A class of exact solutions of the Liénard type ordinary non-linear differential equation

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Abstract. A class of exact solutions is obtained for the Liénard type ordinary non-linear differential equation. As a first step in our study the second order Liénard type equation is transformed into a second kind Abel type first order differential equation. With the use of an exact integrability condition for the Abel equation (the Chiellini lemma), the exact general solution of the Abel equation can be obtained, thus leading to a class of exact solutions of the Liénard equation, expressed in a parametric form. We also extend the Chiellini integrability condition to the case of the general Abel equation. As an application of the integrability condition the exact solutions of some particular Liénard type equations, including a generalized van der Pol type equation, are explicitly obtained.

Keywords: Liénard equation: Abel equation: integrability condition: exact solutions

1. Introduction

The Liénard type second order nonlinear differential equation of the form (Liénard, 1928a; Liénard, 1928b)

\[ \ddot{x}(t) + f(x)\dot{x}(t) + g(x) = 0, \]  

where \( f(x) \) and \( g(x) \) are arbitrary real functions of \( x \), with \( f(x), g(x) \in C^\infty(I) \), defined on a real interval \( I \subseteq \mathbb{R} \), as well as its generalization, the Levinson-Smith type equation (Levinson and Smith, 1942)

\[ \ddot{x}(t) + f(x, \dot{x}) \dot{x}(t) + g(x) = 0, \]
where a dot represents the derivative with respect to the time \( t \), and \( f \) is a function of \( x \) and \( \dot{x} \), plays an important role in many areas of electronics (where Eq. (1) appears as the Rayleigh or van der Pol equation), cardiology (modeling the electric heart activity), neurology (modeling neurons activity), biology, mechanics, seismology, chemistry, physics and cosmology (van der Pol, 1927; Andronov et al., 1973; Strogatz, 1994; van der Pol and Van der Mark, 1928; Fitzhugh, 1961; Nagumo et al., 1962; Glass, 1990; Nayfeh and Balachandran, 1995; Edelstein-Keshet, 1988; Poland, 1994; Salasnich, 1995; Salasnich, 1997).

A particular type of the general Liénard equation, the van der Pol equation (van der Pol, 1927; Andronov et al., 1973; Strogatz, 1994)

\[
\ddot{x}(t) - \mu \left[ 1 - x^2(t) \right] \dot{x}(t) + x(t) = 0,
\]

where \( \mu \) is a positive parameter, describing a non-conservative oscillator with non-linear damping, is extensively applied in both the physical and biological sciences. In the 1920’s the Dutch physicist van der Pol, when he was an engineer working for Philips Company, studied the differential equation (3), which describes the circuit of a vacuum tube. A few years after, the electric activity of the heart rate was modeled by using a Liénard type model (van der Pol and Van der Mark, 1928). In the 1960’s Fitzhugh (Fitzhugh, 1961) and Nagumo et al. (Nagumo et al., 1962) extended the van der Pol equation in a planar field as a model for action potentials of neurons. The van der Pol equation, originally introduced to describe relaxation oscillators in electronic circuits, has been frequently used in theoretical models of the heart function (Glass, 1990; Nayfeh and Balachandran, 1995; Edelstein-Keshet, 1988). The van der Pol equation is a useful phenomenological model for the heartbeat, since it displays many of those features supposedly occurring in the biological setting, as complex periodicity, entrainment, and chaotic behavior (Glass, 1990; Nayfeh and Balachandran, 1995; Edelstein-Keshet, 1988).

Most models based on chemical kinetics can be formulated as Liénard type coupled nonlinear first-order rate equations in several variables (Poland, 1994). The first-order approximation for the Liénard system works well near a bifurcation point, with higher-order terms being required the further the system is from the bifurcation point. The dynamics of a scalar inflaton field with a symmetric double–well potential can also be formulated mathematically as a Liénard system (Salasnich, 1995; Salasnich, 1997). For this case one can prove rigorously the existence of a limit cycle in its phase space, and, by using analytical and numerical arguments one can show that the limit cycle is stable, and its period can be obtained by an analytical formula.
The Liénard type equations can also be used to model fluid mechanical phenomena. The linearly forced isotropic turbulence can be described in terms of a cubic Liénard equation with linear damping of the form (Ran, 2009)

\[ \ddot{x}(t) + [ax(t) + b] \dot{x}(t) + cx(t) - x^3(t) + d = 0, \]  

where \(a, b, c\) and \(d\) are constants, also naturally appear in the mathematical description of some important astrophysical phenomena. For example, the time-dependence \(\phi(t)\) of the perturbations of the stationary solutions of spherically symmetric accretion processes can be described by a generalized Liénard type equation of the form (Sen and Ray, 2012)

\[ \dddot{\phi} + \epsilon f(\phi, \dot{\phi}) \dot{\phi} + V'(\phi) = 0, \]

where \(\epsilon\) is a small parameter, and \(V(\phi)\) is the potential of the system, with the prime indicating the derivative with respect to \(\phi\). A dynamical systems analysis of this Liénard equation reveals a saddle point in real time, with the implication that when the perturbation is extended into the nonlinear regime, instabilities will develop in the accreting system.

From a physical point of view, the Liénard equation represents the generalization of the equation of damped oscillations, \(\ddot{x} + \gamma \dot{x} + \omega^2 x = 0\), where \(\gamma\) and \(\omega^2\) are constant parameters, respectively (DiBenedetto, 2011). For \(\gamma = 0\) we obtain the equation of the linear harmonic oscillator, which represents one of the fundamental equations of both classical and quantum physics. Generally, a linear oscillation can be described by the equation \(\ddot{x} + f(t) \dot{x} + g(t) x = 0\).

The mathematical properties of the Liénard type of equations have been intensively investigated from both mathematical and physical points of view, and their study remains an active field of research in mathematical physics (Dumortier and Rousseau, 1990; Dumortier et al., 2000; Cheb-Terrab and Roche, 2000; Chandrasekar et al., 2006; Liu et al., 2008; Zhou et al., 2008; Pradeep et al., 2009; Pandey et al., 2009a; Pandey et al., 2009b; Banerjee and Bhattacharjee, 2010; Messias and Gouveia, 2011). Several methods of integrability, like the Lie symmetries method (Garcia et al., 2008; Carinena and de Lucas, 2011) and the Weierstrass integrability, introduced in (Giné and Grau, 2010), were used to study the Liénard equation, and the relations between the Riccati and Liénard equations, respectively. Liénard systems which have a generalized Weierstrass first integral, or a generalized Weierstrass inverse integrating factor, were studied in (Giné and Llibre, 2011).

It is the purpose of the present paper to introduce some exactly integrable classes of the Liénard equation, Eq. (1), whose solutions can be obtained in an exact analytical form, and to formulate the integrability
condition for this class of differential equations. To obtain the functional form of the integrable Liénard type equations we reduce them first to an Abel type equation of the form \( y' = p(x)y^3 + q(x)y^2 \) (Polyanin and Zaitsev, 2003; Kamke, 1959). Then, we apply to the latter Abel equation an integrability condition, equivalent to the initial Liénard equation, that was obtained by Chiellini (Chiellini, 1931; Kamke, 1959). In fact, the Chiellini condition has been recently used for the study of the Abel differential equations, as well as for the second order differential equations reducible to an Abel type equation, in (Bandic, 1961; Mak et al., 2001; Mak and Harko, 2002; Harko and Mak, 2003; Mancas and Rosu, 2013a; Mancas and Rosu, 2013b).

Bandic (Bandic, 1961) studied the non-linear differential equation \( y'' + \psi(y)y^2 + \phi(y)y' + f(y) = 0 \), and he did show that it can be solved by using quadratures if \( f(y) = F(y)\exp\left(-2 \int \psi(y)dy\right) \) and \( \phi(y) = G(y)\exp\left(-\int \psi(y)dy\right) \), where \( F(y) \) and \( G(y) \) are the coefficients of the integrable Abel equation \( w' = F(y)w^3 + G(y)w^2 \). The integrability of this Liénard type equation was obtained by using the Chiellini condition.

In (Mancas and Rosu, 2013a) the differential Chiellini integrability condition was reformulated in an integral form, and the general form of the solution of the Abel equation was obtained in a simpler form. The Chiellini integrability condition of the first order first kind Abel equation \( dy/dx = f(x)y^2 + g(x)y^3 \) was extended to the case of the general Abel equation of the form \( dy/dx = a(x) + b(x)y + f(x)y^{\alpha-1} + g(x)y^\alpha \), where \( \alpha \in \mathbb{R} \) and \( \alpha > 1 \), in (Harko et al., 2013).

There are several other methods that can be used for the integration of the general Abel type equation \( y' = p(x)y^3 + q(x)y^2 + r(x)y + s(x) \) (Kamke, 1959). If \( y = y_1(x) \) is a particular solution of the general Abel equation, then by means of the transformations \( u(x) = E(x)/[y(x) - y_1(x)] \), where \( E(x) = \exp\left\{ \int \left[ 3p(x)y_1^2 + 2q(x)y_1 + r(x) \right] dx \right\} \), the Abel can be transformed into \( du/dx + \Phi_1/u + \Phi_2 = 0 \), where \( \Phi_1(x) = p(x)E^2(x) \), and \( \Phi_2(x) = \left[ 3p(x)y_1^2 + q(x)x \right] E(x) \). Therefore, if \( y_1 = -q(x)/3p(x) \), then \( \Phi_2 = 0 \), and the general solution of the Abel equation can be obtained from the integration of a differential equation with separable variables.

Therefore it turns out that, if the coefficients \( f(x) \) and \( g(x) \) of the Liénard equation satisfy two specific conditions, then the general solution of the Liénard equation can be obtained in an exact analytical form. Some examples of exactly integrable Liénard equations, of physical interest, are also considered. The generalization of the method to the case of the Levinson-Smith type equations of the form (2) is briefly discussed.
The present paper is organized as follows. In Section 2, we introduce the Abel equation representation for the Liénard equation, and we formulate the integrability condition of the first order Abel equation. In Section 3, we obtain the general solution of the Liénard type equations satisfying the integrability condition of the Abel equation. The exact solutions of some non-linear Liénard type differential equations are obtained in Section 4. We discuss and conclude our results in Section 5.

2. Reduction of the Liénard equation to an integrable Abel type equation

As a first step in our study of the Liénard equation (1) we reduce it to an Abel type first order non-linear differential equation. Then, by using an integrability condition for this equation, which involves a differential relation between the coefficients \( f(x) \) and \( g(x) \) of the equations, we obtain the general solution of the Liénard equation in an exact parametric form.

By denoting \( \dot{x} = u \), the Liénard equation (1) can be written as

\[
 u \frac{du}{dx} + f(x)u + g(x) = 0. \tag{6}
\]

By introducing a new dependent variable \( v = 1/u \), Eq. (6) takes the form of the standard first kind Abel differential equation,

\[
 \frac{dv}{dx} = f(x)v^2 + g(x)v^3. \tag{7}
\]

2.1. The Chiellini integrability condition for the reduced Abel equation

In this context, an exact integrability condition for the Abel equation Eq. (7) was obtained by Chiellini (Chiellini, 1931) (see also (Kamke, 1959)), and can be formulated as the Chiellini Lemma as follows:

**Chiellini Lemma.** If the coefficients \( f(x) \) and \( g(x) \) of a first kind Abel type differential equation of the form

\[
 \frac{dv}{dx} = f(x)v^2 + g(x)v^3, \tag{8}
\]

satisfy the condition

\[
 \frac{d}{dx} \frac{g(x)}{f(x)} = kf(x), \tag{9}
\]
where \( k = \text{constant} \neq 0 \), then the Abel Eq. (8) can be exactly integrated.

In order to prove the Chiellini Lemma we introduce a new dependent variable \( w \) defined as (Chiellini, 1931; Kamke, 1959)

\[
v = \frac{f(x)}{g(x)}w.
\]

(10)

Then Eq. (8) can be written as

\[
\left[ \frac{1}{g(x)} \frac{df(x)}{dx} - \frac{f(x)}{g^2(x)} \frac{dg(x)}{dx} \right] w + \frac{f(x)}{g(x)} \frac{dw}{dx} = \frac{f^3(x)}{g^2(x)} (w^3 + w^2).
\]

(11)

On the other hand, the condition given by Eq. (9) can be written in an equivalent form as

\[
\frac{f(x)}{g^2(x)} \frac{dg(x)}{dx} - \frac{1}{g(x)} \frac{df(x)}{dx} = \frac{k^3}{g^2(x)}.
\]

(12)

Therefore Eq. (11) becomes

\[
\frac{dw}{dx} = \frac{f^2(x)}{g(x)}w (w^2 + w + k),
\]

(13)

which is a first order separable differential equation, with the general solution given by

\[
\int \frac{f^2(x)}{g(x)} \, dx = \int \frac{dw}{w(w^2 + w + k)} \equiv F(w, k).
\]

(14)

With the use of the condition (9), the left hand side of Eq. (14) can be written as (Mancas and Rosu, 2013a)

\[
\int \frac{f^2(x)}{g(x)} \, dx = \frac{1}{k} \int \frac{d}{dx} \ln \left| \frac{g(x)}{f(x)} \right| \, dx = \frac{1}{k} \ln \left| \frac{g(x)}{f(x)} \right| + C_0,
\]

(15)

where \( C_0 \) is an arbitrary constant of integration. Therefore the general solution of Eq. (13) is obtained as

\[
\frac{g(x)}{f(x)} = C^{-1} e^{F(w, k)},
\]

(16)

where \( C^{-1} = \exp \left( -kC_0 \right) \) is an arbitrary constant of integration, and

\[
e^{F(w, k)} = \begin{cases} 
\frac{w}{\sqrt{w^2+w+k}} \exp \left( -\frac{1}{\sqrt{4k-1}} \arctan \frac{1+2w}{\sqrt{4k-1}} \right), & k > \frac{1}{4}, \\
\exp \left[ \frac{1}{1+2w} - 2\arctanh(1+4w) \right], & k = \frac{1}{4}, \\
\frac{w}{\sqrt{w^2+w+k}} \left( 1 - \frac{1+2w}{\sqrt{1-4k}} \right)^{-1/2\sqrt{1-4k}} \left( 1 + \frac{1+2w}{\sqrt{1-4k}} \right)^{1/2\sqrt{1-4k}}, & k < \frac{1}{4},
\end{cases}
\]

(17)
respectively. Eq. (16) determines $w$ as a function of $x$.

The integrability condition given by Eq. (9) can be written as

$$\frac{dg(x)}{dx} = \frac{1}{f(x)} \frac{df(x)}{dx} - g(x) + kf^2(x),$$

representing a first order linear differential equation in $g$. As a function of $g$, the function $f$ satisfies the differential equation

$$\frac{1}{f(x)} \frac{df(x)}{dx} = -k \frac{1}{g(x)} f^2(x) + \frac{1}{g(x)} \frac{dg(x)}{dx}.$$  

In order to solve Eq. (19) we introduce a new dependent variable $f(x) = 1/\sigma(x)$, and by denoting $\sigma^2(x) = \xi(x)$, we obtain a first order differential equation for $\xi$,

$$\frac{d\xi(x)}{dx} = -2 \frac{1}{g(x)} \frac{dg(x)}{dx} \xi(x) + \frac{2k}{g(x)}.$$  

Therefore the Chiellini Lemma can be reformulated as:

**Lemma 1.** If the coefficients $f(x)$ and $g(x)$ of the Abel Eq. (8) satisfy the conditions

$$g(x) = f(x) \left[ C_1 + k \int f(x) dx \right],$$

or

$$f(x) = \pm \frac{g(x)}{\sqrt{C_2 + 2k \int g(x) dx}},$$

where $C_1$, $C_2$, and $k$ are arbitrary constants, the Abel equation is exactly integrable, and its solution is given by

$$v(x) = Ce^{-F(w(x),k)}w(x),$$

where the functions $F(w,k)$ are given by Eqs. (17). A similar result was obtained in (Mancas and Rosu, 2013a).

### 2.2. The Chiellini Integrability Condition for the General Abel Equation

The Chiellini Lemma can be extended to the general Abel equation of the form

$$\frac{dv}{dx} = a(x) + b(x)v + f(x)v^2 + g(x)v^3,$$

where $a(x), b(x), f(x), g(x) \in C^\infty(I)$ are defined on a real interval $I \subseteq \mathbb{R}$, and $a(x), b(x) \neq 0, \forall x \in I$, as follows. By introducing a new function $p(x)$, defined as

$$v(x) = e^{\int b(x)dx}p(x),$$
Eq. (24) becomes
\[
\frac{dp}{dx} = a(x)e^{-\int b(x)dx} + f(x)e^{\int b(x)dx}p^2 + g(x)e^{2\int b(x)dx}p^3. \tag{26}
\]
We assume now that the functions \( b(x) \), \( f(x) \) and \( g(x) \) satisfy the condition
\[
\frac{d}{dx} \left( g(x)e^{\int b(x)dx} \right) = k_1 f(x)e^{\int b(x)dx}, \tag{27}
\]
where \( k_1 \) is an arbitrary constant. Then, by introducing the transformation
\[
p(x) = \frac{f(x)}{g(x)e^{\int b(x)dx}}s(x), \tag{28}
\]
Eq. (26) becomes
\[
\frac{ds}{dx} = \frac{g(x)}{f(x)} a(x) + \frac{f^2(x)}{g(x)} s \left( s^2 + s + k_1 \right). \tag{29}
\]
Hence we have obtained the following generalization of the Chiellini Lemma:

**Lemma 2.** If the coefficients of the general Abel Eq. (24) satisfy the conditions (27) and
\[
a(x) = k_2 \frac{f^3(x)}{g^2(x)}, \tag{30}
\]
where \( k_2 \) is an arbitrary constant, then the Abel equation can be exactly integrated, and its general solution is given by
\[
v(x) = \frac{f(x)}{g(x)}s(x), \tag{31}
\]
with \( s(x) \) a solution of the equation
\[
G_0(s, k_1, k_2) = \int \frac{f^2(x)}{g(x)} dx = \frac{1}{k_1} \ln \left| \frac{g(x)e^{\int b(x)dx}}{f(x)} \right| + K_0, \tag{32}
\]
where \( K_0 \) is an arbitrary constant of integration, and
\[
G_0(s, k_1, k_2) = \int \frac{ds}{s^3 + s^2 + k_1 s + k_2}. \tag{33}
\]
By using Eq. (27), Eq. (32) becomes
\[
\frac{g(x)e^{\int b(x)dx}}{f(x)} = K_1 e^{G(s, k_1, k_2)}, \tag{34}
\]

where \( K_1 = \exp(-k_1 K_0) \) is an arbitrary constant of integration, and \( G(s, k_1, k_2) = k_1 G_0(s, k_1, k_2) \), respectively.

### 3. A class of exact solutions of the Liénard equation

As we have already seen, the second order non-linear Liénard Eq. (1) can be reduced to an Abel type equation of the form given by Eq. (8), with the general solution given by \( v(x) = C \exp[-F(w(x), k) w(x)] \), where \( w(x) \) is determined, as a function of \( x \), by Eq. (16). Alternatively, Eq. (16) fixes \( x \) as a function of \( w \),

\[ x = x(w). \]  

(35)

To find the time dependence of \( x \), we start from

\[ \frac{dx}{dt} = \frac{dx}{dw} \frac{dw}{dt} = \frac{1}{v} = \frac{g(x)}{f(x) w}, \]  

(36)

which gives

\[ \frac{dw}{dt} = \frac{dw}{dx} \frac{g(x)}{f(x) w}. \]  

(37)

With the use of Eq. (13), satisfied by the function \( w(x) \), we obtain for \( dw/dt \) the equivalent expression,

\[ \frac{dw}{dt} = f(x) (w^2 + w + k). \]  

(38)

Therefore we have obtained the following:

**Theorem.** If the coefficients of the Liénard equation (1) satisfy the conditions

\[ g(x) = f(x) \left[ C_1 + k \int f(x) dx \right], \]  

(39)

or

\[ f(x) = \pm \frac{g(x)}{\sqrt{C_2 + 2k \int g(x) dx}}, \]  

(40)

where \( C_1, C_2 \) and \( k \) are arbitrary constants, then the general solution of the Liénard equation Eq. (1) can be obtained in an exact parametric form, with \( w \) taken as a parameter, as

\[ t - t_0 = \int \frac{dw}{f(x(w)) (w^2 + w + k)}, \quad x = x(w), \]  

(41)

with \( x(w) \) obtained as a solution of the equation

\[ \frac{g(x)}{f(x)} = C^{-1} e^{F(w,k)}, \]  

(42)
and
\[ F(w, k) = k \int \frac{dw}{w(w^2 + w + k)}. \]  

Similar results were previously obtained in (Mancas and Rosu, 2013a), where the integral form of the Chiellini integrability condition were explicitly formulated. A particular integrable case of the Liénard equation can be obtained for the case \( k = 0 \). In this case, the Chiellini condition given by Eq. (9) immediately provides
\[ g(x) = Af(x), \]
with \( A \) an arbitrary constant. Thus, the Liénard equation takes the particular form
\[ \ddot{x} + f(x)\dot{x} + Af(x) = 0, \]
with the associated Abel equation given by
\[ \frac{dv}{dx} = f(x)v^2(1 + Av), \]
where \( \dot{x} = 1/v \). The general solution of Eq. (46) is given by
\[ \int f(x)dx = A \ln \left| \frac{1}{v} + A \right| - \frac{1}{v} + K_1, \]
where \( K_1 \) is an arbitrary constant of integration. Therefore, the general solution of Eq. (45) can be written in a parametric form, with \( v \) taken as a parameter, in the following form
\[ t - t_0 = \int \frac{dv}{f(x(v))v(1 + Av)}, \quad x = x(v), \]
where \( x = x(v) \) is the solution of Eq. (47).

In the general solution for the time, given by Eqs. (41) and (48), one can take the arbitrary integration constant \( t_0 \) as zero, without any loss of generality. This choice fixes the origin of time at \( t = 0 \). The arbitrary integration constant \( C \), as well as the initial value \( w_0 \) of the parameter \( w \) can be determined from the initial conditions at \( t = 0 \) for the position \( x \) and the velocity \( \dot{x} \), given by
\[ x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \]
where \( x_0 \) and \( \dot{x}_0 \) are the initial values of \( x \) and \( \dot{x} \) at \( t = 0 \). By evaluating Eq. (42) for \( x = x_0 \), we obtain
\[ \frac{g(x_0)}{f(x_0)} = C^{-1}e^{F(w_0, k)}, \]
while evaluating Eq. (36) at \( t = 0 \) gives the equation 
\[
\dot{x}_0 = \left[ g(x_0) f(x_0) \right] w_0^{-1},
\]
which determines the initial value of the parameter \( w_0 \) as
\[
w_0 = \frac{1}{\dot{x}_0 f(x_0)}.
\] (51)

Once the initial value of the parameter \( w_0 \) is known, the value of the integration constant \( C^{-1} \) is obtained as
\[
C^{-1} = \frac{g(x_0)}{f(x_0)} e^{-F(w_0,k)}.
\] (52)

### 3.1. An integrability condition for the Levinson-Smith equation

The procedure for the exact integration of the Liénard type equations based on the Chiellini Lemma can be easily extended to the generalized Liénard equations of the Levinson-Smith form, given by Eq. (2), if they can be transformed to an Abel type equation. As a particular case of the integrable Levinson-Smith type non-linear differential equations we consider the equation
\[
\ddot{x} + \left[ \gamma(x) \dot{x}^2 + \delta(x) \dot{x} + f(x) \right] \dot{x} + g(x) = 0,
\] (53)

where \( \gamma(x) \) and \( \delta(x) \) are some arbitrary functions of the variable \( x \). By denoting \( \dot{x} = 1/v \), Eq. (53) takes the form of the general Abel equation
\[
\frac{dv}{dx} = \gamma(x) + \delta(x)v + f(x)v^2 + g(x)v^3 = 0.
\] (54)

If \( \gamma(x) = 0 \), by means of the transformation \( v(x) = e^{\int \delta(x) dx} h(x) \), Eq. (54) can be written in the standard form of the Abel equation,
\[
\frac{dh}{dx} = A(x)h^2 + B(x)h^3,
\] (55)

where \( A(x) = f(x)e^{\int \delta(x) dx} \), and \( B(x) = g(x)e^{2\int \delta(x) dx} \). If the coefficients \( A(x) \) and \( B(x) \) of the equation satisfy the conditions of Lemma 1, then the general solution of Eq. (55) can be obtained through quadratures. If \( \gamma(x) \neq 0 \), then from Lemma 2 it follows that if the functions \( \gamma(x) \), \( \delta(x) \), \( f(x) \) and \( g(x) \) satisfy the conditions
\[
\frac{d}{dx} \frac{g(x)e^{\int \delta(x) dx}}{f(x)} = k_1 f(x)e^{\int \delta(x) dx}, \quad \gamma(x) = k_2 \frac{f^3(x)}{g^2(x)},
\] (56)

where \( k_1 \) and \( k_2 \) are two arbitrary constants, then the generalized Liénard type equation (53) can be integrated exactly. Therefore all the
integrability results obtained for the Liénard equation can be applied for the Levinson-Smith type equations of the form (53).

4. Examples of exactly integrable Liénard type equations

In the present Section, we consider some exactly integrable Liénard type equations, which represent the generalizations of Eqs. (3) and (4), respectively. As a first case we assume that the functional form of the function \( f(x) \) is known. Then the Chiellini integrability condition fixes the form of the function \( g(x) \), and allows to find the general solution of the Liénard equation in an exact parametric form. The case in which the function \( g(x) \) is fixed is also considered. Furthermore, an integrable generalization of the van der Pol equation is also explored.

4.1. First case: \( f(x) = ax + b \).

As a first case we assume that the function \( f(x) \) is given by

\[
 f(x) = ax + b, \tag{57}
\]

where \( a \) and \( b \) are arbitrary constants. Then from the first integrability condition, given by Eq. (39) we obtain the function \( g(x) \) as

\[
 g(x) = \frac{1}{2} a^2 k x^3 + \frac{3}{2} abk x^2 + (aC_1 + b^2 k) x + bC_1, \tag{58}
\]

where \( C_1 \) and \( k \) are arbitrary integration constants. Therefore the exactly integrable Liénard equation is given by

\[
 \ddot{x} + (ax + b) \dot{x} + \frac{1}{2} a^2 k x^3 + \frac{3}{2} abk x^2 + (aC_1 + b^2 k) x + bC_1 = 0. \tag{59}
\]

As a function of the parameter \( w \), \( x \) is determined by Eq. (42), which gives for \( x \) the quadratic algebraic equation

\[
 \frac{ak}{2} x^2 + bkx + C_1 = C_1^{-1} e^{F(w,k)}, \tag{60}
\]

which determines \( x \) as a function of \( w \) as

\[
 x(w) = \frac{-bk \pm \sqrt{b^2 k^2 - 2ak \left[ C_1 - C_1^{-1} e^{F(w,k)} \right]}}{ak}. \tag{61}
\]

The time dependence of \( x \) is determined as a function of \( w \) as

\[
 t - t_0 = \pm \int \frac{kdw}{\sqrt{b^2 k^2 - 2ak \left[ C_1 - C_1^{-1} e^{F(w,k)} \right] (w^2 + w + k)}}. \tag{62}
\]
Eqs. (61) and (62) give the exact solution of the Liénard Eq. (59). Depending on the value of the constant \(k\) there are three distinct classes of solutions of this equation.

4.2. Second case: \(g(x) = cx + d\)

Secondly, we consider the case in which the function \(g(x)\) is fixed. By analogy with the van der Pol Eq. (3), we assume that

\[
g(x) = cx + d, \tag{63}
\]

where \(c\) and \(d\) are arbitrary constants. Then, after determining the function \(f(x)\) from the integrability condition Eq. (40), we obtain the Liénard equation

\[
\ddot{x} \pm \frac{cx + d}{\sqrt{ckx^2 + 2dkx + C_2}} \dot{x} + cx + d = 0. \tag{64}
\]

Eq. (42) gives the equation

\[
ckx^2 + 2dkx + C_2 = C^{-2}e^{2F(w,k)}, \tag{65}
\]

with the solution

\[
x(w) = \frac{-dk \pm \sqrt{d^2k^2 - ck \left[ C_2 - C^{-2}e^{2F(w,k)} \right]}}{ck}. \tag{66}
\]

The parametric time dependence of the solution is obtained as

\[
t - t_0 = \pm \frac{k}{C} \int \frac{e^{F(w,k)}dw}{\sqrt{d^2k^2 - ck \left[ C_2 - C^{-2}e^{2F(w,k)} \right] (w^2 + w + k)}}. \tag{67}
\]

Eqs. (66) and (67) give the exact analytic solution, in a parametric form, of the Liénard Eq. (64).

4.3. Third case: the generalization of the van der Pol equation

Finally, we consider the integrable generalization of the van der Pol Eq. (3), in which we fix the function \(f(x)\) as \(f(x) = -\mu (1 - x^2)\), and obtain the function \(g(x)\) from the integrability condition Eq. (39). Therefore the integrable generalization of the van der Pol equation is given by

\[
\ddot{x} - \mu (1 - x^2) \dot{x} + \frac{1}{3}k\mu^2 x^5 - \frac{4}{3}k\mu^2 x^3 + C_1\mu x^2 + k\mu^2 x - C_1\mu = 0. \tag{68}
\]
The parametric dependence of $x$ is determined from the algebraic equation

$$C_1 - k\mu x + \frac{1}{3}k\mu x^3 = C^{-1}e^{F(w,k)}.$$  \hfill (69)

Equation (69) can be rewritten in the form

$$x^3 - 3x + H(F) = 0.$$  \hfill (70)

where we have denoted

$$H(F(w,k)) = \frac{3[CC_1 - e^{F(w,k)}]}{Ck\mu}.$$  \hfill (71)

The solution of the algebraic Eq. (70) is given by

$$x(w) = \frac{2^{1/3}}{2^{1/3}} \left\{ \sqrt{H^2(F(w,k)) - 4} - H(F(w,k)) \right\}^{1/3} +$$

$$\frac{1}{2^{1/3}} \left\{ \sqrt{H^2(F(w,k)) - 4} - H(F(w,k)) \right\}^{1/3}.$$  \hfill (72)

In order to have a real solution of the cubic Eq. (70), the conditions

$$H^2(F(w,k)) - 4 > 0,$$  \hfill (73)

and

$$\sqrt{H^2(F(w,k)) - 4} - H(F(w,k)) > 0,$$  \hfill (74)

must be satisfied for all $w$ and $k$.

The parametric time dependence of the solution of the generalized van der Pol equation is obtained as

$$t - t_0 = \frac{1}{\mu} \int \frac{\phi^2(w,k)dw}{[\phi^4(w,k) + \phi^2(w,k) + 1](w^2 + w + k)},$$  \hfill (75)

where we have denoted

$$\phi(w,k) = \frac{2^{1/3}}{\left(\sqrt{H^2(F(w,k)) - 4} - H(F(w,k))\right)^{1/3}}.$$  \hfill (76)

Depending on the numerical values of the parameters $k, \mu, C_1, C$ a large class of dynamical evolutions of the solutions of the generalized van der Pol equation can be obtained.
Exact solutions of the Liénard type differential equation

5. Discussions and final remarks

In the present paper we have introduced a class of exactly integrable Liénard, and generalized Liénard type equations. If the coefficients of the second order non-linear equations satisfy some specific conditions, which follow from the Chiellini Lemma, then the general solution of the Liénard differential equation can be obtained in an exact parametric form. As an application of the integrability procedure obtained, we have considered some specific examples of exactly integrable non-linear differential equations that could be of physical interest. One of these equations, Eq. (59) is similar in form with Eq. (4), and in fact represents the exactly solvable generalization of the equation describing the linearly forced isotropic turbulence (Dumortier and Rousseau, 1990). We have also considered an exactly solvable generalization of the classical van der Pol oscillator equation, in which higher order force terms are also present. In all these cases of physical interest the general solution of the corresponding Liénard equation can be obtained in an exact parametric form. The existence of an analytical solution may allow a deeper understanding of the highly non-linear physical processes that govern most of the natural phenomena.

The exact solutions also allow us to obtain some approximate solutions of the considered differential equations, corresponding to the small and large values of the parameter \( w \), respectively. In the limit of small \( w \), i.e., \( w \ll k \), giving \( \exp[F(w,k)] \approx w \), Eq. (42) takes the simple form

\[
\frac{g(x)}{f(x)} \approx C^{-1}w,
\]

while the parametric time evolution can be obtained as

\[
t - t_0 \approx \frac{1}{k} \int \frac{dw}{f(x(w))}.
\]

In the limit of large \( w \), so that \( w \gg k \), and \( w^2 \gg w \), \( \exp(F(w,k)) \approx \exp(k \int dw/w^3) = \exp(-k/2w^2) \), and the approximate asymptotic solution of the exactly integrable Liénard equation is given by

\[
\frac{g(x)}{f(x)} \approx C^{-1}e^{-k/2w^2},
\]

and

\[
t - t_0 \approx \int \frac{dw}{f(x(w))w^2}.
\]

As an application of the previous asymptotic equations we consider the case \( f(x) = ax + b \), with \( a, b = \text{constant} \), giving \( g(x)/f(x) = C_1 + kax^2/2 + kbx \).
In the limit of small $x$, by neglecting the $x^2$ term, we obtain
\begin{equation}
    x(w) \approx \frac{C^{-1}w - C_1}{bk},
\end{equation}
and
\begin{equation}
    t - t_0 \approx \frac{bC}{a} \ln \left| \frac{aw}{C} - aC_1 + b^2k \right|,
\end{equation}
respectively, giving
\begin{equation}
    x(t) \approx \frac{1}{abk} e^{a(t-t_0)/bC} - \frac{b}{a}.
\end{equation}

In the limit of large $x$, so that $kbx \gg C_1$, and $ax/2 \gg b$, respectively, we obtain
\begin{equation}
    g(x)/f(x) \approx kax^2/2, \quad \text{and} \quad \sqrt{\frac{2C^{-1}}{ka}} e^{-k/4w^2},
\end{equation}
\begin{equation}
    t - t_0 \approx \int \frac{dw}{w^2 \left( \sqrt{2a/kC} e^{-k/4w^2} + b \right)}.
\end{equation}

In the range of the values of $w$ for which $\sqrt{2a/kC} e^{-k/4w^2} \gg b$, we obtain
\begin{equation}
    t - t_0 \approx -\sqrt{\frac{\pi C}{2a}} \text{erfi} \left( \sqrt{\frac{K}{2w}} \right),
\end{equation}
where erfi($z$) gives the imaginary error function $\text{erfi}(z) = \text{erf}(iz)/i$. In the large time limit the solution of the Liénard Eq. (59) can be obtained only in a parametric form.

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**References**

Andronov, A. A., Leontovich, E. A., Gordon, I. I., and Maier, A.G. Qualitative Theory of Second Order Dynamic Systems. Wiley, New York, 1973.

Bandic, I. Sur le critère d’intégrabilité de l’équation différentielle généralisée de Liénard. *Bollettino dell Unione Matematica Italiana*, 16: 59-67, 1961.

Banerjee, D. and Bhattacharjee, J. K. Renormalization group and Liénard systems of differential equations. *Journal of Physics A: Mathematical and Theoretical*, 43: 062001, 2010.
Carinena, J. F. and de Lucas, J. Lie systems: theory, generalisations, and applications. *Dissertationes Mathematicae (Rozprawy Matematyczne)*, 479: 1-162, 2011.

Chandrasekar, V. K., Senthilvelan, M., Kundu, A., and Lakshmanan, M. A nonlocal connection between certain linear and nonlinear ordinary differential equations/oscillators. *Journal of Physics A: Mathematical and Theoretical*, 39: 9743-9754, 2006.

Cheb-Terrab, E. S. and Roche, A. D. Abel ODEs: Equivalence and integrable classes. *Computer Physics Communications*, 130: 204-231, 2000.

Chiellini, A. Sull’integrazione dell’equazione differenziale $y' + Py^2 + Qy^3 = 0$. *Bollettino dell’Unione Matematica Italiana*, 10: 301-307, 1931.

DiBenedetto, E. Classical mechanics: theory and mathematical modeling. *Birkhäuser, Springer*, New York, N. Y., 2011.

Dumortier, F., and Rousseau, C. Cubic Liénard equations with linear damping. *Nonlinearity*, 3: 1015-1039, 1990.

Dumortier, F., Kooij, R. E., and Li, C. Z. Cubic Liénard equations with quadratic damping having two antisaddles. *Qualitative Theory of Dynamical Systems*, 3: 1: 163-209, 2000.

Edelstein-Keshet, L. Mathematical Models in Biology. *Random House*, New York, 1988.

Fitzhugh, F. Impulses and physiological states in theoretical models of nerve membranes. *Biophysics Journal*, 1: 445-466, 1961.

Garcia, I. A., Giné, J., and Llibre, J. Liénard and Riccati differential equations related via Lie algebras. *Discrete Continous Dynamical Systems B*, 10: 485-494, 2008.

Giné, J. and Grau, M. Weierstrass integrability of differential equations. *Applied Mathematics Letters*, 23: 523-526, 2010.

Giné J. and Llibre, J. Weierstrass integrability in Liénard differential systems. *Journal of Mathematical Analysis and Applications*, 377: 362-369, 2011.

Glass, L. Theory of Heart. *Springer*, New York-Heidelberg-Berlin, 1990.

Harko, T. and Mak, M. K. Relativistic dissipative cosmological models and Abel differential equation. *Computers & Mathematics with Applications*, 46: 849-853, 2003.

Harko, T., Lobo F. S. N., and Mak, M. K. A Chiellini type integrability condition for the generalized first kind Abel differential equation. *Universal Journal of Applied Mathematics*, 1: 101 - 104, 2013.

Kamke, E. Differentialgleichungen: Lösungsmethoden und Lösungen. *Chelsea*, New York, 1959.

Levinson, N. and Smith, O. A general equation for relaxation oscillations. *Duke Mathematical Journal*, 9: 382-403, 1942.

Liénard, A. Étude des oscillations entreténues. *Revue générale de l’électricité*, 23: 901-912, 1928.

Liénard, A. Étude des oscillations entreténues. *Revue générale de l’électricité*, 23: 946-954, 1928.

Liu, X. G., Tang, M. L., and Martin, R. R. Periodic solutions for a kind of Liénard equation. *Journal of Computational and Applied Mathematics*, 219: 263-275, 2008.

Messias, M. and Gouveia, M. R. A. Time-periodic perturbation of a Liénard equation with an unbounded homoclinic loop. *Physica D: Nonlinear Phenomena*, 240: 1402-1409, 2011.
Mak, M. K., Chan, H. W., and Harko, T. Solutions generating technique for Abel-type nonlinear ordinary differential equations. *Computers & Mathematics with Applications*, 41: 1395-1401, 2001.

Mak, M. K. and Harko, T. New method for generating general solution of Abel differential equation. *Computers & Mathematics with Applications*, 43: 91-94, 2002.

Mancas, S. C. and Rosu, H. C. Integrable dissipative nonlinear second order differential equations via factorizations and Abel equations. *Physics Letters A*, 377: 1234-1238, 2013.

Mancas, S. C. and Rosu, H. C. Integrable Ermakov-Pinney equations with nonlinear Chiellini 'damping'. *arXiv:1301.3567*, 2013.

Nagumo, J., Arimoto, S., and Yoshizawa, S. An active pulse transmission line simulating nerve axon. *Proceedings of the Institute of Radio Engineers*, 50: 2061-2070, 1962.

Nayfeh, A. and Balachandran, B. *Applied Nonlinear Dynamics*. Wiley, New York, 1995.

Pandey, S. N., Bindu, P. S., Senthilvelan, M., and Lakshmanan, M. A group theoretical identification of integrable cases of the Linard-type equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. I. Equations having nonmaximal number of Lie point symmetries. *Journal of Mathematical Physics*, 50: 082702, 2009.

Pandey, S. N., Bindu, P. S., Senthilvelan, M., and Lakshmanan, M. A group theoretical identification of integrable equations in the Liénard-type equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. II: Equations having maximal Lie point symmetries. *Journal of Mathematical Physics*, 50: 102701, 2009.

Poland, D. Loci of limit cycles. *Physical Review E*, 49: 157-165, 1994.

Polyanin, A. D. and Zaitsev, V. F. *Handbook of Exact Solutions for Ordinary Differential Equations*. Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D. C., 2003.

Pradeep, R. G., Chandrasekar, V. K., Senthilvelan, M., and Lakshmanan, M. Non-standard conserved Hamiltonian structures in dissipative/damped systems: nonlinear generalizations of damped harmonic oscillator. *Journal of Mathematical Physics*, 50: 052901, 2009.

Ran, Z. One exactly soluble model in isotropic turbulence. *Advances and Applications in Fluid Mechanics*, 5: 41-47, 2009.

Salasnich, L. Instabilities, point attractors and limit cycles in an inflationary universe. *Modern Physics Letters A*, 10, 3119-3127, 1995.

Salasnich, L. On the Limit Cycle of an Inflationary Universe. *Nuovo Cimento B*, 112, 873-880, 1997.

Sen, S. and Ray, A. K. Implications of nonlinearity for spherically symmetric accretion. *arXiv:1207.1070*, 2012.

Strogatz, S. H. *Nonlinear Dynamics and Chaos*. Addison-Wesley, Reading, Massachussets, 1994.

van der Pol, B. On relaxation-oscillations. *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*, 2: 978-992, 1927.

van der Pol, B. and van der Mark, J. The heartbeat considered as a relaxation oscillation, and an electrical model of the heart. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 6: 763-775, 1928.

Zou, L., Chen, X. W., and Zhang, W. N. Local bifurcations of critical periods for cubic Liénard equations with cubic damping. *Journal of Computational and Applied Mathematics*, 222: 404-410, 2008.