A Note on Preconditioning by Low-Stretch Spanning Trees*

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Abstract

Boman and Hendrickson [BH01] observed that one can solve linear systems in Laplacian matrices in time $O\left(m^{3/2+o(1)}\log(1/\epsilon)\right)$ by preconditioning with the Laplacian of a low-stretch spanning tree. By examining the distribution of eigenvalues of the preconditioned linear system, we prove that the preconditioned conjugate gradient will actually solve the linear system in time $\tilde{O}\left(m^{4/3}\log(1/\epsilon)\right)$.

1 Introduction

For background on the support-theory approach to solving symmetric, diagonally dominant systems of linear equations, we refer the reader to one of [BGH+06, BH03, ST08].

Given a weighted, undirected graph $G = (V,E,w)$, we recall that the Laplacian of $G$ may be defined by

$$L_G = \sum_{(u,v) \in E} w(u,v)L_{(u,v)},$$

where $L_{(u,v)}$ is the Laplacian of the weight-1 edge from $u$ to $v$. This is, $L_{(u,v)}$ is the matrix that is zero everywhere, except for the submatrix in rows and columns $\{u,v\}$ which has form:

$$\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}.$$ 

Note that this last matrix may be written as the outer product of the vector $\psi_u - \psi_v$ with itself, where we let $\psi_u$ denote the elementary unit vector with a 1 in its $u$-th component.

For a connected graph $G$, we recall that a spanning tree of $G$ is a connected graph $T = (V,F,w)$ where $F$ is a subset of $E$ having exactly $n-1$ edges. As we intend for the edges that appear in $T$ to have the same weight as they do in $G$, we use the same weight function $w$. As $T$ is a tree, every pair of vertices of $V$ is connected by a unique path in $T$.

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For any edge \( e \in E \), we now define the \textit{stretch} of \( e \) with respect to \( T \). Let \( e_1, \ldots, e_k \in F \) be the edges on the unique path in \( T \) connecting the endpoints of \( e \). The \textit{stretch} of \( e \) with respect to \( T \) is given by

\[
st_T(e) = w(e) \left( \frac{1}{w(e_i)} \right). \]

The stretch of the graph \( G \) with respect to \( T \) was defined by Alon, Karp, Peleg, and West \cite{AKPW95} to be

\[
st_T(G) \overset{\text{def}}{=} \sum_{e \in E} st_T(e). \]

A \textit{low-stretch spanning tree} of \( G \) is a graph for which the above quantity is reasonably small. The best known bound on attainable stretch was obtained by Abraham, Bartal and Neiman \cite{ABN08}, who present an algorithm that, on input a graph with \( n \) vertices and \( m \) edges, runs in time \( \tilde{O}(m) \) and produces a spanning tree \( T \) of stretch \( O(m \log n \log \log n (\log \log \log n)^3) \).

The advantage of using a spanning tree as a preconditioner is that (after a permutation) one can compute an LU-factorization of the Laplacian of a tree in time \( O(n) \), and that one can use this LU-factorization to solve linear systems in the Laplacian of the tree in linear time as well.

## 2 Preconditioning

We prove the following three results.

\textbf{Theorem 2.1.} Let \( G = (V, E, w) \) be a connected graph and let \( T = (V, F, w) \) be a spanning tree of \( G \). Let \( L_G \) and \( L_T \) be the Laplacian matrices of \( G \) and \( T \), respectively. Then,

\[
\text{Tr} \left( L_G L_T^\dagger \right) = st_T(G),
\]

where \( L_T^\dagger \) denotes the pseudo-inverse of \( L_T \).

As \( T \) is a subgraph of \( G \), all the nonzero eigenvalues of \( \text{Tr} \left( L_G L_T^\dagger \right) \) are at least 1. The analysis of Boman and Hendrickson \cite{BH01} followed from the fact that the largest eigenvalue of \( L_G L_T^\dagger \) is at most \( st_T(G) \). We use the bound on the trace to show that not too many of these eigenvalues are large.

\textbf{Corollary 2.2.} For every \( t > 0 \), the number of eigenvalues of \( L_G L_T^\dagger \) greater than \( t \) is at most \( st_T(G)/t \).

\textbf{Theorem 2.3.} If one uses the preconditioned conjugate gradient (PCG) to solve a linear equation in \( L_G \) while using \( L_T \) as a preconditioner, it will find a solution of accuracy \( \epsilon \) in at most \( O \left( \frac{st_T(G)^{1/3}}{\ln(1/\epsilon)} \right) \) iterations.

As the dominant cost of each iteration of PCG is the time required to multiply a vector by \( L_G \), which is \( O(m) \), and the time required to solve a system of equations in \( L_T \), which is \( O(n) \), the low-stretch spanning trees of Abraham, Bartal and Neiman enable PCG to run in time

\[
O \left( m^{4/3}(\log n)^{1/3}(\log \log n)^{2/3}(\log 1/\epsilon) \right).
\]

The following lemma is the key to the proof of Theorem 2.1.
Lemma 2.4. Let $T = (V,F,w)$ be a tree, let $u,v \in V$, and let $x = \psi_u - \psi_v$. Then,

$$x^T L_T^\dagger x = \sum_{i=1}^{k} 1/w(e_i),$$

where $e_1,\ldots,e_k$ are the edges on the unique simple path in $T$ from $u$ to $v$.

Proof. The quantity $x^T L_T^\dagger x$ is known to equal the effective resistance in the electrical network corresponding to $T$ in which the resistance of every edge is the reciprocal of its weight (see, for example, [SS08]). As only edges on the path from $u$ to $v$ can contribute to the effective resistance in $T$ from $u$ to $v$, the effective resistance is the same as the effective resistance of the path in $T$ from $u$ to $v$. As the effective resistance of resistors in serial is just the sum of their resistances, the lemma follows. $\square$

Proof of Theorem 2.1. We compute

$$\text{Tr} \left( L_G L_T^\dagger \right) = \sum_{(u,v) \in E} w(u,v) \text{Tr} \left( L_{(u,v)} L_T^\dagger \right)$$

$$= \sum_{(u,v) \in E} w(u,v) \text{Tr} \left( (\psi_u - \psi_v)(\psi_u - \psi_v)^T L_T^\dagger \right)$$

$$= \sum_{(u,v) \in E} w(u,v) \text{Tr} \left( (\psi_u - \psi_v)^T L_T^\dagger (\psi_u - \psi_v) \right)$$

$$= \sum_{(u,v) \in E} w(u,v) \sum_{i=1}^{k} 1/w(e_i)$$

(where $e_1,\ldots,e_k$ are the edges on the simple path in $T$ from $u$ to $v$)

$$= \sum_{(u,v) \in E} \text{st}_T(u,v)$$

$$= \text{st}_T(G).$$

$\square$

Proof of Corollary 2.2. As both $L_G$ and $L_T$ are positive semi-definite, all the eigenvalues of $L_G L_T^\dagger$ are real and non-negative. The corollary follows immediately. $\square$

To show that the PCG will quickly solve linear systems in $L_G$ with $L_T$ as a preconditioner, we use the analysis of Axelsson and Lindskog [AL86 (2.4)], which we summarize as Theorem 2.5.

Theorem 2.5. Let $A$ and $C$ be positive semi-definite matrices with the same nullspace such that all but $q$ of the eigenvalues of $AC^\dagger$ lie in the interval $[l,u]$, and the remaining $q$ are larger than $u$. If $b$ is in the span of $A$ and one uses the Preconditioned Conjugate Gradient with $C$ as a preconditioner to solve the linear system $Ax = b$, then after

$$k = q + \left\lceil \frac{\ln(2/\epsilon)}{2} \sqrt{\frac{u}{l}} \right\rceil$$
iterations, the algorithm will produce a solution $x$ satisfying
\[ \|x - A^\dagger b\|_A \leq \epsilon \|A^\dagger b\|_A. \]

We recall that $\|x\|_A \overset{\text{def}}{=} \sqrt{x^T Ax}$. While Axelsson and Lindskog do not explicitly deal with the case in which $A$ and $C$ are positive-semidefinite with the same nullspace, the extension of their analysis to this case is immediate if one applies the pseudo-inverse of $C$ whenever they refer to the inverse.

**Proof of Theorem 2.2.** As $G$ and $T$ are connected, both $L_G$ and $L_T$ have the same nullspace: the span of the all-1s vector.

Set $u = (\text{st}_T(G))^{2/3}$ and $l = 1$. Corollary 2.2 tells us that $L_GL_T^\dagger$ has at most $q = (\text{st}_T(G))^{1/3}$ eigenvalues greater than $u$. The theorem now follows from Theorem 2.5.

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