Numerical Treatment of Mixed Volterra-Fredholm Integral Equations Using Trigonometric Functions and Laguerre Polynomials

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A R T I C L E  I N F O

A B S T R A C T

In this paper, numerical solution of linear mixed Volterra-Fredholm integral equations of the second kind by using trigonometric functions and Laguerre polynomials approximation accompanied with the least square technique is presented. For the explanation of the idea and more illustration, an algorithm is introduced, and several examples are solved. Also, comparison between the exact and the approximate solutions are given to show the efficiency of the methods and accuracy of the results.

1. Introduction

Many problems in mathematical physics, mechanics, many related fields of engineering, mixed boundary value problems, and contact problems in the theory of elasticity lead to integral equations, (Abdou, M., 2002), (Mkhitarian, S. and Abdou, M., 1990) and (Aleksandrow, V., 1968). These integral equations have received considerable interest in a different area of sciences in mathematics Applications, (Wazwaz, A., 2011) and (Constanda, C., 1995). The solution of Volterra-Fredholm integral equation can be obtained theoretically in (Abdou, M., 2003), (Aghajani, A., et al., 2012) and (Abdou, M., 2002) . At the same time, in (Maleknejad, K., Hadizadeh, M., 1999) and (Shehab, S. et al., 2010) the sensing of numerical methods takes an important place in solving this type of equations. Double integral equations had been
treated numerically by (Abdou, M. 2005), (Kauthen, P., 1989), (Wang, Q. and Wang, K.), (Ezza
ti, R. and Najafalizadeh, S., 2012), and (Ahmed, S., 2011) where different methods are
used.

Throughout this paper, we consider the integral equation of the form
\[
\int_{a}^{b} k(r,t)u(t)dt = 0; \quad x \in [a, b]
\]
where \( f(x) \) and \( k(x,t) \) are known on the interval \([a, b]\), and \( u(x) \) is the continuous
function to be determined.

Orthogonal functions and polynomials receive attention in dealing with various problems. One of those is integral equations. The main property of using orthogonal basis is that it reduces these problems to a system of linear algebraic equations by seeking a solution of the form:
\[
\sum_{i=0}^{n} a_i \phi_i(x)
\]
where \( \phi_0(x), \phi_1(x), \ldots, \phi_n(x) \) are the orthogonal functions defined on a certain
interval \([a, b]\). Here we would like to choose \( \phi_i(x) \) as trigonometric functions on \([-\pi, \pi]\) or Laguerre polynomials on \([0, \infty)\) with the least
square approximation for solving this type of integral equation.

2. Principal Concepts

This section deals with some definitions and concepts (Atkinson, K., 1997) and (Delves, L.
and Walsh, J., 1974) which are used in this work.

2.1. Orthogonal Polynomials

Two functions \( p(x) \) and \( q(x) \) are orthogonal over the interval \([a, b]\) with weight
function \( w(x) \) if
\[
\int_{a}^{b} p(x)q(x)w(x)dx = 0
\]

2.2. Trigonometric Functions (T-F)

A trigonometric polynomial of order \( n \) is a function of \( x \) of the form
\[
p(x) = a_0 \tau_0(x) + \sum_{i=1}^{n} (a_i \tau_{2i-1}(x) + b_i \tau_{2i}(x))
\]
where \( \tau_0(x) = 1 \), \( \tau_{2i-1}(x) = \cos ix \), and \( \tau_{2i}(x) = \sin ix \) for \( i = 1, \ldots, n \), while the
coefficients \( a_0, a_1, \ldots, a_n, b_1, \ldots, b_n \) are real numbers such that \( a_n \neq 0 \), or \( b_n \neq 0 \).

The set \( \{\tau_0(x), \tau_1(x), \ldots, \tau_{2n}(x)\} \) is orthogonal
set on \([-\pi, \pi]\) with respect to the weight
function \( w(x) = 1 \). This orthogonally follows
from the fact that for every integer \( i \) the integrals of \( \sin ix \) and \( \cos ix \) over \([-\pi, \pi]\) are
zero. They are periodic functions with period \(2\pi\). Hence we can regard them as element of
the space \([-\pi, \pi]\), then the space of trigonometric polynomials of order \( n \) will be
denoted by \( T_n \) that is
\[
T_n = \{ p \in C[-\pi, \pi], p(x) = a_0 \tau_0(x) + \sum_{k=1}^{n} (a_k \tau_{2k-1}(x) + b_k \tau_{2k}(x)) \}
\]

2.3. Laguerre Polynomials (L-P)

The general form of the Laguerre polynomials of \( nth \) degree is defined by
\[
L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k
\]
The Rodrigues representation for them is
they are orthogonal on \([0, \infty)\) with the weigh function \(w(x) = e^{-x}\), and satisfies recurrence relation

\[
L_{n+1}(x) = \frac{2n+1-x}{n+1} L_n(x) - \frac{n}{n+1} L_{n-1}(x); \quad n \geq 2
\]

where \(L_0(x) = 1\) and \(L_1(x) = 1 - x\).

### 3. Function Approximation

It is sometimes more comfortable to seek an approximate solution \(\tilde{u}_n(x)\) in terms of delineative functions \(\Phi_0(x), \Phi_1(x), \ldots, \Phi_n(x)\), which depends on \(n\) and independent on the kernel \(k(x, t)\).

In this section, we find an approximate solution of equation (1) in the form that defined in equation (2). The substitution of \(\tilde{u}_n(x)\) in the presented integral equation will gives

\[
\tilde{u}_n(x) = f(x) + \int_0^x \int_a^b k(r,t)\tilde{u}_n(t) dt dr
+ e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n)
\]  

where \(e_n\) is the error term which depends on \(x\) and on the way that \(\alpha_i\)’s are chosen, then

\[
\sum_{i=0}^n \alpha_i \Phi_i(x) = f(x) + \int_0^x \int_a^b k(r,t) \sum_{i=0}^n \alpha_i \Phi_i(t) dt dr
+ e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n)
\]

Thus

\[
e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n) =
\sum_{i=0}^n \alpha_i \left( \Phi_i(x) - \int_0^x \int_a^b k(r,t) \Phi_i(t) dt dr \right) - f(x)
\]

Put

\[
y_i(x) = \Phi_i(x) - \int_0^x \int_a^b k(r,t) \Phi_i(t) dt dr; \quad i = 0, 1, \ldots, n
\]

So equation (7) becomes:

\[
e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n)
= \sum_{i=0}^n \alpha_i y_i(x) - f(x)
\]  

Let

\[
E(\alpha_0, \alpha_1, \ldots, \alpha_n) = \int_a^b [e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n)]^2 w(x) dx
\]

where \(w(x)\) is any positive function defined on \([a, b]\) which is called the weight function [15], then

\[
E(\alpha_0, \alpha_1, \ldots, \alpha_n) = \int_a^b \left( \sum_{i=0}^n \alpha_i \phi_i(x) \right)^2 w(x) dx
- \int_a^b \int_a^b k(r,t) \phi_i(t) f(x) w(x) dt dx
\]

The main point here is how to find the coefficients \(\alpha_i\)’s, \(i = 0, 1, \ldots, n\) such that the error is minimized, and this equivalent to finding best approximation of the solution of the presented integral equation. Here the necessary condition for obtaining the minimum value of \(E\) is

\[
\frac{\partial E}{\partial \alpha_i} = 0, \quad i = 0, 1, \ldots, n
\]

That is for each \(i = 0, 1, \ldots, n\), we can get

\[
\frac{\partial E}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} \int_a^b [e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n)]^2 w(x) dx
= 2 \int_a^b w(x) e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n) \frac{\partial}{\partial \alpha_i} [e_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n)] dx
\]
Thus, we have

\[
\sum_{j=0}^{n} \alpha_j \left( \int_{a}^{b} k(r, t) \phi_j(t) dt \right) dx
\]

Finally by putting the last equation as a system of linear algebraic equations the following results will be concluded

\[
\sum_{j=0}^{n} \alpha_j y_j(x) y_i(x) w(x) dx = \int_{a}^{b} f(x) y_i(x) w(x) dx
\]

\[
\sum_{j=0}^{n} \alpha_j y_{ij} = \beta_i \quad i = 0, 1, \ldots, n.
\]

for \( i = 0, 1, \ldots, n \) and \( j = 0, 1, \ldots, n \)

\[
Y_n A = B
\]

where,

\[
Y_n = \begin{bmatrix}
y_{00} & y_{01} & \cdots & y_{0n} \\
y_{10} & y_{11} & \cdots & y_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
y_{n0} & y_{n1} & \cdots & y_{nn}
\end{bmatrix}
\]

\[
A = [\alpha_0, \alpha_1, \ldots, \alpha_n]^T,
\]

and

\[
B = [\beta_0, \beta_1, \ldots, \beta_n]^T.
\]

By applying the collocation of points in the form \( x_i = a + ih; \) for \( i = 0, 1, \ldots, n \) and \( h = (b - a)/n \) and solving the system of linear equations, the unknown points \( \alpha_0, \alpha_1, \ldots, \alpha_n \) will be obtained, then they will be substituted in equation (2) to get the approximate solution of integral equation (1).

4. The Solution of LMV-FIE2nd with (TF) and (LP)

In this section, we solve equation (1) by using the trigonometric functions and Laguerre polynomials based on the above technique.

4.1. Using Trigonometric Functions (TF)

Here the unknown function \( u(x) \) in Equation (1) can be expanded in terms of trigonometric functions as follows:

\[
\hat{u}_n(x) = \sum_{i=0}^{n} \alpha_i \tau_i(x)
\]

where \( \tau_i(x) \) are as defined in section 2.2., by applying the steps which described in section 3 we will get the system (9) that could be easily solved to get the values of \( \alpha_i \)'s. In this case, \( w(x) = 1 \) and equation (9a-9b) will be replaced by

\[
\begin{align*}
y_{ij} &= \int_{a}^{b} y_i(x) y_j(x) w(x) dx \\
\beta_i &= \int_{a}^{b} y_i(x) f(x) w(x) dx
\end{align*}
\]

while

\[
y_i(x) = \varphi_i(x)
\]

\[
\int_{0}^{x} \int_{a}^{b} k(r, t) \varphi_i(t) dt dr
\]

Rewriting equation (9) in the matrix form yields:

\[
Y_n A = B
\]

where,
and 
\[ y_i(x) = \tau_i(x) - \int_0^x \int_a^b k(r,t) \tau_i(t) dt dr \]  
(11b)

At last, by substituting these values of \( \alpha_i \)'s in equation (10) we obtain the approximate solution \( \tilde{u}_n(x) \) of \( u(x) \).

### 4.2. Using Laguerre Polynomials (L-P)

By substituting an approximate solution of the form
\[ \tilde{u}_n(x) = \sum_{i=0}^{n} \alpha_i L_i(x) \]  
(12)

in equation (1), where \( L_i(x) \), \( i = 0, 1, \ldots, n \) are Laguerre polynomials defined in equation (4), then by following the same description in section (3), we get the system of equations (9), while here

\[
\begin{align*}
    y_{ij} &= \int_0^b y_i(x) y_j(x) e^{-x} dx \\
    \beta_j &= \int_0^b y_i(x) f(x) e^{-x} dx 
\end{align*}
\]  
(12a)

and
\[ y_i(x) = L_i(x) - \int_0^x \int_a^b k(r,t) L_i(t) dt dr \]  
(12b)

then the values of \( \alpha_0, \alpha_1, \ldots, \alpha_n \) will be founded and then substituted in equation (11) to get the solution of equation (1).

### 5. The Algorithm

In this section, we will consider each step of the method which is solved by MATLAB and we distinguish the algorithm in six steps:

**Step 1:** Input the value of \( n \) (the number of participant functions \( \phi_i(x) \), \( i = 0, 1, \ldots, n \)).

**Step 2:** Let
\[ \tilde{u}_n(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x) \]
be an approximate solution of equation (1), where \( \phi_i(x) \) are Trigonometric functions (T-F) or Laguerre polynomials (L-P).

**Step 3:** Input \( \tau_i(x) \) (the trigonometric functions), or \( L_i(x) \) (the Laguerre polynomials), where \( i = 0, 1, \ldots, n \).

**Step 4:** For all \( i = 0, 1, \ldots, n \) and \( j = 0, 1, \ldots, n \)

In the first case (T-F), evaluate
- \( y_i(x) \) from equation (11b).
- \( y_{ij} \) and \( \beta_i \) in equation (11a).

while in the second case (L-P), compute
- \( y_i(x) \) in equation (12b).
- \( y_{ij} \) and \( \beta_i \) using equation (12a).

**Step 5:** In both cases, solve the system of equations (10) to get the values of \( \alpha_0, \alpha_1, \ldots, \alpha_n \).

**Step 6:** substitute these values of \( \alpha_i \)’s in the assumption that we have in the first step to get the approximate solution \( \tilde{u}_n(x) \) of equation (1).

### 6. Numerical examples

In this section, we study some examples and demonstrate that, in spite of the above mentioned method in the previous section, the computations, associated with the examples are performed by MATLAB.

**Example 1:** Consider the LMV-FIE2
\[ u(x) = 2 + 4x - \frac{9}{5}x^2 - 5x^3 \]
\[ + \int_0^x \int_0^1 (r-t)u(t) dt dr; 0 \leq x \leq 1 \]
the exact solution is
\[ u(x) = 2 + 3x - 5x^3 \]

Using the formula derived in the previous section and solving the system of equations for \( n = 4 \), we get the following approximate solutions:

1- By using Trigonometric Functions:

Here we take equation (11) as an approximation of \( u(x) \), by applying the algorithm with its program and taking the value of \( n = 4 \), we have the formula

\[
\tilde{u}_5(x) = (-16.0447931435) + (23.3670600343) \cos(x) - (0.5810322888) \sin(x) - (5.3274104457) \cos(2x) + (1.8660347551) \sin(2x)
\]

and we get the results of the solution and consequently the least square error (L.S.E.) that shown in the Table 1, where \( L.S.E. = \sum_i (u(x_i) - u_n(x_i))^2 \) for some \( i \).

1- By Laguerre Polynomials:

In equation (11), put \( n = 4 \) and use the algorithm described in section 5 with its program to get the value of \( \alpha_i's \), then we get the related approximate solution

\[
\tilde{u}_4(x) = -25 + 87(-x + 1) - 90 \left( \frac{1}{2} (x^2 - 4x + 2) \right) + 30 \left( \frac{1}{6} (-x^3 + 9x^2 - 18x + 6) \right) + 0 \left( \frac{1}{24} (x^4 - 16x^3 + 72x^2 - 96x + 24) \right).
\]

Table 1 gives a comparison between the numerical results using (T-F) and (L-p) methods with \( n = 4 \), while Figure 1 gives a comparison between the exact and the approximate solution using (T-F) and (L-p) solution for different values of \( n \).

**Example 2:** Consider the LMV-FIE

\[
u(x) = \sin(x) - \frac{1}{2} (x^2 + 2x)
\]

\[
+ \int_0^x \int_0^{\pi/2} (1 + rt)u(t)dt\,dr \quad 0 \leq x \leq \frac{\pi}{2}
\]

the exact solution is

\[
u(x) = \sin(x).
\]

In the first case, we use Trigonometric functions and as defined in equation (11) with \( n = 2 \), then the values \( \alpha_0 = 0, \alpha_1 = 0, \text{and} \alpha_2 = 1 \) will be obtained, which acquires the exact solution of the problem, thus the \( L.S.E = 0 \).

While in the second case, the Laguerre Polynomials will be taken in the approximate solution as in equation (12), different values of \( n \) have been chosen in some distinct points and the results are listed in Table 2. Exact solutions and numerical results using (L-P) are given in Figure 2 for \( n = 1, 2, 3, \text{and} 4 \).

In both cases, our results are compared with the exact solutions by computing the absolute error and the L.S.E. of them.

**Example 3:** Consider the LMV-FIE

\[
u(x) = x e^x - \frac{x^2}{2}
\]

\[
+ \int_0^x \int_0^1 ru(t)dt\,dr \quad 0 \leq x \leq 1
\]

the exact solution \( u(x) = xe^x \).
Perform the prescribed steps in the above algorithm with \( n=4 \), we get the values of \( \alpha_0, \ldots, \alpha_4 \) in both cases as shown in Table 3, while Table 4 and Figure 3 present the comparison between the exact and the numerical solutions.

**Table 1.** The (T-F) & (L-P) results with \( n=4 \) compared with exact solution of example 1.

| \( x \) | Exact solution \( u(x) \) | Approximate solutions with \( n=4 \) |
|---|---|---|
| | | using (T-F) | using (L-P) |
| 0 | 2 | 1.9948564450 | 2 |
| 0.1 | 2.295 | 2.2970294341 | 2.295 |
| 0.2 | 2.560 | 2.5698463229 | 2.560 |
| 0.3 | 2.765 | 2.7636460722 | 2.765 |
| 0.4 | 2.880 | 2.8783985642 | 2.880 |
| 0.5 | 2.875 | 2.8749715021 | 2.875 |
| 0.6 | 2.720 | 2.7215870510 | 2.720 |
| 0.7 | 2.385 | 2.3864075992 | 2.385 |
| 0.8 | 1.840 | 1.8391842323 | 1.840 |
| 0.9 | 1.055 | 1.0528984137 | 1.055 |
| 1.0 | 0 | 0.0053270672 | 0 |

L.S.E. 7.364991e-05 0

**Table 2.** Absolute errors and L.S.E of example 2 using (L-P) with distinct values of \( n \).

| \( x \) | \( \alpha_0 \) | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \alpha_4 \) |
|---|---|---|---|---|---|
| 0 | 8.01471254 | -9.819635818 | 0.839835858 | 1.806476364 | 0.057682302 |
| 0.1 | 12.705876238 | -1.454772743 | 53.211914728 | -31.88929412 | 0.426502596 |

**Table 3.** The value of \( \alpha_i 's \) in (T-F) and (L-P) methods for \( n=4 \) for example 3.

| Method | (T-F) | (L-P) |
|---|---|---|
| \( \alpha_i \) | \( \alpha_0 \) | 8.01471254 | 12.705876238 |
| | \( \alpha_1 \) | -9.819635818 | -1.454772743 |
| | \( \alpha_2 \) | 0.839835858 | 53.211914728 |
| | \( \alpha_3 \) | 1.806476364 | -31.88929412 |
| | \( \alpha_4 \) | 0.057682302 | 0.426502596 |

**Table 4.** The comparison between the exact and the approximate (T-F) & (L-P) solutions of example 3 for \( n=4 \).
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### Table 1

| x   | Exact values \( u(x) \) | Approximate values \( u_n(x) \) using(T-F) | using(L-P) |
|-----|--------------------------|---------------------------------|------------|
| 0   | 0                        | 0.00155309                      | 0.00022670 |
| 0.1 | 0.11051709               | 0.10990450                      | 0.11041829 |
| 0.2 | 0.24428055               | 0.24400276                      | 0.24425685 |
| 0.3 | 0.40495764               | 0.40536376                      | 0.40503966 |
| 0.4 | 0.59672988               | 0.59723946                      | 0.59680660 |
| 0.5 | 0.82436064               | 0.82439147                      | 0.82434025 |
| 0.6 | 1.09327128               | 1.09277725                      | 1.09316582 |
| 0.7 | 1.40962690               | 1.40916255                      | 1.40955115 |
| 0.8 | 1.78043274               | 1.78067721                      | 1.78050678 |
| 0.9 | 2.21364280               | 2.21433362                      | 2.21378584 |
| 1   | 2.71828183               | 2.71652912                      | 2.71788417 |

**L. S. E** 7.35874e-06 2.75680e-07

### Figure 1

Comparison between the numerical results of example 1 using (T-F) and (L-P) for \( n = 2, 3, \) and 4.

### Figure 2

Exact solutions and numerical results of example 2 using (L-P) for \( n = 1, 2, 3, \) and 4.

### Figure 3

Exact solutions and numerical solutions of example 3 using (T-F) and (L-P) for \( n = 4. \)

### 7. Conclusion

In this paper, the trigonometric functions and Laguerre polynomials depending on the principle of the least square technique are introduced to solve the second kind linear mixed Volterra–Fredholm integral equations. Several examples are applied for illustration and good approximate (sometimes exact) results are found. Moreover, the results of (T-F) and (L-P) are compared to each other and with the exact solutions to demonstrate the propriety and implementation of the method. Also, better results have been obtained by increasing the value of \( n \) which represents the number of basis functions. The given numerical examples and the outcomes in Tables 1-4 and Figures 1-3 are supported these claims.

### 8. Appendix A

```matlab
disp('Trigonometric Polynomials ')
```
for n=2:4; 
LSE=0; 
bb=1;%.5*pi;
syms r x 
for i=1:n 
q(1)=1; 
q((2*i)-1)=cos(i*x); 
q(2*i)=sin(i*x); 
end 
q 
q=[1,cos(x),sin(x),cos(2*x),sin(2*x),cos(3*x),sin(3*x),cos(4*x),sin(4*x),cos(5*x),sin(5*x)]; 
%Ex -1- 
f=2+4*x-(9/8)*(x)^2-5*(x)^3; 
k=r-x; 
E=2+3*x-5*(x)^3; 
%Ex -2- 
%f=x*(exp(x))-(1/2)*x^2; 
%Ex -3- 
%f=sin(x)-(x*(x+2))/2; 
for i=1:n 
y(i)=q(i)-int(int(k*q(i),x,0,bb),0,x); 
end 
for i=1:n 
for j=1:n 
Y(i,j)=int(y(i)*y(j),0,bb); 
end 
end 
for i=1:n 
b(i)=int(y(i)*f,0,bb); 
end 
format long 
for p=0:1:bb; 
for i=1:n 
b(i)=subs(b(i),x,p); 
for j=1:n 
a(i,j)=subs(Y(i,j),x,p); 
end 
end 
A=inv(a); 
B=(b'); 
c=mtimes(A,B); 
% c=inv(a)*b' 
double(c) 
ss=c(1)*q(1)+c(2)*q(2)+c(3)*q(3) +c(4)*q(4)+c(5)*q(5) +c(6)*q(6); 
Ex=subs(E,x,p); 
er=Ex-s; 
e=(er)^2; 
%disp(double([Ex s er])) 
%disp(double([s Ex])) 
end 
for p=.5 
s=subs(ss,x,p); 
Ex=subs(E,x,p); 
er=Ex-s; 
e=(er)^2; 
end 

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