Gravitational pressure on event horizons and thermodynamics in the teleparallel framework

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Abstract

The concept of gravitational pressure is naturally defined in the context of the teleparallel equivalent of general relativity. Together with the definition of gravitational energy, we investigate the thermodynamics of rotating black holes in the teleparallel framework. We obtain the value of the gravitational pressure over the external event horizon of the Kerr black hole, and write an expression for the thermodynamic relation $TdS = dE + pdV$, where the variations refer to the Penrose process for the Kerr black hole. We employ only the notions of gravitational energy and pressure that arise in teleparallel gravity, and do not make any consideration of the area or the variation of the area of the event horizon. However, our results are qualitatively similar to the standard expression of the literature.

PACS numbers: 04.20.-q, 04.20.Cv, 04.70.Dy

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1 Introduction

The discovery by Hawking [1] of the result that the area $A$ of an arbitrary black hole never decreases, together with the idea of the Penrose process [2] of extraction of energy of rotating black holes, led to the establishment of the thermodynamics of the gravitational field. The analysis in the literature is normally restricted to the geometrical and dynamical behaviour of the area of the external event horizon of rotating black holes. By means of the Penrose process the initial mass $m$ and angular momentum $J$ of the Kerr black hole vary by $dm$ and $dJ$, respectively, such that $dm - \Omega_H dJ > 0$, where $\Omega_H$ is the angular velocity of the external event horizon of the black hole. In this process, the variation of the area $A$ of the black hole satisfies $dA > 0$. In the final stage of an idealized Penrose process, the mass of the black hole becomes the irreducible mass $m_{irr}$ [3] defined by the relation $m^2 = m_{irr}^2 + J^2/(4m_{irr}^2)$.

Let us denote, as usual, the parameter $a$ as the angular momentum per unit mass of the Kerr black hole, $a = J/m$, and $S$ and $T$ as the gravitational entropy and temperature, respectively. The ordinary identification of the gravitational thermodynamic quantities with the parameters $(m, a)$ of the Kerr space-time is the following: $S \rightarrow A/4$ and $T \rightarrow k_s/(2\pi)$; $k_s$ is the surface gravity defined by $k_s = \sqrt{m^2 - a^2/(2mr_+)}$, and $r_+ = m + \sqrt{m^2 - a^2}$ is the radius of the external event horizon of the black hole. The ordinary thermodynamic relation for the Kerr black hole reads

$$TdS = dm - \Omega_H dJ. \quad (1)$$

In this paper, we will obtain an expression for the thermodynamic relation $TdS = dE + pdV$ in the context of the teleparallel equivalent of general relativity (TEGR). The expression for the gravitational energy $E$ is well defined in the TEGR. We will show in the present analysis that the notion of gravitational pressure $p$ over the external event horizon of the black hole is also well defined, and naturally arises from the field equations and from the gravitational energy-momentum tensor defined in the realm of the TEGR. The spatial components of the energy-momentum (or stress-energy) tensor yield the standard definition of the gravitational pressure, as we will explain. The variation $dV$ is obtained by means of the variation of $r_+$, when the parameters $m$ and $a$ of the black hole vary by the amounts $dm$ and $dJ$ ($p$ turns out to be a density, and therefore $dV$ is actually given by $dV = drd\theta d\phi$). We will show that our result for $TdS = dE + pdV$ is qualitatively and strikingly
similar to eq. (1). We will restrict our analysis to the external event horizon of the Kerr black hole (defined by \( r = r_+ \)), but will not consider any property of the area \( A \) (or \( dA \)) of the horizon. The indications are that the efficiency of the Penrose process in the present context is lower than in the ordinary thermodynamic formulation. We mention that the use of the concept of pressure in the first law of black hole thermodynamics has been made in two recent investigations \[4\], by considering that the cosmological constant plays the role of pressure.

In section 2 we present the Lagrangian formulation of the TEGR and indicate how the definitions of the gravitational energy-momentum \( P^a \), of the gravitational energy-momentum tensor \( t^{\lambda \mu} \) and of the gravitational pressure arise from the field equations of the theory. In section 3 we apply the definition of the gravitational pressure to the external event horizon of the Kerr black hole. In section 4 we obtain the expression for \( TdS \) that emerges in the present framework, and compare it with eq. (1). Finally, in section 6 we present our conclusions.

Notation: space-time indices \( \mu, \nu, \ldots \) and SO(3,1) indices \( a, b, \ldots \) run from 0 to 3. Time and space indices are indicated according to \( \mu = 0, i, \) \( a = (0), (i) \). The tetrad field is denoted \( e^a_{\mu} \), and the torsion tensor reads \( T_{\alpha \mu \nu} = \partial_{\mu} e_{\alpha \nu} - \partial_{\nu} e_{\alpha \mu} \). The flat, Minkowski spacetime metric tensor raises and lowers tetrad indices and is fixed by \( \eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu \nu} = (-1, +1, +1, +1) \). The determinant of the tetrad field is represented by \( e = \det (e^a_{\mu}) \).

2 The gravitational energy-momentum and pressure in the TEGR

In this section, we present a brief summary of the formulation of the teleparallel equivalent of general relativity. In the TEGR the gravitational field is represented by the tetrad field \( e^a_{\mu} \) only, and the Lagrangian density is written in terms of the torsion tensor \( T_{\alpha \mu \nu} = \partial_{\mu} e_{\alpha \nu} - \partial_{\nu} e_{\alpha \mu} \). This tensor is related to the antisymmetric part of the Weitzenböck connection \( \Gamma^\lambda_{\mu \nu} = e^{a \lambda} \partial_{\mu} e_{a \nu} \). However, the dynamics of the gravitational field in the TEGR is essentially the same as in the usual metric formulation. The physics in both formulations is the same.

We first consider the torsion-free, Levi-Civita connection \( ^0\omega_{\mu ab} \),
\begin{equation}
0 \omega_{\mu ab} = -\frac{1}{2} e^{c} \mu (\Omega_{abc} - \Omega_{bac} - \Omega_{cab}),
\end{equation}

\begin{equation}
\Omega_{abc} = e_{ab} (e^{\mu}_{\nu} \partial_{\mu} e_{c}^{\nu} - e_{c}^{\mu} \partial_{\mu} e_{b}^{\nu}).
\end{equation}

The Christoffel symbols $0 \Gamma_{\mu \nu}^{\lambda}$ and the Levi-Civita connection are identically related by

\begin{equation}
0 \Gamma_{\mu \nu}^{\lambda} = e^{a \lambda} \partial_{\mu} e_{a \nu} + e^{a \lambda} (0 \omega_{\mu ab}) e^{b \nu}.
\end{equation}

In view of this expression, an identity arises between the Levi-Civita connection and the contorsion tensor $K_{\mu ab}$,

\begin{equation}
0 \omega_{\mu ab} = -K_{\mu ab},
\end{equation}

where

\begin{align*}
K_{\mu ab} &= \frac{1}{2} e^{a \lambda} e^{b \nu} (T_{\lambda \mu \nu} + T_{\nu \lambda \mu} + T_{\mu \lambda \nu}), \\
T_{\lambda \mu \nu} &= e^{a \lambda} T_{a \mu \nu}.
\end{align*}

The identity given by eq. (3) may be used to show that the scalar curvature $R(e)$ may be identically written as

\begin{equation}
e R(0 \omega) = -e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^{a} T_{a} \right) + 2 \partial_{\mu} (e T^{\mu}),\end{equation}

where $e$ is the determinant of the tetrad field. Therefore in the framework of the TEGR the Lagrangian density for the gravitational and matter fields reads

\begin{equation}
L = -k e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^{a} T_{a} \right) - \frac{1}{c} L_{M}
\equiv -k e \Sigma^{abc} T_{abc} - \frac{1}{c} L_{M},
\end{equation}

where $k = c^{3}/16\pi G$, $T_{a} = T_{b a}$, $T_{abc} = e^{b \mu} e^{c \nu} T_{a \mu \nu}$ and

\begin{equation}
\Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^{b} - \eta^{ab} T^{c}).
\end{equation}

$L_{M}$ stands for the Lagrangian density for the matter fields. The Lagrangian density $L$ is invariant under the global SO(3,1) group. The absence in the Lagrangian density of the divergent term on the right hand side of eq. (4) prevents the invariance of (5) under arbitrary local SO(3,1) transformations.

The field equations derived from (5) are equivalent to Einstein’s equations. They are given by
\[ e_{a\lambda}e_{b\mu}\partial_{\nu}(e\Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}_{\ a}T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}) = \frac{1}{4kc}eT_{a\mu}, \]  \hspace{1cm} (7)

where \( \delta L_M/\delta e^{a\mu} = eT_{a\mu} \). From now on we will make \( c = 1 = G \).

The definition of the gravitational energy-momentum may be established in the framework of the Lagrangian formulation defined by (5), according to the procedure of ref. [5]. Equation (7) may be rewritten as

\[ \partial_{\nu}(e\Sigma^{a\lambda\nu}) = \frac{1}{4k}e e^{a\mu}(t^{\lambda\mu} + T^{\lambda\mu}) , \]  \hspace{1cm} (8)

where \( T^{\lambda\mu} = e_{a\lambda}T^{a\mu} \) and \( t^{\lambda\mu} \) is defined by

\[ t^{\lambda\mu} = k(4\Sigma^{b\lambda\mu}T_{b\mu\nu} - g^{\lambda\mu}\Sigma^{bcd}T_{bcd}) . \]  \hspace{1cm} (9)

In view of the antisymmetry property \( \Sigma^{a\mu\nu} = -\Sigma^{a\nu\mu} \), it follows that

\[ \partial_{\lambda} \left[ e e^{a\mu}(t^{\lambda\mu} + T^{\lambda\mu}) \right] = 0 . \]  \hspace{1cm} (10)

The equation above yields the continuity (or balance) equation,

\[ \frac{d}{dt} \int_V d^3x e e^{a\mu}(t^{0\mu} + T^{0\mu}) = - \int_S dS_j \left[ e e^{a\mu}(t^{j\mu} + T^{j\mu}) \right] . \]  \hspace{1cm} (11)

Therefore we identify \( t^{\lambda\mu} \) as the gravitational energy-momentum tensor [5],

\[ P^a = \int_V d^3x e e^{a\mu}(t^{0\mu} + T^{0\mu}) , \]  \hspace{1cm} (12)

as the total energy-momentum contained in a volume \( V \) of the three-dimensional space,

\[ \Phi^a_g = \oint_S dS_j \left( e e^{a\mu}t^{j\mu} \right) , \]  \hspace{1cm} (13)

as the gravitational energy-momentum flux [6], and

\[ \Phi^a_m = \oint_S dS_j \left( e e^{a\mu}T^{j\mu} \right) , \]  \hspace{1cm} (14)

as the energy-momentum flux of matter. In view of (8) eq. (12) may be written as

\[ P^a = - \int_V d^3x \partial_j \Pi^{aj} = - \oint_S dS_j \Pi^{aj} , \]  \hspace{1cm} (15)
where $\Pi^{aj} = -4ke \Sigma^{a0j}$. The expression above is the definition for the gravitational energy-momentum presented in ref. [7], obtained in the framework of the vacuum field equations in Hamiltonian form. It is invariant under coordinate transformations of the three-dimensional space and under time reparametrizations. We note that (10) is a true energy-momentum conservation equation.

By substituting eq. (12) in the left hand side of eq. (11), and assuming that the energy-momentum tensor for matter fields $T^{\lambda\mu}$ vanishes, which is the case for the Kerr space-time, we find

$$\frac{dP^a}{dt} = - \oint_S dS_j \left[ e \ e_{\mu}^a t^{j\mu} \right].$$

(16)

Considering now eq. (8), we rewrite the right hand side of the equation above in the form

$$\frac{dP^a}{dt} = -4k \oint_S dS_j \partial_{\nu}(e \Sigma^{aj\nu}).$$

(17)

Now we restrict the Lorentz index $a$ to be $a = (i)$, where $i = 1, 2, 3$, and write eq. (17) as

$$\frac{dP^{(i)}}{dt} = - \oint_S dS_j \phi^{(i)j} = - \oint_S dS_j \left[ e \ e_{\mu}^{(i)} t^{j\mu} \right],$$

(18)

where

$$\phi^{(i)j} = 4k \partial_{\nu}(e \Sigma^{(i)j\nu}).$$

(19)

Equation (18) is precisely eq. (39) of Ref. [5].

The left hand side of eq. (18) represents the momentum of the field divided by time, and therefore has the dimension of force. Since on the right hand side $dS_j$ is an element of area, we see that $-\phi^{(i)j}$ represents the pressure along the $(i)$ direction, over and element of area oriented along the $j$ direction. In cartesian coordinates the index $j = 1, 2, 3$ represents the directions $x, y, z$ respectively.

In the next section we will consider the Kerr solution in terms of the Boyer-Lindquist coordinates, which are spherical type coordinates. In this case, we have $j = r, \theta, \phi$. In order to obtain the radial pressure over the event horizon we need to consider only the index $j = 1$, which represents the radial direction. Thus in spherical type coordinates we define $-\phi^{(r)1}$ as
and from the expression above we define the radial pressure $p$ according to

$$p(r) = \int_0^{2\pi} d\phi \int_0^\pi d\theta [-\phi^{(r)1}].$$

(21)

In view of eqs. (19) and (20), we see that $\phi^{(r)1}$ is a density.

In ref. [5] we have written eq. (18) in vectorial form as

$$\frac{dP}{dt} = \int_{\Delta S} d\theta d\phi [-\phi^{(r)1}] \hat{r},$$

where $\Delta S$ is a spherical open surface of constant radius in the Schwarzschild space-time with mass $m = GM/c^2$, according to eq. (44) of ref. [5]. Considering the surface $\Delta S$ to be $\Delta S = r^2 d\Omega$, where $d\Omega$ is a solid angle, we have shown that in the limit $r >> m$ the equation above is simplified as

$$\frac{d}{dt} \left( \frac{P}{M} \right) = -\frac{GM}{r^2} d\Omega \hat{r}.$$

Finally, we note that all definitions presented in this section follow exclusively from the field equations (8).

3 Radial pressure over the external event horizon of the Kerr black hole

In terms of the Boyer-Lindquist coordinates, the Kerr solution is given by

$$ds^2 = -\frac{\psi^2}{\rho^2} dt^2 - \frac{2\chi \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2$$

$$+ \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\phi^2,$$

(22)

with the following definitions:

$$\Delta = r^2 + a^2 - 2mr,$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$
$$\Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta,$$
$$\psi^2 = \Delta - a^2 \sin^2 \theta,$$
$$\chi = 2amr. \tag{23}$$

The tetrad field is obtained out of the metric tensor according to the relation $e^a \mu e^b \nu \eta_{ab} = g_{\mu \nu}$. The two properties that may determine the tetrad field are the following. First, the tetrad field must be adapted to a field of observers whose trajectories and velocities in space-time are given by $x^\mu(s)$ and $u^\mu(s) = dx^\mu/ds$, respectively, where $s$ is the proper time. Then we identify $e_{(0)}^\mu = u^\mu$. This equation fixes three conditions on the tetrad field: $e_{(0)}^i = u^i$. And second, one may choose the unit vectors $e_{(1)}^\mu$, $e_{(2)}^\mu$ and $e_{(3)}^\mu$ to be oriented asymptotically along the unit cartesian vectors $\hat{x}$, $\hat{y}$, $\hat{z}$.

Since the tetrad field is a kind of square root of the metric tensor, it may not be defined in every region of the space-time. For instance, if we choose the observer to be static in space-time, then the tetrad field is defined only in the region $r > r_+^*$, where $r_+^* = m + \sqrt{m^2 - a^2 \cos^2 \theta}$ represents the external boundary of the ergosphere of the Kerr space-time. This result is justified because inside the ergosphere it is not possible to maintain any observer in static regime. Inside the ergosphere, all observers are necessarily dragged in circular motion by the gravitational field. The four-velocity of observers that circulate around the black hole inside the ergosphere, under the action of the gravitational field of the Kerr space-time, is given by

$$u^\mu(t, r, \theta, \phi) = \frac{\chi \Sigma}{(\psi^2 \Sigma^2 + \chi^2 \sin^2 \theta)^{1/2}} (1, 0, 0, \frac{\chi}{\Sigma^2}), \tag{24}$$

where all functions are defined in eq. (23). It is possible to show that if we restrict the radial coordinate to $r = r_+$, the $\mu = 3$ component of eq. (24) becomes

$$\frac{\chi}{\Sigma^2} = \frac{a}{2mr_+} = \frac{a}{a^2 + r_+^2} = \Omega_H,$$

and $\Omega_H$ is the angular velocity of the external event horizon of the Kerr space-time. The quantity

$$\omega(r) = - \frac{g_{03}}{g_{33}} = \frac{\chi}{\Sigma^2}$$

is the dragging velocity of inertial frames.
The tetrad field (i) that is adapted to observers whose four-velocities are given by eq. (24), i.e., for which \( e^{(0)}_{\mu} = u^\mu \), and consequently is defined in the region \( r \geq r_+ \), (ii) whose \( e^{(i)}_{\mu} \) components in cartesian coordinates are oriented along the unit vectors \( \hat{x}, \hat{y}, \hat{z} \), and (iii) that is asymptotically flat, is given by

\[
e_{a\mu} = \begin{pmatrix}
- A & 0 & 0 & 0 \\
B \sin \theta \sin \phi & C \sin \theta \cos \phi & D \cos \theta \cos \phi & -E \sin \theta \sin \phi \\
-B \sin \theta \cos \phi & C \sin \theta \sin \phi & D \cos \theta \sin \phi & E \sin \theta \cos \phi \\
0 & C \cos \theta & -D \sin \theta & 0
\end{pmatrix},
\]

(25)

where

\[
\begin{align*}
A &= \frac{(g_{03}g_{30} - g_{00}g_{33})^{1/2}}{(g_{33})^{1/2}}, \\
B &= -\frac{g_{03}}{(g_{33})^{1/2} \sin \theta}, \\
C &= (g_{11})^{1/2}, \\
D &= (g_{22})^{1/2}, \\
E &= \frac{(g_{33})^{1/2}}{\sin \theta}.
\end{align*}
\]

(26)

We emphasize that a relevant feature of the tetrad field above is that it is defined from the spatial infinity up to the external event horizon of the Kerr black hole. It is the unique configuration that satisfies the conditions stated above, since six conditions are imposed on \( e_{a\mu} \). Therefore we may evaluate the gravitational pressure on the external event horizon.

Now we have all necessary quantities to evaluate \( \phi^{(r)1} \) given by eq. (20). For this purpose we need eqs. (6), (25) and (26). The calculations are rather lengthy, but otherwise straightforward. We find it relevant to show some intermediate steps. First, we present the non-vanishing contributions to \( \phi^{(r)1}(t, r, \theta, \phi) \). They are given by

\[
\begin{align*}
\phi^{(r)1} &= 4k \cos \theta \partial_2 (eC \cos \theta \Sigma^{112} - eD \sin \theta \Sigma^{212}) \\
&\quad + 4k \sin \theta \partial_2 (eC \sin \theta \Sigma^{112} + eD \cos \theta \Sigma^{212}) \\
&\quad + 4k \epsilon \sin^2 \theta (B \Sigma^{013} - E \Sigma^{313}),
\end{align*}
\]

(27)
recalling that $e$ is the determinant of $e^{\alpha}_{\mu}$. In the expression of $\phi^{(r)}_1$ we have neglected all terms that are linear in $\sin \phi$ and $\cos \phi$, since by integration these terms will not contribute to $p$ given by eq. (21). In order to evaluate the expression above we need the following relations:

$$
\Sigma^{112} = \frac{1}{2} g^{11} g^{22} \left( -g^{00} T_{002} - g^{03} T_{302} + g^{03} T_{023} + g^{33} T_{323} \right),
$$

$$
\Sigma^{212} = \frac{1}{2} g^{11} g^{22} \left( g^{00} T_{001} + g^{03} T_{301} - g^{03} T_{013} - g^{33} T_{313} \right),
$$

$$
B \Sigma^{013} - E \Sigma^{313} = \frac{g^{11}}{\sqrt{g^{33}} \sin \theta D} \left[ \frac{1}{4} g^{03} (-T_{301} + T_{013} + T_{103}) 
\right.
\left. + \frac{1}{2} (g_{33} T_{001} - g^{22} D T_{212}) \right].
$$

Finally, the torsion tensor components $T_{\lambda\mu\nu} = e^{\alpha}_{\lambda} T_{a\mu\nu}$ are given by

$$
T_{001} = \frac{1}{2} \partial_1 (A^2 - B^2 \sin^2 \theta),
$$

$$
T_{301} = E \partial_1 B \sin^2 \theta,
$$

$$
T_{002} = \frac{1}{2} \partial_2 (A^2 - B^2 \sin^2 \theta),
$$

$$
T_{302} = E \partial_2 (B \sin^2 \theta),
$$

$$
T_{103} = -BC \sin^2 \theta,
$$

$$
T_{212} = D \partial_1 D \cos^2 \theta - DC \cos^2 \theta - D (\partial_2 C) \sin \theta \cos \theta,
$$

$$
T_{013} = -B (\partial_1 E - C) \sin^2 \theta,
$$

$$
T_{313} = E (\partial_1 E - C) \sin \theta,
$$

$$
T_{023} = -B (\partial_2 E) \sin^2 \theta - B (E - D) \sin \theta \cos \theta,
$$

$$
T_{323} = E (\partial_2 E) \sin^2 \theta + E (E - D) \sin \theta \cos \theta.
$$

After a large number of calculations and simplifications we evaluate expression (20), and arrive at

$$
\phi^{(r)}_1 = 2k (r_+ - m) \frac{\sin \theta}{\rho_+} + 2k \frac{\sin \theta}{\rho_+^2} \left[ ma^2 \sin^2 \theta + 2mr_+ (r_+ - m) \right],
$$

where $\rho_+ = \rho(r_+)$. 

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Let us analyze the case $a = J/m = 0$ and obtain the radial pressure over the event horizon of the Schwarzschild black hole. By making $a = 0$ we obtain $\phi^{(r)1} = 2k \sin \theta$, when $r_+ = 2m$. Applying this expression in eq. (21) yields $p = -c^3/2G$. The latter is the gravitational pressure over the event horizon of the Schwarzschild black hole. The negative sign means that the pressure is exerted towards the centre of the black hole. Recall that $\phi^{(r)1}$ is a density. Therefore it incorporates a quantity of the type $r^2 \sin \theta$.

In the limit $a \to 0$, eq. (27) reduces to

$$
\phi^{(r)1} = 2k \sin \theta \left[ \frac{2m}{r} (1 - e^\lambda) - e^\lambda (1 - e^\lambda)^2 \right],
$$

where $e^\lambda = \sqrt{-g_{00}}$. This expression was obtained in ref. [5], in the context of the Schwarzschild black hole. We note, however, that in eqs. (43) and (44) for $\phi^{(r)1}$ in ref. [5], the quantity $e^\lambda$, that multiplies the term $(1 - e^\lambda)^2$ on the right hand side of the expression above, is missing.

The radial pressure over the external event horizon of the Kerr black hole is obtained by integration of the angular variables in eq. (30), according to eq. (21), and making $r = r_+$. The integration is not difficult, and eventually we obtain

$$
p = \frac{1}{4} \left[ - \frac{4m}{\sqrt{2mr_+}} + \frac{(2m - r_+)}{a} \ln \left( \frac{\sqrt{2mr_+} + a}{\sqrt{2mr_+} - a} \right) \right].
$$

(31)

It is not difficult to see that the value of the pressure is negative. This is the expression that we will use in the next section in the analysis of the thermodynamics of the Kerr black hole.

4 The thermodynamic relation for $TdS$

In this section, we will consider a Kerr black hole of mass $m$ and angular momentum $J$ in stationary state. Then by means of the Penrose process we consider that $m$ and $J$ vary according to $m \to m + dm$ and $J \to J + dJ$. The standard thermodynamic relation for black holes is given by eq. (1). We will obtain here the thermodynamic relation $TdS$ entirely within the framework of the TEGR, with no identification between $TdS$ and the variation $dA$ of the area of the event horizon. Since the first law of thermodynamics is given by $TdS = dE + pdV$, we need to calculate separately the variations $dE$ and $pdV$ in the Penrose process. Let us begin with $dE$. 

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In ref. [7] we have calculated the energy contained within the external event horizon of the Kerr black hole, making use of the tetrad field given by eqs. (25) and (26). The latter is precisely equal to the set of tetrad fields given by expression (4.9) of ref. [7]. The resulting value for the gravitational energy contained within a surface of constant radius \( r = r_+ \) is given by eq. (5.4) of the latter reference. It reads

\[
E = m \left[ \frac{\sqrt{2p}}{4} + \frac{6p - \lambda^2}{4\lambda} \ln \left( \frac{\sqrt{2p} + \lambda}{p} \right) \right],
\]

where

\[
p = 1 + \sqrt{1 - \lambda^2}, \quad a = \lambda m, \quad 0 \leq \lambda \leq 1.
\]

The parameter \( p \) used above (and in ref. [7]) is not to be confused with radial pressure \( p \). The remarkable feature of the expression above is that, for any value of the parameter \( \lambda \), it is strikingly close to \( 2m_{\text{irr}} \). We recall that the energy contained within the event horizon of the Schwarzschild black hole of mass \( m \) is given by \( 2m \). For the Kerr black hole one expects that the energy contained within the external event horizon is given by \( 2m_{\text{irr}} \), which is the amount of gravitational energy that cannot be extracted from the black hole in the Penrose process. An analysis of the various gravitational energy expressions for the Schwarzschild and Kerr black holes has been made in ref. [8]. It was concluded that none of the known expressions yield the value \( 2m_{\text{irr}} \) for the Kerr black hole, but all of them yield \( 2m \) for the Schwarzschild black hole. The almost coincidence between our expression and \( 2m_{\text{irr}} \), for any value of \( \lambda \), is displayed by Fig. 1 of [7]. Rewriting \( E \) given by eq. (32) in terms of \((m, a)\) only we have

\[
E = \frac{m}{4} \left[ \sqrt{\frac{2r_+}{m}} + \frac{(6mr_+ - a^2)}{ma} \ln \left( \frac{\sqrt{2mr_+} + a}{r_+} \right) \right],
\]

We will obtain \( dE \) and write it in terms of the differentials \( dm \) and \( dJ \), which are the differentials that arise in eq. (1). Thus we will calculate

\[
dE = \frac{\partial E}{\partial m} dm + \frac{\partial E}{\partial J} dJ.
\]

Recall that \( r_+ = m + \sqrt{m^2 - J^2/m^2} \). The calculation is rather lengthy. We obtain
\[ dE = \frac{1}{4} \sqrt{\frac{2r_+}{m}} dm + \frac{1}{8} \left( \frac{6r_+}{a} - \frac{a}{m} \right) \ln \left( \frac{\sqrt{2mr_+ + a}}{\sqrt{2mr_+ - a}} \right) dm + \frac{m}{8} \left( \frac{12r_+}{a\sqrt{m^2 - a^2}} + \frac{2a}{m^2} \right) \ln \left( \frac{\sqrt{2mr_+ + a}}{\sqrt{2mr_+ - a}} \right) dm + \frac{a^2 - 3mr_+}{\sqrt{2mr_+(m^2 - a^2)}} dm - \frac{m}{8} \left( \frac{6r_+}{a^2\sqrt{m^2 - a^2}} + \frac{1}{m^2} \right) \ln \left( \frac{\sqrt{2mr_+ + a}}{\sqrt{2mr_+ - a}} \right) dJ + \left( \frac{3mr_+}{2a} - \frac{a}{2} \right) \frac{1}{\sqrt{2mr_+(m^2 - a^2)}} dJ. \quad (34) \]

Next we will calculate \( pdV \). Since \( \phi^{(r)1} \) is a density, the differential \( pdV \) is evaluated as

\[ pdV = \left[ \int_S (-\phi^{(r)1}) \, d\theta d\phi \right] dr_+ = p \, dr_+, \quad (35) \]

where \( S \) is the surface of constant radius \( r = r_+ \), and \( p \) is given by eq. (31). The differential \( dr_+ \) is given by

\[ dr_+ = \frac{(r_+ + a^2/m)}{\sqrt{m^2 - a^2}} dm - \frac{a/m}{\sqrt{m^2 - a^2}} dJ. \quad (36) \]

Since we are assuming that \( dr_+, \, dm \) and \( dJ \) are infinitesimals, the present analysis is not valid when the square root \( \sqrt{m^2 - a^2} \) approaches zero, i.e., when \( a \) is very close to \( m \). After a number of calculations and simplifications we obtain

\[ pdV = - \frac{(mr_+ + a^2)}{\sqrt{2mr_+(m^2 - a^2)}} dm + \frac{1}{4} \left( \frac{2a}{\sqrt{m^2 - a^2}} - \frac{a}{m} \right) \ln \left( \frac{\sqrt{2mr_+ + a}}{\sqrt{2mr_+ - a}} \right) dm + \frac{a}{\sqrt{2mr_+(m^2 - a^2)}} dJ + \frac{1}{4} \left[ \frac{2 - (r_+/m)}{\sqrt{m^2 - a^2}} \right] \ln \left( \frac{\sqrt{2mr_+ + a}}{\sqrt{2mr_+ - a}} \right) dJ. \quad (37) \]
We are now in a position to write the first law of thermodynamics $TdS = dE + pdV$ out of eqs. (34) and (37). We will combine the latter equations and write $TdS$ in the form

$$TdS = f_1(m, a)dm + f_2(m, a)dJ.$$  \hfill (38)

The functions $f_1$ and $f_2$ are obtained by means of straightforward algebra. They read

$$f_1(m, a) = \frac{r_+}{2\sqrt{m^2 - a^2}} \left[ \left( \frac{r_+ - 9m}{\sqrt{2mr_+}} \right) + \left( \frac{r_+^2 + mr_+ + 16m^2}{4ma} \right) \ln \left( \frac{\sqrt{2mr_+} + a}{\sqrt{2mr_+} - a} \right) \right],$$ \hfill (39)

$$f_2(m, a) = \frac{r_+}{2a} \left[ \left( \frac{3}{\sqrt{2mr_+}} \right) + \left( \frac{3r_+^2 - 11mr_+ + 4m^2}{4ma\sqrt{m^2 - a^2}} \right) \ln \left( \frac{\sqrt{2mr_+} + a}{\sqrt{2mr_+} - a} \right) \right].$$ \hfill (40)

Expressions (38), (39) and (40) should be compared with the standard expression for $TdS$ given by eq. (1), which may be written as

$$TdS = f_3(m, a)dm + f_4(m, a)dJ,$$ \hfill (41)

where

$$f_3(m, a) = 1,$$ \hfill (42)

$$f_4(m, a) = -\Omega_H = -\frac{a}{2mr_+}.$$ \hfill (43)

In order to compare eqs. (38-40) and (41-43), we will plot in the same graph the functions $f_1$ and $f_3$, and $f_2$ and $f_4$. For this purpose, we choose three arbitrary values of the parameter $\lambda$, where $a = \lambda m$. We have chosen $\lambda = 0.2$, $\lambda = 0.5$ and $\lambda = 0.9$. As we explained earlier, the values of $\lambda$ close to 1 are not allowed in the analysis, since in this case the differential $dr_+$ is no longer an infinitesimal. All curves in all graphs are functions of the mass parameter $m$.

In all figures, the dotted lines represent $f_3$ and $f_4$, namely, they represent the standard thermodynamic relation given by eq. (1). The regular (continuous) lines represent our result given by eqs. (38-40). The main results
are that (i) the qualitative behaviour of $f_1$ and $f_2$ is the same in all figures, and (ii) $f_1$ and $f_2$ are qualitatively similar to $f_3$ and $f_4$, respectively, in all figures. At first sight it is surprising that $f_1$ yields a horizontal line, and that $f_2$ looks like $f_4$. We will explain this qualitative behaviour in the next section. Our first conclusion is that the thermodynamic relation given by eq. (38), obtained entirely out of the definitions in the TEGR, is realistic and comparable to eq. (1).

Figure 1: $\lambda = 0.2$

In the standard treatment of the thermodynamics of black holes one, does not make use of the concept of pressure. We find it appropriate to analyse the thermodynamic relation for $TdS$, when the latter is due the variation of the energy only. Thus let us analyse

$$TdS = dE \equiv f_{0m}(m,a)dm + f_{0J}(m,a)dJ.$$  \hfill (44)

In view of eq. (34) we find

$$f_{0m}(m,a) = -\frac{r_+}{2\sqrt{m^2-a^2}}\left[\frac{r_+ + 3m}{\sqrt{2mr_+}}\right] + \frac{\left(\frac{r_+^2 - 9mr_+ - 4m^2}{4ma}\right)}{\ln\left(\frac{\sqrt{2mr_+} + a}{\sqrt{2mr_+} - a}\right)}.$$  \hfill (45)
and

\[ f_{0J}(m, a) = \frac{r_+(r_+ + m)}{2a\sqrt{m^2 - a^2}} \left[ \left( \frac{1}{\sqrt{2mr_+}} \right) + \left( \frac{r_+ - 4m}{4ma} \right) \ln \left( \frac{\sqrt{2mr_+} + a}{\sqrt{2mr_+} - a} \right) \right]. \] (46)

We need to compare \( f_{0m} \) with \( f_3 \), and \( f_{0J} \) with \( f_4 \). Again we chose the values 0.2, 0.5, and 0.9 for \( \lambda = a/m \), and therefore all curves will be functions of the parameter \( m \). Once again we observe that \( f_{0m} \) is qualitatively similar to \( f_3 \), and that \( f_{0J} \) is qualitatively similar to \( f_4 \).

An immediate conclusion about these figures is the following. By comparing the horizontal lines in Fig. 3 (with pressure, \( \lambda = 0.9 \)) and Fig. 6 (with no pressure, \( \lambda = 0.9 \)) we observe that the numerical, constant value of the straight line is lower in Fig. 3 compared to Fig. 6. Admitting that the term \( f_1 dm \) overweights the effect of the term \( f_2 dJ \) in the Penrose process (since the particle may have vanishing angular momentum), then we see that the effect of the presence of the gravitational pressure is to increase the efficiency of the Penrose process. The reason is that the smaller the value of \( dE \), more energy can be extracted from the black hole. However, by inspecting figures 1, 2 and 3 we see that the efficiency of the Penrose process is lower in the thermodynamic process dictated by relation (38), compared with the standard relation (1) or (41), since after the variations \( m \to m + dm \) and
Figure 3: $\lambda = 0.9$

$J \rightarrow J + dJ$, the energy variation $dE$ is higher in the context of the present analysis, namely, in the context of eq. (38). The higher is $dE$, less efficient is the Penrose process.

5 The dependence of the figures on the parameter $m$

In this section, we will explain the reason for the similarity between $f_1$, $f_3$ and $f_{0m}$, and between $f_2$, $f_4$ and $f_{0J}$. The idea is very simple. We just substitute $\lambda m$ for $a$ in all expressions, and factorize the mass parameter $m$. After this procedure we see that $f_1$ and $f_{0m}$ depend only on $\lambda$, i.e., they are independent of $m$, and that $f_2$, $f_{0J}$ and $f_4$ all depend on $1/m$ times a function that depends only on $\lambda$. We find

$$f_1(m, \lambda) = \left(\frac{1 + \sqrt{1 - \lambda^2}}{2\sqrt{1 - \lambda^2}}\right) \left\{ \frac{\sqrt{1 - \lambda^2} - 8}{\sqrt{2 + 2\sqrt{1 - \lambda^2}}} \right\}$$

$$+ \left(\frac{19 + 3\sqrt{1 - \lambda^2} - \lambda^2}{4\lambda}\right) \ln \left[ \frac{\sqrt{2 + 2\sqrt{1 - \lambda^2} + \lambda}}{\sqrt{2 + 2\sqrt{1 - \lambda^2} - \lambda}} \right],$$

(47)
\( f_2(m, \lambda) = \frac{1}{m} \left( \frac{1 + \sqrt{1 - \lambda^2}}{2\lambda} \right) \left\{ \frac{3}{\sqrt{2 + 2\sqrt{1 - \lambda^2}}} - \left( \frac{1 + 5\sqrt{1 - \lambda^2} + 3\lambda^2}{4\lambda\sqrt{1 - \lambda^2}} \right) \ln \left[ \frac{\sqrt{2 + 2\sqrt{1 - \lambda^2} + \lambda}}{\sqrt{2 + 2\sqrt{1 - \lambda^2} - \lambda}} \right] \right\}, \) \hspace{1cm} (48)

\( f_{0m}(m, \lambda) = - \left( \frac{1 + \sqrt{1 - \lambda^2}}{2\lambda\sqrt{1 - \lambda^2}} \right) \left\{ \frac{\sqrt{1 - \lambda^2} + 4}{\sqrt{2 + 2\sqrt{1 - \lambda^2}}} - \left( \frac{11 + 7\sqrt{1 - \lambda^2} + 3\lambda^2}{4\lambda} \right) \ln \left[ \frac{\sqrt{2 + 2\sqrt{1 - \lambda^2} + \lambda}}{\sqrt{2 + 2\sqrt{1 - \lambda^2} - \lambda}} \right] \right\}, \) \hspace{1cm} (49)

\( f_{0J}(m, \lambda) = \frac{1}{m} \left( \frac{(1 + \sqrt{1 - \lambda^2})(2 + \sqrt{1 - \lambda^2})}{2\lambda\sqrt{1 - \lambda^2}} \right) \left\{ \frac{1}{\sqrt{2 + 2\sqrt{1 - \lambda^2}}} + \left( \frac{1}{4\lambda} - \frac{3}{4\lambda} \right) \ln \left[ \frac{\sqrt{2 + 2\sqrt{1 - \lambda^2} + \lambda}}{\sqrt{2 + 2\sqrt{1 - \lambda^2} - \lambda}} \right] \right\}, \) \hspace{1cm} (50)

\( f_4(m, \lambda) = -\frac{\lambda}{2m(1 + \sqrt{1 - \lambda^2})}. \) \hspace{1cm} (51)
Thus we see that \( f_1 \) and \( f_{0m} \) are independent of \( m \), and \( f_2 \), \( f_{0J} \) and \( f_4 \) depend on \( 1/m \). These features explain the shape of the curves in the graphs.

6 Conclusions and summary of the results

In this article, we have investigated the thermodynamics of the Kerr black hole in the context of the teleparallel equivalent of general relativity. We have considered: (i) a stationary black hole with mass and angular momentum per unit mass \((m, a = J/m)\); (ii) the Penrose process by means of which \( m \) and \( J \) vary according to \( m \to m + dm, \ J \to J + dJ \); (iii) the expressions for gravitational energy and pressure, \( E \) and \( p \) respectively, that arise in teleparallel equivalent of general relativity. In particular, we have considered the expression for \( E \) given by eq. (33). The numerical values of the latter, for specific values of \( m \) and \( a \), are strikingly close to \( 2m_{\text{err}} \), as shown in ref. [7]. With all these quantities we evaluated \( TdS = dE + pdV \).

There are two main results in the article. First, we have evaluated the gravitational pressure over the external event horizon of the black hole. The radial pressure, with no dependence on the angular variables, is given by eqs. (21) and (31). The second result is the thermodynamic relation (38), with the definitions (39) and (40). This relation is obtained without any consideration of the area of the external event horizon of the black hole. It follows entirely from the definitions that arise in the TEGR. Although relations (38-40) look
like quite different from the standard thermodynamic relation (41-43), we see from figures 1-3 that both thermodynamic relations are qualitatively similar. Note, however, that in the present analysis the variations of the gravitational energy and pressure that occur in view of the Penrose process are necessarily subject to the conservation equation (11). Admitting that the higher is $TdS$, the lower is the efficiency of the Penrose process, then it seems that in the context of the TEGR the Penrose process is less efficient than in the standard process. Finally, by analysing Figs. 3-6 we see that the presence of the gravitational pressure in the thermodynamic relation for $TdS$ does not imply a significant change. We believe that both the concept of gravitational pressure and our thermodynamic relation $TdS = dE + pdV$ given by eq. (38) are physically consistent and realistic results.

References

[1] S. W. Hawking, Phys. Rev. Lett. 26, 1344 (1971).
[2] R. Penrose, Riv. Nuovo Cimento 1, 252 (1969).
[3] D. Christodoulou and R. Ruffini, Phys. Rev. D 4, 2552 (1971).
[4] B. P. Dolan, Classical Quantum Gravity 28, 235017 (2011); Phys. Rev. D 84, 127503 (2011).
[5] J. W. Maluf, Ann. Phys. (Berlin) 14 (2005) 723 [gr-qc/0504077].

[6] J. W. Maluf, F. F. Faria and K. H. Castello-Branco, Class. Quantum Grav. 20 (2003) 4683.

[7] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toríbio and K. H. Castello-Branco, Phys. Rev. D 65 (2002) 124001.

[8] G. Bergqvist, Classical Quantum Grav. 9, 1753 (1992).