On Concavity of Solutions of the Nonlinear Poisson Equation

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Abstract

We consider the nonlinear Poisson equation $-\Delta u = f(u)$ in domains $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions on $\partial \Omega$. We show (for monotonically increasing concave $f$ with small Lipschitz constant) that if $D^2u$ is negative semi-definite on the boundary, then $u$ is concave. A conjecture of Saint Venant from 1856 (proven by Polya in 1948) is that among all domains $\Omega$ of fixed measure, the solution of $-\Delta u = 1$ assumes its largest maximum when $\Omega$ is a ball. We extend this to $-\Delta u = f(u)$ for monotonically increasing $f$ with small Lipschitz constant.

1. Introduction

We study the behavior of solutions of

$$-\Delta u = f(u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is assumed to be a set with smooth boundary and $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is assumed to be a nonnegative function with $f(0) > 0$. One natural assumption is that if $\Omega$ is ‘simple’ (here: convex), then solutions of this equation should ‘inherit’ the simplicity of the underlying domain and also be ‘simple’. An optimistic guess is that such functions are concave but this is already violated for very simple equations such as $-\Delta u = 1$ (see Fig. 1). Another simple property that one might hope for is perhaps that the level sets are convex (conjectured by Lions [31]): this was disproven only rather recently by Hamel et al. [19].

Nonetheless, it is known that the level sets are convex for some special cases: Makar–Limanov [34] proved that solutions of $-\Delta u = 1$ have the property that $\sqrt{u}$

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is convex (and thus \( \{ x \in \Omega : u(x) \geq t \} \) is a convex set). Brascamp and Lieb [10] showed that if \( u \) is the first Laplacian eigenfunction on \( \Omega \), that is the solution of \( -\Delta u = \lambda_1(\Omega)u \), then \( \log u \) is concave. Keady [24] proved that for

\[
f(u) = u^\gamma \quad \text{and} \quad 0 < \gamma < 1,
\]

the function \( u^{(1-\gamma)/2} \) is concave, this was extended by Kennington [26] to higher dimensions. There are many results of such a flavor: we refer to [5,21,22,25,30,32,34,36] for solutions of the torsion problem \( -\Delta u = 1 \) alone and to [1,3,4,6–9,11–16,18,20,23,27,28,33] for general equations.

1.1. The problem

We were motivated by two simple arguments. First, if \( \Omega \) is a disk, then solutions of such equations tend to be radially symmetric (this is very well understood [17]). Secondly, there is a simple reason why convexity of \( \Omega \) might not be ‘right’ condition. Consider the equation \( -\Delta u = 1 \) which can be non-concave close to corners (see Fig. 1). One would expect something similar to be true for \( -\Delta u = f(u) \) in general: when \( x \) is close to the boundary, then \( u \) is close to 0 because of the Dirichlet boundary conditions and \( f(u(x)) \sim f(0) \) and the solution should behave, at least locally, a bit like a multiple of \( -\Delta u = 1 \). Convexity of the domain \( \Omega \) is a natural assumption but it should be equally important to rule out corners and, by approximation arguments, regions of the boundary with large curvature. This motivates the following (intentionally vague)

**Question.** Under what conditions on \( f > 0 \) (‘sufficiently nice’) and under what conditions on the convex domain \( \Omega \subset \mathbb{R}^n \) (sufficiently ‘round’) is it true that solutions of

\[
-\Delta u = f(u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega,
\]

are concave everywhere in \( \Omega \)?

We are not aware of any results of this flavor with the exception of one: Kosmodemyanskii [29] showed that if every parabola osculating with the boundary of \( \Omega \subset \mathbb{R}^2 \) contains \( \Omega \) in its interior, then the solution of \( -\Delta u = 1 \) is concave.
2. The Results

2.1. Propagating concavity

We start, for illustrative purposes, by considering a simple equation which has been studied for a long time

\[-\Delta u = 1 \quad \text{in } \Omega \]
\[u = 0 \quad \text{on } \partial \Omega.\]

It is known that if $D^2 u$ is negative semi-definite on the boundary, then $u$ is concave inside $\Omega$ (a 'propagation of concavity' phenomenon). We will illustrate our main idea by giving a new proof of this result (our general approach becomes particularly simple when $f \equiv 1$ which is thus well suited for outlining the idea).

**Theorem.** [Keady & McNabb,] [25] Let $\Omega \subset \mathbb{R}^2$ and let $u$ be the solution of $-\Delta u = 1$ with Dirichlet boundary conditions. If $D^2 u$ is negative semi-definite on $\partial \Omega$, then $u$ is concave on all of $\Omega$.

We emphasize that our approach is quite different and uses a stochastic representation of the solution together with a bootstrapping argument. Our first main result is a generalization of this result to equations of the type $-\Delta u = f(u)$.

**Theorem 1.** (Propagating concavity from the boundary) Let $\Omega \subset \mathbb{R}^n$ have a smooth boundary, let $-\Delta u = f(u)$ with Dirichlet boundary condition $u = 0$ on the boundary $\partial \Omega$ and assume $f > 0$, $f' \geq 0$, $f'' \leq 0$ and

\[f'(0) \leq \frac{2n\omega_n^{2/n}}{2^{2/n}} \left(\frac{1}{|\Omega|^{2/n}}\right),\]

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. If $D^2 u$ is negative semi-definite for all $x \in \partial \Omega$, then $u$ is concave in $\Omega$.

In the special case of the torsion function $f \equiv 1$, the condition is trivially satisfied independently of the domain. One of the reasons the concavity of such solutions is difficult to quantify is that sometimes they are 'barely' concave. Consider the solution of $-\Delta u = 1$ in a long rectangle. An explicit computation (see [36]) shows that if $x_0$ is the point of the rectangle in which $u$ assumes its maximum, then the largest eigenvalue of the Hessian satisfies, for some $c_1, c_2 > 0$,

\[\lambda_{\max}(D^2 u(x_0)) \sim -c_1 \exp \left(-c_2 \frac{\text{diam}(\Omega)}{\text{inrad}(\Omega)}\right).\]

As the rectangle becomes more eccentric, this eigenvalue is exponentially small in the eccentricity: the function is concave but it becomes harder to tell from the behavior of the eigenvalues of the Hessian. Our argument will recover this type of phenomenon and lead to a new explanation why this is the case (see §3.2).
2.2. A rearrangement result

Upper bounds on $\|u\|_{L^\infty(\Omega)}$ are a classical subject. It is generally understood that symmetric decreasing rearrangement is a process that tends to increase $L^p$ norms of solutions. The most fundamental formulation of this statement is perhaps Talenti’s rearrangement principle \cite{38,39}. If

$$-\Delta u = f \geq 0 \text{ in } \Omega$$

with Dirichlet boundary conditions on $\partial \Omega$, then solving the same problem with $f$ rearranged on a ball $B$ with the same measure as $\Omega$, that is

$$-\Delta v = f^* \text{ in } B = \Omega^*,$$

leads to a solution for which $v \geq u^*$ in a pointwise sense. Since $u^*(0) = \|u\|_{L^\infty(\Omega)}$, we conclude that $v$ has a larger maximum. One could wonder whether there is a similar result for $-\Delta u = f(u)$ but little seems to be known. The book of Baernstein \cite{2} discusses the case of decreasing $f$ following a result of Weitsman \cite{40}.

**Theorem 2.** (Rearrangement theorem) Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary, let $f > 0$, $f' \geq 0$ and

$$\max_{t>0} f'(t) < \frac{2n\omega_n^{2/n}}{|\Omega|^{2/n}},$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. The solution of

$$-\Delta u = f(u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

satisfies $u^* \leq v$, where $v$ is the solution on the ball with same measure as $\Omega$.

We note that in the case of $f \equiv 1$, this result was first suggested by Saint Venant 1856. The first proof was given by Polya \cite{35}. Nowadays, Talenti’s rearrangement principle \cite{38,39} can be used to give a very simple proof. Theorem 2 can be considered a nonlinear analogue of these ideas. It is not clear to us whether the upper bound on $f'$ is necessary.

3. Proof of Theorem 1

This section gives a proof of Theorem 1. We first discuss the case of $-\Delta u = 1$ which is particularly well suited to illustrate the main ideas behind the argument.
3.1. An identity

We start with the following identity. If

$$-\Delta u = 1 \quad \text{in } \Omega$$

and $u = 0$ on the boundary, then for any $x \in \Omega$ and any unit vector $\|n\| = 1$

$$\frac{\partial^2 u}{\partial n^2}(x) = \int_{\partial\Omega} \frac{\partial^2 u}{\partial n^2}(y) \, d\omega_x(y),$$

where $\omega_x$ is the harmonic measure on the boundary $\partial \Omega$ induced by the point $x \in \Omega$. In particular, it suffices to prove that $D^2 u$ is negative definite on the boundary to deduce global concavity. Theorem 2 will establish a more general form of the identity. Since the case $f \equiv 1$ is particularly simple, we give an independent and much simpler argument here. In particular, the proof uses so little that it also naturally extends to higher derivatives.

Proof. (Proof of the Identity.) The idea is to use the differential quotient

$$\frac{\partial^2 u}{\partial n^2}(x) = \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon n) - 2u(x) + u(x - \varepsilon n)}{\varepsilon^2}.$$

Instead of interpreting this as one function evaluated at three nearby points, we will consider it as the evaluation of three different functions in the same point. The two additional functions are the solutions of the same PDE in the shifted domains

$$-\Delta u = 1 \quad \text{in } \Omega \pm \varepsilon n.$$ 

We denote the three functions by $u_+, u, u_-$. Naturally, since

$$\Delta \left( \frac{u_+ - 2u + u_-}{\varepsilon^2} \right) = 0 \quad \text{in } \Omega_+ \cap \Omega \cap \Omega_-,$$

the differential quotient is a harmonic function in the intersection of the three domains. We know how to solve Laplace’s equation inside a domain, the solution is given by integrating harmonic measure against boundary data. It thus remains to determine boundary data but this is simply given by the second directional derivative at the boundary in direction $n$. Letting $\varepsilon \to 0$ leads to the result. 

This identity immediately implies Theorem 1 for the special case $f \equiv 1$. 

Fig. 2. Shifting the domain by $\pm \varepsilon$ in direction $n$
3.2. A remark on eccentric level sets

It is known [36] that if $\Omega \subset \mathbb{R}^2$ is a convex domain, then the (unique) maximum of the solution of

$$-\Delta u = 1 \quad \text{in } \Omega$$

has the property that the determinant of its Hessian cannot be too small: level sets close to the maximum of the solution are asymptotically ellipses whose eccentricity can be bounded in terms of the geometry of the domain: for some universal $c_1, c_2 > 0$, the largest eigenvalue of the Hessian on the maximum $x_0 \in \Omega$ satisfies

$$\lambda_{\max}(D^2 u(x_0)) \leq -c_1 \exp\left(-c_2 \frac{\text{diam}(\Omega)}{\text{inrad}(\Omega)}\right).$$

The proof uses conformal mapping and only works in two dimensions. The formula

$$\frac{\partial^2 u}{\partial n^2}(x) = \int_{\partial \Omega} \frac{\partial^2 u}{\partial u^2}(y) \, d\omega_x(y)$$

is more generally applicable and may be helpful in explaining why it is possible that the level sets close to the maximum are so remarkably eccentric. In long domains, the harmonic measure is rather small (exponentially in eccentricity) at the endpoints.

However, if $n$ is chosen to be parallel to the long direction, then most contributions in the integral are going to be very small (see Fig. 5). It is conceivable that the representation formula in §3.3 could be used to prove that the same phenomenon also occurs for $-\Delta u = f(u)$ for suitable $f$ and to provide bounds for the largest eigenvalue of the Hessian; this might be an interesting avenue for future research.

3.3. A representation formula

The purpose of this section is to prove an explicit representation formula for the second derivative of a solution of $-\Delta u = f(u)$. More precisely, we assume that $\Omega \subset \mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ solve

$$-\Delta u = f(u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega.$$
The main goal of this section is to show that for any \( x \in \Omega \) and any unit length vector \( \|n\| = 1 \), we have the identity
\[
\frac{\partial^2 u}{\partial n^2}(x) = \mathbb{E} \left\{ \frac{1}{2} \int_0^\infty f''(u(\omega_x(s))) \left( \frac{\partial u}{\partial n}(\omega_x(s)) \right)^2 + f'(u(\omega_x(s))) \frac{\partial^2 u}{\partial n^2}(\omega_x(s))ds \right. \\
\left. + \int_{\partial \Omega} \frac{\partial^2 u}{\partial n^2}(y)d\omega_x(y) \right\},
\]
where \( \omega_x(s) \) denotes a Brownian motion started in \( x \) after \( s \) units time with the convention that it gets absorbed by the boundary \( \partial \Omega \) and stops there (in particular, the first integral could have also been written with the upper limit \( \infty \wedge \tau \), where \( \tau \) is the first hitting time). In the second integral, \( d\omega_x \) denotes the harmonic measure induced by \( x \) on the boundary \( \partial \Omega \). We use a stochastic representation of the solution – since this may be hard to find in the literature and since precise constants matter for our approach, we quickly discuss a self-contained heuristic derivation. Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a smooth function and suppose \( \omega(t) \) is an \( n \)-dimensional Brownian motion started in the origin. Each coordinate of \( \omega(t) \) is a one-dimensional Brownian motion. In particular, we have
\[
\mathbb{E} \|\omega(t)\|^2 = \mathbb{E} \omega_1(t)^2 + \cdots + \mathbb{E} \omega_n(t)^2 = nt.
\]
We will now compute \( \mathbb{E} u(\omega(t)) \) for small values of \( t \). Using a Taylor expansion of \( u \), we have up to lower order terms
\[
u(x) = u(0) + (\nabla u, x) + \frac{1}{2} \left( x, (D^2 u(0))x \right) + \text{l.o.t.}
\]
It remains to set \( x = \omega(t) \) and take an expectation with respect to \( \omega \). The symmetry of Brownian motion guarantees that the linear/gradient term does not play any role. It remains to understand the quadratic term
\[
\mathbb{E} u(\omega(t)) = u(0) + \frac{1}{2} \cdot \mathbb{E} \left\{ \omega(t), (D^2 u(0))\omega(t) \right\} + \text{l.o.t.}
\]
Brownian motion is radially distributed around its point of origin. In terms of probability, we can thus write
\[
\omega(t) = \|\omega(t)\| \cdot X, \quad \text{where } X \sim U(S^{n-1})
\]
is a uniformly chosen random point on the unit sphere. Then, since radial and angular distribution of a Gaussian can be decoupled,
\[
\mathbb{E}_\omega \left\{ \omega(t), (D^2 u(0))\omega(t) \right\} = \mathbb{E}_\omega \left\{ \|\omega(t)\|^2 \cdot \left( X, (D^2 u(0))X \right) \right\}
\]
\[
= \left( \mathbb{E}_\omega \|\omega(t)\|^2 \right) \cdot \left( \mathbb{E}_X \left\{ X, (D^2 u(0))X \right\} \right).
\]
The first of these terms was already computed above. As for the second quantity, using symmetry of \( X \) to expand \( D^2(u(0)) \) into its eigenvectors and eigenvalues. Then
\[
\mathbb{E}_X \left\{ X, (D^2 u(0))X \right\} = \mathbb{E}_X \sum_{k=1}^n \lambda_k \langle v_k, X \rangle^2.
\]
However, the distribution of $X$ is rotationally invariant and thus $\mathbb{E}_X(v, X)^2$ is independent of the vector $v$ (as long as $v$ has norm 1). Thus, by linearity of expectation,

$$\mathbb{E}_X \sum_{k=1}^n \lambda_k \langle v_k, X \rangle^2 = \sum_{k=1}^n \lambda_k \mathbb{E}_X(v_k, X)^2 = \frac{1}{n} \sum_{k=1}^n \lambda_k$$

$$= \frac{1}{n} \text{Tr} D^2 u = \frac{1}{n} (\Delta u)(0).$$

Altogether, we see that

$$\left(\mathbb{E}_\omega \|\omega(t)\|^2\right) \cdot \left(\mathbb{E}_X \{X, (D^2 u(0))X\}\right) = t \cdot (\Delta u)(0)$$

and therefore

$$\mathbb{E} u(\omega(t)) = u(0) + (\Delta u(0)) \frac{t}{2} + \text{l.o.t.}$$

Using the Markov property of Brownian motion and letting time run until the Brownian motion impacts the boundary then suggests the representation formula

$$u(x) = \frac{1}{2} \int_0^\infty (-\Delta u)(\omega_x(s)) ds.$$

Since $\Delta u = f(u)$, we can rewrite this as

$$u(x) = \mathbb{E} \frac{1}{2} \int_0^\infty f(u(\omega_x(s))) ds.$$

A formal justification is, of course, given by the Feynman-Kac formula. This also explains why the expected lifetime of Brownian motion in a domain is given by the solution of the equation $-\Delta u = 2$: the 2 cancels in the formula.

**Proof.** (Proof of the Representation Formula) We will now give a proof of the formula for the second derivatives. Our first proof will be stochastic, we will give a sketch of a non-probabilistic argument that is more along the lines of §3.1 below. Using

$$u(x) = \mathbb{E} \frac{1}{2} \int_0^\infty f(u(\omega_x(s))) ds$$

and the differential quotient

$$\frac{\partial^2 u}{\partial n^2}(x) = \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon n) - 2u(x) + u(x - \varepsilon n)}{\varepsilon^2}.$$

This leads to

$$\frac{\partial^2 u}{\partial n^2}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \frac{1}{2} \mathbb{E} \int_0^\infty f(u(\omega_{x+\varepsilon n}(s))) - 2f(u(\omega_x(s))) + f(u(\omega_{x-\varepsilon n}(s))) ds.$$

We note that this is one expectation over three different random processes: a Brownian motion started in $x$ and two more started in $x + \varepsilon n$ and $x - \varepsilon n$. One would
assume that these should be related somehow and this can be made precise. We adapt an argument given in [37] which is related in spirit. Let the set $A$ denote all Brownian motion paths started in the origin and running for $t$ units of time. Then, for each $x \in \Omega$, we can use translation invariance of Brownian motion in $\mathbb{R}^n$ to write the expectations as an expectation over the set $A$ via

$$E(f(u(\omega x(t)))) = E_{a \in A}(f(u(a(t) + x)) \cdot 1_{\{a(s) + x \in \Omega \text{ for all } 0 \leq s \leq t\}}).$$

This has the advantage of being able to take the expectation with respect to one universal set $A$ shared by all three Brownian motions which differ only by translation invariance. We can use this coupling until the stopping time given by the first time one of the three particles hits the boundary (because it is then absorbed by the boundary and frozen)

$$\tau = \inf \{t > 0 : \{a(t) + x, a(t) + x + \varepsilon n, a(t) - \varepsilon n\} \cap \Omega^c \neq \emptyset\}.$$

Until that time, the integral

$$X_\varepsilon = \frac{1}{\varepsilon^2} \frac{1}{2} \mathbb{E} \int_0^\tau f(u(\omega_{x+\varepsilon n}(s))) - 2f(u(\omega_x(s))) + f(u(\omega_{x-\varepsilon n}(s)))ds$$

is actually not difficult to analyze: as $\varepsilon \to 0$, we can use $f(u(x)) \in C^2(\Omega)$ and the chain rule

$$\frac{\partial^2}{\partial n^2} f(u(x)) = f''(u(x)) \left(\frac{\partial u}{\partial n}\right)^2 + f'(u(x)) \frac{\partial^2 u}{\partial n^2},$$

then leads to

$$\lim_{\varepsilon \to 0} X_\varepsilon = \mathbb{E} \frac{1}{2} \int_0^\tau f''(\omega_x(s)) \left(\frac{\partial u}{\partial n}(\omega_x(s))\right)^2 + f'(\omega_x(s)) \frac{\partial^2 u}{\partial n^2}(\omega_x(s)) ds.$$

It remains to control the total contribution of the integral after the first of the three particles has hit the boundary. For the remainder of the argument it will not matter which of the three points has impacted first, the argument is always the same. We illustrate the argument using the example shown in Fig. 4. In that example, the middle particle has impacted the boundary. Our goal is to understand the remaining integral

$$Y_\varepsilon = \frac{1}{\varepsilon^2} \frac{1}{2} \mathbb{E} \int_\tau^\infty f(u(\omega_{x+\varepsilon n}(s))) - 2f(u(\omega_x(s))) + f(u(\omega_{x-\varepsilon n}(s)))ds.$$
Fig. 5. Shifting the domain by $\pm \varepsilon$ in direction $n$

The middle term in the integral has impacted on the boundary and will stay there for all time and not contribute. Thus

$$E \int_{\tau}^{\infty} -2f(u(\omega_x(s))) ds = 0 = u(\omega_x(\tau)).$$

As for the two remaining terms, we realize that they correspond to the stochastic representation of the solution and thus

$$E \frac{1}{2} \int_{\tau}^{\infty} f(u(\omega_{x+\varepsilon n}(s))) = u(\omega_{x+\varepsilon n}(\tau))$$
$$E \frac{1}{2} \int_{\tau}^{\infty} f(u(\omega_{x-\varepsilon n}(s))) = u(\omega_{x-\varepsilon n}(\tau)).$$

Altogether, we see that $Y_{\varepsilon}$ is simply the second differential quotient in direction $n$ evaluated at $\omega_x(\tau)$. The harmonic measure governs the distribution of impact points and from this the desired result follows. $\square$

**Proof (Sketch of Second Proof).** We quickly sketch another proof that is closer to the argument in §3.1 and uses the fact that solutions of $-\Delta u = f(u)$ are invariant under translations of the domain $\Omega$. We use the same approach as in §3.1 and consider a total of three different domains: $\Omega$ together with $\Omega_+$ and $\Omega_-$ (see Fig. 7). We solve $-\Delta u = f(u)$ in each domain and obtain three solutions: $u$, $u_+$ and $u_-$. 

We use again that

$$\frac{\partial^2 u}{\partial n^2}(x) = \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon n) - 2u(x) + u(x - \varepsilon n)}{\varepsilon^2}.$$

Let us fix some small $\varepsilon > 0$ and let us consider the behavior of

$$w = \frac{u_+(x) - 2u(x) + u_-(x)}{\varepsilon^2} \quad \text{in } \Omega_+ \cap \Omega \cap \Omega_-.$$

We see that, as $\varepsilon \to 0$, for any $x \in \Omega$,

$$(-\Delta w)(x) = (1 + o(1)) \left[ f''(u(x)) \left( \frac{\partial u}{\partial n} \right)^2 + f'(u(x)) \frac{\partial^2 u}{\partial n^2} \right].$$
Moreover, we also understand the behavior at the boundary since, again as \( \varepsilon \to 0 \),
\[
w(x) = (1 + o(1)) \frac{\partial^2 u}{\partial n^2} \quad \text{on } \partial \Omega.
\]

We write \( w \) as the superposition of two solutions \( w = w_1 + w_2 \), where \( w_1 \) has the correct boundary data while being harmonic
\[
\Delta w_1 = 0 \quad \text{in } \Omega
\]
\[
w_1 = \frac{\partial^2 u}{\partial n^2} \quad \text{on } \partial \Omega
\]
and where \( w_2 \) has the correct Laplacian in the domain while satisfying the Dirichlet boundary conditions
\[
-\Delta w_2 = f''(u(x)) \left( \frac{\partial u}{\partial n} \right)^2 + f'(u(x)) \frac{\partial^2 u}{\partial n^2} \quad \text{in } \Omega
\]
\[
w_2 = 0 \quad \text{on } \partial \Omega.
\]

We can represent \( w_1 \) using harmonic measure
\[
w_1(x) = \int_{\partial \Omega} \frac{\partial^2 u}{\partial n^2}(y) d\omega_x(y)
\]
and we already discussed the stochastic representation formula for Poisson equations with Dirichlet boundary conditions and thus
\[
w_2(x) = \mathbb{E} \frac{1}{2} \int_0^\infty f''(u(\omega_x(s))) \left( \frac{\partial u}{\partial n}(\omega_x(s)) \right)^2 + f'(u(\omega_x(s))) \frac{\partial^2 u}{\partial n^2}(\omega_x(s)) ds.
\]
Then \( w = w_1 + w_2 \) gives the desired representation.

\[\square\]

### 3.4. Proof of Theorem 1

**Proof.** Our goal is to show that for each \( x \in \Omega \) and each unit norm vector \( n \)
\[
\frac{\partial^2 u}{\partial n^2}(x) < 0.
\]

We recall the representation formula proved in §4.2 which reads
\[
\frac{\partial^2 u}{\partial n^2}(x) = \mathbb{E} \frac{1}{2} \int_0^\infty f''(u(\omega_x(s))) \left( \frac{\partial u}{\partial n}(\omega_x(s)) \right)^2 + f'(u(\omega_x(s))) \frac{\partial^2 u}{\partial n^2}(\omega_x(s)) ds
\]
\[
+ \int_{\partial \Omega} \frac{\partial^2 u}{\partial n^2}(y) d\omega_x(y).
\]

We want to argue that the maximum value on the left-hand side,
\[
M = \max_{x \in \Omega, \|n\| = 1} \frac{\partial^2 u}{\partial n^2}(x),
\]
is negative: \( M < 0 \). The proof is by contradiction. For this, we use the representation formula in the point \( x \) and with the vector \( n \) with which \( M \) is assumed. We recall that, by assumption, \( f' \geq 0 \) and \( f'' \leq 0 \). Thus

\[
M \leq \mathbb{E} \int_0^\infty \frac{1}{2} \cdot f'(u(\omega_x(s))) M ds + \int_{\partial \Omega} \frac{\partial^2 u}{\partial n^2}(y) d\omega_x(y).
\]

Since \( f'' \leq 0 \), we have \( 0 \leq f'(t) \leq f'(0) \) and thus

\[
\mathbb{E} \int_0^\infty \frac{1}{2} \cdot f'(u(\omega_x(s))) M ds \leq M f'(0) \cdot \frac{1}{2} \mathbb{E} \int_0^\infty 1(\omega_x(s)) ds.
\]

Altogether

\[
M \leq M f'(0) \cdot \frac{1}{2} \mathbb{E} \int_0^\infty 1(\omega_x(s)) ds + \int_{\partial \Omega} \frac{\partial^2 u}{\partial n^2}(y) d\omega_x(y) \cdot \left< \frac{1}{2} \right>.
\]

In order for this inequality to be satisfied for some \( M > 0 \), we require

\[
f'(0) \cdot \frac{1}{2} \mathbb{E} \int_0^\infty 1(\omega_x(s)) ds > 1.
\]

We will now prove an upper bound on the integral that will lead to a contradiction. It is known that among all domains with fixed measure, the maximum lifetime is maximized for a ball and can be bounded from above by the maximum of the solution of \(-\Delta \psi = 2\) in a ball with the same radius as \( \Omega \). The solution is given by

\[
\psi(x_1, \ldots, x_n) = \frac{r^2}{n} - \left( \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \right)
\]

and thus

\[
\max_{x \in \Omega} \mathbb{E} \int_0^\infty 1(\omega_x(s)) ds \leq \frac{r^2}{n} = \frac{|\Omega|^{2/2}}{n\omega_n^{2/n}}.
\]

As noted, it would suffice to show that

\[
\max_{x \in \Omega} f'(0) \cdot \frac{1}{2} \mathbb{E} \int_0^\infty 1(\omega_x(s)) ds \leq 1,
\]

which, using the estimate on the expected lifetime, follows as soon as

\[
f'(0) \leq \frac{2n\omega_n^{2/n}}{|\Omega|^{2/n}}.
\]

\( \square \)
3.5. Proof of Theorem 2

Proof. Let

\[-\Delta u = f(u) \quad \text{in } \Omega.\]

We can then consider the solution (subject to Dirichlet boundary conditions) of

\[-\Delta v = f(u)^* \quad \text{in } B,\]

where $B$ is the ball centered at the origin having the same measure as $\Omega$, $|B| = |\Omega|$, and $f(u)^*$ denotes the symmetric decreasing rearrangement of $f(u)$ (we refer to the book of Baernstein [2] for a detailed introduction to symmetrization techniques). By Talenti’s rearrangement principle [38, 39], we have

\[v \geq u^*\]

and, in particular, we have $\|v\|_{L^\infty(B)} \geq \|u\|_{L^\infty(\Omega)}$. We will compare this to the solution (subject to Dirichlet boundary conditions) of

\[-\Delta \psi = f(\psi) \quad \text{in } B.\]

We will show that $v(x) \leq \psi(x)$ for all $x \in B$. Observe that both $v$ and $\psi$ are nonnegative functions. Moreover, since $v \geq u^*$ and $f$ is non-decreasing,

\[-\Delta v = f(u)^* = f(u^*) \leq f(v).\]

We define the operator $(-\Delta)^{-1}f$ as mapping a given function $f : B \to \mathbb{R}_{\geq 0}$ to the solution $w : B \to \mathbb{R}$ of $-\Delta w = f$ with Dirichlet boundary conditions. The maximum principle implies that if, for all $x \in B$, we have $g_1(x) \leq g_2(x)$, then, for all $x \in B$,

\[
\left[(-\Delta)^{-1}g_1\right](x) \leq \left[(-\Delta)^{-1}g_2\right](x).
\]

From this we obtain, for all $x \in \Omega$, the inequality $v(x) \leq \left[(-\Delta)^{-1}f(v)\right](x)$. Our goal is to show that, for all $x \in B$

\[\psi(x) \geq v(x).\]

Assume the statement is false and that, for $x_0 \in B$, we have $v(x_0) > \psi(x_0)$. Then

\[
0 < v(x_0) - \psi(x_0) \leq \left[(-\Delta)^{-1}f(v) - \psi\right](x_0)
= \left[(-\Delta)^{-1}(f(v) - f(\psi))\right](x_0)
\leq \left[(-\Delta)^{-1}\max_{y \in B}[f(v(y)) - f(\psi(y))]\right](x_0)
\leq \max_{y \in B}[f(v(y)) - f(\psi(y))] \cdot \left[(-\Delta)^{-1}1\right](x_0).
\]
The mean-value theorem yields
\[ \max_{y \in B} [f(v(y)) - f(\psi(y))] \leq \left( \max_{t > 0} f'(t) \right) \cdot \max_{y \in B} [v(y) - \psi(y)]. \]

The function \((-\Delta)^{-1} 1\) is merely the solution of \(-\Delta u = 1\) and well understood. In the proof of Theorem 3, we showed the inequality
\[ \left[ (-\Delta)^{-1} 1 \right](x) \leq \frac{|\Omega|^{\frac{2}{n}}}{2n\omega_n^{2/n}}. \]

Hence,
\[ (v - \psi)(x_0) \leq \frac{|\Omega|^{\frac{2}{n}}}{2n\omega_n^{2/n}} f'(0) \cdot \max_{y \in B} [v(y) - \psi(y)]. \]

However, since \(x_0\) was arbitrary (as long as \(v(x_0) > \psi(x_0)\)), we may as well pick \(x_0\) to be the point in which \(v(x) - \psi(x)\) assumes its maximum. This leads to a contradiction as soon as
\[ \frac{|\Omega|^{\frac{2}{n}}}{2n\omega_n^{2/n}} \cdot f'(0) < 1. \]

This contradiction then shows that
\[ \|u\|_{L^\infty(\Omega)} = \|u^*\|_{L^\infty(B)} \leq \|v\|_{L^\infty(\Omega)} \leq \|\psi\|_{L^\infty(\Omega)}. \]

\[ \square \]

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