Charged Lepton Flavour Violating Radiative Decays $\ell_i \rightarrow \ell_j + \gamma$

in See-Saw Models with $A_4$ Symmetry

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Abstract

The charged lepton flavour violating (LFV) radiative decays, $\mu \rightarrow e + \gamma$, $\tau \rightarrow \mu + \gamma$ and $\tau \rightarrow e + \gamma$ are investigated in a class of supersymmetric $A_4$ models with three heavy right-handed (RH) Majorana neutrinos, in which the lepton (neutrino) mixing is predicted to leading order (LO) to be tri-bimaximal. The light neutrino masses are generated via the type I see-saw mechanism. The analysis is done within the framework of the minimal supergravity (mSUGRA) scenario, which provides flavour universal boundary conditions at the scale of grand unification $M_X \approx 2 \times 10^{16}$ GeV. Detailed predictions for the rates of the three LFV decays are obtained in two explicit realisations of the $A_4$ models due to Altarelli and Feruglio and Altarelli and Meloni, respectively.
1 Introduction

Neutrino experiments revealed a totally different mixing pattern in the lepton sector compared to the one observed in the quark sector and encoded in the CKM matrix. Indeed, two lepton (neutrino) mixing angles are large \( \sin^2 \theta_{12} \approx 0.304 \) and \( \sin^2 \theta_{23} \approx 0.506 \) (3\( \sigma \)),

while the third one, the CHOOZ angle, is limited by the upper bound \( \sin^2 \theta_{13} < 0.056 \) (3\( \sigma \)).

The mixing observed in the lepton sector is remarkably close to the tri-bimaximal (TB) one proposed in \[2\]:

\[
U_{TB} = \begin{pmatrix}
\frac{\sqrt{2}}{3} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

In the case of TB mixing, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino (lepton) mixing matrix \( U \) is given, in general, by:

\[
U = U_{TB} \begin{pmatrix}
1 & \frac{\alpha_{21}}{2} & 0 \\
0 & e^{i \frac{\alpha_{31}}{2}} & 0 \\
0 & 0 & e^{i \frac{\alpha_{23}}{2}}
\end{pmatrix},
\]

where \( \alpha_{21} \) and \( \alpha_{31} \) are two Majorana CP violating phases \[3\]. The three neutrino mixing angles in the standard parametrisation of the PMNS matrix (see, e.g. \[4\]) in this case have fixed values determined by:

\[
\sin^2 \theta_{12}^{TB} = 1/3, \quad \sin^2 \theta_{23}^{TB} = 1/2 \quad \text{and} \quad \sin^2 \theta_{13}^{TB} = 0.
\]

Thus, in the case of exact TB neutrino mixing, the Dirac CP violating phase is not present in the PMNS matrix and the CP symmetry can be violated only by the Majorana phases.

It has been shown in \[5\]-\[8\] that TB mixing can arise in a class of models in which the alternating (tetrahedral) group \( A_4 \) serves as flavour symmetry group \[1\]. The light neutrino masses are generated in these models either through dimension-5 operators or through the type I see-saw mechanism. In the following we concentrate on models which contain three right-handed (RH) neutrinos and employ the latter mechanism for neutrino mass generation. The light massive neutrinos are Majorana particles. The \( A_4 \) symmetry is spontaneously broken at some high energy scale by the vacuum expectation values (VEVs) of a set of scalar fields, called flavons, which transform trivially under the gauge symmetry group of the model. In this approach, the TB mixing appears at leading order (LO). The LO results get corrected by higher order terms which can play important role in the phenomenology of the models. Such corrections are suppressed, in general, by a factor \( \epsilon \) with respect to the LO results, where \( \epsilon \) denotes the ratio of a generic flavon VEV and the cutoff scale \( \Lambda \) of the theory. The scale \( \Lambda \) is assumed to be close to the scale of grand unification. The parameter \( \epsilon \) typically takes values between a few \( \times 10^{-3} \) and 0.05.

Apart from predictions for the mixing angles, in \( A_4 \) models one can obtain also predictions for the light and heavy Majorana neutrino mass spectrum \[7\]-\[10\] and constraints on the Majorana CP violating phases. More specifically, both types of light neutrino mass spectrum -
with normal and inverted ordering, are possible, but in both cases the lightest neutrino mass is constrained to lie in certain intervals. As a consequence, one has specific testable predictions for the magnitudes of the sum of neutrino masses and of the effective Majorana mass, measured in neutrinoless double beta ($\beta \beta_{0
u}$) decay (see, e.g. [4,11]). Recently, it has been shown that the observed baryon asymmetry of the Universe can be generated in the $A_4$ models under discussion through the leptogenesis mechanism [10] (see also [12]; for earlier discussions see [13]). The CP (lepton charge) asymmetry, leading to the requisite baryon asymmetry, is proportional to the forth power of the $A_4$ flavour symmetry breaking parameter $\epsilon$ [10]. The observed relatively small value of the baryon asymmetry [14] is thus to large extent due to the smallness of $\epsilon$. Furthermore, the sign of one of the fundamental parameters of the model, which determines the light and the heavy neutrino mass spectrum, can be uniquely fixed by the requirement that the generated baryon asymmetry has the correct sign [10].

Other phenomenological predictions of this type of $A_4$ models for, e.g. the rates of charged lepton flavour violating (LFV) radiative decays $\ell_i \rightarrow \ell_j + \gamma$, $i > j$, $i,j = 1,2,3$, where $\ell_1 \equiv e$, $\ell_2 \equiv \mu$, $\ell_3 \equiv \tau$, the electric dipole moments (EDMs) and magnetic dipole moments (MDMs) of the charged leptons, have been studied using effective field theory methods in [15]. In this approach [15], a new physics scale $M$ is assumed to exist at (1 ÷ 10) TeV. The effective dimension-6 operator mediating the charged LFV radiative decays and generating new contributions to charged lepton EDMs and MDMs, is suppressed by this scale $M$, instead of the high energy scale $\Lambda$. Thus, the rates of the LFV decays $\ell_i \rightarrow \ell_j + \gamma$ and the EDM of the electron, $d_e$, can have values close to and even above the existing experimental upper limits [2]. Assuming that the flavour structure of the indicated dimension-6 operator is also determined by the $A_4$ symmetry, one finds that its form in flavour space is similar to the one of the charged lepton mass matrix. In [15] the dependence of the branching ratios $B(\ell_i \rightarrow \ell_j + \gamma)$, the EDMs and MDMs on $\epsilon$ has been analyzed in detail. It was found that the contributions of the new physics to the EDMs and MDMs arise at LO in $\epsilon$, whereas the LFV transitions are generated only at next-to-leading order (NLO). It was shown that $B(\ell_i \rightarrow \ell_j + \gamma)$ scales as $\epsilon^2$, independently of the type of the decaying lepton. Correspondingly, all charged LFV radiative decays are predicted to have similar branching ratios. The existing stringent experimental upper bound on $B(\mu \rightarrow e + \gamma)$ can be satisfied if the new physics scale $M > 10$ TeV [4]. These results were shown to be independent of the generation mechanism of the light neutrino masses.

In this paper we follow a different approach and compute the branching ratios of charged LFV radiative decays in two $A_4$ models which are based on the Minimal Supersymmetric extension of the Standard Model (MSSM) with three RH Majorana neutrinos (for an extensive list of articles in which the charged LFV radiative decays were studied see [18,19]). We choose as framework the minimal supergravity (mSUGRA) scenario, which provides flavour universal boundary conditions at the scale of grand unification $M_X \approx 2 \times 10^{16}$ GeV [5]. The SUSY breaking and the sparticle masses are completely specified by the flavour universal mass parameters $m_0$,
Dirac mass matrix. In Section 3 we identify the LO and NLO contributions relevant for our study and discuss the generic structure of the NLO corrections to the neutrino mass spectrum. In Section 4 we present results of the numerical calculations of the branching ratios of neutrino decays in the case of light neutrino mass spectrum with inverted ordering (IO) in the AF model. As a consequence, the branching ratios in the AF and AM models are significantly different from those obtained in the AM model.

In the numerical study we perform we obtain predictions for the light neutrino mass spectrum with normal ordering (NO), all branching ratios are predicted to be independent of the symmetry breaking parameter $\epsilon$, in contrast to the results found in the effective field theory approach in [15]. The suppression of the branching ratios $B(\ell_i \to \ell_j + \gamma)$ in our approach is thus a consequence of the off-diagonal elements of the slepton mass matrices being generated through RG effects and not of the smallness of $\epsilon$. A crucial difference between the predictions for $B(\ell_i \to \ell_j + \gamma)$ of the AF and the AM models is related to the fact that in the AF model there exists an additional source of suppression of $B(\ell_i \to \ell_j + \gamma)$. This suppression is due to the specific pattern of the NLO corrections (or equivalently due to the specific form of the VEVs of the flavons contributing at NLO to the neutrino Dirac mass matrix) in the AF model.

In the numerical study we perform we obtain predictions for the $\ell_i \to \ell_j + \gamma$ decay branching ratios in the AF and AM models for one specific point in the mSUGRA parameter space, which is compatible with direct bounds on sparticle masses and the requirement of having a viable dark matter (DM) candidate and the necessary amount of DM. Results for other points in the mSUGRA parameter space can be easily derived by modifying a certain rescaling function and adjusting $\tan \beta$, present in the expressions for $B(\ell_i \to \ell_j + \gamma)$. We discuss also the possibility of having $B(\mu \to e + \gamma) > 10^{-13}$ and $B(\tau \to \mu + \gamma) \approx 10^{-9}$ in the case of a relatively light sparticle mass spectrum, which is easily accessible to LHC [6]. We also show plots for specific values of the parameters of an A4 model with generic NLO corrections, which illustrate aspects of the predictions for $B(\mu \to e + \gamma)$ that cannot be seen in the scatter plots.

The paper is organized as follows: in Section 2 we present the aspects of the two A4 models, relevant for our study and discuss the generic structure of the NLO corrections to the neutrino Dirac mass matrix. In Section 3 we identify the LO and NLO contributions in $B(\ell_i \to \ell_j + \gamma)$ in the AF and AM models. This is done within the framework of mSUGRA and for the two possible types - with normal or inverted ordering, of the light neutrino mass spectrum. In Section 4 we present results of the numerical calculations of the branching ratios $B(\ell_i \to \ell_j + \gamma)$. We briefly

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$^6$For recent calculations of the rates of charged LFV decays and $\mu \to e$ conversion rates within the mSUGRA framework with three RH neutrinos and SO(10)-inspired fermion mass matrices, in which also the DM constraints are satisfied, see e.g. [25].
comment on results for $\mu - e$ conversion and the decays $\ell_i \to 3\ell_j$ in Section 5. Finally, Section 6 contains a summary of the results obtained in the present work and conclusions.

2 $A_4$ Models

The $A_4$ models [7, 8] we discuss in this paper share several features: the left-handed lepton doublets $l$ and the three RH neutrinos $\nu^c$ transform as triplets under $A_4$. In contrast, the right-handed charged lepton fields $e^c$, $\mu^c$ and $\tau^c$ are singlets under $A_4$. The Majorana mass matrix $m_M$ of the RH neutrinos is generated through the couplings:

$$a\xi(\nu^c\nu^c) + b(\nu^c\nu^c\varphi_S)$$

(2.6)

where ($\cdots$) denotes the contraction to an $A_4$ invariant and $\varphi_S \sim 3$ and $\xi \sim 1$ under $A_4$. Here and in the following we adopt the convention for the group $A_4$ as given in [7, 8]. The vacuum alignment of $\xi$ and $\varphi_S$ achieved, e.g. in [7, 8], is given by:

$$\langle \varphi_S \rangle = v_S \epsilon(1, 1, 1)^t \text{ and } \langle \xi \rangle = u \epsilon \Lambda$$

(2.7)

where $v_S$ and $u$ are assumed to be complex numbers having an absolute value of order one. The (real and positive) parameter $\epsilon$ is associated with the ratio of a typical VEV of a flavon and the cutoff scale $\Lambda$ of the theory. The generic size of $\epsilon$ is around 0.01. At the end of this section we will specify the range of $\epsilon$ in greater detail. The matrix $m_M$ can be parametrized as:

$$m_M = \begin{pmatrix} X + 2Z & -Z & -Z \\ -Z & 2Z & X - Z \\ -Z & X - Z & 2Z \end{pmatrix}.$$  

(2.8)

It contains two complex parameters $X$ and $Z$ which are conveniently expressed through their ratio $\alpha = |3Z/X|$, their relative phase $\phi = \arg(Z) - \arg(X)$ and $|X|$. The parameter $|X|$ determines the absolute mass scale of the RH neutrinos. The matrix $m_M$ is diagonalized by $U_{TB}$ so that:

$$\hat{U}_{TB} = U_{TB} \Omega \text{ with } \Omega = \text{diag}(e^{-i\varphi_1/2}, e^{-i\varphi_2/2}, e^{-i\varphi_3/2})$$

(2.9)

leads to

$$\hat{U}_{TB}^T m_M \hat{U}_{TB} = \text{diag}(M_1, M_2, M_3),$$

(2.10)

$M_i$ being the physical RH neutrino masses. It has been shown in [10] that one can set $\varphi_1 = 0$ without loss of generality. Then the phases $\varphi_2$ and $\varphi_3$ coincide at LO in the expansion parameter $\epsilon$ with the low energy Majorana phases $\alpha_{21}$ and $\alpha_{31}$ as defined in eq. (1.4).

The neutrino Yukawa couplings in the models considered read:

$$y_\nu(\nu^c l)h_u$$

(2.11)

so that the neutrino Dirac mass matrix has the simple form:

$$m_D = y_\nu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u$$

(2.12)

\footnote{In the AF model they transform as the three inequivalent one-dimensional representations 1, 1$'$ and 1$''$, whereas in the AM model all three right-handed charged lepton fields transform trivially under $A_4$. This, however, is irrelevant for our discussion here.}

\footnote{There might exist an additional direct mass term, as in the AM model, $M(\nu^c\nu^c)$. However, this term leads to the same contribution as the term $\xi(\nu^c\nu^c)$.}

\footnote{We use the convention in which the RH neutrino fields are on the left-hand side of the mass matrix and the left-handed fields are on the right-hand side.}
where $v_u$ denotes the VEV of the MSSM Higgs doublet $h_u$. We can define the matrix of neutrino Yukawa couplings as:

$$Y_\nu = \frac{m_D}{v_u}.$$  \hfill (2.13)

The light neutrino mass matrix arises from the type I see-saw mechanism:

$$m_\nu = -m_D^T m_M^{-1} m_D.$$  \hfill (2.14)

It is diagonalized by $U_{TB}$. The light neutrino masses $m_i$, $i = 1, 2, 3$, are given by:

$$m_i = \frac{y_i^2 v_\nu^2}{M_i}.$$

At LO, the charged lepton mass matrix $m_l$ is diagonal in these models. In the AF model the charged lepton masses are generated by the coupling to the flavon $\varphi_T$ with its alignment $\langle \varphi_T \rangle \propto (1, 0, 0)^t$ (and the coupling to a Froggatt-Nielsen field), whereas in the AM model they appear due to the couplings with the flavons $\varphi_T$ and $\xi'$ having the alignments $\langle \varphi_T \rangle \propto (0, 1, 0)^t$ and $\langle \xi' \rangle \neq 0$. Note, in particular, that the mass of the $\tau$ lepton stems from a non-renormalizable coupling:

$$y_\tau (\tau^c l \varphi_T) h_d / A.$$  \hfill (2.16)

Since $m_l$ is diagonal at this level, the lepton mixing originates only from the neutrino sector and is given by eq. (1.4).

This LO result gets corrected by multi-flavon insertions, as well as by shifts in the VEVs of the flavons. As a consequence, the matrices $m_{M_1}, m_{M_2}$ and $m_l$ receive corrections. Correspondingly, the lepton masses and mixings receive relative corrections of order $\epsilon$. For our study of LFV decays, the form of the corrections of the neutrino Yukawa couplings is of special interest. Instead of discussing these for the two specific models, the AF and the AM models, we give here a general parametrization of the form of these corrections. We start by writing down the co-variants for the case $\nu^c \sim 3$ and $l \sim 3$ under $A_4$:

$$(\nu^c l) = \nu_1^c l_1 + \nu_2^c l_2 + \nu_3^c l_3 \sim 1$$  \hfill (2.17)

$$(\nu^c l)' = \nu_2^c l_3 + \nu_3^c l_1 + \nu_1^c l_2 \sim 1'$$  \hfill (2.18)

$$(\nu^c l)'' = \nu_3^c l_2 + \nu_1^c l_1 + \nu_2^c l_3 \sim 1''$$  \hfill (2.19)

$$(\nu^c l)_S = \begin{pmatrix}
2\nu_1^c l_1 - \nu_2^c l_2 - \nu_3^c l_3 \\
2\nu_3^c l_3 - \nu_2^c l_1 - \nu_1^c l_2 \\
2\nu_2^c l_2 - \nu_3^c l_1 - \nu_1^c l_3 
\end{pmatrix} \sim 3_S$$  \hfill (2.20)

$$(\nu^c l)_A = \begin{pmatrix}
\nu_3^c l_2 - \nu_2^c l_3 \\
\nu_2^c l_1 - \nu_1^c l_2 \\
-\nu_3^c l_1 + \nu_1^c l_3 
\end{pmatrix} \sim 3_A$$  \hfill (2.21)

where $3_{S(A)}$ is the (anti-)symmetric triplet in the product $3 \times 3$. As one can see, the structure of $m_D$ at LO coincides with the structure coming from the $A_4$ invariant.

We shall discuss first the contributions which arise at the NLO level through multi-flavon insertions. We assume that such contributions arise at the level of one flavon insertions and are thus suppressed by $\epsilon$ relative to the LO result. This is true in the two realizations which we study below numerically.\footnote{One could also imagine models in which such contributions are suppressed stronger by higher powers of $\epsilon$. However, in the majority of the models, the NLO corrections are suppressed by $\epsilon$ only compared to the LO result.} All NLO contributions which are of the same form as the LO result
can be simply absorbed into the latter. Contributions which cannot be absorbed give rise to NLO terms of the form:

$$y_{l1}^I (\nu^c I)' \psi'' h_u / \Lambda + y_{l4}^I (\nu^c I)'' \psi' h_u / \Lambda + y_S^I (\nu^c I)_S \phi h_u / \Lambda + y_A^I (\nu^c I)_A \phi h_u / \Lambda$$ (2.22)

where $\psi'$ and $\psi''$ stand for flavons which transform as $1'$ and $1''$ under $A_4$, respectively. Here $\phi$ denotes a triplet under $A_4$ and, for simplicity, we assume that there is only one such contribution. For $\langle \psi' \rangle = w' \phi \Lambda$, $\langle \psi'' \rangle = w'' \phi \Lambda$ and $\langle \phi \rangle = (x_1, x_2, x_3)^T \epsilon \Lambda$ (with $w', w''$ and $x_i$ being complex numbers whose absolute value is of order one) we find that these induce matrix structures of the type:

$$\delta m_D = y_{l1}^I w'' \epsilon \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) v_u + y_{l4}^I w' \epsilon \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right) v_u + y_S^I \epsilon \left( \begin{array}{ccc} 2x_1 & -x_3 & -x_2 \\ -x_3 & 2x_2 & -x_1 \\ -x_2 & -x_1 & 2x_3 \end{array} \right) v_u + y_A^I \epsilon \left( \begin{array}{ccc} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{array} \right) v_u .$$ (2.23)

Apart from this type of contribution we could, in principle, find contributions arising at the relative order $\epsilon$ due to the perturbation of the VEVs of the flavons at this relative order, when NLO corrections are included into the flavon (super-)potential. However, the coupling from which the LO term in eq. (2.11) originates is generated at the renormalizable level, i.e. without involving a flavon. Thus, the most general NLO corrections to the neutrino Dirac mass matrix, $\delta m_D$, are of the form given in eq. (2.23). In explicit models the term $\delta m_D$ has usually a special form. On the one hand, the flavons in triplet representations have a certain alignment, such as $(1,1,1)^T$, $(1,0,0)^T$, $(0,1,0)^T$ or $(0,0,1)^T$. On the other hand, in such models usually there exist two different flavour symmetry breaking sectors which are separated by an additional cyclic symmetry. In most cases each of these sectors contains one triplet of flavons. Considering NLO corrections arising at the level of one flavon insertions, we expect that at least fields from one of the two flavour symmetry breaking sectors can couple at the NLO level to give rise to corrections to the neutrino Dirac mass matrix. Thus, there is only one flavon triplet contributing to $\delta m_D$ at this level. In the specific framework of the AF model, the NLO terms are given by the triplet flavon $\varphi_T$ with $\langle \varphi_T \rangle = v_T \phi \Lambda (1,0,0)^T$ ($v_T$ is complex with $|v_T| \sim O(1)$), so that we find:

$$\delta m_D = y_S^I v_T \epsilon \left( \begin{array}{ccc} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array} \right) v_u + y_A^I v_T \epsilon \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right) v_u .$$ (2.24)

In contrast, in the AM model we find that the triplet $\varphi_S$ with $\langle \varphi_S \rangle = v_S \phi \Lambda (1,1,1)^T$ gives rise to the NLO terms such that:

$$\delta m_D = y_S^I v_S \epsilon \left( \begin{array}{ccc} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{array} \right) v_u + y_A^I v_S \epsilon \left( \begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array} \right) v_u .$$ (2.25)

Similar to the neutrino Dirac mass matrix, the matrices $m_M$ and $m_l$ also receive corrections at the NLO level through multi-flavon insertions and shifts in the flavon VEVs. These corrections

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11 A contribution from the flavon $\xi$ transforming as a trivial singlet under $A_4$ can be absorbed into the LO result, as we have already indicated.
generate small off-diagonal elements in the charged lepton mass matrix \( m_L \). If the corrections are of general type, the matrix \( V_{eL} \) satisfying:

\[
V_{eL}^\dagger m_L^i m_L V_{eL} = \text{diag}(m_e^2, m_\mu^2, m_\tau^2),
\]

has the form:

\[
V_{eL} \approx \begin{pmatrix}
1 & z_{A\epsilon} & z_{B\epsilon} \\
-z_{A\epsilon} & 1 & z_{C\epsilon} \\
-z_{B\epsilon} & -z_{C\epsilon} & 1
\end{pmatrix}
\]

(2.27)

where \( \overline{z} \) denotes the complex conjugate of \( z \). The parameters \( z_i \) are, in general, complex numbers and \( |z_i| \sim O(1) \). The Majorana mass matrix \( m_M \) of the RH neutrinos also gets contributions from NLO corrections \( \delta m_M \), so that it is no longer exactly diagonalized by \( U_TB \), i.e. we have:

\[
V_R^T U_T^T (m_M + \delta m_M) U_T B V_R = \text{diag}(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3),
\]

(2.28)

where \( V_R \) is defined by:

\[
V_R \approx \begin{pmatrix}
1 & w_{A\epsilon} & w_{B\epsilon} \\
-w_{A\epsilon} & 1 & w_{C\epsilon} \\
-w_{B\epsilon} & -w_{C\epsilon} & 1
\end{pmatrix}
\]

(2.29)

The mass eigenvalues \( \tilde{M}_i \) are expected to differ from those calculated at LO, \( M_i \), by relative corrections of order \( \epsilon \). Also here the complex parameters \( w_i \) have absolute values \( |w_i| \sim O(1) \). We show in both matrices, \( V_{eL} \) and \( V_R \), the leading term in the expansion in \( \epsilon \) for each matrix element. In the two models we discuss in more detail one finds [7,8] that due to the structure of the NLO terms, not all parameters \( z_i \) and \( w_i \) in \( V_{eL} \) and \( V_R \), respectively, are arbitrary: in the AF model we have \( z_A = z_B = z_C \) with no constraints on \( w_i \), while in the AM model \( z_i \) are not related, but \( w_A = 0 \) and \( w_C = 0 \).

Finally, we comment in a more quantitative way on the size of the expansion parameter \( \epsilon \), which in turn entails constraints on the possible size of \( \tan \beta = \langle h_u \rangle / \langle d_u \rangle = v_u / v_d \). The upper bound on \( \epsilon \) comes from the requirement that the discussed NLO corrections to the lepton mixing angles do not lead to too large deviations from the experimental best fit values. The strongest constraint results from the data on the solar neutrino mixing angle and implies \( \epsilon \lesssim 0.05 \). A lower bound on \( \epsilon \) can be obtained by taking into account the fact that the Yukawa coupling of the \( \tau \) lepton should not be too large. As mentioned, a rather generic feature of the models of interest is that the \( \tau \) lepton mass is generated through a non-renormalizable operator involving one flavon. As a consequence, the following relation holds:

\[
m_{\tau} \approx |y_{\tau}| e \langle h_d \rangle = |y_{\tau}| e \frac{v}{\sqrt{2}} \cos \beta \approx |y_{\tau}| \frac{v}{\sqrt{2}} \frac{1}{\tan \beta}
\]

(2.30)

where \( v \approx 246 \text{ GeV} \). Taking \( m_{\tau} \) at the \( Z \) mass scale, \( m_{\tau}(M_Z) \approx 1.74 \text{ GeV} \) [26], we find:

\[
0.01 \approx |y_{\tau}| \frac{\epsilon}{\tan \beta}.
\]

(2.31)

Reasonable values for \( |y_{\tau}| \) are between 1/3 and 3. Using \( |y_{\tau}| = 3 \) and \( \tan \beta = 2 \) gives:

\[
\epsilon \approx 0.007.
\]

(2.32)

This is the minimal value of \( \epsilon \) in this type of models. For \( \epsilon \approx 0.05 \) one finds that \( |y_{\tau}| = 3 \) corresponds to the largest allowed value of \( \tan \beta = 15 \). All smaller values of \( \tan \beta \approx 2 \) are possible as well. In the numerical analysis we fix \( \epsilon = 0.04 \). In this case the corresponding allowed range of \( \tan \beta \) is \( 2 \lesssim \tan \beta \lesssim 12 \).

\[\text{As is well known, } \tan \beta \text{ cannot be too small [27]. We allow here for the rather low value of } \tan \beta = 2.\]
3 Charged Lepton Flavour Violating Radiative Decays

3.1 Basic Formulae

We calculate the branching ratios of the LFV processes $\ell_i \rightarrow \ell_j + \gamma$ ($m_{\ell_i} > m_{\ell_j}$) using the following expression [23,24]:

$$B(\ell_i \rightarrow \ell_j + \gamma) \approx B(\ell_i \rightarrow \ell_j + \nu_i + \bar{\nu}_j) B_0(m_0, m_{1/2}) \left| \sum_k (\hat{Y}_\nu^T)_{ik} \log \left( \frac{M_X}{M_k} \right) (\hat{Y}_\nu k)_{kj} \right|^2 \tan^2 \beta$$  (3.1)

where $\ell_1 = e$, $\ell_2 = \mu$ and $\ell_3 = \tau$. In eq. (3.1) $\hat{Y}_\nu$ is the matrix of neutrino Yukawa couplings, computed taking into account all NLO effects in the basis in which the charged lepton and RH neutrino mass matrices are diagonal and have positive eigenvalues:

$$\hat{Y}_\nu = V_R^T \Omega U_{TB}^T Y_\nu V_{eL}$$  (3.2)

where $\Omega$ is introduced in eq. (2.9) and $Y_\nu$ represents the matrix of neutrino Yukawa couplings in the basis in which the superpotential is defined (see eq. (2.13)). We consider the neutrino Dirac mass matrix $m_D$, including the generic NLO corrections given in eq. (2.28). The unitary matrices $V_R$ and $V_{eL}$ are given in eqs. (2.29) and (2.27), respectively.

According to the mSUGRA scenario we consider, at the scale of grand unification $M_X \approx 2 \times 10^{16}$ GeV, the slepton mass matrices are diagonal and universal in flavour and the trilinear couplings are proportional to the Yukawa couplings:

$$(m_{\tilde{L}}^2)_{ij} = (m_{\tilde{e}}^2)_{ij} = (m_{\tilde{\nu}}^2)_{ij} = \delta_{ij} m_0^2,$$  (3.3)

$$(A_{\nu})_{ij} = A_0 (Y_\nu)_{ij},$$  (3.4)

$$(A_{e})_{ij} = A_0 (Y_e)_{ij}, \quad A_0 = a_0 m_0,$$  (3.5)

where $m_{\tilde{L}}^2$ and $m_{\tilde{e}}^2$ are the left-handed and right-handed charged slepton mass matrices, respectively, while $m_{\tilde{\nu}}^2$ is the right-handed sneutrino soft mass term. The gaugino masses are assumed to have a common value at the high scale $M_X$:

$$M_{\tilde{B}} = M_{\tilde{W}} = M_{\tilde{g}} = m_{1/2}.$$  (3.6)

The scaling function $B_0(m_0, m_{1/2})$ contains the dependence on the SUSY breaking parameters:

$$B_0(m_0, m_{1/2}) \approx \frac{\alpha_{em}^2}{G_F^2 m_S^8} \left. \left[ \frac{(3 + a_0^2) m_0^2}{8 \pi^2} \right] \right|^2.$$  (3.7)

In eq. (3.7), $G_F$ is the Fermi constant and $\alpha_{em} \approx 1/137$ is the fine structure constant. The SUSY mass parameter $m_S$ in eq. (3.7) was obtained by performing a fit to the exact RG results [23].

The resulting analytic expression in terms of $m_0$ and $m_{1/2}$ has the form [23]:

$$m_S^8 \approx 0.5 m_0^2 m_{1/2}^2 \left( m_0^2 + 0.6 m_{1/2}^2 \right)^2.$$  (3.8)

According to [23], deviations from the exact RG result can be present in the region of relatively large (small) $m_{1/2}$ and small (large) $m_0$. In Fig. 1 we show the dependence of $B_0(m_0, m_{1/2})$ on $m_{1/2}$ for fixed values of $m_0$ and $A_0 = 0$. We notice that the function $B_0$ and, consequently, the branching ratio in eq. (3.1), can vary up to three to four orders of magnitude, depending on which point $(m_0, m_{1/2})$ of the parameter space is considered. For smaller values of $m_{1/2}$ we have larger LFV branching ratios or, equivalently, the larger is the SUSY mass parameter $m_S$, the stronger is the suppression of the predicted branching ratio. Values of $A_0 \neq 0$ lead to larger values of $B_0(m_0, m_{1/2})$ (see eq. (3.7)) and to an increase of the branching ratios, see eq. (3.1).

In the numerical analysis we set $A_0 = 0$, unless otherwise specified.
Figure 1: The dependence of the scaling function $B_0(m_0,m_{1/2})$ (see eq. (3.7)) on $m_{1/2}$ for $A_0 = 0$ and fixed $m_0$: i) $m_0 = 100$ GeV (black, dotted line), ii) $m_0 = 400$ GeV (red, dot-dashed line), iii) $m_0 = 700$ GeV (green, dashed line) and iv) $m_0 = 1000$ GeV (blue, continuous line).

3.2 Identifying the Leading Order Contributions in $B(\ell_i \to \ell_j + \gamma)$

We work for convenience with LFV branching ratios given in eq. (3.1), normalized to the partial branching ratios of the $\mu$ or $\tau$ decays into lighter charged lepton and two neutrinos:

$$B'(\ell_i \to \ell_j + \gamma) = \frac{B(\ell_i \to \ell_j + \gamma)}{B(\ell_i \to \ell_j + \nu_i + \bar{\nu}_j)}.$$  (3.9)

We have: $B(\mu \to e + \gamma) \approx B'(\mu \to e + \gamma)$, $B(\tau \to e + \gamma) \approx 0.18 B'(\tau \to e + \gamma)$ and $B(\tau \to \mu + \gamma) \approx 0.17 B'(\tau \to \mu + \gamma)$ [28].

It proves useful to analyze separately the contributions in the LFV branching ratios, which are associated with each of the three heavy RH Majorana neutrinos. For this purpose we rearrange the terms in eq. (3.1) in the following way:

$$B'(\ell_i \to \ell_j + \gamma) \propto |(\hat{Y}_i^\dagger \hat{Y}_j)|_{ij} \log \left( \frac{m_1}{m_{A_4}} \right) + (\hat{Y}_i^\dagger)_{i2}(\hat{Y}_2^\dagger)_{2j} \log \left( \frac{m_2}{m_{A_4}} \right) + (\hat{Y}_i^\dagger)_{i3}(\hat{Y}_3^\dagger)_{3j} \log \left( \frac{m_3}{m_{A_4}} \right) |^2$$  (3.10)

with

$$m_{\ast} = \frac{v_2 y_{\nu}^2}{M_X} \approx (1.5 \times 10^{-3} \text{ eV}) y_{\nu}^2 \sin^2 \beta \approx 1.5 \times 10^{-3} \text{ eV}$$  (3.11)

where we have fixed $y_{\nu} = 1$ and used $\sin^2 \beta \approx 1$, which is a good approximation given the fact that $\tan \beta \gtrsim 2$. In eq. (3.10), $m_k$, $k = 1, 2, 3$, are the LO neutrino masses in the $A_4$ models, given by eq. (2.15) and generated via the type I see-saw mechanism. We neglect contributions to $m_k$ which arise from NLO corrections in the superpotential since the branching ratio depends only logarithmically on the light neutrino masses and typically all such relative corrections are of order $\epsilon \approx (0.007 \div 0.05)$, as discussed in the previous section. We also do not consider RG effects in the calculation of the neutrino masses and mixings. Such corrections can be relevant in the
the off-diagonal elements of the hermitian matrix \( \hat{\mathcal{V}} \) always originate from NLO corrections in the superpotential. These coefficients correspond to the expression for the neutrino Yukawa couplings \( \hat{\mathcal{V}} \) and depend on the expansion parameter \( \epsilon \) (eqs. (2.12) and (2.23)), as reported in Table 1. In what concerns the coefficients of the term \( O(\epsilon) \) terms in \( \hat{\mathcal{V}} \), associated with the diagonalisation of the RH neutrino Majorana mass term, do not appear. At order \( \epsilon \), they depend only on the parameters of the neutrino Dirac mass matrix (see eqs. (2.12) and (2.23)), as reported in Table 1.

Neutrino Mass Spectrum with Normal Ordering

In the case of NO mass spectrum, the three logarithms in eq. (3.10) are all positive and of the same order (see Fig. 2 left panel). The dominant contribution to the decay amplitude depends strongly on the combination of \( \hat{\mathcal{Y}} \) matrix elements in eq. (3.10). Note that the matrix elements of \( \hat{\mathcal{Y}} \) take all \( O(1) \) values, except for the (31) entry which typically scales as the expansion parameter \( \epsilon \). This is due to the presence of the TB mixing matrix \( U_{TB} \) in the expression for the neutrino Yukawa couplings \( \hat{\mathcal{Y}}_{\nu} \), eq. (3.2).

In Table 1 we give the order of magnitude in \( \epsilon \) of the coefficients of the three logarithms, \( (\hat{Y}_{\nu})_{ij}, (\hat{Y}_{\nu})_{i2}(\hat{Y}_{\nu})_{2j} \) and \( (\hat{Y}_{\nu})_{i3}(\hat{Y}_{\nu})_{3j} \), which appear in the three branching ratios \( B(\ell_i \rightarrow \ell_j + \gamma) \) of interest. The VEVs of the flavon fields are assumed to be real for simplicity. As Table 1 shows, the coefficient of the log\((m_2/m_1)\) term in each of the three LFV branching ratios under discussion is of order one. The same conclusion is valid for the coefficient of the log\((m_3/m_1)\) term in \( B(\tau \rightarrow \mu + \gamma) \). In what concerns the coefficients of the term \( \propto \log(m_1/m_*) \), they always originate from NLO corrections in the superpotential. These coefficients correspond to the off-diagonal elements of the hermitian matrix \( \hat{\mathcal{V}} \), in which the rotation matrices, \( U_{TB} \), \( \Omega \) and \( V_R \), associated with the diagonalisation of the RH neutrino Majorana mass term, do not appear. At order \( \epsilon \), they depend only on the parameters of the neutrino Dirac mass matrix (see eqs. (2.12) and (2.23)), as reported in Table 1.

| \( \mu \rightarrow e + \gamma \) | \( \tau \rightarrow e + \gamma \) | \( \tau \rightarrow \mu + \gamma \) |
|---|---|---|
| \((\hat{Y}_{\nu})_{12}(\hat{Y}_{\nu})_{21}\), \((\hat{Y}_{\nu})_{13}(\hat{Y}_{\nu})_{31}\) | \((\hat{Y}_{\nu})_{13}(\hat{Y}_{\nu})_{32}\), \((\hat{Y}_{\nu})_{32}(\hat{Y}_{\nu})_{22}\), \((\hat{Y}_{\nu})_{33}(\hat{Y}_{\nu})_{32}\) | |
| \( y_{\nu} \left( w'' \hat{y}_{1\nu}'' + w'y_{1\nu}'' - x_2(y''_{1\nu} + y''_{2\nu}) - x_3(y''_{1\nu} - y''_{2\nu}) \right) \epsilon + O(\epsilon^2) \) | \( y_{\nu} \left( w'' y_{1\nu}'' + y' \hat{y}_{1\nu}' - x_2(y''_{1\nu} - y''_{2\nu}) + x_3(y''_{1\nu} - y''_{2\nu}) \right) \epsilon + O(\epsilon^2) \) | \( y_{\nu} \left( w'' \hat{y}_{1\nu}'' + w'y_{1\nu}'' - 2x_2 y''_{2\nu} + 2x_3 \hat{y}_{2\nu}' \right) \epsilon + O(\epsilon^2) \) |
| \( \frac{1}{2} y_{\nu}^2 + O(\epsilon) \) | \( \frac{1}{2} y_{\nu}^2 + O(\epsilon) \) | \( -\frac{1}{2} y_{\nu}^2 + O(\epsilon) \) |

Table 1: Combination of elements of the matrix of neutrino Yukawa couplings, \( \hat{\mathcal{Y}}_{\nu} \), which enter into the expression for the branching ratios of the LFV decay \( \ell_i \rightarrow \ell_j + \gamma \) (see eq. (3.10)). The expression for the relevant \( O(\epsilon) \) terms in \( (\hat{Y}_{\nu})_{ij} \) (i \( \neq \) j) is also given (see text for details).
The sum of the terms proportional to \( \log(m_3/m_1) \) vs \( m_1 \) (continuous line), \( \log(m_2/m_1) \) vs \( m_1 \) (dotted line) and \( \log(m_3/m_1) \) vs \( m_1 \) (dotted line). The results shown correspond to the best fit values \[ \Delta m^2_\text{eff} = 2.40 \times 10^{-3} \text{ eV}^2 \text{ and } r = \Delta m^2_\odot/\Delta m^2_\text{A} = 0.032. \]

Taking into account the magnitude of the different terms shown in Table I, we expect, in general, that in the case of NO neutrino mass spectrum:

\[
B'(\mu \to e + \gamma) \approx B'(\tau \to e + \gamma) \approx B_0(m_0, m_{1/2}) \left[ \frac{1}{3} y_{\nu}^2 \log \left( \frac{m_2}{m_1} \right) \right]^2 \tan^2 \beta \propto 0.1 |y_{\nu}|^4 , \quad (3.12)
\]

\[
B'(\tau \to \mu + \gamma) \approx B_0(m_0, m_{1/2}) \left[ \frac{1}{3} y_{\nu}^2 \log \left( \frac{m_2}{m_1} \right) - \frac{1}{2} y_{\nu}^2 \log \left( \frac{m_3}{m_1} \right) \right]^2 \tan^2 \beta \propto |y_{\nu}|^4 . \quad (3.13)
\]

From the analytical estimates, eqs. (3.12) and (3.13), we conclude that in the case of NO mass spectrum, \( B'(\tau \to \mu + \gamma) \) is approximately by one order of magnitude larger than \( B'(\tau \to e + \gamma) \) and \( B'(\mu \to e + \gamma) \).

**Neutrino Mass Spectrum with Inverted Ordering**

As can be seen in Fig. 2 (right panel), the term proportional to \( \log(m_2/m_1) \) is strongly suppressed with respect to the other terms in the case of IO mass spectrum. This is valid for all values of the lightest neutrino mass allowed in the models of interest, \( m_3 \gtrsim 0.02 \text{ eV} \). More specifically, one has: \( \log(m_2/m_1) \approx 0.014 \) (0.003) for \( m_3 = 0.02 \text{ eV} \). In the case of non-QD neutrino mass spectrum (\( m_3 \lesssim 0.1 \text{ eV} \)), \( B'(\ell_i \to \ell_j + \gamma) \) are determined practically by the sum of the terms proportional to \( \log(m_1/m_\ell) \) and \( \log(m_3/m_1) \). The second term increases as \( m_3 \) decreases towards the minimal allowed value \( m_3 \approx 0.02 \text{ eV} \), so that \( |\log(m_3/m_1)| \approx 1 \) (0.1) for \( m_3 = 0.02 \text{ eV} \). Thus, taking into account the results reported in Table I, we have in the LO approximation in \( \epsilon \):

\[
B'(\mu \to e + \gamma) \approx B'(\tau \to e + \gamma) \propto \mathcal{O}(\epsilon^2) , \quad (3.14)
\]

\[
B'(\tau \to \mu + \gamma) \propto \left| \frac{1}{2} y_{\nu}^2 \log \left( \frac{m_3}{m_1} \right) \right|^2 \approx \begin{cases} 0.25 |y_{\nu}|^4 , & \text{for } m_3 = 0.02 \text{ eV} , \\ 0.0025 |y_{\nu}|^4 , & \text{for } m_3 = 0.1 \text{ eV} . \end{cases} \quad (3.15)
\]
The shown order of magnitude estimates for $B'(\mu \to e + \gamma)$ and $B'(\tau \to e + \gamma)$ in eq. (3.14) can be significantly modified by the rather large contribution of the term containing the factor $\log(m_1/m_\nu) \approx (3.5 \div 4.5)$. It follows from Table I that for, e.g. $\epsilon \approx 0.04$, the contribution in the LFV branching ratios due to the indicated term can be $\sim \epsilon \log(m_1/m_\nu) \approx 1/5 \sim \sqrt{\epsilon}$ such that the branching ratios of the decays $\mu \to e + \gamma$ and $\tau \to e + \gamma$ scale as $\mathcal{O}(\epsilon)$. For $m_3 \approx 0.1$ eV, $B'(\mu \to e + \gamma)$ and $B'(\tau \to e + \gamma)$ can be comparable to the normalized branching ratio of $\tau \to \mu + \gamma$ decay, eq. (3.15). Indeed, for $m_3 \approx 0.1$ eV and $\epsilon \approx 0.04$, owing to the interplay between the leading term in the expansion parameter $\epsilon$, $\log(m_3/m_\nu)$, whose absolute value decreases with increasing of $m_3$, and the contribution from $\log(m_1/m_\nu)$, $B'(\tau \to \mu + \gamma)$ scales as few times $\epsilon$.

Comparing the results for the NO and the IO neutrino mass spectrum we see that in a model with generic NLO corrections to the matrix of neutrino Yukawa couplings, the magnitude of the branching ratio $B'(\tau \to \mu + \gamma)$ practically does not depend on the type of neutrino mass spectrum. For $\epsilon \approx 0.007$, $B'(\mu \to e + \gamma)$ and $B'(\tau \to e + \gamma)$ in the case of IO spectrum can be by one order of magnitude smaller than in the case of NO spectrum, while if $\epsilon \approx 0.04$, these two branching ratios are predicted to be essentially the same for the two types of spectrum. We always have (independently of the type of the spectrum and of the value of $\epsilon$) $B'(\mu \to e + \gamma) \approx B'(\tau \to e + \gamma)$.

In the next section we study numerically the LFV processes in the AF and AM models. One important difference between the two models is in the predicted off-diagonal elements of the hermitian matrix $\hat{Y}_\nu^\dagger \hat{Y}_\nu$. In the AM model they are all of $\mathcal{O}(\epsilon)$ and originate from the NLO corrections to the Dirac mass matrix, eq. (2.23). The exact expressions for the matrix elements can be derived using Table I and setting $w' = w'' = 0$ and $(x_1, x_2, x_3) = (1, 1, 1) v_S$. In what concerns the AF model, the VEV structure of the flavon fields, $w' = w'' = 0$ and $(x_1, x_2, x_3) \propto (1, 0, 0)$, implies that the leading term in the off-diagonal elements of the matrix $\hat{Y}_\nu^\dagger \hat{Y}_\nu$ is of $\mathcal{O}(\epsilon^2)$. This receives contributions from the Dirac mass term, see eq. (2.24), as well as from the charged lepton sector (through $V_{\nu L}$, see eq. (2.27)). This difference in the $\epsilon$ dependence of the elements of $\hat{Y}_\nu^\dagger \hat{Y}_\nu$ in the two models leads to different predictions for the LFV branching ratios for the IO neutrino mass spectrum. As a consequence, in the AM model the branching ratios of the decays $\mu \to e + \gamma$ and $\tau \to e + \gamma$ are up to two orders of magnitude larger than those in the AF model. In contrast, for $m_3 < 0.1$ eV, we expect similar results in both models for the decay $\tau \to \mu + \gamma$ since the coefficient of the term proportional to $\log(m_3/m_1)$ is of order $\epsilon^0$.

We note that in the case of a QD light (heavy) neutrino mass spectrum, $m_3 \gtrsim 0.1$ eV, the term proportional to $\log(m_1/m_\nu)$ in eq. (3.10) gives the dominant contribution and thus the magnitude of the non-diagonal elements of $\hat{Y}_\nu^\dagger \hat{Y}_\nu$ determines the magnitude of the branching ratios of the LFV decays.

The preceding discussion shows that in the $A_4$ models, the LO structure of the matrix of neutrino Yukawa couplings $\hat{Y}_\nu$, which is determined by $U_{TB}$, together with the possibility of having a heavy RH neutrino mass spectrum with partial hierarchy, leads to LFV decay rates scaling as $\mathcal{O}(\epsilon^0)$. This prediction differs significantly from the one obtained in the effective field theory approach. In [15] the branching ratios of the charged LFV radiative decays were shown to scale as $\epsilon^2$ in a generic effective field theory framework, and could even be stronger suppressed (scaling as $\epsilon^4$) in a specific supersymmetric scenario.

Concerning the absolute magnitude of the branching ratios we remark that these are expected to be of similar size in both approaches, because the suppression due to (positive) powers of $\epsilon$ present in the effective field theory approach corresponds in our case to the suppression factor associated to the fact that flavour violating soft slepton masses are generated only through RG
running. The scales \( m_S \) and \( M \), which are the relevant scales for charged LFV radiative decays, in our approach and in the effective field theory one \([15]\), respectively, can be related to each other. Assuming that the mass scale \( M \) arises from one-loop effects of new particles, such as SUSY particles, we see that the mass \( m_S \) of these new particles is identified with \( M \) weighted with the coupling \( g \) of these particles to the charged leptons and divided by the loop factor \( 4\pi \). Thus, we roughly have: \( m_S \sim gM/(4\pi) \).

4 Numerical Results

In this section we report results of the calculations of branching ratios of the LFV decays \( \mu \to e + \gamma \), \( \tau \to e + \gamma \) and \( \tau \to \mu + \gamma \). This is done in the form of scatter plots showing the correlations between each two of the indicated branching ratios. The calculations are performed in the framework of the AF and AM models. The expansion parameter \( \epsilon \) is set equal to 0.04 in the numerical analyses.

We consider a scenario in which the sparticle mass spectrum is moderately heavy:

\[
\begin{align*}
  m_0 &= 150 \text{ GeV}, \quad m_{1/2} = 700 \text{ GeV}, \quad A_0 = 0 \text{ GeV}, \quad \tan \beta = 10. \tag{4.1}
\end{align*}
\]

The parameters in eq. (4.1) lead to squark masses between 1.1 TeV and 1.5 TeV, gluino masses around 1.6 TeV, and masses of right-handed sleptons are 300 GeV. Thus, these sparticles are accessible at LHC. This point in the mSUGRA parameter space belongs to the stau co-annihilation region \([30–32]\), in which the amount of DM in the Universe can be explained through the lightest sparticle (LSP). The latter is a bino-like neutralino and has a mass of approximately 280 GeV \([34]\). As has been shown in \([31]\), the stau co-annihilation and the bulk regions are hardly affected, if RH neutrinos are included into the mSUGRA context. For the set of parameters in eq. (4.1), all decay rates scale with the factor \( B_0(m_0, m_{1/2}) \tan^2 \beta \approx 3.8 \times 10^{-10} \).

The scatter plots are obtained by varying all the \( O(1) \) parameters that enter in the matrix of neutrino Yukawa couplings \( \hat{Y}_\nu \), defined in eq. (3.2). Some of these parameters are equal to zero or have a common value. More specifically, in the AM model \( w_A = w_C = 0 \) and in the AF model \( z_A = z_B = z_C \). The NLO corrections to the Dirac mass matrices for the AF and AM models are given in eqs. (2.24) and (2.25), respectively. In the calculations of the normalized branching ratios \( B'(\ell_i \to \ell_j + \gamma) \), we set \( y_\nu = 1 \) and the absolute values of all the other (complex) parameters in \( \hat{Y}_\nu \) are varied in the interval \([0.5, 2]\). The corresponding phases are varied between 0 and \( 2\pi \).

The results obtained for the AF and the AM models and for both the NO and IO light neutrino mass spectrum are presented graphically in Figs. 3 and 4, respectively. The scatter plots correspond to three values of the lightest neutrino mass: i) \( m_1 = 3.8 \times 10^{-3} \) eV, \( 5 \times 10^{-3} \) eV and \( 7 \times 10^{-3} \) eV (NO spectrum); ii) \( m_3 = 0.02 \) eV, \( 0.06 \) eV and \( 0.1 \) eV (IO spectrum). In all numerical calculations we neglect the RG effects on neutrino masses and mixings. This is a sufficiently good approximation provided the light neutrino mass spectrum is not QD \([29]\). In the class of \( A_4 \) models we consider, the latter condition is fulfilled for the NO spectrum since the lightest neutrino mass \( m_1 \) is constrained to lie in the interval (3.8 ÷ 7) \times 10^{-3} \) eV. In the case of IO spectrum, the condition is approximately satisfied for \( m_3 \lesssim 0.1 \) eV. Correspondingly, for IO spectrum we present results for \( 0.02 \) eV \( \lesssim m_3 \lesssim 0.1 \) eV.

\( ^{13} \)The sparticle masses quoted above have been calculated with ISAJET 7.69 \([33]\).
Figure 3: Correlation between $B'(\mu \to e + \gamma)$, $B'(\tau \to e + \gamma)$ and $B'(\tau \to \mu + \gamma)$, calculated in the AF model. The results shown are obtained for three different values of the lightest neutrino mass for both types of neutrino mass spectrum: $i$) with normal ordering (left panels), $m_1 = 3.8 \times 10^{-3}$ eV (red ×), $m_1 = 5 \times 10^{-3}$ eV (green +) and $m_1 = 7 \times 10^{-3}$ eV (blue ◦); $ii$) with inverted ordering (right panels), $m_3 = 0.02$ eV (red ×), $m_3 = 0.06$ eV (green +) and $m_3 = 0.1$ eV (blue ◦). The horizontal dashed line corresponds to the MEGA bound \cite{35}, $B'(\mu \to e + \gamma) \leq 1.2 \times 10^{-11}$. The horizontal continuous line corresponds to $B'(\mu \to e + \gamma) = 10^{-13}$, which is the prospective sensitivity of the MEG experiment \cite{36}.

4.1 Predictions of the AF model

The results for the AF model are shown in Fig. 3. In the case of NO spectrum (left panels in Fig. 3), the normalized branching ratios $B'(\mu \to e + \gamma)$ and $B'(\tau \to e + \gamma)$, defined in eq. (3.9), are approximately the same, as the analysis performed in Section 3 suggested. The branching ratios are larger for smaller values of the lightest neutrino mass $m_1$, the dominant contribution being due to the term $\propto \log(m_2/m_1)$ which is a decreasing function of $m_1$ (Fig. 2 left panel). The same feature is exhibited by the term $\propto \log(m_3/m_1)$. The latter is multiplied by a coefficient of $O(\epsilon)$. As we have already indicated, the term $\propto \log(m_1/m_2)$ in the AF model is suppressed, being of $O(\epsilon^2)$, and has a negligible effect on the results. Due to the fact that the coefficient
of the term \( \propto \log(m_3/m_1) \) in \( B'(\tau \to \mu + \gamma) \) is of order one, the normalized branching ratio of \( \tau \to \mu + \gamma \) decay is approximately by a factor of ten larger than those of \( \mu \to e + \gamma \) and \( \tau \to e + \gamma \) decays, which is consistent with the analytic estimates given in eqs. (3.12) and (3.13).

We observe that, for the set of boundary conditions we have chosen, eq. (4.1), the MEGA upper limit \( m_1 = 3.8 \times 10^{-3} \) eV. This important experimental constraint can be satisfied for larger values of the lightest neutrino mass and, in particular, for the two other chosen values of \( m_1 \), \( m_1 = 5 \times 10^{-3} \) eV and \( m_1 = 7 \times 10^{-3} \) eV. However, \( B(\mu \to e + \gamma) \) is always larger than \( 10^{-12} \) and thus is within the range of sensitivity of the MEG experiment \( m_1 = 150 \) GeV, \( m_{\mu_3} = 700 \) GeV, \( A_0 = 0, \tan \beta = 10 \)

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We observe that, for the set of boundary conditions we have chosen, eq. (4.1), the MEGA upper limit \( m_1 = 3.8 \times 10^{-3} \) eV. This important experimental constraint can be satisfied for larger values of the lightest neutrino mass and, in particular, for the two other chosen values of \( m_1 \), \( m_1 = 5 \times 10^{-3} \) eV and \( m_1 = 7 \times 10^{-3} \) eV. However, \( B(\mu \to e + \gamma) \) is always larger than \( 10^{-12} \) and thus is within the range of sensitivity of the MEG experiment \( m_1 = 150 \) GeV, \( m_{\mu_3} = 700 \) GeV, \( A_0 = 0, \tan \beta = 10 \)

Figure 4: The same as in Fig. 3 but for the AM model.
term $\propto \log(m_2/m_1)$, which in the case of NO mass spectrum gives the dominant contribution, is strongly suppressed since $m_2$ and $m_1$ are nearly equal, $m_2 \cong m_1$, and partly due to the fact that the coefficient of the term proportional to $\log(m_1/m_*)$ is of order $\epsilon^2$. This conclusion is valid for all allowed values of the lightest neutrino mass, $m_3 \gtrsim 0.02$ eV (Fig. 2 right panel). In contrast to the case of a NO neutrino mass spectrum, the branching ratios of $\mu \to e + \gamma$ and $\tau \to e + \gamma$ decays do not show any significant dependence on the lightest neutrino mass, $m_3$. At the same time, the $\tau \to \mu + \gamma$ decay branching ratio exhibits a strong dependence on the value of $m_3$. Indeed, it varies by up to two orders of magnitude when $m_3$ is varied from 0.02 eV to 0.1 eV (Fig. 3 right bottom panel). The magnitude and the behaviour of $B'(\tau \to \mu + \gamma)$ as a function of $m_3$ is determined by the term proportional to $\log(m_3/m_1)$ in the right-hand side of eq. (3.10). It has a maximal value for $m_3 = 0.02$ eV and decreases as $m_3$ increases, following the decreasing of $\log(m_3/m_1)$. As a consequence of the suppression of the coefficient of the $\log(m_1/m_*)$ term, the analytic estimates reported in eqs. (3.11) and (3.15) are valid. Thus, the $\tau \to \mu + \gamma$ decay has a branching ratio which, at least for $m_3 \approx 0.02$ eV, is by approximately two orders of magnitude larger than those of the two other charged LFV radiative decays. For $m_3 = 0.02$ eV we have $B'(\tau \to \mu + \gamma) \approx 10^{-10}$. Therefore as like in the case of NO spectrum, the predicted $B'(\tau \to \mu + \gamma)$ for the values of mSUGRA parameters considered is below the sensitivity range of the currently planned experiments.

4.2 Predictions of the AM model

Our results for the AM model and both types of neutrino mass spectrum are illustrated in Fig. 4. As was discussed above, the main difference with the AF model is in the prediction for the coefficient of the term $\log(m_1/m_*)$. In the AM model this coefficient is of $O(\epsilon)$ for the three radiative decays and, therefore, the term $\propto \log(m_1/m_*)$ is not negligible. Obviously, $\log(m_1/m_*)$ is a monotonically increasing function of the lightest neutrino mass (Fig. 2). Since the coefficient of this logarithm is a number with absolute value of order one, for both types of neutrino mass spectrum the $\mu \to e + \gamma$, $\tau \to e + \gamma$ and $\tau \to \mu + \gamma$ decay branching ratios exhibit much weaker dependence on the lightest neutrino mass compared to the dependence they show in the AF model. Most importantly, as a consequence of the contribution due to the term $\propto \log(m_1/m_*)$, $B'(\mu \to e + \gamma)$ and $B'(\tau \to e + \gamma)$ in the case of IO spectrum are predicted to be of the same order of magnitude as in the case of NO spectrum (Fig. 4). This is in sharp contrast to the predictions of the AF model.

We find that the predictions for $B'(\tau \to \mu + \gamma)$ in the cases of NO and IO spectrum essentially do not differ and are similar to those obtained in the AF model. As Fig. 4 shows, for both the NO and IO mass spectrum we get $B(\mu \to e + \gamma) < 1.2 \times 10^{-11}$ in roughly half of the parameter space explored. At the same time, in practically all the parameter space considered we find that $B(\mu \to e + \gamma) \gtrsim 10^{-13}$. The $\tau$ LFV radiative decays are predicted to proceed with rates which are below the sensitivity range of the planned experiments.

4.3 On the Possibility of Large $B(\mu \to e + \gamma) > 10^{-13}$ and $B(\tau \to \mu + \gamma) \approx 10^{-9}$ in the AF Model

As we have seen, for the point in the mSUGRA parameter space considered the $\tau \to e + \gamma$ and $\tau \to \mu + \gamma$ decay branching ratios are predicted to be compatible with the existing experimental upper bounds and below the sensitivity of the future planned experiments. However, the decay $\tau \to \mu + \gamma$ might have a rate within the sensitivity range of the future experiments if the SUSY particle masses are smaller (i.e., the effective SUSY mass scale $m_S$, eq. (3.5) is lower) than those
resulting from eq. (4.1). This possibility can be realized for smaller values (than those we have employed) of the mass parameters $m_0$ and $m_{1/2}$. Indeed, consider the following set of values:

$$m_0 = 70 \text{ GeV}, \quad m_{1/2} = 300 \text{ GeV}, \quad A_0 = 70 \text{ GeV}, \quad \tan \beta = 10.$$

(4.2)

For the values given in eq. (4.2) squarks can be as light as 500 GeV, gluinos have masses of approximately 700 GeV and all sleptons have masses smaller than 250 GeV. The LSP providing the correct amount of DM in the Universe is bino-like and has a mass of 115 GeV. The parameters given in eq. (4.2) correspond also to a point in the stau co-annihilation region, very close to the region excluded by the LEP2 data \cite{30,32}: the mass of the lightest Higgs boson is near 114.4 GeV \cite{14}. For the indicated values of the SUSY breaking parameters the predicted LFV branching ratios are larger than those corresponding to the mSUGRA point in eq. (4.1) since $B_{0}(m_0, m_{1/2}) \tan^2 \beta \approx 2.3 \times 10^{-8}$. As a result, the AM model is strongly disfavored by the experimental limit on $B(\mu \rightarrow e + \gamma)$. In the AF model the latter constraint cannot be satisfied, if the neutrino mass spectrum is with NO. In the case of IO mass spectrum, however, the predicted $B(\mu \rightarrow e + \gamma)$ is compatible with the MEGA bound in nearly half of the region of the relevant parameter space and (with the exception of singular specific points) is within the sensitivity reach of the MEG experiment. We show in Fig. 5 left panel, the correlation between the normalized branching ratios of the decays $\mu \rightarrow e + \gamma$ and $\tau \rightarrow \mu + \gamma$ in the AF model, assuming IO light neutrino mass spectrum. The prospective sensitivity of the searches for the $\tau \rightarrow \mu + \gamma$ decay, which can be reached at a SuperB factory, $B(\tau \rightarrow \mu + \gamma) \approx 10^{-9}$ \cite{38}, is also indicated.

\footnote{All masses have been calculated again by using the program ISAJET 7.69 \cite{33}.}
Assuming a scenario in which in the MEG experiment it is found that $B(\tau \rightarrow e + \gamma)$ might be detectable at a SuperB factory if the lightest neutrino mass $m_3 \approx 0.02$ eV. For $m_3 = 0.02$ eV, the $(\beta\beta)_{0v}$-decay effective Majorana mass is predicted [10] to lie in the interval $m_{ee} \approx (0.018 \pm 0.054)$ eV. Values of $m_{ee}$ in the indicated interval might be probed in some of the next generation of $(\beta\beta)_{0v}$-decay experiments (see, e.g. [39, 40]). In Fig. 5, right panel, we show the correlation between the normalized branching ratio of $\tau \rightarrow \mu + \gamma$ decay and the effective Majorana mass $m_{ee}$. The relation between $m_{ee}$ and the lightest light neutrino mass is discussed in [8, 10, 12]:

$$m_{ee} \approx \frac{\sqrt{m_3^2 + |\Delta m_A^2|} + 2 + e^{i\alpha_{21}}}{3},$$

where $m_3$ and $\alpha_{21}$ are both functions of one parameter and thus their values are correlated. We indicate the prospective sensitivity of the GERDA II and GERDA III phase, $m_{ee} = 0.09$ eV and $m_{ee} = 0.02$ eV, respectively [39]. As one can see, with a positive signal of $B(\tau \rightarrow e + \gamma) \approx 10^{-9}$ at a SuperB factory values of $m_{ee}$ up to $m_{ee} \approx 0.04$ eV can be probed. The sum of neutrino masses reads $[10] \sum m_i \approx 0.125$ eV for $m_3 \approx 0.02$ eV. This value is smaller than the current cosmological bounds (see, e.g. [41]), but is within the sensitivity expected to be reached by combining data on weak lensing of galaxies by large scale structure with data from WMAP and PLANCK experiments (see, e.g. [12]).

### 4.4 Specific Features of the Predictions for $B(\mu \rightarrow e + \gamma)$

Apart from discussing generic results for the AF and the AM model it is also interesting to have a closer look at particular points in the parameter space of the $A_4$ models (with generic NLO corrections). In order to do so we use the analytic formula given in Section 3.2., eq.(3.10), for the branching ratio of the decay $\mu \rightarrow e + \gamma$ together with the results given in Table 1 and assume
that the coefficients of the $O(\epsilon)$ terms are real and have the same absolute value $C > 0$:

$$B'(\mu \rightarrow e + \gamma) \propto \left| \frac{1}{3} y_{\nu}^2 \log \left( \frac{m_2}{m_1} \right) + C \epsilon \left( \sigma_1 \log \left( \frac{m_1}{m_*} \right) + \sigma_2 \log \left( \frac{m_2}{m_1} \right) + \sigma_3 \log \left( \frac{m_3}{m_1} \right) \right) \right|^2 .$$

(4.3)

We do not fix the relative sign of these terms and allow for all eight combinations $\sigma_{1,2,3} = \pm 1$. We choose for the mSUGRA parameters the values given in eq. (4.1), set $r = y_{\nu} = 1$ and take again best fit values for $r$ and $|\Delta m_3^2|$. In Fig. 6 left panel, we plot the result for $B(\mu \rightarrow e + \gamma)$ for $C = 1.3$ in the case of a NO light neutrino mass spectrum with respect to the effective Majorana mass $m_{ee}$, $m_{ee} \approx \sqrt{2m_1^2 + \Delta m_3^2}/3$. We show only the two curves which correspond to the upper and lower bound that can be reached for the eight different combinations of $\sigma_{1,2,3}$. As one can see, there exists the possibility of cancellations between the terms contributing to the branching ratio of the $\mu \rightarrow e + \gamma$ decay, so that the value of the latter can be strongly suppressed $^{16}$. The value of $m_{ee}$ at which the suppression takes place depends on the value of $C$. In Fig. 6 right panel, we show the corresponding plot for IO neutrino mass spectrum. We choose $C = 1$. In contrast to the case of NO spectrum, no strong suppression of $B(\mu \rightarrow e + \gamma)$ is possible, because the term $\propto \log(m_1/m_*)$ always dominates (see Fig. 2 right panel). This result holds for all values of the constant $C$ from the interval $0.1 \lesssim C \lesssim 6$. Allowing for arbitrary relative phases between the different contributions in the right-hand side of eq. (4.3), we find that for a NO light neutrino mass spectrum the curve for $\sigma_{1,2,3} = +1$ corresponds to an upper bound on $B(\mu \rightarrow e + \gamma)$, whereas the curve for $\sigma_{1,2,3} = -1$ is an absolute lower bound with the exception of few points in the parameter space. For the IO spectrum, the bounds obtained for real coefficients are also upper and lower bounds in the case of arbitrary relative phases between the different terms in eq. (4.3).

As mentioned earlier, the preceding analysis holds for an $A_4$ model with generic NLO corrections, as is the case of the AM model. In order to perform a similar analysis for the AF model $^{16}$ we replace $\sigma_1 \log(m_1/m_*)$ with $\epsilon \sigma_1 \log(m_1/m_*)$ in eq. (4.3). We find that deep cancellations between the different contributions in $B(\mu \rightarrow e + \gamma)$ can occur in both the cases of NO and IO neutrino mass spectrum. For the IO spectrum, the cancellations leading to a strong suppression of branching ratio $B(\mu \rightarrow e + \gamma)$ take place for $m_{ee}$ around 0.09 eV for almost all values of $C$ in the range considered, $0.1 \lesssim C \lesssim 6$.

5 The $\mu - e$ Conversion and $\ell_i \rightarrow 3\ell_j$ Decay Rates

We briefly discuss in this section the experimental constraints that can be imposed on the $A_4$ models from data on $\mu - e$ conversion and the decays $\ell_i \rightarrow 3\ell_j$. In the mSUGRA scenario, these LFV processes are dominated by the contribution coming from the $\gamma$-penguin diagrams. As a consequence, for $\mu - e$ conversion, the following relation holds with a good approximation $^{22}$:

$$CR(\mu N \rightarrow e N) \equiv \frac{\Gamma(\mu N \rightarrow e N)}{\Gamma_{\text{capt}}} = \frac{a_{em}^4 G_F^2 m_\mu^5 Z}{12\pi^3 \Gamma_{\text{capt}}} Z_{e,e}^4 F(q^2)^2 B(\mu \rightarrow e + \gamma).$$

(5.4)

In eq. (5.4) $Z$ is the proton number in the nucleus $N$, $F(q^2)$ is the nuclear form factor at momentum transfer $q$, $Z_{e,e}$ is an effective atomic charge and $\Gamma_{\text{capt}}$ is the experimentally known

$^{15}$Note that the value of $B(\mu \rightarrow e + \gamma)$ will still be non-zero in general, because we expect corrections to the coefficients of the different logarithms of order $\epsilon^2$.

$^{16}$We remind the reader that in the AF model the coefficient of the logarithm $\log(m_1/m_*)$ is of order $\epsilon^2$ rather than $\epsilon$. 

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are characterised by the parameters \( m \) this scenario the flavour universal slepton masses, trilinear ar couplings and the gaugino masses \( \beta \) are \( \tan \beta \) to \( \tau \). The searches for \( \mu \) article the branching ratios work within the framework of the minimal supergravity (mSUGRA) scenario, which provides bimaximal (TB). The light neutrino masses are generated via the type I see-saw mechanism. We neutrinos, in which the lepton (neutrino) mixing is predicted to leading order (LO) to be tri-

\[ \frac{m_{\tau}}{m_{\mu}} \approx 0.1 \]

(5.5)

Future experimental searches for \( \mu - e \) conversion can reach the sensitivity: \( CR(\mu_{22}^{48}Ti \rightarrow e_{22}^{48}Ti) \approx 10^{-18} \) [43], and \( CR(\mu_{13}^{27}Al \rightarrow e_{13}^{27}Al) \approx 10^{-16} \) [44]. The upper bound \( B(\mu \rightarrow e + \gamma) < 10^{-13} \) which can be obtained in the MEG experiment would correspond to the following upper bounds on the \( \mu - e \) conversion ratios under discussion: \( CR(\mu_{22}^{48}Ti \rightarrow e_{22}^{48}Ti) < 5 \times 10^{-16} \) and \( CR(\mu_{13}^{27}Al \rightarrow e_{13}^{27}Al) < 2.7 \times 10^{-16} \). The latter could be probed by future experiments on \( \mu - e \) conversion, which have higher prospective sensitivity. We see that for the mSUGRA points considered in Section 4, both the AF and AM models can be further constrained by the experiments on \( \mu - e \) conversion if the \( \mu \rightarrow e + \gamma \) decay will not be observed in the MEG experiment.

In what concerns the decay of a charged lepton into three charged leptons, the branching ratio is approximately given by [22]:

\[ B(\ell_i \rightarrow 3\ell_j) \approx \frac{\alpha}{3\pi} \left( \log \left( \frac{m_{\ell_i}^2}{m_{\ell_j}^2} \right) - \frac{11}{4} \right) B(\ell_i \rightarrow \ell_j + \gamma) . \]  

The searches for \( \tau \rightarrow \mu + \gamma, \tau \rightarrow 3\mu \) and \( \tau \rightarrow 3e \) decays at SuperB factories [38] will be sensitive to \( B(\tau \rightarrow \mu + \gamma), B(\tau \rightarrow 3\mu), B(\tau \rightarrow 3e) \geq 10^{-9} \). Therefore, if in the experiments at SuperB factories it is found that \( B(\tau \rightarrow \mu + \gamma) < 10^{-9} \), obtaining the upper limits \( B(\tau \rightarrow 3\mu), B(\tau \rightarrow 3e) < 10^{-9} \) would not constrain further the \( A_4 \) models considered here. However, the observation of the \( \tau \rightarrow 3\mu \) decay with a branching ratio \( B(\tau \rightarrow 3\mu) \geq 10^{-9} \), combined with the upper limit \( B(\tau \rightarrow \mu + \gamma) < 10^{-9} \), or the observation of the \( \tau \rightarrow \mu + \gamma \) decay with a branching ratio \( B(\tau \rightarrow \mu + \gamma) \geq 10^{-9} \), would rule out the \( A_4 \) models under discussion.

The current limit on the \( \mu \rightarrow 3e \) decay branching ratio is \( B(\mu \rightarrow 3e) < 10^{-12} \) [45]. There are no plans at present to perform a new experimental search for the \( \mu \rightarrow 3e \) decay with higher precision.

6 Conclusions

In this paper we have studied charged lepton flavour violating (LFV) radiative decays, \( \mu \rightarrow e + \gamma, \tau \rightarrow \mu + \gamma \) and \( \tau \rightarrow e + \gamma \) in a class of supersymmetric \( A_4 \) models with three heavy RH Majorana neutrinos, in which the lepton (neutrino) mixing is predicted to leading order (LO) to be tri-bimaximal (TB). The light neutrino masses are generated via the type I see-saw mechanism. We work within the framework of the minimal supergravity (mSUGRA) scenario, which provides flavour universal boundary conditions at the scale of grand unification \( M_X \approx 2 \times 10^{16} \) GeV. In this scenario the flavour universal slepton masses, trilinear couplings and the gaugino masses are characterised by the parameters \( m_0, A_0 \) and \( m_{1/2} \) respectively. The other free parameters are \( \tan \beta \) and sign(\( \mu \)). Flavour off-diagonal elements in the slepton mass matrices are generated through RG effects (from the scale \( M_X \) to the scale of heavy RH Majorana neutrino masses). The former induce the LFV decays \( \mu \rightarrow e + \gamma, \tau \rightarrow \mu + \gamma \) and \( \tau \rightarrow e + \gamma \). In the present article the branching ratios \( B(\ell_i \rightarrow \ell_j + \gamma) \) are calculated using the analytic approximations developed in [22][24]. In this approach \( B(\ell_i \rightarrow \ell_j + \gamma) \) depend only on the matrix of neutrino
Yukawa couplings $\tilde{Y}_\nu$ defined in the flavour basis, on the three heavy RH Majorana neutrino masses, and on an “average” SUSY mass scale, $m_S$ (see eq. (3.8)). In the class of $A_4$ models considered, each of the three light Majorana neutrino masses $m_i$ is directly related at LO to the corresponding heavy Majorana neutrino mass $M_i$, eq. (2.15). Both types of light neutrino mass spectrum (with normal ordering (NO) and inverted ordering (IO)), are possible. The lightest neutrino masses, corresponding to the two types of spectrum, are constrained to lie in the intervals: $m_1 \approx (3.8 \div 7.0) \times 10^{-3}$ eV (NO) and $m_3 \gtrsim 0.02$ eV (IO). The RG effects on the neutrino masses and mixing angles were not taken into account. As is well known, these effects are relatively small if the lightest neutrino mass satisfies $\text{min}(m_j) < 0.10$ eV. For this reason we limited our analysis in the case of IO spectrum to $m_3 \lesssim 0.10$ eV.

The analytic estimates of the branching ratios $B(\ell_i \rightarrow \ell_j + \gamma)$ we have given were made for the case of generic next-to-leading order (NLO) corrections to the neutrino Yukawa matrix. The numerical results we presented, however, are obtained for two explicit realizations of the $A_4$ models, those by Altarelli and Feruglio (AF) [7] and by Altarelli and Meloni (AM) [8], respectively. In these models the $A_4$ symmetry is broken spontaneously at high energies and the symmetry breaking parameter $\epsilon$ has a value in the interval $0.007 \lesssim \epsilon \lesssim 0.05$. In the numerical calculations we set to $\epsilon = 0.04$. In this case the allowed range of $\tan \beta$ is $2 \lesssim \tan \beta \lesssim 12$. We used the value of $\tan \beta = 10$ in the numerical calculations.

The predictions for the $\ell_i \rightarrow \ell_j + \gamma$ decay branching ratios, $B(\ell_i \rightarrow \ell_j + \gamma)$, in the AF and AM models are derived for one specific point in the mSUGRA parameter space lying in the stau co-annihilation region, which is compatible with direct bounds on sparticle masses and the requirement of explaining the amount of dark matter (DM) in the Universe: $m_0 = 150$ GeV, $m_{1/2} = 700$ GeV and $A_0 = 0$ GeV. These values correspond to a bino-like LSP with a mass of approximately 280 GeV, which makes up the DM of the Universe, gluino masses of 1.6 TeV and squark masses between 1.1 TeV and 1.5 TeV. Having the indicated masses, the SUSY particles can be observed at LHC. Results for other points in the mSUGRA parameter space can be easily obtained by modifying the rescaling function present in the expressions for $B(\ell_i \rightarrow \ell_j + \gamma)$ (see eqs. (3.1) and (3.7)).

We have found that in the case of NO light neutrino mass spectrum, both the AF and AM models predict $B(\mu \rightarrow e + \gamma) > 10^{-13}$ in practically all the parameter space considered (Figs. 3 and 4). The same conclusion is valid for the IO mass spectrum in the case of the AM model, whereas for the AF model this result holds roughly in half of the parameter space of the model. Values of $B(\mu \rightarrow e + \gamma) \gtrsim 10^{-13}$ can be probed in the MEG experiment which is taking data at present.

The predictions of the AF model for all the three branching ratios $B(\ell_i \rightarrow \ell_j + \gamma)$ in the case of NO spectrum show a noticeable dependence on the value of the lightest neutrino mass. The dependence of $B(\tau \rightarrow \mu + \gamma)$ on $\text{min}(m_j)$ is particularly strong in the case of IO spectrum. In contrast, $B(\mu \rightarrow e + \gamma)$ and $B(\tau \rightarrow e + \gamma)$ in this case vary relatively little with $\text{min}(m_j)$. The predictions for $B(\ell_i \rightarrow \ell_j + \gamma)$ in the AM model do not exhibit significant dependence on $\text{min}(m_j)$. The branching ratios $B(\tau \rightarrow e + \gamma)$ and $B(\tau \rightarrow \mu + \gamma)$ are always predicted to be below the sensitivity of the present and future planned experiments. We have shown, however, that if the SUSY particles are lighter, e.g. the LSP has a mass of 115 GeV, squarks are as light as 500 GeV and gluinos have masses $\approx 700$ GeV, one can have $B(\mu \rightarrow e + \gamma) \gtrsim 10^{-13}$ and $B(\tau \rightarrow \mu + \gamma) \approx 10^{-9}$ in the AF model with IO spectrum (Fig. 5 left panel). A value of $B(\tau \rightarrow \mu + \gamma) \approx 10^{-9}$ requires the lightest neutrino mass to be $m_3 \approx 0.02$ eV. Sensitivity to such a value of $B(\tau \rightarrow \mu + \gamma)$ can be achieved, in principle, at a SuperB factory [38].

We found that the dependence of the branching ratios $B(\ell_i \rightarrow \ell_j + \gamma)$ on the $A_4$ flavour symmetry breaking parameter $\epsilon$ in the case of NO spectrum differs from that in the case of
IO spectrum. For the NO spectrum, all the three branching ratios are predicted to scale as $\epsilon^0$. The normalized branching ratios $B'(\mu \rightarrow e + \gamma) \approx B(\mu \rightarrow e + \gamma)$ and $B'(\tau \rightarrow e + \gamma) = B(\tau \rightarrow e + \gamma)/B(\tau \rightarrow e + \nu_\tau + \bar{\nu}_e)$, are found to be equal at LO, while $B'(\tau \rightarrow \mu + \gamma) \equiv B(\tau \rightarrow \mu + \gamma)/B(\tau \rightarrow \mu + \nu_\tau + \bar{\nu}_\mu)$ is typically larger by one order of magnitude. In the case of IO spectrum we have at LO $B(\mu \rightarrow e + \gamma) \propto \epsilon^2$, $B'(\tau \rightarrow e + \gamma) \propto \epsilon^2$. However, for $\epsilon \approx 0.04$, this $\epsilon$ suppression is compensated by relatively large values of the logarithm $\log(m_1/m_\tau)$. Therefore in a model with generic NLO corrections to the matrix of neutrino Yukawa couplings, $B(\mu \rightarrow e + \gamma)$ and $B'(\tau \rightarrow e + \gamma)$ can have the same magnitude for the two types - NO and IO, of neutrino mass spectrum. The specific structure of the NLO contributions to the matrix of neutrino Yukawa couplings in the AF model leads to an additional suppression of the branching ratios $B(\mu \rightarrow e + \gamma)$ and $B'(\tau \rightarrow e + \gamma)$. As a consequence, $B(\mu \rightarrow e + \gamma)$ and $B'(\tau \rightarrow e + \gamma)$ in the case of IO spectrum are predicted to be smaller in the AF model than in the AM model by up to two orders of magnitude. Since $B'(\tau \rightarrow \mu + \gamma)$ always receives a contribution of order $\epsilon^0$, it has the same magnitude for the NO and IO spectrum, and in the case of IO spectrum and for $m_3 \approx 0.02$ eV it is not affected by the indicated additional suppression. For the AF model thus $B'(\tau \rightarrow \mu + \gamma)$ can be by up to three orders of magnitude larger than $B(\mu \rightarrow e + \gamma)$ and $B'(\tau \rightarrow e + \gamma)$.

The type of dependence of the branching ratios on the $A_4$ symmetry breaking parameter $\epsilon$ is in contrast to that found in the framework of an effective field theory approach in \cite{15}, in which the setup of the AF model was used. This approach suggests that $B(\ell_i \rightarrow \ell_j + \gamma)$ scale generically as $\epsilon^2$, and as $\epsilon^4$ in a specific supersymmetric version of the theory. Moreover, the results found in the effective field theory approach do not show any dependence on the type of the light neutrino mass spectrum and on the value of the lightest neutrino mass.

In \cite{10} we studied versions of the AF and AM models, in which the neutrino Yukawa couplings are suppressed, being proportional to $\epsilon$. As a consequence, the scale of the RH neutrino masses is lowered. Since the branching ratios of the charged LFV radiative decays studied in the present article are proportional to the Yukawa couplings to the forth power, suppressing the latter also efficiently suppresses the $\ell_i \rightarrow \ell_j + \gamma$ decay branching ratios. Thus, for the mSUGRA points chosen in our numerical study and for most of the relevant parameter space of the version of $A_4$ models considered in \cite{10} we expect $B(\mu \rightarrow e + \gamma)$ to be below the sensitivity of the on-going MEG experiment.

We have shown that for specific values of the parameters in an $A_4$ model with generic NLO corrections, “accidental” cancellations between the different contributions in the $\mu \rightarrow e + \gamma$ decay rates and a strong suppression of the branching ratio $B(\mu \rightarrow e + \gamma)$ are possible in the case of NO neutrino mass spectrum. The same result holds in the AF model for both types - NO and IO, of spectrum. No similar suppression is found to occur for the IO spectrum in the $A_4$ model with generic NLO corrections.

Our estimates of the predicted rate of $\mu - e$ conversion in the $A_4$ models considered show that future experiments can further constrain these models if the $\mu \rightarrow e + \gamma$ decay will not be observed in the MEG experiment. The observation at the SuperB factories of the $\tau \rightarrow 3\mu$ decay with a branching ratio $B(\tau \rightarrow 3\mu) \geq 10^{-9}$, combined with the upper limit $B(\tau \rightarrow \mu + \gamma) < 10^{-9}$, or the observation of the $\tau \rightarrow \mu + \gamma$ decay with branching ratio $B(\tau \rightarrow \mu + \gamma) \geq 10^{-9}$, would rule out the $A_4$ models under discussion. If $B(\tau \rightarrow \mu + \gamma)$ is found to satisfy $B(\tau \rightarrow \mu + \gamma) < 10^{-9}$, the prospective sensitivity of SuperB factories to the decay modes $\tau \rightarrow 3\mu$ and $\tau \rightarrow 3e$ would not allow to obtain additional constraints on the parameter space of the $A_4$ models from non-observation of the $\tau \rightarrow 3\mu$ and $\tau \rightarrow 3e$ decays.

The results of the MEG experiment which is taking data at present, and of the upcoming experiments at LHC can provide significant tests of, and can severely constrain, the class of
$A_4$ models predicting tri-bimaximal neutrino mixing, in general, and the Altarelli-Feruglio and Altarelli-Meloni models, in particular.

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References

[1] T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. 10 (2008) 113011 [arXiv:0808.2016 [hep-ph]].

[2] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530 (2002) 167 [arXiv:hep-ph/0202074]; P. F. Harrison and W. G. Scott, Phys. Lett. B 535 (2002) 163 [arXiv:hep-ph/0203209]; Z. Z. Xing, Phys. Lett. B 533 (2002) 85 [arXiv:hep-ph/0204049]; P. F. Harrison and W. G. Scott, Phys. Lett. B 547 (2002) 219 [arXiv:hep-ph/0210197]; P. F. Harrison and W. G. Scott, Phys. Lett. B 557 (2003) 76 [arXiv:hep-ph/0302025]; P. F. Harrison and W. G. Scott, [arXiv:hep-ph/0402006]; P. F. Harrison and W. G. Scott, Phys. Lett. B 594 (2004) 324 [arXiv:hep-ph/0403278].

[3] S. M. Bilenky, J. Hošek and S. T. Petcov, Phys. Lett. B 94, 495 (1980).

[4] S. M. Bilenky, S. Pascoli and S. T. Petcov, Phys. Rev. D 64 (2001) 053010 [arXiv:hep-ph/0102265]; S. T. Petcov, Nucl. Phys. Proc. Suppl. 143 (2005) 159.

[5] E. Ma and G. Rajasekaran, Phys. Rev. D 64 (2001) 113012 [arXiv:hep-ph/0106291]; K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552 (2003) 207 [arXiv:hep-ph/0206292]; M. Hirsch et al., arXiv:hep-ph/0312244; Phys. Rev. D 69 (2004) 093006 [arXiv:hep-ph/0312265]; S. L. Chen, M. Frigerio and E. Ma, Nucl. Phys. B 724 (2005) 423 [arXiv:hep-ph/0504181]; K. S. Babu and X. G. He, [arXiv:hep-ph/0507217]; X. G. He, Y. Y. Keum and R. R. Volkas, JHEP 0604 (2006) 039 [arXiv:hep-ph/0601001]; S. F. King and M. Malinsky, Phys. Lett. B 645 (2007) 351 [arXiv:hep-ph/0610250]; F. Bazzocchi, S. Kaneko and S. Morisi, JHEP 0803 (2008) 063 [arXiv:0707.3032 [hep-ph]]; G. Altarelli, F. Feruglio and C. Hagedorn, JHEP 0803 (2008) 052 [arXiv:0802.0090 [hep-ph]]; Y. Lin, Nucl. Phys. B 824 (2010) 95 [arXiv:0905.3534 [hep-ph]].

[6] G. Altarelli and F. Feruglio, Nucl. Phys. B 720 (2005) 64 [arXiv:hep-ph/0504165].

[7] G. Altarelli and F. Feruglio, Nucl. Phys. B 741 (2006) 215 [arXiv:hep-ph/0512103].

[8] G. Altarelli and D. Meloni, J. Phys. G 36 (2009) 085005 [arXiv:0905.0620 [hep-ph]].

[9] C. S. Lam, Phys. Rev. Lett. 101 (2008) 121602 [arXiv:0804.2622 [hep-ph]]; C. S. Lam, Phys. Rev. D 78 (2008) 073015 [arXiv:0809.1185 [hep-ph]]; F. Bazzocchi, L. Merlo and S. Morisi, Nucl. Phys. B 816 (2009) 204 [arXiv:0901.2080 [hep-ph]]; C. S. Lam, arXiv:0907.2206 [hep-ph]; H. Ishimori, Y. Shimizu and M. Tanimoto, Prog. Theor. Phys. 121 (2009) 769 [arXiv:0812.5031 [hep-ph]].
[10] C. Hagedorn, E. Molinaro and S. T. Petcov, JHEP **0909** (2009) 115 [arXiv:0908.0240 [hep-ph]].

[11] S. T. Petcov, Phys. Scripta **T121** (2005) 94 [arXiv:hep-ph/0504166]; S. Pascoli and S. T. Petcov, Phys. Rev. D **77** (2008) 113003 [arXiv:0711.4993 [hep-ph]]; C. Aalseth et al., hep-ph/0412300.

[12] E. Bertuzzo, P. Di Bari, F. Feruglio and E. Nardi, JHEP **0911** (2009) 036 [arXiv:0908.0161 [hep-ph]].

[13] E. E. Jenkins and A. V. Manohar, Phys. Lett. B **668** (2008) 210 [arXiv:0807.4176 [hep-ph]]; B. Adhikary and A. Ghosal, Phys. Rev. D **78** (2008) 073007 [arXiv:0803.3582 [hep-ph]]; G. C. Branco, R. Gonzalez Felipe, M. N. Rebelo and H. Serodio, Phys. Rev. D **79** (2009) 093008 [arXiv:0904.3076 [hep-ph]]; D. Aristizabal Sierra, F. Bazzocchi, I. de Medeiros Varzielas, L. Merlo and S. Morisi, arXiv:0908.0907 [hep-ph].

[14] J. Dunkley et al. [WMAP Collaboration], Astrophys. J. Suppl. **180** (2009) 306 [arXiv:0803.0586 [astro-ph]].

[15] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B **809** (2009) 218 [arXiv:0807.3160 [hep-ph]]; F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, arXiv:0808.0812 [hep-ph].

[16] S. T. Petcov, Sov. J. Nucl. Phys. **25** (1977) 340.

[17] S. M. Bilenky, S. T. Petcov and B. Pontecorvo, Phys. Lett. B **67** (1977) 309; T. P. Cheng and L. F. Li, Phys. Rev. Lett. **45** (1980) 1908.

[18] M. Raidal et al., Eur. Phys. J. C **57** (2008) 13 [arXiv:0801.1826 [hep-ph]].

[19] S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. **59** (1987) 671.

[20] F. Feruglio, C. Hagedorn and L. Merlo, arXiv:0910.4058 [hep-ph].

[21] F. Borzumati and A. Masiero, Phys. Rev. Lett. **57** (1986) 961.

[22] J. Hisano et al., Phys. Lett. B **357** (1995) 579 [arXiv:hep-ph/9501407]; J. Hisano and D. Nomura, Phys. Rev. D **59** (1999) 116005 [arXiv:hep-ph/9810479].

[23] S. T. Petcov, S. Profumo, Y. Takanishi and C. E. Yaguna, Nucl. Phys. B **676** (2004) 453 [arXiv:hep-ph/0306195].

[24] S. T. Petcov, W. Rodejohann, T. Shindou and Y. Takanishi, Nucl. Phys. B **739** (2006) 208 [arXiv:hep-ph/0510404].

[25] V. Barger, D. Marfatia, A. Mustafayev and A. Soleimani, Phys. Rev. D **80** (2009) 076004 [arXiv:0908.0911 [hep-ph]].

[26] Z. z. Xing, H. Zhang and S. Zhou, Phys. Rev. D **77** (2008) 113016 [arXiv:0712.1419 [hep-ph]].

[27] S. Schael et al. [ALEPH Collaboration and DELPHI Collaboration and L3 Collaboration and OPAL Collaborations and LEP Working Group for Higgs Boson Searches], Eur. Phys. J. C **47** (2006) 547 [arXiv:hep-ex/0602042].

24
[28] C. Amsler et al. [Particle Data Group], Phys. Lett. B 667 (2008) 1.

[29] P. H. Chankowski and Z. Pluciennik, Phys. Lett. B 316 (1993) 312 [arXiv:hep-ph/9306333]; K. S. Babu, C. N. Leung and J. T. Pantaleone, Phys. Lett. B 319 (1993) 191 [arXiv:hep-ph/9309223]; S. Antusch et al., Phys. Lett. B 519 (2001) 238 [arXiv:hep-ph/0108005]; S. Antusch, M. Drees, J. Kersten, M. Lindner and M. Ratz, Phys. Lett. B 525 (2002) 130 [arXiv:hep-ph/0110366]; S. Antusch et al., JHEP 0503 (2005) 024 [arXiv:hep-ph/0501272]; see also, e.g. S. T. Petcov, T. Shindou and Y. Takanishi, Nucl. Phys. B 738 (2006) 219 [arXiv:hep-ph/0508243].

[30] H. Baer, C. Balazs, A. Belyaev, T. Krupovnickas and X. Tata, JHEP 0306 (2003) 054 [arXiv:hep-ph/0304303].

[31] J. R. Ellis, K. A. Olive, Y. Santoso and V. C. Spanos, Phys. Lett. B 565 (2003) 176 [arXiv:hep-ph/0303043].

[32] H. Baer and A. D. Box, arXiv:0910.0333 [hep-ph].

[33] F. E. Paige, S. D. Protopopescu, H. Baer and X. Tata, arXiv:hep-ph/0312045.

[34] V. Barger, D. Marfatia and A. Mustafayev, Phys. Lett. B 665 (2008) 242 [arXiv:0804.3601 [hep-ph]].

[35] M. L. Brooks et al. [MEGA Collaboration], Phys. Rev. Lett. 83 (1999) 1521 [arXiv:hep-ex/9905013].

[36] A. Maki, AIP Conf. Proc. 981 (2008) 363.

[37] B. Aubert [The BABAR Collaboration], arXiv:0908.2381 [hep-ex].

[38] M. Bona et al., arXiv:0709.0451 [hep-ex]; A. G. Akeroyd et al. [SuperKEKB Physics Working Group], arXiv:hep-ex/0406071.

[39] A. A. Smolnikov et al. [GERDA Collaboration], arXiv:0812.4194 [nucl-ex].

[40] A. Giuliani [CUORE Collaboration], J. Phys. Conf. Ser. 120 (2008) 052051.

[41] G. L. Fogli et al., Phys. Rev. D 78 (2008) 033010 [arXiv:0805.2517 [hep-ph]].

[42] M. Tegmark, Phys. Scripta T121 (2005) 153 [arXiv:hep-ph/0503257]; S. Hannestad, H. Tu and Y. Y. Y. Wong, JCAP 0606 (2006) 025 [arXiv:astro-ph/0603019]; J. Lesgourgues et al., Phys. Rev. D 73 (2006) 045021 [arXiv:astro-ph/0511735].

[43] Y. Mori et al. [The PRIME Working Group], “An Experimental Search for $\mu^- \rightarrow e^-$ Conversion Process at an Ultimate Sensitivity of the Order of $10^{-18}$ with PRISM”, LOI-25.

[44] E. C. Dukes et al. [Mu2e Collaboration], “Proposal to Search for $\mu^- N \rightarrow e^- N$ with a Single Event Sensitivity Below $10^{-16}$”, FERMILAB-PROPOSAL-0973.

[45] U. Bellgardt et al. [SINDRUM Collaboration], Nucl. Phys. B 299 (1988) 1.