Non-perturbative production of multi-boson states and quantum bubbles

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Abstract

The amplitude of production of \( n \) on-mass-shell scalar bosons by a highly virtual field \( \phi \) is considered in a \( \lambda \phi^4 \) theory with weak coupling \( \lambda \) and spontaneously broken symmetry. The amplitude of this process is known to have an \( n! \) growth when the produced bosons are exactly at rest. Here it is shown that for \( n \gg 1/\lambda \) the process goes through ‘quantum bubbles’, i.e. quantized droplets of a different vacuum phase, which are non-perturbative resonant states of the field \( \phi \). The bubbles provide a form factor for the production amplitude, which rapidly decreases above the threshold. As a result the probability of the process may be heavily suppressed and may decrease with energy \( E \) as \( \exp(-\text{const} \cdot E^a) \), where the power \( a \) depends on the number of space dimensions. Also discussed are the quantized states of bubbles and the amplitudes of their formation and decay.
1 Introduction

The problem of calculating amplitudes of production of a large number \( n \) of weakly interacting bosons has received a close attention in connection with the observation\([1, 2, 3]\) that the instanton-induced amplitudes in the standard electroweak model, when calculated to lowest orders of the perturbation theory (in the instanton background), display a rapid growth with energy, associated with the growing multiplicity of gauge and Higgs bosons in the final state. Subsequently Cornwall\([4]\) and Goldberg\([5]\) have pointed out that a similar growth takes place for the amplitudes of processes, in which many bosons are produced by few initial particles in a simpler setting of a \( \lambda \phi^4 \) theory of a scalar field \( \phi \). The reason for this growth is that in an amplitude of a process involving large number \( n \) of weakly interacting bosons the smallness of the coupling constant is compensated by a large number of perturbation theory graphs, which typically grows as \( n! \). This growth is a manifestation of the well known factorial divergence of the coefficients of the perturbation theory\([6]\), and thus for \( n > O(1/\lambda) \), at which \( n \) the compensation takes place, the perturbation theory becomes unreliable.

In this paper we consider the amplitudes of production of \( n \) slow on-mass-shell bosons by a virtual field \( \phi \), \( 1 \rightarrow n \) process', in a theory of one real scalar field with spontaneously broken symmetry with respect to the reflection \( \phi \rightarrow -\phi \). The Lagrangian of this theory in Minkowski space-time has the well known form

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\lambda}{4} (\phi^2 - v^2)^2 ,
\]

where \( \lambda \) is a small coupling constant and \( v \) is the vacuum expectation value of the field. The mass \( m \) of the bosons propagating in either of the vacua at +\( v \) or −\( v \) is \( m = \sqrt{2\lambda} v \). A number of exact results has been obtained recently related to the amplitudes of the \( 1 \rightarrow n \) process at the threshold, i.e. when the final \( n \) particles are produced at rest. The sum of the tree graphs for these amplitudes was originally found explicitly\([7]\) by using recursion relations\([8]\):\

\[
\langle n | \phi(0) | 0 \rangle = n! (-2v)^{1-n}
\]

and then reproduced within a functional technique, suggested by Brown\([9]\), an extension of which technique will be heavily used throughout this paper. Within this extension the problem of calculating the threshold production amplitudes for the theory with the Lagrangian \([1]\) reduces to a Euclidean space-time calculation of the quantum average
of the field $\phi$ with the kink-type boundary conditions in (Euclidean) time: $\phi \to -v$ at $t \to -\infty$ and $\phi \to +v$ at $t \to +\infty$. The tree-level expression (2) for the threshold production amplitudes is determined by the coefficients of expansion in powers of $e^{mt}$ at $t \to -\infty$ of the classical kink profile

$$\phi_0(x, t) = v \tanh(mt/2) \ ,$$

which provides the saddle point for the classical action $S[\phi]$. The first correction to the result (2) which amounts to summing one loop graphs has been found and reduces to multiplying the lowest order expression (2) by the factor $(1 + c(d)n(n-1)\lambda)$, where $c(d)$ is a coefficient depending on the number of space dimensions $d$,

$$\langle n | \phi(0) | 0 \rangle_{\text{tree+1\text{loop}}} = n! (-2v)^{1-n} (1 + c(d) n(n-1)\lambda) \ .$$

The $n^2\lambda$ behavior of the relative magnitude of the correction suggests that at large $n$ the saddle point configuration in calculation of the $n$-th coefficient of the expansion at $t \to -\infty$ of the mean field, i.e. of the quantity $\phi(x)e^{-S}$, is driven far away from the saddle point configuration (3) of the action alone. Once the correct saddle point configuration is found, the quantum fluctuations around it should produce only subleading in $n$ corrections. (A similar situation in a (1+1)-dimensional toy model was discussed in Ref. [12].)

Here we find that in the limit of large $n$ the correct saddle point configuration in calculation of the threshold production amplitudes for large $n$ is determined by dynamics of the surface of the inter-phase boundary (‘domain wall’) separating the phases with $\phi = -v$ at large negative $t$ and with $\phi = +v$ at large positive $t$. A WKB treatment of this dynamics relates it to the semiclassical properties of the spherical bubbles filled with the phase $\phi = +v$ in the vacuum with $\phi = -v$. At the classical level such configurations were studied some time ago [13]. In particular it was found that these configurations are reasonably long-lived: a large bubble undergoes several pulsations of its radius before decaying into outgoing waves.

In what follows it will be shown that in a theory in $(d+1)$ dimensions (thus $d$ being the number of spatial dimensions) at $n \gg 1/\lambda$ the WKB result for the amplitude $\langle n | \phi(0) | 0 \rangle$ with all particles having exactly zero momenta can be interpreted as given by a two-stage process. The field operator $\phi(0)$ produces a bubble and then the bubble couples to $n$ particles. The resulting amplitude is large:

\[2\]
\[ \langle n | \phi(0) | 0 \rangle \sim n!(-2\nu)^{1-n} \exp(b(d) \pi^{d-1}) \] (5)

with positive coefficient \( b(d) \) depending on \( d \). This growth is in agreement with the general result that in this theory the threshold amplitudes should grow not slower than as \( n! \). However exactly at the threshold the phase space of the final \( n \) particles is vanishing and to estimate the total probability of the process one needs to know the behavior of the amplitude above the threshold. The presence of the bubble in the intermediate state implies existence of a form factor, which cuts off the amplitude above the threshold when any of the momenta of the final particles is larger than the inverse of the radius of the bubble, \( r^{-1} \). An estimate of the total probability of the process \( 1 \to n \) with the two-stage picture can be done by evaluating the probability of creation of a bubble. The probability of creation of a bubble with energy \( E \) evaluated by means of the Landau-WKB technique is found in this paper to be given by

\[ | \langle B(E) | \phi(0) | 0 \rangle |^2 \sim \exp(-2b(d)(E/m)^{d-1}) \] (6)

with precisely the same coefficient \( b(d) \) as in eq.(5), and \( B(E) \) stands for the state of the bubble with energy \( E \).

Equation (6) shows that due to the form factor provided by the bubble not only the growth of the probability with energy is eliminated but in fact the total probability associated with multi-boson states rapidly falls with energy in the non-perturbative asymptotic regime. Therefore we conclude that it is extremely plausible that in this theory the non-perturbative contribution to the processes with production of many soft final particles by few initial ones does not become large at high energy in spite of the indications to the contrary in lowest orders of the loop expansion. Whether this behavior is universal and is valid for other theories, in particular for the electroweak theory, is yet to be studied.

The suppression of the non-perturbative processes of the type \( few \to many \) at high energy does not contradict to a possible growth with energy of the probability of the processes \( many \to many \), as is discussed in the concluding section.

The rest of this paper is organized as follows. In Section 2 a brief review is given of the standard perturbative calculation of the amplitudes of the \( 1 \to n \) processes within the equivalence to the problem of calculating the quantum average of the field with the kink-type boundary conditions in the Euclidean space. In Section 3 the problem of finding the proper saddle point configuration for calculation of the amplitudes is formulated.
within the so-called thin wall approximation. The search for this configuration leads to considering dynamics of bubbles in the Minkowski space-time. This dynamics is discussed within the Bohr-Sommerfield quantization in Section 4. The properties of bubbles are quantitatively related to the $n$ particle production amplitudes in Section 5 and in Section 6 the probability of creation of a bubble by a highly virtual field is estimated by the Landau-WKB method. Section 7 contains a discussion of the results and of the validity of the approximations made in present calculations.

2 Perturbative calculation

In this section we recapitulate the perturbative calculation of the threshold production amplitudes within the equivalence\(^{14}\) of this problem for the theory described by the Lagrangian \(^{(1)}\) to the Euclidean-space calculation of the quantum mean field with boundary conditions corresponding to a domain wall separating vacua $+v$ and $-v$. Fixing for definiteness that the amplitudes are calculated in the ‘left’ vacuum, i.e. at $\langle \phi \rangle = -v$ the equivalent problem can be formulated as follows.

One first calculates in the Euclidean space-time the quantum mean field

$$
\Phi(t) = \frac{\int \left( \int \phi(x, t) \, d^d x \right) e^{-S[\phi]} \, D\phi}{V \int e^{-S[\phi]} \, D\phi},
$$

(7)

where $d$ is the number of spatial dimensions in the problem, $V$ is the $d$-dimensional spatial normalization volume, and the boundary condition in the path integral at $t \to -\infty$ is specified by the requirement that the asymptotic behavior of $\Phi(t)$ there is given by

$$
\Phi(t) \to -v + z e^{mt} + O(e^{2mt})
$$

(8)

with $z$ being a constant and $v$ and $m$ are the renormalized v.e.v. and the boson mass. The production amplitudes are then given\(^{3, 4}\) by the coefficients of the expansion of $\Phi(t)$ in powers of $e^{mt}$ at large negative $t$, i.e. if one writes the expansion as

$$
\Phi(t) = \sum_{n=0}^{\infty} c_n e^{nt} ,
$$

(9)

then

$$
\langle n|\phi(0)|0 \rangle = n! \frac{c_n}{z^n} .
$$

(10)
The coefficient \( z = c_1 \) in the leading asymptotic behavior (8) is thus a normalization of a one particle state: without dividing by the factor \( z^n \) in eq. (10) one would obtain the amplitudes with the particle states normalized as \( \langle 1|\phi(0)|0 \rangle = z \). As it is written the equation (10) gives the amplitudes with the standard normalization of the particle states. It is also assumed here that the normalization volume \( V \) is set to unity.

The application of the described equivalence is straightforward at the tree level, which corresponds to substituting for the field \( \phi \) the uniform in space (but not time) solution (3) of the classical Euler-Lagrange equations for the Lagrangian (1). Expanding this solution in powers of \( e^{mt} \) reproduces through eq. (10) the tree-level result (2) for the production amplitudes.

The solution (3) is the familiar kink profile of the inter-phase boundary ('domain wall') separating the vacua \( -v \) at \( t \rightarrow -\infty \) and \( +v \) at \( t \rightarrow +\infty \). A particular choice of the normalization factor \( z \) fixes the position of the domain wall in time, i.e. fixes the translational zero mode of the field, which corresponds to an overall shift in the time direction. Therefore one can either choose a particular value of \( z \) or alternatively fix the overall position of the field in time. Throughout this paper we choose to fix the position in time by requiring that the integration in the path integral in eq. (7) runs over field configurations such that \( \phi(x, 0) \) is zero at the boundary of the normalization spatial bounding box. At the classical level this fixes the center of the domain wall at \( t = 0 \) and thus also sets \( z = 2v \). The quantum corrections to the mean field (7) in general renormalize the factor \( z \).

To account for the quantum effects in the mean field (7) one writes the full field \( \phi \) as a sum of classical and quantum parts

\[
\phi(x, t) = \phi_0(x, t) + \phi_q(x, t)
\]

and evaluates the mean value of the quantum part of the field by perturbation theory in the background field \( \phi_0 \). At the one-loop level this calculation \([10, 11]\) leads to the result in eq. (11).
3 Semiclassical configurations for quantum effects at large \( n \)

As is discussed in the Introduction the growth with \( n \) of the loop corrections within the quantization around the saddle point configuration (3) of the action indicates that at large \( n \) the \( n \)-th coefficient of the expansion of \( \Phi(t) \) in powers of \( e^{mt} \) at \( t \to -\infty \) is contributed by a semiclassical configuration which strongly differs from that in eq.(3). To find this appropriate ‘distorted’ configuration we first consider the one shown in Fig.1, where the domain wall (the surface corresponding to \( \phi(x,t) = 0 \)) assumes a non-flat shape slowly varying with \( x \). The surface can be described by its \( x \)-dependent deviation from \( t = 0 \), i.e. by the solution \( t = -h(x) \) of the equation \( \phi(x,t) = 0 \). Assume now that one fixes the shape of the boundary corresponding to a particular function \( h(x) \) and minimizes the action with respect to all other variables of the field by solving the Euler-Lagrange equations with the surface of zeros being fixed. Then for large negative \( t \) at the point \( x_0 \) corresponding to the maximum of \( h(x) \) the \( n \)-th harmonics of the field is given in the leading exponential approximation by

\[
2v(-1)^{n-1} \exp (nm(t + h(x_0) + O(nm\delta)) \sim 2v(-1)^{n-1} \exp(nmh(x_0))e^{nm} . \tag{12}
\]

This behavior can be understood by considering that at the point \( x_0 \) the evolution of the field in time from \( \phi = 0 \) towards \( \phi = -v \) proceeds over the time \( |t| - h(x_0) + \delta \), where \( \delta \) reflects the uncertainty related to the curvature of the surface of the inter-phase boundary. This uncertainty is of a subleading importance in situations where \( h(x_0) \) is large, which as will be seen is the case for \( n \gg 1/\lambda \).

Equation (12) tells that the coefficient \( c_n \) gets multiplied by the factor \( \exp(nmh(x_0)) \). Thus in the leading WKB approximation this coefficient can be evaluated as

\[
c_n \sim \max \left[ \exp(nmh(x_0) - S[H] + S_0) \right] , \tag{13}
\]

where \( S[H] \) is the action for the field configuration described by the shape \( h(x) \) of the inter-phase boundary and \( S_0 = S[h = 0] \) is the action of the unperturbed classical solution (3), and all the pre-exponential factors are omitted.

The appearance of the action \( S_0 \) with the plus sign in eq.(13) is due to the fact that in the expression (7) for the mean field the path integrals in both numerator and the
denominator are calculated with the kink-type boundary conditions. Thus the factor \( \exp(S_0) \) appears from the saddle point value of the denominator.

If \( h(x) \) varies at scale larger than the thickness of the wall, \( m^{-1} \) the action \( S[h] \) can be calculated in the thin-wall approximation as the surface tension of the wall \( \mu \) times its area \( A \):

\[
S[h] = \mu A[h] ,
\]

where

\[
\mu = \int_{-\infty}^{+\infty} \left[ \frac{1}{2} \left( \frac{d}{dt} \phi_0(t) \right)^2 + \frac{\lambda}{4} (\phi_0^2 - v^2)^2 \right] dt = \frac{2}{3} \sqrt{2\lambda v^3} = \frac{m^3}{3\lambda}.
\]

Finding the maximum of the expression (13) with the action (14) is equivalent to solving a surface tension problem for a \( d \)-dimensional film in \((d + 1)\)-dimensions. The edges of the film are fixed at the boundary of the bounding box: \( h(\text{boundary}) = 0 \), and at the point \( x_0 \) the force equal to \( nm \) is applied downwards. The maximal deviation \( h_0 = h(x_0) \) of the film will be largest if the force is applied to the center of the film, therefore we set \( x_0 = 0 \). (In fact for \( d > 2 \) the equilibrium shape of the film does not depend on \( x_0 \) if this point is sufficiently far from the edges. For \( d \leq 2 \) there is an infrared behavior in this problem, so that the equilibrium deviation explicitly depends on the size of the bounding box. Also it is explicitly assumed throughout this paper that \( d > 1 \).

A one-dimensional ‘film’ lacks intrinsic curvature, which makes most of the formulas in this paper singular in the formal limit \( d \to 1 \), i.e. that of a \((1+1)\)-dimensional field theory, for which the present analysis is thus not directly applicable.) Assuming that the bounding box is spherically symmetrical in the spatial \( d \) dimensions with a large radius \( R \), one concludes that the shape, which the film takes under the force applied at the center, is also spherically symmetrical and can be characterized by the radius \( r(t) \) of its slice at \( t = \text{const} \) if the slice is positioned at an instant \( t \) such that \(-h(0) < t < 0\). The boundary conditions for \( r(t) \) being \( r(-h_0) = 0 \) and \( r(0) = R \).

In terms of \( r(t) \) the quantity \( S[h] - nm h_0 \) entering the expression (13) can be written as

\[
s[r] = \int_{-h_0}^{0} l_d \mu r^{d-1} \sqrt{1 + \dot{r}^2} dt - Eh_0 ,
\]

where the factor \( nm \) is identified with the total energy \( E \), \( \dot{r} = dr/dt \), and \( l_d \) is the \((d-1)\) dimensional volume of unit sphere \( S_{d-1} \): \( l_d = 2 \pi^{d/2}/\Gamma(d/2) \). The integral in
eq. (16) is nothing else than the Euclidean action for a spherical bubble in the thin wall approximation in the theory with degenerate vacua. Since $E$ is the conserved value of the Hamiltonian for the classical trajectory $r(t)$, the functional $s[r]$ is identified as the truncated action

$$s[r] = \int p_E \, dr ,$$

where $p_E$ is the Euclidean momentum conjugate of $r$:

$$p_E = \frac{l_d \mu r^{d-1} \dot{r}}{\sqrt{1 + \dot{r}^2}} .$$

On an Euclidean-space classical trajectory the value $E$ of the Hamiltonian is related to $p_E$ and $r$ as

$$E^2 + p_E^2 = (l_d \mu r^{d-1})^2 .$$

The solution of the latter equation for $p_E$ in terms of $E$ and $r$ reads as

$$p_E = \sqrt{(l_d \mu r^{d-1})^2 - E^2} = l_d \mu \sqrt{r^{2d-2} - r_0^{2d-2}} ,$$

where

$$r_0 = (E/(l_d \mu))^{\frac{1}{d-1}} .$$

This solution immediately reveals an important point: there is no real solution for $p_E$ and thus for $r(t)$ at $r < r_0$. In the equivalent surface tension problem the origin of this behavior is obvious: there is a minimal radius equal to $r_0$ of a slice of the surface with surface tension $\mu$ that can support the force $E$. In terms of quantum mechanics with the action (17) the point $r_0$ corresponds to the classical turning point, and for $r < r_0$ the evolution of the system proceeds in the Minkowski time. We are thus compelled to consider the evolution of the radius of the bubble along the complex time trajectory shown in Fig.2, on which the part of the trajectory with $r < r_0$ evolves along imaginary Euclidean, i.e. real Minkowski, time. In the Minkowski space this part of the trajectory describes the bubble expanding from size $r = 0$ to the classical turning point $r = r_0$. Therefore before proceeding with further evaluating the coefficients $c_n$ by eq. (13) it is appropriate to discuss few properties of the bubbles in the Minkowski space-time.
4 Quantizing the bubbles

The Hamiltonian dynamics of the bubble in the Minkowski space-time in the thin wall approximation is described by the simple substitution $t \rightarrow it$ in the previous Euclidean space formulas. In particular the analog of the equation (19) for the Hamiltonian is

$$H^2 - p_M^2 = (l_d \mu r^{d-1})^2$$

(22)

with the Minkowski momentum $p_M$. The classical Minkowski-space trajectory with energy $E$ corresponds to oscillations of the bubble between the turning point $r = r_0$ (eq.(21)) and $r = 0$. Naturally it could be that instead of oscillating the bubble would quickly dissipate into outgoing waves. However a numerical study of the classical evolution of the field of the bubble-type configuration \[13\] (not constrained by the thin wall approximation) has revealed that the bubbles undergo at least several oscillations before they emit a larger portion of their energy in outgoing waves. This implies that the lifetime of a bubble is at least longer than the period of oscillation $T \sim r_0$. Therefore we start with discussing the bubbles as if they were stable and later take into account their slow decay.

The part of the trajectory near zero radius, $r < m^{-1}$, cannot be described within the thin wall approximation since the thickness of the wall is of order $m^{-1}$. However, at large energy, corresponding to the turning radius $r_0 \gg m^{-1}$, most of the evolution of the bubble proceeds within the applicability of the thin wall approximation, and this is the part from which most of the action comes. According to equation (21) the condition $r_0 \gg m^{-1}$ implies that

$$n = E/m \gg m^{3-d}/\lambda ,$$

(23)

where the right hand side is the inverse of the dimensionless coupling in the theory and thus is assumed to be much bigger than 1. In particular for $d = 3$ (the normal (3+1) dimensional theory) the condition (23) translates into $n \gg 1/\lambda$. Thus the applicability of the present calculations lies within an essentially non-perturbative domain.

The oscillatory motion of the bubbles can be quantized and the discrete energy levels found by applying the Bohr - Sommerfield quantization rule:

$$I(E) \equiv \int p_M dr - 2\pi \nu(E) = 2\pi N ,$$

(24)
where the integral runs over one full period of oscillation and contains the momentum \( p_M \) determined by eq.(22) in the thin wall approximation. The quantity \( \nu(E) \) is a correction to the thin wall limit, which arises from the contribution to the action of the motion at short distances \( r \sim m^{-1} \), where the latter limit is not applicable. Since at such distances \( p_M \sim E \), by order of magnitude \( \nu(E) \) can be estimated as \( \nu(E) \sim E/m \). The integral in eq.(24) is of the order of \( E r_0 \), and is thus much larger than \( \nu(E) \) once the condition \( r_0 \gg m^{-1} \) is satisfied. In terms of the turning radius \( r_0 \) the quantization relation (24) reads as

\[
k_d \mu r_0^d = 2\pi (N + \nu(E)) , \tag{25}
\]

with \( k_d \) being a numerical coefficient,

\[
k_d = l_d \frac{\sqrt{\pi} \Gamma[1/(2d - 2)]}{2(d - 1) \Gamma[3/2 + 1/(2d - 2)]} . \tag{26}
\]

At a large energy, corresponding to the condition (23), the correction term with \( \nu(E) \) can be neglected and one finds the expression for an energy level \( E_N \) in terms of the number \( N \) of the level:

\[
E_N = \mu \frac{1}{d} \left( \frac{2\pi N}{k_d} \right)^{d-1} . \tag{27}
\]

By the relation (23) the condition \( r_0 \gg m^{-1} \) requires that

\[
N \gg m^{3-d}/\lambda , \tag{28}
\]

where the right hand side by itself is assumed to be a large number. The energy \( E_N \) of the level then satisfies the condition (23). It can be readily noticed that for such \( N \) the spacing between consecutive levels is small in comparison with the mass \( m \) of the bosons: \( \Delta E_N \sim r_0^{-1} \ll m \). This, perhaps, in addition to the reflectionless property of the wall explains the relative stability of large bubbles. Indeed, emission of individual quanta would require transitions between states with a large difference of their quantum numbers \( N \): \( \Delta N \simeq m r_0 \gg 1 \). The overlap integral for the wave functions of the levels with large difference of their numbers is exponentially small in \( \Delta N \). Therefore the probability of emission of the bosons by large bubbles should be strongly suppressed by the parameter \( m r_0 \). However there perhaps can be a larger probability of simultaneous emission of many bosons at short distances, i.e. when the bubble contracts to \( r \ll m^{-1} \).
during the oscillations. Admittedly at present there is little understanding of the decay of the bubbles besides the study\textsuperscript{13} by means of classical field equations.

5 Quantized bubbles and multi-boson processes

Returning now to evaluation of the coefficients $c_n$ by eq.(13), we notice that the exponent there receives a real contribution from the action on the Euclidean part of the trajectory, i.e. when $r$ changes between $r = r_0$ and $r = R$, and an imaginary part on the Minkowski part of the trajectory, i.e. between $r = 0$ and $r = r_0$. Let us first evaluate the Euclidean part, where the truncated action (eq.(17)) is given by

$$s[r] = \int_{r_0}^{R} p_E \, dr = \int_{r_0}^{R} \sqrt{(l_d \mu r^{d-1})^2 - E^2} \, dr .$$

(29)

The latter integral diverges in the limit $R \to \infty$. However the quantity of interest is the difference $S_0 - s[r]$, which enters the exponent in eq.(13). Since the action $S_0$ of the unperturbed configuration formally corresponds to the truncated action $s[r]$ at zero energy $E = 0$ and has the same leading divergence, one can first calculate the derivative of $s[r]$ with respect to $E$ and then find the difference of the action as

$$S_0 - s[r] = \int_{0}^{E} \left( -\frac{ds[r]}{dE} \right) \, dE .$$

(30)

For the derivative with respect to energy one finds from eq.(29)

$$-\frac{ds[r]}{dE} = \frac{E}{l_d \mu} \int_{r_0}^{R} \frac{dr}{\sqrt{r^{2d-2} - r_0^{2d-2}}} .$$

(31)

Before discussing this integral for large energy, it can be noted that quantitatively this expression can be used to estimate the quantum effects in the coefficients $c_n$ also at low energies where formally $r_0$ is less than thickness of the wall. Then the integral should be cut off at a lower limit $r = r_1 \sim m^{-1}$, which does not depend on $E$ if $E/m \ll m^{3-d}/\lambda$. Then one finds

$$c_n \approx c_n^{(0)} \exp \left( \frac{E^2}{2l_d \mu} \int_{r_1}^{R} \frac{dr}{r^{d-1}} \right),$$

(32)

where $c_n^{(0)}$ is the tree-level expression for the coefficient $c_n$. When the exponent is expanded in powers of its argument this gives the $(1 + \text{const} \cdot n^2 \lambda)$ behavior of the first
quantum correction to the production amplitudes, which one finds by a calculation of one-loop graphs \[^{10,11}\](eq. (4)). Moreover for \(d \leq 2\) the integral in equations (31) and (32) still diverges in the limit \(R \to \infty\). Therefore the dependence on the infrared cut off \(R\) dominates the quantum effects and makes them insensitive to the region of small \(r\). In this case in the leading at large \(R\) approximation the exponent in eq. (13) can be found for any \(E\) with the obvious result

\[
c_n = c_n^{(0)} \exp \left(\frac{E^2 R^{2-d}}{2(2-d) l_d \mu}\right) = c_n^{(0)} \exp \left(\frac{n^2 R^{2-d}}{6 (2-d) l_d m}\right). \tag{33}\]

It is also clear that the Minkowski part of the trajectory \(r(t)\) contains no infrared dependence and thus does not contribute to this result in the leading in \(R\) approximation. It can be readily verified by the technique of Refs. [10, 11] that the infrared behavior of the one-loop corrections to the coefficients \(c_n\) exactly reproduces the expansion in the latter equation of the exponent up the first power of its argument.

Therefore considering the case where the equation (31) is applicable down to \(r_0\) with the condition that \(r_0 \gg m^{-1}\) is sensible only when the integral in that equation is finite in the limit \(R \to \infty\), i.e. when \(d > 2\), which includes the most interesting case of \(d = 3\). Setting \(R = \infty\) in eq. (31) one finds for \(d > 2\)

\[
- \frac{ds[r]}{dE} = f_d r_0(E) = f_d \left(\frac{E}{l_d \mu}\right)^{\frac{1}{d-1}} \tag{34}\]

with the dimensionless factor \(f_d\) given by

\[
f_d = \frac{\sqrt{\pi} \Gamma[1/2 - 1/(2d-2)]}{2(d-1) \Gamma[1 - 1/(2d-2)]}. \tag{35}\]

Therefore one finds the real part of the difference \(S_0 - s[r]\) associated with the Euclidean part of the trajectory \(r(t)\) to be given by

\[
(S_0 - s[r])_E = f_d \frac{d-1}{d} E \left(\frac{E}{l_d \mu}\right)^{\frac{1}{d-1}}. \tag{36}\]

Let us now evaluate the contribution to the coefficients \(c_n\) of the Minkowski part of the evolution of the bubble. The simplest trajectory, which links \(r = 0\) with \(r = r_0\) consists of one half of the period which contributes to the coefficient \(c_n\) the factor

\[
F_0 = \exp(i I(E)/2), \tag{37}\]
where the action integral $I(E)$ over a full period is defined in eq. (24). However there are also trajectories connecting $r = 0$ with $r = r_0$, which differ from this one by an integer number $k$ of full periods of oscillation. Each of these trajectories provides a saddle point. Therefore one has to take the sum of contributions of these saddle points. The sum has the form

$$F(E) = \sum_{k=0}^{\infty} \exp[i(k + 1/2)I(E)] = \frac{\exp(iI(E)/2)}{1 - \exp(iI(E))} \tag{38}$$

It is clear that this expression sums to infinity when the Bohr-Sommerfield relation (eq. (24)) is satisfied, i.e. $I(E) = 2\pi N$. In other words the factor $F(E)$ develops poles at the values of energy coinciding with the positions of the bubble levels $E_N$ given by eq. (27). It can be reminded that thus far the decay of the bubble levels is completely ignored. Therefore we come to the conclusion that in this approximation the amplitude of production of $n$ static bosons consists of poles at the energies $E = E_N$ with the residues proportional to $(-1)^N \exp(S_0 - s[r])_E$, the latter being given by the equation (36). The sign alternating factor $(-1)^N$ arises from $\exp(iI(E_N)/2) = \exp(i\pi N)$ in the numerator of eq. (38). The decay width of the bubble levels can be taken into account in the Breit-Wigner approximation by shifting the positions of the poles into the complex plane: $E_N \rightarrow E_N - i\Gamma_N/2$.

To conclude this section we collect all the factors in our estimate of the production amplitudes of $n$ bosons, all being at rest, at energy $E$ corresponding to $n = E/m \gg m^{3-d}/\lambda$ and write the final result in the form

$$\langle n|\phi(0)|0\rangle \sim n! \frac{\exp(iI(E)/2)}{1 - \exp(iI(E))} \exp \left[ f_d d - 1 \frac{d}{d} E \left( \frac{E}{l_d \mu} \right)^{\frac{1}{d-1}} \right]. \tag{39}$$

This expression consists of poles corresponding to the energy levels $E_N$ of quantized bubble, i.e. corresponding to $I(E) = 2\pi N$. Therefore one can conclude that the production of the bosons in ultra-high-energy limit goes through intermediate states, which are the quantum levels of a bubble.

6 Amplitudes of bubble formation and decay

The residue of an individual pole in eq. (38) is given by
\[ \text{res}_N = (-1)^N \left( \frac{E_N}{m} \right)! \exp(D(E_N)), \]  
(40)

where the shorthand notation \( D(E) \) is used for the expression in eq.\( \text{(36)} \), which also enters the exponent in eq.\( \text{(39)} \). The residue can in general be written as

\[ \text{res}_N = A(1 \to B_N) \cdot A(B_N \to n), \]  
(41)

where \( A(1 \to B_N) = \langle B_N | \phi(0) | 0 \rangle \) is the amplitude of production of the \( N \)-th state of the bubble, \( B_N \) by a virtual field \( \phi \) and \( A(B_N \to n) \) is the amplitude of transition of this state into \( n \) bosons, all being at rest. The former amplitude can be evaluated by the Landau-WKB technique\(^{15} \) for calculating transition matrix elements. According to this technique in the exponential approximation the matrix element of an operator \( f \) between states with energy \( E_1 = 0 \) and \( E_2 = E \), \( \langle E | f | 0 \rangle \), is found by matching the Euclidean classical trajectories, one with the energy of the initial state, i.e. \( E_1 = 0 \), and the other with the energy of the final state \( E_2 = E \), which runs between the matching point and the turning point. The matrix element is given in the exponential approximation by

\[ \langle E | f | 0 \rangle \sim \exp(s(E) - s(0)), \]  
(42)

where \( s(E) \) is the Euclidean-space truncated action on the trajectory. The specific form of the operator \( f \) enters only the pre-exponential factor (unless the operator \( f \) itself is exponential, which is not the case in the problem under discussion) and can be ignored.

The configuration shown in Fig.3 displays such matching of the evolution in the Euclidean space. The evolution starts with zero energy, which corresponds to a flat domain wall. The field then matches that of a bubble with energy \( E \), whose radius then contracts down to the turning point \( r_0 \). The configuration is then symmetrically extended beyond the turning point, so that it gives the square of the matrix element. Clearly, the difference of the truncated action \( (s(E) - s(0)) \) on one half of this configuration, i.e. from \( t = -\infty \) to the turning point, is equal to minus that given by equation \( \text{(36)} \). Thus one immediately finds the estimate

\[ |A(1 \to B_N)| = |\langle B_N | \phi(0) | 0 \rangle| \sim \exp( -D(E_N) ). \]  
(43)

A comparison of the formulas \( \text{(40)}, \text{(41)} \) and \( \text{(43)} \) leads to the following reasoning. The product \( \text{(41)} \) of the amplitudes of formation and decay of the bubble is exponentially large in \( E^{d/(d-1)} \) (eq.\( \text{(40)} \)) while the formation amplitude given by eq.\( \text{(43)} \) is exponentially
small. Thus the amplitude of the coupling of the bubble to \( n \) bosons, which are all being at rest contains the doubled positive exponent:

\[
|A(B_N \rightarrow n)| \sim \left( \frac{E_N}{m} \right)! \exp(2D(E_N)) .
\]

To reconcile this extremely strong coupling of the bubble to the state of \( n \) bosons, in which they all have exactly zero spatial momenta, with a total decay rate that is not exponentially large, one inevitably has to assume that the coupling of the bubble to bosons develops a form factor which sharply decreases above the threshold and thus is capable of suppressing the double-exponential and the factorial energy growth of the amplitude at the threshold. In view of this observation it is extremely likely that in the processes \( 1 \rightarrow n \), whose amplitude at the threshold has only single exponential growth factor, the same form factor makes the total probability exponentially suppressed at high energy.

7 Discussion and conclusions

The results of the search for the correct saddle point for calculation of the \( n \)-th coefficient \( c_n \) in the expansion of the mean field \( \phi \) under the kink type boundary conditions justify the thin wall approximation used in this paper in the limits considered. Indeed, the uncertainty of this approximation, expressed by \( \delta \) in eq.(11), which is of the order of the thickness of the wall, becomes small in comparison with the main term if the maximal deviation of the wall \( h_0 \) is much larger than \( m^{-1} \). One can readily see that this is indeed the case in the calculations of this paper. For the number of space dimensions \( d \leq 2 \) the maximal deviation grows with the size \( R \) of the bounding box and thus for the leading infrared terms the thin wall approximation is applicable at any \( n \). As mentioned in connection with the equation (33) the result for this case can be checked against a direct calculation \([10,11]\) of the infrared behavior of the amplitudes at the one-loop level. In the infrared finite case of \( d > 2 \) in particular for \( d = 3 \) the applicability of the thin wall approximation is guaranteed in the limit of large \( n \): \( n \gg m^{3-d}/\lambda \). This follows from that the maximal deviation \( h_0 \) is of the order of \( r_0 \), thus according to equation (21) \( h_0 m \sim (n \lambda/m^{3-d})^{1/(d-1)} \).

There is possibly not a coincident correlation between the qualitative dependence of the present calculations on the number of spatial dimensions and a similar dependence of the spectra of states of many soft bosons with point-like attraction. Noticing in this
connection that the interaction between soft bosons near the threshold in the theory with the Lagrangian (1) is attractive, we recall that for \( d \leq 2 \) an attraction at short distances produces bound states for any number of bosons, starting from \( n = 2 \). The existence of such bound states strongly changes the spectral density of the \( n \)-particle scattering states, which is another way to formulate that there is a strong behavior of the form factor for production amplitudes near the threshold. For \( d > 2 \), in particular for \( d = 3 \), a weak attraction at short distances is insufficient to bind few bosons. However if the number of bosons is non-perturbatively large, \( n \gg m^{3-d}/\lambda \) they can form collective multi-particle bound states, which in the terminology of the present paper are the bubbles of the field \( \phi \). Though a detailed relation to the present calculations is yet to be understood, this may provide another explanation of why the semiclassical analysis can be applied for any \( n \) if \( d \leq 2 \) and why it becomes applicable in the case of \( d > 2 \) only when the number of produced bosons is non-perturbatively large, \( n \gg 1/\lambda \).

The main result of the present paper is the following hierarchy of the amplitudes for transitions between one highly virtual particle, the state of \( n \) bosons, all having zero spatial momenta, and a state of a spherical bubble with energy \( E = nm \):

\[
A(1 \rightarrow B) \sim e^{-D(E)},
\]
\[
A(1 \rightarrow n) \sim A(1 \rightarrow B) \cdot A(B \rightarrow n) \sim e^{D(E)},
\]
\[
A(B \rightarrow n) \sim e^{2D(E)}.
\] (45)

When the particles have non-zero momenta a form factor arises, due to the size \( r_0 \) of the bubble which cuts off the phase space integration. The suppression due to the form factor can be evaluated from the fact that the decay rate of a large bubble is not exponentially large in its energy, i.e. in this exponential scale the rate is \( \Gamma(B \rightarrow many) \sim O(1) \). If the processes \( 1 \rightarrow many \) at ultra-high energy are going through the bubbles, as it is strongly indicated by the calculations in this paper, their rate should be cut off by the same form factor, which implies

\[
\Gamma(1 \rightarrow many) \sim |A(1 \rightarrow B)|^2 \Gamma(B \rightarrow many) \sim e^{-2D(E)}
\] (46)

and thus these processes are strongly suppressed.

This hierarchy can be extended to processes \( many \rightarrow many \), which potentially can go at high temperature. If such scattering processes are also mediated by the bubbles, one can estimate
\[ A(n \to n) \sim A(n \to B) \cdot A(B \to n) \sim e^{A D(E)} . \] (47)

A kinetical calculation of the rate in a thermal equilibrium in this case involves the form factor for both the final and the initial states. Thus the rate of the multi-particle processes at high temperature should be \( \Gamma(many \to many) \sim O(1) \), which is in agreement with standard thermodynamical calculations.

For a very approximate understanding of the role of the form factor one can invoke the following reasoning. The amplitude of the process \( 1 \to n \) above the threshold is a function of all the \( n \) spatial momenta \( p_i \) of the final bosons, \( A(p_i) \), and the total probability is given by

\[ \Gamma = \frac{1}{n!} \int |A(p_i)|^2 \prod_{i=1}^{n} \frac{d^d p_i}{2 \varepsilon_i (2\pi)^d} \] (48)

with the amplitude \( A(0) \) corresponding to all the momenta vanishing being given by eq. (39). The presence of the bubble with the radius \( r_0 \) in the intermediate state makes the amplitude \( A(p_i) \) to sharply decrease when any of the momenta \( p_i \) is of the order of \( 1/r_0 \). Therefore the phase space corresponding to this region in the momentum of each particle behaves parametrically as \( (r_0)^{-nd} \sim E^{-E d/(d-1)} \). This suppression is sufficient to eliminate the factorial growth of the amplitude \( A(0) \), but is not sufficient to overcome the \( \exp(-const \cdot E^{d/(d-1)}) \) enhancement. The estimate of the total rate in eq. (46) shows that it is most likely that the actual suppression due to the finite size of the bubble is somewhat stronger than in this simplistic reasoning, perhaps, due to coherence effects.

As is mentioned before our estimate of the form factor suppression relies on that the decay rate of a bubble does not grow exponentially as some power of the energy of the bubble. Therefore it is appropriate to emphasize once again the arguments in favor of this behavior. One argument is based on the numerical observation of the classical relative stability of the bubble: it dissipates its energy over several oscillations, thus its lifetime is at least not shorter than of the order of the period of oscillations \( T \sim r_0 \). In the quantum theory the suppression of emission of the bosons from large distances follows from the fact, discussed in Section 4, that the splitting of the levels of the bubble at high energy is much smaller than the mass \( m \) of the quantum. Therefore emission of a boson requires a transition between levels with a large difference \( \Delta N \) of their quantum numbers, which is exponentially suppressed in \( \Delta N \). Thus the emission can take place during the time when the radius of the bubble is small \( r < O(m^{-1}) \) and
the walls collide into each other. However even if the probability of annihilation of the walls into outgoing bosons in this region is of order one, the bubble spends only a small fraction of time at such small radius, so that the total decay rate is proportional to the inverse of the period $T$. Therefore there seems to be every reason to assume that the decay rate of a bubble is limited by $const/T$.

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References

[1] A. Ringwald, Nucl. Phys. B330, 1 (1990).
[2] O. Espinosa, Nucl. Phys. B343, 310 (1990).
[3] For a recent review see M.P. Mattis, Phys. Rep. 214, 159 (1992).
[4] J.M. Cornwall, Phys. Lett. 243B, 271 (1990).
[5] H. Goldberg, Phys. Lett. 246B, 445 (1990).
[6] For a review see e.g. J. Zinn-Justin, Phys. Rept. 70, 109 (1981).
[7] E.N. Argyres, R.H.P. Kleiss and C.G. Papadopoulos, Nucl. Phys. B391, 42 (1993)
[8] M.B. Voloshin, Nucl. Phys. B383, 233 (1992).
[9] L.S. Brown, Phys. Rev. D 46, 4125 (1992).
[10] M.B. Voloshin, Phys. Rev. D 47, R357 (1993).
[11] B.H. Smith, Phys. Rev. D 47, (1993).
[12] M.B. Voloshin, Nucl. Phys. B363, 425 (1991).
[13] N.A. Voronov and I. Yu. Kobzarev, JETP Lett. 24, 532 (1976);
I.L. Bogolyubskii and V.G. Makhan’kov, JETP Lett. 24, 12 (1976);
T.I. Belova, N.A. Voronov, I.Yu. Kobzarev and N.B. Konyukhova, Sov. Phys. JETP 46, 846 (1977).

[14] M.B. Voloshin, Minnesota preprint TPI-MINN-92/61-T, November 1992. To be published in Phys. Rev. D.

[15] L.D. Landau, Phys. Zs. Sowiet. 1, 88 (1932)
L.D. Landau and E.M. Lifshits, Quantum Mechanics, Non-Relativistic Theory, Third edition. Pergamon Press, 1977; section 52.

[16] M.B. Voloshin, I.Yu. Kobzarev and L.B. Okun, Sov. J. Nucl. Phys. 20, 644 (1975)

Figure captions

Figure 1. A configuration of the field, corresponding to the inter-phase boundary bent to the maximal deviation $h_0$. The evolution of the field between the point at a negative time $t$ (heavy dot) and the boundary proceeds over the time $|t| - h_0 + \delta$, where $\delta$ is of the order of the thickness of the domain wall. The bold vertical lines at the edges are the world lines of the boundaries of the spatial bounding box.

Figure 2. The world line of the bubble walls in a complexified space-time. The bubble evolves in the Euclidean space-time at $r > r_0$ and in the Minkowski one, when $r < r_0$.

Figure 3. The Euclidean field configuration for calculating the square of the matrix element $\langle B|\phi|0 \rangle$ by the Landau-WKB formula. At large negative time the field evolves by the classical solution with zero energy, corresponding to a flat domain wall (lower horizontal line). Then it matches on the configuration with a large energy $E$, corresponding to a bubble, which contracts down to the turning radius $r_0$. Beyond the turning point the configuration is symmetrically reflected in time, hence it represents the square of the matrix element. The plus and minus signs indicate the phases, corresponding to the field approaching $+v$ or $-v$. 

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