The Analytical Solution of Incomplete Gamma Function to Determine the Electrical Resistivity at Normal State for MgB$_2$ Superconductor

Intikhab A Ansari

Department of General Studies, Jubail Industrial College, P. O. Box - 10099, Jubail Industrial City-31961, Saudi Arabia

Email: ansari_i@jic.edu.sa

Abstract
In the present study, the series representation of the generalized incomplete Gamma function is described. The theoretical results achieved by calculating the incomplete Gamma function are in agreement with the experimental data of the previous reports as discussed in this study. At different integer values of $m$, the generalized Bloch-Gruneisen function is presented in this work. The theoretical temperature dependence of resistivity at normal state correlates the experimental outcomes for pristine MgB$_2$ superconductor under the range of 40 – 175K. In addition, the incomplete Gamma function that has proposed for the evaluation of temperature dependence of resistivity reveals the validity and precision of the method.

Keywords: Electrical resistivity; incomplete Gamma function; MgB$_2$ superconductor; Debye temperature

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1. Introduction
The present study describes the series representations of the generalized incomplete Gamma function which occurs in variety of physical applications. The Gamma function plays the vital role in describing the contribution of electron-phonon (e-ph) interaction to the resistivity behavior at normal state of the superconductors [1]. Several attempts have been made to solve this function for the temperature dependence of resistivity, $\rho$ (T) of metals [2–4]. In addition, this function has widely used in order to describe the experimental as well as theoretical outcomes to analyze the different superconductors for temperature dependence at normal state of resistivity [5–9].

In order to understand the resistivity behavior in normal-state when temperature is cooled down, this Gamma function widely contributes the e-ph interaction. The main intention of the present study is to derive new series for incomplete Gamma function. After that the numerical solution for the Gamma...
function is presented for different integer values of \( m \). Some reports generalized this function for MgB\(_2\) superconductors to show the preciseness with different methods [10, 11].

The paper is structured as follows: In section 2, we evaluate the incomplete Gamma function by using the Bloch-Gruneisen (BG) integral. This BG integral precisely demonstrate the contribution of electron-phonon interaction at normal-state. In section 3, the numerical outcomes obtained from the Eq. (2) are discussed and compared with experimental data of the temperature dependence of resistivity from the previous reports. Finally in section 4 we express our conclusions.

2. Evaluation of incomplete gamma function

The MgB\(_2\) superconductor have the simple crystal structure and belongs to the AlB\(_2\) family with space group P6\(/
\text{mmm}\). The MgB\(_2\) possess hexagonal unit cell with \( a = b = 3.086 \) Å and \( c = 3.524 \) Å [12]. The valency of MgB\(_2\) can be well characterized by the ionic form, Mg\(^{2+}\)(B\(_2\))\(^{-2}\). The crystal structure of MgB\(_2\) is layered. The structure of the boron layers is the same as that of a layer in a graphite structure. The temperature dependence of electrical resistivity of MgB\(_2\) is a key-element to expose the \( e\)-\( ph\) interaction associated. The Bloch-Gruneisen (BG) integral precisely demonstrate the contribution of electron-phonon interaction in normal-state by the following equation

\[
\rho_{\text{e-ph}}(T) = \frac{A T^{5}}{\theta_D^{2}} \frac{\thetaD}{T} \int_{0}^{\thetaD/T} \frac{x^m}{(e^x - 1)(1 - e^{-x})} \, dx
\]

This Eq. (1) can be transform into other form as shown below

\[
\rho_{\text{e-ph}}(T) = \frac{A T^{5}}{\theta_D^{2}} \frac{\thetaD}{T} \int_{0}^{\thetaD/T} \frac{x^m e^{-x}}{(1 - e^{-x})^2} \, dx
\]

In this equation, \( A \) is a constant independent of both temperature \( T \) and Debye temperature \( \theta_D \), where \( x \) referred to the \( \thetaD/T \).

One can generalized the complete gamma function \( \Gamma(a) \) with the help of upper incomplete gamma function \( \Gamma(a,x) \) such that \( \Gamma(a) = \Gamma(a,0) \). The upper incomplete gamma function is expressed by

\[
\Gamma(a,x) = \int_{x}^{\infty} t^{a-1} e^{-t} \, dt.
\]

For \( a \) an integer \( n \)

\[
\Gamma(n,x) = (n-1)! \, e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}
\]

\[
= (n-1)! \, e^{-x} \, e_{n-1}(x),
\]

Where \( e_{n}(x) \) is denoted as an exponential sum function.

The incomplete gamma function \( \Gamma(a,x) \) is described in terms of Legendre’s continued fraction [13]
\[ \Gamma(a, x) = \frac{e^{-x} x^a}{1 - a} + \frac{1}{1 + \frac{2 - a}{x + \frac{2}{1 + \frac{3 - a}{x + \ldots}}}} \]  

This expansion converges for all \( x \neq 0, |\arg x| < \pi \) and any complex value of \( a \). The continued fraction converges better as the ratio \( |x/a| \) increases.

The \( \gamma(a, x) \) is well-known lower incomplete gamma function [14, 15] as described by

\[
\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt \quad (a > 0)
\]

\[ = a^{-1} x^a e^{-x} F_1(1; 1 + a; x) \]

\[ = a^{-1} x^a F_1(a; 1 + a; -x), \]  

where \( F_1(a; b; x) \) is the Kummer function of the first kind. The series development of \( \gamma(a, x) \) for \( a \) an integer \( n \),

\[
\gamma(n, x) = (n-1)! \left[ 1 - e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!} \right] 
\]

\[ = (n-1)! \left[ 1 - e^{-x} e_{n-1}(x) \right]. \]  

Important series expansions are

\[ \gamma(a, x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^{a+n}}{(a)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{a+n}}{a+n} \]  

The series are convergent for each complex \( a \) and \( x \), with the exception of \( a = 0, -1, -2, \ldots \). Speed of convergence of the first series depends on the ratio \( |x/a| \); when \( |x/a| < 1 \) the convergence is quite fast.

The lower incomplete gamma function can be expressed by iterative method:

\[
\gamma(a, x) = x^a e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{a(a+1)\ldots(a+n)}. \]  

By definition, the gamma function is the summation of upper and lower and incomplete gamma function as described follows

\[
\Gamma(a, x) + \gamma(a, x) = \Gamma(a). \]  

One more series representation of such a formula, involving the polylogarithmic functions is reported in ref [16] as described below
\[
\frac{F_n(x)}{n!} = \zeta(n) - \sum_{j=0}^{n} \frac{x^j}{j!} Li_{n-j}(e^{-x}), \quad n = 2, 3, \ldots,
\]  
(11)

where \(Li_{i}(x)\) denotes the polylogarithm function for integer \(s\), as given underneath

\[
Li_{i}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^i}
\]

(12)
in its convergence region and by analytic continuation in the rest of the complex plane.

In order to calculate the integral of Eq (2) for \(m = 4\), the online wolfram mathematicaintegral calculator gives the following result

\[
\int \frac{x^m e^{-x}}{(1-e^{-x})^2} dx = 12x^2 Li_2(e^x) - 24x Li_3(e^x) + 24 Li_4(e^x) - \frac{e^x}{e^x - 1} - x^4 + 4x^3 \log(1-e^x) + \text{constant}
\]

(13)

where \(\log(x)\) is the natural logarithm and \(Li_{i}(x)\) is the polylogarithm function.

Substitute the obtained integral value in Eq. (2), gives the total resistivity of the sample.

### 3. Results and discussion

This section shows the numerical result of the generalized gamma function for Eq. (13) for \(m = 4\) with some accuracy. The program for precise value of generalized gamma function in Fortran 90 language is constructed with an Intel core i7 computer for Eq. (13) to evaluate the total resistivity of the MgB\(_2\) sample by applying Eq (2). Furthermore, the temperature dependence of resistivity at normal-state was calculated theoretically by using the Eq. (2) for integer value of \(m = 3, 4 \text{ and } 5\) as shown in the figure 1. The Debye temperature, \(\theta_d = 1100 \text{ K}\) is taken into account as a fitting parameter for pure MgB\(_2\) superconductor. These theoretical results are compared with the experimental temperature dependence of resistivity data of the previous report for pristine MgB\(_2\) sample \[17\]. The proposed algorithm is relevant for evaluating the generalized gamma function up to more decimal points with high precision for integer value of \(m\). Figure 1 illustrates that at low value of integer \(m = 4 \text{ and } 5\), the theoretical results of temperature dependence of resistivity at normal-state are in agreement with the experimental results of \(\rho (T)\) plot. It is remarkable that at low integer value of \(m = 3\), the temperature dependence of resistivity is diverting from the experimental results. In this study, the figure 1 demonstrates the variation of resistivity with temperature from room temperature to the 10 K. In addition, the theoretical temperature dependence of resistivity elucidates the variation with different values of integer \(m\) at normal-state. It is evident from the figure 1 that the value of integer \(m = 4\) is fitted well up to 175 K. The generalized BG formula analyzed in the present work to calculate the temperature dependence of resistivity is expressed by Eq. (2) for any integer value of \(m\). In addition, this Eq. (2) includes the lower incomplete gamma function which have different series representation that are discussed in this study.
Figure 1: Comparison of experimental and theoretical calculated temperature dependence of resistivity with varying $m = 3$ to 5 for pure MgB$_2$ superconductor.

4. Conclusion
In the present work, we have analyzed the generalized incomplete gamma function with integer value of $m = 3, 4$ & 5 for temperature dependence of resistivity for pure MgB$_2$ superconductor at normal-state. The experimental results are in agreement with the theoretical outcomes for integer value of $m = 4$ under the range of 40–175K at normal state. The proposed generalized gamma function is applicable for different integer values of $m$ by using the Eq. (2) to calculate the temperature dependence of resistivity for pure MgB$_2$ superconductor. The present study exhibits the easy method for the calculation of generalized gamma function in comparison with the other prior reports.

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