Lyapunov-Razumikhin techniques for state-dependent delay differential equations

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Abstract

We present Lyapunov stability and asymptotic stability theorems for steady state solutions of general state-dependent delay differential equations (DDEs) using Lyapunov-Razumikhin methods. Our results apply to DDEs with multiple discrete state-dependent delays, which may be nonautonomous for the Lyapunov stability result, but autonomous (or periodically forced) for the asymptotic stability result. Our main technique is to replace the DDE by a nonautonomous ordinary differential equation (ODE) where the delayed terms become source terms in the ODE. The asymptotic stability result and its proof are entirely new, and based on a contradiction argument together with the Arzelá-Ascoli theorem. This approach alleviates the need to construct auxiliary functions to ensure the asymptotic contraction, which is a feature of all other Lyapunov-Razumikhin asymptotic stability results of which we are aware.

We apply our results to a state-dependent model equation which includes Hayes equation as a special case, to directly establish asymptotic stability in parts of the stability domain along with lower bounds on the size of the basin of attraction.

Keywords: delay differential equations, asymptotic stability, Lyapunov-Razumikhin theorem

1. Introduction

We consider the following general delay differential equation (DDE) in $d$ dimensions with $N$ discrete state-dependent delays,

\begin{equation}
\begin{aligned}
\dot{u}(t) &= f(t, u(t), u(t - \tau_1(t, u(t))), \ldots, u(t - \tau_N(t, u(t)))), \quad t \geq t_0, \\
\quad u(t) &= \varphi(t), \quad t \leq t_0,
\end{aligned}
\end{equation}

and prove Lyapunov stability and asymptotic stability results using Lyapunov-Razumikhin techniques. We apply our results to the model state-dependent DDE

\begin{equation}
\begin{aligned}
\dot{u}(t) &= \mu u(t) + \sigma u(t - a - cu(t)), \quad t \geq 0, \\
\quad u(t) &= \varphi(t), \quad t \leq 0,
\end{aligned}
\end{equation}

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with $a > 0$, as an example of (1.1), to directly show asymptotic stability in parts of the stability domain, and derive bounds on the basin of attraction.

Differential equations with state-dependent delays arise in many applications including milling [21], control theory [41], haematopoiesis [6] and economics [29]. There is a well-established theory for retarded functional differential equations (RFDEs) as infinite-dimensional dynamical systems on function spaces [15, 7, 10], which encompasses problems with constant or prescribed delay, but very little of this theory is directly applicable to state-dependent delay problems. Extending the theory to state-dependent DDEs, including equations of the form (1.1) is the subject of ongoing study. See [16] for further examples and a review of recent progress.

The model state-dependent DDE (1.2) includes the constant delay DDE often known as Hayes equation [17] as a special case when $c = 0$. Hayes equation is a standard model problem used to illustrate stability theory for constant delay DDEs in most texts on the subject including [15, 39, 20], as well as being a standard numerical analysis test problem [2]. Hayes equation is also used to illustrate Lyapunov-Razumikhin stability results in [1, 15, 37].

The state-dependent DDE (1.2) was introduced by Mallet-Paret and Nussbaum, and is a natural generalisation of Hayes equation to a state-dependent DDE with a single delay which is linearly state-dependent. But whereas Hayes equation is linear, the DDE (1.2) is nonlinear and can admit bounded periodic solutions. Mallet-Paret and Nussbaum investigate the existence and form of the slowly oscillating periodic solutions of a singularly perturbed version of (1.2) in detail in [35] and use it as an illustrative example for more general problems in [32, 33, 34]. This DDE is also studied in [3, 18, 19, 24, 31].

Following [15], to define an RFDE let $\mathbb{R}^d$ be the $d$-dimensional linear vector space over the real numbers equipped with the Euclidean inner product $\cdot$ and Euclidean norm $|\cdot|$. Let $r \geq 0$ and $C = C([-r, 0], \mathbb{R}^d)$ be the Banach space of continuous functions mapping $[-r, 0]$ to $\mathbb{R}^d$ with the supremum norm denoted $\|\cdot\|$. If $u \in C([t_0 - r, t_f], \mathbb{R}^d)$ then for every $t \in [t_0, t_f]$ define $u_t \in C$ by

$$u_t(\theta) = u(t + \theta), \quad \theta \in [-r, 0].$$  \hspace{1cm} (1.3)

Then for $F : \mathbb{R} \times C \to \mathbb{R}^d$, with the dot denoting a right-derivative, an RFDE is defined by

$$\dot{u}(t) = F(t, u_t), \quad u_{t_0} = \varphi \in C.$$  \hspace{1cm} (1.4)

A solution of the RFDE (1.4) is a function $u \in C([t_0 - r, t_f], \mathbb{R}^d)$ which satisfies (1.4) for $t \in [t_0 - r, t_f)$. If $F$ is Lipschitz on $\mathbb{R} \times C$ then existence and uniqueness of solutions is assured [15], and the RFDE (1.4) defines a dynamical system with the function space $C$ as its phase space.

If a constant function $\varphi \in C$ yields $F(t, \varphi) = 0$ for all $t$ then $\varphi$ is a steady state of the RFDE. To study the stability of steady states of RFDEs, the method of Lyapunov functions for ODEs was first extended to Lyapunov functionals $\bar{V} : \mathbb{R} \times C \to \mathbb{R}$ for RFDEs by Krasovskii [25]. Lyapunov theorems for stability of RFDEs require the time derivative of the functional along a solution of (1.4) to be nonpositive or strictly negative, similar to the theorems for ODEs using Lyapunov functions [15, 25]. However, finding functionals $\bar{V} : \mathbb{R} \times C \to \mathbb{R}$ with this property for RFDEs is much harder than in the ODE case where the Lyapunov functions have the form $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$.

Razumikhin [38] developed the theory on how one might go from using the more difficult Lyapunov functionals for RFDEs back to Lyapunov functions again. His fundamental idea is that it is only necessary to require a constraint on the derivative of $V$ whenever the solution is about to exit a ball centered at the steady state. Following this approach, Barnea [1] presents a Lyapunov stability theorem for RFDEs and also considers Hayes equation. A comprehensive discussion of Lyapunov functionals and functions for general RFDEs is presented by Hale and
Verduyn Lunel in chapter 5 of [15]. Other works with Razumikhin-type results include [14, 22, 23, 25, 26, 27, 37, 43]. Of these [27, 37, 43] include results tailored for time-dependent delays.

State-dependent DDEs of the form (1.1) can easily be written as RFDEs. For example, with

\[ F(t, \varphi) = \mu \varphi(0) + \sigma \varphi(-a - c \varphi(0)), \]

the model problem (1.2) is an RFDE of the form (1.4). However, two problems arise. Firstly, we need an \textit{a priori} bound \( r \) on the delays in (1.1) to consider it as an RFDE. This can be overcome for specific problems such as (1.2) for which bounds can be derived under certain parameter conditions. The second difficulty is more fundamental; it is well known that state-dependency of the delays results in \( F \) not being Lipschitz on \( \mathbb{R} \times C \) [16]. This is easily shown directly for (1.5).

The Lyapunov-Razumikhin results of Barnea [1] only establish Lyapunov stability and assume that \( F \) is Lipschitz and so are not applicable to state-dependent DDEs of the form (1.1) when they are written as RFDEs. Other authors, such as Hale and Verduyn Lunel [15], make the weaker assumptions on \( F \), but use auxiliary functions to establish Lyapunov stability and uniform asymptotic stability. The construction of these functions is nontrivial in all but the simplest examples, and we have not seen such functions constructed for a state-dependent problem. Rather than try to circumvent these problems for RFDEs, in Section 2 we will develop new proofs of Lyapunov stability and asymptotic stability for the state-dependent DDE (1.1) with \( N \) discrete delays.

In Section 2 we present our main Lyapunov-Razumikhin stability results. In Assumption 2.1 we state the assumptions that we make on the nonautonomous DDE (1.1) with \( N \) (state-dependent) delays, the main ones being that \( f \) is locally Lipschitz with respect to its arguments in \( \mathbb{R}^d \) and the delays are locally bounded near to the steady state. Then in Theorem 2.5 we provide sufficient conditions for Lyapunov stability of a steady state of the DDE. The main idea behind the proof is the conversion of the DDE into an auxiliary ODE problem where the delayed terms are regarded as source terms. In Theorem 2.7 we establish asymptotic stability of the steady state when the DDE (1.1) is autonomous. For simplicity of exposition we present the proof for the case of a single delay, but the result remains true for multiple delays or periodically forced problems. This result is significantly different to previous Lyapunov-Razumikhin asymptotic stability results which require auxiliary functions to establish uniform asymptotic stability. In contrast, Theorem 2.7 does not require the construction of any auxiliary functions, and is proved using the auxiliary ODE by a contradiction argument, which shows there cannot exist a solution which is not asymptotic to the steady state.

Theorems 2.5 and 2.7 establish Lyapunov stability and asymptotic stability when the solutions of the auxiliary ODE have certain properties, but to determine those properties exactly would require the solutions of the DDE. So, in Section 2, we also show how to define a family of ODE problems that are subject to constraints defined by bounds on the DDE solution and its derivatives, which can be determined without solving the DDE. Lemma 2.3 establishes bounds on the growth of solutions to (1.1) which is used to ensure solutions remain bounded for sufficiently long (\( k \) times the largest delay for some integer \( k \)) to acquire \( k \) bounded derivatives. Stability is then established from the solution properties of this family of constrained ODE problems.

In Section 3 we review the stability region of the model equation (1.2) which is known [12] to be the same for the state-dependent \( (c \neq 0) \) and constant delay cases \( (c = 0) \). We also consider the properties of the auxiliary ODE we define for this problem, and define sets and functions that are required in the following sections to apply our Lyapunov-Razumikhin results.

In Section 4 we apply Theorem 2.7 to provide a proof of asymptotic stability of the steady state of model equation (1.2) in subsets of the known stability region, together with lower bounds on
the size of the basin of attraction. This result is given as Theorem 4.7. Since the delayed inputs to the auxiliary ODE have \( k - 1 \) bounded derivatives, for the \( k = 2 \) and \( k = 3 \) results we establish suitable bounds on the first and second derivatives of the ODE source terms, while the \( k = 1 \) result, does not require any differentiability of these terms.

The expressions for the stability regions derived in Sections 4 all involve a term that needs to be maximized over a closed interval (see the definitions of the \( P(\delta, c, k) \) functions in Definition 3.5). For \( k = 1 \) this maximum is readily evaluated, while for \( k = 2 \) an expression for the maximum is established in Theorem 4.2 (whose proof is given in Appendix A). Plots and measurements of the derived asymptotic stability regions in \((\mu, \sigma)\) parameter space are given in Section 5, where it is seen that these regions grow with the integer \( k \), but do not appear to fill out the entire stability region in the case \( \mu \neq 0 \). In Section 5, we also briefly review previous work on the \( c = 0 \) constant delay case of (1.2) with \( \mu = 0 \) and \( \mu \neq 0 \) and point out an error in the results of Barnea [1].

In Section 6 we present two examples of solutions which are not asymptotic to the steady state of the model equation (1.2) when \( \mu \geq 0 > \sigma \), and which hence give upper bounds on the radius of the largest ball contained in the basin of attraction of the steady state. We compare these with the lower bounds on the basin of attraction given by Theorem 4.7 for \( k = 1, 2 \) and 3. In Section 7 we present brief conclusions, and compare and contrast our approach with linearization.

2. Lyapunov-Razumikhin techniques for state-dependent DDEs

Here we state and prove our main theorems to establish the Lyapunov stability and asymptotic stability of steady state solutions to state-dependent DDEs of the form (1.1).

We will consider continuous initial functions \( \varphi \in C \), where for the state-dependent DDE (1.1) we let \( C = C([-r(\delta), 0], \mathbb{R}^d) \), with \( r(\delta) \) defined by (2.1) below. By a solution of (1.1) we mean a function \( u \in C^1([t_0, t_f], \mathbb{R}^d) \) which satisfies (1.1) for \( t \in [t_0, t_f] \) for \( t_f > t_0 \) and \( u(t_0) = \varphi(t_0) \). We denote by \( B(0, \delta) \) the closed ball centred at zero with radius \( \delta \) in \( \mathbb{R}^d \). We will assume that (1.1) has a steady state at \( u = 0 \) and make the following assumptions throughout.

**Assumption 2.1.** Let \( d, N \) and \( k \in \mathbb{Z}, d \geq 1, N \geq 1, k \geq 1 \) and \( t_0 \in \mathbb{R} \).

1. \( f: \mathbb{R} \times \mathbb{R}^{(N+1)d} \to \mathbb{R}^d \) and \( \tau_i: [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) for \( i = 1, \ldots, N \) are continuous functions of their variables.
2. \( f(t, 0, 0, \ldots, 0) = 0 \) for all \( t \geq t_0 \).
3. \( \tau_i(t_0) \geq 0 \) for all \( t \geq t_0 \) and \( i = 1, \ldots, N \) and the constant \( \tau_{\text{max}} = \max_{i=1,\ldots,N} \sup_{t\geq t_0} \tau_i(t, 0) \) satisfies \( \tau_{\text{max}} \in (0, \infty) \).
4. There exist positive constants \( L_0, L_1, \ldots, L_N \) and \( \delta_0 \) such that

\[
|f(t, u, v_1, \ldots, v_N) - f(t, \tilde{u}, \tilde{v}_1, \ldots, \tilde{v}_N)| \leq L_0|u - \tilde{u}| + L_1|v_1 - \tilde{v}_1| + \cdots + L_N|v_N - \tilde{v}_N|
\]

for all \( t \geq t_0 \) and \( u, v_1, \ldots, v_N, \tilde{u}, \tilde{v}_1, \ldots, \tilde{v}_N \in B(0, \delta_0) \subset \mathbb{R}^d \). Let \( L = L_0 + L_1 + \cdots + L_N \).
5. The delay terms \( \tau_i(t, u) \) are nonnegative and Lipschitz continuous in \( u \) on \([t_0, \infty) \times B(0, \delta_0)\) with Lipschitz constants \( L_{\tau_i} \).
6. \( f(t, u, v_1, \ldots, v_N) \) is at least \( \max\{k - 2, 0\} \) times differentiable in its \( u \) and \( v \) variables, and \( \tau_i(t, u) \) is at least \( \max\{k - 1, 0\} \) times differentiable in \( u \).

**Definition 2.2.** For any \( \delta \in (0, \delta_0) \) define

\[
r(\delta) = \sup_{t \geq t_0, |u| \leq \delta} \frac{\tau_i(t, u)}{i = 1, \ldots, N}.
\]
Assumption 2.1, items 3 and 5 imply that \( 0 < \tau_{\text{max}} \leq r(\delta) \leq \tau_{\text{max}} + \delta \max_{i} L_{\tau_{i}} < \infty \) with \( \lim_{\delta \to 0} r(\delta) = \tau_{\text{max}} > 0 \). In contrast, the bound on the delay terms used in RFDE theory by Hale and Verduyn Lunel [15] is a global bound on \( \tau_{i}(t, u) \) over all \( t \) and \( u \).

The Lipschitz continuity conditions in Assumption 2.1, items 4–5 are based on those of Driver [8]. From the results in [8], these conditions ensure local existence and uniqueness of a solution to (1.1) close to the steady state solution for all sufficiently small Lipschitz continuous initial functions \( \varphi \), and existence of a solution for all sufficiently small continuous initial functions \( \varphi \).

We will prove Lyapunov stability and asymptotic stability by contradiction, by showing that there is no solution of (1.1) that does not have the required stability property. Thus we do not need to make assumptions specifically to ensure existence or uniqueness of solutions. Rather, we make sufficient assumptions so solutions that do exist have the properties we require.

Since with state-dependent delays it can be hard ensure a priori that the delays \( \tau_{i}(t, u(t)) \) are strictly positive along solutions, and problems with vanishing delays can be interesting, we do not assume that delays \( \tau_{i}(t, u) \) are strictly positive in Assumption 2.1. This makes our results more widely applicable, allowing us to deal with state-dependent DDEs without having to first impose or prove that the delays do not vanish along solutions. The possibility of vanishing delays will on the other hand complicate some of our proofs, starting with Lemma 2.3 below. Similar results to Lemma 2.3 are well known for time-varying and nonvanishing delays and proved using a Gronwall lemma and the method of steps; however the method of steps is not applicable with vanishing delays, and we will need a more technical proof.

The following lemma provides bounds on the growth of solutions to (1.1). For initial functions \( \varphi \in C \) we will use Part II of Lemma 2.3 to ensure that solutions \( u(t) \) remain bounded for a sufficiently long time interval so that \( u(t) \) acquires the differentiability that we require. Part I of Lemma 2.3 will be essential in the proof of asymptotic stability in Theorem 2.7.

Lemma 2.3. Suppose that Assumption 2.1 is satisfied.

I. Let \( T > t_{0} \) and \( \varepsilon > 0 \). Suppose that \( u(t) \) and \( \tilde{u}(t) \) are solutions to (1.1) with initial functions \( \varphi \) and \( \tilde{\varphi} \in C \) respectively, where both \( |\varphi(t)| \leq \delta_{0} \) and \( |\tilde{\varphi}(t)| \leq \delta_{0} \) for \( t \leq t_{0} \), and \( |u(t)| \leq \delta_{0} \) and \( |\tilde{u}(t)| \leq \delta_{0} \) for all \( t \in [t_{0}, T] \). If also \( |\varphi(t) - \tilde{\varphi}(t)| \leq \epsilon \) for \( t \leq t_{0} \) then

\[
|u(t) - \tilde{u}(t)| \leq \varepsilon e^{Lt - \delta_{0}} \quad \forall t \in [t_{0}, T].
\]

II. Let \( T > t_{0} \) and \( \delta \in (0, \delta_{0}] \). Let \( |\varphi(t)| \leq \delta e^{L(t_{0} - T)} \) for \( t \leq t_{0} \). Then any solution of (1.1) defined for \( t \in [t_{0}, T] \) satisfies

\[
|u(t)| \leq \delta e^{Lt - \delta_{0}} \leq \delta \quad \forall t \in [t_{0}, T].
\]

Proof: This proof is similar to the derivation given in Halanay [13] for RFDEs of the form (1.4), and which Halanay credits to Krasovskii. We prove part I first. For \( \bar{L} > L \) suppose there exists \( t_{*} \in [t_{0}, T] \) such that Part I holds with \( L \) replaced by \( \bar{L} \) only up to \( t = t_{*} \). That is

\[
|u(t) - \tilde{u}(t)| \leq \varepsilon e^{\bar{L}t - \delta_{0}}
\]

for all \( t \in [t_{0}, t_{*}] \), but that for every \( \tilde{t} > t_{*} \) there exists \( t \in (t_{*}, \tilde{t}) \) such that \( |u(t) - \tilde{u}(t)| > \varepsilon e^{\bar{L}(t - t_{0})} \). The simplest situation would be where for some \( h > 0 \) we have \( |u(t) - \tilde{u}(t)| > \varepsilon e^{\bar{L}(t - t_{0})} \) for \( t \in (t_{*}, t_{*} + h) \), but the existence of such an interval is neither guaranteed by the conditions of the lemma, nor needed in this proof. First we note that

\[
|u(t) - \tilde{u}(t)| \frac{d}{dt}|u(t) - \tilde{u}(t)| = \frac{1}{2} \frac{d}{dt}|u(t) - \tilde{u}(t)|^{2} = |u(t) - \tilde{u}(t)| \cdot |\dot{u}(t) - \dot{\tilde{u}}(t)| \leq |u(t) - \tilde{u}(t)| |\dot{u}(t) - \dot{\tilde{u}}(t)|.
\]
Since \(|u(t_{s}) - \tilde{u}(t_{s})| = \varepsilon e^{L_{s}(t_{s} - t_{0})} > 0\), dividing by the common factor leads to

\[
|\dot{u}(t_{s}) - \tilde{\dot{u}}(t_{s})| \geq \frac{d}{dt}|u(t) - \tilde{u}(t)| \big|_{t=t_{s}}.
\] (2.2)

Since the solutions to (1.1) have right derivatives for \(t \geq t_{0}\), we can also compute

\[
\frac{d}{dt}|u(t) - \tilde{u}(t)| \big|_{t=t_{s}} = \lim_{t \to t_{s}^{-}} \frac{|u(t) - \tilde{u}(t)| - |u(t_{s}) - \tilde{u}(t_{s})|}{t - t_{s}} \geq \lim_{t \to t_{s}^{-}} \left( \frac{\tilde{e}^{L_{s}(t_{s} - t_{0})} - e^{L_{s}(t_{s} - t_{0})}}{t - t_{s}} \right) \varepsilon = \tilde{L} \varepsilon e^{L(t_{s} - t_{0})}.
\]

Since \(t_{s} - \tau(t_{s}, u(t_{s})) \leq t_{s}\), it follows that \(|u(t_{s} - \tau(t_{s}, u(t_{s}))) - \tilde{u}(t_{s} - \tau(t_{s}, \tilde{u}(t_{s})))| \leq \varepsilon e^{L(t_{s} - t_{0})}\). But, using Assumption 2.1, item 4 we see that

\[
|\dot{u}(t_{s}) - \tilde{\dot{u}}(t_{s})| = \left| \int f(t_{s}, u(t_{s}), u(t_{s} - \tau_{1}(t_{s}, u(t_{s}))), \ldots, u(t_{s} - \tau_{N}(t_{s}, u(t_{s})))) \right|
\]

\[
- f(t_{s}, \tilde{u}(t_{s}), \tilde{u}(t_{s} - \tau_{1}(t_{s}, \tilde{u}(t_{s}))), \ldots, \tilde{u}(t_{s} - \tau_{N}(t_{s}, \tilde{u}(t_{s})))) |)
\]

\[
\leq L_{0}|u(t_{s}) - \tilde{u}(t_{s})| + L_{1}|u(t_{s} - \tau_{1}(t_{s}, u(t_{s}))) - \tilde{u}(t_{s} - \tau_{1}(t_{s}, \tilde{u}(t_{s})))| + \ldots + L_{N}|u(t_{s} - \tau_{N}(t_{s}, u(t_{s}))) - \tilde{u}(t_{s} - \tau_{N}(t_{s}, \tilde{u}(t_{s})))|
\]

\[
\leq (L_{0} + L_{1} + \ldots + L_{N}) \varepsilon e^{L(t_{s} - t_{0})} = \tilde{L} \varepsilon e^{L(t_{s} - t_{0})} < \tilde{L} \varepsilon e^{L(t_{s} - t_{0})}.
\]

Thus \(|\dot{u}(t_{s}) - \tilde{\dot{u}}(t_{s})| < \frac{d}{dt}|u(t) - \tilde{u}(t)| \big|_{t=t_{s}}\), but this contradicts (2.2), hence there does not exist such a \(t_{s}\), and so \(|u(t) - \tilde{u}(t)| \leq \varepsilon e^{L(t - t_{0})}\) for all \(t \in [t_{0}, T]\). Since this holds for all \(\tilde{L} > L\), part I follows.

We prove part II by contradiction. If part II is false then for some initial function \(\varphi(t) \leq \delta e^{L(t_{0} - T)}\) for \(t \leq t_{0}\) there exists \(T^{*} \in (t_{0}, T)\) and a solution \(u(t)\) such that \(|u(T^{*})| > \delta e^{L(T^{*} - T)}\) but \(|u(t)| \leq \delta_{0}\) for all \(t \in [t_{0}, T^{*}]\). Letting \(\tilde{\varphi}(t) = 0\) for all \(t < 0\) from Assumption 2.1, item 2 we also have the solution \(\tilde{u}(t) = 0\) for all \(t \geq t_{0}\). Now applying part I on the interval \([t_{0}, T^{*}]\) with these initial conditions and \(\varepsilon = \delta e^{L(t_{0} - T)}\) we find that

\[
|u(T^{*})| = |u(T^{*}) - \tilde{u}(T^{*})| \leq \delta e^{L(t_{0} - T)} e^{L(T^{*} - t_{0})} = \delta e^{L(T^{*} - T)},
\]

which contradicts \(|u(T^{*})| > \delta e^{L(T^{*} - T)}\) and so part II of the lemma holds.

The growth bounds on the solution in Lemma 2.3 are valid even in the case of vanishing delays. If \(\tau_{i}(t, u)\) were bounded away from zero, a Gronwall argument could be used to obtain a tighter bound, but we allow for the possibility of vanishing delays in the state-dependent case.

In this section we will develop a constructive technique for showing Lyapunov stability and asymptotic stability based on Lyapunov-Razumikhin ideas. To establish Lyapunov stability in Theorem 2.5 we will show that solutions remain in a closed ball \(B(0, \delta)\) of radius \(\delta\) about the steady state. We are thus implicitly using the Lyapunov function \(V(u) = |u|^{2}/2\), though it will not appear directly in our results. Asymptotic stability is established in Theorem 2.7, by showing that all solutions that remain in \(B(0, \delta)\) must converge to the steady state. The application of these results to the model problem (1.2) is demonstrated in Sections 3–4.

We will prove Lyapunov stability (and later asymptotic stability) by contradiction. Suppose the DDE (1.1) has a solution which escapes the ball \(B(0, \delta)\) at time \(t\), then \(u(t) = x\) with \(|x| = \delta\). Now letting \(v(\theta) = u(t + \theta)\) and

\[
\eta_{i}(\theta) = u(t + \theta - \tau_{i}(t + \theta, u(t + \theta))), \quad \theta \in [-r(\delta), 0], \quad i = 1, \ldots, N
\]

for this solution \(u(t)\), we can rewrite the DDE (1.1) as a nonautonomous ODE

\[
\dot{v}(\theta) = f(t + \theta, v(\theta), \eta_{1}(\theta), \ldots, \eta_{N}(\theta))
\]
where
\[ v(-\tau_i(t, x)) = u(t - \tau_i(t, x)) = \eta_i(0), \quad i = 1, \ldots, N. \]

Hence for each \( i \) such that \( \tau_i(t, x) > 0 \) the escaping solution of the DDE (1.1) corresponds a solution of an ODE boundary value problem (BVP) with \( v(-\tau_i(t, x)) = \eta_i(0) \) and \( v(0) = x \). To establish stability it is sufficient to show that the ODE BVP does not have any such solutions. We will use Lemma 2.3 part II to ensure that \( |u(t)| \leq \delta \) for \( t \leq t_0 + kr(\delta) \) so that the forcing functions \( \eta_i(\theta) \) acquire the regularity we require. Hence we define the set of forcing functions \( \eta_i \) which could correspond to an escaping trajectory as follows.

**Definition 2.4.** Suppose that Assumption 2.1 is satisfied for (1.1) and \( k \geq 1 \). Let \( \delta \in (0, \delta_0] \), \( |x| = \delta \) and \( t \geq t_0 + kr(\delta) \). Define the set,
\[ E_{(k)}(\delta, x, t) = \left\{ (\eta_1, \ldots, \eta_N) : \eta_i \in C^{k-1}([-r(\delta), 0], B(0, \delta)), \quad \text{and conditions 1 and 2 are satisfied.} \right\}. \] (2.3)

1. \( x \cdot f(t, x, \eta_1(0), \ldots, \eta_N(0)) \geq 0 \).
2. For some initial function \( \varphi \in C \), equation (1.1) has solution \( u(t) \) such that \( \eta_i(\theta) = u(t + \theta - \tau_i(t, u(t + \theta))) \) for \( \theta \in [-r(\delta), 0] \) for each \( i \in [1, \ldots, N] \).

Condition (1) in the definition is equivalent to \( \frac{d}{dt}|u(t)| \geq 0 \), a necessary condition for the solution to escape the ball \( B(0, \delta) \) at time \( t \). In the following theorem we prove the first of our main results, that if certain conditions hold for all functions in the sets \( E_{(k)}(\delta, x, t) \) then the steady state of (2.9) is Lyapunov stable. The condition (2.5) implies that the ODE BVP discussed above has no solutions, while (2.6) allows solutions with \( \frac{d}{dt}|v(\theta)|_{\theta=0} \leq 0 \).

The sets \( E_{(k)}(\delta, x, t) \) however cannot be determined without solving (1.1), so it is not practical to actually solve for them. Instead, the conditions of the theorems can be shown to hold for larger sets that contain \( E_{(k)}(\delta, x, t) \). We prove the theorems first and then consider such larger sets.

**Theorem 2.5.** Suppose that Assumption 2.1 is satisfied for (1.1). For \( \delta \in (0, \delta_0] \), \( x \in \mathbb{R}^d \), \( |x| = \delta \), define \( E_{(k)}(\delta, x, t) \) as in Definition 2.4. Consider the family of auxiliary ODE problems,
\[ \left\{ \begin{array}{l}
\psi(\theta) = f(t + \theta, v(\theta), \eta_1(\theta), \ldots, \eta_N(\theta)), \quad \theta \in [-\tau_i(t, x), 0], \\
v(-\tau_i(t, x)) = \eta_i(0) \in \mathbb{R}^d,
\end{array} \right. \] (2.4)
for \( i = 1, \ldots, N \) and \( t \geq t_0 + kr(\delta) \). We denote the solution of (2.4) by \( v(x, \eta_1, \ldots, \eta_N)(\theta) \) if we want to emphasize the dependence on \( x \) and \( \eta_i \), or just \( v(\theta) \) otherwise. Suppose there exists \( \delta_1 \in (0, \delta_0] \) such that for all \( \delta \in (0, \delta_1) \), and for every \( x \) such that \( |x| = \delta \) and \( t \geq t_0 + kr(\delta) \), for all \( (\eta_1, \ldots, \eta_N) \in E_{(k)}(\delta, x, t) \) the solution of (2.4) for some \( I \in [1, \ldots, N] \) satisfies \( \tau_I(t, x) > 0 \) and either
\[ \frac{1}{\delta} v(x, \eta_1, \ldots, \eta_N)(0) \cdot x < \delta, \] (2.5)
or
\[ \frac{1}{\delta} v(x, \eta_1, \ldots, \eta_N)(0) \cdot x = \delta, \quad \text{and} \quad v(x, \eta_1, \ldots, \eta_N)(0) \cdot x \leq 0, \] (2.6)
then the zero solution to (1.1) is Lyapunov stable. Moreover if \( \delta \in (0, \delta_1) \) and \( |\varphi(s)| < \delta e^{-Lkr(\delta)} \) for \( s \in [t_0 - r(\delta), t_0] \) then the solution of (1.1) satisfies \( |u(t)| \leq \delta \) for all \( t \geq t_0 \).

**Proof.** Let the hypothesis of the theorem hold and let \( \delta \in (0, \delta_1) \). For \( \lambda \in (0, 1) \) suppose \( |\varphi(s)| \leq \lambda \delta e^{-Lkr(\delta)} \) for \( s \in [t_0 - r(\delta), t_0] \). By Lemma 2.3 part II, the solution of (1.1) satisfies \( u(t) \in B(0, \lambda \delta) \) for all \( t \leq t_0 + kr(\delta) \). We will prove that \( u(t) \in B(0, \delta) \) for all \( t \geq t_0 \) by contradiction.
Assume that a solution escapes the ball \(B(0, \delta)\) for the first time at some time \(t^* > t_0 + kr(\delta)\), then there exists \(\delta \in (\delta, \delta_1)\) and \(\bar{t} \in (t^*, \infty)\) such that \(|u(\bar{t})| = \delta \geq \delta\). But \(|u(t^*)| = \delta\) so by the continuity of \(u(t)\), for each \(\delta \in (\delta, \delta_1)\) there exists \(\bar{t} \in (t^*, \bar{t})\) such that \(|u(\bar{t})| = \delta\).

It follows from Assumption 2.1 and equation (2.1) that \(r(\delta)\) is a continuous function of \(\hat{\delta}\) and hence \(\delta e^{-Lkr(\delta)}\) is a continuous function of \(\hat{\delta}\) as well. Thus we can choose \(\delta \in (\delta, \delta_1)\) sufficiently small so that \(\lambda \delta e^{-Lkr(\delta)} < \delta e^{-Lkr(\delta)}\) for all \(\hat{\delta} \in (\delta, \delta_1)\).

For such a \(\delta\), noting that \(u(t) \in C^{2,1} \cup C^1\) for \(t \geq t^*\), and \(|u(\bar{t})| > |u(t^*)| = \delta\), there must exist \(t^{**} \in (t^*, \bar{t})\) such that \(\frac{d}{dt}|u(t^{**})| > 0\) and \(|u(t^{**})| = \delta^{**} = \sup_{t \in [t^*, \bar{t}]} |u(t)| \in (\delta, \Delta)\). This solution escapes the ball \(B(0, \delta^{**})\) for the first time at \(t^{**} > t^* > t_0 + kr(\delta)\). Let \(x = u(t^{**})\) so \(|x| = \delta^{**}\) and

\[
0 < \frac{1}{2} \frac{d}{dt}|u(t^{**})|^2 = u(t^{**}) \cdot \dot{u}(t^{**}),
\]

\[
= u(t^{**}) \cdot f(t^{**}, u(t^{**}), u(t^{**} - \tau_1(t^{**}, u(t^{**}))), \ldots, u(t^{**} - \tau_N(t^{**}, u(t^{**}))))
\]

\[
= x \cdot f(t^{**}, x, u(t^{**} - \tau_1(t^{**}, u(t^{**}))), \ldots, u(t^{**} - \tau_N(t^{**}, u(t^{**}))))
\]

(2.7)

Since \(|x(s)| \leq \lambda \delta e^{-Lkr(\delta)} < \delta^{**} e^{-Lkr(\delta^{**})}\), it follows from Lemma 2.3 part II that \(t^{**} \geq t_0 + kr(\delta^{**})\).

Now consider the auxiliary ODE problem (2.4) for \(i \in \{1, \ldots, N\}\) such that either (2.5) or (2.6) holds. Let \(v(\theta) = u(t^{**} + \theta)\), and noting that \([-\tau_i(t^{**}, x), 0] \subseteq [-r(\delta^{**}), 0]\), let \(\eta_i(\theta) = u(t^{**} + \theta - \tau_i(t^{**} + \theta, u(t^{**} + \theta)))\) for \(\theta \in [-r(\delta^{**}), 0]\). Then

\[
t^{**} + \theta - \tau_i(t^{**} + \theta, u(t^{**} + \theta)) \geq t_0 + kr(\delta^{**}) - 2r(\delta^{**}) = t_0 + (k-2)r(\delta^{**}), \quad \text{for } \theta \in [-r(\delta^{**}), 0].
\]

It follows that \(\eta_i \in C^{k-1}([-r(\delta^{**}), 0], B(0, \delta^{**}))\) for \(i = 1, \ldots, N\). From (2.7) we deduce that

\[
x \cdot f(t^{**}, x, \eta_1(0), \ldots, \eta_N(0)) > 0.
\]

(2.8)

Furthermore, we can consider

\[
\eta_i(\theta) = U(t_0 + kr(\delta^{**}) + \theta - \tau_i(t_0 + kr(\delta^{**}) + \theta, U(t_0 + kr(\delta^{**}) + \theta)), \quad \text{for } \theta \in [-r(\delta^{**}), 0],
\]

where \(U\) is the solution to (1.1) with initial function \(\Phi\) defined as

\[
\Phi(s) = \begin{cases} 
\varphi(s - kr(\delta^{**}) + t^{**}), & s \leq t_0 + kr(\delta^{**}) - t^{**}, \\
u(s - kr(\delta^{**}) + t^{**}), & s \in [t_0 + kr(\delta^{**}) - t^{**}, t_0].
\end{cases}
\]
Thus, \( (\eta_1, \ldots, \eta_N) \in E_{(k)}(\delta^{**}, x, t^{**}) \). Moreover,

\[
\begin{align*}
\dot{v}(\theta) &= \dot{u}(t^{**} + \theta), \\
&= f(t^{**} + \theta, u(t^{**} + \theta), u(t^{**} + \theta - \tau_1(t^{**} + \theta, u(t^{**} + \theta))), \ldots, u(t^{**} + \theta - \tau_N(t^{**} + \theta, u(t^{**} + \theta))) \\
&= f(t^{**} + \theta, v(\theta), \eta_1(\theta), \ldots, \eta_N(\theta)).
\end{align*}
\]

Also,

\[
v(-\tau_j(t^{**}, x)) = u(t^{**} - \tau_j(t^{**}, x)) = u(t^{**} - \tau_j(t^{**}, u(t^{**}))) = \eta_j(0).
\]

Thus \( v \) is a solution to the ODE system (2.4) with \( i = I \) and \( (\eta_1, \ldots, \eta_N) \in E_{(k)}(\delta^{**}, x, t^{**}) \). But \( v(0) = u(t^{**}) = x \) with \( |x| = \delta^{**} \) so \( \frac{1}{\delta^{**}} v(x, \eta_1, \ldots, \eta_N)(0) \cdot x = \delta^{**} \) which contradicts (2.5).

Moreover, equation (2.8) implies that \( v(x, \eta_1, \ldots, \eta_N)(0) \cdot x > 0 \) which contradicts (2.6). Thus any solution which escapes the ball \( B(0, \delta^{**}) \) violates the conditions of the theorem, so \( u(t) \in B(0, \delta^{**}) \subseteq B(0, \hat{\delta}) \) for all \( t \geq t_0 \). But this is true for all \( \hat{\delta} \in (\delta, \hat{\delta}) \) and hence if \( |\varphi(s)| \leq \lambda \delta e^{-Lkr(\delta)} \) then \( |u(t_0, \varphi)(t)| \leq \delta \) for all \( t \geq t_0 \), establishing Lyapunov stability. Since this holds for all \( \lambda \in (0, 1) \) the result follows.

The proof of Theorem 2.5 is complicated by the auxiliary ODE (2.4) being nonautonomous. The solution of a nonautonomous ODE escaping the ball \( B(0, \delta) \) for the first time at \( t^* \) neither implies that \( \frac{d}{dt}(u(t^*)) > 0 \) nor that there exists \( t^{**} > t^* \) such that \( |u(t)| > \delta \) for all \( t \in (t^*, t^{**}) \). As an illustration of this, consider the function \( y(t) = \delta + t^3 \sin(2\pi t) \) which is easily seen to be continuously differentiable and crosses \( \delta \) at \( t = 0 \) with \( y'(0) = 0 \), and for which there does not exist any \( \varepsilon > 0 \) such that \( y(t) > \delta \) for all \( t \in (0, \varepsilon) \).

Our second main result is to show asymptotic stability of the steady state \( u = 0 \) if the auxiliary ODE (2.4) satisfies the strict inequality (2.6). We will do this for autonomous DDEs, and for simplicity of notation we only present the derivation for problems with one delay term (\( N = 1 \)). The extension to multiple delays is straightforward, and we discuss the extension to periodically forced nonautonomous DDEs after Theorem 2.7. Hence we consider autonomous DDEs of the form (1.1) with \( N = 1 \) for which \( f(t, u, v) = f(0, u, v) \) and \( \tau_1(t, u) = \tau(0, u) \) for all \( t \). In this case we may set \( t_0 = 0 \) and rewrite (1.1) as

\[
\begin{align*}
\begin{cases}
\dot{u}(t) = f(0, u(t), u(t - \tau(0, u(t)))), & t \geq 0, \\
\dot{u}(t) = \varphi(t) \in C, & t \leq 0,
\end{cases}
\end{align*}
\]

where \( C = C([-r(\delta), 0], \mathbb{R}^d) \). By Assumption 2.1 item 2 the DDE (2.9) has the trivial steady state solution \( u = 0 \). We write the solution to (2.9) as \( u(\varphi)(t) \) when we want to emphasize initial conditions, or just \( u(t) \) otherwise. For equation (2.9) the auxiliary ODE introduced in (2.4) becomes

\[
\begin{align*}
\begin{cases}
\dot{v}(\theta) &= f(0, v(\theta), \eta(\theta)), & \theta \in [-\tau(0, x), 0], \\
v(-\tau(0, x)) &= \eta(0),
\end{cases}
\end{align*}
\]

and since we consider a single delay, there is one such auxiliary ODE associated with (2.9). We write the solution of (2.10) as \( v(x, \eta)(\theta) \) if we want to emphasize the dependence on \( x \) and \( \eta \), or just \( v(\theta) \) otherwise. The sets \( E_{(k)}(\delta, x, t) \) defined in (2.3) are no longer dependent on \( t \) for an autonomous DDE, and so we denote them by \( E_{(k)}(\delta, x) \) which for (2.9) is defined as follows.

**Definition 2.6.** Suppose that Assumption 2.1 is satisfied for (2.9) and \( k \geq 1 \). Let \( \delta \in (0, \delta_0] \) and \( |x| = \delta \). Define the set

\[
E_{(k)}(\delta, x) = \left\{ \eta \in C^{k-1}([-\tau(0, x), 0], B(0, \delta)) \mid \text{such that } x \cdot f(0, x, \eta(0)) \geq 0, \right. \\
\left. \text{and for some initial function } \varphi \in C \text{ the solution } u(t) \text{ of (2.9) satisfies } \eta(\theta) = u(\kappa r(\delta) + \theta - \tau(0, u(\kappa r(\delta) + \theta))) \text{ for } \theta \in [-\tau(0, x), 0] \right\}.
\]
To show asymptotic stability of the zero solution to (2.9) it is sufficient to strengthen the conditions of Theorem 2.5 by requiring that solutions of the auxiliary ODE problem satisfy the strict inequality (2.5) and not the weaker condition (2.6). For the DDE (2.9) the condition (2.5) becomes
\[
\frac{1}{\delta}v(x, \eta)(0) \cdot x < \delta, \quad (2.12)
\]
and we now show asymptotic stability when all solutions of the auxiliary ODE satisfy (2.12).

**Theorem 2.7** (Asymptotic stability). Suppose that Assumption 2.1 is satisfied for (2.9). For \( \delta \in (0, \delta_0] \), \( x \in \mathbb{R}^d \), \( |x| = \delta \), define \( E_{(k)}(\delta, x) \) as in Definition 2.6. If there exists \( \delta_1 \in (0, \delta_0] \) such that for all \( \delta \in (0, \delta_1) \), and for every \( x \) such that \( |x| = \delta \), for all \( \eta \in E_{(k)}(\delta, x) \) the solution \( v(x, \eta)(\theta) \) of the auxiliary ODE problem (2.10) satisfies (2.12) then the results of Theorem 2.5 hold and moreover, the zero solution of (2.9) is asymptotically stable. Furthermore, if \( |\varphi(s)| < \delta_1 e^{-Lkr(\delta_1)} \) for \( s \in [-r(\delta_1), 0] \) then \( u(t) \to 0 \) as \( t \to \infty \).

**Proof.** The only differences between the conditions of Theorem 2.5 and Theorem 2.7 is that Theorem 2.7 allows a finite number of delays and nonautonomous \( f \) and requires the solution of the auxiliary ODE problem (2.4) to satisfy (2.5) or (2.6), while Theorem 2.7 assumes autonomous \( f \), one delay, and requires that the strict inequality (2.12) hold. Thus it trivially follows that the requirements of Theorem 2.5 are satisfied, and the results of Theorem 2.5 hold.

Let \( \delta \in (0, \delta_1), r = r(\delta) \) and \( |\varphi(s)| \leq \delta e^{-Lkr} \) for \( s \in [-r, 0] \). Then by Theorem 2.5 we have \( |u(t)| \leq \delta \) for all \( t \geq 0 \). Consider such a solution. Since \( |u(t)| \leq \delta \) for all \( t \geq 0 \) we have \( \limsup_{t \to \infty} |u(t)| = \delta_\infty \) with \( \delta_\infty \in [0, \delta] \), and it remains only to show that \( \delta_\infty = 0 \).

Since \( \limsup_{t \to \infty} |u(t)| = \delta_\infty \) and \( \{u : |u| = \delta_\infty\} \) is compact in \( \mathbb{R}^d \) there exists \( t_i \) such that \( \lim_{i \to \infty} t_i = \infty \) and \( \lim_{i \to \infty} u(t_i) = x_\infty \) with \( |x_\infty| = \delta_\infty \). Assume without loss of generality that \( t_i \geq (k + 1)r \) for all \( i \).

Since \( |u(t)| \leq \delta \) for all \( t \geq 0 \), and \( |\varphi(t)| \leq \delta e^{-Lkr} \leq \delta \) for \( t \leq 0 \) it follows from Assumption 2.1 items 2 and 4 that \( |\frac{d}{dt} u(t)| \leq (L_0 + L_1)\delta \) for all \( t \geq 0 \).

Now, consider the sequence of functions \( v_i(\theta) = u(t_i + \theta) \) for \( \theta \in [-k(1 + r), 0] \). These functions and their derivatives are uniformly bounded with \( |v_i| \leq \delta \) and \( |\frac{d}{dt} v_i| \leq (L_0 + L_1)\delta \). The set of all \( C^1 \) functions satisfying these bounds forms a uniformly bounded and equicontinuous closed family of functions defined on compact set \( [-k(1 + r), 0] \). By the Arzelà–Ascoli theorem the sequence of functions \( v_i(\theta) \) has a uniformly convergent subsequence. Let \( \{v_i\} \) denote this subsequence and let \( v(\theta) \) be the limiting function, which has \( v(0) = x_\infty \). Note that \( |v(\theta)| \leq \delta_\infty \) for all \( \theta \in [-k(1 + r), 0] \), since the existence of a point with \( |v(\theta)| > \delta_\infty \) would contradict that \( \limsup_{t \to \infty} |u(t)| = \delta_\infty \).

Let \( \varphi(\theta) = \varphi(-k(1 + r) + \theta) \) for \( \theta \in [-r, 0] \) then we claim that the solution of (2.9) with initial function \( \varphi^* \) is \( \frac{d}{dt} v(t) = v(t - kr) \) for \( t \in [0, kr] \). To see that this is true, let \( \sup_{t \in [0, kr]} |u^*(t) - v(t - kr)| = \varepsilon \). Now let \( u_0(t) \) solve (2.9) with corresponding initial functions \( \varphi_0(\theta) = v(-k(1 + r) + \theta) \) for \( \theta \in [-r, 0] \), so \( u_0(t) = v(t - kr) \) for \( t \in [0, kr] \). For all \( i \) sufficiently large we have \( \sup_{t \in [0, kr]} |u_i(t) - v(t - kr)| = \sup_{t \in [0, kr]} |v_i(t - kr) - v(t - kr)| \leq \frac{1}{2} \varepsilon \) by the uniform convergence of the \( v_i \) to \( v \). But also by the uniform convergence for all \( i \) sufficiently large we have \( |\varphi_i(\theta) - \varphi^*| = |v_i(-k(1 + r) + \theta) - v(-k(1 + r) + \theta)| \leq \frac{1}{2} \varepsilon e^{-Lkr} \) for all \( \theta \in [-r, 0] \), and hence by Lemma 2.3 part I we have \( \sup_{t \in [0, kr]} |u_0(t) - u^*(t)| = \sup_{t \in [0, kr]} |v_i(t - kr) - u^*(t)| \leq \frac{1}{2} \varepsilon \). But now
\[
\varepsilon = \sup_{t \in [0, kr]} |u^*(t) - v(t - kr)| \leq \sup_{t \in [0, kr]} |v_i(t - kr) - u^*(t)| + \sup_{t \in [0, kr]} |v_i(t - kr) - v(t - kr)| = \frac{\varepsilon}{2},
\]
which can only be true if $\varepsilon = 0$ so the solution of (2.9) with $\varphi^*(\theta) = v(-kr + \theta)$ for $\theta \in [-r, 0]$ is indeed $u^*(t) = v(t - kr)$ for $t \in [0, kr]$.

Now let $\eta(\theta) = v(\theta - \tau(0, v(\theta)))$ for $\theta \in [-\tau(0, x_\infty), 0]$ which implies that $\eta(\theta) = u^*(kr + \theta - \tau(0, u^*(kr + \theta)))$. Moreover $|v(\theta)| \leq \delta_\infty$ for all $\theta \in [-(k + 1)r, 0]$ implies that $|\eta(\theta)| \leq \delta_\infty$ for $\theta \in [-\tau(0, x_\infty), 0]$ and hence $\eta \in C^{k-1}([-\tau(0, x_\infty), 0], B(0, \delta_\infty))$. To show that $\eta \in E_{(k)}(\delta_\infty, x_\infty)$ it remains only to show that $x_\infty \cdot f(0, x_\infty, \eta(0)) \geq 0$. But if this is false then

$$0 > x_\infty \cdot f(0, x_\infty, \eta(0)) = v(0) \cdot f(0, v(0), \eta(0)) = u^*(kr) \cdot f(0, u^*(kr), u^*(kr - \tau(0, u^*(kr)))) = u^*(kr) \cdot u^*(kr) = \frac{1}{2} \frac{d^2}{dt^2} |u^*(kr)|.$$ 

But, $|u^*(kr)| = \delta_\infty$ and $\frac{d^2}{dt^2} |u^*(kr)| < 0$ implies that there exists $\varepsilon > 0$ such that $|u^*(t)| > \delta_\infty$ for $t \in (kr - \varepsilon, kr)$, or equivalently $|v(t)| > \delta_\infty$ for $t \in (-\varepsilon, 0)$. But this contradicts $|v(\theta)| \leq \delta_\infty$ for all $\theta \in [-(k + 1)r, 0]$, so we must have $x_\infty \cdot f(0, x_\infty, \eta(0)) \geq 0$ and $\eta \in E_{(k)}(\delta_\infty, x_\infty)$.

Now $v(0) = x_\infty$ implies $v(0) \cdot x_\infty = \delta_\infty^2$. But unless $\delta_\infty = 0$ this contradicts that (2.12) holds for all $\delta \in (0, \delta_1)$. The result follows.

Notice that Theorem 2.7 not only establishes asymptotic stability of the steady state, but also shows that the basin of attraction of the steady state contains the ball

$$\{ \varphi : \|\varphi\| < \delta_1 e^{-tLkr(\delta_1)} \}. \quad (2.13)$$

We will consider the basin of attraction of the steady state of the model problem (1.2) in Section 6.

The extension of Theorem 2.7 to multiple delays is straightforward. The proof given above would not be valid for nonautonomous DDEs. However the proof would only fail in one crucial step; for a general nonautonomous DDE (1.1), the limiting function $v(t)$ would not in general define a solution of the DDE. The result is easily extended to periodically nonautonomous DDEs by choosing the initial sequence $t_i$ to be $t_i = (k + 1)r + iT$ where $T$ is the period of the nonautonomous function $f$, and if necessary taking a subsequence so that $u(t_i)$ converges to $x_\infty$.

Our asymptotic stability result and its proof differs very significantly from other asymptotic stability results for RFDEs which are all similar to Theorem 4.2 of Hale and Verduyn Lunel [15]. Beyond the technical differences in continuity assumptions, and whether delays are locally or globally bounded, there are two fundamental but related differences between our result and results such as those in [15]. Firstly, in Theorem 2.7 we establish asymptotic stability, but in Theorem 4.2 of [15] the stronger property of uniform asymptotic stability is obtained. But secondly, auxiliary functions with specific properties are required (in Theorem 4.2 of [15] four auxiliary functions, $u$, $v$, $\omega$ and $p$ appear) to obtain the contraction that leads to the uniform asymptotic stability. Construction of such functions is difficult even for constant delay DDEs, and a major obstacle to the application of these theorems. In contrast, we use a proof by contradiction which shows that there does not exist a trajectory which is not asymptotic to the steady state. The contradiction argument establishes asymptotic stability rather than uniform asymptotic stability, but does not require any troublesome auxiliary functions, and thus is much easier to apply. In the following sections we will use Theorem 2.7 to study the asymptotic stability of the steady state of the model state-dependent DDE (1.2).

We next define the larger sets containing $E_{(k)}(\delta, x)$ in which we will later show that conditions of Theorem 2.7 hold to establish asymptotic stability for the model problem (1.2). By items 4–6 in Assumption 2.1, if a bound on $u(t)$ is given for $t \in [-r(\delta), (k - 1)r(\delta)]$ we can also find bounds on up to the $k - 1$ order derivatives of $u(t - \tau(0, u(t)))$ for $t \in [(k - 1)r(\delta), kr(\delta)]$. These bounds can
be derived from the bounds on $f$, $\tau$ and their derivatives. Recalling the definition of $E_{(k)}(\delta, x)$ in Definition 2.6 this leads us to the following definition.

**Definition 2.8.** Suppose that Assumption 2.1 is satisfied for (2.9) and $k \geq 1$. Let $\delta \in (0, \delta_0]$ while $|x| = \delta$. Let the functions $D_j(\delta)$ be Lipschitz continuous in $\delta$ for $j = 0, \ldots, k-1$ and satisfy

$$D_j(\delta) \geq \sup_{t \in [\delta - 1, \delta]} \left| \frac{d}{dt} u(t - \tau(t, u(t))) \right|. \tag{2.14}$$

given that $|u(t)| \leq \delta$ for all $t \in [-r(\delta), kr(\delta)]$, where $u(t)$ is a solution to (2.9). Define the set

$$E_{(k)}(\delta, x) = \left\{ \eta : \eta \in PC^{k-1}([-\tau(0, x), 0], B(0, \delta)), \right. \left. x \cdot f(0, x, \eta(0)) \geq 0, \right. \left. \left| \frac{d}{dt} \eta(\theta) \right| \leq D_j(\delta) \text{ for } \theta \in [-\tau(0, x), 0], j = 0, \ldots, k-1 \right\} \tag{2.15}$$

where $PC^{k-1}([-\tau(0, x), 0], B(0, \delta))$ denotes the space of $C^{k-2}$ functions which are piecewise $C^{k-1}$.

Clearly, $E_{(k)}(\delta, x) \subseteq E_{(k)}(\delta, x)$. It is convenient to consider piecewise $C^{k-1}$ functions in Definition 2.8 because we will later seek the supremum of an integral over the set $E_{(k)}(\delta, x)$. Even if all the functions in $E_{(k)}(\delta, x)$ were $C^{k-1}$, in general the maximiser could still be piecewise $C^{k-1}$.

In Section 4 we derive bounds $D_j(\delta)$ for the model problem (1.2), and use these to identify parameter regions for which all $\eta \in E_{(k)}(\delta, x)$ satisfy (2.12), and hence the steady state of (1.2) is asymptotically stable by Theorem 2.7. For $\delta \in (0, \delta_0]$, $x \in \mathbb{R}^d$ and $|x| = \delta$, it is useful to define

$$G(\delta, x) = \sup_{\eta \in E_{(k)}(\delta, x)} \frac{1}{\delta} v(x, \eta)(0) \cdot x, \quad F(\delta) = \sup_{|x| = \delta} G(\delta, x), \tag{2.16}$$

where $v(x, \eta)$ is the solution to (2.10). Notice that for $\delta \in (0, \delta_0]$ and $|x| = \delta$ we have

$$\sup_{\eta \in E_{(k)}(\delta, x)} \frac{1}{\delta} v(x, \eta)(0) \cdot x \leq \sup_{\eta \in E_{(k)}(\delta, x)} \frac{1}{\delta} v(x, \eta)(0) \cdot x = G(\delta, x) \leq \sup_{|x| = \delta} G(\delta, x) = F(\delta). \tag{2.17}$$

Thus if $F(\delta) < \delta$ for all $\delta \in (0, \delta_1)$ then (2.12) holds for all $\delta \in (0, \delta_1)$ and Theorems 2.5 and 2.7 can be applied. Although $F(\delta) < \delta$ is a somewhat stronger condition than (2.12) we will find it convenient to work with when considering the model problem (1.2).

The set $E_{(k)}(\delta, x, t)$ given by (2.3) for the DDE (1.1) can be easily generalised to a larger set $E_{(k)}(\delta, x, t)$, in a similar manner. For $t \geq t_0 + kr(\delta)$ we let

$$E_{(k)}(\delta, x, t) = \left\{ (\eta_1, \ldots, \eta_N) : \eta_i \in PC^{k-1}([-r(\delta), 0], B(0, \delta)), \right. \left. x \cdot f(t, x, \eta_1(0), \ldots, \eta_N(0)) \geq 0, \right. \left. \left| \frac{d}{dt} \eta_i(\theta) \right| \leq D_{ij}(\delta, t) \text{ for } \theta \in [-r(\delta), 0], i = 1, \ldots, N, j = 0, \ldots, k-1 \right\} \tag{2.18}$$

where for all solutions $u$ to (1.1) which satisfy $|u(s)| \leq \delta$ for $s \in [t - (k + 1)r(\delta), t]$,

$$D_{ij}(\delta, t) \geq \sup_{s \in [t - r(\delta), t]} \left| \frac{d}{dt} u(s - \tau_i(s, u(s))) \right|. \tag{2.19}$$

It follows that $E_{(k)}(\delta, x, t) \subseteq E_{(k)}(\delta, x, t)$, and hence establishing properties on the set $E_{(k)}(\delta, x, t)$ is sufficient to apply Theorem 2.5. However, we will consider the autonomous model problem (1.2) in the following sections, and so will not need to consider $E_{(k)}(\delta, x, t)$ or $E_{(k)}(\delta, x, t)$ further.
3. Model Equation Properties

In the following sections we will apply the Lyapunov-Razumikhin theory of Section 2 to the model state-dependent DDE given in (1.2). In this section we consider the properties of the DDE (1.2) and its auxiliary ODE (2.10), and will define the sets and functions that we will use to apply our results to this model problem. We begin by considering boundedness and, existence and uniqueness of solutions of the DDE (1.2) with \( \mu + \sigma < 0 \), which generalise the results of Mallet-Paret and Nussbaum in [35] for \( \sigma < \mu < 0 \).

**Lemma 3.1.** With \( c \neq 0 \), let \( \mu + \sigma < 0 < a \) and suppose \( u \in C^1([0, \infty), \mathbb{R}) \) solves (1.2) for \( t \geq 0 \) with \( c\phi(0) \geq -a \) then \( t - a - cu(t) < t \) for all \( t > 0 \).

**Proof.** The model DDE (1.2) is invariant under the transformation \( u \mapsto -u, \; c \mapsto -c \), so we consider only the case \( c > 0 \). Suppose \( \phi(0) > -a/c \) and let \( t^* > 0 \) be the first time for which \( u(t^*) = -a/c \). Then \( u(t) > -a/c \) for \( t < t^* \) implies \( \dot{u}(t^*) \leq 0 \), but from (1.2) with \( u(t^*) = -a/c \) we have \( \dot{u}(t^*) = (\mu + \sigma)u(t^*) = -\frac{\mu}{c}(\mu + \sigma) > 0 \), supplying the required contradiction. If \( \phi(0) = -a/c \) then \( \dot{u}(0) > 0 \) and the result follows similarly.

We will always consider the DDE (1.2) with \( a > 0 \) and \( \mu + \sigma < 0 \), then Lemma 3.1 assures that the deviating argument is always a delay. The lemma also gives the lower bound \( u(t) > -a/c \) on solutions when \( c > 0 \) (or an upper bound on solutions when \( c < 0 \)). When \( \mu < 0 \) we can bound solutions above and below. It is convenient to define

\[
M_0 = \frac{-a}{c}, \quad N_0 = \frac{a\sigma}{c\mu}, \quad \tau = a + cN, \quad \tau_0 = a + cN_0.
\]

(3.1)

We will use \([M_0, N_0]\) and also \([M, N]\) as bounds on solutions of the single delay DDE (1.2) (in contrast to the multiple delay DDE (1.1) for which we used \( N \) to denote the number of delays).

**Lemma 3.2.** Let \( \mu + \sigma < 0 < a \) and suppose \( u \in C^1([0, \infty), \mathbb{R}) \) solves (1.2) for \( t \geq 0 \). If \( \sigma > 0 \) let sign\((c)M \in \text{sign}(c)[M_0, 0) \) and sign\((c)N > 0 \), and suppose that sign\((c)\phi(t) \in \text{sign}(c)[M, N] \) for all \( t \in [-\tau, 0) \). If \( \sigma \leq 0 \) let \( M = M_0 \) and \( N = \max\{N_0, \phi(0)\} \) and suppose sign\((c)\phi(t) \geq \text{sign}(c)M_0 \) for all \( t \in [-\tau, 0) \). Then

\[
\text{sign}(c)u(t) \in \text{sign}(c)(M, N), \quad \forall t > 0.
\]

(3.2)

**Proof.** Again, we consider the \( c > 0 \) case, then it is sufficient to show that \( \dot{u}(t) > 0 \) if \( u(t) = M \), and \( \dot{u}(t) < 0 \) if \( u(t) = N \) given that \( u(s) \in (M, N) \) for \( s \in (0, t) \). The case where \( u(t) = M_0 \) is dealt with in the proof of Lemma 3.1, the other cases are straightforward.

**Theorem 3.3.** Let \( \mu + \sigma < 0 < a \). Let the initial history function \( \phi \) be continuous and for \( \mu < 0 \) satisfy the bounds given in Lemma 3.2. For \( \mu \geq 0 \) let sign\((c)\phi(t) \geq \text{sign}(c)M_0 \) for all \( t \in (-\infty, 0) \). Then there exists at least one solution \( u \in C^1([0, \infty), \mathbb{R}) \) which solves (1.2) for all \( t \geq 0 \). If \( \mu < 0 \) any solution satisfies the bounds (3.2), while if \( \mu \geq 0 \) any solution satisfies sign\((c)u(t) \geq \text{sign}(c)M_0 \) for all \( t \geq 0 \). If \( \phi \) is locally Lipschitz the solution is unique.

**Proof.** Local existence and uniqueness follows directly from the results of Driver [8], and for \( \mu < 0 \) global existence and uniqueness follows from the extended existence result of Driver [8] using the bounds on the delay and solution given by Lemma 3.1 and 3.2. The only delicate case
Figure 2: The analytic stability region $\Sigma_*$ in the $(\mu, \sigma)$-plane, divided into the delay-independent cone $\Sigma_\Delta$, and the delay-dependent wedge $\Sigma_w$ and cusp $\Sigma_c$.

is for $-\sigma > \mu > 0$ for which (considering the case $c > 0$) Lemma 3.1 gives only a lower bound, $u(t) \geq M_0$. But then $\dot{u}(t) \leq \mu u(t) + \sigma M_0$ and the Gronwall lemma implies that

$$u(t) \leq (\varphi(0) + \frac{\sigma}{\mu} M_0)e^{\mu t} - \frac{\sigma}{\mu} M_0 = (\varphi(0) - N_0)e^{\mu t} + N_0.$$  \hspace{1cm} (3.3)

Since $\varphi(0) \geq M_0 > N_0$ in this case, solutions cannot become unbounded in finite time, and global existence again follows. For this case $\varphi(t)$ should be defined for all $t \leq 0$ since with the exponentially growing bound (3.3) on $u(t)$ it is possible that $t - a - cu(t) \to -\infty$ as $t \to +\infty$. $\square$

As already mentioned in the introduction, the constant delay DDE known as Hayes equation, which corresponds to (1.2) with $c = 0$ has been much studied. The $(\mu, \sigma)$ values for which its steady state is asymptotically stable when $a > 0$ and $c = 0$ are well known (see eg. [15]) and given in Definition 3.4.

**Definition 3.4 (Stability region $\Sigma_*$).** Let $a > 0$ and $c = 0$. Let $\Sigma_*$ be the open set of the $(\mu, \sigma)$-parameter space between the curves

$$\ell_* = \{(s, -s) : s \in (-\infty, 1/a]\}, \quad g_* = \{(\mu(s), \sigma(s)) : s \in (0, \pi/a]\}$$

where the functions $\mu(s)$ and $\sigma(s)$ are given by

$$\mu(s) = s \cot(as), \quad \sigma(s) = -s \csc(as).$$ \hspace{1cm} (3.4)

The stability region $\Sigma_*$ is further divided into three subregions: the cone $\Sigma_\Delta = \{ (\mu, \sigma) : |\sigma| < -\mu \}$, the wedge $\Sigma_w = (\Sigma_* \setminus \Sigma_\Delta) \cap \{ \mu < 0 \}$ and the cusp $\Sigma_c = \Sigma_* \cap \{ \mu \geq 0 \}$, which are shown in Figure 2.

$\Sigma_*$ is the parameter region in the $(\mu, \sigma)$-plane for which the zero solution to the DDE (1.2) is locally asymptotically stable in both the constant and state-dependent delay cases. The cone $\Sigma_\Delta$ forms the delay-independent stability region (because this does not change when $a$ is changed) while $\Sigma_w \cup \Sigma_c$ is often referred to as the delay-dependent stability region. For the constant delay
case \((c = 0)\) this region is found from the characteristic equation \([9]\). The results of Györi and Hartung \([12]\) show the state-dependent case \((c \neq 0)\) of \((1.2)\) has the same (exponentially) asymptotic stability region. On the boundary of \(\Sigma_*\) the steady-state is Lyapunov stable for the constant delay case, and the stability is delicate in the state-dependent case \([40]\).

In this paper we derive new proofs of stability in parts of \(\Sigma_*\) for the state-dependent case using Theorem 2.7. The asymptotic stability of the zero solution to \((1.2)\) in all of the delay-independent region \(((\mu, \sigma) \in \Sigma_\Lambda)\) will be shown in Theorem 4.1. In Theorem 4.7 we will also show asymptotic stability of the steady state of the model problem \((1.2)\) for \((\mu, \sigma)\) in subsets of \(\Sigma_w \cup \Sigma_c\), by applying Theorem 2.7 with \(k = 1\) to 3. Here we define some notation that will be required. Let \((\mu, \sigma) \in \Sigma_w \cup \Sigma_c, k \in \mathbb{Z}, k \geq 1, \delta_0 \in (0,|a/c|)\) and \(\delta \in (0, \delta_0)\). It is easy to see that Assumption 2.1 is satisfied for \((1.2)\) with \(L_0 = |\mu|, L_1 = |\sigma|, \tau_{\text{max}} = a\) and \(r(\delta) = a + |c|\delta\). Thus for the model problem \((1.2)\) the sets \(E_{(\delta)}(\delta, x)\) from Definition 2.6 are given by

\[
E_{(\delta)}(\delta, x) = \left\{ \eta : \eta \in C^{k-1}([-a - cx, 0], [-\delta, \delta]), \mu x^2 + \sigma x\eta(0) \geq 0, \right. \\
\left. \text{and for some initial function } \varphi \in C \right\}
\]

To apply the stability theorems in the next section we will derive bounds \(D_j(\delta)\) for \(j = 0, \ldots, k-1\) and \(\delta \in (0, \delta_0)\) as in Definition 2.8. Once these bounds are determined, the sets \(E_{(\delta)}(\delta, x)\) from Definition 2.8 are given by

\[
E_{(\delta)}(\delta, x) = \left\{ \eta : \eta \in PC^{k-1}([-a - cx, 0], [-\delta, \delta]), \left| \frac{\mu}{dx} \eta(\theta) \right| \leq D_j(\delta) \text{ for } \theta \in [-a - cx, 0], j = 0, \ldots, k-1 \right\}.
\]

Let \(r_+ = a + c\delta\) then the auxiliary ODE problem \((2.10)\) becomes

\[
\left\{ \begin{array}{l}
\dot{v}(\theta) = \mu v(\theta) + \sigma \eta(\theta), \quad \theta \in [-r_+, 0], \\
v(-r_+) = \eta(0).
\end{array} \right.
\]

Integrating \((3.7)\) yields,

\[
v(0) = \eta(0)e^{\mu r_+} + \sigma \int_{-r_+}^{0} e^{-\mu \theta} \eta(\theta) d\theta.
\]

Since the DDE \((1.2)\) is scalar the set of \(x\) such that \(|x| = \delta\) consists of just two points \(x = \delta\) and \(x = -\delta\). Suppose first that \(x = \delta\), then \((3.6)\) implies that \(\eta(0) \in [-\delta, -\delta\mu/\sigma]\).

**Definition 3.5.** Let \(a > 0, c \neq 0, \sigma \leq \mu\) and \(\sigma < -\mu\). For any \(\delta \in (0, |a/c|)\) and \(\hat{u} \in [-\delta, -\delta\mu/\sigma]\), define \(r_+ = a + c\delta\) and \(\eta_{(\delta)}(\theta)\) for \(\theta \in [-r_+, 0]\) by

\[
\eta_{(\delta)}(\theta) = \inf_{\eta \in \hat{E}_{(\delta)}(\delta, \delta)} \eta(\theta).
\]

We also define the function \(I(\hat{u}, \delta, c, k)\) to be

\[
I(\hat{u}, \delta, c, k) = \hat{u}e^{\mu r_+} + \sigma \int_{-r_+}^{0} e^{-\mu \theta} \eta_{(\delta)}(\theta) d\theta.
\]

The function \(\eta_{(\delta)}\) given by \((3.9)\) is the most negative one in \(\hat{E}_{(\delta)}(\delta, \delta)\) satisfying \(\eta(0) = \hat{u}\), and so since \(\sigma < 0\), this function maximizes \(v(0)\) for fixed \(\eta(0)\) by maximising the second term in \((3.8)\). This is the reason for considering \(\eta \in PC^{k-1}([-a - cx, 0], [-\delta, \delta])\) in the definition of \(E_{(\delta)}(\delta, x)\).
We really want to maximise \( v(0) \) for \( \eta \in E_{(k)}(\delta, x) \), where the smaller set \( E_{(k)}(\delta, x) \) is defined in (2.11). Even though all the functions \( \eta \in E_{(k)}(\delta, x) \) satisfy \( \eta \in C^{k-1}([-a - cx, 0], [-\delta, \delta]) \), the maximiser will in general only be piecewise \( C^{k-1} \). With Definition 3.5 we can derive bounds on the solution \( v(0) \) of the auxiliary ODE (3.7) for all \( \eta \in E_{(k)}(\delta, x) \) in both cases where \( x = \pm \delta \).

**Lemma 3.6.** Let \( a > 0, \ c \neq 0, \ \sigma \leq \mu \) and \( \sigma < -\mu \). Let \( \delta \in (0, |a/c|) \). The solution of the auxiliary ODE system (3.7) satisfies

\[
    v(0) \leq \sup_{\hat{u} \in [-\delta, \sigma \delta]} i(\hat{u}, \delta, c, k), \ \forall \eta \in E_{(k)}(\delta, \delta), \ \ v(0) \geq - \sup_{\hat{u} \in [-\delta, \sigma \delta]} i(\hat{u}, \delta, -c, k), \ \forall \eta \in E_{(k)}(\delta, -\delta).
\]

**Proof.** First consider \( x = \delta \). The function \( i(\hat{u}, \delta, c, k) \) comes from (3.8) and depends on \( c \) and \( \delta \) through \( r_* \). Since, as noted above, the choice of \( \eta(k) \) maximizes (3.10) for fixed \( \hat{u} \), the first inequality in the statement of the lemma follows.

Next consider \( x = -\delta \), then (3.6) implies that \( \eta(0) \in [\delta \mu/\sigma, \delta] \). This time we should consider the most positive function in \( E_{(k)}(\delta, -\delta) \) satisfying \( \eta(0) = \hat{u} \in [\delta \mu/\sigma, \delta] \), to obtain a lower bound on \( v(0) \) for all \( \eta \in E_{(k)}(\delta, -\delta) \). However, the model DDE (1.2) is invariant under the transformation \( (u, c) \mapsto (-u, -c) \), so this function is \(-\eta(k)(\theta)\) and the second inequality follows.

Notice from (3.10) that the functions \( i(\hat{u}, \delta, c, k) \) and \( i(\hat{u}, \delta, -c, k) \) only differ in their integration limits with \( i(\hat{u}, \delta, c, k) \) integrating \( \eta(k) \) over the interval \([-a - c\delta, 0]\) and \( i(\hat{u}, \delta, -c, k) \) integrating over \([-a + c\delta, 0]\). The integration over the larger of these intervals will be important in the following sections and so it is convenient to define

\[
    P(\delta, c, k) = \sup_{\hat{u} \in [-\delta, \sigma \delta]} i(\hat{u}, \delta, |c|, k). \tag{3.11}
\]

Comparing the cases when \( x = \delta \) and \( -\delta \) has to be done separately for each value of \( k \), and we can also explicitly define the functions \( \eta(k) \) for each \( k \). This is handled in the following section where we show that \( P(\delta, c, k) < \delta \) implies \( F(\delta) < \delta \), and apply Theorem 2.7 to obtain asymptotic stability for \( (\mu, \sigma) : P(\delta, c, k) < \delta \).

Barnea [1] applied Lyapunov-Razumikhin techniques to the \( c = 0 \) constant delay case of the model DDE (1.2). His results do not apply to state-dependent case, as they were based on a result for autonomous RFDEs which assumed \( F \) was Lipschitz, and he did not define an auxiliary ODE, nor sets similar to \( E_{(k)}(\delta, x) \) or \( E_{(k)}(\delta, x) \). However, he did define functions \( \eta(k) \) for the constant delay case by considering the most negative bounded functions with \( k - 1 \) bounded derivatives as the function segments in the RFDE. In the limit as \( c \to 0 \) our \( \eta(k) \) functions reduce to those found by Barnea for the constant delay case. Because of the linearity of (1.2) with \( c = 0 \), Barnea did not have to consider the upper and lower bounds separately as we did in Lemma 3.6, but did define a function which is equivalent to \( P(\delta, 0, k) \) in (3.11). Our asymptotic stability results for the state-dependent model DDE (1.2) constitute a significant generalisation of the Lyapunov stability results of Barnea [1] for the constant delay case, and moreover in Section 5 we will correct an error of Barnea for the \( k = 2 \) constant delay case.

### 4. Asymptotic stability for \( \dot{u}(t) = \mu u(t) + c u(t - a - c u(t)) \) using \( E_{(k)}(\delta, x) \)

In this section we consider the model state-dependent DDE (1.2) and use Theorem 2.7 to show that the steady state is asymptotically stable in various parameter sets. In Theorem 4.1 we use the set \( E_{(1)}(\delta, x) \) to show that the steady state is asymptotically stable whenever the parameters
values \((\mu, \sigma)\) are in the cone \(\Sigma_\Delta\). In the rest of the section we consider parameters in the wedge and the cusp \((\Sigma_w \cup \Sigma_c)\), and use Theorem 2.7 with \(k = 1, 2\) and 3 to show the steady state is asymptotically stable for \((\mu, \sigma) \in P(1, 0, k < 1)\), where \(P(\delta, c, k)\) is defined in (3.11). The sets \(P(1, 0, k < 1)\) are nested in \(\Sigma_w \cup \Sigma_c\), becoming larger with \(k\). We also find lower bounds on the basin of attraction of the steady state. For the constant delay case \((c = 0)\) the parameter regions found in \(\Sigma_w \cup \Sigma_c\) are independent of the choice of \(\delta\) in \(\mathcal{E}(k)\). For the state-dependent case, these regions change with \(c\) and \(\delta\) (see Figure 4) and converge to the region for the constant delay case as \(\delta \to 0\).

We begin by showing asymptotic stability in the cone \(\Sigma_\Delta\). The following result could also be shown by adapting a stability result for time-dependent delays, such as that of Yorke [43]. Recall that \(M_0\) is defined by (3.1).

**Theorem 4.1** (Asymptotic stability for (1.2) in \(\Sigma_\Delta\)). Let \(a > 0, c \neq 0\) and \(|\sigma| < -\mu\) so \((\mu, \sigma) \in \Sigma_\Delta\). If \(|\varphi(t)| \leq |M_0|\) for \(t \in [-a - c|M_0|, 0]\) then the solution \(u(t)\) to (1.2) satisfies \(u(t) \to 0\) as \(t \to \infty\).

**Proof.** With \(|x| = \delta\) and \(\mu < -|\sigma|\) it is impossible to satisfy \(\mu x^2 + \sigma x \eta(0) \geq 0\) with \(|\eta(0)| \leq \delta\) and so \(\mathcal{E}(1)\) is empty and asymptotic stability of the steady state follows from Theorem 2.7. This holds for all \(\delta \in (0, |M_0|]\) and it follows directly from Theorem 2.7 that \(u(t) \to 0\) as \(t \to \infty\) for \(\delta \in (0, |M_0|]\) and so \(\mathcal{E}(1)\) is empty and asymptotic stability of the steady state follows from Theorem 2.7. This holds for all \(\delta \in (0, |M_0|]\) and it follows directly from Theorem 2.7 that \(u(t) \to 0\) as \(t \to \infty\) provided \(|\varphi(t)| < |M_0|e^{-L_r(M_0)}\) for \(t \in [-a - c|M_0|, 0]\). However, the exponential correction term \(e^{-L_r(M_0)}\) comes from using Lemma 2.3 in the proof of Theorem 2.7 to ensure that \(|u(t)| < |M_0|\) for \(t \in [0, r(|M_0|)]\). But Lemma 3.2 already ensures that \(|u(t)| < |M_0|\) for all \(t > 0\) if \(|\varphi(t)| \leq |M_0|\) for the model DDE (1.2); the result follows.

We already noted in Section 3 that Assumption 2.1 is satisfied for (1.2), and derived an expression for \(\mathcal{E}(k)\) and indicated which are the most relevant functions in these sets. For \(k = 1\) we do not need any bounds \(D_1(\delta)\) and the members of the set \(\mathcal{E}(1)\) need not be continuous. Then the function \(\eta(1)\) from (3.9) is given by

\[
\eta(1)(\theta) = \begin{cases} 
-\delta, \quad &\theta \in [-a - c\delta, 0), \\
\hat{\theta}, \quad &\theta = 0.
\end{cases}
\]  

(4.1)

For \(k = 2\) we need to find \(D_1(\delta)\) such that \(|\frac{d}{d\theta} \eta(\theta)| \leq D_1(\delta)\) for all \(\eta \in E(2)\), where \(E(2)\) is defined by (3.5). For \(\eta \in E(2)\) we have

\[
\eta(\theta) = u(2r + \theta - a - cu(2r + \theta)) \quad \text{for} \quad \theta \in [-a - cx, 0] \subseteq [-r, 0],
\]

where \(|u(t)| \leq \delta\) for \(t \in [-r, 2r]\) and solves (1.2) for \(r \geq 0\). We easily derive that \(|\hat{u}(t)| \leq |\mu u(t)| + |\sigma u(t - a - cu(t))| \leq (|\mu| + |\sigma|)\delta\) for \(t \in [0, 2r]\). Then

\[
\eta'(\theta) = \frac{d}{d\theta} u(2r + \theta - a - cu(2r + \theta)) = (1 - cu(2r + \theta))\hat{u}(2r + \theta - a - cu(2r + \theta)).
\]

Hence \(|\eta'(\theta)| \leq D_1(\delta)\) for \(\theta \in [-r, 0]\) where

\[
D_1 = (|\mu| + |\sigma|)(1 + (|\mu| + |\sigma|)c|\delta|).
\]  

(4.2)

Thus we can choose \(D_1(\delta) = D_1(\delta)\) (note that \(D_1\) also depends on \(\delta\)) to define \(\mathcal{E}(2)\) and we obtain that \(E(2) \subseteq \mathcal{E}(2)\). The function \(\eta(2)\) is given by (3.9), as

\[
\eta(2)(\theta) = \begin{cases} 
\hat{\theta} + D_1 \delta \theta, \quad &\theta \in [-\frac{\delta + \hat{\theta}}{D_1 \delta}, 0], \\
-\delta, \quad &\theta \in [-a - c\delta, -\frac{\delta + \hat{\theta}}{D_1 \delta}],
\end{cases}
\]  

(4.3)

where \(\delta + \hat{\theta} < r_+ = a + c\delta\).
where $D_1$ is defined by (4.2).

For $\eta \in E_{(3)}(\delta, x) \subseteq E_{(2)}(\delta, x)$, the same bound on the first derivative of $\eta$ applies, and we also bound the second derivative as follows. We have

$$\eta(\theta) = u(3r + \theta - a - cu(3r + \theta)) \text{ for } \theta \in [-a - cx, 0] \subseteq [-r, 0],$$

where $|u(t)| \leq \delta$ for $t \in [-r, 3r]$ and solves (1.2) for $r \geq 0$. As above we have that $|\dot{u}(t)| \leq (|\mu| + |\sigma|)\delta$ for $t \in [0, 3r]$. Now noting that $t - a - cu(t) \in [0, 2r]$ for $t \in [r, 3r]$ we have

$$|\ddot{u}(t)| = |\dot{\mu}(t) + \sigma\dot{u}(t - a - cu(t))(1 - cu(t))| \leq (|\mu| + |\sigma|)(1 + |\sigma|c)\delta,$$

for $t \in [r, 3r]$. Then, since $3r + \theta - a - cu(3r + \theta) \in [r, 2r]$ for $\theta \in [-r, 0]$ it follows that

$$|\eta''(\theta)| = \left|\frac{d^2}{d\theta^2}u(3r + \theta - a - cu(3r + \theta))\right|$$

$$= \left|(1 - c\bar{u}(3r + \theta))\ddot{u}(3r + \theta - a - cu(3r + \theta)) - cu(3r + \theta)\dot{\eta}(3r + \theta - a - cu(3r + \theta))\right|$$

$$\leq (D_1^2 + (|\mu| + |\sigma|)^3|\sigma|\delta)(1 + |\sigma|c)\delta = D_2\delta.$$

Hence for all $\eta \in E_{(3)}(\delta, x)$ we have $|\eta''(\theta)| \leq D_2(\delta) = D_2\delta$ where

$$D_2 = (D_1^2 + (|\mu| + |\sigma|)^3|\sigma|\delta)(1 + |\sigma|c)\delta,$$

and $\lim_{\delta \to 0} D_2 = \lim_{\delta \to 0} (D_1^2 + (|\mu| + |\sigma|)^2) = (|\mu| + |\sigma|)^2$. Taking $D_1$ and $D_2$ to satisfy (4.2) and (4.4) ensures that $E_{(3)}(\delta, x) \subseteq E_{(2)}(\delta, x)$. Then the $\eta(k)$ function from (3.9) for $k = 3$ can be defined by

$$\eta_3(\theta) = \bar{\eta}_3(\theta + \theta_{\text{shift}}), \quad \theta \in [-r_+, 0]$$

where

$$\bar{\eta}_3(\theta) = \begin{cases} -\delta, & \theta \leq 0, \\ -\delta + \frac{\delta}{2}D_2\theta^2, & \theta \in (0, D_2), \\ -\delta - \frac{\delta}{2}D_1^2\delta + \delta D_1\theta, & \theta > D_2. \end{cases}$$

$$\theta_{\text{shift}} = \begin{cases} \frac{2(\hat{u} + \delta)}{D_2\delta}, & \hat{u} \in [-\delta, -\delta + \frac{\delta D_1^2}{2D_2}], \\ \hat{u} + \delta + \frac{\delta D_1^2}{D_2\delta}, & \hat{u} > -\delta + \frac{\delta D_1^2}{2D_2}. \end{cases}$$

Here $\theta_{\text{shift}}$ is a convenient device which allows us to define $\eta_3(\theta)$ for all values of $\hat{u}$ by the single function $\bar{\eta}_3(\theta)$ with the shift used to obtain the correct value of $\hat{u}$.

The $\eta(k)$ functions define $I(\hat{u}, \delta, c, k)$ via equation (3.10) and $P(\delta, c, k)$ through equation (3.11). For $k = 1$, using (4.1) we easily evaluate

$$P(\delta, c, 1) = \int \left(-\mu\delta / \sigma, \delta, |\sigma|, 1\right) = \begin{cases} \frac{-\mu}{\sigma}\delta e^{\mu r} + \delta^\sigma \mu (1 - e^{\mu r}), & \mu \neq 0, \\ -\delta \sigma r, & \mu = 0. \end{cases}$$

For $k = 2$, from (3.10) and (4.3), if $\frac{\delta}{D_2}\delta \geq r_+$ then

$$I(\hat{u}, \delta, c, 2) = \hat{u} e^{\mu r_*} + \sigma \int_{-r_*}^{0} e^{-\mu \theta}(\hat{u} + D_1\delta) d\theta$$

$$= \begin{cases} \hat{u} \left[\sigma \mu (e^{\mu r_*} - 1)\right] + \sigma \mu D_1\delta \left[\frac{1}{\mu}(e^{\mu r_*} - 1) - r_+ e^{\mu r_*}\right], & \mu \neq 0, \\ \hat{u} + \sigma r_+ \hat{u} - \frac{\sigma D_1 r_*^2}{2\delta}, & \mu = 0. \end{cases}$$
while if $\frac{\delta \hat{u}}{D_1 \delta} < r_+$ then we have to split the integral into two parts and

$$I(\hat{u}, \delta, c, 2) = \hat{u} e^{\mu r_+} + \sigma \int_{-r_+}^{-\frac{\delta \hat{u}}{D_1 \delta}} e^{-\mu \theta} (-\delta) d\theta + \sigma \int_0^0 e^{-\mu \theta} (\hat{u} + D_1 \delta \theta) d\theta$$

(4.10)

$$= \left\{ \begin{array}{l}
\hat{u} \left[ e^{\mu r_+} - \frac{\sigma}{\mu} \int_{-r_+}^{-\frac{\delta \hat{u}}{D_1 \delta}} e^{\mu \theta} (-\delta) d\theta + \sigma \int_0^0 e^{\mu \theta} (\hat{u} + D_1 \delta \theta) d\theta \right], \\
\hat{u} - \sigma \delta r_+ + \sigma \frac{\delta}{2 D_1 \delta} (\delta + \hat{u})^2,
\end{array} \right. \mu \neq 0, \mu = 0.\quad (4.11)$$

To determine $P(\delta, c, 2)$ we perform the integration in $I(\hat{u}, \delta, |c|, 2)$ and find $\hat{u}$ to maximise this function. If $\frac{\delta \hat{u}}{D_1 \delta} \geq r$ then $\hat{u} \in [(rD_1 - 1)\delta, -\delta \mu / \sigma]$. This is only possible in the region where $rD_1 - 1 \leq -\mu / \sigma$. From (4.9) we have

$$I(\hat{u}, \delta, |c|, 2) = I_1(\hat{u}, \delta) := \left\{ \begin{array}{l}
\hat{u} \left[ e^{\mu r_+} - \frac{\sigma}{\mu} \int_{-r_+}^{-\frac{\delta \hat{u}}{D_1 \delta}} e^{\mu \theta} (-\delta) d\theta + \sigma \int_0^0 e^{\mu \theta} (\hat{u} + D_1 \delta \theta) d\theta \right], \\
\hat{u} + \sigma \delta r_+ - \frac{\sigma}{2 D_1 \delta} (\delta + \hat{u})^2,
\end{array} \right. \mu \neq 0, \mu = 0.\quad (4.12)$$

If $\frac{\delta \hat{u}}{D_1 \delta} < r$ then $\hat{u}$ has another upper bound $\hat{u} < (rD_1 - 1)\delta$ so $\hat{u} \in [-\delta, \min\{(rD_1 - 1)\delta, -\frac{\mu}{\sigma} \delta\}]$. Since the integration is broken down into two parts in this case we label the expression we derive as $I_2$, and from (4.11) we have

$$I(\hat{u}, \delta, |c|, 2) = I_2(\hat{u}, \delta) := \left\{ \begin{array}{l}
\hat{u} \left[ e^{\mu r_+} - \frac{\sigma}{\mu} \int_{-r_+}^{-\frac{\delta \hat{u}}{D_1 \delta}} e^{\mu \theta} (-\delta) d\theta + \sigma \int_0^0 e^{\mu \theta} (\hat{u} + D_1 \delta \theta) d\theta \right], \\
\hat{u} - \sigma \delta r_+ + \sigma \frac{\delta}{2 D_1 \delta} (\delta + \hat{u})^2,
\end{array} \right. \mu \neq 0, \mu = 0.\quad (4.13)$$

The main differences between the expressions for $I(\hat{u}, \delta, c, 2)$ and $I(\hat{u}, \delta, |c|, 2)$ are that the former involve $r_+ = a + c \delta$, and the latter use $r = a + |c| \delta$ as well as being subject to different restrictions on the values of $\hat{u}$ for which they apply. In (4.9),(4.11),(4.12) and (4.13) the $\mu = 0$ expressions equal the $\mu \to 0$ limit of the $\mu \neq 0$ expressions. Results for $\mu = 0$ thus follow from those for $\mu \neq 0$, and so we do not treat these cases separately below.

**Theorem 4.2.** Let $a > 0, c \neq 0, \sigma \leq \mu$ and $\sigma < -\mu$. Let $\delta \in (0, |a/c|)$. If $P(\delta, c, 2) < \delta$ then

$$P(\delta, c, 2) = \left\{ \begin{array}{l}
I_1(-\delta \mu / \sigma, \delta), \quad \text{if} \quad rD_1 - 1 \leq -\mu / \sigma, \\
I_2(-\delta \mu / \sigma, \delta), \quad \text{if} \quad rD_1 - 1 > -\mu / \sigma,
\end{array} \right.\quad (4.14)$$
where $I_1$ is defined by (4.12) and $I_2$ is defined by (4.13).

**Proof.** See Appendix A.

We will not state an explicit expression for $P(\delta, c, 3)$. When needed, this can determined by evaluating (3.11) numerically for $k = 3$.

We now prove four lemmas which will be needed for the proof of Theorem 4.7 where we show asymptotic stability in the set $\{(\mu, \sigma) : P(1, 0, k) < 1\}$.

**Lemma 4.3.** Let $a > 0$, $c \neq 0$, $\sigma \leq \mu$ and $\sigma < -\mu$. Let $\delta \in (0, |a/c|)$ and $\hat{u} \in [-\delta, -\delta\mu/\sigma]$ be fixed. Then $\mathcal{I}(\hat{u}, \delta, c, 1)$ decreases with decreasing $r_+$.

**Proof.** For $r_+ > 0$,

$$\frac{\partial}{\partial r_+} \left( \hat{u} e^{\mu r_+} + \sigma \int_{-r_+}^{0} e^{-\mu \theta} \eta(\theta) d\theta \right) = e^{\mu r_+} [\mu \hat{u} + \sigma \eta(0)(-r_+)] = e^{\mu r_+} [\mu \hat{u} - \sigma \delta] > 0,$$

since $\sigma < -\mu$ and $\hat{u} \in [-\delta, -\delta\mu/\sigma]$.

**Lemma 4.4.** For $k = 2$ or $3$, let $a > 0$, $c \neq 0$, $\sigma \leq \mu$ and $\sigma < -\mu$. Let $\delta \in (0, |a/c|)$ and $\hat{u} \in [-\delta, -\delta\mu/\sigma]$ be fixed. Let $\mathcal{I}(r_+)$ be the expression for $\mathcal{I}(\hat{u}, \delta, c, k)$ as a function of only $r_+$;

$$\mathcal{I}(r_+) = \hat{u} e^{\mu r_+} + \sigma \int_{-r_+}^{0} e^{-\mu \theta} \eta(\theta) d\theta, \quad \frac{\partial}{\partial r_+} \mathcal{I}(r_+) = e^{\mu r_+} [\mu \hat{u} + \sigma \eta(0)(-r_+)]. \quad (4.15)$$

(A) If $\mu \leq 0$, then $\frac{\partial}{\partial r_+} \mathcal{I}(r_+) > 0$.

(B) If $\mu > 0$ and $\eta(0)(-r_+) \leq 0$, then $\frac{\partial}{\partial r_+} \mathcal{I}(r_+) > 0$.

(C) If $\frac{\partial}{\partial r_+} \mathcal{I}(r_+) \leq 0$, then $\mu > 0$, $\eta(0)(-r_+) > 0$, and $\mathcal{I}(\hat{u}, \delta, c, k) < \delta$.

**Proof.** Parts (A) and (B) are easy to show. Let $\frac{\partial}{\partial r_+} \mathcal{I}(r_+) \leq 0$. From the first two cases, this is only possible if $\mu > 0$ and $\eta(0)(-r_+) > 0$.

Consider $k = 2$ first. Since $\eta(0)(-r_+) \neq -\delta$ we are in the case $\frac{\delta + \hat{u}}{\mu} r_+ > \frac{\delta}{\mu}$, $\mathcal{I}(\hat{u}, \delta, c, 2)$ is given by (4.9). Since $\eta(0)(-r_+) = \hat{u} - D_1 \delta r_+ > 0$ implies $-\frac{\sigma}{\mu} D_1 \delta r_+ < \frac{\sigma}{\mu} \hat{u}$, we deduce

$$\mathcal{I}(\hat{u}, \delta, c, 2) \leq \hat{u} \left[ e^{\mu r_+} + \frac{\sigma}{\mu} (e^{\mu r_+} - 1) \right] + \frac{\sigma D_1}{\mu^2} (e^{\mu r_+} - 1) + \frac{\sigma}{\mu} \hat{u} e^{\mu r_+},$$

$$= -\hat{u} \frac{\sigma}{\mu} + (1 + 2 \frac{\sigma}{\mu}) \hat{u} e^{\mu r_+} + \frac{\sigma D_1}{\mu^2} (e^{\mu r_+} - 1),$$

$$\leq -\hat{u} \frac{\sigma}{\mu} + \frac{\sigma}{\mu} \hat{u} e^{\mu r_+} + \frac{\sigma D_1}{\mu^2} (e^{\mu r_+} - 1), \quad \text{since} \quad \frac{\sigma}{\mu} < -1 \text{ and } \hat{u} \geq \eta(0)(-r_+) > 0,$$

$$= \frac{\sigma}{\mu} (e^{\mu r_+} - 1)(\hat{u} + \frac{\delta}{\mu}) < 0 < \delta.$$
Lemma 4.5. For $k = 1, 2$ or $3$, let $a > 0$, $c \neq 0$, $\sigma \leq \mu$ and $\sigma < -\mu$. Let $\delta \in (0, |a/c|)$. Then
\[
\left\{ (\mu, \sigma) : P(\delta, c, k) = \sup_{\hat{u} \in [-\delta, -\delta \mu/\sigma]} I(\hat{u}, \delta, |c|, k) < \delta \right\} \subseteq \left\{ (\mu, \sigma) : \sup_{\hat{u} \in [-\delta, -\delta \mu/\sigma]} I(\hat{u}, \delta, -|c|, k) < \delta \right\}.
\]

Proof. Recall from Section 3 that changing the sign of $c$ in $I(\hat{u}, \delta, c, k)$ only changes the value of $r_+ = a + c\delta$. For $k = 1$ the result follows from Lemma 4.3.

For $k = 2$ or $3$, let $(\mu, \sigma) \in \{ P(\delta, c, k) < \delta \}$ and $\hat{u} \in [-\delta, -\delta \mu/\sigma]$. Recall that $\sigma < 0$, while $\eta(k)(\theta)$ is a nondecreasing function in $\theta$. There are two cases to consider:

(i) If $\mu \hat{u} + \sigma \eta(k)(-(a - |c|\delta)) \leq 0$ then by Lemma 4.4(C), $I(\hat{u}, \delta, -|c|, k) < \delta$.

(ii) If $\mu \hat{u} + \sigma \eta(k)(-(a + |c|\delta)) \geq \mu \hat{u} + \sigma \eta(k)(-(a - |c|\delta)) > 0$, then $\hat{u} \mu + \sigma \eta(k)(-\tau) > 0$ for all $\tau \in (a - |c|\delta, a + |c|\delta)$. By equation (4.15), the expression for $I(r_+)$ is increasing over this interval and thus, $I(\hat{u}, \delta, -|c|, k) \leq I(\hat{u}, \delta, |c|, k) \leq P(\delta, c, k) < \delta$.

Thus $I(\hat{u}, \delta, -|c|, k) < \delta$ and the result follows. \[\square\]

Lemma 4.6. For $k = 1, 2$ or $3$, let $a > 0$, $c \neq 0$, $\sigma \leq \mu$ and $\sigma < -\mu$. If $0 < \delta_* \leq \delta_{**} < |a/c|$ then
\[
\left\{ (\mu, \sigma) : P(\delta_{**}, c, k) < \delta_{**} \right\} \subseteq \left\{ (\mu, \sigma) : P(\delta_{**}, c, k) < \delta_* \right\}.
\]

Proof. Increasing $\delta$ increases $r = a + |c|\delta$ which is the only source of nonlinearity in $\delta$ in the first expression (4.7) for $P(\delta, c, 1)$. Thus for $\mu \neq 0$
\[
\frac{\partial}{\partial \delta} \left( \frac{P(\delta, c, 1)}{\delta} \right) = -e^{\mu r} \left( \frac{\mu}{\sigma} + \frac{\sigma}{\mu} \right) \frac{\partial}{\partial \delta} (\mu r) = \mu |c| \left( \frac{\mu^2 + \sigma^2}{-\sigma^2} \right) > 0. \tag{4.16}
\]

Positivity also follows trivially from (4.7) when $\mu = 0$. The result follows for $k = 1$.

For $k = 2$ or $3$, consider $I(s\delta, \delta, |c|, k)/\delta$ and note that $r, D_1, \ldots, D_{k-1}$ are the only terms in the expression that depend on $\delta$, and that increasing $\delta$ increases $r$, $D_1$ and $D_2$. Thus
\[
\frac{\partial}{\partial \delta} \left( \frac{I(s\delta, \delta, |c|, k)}{\delta} \right) = \frac{\partial}{\partial r} \left( \frac{I(s\delta, \delta, |c|, k)}{\delta} \right) \left| c \right| + \sum_{j=1}^{k-1} \frac{\partial}{\partial D_j} \left( \frac{I(s\delta, \delta, |c|, k)}{\delta} \right) \frac{\partial D_j}{\partial \delta}.
\]

We focus on the first term on the left-hand side, since all the remaining terms are positive. From (3.10) we can write
\[
\frac{\partial}{\partial r} \left( \frac{I(s\delta, \delta, |c|, k)}{\delta} \right) = e^{\mu r} \left[ \mu s + \frac{\sigma}{\delta} \eta(k)(-r) \right] = e^{\mu r} \left[ \mu s + \frac{\sigma}{\delta} \eta(k)(-(a + |c|\delta)) \right].
\]

Let $r^* = a + |c|\delta_*$, $r^{**} = a + |c|\delta_{**}$ and $(\mu, \sigma) \in \{ P(\delta_{**}, c, k) < \delta_{**} \}$. Let $s \in [-1, -\mu/\sigma]$ and use the notation $\eta(k)(\delta, \theta)$ to denote the function $\eta(k)$ as a function of both $\theta$ and $\delta$. Note that $\eta(k)(\delta, -(a + |c|\delta))/\delta$ is always decreasing with $\delta$. Consider the following cases:

(i) If $\mu s + \sigma \eta(k)(\delta_*, r^*)/\delta \leq 0$ then by Lemma 4.4(C), $I(s\delta_*, \delta_*, |c|, k) < \delta_*$.

(ii) If $\mu s + \sigma \eta(k)(\delta_{**}, r^{**})/\delta_{**} \geq \mu s + \sigma \eta(k)(\delta_*, r^*)/\delta_* > 0$ then $\frac{\partial}{\partial \delta} \left( \frac{I(s\delta\delta_*, k)}{\delta} \right) \geq 0$ for $\delta \in [\delta_*, \delta_{**}]$. Thus, $\frac{\partial}{\partial \delta} \left( \frac{I(s\delta\delta_{**}, k)}{\delta} \right) \geq 0$ for $\delta \in [\delta_*, \delta_{**}]$ and,
\[
\frac{I(s\delta_*, \delta_*, c, k)}{\delta_*} \leq \frac{I(s\delta_{**}, \delta_{**}, c, k)}{\delta_{**}} \leq \frac{P(\delta_{**}, c, k)}{\delta_{**}} < 1.
\]
Establishes asymptotic stability for (3.6) shows that 2.12 completes the proof.

If \( (\mu, \sigma) \in \{P(1,0, k) < 1 \} \) then the zero solution to (1.2) is asymptotically stable by Theorem 4.7. As \( \delta \to 0 \) the proof of Theorem 4.7 shows that \( \{P(\delta, c, k) < \delta \} \) converges to \( \{P(1,0, k) < 1 \} \).

Cases (i) and (ii) both yield \( I(s\delta, \delta, |c|, k) < \delta_s \). Since this holds for all \( s \in [-1, -\mu/\sigma] \), \( P(\delta_s, |c|, k) < \delta_s \) follows.

With these lemmas we can prove our main result.

**Theorem 4.7** (Asymptotic stability for (1.2) using \( E_{\langle k \rangle}(\delta, x) \)). For \( k = 1, 2 \text{ or } 3 \), let \( a > 0 \), \( c \neq 0 \), and \( (\mu, \sigma) \in \{P(1,0, k) < 1 \} \) where \( P(\delta, c, k) \) is defined by (3.11). Then \( (\mu, \sigma) \in \{P(\delta_1, c, k) < \delta_1 \} \) for some \( \delta_1 \in (0, |a/c|) \). Furthermore, for \( \delta \in (0, \delta_1) \) let \( \delta_2 = \delta e^{-k(|\mu| + |\sigma|)(|a| + |\delta|)} \) and \( |\varphi(t)| < \delta_2 \) for all \( t \in [-a - |c|\delta, 0] \), then the solution to (1.2) satisfies \( |u(t)| \leq \delta \) for all \( t \geq 0 \) and \( \lim_{t \to \infty} u(t) = 0 \).

**Proof.** For this proof define

\[
J = \bigcup_{\delta \in (0, |a/c|)} \{P(\delta, c, k) < \delta\}.
\]

First we show that \( J = \{P(1,0,1) < 1 \} \). When \( c = 0 \) it is seen that \( I(s\delta, \delta, 0, k) / \delta \) is independent of \( \delta \) for \( k = 1, 2 \text{ or } 3 \). From this it follows that \( P(\delta, 0, k) / \delta = P(1,0, k) \). Moreover, for all \( c \), when \( \delta \to 0 \) then \( r \to a \), and \( I(s\delta, \delta, |c|, k) / \delta \to I(s\delta, \delta, 0, k) / \delta \), since \( c \) only appears multiplied by \( \delta \) in these expressions. Thus \( P(\delta, c, k) / \delta \to P(1,0, k) \) as \( \delta \to 0 \). Because of this and Lemma 4.6, \( J = \{P(1,0, k) < 1 \} \).

Let \( (\mu, \sigma) \in \{P(1,0, k) < 1 \} \). The existence of \( \delta_1 \) such that \( (\mu, \sigma) \in \{P(\delta_1, c, k) < \delta_1 \} \) follows from the above discussion. It also follows that \( (\mu, \sigma) \in \{P(\delta, c, k) < \delta \} \) for all \( \delta \in (0, \delta_1) \).

Let \( \delta \in (0, \delta_1] \). Consider the auxiliary ODE (3.7). For all \( \eta \in E_{\langle k \rangle}(\delta, \delta) \) it follows from Lemmas 3.6 and 4.5 that \( v(0) \leq \sup_{\delta \in [-\delta, -\delta] \eta} I(\dot{u}, \delta, c, k) \leq P(\delta, c, k) < \delta \). Similarly, for all \( \eta \in E_{\langle k \rangle}(\delta, -\delta) \) we obtain \( v(0) \geq -\sup_{\delta \in [-\delta, -\delta] \eta} I(\dot{u}, \delta, -c, k) \geq -P(\delta, c, k) > -\delta \). Thus (2.12) holds for all \( \eta \in E_{\langle k \rangle}(\delta, x) \) or any \( |x| = \delta \). This is true for all \( \delta \in (0, \delta_1] \). Since \( E_{\langle k \rangle}(\delta, x) \subseteq E_{\langle k \rangle}(\delta, x) \), applying Theorem 2.7 completes the proof.

For given \( (\mu, \sigma) \) the condition \( P(1,0, k) < 1 \) ensures (2.12) is satisfied for all \( \eta \in E_{\langle k \rangle}(\delta, x) \) and hence Theorem 4.7 establishes asymptotic stability for \( (\mu, \sigma) \) in the part of \( \Sigma_w \cup \Sigma_c \) for which
P(1, 0, k) < 1. These sets are shown in Figure 4 for k = 1 and k = 2. The stability region \{P(1, 0, 1) < 1\} shown in Figure 4(a) for the DDE (1.2) comprises a relatively small part of \(\Sigma_w \cup \Sigma_c\), because it is derived by requiring that (2.12) holds for all \(\eta \in E_1(\delta, x)\). But \(E_1(\delta, x)\) is a very large set, with the main restrictions on \(\eta\) being that it is merely piecewise continuous with \(\|\eta\| \leq \delta\).

We obtain a larger stability region by increasing \(k\). This is seen in Figure 4(b) where \(P(1, 0, 2) < 1\) ensures that (2.12) is satisfied for all \(\eta \in E_2(\delta, x)\) results in a significantly larger stability region than seen in Figure 4(a). Since \(E_2(\delta, x) \subset E_1(\delta, x)\), with all functions \(\eta \in E_2(\delta, x)\) satisfying the derivative bound \(|\eta'(\theta)| \leq D_1 \delta\), the set \(E_2(\delta, x)\) is smaller than \(E_1(\delta, x)\) and it is possible to satisfy (2.12) over a larger region of \((\mu, \sigma)\) parameter space. We will compare the sizes of the stability regions \{\(P(1, 0, k) < 1\)\} for different \(k\) in Section 5.

The boundary of \{\(P(1/2, 1, k) < 1/2\)\} is also shown in Figure 4 for \(k = 1\) and \(k = 2\). As \(\delta \to 0\) the sets \{\(P(\delta, c, k) < \delta\)\} converge to the set \{\(P(1, 0, k) < 1\)\}, and for \((\mu, \sigma) \in \{\(P(1, 0, k) < 1\)\}\) the inequality \(P(\delta, c, k) < \delta\) can be used to determine the largest \(\delta_1\) and hence the largest \(\delta_2\) for which Theorem 4.7 applies. This determines a ball which is contained in the basin of attraction of the steady state, and in Section 6 we consider how the size of this lower bound on the basin of attraction varies with \(k\).

5. Comparison of the stability regions

In this section we compare the sets in which we can establish asymptotic stability of the steady state of the state-dependent DDE (1.2) using Lyapunov-Razumikhin techniques. In Sections 4 we showed asymptotic stability for \((\mu, \sigma) \in \Sigma_\lambda\), and for \((\mu, \sigma)\) in the parts of the cusp \(\Sigma_c\) and wedge \(\Sigma_w\) for which \(P(1, 0, k) < 1\) for \(k = 1, 2, 3\). Measurements of these sets and the exact stability region \(\Sigma_*\) are presented in Tables 1–3, and they are illustrated in Figure 5.

To compute these stability regions, from Theorems 4.7 we need to compute \(P(1, 0, k)\) in the limiting case \(c = 0, \delta = 1\). This was done in MATLAB [36]. For \(k = 1\) and \(2\) we have exact expressions for \(P(1, 0, k)\) given by (4.7) and (4.14). Noting that \(\sigma < 0\) in \(\Sigma_w \cup \Sigma_c\), from (4.7) we find that \((\mu, \sigma)\) satisfies \(P(1, 0, 1) < 1\) when

\[
\frac{\sigma}{\mu}(1 - e^{\mu a}) - \sigma - \mu e^{\mu a} > 0. \tag{5.1}
\]

The boundary of \{\(P(1, 0, 1) < 1\)\} is defined by equality in (5.1).

For \(k = 3\) the value of \(P(1, 0, 3)\) was calculated by maximizing the function \(I(\hat{u}, 1, 0, k)\) over \(\hat{u} \in [-1, -\mu/\sigma]\) using the MATLAB \texttt{fminbnd} function.

The boundary of \{\(P(1, 0, 1) < 1\)\} is then found by fixing one of \(\mu\) or \(\sigma\) and using the \texttt{fzero} function to find the value of the other one which solves \(P(1, 0, k) - 1 = 0\) (except in the case \(k = 1\) where for given \(\mu\), applying the quadratic formula to (5.1) determines \(\sigma\)). The largest value of \(\mu\) for each region (shown in Table 3) is then found by regarding the \(\mu\) that solves \(P(1, 0, k) = 1\) as a function of \(\sigma\) and using \texttt{fminsearch} to find the \(\sigma\) that maximises \(\mu\). The boundary of the full stability domain \(\Sigma_*\), found by linearization, is given by Definition 3.4.

Since by Theorem 4.7, at least for \(k \leq 3\), the Lyapunov-Razumikhin stability regions are given by \(P(1, 0, k) < 1\) irrespective of the value of \(c\), we obtain the same regions in the constant \(c = 0\) and variable \(c \neq 0\) delay cases. This is consistent with the linearization theory of Győri and Hartung [12] who showed that \(\Sigma_*\) is the exponential stability region for both \(c = 0\) and \(c \neq 0\).

When \(\mu = 0\) the DDE (1.2) becomes

\[
\dot{u}(t) = \sigma u(t - a - cu(t)) \tag{5.2}
\]
and the intervals of $\sigma$ values in the stability regions, (shown in the last column of Table 1) can be found exactly. From (4.7) we have $P(1, 0, 1) = -\sigma a$ when $\mu = 0$ which implies $\sigma \in (-1/a, 0)$ for $P(1, 0, 1) < 1$. Similarly, when $\sigma < -1/a$ from (4.14) we have $P(1, 0, 2) = -\sigma a - 1/2$ and hence $\sigma = -3/(2a)$ on the boundary of $\{P(1, 0, 2) < 1\}$. Magpantay [30] shows that for $\mu = 0$ we require $\sigma \in (-37/(24a), 0)$ for $P(1, 0, 3) < 1$. For the constant delay case of (5.2), with $c = 0$, Barnea [1] showed Lyapunov stability for $\sigma \in [-3/(2a), 0]$, by applying Lyapunov-Razumikhin techniques with $k = 2$. The stability bound $\sigma \geq -37/(24a)$ seems not to have been derived for the constant delay case of (5.2), but is well-known for Wright’s equation [42] which is a nonlinear constant delay DDE whose linear part corresponds to (5.2) with $c = 0$ (see [28, 42]).

When $\mu = 0$, the boundary of $\{P(1, 0, k) < 1\}$ seems to converge rapidly to $-\pi/(2a)$, the boundary of $\Sigma_*$, as $k \to \infty$, suggesting that the full stability interval can be recovered. Indeed, for constant delay with $\mu = c = 0$ Krisztin [26] showed Lyapunov stability for $\sigma \in (-\pi/(2a), 0]$ by considering $k \to \infty$.

Barnea [1] and Myshkis [37] also applied Razumikhin techniques to establish Lyapunov stability for (1.2) in the case of constant delay ($c = 0$) with $\mu \neq 0$. The regions in which they claim stability are shown in Figure 6. The region found by Myshkis [37] has $\sigma \geq -1/a$ and $\mu \leq \sigma(a\sigma + 1)/(a\sigma - 1)$, which for $a = 1$ always has $\mu \leq 3 - 2\sqrt{2}$ and is contained in $\{P(1, 0, 1) < 1\}$.

Barnea [1] claimed that the Lyapunov stability region of (1.2) with $c = 0$ contains the region $X_2 = \{\mu, \sigma : 0 \leq s^* \leq a, P < 1\}$ where

$$s^* = -\frac{e^{\mu a}}{\sigma}, \quad P = \frac{\sigma(\mu + \sigma)}{\mu^2} \left[ e^{\mu s^*} - \frac{\sigma}{\mu + \sigma} \right].$$
We show this region in Figure 6(a), but Barnea did not actually graph \( X_2 \) or give its derivation in [1]. He noted that setting \( P = 1 \) and letting \( \mu \to 0 \) yields that the point \( \sigma = -3/2a \) is a boundary of \( X_2 \) on the \( \sigma \)-axis. We observe that setting \( s^* = 0 \) and \( \mu \to 0 \) yields \( \sigma = -1/a \) as the other boundary of \( X_2 \) on the \( \sigma \)-axis. Thus the region \( X_2 \) does not include the whole interval \( \sigma \in (-3/2a, 0] \) on the \( \sigma \)-axis which Barnea had proven to be Lyapunov stable in the \( \mu = 0 \) case in the same paper [1]. Barnea’s stability region \( X_2 \) is hence incomplete. Although the \( \eta_{(2)} \) function used by Barnea to show Lyapunov stability corresponds to (4.3) with \( c = 0 \), it appears that Barnea performed his integration assuming that \( \frac{\delta + \delta^*}{D_{\delta}} \leq r^* \) in all cases. The case when \( \frac{\delta + \delta^*}{D_{\delta}} > r^* \) occurs in the \( \mu = c = 0 \) case (as well as the general case \( c \neq 0, \mu \neq 0 \) considered in (4.12),(4.13)). Omitting this case results in the incorrect stability region \( X_2 \). The correct region is \((\mu, \sigma) \in \{P(1, 0, 2) < 1\} \) as illustrated in Figure 5(b). Moreover within this region we show the stronger property of asymptotic stability for both the constant delay (\( c = 0 \)) and state-dependent delay (\( c \neq 0 \)) cases.

Tables 1 and 2 also show that for \( \mu < 0 \) we can show asymptotic stability in a larger part of \( \Sigma^* \) by increasing \( k \). However when \( \mu \ll 0 \) the improvement in going from \( k = 1 \) to 2 to 3 is very marginal and we can only show asymptotic stability in a slice of the wedge \( \Sigma_w \) whose width appears to go to zero as \( \mu \to -\infty \). The problem here is that as \( \mu \to -\infty \) the DDE (1.2) is singularly perturbed and can be written as the so-called saw-tooth equation

\[
\varepsilon u(t) = u(t) + Ku(t - a - cu(t))
\]

where \( \varepsilon = 1/\mu \) and \( K = \sigma/\mu \). This DDE had been studied in detail in [35] and for \( K > 1 \) sufficiently large (corresponding to \((\mu, \sigma) \) outside \( \Sigma^* \)) the steady state is unstable, but there is an asymptotically stable slowly oscillating periodic solution. This periodic solution, known as the sawtooth solution, has unbounded gradient and a discontinuous profile in the singular limit. For parameter values inside the wedge \( \Sigma_w \) the steady state is asymptotically stable, and for large and negative \( \mu \) there are no periodic solutions but a slowly decaying sawtooth-like oscillation can occur. Lyapunov-Razumikhin techniques based on bounding derivatives of solutions cannot perform well when those derivatives can be arbitrarily large. To improve the results in this case it would be necessary to define different sets \( E^{\varepsilon}_{(k)}(\delta, x) \) which take into account the structure of the oscillations and are hopefully much closer to \( E^{\varepsilon}_{(k)}(\delta, x) \) than the sets \( E_{(k)}(\delta, x) \) that we use here.

| Region         | \( \mu \) at \( \sigma = -5 \) | \( \mu \) at \( \sigma = -2 \) | \( \mu \) at \( \sigma = -1 \) |
|----------------|---------------------------------|---------------------------------|---------------------------------|
| \( \{P(1, 0, 1) < 1\} \) | -4.9286634                    | -1.16401463                     | 0                               |
| \( \{P(1, 0, 2) < 1\} \) | -4.9283941                    | -1.0040856                      | 0.38774807                      |
| \( \{P(1, 0, 3) < 1\} \) | -4.9281247                    | -0.93607885                     | 0.39925645                      |

\( \Sigma^* \)

\[
\mu > -4.2734224, \quad \sigma > -0.63804505
\]

Table 2: Boundaries of the stability regions: Values of \( \mu \) for fixed \( \sigma \) with \( a = 1 \).

| Region         | Supremum of \( \mu \)         | Corresponding value of \( \sigma \) |
|----------------|-------------------------------|------------------------------------|
| \( \{P(1, 0, 1) < 1\} \) | 0.18822641 = \ln((1 + \sqrt{2})/2)) | -0.45439453                       |
| \( \{P(1, 0, 2) < 1\} \) | 0.45697166                    | -0.73935547                       |
| \( \{P(1, 0, 3) < 1\} \) | 0.45700462                    | -0.74059482                       |
| \( \Sigma^* \) | 1                             | -1                                |

Table 3: The values of \( \mu \) and \( \sigma \) at the rightmost boundary point of each stability region with \( a = 1 \).
Figure 6: (a) The set $X_2 \setminus \Sigma$ which is part of the stability region of (1.2) for the constant delay case ($c = 0$) according to Barnea [1]. (b) The part of the stability domain outside $\Sigma$ for the same problem found by Myshkis [37].

For $\mu > 0$ there is a significant improvement in the computed stability domain in going from $k = 1$ to $k = 2$ and a smaller improvement using $k = 3$. The largest value of $\mu$ which satisfies $P(1, 0, 1) \leq 1$ can be computed from (5.1) which is quadratic in $\sigma$. Then non-negativity of the discriminant imposes the bound that $\mu < (1/\alpha) \ln((1 + \sqrt{2})/2) \approx 0.1882/\alpha$, as seen in Table 3.

Although the parameter regions in which we can show asymptotic stability are independent of $c$, we will see in Section 6 that the basins of attraction do depend on $c$.

6. Basins of attraction

Theorem 4.7 shows that for $(\mu, \sigma) \in \Sigma_w \cup \Sigma_c$ the ball

$$\{ \varphi : ||\varphi|| < \delta_2 = \delta_1 e^{-k(|\mu| + |\sigma|)(\alpha + |c|\delta_1)} \}$$

(6.1)

is contained in the basin of attraction of the steady-state of the state-dependent DDE (1.2) for $k = 1, 2, 3$. For fixed $\delta_1$ the radius of this ball gets smaller as $k$ increases, but the value of $\delta_1$ depends on $k$, $\mu$ and $\sigma$, and some work is required to determine the largest such ball that is contained in the basin of attraction. In [31] we show that (6.1) can be improved when $\mu < 0$, so here we will consider $(\mu, \sigma) \in \Sigma_c$, where $\sigma < 0 \leq \mu$. Lemma 3.2 does not apply when $\mu \geq 0$, so there is no a priori bound on the solutions to (1.2) in this case. We present two examples which show that (1.2) can have unbounded solutions when $\mu \geq 0$, which also shows that the steady-state is not globally asymptotically stable when $(\mu, \sigma) \in \Sigma_c$ and gives an upper bound on the largest ball contained in its basin of attraction. For simplicity of exposition we suppose $c > 0$ in this section, but the results can easily be extended to $c < 0$. We first consider $\mu = 0$.

Example 6.1. Consider (1.2) with $c > 0$, $a > 0$, $\mu = 0$ and $\sigma \in [-\pi/2a, -1/a)$ and for $\delta \in [-1/(c\sigma), a/c)$ let $\varphi(t)$ be Lipschitz continuous with

$$\varphi(0) = \delta, \quad \varphi(t) = -\delta \text{ for } t \leq -a - c\delta,$$

$$\varphi(t) \in (-\delta, \delta) \text{ for } t \in (-a - c\delta, 0).$$
Then while the deviated argument \( \alpha(t, u(t)) = t - a - cu(t) \leq -a - c\delta \) we have
\[
u(t - a - cu(t)) = -\delta \quad \text{and} \quad \dot{u}(t) = -\sigma\delta.
\]
Hence,
\[
u(t) = \delta(1 - \sigma t) \geq \delta + \frac{t}{\sigma}, \quad \text{for } t \geq 0.
\]
But now \( \alpha(t, u(t)) \leq -a - c\delta \) for all \( t \geq 0 \) and (6.2) is valid for all \( t \geq 0 \). Thus for \( \mu = 0, \sigma \in [-\pi/2a, -1/a) \) on the axis between the third and fourth quadrants of the stability region we have \( ||\varphi|| = \delta \) and \( |u(t)| \to \infty \) as \( t \to \infty \). It follows that the steady state is not globally asymptotically stable and also that \( B(0, \delta) \) is not contained in its basin of attraction. Thus \( \delta^* = -1/c\sigma \) provides an upper bound on the radius of the largest ball contained in the basin of attraction.

Figure 7 shows three bounds on the basin of attraction of the steady state of (1.2). The value of \( \delta^* \) from Example 6.1 gives an upper bound on the radius of the largest ball contained in the basin of attraction. Two lower bounds on the radius of the largest ball are also shown. The larger bound \( \delta_1 \) gives the radius of the ball that Theorem 4.7 shows is contracted asymptotically to the steady state provided the solution is sufficiently differentiable. Lemma 2.3 is used to ensure that the solution remains bounded long enough to acquire sufficient regularity, and the growth in the solution allowed by that lemma results in the smaller radius \( \delta_2 \) (as defined by (6.1)) of the ball that is contained in the basin of attraction for general continuous initial functions \( \varphi \). We see that the bounds \( \delta_1 \) increase monotonically with \( k \), but because of the exponential term in (6.1), the largest value of \( \delta_2 \) is achieved with \( k = 1 \) in most of the interval for which \( P(1, 0, 1) < 1 \).

Now consider the case of \( \mu > 0 \). We can again derive an upper bound on the basin of attraction of the steady state when \((\mu, \sigma) \in \Sigma_c\).

**Example 6.2.** Let \( a > 0, c > 0, \mu > 0 \) and \((\mu, \sigma) \in \Sigma_c\) so \( \sigma < -\mu < 0 \). Also let
\[
q(\delta) = -a\mu - 1 - c\sigma\delta - \ln(c\delta(\mu - \sigma)).
\]
Note that \( q(1/(c(\mu - \sigma))) = -\mu(a + 1/(\mu - \sigma)) < 0 \), while \( q'(\delta) = -c\sigma - 1/\delta < 0 \) for all \( \delta \in (0, 1/(c(\mu - \sigma))) \). Also \( q(\delta) \to \infty \) as \( \delta \to 0 \), hence there exists \( \delta^* \in (0, 1/(c(\mu - \sigma))) \) such that \( q(\delta^*) = 0 \) and \( \delta^* \) is unique in this interval.
Suppose that the parameters are chosen so that $\delta^* < a/c$. A sufficient (but not necessary) condition for this is $\sigma < \mu - 1/a$ since this implies $1/(c(\mu - \sigma)) \leq a/c$. Now let $\delta \in (\delta^*, a/c)$ so $q(\delta) < 0$ and consider (1.2) with $\varphi(t)$ Lipschitz continuous and

$$\varphi(0) = \delta, \quad \varphi(t) = -\delta \text{ for } t \leq \mu q(\delta),$$

$$\varphi(t) \in (-\delta, \delta) \text{ for } t \in (\mu q(\delta), 0),$$

Then (1.2) has solution

$$u(t) = \frac{\sigma \delta}{\mu} + \delta e^{\mu t} \left[ \frac{\mu - \sigma}{\mu} \right]$$

with $u(t - a - cu(t)) = -\delta$ for all $t \geq 0$. To see this note that

$$\alpha(t, u(t)) = t - a - cu(t) = t - a - \frac{c\sigma \delta}{\mu} - \delta c e^{\mu t} \left[ \frac{\mu - \sigma}{\mu} \right],$$

with $\alpha(t, u(t)) \to -\infty$ as $t \to \infty$. Differentiating the expression for $\alpha(t, u(t))$ shows that $\alpha(t, u(t)) \leq \mu q(\delta) < 0$ for all $t \geq 0$, with $\alpha(t, u(t)) = \mu q(\delta)$ when $t = -\mu^{-1} \ln (c\delta(\mu - \sigma))$. Hence, as in Example 6.1, we have $\|\varphi\| = \delta$ and $|u(t)| \to \infty$ as $t \to \infty$. The steady state is not globally asymptotically stable and the ball $B(0, \delta)$ is not contained in its basin of attraction. Thus $\delta^*$ provides an upper bound on the radius of the largest ball contained in the basin of attraction.

For $\sigma = -1$ and $\mu \geq 0$, Figure 8 shows the same bounds $\delta_1$, $\delta_2$ and $\delta^*$ on the radius of the basin of attraction of the steady state as were shown in Figure 7. Since these parameters are outside the set $\{P(1, 0, 1) < 1\}$ no bound is shown for $k = 1$. On nearly all of this interval $k = 2$ gives the largest lower bound $\delta_2$ on the radius of a ball contained in the basin of attraction.

Figure 9 shows these bounds on the basin of attraction in the cusp $\Sigma_\epsilon$. The shaded region in Figure 9(c) denotes the portion of $\Sigma_\epsilon$ for which $\delta^* \leq a/c$ when $a = c = 1$, and hence $\delta^*$ from Example 6.2 gives an upper bound on the radius of the largest ball contained in the basin of attraction. The corresponding bounds $\delta^*$ are shown as contours within this region. Figure 9(a) and (b) shows the lower bounds $\delta_1$ and $\delta_2$, along with the value of $k$ that achieves the bound.

In all three figures in this section, $\delta_2$ is computed using (6.1) and $\delta_1$ is obtained from solving $P(\delta_1, c, k) = \delta_1$ similarly to computations described in Section 5.
Figure 9: Plot of $\Sigma_c$ for fixed $a = c = 1$ with contour plots of (a) the maximum $\delta_1 \in (0, a/c]$ such that $(\mu, \sigma) \in \{P(\delta_1, c, k) < \delta_1\}$, and (b) the $\delta_2$ that maximizes $\delta_2 = \delta e^{-k(|\mu|+|\sigma|)(a+|c|)}$ for $\delta \in (0, \delta_1]$. Shading shows the value $k \in \{1, 2, 3\}$ for which the maximum is achieved. (c) The upper bound $\delta^*$ from Example 6.2 for the radius of the largest ball $B(0, \delta)$ contained in the basin of attraction of the zero solution to (1.2).

7. Conclusions

In this paper we have expanded upon the existing work on Lyapunov-Razumikhin techniques by providing results specifically tailored to DDEs with time-varying discrete delays including problems with state-dependent delays and vanishing delays. Our main results provide sufficient conditions for Lyapunov and asymptotic stability of steady state solutions of DDEs in Theorems 2.5 and 2.7 respectively. These conditions involve converting the DDE into a corresponding ODE problem with the delay terms treated as source terms that satisfy constraints. Our results require a Lipschitz condition on the right-hand side function $f$ in (1.1) instead of the more restrictive Lipschitz condition on $F$ in (1.4) required in Barnea [1], and do not require the construction of auxiliary functions as required by Hale and Verduyn Lunel [15]. Nevertheless we are able to show asymptotic stability, using a proof by contradiction showing that there cannot exist a solution which is not asymptotic to the steady state.

We apply our results to the model state-dependent DDE (1.2) in Sections 4–6. The main result of the application of Lyapunov-Razumikhin techniques to (1.2) is given as Theorems 4.7 where
we prove that the zero solution to (1.2) is asymptotically stable if \((\mu, \sigma) \in \{P(1, 0, k) < 1\}\), for \(k = 1, 2\) or 3 and provide lower bounds on the basin of attraction.

The parameter regions in which stability is proven in these theorems are compared in Section 5. As shown in Figure 5, the derived parameter regions grow as larger values of \(k\) are used, though for \(\mu \neq 0\), the derived stability region does not approach the entire known stability region \(\Sigma^*\) as \(k \to \infty\) (for reasons discussed in in Section 5).

In Section 6 we consider (1.2) in the cusp \(\Sigma_c\) where \(\mu > 0\) and the steady state would be unstable without the delay term. In Examples 6.1 and 6.2 we constructed solutions which do not converge to the steady state for \((\mu, \sigma) \in \Sigma_c\). These solutions provide us with an upper bound \(\delta^*\) on the radius of the largest ball about the zero solution contained in the basin of attraction. In Figures 7–9 these upper bounds were compared with the lower bound \(\delta_2\) on the basin of attraction from (6.1).

In the current work have studied stability through Lyapunov-Razumikhin techniques, but let us briefly compare and contrast this approach to the alternative, namely linearization. State-dependent DDEs have long been linearized by freezing the delays at their steady-state values and linearizing the resulting constant delay DDE [4, 5]. This heuristic approach has recently been put on a rigorous footing. For a class of state-dependent DDEs which includes (1.2) with \(\mu = 0\), Győri and Hartung [11] proved that the steady state of the state-dependent DDE is exponentially stable if and only if the steady state of the corresponding frozen-delay DDE is exponentially stable. In [12] they generalise this result to a class of nonautonomous problems which are linear except for the state-dependency.

To compare and contrast our results with the linearization results of [12], we note that our results apply to a larger class of problems (1.1) than was considered in [12], and we prove both Lyapunov stability and asymptotic stability results, whereas [12] is concerned with exponential stability. The results in [12] do apply directly to our model problem (1.2), and reveal the parameter region for which the steady state is exponentially stable. In contrast our Lyapunov-Razumikhin techniques are only able to deduce stability in part of this parameter region.

Even though Lyapunov-Razumikhin techniques do not provide a proof of stability in the entire known stability region for (1.2), just as Lyapunov functions for ODEs do not always do so, they can nevertheless still be a very useful tool for studying stability in state-dependent DDEs. In particular our Lyapunov stability result is applicable to nonautonomous problems (for some of which rigorous linearization has yet to be derived) and the asymptotic stability result yield bounds on the basins of attraction which cannot be derived through linearization.

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References

[1] D.I. Barnea. A method and new results for stability and instability of autonomous functional differential equations. *SIAM J. Appl. Math.*, 17:681–697, 1969.
Appendix A. An explicit expression for the region $P(\delta, c, 2) < \delta$

Here we prove Theorem 4.2. Let $I_1$ and $I_2$ be defined by (4.12) and (4.13). Recall that $I_1$ only applies when $\frac{\hat{u}}{\hat{u}} r \geq r$, in which case the integration does not have to be split into two intervals. For this case to occur we require $\hat{u} \in [(rD_1 - 1)\delta, -\mu\delta/\sigma]$, which is only possible in the region where $rD_1 - 1 \leq -\mu/\sigma$. When $\frac{\hat{u}}{\hat{u}} r < r$ the integration has to be broken into two parts and $I_2$ applies. In that case $\hat{u} \in [-\delta, \min((rD_1 - 1)\delta, -\mu\delta/\sigma)]$. We require the following lemmas.

**Lemma A.1.** Let $a > 0$, $c \neq 0$, $\sigma < \mu$ and $\sigma < -\mu$. Let $\delta \in (0, |a/c|)$ and $rD_1 - 1 \leq -\mu/\sigma$. If $\mu > 0$ then $\sigma > -1/r$. If $\mu < 0$ then $\mu \in [(-3 + 2 \sqrt{2})/r, 0]$ and

$$\sigma \geq -\frac{1}{r}[\frac{1}{2} (1 + \mu r) + \frac{1}{2} \sqrt{1 + 6\mu r + (\mu r)^2}] \geq -\frac{1}{r}. \quad (A.1)$$

**Proof.** Let $rD_1 - 1 \leq -\mu/\sigma$. Then

$$r(\text{sign}(\mu)\mu - \sigma) - 1 = r(|\mu| + |\sigma|) - 1 \leq rD_1 - 1 \leq -\frac{\mu}{\sigma},$$

$$\Rightarrow \quad r\sigma^2 + (1 - \text{sign}(\mu)\mu)\sigma - \mu \leq 0. \quad (A.2)$$

The boundary of the region where this inequality holds is

$$\sigma = -\frac{1}{r}[\frac{1}{2} (1 - \text{sign}(\mu)\mu) r \pm \frac{1}{2} \sqrt{1 - \text{sign}(\mu)\mu)^2 + 4\mu r}]. \quad (A.3)$$

If $\mu \geq 0$ then this simplifies to $\sigma = -1/r$ and $\sigma = \mu$. Since $\sigma \leq 0$, and also $\mu = \sigma = 0$ satisfies (A.2) it follows that (A.1) holds for $\sigma \in [-1/r, 0]$ for the case $\mu \geq 0$.

If $\mu < 0$ then (A.3) simplifies to

$$\sigma = -\frac{1}{r}[\frac{1}{2} (1 + \mu r) \pm \frac{1}{2} \sqrt{1 + 6\mu r + (\mu r)^2}]. \quad (A.4)$$

Requiring $1 + 6\mu r + (\mu r)^2 \geq 0$ yields $\mu r \in [-3 + 2 \sqrt{2}, 0]$. The lower bound on $\sigma$ can be found by taking the lower boundary in (A.4) which attains its minimum at $\mu = 0$. This yields (A.1). 

[31] F.M.G. Magpantay and A.R. Humphries. Generalised Lyapunov–Razumikhin techniques for scalar state-dependent delay differential equations. 2017. arXiv:1703.08638.
[32] J. Mallet-Paret and R.D. Nussbaum. Boundary layer phenomena for differential-delay equations with state-dependent time lags. I. Arch. Ration. Mech. Anal., 120:99–146, 1992.
[33] J. Mallet-Paret and R.D. Nussbaum. Boundary layer phenomena for differential-delay equations with state-dependent time lags: II. J. Reine Angew. Math., 477:129–197, 1996.
[34] J. Mallet-Paret and R.D. Nussbaum. Boundary layer phenomena for differential-delay equations with state-dependent time lags: III. Discrete Contin. Dyn. Syst. Ser. A, 189:640–692, 2003.
[35] J. Mallet-Paret and R.D. Nussbaum. Superstability and rigorous asymptotics in singularly perturbed state-dependent delay-differential equations. J. Diff. Eqns., 250:4037–4084, 2011.
[36] Mathworks. MATLAB 2014a. Mathworks, Natick, Massachusetts, 2014.
[37] A. Myshkis. Razumikhin’s method in the qualitative theory of processes with delay. J. Appl. Math. Stoch. Anal., 8:233–247, 1995.
[38] B.S. Razumikhin. An application of Lyapunov method to a problem on the stability of systems with a lag. Autom. Remote Control, 21:740–748, 1960.
[39] H. Smith. An Introduction to Delay Differential Equations with Applications to the Life Sciences. Texts in Applied Mathematics. Springer, New York, 2011.
[40] E. Stumpf. Local stability analysis of differential equations with state-dependent delay. Discrete Contin. Dyn. Syst. Ser. A, 36:3445–3461, 2016.
[41] H.-O. Walther. On a model for soft landing with state dependent delay. J. Dyn. Diff. Eqns., 19:593–622, 2003.
[42] E.M. Wright. A non-linear difference-differential equation. Journal für die reine und angewandte Mathematik, 194:66–87, 1955.
[43] J.A. Yorke. Asymptotic stability for one dimensional differential-delay equations. J. Diff. Eqns., 7:189–202, 1970.
Lemma A.2. Let \( a > 0, \sigma \leq \mu \) and \( \sigma < -\mu \). Let \( \delta \in (0, |a/c|) \). Define
\[
\hat{u}_* = \left[ \frac{1}{\mu} D_1 \ln \left( 1 - \frac{\mu}{\sigma} e^{\mu r} \right) - 1 \right] \delta.
\]

The following statements are true:

(A) If \( rD_1 - 1 > -\mu/\sigma \) then the maximum of \( I_2(\hat{u}, \delta) \) over \( \hat{u} \in [-\delta, -\mu\delta/\sigma] \) occurs at \( \hat{u}_* \) if \( \hat{u}_* < -\mu\delta/\sigma \), and at \( -\mu\delta/\sigma \) otherwise.

(B) If \( rD_1 - 1 > -\mu/\sigma \) and \( u_* < -\mu\delta/\sigma \) then \( P(\delta, c, 2) \geq \delta \).

(C) If \( rD_1 - 1 \leq -\mu/\sigma \) then \( \sup_{\delta \in [-\delta, (rD_1 - 1)\delta]} I_2(\hat{u}, \delta) = I_2((rD_1 - 1)\delta, \delta) \).

(D) If \( rD_1 - 1 \leq -\mu/\sigma \) then \( \sup_{\delta \in [(rD_1 - 1)\delta, -\mu\delta/\sigma]} I_2(\hat{u}, \delta) = I_1(-\mu\delta/\sigma, \delta) \).

(E) If \( rD_1 - 1 \leq -\mu/\sigma \) then \( I_2((rD_1 - 1)\delta, \delta) \leq I_1(-\mu\delta/\sigma, \delta) \).

Proof of (A). To find \( \hat{u}_* \) which maximises \( I_2(\hat{u}_*, \delta) \) consider the derivative
\[
\frac{\partial}{\partial \hat{u}} I_2(\hat{u}, \delta) = e^{\mu r} + \frac{\sigma}{\mu} \left( e^{\hat{u} \sigma} D_1 e^{\hat{u} \delta} - 1 \right) \quad (A.5)
\]

At \( \hat{u} = -\delta \) this is positive. Setting the derivative equal to zero in (A.5) yields
\[
\hat{u}_* = \left[ \frac{1}{\mu} D_1 \ln \left( 1 - \frac{\mu}{\sigma} e^{\mu r} \right) - 1 \right] \delta.
\]

Since \( 1 - \frac{\mu}{\sigma} e^{\mu r} \in [0, 1] \) if \( \mu < 0 \) and \( 1 - \frac{\mu}{\sigma} e^{\mu r} > 1 \) if \( \mu > 0 \) then \( \hat{u}_* > -\delta \) in both cases.

Proof of (B). First we need to prove that if \( \hat{u}_* < -\mu\delta/\sigma \) and \( \mu > 0 \) then \( rD_1 - 1 \leq -\mu/\sigma \). Let \( \hat{u}_* < -\mu\delta/\sigma \) and \( \mu > 0 \). Then \( \frac{\partial}{\partial \hat{u}} I_2(-\mu\delta/\sigma, \delta) \leq 0 \). Consider the term \( D_1/\mu \),
\[
\frac{D_1}{\mu} = \frac{|\mu| + |\sigma|}{\mu} (1 + (|\mu| + |\sigma|)|c|\delta) \geq \frac{|\mu| + |\sigma|}{\mu} = \left( 1 - \frac{\sigma}{\mu} \right).
\]

Now consider the exponent of the second term in (A.5) with \( \hat{u} = -\mu\delta/\sigma \),
\[
\mu \frac{1 - \mu/\sigma}{D_1} = \left( 1 - \frac{\mu}{\sigma} \right) \frac{\mu}{D_1} \leq \frac{1 - \mu/\sigma}{1 - \sigma/\mu} = -\frac{\mu}{\sigma}.
\]

Thus,
\[
e^{\mu r} + \frac{\sigma}{\mu} \left( e^{-\mu/\sigma} - 1 \right) \leq \frac{\partial}{\partial \hat{u}} I_2(-\mu\delta/\sigma, \delta) < 0.
\]

Isolating \( r \) in this expression yields \( r < \frac{1}{\mu} \ln \left[ \frac{\sigma}{\mu} (1 - e^{-\mu/\sigma}) \right] \). Let \( x = \mu/\sigma \), then \( x \in (-1, 0) \) and \( (1 - e^{-x})/x > 1 \). Also, \( (1 - 1/x) \ln((1 - e^{-x})/x) - 1 \leq -x \). These inequalities and (A.6) imply
\[
rD_1 - 1 \leq \frac{D_1}{\mu} \ln \left[ \frac{\sigma}{\mu} (1 - e^{-\mu/\sigma}) \right] - 1 \leq \left( 1 - \frac{\sigma}{\mu} \right) \ln \left[ \frac{\sigma}{\mu} (1 - e^{-\mu/\sigma}) \right] - 1 \leq -\frac{\mu}{\sigma}.
\]

Thus if \( rD_1 - 1 > -\mu/\sigma \) then \( \hat{u}_* < -\mu\delta/\sigma \) can only occur if \( \mu < 0 \).

Now let \( \mu < 0 \) and \( \hat{u}_* < -\mu\delta/\sigma \). Then by setting \( \frac{\partial}{\partial \hat{u}} I_2(\hat{u}_*, \delta) = 0 \) in (A.5), \( e^{\hat{u} \sigma} D_1 e^{\hat{u} \delta} - 1 = -\frac{\mu}{\sigma} e^{\mu r} \).

Also, \( e^{\mu r} - \sigma/\mu < 0 \) because \( \mu < 0 \) and \( \sigma \leq \mu < 0 \). Thus,
\[
I_2(\hat{u}_*, \delta) = \hat{u}_* \left( e^{\mu r} - \frac{\sigma}{\mu} \right) + \frac{\sigma}{\mu} \left[ \frac{D_1}{\mu} \left( e^{\hat{u} \sigma} D_1 e^{\hat{u} \delta} - 1 \right) - e^{\mu r} \right]
\]
\[
= -\frac{\mu}{\sigma} \delta \left( e^{\mu r} - \frac{\sigma}{\mu} \right) + \frac{\sigma}{\mu} \delta \left[ -\frac{D_1}{\mu} e^{\mu r} - e^{\mu r} \right] = \delta - \left( \frac{D_1}{\mu} + \frac{\mu}{\sigma} + \frac{\sigma}{\mu} \right) | \delta e^{\mu r}.
\]

But \( D_1 \geq |\mu| + |\sigma| \geq |\mu/\sigma| |\mu| + |\sigma| = -|\mu^2/\sigma + \sigma| \) which implies \( \frac{D_1}{\mu} + \frac{\mu}{\sigma} + \frac{\sigma}{\mu} \leq 0 \). Hence \( I_2(\hat{u}_*, \delta) \geq \delta \) as required.
Proof of (C). Let $\frac{\partial}{\partial x} I_2((rD - 1)\delta, \delta) < 0$. Then, $e^{\mu r} + \frac{\sigma}{\mu}(e^{\mu r} - 1) < 0$, which can be rewritten as

$$\sigma < \frac{\mu e^{\mu r}}{1 - e^{\mu r}} = -\frac{1}{r} \left(\frac{-\mu e^{\mu r}}{1 - e^{\mu r}}\right).$$

(A.7)

We show that the expression on the right-hand-side is continuous and decreases as $\mu$ increases. Let $x = \mu r$, then

$$\lim_{\mu \to 0} -\frac{1}{r} \left(\frac{-\mu e^{\mu r}}{1 - e^{\mu r}}\right) = \lim_{x \to 0} -\frac{1}{r} \left(\frac{-xe^x}{1 - e^x}\right) = -\frac{1}{r},$$

$$\frac{d}{d\mu} \left[-\frac{1}{r} \left(\frac{-\mu e^{\mu r}}{1 - e^{\mu r}}\right)\right] = \frac{d}{dx} \frac{xe^x}{(1 - e^x)^2} = \frac{e^x(1 + x - e^x)}{(1 - e^x)^2} \leq 0.$$

So when $\mu > 0$, a necessary condition for (A.7) to hold is $\sigma < -1/r$. From Lemma A.1, if $rD - 1 \leq -\mu/\sigma$ and $\mu > 0$ then $\sigma \geq -1/r$. Thus $\frac{\partial}{\partial x} I_2((rD - 1)\delta, \delta) \geq 0$ if $rD - 1 \leq -\mu/\sigma$ and $\mu > 0$.

Now let $rD - 1 \leq -\mu/\sigma$ and $\mu < 0$. From Lemma A.1, $\mu r \in [-3 + 2\sqrt{2}, 0]$ and $\sigma \geq -\frac{1}{r} \left[\frac{1}{2}(1 + \mu r) - \frac{1}{2}\sqrt{1 + 6\mu r + (\mu r)^2}\right]$. Since $\frac{e^x}{1 - e^x} \geq \frac{1}{2}(1 + x) + \frac{1}{2}\sqrt{1 + 6x + x^2}$ for $x \in [-3 + 2\sqrt{2}, 0]$, then $\sigma \geq -\frac{1}{r} \left(\frac{-\mu e^{\mu r}}{1 - e^{\mu r}}\right)$. This contradicts (A.7). Thus $\frac{\partial}{\partial x} I_2((rD - 1)\delta, \delta) \geq 0$ if $rD - 1 \leq -\mu/\sigma$ and $\mu < 0$. Thus,

$$rD - 1 \leq -\frac{\mu}{\sigma} \quad \Rightarrow \quad \frac{\partial}{\partial x} I_2((rD - 1)\delta, \delta) = e^{\mu r} + \frac{\sigma}{\mu}(e^{\mu r} - 1) \geq 0. \quad (A.8)$$

This is shown in Figure 10. To finish the proof of (C), observe from (A.5) that $\frac{\partial}{\partial x} I_2(\hat{u}, \delta)\delta$ decreases as $\hat{u}$ increases. Then by (A.8), $\sup_{\hat{u} \in [(-\delta, (rD - 1)\delta]} I_2(\hat{u}, \delta) = I_2((rD - 1)\delta, \delta)$. \hfill \Box

Proof of (D). Let $rD - 1 \leq -\mu/\sigma$. For all $\hat{u} \in [(rD - 1)\delta, -\mu\delta/\sigma]$,

$$\frac{\partial}{\partial x} I_1(\hat{u}, \delta) = e^{\mu r} + \frac{\sigma}{\mu}(e^{\mu r} - 1) = \frac{\partial}{\partial x} I_2((rD - 1)\delta, \delta).$$

From (A.8), $\frac{\partial}{\partial x} I_1(\hat{u}, \delta) > 0$ for all $\hat{u} \in [(-\delta, (rD - 1)\delta]$. Thus, $\sup_{\hat{u} \in [(rD - 1)\delta, -\mu\delta/\sigma]} I_1(\hat{u}, \delta) = I_1(-\mu\delta/\sigma, \delta)$. \hfill \Box
Proof of (E). From (4.12) and (4.13) we find that

\[ I_1(-\mu\delta/\sigma, \delta) - I_2((rD_1 - 1)\delta, \delta) = \left(-\frac{\mu}{\sigma} - (rD_1 - 1)\right)\delta \left[ e^{\mu r} + \frac{\xi}{\mu}(e^{\mu r} - 1) \right]. \]

For \( rD_1 - 1 \leq -\mu/\sigma \) it follows from (A.8) that this expression is non-negative. Thus, \( I_1(-\mu\delta/\sigma, \delta) \geq I_2((rD_1 - 1)\delta, \delta) \).

Finally we can prove Theorem 4.2.

Proof of Theorem 4.2. Note that this expression does not hold outside of \( \{p(\delta, c, 2) < \delta\} \). In order to prove this theorem, we need items (A)-(E) in Lemma A.2 which require Lemma A.1.

Let \( rD_1 - 1 > -\mu/\sigma \). Then we can only have the two-part integration so \( I(\hat{u}, \delta, c, 2) = I_2(\hat{u}, \delta) \).
From (A) and (B), \( p(\delta, c, 2) = I(-\mu/\sigma, \delta) \) if \( p(\delta, c, 2) < \delta \).

Let \( rD_1 - 1 \leq -\mu/\sigma \). Then we can have either the one-part or the two-part integration.
From (C) and (D), \( I(\hat{u}, \delta, c, 2) = \max\{I_2((rD_1 - 1)\delta, \delta), I_1(-\mu\delta/\sigma, \delta)\} \).
From (E), \( p(\delta, c, 2) = I_1(-\mu\delta/\sigma, \delta) \).