-alternating, either real or imaginary. Saddle singular points correspond to real sign-
alternating eigenvalues, while attracting centers correspond to imaginary sign-alternating ones. This shows
that in the two-dimensional directions \( \{ \xi, H \} \) and \( \{ \Phi, Z \} \) (classical subspace), the phase trajectories rotate
around the singular points \( M_\xi \). The radius of the phase trajectory in the \( \{ \xi, H \} \) projection decreases with
time and tends to zero as \( \xi \to +\infty \). The radius of the phase trajectory in the \( \{ \Phi, Z \} \) projection remains
constant. In the two-dimensional direction \( \{ \varphi, z \} \) (phantom subspace), the singular point is a saddle point.

At all singular points at infinity \( \xi \to +\infty \) in the two-dimensional direction \( \{ \xi, H \} \), the eigenvalues of the
matrix of the dynamical system are zero. Moreover, at the singular points of \( M_{\infty,\pm,\pm} \), the matrix of the
dynamical system has only zero eigenvalues. This means that the solution of dynamical equations tends to
the coordinates of this point, i.e., is an asymptotically exact solution:

\[
\begin{align*}
\xi &\to +\infty, \quad H \to 0, \quad \Phi \to \pm \frac{m}{\sqrt{\alpha}}, \\
Z &\to 0, \quad \varphi \to \pm \frac{m}{\sqrt{\beta}}, \quad z \to 0 \quad (\Lambda_0 = 0).
\end{align*}
\]  

(206)

At the singular points of \( M_{\pm,0} \), the solution is unstable in the phantom plane, and it therefore makes no
sense to consider it. At the singular points of \( M_{0,\pm} \), the solution is stable, and therefore the cosmological
model can reach a phantom end in the infinite future:

\[
\begin{align*}
\xi &\to +\infty, \quad H \to 0, \quad \Phi \to 0, \quad Z \to 0, \quad \varphi \to \pm \frac{m}{\sqrt{\beta}}, \\
z &\to 0, \quad \Lambda_0 = \frac{m^4}{4\alpha}, \quad \Lambda = -\frac{m^4}{4\beta}.
\end{align*}
\]  

(207)
It is easy to see that both on the line of singular point (202) and at all singular points (203)–(205), the expression for $H^2$ in (191), containing the cosmological constant and the Higgs potential, vanishes. Thus, at all singular points of the infinite future, we have

$$\Omega_{|\xi\to+\infty} = -\frac{1}{2}, \quad [M_{\xi|\xi\to+\infty, M_{\infty, \pm, \pm, M_{\infty, 0, \pm}}].$$  

(208)

The following statement therefore holds.

**Statement 11.** All stable singular points in model 1 correspond to the effective total nonrelativistic equation of state in the infinite future.

### 7. Conclusion

The following results have been obtained in this paper.

1. Based on both the Lagrangian and Hamiltonian formalisms, the equations of motion are obtained for a relativistic scalar-charged particle in gravitational and scalar fields. An expression is obtained for the particle dynamical mass that satisfies the principles of additivity of the Lagrange function and charge conjugation. It is shown that these fundamental principles require the removal of the restriction on the nonnegativity of the dynamical mass of a scalar-charged particle. It is shown that when passing to the sector of negative values of the dynamical mass of a particle, its four-dimensional velocity vector changes its orientation, but the observable physical quantities—the total mass of the particle, its momentum vector, and the three-dimensional velocity vector—preserve their classical values.

2. A self-consistent system of Einstein equations and scalar field equations with microscopic singular densities of a system of scalar-charged particles is derived from the Lagrangian formalism. Expressions are obtained for the microscopic energy density of the scalar charge, the microscopic vector of the scalar current density, and the microscopic energy–momentum tensor of particles. The conservation laws for the scalar charge and the total microscopic energy–momentum tensor of a system of self-gravitating scalar-charged particles are obtained. The equations of motion for scalar-charged particles are obtained from the microscopic Einstein equations. The obtained formula for the dynamical mass is shown to provide the correct expression for the energy–momentum tensor of a system of particles.

3. On the 8-dimensional phase space of particles given by a fiber bundle, the distribution function of particles is determined that is singular on the mass shell in general; given the distribution function and a timelike field of observers, integration over the phase space defines the macroscopic averages of dynamical functions. The connection between the proper time of particles and the proper time of observers is expressed by the $\delta$-function, and the transition to the 7-dimensional phase space and the distribution function that is nonsingular on the mass shell is carried out by means of the $\delta$-function of Hamiltonian (42). As a result, the macroscopic densities—the scalar charge density, the current vector, and the energy–momentum tensor of particles—are correctly and invariantly determined. Self-consistent macroscopic equations for gravitational and scalar fields, and the charge and total energy–momentum conservation laws are obtained for the system. Thus, from the microscopic Lagrangian or Hamiltonian particle dynamics, a macroscopic description of the statistical system of scalar-charged particles is derived.

4. The local thermodynamic equilibrium of a statistical system of scalar-charged particles is investigated, formulas are obtained for the equilibrium values of macroscopic scalars—the energy density of particles, pressure, particle the number density, and the scalar charge density. The transformation properties of these scalars under charge conjugation and the reflection of scalar fields are studied. It is shown that due to the invariant properties of these transformations, the signs of the scalar fields of the statistical system are independent of the signs of the scalar charges of the particles, but are determined only by
the initial conditions. On the basis of the transport equations, which are a rigorous consequence of the relativistic kinetic equations, the constitutive equations are obtained (the equations of hydrodynamics and the conservation of the current of scalar charges). Expressions for all macroscopic scalars of the completely degenerate system of fermions are found in elementary functions.

5. Based on the obtained macroscopic equations for the degenerate system of scalar-charged fermions, the cosmological model of a spatially flat Universe is investigated, for which a complete system of \( n + N + 2 \) ordinary differential equations is formulated (\( n \) is the number of types of scalar-charged particles and \( N \) is the number of scalar fields). In addition, there is one integral condition for the solution of the system. An exact solution of these \( n \) constitutive equations is found, which expresses the Fermi momentum conservation law. As a result, for an asymmetric scalar doublet \( N = 2 \), a normal autonomous system of six ordinary differential equations for six dynamical functions follows. Two simple models of the Fermi system are proposed: in the first model, there are two sorts of fermions, one of which carries a classical charge and the other carries a phantom charge; in the other model, there are fermions of only one type, charged simultaneously by two charges.

6. A complete system of dynamical equations for model 1 is obtained and its qualitative analysis is carried out, the results of which can be briefly described as follows. For arbitrary values of the fundamental constants (the structure constants of the Higgs potentials and the cosmological constant), the dynamical system has no singular points. At special ratios between the fundamental constants, the model has singular points in the infinite future or straight lines of singular points. This property makes the considered cosmological models substantially different from simpler models with a vacuum scalar doublet (see [31]).

7. It is shown that all stable singular points in the infinite future \( t \to 0, a \to \infty, H \to 0 \) correspond to the full effective nonrelativistic equation of state \( \Omega = -1/2 \Rightarrow w = 0 \). Thus, coupling the scalar fields to degenerate fermions can lead to deceleration of the cosmological acceleration in the future history of the Universe. However, it must be emphasized that this conclusion is valid only with a special choice of the fundamental constants of the model.

Because the real Universe can hardly be assumed to correspond to the above specific sets of fundamental constants, it is unlikely that the real cosmological model has stable singular points. In this regard, one can hope that the consideration of models with scalar-charged particles would remove the conclusion that \( \Omega|_{t \to +\infty} \to 1 \), which is unacceptable for observational cosmology. Therefore, it is necessary to carry out numerical integration of the presented mathematical models in a wide range of fundamental constants. We intend to publish the results of such a study in the near future.

**Conflicts of interest.** The authors declare no conflicts of interest.

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