Multiplicative Zagreb indices of k-trees

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Abstract

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The first generalized multiplicative Zagreb index of $G$ is $\prod_{1,c}(G) = \prod_{v \in V(G)} d(v)^c$, for a real number $c > 0$, and the second multiplicative Zagreb index is $\prod_2(G) = \prod_{uv \in E(G)} d(u)d(v)$, where $d(u), d(v)$ are the degrees of the vertices of $u, v$. The multiplicative Zagreb indices have been the focus of considerable research in computational chemistry dating back to Narumi and Katayama in 1980s. In this paper, we generalize Narumi-Katayama index and the first multiplicative index, where $c = 1, 2$, respectively, and extend the results of Gutman to the generalized tree, the $k$-tree, where the results of Gutman are for $k = 1$. Additionally, we characterize the extremal graphs and determine the exact bounds of these indices of $k$-trees, which attain the lower and upper bounds.

Keywords: Multiplicative Zagreb indices, k-trees

1 Introduction

Throughout this paper $G = (V, E)$ is a connected finite simple undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $|G|$ or $|V|$ denote the cardinality of $V$. For $S \subseteq V(G)$ and $F \subseteq E(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$, $G - S$ for the subgraph induced by $V(G) - S$ and $G - F$ for the subgraph of $G$ obtained by deleting $F$. Let $w(G - S)$ be the number of components of $G - S$, and $S$ be a cut set if $w(G - S) \geq 2$. For a vertex
In the 1980s, Narumi and Katayama [7] considered the product

\[ NK = \prod_{v \in V(G)} d(v) \]

which is the "Narumi-Katayama index". And recently, Todeschini and Gutman et al [4, 10, 11] studied the first and second multiplicative Zagreb indices defined as follow:

\[
\begin{align*}
\Pi_1(G) & = \prod_{v \in V(G)} d(v)^2, \\
\Pi_2(G) & = \prod_{uv \in E(G)} d(u)d(v).
\end{align*}
\]

Obviously, the first multiplicative Zagreb index is just the square of the NK index. Gutman [4] in 2011 characterized the multiplicative Zagreb indices for trees and determined the unique trees that obtained maximum and minimum values for \(\Pi_1(G)\) and \(\Pi_2(G)\), respectively.

**Theorem 1 (Gutman 2011)** Let \(n \geq 5\) and \(T_n\) be any tree with \(n\) vertices, then

\[
\begin{align*}
(i) & \quad \Pi_1(S_n) \leq \Pi_1(T_n) \leq \Pi_1(P_n); \\
(ii) & \quad \Pi_2(P_n) \leq \Pi_2(T_n) \leq \Pi_2(S_n).
\end{align*}
\]

In this paper, we consider the first generalized multiplicative Zagreb index defined in (1) below and the second multiplicative Zagreb index: for any real number \(c > 0\),

\[
\begin{align*}
(1) & \quad \Pi_{1,c}(G) = \prod_{v \in V(G)} d(v)^c; \\
(2) & \quad \Pi_2(G) = \prod_{uv \in E(G)} d(u)d(v).
\end{align*}
\]

Eventually, for \(c = 1, 2\), (1) is just the NK index and the first multiplicative Zagreb, respectively. For (2), it is easy to see that \(\Pi_2(G) = \prod_{v \in V(G)} d(v)^{d(v)}\). Also we will find the bounds of the values of \(\Pi_{1,c}(G)\), \(\Pi_2(G)\) for \(k\)-trees, respectively, and determine the extremal graphs which attain the bounds. Our main results are as follows:

**Theorem 2** Let \(T^k_n\) be a \(k\)-tree on \(n \geq k\) vertices, then

\[ \Pi_{1,c}(S_{k,n-k}) \leq \Pi_{1,c}(T^k_n) \leq \Pi_{1,c}(P^k_n), \]

the left-side and the right-side equalities are reached if and only if \(T^k_n \cong S_{k,n-k}\) and \(T^k_n \cong P^k_n\), respectively.
Theorem 3 Let $T_n^k$ be a $k$-tree on $n \geq k$ vertices, then

$$\Pi_2(P_n^k) \leq \Pi_2(T_n^k) \leq \Pi_2(S_{k,n-k}).$$

the left-side and the right-side equalities are reached if and only if $T_n^k \cong P_n^k$ and $T_n^k \cong S_{k,n-k}$, respectively.

2 Preliminary

It is commonly known that the class of $k$-trees is an important subclass of triangular graphs. Harry and Plamer [5] first introduced the 2-tree in 1968, which is showed to be maximal outerplanar graphs in [3, 6]. Beineke and Pippert [1] gave the definition of $k$-trees in 1969. Relating to $k$-trees, there are many interesting applications to the study of a computational complexity and the intersection between graph theory and chemistry [2, 9]. We will just give some notations and definitions below.

**Notation 1.** Let $[a, b]$ be the set of all the integers between $a$ and $b$ with $a \leq b$ including $a, b$, where $a, b$ are integers. Also, let $(a, b) = [a, b] - \{a\}$ and $[a, b) = [a, b] - \{b\}$. In particular, $[a, b) = \phi$ for $a > b$.

**Notation 2.** For any integer $p$, if $p \geq 0$, we denote $x_{\max\{0,p\}} = x_p$; If $p < 0$, we say $x_{\max\{0,p\}}$ does not exist.

**Definition 1.** The $k$-tree, denoted by $T_n^k$, for positive integers $n, k$ with $n \geq k$, is defined recursively as follows: The smallest $k$-tree is the $k$-clique $K_k$. If $G$ is a $k$-tree with $n \geq k$ vertices and a new vertex $v$ of degree $k$ is added and joined to the vertices of a $k$-clique in $G$, then the obtained graph is a $k$-tree with $n + 1$ vertices.

**Definition 2.** The $k$-path, denoted by $P_n^k$, for positive integers $n, k$ with $n \geq k$, is defined as follows: Starting with a $k$-clique $G[\{v_1, v_2, \ldots v_k\}]$. For $i \in [k + 1, n]$, the vertex $v_i$ is adjacent to vertices $\{v_{i-1}, v_{i-2}, \ldots v_{i-k}\}$ only.

**Definition 3.** The $k$-star, denoted by $S_{k,n-k}$, for positive integers $n, k$ with $n \geq k$, is defined as follows: Starting with a $k$-clique $G[\{v_1, v_2, \ldots v_k\}]$ and an independent set $S$ with $|S| = n - k$. For $i \in [k + 1, n]$, the vertex $v_i$ is adjacent to vertices $\{v_1, v_2, \ldots v_k\}$ only.

**Definition 4.** A vertex $v \in V(T_n^k)$ is called a $k$-simplicial vertex if $v$ is a vertex of degree $k$ whose neighbors form a $k$-clique of $T_n^k$. Let $S_1(T_n^k)$ be the set of all $k$-simplicial vertices of $T_n^k$, for $n \geq k + 2$, and set $S_1(K_k) = \phi, S_1(K_{k+1}) = \{v\}$, where $v$ is any vertex of $K_{k+1}$.
$G = G_0, G_i = G_{i-1} - v_i$, where $v_i$ is a $k$-simplicial vertex of $G_{i-1}$, then \{v_1, v_2…v_n\} is called a
simplicial elimination ordering of the $n$-vertex graph $G$.

**Definition 5.** If $w(G - S) \leq 2$ for any $k$-clique $G[S]$ of $T^k_n$, we say $T^k_n$ is a hyper pendent
edge; If there exists a $k$-clique $G[S]$ with $w(G - S) \geq 3$, let $C$ be a component of $T^k_n - S$ and
contain a unique vertex belonging to $S_1(G)$, then we say that $G[V(S)\cup V(C)]$ is a hyper pendent
edge of $T^k_n$, denoted by $\mathcal{P}$. In particular, a $k$-path is a hyper pendent edge.

Moreover, let $G[\{v_1, v_2…v_k\}]$ denote the initial $k$-clique, then just by the definition of $k$-trees,
one can get

**Fact 1.** For the $k$-star, the degree of vertex $v_i$ can be characterized as follows: $d(v_i) = n-k$,
for $i \in [1, k]; d(v_i) = k$, for $i \in [k + 1, n]$.

**Fact 2.** For the $k$-path, the degree of vertex $v_i$ can be characterized as follows:
(1) If $4 \leq n \leq 2k$, $d(v_i) = k + i - 1$, for $i \in [1, n - k - 1]; d(v_i) = n - 1$, for $i \in [n - k, k + 1];$
    $d(v_i) = k + n - i$, for $i \in [k + 2, n]$.
(2) If $n \geq 2k + 1$, $d(v_i) = k + i - 1$, for $i \in [1, k]; d(v_i) = 2k$, for $i \in [k + 1, n - k]; d(v_i) = k + n - i$,
    for $i \in [n - k + 1, n]$.

Easily verified through induction by using the above obseaverations, one can deduce the
first generalized multiplicative Zagreb indices and second multiplicative Zagreb indices of
the $k$-path and $k$-star as follows.

**Fact 3.** Let $S_{k,n-k}$ be a $k$-star on $n \geq k + 1$ vertices, then
(1) $\prod_{1,c}(S_{k,n-k}) = (n - k)^{ck}k^{(n-k)}$;
(2) $\prod_{2}(S_{k,n-k}) = (n - k)^{k(n-k)}k^{(n-k)}$.

**Fact 4.** Let $P_n^k$ be a $k$-path on $n \geq k + 1$ vertices, then
(1.1) $\prod_{1,c}(P_n^k) = (n - 1)^c \prod_{i=k}^{n-2}i^{2c}$, if $n \in [k + 1, 2k]$;
(1.2) $\prod_{1,c}(P_n^k) = (2k)^{c(n-2k)} \prod_{i=k}^{2k-1}i^{2c}$, if $n \geq 2k + 1$;
(2.1) $\prod_{2}(P_n^k) = (n - 1)^{n-1} \prod_{i=k}^{n-2}i^{2i}$, if $n \in [k + 1, 2k]$;
(2.2) $\prod_{2}(P_n^k) = (2k)^{2k(n-2k)} \prod_{i=k}^{2k-1}i^{2i}$, if $n \geq 2k + 1$.

By considering the derivatives of the following functions, one can get

**Fact 5.** The function $f(x) = \frac{x}{x + m}$ is strictly increasing for $x \in [0, \infty)$, where $m$ is a
positive integer.

**Fact 6.** The function $f(x) = \frac{x^x}{(x + m)^{x+m}}$ is strictly decreasing for $x \in [0, \infty)$, where $m$
is a positive integer.
3 Main proofs

Firstly, we give some lemmas that are critical in the proof of our main results.

**Lemma 1**  For any $k$-tree $G \not\cong S_{k,n-k}$, let $u \in S_2$, $N(u) \cap S_1 = \{v_1, v_2...v_s\}$, where $s \geq 1$ is an integer, then

(1) For any $i$ with $1 \leq i \leq s$, there exists a vertex $v \in N(u) - \{v_1, v_2...v_s\}$ of degree at least $k$ in $G[V(G) - \{v_1, v_2...v_s\}]$ such that $vv_i \notin E(G)$.

(2) There exists a $k$-tree $G^*$ such that $\Pi_{1,c}(G^*) < \Pi_{1,c}(G)$ and $\Pi_{2}(G^*) > \Pi_{2}(G)$.

**Proof.** For (1), let $G' = G[V(G) - \{v_1, v_2...v_s\}]$ and $S = N(u) - \{v_1, v_2...v_s\}$, we obtain that $d_{G'}(u) = |S| = k$ and $G[S]$ is a $k$-clique by $u \in S_2$. Since $G \not\cong S_n^k$, $d_{G'}(v) \geq k$ for all $v \in S$. And by the facts that $N(v_i) \subseteq (N(u) - \{v_1, v_2...v_s\}) \cup \{u\}$ with $|N(v_i)| = k$ and $|(N(u) - \{v_1, v_2...v_s\}) \cup \{u\}| = k + 1$, we have for any $i \in [1, s]$, there exists a vertex $v \in S$ such that $vv_i \notin E(G)$.

For (2), choose $v_1$ and by (1) there exists a vertex $v \in N(u) - \{v_1, v_2...v_s\}$ with $d_{G'}(v) \geq k$ such that $vv_1 \notin E(G)$. If $d_{G'}(v) = k$, and by $uv \in E(G')$, we obtain $G'$ is a $k + 1$-clique. Let $x \in S$ be the vertex such that $d(x) = \min_{v \in S}\{d(v)\}$, and let $v_t$ be the vertex such that $v_t \notin E(G)$, $v_t y \notin E(G)$ for some $t \in [1, s]$ and $y \in S$, that is, $d(x) - 1 < d(y)$. Construct a new graph $G^*$ such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{v_t x\} + \{v_t y\}$. Denote $G_0 = G[V(G) - \{x, y\}]$, since $d(x) - 1 < d(y)$, and by the definition of $\Pi_{1,c}(G), \Pi_{2}(G)$ and Fact 5, Fact 6, we have

\[
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} = \frac{[\Pi_{w \in V(G_0)} d(w)^c]d(y)^c d(x)^c}{[\Pi_{w \in V(G_0)} d(w)^c] [d(y) + 1]^c [d(x) - 1]^c} \frac{d(y)^c d(x)^c}{d(y)^c d(x)^c} = \frac{\frac{[d(y) + 1]^c [d(x) - 1]^c}{[d(x) - 1]^c}}{[d(x) - 1]^c} > 1.
\]

Also,
\[
\frac{\Pi_2(G)}{\Pi_2(G^*)} = \frac{[\prod_{w \in V(G_0)} d(w)^{d(w)}]d(y)^{d(y)}d(x)^{d(x)}}{[\prod_{w \in V(G_0)} d(w)^{d(w)}][d(y) + 1]^{d(y)+1}[d(x) - 1]^{d(x)-1}}
\]
\[
= \frac{[d(y) + 1]^{d(y)+1}[d(x) - 1]^{d(x)-1}}{d(y)^{d(y)}}
\]
\[
< 1.
\]

Thus, we find that the \(k\)-tree \(G^*\) satisfies \(\Pi_{1,c}(G^*) < \Pi_{1,c}(G)\) and \(\Pi_2(G^*) > \Pi_2(G)\), we are done.

If \(d_G(v) \geq k + 1\), reorder the subindices of \(\{v_1, v_2 \ldots v_s\}\) such that \(vv_i \notin E(G)\) with \(i \in [1, s_1]\), where \(s_1 \leq s\), and by the fact that \(G[N(u) - \{v_1, v_2 \ldots v_s\}]\) is a \(k\)-clique, we have \(d(u) = k + s\) and \(d(v) \geq k + 1 + s - s_1\), that is, \(d(v) \geq d(u) - s_1 + 1\). Construct a new graph \(G^*\) such that \(V(G^*) = V(G)\), and \(E(G^*) = E(G) - \{uv_i\} + \{vv_i\}\), for all \(i \in [1, s_1]\). Since \(G[N(u) - \{v_1, v_2 \ldots v_s\} + \{u\}]\) is a \(k + 1\)-clique, and for any \(i\), \(N(v_i) \subseteq N_{G - \{v_1, v_2 \ldots v_s\}}(u) \cup \{u\}\), then \(G^*\) is a \(k\)-tree. Denote \(G_0 = G[V(G) - \{u, v\}]\), since \(d(v) \geq d(u) - s_1 + 1\), and by the definition of \(\Pi_{1,c}(G)\), \(\Pi_2(G)\) and Fact 5, Fact 6, we have

\[
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} = \frac{[\prod_{w \in V(G_0)} d(w)^c]d(v)^cd(u)^c}{[\prod_{w \in V(G_0)} d(w)^c][d(v) + s_1]^c[d(u) - s_1]^c}
\]
\[
= \frac{[d(v) + s_1]^c[d(u) - s_1]^c}{d(v)^c}
\]
\[
= \frac{[d(v) + s_1]^c}{d(u)^c}
\]
\[
> 1.
\]

Also,
$$\Pi_2(G) = \frac{\prod_{w\in V(G_0)} d(w)d(w)\prod_{v\in V(G_0)} d(v)v\prod_{u\in V(G_0)} d(u)u}{\prod_{w\in V(G_0)} d(w)d(w)\prod_{v\in V(G_0)} d(v)v\prod_{u\in V(G_0)} d(u)u}$$

$$= \frac{[d(v) + s_1][d(v) - s_1]}{d(d(v))d(d(u))}$$

$$< 1.$$

Hence, we find that the k-tree $G^*$ satisfies $\Pi_1,c(G^*) < \Pi_1,c(G)$ and $\Pi_2(G^*) > \Pi_2(G)$, we are done.

Lemma 2 Let $G$ be a k-tree, if either $\Pi_1,c(G)$ attains the maximal or $\Pi_2(G)$ attains the minimal, then every hyper pendent edge is a k-path.

Proof. Let $P = G[V(S) \cup V(C)]$ be a hyper pendent edge, where $G[S] = G[\{x_1, x_2, \ldots, x_k\}]$ is a cut k-clique and $V(C) = \{u_1, u_2, \ldots, u_p\}$ with $p$ is a positive integer such that $u_1$ is the only vertex of $P$ in $S_1(G)$ and for $i \in [1, p-1], u_i$ is the vertex added following by $u_{i+1}$ through the process of Definition 1.

Fact 7. For any hyper pendent edge $P = G[V(S) \cup V(C)]$ as represented above, $\{u_1, u_2, \ldots, u_p\}$ is a simplicial elimination ordering of $P$.

Proof. By contradiction, assume that $\{u_1, u_2, \ldots, u_p\}$ is not a simplicial elimination ordering of $P$. Let $u_t$ be the first vertex from $u_1$ to $u_p$ such that $\{u_t, u_{t+1}\} \in E$ for $t \in [2, p-1]$, then $u_t u_{t+1} \notin E(G)$ and $\{u_t, u_{t+1}\}$ can not be in some k-cliques. And by Definition 1, there must be at least two vertices that belongs to $S_1$ in $V(C)$, a contradiction. □

By Fact 7, we know $\{u_1, u_2, \ldots, u_p\}$ is a simplicial elimination ordering of $P$. For $p \leq 2$, $P$ is a k-path by Definition 2; For $p \geq 3$, if $P$ is a k-path, then we are done. Otherwise, let $u_s$ be the first vertex from $u_p$ to $u_1$ such that $G[V(S) \cup \{u_p, u_{p-1}, \ldots, u_{s+1}, u_s\}]$ is not a k-path. Since $G[V(S) \cup \{u_p, u_{p-1}, \ldots, u_{s+1}\}]$ is a k-path, for each $i \in [s+1, p]$, let $N_{G-[u_1, u_2, \ldots, u_{i-1}]}(u_i) = \{u_{i+1}, u_{i+2}, \ldots, u_{\min\{p, i+k\}}, x_1, x_2, \ldots, x_{\max\{0, k-p+i\}}\}$, and by Definition 2 and the symmetry of $G[S]$, we have $|N(u_s) \cap \{u_{s+1}, u_{s+2}, \ldots, u_{\min\{p, s+k\}}\}| = \min\{p-s-1, k-1\}$, where $1 \leq s \leq p - 1$.

For $p \leq k + s$, suppose that $u_t$ is the vertex such that $u_t \notin N(u_s)$ with $s + 2 \leq t \leq p$, let $N_{G-[u_1, u_2, \ldots, u_{s-1}]}(u_s) = \{u_{s+1}, u_{s+2}, \ldots, u_{t-1}, u_{t+1}, u_{t+2}, \ldots, u_p, x_1, x_2, \ldots, x_{k-p+s+1}\}$, and let $|N(x_{k-p+s+1}) \cap \{u_1, u_2, \ldots, u_{s-1}\}| = m$ for $m \in [0, s-1]$. By Definition 2, we have $u_t u_i \notin E(G)$ for $i \in [1, s]$, and then $d(u_t) = k + t - s - 1$ and $d(x_{k-p+s+1}) > k + p - s + m - 1$. Now construct a new graph $G^*$ such that $V(G^*) = V(G), E(G^*) = E(G) - \{u_s x_{k-p+s+1}, u_s x_{k-p+s+1}\} + \{u_s u_t, u_t u_t\}$
with \( i \in [0, m] \), then \( G^* \) is a \( k \)-tree. Since \( t \leq p \), we have \( d(x_{k-p+s+1}) > d(u_l) + m + 1 \), and by the definition of \( \Pi_{i,c}(G) \), \( \Pi_2(G) \) and Fact 5, Fact 6, we get

\[
\frac{\Pi_{i,c}(G)}{\Pi_{i,c}(G^*)} = \frac{d(u_t)^c d(x_{k-p+s+1})^c}{[d(u_t) + m + 1]^c[d(x_{k-p+s+1}) - m - 1]^c} = \frac{[d(u_t) + m + 1]^c}{d(x_{k-p+s+1})} < 1, \]

\[
\frac{\Pi_2(G)}{\Pi_2(G^*)} = \frac{d(u_t)^c d(x_{k-p+s+1})^c}{[d(u_t) + m + 1]^c[d(x_{k-p+s+1}) - m - 1]^c} = \frac{[d(u_t) + m + 1]^c}{d(x_{k-p+s+1})} > 1. \]

Thus, \( \Pi_{i,c}(G^*) > \Pi_{i,c}(G) \) and \( \Pi_2(G^*) < \Pi_2(G) \), a contradiction.

For \( p \geq k + s + 1 \), let \( |N(u_{k+s+1}) \cap \{u_1, u_2 \ldots u_{s+1}\}| = m \) for \( m \in [0, s - 1] \). Since \( G[V(S) \cup \{u_p, u_{p-1} \ldots u_{s+1}\}] \) is a \( k \)-path, we have \( G[\{u_{s+1}, u_{s+2} \ldots u_{s+k+1}\}] \) is a \( k + 1 \)-clique. Suppose that \( u_t \) is the vertex such that \( u_t \notin N(u_s) \) with \( s + 2 \leq t \leq s + k \), let \( N_{G-\{u_1, u_2 \ldots u_{s+1}\}}(u_s) = \{u_{s+1}, u_{s+2} \ldots u_{t-1}, u_{t+1} \ldots u_{s+k+1}\} \). Now we construct a new graph \( G^* \) such that \( V(G^*) = V(G) \), \( E(G^*) = E(G) - \{u_s u_{k+s+1}, u_i u_{k+s+1}\} + \{u_i u_t, u_t u_t\} \) for \( i \in [0, m] \), then \( G^* \) is a \( k \)-tree and \( d(u_{k+s+1}) = 2k + m, d(u_t) = k + t - s - 1 \). Since \( t \leq s + k \), we have \( d(u_{k+s+1}) > d(u_t) + m + 1 \), and by the definition of \( \Pi_{i,c}(G) \), \( \Pi_2(G) \) and Fact 5, Fact 6, we get

\[
\frac{\Pi_{i,c}(G)}{\Pi_{i,c}(G^*)} = \frac{d(u_t)^c d(u_{k+s+1})^c}{[d(u_t) + m + 1]^c[d(u_{k+s+1}) - m - 1]^c} = \frac{[d(u_t) + m + 1]^c}{d(u_{k+s+1})} < 1, \]

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\[ \frac{\Pi_2(G)}{\Pi_2(G^*)} = \frac{d(u_t)d(u_i)d(u_{k+s+1})}{d(u_t)d(u_i)} \times \frac{d(u_{k+s+1})}{d(u_{k+s+1})} \]

Thus, \( \Pi_{1,c}(G^*) > \Pi_{1,c}(G) \) and \( \Pi_2(G^*) < \Pi_2(G) \), a contradiction. Hence, for any \( s \in [1,p] \)
\[ N_{G-\{u_1,u_2...u_{s-1}\}}(u_s) = \{u_{s+1}, u_{s+2}...u_{\min\{p,k+s\}}, x_1, x_2...x_{\max\{0,k-p+s\}}\} \], that is, \( \mathcal{P} \) is a \( k \)-tree. \( \square \)

**Lemma 3** Let \( G \) be a \( k \)-tree, if either \( \Pi_{1,c}(G) \) attains the maximal or \( \Pi_2(G) \) attains the minimal, then \( |S_1(G)| = 2 \).

**Proof.** We know that \( |S_1(G)| \geq 2 \) for \( n \geq k + 2 \), and by Lemma 2, every hyperpendent edge is a \( k \)-path for \( \Pi_{1,c}(G) \) to attain the maximal or \( \Pi_2(G) \) to attain the minimal. If \( |S_1(G)| = 2 \), we are done; Suppose that \( |S_1(G)| \geq 3 \), it suffices to prove that there exists a graph \( G' \) such that \( |S_1(G')| = |S_1(G)| - 1 \) with \( \Pi_{1,c}(G') > \Pi_{1,c}(G) \) and \( \Pi_2(G') < \Pi_2(G) \).

**Fact 8.** For any \( k \)-tree \( G \) satisfying the conditions of Lemma 3, if \( |S_1(G)| \geq 3 \), then there exists a \( k \)-clique \( G[S] \) such that \( w(G - S) \geq 3 \).

**Proof.** We will proceed by induction on \( n = |G| \). For \( n = k + 3 \), it’s trivial; For \( n \geq k + 4 \), assume that the fact is true for the \( k \)-tree \( G \) with \( n < k + p \), and consider \( n = k + p \). If \( |S_1(G)| \geq 4 \), choose any vertex \( v \in S_1(G) \), or \( |S_1(G)| = 3 \) and \( |S_2(G)| \geq 2 \), choose the vertex \( v \in S_1(G) \) such that \( N(w) \cap S_1(G) = \{v\} \) for some \( w \in S_2(G) \), then construct a new graph \( G' \) such that \( G' = G - v \). Since \( S_2(G) \) is an dependent set and \( G'[N(v)] \) is a \( k \)-clique for any \( v \in S_1(G) \), we obtain \( |S_1(G')| \geq 3 \). By the induction hypothesis, there exists a \( k \)-clique \( G[S] \) in \( G' \) such that \( w(G' - S) \geq 3 \). Thus, by adding back \( v \), \( G[S] \) is still a \( k \)-clique in \( G \) and \( w(G - S) \geq 3 \), we are done. Next, we only consider \( |S_1(G)| = 3 \) and \( |S_2(G)| = 1 \).

Let \( S_1(G) = \{v_1, v_2, v_3\} \) and \( G_0 = G - \{v_1, v_2, v_3\} \), by Definition 4, we have \( G_0 \) is a \( k \)-clique, denoted \( G'[\{x_1, x_2...x_{k+1}\}] \). If there exists \( N(v_i) = N(v_j) \), for some \( i, j \in [1,3] \) with \( i \neq j \), and take \( S = N(v_i) \), then \( w(G - S) \geq 3 \), we are done; If \( N(v_i) \neq N(v_j) \), for any \( i, j \in [1,3] \) with \( i \neq j \), then reorder the index of \( x_i \) such that \( N(v_1) = \{x_1, x_2...x_k\}, N(v_2) = \{x_2, x_3...x_{k+1}\} \) and \( N(v_3) = \{x_1, x_3...x_{k+1}\} \). Construct a new graph \( G^* \) such that \( V(G^*) = V(G), E(G^*) = E(G) - \{v_1x_1\} + \{v_1v_2\} \), then \( G^* \) is still a \( k \)-tree and \( d_G(x_1) = k + 2 \), \( d_{G^*}(x_1) = k + 1 \), \( d_G(v_1) = d_G(v_2) = k \) and \( d_{G^*}(v_2) = k + 1 \). By the definition of \( \Pi_{1,c}(G), \Pi_2(G) \) and Fact 6, we...
\[
\frac{\Pi_1(c)(G)}{\Pi_1(c)(G^*)} = \frac{d(v_2)^c d(x_1)^c}{[d(v_2) + 1]^c[d(x_1) - 1]^c} = \left[\frac{k(k+2)}{(k+1)^2}\right]^c < 1,
\]

\[
\frac{\Pi_2(G)}{\Pi_2(G^*)} = \frac{d(v_2)^{d(v_2)} d(x_1)^{d(x_1)}}{[d(v_2) + 1]^{d(v_2)+1}[d(x_1) - 1]^{d(x_1)-1}} = \left[\frac{(k+1)^{k+1}}{(k+2)^{k+2}}\right] > 1.
\]

Thus, we find a graph \(G^*\) with \(\Pi_1(c)(G^*) > \Pi_1(c)(G)\) and \(\Pi_2(G^*) < \Pi_2(G)\), a contradiction with that \(\Pi_1(c)(G)\) attains the maximal or \(\Pi_2(G)\) attains the minimal, we are done. \(\square\)

Choose a \(k\)-clique \(G[S]\) with \(w(G - S) \geq 3\) such that there are two components of \(G - S\): \(C_1, C_2\) with \(|C_1| = p, |C_2| = q\) and \(p + q\) being minimal, for \(p \geq q\). Let \(u_1 \in V(C_1), v_1 \in V(C_2)\) with \(\{u_1, v_1\} \subseteq S_1(G)\). Let \(N_{G-\{v_1,v_2,...,v_{i-1}\}}(v_i) = \{v_{i+1}, v_{i+2}...v_{\min\{k+1,q\}}, x_1, x_2...x_{\max\{0,k-q+i\}}\}, \)

\(N_{G-\{u_1,u_2,...,u_{j-1}\}}(u_j) = \{u_{j+1}, u_{j+2}...u_{\min\{k+1,p\}}, y_1, y_2...y_{\max\{0,k-p+i\}}\}\) for \(i \geq 1, j \geq 1,\) where \(\{v_1, v_2...v_q\}\) and \(\{u_1, u_2...u_p\}\) are simplicial elimination orderings of \(G[S \cup V(C_1)]\) and \(G[S \cup V(C_2)]\), respectively. We will prove Lemma 3 by induction on \(q\).

(1) If \(q = 1\), then \(d(v_1) = k\). Choose \(x_i \in N(v_1),\) let \(|N(x_i) \cap \{u_1, u_2...u_p\}| = m\) for \(m \in [1,k]\), we get \(d(x_i) > k + 1 + m\) by \(w(G - S) \geq 3\), and then \(d(x_i) > d(v_1) + m + 1\). Now construct a new graph \(G^*\) such that \(V(G^*) = V(G), E(G^*) = E(G) - \{u_i x_i\} + \{u_i v_1\}\) for \(i \in [1,m]\), then \(G^*\) is a \(k\)-tree and \(|C_1| + |C_2| = p\) with \(G[\{x_1, x_2...x_{t-1}, x_{t+1}...x_k, v_1\}]\) is a \(k\)-clique in \(G^*\). Since \(d(x_i) > d(v_1) + m + 1\), by the definition of \(\Pi_1(c)(G), \Pi_2(G)\) and Fact 5, Fact 6, we have

\[
\frac{\Pi_1(c)(G)}{\Pi_1(c)(G^*)} = \frac{d(v_1)^c d(x_t)^c}{[d(v_1) + m]^c[d(x_t) - m]^c} = \frac{\left[\frac{d(v_1) + m}{d(x_t)}\right]^c}{\left[\frac{d(v_1) + m}{d(x_t) - m}\right]^c} < 1.
\]
$$\frac{\Pi_2(G)}{\Pi_2(G^*)} = \frac{\frac{d(v_1)^d(x_1)d(x_t)}{[d(v_1) + m]^{d(v_1)/d(x_t) - m}}}{\frac{d(v_1)^d(x_t)}{d(x_t)^{d(x_t)}}}$$

$$= \frac{\frac{[d(v_1) + m]^{d(v_1)/d(x_t)} - m]}{[d(v_1) + m]^{d(v_1)/d(x_t) - m}}$$

$$> 1.$$ 

Then, $\Pi_{1,c}(G^*) > \Pi_{1,c}(G)$ and $\Pi_2(G^*) < \Pi_2(G)$. Thus, let $G' = G^*$, $|S_1(G')| = |S_1(G)| - 1$, $\Pi_{1,c}(G') > \Pi_{1,c}(G)$ and $\Pi_2(G') < \Pi_2(G)$, and we are done.

(2) Assume that $q = s$, there exists a $k$-tree $G'$ such that $|S_1(G')| = |S_1(G)| - 1$, $\Pi_{1,c}(G') > \Pi_{1,c}(G)$, $\Pi_2(G') < \Pi_2(G)$ and we consider $q = s + 1$.

If $q \leq k$, we have $d(v_q) = k + q - 1$ by the fact that $G[S \cup V(C_2)]$ is a $k$-path. Choose $x_t \in N(v_1)$, we know $x_t \in N(v_i)$ for all $i \in [1, p]$ by $G[S \cup V(C_2)]$ is a $k$-path. Let $|N(x_t) \cap \{u_1, u_2...u_p\}| = m$ for $m \in [1, k]$, we have $d(x_t) > k + q + m$ by $w(G - S) \geq 3$, and then $d(x_t) > d(v_q) + m + 1$. Now construct a new graph $G^*$ such that $V(G^*) = V(G), E(G^*) = E(G) - \{u_i x_t \} + \{u_i v_q\}$ for $i \in [1, m]$, then $G^*$ is a $k$-tree and $|C_1| + |C_2| = p + q - 1$ with $G'\{x_1, x_2...x_{t-1}, x_{t+1}...x_k, v_q\}$ is a $k$-clique in $G^*$. Since $d(x_t) > d(v_q) + m + 1$, by the definition of $\Pi_{1,c}(G), \Pi_2(G)$ and Fact 5, Fact 6, we have

$$\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} = \frac{\frac{d(v_q)^c d(x_t)^c}{[d(v_q) + m]^c [d(x_t) - m]^c}}{\frac{d(v_q)^c}{d(x_t)^c}}$$

$$= \frac{\frac{[d(v_q) + m]^c - m}{[d(v_q) + m]^c - m}}{d(x_t)}$$

$$< 1,$$

$$\frac{\Pi_2(G)}{\Pi_2(G^*)} = \frac{\frac{d(v_q)^d(x_t)d(x_t)}{[d(v_q) + m]^{d(v_q)/d(x_t) - m}}}{\frac{d(v_q)^d(x_t)}{d(x_t)^{d(x_t)}}}$$

$$= \frac{\frac{[d(v_q) + m]^{d(v_q)/d(x_t)}}{[d(v_q) + m]^{d(v_q)/d(x_t) - m}}}{d(x_t)^{d(x_t)}}$$

$$> 1.$$ 

Then, $\Pi_{1,c}(G) < \Pi_{1,c}(G^*)$, $\Pi_2(G) > \Pi_2(G^*)$ and $q = s$ in $G^*$, then by the induction hypothesis, there exists a $k$-tree $G'$ such that $|S_1(G')| = |S_1(G)| - 1$, $\Pi_{1,c}(G') > \Pi_{1,c}(G)$ and $\Pi_2(G') < \Pi_2(G)$, we are done.

If $q \geq k + 1$, we have $N(u_1) = \{u_2, u_3...u_{k+1}\}$, $N(v_1) = \{v_2, v_3...v_{k+1}\}$ by the facts that
$p \geq q$ and $G[S \cup V(C_1)]$, $G[S \cup V(C_2)]$ are $k$-paths. We construct a new graph $G^*$ such that $V(G^*) = V(G)$, $E(G^*) = E(G) - \{v_i v_j\} + \{u_j v_1\}$ for $i \in [2, k + 1]$, $j \in [1, k]$. And by Fact 2 and the definition of $\prod_{1,c}(G), \prod_2(G)$, we obtain

\[
\frac{\prod_{1,c}(G)}{\prod_2(G)} = \frac{\prod_{i=2}^{k+1} d(v_i)\prod_{i=1}^{k} d(u_j)}{\prod_{i=2}^{k} [d(v_i) - 1]\prod_{i=1}^{k} [d(u_j) + 1]} = 1,
\]
\[
\frac{\prod_{1,c}(G^*)}{\prod_2(G^*)} = \frac{\prod_{i=2}^{k+1} d(v_i)\prod_{i=1}^{k} d(u_j)}{\prod_{i=2}^{k} [d(v_i) - 1]\prod_{i=1}^{k} [d(u_j) + 1]} = 1.
\]

Then, $\prod_{1,c}(G) = \prod_{1,c}(G^*)$, $\prod_2(G) = \prod_2(G^*)$ and $q = s$ in $G^*$, then by the induction hypothesis, there exists a $k$-tree $G'$ such that $|S_1(G')| = |S_1(G)| - 1$, $\prod_{1,c}(G') > \prod_{1,c}(G)$ and $\prod_2(G') < \prod_2(G)$, we are done. $\square$

Now, we turn to prove the main results of the paper.

**Proof of Theorem 2.** For any $k$-tree $T_n^k$, if $|S_1(T_n^k)| = n - k$, then $T_n^k \cong S_{k,n-k}$, we are done. And if $|S_1(T_n^k)| \leq n - k - 1$, we can recursively use Lemma 1 to make $\prod_{1,c}(T_n^k)$ decreasing until $|S_1(T_n^k)| = n - k$. Thus, we have $T_n^k \cong S_{k,n-k}$ for $\prod_{1,c}(T_n^k)$ to arrive the minimal value.

By Lemma 2, if $\prod_{1,c}(T_n^k)$ get the maximal, then every hyper pendent edge is a $k$-path, and by Lemma 3, $|S_1(T_n^k)| = 2$, implying that $T_n^k \cong P_n^k$ for $\prod_{1,c}(T_n^k)$ to arrive the maximal value. $\square$

**Proof of Theorem 3.** For any $k$-tree $T_n^k$, if $|S_1(T_n^k)| = n - k$, then $T_n^k \cong S_{k,n-k}$, we are done. And if $|S_1(T_n^k)| \leq n - k - 1$, we can recursively use Lemma 1 to make $\prod_2(T_n^k)$ increasing until $|S_1(T_n^k)| = n - k$, then we have $T_n^k \cong S_{k,n-k}$ for $\prod_2(T_n^k)$ to arrive the maximal value.

By Lemma 2, if $\prod_2(T_n^k)$ get the minimal, every hyper pendent edge is a $k$-path, and by Lemma 3, $|S_1(T_n^k)| = 2$. Then this $k$-tree is a $k$-path, that is, $T_n^k \cong P_n^k$ for $\prod_2(T_n^k)$ to arrive the minimal value. $\square$

**References**

[1] L. W. Beineke, R. E. Pippert, The number labeled $k$-dimentional trees, J. Combin. Theory 6 (1969), 200-205.

[2] D. de Cacn D, An upper bound on the sum of squares of degrees in a graph. Discrete Math 185 (1998), 245-248.

[3] J. Estes, B. Wei, Sharp bounds of the Zagreb indices of k-trees. J Comb Optim (2012), DOI 10.1007/s10878-012-9515-6.
[4] I. Gutman, Multiplicative Zagreb indices of trees, Bull.Soc.Math. Banja Luka, ISSN 0354-5792 (p), ISSN 1986-521X (o), Vol. 18 (2011), 17-23.

[5] F. Harary, E. M. Plamer, On acyclic simplicial complexes, Mathematika 15 (1968), 115-122.

[6] A. Hou, S. Li, l. Song, B. Wei, Sharp bounds for Zagreb indices of maximal outerplanar graphs. J Comb Optim 22 (2011), 252-269.

[7] H. Narumi, M. Hatayama, Simple topological index. A newly devised index characterizing the topological nature of structural isomers of saturated hydrocarbons. Mem. Fac. Engin. Hokkaido Univ. 16 (1984), 209-214.

[8] s. Nikolić, G. Kovačević, Milicčević, A., Trinajstić, N., The Zagreb indices 30 years after. Croat. Chem. Acta 76 (2003), 113-124.

[9] L. Song, W. Staton, B. Wei, Independence polynomials of $k$-tree related graphs. Discrete Appl Math 158(2010), 943-950.

[10] R. Todeschini, D. Ballabio, V. Consonni, Novel molecular descriptors based on functions of new vertex degrees. In: Novel molecular structure descriptors - Theory and applications I. (I. Gutman, B. Furtula, eds.), Univ. Kragujevac (2010), pp. 73 - 100.

[11] R. Todeschini, V. Consonni, New local vertex invariants and molecular descriptors based on functions of the vertex degrees. MATCH commun. Math. Comput. Chem. 64 (2010), 359-372.

[12] Q, Zhao, S. Li, On the maximum Zagreb index of graphs with $k$ cut vertices. Acta Appl. Math. 111 (2010), 93-106.