GROUP ISOMORPHISM IS NEARLY-LINEAR TIME FOR MOST ORDERS

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ABSTRACT. We show that there is a dense set $\Upsilon \subseteq \mathbb{N}$ of group orders and a constant $c$ such that for every $n \in \Upsilon$ we can decide in time $O(n^2 (\log n)^c)$ whether two $n \times n$ multiplication tables describe isomorphic groups of order $n$. This improves significantly over the general $n^{O(\log n)}$-time complexity and shows that group isomorphism can be tested efficiently for almost all group orders $n$. We also show that in time $O(n^2 (\log n)^2)$ it can be decided whether an $n \times n$ multiplication table describes a group; this improves over the known $O(n^3)$ complexity.

1. Introduction

Given a natural number $n$, there are many structures that can be recorded by an $n \times n$ table $T_i$ taking values $T_{ij}$ in $[n] = \{1, \ldots, n\}$. Isomorphisms of these tables are permutations $\sigma$ on $[n]$ with $T_{\sigma(i)\sigma(j)} = \sigma(T_{ij})$ for all $i, j \in [n]$. It is convenient to assign these tables either a geometric or algebraic interpretation. A geometric view treats these as edge colored directed graphs or as the relations of an incidence structure. We will consider the algebraic interpretation where the table describes a binary product $\ast : [n] \times [n] \rightarrow [n]$.

The upper bound complexity to decide isomorphism is $O(n!)$ by testing all permutations. Better timings arise when we consider subfamilies of structures, for example, by imposing equational laws on the product such as associativity $a \ast (b \ast c) = (a \ast b) \ast c$ (i.e. semigroups) or the existence of left and right fractions (i.e. quasigroups or latin squares). Booth [19, p. 132] observed that the complexity of isomorphism testing of semigroups is polynomial-time equivalent to the complexity of graph isomorphism. At the time, that complexity was subexponential, but it has since been shown by Babai [1] to be in quasi-polynomial time with a highly inventive algorithm. Meanwhile Miller [18, 19] observed that the complexity of quasigroup isomorphism is in quasi-polynomial time $n^{O(\log n)}$ through an almost brute-force algorithm: since quasigroups of order $n$ are generated by $\log_2 n$ elements, a brute-force comparison of all $[\log_2 n]$-tuples either finds an isomorphism between two quasigroups or determines that no isomorphism exists.

An intriguing bottleneck to further improvement has been the case of groups that have associative products with an identity and left and right inverses. Because these are quasigroups, they have a brute-force isomorphism test with complexity $n^{O(\log n)}$; Miller gave credit to Tarjan for this complexity. Guralnick and Lucchini [4, Theorem 16.6] showed independently that every finite group of order $n$ can be generated by at most $d(n)$ elements and $d(n) \leq \mu(n) + 1$, where $\mu(n)$ is the largest exponent of a prime power divisor $p^{\mu(n)}$ of $n$. Thus, the complexity of brute-force group isomorphism testing is more accurately described as $n^{O(\mu(n))}$. Since $\mu(n) \in \Theta(\log n)$ when $n = 2^m$ with $m \in O(1)$, this does not improve on the generic $n^{O(\log n)}$ bound. Group isomorphism testing seems to be a leading bottleneck to improving the complexity of graph isomorphism, see Babai [1, Section 13.2]. Even so, we prove here that for most orders, group isomorphism is in nearly-linear time compared to the input size; this also shows that groups of general orders are not a bottleneck to graph isomorphism.
1.1. Current state of group isomorphism. Surprisingly, the brute-force \( n^O(\log n) \) complexity of group isomorphism has been resilient. Progress has fragmented into work on numerous subclasses \( \mathcal{X} \) of groups; the precise problem studied today is:

\[
\text{\textbf{\( \mathcal{X} \)-GROUPIso}}
\]

\textbf{Given} a pair \((T, T')\) of \( n \times n \) tables with entries in \([n]\) representing groups in \( \mathcal{X} \),

\textbf{Decide} if the groups are isomorphic.

Grochow-Qiao \cite{GQ12} give a detailed survey of recent progress; here we summarize a few results related to our setting. Iliopoulos, Karagiorgos-Poulakis, Vikas, and Kavitha \cite{IKVP14} progressively improved the complexity for the class \( \mathfrak{A} \) of abelian groups (where the product satisfies \( a \ast b = b \ast a \) for all \( a, b \)), resulting in a linear-time algorithm for \( \mathfrak{A}\)-GROUPIso in a RAM model (more on this below). Wagner-Rosenbaum \cite{WR15} gave an \( n^{0.25 \log n + O(1)} \) time algorithm for the class \( \mathfrak{A}_p \) of groups of order a power of a prime \( p \), and later generalized this to the class of solvable groups. Li-Qiao \cite{LQ15} proved an average run time of \( n^{O(1)} \) for an essentially dense subclass \( \mathfrak{A}_{p,2} \subset \mathfrak{A}_p \). Babai-Codenotti-Qiao \cite{BCQ17} proved an \( n^{O(1)} \) bound for the class \( \mathfrak{T} \) of groups with no nontrivial abelian normal subgroups.

Other research attempts to combine results for separate classes by considering isomorphism between groups that decompose into a subgroup in class \( \mathcal{X} \) and a quotient in class \( \mathcal{Y} \), see also Section 2; we call this the \((\mathcal{X}, \mathcal{Y})\)-extension problem. Le Gall \cite{LG14} studied \((\mathfrak{A}, \mathfrak{C})\)-extensions, where \( \mathfrak{C} \) consists of cyclic groups. Grochow-Qiao \cite{GQ12} considered \((\mathfrak{A}, \mathfrak{T})\)-extensions, and outlined a general framework for solving extension problems.

A further class of algorithms considers terse input models, such as black-box models or groups of matrices or permutations; we refer to Seress \cite{S17} Section 2 for details of those models. In this format, groups can be exponentially larger than the data it takes to specify the group. Using this framework for solving extension problems.

Some of the motivation of this and earlier work \cite{GQ12} has been the observation that, in contrast to graph isomorphism, the difficulty of group isomorphism is influenced by the prime power factorization of the group orders \( n \). For example, if \( n = 2^c \pm 1 \) is a prime, then there is exactly one isomorphism type of groups of prime order \( n \) and isomorphism can be tested by comparing orders. Yet, there are \( n^{2^{2^c} / 27 - O(e)} \) isomorphism types of groups of prime power order \( n = 2^c \), see \cite[p. 23]{GQ12}. As of today, isomorphism testing of groups of order \( 2^c \) has the worst-case complexity.

1.2. Main results. The main result of this paper is a proof that group isomorphism can be tested efficiently \textit{for almost all} group orders \( n \) in time \( O(n^2(\log n)^c) \) for some constant \( c \), if the groups are input by their Cayley tables, that is, by \( n \times n \) tables describing their multiplication maps \( [n] \times [n] \rightarrow [n] \). To make “almost all” specific, we define the \textit{density} of a set \( \Omega \subseteq \mathbb{N} \) to be the limit

\[
\delta(\Omega) = \lim_{n \to \infty} \frac{|\Omega \cap [n]|}{n};
\]

the set \( \Omega \) is \textit{dense} if \( \delta(\Omega) = 1 \). By abuse of notation, \( \Omega\)-GROUPIso denotes the isomorphism problem for the class of groups whose orders lie in \( \Omega \). All our complexities are stated for deterministic Turing machines; we give details in Section 2.

\textbf{Theorem 1.1.} There is a dense subset \( \Upsilon \subset \mathbb{N} \) and a deterministic Turing machine that decides \( \Upsilon\)-GROUPIso in time \( O(n^2(\log n)^c) \) for some constant \( c \).

We provide a proof in Section 4. Since every multiplication table \([n] \times [n] \rightarrow [n]\) can be encoded and recognized from a binary string of length \( \Theta(n^2 \log n) \), the algorithm of Theorem 1.1 is nearly-linear time in the input size. The dense set \( \Upsilon \) is specified in Definition 2.3; here we remark that we can determine in time \( O(n) \) whether \( n \in \Upsilon \), and the complexity for brute-force isomorphism
testing of groups of order $n \in \mathcal{Y}$ is $n^{O(\log \log n)}$. Because of this, we would have been content with a polynomial-time bound; being able to prove nearly-linear time bound was a surprise.

Our set $\mathcal{Y}$ excludes an important but difficult class of group orders, specifically orders that have a large power of a prime as a divisor. Theorem 1.1 therefore goes some way towards confirming the expectation that groups of prime power order are the essential bottleneck to group isomorphism testing. Indeed, examples such as provided in [27] show that large numbers of groups of prime power order can appear identical and yet be pairwise non-isomorphic. In fact, known estimates on the proportions of groups show that most isomorphism types of groups accumulate around orders with large prime powers, see [4, pp. 1–2]. So our Theorem 1.1 should not be misunderstood as saying that group isomorphism is efficient on most groups, just on most orders. Even so, we see in results like Li-Qiao [16] and Theorem 1.1 the beginnings of an approach to show that group isomorphism is polynomial-time on average, and we encourage work in this direction.

The solutions of $\mathcal{X}$-GROUPISO cited so far deal with the problem in the promise polynomial hierarchy [11] where one promises that inputs are known to be groups and that they lie in $\mathcal{X}$. To relate those solutions to the the usual deterministic polynomial-time hierarchy forces us to consider the complexity of the associated membership problem:

$$\begin{align*}
\text{Given:} & \quad \text{a binary string } T, \\
\text{Decide:} & \quad \text{if } T \text{ encodes the Cayley table of a group contained in } \mathcal{X}.
\end{align*}$$

While it is straightforward to verify that an input encodes an $n \times n$ Latin square (i.e. quasigroup), current methods available in the literature seem to require $O(n^3)$ steps to verify that the binary product is associative, see [10, Chapter 2]. Here we present an improvement that solves $\mathcal{G}$-GROUP for the class $\mathcal{G}$ of all finite groups in time $O(n^2(\log n)^2)$.

**Theorem 1.2.** There is a deterministic Turing machine that decides in time $O(n^2(\log n)^2)$ whether a multiplication table on $[n]$ describes a group and, if so, returns a homomorphism $[n] \to \text{Sym}_n$ into the group $\text{Sym}_n$ of permutations on $[n]$.

We prove Theorem 1.2 in Appendix 4.2. Thus, we may cast Theorem 1.1 as nearly-linear time in the deterministic polynomial-time hierarchy, that is, it properly accepts or rejects all strings without assuming external promises on these inputs. Theorem 1.2 also offers a hint that our strategy partly entails working with data structures for permutation groups, instead of working with the multiplication tables directly. This is responsible for much of the nearly-linear time complexity of the various group theoretic routines upon which we build our algorithm for Theorem 1.1.

While we provide a self-contained proof, Theorem 1.2 is an example of a general approach we are developing for shifting promise problems to deterministic problems, see also Section 5. Promise problems are especially common when Huffman inputs are given by compact encodings such as black-box inputs, see Goldreich [11]. In ongoing work [8], we introduce a more general process for verifying promises by specifying inputs not as strings for a Turing machine, but rather as types in a sufficiently strong Type Theory. Theorem 1.2 can be interpreted as an example of such an input where the rows of the multiplication table are themselves treated as inhabitants of a permutation type. The algorithm then effectively type-checks that these rows satisfy the required introduction rules for a permutation group type. Type-checking is not in general decidable so the effort is to confirm an efficient complexity for specific settings. As a by-product of such models, the subsequent algorithms also profit from using these rich data types; for more details on this topic we refer to [8].

1.3. **Structure.** In Section 2 we introduce relevant notation and state Theorems 2.4–2.6, which are the main ingredients for our proof of Theorem 1.1. Since the proofs of Theorems 2.4–2.6 are more involved and partly depend on technical Group Theory results, we delay them until Appendix A. In Section 3 we discuss some algorithmic results required for our proof of Theorem 2.6. Proofs for our main results are provided in Section 4. A conclusion and outlook are given Section 5.
2. Notation and preliminary results

2.1. Computational model. Throughout \( n \) is the order of the multiplication tables used as input to programs, so input lengths are \( \Theta(n^2 \log n) \). We write \( \tilde{O}(n^d) \) for \( O(n^d(\log n)^c) \), where \( c \) and \( d \) are constants, and we note that \( O(n(\log n)^{O((\log \log n)^{1+\varepsilon})}) \subset O(n^{1+\varepsilon}) \) for any \( \varepsilon > 0 \); below we use \( \varepsilon = 0.1 \) for convenience. Computations are carried out on a Turing machine (TM) with separate tapes for each input group, an output tape, and a pair of spare tapes of length \( O(n \log n) \) to store our associated permutation group representations developed in the course of Theorem 1.2. In this model, carrying out a group multiplication requires one to reposition the tape head to the correct product, at the cost of \( O(n^2 \log n) \). That will be prohibitive for our given timing, so our first order of business will be to replace the input with an efficient \( \tilde{O}(n) \)-time multiplication, cf. Remark 1.1. For comparison, work of Vikas and Kavitha [24] provide isomorphism tests for abelian groups using \( O(n) \) group operations. Such an algorithm can be considered as linear time in a random access memory (RAM) model where group and arithmetic operations are stated as a unit cost. This is partly how it is possible to produce a running time shorter than the input length. In general, an \( f(n) \)-time algorithm in a RAM model produces an \( O(f(n)^3) \)-time algorithm on a TM, see [21, Section 2], although a lower complexity reduction may exist for specific programs.

2.2. Group theory preliminaries. We follow most common conventions in group theory, e.g. as in [22, 24]. Given a group \( G \), a subgroup \( H \leq G \) is a nonempty subset which is a group with the inherited operations from \( G \). Homomorphisms between groups \( G \) and \( K \) are functions \( f : G \rightarrow K \) with \( f(xy) = f(x)f(y) \) for all \( x, y \in G \). For \( S \subseteq G \), let \( (S) \) be the intersection of all subgroups containing \( S \); it is the smallest subgroup of \( G \) containing \( S \), also called the subgroup generated by \( S \). The commutator of group elements \( x, y \) is \( [x, y] = x^{-1}y^{-1}xy \), and conjugation is written as \( x^y = x[x, y] \). For \( X, Y \subseteq G \), let \( [X, Y] = \langle [x, y] : x \in X, y \in Y \rangle \). The number of elements in \( G \), the order of \( G \), is denoted \( |G| \); in this work, \( G \) always is a finite group.

Given a set \( \pi \) of primes, a subgroup \( H \leq G \) is a Hall \( \pi \)-subgroup if for every \( p \in \pi \) dividing \( |G| \), we have that \( p \) divides \( |H| \), but not the index \( |G : H| = |G|/|H| \). If \( \pi = \{ p \} \), then \( H \) is a Sylow \( p \)-subgroup. A further convention is to let \( \pi' \) denote the complement of \( \pi \) in the set of all primes and to speak of Hall \( \pi' \)-subgroups. The \( \pi \)-factorization of an integer \( n > 1 \) is \( n = ab \), where every prime divisor of \( b \) lies in \( \pi \), and no prime divisor of \( a \) lies in \( \pi \).

A subgroup \( B \) is normal in \( G \), denoted \( B \trianglelefteq G \), if \( |G : B| = |G|/|B| \). The group \( G \) is simple if its only normal subgroups are \( \{1\} \) and \( G \). A composition series for \( G \) is a series \( G = G_1 > \ldots > G_m = \{1\} \) of subgroups with each \( G_{i+1} \trianglelefteq G_i \) and each composition factor \( G_i/G_{i+1} \) is simple. The group \( G \) is solvable if every composition factor is abelian.

A normal subgroup \( B \leq G \) splits in \( G \), denoted \( G = H \rtimes B \), if there is \( H \leq G \) with \( H \cap B = \{1\} \) and \( G = \langle H, B \rangle \). Let \( \text{Aut}(B) \) be the set of invertible homomorphisms \( B \rightarrow B \). If \( G = H \rtimes B \) and \( h \in H \), then conjugation \( b \mapsto h^b \) defines \( \theta(h) \in \text{Aut}(B) \), and \( \theta : H \rightarrow \text{Aut}(B), h \mapsto \theta(h) \), is a homomorphism. Conversely, given \( (H, B, \theta) \) with homomorphism \( \theta : H \rightarrow \text{Aut}(B) \), there is a group \( H \rtimes \theta B \) on the set \( \{ (h, b) : h \in H, b \in B \} \) with product \( (h_1, b_1)(h_2, b_2) = (h_1h_2, b_1^\theta(b_2)) \); here we abbreviate \( b_1^\theta = \theta(h_2)(b_1) \). If conjugation in \( G = H \rtimes B \) induces \( \theta : H \rightarrow \text{Aut}(B) \), then \( G \) is isomorphic to \( H \rtimes \theta B \). In fact, the following observation holds; we will use this lemma only in the situation that \( B \) and \( \tilde{B} \) are cyclic groups; this case of Lemma 2.1 is proved in [6, Lemma 2.8].

Lemma 2.1. Let \( G = H \rtimes \theta B \) and \( \tilde{G} = \tilde{H} \rtimes \tilde{\theta} \tilde{B} \). If \( \alpha : H \rightarrow \tilde{H} \) and \( \beta : B \rightarrow \tilde{B} \) are isomorphisms such that for all \( h \in H \)

\[
(1) \quad \tilde{\theta}(\alpha(h)) = \beta \circ \theta(h) \circ \beta^{-1},
\]

then \( (h, b) \mapsto (\alpha(h), \beta(b)) \) is an isomorphism \( G \rightarrow \tilde{G} \). Conversely, if \( G \) and \( \tilde{G} \) are isomorphic and \( H \) and \( B \) have coprime orders, then there is an isomorphism \( G \rightarrow \tilde{G} \) of this form.
2.3. Number theory preliminaries. Our algorithm for Theorem 1.1 depends on crucial number theoretic observations. For integers $n$, we characterize a family of prime divisors we call strongly isolated, and we show that any group $G$ of order $n$ in our dense set $\Upsilon$ decomposes as $G = H \rtimes B$ such that the prime factors of $|B|$ are exactly the strongly isolated prime divisors of $n$ that are larger than $\log \log n$; we also prove that $B$ is cyclic. This reduces our isomorphism test to considering the data $(H, B, \theta)$ and Lemma 2.2. Not just isomorphism testing, but many natural questions of finite groups reduce to properties of $(H, B, \theta)$, and so this decomposition has interesting implications for computing with groups generally. Note that for a group $G$ with order $n \in \Upsilon$, the decomposition described above defines integers $a = |H|$ and $b = |B|$ that depend only on $n$. Our definition of $\Upsilon$ will also imply that $b$ is square-free, and if a prime power $p^e$ with $e > 1$ divides $n$, then $p^e \leq \log n$ and $p^e \mid a$. With $B$ being cyclic, the group theory of $B$ is elementary, and while the group theory of $H$ can be quite complex, we will see in Theorem 2.5 that $H$ has relatively small size, meaning that brute-force becomes an efficient solution. The following definition is central for our work.

**Definition 2.2.** Let $n \in \mathbb{N}$. Write $q_2(n)$ for the largest 2-power dividing $n$. A prime $p \mid n$ is isolated if $k = 0$ for every prime power $q^k$ with $q^k \mid n$ and $p \mid (q^k - 1)$. If, in addition, $p \mid |T|$ for every non-abelian simple group $T$ of order dividing $n$, then $p$ is strongly isolated. We write $\pi_n$ for the set of strongly isolated prime divisors of $n$ and define

$$\pi_n^{\text{big}} = \{p \in \pi_n : p > \log \log n\}.$$ 

For example, 31 is isolated in $2^4 \cdot 5^2 \cdot 31$ but not in $2^5 \cdot 5^2 \cdot 31$ or in $2^4 \cdot 5^3 \cdot 31$. As indicated above, the properties of our dense set $\Upsilon$ are a critical ingredient in our algorithm for Theorem 1.1; we are now in the situation to give the formal definition.

**Definition 2.3.** Let $\Upsilon \subseteq \mathbb{N}$ be the set of all integers $n$ that factor as $n = ab$ such that:

- a) if $p \mid a$ is a prime divisor, then $p \leq \log \log n$ and, if $p^e \mid a$, then $p^e \leq \log n$;
- b) if $p \mid b$ is a prime divisor, then $p > \log \log n$ and $p \mid n$ is isolated;
- c) the factor $b$ is square-free, that is, if $p^e \mid b$ is a prime power divisor, then $e \in \{0, 1\}$.

**Theorem 2.4.** The set $\Upsilon$ is a dense subset of $\mathbb{N}$.

A proof of Theorem 2.4 and more details on $\Upsilon$ are given in Appendix A.1 and Section 5.

2.4. Splitting results. As mentioned in Section 2.3, the properties of $n \in \Upsilon$ impose limits on the structure of groups of order $n$; we prove the next theorem in Appendix A.2.

**Theorem 2.5.** Every group $G$ of order $n \in \Upsilon$ has a unique Hall $\pi_n^{\text{big}}$-subgroup $B$, which is cyclic, and $G = H \rtimes B$ for some subgroup $H \leq G$ of small order $|H| \in (\log n)^{\mathcal{O}(\log \log n^2)}$.

**Remark 2.1.** In fact, our proof of Theorem 2.5 also shows the following result for any group $G$ of any order $n \in \mathbb{N}$: If $G$ is solvable and $p \mid n$ is an isolated prime, or if $G$ is non-solvable, $p \mid n$ is a strongly isolated prime, and $p > \nu_2(n)$, then $G$ has a normal Sylow $p$-subgroup $S \leq G$; in both cases, $G = H \rtimes S$ for some $H \leq G$ by the Schur-Zassenhaus Theorem [22] (9.1.2)].

The next theorem shows that we can construct generators for the decomposition in Theorem 2.5; we discuss the proof of Theorem 2.6 in Appendix A.2.

**Theorem 2.6.** There is an $\tilde{O}(n^{1.1})$-time algorithm that, given a group $G$ of order $n \in \Upsilon$, returns generators for $H, B \leq G$ such that $G = H \rtimes B$ and $B$ is a Hall $\pi_n^{\text{big}}$-subgroup.

3. Algorithmic preliminaries: presentations and complements

We assume now that our input has been pre-processed and confirmed to be a group by our algorithm for Theorem 1.2. In so doing, we also produce a series of important data types and accompanying routines, including the following for each input group of order $n$:
• a permutation group representation,
• a generating set of size $O(\log n)$,
• an algorithm to write group elements as a product of the generators in time $\tilde{O}(n)$,
• an algorithm to multiply two group elements in time in $(\log n)^{O(n)}$, and
• an algorithm to test equality of elements in the group in time $\tilde{O}(n)$.

Remark 4.1 explains some of these routines in more detail. The advantage is that we can now multiply group elements without moving the Turing machine head over the original Cayley tables which would cost us $O(n^2)$ steps for each group operation.

Many of the following observations are variations on classical techniques designed originally for permutation groups, e.g. as in [17, 24, 25]. We include proofs here to demonstrate nearly-linear time when applied to the Cayley table model. For simplicity we assume that all generating sets contain 1. All our lists have size $O(n)$, so all searches can be done in nearly-linear time.

A membership test for a subgroup $L \leq G$ is a function that, given $g \in G$, decides whether $g \in L$. For example, a membership test for the center $Z(G) \leq G$ is to report the outcome of the test whether $g \in G$ satisfies $g = s = s \cdot g$ for all $s \in S$. This test defines $Z(G)$ without having specified a generating set for it. The next result shows that the Cayley graph of $G$ can be computed in nearly-linear time; note that Cay$(S, \{1\})$ is indeed the usual Cayley graph with respect to $S$.

**CAYLEYGRAPH**

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Given a group $G$ generated by $S \subseteq G$ and a membership test for a subgroup $L$.
Return the Cayley graph Cay$(S, L) = \{(xL, sxL, s) \mid s \in S, x \in G\}$ with a spanning tree (a so-called Schreier tree).
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**Proposition 3.1.** Let $G = \langle S \rangle$ with $|G| = n$ and $|S| \in O(\log n)$, and let $L \leq G$ be given by a membership test. CAYLEYGRAPH can be solved using $O(|S||G : L|) \subseteq \tilde{O}(n)$ group multiplications and membership test applications. Once solved, there is an $\tilde{O}(n)$-time algorithm to compute generators for $L$, and a $\tilde{O}(|G : L|)$-time algorithm that given $g \in G$ finds a word $\bar{g}$ in $S$ with $\bar{g}L = gL$.

**Proof.** We initialize graphs $G$ and $T$, both with vertex set $V = \{L\}$ and empty edge sets. Now use the orbit-stabiliser algorithm [24 Section 4]: While there is $s \in S$ and a vertex $xL \in V$ such that $sxL \notin V$, add $sxL$ to $V$ and add $(xL, sxL, s)$ to the edge set of $C$ and $T$. Otherwise, $sxL \in V$ and add $(xL, sxL, s)$ only to the edge set of $C$. One compares $xL = yL$ by deciding $x^{-1}y \in L$ via the membership test. The algorithm terminates after $O(|S||G : L|)$ steps, and then $|V| = |G : L|$ and $V = \text{Cay}(S, L)$ with Schreier tree $T$. This spanning tree determines a transversal $T$ for $L$ in $G$, that is, $G$ is the disjoint union of all $tL$ with $t \in T$. Given $g \in G$, there is a unique vertex $xL$ in $T$ with $gL = xL$ and unique labeled path $L \xrightarrow{t_1} t_1L \xrightarrow{t_2} \cdots \xrightarrow{t_k} xL$ with each $t_i \in S$; that is, $\bar{g}L = gL$ for $\bar{g} = t_k \cdots t_2 \xrightarrow{t_1}$. Schreier’s lemma [24 Lemma 4.2.1] shows that $L$ is generated by the set of all $(st)^{-1}st$ where $s \in S$ and $t \in T$. The number of such generators is $O(|S||G : L|)$ and the products cost $(\log n)^{O(1)}$ each. The computation of $\bar{g}$ in time $\tilde{O}(|G : L|)$ is as described in Remark 4.1. □

**Corollary 3.2.** Let $G = \langle S \rangle$ be a group of order $n$ and let $\pi$ be a set of primes. If $G$ has a normal Hall $\pi$-subgroup $B$, then there is an $\tilde{O}(n^{1-\epsilon})$-time algorithm that returns generators for $B$, and a membership test for $B$ that decides in poly-logarithmic time.

**Proof.** Since $B$ is the unique Hall $\pi$-subgroup, $g \in G$ lies in $B$ if and only if the order of $g$ divides $b = |B|$; therefore we may test membership in $B$ by testing if $g^b = 1$. This can be done in $O(\log b)$ group products using fast exponentiation, followed by a comparison with the identity 1. From our foregoing assumptions on $G$, this can be done in time $\tilde{O}(n)$. Finally, as $|S| \in O(\log n)$, generators of $B$ can be obtained from $\tilde{O}(|G : B|)$ group products and membership tests, using CAYLEYGRAPH$(S, B)$; all this can be done in time $\tilde{O}(|G : B|n) \subseteq \tilde{O}(n^{1-\epsilon})$ since $|G : B| \in (\log n)^{O((\log \log n)^2)}$ by Theorem 2.5. □
We now introduce a tool that lets us find a complement to a normal Hall \( \pi_n^{\text{big}} \)-subgroup.

**BigSplit**

- **Given** a group \( G \) of order \( n \in \mathbb{N} \),
- **Return** a subgroup \( H \leq G \) such that \( G = H \rtimes B \), where \( B \) is the Hall \( \pi_n^{\text{big}} \)-subgroup.

To solve BigSplit we need a brief detour into a generic model for encoding groups via presentations, cf. [25, Section 1.4]: The free group \( F[X] \) on a given alphabet \( X \) is formed by creating a disjoint copy \( X^- \) of the alphabet and treating the elements of \( F[X] \) as words over the disjoint union \( X \cup X^- \), including the empty word 1. The latter serves as the identity and word concatenation is the group product; to impose the existence of inverses we apply rewriting rules finds a word \( \bar{w} \) as words over the disjoint union \( X \cup X^- \), including the empty word 1. The latter serves as the identity and word concatenation is the group product; to impose the existence of inverses we apply rewriting rules $w^\pm \to 1$ for each $x \in X$ and corresponding $x^- \in X^-$. For a group $G$, tuple $g \in G^X$, and word $w \in F[X]$, we assign an element $w(g) \in G$ by replacing each variable $x^\pm$ in $w$ with the value $g_x \in G$ and $g_x^{-1} \in G$, respectively, and then evaluating the corresponding product in $G$. The mapping $w \mapsto w(g)$ is a homomorphism $\hat{g} : F[X] \to G$, whose kernel \( \ker \hat{g} = \{ w \in F[X] : w(g) = 1 \} \) is a normal subgroup of \( F[X] \). If $G$ is generated by the image \( S = \{ g_x : x \in X \} \) and \( R \) generates \( \ker \hat{g} \) as a normal subgroup, then the pair \( (S \mid R) \) is a presentation of \( G \), where \( R \) is a set of relations for \( G \) relative to \( S \). Note that \( \langle S \mid R \rangle \) carries all the information necessary to describe \( G \) up to isomorphism; however, in such an encoding isomorphism testing may be even become undecidable, see [25, Section 1.9].

Our interest in presentations is to produce a relatively small number of equations whose solutions help to solve BigSplit; for that purpose the following will suffice.

**Proposition 3.3.** Let \( G \) be a group of order \( n \in \mathbb{N} \) with Hall \( \pi_n^{\text{big}} \)-subgroup \( B \). There is an \( \tilde{O}(n^{1.1}) \)-time algorithm to compute a presentation \( (S \mid R) \) of the quotient \( G/B \) such that \( |S| \in O(\log n) \) and \( |R| \in (\log n)^{O((\log \log n)^2)} \), and each \( w \in R \) is a word in \( S \) of length \( O(\log n) \).

**Proof.** As shown above, we find \( G = \langle S \rangle \) with \( |S| \in O(\log n) \). Use CayleyGraph and Corollary 3.2 to get a transversal \( T \) for \( B \) in \( G \), generators for \( B \), and a rewriting algorithm that given \( g \in G \), finds a word \( \bar{g} \in S \) with \( \bar{g}B = gB \). Choose a set \( X = \{ x_g : g \in S \} \) of variables, and for each \( g \in G \) define \( w_g \in F[X] \) as the word in \( X \cup X^- \) produced by replacing each \( u \in S \) in \( \bar{g} \) with \( x_u \). Now \( R = \{ w_{x_1}w_{x_2}\ldots w_{x_s} : t \in T, s \in S \} \) is a set of relations for \( G/B \) relative to \( \{ sB : s \in S \} \), cf. [24, p. 112]. Note that \( |R| \leq |G : B| \cdot |S| \) and \( |G : B| \in (\log n)^{O((\log \log n)^2)} \) by Theorem 2.5. Also computing \( w_{g} \) is dominated by the time \( \tilde{O}(|G : B|) \) it takes to compute \( \bar{g} \). We do this on \( O(|S| \cdot |G : B|) \) elements for a total time of \( (\log n)^{O((\log \log n)^2)} \in O(n^{0.1}) \). Our relations have length \( O(|G : B|) \), but there is a \( \tilde{O}(n) \)-time algorithm to replace such relators with ones of length \( O(\log n) \), see [24, Lemma 4.4.2].

**Remark 3.1.** Babai-Luks-Seress, Kantor-Luks-Marks and others (see the bibliography in [13] and [24, Section 6]) developed various algorithms to construct presentations of (quotient) groups of permutations. Their complexities range from polynomial-time, to polylogarithmic-parallel (NC), to Monte-Carlo nearly-linear time, and they produce presentations that can be considerably smaller than what we obtain in Proposition 3.3. Though it is not necessary for our complexity goals, we expect that a better analysis and a better performing implementation would use such methods instead of our above brute-force approach.

**Proposition 3.4.** BigSplit is in time \( \tilde{O}(n^{1.1}) \).

**Proof.** BigSplit is solved via the function Complement discussed in [13, Section 3.3]; we briefly sketch the approach. Let \( G = \langle S \rangle \) with \( S = \{ s_1, \ldots, s_d \} \) and \( d \in O(\log n) \). Use the algorithm of Proposition 3.3 to get a presentation \( \langle x_1, \ldots, x_d \mid R \rangle \) for \( G/B \), such that each \( x_i = x_{s_i} \) as defined in the proof of Proposition 3.3. Every complement \( H \) to \( B \), if it exists, is generated by \( \{ s_1 m_1, \ldots, s_d m_d \} \) for some \( m_1, \ldots, m_d \in B \), and such a generating set satisfies the relations in \( R \),
4. Proofs of the main results

4.1. Proof of Theorem 1.1: isomorphism testing.

Proof of Theorem 1.1. Given two binary maps \([n] \times [n] \rightarrow [n]\), we decide that \(n \in \mathcal{Y}\) in time \(O(n)\), and we use Theorem 1.2 to decide whether these maps describe Cayley tables. If so, we have been given two groups \(G\) and \(\tilde{G}\) of order \(n \in \mathcal{Y}\), and we can use Theorem 2.6 to find generators for subgroups \(H, B \leq G\) and \(\tilde{H}, \tilde{B} \leq \tilde{G}\) with \(G = H \times B\) and \(\tilde{G} = \tilde{H} \times \tilde{B}\). Having generators of these subgroups, we can define homomorphisms \(\theta\) and \(\tilde{\theta}\) such that \(G = H \times_B B\) and \(\tilde{G} = \tilde{H} \times_{\tilde{B}} \tilde{B}\). Since \(n \in \mathcal{Y}\), we know that \(B\) and \(\tilde{B}\) are cyclic, hence \(B\) and \(\tilde{B}\) are isomorphic if and only if \(|B| = |\tilde{B}|\). Moreover, since \(|H|, |\tilde{H}| \leq (\log n)^{O((\log \log n)^2)}\) we can test isomorphism \(H \cong \tilde{H}\) using brute-force methods in time \((\log n)^{O((\log \log n)^2)}\). If \(H \cong \tilde{H}\) and \(B \cong \tilde{B}\) is established, then we can identify \(H = \tilde{H}\) and \(B = \tilde{B}\), and test \(G \cong \tilde{G}\) by using Lemma 2.1 since \(B\) is cyclic, \(\text{Aut}(B)\) is abelian, and so Condition (1) reduces to \(\tilde{\theta}(\alpha(h)) = \theta(h)\) for all \(h \in H\), which we can test by enumerating \(\text{Aut}(H)\) and looking for a suitable \(\alpha\); since \(|H|\) is small, such a brute-force enumeration is efficient.

4.2. Proof of Theorem 1.2: recognizing groups. Similar to [10], Chapter 2, our strategy for recognizing groups uses Cayley’s Theorem [22, (1.6.8)]. The latter implies that the rows of an \(n \times n\) group table can be interpreted as permutations which form a regular permutation group on \([n]\), that is, the group is transitive on \([n]\) and has trivial point-stabilizers, see [24, Section 1.2.2].

A new idea is to exploit that groups of order \(n\) can be specified by generating sets of size \(\log n\), so some \(\log n\) rows determine the entire table. Once the input is verified to be a latin square, our approach is to define a permutation group generated by \(O(\log n)\) rows, and then compare its Cayley table with the original table. In more abstract terms, our algorithm creates an instance of an abstract permutation group data type, as defined in [24, Section 3]. That data type is guaranteed to be a group and so the promise is converted into a computable type-check: We confirm that the group we create in this new data type is the one specified by the original table; the proof given below makes this argument specific. This methodology of removing a promise by appealing to a type-checker generalizes; we refer to our forthcoming work [3] for more details.

Proof of Theorem 1.2. Let \(* : [n] \times [n] \rightarrow [n]\) be the multiplication defined by the table \(T\). In time \(O(n^2(\log n)^2)\) we verify that \(T\) is a latin square (if not, return false) and arrange that \(T\) is reduced, that is, its first row and column have 1, \ldots, \(n\) in order. Now \(T\) describes a loop \(L = ([n], *)\) with identity 1.

For \(i \in [n]\) denote by \(\lambda_i \in \text{Sym}_n\) the map \([n] \rightarrow [n]\) defined by left multiplication \(\lambda_i(a) = i * a\). Since \(T\) is reduced, \(\lambda_i\) is given by the \(i\)-th row of \(T\). Note that \(\Lambda = \{\lambda_i : i \in [n]\}\) is a subgroup of \(\text{Sym}_n\), and \(L\) is a group if and only if \(\Lambda\) is a regular permutation group on \([n]\), if and only if \(\lambda_i\lambda_j = \lambda_{i+j}\) for all \(i, j \in [n]\), see [10] Theorems 2.16 & 2.17. Since \(T\) is reduced, it follows that \(\Lambda\) is a transitive subgroup of \(\text{Sym}_n\). Thus, \(\Lambda\) is regular if and only if the stabiliser \(\Lambda_1\) of 1 in \(\Lambda\) is trivial. We now show how to find a generating set of \(\Lambda\) of size \(O(\log n)\) and prove that \(\Lambda\) is regular, or prove that \(L\) is not a group and return false.

For a subset \(S \subseteq [n]\) define \(\Lambda(S) = \{\lambda_i : i \in S\}\). Since \(T\) is reduced, \(\lambda_i(1) = i\) for every \(i \in [n]\). Let \(\Lambda(S)(1) = \{\lambda(1) : \lambda \in \Lambda(S)\}\) and \(\Lambda(S)_1 = \{\lambda \in \Lambda(S) : \lambda(1) = 1\}\) be the orbit and stabiliser of 1.
in $\Lambda(S)$, respectively. Both can be computed in time $O(n|S|)$ using the orbit-stabiliser algorithm \cite[Section 4]{24}, see also \textsc{CayleyGraph}, which returns $\{\lambda_i\lambda_j^{-1} : i \in \Lambda(S)(1), j \in S\}$ as the Schreier generating set of $\Lambda(S)_1$; it also returns, for each $i \in [n]$, an element $\lambda_i \in \Lambda(S)$ with $\lambda_i(1) = i$. If we find a Schreier generator $\lambda_i\lambda_j^{-1}$ that is not trivial, or an element $\lambda_i \neq \lambda_i$, then we return false because we have proved that $\Lambda$ is not regular and therefore $L$ is not a group. For given $i, j \in [n]$, it can be tested in time $O(n)$ whether $\lambda_i\lambda_j^{-1}$ is trivial; the construction of all the $\lambda_i$ takes time $O(n|S|)$.

We want to construct $S \subseteq [n]$ of size $O(\log n)$ such that $\Lambda(S)$ is transitive on $[n]$ and such that all Schreier generators are trivial; this implies that $\Lambda(S)$ is a regular permutation subgroup on $[n]$. If at any point we find a nontrivial Schreier generator, then we return false. We start with an arbitrarily chosen subset $S$ of size at most $\lceil \log n \rceil$ and check whether the orbit $\Lambda(S)(1)$ has size $n$; this can be done in time $O(n \log n)$. If so, then $\Lambda(S)$ is transitive, hence regular (or else we will have aborted). If not, then choose $k \in [n] \setminus \Lambda(S)(1)$ and replace $S$ by $S \cup \{k\}$. By construction, $\lambda_k(1) = k$, so the new orbit and hence the new group $\Lambda(S)$ have increased in size. When we have not aborted, then in each step $\Lambda(S)$ is a subgroup of a regular permutation group on $[n]$, hence $|\Lambda(S)| \leq n$. We extend $\Lambda(S)$ at most $\log_2 n$ times until we either abort or we have found a regular group $\Lambda(S)$ on $[n]$ with $|S| \in O(\log n)$. By construction, $\Lambda(S) \leq \Lambda$, and $L$ is a group if and only if $\Lambda(S) = \Lambda$. This holds since $\lambda_i \in \Lambda(S)$ for each $i$, namely, $\lambda_i = \lambda_i(1) \in \Lambda(S)$ as found above. 

\begin{remark}
Having converted our Cayley table into a regular permutation group $G$, we now have a generating set $S$ of size $O(\log n)$ and a corresponding Schreier tree; both are stored in a separate tape of length $O(n \log n)$. This is the tape that is used whenever we operate in the group; the original input tapes will never be revisited. As a further pre-processing step we may assume that the Schreier trees are shallow, that is, they have depth bounded by $O(\log n)$, see \cite[Lemma 4.4.2]{24}. The algorithms we now use are as follows. Each $g \in G$ is a node in the Schreier tree and there is a unique path from the origin to $g$; if $g_1, \ldots, g_k \in S$ are the labels of that path, then $g = g_k \cdots g_1$; note that $k \in O(\log n)$ since the Schreier tree is shallow. We now describe how to compute the product of these labels. Recall that $g_1, \ldots, g_k$ are permutations on $[n]$, so we can compute the image of $1 \in [n]$ under $g_k \cdots g_1$, by looking up $i_1 = g_1(1), i_2 = g_2(i_1), \ldots$, until we obtain $u = g_k \cdots g_1(1)$. This scan occurs on the short tape in time $O(\log n)^2 \cdot n^2$. Since the group is regular, there is a unique $g \in G$ with $u = g(1)$, which determines $g = g_k \cdots g_1$. To multiply elements $g_k \cdots g_1$ and $g'_j \cdots g'_1$, we merely concatenate the generators, and continue with this word, yielding a $O(\log n)$-time multiplication. We note that none of our product lengths exceed $O(\log n)^2$. To compare $g_k \cdots g_1$ and $g'_j \cdots g'_1$ with $k, j \in O(\log n)$, we determine and compare $g_k \cdots g_1(1)$ and $g'_j \cdots g'_1(1)$ in time $O(n)$. More details of these methods are given in \cite[p. 85–86]{24}.
\end{remark}

5. Conclusion and outlook

We have shown that when restricted to a dense set $\Upsilon$ of group orders, testing isomorphism of groups of order $n \in \Upsilon$ given by Cayley tables can be done in time $\tilde{O}(n^2)$; this significantly improves the known general bound of $n^{O(\log n)}$. The set $\Upsilon$ includes all square-free integers ($\approx 60\%$ of all orders), and we note that $|\Upsilon \cap \{1, 2, \ldots, 10^k\}|/10^k$ is already approximately $0.535, 0.618$, and $0.702$ for $k = 8, 9, 10$, respectively; moreover, one can decide if $n \in \Upsilon$ in time $O(n)$.

We have proved that groups of these orders admit a computable factorisation $G = H \ltimes B$ with the following useful property: firstly, the \textbf{Hard} group theory of $G$ is captured in $H$, but $|H|$ is small compared to $|G|$ so brute-force methods can be applied to $H$; secondly, the \textbf{Easy} number theory of $G$ is captured by $|B|$, but $B$ is cyclic, hence its group theory is \textbf{Easy}. These decompositions exist for a dense set of group orders, so we expect this will be useful for other computational tasks as well. In fact, we will exploit properties of these decompositions in future work: This paper is part of our program to enhance group isomorphism, see \cite{6,7} for recent work, and we plan to extend
the present results to other input models. Specifically, in our current work [8] we develop a new black-box input model for groups (based on Type Theory) that does not need a promise that the input really encodes a group, so algorithms for this model can be implemented within the usual polynomial-time hierarchy. Due to Theorem [1.2] the algorithms presented here do also not require a promise that the input tables describe groups. We conclude by mentioning that our algorithm for isomorphism testing can be adapted to find a single isomorphism, generators for the set of all isomorphisms, or to prescribe a canonical representative of the isomorphism type of a single group.

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Appendix A. Proofs of Theorems 2.4, 2.5

A.1. Number theory: Proof of Theorem 2.4

Proof of Theorem 2.4. Erdős-Pálfy [9, Lemma 3.5] showed that almost every \( n \in \mathbb{N} \) has the property that if a prime \( p > \log \log n \) divides \( n \), then \( p^2 \nmid n \); thus, \( n = p_1^{e_1} \cdots p_k^{e_k} b \) with \( b \) square-free, every prime divisor of \( b \) is greater than \( \log \log n \), and \( p_1, \ldots, p_k \leq \log \log n \) are distinct primes. Let \( x > 0 \) be an integer. We now compute an estimate for the number \( N(x) \) of integers \( 0 < n \leq x \) which are divisible by a prime \( p \leq \log \log n \) such that the largest \( p \)-power \( p^e \) dividing \( n \) satisfies \( p^e > \log n \).

We want to show that \( N(x)/x \to 0 \) for \( x \to \infty \); this proves that for almost all integers \( n \), if \( p^e \mid n \) with \( p \leq \log \log n \), then \( p^e \leq \log n \). To get an upper bound for \( N(x) \), we consider integers between \( \sqrt{x} \) and \( x \) with respect to the above property, and add \( \sqrt{x} \) for all integers between 1 and \( \sqrt{x} \). Note that if \( p^e \geq \log n \), then \( e \geq \log \log n/\log p \). Since we only consider \( \sqrt{x} \leq n \leq x \), this yields \( e \geq c(x) \) where \( c(x) = \log \log \sqrt{x}/\log \log x \).

Note that \( c(x) \to \infty \) if \( x \to \infty \), thus

\[
N(x) \leq \sqrt{x} + \sum_{k=2}^{[\log \log \sqrt{x}]} \frac{x}{k^c(x)} \leq \sqrt{x} + x \int_2^{\log \log \sqrt{x}} \frac{1}{y^{c(x)-1}} l\right] dy
\]

Since \( 1/(1 - c(x)) \to 0 \) from below, we can estimate:

\[
N(x) \leq \sqrt{x} + x \left[ \frac{1}{1 - c(x)} \right] \left[ -\frac{1}{(\log \log \sqrt{x})^{c(x)-1}} + \frac{1}{2^{c(x)-1}} \right] \leq \sqrt{x} + x \left[ \frac{1}{1 - c(x)} \right] \left[ \frac{1}{2^{c(x)-1}} \right];
\]

thus \( N(x) = o(x) \), since \( N(x)/x \leq \sqrt{x}/x + |1/(1 - c(x))2^{c(x)-1}| \to 0 \) if \( x \to \infty \). This proves that the set \( \mathcal{Y}_1 \) of all positive integers satisfying conditions a,c) in Definition 2.3 is dense. By [9, Lemmas 3.5 & 3.6], the set \( \mathcal{Y}_2 \) of positive integers \( n \) satisfying conditions b,c) is dense as well. An inclusion-exclusion argument proves that \( \mathcal{Y} = \mathcal{Y}_1 \cap \mathcal{Y}_2 \) is dense. \( \square \)

A.2. Splitting theorems: Proofs of Theorems 2.5 & 2.6

Proof of Theorem 2.5. Let \( G \) be a group of order \( n \in \mathcal{Y} \); we first show that \( G \) has a normal Hall \( \pi_n^{\text{bip}} \)-subgroup; in fact, we show that if \( G \) is solvable, then there is a normal Hall \( \pi_n \)-subgroup.

First, suppose that \( G \) is solvable. We show that \( G \) has a normal Sylow \( p \)-subgroup for every \( p \in \pi_n \). Let \( q \neq p \) be a prime dividing \( n \), and let \( A \) be a Hall \( \{p, q\} \)-subgroup of \( G \) of order \( p^e q^f \); see [22, Section 9.1] The Sylow Theorem [22 (1.6.16)] shows that the number \( h_p \) of Sylow \( p \)-subgroups of \( A \) divides \( q^f \) (and hence \( n \)) and \( p \mid (h_p - 1) \). Since \( p \mid n \) is isolated, we have \( h_p = 1 \) and \( A \) has a normal Sylow \( p \)-subgroup. Now fix a Sylow basis \( \mathcal{P} = \{P_1, \ldots, P_s\} \) for \( G \), that is, a set of Sylow
subgroups, one for each prime dividing \( n \), such that \( P_i P_j = P_j P_i \) for all \( i \) and \( j \); see [22 Section 9.2]. Let \( P = P_u \) be the Sylow \( p \)-subgroup for \( G \) in \( \mathcal{P} \). Since \( G = P_1 \cdots P_u \), every \( g \in G \) can be written as \( g = g_1 \cdots g_u \) with each \( g_j \in P_j \). Since \( PP_j = P_j P \), the group \( PP_j \) is a Hall \( \{ p, p_j \} \)-subgroup. As shown above, \( P \triangleleft PP_j \), so each \( g_j P = Pg_j \). Thus, \( gP = g_1 \cdots g_u P = Pg_1 \cdots g_u P = Pg \), so \( P \trianglelefteq G \).

Second, suppose that \( G \) is non-solvable. We show that \( G \) has a normal Sylow \( p \)-subgroup for every \( p \in \pi_n^\text{big} \). Being non-solvable, \( G \) has a non-abelian simple composition factor, so \( [G] \) is divisible by 4, see [5, p. 155]. Since \( n = \Upsilon \), this implies that \( 2^{\nu_2(n)} \leq \log n \), so \( \nu_2(n) \leq \log \log n < p \) for every \( p \in \pi_n^\text{big} \). We claim that \( p \mid |G : O_\infty(G)| \) where \( O_\infty(G) \triangleleft G \) is the largest normal solvable subgroup. The socle of \( G/O_\infty(G) \) decomposes as \( T_1 \times \cdots \times T_\ell \) where each \( T_i \) is non-abelian simple and a minimal normal subgroup of \( G/O_\infty(G) \), see [24 pp. 157–159]. Note that \( p \mid |T_i| \) for each \( i \) since \( p \in \pi_n^\text{big} \). By the Classification of Finite Simple Groups, a prime \( r \) divides \( |\text{Aut}(T_i)| \) only if \( r \) divides \( |T_i| \), as seen from the list of known orders of simple groups and their outer automorphism groups. Let \( \text{PKer}(G)/O_\infty(G) \) be the kernel of the permutation representation \( G/O_\infty(G) \to \text{Sym}_\ell \) induced by the conjugation action on \( \{ T_1, \ldots, T_\ell \} \); this kernel embeds into \( \text{Aut}(T_1) \times \cdots \times \text{Aut}(T_\ell) \).

Assume, for a contradiction, that \( p \) divides \( |G : O_\infty(G)| \). By assumption, \( p \mid |\text{Aut}(T_i)| \) for each \( i \), which forces \( p \mid |G : \text{PKer}(G)| \) and \( p \leq \ell \). Every \( T_i \) has even order by the Odd-Order Theorem, see [28, p. 2], so \( 2^{\ell} \mid n \) and \( \nu_2(n) \geq \ell \); now \( p \leq \ell \) contradicts \( p > \nu_2(n) \), which we have shown above. This forces \( p \mid |G : O_\infty(G)| \), so the Sylow \( p \)-subgroup \( P \) lies in \( O_\infty(G) \). Since \( p \) is also isolated in \( |O_\infty(G)| \), the proof of the solvable case shows \( P \trianglelefteq O_\infty(G) \). Since \( O_\infty(G) \) is characteristic in \( G \), we know that \( P \) is normal in \( G \).

In conclusion, for every \( p \in \pi_n^\text{big} \) there is a normal Sylow \( p \)-subgroup \( G_p \) in \( G \). Since \( n = \Upsilon \) and \( p > \log \log n \), this subgroup is cyclic of order \( p \). If \( p, q \in \pi_n^\text{big} \) are distinct, then \( G_p, G_q \trianglelefteq G \) implies \( G_p G_q \cong G_p \times G_q \cong C_{pq} \), a cyclic group of order \( pq \). It follows that \( G \) has a normal cyclic Hall \( \pi_n^\text{big} \)-subgroup \( B \), and \( G = H \rtimes B \) for some \( H \leq G \) by the Schur-Zassenhaus Theorem [22 (9.1.2)].

It remains to prove that \( |H| \leq (\log n)^{O((\log \log n)^2)} \). Recall from Definition [23] that \( n = ab \) such that \( a \leq (\log n)^{\log \log n} \) and if \( p \in \pi_n^\text{big} \), then \( p \mid b \) and \( p^2 \nmid n \). In particular, \( |H| = ab/\!\!/z \), where \( z \) is the product of the primes in \( \pi_n^\text{big} \). It remains to show that \( b/\!\!/z \leq (\log n)^{O((\log \log n)^2)} \). If \( p \) is a prime divisor of \( b/\!\!/z \), then \( p > \log \log n \) and \( p \notin \pi_n^\text{big} \), that is, \( p^2 \nmid n \) and \( p \mid n \) is isolated, but not strongly isolated. Thus, there is some non-abelian simple group \( T \) of order dividing \( n \) with \( p \mid |T| \). In the remainder of this proof we show that the non-abelian composition factors of \( H \) all together contribute at most \( (\log n)^{O((\log \log n)^2)} \) to the order of \( H \); this will imply that \( b/\!\!/z \leq (\log n)^{O((\log \log n)^2)} \), which then completes the proof.

We first show that the number \( f \) of non-abelian composition factors of \( H \) satisfies \( f \leq \log \log n \). Distinct simple groups intersect trivially and every non-abelian simple group has order divisible by 4, see [5, p. 155], so \( 4^{\ell} \mid n \). Now \( n = \Upsilon \) forces \( 4^{\ell} \leq \log n \), hence \( f \leq \log \log n \). Thus, the proof is complete if every non-abelian composition factor of \( H \) has order at most \( (\log n)^{O((\log \log n))} \).

One can see from the known factorized orders of the finite non-alternating simple groups that every such group \( T \) has a distinguished prime power divisor \( r^m \mid |T| \) with \( m > 1 \) and \( |T| \leq (r^m)^{O(m)} \); this is trivially true for the 26 sporadic groups; for the other non-alternating simple groups this follows because they are representable as quotients of groups of \( d \times d \) matrices over a field of order \( r^e \), and then \( m = de \). Now if the order of \( T \) divides \( n \in \Upsilon \), then \( m > 1 \) forces \( r \leq \log \log n \) and \( r^m \leq \log n \), hence \( m \leq \log \log n \), and \( |T| \leq (\log n)^{O((\log \log n))} \), as claimed.

If \( T \cong \text{Alt}_k \) is alternating of order \( k! / 2 \leq k^k \) then the distinguished prime power divisor is \( 2^{\nu_2(k!)} \). Legendre’s formula [20 Theorem 2.6.4] shows that \( \nu_2(k!) = k^2 - s_2(k) \) where \( s_2(k) \leq \log k \) is the number of 1’s in the 2-adic representation of \( k \). Since \( n \in \Upsilon \), we have \( 2^{\nu_2(k!)} - 1 \leq \log n \), so

\footnote{The finite simple groups (Classification Theorem of Finite Simple Groups) are listed in [28 Section 1.2]; the orders of these groups and the orders of their automorphism groups are described in various places in said book. Simply for the convenience of the reader, we refer to \textit{en.wikipedia.org/wiki/List_of_finite_simple_groups}#Summary for a concise list of these orders.}
$k - \log(k) - 1 \leq \nu_2(k!) - 1 \leq \log \log n$. This shows that $2^k \leq 2k \log n$, and so $|T| \leq (2k \log n)^{\log k}$. Note that $|T| = k!/2$ divides $n$, and so Stirling’s formula $\ln(k!) = k \ln(k) - k + O(\ln(k))$ shows that $k \leq \log 2n$ for large enough $k$. This yields $|T| \leq (\log n)^{O(\log \log n)}$, as claimed. □

Proof of Theorem 2.6. Let $G$ be a group of order $n \in \mathbb{T}$. By Theorem 2.5 and Corollary 3.2 we can construct generators and a membership test for the cyclic Hall $\pi_{n^{\text{bis}}}^b$-subgroup $B$ of $G$. With this we can use Proposition 3.3 to construct a complement $H$, thus $G = H \times B$. The complexity statement follows from the results we have used. □

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