Differential algebras with Banach-algebra coefficients II: 
The operator cross-ratio tau-function
and the Schwarzian derivative

Maurice J. Dupré  
Department of Mathematics, Tulane University  
New Orleans, LA 70118 USA  
mdupre@tulane.edu

James F. Glazebrook  
(Primary Inst.)  
Department of Mathematics and Computer Science  
Eastern Illinois University  
600 Lincoln Ave., Charleston, IL 61920–3099 USA  
jfglazebrook@eiu.edu  
(Adjunct Faculty)

Department of Mathematics  
University of Illinois at Urbana–Champaign  
Urbana, IL 61801, USA

Emma Previato*  
Department of Mathematics and Statistics, Boston University  
Boston, MA 02215–2411, USA  
ep@math.bu.edu

Abstract

Several features of an analytic (infinite-dimensional) Grassmannian of (commensurable) subspaces of a Hilbert space were developed in the context of integrable PDEs (KP hierarchy). We extended some of those features when polarized separable Hilbert spaces are generalized to a class of polarized Hilbert modules, in particular the Baker and τ-functions, which become operator-valued. Following from Part I we produce a pre-determinant structure for a class of τ-functions defined in the setting of the similarity class of projections of a certain Banach *-algebra. This structure is explicitly derived from the transition map of a corresponding principal bundle. The determinant of this map gives a generalized, operator-valued τ-function that takes values in a commutative C*-algebra. We extend to this setting the operator cross-ratio which had been used to produce the scalar-valued τ-function, as well as the associated notion of a Schwarzian derivative along curves inside the space of similarity classes. We link directly this cross-ratio with Fay’s trisecant identity for the τ-function (equivalent to the KP hierarchy). By

*Partial research support under grant NSF-DMS-0808708 is very gratefully acknowledged.
restriction to the image of the Krichever map, we use the Schwarzian to introduce the notion of operator-valued projective structure on a compact Riemann surface: this allows a deformation inside the Grassmannian (as it varies its complex structure). Lastly, we use our identification of the Jacobian of the Riemann surface in terms of extensions of the Burchnall-Chaundy C*-algebra (Part I) to describe the KP hierarchy.

Mathematics Subject Classification (2010): 46L08, 53B10, 53C30, 14H70

Keywords: Hilbert module, polarization, tau-function, projective structure, cross-ratio, Schwarzian derivative, KP hierarchy, Fay trisecant identity.

1 Introduction

This paper continues from Part I [9] of this work where we focused on extensions of the algebra of continuous functions $C(X)$ by compact operators, classified via the Calkin algebra using Brown-Douglas-Fillmore extension theory and KK-theory to obtain results on the K-homology of $X$ and its Jacobian. In this Part II, we turn our attention to the connection on the determinant bundle over the Grassmannian, whose curvature is related to Sato’s $\tau$-function in the scalar case, as well as in our operator-coefficient case [10], and tie these in with several concepts developed in Part I in relationship to the KP equation. Accordingly, this Part II, as for Part I, continues an operator-theoretic approach to the Sato-Segal-Wilson theory [24, 26].

We recall in §3 that the classical $\tau$-function can be realized as a specific cocycle of the determinant line bundle of the universal bundle pulled back over the space of restricted polarizations from the restricted Grassmannian and its dual [24, 26]. The specific cocycle is determined by two non-vanishing sections. The underlying cocycle for the universal bundle is used to arrive at the pre-determinant structure of the $\tau$-function which we refer to as the $\Sigma$-function; accordingly, its determinant $\text{Det}(\Sigma)$ is the $\tau$-function (see [10, 18, 28]). The important structure lies within the geometry of this determinant bundle, its connection and its curvature. But the $\Sigma$-function is the cocycle for the universal bundle over a space of restricted polarizations $\mathfrak{P}$, relating essentially the same two underlying sections, in this case, of the associated principal bundle. Hence the interest is in the calculation of the geometry, connection, and curvature of the universal bundle using the two sections which are each covariantly constant over two complementary subbundles of the tangent bundle of the space of restricted polarizations $\mathfrak{P}$.

In [8, 9] we considered a certain (complex) Banach *-algebra $A$ modeled on the linear operators of a Hilbert module denoted $H_A$, where $A$ is a commutative separable C*-algebra. Letting $P(A)$ denote the idempotents in $A$, in [10] we considered the geometry of the space $\Lambda = \text{Sim}(p, A)$, the similarity class of $p \in P(A)$, which is closely related to the Grassmanian $\text{Gr}(p, A)$ of Part I [9] (see also [8, 10]). From the transition map of a principal bundle $V_\Lambda \rightarrow \Lambda$, we deduced a corresponding pre-determinant denoted $T$. We identify in §3 a relationship between $T$ and $\Sigma$ obtained via a diffeomorphism between $\Lambda$ and $\mathfrak{P}$. In fact, as shown in [10], certain calculations involving the connection and curvature are more straightforward when performed on $\Lambda$, and that is why we choose to work mainly with predeterminants $T_\lambda$ and the ensuing determinants $\tau_\lambda$.

We revisit the construction of the $\tau$-function in terms of a cross-ratio (§2.3), and of the Schwarzian derivative that preserves it (§4.1). In doing so, we need to extend to the C*-algebra case an (abelian) group action on the Grassmannian (§3.1), study the pull-back of the universal bundle through it, compare this to the Poincaré bundle (§3.2), and relate the operator cross-ratio to
the projective structure on the Riemann surface associated to the Schwarzian derivative ([12]). We then switch ([4,3]) to Raina’s interpretation of the KP hierarchy (Fay’s trisecant identity) in model quantum-field theory [23]: he reduced the identity to Wick’s theorem, by writing the 4-point function in terms of theta functions and prime forms, essentially a (generalized) cross-ratio on a Riemann surface. As a result, we are able to give in §4.4 a KP hierarchy satisfied on the space of extensions of the Burchnall-Chaundy C* algebra.

We retain the notation of Part I and refer there to the relevant concepts and results obtained. The Appendix to Part I recalls much of the background material which is used here. The new results obtained in this Part II are Lemma 3.1, Propositions 3.1, 3.2, 4.1, 4.2, 4.3, Theorems 4.1, 4.2.

2 Universal bundles with connection and the \( \mathcal{T} \)-function

2.1 The space of polarizations

Let \( A \) be a unital complex Banach(able) algebra with group of units \( G(A) \) and space of idempotents \( P(A) \). Let \( H \) be a separable (infinite dimensional) Hilbert space (here as in Part I we take \( H = L^2(S^1, \mathbb{C}) \)). Given a unital separable C*-algebra \( \mathcal{A} \), we take the standard (free countable dimensional) Hilbert module \( H_\mathcal{A} \) over \( \mathcal{A} \) and consider a polarization of \( H_\mathcal{A} \) given by a pair of submodules \( (H_+, H_-) \), such that

\[
\begin{align*}
H_\mathcal{A} &= H_+ \oplus H_- , \quad \text{and} \quad H_+ \cap H_- = \{0\}. 
\end{align*}
\]

(2.1)

Recall also from [8, 9] the (restricted) Banach *-algebra \( A = \mathcal{L}_J(H_\mathcal{A}) \) (\( J \) being a unitary \( \mathcal{A} \)-module map satisfying \( J^2 = 1 \)) which henceforth we use. As in [9], we assume \( \mathcal{A} \) to be commutative (and separable). The Gelfand transform implies there exists a compact metric space \( Y = \text{Spec}(\mathcal{A}) \) such that

\[ Y = \text{Spec}(\mathcal{A}) \Rightarrow \mathcal{A} \cong \mathcal{C}(Y), \]

(2.2)

where \( \mathcal{C}(Y) \) is the space of continuous functions \( Y \rightarrow B \).

We recall the Grassmannian \( \text{Gr}(p, A) \) and refer to [9, A.2] for the necessary background. For a given \( p \in P(A) \), there is associated to \( \text{Gr}(p, A) \) its dual Grassmannian \( \text{Gr}^*(p, A) \) [10, §3]. Let \( \mathfrak{P} \) denote the space of polarizations \( (H_+, H_-) \) on \( H_\mathcal{A} \). Then as shown in [10, §3], the space \( \mathfrak{P} \) can be regarded as a subspace

\[ \mathfrak{P} \subset \text{Gr}(p, A) \times \text{Gr}^*(p, A). \]

(2.3)

A significant observation is that \( \mathfrak{P} \) can be closely related to the similarity class \( \Lambda = \text{Sim}(p, A) \) of \( A \) where \( \Lambda \) consists of the elements of \( \mathfrak{P} \) expressed in terms of projections. In fact, \( \Lambda \) admits a natural complex analytic structure induced from that of \( A \) via \( P(A) \) (see [7, 10] and Proposition 4.2 below). Further, from [10, Theorem 4.1(3)] there exists an analytic diffeomorphism

\[ \phi : \mathfrak{P} \rightarrow \Lambda \subset P(A). \]

(2.4)

Remark 2.1. Given that \( \Lambda \), as a set of idempotents in the Banach *-algebra \( A \), can admit non self-adjoint elements (see [9, Remark 2.1]), we note that it will not be diffeomorphic to \( \text{Gr}(p, A) \) under the natural quotient map of equivalence relations \( \Pi : P(A) \rightarrow \text{Gr}(A) = P(A)/\sim \). However, restriction of \( \Pi \) to the self-adjoint elements \( \Lambda^{sa} \) of \( \Lambda \) can be shown to establish a bijective
diffeomorphism between $\Lambda^{sa}$ and $\text{Gr}(p, A)$, and moreover, there exists a smooth retraction of $P(A)$ onto $P^{sa}(A)$ (as can be shown using the technique of [6, Proof of Proposition 4.6.2]).

2.2 The universal bundle with connection over $\Lambda$

Here we briefly describe part of the basic geometry of [10]. Firstly, let $\pi_\Lambda = \Pi|\Lambda$ and $\pi_V = \Pi|V(p, A)$. Let $V_\Lambda = \pi_\Lambda^*(V(p, A))$; specifically,

\[ V_\Lambda = \{(r, u) \in \Lambda \times V(p, A) : \pi_\Lambda(r) = \pi_V(u)\}. \]  

In [10] we constructed an analytic principal right $G(pAp)$-bundle with connection

\[ (V_\Lambda, \omega_\Lambda) \rightarrow \Lambda, \]  

whose (analytic) $G(pAp)$-valued transition map $t_\Lambda$ is given by the formula

\[ t_\Lambda((r, u), (r, v)) = t_V(u, v), \]  

where $t_V$ is the transition map for the $G(pAp)$-bundle $V(p, A) \rightarrow \text{Gr}(p, A)$ (see [9, A.2]).

By the standard means we have the associated vector bundle with (Koszul) connection, namely the universal bundle with connection over $\Lambda$

\[ (\gamma_\Lambda, \nabla_\Lambda) \rightarrow \Lambda, \]  

for which the curvature operator $R_\nabla$ was computed explicitly in [10, §8].

2.3 The $T$-function

For given parallel (covariantly constant) sections $\alpha_p, \beta_p$ of (2.6), we defined the $T$-function in terms of the transition map $t_\Lambda$ of (2.7) by

\[ T(r) = t_\Lambda(\alpha_p(r), \beta_p(r)), \]  

which can be expressed more conveniently as $T = t_\Lambda(\alpha_p, \beta_p)$ (see [10, (8.1)]).

Returning to the space of polarizations $\mathcal{P}$, there are several closely associated objects as described in [10, 28]. Firstly, there is the principal bundle with connection

\[ (V_\mathcal{P}, \omega_\mathcal{P}) \rightarrow \mathcal{P}, \]  

and associated universal (vector) bundle with connection

\[ (\gamma_\mathcal{P}, \nabla_\mathcal{P}) \rightarrow \mathcal{P}. \]  

Observe that there is a splitting

\[ d = \partial_+ + \partial_-, \]  

of the exterior derivative of $\omega_\Lambda$ induced in the following way. For a given polarization $\mathcal{P} = (H_+, H_-) \in \mathcal{P}$, the exterior derivative of $\omega_\mathcal{P}$ splits as $d = \partial_+ + \partial_-$ where $\partial_+$ (respectively, $\partial_-$) denotes the covariant derivative in the directions from $H_+$ (respectively, $H_-$). The induced splitting (2.12) thus follows from the analytic diffeomorphism in (2.4) for which $\omega_\Lambda = (\phi^{-1})^*\omega_\mathcal{P}$. 

4
In [28], the function $\Sigma$ constructed via operator cross-ratio, is likewise shown to be derived from the transition map $\gamma$ for $V_\Psi \to \Psi$. This is closely related to our $\mathcal{T}$-function, and in [10], we showed that the geometry of $(V_\Lambda, \omega_\Lambda) \to \Lambda$ is also closely related to that of $(V_\Psi, \omega_\Psi) \to \Psi$, where local coordinates for $\Lambda$ and $\Psi$ can be expressed in terms of an operator cross-ratio (see [11] below). In particular, with reference to [28] p. 47 and recalling the analytic diffeomorphism $\phi : \Psi \to \Lambda$ in (2.4), we have in terms of the parallel sections $\alpha_p$ and $\beta_p$, the relationship

$$\phi^*\mathcal{T} = \phi^*(t_\Lambda(\alpha_p, \beta_p)) = \beta_p^{-1}\alpha_p = \Sigma.$$  

(2.13)

Consider a pair of polarizations $(H_+, H_-), (K_+, K_-) \in \Psi$. Let $H_\pm$ and $K_\pm$ be ‘coordinatized’ via maps $P_\pm : H_\pm \to H_\mp$, and $Q_\mp : K_\pm \to K_\mp$. The composite map

$$H_+ \xrightarrow{K_+} K_+ \xrightarrow{H_+} H_+,$$

(2.14)

allows us to take the operator cross-ratio [28] (cf. [10]):

$$\Sigma(H_+, H_-; K_+, K_-) = (P_-P_+ - 1)^{-1}(P_-Q_+ - 1)(Q_-Q_+ - 1)^{-1}(Q_-P_+ - 1).$$  

(2.15)

For this construction there is no essential algebraic change in generalizing from polarized Hilbert spaces to polarized Hilbert modules. The principle here is that the transitions between charts define endomorphisms of $W \in \text{Gr}(p, A)$ that will become the transition functions of the universal bundle $\gamma_\Psi \to \Psi$. The main properties of $\mathcal{T}$ and $\Sigma$ are that they are pre-determinants for various classes of $\tau$-functions, as we shall see in [33].

### 2.4 Trace-class operators and the determinant

An equivalent, operator, description leading to the functions $\mathcal{T}$ and $\Sigma$ above, can be obtained along the lines of e.g. [18, 26, 28]. Here, as in [9], we make use of the nested sequence of Schatten ideals definable in $\mathcal{L}(H_A)$ (see e.g. [27]). Suppose $(H_+, H_-), (K_+, K_-) \in \Psi$ are such that $H_+$ is the graph of a linear map $S : K_+ \to K_-$ and $H_-$ is the graph of a linear map $T : K_- \to K_+$. Then on $H_A$ we consider the identity map $H_+ \oplus H_- \to K_+ \oplus K_-$, as represented in the block form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$  

(2.16)

where $a : H_+ \to K_+$, $d : H_+ \to K_-$ are zero–index Fredholm operators, and $b : H_+ \to K_+$, $c : H_+ \to K_-$ are in $K(H_A)$ (the compact operators), such that $S = ca^{-1}$ and $T = bd^{-1}$. Initially, one considers the operator $1 - ST = 1 - ca^{-1}bd^{-1}$. In particular, with a view to defining a generalized determinant leading to an operator–valued $\tau$-function, we consider cases where ST is of trace class. If we take $b, c$ to be Hilbert–Schmidt operators (as for the case $A = \mathbb{C}$ as in [18, 26, 28]), then $ST$ is of trace class, the operator $(1 - ST)$ is $\Sigma(H_+, H_-; K_+, K_-)$ above, and the $\tau$-function is defined as

$$\det \Sigma(H_+, H_-; K_+, K_-) \otimes 1_A = \det(1 - ca^{-1}bd^{-1}) \otimes 1_A = \det(\alpha_p\beta_p^{-1}).$$  

(2.17)

Starting from the universal bundle $\gamma_{\text{Gr}} \to \text{Gr}(p, A)$, then with respect to an ‘admissible basis’ (for defining determinants) in the Stiefel bundle $V(p, A) \to \text{Gr}(p, A)$ (see [7] and [9, A.2]), the $\tau$-function in (2.17) is equivalently derived from the canonical section of the determinant line bundle $\det(\gamma_{\text{Gr}}) \to \text{Gr}(p, A)$ (cf. [18, 26, 28]).
3 Relationship between the predeterminants and $\tau$-functions

3.1 Relationship between the predeterminants

We start with the following lemma:

**Lemma 3.1.** Let $\Gamma \subset G(pAp)$ be a group acting on the subspace $H_+$. Then there exists a natural action of $\Gamma$ on the spaces $\text{Gr}(pA)$, $\Lambda = \text{Sim}(pA)$ and $\mathfrak{P}$.

**Proof.** Firstly, $H_+$ is a splitting subspace for $H_A$ and thus determines the polarization $H_A = H_+ \oplus H_-$, and likewise for any other polarizing pair $(K_+, K_-)$.

We recall from [7, §4] that Gr is a functor, and if $g$ is a linear automorphism of $H_\pm$, it thus defines an element of $G(pAp)$. For $p \in P(A)$, let $\hat{p} = 1 - p$. Then $g + \hat{p} \in G(A)$, and therefore this term defines an inner automorphism of $A$ taking $\Lambda$ to itself, fixing $p$ and inducing an analytic diffeomorphism of $\text{Gr}(pA)$ with itself. Thus by functorial properties of inner automorphisms and the functorial properties of $\text{Gr}$, the group $\Gamma$ acting on $H_+$, viewed as a subgroup of $G(pAp)$, induces actions on $\Lambda$ and $\text{Gr}(pA)$.

Further, given an action of $\Gamma$ on $\Lambda$, there is an induced action on the space of polarizations $\mathfrak{P} \subset \text{Gr}(pA) \times \text{Gr}'(pA)$, since $\Lambda$ simply consists of the elements of $\mathfrak{P}$ viewed as projections.

The induced action of $\Gamma$ gives rise to following commutative diagram

$$
\begin{array}{ccc}
\Gamma \times \mathfrak{P} & \xrightarrow{\mu_{\mathfrak{P}}} & \mathfrak{P} \\
\tilde{\phi} \downarrow & & \downarrow \phi \\
\Gamma \times \Lambda & \xrightarrow{\mu_{\Lambda}} & \Lambda \\
\tilde{\pi}_\Lambda \downarrow & & \downarrow \pi_{\Lambda} \\
\Gamma \times \text{Gr}(pA) & \xrightarrow{\mu_{\text{Gr}}} & \text{Gr}(pA)
\end{array}
$$

(3.1)

where we note that the action of $\Gamma$ as a subgroup of $G(pAp)$ on $\text{Gr}(pA)$ is essentially trivial since $G(pAp)$ is the structure group of the principal bundle $V(pA) \rightarrow \text{Gr}(pA)$. An example of such a group $\Gamma$ is given by the group of multiplication operators $\Gamma_+(\mathcal{A})$ in [9] (in particular $\Gamma_+$ in [26] for $\mathcal{A} = \mathbb{C}$). In the following we will make use of the various maps appearing in (3.1) besides recalling the role of canonical sections for the universal bundles $\gamma_{\text{Gr}} \rightarrow \text{Gr}(pA)$ (the associated vector bundle to $V(pA) \rightarrow \text{Gr}(pA)$, see [9, A.2]) and $\gamma_\Lambda \rightarrow \Lambda$, introduced in [10, §7]. Let $S_p$ be the canonical section of $\gamma_{\text{Gr}}$ and set $\tilde{S}_p = \mu_{\text{Gr}}(S_p)$. Likewise, let $S'_\Lambda$ be the canonical section of $\gamma_\Lambda$ and set $\tilde{S}'_{\Lambda} = \mu_{\Lambda}(S'_\Lambda)$. The following implements the sections functor $H^0$.

**Proposition 3.1.** Let $W \in \text{Gr}(pA)$ and $\lambda \in \Lambda$. In terms of the maps in (3.1) the following diagram is commutative where the horizontal maps $\rho_1, \rho_2$ are homomorphisms and the vertical maps $\tilde{\pi}_\Lambda, (\tilde{\pi}_\Lambda)_{\text{res}}$ are isomorphisms:

$$
\begin{array}{ccc}
H^0(\Gamma \times \text{Gr}(pA), \mu_{\text{Gr}}^* \gamma_{\text{Gr}}) & \xrightarrow{\rho_1} & H^0(\Gamma \times \{W\}, \mu_{\text{Gr}}^* \gamma_{\text{Gr}}|\Gamma \times \{W\}) \\
\tilde{\pi}_\Lambda \circ & & \downarrow (\tilde{\pi}_\Lambda)_{\text{res}} \circ \\
H^0(\Gamma \times \Lambda, \mu_{\Lambda}^* \gamma_\Lambda) & \xrightarrow{\rho_2} & H^0(\Gamma \times \{\lambda\}, \mu_{\Lambda}^* \gamma_\Lambda|\Gamma \times \{\lambda\})
\end{array}
$$

(3.2)

In particular, we have $\tilde{\pi}_\Lambda(\tilde{S}_\Lambda) = \tilde{S}'_p$. 

6
Proof. The commutativity follows by applying the sections functor $H^0$ to the lower square in (3.1). Since we have $\gamma_\Lambda \cong \pi^*_{\Lambda,\Gr}$, it follows that

\[
(m_{\Gr} \circ \pi_{\Lambda})^* \gamma_{\Gr} \cong (\pi_{\Lambda} \circ \mu)^* \gamma_{\Gr} \\
\cong \mu_{\Lambda}^*(\pi_{\Lambda,\Gr}) \cong \mu_{\Lambda}^* \gamma_{\Lambda}.
\] (3.3)

Thus $\bar{\pi}_{\Lambda}^*(\mu_{\Gr}^* \gamma_{\Gr}) \cong \mu_{\Lambda}^* \gamma_{\Lambda}$, and likewise for the restriction to $\Gamma \times \{W\}$.

With regards to the maps in Proposition 3.1, we next introduce the following:

**Definition 3.1.** For $W \in \Gr(p, A)$, we denote by $T^W$ the $T$-function of the point $W$ over $\Gamma$, as defined to be the image of the section $\tilde{S}_{\gamma}^W = \mu_{\Gr}^*(\gamma_{\Gr})$ under $\rho_1$.

**Definition 3.2.** For $\lambda \in \Lambda$, we denote by $T^\lambda$ the $T$-function of the point $\lambda$ over $\Lambda$, as defined to be the image of the section $\tilde{S}_{\gamma}^\lambda = \mu_{\Lambda}^*(\gamma_{\Lambda})$ under $\rho_2$.

Then it is easily seen from Proposition 3.1 that we have

\[
(\bar{\pi}_{\Lambda})_{\text{res}}(T^W) = T^\lambda.
\] (3.4)

Recalling the analytic diffeomorphism $\phi: \mathcal{T} \rightarrow \Lambda$ in (2.4), the relationship (3.4) is another way of interpreting our observation that $\phi^* T = \mathcal{T}$.

In terms of the group multiplication $m: \Gamma \times \Gamma \rightarrow \Gamma$ in $\Gamma$, we have the following commutative diagram

\[
\begin{array}{ccc}
\Gamma \times \Gamma \times \Lambda & \xrightarrow{m \times \text{Id}} & \Gamma \times \Lambda \\
\downarrow & & \downarrow \\
\Gamma \times \Gamma \times \Gr(p, A) & \xrightarrow{m \times \text{Id}} & \Gamma \times \Gr(p, A)
\end{array}
\] (3.5)

(1) For $W \in \Gr(p, A)$, let $\mu_W: \Gamma \times \{W\} \rightarrow \Gr(p, A)$ denote the map induced by $\mu_{\Gr}$ in (3.1).

(2) For $\lambda \in \Lambda$, let $\mu_{\lambda}: \Gamma \times \{\lambda\} \rightarrow \Lambda$ denote the map induced by $\mu_{\Lambda}$ in (3.1).

Thus for each point $W \in \Gr(p, A)$ (respectively, $\lambda \in \Lambda$) we obtain vector bundles over $\Gamma \times \Gamma$ associated with $W$ (respectively, $\lambda$), as given by

\[
E^W = (1 \times \mu_W)^* (\mu_{\Gr}^* \gamma_{\Gr}) = m^*(\mu_{\Gr}^* \gamma_{\Gr}),
\]

\[
E^\lambda = (1 \times \mu_{\lambda})^* (\mu_{\Lambda}^* \gamma_{\Lambda}) = m^*(\mu_{\Lambda}^* \gamma_{\Lambda}).
\] (3.6)

### 3.2 The Poincaré bundles and the $\tau$-function

Returning to the $\tau$-functions considered in Part I [9], we next make some corresponding observations for determinants using admissible bases in $V(p, A)$ (cf. [2]). Here, for ease of notation, we set $\Gamma_1 = \Gamma_+(\mathcal{A})$, the action we are concerned with (cf Lemma 3.1). In this case it will be convenient to use the following notation:

\[
\Det_{\Gr} = \Det(\gamma_{\Gr}^*) \quad \text{and} \quad \Det_{\Lambda} = \Det(\gamma_{\Lambda}^*),
\] (3.7)

\[
\Det_{\Gr} = \Det(\gamma_{\Gr}^*) \quad \text{and} \quad \Det_{\Lambda} = \Det(\gamma_{\Lambda}^*).\]
noting that $\text{Det}_\Lambda = \pi^*_\Lambda \text{Det}_{Gr}$. These we will pull-back by maps

$$
\begin{align*}
\hat\mu_{Gr} : J_A(X) \times \text{Gr}(p, A) &\longrightarrow \text{Gr}(p, A), \\
\hat\mu_\Lambda : J_A(X) \times \Lambda &\longrightarrow \Lambda,
\end{align*}
$$

where we recall the space $J_A(X)$ of monomorphisms $\Lambda \otimes A \longrightarrow B_W$ with respect to the spectral curve $X$ (see [9 Appendix A.4]). In keeping with some standard algebraic-geometric terminology (cf. [2]), the line bundles

$$
\begin{align*}
\hat\mu^*_\Lambda \text{Det}_{Gr} \longrightarrow J_A(X) \times \text{Gr}(p, A), \\
\hat\mu^*_\Lambda \text{Det}_\Lambda \longrightarrow J_A(X) \times \Lambda,
\end{align*}
$$

are referred to as Poincaré bundles. Next we pull-back the maps in (3.9) along the map $\Gamma_1 \longrightarrow J_A(X)$ in [9 A.19] to obtain

$$
\begin{align*}
\mu^*_\Lambda \text{Det}_{Gr} \longrightarrow \Gamma_1 \times \text{Gr}(p, A), \\
\mu^*_\Lambda \text{Det}_\Lambda \longrightarrow \Gamma_1 \times \Lambda.
\end{align*}
$$

Remark 3.1. By incorporating the group multiplication, we have in a similar way to (3.6) the Poincaré bundles over $\Gamma_1 \times \Gamma_1$ associated with $W \in \text{Gr}(p, A)$ (respectively, $\lambda \in \Lambda$)

$$
\begin{align*}
\mathcal{B}_W = (1 \times \mu_W)^*(\mu^*_\Lambda \text{Det}_{Gr}) &= m^*(\mu^*_\Lambda \text{Det}_{Gr}), \\
\mathcal{B}_\lambda = (1 \times \mu_\Lambda)^*(\mu^*_\Lambda \text{Det}_\Lambda) &= m^*(\mu^*_\Lambda \text{Det}_\Lambda).
\end{align*}
$$

The next step is to give the analogous statement to Proposition 3.1 for determinants and thus relate the $\tau$-functions corresponding to respective points of $\text{Gr}(p, A)$ and $\Lambda$. Specifically, for fixed $W \in \text{Gr}(p, A)$, let us set $L_r(W) = \mu^*_\Lambda \text{Det}_{Gr}|_{\Gamma_1 \times \{W\}}$, and likewise, for fixed $\lambda \in \Lambda$, let us set $\tilde{L}_r(\lambda) = \mu^*_\Lambda \text{Det}_\Lambda|_{\Gamma_1 \times \{\lambda\}}$.

Let $Q_p$ be the canonical section of $\text{Det}_{Gr}$ and set $\tilde{Q}_p = \mu^*_\Lambda (Q_p)$. Likewise, we take $Q'_\Lambda$ to be the canonical section of $\text{Det}_\Lambda$ and set $\tilde{Q}'_\Lambda = \mu^*_\Lambda (Q'_\Lambda)$. Motivated by [2 §5] and [26], we introduce a $\tau$-function $\tilde{\tau}_W$ associated to $W \in \text{Gr}(p, A)$ defined by $\tilde{\tau}_W = \rho_1(\tilde{Q}_p)$, and associated to $\lambda \in \Lambda$, we likewise define $\tilde{\tau}_\lambda = \rho_2(\tilde{Q}'_\Lambda)$.

Proposition 3.2. Let $W \in \text{Gr}(p, A)$ and $\lambda \in \Lambda$. With regards to the maps in (3.11) we have the following commutative diagram in which the horizontal maps $\rho_1, \rho_2$ are homomorphisms and the vertical maps $\tilde{\pi}_\Lambda, (\tilde{\pi}_\Lambda)_{\text{res}}$ are isomorphisms:

$$
\begin{align*}
&\xymatrix{ H^0(\Gamma_1 \times \text{Gr}(p, A), \mu^*_\Lambda \text{Det}_{Gr}) \ar[r]^-{\rho_1} & H^0(\Gamma_1 \times \{W\}, \tilde{L}_r(W)) \\
\tilde{\pi}_\Lambda & (\tilde{\pi}_\Lambda)_{\text{res}} \ar[l]}
\end{align*}
$$

$$
\begin{align*}
&\xymatrix{ H^0(\Gamma_1 \times \Lambda, \mu^*_\Lambda \text{Det}_\Lambda) \ar[r]^-{\rho_2} & H^0(\Gamma_1 \times \{\lambda\}, \tilde{L}_r(\lambda)) \\
\tilde{\pi}_\Lambda & (\tilde{\pi}_\Lambda)_{\text{res}} \ar[l]}
\end{align*}
$$

In particular, we have $\tilde{\tau}_W = \text{Det} \tilde{\pi}_W = \rho_1(\tilde{Q}_p)$ and $\tilde{\tau}_\lambda = \text{Det} \tilde{\tau}_\lambda = \rho_2(\tilde{Q}'_\Lambda)$.

Proof. This follows immediately from Proposition 3.1 when the coefficients are in the respective determinant bundle, and from the definitions of the $\tilde{\tau}_W$ and $\tilde{\tau}_\lambda$-functions once their respective determinants are taken. \qed
We also recall the notion of ‘transverse subspace’ from [9] [26]. We note that one way of characterizing W’s transverse to $H_-$ is that the orthogonal projection $W \to H_+$ should be an isomorphism; this property is not preserved in general under multiplication by an element of the group $\Gamma_1 = \Gamma_+ (\mathcal{A})$, but it is preserved over a dense subset which we denote by $\Gamma^W_1 = \Gamma^W (\mathcal{A})$ (see [9 Appendix A.5]). Likewise, we set $\Gamma_1^\lambda = \Gamma_1^\lambda (\mathcal{A})$.

In a similar way to [26] (cf. [2]) we define operator-valued $\tau$-functions relative to such subspaces $W \in \text{Gr}(p, \mathcal{A})$ and $\lambda \in \Lambda$ as follows:

1. Fix a transverse subspace $W \in \text{Gr}(p, \mathcal{A})$ and define
   \[ \mathcal{L}_\tau(W) = \tilde{\mathcal{L}}_\tau(W) | \Gamma^W_1 \times \{W\}. \] (3.13)

   Let $\sigma_W$ be a constant section trivializing $\mathcal{L}_\tau(W)$ over $\Gamma^W_1$ (it is to ensure the existence of this section that we take $W$ transverse [26 Prop. 3.3]). Recalling from Proposition 3.2 that we have $\tilde{\tau}_W = \text{Det} \Sigma_W = \rho_1(\bar{Q}_p)$, and for $g \in \Gamma^W_1$, we define $\tau_W : \Gamma^W_1 \to \mathbb{C} \otimes 1_{\mathcal{A}}$ by
   \[ \tau_W(g) = \tilde{\tau}_W(g)(\sigma_W(g))^{-1}, \] (3.14)

   which simply recovers the tau-function $\tau_W$ of [9] (cf. [26]) with $\mathcal{A}$-valued coefficients.

2. Fix a transverse subspace $\lambda \in \Lambda$ and define
   \[ \mathcal{L}_\tau(\lambda) = \tilde{\mathcal{L}}_\tau(\lambda) | \Gamma^\lambda_1 \times \{\lambda\}. \] (3.15)

   Let $\sigma_\lambda$ be a constant section trivializing $\mathcal{L}_\tau(\lambda)$ over $\Gamma^\lambda_1$. Recalling from Proposition 3.2 that we have $\tilde{\tau}_\lambda = \text{Det} \Sigma_\lambda = \rho_2(\bar{Q}_\lambda)$, and for $g \in \Gamma^\lambda_1$, we define $\tau_\lambda : \Gamma^\lambda_1 \to \mathbb{C} \otimes 1_{\mathcal{A}}$ by
   \[ \tau_\lambda(g) = \tilde{\tau}_\lambda(g)(\sigma_\lambda(g))^{-1}. \] (3.16)

   This leads to the straightforward relationship
   \[ (\tilde{\pi}_\lambda^\lambda)_{\text{res}}(\tau_W(g)) = \tau_\lambda(g). \] (3.17)

   At this stage it should be clear from the above results that the geometry of $\Lambda$ and $\text{Gr}(p, \mathcal{A})$, as well as the functions $\tau_\lambda, \tau_W$, are closely related via $\pi_\lambda$ and (2.4). In particular, by following standard procedures, $(\gamma_\Lambda, \nabla_\Lambda) \to \Lambda$ induces the determinant line bundle with its connection $(\text{Det}_\Lambda, \nabla(\text{Det}_\Lambda)) \to \Lambda$, and likewise for $(\text{Det}_{\text{Gr}}, \nabla(\text{Det}_{\text{Gr}})) \to \text{Gr}(p, \mathcal{A})$. Here the $\tau$-function serves as a ‘logarithmic potential’ for the curvature of the connection, and from [13] [28] (cf. [10]), we have recalling (2.12) for the curvature 2-forms:
   \[ \Omega(\text{Det}_\Lambda) = \frac{1}{2\pi i} \partial_u \partial_w \log |\tau_\lambda|, \quad \Omega(\text{Det}_{\text{Gr}}) = \frac{1}{2\pi i} \partial_u \partial_w \log |\tau_W|. \] (3.18)

   Since the corresponding calculations of the connection and curvature of $(\gamma_{\text{Gr}}, \nabla_{\text{Gr}}) \to \text{Gr}(p, \mathcal{A})$ are more straightforward on passing to $(\gamma_{\Lambda}, \nabla_{\Lambda}) \to \Lambda$ (see [10] for details), we choose to emphasize objects relative to $\Lambda$, such as $\tau_\lambda, \tau_W$, etc. in the following.

   Specifically, we can use the canonical section $S^\lambda_{\text{Gr}}$ to lift the action of $\Gamma_1$ to the universal bundle $\gamma_\Lambda$. Moreover, in order to give a more explicit expression for $\tau_\lambda$, we use parallel sections $\alpha_\lambda, \beta_\lambda$ as in (2.9), and then $\tau_\lambda : \Gamma_1 \to G(\Lambda A \Lambda)$ is equivalently defined by
   \[ \tau_\lambda(g) = t_\lambda(g^{-1} \alpha_\lambda(r), \beta_\lambda(rg)), \] (3.19)
for $g \in \Gamma$. In the following we shall simply drop the argument in $r$ since this is understood.

On recalling the element $q_\zeta \in \Gamma_1$ as given by a map $q_\zeta(z) = (1 - z\zeta^{-1})$ in [9 §4.4], we proceed, in view of the ‘transversality’ to define for $g \in \Gamma_1$,

$$\Psi_\lambda(g, \zeta) = T_\lambda(g \cdot q_\zeta)(T_\lambda(g))^{-1}.$$  \hspace{1cm} (3.20)

Observe that the function $\Psi_\lambda$ is a ‘predeterminant’ for a Baker function $\psi_\lambda$ in the following sense. On taking determinants of (3.20), we obtain

$$\psi_\lambda(g, \zeta) := \text{Det}(\Psi_\lambda(g, \zeta)) = \tau_\lambda(g \cdot q_\zeta)(\tau_\lambda(g))^{-1},$$  \hspace{1cm} (3.21)

which is simply a ‘lifted-to-$\Lambda$’ version of the relationship between the Baker $\psi_W$ and $\tau_W$ functions under pull-back by $\pi_\lambda$ (see [9 §4.4] and [26]):

$$\psi_W(g, \zeta) = \tau_W(g \cdot q_\zeta)(\tau_W(g))^{-1}.$$  \hspace{1cm} (3.22)

4 Applications

4.1 Operator cross-ratio and the Schwarzian derivative

Smooth and analytic parametrizations of subspaces of a Banach space were studied in [7, 11] (cf [13]). Using the techniques in question we can regard spaces such as $\Lambda$ (and likewise, $\text{Gr}(p, A)$, $\mathcal{P}$, etc.) as analytically parametrized in terms of analytic maps $D_0 \to \Lambda$ to the underlying Banach space (of $\Lambda$), where $D_0 \subset \mathbb{C}$ denotes the open unit disk. Thus taking $w \in D_0$ as a local parameter, one can assign (operator) $\Lambda$-valued functions $\zeta_\lambda(w)$ parametrizing $\lambda \in \Lambda$ (and likewise for, e.g., $W \in \text{Gr}(p, A)$). With this understood, we shall simply write, as a convention, $\zeta$ for $\zeta(w)$ and $z$ for $z = f(w)$, etc., in the following. We also take Hol($D_0, \Lambda$) to denote the space of holomorphic (analytic) $\Lambda$-valued functions on $D_0$.

Following [28], we assume that (commensurable) subspaces in $\text{Gr}(p, A)$ are isomorphic to those of $\text{Gr}^*(p, A)$ and consider a smooth family of subspaces $\mathcal{H}(s) \in \text{Gr}(p, A)$ parametrized by one real parameter $s$, where $\mathcal{H}(u) \cong \mathcal{H}_+$ for $u > 0$, and $\mathcal{H}(v) \cong \mathcal{H}_-$ for $v \leq 0$, so that the pair $(\mathcal{H}(u), \mathcal{H}(v))$ defines a polarization of $\mathcal{H}_A$. More specifically, consider an ordering $s_2 < 0 < s_1 < s_3$, and a pair of polarizations identified with points $(\mathcal{H}_+, \mathcal{H}_-), (K_+, K_-) \in \mathcal{P}$:

$$(\mathcal{H}_2, \mathcal{H}_1) := (\mathcal{H}(s_2), \mathcal{H}(s_1)) \cong (\mathcal{H}_+, \mathcal{H}_-),$$

$$(\mathcal{H}, \mathcal{H}_3) := (\mathcal{H}(0), \mathcal{H}(s_3)) \cong (K_+, K_-).$$  \hspace{1cm} (4.1)

For spaces such as $\text{Gr}(p, A), \Lambda$ and $\mathcal{P}$ (we recall that $\Lambda$ and $\mathcal{P}$ are analytically diffeomorphic), we define the **operator cross-ratio** (‘$\text{cr}$’) in terms of projection ($\Lambda$-valued) affine coordinates as

$$\text{cr}(a, b; c, d) = (a - c)(a - b)^{-1}(b - d)(c - d)^{-1}. \hspace{1cm} (4.2)$$

Let $z$ be such a $\Lambda$-valued variable in $\mathcal{H}(s)$, and letting $z_i$ denote variables with respect to $\mathcal{H}_i$ ($z$ corresponds to $\mathcal{H}(0)$), we apply (4.2) to the polarizing pair in (4.1) to obtain

$$\text{cr}(\mathcal{H}_2, \mathcal{H}_1; \mathcal{H}, \mathcal{H}_3) = \text{cr}(z_1, z_2; z, z_3) = (z_1 - z)(z_1 - z_3)^{-1}(z_2 - z)(z_2 - z_3)^{-1}.$$  \hspace{1cm} (4.3)
which is defined on projection $(A$-valued) analytic coordinates of $\mathcal{Q}$. As shown in [28], this yields a $\text{End}(\gamma_{Gr})$-valued 1-cocycle $\{\mathcal{c}r\} \in H^1(\text{Gr}(p, A), \text{End}(\gamma_{Gr}))$, and hence under the pullback

\[ \pi^*_A : H^1(\text{Gr}(p, A), \text{End}(\gamma_{Gr})) \rightarrow H^1(A, \text{End}(\gamma_A)), \]  

we regard $\{\mathcal{c}r\}$ as also an $\text{End}(\gamma_A)$-valued 1-cocycle on $A$. Likewise the $T$-function can be viewed as a $G(pAp)$-valued 1-cocycle in the transition function $t_A$ as given by (2.9) (see [10] §8 for details). The operator cross-ratio is used by Zelikin [28], [29] to introduce an operator analogous to the Schwarzian derivative [28], §4. The key idea is that, though operators do not commute, one can take limits within the cross-ratio along the real parameter $s$, one at a time for $s_2, s_3$, and $s_1$ as in (4.1), checking at every step an asymptotic polarization consistency in the process. We can apply Zelikin’s result verbatim for $z$-curves ($z \in \mathbb{C}$) in $A$, and write (with respect to the 1-dimensional parameter $s$):

\[ \mathcal{S}_A(z) = (z')^{-1}z'' - \frac{3}{2}((z')^{-1}z'')^2. \]  

(4.5)

**Remark 4.1.** The Schwarzian derivative, in the classical case of scalar-valued functions, arises naturally (cf. [29]), where it is derived from the Wronskian for a basis of solutions of a third-order differential equation obtained by writing the invariance of the cross-ratio under linear-fractional transformation, differentiating both sides, and eliminating the three parameters of $\text{PSL}(2)$ and in fact the only projectively invariant 1-cocycle on $\text{Diff}(\mathbb{R}P^1)$ [21], [10]. The significance of Zelikin’s definition [28] rests partly on the fact that, for matrix-valued deformations $z(s)$, he was able to show that the Schwarzian operator preserves the operator cross-ratio (for extensions of linear fractional transformations/cross-ratio to the operator-valued setting, along with applications, see e.g. [10] Ch. 3).

We now apply properties of this Schwarzian to our setting and proceed to define:

\[ \text{Hol}^{(3)}(D_0, A) = \{f \in \text{Hol}(D_0, A) : \text{values of the derivatives } f^{(n)} \text{ commute for } n \leq 3, f'(z) \neq 0\}. \]  

(4.6)

Taking $f \in \text{Hol}^{(3)}(D_0, A)$ with $z = f(w)$ as before, we next consider

\[ (\mathcal{S}_A f)(w, t) = \mathcal{c}r(f(w + ta), f(w + tb); f(w + tc), f(w + td)), \]  

(4.7)

Following [1] (cf. [21]), we deduce formally from the vector-valued case that, to second order,

\[ (\mathcal{S}_A f)(w, t) = \mathcal{c}r(a, b, c, d)(1 + \frac{1}{6}(a - b)(c - d)(\mathcal{S}_A f)(w)t^2 + o(t^2)), \]  

(4.8)

confirming the Schwarzian $\mathcal{S}_A f$ to be the infinitesimal version of the cross-ratio that does not change in first order (cf. [28]).

**Remark 4.2.** In direct analogy with the classical case [13] (see also [12], we may regard those $f \in \text{Hol}^{(3)}(D_0, A)$ for which $\mathcal{S}_A f = 0$ as *projective transformations* and in particular, a coordinate system for which $\mathcal{S}_A(z) = 0$ could be taken to define a *projective structure on $A$*.

Let us now recall the function $\mathcal{T}_A$ from Definition 3.2. In view of (2.15), and on setting $\Delta \mathcal{T}_A(t) = (1 + \frac{1}{6}(a - b)(c - d)(\mathcal{S}_A f)(w)t^2 + o(t^2))$, we directly deduce from (1.8) on applying the $\mathcal{T}_A$-function to (4.3), the relationship

\[ \mathcal{T}_A(\mathcal{H}_2, \mathcal{H}_1; \mathcal{H}, \mathcal{H}_3) \Delta \mathcal{T}_A(t) = \mathcal{S}_A(\mathcal{H}_2, \mathcal{H}_1; \mathcal{H}, \mathcal{H}_3)(t), \]  

(4.9)
where we have implicitly used the pull-back of the cross-ratio under \( \pi \Lambda \) in (4.4) above, along with (2.13). In view of the formulas in (3.18), we next consider the infinitesimal deformation of the curvature 2-form of \( \nabla (\text{Det} \Lambda) \) under an element of \( \text{Hol}^{(3)}(D_0, \Lambda) \):

**Proposition 4.1.** Let \( f \in \text{Hol}^{(3)}(D_0, \Lambda) \). Then we have the following ‘asymptotic’ relationship under \( f \) between 2-forms in (3.18):

\[
\Omega(\text{Det} \Lambda) - \Omega(\text{Det} f(\Lambda)) = \frac{1}{2\pi i} \partial_+ \partial_- \log |\tau_\lambda| - \frac{1}{2\pi i} \partial_+ \partial_- \left[ \log |\Delta T_\lambda(t)^{-1}| + \log |\det((S_\lambda) f(w,t))| \right].
\]

**Proof.** From (3.18), we have on applying ‘det’ to (4.9) the following:

\[
\Omega(\text{Det} \Lambda) = \frac{1}{2\pi i} \partial_+ \partial_- \log |\tau_\lambda|
= \frac{1}{2\pi i} \partial_+ \partial_- \left[ \log |\Delta T_\lambda(t)^{-1}| + \log |\det((S_\lambda) f(w,t))| \right].
\]

But by definition, \( (S_\lambda) f(w,t) = f \circ T_\lambda \), and so we have

\[
\det((S_\lambda) f(w,t)) = \det(f \circ T_\lambda) = \tau_{f(\Lambda)}.
\]

Thus the right-hand side of (4.11) becomes

\[
\frac{1}{2\pi i} \partial_+ \partial_- \left[ \log |\Delta T_\lambda(t)^{-1}| + \log |\tau_{f(\Lambda)}| \right],
\]

from which the result follows in view of (3.18) (observe that essentially the same applies to \( \Omega(\text{Det}_{Gr}) \) in (3.18)).

**Remark 4.3.** In view of his definition of the operator Schwarzian derivative on one-dimensional submanifolds of the Grassmannian, Zelikin posits, in a speculative manner, that the operator-valued KP deformations (whose Baker functions satisfy Riccati’s equation, cf. also [25]) might be studied using the Schwarzian (of which he proved that a quotient of solutions, which defines a projective structure, also satisfies a Riccati equation); but, quote [28, p. 51]: “Unfortunately, the trajectories of [the Riccati] fields do not lie in [the restricted Grassmannian], not even on arbitrarily small time intervals”. Here we take a different route. Raina [23], motivated by conformal-field theory, re-wrote Fay’s trisecant identity (the Riemann-surface version of the KP equation) as a generalized cross-ratio; below (4.2) we use that strategy, together with the operator cross-ratio for the Riemann surface, to arrive at the operator KP equations.

### 4.2 Relationship with projective structures on \( X \)

We turn to the work in [5, 23]; we apply Zelikin’s Schwarzian operator (4.1), extended to our setting, to the description of the projective structure on a compact Riemann surface \( X \) given in [5]. We are ultimately able (4.3) to re-derive the KP hierarchy with operator coefficients that was the motivation behind [9].

Let \( K_X \) denote the canonical line bundle of \( X \). The space of all projective structures on \( X \) is an affine space for the complex vector space \( H^0(X, K_X^2) \) of global holomorphic quadratic differentials, in which the classical Schwarzian \( S_X \) determines a cocycle (see [15] pp. 170–172)).
More explicitly, take a covering \( \{ U_i \}_{i \in \mathbb{Z}} \) for \( X \) and let \( \phi_i \) be a holomorphic function on \( \zeta_i \in U_i \), so that the transition function \( h_i \) for \( H^0(X, K_X^2) \), satisfies (see e.g. [5, 15])

\[
h_i = \mathcal{S}_X(\phi_i)\zeta_i.
\] (4.14)

Observe that if \( \phi'_i \) is another function satisfying (4.14), then \( \phi'_i(\zeta_i) = \Xi \circ \phi_i(\zeta_i) \) where \( \Xi \) denotes a Möbius transformation.

Referring back to [4.1] as noted in [28] (in the matrix-valued case), the definition of cross-ratio and Schwarzian can be extended to complex deformations in one parameter \( s \). In such a situation we denote the Schwarz operator by \( \mathcal{S}_{\Lambda}(s) \). We focus on the deformations taking place along the curve (the compact Riemann surface) \( X \) embedded in the \( \text{Gr}(p, A) \) by the Krichever map: recall [9] Appendix §A.4 that to define the Krichever map we need to fix a local parameter at a point \( \infty \), a (generic) line bundle and a local trivialization; with these data, we produce a point \( W \in \text{Gr}(p, A) \); the action of \( \Gamma_1 = \Gamma_+(\mathcal{A}) \) on \( W \) sweeps out \( J_{\mathcal{A}}(X) \), so by the Abel map we have (up to several choices) a holomorphic embedding \( X \rightarrow \Lambda \) (see Proposition 4.3 below). In analogy with the classical case [15] (see also Remark 4.2) we regard those \( z \) for which \( \mathcal{S}_{\Lambda}(z) = 0 \) as projective coordinates, thereby defining a projective structure on the embedded copy of \( X \).

The following proposition summarizes certain analytic properties of \( \Lambda \) (which like \( \text{Gr}(p, A) \) is modeled on a complex Banach space):

**Proposition 4.2.** The space \( \Lambda \) is an open and closed holomorphic (Banach) submanifold of \( P(A) \) which is a holomorphic (Banach) submanifold of \( A \).

**Proof.** That \( \Lambda \) is an open and closed holomorphic submanifold of \( P(A) \) has been shown to be the case in [7, 10](cf. [22]). In fact, \( \Lambda \) is locally a holomorphic retract of \( A \), as seen as follows. For \( x, y \in A \), we define \( g(x, y) = xy + (1 - x)(1 - y) \), noting that \( g(p, p) = 1 \) and therefore invertible for all \( x, y \) in some open subset \( U \) of \( A \) containing \( p \). We note then for \( q \in \Lambda \), that \( pg(p, q) = g(p, q)q \), so for \( q \in \Lambda \cap U \), we have \( g(p, q)^{-1}pg(p, q) = q \). Thus \( r(x) = g(p, x)^{-1}pg(p, x) \) is a holomorphic retraction of \( U \) onto its overlap with \( \Lambda \), on shrinking \( U \) further if necessary. This also shows that \( \Lambda \) is a holomorphic submanifold of \( A \).

We summarize the above facts relating to holomorphic/projective structures in the following proposition which compares, via \( \mathcal{S}_{\Lambda} \), the complex structure induced from \( \Lambda \) with the one intrinsic to \( X \).

**Proposition 4.3.** Let \( \eta : X \rightarrow \Lambda \) be a holomorphic embedding with respect to the natural (complex) analytic structure of \( \Lambda \). Then the Schwarzian operator intrinsic to \( X \) and the one induced from \( \Lambda \), with respect to the holomorphic deformation along the embedded curve, correspond.

Returning to the cross-ratio class \( \{ cr \} \in H^1(\Lambda, \text{End}(\gamma_{\Lambda})) \) in [4.1] leading to the Schwarzian \( \mathcal{S}_{\Lambda} \), we see that for functions \( f, h \in \text{Hol}^{(3)}(D_0, \Lambda) \), it is formally deduced from the classical case that

\[
\mathcal{S}_{\Lambda}(h \circ f) = (\mathcal{S}_{\Lambda}(h) \circ f)(f')^2 + \mathcal{S}_{\Lambda}(f),
\] (4.15)

(see e.g. [1, 20, 21]). Here the first right-hand summand is the action of \( f \) on a quadratic differential

\[
(u \circ f)(z) = u(f(z) \cdot f'(z))^2,
\] (4.16)
in terms of the \( z \)-coordinate above, in turn leading to a transition function for the vector bundle \( K_\Lambda^2 \) in an analogous way to the classical situation. We thus have arrived at the novel concept of an \emph{operator-valued projective structure} induced on a one-dimensional complex submanifold from a space such as \( \Lambda \).

**Remark 4.4.** It would be interesting to apply the operator Schwarzian derivative to compare the deformations of \( X \) inside \( \Lambda \) with the deformations of \( X \) parametrized by \( H^0(X, K_X^2) \), especially since the former should be unobstructed (Krichever’s map can be applied to any Riemann surface, and locally it should be possible to make consistent choices of a defining quintuple, cf. [26]). However, the tangent bundle to \( \Lambda \), whose first cohomology gives the deformations, is of infinite rank, and in order to define a canonical line bundle over \( \Lambda \) requires different techniques (cf. [4]), as \( \Lambda \) is an infinite-dimensional Banach manifold which is a holomorphic submanifold of \( \Lambda \), thus in general, since we are in infinite dimensions, the top exterior power is not formed in the usual way. However, in our specific situation, where \( A \) is the restricted algebra, we have a group transforming the restricted frames in \( V(p, A) \) ("admissible bases" in [26] §3) on which the determinant is defined, and a central extension of it (\( \mathcal{E} \) in [26] §3) where the function \( g \) giving the retraction in Proposition 4.2 takes values, so that coordinate transformations in effect have derivatives which have values in that central extension (\( \mathcal{E} \)) as well. With this in hand, we would need to restrict the line bundle thus obtained to \( X \) and see if each projective structure of \( X \) can be extended away from it, not only along the one-dimensional deformations controlled by the Schwarzian derivative.

### 4.3 The Riemann theta function and Wick’s theorem

Our next main observation concerns (generalized) projective structures on \( X \) using a ‘correlation function’ approach motivated by Wick’s theorem and a (generalized) cross-ratio (see [5, 23]). The point of this subsection is to implement Raina’s rendition of Fay’s trisecant identity in terms of the cross-ratio we developed ([41, 42]), and obtain the KP hierarchy in §4 below, in our extension-group model. Here we utilize the Burchnall-Chaundy C*-algebra \( \mathfrak{A} \) and the extension group \( \text{Ext}(\mathfrak{A}) \) from [9, §4].

**Theorem 4.1.** For genus \( g_X \geq 2 \), the action of the group \( \Gamma_1 = \Gamma_+(\mathfrak{A}) \) on \( \text{Ext}(\mathfrak{A}) \) corresponds to translating the theta function of \( X \) on the Jacobian.

**Proof.** Let \( \mathfrak{L}_\theta \rightarrow J\mathfrak{A}(X) \) be the holomorphic line bundle whose sections are theta functions \( \theta[\xi](z) \) of characteristic \( \xi \) (see e.g. [12, 14]) taken as \( \mathbb{C} \otimes 1\mathfrak{A} \)-valued. Similar to before, consider a \( \Lambda \)-parametrization of the surjective homomorphism of [9, A(19)], giving a commutative diagram

\[
\begin{array}{ccc}
\Gamma_1 \times \Lambda & \overset{\tau_\Lambda}{\longrightarrow} & \mathbb{C} \otimes 1\mathfrak{A} \\
\downarrow & & \downarrow \\
J\mathfrak{A}(X) \times \Lambda & \overset{\theta_\Lambda[\xi]}{\longrightarrow} & \mathbb{C} \otimes 1\mathfrak{A}
\end{array}
\]  

(4.17)

The commutativity of this diagram reveals that \( \tau_\Lambda \) is ‘proportional’ to \( \theta_\Lambda[\xi] \) following the classical case (cf. [23, 20]), and hence from [9, Theorem 4.5], \( \text{Ext}(\mathfrak{A}) \) parametrizes a family of (translations of the) theta function(s) via extensions by the compact operators.

Next, consider an (operator)-valued spinor field \( \psi \) on \( X \), and points \( a, b \in X \) suitably chosen to lie in the same coordinate patch. In [23] it is shown that the Fay trisecant identity (see ([1, 23])) is
equivalent to Wick’s theorem, which in terms of a left-hand-side ‘correlation function’ below (see \[5\] [23]), can be expressed in the form
\[
\langle \bar{\psi}^\star(b_1)\bar{\psi}(a_1)\bar{\psi}^\star(b_2)\bar{\psi}(a_2) \rangle = \det \begin{bmatrix} S_\alpha(b_1, a_1) & S_\alpha(b_1, a_2) \\ S_\alpha(b_2, a_1) & S_\alpha(b_2, a_2) \end{bmatrix}
\]
where $S_\alpha$ is the Szeg\"{o} kernel of a theta prime-form [12]. Moreover, in view of \[5\] §5 the ‘correlation function’ of \(4.18\) defines a projective connection and hence a projective structure [15].

**Remark 4.5.** In terms of the Cauchy kernels of flat vector bundles of arbitrary rank on $X$ (for $g_X \geq 1$), explicit formulas in \[3\] interpolate homomorphisms of these bundles leading to generalizations of the Fay trisecant identity. We note that flat rank-2 bundles over $X$ (for $g_X \geq 2$) have been related \[5\] to quadratic differentials (in both cases the moduli spaces have the same dimension, $3g - 3$). It is possible this may lead to further applications of operator projective structures on $X$ as well.

### 4.4 The KP tau-function and trisecant identity

We briefly recall the concept of the KP-hierarchy starting from \[9\] Appendix §A4. Consider a formal pseudodifferential operator of the form
\[
L = \partial + a_0 \partial^{-1} + a_1 \partial^{-2} + \cdots,
\]
where as in \[9\] §A4 we take the coefficients $a_i = a_i(t_1, t_2, \ldots)$ to be $\mathcal{A}$-valued functions. Next, let us set $P^{(k)} = (L^k)_+$. Then for each $k \in \mathbb{N}$, the **KP hierarchy** (see e.g. \[17\] 24 \[25\] 26) consists of partial differential equations of the type
\[
\frac{\partial L}{\partial t_k} = [P^{(k)}, L],
\]
according to which
\[
\frac{\partial}{\partial t_k} P^{(\ell)} - \frac{\partial}{\partial t_\ell} P^{(k)} + [P^{(\ell)}, P^{(k)}] = 0.
\]

Since we are using $\mathcal{A}$-valued coefficients, we choose to denote this hierarchy by KP($\mathcal{A}$). Once more we apply the extension group Ext($\mathcal{A}$) and establish the following for the KP($\mathcal{A}$) flows with respect to the $\Gamma_1$-action:

**Theorem 4.2.** The KP($\mathcal{A}$) flows evolve on the group Ext($\mathcal{A}$) via extensions by compact operators.

**Proof.** Recalling the $z$-coordinate of \[4.1\] we set $[z] = (z, z^2/2, z^3/3, \ldots)$. Then from the operator-valued $\tau$-function $\tau_\lambda : \Gamma_1 \rightarrow \mathbb{C} \otimes 1_\mathcal{A}$ in \[3.10\] (cf. \[9\]), it follows that (cf. \[26\] 24)
\[
\tau_\lambda(t + [z]) = \tau_\lambda(t_1 + z, t_2 + z^2/2, t_3 + z^3/3, \ldots),
\]
that satisfies $\tau_\lambda(t + [-z]) = \tau_\lambda(t - [z])$. Further, on setting $C(z_i, z_j, z_k, z_\ell) = (z_i - z_j)(z_k - z_\ell)$ in $\mathcal{A}$-valued variables, for $0 \leq i, j, k, \ell \leq 3$, we formally deduce from references [12] 19 the Fay trisecant identity:
\[
\begin{align*}
C(z_0, z_1, z_2, z_3)\tau_\lambda(t + [z_0] + [z_1])\tau_\lambda(t + [z_2] + [z_3]) &+ C(z_0, z_2, z_3, z_1)\tau_\lambda(t + [z_0] + [z_2])\tau_\lambda(t + [z_3] + [z_1]) \\
C(z_0, z_3, z_1, z_2)\tau_\lambda(t + [z_0] + [z_3])\tau_\lambda(t + [z_1] + [z_2]) &+ C(z_0, z_1, z_2, z_3)\tau_\lambda(t + [z_0] + [z_1] + [z_2]) = 0.
\end{align*}
\]
from the analogous identity for $\tau_W$ and then using (3.17).

From [9, Theorem 4.5], elements of $\text{Ext}(\mathcal{A})$, extensions by the Burchnell-Chaundy C*-algebra $\mathcal{A}$ of the compact operators, lead to the map

$$\Upsilon^{-1}_A : \text{Ext}(\mathcal{A}) \to \Gamma_1$$

(4.24)

(the inverse of the map $\Upsilon_A$ as in the proof of [9, Theorem 4.5, (4.20)]), which yields a family of $\tau$-functions, one for each such extension. Hence, in each case a corresponding trisecant identity as in (4.23) follows. But (4.23) has been shown to be equivalent to the KP-hierarchy (see e.g. [19]). Hence we conclude that the KP-hierarchy, implemented by the $\Gamma_1$-action, flows on these extensions by compact operators as derived from $\text{Ext}(\mathcal{A})$. \hfill $\square$

References

[1] L. V. Ahlfors: Cross-ratios and Schwarzian derivatives in $\mathbb{R}^n$. *Complex Analysis*, 1–15, Birkhäuser, Basel, 1988.

[2] A. Álvarez Vásquez, J. M. Muñoz Porras and F. J. Plaza Martín: The algebraic formalism of soliton equations over arbitrary base fields. In ‘Workshop on Abelian varieties and Theta Functions’ (Morelia MX, 1996), 3–40, *Aportaciones Mat. Investig., 13, Soc. Mat. Mexicana*, Mexico 1998.

[3] J. Ball and V. Vinnikov: Zero-pole interpolation for matrix meromorphic functions on a compact Riemann surface and a matrix Fay trisecant identity. *Amer. J. Math* 121 (1999), no. 4, 841–888.

[4] D. Beltiţă and J. E. Galé: Holomorphic geometric models for representations of C*-algebras. *J. Funct. Anal.* 255 (2008), no. 10, 2888–2932.

[5] I. Biswas and A. K. Raina: Projective structures on a Riemann surface. *Internat. Math. Res. Notices* 15 (1996), 753–768.

[6] B. Blackadar: *K–Theory for Operator Algebras*. Springer Verlag 1986.

[7] M. J. Dupré and J. F. Glazebrook: The Stiefel bundle of a Banach algebra. *Integral Equations Operator Theory* 41 No. 3, (2001), 264–287.

[8] M. J. Dupré, J. F. Glazebrook and E. Previato: A Banach algebra version of the Sato Grassmannian and commutative rings of differential operators. *Acta Applicandae Math.* 92 (3) (2006), 241–267.

[9] M. J. Dupré, J. F. Glazebrook and E. Previato: Differential algebras with Banach-algebra coefficients I: From C*-algebras to the K-theory of the spectral curve. *Max Planck Institut für Mathematik*, Preprint Series 73 (2008) (revised and extended version submitted for publication).

[10] M. J. Dupré, J. F. Glazebrook and E. Previato: The curvature of universal bundles of Banach algebras. In, ‘Proceedings of the International Workshop on Operator Theory and Applications–2008’, eds. J. Ball et al., *Operator Theory: Advances and Applications* 202 (2009), 195–222.
[11] M. J. Dupré, J.-Cl. Evard and J. F. Glazebrook: Smooth parametrization of subspaces of a Banach space. *Rev. Un. Mat. Argentina* 41 No. 2 (1998), 1–13.

[12] J. Fay: *Theta functions on Riemann surfaces*. Lect. Notes in Math. 352, Springer-Verlag, Berlin 1973.

[13] I. Gohberg and J. Leiterer: On cocycles, operator functions and families of subspaces. *Mat. Issled (Kishinev)* 8 (1973) 23–56.

[14] P. A. Griffiths and J. Harris: *Principles of Algebraic Geometry*. J. Wiley & Sons, New York, 1978.

[15] R. Gunning, *Lectures on Riemann surfaces*. Princeton Univ. Press, 1966.

[16] J. W. Helton, *Operator Theory, Analytic Functions, Matrices and Electrical Engineering*. CBMS Reg. Conf. Ser. 68, Amer. Math. Soc., Providence RI, 1987.

[17] I. M. Krichever: Integration of nonlinear equations by the methods of algebraic geometry. *Funkcional. Anal. i Priložen*. 11 No. 1 (1977), 15–31.

[18] L. J. Mason, M. A. Singer and N. M. J. Woodhouse: Tau functions and the twistor theory of integrable systems. *J. Geom. and Phys.* 32 (2000), 397–430.

[19] T. Miwa: On Hirota’s difference equations. *Proc. Japan Acad. Ser. A Math. Sci.* 58 (1982), no. 1, 9–12.

[20] B. Osgood: Old and new on the Schwarzian derivative. In *Quasiconformal mappings and analysis (Ann Arbor, MI, 1995)*, 275–308, Springer, New York, 1998.

[21] V. Ovsienko and S. Tabachnikov: What is ... the Schwarzian derivative? *Notices Amer. Math. Soc.* 56 (2009), no. 1, 34–36.

[22] I. Raeburn: The relationship between a commutative Banach algebra and its maximal ideal space. *J. of Functional Analysis* 25 (1977), 366–390.

[23] A. K. Raina: Fay’s trisecant identity and conformal field theory. *Commun. Math. Phys.* 122 (1989) (4), 625–641.

[24] M. Sato: The KP hierarchy and infinite dimensional Grassmann manifolds. *Proc. Sympos. Pure Math.* 49 Part 1., Amer. Math. Soc., 1989.

[25] G. Segal: The geometry of the KdV equation, in *Topological methods in quantum field theory (Trieste, 1990)* Internat. J. Modern Phys. A 6 (1991), no. 16, 2859–2869.

[26] G. Segal and G. Wilson: Loop groups and equations of KdV type. *Publ. Math. IHES* 61, (1985), 5–65.

[27] J. F. Smith: The $p$-classes of a Hilbert module. *Proc. Amer. Math. Soc.* 36 (2) (1972), 428–434.

[28] M. I. Zelikin: Geometry of operator cross ratio. *Math. Sbornik* 197 (1) (2006), 39–54.

[29] M. I. Zelikin: *Control Theory and Optimization I*, Encyclopedia of Mathematical Sciences 86, Springer Heidelberg-New York, 1998.