We consider a model for tensionless (null) $p$-branes with $N = 1$ global supersymmetry in 10-dimensional Minkowski space-time. We give an action for the model and show that it is reparametrization and kappa-invariant. We also find some solutions of the classical equations of motion. In the case of null superstring ($p = 1$), we obtain the general solution in arbitrary gauge.

1 Introduction

The null $p$-branes are the zero tension limit of the tensionful ones. The correspondence between this two types of branes may be regarded as a generalization of the massless-massive particles relationship. Null branes with manifest space-time or world-volume supersymmetry are considered in [1] and [2] respectively. In a previous paper [3], we began the investigation of a tensionless $p$-brane model with $N = 1$ supersymmetry in ten dimensional flat space-time. Starting with a Hamiltonian which is a linear combination of first and mixed (first and second) class constraints, we succeed to obtain a new one, which is a linear combination of first class, BFV-irreducible and Lorentz-covariant constraints only. This was done with the help of the introduced auxiliary harmonic variables [4], [5]. Then we gave manifest expressions for the classical BRST charge, the corresponding total constraints and BRST-invariant Hamiltonian.

In this letter, we continue the investigation of the model. Here, we consider the corresponding action, establish its symmetries, and present some solutions of the classical equations of motion.

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Our initial Hamiltonian is\footnote{3}

\[ H_0 = \int d^p \sigma [\mu^0 T_0 + \mu^j T_j + \mu^\alpha D_\alpha], \] \tag{1} \]

where the constraints $T_0$, $T_j$ and $D_\alpha$ are defined by the equalities:

\[
\begin{align*}
T_0 &= p_\mu p_\nu \eta^{\mu\nu}, \\
T_j &= p_\nu \partial_j x^\nu + p_\theta \alpha \partial_j \theta^\alpha, \\
D_\alpha &= -ip_\theta - (\hat{\theta} \theta)^\alpha = p_\nu \sigma_{\nu\alpha},
\end{align*}
\]

Here $(x^\nu, \theta^\alpha)$ are the superspace coordinates, $(p_\nu, p_\theta)$ are their canonically conjugated momenta, $\theta^\alpha$ is a left Majorana-Weyl space-time spinor, and $\sigma^{\mu}$ are the ten dimensional Pauli matrices (our spinor conventions are as in \footnote{3}). The Hamiltonian \footnote{3} is a generalization of the Hamiltonians for the bosonic null p-brane and for the $N=1$ Brink-Schwarz superparticle.

2 Solutions of the equations of motion

The equations of motion which follow from the Hamiltonian $H_0$ are ($\partial_\tau = \partial/\partial \tau$):

\[
\begin{align*}
(\partial_\tau - \mu^j \partial_j) x^\nu &= 2\mu^0 p^\nu - (\mu \sigma^\nu \theta), \\
(\partial_\tau - \mu^j \partial_j) p_\nu &= (\partial_j \mu^j) p_\nu, \\
(\partial_\tau - \mu^j \partial_j) \theta^\alpha &= i\mu^\alpha, \\
(\partial_\tau - \mu^j \partial_j) p_\theta &= (\partial_j \mu^j) p_\theta + (\mu \hat{\theta})^\alpha.
\end{align*}
\]

In \footnote{3}, one can consider $\mu^0$, $\mu^j$ and $\mu^\alpha$ as depending only on $\sigma = (\sigma^1, ..., \sigma^p)$ but not on $\tau$ (this is a consequence from their equations of motion).

In the gauge when $\mu^0$, $\mu^j$ and $\mu^\alpha$ are constants, the general solution of \footnote{3} is

\[
\begin{align*}
x'^\nu(\tau, \underline{z}) &= x^\nu(t, \underline{z}) + \tau [2\mu^0 p^\nu(t, \underline{z}) - (\mu \sigma^\nu \theta(t, \underline{z}))], \\
&= x^\nu(t, \underline{z}) + \tau [2\mu^0 p^\nu(t, \underline{z}) - (\mu \sigma^\nu \theta(t, \underline{z}))] \\
p_\nu(\tau, \underline{z}) &= p_\nu(t, \underline{z}), \\
\theta^\alpha(\tau, \underline{z}) &= \theta^\alpha(t, \underline{z}) + i\tau \mu^\alpha, \\
p_{\theta \alpha}(\tau, \underline{z}) &= p_{\theta \alpha}(t, \underline{z}) + \tau (\mu \sigma^\nu \alpha) p_\nu(t, \underline{z}),
\end{align*}
\]

where $x^\nu(t, \underline{z})$, $p_\nu(t, \underline{z})$, $\theta^\alpha(t, \underline{z})$ and $p_{\theta \alpha}(t, \underline{z})$ are arbitrary functions of their arguments

$$z^j = \mu^j \tau + \sigma^j.$$

In the case of tensionless strings ($p=1$), one can write explicitly the general solution of the equations of motion in arbitrary gauge: $\mu^0 = \mu^0(\sigma)$, $\mu^1 \equiv \mu = \mu(\sigma)$,
\[ \mu^\alpha = \mu^\alpha(\sigma). \] This solution is given by
\[ x^\nu(\tau, \sigma) = g^\nu(w) - 2 \int \frac{\mu^0(s)}{\mu(s)^2} ds f^\nu(w) + \int \frac{\mu^\alpha(s)}{\mu(s)} ds [\sigma^\nu \zeta(w)]_\alpha \]
\[ - i \int \frac{\mu^\alpha(s)}{\mu(s)} ds_1 \frac{(\mu^\nu_\alpha(s_1))}{\mu(s_1)} \int \frac{\mu^\alpha(s)}{\mu(s)} ds, \]
\[ p_\nu(\tau, \sigma) = \mu^{-1}(\sigma) f_\nu(w), \]
\[ \theta^\alpha(\tau, \sigma) = \zeta^\alpha(w) - i \int \frac{\mu^\alpha(s)}{\mu(s)} ds, \]
\[ p_{\theta\alpha}(\tau, \sigma) = \mu^{-1}(\sigma) \left[ h_\alpha(w) - \int \frac{\mu^\nu_\alpha(s)}{\mu(s)} ds f_\nu(w) \right]. \]

Here \( g^\nu(w), f_\nu(w), \zeta^\alpha(w) \) and \( h_\alpha(w) \) are arbitrary functions of the variable
\[ w = \tau + \int \frac{\sigma}{\mu(s)} ds. \]

When \( p = 1 \), the solution (3) differs from (4) by the choice of the particular solutions of the inhomogeneous equations. As for \( z \) and \( w \), one can write for example \( (\mu^0, \mu, \mu^\alpha \) are now constants)
\[ p_\nu(\tau, \sigma) = \mu^{-1} f_\nu(\tau + \sigma/\mu) = \mu^{-1} f_\nu[\mu^{-1}(\mu\tau + \sigma)] = p_\nu(z) \]
and analogously for the other arbitrary functions in the general solution of the equations of motion.

### 3 Lagrangian formulation

Taking into account the equations of motion for \( x^\nu \) and \( \theta^\alpha \), one obtains the corresponding Lagrangian density
\[ L = \frac{1}{4\mu^0} [(\partial_\tau - \mu^j \partial_j)x + i\theta^\sigma(\partial_\tau - \mu^j \partial_j)\theta]^2. \]

Indeed, one verifies that the equations of motion for the Lagrange multipliers \( \mu^0 \) and \( \mu^j \) give the constraints \( T_0 \) and \( T_j \). The remaining constraints follow from the definition of the momenta \( p_{\theta\alpha} \).

To establish the invariances of the action, it is useful to rewrite \( L \) in the form
\[ L = V^J V^K Y^\nu_J Y^K_\nu, \quad (J, K = 0, 1, ..., p), \]
where
\[ V^J = (V^0, V^j) = \left( -\frac{1}{2\sqrt{\mu^0}}, \frac{\mu^j}{2\sqrt{\mu^0}} \right) \]
Thus, the action

$$S = \int d^{p+1}\xi V^{J} V^{K} Y_{\mu}^{J} Y_{\nu}^{K} \quad , \quad \xi^{J} = (\xi^{0}, \xi^{J}) = (\tau, \varphi),$$

has global super-Poincaré symmetry, local world-volume reparametrization and \(\kappa\)-invariances. Let us show that this is indeed the case. Before doing this, we note that actions of this type are first given in [6] for the case of tensionless superstring \((p=1)\) and in [7] for the bosonic case \((N=0)\).

The global Poincaré invariance is obvious. Under global infinitesimal supersymmetry transformations, the fields \(x^{\mu}(\xi), \theta^{a}(\xi)\) and \(V^{J}(\xi)\) transform as follows

$$\delta_{\eta}\theta^{a} = \eta^{a} \quad , \quad \delta_{\eta}x^{\mu} = i(\theta\sigma^{\mu}\delta_{\eta}\theta) \quad , \quad \delta_{\eta}V^{J} = 0.$$ 

As a consequence, \(\delta_{\eta}Y^{\nu}_{J} = 0\) and hence \(\delta_{\eta}L = \delta_{\eta}S = 0\) also.

To establish the invariance of the action under infinitesimal diffeomorphisms, we first write down the corresponding transformation law for the \((r,s)\)-type tensor density of weight \(a\)

$$\delta_{\varepsilon}T^{J_{1}\ldots J_{r}}_{K_{1}\ldots K_{s}}[a] = L_{\varepsilon}T^{J_{1}\ldots J_{r}}_{K_{1}\ldots K_{s}}[a] = \varepsilon^{L}\partial_{L}T^{J_{1}\ldots J_{r}}_{K_{1}\ldots K_{s}}[a]$$

$$+ \ T^{J_{1}\ldots J_{r}}_{K_{1}\ldots K_{s}}[a]\partial_{K_{1}}\varepsilon^{K} + \ldots + \ T^{J_{1}\ldots J_{r}}_{K_{1}\ldots K_{s+1}}[a]\partial_{K_{s}}\varepsilon^{K}$$

$$- \ T^{J_{1}\ldots J_{r}}_{K_{1}\ldots K_{s}}[a]\partial_{L}\varepsilon^{J_{1}} + \ldots - \ T^{J_{1}\ldots J_{r-1}}_{K_{1}\ldots K_{s}}[a]\partial_{J_{r}}\varepsilon^{J_{1}}$$

$$+ \ aT^{J_{1}\ldots J_{r}}_{K_{1}\ldots K_{s}}[a]\partial_{L}\varepsilon^{L},$$

where \(L_{\varepsilon}\) is the Lie derivative along the vector field \(\varepsilon\). Using (5), one verifies that if \(x^{\mu}(\xi), \theta^{a}(\xi)\) are world-volume scalars \((a=0)\) and \(V^{J}(\xi)\) is a world-volume \((1,0)\)-type tensor density of weight \(a = 1/2\), then \(Y^{\nu}_{J}\) is a \((0,1)\)-type tensor, \(Y^{\nu}_{J}Y_{K\nu}\) is a \((0,2)\)-type tensor and \(L\) is a scalar density of weight \(a = 1\). So,

$$\delta_{\varepsilon}S = \int d^{p+1}\xi \partial_{J}(\varepsilon^{J}L)$$

and the variation \(\delta_{\varepsilon}S\) of the action vanishes under suitable boundary conditions.

Let us now check the kappa-invariance. We define the \(\kappa\)-variations of \(\theta^{a}(\xi), x^{\nu}(\xi)\) and \(V^{J}(\xi)\) as follows:

$$\delta_{\kappa}\theta^{a} = i(G\kappa)^{a} = iV^{J}(X_{J}\kappa)^{a},$$

$$\delta_{\kappa}x^{\nu} = -i(\theta\sigma^{\nu}\delta_{\kappa}\theta),$$

$$\delta_{\kappa}V^{K} = 2V^{K}V^{L}(\partial_{L}\kappa).$$

Therefore, \(\kappa^{a}(\xi)\) is a left Majorana-Weyl space-time spinor and world-volume scalar density of weight \(a = -1/2\).

From (6) we obtain:

$$\delta_{\kappa}(Y^{\nu}_{J}Y_{K\nu}) = -2i[\partial_{J}\theta \ Y_{K} + \partial_{K}\theta \ Y_{J}]\delta_{\kappa}\theta$$
and
\[ \delta_\kappa L = 2 V^J Y^\nu Y_K^{\nu} [\delta_\kappa V^K - 2 V^K V^L (\partial_L \theta \kappa)] = 0. \]

The algebra of kappa-transformations closes only on the equations of motion, which can be written in the form:

\begin{align*}
\partial_I (V^J V^K Y_{K\nu}) &= 0, \\
V^J V^K (\partial_I \theta Y_K^\alpha) &= 0, \\
V^J Y^\nu_{J\nu} Y_{K\nu} &= 0.
\end{align*}
(7)

As usual, an additional local bosonic world-volume symmetry is needed for its closure. In our case, the Lagrangian, and therefore the action, are invariant under the following transformations of the fields:

\[ \delta_\lambda \theta (\xi) = \lambda V^J \partial_J \theta, \quad \delta_\lambda x^\nu (\xi) = -i (\theta \sigma^\nu \delta_\lambda \theta), \quad \delta_\lambda V^J (\xi) = 0. \]

Now, checking the commutator of two kappa-transformations, we find:

\begin{align*}
[\delta_{\kappa_1}, \delta_{\kappa_2}] \theta^\alpha (\xi) &= \delta_\kappa \theta^\alpha (\xi) + \text{terms } \propto \text{eqs. of motion}, \\
[\delta_{\kappa_1}, \delta_{\kappa_2}] x^\nu (\xi) &= (\delta_\kappa + \delta_\epsilon + \delta_\lambda) x^\nu (\xi) + \text{terms } \propto \text{eqs. of motion}, \\
[\delta_{\kappa_1}, \delta_{\kappa_2}] V^J (\xi) &= \delta_\epsilon V^J (\xi) + \text{terms } \propto \text{eqs. of motion}.
\end{align*}

Here \( \kappa (\xi), \lambda (\xi) \) and \( \epsilon (\xi) \) are given by the expressions:

\[ \kappa^\alpha = -2 V^K [ (\partial_K \theta \kappa_1) \kappa_2^\alpha - (\partial_K \theta \kappa_2) \kappa_1^\alpha], \]
\[ \lambda = 4i V^K (\kappa_1 Y_K^{\kappa_2}), \quad \epsilon^J = -V^J \lambda. \]

We stress that
\[ \Gamma_{\alpha\beta} = (V^J Y_J^J)_{\alpha\beta} \]

in (8) has the following property on the equations of motion

\[ \Gamma^2 = 0. \]

This means that the kappa-invariance of the action indeed halves the fermionic degrees of freedom as is needed.

Finally, we give the expression for world-volume stress-energy tensor

\[ T^J_K = (2V^J Y^\nu_K - \delta^J_K V^L Y^\nu_L) V^M Y_{M\nu}, \quad Tr(T) = (1 - p) L. \]
(8)

From (7) and (8) it is clear, that

\[ T^J_K = 0 \]
(9)

on the equations of motion. It is natural, because the equality (9) is a consequence of \( p + 1 \) of the constraints.
4 Conclusions

In this letter we consider a model for tensionless (null) $p$-branes with $N = 1$ global supersymmetry in 10-dimensional Minkowski space-time. We give an action for the model and show that it is reparametrization and kappa-invariant. As usual, the algebra of kappa-transformations closes only on-shell and it halves the fermionic degrees of freedom. In proving the kappa-invariance, we do not use any specific ten dimensional properties of the spinors. Hence, the model is extendable classically to other space-time dimensions. There exist also the possibility of its generalization to $N$ supersymmetries \[8\]. The properties of the model in nontrivial backgrounds are also under investigation \[9\].

In this letter we also find some solutions of the classical equations of motion. In the case of null superstring ($p = 1$), we obtain the general solution in arbitrary gauge.

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