Mode-entanglement of Gaussian fermionic states

C. Spec, K. Schwaiger, G. Giedke, and B. Kraus

1 Naturwissenschaftlich-Technische Fakultät, Universität Siegen, 57068 Siegen, Germany
2 Institute for Theoretical Physics, University of Innsbruck, 6020 Innsbruck, Austria
3 Donostia International Physics Center, 20013 Bilbao, Spain
4 Ikerbasque Foundation for Science, 48013 Bilbao, Spain

Dated: 2017/12/21

We investigate the entanglement of n-mode n-partite Gaussian fermionic states (GFS). First, we identify a reasonable definition of separability for GFS and derive a standard form for mixed states, to which any state can be mapped via Gaussian local unitaries (GLU). As the standard form is unique two GFS are equivalent under GLU if and only if their standard forms coincide. Then, we investigate the important class of local operations assisted by classical communication (LOCC). These are central in entanglement theory as they allow to partially order the entanglement contained in states. We show, however, that there are no non-trivial Gaussian LOCC (GLOCC). That is, any GLOCC transformation can be accomplished via GLUs. To still obtain insights into the various entanglement properties of n-mode n-partite GFS we investigate the richer class of Gaussian stochastic LOCC. We characterize Gaussian SLOCC classes of pure states and derive them explicitly for few-mode states. Furthermore, we consider certain fermionic LOCC and show how to identify the maximally entangled set (MES) of pure n-mode n-partite GFS, i.e., the minimal set of states having the property that any other state can be obtained from one state inside this set via fermionic LOCC. We generalize these findings also to the pure m-mode n-partite (for m > n) case.

I. INTRODUCTION

Entanglement plays a crucial role in understanding the quantum physics of systems composed of many subsystems or many particles. It is the primary resource of many applications in quantum computation and communication, and is the basis of many of the intriguing effects of quantum many-body physics.

In multipartite systems there are various qualitatively different kinds of entanglement. Relating them to physical properties or to performable tasks, contributes to elucidating the role of entanglement in nature and as a resource for quantum technologies.

One very successful approach to identify different classes of entanglement is to consider whether states can be converted into each other using some naturally restricted set of quantum operations, defining states for which such conversion is mutually impossible to belong to distinct classes. This has lead to the discovery of inequivalent kinds of entanglement, and to their classification. Furthermore, the maximally entangled states and sets, which are the most relevant states regarding local state transformations, have been identified.

Most of these notions have been developed considering systems of distinguishable particles, and with system Hilbert spaces that have a natural tensor-product structure imposed by the spatial separation of subsystems. When applying them to systems of indistinguishable particles, central notions of entanglement theory have to be adapted to account for (anti)commutation relations and superselection rules, that restrict the set of allowed operations and modify the structure of “local” operations.

In the present article, we investigate the entanglement properties of multipartite fermionic states. There are both fundamental and practical reasons to do so: on the one hand, fermions are the fundamental constituents of matter, hence to understand the entanglement properties of quantum many-body systems the fermionic perspective is indispensable. This has motivated a broad effort to study fermionic entanglement and work out the differences with qubit systems, see, e.g., [12,23]. Even in quantum information, where bosonic or effectively distinguishable particles play the important role, genuinely fermionic systems such as single semiconductor electrons or holes in quantum dots, or ballistic electrons in quantum wires or edge channels or Majorana fermions in quantum wires are of increasing interest. On the other hand, this analysis gives new insights into the nature of entanglement in general and the comparison of fermionic, bosonic, and distinguishable systems affords a clearer picture of the role of statistics.

Here we apply this state-conversion-based entanglement classification to multipartite Gaussian fermionic states. This important family of states contains the eigen- and thermal states of quadratic Hamiltonians, i.e., those describing quasi-free single-particle dynamics. Despite their simplicity, these states comprise a large range of different kinds of entangled states, including GHZ-like states, spin-squeezed states, paired states, and topological states, thus serving as a convenient test-bed for entanglement studies and can be used for basic quantum information processing tasks such as probabilistic teleportation, entanglement distillation, or metrology, while for universal quantum computation, the Gaussian states and operations have to be augmented by a non-Gaussian measurement. In the present work, we focus first on pure n-partite states with a single mode per party, and investigate their transformation properties under different kinds of local fermionic operations.
operations. Then we generalize some of the results to m-mode n-partite (for $m > n$) states.

The outline of the remainder of the paper is the following. In Sec. IV we recall the definition and some properties of fermionic states (FS), Gaussian fermionic states (GFS) and Gaussian operations. Moreover, we recall the mapping between GFS and spin states using the Jordan-Wigner transformation. In Sec. V we consider mixed GFS and first recall the various definitions of separability for FS [15]. We identify a reasonable definition of separability of GFS. Then, we introduce a standard form for the CM, which is invariant under GLU. In the last two sections, Sec. VI and Sec. VII, we investigate the entanglement properties of pure GFS considering GLOCC. As this class of operations turns out to be trivial for $n$-mode $n$-partite as well as multipartite multimode fully entangled GFS we study also GSLOCC and FLOCC to still obtain insights into the entanglement of GFS. In particular, the following results are presented: (i) We characterize the separable Gaussian fermionic states and different kinds of local Gaussian fermionic operations (GLOCC, GSLOCC, GSEP); (ii) we derive a standard form for $n$-mode, $n$-partite GFS into which any such state can be transformed by GLU; as this standard form is unique, two GFS are GLU-equivalent if their standard forms coincide; (iii) we show that there are no non-trivial Gaussian fermionic LOCC transformations between fully entangled pure $n$-mode GFS; (iv) we characterize the Gaussian SLOCC classes for pure $n$-mode, $n$-partite GFS; (v) we consider general fermionic LOCC between Gaussian states and identify the corresponding maximally entangled set (MES), and, finally, (vi) we generalize these findings to the $m$-mode $n$-partite ($m > n$) case.

II. PRELIMINARIES

We summarize here some results concerning GFS and introduce our notation. We consider systems composed of $n$ fermionic modes. To each mode $k = 1, \ldots, n$ belongs a creation and an annihilation operator $b_k, b_k^\dagger$, obeying the anticommutation relations $\{b_k^\dagger, b_l^\dagger\} = \{b_k, b_l\} = 0, \{b_k, b_l^\dagger\} = \delta_{kl}$. The antisymmetric Fock space over $n$ modes is spanned by the Fock basis defined as

$$|k_1, \ldots, k_n\rangle = (b_1^\dagger)^{k_1} \cdots (b_n^\dagger)^{k_n} |0\rangle,$$

where $k_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$ and the vacuum state $|0\rangle$ obeys $b_i |0\rangle = 0 \forall i$. Note that $|k_1, \ldots, k_n\rangle$ is an eigenstate of all number operators $n_i = b_i^\dagger b_i$ to eigenvalue $k_i$.

It is sometimes more convenient to consider the $2n$ hermitian fermionic Majorana operators,

$$\tilde{c}_{2k-1} = b_k + b_k^\dagger, \quad \tilde{c}_{2k} = -i(b_k - b_k^\dagger)$$

instead of the creation and annihilation operators. The anticommutation relations are then equivalent to

$$\{\tilde{c}_k, \tilde{c}_l\} = 2\delta_{kl}.$$

For any Clifford algebra satisfying the relation above, the operators $b_k = \frac{1}{2}(\tilde{c}_{2k-1} + \i \tilde{c}_{2k})$ obey the anticommutation relation and vice versa.

A linear transformation of the fermionic operators $\{\tilde{c}_k\}$, i.e., $\tilde{c}_k \to \tilde{c}'_k = \sum_l O_{kl} \tilde{c}_l$, preserves the canonical anticommutation relations iff $O \in O(2n, \mathbb{R})$, i.e., iff $O$ is a real orthogonal matrix. These are called canonical transformations or Bogoliubov transformations. They realize a basis change in the fermionic phase space and can be implemented by Gaussian operations (see below).

A. Gaussian States

A GFS of $n$ modes is defined as the thermal (Gibbs) state of a quadratic Hamiltonian, $H = \frac{1}{2} \tilde{G} \tilde{c}^\dagger \tilde{c}$ with $\tilde{G}$ a real antisymmetric $2n \times 2n$ matrix and $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{2n})$, i.e.,

$$\rho = Ke^{-\frac{1}{2} \tilde{G} \tilde{c}^\dagger \tilde{c}},$$

where $K$ denotes a normalization constant (or, to include states of non-maximal rank, can be expressed as a limit of such expressions). Equivalently, they can be characterized as those states satisfying Wick’s theorem, i.e., for which all cumulants vanish [33] [37].

It is well known that any real antisymmetric $2n \times 2n$ matrix can be transformed into a normal form via a real special orthogonal matrix [38]. More precisely, there exists a matrix $O \in O(2n, \mathbb{R})$ such that

$$O \tilde{G} O^T = n_{\delta_{k=1}}^2 \beta_k J_2,$$

where $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\beta_k \in \mathbb{R}$ (5)

Hence, a GFS is a state of the form

$$\rho = \tilde{\otimes}_{k=1}^n \rho_k,$$

where $\rho_k = \frac{1}{2} \left( \mathbb{1} - \mu_k b_k^\dagger b_k^\dagger \right)$ for $\mu_k = \tanh(\beta_k/2)$.

Note that here and in the following $\tilde{\otimes}$ denotes a product of operators which are acting only on distinct sets of modes. However, we only use this notation if the operators fulfill a commutation relation. Here, the operators $b_k^\dagger = \sum_i u_{ki} b_i$ obey again the anticommutation relations, i.e., they are fermionic annihilation operators [38]. Thus, $\rho$ can be written as

$$\rho = \frac{1}{N} e^{-\sum_k \beta_k b_k^\dagger b_k},$$

where $N = \prod_k (1 + e^{-\beta_k})$ denotes a normalization constant. It is evident that a Gaussian state is completely determined by its second moments, which are usually collected in the covariance matrix (CM). In terms of the Majorana operators the CM of a GFS, $\rho$, which we denote by $\gamma$, is defined as

$$\gamma_{kl} = -\frac{1}{2i} \text{tr}(\rho [\tilde{c}_k, \tilde{c}_l]).$$
As can be easily seen from this definition, the CM is an antisymmetric $2n \times 2n$ real matrix, which can be transformed by a real special orthogonal matrix, $O$, into the normal form
\[ O \gamma O^T = \oplus_{k=1}^{n} ( -\mu_k J_2 ). \]

Note that $\mu_k = \tanh(\beta_k/2)$ [29]. This normal form is referred to as the (fermionic) Williamson normal form [38]. Note that in contrast to the case of bosons, no first moments have to be specified for fermions since due to the parity superselection rule all physical states have $\text{tr}(x_\rho) = 0$. Thus, GFS are completely characterized by their second moments, i.e., their CM, due to Wick’s theorem [39].

\[ i^p \text{tr}(\rho c_{j_1} \cdots c_{j_2p}) = \text{Pf}(\gamma_{j_1,\ldots,j_{2p}}), \]

where $1 \leq j_1 < \cdots < j_{2p} \leq 2n$ and $\gamma_{j_1,\ldots,j_{2p}}$ is the $2p \times 2p$ submatrix of $\gamma$ with rows and columns $j_1,\ldots,j_{2p}$. Here Pf denotes the Pfaffian which for a $2n \times 2n$ matrix $A = (a_{ij})$ is defined as $\text{Pf}(A) = \frac{1}{n!} \sum_{\pi \in S_{2n}} \text{sgn}(\pi) \prod_{i=1}^{n} a_{\pi(2i-1),2}\pi(2i))$, where the sum is over all permutations $\pi$ and $\text{sgn}(\pi)$ the signature of $\pi$ and satisfies $\text{Pf}(A)^2 = \det(A)$.

Note that an antisymmetric real matrix $\gamma$ corresponds to the CM of a GFS, in particular to a normalized positive semidefinite operator, iff $\gamma^2 \preceq -1$, i.e., iff all the eigenvalues of $\gamma$, which are all purely imaginary, have modulus smaller or equal to one. The CM corresponds to a pure state given in Eq. (12) the $\text{CM}$ corresponds to a pure state of a GFS, in particular to a normalized positive semidefinite operator, iff $\text{Pf}(A)^2 = \det(A)$.

Note, however, that the parties are ordered and one cannot simply reorder them, as the order is fixed due to the commutation relations. To give an example, the state $|0\rangle_2 + |1\rangle_2 = |0\rangle_1 - |1\rangle_1$, where the minus sign results from permuting particle one and two. To be more precise, the operation which has to be performed on the qubit state in order to swap two systems is the fermionic swap, which is the mapping $|ij\rangle \rightarrow (-1)^{ij} |ji\rangle$. In order to perform, for instance, a partial trace, also these commutation relations have to be taken into account. That is, first the party over which the trace is performed has to be swapped (with a fermionic swap) to the last position [40]. After that, the partial trace can be performed as usual. Fermionic mixed states are then convex combinations of fermionic pure states.

Note that the parity conservation implies that a FS is always a direct sum of states whose support is only in the subspace with even parity and states whose support is only in the subspace with odd parity. Here, the subspace with even (odd) parity coincides with the set of states which are a superposition of Fock states which have all an even (odd) number of 1’s, respectively. Denoting by $P_e (P_o)$ the projector onto the even (odd) subspace we hence have that a state with density matrix $\rho$ is fermionic iff $\rho = P_e \rho P_e + P_o \rho P_o$, i.e., iff $P_e \rho P_o = P_o \rho P_e = 0$ [41].

Especially in case one is working with this representation it is important to be able to identify which of the FS are Gaussian. Fortunately, given a FS, the following result can be used to decide whether it is Gaussian or not. Recall that any operator in the Clifford algebra generated by the Majorana operators $c_i$ ($i = 1,\ldots,2n$) can be written as

\[ x = \alpha \mathbb{I} + \sum_{p=1}^{2n} \sum_{1 \leq a_1 < \cdots < a_p \leq 2n} \alpha_{a_1,\ldots,a_p} c_{a_1} \cdots c_{a_p}. \]

An operator is called even if it involves only even powers of the generators, or stated differently and using the Jordan-Wigner transformation, if the number of $X$’s plus the number of $Y$’s occurring in the sum is even. As any odd operator changes the parity, it is easy to see that $x$ is even iff $P_e x P_e = P_e x P_e = 0$. Thus, in particular, all FS have even density matrices.

It has been shown in [42] that an even operator, $x$, is Gaussian iff

\[ [\Lambda, x \otimes x] = 0, \]

The Jordan-Wigner transformation is a unitary mapping between the antisymmetric Fock space of $n$ modes and the Hilbert space of $n$ qubits, relating Fermi operators $\tilde{c}_i$ with qubit operators in Eq. (11) and the states in Eq. (12) with the ones in Eq. (13).
where
\[ \Lambda = \sum_{i=1}^{2n} c_i \otimes c_i. \]  
(16)

Thus, we have that a FS, \( \rho \), is Gaussian iff
\[ [\Lambda, \rho^\otimes 2] = 0. \]  
(17)

Let us give some examples. For a single mode, a state is fermionic if its density matrix is diagonal in the computational basis. Any such state is also Gaussian. For two modes, any FS is of the form \( \rho = \rho_e \oplus \rho_o \) where \( \rho_e \) (\( \rho_o \)) are density operators in the two-dimensional even (odd) parity subspace spanned by \( \{|00\rangle, |11\rangle\} \) \( \{|01\rangle, |10\rangle\} \) respectively. It can be easily seen that such a state is then Gaussian, i.e., fulfills the condition given in Eq. (17) iff \( |\rho_e| = |\rho_o| \), where \( | \cdot | \) denotes the determinant. An example of such a state would be \( e^{i\alpha(b_1b_2^\dagger)} \), where \( \rho_e = \left( \begin{array} {cc} \cosh(\alpha) & -i\sinh(\alpha) \\ -i\sinh(\alpha) & \cosh(\alpha) \end{array} \right) \) and \( \rho_o = \mathbb{1} \). In particular, all pure two-mode FS are Gaussian. However, not all mixed two-mode FS are: Examples of non-Gaussian FS are the Werner states, \( \rho_W = \frac{1}{F^2-1} |\psi^-\rangle \langle \psi^-| + \frac{1-F}{F^2-1} \mathbb{1}, \) for \( F \in (1/4, 1) \). Moreover, as we will see later, any pure FS of three modes is Gaussian. However, this is not the case for four modes.

When discussing pure GFS we either consider the Jordan-Wigner representation of the FS, or the CM of the state.

C. Gaussian operations

Let us now briefly recall the definitions and properties of Gaussian unitary operations, general Gaussian operators and Gaussian maps in the fermionic case. First note that all quantum operations (completely positive maps) that respect parity are considered as valid physical operations here, and referred to as fermionic operations.

Gaussian operations are those that can be realized with Gaussian means: evolution under quadratic Hamiltonians, adjoining of systems in Gaussian states, discarding of subsystems, measuring Gaussian POVMs, and projecting on pure Gaussian states. A Gaussian fermionic unitary, \( U \), acting on \( n \) modes can be written as \( U = e^{-iH} \), where \( H \) is quadratic in the Majorana operators, that is,
\[ H = i \sum_{kl} h_{kl} \tilde{c}_k \tilde{c}_l, \]  
(18)

with \( h \) being a real antisymmetric \( 2n \times 2n \) matrix [44]. In it was shown that these unitaries effect a canonical transformation of the Majorana operators
\[ U^\dagger \tilde{c}_k U = \sum_{k=1}^{2n} O_{jk} \tilde{c}_k, \]  
(19)

where \( O = e^{ih} \in \text{SO}(2n) \) is a real special orthogonal \( 2n \times 2n \) matrix. Hence, a fermionic Gaussian unitary maps the CM to \( O \gamma O^T \), where \( O \in \text{SO}(2n) \) [45]. All Gaussian unitaries preserve the parity, i.e., they commute with the parity operator \( P = (-1)^{\sum_k \nu_k} \). However, the parity-flipping transformation of mode \( k \), which corresponds to an (non-special) orthogonal transformation \( O = \oplus_{i=1}^k \mathbb{1} \oplus Z \oplus_{i=k+1}^n \mathbb{1} \oplus Z \) on the Majorana operators of the system [10] (here and in the following \( X,Y,Z \) denote the Pauli operators) also has a (local) physical realization. The transformation can be achieved for example by adjoining an ancillary mode in a Fock state and then acting on the Majorana operators of the system modes and the ancillary mode with the \( \text{SO}(2n+2) \) operation \( O = \oplus_{i=1}^k \mathbb{1} \oplus Z \oplus_{i=k+1}^n \mathbb{1} \oplus Z \) [17]. This exchanges particles with holes both in mode \( k \) and in the ancilla and leaves the latter unentangled, i.e., after discarding the ancilla it realizes \( Z \) on mode \( k \). Since for any \( O \in \text{SO}(2n) \) there exists a \( O' \in \text{SO}(2n) \) such that \( O = (\oplus_{i=1}^n \mathbb{1} \oplus Z)O' \) we can allow for all orthogonal operations. That adjoining local ancillas enlarges the set of implementable unitaries is in contrast to the Gaussian bosonic states and also to systems consisting of qudits. Hence, the most general operation on a single mode can be written as \( \tilde{O} = Z^mO, \) where \( m \in \{0,1\}, \) i.e., an arbitrary real orthogonal matrix. Clearly these operations no longer correspond to unitaries which are generated by quadratic Hamiltonians on the system modes alone [see Eq. (19)]. However, as they can be implemented using a quadratic Hamiltonian and ancillas in a Gaussian state we consider them as GLUs and take them into account in the following. If it is, however, the case that a particle-number superselection rule would forbid these kind of transformations, it would be straightforward to slightly modify the results derived here to exclude any operation which is not of the form given in Eq. (19).

Let us also note here that the action of any Gaussian unitary in the Jordan-Wigner representation corresponds to a product of nearest neighbor match gates [43], which are unitaries of the form \( \tilde{U} = U_e \oplus U_o \), where both, \( U_e \) and \( U_o \) are \( 2 \times 2 \) unitary operators acting on the even and odd subspace, respectively; and moreover, \( |U_e| = |U_o| \).

A general Gaussian operator is any operator of the form \( x = e^{i \sum_{i,j} x_{ij} \tilde{c}_i \tilde{c}_j} \) for a complex antisymmetric matrix \( \chi \).

In the most general Gaussian maps have been characterized via the Choi-Jamiolkowski (CJ) isomorphism. Recall that a completely positive (CP) map is called Gaussian if it maps Gaussian states to Gaussian states. We reconsider in Appendix A the CJ isomorphism for GFS. It follows that a map \( \mathcal{E} \) mapping \( n \) to \( m \) modes is Gaussian iff the corresponding CJ state is Gaussian (see also [42]), i.e., if it is given by the CM \( E_\mathcal{E} = \left( \begin{array} {cc} A & B \\ -B^T & D \end{array} \right) \), with \( A,B,D \in \text{Mat}(2n \times 2n, 2n \times 2n, 2n \times 2n, 2n \times 2n) \), respectively (for more details see Appendix A and also [42]).

Note that the condition for Gaussian maps to map every Gaussian state to a Gaussian is very stringent. Consider
for instance the situation where one wants to transform the state $|00⟩ + |11⟩$ into a state $α|00⟩ + β|11⟩$. Note that these are 2-mode GFS in the Jordan-Wigner representation and such a transformation is always possible for 2-qubit states via LOCC. The local operations accomplishing this transformation, i.e., $A_1 = \text{diag}(α, β)$, $A_2 = \text{diag}(β, α)$, are Gaussian, however, the map

$$E(ρ) = A_1 ⊗ \mathbb{I} (ρ) A_1^† ⊗ \mathbb{I} + X A_2 ⊗ Χ (ρ) A_2^† X ⊗ X(20)$$

is non-Gaussian even though both terms in the sum are. A simple example of a GFS that is not mapped to a GFS by $E$ is the 2-mode GFS $ρ = ρ_o ⊕ ρ_e$, with

$$ρ_e = \begin{pmatrix} z_e + 1/4 & 0 \\ 0 & 1/4 - z_e \end{pmatrix}, \quad ρ_o = \begin{pmatrix} z_o + 1/4 & x_o \\ x_o & 1/4 - z_o \end{pmatrix}$$

for $z_e ≤ 1/4$, $\sqrt{x_e^2 + x_o^2} ≤ 1/4$ and $z_o^2 = x_o^2 + z_e^2$, $x_o \neq 0$. Note that $E$ is FLOCC, i.e., it is a local map which maps FS to FS (as it preserves parity), and would accomplish the desired transformation. Due to that, we consider in Sec. IV not only GLOCC, but also the richer class of FLOCC.

### III. Separability of Gaussian Fermionic States and Operations

Here we specialize the three definitions of separability of general FS presented in [15] to the case of GFS. We show that they do not all coincide even for Gaussian states and that one of them is not stable when considering multiple copies of a state. We show that one of the two remaining definitions of separability is also consistent with the desired property that any separable state can be generated by a local operation. Furthermore, we derive a standard form for mixed $n$-mode $n$-partite states into which any GFS can be transformed via GLU.

#### A. Mixed Gaussian Fermionic Separable States

The notion of entanglement is complicated for fermions (compared to bosons or qubits) due to superselection rules and anticommutation relations. The former enforces that all physical states have to commute with the parity operator but prevents that all states can be uniquely characterized by local measurements of “physical” observables, i.e., those commuting with the parity operator. The latter implies that observables acting on different sites (disjoint sets of modes) do not, in general, commute.

In [15] several notions of product state and separable state were discussed for arbitrary FS, i.e., not necessarily Gaussian states. There, the set of physical states was defined as $Π := \{ ρ : [ρ, P] = 0 \}$, with $P$ the parity operator. This gave rise to two notions of “product states”: The set of physical states for which the expectation values of all products of physical observables factorize, i.e., $ρ(A_x B_z) = ρ(A_x) ρ(B_z)$, was denoted by $Π^{1}_π$. $Π^{2}_π$ ($Π^{2}_π$) is the set of all states of the form $ρ = ρ_A ⊕ ρ_B$ with (without) the parity restriction, respectively.

Then the three separable sets $S^{1}_π, S^{2}_π, S^{2}_π'$ can be defined via the convex hull of the different product sets together with the requirement that the final state commutes with the local parity. Specifically: $S^{1}_π = co(Π^{1}_π)$, $S^{2}_π = co(Π^{2}_π)$ and $S^{2}_π' = co(Π^{2}_π) \cap Π$.

Let us now investigate these definitions further by considering GFS. In order to identify the set of separable GFS one might want to define the separable states as those that are not useful for any quantum information task even if arbitrarily many copies of the state are given. Another reasonable choice would be to define the set of separable states to be those that can (at least asymptotically) be prepared by LOCC. In the single copy case [15] shows that these two notions do not coincide for fermions: $Π$ contains states that cannot be prepared locally but the set $Π^{1}_π$ of states that are not useful (considering only a single copy) is strictly larger. Before we focus on the first choice, i.e., on $Π$, and show that the definition using $Π^{1}_π$ can be ruled out, let us present some observations about these sets.

**Observation 1.** A GFS is in the set $S^{2}_π$ iff its covariance matrix takes direct-sum form.

This can be easily seen by noting that all states in $S^{2}_π$ are convex combinations of products of states that each commute with the local parity; i.e., all terms in the mixture have a CM that is block diagonal (and all first moments vanish), hence, the CM of the mixture is also block diagonal. In contrast, even the states in $Π^{1}_π$ can have non-zero correlations between $A$ and $B$ as stated in the next observation.

**Observation 2.** A state in $Π^{1}_π$ can have non-zero correlations between $A$ and $B$. However, in that case the block of the CM containing the correlations between $A$ and $B$ has at most one non-vanishing singular value.

For a proof of the above Observation see Appendix [3].

An example of a Gaussian state, which is separable according to definition $S^{1}_π$ but not according to $S^{2}_π$, is the 2-mode Gaussian state with CM

$$γ_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

It describes a state in which one (non-local and paired) mode is prepared in a pure Fock state and the other in the maximally mixed one. In general, we could consider the first mode to be in a (finite temperature) thermal state (e.g., being occupied with probability $p$), then

$$γ_p = (1 - 2p)p_0.$$  

The two modes are defined by the non-local $SO(4)$ matrix

$$O = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
which maps $O_{\gamma_0}^{OT} = [-J_2 \oplus 0_2]$. The mode operators of the transformed state are $(\hat{c}_3, \hat{c}_1)$ and $(\hat{c}_2, \hat{c}_4)$, i.e., the new annihilation operators are given in terms of the old ones as $b'_1 = \frac{1}{2}(b_2 + b_1^* + i(b_1 + b_2^*))$ and $b'_2 = \frac{1}{2}(b_2 - b_1^* - i(b_1 - b_2^*))$. It is readily checked that the vacuum for these two modes in the original basis is $|0_{\gamma_0}0_{\gamma_0}\rangle = |(0_1, 0_2) + i(1_1, 0_2)\rangle/\sqrt{2}$. Therefore the mixed Gaussian state with CM $\gamma_0$ is given by the mixture of $|0_{\gamma_0}0_{\gamma_0}\rangle$ and $|0_{\gamma}1_{\gamma}\rangle = |(0_1, 1_2) + i(1_1, 0_2)\rangle/\sqrt{2}$ (each with probability $(1-p)/2$) and $|1_{\gamma_0}0_{\gamma_0}\rangle, |1_{\gamma}1_{\gamma}\rangle$ (each with probability $p/2$). Since the Fock states in the $b'_1, b'_2$ basis correspond up to a local phase gate to Bell states in the local basis the state can be seen as being GL-equivalent to a Bell-diagonal state with entries $((1-p)/2, (1-p)/2, p/2, p/2)$ in the $(\Phi^+, \Psi^+, \Phi^-, \Psi^-)$ basis. For qubits, we would argue that for all $p = 0$ even to distill pure singlets. Consequently, separability should be defined in a way that does not include these states. Note that this has already been shown for FS in [13]. The following theorem proves that the statement also holds for the restricted set of GFS.

**Theorem 3.** The set of Gaussian states in $S_{1\pi}$ is not stable. That is, there exists a GFS, $\rho$ such that $\rho \in S_{1\pi}$ (even in $P_{1\pi}$), however, $\rho \otimes \rho \not\in S_{1\pi}$.

**Proof.** Given two copies of a Gaussian state, $\rho$, with CM $\Gamma_{\rho} = \begin{pmatrix} \Gamma_A & C \\ -C^T & \Gamma_B \end{pmatrix}$ and rank $C = 1$ then the full state now has a rank-2 matrix $C$ and therefore is no longer in $P_{1\pi}$, since we can find a pair of local observables (commuting with local parity) for which the expectation value does not factorize. That is, assuming $(\Gamma_{\rho})_{kl} \propto \rho(c_k c_l^\dagger) \neq 0$ and using Wick’s theorem implies $\rho(c_k c_l^\dagger c_l c_k^\dagger) = -\rho(c_k c_l^\dagger)c_l c_k^\dagger) \neq 0$, where the primed operators refer to the second copy. Hence, $\rho \otimes \rho \not\in S_{1\pi}$. \hfill $\Box$

This shows that any Gaussian state $\rho$ for which $\rho \otimes^n (A_n B_n) = \rho \otimes^n (A_n)\rho \otimes^n (B_n) \forall A_n, B_n, n$ must have a CM $\Gamma_{\rho} = \Gamma_A \oplus \Gamma_B$. We are going to show next that $\rho \otimes^2$ is not only no longer in the set $S_{1\pi}$, but that it can also be useful for quantum information theoretical tasks.

**Observation 4.** Some states in $S_{1\pi}$ can be useful for quantum information processing.

Given two copies of a Gaussian state with CM $\gamma_0$, we can use local Gaussian unitaries to transform it to the form (now written in $2 \times 2$ block form)

$$
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

which now contains a pure, maximally entangled two-mode state in the first modes of $A$ and $B$ [31]. These states can be used for teleportation (though only probabilistically). This shows, that $S_{1\pi}$ is not a viable definition of separability.

Due to these observations it is clear that one relevant set of separable states is defined via $S_{2\pi}$. Hence, we use this definition in the following. In that case the CM of any $n$-partite mixture of product states has direct-sum form, i.e., $\gamma = \otimes_i \gamma_i$. That is a GFS is separable iff its CM is of that form. Moreover, this definition of separability is meaningful in the context of the generation of separable states, as all these states can be prepared locally. To be more precise, let us note that as separability does not have such a clear meaning for FS, as it has, e.g., in the bosonic, or finite dimensional case, it is a priori not clear how separable maps ought to be defined. This is especially due to the fact that the set of separable maps (SEP) does not have a clear physical meaning. In contrast to that, LOCC transformations, even if restricted to certain local operations, such as (Gaussian) fermionic operations, are operationally defined. It is the set of transformations which can be implemented by local (Gaussian) fermionic operations assisted by classical communication. LOCC is strictly contained in SEP and is mathematically usually much harder to characterize. However, in a situation as here, where the definition of the larger set is not clear, one is forced to deal with LOCC. Hence, we consider in Appendix A FLOCC transformations and show that this leads to a natural choice of the definition of FSEP. Moreover, we show that any separable state (according to $S_{2\pi}$) can be generated via a FLOCC transformation. Hence, the definition of separable states being those which are elements of $S_{2\pi}$ meets all the necessary requirements. Note that it is, however, not clear if for every Gaussian state in $S_{2\pi}$ there exists a decomposition into physical product states, i.e., it is not clear whether for Gaussians the sets $S_{2\pi}$ and $S_{2\pi}^{\prime}$ coincide or not (in general they do not [15]). However, as mentioned above and as shown in Appendix A all states which can be reasonably prepared locally must belong to $S_{2\pi}$.

**B. Gaussian Fermionic Separable Operations**

As recalled in Sec. [13] the CJ isomorphism provides a one-to-one mapping between quantum states and quantum operations. Moreover, it has been shown in the finite-dimensional case that a map is separable, i.e., it can be written as a convex combination of local operators if the corresponding CJ state is separable [15].
Appendix A we show that the CJ state of a Gaussian separable map has a CM of the form
\[ \Gamma_{AB} = \Gamma_A \oplus \Gamma_B, \]
with a natural generalization to more systems. As a consequence of the previous section this state is a separable GFS according to $S_2$. Thus, this definition of separability agrees with the operational viewpoint that all separable states can be generated locally (an agreement which is not maintained for all definitions in the presence of superselection rules, see, e.g., [49]). Moreover, this definition can be naturally generalized to Gaussian separable maps (GSEP) (see Appendix A.2 for more details).

As stated in the following lemma, fermionic completely positive maps (FCPM), i.e., CP maps that map FSs onto FSs, can be written in Kraus decomposition with special Majorana operators (i.e., that are either sums of only even monomials in the Majorana operators $\tilde{\pi}$, i.e., sums of only odd monomials).

**Lemma 5.** All fermionic completely positive maps can be written using only Kraus operators with definite parity (i.e., that are either sums of only even monomials in the Majorana operators $\tilde{\pi}$, i.e., sums of only odd monomials).

**Proof:** Let $\mathcal{E}$ denote a FCPM with Kraus operators $\{A_k\}$, i.e., $\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$ for all $\rho$. In general, the $A_k$ are sums of even and odd terms, that is $A_k = A_k^{(e)} + A_k^{(o)}$. FCMs map FSs to (unnormalized) FSs, i.e., both $\rho$ and $\mathcal{E}(\rho)$ are even sums of even monomials in the Majorana operators $\tilde{\pi}$. Using the Kraus representation, this implies that $\sum_k A_k^{(e)} \rho A_k^{(e)} + A_k^{(o)} \rho A_k^{(o)\dagger} = 0$ for all $\rho$. Consequently, the FCPM $\tilde{\mathcal{E}}$ with Kraus operators $\{A_e, A_o\}$, which we denote by $\tilde{A}_k$ in the following, represents the same channel as $\mathcal{E}(\rho) = \tilde{\mathcal{E}}(\rho)$ for all FSs $\rho$. To show that $\sum_k \tilde{A}_k \tilde{A}_k^\dagger = 1$ on the whole state space, note that $\text{tr}(Y) = 1 \Leftrightarrow Y = 1_{Y_e}$. For $Y_{ee} = P_e Y_{ee} P_e$ ($Y_{oe} = P_e Y_{oe} P_e$) and both the even and odd part of $Y$ have to be equal to the identity. Moreover, for $Y = \sum_k \tilde{A}_k \tilde{A}_k^\dagger$, it follows immediately that $Y_{ee} = Y_{oe} = 0$. Thus, the Kraus operators of the FCPM $\tilde{\mathcal{E}}$ also satisfy $\sum_k \tilde{A}_k \tilde{A}_k^\dagger = 1$. \hfill $\Box$

**C. Standard Form and GLU–Equivalence for n-mode n-partite States**

Here, we consider $n$-mode $n$-partite fermionic systems. That is, each mode is spatially separated from the others. We derive a standard form $S(\gamma)$ into which the CM $\gamma$ can be transformed via GLU. As the standard form is unique, we have that two GFS are GLU equivalent iff their CMs in standard form coincide.

Let us start by recalling that the most general GLU operation corresponds to an arbitrary real orthogonal matrix on each mode. Hence, the CM $\gamma$ is transformed to
\[ S(\gamma) = (\otimes_i Z_t^m O_i) \gamma (\otimes_i O_i^T Z_t^m), \]
via GLU with $O_i \in SO(2,\mathbb{R})$ and $m_i \in \{0,1\}$. We denote in the following by $\gamma_{jk}$ the $2 \times 2$ matrix describing the covariances between modes $j$ and $k$. Due to the fact that $\gamma = -\gamma^T$ we have for $i \leq n$
\[ \gamma_{ii} = \left( \begin{array}{cc} 0 & \lambda_i \\ -\lambda_i & 0 \end{array} \right) = \lambda_i J_z, \]
where $\lambda_i \in [-1,1]$. As $A_{j} J_z A_{k} = A_{k} J_z A_{j}$, for any $2 \times 2$ matrix $A$, $\gamma_{ij}$ transforms to $Z_{ij} \gamma_{ij} Z_{ij}^T = (-1)^{m_i} \gamma_{ij}$. If $\lambda_i \neq 0$ we chose $m_i$ such that $Z_{ij} \gamma_{ij} Z_{ij}^T = \lambda_i J_z$, where $\lambda_i > 0$. In case $\lambda_i = 0$, i.e., mode $i$ is completely mixed, we show below how the bit value $m_i$ can be uniquely defined. In order to uniquely define $O_i = e^{\gamma_{i,j} Y}$ we proceed as follows. Consider the first index $j$ with $i < j$ such that the off-diagonal matrix $\gamma_{ij}$ is not vanishing. If the singular values of $\gamma_{ij}$ are non-degenerate ($d_j \neq |d_j'|$) we define $O_i$, $O_j \in SO(2,\mathbb{R})$ by
\[ O_{ij} O_{ij}^T = D_{ij} = \text{diag}(d_{ij}, d_{ij}', d_{ij} > |d_{ij}'|) \]
If the singular values of $\gamma_{ij}$ are degenerate, $\gamma_{ij}$ is itself proportional to an orthogonal matrix. In case $|\gamma_{ij}| > 0$, $\gamma_{ij}$ is proportional to a special orthogonal matrix, $e^{\alpha_{ij} Y}$. Then, we define $O_j \propto Z \gamma_{ij}$, that is we set $\alpha_{ij} = \alpha_{ij} + \alpha_{ij}$. In case $|\gamma_{ij}| < 0$, $\gamma_{ij}$ is proportional to a matrix $Ze^{\alpha_{ij} Y}$. Then, we define $O_j \propto Z \gamma_{ij}$, that is we set $\alpha_{ij} = \alpha_{ij} - \alpha_{ij}$. In all cases $S(\gamma_{ij})$ is diagonal. We proceed in the same way for $\gamma_{ij+1}$ (and then any subsequent $\gamma_{ij}$).

If $\alpha_{ij}$ has already been determined in a previous step, $\alpha_{ij}$ is determined by diagonalizing $\gamma_{jk}$. More precisely, $\alpha_{ij}$ is chosen such that $O_j \gamma_{jk} O_j^T = \tilde{O}_j \gamma_{jk} \tilde{O}_j^T$, where $\tilde{O}_j \gamma_{jk} \tilde{O}_j^T = \text{diag}(d_{jk}, d_{jk}', d_{jk} > |d_{jk}'|)$. Then, we define $O_j \propto Z \gamma_{ij}$, that is we set $\alpha_{ij} = \alpha_{ij} - \alpha_{ij}$.

It is easy to see that in this way any $\alpha_{ij}$ is uniquely determined unless the CM is invariant under the conjugation with $O_j$, that is, the mode $j$ is decoupled from all other modes, in which case we set $\alpha_{ij} = 0$. At this point all the operators which are not symmetry of the CM are determined. Those which leave the CM invariant can be chosen to be equal to the identity, e.g., if for 3-modes $\gamma_{12} = O_{12}$, $\gamma_{13} = O_{13}$ and $\gamma_{23} = O_{23}$ with $O_{ij} \in SO(2,\mathbb{R})$, i.e., all of them are special orthogonal matrices and invariant under $O_i$, we choose $O_i = 1$.

It remains to consider the case where $\lambda_i = 0$. If there is no index $j$ such that $\gamma_{ij} \neq 0$ then the mode $i$ factorizes and we set $m_i = 0$. Hence, let us assume that $\gamma_{ij} \neq 0$.
therefore, as explained above the standard form looks derived. More precisely, if one of the conditions do not hold a similar standard form can be obtained in GFS, Theorem 6 presents a criterion for GLU–equivalence of GFS.

Let us consider now some examples, where we explicitly compute the standard form for the CM. As mentioned above we consider here $n$-mode $n$-partite systems, i.e., the $1\times 1\times \cdots \times 1$ case. Here, we compute the standard form of 2- and 3-mode states.

### 1. $1 \times 1$

Using the definition of the standard form introduced above, it is straightforward to see that any 2-mode state can be written (up to GLU) as

$$S(\gamma) = \begin{pmatrix}
0 & \lambda_1 & a_{1a4} + a_{2a3} & 0 & 0 & -a_{1a3} + a_{2a4} \\
-\lambda_1 & 0 & 0 & 0 & -a_{1a4} + a_{2a3} & a_{1a3} + a_{2a4} \\
-a_{1a4} + a_{2a3} & 0 & 0 & \lambda_2 & a_{3a4} - a_{1a2} & 0 \\
a_{1a3} + a_{2a4} & -a_{1a3} + a_{1a2} & -a_{3a4} + a_{1a2} & 0 & 0 & \lambda_3 \\
a_{1a3} + a_{2a4} & 0 & 0 & (a_{3a4} + a_{1a2}) & 0 & 0
\end{pmatrix}.$$  

(24)

with $\lambda_1 = 1/2(a_3^2 + a_4^2 - a_1^2 - a_2^2)$, $\lambda_2 = 1/2(a_3^2 + a_4^2 - a_1^2 - a_2^2)$, $\lambda_3 = 1/2(a_3^2 + a_4^2 - a_1^2 - a_2^2)$. If the above stated conditions do not hold a similar standard form can be derived. More precisely, if one of the $a_i$'s is equal to zero at least one of the off-diagonal blocks is degenerate and therefore, as explained above the standard form looks slightly different. Note that as in the bosonic Gaussian case and in contrast to the qubit case it can be easily seen that the purities of the reduced states, that is the $\lambda_i$'s, uniquely define the state. Let us remark here, that there exists only one GFS (up to GLU) with $\rho_i \propto 1$ for each subsystem $i$, namely $|000\rangle + |011\rangle + |101\rangle + |110\rangle$.
Note that—although it is known that any three-qubit state whose single-qubit reduced density operators are completely mixed is LU-equivalent to the GHZ state—this does not immediately imply the same for GFS due to the restriction to GLU.

IV. PURE GAUSSIAN FERMIONIC STATES AND LOCAL TRANSFORMATIONS FOR n-MODE n-PARTITE STATES

Let us now investigate in more detail the entanglement contained in pure GFS. For this purpose we consider the class of Gaussian separable operations (GSEP). In general, SEP contains LOCC but is a strictly larger class \[ \text{LOCC} \subset \text{SEP}. \] We show, however, that for Gaussian operations on n-mode n-partite systems any transformation among pure fully entangled states via Gaussian SEP (GSEP) can be performed via GLU. Hence, in particular, only trivial Gaussian LOCC (GLOCC) transformations exist for single modes. Note that here and in the following we consider only fully entangled states, i.e., states where no subset of modes factorizes from the remainder. Due to the triviality of GLOCC we study then Gaussian stochastic LOCC (GSLOCC) and certain fermionic LOCC (FLOCC), see Section IV C, which map FSs to FSs. We characterize the various GSLOCC classes, which are, in contrast to the bosonic case, indeed equivalence classes. We then show that there exist non-trivial FLOCC transformations that map a pure GFS to some other pure GFS and demonstrate how to identify all possible transformations of that kind. Interestingly, many of the pure GFS belong to the Maximally Entangled Set (MES) \[ \text{MES}. \] That is, they cannot be obtained from any other state via local deterministic transformations. For other states we derive a very simple local protocol which can be used to reach the state from a state in the MES.

Let us first of all show that Cond. (17), which is a prerequisite for reaching the state from a state in the MES, can be used to reach the state from a state in the MES. That is, they cannot be obtained from any other pure state via Gaussian SEP (GSEP) and therefore there exists at least one pure state \[ |\Psi_i\rangle \] for which

\[ |\Phi\rangle \propto A_k \otimes \tilde{I} \otimes \cdots \otimes \tilde{I} \otimes |\Psi_i\rangle \propto A_l \otimes \tilde{I} \otimes \cdots \otimes \tilde{I} \otimes |\Psi_i\rangle, \]

where by \( A_k \) we denote the Kraus operators of \( \mathcal{E}_k \). Note that \( |\Psi_i\rangle \) has to be entangled in the splitting mode 1 versus the remaining modes as \( |\Phi\rangle \) is entangled in this splitting and \( \mathcal{E}_k \) cannot generate entanglement. Hence, considering \( |\Psi_i\rangle \) in its Jordan-Wigner representation its Schmidt decomposition can be written as

\[ |\Psi_i\rangle = \sum_{j=0}^{n-1} \lambda_j |j\rangle \otimes |\psi_i^j\rangle \]

with \( \lambda_j, \lambda_1 \neq 0 \). Using this in Eq. (27) as well as that due to Lemma 6 the Kraus operators of \( \mathcal{E}_k \) can be chosen such that each of them commutes with \( |\psi_i^j\rangle\langle \psi_i^j| \) (which is a sum of only even monomials in the Majorana operators acting on the modes \( 2, \ldots, n \)) it is easy to see that

\[ A_k |j\rangle \otimes |\psi_i^j\rangle = c_k |j\rangle \otimes |\psi_i^j\rangle \]

for \( j \in \{ 0, 1 \} \) and \( c_k \in \mathbb{C} \). [22]. As the action of the different Kraus operators on a basis leads to the same states (up to a constant proportionality factor) we have that \( A_k \propto A_l \). Moreover, as this holds true for all possible pairs of Kraus operators one obtains from \( \sum_j A_j^\dagger A_j = 1 \) that \( A_j^\dagger A_j \propto 1 \) and hence the map \( \mathcal{E}_1 \) corresponds to the application of a GLU on mode 1. Rearranging of the modes such that \( \mathcal{E}_j \) is a GLU transformation on mode \( j \) for all \( j \). Note that here we make use of the fact that the maps \( \mathcal{E}_j \) commute with each other.

A. Gaussian Separable Operations and Gaussian LOCC

Let us start with the investigation of GSEP transformations. As argued in Appendix A 2 GSEP is defined as the class of operations for which the CJ state is Gaussian and has a CM of the form \( \Gamma = \oplus_{1}^{n} \Gamma_i \). We show here that any GSEP acting on \( n \) separated modes, which maps at least one pure (fully entangled) state into a different pure (fully entangled) state is a GLU transformation. Hence, non-trivial state transformation is possible. The following lemma allows us to show in the end that GLOCC on pure states are trivial, as GSEP strictly includes GLOCC (see Appendix A 2).

**Lemma 8.** Let \( \mathcal{E}_{\text{sep}} \) denote a Gaussian trace preserving separable map which transforms at least one pure n-partite n-mode FS, \( |\Psi\rangle \), into another pure n-partite n-mode fully entangled FS, \( |\Phi\rangle \). Then, it holds that

\[ \mathcal{E}_{\text{sep}}(\rho) = (U_1 \otimes U_2 \cdots \otimes U_n) \rho (U_1^\dagger \otimes U_2^\dagger \cdots \otimes U_n^\dagger) \]

for all \( \rho \).

**Proof.** Every separable Gaussian CP trace-preserving map (GCPTM) \( \mathcal{E}_{\text{sep}} \) has a separable Gaussian CJ state \( \mathcal{E}_{\text{sep}}(\psi) \), i.e., \( \mathcal{E}_{\text{sep}} \) is of the form \( p_1 \otimes \cdots \otimes p_N \), and, consequently, \( \mathcal{E}_{\text{sep}} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N \) is a product operation with GCPTMs \( \mathcal{E}_i \) (see Appendix A 2). Let us denote \( \mathcal{E}_k(\psi) \) by \( \mathcal{E}_k \) and write it in its spectral decomposition \( \rho = \sum_j p_j |\psi_i\rangle \langle \psi_i| \). It follows from \( \mathcal{E}_{\text{sep}}(\psi) = \psi \) that \( \sum_j p_j \mathcal{E}_k(\psi_i) = |\psi_i\rangle \langle \psi_i| \) for all \( p_i \neq 0 \). Let \( \mathcal{E}_i \otimes \cdots \otimes \mathcal{E}_N \) be a Gaussian trace preserving map (GSEP). As argued in Appendix A 2, GSEP is defined as the class of operations for which the CJ state is Gaussian and has a CM of the form \( \Gamma = \oplus_{1}^{n} \Gamma_i \). We show here that any GSEP acting on \( n \) separated modes, which maps at least one pure (fully entangled) state into a different pure (fully entangled) state is a GLU transformation. Hence, non-trivial state transformation is possible. The following lemma allows us to show in the end that GLOCC on pure states are trivial, as GSEP strictly includes GLOCC (see Appendix A 2).
Hence, we can apply the $E_i$ sequentially in any order. This implies that under rearranging the modes the product structure of the map $E_{sep}$ and the Kraus operators of the local maps $E_i$ are preserved [64]. Hence, we have that $E_{sep}$ is a GLU transformation.

As mentioned above, Lemma 8 allows us to directly obtain the following corollary.

**Corollary 9.** There exists no non–trivial GLOCC operation mapping a pure $n$–mode $n$–partite FS $|\psi\rangle$ into another pure $n$–mode $n$–partite FS $|\phi\rangle$.

Let us note here that a very similar result has recently been proven for finite dimensional Hilbert spaces [12, 13].

There, it has been shown that generically, i.e., for a full–measure set of states, there exists no LOCC (even SEP) transformation, which transforms one pure (fully entangled) state into another, which is not LU–equivalent. In strong contrast to the scenario considered here, the reason for that is however not that all separable maps are particularly restricted, but that generically a state has no local symmetry. The relevance of local symmetries for local state transformation is recalled in Sec. [IV C]. Note, however, that in the qubit case, the result only holds generically and that there exists a zero–measure set of states which can be transformed via LOCC, whereas for FS the result holds for any state.

**B. Gaussian Stochastic LOCC**

In the previous subsection we have shown that GLOCC transformations among pure GFS are trivial. Thus, to quantify and qualify entanglement properties of pure GFS we have to turn to a larger class of local operations. To that end, we now consider Gaussian stochastic LOCC (GSLLOCC) [65].

As mentioned before, the most general Gaussian operation consists of attaching an auxiliary system by applying a Gaussian unitary to it and the system mode and the auxiliary system mode can be changed with the auxiliary system generators possible due to the fact that the parity of the system mode can be changed with the auxiliary system (for total parity–preserving operations we have $k_i = 1$ for an even number of $k_i$’s). Note, furthermore, that for a single mode the Gaussian operations coincide with the fermionic operations (see Sec. [IV C]). Given the fact that these are the most general Gaussian local operations we have that two states can be transformed into each other via GSLLOCC if there exists an invertible operator of the form given in Eq. (28) which transforms one state into the other (in the Jordan-Wigner representation). In particular, we have that GSLLOCC is indeed an equivalence relation.

Before studying now the possible GSLLOCC classes let us introduce a standard form for FS. We consider a FS in Jordan-Wigner representation. Note again that as shown in Lemmas a pure FS is Gaussian iff $\Lambda(|\Psi\rangle \otimes |\varphi\rangle) = 0$. Using the standard form of FS explained below together with this condition one obtains a characterization of the GSLLOCC classes. We then present the different GSLLOCC classes for up to four mode GFS.

The following lemma states that by consecutive application of diagonal matrices any FS can be transformed into a normal form, which can, however, also vanish. For this we need the notion of a critical state, i.e., a state whose single system reduced states are all proportional to the identity.

**Lemma 10.** Let $|\Psi\rangle$ be a fully entangled FS. Then $|\Psi\rangle$ can constructively (by applying invertible diagonal matrices) be transformed into a unique (up to LUs) critical state, $|\Psi_s\rangle$ (up to a proportionality factor $\lambda \in \mathbb{C}$ which can tend to 0).

**Proof.** The lemma follows from the normal form of multipartite states describing finite dimensional systems presented in [69]. There, it has been shown that any state can be transformed via (a sequence of) local operations into a state whose single system reduced state is completely mixed. In the algorithm presented in [69], which achieves this transformation, the local determinant 1 operations are\footnote{Glos.} $X_i^{(k)} = |\rho_i^{(k)}|^{1/2d_i} |\sqrt{\rho_i^{(k)}}|^{-1}$, where $d_i$ denotes the local dimension of system $i$ and $\rho_i^{(k)}$ the reduced state of party $i$ in the $k$–th step of the algorithm. In order to apply this result to FS note that the reduced state of a FS has to be fermionic and hence diagonal. Moreover, as local diagonal operators are fermionic operations (even Gaussian), each state during the algorithm is a FS. Hence, in each step $k$ and for each party $i$, the operators $X_i^{(k)}$ are diagonal, which proves the statement.

The normal form of $|\Psi\rangle$ is given by $\lambda |\Psi_s\rangle$ (where $\lambda$ can tend to 0).

Depending on the normal form one can group states in the following three (disjoint) classes of states: (i) stable states: These are states belonging to a SLOCC class which contains a critical state, which then is their normal form. Due to the Kempf–Ness theorem [67], there exists only one critical state in a SLOCC class (up to LUs). In the following we will call this state seed state and denote it by $|\Psi_s\rangle$. That the normal form of any stable state is the corresponding seed state follows also from the Kempf–Ness theorem. The GHZ–state, $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, is an example of a critical and therefore a stable state; (ii) semi–stable states: belong to a SLOCC class without critical state: The normal form of these states tends to a non–zero normal form. More precisely, it tends to a seed state...
of a different SLOCC class \((59)\). The 4-qubit state \(|\psi\rangle = a(0000) + |1111\rangle + b(0110) + |0101\rangle\) is an example of a semi-stable state, whose normal form tends to the 4-qubit GHZ-state (see \((56)\)); (ii) states in the null cone: The normal form of these states vanishes. An example of such a state is the \(W\)-state.

In the Hilbert space \(\mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d\) the union of stable states is of full measure and dense \([8]\). Hence, for almost all states the normal form is not vanishing. Whether the same holds true for FS is currently not clear. Despite this, we will focus now on stable FS. However, in the more detailed investigations of few-mode states we will also consider semi-stable states and states in the null cone.

It follows straightforwardly from the lemma above that stable FSs can be written as \(X^m D_1 \otimes X^m D_2 \ldots \otimes X^m D_n |\Psi_s\rangle\) with \(|\Psi_s\rangle\) being critical. Note, however, that any GFS can be written as \(X^m D_1 \otimes X^m D_2 \ldots \otimes X^m D_n |\Psi_f\rangle\) where \(|\Psi_f\rangle\) is some representative (not necessarily critical) of the GSLOCC class and \(m_i \in \{0, 1\}\). This follows from the fact that the most general Gaussian operations are of the form \(X^m D_1 \otimes X^m D_2 \ldots \otimes X^m D_n\). The subsequent corollary allows to characterize the GSLOCC classes of stable GFS.

**Corollary 11.** Let \(|\Psi\rangle\) be a stable FS and \(|\Psi\rangle = D_1 \otimes D_2 \ldots \otimes D_n |\Psi_s\rangle\). Then, \(|\Psi\rangle\) is GFS iff \(|\Psi_s\rangle\) is GFS.\[\]

**Proof.** The “if”-part follows from the fact that local diagonal matrices are Gaussian operations. The “only if”-part can be seen as follows. Due to Lemma 7 we have that \(|\Psi\rangle\) is GFS iff \(A(|\Psi\rangle \otimes |\Psi\rangle) = 0\), which is equivalent to \(A(|\Psi_s\rangle \otimes |\Psi_s\rangle) = 0\). Hence, \(|\Psi\rangle\) is a GFS iff \(|\Psi_s\rangle\) is.\[\]

An interesting example of a critical GFS state is the \(n\)-mode state \(|\Psi\rangle = H^n |\text{GHZ}\rangle [\text{with } |\text{GHZ}\rangle = 1/\sqrt{2}(|00\ldots0\rangle + |11\ldots1\rangle)]\). To see that \(|\Psi\rangle\) is a GFS, note that \(|\Psi\rangle \propto \sum_{k \in \{0,1\}^n} (1 + (-1)^{h(k)}) |k\rangle\) with \(h(k)\) being the Hamming weight of the bitstring \(k\). Therefore, \(|\Psi\rangle\) is a GFS. That \(A(|\Psi\rangle \otimes |\Psi\rangle) = 0\) can be easily verified by direct computation. The fact that the state is critical follows from the criticality of the GHZ state. Note that the GHZ state itself is only a FS for even \(n\). Moreover, the fermionic swap applied to any two modes of \(|\Psi\rangle\) (or of any critical state) is also critical. As there exists only one critical state in a SLOCC class, this state is either LU-equivalent to \(|\Psi\rangle\) or in a different SLOCC class \((58)\).

Let us now explicitly compute the GSLOCC classes of up to 4-mode GFS.

1. **\(1 \times 1\) case**

We start with the simplest case of pure 2-mode 2-party systems. First note that the spin representation of any FS of two modes is either of the form \(|\Psi_1\rangle = \alpha |00\rangle + \beta |11\rangle\) of the form \(|1 \otimes X\rangle |\Psi_1\rangle = \alpha |01\rangle + \beta |10\rangle\). As \(|\Psi_1\rangle \propto D \otimes I |\Psi_+\rangle\), where \(|\Psi_+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle\) denotes the critical seed state of two qubits and \(D = \text{diag}(\alpha, \beta)\) there is only one entangled GSLOCC class. It is easy to see that these states are all Gaussian, as \(A(|\Phi^+\rangle \otimes |\Phi^+\rangle) = 0\) (see Lemma 4 and Corollary 11).

2. **\(1 \times 1 \times 1\) case**

For 4 modes it is no longer true that any pure FS is a GFS. In fact, from Lemma 4 one easily derives the following observation.

**Lemma 12.** There are two 3–mode entangled GSLOCC classes, the GHZ and the \(W\) class. The state \(|\Psi(a_1, a_2, a_3, a_4)\rangle\) belongs to the GHZ class iff \(a_i \neq 0 \forall i\). It belongs to the \(W\)-class iff there exists exactly one \(i\) such that \(a_i = 0\). Moreover, the state is biseparable iff exactly two \(a_i = 0\) (else it is separable).

**Proof.** First consider the case where \(a_i \neq 0 \forall i\). It can be easily seen that the state can be written as \(D_1 \otimes D_2 \otimes D_3 |GHZ\rangle_3\) with \(D_i\) invertible and hence it belongs to the GHZ class. Let us denote by \(|W\rangle_3 = 1/\sqrt{3}(|011\rangle + |101\rangle + |110\rangle)\) the \(W\)-state. Then it is easy to see that any state \(|\Psi(a_1, a_2, a_3, a_4)\rangle\) with exactly one \(i\) such that \(a_i = 0\) can be written as \(X^k D_1 \otimes X^k D_2 \otimes X^k |W\rangle_3\), where \(k + k_2 + k_3 = 0 \mod 2\) and \(D_i\) diagonal and invertible. If two coefficients vanish the state can be written as \(X^k D \otimes X^k |0\rangle\) (up to particle permutation), where \(k + k_2 + k_3 = 0 \mod 2\) and \(D\) is invertible, which proves the statement.\[\]

Note that this implies that a tripartite entangled 3–mode GFS is of the form \(D_1 \otimes D_2 \otimes D_3 |\Psi_f\rangle_3\) (up to GLUs), where \(|\Psi_f\rangle_3\) is either the GHZ– or the \(W\)-state and all \(D_i\)’s are invertible. Hence, there exist, as in the qubit case, two fully entangled GSLOCC classes.

The standard forms of the corresponding CM are given in Sec. 11C. To give an example for the GHZ-state with \(a_1 = 1/2 \forall i\) the standard form is given in Eq. (25). A similar standard form for the \(W\)-state \((a_1 = 0, a_2 = a_3 = a_4 = 1/\sqrt{3})\) can be determined. However, it is slightly different, as in this case \(\gamma_{12}, \gamma_{13}, \gamma_{23} \in SO(2, \mathbb{R})\) in Eq. (25).

3. **\(1 \times 1 \times 1 \times 1\) case**

For 4 modes it is no longer true that any pure FS is a GFS. In fact, from Lemma 4 one easily derives the following observation.
Observation 13. A pure 4-mode FS, $|\Psi\rangle$ (in Jordan-Wigner representation) is Gaussian iff
\[ \langle \Psi^* | (X \otimes Y \otimes X \otimes Y) | \Psi \rangle = 0, \tag{29} \]
where $X, Y$ denote the Pauli operators.

This condition, which resembles the SL-invariant polynomials [60] defined for qubit-states, is in fact equivalent to the condition that all reduced 3-mode states of $|\Psi\rangle$ (taking the partial trace of one party) are Gaussian. An arbitrary 4-mode (even parity) FS is given by $|\Psi\rangle = a_1|0000\rangle + a_2|0011\rangle + a_3|0110\rangle + a_4|1100\rangle + a_5|1010\rangle + a_6|0101\rangle + a_7|1001\rangle + a_8|1111\rangle$. It can be easily seen (analogously to the 3-mode case) that any such state can be written as in the following lemma [69].

**Lemma 14.** A pure 4-mode FS, $|\Psi\rangle$ can be written as
\[ |\Psi\rangle = X^{k_1}D_1 \otimes X^{k_2}D_2 \otimes X^{k_3}D_3 \otimes X^{k_4}D_4 |\Psi_f\rangle, \tag{30} \]
with $|\Psi_f\rangle$ an appropriate representative of each SLOCC class, $k_i \in \{0, 1\}$ and $k_1 + k_2 + k_3 + k_4 = 0 \mod 2$. Moreover, the state is GFS iff the FS $|\Psi\rangle$ is.

The last conclusion follows directly from Corollary [11] as in the proof it has not been used that $|\Psi_f\rangle$ is critical and the local $X^{k_i}D_i$ are Gaussian operators. Note that, as in the 3-mode case, some GSLOCC classes contain a critical state, whereas others do not. Moreover, in the 4-mode case there also exist semi-stable states, i.e., states that tend to a non-vanishing normal form, even though they are not stable. Let us state the different GSLOCC classes in more detail based on the results on 4-qubit SLOCC classes in [7].

- **GSLOCC classes containing a critical state:** These are states from the SLOCC classes $G_{abcd}$ (7), with representatives
\[ |\Psi_f\rangle = a|\Phi^+\rangle^\otimes 2 + b|\Phi^-\rangle^\otimes 2 + c|\Psi^+\rangle^\otimes 2 + d|\Psi^-\rangle^\otimes 2. \tag{31} \]
Note that the states $|\Psi_f\rangle$ are critical. Due to Observation [13] we can easily see that the FS in Eq. (31) are Gaussian iff $ab + cd = 0$. Hence, either two or three of the parameters of $|\Psi_f\rangle$ can vanish, according to this necessary and sufficient condition. Whereas the states where two of the four parameters are equal to zero are still 4-partite entangled, states with three parameters being equal to zero are biseparable states.

- **GSLOCC classes containing semi-stable states:** As mentioned above there exist classes that contain semi-stable states (see [20] for results on semi-stable 4-qubit states). The SLOCC classes containing 4-mode entangled GFS are $L_{abc_2}$ and $L_{a_2b_2}$ (see [7]) with representatives
\[
|\Psi_f(abc_2)\rangle = \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle) + |0110\rangle + |1010\rangle,
\]
\[
|\Psi_f(a_2b_2)\rangle = a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle,
\]
\[ |\Psi_f(a_2b_2\rangle) \]
Note that neither $|\Psi_f(abc_2)\rangle$ nor $|\Psi_f(a_2b_2)\rangle$ are critical. Using Lemma [14] and Observation [13] we find that the FS are also Gaussian iff either $ab = -c^2$ for states in $L_{abc_2}$ or $a^2 + b^2 = 0$ for states in $L_{a_2b_2}$. Note that if all of the parameters of a state in $L_{abc_2}$ ($L_{a_2b_2}$) are equal to zero, the state is a product state (biseparable state) respectively.

- **GSLOCC classes containing states in the null cone:** The states in the null cone are the ones for which the normal form vanishes. For 4-mode GFS there exists, as in the 3-mode case, exactly one GSLOCC class containing these states, which is the class $L_{abs}$ of [7] with $a = b = 0$. The representative is of the form
\[ |\Psi_f\rangle = |1100\rangle + |1111\rangle + |1010\rangle + |0110\rangle. \tag{33} \]
This state is Gaussian and GLU-equivalent to the 4-qubit W-state.

Hence, for 4-mode GFS there exist infinitely many entangled GSLOCC classes. More precisely, there are infinitely many GSLOCC classes that contain a critical state, that is the states in these classes can be transformed into the normal form. Furthermore, there exist infinitely many GSLOCC classes of semi-stable states, which tend to a non-zero normal form without being stable. There exists also a single GSLOCC class containing states in the null cone for which the normal form vanishes.

There are less Gaussian GSLOCC classes for 4-mode GFS than there are for FS, which is not surprising as not all FS are GFS, due to the condition given in Eq. (29) on $|\Psi_f\rangle$. This also implies that there exist less GSLOCC classes than SLOCC classes in the qubit case (see [7]). However, as mentioned above, there are still infinitely many such classes. Examples of SLOCC classes that contain FS but no GFS are those denoted by $L_{a_2b_2}$ in [7] for $a \neq 0$ [71].

### C. Fermionic LOCC operations

As for transformations of pure $n$-mode $n$-partite GFS there exist no non-trivial GLOCC transformations, we consider here a larger class of deterministic transformations and study fermionic LOCC (FLOCC) transformations. For such transformations the local maps that are applied have to be fermionic and the measurement operators that are implemented in each round have to be parity-respecting and local, i.e., they have to be of the form $X^kD$ (in Jordan-Wigner representation), where $k \in \{0, 1\}$ and $D$ denotes here and in the following a diagonal matrix [72]. More precisely, in each round of an FLOCC transformation one implements locally a fermionic POVM measurement with measurement operators that are of the form $X^kD$, possibly discards some
consider classical information about the outcome, then communicates the relevant information to the other parties. These apply depending on the measurement outcome an arbitrary local completely positive trace-preserving (CPT) fermionic map. Note that the Kraus operators of such maps can be chosen to be of the form $X^k D$ (cf. Lemma 9). Note further that the operations that are implemented in a subsequent round might depend on the information about the prior outcomes.

For a concatenation of finitely many of such rounds the Kraus operators of the map that is implemented in each branch of the protocol, i.e., for a specific sequence of outcomes (taking into account that some information might have been discarded), are of the form $X^{k_1} D_1 \otimes X^{k_2} D_2 \otimes \ldots X^{k_n} D_n$. This can be easily seen as a finite product of operators of this form results in an operator of the same form.

In order to provide a rigorous definition of FLOCC protocols which can also involve infinitely many rounds (in analogy to the one given in [55] for LOCC protocols) let us use the description of a protocol in terms of a quantum instrument, i.e., by the family of CP maps $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$. Here, $\mathcal{E}_i$ is the CP map that is implemented in a specific branch of the protocol denoted by $i$ and it holds that $\sum_{i=1}^m \mathcal{E}_i$ is a trace-preserving map. Moreover, a quantum instrument $\mathcal{P}$ will be called FLOCC-linked to an instrument $\mathcal{P}$ if $\mathcal{P}$ can be implemented by first implementing $\mathcal{P}$ followed by exactly one more round of an FLOCC protocol as defined before (where again the operations that are implemented in each branch $i$ can depend on all previous outcomes) and then possibly by some discarding of classical information. With all that, $\mathcal{F}$ is defined as the instrument of a FLOCC transformation if there exists a sequence of instruments of finite-round FLOCC protocols where each element of the sequence is FLOCC-linked to its preceding element. Furthermore, for each element there exists a way to discard information in the final round such that the resulting sequence of instruments converges to $\mathcal{F}$. In the following we consider also infinite-round FLOCC, however, only those for which all Kraus operators are of the form $X^k D$. In order to highlight that there might be a difference to FLOCC as defined above, we denote this set of operations by FLOCC'. Note that, of course, any finitely-many-rounds FLOCC is contained in FLOCC'. We are interested in FLOCC' transformations among pure GFS, and, in particular, in the maximally entangled set for this scenario. We first review the concept of the maximally entangled set and then explain how it can be determined for GFS when one considers FLOCC' transformations.

1. The maximally entangled set

In [10] some of us introduced the Maximally Entangled Set (MES) as the minimal set of $n$-partite entangled states that has the property that any pure $n$-partite entangled state can be obtained via LOCC from a state within this set. That is the states in the MES are those which cannot be reached via LOCC from some state that is not LU-equivalent. In [54] FLOCC transformations among Gaussian states of two or three bosonic modes have been considered. There, it has been shown that not all pure bosonic three-mode Gaussian states can be obtained via GLOCC from a symmetric Gaussian state, i.e., the MES of bosonic three-mode Gaussian states under GLOCC cannot consist only of symmetric Gaussian states. In the following, we are interested in the MES of GFS under FLOCC'. It is defined analogously to before as the minimal set of $n$-partite $n$-mode entangled GFS for which it holds that any pure $n$-partite $n$-mode entangled GFS can be obtained via FLOCC' from a state within this set.

As we explain in the next section using the Jordan-Wigner representation, FLOCC' reachability of GFS can be studied in a way analogous to qubit systems. There, we used the necessary and sufficient conditions of convertibility via separable maps (SEP) of [73] to identify the states that cannot be reached via SEP from a state that is not LU-equivalent. As separable maps (strictly) include LOCC transformations it follows that these states are not reachable via LOCC. We outline here the basic idea of the proof of the necessary and sufficient condition derived in [73] for qudits in order to explain how this result can also be applied to study FLOCC' transformations of GFS.

The initial state of the transformation is denoted by $g|\Psi_s\rangle$ and the final state by $h|\Psi_s\rangle$, where $g,h$ are invertible local operators [74]. In order to perform this transformation it has to hold for all the Kraus operators of the separable map, $A_i = A_i^{(1)} \otimes A_i^{(2)} \otimes \ldots \otimes A_i^{(n)}$, that $A_i g |\Psi_s\rangle \propto h |\Psi_s\rangle$ and therefore $(h^{-1} A_i g) |\Psi_s\rangle \propto |\Psi_s\rangle$. Using the definition for the local symmetries of a state $S_{\Psi} = \{S : S |\Psi\rangle = |\Psi\rangle, S = S^{(1)} \otimes S^{(2)} \otimes \ldots \otimes S^{(n)}, S^{(j)} \in GL(d_j, \mathbb{C})\}$, where $d_j$ denotes the local dimension of system $j$, we have that $h^{-1} A_i g \propto S_i$ where $S_i \in S_{\Psi_s}$. That is, the measurement operators $A_i$ are proportional to $h S_i g^{-1}$. Taking into account the proper proportionality factors and using that the separable map has to be trace-preserving one obtains the following necessary condition for transforming $g |\Psi_s\rangle$ into $h |\Psi_s\rangle$ via SEP. There has to exist a probability distribution $\{p_i\}_{i=1}^m$ and local symmetries $S_i \in S_{\Psi_s}$, such that

$$\sum_{i=1}^m p_i S_i^\dagger H S_i = r G,$$

where $H = h^\dagger h$, $G = g^\dagger g$ and $r = \frac{\langle \Psi_s | H |\Psi_s\rangle}{\langle \Psi_s | G |\Psi_s\rangle}$. Moreover, it is straightforward to see that this condition is also sufficient [73]. Using this criterion one can determine the states that are not reachable via a SEP transformation and hence, not via LOCC.

In the subsequent subsection we discuss how one can in an analogous way obtain necessary and sufficient conditions for transformations among pure GFS via CPT maps with local fermionic Kraus operators.
2. The maximally entangled set of GFS under FLOCC' transformations

As mentioned before the MES of GFS under FLOCC' corresponds to the minimal set of n-partite n-mode GFS with the property that any pure n-partite n-mode entangled GFS can be obtained via FLOCC' from a state within this set. Hence, this set corresponds to the optimal resource under the restriction to pure GFS and FLOCC' transformations. As we will see, it can be determined using a similar method as has been employed to characterize the MES for 3- and 4-qubit states. In particular, using the Jordan-Wigner representation one can find analogously to the qudit case [73], which we reviewed in the previous subsection, the necessary and sufficient condition for transformations among GFS via separable maps whose Kraus operators are of the form
\[ X^{m_1} D_1 \otimes X^{m_2} D_2 \otimes \ldots \otimes X^{m_n} D_n, \]
with \( m_i \in \{0,1\} \) and \( D_i \) is diagonal. Note that this class of separable maps includes all FLOCC' transformations, as all local fermionic operators can be written like that (in Jordan-Wigner representation).

Before proceeding studying the separable maps, let us briefly recall the relation between the operator \( X \) and the Majorana operators. \( X \) corresponds to the minimal set of the Majorana operators and hence, it either commutes or anticommutes with the application of \( (X)^{m_j} D_j \) for \( j \neq i \). Note that as \( X_i \) (in Jordan-Wigner representation) corresponds in the Majorana operators to \(-i\bar{c}_i\bar{c}_2(-i\bar{c}_3\bar{c}_4)\ldots(-i\bar{c}_{2i-1}\bar{c}_{2i-2})\bar{c}_{2i-1} \) it follows that despite the fact that this operator is acting locally on the modes it is not only acting on mode \( i \). Its implementation requires also other parties to apply a local unitary. Any diagonal matrix \( D_i \) can be written in the Majorana operators (up to a proportionality factor) as \( e^{-i\alpha \bar{c}_i\bar{c}_2} \) for some \( \alpha \in \mathbb{C} \) and therefore only acts on mode \( i \).

In the previous subsection we have seen that all Kraus operators \( A_i \) of a separable map transforming \( \gamma|\Psi_s\rangle \) to \( h|\Psi_s\rangle \) have to be proportional to \( hS_i g^{-1} \). Recall that \( S_i \) denotes a local symmetry of \( |\Psi_s\rangle \). As for the transformations we are interested in the operators \( h, g \) and the Kraus operators \( A_i \) are local fermionic operators this implies that also any symmetry \( S_i \propto h^{-1}A_i g \) contributes to the transformation of the form \( (X)^{m_1} D_1 \otimes (X)^{m_2} D_2 \ldots \otimes (X)^{m_n} D_n \). Hence, only symmetries of this form appear in the necessary and sufficient condition given by Eq. \( (34) \) if one considers transformations among GFS via the considered class of separable maps.

Thus, the local symmetries that can contribute to such transformations are a subset of the local symmetries that are available for transformations among qubit states. It follows straightforwardly that if the qubit state corresponding to the GFS (in Jordan-Wigner representation) is not reachable via a non-trivial SEP transformation then the GFS is not reachable via a separable map with the specific form of Kraus operators that we impose.

Moreover, as exactly the same methods can be applied that we used to determine the MES for three- and four-qubits one can infer from these results the MES for 3- and 4-mode GFS under FLOCC' [76].

In [74] and [75] finite round LOCC transformations among pure n-qudit states have been investigated. Restricting the measurement operators, local unitaries and SLOCC operators to local fermionic operators one can use an analogous argumentation to obtain the corresponding results for finite-round FLOCC transformations among GFS. In the following subsections we discuss explicitly the MES for 3- and 4-mode GFS under FLOCC'.

As shown in [10] the MES of three-qubit states is given (up to LUs) by
\[ \{D_1 \otimes D_2 \otimes D_3 |GHZ\rangle_3, |GHZ\rangle_3 \}, \]
where for the GHZ-class none of the \( D_i \)'s is proportional to the identity and all of them are real and invertible. Note that all these states are Gaussian and it follows directly that these states also have to be in the MES of 3-mode GFS. As any GFS in the W-class can be written (up to GLUs) as given in Eq. \( (35) \) we have that any tripartite entangled 3-mode GFS is either in the MES or it is of the form \( D_1 \otimes D_2 \otimes |GHZ\rangle_3 \), where at least one \( D_i \) is not proportional to the identity (up to GLUs and particle permutations). In the first case, the state cannot be reached from any other state (even if one would allow the most general LOCC transformation). In the second case it can be easily reached from the GHZ state with the following FLOCC' protocol. Party 1 applies the measurement consisting of the measurement operators \( D_1, D_1 X \) and party 2 applies a measurement consisting of the measurement operators \( D_2, D_2 X \). Hence, the resulting state is \( D_1 X^{k_1} \otimes D_2 X^{k_2} \otimes |GHZ\rangle_3 \). Using that \( X^{k_1} \otimes X^{k_2} \otimes X^{k_1+k_2} |GHZ\rangle_3 = |GHZ\rangle_3 \), we have that if party 3 applies the GLU \( X^{k_1+k_2} \) the resulting state is for any outcome the desired state and hence, the transformation is deterministic.

4. 1 × 1 × 1 × 1 case

The 4-mode case is very similar to the previously discussed 3-mode case. In order to illustrate this, let us consider a few examples of possible transformations among 4-mode GFS via FLOCC'. Note that we consider here only GLOCC classes with non-degenerate and non-cyclic seed states as in Eq. \( (51) \) and \( (52) \). Due to Lemma \( (14) \) any 4-mode GFS with a seed state of the above form is either a state in the MES (see \( (19) \) or of the form (up to permutations) \( |\Psi\rangle = D_1 \otimes D_2 \otimes |\Psi_s\rangle \). If the state is in the MES, it cannot be reached by any other state (even if LOCC would be allowed). Moreover, apart from the seed states all other states in the MES are isolated.
i.e., they cannot be transformed into any other state via FLOCC’. Note that this is in contrast to the qubit case, where the states in Eq. (30) are states in the MES that are non-isolated, i.e., they can be transformed into a state with exactly one local non-diagonal operator (see [10]) via LOCC. These states are, however, no GFS. In case the 4-mode GFS is not in the MES it can be easily reached from the GFS seed state via the following FLOCC’ protocol (for more sophisticated protocols see below). Party 1 applies the measurement consisting of the measurement operators $D_1, D_1 X$. In case of the first outcome, the other parties do not need to apply any transformation. In case of the second outcome all three apply $D$. Hence, as before we consider larger classes among pure fully entangled GFS can be implemented via GLUs. Hence, as before we consider larger classes.

We conclude this section by briefly discussing non-trivial FLOCC’ transformations among pure multimode states. We investigate Gaussian separable transformations (GSEP) among pure fully entangled multimode states, i.e., multimode FS with the property that the Schmidt decomposition (of the state in its Jordan-Wigner representation) with respect to the splitting of one party versus the rest has no zero Schmidt coefficients. As stated in the following Lemma we show that such transformations are only possible if the map corresponds to applying GLUs.

**Lemma 15.** Let $\mathcal{E}_{sep}$ denote a Gaussian trace preserving separable map which transforms at least one pure fully entangled $m_1 \times m_2 \times \ldots \times m_N$-mode FS, $|\Psi\rangle$, into another pure fully entangled $m_1 \times m_2 \times \ldots \times m_N$-mode FS, $|\tilde{\Psi}\rangle$. Then, it holds that $\mathcal{E}_{sep}(\rho) = (U_1 \otimes U_2 \ldots \otimes U_N)(|\Psi\rangle \langle \Psi|)U_1^\dagger \otimes U_2^\dagger \ldots \otimes U_N^\dagger)$ for all $\rho$.

Note that this lemma holds, as in the $n$-partite $n$-mode case for all FS (not only GFS).

**Proof.** This lemma can be shown using an analogous argumentation as in the proof of Lemma 8. We recall here the main steps of the proof and comment on its generalization to the multimode case. As argued in Appendix A2 Gaussian separable maps correspond to product operations, i.e., they are of the form $\mathcal{E}_{sep} = \mathcal{E}_1 \otimes \mathcal{E}_2 \ldots \otimes \mathcal{E}_N$ with GCTPs $\mathcal{E}_i$ which act now on $m_i$ modes. Analogously to the case of a single mode per site we consider $\rho = I \otimes_{k \neq 1} \mathcal{E}_k(|\Psi\rangle \langle \Psi|)$ with spectral decomposition $\sum_i \rho_i |\Psi_i\rangle \langle \Psi_i|$. As before it follows straightforwardly that for $\rho_i \neq 0$

$$|\Psi\rangle \propto A_k \otimes_j I_{m_j} |\Psi_i\rangle \propto A_l \otimes_j I_{m_j} |\Psi_i\rangle,$$

where the operators $A_i$ are the Kraus operators of $\mathcal{E}_1$ and $I_{m_j}$ denotes here the identity on $m_j$ modes. We show next that there exists a Schmidt decomposition of the Jordan-Wigner representation of $|\Psi_i\rangle$ in the splitting of the first $m_1$ modes versus the remaining modes such that all involved local (with respect to that splitting) states are fermionic. In order to do so note that the reduced state of the first $m_1$ modes has to be fermionic and therefore the range of the reduced state is spanned by FSs. Hence, any purification of this state (in particular $|\Psi_i\rangle$) is given by $\sum_{j=1}^{2^{m_1} - 1} \lambda_j |\eta_j\rangle |\nu_j\rangle$, where $|\eta_j\rangle$ are orthogonal FSs of $m_1$ modes. That the $n_1 \equiv \sum_{j=2}^N m_j$-mode states $|\nu_j\rangle$ are also fermionic, follows from the facts that the projector onto the states $|\eta_j\rangle$ are fermionic operators (as they are sums of only even monomials in the Majorana operators) and that $|\Psi_1\rangle$ is a FS. Moreover, as the final state $|\Phi\rangle$ is fully entangled all Schmidt coefficients of $|\Psi_i\rangle$ have to be unequal to zero (see Eq. (30)), i.e., $\lambda_j \neq 0 \forall j \in \{1, \ldots, 2^{m_1} - 1\}$. Analogous to the case of a single mode per site one can now apply $|\nu_j\rangle |\nu_j\rangle$ on both sides of Eq. (30) in order to see that the action of $A_k$ on a basis is the same (up to a proportionality factor) for all Kraus operators $A_k$ and $A_l$. A.

**A. Gaussian separable transformations**

In this section we consider pure $N$-partite GFS where each party holds $m_i$ modes. We first investigate transformations among fully entangled multimode GFS (for the definition see below) via Gaussian trace preserving separable transformations (GSEP), i.e., Gaussian transformations for which the CM of the CJ state is of direct sum form. We show that also in this more general setting such transformations are only possible if the map is a GLU transformation. As GSEP includes GLOCC transformations (see Appendix A2) this implies that any GLOCC transformation that is possible among pure fully entangled GFS can be implemented via GLUs. Hence, as before we consider larger classes of operations, namely probabilistic transformations and FLOCC’ transformations. More precisely, we briefly explain how the GSLOCC classes can be characterized in the multimode case for classes which contain a critical state. We conclude this section by briefly discussing non-trivial FLOCC’ transformations among pure multimode GFS.
hence $\mathcal{E}_1$ is a Gaussian unitary operation. Rearranging the modes \[31\] and applying the same argumentation for the various parties proves the lemma. \qed

As GSEP is defined such that it includes all GLOCC transformations (see Appendix \[12\]) this lemma implies that non-trivial GLOCC transformations among pure fully entangled GFS are not possible even if one considers the case of an arbitrary (finite) number of modes per site. Hence, in the following section we will consider probabilistic local transformations and comment on the characterization of the GSLOCC classes for multimode states.

### B. Gaussian Stochastic LOCC

As deterministic transformation are not possible among pure fully entangled GFS we will consider next stochastic GLOCC operations. We distinguish between bipartite and multipartite GFS, as in \[38\] a decomposition for bipartite states was introduced. For multipartite states we show similar to the single-mode per site case that stable states can be brought into a normal form.

1. **Bipartite case**

For bipartite pure multimode states, i.e., party A (B) holds $d_1$ ($d_2$) modes respectively, it was shown in \[38\] that one can consider without loss of generality two subsystems of $d$ modes each, where $d = \min(d_1, d_2)$, that is the two parties hold the same number of modes. Thus, we only consider $d \times d$ states here. A direct consequence of the results obtained in \[38\] is the following observation for bipartite multimode GSLOCC classes.

**Observation 16.** For $d \times d$ modes (GFS) there exist $d$ different GSLOCC classes.

Proof. This can be easily shown by using that any such state is up to GLU equivalent to $\otimes_{i=1}^d |\Psi_i\rangle_{AB}$, with $|\Psi_i\rangle_{AB} = \cos \theta_i |00\rangle_{AB} + \sin \theta_i |11\rangle_{AB}$ \[38\]. Thus, A and B share $d$ 2-mode states $|\Psi_i\rangle_{AB}$, which are entangled for $\theta_i \neq 0, \pi/2$. Moreover, each GSLOCC class is characterized by the local rank of the states (the rank of the reduced states $\rho_A, \rho_B$ does not increase under GSLOCC) \[32\] and, hence, we immediately arrive at the above stated result. \qed

Thus, there exist as many GSLOCC classes for bipartite GFS as SLOCC classes for bipartite qudit states.

2. **Multipartite case**

Analogously to the case of a single mode per site one can transform any multi-mode FS into a normal form by consecutively applying fermionic local invertible operators. Note again that this normal form vanishes for states in the null cone. Moreover, there exist semi-stable states that tend to a non-zero normal form but their SLOCC class does not contain a critical state \[5\].

**Lemma 17.** Let $|\Psi\rangle$ be an entangled $m_1 \times m_2 \times \ldots \times m_N$-mode FS. Then $|\Psi\rangle$ can be constructively transformed (by applying invertible fermionic operators) into a unique (up to LUs) critical FS, $|\Psi_s\rangle$ (up to a proportionality factor which can tend to 0).

The lemma can be proven by using the same argumentation as in the case of a single mode per site (see Lemma \[10\]). Note that the only difference is that the local invertible operators, i.e., the reduced states, are no longer diagonal and thus, not automatically also Gaussian. However, they are general fermionic operators. Note, furthermore, that any GSLOCC class containing a critical state can be easily characterized via this state. That is, if $|\Psi_s\rangle$ is a critical GFS then any other state $|\Psi\rangle$ in the same GSLOCC class is given by $M_1 \otimes M_2 \ldots \otimes M_n |\Psi_s\rangle = |\Psi\rangle$. Here, the operators $M_i$ are Gaussian invertible operators.

### C. Fermionic LOCC

Transformations among fully entangled multimode GFS via FLOCC \[38\] work analogously to the $n$-mode $n$-partite case. Note, however, that in this setting there is an additional freedom when one considers transformations to not fully entangled states. Similar to the finite dimensional qudit case and contrary to the single-mode case it is possible to reduce the local rank of the parties via FLOCC’, leaving still all parties entangled with each other.

### VI. CONCLUSION

We investigated the entanglement of GFS. For this purpose, we first derived a standard form of the CM for mixed $n$-mode $n$-partite GFS. Any CM can be brought into this standard form via GLU. As the standard form is unique, any two GFS are GLU-equivalent iff their CMs in standard form coincide. Furthermore, we showed that only two of the definitions of separable FS from \[15\] are reasonable for GFS. This is due to the fact that any separable state should have the property that also two copies of this state are again separable. For our derivations we used the definition of separability which declares a state separable if it is given by a convex combination of product states which commute with the local parity operator. According to this physically meaningful definition any separable state can be prepared locally. Using this definition we showed that for pure fully entangled $n$-mode $n$-partite as well as multimode GFS any GSEP is equivalent to a GLU. Thus, there exist no non-trivial GLOCC transformations among pure fully entangled GFS. Due to
this fact we consider then the larger class of GSLOCC. With the help of a result on normal forms of states from 66 we also characterized the GSLOCC classes in the Jordan-Wigner representation and furthermore, explicitly derive them for few-mode systems. Then, we investigated the more general FLOCC’, which contains in particular finitely-many rounds FLOCC (see Sec. IV C), to obtain insights into the various entanglement properties of GFS and we show how to identify the MES of pure $n$-mode $n$-partite GFS under FLOCC’.

Let us finally compare the fermionic case investigated here with the bosonic and the finite dimensional scenarios. In all three cases a computable condition for two ($n$–partite $n$–modes or $n$–qubit) states to be (G)LU–equivalent has been presented 54–84. Regarding the bosonic Gaussian case, we have that GSLOCC coincide with GLOCC transformations. This follows from the fact that any GSLOCC operation can be completed to a deterministic transformation. Moreover, there exist GLOCC transformation among pure bosonic Gaussian states which are not just GLU transformations (see e.g. 54). The MES for bosonic Gaussian states is not known, however, in 54 a class of three–mode states has been identified which can reach states which cannot be reached from any symmetric three–mode state (including the GHZ and W states). Regarding the finite dimensional case, there exist (not surprisingly) more SLOCC classes than for GFS. Moreover, for Hilbert spaces composed of local Hilbert spaces of equal dimensions, it has been shown that almost all states are isolated, i.e., the state can neither be reached, nor transformed into any other (not LU–equivalent) state via LOCC 12–13. This resembles the fermionic case. However, as mentioned before, the reason for this to be true stems from the fact that almost no state possesses a local symmetry.

It would be interesting to investigate another physically relevant scenario by imposing a (global) particle-number selection rule (as it is observed by elementary fermions in nature) on the states considered and studying state transformations via number-preserving local operations. Moreover, as in the qudit case, the transformations from a multipartite state, where each party holds more than a single mode (a single qubit) to a state whose local rank is smaller might well allow (more) non-trivial transformations, respectively. Physically motivated, restricted set of states, such as FS or GFS, are ideally suited for this investigation, as it will be more trackable than the general qudit case. Moreover, this class of states is rich enough so that the results derived for them have the potential to lead also to new insight into state transformations among qudit states.

ACKNOWLEDGMENTS

The research of KS and BK was funded by the Austrian Science Fund (FWF) Grant No. Y535-N16, the DFG and the ERC (Consolidator Grant 683107/TempoQ).

Appendix A

In this appendix we study first the Choi-Jamiolkowski (CJ) isomorphism 15–53–56 among Gaussian states and Gaussian CP maps. Note that similar aspects of Gaussian CP maps have already been studied in 42. However, there the author was using a different definition of the “tensor product” ($\otimes_f$) in the calculation. We summarize here the results using our notation. Then, we consider Gaussian LOCC (GLOCC) transformations and show that any GLOCC corresponds via the CJ isomorphism to a separable state. These investigations lead to the natural definition of fermionic separable maps (FSEP). Considering then the possible states which can be generated via GLOCC enables us to rule out the definition $S_{2\pi}$ for separable states. That is, if $S_{2\pi}$ does not coincide with $S_{2\pi}$ for GFS there exist states in $S_{2\pi}$ which can neither be prepared locally by Gaussian operations, nor do they belong to the limit of such a preparation scheme.

1. Choi-Jamiolkowski isomorphism in the Gaussian case

The CJ isomorphism is a one to one mapping between CP maps and positive semidefinite operators. Denoting by $E$ the CP map that is acting on $n$ modes and by $\rho_E$ the corresponding operator we have

$$\rho_E = E(\rho) = tr_{23}(\rho_2^{12}\rho_3^{23}\Phi_{2n}^+\Phi_{2n}^-), \quad (A1)$$

where $\Phi_{2n}^+ = \otimes_{i=1}^{2n}(1 + \i\epsilon_{2n}a_{2n+i})$. In 48 it has been shown that separable maps correspond to separable operations and that several other properties of the operators can be inferred from the maps and vice versa. The aim of this section is to show that the same isomorphism holds for Gaussian states. In the subsequent subsection we will then investigate the relation between separable operators and the corresponding maps. Note that we write Gaussian states and operators in this section in the Grassmann representation, see 12 for more details. Note further that $\rho_E$ is a GFS iff $E$ is a Gaussian map. It is obvious that $\rho_E$ is a Gaussian state if $E$ is Gaussian as $\Phi_{2n}^+$ is a GFS. Moreover, due to $E(\rho) = tr_{23}(\rho_2^{12}\rho_3^{23}\Phi_{2n}^+\Phi_{2n}^-)$ one obtains that if $\rho_E$ is a GFS then also $E(\rho)$ is Gaussian for all GFS $\rho$ and therefore $E$ corresponds to a Gaussian map.

In 42 it was shown that a linear CP map on $n$ fermionic
modes is Gaussian iff it has a (Grassmann) integral representation
\[ \mathcal{E}(X)(\theta) = C \int D\eta D\mu \exp[S(\theta, \eta) + i\eta^T \mu] X(\mu), \]  
(A2)
where
\[ S(\theta, \eta) = \frac{i}{2} \left( \begin{array}{c} \theta \\ \eta \end{array} \right)^T \left( \begin{array}{cc} A & B \\ -B^T & D \end{array} \right) \left( \begin{array}{c} \theta \\ \eta \end{array} \right) \equiv \tilde{\theta}^T M \tilde{\theta}, \]
with \( C \geq 0 \), real \( 2n \times 2n \) matrices \( A, B, D \) and \( M^T \tilde{\theta} M \leq 1 \). The identity map on \( n \) modes is given by \( A = D = 0 \) and \( B = I \). Thus for a map \( \mathcal{E}^r \) on \( n+m \) modes that acts non-trivially only on the first \( n \) modes we take \( A' = A \oplus 0, B' = D \oplus 0, B' = B \oplus I \). Applying this map (for \( m = n \)) to the maximally entangled state of 2n modes, we get as the CM of the output state (with \( \tilde{\theta} = (\theta, \theta') \) (and same for \( \tilde{\eta}, \tilde{\mu} \))
\[ \tilde{x}_{12} = (\tilde{\theta}, \tilde{\eta}), \tilde{x}_{23} = (\tilde{\eta}, \tilde{\mu}) \]
and \( \int D\eta D\mu D\rho e^{i\tilde{x}_{12}^T (A_{t+} B') \tilde{x}_{12} + i\tilde{\eta}^T \tilde{\mu} e^{i\tilde{x}_{23}^T (0_{11}) \tilde{\mu}} = e^{i\tilde{\theta}^T (A_{t+} B') \tilde{\mu} e^{i\tilde{x}_{23}^T (0_{11}) \tilde{\mu}}} \]
In the last step we used the Gaussian integration rule (see Eq. (13) of [42]), \( y = (iB')^{-1} \tilde{\theta}, \tilde{\eta} \) and \( \tilde{M} = \left( \begin{array}{cc} D & 0 \\ 0 & I \end{array} \right) \).

Since \( y \) is non-zero only in the first two components, we only need the upper diagonal block of the \( 2 \times 2 \) block-matrix \( \tilde{M}^{-1} \), which is given by the Schur complement as
\[ \left( \begin{array}{cc} D & 0 \\ 0 & 0 \end{array} \right)^{-1} = \left( \begin{array}{cc} 0 & 1 \\ 1 & D \end{array} \right). \]
Thus, we end up with a Gaussian Grassmann representation with CM
\[ \left( \begin{array}{cc} A & B \\ -B^T & D \end{array} \right). \]  
(A3)
Hence, the GFS with this CM is the CJ-state \( \rho_C \) of the map \( \mathcal{E} \). Note that by using the above mentioned definition of a tensor product \( \otimes_f \) (see Def. 5 in [42]) for the computation of the CJ-state we obtain a CM \( \left( \begin{array}{cc} A & B \\ -B^T & D \end{array} \right) \).

The corresponding state is obtained by applying the local operator \( \prod_{i=1}^{2n} \tilde{c}_i \) to \( \rho_C \).

In order to confirm that the state \( \rho_C \) with CM given in Eq. (A3) allows for the physical interpretation, which is characteristic for the CJ-state, that it can be used to realize the map \( \mathcal{E} \) via teleportation, we compute
\[ \text{tr}_{23}(\rho_C^{12} \hat{A}^T \phi^*_2 \phi_2^{23} \langle \phi^*_2 \phi_2 \rangle). \]
Here, the superscripts indicate on which of the three different blocks of modes the state is nontrivial. Using the formula for the trace of two operators \( X, Y \) in Grassmann variables [7] (see also Eq. (15) in [42]) and with \( X = \rho_C^{12} \hat{A}^T, Y = \phi^*_2 \phi_2^{23} \langle \phi^*_2 \phi_2 \rangle \) the trace is given by
\[ \text{tr}_{23}(XY) \propto \int D\eta D\mu e^{(iB')^T \eta + \tilde{\theta}^T (A_{t+} \eta + \eta'^T D \eta' + \tilde{\mu}^T (0_{11} \tilde{\mu}) e^{i\tilde{\theta}^T \eta} = e^{i\tilde{\theta}^T A_{t+} \eta} \int D\tilde{x}_{23}^T \tilde{x}_{23} + i\tilde{\eta}^T \tilde{\mu} \tilde{\mu}. \]

Here, again \( \tilde{x}_{23} = (\tilde{\eta}, \tilde{\mu}) \) and
\[ \xi^T = ((iB')^T \theta, 0, 0, 0), \]
\[ M' = \left( \begin{array}{cccc} D & 0 & -iI & 0 \\ 0 & \Gamma & 0 & -iI \\ iI & 0 & 0 & I \\ 0 & iI & -I & 0 \end{array} \right). \]
Using again the Gaussian integration rule (Eq. (13) in [42]) we obtain as a result a Gaussian state with CM
\[ \Gamma_{\text{out}} = A - (iB) \left( \left( \begin{array}{c} D \\ \Gamma \end{array} \right) - \left( \begin{array}{c} 0 \\ -I \end{array} \right) \right)^{-1} \right)_{11} \]
\[ = A + B \left( \left( \begin{array}{c} D \\ \Gamma \end{array} \right)^{-1} \right)_{11} B^T, \]  
(A4)
which is just \( \xi^T \).

Summarizing, we have shown that the state \( \rho_C = (\mathcal{E} \otimes \mathbb{I})(\Phi^{2n}_2 \langle \Phi^{2n}_2 \rangle) = \rho_M \), where the GFS \( \rho_M \) with CM \( M = \left( \begin{array}{cc} A & B \\ -B^T & D \end{array} \right) \) is the CJ-state of the Gaussian map \( \mathcal{E} \) given in Eq. (A2) or equivalently as the Gaussian map which maps the CM \( \Gamma \) to \( \Gamma_{\text{out}} \) as given in Eq. (A4).

2. Gaussian LOCC (GLOCC)

Let us now investigate the relation of the entanglement properties of CJ-state and the entanglement properties of the corresponding CP map. We will consider here only bipartite systems, however, all arguments hold also for the multipartite setting. In case of finite dimensional systems a CPTM, \( \mathcal{E} \), is called separable if it can be written as
\[ \mathcal{E}(\rho) = \sum_k A_k \otimes B_k \rho A_k^T \otimes B_k^T. \]  
(A5)
As the set of separable maps (SEP) strictly contains the set of LOCC, i.e., the set of maps which can be realized via local operations and classical communication, SEP lacks a clear physical meaning. Hence, when considering restricted sets of maps, such as here fermionic or Gaussian maps, there is no clear way of specializing the notion of SEP to these sets. This is why we consider here the physically meaningful, however, mathematically generic much less tractable set of LOCC, for which this
specialization is obvious. We will then show that this consideration suggests the natural definition of fermionic SEP (FSEP).

Let us first consider the CJ-state of a local map, \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \), i.e., a composition of two maps, \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), which act on the first and second system nontrivially, respectively. In this case, the CM of the CJ-state splits in the form \( A = A_1 \oplus A_2, B = B_1 \oplus B_2, D = D_1 \oplus D_2 \). One can easily check that \( \mathcal{E}(\Phi_{-}) \) is separable with respect to the splitting \( 13/24 \) according to our definition (see Sec. IIIA). Hence, using our definition of separability (\( S_{2x} \)), the CJ isomorphism maps local maps to separable states.

Let us next show that the CJ-state of any Gaussian LOCC is separable according to the definition \( S_{2x} \). That is, we show that any map which describes a Gaussian state, \( \rho \), is separable. Any map, \( \mathcal{E} \), corresponding to a finitely-many-rounds FLOCC protocol together with those, which are parity respecting, i.e., is fermionic, \( \mathcal{E}_A \otimes \mathcal{E}_B \), can have non-zero correlation between \( A \) and \( B \) products of locally measurable observables factorize, by definition. In case the set \( S_{2x} \) contains such a state, then calling states in \( S_{2x} \), separable does not conform to the usual operational definition.

Note that in the argument above the restriction to locally realizable maps has never been used. Hence, a natural definition of Gaussian separable maps (GSEP) is the set of CPTMs whose CJ-state is a separable Gaussian state, i.e., \( \rho_{GSEP} = \rho_{A} \otimes \rho_{B} \) (which for GFS is equivalent to \( \Gamma_{GSEP} = \Gamma_{A} \oplus \Gamma_{B} \)). Note that this implies that \( \mathcal{E}_{GSEP} = \mathcal{E}_{A} \otimes \mathcal{E}_{B} \). FSEP is then defined as the set of CPT maps that can be written as \( \mathcal{E}(\rho) = \sum_{k} (A_{k} \otimes B_{k}) \rho (A_{k} \otimes B_{k})^{\dagger} \) where all the \( A_{k}, B_{k} \) are parity respecting operators.

### Appendix B: Proof of Observation 2

Here, we prove the observation that a product state according to definition \( \mathcal{T}_{1x} \), i.e., the set of states for which the expectation values of all products of physical observables factorize, can have non-zero correlation between \( A \) and \( B \).

**Proof.** Let us denote by \( \mathcal{T}_{1x} \) the set of states for which all products of locally measurable observables factorize, by \( S_{1x} \) its convex hull, and by \( S_{G} \) the set of Gaussian states.

We show that \( \rho \in S_{1x} \cap S_{G} \) implies \( \Gamma_{\rho} = \left( \begin{array}{cc} \Gamma_{A} & C \\ -C^{T} & \Gamma_{B} \end{array} \right) \) with rank \( C \leq 1 \) and that there are such states with rank \( C = 1 \).

We consider observables of the form \( \Pi_{m=1}^{m=n} c^{a}_{m} \) and \( \Pi_{j=1}^{m=n} \tilde{c}_{j} b^{a}_{j} \), where \( c^{a}_{m} \) refer to Majorana operators on Alice’s (Bob’s) modes. We exploit the fact that we can compute their expectation values in two ways: either by using the Wick formula for the \( n + m \)-mode Gaussian state or by using the separability condition and the Wick formula twice for the \( n \) and \( m \) local modes separately. We show that these only coincide for all observables if the rank of the off-diagonal block \( C \) of the full CM is not larger than 1.
Considering the observable $c_k^c c_k^b c_i^c c_i^b$, we find that $C_{k_1i_2} C_{k_2i_2} = C_{k_1i_1} C_{k_2i_2}$ where $C = (C_{ij})_{ij}$. W.l.o.g. we can choose to work in the basis in which $C$ takes diagonal form (i.e., apply local basis changes $O_a, O_b$ such that $O_a C O_b^T$ is diagonal (singular value decomposition)). Then, considering $k_1 = l_1, k_2 = l_2$ one obtains that the rank of $C$ can be at most one since two non-zero singular values would lead to a contradiction. This single non-zero entry, however, can not lead to any difference between the two ways of computing expectation values of products of even observables and thus there can be (and are) Gaussian states in $S_{21}$ with $C \neq 0$. For example, consider any Gaussian state with CM such that $C_{k_1i_1} \neq 0$ is the only non-zero entry of $C$ and consider any pair of even observables $A = \Pi, c_k^c B = \Pi, \cd_k^b$, then $\rho_r(AB) = \rho_r(A) \rho_r(B) = \rho_r(A) \rho_r(B)$, since, using Wick’s formula any term that contains a pairing $(k_1, k_2)$ must necessarily contain another AB-correlating pair $(k_2, k_2')$ with $k_1 \neq k_2, l_1 \neq l_2$ since no index appears twice in the same subsystem. However, since $C_{k_1i_1}$ is the only non-vanishing entry of $C$ the corresponding term is zero and only the local blocks $\Gamma_A, \Gamma_B$ contribute to $\rho_r(AB)$.

Appendix C: Standard form of the CM of $1 \times 1 \times 1$ states

Here, we state the conditions on the parameters of the standard form for mixed 3 modes GFS, i.e.,

$$S(\gamma) = \begin{pmatrix}
0 & \lambda_1 & d_{12} & 0 & l_1 d_{13} & l_2 d'_{13} \\
-\lambda_1 & 0 & 0 & \lambda_2 & m_1 & m_2 \\
-\lambda_2 & 0 & 0 & 0 & \lambda_3 & m_1 & m_2 \\
-\lambda_1 d_{12} & \lambda_2 & 0 & m_1 & m_2 & 0 \\
-l_1 d_{12} & -\lambda_1 & m_1 & m_2 & 0 & \lambda_3 \\
-l_2 d'_{12} & -l_1 d'_{13} & -m_1 & -m_2 & -\lambda_1 & 0 \\
-l_2 d'_{13} & -l_1 d'_{13} & -m_1 & -m_2 & -\lambda_1 & 0
\end{pmatrix}.$$  

in more detail. If no mode factorizes we have for $\lambda_i > 0$ for $i \in \{1, 2, 3\}$ the following cases:

- $d_{12} > |d'_{12}|$ and $-d_{13} > |d'_{13}|$ and $l_1^2 + l_2^2 = 1$ with either $l_1 > 0$ or $l_2 > 0$ or $l_1 = 0$ and $l_2 > 0$ or $l_1 > 0$ and $l_2 = 0$ or $l_1 = 0$ and $l_2 = 0$.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1$ and $l_2 = 0$ or $l_1 = 0$ and $l_2 = 0$.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = \frac{1}{2}, m_2 = \frac{1}{2}, m_1 = \frac{1}{2}, m_2 = \frac{1}{2}$ with $l_1 = 0$ and $l_2 = 0$.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = |m_2|, m_1 = |m_2|, m_2 = |m_2|, m_2 = |m_2|.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, l_1 = |m_2|, l_2 = |m_2|, l_1 = |m_2|, l_2 = |m_2|.

- $l_1 = l_2 = 0, m_1 = |m_2| \neq 0, m_2 = 0$ and $m_2 = 0$.

- $l_1 = l_2 = 0, m_1 = |m_2| \neq 0, m_2 = 0$ and $m_2 = 0$.

- $d_{12} = |d'_{12}| \neq 0$ and $-d_{13} > |d'_{13}|$, $l_1 = 1$ and $l_2 = 0$ or $l_1 = 0$ and $l_2 = 0$.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = \frac{1}{2}, m_2 = \frac{1}{2}, m_1 = \frac{1}{2}, m_2 = \frac{1}{2}$ with $l_1 = 0$ and $l_2 = 0$.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = |m_2| \neq 0, m_2 = 0$ and $m_2 = 0$.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = |m_2|, m_2 = 0$ and $m_2 = 0$.

- $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = |m_2| \neq 0, m_2 = 0$ and $m_2 = 0$.

In case $\lambda_i = 0$ for some $i \in \{1, 2, 3\}$ the standard form can be obtained analogously. However, in this case $m_i$ is not determined by $\gamma_{ii}$.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009)
[2] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008)
[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993)
[4] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, Proc. R. Soc. A 465, 2537 (2009)
[5] A. Acín, I. Bloch, H. Buhrman, T. Calarco, C. Eichler, J. Eisert, D. Esteve, N. Gisin, S. J. Glaser, F. Jelezko, S. Kuhr, M. Lewenstein, M. F. Riedel, P. O. Schmidt, R. Thew, A. Wallraff, I. Walmsley, and F. K. Wilhelm, (2017), arXiv:1712.03773
[6] W. Dürr, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000)
[7] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002)
[8] G. Gour and N. R. Wallach, Phys. Rev. Lett. 111, 060502 (2013)
[9] C. Eltschka, T. Bastin, A. Osterloh, and J. Siewert, Phys. Rev. A 85, 022301 (2012)
[10] J. I. de Vicente, C. Spee, and B. Kraus, Phys. Rev. Lett. 111, 110502 (2013)
[11] C. Spec, J. de Vicente, and B. Kraus, J. Math. Phys. 57, 052201 (2016) arXiv:1510.09164
[12] G. Gour, B. Kraus, and N. R. Wallach, J. Math. Phys. 58, 092204 (2017)
[13] D. Sauerwein, N. R. Wallach, G. Gour, and B. Kraus, (2017), arXiv:1711.10506
[14] K. Eckert, J. Schliemann, D. Bruss, and M. Lewenstein, Ann. Phys. 299, 88 (2002)
[15] M.-C. Bahnuls, J. I. Cirac, and M. M. Wolf, Phys. Rev. A 76, 022311 (2007)
[16] L. Campos Venuti, M. Cozzini, M. Keyl, and D.-M. Schlingemann, Phys. Rev. A 78, 032301 (2007)
[17] A. R. Plastino, D. Manzano, and J. S. Dehesa, Eur. Phys. Lett. 86, 20005 (2009)
[18] N. Friis, A. R. Lee, and D. E. Bruschi, Phys. Rev. A 87, 022338 (2013)
[19] F. Benatti, R. Floreanini, and K. Titimbo, Open Systems & Information Dynamics, 21, 1440003 (2014), 1303.3178
[20] G. Sárosi and P. Lévay, J. Math. Phys. 47, 115304 (2014)
[21] G. M. D’Ariano, F. Manessi, P. Perinotti, and A. Tosini, Int. J. Mod. Phys. A 29, 1430025 (2014)
[22] V. Eisler and Z. Zimborás, New J. Phys. 17, 053048 (2015)
[23] R. Lo Franco and G. Compagno, Sci. Rep. 6, 20603 (2016)
[24] C. Kloefell and D. Loss, Annu. Rev. Condens. Matter Phys. 4, 51 (2013)
[25] S. Hermelin, S. Takada, M. Yamamoto, S. Tarucha, A. D. Wieck, L. Saminadayar, C. Bäuerle, and T. Meunier, Nature 477, 435 (2011) arXiv:1107.4759
[26] R. P. G. McNeil, M. Kataoka, C. J. B. Ford, C. H. W. Barnes, D. Anderson, G. A. C. Jones, I. Farrer, and D. A. Ritchie, Nature 477, 439 (2011)
[27] E. Bocquillon, F. D. Parmentier, C. Grenier, J.-M. Berroir, P. Degiovanni, D. C. Glattli, B. Plaïais, A. Cavanana, Y. Jin, and G. Fève, Phys. Rev. Lett. 108, 196803 (2012)
[28] S. Das Sarma, M. Freedman, and C. Nayak (2015) npj Quant. Inf. 1, 15001
[29] C. V. Kraus, M. M. Wolf, J. I. Cirac, and G. Giedke, Phys. Rev. Lett. 79, 012306 (2000)
[30] T. B. Wahl, H.-H. Tu, N. Schuch, and J. I. Cirac, Phys. Rev. Lett. 111, 236805 (2013) arXiv:1308.0316
[31] O. Morgenschweis, B. Reznik, and I. Zalzberg, (2008), arXiv:quant-ph/0807.0850
[32] Z. Kadar, M. Keyl, and D. Schlingemann, J. Quant. Inf. Comp. 12, 74 (2012), arXiv:1003.2797
[33] B. Yurke, Phys. Rev. Lett. 56, 1515 (1986)
[34] F. Benatti, R. Floreanini, and U. Marzolino, Phys. Rev. A 89, 032326 (2014)
[35] D. DiVincenzo and B. Terhal, Found. Phys. 35, 1967 (2005)
[36] K. Baumann and G. C. Hegerfeldt, Pub Res Inst Math Sci 21, 191 (1985)
[37] D. W. Robinson, Comm. Math. Phys. 1, 89 (1965)
[38] A. Botero and B. Reznik, Phys. Lett. A 331, 39 (2004)
[39] V. Bach, E. H. Lieb, and J. P. Solovej, J. Stat. Phys. 76, 3 (1994)
[40] Note that the party over which one performs the trace has to be mapped to the last position. This ensures that the expectation value of all operators A acting on only the first part of a bipartition A|B fulfills (A) = tr(AρA) = tr(AρA), with ρA = trB(ρ), see [18, 91].
[41] This can be easily seen by writing the parity operator, P = i^n k c_k, as P = P_k = P and thus, \{P_i, P\} = 0 iff \rho = P_kρP_k + P\rho P_k.
[42] S. Bravyi, J. Quant. Inf. Comp. 5, 216 (2005), arXiv:quant-ph/0404180
[43] see R. Jozsa and A. Miyake, Proc. Roy. Soc. A 464, 3089 (2008) and references therein
[44] B. M. Terhal and D. P. DiVincenzo, Phys. Rev. A 65, 032325 (2002)
[45] Note that the operator O can be written as O = O_1O_2O_2, with O_i ∈ SO(2n, R) ∩ SP(2n, R), for i = 1, 2 and O_2 ∈ SO(2n, R) the so called pairing operator. The real orthogonal matrices O_i are passive transformations, i.e., they commute with the number operator, whereas the pairing operator stems from, e.g., b_i a_i. Note that for a single mode, i.e., n = 1, any real symplectic matrix is a special orthogonal matrix.
[46] The order of the Majorana operators is here c_1 c_2 c_3 c_4 . . . c_{2n-1} c_{2n}.
[47] The corresponding unitary is proportional to c_{2n} c_{2n+1}. Note that c_{2n+1} is a Majorana operator of ancillary mode which has to be ranked last. Note that the operator c_{2n} c_{2n+1} c_{2n+i} . . . c_{2n-1} c_{2n+2}, which is GLU to c_{2n} c_{2n+2}, corresponds to a X on particle k in the JW-representation (see Sec. [11B]).
[48] J. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. 86, 544 (2001) quant-ph/0007057
[49] N. Schuch, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. 92, 087004 (2004) quant-ph/0310124
[50] O_i diagonalizes γ_k γ_{k+1} and O_j diagonalizes γ_k γ_{k+1}. Note that due to the restriction O_i, O_j ∈ SO(2, R) D_{ij} cannot be chosen non-negative.
[51] If d_{jk} = d_{kj} then γ_{k+1} is proportional to an orthogonal matrix and α_k is determined as explained before.
[52] Analogously one determines α_k by diagonalizing γ_{k+1} γ_{k+1} (and imposing the same conditions on the singular values and the orthogonal matrix) if only α_k has been already determined.
[53] One applies an analogous procedure if α_k is expressed as a function of α_n.
[54] G. Giedke and B. Kraus, Phys. Rev. A 89, 012335 (2014)
[55] Le Sawicki, M. Walter, and M. Knä, Journal of Physics A: Mathematical and Theoretical 46, 055304 (2013).
[56] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 50, 2170 (1999) arXiv:quant-ph/9804053
[57] M. Kleinmann, H. Kampermann, and D. Bruß, Phys. Rev. A 84, 042326 (2011)
[58] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter, Comm. Math. Phys. 328, 303 (2014) arXiv:1210.5383
[59] M. Heinonen, C. Spee, and B. Kraus, Phys. Rev. A 93, 012339 (2016)
[60] In analogy to the finite dimensional case and in contrast to the bosonic case all operators are bounded and hence invertible.
[61] F. de Melo, P. Čwikliński, and B. M. Terhal, New J. Phys. 15, 013015 (2013) arXiv:1208.5334
[62] It straightforwardly follows from this equation that the action of A_2 is the same (up to a proportionality factor) as the action of A_1 on the whole Fock basis. In order to see this, apply for j ∈ {0, 1} the projector |v_j⟩⟨v_j| where |v_j⟩ is a Fock state of modes 2, . . . , n for which |v_j⟩⟨v_j| (Ψ∗) ̸= 0 to both sides of the equation.
A simple example is given by the 4-mode GHZ, i.e.,

\[ G. \text{ Kempf and L. Ness, } \text{"The length of vectors in representation spaces," in} \text{Algebraic Geometry: Summer Meeting, Copenhagen, August 7–12, 1978 edited by K. Lønsted (Springer, 1979) pp. 233–243.} \]

Due to Lemma 5, the Kraus operators can be chosen with different parity and therefore the operations on the different modes commute. That is, \((E_1 \otimes E_2)(\cdots) = \sum_{k,l,} (\otimes^{k,l=1}) (\otimes^{k,l=2}) (\otimes^{k,l=3}) (\otimes^{k,l=4}) = \sum_{k,l} (\otimes^{k=1,l=2}) (\otimes^{k=1,l=3}) (\otimes^{k=1,l=4}) (\otimes^{k=2,l=3}) (\otimes^{k=2,l=4}) (\otimes^{k=3,l=4})\). This follows from the fact that we get either no or two phase factors of \(-1\) when commuting the Kraus operators.

[52] G. Gour and N. R. Wallach, New J. Phys. 24, 020504 (2010).

Note that if we consider more modes per site, the operators \((E_1 \otimes E_2)(\cdots) \neq \sum_{k,l} (\otimes^{k,l=1}) (\otimes^{k,l=2}) (\otimes^{k,l=3}) (\otimes^{k,l=4}) = \sum_{k,l} (\otimes^{k=1,l=2}) (\otimes^{k=1,l=3}) (\otimes^{k=1,l=4}) (\otimes^{k=2,l=3}) (\otimes^{k=2,l=4}) (\otimes^{k=3,l=4})\). This leads to the continuity of the expectation value of the local fermionic operators. Hence, the local operators commute and partial traces can be performed without any additional reordering. It can also easily be seen that any operator \(X\) acting on system \(i\) that appears in this equation can be represented there in terms of Majorana operators by \(\tilde{x}\).

[53] G. Gour and N. R. Wallach, New J. Phys. 24, 020504 (2010).

as we do not consider transformations reducing the local rank of the states, i.e., we consider only transformations among truly multipartite entangled states.

For some operators \(L\), the local symmetries appearing in this equation are of the form \((X)^m D_1 \otimes (X)^m D_2 \otimes \cdots \otimes (X)^m D_n\), it can be easily seen that if one considers this equation in terms of Majorana operators it only involves even powers of the Majorana operators of a single mode.

[54] G. Gour and N. R. Wallach, New J. Phys. 24, 020504 (2010).

Hence, the local operators commute and partial traces can be performed without any additional reordering. It can also easily be seen that any operator \(X\) acting on system \(i\) that appears in this equation can be represented there in terms of Majorana operators by \(\tilde{x}\).

[55] G. Gour and N. R. Wallach, New J. Phys. 24, 020504 (2010).

Note that exchanging the order of the parties (but keeping the relative order among the modes belonging to one party) neither changes the product structure of the maps due to Lemma 4 nor the Krauss operators. Moreover, the Schmidt coefficients of the state in Jordan-Wigner representation are not changed by such a rearranging.

[56] M. A. Nielsen, Phys. Rev. Lett. 83, 436 (1999).

Note that if we consider more modes per site, the operators \((E_1 \otimes E_2)(\cdots) \neq \sum_{k,l} (\otimes^{k,l=1}) (\otimes^{k,l=2}) (\otimes^{k,l=3}) (\otimes^{k,l=4}) = \sum_{k,l} (\otimes^{k=1,l=2}) (\otimes^{k=1,l=3}) (\otimes^{k=1,l=4}) (\otimes^{k=2,l=3}) (\otimes^{k=2,l=4}) (\otimes^{k=3,l=4})\). This leads to the continuity of the expectation value of the local fermionic operators. Hence, the local operators commute and partial traces can be performed without any additional reordering. It can also easily be seen that any operator \(X\) acting on system \(i\) that appears in this equation can be represented there in terms of Majorana operators by \(\tilde{x}\).

[57] M. A. Nielsen, Phys. Rev. Lett. 83, 436 (1999).

Note that if we consider more modes per site, the operations which can be applied would be of the form \(X^a \otimes X^b T\), where \(T\) has to commute with \(Z \otimes Z\).

[58] M. A. Nielsen, Phys. Rev. Lett. 83, 436 (1999).

Note that if we consider more modes per site, the operations which can be applied would be of the form \(X^a \otimes X^b T\), where \(T\) has to commute with \(Z \otimes Z\).

[59] M. A. Nielsen, Phys. Rev. Lett. 83, 436 (1999)

Note that if we consider more modes per site, the operations which can be applied would be of the form \(X^a \otimes X^b T\), where \(T\) has to commute with \(Z \otimes Z\).

[60] M. A. Nielsen, Phys. Rev. Lett. 83, 436 (1999)

Note that if we consider more modes per site, the operations which can be applied would be of the form \(X^a \otimes X^b T\), where \(T\) has to commute with \(Z \otimes Z\).

[61] M. A. Nielsen, Phys. Rev. Lett. 83, 436 (1999)