On the distribution of Lagrangian accelerations in turbulent flows

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Abstract. Superstatistical Lagrangian stochastic models are shown to predict accurately the distribution of the magnitude of the acceleration vector in three-dimensional high Reynolds-number turbulence. Distributions are closely log-normal having high tails that are nearly coincident with measured distributions of enstrophy. The findings support the view that the dominant contribution to extreme accelerations comes from centripetal accelerations induced by vortex filaments.

Following the seminal experiments of Bodenschatz and co-workers [1, 2], there has been considerable interest in distributions of components of the Lagrangian acceleration vector, A, in turbulence and the origin of the observed extreme intermittency. The interpretation of this data has spawned approaches related to multi-fractal scaling [3], multi-fractal random walks [4], non-extensive entropy [5], and more recently super-statistics [6, 7]. Superstatistical approaches are the most ambitious and attempt to model stochastically the evolution of Lagrangian accelerations and Lagrangian velocity on all scales.

In this paper, the superstatistical approach is used to predict distributions of magnitude of the three-dimensional acceleration vector |A|. The observed log-normality of |A| [8, 9] is shown to be well reproduced by the approach. The distribution of the acceleration magnitude is also found to be well approximated by the observed distribution of enstrophy [10]. These findings

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support the view [11]–[13] that fluid-motions in the vicinity of vortex filaments are responsible for extreme intermittency of Lagrangian accelerations.

Our experimental data of Lagrangian accelerations in highly turbulent flows was obtained by using silicon strip detectors [14, 15]. Each detector recorded a one-dimensional projection of the trajectories of small solid tracer particles advected by the flow with both high spatial-resolution (512 pixels and 0.1 sub-pixel resolution) and high sampling rates (up to 70 000 measurements s\(^{-1}\)). The experiments were conducted in a von Karman-type flow: two coaxial, counter-rotating disks drove water into turbulent motion in a cylindrical container (see [14] for details of the experimental setup). A pulsed laser illuminated the centre of the flow, which was imaged onto four detectors. Two pairs of detectors imaged the illuminated region from orthogonal directions so as to record all the three components. The one-dimensional components were then matched to recover the three-dimensional trajectories [14]. In this way, we were able to obtain sufficiently large data sets from which the Lagrangian accelerations were computed by differentiating the trajectories and low-pass filtering to reduce the experimental noise. The experiments were conducted up to a Taylor-microscale Reynolds number \(R_\lambda = 970\).

In the superstatistical approach, acceleration statistics conditioned on the dissipation rate, \(\varepsilon\), are taken to be Gaussian. Intermittency arises from broad distributions of dissipation rates. In accordance with Kolmogorov’s (1962) refined similarity theory, the conditional acceleration variance is prescribed by \(\sigma^2_{A|\varepsilon} = a_0^\ast \varepsilon^{3/2} \nu^{-1/2}\), where \(\nu\) is the kinematic viscosity. Optimal model agreement with experimental data for distributions of Lagrangian accelerations in turbulent flows with Reynolds numbers between \(R_\lambda = 140\) and 970 was obtained when \(a_0^\ast \approx 3.3\) [6]. With these assumptions, model predictions for unconditional distributions of single components of the Lagrangian acceleration vector are determined by

\[
P(A) = \int \frac{1}{(2\pi\sigma^2_{A|\varepsilon})^{1/2}} \exp \left( -\frac{A^2}{2\sigma^2_{A|\varepsilon}} \right) p(\varepsilon) \, d\varepsilon, \tag{1}
\]

where \(p(\varepsilon)\) is the distribution of dissipation rates. By focusing upon the dynamical properties of a single degree of freedom, the modelling approach circumvents the strong criticisms made recently by Gotoh and Kraichnan [16] of the application of Tsallis statistics and superstatistics to turbulence.

For sufficiently large Reynolds numbers, the results of direct numerical simulations (DNS) [9] indicate, in accordance with the conjecture of Oboukhov [17], that distributions of \(\chi = \ln(\varepsilon/\langle \varepsilon \rangle)\) are approximately Gaussian. Although not exactly correct [18] when utilized within the superstatistical approach, the assumption of log-normality is sufficient to account accurately for all of the currently available experimental data for intermittency exponents that characterize the observed extended self-similarity of Lagrangian velocity structure functions [4, 6]. Reliable experimental data for exponents characterizing sixth- and higher-order Lagrangian velocity structure functions, which may indicate the need for a refinement of the log-normal assumption for high Reynolds-number turbulence, are yet to be established. Consistency between (1) and the mean rate of dissipation \(\langle \varepsilon \rangle\) requires that the mean and variance of \(\chi\) be related by \(\langle \chi \rangle = -1/2\sigma^2_\chi\). Data from DNS [9] indicate that \(\sigma^2_\chi = 0.35 + 0.29 \ln R_\lambda\). Model predictions for distributions of one-component of acceleration are found to be in nearly precise agreement with experimental data acquired for \(|A|/\sigma_A \leq 30\) [1] and, as shown in figure 1, are in good agreement with simulation data for the pressure gradient acquired for \(|\nabla P|/\sigma_{\nabla P} \leq 150\) [16].
Figure 1. Comparison of the predicted distribution of one-component of acceleration for $R_\lambda = 381$ (——) and data for the pressure gradient $|\nabla P|$ extracted from high-resolution direct numerical simulations [16].

Predictions for the distribution of magnitude of the acceleration vector $|A|$, and so $\ln |A|$, can be obtained directly from the superstatistical approach using

$$P(|A|) = 4\pi A^2 \int \frac{1}{(2\pi \sigma_{A|e}^2)^{3/2}} \exp \left( -\frac{A^2}{2\sigma_{A|e}^2} \right) p(\varepsilon) \, d\varepsilon. \quad (2)$$

Figure 2 shows that, except for the core, (2) is in close agreement with experimental data acquired for $R_\lambda = 690$. Model predictions for the central moments of $\ln |A|$, $\mu_4 = 3.07$ and $\mu_6 = 16.3$, are very close to the central moments, $\mu_4 = 3$ and $\mu_6 = 15$, of a Gaussian distribution. The tails retain their log-normal form when Lagrangian accelerations corresponding to (2) are simulated stochastically and then filtered in the same manner as the experimental data. Model predictions for $\sigma_{\ln |A|}^2$ also compare favourably with experiment. For $R_\lambda = 285, 690$ and 960, the model predicts that $\sigma_{\ln |A|}^2 = 0.96, 1.10$ and 1.14 respectively whilst experimental estimates based typically on data for $|A| \leq 25\sigma_A$ are 0.90, 0.95 and 1.0. The apparent under-prediction of the exponential tails of the distribution for small accelerations can be attributed, at least in part, to the presence of noise in the experiment. The under-prediction of the core can be attributed, in part, to the neglect of the observed conditional dependence of the acceleration variance upon velocity [19]. When the acceleration variances are isotropic, they must take the form of a general second-order isotropic tensor,

$$[\sigma_{A|u}]_{ij} = [f(|u|) - g(|u|)] \frac{u_i u_j}{u^2} + g(|u|)\delta_{ij}, \quad (3)$$

where the longitudinal and the transverse functions, $f(|u|)$ and $g(|u|)$, describe the covariance of components in acceleration parallel and orthogonal to the velocity vector. Data from DNS
Figure 2. Comparison of predicted (——) and measured (●) distributions of \( \ln |A| \) for \( R_\lambda = 690 \) on a linear–linear scale (left) and on a log-linear scale (right). Also shown (-----) is a normal distribution for \( \ln |A| \) with mean and variance equivalent to the model predictions. Model predictions are based upon a log-normal distribution for dissipation and velocity-independent acceleration statistics.

for these functions are found to be well represented by cubic polynomials of \( u^2 \); namely by \( f = f_0 + f_1 u^2 + f_2 u^4 + f_3 u^6 \) and \( g = g_0 + g_1 u^2 + g_2 u^4 + g_3 u^6 \) [19], where the coefficients depend only weakly on Reynolds number. Figure 3 shows that, when these velocity-dependent acceleration variances are incorporated into the modelling approach, model predictions for the
core of $P(|A|)$ are brought into nearly precise agreement with the experimental data whilst the tails of $P(|A|)$ remain closely exponential and log-normal in form. Predictions were obtained using the approach of Reynolds [20] and data for $\sigma_A|u|$ acquired in DNS for $R_\lambda = 140$ [19]. Model predictions obtained using recently published experimental data for $\sigma_A|u|$ acquired for a turbulent flow with $R_\lambda = 690$ [21] are barely discernible from those presented in figure 3.

The modelled distribution of the acceleration magnitude, $P(|A|)$, is seen in figure 4 to account accurately for the observed inter-dependence of the components of acceleration. This inter-dependence is further illustrated in figure 5 which shows the predicted ratio of probabilities, $P(A_y,A_z)/P(A_y)P(A_z)$ for $R_\lambda = 690$. For independent components, $P(A_y,A_z)/P(A_y)P(A_z) = 1$. In accordance with experimental observations [8], it is seen that the predicted joint probability of simultaneously observing two components taking high absolute values is higher than the product of the probabilities, $P(A_y)$ and $P(A_z)$ by more than three orders of magnitude. It is less probable to observe one component taking a small value and another taking a large absolute value.

The predicted and observed log-normality of $|A|$ suggests that intermittency of the acceleration is closely related to the multiplicative cascade process and vortex filament stretching. This scenario provides a natural explanation for the autocorrelation timescales for one-component of acceleration and for $|A|$ being comparable with Kolmogorov dissipation timescale and integral Lagrangian timescales, respectively [5]. This is because the radii of vortex filaments are comparable with the dissipation length scale and because their lifetimes (i.e. the timescales on which fluid-particle remains trapped) extend up to the largest timescales in the flow [22]–[24]. Vortex filaments also account naturally for the velocity-dependence of acceleration statistics [19]. Vortex filaments associated with vorticity $\omega$, produce velocities $u = 1/2\omega \wedge l$ at their circumferences, located at a distance $l$ from their cores. The associated centripetal acceleration,
Figure 5. Predicted ratio of probabilities, $P(A_y, A_z)/P(A_y)P(A_z)$ for $R_\lambda = 690$.

$A_c$, of a fluid-particle around a filament is, therefore, proportional to the enstrophy $\Omega = \omega^2/2$. Zeff et al [10] measured enstrophy and dissipation in a laboratory-scale experiment with $R_\lambda = 54$. For this small Reynolds number, the results of DNS indicate that departures from log-normality are significant [9]. For example, the central moments of $\chi$ extracted from the results of DNS for isotropic turbulence with $R_\lambda = 54$, $\mu_3 = -0.22$, $\mu_4 = 3.23$ and $\mu_6 = 19.8$ are markedly different from the log-normal values of 0, 3 and 15. The impact of these departures from log-normality upon the modelled distributions of Lagrangian accelerations can be calculated by first constructing a distribution, $P(\chi)$, that is consistent with the DNS data for the central moments. The least-biased choice for $P(\chi)$ and the one adopted here is

$$p(\chi) = \exp(c_0 + c_1\chi + c_2\chi^2 + c_3\chi^3 + c_4\chi^4 + c_6\chi^6).$$

This distribution maximizes the uncertainty about the missing information contained in fifth, seventh and higher-order moments. The six coefficients, $c_i$, are determined from the conditions imposed by normalization and consistency with the DNS data for $\langle \chi \rangle$, $\langle \chi^2 \rangle$, $\mu_3$, $\mu_4$ and $\mu_6$. Figure 6 shows that measured distribution of dissipation rates is well represented by (4) when moments of the distributions are drawn from the results of direct numerical simulations [9]. The high-tail of the corresponding distribution of acceleration magnitudes are seen in figure 6 to mimic the observed distribution of enstrophy. The predicted correlation, $\langle \varepsilon |A| \rangle \approx 1.8 \langle |A| \rangle \langle \varepsilon \rangle$, between dissipation rates and accelerations is also comparable with the measured correlation between dissipation rates and enstrophy $\langle \varepsilon \Omega \rangle \approx 1.7 \langle \Omega \rangle \langle \varepsilon \rangle$. The interdependence of $\varepsilon$ and $|A|$ is further illustrated in figure 7 which shows the joint probability density $P(\varepsilon, |A|)$ divided by the corresponding marginal distributions $P(\varepsilon)$ and $P(|A|)$. It is
Figure 6. Comparison of predicted distributions of $|A|$ (——) and dissipation (-----) and measured distributions of enstrophy (○) and dissipation (×) for $R_{\lambda} = 54$. Model predictions are based upon a maximum-missing information distribution for dissipation and velocity-independent acceleration statistics.

Figure 7. Logarithm of the ratio of the joint probability to the product of the marginal probabilities for $\varepsilon$ and $|A|$ for $R_{\lambda} = 54$. 
evident from figure 7 that while values of $\varepsilon$ and $|A|$ close to their mean values are uncorrelated, the occurrence of large values of $\varepsilon$ and $|A|$ are strongly correlated (in figure 7 the highest values of $\varepsilon$ and $|A|$ occur simultaneously about 30 times more often than they would if they were uncorrelated). The converse is also evident. This predicted behaviour is mirrored by the observed ratio of the joint probability and by the joint probability function for $\varepsilon$ and $\Omega$ observed in experiment [10] and extracted from DNS [9].

These findings suggest that the dominant contribution to the extreme accelerations predicted by the superstatistical modelling approach correspond to centripetal accelerations occurring on the Kolmogorov dissipation length-scale that are associated with highly dissipative vortex filaments. The modelled dynamics associated with these extreme events coincides with the dynamics observed in experiment [10]. In both cases, gradients in the dissipation rate first grow, in the absence of large accelerations. Once the gradients are large, the magnitude of the acceleration shows a sudden, rapid increase. The peaking of the acceleration is then accompanied by a rapid decline of the dissipation. In the stochastic model, these dynamics may simply be a consequence of the finite-time response of accelerations to changes in the dissipation rate. Model predictions for the cross-correlation of acceleration and Lagrangian analogies of enstrophy, $\langle |A(0)||\Omega(\tau) \rangle$, are, however, symmetric about $\tau = 0$ and consequently at variance with the results of DNS [9] which indicate that acceleration is ahead of enstrophy.

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References

[1] La Porta A, Voth G A, Crawford A M, Alexander J and Bodenschatz E 2001 Nature 409 1017
[2] Mordant N, Metz P, Michel O and Pinton J-F 2001 Phys. Rev. Lett. 87 214501
[3] Biferale L, Boffetta G, Celani A, Devenish B J, Lanotte A and Toschi F 2004 Phys. Rev. Lett. 93 064502
[4] Mordant N, Delour J, Lévéque E, Arnéodo A and Pinton J-F 2002 Phys. Rev. Lett. 89 254502
[5] Beck C 2001 Phys. Rev. Lett. 87 180601
[6] Reynolds A M 2003 Phys. Rev. Lett. 91 084503
[7] Beck C and Cohen E G D 2003 Physica A 322 267
[8] Mordant N, Crawford A M and Bodenschatz E 2004 Phys. Rev. Lett. 93 214501
[9] Yeung P K and Pope S B 1989 J. Fluid Mech. 207 531
[10] Zeff B W, Lanterman D L, McAllister R, Roy R, Kostelich E J and Lathrop D P 2003 Nature 421 146
[11] Biferale L, Boffetta G, Celani A, Lanotte A and Toschi F 2005 Phys. Fluids 17 021701
[12] Lee C, Yeo K and Choi J-I 2004 Phys. Rev. Lett. 92 144502
[13] Mordant N, Lévéque E and Pinton J-F 2004 New J. Phys. 6 116
[14] Voth G A, La Porta A, Crawford A M, Alexander J and Bodenschatz E 2002 J. Fluid Mech. 469 121
[15] Voth G A, La Porta A, Crawford A M, Bodenschatz E and Alexander J 2001 Rev. Sci. Instrum. 72 4348
[16] Gotoh T and Kraichnan R H 2004 Physica D 193 231
[17] Oboukhov A M 1962 J. Fluid Mech. 13 77
[18] Monin A S and Yaglom A M Statistical Fluid Mechanics vol 2, ed J L Lumley (Cambridge, MA: MIT Press)
[19] Sawford B L, Yeung P K, Borgas M S, Vedula P, La Porta A, Crawford A M and Bodenschatz E 2003 Phys. Fluids 15 3478
[20] Reynolds A M 2004 *Physica A* **340** 298
[21] Crawford A M, Mordant N and Bodenschatz E 2005 *Phys. Rev. Lett.* **94** 024501
[22] Siggia E D 1981 *J. Fluid Mech.* **107** 375
[23] She Z S, Jackson E and Orszag S A 1990 *Nature* **344** 226
[24] Vincent A and Meneguzzi M 1991 *J. Fluid Mech.* **225** 1