Abstract
In this paper we shall investigate the asymptotic behaviour of Weyl numbers of embeddings of tensor product Besov spaces into Lebesgue spaces. We shall compare these results with the known behaviour of entropy, approximation and Kolmogorov numbers.

1 Introduction
Weyl numbers have been introduced by Pietsch [36]. They belong to the family of s-numbers as, e.g., Kolmogorov, Gelfand and approximation numbers, we refer to [12, 29, 37, 38]. The particular interest in Weyl numbers stems from the fact that they are the smallest known s-number satisfying the famous Weyl-type inequalities, i.e.,
\[
\left( \prod_{k=1}^{2n-1} |\lambda_k(T)|^{1/(2n-1)} \right) \leq \sqrt{2e} \left( \prod_{k=1}^{n} x_k(T) \right)^{1/n} \quad (1.1)
\]
holds for all \( n \in \mathbb{N} \), in particular,
\[
|\lambda_{2n-1}(T)| \leq \sqrt{2e} \left( \prod_{k=1}^{n} x_k(T) \right)^{1/n},
\]
see Pietsch [36] and Carl, Hinrichs [11]. Here \( T : X \to X \) is a compact linear operator in a Banach space \( X \) and \( (\lambda_n(T))_{n=1}^{\infty} \) denotes the sequence of all non-zero eigenvalues of \( T \), repeated according to algebraic multiplicity and ordered so that
\[
|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq 0.
\]
Also as a consequence of (1.1) one obtains, for all \( p \in (0, \infty) \), the existence of a constant \( c_p \) (independent of \( T \)) s.t.
\[
\left( \sum_{n=1}^{\infty} |\lambda_n(T)|^p \right)^{1/p} \leq c_p \left( \sum_{n=1}^{\infty} x_n^p(T) \right)^{1/p}. \quad (1.2)
\]
Hence, Weyl numbers may be seen as an appropriate tool to control the eigenvalues of $T$. Many times operators of interest can be written as a composition of an identity between appropriate function spaces and a further bounded operator, see, e.g., the monographs of König [29] and of Edmunds, Triebel [20]. This motivates the study of Weyl numbers of identity operators. Lubitz [32], König [29] and Caetano [8, 9, 10] studied the Weyl numbers of $id : \mathcal{B}_{p_1,p_1}^t((0,1)^d) \to L_{p_2}((0,1)^d)$, where $\mathcal{B}_{p_1,q_1}^t((0,1)^d)$ denotes the isotropic Besov spaces. Zhang, Fang and Huang [68] and Gasiorowska and Skrzypczak [24] investigated the case of embeddings of weighted Besov spaces, defined on $\mathbb{R}^d$, into Lebesgue spaces. Here we are interested in the investigation of the asymptotic behaviour of Weyl numbers of the identity $\underbrace{id : \mathcal{B}_{p_1,p_1}^t(0,1) \otimes \gamma_p \cdots \otimes \gamma_p \mathcal{B}_{p_1,p_1}^t(0,1)}_{\text{d-fold}} \to L_{p_2}((0,1)^d)$.

The symbol $\gamma_p$ refers to some tensor norm including the $p$-nuclear norm (if $1 < p < \infty$) and $\mathcal{B}_{p_1,p_1}^t(0,1) \otimes \gamma_p \cdots \otimes \gamma_p \mathcal{B}_{p_1,p_1}^t(0,1)$ denotes the $d$-fold tensor product of the univariate Besov space $\mathcal{B}_{p_1,p_1}^t(0,1)$, see Subsections 4.2 and 8.3 for more details. For brevity we put $S_{p_1,p_1}^t B((0,1)^d) := \underbrace{\mathcal{B}_{p_1,p_1}^t(0,1) \otimes \gamma_p \cdots \otimes \gamma_p \mathcal{B}_{p_1,p_1}^t(0,1)}_{\text{d-fold}}$.

This notation is chosen in accordance with the fact that $S_{p_1,p_1}^t B((0,1)^d)$ can be interpreted as a special case of the scale of Besov spaces of dominating mixed smoothness, see Subsection 4.2.

The behaviour of $x_n(id : S_{p_1,p_1}^t B((0,1)^d) \to L_{p_2}((0,1)^d))$, $1 < p_2 < \infty$, will be discussed in Subsection 2.2. Here, up to some limiting situations, we have the complete picture, i.e., we know the exact asymptotic behaviour of the Weyl numbers. For the extreme cases $p_2 = \infty$ and $p_2 = 1$, see Subsection 2.3 and Subsection 2.4, we are also able to describe the behaviour in any meaningful situation. It might be of some interest that we use here the interpolation properties of Weyl numbers. For $p_2 = \infty$, as a by-product, we can also deal with the characterization of the asymptotic behaviour of the approximation numbers, at least, if $1 \leq p_1 \leq \infty$. In Subsection 2.5 we discuss the behaviour of $x_n(id : S_{p_1,p_1}^t B((0,1)^d) \to \mathcal{Z}_{\text{mix}}^t((0,1)^d))$, where $\mathcal{Z}_{\text{mix}}^t((0,1)^d)$ denotes a space of Hölder-Zygmund type. This will be applied when dealing with the tensor product Riemann-Liouville operator in Subsection 2.7. It will be also applied when dealing with the approximation numbers $a_n(id : S_{p_1,p_1}^t B((0,1)^d) \to L_{\infty}((0,1)^d))$ in Subsection 2.3. Finally, in Subsection 2.6, we compare the behaviour of the Weyl numbers with that one of entropy, approximation and Kolmogorov numbers.

The paper is organized as follows. Our main results are discussed in Section 2. In Section 3 we recall the definition and some properties of Weyl numbers. The next Section 4 is
devoted to the function spaces under consideration. In Section 5 we discuss the Weyl numbers of embeddings of certain sequence spaces associated to tensor product Besov spaces and spaces of dominating mixed smoothness. This will be followed by Section 6, where, beside others, Theorem 2.3 (our main result) will be proved. In Appendix A we recall the behaviour of the Weyl numbers of embeddings $i\ell^m_{p_1,p_2} : \ell^m_{p_1} \to \ell^m_{p_2}$. Finally, in Appendix B, a few more facts about the Lizorkin-Triebel spaces of dominating mixed smoothness $S^t_{p,q}F(\mathbb{R}^d)$, $S^t_{p,q}F((0,1)^d)$ and the Besov spaces of dominating mixed smoothness $S^t_{p,q}B(\mathbb{R}^d)$, $S^t_{p,q}B((0,1)^d)$ are collected.

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. For a real number $a$ we put $a_+ := \max(a,0)$. By $[a]$ we denote the integer part of $a$. If $\vec{j} \in \mathbb{N}_0^d$, i.e., if $\vec{j} = (j_1, \ldots, j_d)$, $j_\ell \in \mathbb{N}_0$, $\ell = 1, \ldots, d$, then we put

$$|\vec{j}| := j_1 + \ldots + j_d.$$

By $\Omega$ we denote the unit cube in $\mathbb{R}^d$, i.e., $\Omega := (0,1)^d$. If $X$ and $Y$ are two quasi-Banach spaces, then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. As usual, the symbol $c$ denotes positive constants which depend only on the fixed parameters $t, p, q$ and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols “$\lesssim$” and “$\gtrsim$” instead of “$\leq$” and “$\geq$”, respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \leq cB$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. For a discrete set $\nabla$ the symbol $|\nabla|$ denotes the cardinality of this set. Finally, the symbols $id, id^*$ will be used for identity operators, $id^*$ mainly in connection with sequence spaces. The symbol $id^m_{p_1,p_2}$ refers to the identity

$$id^m_{p_1,p_2} : \ell^m_{p_1} \to \ell^m_{p_2}. \quad (1.3)$$

Tensor products of Besov and Sobolev spaces are investigated in [53], [51], [52] and Hansen [26]. General information about Besov and Lizorkin-Triebel spaces of dominating mixed smoothness can be found, e.g., in [1, 2, 3, 4, 50, 49, 64] ($S^t_{p,q}B(\mathbb{R}^d)$, $S^t_{p,q}F(\mathbb{R}^d)$). The (Fourier analytic) definitions of these spaces are reviewed in the Appendix B. The reader, who is interested in more elementary descriptions of these spaces, e.g., by means of differences, is referred to [1, 50] and [63].

Agreement: To avoid inconvenient repetitions we require for the rest of this paper the following restrictions: $p_2$ is always a parameter ranging in $[1, \infty]$, whereas $p_1$ is allowed to vary in $(0, \infty]$.

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a misprint in the phd-thesis of Lubitz [32], Thomas Kühn for an explanation how to use (6.5) and Dachun Yang and Wen Yuan for a nice new proof of the continuous embedding $F_{1,2}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$.

2 The main results

2.1 Preparations

As a preparation for the main results we recall under which conditions the identity $S_{t,p_1}^1 B((0,1)^d) \hookrightarrow L_{p_2}((0,1)^d)$ is compact, see Vybiral [64, Thm. 3.17].

Proposition 2.1. The following assertions are equivalent:
(i) The embedding $S_{t,p_1}^1 B((0,1)^d) \hookrightarrow L_{p_2}((0,1)^d)$ is compact;
(ii) We have
\[ t > \max \left(0, \frac{1}{p_1} - \frac{1}{p_2}\right). \tag{2.1} \]

Since we are exclusively interested in compact embeddings the restriction (2.1) will be always present. Also for later use, we recall the Weyl numbers of the embedding $id : B_{p_1,q_1}^t (0,1) \hookrightarrow L_{p_2}(0,1)$. For all $0 < p_1, q_1 \leq \infty$, $1 \leq p_2 \leq \infty$ and $t > \max(0,1/p_1 - 1/p_2)$ we have
\[ x_n(id) = x_n(id : B_{p_1,q_1}^t (0,1) \hookrightarrow L_{p_2}(0,1)) \asymp n^{-\alpha}, \quad n \in \mathbb{N}, \tag{2.2} \]
where $\alpha = \alpha(t, p_1, p_2)$.

Proposition 2.2. The value of $\alpha$ in (2.2) is given by

I $\quad \alpha = t$ \quad if $p_1, p_2 \leq 2$;
II $\quad \alpha = t + \frac{1}{p_2} - \frac{1}{2}$ \quad if $p_1 \leq 2 \leq p_2$;
III $\quad \alpha = t + \frac{1}{p_2} - \frac{1}{p_1}$ \quad if $2 \leq p_1 \leq p_2$;
IV $\quad \alpha = t + \frac{1}{p_2} - \frac{1}{p_1}$ \quad if $2 \leq p_2 \leq p_1$ and $t > \frac{1}{p_2 - 1/p_1}$;
IV $\quad \alpha = \frac{tp_1}{2}$ \quad if $2 \leq p_2 \leq p_1$ and $t < \frac{1}{p_2 - 1/p_1}$;
V $\quad \alpha = t - \frac{1}{p_1} + \frac{1}{2}$ \quad if $p_2 \leq 2 \leq p_1$ and $t > \frac{1}{p_1}$;
V $\quad \alpha = \frac{tp_1}{2}$ \quad if $p_2 \leq 2 \leq p_1$ and $t < \frac{1}{p_1}$.

The above results indicate a decomposition of the $(1/p_1, 1/p_2)$-plane into five parts.
In regions $IV$ and $V$ we have a further splitting into the cases of small ($IV_*, V_*$) and large smoothness ($IV^*, V^*$). Proposition 2.2 has been proved by Lubitz [32, Satz 4.13] in case $1 \leq p_1, q_1, p_2 \leq \infty$, we refer also to König [29] in this context. The proof of the general case, i.e., if $p_1, q_1$ can be also less than 1, can be found in the thesis of Cae-tano [10], see also [9].

Obviously we do not have a dependence on the fine-index $q$. This will be different in the dominating mixed case with $d > 1$.

### 2.2 The Littlewood-Paley case

Littlewood-Paley analysis is one of the main tools to understand the behaviour of the Weyl numbers if $1 < p_2 < \infty$ (i.e., the target space $L_{p_2}$ allows a Littlewood-Paley-type decomposition). The cases $p_2 = 1$ and $p_2 = \infty$ require different techniques and will be treated in the next subsection. As in the isotropic case the results suggest to work with the same decomposition of the $(1/p_1, 1/p_2)$-plane as in Proposition 2.2. But this time we have in some more cases a splitting into small and large smoothness. For all $0 < p_1 \leq \infty$, $1 < p_2 < \infty$ and $t > \max(0, 1/p_1 - 1/p_2)$ we have

\[
x_n(id) = x_n(id : S_{p_1,p_2}^d B((0,1)^d) \to L_{p_2}((0,1]^d)) \asymp n^{-\alpha} (\log n)^{(d-1)\beta}, \quad n \geq 2,
\]

where $\alpha = \alpha(t,p_1,p_2)$ and $\beta = \beta(t,p_1,p_2)$. The symbols $I, II, \ldots$ have the same meaning as in Figure 1.

**Theorem 2.3.** Let $1 < p_2 < \infty$ and $t > \max(0, \frac{1}{p_1} - \frac{1}{p_2})$. Then the values of $\alpha$ and $\beta$ in (2.3) are given by

- $I^*$: $\alpha = t$ and $\beta = t + \frac{1}{2} - \frac{1}{p_1}$ if $t > \frac{1}{p_1} - \frac{1}{2}$;
- $I_*$: $\alpha = t$ and $\beta = 0$ if $t < \frac{1}{p_1} - \frac{1}{2}$;
- $II$: $\alpha = t - \frac{1}{2} + \frac{1}{p_2}$ and $\beta = t + \frac{1}{p_2} - \frac{1}{p_1}$;
- $III$: $\alpha = t - \frac{1}{p_1} + \frac{1}{p_2}$ and $\beta = t + \frac{1}{p_2} - \frac{1}{p_1}$;
- $IV^*$: $\alpha = t - \frac{1}{p_1} + \frac{1}{p_2}$ and $\beta = t + \frac{1}{p_2} - \frac{1}{p_1}$ if $t > \frac{1}{p_2} - \frac{1}{p_1}$.
IV \* \alpha = \frac{tp_1}{2} and \beta = t + \frac{1}{2} - \frac{1}{p_1} \text{ if } t < \frac{1}{p_1} - \frac{1}{2} ; \\
V^* \alpha = t - \frac{1}{p_1} + \frac{1}{2} and \beta = t + \frac{1}{2} - \frac{1}{p_1} \text{ if } t > \frac{1}{p_1} ; \\
V \alpha = \frac{tp_1}{2} and \beta = t + \frac{1}{2} - \frac{1}{p_1} \text{ if } t < \frac{1}{p_1} .

Thm. \ref{thm:2.3} gives the final answer about the behaviour of the \( x_n \) in almost all cases. It is interesting to notice that in regions \( I, IV \) and \( V \) we have a different behaviour for small smoothness compared with large smoothness. Only in the resulting limiting cases we are not able to characterize the behaviour of the \( x_n(id) \). However, estimates from below and above also for these limiting situations will be given in Subsection \ref{subsection:5.3}.

We comment on the proof. The proof is in some sense standard. Concerning the estimate from above the first step consists in a reduction step. By means of wavelet characterizations we switch from the consideration of \( id : S_{p_1,p_1}^t B((0,1)^d) \to L_{p_2}((0,1)^d) \) to \( id^* : s_{p_1,p_1}^{t,\Omega}b \to s_{p_2,2}^{0,\Omega}f \), where \( s_{p_1,p_1}^{t,\Omega}f \) and \( s_{p_2,2}^{0,\Omega}f \) are appropriate sequence spaces. Next, this identity is splitted into \( id^* = \sum_{\mu=0}^{\infty} id_{\mu}^* \) (the \( id_{\mu}^* \) are identities with respect to certain subspaces) which results in an estimate of \( x_n(id^* : s_{p_1,p_1}^{t,\Omega}b \to s_{p_2,2}^{0,\Omega}f) \)

\[ x_n^p(id^*) \leq \sum_{\mu=0}^{J} x_{n_{\mu}}^p(id_{\mu}^*) + \sum_{\mu=J+1}^{L} x_{n_{\mu}}^p(id_{\mu}^*) + \sum_{\mu=L+1}^{\infty} ||id_{\mu}^*||^p, \quad \rho := \min(1,p_2), \]

and \( n - 1 = \sum_{\mu=0}^{L}(n_{\mu} - 1) \), see \ref{eq:5.2}. Till this point we would call the proof standard, compare, e.g., with Vybiral \cite{55}. But now the problem consists in choosing \( J,L \) and \( n_{\mu} \) in a way leading to the desired result. This is the real problem which we solved in Subsection \ref{subsection:5.3}. In a further reduction step estimates of \( x_{n_{\mu}}(id_{\mu}^*) \) are traced back to estimates of \( x_{n_{\mu}}(id_{p_1,p_1}^{D_{\mu}}) \), see \ref{eq:1.3}. All what is needed about these number is collected in Appendix A. Concerning the estimate from below one has to figure out appropriate subspaces of \( S_{p_1,p_1}^t B((0,1)^d) (s_{p_1,p_1}^{t,\Omega}b) \). Then, also in this case, all can be reduced to the known estimates of \( x_n(id_{p_1,p_1}^{D_{\mu}}) \).

### 2.3 The extreme case \( p_2 = \infty \)

Let us recall a result of Temlyakov \cite{56}, see also \cite{14}.

**Proposition 2.4.** Let \( t > \frac{1}{2} \). Then we have

\[ x_n(id : S_{2,2}^t B((0,1)^d) \to L_{\infty}((0,1)^d)) \asymp \frac{(\log n)^{\frac{d-1}{2}}}{n^{t - \frac{1}{2}}} , \quad n \geq 2 . \quad (2.4) \]

**Remark 2.5.** (i) In the literature many times the notation \( H_{mix}^t((0,1)^d) \) and \( MW_2^t((0,1)^d) \) are used instead of \( S_{2,2}^t B((0,1)^d) \).

(ii) In \cite{56} and \cite{14} the authors deal with approximation numbers \( a_n(id : S_{2,2}^t B((0,1)^d) \to \)
\( L_\infty((0,1)^d) \), see Section 3. However, for Banach spaces \( Y \) and Hilbert spaces \( H \) we always have
\[
x_n(T : H \to Y) = a_n(T : H \to Y),
\]
see [38, Prop. 2.4.20].

By using abstract properties of Weyl numbers, see Section 3, we will extent Prop. 2.4 to the following quite satisfactory result.

**Theorem 2.6.** Let \( 0 < p_1 \leq \infty \) and \( t > \frac{1}{p_1} \). Then we have
\[
x_n(id : S_{p_1,p_1}^{t} B((0,1)^d) \to L_\infty((0,1)^d)) \approx \begin{cases} 
\frac{(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})}}{n^{t-\frac{1}{2}}} & \text{if } 0 < p_1 \leq 2, \\
\frac{(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})}}{n^{t-\frac{1}{p_1}}} & \text{if } 2 \leq p_1 \leq \infty, 
\end{cases}
\]
\[
\times \begin{cases} 
\frac{(\log n)^{(d-1)(t-s - \frac{1}{p_1})}}{n^{t-s - \frac{1}{2}}} & \text{if } 1 \leq p_1 < 2, \quad t - s > 1, \\
\frac{(\log n)^{(d-1)(t-s - \frac{1}{p_1})}}{n^{t-s - \frac{1}{p_1}}} & \text{if } 2 \leq p_1 \leq \infty, \quad t - s > \frac{1}{p_1}, 
\end{cases}
\]
for all \( n \geq 2 \).

**Remark 2.7.** (i) Recall that \( S_{p_1,p_1}^{t} B((0,1)^d) \) is compactly embedded into \( L_\infty((0,1)^d) \) if and only if \( t > \frac{1}{p_1} \), see Prop. 2.1. (ii) Considering \( p_2 \to \infty \) in parts II and III of Thm. 2.3 then it turns out that in (2.6) there is an additional log factor, more exactly \( (\log n)^{(d-1)/2} \).

(iii) Beside Prop. 2.4 and Lemma 6.7 we are not aware of any other result concerning \( s \)-numbers (Kolmogorov numbers, approximation numbers, ...) where the exact order of \( s_n(id : S_{p_1,q_1}^{t} B((0,1)^d) \to L_\infty((0,1)^d)), p_1 \neq 2 \), if \( n \) tends to infinity, has been found. For some partial results with respect to Kolmogorov numbers we refer to Romanyuk [41].

It is quite interesting that the same methods as above apply to approximation numbers. As a result we get the following.

**Theorem 2.8.** Let \( n \in \mathbb{N} \) and \( n \geq 2 \). Then we have
\[
a_n(id : S_{p_1,p_1}^{t} B((0,1)^d) \to Z_{mix}^s((0,1)^d)) \approx \begin{cases} 
\frac{(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})}}{n^{t-\frac{1}{2}}} & \text{if } 1 \leq p_1 < 2, \quad t > 1, \\
\frac{(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})}}{n^{t-\frac{1}{p_1}}} & \text{if } 2 \leq p_1 \leq \infty, \quad t > \frac{1}{p_1}, 
\end{cases}
\]
as well as
\[
a_n(id : S_{p_1,p_1}^{t} B((0,1)^d) \to L_\infty((0,1)^d)) \approx \begin{cases} 
\frac{(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})}}{n^{t-\frac{1}{2}}} & \text{if } 1 \leq p_1 < 2, \quad t > 1, \\
\frac{(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})}}{n^{t-\frac{1}{p_1}}} & \text{if } 2 \leq p_1 \leq \infty, \quad t > \frac{1}{p_1}, 
\end{cases}
\]
Remark 2.9. (i) Theorem 2.8 gives a quite satisfactory answer concerning the asymptotic behaviour of $a_n(id : S_{p_1,p_1}^t B((0,1)^d) \to L_\infty((0,1)^d))$. There are two open questions we could not answer.

- What about $a_n(id : S_{p_1,p_1}^t B((0,1)^d) \to L_\infty((0,1)^d))$ if $1 < p_1 < 2$ and $1/p_1 < t \leq 1$?

  In case of isotropic spaces it is well-known, see [65], that the behaviour in case $t > 1$ differs from the behaviour for $1/p_1 < t < 1$. So we conjecture that this happens also in the dominating mixed case.

- What about $a_n(id : S_{p_1,p_1}^t B((0,1)^d) \to L_\infty((0,1)^d))$ if $0 < p_1 < 1$?

(ii) The cases $p_1 = 2$ and $p_1 = \infty$ have been known before, see Prop. 2.4 and Lemma 6.7

We refer to Temlyakov [54, 56].

(iii) Bazarkhanov [5], Dinh Dung [16] and Romanyuk [39, 40, 47] are dealing with the behaviour of approximation numbers in the general case of $a_n(id : S_{p_1,p_1}^t B((0,1)^d) \to L_{p_2}((0,1)^d))$.

2.4 The extreme case $p_2 = 1$

Let us recall a result obtained by Romanyuk [47]. Again Romanyuk has dealt with approximation numbers, but see Remark 2.5 for this.

Proposition 2.10. Let $t > 0$. Then we have

$$x_n(id : S_{2,2}^t B((0,1)^d) \to L_1((0,1)^d)) \asymp \frac{(\log n)^{(d-1)t}}{n^t}, \quad n \geq 2. \quad (2.8)$$

By making use of the embedding $S_{1,2}^0 F((0,1)^d) \hookrightarrow L_1((0,1)^d)$ we are able to extend Prop. 2.10 to the following.

Theorem 2.11. Let $0 < p_1 \leq \infty$ and $t > (\frac{1}{p_1} - 1)_+$. Then

$$x_n(id : S_{p_1,p_1}^t B((0,1)^d) \to L_1((0,1)^d))$$

is

$$\begin{cases}
  n^{-t} & \text{if } p_1 < 2, \ t < \frac{1}{p_1} - \frac{1}{2}, \\
  n^{-t}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})} & \text{if } p_1 \leq 2, \ t > \frac{1}{p_1} - \frac{1}{2}, \\
  n^{-t+\frac{1}{p_1} - \frac{1}{2}}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})} & \text{if } p_1 > 2, \ t > \frac{1}{p_1}, \\
  n^{-\frac{np_1}{2}}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_1})} & \text{if } p_1 > 2, \ t < \frac{1}{p_1}.
\end{cases}$$

Remark 2.12. The most interesting case is given by $p_1 = 1$. It follows that we have

$$x_n(id : S_{1,1}^t B((0,1)^d) \to L_1((0,1)^d)) \asymp \begin{cases}
  n^{-t} & \text{if } t < \frac{1}{2}, \\
  n^{-t}(\log n)^{(d-1)(t-\frac{1}{2})} & \text{if } t > \frac{1}{2}.
\end{cases}$$
for all \( n \geq 2 \). We are not aware of any other result concerning \( s \)-numbers (Kolmogorov numbers, approximation numbers, ...) where the exact order of \( s_0(id : S^t_{1,1}B((0,1)^d) \to L_1((0,1)^d)) \) if \( n \) tends to infinity, has been found. A few more results concerning approximation and Kolmogorov numbers are known in case of the embeddings \( id : S^t_{p_1,p_1}B((0,1)^d) \to L_1((0,1)^d) \), \( p_1 > 1 \), and \( id : S^t_{1,1}B((0,1)^d) \to L_{p_2}((0,1)^d) \), \( 1 < p_2 < \infty \). E.g., in \([47]\) Romanyuk has proved for \( 2 \leq p_1 < \infty \) and \( t > 0 \)

\[
a_n(id : S^t_{p_1,p_1}B((0,1)^d) \to L_1((0,1)^d)) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2})-\frac{1}{m}}
\]

for all \( n \geq 2 \).

### 2.5 A version of Hölder-Zygmund spaces (related to tensor products) as target spaces

As a supplement we investigate the Weyl numbers of the embeddings \( id : S^t_{p_1,p_1}B((0,1)^d) \to Z^s_{\text{mix}}((0,1)^d) \), where the spaces \( Z^s_{\text{mix}}((0,1)^d) \) are versions of Hölder-Zygmund spaces. Let \( j \in \{1, \ldots , d\} \). For \( m \in \mathbb{N} \), \( h_j \in \mathbb{R} \) and \( x \in \mathbb{R}^d \) we put

\[
\Delta^m_{h_{j,d}}f(x) := \sum_{\ell = 0}^{m} (-1)^{m-\ell} \binom{m}{\ell} f(x_1, \ldots , x_{j-1}, x_j + \ell h_j, x_{j+1}, \ldots , x_d).
\]

This is the \( m \)-th order difference in direction \( j \). Mixed differences are defined as follows. Let \( e \) be a non-trivial subset of \( \{1, \ldots , d\} \). For \( h \in \mathbb{R}^d \) we define

\[
\Delta^m_{h,e} := \prod_{j \in e} \Delta^m_{h_{j,d}}.
\]

Of course, here \( \Delta^m_{h_{j,d}} \cdot \Delta^m_{h_{\ell,d}} \) has to be interpreted as \( \Delta^m_{h_{j,d}} \circ \Delta^m_{h_{\ell,d}} \).

**Definition 2.13.** Let \( s > 0 \). Let \( m \in \mathbb{N} \) s.t. \( m - 1 \leq s < m \). Then \( f \in Z^s_{\text{mix}}((0,1)^d) \) if

\[
\| f \|_{Z^s_{\text{mix}}((0,1)^d)} := \| f \|_{C((0,1)^d)} + \max_{e \subseteq \{1, \ldots , d\}} \sup_{\| h \|_\infty \leq 1} \prod_{j \in e} |h_j|^{-s} \sup_{x \in \Omega_{m,e,h}} \| \Delta^m_{h,e} f(x) \| < \infty,
\]

where

\[
\Omega_{m,e,h} := \{ x \in (0,1)^d : (x_1 + \varepsilon_1 \ell_1 h_1, \ldots , x_d + \varepsilon_d \ell_d h_d) \in (0,1)^d \ \forall \ell \in \mathbb{N}_0^d , \| \ell \|_\infty \leq m \},
\]

and

\[
\varepsilon_j := \begin{cases} 1 & \text{if } j \in e, \\ 0 & \text{if } j \notin e. \end{cases}
\]

A few properties of these spaces are obvious:

- Let \( d = 1 \) and \( m = 1 \), i.e., \( 0 < s < 1 \). Then \( Z^s_{\text{mix}}(0,1) \) is the classical space of Hölder-continuous functions of order \( s \).
• Let $d = 1$, $s = 1$ and $m = 2$. Then $Z_{\text{mix}}^1(0, 1)$ is the classical Zygmund space.

• If $f(x) = f_1(x_1) \cdots f_d(x_d)$ with $f_j \in Z_{\text{mix}}(0, 1)$, $j = 1, \ldots, d$, then $f \in Z^s_{\text{mix}}((0, 1)^d)$ follows and
  \[
  \| f \|_{Z^s_{\text{mix}}((0, 1)^d)} \asymp \prod_{j=1}^d \| f_j \|_{Z^s_{\text{mix}}(0, 1)}.
  \]

• Let $f \in Z^s_{\text{mix}}((0, 1)^d)$ and define $g(x') := f(x', 0)$, where $x = (x', x_d)$, $x' \in \mathbb{R}^{d-1}$. Then $g \in Z^s_{\text{mix}}((0, 1)^{d-1})$ follows.

• We define $Z^s_{\text{mix}}(\mathbb{R}^d)$ by replacing $(0, 1)^d$ by $\mathbb{R}^d$ in the Def. \ref{def:zs_mix}. Let $E : Z^s(0, 1) \to Z^s(\mathbb{R})$ be a linear and bounded extension operator such that $E$ maps $C(0, 1)$ into itself. Then $E \otimes \cdots \otimes E$ ($d$-fold tensor product) is well-defined on $C((0, 1)^d)$ and maps this space into itself. Observe that $\Delta_{h,e}^m f(x)$ can be written as the $|e|$-fold iteration of a directional difference. As a consequence we obtain that $\mathcal{E}_d := E \otimes \cdots \otimes E$ maps $Z^s_{\text{mix}}((0, 1)^d)$ into itself.

Less obvious is the following, see [50, Rem. 2.3.4/3] or [63].

Lemma 2.14. Let $s > 0$. Then

\[
Z^s_{\text{mix}}((0, 1)^d) = S^s_{\infty, \infty}B((0, 1)^d)
\]

holds in the sense of equivalent norms.

Essentially by the same methods as used for the proof of Thm. \ref{thm:zs_limit} one obtains the following.

Theorem 2.15. Let $s > 0$ and $t > s + \frac{1}{p_1}$. Then it holds

\[
\begin{align*}
  x_n(id : S^t_{p_1, p_1} B((0, 1)^d) \to Z^s_{\text{mix}}((0, 1)^d)) \\ &\asymp \left\{ \begin{array}{ll}
  \frac{(\log n)^{(d-1)(-s-\frac{1}{p_1})}}{n^{t-s-\frac{1}{p_1}}} & \text{if } 0 < p_1 \leq 2, \\
  \frac{(\log n)^{(d-1)(-s-\frac{1}{p_1})}}{n^{t-s-\frac{1}{p_1}}} & \text{if } 2 \leq p_1 \leq \infty,
  \end{array} \right. 
\end{align*}
\]

for all $n \geq 2$.

Remark 2.16. (i) Recall that $S^t_{p_1, p_1} B((0, 1)^d)$ is compactly embedded into $Z^s_{\text{mix}}((0, 1)^d)$ if and only if $t > s + 1/p_1$, see [64].

(ii) Observe, that Thm. 2.15 is not the limit of Thm. 2.6 for $s \downarrow 0$. There, in Thm. 2.6 is an additional factor $(\log n)^{(d-1)/2}$ as many times in this field.
2.6 A comparison with entropy, approximation and Kolmogorov numbers

There are good reasons to compare Weyl numbers with entropy numbers. As explained above, both, entropy and Weyl numbers, are tools to control the behaviour of eigenvalues of linear operators. Approximation and Gelfand numbers are known to dominate Weyl numbers. However, it seems nothing is known in case of Gelfand numbers. So we compare Weyl numbers with approximation numbers and as a supplement we also have a look onto a comparison with Kolmogorov numbers.

Entropy numbers

Let us recall the definition of entropy numbers.

**Definition 2.17.** Let $T : X \rightarrow Y$ be a bounded linear operator between complex quasi-Banach spaces, and let $n \in \mathbb{N}$. Then the $n$-th (dyadic) entropy number of $T$ is defined as

$$e_n(T : X \rightarrow Y) := \inf \{ \varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{n-1} \text{ balls in } Y \text{ of radius } \varepsilon \},$$

where $B_X := \{ x \in X : \| x \|_X \leq 1 \}$ denotes the closed unit ball of $X$.

In particular, $T : X \rightarrow Y$ is compact if and only if $\lim_{n \rightarrow \infty} e_n(T) = 0$. For details and basic properties like multiplicativity, additivity, behaviour under interpolation etc. we refer to the monographs [12, 20, 29, 36]. Most important for us is the Carl-Triebel inequality which states

$$|\lambda_n(T)| \leq \sqrt{2} e_n(T), \quad (2.10)$$

cf. Carl, Triebel [13] (see also the monographs [12] and [20]).

Entropy numbers of embeddings $id : S_{p_1,p_1}^d B((0,1)^d) \hookrightarrow L_{p_2}((0,1)^d)$ have been investigated in Vybiral [64]. The picture is less complete than in case of Weyl numbers. Only for sufficiently large smoothness the behaviour is exactly known. We have

$$e_n(id) \asymp n^{-t} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_1})} \quad \text{if} \quad t > \max \left( 0, \frac{1}{p_1} - \frac{1}{2}, \frac{1}{p_1} - \frac{1}{p_2} \right). \quad (2.11)$$
We use Figure 2 to explain the different behaviour of entropy and Weyl numbers. Weyl numbers are essentially smaller than entropy numbers in regions IV and V, entropy numbers are essentially smaller than Weyl numbers in regions II and III, and they show a similar behaviour in region I∗.

Remark 2.18. Further estimates of the decay of entropy numbers related to embeddings 
\[id : S^t_{p_1,q_1}B((0,1)^d) \hookrightarrow L^p_{p_2}((0,1)^d) \text{ (id : } S^t_{p_1}W((0,1)^d) \hookrightarrow L^p_{p_2}((0,1)^d))\] can be found in Belinsky [6], Dinh Dung [15], and Temlyakov [54].

Approximation numbers

Definition 2.19. Let \( n \in \mathbb{N} \). The \( n \)-th approximation number of the linear operator \( T \in \mathcal{L}(X,Y) \) is defined to be

\[a_n(T) := \inf \{ \| T - A \| : A \in \mathcal{L}(X,Y), \ rank(A) < n \} .\]

Weyl numbers are, in general, smaller than approximation numbers, i.e.,

\[x_n(T) \leq a_n(T), \quad n \in \mathbb{N} ,\]

see [38, Thm. 2.10.1]. Concerning approximation numbers of embeddings 
\[S^t_{p_1,q_1}B((0,1)^d) \hookrightarrow L^p_{p_2}((0,1)^d)\] the picture is less complete than in case of Weyl numbers, we refer to Bazarkhanov [5] for the most recent publication in this direction. The topic itself has been investigated at various places over the last 30 years, we refer to Temlyakov [54, 55], Galeev [21, 23] and Romanyuk [39, 40, 47]. For a majority of cases we have

\[\lim_{n \to \infty} a_n(id : S^t_{p_1,q_1}B((0,1)^d) \to L^p_{p_2}((0,1)^d)) = 0 .\]

It is easier to describe those situations where we have

\[\frac{x_n(id : S^t_{p_1,q_1}B((0,1)^d) \to L^p_{p_2}((0,1)^d))}{a_n(id : S^t_{p_1,q_1}B((0,1)^d) \to L^p_{p_2}((0,1)^d))} \times 1 .\]  \( (2.12) \)

(2.12) takes place if one of the following conditions is satisfied:
\[ 2 \leq p_1 < p_2 < \infty; \]
\[ 1 \leq p_1 < 2, \quad p_2 = \infty \quad \text{and} \quad t > 1; \]
\[ 2 \leq p_1 \leq \infty, \quad p_2 = \infty \quad \text{and} \quad t > 1/p_1. \]

**Kolmogorov numbers**

We follow \[36, 38, 12\].

**Definition 2.20.** Let \( T : X \to Y \) be a bounded linear operator between complex quasi-Banach spaces and let \( n \in \mathbb{N} \). The \( n \)-th Kolmogorov number of \( T \) is defined

\[ d_n(T) = \inf \{ \|Q_N T\| : \dim(N) < n \}, \]

where \( Q_N \) canonical surjection from \( F \) onto quotation space \( F/N \).

Many times in approximation theory people prefer to work with the notion Kolmogorov width, see, e.g., \[5, 22, 41, 42, 43, 44, 45, 55, 56\]. It is not difficult to see that the Kolmogorov numbers of the operator \( T \) coincide with the Kolmogorov widths of the set \( T(B_X) \) in the space \( Y \).

In general one knows the following

\[ \max(x_n(T), d_n(T)) \leq a_n(T), \quad n \in \mathbb{N}, \]

see \[36\] Thm. 11.2.3 and \[38\] Thm. 2.10.1. Weyl numbers and Kolmogorov numbers are incomparable. This turns out to be true also in our situation.

Let us recall some results obtained by Galeev \[22\], Romanyuk \[42, 43, 44, 45\] and Bazarkhanov \[5\]. For \( t > (1/p_1 - 1/p_2)_+ \) it holds

\[ d_n(id : S_{p_1,p_2}^d ((0,1]^d) \hookrightarrow L_{p_2}^2((0,1]^d)) \asymp \frac{(\log n)^{(d-1)/\beta}}{n^{\alpha}}, \quad n \geq 2, \]

where

- \( \alpha = t \) and \( \beta = t + (1/p_1)_+ \) if \( 1 < p_2 \leq p_1 < \infty \),
- \( \alpha = t - \frac{1}{p_1} + \frac{1}{p_2} \) and \( \beta = t - \frac{1}{p_1} + \frac{1}{p_2} \) if \( 1 < p_1 < p_2 \leq 2 \),
- \( \alpha = t - \frac{1}{p_1} + \frac{1}{2} \) and \( \beta = t - \frac{1}{p_1} + \frac{1}{2} \) if \( 1 < p_1 \leq 2 < p_2 < \infty \) and \( t > \frac{1}{p_1} \),
- \( \alpha = t \) and \( \beta = t + \frac{1}{2} - \frac{1}{p_1} \) if \( 2 \leq p_1 \leq p_2 < \infty \) and \( t > \frac{1/p_1 - 1/p_2}{1-2/p_2} \),
- \( \alpha = p_2/2(t - \frac{1}{p_1} + \frac{1}{p_2}) \) and \( w_3 \leq \beta \leq w_5 \) if \( 1 < p_1 \leq 2 < p_2 < \infty \) and \( t < \frac{1}{p_1} \),
- \( \alpha = p_2/2(t - \frac{1}{p_1} + \frac{1}{p_2}) \) and \( w_3 \leq \beta \leq w_4 \) if \( 2 \leq p_1 < p_2 < \infty \) and \( t < \frac{1/p_1 - 1/p_2}{1-2/p_2} \).
Here

\[ w_3 := \max \left( \frac{p_2}{2} \left( t - \frac{1}{p_1} + \frac{1}{p_2} \right) + \frac{1}{p_2} - \frac{1}{p_1}, t - \frac{1}{p_1} + \frac{1}{p_2} \right), \]
\[ w_4 := \left( \frac{1/2 - 1/p_1}{1/2 - 1/p_1 - 1/p_2} \right) \left( \frac{p_2}{2} \left( t - \frac{1}{p_1} + \frac{1}{p_2} \right) \right), \]

and \[ w_5 := \frac{p_2}{2} \left( t - \frac{1}{p_1} + \frac{1}{p_2} \right). \]

To our best knowledge the exact asymptotic behaviour of the \( d_n \) is not known in the last two cases related to small smoothness. It is much easier to compare Weyl and Kolmogorov numbers with a figure as already used above.

We use Figure 3 to explain the different behaviour of Weyl and Kolmogorov numbers. Region II_1 is defined to be

\[ p_2 \leq p_1' = 1 - \frac{1}{p_1} \quad \text{and} \quad t > \frac{1}{p_1} \]

or

\[ t < \min \left( \frac{1}{p_1}, \frac{p_2/(2p_1) + 1/p_2 - 1}{p_2/2 - 1} \right). \]

Region II_2 is given by

\[ p_1' < p_2 < \infty \quad \text{and} \quad t > \frac{1}{p_1} \]

or

\[ \frac{p_2/(2p_1) + 1/p_2 - 1}{p_2/2 - 1} < t < \frac{1}{p_1}. \]

Figure 3 shows that in most of the cases Weyl numbers tend to zero faster than Kolmogorov numbers as \( n \to \infty \). Exceptional cases are regions III and II_2.

2.7 The tensor product Riemann-Liouville operator

The identity \( \text{id} : S_t^{t} B((0,1)^d) \to L_{p_2}((0,1)^d) \) in regions IV and V, see Figure 2, is a good example for an linear operator s.t.

\[ x_n(\text{id} : S_t^{t} B((0,1)^d) \to L_{p_2}((0,1)^d)) \lesssim e_n(\text{id} : S_t^{t} B((0,1)^d) \to L_{p_2}((0,1)^d)), \quad n \in \mathbb{N}. \]

Already Carl and Stephani \[12\] 1.5 constructed a linear and continuous operator \( T \) such that

\[ x_n(T) = 2^{-n}, \quad n \in \mathbb{N}, \]
and
\[ \frac{1}{2} 2^{-\sqrt{n}} \leq e_{n+1}(T) \leq 3 \sqrt{2} 2^{-\sqrt{n}}, \quad n \in \mathbb{N}, \]

see [12 (1.5.21)]. The advantage of our example consists in the polynomial behaviour of
the Weyl and entropy numbers, the disadvantage in the complexity.

It will be not our aim here to continue in this direction. Here we would like to investigate
the tensor product Riemann-Liouville operator.

Let \( \alpha > 0 \). Then for \( f \in L_1(0, 1) \) we define the Riemann-Liouville operator as
\[
\mathcal{R}_\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt, \quad x \in (0, 1).
\]

Hence, for \( f \in L_1((0,1)^d) \) we can consider the tensor product Riemann-Liouville operator
given by
\[
\mathcal{R}_\alpha^d f := \underbrace{(\mathcal{R}_\alpha \otimes \ldots \otimes \mathcal{R}_\alpha)}_{d\text{-fold}} f.
\]

Approximation and entropy numbers of the tensor product Riemann-Liouville operator
have been considered in Kühn and Linde [30] with motivation traced back to optimal
series representations of the fractional Brownian sheet. They proved

\[
a_n(\mathcal{R}_\alpha^d : L_2((0,1)^d) \to L_2((0,1)^d)) \lesssim n^{-\alpha}(\log n)^{(d-1)\alpha}, \quad n \geq 2, \quad (2.13)
\]

if \( \alpha > 0 \) and
\[
a_n(\mathcal{R}_\alpha^d : L_2((0,1)^d) \to C((0,1)^d)) \lesssim n^{-\alpha+\frac{1}{2}}(\log n)^{(d-1)\alpha}, \quad n \geq 2, \quad (2.14)
\]

if \( \alpha > 1/2 \). Based on what has been done so far it is now easy to derive the following
results concerning Weyl numbers of \( \mathcal{R}_\alpha^d \).

**Theorem 2.21.** Let \( 0 < \alpha < 1 \).

(i) Then
\[
\sigma_n(\mathcal{R}_\alpha^d : L_2((0,1)^d) \to L_2((0,1)^d)) \lesssim n^{-\alpha}(\log n)^{(d-1)\alpha}
\]

holds for all \( n \geq 2 \).

(ii) Let \( 1/2 < \alpha < 1 \) and \( 0 < s < \alpha - 1/2 \). Then
\[
\sigma_n(\mathcal{R}_\alpha^d : L_2((0,1)^d) \to C((0,1)^d)) \lesssim n^{-\alpha+\frac{1}{2}}(\log n)^{(d-1)\alpha}
\]

and
\[
\sigma_n(\mathcal{R}_\alpha^d : L_2((0,1)^d) \to Z_{\text{mix}}^s((0,1)^d)) \lesssim n^{-\alpha+s+\frac{1}{2}}(\log n)^{(d-1)(\alpha-s-\frac{1}{2})}
\]

hold for all \( n \geq 2 \).
Remark 2.22. (i) Because of $x_n(T : H \to Y) = a_n(T : H \to Y)$ in case that $H$ is a Hilbert space, see (2.23), we conclude that we can replace $x_n$ by $a_n$ in Thm. 2.21. With other words, we give a different proof of (2.13), (2.14) and (2.17) is a supplement to (2.13), (2.14).

(ii) Our methods also allow to deal with $x_n(R_d : \mathcal{L}_{p_1}((0,1)^d) \to C((0,1)^d))$ and $x_n(R_d : \mathcal{L}_{p_1}((0,1)^d) \to Z_{\text{mix}}^s((0,1)^d))$, $1 < p_1 < \infty$. But then we will loose the relation to approximation numbers, see part (i). So we omit details.

3 Weyl numbers - basic properties

Weyl numbers are special $s$-numbers. For later use we recall this general notion following Pietsch [38, 2.2.1] (note that this differs slightly from earlier definitions in the literature).

Let $X, Y, X_0, Y_0$ be quasi-Banach spaces. As usual, $\mathcal{L}(X,Y)$ denotes the space of all continuous linear operators from $X$ to $Y$. Finally, let $Y$ be $p$-Banach space for some $p \in (0,1]$, i.e.,

$$\|x + y|Y|^p \leq \|x|Y|^p + \|y|Y|^p$$

for all $x, y \in Y$. (3.1)

An $s$-function is a map $s$ assigning to every operator $T \in \mathcal{L}(X,Y)$ a scalar sequence $(s_n(T))$ such that the following conditions are satisfied:

s1. $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$ for all $T \in \mathcal{L}(X,Y)$;

s2. $s_{n+m-1}(S + T) \leq s_n(S) + s_m(T)$ for $S, T \in \mathcal{L}(X,Y)$ and $m, n = 1, 2, ...$;

s3. $s_n(BTA) \leq \|B\| \cdot s_n(T) \cdot \|A\|$ for $A \in \mathcal{L}(X_0, X)$, $T \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Y_0)$;

s4. $s_n(T) = 0$ if rank$(T) < n$ for all $n \in \mathbb{N}$;

s5. $s_n(id : \ell^n_2 \to \ell^n_2) = 1$ for all $n \in \mathbb{N}$.

We will refer to (s1) as monotonicity, to (s2) as additivity, to (s3) as ideal property, to (s4) as the rank property and to (s5) as normalization (norm-determining property) of the $s$-numbers.

Sometimes a further property is of some use. Let $Z$ be a quasi-Banach space. An $s$-function is called multiplicative if

s6. $s_{n+m-1}(ST) \leq s_n(S) s_m(T)$ for $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,Z)$ and $m, n = 1, 2, ...$.

Examples

The following numbers are $s$-numbers:
(i) Kolmogorov numbers are multiplicative $s$-numbers, see, e.g., [36, Thm. 11.9.2].

(ii) Approximation numbers are multiplicative $s$-numbers, see, e.g., [38, 2.3.3].

(iii) The $n$-th Gelfand number of the linear operator $T \in \mathcal{L}(X,Y)$ is defined to be

$$c_n(T) := \inf \left\{ \| T J^X_M \| : \text{codim}(M) < n \right\},$$

where $J^X_M : M \to X$ refers to the canonical injection of $M$ into $X$. Gelfand numbers are multiplicative $s$-numbers, see, e.g., [38, Prop. 2.4.8].

Entropy numbers do not belong to the class of $s$-numbers since they do not satisfy (s4). Now we are in position to define Weyl numbers.

**Definition 3.1.** Let $n \in \mathbb{N}$. Then the $n$-th Weyl number of the linear operator $T \in \mathcal{L}(X,Y)$ is defined to be

$$x_n(T) := \sup \{ a_n(TA) : A \in \mathcal{L}(\ell_2, X), \|A\| \leq 1 \}.$$

**Remark 3.2.**

(i) Weyl numbers are multiplicative $s$-numbers, see [38, 2.4.14, 2.4.17].

(ii) There is an alternative way to calculate the $n$-th Weyl number. Indeed, for $T \in \mathcal{L}(X,Y)$ it holds

$$x_n(T) := \sup \left\{ c_n(TA) : A \in \mathcal{L}(\ell_2, X), \|A\| \leq 1 \right\},$$

see Pietsch [37].

**Interpolation properties of Weyl numbers**

For later use we add the following assertion concerning interpolation properties of Weyl numbers. For the basics in interpolation theory we refer to the monographs [7, 33, 58]. Let $\mathcal{K}(X,Y)$ denote the collection of all compact linear operators belonging to $\mathcal{L}(X,Y)$.

**Theorem 3.3.** Let $X$ be a Banach space and let $(Y_0, Y_1)$ be an interpolation couple of Banach spaces. Let $Y$ be an intermediate Banach space with respect to $(Y_0, Y_1)$, i.e.,

$$Y_0 \cap Y_1 \hookrightarrow Y \hookrightarrow Y_0 + Y_1.$$

Further we assume that for some $0 < \theta < 1$ there exists a positive constant $C$ such that

$$\|y\|_Y \leq C \|y\|_{Y_0}^{1-\theta} \|y\|_{Y_1}^\theta$$

for all $y \in Y_0 \cap Y_1$. (3.2)

Then, if $T \in \mathcal{K}(X,Y_0)$ and $T \in \mathcal{K}(X,Y_1)$, we have $T \in \mathcal{K}(X,Y)$ as well as

$$x_{n+m-1}(T : X \to Y) \leq C x_n^{1-\theta}(T : X \to Y_0) x_m^\theta(T : X \to Y_1)$$

for all $n, m \in \mathbb{N}$. Here $C$ is the same constant as in (3.2).

**Remark 3.4.** The interpolation properties of Kolmogorov and Gelfand numbers have been studied by Triebel [57]. Theorem 3.3 shows that Gelfand and Weyl numbers share the same interpolation properties.
4 Tensor product Besov spaces and spaces of dominating mixed smoothness

In this section we explain how to switch from the tensor product of univariate Besov spaces to more convenient descriptions of these spaces.

4.1 Besov and Lizorkin-Triebel spaces of dominating mixed smoothness and wavelets

For us it will be convenient to introduce Besov and Lizorkin-Triebel spaces of dominating mixed smoothness by means of wavelets. In the Appendix below we recall the probably better known Fourier-analytic definition. Lizorkin-Triebel spaces of dominating mixed smoothness will be used in our proofs of the main results for Besov spaces.

Let $\bar{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ and $\bar{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$. Then we put

$$2^{-\bar{\nu} \cdot \bar{m}} = (2^{-\nu_1 m_1}, \ldots, 2^{-\nu_d m_d})$$

and

$$Q_{\bar{\nu}, \bar{m}} := \{ x \in \mathbb{R}^d : 2^{-\nu_\ell} m_\ell < x_\ell < 2^{-\nu_\ell} (m_\ell + 1), \ell = 1, \ldots, d \}.$$ 

By $\chi_{\bar{\nu}, \bar{m}}(x)$ we denote the characteristic function of $Q_{\bar{\nu}, \bar{m}}$. First we have to introduce some sequence spaces.

Definition 4.1. If $0 < p, q \leq \infty$, $t \in \mathbb{R}$ and $\lambda := \{ \lambda_{\bar{\nu}, \bar{m}} \in \mathbb{C} : \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d \}$, then we define

$$s_{t, p, q}^b := \{ \lambda : \| \lambda |s_{t, p, q}^b\| = \left( \sum_{\bar{\nu} \in \mathbb{N}_0^d} 2^{[\bar{\nu}] \cdot (t - \frac{1}{p}) q} \left( \sum_{\bar{m} \in \mathbb{Z}^d} |\lambda_{\bar{\nu}, \bar{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \}$$

and, if $p < \infty$,

$$s_{t, p, q}^f = \{ \lambda : \| \lambda |s_{t, p, q}^f\| = \left\| \left( \sum_{\bar{\nu} \in \mathbb{N}_0^d} \sum_{\bar{m} \in \mathbb{Z}^d} 2^{[\bar{\nu}] \cdot t} |\lambda_{\bar{\nu}, \bar{m}} \chi_{\bar{\nu}, \bar{m}}(.)|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^d)} < \infty \}$$

with the usual modification for $p$ or/and $q$ equal to $\infty$.

Remark 4.2. Let $\sigma \in \mathbb{R}$. For later use we mention that the mapping

$$J_{\sigma} : (\lambda_{\bar{\nu}, \bar{m}})_{\bar{\nu}, \bar{m}} \mapsto (2^{\sigma [\bar{\nu}]_1} \lambda_{\bar{\nu}, \bar{m}})_{\bar{\nu}, \bar{m}}$$

yields an isomorphism of $s_{t, p, q}^a$ onto $s_{t, p, q}^{a - \sigma} a$, $a \in \{b, f\}$.

Now we recall wavelet bases of Besov and Lizorkin-Triebel spaces of dominating mixed smoothness. Let $N \in \mathbb{N}$. Then there exists $\psi_0, \psi_1 \in C^N(\mathbb{R})$, compactly supported,

$$\int_{-\infty}^{\infty} t^m \psi_1(t) \, dt = 0, \quad m = 0, 1, \ldots, N,$$
such that \( \{2^{j/2} \psi_{j,m} : j \in \mathbb{N}_0, m \in \mathbb{Z}\} \), where
\[
\psi_{j,m}(t) := \begin{cases} 
\psi_0(t-m) & \text{if } j = 0, m \in \mathbb{Z}, \\
\sqrt{1/2} \psi_1(2^{j-1}t-m) & \text{if } j \in \mathbb{N}, m \in \mathbb{Z},
\end{cases}
\]
is an orthonormal basis in \( L_2(\mathbb{R}) \), see \[66\]. We put
\[
\Psi_{\nu,\bar{m}}(x) := \prod_{\ell=1}^d \psi_{\nu,\bar{m}_\ell}(x_\ell).
\]
Then \( \Psi_{\nu,\bar{m}}, \bar{\nu} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d \) is a tensor product wavelet basis of \( L_2(\mathbb{R}^d) \). Vybiral \[64\] has proved the following.

**Lemma 4.3.** Let \( 0 < p, q \leq \infty \) and \( t \in \mathbb{R} \).

(i) There exists \( N = N(t,p) \in \mathbb{N} \) s.t. the mapping
\[
W : f \mapsto (2^{[\nu]}(f, \Psi_{\nu,\bar{m}}))_{\nu,\bar{m} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d}
\]
is an isomorphism of \( S^t_{p,q}B(\mathbb{R}^d) \) onto \( s^t_{p,q}B \).

(ii) Let \( p < \infty \). Then there exists \( N = N(t,p,q) \in \mathbb{N} \) s.t. the mapping \( W \) is an isomorphism of \( S^t_{p,q}F(\mathbb{R}^d) \) onto \( s^t_{p,q}F \).

**Spaces on** \( \Omega \)

We put \( \Omega := (0,1)^d \). For us it will be convenient to define spaces on \( \Omega \) by restrictions. We shall need the set \( D'(\Omega) \), consisting of all complex-valued distributions on \( \Omega \).

**Definition 4.4.** (i) Let \( 0 < p, q \leq \infty \) and \( t \in \mathbb{R} \). Then \( S^t_{p,q}B((0,1)^d) \) is the space of all \( f \in D'(\Omega) \) such that there exists a function \( g \in S^t_{p,q}B(\mathbb{R}^d) \) satisfying \( f = g|_\Omega \). It is endowed with the quotient norm
\[
\| f |_{S^t_{p,q}B((0,1)^d)} \| = \inf \left\{ ||g|_{S^r_{p,q}B(\mathbb{R}^d)}| : g|_\Omega = f \right\}.
\]

(ii) Let \( 0 < p < \infty, 0 < q \leq \infty \) and \( t \in \mathbb{R} \). Then \( S^t_{p,q}F((0,1)^d) \) is the space of all \( f \in D'(\Omega) \) such that there exists a function \( g \in S^t_{p,q}F(\mathbb{R}^d) \) satisfying \( f = g|_\Omega \). It is endowed with the quotient norm
\[
\| f |_{S^t_{p,q}F((0,1)^d)} \| = \inf \left\{ ||g|_{S^r_{p,q}F(\mathbb{R}^d)}| : g|_\Omega = f \right\}.
\]

Several times we shall work with the following consequence of this definition in combination with Lemma \[4.3\]. Let \( t, p \) and \( q \) be fixed. Let the wavelet basis \( \Psi_{\nu,\bar{m}} \) be admissible in the sense of Lemma \[4.3\]. We put
\[
A_{\nu,\bar{m}} := \left\{ \bar{m} \in \mathbb{Z}^d : \text{supp} \Psi_{\nu,\bar{m}} \cap \Omega \neq \emptyset \right\}, \quad \bar{\nu} \in \mathbb{N}_0^d.
\]
For given \( f \in S^t_{p,q} A(\Omega), A \in \{B, F\} \), let \( \mathcal{E} f \) be an element of \( S^t_{p,q} A(\mathbb{R}^d) \) s.t.
\[
\| \mathcal{E} f \|_{S^t_{p,q} A(\mathbb{R}^d)} \leq 2 \| f \|_{S^t_{p,q} A(\Omega)} \quad \text{and} \quad (\mathcal{E} f)_{|\Omega} = f.
\]

We define
\[
g := \sum_{\nu \in \mathbb{N}_0^d} \sum_{m \in A^d} 2^{j|\nu|} \langle \mathcal{E} f, \Psi_{\nu, m} \rangle \Psi_{\nu, m}.
\]

Then it follows that \( g \in S^t_{p,q} A(\mathbb{R}^d) \), \( g_{|\Omega} = f \),
\[
supp g \subset \{ x \in \mathbb{R}^d : \max_{j=1,\ldots,d} |x_j| \leq c_1 \} \quad \text{and} \quad \| g \|_{S^t_{p,q} A(\mathbb{R}^d)} \leq c_2 \| f \|_{S^t_{p,q} A(\Omega)}.
\]
Here \( c_1, c_2 \) are independent of \( f \).

### 4.2 Tensor products of Besov spaces

Tensor products of Besov spaces have been investigated in \([27, 51]\) and \([52]\). First we recall some results from \([51]\) and \([52]\). For the basic notions of tensor products used here we refer to the Appendix B, in particular, what concerns the meaning of the tensor norms \( \alpha_p, \gamma_p \) and \( \lambda \).

**Theorem 4.5** (Tensor products of Besov spaces).

*Let \( d \geq 1 \) and let \( t \in \mathbb{R} \).*

(i) Let \( 1 < p < \infty \). Then the following formula
\[
B^t_{p,p}(\mathbb{R}) \otimes_{\alpha_p} S^t_{p,p} B(\mathbb{R}^d) = S^t_{p,p} B(\mathbb{R}^d) \otimes_{\alpha_p} B^t_{p,p}(\mathbb{R})
\]
\[
= S^t_{p,p} B(\mathbb{R}^{d+1})
\]
holds true in the sense of equivalent norms.

(ii) Let \( p = \infty \). Then we have
\[
\hat{B}^t_{\infty, \infty}(\mathbb{R}) \otimes_{\lambda} \hat{S}^t_{\infty, \infty} B(\mathbb{R}^d) = \hat{S}^t_{\infty, \infty} B(\mathbb{R}^d) \otimes_{\lambda} \hat{B}^t_{\infty, \infty}(\mathbb{R})
\]
\[
= \hat{S}^t_{\infty, \infty} B(\mathbb{R}^{d+1})
\]
in the sense of equivalent norms.

(iii) Let \( 0 < p \leq 1 \). Then the following formula
\[
B^t_{p,p}(\mathbb{R}) \otimes_{\gamma_p} S^t_{p,p} B(\mathbb{R}^d) = S^t_{p,p} B(\mathbb{R}^d) \otimes_{\gamma_p} B^t_{p,p}(\mathbb{R})
\]
\[
= S^t_{p,p} B(\mathbb{R}^{d+1})
\]
holds true in the sense of equivalent quasi-norms.

**Remark 4.6.** (i) For a quasi-Banach space \( X(\Omega) \) of distributions defined on \( \Omega \) and satisfying \( D(\Omega) \subset X(\Omega) \) we denote by \( A(\Omega) \) the closure of the test functions \( D(\Omega) \) in the quasi-norm of \( X(\Omega) \).

(ii) We identify \( S^t_{p,p} B(\mathbb{R}^d) \) with \( B^t_{p,p}(\mathbb{R}) \) if \( d = 1 \).
Theorem 4.7 (Tensor products of Besov spaces on the interval).

Let $d \geq 1$ and let $t \in \mathbb{R}$.

(i) Let $1 < p < \infty$. Then the following formula
\[
B^t_{p,p}(0,1) \otimes \alpha_p S^t_{p,p}B((0,1)^d) = S^t_{p,p}B((0,1)^d) \otimes \alpha_p B^t_{p,p}(0,1) = S^t_{p,p}B((0,1)^{d+1})
\]
holds true in the sense of equivalent norms.

(ii) Let $p = \infty$. Then we have
\[
\mathring{B}^t_{\infty,\infty}(0,1) \otimes \lambda \mathring{S}^t_{\infty,\infty}B((0,1)^d) = \mathring{S}^t_{\infty,\infty}B((0,1)^d) \otimes \lambda \mathring{B}^t_{\infty,\infty}(0,1)
\]
in the sense of equivalent norms.

(iii) Let $0 < p \leq 1$. Then the following formula
\[
B^t_{p,p}(0,1) \otimes \gamma_p S^t_{p,p}B((0,1)^d) = S^t_{p,p}B((0,1)^d) \otimes \gamma_p B^t_{p,p}(0,1) = S^t_{p,p}B((0,1)^{d+1})
\]
holds true in the sense of equivalent quasi-norms.

Remark 4.8. For easier notation we define
\[
\gamma_p := \begin{cases} 
\alpha_p & \text{if } 1 < p < \infty, \\
\lambda & \text{if } p = \infty.
\end{cases}
\]

In both situations, the local and the non-local, we may iterate the process of taking tensor products. Defining for $m > 2$
\[
X_1 \otimes \gamma_p X_2 \otimes \gamma_p \ldots \otimes \gamma_p X_m := X_1 \otimes \gamma_p \left( \ldots X_{m-2} \otimes \gamma_p (X_{m-1} \otimes \gamma_p X_m) \right)
\]
we obtain an interpretation of $S^t_{p,p}B(\mathbb{R}^d)$, $0 < p < \infty$, as an iterated tensor product of univariate Besov spaces, namely
\[
S^t_{p,p}B(\mathbb{R}^d) = B^t_{p,p}(\mathbb{R}) \otimes \gamma_p \ldots \otimes \gamma_p B^t_{p,p}(\mathbb{R}), \quad 0 < p < \infty.
\]

Similarly we obtain for $S^t_{p,p}B((0,1)^d)$
\[
S^t_{p,p}B((0,1)^d) = B^t_{p,p}(0,1) \otimes \gamma_p \ldots \otimes \gamma_p B^t_{p,p}(0,1), \quad 0 < p < \infty.
\]

The iterated tensor products, considered in this paper, do not depend on the order of the tuples which are formed during the process of calculating $X_1 \otimes \gamma_p X_2 \otimes \gamma_p \ldots \otimes \gamma_p X_m$, i.e.,
\[
(X_1 \otimes \gamma_p X_2) \otimes \gamma_p X_3 = X_1 \otimes \gamma_p (X_2 \otimes \gamma_p X_3).
\]
Consequently, if $p < \infty$, we may deal with $S^t_{p,p}B((0,1)^d)$ instead of $B^t_{p,p}(0,1) \otimes \gamma_p \ldots \otimes \gamma_p B^t_{p,p}(0,1)$ in what follows.
5 Weyl numbers of embeddings of sequence spaces

In this section we will estimate the behavior of Weyl numbers of the identity mapping

\[ \text{id}^*: s_{p_0,p_0}^0 \rightarrow s_{p_2}^0. \]

To avoid any conflict with our agreement in Section 1 we switch to pairs \((p_0, p)\) instead of \((p_1, p_2)\). Now \(p_0\) may vary in \((0, \infty]\) and \(p\) in \((0, \infty)\).

5.1 Preparations

For technical reasons we need a few more sequence spaces. Recall, \(A_\nu^\Omega\) has been defined in [4.3].

**Definition 5.1.** If \(0 < p \leq \infty, 0 < q \leq \infty, t \in \mathbb{R}\) and

\[ \lambda = \{ \lambda_{\nu, \bar{m}} \in \mathbb{C} : \nu \in \mathbb{N}_0^d, \; \bar{m} \in A_\nu^\Omega \}, \]

then we define

\[ s_{p,q}^t \Omega^\nu = \left\{ \lambda : \| \lambda \|_{s_{p,q}^t \Omega^\nu} = \left( \sum_{\nu \in \mathbb{N}_0^d} 2^{\nu_1 (t-\frac{1}{p})/q} \left( \sum_{\bar{m} \in A_\nu^\Omega} |\lambda_{\nu, \bar{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\} \]

and, if \(p < \infty\),

\[ s_{p,q}^t \Omega^\nu f = \left\{ \lambda : \| \lambda \|_{s_{p,q}^t \Omega^\nu f} = \left\| \left( \sum_{\nu \in \mathbb{N}_0^d} \sum_{\bar{m} \in A_\nu^\Omega} |2^{\nu_1 t} \lambda_{\nu, \bar{m}} \chi_{\nu, \bar{m}}(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_p^\nu(\mathbb{R}^d)} < \infty \right\}. \]

In addition we need the following sequence of subspaces.

**Definition 5.2.** If \(0 < p \leq \infty, 0 < q \leq \infty, t \in \mathbb{R}, \mu \in \mathbb{N}_0\) and

\[ \lambda = \{ \lambda_{\nu, \bar{m}} \in \mathbb{C} : \nu \in \mathbb{N}_0^d, \; \nu_1 = \mu, \; \bar{m} \in A_\nu^\Omega \}, \]

then we define

\[ (s_{p,q}^t \Omega^\nu b)_\mu = \left\{ \lambda : \| \lambda \|_{(s_{p,q}^t \Omega^\nu b)_\mu} = \left( \sum_{\nu_1 = \mu} 2^{\nu_1 (t-\frac{1}{p})/q} \left( \sum_{\bar{m} \in A_\nu^\Omega} |\lambda_{\nu, \bar{m}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\} \]

and, if \(p < \infty\),

\[ (s_{p,q}^t \Omega^\nu f)_\mu = \left\{ \lambda : \| \lambda \|_{(s_{p,q}^t \Omega^\nu f)_\mu} = \left\| \left( \sum_{\nu_1 = \mu} \sum_{\bar{m} \in A_\nu^\Omega} |2^{\nu_1 t} \lambda_{\nu, \bar{m}} \chi_{\nu, \bar{m}}(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_p^\nu(\mathbb{R}^d)} < \infty \right\}. \]

The two following lemmas are taken from [64] and [26].
Lemma 5.3. (i) We have

\[ |A_\nu^\nu| \lesssim 2^{\nu_1}, \quad D_\mu := \sum_{|\nu|=\mu} |A_\nu^\nu| \lesssim \mu^{d-1} \mu \]

with equivalence constants independent of \( \bar{\nu} \in \mathbb{N}_0^d \) and \( \mu \in \mathbb{N}_0 \).

(ii) Let \( 0 < p_0, p \leq \infty \) and \( m \in \mathbb{N} \). Then

\[ \| id : \ell_{p_0}^m \rightarrow \ell_p^m \| = \begin{cases} 1, & p_0 \leq p \\ \frac{m^{\frac{1}{p_0} - \frac{1}{p}}}{m}, & p_0 > p \end{cases} \]

(iii) Let \( 0 < p < \infty \) and \( t \in \mathbb{R} \). Then

\[ s_{p,p}^{t,\Omega} = s_{p,p}^{t,\Omega} \]

and

\[ (s_{p,p}^{t,\Omega} f)_\mu = (s_{p,p}^{t,\Omega} b)_\mu = 2^{\mu(t-\frac{1}{p})} \ell_{p}^D, \quad \mu \in \mathbb{N}_0, \]

with the obvious interpretation for the quasi-norms.

To avoid repetitions we shall use \( s_{p,q}^{t,\Omega} a, \ s_{p,q}^{t,\Omega} a, \ (s_{p,q}^{t,\Omega} a)_\mu \) with \( a \in \{ b, f \} \) in case that an assertion holds for both scales simultaneously. Here we always use of convention that \( p = \infty \) is excluded for any type of \( f \)-space.

Lemma 5.4. (i) Let \( 0 < p_0, p \leq \infty \) and \( 0 < q \leq \infty \). Then

\[ \| id^*_\mu : (s_{p_0,q}^{t,\Omega} a)_\mu \rightarrow (s_{p,q}^{t,\Omega} a)_\mu \| \lesssim 2^{\mu(t-\frac{1}{p_0} - \frac{1}{p})} \]

with equivalence constants independent of \( \mu \in \mathbb{N}_0 \).

(ii) Let \( 0 < q_0, q \leq \infty \) and \( 0 < p \leq \infty \). Then

\[ \| id^*_\mu : (s_{p,q_0}^{t,\Omega} a)_\mu \rightarrow (s_{p,q}^{t,\Omega} a)_\mu \| \lesssim \mu^{(d-1)(\frac{1}{q} - \frac{1}{q_0})} \]

with equivalence constants independent of \( \mu \in \mathbb{N}_0 \).

Corollary 5.5. Let \( 0 < p_0, p, q_0, q \leq \infty \) and \( t \in \mathbb{R} \). Then

\[ \| id^*_\mu : (s_{p_0,q_0}^{t,\Omega} a)_\mu \rightarrow (s_{p,q_0}^{0,\Omega} a)_\mu \| \lesssim 2^{\mu(-t+(\frac{1}{p_0} - \frac{1}{p})_+)} \mu^{(d-1)(\frac{1}{q} - \frac{1}{q_0})_+}, \]

with a constant behind \( \lesssim \) independent of \( \mu \).

Proof. This is an immediate consequence of Lemma 5.4. ■

Sometimes the previous estimate can be improved.
Lemma 5.6. Let $0 < p_0 < p < \infty$, $0 < q_0, q \leq \infty$ and $t \in \mathbb{R}$. Then

$$\|id^*_\mu : (s_{p_0,q_0}^t f)_\mu \to (s_{p,q}^0 f)_\mu \| \lesssim 2^{\mu(\frac{t}{p_0} - \frac{1}{p})}$$

Proof. This assertion is contained in [26]. Since this PhD is not published we give a proof. Let $\lambda$ be a sequence such that $\lambda_{\bar{\nu}, \bar{m}} = 0$ if $|\bar{\nu}| \neq \mu$. Since $p_0 < p$ the Sobolev-type embedding yields

$$s_{p_0,q_0}^t f \hookrightarrow s_{p,q}^0 f,$$

see [50, Thm. 2.4.1] $(d = 2)$ or [28], we have

$$\|\lambda(s_{p,q}^0 f)_\mu\| = \|\lambda|s_{p,q}^0 f\| = 2^{\mu(\frac{t}{p_0} - \frac{1}{p})}\|\lambda|s_{p,q}^0 f\| \lesssim 2^{\mu(\frac{t}{p_0} - \frac{1}{p})}\|\lambda|s_{p_0,q_0}^t f\| = 2^{\mu(\frac{t}{p_0} - \frac{1}{p})}\|\lambda|s_{p_0,q_0}^t f\|.$$  

This proves the claim.  

5.2 Weyl numbers of embeddings of sequence spaces related to spaces of dominating mixed smoothness - preparations

Here we allow a greater generality than originally needed. We define the operators

$$id^*_\mu : s_{p_0,p_0}^t b \to s_{p,2}^0 f,$$

where

$$(id^*_\mu \lambda)_{\bar{\nu}, \bar{m}} := \begin{cases} 
\lambda_{\bar{\nu}, \bar{m}} & \text{if } |\bar{\nu}| = \mu, \\
0 & \text{otherwise}.
\end{cases}$$

The main idea of our proof is the following splitting of $id^*_\mu : s_{p_0,p_0}^t b \to s_{p,2}^0 f$ into a sum of identities between building blocks

$$id^*_\mu = \sum_{\mu=0}^{\infty} id^*_\mu = \sum_{\mu=0}^{J} id^*_\mu + \sum_{\mu=J+1}^{L} id^*_\mu + \sum_{\mu=L+1}^{\infty} id^*_\mu, \quad (5.1)$$

where $J$ and $L$ are at our disposal. These numbers $J$ and $L$ will be chosen in dependence on the parameters. Let us mention that a similar splitting has been used by Vybiral [64] for the estimates of related entropy numbers.

The additivity and the monotonicity of the Weyl numbers and the quasi-triangle inequality (3.1) yield

$$x^\rho_n(id^*_\mu) \leq \sum_{\mu=0}^{J} x^\rho_n(id^*_\mu) + \sum_{\mu=J+1}^{L} x^\rho_n(id^*_\mu) + \sum_{\mu=L+1}^{\infty} \|id^*_\mu\|^\rho, \quad \rho := \min(1,p), \quad (5.2)$$

where $n - 1 = \sum_{\mu=0}^{L} (n_{\mu} - 1)$. Of course, $\|id^*_\mu\| = \|id^*_\mu : (s_{p_0,p_0}^t f)_\mu \to (s_{p,2}^0 f)_\mu\|$. For brevity we put

$$\alpha = t - \left(\frac{1}{p_0} - \frac{1}{p}\right) +.$$
Then by Corollary 5.5 we have
\[ \|\text{id}_\mu^p\| \lesssim 2^{-\mu\alpha} \mu^{(d-1)(\frac{1}{2} - \frac{1}{p_0})_+}, \]
which results in the estimate
\[ \sum_{\mu = L+1}^{\infty} \|\text{id}_\mu^p\|^{\rho} \lesssim 2^{-L\alpha\rho} L^{(d-1)\rho(\frac{1}{2} - \frac{1}{p_0})_+}. \] (5.3)

Now we choose \( n_\mu \)
\[ n_\mu := D_\mu + 1, \quad \mu = 0, 1, \ldots, J. \] (5.4)
Then we get
\[ \sum_{\mu = 0}^{J} n_\mu \times \sum_{\mu = 0}^{J} \mu^{(d-1)2\mu} \times J^{d-1}2^J \] (5.5)
and \( x_{n_\mu}(\text{id}_\mu^p) = 0 \), see (s4), which implies
\[ \sum_{\mu = 0}^{J} x_{n_\mu}^p(\text{id}_\mu^p) = 0. \] (5.6)

Summarizing (5.2)-(5.6) we have found
\[ x_n^p(\text{id}^*) \lesssim \sum_{\mu = J+1}^{L} x_{n_\mu}^p(\text{id}_\mu^p) + 2^{-L\alpha\rho} L^{(d-1)\rho(\frac{1}{2} - \frac{1}{p_0})_+}. \] (5.7)

Now we turn to the problem to reduce the estimates for the Weyl numbers \( x_{n_\mu}(\text{id}_\mu^p) \) to estimates for \( x_n(\text{id}^{m}_{p_0,p}) \).

**Proposition 5.7.** Let \( 0 < p_0 \leq \infty \) and \( t \in \mathbb{R} \). Then we have the following assertions.

(i) If \( 0 < p \leq 2 \), then
\[ \mu^{(d-1)(\frac{1}{2} - \frac{1}{p} - \frac{1}{p_0})} x_n(\text{id}^{D\mu}_{p_0,p}) \lesssim x_n(\text{id}_\mu^p) \lesssim 2\mu^{(d-1)(\frac{1}{2} - \frac{1}{p} - \frac{1}{p_0})} x_n(\text{id}^{D\mu}_{p_0,2}) \] (5.8)

(ii) If \( 2 \leq p < \infty \), then
\[ 2\mu^{(d-1)(\frac{1}{2} - \frac{1}{p} - \frac{1}{p_0})} x_n(\text{id}^{D\mu}_{p_0,2}) \lesssim x_n(\text{id}_\mu^p) \lesssim \mu^{(d-1)(\frac{1}{2} - \frac{1}{p} - \frac{1}{p_0})} x_n(\text{id}^{D\mu}_{p_0,p}). \] (5.9)

**Proof.** Step 1. Estimate from above. We define \( \delta := \max(p, 2) \) and consider the following diagram:

\[ \begin{array}{ccc}
(s^{t,\Omega}_{p_0,p_0}b)_{\mu} & \overset{\text{id}_\mu^p}{\longrightarrow} & (s^{0,\Omega}_{p_2}f)_{\mu} \\
\text{id}^2 & \downarrow & \text{id}^1 \\
(s^{0,\Omega}_{\delta,\delta}f)_{\mu}
\end{array} \]
Using property (s3) of Weyl numbers we conclude
\[ x_n(id_\mu^\ast) \leq \| id^1 \| x_n(id^2). \]

By Corollary 5.5, we have
\[ \| id^1 \| \lesssim \mu(d-1)(\frac{1}{2} - \frac{1}{p}). \]

From Lemma 5.3 (iii), we derive
\[ x_n(id^2) \lesssim 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{\gamma})} x_n(id_{D^\mu_{p_0,\delta}}), \]

taking into account (s3) and the commutative diagram
\[ 2^{\mu(t-\frac{1}{p_0})}(l_{p_0}^\mu) \xrightarrow{id^3} l_{p_0}^\mu \]
\[ id^2 \downarrow \quad \downarrow id_{p_0,\delta}^D \]
\[ 2^{-\frac{\mu}{2}}(l_{\delta}^D) \xleftarrow{id^4} l_{\delta}^D, \]
(i.e., \( id^2 = id^4 \circ id_{p_0,\delta}^D \circ id^3),
\[ \| id^3 \| = 2^{-\mu(t-\frac{1}{p_0})} \quad \text{and} \quad \| id^4 \| = 2\frac{\mu}{2}. \]

Altogether this implies
\[ x_n(id_\mu^\ast) \lesssim \mu^{(d-1)(\frac{1}{2} - \frac{1}{\gamma})} 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{\gamma})} x_n(id_{D^\mu_{p_0,\delta}}). \]

\[ (5.11) \]

**Step 2.** Now we turn to the estimate from below. We define \( \gamma := \min(p, 2) \) and use the following commutative diagram
\[ (s_{p_0,\mu,\gamma}^\mu b)_\mu \xrightarrow{id^2} (s_{\gamma,\gamma}^\mu f)_\mu \]
\[ id^2 \downarrow \quad \downarrow id_{p_0,\delta}^D \]
\[ (s_{p,\delta}^\mu f)_\mu \]
\[ id_\mu^\ast \]

This time we have \( x_n(id_2) \leq \| id^1 \| x_n(id_\mu^\ast) \) and by Corollary 5.5, we get
\[ \| id^1 \| \lesssim \mu^{(d-1)(\frac{1}{2} - \frac{1}{\gamma})}. \]

Similarly as in Step 1 Lemma 5.3 (ii) yields
\[ x_n(id^2) \gtrsim 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{\gamma})} x_n(id_{D^\mu_{p_0,\gamma}}). \]

Inserting this in our previous estimate we find
\[ x_n(id_\mu^\ast) \gtrsim \mu^{(d-1)(\frac{1}{2} - \frac{1}{\gamma})} 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{\gamma})} x_n(id_{D^\mu_{p_0,\gamma}}). \]

\[ (5.12) \]

The proof is complete. \( \blacksquare \)
Proposition 5.8. Let $0 < p_0 \leq \infty$ and $t \in \mathbb{R}$. Then we have the following assertions.

(i) If $0 < p \leq 2$, then
\[ x_n(id^*_\mu) \lesssim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,p}^D). \] \hspace{1cm} (5.13)

(ii) If $2 \leq p < \infty$, then
\[ 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,p}^D) \lesssim x_n(id^*_\mu). \] \hspace{1cm} (5.14)

Proof. Step 1. Proof of (i). We consider the following diagram
\[
\begin{array}{ccc}
(s_{p_0,p_0} \circ b)_\mu & \xrightarrow{id^*_\mu} & (s_{p_2,p_2} \circ f)_\mu \\
& \downarrow id^2 & \downarrow id^1 \\
& (s_{p_0,p_0} \circ f)_\mu & (s_{p_2,p_2} \circ f)_\mu
\end{array}
\]
This implies $x_n(id^*_\mu) \lesssim \|id^1\| x_n(id^{2})$. Corollary 5.5 yields $\|id^1\| \lesssim 1$ and from Lemma 5.3 we derive
\[ x_n(id^2) \lesssim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,p}^D). \]
Altogether we have found
\[ x_n(id^*_\mu) \lesssim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,p}^D). \]

Step 2. Proof of (ii). We use the following diagram
\[
\begin{array}{ccc}
(s_{p_0,p_0} \circ b)_\mu & \xrightarrow{id^2} & (s_{p_2,p_2} \circ f)_\mu \\
& \downarrow id^*_\mu & \downarrow id^1 \\
& (s_{p_0,p_0} \circ f)_\mu & (s_{p_2,p_2} \circ f)_\mu
\end{array}
\]
Because of $x_n(id^2) \lesssim \|id^1\| x_n(id^*_\mu)$, $\|id^1\| \lesssim 1$, see Corollary 5.5 and
\[ x_n(id^2) \gtrsim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,p}^D), \]
(see Lemma 5.3), we obtain
\[ x_n(id^*_\mu) \gtrsim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,p}^D). \]
The proof is complete. $\blacksquare$

We need a few more results of the above type.
Lemma 5.9. Let $0 < p_0, p < \infty$ and $0 < \epsilon < p$. Then

\[ x_n(id^*_\mu) \leq 2^{\mu(-t+ \frac{1}{p_0} - \frac{1}{p})} x_n(id^D_{p_0,p-\epsilon}). \] (5.15)

Proof. We consider the following diagram

Clearly, $x_n(id^*_\mu) \leq \| id_1 \| x_n(id^2)$ and by Lemma 5.6 we have

\[ \| id_1 \| \leq 2^{\mu(-r+ \frac{1}{p_0} - \frac{1}{p})}. \]

Further we know

\[ x_n(id^2) = 2^{\mu(r- \frac{1}{p_0} - t+ \frac{1}{p})} x_n(id^D_{p_0,p-\epsilon}). \]

Inserting the previous inequality in this identity we obtain (5.15). \(\blacksquare\)

Lemma 5.10. For all $\mu \in \mathbb{N}_0$ and all $n \in \mathbb{N}$ we have

\[ x_n(id^*_\mu) \leq x_n(id^*). \] (5.16)

Proof. We consider the following diagram

Here $id^1$ is the canonical embedding and $id^2$ is the canonical projection. Since $id^*_\mu = id_2 \circ id^* \circ id_1$ the property (s3) yields

\[ x_n(id^*_\mu) \leq \| id_1 \| \| id^2 \| x_n(id^*) = x_n(id^*). \]

This completes the proof. \(\blacksquare\)

5.3 Weyl numbers of embeddings of sequence spaces related to spaces of dominating mixed smoothness - results

Now we are in position to deal with the Weyl numbers of $id^* : s^t_{p_0,p_0}b \rightarrow s_{p,2}^0 f$. We have to continue with the proof already started in (5.1)-(5.7). Therefore we need to distinguish several cases. Always the positions of $p_0$ and $p$ relative to 2 are of importance.
5.3.1 The case $0 < p_0 \leq 2 \leq p < \infty$

**Theorem 5.11.** Let $0 < p_0 \leq 2 \leq p < \infty$ and $t > \frac{1}{p_0} - \frac{1}{p}$. Then

$$x_n(id^*) \asymp n^{-t + \frac{1}{2} - \frac{1}{p} (\log n)^{(d-1)(t+\frac{1}{p} - \frac{1}{p_0})}}, \quad n \geq 2.$$  

**Proof.** *Step 1.* Estimate from below. Since $p \geq 2$, from (5.16) and (5.14) we derive

$$x_n(id^*) \gtrsim 2^{\mu(-t + \frac{1}{2} - \frac{1}{p})} x_n(id_{D\mu_0,p}).$$

Next we choose $n = \lfloor D\mu_2 \rfloor$ (here $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$). Then from property (a) in Appendix A we get

$$x_n(id_{D\mu_0,p}) \gtrsim (D\mu)^{\frac{1}{2} - \frac{1}{p_0}} (2\mu^{d-1})^{\frac{1}{2} - \frac{1}{p_0}},$$

which implies

$$x_n(id^*) \gtrsim 2^{\mu(-t + \frac{1}{2} - \frac{1}{p})} (\mu^{d-1})^{\frac{1}{2} - \frac{1}{p_0}}.$$  

Because of

$$\mu \asymp \log n \quad \text{and} \quad 2^{\mu} \asymp \frac{n}{\log^{d-1} n},$$

we conclude

$$x_n(id^*) \gtrsim n^{-t + \frac{1}{2} - \frac{1}{p} (\log n)^{(d-1)(t+\frac{1}{p_0} - \frac{1}{p})}}.$$  

*Step 2.* Estimate from above. Let $L, J$ and $\alpha$ as in (5.1)-(5.3). By our assumptions we obviously have

$$2^{-\alpha L} L^{d-1}(\frac{1}{2} - \frac{1}{p_0}) = 2^L(-t + \frac{1}{2} - \frac{1}{p}).$$

For given $J$ we choose $L > J$ large enough such that

$$2^L(-t + \frac{1}{2} - \frac{1}{p}) \leq (\frac{1}{2} - \frac{1}{p_0}) 2^J(-t + \frac{1}{2} - \frac{1}{p}).$$  

(5.18)

For the sum in (5.7), we define

$$n_\mu := \lfloor D\mu 2^{(J-\mu)\lambda} \rfloor \leq \frac{D\mu}{2}, \quad J + 1 \leq \mu \leq L,$$

where $\lambda > 1$ is at our disposal. We choose $\lambda$ such that

$$t - \frac{1}{2} + \frac{1}{p} > \lambda \left( \frac{1}{p_0} - \frac{1}{2} \right)$$  

(5.19)

which is always possible under the given restrictions. Then

$$\sum_{\mu=J+1}^{L} n_\mu \asymp J^{d-1} 2^J$$  

(5.20)

follows. If $p > 2$, we choose $\epsilon > 0$ such that $2 \leq p - \epsilon$. From (5.15) we obtain

$$x_{n_\mu}(id^*) \lesssim 2^{\mu(-t + \frac{1}{2} - \frac{1}{p})} x_{n_\mu}(id_{p_0,p-\epsilon}).$$

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If $p = 2$, then (5.9) implies
\[
x_{\mu}(id_{\mu}^n) \lesssim 2^{\mu\left(-t + \frac{1}{p_0} - \frac{1}{p}\right)} x_{\mu}(id_{p_0}^{D_{\mu}}).
\]

Employing property (a) in Appendix A we obtain
\[
x_{\mu}(id_{\mu}^n) \lesssim 2^{\mu\left(-t + \frac{1}{p_0} - \frac{1}{p}\right)} \left(\mu^{d-1} 2^{\mu 2(J-\mu)\lambda}\right)^{\frac{1}{2} - \frac{1}{p_0}}
\]
\[
= \mu^{(d-1)\left(\frac{1}{2} - \frac{1}{p_0}\right)} 2^{\mu\left(-t + \frac{1}{2} - \frac{1}{p}\right)} 2^{(J-\mu)\lambda\left(\frac{1}{2} - \frac{1}{p_0}\right)}.
\]

Our special choice of $\lambda$ in (5.19) yields
\[
\sum_{\mu=J+1}^{L} x_{\mu}^n(id_{\mu}^n) \lesssim J^{(d-1)\rho\left(\frac{1}{2} - \frac{1}{p_0}\right)} 2^{J\rho\left(-t + \frac{1}{2} - \frac{1}{p}\right)}.
\] (5.21)

Inserting (5.18) and (5.21) into (5.7) leads to
\[
x_{n}(id^*) \lesssim J^{(d-1)\rho\left(\frac{1}{2} - \frac{1}{p_0}\right)} 2^{J\rho\left(-t + \frac{1}{2} - \frac{1}{p}\right)}.
\]

Notice
\[
n = 1 + \sum_{\mu=0}^{L}(n_{\mu} - 1) = 1 + \sum_{\mu=0}^{J} D_{\mu} + \sum_{\mu=J+1}^{L} ([D_{\mu} 2^{(J-\mu)\lambda}] - 1) \asymp J^{d-1} 2^J,
\]
see (5.4), (5.5) and (5.20). Hence, our proof works for a certain subsequence $(n_{J})_{J=1}^{\infty}$ of the natural numbers. More exactly, with
\[
n_{J} := 1 + \sum_{\mu=0}^{J} D_{\mu} + \sum_{\mu=J+1}^{L} ([D_{\mu} 2^{(J-\mu)\lambda}] - 1), \quad J \in \mathbb{N},
\]
and $L = L(J)$ chosen as the minimal admissible value in (5.18) we find
\[
x_{n_{J}}(id^*) \lesssim J^{(d-1)\rho\left(\frac{1}{2} - \frac{1}{p_0}\right)} 2^{J\rho\left(-t + \frac{1}{2} - \frac{1}{p}\right)}.
\]

We already know
\[
A J^{d-1} 2^J \leq n_{J} \leq B J^{d-1} 2^J, \quad J \in \mathbb{N},
\]
for suitable $A, B > 0$. Without loss of generality we assume $B \in \mathbb{N}$. Then we conclude from the monotonicity of the Weyl numbers
\[
x_{B J^{d-1} 2^J}(id^*) \lesssim \log(B J^{d-1} 2^J)^{(d-1)\left(\frac{1}{2} - \frac{1}{p_0}\right)} \left(\frac{B J^{d-1} 2^J}{\log^{d-1}(B J^{d-1} 2^J)}\right)^{-t + \frac{1}{2} - \frac{1}{p}}.
\]

Employing one more times the monotonicity of the Weyl numbers and in addition its polynomial behaviour we can switch from the subsequence $(B J^{d-1} 2^J)_{J}$ to $n \in \mathbb{N}$ in this formula by possibly changing the constant behind $\lesssim$. This finishes our proof.
5.3.2 The case $2 \leq p_0 < p < \infty$

**Theorem 5.12.** Let $2 \leq p_0 < p < \infty$ and $t > \frac{1}{p_0} - \frac{1}{p}$. Then

$$x_n(id^*) \asymp n^{-t+\frac{1}{p_0} - \frac{1}{p}} (\log n)^{(d-1)(t+\frac{1}{p} - \frac{1}{p_0})}, \quad n \geq 2.$$  

**Proof.** Step 1. Estimate from below. We apply the same arguments as in proof of the previous theorem. However, notice that $x_n(id^*_{p_0,p})$ has a different behaviour, see property (a) in Appendix A. With $n = [D\mu/2]$ and $x_n(id^*_{p_0,p}) \geq 1$ we conclude that

$$x_n(id^*) \geq 2^{\mu(t+\frac{1}{p_0} - \frac{1}{p})} x_n(id^*_{p_0,p}) \geq 2^{\mu(t+\frac{1}{p_0} - \frac{1}{p})}.$$  

Because of $2^\mu \asymp \frac{n}{\log^{d-1} n}$ this results in the estimate

$$x_n(id^*) \geq n^{-t+\frac{1}{p_0} - \frac{1}{p}} (\log n)^{(d-1)(t+\frac{1}{p} - \frac{1}{p_0})}.$$  

Step 2. Estimate from above. For $J \in \mathbb{N}$ and $\lambda \in s_{p_0,p_0}^{t,\Omega}$ we put

$$S_J \lambda := \sum_{\mu=0}^J \sum_{|\nu|=\mu} \sum_{\tilde{m} \in A^J_\nu} \lambda_{\nu,\tilde{m}} e^{\nu,\tilde{m}},$$

where $\{e^{\nu,\tilde{m}} : \nu \in \mathbb{N}_0^d, \tilde{m} \in A^J_\nu\}$ is the canonical orthonormal basis of $s_{2,2}^{0,\Omega}$. Obviously

$$\|id^* - S_J : s_{p_0,p_0}^{t,\Omega} b \rightarrow s_{p_2,2}^{0,\Omega} f \| \leq \sum_{\mu=J+1}^\infty \|id^* : (s_{p_0,p_0}^{t,\Omega} b)_{\mu} \rightarrow (s_{p_2,2}^{0,\Omega} f)_{\mu} \|.$$  

Using Lem. 5.6 and $(s_{p_0,p_0}^{t,\Omega} b)_{\mu} = (s_{p_0,p_0}^{t,\Omega} f)_{\mu}$ we get

$$\|id^* - S_J : s_{p_0,p_0}^{t,\Omega} b \rightarrow s_{p_2,2}^{0,\Omega} f \| \leq \sum_{\mu=J+1}^\infty 2^{-\mu(t+\frac{1}{p_0} - \frac{1}{p})} \lesssim 2^{-J(t+\frac{1}{p_0} - \frac{1}{p})}. $$

Because of rank$(S_J) \asymp 2^J J^{d-1}$ we conclude in case $n = 2^J J^{d-1}$ that

$$a_n(id^*) \lesssim 2^{-J(t+\frac{1}{p_0} - \frac{1}{p})}.$$  

Since $x_n \leq a_n$ we can complete the proof of the estimate from above by arguing as at the end of the proof of Thm. 5.11. \[\blacksquare\]

5.3.3 The case $2 \leq p \leq p_0 \leq \infty$

**Theorem 5.13.** Let $2 \leq p \leq p_0 \leq \infty$ and $t > \frac{1}{p} - \frac{1}{p_0}$. Then

$$x_n(id^*) \asymp n^{-t+\frac{1}{p_0} - \frac{1}{p}} (\log n)^{(d-1)(t+\frac{1}{p} - \frac{1}{p_0})}, \quad n \geq 2.$$
Proof. Step 1. Estimate from below. Because of $p > 2$, (5.16) and (5.14) imply
\[ x_n(id^x) \gtrsim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,p}^\nu). \]
We choose $n = [D_\mu/2]$. Then property (f) in Appendix A yields $x_n(id_{p_0,p}^\nu) \gtrsim 1$. Hence
\[ x_n(id^x) \gtrsim 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{p})}. \]
Because of $2^\mu \approx \frac{n}{\log_d n}$ this implies the desired estimate.

Step 2. Estimate from above. Since $2 \leq p \leq p_0$ we obtain
\[ 2^{-\alpha L} L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} = 2^{-\alpha L} L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})}. \]
For given $J$ we choose $L$ large enough such that
\[ 2^{-\alpha L} L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} \leq 2^{-\gamma J L} \quad (5.22) \]
for some $\gamma \geq 1$ (to be chosen later on). We define
\[ n_\mu := [D_\mu 2^{(J-\mu)\beta}] \leq D_\mu, \quad J + 1 \leq \mu \leq L, \]
where the parameter $\beta > 1$ will be also chosen later on. Hence
\[ \sum_{\mu=J+1}^{L} n_\mu \approx 2^{J}J^{d-1}. \]
The restriction $t > \frac{1}{p_0} - \frac{1}{2} - \frac{1}{p}$ implies
\[ -t + \frac{1}{p_0} - \frac{1}{p} + \frac{1}{p_0} - \frac{1}{2} - \frac{1}{p} < 0. \]
If $p > 2$ we choose $\epsilon > 0$ such that $2 \leq p - \epsilon$ and
\[ -t + \frac{1}{p_0} - \frac{1}{p} + \frac{1}{p_0} - \frac{1}{2} - \frac{1}{p} < 0. \quad (5.23) \]
In this situation we derive from property (b)(ii) in Appendix A
\[ x_{n_\mu}(id_{p_0,p-\epsilon}^\nu) \lesssim \left( \frac{D_\mu}{n_\mu} \right)^\frac{1}{r} \times 2^{-\frac{(J-\mu)\beta}{r}}, \quad \frac{1}{r} := \frac{1}{p_0} - \frac{1}{2} - \frac{1}{p_0}. \]
The estimate (5.15) guarantees
\[ \sum_{\mu=J+1}^{L} x_{n_\mu}(id_{p_0,p-\epsilon}^\nu) \lesssim \sum_{\mu=J+1}^{L} 2^{\mu\mu(-t + \frac{1}{p_0} - \frac{1}{p})} x_{n_\mu}(id_{p_0,p-\epsilon}^\nu). \quad (5.24) \]
In case $p = 2$, property (b)(iii) in Appendix A yields
\[
x_{n_{\mu}}(id_{p_{0},2}^{D_{\mu}}) \lesssim \left( \frac{D_{\mu}}{n_{\mu}} \right)^{\frac{1}{\beta}} \times 2^{-\frac{(J_{-\mu})\beta}{r}} \times 2^{-\frac{1}{r}} = \frac{1}{2}.
\]

From (5.13) we obtain
\[
\sum_{\mu = J+1}^{L} x_{n_{\mu}}^{\rho}(id_{\mu}^{*}) \lesssim \sum_{\mu = J+1}^{L} 2^{\mu \rho(-t+\frac{1}{p_{0}}-\frac{1}{p})} x_{n_{\mu}}^{\rho}(id_{p_{0},2}^{D_{\mu}})
\]
(5.25)

Now (5.24) and (5.25) yield
\[
\sum_{\mu = J+1}^{L} x_{n_{\mu}}^{\rho}(id_{\mu}^{*}) \lesssim \sum_{\mu = J+1}^{L} 2^{\mu \rho(-t+\frac{1}{p_{0}}-\frac{1}{p}+\frac{\beta}{r})} 2^{-\frac{J_{-\mu}\beta}{r}} = 2^{J \rho(-t+\frac{1}{p_{0}}-\frac{1}{p}+\frac{\beta}{r})} 2^{-\frac{J_{-\mu}\beta}{r}}.
\]

The condition (5.23) can be rewritten as
\[-t + \frac{1}{p_{0}} - \frac{1}{p} + \frac{1}{r} < 0.\]

Now we choose $\beta > 1$ such that $-t + \frac{1}{p_{0}} - \frac{1}{p} + \frac{\beta}{r} < 0$. Then
\[
\sum_{\mu = J+1}^{L} x_{n_{\mu}}^{\rho}(id_{\mu}^{*}) \lesssim 2^{J \rho(-t+\frac{1}{p_{0}}-\frac{1}{p}+\frac{\beta}{r})} 2^{-\frac{J_{-\mu}\beta}{r}} = 2^{J \rho(-t+\frac{1}{p_{0}}-\frac{1}{p})}
\]
follows. Inserting this and (5.22) into (5.17) we find
\[
x_{n}(id^{*}) \lesssim \left( 2^{J \rho(-t+\frac{1}{p_{0}}-\frac{1}{p})} + 2^{-\gamma J \rho} \right).
\]

Choosing
\[
\gamma := \frac{-t + \frac{1}{p_{0}} - \frac{1}{p}}{-t} \geq 1
\]
then we conclude
\[
x_{n}(id^{*}) \lesssim 2^{J(-t+\frac{1}{p_{0}}-\frac{1}{p})}
\]
and this is enough to prove the estimate from above, compare with the end of the proof of Thm. 5.11.

\subsection*{Theorem 5.14.}

Let $2 \leq p < p_{0} < \infty$ and $0 < t < \frac{1}{p_{0}} - \frac{1}{p}$. Then
\[
x_{n}(id^{*}) \asymp n^{-\frac{p_{0}}{2}} \left( \log n \right)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_{0}})}, \quad n \geq 2.
\]
\textbf{Proof.} Step 1. Estimate from below. From \((5.9)\) and \((5.16)\) we derive
\[2^{\mu(-t+\frac{1}{p_0} - \frac{1}{2})} x_n(id_{p_0,2}^D) \lesssim x_n(id^*).\]
Now we choose \(n = [D_\mu^{\frac{2}{p_0}}]\). Then it follows from property (c) (part (i)) in Appendix A that
\[x_n(id_{p_0,2}^D) \gtrsim D_\mu^{\frac{1}{p_0}} \gtrsim (\mu^{d-1}2^{\mu})^{\frac{1}{2} - \frac{1}{p_0}}.\]
This implies
\[x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} 2^{-t\mu}.\]
Rewriting the right-hand side in dependence on \(n\) we obtain
\[x_n(id^*) \gtrsim n^{-\frac{t\mu}{2}} (\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_0})}.\]

Step 2. Estimate from above. Since \(2 \leq p < p_0\) we have
\[2^{-\alpha L} L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} = 2^{-t L} L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})}.\]

For fixed \(J \in \mathbb{N}\) we choose
\[L := \left[\frac{p_0}{2} J + (d-1)(\frac{p_0}{2} - 1) \log J \right].\]
Hence
\[2^{-Lt} = 2^{-t((\frac{p_0}{2} J + (d-1)(\frac{p_0}{2} - 1) \log J))} \lesssim 2^{-\frac{p_0}{2} J t (d-1)(t - \frac{1}{p_0})}\]
and
\[L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} = \left(\frac{p_0}{2} J + (d-1)(\frac{p_0}{2} - 1) \log J \right)^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} \lesssim J^{(d-1)(\frac{1}{2} - \frac{1}{p_0})}.\]
This results in the estimate
\[2^{-Lt} L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} \lesssim 2^{-\frac{p_0}{2} J t (d-1)(t - \frac{1}{p_0})} J^{(d-1)(\frac{1}{2} - \frac{1}{p_0})}.\]
\[\text{(5.26)}\]

We define
\[n_\mu := [D_\mu 2^{((\mu-L)\beta+J-\mu)}] \leq D_\mu, \quad J + 1 \leq \mu \leq L,\]
where \(\beta > 0\) will be fixed later on. Consequently
\[\sum_{\mu=J+1}^{L} n_\mu \lesssim 2^J J^{d-1}.\]
\[\text{(5.27)}\]
Employing property (b) (part (iii)) in Appendix A we get
\[x_n(id_{p_0,p}^D) \lesssim \left(\frac{D_\mu}{n_\mu}\right)^{\frac{1}{r}} \lesssim 2^{-\frac{(\mu-L)\beta+J-\mu}{r}}, \quad \frac{1}{r} := \frac{1}{r} = \frac{1}{1 - \frac{1}{2}}.\]
\[\text{(5.28)}\]
We continue by applying (5.29)

$$
\sum_{\mu=J+1}^L x_{n\mu}^0 (id_{\mu}^n) \lesssim \sum_{\mu=J+1}^L \mu^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{\mu\rho(-t+\frac{1}{p} - \frac{1}{r})} x_{n\mu}^0 (id_{D_{0,p}}^\mu)
$$

$$
\lesssim \sum_{\mu=J+1}^L \mu^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{\mu\rho(-t+\frac{1}{p} - \frac{1}{r})} 2^{-(\frac{\mu-L+J+\mu}{r})}
$$

$$
= \sum_{\mu=J+1}^L \mu^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{\mu\rho(-t+\frac{1}{p} - \frac{1}{r})} 2^{(\frac{L-J}{r})}
$$

Because of

$$
t < \frac{\rho - \frac{1}{p}}{\frac{2}{r}-1} \iff -t + \frac{1}{p_0} - \frac{1}{p} + \frac{1}{r} > 0
$$

we can choose $\beta > 0$ such that $-t + \frac{1}{p_0} - \frac{1}{p} + \frac{1}{r} > 0$. Then

$$
\sum_{\mu=J+1}^L x_{n\mu}^0 (id_{\mu}^n) \lesssim L^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{L\rho(-t+\frac{1}{p} - \frac{1}{r})} 2^{(\frac{L-J}{r})}
$$

$$
= L^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} 2^{L\rho(-t+\frac{1}{p} - \frac{1}{r})} 2^{(\frac{L-J}{r})}
$$

(5.29)

follows. Inserting the definition of $L$ we conclude

$$
L^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} \lesssim J^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})}
$$

and

$$
2^{L(-t+\frac{1}{p} - \frac{1}{r})} 2^{(\frac{L-J}{r})} \lesssim 2^{\left[\frac{\rho-1}{2} J + (d-1)\rho(\frac{1}{2}-\frac{1}{p})\right] \left[-t+\frac{1}{p} - \frac{1}{r} - \frac{1}{r}\right]} 2^{(\frac{L-J}{r})}
$$

$$
\lesssim 2^{\frac{\rho-1}{2} J} J^{(d-1)\rho(\frac{1}{2}-\frac{1}{p})} (-t+\frac{1}{p} - \frac{1}{r})
$$

$$
= 2^{\frac{\rho-1}{2} J} J^{(d-1)(-t+\frac{1}{p} - \frac{1}{r})}
$$

Now (5.29) yields

$$
\sum_{\mu=J+1}^L x_{n\mu}^0 (id_{\mu}^n) \lesssim J^{(d-1)\rho(t-\frac{\rho-1}{2}\frac{1}{p} + \frac{1}{2})} 2^{-\frac{\rho-1}{2} J}
$$

This, together with (5.26), has to be inserted into (5.7)

$$
x_n (id^r) \lesssim J^{(d-1)(-\frac{\rho-1}{2}\frac{1}{p} + \frac{1}{2})} 2^{-\frac{\rho-1}{2} J}
$$

The same type of arguments as at the end of the proof of Thm. 5.11 complete the proof. ■

**Remark 5.15.** Without going into details we mention the following estimate for the limiting case $t = \frac{p-\frac{1}{p_0}}{\frac{2}{r}-1}$. For all $n \geq 2$ we have

$$
n^{-\frac{\rho-1}{2} \frac{1}{r}} (\log n)^{d-1)(t+\frac{1}{2}-\frac{1}{p_0})} \lesssim x_n (id^r) \lesssim n^{-\frac{\rho-1}{2} \frac{1}{r}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0}) (\log n)^{\frac{1}{2} + 1}},
$$

where $r$ is as in (5.28).
5.3.4 The case $0 < p_0, p \leq 2$

We need some preparations.

**Lemma 5.16.** Let $0 < p_0, p \leq 2$ and $t > (\frac{1}{p_0} - \frac{1}{p})_+$. Then

$$n^{-t}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_0})+} \lesssim x_n(id^*)$$

holds for all $n \geq 2$.

**Proof.** Step 1. We consider the following commutative diagram

$$\begin{array}{c}
\varepsilon^{t,\Omega}_{p_0,0} b & \longrightarrow & \varepsilon^{0,\Omega}_{p,2} f \\
\uparrow \scriptstyle{id^1} & & \downarrow \scriptstyle{id^2} \\
2^{\mu(t+\frac{1}{2} - \frac{1}{p_0})} \ell_{A_\mu} & \longrightarrow & 2^{\mu(0-\frac{1}{p})} \ell_{A_\mu}.
\end{array}$$

Here $A_\mu = |A_\mu^2|$ for some $\bar{\nu}$ with $|\bar{\nu}|_1 = \mu$, $id^1$ is the canonical embedding, whereas $id^2$ is the canonical projection. From (s3) we derive

$$x_n(I_\mu) = x_n(id^2 \circ id^* \circ id^1) \leq \|id^1\| \|id^2\| x_n(id^*) = x_n(id^*).$$

Also (s3) guarantees

$$x_n(I_\mu) = 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,0}^{A_\mu}).$$

We choose $n = [A_\mu/2]$. Then property (a) in Section Appendix A yields

$$x_n(I_\mu) \gtrsim 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,0}^{A_\mu}) \gtrsim 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{p})} 2^{\mu(\frac{1}{p} - \frac{1}{p_0})} = 2^{-t \mu} \times n^{-t},$$

which implies $x_n(id^*) \gtrsim n^{-t}$. This proves the lemma if $t + \frac{1}{2} - \frac{1}{p_0} \leq 0$.

Step 2. From (5.16) and (5.8) we have

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2} - \frac{1}{p})} 2^{\mu(-t+\frac{1}{p_0} - \frac{1}{p})} x_n(id_{p_0,0}^{A_\mu}).$$

We choose $n := [D_\mu/2]$. Then property (a) in Appendix A leads to

$$x_n(id_{p_0,0}^{A_\mu}) \gtrsim D_\mu^{\frac{1}{p} - \frac{1}{p_0}} \gtrsim (\mu^{d-1} 2^\mu)^{\frac{1}{p} - \frac{1}{p_0}},$$

which implies

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} 2^{-t \mu}.$$

Because of $\mu \asymp \log n$ and $2^\mu \asymp \frac{n}{\log^{d-1} n}$ this yields

$$x_n(id^*) \gtrsim n^{-t}(\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_0})}.$$

The proof is complete. 

\[\Box\]
Lemma 5.17. If $0 < p_0, p \leq 2$ and $t > \frac{1}{p_0} - \frac{1}{2}$, then

$$x_n(id^*) \lesssim n^{-t} (\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_0})}$$

holds for all $n \geq 2$.

Proof. The restriction $t > \frac{1}{p_0} - \frac{1}{2}$ implies the following chain of continuous embeddings

$$s_{p_0, p_0} f \hookrightarrow s_{2, 2} f \hookrightarrow s_{p, 2} f.$$  

Now we consider the commutative diagram

$$\begin{array}{ccc}
\scriptsize s_{p_0, p_0} f & \xrightarrow{id^*} & \scriptsize s_{p, 2} f \\
\downarrow{id^1} & & \downarrow{id^2} \\
\scriptsize s_{2, 2} f & & \scriptsize s_{2, 2} f
\end{array}$$

Property (s3) and Thm. 5.11 (applied with $p = 2$) yield the claim.  

Lemma 5.18. Let $0 < p \leq p_0 < 2$ and $0 < t < \frac{1}{p_0} - \frac{1}{2}$. Then

$$x_n(id^*) \lesssim n^{-t}$$

holds for all $n \geq 1$.

Proof. For given $J \in \mathbb{N}$ we choose $L := J + (d - 1)[\log J]$. Then

$$2^{-L\alpha} L^{(d-1)(\frac{1}{2} - \frac{1}{p_0})+} = 2^{-Lt} \ll 2^{-tJ} J^{(d-1)(-t)}.$$  \hspace{1cm} (5.30)

We define

$$n_{\mu} := \left[ D_{\mu} 2^{(\mu-L)\beta + J-\mu} \right], \quad J + 1 \leq \mu \leq L,$$

for some $\beta > 0$. Then (5.27) follows.

Property (a) in Appendix A yields

$$x_{n_{\mu}}(id_{p_0, 2}) \lesssim \left( D_{\mu} 2^{(\mu-L)\beta + J-\mu} \right)^{\frac{1}{2} - \frac{1}{p_0}}.$$  

This, in connection with (5.3), leads to

$$\sum_{\mu=J+1}^L x_{n_{\mu}}^2(id_{p_0}^*) \lesssim \sum_{\mu=J+1}^L 2^{\mu p(-t+\frac{1}{p_0} - \frac{1}{2})} \left( D_{\mu} 2^{(\mu-L)\beta + J-\mu} \right)^{\beta\left(\frac{1}{2} - \frac{1}{p_0}\right)}$$

$$\lesssim \sum_{\mu=J+1}^L 2^{\mu p(-t+\frac{1}{p_0} - \frac{1}{2} + (\frac{1}{2} - \frac{1}{p_0})\beta)} \left( \mu^{(d-1)2-L\beta + J} \right)^{\beta\left(\frac{1}{2} - \frac{1}{p_0}\right)}.$$  

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Because of \( t < \frac{1}{p_0} - \frac{1}{2} \) we can select \( \beta > 0 \) such that

\[
-t + \frac{1}{p_0} - \frac{1}{2} + \left( \frac{1}{2} - \frac{1}{p_0} \right) \beta > 0.
\]

Consequently

\[
\sum_{\mu=J+1}^{L} x_n^\mu (id_\mu^*) \lesssim 2^{L \rho (-t + \frac{1}{p_0} - \frac{1}{2} + \left( \frac{1}{2} - \frac{1}{p_0} \right) \beta)} \left( L^{\rho \left( \frac{1}{2} - \frac{1}{p_0} \right)} \right)
\]

\[
= 2^{L \rho (-t + \frac{1}{p_0} - \frac{1}{2} + \left( \frac{1}{2} - \frac{1}{p_0} \right) \beta)} \left( L^{\rho \left( \frac{1}{2} - \frac{1}{p_0} \right)} \right)
\]

\[
\lesssim 2^{L \rho (-t + \frac{1}{p_0} - \frac{1}{2} + \left( \frac{1}{2} - \frac{1}{p_0} \right) \beta)} \left( J^{\rho \left( \frac{1}{2} - \frac{1}{p_0} \right)} \right).
\]

(5.31)

Observe

\[
2^{L \left( \frac{1}{p_0} - \frac{1}{2} \right)} \left( J^{\rho \left( \frac{1}{2} - \frac{1}{p_0} \right)} \right) = 2^{J + \rho \left( \frac{1}{2} - \frac{1}{p_0} \right)} \left( J^{\rho \left( \frac{1}{2} - \frac{1}{p_0} \right)} \right) \approx 1.
\]

Replacing \( L \) by \( J + (d - 1) \log J \) in (5.31) we obtain

\[
\sum_{\mu=J+1}^{L} x_n^\mu (id_\mu^*) \lesssim 2^{- \rho t} \lesssim (2^J J^{d-1})^{-\rho t}.
\]

This inequality, together with (5.30), yield

\[
x_n^J (id^*) \lesssim n^{-t},
\]

where

\[
n_J := 1 + \sum_{\mu=0}^{J} D_\mu + \sum_{\mu=J+1}^{L} \left( [D_\mu 2^{\rho \left( \frac{1}{2} - \frac{1}{p_0} \right)} \beta] - 1 \right), \quad J \in \mathbb{N}.
\]

Now we can continue as at the end of the proof of Thm. 5.11.

It remains to investigate the following situation: \( 0 < p_0 < p < 2 \) and \( \frac{1}{p_0} - \frac{1}{p} < t < \frac{1}{p_0} - \frac{1}{2} \).

The estimates of the Weyl numbers \( x_n(id^*) \) from above will be the most complicated part within this paper.

**Lemma 5.19.** Let \( 0 < p_0 < p < 2 \) and \( \frac{1}{p_0} - \frac{1}{p} < t < \frac{1}{p_0} - \frac{1}{2} \). Then

\[
x_n(id^*) \lesssim n^{-t}
\]

holds for all \( n \geq 1 \).

**Proof.** Step 1. We need to replace the decomposition of \( id^* \) from (5.1) by a more sophisticated one:

\[
id^* = \sum_{\mu=0}^{J} id_\mu^* + \sum_{\mu=J+1}^{L} id_\mu^* + \sum_{\mu=L+1}^{K} id_\mu^* + \sum_{\mu=K+1}^{\infty} id_\mu^* \quad \text{with} \quad J < L < K.
\]

(5.32)
Here $J, L$ and $K$ will be chosen later on. As in (5.2) this decomposition results in the estimate

$$x^n(\text{id}^*) \leq \sum_{\mu=0}^J x^n_{\mu}(\text{id}^*_\mu) + \sum_{\mu=J+1}^L x^n_{\mu}(\text{id}^*_\mu) + \sum_{\mu=L+1}^K x^n_{\mu}(\text{id}^*_\mu) + \sum_{\mu=K+1}^\infty \|\text{id}^*_\mu\|^\rho, \quad \rho = \min\{1, p\},$$

where $n - 1 = \sum_{\mu=0}^K (n_\mu - 1)$. Cor. 5.5 yields

$$\|\text{id}^*_\mu\| \lesssim 2^{-\mu\alpha} \mu^{(d-1)(\frac{1}{2} - \frac{1}{p_0}) +} = 2^{\mu(-t + \frac{1}{p_0} - \frac{1}{2})}$$

and therefore

$$\sum_{\mu=K+1}^\infty \|\text{id}^*_\mu\|^\rho \lesssim 2^K \rho(-t + \frac{1}{p_0} - \frac{1}{2}).$$

As above we choose

$$n_\mu := D_\mu + 1, \quad \mu = 0, 1, ..., J,$$

see (5.4). Hence

$$\sum_{\mu=0}^J n_\mu \asymp J^{d-1} 2^J \quad \text{and} \quad \sum_{\mu=0}^J x^n_{\mu}(\text{id}^*_\mu) = 0,$$

see (5.5) and (5.6). Inserting this into (5.33) we obtain

$$x^n(\text{id}^*) \lesssim \sum_{\mu=J+1}^L x^n_{\mu}(\text{id}^*_\mu) + \sum_{\mu=L+1}^K x^n_{\mu}(\text{id}^*_\mu) + 2^K \rho(-t + \frac{1}{p_0} - \frac{1}{2}).$$

**Step 2.** For given $J$ we choose $K$ large enough such that

$$2^K(-t + \frac{1}{p_0} - \frac{1}{2}) \leq 2^{-Jt} J^{(d-1)(-t)}.$$  

(5.36)

Furthermore, we choose $L := J + (d - 1)[\log J]$ also in dependence on $J$. This implies

$$2^{-Lt} \lesssim 2^{-Jt} J^{(d-1)(-t)}.$$  

(5.37)

Now we fix our remaining degrees of freedom by defining

$$n_\mu := \begin{cases} 
[D_\mu 2^{(\mu-L)\beta + J - \mu}] & \text{if} \quad J + 1 \leq \mu \leq L, \\
[J^{d-1} 2^{J (L-\mu)\gamma}] & \text{if} \quad L + 1 \leq \mu \leq K. 
\end{cases}$$

Here $\beta, \gamma > 0$ will be fixed later. Since $\gamma > 0$, applying (5.27), we have

$$\sum_{\mu=J+1}^K n_\mu \approx J^{d-1} 2^J.$$  

(5.38)
Substep 2.1. We estimate the first sum in (5.35). Making use of the same arguments as in proof of Lemma 5.18 we find
\[
\sum_{\mu = J+1}^{L} x_{\mu}^{p}(\text{id}^{p}_{\mu}) \lesssim 2^{-L \rho t} \lesssim 2^{-t J \rho} J^{-(d-1) \rho t} .
\] (5.39)

Substep 2.2. Now we estimate the second sum in (5.35). Therefore we consider the following splitting of \( n_{\mu}, \) \( L + 1 \leq \mu \leq K \)
\[
n_{\mu} \asymp J^{d-1} 2^{(L-\mu)\gamma} = J^{d-1} 2^{\mu} 2^{L-\mu} 2^{-(d-1)\log J} 2^{(L-\mu)\gamma} = 2^{\mu} 2^{(L-\mu)(\gamma+1)} ,
\]
where we used the definition of \( L. \) Observe \( n_{\mu} \lesssim D_{\mu}/2. \) The inequality (5.13) and property (a) in Appendix A lead to the estimate
\[
x_{n_{\mu}}^{p}(\text{id}^{p}) \lesssim 2^{\mu \left( -t + \frac{1}{p_{0}} \right)} x_{n_{\mu}}^{p}(\text{id}_{p_{0},p}^{D_{\mu}}) \lesssim 2^{\mu \left( -t + \frac{1}{p_{0}} \right)} (2^{\mu} 2^{(L-\mu)(\gamma+1)}) \left( \frac{1}{p_{0}} \right)
\]
This implies
\[
\sum_{\mu = L+1}^{K} x_{n_{\mu}}^{p}(\text{id}^{p}_{\mu}) \lesssim \sum_{\mu = L+1}^{K} 2^{-\mu \rho t} 2^{(L-\mu)(\gamma+1)\left( \frac{1}{p_{0}} \right)} .
\]
Choosing \( \gamma > 0 \) such that
\[
t > (\gamma + 1) \left( \frac{1}{p_{0}} - \frac{1}{p} \right)
\]
we conclude
\[
\sum_{\mu = L+1}^{K} x_{n_{\mu}}^{p}(\text{id}^{p}_{\mu}) \lesssim 2^{-L \rho t} \asymp 2^{-t J \rho} J^{-(d-1) \rho t} .
\]
Hence, inserting the previous inequality and (5.39) into (5.35),
\[
x_{n}^{p}(\text{id}^{p}) \lesssim 2^{-t J} J^{(d-1)(-t)}
\]
follows. Based on this estimate and (5.38) one can finish the proof as before. ■

As a corollary of Lem. 5.16 - Lem. 5.19 we obtain the main result of this subsection.

**Theorem 5.20.** Let \( 0 < p_{0}, p \leq 2 \) and \( t > \left( \frac{1}{p_{0}} - \frac{1}{p} \right)_{+} . \)
(i) If \( t > \frac{1}{p_{0}} - \frac{1}{p} \), then
\[
x_{n}^{p}(\text{id}^{p}) \asymp n^{-t} (\log n)^{(d-1)(t+\frac{1}{p_{0}} - \frac{1}{p})}
\]
holds for all \( n \geq 2. \)
(ii) If \( t < \frac{1}{p_{0}} - \frac{1}{p} \), then
\[
x_{n}^{p}(\text{id}^{p}) \asymp n^{-t}
\]
holds for all \( n \geq 1. \)

**Remark 5.21.** Again we comment on the limiting situation \( t = \frac{1}{p_{0}} - \frac{1}{p} \). For \( 0 < p_{0}, p < 2 \) and \( t = \frac{1}{p_{0}} - \frac{1}{2} \) it follows
\[
n^{-t} \lesssim x_{n}^{p}(\text{id}^{p}) \lesssim n^{-t} (\log \log n)^{t+1}, \quad n \geq 3.
\]
5.3.5 The case $0 < p \leq 2 \leq p_0 \leq \infty$

This is the last case we have to consider.

**Theorem 5.22.** Let $0 < p \leq 2 \leq p_0 \leq \infty$ and $t > \frac{1}{p_0}$. Then

$$x_n(id^x) \asymp n^{-t + \frac{1}{p_0} - \frac{1}{2} (log n) (d-1)(t + \frac{1}{2} - \frac{1}{p_0})}$$

holds for all $n \geq 2$.

**Proof.** *Step 1.* Estimate from below. Since $p \leq 2$, from (5.16) and (5.8) we derive

$$x_n(id^x) \geq \mu^{(d-1)(\frac{1}{2} \frac{1}{p} \frac{1}{p_0} - \frac{1}{2})} 2^{\mu(-t + \frac{1}{p} - \frac{1}{p_0})} x_n(id^{D_\mu}_{p_0,p}).$$

We choose $n := [D_\mu/2]$ and obtain from property (d)(part(i)) in Appendix A that

$$x_n(id^{D_{p_0}_p}) \geq (D_\mu)^{\frac{1}{p} \frac{1}{p_0} - \frac{1}{2}} \geq (\mu^{d-1} 2^{\mu})^{\frac{1}{p} \frac{1}{p_0} - \frac{1}{2}}.$$

This implies

$$x_n(id^x) \geq 2^{\mu(-t \frac{1}{2} + \frac{1}{p_0})}.$$

Using $\mu \asymp log n$ and $2^{\mu} \asymp \frac{n}{log^{d-1} n}$ we conclude

$$x_n(id^x) \asymp n^{t \frac{1}{2} + \frac{1}{p_0} (log n) (d-1)(t + \frac{1}{2} - \frac{1}{p_0})}.$$

*Step 2.* Estimate from above. We consider the commutative diagram

$$s_{p_0,p_0} f \xrightarrow{id^x} s^{0,\Omega}_p f \xleftarrow{id^2} s^{0,\Omega}_p f \xrightarrow{id^1} s^{0,\Omega}_p f$$

From $p < 2$ we derive $s^{0,\Omega}_p f \rightarrow s^{0,\Omega}_{p,p} f$ which implies $\|id^2\| \asymp \infty$. Property (s3) in combination with Thm. 5.13 yield

$$x_n(id^x) \leq n^{-t + \frac{1}{p_0} - \frac{1}{2} (log n) (d-1)(t + \frac{1}{2} - \frac{1}{p_0})}$$

if $t > \frac{1}{2} \frac{1}{p_0} - \frac{1}{2} = \frac{1}{p_0}$. □

**Theorem 5.23.** Let $0 < p \leq 2 < p_0 \leq \infty$ and $0 < t < \frac{1}{p_0}$. Then

$$x_n(id^x) \asymp n^{-\frac{t p_0}{2} (log n) (d-1)(t + \frac{1}{2} - \frac{1}{p_0})}$$

holds for all $n \geq 2$. 41
Proof. Step 1. Estimate from below. Since $p \leq 2$, from (5.16) and (5.8) we obtain

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2} - \frac{1}{p})} 2^{\mu t + \frac{1}{p_0} - \frac{1}{p}} x_n(id_{p_0, p}).$$

With $n := [D_{\mu}^{p_0}]$ property (d)(part(ii)) in Appendix A yields

$$x_n(id_{p_0, p}) \gtrsim D_{\mu}^{\frac{1}{2} - \frac{1}{p_0}} \gtrsim \mu^{(d-1)\frac{1}{2} - \frac{1}{p_0}}.$$

Hence

$$x_n(id^*) \gtrsim \mu^{(d-1)(\frac{1}{2} - \frac{1}{p_0})} 2^{-t \mu}.$$\[\text{Since } \mu \propto \log(n^{\frac{p_0}{2}}) \text{ and } 2^\mu \propto \frac{n^{\mu}}{\log^{d-1}(n^{\frac{p_0}{2}})} \text{ we conclude} \]

$$x_n(id^*) \gtrsim n^{-\frac{t \mu}{2}} (\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_0})}.$$\[\text{Step 2. Estimate from above. As a consequence of Thm. 5.14 and (s3) in combination} \]

with the same diagram as in Step 2 of the proof of Thm. 5.22 we obtain

$$x_n(id^*) \lesssim n^{-\frac{t \mu}{2}} (\log n)^{(d-1)(t+\frac{1}{2} - \frac{1}{p_0})}, \quad n \geq 2$$

if $0 < t < \frac{1}{p_0}$.

Remark 5.24. In the limiting situation $t = \frac{1}{p_0} > 0$ we have

$$n^{-\frac{1}{2}}(\log n)^{(d-1)\frac{1}{2}} \lesssim x_n(id^*) \lesssim n^{-\frac{1}{2}}(\log n)^{(d-1)\frac{1}{2}}(\log n)^{\frac{1}{2}}$$

for all $n \geq 2$.

6 Proofs

Here we will give proofs of the assertions in Section 2. For better readability we continue to work with $(p_0, p)$ instead of $(p_1, p_2)$.

6.1 Proof of the main Theorem 2.3

The heart of the matter is the following well-known lemma.

Lemma 6.1. Let $0 < p_0 \leq \infty$, $0 < p < \infty$ and $t \in \mathbb{R}$. Then

$$x_n(id^* : s_{p_0, p_0}^t \rightarrow s_{p, 2}^{0, \Omega} f) \asymp x_n(id : S_{p_0, p_0}^t B(\Omega) \rightarrow S_{p, 2}^{0, \Omega} F(\Omega))$$

holds for all $n \in \mathbb{N}$.
**Proof. Step 1.** Let $0 < p_0 < \infty$. Let $E : B^t_{p_0,p_0}(0,1) \to B^t_{p_0,p_0}(\mathbb{R})$ denote a linear and continuous extension operator. For existence of those operators we refer, e.g., to [59] 3.3.4 or [48]. Without loss of generality we may assume that $\text{supp} E f \subset [-1,2]$ for all $f$. Then the $d$-fold tensor product operator

$$\mathcal{E}_d := E \otimes \ldots \otimes E$$

maps the tensor product space $B^t_{p_0,p_0}(0,1) \otimes \gamma_{p_0} \ldots \otimes \gamma_{p_0} B^t_{p_0,p_0}(0,1)$ into the tensor product space $B^t_{p_0,p_0}(\mathbb{R}) \otimes \gamma_{p_0} \ldots \otimes \gamma_{p_0} B^t_{p_0,p_0}(\mathbb{R})$ and is again a linear and continuous extension operator. This follows from the fact that $\gamma_{p_0}$ is an uniform quasi-norm. Thm. 4.7 and Thm. 4.7 imply that $\mathcal{E}_d \in \mathcal{L}(S^t_{p_0,p_0} B(\Omega), S^t_{p_0,p_0} B(\mathbb{R}^d))$.

**Step 2.** Let $p_0 = \infty$. We discussed extension operators in this case in Subsection 2.5. Now we can argue as in Step 1.

**Step 3.** We follow [64] and consider the commutative diagram

$$\begin{array}{ccc}
S_{p_0,p_0}^t B(\Omega) & \xrightarrow{\mathcal{E}_d} & S_{p_0,p_0}^t B(\mathbb{R}^d) \\
\downarrow{id} & & \downarrow{id^*} \\
S_0^0 F(\Omega) & \xleftarrow{R_{\Omega}} & S_0^0 F(\mathbb{R}^d) \\
& & \downarrow{id^*} \\
& & S_0^0 \Omega^* f
\end{array}$$

Here $\Omega^* = (-1,2)^d$ and the mapping $\mathcal{W}$ is defined as

$$\mathcal{W} f := \left( 2^{|\nu|} \langle f, \Psi_{\nu,k} \rangle \right)_{\nu \in \mathbb{N}_0^d, k \in A^{\Omega^*}_\nu},$$

where

$$A^{\Omega^*}_\nu := \{ k \in \mathbb{Z}^d : \text{supp} \Psi_{\nu,k} \cap (-1,2)^d \neq \emptyset \}.$$  

Furthermore, $\mathcal{W}^*$ is defined as

$$\mathcal{W}^* \lambda := \sum_{\nu \in \mathbb{N}_0^d} \sum_{k \in A^{\Omega^*}_\nu} \lambda_{\nu,k} \Psi_{\nu,k}$$

and $R_{\Omega}$ means the restriction to $\Omega$. The boundedness of $\mathcal{E}_d, \mathcal{W}, \mathcal{W}^*, R_{\Omega}$ and property (s3) yield $x_n(id) \lesssim x_n(id^*)$. A similar argument with a slightly modified diagram yields $x_n(id^*) \lesssim x_n(id)$ as well. Finally, we mention that

$$x_n(id^* : s_{p_0,p_0}^t b \to s_{p_0,p_0}^t f) \asymp x_n(id^* : s_{p_0,p_0}^t b \to s_{p_0,p_0}^t f), \quad n \in \mathbb{N},$$

since the only property of $A^{\Omega^*}_\nu$ we had used consisted in $|A^{\Omega^*}_\nu| \lesssim 2^{|\nu|}$, see Lemma 5.3(i). □

Next we need to recall an adapted Littlewood-Paley assertion, see Nikol’skij [34] 1.5.6.

**Lemma 6.2.** Let $1 < p < \infty$. Then

$$S^0_{p_2} F(\mathbb{R}^d) = L_p(\mathbb{R}^d) \quad \text{and} \quad S^0_{p_2} F(\Omega) = L_p(\Omega)$$

in the sense of equivalent norms.
Proof of Thm. 2.3: Lemma 6.1 and Lemma 6.2 allow to carry over the results obtained in Section 5 to the level of function spaces. Theorem 2.3 becomes a consequence of Theorems 5.11 - 5.14, Theorem 5.20 and Theorems 5.22, 5.23.

6.2 Proofs of the results in Subsections 2.3 and 2.5

Lemma 6.3. Let \( t, r \in \mathbb{R} \) and \( 0 < p, q, p_0, q_0 \leq \infty \). Then

\[
x_n(id^1: s_{p_0, q_0}^t, \Omega \rightarrow s_{p, q}^r, \Omega) \times x_n(id^2: s_{p_0, q_0}^{t-r}, \Omega \rightarrow s_{p, q}^{0}, \Omega), \quad n \in \mathbb{N}.
\]

Proof. We consider commutative diagram

\[
\begin{array}{ccc}
s_{p_0, q_0}^t a & \xrightarrow{id^1} & s_{p, q}^r a \\
\downarrow J_r & & \uparrow J_r \\
s_{p_0, q_0}^{t-r} a & \xrightarrow{id^2} & s_{p, q}^{0} a
\end{array}
\]

Here \( J_r \) is the isomorphism defined in (4.1). Hence \( x_n(id^1) \lesssim x_n(id^2) \). But

\[
\begin{array}{ccc}
s_{p_0, q_0}^{t-r} a & \xrightarrow{id^2} & s_{p, q}^{0} a \\
\downarrow J_r & & \uparrow J_r \\
s_{p_0, q_0}^t a & \xrightarrow{id^1} & s_{p, q}^{r} a
\end{array}
\]

yields \( x_n(id^2) \lesssim x_n(id^1) \) as well. The proof is complete.

For later use, we prove Thm. 2.15 first.

Proof of Theorem 2.15 Define \( id^* : s_{p_0, p_0}^{t, \Omega} b \rightarrow s_{\infty, \infty}^{0, \Omega} b \) and \( id_{\mu}^{*} : (s_{p_0, p_0}^{t, \Omega} b)_{\mu} \rightarrow (s_{\infty, \infty}^{0, \Omega} b)_{\mu} \).

Cor. 5.5 yields

\[
\| id_{\mu}^{*} \| \lesssim 2^\mu \left( \frac{1}{p_0} - t \right).
\]

Arguing as in proof of Prop. 5.7 one can establish the following.

Lemma 6.4. Let \( 0 < p_0 \leq \infty \) and \( t \in \mathbb{R} \). Then

\[
x_n(id_{\mu}^{*}) \times x_n(id_{\mu}^{**} : 2^\mu(t-\frac{1}{p_0}) \ell_{p_0}^{D_{\mu}} \rightarrow \ell_{\infty}^{D_{\mu}}) \times 2^\mu(-t+\frac{1}{p_0}) x_n(id_{p_0, \infty}^{D_{\mu}})
\]

for all \( n \in \mathbb{N} \).

Property (a) in Appendix A yields

\[
x_n(id_{p_0, \infty}^{D_{\mu}}) \asymp \begin{cases} 1 & \text{if } 2 \leq p_0 \leq \infty, \\
\frac{1}{n^2 - \frac{1}{p_0}} & \text{if } 0 < p_0 \leq 2,
\end{cases}
\]

if \( 2n \leq D_{\mu} \). Now we may follow the proof of Thm. 5.11. This results in the following useful statement.
Theorem 6.5. (i) Let \( 0 < p_0 \leq 2 \) and \( t > \frac{1}{p_0} \). Then
\[ x_n(id^* : s_{p_0,p_0}^t \cdot b \to s_{\infty, \infty}^0 \cdot b) \asymp n^{-t + \frac{1}{2} \left( \log n \right)^{(d-1) \left( t - \frac{1}{p_0} \right)}} , \quad n \geq 2. \]

(ii) Let \( 2 \leq p_0 \leq \infty \) and \( t > \frac{1}{p_0} \). Then
\[ x_n(id^* : s_{p_0,p_0}^t \cdot b \to s_{\infty, \infty}^0 \cdot b) \asymp n^{-t + \frac{1}{p_0} \left( \log n \right)^{(d-1) \left( t - \frac{1}{p_0} \right)}} , \quad n \geq 2. \]

By making use of a lifting argument, see Lemma 6.3, and the counterpart of Lemma 6.1 for this situation, i.e.,
\[ x_n(id^* : s_{p_0,p_0}^t \cdot b \to s_{\infty, \infty}^0 \cdot b) \asymp x_n(id : S_{p_0,p_0}^t B(\Omega) \to S_{\infty, \infty}^0 B(\Omega)) , \quad n \in \mathbb{N}, \]
we immediately get the following corollary.

Corollary 6.6. Let \( s \in \mathbb{R} \).

(i) Let \( 0 < p_0 \leq 2 \) and \( t > s + \frac{1}{p_0} \). Then
\[ x_n(id : S_{p_0,p_0}^t B((0,1)^d) \to S_{\infty, \infty}^s B((0,1)^d)) \asymp n^{-t + s + \frac{1}{2} \left( \log n \right)^{(d-1) \left( t - s - \frac{1}{p_0} \right)}} , \quad n \geq 2. \]

(ii) Let \( 2 \leq p_0 \leq \infty \) and \( t > s + \frac{1}{p_0} \). Then
\[ x_n(id : S_{p_0,p_0}^t B((0,1)^d) \to S_{\infty, \infty}^s B((0,1)^d)) \asymp n^{-t + s + \frac{1}{p_0} \left( \log n \right)^{(d-1) \left( t - s - \frac{1}{p_0} \right)}} , \quad n \geq 2. \]

Now Thm. 2.15 follows from \( Z_{\text{mix}}^s((0,1)^d) = S_{\infty, \infty}^s B((0,1)^d) \), see Lemma 2.14.

Before proving Thm. 2.6 let us recall a result obtained by Temlyakov [54].

Lemma 6.7. Let \( r > 0 \). Then we have
\[ a_n(id : S_{\infty, \infty}^r ((0,1)^d) \to L_{\infty}((0,1)^d)) \asymp n^{-r \left( \log n \right)^{(d-1) \left( r + \frac{1}{2} \right)}} \]

Proof of Theorem 2.6 Step 1. Estimate from above. Under the given restrictions there always exists some \( r \) such that \( r > 0 \) and \( t > r + \frac{1}{p_0} \). We consider the commutative diagram
\[
\begin{array}{ccc}
S_{p_0,p_0}^t B((0,1)^d) & \xrightarrow{id_1} & L_{\infty}((0,1)^d) \\
\downarrow{id_2} & & \downarrow{id_3} \\
S_{\infty, \infty}^r B((0,1)^d) & &
\end{array}
\]
The multiplicativity of the Weyl numbers yields

\[ x_{2n-1}(id_1) \leq x_n(id_2) x_n(id_3). \]

Cor. 6.6 Lem. 6.7 and \( x_n \leq a_n \) lead to

\[ x_{2n-1}(id_1) \lesssim \frac{(\log n)^{(d-1)(r-\frac{1}{p_0})}}{n^{t-r}} \]

\[ \begin{cases} \frac{(\log n)^{(d-1)(t-r-\frac{1}{p_0})}}{n^{t-r}} & \text{if } 0 < p_0 \leq 2, \ t - r > \frac{1}{p_0}, \\ \frac{(\log n)^{(d-1)(t-r-\frac{1}{p_0})}}{n^{t-r}} & \text{if } 2 \leq p_0 \leq \infty, \ t - r > \frac{1}{p_0}. \end{cases} \]

Finally, the monotonicity of the Weyl numbers yields the claim for all \( n \geq 2 \).

**Step 2. Estimate from below.** The claim will follow from the next proposition.

**Proposition 6.8.** Let \( t > \frac{1}{p_0} \). As estimates from below we get

\[ x_n(id : S_{p_0,p_0}^t B((0,1)^d)) \to L_\infty((0,1)^d)) \]

\[ \lesssim \begin{cases} \frac{(\log n)^{(d-1)(t+\frac{1}{p_0})}}{n^{t}} & \text{if } 0 < p_0 \leq 2, \\ \frac{(\log n)^{(d-1)(t+\frac{1}{p_0})}}{n^{t}} & \text{if } 2 \leq p_0 \leq \infty, \end{cases} \]

for all \( n \geq 2 \).

**Proof.** Again we shall use the multiplicativity of the Weyl numbers, but this time in connection with its relation to the 2-summing norm [37, Lemma 8]. Let us recall this notion.

An operator \( T \in \mathcal{L}(X,Y) \) is said to be **absolutely 2-summing** if there is a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in X \) the inequality

\[ \left( \sum_{j=1}^{n} \|Tx_j\|Y^2 \right)^{1/2} \leq C \sup_{x^* \in X^*, \|x^*\| \leq 1} \left( \sum_{j=1}^{n} |<x_j,x^*>|^2 \right)^{1/2} \]

holds (see [36, Chapter 17]). The norm \( \pi_2(T) \) is given by the infimum of all \( C > 0 \) satisfying (6.4). \( X^* \) refers to the dual space of \( X \). Pietsch [37] has proved

\[ n^{1/2} x_n(S) \leq \pi_2(S), \quad n \in \mathbb{N}, \]

for any linear operator \( S \). Using this inequality with respect \( S = id \) we conclude

\[ x_{2n-1}(id : S_{p_0,p_0}^t B((0,1)^d)) \to L_2((0,1)^d)) \]

\[ \leq x_n(id : S_{p_0,p_0}^t B((0,1)^d)) x_n(id : L_\infty((0,1)^d) \to L_2((0,1)^d)) \]

\[ \leq x_n(id : S_{p_0,p_0}^t B((0,1)^d)) n^{-1/2} \pi_2(id : L_\infty((0,1)^d) \to L_2((0,1)^d)) \]

\[ = x_n(id : S_{p_0,p_0}^t B((0,1)^d) \to L_\infty((0,1)^d)) n^{-1/2}; \]
where in the last equality we have used that

\[ \pi_2(id : L_\infty((0,1)^d) \to L_2((0,1)^d)) = \|id : L_\infty((0,1)^d) \to L_2((0,1)^d)\| = 1, \]

see [38, Example 1.3.9]). Since

\[
n^{1 \over 2} x_{2n-1}(id : S_{p_0,p_0}^t B((0,1)^d) \to L_2((0,1)^d)) \\
\leq \begin{cases} 
\frac{(\log n)^{(d-1)(t+1/-p_0)}}{n^{t-1 \over 2}} & \text{if } 0 < p_0 \leq 2, \ t > \frac{1}{p_0} - \frac{1}{2}, \\
\frac{(\log n)^{(d-1)(t+1/-p_0)}}{n^{t-1 \over 2}} & \text{if } 2 \leq p_0 \leq \infty, \ t > \frac{1}{p_0},
\end{cases}
\]

see Thm. 5.20 Thm. 5.13 this proves the claimed estimate from below. \[ \square \]

**Proof of Theorem 2.8** The lower estimate is direct consequence of inequality \( x_n \leq a_n \) and Theorems 2.6 2.15.

**Step 1.** We prove the upper bound of \( a_n(id : S_{p_0,p_0}^t B((0,1)^d) \to Z_{mix}((0,1)^d)) \) in case \( p_0 > 1 \). First, recall

\[
a_n(id_{D_{p_0,\infty}}) \propto \begin{cases} 
1 & \text{if } 2 \leq p_0 \leq \infty, \\
\min(1, D_{\mu}^{1-1/p_0 n^{-1/2}}) & \text{if } 1 < p_0 < 2,
\end{cases}
\]

if \( 2n \leq D_\mu \), see [25] 65. To avoid nasty calculations by checking this behaviour for \( p_0 \geq 2 \) one may use the elementary chain of inequalities

\[
x_n(id_{D_{p_0,\infty}}) \leq a_n(id_{p_0,\infty}) \leq 1
\]

in combination with property (a) in Appendix A.

Let \( 2 \leq p_0 \leq \infty \). Because of \( a_n(id_{p_0,\infty}) \times x_n(id_{p_0,\infty}) \times 1 \) if \( 2n \leq D_\mu \) we may argue as in case of Weyl numbers, see the proof of Thm. 2.15 given above.

Now we consider the case \( 1 < p_0 < 2 \). We define

\[
id^s : s_{p_0,p_0}^{t,\Omega} b \to s_{\infty,\infty}^{0,\Omega} b \quad \text{and} \quad id^s : (s_{p_0,p_0}^{t,\Omega} b)_\mu \to (s_{\infty,\infty}^{0,\Omega} b)_\mu.
\]

(6.1) and Lemma 6.4 yield

\[
\|id^s_{\mu}\| \lesssim 2^{\mu(1 \over p_0 - t)}
\]

and

\[
a_n(id^s_{\mu}) \asymp a_n(id^s_{\mu} : 2^{\mu(t-1/p_0)} \ell_{p_0}^{D_\mu} \to \ell_{\infty}^{D_\mu}) \asymp 2^{\mu(t+1/p_0)} a_n(id_{p_0,\infty}^{D_\mu}) (6.6)
\]

for all \( n \in \mathbb{N} \). Now we get as in Subsection 5.2, formula (5.7),

\[
a_n(id^s) \lesssim \sum_{\mu=J+1}^L a_n(id^s_{\mu}) + 2^{L(-t+1/p_0)}, (6.7)
\]

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since $\rho = 1$ here. For
\[ 1 < \lambda < \frac{1}{2} + \frac{t}{2} \quad (6.8) \]
we define
\[ n_\mu := D_\mu 2^{(J-\mu)\lambda}, \quad J + 1 \leq \mu \leq L. \]

Then, as above,
\[ n_\mu \leq \frac{D_\mu}{2} \quad \text{and} \quad \sum_{\mu=J+1}^{L} n_\mu \asymp J^{d-1}2^{J} \]
follows. From (6.6) and $a_n(id_{p_0,\infty}) \asymp \min(1, D_\mu^{1-\frac{1}{p_0}} n^{-\frac{1}{2}})$ we conclude
\[
\begin{align*}
  a_n(id^*_\mu) &\lesssim 2^{\mu(-t+\frac{1}{p_0})} a_n(id_{p_0,\infty}) \\
  &\lesssim 2^{\mu(-t+\frac{1}{p_0})} D_\mu^{1-\frac{1}{p_0}} (D_\mu 2^{(J-\mu)\lambda})^{-\frac{1}{2}} \\
  &\lesssim 2^{\mu(-t+\frac{1}{2}) (d-1)\frac{1}{2} - \frac{1}{2} (J-\mu)\lambda}.
\end{align*}
\]
This leads to
\[
\sum_{\mu=J+1}^{L} a_n(id^*_\mu) \lesssim \sum_{\mu=J+1}^{L} 2^{\mu(-t+\frac{1}{2}) (d-1)\frac{1}{2} - \frac{1}{2} (J-\mu)\lambda} \\
\lesssim 2^{J(-t+\frac{1}{2}) J^{d-1}(\frac{1}{2} - \frac{1}{p_0})},
\]
since $\lambda$ satisfies $\lambda < \frac{1}{2} + \frac{t}{2}$, see (6.8), guaranteeing the convergence of the series in that way. Now we choose $L$ large enough such that
\[ 2^{L(-t+\frac{1}{p_0})} \lesssim 2^{J(-t+\frac{1}{2}) J^{d-1}(\frac{1}{2} - \frac{1}{p_0})}. \]
In view of (6.7) this yields
\[ a_n(id^*_\mu) \lesssim 2^{J(-t+\frac{1}{2}) J^{d-1}(\frac{1}{2} - \frac{1}{p_0})}. \]
This proves the estimate from above.

Step 2. Let $p_0 = 1$. Then we use
\[ a_n(id_{p_0,\infty}^{D_\mu}) \lesssim n^{-\frac{1}{2}} \]
if $2n \leq D_\mu$, see [65]. This is just the limiting case of Step 1. So we argue as there.

Step 3. To prove the upper bound in $a_n(id : S^t_{p_0,p_0} B((0,1)^d) \to L_\infty((0,1)^d))$ we use Step 1 and the commutative diagram
\[
\begin{array}{c}
\text{Step 2. Let } p_0 = 1. \text{ Then we use} \\
\text{Step 3. To prove the upper bound in } a_n(id : S^t_{p_0,p_0} B((0,1)^d) \to L_\infty((0,1)^d)) \text{ we use Step 1 and the commutative diagram}
\end{array}
\]

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Making use of the multiplicativity of the approximation numbers, Lemma 6.7 and the results in Step 1,2 we find
\[ a_{2n-1}(id_1) \leq a_n(id_2) a_n(id_3) \]
\[ \lesssim n^{-r} (\log n)^{(d-1)(r+1/2)} \]
\[ \begin{cases} (\log n)^{(d-1)(t-r-\frac{1}{2})} & \text{if } 1 \leq p_0 < 2, \quad t - r > 1, \\ (\log n)^{(d-1)(t-r-\frac{1}{p_0})} & \text{if } 2 \leq p_0 \leq \infty, \quad t - r > \frac{1}{p_0}. \end{cases} \]
Choosing \( r > 0 \) small enough the claim follows.

6.3 Proof of the results in Subsection 2.4

As a preparation we need the following counterpart of the classical result \( F_{1,2}^0(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) \) in the dominating mixed situation. The following proof we learned from Dachun Yang and Wen Yuan [67].

Lemma 6.9. We have
\[ S_{1,2}^1 F(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d). \]

Proof. Let \( f \in S_{1,2}^1 F(\mathbb{R}^d) \). We may assume that \( f \) is a Schwartz function, due to the density of \( S(\mathbb{R}^d) \) in \( S_{1,2}^1 F(\mathbb{R}^d) \). Let \( (\varphi_j)_j \) be the smooth dyadic decomposition of unity defined in (8.2). Let \( \phi_0, \phi \in C_0^\infty(\mathbb{R}) \) be functions s.t.
\[ \phi_0(t) = 1 \quad \text{on supp} \varphi_0 \]
\[ \phi(t) = 1 \quad \text{on supp} \varphi_1. \]

We put \( \phi_j(t) := \phi(2^{-j+1}t), \ j \in \mathbb{N}, \) and
\[ \phi_j := \phi_{j_1} \otimes \ldots \otimes \phi_{j_d}, \quad j = (j_1, \ldots, j_d) \in \mathbb{N}_0^d. \]

It follows
\[ \sum_{j \in \mathbb{N}_0^d} \varphi_{\tilde{j}}(x) \cdot \phi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}^d, \]
see (8.1) and (8.2). This implies
\[ f = \sum_{j \in \mathbb{N}_0^d} F^{-1}[\varphi_j(\xi) \phi_j(\xi) Ff(\xi)] \quad \text{(convergence in } S'(\mathbb{R}^d)). \]

Let \( g \in L_\infty(\mathbb{R}^d) \). Hölder’s inequality yields
\[ \|f, g\| \leq \sum_{j \in \mathbb{N}_0^d} |\langle F^{-1}[\varphi_j Ff], F^{-1}[\phi_j Fg] \rangle| \]
\[ \leq \left\| \left( \sum_{j \in \mathbb{N}_0^d} |F^{-1}[\varphi_j Ff]|^2 \right)^{\frac{1}{2}} \right\|_{L_1(\mathbb{R}^d)} \cdot \left\| \left( \sum_{j \in \mathbb{N}_0^d} |F^{-1}[\phi_j Fg]|^2 \right)^{\frac{1}{2}} \right\|_{L_\infty(\mathbb{R}^d)}. \]
Next we are going to use the tensor product structure of $\mathcal{F}^{-1}\phi_j$ and the fact that $\mathcal{F}^{-1}\phi_{jt}$ are Schwartz functions. For any $M > 0$, we have

$$|\mathcal{F}^{-1}[\phi_j\mathcal{F}g](x)| \lesssim \int_{\mathbb{R}^d} g(y) (\mathcal{F}^{-1}\phi_j)(x+y) \, dy$$

$$\lesssim \|g\|_{L_\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \prod_{l=1}^d \frac{2^{|l|_1}}{(1 + 2^{|l|_1|x_l-y_l|}(1+M))} \, dy$$

$$= \|g\|_{L_\infty(\mathbb{R}^d)} \prod_{l=1}^d \int_{\mathbb{R}} \frac{2^{|l|}}{(1 + 2^{|l|_1|x_l-y_l|}(1+M))} \, dy.$$ 

Some elementary calculations yield

$$\int_{\mathbb{R}} \frac{2^{|l|}}{(1 + 2^{|l|_1|x_l-y_l|}(1+M))} \, dy \lesssim 2^{-|j|_1 M}$$

with constants independent of $j_l$. Inserting this in our previous estimate we obtain

$$|\mathcal{F}^{-1}[\phi_j\mathcal{F}g](x)| \lesssim \|g\|_{L_\infty(\mathbb{R}^d)} \cdot 2^{-|j|_1 M}.$$ 

Hence

$$\left\| \left( \sum_{j \in \mathbb{N}_0^d} |\mathcal{F}^{-1}[\phi_j\mathcal{F}g](x)|^2 \right)^{\frac{1}{2}} \right\|_{L_\infty} \lesssim \|g\|_{L_\infty(\mathbb{R}^d)} \sum_{j \in \mathbb{N}_0^d} 2^{-|j|_1 M}$$

$$\lesssim \|g\|_{L_\infty(\mathbb{R}^d)} \sum_{\mu=0}^{\infty} \sum_{|j|_1=\mu} 2^{-\mu M}$$

$$\lesssim \|g\|_{L_\infty(\mathbb{R}^d)}.$$ 

Therefore, we obtain

$$\|f\|_{L_1(\mathbb{R}^d)} = \sup_{\|g\|_{L_\infty(\mathbb{R}^d)} = 1} \left| \langle f, g \rangle \right| \lesssim \left\| \left( \sum_{j \in \mathbb{N}_0^d} |\mathcal{F}^{-1}[\phi_j\mathcal{F} f](x)|^2 \right)^{\frac{1}{2}} \right\|_{L_1(\mathbb{R}^d)}.$$ 

That completes our proof.

**Proof of Theorem 2.11** Step 1. Estimate from above. From the chain of embeddings

$$S_{p_0,p_0}^t B((0,1)^d) \hookrightarrow S_{1,2}^0 F((0,1)^d) \hookrightarrow L_1((0,1)^d),$$

together with Lem. 6.1, Thm. 5.20, Thm. 5.22, Thm. 5.23 and the abstract properties of Weyl numbers, see Section 3, we derive the upper bound.

Step 2. We prove the lower bound for the case $p_0 < 2$ and $t < \frac{1}{p_0} - \frac{1}{2}$. First we note that, under the condition $t > \max(0, \frac{1}{p_0} - \frac{1}{2})$, the chain of embeddings holds true

$$S_{p_0,p_0}^t B((0,1)^d) \hookrightarrow L_1((0,1)^d) \hookrightarrow S_{1,\infty}^0 B((0,1)^d).$$

Then property (s3) yields

$$x_n(id : S_{p_0,p_0}^t B((0,1)^d) \hookrightarrow S_{1,\infty}^0 B((0,1)^d)) \lesssim x_n(S_{p_0,p_0}^t B((0,1)^d) \hookrightarrow L_1((0,1)^d)). \quad (6.9)$$
Next we consider the commutative diagram
\[
\begin{array}{ccc}
B^{t+r}_{p_0,p_0}(0,1) & \xrightarrow{id_1} & B^r_{1,\infty}(0,1) \\
\Ext & \searrow & \nearrow \Tr \\
S^{t+r}_{p_0,p_0}B((0,1)^d) & \xrightarrow{id} & S^r_{1,\infty}B((0,1)^d)
\end{array}
\]
Here the linear operators \(\Ext\) and \(\Tr\) are defined as follows. For \(g \in B^{t+r}_{p_0,p_0}(0,1)\), we put
\[
(\Ext g)(x_1,\ldots,x_d) = g(x_1), \quad x = (x_1,\ldots,x_d) \in \mathbb{R}^d.
\]
In case of \(f \in S^r_{1,\infty}B((0,1)^d)\) we define
\[
(\Tr f)(x_1) = f(x_1,0,\ldots,0), \quad x_1 \in \mathbb{R}.
\]
Note that the condition \(r > 1\) guarantees that the operator \(\Tr\) is well defined, see [50, Thm. 2.4.2]. Furthermore, \(\Ext\) maps \(B^{t+r}_{p_0,p_0}(0,1)\) continuously into \(S^{t+r}_{p_0,p_0}B((0,1)^d)\). This follows from the fact that \(\| \cdot \|_{S^{t+r}_{p_0,p_0}B((0,1)^d)}\) is a cross-quasi-norm, see the formula in Rem. 84(i). Hence \(id_1 = \Tr \circ id \circ \Ext\) and
\[
x_n(id_1 : B^{t+r}_{p_0,p_0}(0,1) \to B^r_{1,\infty}(0,1)) \leq x_n(id : S^{t+r}_{p_0,p_0}B((0,1)^d) \to S^r_{1,\infty}B((0,1)^d)) \tag{6.10}
\]
Making use of a lifting argument, see Lem. 6.3 we conclude that
\[
x_n(id : S^{t+r}_{p_0,p_0}B((0,1)^d) \to S^r_{1,\infty}B((0,1)^d)) \leq x_n(id : S^t_{p_0,p_0}B((0,1)^d) \to S^0_{1,\infty}B((0,1)^d)). \tag{6.11}
\]
The lower bound is now obtained from (6.9), (6.10), (6.11) and
\[
x_n(id_1 : B^{t+r}_{p_0,p_0}(0,1) \to B^r_{1,\infty}(0,1)) \asymp n^{-t}, \quad n \in \mathbb{N},
\]
if \(0 < p_0 \leq 2\) and \(t > \max(0, \frac{1}{p_0} - 1)\), see Lubitz [32] and Caetano [10].

**Step 3.** Now we turn to the proof of the lower bound for cases of high smoothness. Note that there always exists some \(r\) (large enough) such that the following diagram makes sense and becomes commutative
\[
\begin{array}{ccc}
S^r_{2,2}B((0,1)^d) & \xrightarrow{id_1} & L_1((0,1)^d) \\
\id_2 & & \id \\
S^r_{p_0,p_0}B((0,1)^d)
\end{array}
\]

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The multiplicity of the Weyl numbers and Prop. 2.10 yield
\[ x_n(id) \gtrsim \frac{x_{2n-1}(id)}{x_n(id_2)} \gtrsim \frac{(\log n)^{(d-1)r}}{n^r x_n(id_2)}. \] (6.12)

Clearly, if \( 2n \leq D_\mu \) we have
\[
x_n(id^*: s_{2,2}^\mu \to s_{p_0,p_0}^\mu) \times 2^{\mu(t-r-\frac{1}{p_0}+\frac{1}{2})} x_n(id_{2,p_0}^{D_\mu}) \times 2^{\mu(t-r-\frac{1}{p_0}+\frac{1}{2})} \begin{cases} 1 & \text{if } 2 \leq p_0 \leq \infty, \\ (D_\mu)^{\frac{1}{p_0} - \frac{1}{2}} & \text{if } 0 < p_0 \leq 2, \end{cases}
\]
where we used Appendix A(a). Now we follow the arguments in Step 2 in the proof of Thm. 5.11 resulting in
\[ x_n(id_2) \gtrsim \frac{(\log n)^{(d-1)(r-t+\frac{1}{p_0}-\frac{1}{2})}}{n^{r-t+\frac{1}{p_0}-\frac{1}{2}}} \] (6.13)
if \( 2 \leq p_0 \leq \infty \). Next we employ the method from the proof of Thm. 5.20(i). This time we find
\[ x_n(id_2) \gtrsim \frac{(\log n)^{(d-1)(r-t+\frac{1}{p_0}-\frac{1}{2})}}{n^{r-t}} \] (6.14)
if \( 0 < p_0 \leq 2 \). Inserting (6.13) and (6.14) into (6.12) we obtain the desired results.

Step 4. Finally, we have to consider the case \( p_0 > 2 \) and \( 0 < t < \frac{1}{p_0} \). Because of these restrictions there always exists a pair \((\theta, p)\) such that
\[ 0 < \theta < 1, \quad 2 < p < p_0, \quad 0 < t < \frac{\frac{1}{p} - \frac{1}{p_0}}{\frac{p}{2} - \frac{1}{2}} \quad \text{and} \quad \frac{1}{2} = (1 - \theta) + \frac{\theta}{p}. \]

The Lyapunov inequality yields
\[ \|f\|_{L_2((0,1)^d)} \leq \|f\|_{L_1((0,1)^d)}^{1-\theta} \|f\|_{L_p((0,1)^d)}^\theta \quad \text{for all } f \in L_p((0,1)^d). \]

Next we employ the interpolation property of the Weyl numbers, see Thm. 3.3 and obtain
\[
x_{2n-1}(id : S^t_{p_0,p_0} B((0,1)^d) \to L_2((0,1)^d)) \lesssim x_n^{1-\theta}(id : S^t_{p_0,p_0} B((0,1)^d) \to L_1((0,1)^d)) x_n^\theta(id : S^t_{p_0,p_0} B((0,1)^d) \to L_p((0,1)^d)).
\]

Note that \( 2 < p < p_0 \) and \( 0 < t < \frac{\frac{1}{p} - \frac{1}{p_0}}{\frac{p}{2} - \frac{1}{2}} \) imply
\[
x_n(id : S^t_{p_0,p_0} B((0,1)^d) \to L_2((0,1)^d)) \times x_n(id : S^t_{p_0,p_0} B((0,1)^d) \to L_p((0,1)^d)) \times n^{-\frac{t}{p_0}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})},
\]
see Thm. 2.3. This leads to
\[ x_n(id : S^t_{p_0,p_0} B((0,1)^d) \to L_1((0,1)^d)) \gtrsim n^{-\frac{t}{p_0}} (\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})}. \]

The proof is complete.
6.4 Proofs of the results in Subsection 2.7

First we recall a result from [18].

Lemma 6.10. Let $0 < \alpha < 1$. Then $R_\alpha$ maps $L_2(0,1)$ continuously into $B_{2,2}^\alpha(0,1)$.

Proof of Theorem 2.21. Since $\alpha_2$ is an uniform tensor norm it follows that $R_\alpha^d$ maps $L_2(0,1) \otimes_\alpha ... \otimes_\alpha L_2(0,1)$ continuously into $B_{2,2}^\alpha(0,1) \otimes_\alpha ... \otimes_\alpha B_{2,2}^\alpha(0,1)$.

Clearly, $L_2(0,1) \otimes_\alpha ... \otimes_\alpha L_2(0,1) = L_2((0,1)^d)$. Thm. 4.7 yields

$$B_{2,2}^\alpha(0,1) \otimes_\alpha ... \otimes_\alpha B_{2,2}^\alpha(0,1) = S_{2,2}^\alpha B((0,1)^d).$$

From the commutative diagram

$$\begin{array}{ccc}
L_2((0,1)^d) & \xrightarrow{id \circ R_\alpha^d} & L_2((0,1)^d) \\
\downarrow{R_\alpha^d} & & \downarrow{id} \\
S_{2,2}^\alpha B((0,1)^d) & & \\
\end{array}$$

we derive in view of Thm 2.3

$$x_n(id \circ R_\alpha^d) \lesssim x_n(id) \lesssim n^{-\alpha} (\log n)^{(d-1)\alpha}, \ n \geq 2.$$ 

This proves Thm. 2.21(i).

Next we consider

$$\begin{array}{ccc}
L_2((0,1)^d) & \xrightarrow{id \circ R_\alpha^d} & C((0,1)^d) \\
\downarrow{R_\alpha^d} & & \downarrow{id} \\
S_{2,2}^\alpha B((0,1)^d) & & \\
\end{array}$$

Here $id$ is bounded since $\alpha > 1/2$. By means of Prop. 2.4 we deduce

$$x_n(id \circ R_\alpha^d) \lesssim x_n(id) \lesssim n^{-\alpha + \frac{1}{2}} (\log n)^{(d-1)\alpha}, \ n \geq 2.$$ 

This proves (2.16) except the change from $L_\infty((0,1)^d)$ to $C((0,1)^d)$. But this can be done by using (s3).

Finally, we consider
The boundedness from $id$ is guaranteed by $\alpha - 1/2 > s$, see [28] or [50]. Employing Cor. 6.6 we obtain
\[ x_n(id \circ R_d^\alpha) \lesssim x_n(id) \lesssim n^{-\alpha+s+\frac{1}{2}} (\log n)^{(d-1)(\alpha-s-\frac{1}{2})}, \quad n \geq 2. \]

The proof of Thm. 2.21 is complete.  

6.5 Proof of interpolation properties of Weyl numbers

For the basics in interpolation theory we refer to the monographs [7, 33, 58]. Next we recall the interpolation properties of Gelfand numbers, see Triebel [57].

**Theorem 6.11.** Let $X$ be a Banach space and let $(Y_0, Y_1)$ be an interpolation couple of Banach spaces. Let $0 < \theta < 1$ and let $Y$ be an intermediate Banach space with respect to $(Y_0, Y_1)$ such that there exists a positive constant $C$ with
\[ \|y|Y\| \leq C \|y|Y_0\|^{1-\theta} \|y|Y_1\|^\theta \quad \text{for all} \quad y \in Y_0 \cap Y_1. \] (6.15)

Then, if $T \in K(X, Y_0)$ and $T \in K(X, Y_1)$, it follows $T \in K(X, Y)$. Moreover,
\[ c_{n+m-1}(T : X \to Y) \leq C c_n^{1-\theta}(T : X \to Y_0) c_m^\theta(T : X \to Y_1) \]

for all $n, m \in \mathbb{N}$. Here $C$ is the same constant as in (6.15).

**Remark 6.12.** Triebel [57] works with Gelfand widths. However, for compact operators Gelfand widths and Gelfand numbers coincide, see also [57]. Without extra conditions Gelfand widths and Gelfand numbers may not coincide, see Edmunds and Lang [19] for some sufficient conditions.

Now we ready prove the Thm. 3.3

**Proof of Theorem 3.3.** Let $A \in L(l_2, X)$ such that $\|A\| \leq 1$. Then from Thm. 6.11 we conclude
\[ c_{n+m-1}(TA : l_2 \to Y) \leq C c_n^{1-\theta}(TA : l_2 \to Y_0) c_m^\theta(TA : l_2 \to Y_1). \]

Employing Remark 3.2(ii) we obtain
\[ c_{n+m-1}(TA : l_2 \to Y) \leq C x_n^{1-\theta}(T : X \to Y_0) x_m^\theta(T : X \to Y_1). \]
Now taking the supremum with respect to $A$ we find

$$x_{n+m-1}(T : X \to Y) \leq C x_n^{1-\theta}(T : X \to Y_0) x_m^{\theta}(T : X \to Y_1).$$

The proof is complete.

7 Appendix A - Weyl numbers of the embeddings $\ell_{p_0}^m \to \ell_p^m$

The Weyl numbers of $id : \ell_{p_0}^m \to \ell_p^m$ have been investigated at various places, we refer to Lubitz [32], König [29], Caetano [8, 9] and Zhang, Fang, Huang [68]. We shall need the following.

(a) \([68]\) Let $2n \leq m$. Then we have

$$x_n(id_{p_0,p}^m) \asymp \begin{cases} 1 & \text{if } 2 \leq p_0 \leq p \leq \infty, \\ \frac{1}{n^{\frac{1}{p-p_0}}} & \text{if } 0 < p_0 \leq p \leq 2, \\ \frac{1}{n^{\frac{1}{2-p_0}}} & \text{if } 0 < p_0 \leq 2 \leq p \leq \infty, \\ \frac{1}{m^{\frac{1}{p}}} & \text{if } 0 < p < p_0 \leq 2. \end{cases}$$

(b) \([32]\) For $1 \leq p < p_0 \leq \infty$ we have

(i) $x_n(id_{p_0,p}^m) \leq m^{\frac{p}{2} - \frac{1}{p_0}}$ if $1 \leq p < p_0 \leq \infty$, 

(ii) $x_n(id_{p_0,p}^m) \leq \frac{m^{\frac{p}{2}}}{n^{\frac{1}{2}}}$ if $1 \leq p \leq 2 < p_0 \leq \infty$, 

(iii) $x_n(id_{p_0,p}^m) \leq \left(\frac{m}{n}\right)^{\frac{1}{r}}$ if $2 \leq p < p_0 \leq \infty$, 

\[ \frac{1}{r} = \frac{\frac{1}{p} - \frac{1}{p_0}}{1 - \frac{2}{p_0}}. \]

(c) \([32]\) Let $1 \leq p \leq \infty$ and $\max(p, 2) \leq p_0$. Then

(i) $x_n(id_{p_0,p}^m) \geq m^{\frac{p}{2} - \frac{1}{p_0}}$ if $n \leq \left[m^{\frac{2}{p_0}}\right]$, 

(ii) $x_n(id_{p_0,p}^m) \geq \frac{m^{\frac{p}{2}}}{n^{\frac{1}{2}}}$ if $\left[m^{\frac{2}{p_0}}\right] \leq n \leq m$. 

(d) \([68]\) Let $0 < p \leq 2 < p_0 \leq \infty$ and $m \in \mathbb{N}$. Then

(i) $x_n(id_{p_0,p}^m) \geq m^{\frac{p}{2} - \frac{1}{p_0}}$ if $n \leq \frac{m}{2}$, 

(ii) $x_n(id_{p_0,p}^m) \geq \frac{m^{\frac{p}{2}}}{n^{\frac{1}{2}}}$ if $n \leq \frac{m}{2^{p_0}}$. 

(e) \([68]\) Let $0 < p \leq 2 < p_0 \leq \infty$ and $m \in \mathbb{N}$. Then

(i) $x_n(id_{p_0,p}^m) \leq m^{\frac{p}{2} - \frac{1}{p_0}}$ if $1 \leq n \leq m^{\frac{2}{p_0}}$, 

(ii) $x_n(id_{p_0,p}^m) \leq m^{\frac{p}{2} - \frac{1}{p_0}}$ if $1 \leq n \leq m^{\frac{2}{p_0}}$. 

(f) \([32]\) Let $2 \leq p \leq p_0 \leq \infty$ and $k \in \mathbb{N}, k \geq 2$. Then $x_n(id_{p_0,p}^m) \asymp 1$. 

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8 Appendix B - Function spaces of dominating mixed smoothness

8.1 Besov and Lizorkin-Triebel spaces on $\mathbb{R}$

Here we recall the definition and a few properties of Besov and Sobolev spaces defined on $\mathbb{R}$. We shall use the Fourier analytic approach, see e.g. [59]. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi(t) = 1$ in an open set containing the origin. Then by means of

$$
\varphi_0(t) = \varphi(t), \quad \varphi_j(t) = \varphi(2^{-j}t) - \varphi(2^{-j+1}t), \quad t \in \mathbb{R}, \ j \in \mathbb{N}, \quad (8.1)
$$

we get a smooth dyadic decomposition of unity, i.e.,

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{for all} \ t \in \mathbb{R},$$

and supp $\varphi_j$ is contained in the dyadic annulus $\{ t \in \mathbb{R} : a 2^j \leq |t| \leq b 2^j \}$ with $0 < a < b < \infty$ independent of $j \in \mathbb{N}$.

**Definition 8.1.** Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

(i) The Besov space $B_{p,q}^s(\mathbb{R})$ is then the collection of all tempered distributions $f \in S'(\mathbb{R})$ such that

$$
\| f \|_{B_{p,q}^s(\mathbb{R})} := \left( \sum_{j=0}^{\infty} 2^{jsq} \| \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot) \|_{L_p(\mathbb{R})} \right)^{1/q}
$$

is finite. By $\hat{B}_{p,q}^s(\mathbb{R})$ we denote the closure of $C_0^\infty(\mathbb{R})$ with respect to the quasi-norm $\| \cdot \|_{B_{p,q}^s(\mathbb{R})}$.

(ii) Let $p < \infty$. The Lizorkin-Triebel space $F_{p,q}^s(\mathbb{R})$ is then the collection of all tempered distributions $f \in S'(\mathbb{R})$ such that

$$
\| f \|_{F_{p,q}^s(\mathbb{R})} := \left( \sum_{j=0}^{\infty} 2^{jsq} \| \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot) \|_{L_p(\mathbb{R})} \right)^{1/q}
$$

is finite.

**Remark 8.2.** (i) There is an extensive literature about Besov and Lizorkin-Triebel spaces, we refer to the monographs [34], [59], [60] and [61]. These quasi-Banach spaces $B_{p,q}^s(\mathbb{R})$ and $F_{p,q}^s(\mathbb{R})$ can be characterized in various ways, e.g. by differences and derivatives, whenever $s$ is sufficiently large, i.e., $s > \max(0, 1/p - 1)$ in case of Besov spaces and $s > \max(0, 1/p - 1, 1/q - 1)$ in case of Lizorkin-Triebel spaces. We refer to [59] for details.

(ii) The spaces $B_{p,q}^s(\mathbb{R})$ and $F_{p,q}^s(\mathbb{R})$ do not coincide as sets except the case $p = q$. 

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8.2 Besov and Lizorkin-Triebel spaces of dominating mixed smoothness

Detailed treatments of Besov and Lizorkin-Triebel spaces of dominating mixed smoothness are given at various places, we refer to the monographs [1, 50], the survey [49] as well as to the booklet [64].

If $\varphi_j, j \in \mathbb{N}_0$, is a smooth dyadic decomposition of unity as in (8.1), then by means of

$$\varphi_{\vec{j}} := \varphi_{j_1} \otimes \ldots \otimes \varphi_{j_d}, \quad \vec{j} = (j_1, \ldots, j_d) \in \mathbb{N}_0^d,$$

we obtain a smooth decomposition of unity on $\mathbb{R}^d$.

**Definition 8.3.** Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$.

(i) The Besov space of dominating mixed smoothness $S_{t,p,q}^d(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that

$$\| f \|_{S_{t,p,q}^d(\mathbb{R}^d)} := \left( \sum_{\vec{j} \in \mathbb{N}_0^d} 2^{|\vec{j}|tq} \| F^{-1}[\varphi_{\vec{j}} Ff](\cdot) \|_{L_p(\mathbb{R}^d)} \right)^{1/q}$$

is finite. By $\hat{S}_{t,p,q}^d(\mathbb{R}^d)$ we denote the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the quasi-norm $\| \cdot \|_{S_{t,p,q}^d(\mathbb{R}^d)}$.

(ii) Let $0 < p < \infty$. The Lizorkin-Triebel space of dominating mixed smoothness $S_{t,p,q}^d(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that

$$\| f \|_{S_{t,p,q}^d(\mathbb{R}^d)} := \left( \sum_{\vec{j} \in \mathbb{N}_0^d} 2^{|\vec{j}|tq} \| F^{-1}[\varphi_{\vec{j}} Ff](\cdot) \|_{L_p(\mathbb{R}^d)} \right)^{1/q}$$

is finite.

**Remark 8.4.** (i) The most interesting property of these classes for us consists in the following: if

$$f(x) = \prod_{j=1}^d f_j(x_j), \quad x = (x_1, \ldots, x_d), \quad f_j \in A_{t,p,q}^e(\mathbb{R}), \quad j = 1, \ldots, d,$$

then $f \in S_{t,p,q}^d(\mathbb{R}^d)$ and

$$\| f \|_{S_{t,p,q}^d(\mathbb{R}^d)} = \prod_{j=1}^d \| f_j \|_{A_{t,p,q}^e(\mathbb{R})}, \quad A \in \{ B, F \}.$$

I.e., Lizorkin-Triebel and Besov spaces of dominating mixed smoothness have a cross-quasi-norm.

(ii) These classes $S_{t,p,q}^d(\mathbb{R}^d)$ as well as $S_{t,p,q}^d(\mathbb{R}^d)$ are quasi-Banach spaces. If either $t > \max(0, (1/p) - 1)$ (B-case) or $t > \max(0, 1/p - 1, 1/q - 1)$ (F-case), then they can be characterized by differences, we refer to [50] and [63] for details.
(iii) Again the spaces $S_{p,q}^t B(\mathbb{R}^d)$ and $S_{p,q}^t F(\mathbb{R}^d)$ do not coincide as sets except the case $p = q$.

(iv) For $d = 1$ we have

$$S_{p,q}^t A(\mathbb{R}) = A_{p,q}^t(\mathbb{R}), \quad A \in \{B, F\}.$$ 

8.3 Tensor products of Banach spaces

For the convenience of the reader we recall some notion concerning the tensor products of quasi-Banach spaces. We follow \[31\], but see also \[17\].

Let $X$ and $Y$ be Banach spaces. $X'$ denotes the dual of $X$. Consider the set of all formal expressions

$$\sum_{i=1}^n f_i \otimes g_i, \quad n \in \mathbb{N}, \quad f_i \in X \text{ and } g_i \in Y.$$ 

We introduce an equivalence relation by means of

$$\sum_{i=1}^n f_i \otimes g_i \sim \sum_{j=1}^m u_j \otimes v_j$$

if both expressions generate the same operator $A : X' \to Y$, i.e.

$$\sum_{i=1}^n \varphi(f_i) g_i = \sum_{j=1}^m \varphi(u_j) v_j \quad \text{for all } \varphi \in X'.$$  \hspace{1cm} (8.3)

Then the algebraic tensor product $X \otimes Y$ of $X$ and $Y$ is defined to be the set of all such equivalence classes. One can equip this set with several different norms. We are interested in so-called uniform norms only. Let $X_1, X_2, Y_1, Y_2$ be Banach spaces. For $T_i \in \mathcal{L}(X_i, Y_i)$, $i = 1, 2$, we define their tensor product by

$$(T_1 \otimes T_2)h := \sum_{i=1}^n (T_1 f_i) \otimes (T_2 g_i), \quad h = \sum_{i=1}^n f_i \otimes g_i \in X_1 \otimes X_2.$$  \hspace{1cm} (8.4)

We call a norm $\alpha(\cdot, X, Y)$ on $X \otimes Y$ a uniform tensor norm if it satisfies

$$\alpha((T_1 \otimes T_2)h, Y_1, Y_2) \leq \|T_1\|\mathcal{L}(X_1, Y_1)\| \cdot \|T_2\|\mathcal{L}(X_2, Y_2)\| \cdot \alpha(h, X_1, X_2).$$

for all $h = \sum_{j=1}^n f_j \otimes g_j \in X_1 \otimes X_2$ and all $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$. The completion of $X \otimes Y$ with respect to the tensor norm $\alpha$ will be denoted by $X \otimes_\alpha Y$. If $\alpha$ is uniform then $T_1 \otimes T_2$ has a unique extension to $X_1 \otimes_\alpha X_2$ which we again denote by $T_1 \otimes T_2$.

Simple, but important, is the next property we need.

Next we recall three well-known constructions of tensor norms, namely the injective, the projective and the $p$-nuclear norm.

**Definition 8.5.** Let $X$ and $Y$ be Banach spaces.

(i) Let $h \in X \otimes Y$ be given by

$$h = \sum_{j=1}^n f_j \otimes g_j, \quad f_j \in X, \quad g_j \in Y.$$
Then the injective tensor norm $\lambda(\cdot, X, Y)$ is defined as
\[
\lambda(h, X, Y) = \sup \left\{ \left\| \sum_{j=1}^{n} \psi(f_j) \cdot g_j \right\| : \psi \in X', \left\| \psi |X'\right\| \leq 1 \right\} .
\]
(ii) The projective tensor norm $\gamma(\cdot, X, Y)$ is defined by
\[
\gamma(h, X, Y) = \inf \left\{ \sum_{j=1}^{n} \| f_j |X \| \| g_j |Y \| : f_j \in X, g_j \in Y, h = \sum_{j=1}^{n} f_j \otimes g_j \right\} .
\]
(iii) Let $1 \leq p \leq \infty$ and let $1/p + 1/p' = 1$. Then the $p$-nuclear tensor norm $\alpha_p(\cdot, X, Y)$ is given by
\[
\alpha_p(h, X, Y) := \inf \left\{ \left( \sum_{i=1}^{n} \| f_i |X \|^p \right)^{1/p} \sup \left\{ \left( \sum_{j=1}^{n} \| \psi(g_j) |Y' \|^{p'} \right)^{1/p'} : \psi \in Y', \| \psi |Y'\| \leq 1 \right\} ,
\]
where the infimum is taken over all representations of $h$ (as in (ii)).

**Remark 8.6.** We refer to [31, Chapt. 1] for basic properties of these norms.

### 8.4 Tensor products of certain quasi-Banach spaces

The concept of the projective tensor-norm $\gamma_1$ (see Definition 8.5 (ii)) can be extended in order to include also special quasi-Banach spaces either of type $X \hookrightarrow S'(\mathbb{R}^d)$ or of type $\ell_p(w)$. In part (i) of the following definition $f \otimes D g$ stands for the tensor product of the distributions $f$ and $g$, whereas in part (ii) $f \otimes^d g$ refers to the tensor product of sequences.

**Definition 8.7.** Let $0 < p < 1$.

(i) Let $X$ and $Y$ be quasi-Banach spaces such that $X \hookrightarrow S'(\mathbb{R}^{d_1})$ and $Y \hookrightarrow S'(\mathbb{R}^{d_2})$. Then we define the projective tensor $p$-norm $\gamma_p$ by
\[
\gamma_p(h, X, Y) := \inf \left\{ \left( \sum_{j=1}^{n} \| f_j |X \|^p \right)^{1/p} \sup \left\{ \left( \sum_{i=1}^{n} \| \psi(g_j) |Y' \|^{p'} \right)^{1/p'} : \psi \in Y', \| \psi |Y'\| \leq 1 \right\} ,
\]

(ii) Let $X$ and $Y$ be quasi-Banach spaces such that $X = \ell_{q_1}(w_1)$ and $Y = \ell_{q_2}(w_2)$ for some $q_1, q_2 \in (0, \infty]$. Then the projective tensor $p$-norm is defined as
\[
\gamma_p(h, X, Y) := \inf \left\{ \left( \sum_{j=1}^{n} \| f_j |X \|^p \right)^{1/p} \sup \left\{ \left( \sum_{i=1}^{n} \| \psi(g_j) |Y' \|^{p'} \right)^{1/p'} : \psi \in Y', \| \psi |Y'\| \leq 1 \right\} ,
\]

**Remark 8.8.** (i) $\gamma_p$ defines an uniform quasi-norm on $X \otimes Y$. The inequality
\[
\gamma_p(h_1 + h_2, X, Y)^p \leq \gamma_p(h_1, X, Y)^p + \gamma_p(h_2, X, Y)^p
\]
as well as the uniformness are obvious.

(ii) Different attempts to introduce tensor products of quasi-Banach spaces have been undertaken by Turpin [62] and Nitsche [33]. In particular the approach of Nitsche applies to so-called placid $q$-Banach spaces. Let us mention that $\ell_q, B_{\ell,q}^r(\mathbb{R})$ as well as $S_{\ell,q}^{r_1 \ldots r_d}(\mathbb{R}^d)$ are placid $q$-quasi-Banach spaces if $0 < q < 1$. 59
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