PARциально DISSIPATIVE HYPERBOLIC SYSTEMS IN THE CRITICAL REGULARITY SETTING: THE MULTI-DIMENSIONAL CASE

TIMOTHÉE CRIN-BARAT, RAPHAËL DANCHIN

Abstract. We are concerned with quasilinear symmetrizable partially dissipative hyperbolic systems in the whole space $\mathbb{R}^d$ with $d \geq 2$. Following our recent work [10] dedicated to the one-dimensional case, we establish the existence of global strong solutions and decay estimates in the critical regularity setting whenever the system under consideration satisfies the so-called (SK) (for Shizuta-Kawashima) condition. Our results in particular apply to the compressible Euler system with damping in the velocity equation.

Compared to the papers by Kawashima and Xu [27, 28] devoted to similar issues, our use of hybrid Besov norms with different regularity exponents in low and high frequency enable us to pinpoint optimal smallness conditions for global well-posedness and to get more accurate information on the qualitative properties of the constructed solutions.

A great part of our analysis relies on the study of a Lyapunov functional in the spirit of that of Beauchard and Zuazua in [2]. Exhibiting a damped mode with faster time decay than the whole solution also plays a key role.

Introduction

We are concerned with first order $n$-component systems in $\mathbb{R}^d$ of the type:

$$A^0(V) \frac{\partial V}{\partial t} + \sum_{j=1}^{d} A^j(V) \frac{\partial V}{\partial x_j} = H(V)$$

where the (smooth) matrices valued functions $A^j$ $(j = 0, \cdots, d)$ and vector valued function $H$ are defined on some open subset $\mathcal{O}_V$ of $\mathbb{R}^n$ and the unknown $V = V(t,x)$ depends on the time variable $t \in \mathbb{R}_+$ and on the space variable $x \in \mathbb{R}^d$ $(d \geq 2)$. We assume that the system is symmetrizable and satisfies additional structure assumptions that will be specified in the next section.

System (1) is supplemented with initial data $V_0 \in \mathcal{O}_V$ at time $t = 0$. We are concerned with the existence of global strong solutions in the case where $V_0$ is close to some constant state $\bar{V}$ such that $H(\bar{V}) = 0$.

In the nondissipative case, that is if $H \equiv 0$, it is classical that symmetrizable quasilinear hyperbolic systems supplemented with initial data with Sobolev regularity $H^s$ such that $s > 1 + d/2$ admit local-in-time strong solutions (see e.g. [3]), that may develop singularities in finite time even if the initial data are small (see for instance the works by Majda in [18] or Serre in [21]). By contrast, if in the neighborhood of $\bar{V}$, the term $H(V)$ has the ‘good’ sign and acts on each component of the solution (like e.g. $H(V) = D(V - \bar{V})$ for some matrix $D$ having all its eigenvalues with positive real part), then smooth perturbations of $\bar{V}$ give rise to global-in-time solutions that tend exponentially fast to $\bar{V}$ when time goes to $\infty$.

In most physical situations that may be modelled by systems of the form (1) however, some components of the solution satisfy conservation laws and only partial dissipation occurs, that is to say, the term $H(V)$ acts only on a part of the solution. Typically, this happens in gas dynamics where the mass density and entropy are conserved, or in numerical schemes involving conservation laws with relaxation. A well known example is the damped compressible Euler system for isentropic flows that will be addressed at the end of the paper. For this system, it is known from the works of Wang and Tang [24] or Sideris, Thomases and Wang [23] that the
dissipative mechanism, albeit only present in the velocity equation, can prevent the formation of singularities that would occur for $H \equiv 0$.

Looking for sufficient conditions on the dissipation term $H$ guaranteeing the global existence of strong solutions for perturbations of a constant state $\bar{V}$ goes back to the thesis of Kawashima [15] and to the more recent work by Yong in [29]. Two main conditions arise. The first one is the so-called (SK) (for Shizuta-Kawashima) stability condition, see [22], that ensures that the damping is strong enough to prevent the solutions emanating from small perturbations of $\bar{V}$ from blowing up. The second one is the existence of a (dissipative) entropy which provides a suitable symmetrisation of the system compatible with $H$. Thanks to those two conditions, Yong [29] obtained a global existence result for systems that are more general than those that have been considered by Kawashima.

More recently, by taking advantage of the properties of the Green kernel of the linearized system around $\bar{V}$ and on the Duhamel formula, Bianchini, Hanouzet and Natalini in [5] pointed out the convergence in $L^p$ of global solutions to $\bar{V}$, with the rate $O(t^{-(d+1)/p})$ when $t \to +\infty$, for all $p \in [\min\{d, 2\}, \infty]$. Let us further mention that Kawashima and Yong proved decay estimates in regular Sobolev space in [17].

A few years ago, Kawashima and Xu in [27] and [28] extended the prior works on partially dissipative hyperbolic systems satisfying the (SK) and entropy conditions to critical non-homogeneous Besov spaces. To obtain their results, they used the symmetrisation from [16] and applied a frequency localisation argument relying on the Littlewood-Paley decomposition. In their work, the equivalence between Condition (SK) and the existence of a compensating function allows to exhibit the global-in-time $L^2$ integrability properties of all the components of the solution.

However, it is known that the condition (SK) is not optimal in the sense that there exist many systems that do not verify it but for which one can prove global well-posedness results, see e.g. [20, 4, 14]. In [2], Beauchard and Zuazua developed a new and systematic approach that allows to establish global existence results and to describe large time behavior of solutions to partially dissipative systems that need not satisfy Condition (SK). Looking at the linearization of System (1) around a constant solution, namely (denoting from now on $\partial_t \triangleq \frac{\partial}{\partial t}$ and $\partial_j \triangleq \frac{\partial}{\partial x_j}$),

$$\partial_t Z + \sum_{j=1}^m A^j \partial_j Z = -LZ,$$

(2)

they show that Condition (SK) is equivalent to the Kalman maximal rank condition on the matrices $A^j$ and $L$. More importantly, they introduce a Lyapunov functional equivalent to the $L^2$ norm that encodes enough information to recover dissipative properties of (2). Considering such a functional is motivated by the classical (linear) control theory of ODEs, and is also related to Villani’s paper [24]. Back to the nonlinear system (1), Beauchard and Zuazua obtained the existence of global smooth solutions for perturbations of a constant equilibrium $\bar{V}$ that satisfies (SK) Condition. Furthermore, using arguments borrowed from Coron’s return method [9], they were able to achieve certain cases where (SK) does not hold.

Our aim here is to extend the results we obtained recently in the one-dimensional case [10] to multi-dimensional partially dissipative hyperbolic systems (see also the on-going work [6] by the first author dedicated to the relaxation limit of a non conservative multi-fluid system that does not satisfy the (SK) condition). More precisely, under Condition (SK), we shall develop Beauchard and Zuazua’s approach as suggested by the second author in [12] and prove the global well-posedness of (1) supplemented with data that are close to $\bar{V}$ in an optimal critical regularity setting. As in the study of the compressible Navier-Stokes system and related models (see e.g. [7, 8, 11, 13]) it will appear naturally that in order to get optimal results, one has to use functional spaces with different regularity exponents in low and high frequencies. Here,
Beauchard and Zuazua’s approach will give us the information that the low frequencies (resp. high frequencies) of the solution of the linearized system behave like the heat flow (resp. are exponentially damped). Furthermore, in order to improve our low frequency analysis, we will exhibit a damped mode with better decay properties than the whole solution. Thanks to that, we will end up with more accurate estimates and a weaker smallness condition that in prior works (in particular [27]) and refine the decay estimates that were obtained in [28].

The paper is arranged as follows. In the first section, we specify the structure of the class of partially dissipative hyperbolic systems we aim at considering, and explain the construction of a Lyapunov functional that will be the key to our global results. In Section 2 we state the main results of the paper. Section 3 is devoted to the proof of a first global existence result and time decay estimates for general partially dissipative systems satisfying the Shizuta-Kawashima condition. In section 4 under additional structure assumptions (that are satisfied by the compressible Euler system with damping), we obtain a more accurate global existence result. Some technical results are proved or recalled in Appendix.

1. Hypotheses and method

In this section, we specify our assumptions on the system under consideration, and explain the main steps of our approach.

1.1. Friedrichs-symmetrizability. First, we fix some constant solution \( \bar{V} \in \mathcal{O}_V \) of (1) (thus satisfying \( H(\bar{V}) = 0 \)). To ensure the local well-posedness, we assume that (1) is Friedrichs-symmetizable, namely that there exists a smooth function \( S : V \mapsto S(V) \) defined on \( \mathcal{O}_V \), valued in the set of symmetric and positively definite matrices such that for all \( V \in \mathcal{O}_V \), the matrices \((SA^0)(V), \ldots, (SA^d)(V)\) are symmetric and, in addition, \((SA^0(V))\) is definite positive.

Denoting \( \bar{H} \triangleq SH \) and \( \bar{A}^j \triangleq SA^j \) for \( j \in \{0, \ldots, d\} \), System (1) rewrites

\[
\bar{A}^0(V)\partial_t V + \sum_{j=1}^d \bar{A}^j(V)\partial_j V = \bar{H}(V).
\]

Then, setting \( Z \triangleq V - \bar{V}, L \triangleq -D_V \bar{H}(\bar{V}) \) and \( r(Z) \triangleq \bar{H}(\bar{V} + Z) + LZ \), we get

\[
\bar{A}^0(V)\partial_t Z + \sum_{j=1}^d \bar{A}^j(V)\partial_j Z + LZ = r(Z).
\]

By construction, the remainder \( r \) is at least quadratic with respect to \( Z \).

Second, we assume that System (1) is partially dissipative in the following meaning:

(i) The whole space \( \mathbb{R}^n \) may be decomposed into \( \mathbb{R}^n = \mathcal{M} \bigoplus \mathcal{M}^\perp \) where

\[
\mathcal{M} = \{ \phi \in \mathbb{R}^n, \langle \phi, \bar{H}(V) \rangle = 0 \text{ for all } V \in \mathcal{O}_V \}.
\]

Hence, denoting by \( \mathcal{P} \) the orthogonal projection on \( \mathcal{M} \), we may write

\[
V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad \text{and} \quad H(V) = \begin{pmatrix} 0 \\ H_2(V) \end{pmatrix}
\]

where \( V_1 = \mathcal{P} V \in \mathbb{R}^{n_1}, V_2 = (I - \mathcal{P}) V \in \mathbb{R}^{n_2} \) and \( n_1 + n_2 = n \).

(ii) The linear map \( L \triangleq -D_V \bar{H}(\bar{V}) \) is an isomorphism on \( \mathcal{M}^\perp \) such that for some \( c > 0, \)

\[
\forall \eta \in \mathbb{R}^n, \quad \langle L\eta | \eta \rangle \geq c |L\eta|^2.
\]

(iii) System (1) has a block structure that is compatible with decomposition (1), namely all the matrices \( \bar{A}^j \) are diagonal by blocks (first block being of size \( n_1 \times n_1 \) and second one of size \( n_2 \times n_2 \)) and we have \( r(Z_1, 0) = 0 \) for all \( Z_1 \) close to 0. This entails that \( r \) is at least linear with respect to \( Z_2 \).
According to the above assumptions and introducing the decompositions:

\[
\begin{pmatrix}
A_0 & 0 \\
0 & A_2
\end{pmatrix}, \quad \nabla^j = \begin{pmatrix}
\nabla_{1,1}^j & \nabla_{1,2}^j \\
\nabla_{2,1}^j & \nabla_{2,2}^j
\end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ L \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 0 \\ Q \end{pmatrix},
\]

System (3) may thus be rewritten as:

\[
\begin{align*}
\nabla_{1,1}^0(V)\partial_t Z_1 + \sum_{j=1}^d \left( \nabla_{1,1}^j(V)\partial_j Z_1 + \nabla_{1,2}^j(V)\partial_j Z_2 \right) &= 0, \\
\nabla_{2,2}^0(V)\partial_t Z_2 + \sum_{j=1}^d \left( \nabla_{2,1}^j(V)\partial_j Z_1 + \nabla_{2,2}^j(V)\partial_j Z_2 \right) + L Z_2 &= Q(Z).
\end{align*}
\]

As we shall see in Section 4, the compressible Euler equations with damping, rewritten in suitable variables, satisfies the above assumptions about any constant state with positive density and null velocity.

1.2. The Shizuta-Kawashima and Kalman rank conditions. In order to specify the supplementary conditions on the structure of the system ensuring global well-posedness and present the overall strategy, let us consider the linearization of (1) about \( \bar{V} \), namely:

\[
\nabla^0 \partial_t Z + \sum_{j=1}^d \nabla^j \partial_j Z + L Z = G \quad \text{with} \quad \nabla^j := \nabla^j(\bar{V}) \quad \text{for} \quad j = 0, \ldots, d.
\]

Then, owing to the symmetry of the matrices \( \nabla^j \), the classical energy method leads to

\[
\frac{1}{2} \frac{d}{dt} \|Z\|^2_{L^2_{\Lambda_0}} + (LZ|Z) = 0 \quad \text{with} \quad \|Z\|^2_{L^2_{\Lambda_0}} \triangleq (A^0_0 Z|Z).
\]

On the one hand, since the matrix \( A_0 \) is symmetric and positive definite, we have

\[
\|Z\|^2_{L^2_{\Lambda_0}} \simeq \|Z\|^2_{L^2}.
\]

On the other hand, (5) and the definition of \( Z_2 \) guarantee that there exists \( \kappa_0 > 0 \) such that

\[
(LZ|Z) \geq \kappa_0 \|Z_2\|^2_{L^2} \quad \text{for all} \quad Z \in L^2(\mathbb{R}^d; \mathbb{R}^n).
\]

Hence, (8) yields \( L^2 \)-in-time integrability on the components of \( Z \) experiencing direct dissipation, but not on the whole solution. To compensate this lack of coercivity, following Beauchard and Zuazua in [2], we are going to introduce a lower order corrector \( \tilde{I} \) to track the optimal dissipation of the solution to (7). Since it is more natural to define that corrector on the Fourier side, let us look at (7) in the Fourier space, that is, denoting by \( \xi \in \mathbb{R}^d \) the Fourier variable,

\[
\begin{align*}
\nabla^0 \partial_t \tilde{Z} + i \sum_{j=1}^d \nabla^j \xi_j \tilde{Z} + L \tilde{Z} &= \tilde{G}.
\end{align*}
\]

Let us write \( \xi = \rho \omega \) with \( \omega \in S^{d-1} \) and \( \rho = |\xi| \). Then, the above system rewrites

\[
\begin{align*}
\partial_t \tilde{Z} + i \rho M_{\omega} \tilde{Z} + N \tilde{Z} &= \nabla^0_0^{-1} \tilde{G} \quad \text{with} \quad M_{\omega} := \nabla^0_0^{-1} \sum_{j=1}^d \omega_j \nabla^j \quad \text{and} \quad N := \nabla^0_0^{-1} L.
\end{align*}
\]

Clearly, since \( \nabla^0_0^{-1} \) is positive definite, (5) implies that there exists a positive constant (still denoted by \( \kappa_0 \)) so that

\[
\forall \eta \in \mathbb{R}^n, \quad (N \eta | \eta) \geq \kappa_0 |N \eta|^2.
\]
Fix $n - 1$ positive parameters $\varepsilon_1, \ldots, \varepsilon_{n-1}$ (bound to be small), and set
\begin{equation}
I \triangleq \Re \sum_{k=1}^{n-1} \varepsilon_k (NM_{\omega}^{k-1} \widehat{Z} \cdot NM_{\omega}^k \widehat{Z})
\end{equation}
where $\cdot$ designates the Hermitian scalar product in $\mathbb{C}^n$.

For expository purpose, assume that $G \equiv 0$. Then, differentiating $I$ with respect to time and using (11) yields
\begin{equation}
\frac{d}{dt} I + \sum_{k=1}^{n-1} \varepsilon_k |NM_{\omega}^k \widehat{Z}|^2 = -3 \sum_{k=1}^{n-1} \varepsilon_k (NM_{\omega}^{k-1} N \widehat{Z} \cdot NM_{\omega}^k \widehat{Z})
\end{equation}
\begin{equation*} + \Re \sum_{k=1}^{n-1} \varepsilon_k (NM_{\omega}^{k-1} \widehat{Z} \cdot NM_{\omega}^{k+1} \widehat{Z}) - 3 \sum_{k=1}^{n-1} \varepsilon_k (NM_{\omega}^{k-1} \widehat{Z} \cdot NM_{\omega}^k N \widehat{Z}).
\end{equation*}
As pointed out in [2] (and recalled in Appendix for the reader’s convenience), it is possible to choose positive and arbitrarily small parameters $\varepsilon_1, \ldots, \varepsilon_{n-1}$ so that (14) implies for some $C > 0$,
\begin{equation}
\frac{d}{dt} I + \frac{1}{2} \sum_{k=1}^{n-1} \varepsilon_k |NM_{\omega}^k \widehat{Z}|^2 \leq \frac{\kappa_0}{2(2\pi)^d \rho} |N \widehat{Z}|^2 + C\varepsilon_1 |N \widehat{Z}|^2.
\end{equation}
Setting $\varepsilon_0 = (2\pi)^{-d}\kappa_0/2$, taking $\varepsilon_1$ small enough, integrating on $\mathbb{R}^d$, using Fourier-Plancherel theorem and combining with (9), we end up with
\begin{equation}
\frac{d}{dt} \mathcal{L} + \mathcal{H} \leq 0 \quad \text{with} \quad \mathcal{H} \triangleq \int_{\mathbb{R}^d} \sum_{k=0}^{n-1} \varepsilon_k \min(1, |\xi|^2) |NM_{\omega}^k \widehat{Z}(\xi)|^2 \, d\xi
\end{equation}
and \( \mathcal{L} \triangleq \|Z\|_{L^2_{\omega}}^2 + \int_{\mathbb{R}^d} \min(|\xi|, |\xi|^{-1}) I(\xi) \, d\xi. \)
Clearly, if $\varepsilon_1, \ldots, \varepsilon_{n-1}$ are small enough, then $\mathcal{L} \simeq \|Z\|_{L^2_{\omega}}^2$. The question now is whether $\mathcal{H}$ may be compared to $\|Z\|_{L^2_{\omega}}^2$. The answer depends on the properties of the support of $\widehat{Z}_0$ and on the possible cancellation of the following quantity:
\begin{equation}
\mathcal{N}_\Phi := \inf \left\{ \sum_{k=0}^{n-1} \varepsilon_k |NM_{\omega}^k x|^2; \ x \in \mathbb{S}^{n-1}, \omega \in \mathbb{S}^{d-1} \right\}.
\end{equation}
At this very point, the (SK) (for Shizuta and Kawashima) condition comes into play:

**Definition 1.1.** System (11) verifies the (SK) condition at $\widehat{V} \in \mathcal{M}$ if, for all $\omega \in \mathcal{S}^{d-1}$, whenever $\phi \in \mathbb{R}^n$ satisfies $\widehat{N}\phi = 0$ and $\lambda \phi + M_\omega \phi = 0$ for some $\lambda \in \mathbb{R}$, we must have $\phi = 0$.

It is clear that Condition (SK) at $\widehat{V}$ is equivalent to:
\[ \forall \omega \in \mathbb{S}^{d-1}, \ ker N \cap \{\text{eigenvectors of } M_\omega\} = \{0\}. \]
In order to pursue our analysis, we need the following key result (see the proof in e.g. [2]).

**Proposition 1.1.** Let $M$ and $N$ be two matrices in $\mathcal{M}_n(\mathbb{R})$. The following assertions are equivalent:
(1) $N\phi = 0$ and $\lambda \phi + M\phi = 0$ for some $\lambda \in \mathbb{R}$ implies $\phi = 0$;
(2) For every $\varepsilon_0, \ldots, \varepsilon_{n-1} > 0$, the function
\[ y \mapsto \sqrt{\sum_{k=0}^{n-1} \varepsilon_k |NM^k y|^2} \]
defines a norm on $\mathbb{R}^n$;
Thanks to the above proposition and observing that the unit sphere $S^{d-1}$ is compact, one may conclude that Condition (SK) is satisfied by the pair $(M_\omega, N)$ for all $\omega \in S^{d-1}$ if and only if $N_\gamma > 0$. Furthermore, we note that:

- if $\tilde{Z}_0$ is compactly supported then, $\mathcal{H} \gtrsim \|\nabla Z\|_{L^2}^2$, which reveals a parabolic behavior of all components of the solution;
- if the support of $\tilde{Z}_0$ is away from the origin, then $\mathcal{H} \gtrsim \|Z\|_{L^2}^2$, which corresponds to exponential decay.

Therefore, at the linear level, in order to get optimal dissipative estimates, it is suitable to split the solution into low and high frequencies parts. This will actually be achieved by means of a Littlewood-Paley decomposition (introduced in the next section). Then, a great part of our analysis will consist in localizing (3) on the Fourier side by means of this decomposition, and to study the evolution of the functional $\mathcal{L}$ pertaining to each part.

1.3. The damped mode. Another important ingredient of our analysis is the use of a ‘damped mode’ that, somehow, may be seen as an eigenmode corresponding to the part of the solution that experiences maximal dissipation in low frequencies. It is defined as follows:

$$W \triangleq -L^{-1}\tilde{A}^0_{2,2}(V)\partial_tZ_2 = Z_2 + \sum_{j=1}^d L^{-1}(\tilde{A}^2_{2,1}(V)\partial_jZ_1 + \tilde{A}^j_{2,2}(V)\partial_jZ_2) - L^{-1}Q(Z).$$

Note that

$$\tilde{A}^0_{2,2}(V)\partial_tW + LW = \tilde{A}^0_{2,2}(V)L^{-1}\sum_{j=1}^d \partial_t(\tilde{A}^2_{2,1}(V)\partial_jZ_1 + \tilde{A}^j_{2,2}(V)\partial_jZ_2)$$

$$- \tilde{A}^0_{2,2}(V)L^{-1}\partial_tQ(Z).$$

On the left-hand side, Property (5) ensures maximal dissipation on $W$. As the right-hand side of (19) contains only at least quadratic terms, or linear terms with one derivative, it can be expected to be negligible in low frequencies if $Z$ is small enough. Furthermore, (18) reveals that $W$ is comparable to $Z_2$ in low frequencies. This will ensure better integrability for $Z_2$ than for the whole solution $Z$.

2. Main results

Before stating our main results, introducing a few notations is in order.

First, we fix a homogeneous Littlewood-Paley decomposition $(\tilde{\Delta}_q)_{q \in \mathbb{Z}}$ that is defined by

$$\tilde{\Delta}_q \triangleq \varphi(2^{-q}D) \quad \text{with} \quad \varphi(\xi) \triangleq \chi(\xi/2) - \chi(\xi)$$

where $\chi$ stands for a smooth function with range in $[0, 1]$, supported in the open ball $B(0, 4/3)$ and such that $\chi \equiv 1$ on the closed ball $\bar{B}(0, 3/4)$. We further state

$$\hat{\Delta}_q \triangleq \chi(2^{-q}D) \quad \text{for all} \quad q \in \mathbb{Z}$$

and define $S'_h$ to be the set of tempered distributions $z$ such that

$$\lim_{q \to -\infty} \|\hat{\Delta}_q z\|_{L^\infty} = 0.$$

Following [1], we introduce the homogeneous Besov semi-norms:

$$\|z\|_{\dot{B}_{p,r}^s} \triangleq \|2^{qs}\|\tilde{\Delta}_q z\|_{L^p(\mathbb{R}^d)}\|_{l^r(\mathbb{Z})},$$

then define the homogeneous Besov spaces $\dot{B}_{p,r}^s$ (for any $s \in \mathbb{R}$ and $(p, r) \in [1, \infty)^2$) to be the subset of $z$ in $S'_h$ such that $\|z\|_{\dot{B}_{p,r}^s}$ is finite.
Using from now on the shorthand notation

\((20)\) \[ \tilde{\Delta}_q z \triangleq z_q, \]

we associate to any element \( z \) of \( \mathcal{S}_h \), its low and high frequency parts through

\[ z^\ell \triangleq \sum_{q \leq 0} z_q = \hat{S}_1 z \quad \text{and} \quad z^h \triangleq \sum_{q > 0} z_q = (\text{Id} - \hat{S}_1)z. \]

We shall constantly use the following Besov semi-norms for low and high frequencies:

\[ \|z\|_{B^s_{2,1}}^\ell \triangleq \sum_{q \leq 0} 2^{qs}\|z_q\|_{L^2} \quad \text{and} \quad \|z\|_{B^s_{2,1}}^h \triangleq \sum_{q > 0} 2^{qs}\|z_q\|_{L^2}, \]

\[ \|z\|_{B^s_{2,\infty}}^\ell \triangleq \sup_{q \leq 0} 2^{qs}\|z_q\|_{L^2} \quad \text{and} \quad \|z\|_{B^s_{2,\infty}}^h \triangleq \sup_{q > 0} 2^{qs}\|z_q\|_{L^2}. \]

Throughout the paper, we shall use repeatedly the following obvious fact:

\[(21) \quad \|z\|_{B^s_{2,r}} \leq \|z\|_{B^s_{2,r}}^\ell \quad \text{and} \quad \|z\|_{B^s_{2,r}}^h \geq \|z\|_{B^s_{2,r}}^h \quad \text{for} \quad r = 1, \infty, \quad \text{whenever} \quad s \leq s'. \]

For any Banach space \( X \), index \( \rho \) in \([1, \infty]\) and time \( T \in [0, \infty) \), we use the notation \( \|z\|_{L^p_t(X)} \triangleq \|z\|_{L^p([0,T],X)} \). If \( T = +\infty \), then we just write \( \|z\|_{L^p(X)} \). Finally, in the case where \( z \) has \( n \) components \( z_j \) in \( X \), we keep the notation \( \|z\|_X \) to mean \( \sum_{j=1}^n \|z_j\|_X \).

We can now state our main global existence result for System \((11)\), rewritten as \((3)\).

**Theorem 2.1.** Let \( \bar{V} \) be an equilibrium state such that \( H(\bar{V}) = 0 \) and suppose that the structure assumptions of paragraph \([17]\) and \((SK)\) condition are satisfied. Then, there exists a positive constant \( \alpha \) such that for all \( Z_0 \in B^d_{2,1} \cap B^{d+1}_{2,1} \) satisfying

\[(22) \quad Z_0 \triangleq \|Z_0\|_{B^d_{2,1}}^\ell + \|Z_0\|_{B^{d+1}_{2,1}}^h \leq \alpha, \]

System \((3)\) supplemented with initial data \( Z_0 \) admits a unique global-in-time solution \( Z \) in the space \( E \) defined by

\[ Z \in C_0(\mathbb{R}^+; B^d_{2,1} \cap B^{d+1}_{2,1}), \quad Z^h \in L^1(\mathbb{R}^+; B^{d+1}_{2,1}), \quad Z^\ell \in L^1(\mathbb{R}^+; B^{d+1}_{2,1}) \quad \text{and} \quad W \in L^1(\mathbb{R}^+; B^d_{2,1}), \]

with \( W \) defined according to \((18)\).

Moreover, there exists a Lyapunov functional that is equivalent to \( \|Z\|_{B^d_{2,1}} \cap B^{d+1}_{2,1} \), and a constant \( C \) depending only on the matrices \( A_j \) and on \( H \), such that

\[(23) \quad \mathcal{Z}(t) \leq C \mathcal{Z}_0 \quad \text{for all} \quad t \geq 0 \]

where

\[(24) \quad \mathcal{Z}(t) \triangleq \|Z\|_{L^\infty_t(B^d_{2,1})}^\ell + \|Z\|_{L^\infty_t(B^{d+1}_{2,1})}^h + \|Z\|_{L^1_t(B^d_{2,1})}^\ell + \|Z\|_{L^1_t(B^{d+1}_{2,1})}^h + \|W\|_{L^1_t(B^d_{2,1})}^\ell + \|Z_2\|_{L^1_t(B^{d+1}_{2,1})}^\ell. \]

**Remark 2.1.** As is, the above theorem does not extend to the case \( d = 1 \). The reason why is that the low frequency regularity index then becomes negative, so that some nonlinear terms cannot be bounded in the proper spaces. For more details, the reader may refer to \([10]\).

Our second result concerns the time-decay estimates of the solution we constructed in Theorem \(2.1\).
Theorem 2.2. Under the hypotheses of Theorem 2.1 and if, additionally, \( Z_0 \in \tilde{H}^{-\sigma_1}_{2,\infty} \) for some \( \sigma_1 \in [-\frac{d}{2}, \frac{d}{2}] \) then, there exists a constant \( C \) depending only on \( \sigma_1 \) and such that
\[
\| Z(t) \|_{\tilde{H}^{-\sigma_1}_{2,\infty}} \leq C \| Z_0 \|_{\tilde{H}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0.
\]
Furthermore, if \( \sigma_1 > 1 - d/2 \) then, denoting
\[
\langle t \rangle \triangleq \sqrt{1 + t^2}, \quad \alpha_1 \triangleq \sigma_1 + \frac{d}{2} - 1
\]
and \( C_0 \triangleq \| Z_0 \|_{\tilde{H}^{-\sigma_1}_{2,\infty}} + \| Z_0 \|_{L^{2,1}_{2,1}} \),
we have the following decay estimates:
\[
\sup_{t \geq 0} \| (t)^{\sigma_1} Z(t) \|_{\tilde{H}^{1}_{2,1}} \leq C C_0 \quad \text{if} \quad -\sigma_1 < \sigma \leq d/2 - 1,
\]
\[
\sup_{t \geq 0} \| (t)^{\sigma_1} Z_2(t) \|_{\tilde{H}^{1}_{2,1}} \leq C C_0 \quad \text{if} \quad -\sigma_1 < \sigma \leq d/2 - 2,
\]
\[
\sup_{t \geq 0} \| (t)^{\alpha_1} Z_2(t) \|_{\tilde{H}^{1}_{2,1}} \leq C C_0 \quad \text{if} \quad \min(d/2 - 2, -\sigma_1) < \sigma \leq d/2 - 1.
\]

Remark 2.2. Since we have the embedding \( L^1 \hookrightarrow \tilde{H}^{-\frac{d}{2}}_{2,\infty} \), the above statement encompasses the classical decay assumption \( Z_0 \in L^1 \) (see e.g. [10] in a slightly different context). 

Remark 2.3. Owing to the presence of "direct" dissipation in the equation of \( Z_2 \), the decay of the low frequencies of \( Z_2 \) is stronger by a factor 1/2 than the decay of the whole solution.

If we assume in addition that:
\[
\begin{cases} 
\text{For all } j \in \{1, \ldots, d\}, \quad A^j_{1,1}(\overline{\Sigma}) = 0 \quad \text{and} \quad D V_1 A^j_{1,1}(\overline{\Sigma}) = 0; \\
\text{For all } j \in \{1, \ldots, d\}, \quad D V_1 A^j_{2,1}(\overline{\Sigma}) = 0 \quad \text{and} \quad D V_1 A^j_{1,2}(\overline{\Sigma}) = 0; \\
\text{The function } r \text{ is quadratic with respect to } Z_2 \ (\text{i.e. } D V_{i,j}^2 \tau(0) = 0 \text{ for } (i,j) \neq (2,2)),
\end{cases}
\]
then one can weaken the low frequency assumption, as we did in our work [10] dedicated to one-dimensional case, and get:

Theorem 2.3. Let the assumptions of Theorem 2.1 concerning system (11) be in force and assume in addition that (26) holds true.

Then, there exists a positive constant \( \alpha \) such that for all \( Z_0 \in \tilde{H}^{d}_{2,1} \cap \tilde{H}^{d+1}_{2,1} \) satisfying
\[
\sum^d_{j=1} \| Z_j \|_{L^\infty_{1,2} \cap L^{d+1}_{1,2}} \leq \alpha,
\]
System (3) supplemented with initial data \( Z_0 \) admits a unique global-in-time solution \( Z \) in the space \( F \) defined by \( Z \in C_0(\mathbb{R}^+; \tilde{H}^{d}_{2,1} \cap \tilde{H}^{d+1}_{2,1}) \), \( Z^h \in L^1(\mathbb{R}^+; \tilde{H}^{d+2}_{2,1}) \) and \( W \in L^1(\mathbb{R}^+; \tilde{H}^{d}_{2,1}) \).

Moreover, there exists a Lyapunov functional that is equivalent to \( \| Z \|_{L^\infty_{1,2} \cap L^{d+1}_{1,2}} \), and we have the following a priori estimate:
\[
\sum^d_{j=1} \| Z_j \|_{L^\infty_{1,2} \cap L^{d+1}_{1,2}} + \| Z \|_{L^\infty_{1,2} \cap L^{d+1}_{1,2}} \leq \alpha,
\]

(28) \( \sum^d_{j=1} \| Z_j \|_{L^\infty_{1,2} \cap L^{d+1}_{1,2}} + \| Z \|_{L^\infty_{1,2} \cap L^{d+1}_{1,2}} \)
Finally, if, additionally, \( Z_0 \in \dot{B}^{-\sigma_1}_{2,\infty} \) for some \( \sigma_1 \in \left[-\frac{d}{2}, \frac{d}{2}\right] \), then (25) is satisfied as well as the decay estimates that follow, up to \( \sigma = d/2 \) for the first one, and with \( d/2 - 1 \) and \( d/2 \) instead of \( d/2 - 2 \) and \( d/2 - 1 \) for the next two ones, with \( \alpha_1 \) replaced by \((\sigma_1 + d/2)/2\).

**Remark 2.4.** As will be shown in the last section, this theorem applies to the compressible Euler with damping (see Theorem 4.1).

**Remark 2.5.** In contrast with Theorem 2.1, the functional setting of Theorem 2.3 allows to obtain uniform estimates in the asymptotic \( \lambda \to +\infty \) if the dissipative term is \( \lambda H \). This is the first step for studying the high relaxation limit.

### 3. Proof of Theorems 2.1 and 2.2

This section is devoted to proving the global existence of strong solutions and decay estimates for System (11) supplemented with initial data that are close to the reference solution \( \tilde{V} \), in the general case where the structural assumptions listed in Subsection 1.1 and (SK) condition are satisfied.

The bulk of the proof consists in establishing a priori estimates, the other steps (proving existence and uniqueness) being more classical. As explained before, our strategy is to first work out a Lyapunov functional in Beauchard-Zuazua’s style, that is equivalent to the norm that we aim at controlling, then to combine with the study of the damped mode \( W \) defined in (18) so as to close the estimates.

#### 3.1. Establishing the a priori estimates.

Throughout this part, we assume that we are given a smooth (and decaying) solution \( Z \) of (3) on \([0,T] \times \mathbb{R}^d\) with \( Z_0 \) as initial data, satisfying

\[
\sup_{t \in [0,T]} \|Z(t)\|_{\dot{B}^\frac{d}{2}} \ll 1.
\]

We shall use repeatedly that, owing to the embedding \( \dot{B}^\frac{d}{2}_{2,1} \hookrightarrow L^\infty \), we have also

\[
\sup_{t \in [0,T]} \|Z(t)\|_{L^\infty} \ll 1.
\]

From now on, \( C > 0 \) designates a generic harmless constant, the value of which depends on the context and we denote by \((c_q)_{q \in \mathbb{Z}}\) nonnegative sequences such that \( \sum_{q \in \mathbb{Z}} c_q = 1 \).

To start with, let us rewrite (3) as follows:

\[
\tilde{A}^0 \partial_t Z + \sum_{j=1}^d \tilde{A}^j \partial_j Z + LZ = G
\]

with \( G \triangleq G_1 + G_2 + G_3 \) and

\[
G_1 \triangleq -\sum_{j=1}^d \tilde{A}^0 \left((\tilde{A}^0(V))^{-1} \tilde{A}^j(V) - (\tilde{A}^0)^{-1} \tilde{A}^j\right) \partial_j Z,
\]

\[
G_2 \triangleq -\tilde{A}^0 \left((\tilde{A}^0(V))^{-1} - (\tilde{A}^0)^{-1}\right) LZ,
\]

\[
G_3 \triangleq \tilde{A}^0 (\tilde{A}^0(V))^{-1} r(Z).
\]

For \( q \in \mathbb{Z} \), applying \( \dot{\Delta}_q \) to (31) yields

\[
\tilde{A}^0 \partial_t Z_q + \sum_{j=1}^d \tilde{A}^j \partial_j Z_q + LZ_q = \dot{\Delta}_q G \quad \text{with} \quad Z_q \triangleq \dot{\Delta}_q Z.
\]

Our analysis will mainly consist in estimating for all \( q \in \mathbb{Z} \) a functional \( \mathcal{L}_q \) that is equivalent to the \( L^2(\mathbb{R}^d; \mathbb{R}^n) \) norm of \( Z_q \) and encodes informations on the dissipative properties of the system. That functional will be built from (16) and, since Condition (SK) is satisfied, the number \( \mathcal{N}_V \)
defined in (17) will be positive. Furthermore, since the Fourier transform of \( Z_q \) is localized near the frequencies of magnitude \( 2^q \), the corresponding dissipation term \( \mathcal{H}_q \) will satisfy

\[
\mathcal{H}_q \geq \min(1, 2^{2q}) \mathcal{L}_q.
\]

The prefactor \( \min(1, 2^{2q}) \) may be seen as a gain of two derivatives in low frequencies after time integration (like for the heat equation) whereas it corresponds to exponential decay for high frequencies. In our setting where the low and high frequencies of \( Z_0 \) belong to the spaces \( \frac{\mathbb{H}^{-1}}{2, 1} \) and \( \frac{\mathbb{H}^{\frac{d+1}{2}}}{2, 1} \), respectively, we thus have

\[
\| Z(t) \|_{\frac{\mathbb{H}^{-1}}{2, 1}} + \int_0^t \| Z \|_{\frac{\mathbb{H}^{\frac{d+1}{2}}}{2, 1}} \leq \| Z_0 \|_{\frac{\mathbb{H}^{-1}}{2, 1}} + \int_0^t \| G \|_{\frac{\mathbb{H}^{\frac{d+1}{2}}}{2, 1}},
\]

\[
\| Z(t) \|_{\frac{\mathbb{H}^{\frac{d+1}{2}}}{2, 1}} + \int_0^t \| Z \|_{\frac{\mathbb{H}^{-1}}{2, 1}} \leq \| Z_0 \|_{\frac{\mathbb{H}^{-1}}{2, 1}} + \int_0^t \| G \|_{\frac{\mathbb{H}^{-1}}{2, 1}}.
\]

A rapid examination reveals that the part \( G_1 \) of \( G \) may entail a loss of one derivative (since it is a combination of components of \( \nabla Z \)) while \( G_2 \) and \( G_3 \) contain products of components of \( Z \) and \( Z_2 \). Overcoming the difficulty with \( G_1 \) will be achieved by exploiting the symmetrizable character of the system under consideration and changing slightly the weight \( \tilde{A}_0 \) in the definition of \( \mathcal{L}_q \) for the high frequencies: we shall take

\[
\mathcal{L}_q \triangleq \| Z_q \|_{L^2_{\tilde{A}_0(V)}}^2 + 2^{-q} \mathcal{I}_q \quad \text{if } q \geq 0,
\]

with

\[
\mathcal{I}_q \triangleq \int_{\mathbb{R}^d} \sum_{k=1}^{n-1} \varepsilon_k \Re \left( (NM_k^{-1} \hat{Z}_q) \cdot (NM_k \hat{Z}_q) \right),
\]

where \( \varepsilon_1, \ldots, \varepsilon_{n-1} > 0 \) will be chosen small enough (according to the Appendix).

For the low frequencies, we shall keep the original definition that we proposed in the analysis of (7), that is to say, after integrating on the whole space and using Fourier-Plancherel theorem,

\[
\mathcal{L}_q \triangleq \| Z_q \|_{L^2_{\tilde{A}_0}}^2 + 2^q \mathcal{I}_q \quad \text{if } q < 0.
\]

However, we will discover that the terms \( G_2 \) and \( G_3 \) cannot be controlled properly in the space \( L^1_L(B_2^{-1}) \) because \( Z_2 \) is, somehow, too regular! The way to overcome the difficulty is to look for an estimate of the low frequencies of the damped mode \( W \), then to compare with \( Z_2 \).

We shall keep in mind all the time that if choosing the coefficients \( \varepsilon_k \) small enough, then we have

\[
\sum_{k=1}^{n-1} \varepsilon_k |((M_\omega \tau)^k N^t NM_\omega^{-1})| \leq \frac{1}{2} \frac{1}{(2\pi)^d},
\]

whence, owing to Fourier-Plancherel theorem,

\[
|\mathcal{I}_q| \leq \frac{1}{2} \| Z_q \|_{L^2}.
\]

Furthermore, as \( \tilde{A}_0 = A_0(V) \) is definite positive and \( V \mapsto \tilde{A}_0(V) \), continuous, Condition (30) ensures that \( \| Z_q \|_{L^2_{\tilde{A}_0(V)}} \simeq \| Z_q \|_{L^2} \) and \( \| Z_q \|_{L^2_{\tilde{A}_0(V)}} \simeq \| Z_q \|_{L^2} \). Therefore, we have

\[
\mathcal{L}_q \simeq \| Z_q \|_{L^2}^2 \quad \text{for all } q \in \mathbb{Z}.
\]
3.1.1. Basic energy estimates. The first step is devoted to studying the time evolution of \( \|Z_q\|_{L^2_{\chi_0(V)}}^2 \) and \( \|Z_q\|_{L^2_{\chi_0}}^2 \). The outcome is given in the following proposition.

**Proposition 3.1.** Let \( Z \) be a smooth solution of (32) on \([0, T] \times \mathbb{R}^d\) satisfying (25). Then, for all \( s \in \left[ \frac{d}{2}, \frac{d}{2} + 1 \right] \) and \( q \geq 0 \), we have:

\[
\frac{1}{2} \frac{d}{dt} \|Z_q\|_{L^2_{\chi_0(V)}}^2 + \kappa_0 \|Z_2,q\|_{L^2}^2 \lesssim \|\nabla Z, Z_2\|_{L^\infty} \|Z_q\|_{L^2}^2 + c_q 2^{-qs} \|\nabla Z\|_{B^s_{2,1}} \|Z\|_{B^s_{2,1}} \|Z_q\|_{L^2}^2
+ c_q 2^{-qs} \|\nabla Z\|_{B^s_{2,1}} \|Z_2\|_{L^2} + c_q 2^{-qs} \|Z_2\|_{B^s_{2,1}} \|Z\|_{B^s_{2,1}} \|Z_q\|_{L^2}^2.
\]

Furthermore, for all \( s' \in \left[ \frac{d}{2} - 1, \frac{d}{2} \right] \) and \( q \leq 0 \), we have:

\[
\frac{1}{2} \frac{d}{dt} \|Z_q\|_{L^2_{\chi_0}}^2 + \kappa_0 \|Z_2,q\|_{L^2}^2 \lesssim \|\nabla Z\|_{L^\infty} \|Z_q\|_{L^2}^2 + c_q 2^{-qs'} \|\nabla Z\|_{B^s_{2,1}} \|Z\|_{B^s_{2,1}} \|Z_q\|_{L^2}^2
+ c_q 2^{-qs'} \|Z\|_{B^s_{2,1}} \|Z_2\|_{L^2} + c_q 2^{-qs'} \|Z_2\|_{B^s_{2,1}} \|Z\|_{B^s_{2,1}} \|Z_q\|_{L^2}^2.
\]

**Proof.** It relies on an energy method implemented on (3) after localization in the Fourier space, and on classical commutator estimates.

In order to prove (37), apply operator \( \hat{\Delta}_q \) to (3) to get:

\[
\tilde{A}^0(V) \partial_t Z_q + \sum_{j=1}^{d} \tilde{A}^j(V) \partial_j Z_q + LZ_q = R^1_q + R^2_q + \hat{\Delta}_q(r(Z))
\]

with \( R^1_q \triangleq \sum_{j=1}^{d} [\tilde{A}^j(V), \hat{\Delta}_q] \partial_j Z \) and \( R^2_q \triangleq [\tilde{A}^0(V), \hat{\Delta}_q] \partial_t Z \).

Taking the \( L^2(\mathbb{R}^d; \mathbb{R}^n) \) scalar product with \( Z_q \), integrating by parts in the second term and using the fact that \( \tilde{A}^j(V) \) is symmetric yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \tilde{A}_0(V) Z_q \cdot Z_q + \int_{\mathbb{R}^d} LZ_q \cdot Z_q = \frac{1}{2} \int_{\mathbb{R}^d} \left( \partial_t \tilde{A}^0(V) + \sum_{j} \partial_j (\tilde{A}^j(V)) \right) Z_q \cdot Z_q
+ \int_{\mathbb{R}^d} (R^1_q + R^2_q) \cdot Z_q + \int_{\mathbb{R}^d} \hat{\Delta}_q r(Z) \cdot Z_q.
\]

Hence, thanks to Property (12), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|Z_q\|_{L^2_{\chi_0(V)}}^2 + \kappa_0 \|NZ_q\|_{L^2}^2 \leq \frac{1}{2} \int_{\mathbb{R}^d} \left( \partial_t \tilde{A}^0(V) + \sum_{j} \partial_j (\tilde{A}^j(V)) \right) Z_q \cdot Z_q
+ \int_{\mathbb{R}^d} (R^1_q + R^2_q) \cdot Z_q + \int_{\mathbb{R}^d} \hat{\Delta}_q r(Z) \cdot Z_q.
\]

For the first term in the right-hand side, we have

\[
\int_{\mathbb{R}^d} \partial_t (\tilde{A}^0(V)) Z_q' Z_q \lesssim \|\partial_t Z\|_{L^\infty} \|Z_q\|_{L^2}^2.
\]

Hence, using the fact that

\[
\partial_t Z = (\tilde{A}^0(V))^{-1} \left( \tilde{H}(V + Z) - \sum_{j=1}^{d} \tilde{A}^j(V) \partial_j Z \right),
\]
the smallness condition (30) and the structure of $\tilde{H}$, we get
\begin{equation}
\int_{\mathbb{R}^d} \partial_t (\tilde{A}^0(V)) Z_q \cdot Z_q \lesssim \| (\nabla Z, Z_2) \|_{L^\infty} \| Z_q \|_{L^2}^2.
\end{equation}

For the second term in the right-hand side of (39), we may write
\begin{equation}
\int_{\mathbb{R}^d} \sum_{j=1}^d \partial_j (\tilde{A}^j(V)) Z_q \cdot Z_q \lesssim \| \nabla Z \|_{L^\infty} \| Z_q \|_{L^2}^2.
\end{equation}

Bounding the commutators terms in (39) relies on Cauchy-Schwarz inequality and Inequality (57) that give
\begin{equation}
\int_{\mathbb{R}^d} (R_q^1 + R_q^2) : Z_q \lesssim c_1 q^{-q_1} \left( \sum_j \| \nabla (\tilde{A}^j(V)) \|_{B^d_{1,1}} \| \nabla Z \|_{B^d_{1,1}}^{-1} + \| \nabla (\tilde{A}^0(V)) \|_{B^d_{1,1}} \| \partial_t Z \|_{B^d_{1,1}^{-1}} \right) \| Z_q \|_{L^2}^2
\lesssim c_1 q^{-q_1} \| \nabla Z \|_{B^d_{1,1}} \left( \| Z \|_{B^d_{1,1}} \| Z_2 \|_{B^d_{1,1}}^{-1} \right) \| Z_q \|_{L^2}^2.
\end{equation}

To bound $\partial_t Z$, we need the following lemma.

**Lemma 3.1.** Under assumption (29), we have for all $\sigma \in [-d/2, d/2]$,
\begin{equation}
\| \partial_t Z \|_{B^d_{1,1}} \lesssim \| \nabla Z \|_{B^d_{1,1}} + \min \left( \| W \|_{B^d_{1,1}}, \| Z_2 \|_{B^d_{1,1}} \right).
\end{equation}

**Proof.** Using (10), Propositions 5.2, 5.3 and 5.4 yields
\begin{equation}
\| \partial_t Z \|_{B^d_{1,1}} \lesssim \left( \sum_{j=1}^d \tilde{A}^j(V) \partial_j Z \right) \| \tilde{A}^j(V) \|_{B^d_{1,1}} + \| LZ \|_{B^d_{1,1}} + \| r(Z) \|_{B^d_{1,1}}
\lesssim (1 + \| Z \|_{B^d_{1,1}}) \| \nabla Z \|_{B^d_{1,1}} + \| Z_2 \|_{B^d_{1,1}} + \| Z \|_{B^d_{1,1}} \| Z_2 \|_{B^d_{1,1}}.
\end{equation}

Since we assumed that $\| Z \|_{B^d_{1,1}}$ is small, we have
\begin{equation}
\| \partial_t Z \|_{B^d_{1,1}} \lesssim \| \nabla Z \|_{B^d_{1,1}} + \| Z_2 \|_{B^d_{1,1}}.
\end{equation}

Note that, actually, $\partial_t Z_1$ can be bounded by just $\nabla Z$ and that we have $\partial_t Z_2 = - (\tilde{A}^0_{1,2}(V))^{-1} L W$ by definition of $W$, whence the final result.

Finally, Proposition 5.4 ensures that
\begin{equation}
\int_{\mathbb{R}^d} \tilde{\Delta}_q r(Z) : Z_q \lesssim c_1 q^{-q_1} \| r(Z) \|_{B^d_{1,1}} \| Z_q \|_{L^2}
\lesssim c_1 q^{-q_1} \left( \| Z_2 \|_{B^d_{1,1}} + \| Z \|_{B^d_{1,1}} \| Z_2 \|_{B^d_{1,1}} \right) \| Z_q \|_{L^2}^2.
\end{equation}

Putting all the above estimates together completes the proof of (37).

For proving (38), since we do not know how to control $\partial_t Z$ in $L^1_T (B^s_{2,1})$ for $s' = d/2 - 1$ (which is the value that we will take eventually), we proceed slightly differently, writing the equation satisfied by $Z_q$ as follows:
\begin{equation}
\tilde{A}^0 \partial_t Z_q + \sum_{j=1}^d \tilde{A}^j(V) \partial_j Z_q + L Z_q = R_q^1 + R_q^2 + \Delta_q (r(Z)) \quad \text{with} \quad R_q^1 \triangleq \tilde{\Delta}_q \left( (\tilde{A}^0 - \tilde{A}^0(V)) \partial_t Z \right).
\end{equation}
Arguing as for proving (39), we now get
\[
\frac{1}{2} \frac{d}{dt} \|Z_q\|_{L^2}^2 + \kappa_0 \|N Z_q\|_{L^2}^2 \leq \frac{1}{2} \int_{\mathbb{R}^d} \left( \sum_j \partial_j (\tilde{A}_j(V)) \right) Z_q \cdot Z_q \\
+ \int_{\mathbb{R}^d} (R_1^q + R_3^q) \cdot Z_q + \int_{\mathbb{R}^d} \hat{\Delta}_q r(Z) \cdot Z_q.
\]

The term $R_1^q$ may be estimated as above (with $s'$ instead of $s$), and $\hat{\Delta}_q (r(Z))$ may be bounded by means of (103). As regards $R_3^q$, we write that
\[
\|R_3^q\|_{L^2} \lesssim c_2 2^{-q s'} \|A^0(V) - \tilde{A}^0(V)\|_{\mathcal{B}^{\frac{d}{2},1}} \|\partial_s Z\|_{\mathcal{B}^{s'}_{\frac{d}{2},1}}.
\]

Thus, using composition, product estimates and Lemma 3.1 we obtain
\[
\left| \int_{\mathbb{R}^d} R_3^q Z_q \right| \lesssim c_2 2^{-q s'} \|Z\|_{\mathcal{B}^{\frac{d}{2},1}} \|\nabla Z, Z_2\|_{\mathcal{B}^{s'}_{\frac{d}{2},1}} \|Z_q\|_{L^2},
\]
which leads to the desired estimate.

3.1.2. Cross estimates. Proposition 3.1 only allows to exhibit the integrability properties of the components of $Z$ experiencing direct dissipation. To recover the dissipation for all the components, we have to look at the time derivative of the quantity $I_q$ defined in (44). To achieve it, we apply to (32) the method that has been explained in Section 3.1 and leads to (15). The only change lies in the (harmless) additional source term $G_q$. In the end, integrating on $\mathbb{R}^d$ the obtained identity, then using the fact that $\text{Supp} \tilde{Z}_q \subset \{ 3 \cdot 2^q / 4 \leq |\xi| \leq 8 \cdot 2^q / 3 \}$ yields
\[
\frac{d}{dt} I_q + \frac{2^q}{2} \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |N M_k \tilde{Z}_q|^2 \, d\xi \leq \frac{2^{-q K_0}}{2} \|N Z_q\|_{L^2}^2 + C \|\hat{\Delta}_q G\|_{L^2} \|Z_q\|_{L^2}.
\]

The last term may be bounded by means of Propositions 5.2 and 5.3 (keeping all the time in mind that (29) is satisfied). More precisely, for $G_1$, we have for all $\sigma \in [-d/2, d/2]$,
\[
\|G_1\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}} \lesssim \sum_{j=1}^d \left\| \left( \tilde{A}^0(V)^{-1} \tilde{A}_j(V) - (\tilde{A}^0)^{-1} \tilde{A}_j \right) \partial_s Z \right\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}}
\leq \|Z\|_{\mathcal{B}^{\frac{d}{2},1}} \|\nabla Z\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}}.
\]

Similarly,
\[
\|G_2\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}} \lesssim \left\| \left( \tilde{A}^0(V)^{-1} - (\tilde{A}^0)^{-1} \right) L Z \right\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}} \lesssim \|Z\|_{\mathcal{B}^{\frac{d}{2},1}} \|Z_2\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}},
\]

and, using Proposition 5.4
\[
\|G_3\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}} = \left\| \tilde{A}^0 \tilde{A}^0(V)^{-1} r(Z) \right\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}} \lesssim \|Z\|_{\mathcal{B}^{\frac{d}{2},1}} \|Z_2\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}}.
\]

Hence, one can conclude that for all $\sigma \in [-d/2, d/2]$, we have
\[
\frac{d}{dt} I_q + \frac{2^q}{2} \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |N M_k \tilde{Z}_q|^2 \, d\xi \\
\leq \frac{2^{-q K_0}}{2} \|N Z_q\|_{L^2}^2 + C c_4 2^{-q \sigma} \|\nabla Z, Z_2\|_{\mathcal{B}^{\sigma}_{\frac{d}{2},1}} \|Z\|_{\mathcal{B}^{\frac{d}{2},1}} \|Z_q\|_{L^2}.
\]
3.1.3. Closure of the estimates: a first attempt. Remember that since Condition (SK) is satisfied, the quantity $N\mathcal{F}_{\gamma}$ defined in (17) is positive for any choice of positive parameters $\varepsilon_0, \ldots, \varepsilon_{n-1}$. Consequently, if we set

$$\mathcal{H}_q := \frac{\kappa_0}{2} \|NZ_q\|^2 + \min(1, 2^{2q}) \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |NM_{\omega}^{k} Z_q|^2 \, d\xi$$

and use Fourier-Plancherel theorem and the equivalence (36), we see that (up to a change of $\kappa_0$), we have for all $q \in \mathbb{Z}$,

(48) $$\mathcal{H}_q \geq \kappa_0 \min(1, 2^{2q}) \mathcal{L}_q.$$ 

Our goal is to use this inequality to bound the quantity $Z$ defined in (24) in terms of $Z_0$ only.

Let us start with the bounds for the low frequencies. Putting together Inequality (48) with $s' = d/2 - 1$ and the cross estimate (47), then, using (48), we get for all $q < 0$,

$$\frac{d}{dt} \mathcal{L}_q + \kappa_0 2^{2q} \mathcal{L}_q \lesssim \|\nabla Z\|_{L^\infty} \|Z_q\|^2_{L^2} + c_q 2^{-q(d-1)} \left(\|\nabla Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right. + \left. \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right)\|Z_q\|_{L^2}.$$ 

Hence, using (36), applying Lemma 5.1 multiplying by $2^{q(d-1)}$, using the embedding $\dot{B}_{2,1}^{\frac{d}{2}+1} \hookrightarrow L^\infty$ and summing up on $q < 0$ gives

(49) $$\|Z(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \kappa_0 \int_0^t \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \, dt \leq \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \int_0^t \left(\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \, dt.$$ 

To handle the high frequencies, we combine Inequality (37) with $s = d/2 + 1$, the cross estimate (47) and (48), to get for all $q \geq 0$,

(50) $$\frac{d}{dt} \mathcal{L}_q + \kappa_0 \mathcal{L}_q \lesssim \|\nabla Z, Z_2\|_{L^\infty} \|Z_q\|_{L^2}^2 + c_q 2^{-q(d+1)} \left(\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|Z_q\|_{L^2}.$$ 

Hence, using the equivalence (36), Lemma 5.1 multiplying by $2^{q(d+1)}$ and summing up on $q \geq 0$ gives

$$\|Z(t)\|_{\dot{B}_{2,1}^h} + \kappa_0 \int_0^t \|Z\|_{\dot{B}_{2,1}^h} \leq \|Z_0\|_{\dot{B}_{2,1}^h} + \int_0^t \left(\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \, dt$$

where we used the notation

$$\|Z\|_{\dot{B}_{2,1}^h} \triangleq \sum_{q \geq 0} 2^{q\sigma} \sqrt{\mathcal{L}_q}.$$ 

Let us introduce the functional

(51) $$\mathcal{L} \triangleq \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$$ 

which, in light of (36), is equivalent to $\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}$, and thus to $\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}}$. 
Adding up the above inequalities for the low and high frequencies, we get up to a change of $\kappa_0$ and for all $t \in [0, T]$,
\[
\mathcal{L}(t) + \kappa_0 \int_0^t \|Z\|_{B_{2,1}^{2+1}}^4 \leq \mathcal{L}(0) + C \int_0^t \left( \|Z\|_{B_{2,1}^{2+1}}^4 + \|Z\|_{B_{2,1}^{2+1}}^4 + \|Z\|_{B_{2,1}^{2+1}}^4 + \|Z\|_{B_{2,1}^{2+1}}^4 \right).
\]
Hence, using the interpolation inequality
\[
\|Z\|_{B_{2,1}^4} \leq \sqrt{\|Z\|_{B_{2,1}^{2+1}}^4 \|Z\|_{B_{2,1}^{2+1}}^4} \leq \mathcal{L},
\]
and eliminating some redundant terms, we end up with
\[
\mathcal{L}(t) + \kappa_0 \int_0^t \|Z\|_{B_{2,1}^{2+1}}^4 \leq \mathcal{L}(0) + C \int_0^t \|Z\|_{B_{2,1}^{2+1}}^4 \mathcal{L} + C \int_0^t \|Z\|_{B_{2,1}^{2+1}}^4 \cdot \mathcal{L}.
\]
As we start from small data, we expect $\mathcal{L}$ to be small as well so that the first term in the first integral in the right-hand side may be absorbed by the second term on the left. However, at this stage, we have no proper control on $\|Z\|_{B_{2,1}^{2+1}}^4$. Studying the evolution of the damped mode $W$, which is the aim of the next section, will enable us to overcome the difficulty.

3.1.4. The damped mode. As underlined in the introduction, the function
\[
W \triangleq -L^{-1}A_{2,2}^0(V)\partial_t Z_2 = Z_2 + \sum_{j=1}^d L^{-1}(\tilde{A}_{2,1}^j(V)\partial_j Z_1 + \tilde{A}_{2,2}^j(V)\partial_j Z_2) - L^{-1}Q(Z).
\]
is expected to have better integrability properties in low frequencies than the whole solution. This will be a consequence of the following proposition.

**Proposition 3.2.** Let $Z$ be a smooth solution of (33) on $[0, T] \times \mathbb{R}^d$ satisfying (29), and denote $\tilde{A}_{2,2}^0 \triangleq A_{2,2}^0(V)$. Assume that $\sigma \in [-d/2, d/2]$, then we have for all $q < 0$,
\[
\frac{1}{2} \frac{d}{dt} \|W_q\|_{L_{q,2}^2}^2 + \kappa_0 \|W_q\|_{L^2}^2 \lesssim \left( \|\nabla^2 Z_q\|_{L^2} + \|\nabla W_q\|_{L^2} \right) \|W_q\|_{L^2}^2 + c_q 2^{-q\sigma} \|W, Z_2\|_{B_{2,1}^{2+1}} \|Z\|_{B_{2,1}^{2+1}} \|W_q\|_{L^2} + c_q 2^{-q\min(\sigma, \frac{d}{2} - 1)} \|\nabla Z, W\|_{B_{2,1}^{2+1}} \|Z\|_{B_{2,1}^{2+1}} \|W_q\|_{L^2}.
\]

**Proof.** From (19), we gather that
\[
\tilde{A}_{2,2}^0 \partial_t W + LW = h
\]
with $h \triangleq h_1 + \tilde{A}_{2,2}^0 L^{-1}(h_2 + h_3)$ and
\[
h_1 \triangleq (\text{Id} - \tilde{A}_{2,2}^0(A_{2,2}^0(V))^{-1})LW,
\]
\[
h_2 \triangleq \sum_{j=1}^d \partial_j (A_{2,1}^j(V)\partial_j Z_1 + A_{2,2}^j(V)\partial_j Z_2),
\]
\[
h_3 = -\partial_t Q(Z).
\]
Applying $\hat{\Delta}_q$ to (54) and taking the scalar product with $W_q \triangleq \hat{\Delta}_q W$ yields, thanks to (10),
\[
\frac{1}{2} \frac{d}{dt} \|W_q\|_{L_{q,2}^2}^2 + \kappa_0 \|W_q\|_{L^2}^2 \leq \left( \|\hat{\Delta}_q h_1\|_{L^2} + C\|\hat{\Delta}_q h_2\|_{L^2} + C\|\hat{\Delta}_q h_3\|_{L^2} \right) \|W_q\|_{L^2}.
\]
As (29) is satisfied, Composition estimates readily give that for all $\sigma \in [-d/2, d/2]$,
\[
\|h_1\|_{B_{2,1}^{2+1}} \lesssim \|Z\|_{B_{2,1}^{2+1}} \|W\|_{B_{2,1}^{2+1}}.
\]
For bounding $h_2$, we use that for all $j \in \{1, \cdots, d\}$,
\[ \partial_t (A^j_{2,1}(V) \partial_j Z_1 + A^j_{2,2}(V) \partial_j Z_2) = D_V A^j_{2,1}(V) \partial_t Z \partial_j Z_1 + A^j_{2,1} \partial_t \partial_j Z_1 + (A^j_{2,1}(V) - A^j_{2,1}(V)) \partial_t \partial_j Z_1 + A^j_{2,2} \partial_t \partial_j Z_2 + (A^j_{2,2}(V) - A^j_{2,2}(V)) \partial_t \partial_j Z_2. \]
For $k = 1, 2$, we have, according to product and composition laws, and Lemma 3.1
\[ \|D_V A^j_{2,k}(V) \partial_t Z \partial_j Z_k\|_{B^2_2} \lesssim \|\partial_t Z\|_{B^4_2} \|\nabla Z\|_{B^2_2} \lesssim \|\nabla Z, W\|_{B^{4_2}} \|Z\|_{B^{2_2}+1}, \]
as well as (provided we also have $\sigma \leq d/2 - 1$):
\[ \|A^j_{2,k}(V) - A^j_{2,k}(V)\|_{B^2_2} \lesssim \|\partial_t \nabla Z\|_{B^{4_2}} \|\nabla Z\|_{B^{2_2}} \lesssim \|\nabla Z, W\|_{B^{4_2}} \|Z\|_{B^{2_2}+1}. \]
Multiplying the first equation of (56) (on the left) by the matrix $(\tilde{A}^0_{2,1}(V))^{-1}$ then differentiating with respect to $x_j$, we discover that $\partial_t \partial_j Z_1$ is a combination of terms of type \( A(Z) D^2 Z \) and $B(Z) DZ \otimes DZ$. Consequently, we have for all $q \leq 0$ (still if $\sigma \leq d/2 - 1$):
\[ \|\tilde{\Delta}_q(\partial_t \partial_j Z_1)\|_{L^2} \lesssim \|D^2 Z\|_{L^2} + c_q 2^{-q\sigma} \|Z\|_{B^{2_2}+1} \|\nabla Z\|_{B^{4_2}}. \]
Note that if $\sigma \in [d/2 - 1, d/2]$, then the above inequalities are valid (owing to (21)) if we change $\sigma + 1$ to $d/2$.

Finally, we have
\[ \partial_t \partial_j Z_2 = -\partial_j ((\tilde{A}^0_{2,2})^{-1} LW) + \partial_j ((\tilde{A}^0_{2,2})^{-1} - (\tilde{A}^0_{2,2}(V))^{-1}) LW. \]
Hence, for all $q \leq 0$, and thanks to (21),
\[ \|\tilde{\Delta}_q(\partial_t \partial_j Z_2)\|_{L^2} \lesssim \|\nabla W_q\|_{L^2} + c_q 2^{-q\sigma} \|((\tilde{A}^0_{2,2})^{-1} - (\tilde{A}^0_{2,2}(V))^{-1}) LW\|_{B^{2_2}}, \]
\[ \lesssim \|\nabla W_q\|_{L^2} + c_q 2^{-q\sigma} \|Z\|_{B^4_2} \|W\|_{B^{2_2}}. \]
To bound $h_3$, we use the fact that $\partial_t Q(Z) = D_Z Q(Z) \partial_t Z$. Hence, as $Q(Z)$ is at least quadratic, we easily obtain from Propositions 5.2 and 5.3 that
\[ \|h_3\|_{B^{2_2}} \lesssim \|Z\|_{B^4_2} \|\nabla Z, W\|_{B^{2_2}} \|Z\|_{B^{2_2}}, \]
which concludes the proof. \[ \Box \]

It is now easy to obtain dissipative estimates for the low frequencies of $W$. Indeed, starting from the inequality of Proposition 3.2 taking advantage of Lemma 5.1, multiplying the resulting inequality with $2^q\sigma$ and summing up on $q < 0$, we get whenever $\sigma \in [-d/2, d/2]$,
\[ W^\sigma(t) + \kappa_0 \int_0^t \|W\|_{B^{2_2}}^d \leq W^\sigma(0) + C \int_0^t \|\nabla^2 Z, \nabla W\|_{B^{2_2}}^d \]
\[ + C \int_0^t \|W, Z\|_{B^{2_2}} \|Z\|_{B^4_2} + C \int_0^t \|\nabla Z, W\|_{B^{2_2}} \|Z\|_{B^{2_2}}, \]
with $W^\sigma \triangleq \sum_{q<0} 2^q\sigma \|\tilde{\Delta}_q W\|_{L^2_{A^0_{2,2}}}$. \[ (57) \]

\[ \Box \]
Let us first apply (57) with \( \sigma = d/2 \). Then we get (discarding the redundant terms):

\[
W^d(t) + \kappa_0 \int_0^t \| W \|^\ell_{B^d_{2,1}} \leq W^d(0) + C \int_0^t \| (\nabla^2 Z, \nabla W) \|^\ell_{B^d_{2,1}} + C \int_0^t \| (\nabla Z, Z_2, W) \|^\ell_{B^d_{2,1}} + \| Z \|^\ell_{B^d_{2,1}}.
\]

In order to close the estimates, we also need the inequality corresponding to \( \sigma = d/2 - 1 \), namely

\[
W^{d-1}(t) + \kappa_0 \int_0^t \| W \|^{\ell_{B^{d-1}_{2,1}}} \leq W^{d-1}(0) + C \int_0^t \| (\nabla Z, W) \|^\ell_{B^d_{2,1}} + C \int_0^t \| (W, \nabla Z) \|^\ell_{B^{d-1}_{2,1}} + C \int_0^t \| (Z_2, W) \|^\ell_{B^d_{2,1}} + \| Z \|^\ell_{B^d_{2,1}}.
\]

Since

\[
Z_2 = W - \sum_{j=1}^d \mathcal{L}^{-1}(A_j(V) \partial_j Z_1 + A_j(V) \partial_j Z_2) + \mathcal{L}^{-1}Q(Z)
\]

and \( \| Z \|^{\ell_{B^d_{2,1}}} \) is small, we have for all \( \sigma \in ]-d/2, d/2[ \),

\[
\| W - Z_2 \|^{\ell_{B^d_{2,1}}} \lesssim \| \nabla Z \|^{\ell_{B^{d-1}_{2,1}}} + \| Z \|^{\ell_{B^{d-1}_{2,1}}} + \| Z_2 \|^{\ell_{B^{d-1}_{2,1}}}.
\]

Hence, \( W \) may be omitted in the last term of Inequality (58), and (59) becomes

\[
W^{d-1}(t) + \kappa_0 \int_0^t \| W \|^{\ell_{B^{d-1}_{2,1}}} \leq W^{d-1}(0) + C \int_0^t \| (\nabla Z, W) \|^\ell_{B^d_{2,1}} + C \int_0^t \| \nabla Z \|^\ell_{B^{d-1}_{2,1}} + C \int_0^t \| Z \|^\ell_{B^{d-1}_{2,1}} + C \int_0^t \| Z_2 \|^\ell_{B^{d-1}_{2,1}} + \| Z \|^{\ell_{B^d_{2,1}}}.
\]

3.1.5. **Global a priori estimates.** We are now ready to establish the following proposition which will be the key to the proof of the existence part of Theorem 2.1.

**Proposition 3.3.** Let \( Z \) be a smooth solution of (3) on \([0, T]\) satisfying the smallness condition (29). Then, there exist three (small) positive parameters \( \kappa_0, \varepsilon \) and \( \varepsilon' \) such that

\[
\tilde{\mathcal{L}} \triangleq \mathcal{L} + \varepsilon W^d + \varepsilon' W^{d-1}
\]

with \( \mathcal{L} \) and \( W^\sigma \) defined in (53) and (57), respectively, satisfies for all \( 0 \leq t_0 \leq t \leq T \),

\[
\tilde{\mathcal{L}}(t) + \kappa_0 \int_{t_0}^t \left( \| Z \|^{\ell_{B^{d+1}_{2,1}}} + \| W \|^{\ell_{B^{d}_{2,1}}} + \varepsilon' \| W \|^{\ell_{B^{d-1}_{2,1}}} \right) \leq \tilde{\mathcal{L}}(t_0) + C(\varepsilon + \varepsilon') \int_{t_0}^t \| Z \|^{\ell_{B^{d+1}_{2,1}}} + C \varepsilon' \int_{t_0}^t \| W \|^{\ell_{B^{d}_{2,1}}} + C \int_{t_0}^t \| Z \|^{\ell_{B^{d+1}_{2,1}}} + C \int_{t_0}^t \| Z_2 \|^{\ell_{B^{d-1}_{2,1}}} + \| Z \|^{\ell_{B^d_{2,1}}}.
\]

Furthermore, there exists a positive constant \( C \) such that

\[
Z(t) \leq C Z_0 \quad \text{for all } t \in [0, T],
\]

where \( Z_0 \) and \( Z \) have been defined in (22) and (21), respectively.

**Proof.** From (53), (58), (62) and (60), we get after a few simplifications,

\[
\tilde{\mathcal{L}}(t) + \kappa_0 \int_{t_0}^t \left( \| Z \|^{\ell_{B^{d+1}_{2,1}}} + \| W \|^{\ell_{B^{d}_{2,1}}} + \varepsilon' \| W \|^{\ell_{B^{d-1}_{2,1}}} \right) \leq \tilde{\mathcal{L}}(t_0) + C(\varepsilon + \varepsilon') \int_{t_0}^t \| Z \|^{\ell_{B^{d+1}_{2,1}}} + C \varepsilon' \int_{t_0}^t \| W \|^{\ell_{B^{d}_{2,1}}} + C \int_{t_0}^t \| Z \|^{\ell_{B^{d+1}_{2,1}}} + C \int_{t_0}^t \| Z_2 \|^{\ell_{B^{d-1}_{2,1}}} + \| Z \|^{\ell_{B^d_{2,1}}}.
\]
Hence, choosing (positive) \( \varepsilon \) and \( \varepsilon' \) so that

\[
2C\varepsilon' \leq \kappa_0 \varepsilon \quad \text{and} \quad 2C(\varepsilon + \varepsilon') \leq \kappa_0,
\]

using again (61) and the interpolation inequality (52) eventually yields:

\[
(65) \quad \tilde{L}(t) + \kappa_0 \int_0^t \left( \|Z\|_{B^{d+1}_{2,1}} + \varepsilon \|W\|_{B^d_{2,1}} + \varepsilon' \|W\|_{B^{d-1}_{2,1}} \right) \leq \tilde{L}(0)
\]

\[
+ C \int_0^t \left( \|Z\|_{B^{d+1}_{2,1}} + \|W\|_{B^{d-1}_{2,1}} \right) L.
\]

Let us denote

\[
T_0 \triangleq \sup \{ t \in [0, T], \sup_{\tau \in [0, t]} \tilde{L}(\tau) \leq 2\tilde{L}(0) \}.
\]

Discarding the trivial case \( \tilde{L}(0) = 0 \) (corresponding to the stationary solution \( \tilde{V} \)), the continuity of \( \tilde{L} \) ensures that \( T_0 > 0 \). Now, for all \( t \in [0, T_0] \), Inequality (65) ensures that

\[
\tilde{L}(t) + \kappa_0 \int_0^t \left( \|Z\|_{B^{d+1}_{2,1}} + \varepsilon \|W\|_{B^d_{2,1}} + \varepsilon' \|W\|_{B^{d-1}_{2,1}} \right) \leq \tilde{L}(0) + 2C\tilde{L}(0) \int_0^t \left( \|Z\|_{B^{d+1}_{2,1}} + \|W\|_{B^{d-1}_{2,1}} \right).
\]

Consequently, if the initial data are so small that \( 4C\tilde{L}(0) \leq \varepsilon' \kappa_0 \), then we deduce that

\[
\tilde{L}(t) + \frac{\kappa_0}{2} \int_0^t \left( \|Z\|_{B^{d+1}_{2,1}} + \varepsilon \|W\|_{B^d_{2,1}} + \varepsilon' \|W\|_{B^{d-1}_{2,1}} \right) \leq \tilde{L}(0),
\]

and thus \( T_0 = T \). Hence (63) holds (with \( \kappa_0/2 \)) on \( [0, T] \). Clearly, the argument may be started from any time \( t_0 \in [0, T] \), which gives (63) in full generality.

Let us finally establish (64). First, since \( L \) is equivalent to \( \|Z\|_{B^d_{2,1}} + \|Z\|_{B^d_{2,1}} \), it is easy to see that, under Assumption (29), we also have \( \tilde{L} \simeq \|Z\|_{B^d_{2,1}} + \|Z\|_{B^d_{2,1}} \). Combining with (63), we thus already get

\[
\|Z\|_{L^2_t(B^{d+1}_{2,1})} + \|Z\|_{L^2_t(B^{d+1}_{2,1})} + \|Z\|_{L^1_t(B^{d+1}_{2,1})} + \|W\|_{L^1_t(B^{d+1}_{2,1})} \leq C \|Z\| \quad \text{for all} \quad t \in [0, T].
\]

Combining with (61), we discover that

\[
\|Z\|_{L^2_t(B^{d+1}_{2,1})} \lesssim \|W\|_{L^2_t(B^{d+1}_{2,1})} + C \|\nabla Z\|_{L^1_t(B^{d+1}_{2,1})}
\]

and

\[
\|Z\|_{L^2_t(B^{d+1}_{2,1})} \lesssim \|W\|_{L^1_t(B^{d+1}_{2,1})} + C \|\nabla Z\|_{L^2_t(B^{d+1}_{2,1})} + C \|Z\|_{L^1_t(B^{d+1}_{2,1})} \lesssim 0.
\]

Owing to (29), the last term may be absorbed by the left-hand side. Furthermore, one can bound the last but one thanks to (21) and, by Hölder inequality, interpolation and (63),

\[
\|\nabla Z\|_{L^2_t(B^{d+1}_{2,1})} \lesssim \sqrt{\|Z\|_{L^2_t(B^{d+1}_{2,1})} \|Z\|_{L^1_t(B^{d+1}_{2,1})}} \lesssim 0,
\]

\[
\|W\|_{L^2_t(B^{d+1}_{2,1})} \lesssim \sqrt{\|W\|_{L^2_t(B^{d+1}_{2,1})} \|W\|_{L^1_t(B^{d+1}_{2,1})}} \lesssim 0,
\]

which completes the proof of the proposition. \( \square \)
3.2. Proof of Theorem 2.1. The starting point of the proof of existence is the following local well-posedness result that may be found in \[27\].

Proposition 3.4. For any data \(Z_0\) in the nonhomogeneous Besov space \(B^{d+1}_{2,1}\), the following results hold true:

1. Existence: there exists a positive time \(T_1\), depending only the coefficients of the matrices \(A^j\), on \(H\) and on \(\|Z_0\|_{B^{d+1}_{2,1}}\) such that System (3) has a unique classical solution \(Z\) with

\[Z \in C^1([0,T_1] \times \mathbb{R}^d) \quad \text{and} \quad Z \in C([0,T_1]; B^{d+1}_{2,1}) \cap C^1([0,T_1]; B^d_{2,1}).\]

2. Blow-up criterion: If \(T^*\) is finite, then

\[\int_0^{T^*} \| \nabla Z \|_{L^\infty} \, dt = \infty.\]

The proof of the existence part of Theorem 2.1 is structured as follows. First, we truncate the low frequencies of the data and use the above theorem to construct a sequence \((Z^n)_{n \in \mathbb{N}}\) of (a priori local) approximate solutions. Then we use the previous part to establish that those solutions are actually global and uniformly bounded in \((a \text{ priori local})\) approximate solutions. Taking advantage of Proposition 3.3 and denoting by \(n\) the maximal solution \(Z^n \in C([0,T_n];B^{d+1}_{2,1}) \cap C^1([0,T_n];B^d_{2,1})\).

Second step. Uniform estimates. Taking advantage of Proposition 3.3 and denoting by \(Z^n\) the function \(Z\) pertaining to \(Z^n\), we get \(Z^n \leq CZ^n_0\) as long as \(Z^n\) satisfies the smallness condition (29). Owing to the definition of \(Z^n_0\), we have \(Z^n_0 \leq Z_0\) and we clearly have \(\|Z^n(t)\|_{B^{d+1}_{2,1}} \lesssim Z^n(t)\).

Hence using a classical bootstrap argument, one can conclude that, if \(Z_0\) is small enough, then

\[Z^n(t) \leq CZ_0, \quad \text{for all } t \in [0,T_n].\]

In order to show that the solution \(Z^n\) is global (that is \(T_n = +\infty\)), one can use the blow-up criterion of Theorem 3.4. However, we first have to justify that the nonhomogeneous Besov norm \(B^{d+1}_{2,1}\) of the solution is under control up to time \(T_n\). Indeed, using the classical energy method for (3), then the Gronwall lemma, we discover that for all \(t < T_n\),

\[\|Z^n(t)\|_{L^2} \leq C \|Z^n_0\|_{L^2} \exp\left(C \int_0^t \|\nabla Z^n\|_{L^\infty}\right).\]

Now, (66) and the embedding of \(B^d_{2,1}\) in \(L^\infty\) ensure that \(\nabla Z^n\) is in \(L^1_{T_n}(L^\infty)\), from which we deduce that \(Z^n\) is in \(L^\infty_{T_n}(L^2)\), and thus in \(L^\infty_{T_n}(B^{d+1}_{2,1})\) owing, again, to (66). It is now easy to conclude: we have \(\nabla Z^n \in L^1_{T_n}(L^\infty)\), and \(Z^n\) is in \(C([0,T];B^{d+1}_{2,1}) \cap C^1([0,T];B^d_{2,1})\) for all \(T < T_n\). Hence \(T_n = +\infty\) and (66) is satisfied for all time.
**Proposition 3.5.** Let $\tilde{Z} = Z^1 - Z^2$ where $Z^1$ and $Z^2$ are two solutions of (3), having respectively $Z^1_0$ and $Z^2_0$ as initial data, and belonging to the space $E$. There exists a constant $c$ such that if both $\|Z^1\|_{L_\infty^2(B_{\mathbb{Z},1}^2)}$ and $\|Z^2\|_{L_\infty^2(B_{\mathbb{Z},1}^2)}$ are smaller than $c$, then we have for all $t \in [0,T]$,

$$
\|\tilde{Z}\|_{L_\infty^2(B_{\mathbb{Z},1}^2)} \lesssim \|\tilde{Z}_0\|_{B_{\mathbb{Z},1}^2} + \int_0^t \left( \|(Z^1, Z^2)\|_{B_{\mathbb{Z},1}^4}^4 + \|(Z^1, Z^2)\|_{B_{\mathbb{Z},1}^2}^2 \right) \|\tilde{Z}\|_{B_{\mathbb{Z},1}^4}^4.
$$

**Proof.** Let $V^1 \triangleq \tilde{V} + Z^1$ and $V^2 \triangleq \tilde{V} + Z^2$. Observe that $\tilde{Z}$ is a solution of

$$
\tilde{A}^0(V^1)\partial_t \tilde{Z} + \sum_{j=1}^d \tilde{A}^j(V^1)\partial_j \tilde{Z} = -\tilde{A}^0(V^1)\sum_{j=1}^d \left( \tilde{A}^0(V^1)^{-1} \tilde{A}^j(V^1) - \tilde{A}^0(V^2)^{-1} \tilde{A}^j(V^2) \right) \partial_j Z^2 - L\tilde{Z} + r(Z^1) - r(Z^2).
$$

Applying $\tilde{A}_q$, taking the scalar product with $\tilde{Z}_q$, integrating on $\mathbb{R}_+ \times \mathbb{R}^d$ and using Lemma 5.1 we get for all $q \in \mathbb{Z}$,

$$
\left\| \tilde{Z}_q \right\|_{L_2^2(A_0^0(V^1))} + \kappa_0 \int_0^t \left\| L\tilde{Z}_q \right\|_{L_2^2} \leq \left\| \tilde{Z}_{0,q} \right\|_{L_2^2} + \int_0^t \left\| \nabla \tilde{A}^j(V^1) \right\|_{L_\infty^2} \left\| \tilde{Z}_q \right\|_{L_2^2} + \int_0^t \left\| \tilde{A}_q \sum_{j=1}^d \left( \tilde{A}^0(V^1)^{-1} \tilde{A}^j(V^1) - \tilde{A}^0(V^2)^{-1} \tilde{A}^j(V^2) \right) \partial_j Z^2 \right\|_{L_2^2} + \int_0^t \left\| \tilde{A}_q(r(Z^1) - r(Z^2)) \right\|_{L_2^2} + \sum_j \left\| \tilde{A}_q, \tilde{A}_j(V^1) \right\|_{\partial_j L_2^2}.
$$

Multiplying this inequality by $2^{q/2}$ and using commutator estimates, we get

$$
2^{q/2} \left\| \tilde{Z}_q \right\|_{L_2^2} \lesssim 2^{q/2} \left\| \tilde{Z}_{0,q} \right\|_{L_2^2} + \int_0^t \left\| \nabla Z^1 \right\|_{B_{\mathbb{Z},1}^4} 2^{q/2} \left\| \tilde{Z}_q \right\|_{L_2^2} + \int_0^t 2^{q/2} \left\| \tilde{A}_q \sum_{j=1}^d \left( \tilde{A}^0(V^1)^{-1} \tilde{A}^j(V^1) - \tilde{A}^0(V^2)^{-1} \tilde{A}^j(V^2) \right) \partial_j Z^2 \right\|_{L_2^2} + \int_0^t 2^{q/2} \left\| \tilde{A}_q(r(Z^1) - r(Z^2)) \right\|_{L_2^2}.
$$

Thanks to Propositions 5.2 and Inequality (102),

$$
\left\| \left( \tilde{A}^0(V^1)^{-1} \tilde{A}^j(V^1) - \tilde{A}^0(V^2)^{-1} \tilde{A}^j(V^2) \right) \partial_j Z^2 \right\|_{B_{\mathbb{Z},1}^4} \lesssim \left\| \tilde{Z} \right\|_{B_{\mathbb{Z},1}^4} \left\| \nabla Z^2 \right\|_{B_{\mathbb{Z},1}^4},
$$

and, according to Inequality (105), we have

$$
\left\| r(Z^1) - r(Z^2) \right\|_{B_{\mathbb{Z},1}^4} \lesssim \left\| \tilde{Z} \right\|_{B_{\mathbb{Z},1}^4} \left\| (Z^1, Z^2) \right\|_{B_{\mathbb{Z},1}^4}.
$$

Hence, summing (68) on $q \in \mathbb{Z}$, we end up with

$$
\left\| \tilde{Z} \right\|_{L_\infty^2(B_{\mathbb{Z},1}^2)} \lesssim \left\| \tilde{Z}_0 \right\|_{B_{\mathbb{Z},1}^2} + \int_0^t \left\| Z^1 \right\|_{B_{\mathbb{Z},1}^4} \left\| \tilde{Z} \right\|_{B_{\mathbb{Z},1}^4} + \int_0^t \left\| \nabla Z^2 \right\|_{B_{\mathbb{Z},1}^4} + \int_0^t \left\| \tilde{Z} \right\|_{B_{\mathbb{Z},1}^4} \left\| (Z^1, Z^2) \right\|_{B_{\mathbb{Z},1}^4}.
$$
Splitting in low and high frequencies yields the desired estimate.

The above lemma combined with the fact that \((Z^0_n)_{n \in \mathbb{N}}\) converges to \(Z_0\) in \(\dot{B}^{-\frac{d}{2}}_{2,1}\) ensures that \((Z^\alpha_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^\infty_T(\dot{B}^{-\frac{d}{2}}_{2,1})\) and thus has a limit \(Z\) in that space, and passing to the limit in (3) is straightforward. Furthermore, using the Fatou property of Besov spaces, we obtain that \(Z^\ell \in L^\infty_T(\dot{B}^{-\frac{d}{2}}_{2,1}) \cap L^1_T(\dot{B}^{-\frac{d}{2}+1}_{2,1})\) and \(Z^h \in L^\infty_T(\dot{B}^{-\frac{d}{2}+1}_{2,1}) \cap L^1_T(\dot{B}^{-\frac{d}{2}+1}_{2,1})\) for all \(T > 0\), together with the desired bounds. Time continuity of the solution may be obtained by adapting the arguments of [11 Chap. 4].

**Fourth step. Uniqueness.** Knowing that \(Z^1\) and \(Z^2\) are in \(E\), we have for all \(T > 0\),

\[
\int_0^T \left( \|Z^1, Z^2\|_{\dot{B}^{-\frac{d}{2}}_{2,1}} + \|Z^1, Z^2\|_{\dot{B}^{-\frac{d}{2}+1}_{2,1}} \right) \, dt < \infty.
\]

Furthermore, one can assume with no loss of generality that \(Z^1\) is the solution that we constructed before and thus satisfies the smallness assumption (29). Owing to time continuity and since \(Z^2(0) = Z^1(0)\), the solution \(Z^2\) also satisfies (29) on some nontrivial time interval \([0, T]\), and combining Inequality (67) with Gronwall lemma allows to conclude that \(Z^1\) and \(Z^2\) coincide on \([0, T]\). A bootstrap argument then yields uniqueness on the whole half-line \(\mathbb{R}_+\).

### 3.3. Proof of Theorem [2.2]

The overall strategy is taken from the work by Z. Xin and J. Xu in [26].

**First step : uniform bound in \(\dot{B}^{-\sigma_1}_{2,\infty}\).** In order to establish (25), one can look at System (3) as

\[
\tilde{A}^0 \partial_t Z + \sum_{j=1}^d \tilde{A}^j \partial_j Z + LZ = f + g + h
\]

with \(f \triangleq \sum_{j=1}^d (\tilde{A}^j - \tilde{A}^j(V)) \partial_j Z\), \(g \triangleq r(Z)\) and \(h \triangleq \tilde{A}^0 - \tilde{A}^0(V)\) \(\partial_t Z\),

then apply \(\tilde{\Delta}_q\) and perform \(L^2\) estimates for each \(Z_q\).

After using Lemma 5.1 multiplying by \(Z^{-\eta \sigma_1}\) then taking the supremum on \(Z\), we end up (omitting the term coming from \(L\) that has the ‘good’ sign) with

\[
\|Z(t)\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \lesssim \|Z_0\|_{\dot{B}^{-\sigma_1}_{2,\infty}} + \int_0^t \|(f, g, h)\|_{\dot{B}^{-\sigma_1}_{2,\infty}}. \tag{69}
\]

Setting \(f_j = (\tilde{A}^j(V) - \tilde{A}^j) \partial_j Z\) and using Inequality (101) yields

\[
\|f_j\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \lesssim \left\| \tilde{A}^j(V) - \tilde{A}^j \right\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \left\| \partial_j Z \right\|_{\dot{B}^{-\frac{d}{2}+1}_{2,1}}.
\]

In order to bound \(\tilde{A}^j(V) - \tilde{A}^j\) in \(\dot{B}^{-\sigma_1}_{2,\infty}\), one cannot use directly Proposition 5.3 as \(-\sigma_1\) may be negative. However, applying Taylor formula, product laws and a composition estimate (see the details in the proof of [10 Th. 4.1]), we can still obtain if (29) is satisfied,

\[
\left\| \tilde{A}^j(V) - \tilde{A}^j \right\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{B}^{-\sigma_1}_{2,\infty}},
\]

whence

\[
\|f\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \|Z\|_{\dot{B}^{-\sigma_1}_{2,\infty}}.
\]

For \(g\), using similar arguments as in Proposition 5.4 combined with Inequality (101) yield

\[
\|g\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \|Z\|_{\dot{B}^{-\sigma_1}_{2,\infty}} \|
\]
Concerning $h$, we have, keeping Lemma 3.1 in mind, that
\[ \|h\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \lesssim \|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \|\partial_t Z\|_{\dot{B}_{2,1}^\frac{d}{2} + 1} \lesssim \|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \|(\nabla Z, Z_2)\|_{\dot{B}_{2,1}^\frac{d}{2}}. \]

Thus, regrouping all those estimates, we obtain
\[ \|Z(t)\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \lesssim \|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}} + \int_0^t \|(\nabla Z, Z_2)\|_{\dot{B}_{2,1}^\frac{d}{2}} \|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}}, \quad t \geq 0. \]

Since, as pointed out before, we have the proof of (25).

For the sake of completeness, one has to justify that if $Z_0$ is in $\dot{B}_{2,\infty}^{-\sigma_1}$ (in addition to $Z_0^n$), then the solution constructed in Theorem 2.1 is in $\dot{B}_{2,\infty}^{-\sigma_1}$ for all time. This may be checked by following the construction scheme of the previous subsection. Indeed, recall that the approximated solutions $Z^n$ are in $C^1(\mathbb{R}^+; \dot{B}_{2,\infty}^{-\sigma_1})$. Then, discarding the linear term $LZ^n$ (that may be handled by suitable conjugation), we get $\partial_t Z^n \in C(\mathbb{R}^+; L^1)$. As $L^1 \hookrightarrow \dot{B}_{2,\infty}^{d/2}$ and $\sigma_1 \geq d/2$, the low frequencies of $\partial_t Z^n$ (and thus the whole $\partial_t Z^n$) are in $C(\mathbb{R}^+; \dot{B}_{2,\infty}^{-\sigma_1})$. As $Z^n_0$ itself is in $\dot{B}_{2,\infty}^{-\sigma_1}$ (since $F(Z^n_0)$ is supported away from 0), we have $Z^n \in C^1(\mathbb{R}^+; \dot{B}_{2,\infty}^{-\sigma_1})$. Consequently, (25) holds for $Z^n$ and, passing to the limit, ensures that it holds for $Z$, too.

Second step : proof of generic decay estimates. According to Proposition 3.3, the functional $\tilde{L}$ introduced therein is nonincreasing and equivalent to $\|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} + \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}}$. Furthermore, there exist positive $\kappa_0, \varepsilon$ and $\varepsilon'$ such that denoting $\tilde{H} \triangleq \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} + \varepsilon \|W\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} + \varepsilon' \|W\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}}$, we have

\[ \frac{d}{dt} \tilde{L} + \varepsilon \tilde{H} \leq 0 \quad \text{for all } \ 0 \leq t_0 \leq t. \]

Hence and one may conclude as in [10] that $\tilde{L}$ is differentiable almost everywhere and satisfies
\[ \frac{d}{dt} \tilde{L} + \varepsilon \tilde{H} \leq 0 \quad \text{a. e. on } \mathbb{R}^+. \]

Granted with this information and (25), one can prove the first decay estimate of Theorem 2.2 by following the general argument of [26]. The starting point is that, provided $-\sigma_1 < d/2 - 1$,
\[ \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} \lesssim \left( \|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \right)^{\theta_0} \left( \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} \right)^{(1 - \theta_0)} \quad \text{with } \theta_0 = \frac{2}{d/2 + 1 + \sigma_1}. \]

Inequality (25) thus implies that
\[ \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} \gtrsim \left( \|Z\|_{\dot{B}_{2,\infty}^{-\sigma_1}} \right)^{\frac{1 - \theta_0}{1 - \theta_0}} \|Z_0\|_{\dot{B}_{2,\infty}^{-\sigma_1}}. \]

For the high frequencies term, using the estimate of Theorem 2.1, one can just write:
\[ \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} \gtrsim \left( \|Z\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1}} \right)^{-\frac{\theta_0}{\theta_0}} \|Z_0\|_{\dot{B}_{2,1}^{\frac{d}{2} + 1} \cap \dot{B}_{2,1}^{\frac{d}{2} + 1}}. \]
Hence, there exists a (small) constant $c$ such that
\[
\frac{d}{dt} \bar{L} + cC_0 \frac{\theta_0}{\bar{L}^{1 - \theta_0} - \theta_0} \leq 0 \quad \text{with} \quad C_0 \triangleq \|Z_0\|_{B^{\sigma_1}_{2,1} \cap \bar{B}^{\frac{d}{2}+1}_{2,1}}.
\]
Integrating, this gives us
\[
\bar{L}(t) \leq \left(1 + c \frac{\theta_0}{1 - \theta_0} \left(\frac{\bar{L}(0)}{C_0}\right)^{\frac{\theta_0}{1 - \theta_0}}\right)^{-\frac{1}{\theta_0}} \bar{L}(0)
\]
whence, since $\bar{L} \leq \bar{L}(0) \lesssim Z_0 \lesssim C_0$,
\[
\|Z(t)\|_{\bar{B}^{\frac{d}{2}+1}_{2,1}} + \|Z(t)\|_{\bar{B}^{\sigma}_{2,1}} \lesssim (1 + t)^{-\alpha_1} Z_0 \quad \text{with} \quad \alpha_1 = \frac{d/2 - 1 + \sigma_1}{2}.
\]
The decay rates in $\bar{B}^{\sigma}_{2,1}$ for all $\sigma \in ] - \sigma_1, d/2 - 1]$ follow from Inequalities (25) and (72), and interpolation inequalities.

**Third step: decay enhancement for the damped mode.** From (55) and Lemma 5.1, one can get for all $\sigma \in ] - \sigma_1, d/2 - 1]$,\[
\mathcal{W}^\sigma(t) \leq e^{-ct} \mathcal{W}^\sigma(0) + C \int_0^t e^{-c(t-\tau)} \|h(\tau)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \, d\tau.
\]
Hence, since $\mathcal{W}^\sigma \approx \|W\|_{\bar{B}^{\sigma}_{2,1}}^\ell$ and using the estimates of $h$ pointed out in the proof of Proposition 3.2, we get
\[
\|W(t)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \lesssim e^{-t} \|W(0)\|_{\bar{B}^{\sigma}_{2,1}}^\ell + \int_0^t e^{-(t-\tau)} \left(\|Z\|_{\bar{B}^{\frac{d}{2}+1}_{2,1}}^\ell \|Z_2, W\|_{\bar{B}^{\sigma}_{2,1}}^\ell + \|(\nabla Z, W)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \|Z\|_{\bar{B}^{\sigma+1}_{2,1}}^\ell + \|(\nabla^2 Z, \nabla W)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \right),
\]
whence
\[
\|W(t)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \lesssim e^{-t} \|W(0)\|_{\bar{B}^{\sigma}_{2,1}}^\ell + \int_0^t e^{-(t-\tau)} \left(\|Z\|_{\bar{B}^{\sigma}_{2,1}}^\ell \|Z\|_{\bar{B}^{\frac{d}{2}+1}_{2,1}}^\ell + \|(\nabla Z, Z_2)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \|Z\|_{\bar{B}^{\sigma+1}_{2,1}}^\ell + \|\nabla Z\|_{\bar{B}^{\sigma}_{2,1}}^\ell \right).
\]
In light of the previous step, the worst decay comes from the last term. In order to be allowed to use the corresponding estimate however, we need $\sigma + 1 \leq d/2 - 1$. If that condition is satisfied then, setting $\beta = (\sigma + \sigma_1 + 1)/2$, the above inequality implies that
\[
\langle t \rangle^{\beta} \|W(t)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \lesssim \langle t \rangle^{\beta} e^{-ct} \|W(0)\|_{\bar{B}^{\sigma}_{2,1}}^\ell + C_0 \int_0^t \langle \tau \rangle^{\beta} e^{-c(t-\tau)} \, d\tau \lesssim C_0.
\]
In the case $\sigma + 1 > d/2 - 1$, one can use the fact that $\|W(t)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \lesssim \|W(t)\|_{\bar{B}^{\frac{d}{2}-2}_{2,1}}^\ell$, and the above argument thus just implies that
\[
\|W(t)\|_{\bar{B}^{\sigma}_{2,1}}^\ell \lesssim (1 + t)^{-\alpha_1}.
\]
Keeping in mind (61), one can conclude that $\|Z_2\|_{\bar{B}^{\sigma}_{2,1}}^\ell$ satisfies the same decay estimates as $W$. 

---

**PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS**

---

**PoP**
Last step: high frequencies decay. Let us start from \((50)\). The usual method based on Lemma 5.1 leads after multiplying by \((t)^{2\alpha_1}\) (where \(\alpha_1\) comes from \((72)\)) yields

\[
(73) \quad \| (t)^{2\alpha_1} Z(t) \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^h \leq e^{-c t} \| Z_0 \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^h + \int_0^t (t)^{2\alpha_1} e^{-c(t-\tau)} \| Z \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^h + \| Z \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^\ell \, d\tau + \int_0^t (t)^{2\alpha_1} e^{-c(t-\tau)} \| Z \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^\ell \| Z_2 \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^\ell \, d\tau.
\]

Thanks to \((72)\), the first quadratic term may be bounded as follows:

\[
\int_0^t (t)^{2\alpha_1} e^{-c(t-\tau)} \| Z \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^h \leq \int_0^t \left( \frac{(t)}{(\tau)} \right)^{2\alpha_1} e^{-c(t-\tau)} (\langle \tau \rangle^{\alpha_1} \| Z \|_{\dot{B}^\frac{d}{2}+1}_{2,1}^h)^2 \, d\tau \lesssim C_0,
\]

and the other terms of the right-hand side of \((73)\) may be bounded similarly. This completes the proof of Theorem 2.2.

4. The proof of Theorem 2.3 and application to the compressible Euler system

This section is devoted to the proof of Theorem 2.3 that is to say to a refinement of Theorem 2.1 corresponding to the case where System \((11)\) satisfies the extra conditions listed in \((26)\). As an application, we shall obtain a global existence statement for the compressible Euler with damping, in a new functional framework, and will specify the dependency of the estimates with respect to the relaxation (or damping) parameter.

4.1. Proof of Theorem 2.3. Proving existence and uniqueness being very similar to what we did before, we focus on establishing a priori estimates for a smooth solution \(Z\) of \((3)\) on \([0, T] \times \mathbb{R}^d\), satisfying the smallness condition \((29)\). The general strategy is the same as in the previous section, and we shall mainly underline the places where having the structure \((26)\) comes into play.

The first difference is in the following refinement of Lemma 3.1

Lemma 4.1. Under hypotheses \((26)\) and \((29)\), we have for all \(\sigma \in \mathbb{R}\)

\[
\| \partial_t Z_1 \|_{\dot{B}^\sigma_{2,1}} \lesssim \| \nabla Z_2 \|_{\dot{B}^\sigma_{2,1}} + \| Z_2 \|_{\dot{B}^\frac{d}{2}+1}_{2,1} \| \nabla Z_1 \|_{\dot{B}^\sigma_{2,1}}
\]

\[
\| \partial_t Z_2 \|_{\dot{B}^\sigma_{2,1}} \lesssim \| W \|_{\dot{B}^\sigma_{2,1}}.
\]

Proof. The second inequality has been proved before (see Lemma 3.1). The first one relies on the decomposition

\[
(74) \quad \partial_t Z_1 = -\sum_{j=1}^d (\bar{A}^{ij}_{1,1}(V))^{-1} \left( \bar{A}^{ij}_{1,1}(V) \partial_j Z_1 + \bar{A}^{ij}_{1,2}(V) \partial_j Z_2 \right).
\]

As the function \(V \mapsto \bar{A}^{ij}_{1,1}(V)^{-1} \bar{A}^{ij}_{1,1}(V)\) vanishes at \(\bar{V}\) and is linear with respect to \(Z_2\), Propositions 5.2, 5.3 and Condition \((29)\) guarantee the desired inequality.

4.1.1. Basic energy estimates. As for Theorem 2.1, the first step consists in proving estimates for \(\| Z_2 \|_{L^2_{\bar{A}_0(V)}}^2\) and \(\| Z_2 \|_{L^2_{\bar{A}_0}}^2\).

Proposition 4.1. Let \(Z\) be a smooth solution \((3)\) on \([0, T]\) satisfying \((29)\). Then, under Condition \((26)\), we have for all \(q \geq 0\),

\[
(75) \quad \frac{1}{2} \frac{d}{dt} \| Z_q \|_{L^2_{\bar{A}_0(V)}}^2 + \kappa_0 \| Z_{2,q} \|_{L^2}^2 \lesssim c_q 2^{-q(\frac{d}{2}+1)} \| (W, \nabla Z) \|_{\dot{B}^\frac{d}{2}+1}_{2,1} \| Z \|_{\dot{B}^\frac{d}{2}+1}_{2,1} \| Z_q \|_{L^2}.
\]
and for all $q \leq 0$,

\begin{align}
\frac{1}{2} \frac{d}{dt} \| Z_q \|_{L^2_{\mathcal{X}_0}}^2 + \kappa_0 \| Z_{2,q} \|_{L^2}^2 \lesssim c_q 2^{-q\frac{d}{2}} \left( \| Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| (W, \nabla Z_2) \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| \nabla Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| Z_2 \|_{L^2}^2 \right) \| Z_q \|_{L^2}.
\end{align}

**Proof.** The starting point is still (39) but we now take advantage of Lemma 4.1 and refine the estimates for $R_q^1$ and (42). More precisely, we have

\[ R_q^1 = \sum_{j=1}^d \left( [\tilde{A}_{1,1}^j(V), \tilde{\Delta}_q] \partial_j Z_1 + [\tilde{A}_{1,2}^j(V), \tilde{\Delta}_q] \partial_j Z_2 \right). \]

Hence, using Inequality (97), we get for all $\sigma \in [-d/2, d/2 + 1]$,

\[ \| R_q^1 \|_{L^2} \lesssim c_q 2^{-q\sigma} \left( \| \nabla (\tilde{A}_{1,1}^j(V)), \nabla (\tilde{A}_{2,1}^j(V)) \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| Z_1 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \right. \\
\left. + \| \nabla (\tilde{A}_{1,2}^j(V)), \nabla (\tilde{A}_{2,2}^j(V)) \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \right). \]

At this point, one can use that (26) ensures that for all $j \in \{1, \cdots, d\}$ and $k \in \{1, 2\}$, there exist a linear map $h$ and a smooth map $F$ such that $\tilde{A}_{k,1}^j(V) - \tilde{A}_{k,2}^j(V) = h(Z_2) F(Z)$. Consequently, product laws and composition estimates give us

\[ \left( \| \nabla (\tilde{A}_{k,1}^j(V)) \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \right)^2 \lesssim \| \nabla (h(Z_2)) \otimes F(Z) \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| h(Z_2) \otimes \nabla (F(Z)) \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \]

\[ \lesssim \| \nabla Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}}(1 + \| Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}}) + \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| \nabla Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \]

\[ \lesssim \| \nabla Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| \nabla Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}}, \]

whence

\begin{align}
\| R_q^1 \|_{L^2} \lesssim c_q 2^{-q\sigma} \left( \| \nabla Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \right) \| Z_1 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| \nabla Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}}.
\end{align}

Let us also observe that Inequality (107) of Proposition 5.4 gives us

\[ \| r(Z) \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \lesssim \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| \nabla Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \]

\[ \| Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| \nabla Z_1 \|_{\mathcal{B}_{2,1}^\frac{d}{2}}. \]

Remembering that

\[ \| R_q^2 \|_{L^2} \lesssim c_q 2^{-q(d+1)} \| \nabla Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| \partial_t Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}}, \]

and using Lemma 4.1 as well as (32) and (11), we eventually get (75).

For proving (76), the starting point is (43). The term corresponding to $R_q^1$ (resp. $r(Z)$) can be bounded according to (77) (resp. (107)) with $\sigma = d/2$. In order to bound the term corresponding to $R_q^2$, we observe that, in light of Lemma 4.1

\[ \| (\tilde{A}_{1,1}^0(V) - \tilde{A}_{1,1}^0(V)) \partial_t Z_1 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \lesssim \| Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| \partial_t Z_1 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \]

\[ \lesssim \| Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \left( \| \nabla Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} + \| Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| \nabla Z_1 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \right), \]

\[ \| (\tilde{A}_{2,2}^0(V) - \tilde{A}_{2,2}^0(V)) \partial_t Z_2 \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \lesssim \| Z \|_{\mathcal{B}_{2,1}^\frac{d}{2}} \| W \|_{\mathcal{B}_{2,1}^\frac{d}{2}}. \]
Finally, we have to refine Inequality (42). To this end, we use the decomposition
\[
\int \mathbb{R}^d \sum_{j=1}^d \partial_j (\tilde{A}_j^i(V)) Z_q \cdot Z_q = \sum_{j=1}^d \int \mathbb{R}^d \left( \partial_j (\tilde{A}_{1,1}^j(V)) Z_{1,q} \cdot Z_{1,q} + \partial_j (\tilde{A}_{2,1}^j(V)) Z_{2,q} \cdot Z_{1,q} \\
+ \partial_j (\tilde{A}_{1,2}^j(V)) Z_{1,q} \cdot Z_{2,q} + \partial_j (\tilde{A}_{2,2}^j(V)) Z_{2,q} \cdot Z_{2,q} \right).
\]

The structure assumptions (26) and the symmetry of the system ensure that
\[
\| \partial_j (\tilde{A}_{1,1}^j(V)) \|_{L^\infty} + \| \partial_j (\tilde{A}_{1,2}^j(V)) \|_{L^\infty} + \| \partial_j (\tilde{A}_{2,1}^j(V)) \|_{L^\infty} \lesssim \| Z_2 \|_{L^\infty} \| \nabla Z \|_{L^\infty} + \| \nabla Z_2 \|_{L^\infty}.
\]
Hence, remembering (30),
\[
\int \mathbb{R}^d \sum_{j=1}^d \partial_j (\tilde{A}_j^i(V)) Z_q \cdot Z_q \lesssim (\| Z_2 \|_{L^\infty} \| \nabla Z \|_{L^\infty} + \| \nabla Z_2 \|_{L^\infty}) \| Z_{1,q} \|^2 + \| \nabla Z \|_{L^\infty} \| Z_{2,q} \|^2.
\]
Plugging all the above inequalities in (43), we end up with (76).

4.1.2. Cross estimates. Remember that for all \( q \in \mathbb{Z} \), we have
\[
(78) \quad \frac{d}{dt} I_q + \frac{2^n}{2} \sum_{k=1}^{n-1} e_k \int \mathbb{R}^d |N M^k \partial \tilde{Z}_q|^2 \, d\xi \leq \frac{2^{-q} K_0}{2} \| NZ_q \|^2_{L^2} + C \| \hat{A}_q G \|_{L^2} \| Z_q \|_{L^2}.
\]
In our new regularity context, we have to add up \( 2^n I_q \) if \( q < 0 \) (resp. \( 2^{-q} I_q \) if \( q \geq 0 \)) to \( L_q \), then to multiply by \( 2^q \) (resp. \( 2^{-q} (q+1) \)). This amounts to bounding \( \| G \|^{\ell}_{B^{2,1}_{2,1}} \) and \( \| G \|^{h}_{B^{2,1}_{2,1}} \). To this end, we have to refine the estimates (44), (45) and (46) taking our structure assumption (26) into account.

As a first, we see that (21) and Proposition 5.4 ensure that
\[
(79) \quad \| G_3 \|^{\ell}_{B^{2,1}_{2,1}} + \| G_3 \|^{h}_{B^{2,1}_{2,1}} \lesssim \| G_3 \|^{\ell}_{B^{2,1}_{2,1}} + \| G_3 \|^{h}_{B^{2,1}_{2,1}} \lesssim \| Z_2 \|^2_{B^{2,1}_{2,1}}.
\]
Next, we have, thanks to Propositions 5.2 and 5.4
\[
(80) \quad \| G_2 \|^{\ell}_{B^{2,1}_{2,1}} + \| G_2 \|^{h}_{B^{2,1}_{2,1}} \lesssim \| G_2 \|^{\ell}_{B^{2,1}_{2,1}} + \| G_2 \|^{h}_{B^{2,1}_{2,1}} \lesssim \| Z \|^{\ell}_{B^{2,1}_{2,1}} \| Z \|^{h}_{B^{2,1}_{2,1}} + \| Z \|^{\ell}_{B^{2,1}_{2,1}} \| Z \|^{h}_{B^{2,1}_{2,1}}.
\]
In order to improve the estimate for \( G_1 \), we use that \( \tilde{A}_0^{-1} G_1 \) is the sum for \( j = 1 \) to \( d \) of
\[
\left( ((\tilde{A}_{1,1}^0(V))^{-1} \tilde{A}_{1,1}^j(V) - (\tilde{A}_{1,1}^0(V))^{-1} \tilde{A}_{1,2}^j(V)) \partial_j Z_1 + ((\tilde{A}_{1,1}^0(V))^{-1} \tilde{A}_{1,2}^j(V) - (\tilde{A}_{1,2}^0(V))^{-1} \tilde{A}_{2,1}^j(V) \partial_j Z_2 \right)
\]
\[
\left( ((\tilde{A}_{2,2}^0(V))^{-1} \tilde{A}_{2,1}^j(V) - (\tilde{A}_{2,2}^0(V))^{-1} \tilde{A}_{2,2}^j(V)) \partial_j Z_1 + ((\tilde{A}_{2,2}^0(V))^{-1} \tilde{A}_{2,2}^j(V) - (\tilde{A}_{2,2}^0(V))^{-1} \tilde{A}_{2,2}^j(V) \partial_j Z_2 \right).
\]
Hence, owing to (26), we just have
\[
\| G_1 \|^{\ell}_{B^{2,1}_{2,1}} + \| G_1 \|^{h}_{B^{2,1}_{2,1}} \lesssim \| Z_2 \|^{\ell}_{B^{2,1}_{2,1}} \| Z \|^{\ell}_{B^{2,1}_{2,1}} \| Z \|^{h}_{B^{2,1}_{2,1}} \| Z \|^{h}_{B^{2,1}_{2,1}} \| \nabla Z_2 \|^{\ell}_{B^{2,1}_{2,1}} \| \nabla Z_2 \|^{h}_{B^{2,1}_{2,1}}.
\]
Together with (79) and (80), we can conclude that
\[
(81) \quad \| G \|^{\ell}_{B^{2,1}_{2,1}} + \| G \|^{h}_{B^{2,1}_{2,1}} \lesssim \| Z_2 \|^2_{B^{2,1}_{2,1}} + \| Z \|^{\ell}_{B^{2,1}_{2,1}} \| Z \|^{h}_{B^{2,1}_{2,1}} + \| Z \|^{\ell}_{B^{2,1}_{2,1}} \| Z \|^{h}_{B^{2,1}_{2,1}} \| Z \|^{\ell}_{B^{2,1}_{2,1}} \| Z \|^{h}_{B^{2,1}_{2,1}}.
\]
4.1.3. Provisional assessment. Let $\mathcal{L}' \triangleq \sum_{q<0} 2^{\frac{4}{d}} \sqrt{\mathcal{L}_q} + \sum_{q \geq 0} 2^{\frac{4}{d} + 1} \sqrt{\mathcal{L}_q}$. Putting together Inequalities (75), (76), (78) and (81), using Lemma 5.1 and discarding the redundant terms, we end up with

$$\mathcal{L}'(t) + \kappa_0 \int_0^t (\|Z\|_{L^2_{B_2^4}}^{\frac{\ell}{d}} + \|Z\|_{L^2_{B_2^4}}^{\frac{h}{d} + 1}) \leq \mathcal{L}'(0)$$

$$+ C \int_0^t (\|(W, \nabla Z_2)\|_{L^2_{B_2^4}} + \|Z_2\|_{L^2_{B_2^4}}^{\frac{\ell}{d}} \|Z_2\|_{L^2_{B_2^4}} + \|Z_2\|_{L^2_{B_2^4}}^{\frac{h}{d} + 1} + \|Z\|_{L^2_{B_2^4}}^{\frac{2}{d}} + \|Z_2\|_{L^2_{B_2^4}}^{\frac{2}{d}} \|Z_2\|_{L^2_{B_2^4}}^{\frac{2}{d} + 1} + \|Z\|_{L^2_{B_2^4}}^{\frac{2}{d} + 1}) \cdot$$

In order to close the estimates, we need to exhibit the $L^1$-in-time integrability of $W$ and $\nabla Z_2$ in $B_2^4_{d,1}$ and the $L^2$-in-time integrability of $Z_2$ in $B_2^4_{d,1}$.

4.1.4. Bounds for the damped mode. With the notations we used to prove (55), remember that

$$\frac{1}{2} \frac{d}{dt} \|W_q\|_{L^2_{A_2^2}}^2 + \kappa_0 \|W_q\|_{L^2}^2 \leq \left(\|\hat{\Delta} q h_1\|_{L^2} + C \|\hat{\Delta} q h_2\|_{L^2} + C \|\hat{\Delta} q h_3\|_{L^2}\right) \|W_q\|_{L^2}.$$

From Lemma 4.1, Propositions 5.2, 5.3 and 5.4 and, since

$$\partial_t Q(Z) = D_{Z_1} Q(Z) \partial_t Z_1 + D_{Z_2} Q(Z) \partial_t Z_2,$$

we readily get

$$\|h_1\|_{L^2_{B_2^4}}^{\frac{\ell}{d}} + \|h_1\|_{L^2_{B_2^4}}^{\frac{h}{d} + 1} \leq \|h_1\|_{L^2_{B_2^4}}^{\frac{\ell}{d}} \|W\|_{L^2_{B_2^4}}^{\frac{h}{d} + 1},$$

$$\|h_3\|_{L^2_{B_2^4}}^{\frac{\ell}{d}} + \|h_3\|_{L^2_{B_2^4}}^{\frac{h}{d} + 1} \leq \|h_3\|_{L^2_{B_2^4}}^{\frac{\ell}{d}} \|\nabla Z\|_{L^2_{B_2^4}}^{\frac{h}{d} + 1} + \|Z_2\|_{L^2_{B_2^4}}^{\frac{2}{d}} \|W\|_{L^2_{B_2^4}}^{\frac{2}{d} + 1}.$$

For bounding $h_2$, we need to refine the decomposition we did in the previous section. More precisely, we now write that for all $j \in \{1, \cdots, d\}$,

$$\partial_t (A_{2,1}^j(V) \partial_j Z_1 + A_{2,2}^j(V) \partial_j Z_2) = D_{V_1} A_{2,1}^j(V) \partial_j Z_1 \partial_j Z_1 + D_{V_2} A_{2,1}^j(V) \partial_j Z_2 \partial_j Z_1$$

$$+ A_{2,1}^j(V) \partial_j Z_1 + (A_{2,1}^j(V) - A_{2,1}^j(V)) \partial_j Z_1$$

$$+ D_V A_{2,2}^j(V) \partial_j Z_2 + A_{2,2}^j(V) \partial_j Z_2 + (A_{2,2}^j(V) - A_{2,2}^j(V)) \partial_j Z_2.$$

Since $A_{2,1}^j$ is linear with respect to $V_2$, we get after using (74) and Lemma 5.1 that

$$\|\hat{\Delta} q h_2\|_{L^2} \lesssim \|\nabla^2 Z_2 q\|_{L^2} + c q 2^{-q \frac{d}{2}} \left(\|\nabla Z\|_{L^2_{B_2^4}}^{\frac{2}{d}} + \|Z_2\|_{L^2_{B_2^4}}^{\frac{2}{d}} \|Z_2\|_{L^2_{B_2^4}}^{\frac{2}{d}} \|\nabla Z\|_{L^2_{B_2^4}}^{\frac{2}{d}} \right).$$

Hence, reverting to (83), using Lemma 5.1 and keeping the notation (57), we end up for $\sigma \in [d/2, d/2 + 1]$ with

$$W(\sigma) \leq \kappa_0 \int_0^\tau \|W\|_{L^2_{B_2^4}}^{\frac{\ell}{d}} \leq W(0) + C \int_0^\tau \left(\|W\|_{L^2_{B_2^4}}^{\frac{2}{d}} + \|\nabla Z\|_{L^2_{B_2^4}}^{\frac{2}{d}} \|Z\|_{L^2_{B_2^4}}^{\frac{2}{d}} + \|Z\|_{L^2_{B_2^4}}^{\frac{2}{d} + 1} \|\nabla Z\|_{L^2_{B_2^4}}^{\frac{2}{d} + 1} \right).$$

In order to compare $W$ with $Z_2$, one can use the decomposition:

$$W - Z_2 = L^{-1} \sum_{j=1}^d \left(\tilde{A}_{2,1}^j \partial_j Z_1 + (\tilde{A}_{2,1}^j(V) - \tilde{A}_{2,1}^j) \partial_j Z_1 + \tilde{A}_{2,2}^j \partial_j Z_2 + (\tilde{A}_{2,2}^j(V) - \tilde{A}_{2,2}^j) \partial_j Z_2 \right) - L^{-1} Q(Z),$$
which implies that
\begin{equation}
\|W - Z\|_{B_{2,1}^{s}} \lesssim \|\nabla Z\|_{B_{2,1}^{s}} + \|Z_2\|_{B_{2,1}^{s}} \|\nabla Z_1\|_{B_{2,1}^{s}} + \|Z\|_{B_{2,1}^{s}} \|\nabla Z_2\|_{B_{2,1}^{s}} + \|Z_2\|_{B_{2,1}^{s}}^2.
\end{equation}
and that, for all \(s \geq d/2\),
\begin{equation}
\|W - Z\|_{B_{2,1}^{s}} \lesssim \|\nabla Z\|_{B_{2,1}^{s}} + \|Z_2\|_{B_{2,1}^{s}} \|\nabla Z_1\|_{B_{2,1}^{s}} + \|Z\|_{B_{2,1}^{s}} \|\nabla Z_2\|_{B_{2,1}^{s}} + \|Z_2\|_{B_{2,1}^{s}}^2.
\end{equation}

### 4.1.5. Closure of the estimates.
As in the previous section, if we set
\[\tilde{\mathcal{L}}' \triangleq \mathcal{L}' + \varepsilon W_{2}^{\frac{d}{2}+1} + \varepsilon' W_{2}^{\frac{d}{2}}\]
and
\[\tilde{\mathcal{H}}' \triangleq \|Z\|_{B_{2,1}^{s}}^{\ell} + \|Z\|_{B_{2,1}^{s}}^{h} + \varepsilon' \|W\|_{B_{2,1}^{s}}^{\ell} + \varepsilon' \|W\|_{B_{2,1}^{s}}^{h},\]
with suitable \(\varepsilon\) and \(\varepsilon'\), then putting together (83) and (84) yields
\begin{equation}
\tilde{\mathcal{L}}'(t) + \kappa_0 \int_0^t \tilde{\mathcal{H}}'(0) + C \int_0^t (\|Z_2\|_{B_{2,1}^{s}}^{h} + \|Z_2\|_{B_{2,1}^{s}}^{h} + \|Z\|_{B_{2,1}^{s}}^{h} + \|Z\|_{B_{2,1}^{s}}^{h}) \mathcal{L}'.
\end{equation}
Note that we have
\begin{equation}
\|Z_2\|_{B_{2,1}^{s}}^{h} \lesssim \|Z_2\|_{B_{2,1}^{s}}^{h} \mathcal{L}'.
\end{equation}
and
\begin{equation}
\|Z\|_{B_{2,1}^{s}}^{2} + \|Z_2\|_{B_{2,1}^{s}}^{h} \|Z\|_{B_{2,1}^{s}}^{h} \lesssim \|Z\|_{B_{2,1}^{s}}^{2} \mathcal{L}'.
\end{equation}

Hence, using also (85) and (86) with \(s = d/2 + 1\), we see that (87) becomes just
\[\tilde{\mathcal{L}}'(t) + \kappa_0 \int_0^t \tilde{\mathcal{H}}'(0) + C \int_0^t \tilde{\mathcal{H}}'\mathcal{L}'.
\]
To handle the last integral, let us write that, by virtue of (86) with \(s = d/2\), we have
\begin{equation}
(\|Z_2\|_{B_{2,1}^{s}}^{h})^2 \lesssim (\|W\|_{B_{2,1}^{s}}^{h})^2 + \|Z_2\|_{B_{2,1}^{s}}^{4} + (\|Z\|_{B_{2,1}^{s}}^{h})^2 + \|Z_2\|_{B_{2,1}^{s}}^{2} \|\nabla Z_1\|_{B_{2,1}^{s}}^{2} + \|Z\|_{B_{2,1}^{s}}^{2} \|\nabla Z_2\|_{B_{2,1}^{s}}^{2}.
\end{equation}
The last three terms of the right-hand side may be bounded (owing to (88) and to (29)) by \(\tilde{\mathcal{H}}'\mathcal{L}'\), and we have
\begin{equation}
(\|W\|_{B_{2,1}^{s}}^{h})^2 \lesssim \tilde{\mathcal{H}}'\tilde{\mathcal{L}}'.
\end{equation}
Finally, we have
\begin{equation}
\|Z_2\|_{B_{2,1}^{s}}^{4} \|Z\|_{B_{2,1}^{s}}^{h} \|Z_2\|_{B_{2,1}^{s}}^{2} \|Z\|_{B_{2,1}^{s}}^{h} \lesssim (\mathcal{L}')^2 \|Z_2\|_{B_{2,1}^{s}}^{2}.
\end{equation}
Hence there exists a constant \(C\) (that may depend on \(\varepsilon\) and \(\varepsilon'\) but not on the solution) such that for all \(t \in [0, T]\), we have
\[\tilde{\mathcal{L}}'(t) + \kappa_0 \int_0^t \tilde{\mathcal{H}}' \leq \tilde{\mathcal{L}}'(0) + C \int_0^t \tilde{\mathcal{H}}'\tilde{\mathcal{L}}' + C \int_0^t (\mathcal{L}')^2 \|Z_2\|_{B_{2,1}^{s}}^{2}.
\]
Then, one can conclude exactly as in the previous section that if \( \mathcal{L}'(0) \) (or, equivalently, \( \mathcal{Z}'_0 \)) is small enough, then \( \mathcal{L}' \) is a Lyapunov functional such that for some (new) positive real numbers \( \kappa_0 \) and \( C \),

\[
\mathcal{L}'(t) + \kappa_0 \int_0^t \mathcal{H}' \leq \mathcal{L}'(0) + C \mathcal{L}'(0) \int_0^t \| Z_2 \|_{L^2_2}^2.
\]

Furthermore, \( \mathcal{L}' \) with \( s = d/2 \) ensures that

\[
\| Z_2 \|_{L^2_1(\mathbb{R}^{d+1}_1)} \leq \| W \|_{L^2_1(\mathbb{R}^{d+1}_1)} + \| \nabla Z \|_{L^2_1(\mathbb{R}^{d+1}_1)} + \| Z_2 \|_{L^2_1(\mathbb{R}^{d+1}_1)} \| \nabla Z \|_{L^2_1(\mathbb{R}^{d+1}_1)}
+ \| \nabla Z \|_{L^2_2(\mathbb{R}^{d+1}_1)} \| \nabla Z \|_{L^2_1(\mathbb{R}^{d+1}_1)}.
\]

Inequality \( \mathcal{L}' \) combined with \( \mathcal{L}' \) and an obvious interpolation inequality thus yields

\[
\| Z_2 \|_{L^2_1(\mathbb{R}^{d+1}_1)} \leq Z'(0).
\]

Similarly, using again \( \mathcal{L}' \) but with \( s = d/2 + 1 \), we see that

\[
\| Z_2 \|_{L^2_1(\mathbb{R}^{d+1}_1)} \leq \| W \|_{L^2_1(\mathbb{R}^{d+1}_1)} + \| \nabla Z \|_{L^2_1(\mathbb{R}^{d+1}_1)} + \| Z_2 \|_{L^2_1(\mathbb{R}^{d+1}_1)} \| \nabla Z \|_{L^2_1(\mathbb{R}^{d+1}_1)}
+ \| \nabla Z \|_{L^2_2(\mathbb{R}^{d+1}_1)} \| \nabla Z \|_{L^2_1(\mathbb{R}^{d+1}_1)}.
\]

In light of \( \mathcal{L}' \), the last term may be absorbed by the left-hand side and all the other terms may be bounded either through \( \mathcal{L}' \) or through \( \mathcal{L}' \).

From this point, the rest of the proof of this theorem essentially follows the lines of the previous section. \( \square \)

### 4.2. The isentropic compressible Euler System with damping

We consider

\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P + \lambda \rho u = 0,
\end{cases}
\]

with \( \lambda > 0 \) and where \( P \) is a (smooth) pressure law satisfying

\[
P'(\rho) > 0 \quad \text{for } \rho \text{ close to } 1 \quad \text{and} \quad P'(1) = 1.
\]

Considering the new unknown \( n(\rho) = \int_1^\rho \frac{P'(s)}{s} ds \), we can rewrite \( \mathcal{L}' \) under the form

\[
\begin{cases}
\partial_t n + u \cdot \nabla n + \text{div} u + G(n) \text{div} u = 0, \\
\partial_t u + u \cdot \nabla u + \nabla n + \lambda u = 0,
\end{cases}
\]

where \( G(n) \) is defined by the relation \( G(n) = P'(\rho) - 1 \).

In order to state our global existence for \( \mathcal{L}' \), we need to introduce the following notations:

\[
\begin{align*}
\| z \|_{L^2_1}^{\ell, \lambda} & \triangleq \sum_{2^\ell \leq \lambda} \hat{A}_{q} z, & \| z \|_{L^2_1}^{h, \lambda} & \triangleq \sum_{2^\ell > \lambda} \hat{A}_{q} z, \\
\| z \|_{L^2_1}^{\ell, \lambda} & \triangleq \sum_{2^\ell \leq \lambda} 2^{2q} \| \hat{A}_{q} z \|_{L^2} & \| z \|_{L^2_1}^{h, \lambda} & \triangleq \sum_{2^\ell > \lambda} 2^{2q} \| \hat{A}_{q} z \|_{L^2}.
\end{align*}
\]

\(^1\)For simplicity we assume that the reference density is 1 so that the steady state is \( \bar{V} = (1, 0) \).

\(^2\)Observe that \( \rho \mapsto n(\rho) \) is a smooth diffeomorphism from a neighborhood of 1 to a neighborhood of 0.
Theorem 4.1. Let \((n_0, u_0)\) be in \(\mathcal{B}_{2,1}^d \cap \mathcal{B}_{2,1}^{d+1}\). Then, there exist two positive constants \(c\) and \(C\) depending only on \(G\) and on \(d\), such that if

\[
\|(n_0, u_0)\|_{\mathcal{B}_{2,1}^d} + \lambda^{-1} \|(n_0, u_0)\|_{\mathcal{B}_{2,1}^{d+1}} \leq c,
\]

then System \([33]\) supplemented with initial data \((n_0, u_0)\) admits a unique global-in-time solution \((n, u)\) in the space defined by

\[
(n, u) \in C_0(\mathbb{R}^+; \mathcal{B}_{2,1}^d \cap \mathcal{B}_{2,1}^{d+1}), \quad (n^{h,\lambda}, u^{h,\lambda}) \in L^1(\mathbb{R}^+; \mathcal{B}_{2,1}^d \cap \mathcal{B}_{2,1}^{d+2}),
\]

\[u^{\ell,\lambda} \in L^1(\mathbb{R}^+; \mathcal{B}_{2,1}^{d+1}), \quad u \in L^2(\mathbb{R}^+; \mathcal{B}_{2,1}^d) \quad \text{and} \quad \nabla n + \lambda u \in L^1(\mathbb{R}^+; \mathcal{B}_{2,1}^d).\]

Moreover we have the following a priori estimate:

\[
\mathcal{Z}_\lambda(t) \lesssim \|(n_0, u_0)\|_{\mathcal{B}_{2,1}^d} + \lambda^{-1} \|(n_0, u_0)\|_{\mathcal{B}_{2,1}^{d+1}} \quad \text{for all } t \geq 0
\]

where

\[
\mathcal{Z}_\lambda(T) \triangleq \|(n, u)\|_{L^\infty_T(\mathcal{B}_{2,1}^d)} + \lambda^{-1} \|(n, u)\|_{L^1_T(\mathcal{B}_{2,1}^{d+1})} + \lambda^{-1} \|n\|_{L^1_T(\mathcal{B}_{2,1}^{d+2})} + \|(n, u)\|_{L^1_T(\mathcal{B}_{2,1}^d)} + \|u\|_{L^1_T(\mathcal{B}_{2,1}^{d+1})} + \lambda^{1/2} \|u\|_{L^2_T(\mathcal{B}_{2,1}^d)} + \|\nabla n + \lambda u\|_{L^1_T(\mathcal{B}_{2,1}^d)}.
\]

If furthermore, \((n_0, u_0)\) belongs to \(\mathcal{B}_{2,\infty}^{-\sigma_1}\) for some \(\sigma_1 \in ]-d/2, d/2]\), then the solution \((n, u)\) satisfies \([25]\) and the decay estimates mentioned at the end of Theorem 2.3 hold true.

Proof. Performing the rescaling

\[
(n, u)(t, x) \triangleq (\bar{n}, \bar{u})(\lambda t, \lambda x)
\]

reduces the proof to \(\lambda = 1\) (and the inverse scaling will eventually give the desired dependency with respect to \(\lambda\) in the above statement). Then, the whole result is a corollary of Theorem 2.3 provided System \([33]\) satisfies the structural assumption \([26]\) at 0. Indeed, one can take as a symmetrizer the matrix \( \left( \begin{array}{cc} 1 + G(n)^{-1} & 0 \\ 0 & I_d \end{array} \right) \) where the first diagonal block is of size 1 \( \times \) 1 and the second one, of size \( d \times d \). The blocks of type \( A_{1,1}^j \) and \( A_{2,1}^j \) depend only (and linearly) on \( u \), which is indeed the damped component. Finally, the damped mode (in the case \( \lambda = 1\)) is \( W = u + \nabla n + u \cdot \nabla u \). Now, by virtue of \([34]\),

\[
\|W - (u + \nabla n)\|_{L^\infty_T(\mathcal{B}_{2,1}^d)} \lesssim \|u\|_{L^\infty_T(\mathcal{B}_{2,1}^d)} \|\nabla u\|_{L^1_T(\mathcal{B}_{2,1}^d)} \lesssim \left(\|(n_0, u_0)\|_{\mathcal{B}_{2,1}^d} + \|(n_0, u_0)\|_{\mathcal{B}_{2,1}^{d+1}}\right)^2,
\]

hence \( u + \nabla n \) satisfies the same estimates as \( W \), which completes the proof.

\[\square\]

5. Appendix

Here we gather a few technical results that have been used repeatedly in the paper.

The first one is the justification that one may choose (arbitrarily small) positive parameters \(\varepsilon_1, \cdots, \varepsilon_{n-1}\) so that, whenever \( \mathcal{Z} \) satisfies \([11]\), Inequality \([15]\) holds true. The proof just consists in bounding suitably the terms of the right-hand side of \([14]\).

- Terms \(\mathcal{I}_k^1 \triangleq (NM_{\omega}^{k-1}N\mathcal{Z} \cdot NM_{\omega}^{k}\mathcal{Z})\) with \( k \in \{1, \cdots, n-1\} \).

Since matrices \( M_{\omega} \) are bounded on \( S^{d-1} \), we may write

\[
\varepsilon_k |\mathcal{I}_k^1| \lesssim \varepsilon_k |N\mathcal{Z}| |NM_{\omega}^{k}\mathcal{Z}| \leq \frac{|N\mathcal{Z}|^2}{4\pi} + C\rho \varepsilon_k^2 |NM_{\omega}^{k}\mathcal{Z}|^2.
\]

- Terms \(\varepsilon_k (NM_{\omega}^{k-1}\mathcal{Z} \cdot NM_{\omega}^{k}N\mathcal{Z})\) with \( k \in \{2, \cdots, n-1\} \) may be bounded similarly.
Then, for all $t \in (95)$

**Lemma 5.1.** Let $\varepsilon$ can be taken arbitrarily small): The following inequalities hold true:

Assume that there exists a constant $B$ such that

Clearly, one is done if it is possible to find $\epsilon$ for instance

We often used the following well known result (see e.g. [10] for the proof).

The following estimates are proved in e.g. [1, Chap. 2].

• Terms $I^2_k \triangleq \rho(N M_k^{k-1} \hat{Z} - N M_k^{k+1} \hat{Z})$ with $k \in \{1, \cdots, n-2\}$. We have

Therefore one needs to assume in addition that

$$4\varepsilon_j^2 \leq \varepsilon_{k-1} \varepsilon_{k+1}. \quad j = 0, \cdots, n-1.$$  

Therefore we have

Consequently, one may write

$$|I^2_{n-1}| \leq \varepsilon_{n-1} \rho \sum_{j=0}^{n-1} |N M_j^{n-2} \hat{Z}||N M_j^{n} \hat{Z}|$$

Therefore one needs to assume in addition that

Clearly, one is done if it is possible to find $\epsilon_1, \cdots, \epsilon_{n-1}$ fulfilling (95) and (96). One can take for instance $\varepsilon_k = \epsilon^{m_k}$ with $\epsilon$ small enough and $m_1, \cdots, m_{n-1}$ satisfying for some $\delta > 0$ (that can be taken arbitrarily small):

$$m_k \geq \frac{m_{k-1} + m_{k+1}}{2} + \delta \quad \text{and} \quad m_{n-1} \geq \frac{m_k + m_{n-2}}{2} + \delta, \quad k = 1, \cdots, n-2.$$  

We often used the following well known result (see e.g. [10] for the proof).

**Lemma 5.1.** Let $X : [0, T] \to \mathbb{R}^+$ be a continuous function such that $X^2$ is differentiable. Assume that there exists a constant $B \geq 0$ and a measurable function $A : [0, T] \to \mathbb{R}^+$ such that

$$\frac{1}{2} \frac{d}{dt} X^2 + BX^2 \leq AX \quad \text{a.e. on} \quad [0, T].$$

Then, for all $t \in [0, T]$, we have

$$X(t) + B \int_0^t X \leq X_0 + \int_0^t A.$$  

The following estimates are proved in e.g. [1] Chap. 2.

**Proposition 5.1.** The following inequalities hold true:

• If $-d/2 < s \leq d/2 + 1$, then

$$2^s \| \{ w, \Delta q \} \nabla v \|_{L^2} \leq C c_q \| \nabla w \|_{\mathbb{B}^{2,1}_2} \| v \|_{\mathbb{B}^{2,1}_2} \quad \text{with} \quad \sum_{q \in \mathbb{Z}} c_q = 1.$$
If \(-d/2 \leq s < d/2 + 1\), then
\[
\sup_{q \in \mathbb{Z}} 2^{qs} \| [w, \hat{\Delta}_q] \nabla v \|_{L^2} \leq C \| \nabla w \|_{\dot{B}^2_{2,1}} \| v \|_{\dot{B}^2_{2,\infty}}.
\]

The following product laws in Besov spaces have been used several times.

**Proposition 5.2.** Let \((s, r)\) be in \([0, \infty) \times [1, \infty)\). Then, \(\dot{B}^s_{2,r} \cap L^\infty\) is an algebra and we have
\[
\| ab \|_{\dot{B}^s_{2,r}} \leq C \left( \| a \|_{L^\infty} \| b \|_{\dot{B}^s_{2,r}} + \| a \|_{\dot{B}^s_{2,r}} \| b \|_{L^\infty} \right).
\]
If, furthermore, \(-d/2 < s \leq d/2\), then the following inequality holds:
\[
\| ab \|_{\dot{B}^s_{2,1}} \leq C \| a \|_{\dot{B}^s_{2,1}} \| b \|_{\dot{B}^s_{2,1}}.
\]
Finally, if \(-d/2 < \sigma_1 \leq d/2\), then the following inequality holds true:
\[
\| fg \|_{\dot{B}^{-\sigma_1}_{2,\infty}} \leq C \| f \|_{\dot{B}^s_{2,1}} \| g \|_{\dot{B}^{-\sigma_1}_{2,\infty}}.
\]

The next proposition can be found in [1].

**Proposition 5.3.** Let \(f\) be a function in \(C^\infty(\mathbb{R})\) such that \(f(0) = 0\). Let \((s_1, s_2) \in [0, \infty]^2\) and \((r_1, r_2) \in [1, \infty]^2\). We assume that \(s_1 < d/2\) or that \(s_1 = d/2\) and \(r_1 = 1\).

Then, for every real-valued function \(u \in \dot{B}^{s_1}_{2,r_1} \cap \dot{B}^{s_2}_{2,r_2} \cap L^\infty\), the function \(f \circ u\) belongs to \(\dot{B}^{s_1}_{2,r_1} \cap \dot{B}^{s_2}_{2,r_2} \cap L^\infty\) and we have
\[
\| f \circ u \|_{\dot{B}^{s_k}_{2,r_k}} \leq C \left( f', \| u \|_{L^\infty} \right) \| u \|_{\dot{B}^{s_k}_{2,r_k}} \quad \text{for} \quad k \in \{1, 2\}.
\]

As a consequence (see [1 Cor. 2.66]), if \(g\) is a \(C^\infty(\mathbb{R})\) function such that \(g'(0) = 0\). Then, for all \(u, v \in \dot{B}^2_{2,1} \cap L^\infty\) with \(s > 0\), we have
\[
\| g(v) - g(u) \|_{\dot{B}^{2}_{2,1}} \leq C \left( \| v - u \|_{L^\infty} \| (u, v) \|_{\dot{B}^{2}_{2,1}} + \| v - u \|_{\dot{B}^{2}_{2,1}} \| (u, v) \|_{L^\infty} \right).
\]

We used the following result to estimate the remainder of the dissipative term.

**Proposition 5.4.** Let \(\tilde{V} \in \mathcal{M}\) and \(Z \triangleq V - \tilde{V}\). Define \(r(Z) \triangleq \tilde{H}(V + Z) + L Z\), \(L \triangleq -D_V \tilde{H}(V)\) and \(Z_2 \triangleq (I_d - \mathcal{P}) Z\), and assume that \(r(Z_1, 0) = 0\) for \(Z_1\) in a neighborhood of \(0\). Then, provided \(\| Z_1 \|_{\dot{B}^{d}_{2,1}}\) is sufficiently small, the following inequalities hold true:
\[
\| r(Z) \|_{\dot{B}^{\sigma}_{2,1}} \lesssim \| Z \|_{\dot{B}^{\sigma}_{2,1}} \| Z_2 \|_{\dot{B}^{d}_{2,1}} \quad \text{for} \quad \sigma \in [-d/2, d/2]
\]
and, for \(\sigma > d/2\),
\[
\| r(Z) \|_{\dot{B}^{\sigma}_{2,1}} \lesssim \| Z_2 \|_{\dot{B}^{\sigma}_{2,1}} \| Z \|_{\dot{B}^{d}_{2,1}} + \| Z \|_{\dot{B}^{\sigma}_{2,1}} \| Z_2 \|_{\dot{B}^{d}_{2,1}}.
\]
Furthermore, if both \(Z^1\) and \(Z^2\) are sufficiently small in \(\dot{B}^{d}_{2,1}\) then we have the following estimate for \(\widetilde{Z} := Z^1 - Z^2\):
\[
\| r(Z^1) - r(Z^2) \|_{\dot{B}^{\sigma}_{2,1}} \lesssim \| Z^1 \|_{\dot{B}^{d}_{2,1}} \| \widetilde{Z} \|_{\dot{B}^{\sigma}_{2,1}} + \| Z \|_{\dot{B}^{\sigma}_{2,1}} \| Z^2 \|_{\dot{B}^{d}_{2,1}}, \quad \sigma \in [0, d/2).
\]
Finally, if \(r\) is at least quadratic with respect to \(Z_2\) (that is \(D^2_{V^1, V^j} r(0) = 0\) for \((i, j) \neq (2, 2)\)), then we have
\[
\| r(Z) \|_{\dot{B}^{\sigma}_{2,1}} \lesssim \| Z_2 \|_{\dot{B}^{\sigma}_{2,1}} \| Z \|_{\dot{B}^{d}_{2,1}} \| Z_2 \|_{\dot{B}^{d}_{2,1}}, \quad \text{for} \quad \sigma \in [-d/2, d/2]
\]
and
\[
\| r(Z) \|_{\dot{B}^{\sigma}_{2,1}} \lesssim \| Z_2 \|_{\dot{B}^{\sigma}_{2,1}} \| Z \|_{\dot{B}^{d}_{2,1}} \| Z_2 \|_{\dot{B}^{d}_{2,1}}, \quad \text{for} \quad \sigma > d/2.
\]
Proof. Since \( r(Z_1, 0) = 0 \) for \( Z_1 \) close to 0, the mean value formula gives
\[
r(Z_1, Z_2) = \int_0^1 DZ_\tau r(Z_1, \tau Z_2) \cdot Z_2 \, d\tau.
\]
Furthermore, we have \( Dr(0) = 0 \) and thus \( DZ_\tau r(0, 0) = 0 \). Hence there exists a smooth function \( F \) defined near 0 and such that \( DZ_\tau r(Z) = F(Z) \cdot Z \). Consequently, there exists a smooth function \( G \) vanishing at 0, and such that
\[
r(Z_1, Z_2) = G(Z) \cdot Z_2.
\]
Granted with the above decomposition, the first two inequalities readily follow from Propositions 5.2 and 5.3.

To prove (105), we use the decomposition
\[
r(Z^1) - r(Z^2) = G(Z^1) \cdot (Z_2^2 - Z_1^2) + (G(Z^2) - G(Z^1)) \cdot Z_2^2,
\]
then Propositions 5.2 and 5.3 combined with Corollary 2.66 from [1].

Finally, if \( r \) is quadratic with respect to \( Z_2 \) there exists a quadratic form \( \tilde{Q} \) and a smooth function \( F \) such that \( r(Z) = \tilde{Q}(Z_2)F(Z) \), whence
\[
r(Z) = F(0)\tilde{Q}(Z_2) + G(Z)\tilde{Q}(Z_2) \quad \text{with} \quad G(Z) \equiv F(Z) - F(0).
\]
In the case \( \sigma \in [-d/2, d/2] \), we can thus write by virtue of Propositions 5.2 and 5.3
\[
\|r(Z)\|_{\tilde{B}^{q}_{2,1}} \lesssim \|\tilde{Q}(Z_2)\|_{\tilde{B}^{q}_{2,1}} (1 + \|Z\|_{\tilde{B}^{q}_{2,1}}) \lesssim \|Z_2\|_{\tilde{B}^{1/2}_{2,1}} \|Z_2\|_{\tilde{B}^{1/2}_{2,1}}
\]
while, if \( \sigma > d/2 \),
\[
\|r(Z)\|_{\tilde{B}^{q}_{2,1}} \lesssim \|\tilde{Q}(Z_2)\|_{\tilde{B}^{q}_{2,1}} (1 + \|Z\|_{\tilde{B}^{1/2}_{2,1}}) + \|Z\|_{\tilde{B}^{1/2}_{2,1}} \|\tilde{Q}(Z_2)\|_{\tilde{B}^{1/2}_{2,1}}
\]
whence the last inequality. \( \square \)

REFERENCES

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. Fourier Analysis and Nonlinear Partial Differential Equations, volume 343 of Grundlehren der Mathematischen Wissenschaften. Springer, Heidelberg, 2011.
[2] K. Beauchard and E. Zuazua. Large time asymptotics for partially dissipative hyperbolic systems. Arch. Rational Mech. Anal, 199, 177–227, 2011.
[3] S. Benzoni-Gavage and D. Serre. Multi-dimensional Hyperbolic Partial Differential Equations : First-order Systems and Applications. Oxford Science Publications, New-York, 2007.
[4] R. Bianchini and R. Natalini. Nonresonant bilinear forms for partially dissipative hyperbolic systems violating the shizuta-kawashima condition. arXiv:1906.02767, 2019.
[5] S. Bianchini, B. Hanouzet, and R. Natalini. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. Comm. Pure and Appl. Math., 60, 1559-1622, 2007.
[6] J. Tan C. Burtea, T. Crin-Barat. Relaxation limit of a damped baer-nunziato model. Work in progress, 2021.
[7] F. Charve and R. Danchin. A global existence result for the compressible Navier-Stokes equations in the critical L^p framework. Arch. Rational Mech. Anal, 198, 233-271, 2010.
[8] Q. Chen, C. Miao, and Z. Zhang. Global well-posedness for compressible Navier-Stokes equations with highly oscillating initial velocity. Comm. Pure Appl. Math., 63(9):1173–1224, 2010.
[9] J.-M. Coron. Control and nonlinearity, volume 136. Mathematical Surveys and Monographs. American Mathematical Society, 2007.
[10] T. Crin-Barat and R. Danchin. Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications. arXiv:2101.05391, 2020.
[11] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. Inventiones Mathematicae, 141, 579-614, 2000.
[12] R. Danchin. Fourier analysis methods for the compressible Navier-Stokes equations. in : Giga Y., Novotný A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham, 2018.
[13] R. Danchin and L. He. The incompressible limit in L^p type critical spaces. Math. Ann., 366 (3-4), 1365-1402, 2016.
[14] B. Hanouzet G. Carbou. Relaxation approximation of Kerr model for the three dimensional initial boundary value problem. *Journal of Hyperbolic Differential Equations*, 2010.
[15] S. Kawashima. Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics. *Doctoral Thesis*, 1983.
[16] S. Kawashima and W.-A. Yong. Dissipative structure and entropy for hyperbolic systems of balance laws. *Arch. Rational Mech. Anal.*, 174, 345–364, 2004.
[17] S. Kawashima and W.-A. Yong. Decay estimates for hyperbolic balance laws. *Journal for Analysis and its Applications*, 28, 1–33, 2009.
[18] A. Majda. *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variable*. Springer, New-York, 1984.
[19] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20, 67-104, 1980.
[20] P. Qu and Y. Wang. Global classical solutions to partially dissipative hyperbolic systems violating the Kawashima condition. *Journal de Mathématiques Pures et Appliquées*, 109:93–146, 2018.
[21] D. Serre. *Systèmes de lois de conservation, tome 1*. Diderot editeur, Arts et Sciences, Paris, New-York, Amsterdam, 1996.
[22] S. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14, 249-275, 1985.
[23] T. Sideris, B. Thomases, and D. Wang. Long time behavior of solutions to the 3d compressible Euler equations with damping. *Comm. Partial Differential Equations*, 28, 795-816, 2003.
[24] C. Villani. Hypocoercivity. *Mem. Am. Math. Soc.*, 2010.
[25] W. Wang and T. Yang. The pointwise estimates of solutions for Euler equations with damping in multi-dimensions. *J Diff. Eqs.*, 173, Issue 2, 410-450, 2001.
[26] Z. Xin and J. Xu. Optimal decay for the compressible Navier-Stokes equations without additional smallness assumptions. *Journal of Differential Equations*, 274, 543-575, 2021.
[27] J. Xu and S. Kawashima. Global classical solutions for partially dissipative hyperbolic system of balance laws. *Arch. Rational Mech. Anal.*, 211, 513–553, 2014.
[28] J. Xu and S. Kawashima. The optimal decay estimates on the framework of Besov spaces for generally dissipative systems. *Arch. Rational Mech. Anal.*, 218, 275–315, 2015.
[29] W.-A Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Rational Mech. Anal.*, 172, 47–266, 2004.