Constructing a science of cities has become a crucial task for our societies, which are growing ever more concentrated in urban systems. Better planning could be achieved with a better understanding of city growth and how it affects society and the environment. Various important aspects of cities such as urban sprawl, infrastructure development or transport planning depend on the population evolution over time, and multiple theoretical attempts have been made in order to understand this crucial phenomenon.

**Growth of cities and Zipf’s law**

So far, most research in city growth has been done with the idea that the stationary state for a set of cities is described by Zipf’s law. This law is considered to be a cornerstone of urban economics and geography, and states that the population distribution of urban areas in a given territory (or country) displays a Pareto law with exponent equal to 2 or, equivalently, that the city populations sorted in decreasing order versus their ranks follow a power law with exponent 1. This alleged regularity through time and space is probably the most striking fact in the science of cities and for more than a century has triggered intense debate and many studies. A theoretical model must also be able to explain the relatively frequent rises and falls of cities and civilizations, but despite many attempts these fundamental questions have not yet been satisfactorily answered. Here we introduce a stochastic equation for modelling population growth in cities, constructed from an empirical analysis of recent datasets (for Canada, France, the UK and the USA). This model reveals how rare, but large, interurban migratory shocks dominate city growth. This equation predicts a complex shape for the distribution of city populations and shows that, owing to finite-time effects, Zipf’s law does not hold in general, implying a more complex organization of cities. It also predicts the existence of multiple temporal variations in the city hierarchy, in agreement with observations. Our result underlines the importance of rare events in the evolution of complex systems and, at a more practical level, in urban planning.
As discussed in that same work\(^5\), cities and civilizations rise and fall many times on a large range of time scales, and Gabaix’s model is both quantitatively and qualitatively unable to explain these specific chaotic dynamics. Therefore, a model able to simultaneously explain observations about the stationary population distribution and the temporal dynamics of systems of cities is missing. In particular, we are not at this point able to identify the causes of the diversity of empirical flows $\xi$ compared to Lévy (continuous red lines) and normal distributions (green dashed lines). See Extended Data Fig. 4 for the left-cumulative distribution function.

Fig. 1 | Migration flow analysis. a–d. Analysis for France (a), the USA (b), the UK (c) and Canada (d). Left, migration-rate ratio versus the ratio of populations. The straight line is a power-law fit that gives an exponent equal to one. Right, empirical right-cumulative distribution function of renormalized migrations flows $\xi$. As seen in the graph, the Lévy stable law shows a straight line, whereas the normal law deviates significantly from the power-law fit.
observations about the hierarchical organization of cities, the occurrence of megacities, and the empirical instability in city dynamics seen in the births and deaths of large cities on short time scales. In this respect, we do not need just a quantitative improvement of models but a shift of paradigm.

In this paper, we show that city growth is dominated by rare events—namely large interurban migratory shocks—rather than by the average growth rate. Rare but large positive or negative migratory flows can destabilize the hierarchy and the dynamics of a city on very short time scales, leading to the disordered dynamics of cities observed throughout history. On the basis of an empirical analysis of migrations flows in four countries, in the following we derive a stochastic equation of city growth that is able to explain empirical observations of the statistics and temporal dynamics of cities.

Deriving the equation of city growth

To understand city growth, we require a robust, bottom-up approach, starting from elementary mechanisms governing the evolution of cities. Without loss of generality, the growth dynamics of a system (such as a country) of cities of size $S_i$ can be decomposed into the sum of an interurban migration term between metropolitan areas and an ‘out-of-system’ term that combines other sources of growth: natural birth, death, and interurban migration term between metropolitan areas and an average equal to 1 and encode the noise as well as multiple other effects, including distance. We denote by $I_{ij} = J_{ij}/S_i$, the probability per unit time and per capita of moving from city $i$ to city $j$. The left panel of Fig. 1 shows that the ratio $I_{ij}$ versus the population $S_i/S_j$ displays, on average, linear behaviour. This implies that $\mu = \nu$, and that we have, on average, a sort of detailed balance $\langle J_{ij} \rangle = \langle J_{ji} \rangle$ (where the angled brackets here denote the average over cities), but that crucially, fluctuations are non-zero. More precisely, if we denote by $X_{ij} = (J_{ij} - \langle J_{ij} \rangle)/I_{ij}S_i$, we observe that these random variables $X_{ij}$ are heavy-tailed—that is, they are distributed according to a broad law that decreases asymptotically as a power law with exponent $\alpha < 2$ (see Supplementary Information for more details and empirical evidence). The sum in the second term of the right-hand side of equation (1) can then be rewritten as

$$
\sum_{j \in N(i)} J_{ij} - I_{ij} = I_{0}S_i \sum_{j \in N(i)} X_{ij},
$$

and, according to the generalized version of the central limit theorem (assuming that correlations between the variables $X_{ij}$ are negligible), the random variable

$$
\zeta_i = \frac{1}{|N(i)|^{1/2}} \sum_{j \in N(i)} X_{ij}
$$

follows a Lévy stable law $L_\alpha$ with parameter $\alpha$ (for large enough $N(i)$). This is empirically confirmed in Fig. 1 (right panel): French, US, British and Canadian data are better fitted by a Lévy stable law than by any other distribution and the estimates of $\alpha$ (using different methods) are given in Table 1. We are led to the conclusion that the growth of systems of cities is governed by a stochastic differential equation with two independent noises, which reads as follows

$$
\frac{dS_i}{dt} = \eta_i S_i + D S_i^\beta \xi_i,
$$

where $D = I_{0}S_i \nu = \nu/\alpha$ and $\eta_i$ is a Gaussian noise with mean the average growth rate $r$ and a dispersion $\sigma$. This is the growth equation of cities that governs the dynamics of large urban populations; it is our main result here. In equation (3) both noises are uncorrelated and multiplicative, and Itô’s convention here seems to be more appropriate than Stratonovich’s because population sizes at time $t$ are computed independently from interurban migration terms at time $t + dt$. Estimates for the various parameters together with the prediction for the value of $\beta$ are given in Table 2.

### Table 1 | Estimates of parameter $\alpha$

| Dataset                | MLE          | Kolmogorov–Smirov test | Log-moments | Hill |
|------------------------|--------------|------------------------|-------------|------|
| France, 2003–2008      | $1.43 \pm 0.07$ | $1.2 < \alpha < 1.8$   | $1.3$       | $1.4 \pm 0.3$ |
| USA, 2013–2017         | $1.27 \pm 0.07$ | $1.15 < \alpha < 1.20$ | $1.2$       | Inconclusive |
| UK, 2012–2016          | $1.32 \pm 0.26$ | Inconclusive           | $1.0$       | $1.2 \pm 0.8$ |
| Canada, 2012–2016      | $1.69 \pm 0.12$ | Inconclusive           | $1.9$       | $1.4 \pm 0.6$ |

We used four different methods of estimation: maximum-likelihood estimation, Kolmogorov–Smirov test, log-moments and Hill estimates (see, for example, ref. 4).

### Table 2 | Estimates of parameters for the four datasets

| Dataset                | $\gamma$ | $\nu$ | $\beta = \nu + \gamma/\alpha$ | $P_{\text{measured}}$ |
|------------------------|----------|-------|-------------------------------|------------------------|
| France, 2003–2008      | $0.55 \pm 0.06$ | $0.4 \pm 0.3$ | $0.8 \pm 0.4$ | $0.75 \pm 0.07$ |
| USA, 2013–2017         | $0.41 \pm 0.05$ | $0.4 \pm 0.4$ | $0.7 \pm 0.5$ | $0.93 \pm 0.07$ |
| UK, 2012–2016          | $0$      | $0.7 \pm 0.3$ | $0.7 \pm 0.3$ | $0.51 \pm 0.05$ |
| Canada, 2012–2016      | $0.5 \pm 0.4$ | $0.5 \pm 0.4$ | $0.78 \pm 0.06$ | |

We observe good agreement between the measured and predicted values of $\beta$ in the USA and Canada datasets are small and hence fully connected (implying $\gamma = 0$), and are very noisy.
The central limit theorem, together with the broadness of interurban migration flow, enables us to show that many details in equation (1) are unnecessary and that the dynamics can be described by the more universal equation (3). We conclude that starting from equation (1) is thus less useful than previously thought. The importance of migrations has been previously noted\textsuperscript{25}, but in that work the authors derived a stochastic differential equation with multiplicative Gaussian noise, which we show here to be incorrect: we indeed have a first term with multiplicative noise but also, crucially, we obtain another term that is a multiplicative Lévy noise with zero average. This is a major theoretical shift that is not included in previous studies on urban growth and which has many crucial implications in understanding both the stationary and dynamic properties of cities.

No stationary distribution for cities

Equation (3) governs the evolution of urban populations and analysing it at large times gives indications about the stationary distribution of cities. To discuss the analytical properties of equation (3), we assume that Gaussian fluctuations are negligible compared to the Lévy noise and write \( \eta_i = r \) (see Extended Data Fig. 5). The corresponding Fokker–Planck equation (with Itô’s convention) can be solved using the formalism of fractional-order derivatives and Fox functions\textsuperscript{38–41}, leading to the general distribution at time \( t \) that can be expanded in powers of \( S \) as (see Supplementary Information for derivation and complete expressions of all terms):

\[
P(S, t) = \sum_{k=1}^{\infty} C_k \frac{a^k}{(S^\alpha - (\eta - r)^k)}
\]

where \( C_k \) is a prefactor that is a function of \( \alpha, \beta \) and \( k \) and independent of \( t \) and \( S \), and where \( a(t) = \frac{\alpha/\beta^\alpha}{(\alpha - 1/(1-\beta))} \) decreases exponentially at large times. This expansion shows that the probability distribution of city sizes is dominated at large \( S \) by the order \( k = 1 \) and converges towards a Pareto distribution with exponent \( \alpha = 1 \). The speed of convergence towards this power law can be estimated with the ratio \( \lambda(S, t) \) of the first and second terms of the expansion equation (4) and leads to:

\[
\lambda(S, t) = \left( \frac{S(t)}{S} \right)^{\alpha(1-\beta)}
\]

where \( S(t) \) is the mean city size. If \( \lambda(S) > 1 \), the \( \alpha \)-exponent regime is not valid in the right tail with threshold \( S \) at time \( t \). Estimates of \( \alpha \) and \( \beta \) for the four datasets show that finite-time effects are very important in all cases and that a power-law regime is only reached for unrealistically large city sizes (see discussion in Supplementary Information). Hence, the range of city sizes for which we can observe a power-law distribution may not exist in practice and there is no reason in general to observe Zipf’s law or any other stationary distribution. We also note that from equation (4) there is a scaling of the form \( P(S, t) = \frac{1}{t} F(S) \) with a scaling function \( F \) that depends on the country. We confirmed this scaling form for France (the only country for which we had sufficient data); details can be found in Supplementary Information (see also Extended Data Fig. 6).

In addition, if we perform a power-law fit of the expansion (equation (4)), the upper tail of the city-size distributions may be mistaken for a Pareto tail with a spurious exponent that changes with the definition of the upper tail (Extended Data Fig. 7). This might explain the discrepancies observed in the literature on Zipf’s law. As city sizes increase, the apparent exponent changes and can dramatically deviate from 1, as we initially observe in Extended Data Fig. 1. Following our analysis, the apparent exponent should converge towards the value given by \( \alpha \), as is indeed observed in, for example, France (\( \alpha = 1.4 \)) and the USA (\( \alpha = 1.3 \)).

Dynamics: splendour and decline of cities

The validity of our model (equation (3)) can be further tested on the dynamics of systems of cities over large periods of time. This can be done by following the populations and ranks of the system’s cities at different times with the help of ‘rank clocks’, as previously proposed\textsuperscript{5}. In that work, it was proven that the micro-dynamics of cities is very turbulent, with many rises and falls of entire cities that cannot result from Gabaix’s model (which is, in essence, Gibrat’s model with a non-zero minimum for city sizes). We show in Fig. 2 the empirical rank clock for France (from 1876 to 2015) and for the results obtained with Gabaix’s model and ours (for the other countries, see Extended Data Fig. 8).

We see that in Gabaix’s model (middle), the city rank is stable on average, and not turbulent: the rank trajectories are concentric and the rank of a city oscillates around its average position. In the real dynamics (left), cities can emerge or die. Very fast changes in rank order can occur, leading to much more turbulent behaviour. In our model (right), the large fluctuations of Lévy noise are able to statistically reproduce such ebbs and flows of cities. More quantitatively, we first compare the average shift per time \( d = \langle \sum_{i=1}^{N} |r_i(t) - r_i(t-1)| \rangle / NT \) over \( T \) years and for \( N \) cities in the three cases (Table 3) and look at the statistical fluctuations of the rank (see Extended Data Fig. 9): we note that Lévy fluctuations are much more able to reproduce the turbulent properties of the dynamics of cities through time. Indeed, the fast births and deaths of cities—due, for example, to wars, discoveries of new resources, incentive settlement policies, and so on—are statistically explained by broadly distributed migrations and are incompatible with a Gaussian noise. Second, we can compare with the empirical data the predictions of the different models for the time needed to make the largest rank jump (see Extended Data Fig. 10 for France, which typically predicts a duration of order 80 years to make a very large jump).
confirm that Gabaix’s model is unable to reproduce these very large fluctuations and that our equation agrees very well with the data.

A new paradigm

In this Article, we build a stochastic equation of growth for cities on the basis of microlevel considerations that is empirically sound and that challenges the paradigm of Zipf’s law and current models of urban growth. We show that microscopic details are irrelevant and that the growth equation obtained is universal. A crucial point in this reasoning is that, although we have on average a sort of detailed balance that would lead to a Gaussian multiplicative-growth process, it is the existence of non-universal and broadly distributed fluctuations of the microscopic migration flows between cities that govern the statistics of city populations. We introduce here a stochastic equation that describes city growth that includes two sources of noise and that predicts an asymptotic power-law regime. However, this stationary regime is not generally reached and finite-time effects cannot be discarded. Our model is also able to statistically reproduce the turbulent micro-dynamics of cities that rapidly rise and fall, in contrast with previous Gaussian-based models of growth. In addition, our fundamental result exhibits an interesting connection between the behaviour of complex systems and non-equilibrium statistical physics for which microscopic currents and the violation of detailed balance seem to be the rule rather than the exception. At a practical level, this result also highlights the critical effect of not only interurban migration flows (an ingredient that is not generally considered in urban-planning theories) but also, more importantly, their large fluctuations—which are ultimately connected to the capacity of a city to attract a large number of new citizens. Our approach, which relies in essence on the population budget description and empirical results, provides a solid ground for future research on the temporal evolution of cities, a central problem in urban science.

Online content

Any methods, additional references, Nature Research reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41586-020-2900-x.

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Methods

For each of the four countries we build a graph of migration flows between metropolitan areas. We have (1) the populations of metropolitan areas and (2) the migration flows between metropolitan areas (described in more detail below).

US migrations
Data of migrations in the USA are taken from the 2013–2017 American Community Survey (ACS). Aggregated metro-area-to-metro-area migration flows and counterflows are directly given between 389 metropolitan statistical areas in the USA. More precisely, the ACS asked respondents whether they lived in the same residence one year ago; for people who lived in a different residence, the location of their previous residence was collected.

French interurban migrations
Data of migrations in France are taken from the 2008 INSEE report for residential migrations at the town (commune) level for each individual household. The main residence in 2008 is compared to the main residence in 2003. In order to work at the urban area level, we used the 1999 INSEE list of urban areas and aggregate residential migrations at the metropolitan level, enabling us to analyse migration flows between the 500 largest urban areas in France.

UK interurban migrations
Data of migrations in the UK are taken from 2012–2016 ONS reports on internal migration between English and Welsh local authorities, giving the square matrix of moves each year. In order to work at the urban area level, we used the list of local authorities by OECD functional urban areas and aggregate residential migrations at the metropolitan level, enabling us to analyse migration flows between the 41 largest urban areas in England and Wales.

Canadian interurban migrations
Data of migrations in Canada are taken from 2012–2016 census reports on internal migration between Canadian metropolitan areas. Flows between these areas are given city-to-city for each year between 2012 and 2016 for the top-160 largest cities in Canada.

Data availability
The datasets used in this study are freely available from public repositories.

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Author contributions
V.V. and M.B. designed the study, V.V. acquired the data, and V.V. and M.B. analysed and interpreted the data and wrote the manuscript.

Competing interests
The authors declare no competing interests.

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Extended Data Fig. 1 | No universal exponent. We show here the measured Pareto exponent of the upper tail of city-size distributions as a function of the lower threshold defining the tail for the largest cities of eight different countries. The exponents are obtained with a maximum-likelihood estimate (data from https://simplemaps.com/data/world-cities).
Extended Data Fig. 2 | In- and out-neighbours. a, b, Number of in- and out-neighbours (in the sense of graph theory) for the USA (a) and France (b). The red lines correspond to the equality $N_{\text{in}} = N_{\text{out}}$. In the UK and Canada, we have a fully connected dataset and $N_{\text{in}} = N_{\text{out}} = \text{constant}$. c, d, Number of neighbours for each city as a function of population for the USA (c) and France (d). The dotted red lines indicate the power-law fit $|N_i| \propto S_i^\gamma$. In the UK and Canada, we have a variance of the normalized quantity fully connected dataset and $\gamma = 0$. 
Extended Data Fig. 3 | Density function of the out-of-system growth rate.

Natural growth and out-of-system migrations include international migrations and exchanges with small towns. The data shown are for US cities in 2013–2017 (top) and French cities (bottom) in 2003–2008, compared to a normal distribution. We note that a power-law fit of the right or the left tail would lead to a Pareto exponent of $\beta \approx 1$. For French cities, we extrapolated the 2003 population in each city from the 1999 and 2006 censuses to test our assumption on the period 2003–2008.
Extended Data Fig. 4 | Migration-flow analysis. Complementary figure to Fig. 2. Empirical left-cumulative distribution function of renormalized migration flows compared to Lévy (continuous red line) and normal (green dashed lines) distributions. Clockwise from top left, distributions are given for France, the USA, Canada and the UK.
Extended Data Fig. 5 | Average distribution of city sizes. Data obtained by 10 numerical runs of the stochastic differential equation (10) with a Gaussian noise with finite variance \( \sigma_\eta \), compared with the numerical solution of equation (10) where \( \eta \sim N(r, \sigma_\eta) \). Parameters here are \( \alpha = 1.3, \beta = 0.8, r = 0.01, \sigma_\eta = 0.06, D = 0.06 \) and \( t = 500 \).
Extended Data Fig. 6 | Scatterplot of the quantity $P(S, t) \times S$ versus the ratio $S/\bar{S}(t)$ for France’s top-500 largest cities between 1875 and 2016. Each colour is a different year. We observe that the plots of all years collapse towards a unique universal function of the ratio in agreement with the result of equation (33) in Supplementary Information.
Extended Data Fig. 7 | Power-law fit of the expansion with $\alpha = 1.3$, as a function of the lower threshold of city sizes, $S_{\text{min}}$. The expansion is described in equation (25) in Supplementary Information. The fit gives an apparent exponent $\alpha(S_{\text{min}})$ with very good quality ($R^2 \approx 1$), although the expansion itself is not a power law. The apparent exponent is smaller than $\alpha$, but slowly converges towards $\alpha = 1.3$ as the value of the threshold $S_{\text{min}}$ increases. Parameters here are $\alpha = 1.3, \beta = 0.8, r = 0.01, D = 0.06$ and $t = 500$. 
Extended Data Fig. 8 | Rank clocks of the USA and the UK. Top, USA; bottom, UK. The left panels display real data, the middle panels show Gabaix's model of growth and the right panels give our model of growth. Parallel lines for earlier years are spurious effects resulting from the absence of data for cities out of the top-100 largest in the USA or the top-40 largest in the UK (for these rank clocks, we assigned a random increasing radius to cities without data).
Extended Data Fig. 9 | Microdynamics of city rank through time for the largest cities in France, the USA and the UK. Data is given for the 500 largest cities in France between 1875 and 2016 (top), the 100 largest cities in the United States between 1790 and 1990 (middle) and the 40 largest cities in the UK between 1861 and 1991 (bottom). The left panels display the right-cumulative distribution of the maximal variation of the rank $r_i(t)$ for city, that is, the difference between the highest and the lowest rank in population for each city. The right panels display the typical fluctuations of the rank $r_i(t)$ through time. In the three cases, the Lévy model is able to predict rare but non-negligible large variations of rank such as the sudden birth or death of city, in contrast to Gabaix’s model or Gibrat’s model for growth, for which large fluctuations of rank order do not occur.
Extended Data Fig. 10 | Average number of years (and standard dispersion) taken to observe the maximal rank variation $\Delta r$ as a function of $\Delta r$. Although the dispersion is large, Lévy’s model is compatible with real data, in contrast to Gabaix’s model of growth.