Resolutions of free partially commutative monoids

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**Abstract.** A free resolution of free partially commutative monoids is constructed and with its help the homological dimension of these monoids is calculated.

A free partially commutative monoid is a monoid generated by elements some of which commute (see the precise definition in Section 1). Homology of these monoids appeared in an article [1] by A. Husainov and V. Tkachenko in connection with constructing the homology groups of asynchronous transition systems. In [2] Husainov proposes the following Conjecture:

**Conjecture.** Let $\Sigma$ be a finite set and $M$ be a free partially commutative monoid whose generating set is $\Sigma$. If there are no distinct letters $a_1, a_2, \ldots, a_{n+1} \in \Sigma$ such that $a_i a_j = a_j a_i$ for every $1 \leq i < j \leq n+1$, then the monoid $M$ has the homological dimension $\leq n$.

In this paper we construct a free resolution for a free partially commutative monoid and with its help prove the Husainov’s Conjecture. We follow the ideas of D. Cohen who built in [3] a resolution for the so-called graph product of groups, given resolutions for factors. The presentation of the graph product with the help of direct and free amalgamated products played the leading role at that. However the additional difficulties appear while using this method for monoids.

Section 1 is devoted to basic definitions and facts concerned with free partially commutative monoids and free amalgamated products. In Section 2 the desired resolution is constructed. If the opposite is not specified all considered modules are right.

1 Preliminaries

In the subsequent text we follow mainly [4] in considering free partially commutative monoids and [5] in considering monoid free amalgamated products.

Let $\Sigma$ be a finite set called the alphabet. We denote by $\Sigma^*$, and its elements are called the words, the free monoid generated by $\Sigma$. The notation $\text{alph}(x)$ is used for all letters of $\Sigma$ that appear in a word $x \in \Sigma^*$.

Let $I \subseteq \Sigma \times \Sigma$ be a symmetric irreflexive binary relation over the alphabet $\Sigma$ called the commutation relation. The complement of $I$ is denoted by $D = \Sigma \times \Sigma \setminus I$. 
A monoid $M(\Sigma, I)$, which has a presentation $< \Sigma | \{ab = ba, (a, b) \in I\}$, is called a free partially commutative monoid.

An undirected graph without loops $\Gamma(M)$ can be uniquely compared to the free partially commutative monoid $M(\Sigma, I)$ in the following way: the vertex set of $\Gamma(M)$ is $\Sigma$, and the edges connect commuting vertices.

Important tools to work with free partially commutative monoids are the Projection Lemma and Levi’s Lemma ([4]). To formulate the former we need a definition. Let $A \subseteq \Sigma$ and $I_A = (A \times A) \cap I$ be an induced commutation relation. The projection is a homomorphism $\pi_A : M(\Sigma, I) \to M(A, I_A)$ which erases all the letters from a word which do not belong to $A$. In other words, for $a \in \Sigma$ we have

$$\pi_A(a) = \begin{cases} a, & a \in A, \\ 1, & a \notin A. \end{cases}$$

If $I_A = \emptyset$, then $\pi_A$ is a projection of $M(\Sigma, I)$ onto the free monoid $A^*$. If $A = \{a, b\}$, we write $\pi_{a,b}$ instead of $\pi_{\{a,b\}}$.

**Lemma 1.1 (Projection Lemma, [4])** Elements $u, v \in M(\Sigma, I)$ are equal if and only if $\pi_{a,b}(u) = \pi_{a,b}(v)$ for all $(a, b) \in D$.

**Lemma 1.2 (Levi, [4])** Let $t, u, v, w \in M(\Sigma, I)$. The following assertions are equivalent:

1) $tu = vw$;

2) there exist $p, q, r, s \in M(\Sigma, I)$ such that $t = pr$, $u = sq$, $v = ps$, $w = rq$ with $rs = sr$ and $\text{alph}(r) \cap \text{alph}(s) = \emptyset$.

To write down the elements of the free partially commutative monoid the so called Foata normal form is used. It is defined in the following way. Let $\Sigma$ be totally ordered. A word $x \in M(\Sigma, I)$ is in the Foata normal form if either it is the empty word or there exist an integer $n > 0$ and non-empty words $x_i$ ($1 \leq i \leq n$), such that

1) $x = x_1 x_2 \ldots x_n$;

2) for each $i$ the word $x_i$ is a product of distinct pairwise commuting letters, the letters of $x_i$ being written with regard to ordering introduced on $\Sigma$;

3) for each $1 \leq i < n$ and for each letter $a$ of $x_{i+1}$ there exists a letter $b$ of $x_i$, such that $(a, b) \in D$.

The following theorem holds:

**Theorem 1.3 ([4], [6])** Every element of $M(\Sigma, I)$ has a unique Foata normal form.

For example, the Foata normal form of the $n$-th power $a^n$ ($a \in \Sigma$) consists of $n$ factors equal $a$.

We recall briefly the concepts of the monoid free product and the free amalgamated product.
A monoid free product $\Pi^*\{M_j, j \in J\}$ or simple $\Pi^*M_j$ is built for a monoid family $\{M_j, j \in J\}$ provided $M_i \cap M_j = 1$, $i \neq j$ (see, for instance, [5] V.2., §9.4). It consists if the single-element sequence (1) and all nonempty sequences $(a_1, \ldots, a_k)$, such that $a_j \neq 1$, $a_j \in M_{(j)}$, $j = 1, \ldots, k$ and $i(j) \neq i(j + 1)$, $j = 1, \ldots, k - 1$.

For each $j \in J$ a canonical isomorphic embedding $\chi_j : M_j \to \Pi^*M_j$ can be defined as follows: $\chi_j(a) = a$; and $M_j$ can be identified with its canonical image. Hence, we can suppose that $\Pi^*M_j$ is generated by its submonoids $M_j$. The element $(a_1, \ldots, a_k)$ in $\Pi^*M_j$ can be written as $a_1 \ldots a_k$.

A monoid free amalgamated product is built for a family $\{\{M_j, j_j\}; U; \{\varphi_j, j_j\}\}$ which is called a monoid amalgam. Here $\{M_j, j \in J\}$ and $U$ are monoids. We assume again that $M_i \cap M_j = 1$, $i \neq j$, and in the free product $\Pi^*M_j$ each monoid $M_j$ is identified with its canonical image. Homomorphisms $\varphi_j, j \in J$ are the embeddings of the monoid $U$ into the monoids $M_j$, such that the unit of $U$ is mapped onto the common unit of the monoids $M_j$.

We define the relation $\nu$ over $\Pi^*M_j$ assuming

$$\nu = \{(u_i, u_j)|u_i = \varphi_i(u), u_j = \varphi_j(u), \text{ for some } i, j \in J, u \in U\}.$$ 

Let $\sim_\nu$ be the least congruence containing $\nu$. A monoid $\Pi^*\{M_j, j \in J\}/\sim_\nu$ is called a free product of the amalgam $\{\{M_j, j_j\}; U; \{\varphi_j, j_j\}\}$ or a free amalgamated product and is denoted by $\Pi^*_U M_j$.

The free amalgamated product can be described in terms of generators and defining relations, namely, the following proposition holds:

**Proposition 1.4** ([5]) Let $[M_j; U; \varphi_j]$ be a monoid amalgam and the monoid $U$ has a presentation $< Y|\pi >$ where $\pi \subseteq Y^* \times Y^*$. Then there exist such sets $X_j$ that $Y \subseteq X_j$, $X_i \cap X_j = Y$, if $i \neq j$, and there exist such relations $\sigma_j \subseteq X_j^* \times X_j^*$ that $M_j = (X_j, \sigma_j)$, $\sim_{\pi} = \sim_{\sigma_j} \cap (Y^* \times Y^*)$ for each $j$. If $X = \bigcup_{j \in J} X_j$, $\sigma = \bigcup_{j \in J} \sigma_j$, then the free amalgamated product $\Pi^*_U M_j$ has a presentation $< X|\sigma >$.

2 Constructing resolutions

We assume that the tensor product is considered over the ring $\mathbb{Z}$ if it is not specified. Also, for the monoid $M$ we write “$M$-module” instead of “$\mathbb{Z}M$-module” and $\otimes_M$ means that the tensor product is considered over the ring $\mathbb{Z}M$.

First, we discuss how a resolution for a free commutative monoid looks like. The free commutative monoid $M$ with $n$ generators $a_1, a_2, \ldots, a_n$ is a direct product of $n$ infinite cyclic monoids $M^1, M^2, \ldots, M^n$ with generators $a_1, a_2, \ldots a_n$ respectively. For each of them the resolution looks like

$$0 \rightarrow [a_j]\mathbb{Z}M^j \overset{\partial_j}{\rightarrow} \mathbb{Z}M^j \overset{\epsilon_j}{\rightarrow} \mathbb{Z} \rightarrow 0$$

where $[a_j]\mathbb{Z}M^j$ is a free $M^j$-module with one generator $[a_j]$ and $\partial_j[a_j] = a_j - 1$.

Denote by $X^j$ the complex $0 \rightarrow [a_j]\mathbb{Z}M^j \rightarrow \mathbb{Z}M^j$. Reasoning similarly for monoids as in [7] IV, §6 we obtain the resolution for $M$ as the tensor product of complexes $X^j$ with the augmentation $\varepsilon = \varepsilon^1 \otimes \cdots \otimes \varepsilon^n$:

$$0 \rightarrow X_n \overset{\delta_n}{\rightarrow} \cdots \rightarrow X_2 \overset{\delta_2}{\rightarrow} X_1 \overset{\delta_1}{\rightarrow} X_0 = \mathbb{Z}M \overset{\varepsilon}{\rightarrow} \mathbb{Z} \rightarrow 0,$$  

(*)
where
\[ X_k = \sum_{m_1 + \cdots + m_k = n} X_{m_1}^1 \otimes \cdots \otimes X_{m_n}^n. \]

The number of the summands in this sum equals \( C_n^k \), and each of them can be identified with the free \( M \)-module with one generator \([a_i a_{i_2} \ldots a_{i_k}]\) where \( 1 \leq i_1 < i_2 < \ldots i_k \leq n \) and \( X_{m_j}^{i_j} = [a_j] \mathbb{Z} M^{i_j}, \quad j = 1, 2, \ldots, k \).

The boundary homomorphisms \( \delta_k \) are of the form:
\[ \delta_k[a_{i_1} \ldots a_{i_k}] = \sum_{j=1}^k [a_{i_1} \ldots \widehat{a}_{i_j} \ldots a_{i_k}] (a_{i_j} - 1)(-1)^{j-1}. \]

Before we turn to the main Theorem we prove two lemmas.

**Lemma 2.1** Let \( M = M(\Sigma, I) \) be a free partially commutative monoid, \( \Sigma_0 \subset \Sigma \), \( I_0 = (\Sigma_0 \times \Sigma_0) \cap I \) and \( M_0 = M(\Sigma_0, I_0) \). Then the monoid ring \( \mathbb{Z} M \) is a free (left) \( \mathbb{Z}M_0 \)-module.

**Proof.** To make sure that \( M_0 \) is really a submonoid of \( M \) we build an embedding \( i : M_0 \to M \). Let \([x]_{M_0}\) be an element of \( M_0 \) defined by \( x \in \Sigma_0 \), and \([y]_M\) be an element of \( M \) defined by \( y \in \Sigma^* \). Now set \( i([x]_{M_0}) = [x]_M \).

This mapping is correct and is a homomorphism, for instance, in view of [5, V.1, §1.12, corollary 1.29]. We prove that \( i \) is an injection. If \( x, y \in \Sigma_0 \) and \( i([x]_{M_0}) = [x]_M = [y]_M = i([y]_{M_0}) \), then \( x \) can be obtained from \( y \) by successive transpositions of neighboring letters \( a, b \), such that \((a, b) \in I \). However \((a, b) \in I_0 \) since \( x, y \in \Sigma_0 \). Hence, \([x]_{M_0} = [y]_{M_0} \).

We order the elements of \( \Sigma \) in such a way that all the element of \( \Sigma_0 \) precede all the elements of \( \Sigma \setminus \Sigma_0 \).

To prove that \( \mathbb{Z} M \) is a free module we construct its basis. Consider the set \( B \) which consists of the monoid unit and all the elements \( t \in M \), such that the presentation \( t \) in the Foata normal form \( t = w_1 w_2 \ldots w_n \) has the following property: \( w_1 \) consists only of those letters which belong to \( \Sigma \setminus \Sigma_0 \).

Notice that if \( u \in B \), then it cannot be presented in the form \( u = su_0 \) where \( s \in M_0 \setminus 1 \). Indeed, suppose such a presentation exists. Consider the presentation of \( u \) in the Foata normal form \( u = u_1 u_2 \ldots u_n \) and the letter \( x \) which is the first letter of the word \( s \). The first occurrence of \( x \) in the word \( u_1 u_2 \ldots u_n \) belongs to some \( u_j \), \( j > 1 \). Then by the definition of the Foata normal form there exists such a letter \( y \) of \( u_{j-1} \) that \((x, y) \in D\). Two projections \( \pi_{x,y}(u_1 u_2 \ldots u_n) \) and \( \pi_{x,y}(su_0) \) do not coincide in the free monoid \((x, y)^*\), since the first letter of \( \pi_{x,y}(u_1 u_2 \ldots u_n) \) is \( y \), but the first letter of \( \pi_{x,y}(su_0) \) is \( x \), that contradicts Lemma 1.1.

Show that each element \( w \in M \) can be presented in the form \( w = au \) where \( a \in M_0, \ u \in B \). To find such a presentation it is sufficient to consider the following procedure. Present \( w \) in the Foata normal form: \( w = w_1 w_2 \ldots w_n \). If the letters of the alphabet \( \Sigma_0 \) do not occur in the word \( w_1 \), then \( w = 1 \cdot w \) where \( w \in B \). Otherwise, \( w_1 = a_1 u_1 \), where, in view of the introduced on \( \Sigma \) order, the words \( a_1 \) and \( u_1 \) can be chosen in such a way that \( a_1 \in M_0 \) and \( u_1 \) does not contain the letters

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of the alphabet \( \Sigma_0 \). Consider the word \( u_1w_2\ldots w_n \), present it in the Foata normal form \( u_1w_2\ldots w_n = w_1^1w_2^1\ldots w_n^1 \) and again “separate” the element of \( M_0 \) in the word \( w_1^1: w_1^1 = a_2u_2 \). Continuing similarly we obtain finally the decomposition of the form \( w = a_1\ldots a_kw_kw_{k+1}\ldots w_{n_k} = au \) where \( a = a_1\ldots a_n \in M_0 \) and \( u = u_kw_kw_{k+1}\ldots w_{n_k} \) is an element of the set \( B \).

Thus we know how to decompose the elements of the monoid \( M \) by the elements of \( B \). Now obviously we can decompose the elements of the ring \( ZM \) by the elements of \( B \) with the coefficients in the ring \( ZM_0 \).

To show that such a decomposition is unique it is sufficient to prove that the element \( w \) of the monoid \( M \) does not have two different decompositions. Suppose \( w = au = bv \) where \( a, b \in M_0 \), \( \{u, v\} \in B \). Then Lemma 2.2 implies that there exist \( p, q, r, s \in M \), such that \( a = pr, b = ps, u = sq, v = rq \). From \( a = pr, b = ps \) it follows particular that \( r, s \in M_0 \). But then the presentations \( u = sq \) and \( v = rq \) contradict the fact \( u, v \in B \) if \( r = s = 1 \) does not hold. In this case \( a = b = p, u = v = q \) and the decompositions \( a \cdot u = b \cdot v \) coincide.

Therefore, the set \( B \) is a basis that proves the lemma. 

Remark. Note that Lemma 2.1 always holds for groups, i.e. if \( G \) is an arbitrary group and \( G_0 \) is its subgroup, then the group ring \( ZG \) is a free \( ZG_0 \)-module (see., for instance, [8, I, §3]). At the same time, if we choose a submonoid of \( M(\Sigma, I) \) which is not free partially commutative, the statement generally does not hold.

Example. Let \( M = < a > \) be an infinite cyclic monoid and \( M_0 = \{1, a^2, a^3, \ldots, a^n, \ldots \} \) be a submonoid of \( M \). Then \( ZM \) is not a free \( M_0 \)-module. Suppose the contrary, then there exists a basis \( B \) in \( ZM \) and each of the elements of \( ZM \) has the unique decomposition by this basis over the ring \( ZM_0 \). Consider the elements \( 1, a \). None of them can be presented as \( xu \) where \( x \in M_0 \setminus 1, u \in M \). Hence, both of them are contained in \( B \). But then the element \( a^3 = a^2 \cdot a = a^3 \cdot 1 \) has two decompositions by the elements of the basis.

Observe that the ring \( Z \) becomes a trivial \( M_0 \)-module if we set \( n \cdot a = n \) for each \( a \in M_0, n \in Z \). Then the tensor product \( Z \otimes_{M_0} ZM \) exists and is an \( M \)-module, since \( ZM \) is an \( M \)-module. This remark allows us to formulate the following Lemma.

Lemma 2.2 Let \( M \) be a monoid, \( M_0, M_1, M_2 \) be its submonoids and \( M = M_1 \ast_{M_0} M_2 \). Then the following sequence of \( M \)-modules

\[
0 \rightarrow Z \otimes_{M_0} ZM \overset{i}{\rightarrow} Z \otimes_{M_1} ZM \oplus Z \otimes_{M_2} ZM \overset{p}{\rightarrow} Z \rightarrow 0.
\]

is exact.

Proof. Define \( i \) and \( p \) in the following way. For arbitrary \( w, u, v \in M \) set

\[
i(1 \otimes_{M_0} w) = 1 \otimes_{M_1} w + 1 \otimes_{M_2} w, \quad p(1 \otimes_{M_1} v) = 1, \quad p(1 \otimes_{M_2} u) = -1.
\]

It is easy to check that in terms \( Z \) and \( Z \otimes_{M_0} ZM \) the sequence is exact. We prove the exactness in the middle term. Since

\[
p(i(1 \otimes_{M_0} \sum_{w \in \Sigma} l_w w)) = \sum_{w \in \Sigma} l_w - \sum_{w \in \Sigma} l_w = 0,
\]

then \( \text{Im} i \subseteq \text{Ker} p \).

The inverse involving must be proved. An element \( 1 \otimes_{M_1} \sum n_uu + 1 \otimes_{M_2} \sum m_vv \) belongs to \( \text{Ker} p \) if and only if \( \sum n_u = \sum m_v \). These sums can be presented in the form:

\[
\sum n_u = n_+ - n_-, \quad \sum m_v = m_+ - m_-\]
where \( n_+, m_+ \) are the sums of all the positive coefficients and \( n_-, m_- \) the sums of all the negative ones respectively. If \( n_+ = m_+ \), then \( n_- = m_- \). It means that the sum 
\[
1 \otimes_{M_1} \sum n_a u \text{ contains the same quantity of the “plus”-sign summands of the form } 1 \otimes_{M_1} a (a \in M) \text{ as the sum } 1 \otimes_{M_2} \sum m_v v.
\]
Similarly for the “minus”-sign summands. Suppose \( n_+ \neq m_+ \), for instance \( n_+ > m_+ \). Then we add \( n_+ - m_+ \) summands of the form \( 1 \otimes_{M_2} b \) for some \( b \in M \) to the sum \( 1 \otimes_{M_2} \sum m_v v \) and subtract them. Then the numbers of “plus”-sign summands and “minus”-sign summands coincide for both of the sums, since \( m_+ + (n_+ - m_+) - (m_- + (n_+ - m_+)) = n_+ - n_- \).

The following step is to construct a preimage under the action of \( i \) for the element of the form \( 1 \otimes_{M_1} u + 1 \otimes_{M_2} v \) where \( u, v \in M \). After solving this problem we will be able to find a preimage for all elements of \( \text{Ker } p \) due to the reasoning stated above, i.e. to show that \( \text{Ker } \subseteq \text{Im } i \) and finish the proof of the Lemma.

To find the inverse image we present \( u \) and \( v \) in the form:

\[
u = a_1 b_1 a_2 b_2 \ldots a_i b_i; \quad v = c_1 d_1 c_2 d_2 \ldots c_s d_s
\]

where \( a_i, c_j \in M_1, b_i, d_j \in M_2, i = 1, 2, \ldots, l, j = 1, 2, \ldots, s \). Then the element

\[
i(1 \otimes_{M_0} w) = 1 \otimes_{M_1} [b_1 a_2 b_2 \ldots a_l b_l] + \sum_{k=1}^{l-1} (-a_{k+1} b_{k+1} \ldots a_l b_l + b_{k+1} a_{k+2} b_{k+2} \ldots a_l b_l) +
\]

\[
+ \sum_{j=1}^{s} (c_j d_j c_{j+1} d_{j+1} \ldots c_s d_s - d_j c_{j+1} d_{j+1} \ldots c_s d_s) +
\]

\[
+ 1 \otimes_{M_2} [\sum_{k=1}^{l-1} (b_{k+1} b_{k+1} \ldots a_l b_l - a_{k+1} b_{k+1} \ldots a_l b_l) + b_l +
\]

\[
+c_1 d_1 c_2 d_2 \ldots c_s d_s + \sum_{j=1}^{s-1} (-d_{j+1} d_{j+1} \ldots c_s d_s + c_{j+1} d_{j+1} \ldots c_s d_s) - d_s] =
\]

\[
= 1 \otimes_{M_1} b_1 a_2 b_2 \ldots a_l b_l + 1 \otimes_{M_2} (b_l + c_1 d_1 c_2 d_2 \ldots c_s d_s - d_s) =
\]

\[
= 1 \otimes_{M_1} a_1 b_1 a_2 b_2 \ldots a_l b_l + 1 \otimes_{M_2} c_1 d_1 c_2 d_2 \ldots c_s d_s = 1 \otimes_{M_1} u + 1 \otimes_{M_2} v,
\]

what required. \( \blacksquare \)

Now we are ready to prove the main theorem. Let \( M(\Sigma, I) \) be a free partially commutative monoid with a totally ordered generating set \( \Sigma \). Let \( \Gamma(M) \) be its graph and \( r_k \) be the number of complete subgraphs with \( k \) vertices in graph \( \Gamma(M) \). Let \( F_k \) be a free \( M \)-module with \( r_k \) generators. We denote each of such generators as \( [a_1 \ldots a_k] \) by putting the ascending sequence of vertices of corresponding subgraph. We denote the \( M \)-module homomorphisms \( \delta_k : F_k \rightarrow F_{k-1}, k > 1 \) by setting for
 generators

\[ \delta_k[a_1 \ldots a_k] = \sum_{j=1}^{k} [a_1 \ldots \widehat{a_j} \ldots a_k](a_j - 1)(-1)^{j-1}. \]

Besides we set \( \delta_1[a] = a - 1 \) and \( \varepsilon(a) = 1 \) for all \( a \in \Sigma \), and thus define homomorphisms \( \delta_1 : F_1 \to \mathbb{Z}M \) and \( \varepsilon : \mathbb{Z}M \to \mathbb{Z} \). The following theorem holds:

**Theorem 2.3** The sequence of \( M \)-modules and their homomorphisms

\[ \cdots \to F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} \mathbb{Z}M \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \quad (**) \]

is a free resolution of module \( \mathbb{Z} \) over \( \mathbb{Z}M \).

**Proof.** It has to be proved only the exactness of this sequence. We use the induction on the number of generators of \( M \).

If \( M \) is a free commutative monoid (i.e. its graph is complete), then the resolution \( (**) \) coincides with a sequence \( (***) \) for this monoid, since the generators \( [a_{i_1} \ldots a_{i_k}] \), \( 1 \leq i_1 < \cdots < i_k \leq n \) of modules \( X_k \) are in one-one correspondence with complete subgraphs with the vertices \( a_{i_1}, \ldots, a_{i_k} \). Particulary, in case when \( M \) is generated only by one generator, we obtain the induction assumption.

Suppose \( \Gamma(M) \) is not complete. Then there exist two vertices \( x \) and \( y \) which are not adjacent. Consider the subgraphs \( \Gamma_0 = \Gamma \setminus \{x, y\} \), \( \Gamma_1 = \Gamma \setminus x \), \( \Gamma_2 = \Gamma \setminus y \) and their corresponding submonoids of the monoid \( M \): \( M_0(\Sigma_0, I_0) \), \( M_1(\Sigma_1, I_1) \) and \( M_2(\Sigma_2, I_2) \). We have for them:

\[
\Sigma_0 = \Sigma \setminus \{x, y\}; \quad \Sigma_1 = \Sigma \setminus x; \quad \Sigma_2 = \Sigma \setminus y; \quad I_j = (\Sigma_j \times \Sigma_j) \cap I, \ j = 0, 1, 2.
\]

Since for monoids \( M_0, M_1, M_2 \) the relations \( \Sigma_1 \cap \Sigma_2 = \Sigma_0; I_0 = I_1 \cap (\Sigma_0 \times \Sigma_0) = I_2 \cap (\Sigma_0 \times \Sigma_0) \) hold, then by Proposition 1.4 the free amalgamated product \( M_1 \ast_{M_0} M_2 \) has a presentation \(< \Sigma_1 \cup \Sigma_2 | \{ab = ba, (a, b) \in I_1 \cup I_2\} > = \Sigma, \{ab = ba, (a, b) \in I\} >\), i.e. it coincides with the monoid \( M \).

Further we apply the induction assumption to \( M_0, M_1, M_2 \). Let

\[ \cdots \to F_j^0 \xrightarrow{\delta_j^0} F_j^1 \xrightarrow{\delta_j^1} \mathbb{Z}M_j \xrightarrow{\delta_j^j} \mathbb{Z} \to 0, \ j = 0, 1, 2 \]

be the resolutions for these monoids. We consider their tensor product with \( \mathbb{Z}M \) over \( \mathbb{Z}M_j \), \( j = 0, 1, 2 \) respectively. By Lemma 2.1, \( \mathbb{Z}M \) is a free \( M_j \)-module \( (j = 0, 1, 2) \), thus the functors \( \otimes_{M_j} \mathbb{Z}M \) are exact, therefore, the sequences remain exact, the modules \( F_j^0 \otimes_{M_j} \mathbb{Z}M \) being free \( M \)-modules. Further consider the commutative diagram consisting of free \( M \)-modules:
As it has been already noticed the left and the middle columns are exact. The bottom row is exact by Lemma 2.2. The second (from the bottom) row is a sequence of modules

\[ 0 \to Z \otimes M \xrightarrow{i_n} Z \otimes M \oplus ZM \xrightarrow{p_n} ZM \to 0 \]

where \( i_0(a) = a \oplus a \) and \( p_0(b \oplus c) = b - c \) for all \( a, b, c \in M \). From here it can be easy shown that it is also exact.

To prove the exactness of other rows we note that each complete subgraph, which is contained in \( \Gamma_0 \), is contained in \( \Gamma_1 \) and in \( \Gamma_2 \) simultaneously, and each complete subgraph, which is contained in \( \Gamma \), is contained either in \( \Gamma_1 \) or in \( \Gamma_2 \), since the vertices \( x \) and \( y \) are not adjacent.

Let elements \( [c_1], \ldots, [c_l] \) be generators of \( F_n^0 \); \( [a_1], \ldots, [a_m], [c'_1], \ldots, [c'_l] \) be generators of \( F_n^1 \); \( [b_1], \ldots, [b_k], [c''_1], \ldots, [c''_l] \) be generators of \( F_n^2 \) (for the sake of simplicity we denote the generators with a single letter and give them accents for not to mistake what direct summand they belong to). Then the generators of \( F_n \) are \( [a_1], \ldots, [a_m], [b_1], \ldots, [b_k], [c_1], \ldots, [c_l] \). The homomorphisms \( i_n \) and \( p_n \) look like:

\[ i_n \left( \sum_{j=1}^{l} [c_j] \otimes M_0 \gamma_j \right) = \sum_{j=1}^{l} [c'_j] \otimes M_1 \gamma_j + \sum_{j=1}^{l} [c''_j] \otimes M_2 \gamma_j \]

where \( \gamma_j \in ZM, j = 1, 2, \ldots, l \),

\[ p_n \left( \sum_{i=1}^{m} [a_i] \otimes M_1 \alpha_i + \sum_{j=1}^{l} [c'_j] \otimes M_1 \delta'_j + \sum_{i=1}^{k} [b_i] \otimes M_2 \beta_i + \sum_{j=1}^{l} [c''_j] \otimes M_2 \delta''_j \right) = \]

\[ = \sum_{i=1}^{m} [a_i] \alpha_i - \sum_{i=1}^{k} [b_i] \beta_i + \sum_{j=1}^{l} [c_j] (\delta'_j - \delta''_j). \]

Evidently, for each \( n \) the identity \( p_n i_n = 0 \) holds. Besides, the element \( x = \sum_{i=1}^{m} [a_i] \otimes M_1 \alpha_i + \sum_{j=1}^{l} [c'_j] \otimes M_1 \delta'_j + \sum_{i=1}^{k} [b_i] \otimes M_2 \beta_i + \sum_{j=1}^{l} [c''_j] \otimes M_2 \delta''_j \) belongs to
Ker $p_n$ if and only if $\alpha_i = 0$, $(i = 1, \ldots, m)$, $\beta_i = 0$, $(i = 1, \ldots, k)$ and $\delta_j^i = \delta_j^i$, $(j = 1, \ldots, l)$, i.e. $x = i_n(\sum_{j=1}^l [c_j] \otimes_{M_0} \delta_j^i)$. Hence, Im $i_n = \text{Ker } p_n$ holds.

The exactness of all the rows, the left and the middle columns implies the exactness of the right column what can be shown by standard diagram search method. Thus the theorem is proved. ■

**Corollary 2.4** If the graph $\Gamma(M)$ of free partially commutative monoid $M$ does not contain complete subgraphs with more than $n$ vertices, then the homological dimension of $M$ does not exceed $n$.

Corollary 2.4 proves the Husainov’s Conjecture formulated in the introduction.

**Corollary 2.5** Let $M$ be a free partially commutative monoid, $A$ be a trivial left $M$-module. Then for $n \geq 1$ homology groups $H_n(M, A) \cong A \oplus \cdots \oplus A$ where $r_n$ is the number of complete subgraphs with $n$ vertices in graph $\Gamma(M)$.

**Proof.** Denote $C_n = A \oplus \cdots \oplus A$. Notice that there exists a homomorphism $F_n \otimes A \cong C_n$. Indeed the mapping $\varphi : F_n \times A \to C_n$ can be constructed in the following way. Let $[x_1], \ldots, [x_{r_n}]$ be the generators of $F_n$. For each $j = 1, \ldots, r_n$, $\alpha \in \mathbb{Z}M$, $a \in A$ we set $\varphi([x_j] \alpha \cdot a) = (0, \ldots, \alpha a, \ldots, 0)$, where $\alpha a$ is put on the $j$-th place, and extend this mapping linearly. It is easy to check that the Abelian group $C_n$ together with the mapping $\varphi$ satisfies the universal property of the tensor product $F_n \otimes_M A$ and, therefore, it is isomorphic to this tensor product.

The groups $H_n(M, A)$ are the homology groups of the complex

$$\cdots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} A$$

where $\partial_n = \delta_n \otimes_M A$. Since $M$-module $A$ is trivial, then

$$\partial_k([a_1 a_2 \ldots a_k] \otimes_M 1) = \sum_{j=1}^k [a_1 \ldots \hat{a}_j \ldots a_k](a_j - 1)(-1)^{j-1} \otimes_M 1 = 0.$$ 

Hence,

$$H_n(M, A) = \text{Ker } \partial_n / \text{Im } \partial_{n+1} \cong C_n.$$ 

■

The particular case of this Corollary is

**Corollary 2.6** Let $M$ be a free partially commutative monoid. Then the homology groups $H_n(M, \mathbb{Z})$, $n \geq 1$ are the free Abelian groups of rank $r_n$ where $r_n$ is the number of complete subgraphs with $n$ vertices in graph $\Gamma(M)$. 

9
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