THE CHARACTER THEORY OF A COMPLEX GROUP

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Abstract. We apply the ideas of derived algebraic geometry and topological field theory to the representation theory of reductive groups. We first establish good functoriality properties for ordinary and equivariant \( \mathcal{D} \)-modules on schemes, in particular showing that the integral transforms studied in algebraic analysis give all continuous functors on \( \mathcal{D} \)-modules. We then focus on the categorified Hecke algebra \( \mathcal{H}_G \) of Borel bi-equivariant \( \mathcal{D} \)-modules on a complex reductive group \( G \). We show that its monoidal center and abelianization (Hochschild cohomology and homology categories) coincide and are identified through the Springer correspondence with the derived version of Lusztig’s character sheaves. We further show that \( \mathcal{H}_G \) is a categorified Calabi-Yau algebra, and thus satisfies the strong dualizability conditions of Lurie’s proof of the cobordism hypothesis. This implies that \( \mathcal{H}_G \) defines the \((0,1,2)\)-dimensional part of a three-dimensional topological field theory which we call the character theory \( \chi_G \). It organizes much of the representation theory associated to \( G \). For example, categories of Lie algebra representations and Harish Chandra modules for \( G \) and its real forms give natural boundary conditions in the theory. In particular, they have characters (or charges) as Hecke modules which are character sheaves. The Koszul duality for Hecke categories provides an equivalence between character theories for Langlands dual groups, and in particular a duality of character sheaves. It can be viewed as a dimensionally reduced version of the geometric Langlands correspondence, or as \( S \)-duality for a generically twisted maximally supersymmetric gauge theory in three dimensions.

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1. Introduction

Algebraic systems of differential equations in the form of \( D \)-modules are central objects in algebraic analysis. They allow one to separate out the algebraic and geometric structures of a system of differential equations from the analytic issues of solvability in specific functions spaces. Categorical tools such as functoriality and adjunctions, and geometric tools such as integral transforms allow one to reduce complicated \( D \)-modules to simpler ones. In this paper, we establish foundational results relating these categorical and geometric tools, and apply them to categories of \( D \)-modules arising in geometric representation theory. (The reader primarily interested in algebraic analysis and results on integral transforms could skip to Section 1.5 where our results are discussed from that perspective.)

The seminal works of Harish Chandra, Kashiwara, Kazhdan, Lusztig, Beilinson, Bernstein and other early pioneers place \( D \)-modules at the heart of representation theory. The fundamental categories in the representation theory of reductive groups – such as Harish Chandra modules for real and complex Lie groups and the highest weight representations of Category \( O \) – are equivalent to categories of \( D \)-modules on flag varieties and related spaces. Moreover, the natural symmetries of representations – such as the intertwining operators for principal series representations – correspond to integral transforms acting on \( D \)-modules. For example, for \( D \)-modules on the flag variety \( G/B \) of a complex reductive group \( G \), the collection of all such integral transforms form the (differential graded) Hecke category \( \mathcal{H}_G = \mathcal{D}(B \setminus G/B) \) of Borel bi-equivariant (complexes of) \( D \)-modules on \( G \).

Composition of integral transforms (often called convolution) equips the Hecke category \( \mathcal{H}_G \) with a monoidal structure that categorifies the classical Hecke algebra associated to the Weyl group of \( G \). In another direction, Harish Chandra established his celebrated regularity results for distributional characters of admissible representations of real Lie groups by showing that they satisfy an adjoint-equivariant regular holonomic system of differential equations. The resulting \( D \)-module is the fundamental example of a character sheaf in the sense of Lusztig and also is the basic object in Springer theory.

Our main results are best understood (or at least most succinctly stated) in the framework of extended topological field theory. We show that there is a topological field theory (TFT for short), which we call the character theory and denote by \( \chi_G \), that organizes much of the geometric representation theory associated to \( G \). The character theory can be viewed as an extended three-dimensional TFT, which we only define on 0,1 and 2 dimensional manifolds, or alternatively as a categorified two-dimensional TFT. It assigns the Hecke category \( \mathcal{H}_G \) of Borel equivariant \( D \)-modules on the flag variety to a point. (Equivalently, we can say that \( \chi_G \) assigns the 2-category of \( \mathcal{H}_G \)-module categories to a point.) The character theory \( \chi_G \) assigns to the circle the derived Drinfeld center of \( \mathcal{H}_G \), which we show is a derived version of the category of Lusztig’s character sheaves on the
group $G$ itself. This implies that character sheaves admit the rich operadic structure of Drinfeld centers (topological Hochschild cohomology), and also admit a dual description via the deep theory of Langlands duality. In order to construct the character theory $\chi_G$, we develop the functional analysis of categories of $D$-modules, studying functoriality, integral transforms and various forms of duality.

Before explaining our results further, we begin with a brief prelude about finite groups to acclimate the reader to the structures of two dimensional TFT. Throughout we will use the following idiosyncratic notation: we will write $G/\text{ad}G$ to denote the adjoint quotient of a group $G$, or in other words, the quotient of $G$ by itself acting by conjugation. More generally, for a subgroup $H \subset G$, we will write $G/\text{ad}H$ to denote the quotient of $G$ by the subgroup $H$ acting by conjugation. For example, for a reductive group $G$ and Borel subgroup $B \subset G$, we can write $G/\text{ad}B$ and not confuse the quotient with the flag variety $G/B$.

1.1. Toy model: finite group gauge theory. We will not give a formal introduction to extended two dimensional topological field theory but rather remind the reader of its broad outline informally via the example of finite group gauge theory. The character theory we introduce below for a complex reductive group will have categorified analogues for all of the structures discussed here.

Let $G$ be a finite group. There is an extended two dimensional TFT called topological gauge theory or Dijkgraaf-Witten theory after [DW] and denoted by $Z_G$ that encodes all of the familiar structures in the complex representation theory of $G$ (see [F] for a detailed study). It is a quantum gauge theory with gauge group $G$ over 0, 1, and 2-manifolds (with boundaries and corners).

An extended two dimensional TFT is a symmetric monoidal functor $Z$ with domain the 2-category $2\text{Bord}$ with the following structure:

- objects: 0-manifolds,
- 1-morphisms: 1-dimensional bordisms between 0-manifolds,
- 2-morphisms: 2-dimensional bordisms between 1-bordisms, up to diffeomorphism.

The symmetric monoidal structure of $2\text{Bord}$ is given by disjoint union, with unit the empty 0-manifold.

The target of the functor $Z_G$ is the 2-category $\text{AbCat}_\mathbb{C}$ with the following structure:

- objects: small $\mathbb{C}$-linear abelian categories,
- 1-morphisms: exact functors,
- 2-morphisms: natural transformations.

The symmetric monoidal structure of $\text{AbCat}_\mathbb{C}$ is given by tensor product with unit the category of finite dimensional complex vector spaces $\text{Vect}^{fd}_\mathbb{C}$.

We will be interested in $\mathbb{C}$-linear abelian categories that can be written as module categories over an associative algebra over $\mathbb{C}$. Thus we can restrict our attention to the the Morita 2-category of algebras $\text{Alg}_\mathbb{C}$ with the following structure:

- objects: associative algebras over $\mathbb{C}$,
- 1-morphisms: bimodules,
- 2-morphisms: morphisms of bimodules.

(More precisely, exact functors between module categories of algebras correspond to bimodules which are flat over the source algebra.) We will identify $\text{Alg}_\mathbb{C}$ with the full subcategory of $\text{AbCat}_\mathbb{C}$ given by assigning to an algebra $A$ its category of modules $\text{Mod}_A$.

Now given a finite group $G$, the field theory $Z_G$ assigns to each cobordism $M$ a linearization of the space of the space of gauge fields on $M$ (or in other words, a linearization of the orbifold of principal $G$-bundles or Galois $G$-covers over $M$). In particular, $Z_G$ assigns the following to closed 0, 1 and 2-manifolds:

- To a point, $Z_G$ assigns the group algebra of distributions:

$$Z_G(\text{pt}) = \mathbb{C}[G] \in \text{Ob}(\text{Alg}_\mathbb{C}).$$
Equivalently, we can view $Z_G$ as assigning the category of finite dimensional complex $G$-modules:

$$Z_G(pt) = \text{Rep}_\mathbb{C}^{fd}(G) = \text{Mod}_\mathbb{C}^{\text{fd}} = \text{Vect}_\mathbb{C}^{\text{fd}}(BG) \in \text{Ob}(\text{AbCat}_\mathbb{C}),$$

or in other words, finite dimensional complex algebraic vector bundles on the orbifold of $G$-bundles on a point.

- To a circle, $Z_G$ assigns the vector space of complex class functions on $G$:
  $$Z_G(S^1) = \mathbb{C}[G]^G = \mathbb{C}[G/_{\text{ad}} G] \in \text{Vect}_\mathbb{C} = 1 \text{Hom}_{\text{AbCat}_\mathbb{C}}(\text{Vect}_\mathbb{C}, \text{Vect}_\mathbb{C}),$$

or in other words, complex functions on the orbifold of $G$-bundles on the circle.

- To a closed surface, $Z_G$ counts the number of points of the orbifold of $G$-bundles over the surface:
  $$Z_G(\Sigma) = \#\{\text{Hom}(\pi_1(\Sigma), G)/G\} \in \mathbb{C} = 2 \text{Hom}_{\text{AbCat}_\mathbb{C}}(\mathbb{C}, \mathbb{C})$$

where as usual, a bundle $\mathcal{P}$ is weighted by $1/\text{Aut}(\mathcal{P})$.

The field theory $Z_G$ encodes many of the salient features of finite group representation theory. Let us mention several that will arise again in the case of complex reductive groups:

- The group algebra $Z_G(pt) = \mathbb{C}[G]$ is a noncommutative Frobenius algebra with nondegenerate trace
  $$\tau : \mathbb{C}[G] \to \mathbb{C} \quad \tau(f) = f(e)/|G|$$

where $e \in G$ is the unit. The class functions $Z_G(S^1) = \mathbb{C}[G]^G$ are a commutative Frobenius algebra with trace induced by $\tau$ under a trace map (see below).

- The class functions $Z_G(S^1) = \mathbb{C}[G]^G$ are the center of the group algebra $Z_G(pt) = \mathbb{C}[G]$. Equivalently, they are the endomorphisms of the identity functor of $\text{Rep}_\mathbb{C}^{fd}(G)$ so act functorially on any representation $M \in \text{Rep}_\mathbb{C}^{fd}(G)$.

- The class functions $Z_G(S^1) = \mathbb{C}[G]^G$ are also the abelianization of the group algebra $Z_G(pt) = \mathbb{C}[G]$, or in other words, the target of a universal trace map
  $$\text{tr} : \mathbb{C}[G] \to \mathbb{C}[G]^G$$

(an initial map among all maps that are equal on $ab$ and $ba$, for all $a, b \in \mathbb{C}[G]$). Equivalently, they are the recipient of a functorial trace map from endomorphisms of any representation $M \in \text{Rep}_\mathbb{C}^{fd}(G)$ which assigns to $id_M$ the character $\chi_M \in \mathbb{C}[G]^G$.

As an example of the last highlighted property, consider a subgroup $H \subset G$ and the induced representation $\mathbb{C}[G/H] \in \text{Rep}_\mathbb{C}^{fd}(G)$. Its endomorphisms are the Hecke algebra $\mathbb{C}[H\backslash G/H]$ of $H$-biinvariant functions on $G$. The trace of an element of the Hecke algebra can be calculated by pullback and pushforward along the diagram

$$G/_{\text{ad}} G \leftarrow G/_{\text{ad}} H \to H\backslash G/H.$$

1.2. **Background: geometric representation theory.** We state our main results in the next section, but provide here a brief reminder on the representation theoretic significance of the categories of $D$-modules that will appear.

Geometric approaches to the representation theory of Lie groups are intimately linked with the study of $D$-modules. For example, a traditional starting point for geometric representation theory is the study of linear actions of real Lie groups $G(\mathbb{R})$ on topological vector spaces such as function spaces of homogeneous spaces. To capture the underlying algebraic structure of the symmetries while avoiding the analytic intricacies involving the function spaces, one replaces such representations with their infinitesimal equivalence classes in the form of Harish Chandra $(\mathfrak{g}, K)$-modules. Then to apply the powerful sheaf-theoretic techniques of algebraic geometry, one uses the localization theorem of Beilinson-Bernstein to identify Harish Chandra modules with $K$-equivariant $D$-modules on the flag variety $G/B$. With this approach, one sees that questions in the representation theory of real Lie groups can be reformulated in the geometric language of $D$-modules.
We will single out two categories of $\mathcal{D}$-modules of fundamental interest in geometric representation theory: the Hecke category of Borel bi-equivariant $\mathcal{D}$-modules on a complex reductive group $G$, and Lusztig’s character sheaves (with unipotent central character) on the adjoint quotient $G/_{ad}G$. One of the main motivations for this paper was to understand the precise relation between the two.

Since our methods and results will involve homotopical algebra and topological field theory, we find it most natural to work in the context of derived categories. More precisely (as explained below in the preliminaries), we will work in the underlying $\infty$-categories (or differential graded categories) of complexes of $\mathcal{D}$-modules whose homotopy categories are the derived categories one usually encounters. Thus by a $\mathcal{D}$-module, we will usually mean a complex of $\mathcal{D}$-modules and not specify that further. Given a smooth scheme or stack $X$, we will denote by $\mathcal{D}(X)$ the $\infty$-category of all complexes of $\mathcal{D}$-modules on $X$.

1.2.1. Hecke categories. Let $G$ be a complex reductive group with Borel subgroup $B \subset G$.

The intertwiners acting on Harish Chandra modules of real forms of $G$ (such as principal series representations) or on any category of $\mathcal{D}$-modules on the flag variety $G/B$ are realized by Hecke correspondences. For example, there are the basic integral transforms associated to pairs of flags in a fixed relative position indexed by an element of the Weyl group of $G$. (These are closely related to the Radon and Penrose transforms of algebraic analysis.) More generally, all $B$-equivariant $\mathcal{D}$-modules on the flag variety $G/B$ provide integral kernels and thus integral transforms. The collection of all such $\mathcal{D}$-modules is often called the Hecke category $\mathcal{H}_G = \mathcal{D}(B \backslash G/B)$ of the group $G$. It is a categorification of the finite Hecke algebra appearing in Kazhdan-Lusztig theory. By Beilinson-Bernstein localization, $\mathcal{H}_G$ is a derived version of the category of Harish Chandra bimodules for the Lie algebra $\mathfrak{g}$ – or Harish Chandra modules for the complex group $G$ – with trivial infinitesimal character. We will also discuss the closely related monodromic Hecke category $\mathcal{H}_G^{mon} \subset \mathcal{D}(N \backslash G/N)$, consisting of $\mathcal{D}$-modules which have generalized unipotent monodromy with respect to the actions of the torus $T = B/N$ on the left and right. This category is also a categorification of the finite Hecke algebra, and is equivalent by localization to Harish Chandra bimodules with generalized trivial infinitesimal character.

1.2.2. Character sheaves. As mentioned above, Harish Chandra’s theory of characters for representations of real Lie groups relies on an adjoint-equivariant differential equation satisfied by the characters. The resulting $\mathcal{D}$-module, known also as the Springer sheaf [HK], is the fundamental example of a character sheaf in the sense of Lusztig [Lu]. See also [La] for a review of character sheaves and [Gi,MV] for geometric approaches to the theory. (As far as technical results needed, we will work within the framework of [MV].)

The traditional definition [Lu] of character sheaf (as reformulated in [Gi,MV]) is based on the horocycle correspondence

$$G/_{ad}G \xleftarrow{p} G/_{ad}B \xrightarrow{\delta} B \backslash G/B.$$ 

The map $p$ is the quotient with fibers isomorphic to the flag variety $G/B$, and the map $\delta$ is the quotient with fibers isomorphic to $B$. Pulling back and pushing forward $\mathcal{D}$-modules gives a functor called the Harish Chandra transform (in the terminology of [Gi])

$$F = p_* \delta^! : \mathcal{H}_G = \mathcal{D}(B \backslash G/B) \to \mathcal{D}(G/_{ad}G).$$

An unipotent character sheaf is a $G$-equivariant $\mathcal{D}$-module on $G$ that is a simple constituent of a $\mathcal{D}$-module obtained by applying the transform $F$ to a simple $\mathcal{D}$-module (in the heart of the standard $t$-structure on $\mathcal{H}_G$). They also can be characterized geometrically as the simple adjoint-equivariant $\mathcal{D}$-modules on $G$ with singular support in the nilpotent cone and unipotent central character. Lusztig showed that the collection of all character sheaves provides a construction of the characters of finite groups of Lie type, and gave a detailed classification in relation with the structure of the finite Hecke algebra.

With the above in mind, we make the following definition in the homotopical setting.
Definition 1.1. The $\infty$-category $\text{Ch}_G$ of unipotent character sheaves is defined to be the full subcategory of $\mathcal{D}(G/\text{ad} G)$ generated (under colimits) by holonomic objects with nilpotent singular support and unipotent central character. Equivalently, it is the full subcategory generated (under colimits) by the image of the Harish Chandra transform $F$.

Note that traditional character sheaves are simple objects of the heart of the character category $\text{Ch}_G$ with respect to the standard $t$-structure.

Example 1.2. The above correspondence can be viewed as a collection of Weyl group twisted versions of the Grothendieck-Springer simultaneous resolution. Namely, if we restrict to the support $pt/B = B\backslash B/B \subset B\backslash G/B$ of the monoidal unit of $\mathcal{H}_G$, then we recover (an equivariant global version of) the Grothendieck-Springer correspondence

$$G/\text{ad} G \xrightarrow{p} \tilde{G}/G \xrightarrow{\delta} B\backslash B/B \simeq pt/B.$$ 

Here $\tilde{G}/G$ denotes the simultaneous resolution

$$\tilde{G}/G = \{(g, B') | g \in B'\}/G \simeq B/\text{ad} B,$$ 

and the map $p$ forgets the flag $B'$, while the map $\delta$ forgets the group element $g$.

The global version of the Springer sheaf

$$S_G = F(O_{pt/B}) = p_* \delta^! O_{pt/B}$$

is the fundamental example of a character sheaf. According to [HK], it coincides with Harish Chandra’s adjoint-equivariant holonomic system of differential equations obtained by setting all of the Casimir operators to zero.

1.3. Main results: character theory for complex reductive groups. We continue to fix a complex reductive group $G$ with Borel subgroup $B \subset G$. (Our results generalize easily to any parabolic subgroup $P \subset G$).

1.3.1. Representation theory formulation. Our results center around the Hecke category $\mathcal{H}_G = \mathcal{D}(B\backslash G/B)$ of Borel equivariant (complexes of) $\mathcal{D}$-modules on the flag variety $G/B$ equipped with its natural convolution structure.

As mentioned above (and explained below in the preliminaries), we will work in the framework of $\infty$-categories (or differential graded categories) whose homotopy categories are familiar derived categories. For foundations on $\infty$-categories (in the sense of Joyal [Jo]), we refer to the comprehensive work of Lurie [L1, L2, L3, L4]. (See Sections 2.1, 2.2, 2.3 for a summary of what we will need of this theory.) In organizing our results, it will also be useful to adopt the language of homotopical 2-categories in the form of $(\infty, 2)$-categories. Roughly speaking, an $(\infty, 2)$-category is an $(\infty, 1)$-category tensored and enriched over $(\infty, 1)$-categories. We refer the reader to [L6] for a general study of $(\infty, 2)$-categories, as well as to [L5] where their role in extended TFT is explained.

Though the above foundations can be daunting, it is worth commenting that our results and proofs involve familiar structures.

We can view the monoidal category $\mathcal{H}_G$ as an algebra object in the $\infty$-category $St$ of stable presentable $\infty$-categories with morphisms given by continuous (colimit-preserving) functors. Going one step further, we can view $\mathcal{H}_G$ as an object in the Morita $(\infty, 2)$-category $2\text{Alg}$ of algebra objects in $St$ with 1-morphisms given by bimodules, and 2-morphisms given by natural transformations.

Here is a summary of some of our main results (see Section 1.3.3 for other variants):

- [Theorem 5.8] The Hecke category $\mathcal{H}_G$ is a Calabi-Yau object of $2\text{Alg}$ with trace $\tau : \mathcal{H}_G \to \text{Vect}$

the natural continuous extension of the hom-pairing on compact objects

$$\tau(\mathcal{M}) = \text{Hom}_{\mathcal{H}_G}(u, \mathcal{M}), \quad \text{for } u \in \mathcal{H}_G \text{ the monoidal unit.}$$
• [Theorem 6.10] The $\infty$-category $\text{Ch}_G$ of unipotent character sheaves is the Drinfeld center (or categorical Hochschild cohomology) of the Hecke category $\mathcal{H}_G$ with central action

$$\mathcal{J} : \text{Ch}_G \to \mathcal{H}$$

given by the horocycle correspondence (or twisted simultaneous resolution). In particular, $\text{Ch}_G$ is naturally an $E_2$-algebra object in $\text{St}$, and acts functorially as endomorphisms of any Hecke module $M \in \text{Mod}_{\mathcal{H}_G}(\text{St})$.

• [Theorem 6.10] The $\infty$-category $\text{Ch}_G$ is also the abelianization (or categorical Hochschild homology) of the Hecke category $\mathcal{H}_G$ with universal trace

$$\text{tr} : \mathcal{H} \to \text{Ch}_G$$

again given by the horocycle correspondence (or twisted simultaneous resolution). In particular, $\text{Ch}_G$ comes equipped with an $S^1$-action compatible with that on $D$-modules on the loop space $G_{/\text{ad}} = L(BG)$, as well as an $S^1$-invariant trace

$$\widetilde{\tau} : \text{Ch}_G \to \text{Vect}$$

descended from the trace on $\mathcal{H}_G$. Moreover, $\text{Ch}_G$ is the recipient of a universal trace from endomorphisms of any Hecke module, and thus every Hecke module $M \in \text{Mod}_{\mathcal{H}_G}(\text{St})$ has a character $\chi_M \in \text{Ch}_G$ associated to the identity $\text{id}_M$.

Let us say a few words about a particular instance of the last highlighted result (we refer to [TV] for the general theory of characters of module $\infty$-categories as objects in the Hochschild homology category). Consider a spherical subgroup $K \subset G$ (for example, $K$ a parabolic subgroup, or the fixed points of an involution). Harish Chandra $(\mathfrak{g}, K)$-modules with trivial infinitesimal character are equivalent to $K$-equivariant $D$-modules on $G/B$. The latter naturally provide a Hecke module where the Hecke category acts via convolution on the right. The character of this Hecke module is the object of $\text{Ch}_G$ given by pushing forward the structure sheaf of $G_{/\text{ad}}K$ along the natural project to $G_{/\text{ad}}G$, and then projecting onto $\text{Ch}_G$ (the central functor $\mathcal{J}$ given by the horocycle correspondence naturally extends to a functor from $\mathcal{D}(G_{/\text{ad}}G)$ to the Drinfeld center of $\mathcal{H}$ and hence provides a projection to $\text{Ch}_G$). For example, the Springer sheaf $S_G$ discussed in Example 1.2 is the character of the left regular representation of the Hecke category.

Remark 1.3. It is interesting to compare this with a recent independent result of Bezrukavnikov, Finkelberg and Ostrik [BFO] who prove that the Drinfeld center of the abelian category of Harish Chandra bimodules is equivalent to the abelian category generated by character sheaves. The $\infty$-categorical version of this result, identifying the center of the monodromic Hecke category $\mathcal{H}_G^{\text{mon}}$ with character sheaves, is discussed in Section 1.3.3 below.

1.3.2. Topological field theory reformulation. Baez and Dolan [BDa] formulated a principle, the cobordism hypothesis, which asserts that extended $d$-dimensional TFTs (which assign invariants to arbitrary manifolds with boundaries and corners of dimension at most $n$) should be completely determined by what they assign to a point. Conversely, an object of an $n$-category satisfying strong finiteness assumptions should freely determine an $n$-dimensional extended TFT. In two dimensions, the notion of extended TFT is closely related to that of open-closed field theory as studied in detail by Moore and Segal [MS] in the semisimple setting (such as the finite group gauge theory recalled above) and by Costello [C] in the derived setting. (Costello’s arguments were used by Hopkins and Lurie to give a proof of the cobordism hypothesis in dimension two.) Lurie has given a precise formulation of the general cobordism hypothesis in an $\infty$-categorical setting in [L5], and presented a detailed outline of a proof and a variety of generalizations. In reformulating our results in the framework of TFT, we adopt below the geometric constructions and categorical language (such as bordism categories, and general $(\infty, 2)$-categories) explained in [L5].

Our results outlined above allow us to show that $\mathcal{H}_G$ satisfies the conditions of Lurie’s proof of the cobordism hypothesis, and thus conclude that $\mathcal{H}_G$ defines an extended categorified two-dimensional
topological field theory. We call the resulting field theory the (unipotent) character theory $\chi_G$ associated to $G$. It is a symmetric monoidal functor

$$\chi_G : 2\text{Bord} \to 2\text{Alg}$$

from the $(\infty, 2)$-category of unoriented bordisms:

- objects: 0-manifolds,
- 1-morphisms: 1-dimensional bordisms between 0-manifolds,
- 2-morphisms: classifying spaces of 2-dimensional bordisms between 1-bordisms,

to the Morita $(\infty, 2)$-category of algebras $2\text{Alg}$:

- objects: algebra objects in stable presentable $\infty$-categories,
- 1-morphisms: bimodule objects in stable presentable $\infty$-categories,
- 2-morphisms: classifying spaces of morphisms of bimodules.

Note that given $A \in 2\text{Alg}$, we can pass to the $(\infty, 2)$-category of modules $\text{Mod}_A(\text{St})$ and bimodules correspond to functors between $(\infty, 2)$-categories of modules. Thus we can think of the corresponding field theory as assigning $\text{Mod}_A(\text{St})$ to a point.

The field theory $\chi_G$ assigns the following to closed 0, 1 and 2-manifolds:

- To a point, $\chi_G$ assigns the Hecke category $\chi_G(\text{pt}) = \mathcal{H}_G \in \text{Ob}(2\text{Alg})$.
  
  Equivalently, $\chi_G(\text{pt})$ is the $(\infty, 2)$-category of Hecke modules $\text{Mod}_{\mathcal{H}_G}(\text{St})$. Examples of objects include $\infty$-categories of Harish Chandra modules for subgroups $K \subset G$, the BGG Category $\mathcal{O}$, and in general, the $\infty$-category of $\mathcal{D}$-modules $\mathcal{D}(X/B)$, for any scheme $X$ with a $G$-action.

- To a circle, $\chi_G$ assigns the Hochschild homology of $H_G$, which by the above results is the $\infty$-category of unipotent character sheaves $\chi_G(S^1) = \mathcal{C}h_G \subset \mathcal{D}(G/\text{ad}G)$.

- To the 2-sphere, $\chi_G$ assigns the $T$-equivariant cohomology of a point $\chi_G(S^2) = H^*_T(\text{pt})$.

  and to the 2-torus, it assigns the Hochschild chain complex of $\mathcal{C}h_G$.

The identification of the abelianization of the Hecke category $\mathcal{H}_G$ with character sheaves implies that any module category $M \in \text{Mod}_{\mathcal{H}_G}(\text{St})$ has a character object in $\mathcal{C}h_G$. In other words, character sheaves are precisely the categorized characters of Hecke modules. This is particularly interesting in the case of the Hecke modules arising naturally in geometric representation theory, in particular the categories of Harish Chandra modules for real forms of the group $G$. We will return to this in future work.

1.3.3. Monodromic character theory and Langlands duality. We introduce here the monodromic character theory, which is a Langlands dual counterpart to the (equivariant) character theory discussed above. We continue to fix a complex reductive group $G$ with Borel subgroup $B \subset G$.

Recall that the Hecke category $\mathcal{H}_G = \mathcal{D}(B\backslash G/B)$ acts by integral transforms on any category of $\mathcal{D}$-modules on the flag variety $G/B$. We may also consider the natural integral transforms that act on twisted $\mathcal{D}$-modules on the flag variety. More precisely, for any element $\lambda$ of the dual Cartan $\mathfrak{h}^\vee$, we can consider the category of $\lambda$-monodromic $\mathcal{D}$-modules on the flag variety $G/B$. These are $\mathcal{D}$-modules on $G/N$ which have monodromy along the fibers of $G/N \to G/B$ with generalized eigenvalue $\lambda$. Here the natural integral transforms are objects of the monoidal $\infty$-category $\mathcal{H}_{G,\lambda}^{\text{mon}}$ of $\lambda$-bimonodromic $\mathcal{D}$-modules on $B\backslash G/B$. We will focus on the case $\lambda = 0$ and denote the corresponding (unipotent) monodromic Hecke category by $\mathcal{H}_{G,\lambda}^{\text{mon}}$. (See Section 6.3 for a more detailed discussion and technical approach to this category.)

Our main motivation for introducing the monodromic Hecke category $\mathcal{H}_{G,\lambda}^{\text{mon}}$ is the following Koszul duality theorem of Beilinson-Ginzburg-Soergel [BGS], in the Langlands dual form developed
by Soergel [S] to prove the complex case of his categorical real local Langlands conjecture. It is most naturally stated in the mixed setting, but since such structures are beyond the scope of this paper, we will state an equivalence of underlying two-periodic categories. In general, given a stable presentable ∞-category \( C \), we will write \( C_{\text{per}} \) for its two-periodic localization where we invert even shifts. Now for \( G \) and its Langlands dual group \( G^\vee \), the Koszul duality of [BGS, S] localizes to an equivalence of two-periodic Hecke categories:

\[
H_{G^\vee,\text{per}} \simeq H_{G,\text{per}}^{\text{mon}}.
\]

Further arguments of Bezrukavnikov and Yun [BY] show that this is a canonical monoidal equivalence. In other words, in the setting of two-periodic stable presentable ∞-categories, we have a canonical equivalence of algebra objects.

It immediately follows that the monodromic Hecke category \( H_{G,\text{per}}^{\text{mon}} \) satisfies all of our results for the Hecke category \( H_{G^\vee,\text{per}} \). Therefore \( H_{G,\text{per}}^{\text{mon}} \) is a Calabi-Yau algebra, and we have a two dimensional monodromic character theory \( \chi_{G,\text{per}}^{\text{mon}} \) which assigns \( H_{G,\text{per}}^{\text{mon}} \) to a point. (In fact, our arguments for \( H_G \) can be directly adapted to the monodromic setting to give an independent and non-periodic proof of this result.)

**Corollary 1.4.** There is a canonical equivalence of two-periodic topological field theories

\[
\chi_{G^\vee,\text{per}} \simeq \chi_{G,\text{per}}^{\text{mon}}
\]

between the equivariant character theory for \( G^\vee \) and the monodromic unipotent character theory for the Langlands dual group \( G \).

We can now deduce a Langlands duality relating character sheaves on the group \( G \) with those on the dual group \( G^\vee \). To our knowledge (and that of experts we have asked), this duality was not previously known. In Section 1.3.4, we explain how to adapt our results to show that the ∞-category of unipotent character sheaves \( \text{Ch}_G \) is equivalent to the abelianization \( \text{Ab}(H_G^{\text{mon}}) \) of the monodromic Hecke category \( H_G^{\text{mon}} \). This is sufficient to establish the following.

**Corollary 1.5.** There is a canonical equivalence of two-periodic ∞-categories of unipotent character sheaves

\[
\text{Ch}_G,\text{per} \simeq \text{Ch}_{G^\vee,\text{per}}^{\text{mon}}.
\]

1.3.4. General Monodromicity. We briefly comment on the natural extension of our results from trivial infinitesimal character to other monodromicities. (While we do not go into details, the proofs are natural extensions of those presented.) The field theories \( \chi_{G^\vee} \) and \( \chi_{G}^{\text{mon}} \) both fit into families of topological field theories labelled by a parameter \( \lambda \in \mathfrak{h}^\vee \).

On the one hand, let \( G^\vee_\lambda \subset G^\vee \) be the centralizer of a semisimple representative of \( \lambda \), and let \( B^\vee_\lambda \subset G^\vee_\lambda \) be a Borel subgroup. Consider the monoidal Hecke category \( H_{G^\vee_\lambda} \) of \( D \)-modules on \( B^\vee_\lambda \backslash G^\vee_\lambda / B^\vee_\lambda \), and the corresponding equivariant character theory \( \chi_{G^\vee_\lambda} \).

On the other hand, consider the monoidal Hecke category \( H_{G,\text{per}}^{\text{mon},\lambda} \) of \( \lambda \)-bimonodromic \( D \)-modules on \( B \backslash G / B \), and the corresponding \( \lambda \)-monodromic character theory \( \chi_{G,\text{per}}^{\text{mon},\lambda} \). The results of [BGS, S] [BY] apply in this twisted setting as well, giving rise to a Langlands duality for a family of TFTs:

**Corollary 1.6.** There is a canonical equivalence of two-periodic topological field theories

\[
\chi_{G^\vee_\lambda,\text{per}} \simeq \chi_{G,\text{per}}^{\text{mon},\lambda}
\]

between the equivariant character theory for \( G^\vee_\lambda \) and the \( \lambda \)-monodromic character theory for the Langlands dual group \( G \). Evaluated on the circle \( S^1 \), the equivalence gives a canonical identification between the two-periodic ∞-categories of unipotent character sheaves on \( G^\vee_\lambda \) and character sheaves with central character \( \lambda \) on \( G \).

As we explain in the next section, this duality of families of partial extended three-dimensional TFTs can be viewed as the dimensional reduction of the geometric Langlands correspondence.
1.4. Perspectives: gauge theory, geometric Langlands, and the character theory. In this section, we explain informally how the character theory $\chi_G$ fits into three dimensional gauge theory, as well as its relation to four dimensional gauge theory and the Geometric Langlands Program.

1.4.1. Topological gauge theories. We wish to place the character theory $\chi_G$ in the context of three dimensional topological gauge theories, by which we mean very informally a field theory which studies quantizations of moduli of principal bundles (see Section 14.2 for a more physical perspective).

For a finite group $G$, complex representation theory is captured efficiently by the two-dimensional field theory $Z_G$ discussed above. For $G$ an affine algebraic group, for example for $G$ reductive, representation theory is better captured by three-dimensional field theories.

Given a scheme or stack $X$, consider the $\infty$-category $Q(X)$ of quasicoherent sheaves on $X$. Given an affine algebraic group $G$, we form the quasicoherent group algebra $Q(G)$ under convolution. Module categories for $Q(G)$, which we call quasicoherent $G$-categories, are equivalently $\infty$-categories with an algebraic action of $G$ (as considered for example in [BFN]). The object $Q(G) \in 2Alg$ is fully dualizable and defines a $(0,1,2)$-dimensional topological field theory $Z_G$ (see [BFN] for a detailed discussion prior to Lurie’s proof of the cobordism hypothesis). It has the following features:

- To the point, $Z_G^0$ assigns the monoidal $\infty$-category $Q(G) \in 2Alg$ of “quasicoherent distributions”. Equivalently, $Z_G^0(pt)$ is the $(\infty,2)$-category of quasicoherent $G$-categories $\text{Mod}_{Q(G)}(St)$. Examples include $Q(X)$ for any $G$-variety $X$.
- To the circle, $Z_G^0$ assigns the stable $\infty$-category $Q(G_{/ad}G) \in St$ of “quasicoherent class functions”. It is the Drinfeld center of $Q(G)$ (see [BFN]).
- To a closed surface $\Sigma$, $Z_G^0$ assigns the cochain complex of derived functions or coherent cohomology $R\Gamma(\text{Char}_G(\Sigma),\mathcal{O}) \in \text{Mod}_{\mathbb{C}}$ of the character variety of $G$-local systems

$$\text{Char}_G(\Sigma) = BG^\Sigma = \{\pi_1(\Sigma) \to G\}/G.$$

In addition to the quasicoherent group algebra $Q(G)$, we could also consider any quasicoherent Hecke algebra. Namely, for $H \subset G$ an algebraic subgroup, we have a monoidal $\infty$-category

$$Q(H\backslash G/H) \simeq \text{End}_{\text{Mod}_{Q(G)}} Q(G/H)$$

where the identification is a result of [BFN]. The two extremes $H = G$ and $H = \{e\}$ correspond to the symmetric monoidal $\infty$-category $Q(BG) = \text{Rep}(G)$ and the quasicoherent group algebra $Q(G)$ itself. Examples of Hecke modules include $Q(X/H)$ for any $G$-variety $X$.

The quasicoherent Hecke category $Q(H\backslash G/H)$ can also be used to build a two dimensional TFT, but it turns out to be very similar to $Z_G^0$. More precisely, a special case of a theorem of [BFN] asserts that we have equivalences

$$\mathbb{A}(Q(H\backslash G/H)) \sim \widetilde{Q(G_{/ad}G)} \sim \mathcal{Z}(Q(H\backslash G/H))$$

In particular, the abelianization and center are independent of $H$. In work in progress with John Francis [BFN2], we show that $Q(H\backslash G/H)$ is in fact Morita equivalent to $Q(G)$ in the sense that they have equivalent $(\infty,2)$-categories of dualizable modules. For example, the modules $Q(X)$ and $Q(X/H)$ for any $G$-variety $X$ correspond under this equivalence.

Remark 1.7. This Morita equivalence is a natural categorified statement about matrix algebras. Namely, if we let $X = BH$ and $Y = BG$, then $X \times_Y X = H\backslash G/H$ is an example of a convolution algebra (as studied in [BFN]). In the case when $Y$ is a point, $Q(X \times_Y X)$ is a categorified version of the algebra of matrices with entries labelled by $X$. In general, $Q(X \times_Y X)$ is a categorified version of block-diagonal matrices with blocks labelled by $Y$. Thus one expects the center of $Q(X \times_Y X)$ to be a categorified version of block-scalar matrices. Such matrices are nothing more than functions on $Y$, and one arrives at $Q(\mathcal{L}Y)$ as its appropriate categorified version (when $Y = BG$, we have $\mathcal{L}Y = G_{/ad}G$). In particular, the center is independent of $X$: it is always given by quasicoherent sheaves on the loop space of the base.
Now let’s compare with the situation for \( \mathcal{D} \)-modules. Naively, given an affine algebraic group \( G \), one could expect to define a categorified two-dimensional TFT or truncated three-dimensional TFT \( Z^G_\mathcal{D} \) with the following features:

- To a point, \( Z^G_\mathcal{D} \) would assign the “smooth group algebra” \( \mathcal{D}(G) \in 2\text{Alg} \). Equivalently, \( Z^G_\mathcal{D}(\text{pt}) \) would be the \((\infty, 2)\)-category \( \text{Mod}_{\mathcal{D}(G)}(\text{St}) \) of \( \infty \)-categories with an infinitesimally trivialized \( G \)-action.
- To a line, \( Z^G_\mathcal{D} \) would assign the \( \infty \)-category of “smooth class functions” \( \mathcal{D}(G_{\text{ad}}) \).
- To a closed surface \( \Sigma \), \( Z^G_\mathcal{D} \) would assign topological cochains on the character variety \( \text{Char}_G(\Sigma) \) of \( G \)-local systems on \( \Sigma \).

One might also hope that a Morita equivalence relates this theory for \( G \) with analogous theories built out of Hecke categories \( \mathcal{D}(H \backslash G / H) \), for subgroups \( H \subseteq G \).

Unfortunately, such a field theory does not exist: \( \mathcal{D}(G) \) is not a fully dualizable object of \( 2\text{Alg} \). Like any algebra object in an \( \infty \)-category, it does define a one-dimensional extended TFT \cite[Section 4.1]{L5}. However, the non-properness of the convolution morphism prevents \( \mathcal{D}(G) \) from satisfying the strict conditions of two-dimensional dualizability.

Moreover, in the setting of \( \mathcal{D} \)-modules the above Morita picture breaks down dramatically. The representation theory of the Hecke categories \( \mathcal{D}(H \backslash G / H) \) is far from independent of the subgroup \( H \subseteq G \). For example, it is not difficult to check that the center of the “smooth” group algebra \( \mathcal{D}(G) \) is indeed adjoint-equivariant \( \mathcal{D} \)-modules \( \mathcal{D}(G_{\text{ad}}) \) on the group as expected. However, on the other extreme the Tannakian theory of the symmetric monoidal category \( \mathcal{D}(BG) \) is far less rich. This latter category can be identified with representations of \( G \) which are infinitesimally trivialized, or as representations of \( G \) as a homotopy type. For \( G \) connected, we can identify \( \mathcal{D}(BG) \) with modules over topological chains on \( G \) with its convolution product. From this, we can then identify its center with the full \( \infty \)-subcategory \( \text{Loc}(G/G) \subset \mathcal{D}(G/G) \) consisting of adjoint-equivariant \( \mathcal{D} \)-modules on \( G \) with locally constant cohomology. Thus the centers of \( \mathcal{D}(G) \) and \( \mathcal{D}(BG) \) differ dramatically: the equivariance evident in the presentation \( BG = G \backslash G / G \) leads to microlocal equations satisfied by the center.

The family of character theories \( \chi^G_\lambda \), for \( \lambda \in \mathfrak{h}^\vee \), is a well defined substitute for the nonexistent field theory \( Z^G_\mathcal{D} \). More generally, for any affine group \( G \) and subgroup \( P \) with \( G/P \) proper, we have a two-dimensional theory attached to \( \mathcal{D}(P \backslash G / P) \). In particular the Drinfeld center of the Hecke category \( \mathcal{H}_G = \mathcal{D}(B \backslash G / B) \) is cut out of \( \mathcal{D}(G_{\text{ad}}) \) by the condition of nilpotency of the characteristic variety (together with the restriction to unipotent generalized monodromy along tori). One might expect that in some weak sense the integral over the family of character theories recovers the information in the putative \( Z^G_\mathcal{D} \). In other words, the failure of \( Z^G_\mathcal{D} \) to be a field theory should be accounted for by the existence of natural parameter spaces, the dual Cartan subalgebras, over which we need to decompose the theory.

One interesting consequence of this picture is that it suggests a close connection between the character theory \( \chi^G_\Sigma \) for a closed surface \( \Sigma \) and the cohomology of the character variety \( \text{Char}_G(\Sigma) \), which is the subject of fascinating results and conjectures of Hausel and Rodriguez-Villegas \cite{HRV}. We hope to revisit this in future works.

1.4.2. Supersymmetric gauge theory and geometric Langlands. A guiding principle behind this work is the seminal work of Kapustin and Witten \cite{KW}, which uncovered the structure of four-dimensional topological field theory underlying the geometric Langlands correspondence. This field theory is given by a topological twist (the “GL twist”) of the maximally supersymmetric \((N = 4)\) \( G_{\text{cpt}} \)-gauge theory in four dimensions. In topologically twisting the theory, Kapustin and Witten find that there is a \( \mathbb{P}^1 \)-family of choices of nilpotent supercharges, which combines with the gauge coupling constant to define a parameter \( \Psi \in \mathbb{P}^1 \) on which the topological theory depends. Thus Kapustin and Witten
show that we have a four-dimensional topological field theory \( Z_{G, \Psi} \) attached to a reductive group \( G \) and parameter \( \Psi \in \mathbb{P}^1 \). Moreover the famous Montonen-Olive electric/magnetic S-duality of \( \mathcal{N} = 4 \) super-Yang Mills theory has a topological shadow, which is an equivalence

\[
Z_{G, \Psi} \simeq Z_{G^\vee, -1/\Psi}.
\]

Kapustin and Witten further analyze the reduction of the twisted gauge theory along a Riemann surface \( \Sigma \), which in the language of extended TFT is the assignment of the field theory on \( \Sigma \). Namely, the theory \( Z_{G,0} \) assigns to a Riemann surface \( \Sigma \) an \( \infty \)-category \( D(Bun_G(\Sigma)) \) of \( D \)-modules on the moduli stack of \( G \)-bundles, while \( Z_{G^\vee, \infty} \) assigns an \( \infty \)-category of coherent sheaves on the stack \( \text{Loc}_{G^\vee} \) of \( G^\vee \)-local systems on \( \Sigma \). The asserted equivalence \( Z_{G,0}(\Sigma) \simeq Z_{G^\vee, \infty}(\Sigma) \) is the geometric Langlands correspondence. Moreover, the general structure of field theory provides both categories with actions (for every point \( x \in \Sigma \)) of loop operators (‘t Hooft operators for \( \Psi = 0 \) and Wilson operators for \( \Psi = \infty \)). These loop operators are given by the monoidal \( \infty \)-category assigned to the two-sphere, which by the geometric Satake correspondence can be identified with a derived version of \( \text{Rep}(G^\vee) \). On the circle \( S^1 \), one expects the duality to relate to the local geometric Langlands conjecture of Frenkel and Gaitsgory \( \text{[FG]} \). Namely, \( Z_{G,0} \) should assign an \( (\infty,2) \)-category of stable \( \infty \)-categories with a smooth action of the loop group (or modules for the “smooth group algebra” \( D(LG) \)), while \( Z_{G^\vee, \infty} \) should assign an \( (\infty,2) \)-category of stable \( \infty \)-categories over the stack of \( G^\vee \)-local systems on the punctured disc.

The family of character theories introduced above are closely related to the dimensional reduction of this picture to three dimensions. Compactifying an extended topological field theory \( Z \) on \( S^1 \) means defining a new theory \( Z_{S^1} \) by the assignment \( Z_{S^1}(X) = Z(X \times S^1) \). For example, compactifying \( Z_{G,0} \) on the circle, we obtain a field theory which assigns to the point the “smooth group algebra” \( D(LG) \) of the loop group. To complete the dimensional reduction, we now pass to modes invariant along the circle – or mathematically, to the localized \( S^1 \)-equivariant category (see \( \text{[BN1]} \) for a discussion). Using the localization theorem of \( \text{[GKM]} \), we end up replacing \( D \)-modules on \( LG \) by \( D \)-modules on the fixed points \( G \). The expected result is a two-periodic version of the naive three-dimensional gauge theory \( Z_G^D \). A mathematically precise substitute for this theory is the monodromic family \( \chi_G^{\text{mod}, \lambda} \) of character theories labeled by the dual Cartan \( \mathfrak{h}^\vee \). As explained in \( \text{[BN1]} \), these are obtained by equivariant localization from the affine Hecke categories studied in \( \text{[Be]} \). A remarkable feature of this study (discovered in \( \text{[BN1]} \)) is that the result of this reduction process is independent of \( \Psi \). The dimensional reduction of \( Z_{G^\vee, \infty} \), which is described in terms of quasicoherent sheaves, gives rise to the equivariant family of character theories \( \chi_{G^\vee}(\lambda) \) over \( \mathfrak{h}^\vee \), which are described in terms of \( D \)-modules. The dimensional reduction of geometric Langlands duality is then the Langlands duality for character theories discussed in the previous section.

From the point of view of supersymmetric gauge theory, this picture takes the following form (which was explained to us by Edward Witten). Compactification of \( \mathcal{N} = 4 \) Yang-Mills gives rise to a maximally supersymmetric \( \mathcal{N} = 8 \) gauge theory. S-duality in four dimensions gives rise to a duality of \( \mathcal{N} = 8 \) gauge theories in four dimensions studied in \( \text{[Se]} \text{[BK]} \). The GL-type topological twistings of this theory now contain in addition to the parameter \( \Psi \in \mathbb{P}^1 \) coming from four dimensions also one more “generic” twist, which can be obtained by perturbation of any of the previous twists. This generic theory is necessarily preserved by S-duality – in other words, we have a single three-dimensional field theory with an equivalence for Langlands dual groups. It is this theory (or rather its decomposition according to a parameter \( \lambda \in \mathfrak{h}^\vee \)) that we are studying in this paper.

1.5. Techniques: integral transforms in algebraic analysis. In this section, we state some of the foundational results we develop in order to study Hecke categories.

A key idea of algebraic analysis is to replace the functions and distributions of harmonic analysis by the algebraic systems of differential equations that they satisfy. One can view this as a form of categorification where the resulting \( D \)-modules play the role of the original functions and distributions, and categories of \( D \)-modules play the role of generalized function spaces. For example, the exponential function \( f(x) = e^{\lambda x} \) is characterized by the algebraic equation \((\partial_x - \lambda)f(x) = 0,\)
and hence is a solution of the $\mathcal{D}$-module $\mathcal{D}_{\mathbb{A}^1}/\mathcal{D}_{\mathbb{A}^1}(\partial_x - \lambda)$. Similarly, the delta distribution $\delta_\lambda$ is characterized by the algebraic equation $(x - \lambda)f(x) = 0$, and hence is a solution of the $\mathcal{D}$-module $\mathcal{D}_{\mathbb{A}^1}/\mathcal{D}_{\mathbb{A}^1}(x - \lambda)$. 

Natural operations in harmonic analysis are given by integral transforms acting on function spaces

$$f(x) \mapsto (K * f)(y) = \int f(x)K(x, y)dx$$

where $K(x, y)$ is an integral kernel. For example, (one normalization of) the Fourier transform on the real line is given by the integral kernel $K(x, y) = e^{-2\pi ixy}$. In continued analogy, natural operations in algebraic analysis are given by integral transforms acting as functors between categories of $\mathcal{D}$-modules. In this context, derived versions of tensor product and pushforward replace multiplication and integration respectively. To be more precise, given varieties $X, Y$ and a $\mathcal{D}$-module $\mathcal{K}$ on the product $X \times Y$, one defines a functor on derived categories of $\mathcal{D}$-modules

$$\mathcal{D}(X) \overset{\pi^X}{\longrightarrow} \mathcal{D}(X \times Y) \overset{\pi^Y}{\longrightarrow} \mathcal{D}(Y) \overset{\mathcal{F}}{\longrightarrow} \pi_Y^* (\pi_X^* \mathcal{F} \otimes \mathcal{K}).$$

by pulling back from $X$ to the product $X \times Y$, tensoring with the integral kernel $\mathcal{K}$, and then pushing forward to $Y$ via the natural diagram

$$X \overset{\pi^X}{\longrightarrow} X \times Y \overset{\pi^Y}{\longrightarrow} Y.$$  

For example, the geometric Fourier transform of Malgrange, an autoequivalence of $\mathcal{D}$-modules on $\mathbb{A}^1$, is given by the integral kernel $\mathcal{K} = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}(\partial_x - iy)$ with solution $K(x, y) = e^{ixy}$. The classical Fourier transform of a solution of a $\mathcal{D}$-module $\mathcal{F}$ is a solution of the geometric Fourier transform of $\mathcal{F}$. 

For another example, given a correspondence of varieties

$$X \overset{f}{\longleftarrow} Z \overset{g}{\longrightarrow} Y$$

one defines a functor on derived categories of $\mathcal{D}$-modules by the similar formula

$$\mathcal{D}(X) \overset{\mathcal{F}}{\longrightarrow} \mathcal{D}(Y) \overset{g_* f^!}{\longrightarrow} \mathcal{F}.$$ 

By the projection formula, this functor coincides with the integral transform given by the integral kernel $\mathcal{K} = (f \times g)_* \mathcal{O}_Z$ on the product $X \times Y$. 

In general, integral transforms can be interpreted as operations on systems of differential equations, transforming solutions to one system into solutions of a new (and potentially more accessible) system. The theory of integral transforms for $\mathcal{D}$-modules has been developed and applied to a host of problems in integral geometry and analysis, in particular to the study of the Radon, Laplace and Penrose transforms, starting with the influential paper of Brylinski [Br] and continuing in the beautiful work of Goncharov [Go1, Go2], Kashiwara, Schapira, D’Agnolo [KS, DS1, DS2] and others. The $\mathcal{D}$-module approach allows one to separate the algebraic and geometric aspects underlying a system of differential equations from the analytic problems involving solvability in different function spaces, allowing one to obtain powerful general results.

1.5.1. **Review of integral transforms for quasicoherent sheaves.** Integral transforms on derived categories of (quasi)coherent sheaves have been intensely studied since Mukai introduced his analogue of the Fourier transform on abelian varieties. Following Orlov’s theorem, which characterizes all derived equivalences between categories of coherent sheaves on smooth projective varieties as integral transforms, the question arose as to whether all reasonable functors on derived categories are of this form. In order to make such a statement precise one must replace triangulated categories by a more refined setup, such as (pretriangulated) differential graded (dg) categories or stable ∞-categories [L2]. In the former context such results are due to Bondal, Larsen and Lunts [BLL] and Toën [To], who deduced it from a general study of a model structure on dg categories. In particular Toën proves that for $X, X'$ schemes over a ring $k$, the dg category of $k$-linear continuous (that is, colimit
preserving) functors between (dg enhancements of) the derived categories of quasicoherent sheaves is identified with the enhanced derived category of the product.

In our paper [BFN] with John Francis, we applied the formalism of $\infty$-categories [Jo] as developed by Jacob Lurie [L1, L2, L3, L4] to study algebraic operations on the refined versions of derived categories of sheaves provided by stable $\infty$-categories. In particular we derived a general version of Toën’s theorem on the representability of functors by integral kernels, extending the result to a relative situation and from schemes to a broad class of stacks and derived stacks (see [162] for a survey of higher and derived stacks). Namely, we introduce the class of perfect stacks, which are quasi-compact derived stacks $X$ with affine diagonal, and with the property that the $\infty$-category $Q(X)$ of quasicoherent sheaves on $X$ is the inductive limit $Q(X) = \text{Ind Perf}(X)$ of perfect complexes on $X$. The class of perfect stacks includes all quasi-compact and separated schemes as well as quotients of quasiprojective schemes by affine algebraic groups in characteristic zero, and is closed under fiber products, passing to total spaces of quasiprojective morphisms, and quotients by finite group schemes. We showed that for perfect stacks $X_1$, $X_2$ over a perfect stack $Y$, the natural maps are equivalences

$$Q(X_1) \otimes_{Q(Y)} Q(X_2) \sim Q(X_1 \times Y X_2) \sim \text{Fun}_{Q(Y)}(Q(X_1), Q(X_2)).$$

Let $X \to Y$ be a cover of perfect stacks. The $\infty$-category of integral transforms $Q(X \times Y X) \simeq \text{Fun}_{Q(Y)}(Q(X), Q(Y))$ has a natural multiplication, given by convolution with respect to $Y$, or equivalently by composition of functors, making it into a monoidal $\infty$-category. In the case $Y$ is a point, this is a categorified version of the algebra of matrices with entries labelled by $X$, while in general it is a version of $Y$-block diagonal matrices. Thus we expect the center (or Hochschild cohomology) of such an algebra, appropriately defined, to be a categorified version of (block-)scalar matrices, i.e., of functions on $Y$, and likewise for its abelianization (or Hochschild homology). In fact we show that the center (as defined in [BFN]) of $Q(X \times Y X)$ is identified with $Q(LY)$, sheaves on the derived loop space of $Y$. For $X \to Y$ also proper Gorenstein and representable the same holds for the abelianization. The central action $Q(LY) \to Q(X \times Y X)$ and universal trace $Q(X \times Y X) \to Q(LY)$ are both given by integral transforms associated to the correspondence

$$LY \leftarrow \pi_1 \longrightarrow L \times Y X \xrightarrow{\pi_2} X \times Y X$$

In work in progress with John Francis [BFN2] we prove that (in agreement with the intuition from matrix algebra) the monoidal $\infty$-category $Q(X \times Y X)$ is in fact (2-)Morita equivalent to $Q(Y)$ - they have equivalent $(\infty,2)$-categories of dualizable modules.

1.5.2. Setting of schemes. Our first foundational result identifies functors between $D$-modules on smooth schemes with integral transforms. In order to perform algebraic operations on categories of $D$-modules (in particular, to study functor and tensor product categories), it is crucial to work with a refined version of the usual derived category of complexes of $D$-modules with quasicoherent cohomology. Our technical approach to $D$-modules was inspired by the work with Francis on quasicoherent sheaves and by the work of Neeman [N2].

Given a smooth scheme $X$, we will consider the stable $\infty$-category $D(X)$ of $D$-modules on $X$. Equivalently (since we work in characteristic zero), we could consider $D(X)$ as a pre-triangulated differential graded category. We will also consider the full $\infty$-subcategory $D_{coh}(X)$ of $D$-modules with coherent cohomology. For foundations on $\infty$-categories (in the sense of Joyal [Jo], we refer to the comprehensive work of Lurie [L1, L2, L3, L4]. (See Sections 2.1, 2.2, 2.3 for a summary of what we will need of this theory.) When considering $D(X)$, we will work within the $\infty$-category $St$ of stable presentable $\infty$-categories, where morphisms are continuous (colimit-preserving) functors. When considering $D_{coh}(X)$, we will work within the $\infty$-category $St$ of stable idempotent-complete small $\infty$-categories, where morphisms are exact (finite colimit-preserving) functors. Both form symmetric monoidal $\infty$-categories in which it makes sense to consider algebra objects and to perform algebraic operations such as tensor product of module objects.
The stable $\infty$-category $\mathcal{D}(X)$ is naturally symmetric monoidal with respect to tensor product, or in other words, it is an algebra object of $\text{St}$. We will write $\cdot \otimes_{\mathcal{D}(X)} \cdot$ for the tensor product of stable $\mathcal{D}(X)$-module $\infty$-categories, and $\text{Fun}_{\mathcal{D}(Y)}(\cdot, \cdot)$ for the $\mathcal{D}(X)$-linear morphisms between stable $\mathcal{D}(X)$-module $\infty$-categories. The stable $\infty$-category $\mathcal{D}_{\text{coh}}(X)$ is also naturally symmetric monoidal with respect to tensor product, and we will use similar notation when working with it.

By a scheme, we will always mean a quasicompact, separated scheme of finite type.

**Theorem 1.8.** Let $X_1, X_2$ be smooth schemes over a smooth scheme $Y$. Then the natural maps are equivalences

$$\mathcal{D}(X_1) \otimes_{\mathcal{D}(Y)} \mathcal{D}(X_2) \xrightarrow{\sim} \mathcal{D}(X_1 \times_Y X_2) \xrightarrow{\sim} \text{Fun}_{\mathcal{D}(Y)}(\mathcal{D}(X_1), \mathcal{D}(X_2)).$$

$$\mathcal{D}_{\text{coh}}(X_1) \otimes_{\mathcal{D}(Y)} \mathcal{D}_{\text{coh}}(X_2) \xrightarrow{\sim} \mathcal{D}_{\text{coh}}(X_1 \times_Y X_2).$$

If in addition, $X_1$ is proper, then the natural map is also an equivalence

$$\mathcal{D}_{\text{coh}}(X_1 \times_Y X_2) \xrightarrow{\sim} \text{Fun}(\mathcal{D}_{\text{coh}}(X_1), \mathcal{D}_{\text{coh}}(X_2)).$$

The first map is induced by the pullbacks from the factors to the fiber product, the second map is given by associating integral transforms to integral kernels. Thus we see that all linear functors on $\mathcal{D}$-modules on schemes are representable by integral transforms. In particular, we obtain a concrete description of (the $\infty$-categorical enhancement of) Goncharov’s bicategory $\text{Go}_2$ of $\mathcal{D}$-modules with integral transforms as the full subcategory of $\text{St}$ consisting of the stable $\infty$-categories $\mathcal{D}(X)$ for smooth schemes $X$.

1.5.3. **Equivariant setting.** In the setting of algebraic stacks, the natural analogues of Theorem fail in general. The fundamental obstruction is the inherently topological nature of $\mathcal{D}$-modules. Pushing forward along an affine map almost always loses information (that is, it fails to be conservative): for example, unlike quasicoherent sheaves, many nontrivial $\mathcal{D}$-modules have no global flat sections at all. Thus even for stacks with affine diagonal, one can not reconstruct $\mathcal{D}$-modules on a fiber product from algebraic operations on $\mathcal{D}$-modules on the factors. In short, the Tannakian theory of $\mathcal{D}$-modules is not rich enough to capture the geometric theory. As a result, tensor and functor $\infty$-categories are not identified with all integral kernels: there are adjunctions exhibiting the former as pale shadows of the latter.

With applications to representation theory in mind, we will focus on a more precise formulation in the following special setting. We will restrict our attention to finite orbit stacks, which are stacks of the form $Z/G$, where $Z$ is a smooth quasiprojective variety, and $G$ is an affine algebraic group that acts linearly on $Z$ with finitely many orbits. Examples include classifying stacks $BG = pt/G$ and the stacks $\text{P} \setminus \text{G}/\text{P}$, for $\text{P} \subset \text{G}$ a parabolic subgroup of a reductive group $\text{G}$. The key technical advantage of finite orbit stacks is that coherent and holonomic $\mathcal{D}$-modules coincide, and $\mathcal{D}(Z/G)$ is compactly generated by a collection of holonomic $\mathcal{D}$-modules. This implies that all $\mathcal{D}$-modules on finite orbit stacks enjoy the rich functoriality and adjunctions available in the holonomic setting. In this setting, we have the following partial analogue of Theorem tensor products and linear functors are identified with each other via the natural map factoring through $\mathcal{D}$-modules on the fiber product.

**Theorem 1.9.** Let $Z/G$ be a finite orbit stack. Then $\mathcal{D}(Z/G)$ is self-dual as a $\mathcal{D}(BG)$-module. In particular, for any stack $X' = Z'/G$, the natural map (factoring through $\mathcal{D}$-modules on the fiber product) is an equivalence

$$\mathcal{D}(Z/G) \otimes_{\mathcal{D}(BG)} \mathcal{D}(Z'/G) \xrightarrow{\sim} \text{Fun}_{\mathcal{D}(BG)}(\mathcal{D}(Z/G), \mathcal{D}(Z'/G)).$$
1.6. Outline of the paper. We begin in Section 2 with an overview of the technical tools we will need, including basics of stable and monoidal ∞-categories, ind-categories and compact generators, centers and abelianization (or Hochschild cohomology and homology).

In Section 3 we study \(\mathcal{D}\)-module categories on schemes. After reviewing the basic formalism, we show \(\mathcal{D}\)-modules enjoy good functional analytic properties including extended functoriality, self-duality of \(\mathcal{D}\)-module categories, and the representation of continuous functors in an absolute and relative setting by integral transforms.

In Section 4, we discuss properties of \(\mathcal{D}\)-modules on stacks, establishing restricted analogues of some of the functoriality results for schemes, in particular the representation of functors relative to a classifying space by integral transforms.

In Section 5, we examine Hecke categories in detail, working in somewhat greater generality than is needed for reductive groups. We first show some adjunction properties which enable us to identify the abelianization and centers of Hecke categories. These results are then applied to show that Hecke categories are fully dualizable in the two dimensional sense, defining categorified extended 2d TFTs.

Finally, in Section 6, we study the fundamental correspondence between the Hecke category and equivariant sheaves on the group, and use it to identify the abelianizations of equivariant and monodromic Hecke categories with character sheaves.

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2. Preliminaries

In this section, we summarize relevant technical foundations for \(\infty\)-categories and homotopical algebra in \(\infty\)-categories. It is beyond the scope of this paper (or any single paper) to provide a comprehensive discussion of these topics. We will instead focus on highlighting the particular concepts, results and their references that play a role in what follows. For a less condensed overview, one could also consult \[BFN\] from which we have borrowed much of the following account.

Throughout this paper, we work over a fixed algebraically closed field \(\mathbb{C}\) of characteristic zero.

2.1. \(\infty\)-categories. When studying \(\mathcal{D}\)-modules on a scheme or stack, we will work in the setting of stable \(\infty\)-categories (or equivalently, over \(\mathbb{C}\), with pre-triangulated differential graded categories) rather than the more traditional setting of derived categories. It is well established that the theory of triangulated categories, through which derived categories are often viewed, is inadequate to handle many basic operations. The essential problem is that passing to homotopy categories discards the essential information captured by homotopy coherent structures. Furthermore, we will also need the formalism of \(\infty\)-categories to describe collections of \(\infty\)-categories themselves.

Roughly speaking, an \(\infty\)-category (or synonymously \((\infty,1)\)-category) encodes the notion of a category whose morphisms form topological spaces and whose compositions and associativity properties are defined up to coherent homotopies. The theory of \(\infty\)-categories has many alternative formulations (as topological categories, Segal categories, quasicategories, etc; see \[Ber\] for a comparison between the different versions). We will follow the conventions of \[L1\], which is based on Joyal’s quasi-categories \[Jo\]. Namely, an \(\infty\)-category is a simplicial set satisfying a weak version of the Kan condition guaranteeing the fillability of certain horns. The underlying vertices play the role of the set of objects while the fillable horns correspond to sequences of composable morphisms.
The book [L1] presents a detailed study of ∞-categories, developing analogues of many of the common notions of category theory (an overview of the ∞-categorical language, including limits and colimits, appears in [L1, Chapter 1.2]). An important distinction between ∞-categories and the more traditional settings of model categories or homotopy categories is that coherent homotopies are naturally built into the definitions. Thus for example, all functors are derived and the natural notions of limits and colimits correspond to homotopy limits and colimits.

Most of the ∞-categories that we will encounter are presentable [L1 5.5] in the sense that they are closed under all small colimits (as well as limits, by [L1 Proposition 5.5.2.4]), and generated under suitable colimits by a small category. Examples include ∞-categories of spaces and of modules over a ring. Presentable ∞-categories form an ∞-category $\mathcal{P}r$ whose morphisms are left adjoints, or equivalently by the adjoint functor theorem [L1 5.2], functors that preserve all colimits [L1 5.5.3]. Furthermore, many of the ∞-categories that we will encounter are stable [L2 2.4] in the sense that they have a zero object, are closed under finite limits and colimits, and their pushouts and pullbacks coincide. Stable ∞-categories are an analogue of the additive setting of homological algebra: the homotopy category of a stable ∞-category has the canonical structure of a triangulated category [L2 3]. We will denote by $\text{St} \subset \mathcal{P}r$ the full ∞-subcategory of stable presentable ∞-categories as studied in [L2 17]. Finally, we will make frequent use of Lurie’s powerful extension [L3, Theorem 3.4.5] of the Barr-Beck theorem to the setting of ∞-categories.

### 2.2. Monoidal ∞-categories

Tensor product of $\mathcal{D}$-modules on a scheme or stack provides a symmetric monoidal product. We will need the notion of a symmetric monoidal ∞-category, along with algebra and module objects therein. Furthermore, we will also need the symmetric monoidal structure on the ∞-category $\text{St}$ of stable presentable ∞-categories.

The foundations of homotopical algebra in the setting of ∞-categories has been developed by Lurie [L3, L4]. A monoidal ∞-category as defined in [L3 1.1] is an ∞-category equipped with a homotopy coherent associative unital product. Its homotopy category is an ordinary monoidal category. An algebra object $A$ in a monoidal ∞-category as defined in [L3 1.1.14] is an object equipped with a homotopy coherent multiplication. Left and right module objects over an algebra object are defined similarly [L3 2.1], with right modules identified with left modules over the opposite algebra object $A^{\text{op}}$, and there is a relative tensor product $\cdot \otimes_A \cdot$ of left and right modules given by the two-sided bar construction [L3 4.5]. Monoidal ∞-categories, algebra objects in a monoidal ∞-category, and module objects over an algebra object themselves form ∞-categories, some of whose properties (in particular, behavior of limits and colimits) are worked out in [L3 1.2].

The definition of a symmetric monoidal ∞-category is given in [L4 1] modeled on the Segal machine for infinite loop spaces. Its homotopy category is an ordinary symmetric monoidal category. There is the notion of commutative algebra object in a symmetric monoidal ∞-category $\mathcal{C}$ such that its modules form a symmetric monoidal ∞-category with respect to relative tensor product [L4 Proposition 5.7]. Given two associative algebras $A, B \in \mathcal{C}$, their monoidal product $A \otimes B \in \mathcal{C}$ carries a natural associative algebra structure. Furthermore, any associative algebra $A \in \mathcal{S}$ is a left (as well as a right) module object over the associative algebra $A \otimes A^{\text{op}}$ via left and right multiplication.

One of the key developments of [L3] is the monoidal structure on the ∞-category $\mathcal{P}r$ of presentable ∞-categories. The tensor product $\mathcal{C} \otimes \mathcal{D}$ of presentable $\mathcal{C}, \mathcal{D}$ is a recipient of a universal functor from the Cartesian product $\mathcal{C} \times \mathcal{D}$ which is “bilinear” (commutes with colimits in each variable separately). In fact, the tensor product lifts to a symmetric monoidal structure in which the unit object is the ∞-category of spaces [L4 Proposition 6.18]. Furthermore, the symmetric monoidal structure is closed in the sense that $\mathcal{P}r$ admits an internal hom functor compatible with the tensor structure [L1 Remark 5.5.3.9], [L3 Remark 4.1.6]. The internal hom assigns to presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ the ∞-category of colimit-preserving functors $\text{Fun}(\mathcal{C}, \mathcal{D})$ which is presentable by [L1 Proposition 5.5.3.8]. The symmetric monoidal structure on the ∞-category $\mathcal{P}r$ of presentable ∞-categories restricts to one on the full ∞-subcategory $\text{St}$ of stable presentable ∞-categories [L3].
2.3. **Ind-categories and compact generators.** We begin by fixing some terminology which we will use throughout. Recall that an object \( C \) of an \( \infty \)-category \( \mathcal{C} \) is compact if the functor \( \text{Hom}_\mathcal{C}(C, -) \) preserves colimits (see [L1] 5.3.4 for a detailed discussion).

**Definition 2.1.** Given a functor \( f : \mathcal{C} \to \mathcal{D} \) between \( \infty \)-categories, we will say

- \( f \) is continuous if it preserves colimits, and
- \( f \) is proper if it takes compact objects to compact objects.

We continue by briefly recalling the properties of ind-categories from [L1] Section 5.3.5. Given a small \( \infty \)-category \( \mathcal{C} \) which admits finite colimits, we may freely adjoin to \( \mathcal{C} \) all small filtered colimits. The result is a new \( \infty \)-category \( \text{Ind}\mathcal{C} \), called the ind-category of \( \mathcal{C} \), which is presentable (and so in particular, admits all small colimits). By [L2] Section 8, if \( \mathcal{C} \) is stable then so is \( \text{Ind}\mathcal{C} \). In general, \( \text{Ind}\mathcal{C} \) can be identified with the category of those presheaves on \( \mathcal{C} \) taking finite colimits to finite limits (and so in particular, it comes with an embedding \( \mathcal{C} \subset \text{Ind}\mathcal{C} \)). It also satisfies the universal property [L1] Proposition 5.3.5.10 that continuous functors from \( \text{Ind}\mathcal{C} \) to a cocomplete category \( \mathcal{D} \) are identified with functors from \( \mathcal{C} \) to \( \mathcal{D} \) that preserve finite colimits. We record this statement in the stable setting for future reference:

**Lemma 2.2.** Let \( \mathcal{C} \) denote a small stable \( \infty \)-category and \( \mathcal{D} \in \text{St} \) a presentable stable \( \infty \)-category. There is an equivalence of \( \infty \)-categories

\[
\text{Ind} : \text{Fun}^{ex}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^L(\text{Ind}\mathcal{C}, \mathcal{D})
\]

**Remark 2.3.** In the statement of the lemma, we have written \( \text{Fun}^{ex} \) to denote exact functors, and \( \text{Fun}^L \) to denote continuous functors (this is the internal hom in the \( \infty \)-category \( \mathcal{P}r \) of presentable \( \infty \)-categories; the superscript "L" refers to the fact that continuous functors are equivalently left adjoints). Throughout this paper, we will almost exclusively deal with continuous functors and denote them simply by \( \text{Fun} \) when no ambiguity is possible.

**Remark 2.4.** We will have occasion to utilize [L1] Proposition 5.3.6.2 which generalizes Lemma 2.2 from ind-categories to more general cocompletions. Namely, given a collection \( \mathcal{R} \) of small diagrams in a small \( \infty \)-category \( \mathcal{C} \) closed under finite colimits, we may construct a new \( \infty \)-category \( \text{Ind}_\mathcal{R}\mathcal{C} \) in which we add all small colimits to \( \mathcal{C} \) while respecting the diagrams in \( \mathcal{R} \). In other words, the \( \infty \)-category of functors from \( \text{Ind}_\mathcal{R}\mathcal{C} \) to a cocomplete category \( \mathcal{D} \) is identified with the \( \infty \)-category of functors from \( \mathcal{C} \) to \( \mathcal{D} \) which carry the diagrams in \( \mathcal{R} \) to colimit diagrams in \( \mathcal{D} \).

Let us denote by \( st \) the \( \infty \)-category of small stable \( \infty \)-categories that are idempotent-complete [L1] 4.4.5 (see [BFN] 4.1 for a discussion). For \( \mathcal{C} \in \text{st} \), we can recover \( \mathcal{C} \) from its ind-category \( \text{Ind}\mathcal{C} \) as the full \( \infty \)-category \( \mathcal{C} \simeq (\text{Ind}\mathcal{C})_{\text{cpt}} \) of compact objects of \( \text{Ind}\mathcal{C} \) (see [L1] Lemma 5.4.2.4 or [L1] 5.3.4.17, and also Neeman [N1]).

This hints at the intimate relation between ind-categories and the classical notion of compact generation (see [L2] 17 and [L1] 5.5.7 for more details).

**Definition 2.5.** A stable \( \infty \)-category \( \mathcal{C} \) is said to be compactly generated if there is a small \( \infty \)-category \( \mathcal{C}^{\text{cpt}} \) of compact objects \( C_i \in \mathcal{C} \) whose right orthogonal vanishes: if \( M \in \mathcal{C} \) satisfies \( \text{Hom}_\mathcal{C}(C_i, M) \simeq 0 \), for all \( i \), then \( M \simeq 0 \).

Note that an ind-category \( \text{Ind}\mathcal{C} \) is automatically compactly generated (with compact objects the objects of \( \mathcal{C} \)). In fact, any compactly generated stable presentable \( \infty \)-category \( \mathcal{C} \) can be identified with the ind-category \( \text{Ind}\mathcal{C}^{\text{cpt}} \) of its subcategory \( \mathcal{C}^{\text{cpt}} \subset \mathcal{C} \) of compact objects. This is a version of a theorem of Schwede and Shipley [SS] characterizing module categories for \( A_\infty \)-algebras (see [L3] 4.4 for the \( \infty \)-categorical version, and [KL] for the differential graded version). More generally, the functor \( \text{Ind} : \text{st} \to \text{St} \) identifies \( \text{st} \) with the subcategory of \( \text{St} \) consisting of compactly generated \( \infty \)-categories, and with morphisms proper continuous functors; a quasi-inverse is given by \( \mathcal{C} \mapsto \mathcal{C}^{\text{cpt}} \).
2.3.1. Opposite categories and restricted opposites. We denote by \( C^\circ \) the opposite category of an \( \infty \)-category \( C \), reserving the superscript \( \text{op} \) for the opposite monoidal structure on an algebra object (see Section 2.2 above). We will typically consider opposite categories for small categories only. For \( C = \text{Ind} C_{\text{cpt}} \) a compactly generated \( \infty \)-category, we will work with the modified notion of the restricted opposite category defined by

\[
C' = \text{Ind}(C_{\text{cpt}}^\circ).
\]

The terminology is motivated by that of the restricted dual of vector spaces equipped with an extra structure such as a grading.

Example 2.6. For the \( \infty \)-category \( \text{Vect} = \text{Mod}_C \), we have \( \text{Vect}_{\text{cpt}} \) consists of perfect complexes of \( C \)-vector spaces. Duality gives an identification \( \text{Vect}_{\text{cpt}} \cong \text{Vect}_{\text{cpt}}^\circ \), which extends by continuity to an equivalence \( \text{Vect} \cong \text{Vect}' \). Note that the plain opposite category \( \text{Vect}^\circ \) is the \( \infty \)-category of pro-finite dimensional vector spaces, which is quite different from \( \text{Vect}' \).

2.3.2. Adjoint and proper functors. We close this summary by recording a couple of other useful statements:

Lemma 2.7. Let \( f : C \hookrightarrow D : g \) denote an adjoint pair of functors between small \( \infty \)-categories. Then the induced functors \( \text{Ind} f : \text{Ind} C \hookrightarrow \text{Ind} D : \text{Ind} g \) are also adjoint.

Proof. The assertion follows from Lemma 2.2 applied to the adjunction morphisms \( fg \rightarrow \text{id}_D \) and \( gf \rightarrow \text{id}_D \).

Lemma 2.8. Let \( F : C \hookrightarrow D : G \) denote an adjoint pair of functors. If \( G \) is continuous, then \( F \) is proper. Conversely, if \( C = \text{Ind} C_{\text{cpt}} \) is compactly generated and \( F \) is proper, then \( G \) is continuous.

Proof. Suppose \( G \) is continuous and \( N \in C \) is compact. For \( \{ M_i \} \) an arbitrary small diagram in \( D \), we calculate

\[
\text{Hom}_D(F(N), \colim_i M_i) \cong \text{Hom}_C(N, G(\colim_i M_i)) \\
\cong \text{Hom}_C(N, \colim_i G(M_i)) \\
\cong \colim_i \text{Hom}_C(N, G(M_i)) \\
\cong \colim_i \text{Hom}_D(F(N), M_i)
\]

and conclude that \( F(N) \) is compact.

Conversely, suppose \( C = \text{Ind} C_{\text{cpt}} \) is compactly generated and \( F \) is proper. Then any \( N \in C \) can be written as a colimit \( \colim_j N_j \) of compact objects. For \( \{ M_i \} \) an arbitrary small diagram in \( D \), we calculate

\[
\text{Hom}_C(\colim_j N_j, G(\colim_i M_i)) \cong \lim_j \text{Hom}_C(N_j, G(\colim_i M_i)) \\
\cong \lim_j \text{Hom}_D(F(N_j), \colim_i M_i) \\
\cong \lim_j \colim_i \text{Hom}_D(F(N_j), M_i) \\
\cong \lim_j \colim_i \text{Hom}_C(N_j, G(M_i)) \\
\cong \lim_j \text{Hom}_C(N_j, \colim_i G(M_i)) \\
\cong \text{Hom}_C(\colim_j N_j, \colim_i G(M_i))
\]

By the Yoneda lemma, we conclude that \( G \) is continuous.

\[\square\]

2.4. Centers and abelianizations. We continue with a discussion of centers and abelianizations of associative algebra objects in closed symmetric monoidal \( \infty \)-categories, following [BFN] to which we refer for more details. This is a general version of the approach to topological Hochschild homology developed in [EKMM, SH].

Let \( S \) be a closed symmetric monoidal \( \infty \)-category. Then we have internal hom objects [L3, 2.7], and given \( A \otimes A^{\text{op}} \)-modules \( M, N \), we can consider the \( A \otimes A^{\text{op}} \)-linear morphism object \( \text{Hom}_{A \otimes A^{\text{op}}}(M, N) \in S \). Likewise, given left and right \( A \otimes A^{\text{op}} \) modules \( M, N \in S \) we have a pairing \( M \otimes_{A \otimes A^{\text{op}}} N \in S \) defined by the two-sided bar construction [L3, 4.5] over \( A \otimes A^{\text{op}} \).
Definition 2.9. Let $A$ be an associative algebra object in a closed symmetric monoidal $\infty$-category $S$.

(1) The derived center or Hochschild cohomology $Z(A) = \text{HH}^*(A) \in S$ is the endomorphism object $\text{End}_{A \otimes A^{op}}(A)$ of $A$ as an $A$-bimodule.

(2) The derived abelianization or Hochschild homology $\text{Ab}(A) = \text{HH}_*(A) \in S$ is the pairing object $A \otimes_{A \otimes A^{op}} A$ of $A$ with itself as an $A$-bimodule.

In general, the center $Z(A)$ is again an associative algebra object in $S$. Furthermore, $Z(A)$ comes with a canonical central morphism

$$\jmath : Z(A) \longrightarrow A \quad \quad F \longrightarrow F(1_A)$$

while $\text{Ab}(A)$ comes with a canonical trace morphism

$$\text{tr} : A \longrightarrow \text{Ab}(A)$$

coequalizing left and right multiplication.

2.4.1. Cyclic bar construction. It is useful to have versions of the Hochschild chain and cochain complexes that calculate the abelianization and center. In the setting of an associative algebra object $A$ in a monoidal $\infty$-category $S$, they take the form of a simplicial object $N^*_\text{cyc}(A)$ and cosimplicial object $N^\text{cyc}_\text{op}(A)$ such that the geometric realization colim $N^\text{cyc}_\text{op}(A)$ is the abelianization $\text{Ab}(A)$ and the totalization lim $N^*_\text{cyc}(A)$ is the center $Z(A)$.

We construct the simplicial object $N^*_\text{cyc}(A)$ and cosimplicial object $N^\text{cyc}_\text{op}(A)$ as follows. The $A$-bimodule $A$ has a canonical simplicial resolution $C_*(A)$ of the $A$-bimodule $A$ whose terms $C_{n-1}(A) \simeq A^{\otimes n+1}$ are $A$-bimodules which are free as left $A$-modules. (It is defined using the usual formalism of cotriple resolutions, see [BFN].)

Now recall that the abelianization $\text{Ab}(A)$ is defined by the self-pairing $A \otimes_{A \otimes A^{op}} A$. Since the tensor product commutes with colimits, in particular geometric realizations, we calculate

$$\text{Ab}(A) = A \otimes_{A \otimes A^{op}} A \simeq A \otimes_{A \otimes A^{op}} |C_*(A)| \simeq |A \otimes_{A \otimes A^{op}} C_*(A)|.$$

Thus the geometric realization of $A \otimes_{A \otimes A^{op}} C_*(A)$ calculates the abelianization $\text{Ab}(A)$.

We write $N^\text{cyc}_n(A)$ for the simplicial object $A \otimes_{A \otimes A^{op}} C_n(A)$ and refer to it as the Hochschild chain complex. Since $A$ is free as a right $A$-module, the terms of the simplicial object $C_*(A)$ are free as $A$-modules. Thus we can evaluate the terms of the Hochschild chain complex

$$N^\text{cyc}_n(A) = A \otimes_{A \otimes A^{op}} C_n(A) \simeq A \otimes_{A \otimes A^{op}} A^{\otimes n+2} \simeq A^{\otimes n+1}$$

In particular, there are equivalences $N^\text{cyc}_0(A) \simeq A$ and $N^\text{cyc}_1(A) \simeq A \otimes A$, and the two simplicial maps $A \otimes A \to A$ are the multiplication and the opposite multiplication of $A$.

Similarly, recall that the center $Z(A)$ is defined by the endomorphisms $\text{End}_{A \otimes A^{op}}(A)$. Since morphisms take colimits in the domain to limits, in particular geometric realizations to totalizations, we calculate

$$Z(A) = \text{End}_{A \otimes A^{op}}(A) \simeq \text{Hom}_{A \otimes A^{op}}([C_*(A)], A) \simeq |\text{Hom}_{A \otimes A^{op}}(C_*(A), A)|$$

Thus the totalization of $\text{Hom}_{A \otimes A^{op}}(C_*(A), A)$ calculates the center $Z(A)$.

We write $N^\text{cyc}_n(A)$ for the cosimplicial object $\text{Hom}_{A \otimes A^{op}}(C_n(A), A)$ and refer to it as the Hochschild cochain complex. As before, we can evaluate the terms of the Hochschild cochain complex

$$N^\text{cyc}_n(A) = \text{Hom}_{A \otimes A^{op}}(C_n(A), A) \simeq \text{Hom}_{A \otimes A^{op}}(A^{\otimes n+2}, A) \simeq \text{Hom}(A^{\otimes n}, A).$$

In particular, there are equivalences $N^\text{cyc}_0(A) \simeq A$ and $N^\text{cyc}_1(A) \simeq \text{Hom}(A, A)$, and the two cosimplicial maps $A \to \text{Hom}(A, A)$ are induced by the left and right multiplication of $A$. 

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2.4.2. Circle action on Hochschild homology. An important feature of the Hochschild cohomology \( Z(A) \) is its monoidal structure, which is evident from its definition as an endomorphism object. The Hochschild homology \( \text{Ab}(A) \) is not monoidal in general, but rather carries a canonical action of the circle group \( S^1 \), which is captured by the theory of cyclic homology (see [Lo] for a comprehensive overview). In the case of the algebra \( A \) of cochains on a topological space \( X \), this structure captures the \( S^1 \) action on cochains on the free loop space of \( X \) (see [J]). Specifically, the cyclic bar construction calculating Hochschild homology has a canonical structure of cyclic object (as defined by Connes [Co]) extending its definition as a simplicial object. In other words, in addition to the usual face and degeneracy maps, labelled by order preserving maps of finite sets, we have additional morphisms corresponding to cyclic rotations of finite sets. The \( \infty \)-category of cyclic sets is identified by [DHK] with that of \( S^1 \)-spaces (or spaces over \( BS^1 \), the classifying space of the cyclic category). More generally, a cyclic object in any \( \infty \)-category gives rise to an \( S^1 \)-action on the geometric realization of the underlying simplicial object.

3. \( D \)-modules on schemes

Throughout the remainder of this paper, all schemes will be over a fixed algebraically closed field \( C \) of characteristic zero without further comment. By a scheme, we will always mean a quasi-compact, separated scheme.

3.1. Basic properties. We collect some standard definitions and properties of \( D \)-modules. References include Bernstein’s lecture notes [B] and the books by Borel [Bo] and Kashiwara [K].

By a \( D \)-module \( \mathcal{M} \) on a smooth scheme \( X \), we will mean a complex of quasicoherent sheaves with a compatible left action of the sheaf of differential operators \( D_X \). We will write \( D(X) \) for the stable \( \infty \)-category (or equivalently, pre-triangulated differential graded category) of \( D \)-modules on \( X \). It is a symmetric monoidal \( \infty \)-category with product given by the tensor product of the underlying quasicoherent sheaves

\[
\mathcal{M}, \mathcal{N} \mapsto \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}.
\]

Given a map \( f : X \to Y \) of schemes, we have two derived functors

\[
f_* : D(X) \to D(Y) \quad f^! : D(Y) \to D(X).
\]

The pullback \( f^! \) is the usual pullback on the underlying quasicoherent sheaf. In particular, the tensor product of \( D \)-modules is given by the pullback along the diagonal map

\[
\mathcal{M} \otimes \mathcal{N} \simeq \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \simeq \Delta^!(\mathcal{M} \boxtimes \mathcal{N}).
\]

It is also useful to introduce a shift and to define the exceptional pullback

\[
f^! = f^! [2(\dim X - \dim Y)].
\]

The \( D \)-module pushforward \( f_* \) is given by the standard \((f^{-1}D_Y, D_X)\)-bimodule \( D_{Y \leftarrow X} \) and the usual pushforward on the underlying sheaf

\[
f_*(\mathcal{M}) = f_* (D_{Y \leftarrow X} \otimes_{D_X} \mathcal{M})
\]

Here are some standard properties we will use:

- The \( D \)-module pullback \( f^! \) is monoidal with respect to tensor product (since \( f^! \) and the tensor product are given by the usual pullback and tensor product on the underlying quasicoherent sheaves).
- Composition identities: given a sequence of maps

\[
X \xrightarrow{f} Y \xrightarrow{g} Z
\]

there are canonical equivalences of functors \((gf)_* \simeq g_* f_* \), \((gf)^! \simeq f^! g^! \).
• Base change: given a Cartesian square

\[
\begin{array}{ccc}
x \times_Y Z & \xrightarrow{\tilde{g}} & X \\
\downarrow f & & \downarrow f \\
Z & \xrightarrow{g} & Y
\end{array}
\]

there is a canonical equivalence of functors

\[\tilde{g}_* \hat{f}^! \simeq f^! g_*\]

• Projection formula: for a map \(f : X \to Y\), there is a functorial equivalence

\[f_*(f^!(\sM) \otimes \sN) \simeq f_*(\sM) \otimes \sN.\]

This follows from applying base change to the Cartesian diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & Y \\
\downarrow \id_x \times f & & \downarrow \delta \\
x \times Y & \xrightarrow{f \times \id_Y} & Y \times Y
\end{array}
\]

3.2. Coherent and holonomic \(\sD\)-modules. By a coherent \(\sD\)-module on a smooth scheme \(X\), we will mean an object \(\sM \in \sD(X)\) such that it is locally representable by a finite complex of \(\sD\)-modules with finitely generated cohomology. A coherent \(\sD\)-module will be called holonomic if it is trivial or if its cohomology sheaves have Lagrangian singular support. We will write \(\sD_{coh}(X)\) for the full subcategory of \(\sD(X)\) of coherent objects and similarly \(\sD_{hol}(X)\) for the full subcategory of holonomic objects.

Here are some standard properties we will use:

• For smooth maps \(f : X \to Y\), pullback descends to a functor \(f^! : \sD_{coh}(Y) \to \sD_{coh}(X)\).
• For proper maps \(f : X \to Y\), pushforward descends to a functor \(f_* : \sD_{coh}(X) \to \sD_{coh}(Y)\).
• For any morphism \(f : X \to Y\), the pullback and pushforward descend to functors

\[f^! : \sD_{hol}(Y) \to \sD_{hol}(X) \quad f_* : \sD_{hol}(X) \to \sD_{hol}(Y)\]

As a consequence, observe that the tensor product \(\sM \otimes \sN \simeq \Delta^!(\sM \boxtimes \sN)\) descends from \(\sD(X)\) to equip \(\sD_{hol}(X)\) with a natural symmetric monoidal structures. (On a scheme, the tensor product of coherent \(\sD\)-modules is rarely coherent, for example \(\sD_X \otimes \sD_X\) is not finitely generated.)

Let \(\omega_X\) be the canonical bundle, and consider the dualizing object

\[\sD^!_X = \text{Diff}(\omega_X, \sO_X) = \sD_X \otimes_{\sO_X} \omega_X^{-1}\]

It has two canonically interchangeable (left) \(\sD\)-module structures arising via composition of differential operators. It provides the Verdier duality involution

\[\sD_X : \sD_{coh}(X)^\vee \to \sD_{coh}(X) \quad \sD_X(\sM) = \text{Hom}_{\sD_X}(\sM, \sD^!_X)[\dim X]\]

which also descends to an involution on holonomic \(\sD\)-modules.

We have the standard properties:

• Morphisms out of a coherent \(\sD\)-module are calculated locally

\[\text{Hom}_{\sD(X)}(\sM, \sN) \simeq \pi_* (\sD_X(\sM) \otimes \sN)_{[-\dim X]}, \quad \text{for } \sM \in \sD_{coh}(X)^\vee, \sN \in \sD(X)\]

where \(\pi : X \to pt.\) (See [11], [10] Corollry 9.8.)

• For \(f : X \to Y\) is proper, there is a canonical equivalence

\[\sD_Y \circ f_* \simeq f_* \circ \sD_X : \sD_{coh}(X) \to \sD_{coh}(Y)\]

• For \(f : X \to Y\) smooth, there is a canonical equivalence

\[\sD_X \circ f^! \circ \sD_Y \simeq f^!(\dim X - \dim Y) : \sD_{coh}(Y) \to \sD_{coh}(X)\]

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For a map \( f : X \to Y \) of schemes, one defines functors

\[
    f_! : D_{\text{hol}}(X) \to D_{\text{hol}}(Y) \quad f^* : D_{\text{hol}}(Y) \to D_{\text{hol}}(X) \\
    f_! = D_Y \circ f_* \circ D_X \quad f^* = D_X \circ f^! \circ D_Y
\]

They satisfy the standard adjunctions:

- The functors \((f_!, f^!\) and \((f^*, f_*)\) form adjoint pairs on holonomic \(D\)-modules.

The duality involution also equips \(D_{\text{hol}}(X)\) with an alternative symmetric monoidal structure in addition to the usual \(D\)-module or \(t\)-tensor product \( \otimes = \otimes^t \). Namely, we define the \(s\)-tensor product

\[
    \mathcal{M} \otimes^s \mathcal{N} = \Delta^s(\mathcal{M} \boxtimes \mathcal{N}) \simeq \mathcal{D}_X(D_X(\mathcal{M}) \otimes^t D_X(\mathcal{N})).
\]

Under the Riemann-Hilbert correspondence for regular, holonomic \(D\)-modules, the \(s\)-tensor product goes over to the usual tensor product for constructible sheaves.

Note that for a map \( f : X \to Y \), the natural monoidal pullback for \( \otimes^s \) is the functor \( f^* \). In addition, the \(s\)-tensor product satisfies the dual version of the projection formula

\[
    f_!(\mathcal{M} \otimes^s f^*(\mathcal{N})) \simeq f_!(\mathcal{M}) \otimes^s \mathcal{N}.
\]

### 3.3. Compact generators.

We continue with \( X \) a smooth scheme.

The following characterizes the regular \( D\)-module of differential operators \( D_X \): tensoring with \( D_X \) defines an induction functor \( \mathcal{Q}(X) \to D(X) \) which is left adjoint to the forgetful functor \( D(X) \to \mathcal{Q}(X) \).

**Lemma 3.1.** For \( \mathcal{L} \in \mathcal{Q}(X) \) and \( \mathcal{M} \in D(X) \), there is a canonical adjunction equivalence

\[
    \text{Hom}_{D(X)}(D_X \otimes_{\mathcal{O}_X} \mathcal{L}, \mathcal{M}) \simeq \text{Hom}_{\mathcal{Q}(X)}(\mathcal{L}, \mathcal{M}).
\]

**Proof.** Local calculation. □

It is a standard result (see \[N2\], \[BFN\]) that the \(\infty\)-category \( \mathcal{Q}(X) \) of quasicoherent sheaves on a quasi-compact, separated scheme \( X \) is compactly generated with compact objects the perfect (or equivalently, dualizable) complexes.

**Proposition 3.2.** If \( \mathcal{Q}(X) \) is generated by objects \( \mathcal{L} \), then \( D(X) \) is generated by the inductions \( D_X \otimes_{\mathcal{O}_X} \mathcal{L} \). If the objects \( \mathcal{L} \) are compact, then the inductions \( D_X \otimes_{\mathcal{O}_X} \mathcal{L} \) are as well.

**Proof.** The first assertion is an immediate consequence of Lemma 3.1. To see that induction is proper, it suffices by Lemma 2.5 to note that the forgetful functor from \( D\)-modules to \( \mathcal{O}\)-modules is continuous, since it is exact and preserves arbitrary direct sums (or alternatively, since it has a natural right adjoint coming from writing \( D\)-modules as comodules for the coalgebra of jets). □

We see that \( D(X) \) is compactly generated by coherent \( D\)-modules obtained by induction. We next check that pullback and pushforward of \( D\)-modules preserve colimits.

**Proposition 3.3.** Let \( f : X \to Y \) be a map of smooth schemes. Then the \( D\)-module pullback \( f^! : D(Y) \to D(X) \) is continuous.

**Proof.** It suffices to show that the canonical morphism

\[
    \text{colim}_i f^!(\mathcal{M}_i) \to f^!(\text{colim}_i \mathcal{M}_i)
\]

induces an equivalence

\[
    \text{Hom}_{D(X)}(D_X \otimes_{\mathcal{O}_X} \mathcal{L}, f^!(\text{colim}_i \mathcal{M}_i)) \simeq \text{Hom}_{D(X)}(D_X \otimes_{\mathcal{O}_X} \mathcal{L}, \text{colim}_i f^!(\mathcal{M}_i))
\]

where \( \mathcal{L} \) runs through objects of \( \mathcal{Q}(X) \). Using Lemma 3.1 twice, we calculate

\[
    \text{Hom}_{D(X)}(D_X \otimes_{\mathcal{O}_X} \mathcal{L}, f^!(\text{colim}_i \mathcal{M}_i)) \simeq \text{Hom}_{\mathcal{Q}(X)}(\mathcal{L}, f^*(\text{colim}_i \mathcal{M}_i)) \simeq \text{Hom}_{\mathcal{Q}(X)}(\mathcal{L}, \text{colim}_i f^*(\mathcal{M}_i)) \simeq \text{Hom}_{D(X)}(D_X \otimes_{\mathcal{O}_X} \mathcal{L}, \text{colim}_i f^!(\mathcal{M}_i))
\]

□
**Proposition 3.4.** Let $f : X \to Y$ be a map of smooth schemes. Then the $\mathcal{D}$-module pushforward $f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$ is continuous.

*Proof.* It suffices to show that the canonical morphism

$$\text{colim}_i f_*(\mathcal{M}_i) \to f_*(\text{colim}_i \mathcal{M}_i)$$

induces an equivalence

$$\text{Hom}_{\mathcal{D}(X)}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}, f_*(\text{colim}_i \mathcal{M}_i)) \simeq \text{Hom}_{\mathcal{D}(X)}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}, \text{colim}_i f_*(\mathcal{M}_i))$$

where $\mathcal{L}$ runs through objects of $\mathcal{Q}(X)$. Using the fact that tensor product preserves colimits, we calculate

$$\text{Hom}_{\mathcal{D}(X)}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}, f_*(\text{colim}_i \mathcal{M}_i)) \simeq \text{Hom}_{\mathcal{Q}(X)}(\mathcal{L}, f_*(\mathcal{D}_{Y-X} \otimes_{\mathcal{D}_X} \text{colim}_i \mathcal{M}_i))$$

$$\simeq \text{Hom}_{\mathcal{Q}(X)}(\mathcal{L}, \text{colim}_i f_*(\mathcal{D}_{Y-X} \otimes_{\mathcal{D}_X} \mathcal{M}_i))$$

$$\simeq \text{Hom}_{\mathcal{D}(X)}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}, \text{colim}_i f_*(\mathcal{M}_i)).$$

$\square$

**Proposition 3.5.** Let $X$ be a smooth scheme. Then $\mathcal{D}_{coh}(X)$ comprises precisely the compact objects of $\mathcal{D}(X)$.

*Proof.* Let $\pi : X \to pt = \text{Spec}(\mathbb{C})$ be the structure map. Coherent $\mathcal{D}$-modules are compact thanks to the local expression for $\mathcal{D}$-module morphisms out of a coherent $\mathcal{D}$-module

$$\text{Hom}_{\mathcal{D}(X)}(\mathcal{M}, \mathcal{N}) = \pi_* (\mathcal{D}(\mathcal{M}) \otimes \mathcal{N})[-\dim X].$$

It follows that $\mathcal{M}$ is compact since the above expression is a composition of the continuous functors tensor product and pushforward. Conversely, since coherent $\mathcal{D}$-modules generate, all compact objects are finite colimits of coherent $\mathcal{D}$-modules and hence coherent themselves. $\square$

**Corollary 3.6.** For $X$ a smooth scheme, we have an equivalence $\mathcal{D}(X) \simeq \text{Ind} \mathcal{D}_{coh}(X)$.

### 3.4. Extended functoriality.

The identification $\mathcal{D}(X) \simeq \text{Ind} \mathcal{D}_{coh}(X)$ of Corollary 3.6 allows us to analyze arbitrary $\mathcal{D}$-modules on a smooth scheme in terms of coherent $\mathcal{D}$-modules. By Lemma 2.2, any exact functor out of $\mathcal{D}_{coh}(X)$ extends canonically to a continuous functor out of $\mathcal{D}(X)$.

Recall the notation $\mathcal{C}' = \text{Ind}(\mathcal{C}'_{\text{cpt}})$ from Section 2.3.1 for the restricted opposite. The Verdier duality equivalence $\mathcal{D}_X : \mathcal{D}_{coh}(X) \simeq \mathcal{D}_{coh}(X)^\vee$ extends to an equivalence

$$\mathcal{D}_X : \mathcal{D}(X) = \text{Ind} \mathcal{D}_{coh}(X) \simeq \text{Ind}(\mathcal{D}_{coh}(X)^\vee) = \mathcal{D}(X)'$$

which we will denote by the same symbol. Note that $\mathcal{D}(X)'$ has no obvious independent meaning, unlike $\mathcal{D}(X)$ which by Corollary 3.6 represents both the category of all $\mathcal{D}$-modules and also the ind-category of coherent $\mathcal{D}$-modules $\text{Ind} \mathcal{D}_{coh}(X)$. While Verdier duality gives an equivalence $\mathcal{D}_X : \mathcal{D}(X) \simeq \mathcal{D}(X)'$, it is often useful to keep the two distinct.

Proposition 3.3 guarantees that the pullback functor $f^!$ will agree with the continuous extension of its restriction to coherent $\mathcal{D}$-modules (though in general the image of coherents is not coherent unless $f$ is smooth). Similarly, Proposition 3.4 guarantees that the pushforward functor $f_*$ will agree with the continuous extension of its restriction to coherent $\mathcal{D}$-modules (though in general the image of coherents is not coherent unless $f$ is proper).

Furthermore, whenever exact functors on coherent $\mathcal{D}$-modules satisfy an identity, their continuous extensions will as well. We have the following standard examples:

- For $f : X \to Y$ proper, there is a canonical equivalence
  $$\mathcal{D}_Y \circ f_* \simeq f_* \circ \mathcal{D}_X : \mathcal{D}(X) \to \mathcal{D}(Y).$$

- For $f : X \to Y$ smooth, there is a canonical equivalence
  $$\mathcal{D}_X \circ f^! \circ \mathcal{D}_Y \simeq f^! [2(\dim Y - \dim X)] : \mathcal{D}(Y) \to \mathcal{D}(X).$$
Remark 3.7. It is important to note that the continuous extensions of standard functors from coherent to all \(D\)-modules are not in general given by the standard formulas. For example, the continuous Verdier duality \(\mathbb{D}_X : D(X) \simeq D(X)'\) is not given on all \(D\)-modules by a \(Hom\) construction.

We may also consider the full subcategory \(\mathbb{D}_{hol}(X) \subset D(X)\) generated under colimits by holonomic \(D\)-modules. It agrees with the ind-category of \(\mathbb{D}_{hol}(X)\) since holonomic \(D\)-modules are compact:

\[
\mathbb{D}_{hol}(X) \simeq \text{Ind} \mathbb{D}_{hol}(X) \subset \text{Ind} \mathcal{D}_{coh}(X) \simeq D(X).
\]

Here as well, Proposition 3.3 and Proposition 3.4 guarantee that there is no ambiguity in defining pullback and pushforward: the continuous extensions of the usual pullback \(f^!\) and pushforward \(f_*\) to ind-holonomic \(D\)-modules agree with the usual pullback and pushforward on all \(D\)-modules. Furthermore, we immediately have that ind-holonomic \(D\)-modules satisfy the usual functoriality of holonomic \(D\)-modules:

- The adjoint pairs of exact functors \((f_!, f^!\) and \((f^*, f_*)\) on holonomic \(D\)-modules extend to adjoint pairs of continuous functors on ind-holonomic \(D\)-modules.

3.5. Tensors and products.

Lemma 3.8. Let \(X_1, X_2\) be smooth schemes. There is a canonical equivalence

\[
\mathcal{D}_{X_1 \times X_2} \simeq \mathcal{D}_{X_1} \boxtimes \mathcal{D}_{X_2}.
\]

Proof. Local calculation. \(\square\)

Theorem 3.9. Let \(X_1, X_2\) be smooth schemes. Then external tensor product defines an equivalence

\[
\boxtimes : D(X_1) \otimes D(X_2) \simeq D(X_1 \times X_2)
\]

Proof. For notational convenience, set \(X = X_1 \times X_2\), with \(p_1 : X \rightarrow X_1\), \(p_2 : X \rightarrow X_2\) the natural projections. It suffices to show that external tensor product defines an equivalence in \(st\) of \(\infty\)-categories of compact objects

\[
\boxtimes : \mathcal{D}_{cpt}(X_1) \otimes \mathcal{D}_{cpt}(X_2) \simeq \mathcal{D}_{cpt}(X)
\]

Recall from [BFN] that the external products \(\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{Q}(X)\) of perfect generators \(\mathcal{L}_1 \in \mathcal{Q}(X_1)\), \(\mathcal{L}_2 \in \mathcal{Q}(X_2)\) are perfect generators for \(\mathcal{Q}(X)\). Thus by Lemma 3.2 objects of the form

\[
D_X \otimes_{\mathcal{O}_X} (\mathcal{L}_1 \boxtimes \mathcal{L}_2) \simeq (D_{X_1} \otimes_{\mathcal{O}_{X_1}} \mathcal{L}_1) \boxtimes (D_{X_2} \otimes_{\mathcal{O}_{X_2}} \mathcal{L}_2) \in D(X)
\]

provide a collection of compact generators for \(D(X)\). By a result of Neeman, all compact objects can be characterized as summands of finite colimits of compact generators, thus we conclude that the external product preserves all compact objects. Furthermore, by the definition of the tensor product of stable idempotent-complete small \(\infty\)-categories, every compact object is in the image of the external tensor product.

Now we must verify that for \(\mathfrak{M}_i, \mathfrak{N}_i \in \mathcal{D}_{cpt}(X_i)\) we have an equivalence

\[
\text{Hom}_{\mathcal{D}(X)}(\mathfrak{M}_1 \boxtimes \mathfrak{M}_2, \mathfrak{N}_1 \boxtimes \mathfrak{N}_2) \simeq \text{Hom}_{\mathcal{D}(X_1)}(\mathfrak{M}_1, \mathfrak{N}_1) \otimes \text{Hom}_{\mathcal{D}(X_2)}(\mathfrak{M}_2, \mathfrak{N}_2).
\]

It suffices to assume that \(\mathfrak{M}_i\) is of the form \(D_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{L}_i\) for some perfect object \(\mathcal{L}_i \in \mathcal{Q}(X_i)\). Then the above assertion can be rewritten in the form

\[
\text{Hom}_{\mathcal{Q}(X)}(\mathcal{L}_1 \boxtimes \mathcal{L}_2, \mathfrak{M}_1 \boxtimes \mathfrak{M}_2) \simeq \text{Hom}_{\mathcal{Q}(X_1)}(\mathcal{L}_1, \mathfrak{M}_1) \otimes \text{Hom}_{\mathcal{Q}(X_2)}(\mathcal{L}_2, \mathfrak{M}_2).
\]

Now using the dualizability of \(\mathcal{L}_i\) and the projection formula for \(p_2 : X \rightarrow X_2\), we can calculate

\[
\text{Hom}_{\mathcal{Q}(X)}(p_1^! \mathcal{L}_1 \otimes p_2^! \mathcal{L}_2, p_1^* \mathfrak{M}_1 \otimes p_2^* \mathfrak{M}_2) \simeq \text{Hom}_{\mathcal{Q}(X_1)}(\mathcal{L}_1, \mathfrak{M}_1) \otimes \text{Hom}_{\mathcal{Q}(X_2)}(\mathcal{L}_2, \mathfrak{M}_2)
\]

\[
\simeq \text{Hom}_{\mathcal{Q}(X)}(p_2^! \text{Hom}_{\mathcal{Q}(X_1)}(\mathcal{L}_1, \mathfrak{M}_1) \otimes p_1^* \text{Hom}_{\mathcal{Q}(X_2)}(\mathcal{L}_2, \mathfrak{M}_2))
\]

\[
\simeq \text{Hom}_{\mathcal{Q}(X)}(p_1^! \text{Hom}_{\mathcal{Q}(X_1)}(\mathcal{L}_1, \mathfrak{M}_1) \otimes p_2^* \text{Hom}_{\mathcal{Q}(X_2)}(\mathcal{L}_2, \mathfrak{M}_2))
\]

\[
\simeq \text{Hom}_{\mathcal{Q}(X_1)}(\mathcal{L}_1, \mathfrak{M}_1) \otimes \text{Hom}_{\mathcal{Q}(X_2)}(\mathcal{L}_2, \mathfrak{M}_2)
\]

\(\square\)
Corollary 3.10. Let $X_1, X_2$ be smooth schemes. Then external tensor product defines an equivalence

$$
\Xi : \mathcal{D}_{coh}(X_1) \otimes \mathcal{D}_{coh}(X_2) \xrightarrow{\sim} \mathcal{D}_{coh}(X_1 \times X_2)
$$

Proof. The proof of the theorem shows that the assertion is true for compact objects. But we have seen that on a scheme, compact $\mathcal{D}$-modules coincide with coherent $\mathcal{D}$-modules. \qed

3.6. Duality over a point. We now show that $\mathcal{D}(X)$ is dualizable as a $\mathbb{C}$-linear stable presentable $\infty$-category, or in other words, as a $\text{Mod}_C$-module.

Proposition 3.11. For $X$ a scheme, $\mathcal{D}(X)$ is self-dual over $\mathcal{D}(pt) \simeq \text{Mod}_\mathbb{C}$. When $X$ is proper, then $\mathcal{D}_{coh}(X)$ is self-dual over $\mathcal{D}_{coh}(pt) \simeq \text{Perf}_\mathbb{C}$.

Proof. By Theorem 3.9 we have a canonical equivalence

$$
\mathcal{D}(X) \otimes \mathcal{D}(X) \simeq \mathcal{D}(X \times X).
$$

Thus we can define a unit and counit by the correspondences

$$
u = \Delta, f^\dagger : \mathcal{D}(pt) \to \mathcal{D}(X) \otimes \mathcal{D}(X) \quad c = f_! \Delta^\dagger : \mathcal{D}(X) \otimes \mathcal{D}(X) \to \mathcal{D}(pt)
$$

where $\Delta : X \to X \times X$ is the diagonal, and $f : X \to pt$ is the unique morphism.

We must check that the following composition is the identity

$$
\mathcal{D}(X) \xrightarrow{\nu \otimes \text{id}} \mathcal{D}(X) \otimes \mathcal{D}(X) \otimes \mathcal{D}(X) \xrightarrow{\text{id} \otimes c} \mathcal{D}(X).
$$

Consider the following commutative diagram with Cartesian square:

\[\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times X \\
\downarrow{\Delta} & & \downarrow{\pi_1} \\
X \times X & \xrightarrow{\Delta \times \text{id}_3} & X \times X \times X
\end{array}\]

Using standard identities for composition and base change, we calculate

$$(\text{id} \otimes c) \circ (\nu \otimes \text{id}) = \pi_1^* (\text{id}_1 \times \Delta_2 \times \text{id}_3) \Delta_1^\dagger \Delta_2 \pi_2^\dagger = \pi_1^* \Delta \Delta_1^\dagger \pi_2^\dagger 
\simeq \pi_1^* \Delta_1 \Delta_2 \pi_2^\dagger 
\simeq \text{id}_{\mathcal{D}(X)}.$$

The same argument gives the second assertion. \qed

The general $\mathcal{D}$-module formalism allows us to construct functors out of integral kernels. For $X_1, X_2$ schemes, there is a canonical map

$$
\mathcal{D}(X_1 \times X_2) \to \text{Fun}(\mathcal{D}(X_1), \mathcal{D}(X_2)) \quad \mathcal{M} \mapsto \pi_{2*} (\pi_1^\dagger(-) \otimes \mathcal{M})
$$

When $X_1$ is proper, the same formalism provides a canonical map

$$
\mathcal{D}_{coh}(X_1 \times X_2) \to \text{Fun}(\mathcal{D}_{coh}(X_1), \mathcal{D}_{coh}(X_2)) \quad \mathcal{M} \mapsto \pi_{2*} (\pi_1^\dagger(-) \otimes \mathcal{M})
$$

Our preceding results immediately imply that the above maps are equivalences, and thus identify integral transforms with functors.

Corollary 3.12. For $X_1, X_2$ smooth schemes, the natural maps give equivalences

$$
\mathcal{D}(X_1 \times X_2) \simeq \mathcal{D}(X_1) \otimes \mathcal{D}(X_2) \simeq \text{Fun}(\mathcal{D}(X_1), \mathcal{D}(X_2)).
$$

When $X_1$ is proper, the natural maps give equivalences

$$
\mathcal{D}_{coh}(X_1 \times X_2) \simeq \mathcal{D}_{coh}(X_1) \otimes \mathcal{D}_{coh}(X_2) \simeq \text{Fun}(\mathcal{D}_{coh}(X_1), \mathcal{D}_{coh}(X_2)).
$$

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3.6.1. Duality via Verdier duality. We present here an alternative abstract proof of the first assertion of Theorem 3.15 pointed out by Jacob Lurie. It is based on the following general principle:

**Proposition 3.13.** Suppose \( C \in \mathcal{St}_C \) is compactly generated so that \( C = \text{Ind} C_{cpt} \). Then \( C \) is dualizable in \( \mathcal{St}_C \) with dual the restricted opposite \( C' = \text{Ind}(C_{cpt}^\circ) \).

**Proof.** We construct unit and counit functors
\[ u : \text{Mod}_C \to C \otimes C' \quad c : C' \otimes C \to \text{Mod}_C \]
and leave it to the reader to check the usual compatibility. By [BFN Proposition 4.4.], the functor \( \text{Ind} : \text{st}_C \to \text{St}_C \) is symmetric monoidal, so we have an identification \( \text{Ind} C_{cpt} \otimes \text{Ind}(C_{cpt}^\circ) \cong \text{Ind}(C_{cpt} \otimes C_{cpt}^\circ) \). Then we take the counit \( c \) to be the continuous extension of \( \text{Hom} : C_{cpt}^\circ \otimes C_{cpt} \to \text{Mod}_C \) to the ind-categories. We take the unit \( u \) to be the \( \mathcal{C} \)-linear extension of the functor sending the unit \( C \) to the functor \( \text{Hom} : C_{cpt}^\circ \otimes C_{cpt} \to \text{Mod}_C \) considered as an object of \( \text{Ind}(C_{cpt} \otimes C_{cpt}) \) where we think of ind-objects as \( \text{Mod}_C \)-valued functionals on the opposite category \( C_{cpt}^\circ \otimes C_{cpt} \).

Now we can apply the above taking \( C = \mathcal{D}(X), C_{cpt} = \mathcal{D}_{coh}(X) \). Then Verdier duality provides an equivalence \( \mathcal{D} : \mathcal{D}_{coh}(X) \cong \mathcal{D}_{coh}(X)^\circ \), and hence an equivalence between \( \mathcal{D}(X) \) and its restricted opposite \( \mathcal{D}(X)' \). Note that this identification agrees (up to a shift) with that of Theorem 3.15 since the counits are equivalent on compact generators. Namely, for \( \mathcal{M}, \mathcal{N} \in \mathcal{D}_{coh}(X) \), we have an equivalence
\[ \text{Hom}_{\mathcal{D}(X)}(\mathcal{D}(X)(\mathcal{M}), \mathcal{N}) \cong \pi_*([\mathcal{M} \otimes \mathcal{N}] \sim \dim X] : \mathcal{D}(X) \otimes \mathcal{D}(X) \to \text{Mod}_C \]
where \( \pi : X \to pt \). The pairing on the left is the abstract counit of Proposition 3.13 and the pairing on the right is the (shifted) geometric counit of Theorem 3.15.

3.7. Linear functors and fiber products. We continue to work with smooth schemes, but now introduce a nontrivial base.

**Theorem 3.14.** Let \( X_1 \to Y \leftarrow X_2 \) be maps of smooth schemes such that \( X_1 \times_Y X_2 \) is also smooth. Then the canonical maps are equivalences
\[ \mathcal{D}(X_1) \otimes_{\mathcal{D}(Y)} \mathcal{D}(X_2) \sim \mathcal{D}(X_1 \times_Y X_2) \]

**Proof.** The tensor product of \( \mathcal{D}(Y) \)-modules can be calculated ([L4 5]) as the geometric realization of the two-sided bar construction ([L3 4.5]), the simplicial category with \( k \)-simplices
\[ \mathcal{D}(X_1) \otimes \mathcal{D}(Y) \otimes \cdots \otimes \mathcal{D}(Y) \otimes \mathcal{D}(X_2), \]
where the factor \( \mathcal{D}(Y) \) appears \( k \) times and the maps are given by the \( \mathcal{D}(Y) \)-module structures in the usual pattern. Note that by Theorem 3.13, we can rewrite the \( k \)-simplices in the form
\[ \mathcal{D}(X_1 \times Y \times \cdots \times Y \times X_2) \]

Let \( j : X_1 \times_Y X_2 \to X_1 \times X_2 \) be the natural closed embedding. Then the pushforward
\[ j_* : \mathcal{D}(X_1 \times_Y X_2) \to \mathcal{D}(X_1 \times X_2) \]
and its analogues
\[ \mathcal{D}(X_1 \times_Y X_2) \to \mathcal{D}(X_1 \times Y \times \cdots \times Y \times X_2) \]
for higher simplices provides a lift of the above simplicial category to a split augmented simplicial category. To see this, one uses base change and Kashiwara’s lemma: for a closed embedding \( j \) and a \( \mathcal{D} \)-module \( \mathcal{M} \), the natural adjunction \( \mathcal{M} \to j^! j_* \mathcal{M} \) is an equivalence.

Finally, by [L4 Lemma 6.1.3.16], in any \( \infty \)-category, split augmented simplicial diagrams are colimit diagrams. In other words, \( \mathcal{D}(X_1 \times_Y X_2) \) is identified with the colimit \( \mathcal{D}(X_1) \otimes_{\mathcal{D}(Y)} \mathcal{D}(X_2) \) of the diagram.

**Theorem 3.15.** Let \( f : X \to Y \) be a map of schemes. Then \( \mathcal{D}(X) \) is a self-dual module over \( \mathcal{D}(Y) \).
Proof. By Theorem \ref{thm:canonical_equivalence} we have a canonical equivalence
\[ D(X) \otimes_{D(Y)} D(X) \simeq D(X \times_Y X). \]
Thus we can define a unit and counit by the correspondences
\[ u = \Delta_* f^! : D(Y) \to D(X) \otimes_{D(Y)} D(X) \]
\[ c = f_* \Delta^! : D(X) \otimes_{D(Y)} D(X) \to D(Y) \]
where \( \Delta : X \to X \times_Y X \) is the diagonal.

We must check that the following composition is the identity
\[ D(X) \xrightarrow{u \otimes \text{id}} D(X) \otimes_{D(Y)} D(X) \otimes_{D(Y)} D(X) \xrightarrow{\text{id} \otimes c} D(X). \]

Consider the following commutative diagram with Cartesian square:
\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times_Y X \\
\downarrow & & \downarrow \pi_1 \\
X \times_Y X & \xrightarrow{\Delta_{12} \times \text{id}_3} & X \times_Y X \times_Y X
\end{array}
\]

Using standard identities for composition and base change, we calculate
\[
(id \otimes c) \circ (u \otimes \text{id}) = \pi_{1+}(\text{id}_1 \times \Delta_{23})^!((\Delta_{12} \times \text{id}_3)_* \pi_2^!) \simeq \pi_{1*} \Delta_* \Delta^! \pi_2^! \simeq \text{id}_{D(X)}.
\]

By the general \( D \)-module formalism of base change and the projection formula, we can construct linear functors out of integral kernels. Namely, for maps of schemes \( X_1 \to Y \leftarrow X_2 \), there is a canonical map
\[ D(X_1 \times_Y X_2) \to \text{Fun}_{D(Y)}(D(X_1), D(X_2)) \]
\[ \mathcal{M} \mapsto \pi_{2*}(\pi_{1*}(-) \otimes \mathcal{M}) \]

Our preceding results immediately imply that the above map is an equivalence, and thus identifies integral transforms with linear functors.

Corollary 3.16. Let \( X_1 \to Y \leftarrow X_2 \) be maps of smooth schemes. Then the natural maps are equivalences
\[ D(X_1 \times_Y X_2) \simeq D(X_1) \otimes_{D(Y)} D(X_2) \simeq \text{Fun}_{D(Y)}(D(X_1), D(X_2)). \]

4. \( D \)-modules on stacks

4.1. Basic properties. By a stack \( X \), we will always mean a quasi-compact stack with affine diagonal. Throughout this paper, we will also always assume that \( X \) is smooth. The fundamental references for equivariant derived categories and \( D \)-modules on stacks are Bernstein-Lunts \cite{BL}, and Beilinson-Drinfeld \cite{BD} Chapters 1 and 7.

Given a smooth stack \( X \), we can choose a smooth cover \( U \to X \) by a smooth scheme, and consider the induced Čech simplicial smooth scheme \( U_* \to X \). (Note that this provides an unambiguous notion of dimension: \( \dim X = \dim U_0 - \dim U_1 \).) By definition, the \( \infty \)-category of \( D \)-modules on \( X \) is the limit (totalization) in the \( \infty \)-category \( St \) of stable presentable \( \infty \)-categories of the cosimplicial stable presentable \( \infty \)-category of \( D \)-modules on the simplices
\[ D(X) = \lim D(U_*) \in St \]
where the diagram maps are the pullbacks \( \sigma^! \) along the face maps \( \sigma \). Note that by \cite{Lus} Proposition 5.5.3.13, a small limit can be calculated in the \( \infty \)-category \( St \) (which in turn is a full subcategory of \( Pr \) closed under such limits), or equivalently as a limit of plain \( \infty \)-categories. An object of \( D(X) \) can be represented by a collection of \( D \)-modules on the simplices \( U_* \) that are compatible under the
pullbacks $\sigma^\dagger$. Standard arguments show that $D(X)$ is independent of the choice of covers. (See [BD] Section 7.5 for a discussion in the language of ordinary derived categories.)

Given a representable map $f : X \to Y$ of smooth stacks, we have the two $D$-module functors $f^\dagger$, $f_*$ obtained by taking the limits of their local analogues. The pullback $f^\dagger$ is well-defined thanks to the composition identity $\sigma^\dagger f^\dagger \simeq f^\dagger \sigma^\dagger$. The pushforward $f_*$ is well-defined thanks to the base change identity $\sigma^\dagger f_* \simeq f_* \sigma^\dagger$. They can be calculated by applying their local analogues term by term to the cosimplicial categories. As morphisms in the $\infty$-category $St$, the functors $f^\dagger, f_*$ are continuous. They also satisfy the usual local properties such as base change. Note that the shift functor can also be applied locally on the simplices, and the shifted pullback $f^\dagger = f^\dagger[2\dim X - \dim Y]$ can be obtained by shifting by the relative dimension locally on the simplices.

By a coherent or holonomic $D$-module on a stack $X$, we mean an object $M \in D(X)$ such that for any smooth map $u : U \to X$ from a smooth scheme $U$, the pullback $u^! M$ is coherent or holonomic respectively. (Note that this is consistent since smooth pullbacks between schemes preserve coherence and holonomicity.) The subcategories $D_{hol}(X) \subset D_{coh}(X) \subset D(X)$ are identified with the limits of the cosimplicial $\infty$-categories $D_{hol}(U_*) \subset D_{coh}(U_*) \subset D(U_*)$ respectively. Given a representable map $f : X \to Y$ of smooth stacks, $f^\dagger$ and $f_*$ preserve holonomicity, $f^\dagger$ preserves coherence when $f$ is smooth, and $f_*$ preserves coherence when $f$ is proper.

Recall that for a smooth scheme $U$, we have Verdier duality $D_U : D_{coh}(U) \simeq D_{coh}(U)^\diamond$. It is convenient to introduce the normalized duality $D'_U = [-2\dim U] \circ D_U \simeq D_U \circ [2\dim U]$ so that $D_U \circ D'_U \simeq \text{id}$ and for example $D_U((O_U) \simeq O_U$. Then for a smooth morphism $f : U \to V$ the normalized duality commutes with pullback $f^\dagger D'_U = D'_V f^\dagger$. As a result, on a smooth stack $X$, the normalized duality descends to an equivalence

$$D'_{X} : D_{coh}(X) \simeq D_{coh}(X)^\diamond.$$ 

We will write $D_X$ for the Verdier duality obtained by once and for all shifting back the normalized duality into the usual degree $D_X \simeq [2\dim X] \circ D'_X$ Note that this shift is not given locally by shifting by twice the dimension of simplices but always by twice the total dimension.

Finally, given a representable map $f : X \to Y$ of smooth stacks, one defines functors on holonomic $D$-modules by the usual formulas

$$f_! : D_{hol}(X) \to D_{hol}(Y) \quad f^* : D_{hol}(Y) \to D_{hol}(X)$$

$$f_! = D_Y \circ f_* \circ D_X \quad f^* = D_X \circ f^\dagger \circ D_Y$$

They satisfy the usual adjunctions with adjunction maps given by the limits of the compatible adjunction maps for the simplices of a given Čech cover.

4.2. Holonomic stacks. When working with $D$-modules on stacks, one must struggle with the failure of Proposition 3.5. Coherent and in particular holonomic $D$-modules may no longer be compact.

Example 4.1. The structure sheaf $O_X$, while always holonomic, is not necessarily compact. For example, $O_X$ is compact when $X$ is a scheme, but not compact when $X = BG = pt/G$, and $G$ is a (nontrivial) reductive group. Equivalently, the functor of global flat sections on $BG$, in other words equivariant cohomology, is not continuous even for $G$ the multiplicative group. More generally, finite rank vector bundles with flat connection are holonomic but not necessarily compact.

Definition 4.2. The $\infty$-category $\hat{D}_{hol}(X) \subset St$ of holonomically generated $D$-modules is the full subcategory of $D(X)$ generated under colimits by $D_{hol}(X)$.

In general, $\hat{D}_{hol}(X)$ is not equivalent to the ind-category Ind $D_{hol}(X)$ since holonomic $D$-modules are not necessarily compact. More precisely, $\hat{D}_{hol}(X)$ is the localization Ind$_R D_{hol}(X)$, where $R$ is the collection of small diagrams that are colimit diagrams in $D_{hol}(X)$ (see Remark 2.3). Nevertheless, as the following discussion establishes, the rich functoriality of holonomic $D$-modules extends to holonomically generated $D$-modules.
Proposition 4.3. For any representable map \( f : X \to Y \) of smooth stacks, there are adjoint pairs of continuous functors \((f^*, f_*)\) and \((f_!, f!)\) on holonomically generated \(D\)-modules, extending their counterparts on holonomic \(D\)-modules. Moreover, \(f^*\) and \(f_!\) are proper.

Proof. The four functors are all continuous on holonomic \(D\)-modules: as mentioned above, \(f_! = f^!(2\dim X - \dim Y)\) and \(f_*\) are continuous on all \(D\)-modules; on holonomic \(D\)-modules, \(f^*\) and \(f_!\) are left adjoints and hence continuous. Thus all four functors extend canonically and continuously to holonomically generated \(D\)-modules and retain all adjunctions.

The functors \(f^*\) and \(f_!\) have the continuous right adjoints \(f_*\) and \(f^!\) hence are proper. \(\Box\)

Remark 4.4. A curious nonexample is when \(X = BG = pt/G\), where \(G\) is an affine algebraic group (of characteristic zero), and \(Y = pt\). Then (as long as \(G\) is not trivial) the projection \(BG \to pt\) is not representable, though one can check that the \(D\)-module pushforward is continuous.

This may look like a paradox since we have asserted that the structure sheaf \(\mathcal{O}_{BG}\) is not a compact object of \(D(BG)\). In other words, the assignment of flat sections

\[
\mathcal{M} \mapsto \text{Hom}_{\mathcal{C}_{-\times}(G)}(\mathbb{C}, \mathcal{M})
\]

to a \(D\)-module on \(BG\) does not preserve colimits. (Here we identify \(D\)-modules on \(BG\) via descent with modules over the “group algebra” \(\mathcal{C}_{-\times}(G)\) of chains on \(G\).) To resolve this paradox, one should also remember that there is no adjunction available here to calculate the pushforward, and so even for holonomic \(D\)-modules, the pushforward has less to do with flat sections than one might have guessed. Indeed, unwinding the definition, we see that the pushforward is calculated as the coinvariants

\[
\mathcal{M} \mapsto \mathbb{C} \otimes_{\mathcal{C}_{-\times}(G)} \mathcal{M}
\]
rather than as the invariants.

Definition 4.5. A smooth stack \(X\) is said to be holonomic if the canonical inclusion \(\mathcal{D}_{\text{hol}}(X) \subset \mathcal{D}(X)\) is an equivalence, or in other words, if every \(D\)-module is a colimit of holonomic \(D\)-modules.

For holonomic stacks, the rich functoriality of holonomic \(D\)-modules extends to all \(D\)-modules.

Corollary 4.6. For any representable map \(f : X \to Y\) of holonomic stacks, there are adjoint pairs of continuous functors \((f^*, f_*)\) and \((f_!, f!)\) on all \(D\)-modules, extending their counterparts on holonomic \(D\)-modules. Moreover, \(f^*\) and \(f_!\) are proper.

4.3. Finite orbit stacks. To guarantee an easy supply of compact objects, we will restrict to a specialized context adapted to our applications.

Definition 4.7. A finite orbit stack is a quotient stack \(Z/G\), where \(Z\) is a smooth quasiprojective variety, and \(G\) is an affine algebraic group that acts on \(Z\) with finitely many orbits.

Favorite examples include classifying stacks \(BG = pt/G\) of affine algebraic groups, and double quotient stacks \(P^\circ G/P\), where \(G\) is a reductive group, and \(P \subset G\) is a parabolic subgroup.

Remark 4.8. Coherent and holonomic \(D\)-modules on finite orbit stacks coincide, since the singular supports of \(G\)-equivariant \(D\)-modules on \(Z\) are contained in the union of conormals to the \(G\)-orbits and the conormals are Lagrangian.

Proposition 4.9. Let \(Z/G\) be a finite orbit stack. Then \(\mathcal{D}(Z/G)\) is compactly generated by holonomic objects, and all compact objects are holonomic. In particular, \(Z/G\) is a holonomic stack.

Proof. We argue by induction on the number of orbits of \(G\) acting on \(Z\).

Suppose there is a single orbit so \(Z = BH\) where \(H \subset G\) is the stabilizer. By descent, we have a canonical equivalence

\[
\mathcal{D}(BH) \simeq \text{Mod}_{\mathcal{C}_{-\times}(H)}
\]

where \(\mathcal{C}_{-\times}(H)\) denotes the group algebra of singular chains on \(H\) with its natural convolution product. The regular module \(\mathcal{C}_{-\times}(H)\) itself provides a compact holonomic generator.
In general, suppose there are two or more orbits of \( G \) acting on \( Z \). Choose a nonempty open collection of orbits \( i : U \to Z/G \) whose complement is a nonempty closed collection of orbits \( j : V \to Z/G \). Then as explained in [B], any object \( \mathcal{M} \in D(Z/G) \) fits into an exact triangle

\[
j_* i^! \mathcal{M} \to \mathcal{M} \to i_* i^! \mathcal{M} \to [1]
\]

By induction, this immediately implies that \( D(Z/G) \) is generated under colimits by holonomic objects, and so \( Z/G \) is a holonomic stack. Thus we have the functoriality of Corollary 4.10.

Now by induction, both \( D(U) \) and \( D(V) \) are generated by compact holonomic objects. By Corollary 4.10, \( j_* \approx j_*' \) is proper and \( (j_!, j_*') \) form an adjoint pair. Thus to find a compact holonomic object \( \mathcal{R} \in D(Z/G) \) such that \( \text{Hom}_{D(Z/G)}(\mathcal{R}, \mathcal{M}) \not\approx 0 \), it suffices to assume \( \mathcal{M} \cong i_* i^! \mathcal{M} \).

By induction, we can choose a compact holonomic object \( \mathcal{R} \in D(U) \) with \( \text{Hom}_{D(U)}(\mathcal{R}, i^! \mathcal{M}) \not\approx 0 \). Then by Corollary 4.10, \( i_* \mathcal{R} \) is compact, and since the natural morphism \( i_* \mathcal{R} \to i_* \mathcal{R} \to i_* \mathcal{M} \) is nonzero on \( U \), we conclude \( \text{Hom}_{D(Z/G)}(i_* \mathcal{R}, i_* \mathcal{M}) \not\approx 0 \).

Conversely, since we have a collection of holonomic compact generators, all compact objects are finite colimits of holonomic objects and hence holonomic themselves.

We can restate the proposition as asserting that \( D(Z/G) \cong \text{Ind} D_{\text{cpt}}(Z/G) \) where \( D_{\text{cpt}}(Z/G) \subset D_{\text{hol}}(Z/G) \) consists of compact objects.

Now with compact generators in hand, we will focus on Verdier duality and its implications. Recall from Section 2.3.1 the notation \( C' = \text{Ind}(C^\vee) \) for the restricted opposite.

**Proposition 4.10.** On finite orbit stacks \( Z/G \), Verdier duality on holonomic \( D \)-modules extends to a canonical equivalence \( D_{Z/G} : D(Z/G) \cong D(Z/G)^{\vee} \), and exhibits \( D(Z/G) \) as self-dual over \( D(pt) \cong \text{Mod}_C \).

**Proof.** Verdier duality \( D_{Z/G} : D_{\text{hol}}(Z/G) \to D_{\text{hol}}(Z/G)^{\vee} \) is an equivalence, hence preserves all colimits. Thus it agrees on all holonomic objects with its canonical continuous extension from compact objects \( D_{\text{cpt}}(Z/G) \subset D_{\text{hol}}(Z/G) \) to the ind-category \( D(Z/G) \cong \text{Ind} D_{\text{cpt}}(Z/G) \).

Finally, Proposition 4.10 immediately establishes the self-duality.

We also have the familiar local formula for morphisms out of a compact \( D \)-module.

**Proposition 4.11.** For a finite orbit stack \( Z/G \), we have

\[
\text{Hom}_{D(Z/G)}(\mathcal{M}, \mathcal{N}) \cong \pi_*(\text{Proj}(\mathcal{M} \otimes \mathcal{N})[-\dim Z/G]), \quad \text{for } \mathcal{M} \in D_{\text{cpt}}(Z/G)^{\vee}, \mathcal{N} \in D(Z/G)
\]

where \( \pi : Z/G \to pt \).

**Proof.** The proof is a similar induction to the proof of Proposition 4.9. We adopt the setup and notation introduced there.

The case of a single orbit \( BH \) follows from the identification \( D(BH) \cong \text{Mod}_{C_{\text{an}}(H)} \). The assertion is immediate for the algebra \( C_{\text{an}}(H) \), and every compact object is a finite colimit of copies of it.

In general, we have seen that any compact object is a finite colimit of compact objects of the form \( j_* \mathcal{R} \approx j_* \mathcal{R} \) and \( i_* \mathcal{R} \). By standard identities and induction, the assertion is immediate for such objects, and so follows for all finite colimits of them as well.

**Theorem 4.12.** Let \( Z_1/G_1, Z_2/G_2 \) be finite orbit stacks. Then we have canonical equivalences

\[
D(Z_1/G_1) \otimes D(Z_2/G_2) \sim D(Z_1/G_1 \times Z_2/G_2) \sim \text{Fun}(D(Z_1/G_1), D(Z_2/G_2))
\]

**Proof.** Thanks to the compact objects of Proposition 4.9, the equivalence of the tensor product category with \( D \)-modules on the product is similar to Theorem 4.9 and we leave it to the reader.

By Proposition 4.10 \( D(X_1) \) is self-dual and so the tensor product is equivalent to the functor category.
4.4. **Functoriality over classifying stacks.** In this section, we study the structure of equivariant categories of \( \mathcal{D} \)-modules as modules over \( \mathcal{D} \)-modules on classifying stacks. In Proposition 4.14 we establish the self-duality of the \( \infty \)-category of \( \mathcal{D} \)-modules \( \mathcal{D}(Z/G) \) on a finite orbit stack \( Z/G \) as a module category over \( \mathcal{D}(BG) \).

As a consequence, we will immediately obtain the following.

**Theorem 4.13.** Let \( Z/G \) be a finite orbit stack. Then \( \mathcal{D}(Z/G) \) is canonically self-dual as a \( \mathcal{D}(BG) \)-module category, and so for any \( \mathcal{D}(BG) \)-module category \( \mathcal{M} \), there is a canonical equivalence

\[
\mathcal{D}(Z/G) \otimes_{\mathcal{D}(BG)} \mathcal{M} \cong \text{Fun}_{\mathcal{D}(BG)}(\mathcal{D}(Z/G), \mathcal{M}).
\]

When \( \mathcal{M} \) consists of \( \mathcal{D} \)-modules on a stack \( Z'/G \), the above equivalence is realized by the usual formalism of integral transforms.

To clarify the argument, we adopt the notation \( Y = BG \), \( X = Z/G \). The crucial feature of the equivariant setup is that the diagonal map of the base \( Y \) is smooth.

Given two such finite orbit stacks \( X_1, X_2 \), consider the natural product morphism

\[
\pi : X_1 \times_Y X_2 \to X_1 \times X_2.
\]

Since \( \pi \) is a base change of the diagonal of \( Y \), it is a smooth and affine morphism. Note that \( \pi \) is a fibration with fibers (non-canonically) isomorphic to \( G \).

Consider the adjoint pair of functors

\[
\pi^*: \mathcal{D}(X_1 \times X_2) = \mathcal{D}_{hol}(X_1 \times X_2) \xleftarrow{\sim} \mathcal{D}_{hol}(X_1 \times_Y X_2) : \pi_*
\]

Note that \( \mathcal{D}(X_1 \times X_2) \) is compactly generated, and \( \pi^* \) is proper, hence \( \pi_* \) is continuous.

Let \( T = \pi_* \pi^* \) be the resulting monad acting on \( \mathcal{D}(X_1 \times X_2) \). The action of \( T \) is equivalent to tensoring with the commutative algebra object

\[
A_{X_1 \times_Y X_2} = \pi_* \mathcal{O}_{X_1 \times_Y X_2} \in \mathcal{D}(X_1 \times X_2)
\]

of cochains along the fibers, with the commutative algebra structure determining the monad structure. Note that the fiber of this algebra at any point is (non-canonically) isomorphic to the algebra of singular cochains \( C^*(G) \).

Passing from \( \mathcal{D}(X_1 \times_Y X_2) \) to its monadic image \( \text{Mod}_T \) loses a great deal of information, so we cannot hope to recover \( \mathcal{D} \)-modules on \( X_1 \times_Y X_2 \) from the algebra \( A_{X_1 \times_Y X_2} \). However, the algebra does capture the relative tensor product of \( \mathcal{D} \)-module categories.

**Proposition 4.14.** There is a canonical equivalence

\[
\mathcal{D}(X_1) \otimes_{\mathcal{D}(Y)} \mathcal{D}(X_2) \cong \text{Mod}_{A_{X_1 \times_Y X_2}}(\mathcal{D}(X_1 \times X_2)).
\]

**Proof.** We will calculate both sides as geometric realizations of simplicial stable categories that we can compare.

On the left hand side, the tensor product of \( \mathcal{D}(Y) \)-modules is defined ([L4 5]) as the geometric realization of the two-sided bar construction ([L3 4.5]), the simplicial category with \( k \)-simplices

\[
\mathcal{D}(X_1) \otimes \mathcal{D}(Y) \otimes \cdots \otimes \mathcal{D}(Y) \otimes \mathcal{D}(X_2),
\]

where the factor \( \mathcal{D}(Y) \) appears \( k \) times. So its initial terms have the form

\[
\mathcal{D}(X_1) \otimes \mathcal{D}(X_2) \xrightarrow{\sim} \mathcal{D}(X_1) \otimes \mathcal{D}(Y) \otimes \mathcal{D}(X_2) \xrightarrow{\sim} \mathcal{D}(X_1) \otimes \mathcal{D}(Y) \otimes \mathcal{D}(Y) \otimes \mathcal{D}(X_2) \cdots,
\]

and the maps are given by the \( \mathcal{D}(Y) \)-module structures in the usual pattern.

For the right hand side, observe that the fiber product \( X_1 \times_Y X_2 \) is the totalization of the cosimplicial stack with \( k \)-cosimplices

\[
X_1 \times Y \times \cdots \times Y \times X,
\]

where the factor \( Y \) appears \( k \) times. So its initial terms have the form

\[
X_1 \times X_2 \xrightarrow{t_1} X_1 \times Y \times X_2 \xrightarrow{t_2} X_1 \times Y \times Y \times X_2 \cdots
\]
and the maps are induced by the maps $X_1, X_2 \to Y$ and various diagonals.

Since $\pi$ is smooth, we have $\pi^* \simeq \pi^!$, and so by base change, the monad $T = \pi_* \pi^*$ is equivalent to the composition $\Delta \circ \tau_{11}$, coming from the initial maps of the above diagram.

By a repeated application of Theorem 4.12, we see that such modules are calculated as the geometric realization of a simplicial category with the same $k$-simplices

$$D(X_1) \otimes D(Y) \otimes \cdots \otimes D(Y) \otimes D(X_2)$$

as above for the tensor product category. Moreover, it is straightforward to check that the simplicial structure maps in both cases are also identified. Thus the geometric realizations of the two simplicial categories agree. \(\square\)

Now let us specialize to a single finite orbit stack $f : X = Z/G \to Y = BG$.

**Proposition 4.15.** The $\infty$-category $D(X)$ is self-dual as a $D(Y)$-module.

*Proof.* By Proposition 4.14 we have a canonical equivalence

$$D(X) \otimes_{D(Y)} D(X) \simeq \text{Mod}_{A_{X \times Y}X} (D(X \times X)).$$

In addition, the adjoint pair of functors $(\pi^*, \pi_*)$ induce an adjoint pair of functors

$$\tilde{\pi}^* : \text{Mod}_{A_{X \times Y}X} (D(X \times X)) \to \hat{D}_{\text{hol}}(X \times Y X)$$

$$\tilde{\pi}_* : \hat{D}_{\text{hol}}(X \times Y X) \to \text{Mod}_{A_{X \times Y}X} (D(X \times X)).$$

Using the above functors, we can define a unit and counit by the correspondences

$$u = \tilde{\pi}_* \Delta_* f^! : D(Y) \to D(X) \otimes_{D(Y)} D(X)$$

$$c = f_* \Delta^! \tilde{\pi}^* : D(X) \otimes_{D(Y)} D(X) \to D(Y)$$

where $\Delta : X \to X \times Y X$ is the relative diagonal. Recall that $\pi$ is smooth, hence $\pi^* \simeq \pi^!$, and hence the composition $\Delta^! \tilde{\pi}^*$ is not as strange as it may appear.

We must check that the following composition is the identity

$$D(X) \xrightarrow{u \otimes \text{id}} D(X) \otimes_{D(Y)} D(X) \otimes_{D(Y)} D(X) \xrightarrow{\text{id} \otimes \tau} D(X).$$

Consider the following commutative diagram:

![Diagram](https://example.com/image.png)

Pass to $D$-modules to obtain the following commutative diagram:

![Diagram](https://example.com/image.png)
First, thanks to the commutativity of the diagram (and the resulting compatibility of the corresponding monads), we have the identity
\[ \tilde{\pi}_*(\Delta_{12} \times \text{id}_3) \simeq (\Delta_{12} \times \text{id}_3)_* \tilde{\pi}_*, \]
and in addition, since \( \pi^* = \pi^! \), we also have the identity
\[ (\text{id}_1 \times \Delta_{23})^! \tilde{\pi}^* \simeq \tilde{\pi}^*(\text{id}_1 \times \Delta_{23})^! \].

Second, let \( \delta : X \to X \times X \times X \) be the total diagonal. Within \( \text{Mod}_{A_{X \times Y \times X \times Y}}(D(X \times X \times X)) \), we have an equivalence of commutative algebra objects

\[ \delta_* A_X \simeq (\Delta_{12} \times \text{id}_3)_* A_{X \times Y} \otimes A_{X \times Y \times Y} \] \( \text{id}_1 \times \Delta_{23} \) \( A_{X \times Y} \)

which is the monadic image of the underlying statement on stacks, describing the small diagonal as a fiber product of the large diagonals.

Therefore we have the enhanced base change identity (following two arcs of the circumference of the above diagram)
\[ \tilde{\pi}^*(\text{id}_1 \times \Delta_{23})^!(\Delta_{12} \times \text{id}_3)_* \tilde{\pi}_* \simeq \Delta_* \Delta^! \]
for functors \( D(X \times_Y X) \to D(X \times_Y X) \).

Putting the above together, we have equivalences of functors
\[ (\text{id} \otimes \tau) \circ (u \otimes \text{id}) = \pi_1^*(\text{id}_1 \times \Delta_{23})^! \tilde{\pi}_1^* (\Delta_{12} \times \text{id}_3), \pi_2^! \]
\[ \simeq \pi_1^* \tilde{\pi}_1^* (\text{id}_1 \times \Delta_{23})^!(\Delta_{12} \times \text{id}_3) \tilde{\pi}_2 \]
\[ = \pi_1^* \Delta_* \Delta^! \tilde{\pi}_2 \]
\[ \simeq \pi_1^* \Delta_* \Delta^! \tilde{\pi}_2 \]
\[ \simeq \text{id}_{D(X)}. \]

The assertion of Theorem 4.13 is an immediate consequence of the above proposition.

5. Hecke categories

In this section, we consider the homotopical algebra of monoidal categories arising from correspondences. Adopting the familiar name from representation theory, we refer to such monoidal categories as Hecke categories.

Our main interest is in the Hecke category \( D(P \backslash G/P) \) of \( D \)-modules on the double quotient stack of a complex reductive group \( G \) by a parabolic subgroup \( P \subset G \). If we write \( X = BP, Y = BG \) for the corresponding classifying stacks, then we can realize \( P \backslash G/P \) as the fiber product \( X \times_Y X \). The main technical features of this setup are the following:

1. \( X, Y \) and \( X \times_Y X \) are finite orbit stacks.
2. The morphism \( p : X \to Y \) is smooth and proper.
3. The diagonal \( \delta : X \to X \times X \) is smooth.

The main consequence of the above is that we have a sufficient arsenal of duality and adjunctions.

After a brief introduction to convolution categories, we will thereafter restrict our attention to morphisms of stacks \( p : X \to Y \) satisfying the above very restrictive conditions. To reduce clutter, when it is convenient, we will denote the Hecke category by \( \mathcal{H} = D(X \times_Y X) \).

5.1. Convolution. To begin, let \( p : X \to Y \) be an arbitrary representable morphism of smooth stacks. Consider the convolution diagram

\[ \begin{array}{ccc}
X \times_Y X & \xrightarrow{p_{12}} & X \\
\downarrow p_{13} & & \downarrow p_{23} \\
X \times_Y X & \xrightarrow{p_{23}} & X \times_Y X
\end{array} \]

Equip \( D(X \times_Y X) \) with the monoidal product defined by convolution
\[ m : D(X \times_Y X) \otimes D(X \times_Y X) \to D(X \times_Y X) \]
\[ m(\mathcal{M}, \mathcal{N}) = p_{13*}(p_{12}^!(\mathcal{M}) \otimes p_{23}^!(\mathcal{N})) \simeq p_{13*}(p_{12} \times p_{23})^!(\mathcal{M} \boxtimes \mathcal{N}). \]

Similar diagrams provide the usual associativity compatibilities of an algebra object. The unit is given by the pushforward \( u_* \mathcal{O}_X \) along the map \( u : X \to X \times_Y X \) induced by the diagonal \( \delta : X \to X \times X \). By construction, the natural map
\[ \mathcal{D}(X \times_Y X) \to \text{Fun}_{\mathcal{D}(Y)}(\mathcal{D}(X), \mathcal{D}(X)) \]
\[ \mathcal{M} \to p_{2*}(p_1(-) \otimes \mathcal{N}) \]
is monoidal where the endofunctor category is equipped with the usual composition product.

We will hereon specialize to a specific situation of interest in representation theory. First, we will assume that \( X, Y, \) and \( X \times_Y X \) are finite orbit stacks. Next, we will assume that the morphism \( p : X \to Y \) is smooth and proper. Finally, we will assume that the diagonal \( \delta : X \to X \times X \) is smooth. The latter assumption is very restrictive but satisfied for classifying spaces of smooth group schemes. The motivating example is when \( Y = BG \) is the classifying stack of a reductive group, and \( X = B\mathcal{P} \) is the classifying stack of a parabolic subgroup. Then \( p \) is proper with fibers \( G/P, \delta \) is smooth with fibers \( \mathcal{P}, \) and \( X \times_Y X \) is the double quotient stack \( \mathcal{P}\backslash G/\mathcal{P}. \)

Observe that if \( p \) is proper then \( p_{13} \) is also proper (it is a base change of \( p \)), and if \( \delta \) is smooth (note that its relative dimension will be \( -\dim X \)), then \( p_{12} \times p_{23} \) is also smooth (it is a base change of \( \delta \)).

With the above assumptions, we have the following flexibility in how we write the convolution product. Conjugating by the shifted Verdier duality \( \mathcal{D}' = \mathcal{D}_{X \times_Y X}[-2\dim X] \), we can equip \( \mathcal{H} = \mathcal{D}(X \times_Y X) \) with an alternative convolution product
\[ m'(\mathcal{M}, \mathcal{N}) = \mathcal{D}'(m(\mathcal{D}' \mathcal{M}, \mathcal{D}' \mathcal{N})) \]
\[ \simeq p_{13*}(p_{12} \times p_{23})^!(\mathcal{M} \boxtimes \mathcal{N})[2\dim X] \]
\[ \simeq p_{13*}(p_{12} \times p_{23})^*(\mathcal{M} \boxtimes \mathcal{N}). \]

Our assumptions imply we have an equivalence
\[ m(\mathcal{M}, \mathcal{N}) = p_{13*}(p_{12} \times p_{23})^!(\mathcal{M} \boxtimes \mathcal{N}) \simeq p_{13*}(p_{12} \times p_{23})^*(\mathcal{M} \boxtimes \mathcal{N}) = m'(\mathcal{M}, \mathcal{N}) \]
between the two convolution products. In particular, we can explicitly construct left and right adjoints to \( m \simeq m' \) by the following formulas
\[ m^!(\mathcal{M}) \simeq (p_{12} \times p_{23})_!p_{13*}(\mathcal{M}) \quad m^!(\mathcal{N}) \simeq (p_{12} \times p_{23})_!p_{13*}(\mathcal{N})[-2\dim X]. \]

To interpret these as objects of the tensor product category, we use Theorem 4.12 to obtain the identification
\[ \mathcal{D}(X \times_Y X \times X \times_Y X) \simeq \mathcal{D}(X \times_Y X) \otimes \mathcal{D}(X \times_Y X). \]

Finally, note as well that the convolution functor is proper, since by Proposition 4.3 it is the composition of proper functors.

### 5.2. An invariant form on the Hecke category.

The shifted Verdier duality
\[ \mathcal{D}' = \mathcal{D}_{X \times_Y X}[-2\dim X] \]
provides an equivalence \( \mathcal{D}_{\text{c}pt}(X \times_Y X) \simeq \mathcal{D}_{\text{c}pt}(X \times_Y X)^\diamond \) which tautologically extends to an equivalence of the ind-category
\[ \mathcal{H} = \mathcal{D}(X \times_Y X) = \text{Ind} \mathcal{D}_{\text{c}pt}(X \times_Y X) \]
with its restricted opposite
\[ \mathcal{H}' = \mathcal{D}(X \times_Y X)' = \text{Ind}(\mathcal{D}_{\text{c}pt}(X \times_Y X)^\diamond). \]

Note that the opposite label \( \diamond \) refers to the category structure not the algebra structure. By construction, the duality intertwines the two convolutions \( m \) and \( m' \) (viewed as continuous extensions of convolution products on \( \mathcal{D}_{\text{c}pt}(X \times_Y X) \) and \( \mathcal{D}_{\text{c}pt}(X \times_Y X)^\diamond \) respectively). Since we have an equivalence \( m \simeq m' \), the duality \( \mathcal{H} \simeq \mathcal{H}' \) is an equivalence of algebras.
Next consider the swap involution
\[ \sigma : X \times_Y X \sim X \times_Y X \quad \sigma(x_1, p_y, x_2) = (x_2, p_y^{-1}, x_1) \]
where \( p_y^{-1} \) is the path \( p_y \) traced in the opposite direction. The induced equivalence on \( \mathcal{D} \)-modules intertwines the algebra structure and the opposite algebra structure
\[ \sigma : \mathcal{H} \sim \mathcal{H}^{op} \]

Finally, consider the composite of the duality and swap involutions
\[ \iota = \mathcal{D}' \circ \sigma : \mathcal{H} \sim \mathcal{H}^{op}. \]
It intertwines the algebra structure and the opposite algebra structure.

**Proposition 5.1.** For \( \mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathcal{H}_{cpt} \) there are functorial equivalences
\[
\text{Hom}_{\mathcal{H}}(\mathcal{L} \ast \mathcal{M}, \mathcal{N}) \simeq \text{Hom}_{\mathcal{H}}(\mathcal{L}, \iota(\mathcal{M}) \ast \mathcal{N}) \\
\text{Hom}_{\mathcal{H}}(\mathcal{M} \ast \mathcal{L}, \mathcal{N}) \simeq \text{Hom}_{\mathcal{H}}(\mathcal{L}, \mathcal{N} \ast \iota(\mathcal{M}))
\]
monoidal in \( \mathcal{M} \).

**Proof.** We will prove the first equivalence, the second follows by duality.
It suffices to establish a functorial equivalence
\[
(1) \quad \text{Hom}_{\mathcal{H}}(\mathcal{M}, \mathcal{N}) \simeq \text{Hom}_{\mathcal{H}}(u_1 \mathcal{O}_X, \iota(\mathcal{M}) \ast \mathcal{N})
\]
where the pushforward \( u_1 \mathcal{O}_X \simeq u_* \mathcal{O}_X \) along the map \( u : X \to X \times_Y X \) is the monoidal unit.

Consider the following diagram with Cartesian square
\[
\begin{array}{ccc}
X \times_Y X & \xrightarrow{p_2} & X \\
\downarrow \sigma \times \text{id} & & \downarrow u \\
X \times Y \times X \times Y X & \xrightarrow{p_{12} \times p_{23}} & X \times Y \times X \times Y X
\end{array}
\]
Starting with \( \iota(\mathcal{M}), \mathcal{N} \in \mathcal{H}_{cpt} \), the right hand side of Equation (1) can be calculated as follows
\[
\text{Hom}_{\mathcal{H}}(u_1 \mathcal{O}_X, \iota(\mathcal{M}) \ast \mathcal{N}) \simeq \text{Hom}_{\mathcal{D}(X)}(\mathcal{O}_X, u_1^!(\iota(\mathcal{M}) \ast \mathcal{N})) \\
\simeq \text{Hom}_{\mathcal{D}(X)}(\mathcal{O}_X, (u_1^! p_{13})^!(\iota(\mathcal{M}) \otimes \mathcal{N})) \\
\simeq \text{Hom}_{\mathcal{D}(X)}(\mathcal{O}_X, (u_1^! p_{13})^!(\iota(\mathcal{M}) \otimes \mathcal{N}))[2d]
\]
where \( d \) denotes the relative dimension \( \dim X - \dim(X \times_Y X) \).

Similarly, observing that the diagonal map \( \Delta : X \times_Y X \to X \times Y X \times X \times_Y X \) can be rewritten as \( (\sigma \times \text{id}) \circ (p_{12} \times p_{23}) \circ (\sigma \times \text{id}) \), the left hand side can be calculated
\[
\text{Hom}_{\mathcal{H}}(\mathcal{M}, \mathcal{N}) \simeq \text{Hom}_{\mathcal{H}}(\mathcal{O}_X \times_{Y \times X} \mathcal{D}[\mathcal{D}[\mathcal{M}[\mathcal{N}]] \otimes \mathcal{N}) \\
\simeq \text{Hom}_{\mathcal{H}}(p_{13}^! \mathcal{O}_X, \Delta^!(\mathcal{D}[\mathcal{D}[\mathcal{M}[\mathcal{N}]] \otimes \mathcal{N}))) \\
\simeq \text{Hom}_{\mathcal{H}}(\mathcal{O}_X, p_{23}^!(\Delta^!(\mathcal{D}[\mathcal{D}[\mathcal{M}[\mathcal{N}]] \otimes \mathcal{N})))) \\
\simeq \text{Hom}_{\mathcal{H}}(\mathcal{O}_X, p_{23}^!(\Delta^!(\mathcal{D}[\mathcal{D}[\mathcal{M}[\mathcal{N}]] \otimes \mathcal{N}))))[2d]
\]

The assertion now follows by base change. \( \square \)

5.3. **Adjoints to convolution.** Recall that we have an equivalence
\[
m(\mathcal{M}, \mathcal{N}) = p_{13}^!(p_{12} \times p_{23})^!(\mathcal{M} \otimes \mathcal{N}) \simeq p_{13}!(p_{12} \times p_{23})^*(\mathcal{M} \otimes \mathcal{N}) = m'(\mathcal{M}, \mathcal{N})
\]
between the two convolution products and that we used it to construct left and right adjoints to \( m \simeq m' \) by the following formulas
\[
m'(\mathcal{M}) \simeq (p_{12} \times p_{23})_*(p_{13}^!(\mathcal{M})) \\
m'(\mathcal{N}) \simeq (p_{12} \times p_{23})_*(p_{12}^!(\mathcal{N})[-2 \dim X]).
\]

Our aim in this section is to compare the above adjoints with the canonical map given by left convolution
\[
L : \mathcal{H} \to \text{Fun}(\mathcal{H}, \mathcal{H}) \quad \mathcal{M} \mapsto L_{\mathcal{M}}(-) = \mathcal{M} \ast (-).
\]
Recall that Proposition 3.13 provides an identification between the monoidal dual of \(\mathcal{H} = \text{Ind} D_{\text{cpt}}(X \times_Y X)\) and its restricted opposite \(\mathcal{H}' = \text{Ind}(D_{\text{cpt}}(X \times_Y X)^\circ)\), and hence we have

\[
\text{Fun}(\mathcal{H}, \mathcal{H}) \simeq \mathcal{H}' \otimes \mathcal{H}.
\]

By construction, an external product of compact objects \(\mathcal{M} \boxtimes \mathcal{N}\) corresponds to the functor

\[
\mathcal{H} \to \mathcal{H} \quad \mathcal{L} \mapsto \text{Hom}(\mathcal{M}, \mathcal{L}) \otimes \mathcal{N}.
\]

We will use the shifted Verdier duality \(\mathbb{D}'_{X \times_Y X} \mathcal{M} = \mathbb{D}_{X \times_Y X} \mathcal{M}[−2 \dim(X \times_Y X)]\) to define a further identification

\[
\mathcal{H}' \otimes \mathcal{H} \simeq \mathcal{H} \otimes \mathcal{H}.
\]

By construction, under the above identifications, an external product of compact objects \(\mathcal{M} \boxtimes \mathcal{N}\) corresponds to the functor

\[
\mathcal{H} \to \mathcal{H} \quad \mathcal{L} \mapsto \text{Hom}(\mathbb{D}'_{X \times_Y X} \mathcal{M}, \mathcal{L}) \otimes \mathcal{N}.
\]

We will need the following straightforward lemma.

**Lemma 5.2.** Suppose a functor \(\Phi\) corresponds to a kernel \(\mathcal{K}\), then there is a functorial equivalence

\[
\text{Hom}_{\mathcal{H}}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{K}) \simeq \text{Hom}_{\mathcal{D}(\mathcal{X} \times \mathcal{Y})}(\mathcal{M}, \Phi(\mathbb{D}' \mathcal{N}))
\]

**Proof.** It suffices to check the assertion when \(\mathcal{K}\) is an external product of compact generators and \(\mathcal{M}, \mathcal{N}\) are compact generators. This is straightforward. \(\square\)

Now we can compare the left convolution to the adjoints to convolution.

**Proposition 5.3.** Consider the functor

\[
k : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}
\]

sending a \(\mathcal{D}\)-module \(\mathcal{M}\) to the kernel representing left convolution

\[
L : \mathcal{H} \to \text{Fun}(\mathcal{H}, \mathcal{H}) \quad \mathcal{M} \mapsto L\mathcal{M}(-) = \mathcal{M} \ast (-)
\]

Then we have a canonical identification of the right adjoint to convolution

\[
m' \simeq [-2 \dim X/Y](\sigma \times \text{id}) \circ k.
\]

**Proof.** Given compact generators \(\mathcal{M}, \mathcal{N} \in \mathcal{H}_{\text{cpt}}\), it suffices to establish a functorial identification

\[
\text{Hom}_D(\mathcal{M} \boxtimes \mathcal{N}, (\sigma \times \text{id})(k(\mathcal{L}))[−2 \dim X/Y]) \simeq \text{Hom}_D(\mathcal{M} \ast \mathcal{N}, \mathcal{L}),
\]

and it suffices to establish this identification for compact \(\mathcal{L} \in \mathcal{H}_{\text{cpt}}\) (since convolution is proper, and hence all our functors are continuous in \(\mathcal{L}\)). Note that the left hand side can be rewritten in the form

\[
\text{Hom}_D(\mathcal{M} \boxtimes \mathcal{N}, (\sigma \times \text{id})(k(\mathcal{L}))[−2 \dim X/Y]) \simeq \text{Hom}_D(\sigma(\mathcal{M}) \boxtimes \mathcal{N}, k(\mathcal{L}))[−2 \dim X/Y].
\]

Now by the previous lemma we have a functorial equivalence

\[
\text{Hom}_D(\sigma(\mathcal{M}) \boxtimes \mathcal{N}, k(\mathcal{L})) \simeq \text{Hom}_D(\mathcal{N}, \mathcal{L} \ast \sigma(\mathbb{D}' \mathcal{M}))
\]

\[
\simeq \text{Hom}_D(\mathcal{N}, \mathcal{L} \ast \iota(\mathcal{M}))[2 \dim X/Y].
\]

And by Proposition 5.1 of the previous section we have a functorial equivalence

\[
\text{Hom}_D(\mathcal{N}, \mathcal{L} \ast \iota(\mathcal{M})) \simeq \text{Hom}_D(\mathcal{M} \ast \mathcal{N}, \mathcal{L})
\]

\(\square\)
5.4. Abelianizations and Centers of Hecke categories. In this section, we construct a canonical equivalence between the abelianization $\text{Ab}(\mathcal{H})$ and center $\mathcal{Z}(\mathcal{H})$ of the Hecke category $\mathcal{H}$. The proof relies on the following consequence [L1, Corollary 5.5.3.4] of the $\infty$-categorical adjoint functor theorem: there is a canonical equivalence

$$\mathcal{P}r^\wedge \simeq \mathcal{P}r_R$$

between the opposite of the $\infty$-category $\mathcal{P}r$ of presentable $\infty$-categories with morphisms left adjoints and the $\infty$-category $\mathcal{P}r_R$ of presentable $\infty$-categories with morphisms right adjoints. In other words, we can reverse diagrams of presentable $\infty$-categories, in which the functors are all left adjoints, by passing to the corresponding right adjoints.

**Theorem 5.4.** There is a canonical equivalence

$$\mathcal{Z}(\mathcal{H}) \simeq \text{Ab}(\mathcal{H}).$$

**Proof.** As explained in Section 2.4.1, the abelianization $\text{Ab}(\mathcal{H})$ is the colimit in $\mathcal{P}r$ of the simplicial category given by the Hochschild simplicial category

$$\mathcal{N}_{\text{cyc}}^n(\mathcal{H}) \simeq \mathcal{H}^\otimes n+1$$

with chain maps given by multiplication maps in the usual cyclic pattern. This colimit can be calculated as the limit in $\mathcal{P}r_R$ of the cosimplicial $\infty$-category with $n$-cosimplices

$$\mathcal{N}_{\text{cyc}}^n(\mathcal{H}) \simeq \mathcal{H}^\otimes n+1$$

with cochain maps given by the right adjoints to the Hochschild chain maps. Recall by [L1, Theorem 5.5.3.18], a small limit in $\mathcal{P}r_R$ can be calculated as a limit in the $\infty$-category of $\infty$-categories.

On the other hand, the center $\mathcal{Z}(\mathcal{H})$ is the limit in $\mathcal{P}r$ of the Hochschild cosimplicial category

$$\mathcal{N}_{\text{cyc}}^n(\mathcal{H}) \simeq \text{Fun}(\mathcal{H}^\otimes n, \mathcal{H})$$

with cochain maps derived from multiplication maps in the usual pattern. Recall as well that by [L1, Proposition 5.5.3.13], a small limit in $\mathcal{P}r_R$ can be calculated as a limit in the $\infty$-category of $\infty$-categories.

We have seen that the monoidal dual of $\mathcal{H}$ is the ind-category $\text{Ind}(\mathcal{H}^\wedge)$, and that Verdier duality provides an equivalence $\mathbb{D}: \mathcal{H} \simeq \mathcal{H}'$. For each $n$, twist $\mathbb{D}$ by the involution $\sigma$ to obtain $\iota = \sigma \circ \mathbb{D}$, and consider the resulting term-wise equivalences

$$\mathcal{N}_{\text{cyc}}^n(\mathcal{H}) = \mathcal{H}^\otimes n \otimes \mathcal{H} \xrightarrow{\iota \otimes \cdots \otimes \iota \otimes \text{id}} (\mathcal{H}')^\otimes n \otimes \mathcal{H} \xrightarrow{\text{Fun}(\mathcal{H}^\otimes n, \mathcal{H})} \mathcal{N}_{\text{cyc}}^n(\mathcal{H})$$

Proposition 5.3 implies that under the above equivalences, the Hochschild cochain maps are the right adjoints to the corresponding Hochschild chain maps. Thus the colimit of the Hochschild chain complex and the limit of the Hochschild cochain complex are canonically identified. □

5.5. Two-dimensional field theory. In this section, we show that the Hecke category $\mathcal{H}$ satisfies the requirements of the two-dimensional oriented cobordism hypothesis as formulated by Lurie in [L5].

Recall that the center $\mathcal{Z}(\mathcal{H})$ comes equipped with a monoidal structure and a monoidal functor $\mathfrak{z}: \mathcal{Z}(\mathcal{H}) \to \mathcal{H}$ that underlies the central action. In particular, $\mathfrak{z}$ takes the monoidal unit of $\mathcal{Z}(\mathcal{H})$ to the monoidal unit of $\mathcal{H}$. Recall that the monoidal unit of $\mathcal{H}$ is given by $u_!\mathcal{O}_X = u_!\pi^*\mathbb{C}_pt$ where $u: X \to X \times_Y X$ is the relative diagonal, and $\pi: X \to \text{pt}$ is the projection to a point. In summary, we have a commutative diagram of monoidal functors

$$\text{Vect} \xrightarrow{u_!\pi^*} \mathcal{Z}(\mathcal{H}) \xrightarrow{\mathfrak{z}} \mathcal{H}.$$ 

We next introduce a dual structure on the abelianization $\text{Ab}(\mathcal{H})$. It will be induced by the functor

$$\tau: \mathcal{H} \to \text{Vect}, \quad \tau(\mathfrak{M}) = \pi_*u^\dagger\mathfrak{M}.$$
Note that on compact objects, $\tau$ is given by pairing with the unit

$$\tau(M) \simeq \text{Hom}_H(u_O, M),$$

and in general, $\tau$ is the continuous extension of this pairing to all of $H$.

**Proposition 5.5.** The functor $\tau : H \to \text{Vect}$ is a trace: there is a canonical factorization of $\tau$ through a functor $\tilde{\tau}$ on the abelianization

$$H \xrightarrow{\text{tr}} \text{Ab}(H) \xrightarrow{\tilde{\tau}} \text{Vect}.$$ 

Moreover, $\tilde{\tau}$ is $S^1$-invariant with respect to the canonical cyclic structure on Hochschild homology.

**Proof.** By Proposition 5.1 on compact objects, the functor $\tau$ is invariant under cyclic permutations

$$\tau(M \ast M) \simeq \tau(M \ast M),$$

hence also on all objects by continuity. More generally, we can extend $\tau$ to a functor $\tau_*$ from the cyclic bar construction $N_{\text{hc}}(H)$ to the constant simplicial category $\text{Vect}$. On the compact objects of the $n$-simplices, we define

$$\tau_n(M_1 \otimes \cdots \otimes M_{n+1}) = \pi_*u_1(M_1 \ast \cdots \ast M_{n+1})$$

and extend by continuity. It is straightforward to check the requisite compatibilities with the face and degeneracies by the unit property and Proposition 5.1. In fact, it is no more difficult to check that $\tau_*$ extends to the cyclic category underlying the simplicial category $N_{\text{hc}}$. Passing to the colimit, we obtain the sought-after functor $\tilde{\tau} : \text{Ab}(H) \to \text{Vect}$. The fact that $\tau_*$ respects the cyclic structure on $N_{\text{hc}}$ (Section 2.4.2) implies that $\tilde{\tau}$ is $S^1$-invariant. \qed

We will now consider dualizability properties of the Hecke category $H$ as an object of a certain $(\infty, 2)$-category. Recall that $St$ denotes the $\infty$-category whose objects are stable presentable $\infty$-categories and whose morphisms are continuous functors. We will now introduce the $(\infty, 2)$-category in which we wish to consider $H$. (In the notation of [L5, Section 4.1], we will consider $\text{Alg}(1)(St)$, but with its canonical enrichment as an $(\infty, 2)$-category.)

**Definition 5.6.** The symmetric monoidal $(\infty, 2)$ category $2\text{Alg}$ consists of the following:

- **objects:** algebra objects in the $\infty$-category $St$ of presentable stable $\infty$-categories;
- **1-morphisms:** bimodule categories;
- **2-morphisms:** functors of bimodule categories.

In other words, objects of $2\text{Alg}$ are stable presentable monoidal $(\infty, 1)$-categories. For two objects $A, B$, the morphisms $\text{Hom}_{2\text{Alg}}(A, B)$ form the $(\infty, 1)$-category of stable presentable $A$-$B$-bimodule $(\infty, 1)$-categories. The tensor product $A \otimes B$ is the usual tensor of stable presentable $(\infty, 1)$-categories. The unit is the stable presentable monoidal $(\infty, 1)$-category $\text{Vect}$.

Any object $A \in 2\text{Alg}$ is dualizable with dual the opposite algebra $A^{\text{op}}$. The regular $A-A^{\text{op}}$-bimodule $A$ thought of as a $\text{Vect} \otimes (A \otimes A^{\text{op}})$-bimodule gives both the unit and trace

$$\text{Vect} \xrightarrow{A} A \otimes A^{\text{op}} \xrightarrow{A} \text{Vect}$$

satisfying the usual duality identities.

Before preceding, let us collect some of the relevant properties of the Hecke category $H$ that have appeared in previous sections. Recall that we write $H'$ for the ind-category $\text{Ind}(H_{\text{op}})$ with compact objects the opposite of the full subcategory of compact objects of $H$. Then $H$ as an object of the symmetric monoidal $\infty$-category $St$ satisfies the following:

- $H$ is dualizable with dual $H'$: there are canonical functors

$$\text{Vect} \xrightarrow{u_N} H \otimes H' \xrightarrow{c_N} \text{Vect}$$

satisfying the usual duality identities.
The involution \( \iota = \mathcal{D}' \circ \sigma \) gives an equivalence of monoidal categories

\[ \iota : \mathcal{H} \xrightarrow{\sim} (\mathcal{H}')^{op}. \]

We will use the above to show that \( \mathcal{H} \) is in fact a fully dualizable object of the symmetric monoidal \((\infty, 2)\)-category \(2\text{Alg}\). We restate the definition of this notion in a form specialized to the current setting:

**Definition 5.7.** (1) \([L5]\) Proposition 4.2.3] A fully dualizable object of \(2\text{Alg}\) is an object \(A\) such that the evaluation morphism \(\text{ev}_A : A \otimes A^{op} \to \text{Vect}\) has both a right and a left adjoint.

(2) \([L5]\) Definition 4.2.6] A Calabi-Yau object of \(2\text{Alg}\) is an object \(A\) with an \(S^1\)-invariant functional \(A \otimes \sigma : A \otimes A^{op} \to \text{Vect}\) which is the counit of an adjunction between \(\text{ev}_A\) and \(\text{coev}_A\).

Now the main result of this section is the following.

**Theorem 5.8.** The Hecke category \(\mathcal{H}\) is a fully dualizable Calabi-Yau object of \(2\text{Alg}\).

Not only is \(\mathcal{H}\) fully dualizable, but recall that it is also self-dual:

- \(\mathcal{H}\) is equivalent to the opposite algebra: there is a canonical equivalence
  \[ \sigma : \mathcal{H} \xrightarrow{\sim} \mathcal{H}^{op} \]
  compatible with the functors \(u_{\mathcal{H}}, c_{\mathcal{H}}\), and \(\iota\).

It follows that \(\mathcal{H}\) is self-dual as a fully dualizable algebra: it is canonically fixed under the duality \(\mathbb{Z}/2\mathbb{Z}\)-action on \(2\text{Alg}\).

Before giving the proof of Theorem 5.8, let us mention why fully dualizable Calabi-Yau objects of \(2\text{Alg}\) are interesting. First, fully dualizable objects \(M\) of an \((\infty, 2)\)-category \(\mathcal{C}\) extend to functors from the framed bordism category \(2\text{Bord}_{fr} \to \mathcal{C}\) sending a framed point to \(M\). The Calabi-Yau condition translates into \(SO(2)\)-invariance of \(M\), allowing the theory to descend to an oriented field theory \(2\text{Bord} \to \mathcal{C}\). Furthermore, the self-duality of \(M\) translates into the \(O(2)\)-invariance of \(M\), allowing the theory to descend to an unoriented field theory \(2\text{Bord} \to \mathcal{C}\). Thus applying the unoriented form of Lurie’s Cobordism Hypothesis result, announced in \([L5]\), to the Hecke category \(\mathcal{H}\) we immediately deduce the following.

**Corollary 5.9.** There is a unique symmetric monoidal functor \(Z_{\mathcal{H}} : 2\text{Bord} \to 2\text{Alg}\) from the unoriented 2-bordism category to \(2\text{Alg}\) with \(Z(\text{pt}) = \mathcal{H}\). Moreover, we have

1. \(Z_{\mathcal{H}}(S^1) = \mathcal{Z}(\mathcal{H}) = \text{Ab}(\mathcal{H})\), the center and abelianizations of \(\mathcal{H}\),
2. \(Z_{\mathcal{H}}(S^2) = \text{Hom}_{\mathcal{H}}(w_0 \mathcal{O}_X, w_0 \mathcal{O}_X)\), the endomorphisms of the unit of \(\mathcal{H}\), and
3. \(Z_{\mathcal{H}}(S^1 \times S^1) = \text{Hom}_{\mathcal{H}}(\mathcal{Z}(\mathcal{H}))\), the Hochschild chain complex of the center of \(\mathcal{H}\).

**Proof of Theorem 5.8.** Recall that the evaluation map \(\text{ev}_\mathcal{H} : \mathcal{H} \otimes \mathcal{H} \to \text{Vect}\) is given by \(\mathcal{H}\) thought of as an \((\mathcal{H} \otimes \mathcal{H}^{op}) \otimes \text{Vect}\)-bimodule. As our candidate for both the left and right adjoint to \(\text{ev}_\mathcal{H}\), we take the coevaluation morphism \(\text{coev}_\mathcal{H} : \text{Vect} \to \mathcal{H} \otimes \mathcal{H}\) given by \(\mathcal{H}\) itself as a \(\text{Vect} \otimes (\mathcal{H} \otimes \mathcal{H}^{op})\)-bimodule. Observe that the two possible compositions of \(\text{ev}_\mathcal{H}\) and \(\text{coev}_\mathcal{H}\) are easily calculated

\[ \text{coev}_\mathcal{H} \circ \text{ev}_\mathcal{H} \simeq \mathcal{H} \otimes \mathcal{H} \quad \text{ev}_\mathcal{H} \circ \text{coev}_\mathcal{H} \simeq \mathcal{H} \otimes \mathcal{H}^{op} \mathcal{H} = \text{Ab}(\mathcal{H}). \]

To establish that \(\text{coev}_\mathcal{H}\) is the right adjoint, we need to construct a unit \(u_r : \text{Vect} \to \mathcal{H} \otimes \mathcal{H}\) and a counit \(c_r : \mathcal{H} \otimes \mathcal{H}^{op} \mathcal{H} \to \text{Vect}\) such that the following diagram commutes

\[ \begin{array}{ccc}
\mathcal{H} & \xrightarrow{id \otimes u_r} & \mathcal{H} \otimes \mathcal{H}^{op} \mathcal{H} \otimes \mathcal{H} \\
& \mathcal{H} \otimes \mathcal{H}^{op} \mathcal{H} \otimes \mathcal{H} & \mathcal{H} \\
\end{array} \]

For the unit, we take the \(\iota\)-twisted form

\[ u_r : \text{Vect} \xrightarrow{\text{ev}_\mathcal{H}} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\iota \otimes \text{id}} \mathcal{H} \otimes \mathcal{H} \]
of the unit \( u_{\mathcal{H}} \) used to exhibit the dualizability of \( \mathcal{H} \) as an object of \( \text{St} \). For the counit, we take the cyclic trace

\[
c_r = \tilde{\tau} : \text{Ab}(\mathcal{H}) = \mathcal{H} \otimes_{\mathcal{H} \otimes \mathcal{H}^{op}} \mathcal{H} \to \text{Vect}
\]

constructed in Proposition 5.3.

Recall that Equation (11) provides a strong compatibility between the trace \( \tau : \mathcal{H} \to \text{Vect} \) and the hom pairing:

\[
\tau(\mathcal{M} \ast \mathcal{N}) = \text{Hom}_{\mathcal{H}}(\iota(\mathcal{M}), \mathcal{N}), \quad \mathcal{M}, \mathcal{N} \in \mathcal{H}_{\text{cpt}}.
\]

In other words, the two continuous functors \( \mathcal{H}' \otimes \mathcal{H} \to \text{Vect} \) given on compact objects by \( \tau(\iota(\mathcal{M}) \ast \mathcal{N}) \) and \( c_{\mathcal{H}}(\mathcal{M}, \mathcal{N}) = \text{Hom}(\mathcal{M}, \mathcal{N}) \) coincide. Thus we have the following commutative diagram, establishing the sought-after adjunction:

Likewise, to establish that \( \text{coev}_{\mathcal{H}} \) is the left adjoint to \( \text{ev}_{\mathcal{H}} \), we need to construct a counit \( c_{\ell} : \mathcal{H} \otimes \mathcal{H} \to \text{Vect} \) and a unit \( u_{\ell} : \text{Vect} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}^{op} \mathcal{H} \) such that the following diagram commutes

For the counit, we take the \( \iota \)-twisted form

\[
c_{\ell} : \mathcal{H} \otimes \mathcal{H} \xrightarrow{\iota \otimes \text{id}} \mathcal{H}' \otimes \mathcal{H} \xrightarrow{c_{\mathcal{H}}} \text{Vect}
\]

of the counit \( c_{\mathcal{H}} \) used to exhibit the dualizability of \( \mathcal{H} \) as an object of \( \text{St} \). For the unit, we use the identification of the abelianization and center of Theorem 5.4, and take \( u_{\ell} \) to be the monoidal unit of the center

\[
u_{\ell} : \text{Vect} \to Z(\mathcal{H}) \simeq \text{Ab}(\mathcal{H}) = \mathcal{H} \otimes_{\mathcal{H} \otimes \mathcal{H}^{op}} \mathcal{H}.
\]

The adjunction identity for \( c_{\ell} \) and \( u_{\ell} \) expresses the unit property of \( u_{\ell} \) acting centrally on \( \mathcal{H} \). Recall the notations for the left multiplication functor and the corresponding integral kernel

\[
L : \mathcal{H} \to \text{Fun}(\mathcal{H}, \mathcal{H}) \quad k : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}
\]
We will use the notation $L_{\mathcal{M}}$ and $k_{\mathcal{M}}$ for the images of an object $\mathcal{M} \in \mathcal{H}$ under the two functors. They are characterized by the fact that we have a commutative diagram

\[
\begin{array}{ccccccccc}
\mathcal{H} & \xrightarrow{L_{\mathcal{M}} \otimes \text{id}} & \text{Fun}(\mathcal{H}, \mathcal{H}) \otimes \mathcal{H} & \xrightarrow{\sim} & \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H} & \xrightarrow{id \otimes c_{\mathcal{H}}} & \mathcal{H} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\text{id} \otimes \text{id}} & \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H} & \xrightarrow{id \otimes c_{\mathcal{H}}} & \mathcal{H} \\
\end{array}
\]

Suppose $\mathcal{M} \in \mathcal{H}$ is central in the sense that we are given a functor $z_{\mathcal{M}} : \text{Vect} \to \mathcal{Z}(\mathcal{H}) = \text{Fun}(\mathcal{H} \otimes \mathcal{H}^\text{op}, \mathcal{H})$ equipped with an equivalence $\mathcal{Z}(z_{\mathcal{M}}(\mathcal{C})) \simeq \mathcal{M}$. Then the above commutative diagram descends to a commutative diagram

\[
\begin{array}{cccccc}
\mathcal{H} & \xrightarrow{z_{\mathcal{M}} \otimes \text{id}} & \text{Fun}_{\mathcal{H}}(\mathcal{H}, \mathcal{H}) \otimes \mathcal{H} & \xrightarrow{\sim} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}^\text{op} \mathcal{H} \otimes \mathcal{H} & \xrightarrow{id \otimes c_{\mathcal{H}}} & \mathcal{H} \\
\end{array}
\]

The sought-after adjunction is then the special case of this identity when $\mathcal{M}$ is the monoidal unit of $\mathcal{Z}(\mathcal{H})$.

\[\Box\]

6. Character sheaves and loop spaces

In this section, we continue our study of the homotopical algebra of Hecke categories $\mathcal{H} = \mathcal{D}(X \times_Y X)$. In particular, we continue with the following setup:

1. $X, Y$ and $X \times_Y X$ are holonomic stacks.
2. The morphism $p : X \to Y$ is smooth and proper.
3. The diagonal $\delta : X \to X \times X$ is smooth.

Our aim is to relate the abelianization $\text{Ab}(\mathcal{H})$ and center $\mathcal{Z}(\mathcal{H})$ to the $\infty$-category $\mathcal{D}(\mathcal{L}Y)$ of $\mathcal{D}$-modules on the loop space of $Y$. In general, the loop space $\mathcal{L}Y$, and other intermediate stacks appearing in the story, will not be holonomic. But the $\mathcal{D}$-modules which arise via our constructions will be colimits of holonomic $\mathcal{D}$-modules. Thus we will always restrict our attention to the full subcategory $\mathcal{D}_{\text{hol}}(\mathcal{L}Y)$ of holonomically generated $\mathcal{D}$-modules. This will allow us to work freely with the familiar collection of adjunctions and identities that hold for holonomic $\mathcal{D}$-modules and their colimits.

6.1. The fundamental correspondence. In this section, we set up the basic relation between Hecke categories and loop spaces. It is an abstraction of the horocycle correspondence from representation theory.

Consider the fundamental correspondence

\[
\begin{array}{cccccc}
\mathcal{L}Y = Y \times_Y X & \xrightarrow{\pi} & \mathcal{L}Y \times_Y X = X \times_{X \times Y} X & \xrightarrow{\delta} & X \times_Y X \\
\end{array}
\]

where the maps are defined by the formulas

\[
\pi = \text{id}_{\mathcal{L}Y} \times \text{id}_Y \quad p = p \times p \times \text{id}_Y \quad \delta = \text{id}_X \times \pi_Y \text{id}_X
\]

where $\pi_Y : X \times Y \to Y$ is the obvious projection. Note that of the three stacks appearing, only $X \times_Y X$ is holonomic in general. Thus we have $\mathcal{D}(X \times_Y X) = \mathcal{D}_{\text{hol}}(X \times_Y X)$, but we will restrict our attention to $\mathcal{D}_{\text{hol}}(\mathcal{L}Y)$ and $\mathcal{D}_{\text{hol}}(\mathcal{L}Y \times_Y X)$.

We will study how much of the loop space is seen by the Hecke category $\mathcal{H} = \mathcal{D}(X \times_Y X)$ under this correspondence. First, we define the functor

\[
F : \mathcal{D}(X \times_Y X) \xrightarrow{\delta^*} \mathcal{D}_{\text{hol}}(\mathcal{L}Y \times_Y X) \xrightarrow{\pi^*} \mathcal{D}_{\text{hol}}(\mathcal{L}Y)
\]
Since $\pi$ is proper, $\pi_! = \pi_*$, and since $\delta$ is smooth, $\delta^* = \delta^! = \delta^![-\dim X]$. Thus $F$ coincides up to a shift with the functor

$$F' = \mathbb{D} \circ F \circ \mathbb{D} : \mathcal{D}(X \times_Y X) \xrightarrow{\delta^!} \mathcal{D}_{\text{hol}}(\mathcal{L}Y \times_Y X) \xrightarrow{\pi_*} \mathcal{D}_{\text{hol}}(\mathcal{L}Y).$$

We have the right adjoint

$$F^r : \mathcal{D}_{\text{hol}}(\mathcal{L}Y) \xleftarrow{\pi_*} \mathcal{D}_{\text{hol}}(\mathcal{L}Y \times_Y X) \xrightarrow{\delta_*} \mathcal{D}(X \times_Y X)$$

Conjugating by Verdier dual, we have the left adjoint

$$(F')^\ell = \mathbb{D} \circ F^r \circ \mathbb{D} : \mathcal{D}_{\text{hol}}(\mathcal{L}Y) \xrightarrow{\pi_*} \mathcal{D}_{\text{hol}}(\mathcal{L}Y \times_Y X) \xrightarrow{\delta_!} \mathcal{D}(X \times_Y X)$$

Since we have $F \simeq F'[-\dim X]$, we obtain the left adjoint $F^\ell \simeq (F')^\ell[\dim X]$.

The adjoint pair $(F, F^r)$ provides a monad $T = F^r \circ F$ acting on $\mathcal{D}(X \times_Y X)$. Our initial aim is to describe the $\infty$-category of $T$-modules.

Define $\mathcal{K}(F^r)$ to be the kernel of $F^r$, that is the full subcategory of objects $\mathcal{M} \in \mathcal{D}_{\text{hol}}(\mathcal{L}Y)$ such that $F^r(\mathcal{M}) \simeq 0$. Define $\perp \mathcal{K}(F^r)$ to be its left orthogonal, that is the full subcategory of $\mathcal{D}_{\text{hol}}(\mathcal{L}Y)$ of objects $\mathcal{M}$ such that $\text{Hom}_{\mathcal{D}_{\text{hol}}(\mathcal{L}Y)}(\mathcal{M}, \mathcal{N}) \simeq 0$, for all $\mathcal{N} \in \mathcal{K}(F^r)$.

Observe that $\perp \mathcal{K}(F^r)$ is stable and closed under colimits viewed in the ambient category $\mathcal{D}_{\text{hol}}(\mathcal{L}Y)$, and thus in particular closed under colimits viewed intrinsically.

**Proposition 6.1.** The functor $F^r$ induces an equivalence

$$F^r : \perp \mathcal{K}(F^r) \xrightarrow{\sim} \text{Mod}_T(\mathcal{D}(X \times_Y X))$$

Moreover, the category $\text{Mod}_T(\mathcal{D}(X \times_Y X))$ is the cocompletion of the essential image of $F$ inside of $\mathcal{D}_{\text{hol}}(\mathcal{L}Y)$.

**Proof.** The functor $F^r$ preserves colimits and its restriction to $\perp \mathcal{K}(F^r)$ is conservative, since the cone of a morphism that becomes an isomorphism under $F^r$ must lie in $\mathcal{K}(F^r)$ as well as $\perp \mathcal{K}(F^r)$ and hence vanish. It follows that the adjunction

$$F : \mathcal{D}(X \times_Y X) \xrightarrow{\perp \mathcal{K}(F^r)} : F^r$$

satisfies the monadic Barr-Beck hypotheses, so that we obtain the sought after equivalence. $\square$

**Remark 6.2.** The Barr-Beck equivalence is very explicit. On the one hand, any object of the form $F^r(\mathcal{M})$ is naturally a $T$-module. On the other hand, any $T$-module $\mathcal{M}$ admits a simplicial resolution $\mathcal{M} \leftarrow T(\mathcal{M}) \cdots$ with n-simplices $T^n(\mathcal{M})$. This split simplicial object naturally arises by applying $F^r$ to the natural simplicial object with n-simplices $F(T^{n-1}(\mathcal{M}))$. The Barr-Beck hypotheses guarantee that the latter simplicial object has a geometric realization and $F^r$ takes the geometric realization to $\mathcal{M}$. (See [L3 3.4.10].)

In summary, as a general consequence of the Barr-Beck theorem in the context of stable $\infty$-categories, we may informally treat adjunctions of stable $\infty$-categories as analogues of split short exact sequences: the functor $F$ admits a canonical factorization

$$F : \mathcal{D}(X \times_Y X) \xrightarrow{F^r \circ F} \text{Mod}_T(\mathcal{D}(X \times_Y X)) \simeq \perp \mathcal{K}(F^r) \xrightarrow{\perp \mathcal{K}(F^r)^\ell} \mathcal{D}_{\text{hol}}(\mathcal{L}Y).$$

**Corollary 6.3.** The category $\text{Mod}_T(\mathcal{D}(X \times_Y X))$ is the cocompletion of the essential image of $F$ inside of $\mathcal{D}_{\text{hol}}(\mathcal{L}Y)$. 43
6.2. Hochschild shadow of the loop space. One can interpret the $\infty$-category of modules $\text{Mod}_T(\mathcal{D}(X \times Y X))$ appearing in Proposition 6.1 as the shadow of $\mathcal{D}$-modules on the loop space seen by the light of the functor $F^r$. Our aim here is to relate $\text{Mod}_T(\mathcal{D}(X \times Y X))$ to the abelianization of the algebra $\mathcal{D}(X \times Y X)$. By Proposition 6.1, this will provide an intrinsic characterization of the image of the adjoint functor $F$.

We begin with a useful reformulation of the monad $T$ in terms of convolution. Recall that we have a right adjoint to the convolution product

$$m^r : \mathcal{D}(X \times Y X) \to \mathcal{D}(X \times Y X) \otimes \mathcal{D}(X \times Y X)$$

$$m^r(M) \simeq (p_{12} \times p_{23})_\ast p_{13}^\ast (M)$$

Let $\eta$ be the involution of the tensor product

$$\mathcal{D}(X \times Y X) \otimes \mathcal{D}(X \times Y X) \simeq \mathcal{D}(X \times Y X \times X \times Y X)$$

induced by swapping the two factors.

**Lemma 6.4.** There is a canonical equivalence

$$T = F^r \circ F \simeq m \circ \eta \circ m^r$$

**Proof.** This is a diagram chase, using base change, from the lower left corner to the upper right corner of the following diagram with Cartesian squares

$$\begin{array}{c}
X \times Y X \times X \times Y X \leftarrow P_{23} \times P_{12} \rightarrow X \times Y X \times Y X \rightarrow P_{13} \rightarrow X \times Y X \\
p_{12} \times p_{23} \downarrow \delta \downarrow \delta \downarrow \delta \downarrow \delta \downarrow \delta \\
X \times Y X \times Y X \leftarrow X \times Y X \times Y X \rightarrow X \times X \times Y X \rightarrow \pi \rightarrow Y \times Y \times Y \simeq LY
\end{array}$$

\[\square\]

**Theorem 6.5.** Consider the monad $T = F^r \circ F$ acting on $\mathcal{D}(X \times Y X)$. Then there is a natural commutative diagram

$$\begin{array}{c}
\mathcal{D}(X \times Y X) \\
\text{Ab}(\mathcal{D}(X \times Y X)) \longrightarrow \text{Mod}_T(\mathcal{D}(X \times Y X))
\end{array}$$

$$\begin{array}{c}
\text{tr} \quad F^r \circ F
\end{array}$$

**Proof.** Recall that the abelianization $\text{Ab}(\mathcal{D}(X \times Y X))$ is calculated as the geometric realization (colimit) of the simplicial category defined by the Hochschild chain complex $N^{cy}_\ast(\mathcal{D}(X \times Y X)) \simeq \mathcal{D}(X \times Y X)^{\otimes n+1}$. Its boundary maps are the usual cyclic contractions given by the monoidal product. They are base changes of the initial two boundary maps $m_{12}, m_{21}$ given by the monoidal product in the two possible orders

$$m_{12} : \mathcal{D}(X \times Y X) \otimes \mathcal{D}(X \times Y X) \to \mathcal{D}(X \times Y X) \quad m_{12}(M_1 \otimes M_2) = m(M_1, M_2)$$

$$m_{21} : \mathcal{D}(X \times Y X) \otimes \mathcal{D}(X \times Y X) \to \mathcal{D}(X \times Y X) \quad m_{21}(M_1 \otimes M_2) = m(M_2, M_1)$$

Now the opposite of the $\infty$-category $\mathcal{P}_r$ of presentable $\infty$-categories with morphisms left adjoints is the $\infty$-category $\mathcal{P}_r^L$ of presentable $\infty$-categories with morphisms right adjoints. Thus to calculate the colimit of a diagram of functors in $\mathcal{P}_r$ is the same as to calculate the limit of the diagram of right adjoints in $\mathcal{P}_r^L$.

Recall that the right adjoint of $m$, and hence $m_{12}$, is given by the usual functors

$$m^r_{12}(M) \simeq (p_{12} \times p_{23})_\ast p_{13}^\ast (M).$$
By Lemma 6.4, the monad $T$ is equivalent to the composition $m_1^2 \circ m_{21}$. Now by base change, it is straightforward to check that the limit over the right adjoints to the Hochschild boundary maps is canonically equivalent to $\text{Mod}_T(D(X \times_Y X))$. On the one hand, by adjunction, any object of the limit is naturally a $T$-module. On the other hand, any $T$-module $N$ admits a simplicial resolution $N \leftarrow T(N) \cdots$ with $n$-simplices $T^n(N)$. This split simplicial object naturally arises by applying $F^r$ to the natural simplicial object with $n$-simplices $F(T^{n-1}(N))$. Applying the right adjoints to the colimit of this object provides an object of the limit. □

Passing to right adjoints and using Theorem 5.4, we see that the proof of the theorem gives the following.

**Corollary 6.6.** There is a natural commutative diagram

\[
\begin{array}{ccc}
D(X \times_Y X) & \xrightarrow{\delta} & \text{Z}(D(X \times_Y X)) \\
\text{Mod}_T(D(X \times_Y X)) & \xrightarrow{\sim} & \text{Z}(D(X \times_Y X))
\end{array}
\]

where the upward arrow is the obvious forgetful functor.

Finally, let us summarize all of our preceding results without reference to the monad $T$ but rather in terms of the loop space itself. Let us write $\hat{\mathcal{I}}(F)$ for the full subcategory of $\hat{D}_{\text{hol}}(LY)$ obtained by cocompleting the essential image of $F$. Recall that it is equivalent to the left orthogonal $\perp K(F^r)$ to the kernel of the right adjoint $F^r$.

**Theorem 6.7.** There are natural adjoint commutative diagrams

\[
\begin{array}{ccc}
D(X \times_Y X) & \xrightarrow{\delta} & \text{Z}(D(X \times_Y X)) \\
\text{Ab}(D(X \times_Y X)) & \xrightarrow{\sim} & \hat{\mathcal{I}}(F) \simeq \frac{1}{2}K(F^r)
\end{array}
\]

6.3. **Character sheaves.** Now we will apply our previous results, as summarized in Theorem 6.7, to our motivating example. Namely, we will consider the Hecke category $\mathcal{H}_G \simeq D(B\backslash G/B)$ of Borel bi-equivariant $D$-modules on a complex reductive group $G$. This is the case when $X = pt/B$ and $Y = pt/G$ are classifying stacks, and so the fiber product $X \times_Y X$ is the double quotient $B\backslash G/B$.

With this setup, the fundamental correspondence is the horocycle correspondence

\[
\begin{array}{ccc}
G/G & \xrightarrow{\tau} & (G \times G/B)/G & \xrightarrow{\delta} & B\backslash G/B
\end{array}
\]

It contains the traditional Springer correspondence as a subspace

\[
\begin{array}{ccc}
G/G & \xrightarrow{\tau} & \tilde{G}/G & \xrightarrow{\delta} & pt/B
\end{array}
\]

where $\tilde{G} \subset G \times G/B$ is the space of pairs where the group element fixes the flag.

We find that our identifications of the abelianization and center is a well known category in representation theory.

**Definition 6.8.** The $\infty$-category of unipotent character sheaves $\text{Ch}_G$ is defined to be the full subcategory of $\hat{D}_{\text{hol}}(G/G)$ of objects with nilpotent singular support and unipotent central character.

Note that the nilpotent singular locus is Lagrangian so that if a $D$-module with nilpotent singular support is finitely-generated, then it is holonomic. An argument of Mirkovic-Vilonen [MV] shows that for any simple object $\mathfrak{M} \in \text{Ch}_G$ in the heart of the standard $t$-structure, a shift of $\mathfrak{M}$ appears as a summand of the object $F(F^r(\mathfrak{M})) \in \text{Ch}_G$. This provides an alternative constructive definition of character sheaves that is a derived version of Lusztig’s original formulation.
Proposition 6.9. The ∞-category of unipotent character sheaves \( Ch_G \) is the cocompletion of the essential image of the functor

\[
F = \pi_! \delta^* : \mathcal{H}_G \to \mathcal{D}_{hol}(G/G).
\]

Now applying Theorem 6.7, we obtain the following result.

Theorem 6.10. The ∞-category of unipotent character sheaves \( Ch_G \) is equivalent to both the abelianization \( \text{Ab}(\mathcal{H}_G) \) and the center \( Z(\mathcal{H}_G) \) of the Hecke category \( \mathcal{H}_G \). The equivalences fit into natural adjoint commutative diagrams

\[
\begin{tikzcd}
\mathcal{H}_G \ar[d, hookrightarrow] \ar[r, hookrightarrow, \text{tr}] & \text{Ab}(\mathcal{H}_G) \ar[r, hookrightarrow, \sim] & \sim \mathcal{H}_G \ar[r, hookrightarrow, \sim] & \sim Z(\mathcal{H}_G)
\end{tikzcd}
\]

6.4. Monodromic version. In this section, we sketch an analogue of the results of the preceding sections when we replace the equivariant Hecke category with a monodromic version.

Let us begin with a global version of the monodromic Hecke category. We will then replace it with a more technically amenable local version.

6.4.1. Global version. Let \( G \) be a complex reductive group with Borel subgroup \( B \subset G \), unipotent subgroup \( N \subset B \), and universal Cartan \( H = B/N \). A monodromic \( D \)-module on \( G \) is a \( B \times B \)-weakly equivariant \( D \)-module on \( G \), or equivalently, it is an \( H \times H \)-weakly equivariant \( D \)-module on \( N \backslash G/N \). Let \( \mathcal{H}^{mon}_G \) be the full subcategory of \( \mathcal{D}(N \backslash G/N) \) of monodromic \( D \)-modules. The usual convolution formalism equips \( \mathcal{H}^{mon}_G \) with a natural monoidal structure.

Example 6.11. If \( G \) is a torus \( T \), then \( \mathcal{H}^{mon}_T \) is the full subcategory of \( \mathcal{D}(T) \) generated under colimits by regular holonomic \( D \)-modules with singular support contained in the zero section. The monoidal structure is given by the pushforward \( m \) along the multiplication map \( m : T \times T \to T \).

In the monodromic setting, the fundamental correspondence takes the following form

\[
G/G \hat{\leftarrow} \check{\pi}_1 \sim (G \times G/N)/G \hat{\rightarrow} \delta \sim N\backslash G/N.
\]

Observe that the pullback along the relative diagonal is fully faithful

\[
\delta^* : \mathcal{H}^{mon}_G \to \mathcal{D}((G \times G/N)/G).
\]

Thus all of the geometry of the composition \( F = \pi_! \delta^* \) is in the pushforward \( \pi_! \). One can think of \( \pi_! \) as an equivariantization functor from monodromic \( D \)-modules to \( G \)-equivariant \( D \)-modules.

Remark 6.12. It is illuminating to factor the projection \( \pi \) into two distinct maps

\[
G/G \sim \pi_2 (G \times G/B)/G \hat{\rightarrow} \pi_1 (G \times G/N)/G
\]

The resulting factoring of the pushforward \( \pi_1 = \pi_2 \pi_1 \) represents the fact that we can sequentially equivariantize from \( N \) first to \( B \) and then to \( G \). This matches up with the effect of the two functors in the fundamental correspondence for the equivariant Hecke category \( \mathcal{H}_G \).

6.4.2. Trivial generalized eigenvalue. Now let us restrict our attention to monodromic \( D \)-modules whose monodromy has generalized eigenvalue equal to one. Rather than studying the full subcategory of \( \mathcal{H}^{mon}_G \) containing such objects, we will follow tradition in representation theory and consider a completed version of it.

It is convenient to return to the general notation of preceding sections. Let’s take \( X = BB \) and \( Y = BG \) so that \( X \times_Y X = B \backslash G/B \). Now we additionally set \( \check{X} = BN \) so that \( \check{X} \times_Y \check{X} = N \backslash G/N \).

Consider the obvious projection

\[
q : \check{X} \times_Y \check{X} \to X \times_Y X
\]
and the corresponding algebra (with respect to the standard tensor symmetric monoidal structure) of cochains along the fibers

\[ A = q_* \mathcal{O}_{\bar{X} \times Y, \bar{X}} \in \mathcal{D}(X \times Y, X). \]

We define the $\infty$-category of unipotent monodromic $\mathcal{D}$-modules to be the module $\infty$-category

\[ \mathcal{H}^{uni}_G = \text{Mod}_A(\mathcal{D}(X \times Y, X)). \]

**Remark 6.13.** By the Barr-Beck theorem, we can identify $\mathcal{H}^{uni}_G$ with the full subcategory $^\perp \mathcal{K}(q_*) \subset \mathcal{D}_{mon}(\bar{X} \times Y, \bar{X})$ of objects that are in the left orthogonal to the pushforward $q_*$. 

**Example 6.14.** If $G$ is a torus $T$, then $\mathcal{H}^{uni}_G$ is equivalent to $H^*(T)$-bimodules in $\mathcal{D}(BT)$ which in turn is equivalent to $H^*(T)$-modules in $\mathcal{D}(pt)$.

In general, for any stack $\mathcal{S}$ built out of copies of $X$, we can replace it with the analogous stack $\bar{\mathcal{S}}$ built out of copies of $\bar{X}$. Then the obvious projection $q : \mathcal{S} \to \mathcal{S}$ provides an algebra object $A_{\mathcal{S}} \in \mathcal{D}(\mathcal{S})$. We define the monadic shadow of $\mathcal{D}(\mathcal{S})$ along $q_*$ to be the module $\infty$-category

\[ \hat{\mathcal{D}}(\mathcal{S}) = \text{Mod}_{A_{\mathcal{S}}}(\mathcal{D}(\mathcal{S})). \]

Observe that if $\mathcal{D}(\mathcal{S})$ is compactly generated, then $\hat{\mathcal{D}}(\mathcal{S})$ will be compactly generated by inductions.

We can work with such categories as we usually work with modules on a ringed space. Given a map $f : \mathcal{S} \to T$, we have the induced functors $f^*, f_*, f_!$, $f_\dagger$ between $\hat{\mathcal{D}}(\mathcal{S})$ and $\hat{\mathcal{D}}(\mathcal{T})$. It is straightforward to check that standard identities such as base change hold in this context.

As a first instance of this formalism, let’s return to the convolution diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{p_{12}} & X \\
\downarrow p_{13} & & \downarrow p_{23} \\
X \times Y & \xrightarrow{p_{13}} & X \\
\end{array}
\]

If we replace the copies of $X$ with copies of $\bar{X}$, we obtain an analogous diagram that maps to the above diagram. If we pushforward the structure sheaves, we obtain a diagram of algebra objects in $\mathcal{D}$-modules. Thus via the usual formalism, we can equip $\mathcal{H}^{uni}_G = \hat{\mathcal{D}}(X \times Y, X)$ with a monoidal product

\[ m : \hat{\mathcal{D}}(X \times Y, X) \otimes \hat{\mathcal{D}}(X \times Y, X) \to \hat{\mathcal{D}}(X \times Y, X) \]

\[ m(\mathfrak{M}, \mathfrak{N}) = (p_{13}^\dagger p_{12}^\dagger \mathfrak{M}) \otimes (p_{23}^\dagger \mathfrak{N}) \simeq p_{13}^\dagger (p_{12} \times p_{23})^\dagger (\mathfrak{M} \boxtimes \mathfrak{N}). \]

Here is a second instance of this formalism. Recall the fundamental correspondence

\[ \mathcal{L}Y = Y \times Y, Y \xleftarrow{\pi} \mathcal{L}Y \times Y, X \xrightarrow{\delta} X \times Y, X \]

where the maps are defined by the formulas

\[ \pi = \text{id}_{\mathcal{L}Y} \times \text{id}_Y \] \[ p = px_t \times \text{id}_Y \] \[ \delta = \text{id}_X \times \pi_Y \times \text{id}_X \]

where $\pi_Y : X \times Y \to Y$ is the obvious projection. As before, if we replace the copies of $X$ with copies of $\bar{X}$, we obtain an analogous diagram that maps to the above diagram. If we pushforward the structure sheaves, we obtain a diagram of algebra objects in $\mathcal{D}$-modules. Thus we can define a trace functor

\[ m : \hat{\mathcal{D}}(X \times Y, X) \to \mathcal{D}(\mathcal{L}Y) \]

\[ F(\mathfrak{M}) = p_1 \delta^* (\mathfrak{M}). \]

Now formal analogues of the arguments of preceding sections establish the following.
Theorem 6.15. The ∞-category of unipotent character sheaves $Ch_G$ is equivalent to the abelianization $\text{Ab}(\mathcal{H}^\text{uni}_G)$ of the unipotent Hecke category $\mathcal{H}^\text{uni}_G$. The equivalence fits into the commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}^\text{uni}_G & \xrightarrow{\tau} & \text{Ab}(\mathcal{H}^\text{uni}_G) \\
\downarrow & & \downarrow \\
Ch_G & \xrightarrow{\sim} & Ch_G
\end{array}
$$

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