Relation between semi- and fully-device-independent protocols

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We study the relation between semi and fully device independent protocols. As a tool, we use the correspondence between Bell inequalities and dimension witnesses. We present a method for converting the former into the latter and vice versa. This relation provides us with interesting results for both scenarios. First, we find new random number generation protocols with higher bit rates for both the semi and fully device independent cases. As a byproduct, we obtain whole new classes of Bell inequalities and dimension witnesses. Then, we show how optimization methods used in studies on Bell inequalities can be adopted for dimension witnesses.

Introduction - In device-independent (DI) protocols, two distant parties either do not know all the relevant parameters of their machines or do not trust them. This was formally presented in \cite{1}. Initially this approach was very successful in quantum cryptography \cite{2,3}. Later, Colbeck \cite{4,5} proposed a true random number expansion protocol based on the GHZ test, while Pironio \textit{et al.} \cite{8} proposed a protocol based on Bell inequality violations. All these protocols require entanglement, which has a negative effect on the complexity of the devices and the rates of randomness generation \cite{8} and key distribution. To cope with this problem the semi device-independent (SDI) scenario was introduced in \cite{8}. In this approach, we consider prepare, and measure protocols without making any assumptions about the internal operations of the preparation and measurement devices. The only assumption made is about the size of the communicated system. We assume there to be a single qubit in each round of the experiment. This approach is a very good compromise between the fully DI scenario and experimental feasibility. The possibility of using prepare and measure protocols implies no need for entanglement, which makes the experiments easier by several orders of magnitude. However, the price to pay for this is that one extra assumption means the possibility of a loophole if not met. This lowers the overall security of the protocol, albeit not significantly, since it is relatively easy to find the dimension of the system in which Alice’s device encodes information even through superficial inspection of the device. However, it is almost impossible to test each part of the device to check whether it indeed works as advertised. The first SDI protocol, presented in \cite{8}, was for quantum key distribution. Shortly thereafter, the first SDI randomness expansion protocol was proposed \cite{10}. This work studies the relation between DI and SDI protocols. We show how and under what conditions one can be converted into the other and how this change affects their parameters. This relation provides us with interesting results for both scenarios. First, we find new random number generation protocols with higher bit rates for both semi and fully device-independent cases. As a byproduct we obtain whole new classes of Bell inequalities and dimension witnesses. Then, we show how optimization methods used in studies on Bell inequalities can be adopted for dimension witnesses. Our paper is structured as follows. First we describe the method for converting DI protocols to SDI and vice versa. Then we apply our method to SDI random generators to obtain new DI protocols with higher bit rates. We also present a new family of Bell inequalities. Next we take a class of DI protocols and turn these into SDI protocols with better rates. This time our byproduct is a new family of dimension witnesses. Finally, we show how semi definite programming (SDP) methods, which are a powerful tool in the DI scenario, can be used in an SDI one.

Bell inequalities and dimension witnesses - In a DI protocol, distant parties receive systems in an unknown (possibly) entangled state from an untrusted sender. In each round they choose their inputs and make measurements to obtain the outcomes. In our paper we are interested in bi-partite protocols, and thus, we have two parties: Alice and Bob, with their setting choice denoted by $x$ and $y$, respectively, and their outcome by $a$ and $b$, respectively. In some, randomly chosen rounds of the protocol, both parties will publicly compare their settings and outcomes to estimate the conditional probability distribution $P(a, b | x, y)$. From this they can calculate the value of some Bell inequality

$$I = \sum_{a,b,x,y} \alpha_{a,b,x,y} P(a, b | x, y),$$

which is their security parameter. This parameter can then be used as the lower bound on the amount of randomness or secrecy in the remaining rounds. In an SDI
protocol Alice chooses her input $x'$, but she does not have any outcome. Instead, in each round, she prepares a state depending on $x'$ and sends it to Bob. Bob chooses his measurement setting $y$ and obtains outcome $b$. Although the devices that prepare the system and then measure it are not trusted, we assume that the communicated states are described by a Hilbert space with a fixed dimension (here we assume they are qubits) and that there is no entanglement between the devices of Alice and Bob. Again (here we assume they are qubits) and that there is no entanglement between the devices that prepare the system and then measure it.

Both of these scenarios are illustrated in Fig. 1, which has exactly the same function as $I$ in some rounds, $t$ tanglement between the devices of Alice and Bob. Again (here we assume they are qubits) and that there is no entanglement between the devices that prepare the system and then measure it.

In both cases violation of the classical bound is a necessary (though not always sufficient) condition for the protocol to work. Moreover, in both cases the form of $I$ or $W$ is the most important part of the protocol’s description. Therefore, finding the correspondence between these two objects is equivalent to finding the correspondence between the two protocols. Our method for doing so is quite straightforward: Let us rewrite $I$ as a collection of $n$-independent bits $a_0, ..., a_n$, then Alice can choose her setting by taking $a_i = c_i \oplus c_0$ for all $i > 0$ and transmit the message $m = a \oplus c_0$ to Bob. If he XORs his outcome $b$ with the message it is easy to calculate that he obtains the correct value of $a_y$ with average probability $P_n = \frac{1}{2^{n-1}}$. Therefore, we see that there is indeed a correspondence between the dimension witness and the Bell inequality related by our method, also at the level of protocols. In this case they are both a measure of the success probability for the different kinds of random access codes. $I_2$ is equivalent to the CHSH inequality. However, members of this family for $n > 2$ have never been studied. Because it is possible to use them for entanglement assisted random access codes, the bounds on their efficiency derived in [14] apply and they translate to the maximum quantum value of $P_n$, that is, $P_{n}^{\text{max}} = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{n}} \right)$. Now we show how our new Bell inequalities perform in DI randomness generation. The quantity that we wish to optimize is the min-entropy $H_{\infty}(a, b|x, y) = -\log \max_{a, b} P(a, b|x, y)$. To find the lower bound on this for a given value of $P_n$ we use the methods described in [15]. More precisely, we bound the set of allowed probability distributions by the second level of their hierarchy. We obtained the following lower bounds on the min-entropy for the maximal quantum values of $P_n$:

Compared with the randomness obtained from the SDI protocols, the main difference is that it grows with $n$ instead of reaching a maximum at $n = 3$. In fact the upper bound is $H_{\infty}(a, b|x, y) = 1 - \log P_{n}^{\text{max}} = 2 - \log \left( 1 + \frac{1}{\sqrt{n}} \right)$, which approaches $2$ as $n \to \infty$. We
TABLE I: Lower bounds on the min-entropy for the protocols corresponding to the $n \rightarrow 1$ random access codes. The values in the rightmost column are for the family of protocols defined in [12] and are taken from there. The values in the middle column correspond to the min-entropy of the outcomes in Bell inequalities $I_n$ for the maximal quantum values thereof. These were obtained using the SDP methods in [13]. The inequalities $I_n$ were derived from the protocols in [12] using the method depicted in Fig. 1.

| $n$ | DI: $H_\infty(a, b|x, y)$ | SDI: $H_\infty(b|a, x, y)$ |
|-----|--------------------------|-----------------------------|
| 2   | 1.2284                   | 0.2284                      |
| 3   | 1.3421                   | 0.3425                      |
| 4   | 1.4126                   | 0.1388                      |
| 5   | 1.4652                   | 0.1024                      |

Conjecture that this is reached for any $n$ but the second level of the SDP hierarchy form [13] that we use for the lower bound, is sufficient only for $n = 2$. Proving this conjecture is one of the open areas of research. The lower bounds as a function of $P_n$ are plotted in Fig. 2.

**FIG. 2:** (Color Online) The lower bounds on $H_\infty(a, b|x, y)$ as functions of $P_n$.

**From DI to SDI protocols** - Now we apply our method to show that we can go the other way and convert a DI protocol to an SDI one. We start from the randomness generation protocol form [16] based on Bell inequality $I_\alpha$, which expressed in the form (4) is

$$I_\alpha = \sum_{a,b,y} \delta_{a,b} P(a, b|x = 0, y) + \sum_{a,b,y} \delta_{a,b \oplus y} P(a, b|x = 1, y).$$

(4)

Converting this to a dimension witness we get

$$W_\alpha = \sum_{a,b,y} \frac{\alpha \delta_{a,b}}{2} P(a, b|x = 0, y) + \sum_{a,b,y} \frac{\delta_{a,b \oplus y}}{2} P(a, b|x = 1, y).$$

(5)

The lower bound on the min-entropy as a function of coefficient $\alpha$ is plotted in Fig. 3. For large values of $\alpha$ the amount of randomness is clearly greater than that for the best of the protocols described in [12]. The intuitive explanation for this is that $W_\alpha$ also corresponds to a kind of quantum random access code. In this case it is a $2 \rightarrow 1$ code with different weights assigned to the cases with $x = 0$ or $x = 1$. For large $\alpha$ it is much more important for the protocol to be correct when $x = 0$ than in the case of $x = 1$. This means that the protocols reaching maximum quantum value will tend to give the correct value of $b$ for $x = 0$. Here correct means fully specified by $a, y$, and $x$, in other words, deterministic. The price paid for this is that for $x = 1$ the probability of the correct (predetermined by $a, y$, and $x$) value is small, which implies a lot of randomness. Previously in [10, 12], the bounds on the entropy in SDI protocols were calculated using the Levenberg–Marquardt algorithm [17], which is not guaranteed to find global minima. SDP on the other hand always finds these; however, it was previously not known how this could be applied in the SDI case. Below we give a solution to this problem.

**FIG. 3:** (Color Online) Lower bound on the min-entropy $H_\infty(b|a, x = 1, y)$ as a function of coefficient $\alpha$ for the maximal quantum value of $W_\alpha$. In [16] a large amount of randomness is generated only for one setting of $x$. Here we observe the same result with high randomness for $x = 1$ and low randomness for $x = 0$.

**Optimization in SDI protocols** - It is not possible to use SDP optimization directly in the SDI case because of the nonlinear target function. Neither can methods from [15] be applied because they do not allow the dimension of the system to be set. Therefore, we need to find another solution. We do it by proving the following theorem:

**Theorem 1** If $H_\infty(b|a, x, y)$ is the min-entropy obtained in the SDI case and $H_\infty(a, b|x, y)$ the min-entropy obtained in the corresponding DI protocol, then

$$H_\infty(b|a, x, y) \geq H_\infty(a, b|x, y) - 1,$$

for the same value of the security parameter.

**Proof** - See the appendix.

Let us stress that (6) holds only when the values of the dimension witness and the Bell inequality are the
same. Consider Table 1 once again. For \( n = 2 \) we have equality \( H_{\infty}(b|x,a,y) = H_{\infty}(a,b|x,y) - 1 \). For \( n = 3 \) \( H_{\infty}(b|x,a,y) \) is slightly larger than \( H_{\infty}(a,b|x,y) - 1 \). This most probably stems from the fact that the bound in the table is not tight for \( n = 3 \). In fact, the upper bound on \( H_{\infty}(a,b|x,y) \) is exactly \( H_{\infty}(b|x,a,y) + 1 \). The situation changes for \( n = 4, 5 \). In these cases the value in the table is not the same for \( n = 4, 5 \) are not the same. If we calculate the entropy bound for the DI case when the value of the Bell inequality is equal to the maximal quantum value of the dimension witness, then the values are in agreement with Fig. 4. Using this method we were able to refine the results in [10], as shown in Fig. 4.

![FIG. 4: (Color Online) Min-entropy bounds for the SDI randomness generation protocol based on the 2 \( \rightarrow \) 1 quantum random access code. The dots are obtained from the Levenberg–Marquardt algorithm used in [10], which is not guaranteed to find global minima, while the line depicts the SDP method described here. Note that state preparation in the SDI protocol assumes that \( p(a|x,y) = \frac{1}{2} \).

Conclusions - We investigated the relation between DI and SDI protocols. Although our study focused on randomness generation, our results are also applicable to quantum key distribution since all the state-of-the-art proofs of security are based on the randomness of measurement outcomes. To this end we demonstrated a method for converting Bell inequalities into dimension witnesses and vice versa. This allowed us to generate new examples of both types of objects with very interesting properties. Our new family of Bell inequalities gave rise to DI randomness generation protocols with better bit rates, while our family of dimension witnesses did the same for SDI protocols. Finally, using the correspondence between the DI and SDI approach we were able to modify the SDP-based methods, which were proven successful in the former case, to work in the latter one. Apart from the similarities, our study also showed interesting differences such as the completely different dependence on \( n \) in Table 1. It also introduced many new protocols for both scenarios. Comparison of their efficiency with that of existing ones, especially in the presence of noise and imperfect detectors, opened a new area of research.

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I. APPENDIX: PROOF OF THEOREM 1

Every SDI protocol can be realized in the following way. Alice has a pair of systems in the singlet state. If she wishes to prepare state $|\phi\rangle$, she measures one particle in the basis $\{|\phi\rangle, |\phi^{\perp}\rangle\}$ and the other will also collapse to one of these states. Based on her measurement outcome she either sends the other particle to Bob unchanged or performs the unitary that flips $|\phi^{\perp}\rangle$ to $|\phi\rangle$ and then sends it. If Bob’s measurement outcomes are binary Alice does not even have to perform this unitary. She can just send her measurement outcome to Bob (0 denoting $|\phi\rangle$ and 1 denoting $|\phi^{\perp}\rangle$) who after XORing it with his outcome will get exactly the same probability distribution $P(b|a,x,y)$ as in the initial SDI protocol. These two cases are demarcated by letters (A) and (B) in Fig. 1. Obviously, nothing changes if the source of the singlet states is outside Alice’s lab and she is only the receiver of one of the subsystems just like Bob. This is depicted in fragment (C). But a lot changes if we now assume that the state that they receive can be an arbitrary maximally entangled state of any dimension (D). However, this only enlarges the space of allowed probability distributions $P(b|a,x,y)$, so any lower bound on the entropy in case (D) will also hold in (A). Finally, (D) is just the description of a DI protocol with some additional assumptions on the state. We can lower bound the entropy in this case with the SDP-based methods in [15] and the fact that the state is maximally entangled will be reflected by adding constraints

$$\forall_{x,y} P(a|x,y) = \frac{1}{2},$$

(7)

We now have

$$H^\infty_{\infty} (b|x,a,y) \geq H^\infty_{\infty} (b|x,a,y)$$

$$= -\log \max_b P^{(D)}(b|x,a,y),$$

(8)

which with $P^{(D)}(b|x,a,y) = \frac{P^{(D)}(a,b|x,y)}{P^{(D)}(a|x,y)} = 2P^{(D)}(a,b|x,y)$ gives

$$H^\infty_{\infty} (b|x,a,y) \geq -\log \max_b P^{(D)}(a,b|x,y)$$

$$\geq H^\infty_{\infty} (b|x,a,y) - 1$$

(9)

The above formula implies that the randomness obtained in an SDI protocol is greater than or equal to that in its DI counterpart minus 1.