Calculation of a Penning plasma discharge characteristics taking into account the magnetic field produced by annular magnets

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Abstract. Penning discharge represent a gas discharge, that can exist only in presence of a magnetic field of permanent magnets or current solenoids. Based on this at VNIIA a series of experiments were performed where discharge currents were recorded for various cell geometries, voltage, and pressure. For the correct calculation of the current-voltage characteristics of the system it is necessary to calculate the distribution of the magnetic field inside the gas discharge chamber. Thus, a computer program was developed that calculates the magnetic field of a ring magnet with the specified parameters. The results obtained using the program was compared with the experimental and numerical data reported in literature. Using this approach two-dimensional distribution of magnetic field in the Penning cell was calculated and preliminary results on the electrodynamic structure of plasma of the Penning gas discharge corresponding to one studied in VNIIA experiments were obtained.

1. Introduction

Nowadays permanent magnets are used in many applications [1]: source of charged particles, manometers and ion pumps. One of the systems, in which permanent magnets are used, is Penning gas discharge [2]. It is an example of the system in which formation and sustaining of gas discharge plasma is determined by the presence of axial magnetic field created by permanent magnets or solenoids. The distribution of the magnetic field affects the motion of charged particles and thus the structure of the gas discharge plasma in the Penning cell.

Penning gas discharge plasma is rarefied. To model this kind of plasma particle-in-cell method can be used [3], however there are studies [4, 5] devoted to numerical investigation of Penning gas discharge in which other models are adopted.

In some cases Penning gas discharge is sustained by the non-uniform magnetic field. In order to study such systems one has to be able to calculate the distribution of it. An example of such system was studied in the experiments performed in VNIIA [6]. In mentioned experiments in the presence of external magnetic field created by permanent magnets Penning discharge currents were recorded for cells with various geometries, voltages and pressures. Also visual observations of the glowing plasma were presented [6].
In the general case, the problem of finding the magnetic field distribution is solved by numerical methods, but for systems where the permanent magnets have a simple shape (ring topology) (e.g. [6]), it is possible to solve this problem analytically [7, 8].

In the paper based on the model described in [7, 8], computer program was developed for calculation of the distribution of magnetic field created by annular permanent magnet. The computer code was validated against experimental and numerical data presented in literature. Based on the magnetic field distribution calculated using the developed code preliminary results on the electrodynamics structure of Penning discharge studied in [6] were reported.

2. Analytical method

The calculation of the magnetic field generated by the annular permanent magnet can be obtained by the method that was presented in [7]. This method uses a Coulombian model and in the case of axial polarization, the formulas for calculation of the magnetic field are as follows:

$$
\bar{H}(r, z) = \bar{H}^+(r, z) + \bar{H}^-(r, z)
$$

where

$$
P_1, \bar{M} = (r - r_1 \cos \theta) \bar{i}_r - r_1 \sin \theta \bar{j}_\theta + (z - h) \bar{j}_z,
P_2, \bar{M} = (r - r_1 \cos \theta) \bar{i}_r - r_1 \sin \theta \bar{j}_\theta + (z + h) \bar{j}_z
$$

By substituting $\theta = \pi - 2\beta$, where $0 \leq \beta \leq \pi / 2$ and integrating twice, one obtains 3 components of magnetic field (2) for point M. Assumption of axisymmetric geometry (which is valid for most of the Penning cells) leads to $H_0 = 0$.

$$
\begin{align*}
H_i(r, z, \theta) &= H_1^+(r, z, \theta) + H_2^-(r, z, \theta) \\
H_o(r, z, \theta) &= H_1^+(r, z, \theta) + H_2^-(r, z, \theta) \\
H_{io}(r, z, \theta) &= H_1^+(r, z, \theta) + H_2^-(r, z, \theta) = 0
\end{align*}
$$

Expressions for only non-singular cases, where $r \neq r_n, r \neq r_{out}, z \neq h$ are:

$$
\begin{align*}
H^+_i(r, z) &= \frac{\sigma^*}{\pi \mu_0} \sum_{n=1}^{2} \frac{(-1)^{n+1}}{k_n^+} \frac{r_n}{r} \left[ E(k_n^+) - (1 - k_n^{+2}) K(k_n^+) \right] \\
H^-_i(r, z) &= \frac{\sigma^*}{\pi \mu_0} \sum_{n=1}^{2} \frac{(-1)^{n+1}}{k_n^-} \frac{r_n}{r} \left[ E(k_n^-) - (1 - k_n^{-2}) K(k_n^-) \right] \\
H^+_o(r, z) &= \frac{\sigma^*}{4 \pi \mu_0 (z - h)} \sum_{n=1}^{3} \frac{(-1)^{n+1}}{k_n^+} \frac{r_n}{\sqrt{r_n r_z}} \left\{ \left[ \sqrt{r^2 + (z - h)^2 - r_n} \right] \left[ \sqrt{r^2 + (z - h)^2 - r_n} \right] \Pi(h_n^+, k_n^+) \right. \\
&\left. + \left[ \sqrt{r^2 + (z - h)^2 + r_n} \right] \left[ \sqrt{r^2 + (z - h)^2 + r_n} \right] \Pi(h_n^+, k_n^+) \right\}
\end{align*}
$$
\[ H^*_r(r,z) = \frac{\sigma^*}{4\pi\mu_0(z)} \sum_{n=1}^2 (-1)^{n-1} \frac{k^-_{n}}{\sqrt{rr_n}} \left\{ \left[ \sqrt{r^2+z^2} - r \right] \Pi(h^-_n, k^-_n) + \left[ \sqrt{r^2+z^2} + r \right] \Pi(h^-_n, k^-_n) \right\} \]

And for the singular case (where \( r = r_{in}, r = r_{out}, z = h \)) we can use modification of (4), where the integrals of the third kind may be rewritten using the Heuman’s lambda-function [9], which is the sum of elliptic integral of the first kind:

\[ \Lambda_0(\varphi,k) = \frac{F(\varphi,(1-k^2)^{1/2})}{K((1-k^2)^{1/2})} + \frac{2}{\pi} K(k)Z(\varphi,(1-k^2)^{1/2}) \]  

(5)

\[ \Pi(n,k) = K(k) + \frac{\pi}{2} \delta \left[ 1 - \Lambda_0(\varphi,k) \right]; \delta = [n(1-n)^{-1}(n - \sin^2 k)^{-1}]^{1/2}; \]

(6)

\[ \varphi = \arcsin\left(\frac{1-n}{\cos^2 k}\right)^{1/2} \]

The modified expressions are as follows:

\[ H^*_r(r,z) = \frac{\sigma^*}{2\pi\mu_0(z)} \sum_{n=1}^2 (-1)^{n-1} \frac{k^+_{n}(z-h)\sqrt{r^2+(z-h)^2}}{\sqrt{rr_n}(\sqrt{r^2+(z-h)^2} + r)} K(k^+_n) + \]

\[ + \frac{\pi}{2} \text{sign}(z-h)\text{sign}(\sqrt{r^2+(z-h)^2} - r_n)[1 - \Lambda_0(\theta_{in}^*, k^+_n)] + \frac{\pi}{2} \text{sign}(z-h)[1 - \Lambda_0(\theta_{in}^*, k^+_n)] \]  

(7)

\[ H^-_r(r,z) = -\frac{\sigma^*}{2\pi\mu_0(z)} \sum_{n=1}^2 (-1)^{n-1} \frac{k^-_{n}\sqrt{r^2+z^2}}{\sqrt{rr_n}(\sqrt{r^2+z^2} + r)} K(k^-_n) + \]

\[ + \frac{\pi}{2} \text{sign}(z)\text{sign}(\sqrt{r^2+z^2} - r_n)[1 - \Lambda_0(\theta_{in}^*, k^-_n)] + \frac{\pi}{2} \text{sign}(z)[1 - \Lambda_0(\theta_{in}^*, k^-_n)] \]  

(8)

where

\[ k^+_n = \frac{4rr_n}{\left( r + r_n \right)^2 + (z-h)^2}^{1/2} \]

\[ k^-_n = \frac{4rr_n}{\left( r + r_n \right)^2 + z^2}^{1/2} \]

\[ h^+_n = \frac{2r}{r + (r^2 + (z-h)^2)^{1/2}} \]

\[ h^-_n = \frac{2r}{r - (r^2 + (z-h)^2)^{1/2}} \]

\[ h^+_n = \frac{2r}{r + (r^2 + z^2)^{1/2}} \]

\[ h^-_n = \frac{2r}{r - (r^2 + z^2)^{1/2}} \]

\[ r_1 = r_{in} \]

\[ r_2 = r_{out} \]
\[ \theta_{1n} = \arcsin \left( \frac{1 - h_1}{1 - k_n^2} \right)^{1/2} \quad \theta_{2n} = \arcsin \frac{|z - h|}{(r^2 + z^2)^{1/2} + r} \]\[ \theta_{1n} = \arcsin \left( \frac{1 - h_1}{1 - k_n^2} \right)^{1/2} \quad \theta_{2n} = \arcsin \frac{|z|}{(r^2 + z^2)^{1/2} + r} \]

\( K(k) \) – complete elliptic integral of the first kind.

\( E(k) \) – complete elliptic integral of the second kind.

\( \Pi(\varepsilon, k) \) – complete elliptic integral of the third kind.

\( \Lambda_0(\varphi, k) \) – Heuman’s lambda-function

\( Z(\varphi, k) \) – Jacobi zeta-function

### 2.1 Solving elliptic integrals

In order to be able to use formulas reported in previous section one has to implement a procedure for evaluation of various special functions (e.g. elliptic integrals and so on). And in order to solve this problem there are different possibilities: approximation scheme presented in [9] or alternative forms of elliptic integrals. There were decided to use Carlson’s integrals [10]. They are a modern alternative to Legendre forms. The Legendre forms may be expressed in terms of the Carlson forms. The Carlson’s elliptic integrals are:

\[ R_\varphi(x, y, z) = \frac{1}{2} \int_0^\infty \frac{dt}{((t + x)(t + y)(t + z))^{1/2}} \]

\[ R_j(x, y, z, p) = \frac{3}{2} \int_0^z \frac{dt}{t + p} \left( \frac{1}{(t + x)(t + y)(t + z)} \right)^{1/2} \]  \( (9) \)

\[ R_c(x, y) = R_\varphi(x, y, y) = \frac{3}{2} \int_0^y \frac{dt}{t + x(t + y)^{1/2}} \]

\[ R_j(x, y, z, p) = R_j(x, y, z, z) = \frac{3}{2} \int_0^z \frac{dt}{t + z} \left( \frac{1}{(t + x)(t + y)(t + z)} \right)^{1/2} \]

The Carlson’s elliptic integrals are calculated using an iterative scheme described in [10]. In the scheme we use the initial values to calculate the values at which the convergence condition is satisfied. Then, a series of expansions is calculated, which is the output value of the function. Incomplete elliptic integrals, which are used in calculation of Heuman’s lambda-function and formulas (4-8), can be expressed by Carlson’s elliptic integrals as:

\[ F(\varphi, k) = \sin \varphi R_\varphi(\cos^2 \varphi, 1 - k^2 \sin^2 \varphi, 1) \]

\[ E(\varphi, k) = \sin \varphi R_\varphi(\cos^2 \varphi, 1 - k^2 \sin^2 \varphi, 1) - \frac{1}{3} k^2 \sin^3 \varphi R_D(\cos^2 \varphi, 1 - k^2 \sin^2 \varphi, 1) \]  \( (10) \)

\[ \Pi(\varphi, n, k) = \sin \varphi R_\varphi(\cos^2 \varphi, 1 - k^2 \sin^2 \varphi, 1) + \frac{1}{3} n \sin^3 \varphi R_j(\cos^2 \varphi, 1 - k^2 \sin^2 \varphi, 1 - n \sin^2 \varphi, 1) \]

For the case of complete elliptic integrals we put \( \varphi = \frac{\pi}{2} \). Also by Carlson’s elliptic integrals we can obtain values for Heuman’s lambda-function \( \Lambda_0(\varphi, k) \):
\[ \Lambda_\varphi(k) = \frac{2}{\pi} \sin(k) R_p(0, \cos^2 \varphi, 1) - \frac{\sin^2 \varphi R_p(0, \cos^2 \varphi, 1)}{3} R_p(\cos^2 k, 1 - \cos^2 \varphi \sin^2 k, 1) \]

\[ - \frac{\cos^2 \varphi \sin^2 k \ast R_p(0, \cos^2 \varphi, 1)}{3} R_p(\cos^2 k, 1 - \cos^2 \varphi \sin^2 k, 1) \]

(11)

3. Testing analytical method

For the checking the selected algorithm the computer program was developed, which computes complete and incomplete elliptic integrals as well as components for the magnetic field of permanent ring magnets. Further were made comparisons with known data.

3.1 Comparison with numerical data

As the example of numerical data was selected the data from [7]. Further was performed digitization of plots of distribution of the magnetic field components of the upper surface and have made a comparison between last one and our calculation. The parameters of the magnets were chosen according to [7]: \( r_i = 0.025 \text{ m}, r_o = 0.028 \text{ m}, h = 0.003 \text{ m}, \sigma_0 = 1.0 \text{ T} \). The comparison is shown in figures 1–2.

![Figure 1](image1.png)

**Figure 1.** Comparison for radial component of magnetic field with \( z = 0.003 \text{ m} \).

![Figure 2](image2.png)

**Figure 2.** Comparison for axial component of magnetic field with \( z = 0.004 \text{ m} \).

Comparison of the analytical formula derived for the case \( r = 0 \) and the calculated distribution of the magnetic field is presented in figure 3. For this formulas for \( k_i \) and \( k_o \) were rewritten and linked with (4) and (6). Further comparison was performed with results obtained by the following formula:

\[ B_z = \frac{\sigma}{2} \left[ \left( \frac{z}{r_o^2 + z^2} \right)^{1/2} - \frac{z - h}{(r_o^2 + (z - h)^2)^{1/2}} \right] - \left[ \left( \frac{z}{r_i^2 + z^2} \right)^{1/2} - \frac{z - h}{(r_i^2 + (z - h)^2)^{1/2}} \right] \]

(12)

In another case we have used the data reported in [11]. In the figure 4 one can see that 1D distribution obtained by the analytical method agrees well with numerical calculations obtained using finite element method (FEM) \( r_i = 0.0125 \text{ m}, r_o = 0.035 \text{ m}, h = 0.004 \text{ m}, z = 0.006 \text{ m}, \sigma_0 = 1.2 \text{ T} \). As author of [11, 12] has noticed the problem of FEM is a relatively low speed of calculation. The 2D plane calculations with \( 140 \times 140 \) points took about 300 minutes. Code based on analytical formulas performs similar calculations in 1–3 seconds.
3.2 Comparison with the experiment

Further comparison of the magnetic field distribution was made using the experimental data presented in [11]. The permanent magnet parameters are as follows: \( r_{\text{in}} = 0.0125 \, \text{m}, \) \( r_{\text{out}} = 0.035 \, \text{m}, \) \( h = 0.004 \, \text{m}, \) \( \sigma^* = 1.2 \, \text{T}. \) Results of the comparison are given in figure 4.

In [8] the results of experiments performed in VNIIA are reported. In this research magnetic field is produced by 2 SmCo magnets with parameters: \( r_{\text{in}} = 0.010 \, \text{m}, \) \( r_{\text{out}} = 0.013 \, \text{m}, \) \( h = 0.010 \, \text{m}. \) Parameters of magnetic density were selected by SmCo magnetic characteristics, the left magnet had magnetic density \( \sigma = 1.02 \, \text{T} \) and right one \( \sigma = 0.89 \, \text{T}. \) The comparison (figure 5) showed that the computed data are close to the experimental results. Also a 2D distribution was modeled in order to see the full picture of axial and radial components of magnetic field (figures 6–7) that can be used in particle-in-cell calculations.
4. Electrodynamics structure of a plasma

After the calculation of magnetic field in the Penning discharge cell, it became possible to get preliminary results on the electrodynamics structure of plasma. Particle-in-cell method was chosen for the study. This method allows to obtain a lot of information about the structure of plasma in the Penning discharge cell.

For the numerical calculation parameters from Mamedov and Schitov experiment [8] was chosen: \(D_a = L_a = 28\) mm, \(L = 30\) mm, \(p = 0.5\) mTorr, \(U_a = 2\) kV, \(I_{dis} = 1.187\) mA, \(\sigma = 0.88\) T, \(r_{in} = 10\) mm, \(r_{out} = 13\) mm, \(h = 5\) mm, \(L_m = 36\) mm; here \(D_a\) – diameter of anode; \(L_a\) – length of anode; \(L\) – distance between anodes; \(p\) – pressure in discharge cell; \(U_a\) – voltage on anode; \(I_{dis}\) – discharge currents; \(r_{in}\) – inner radius of magnets; \(r_{out}\) – outer radius of magnets; \(h\) – height (length) of magnets; \(L_m\) – distance between two magnets. In the considered case modular ion source with Al anode and cathodes made of different materials was used. The permanent magnets were located behind the cathode and anticathode. The material of permanent magnets is SmCo.

As the result of numerical simulation 2D distributions (figures 8 – 12) of electric potential, electron density, \(H_2^+\) ion density, temperature of electrons and \(H_2^+\) ions were obtained. Also an energy distribution function of electrons and ions was calculated (figure 13).
Figure 9. 2D distribution of electron density[$\text{cm}^3$].

Figure 10. 2D distribution $\text{H}_2$ ion density[$\text{cm}^3$].

Figure 11. 2D distribution of electron’s temperature[eV].
As we can see, the highest ion and electron density (see figure 9–10) are observed at the center of the computational area in the vicinity of the axis of symmetry. The electron density also decreases with distance along the r-coordinate. In this case, a large dependence on the position relative to the center of symmetry is observed.

Analysis of the distribution of ion and electron temperature (see figure 11–12) shows that the lowest temperatures are near the boundaries of the discharge cell. In the case of electron temperature distribution (figure 11), regions with the highest temperature are closer to the middle along the axial component. On the other hand, in the case of ion temperature distribution, we can note that the regions with the lowest temperature are located on the “upper” and “lower” side of the discharge cell with the temperature of about 50 eV. The highest temperature regions are located near the center of symmetry of discharge cell. Looking at the plot of the distribution function, we can say their energy does not exceed 2 keV.

![Figure 12. 2D distribution of H₂ ion's temperature[eV].](image)

![Figure 13. Function of distribution for electrons and H₂ ions by energy.](image)
5. Conclusion
In conclusion computed code for evaluation of two-dimensional distribution of magnetic field created by annular magnets is developed based on the model presented in [7, 8]. The computed code was validated against numerical and experimental data available in literature. Preliminary results on the simulation of electrodynamic structure of Penning gas discharge plasma (at parameters given in [6]) in the presence of non-uniform external magnetic field created by annular permanent magnets were calculated using particle-in-cell method with the distribution of the magnetic field calculated using developed code.

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