Strongly coupled large-angle stimulated Raman scattering of short laser pulse in plasma-filled capillary

Serguei Kalmykov
Centre de Physique Théorique (UMR 7644 du CNRS),
Ecole Polytechnique, 91128 Palaiseau cedex, France, and
Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany

Patrick Mora
Centre de Physique Théorique (UMR 7644 du CNRS),
Ecole Polytechnique, 91128 Palaiseau cedex, France
(Dated: August 31, 2018)

Strongly coupled large-angle stimulated Raman scattering (LA SRS) of a short intense laser pulse develops in a plane plasma-filled capillary differently than in a plasma with open boundaries. Coupling the laser pulse to a capillary seeds the LA SRS in the forward direction (scattering angle smaller than \(\pi/2\)) and can thus produce a high instability level in the vicinity of the entrance plane. In addition, oblique mirror reflections off capillary walls partly suppress the lateral convection of scattered radiation and increase the growth rate of the SRS under arbitrary (not too small) angle. Hence, the saturated convective gain falls with an angle much slower than in an unbounded plasma and even for the near-forward SRS can be close to that of the direct backscatter. At a large distance, the LA SRS evolution in the interior of the capillary is dominated by quasi-one-dimensional leaky modes, whose damping is related to the leakage of scattered radiation through the walls.

PACS numbers: 52.35 Mw, 52.38 Bv, 52.40 Fd

I. INTRODUCTION

The technique of chirped-pulse amplification [1] made sub-picosecond laser pulses of high power \((P > 10^{12} \text{ W})\) available for generation of coherent x-rays [2], high harmonics of radiation [3], and laser wakefield acceleration (LWFA) of electrons [4, 5, 6] in rarefied plasmas [where \(\omega_0 \gg \omega_{pe}\)] in rarefied plasmas [where \(\omega_0 \gg \omega_{pe}\), \(\omega_0\) is a laser frequency, \(\omega_{pe} = (4\pi e^2n_0/m_e)^{1/2}\) is an electron plasma frequency, \(n_0\) is a background electron density, \(m_e\) and \(-|e|\) are the electron mass at rest and charge]. Full potential of these applications can be realized with the laser-plasma interaction length increased beyond the Rayleigh diffraction length, \(z_R = \pi\sigma_0^2/\lambda_0\), by means of external optical guiding [8] (here and later, \(\lambda_0 \approx 2\pi c/\omega_0\) is a laser wavelength, and \(\sigma_0\) is a laser beam waist radius). In particular, a dielectric capillary, where the oblique mirror reflections suppress the laser beam diffraction, can be used as a guiding tool [7, 8, 9]. Plasma can be created in a capillary by an optical field ionization of the filling gas [10, 11, 12], or by a laser ablation of the walls [13]. Then, the large-angle stimulated Raman scattering (LA SRS) starts to challenge the transportation of a laser beam over a long distance [14, 15].

In the standard SRS process [16], the pump electromagnetic wave (EMW) is scattered off spontaneous fluctuations of electron density, which, in turn, can be amplified by the ponderomotive beat wave of pump and scattered light. Appropriate phase matching of the waves results in a positive feedback loop with the onset of a spatio-temporal instability [17]. When the plasma extent is much larger than a laser pulse length, and no reflections off plasma boundaries occur, both scattering electron plasma waves (EPW) and scattered EMW quit the region of amplification, and the convective gain saturates within a time interval of the order of pulse duration [18, 19, 20, 21]. Convection of scattered radiation out of the laser waist may result in a strong pulse depletion [22]. Even when the full depletion does not occur, the LA SRS can produce considerable pulse erosion [23], suppression of the relativistic self-focusing [24], heating and pre-acceleration of plasma electrons [25], and seeding the forward SRS [26, 27]. Thereby, knowing the details of the LA SRS evolution in various physical conditions is a matter of high importance for applications.

Confining plasma between reflecting surfaces deeply modifies the SRS process. In the one-dimensional (1D) geometry, the Raman backscatter changes its nature from convective to absolute [28]: reflections trap the unstable radiation modes inside plasma and give rise to the continuous amplification. When the laser beam is confined between the mirror-reflecting partly transparent walls, and propagates collinearly to them, reflections reduce the sideward convection of scattered light. If the reflective modes dominate in plasma, the LA SRS gain tends to that of the direct backscatter and thus re-
veals a dramatic increase in comparison with the open-boundary system [14]. The LA SRS in this geometry has been considered so far in the regime of weak coupling [14], when the scattering EPW is similar to the plasma natural mode [17], and temporal growth rate is well below the electron plasma frequency. This regime requires fairly low amplitude of a laser pulse, i.e., \( a_0 \ll \sqrt{\omega_{pe}/\omega_0} \ll 1 \), where \( a_0 = eE_0/(m_e\omega_0c) \) is a normalized amplitude of the laser electric field. However, for the efficient LWFA [4, 5, 6], the plasma natural mode [17], and temporal growth rate is large, when the scattering EPW is similar to the boundary system [14]. The LA SRS in this geometry reveals a dramatic increase in comparison with the open-boundary plasma. Independent fluid modelling [32] of those in a slab of rarefied plasma laterally confined between the mirror-reflecting partly transparent flat walls (flat capillary). The unstable plasma modes were primarily transverse and therefore useless for the longitudinal electron acceleration. For the same range of parameters, this effect has never been significant in an open-boundary plasma. Independent fluid modelling [32] verified these observations.

Making a step to understanding this phenomenon we propose a two-dimensional (2D) linear theory of strongly coupled SRS of a short laser pulse under a given angle \( \alpha \) in a slab of rarefied plasma laterally confined between the mirror-reflecting partly transparent flat walls (flat capillary). Boundary conditions for the scattered radiation describe the oblique mirror reflections and the electromagnetic (EM) seed at the entrance plane. We associate the latter with the signal formed of the high-order capillary eigenmodes produced by the laser beam coupling to the capillary (the coupling process is described elsewhere [8, 11] and is outlined in Appendix A). SRS in forward (\( \alpha < \pi/2 \)) and backward (\( \alpha > \pi/2 \)) directions proceed differently. Forward Raman amplification of the EM seed can be dominant within a finite distance from the entrance plane and be responsible for the instability enhancement observed in the modelling. On the other hand, the backward SRS is affected by the reflections only. As the LA SRS of a finite-length laser pulse preserves the convective nature (backward and, partly, sideward convection of radiation is allowed), the gain saturation occurs within a finite distance from the entrance plane. Reflections give the unstable modes additional rise time in any transverse cross-section of a plasma, and, even for relatively small scattering angles (such as \( \alpha = \pi/6 \) taken for numerical examples of this paper), the saturated convective gain can approach that of the backward SRS (BSRS). The field structure is then approximated by a quasi-1D lossy mode whose damping is produced by the leakage of radiation through the walls.

The paper is organized as follows. Section II presents a theoretical model of the strongly coupled LA SRS in a 2D slab geometry. The laser pulse entrance into a plasma and oblique mirror reflections of scattered light are expressed in terms of appropriate boundary-value conditions for the coupled-mode equations. General solution of the boundary-value problem is presented (derivation is given in Appendix A). Section III discusses the spatio-temporal evolution of instability. The case of fully transparent lateral boundaries is considered in subsection III A. Enhancement of the LA SRS in the generic reflective case is considered, and the maximum gain factors are evaluated in terms of appropriate asymptotic solutions in subsection III B. Section IV summarizes the results.

II. BASIC EQUATIONS AND SOLUTION OF BOUNDARY-VALUE PROBLEM

In a 2D plasma slab confined between mirror-reflecting capillary walls, the high-frequency (hf) electric field is a superposition

\[
a(r, t) = \frac{e^{-i\omega_0 t}}{2} \left\{ a_0(r, t)e^{ik_0z} + \sum_{\sigma = \pm} a_{\sigma}(r, t)e^{i(k_{\sigma}r)} \right\} + c.c.
\]

(1)

of that of the laser, \( a_0 \), and of up- and down-going scattered EMWs, \( a_{\pm} \equiv eE_{\pm}/(m_ec\omega_0c) \) \( (|a_\pm| \ll |a_0| < 1) \); in Eq. 1 and below, \( r = (x, z) \) is a radius-vector in a plane geometry. We assume that the waves have the linear polarization with the electric vectors parallel to the walls, and that the scattering occurs under an angle \( \alpha \) in the plane orthogonal to the polarization vector. In the rareified plasma, both \( k_0 = e_xk_0 \) and \( k_{\pm} \) obey the same dispersion relation \( \omega_0^2 = \omega_{pe}^2 + c^2k_0^2(\pm \alpha) \approx c^2k_{0}(\pm \alpha)^2 \); hence, \( |k_{\pm}| = k_0, k_{\pm} = k_{\pm} - k_0 \cos \alpha, \pm k_{\pm} = k_{\pm} = k_0 \sin \alpha \), and the amplitudes \( a_{0}(\pm \alpha) \) vary slowly in time and space on the scales \( \omega_0^{-1}, k_{\pm}^{-1}, \) and \( k_{\pm}^{-1} \). Ions form a homogeneous positive background; this assumption holds for a laser pulse shorter than an ion plasma period, \( t_0 \ll 2\pi\omega_0^{-1} \). The beat wave of incident and scattered radiation excites perturbations of electron density,

\[
\frac{n_e - n_0}{n_0} = \sum_{\sigma = \pm} N^r_\sigma(r, t)e^{i(k_{\sigma}r)} + c.c.,
\]

(2)

whose wave vectors obey the matching conditions \( k_{\pm} = k_0 - k_{\pm} \); hence, \( |k_{\pm}| \equiv k_{\pm} = 2k_0 \sin(\alpha/2) \), \( k_{\pm} \equiv k_{\pm} = k_0(1 - \cos \alpha) \), \( k_{\pm} = \mp k_0 \sin \alpha \). Wave vector diagram of the LA SRS is shown in Fig. 1. The scattering EPW has a longitudinal component of phase velocity small compared to the speed of light, i.e., \( k_{\pm} > k_0 = \omega_{pe}/c \). This restriction eliminates the forward Raman scattering [26, 33] and resonant modulational instability (RMI) [34]. In the rarefied plasma, the amplitudes \( N_\pm \) vary slowly in space on the scales \( k_{\pm}^{-1}, k_{\pm}^{-1} \).
The amplitudes of up- and down-going scattered EMW and scattering EPW obey the coupled-mode equations derived from the equations of non-relativistic hydrodynamics of cold electron fluid in the hf field $\hat{\mathbf{E}}$ and Maxwell’s equations for the scattered radiation,

$$i \left( \frac{\partial}{\partial \xi} + V_z \frac{\partial}{\partial z} \pm V_x \frac{\partial}{\partial x} \right) a_{s\pm} = g_1 N_{s\pm}, \quad (3a)$$

$$- \left( \frac{\partial^2}{\partial \xi^2} + k_p^2 \right) N_{s\pm} = g_2 a_{s\pm}, \quad (3b)$$

where $V_z = \cos \alpha/(1 - \cos \alpha)$, $V_x = \sin \alpha/(1 - \cos \alpha)$, $g_1 = (a_0/2)(k_p^2/k_{sc})$, and $g_2 = a_0^2(k_p/2)^2$. The wave coupling parameter is $G^3 \equiv g_1 g_2 = (a_0/2)^2 k_p^2 k_0$ (strong coupling is the case for $G \gg k_p$). Equations (3a) are expressed through the variables $x$, $z$, and $\xi = ct - z$, that is, the temporal evolution of waves is traced in an $x - y$ cross-section at a longitudinal position $z$.

Figure 2 shows the interaction area. At $z = 0$, $t = 0$, the laser pulse enters a semi-infinite plasma-filled gap between flat mirror-reflecting walls ($x \geq 0, 0 \leq z \leq L_z$) and propagates towards positive $z$. The pulse leading front, $\xi = 0$, encounters the stationary level of electron density perturbations with a constant amplitude $N_0$,

$$N_{s\pm}(x, z, 0) = N_0, \quad (4a)$$

$$\partial N_{s\pm}/\partial \xi(x, z, 0) = 0, \quad (4b)$$

fluctuations of radiation in fresh plasma being neglected,

$$a_{s\pm}(x, z, 0) \equiv 0. \quad (5)$$

At the capillary entrance plane, the transverse profile of radiation can have a significant content of the high-order capillary eigenmodes (coupling the incident laser beam to the capillary is discussed elsewhere [8, 11] and is outlined in Appendix A). The resonant condition for the wave vectors selects the modes that can be amplified by the forward SRS ($\alpha < \pi/2$) thus leading to the formation of an EM seed signal. To facilitate the forthcoming analytic job, we give this signal in a simple parabolic form with an amplitude vanishing at the walls in order to provide the continuity of the solution in the interior of the capillary,

$$a_{s\pm}(x, z = -0, \xi) = a_{s0} \left[ 1 - (1 - 2x/L_z)^2 \right]. \quad (6)$$

Inside the capillary, oblique mirror reflections couple up- and down-going EMW: each reflection converts an up-going wave into a down-going one and vice versa,

$$a_{s+}(0, z, \xi) = r(\alpha) a_{s-}(0, z, \xi), \quad (7a)$$

$$a_{s-}(L_z, z, \xi) = r(\alpha) a_{s+}(L_z, z, \xi). \quad (7b)$$

The conditions (7) set up a quasi-1D exponential behavior of waves at large $z$. The reflectivity coefficient is a known function of scattering angle, $r(\alpha) = \sin \alpha - [(\delta_w/\delta_p)^2 - \cos^2 \alpha]^{1/2}/(\sin \alpha + [(\delta_w/\delta_p)^2 - \cos^2 \alpha]^{1/2})$, where $\delta_w$ and $\delta_p$ are the refraction indexes of walls and plasma. Figure 2 shows $r(\alpha)$ for a glass capillary with $\delta_w \approx 1.5$ and $\delta_p \approx 1$.

The temporal increment of strongly coupled LA SRS exceeds the electron plasma frequency. Hence, we neglect $k_p^2 N_{s\pm}$ in comparison with $\partial^2 N_{s\pm}/\partial \xi^2$ in the left-hand side (LHS) of Eq. (6). With the allowance for not very tight capillary, the pump field envelope $a_0(x, \xi)$ represents a portion of laser radiation coupled to the capillary which experiences mostly paraxial propagation, $k_{\perp}/k_0 \ll \sqrt{\omega_{pe}/\omega_0}$. Assuming that the pump field evolution at a given point $(x, z)$ takes much longer than the SRS growth $(z_H/c \gg t_0)$, and in order to enable the analytic progress, we approximate $a_0(x, \xi)$ with a fixed flat
effective pulse duration \( t_0 \) and amplitude \( a_0 \). Here and below, \( H(y) \) is the Heaviside step-function. Solution of the boundary-value problem,

\[
a_{s+}(R; r) = -a_0 F(R; r) \Phi_1(r, c_j) - \sum_{j=1}^{3} \frac{e^{\epsilon_j \xi}}{c_j} [1 - \Phi_1(R; r, c_j) - \Phi_2(R; r, c_j)],
\]

\[
N_+(R; r) = \frac{1}{4} a_0 g_2 R (\xi - z/V_\xi)^2 \sum_{j=1}^{3} \frac{e^{\epsilon_j \xi}}{c_j} [1 - \Phi_1(R; r, c_j) - \Phi_2(R; r, c_j)],
\]

is then obtained via the 2D Laplace transform; here, \( R = (r, \xi) \), \( c_j = iG \), \( \xi = (G^2/4)(\xi - z/V_\xi)^2/V_\xi, \) \( n F_2(; b_1, b_2; \theta) \) is the regularized generalized hypergeometric function [32],

\[
\Phi_1 = F_s \left( c_j, \frac{z}{V_\xi}, \xi \right) \left\{ H(V_\xi x - V_\xi z) + \sum_{n=1}^{\infty} r^n [H(V_\xi z - V_\xi x_{n-1}) - H(V_\xi z)] \right\},
\]

\[
\Phi_2 = (1 - r) \sum_{n=0}^{\infty} r^n F_s \left( c_j, \frac{x_n}{V_\xi}, \xi \right) H(V_\xi z - V_\xi x_n),
\]

\[
\mathcal{F} = \sqrt{\pi} H \left( \xi - \frac{z}{V_\xi} \right) \left\{ \frac{(V_\xi x - V_\xi z)(V_\xi x - V_\xi z - V_\xi L_x)}{(V_\xi L_x/2)^2} H(V_\xi x - V_\xi z)
\]

\[
+ \sum_{n=1}^{\infty} r^n \frac{(V_\xi z - V_\xi x_{n-1}) - H(V_\xi z - V_\xi x_{n-1})}{(V_\xi L_x/2)^2} \right\},
\]

where \( x_n = x + nL_x \), and the fundamental solution \( F_s(\mu, v, \xi) \) (Ref. 21) is defined by Eq. (9b). The up- and down-going amplitudes are symmetric, \( a_{s+}(x) = a_{s-}(L_x - x) \), \( N_- (x) = N_+ (L_x - x) \), so we consider below the evolution up-going waves only. Our solution possesses the same generic structure as the weakly coupled reflective solution discussed in detail in Ref. 14. However, contrary to the case of semi-infinite laser pulse of Ref. 14, the growth time of unstable waves is now limited by the pulse duration \( t_0 \) and, as shown in the subsection III.B, the gain at a given point \((x, z)\) remains finite and is given by either Eq. (15) or (16).

III. SPIATIO-TEMPORAL EVOLUTION OF UNSTABLE WAVES

A. Evolution of instability in open-boundary system

When the lateral boundaries are fully transparent, \( r = 0 \), the EM seed at the entrance plane vanishes \( (a_{sat} = 0) \), and the instability grows from the electron density noise in a fresh plasma ahead of the pulse. The functions 35 then read

\[
\Phi_1(R; r, c_j) = F_s(c_j, z/V_\xi, \xi) H(V_\xi x - V_\xi z),
\]

\[
\Phi_2(R; r, c_j) = F_s(c_j, x/V_\xi, \xi) H(V_\xi z - V_\xi x).
\]

Plasma is divided by the characteristics \( \xi = z/V_\xi \), \( x = z(V_\xi/V_x) \), into ranges of dependence, where the solution is prescribed by the boundary conditions for radiation posed at the pulse leading edge \( \xi = 0 \) (range I),

\[
\begin{align*}
\xi &< \frac{x}{V_\xi} \\
\xi &< \frac{z}{V_\xi}
\end{align*}
\]

\[\Rightarrow \Phi_1 \equiv 0, \ \Phi_2 \equiv 0 \quad (10)\]

at the wall \( x = 0 \) (range II),

\[
\begin{align*}
x &< \frac{z(V_\xi/V_x)}{V_x} \\
\xi &> \frac{x}{V_\xi}
\end{align*}
\]

\[\Rightarrow \Phi_1 \equiv 0, \ \Phi_2 \neq 0 \quad (11)\]

or at the entrance plane \( z = -0 \) (range III),

\[
\begin{align*}
x &> \frac{z(V_\xi/V_x)}{V_x} \\
\xi &> \frac{z}{V_\xi}
\end{align*}
\]

\[\Rightarrow \Phi_1 \neq 0, \ \Phi_2 \equiv 0 \quad (12)\]

The principal feature that makes the LA SRS of a short pulse [21] different from the case of semi-infinite laser beam [14] is the gain saturation within a finite distance from the entrance plane. At some point, \( z_{sat} < +\infty \), the scattered radiation arriving from the plasma boundary
all the three areas (10)-(12) are available in the pulse angle-independent increment

Thus, the entrance effect does not change the solution with \( z \). The gain saturates differently for the forward (\( \alpha < \pi/2 \)) and backward (\( \alpha > \pi/2 \)) scattering.

When \( \alpha < \pi/2 \), and the distance from the entrance plane is not too large, i.e., \( z < \min \{ V_z c t_0, L_x (V_z/V_x) \} \), all the three areas (10)-(12) are available in the pulse body (that is, within a rectangle \( 0 \leq \xi \leq c t_0, 0 \leq x \leq L_x \)). Given the point \( (x,z) \), waves fall initially within a range \( \text{I} \), where they grow in time exponentially with an angle-independent increment

\[
\gamma_0 = (\sqrt{3}/2) G \approx (\sqrt{3}/2) \sqrt{(a_0/2)^2 \omega_0^2 \omega_{pe}^2}.
\]

Note, that \( \kappa = \gamma_0/c \) is the known “spatial” increment of the strongly coupled BSRS in the co-moving frame [21, 22, 26, 30]. The evolution of waves is strictly 1D in space on this stage. Later, information from the boundaries \( x = 0 \) and \( z = -0 \) reaches the point \( (x,z) \), and the spatial dependence becomes either 2D for \( \xi > x/V_x, x < z(V_x/V_z) \) (range II) or remains 1D for \( \xi > z/V_z, x > z(V_x/V_z) \) (range III). The waves are not exponentially growing at this time. In the range III, the entrance effect dominates: vanishing the scattered EMW at the boundary \( z = -0 \) determines the behavior of 1D amplitudes. Deeply enough in plasma, \( z \geq \min \{ V_z c t_0, L_x (V_z/V_x) \} \), the entrance effect vanishes as the pulse terminates sooner than the scattered EMW from the entrance plane can reach the observer at a given \( z \). The pulse body is then divided between the ranges I and II, and the evolution of LA SRS is the same through the rest of the plasma.

For \( \alpha > \pi/2 \), the boundary-value condition posed for radiation at \( z = -0 \) can only produce the scattered EMW convecting outwards (the EMW characteristic \( \xi = z/V_z \) recasts in the lab frame variables as \( z = -ct | \cos \alpha | \), and corresponds to the wave propagating towards negative \( z \)). Thus, the entrance effect does not change the solution at a positive \( z \), and the boundary-value condition becomes excessive (this is also valid in the reflective case). The spatio-temporal evolution of the LA SRS remains the same at any \( z \geq 0 \).

Given the pulse duration, the maximum of the saturated gain, \( a_{x+}, N_+ \sim e^{\gamma_0 t_0} \), does not depend on the scattering angle, and whether it is achieved or not is determined solely by the pulse aspect ratio [21]. If the pulse is wide, or the scattering angle is sufficiently large, \( \alpha > \alpha_0 = 2 \arctan (c t_0/L_x) \), the maximum gain is achieved at the pulse rear edge \( \xi = c t_0 \) for \( c t_0 \cot(\alpha/2) < x < L_x \). Hence, the angular spectrum of scattered light is prescribed by the pulse aspect ratio rather than the angular dependence of the increment. For \( L_x \gg c t_0 \), scattering within a broad range of angles \( 2 c t_0/L_x < \alpha \leq \pi \) proceeds with the maximum gain. Otherwise, for \( L_x \ll c t_0 \), the highest gain corresponds to the near-backward scattering only, \( \pi - L_x/(c t_0) < \alpha \leq \pi \) (Ref. [21]). To estimate the pulse energy depletion due to the LA SRS, it is sufficient to neglect the radiation scattered under angles smaller than \( \alpha_0 \) (Ref. [21]).

B. LA SRS evolution in the reflective case

In a capillary, the oblique mirror reflections of scattered light off the walls [the boundary condition (7)] contribute to the LA SRS evolution over the whole laser path in plasma, but become actually dominating later, when the entrance effect vanishes (\( z > V_z c t_0 \)). The reflections establish a long-distance asymptotic state of the LA SRS — amplification of a quasi-1D radiation and plasma modes with a temporal increment close to that of the direct backscatter [14]. Besides, the forward SRS is seeded by the EM signal \( a_{x+}(x,-0,\xi) \) at the capillary entrance plane [the boundary condition (7)]. If the signal amplitude \( a_{00} \) exceeds the amplitude of the electron density noise \( N_0 \), the unstable waves will achieve a large amplitude (by virtue of the high seed level) at the transient stage of laser propagation, \( 0 < z < V_z c t_0 \). In this case, contrary to the SRS in the unbounded plasmas described in the preceding subsection, the near-forward SRS exhibits much higher level of amplification than the backward SRS (\( \alpha > \pi/2 \)), and its contribution to the dynamics of electron density perturbations can be dominating. Nonlinearities of plasma response that can then appear are worth investigating and will be addressed in future publications.

The asymptotic \( \sim (2\pi\sqrt{3})^{-1}g(1-b_1-b_2)/3 e^{3\sqrt{3}} + O(\theta^{-1/3}) \) at \( | \theta | \to \infty \) (Ref. [33]) helps to evaluate the level of the EM seed amplification within the interval \( 0 < z < V_z c t_0 \).

\[
|a_{x+}(z < V_z c t_0)| \sim a_{00} \frac{F}{2\sqrt{3} \gamma} e^{3\sqrt{3}/2} \left[ \frac{z}{\xi} \right]^{1/3},
\]

where \( \xi \gg 1 \). Then, \( |N_+| \sim (V_z/g_1)(\xi^{1/3}/z)|a_{x+}| \). At the pulse trailing edge, \( \xi_0 = c t_0 \), the argument of the asymptotic reaches the maximum \( \xi_{\text{max}} = (G c t_0)^3/3 \) at
\[ z_{\text{max}} = V_z c t_0 / 3. \] The EPW amplitude at this point is

\[
|N_+(\xi, z_{\text{max}})| \sim a_{s0} \frac{G \sqrt{3} F}{g_1 \sqrt{8 \pi \gamma^0 t_0}}, \tag{15}
\]

and \( |a_+ (\xi, z_{\text{max}})| \sim (g_1 / G) |N_+(\xi, z_{\text{max}})|. \) Equation 15 shows that the maximum gain on the transient stage is determined by the 1D temporal increment in \( \alpha \) independent on \( \alpha \) (the angular dependence is retained in the pre-exponential factors only). For the minimally allowed scattering angle, \( \alpha_{\text{min}} = \sqrt{2 \omega_p \epsilon / \omega_0} \) (hence, \( V_z = \omega_0 / \omega_p \)), the point of the maximal gain is

\[
z_{\text{max}}(\alpha_{\text{min}}) = (\omega_0 / \omega_p)(c t_0 / 3) > ct_0. \]

Parameters of the following numerical example (Fig. 3) give \( z_{\text{max}}(\alpha_{\text{min}}) \approx 3 \) mm, which is shorter than a typical capillary length used in experiments (\( \geq 1 \) cm).

For \( z > z_{\text{max}} \), the entrance effect becomes less pronounced and finally vanishes at the point \( z = V_z c t_0 = 3z_{\text{max}} \), where the EM signal arriving from \( z = -0 \) drops behind the laser pulse, and the instability growth saturates. Evolution of both forward and backward scattering is then determined by the lateral reflections only and, given the scattering angle, remains the same in any \( x-y \) cross-section for \( z \geq V_z c t_0. \) We show in Appendix 1 that at \( z > V_z c t_0 \), and \( \gamma_0 t_0 > 1 \), a cumbersome exact solution tends asymptotically to a quasi-1D damped mode with a temporal growth rate close to 15 that gives the asymptote for the density perturbation amplitude,

\[
N_+(x, \xi) \sim (N_0 / 3)(1 - r) / \ln |e^{s_0 \xi - \ln r}|. \tag{16}
\]

Here, \( s_0 = (\gamma_0 - \Delta \gamma) / c, \Delta \gamma / c \approx -V_z \ln r / (3L_x) \). Equation 16 is valid under the "low leakage" condition, \( r > \exp (-3GL_x / V_x) \), which indicates that the scattered light is mostly trapped inside a plasma slab: the energy leakage through the wall at one reflection (that produces an effective decrement \( \Delta \gamma \ll \gamma_0 \)) is less than the energy gain on the way between the walls. The lateral growth of the asymptote 16 is much slower than the growth with \( \xi \). Asymptotic solution 16 displays the basic result of the reflective theory of the LA SRS: at large distances from the entrance plane and large coefficients of amplification, the amplitudes of unstable waves tend to quasi-1D leaky modes exponentially growing in time with the increment tending to that of the BSRS 11.

Comparison of Eqs. 15 and 16 shows that the maximum amplitude of the scattering EPW on the transient stage, \( z < V_z c t_0 \), differs from the final asymptotic level of density perturbations roughly by a factor of \( (g_1 / G)(\gamma_0 t_0)^{1/2} / a_{s0} / N_0 \). \) When this factor is larger than unity, and \( |N_+(z \geq V_z c t_0, x, ct_0)| \sim 1 \), the plasma response can become nonlinear in the vicinity of \( z = V_z c t_0 / 3. \) This could be avoided by keeping the ratio of the seed amplitudes \( a_{s0} / N_0 \) below \( (g_1 / G)(\gamma_0 t_0)^{1/2} \). The amplitude of the electron density noise is difficult to control in experiment; however, as shown in Appendix A, the content of the high-order eigenmodes in the laser radiation coupled to the capillary (and, hence, the amplitude \( a_{s0} \) of the EM seed signal), can be effectively reduced by increasing the capillary radius versus the radius of the incident laser beam.

Figure 4 shows the spatio-temporal evolution of an up-going EPW [Eq. 15] for the SRS under the angle \( \alpha = \pi / 6. \) The laser and plasma parameters are \( a_0 = 0.7, \lambda_0 = 0.5 \mu m, c t_0 = 0.5L_x = 6k \rho^{-1}, \omega_p / \omega_0 = 0.007, \) which gives \( n_0 \approx 2.2 \times 10^{17} \) cm\(^{-3} \), the pulse duration \( t_0 \approx 230 \) fs, and the maximum increment \( \gamma_0 \approx 2.25 \beta \omega_p \). The level of EM seed is chosen \( a_{s0} \approx 0.58 \times 10^{-5} a_0 \) (according to Appendix A, it corresponds to a capillary by a factor of two wider than in the case of perfect matching: by the definition, in axi-symmetric geometry, the perfect matching condition provides coupling 98% of energy of an incident Gaussian laser pulse to the fundamental eigenmode EH\(_{11}\) of a capillary 8. The level of the plasma noise evaluated in Appendix A is \( N_0 \approx 1.5 \times 10^{-6} \). The plots (a) and (c) correspond to the plasma confined in a glass capillary with the reflection coefficient \( r = 0.42 \) (see Fig. 3), and (b) — to the unbound plasma \( (r = 0 \) and \( a_{s0} = 0) \); plot (d) shows the long-term asymptotic behavior of the reflective solution. The plasma cross-sections are set at \( z = V_z c t_0 / 3 \approx 2.2 c t_0 \), (c), (d) \( z = V_z c t_0 \approx 6.5 c t_0 \). Given the calculation parameters, the non-reflective solution saturates at \( z = L_x (V_z / V_x) \approx 3.4 c t_0 \) and is exactly
other hand, reduction \(a_{s0}\) to the level \(7 \times 10^{-8}a_0\) (which in the axi-symmetric case would correspond to a capillary tube by a factor 2.5 wider than in the case of perfect matching, see Appendix A), will make the forward Raman amplification of the EM seed almost negligible and thus tolerable on the transient stage.

Plot (c) shows a quasi-1D saturated solution [compare with Fig. 4(b)] thus shaped by the contribution from two reflections [according to Fig. 4(b)], which, in full agreement with the long-scale asymptote (2), demonstrates the growth rate close to that of BSRS. Difference between Figs. (a) and (c) shows that, under the parameters of our example, the forward scattering \((\alpha < \pi/2)\) is characterized at \(z < V_zct_0\) by much higher gain than the SRS in the backward direction \((\alpha > \pi/2)\). This situation is completely reverse of the SRS in an unbounded plasma, where the gain can only fall as an angle drops \(\alpha\). So, the higher the EM seed level produced by the laser beam coupling (that is, the tighter the capillary in a numerical or real-scale experiment), the more important becomes the forward SRS. In such case, a high amplification level of waves may be observed within quite a long distance in plasma, \(z < V_z(\alpha_{\text{min}})ct_0 \approx (\omega_0/\omega_{pr})ct_0\) [see also the discussion following Eq. (14)]. This effect is adverse for such applications as the self-modulated LWFA in capillaries. The way of reducing the excessive forward SRS enhancement can be found in using a wider capillary (the laser focal spot fixed) than the perfect matching requires.

FIG. 5: Ranges of dependence for the SRS under the angle \(\alpha = \pi/6\) in the cross-sections at (a) \(z = V_zct_0/3 \approx 2.2c_t\) and (b) \(z = V_zct_0 \approx 6.5c_t\). The thick solid line \(x = z(V_z/V_0)\) divides the principal ranges prescribed by the non-reflective theory. The dashed lines are the characteristics of reflected waves, \(x + nL_x = z(V_z/V_0)\), \(\xi = z/V_0\), and \(\xi = (x + nL_x)/V_z\). One reflection contributes to the scattering process in the range II in the case (a), and two reflections — in the case (b). Raman amplification of the EM signal given at the boundary \(z = -0\) contributes to the solution in the sub-range IIb and range III.

The same in any plasma cross-section \(x - y\) beyond that point. Figure (b) shows this solution. The ranges of influence of boundary conditions are shown in Fig. 5. In the range I [see Eq. (10)], the electron density noise from \(\xi = 0\) is amplified, and the waves do not experience reflections. In the range IIa the instability is seeded by the free-plasma noise, and yet is enhanced by the reflections; the Raman amplified signal arriving from the entrance plane is added to these waves in the range IIb. The EM seed from the entrance plane \(z = 0\) is amplified by the forward SRS in the range III.

In a capillary, the forward SRS is considerably enhanced at \(z < V_zct_0\) (roughly by a factor of 10 in amplitude) versus the case of unbound plasma [compare Figs. (a) and (b)]. Despite the EM seed amplified in the range IIb is non-exponentially growing, the high ratio of seed amplitudes, \(a_{s0}/N_0 \approx 2.7 \gg (g_1/G)(\gamma_0/\gamma_{10})^{1/2} \approx 0.026\), makes the entrance effect rather pronounced [plot (a)]. For \(N_0 \approx 1.5 \times 10^{-6}\), and \(F \approx 1\), Eq. (14) gives \(\log_{10}[N_+ (\xi_0, z_{\text{max}})/N_0] \approx 7.15\), which agrees with Fig. (a). Hence, parameters of the numerical example lay at the border of validity of the linear approach, and increase in \(a_{s0}\) will result in the nonlinearity of the plasma response on the transient stage \{e.g., perfect matching gives \(a_{s0} \approx 0.012a_0\), hence, according to Eq. (13), \(|N_+ (\xi_0, z_{\text{max}})| \sim 10^6\); this burst of the forward SRS has been the regular feature in our numerical experiments and in the fluid simulations \{22\}. On the other hand, reduction \(a_{s0}\) to the level \(7 \times 10^{-8}a_0\) (which in the axi-symmetric case would correspond to a capillary tube by a factor 2.5 wider than in the case of perfect matching, see Appendix A), will make the forward Raman amplification of the EM seed almost negligible and thus tolerable on the transient stage.

Plot (c) shows a quasi-1D saturated solution [compare with Fig. 4(b)] thus shaped by the contribution from two reflections [according to Fig. 4(b)], which, in full agreement with the long-scale asymptote (2), demonstrates the growth rate close to that of BSRS. Difference between Figs. (a) and (c) shows that, under the parameters of our example, the forward scattering \((\alpha < \pi/2)\) is characterized at \(z < V_zct_0\) by much higher gain than the SRS in the backward direction \((\alpha > \pi/2)\). This situation is completely reverse of the SRS in an unbounded plasma, where the gain can only fall as an angle drops \(\alpha\). So, the higher the EM seed level produced by the laser beam coupling (that is, the tighter the capillary in a numerical or real-scale experiment), the more important becomes the forward SRS. In such case, a high amplification level of waves may be observed within quite a long distance in plasma, \(z < V_z(\alpha_{\text{min}})ct_0 \approx (\omega_0/\omega_{pr})ct_0\) [see also the discussion following Eq. (14)]. This effect is adverse for such applications as the self-modulated LWFA in capillaries. The way of reducing the excessive forward SRS enhancement can be found in using a wider capillary (the laser focal spot fixed) than the perfect matching requires.

Figure (d) traces the temporal evolution of the up-going EPW at \(x = L_x/4\) (near the capillary wall). The asymptote (2) perfectly approximates (and, for applications, can be used instead of) the exact reflector solution \{23\}. The dash-dotted and dotted lines in Fig. (d) correspond to the BSRS solution \(|N_+ (\xi)| \approx (N_0/3)\exp(\gamma_0\xi/c)\) and the exact non-reflective solution, respectively, providing the upper and lower limits of the convective gain variation. In this example, only in the vicinity of the border \(x = 0\) the coefficients of amplification in the reflective and non-reflective cases are considerably different [compare Figs. (b) and (c)]. Contribution from the reflections increases the wave amplitude at \(x = L_x/4\) by roughly an order of magnitude (compare solid and dotted lines at \(k_x\xi \approx 6\)). For the parameters chosen, the scattering EPW remains linear in the convective saturated regime (\(z > V_zct_0\)): the plasma noise level, \(N_0 \approx 1.5 \times 10^{-6}\), substituted into Eq. (14), gives \(|N_+| \leq 0.2\) throughout the whole time interval \(0 < t < t_0\).

In the limit \(r \to 1\) the total suppression of the lateral convection occurs. The up- and down-going amplitudes \{23\} become purely one-dimensional for \(z > V_zct_0\) and completely identical. These amplitudes grow in time exponentially with the BSRS increment \{23\}. 

---

The text above is a natural representation of the given document page.
IV. CONCLUSION

We have proposed a 2D non-stationary linear theory of strongly coupled LA SRS of a short laser pulse in a flat plasma slab confined between mirror-reflecting walls (flat capillary). In a capillary, the lateral convection of scattered light is partly suppressed by the oblique reflections, and the instability experiences an enhancement. Additional enhancement of the SRS in forward direction ($\alpha > \pi/2$) is produced by the amplification of the electromagnetic seed signal that is formed of the high-order capillary eigenmodes at the entrance plane (formation of the signal is a consequence of the laser beam coupling to the capillary). The convective nature of LA SRS does not change. The asymptotic behavior of the waves demonstrates the transition from the set of 2D modes to the dominant quasi-1D mode may be close to the BSRS gain. Even for near-forward scattering the convective gain of the dominant quasi-1D mode may be close to the BSRS gain.

Acknowledgments

This work was started with the aid of a postdoctoral fellowship from the Centre de Physique Théorique, Ecole Polytechnique, and its completion was supported by a fellowship from the Centre de Physique Théorique, Ecole Polytechnique, and its completion was supported by the Fortbildungsstipendium from Max-Planck-Institut für Quantenoptik. We acknowledge useful discussions with N. E. Andreev, B. Cros, L. M. Gorbunov, G. Matthieussent, and J. Meyer-ter-Vehn.

APPENDIX A: SEED SOURCES FOR LA SRS

The LA SRS under arbitrary angle in a strongly rarified plasmas ($\omega_{pe} \ll \omega_0$) is seeded by spontaneous electron density fluctuations ahead of the pulse. A root-mean-square (rms) amplitude of these fluctuations is represented in the equations by the quantity $N_0$, which gives the amount of seed corresponding to the element of solid angle $d\Omega_k$, the direction of the wave vector $k_\perp$ of scattering EPW, and can be expressed as

$$N_0^2(k_\perp) = d\Omega_k \int n_\perp^3 \omega_0^2 k^2 dk \approx n_\perp^3(k) k_\perp^2 \Delta k_\perp d\Omega_k,$$

where $n_\perp^3(k)$ is a spectral density of electron fluctuations integrated over frequencies [37]. The integral is taken over the area of maximal spectral density of scattering EPW $|k| \approx k_\perp$, the amplification bandwidth $\Delta k_\perp \approx 4\sqrt{\gamma_0 t_0}$ is estimated at $\gamma_0 t_0 \gg 1$ with taking account of the gain narrowing (Ref. [36]). Using the ratio of the phase volumes $d\Omega_k/d\Omega_{k_\perp} = 2 \sin^2(\alpha/2)$ (Ref. [35]), we express the seed amplitude through the element of solid angle $d\Omega_k$ in the direction of detector, $N_0^2 \approx 8n_\perp^3(k) k_\perp^2 \gamma_0^2 (\omega_0 t_0)/\omega_0^2 \sin^2(\alpha/2) d\Omega_k$. Our theoretical formalism based on the assumption of quasi-plane interacting waves requires small variation of the scattered wave amplitude across the direction $k_\perp$ in the transversely limited area, $0 \leq x \leq L_x$, which gives an estimate of the angular spread $\Delta \alpha \approx \cos \alpha/(L_x k_0)$. The element of solid angle then estimated as $\Delta \Omega_k \approx 2\pi \sin \alpha \Delta \alpha = \pi \sin 2\alpha/(L_x k_0)$ gives

$$|N_0| \approx \sqrt{\frac{8\pi}{\omega_0^2}} \frac{1+(k_n r_{Dx})^2}{2+(k_n r_{Dx})^2} \frac{\sin^2(\alpha/2) \sin 2\alpha}{k_0 L_x},$$

where the spectral density of low-frequency electron fluctuations $n_\perp^3(k)$ is evaluated using formula (11.2.6.6) of Ref. [37]. Parameters of Fig. 4 and $k_n r_{Dx} \ll 1$ give $|N_0| \approx 1.5 \times 10^{-6}$.

For the strongly coupled SRS in the forward direction ($\alpha < \pi/2$), coupling the laser beam to a capillary creates additional source of instability. The radial profile of an incident radiation with the wings cut off by the edges of the entrance aperture is approximated with an expansion through an infinite number of radial eigenmodes (having the same frequency $\omega_0$). The high-order eigenmodes (characterized by the frequency $\omega_s = \omega_0$ and high transverse wavenumbers, $k_n \perp \sim k_n x$, where the integer $n$ is the mode order) form the seed signal that is further amplified in plasma by the SRS [in the model form, the transverse profile of this signal is given by Eq. (4)]. It should be emphasized that these modes provide no seed for the weakly coupled SRS (including near-forward SRS and RMI) that correspond to small scattering angles, $\alpha \ll \sqrt{\omega_{pe}/\omega_0}$, as this process requires the frequency matching $\omega_s \approx \omega_0 - \omega_{pe}$ between the seed and the pump.

Exact functional form of the capillary eigenmodes depends on the geometry chosen. Despite the LA SRS in a flat capillary is considered in the paper, we suppose that an estimate of the EM seed level will be more useful for applications if inferred from the axi-symmetric theory of the laser beam propagation in a dielectric tube [26,8]. The theory represents an electric field profile at the entrance aperture of the radius $r_0$ as an in-
finite sum \( \tilde{a}(r) = a_0 \sum_{n=1}^{\infty} C_n J_0(k_{n,1} r) \) (Ref. 1), where \( C_n = 2 |r_0 J_1(u_n)|^2 \int_0^{r_0} a(r) J_0(u_n r/r_0) r dr \) is the overlap integral of the hybrid capillary eigenmode \( EH_{n,1} \) with the incident laser profile \( a(r) = \exp(-r^2/\sigma_0^2) \) (Fig. 3). Here, \( u_n \) is the nth zero of the zero-order Bessel function of the first kind, \( J_0(u_n) = 0 \); for \( k_0 r_0 \gg 1 \), and \( n \gg 1 \), \( k_{n,1} r_0 \approx u_n \approx (n + 1/2) \pi \). Figure 3 shows that about 98% of laser energy is coupled to the fundamental mode for \( \sigma_0 = 0.645 r_0 \) (the perfect matching condition); however, the overlap integral decays very slowly as \( n \) grows (the analytic fit, \( C_n \approx F_n = \exp[-(n + 50)^{0.5}] \), is almost exact for \( n > 10 \)). When the ratio \( \sigma_0/\sigma_0 \) drops sharply (e.g., analytic fit \( C_n \approx 3 \times 10^{-6} F_n, n > 10 \)), holds for \( \sigma_0 = 0.258 r_0 \). Therefore, choosing wider capillary is the way of reducing the effect of laser coupling on the forward strongly coupled SRS.

The numerical example of subsection 3.B shows how the SRS under the angle \( \alpha = \pi/6 \) develops in the capillary with an EM seed amplitude characteristic of the SRS under the angle \( \theta; \) and with respect to the seed amplitude with characteristics of the perfect matching condition. However, the overlap integral decays very slowly as \( n \) grows (the analytic fit, \( C_n \approx F_n = \exp[-(n + 50)^{0.5}] \), is almost exact for \( n > 10 \)). When the ratio \( \sigma_0/\sigma_0 \) drops sharply (e.g., analytic fit \( C_n \approx 3 \times 10^{-6} F_n, n > 10 \)), holds for \( \sigma_0 = 0.258 r_0 \). Therefore, choosing wider capillary is the way of reducing the effect of laser coupling on the forward strongly coupled SRS.

\[ \frac{\partial}{\partial x} \tilde{a}_{s,\pm}(x, p, s; r) = \pm K(x), \]  

where \( \Omega = (V_z/V_x) \) is the scattering angle; this spread should not exceed the admissible value established above, \( \Delta \alpha < \cos \alpha / (L_z k_0) \lesssim \cos \alpha / (2 r_0 k_0) \). Fixing \( |\Delta \alpha| \ll \alpha \) and \( |\Delta n| \ll n^* \), one has \( \Delta n \approx (k_0 r_0 / \pi) \cos \alpha \Delta \alpha \ll \cos^2 \alpha / (2 \pi) \ll 1 \), so that only one capillary mode with \( n = n^* \) can contribute to the scattering under the given angle. Therefore, the seed amplitude used for the numerical demonstration of Fig. 3(a) is evaluated as \( a_{s0} = C_{136} \approx 0.4 \times 10^{-5} \).

**APPENDIX B: DERIVATION OF THE EXACT REFLECTIVE SOLUTION**

Omitted \( k_0^2 \) in the LHS of Eq. 30, the Laplace transform (LT) of Eqs. 3 with respect to \( \xi \) (LT variable \( s \)) and with respect to \( z \) (LT variable \( p \)) give the set of ODE for the Laplace images \( \tilde{a}_{s,\pm}(x, p, s; r) \),

\[ \tilde{a}_{s+}(x, p, s; r) = -\left( K_1 / p + K_2 \right) \left[ 1 - \frac{(1 - r) e^{i \Omega x}}{1 - r e^{i \Omega L_x}} \right] \frac{1}{1 + \left[ \Omega \left( L_x / 2 - x \right) - 1 \right]^2} \frac{1}{s \Omega^3} \]

where \( x_n = x + n L_z \). Lateral symmetry gives \( \tilde{a}_{s-}(x) = \tilde{a}_{s+}(L_x - x) \). The Laplace transform inversion of Eq. 32 with respect to \( p \) reads

\[ \tilde{a}_{s+}(x, z, s; r) = -\frac{K_1}{s \Gamma_x} + \frac{K_1}{s \Gamma_x} \frac{e^{\Gamma_x z}}{1 - r} \frac{e^{\Gamma_x z} (x - \tilde{z}) (x - \tilde{z} - L_x)}{(L_x / 2)^2} \left[ H(x - \tilde{z}) - K_1 (1 - r) \sum_{n=0}^{\infty} r^n \frac{e^{\Gamma_x z} (x - \tilde{z}) (x - \tilde{z} - L_x)}{(L_x / 2)^2} \right] H(\tilde{z} - x - n), \]

where, in accordance with Ref. 14, the expansion \( \left( 1 - r \exp(i \Omega L_x) \right)^{-1} = \sum_{n=0}^{\infty} r^n \exp(n i \Omega L_x) \) is used, and \( \tilde{z} \equiv (V_z/V_x) z, \tilde{K}_{1,2} \equiv (V_z/V_x) K_{1,2} \). Equation 33 includes Laplace images of the three types: \( 1/(s \Gamma_x), e^{\Gamma_x z}/(s \Gamma_x) \), and \( e^{\Gamma_x z}/s \). Their inversions read:

\[ L_s^{-1} \left( e^{\Gamma_x z}/s \right) = \sqrt{\pi} H(\xi - z / V_x) \bar{F}_2(1, 1/2; i \xi), \]

expressed through the regularized generalized hypergeometric function 33 of variable \( i \xi = i (G^2/4)(\xi -

---

This text has been formatted into a readable layout. The original text contains mathematical expressions and references, which have been maintained as closely as possible in the natural text format. The content is focused on the derivation of the exact reflective solution for the scattering problem in SRS, including the use of Laplace transforms and the evaluation of certain integrals and expansions.
\[ \frac{z}{V_z^2} = \frac{z}{V_z^2} \]

\[ \mathcal{L}_s^{-1} \left\{ \frac{1}{s \Gamma_s} \right\} = -\frac{V_z^2}{3} \sum_{j=1}^{3} \frac{e^{c_j \xi}}{c_j}, \]  

(B5)

where \( c_3 = iG^3 \), so that \( c_1 = -iG \), \( c_2 = (i + \sqrt{3})G/2 \), \( c_3 = (i - \sqrt{3})G/2 \); and

\[ \mathcal{L}_s^{-1} \left\{ \frac{e^{r \eta}}{s \Gamma_s} \right\} = -\frac{V_z^2}{3} \sum_{j=1}^{3} \frac{e^{c_j \xi}}{c_j} F_s \left( c_j, \frac{y}{V_z}, \xi \right), \]  

(B6)

which is expressed through the fundamental solutions \(^21\).\(^21\)

\[ F_s(\mu, v, \xi) = e^{-\mu \nu} \sum_{n=0}^{\infty} \frac{(\mu \nu)^n}{n!(2n)!} \gamma(2n+1, \mu(\xi-v)) H(\xi-v), \]  

(B7)

where \( \gamma(m, \tau) = \int_{\tau}^{\infty} e^{-\tau' \tau^{-m-1}} d\tau' \) is the incomplete gamma-function of the order \( m \) of a complex variable \( \tau \) (Ref. \(^22\)).\(^22\) The combination expressions \(^{21}\), \(^{20}\), and \(^{24}\) in Eq. \(^{28}\) gives the envelope \( \mathcal{E}_{\text{a}} \) of the up-going EMW. The amplitude \( \mathcal{E}_{\text{a}} \) of the scattering EPW is derived in the similar fashion.

**APPENDIX C: ASYMPTOTIC REFLECTIVE SOLUTION**

The asymptotic can be found by applying the inversion formula, \( \tilde{a}_{s+}(x, z, s) = \frac{2\pi i}{1} \int_{c-i\infty}^{c+i\infty} e^{\mu \nu} \tilde{a}_{s+}(x, p, s) dp \), to the expression \(^{22}\).\(^22\) The asymptotic behavior at \( z \to \infty \) is determined by the singularities of the integrand at \( p = 0 \), \( p = \Gamma_s \), and \( p = n \Gamma_s \), and \( p = n \Gamma_s \) with \( n \) integer and \( \nu = -(V_z/V_s)(\ln r/L_z) \). Expanding \( \tilde{a}_{s+}(x, p, s) \) in the vicinity of the specific points shows that all the singularities at \( p = \Gamma_s \), \( p = n \Gamma_s \) cause the contribution to the asymptotic from these specific points as soon as the distance \( z \) from the entrance plane exceeds \( V_z c \tau_0 \) or 0, for \( \alpha > \pi/2 \). (Actually, the Laplace image singularities at \( p = \Gamma_s \) and \( p = n \Gamma_s \) determine the entrance effect, i.e., the waves produced by the beam at the entrance aperture amplified in the pump field in plasma; these waves will inevitably drop behind the laser pulse and their effect therefore vanishes as \( z \to \infty \).) The contribution from \( p = 0 \), \( \tilde{a}_{s+}(x, z, \to \infty, s) = \frac{1}{s \Gamma_s} \left\{ 1 - (1-r) \exp[(V_z/V_s)\Gamma_s x] \right\}, \]  

thus determines the long-scale evolution of the instability in plasma, which is dominated by the lateral reflections of scattered EMW. Neither \( s = c_1 \) (where \( \Gamma_s = 0 \)) nor \( s = 0 \) are the singularities of the image \( \tilde{a}_{s+}(x, z, \to \infty, s) \), so contributions to the asymptotic originate from the singular points \( s_n \) only, which are the solutions of the equation \( \Gamma_{s_n} = \nu_n \) here, \( \nu_n = (V_z/V_s)(\ln r + \pi i n/L_z) \), or \( s_n^3 + \nu_n V_z s_n^2 - iG^3 = 0, n \) is integer. The fundamental specific point \( s_0 \) with the maximum real part is the root of the cubic equation \( s_n^3 + \nu_0 V_z s_n^2 - iG^3 = 0 \), which, for arbitrary \( \nu_0 \), admits quite a cumbersome explicit expression. We address to the physically interesting limit of “low leakage,” i.e., \( \nu_0 < 3G/V_z \), or \( r > \exp(-3GL_z/V_z) \), which means that the scattered EMW gains more energy between two reflections than loses due to leakage through the wall at one reflection. In this case the lateral convection is mostly suppressed. Solution obtained via the perturbation approach reads \( s_0 \approx (i + \sqrt{3})G/2 - \nu_0 V_z/3 \). Contribution to the asymptotic from that point is of the order of \( e^{i\pi \xi} \), which grows in time with an increment \( \Re s_0 \). However, the absolute value of \( \nu_0 \) grows with \( n \), so an evaluation has to be done of the contribution from the points \( s_n \) with large \( n \). We again use the perturbation approach with the small parameter \( \mu_n = G/|n\Gamma_s| V_z \), that is, \( n > GL_z/(2\pi V_z) \), and find the solutions \( s_n^{(1)} \approx -\nu_n V_z - iG\mu_n, s_n^{(2,3)} \approx \pm G\sqrt{-(\text{sign} n)\mu_n} \). Obviously, \( \Re s_j \ll G \) for \( \mu_n \ll 1 \), so that contribution from these points to the asymptotic is negligible compared to that from the points \( s_n \) with \( n < GL_z/(2\pi V_z) \), which is of the order of \( e^{i\pi \xi} \). Expanding the image \( \tilde{a}_{s+}(x, z, \to \infty, s) \) in the vicinity of \( s = s_0 \), we arrive at the expression asymptotically valid for \( G \gg 1 \), \( a_{s+}(x, \xi) \approx \frac{V_z}{G} \left( \frac{1-r}{\ln r} \right) \frac{i-\sqrt{3}}{6} \exp \left( s_0 \xi - \ln r \frac{x}{L_z} \right), \]  

which represents the unstable EMW as a quasi-1D exponentially growing mode.

---

[1] M. D. Perry and G. Mourou, Science 64, 917 (1994), and references therein.

[2] N. H. Burnett and G. D. Enright, IEEE J. Quantum Electron. QE-26, 1797 (1990).

[3] X. F. Li, A. L'Huillier, M. Ferray, L. A. Lampré, and G. Mainfray, Phys. Rev. A 39, 3751 (1989).

[4] T. Tajima and J. M. Dawson, Phys. Rev. Lett. 43, 267 (1979).

[5] N. E. Andreev, L. M. Gorbunov, V. I. Kirsanov, A. A. Pogosova, and R. R. Ramazashvili, JETP Lett. 55, 571 (1992); P. Sprangle, E. Esarey, J. Krall, and C. Joyce, Phys. Rev. Lett. 69, 2200 (1992); T. Antonsen, Jr. and P. Mora, ibid. 69, 2204 (1992).

[6] E. Esarey, P. Sprangle, J. Krall, and A. Ting, IEEE Trans. Plasma Sci. PS-24, 292 (1996).

[7] E. A. J. Marcatili and R. A. Schmeltzer, Bell Syst. Tech. J. 43, 1783 (1964).

[8] B. Cros, C. Courtois, G. Mattheiuessent, A. D. Bernardo,
