Three Symmetrical Systems of Coupled Sylvester-like Quaternion Matrix Equations

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Abstract: The current study investigates the solvability conditions and the general solution of three symmetrical systems of coupled Sylvester-like quaternion matrix equations. Accordingly, the necessary and sufficient conditions for the consistency of these systems are determined, and the general solutions of the systems are thereby deduced. An algorithm and a numerical example are constructed over the quaternions to validate the results of this paper.

Keywords: Sylvester-like matrix equation; quaternion; matrix equations; rank; generalized inverse

MSC: 15A03; 15A09; 15A24; 15B33

1. Introduction

Sylvester-like matrix equations and their generalizations have applications in many scientific fields such as graph theory [1], image processing [2], output feedback control [3], neural networks [4], eigenvalue assignment problems [5], robust control [6] and so on (for example, see [7–17]). The iterative methods of solving linear and nonlinear Sylvester-like matrix equations are built on numerical solution methods [18–27]. So far, there has not been fruitful information on the solvability conditions and the general solution to systems of mixed and coupled Sylvester-like quaternion matrix equations. For the sake of understanding, analyzing, and developing theoretical studies of generalized systems of Sylvester coupled matrix equations and their applications, we, in this paper, consider the solvability conditions and the general solutions of the following three symmetrical systems of coupled Sylvester-like quaternion matrix equations:

\[
\begin{align*}
A_1V &= C_1, & VB_1 &= C_2, \\
A_2X + YB_3 &= C_3, \\
A_2Y + ZB_2 + A_5VB_5 &= C_5, \\
A_4W + ZB_4 &= C_4, \\
A_1V &= C_1, & VB_1 &= C_2, \\
A_3X + YB_3 &= C_3, \\
A_2Z + YB_2 + A_5VB_5 &= C_5, \\
A_4Z + WB_4 &= C_4, \\
A_1V &= C_1, & VB_1 &= C_2, \\
A_3X + YB_3 &= C_3, \\
A_2Y + ZB_2 + A_5VB_5 &= C_5, \\
A_4Z + WB_4 &= C_4,
\end{align*}
\]

where \( A_i, B_i \) and \( C_i \) (\( i = 1,5 \)) are the given matrices with compatible dimensions, while \( X, Y, Z, W, \) and \( V \) are the unknown matrices over \( \mathbb{H} \). This work is motivated by a variety...
of well-known results on the solvability conditions and the general solutions of linear matrix equations in the literature, especially the findings of the following generalized Sylvester matrix equation:

\[ A_1 X + Y B_1 = C_1. \]  

Baksalary and Kala [28] came up with the general solution to (4). Lee and Vu [29] conducted a thorough examination of the general solution and solvability conditions for the following coupled system of matrix equations:

\[
\begin{cases}
A_1 X + Y B_1 = C_1, \\
A_2 Z + Y B_2 = C_2,
\end{cases}
\]  

where \( X, Y \) and \( Z \) are unknowns over the complex field with compatible sizes. Wang and He [30] derived the solvability conditions and the general solution for the following systems of symmetrical coupled Sylvester-like matrix equations:

\[
\begin{align*}
A_1 X + Y B_1 &= C_1, \\
A_2 Z + Y B_2 &= C_2, \\
A_3 W + Z B_3 &= C_3,
\end{align*}
\]  

Meanwhile, Liu et al. [31] derive the determinantal representations of the general solution to (6). The solvability conditions and the general solution formulas for the matrix equations:

\[ AX = B, \quad XC = D, \]  

and

\[ A_1 X_1 + Y_1 B_1 + C_1 Z D_1 = E_1 \]  

are given by [32,33], respectively, which have a fundamental role in inferring the main results of this paper. Here, we provide proper generalizations to the systems of Sylvester-like matrix Equations (4)–(8). In particular, we reproduce the results of [11,29,33,34], see Section 3.1. In [30,33], He and Wang presented the solvability conditions and the general solution for the following Sylvester-like matrix equation:

\[
A_1 X_1 + X_2 B_1 + C_3 X_3 D_3 + C_4 X_4 D_4 = E_1. \]  

They presented these findings in terms of the Moore–Penrose inverse of certain given matrices. The resulting equation has wide applications in the linear matrix equations field, see [7,34–40]. He and Meng [41] derived the rank equalities solvability conditions for the following quaternion matrix equation with two different restrictions:

\[ A_1 Z_1 B_1 + A_2 Z_2 B_2 + A_3 Z_3 B_3 = C_1, \quad D Z_1 = F, \quad Z_1 H = G. \]  

The solvability conditions and the general solution to the Sylvester-like matrix equation

\[ \hat{A}_1 X_1 + Y_1 \hat{B}_1 + \hat{A}_2 Z_1 \hat{B}_2 + \hat{A}_3 Z_2 \hat{B}_3 + \hat{A}_4 Z_3 \hat{B}_4 = \hat{C}_1 \]  

were given by [42], which is crucial to the proof in this study.

We organize this paper as follows. Some definitions and lemmas are reviewed in Section 2. In Section 3, we carry out the solvability conditions and the general solution of (1). In Section 4, we consider similarly the systems (2) and (3). To close this paper, we give a conclusion in Section 5.
2. Preliminaries

We denote the real number field by \( \mathbb{R} \), while \( \mathbb{H} \) stands for the quaternion algebra, which considered as a division ring \([12,43,44]\). \( \mathbb{H}^{l \times k} \) expresses the set of all \( l \times k \) matrices over \( \mathbb{H} \). The symbol \( A^* \) represents the conjugate transpose of \( A \). The Moore–Penrose inverse of \( A \in \mathbb{H}^{l \times k} \) is denoted by \( A^* \in \mathbb{H}^{k \times l} \). Furthermore, \( L_A = I - AA^* \), \( R_A = I - A^*A \) stand for the two projectors of \( A \) and \( r(A) \) represents the rank of the matrix \( A \).

**Lemma 1** ([45]). Let \( A \in \mathbb{H}^{m \times n} \), \( B \in \mathbb{H}^{m \times k} \) and \( C \in \mathbb{H}^{l \times n} \) be given. Then

\[
(1) \quad r(A) + r(R_AB) = r(B) + r(R_BA) = r(A^*B), \\
(2) \quad r(A) + r(CL_A) = r(C) + r(AL_C) = r(A^*C).
\]

**Lemma 2** ([32]). Let \( A, B, C \) and \( D \) be given. Then the quaternion matrix Equation (7) is solvable if and only if

\[
R_AB = 0, \quad DL_C = 0, \quad AD = BC.
\]

In this case, the general solution of (7) can be expressed as

\[
X = A^*B + L_ADC^* + L_AVRC,
\]

where \( V \) is an arbitrary matrix over \( \mathbb{H} \) with suitable size.

**Lemma 3** ([28]). Let \( A_1, B_1 \) and \( C_1 \) be given. Then the quaternion matrix Equation (4) is solvable if and only if

\[
R_{A_1}C_1L_{B_1} = 0.
\]

In this case, the general solution of (4) can be expressed as

\[
X = A_1^*C_1 + U_1B_1 + L_{A_1}U_2, \\
Y = R_{A_1}C_1B_1^* + A_1U_1 + U_3R_{B_1},
\]

where \( U_i, \ (i = 1,3) \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

**Lemma 4** ([42]). Let \( \bar{A}_i, \bar{B}_i \ (i = \overline{1,4}) \) and \( \bar{C}_1 \) be given. For simplicity, set

\[
A_{ii} = \overline{A}_{1i} \overline{A}_{i1}, \quad B_{ii} = \overline{B}_{i1}L_{\overline{B}_i} \quad (i = \overline{1,3}), \quad M_1 = \overline{R}_{A_{11}}A_{22}, \quad S_1 = \overline{A}_{22}L_{M_1}, \\
N_1 = \overline{B}_{22}L_{\overline{B}_1}, \quad T_1 = \overline{R}_{A_{1i}}\overline{C}_1L_{\overline{B}_i}, \quad T = T_1 - A_{33}Z_3B_{33}, \quad P = (A_{11}, \quad A_{22}), \\
Q = \begin{pmatrix} \overline{B}_{11} \\ \overline{B}_{22} \end{pmatrix}, \quad H_1 = B_{33}, \quad H_2 = B_{33}L_{\overline{B}_2}, \quad H_3 = B_{33}L_{\overline{B}_1}, \quad H_4 = B_{33}L_Q, \quad G_1 = R_{P}A_{33}, \\
G_2 = R_{A_{11}}A_{33}, \quad G_3 = R_{A_{22}A_{33}}, \quad G_4 = A_{33}, \quad L_1 = R_P T_1, \quad L_2 = R_{A_{11}}T_1L_{\overline{B}_2}, \\
L_3 = R_{A_{22}T_1L_{\overline{B}_1}}, \quad L_4 = T_1L_Q, \quad \overline{A}_1 = (L_G, \quad L_{\overline{G}}), \quad \overline{B}_1 = \begin{pmatrix} \overline{R}_{H_1} \\ \overline{R}_{H_2} \end{pmatrix}, \quad \overline{C}_1 = L_{\overline{G}_1}, \\
\overline{D}_1 = R_{H_2}, \quad \overline{A}_2 = R_{A_{11}}\overline{C}_1, \quad \overline{B}_2 = \overline{D}_1L_{\overline{B}_1}, \quad \overline{C}_2 = L_{\overline{G}_1}, \quad \overline{D}_2 = R_{H_4}, \quad \overline{A}_3 = R_{A_{11}}\overline{C}_2, \\
\overline{B}_3 = \overline{D}_2L_{\overline{B}_1}, \quad M = R_{A_{11}}\overline{A}_3, \quad N = \overline{B}_3L_{\overline{B}_1}, \quad S = \overline{A}_3L_M, \quad Q_0^{11} = G_1^*L_1H_1^* + L_G^*G_2^*L_2H_2^*, \\
Q_0^{22} = G_3^*L_3H_3^* + L_G^*G_4^*L_4H_4^*, \quad \overline{Q} = Q_0^{22} - Q_0^{11}, \quad E = \overline{R}_{A_{1i}}\overline{Q}L_{\overline{B}_i}.
\]
Then the following statements are equivalent:

(1) The matrix Equation (11) is solvable.

(2) 
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1),
\]

(3)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(4)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(5)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(6)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(7)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(8)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(9)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(10)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(11)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(12)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(13)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(14)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(15)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(16)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(17)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(18)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(19)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(20)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(21)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

(22)
\[
\begin{vmatrix}
C_1 & A_2 & A_3 & A_4 & A_1 \\
B_1 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
B_2 \\
B_3 \\
B_4 \\
B_1 \\
\end{vmatrix}
= r(A_1) + r(B_1). 
\]

In this case, the general solution of (11) can be expressed as

\[
X_1 = \hat{A}_1^T (C_1 - \hat{A}_2 Z_1 B_2 - \hat{A}_3 Z_2 B_3 - \hat{A}_4 Z_3 B_4) - P_1 B_2 + L_{A_1} P_2,
\]

\[
Y_1 = R_{\hat{A}_1} (C_1 - \hat{A}_2 Z_1 B_2 - \hat{A}_3 Z_2 B_3 - \hat{A}_4 Z_3 B_4) B_1^T + \hat{A}_1 P_1 + P_3 R_{B_1},
\]

\[
Z_1 = A_{11}^T B_{11}^T - A_{11}^T A_{22} M_1^T B_{11}^T - A_{11}^T S_1 A_{22} T N_1^T B_{22} B_{11}^T - A_{11}^T S_1 P_3 R_{N_1} B_{22} B_{11}^T + L_{A_1} P_5 + P_6 R_{B_1},
\]

\[
Z_2 = M_1^T B_{22} + S_1^T S_1 A_{22} T N_1^T + L_{M_1} L_{S_1} P_7 + P_8 R_{B_2} + L_{M_1} U_4 R_{N_1},
\]
Theorem 1. Let $A, B$ and $C$ be given. For simplicity, consider (12) such that

$$
\hat{A}_1 = A_2 A_3, \quad \hat{B}_1 = R_{B_i} B_2, \quad \hat{A}_2 = A_2, \quad \hat{B}_2 = R_{B_3}, \quad \hat{A}_3 = A_4, \quad \hat{B}_3 = B_2, \quad \hat{A}_4 = A_5 L_{A_1}, \\
\hat{B}_4 = R_{B_1} B_3, \quad C_1 = C_5 - [A_2 R_{A_3} C_5 B_3^I + R_{A_3} C_4 B_4 B_2^I + A_3 (A_4 C_1 + L_{A_1} C_2 B_4^I)] B_3.
$$

Then the following statements are equivalent:

(1) The system (1) is solvable.

(2) $R_{A_1} C_1 = 0, \quad C_2 L_{B_i} = 0, \quad A_1 C_2 = C_1 B_1, \quad R_{A_1} C_i L_{B_i} = 0 \quad (i = 3, 4), \\
R_{C_i} L_i = 0, \quad L_i L_{H_i} = 0 \quad (i = 1, 4), \quad R_{A_3} E L_{B_2} = 0.

(3)

$$
\begin{align*}
\text{r}(A_1) & = r(A_1), \quad \text{r}(\frac{C_2}{B_1}) = r(B_1), \quad A_1 C_2 = C_1 B_1, \\
\text{r}(\frac{C_i}{B_i}) & = r(A_i) + r(B_i), \quad (i = 3, 4), \\
\text{r}(\frac{C_5}{B_5}) & = r(B_5), \quad \text{r}(\frac{C_4}{B_4}) = r(B_4), \quad \text{r}(\frac{A_2}{A_1}) = r(B_3), \quad \text{r}(\frac{A_4}{A_3}) = r(B_2), \\
\text{r}(\frac{A_5}{A_4}) & = r(B_1), \quad \text{r}(\frac{A_5}{A_4}) = r(B_1).
\end{align*}
$$
\[
\begin{align*}
& r \left( \begin{array}{cccc}
C_3 B_3 - A_2 C_3 & A_5 & A_4 & A_2 A_3 & C_4 \\
B_2 B_3 & 0 & 0 & 0 & B_4 \\
C_1 B_3 & A_1 & 0 & 0 & 0
\end{array} \right) = r \left( \begin{array}{cccc}
A_5 & A_2 A_3 & A_4 \\
A_1 & 0 & 0 \\
\end{array} \right) + r (B_2 B_3, B_4), \quad (30) \\
& r \left( \begin{array}{cccc}
C_5 B_5 - A_2 C_3 & A_5 & A_2 A_3 \\
B_2 B_3 & 0 & 0 & 0 \\
C_1 B_3 & A_1 & 0
\end{array} \right) = r \left( \begin{array}{cccc}
A_5 & A_2 A_3 \\
A_1 & 0 \\
\end{array} \right) + r (B_2 B_3), \quad (31) \\
& r \left( \begin{array}{cccc}
C_5 B_5 - A_2 C_3 & A_5 & A_2 A_3 & A_4 & C_4 \\
B_2 B_3 & 0 & 0 & 0 & B_4 \\
C_1 B_3 & A_1 & 0
\end{array} \right) = r \left( \begin{array}{cccc}
B_5 B_3 & B_1 & 0 \\
B_2 B_3 & 0 & 0 & B_4
\end{array} \right) + r (A_4, A_2 A_3), \quad (32) \\
& r \left( \begin{array}{cccc}
C_5 & A_2 & A_5 C_2 \\
B_2 & 0 & 0 \\
B_5 & 0 & B_1
\end{array} \right) = r (A_2) + r \left( \begin{array}{cccc}
B_2 & 0 & 0 \\
B_5 & B_1
\end{array} \right), \quad (33) \\
& r \left( \begin{array}{cccc}
C_5 & A_2 & 0 & 0 & A_5 & 0 & 0 \\
B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -C_5 B_3 + A_2 C_3 & A_4 & A_2 A_3 & A_5 & -C_4 & -A_5 C_2 \\
0 & 0 & B_2 B_3 & 0 & 0 & 0 & B_4 & 0 \\
B_5 & 0 & B_2 B_3 & 0 & 0 & 0 & B_1 \\
C_1 B_3 & 0 & 0 & 0 & 0 & A_1 & 0 & 0
\end{array} \right) \nonumber \\
& = r \left( \begin{array}{cccc}
B_2 & 0 & 0 & 0 \\
0 & B_2 B_3 & B_4 & 0 \\
0 & B_5 B_3 & 0 & B_1
\end{array} \right) + r \left( \begin{array}{cccc}
A_2 & 0 & 0 & A_5 \\
A_4 & A_2 A_3 & A_5 \\
0 & 0 & 0 & A_1
\end{array} \right). \quad (34)
\end{align*}
\]

In this case, the general solution of (1) can be expressed as

\[
\begin{align*}
X &= A_1^+ C_3 + U_1 B_3 + L_{\hat{A}} U_2, \quad (35) \\
Y &= R_{A_1} C_3 B_4^+ + A_3 U_1 + U_3 R_{B_3}, \quad (36) \\
W &= A_1^+ C_4 + W_4 B_4 + L_{A_1} W_2, \quad (37) \\
Z &= R_{A_1} C_4 B_4^+ + A_4 W_1 + W_3 R_{B_4}, \quad (38) \\
V &= A_1^+ C_1 + L_{A_1} C_2 B_4^+ + L_{A_1} Q R_{B_4}, \quad (39)
\end{align*}
\]

where

\[
\begin{align*}
W_2 &= A_1^+ \left( C_1 - A_2 W_1 B_2 - A_3 U_1 B_3 - A_4 Q B_4 \right) - P_1 B_1 + L_{A_1} P_2, \\
U_3 &= R_{A_1} \left( C_1 - A_2 W_1 B_2 - A_3 U_1 B_3 - A_4 Q B_4 \right) B_1^+ + A_1 P_1 + P_3 R_{B_4}, \\
W_1 &= A_1^+ T B_1^+ - A_1^+ A_2 M_1^+ T B_1^+ - A_1^+ S_1 A_2^+ T N_1^+ B_2 B_1^+ \\
&\quad - A_1^+ S_1 P_4 R_{B_1} B_2 B_1^+ + L_{A_1} P_5 + P_6 R_{B_1}, \\
U_1 &= M_1^+ T B_2 + S_1 A_2^+ T N_1^+ + L_{A_1} U_{11} + L_{A_1} U_{12}, \\
Q &= Q_1^0 + L_{A_1} U_{11} + J_2 R_{H_1} + L_{A_1} R_{H_2}, \\
or \quad Q &= Q_2^0 - L_{A_1} K_1 - K_2 R_{H_2} - L_{A_1} K_3 R_{H_4}, \\
J_1 &= \left( I_n 0 \right) \left[ A_1^+ \left( \overline{Q} - C_1^+ J_3 D_1 \right) - C_2 K_3 D_2 \right] - U_{11} B_1 + L_{A_1} U_{12}, \\
K_1 &= \left( 0 I_n \right) \left[ A_1^+ \left( \overline{Q} - C_1^+ J_3 D_1 \right) - C_2 K_3 D_2 \right] - U_{11} B_1 + L_{A_1} U_{12}, \\
J_2 &= \left[ R_{A_1} \left( \overline{Q} - C_1^+ J_3 D_1 \right) - C_2 K_3 D_2 \right] B_1^+ + A_1 U_{11} + U_{21} R_{B_1} \left( I_n \right), \\
K_2 &= \left[ R_{A_1} \left( \overline{Q} - C_1^+ J_3 D_1 \right) - C_2 K_3 D_2 \right] B_1^+ + A_1 U_{11} + U_{21} R_{B_1} \left( 0 \right).
\end{align*}
\]
where \( P_i (i = 1, 8) \), \( U_2, W_2, U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41} \) and \( U_{42} \) are arbitrary matrices over \( \mathbb{H} \), \( m \) and \( n \) are the number of columns of \( A_1 \) and the number of rows of \( B_1 \), respectively.

**Proof of Theorem 1.** \((1) \iff (2)\) : Under the conditions \( R_{A_i} C_1 = 0, C_2 L_{B_1} = 0, A_1 C_2 = C_1 B_1 \), the matrix equations \( A_1 V = C_1, VB_1 = C_2 \) have a common solution. By using Lemma 2, we have the general common solution as

\[
V = A_1^\dagger C_1 + L A_i C_2 B_1^\dagger + L A_1 Q R_{B_1}, \quad (40)
\]

where \( Q \) is an arbitrary matrix over \( \mathbb{H} \) with suitable size. For the matrix equations

\[
A_3 X + Y B_3 = C_3, \quad (41)
A_4 Z + W B_4 = C_4, \quad (42)
\]

when the conditions \( R_{A_i} C_i L_{B_1} = 0 (i = 1, 3, 4) \) are met, we have that \((41)\) and \((42)\) are solvable, respectively. By using Lemma 3, their general solutions can be expressed as

\[
X = A_3^\dagger C_3 + U_4 B_1 + L A_4 U_2, \quad Y = R_{A_3} C_3 B_1^\dagger + A_3 U_1 + U_3 R_{B_1}, \quad (43)
Z = A_4^\dagger C_4 + W_4 B_1 + L A_4 W_2, \quad W = R_{A_4} C_4 B_1^\dagger + A_4 W_1 + W_3 R_{B_1}, \quad (44)
\]

where \( W_i \) and \( V_i \) \((i = 1, 3)\) are arbitrary matrices with suitable sizes. Substituting \((40), (43)\) and \((44)\) into the equation

\[
A_2 Z + Y B_2 + A_5 V B_5 = C_5, \quad (45)
\]

we have the following Sylvester-like matrix equation:

\[
A_1 W_2 + U_3 B_1 + A_2 W_1 B_2 + A_3 U_1 B_3 + A_4 Q R_{B_4} = \dot{C}_1. \quad (46)
\]

Now, by applying Lemma 4, the Equation \((46)\) has a solution if and only if

\[
R_{G_i} L_i = 0, \quad L_i L_{H_i} = 0 \quad (i = 1, 4), \quad R_{A_3} E L_{B_2} = 0. \quad (47)
\]

In this case, the expression of the general solution of \((46)\) can be expressed as

\[
W_2 = \dot{A}_1^\dagger (\dot{C}_1 - \dot{A}_2 W_1 B_2 - \dot{A}_3 U_1 B_3 - \dot{A}_4 Q R_{B_4}) - P_1 B_1 + L A_1 P_2, \quad (48)
U_3 = R_{\dot{A}_1} (\dot{C}_1 - \dot{A}_2 Z_1 B_2 - \dot{A}_3 Z_2 B_3 - \dot{A}_4 Z_3 B_4) B_1^\dagger + A_1 P_1 + P_3 R_{B_1}, \quad (49)
W_1 = A_1^\dagger T B_1 + A_1^\dagger T A_2 M_3 T B_1^\dagger - A_1^\dagger T S_1 A_2^\dagger T N_1^\dagger B_2^\dagger \quad (50)
- A_1^\dagger T S_1 P_4 R_{N_1} B_2^\dagger + L A_1 P_3 + P_6 R_{B_1}, \quad (51)
U_1 = M_1^\dagger T B_1 + S_1^\dagger S_1^\dagger T N_1^\dagger + L M_1 L_{S} P_7 + P_8 R_{B_2} + L M_1 U_{4} R_{N_1}, \quad (52)
Q = Q_{11}^\dagger + L G_1 J_1 + J_2 R_{H_1} + L G_1 J_3 R_{H_1}, \quad (53)
\]

or

\[
Q = Q_{52}^\dagger - L G_4 K_1 - K_2 R_{H_3} - L G_3 K_3 R_{H_4}. \quad (54)
\]
where
\[
\begin{align*}
\begin{pmatrix} I_1 \\ K_1 \end{pmatrix} &= \tilde{A}_1^0 \begin{pmatrix} \mathcal{Q} - \tilde{C}_1 \mathcal{D}_1 - \tilde{C}_2 \mathcal{D}_2 \end{pmatrix} - U_{11} \bar{B}_1 + L_{\tilde{A}_1} U_{12}, \\
\begin{pmatrix} j_2 \\ k_2 \end{pmatrix} &= R_{\tilde{A}_1} \left( \mathcal{Q} - \tilde{C}_1 \mathcal{D}_1 - \tilde{C}_2 \mathcal{D}_2 \right) \mathcal{B}_1^0 + \tilde{A}_1 U_{11} + U_{21} R_{\bar{B}_1}, \\
\begin{pmatrix} j_3 \\ k_3 \end{pmatrix} &= \tilde{A}_2^0 \mathcal{Q} \mathcal{B}_2^0 - A_2^0 A_3 M^0 \mathcal{Q} \mathcal{B}_2^0 - A_2^0 S A_3^0 \mathcal{Q} N^0 \mathcal{B}_3^0 + L_{\tilde{A}_2} U_{32} + U_{33} R_{\bar{B}_1}, \\
K_3 &= M^0 \mathcal{Q} \mathcal{B}_3^0 + S^0 S A_3^0 \mathcal{Q} N^0 + L_M L_5 U_{41} + U_{42} R_{\bar{B}_3} + L_M U_{31} R_N,
\end{align*}
\]

and \( P_i (i = 1, 8) \), \( U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41} \) and \( U_{42} \) are arbitrary matrices.

(2) \(\Leftrightarrow\) (3): Under the conditions in (47), we have that the system (1) is solvable and hence there is a special solution \( X_0, Y_0, Z_0, W_0 \) and \( V_0 \) such that
\[
\begin{align*}
A_1 V_0 &= C_1, \quad V_0 B_1 = C_2, \\
A_3 X_0 + Y_0 B_3 &= C_3, \\
A_2 Z_0 + Y_0 B_2 + A_5 V_0 B_5 &= C_5, \\
A_4 Z_0 + W_0 B_4 &= C_4.
\end{align*}
\]

By using Lemma 1, it is easy to check that:
\[
R_{A_i} C_0 = 0 \Leftrightarrow r \begin{pmatrix} C_1 & A_1 \end{pmatrix} = r(A_1), \quad C_2 L_{B_i} = 0 \Leftrightarrow r \begin{pmatrix} C_2 \\ B_1 \end{pmatrix} = r(B_1),
\]
\[
R_{A_i} C_i L_{B_i} = 0 \Leftrightarrow r \begin{pmatrix} C_i \\ B_i \\ 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 3, 4).
\]

By Lemma 4, it suffices to show that the conditions (13)–(21) are equivalent to the conditions (26)–(34), respectively.

(13) \(\Leftrightarrow\) (22)
\[
r \begin{pmatrix} C_1 & A_2 & A_3 & A_5 L_{A_1} & A_2 L_{A_4} \\ R_{B_3} B_2 & 0 & 0 & 0 & 0 \end{pmatrix} = r \left( R_{B_3} B_2 \right) + r \left( A_2 & A_3 \right) \begin{pmatrix} A_5 & A_2 & A_4 \\ A_1 & 0 & 0 \end{pmatrix} \Leftrightarrow (26).
\]

(14) \(\Leftrightarrow\) (22)
\[
r \begin{pmatrix} C_1 & A_2 A_3 \\ R_{B_3} & 0 \\ B_2 & 0 \end{pmatrix} = r(A_2 A_3) + r \begin{pmatrix} R_{B_3} B_2 \\ R_{B_3} B_5 \end{pmatrix} \Leftrightarrow (27).
\]

(15) \(\Leftrightarrow\) (22)
\[
r \begin{pmatrix} C_5 & A_2 A_3 & A_2 A_3 & A_5 C_2 \\ I & 0 & B_3 & 0 \\ B_2 & 0 & 0 & 0 \\ B_5 & 0 & 0 & B_1 \end{pmatrix} = r(A_2 A_3) + r \begin{pmatrix} I & B_3 & 0 \\ B_2 & 0 & 0 \\ B_5 & 0 & B_1 \end{pmatrix} \Leftrightarrow (27).


\begin{align}
(15) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_1 & A_2 & A_4 & A_2A_3 \\
R_{B_1}B_5 & 0 & 0 & 0 \\
R_{B_1}B_2 & 0 & 0 & 0 
\end{pmatrix} = r \begin{pmatrix}
R_{B_1} & B_5 \\
R_{B_1}B_2 & 0 & 0 & 0 
\end{pmatrix} + r(\begin{pmatrix} A_2 & A_4 & A_2A_3 \end{pmatrix}) \\
\end{align}

\begin{align}
(16) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_5 & A_2 & A_4 & A_5C_2 \\
B_5 & 0 & 0 & B_1 & 0 \\
B_2 & 0 & 0 & 0 & B_4 
\end{pmatrix} = r \begin{pmatrix}
B_5 & B_1 & 0 \\
B_2 & 0 & 0 & B_4 
\end{pmatrix} + r(\begin{pmatrix} A_2 & A_4 \end{pmatrix}) \iff (28). \\
\end{align}

\begin{align}
(17) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_1 & A_2 & A_5L_{A_1} & A_2A_3 \\
R_{B_1} & 0 & 0 & 0 \\
R_{B_1}B_2 & 0 & 0 & 0 
\end{pmatrix} = r \begin{pmatrix}
R_{B_1} & B_2 \\
R_{B_1}B_2 & 0 & 0 & 0 
\end{pmatrix} + r(\begin{pmatrix} A_2 & A_5L_{A_1} & A_2A_3 \end{pmatrix}) \\
\end{align}

\begin{align}
(18) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_1 & A_5L_{A_1} & A_2A_3 \\
R_{B_1} & 0 & 0 \\
R_{B_1}B_2 & 0 & 0 
\end{pmatrix} = r \begin{pmatrix}
R_{B_1} & B_2 \\
R_{B_1}B_2 & 0 & 0 
\end{pmatrix} + r(\begin{pmatrix} A_5L_{A_1} & A_2A_3 \end{pmatrix}) \\
\end{align}

\begin{align}
(19) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_1 & A_2 & A_5A_3 \\
R_{B_1} & 0 & 0 \\
R_{B_1}B_5 & 0 & 0 \\
R_{B_1}B_2 & 0 & 0 
\end{pmatrix} = r \begin{pmatrix}
R_{B_1} & B_5 \\
R_{B_1}B_2 & 0 & 0 
\end{pmatrix} + r(\begin{pmatrix} A_4 & A_2A_3 \end{pmatrix}) \\
\end{align}

\begin{align}
(20) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_5 & A_4 & A_2A_3 & A_5C_2 \\
I & 0 & 0 & 0 \\
B_5 & 0 & 0 & B_4 \\
B_2 & 0 & 0 & 0 
\end{pmatrix} = r \begin{pmatrix}
I & B_3 & 0 \\
B_2 & 0 & 0 & B_4 
\end{pmatrix} + r(\begin{pmatrix} A_5 & A_2 & A_3 & A_4 \end{pmatrix}) \iff (30). \\
\end{align}

\begin{align}
(21) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_5 & A_5 & A_3A_2A_3 \\
I & 0 & 0 & B_3 \\
B_5 & 0 & 0 & B_4 \\
B_2 & 0 & 0 & 0 
\end{pmatrix} = r \begin{pmatrix}
I & B_3 \\
B_2 & 0 & 0 & B_4 
\end{pmatrix} + r(\begin{pmatrix} A_5 & A_2A_3 & A_4 \end{pmatrix}) \iff (31). \\
\end{align}

\begin{align}
(22) \quad & \text{Elementary rows (columns) operations, Equation (48)} \\
& r \begin{pmatrix}
\tilde{C}_5 & A_4 & A_2A_3 \\
I & 0 & 0 \\
R_{B_1}B_5 & 0 & 0 \\
R_{B_1}B_2 & 0 & 0 
\end{pmatrix} = r \begin{pmatrix}
R_{B_1} & B_5 \\
R_{B_1}B_2 & 0 & 0 
\end{pmatrix} + r(\begin{pmatrix} A_4 & A_2A_3 \end{pmatrix}) \iff (32). \\
\end{align}
3.1. Some Special Cases to the System (1)

In this section, we consider some special cases of the system (1). The following corollary coincides with one of the essential findings in [33].

**Corollary 1.** Let $A_1, B_1, C_1, D_1$ and $E_1$ be given. Set

$$A_3 = R_{A_1}C_1, \quad B_3 = D_1L_{B_1}, \quad C_3 = R_{A_1}E_1L_{B_1}.$$ 

Then the following statements are equivalent:

1. The matrix Equation (8) is solvable.
2. $R_{A_3}C_3 = 0, \quad C_3L_{B_3} = 0.$
Let $A_i$, $B_i$ and $C_i$ ($i = 1, 3$) be given. Set

$$A = R_{(A_2A_1)}A_2, \quad B = R_{B_2}L_{(R_{B_3}B_2)}, \quad C = R_{(A_2A_1)}A_3, \quad D = B_2L_{(R_{B_3}B_2)},$$

$$C_4 = C_2 - A_2R_{A_1}C_1B_1^\dagger - R_{A_3}C_3B_3^\dagger B_2, \quad E = R_{(A_2A_1)}C_4L_{(R_{B_3}B_2)},$$

$$M = R_{AC}, \quad N = DL_B, \quad S = CL_M.$$

Then the following statements are equivalent:

(1) The system

$$\begin{align*}
A_1X + YB_1 &= C_1, \\
A_2Y + ZB_2 &= C_2, \\
A_3W + ZB_3 &= C_3,
\end{align*}$$

is solvable.

(2) $R_{A_i}C_iL_{B_i} = 0, \quad R_MR_AE = 0, \quad E_LB_LN = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$

(3)

$$r\begin{pmatrix} E_1 & C_1 & A_1 \\ B_1 & 0 & 0 \end{pmatrix} = r(C_1 + r(B_1), \quad r\begin{pmatrix} E_1 & A_1 \\ D_1 & 0 \end{pmatrix} = r(A_1) + r(D_1 + B_1).$$

In this case, the general solution of (8) can be expressed as

$$X = A_1^\dagger(E_1 - C_1ZD_1) + W_1B_1 + L_{A_1}W_2, \quad Y = R_{A_1}(E_1 - C_1ZD_1)B_1^\dagger + A_1W_1 + W_3R_{B_1},$$

$$Z = A_2^\dagger C_3B_3^\dagger + L_{A_3}U + VR_{B_3},$$

where $W_1, W_2, W_3, U$ and $V$ are arbitrary matrices over the $\mathbb{H}$ with suitable sizes.
The following corollary can be considered as an essential finding in [11,29,34].

**Corollary 3.** Let $A_i$, $B_i$ and $C_i$ ($i = 1, 2$) be given. Set

$$D_1 = R_{B_1}B_2, \ A = R_{A_2}A_1, \ B = B_2L_{D_1}, \ C = R_{A_2}(R_{A_1}C_1B_1B_2 + C_2)L_{D_1}$$

Then the following statements are equivalent:

1. The system (5) is solvable.
2. $R_{A_1}C_1L_{B_1} = 0, \ R_AC = 0, \ CL_B = 0.$
3. $r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) \ (i = 1, 2),$
4. $r\begin{pmatrix} C_2 & C_1 & A_1 & A_2 \\ B_2 & B_1 & 0 & 0 \end{pmatrix} = r(A_1 A_2) + r(B_1 B_2).$

In this case, the general solution of (5) can be expressed as

$$X = A_1^tC_1 + U_1B_1 + L_{A_1}W_1,$$
$$Y = R_{A_1}C_1B_1^t + A_1U_1 + V_1R_{B_1},$$
$$Z = A_2^t(C_2 + R_{A_1}C_1B_1^tB_2 + A_1U_1B_2) + W_4D_1 + L_{A_2}W_6,$$
$$U_1 = A_1^tCB_1^t + LAW_2 + W_3R_B,$$
$$V_1 = R_{A_2}(C_2R_{A_1}C_1B_1^tB_2 + A_1U_1B_2)D_1^t + A_2W_4 + W_5R_{D_1},$$

where $W_1, \ldots, W_6$ are arbitrary matrices with suitable sizes over $\mathbb{H}.$

**3.2. Algorithm with a Numerical Example**

In this subsection, we carry out an algorithm (see Algorithm 1) and a numerical example to justify our main findings. The numerical calculations can be done using MATLAB 2018 (R2018b).

**Algorithm 1**

1. **Input** the system (1) with coefficients $A_k, B_k$ and $C_k, \ (k = 1,5)$ with viable dimensions over $\mathbb{H}$.
2. Compute all matrices that appeared in (12) and (22).
3. Check whether the Moore–Penrose inverses conditions in Theorem 1 are satisfied or not. If not, return “The system (1) is unsolvable”;
4. Else compute $X, Y, Z, W$ and $V$ by (35)–(39).
5. **Output** the general solution of the system (1) is $(X, Y, Z, W, V)$.

Note that the conditions in (3) can be exchange by rank equalities conditions (24)–(34).

**Example 1.** Consider the system of Sylvester-like quaternion matrix (1), where

$$A_1 = (i \ 0 \ j), \ A_2 = \begin{pmatrix} i & 0 & -k \\ 0 & 0 & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} 0 & 1 \\ -i & 0 \\ 3 & 0 \end{pmatrix}, \ A_4 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix},$$

$$A_5 = \begin{pmatrix} i & 0 & 0 \\ 0 & 2 & k \end{pmatrix}, \ B_1 = \begin{pmatrix} -j \\ 0 \\ -k \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & -j \\ k & 0 \end{pmatrix}, \ B_3 = \begin{pmatrix} 1-i \\ 0 \end{pmatrix},$$
\[
B_4 = \begin{pmatrix}
i \\
1+k \\
-1 \\
0
\end{pmatrix}, \quad B_5 = \begin{pmatrix}
0 & -j \\
1 & 0 \\
k & -k \\
0
\end{pmatrix}, \quad C_1 = \begin{pmatrix}
0 & -i & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad C_2 = \begin{pmatrix}
0 \\
k \\
0
\end{pmatrix},
\]
\[
C_3 = \begin{pmatrix}
2j + k \\
1 \\
i + j \\
3i
\end{pmatrix}, \quad C_4 = \begin{pmatrix}
i + j \\
2i - j + k
\end{pmatrix}, \quad C_5 = \begin{pmatrix}
-i + k & -k \\
1 & k
\end{pmatrix}.
\]

Straightforward calculations in Algorithm 1, using the quaternion package on MatLab or Maple software, it can be found that these matrices obey all the equalities in (23). Upon computation, we observe that \(A_{11} = 0, B_{11} = 0, B_{22} = 0, B_{33} = 0, T_1 = 0, N_1 = 0, S_1 = 0, Q = 0, H_i = 0,\) \(L_i = 0 (i = \overline{1,4}), G_1 = 0, G_3 = 0, G_2 = G_4 = A_{33}, \) \(\hat{B}_1 = \begin{pmatrix} I_3 \\
I_3
\end{pmatrix}, \hat{C}_1 = \hat{D}_1 = \hat{C}_2 = \hat{D}_2 = I_3,\) \(\hat{A}_2 = \hat{A}_3 = R_{A_1} \) and \(\hat{B}_2 = \hat{B}_3 = L_{B_1}\), where \(I_j\) is the identity matrix of order \(j\). Consequently, the system (1) is solvable. Hence the general solution to (1) is
\[
X = X_0 + P_1 M_{11} + M_{12} P_2 M_{13},
\]
\[
Y = Y_0 + M_{21} P_1 M_{22} + M_{23} P_2 + M_{24} P_3 M_{25} + P_6 M_{26},
\]
\[
W = W_0 + M_{31} P_1 M_{32} + P_8 M_{33} + M_{34} W_2,
\]
\[
Z = Z_0 + M_{41} P_8 + M_{42} P_3 M_{43} + P_3 M_{44},
\]
\[
V = V_0 + M_{51} U_{32} M_{52},
\]

where \(P_1, P_2, P_5, P_6, P_{11} = P_7 + U_4, P_8, W_2, P_3\) and \(U_{32}\) are arbitrary matrices over \(\mathbb{H}\), and
\[
X_0 = \begin{pmatrix}i \\
j \\
0 \\
0
\end{pmatrix}, \quad Y_0 = \begin{pmatrix}j \\
0 \\
-k \\
1
\end{pmatrix}, \quad Z_0 = \begin{pmatrix}1+k \\
0 \\
i \\
0
\end{pmatrix}, \quad W_0 = \begin{pmatrix}0 \\
1-j
\end{pmatrix},
\]
\[
V_0 = \begin{pmatrix}0 & -1 & 0 \\
-i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad M_{11} = -B_1 B_3 = \begin{pmatrix}0 & 0 & 0 \\
0 & 0 & 0 \\
j & k
\end{pmatrix}, \quad M_{12} = L_{A_1} = \begin{pmatrix}0.1000 & 0.3000j \\
-0.3000j & 0.9000
\end{pmatrix},
\]
\[
M_{13} = B_3, \quad M_{21} = -A_3, \quad M_{22} = B_1 = \begin{pmatrix}0 & -0.2500 + 0.2500k \\
0 & -0.5000j \\
0 & -0.2500i - 0.2500j \\
1.000k & 0
\end{pmatrix},
\]
\[
M_{25} = R_{B_3} = \begin{pmatrix}0 & 0 \\
0 & 1.000
\end{pmatrix}, \quad M_{23} = A_3 L_{A_1} = \begin{pmatrix}0.3000j & 0.9000 \\
-0.1000i & 0.3000k \\
0.3000 & -0.9000j
\end{pmatrix},
\]
\[
M_{24} = L_{A_1} = \begin{pmatrix}0.5000 & 0.5000j \\
0 & 1.0000 \\
-0.5000j & 0.5000
\end{pmatrix}, \quad M_{26} = R_{B_3} R_{B_3} = \begin{pmatrix}0 & 0 \\
0 & 0.6667 + 0.4714j
\end{pmatrix},
\]
\[
M_{31} = M_{34} = L_{M_6} = L_{A_4} = \begin{pmatrix}0 & 0 \\
0 & 0
\end{pmatrix}, \quad M_{32} = M_{33} = B_4, \quad M_{41} = A_4, \quad M_{42} = \hat{A}_1,
\]
\[
M_{43} = \begin{pmatrix}
0.7500 & 0.2500i - 0.2500j & 0.2500i & 0 \\
0.2500i + 0.2500j & 0.5000 & 0.2500 + 0.2500k & 0 \\
-0.2500i & 0.2500 & 0.7500 - 0.2500k & 0 \\
0 & 0 & 0 & 1.000
\end{pmatrix},
\]
\[
M_{44} = R_{B_3} R_{B_3} = \begin{pmatrix}0.5000 & 0.5000 & 0.5000i \\
0 & 0 & 0 \\
-0.5000i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad M_{51} = L_{A_1} L_{A_2} = \begin{pmatrix}0.5000 & 0.5000k \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The following statements are equivalent:

4. The Solvability Conditions and the General Solutions to the Systems (2) and (3)

We can easily investigate systems (2) and (3) by the same method and techniques we used in investigating the system (1). We merely present the main results of (2) and (3) in this part, omitting the intricacies of their proofs.

Let $A_i$, $B_i$ and $C_i$ ($i = 1, 2, 3$) be given. For simplicity, consider (12) such that

$$
\begin{align*}
\hat{A}_1 &= A_2L_{A_1}, \quad \hat{B}_1 = R_Bb_2, \quad \hat{A}_2 = A_2, \quad \hat{B}_2 = B_4, \quad \hat{A}_3 = A_3, \quad \hat{B}_3 = B_2, \quad \hat{A}_4 = A_5L_{A_1}, \\
\hat{B}_4 &= R_Bb_4, \quad \hat{C}_1 = C_5 - [A_2\hat{A}_4^t C_4 + R_{A_3} C_3 \hat{B}_2^t B_2 + A_5(A_1 C_1 + L_{A_1} C_2 B_1^t) B_3].
\end{align*}
$$

(50)

**Theorem 2.** The following statements are equivalent:

1. The system (2) is solvable.
2. $R_{A_i} C_1 = 0, \quad C_2 L_{B_i} = 0, \quad A_1 C_2 = C_1 B_1, \quad R_{A_i} C_i L_{B_i} = 0$ ($i = 3, 4$),
3. $R_{C_i} L_i = 0, \quad L_i L_{H_i} = 0$ ($i = \overline{1, 4}$), \quad $R_{A_3} E L_{B_2} = 0$.

(51)

$$
M_{32} = R_{B_i} R_{B_1} = \begin{pmatrix}
0.5000 & 0 & 0.5000i \\
0 & 1.0000 & 0 \\
-0.5000i & 0 & 0.5000
\end{pmatrix}.
$$

4. Let $F_i, G_i$ and $H_i$ ($i = 1, 2, 3$) be given. For simplicity, consider (12) such that

$$
\begin{align*}
\hat{F}_1 &= F_2L_{F_1}, \quad \hat{G}_1 = G_2, \quad \hat{F}_2 = F_2, \quad \hat{G}_2 = G_4, \quad \hat{F}_3 = F_3, \quad \hat{G}_3 = G_2, \quad \hat{F}_4 = F_2, \quad \hat{G}_4 = G_2.
\end{align*}
$$

(50)
\[
\begin{align*}
    r \begin{pmatrix}
        C_5 & A_5C_2 & A_2 \\
        B_2 & 0 & 0 \\
        B_5 & B_1 & 0 
    \end{pmatrix} &= r \begin{pmatrix}
        B_2 & 0 \\
        B_5 & B_1 
    \end{pmatrix} + r(A_2), \\
    r \begin{pmatrix}
        C_5 & A_2 & 0 & 0 & 0 & A_5 & 0 & 0 \\
        B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
        B_5 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & -C_5 & A_3 & A_2 & A_5 & -C_3 & -A_3C_2 \\
        0 & 0 & B_4 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & B_2 & 0 & 0 & 0 & B_3 & 0 \\
        B_5 & B_5 & 0 & 0 & 0 & 0 & B_1 & 0 \\
        0 & 0 & -C_4 & A_4 & 0 & 0 & 0 & 0 \\
        C_1B_5 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 
    \end{pmatrix} \\
    &= r \begin{pmatrix}
        B_2 & 0 & 0 & 0 & 0 \\
        0 & B_4 & 0 & 0 & 0 \\
        0 & B_2 & B_3 & 0 & 0 \\
        B_5 & B_5 & 0 & B_1 & 0 
    \end{pmatrix} + r \begin{pmatrix}
        A_2 & 0 & 0 & 0 & A_5 \\
        0 & A_3 & A_2 & A_5 & 0 \\
        0 & 0 & A_4 & 0 & 0 \\
        0 & 0 & 0 & A_1 & 0 
    \end{pmatrix}.
\end{align*}
\]

In this case, the general solution of (2) can be expressed as

\[
\begin{align*}
    X &= A_3^tC_3 + U_1B_3 + L_{A_3}U_2, \\
    Y &= R_{A_3}C_3B_3^t + A_3U_1 + U_3R_{B_3}, \\
    Z &= A_4^tC_4 + W_1B_4 + L_{A_4}W_2, \\
    W &= R_{A_4}C_4B_4^t + A_4W_1 + W_3R_{B_4}, \\
    V &= A_1^tC_1 + L_{A_1}C_2B_1^t + L_{A_1}QR_{B_1},
\end{align*}
\]

where

\[
\begin{align*}
    W_2 &= \hat{A}_1^t(\hat{C}_1 - \hat{A}_2W_1B_2 - \hat{A}_3U_1\hat{B}_3 - \hat{A}_4Q\hat{B}_4) - P_1\hat{B}_1 + L_{A_1}P_2, \\
    U_3 &= R_{A_1}(\hat{C}_1 - \hat{A}_2W_1B_2 - \hat{A}_3U_1\hat{B}_3 - \hat{A}_4Q\hat{B}_4)\hat{B}_1^t + \hat{A}_1P_1 + P_3R_{\hat{B}_1}, \\
    W_1 &= A_1^tTB_{11} - A_1^tA_22M_1^tTB_{11} - A_1^tA_5A_5^tTN_{11}^tB_{22}B_{11} \\
    &\quad - A_1^tS_1P_4R_{N_1}B_{22}B_{11} + L_{A_1}P_5 + P_6R_{\hat{B}_1}, \\
    U_1 &= M_1^tTB_{22} + S_1^tA_5^tTN_{11}^t + L_{M_1}S_1P_7 + P_8R_{B_2} + L_{M_1}U_4R_{N_1}, \\
    Q &= Q_{11}^t + L_{G_1}J_1 + J_2R_{H_1} + L_{G_1}J_3R_{H_2}, \\
    \text{or } Q &= Q_{22}^t - L_{G_4}K_1 - K_2R_{H_3} - L_{G_3}K_3R_{H_3}, \\
    J_1 &= (I_m \ 0)[A_1^t(\hat{Q} - \hat{C}_1J_3\hat{D}_1 - \hat{C}_2K_3\hat{D}_2) - U_1\hat{B}_1 + L_{\hat{A}_1}U_12], \\
    K_1 &= (0 \ I_m)[A_1^t(\hat{Q} - \hat{C}_1J_3\hat{D}_1 - \hat{C}_2K_3\hat{D}_2) - U_1\hat{B}_1 + L_{\hat{A}_1}U_12], \\
    J_2 &= [R_{\hat{A}_1}(\hat{Q} - \hat{C}_1J_3\hat{D}_1 - \hat{C}_2K_3\hat{D}_2)\hat{B}_1^t + \hat{A}_1U_11 + U_21R_{\hat{B}_1}][I_n \ 0], \\
    K_2 &= [R_{\hat{A}_1}(\hat{Q} - \hat{C}_1J_3\hat{D}_1 - \hat{C}_2K_3\hat{D}_2)\hat{B}_1^t + \hat{A}_1U_11 + U_21R_{\hat{B}_1}][0 \ I_n], \\
    J_3 &= \hat{A}_2^t\hat{Q}\hat{B}_2^t - \hat{A}_2^t\hat{A}_3\hat{M}^t\hat{Q}\hat{B}_2^t - \hat{A}_2^tS\hat{A}_3^t\hat{Q}\hat{N}^t\hat{B}_2^t \\
    &\quad - \hat{A}_2^tSU_31R_{N_2}\hat{B}_2^t + L_{\hat{A}_2}U_32 + U_33R_{\hat{B}_2}, \\
    K_3 &= \hat{M}^t\hat{Q}\hat{B}_3^t + S^t\hat{S}\hat{A}_3^t\hat{Q}\hat{N}^t + L_{M_1}S_1U_41 + U_42R_{\hat{B}_3} + L_{M_1}U_3R_{N_1},
\end{align*}
\]

and \( P_i \ (i = 1, 8) \), \( U_2 \), \( W_3 \), \( U_{11} \), \( U_{12} \), \( U_{21} \), \( U_{31} \), \( U_{32} \), \( U_{33} \), \( U_{41} \) and \( U_{42} \) are arbitrary matrices over \( \mathbb{H} \), \( m \) and \( n \) are the number of columns of \( A_1 \) and the number of rows of \( B_1 \), respectively.
Let $A_i$, $B_i$ and $C_i$ ($i = 1, 5$) be given. For simplicity, consider (12) such that

$$
\begin{align*}
\hat{A}_1 &= A_2 A_3, \quad B_1 = B_4 B_2, \quad \hat{A}_2 = A_2, \quad B_2 = R_{B_i}, \quad \hat{A}_3 = L_{A_1}, \quad B_3 = B_2, \quad \hat{A}_4 = A_5 L_{A_1}, \\
\hat{B}_4 &= R_{B_i} B_5, \quad \hat{C}_1 = C_5 - [A_2 R_{A_1} C_3 B_3^i + A_3^i C_4 B_2 + A_5 (A_1 C_1 + L_{A_1} C_2 B_1^i)] B_5.
\end{align*}
$$

(52)

**Theorem 3.** The following statements are equivalent:

1. The system (3) is solvable.
2. $R_{A_1} C_1 = 0, C_2 L_{B_1} = 0, A_1 C_2 = C_1 B_1, R_{A_1} C_{i L_{B_1}} = 0$ ($i = 3, 4$), $R_{C_{i L_i}} = 0, L_{i L_{H_{i}}} = 0$ ($i = 1, 4$), $R_{H_{3}} E L_{B_2} = 0$.

(3)

$$
\begin{align*}
&\begin{pmatrix}
C_i \\
A_i
\end{pmatrix}
= r(\begin{pmatrix}
C_i \\
A_i
\end{pmatrix}) = r(\begin{pmatrix}
C_i \\
B_i
\end{pmatrix}), \quad A_1 C_2 = C_1 B_1, \\
&\begin{pmatrix}
A_4 C_5 - C_4 B_2 & A_4 A_2 & A_4 A_3 \\
B_1 B_5 & 0 & 0
\end{pmatrix}
= r(B_4 B_2) + r\left(\begin{pmatrix}
A_4 A_2 & A_4 A_5 \\
0 & A_1
\end{pmatrix}\right), \\
&\begin{pmatrix}
A_4 C_5 - C_4 B_2 & A_4 A_2 & A_4 A_3 \\
B_1 B_5 & 0 & 0
\end{pmatrix}
= r(A_2 A_3) + r\left(\begin{pmatrix}
B_3 B_3 & 0 \\
B_1 & 0
\end{pmatrix}\right), \\
&\begin{pmatrix}
A_4 C_5 - C_4 B_2 & A_4 A_2 & A_4 A_3 \\
B_1 B_5 & 0 & 0
\end{pmatrix}
= r(A_4 A_2) + r\left(\begin{pmatrix}
B_3 B_3 & 0 \\
B_1 & 0
\end{pmatrix}\right), \\
&\begin{pmatrix}
A_4 C_5 - A_4 A_2 C_3 - C_4 B_2 B_3 & A_4 A_5 & A_4 A_2 A_3 \\
C_1 B_5 B_3 & A_1 & 0
\end{pmatrix}
= r(A_4 A_5) + r\left(\begin{pmatrix}
A_4 A_2 A_3 \\
0 & A_1
\end{pmatrix}\right) + r(B_4 B_2 B_3), \\
&\begin{pmatrix}
A_4 C_5 - A_4 A_2 C_3 - C_4 B_2 B_3 & A_4 A_5 & A_4 A_2 A_3 \\
C_1 B_5 B_3 & A_1 & 0
\end{pmatrix}
= r(A_5 A_2 A_3) + r(A_2 B_3), \\
&\begin{pmatrix}
A_4 C_5 - A_4 A_2 A_3 - C_4 B_2 B_3 & A_4 A_5 C_2 & A_4 A_2 A_3 \\
B_3 B_3 & B_1 & 0
\end{pmatrix}
= r\left(\begin{pmatrix}
B_3 B_3 & B_1 \\
B_4 B_2 B_3 & 0
\end{pmatrix}\right) + r(A_4 A_2 A_3), \\
&\begin{pmatrix}
A_5 & A_2 & A_3 \\
B_2 & 0 & 0
\end{pmatrix}
= r(A_2) + r\left(\begin{pmatrix}
B_2 & 0 \\
B_5 & B_1
\end{pmatrix}\right), \\
&\begin{pmatrix}
C_5 & A_2 & 0 & 0 & A_5 & 0 \\
B_2 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
= r\left(\begin{pmatrix}
B_2 & 0 & 0 \\
B_4 B_2 B_3 & 0 & 0 \\
B_5 & B_2 B_3 & B_1 & 0
\end{pmatrix}\right) + r\left(\begin{pmatrix}
A_2 & 0 & A_5 \\
0 & A_4 A_2 A_3 & A_4 A_5 \\
0 & 0 & A_1
\end{pmatrix}\right).
\end{align*}
$$
In this case, the general solution of (3) can be expressed as

\[ X = A_1^T C_3 + U_1 B_3 + L_{A_1} U_2, \]
\[ Y = R_{A_1} C_3 B_3^T + A_3 U_1 + U_3 R_{B_3}, \]
\[ Z = A_4^T C_4 + W_1 B_4 + L_{A_1} W_2, \]
\[ W = R_{A_1} C_4 B_4^T + A_4 W_1 + W_3 R_{B_4}, \]
\[ V = A_1^T C_1 + L_{A_1} C_2 B_1^T + L_{A_1} Q R_{B_1}, \]

where

\[ U_1 = A_1^T (C_1 - A_2 U_3 B_2 - A_3 W_2 B_3 - A_4 Q R_{B_4}) - P_1 B_1 + L_{A_1} P_2, \]
\[ W_1 = R_{A_1} (C_1 - A_2 U_3 B_2 - A_3 W_2 B_3 - A_4 Q R_{B_4}) B_1^T + A_1 P_1 + P_3 R_{B_1}, \]
\[ U_3 = A_{11}^T T B_{11}^T - A_{11}^T A_{22} M_{11}^T T B_{11}^T - A_{11}^T S_{11} A_{22}^T T N_{11}^T B_{22} B_{11}^T - A_{11}^T S_{11} P_4 R_{N_1} B_{22} B_{11}^T + L_{A_1} P_3 + P_6 R_{B_1}, \]
\[ W_2 = M_{11}^T B_{22}^T + S_{41} B_{22} T N_{11}^T + L_{M_1} L_{S_4} P_7 + P_8 R_{B_2} + L_{M_1} U_4 R_{N_1}, \]
\[ Q = Q_1^1 + L_{C_2} J_1 + J_2 R_{H_1} + L_{G_1} J_3 R_{H_2}, \]

or \( Q = Q_2^0 - L_{G_2} K_1 - K_2 R_{H_5} - L_{G_2} K_3 R_{H_4}, \)

\[ J_1 = (I_m \ 0) | A_1^T (Q - C_1 J_3 \bar{D}_1 - C_2 K_3 \bar{D}_2) - U_{11} \bar{B}_1 + L_{A_1} U_{12}, \]
\[ K_1 = (0 \ I_m) | A_1^T (Q - C_1 J_3 \bar{D}_1 - C_2 K_3 \bar{D}_2) - U_{11} \bar{B}_1 + L_{A_1} U_{12}, \]
\[ J_2 = [R_{A_1} (Q - C_1 J_3 \bar{D}_1 - C_2 K_3 \bar{D}_2) B_1^T + A_1 U_{11} + U_{21} R_{B_1}] \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \]
\[ K_2 = [R_{A_1} (Q - C_1 J_3 \bar{D}_1 - C_2 K_3 \bar{D}_2) B_1^T + A_1 U_{11} + U_{21} R_{B_1}] \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \]
\[ J_3 = \bar{A}_2^T Q B_2^T - \bar{A}_2^T A_3 M_{11}^T Q B_2^T - \bar{A}_2^T S_{A_3}^T Q N_{11}^T B_3 B_2^T - \bar{A}_2^T SU_{31} R_{N} B_3 B_2^T + L_{\bar{A}_2} U_{32} + U_{33} R_{B_3}, \]
\[ K_3 = M_{11}^T Q B_3^T + S_{A_3}^T Q N_{11}^T + L_{M} L_{S_4} U_{41} + U_{42} R_{B_3} + L_{M} U_{31} R_{N}, \]

and \( P_i, i = 1, 2, U_2, W_3, U_{11}, U_{12}, U_{21}, U_{31}, U_{32}, U_{33}, U_{41} \) and \( U_{42} \) are arbitrary matrices over the quaternion algebra, \( m \) and \( n \) are the number of columns of \( A_1 \) and the number of rows of \( B_1 \), respectively.

5. Conclusions

We have presented a thorough analysis of the three systems of symmetrical coupled Sylvester-like matrix Equations (1)–(3) over the quaternion algebra \( \mathbb{H} \). Certain solvability conditions and the expressions of the general solutions to the systems have been derived. Some specific cases that match well-known results have been examined. Finally, an algorithm and a numerical example have been utilized to check the validity of the main findings. It is worth mentioning that the main results of this paper are valid over an arbitrary division ring.

Author Contributions: All authors have equal contributions for Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Writing an original draft, Writing a review, and Editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the grants from the National Natural Science Foundation of China 11971294; 12171369.

Institutional Review Board Statement: Not applicable.
Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors have declared that there is no conflict of interest.

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