Validity of the linear marginal stability principle for monotonic fronts of the extended Fisher-Kolmogorov equation

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(Dated: March 27, 2022)

The extended Fisher Kolmogorov equation \( u_t = u_{xx} - \gamma u_{xxxx} + f(u) \) with arbitrary positive \( f(u) \), satisfying \( f(0) = f(1) = 0 \), has monotonic traveling fronts for \( \gamma < 1/12 \). We find a simple lower bound on the speed of the fronts which allows to assess the validity of linear marginal stability.

I. INTRODUCTION

The extended Fisher-Kolmogorov equation (EFK),
\[
u_t = u_{xx} - \gamma u_{xxxx} + f(u),
\]
with \( f(u) = u - u^3 \) arises in the description of different systems. It appears for example in the study of phase transitions near a Lifshitz point \[1, 2\]. It has been derived as an amplitude equation at the onset of instabilities near certain degenerate points \[3\]. It has also been proposed as a model for the onset of spatiotemporal chaos in bistable systems \[4\] and as a natural extension to the reaction diffusion equation (\( \gamma = 0 \)) on which to study the dynamics of front propagation and pattern formation \[5, 6\] etc. Its steady version with different functions
\[
u_{xxxx} + qu_{xx} + f(u) = 0,
\]
is of interest in different fields and much work has been devoted to it. A very complete account of its solutions can be found in \[7\].

For \( \gamma < 1/12 \) numerical results indicate that sufficiently localized conditions evolve into a uniform translating front joining the stable point \( u=1 \) to the unstable \( u=0 \) point \[8\]. Similarly to what is found in the reaction diffusion equation, for the Fisher case \[9, 10\] \( f(u) = u - u^2 \) and for \( f(u) = u - u^3 \) the front propagates with the linear speed which now is \[6\]
\[
c_L = \frac{2}{\sqrt{34\gamma}} [1 + 36\gamma - (1 - 12\gamma)^3/2]^{1/2},
\]
obtained from linear analysis near \( u = 0 \). If \( \gamma > 1/12 \) monotonic fronts do not exist, and a pattern may appear.

Numerical results of the integrations of the EFK equation with arbitrary \( f(u) \) show that, as it occurs in the reaction diffusion equation, transition from linear to nonlinear marginal stability will occur as parameters in \( f(u) \) are varied \[8\].

In recent work a sufficient criterion \( f(u) \) for the validity of the linear speed selection mechanism analogous to the KPP \[13\] and Aronson-Weinberger \[14\] criteria for the reaction diffusion equation has been established \[12\]. As for the reaction diffusion equation, this criterion gives sufficient but not necessary conditions on the validity of the linear speed selection mechanism. Numerical results indicate that for small \( \gamma \) fronts of the EFK equation have similar properties to fronts of the reaction diffusion equation. Rigorous existence results of fronts of the EFK equation have been given for general functions \( f(u) \) \[15\]. For \( \gamma \to 0 \) it was proved that there is a minimal speed \( c^* \) for the existence of monotonic fronts, and that the fronts are stable. For \( \gamma = \epsilon^2 \) the minimal speed is given by \( c^* \geq 2 - \epsilon^2 + ... \) \[16\].

The purpose of the present work is to establish a simple lower bound on the speed \( c^* \) for which monotonic fronts exist. This enables to test whether for a given function \( f(u) \) the minimal speed is the linear value \( c_L \) obtained from the linear analysis at the edge of the front. The bound given in this work is not sharp, but the derivation suggests that it is possible to obtain a variational formulation for the minimal speed analogous to that given for the reaction-diffusion equation \[13, 14\]. Future work will address this aspect.

II. MONOTONIC FRONTS

A traveling monotonic front \( u = q(x - ct) \) joining the stable state \( u = 1 \) to \( u = 0 \) satisfies the ordinary differential equation
\[
q_{zz} + c q_z - \gamma q_{zzzz} + f(q) = 0,
\]
with \( \lim_{z \to -\infty} q = 1, \) \( \lim_{z \to \infty} q = 0, \) \( q_z < 0, \)
where \( z = x - ct \) and subscripts denote derivatives.

Following the usual procedure, since the front is monotonic, we may use phase space variables and define \( p(q) = -dq/dz, \) where the minus sign is introduced to have \( p > 0. \) A simple calculation shows that monotonic fronts obey
\[
\gamma p \frac{d}{dq} \left[ p \frac{dp}{dq} \left( \frac{dp}{dq} \right) \right] - p \frac{dp}{dq} + cp - f(q) = 0 \tag{3}
\]
with
\[
p(0) = p(1) = 0, \quad \text{and} \quad p > 0.
\]

Next we obtain the upper bound. Let \( g(q) \) be an arbitrary positive decreasing function. Multiplying Eq. (3) by \( g/p \) and integrating with respect to \( q \) between 0 and 1 we obtain the identity
\[
c \int_0^1 g(q)dq = \int_0^1 \frac{gf}{p} dq + \int_0^1 ph dq + \gamma \int_0^1 \left( \frac{1}{3} g'' p^3 + h p p'^2 \right) dq, \tag{4}
\]
where primes denote derivatives with respect to \( q \) and where we have defined \( h(q) = -g'(q) > 0. \) In obtaining this expression several integrations by parts were performed. Surface terms vanish due to the boundary conditions on \( p. \)

Furthermore we assume that the function \( g \) does not diverge in a manner that prevents the vanishing of surface terms.

Consider now the functional
\[
S_g(p) = \int_0^1 \frac{gf}{p} dq + \int_0^1 ph dq + \gamma \int_0^1 \left( \frac{1}{3} g'' p^3 + h p p'^2 \right) dq. \tag{5}
\]
It can be shown (details will be given elsewhere) that for \( g \in C^3([0,1]), g' < 0, g''' > 0, \) this functional has a unique minimizer which we call \( \hat{p}. \)

Therefore
\[
S_g(p) \geq \min_p S_g(p) = S_g(\hat{p}).
\]
This implies in Eq. (4) that
\[
c \int_0^1 g(q)dq \geq S_g(\hat{p}). \tag{6}
\]
The minimizing \( p, \) \( \hat{p} \) can be obtained by solving the Euler-Lagrange equation for \( S_g(p), \)
\[
\frac{d}{dq} \left( \frac{\partial L}{\partial p'} \right) - \frac{\partial L}{\partial p} = 0.
\]
Recalling that the arbitrary function \( g \) is a function of \( q \) we obtain
\[
2\gamma \frac{d}{dq} [h\hat{p}'] + \frac{gf}{p'} + g' + \gamma(g'\hat{p}'^2 - g'''\hat{p}^2) = 0. \tag{7}
\]
To obtain the minimizing \( p \) for each function \( g(q) \) we should solve this equation. This is not an easy task since \( g(q) \) is an arbitrary unspecified function. However, it follows from this equation, multiplying by \( p(q) \) and integrating in \( q \) that
\[
\int_0^1 \left( \frac{gf}{p'} - h p - \gamma g''' p^3 - 3\gamma h p^2 \right) dq = 0.
\]
Using this result we find that \( S_g(\hat{p}) \) can be written as
\[
S_g(\hat{p}) = \frac{4}{3} \int_0^1 \frac{f g}{p} dq + \frac{2}{3} \int_0^1 \hat{p} h dq.
\]
Inequality (6) is then
\[
c \int_0^1 g(q)dq \geq \frac{4}{3} \int_0^1 \frac{f(q)g(q)}{p(q)} dq + \frac{2}{3} \int_0^1 \hat{p}(q) h(q) dq.
\]
Finally, since \( f > 0, g > 0 \) and \( h = -g' > 0 \) we use the inequality \( a^2 + b^2 \geq 2ab \) in the expression above to obtain our main result,

\[
c \geq \frac{4\sqrt{2}}{3} \int_0^1 \sqrt{fghdq}.
\]

Here we recall that \( g \) is an arbitrary decreasing monotonic function.

Notice that this expression is similar in form to the bound obtained previously for the speed of fronts of the reaction diffusion equation. In that case we proved that the bound is sharp and that it follows from a variational principle. In the present case this bound does not saturate, however a variational principle will follow from (6) if we succeed in proving that there is a certain function \( g(q) \) for which \( \hat{p} \) is the solution of the differential equation (3). This point will be addressed in future work.

### III. CONCLUSION

A lower bound on the speed of monotonic fronts of the EFK equation has been obtained. This bound allows to determine the range of validity of the linear speed selection mechanism. We conjecture that there is a variational principle for the minimal speed of the fronts from which its exact value could be calculated.

### Acknowledgments

R. D. Benguria and M. C. Depassier acknowledge partial support from Fondecyt under grants 1020844 and 1020851.

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