Chordal Komatu–Loewner equation for a family of continuously growing hulls

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Abstract

In this paper, we discuss the chordal Komatu–Loewner equation on standard slit domains in a manner applicable not just to a simple curve but also a family of continuously growing hulls. Especially a conformally invariant characterization of the Komatu–Loewner evolution is obtained. As an application, we prove a sort of conformal invariance, or locality, of the stochastic Komatu–Loewner evolution $\text{SKLE}_{\sqrt{6}-b_{BMN}}$ in a fully general setting, which solves an open problem posed by Chen, Fukushima and Suzuki [Stochastic Komatu–Loewner evolutions and SLEs, Stoch. Proc. Appl. 127 (2017), 2068–2087].

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1. Introduction

The Komatu–Loewner equation is an extension of the celebrated Loewner equation to multiply connected domains. This equation describes the time-evolution of increasing subsets of multiply connected domains, called growing hulls, and was rigorously obtained in the previous studies [1, 4, 3] when the family of growing hulls consist of a trace of a simple curve. In this paper, we shall give a systematic treatment of this equation for a family of growing hulls which are not necessarily induced by a simple curve. In order to describe mathematical details, we begin to recall the Loewner theory briefly. The reader can consult [12] for further detail.

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We denote by \( \mathbb{H} \) the upper half-plane \( \{ z \in \mathbb{C}; \Im z > 0 \} \). Let \( \gamma: [0, t_\gamma) \to \mathbb{H} \) be a simple curve with \( \gamma(0) \in \partial \mathbb{H} \) and \( \gamma(0, t_\gamma) \subset \mathbb{H} \). For each \( t \geq 0 \), there exists a unique conformal map \( g_t \) from \( \mathbb{H} \setminus \gamma(0, t) \) onto \( \mathbb{H} \) with the hydrodynamic normalization

\[
g_t(z) = z + \frac{a_t}{z} + o(z^{-1}), \quad z \to \infty,
\]

for some constant \( a_t > 0 \). This is a version of Riemann’s mapping theorem. If we reparametrize \( \gamma \) so that \( a_t = 2t \) (as mentioned later in Section 4.1), then we obtain the chordal Loewner equation

\[
\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \in \mathbb{H},
\]

(1.1)

where \( \xi(t) = g_t(\gamma(t)) := \lim_{z \to \gamma(t)} g_t(z) \in \partial \mathbb{H} \). We call \( \xi \) the driving function of \( \{ g_t \} \).

Since (1.1) is an ODE satisfying the local Lipschitz condition, the solution \( g_t(z) \) to (1.1) uniquely exists up to its explosion time \( t_z \). If we set \( F_t := \{ z \in \mathbb{H}; t_z \leq t \} \), then \( F_t \) must be the complement of the domain of definition of \( g_t \), that is, \( F_t = \gamma(0, t) \). Thus the information on the curve \( \gamma \) is fully encoded into the driving function \( \xi(t) \) via the Loewner equation. More generally, we can consider (1.1) driven by any continuous function \( \xi \). Even in this case, the solution \( g_t(z) \) defines a unique conformal map \( g_t: \mathbb{H} \setminus F_t \to \mathbb{H} \) with the hydrodynamic normalization, though the resulting family \( \{ F_t \} \) is not necessarily a simple curve but a family of bounded sets called growing hulls. \( \{ F_t \} \), \( \{ g_t \} \) or the couple \( (g_t, F_t) \) is called the Loewner evolution driven by \( \xi \).

In the theory of conformal maps, \( \{ g_t \} \) is usually called the Loewner chain.

Schramm [18] used the Loewner equation (1.1) to define the stochastic Loewner evolution (SLE). For \( \kappa > 0 \), SLE\( _\kappa \) is the random Loewner evolution driven by \( \xi(t) = \sqrt{\kappa} B_t \), where \( B_t \) is the one-dimensional standard Brownian motion (BM). Schramm’s original aim was to describe the scaling limit of two-dimensional lattice models in statistical physics. SLE\( _\kappa \) was actually proven to be the scaling limit of some models according to the value of \( \kappa \). For individual models, we refer the reader to [10, Section 2.5] and the references therein. In addition, recent studies such as [8] reveal the relation between the Loewner equation and integrable systems. We therefore have much interest in the Loewner theory from various points of view.

As seen, for example, from the usage of Riemann’s mapping theorem above, the simple connectivity of \( \mathbb{H} \) is crucial to the Loewner theory. Thus it
is not straightforward to extend the Loewner equation to multiply connected domains (or to Riemann surfaces). This problem was originally proposed by Komatu \cite{11}, who obtained primary expression of corresponding equations on special multiply connected domains. After more than fifty years, Bauer and Friedrich \cite{1} established its definitive expression by means of the Green function and harmonic measures, a standard way in complex analysis used by \cite{11}. Lawler \cite{13} then gave a probabilistic comprehension of the equation in terms of the excursion reflected Brownian motion (ERBM). The idea provided in \cite{13} was implemented by Drenning \cite{7} later in some detail. Motivated by \cite{1} and \cite{13}, Chen, Fukushima and Rohde \cite{4} adopted the notion of the Brownian motion with darning (BMD) to fill missing arguments in the existing proofs.

We now describe the framework where our domain has multiple connectivity. Fix a positive integer \( N \). Let \( C_j \subset \mathbb{H}, 1 \leq j \leq N, \) be mutually disjoint horizontal slits, that is, segments parallel to the real axis. We call \( D := \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j \) a standard slit domain. Any \( N \)-connected domain is conformally equivalent to some standard slit domain. The case of parallel slit plane, namely, the whole plane \( \mathbb{C} \) deleted by some parallel slits, is typically treated in some textbooks, and in the present case the proof is almost the same as explained in \cite[Section 2.2]{1}.

Let \( \gamma: [0, t_\gamma) \to \mathcal{D} \) be a simple curve with \( \gamma(0) \in \partial \mathbb{H} \) and \( \gamma(0, t_\gamma) \subset D \). For each \( t \geq 0 \), there exists a unique conformal map \( g_t \) from \( D \setminus \gamma(0, t] \) onto another standard slit domain \( D_t \) with the hydrodynamic normalization. After the same reparametrization of \( a_t, g_t(z) \) satisfies the chordal Komatu–Loewner equation (\cite[Theorem 3.1]{1}, \cite[Theorem 9.9]{4})

\[
\frac{d}{dt} g_t(z) = -2\pi \Psi_{D_t}(g_t(z), \xi(t)), \quad g_0(z) = z \in D, \quad (1.2)
\]

where \( \xi(t) = g_t(\gamma(t)) \in \partial \mathbb{H} \). \( \Psi_{D_t}(\cdot, \xi_0), \xi_0 \in \mathbb{R}, \) is the conformal map on \( D_t \) defined in Section 2.1.

Here (1.2) differs from (1.1) in that the image \( D_t \) differs from \( \mathbb{H} \) and varies as time passes. Let \( C_j(t) \) be the \( j \)-th slit of \( D_t \) so that \( C_j(0) = C_j \). The left and right endpoints of \( C_j(t) \) are denoted by \( z_j(t) = x_j(t) + iy_j(t) \) and \( z^*_j(t) = x^*_j(t) + iy_j(t) \), respectively. These endpoints then satisfy the
Figure 1: Conformal map $g_t$

Komatu–Loewner equation for slits ([1, Theorem 4.1], [3, Theorem 2.3])

\[
\begin{align*}
\frac{dy_j}{dt}(t) &= -2\pi \Im \Psi_{D_t}(z_j(t), \xi(t)), \\
\frac{dx_j}{dt}(t) &= -2\pi \Re \Psi_{D_t}(z_j(t), \xi(t)), \\
\frac{dx_j^r}{dt}(t) &= -2\pi \Re \Psi_{D_t}(z_j^r(t), \xi(t)).
\end{align*}
\]

Hence the motion of $D_t$ is described by (1.3) in terms of those of the slits $C_j(t)$.

Once we get (1.2) and (1.3), the initial value problem for them, as done for (1.1), is a natural question. Namely, for a given continuous function $\xi$, we look for the solution to (1.2) and (1.3) and then obtain a family $\{F_t\}$ of growing hulls. We shall explain the actual procedure in Section 2.2. As a result, (1.2) generates a family $\{g_t\}$ of conformal maps and $\{F_t\}$ of growing hulls. They are called the Komatu–Loewner evolution driven by $\xi$. Let us call $\{g_t\}$ the Komatu–Loewner chain as well in this paper. In addition, Chen and Fukushima [3] defined the stochastic Komatu–Loewner evolution (SKLE) with the random driving function $\xi$ given by the system of SDEs (2.18) and (2.19), based on the discussion in [1, Section 5]. Its relation to SLE was also examined by Chen, Fukushima and Suzuki [5].

In such a research on SKLE, the trouble often arises concerning the “transformation of the Komatu–Loewner chains.” Here by the term “transformation” we mean the following situation: Let $(g_t, F_t)$ be the Komatu–Loewner evolution in a standard slit domain $D$ and $\tilde{D}$ be another slit domain with $F_t \subset \tilde{D}$. The degree of connectivity of $\tilde{D}$ can be different from that of $D$. There is then a unique conformal map $\tilde{g}_t$ from $\tilde{D} \setminus F_t$ onto a slit domain
with the hydrodynamic normalization by Proposition 2.3. We expect \((\tilde{g}_t, F_t)\) to be the Komatu–Loewner evolution in \(\tilde{D}\), that is, generated by the equation (modulo time-change). This fact however needs proof since we have deduced the equation only for a simple curve, not for a family of growing hulls. From this standpoint, we can say that [5] established exactly the transformation of chains with \(\tilde{D} = \mathbb{H}\) by the hitting time analysis for BM. This method is successful but not applicable to general \(D\) and \(\tilde{D}\), and thus some problems mentioned in [5, Section 5] remain open.

A major purpose of this paper is to settle down these circumstances. To be more precise, we shall deduce the Komatu–Loewner equation for a family of “continuously” growing hulls in Section 4. The continuity of growing hulls is introduced in Definition 4.2 via the kernel convergence of domains, which is a key concept in this paper. In Section 3 we provide a detailed description on the kernel convergence. The continuity of hulls and the existence condition (2.17) of driving function prove to be a complete characteristic of the Komatu–Loewner evolution in Theorem 4.6. Our definition of the continuity is moreover independent of the domain and conformally invariant, and thus the chains can be transformed for any domains (Proposition 4.7 and Theorem 4.8). This systematic treatment of the Komatu–Loewner equation is our main result. We further show that our result extends the previous results on the locality of chordal SKLE\(\sqrt{\pi_b} - \text{bMD}\) in a full generality, which solves an open problem in [5, Section 5]. Roughly speaking, the locality means that the distribution of SKLE\(\sqrt{\pi_b} - \text{bMD}\) is invariant modulo time-change under conformal maps. The precise statement is given in Theorem 4.9.

2. Preliminaries

First of all, let us confirm the usage of basic terms on domains and functions.

- \(\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}\) (the Riemann sphere).
- \(B(a, r) := \{z \in \mathbb{C}; |z - a| < r\}, a \in \mathbb{C}, r > 0\).
- \(\Delta(a, r) := \{z \in \mathbb{C}; |z - a| > r\}\).
- \(\mathbb{D} := B(0, 1), \mathbb{D}^* := \Delta(0, 1)\).
- \(\Pi\) denotes the mirror reflection with respect to the real axis \(\partial\mathbb{H}\).
A non-empty set \( F \subset \mathbb{H} \) is called a \((\text{compact } \mathbb{H})\)-hull if \( F \) is bounded, \( F = \mathbb{H} \cap \overline{F} \), and \( \mathbb{H} \setminus F \) is simply connected.

\[ \{ \Delta(0, r) \cup \{ \infty \}; r > 0 \} \]

is a fundamental neighborhoods system of \( \infty \) in \( \widehat{\mathbb{C}} \).

Suppose that \( D \) and \( \tilde{D} \) are domains in \( \widehat{\mathbb{C}} \). A continuous function \( f: D \to \tilde{D} \) is said to be univalent if it is holomorphic (as a continuous map between two Riemann surfaces) and injective on \( D \). If further \( f \) is surjective, then it is called a conformal map. In other words, \( f \) is conformal if and only if it is a biholomorphic map from \( D \) onto \( \tilde{D} \).

2.1. Brownian motion with darning and conformal maps on multiply connected domains

In this subsection, we summarize the properties of BMD and some of their applications to the theory of conformal maps. In particular, Proposition 2.5 will be a key estimate throughout Section 4.1.

Fix a positive integer \( N \) and a simply connected domain \( E \subset \mathbb{C} \). Let \( A_j \subset E \), \( 1 \leq j \leq N \), be mutually disjoint compact continua such that each \( E \setminus A_j \) is connected. Here, a continuum means a connected closed set consisting of more than one point. The domain \( D := E \setminus \bigcup_j A_j \) is then \( N \)-connected. We “darn” each hole \( A_j \) as follows: Regarding each \( A_j \) as one point \( a^*_j \), we define the quotient topological space \( D^* \) by \( D^* := D \cup \{ a^*_1, \ldots, a^*_N \} \). BMD \((Z^*_t, \mathbb{P}^*_z)\) is defined on \( D^* \) by \([4, \text{Definition 2.1}]\). The harmonicity for BMD is then defined by \([4, (3.2)]\). The next proposition shows that the BMD-harmonicity is a stronger condition than the usual harmonicity for the absorbing Brownian motion (ABM) on \( D \):

**Proposition 2.1** ([2] and [4, Section 3.3]). The following are equivalent for a continuous function \( u: D^* \to \mathbb{R} \):

(i) \( u \) is BMD-harmonic on \( D^* \).

(ii) There is a holomorphic function \( f \) on \( D \) whose real or imaginary part is \( u \);

In particular, a function \( u \) on \( D \) satisfying Condition (ii) extends to a BMD-harmonic function on \( D^* \) if it takes a constant limit value on each \( \partial A_j \), \( 1 \leq j \leq N \).

We define the Green function and Poisson kernel of BMD, like those for ABM. Let \( A_0 \) be a hull with piecewise smooth boundary or an empty set, \( E = \mathbb{H} \setminus A_0 \) and \( D \) be as above. We denote by \( G^*_D \) the 0-order resolvent...
kernel, or Green function of $Z^*$. Taking the normal derivative, we get the Poisson kernel of $Z^*$

$$K^*_D(z, \xi_0) := -\frac{1}{2} \frac{\partial}{\partial n_{\xi_0}} G^*_D(z, \xi_0),$$

where $n_{\xi_0}$ is the outward unit normal at $\xi_0 \in \partial E$. The kernels $G^*_D$ and $K^*_D$ can be expressed by the classical Green function and harmonic measures. See Sections 4 and 5 in [4] for their concrete expressions. The following version of Poisson’s integral formula holds for the kernel $K^*_D$:

Suppose that a BMD-harmonic function $u$ on $D^*$ vanishes at infinity, extends continuously to $\partial D^* = \partial E$ and has a compact support on $\partial E$. Then by [4, (5.5)] and the proof of [4, Theorem 6.4], $u$ satisfies

$$u(z) = \mathbb{E}^*[u(Z^*_{\zeta^*})] = \int_{\partial E} u(\xi_0) K^*_D(z, \xi_0) |d\xi_0|, \quad (2.1)$$

where $\zeta^*$ is the lifetime of $Z^*$. Note that the former equality holds by the maximum value principle for BMD-harmonic functions on $D^*$ even if $\partial A_0$ is not smooth.

When $A_j$ is a horizontal slit $C_j$ for each $1 \leq j \leq N$, we can further define the complex Poisson kernel $\Psi_D$ of $Z^*$ by [4, Lemma 6.1]. Namely, there is a unique holomorphic function $\Psi_D(z, \xi_0), \xi_0 \in \partial E$, such that $\Im \Psi_D(z, \xi_0) = K^*_D(z, \xi_0)$ and $\lim_{z \to \infty} \Re \Psi_D(z, \xi_0) = 0$. This $\Psi_D$ coincides with $\Psi$ in [1, Section 2.2] by construction. If $D = \mathbb{H}$, namely, no slit is present in $D$, then the BMD is reduced to the ABM on $\mathbb{H}$ and its complex Poisson kernel $\Psi_{\mathbb{H}}$ becomes $\frac{1}{\pi} \frac{1}{z - \xi_0}$ whose imaginary part is the usual Poisson kernel $\frac{1}{\pi z - \xi_0}$ on $\mathbb{H}$. We consider the difference between $\Psi_D$ and $\Psi_{\mathbb{H}}$ provided that $E = \mathbb{H}$.

In view of the proof of [3, Lemma 5.6], the function

$$\mathbb{H}_D(z, \xi_0) := \Psi_D(z, \xi_0) + \frac{1}{\pi} \frac{1}{z - \xi_0}, \quad z \in D, \xi_0 \in \partial \mathbb{H}, \quad (2.2)$$

can be extended, for each $\xi_0 \in \partial \mathbb{H}$, to a holomorphic function in $z \in D \cup \Pi D \cup \partial \mathbb{H}$ after making Schwarz’s reflection across $\partial \mathbb{H} \setminus \{\xi_0\}$. The extended function is denoted by $\mathbb{H}_D(z, \xi_0)$ again. Accordingly, $\Psi_D(\cdot, \xi_0)$ extends to a conformal map from $D \cup \Pi D \cup \partial \mathbb{H} \cup \{\infty\}$ onto $\widehat{D} \cup \Pi \widehat{D} \cup \partial \mathbb{H} \cup \{\infty\}$.

**Remark 2.2.** Concerning (2.1) and the definition of $\Psi_D$, Lemma 6.1 and Theorem 6.4 of [4] dealt with only the case where $E = \mathbb{H}$. However, we can easily check that the proof is still valid for $E = \mathbb{H} \setminus A_0$. Indeed, the BMD complex Poisson kernel for $E = \mathbb{H} \setminus \{z; |z| \leq \varepsilon\}$ appeared in Appendix of [3].
As an application of BMD to the theory of conformal maps, [4, Theorem 7.2] constructed the canonical conformal map for a hull in $D$ in a probabilistic manner, which was originally due to [13, Corollary 3.1]. We restate [4, Theorem 7.2] in the next proposition.

**Proposition 2.3.** (i) Let $E = \mathbb{H}$ and $D$ be as above. Suppose that $F$ is a hull contained in $D$ or an empty set. Then, there exists a unique pair of a standard slit domain $\tilde{D}$ and conformal map $f_F : D \setminus F \to \tilde{D}$ with the hydrodynamic normalization $\lim_{z \to \infty} (f_F(z) - z) = 0$.  

(ii) The map $f_F$ in (i) can be extended to a univalent function on $(D \setminus F) \cup \Pi(D \setminus F) \cup (\partial \mathbb{H} \setminus F) \cup \{\infty\}$. This extended map is denoted by $f_F$ again and has the following expansion around $\infty$:

$$f_F(z) = z + \frac{c}{z} + o(z^{-1}), \quad (2.3)$$

where $c$ is a constant which is positive if $F$ is non-empty.

In Proposition 2.3, we refer to $f_F$ as the canonical map from $D \setminus F$ onto $\tilde{D}$. The constant $c$ in Proposition 2.3 (ii) is called the half-plane capacity of $F$ relative to $D$ and denoted by $\text{hcap}_D(F)$. Now a reader familiar with the boundary behavior of conformal maps can skip the following proof and Remark 2.4 which are somewhat lengthy due to the exposition on prime ends.

**Proof of Proposition 2.3.** (i) The existence of such a pair of standard slit domain $\tilde{D}$ and conformal map $f_F$ is ensured by [4, Theorem 7.2] or [1, Section 2.2]. We prove the uniqueness by the same proof as in [19, Theorem IX.23], starting with the summary on the boundary correspondence induced by conformal maps.

Let $D_0$ be a finitely connected domain.

Figure 2: Canonical map $f_F$
A simple curve \( q \) in \( \overline{D_0} \) is called a \textit{cross cut} if both of its end points lie in a single component of \( \partial D_0 \), and the other points of \( q \) lie in \( D_0 \). A cross cut \( q \) obviously separates the domain \( D_0 \) into two components, that is, \( D_0 \setminus q \) consists of two connected components.

A sequence \( \{q_n\} \) of cross cuts is called a \textit{null-chain} if all \( q_n \) are disjoint, there is a component of \( D_0 \setminus q_n \) denoted by \( \text{ins} q_n \) such that \( \text{ins} q_{n+1} \subseteq \text{ins} q_n \) for all \( n \), and \( \text{diam} q_n \to 0 \) as \( n \to \infty \).

Two null-chains \( \{q_n\} \) and \( \{q'_n\} \) is said to be \textit{equivalent} if, for every \( m \), there exists a number \( n \) such that \( \text{ins} q_n' \subseteq \text{ins} q_m \), and the same condition with \( q_n \) and \( q'_n \) exchanged holds. We call a equivalence class by this relation a \textit{prime end} of \( D_0 \).

\( P(D_0) \) denotes the collection of all prime ends of \( D_0 \).

We endow a topology on \( D_0 \cup P(D_0) \) as follows: A subset \( U \subset D_0 \cup P(D_0) \) is open if \( U \cap D_0 \) is open, and for every prime end \( p \in U \cap P(D_0) \), there exists a null-chain \( \{q_n\} \in p \) such that \( \text{ins} q_n \subset U \cap D_0 \) for some \( n \). Then by definition, a sequence \( \{z_m\} \) in \( D_0 \) converges to a prime end \( p \) if and only if, for some null-chain \( \{q_n\} \in p \) and each \( n \), it holds that \( z_m \in \text{ins} q_n \) for sufficiently large \( m \).

For the standard slit domain \( \tilde{D} \), the collection of prime ends \( P(\tilde{D}) \) has a simple expression. Let \( \tilde{C}_j, j = 1, \ldots, N \), be the slits of \( \tilde{D} \) whose left and right end points are \( \tilde{z}_j \) and \( \tilde{z}_j' \), respectively. We use \( \partial_p A, A \subset \mathbb{C} \), to denote the boundary of \( A \) with respect to the path distance topology in \( \mathbb{C} \setminus A \). Then \( \partial_p \tilde{C}_j = \tilde{C}_j^+ \cup \tilde{C}_j^- \cup \{\tilde{z}_j, \tilde{z}_j'\} \), where \( \tilde{C}_j^+ \) are the upper and lower side of the open slit \( \tilde{C}_j^0 := \tilde{C}_j \setminus \{\tilde{z}_j, \tilde{z}_j'\} \), respectively. \( P(\tilde{D}) \) coincides with the boundary of \( \tilde{D} \) in the path distance topology: \( P(\tilde{D}) = \partial \mathbb{H} \cup \bigcup_j \partial_p \tilde{C}_j \).

It is well known as Carathéodory’s theorem that, a conformal map between two finitely connected domains \( D_0 \) and \( D_1 \) extends to a homeomorphism between \( D_0 \cup P(D_0) \) and \( D_1 \cup P(D_1) \). (See [19, Theorem IX.1] or [6, Theorem 14.3.4].) Although \( D_0 \) and \( D_1 \) are originally supposed to be simply connected in Carathéodory’s theorem, we can easily prove this fact even if the domains have finitely multiple connectivity, for instance, via the proof of [6, Theorem 15.3.4]. In our case, the conformal map \( f_F : D \setminus F \to \tilde{D} \) induces a homeomorphism from \( (D \setminus F) \cup P(D \setminus F) \) onto \( \tilde{D} \cup \partial \mathbb{H} \cup \bigcup_j \partial_p \tilde{C}_j \).

Keeping this boundary correspondence in mind, we proceed to the uniqueness of the pair \( (\tilde{D}, f_F) \). To the contrary, we assume that a pair of a standard
slit domain $\tilde{D}_s$ and conformal map $f_s : D \setminus F \to \tilde{D}_s$ distinct from the pair $(\tilde{D}, f_F)$ enjoys the same property. The difference $g(z) := f_F(z) - f_s(z)$ is non-constant, holomorphic on $D \setminus F$ and especially bounded due to the hydrodynamic normalization. By the above correspondence, the boundary of the image $g(D)$ is written as

$$\partial g(D) = \{ f_F(z) - f_s(z) ; z \in P(D \setminus F) \},$$

which consists of finitely many parallel slits and a subset of $\partial \mathbb{H}$. It is however impossible that such a form of boundary surrounds a bounded domain $g(D)$, a contradiction. Thus the uniqueness of the map $f_F$ follows.

(ii) It is obvious from definition that each point in $\partial \mathbb{H} \setminus F$ corresponds to a prime end in $P(D \setminus F)$. Thus by the boundary correspondence we have

$$\lim_{z \to \zeta_0, z \in D \setminus F} \Im f_F(z) = 0, \quad \zeta_0 \in \partial \mathbb{H} \setminus F.$$

The extension of $f_F$ across $\partial \mathbb{H} \setminus F$ is now obtained from Schwarz’s reflection principle. The hydrodynamic normalization implies that $f_F$ has the expansion (2.3). Finally by [3, (A.20)] we have

$$c = 2\frac{R}{\pi} \int_0^\pi \mathbb{E}^*_R e^{i\theta} \left[ \Im Z^*_t; \sigma_F < \infty \right] \sin \theta d\theta \quad (2.4)$$

for any $R > \sup \{ |z| ; z \in F \cup \bigcup_j A_j \}$. Here $\sigma_F := \inf \{ t > 0 ; Z^*_t \in F \}$, and $\mathbb{E}^*_R$ denotes the expectation with respect to $Z^*$ starting at $z \in D^*$. Although [3, (A.20)] was shown when $D$ is a standard slit domain, it is also valid for general $D$ as remarked immediately after [3, (A.21)]. The expression (2.4) implies $c > 0$ for a non-empty $F$ since $F$ is non-polar with respect to the ABM on $D$.

Remark 2.4. If the closure of the hull $F$ is connected, which is the case when $F$ is a trace of a simple curve, then Proposition 2.3 follows from Theorems IX.22 and IX.23 of [19] as follows: Let $D' := (D \setminus F) \cup \Pi(D \setminus F) \cup (\partial \mathbb{H} \setminus F)$. There exists a pair of a parallel slit plane $\tilde{D}'$ and a conformal map $f : D' \to \tilde{D}'$ with the normalization

$$f(z) = z + \frac{c}{z} + o(z^{-1}), \quad z \to \infty,$$

for some $c \in \mathbb{C}$ with $\Re c > 0$ by [19, Theorem IX.22]. Clearly $\hat{f}(z) := \overline{f(\overline{z})}$ satisfies the same condition with $c$ replaced by $\overline{c}$, and so $f = \hat{f}$ by [19].

10
Theorem IX.23] since $D'$ is of finite connectivity. Thus $c = \bar{c} > 0$, and $f_F$ is obtained from the restriction of $f$ on $D \setminus F$.

The crucial point is the finite connectivity of $D'$, which is not necessarily true when $\mathcal{F}$ is not connected. Because the uniqueness theorem [19, Theorem IX.23] does not work for the domain of infinite connectivity, we cannot conclude that $f = \hat{f}$ in the above argument unless the connectivity of $\mathcal{F}$ is assumed. In relation with this remark, we would like to point out that, in the proof of [4, Theorem 7.2], the image of the hull $F$ by the canonical map $\phi: \mathbb{H} \setminus F \to \mathbb{H}$ is stated to be a interval, which is not the case in general. Needless to say, the proof itself is completely valid since [4, Theorem 11.2] used there does not depend on the degree of connectivity of the domains at issue.

In the rest of this subsection, $D$ is a standard slit domain $\mathbb{H} \setminus \bigcup_{j=1}^{N} C_j$. We denote by $C_j^\pm$ the upper and lower side of the slit $C_j^0 := C_j \setminus \{z_j, z'_j\}$, respectively, where $z_j$ and $z'_j$ are the left and right end points of $C_j$, respectively. We set $\partial_p C_j := C_j^+ \cup C_j^- \cup \{z_j, z'_j\}$, which is the boundary of $\mathbb{H} \setminus C_j$ in the path distance topology, as in the proof of Proposition 2.3.

Given the canonical map $f_F$ for a hull $F \subset D$, we can always extend it holomorphically to $\bigcup_j \partial_p C_j$ in the following sense as in [3, Section 2], which will be used extensively throughout this paper. Fix $1 \leq j \leq N$ and consider the open rectangles

$$R_+ := \{z \in \mathbb{C}; x_j < \Re z < x'_j, y_j < \Im z < y_j + \delta\},$$

$$R_- := \{z \in \mathbb{C}; x_j < \Re z < x'_j, y_j - \delta < \Im z < y_j\},$$

and $R := R_+ \cup C_j^0 \cup R_-$, where $\delta > 0$ is taken so small that $R_+ \cup R_- \subset D \setminus F$. Since $\Im f_F$ takes a constant value on the slit $C_j$ by the boundary correspondence, $f_F$ extends to a holomorphic function $f_F^+$ from $R_+$ to $R$ across $C_j^0$ by Schwarz’s reflection. The extension $f_F^-$ of $f_F|_{R_-}$ across $C_j^0$ is defined in the same way. As for the extension of $f_F$ on the left end point $z_j$, we take $\varepsilon > 0$ so small that it is less than one-half of the length of $C_j$ and that $B(z_j, \varepsilon) \setminus C_j \subset D \setminus F$. Then $\psi(z) := (z - z_j)^{1/2}$ maps $B(z_j, \varepsilon) \setminus C_j$ conformally onto $B(0, \sqrt{\varepsilon}) \cap \mathbb{H}$, and $f_F^+ := f_F \circ \psi^{-1}$ extends holomorphically to $B(0, \sqrt{\varepsilon})$ by Schwarz’s reflection. We can also construct the holomorphic function $f_F^-$, the extension of $f_F$ on the right end point $z'_j$. Note that, by the proof of [4, Lemma 6.1], the BMD complex Poisson kernel $\Psi_D(z, \zeta_0)$ extends holomorphically to $\bigcup_j \partial_p C_j$ for each $\zeta_0 \in \partial \mathbb{H}$ in the same manner.
The canonical map \( f_F \) so extended has the following important estimate, which was originally formulated in [7, Proposition 6.12] in terms of ERBM:

**Proposition 2.5.** Let \( D \) be a standard slit domain. Suppose that \( \xi_0 \in \partial \mathbb{H} \) and that \( r_0 > 0 \) satisfies \( B(\xi_0, r_0) \cap \mathbb{H} \subseteq D \). Then for any hull \( F \subseteq D \) with \( r := \inf \{ R > 0; F \subseteq B(\xi_0, R) \} \leq r_0 \), it holds that, for all \( z \in D \cup \bigcup_j \partial \sigma C_j \) with \( |z - \xi_0| > r \),

\[
|z - f_F(z) - \pi h\text{cap}^D(F)\Psi_D(z, \xi_0)| \leq C(z) r h\text{cap}^D(F).
\]

(2.5)

Here \( C(z) = C_{D, \xi_0, r_0}(z) > 0 \) is a locally bounded function depending only on \( D, \xi_0 \) and \( r_0 \).

Proposition 2.5 is a generalization of [14, Lemma 2.7] and [12, Proposition 3.46] for the upper half-plane toward the standard slit domain. Drenning [7] used it to obtain the Komatu–Loewner equation for a simple curve in the right derivative sense. He then discussed the left differentiability by some probabilistic methods based on the fact that the hull at issue was a simple curve. In Section 4, we also establish the right differentiability by Proposition 2.5 as he did, but the subsequent argument is completely different. We employ the kernel convergence condition instead of his methods to examine the left differentiability for a family of “continuously” growing hulls.

In what follows, we give a complete proof of Proposition 2.5 by making use of BMD instead of ERBM. We first quote some estimates on BMD from Appendix of [3]. Let \( D, r_0, F \) and \( r \) be as in the assumption of Proposition 2.5. By horizontal translation, we may and do assume \( \xi_0 = 0 \) without loss of generality. Let \( D_{\varepsilon} := D \setminus B(0, \varepsilon) \). By [3, Proposition A.2], there is a function \( c(z, \theta) \) uniformly bounded in \( |z| > r_0 \) and \( 0 \leq \theta \leq \pi \) such that

\[
K_{D_{\varepsilon}}^*(z, \varepsilon e^{i\theta}) = 2K_D^*(z, 0)(1 + c(z, \theta)\varepsilon) \sin \theta
\]

(2.6)

for \( |z| > r_0, \ 0 < \varepsilon < r_0 \) and \( 0 \leq \theta \leq \pi \). Clearly (2.6) still holds for \( z \in \bigcup_j \partial \sigma C_j \) since \( K_D^*(z, \xi_0) \) extends harmonically as the imaginary part of \( \Psi_D(z, \xi_0) \). By [3, (A.22) and (A.23)],

\[
\text{hcap}^D(F) = \frac{2r}{\pi} \left( 1 + c'(z, \theta)r \right) \int_0^\pi \mathbb{E}_{r\varepsilon, \theta}^* \left[ \Im Z_{\sigma_F^*}^* ; \sigma_F^* < \infty \right] \sin \theta \, d\theta,
\]

(2.7)

where \( c'(z, \theta) \) is a uniformly bounded function in \( z \) and \( \theta \).

Though it is irrelevant to BMD and rather standard, we remark the following:
Lemma 2.6 (cf. [12, Exercise 2.17]). Let $n \in \mathbb{N}$ and $u$ be a bounded harmonic function on a domain $V$. Then, every derivative of $u$ of order $n$ is bounded by $c(n) \text{dist}(z, \partial V)^{-n} \|u\|_{\infty}$ for some constant $c(n)$.

Proof of Proposition 2.5. Let

$$h(z) := z - f_F(z) - \pi \text{hcap}^D(F)\Psi_D(z, 0),$$

$$v(z) := \Im h(z) = \Im (z - f_F(z)) - \pi \text{hcap}^D(F)K_D^*(z, 0).$$

Just as in the proof of [3, Theorem A.1], we have

$$\Im (z - f_F(z)) = E_{z}^{*}\left[\Im Z^{*}_{\sigma_{F}^{*}}; \sigma_{F}^{*} < \infty\right].$$

Denote the right hand side by $v_0(z)$. From the strong Markov property of $Z^{*}$, (2.1), (2.6) and (2.7), we obtain, for $z \in D \cup \bigcup_{j} \partial D_j$ with $|z| > r$,

$$\Im (z - f_F(z)) = E_{z}^{*}\left[\Im v_0(Z^{*}_{\sigma_{F}^{*}}; \sigma_{F}^{*} < \infty)\right].$$

Denote the right hand side by $v_0(z)$. From the strong Markov property of $Z^{*}$, (2.1), (2.6) and (2.7), we obtain, for $z \in D \cup \bigcup_{j} \partial D_j$ with $|z| > r$,

$$\Im (z - f_F(z)) = E_{z}^{*}\left[\Im v_0(Z^{*}_{\sigma_{F}^{*}}; \sigma_{F}^{*} < \infty)\right].$$

Hence for some $M_1$,

$$|v(z)| \leq rM_1 \text{hcap}^D(F)K_D^*(z, 0), \quad z \in D \cup \bigcup_{j} \partial D_j. \quad (2.8)$$

We now fix $L > r_0$ such that $\bigcup_{j} C_j \subset B(0, L)$. Let $\Gamma_z$ be a curve from $iL$ to $z$ in $D \setminus B(0, r)$. Then, $h(z)$ is given by

$$h(z) = h(iL) + \int_{\Gamma_z} h'(w)\,dw, \quad z \in D \cup \bigcup_{j} \partial D_j. \quad (2.9)$$

Further by (2.8), Lemma 2.6 and the Cauchy–Riemann equation, we have

$$|h'(w)| \leq \frac{M_2 \sup_{z' \in N_{\Gamma_z}} K_D^*(z', 0)}{\text{dist}(\Gamma_z, \partial D)} r \text{hcap}^D(F), \quad w \in \Gamma_z, \quad (2.10)$$
for some constant $M_2$. $\mathcal{N}_{\Gamma_z}$ is an appropriate neighborhood of $\Gamma_z$. We describe how to choose it later. Combining (2.10) with (2.9) yields that
\[
|h(z)| \leq |h(iL)| + \frac{M_2|\Gamma_z|\sup_{z' \in \mathcal{N}_{\Gamma_z}} K^*_D(z', 0)}{\text{dist}(\Gamma_z, \partial D)} r \text{hcap}^D(F),
\] (2.11)
where $|\Gamma_z|$ denotes the length of $\Gamma_z$.

It remains to estimate $|h(iL)|$. By (A.21) and (A.23) of [3],
\[
K^*_D(z, 0) = \frac{1}{\pi} \frac{\Im z}{|z|^2} + O\left(\frac{1}{|z|^2}\right),
\]
so that
\[
K^*_D(z, 0) \leq \frac{M_3}{|z|}, \quad |z| > L.
\] (2.12)

Since $v$ is harmonic on $B_y := \{z \in \mathbb{C}; |z - iy| < y/2\}(\subset D)$ for $y > 2L$, it follows from Lemma 2.6, (2.8) and (2.12) that
\[
|v_x(iy)| \leq \frac{2c(1)}{y} \sup_{z \in B_y} |v(z)| \leq 4c(1) M_1 M_3 r \text{hcap}^D(F) \cdot \frac{1}{y^2}, \quad y > 2L.
\]

Now note that, for $u := \Re h$, $\lim_{z \to \infty} u(z) = 0$ by the properties of $f_F$ and $\Psi_D$. Consequently by the Cauchy-Riemann equation we have
\[
|u(iL)| \leq \int_{-L}^{\infty} |v_x(iy)| dy \leq M_4 r \text{hcap}^D(F).
\] (2.13)

Here $M_3$ and $M_4$ are constants. We finally set
\[
C(z) = C_{D, 0, r_0}(z) := \frac{M_2|\Gamma_z|\sup_{z' \in \mathcal{N}_{\Gamma_z}} K^*_D(z', 0)}{\text{dist}(\Gamma_z, \partial D)} + M_4 + M_1 K^*_D(iL, 0).
\] (2.14)

By choosing appropriate $\Gamma_z$ and $\mathcal{N}_{\Gamma_z}$, we can take $C$ as a locally bounded function independent of $F$ and $r$. Thus, (2.11), (2.13) and (2.8) lead us to the desired conclusion. \hfill \Box

Note that (2.5) still holds with $f_F$ replaced by the extended map $f_F^+$ or other extensions, since one may define $C(z)$ by taking some appropriate reflection.
2.2. Initial value problem for the Komatu–Loewner equation

In this subsection, we describe how one obtains a family of growing hulls from the initial value problem for the Komatu–Loewner equation.

Fix $N \in \mathbb{N}$ and let $C_j \subset \mathbb{H}$, $1 \leq j \leq N$ be mutually disjoint horizontal slits. We denote the left and right endpoints of the $j$-th slit $C_j$ by $z_j = x_j + iy_j$ and $z_j^r = x_j^r + iy_j$, respectively. Then, the $N$-tuple $(C_j; 1 \leq j \leq N)$ of the slits are identified with an element $s = (y_1, \ldots, y_N, x_1, \ldots, x_N, x_1^r, \ldots, x_N^r)$ in $\mathbb{R}^{3N}$. We define the open subset $\text{Slit}$ of $\mathbb{R}^{3N}$ consisting of all such elements by

$$\text{Slit} := \{s = (y_1, \ldots, y_N, x_1, \ldots, x_N, x_1^r, \ldots, x_N^r) \in \mathbb{R}^{3N}; y_j > 0, x_j < x_j^r, \text{ either } x_j < x_k^r \text{ or } x_k < x_j^r \text{ whenever } y_j = y_k, j \neq k\}.$$.

We denote by $C_j(s)$ (resp. $D(s)$) the $j$-th slit (resp. the standard slit domain) corresponding to $s \in \text{Slit}$. $\Psi_s := \Psi_{D(s)}$ is the BMD complex Poisson kernel of $D(s)$.

For $\xi_0 \in \mathbb{R}$ and $s \in \text{Slit}$, we put

$$b_l(\xi_0, s) := \begin{cases} -2\pi \Im \Psi_s(z_l, \xi_0), & 1 \leq l \leq N, \\ -2\pi \Re \Psi_s(z_{l-N}, \xi_0), & N + 1 \leq l \leq 2N, \\ -2\pi \Re \Psi_s(z_{l-2N}, \xi_0), & 2N + 1 \leq l \leq 3N, \end{cases}$$

where $z_j$ and $z_j^r$, $1 \leq j \leq N$, are the left and right endpoints of the $j$-th slit $C_j(s)$, respectively. The function $b_l$, $1 \leq l \leq 3N$, has an invariance under horizontal translations, that is,

$$b_l(\xi_0, s) = b_l(0, s - \hat{\xi}_0),$$

where $\hat{\xi}_0$ denotes the vector in $\mathbb{R}^{3N}$ whose first $N$ entries are zero and last $2N$ entries are $\xi_0$. (3) called this property the homogeneity in $x$-direction.) We can easily check this invariance since $\Psi_s(z, \xi_0) = \Psi_{s-\hat{\xi}_0}(z - \xi_0, 0)$.

The Komatu–Loewner equation for slits (1.3) is now written as

$$\frac{d}{dt}s_l(t) = b_l(\xi(t), s(t)), \quad 1 \leq l \leq 3N,$$

where $s_l(t)$ is the $l$-th entry of $s(t)$. Since $b_l$ is locally Lipschitz on $\mathbb{R} \times \text{Slit}$ for each $l$ by (3) Lemma 4.1, (2.15) is solved up to its explosion time $\zeta$. Here we note that Condition (L) on a function $f : \text{Slit} \to \mathbb{R}$ appearing in (3) Lemma 4.1 is equivalent to each of the following conditions:
• the local Lipschitz continuity of \( f(s) \) in \( s \in \text{Slit} \),

• the local Lipschitz continuity of \( f(s - \hat{\xi}_0) \) in \( (\xi_0, s) \in \mathbb{R} \times \text{Slit} \).

Therefore we simply say that \( f \) is locally Lipschitz if one of these conditions holds.

In this context, we introduce a few more notations. For a function \( f : \text{Slit} \rightarrow \mathbb{C} \), we denote \( f(s - \hat{\xi}_0) \) by \( f(\xi_0, s) \) and regard it as a function on \( \mathbb{R} \times \text{Slit} \) with the invariance under horizontal translations. Conversely, for a function \( \tilde{f} : \mathbb{R} \times \text{Slit} \rightarrow \mathbb{C} \) with the invariance \( \tilde{f}(\xi_0, s) = \tilde{f}(0, s - \hat{\xi}_0) \), we denote \( \tilde{f}(0, s) \) by \( \tilde{f}(s) \) and regard it as a function on \( \text{Slit} \).

Returning to the initial value problem, we set \( D_t := D(s(t)) \) for the solution \( s(t), 0 \leq t < \zeta \), to (2.15). The Komatu–Loewner equation (1.2) is written as

\[
\frac{d}{dt}g_t(z) = -2\pi \Psi_{s(t)}(g_t(z), \xi(t)), \quad g_0(z) = z \in D(:= D(s)).
\]

(2.16) has a unique solution \( g_t(z) \) up to \( t = \zeta \wedge \sup \{ t; |g_t(z) - \xi(t)| > 0 \} \) by Theorem 5.5 (i) of [3]. By Theorems 5.5, 5.8 and 5.12 of [3], \( g_t \) is the canonical map from \( D \setminus F_t \) onto \( D_t \) where \( F_t := \{ z \in D; t_z \leq t \}, t \in [0, \zeta) \), \( \{F_t\} \) is a family of growing (i.e. strictly increasing) hulls satisfying

\[
\bigcap_{\delta > 0} g_t(F_{t+\delta} \setminus F_t) = \{ \xi(t) \}
\]

for all \( t < \zeta \), and further \( \text{hcap}^D(F_t) = 2t \). The family \( \{F_t\}, \{g_t\} \) or \( (g_t, F_t) \) here is called the \textit{Komatu–Loewner evolution driven by }\( \xi \). In the present paper, we also refer to \( \{g_t\} \) as the \textit{Komatu–Loewner chain}.

In the same manner, we introduce the stochastic Komatu–Loewner evolution (SKLE) as we defined SLE in Section 1. We say that a function \( f : \text{Slit} \rightarrow \mathbb{R} \) is \textit{homogeneous with degree }\( a \in \mathbb{R} \) if, for any \( c > 0 \),

\[
f(cs) = c^a f(s), \quad s \in \text{Slit}.
\]

Take two functions \( \alpha(s) \) and \( b(s) \) homogeneous with degree 0 and \(-1\), respectively, and suppose that both of them satisfy the local Lipschitz condition. We consider the following SDEs:

\[
\xi(t) = \xi + \int_0^t \alpha(\xi(s), s(s)) \, dB_s + \int_0^t b(\xi(s), s(s)) \, ds, \quad (2.18)
\]

\[
s_l(t) = s_l + \int_0^t b_l(\xi(s), s(s)) \, ds, \quad 1 \leq l \leq 3N, \quad (2.19)
\]

16
where $B_t$ is the one-dimensional standard BM. The second equation (2.19) is the same as (2.15), though we regard it as a part of the system of SDEs instead of a single ODE. By the local Lipschitz condition, this system has a unique strong solution up to its explosion time $\zeta$ (Theorem 4.2). The above-mentioned properties also holds for this solution $(\xi(t), s(t))$. We designate the resulting random evolution $\{F_t\}$ as SKLE$_{a,b}$.

3. Convergence of a sequence of univalent functions

In this section, a version of Carathéodory’s kernel theorem is formulated, which is later used to discuss the continuity of growing hulls. Our discussion seems almost the same as in Chapter V, Section 5 of [9], but we need some modifications, because Goluzin [9] treated domains containing $\infty$ (in their interior) while we deal with domains in $\mathbb{H}$, which does not contain $\infty$. Therefore we provide a detailed description below for the sake of completeness. The following two facts are fundamental to our argument:

**Proposition 3.1** (Montel). A family $\mathcal{H}$ of holomorphic functions on a domain $D \subset \mathbb{C}$ is equicontinuous uniformly on every compact subset of $D$ if it is locally bounded. In this case $\mathcal{H}$ is a normal family on $D$, i.e., relatively compact in the topology of locally uniform convergence.

**Proposition 3.2.** If a sequence $\{f_n\}$ of univalent functions on a domain $D$ converges to a non-constant function $f$ uniformly on every compact subset of $D$, then $f$ is also univalent on $D$.

In addition, the following two classes of univalent functions are significant: First, we define the set $S$ as the totality of univalent functions $f: \mathbb{D} \to \mathbb{C}$ satisfying $f(0) = 0$ and $f'(0) = 1$. In other words, a univalent function $f: \mathbb{D} \to \mathbb{C}$ belongs to $S$ if and only if $f(z)$ has the following power series expansion around the origin:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (3.1)$$

Next, we define the set $\Sigma$ as the totality of univalent functions $f: \mathbb{D}^* \to \mathbb{C}$ satisfying $f(\infty) = \infty$ and $\text{Res}(f, \infty) = 1$. In other words, a univalent function $f: \mathbb{D}^* \to \mathbb{C}$ belongs to $\Sigma$ if and only if $f(z)$ has the following Laurent series expansion around $\infty$:

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}. \quad (3.2)$$
Proposition 3.3 (Area theorem, Gronwall). Suppose \( f \in \Sigma \) with Laurent series expansion \((3.2)\). Then it holds that \( \sum_{n=1}^\infty n|b_n|^2 \leq 1 \).

Proof. See for instance [9, Theorem II.4.1] or Inequality (5) in [16, Section 1.2]. \( \square \)

Proposition 3.4 (Bieberbach). Suppose \( f \in S \) with power series expansion \((3.1)\). Then it holds that \( |a_2| \leq 2 \).

Proof. See Chapter II, Section 4 of [9] or [16, Theorem 1.5]. \( \square \)

Lemma 3.5 ([9, Theorem II.4.3 and Lemma V.2.2]). Suppose \( f \in \Sigma \) with Laurent series expansion \((3.2)\). Then \( \mathbb{C} \setminus f(\mathbb{D}^*) \subset B(b_0,2) \) and \( |f(z) - b_0| \leq 2|z| \) for \( z \in \mathbb{D}^* \).

Proof. For any \( c \in \mathbb{C} \setminus f(\mathbb{D}^*) \), the function

\[
    f_c(z) := \frac{1}{f(1/z) - c} = z + (c - b_0)z^2 + \cdots
\]

belongs to \( S \), and by Bieberbach’s theorem \(3.4\) we have \( |c - b_0| \leq 2 \). Hence the former part of the lemma follows.

To prove the latter, we consider the function

\[
    F_w(z) := \frac{1}{w}f(wz) - \frac{b_0}{w} = z + \frac{b_1}{w^2z} + \cdots
\]

for \( w \in \mathbb{D}^* \). Since \( F_w \in \Sigma \), we have \( \partial F_w(\mathbb{D}^*) \subset B(0,2) \) by the former part of the lemma. In particular \( |F_w(1)| \leq 2 \), that is, \( |f(w) - b_0| \leq 2|w| \). \( \square \)

The following corollary easily follows from Lemma 3.5

Corollary 3.6. Suppose that \( D \subset \mathbb{C} \) is a domain containing \( \Delta(0,r) \) for some \( r > 0 \) and that \( \mathcal{H} = \{f_\lambda; \lambda \in \Lambda\} \) is a family of univalent functions on \( D \) with Laurent series expansion

\[
    f_\lambda(z) = z + b_0^{(\lambda)} + \sum_{n=1}^\infty b_n^{(\lambda)}z^{-n}
\]

around \( \infty \). Then \( \mathcal{H} \) is locally bounded on \( D \) if and only if \( \{b_0^{(\lambda)}; \lambda \in \Lambda\} \) is bounded.
We now turn to the definition of kernel, a key notion throughout our discussion in Section 4. To clarify the role of each hypothesis in the kernel theorem 3.8 below, we mention our hypotheses in a fashion slightly more abstract than we need in this paper. Let \( \{D_n; n \in \mathbb{N}\} \) be a sequence of domains in \( \mathbb{H} \). We assume that

(K.1) there exists a constant \( L > 0 \) such that \( \Delta(0, L) \cap \mathbb{H} \subset D_n \) for all \( n \).

**Definition 3.7.** Under Assumption (K.1), the kernel of \( \{D_n\} \) is defined as the largest unbounded domain \( D \) such that each compact subset \( K \subset D \) is included by \( \bigcap_{n \geq n_K} D_n \) for some \( n_K \in \mathbb{N} \). If every subsequence of \( \{D_n\} \) has the same kernel, then we say that \( \{D_n\} \) converges to \( D \) in the sense of kernel convergence and denote it simply by \( D_n \to D \).

In other words, the kernel \( D \) is an unbounded connected component of the set of all points \( z \) such that \( B(z, r_z) \subset D_n \), \( n \geq n_z \), for some \( r_z > 0 \) and \( n_z \in \mathbb{N} \). By Assumption (K.1), \( D \) always exists, is unique and contains \( \Delta(0, L) \cap \mathbb{H} \).

Let \( D \) be the kernel of \( \{D_n\} \) and \( f \) and \( f_n \), \( n \in \mathbb{N} \), be functions on \( D \) and \( D_n \), respectively. If \( \{f_n\} \) converges to \( f \) uniformly on each compact subset \( K \) of \( D \), then we say as usual that \( \{f_n\} \) converges to \( f \) uniformly on compacta and denote it by \( f_n \to f \) u.c. on \( D \). This convergence makes sense since \( K \) is included by \( D_n \) for sufficiently large \( n \). In what follows, we assume that each \( f_n : D_n \to \mathbb{C} \) is univalent and enjoys the following two conditions:

(K.2) \( \lim_{z \to \infty} (f_n(z) - z) = 0 \);
(K.3) \( \lim_{z \to \xi_0, z \in D_n} f_n(z) = 0 \) for all \( \xi_0 \in \partial \mathbb{H} \cap \Delta(0, L) \);

where \( L \) is the constant in Assumption (K.1). Note that, as is easily seen, Propositions 3.1 and 3.2 and Corollary 3.6 hold even for the moving domains \( D_n \). By Schwarz’s reflection, Assumption (K.3) means that \( f_n \) can be extended to a univalent function on the domain \( D_n \cup \Delta(0, L) \). We denote the extended map by \( f_n \) again. Then by Assumption (K.2), \( f_n \) has the Laurent expansion \( f_n(z) = z + a_n/z + o(z^{-1}) \) around \( \infty \) for some constant \( a_n \), and \( \{f_n\} \) is a normal family on \( D \) by Corollary 3.6 and Montel’s theorem 3.1.

Under (K.1)–(K.3) and some additional assumptions, we prove a version of the kernel theorem, which relates the u.c. convergence of \( \{f_n\} \) to the kernel convergence of \( \{f_n(D_n)\} \), mainly following the proof of [9, Theorem V.5.1].

**Theorem 3.8 (Kernel theorem).** Suppose that \( \{D_n\} \) and \( \{f_n\} \) satisfy Assumptions (K.1)–(K.3) and that there exist mutually disjoint subsets \( A_0, A_1, \ldots, A_N, N \in \mathbb{N} \), of \( \mathbb{H} \) with the following conditions:

19
• $A_0$ is a hull or an empty set;
• Each $A_j$, $1 \leq j \leq N$, is a compact, connected set with $\mathbb{H}\setminus A_j$ connected;
• $D_n \to D := \mathbb{H}\setminus \bigcup_{j=0}^{N} A_j$.

Let $\tilde{D}_n := f_n(D_n)$. Then the following are equivalent:

(i) There exists a univalent function $f$ on $D$ such that $f_n \to f$ u.c. on $D$;
(ii) There exists a domain $\tilde{D}$ such that $\tilde{D}_n \to \tilde{D}$.

If one of these conditions happens, then $\tilde{D} = f(D)$, and $f_n^{-1} \to f^{-1}$ u.c. on $\tilde{D}$.

Lemma 3.9. Under Assumptions (K.1)–(K.3), the sequence $\{\tilde{D}_n\}$ in Theorem 3.8 enjoys Condition (K.1) with the constant $L$ in (K.1) replaced by $2L$.

Proof. Since $L^{-1}f_n(Lz) \in \Sigma$ by (K.1)–(K.3), we have $\mathbb{C}\setminus L^{-1}f_n(L\mathbb{D}^*) \subset B(0,2)$ by Lemma 3.5, that is, $f_n(\Delta(0,L)) \supset \Delta(0,2L)$. By Lemma 3.9, we can define the kernel of $\{\tilde{D}_n\}$, which is denoted by $\tilde{D}$.

Proof of Theorem 3.8. (i) $\Rightarrow$ (ii): Assume (i). Note that $f$ is univalent on $D$ by Proposition 3.2. What we should prove is that any subsequence of $\{\tilde{D}_n\}$ has the same kernel $\tilde{D}$.

We first show that $f(D) \subset \tilde{D}$. Fix an arbitrary compact subset $K$ of $f(D)$. We take a bounded domain $V$ with smooth boundary so that $K \subset V \subset V \subset f(D)$ and put $\delta := \text{dist}(K, \partial V)/2 > 0$. We then have $|f(z) - w| > \delta$ for $w \in K$ and $z \in \partial f^{-1}(V)$. On the other hand, there is some $n_{K,V} \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \delta$ for $z \in f^{-1}(V)$ and $n \geq n_{K,V}$, since $\{f_n\}$ converges to $f$ uniformly on the compact subset $f^{-1}(V)(= f^{-1}(\overline{V}) \subset D)$. Thus by the equation

$$f_n(z) - w = (f_n(z) - f(z)) + (f(z) - w),$$

we can conclude from Rouché’s theorem that all the functions $f_n(z) - w$ for $w \in K$ and $n \geq n_{K,V}$ have exactly one zero in $f^{-1}(V)$. This implies that $K \subset f_n(f^{-1}(V)) \subset \tilde{D}_n$ for $n \geq n_{K,V}$, and so $f(D) \subset \tilde{D}$ by definition.

We next consider the inverse map $f_n^{-1}$. By the Laurent expansion of $f_n$ and Lemma 3.9, $f_n^{-1}$ also has the expansion $f_n^{-1}(z) = z - a_n/z + \cdots$, $z \to \infty$. By Corollary 3.6 and Montel’s theorem 3.1, $\{f_n^{-1}\}$ is a normal family and
so has a subsequence \( \{f_{n_k}^{-1}\} \) converging u.c. on \( \tilde{D} \). We can check that the limiting univalent function \( g := \lim_{k \to \infty} f_{n_k}^{-1} \) is the inverse map of \( f \) on \( f(D) \) as follows: For a fixed \( z \in D \), we take \( N \) so large that \( \{f_{n_k}(z); k \geq N\} \cup \{f(z)\} \) is a bounded subset of \( f(D) \). Since \( \{f_{n_k}^{-1}\}_k \) converges and is equicontinuous uniformly on this compact set, we have

\[
g(f(z)) - z = \{g(f(z)) - f_{n_k}^{-1}(f(z))\} + \{f_{n_k}^{-1}(f(z)) - f_{n_k}^{-1}(f_{n_k}(z))\}
\]

\[
\to 0 \quad \text{as} \quad k \to \infty.
\]

Hence \( g|_{f(D)} = f^{-1} \), independent of the choice of the subsequence \( \{f_{n_k}^{-1}\} \). By the identity theorem, any convergent subsequence of \( \{f_{n_k}^{-1}\} \) has the same limit \( g \) on the whole \( \tilde{D} \). Thus the original sequence \( \{f_n^{-1}\} \) converges to \( g \) u.c. on \( \tilde{D} \).

Reversing the roles of \( f_n \) and \( f_n^{-1} \) at the beginning of this proof, we have \( g(\tilde{D}) \subset D \). In particular, since \( g|_{f(D)} = f^{-1} \) and \( f(D) \subset \tilde{D} \), it follows that \( D = g(f(D)) \subset g(\tilde{D}) \subset D \), which yields \( f(D) = \tilde{D} \). If we repeat the argument so far for any subsequence of \( \{\tilde{D}_n\} \), then we see that it has the same kernel \( f(D) \). Hence \( \tilde{D}_n \to f(D) \), which completes the proof of (i) \( \Rightarrow \) (ii).

Note that this proof also establishes the latter part of the proposition, that is, \( f_n^{-1} \to f^{-1} \) u.c. on \( \tilde{D} = f(D) \).

(ii) \( \Rightarrow \) (i): Assume (ii). Contrary to our claim, we suppose that (i) is false. Since \( \{f_n\} \) is a normal family on \( D \), there are at least two subsequences \( \{f_n^{(1)}\}_n \) and \( \{f_n^{(2)}\}_n \) of \( \{f_n\} \) converging to distinct limits \( f^{(1)} \) and \( f^{(2)} \), respectively, u.c. on \( D \). By the implication (i) \( \Rightarrow \) (ii) already proven, \( \{f_n^{(k)}(D_n)\}_n \), \( k = 1, 2 \), converge in the sense of kernel convergence, and their limits are the same domain \( \tilde{D} \) by our hypothesis (ii). Then the composite \( f^{(1)-1} \circ f^{(2)} \) is a conformal automorphism on \( D \) that is not the identity map \( \text{id}_D \). If \( A_1, \ldots, A_N \) are all continua, then we can take the canonical map \( g: D \to \tilde{D} \), where \( \tilde{D} \) is a standard slit domain. In this case, \( g \circ f^{(1)-1} \circ f^{(2)} \) is also the canonical map on \( D \), and by the uniqueness of canonical map we have \( f^{(1)-1} \circ f^{(2)} = \text{id}_D \), a contradiction. If some of \( A_k \), say \( A_{l+1}, A_{l+2}, \ldots, A_N \), are singletons, then we can easily see, as in [6, Exercise 15.2.1], that \( f^{(1)-1} \circ f^{(2)} \) is extended to a conformal automorphism on \( D' := \mathbb{H} \setminus \bigcup_{j=0}^{l} A_j \) since the singularities \( A_{l+1}, A_{l+2}, \ldots, A_N \) are removable. Hence \( f^{(1)-1} \circ f^{(2)} = \text{id}_{D'} \) by the same argument. Thus in any case we arrive at a contradiction, which yields (i).

Note that, in the proof of the implication (ii) \( \Rightarrow \) (i) above, we do not use the hypothesis that the kernel \( D \) of \( \{D_n\} \) has the form \( \mathbb{H} \setminus \bigcup_j A_j \). We
need this hypothesis only for proving the uniqueness of automorphism on \( D \).

In Goluzin’s proof of [9, Theorem V.5.1], this uniqueness follows from the

property of \( \Sigma \) applied to \( f^{(1)} \circ f^{(2)} \), but in our case, \( f^{(1)} \) and \( f^{(2)} \) extend only to \( \Delta(0, L) \cup D \), not enough to mimic his argument. This is the reason why we have to suppose that \( D = \mathbb{H} \setminus \bigcup_j A_j \).

4. Komatu–Loewner equation for a family of growing hulls

4.1. Deduction of the Komatu–Loewner equation

In this subsection, we define the continuity of a family of growing hulls and deduce the Komatu–Loewner equation for such hulls.

Here is our basic setting throughout this subsection. Let \( \{ F_t; 0 \leq t < t_0 \} \) be a family of growing hulls (i.e. strictly increasing hulls) in a fixed standard slit domain \( D \). For each \( t \), let \( g_t: D \setminus F_t \to D_t \) be the canonical map, \( a_t := \text{hcap}_{D}(F_t) \) and \( s(t) \in \text{Slit} \) correspond to the slits \( \{ C_j(t) \} \) of \( D_t \). \( C_j(t) \) is sometimes denoted by \( C_{j,t} \) as well. We further define, for \( 0 \leq s \leq t < t_0 \),

\[
g_{t,s} := g_s \circ g_t^{-1}: D_t \to D_s \setminus g_s(F_t \setminus F_s).
\]

Clearly \( g_s(F_t \setminus F_s) \) is a hull, and \( g_{t,s} \) is the canonical map on \( D_s \setminus g_s(F_t \setminus F_s) \). Moreover for a fixed \( t_1 \in (0, t_0) \), the family \( \{(D \setminus F_t, g_t); t \in [0, t_1]\} \) satisfies Assumptions (K.1)–(K.3) in Section 3. Indeed, the constant \( L = L_{t_1} \) in (K.1) can be taken so that \( F_{t_1} \cup \bigcup_j C_j \subset B(0, L_{t_1}) \). (K.2) and (K.3) are obvious. Thus we can apply the theory developed in Section 3 to \( g_t \) and \( g_{t,s} \) over each compact subinterval of \([0, t_0)\).

In what follows, several conditions are imposed on \( \{ F_t \} \). If there exists a function \( \xi: [0, t_0) \to \mathbb{R} \) such that (2.17) holds for any \( t \in [0, t_0) \), then we call \( \xi \) the driving function of \( \{ F_t \} \). The condition (2.17) is sometimes called the right continuity of \( \{ F_t \} \) and employed in the existing literature, for example, [12, Section 4.1], [13, Section 4] and [3, Section 6]. One reason is that, for a family of growing hulls having this property, we can obtain the Komatu–Loewner equation in the right derivative sense as in Proposition 4.1. However, it should be noted that we mean a weaker condition than (2.17) by the “right continuity” in Definition 4.2.

**Proposition 4.1.** Let \( \{ F_t \}_{t \in [0, t_0)} \) be a family of growing hulls in \( D \) with driving function \( \xi: [0, t_0) \to \mathbb{R} \).

(i) The half-plane capacity \( a_t \) is strictly increasing and right continuous in \( t \).
(ii) \( g_{t,s}^{-1}(z) \to z \text{ u.c. on } D_s \text{ as } t \downarrow s \text{ for any } s \in [0,t_0). \)

(iii) \( g_t(z) \) is right differentiable in \( a_t \) for each \( z \in D \cup \bigcup_j \partial_p C_j \), and

\[
\frac{\partial^+ g_t(z)}{\partial a_t} = -\pi \Psi_{D_t}(g_t(z), \xi(t)), \quad g_0(z) = z, \quad t \in [0,t_0). \quad (4.1)
\]

Here \( \partial^+ g_t(z)/\partial a_t \) denotes the right derivative of \( g_t(z) \) with respect to \( a_t \).

Proof. (i) Let \( 0 \leq s \leq t < t_0 \). We can easily observe that

\[
a_t - a_s = h\text{cap}^{D_s}(g_s(F_t \setminus F_s)).
\]

Since \( g_s(F_t \setminus F_s) \) is non-polar (with respect to the ABM on \( D_s \)) for \( t > s \),
the right hand side is positive by (2.4). (2.4) also implies \( \lim_{t \downarrow s}(a_t - a_s) = 0 \)
because \( \sup\{\Re z; z \in g_s(F_t \setminus F_s)\} \to 0 \) as \( t \downarrow s \).

(ii) and (iii) are immediate consequences of (i) and Proposition 2.5 \( \square \)

The left continuity of \( a_t \) and left differentiability of \( g_t(z) \) do not follow from (2.17). To proceed further, we define the continuity of \( \{F_t\} \) as the continuity of \( D \setminus F_t \) in the sense of kernel convergence.
Definition 4.2. \(\{F_t\}_{t \in [0,t_0]}\) is said to be (left/right) continuous in \(D\) at \(t \in [0,t_0]\) if \(D \setminus F_u \to D \setminus F_t\) as \(u\) approaches \(t\) (from left/right).

Such a continuity condition did not appear in the recent studies \([1, 13, 7, 4, 3]\), but it is not new in complex analysis. Indeed, a similar condition was imposed when Pommerenke established a version of the radial Loewner equation in [16, Section 6.1]. Below we show that Definition 4.2 works well even when the domain has multiple connectivity.

Lemma 4.3. If \(\{F_t\}\) satisfies (2.17) for some \(\xi(s) \in \mathbb{R}\) at \(s \in [0,t_0]\), then it is right continuous in \(D\) at \(s\).

Proof. By (2.17) and Proposition 4.1 (ii), we get the following two convergences as \(t \downarrow s\):

\[
D_s \setminus g_s(F_t \setminus F_s) \to D_s, \quad g_t^{-1}(z) \to z \text{ u.c.}
\]

Hence \(D_t \to D_s\) and \(g_{t,s}(z) \to z\) u.c. as \(t \downarrow s\) by the kernel theorem 3.8. Since \(g_t^{-1} = g_{t,s} \circ g_s^{-1}\), it also holds that \(g_t^{-1} \to g_s^{-1}\) u.c. as \(t \downarrow s\). By the kernel theorem 3.8 again, we obtain \(D \setminus F_t \to D \setminus F_s\).

Lemma 4.4. (i) Suppose that \(\{F_t\}\) is left continuous in \(D\) at \(t \in (0,t_0)\). Then \(D_s \to D_t\) as \(s \uparrow t\), that is, \(s(s)\) is left continuous at \(t\). Moreover, \(g_{t,s}^{-1}(z) \to z\) u.c. on \(D_t\) as \(s \uparrow t\), and \(a_s\) is left continuous at \(t\).

(ii) Suppose that \(\{F_t\}\) is right continuous in \(D\) at \(s \in [0,t_0]\). Then \(D_t \to D_s\) as \(t \downarrow s\), that is, \(s(t)\) is right continuous at \(s\). Moreover, \(g_{t,s}^{-1}(z) \to z\) u.c. on \(D_s\) as \(t \downarrow s\), and \(a_t\) is right continuous at \(s\).

Proof. We prove only (i) because the proof of (ii) is quite similar.

Since the family \(\{(D \setminus F_s) \cap \mathbb{H}; s \in [0,t]\}\) satisfies (K.1)–(K.3), we have \(D_s \supset \Delta(0,2L_t) \cap \mathbb{H}\) by Lemma 3.9. This implies \(\bigcup_{s \in [0,t]} \bigcup_j C_{s,j} \subset B(0,2L_t)\), that is, \(\{s(s); s \in [0,t]\}\) is bounded. We can thus take a sequence \(\{s_n\}\) with \(s_n \uparrow t\) so that \(\tilde{s} := \lim_{n \to \infty} s(s_n)\) exists in \(\mathbb{R}^3\). Though \(\tilde{s}\) is not necessary in Slit, it is obvious from definition that \(D_{s_n}\) converges to a slit domain \(\tilde{D}\) in the sense of kernel convergence. Some of the slits of \(\tilde{D}\) may degenerate. Since \(D \setminus F_{s_n} \to D \setminus F_t\) by the left continuity of \(\{F_t\}\), we can apply the kernel theorem 3.8 to \(\{g_{s_n}\}\) to obtain the limiting conformal map \(\tilde{g} := \lim_{n \to \infty} g_{s_n}; D \setminus F_t \to \tilde{D}\). Then, all the slits of \(\tilde{D}\) must not degenerate, and \(\tilde{g}\) must be the canonical map on \(D \setminus F_t\), which yields \(\tilde{g} = g_t\) and \(\tilde{D} = D_t\).
by the uniqueness in Proposition 2.3. In particular, this limit is independent of the choice of \( \{s_n\} \). We therefore conclude that \( D_s \to D_t \) as \( s \uparrow t \).

The equivalence between the left continuity of \( \{D_s\} \) and that of \( s(s) \) can be checked easily from definition, and so we omit it.

Since \( D \setminus F_s \to D \setminus F_t \) and \( D_s \to D_t \) as \( s \uparrow t \), the kernel theorem 3.8 implies \( g_s^{-1} \to g_t^{-1} \) u.c., which in turn yields \( g_{t,s}(z)^{-1} \to z \) u.c. as \( s \uparrow t \). To show the left continuity of \( a_s \), we regard \( h_{a}(z) := (2L_t)^{-1} g_{t,s}^{-1} (2L_t z) \) as an element of \( \Sigma \) by Schwarz’s reflection. Writing the Laurent series expansion around infinity as

\[
h_a(z) = z + \frac{a_t - a_s}{4L_t^2} \frac{1}{z} + \sum_{n \geq 2} \frac{c_{n,s}}{z^n},
\]

we get, from the Cauchy–Schwarz inequality and the area theorem 3.3,

\[
|h_a(z) - z| = \left| \frac{a_t - a_s}{4L_t^2} \frac{1}{z} + \sum_{n \geq 2} \frac{c_{n,s}}{z^n} \right| \geq \left| \frac{a_t - a_s}{4L_t^2} \frac{1}{z} \right| - \sum_{n \geq 2} \frac{c_{n,s}}{z^n} \geq \frac{a_t - a_s}{4L_t^2} \frac{1}{|z|} - \frac{|z|^{-4}}{1 - |z|^{-2}}.
\]

Since \( \lim_{s \uparrow t} |h_a(z) - z| = 0 \) for any \( z \in D_t \), we have \( \lim_{s \uparrow t} (a_t - a_s) = 0 \).  \( \square \)

By Lemma 4.4, \( a_t \) is a strictly increasing continuous function on \( [0, t_0] \) if \( \{F_t\} \) is continuous. We can thus reparametrize \( \{F_t\} \) so that \( a_t \) is differentiable in \( t \). As a particular case, we say that \( \{F_t\} \) obeys the half-plane capacity parametrization in \( D \) if \( a_t = \text{hcap}^D(F_t) = 2t \).

**Lemma 4.5.** Suppose that \( \{F_t\} \) is continuous in \( D \) at every \( t \in [0, t_0) \). Then \( g_t(z) \) is jointly continuous in \( (t, z) \in [0, t_0) \times (D \cup \bigcup_j \partial_p C_j) \).

**Proof.** \( g_t(z) \) is jointly continuous on \( [0, t_0) \times D \) since \( g_s \to g_t \) u.c. on \( D \) as \( s \to t \) for any \( t \in [0, t_0] \) by Lemma 4.4. Recall from Section 2.1 that the canonical map \( g_t = f_{F_t} \) can be extended holomorphically to \( \bigcup_j \partial_p C_j \). For a fixed \( j \), let \( g_t^+ \) be the extended map of \( g_t \) from

\[
R_+ = \{ z \in \mathbb{C}; x_j < \Re z < x_j', y_j < \Im z < y_j + \delta \}
\]

to \( R = R_+ \cup C_j \cup R_- \) across \( C_j^0 \). Here we use the notation in Section 2.1. For a fixed \( t_1 \in (0, t_0) \), \( \{g_t^+ \}_{t \in [0, t_1]} \) is locally bounded on \( R_+ \cup R_- \) by the local
boundedness of \( \{g_t\} \), and also on \( C_j^0 \) since \( g_t^+(C_j^0) = C_j \subset B(0, 2L_t) \). Hence \( \{g_t^+\}_{t \in [0, t_1]} \) is a normal family on \( R \). Any sequence \( \{g_{s_n}^+\} \), \( s_n \to t \in [0, t_1] \), converging u.c. when \( n \to \infty \) has the same limit, because it converges to \( g_t \) on \( R_+ \) and so the identity theorem applies to \( \lim_{n \to \infty} g_{s_n}^+ \). Thus \( g_t^+ \to g_t^+ \) u.c. on \( R \) as \( s \to t \in [0, t_0] \), which yields the joint continuity of \( g_t(z) \) on \( [0, t_0] \times C_j^+ \). The joint continuity of \( g_t(z) \) on \( [0, t_0] \times (C_j^- \cup \{z_j, z_j^\prime\}) \) is obtained in the same way.

We now arrive at our main result. Here, the dot ‘denotes the \( t \)-derivative.

**Theorem 4.6.** Suppose that \( \xi(t) \) is continuous and \( a_t \) is strictly increasing and differentiable over \([0, t_0] \). Then the following are equivalent:

1. \( \{F_t\}_{t \in [0, t_0]} \) is a family of continuously growing hulls in \( D \) with driving function \( \xi \) and half-plane capacity \( a_t \).
2. \( s(t) \) and \( g_t(z) \) solve the ODEs

\[
\frac{d}{dt} z_j(t) = -\pi \dot{a}_t \Psi_{s(t)}(z_j(t), \xi(t)), \quad \frac{d}{dt} \dot{z}_j(t) = -\pi \dot{a}_t \Psi_{s(t)}(z_j(t), \xi(t)),
\]

\[
\frac{d}{dt} g_t(z) = -\pi \dot{a}_t \Psi_{s(t)}(g_t(z), \xi(t)), \quad g_0(z) = z \in D.
\]

**Proof.** It is sufficient to prove the theorem when \( a_t = 2t \), for the general case is established by the time-change of \( a_t \). Under this half-plane capacity parametrization, the ODEs (4.2) and (4.3) reduce to (2.15) and (2.16), respectively.

\( \text{(i)} \Rightarrow \text{(ii)} \): Assume \( \text{(i)} \). The conditions (2.17) and \( a_t = 2t \) are already mentioned in Section 2.2. To show the continuity of \( \{F_t\} \), observe that the continuity of \( s(t) \) implies that of \( \{D_t\} \) in the sense of kernel convergence. It suffices to prove the joint continuity of \( f_t(z) := g_t^{-1}(z) \) in \((t, z)\) because then \( f_s \to f_t \) u.c. as \( s \to t \) and the kernel theorem 3.8 implies \( D \setminus F_s \to D \setminus F_t \).

As a solution to the ODE (2.16), \( g_t(z) \) is jointly continuous. We then see from Cauchy’s integral formula that \( g_t'(z) \) is also jointly continuous. Note that it is non-vanishing since \( g_t \) is univalent. The joint continuity of \( f_t(z) \) now follows from the fact that it is a solution to the ODE

\[
\dot{f}_t(z) = 2\pi f_t'(z) \Psi_{s(t)}(z, \xi(t)) = \frac{2\pi}{g_t(f_t(z))} \Psi_{s(t)}(z, \xi(t)).
\]

\( \text{(ii)} \Rightarrow \text{(i)} \): We have already seen that (2.16) holds for \( z \in D \cup \bigcup_j \partial_p C_j \) in the right derivative sense in Proposition 4.1 (iii). Since \( \xi(t) \), \( s(t) \) and \( g_t(z) \)
are continuous in $t$ by (i) and Lemmas 4.4 and 4.5, (2.16) holds as a genuine ODE by the same proof as that of Theorems 9.8 and 9.9 in [4].

To establish (2.15), we can use the same method as in [3, Section 2]. Hence it suffices to prove [3, Lemma 2.1] in our case, because the proof of Lemma 2.2 and Theorem 2.3 in [3] depends only on this lemma, not on whether $F_t$ is a simple curve. Assertions (i), (ii), (iv) and (v) of [3, Lemma 2.1] follows from Cauchy’s integral formula and Lemma 4.5. The proofs of (iii), (vi) and (vii) are quite similar, and so we prove only (iii) here.

For a fixed $j$, let $g^+_t$ be the extended map of $g_t$ from $R_+$ to $R$ as in the proof of Lemma 4.5. We can check that $\eta^+_t(z,\xi_0) := \Psi_s(t)(g^+_t(z),\xi_0)$ can also be extended from $R_+$ to a holomorphic function on $R$, which is denoted by $\eta^+_s(z,\xi_0)$, and that $\eta^+_s(z,\xi_0)$ is continuous in $(t, z, \xi_0)$. By Proposition 2.5 and Cauchy’s integral formula we have, for $0 \leq s < t < t_0$,

$$|(g^+_t)'(z) - (g^+_s)'(z) - 2\pi(t - s)(\eta^+_s(z, \xi(s)))'| \leq 2(t - s)C'r_{s,t},$$

where $r_{s,t} := \inf\{R > 0; g_s(F_t \setminus F_s) \subset B(\xi(s), R)\}$ and $C'$ is a constant. Since $r_{s,t} \to 0$ as $t \downarrow s$, we obtain

$$\frac{\partial^+(g^+_t)'(z)}{\partial t} = -2\pi(\eta^+_s(g^+_s(z),\xi(s)))'.$$  (4.4)

The right hand side of (4.4) is jointly continuous in $(t, z)$ in view of (ii) of [3, Lemma 2.1], and thus the left hand side becomes the genuine derivative by [12, Lemma 4.3]. Therefore, $(g^+_t)'(z)$ is differentiable in $t$ and $\partial_t(g^+_t)'(z)$ is continuous in $(t, z)$.

In this way, we can prove the assertions corresponding to [3, Lemma 2.1] and then Lemma 2.2 and Theorem 2.3 of [3] tell us that (2.15) holds under our assumption (i) of the theorem.

Every assertion in Section 4.1 remains valid for the upper half-plane $\mathbb{H}$ in place of the standard slit domain $D$ by replacing $\Psi_D$ with $\Psi_H = -\frac{1}{\pi(z - \xi_0)}$.

4.2. Transformation of the chains, half-plane capacities and driving functions

From Theorem 4.6, the Komatu–Loewner evolution defined in Section 2.2 proves to be nothing but a family of continuously growing hulls with continuous driving function and differentiable half-plane capacity $2t$. In this subsection, we check that such nice properties on growing hulls are independent of the domain $D$ and conformally invariant. The transformation of the chains described in Section 4.1 is thus always possible. We take over the notations in Section 4.1.

27
Proposition 4.7. \( \{F_t\} \) is continuous with continuous driving function and differentiable half-plane capacity in \( D \) if and only if it has the same property in \( \mathbb{H} \).

Proof. Let \( \iota: D \hookrightarrow \mathbb{H} \) be the inclusion map, \( g_t^0: \mathbb{H} \setminus F_t \to \mathbb{H}, t \in [0, t_0), \) be the canonical map and \( \iota_t := g_t^0 \circ \iota \circ g_t^{-1} \). By Schwarz’s reflection, \( \iota_t: D_t \hookrightarrow \mathbb{H} \) extends to a conformal map from \( D_t \cup \Pi D_t \cup \partial \mathbb{H} \) onto \( g_t^0(D) \cup \Pi g_t^0(D) \cup \partial \mathbb{H} \). It is especially a homeomorphism between these domains.

Assume that \( \{F_t\}_{t \in [0, t_0)} \) is continuous with continuous driving function \( \xi \) and differentiable half-plane capacity \( a_t \) in \( D \). We set \( U(t) := \iota_t(\xi(t)) \). It holds that
\[
\bigcap_{\delta > 0} g_t^0(F_{t+\delta} \setminus F_t) = \bigcap_{\delta > 0} \iota_t \circ g_t(F_{t+\delta} \setminus F_t) = \iota_t \left( \bigcap_{\delta > 0} g_t(F_{t+\delta} \setminus F_t) \right) = \iota_t(\{\xi(t)\}) = \{U(t)\}.
\]
Hence \( \{F_t\} \) has driving function \( U(t) \) in \( \mathbb{H} \). Next we fix \( t \in (0, t_0) \). For any \( z \in \bigcup_j C_j \), there is some \( r > 0 \) such that \( B(z, r) \subset \mathbb{H} \setminus F_t \subset \mathbb{H} \setminus F_s \) for all \( s \leq t \). Combining this with the assumption that \( D \setminus F_s \to D \setminus F_t \) as \( s \uparrow t \), we can conclude that \( \mathbb{H} \setminus F_s \to \mathbb{H} \setminus F_t \). This means the left continuity of \( \{F_t\} \) in \( \mathbb{H} \). By Lemma 4.4, \( \{g_t^0\} \) is continuous in the sense of uniform convergence on compacta. \( \iota_t(z) \) is then jointly continuous in \( (t, z) \in [0, t_0) \times (D_t \cup \Pi D_t \cup \partial \mathbb{H}) \). Hence \( U(t) = \iota_t(\xi(t)) \) is continuous in \( t \). Finally the differentiability of \( a_t^0 := hcap^\mathbb{H}(F_t) \) is obtained from the capacity comparison theorem [3, Theorem A.1]. Thus \( \{F_t\} \) is continuous with continuous driving function \( U \) and differentiable half-plane capacity \( a_t^0 \) in \( \mathbb{H} \).

The proof of the converse implication is quite similar, and we omit it. \( \square \)

Proposition 4.7 implies that, if \( \{F_t\} \) is continuous with continuous driving function and differentiable half-plane capacity in one standard slit domain, then we get the Komatu–Loewner equation on every standard slit domain. In almost the same way, we can also show that this condition is conformally invariant. More precisely, [14, (2.7)] combined with the capacity comparison theorem [3, Theorem A.1] yields the following:

Theorem 4.8. Denote by \( D \) either a standard slit domain or the upper half-plane \( \mathbb{H} \). Denote also by \( \tilde{D} \) a standard slit domain or \( \mathbb{H} \), but the degree of connectivity of \( \tilde{D} \) can be different from that of \( D \). Let \( \{F_t\}_{t \in [0, t_0)} \) be a
family of continuously growing hulls with continuous driving function \( \xi \) and differentiable half-plane capacity \( a_t \) in \( D \). Suppose that a domain \( V \subset D \) and a univalent function \( h: V \leftrightarrow \tilde{D} \) satisfy the following conditions:

(i) \( \bigcup_{t \in [0, t_0)} F_t \subset V \);
(ii) \( \partial V \cap \partial \mathbb{H} \) is locally connected;
(iii) \( h \) maps \( \partial V \cap \partial \mathbb{H} \) into \( \partial \mathbb{H} \), that is, \( \lim_{y \downarrow 0} \Re h(x + iy) = 0 \) for all \( x \in \partial V \cap \partial \mathbb{H} \).

Under these assumptions, \( \{h(F_t)\} \) is a family of continuously growing hulls in \( \tilde{D} \). Further let \( g_t \) and \( \tilde{g}_t \) be the canonical maps on \( D \setminus F_t \) and \( \tilde{D} \setminus h(F_t) \), respectively, and set \( h_t := \tilde{g}_t \circ h \circ g_t^{-1} \) with the domain of definition being \( g_t(V \setminus F_t) \subset \mathbb{H} \). By Schwarz’s reflection, \( h_t \) is extended to be holomorphic on

\[
g_t(V \setminus F_t) \cup \Pi g_t(V \setminus F_t) \cup \left( \partial \mathbb{H} \setminus (D_t \setminus g_t(V \setminus F_t)) \right) \subset \mathbb{C}.
\]

\( h_t(\xi(t)) \) is then the continuous driving function of \( \{h(F_t)\} \). Moreover, \( \tilde{a}_t := \text{hcap}^D(h(F_t)) \) is differentiable with

\[
\hat{a}_t = h_t'(\xi(t))^2 \tilde{a}_t, \quad t \in [0, t_0).
\]

Proof. \( \{h(F_t)\} \) can be shown in the same way as the proof of [3, Theorem 6.8] by using the capacity comparison theorem in it. Note that \( h_t'(\xi(t)) \neq 0 \) because \( h_t \) is univalent. The rest of the assertion can be shown in the same way as the proof of Proposition 4.7 except that \( \iota \) is replaced by \( h \).

We note that the degrees of connectivity of \( D \) and \( \tilde{D} \) can be different in Theorem 4.8. Thus Theorems 4.6 and 4.8 establish the transformation of Komatu–Loewner chains under any possible conformal transformation. More precisely, if \( \{F_t\} \) is a Komatu–Loewner evolution \( D \) driven by \( \xi \), then \( \{h(F_t)\} \) in Theorem 4.8 is a family of continuously growing hulls on \( \tilde{D} \), and we can reparametrize \( \{F_t\} \) so that \( \{h(F_t)\} \) obeys the half-plane capacity parametrization on \( \tilde{D} \) by setting

\[
\tilde{F}_t := h(F_{\tilde{a}^{-1}(2t)}), \quad \tilde{g}_t := \tilde{g}_{\tilde{a}^{-1}(2t)}, \quad \tilde{\xi}(t) := h_{\tilde{a}^{-1}(2t)}(\xi(\tilde{a}^{-1}(2t))).
\]

Now \( \{\tilde{g}_t, \tilde{F}_t\}_{t \in [0, \tilde{a}_{t_0}/2]} \) is a Komatu–Loewner evolution in \( \tilde{D} \) driven by \( \tilde{\xi} \). Note that the half-plane capacity is bounded if the hull is bounded by \( \tilde{2.4} \). Hence,
Figure 4: Conformal maps $h$ and $h_t$

The time-change $\tilde{a}_{t_0}/2$ maps a compact subinterval of $[0, t_0)$ onto a compact. In particular, if $\bigcup_{t \in [0, t_0]} F_t$ is bounded, then $\tilde{a}_{t_0} < \infty$.

Finally, let $\{F_t\}$ be an SKLE$_{a, b}$ defined at the end of Section 2.2. By Theorems 5.8 and 5.12 of [3] and Theorem 4.6, $\{F_t\}$ is a family of continuously growing hulls on $D$ driven by the solution $\xi(t)$ of the SDEs (2.18) and (2.19) with the half-plane capacity $a_t = 2t$. Under the setting of Theorem 4.8 on $D$, $\tilde{D}$, $V$ and $h$, $\{h_t(F_t)\}$ becomes a family of continuously growing hulls in $\tilde{D}$ with the driving function $\tilde{\xi}(t) = h_t(\xi(t))$ and with the half-plane capacity $\tilde{a}_t = 2h'_t(\xi(t))^2t$. Consequently, $\tilde{s}(t)$ with $D(\tilde{s}(t)) = \tilde{g}_t(\tilde{D} \setminus h(F_t))$ and $\tilde{g}_t$ satisfy the ODEs (4.2) and (4.3) for these choices of $\tilde{\xi}(t)$ and $\tilde{a}_t$ by Theorem 4.6 again. Denote these ODEs by (4.2)' and (4.3).'</p>

In a similar manner to the proof of [3, Theorem 6.9], one can then derive from (4.3)' the following semimartingale decomposition of the driving process $\tilde{\xi}(t) = h_t(\xi(t))$ of $\{h(F_t)\}$:

\[
d\tilde{\xi}(t) = h'_t(\xi(t)) \left( b(s(t) - \tilde{\xi}(t)) + b_{BMD}(\xi(t), s(t)) \right) dt \\
+ \frac{1}{2} h''_t(\xi(t)) \left( \alpha(s(t) - \tilde{\xi}(t))^2 - 6 \right) dt \\
- h'_t(\xi(t))^2 b_{BMD}(\xi(t), s(t)) dt + h'_t(\xi(t))\alpha(s(t) - \tilde{\xi}(t)) dB_t, \quad t < T_V,
\]
where \( T_V = \zeta \land \sup \{ t > 0; F_t \subset V \} \). Here \( b_{\text{BMD}}(\xi_0, D) \) for \( \xi_0 \in \partial \mathbb{H} \) and for a standard slit domain \( D \) is defined by

\[
b_{\text{BMD}}(\xi_0, D) := 2\pi \lim_{z \to 0} \left( \Psi_D(z, \xi_0) - \frac{1}{\pi} \frac{1}{z - \xi_0} \right) = 2\pi H_D(\xi_0, \xi_0),
\]

where \( H_D \) is the function appearing in (2.2). We let

\[
b_{\text{BMD}}(\xi_0, \mathbb{H}) = 0, \quad \xi_0 \in \partial \mathbb{H}
\]

accordingly. We also write \( b_{\text{BMD}}(\xi_0, D) \) as \( b_{\text{BMD}}(\xi_0, s) \) in terms of \( s = s(D) \in \text{Slit} \) for \( D \). We set \( b_{\text{BMD}}(s) = b_{\text{BMD}}(0, s) \) and call it the BMD domain constant of \( D = D(s) \). By [3, Lemma 6.1], the BMD domain constant \( b_{\text{BMD}}(s) \) is homogeneous with degree \(-1\) and locally Lipschitz continuous so that

\[
b_{\text{BMD}}(\xi_0, s) = b_{\text{BMD}}(s - \xi_0), \quad s \in \text{Slit}, \xi_0 \in \partial \mathbb{H}.
\]

(4.7) is derived from the same computation as in the proof of [3, Theorem 6.9] based on a generalized Itô formula [17, Exercise IV.3.12]. As pointed out in [3, Remark 2.9], one needs more assumptions than stated in [17] to verify the formula. Accordingly the following property of \( h_t \) is necessary to legitimize (4.7):

(C) \( h_t(z), h'_t(z) \) and \( h''_t(z) \) are jointly continuous in \((t, z)\) for \( z \) in a neighborhood of \( \xi(t) \) in \( \mathbb{C} \).

We here prove Property (C) in a way similar to the proof of Lemma 4.5.

**Proof of Property (C).** Fix \( t \in [0, t_0) \). Since \( g_s \to g_t \) u.c. on \( D \setminus F_t \), we have \( g_s \to g_t \) u.c. on \( \Pi(D \setminus F_t) \) for the maps \( g_s \) extended by Schwarz’s reflection. Since \( \{g_s\}_s \) is a normal family on \( (D \setminus F_t) \cup \Pi(D \setminus F_t) \cup (\partial \mathbb{H} \setminus F_t) \) by Corollary 3.1 and Montel’s theorem \( 3.1 \) \( g_s \to g_t \) u.c. on this domain, which we can check by the identity theorem as in the proof of Lemma 4.5. Hence we can take a bounded open subset \( U \) of \( \mathbb{C} \) so that

\[
\xi(s) \in U \subset g_s(V \setminus F_s) \cup \Pi g_s(V \setminus F_s) \cup \left( \partial \mathbb{H} \setminus \left( D_s \setminus g_s(V \setminus F_s) \right) \right), \quad s \in [t_-, t_+],
\]

for some \( 0 \leq t_- < t < t_+ < t_0 \).

We observe from the same argument as for \( g_s \) that \( g_s^{-1} \to g_t^{-1} \) uniformly on \( U \) as \( s \) tends to \( t \) in \([t_-, t_+]\) and that \( \tilde{g}_s \to \tilde{g}_t \) u.c. on \((\bar{D} \setminus h(F_t)) \cup \Pi(\bar{D} \setminus h(F_t)) \cup (\partial \mathbb{H} \setminus (\bar{D} \setminus h(F_t))) \). Thus the composite \( \tilde{h}_s = \tilde{g}_s \circ h \circ g_s^{-1} \) converges to \( h_t \) as \( s \to t \) in \([t_-, t_+]\) uniformly on \( U \). As a consequence, Cauchy’s integral formulae for \( h'_t \) and \( h''_t \) yield Property (C).
Using the formula (4.7) along with the ODEs (4.2)' and (4.3)', we arrive at the following theorem in exactly the same manner as the proof of [3, Theorem 6.11]:

**Theorem 4.9** (Conformal invariance of Chordal SKLE\(\sqrt{6}, -b\)BMD). Let \(D, \tilde{D}, V\) and \(h\) be as in Theorem 4.8 except for Condition (i) for \(V\). For any SKLE\(\sqrt{6}, -b\)BMD \(\{F_t\}_{t \in [0, \zeta)}\) in \(D\), we set \(T_V := \zeta \wedge \sup\{t > 0; F_t \subset V\}\). Then, \(\{\tilde{F}_t\}\) defined by (4.6) is SKLE\(\sqrt{6}, -b\)BMD in \(\tilde{D}\) up to \(\tilde{a}(T_V-)/2\).

Theorem 4.9 extends [5, Theorem 4.2] in which case \(\tilde{D} = \mathbb{H}\) and \(h\) is the inclusion map from \(D\) into \(\mathbb{H}\). It also extends [3, Theorem 6.11] in which case \(h\) is the canonical map \(\Phi_A\) from \(D \setminus A\) for any hull \(A\) in \(D\) and \(\tilde{D} = \Phi_A(D \setminus A)\). The special case of [3, Theorem 6.11] where \(D = \tilde{D} = \mathbb{H}\) and accordingly \(b\)BMD = 0 was discovered by Lawler, Schramm and Werner [14, 15] and shown recently in [5] more rigorously. Such a property of SLE\(6\) has been called its *locality* under a phrase that SLE\(6\) does not feel the boundary before hitting it. Theorem 4.9 resolves some of the problems posed in [5, Section 5] as well.

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