Comparative Study of Sliding Window Multifractal Detrended Fluctuation Analysis and Multifractal Moving Average Algorithm

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Abstract. Sliding window multifractal detrended fluctuation analysis (W-MF DFA) and multifractal moving average detrended method (MFDMA) are two effective methods to study multifractal characteristics of nonstationary time series. Taking the typical BMS signal model as an example, the selection of parameters, calculation accuracy and noise effects of the two algorithms are analyzed and compared. The results show that the calculation accuracy of MFDMA is better than that of W-MFDFA, but the latter is not sensitive to the changes of parameters, and has stronger anti-interference ability to noise and better stability. It can provide valuable reference for the research of actual data and the selection of internal parameters of the algorithm.

1. Introduction

Multifractal refers to irregular spatio-temporal sequences or bodies with self-similarity, which are mainly characterized by multiple scale indices and long-term correlation. With the continuous improvement and development of fractal theory, the research and application of fractal analysis methods have become a topic of common concern to scholars at home and abroad. At present, the common multifractal analysis methods include partition function method (MF), multifractal detrended fluctuation analysis method (MFDFA), multifractal detrended moving average method (MFDMA), entropy analysis method, and so on [1]. MFDFA has been widely used in multifractal analysis of geology, economy, biology, medicine and other disciplines due to its simple and easy implementation [2-5]. In 2006, zhou extended MFDFA to two-dimensional data analysis by combining MFDFA with cross-correlation analysis, making the application of this method more extensive [6]; However, when using MFDFA method, the non-overlapping segmentation method in the algorithm will produce certain pseudo fluctuation errors, which will affect the analysis results. To this end, Cao Guangxi proposed a sliding window detrended fluctuation analysis method (W-MFDFA) in mid-2007 and used it in the analysis of stock market returns [7]; After that, Xu Lin and others used the W-MFDFA method to study the stock style investment income series, accurately obtained the relevant characteristics and laws of style assets, and confirmed the practicability of the W-MFDFA method [8]. Also after MFDFA, MFDMA method proposed by Gu and Zhou [9] is widely used to study multifractal characteristics of one-dimensional non-stationary spatio-temporal sequences. Wang et al. revealed multifractal characteristics of RMB exchange rate index by using MFDMA method and
explained the main source of multifractal [10]; Malin and Provash analyzed the multifractal structure of the global monthly mean temperature time series from 1850 to 2012 with MFDMA, proving that MFDMA is an effective method to study multifractal [11]. In this paper, the typical Binomial Multifractal Series (BMS) is processed by using W-MFDFA and MFDMA methods. The influence of the selection of internal parameters of the algorithm on the results is analyzed, and the calculation accuracy of the two algorithms is compared. By superimposing random noise and Gaussian noise on BMS signal models generated by different parameters, the sensitivity of W-MFDFA and MFDMA to noise is explored, which provides important theoretical guidance for the analysis of actual data and the application of methods.

2. Introduction of W-MFDFA and MFDMA Algorithms

2.1. Algorithmic steps of W-MFDFA

Assuming a series of non-stationary time series is \( x(t), t = 1,2,3,\ldots,N \), the calculation steps of W-MFDFA are as following [7-8]:

Step 1, the amplitude cumulative dispersion sequence is constructed by

\[
y(\tau) = \sum_{i=1}^{\tau} (x(i)-\bar{x}), \quad t = 1,2,3,\ldots,N
\]

(1)

The average value of the original sequence is \( \bar{x} \) in the formula.

Step 2, symbol \( s \) is taken as the size of sliding window and gradually slide from the left end of the sequence \( \{y(t)\} \) to the last item. From this, \((N-s+1)\) small intervals are obtained. Each interval (represented by the letter \( v \)) contains \( s \) data points. For each subinterval \( v \), the least square method is adopted to carry out \( m \)-order polynomial fitting to obtain a local trend function \( f_v \).

Step 3, the trend function is subtracted from the cumulative dispersion series to calculate the average variance value to eliminate the fluctuation trend of each subinterval:

\[
F^2(v,s) = \frac{1}{s} \sum_{j=1}^{s} [y((v-1)s+j)-f_v(j)]^2, \quad v = 1,2,\ldots,N-s+1
\]

(2)

\[
F^2(v,s) = \frac{1}{s} \sum_{j=1}^{s} [y(N-(v-N_v)s+j)-f_v(j)]^2, \quad v = N_v+1, N_v+2,\ldots,2N_v
\]

(3)

Step 4, the average square of the above formula is calculated on the whole partition interval, and the corresponding order-de-trend fluctuation function is obtained.

\[
F_q(s) = \left\{ \frac{1}{N-s+1} \sum_{v=1}^{N-s+1} (F^2(v,s))^{\frac{q}{2}} \right\}^{\frac{2}{q}}, \quad q \neq 0
\]

(4)

\[
F_0(s) = \exp \left\{ \frac{1}{4N_v} \sum_{v=1}^{2N_v} \ln (F^2(v,s)) \right\}, \quad q = 0
\]

(5)

Step 5, by observing the double logarithmic graph of \( F_q(s) \) and \( s \), observe the variation of the fluctuation function \( F_q(s) \) with \( s \); if the sequence has long-range correlation, then \( F_q(s) \) and \( s \) have the following power-law correlation:

\[
F_q(s) \propto s^{H(q)}
\]

(6)

Logarithms are taken at both ends of the above formula, and the slope obtained by fitting based on the principle of least square method is \( H(q) \) value (\( H(q)>0 \)), also known as generalized Hurst index. It is an important indicator to judge whether a sequence has multifractal characteristics. When \( H(q) \) changes with the change of \( q \) value, the sequence is multifractal, otherwise, the sequence is monofractal.

2.2. Algorithmic steps of MFDMA

The algorithm implementation steps of MFDMA are as following [12-13].
Firstly, to preprocess the time series \( x(t), t = 1, 2, 3, \ldots, N \) with length \( N \) and construct a new sequence according to the formula

\[
y(t) = \sum_{i=1}^{t} x(i), \quad t = 1, 2, \ldots, N
\]

(7)

Secondly, calculate the moving average function \( \tilde{y}(t) \) in the moving window with scale \( n \).

\[
\tilde{y}(t) = \frac{1}{n} \sum_{k=\lceil(n-1)\theta\rceil}^{\lceil n-z \rceil} y(t-k), \quad z = \left\lceil (n-1)\theta \right\rceil, \quad t \in [n-z, N-z]
\]

(8)

In the formula (8), the maximum integer not exceeding \( x \) is represented as \( \lfloor x \rfloor \), and the minimum integer not less than \( x \) is represented as \( \lceil x \rceil \). Parameter \( \theta \) takes a value in the range of 0 to 1, representing the position where the sliding window takes the mean value; \( \theta = 0 \) means moving backward to take the mean, \( \theta = 0.5 \) means moving centrally to take the mean, and \( \theta = 1 \) means moving forward to take the mean.

Thirdly, calculate the residual error by

\[
\varepsilon(t) = y(t) - \tilde{y}(t), \quad z = \left\lceil (n-1)\theta \right\rceil, \quad t \in [n-z, N-z]
\]

(9)

\[
\varepsilon(i) = \varepsilon(i+n-z-1), \quad i = 1, 2, \ldots, N-n+1
\]

(10)

The residual sequence is taken as a new data sequence, and the length of the data is \( L = N-n+1 \), the sequence is divided into an average of non-overlapping intervals \( N_n \), each interval has \( n \) data, \( 5 \leq n \leq L/10 \), and then the sequence is divided in the same way in reverse once to get a total \( 2N_n \) of interval segments.

Fourthly, the square average of each interval segment is calculated, and the square mean function of the interval of the \( v \) segment is

\[
F_v^2(n) = \frac{1}{n} \sum_{i=1}^{n} \varepsilon^2[(v-1)n+i], \quad v = 1, 2, \ldots, N_n
\]

(11)

\[
F_v^2(n) = \frac{1}{n} \sum_{i=1}^{n} \varepsilon^2[(v-N_n)n+i], \quad v = N_n+1, N_n+2, \ldots, 2N_n
\]

(12)

Fifthly, By calculating the global root mean square of order \( q \), the wave function of order \( q \) is obtained.

\[
F_q(n) = \left\{ \frac{1}{2N_n} \sum_{v=1}^{2N_n} [F_v(n)]^q \right\}^{1/q}, \quad q \neq 0
\]

(13)

\[
\ln[F_q(n)] = \frac{1}{2N_n} \sum_{v=1}^{2N_n} \ln[F_v(n)], \quad q = 0
\]

(14)

Sixthly, Change the value of the scale \( n \) to obtain the corresponding wave function \( F_q(n) \). If the sequence has a long-range power law correlation, then,

\[
F_q(n) \propto n^H(q)
\]

(15)

According to Kanterhardt et al., the generalized Hurst index has the following relationship with the quality index

\[
\tau(q) = q \cdot H(q) - 1
\]

(16)

The singularity intensity and multifractal spectrum of time series can be obtained by Legendre transform

\[
\alpha(q) = d \left\{ \tau(q) \right\}/dq
\]

(17)

\[
f(\alpha) = q \cdot \alpha - \tau(q)
\]

(18)

In equations (17) and (18), \( \alpha \) is a singularity index, which is used to describe the singular intensity of different intervals in the time series; \( f(\alpha) \) is a multifractal spectrum whose value corresponds to the fractal dimension of the singularity index.
3. Comparative analysis of W-MFDFA and MFDMA methods

3.1. Comparison of calculation accuracy

Taking the typical multifractal signal sequence BMS as the research object [14], we select the parameter \( a = 0.3 \) to generate a signal sequence with a length of 1024, and calculates its multifractal parameters with W-MFDFA and MFDMA methods respectively. When applying the W-MFDFA method, let the polynomial detrended fitting order \( m \) to be 1, 2, 3 respectively, and the corresponding methods are recorded as W-MFDFA1, W-MFDFA2, MFDFA3; when applying the MFDMA method, three cases of position parameters \( \theta = 0, 0.5, 1 \) are selected for discussion. The corresponding methods are recorded as MFDMA-0, MFDMA-0.5 and MFDMA-1. The value range of \( q \) is in range -10 to 10, and 21 \( q \)-value points are uniformly taken at interval 1, the scale-free area is set to 8-256, and 32 values are equally spaced, as the division scale and sliding window width of the sequence.

Figure 1. shows the calculation results of sequence multifractal index for two algorithms with different parameter settings. Among them, (a),(b) and (c),(d) are the relation graph and fractal spectrum of Hurst index \( q \) obtained by corresponding algorithms. \( \Delta H = H_{\text{max}} - H_{\text{min}} \) indicates the change range of the generalized Hurst index; \( \Delta \alpha = \alpha_{\text{max}} - \alpha_{\text{min}} \) indicates the width of the multifractal spectrum; \( \Delta H_1, \Delta H_2, \Delta H_3 \) and \( \Delta H \) respectively represents the variation amplitude of \( H(q) \) value corresponding to the theoretical value and the W-MFDFA1(MFDMA-0), W-MFDFA2(MFDMA-0.5), W-MFDFA3(MFDMA-1) method. It can be seen that the Hurst index and multispectral curves calculated by the W-MFDFA and MFDMA methods are consistent with the theoretical curves, indicating that both methods are effective methods for analyzing multifractals.

![Figure 1. Comparison of calculation results of different parameters in two algorithm](image)

| Method       | RMS Error of \( H(q) \) | Error fluctuation range |
|--------------|--------------------------|-------------------------|
| W-MFDFA1     | 0.00773                  |                         |
| W-MFDFA2     | 0.00536                  | 0.00237                 |
| W-MFDFA3     | 0.00559                  |                         |
According to (a) and (b) in figure 1., it can be seen that the $H(q)$ values calculated by the three fitting orders are generally smaller than the theoretical values. At the same time, the parts corresponding to the Hurst index $H(q)$ and the theoretical values have higher fitting degrees, which indicates that W-MFDFA tends to depict the small fluctuation behavior of the fluctuation function. At this time, the magnitude of the variation of $H(q)$ values obtained when the orders are 2 and 3 is relatively close to the theoretical magnitude. By observing the multifractal spectrum, it can be found that the deviation mainly occurs in the part, which deviates from the theoretical spectral line on the whole to the left. When the order value is 2, the multifractal spectral width calculated by this method is the closest to the theoretical spectral width. Based on the above analysis, for multifractal signal sequences, the second-order W-MFDFA method can obtain more accurate analysis results.

According to (c) and (d) in Figure 1., when the value of position parameter $\theta$ is 0 and 1, the $H(q)$ value obtained by MFDMA method is basically the same as that of multifractal spectrum curve, but when the value of $\theta$ is 0.5, the $H(q)$ value and multifractal spectrum deviate from the theoretical value to a great extent; the $H(q)$ value obtained by this method is generally small, which makes the phase smaller. The corresponding fractal line deviates from the theoretical line to the left. By further analyzing the change range of $H(q)$ value and the fractal spectrum width, different position parameters can be obtained, and the results calculated by MFDMA-0 method are more accurate.

Table 1 shows the root mean square errors of $H(q)$ and theoretical values calculated by the two algorithms corresponding to different parameters and the corresponding error fluctuations. According to the data in the table, the error of MFDMA-0.5 method is the largest, but the accuracy of MFDMA-0.5 and MFDMA-1 is obviously better than that of W-MFDFA. By comparing the error fluctuation of $H(q)$ values of the two methods, we can see that W-MFDFA is less affected by the internal parameters of the algorithm and has better stability.

### 3.2. Contrastive Analysis of Noise Sensitivity

Noise mainly refers to the abnormal data in the signal sequence. The existence of noise often interferes with the analysis of the signal sequence, and the interference degree of different types of noise to the signal is different. For BMS signals, parameters $a = 0.25$, $a = 0.3$ and $a = 0.35$, length 1024, respectively. The effects of random noise and Gauss noise on W-MFDFA and MFDMA were explored.

#### 3.2.1 Add random noise

In MATLAB environment, 10 groups of random noise sequences with length 1024 and intensity of $1 \times 10^{-4}$ and $5 \times 10^{-4}$ respectively are generated. After taking the mean value, they are added to three groups of BMS signal models. W-MFDFA2 and MFDMA-0 are used to calculate the Hurst index of the original sequence and the random noise sequence. The results are shown in figure 2.
Figure 2. Comparison of analysis results of adding random noise sequences by two methods

Figure 2. shows that, on the one hand, the Hurst exponents calculated by two different methods show a trend of $s'$ with the change of $q$ value, indicating that the sequence after adding random noise is still multifractal. On the other hand, for the BMS signal model with given parameters, noise has little effect on the scaling index of large fluctuations, and the degree of influence on the small fluctuations depends on the parameters of the model. These two methods have some anti-jamming ability to random noise. Through further comparison, it is found that the fluctuation amplitude of W-MFDFA calculation results disturbed by noise is significantly smaller than that of MFDMA, which indicates that W-MFDFA has lower sensitivity to random noise and better anti-jamming ability to random noise than MFDMA.

3.2.2 Add Gaussian noise

The randn function is used to generate 10 Gaussian noise sequences with length 1024 and intensity $1 \times 10^{-5}$ and $5 \times 10^{-5}$ respectively. The mean values are added to the three BMS signal models. The Hurst exponents of the original sequence and the added Gaussian noise sequence are calculated using W-MFDFA2 and MFMA-0 separately. The results are shown in Figure 3.
Figure 3. Comparison of analysis results of adding Gaussian noise sequences by two methods

From Figure 3, we can see that the multifractal of the sequence is weakened after adding Gaussian noise. The results of the analysis using W-MF DFA and MFDMA methods are almost the same. The stronger the intensity of Gaussian noise, the greater the impact on the signal sequence. With the increase of model parameters, the multiplicity of the sequence decreases, and the analysis results are affected by noise. The noise also decreases correspondingly; for the sequence model with different parameters, the large deviation mainly occurs in the front section, which indicates that noise mainly interferes with the characterization of the wavelet fluctuation behavior by different methods; further observation shows that the fluctuation range of MFDMA method is slightly higher than that of W-MF DFA method for adding different intensity noise sequence, indicating that MFDMA is high. The sensitivity of S noise is strong.

4. Conclusion
On the one hand, the calculation accuracy of the two methods is compared by analyzing the influence of the internal parameters of the algorithm. The results show that W-MF DFA has a good stability for the change of fitting order. MFDMA is greatly affected by position parameters. The results calculated by MFDMA-0.5 have a large error, which is not conducive to the analysis and research of high accuracy requirements in actual data. The analysis results using MFDMA-0 and MFDMA-1 have a high accuracy. On the other hand, by adding random noise and Gaussian noise to the BMS signal model generated by different parameters, the sensitivity of the two methods to noise is compared. The results show that adding noise can reduce the multifractal of the sequence. Whether adding random noise or Gaussian noise, it will affect the analysis results of different methods. Compared with MFDMA method, W-MF DFA has better anti-noise ability.

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