PENTAGON AND HEXAGON EQUATIONS

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Abstract. The author will prove that Drinfel’d’s pentagon equation implies his two hexagon equations in the Lie algebra, pro-unipotent, pro-l and pro-nilpotent contexts.

Contents

0. Introduction 1
1. Lie algebra case 3
2. Proof of Theorem 1 5
3. Proof of Theorem 2 8
References 9

0. Introduction

In his celebrated papers on quantum groups [Dr86, Dr90, Dr91] Drinfel’d came to the notion of quasitriangular quasi-Hopf quantized universal enveloping algebra. It is a topological algebra which differs from a topological Hopf algebra in the sense that the coassociativity axiom and the cocommutativity axiom is twisted by an associator and an R-matrix satisfying a pentagon axiom and two hexagon axioms. One of the main theorems in [Dr91] is that any quasitriangular quasi-Hopf quantized universal enveloping algebra modulo twists (in other words gauge transformations [K]) is obtained as a quantization of a pair (called its classical limit) of a Lie algebra and its symmetric invariant 2-tensor. Quantizations are constructed by universal associators. The set of group-like universal associators forms a pro-algebraic variety, denoted $M$. Its non-emptiness is another of his main theorem (reproved in [B]). Our first theorem is on the defining equations of $M$.

Let us fix notation and conventions: Let $k$ be a field of characteristic 0, $k$ its algebraic closure and $U\mathcal{F}_2 = k\langle\langle X,Y \rangle\rangle$ a non-commutative formal power series ring with two variables $X$ and $Y$. Its element $\varphi = \varphi(X,Y)$ is called group-like if it satisfies $\Delta(\varphi) = \varphi \otimes \varphi$ with $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$ and its constant term is equal to 1. Its coefficient of $XY$ is denoted by $c_2(\varphi)$. For any $k$-algebra homomorphism $\iota : U\mathcal{F}_2 \rightarrow S$ the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(X),\iota(Y))$. Let $a_4$ be the completion (with respect to the natural grading) of the Lie algebra over $k$ with generators $t_{ij}$ ($1 \leq i,j \leq 4$) and defining relations $t_{ij} = 0$, $t_{ij} = t_{ji}$, $[t_{ij}, t_{tk} + t_{jk}] = 0$ ($i,j,k$: all distinct) and $[t_{ij}, t_{kl}] = 0$ ($i,j,k,l$: all distinct).

Theorem 1. Let $\varphi = \varphi(X,Y)$ be a group-like element of $U\mathcal{F}_2$. Suppose that $\varphi$ satisfies Drinfel’d’s pentagon equation:

$$\varphi(t_{12}, t_{23} + t_{24})\varphi(t_{13} + t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\varphi(t_{12} + t_{13}, t_{24} + t_{34})\varphi(t_{12}, t_{23}).$$
Then there exists an element (unique up to signature) \( \mu \in \hat{k} \) such that the pair \((\mu, \varphi)\) satisfies his two hexagon equations:

\[
\begin{align*}
(2) \quad & \exp\left(\frac{\mu(t_{13} + t_{23})}{2}\right) = \varphi(t_{13}, t_{12}) \exp\left(\frac{\mu t_{13}}{2}\right) \varphi(t_{13}, t_{23})^{-1} \exp\left(\frac{\mu t_{23}}{2}\right) \varphi(t_{12}, t_{23}), \\
(3) \quad & \exp\left(\frac{\mu(t_{12} + t_{13})}{2}\right) = \varphi(t_{23}, t_{13})^{-1} \exp\left(\frac{\mu t_{13}}{2}\right) \varphi(t_{12}, t_{13}) \exp\left(\frac{\mu t_{12}}{2}\right) \varphi(t_{12}, t_{23})^{-1}.
\end{align*}
\]

Actually this \( \mu \) is equal to \( \pm (24c_2(\varphi))^\chi \).

It should be noted that we need to use an (actually quadratic) extension of a field \( k \) in order to obtain the hexagon equations from the pentagon equation. The theorem says that the pentagon equation is essentially a single defining equation of the associator set. The Drinfel’d associator \( \Phi_{KZ} \in \mathbf{R}\langle\langle X, Y \rangle\rangle \) is a group-like series constructed by solutions of KZ-equation \([Dr91]\). It satisfies (1), (2), and (3) with \( \mu = \pm 2\pi \sqrt{-1} \). Its coefficients are expressed by multiple zeta values \([LM]\) (and also \([F03]\)). The theorem also says that the two hexagon equations do not provide any new relations under the pentagon equation.

The category of representations of a quasitriangular quasi-Hopf quantized universal enveloping algebra forms a quasitensored category \([Dr91]\), in other words, a braided tensor category \([JS]\): its associativity constraint and its commutativity constraint are subject to one pentagon axiom and two hexagon axioms. The Grothendieck-Teichmüller pro-algebraic group \( GT \) is introduced in \([Dr91]\) as a group-like series constructed by solutions of KZ-equation \([Dr91]\). It satisfies (1), (2), and (3) with \( \mu = \pm 2\pi \sqrt{-1} \). Its coefficients are expressed by multiple zeta values \([LM]\) (and also \([F03]\)). The theorem also says that the two hexagon equations do not provide any new relations under the pentagon equation.

The second theorem is on defining equations of \( GT \).

**Theorem 2.** Let \( F_2(k) \) be the the free pro-unipotent algebraic group with two variables \( x \) and \( y \). Suppose that its element \( f = f(x, y) \) satisfies Drinfel’d’s pentagon equation:

\[
\begin{align*}
(4) \quad & f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}) \\
& \text{in } K_4(k). \quad \text{Then there exists an element (unique up to signature) } \lambda \in \hat{k} \text{ such that the pair } (\lambda, f) \text{ satisfies his hexagon equations (3- and 2-cycle relation):} \\
(5) \quad & f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1 \text{ with } xyz = 1 \text{ and } m = \frac{\lambda - 1}{2}, \\
(6) \quad & f(x, y)f(y, x) = 1.
\end{align*}
\]

Actually this \( \lambda \) is equal to \( \pm (24c_2(f) + 1)^\chi/2 \) where \( c_2(f) \) stands for \( c_2(f(e^x, e^y)) \).

Here \( K_4(k) \) stands for the unipotent completion of the pure braid group \( K_4 = \ker\{B_4 \to \mathfrak{S}_4\} \) of 4 strings \( B_4 \): the Artin braid group and \( \mathfrak{S}_4 \): the symmetric group) with standard generators \( x_{ij} \) (1 ≤ \( i, j \) ≤ 4).

It should be noted again that we need to use an (actually quadratic) extension of a field \( k \) in order to obtain the hexagon equations from the pentagon.
equation. The set of pairs \((\lambda, f)\) satisfying \((\ref{1}), \(\ref{3}\)) and \((\ref{4})\) determines a pro-algebraic variety \(\mathcal{GT}\) and \(\mathcal{GT}\) is its open subvariety defined by \(\lambda \neq 0\). The product structure on \(\mathcal{GT}(k)\) is given by \((\lambda_1, f_1) \circ (\lambda_2, f_2) := (\lambda, f)\) with \(\lambda = \lambda_1 \lambda_2\) and \(f(x, y) = f_1(f_2 x ^ {\lambda_1} f_2 ^ {-1}, y ^ {\lambda_2}) f_2\). The theorem says that the pentagon equation is essentially a single defining equation of \(\mathcal{GT}\).

The construction of the paper is as follows: \((\ref{1})\) is a crucial part of the paper. The implication of the pentagon equation is proved for Lie series. In \((\ref{2})\) we give a proof of theorem \((\ref{1})\) by using Drinfel’d’s gadgets. \((\ref{3})\) gives a proof of theorem \((\ref{2})\) and its analogue in the pro-f group and pro-nilpotent group setting.

1. Lie Algebra Case

In this section we prove the Lie algebra version of Theorem \((\ref{1})\) in a rather combinatorial argument.

Let \(\mathfrak{s}_2\) be the set of Lie-like elements \(\varphi\) in \(U\mathfrak{s}_2\) (i.e. \(\Delta(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi\)).

**Theorem 3.** Let \(\varphi\) be a commutator Lie-like element \((\ref{4})\) with \(c_2(\varphi) = 0\). Suppose that \(\varphi\) satisfies the pentagon equation (5-cycle relation):

\[
(7) \quad \varphi(X_{12}, X_{23}) + \varphi(X_{34}, X_{45}) + \varphi(X_{51}, X_{12}) + \varphi(X_{23}, X_{34}) + \varphi(X_{45}, X_{51}) = 0
\]

in \(\mathfrak{s}_5\). Then it also satisfies the hexagon equations (3- and 2-cycle relation):

\[
(8) \quad \varphi(X, Y) + \varphi(Y, Z) + \varphi(Z, X) = 0 \quad \text{with} \quad X + Y + Z = 0,
\]

\[
(9) \quad \varphi(X, Y) + \varphi(Y, X) = 0.
\]

Here \(\mathfrak{s}_5\) stands for the completion (with respect to the natural grading) of the pure sphere braid Lie algebra \(\mathfrak{s}_5\) \((\ref{4})\) with 5 strings; the Lie algebra generated by \(X_{ij}\) \((1 \leq i, j \leq 5)\) with clear relations \(X_{ii} = 0, X_{ij} = X_{ji}, \sum_{j=1}^{5} X_{ij} = 0\) \((1 \leq i, j \leq 5)\) and \([X_{ij}, X_{kl}] = 0\) if \(\{i, j\} \cap \{k, l\} = \emptyset\). It is a quotient of \(a_4\) (cf. \((\ref{2})\)).

**Proof.** There is a projection from \(\mathfrak{s}_5\) to the completed free Lie algebra \(\mathfrak{g}_2\) generated by \(X\) and \(Y\) by putting \(X_{15} = 0, X_{12} = X\) and \(X_{23} = Y\). The image of 5-cycle relation gives 2-cycle relation.

For our convenience we denote \(\varphi(X_{ij}, X_{jk})\) \((1 \leq i, j, k \leq 5)\) by \(\varphi_{ijk}\). Then the 5-cycle relation can be read as

\[
\varphi_{123} + \varphi_{345} + \varphi_{512} + \varphi_{234} + \varphi_{451} = 0.
\]

We denote LHS by \(P\). Put \(\sigma_i\) \((1 \leq i \leq 4)\) to be elements of \(\mathfrak{s}_5\) defined as follows: \(\sigma_1(12345) = (12345), \sigma_2(12345) = (54231), \sigma_3(12345) = (13425)\) and \(\sigma_4(12345) = (43125)\). Then

\[
\sum_{i=1}^{4} \sigma_i(P) = \varphi_{123} + \varphi_{345} + \varphi_{512} + \varphi_{234} + \varphi_{451} + \varphi_{542} + \varphi_{231} + \varphi_{154} + \varphi_{423} + \varphi_{315} + \varphi_{134} + \varphi_{425} + \varphi_{513} + \varphi_{342} + \varphi_{251} + \varphi_{431} + \varphi_{125} + \varphi_{543} + \varphi_{312} + \varphi_{254}.
\]

1. In this paper we call a series \(\varphi = \varphi(X, Y)\) commutator Lie-like if it is Lie-like and its coefficient of \(X\) and \(Y\) are both 0, in other words \(\varphi \in \mathfrak{s}_2 := [\mathfrak{g}_2, \mathfrak{g}_2]\).
By the 2-cycle relation, \( \varphi_{ijk} = -\varphi_{kji} \) (1 \( \leq i, j, k \leq 5 \)). This gives
\[
\sum_{i=1}^{4} \sigma_i(P) = (\varphi_{123} + \varphi_{231} + \varphi_{312}) + (\varphi_{512} + \varphi_{125} + \varphi_{251}) + (\varphi_{342} + \varphi_{423}) + (\varphi_{542} + \varphi_{425} + \varphi_{254}).
\]

By \([X_{23}, X_{12} + X_{23} + X_{31}] = [X_{31}, X_{12} + X_{23} + X_{31}] = [X_{12}, X_{12} + X_{23} + X_{31}] = 0\) and \(\varphi \in \overline{\mathfrak{g}}_2\), \(\varphi_{231} = \varphi(X_{23}, X_{31}) = \varphi(X_{23}, -X_{12} - X_{23})\) and \(\varphi_{312} = \varphi(X_{31}, X_{12}) = \varphi(X_{23}, -X_{23}, X_{12})\).

By \([X_{51}, X_{12} + X_{25} + X_{51}] = [X_{12}, X_{12} + X_{25} + X_{51}] = [X_{25}, X_{12} + X_{25} + X_{51}] = 0\) and \(\varphi \in \overline{\mathfrak{g}}_2\), \(\varphi_{512} = \varphi(X_{51}, X_{12}) = \varphi(-X_{12} - X_{25}, X_{12})\) and \(\varphi_{251} = \varphi(X_{25}, X_{51}) = \varphi(X_{25}, -X_{12} - X_{25})\).

By \([X_{23}, X_{42} + X_{23} + X_{34}] = [X_{34}, X_{42} + X_{23} + X_{34}] = [X_{42}, X_{23} + X_{34}] = 0\) and \(\varphi \in \overline{\mathfrak{g}}_2\), \(\varphi_{234} = \varphi(X_{23}, X_{34}) = \varphi(X_{21}, -X_{42} - X_{23})\) and \(\varphi_{342} = \varphi(X_{34}, X_{42}) = \varphi(-X_{25} - X_{42}, X_{23})\).

By \([X_{54}, X_{42} + X_{25} + X_{54}] = [X_{42}, X_{42} + X_{25} + X_{54}] = [X_{25}, X_{42} + X_{25} + X_{54}] = 0\) and \(\varphi \in \overline{\mathfrak{g}}_2\), \(\varphi_{542} = \varphi(X_{54}, X_{42}) = \varphi(-X_{42} - X_{25}, X_{42})\) and \(\varphi_{545} = \varphi(X_{25}, X_{54}) = \varphi(-X_{42} - X_{25}, X_{42})\).

Put \(R(X, Y) = \varphi(X, Y) + \varphi(Y, -X - Y) + \varphi(-X - Y, X)\). Then
\[
\sum_{i=1}^{4} \sigma_i(P) = R(X_{21}, X_{23}) + R(X_{21}, X_{25}) + R(X_{24}, X_{23}) + R(X_{24}, X_{25}).
\]

The elements \(X_{21}, X_{23}, X_{24}\) and \(X_{25}\) generate a completed Lie subalgebra \(\mathfrak{g}_3\) of \(\mathfrak{g}_5\) which is free of rank 3 and whose set of relations is given by \(X_{21} + X_{23} + X_{24} + X_{25} = 0\). It contains \(\sum_{i=1}^{4} \sigma_i(P)\). Let \(q_1 : \mathfrak{g}_3 \to \mathfrak{g}_2\) be the projection sending \(X_{21} \mapsto X, X_{23} \mapsto Y\) and \(X_{24} \mapsto X\). Then
\[
q_1(\sum_{i=1}^{4} \sigma_i(P)) = R(X, Y) + R(X, -2X - Y) + R(Y, X) + R(X, -2Y - X).
\]
Since \(P = 0\), we have \(R(X, -2X - Y) = -R(X, Y)\). Let \(q_2 : \mathfrak{g}_3 \to \mathfrak{g}_2\) be the projection sending \(X_{21} \mapsto X, X_{23} \mapsto X\) and \(X_{24} \mapsto Y\). Then
\[
q_2(\sum_{i=1}^{4} \sigma_i(P)) = R(X, X) + R(X, -2X - Y) + R(Y, X) + R(Y, -2X - Y).
\]
By \(\varphi \in \overline{\mathfrak{g}}_2\), \(R(X, X) = 0\). By definition, \(R(Y, -2X - Y) = R(2X, Y)\). Since \(P = 0, -R(X, Y) + R(Y, X) + R(Y, 2X) = 0\). The 2-cycle relation gives \(R(X, Y) = -R(Y, X)\). Therefore \(2R(X, Y) = R(2X, Y)\). Expanding this equation in terms of a linear basis, say the Hall basis, we see that \(R(X, Y)\) must be of the form \(\sum_{m=1}^{\infty} a_m(adY)^{m-1}(X)\) with \(a_m \in k\). Since it satisfies \(R(X, Y) = -R(Y, X)\), we have \(a_1 = a_3 = a_4 = a_5 = \cdots = 0\). By our assumption \(c_2(\varphi) = 0\), \(a_2\) must be 0 also. Therefore \(R(X, Y) = 0\), which is the 3-cycle relation.

We note that the assumption \(c_2(\varphi) = 0\) is necessary: e.g. the element \(\varphi = [X, Y]\) satisfies the 5-cycle relation but it does not satisfy the 3-cycle relation.

**Remark 4.** There is partially a geometric picture in the proof: We have a de Rham fundamental groupoid \([\text{Dec}](\text{mod. } \mathcal{M}_{0,n}) = \{(x_1 : \cdots : x_n) \in (\mathbb{P}^1)^n | x_i \neq x_j (j \neq i)\}/\text{PGL}(2)\) for \(n \geq 4\), its central extension given by the normal bundle of \(\mathcal{M}_{0,n-1}\) inside its stable compactification \(\overline{\mathcal{M}}_{0,n}\) and maps between them. An automorphism of the system is determined by considering what happens to the canonical de Rham path from ‘0’ to ‘1’ (loc.cit.) in \(\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}\). Equation (7) reflects the necessary condition that such an automorphism must keep the property that the image of the composite of the path, the boundaries
of the fundamental pentagon $B_5$ \[\text{I}\] formed by the divisors $x_i = x_{i+1}$ $(i \in \mathbb{Z}/5\mathbb{Z})$ in $\overline{M}_{0,5}(\mathbb{R})$, must be a trivial loop. Each $\sigma_i(B_5)$ $(1 \leq i \leq 4)$ is a connected component of $M_{0,5}(\mathbb{R})$. The sum of four 5-cycles $\sum_{i=1}^{4} \sigma_i(P)$ corresponds to a path following the (oriented) boundaries of the four real pentagonal regions $\sigma_i(B_5)$ of $M_{0,5}(\mathbb{R})$. The four 3-cycles correspond to four loops around the four boundary divisors $x_4 = x_5, x_3 = x_4, x_5 = x_1$ and $x_1 = x_3$ in $\overline{M}_{0,5}(\mathbb{R})$. The author expects that the geometric interpretation might help to adapt our proof to the pro-finite context (cf. question \[\text{I}\]).

The equations \[\text{I}, \text{S} \text{ and } \text{F}\] are defining equations of Ihara’s stable derivation (Lie-)algebra \[\text{I}\]. Its Lie bracket is given by $\langle \varphi_1, \varphi_2 \rangle := [\varphi_1, \varphi_2] + D_{\varphi_2}(\varphi_1) - D_{\varphi_1}(\varphi_2)$ where $D_{\varphi}$ is the derivation of $\varphi$ given by $D_{\varphi}(X) = [\varphi, X]$ and $D_{\varphi}(B) = 0$. We note that its completion with respect to degree is equal to the graded Lie algebra $grt_1$ of the Grothendieck-Teichmüller group $GT$ in \[\text{Dr91}\]. Our theorem says that the pentagon equation is its single defining equation and two hexagon equations are needless for its definition when $\deg \varphi \geq 3$.

2. Proof of Theorem \[\text{I}\]

This section is devoted to a proof of Theorem \[\text{I}\].

Between the Lie algebra $a_4$ in theorem \[\text{I}\] and $\mathfrak{P}_5$ in theorem \[\text{S}\] there is a natural surjection $\tau : a_4 \to \mathfrak{P}_5$ sending $t_{ij}$ to $X_{ij}$ $(1 \leq i, j \leq 4)$. Its kernel is generated by $\Omega = \sum_{1 \leq i < j \leq 4} t_{ij}$. We also denote its induced morphism $Uaq_4 \to U\mathfrak{P}_5$ by $\tau$. On the pentagon equation we have

\textbf{Lemma 5.} Let $\varphi$ be a group-like element. Giving the pentagon equation \[\text{I}\], for $\varphi$ is equivalent to giving that $\varphi$ is commutator group-like\[\text{S}\] and $\varphi$ satisfies the 5-cycle relation in $U\mathfrak{P}_5$:

\[\text{(10)} \quad \varphi(X_{12}, X_{23})\varphi(X_{34}, X_{45})\varphi(X_{51}, X_{12})\varphi(X_{23}, X_{34})\varphi(X_{15}, X_{51}) = 1.\]

\textbf{Proof.} Assume \[\text{I}\]. Denote the abelianization of $\varphi(X, Y) \in k\langle\langle X, Y\rangle\rangle$ by $\varphi^{ab} \in k[[X, Y]]$. The series $\varphi$ is group-like, so $\varphi^{ab}$ is also, i.e. $\Delta(\varphi^{ab}) = \varphi^{ab} \otimes \varphi^{ab}$. Therefore $\varphi^{ab}$ must be of the form $\exp\{\alpha X + \beta Y\}$ with $\alpha, \beta \in k$. The equation \[\text{I}\] gives $\alpha X_{12} + \beta X_{14} = 0$. Hence $\alpha = \beta = 0$ which means that $\varphi$ is commutator group-like. Therefore

\[\varphi(X_{12}, X_{51}) = \varphi(X_{12}, -X_{21} - X_{52}) = \varphi(X_{12}, X_{23} + X_{24})\]
by $[X_{12}, X_{51} + X_{21} + X_{52}] = [X_{51}, X_{51} + X_{21} + X_{52}] = 0,$
\[\varphi(X_{45}, X_{34}) = \varphi(-X_{43} - X_{53}, X_{34}) = \varphi(X_{13} + X_{23}, X_{34})\]
by $[X_{45}, X_{45} + X_{43} + X_{53}] = [X_{34}, X_{45} + X_{43} + X_{53}] = 0$ and
\[\varphi(X_{45}, X_{51}) = \varphi(-X_{14} - X_{15}, X_{51}) = \varphi(-X_{14} - X_{15}, -X_{14} - X_{45}) = \varphi(X_{12} + X_{13}, X_{24} + X_{34})\]
by $[X_{45}, X_{45} + X_{14} + X_{51}] = [X_{51}, X_{45} + X_{14} + X_{51}] = 0$ and $[X_{14} + X_{15}, X_{51} + X_{14} + X_{45}] = [X_{51}, X_{51} + X_{14} + X_{15}] = 0.$ (N.B. If $\varphi$ is commutator group-like, $\varphi(A + C, B) = \varphi(A, B + C) = \varphi(A, B)$ with $[A, C] = [B, C] = 0.$) So the image of \[\text{I}\] by $\tau$ is

\[\text{(11)} \quad \varphi(X_{12}, X_{51})\varphi(X_{45}, X_{34}) = \varphi(X_{23}, X_{34})\varphi(X_{45}, X_{51})\varphi(X_{12}, X_{23}).\]

The following lemma below gives \[\text{I}\].

\[\text{In this paper we call a series } \varphi = \varphi(X, Y) \text{ commutator group-like if it is group-like and its coefficient of } X \text{ and } Y \text{ are both 0.}\]
Conversely assume (10) and the commutator group-likeness for \( \varphi \). The lemma below gives the above equality (11). Whence we say (11) modulo ker \( \tau \). That is, the quotient of LHS of (1) by RHS of (1) is expressed as \( \exp \gamma \Omega \) for some \( \gamma \in k \). Since both hand sides of (1) are commutator group-like, \( \exp \gamma \Omega \) must be also. Therefore \( \gamma \) must be 0, which gives (1).

**Lemma 6.** Let \( \varphi \) be a group-like element. If \( \varphi \) is commutator group-like and it satisfies the 5-cycle relation (10), it also satisfies the 2-cycle relation:

\[
\varphi(X,Y)\varphi(Y,X) = 1.
\]

And if \( \varphi \) satisfies the pentagon equation (11), it also satisfies (12).

**Proof.** There is a projection \( U\mathfrak{p}_5^\wedge \to U\mathfrak{g}_2 \) by putting \( X_{15} = 0 \) (1 \( \leq \) i \( \leq \) 5), \( X_{12} = X \) and \( X_{23} = Y \). The image of (10) is (12) by the commutator group-likeness.

As was shown in the previous lemma, the equation (11) for \( \varphi \) in \( U\mathfrak{g}_4 \) implies its commutator group-likeness and (11) in \( U\mathfrak{p}_5 \). The image of (11) by the projection gives (12).

In [11] they showed the equivalence between (11) and (10) assuming the commutativity and the 2-cycle relation in the pro-finite group setting. But by the above argument the latter assumption can be excluded.

As for the hexagon equations we also have

**Lemma 7.** Let \( \varphi \) be a group-like element. Giving two hexagon equations (2) and (3) for \( \varphi \) is equivalent to giving the 2-cycle relation (12) and the 3-cycle relation:

\[
e^{\frac{a}{3}X} \varphi(Z,X)e^{\frac{a}{3}X} \varphi(Y,Z)e^{\frac{a}{3}X} \varphi(X,Y) = 1 \text{ with } X + Y + Z = 0.
\]

**Proof.** We review the proof in [12]. The Lie subalgebra generated by \( t_{12}, t_{13} \) and \( t_{23} \), is the direct sum of its center, generated by \( t_{12} + t_{23} + t_{13} \), and the free Lie algebra generated by \( X = t_{12} \) and \( Y = t_{23} \). The projections of (2) and (3) to the first component are both tautologies but the projections to the second component are \( e^{\frac{a}{3}X} \varphi(Z,X)e^{\frac{a}{3}X} \varphi(Y,Z)e^{\frac{a}{3}X} \varphi(X,Y) = 1 \) and \( e^{\frac{a}{3}X} \varphi(Z,X)e^{\frac{a}{3}X} \varphi(Y,Z)e^{\frac{a}{3}X} \varphi(X,Y) = 1 \). They are equivalent to (12) and (13).

The followings are keys to prove theorem 1.

**Lemma 8.** Let \( \varphi_1 \) and \( \varphi_2 \) be commutator group-like elements. Put \( \varphi_3 = \varphi_2 \circ \varphi_1(X,Y) := \varphi_2(\varphi_1X\varphi_1^{-1},Y) \cdot \varphi_1 \). Assume that \( \varphi_1 \) satisfies (10), (12) and

\[
\varphi(Z,X)\varphi(Y,Z)\varphi(X,Y) = 1 \text{ with } X + Y + Z = 0.
\]

Then \( \varphi_2 \) satisfies (10) if and only if \( \varphi_3 \) satisfies (10).

**Proof.** By the arguments in §1.2, \( \varphi_1 \) determines an automorphism of \( U\mathfrak{p}_5^\wedge \) sending \( X_{12} \to X_{12}, X_{23} \to \varphi_1(X_{12},X_{23})^{-1}X_{23}\varphi_1(X_{12},X_{23}), X_{34} \to \varphi_1(X_{34},X_{45})X_{34}\varphi_1(X_{34},X_{45})^{-1}, X_{45} \to X_{45} \) and \( X_{51} \to \varphi_1(X_{12},X_{23})^{-1}X_{51}\varphi_1(X_{45},X_{51})^{-1}X_{51} \). The direct calculation shows that LHS of (10) for \( \varphi_2 \) maps to LHS of (10) for \( \varphi_3(X,Y) \). This gives the claim.

**Lemma 9.** Let \( \varphi \) be a commutator group-like element with \( c_2(\varphi) = 0 \). Suppose that \( \varphi \) satisfies (10). Then it also satisfies (14).

\[
\varphi(Z,X)\varphi(Y,Z)\varphi(X,Y) = 1 \text{ with } X + Y + Z = 0.
\]
Theorem 10. Let $\varphi$ be a commutator group-like element. Suppose that $\varphi$ satisfies the 5-cycle relation (10). Then there exists an element (unique up to signature) $\mu \in \bar{k}$ such that the pair $(\mu, \varphi)$ satisfies the 3-cycle relation (13). Actually this $\mu$ is equal to $\pm(24c_2(\varphi))^{\frac{1}{2}}$.

Proof. We may assume $c_2(\varphi) \neq 0$ by the previous lemma. Let $\mu$ be a solution of $x^2 = 24c_2(\varphi)$ in $\bar{k}^\times$. Put $M'_\mu$ (resp. $M_\mu$) to be the pro-affine algebraic variety whose $k$-valued points are commutator group-like series $\varphi$ in $\bar{k}((X,Y))$ satisfying (10) and $c_2(\varphi) = \frac{\mu^2}{27}$ (resp. (10), (12) and (13)) for $(\mu, \varphi)$. By calculating the coefficient of $XY$ in (13) for $(\mu, \varphi)$, we get $3c_2(\varphi) - \frac{\mu^2}{9} = 0$. Thus $M'_\mu$ is a pro-subvariety of $M_\mu$. To prove $M'_\mu = M_\mu$, it suffices to show this for $\mu = 1$ because we have a replacement $\varphi(A, B)$ by $\varphi(\frac{A}{\mu}, \frac{B}{\mu})$. In a similar way to [F06] §6 the regular function ring $O(M'_1)$ (resp. $O(M_1)$) is encoded the weight filtration $W = \{W_nO(M_1')\}_{n \in \mathbb{Z}}$ (resp. $\{W_nO(M_1)\}_{n \in \mathbb{Z}}$): the algebra $O(M'_1)$ (resp. $O(M_1)$) is generated by $xW$’s $(W$: word $\frac{5}{3}$) and defined by the commutator group likeness, (10) and $c_2(\varphi) = \frac{\mu^2}{27}$ (resp. (10), (12) and (13)) for $\varphi = 1 + \sum W xW$. Put $\deg xW = \deg W$. Each $W_nO(M'_1)$ (resp. $W_nO(M_1)$) is the vector space generated by polynomials whose total degree is less than or equal to $n$.

The inclusion $M_1 \rightarrow M'_1$ gives a projection $O(M'_1) \rightarrow O(M_1)$ which is strictly compatible with the filtrations. It induces a projection $p: Gr^W_1O(M'_1) \rightarrow Gr^W_1O(M_1)$ between their associated graded quotients. The graded quotient $Gr^W_1O(M_1)$ is isomorphic to $O(GRT_1)$ by [F06] theorem 6.2.2. It is the algebra generated by $\bar{x}W$’s and defined by the commutator group likeness, (10), (12) and (13) for $\bar{\varphi} = 1 + \sum W \bar{x}W$. On the other hand the graded quotient $Gr^W_1O(M'_1)$ is generated by $\bar{x}W$’s. These generators especially satisfy the commutator group likeness, (10) and $c_2(\bar{\varphi}) = 0$ for $\bar{\varphi} = 1 + \sum W \bar{x}W$ among others. By the previous lemmas, $\bar{\varphi}$ must

---

A word means a monic monomial element but 1 in $k((X,Y))$. 

---
also satisfy (12) and (14). Therefore $p$ should be an isomorphism. This implies $M'_1 = M_1$. □

The combination of this theorem with the previous lemmas completes the proof of theorem (1).

3. Proof of Theorem 2

In this section we deduce theorem (2) from the previous theorem and also show its pro-$l$ group analogue (corollary (12)) and its pro-nilpotent group analogue (corollary (13)).

Proof of theorem (2) Let $f$ be an element of $F_2(k)$ satisfying (4). Let $\lambda$ be a solution of $\frac{z^2 - 1}{z} = c_2(f)$. Let $\mu \in k^\times$ and $\varphi \in k(\langle A, B \rangle)$ be a pair such that $\varphi$ is commutator group-like and $(\mu, \varphi)$ satisfies (10), (12) and (13). Put $\varphi' = f(\varphi e^{\mu X}, \varphi^{-1}, e^{\mu Z}) \cdot \varphi \in k(\langle A, B \rangle)$. In the proof of [Dr91], proposition 5.1 it is shown that giving (1) for $f$ is equivalent to giving (1) for $\varphi'$. Hence $\varphi'$ satisfies (10) by lemma (5). Put $\mu' = \lambda \mu$. The equation (13) for $(\mu, \varphi)$ gives $c_2(\varphi) = \frac{\mu^2}{4}$. So $c_2(\varphi') = c_2(\varphi) + \mu^2 c_2(f) = \frac{\mu^2}{4} z$. Since $\varphi'$ satisfies (10), our previous theorem gives (13) for $(\mu', \varphi')$. Consider the group isomorphism from $F_2(k)$ to the set of group-like elements of $U(F_2)$ which sends $x$ to $e^{\mu X}$ and $y$ to $e^{-\frac{\mu}{4} X} \varphi(y, x) e^{\mu Y} \varphi(x, y)^{-1} e^{\frac{\mu}{4} X}$. Consequently $z$ goes to $\varphi(Z, X) e^{\mu Z} \varphi(Z, X)^{-1}$ by (12) and (13) for $(\mu, \varphi)$. The direct calculation shows that LHS of (6) maps to LHS of (13). Therefore giving (5) for $(\lambda, f)$ is equivalent to giving (13) for $(\mu', \varphi')$. This completes the proof of theorem (2). □

Remark 11. By the same argument as lemma (5) giving the pentagon equation (4) for $f$ is equivalent to giving that $f(e^X, e^Y)$ is commutator group-like and $f$ satisfies the 5-cycle relation in $P_3(k)$:

$$f(x_{12}, x_{23}) f(x_{34}, x_{45}) f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{51}) = 1.$$ 

Here $P_3(k)$ means the unipotent completion of the pure sphere braid group with 5 strings and $x_i$ means its standard generator. Occasionally, in some of the literature, the formula is used directly instead of (11) in the definition of the Grothendieck–Teichmüller group.

As a corollary the following pro-$l$ ($l$: a prime) group and pro-nilpotent group version of theorem (2) are obtained by the natural embedding from the pro-$l$ completion $F_2^{(l)}$ to $F_2(Q_l)$ and its associated embedding from the pro-nilpotent completion $F_2^{nil} = \prod_{l \text{prime}} F_2^{(l)}$ to $\prod_l F_2(Q_l)$.

Corollary 12. Let $f = f(x, y)$ be an element of $F_2^{(l)}$ satisfying (1) in $K_4^{(l)}$ (: the pro-$l$ completion of $K_4$). Then there exists $\lambda$ such that the pair $(\lambda, f)$ satisfies (12) and (13). Actually this $\lambda$ is equal to $\pm(24 c_2(f))^{1/2}$.

Corollary 13. Let $f = f(x, y)$ be an element of $F_2^{nil}$ satisfying (1) in $K_4^{nil} = \prod_l K_4^{(l)}$. Then there exists $\lambda$ such that the pair $(\lambda, f)$ satisfies (5) and (6). Actually this $\lambda$ is equal to $\pm(24 c_2(f))^{1/2}$.

It should be noted that though $\lambda$ might lie on a quadratic extension the equation (5) makes sense for such $(\lambda, f)$. In the pro-unipotent context taking a quadratic extension is necessary: the Drinfel’d associator $\Phi_{KZ} \in R(\langle X, Y \rangle)$ satisfies (2) and
with \( \mu = \pm 2\pi \sqrt{-1} \not\in \mathbb{R}^\times \). In the pro-l context the author thinks that it might also happen \( \pm(24c_2(f) + 1) \not\in \mathbb{Z}_l^\times \). In the pro-l context the author thinks that it might also happen

\[
\pm(24c_2(f) + 1) \not\in \mathbb{Z}_l^\times.
\]

We have a group theoretical definition of \( c_2(f) \) (cf. \[LS\] lemma 9): put

\[
F^{(1)}_2 := \left[ F^{(2)}_2, F^{(2)}_2 \right] \quad \text{and} \quad F^{(2)}_2 := \left[ F^{(1)}_2, F^{(1)}_2 \right]
\]

where \([\cdot, \cdot]\) means the topological commutator. The quotient group \( F^{(1)}_2/F^{(2)}_2 \) is cyclic generated by the commutator \((x, y)\). For \( f \in F^{(1)}_2, c_2(f) \in \mathbb{Z}_l \) is defined by \((x, y)^{c_2(f)} = f\) in this quotient.

Posing the following question on a pro-finite group analogue of theorem \[2\] might be particularly interesting:

**Question 14.** Let \( f = f(x, y) \) be an element of the pro-finite completion \( \hat{F}_2 \) satisfying \[4\] (hence \[5\]) in the pro-finite completion \( \hat{K}_4 \). Let \( c_2(f) \) be an element in \( \hat{\mathbb{Z}} \) defined in a similar way to the above. Assume that there exists \( \lambda \) in \( \hat{\mathbb{Z}} \) such that

\[
\lambda^2 = 24c_2(f) + 1.
\]

Then does the pair \((\lambda, f)\) satisfy \[5\]?

**Remark 15.** Although the pentagon equation \[4\] implies the two hexagon equations \[5\] and \[6\] of \( GT \), it does not mean that the pentagon axiom \[Dr91\] (1.7) implies two hexagon axioms, \[Dr91\] (1.9a) and (1.9b), of braided tensor categories. It is because that the pentagon equation \[4\] of \( GT \) is a consequence of the three axioms of braided tensor categories: \( GT \) is interpreted as a group of deformations of braided tensor categories by Drinfel’d in \[Dr91\] §4. The equation \[4\] of \( GT \) is read as a condition to keep the pentagon axiom. However it is formulated in terms of the braid group \( K_4 \), where its generators \( x_{ij} \)'s are subject to the braid relations. In his interpretation the relations are guaranteed by the dodecagon diagram (the Yang-Baxter equation) (see \[JS\] proposition 2.1 and \[K\] theorem XIII.1.3) which is deduced from two hexagon axioms.

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