Coupling Peridynamic Continuum Mechanics with an Analytical Solution

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Peridynamics is a nonlocal theory of continuum mechanics expressing the dynamic equilibrium of forces by using integro-differential equations instead of partial differential equations. Thus, the equilibrium equations are still valid in case of discontinuous displacement fields. In this study, we investigate the coupling between a harmonically excited peridynamic rod with a rod based on classical continuum mechanics by using the Arlequin-method. The peridynamic region is solved by finite elements whereas for the classical region an analytical solution can be used.

1 Introduction

Peridynamics was invited by Silling [1] to circumvent the problems of PDE-based continuum mechanics models with discontinuities of the displacement field, e.g. at a crack tip. In peridynamics, the divergence of the stress tensor in the point of investigation \( \vec{x} \) is replaced by the integral over force densities exerted by every point \( \vec{x}' \) within a sphere of radius \( \delta \) around \( \vec{x} \). Because the force densities are functions of displacements and not of their derivatives and the integral over the finite volume \( \mathcal{H} \), the peridynamic equation of motion remains valid even if the displacement field includes discontinuities.

2 Problem

We investigate a rod that is divided into a peridynamic \( S_{\text{peri}} \) and a classical continuum mechanics \( S_{\text{ccm}} \) subsection. Both regions interleave in a coupling region \( S_g \). The left end of the peridynamic rod and the right end of the classical rod are fixed in longitudinal direction. A harmonically oscillating load in longitudinal direction is applied at some point in the peridynamic region. The Arlequin method [2] is a work-based coupling scheme which introduces a set of Lagrange multipliers acting upon the difference of the displacement field in the coupling region. To apply the method, expressions for the virtual work in the peridynamic and the classical continuum mechanics rod need to be developed.

2.1 Peridynamics

The bond-based peridynamic equation of motion is defined as [4]. \( c \) is a material parameter.

\[
\rho(\vec{x}) \ddot{\vec{u}}(\vec{x}) = \int_{\mathcal{H}} \frac{c}{2} \frac{1}{|\vec{x}^* - \vec{x}|} \left( |\vec{x}^* + \vec{u}(\vec{x}^*) - \vec{x} - \vec{u}(\vec{x})| - |\vec{x}^* - \vec{x}| \right) \frac{\vec{x}^* + \vec{u}(\vec{x}^*) - \vec{x} - \vec{u}(\vec{x})}{|\vec{x}^* + \vec{u}(\vec{x}^*) - \vec{x} - \vec{u}(\vec{x})|} d\mathcal{H} + \vec{b}(\vec{x}) \tag{1}
\]

In one dimension, the peridynamic equation of motion (1) can be simplified [3] with \( E \) being the Young’s Modulus and \( A \) denoting the area of the rod.

\[
\rho \ddot{u}(x) = \int_{x-\delta}^{x+\delta} \frac{2EA}{\delta^2} \frac{|u(x^*) - u(x)|}{|x^* - x|} dx^* + b(x) \tag{2}
\]

2.2 Peridynamic Finite Elements

A finite element formulation of (2) yields an expression for the virtual work in the peridynamic region. According to [3], a regular meshgrid with element length \( h = \frac{1}{2} \delta \) and piecewise linear shape functions \( \Phi_i(x) \) are chosen. The \( S_{\text{peri}} \) domain is discretized into \( N \) elements. Due to the nonlocality, three more elements (\( K \)) have to be attached to the right and left end of \( S_{\text{peri}} \). The discretized form of the virtual work done by the internal forces of (2) reads as follows:

\[
W_{\text{int}_{\text{peri}}} = \frac{2EA}{\delta^2} \int_{0}^{l_{\text{peri}}} \sum_{j=-K}^{N+K+1} \sum_{k=j+1}^{K} \Phi_j(x) u_j \frac{\Phi_k(x^*) u_k}{|x - x^*|} dx^* \Phi_k(x) dx \tag{3}
\]

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The integrals can be either solved analytically or via Gaussian integration. For a horizon $\delta = 3 \cdot h$, the following, non-quadratic stiffness matrix is derived:

$$ K = \frac{2E}{A \delta^2} \begin{bmatrix} -0.05 & -0.21 & -0.45 & 1.03 & -0.33 & -0.16 & -0.01 \\ -0.01 & -0.16 & -0.45 & 0.18 & 1.03 & -0.45 & -0.21 & -0.05 \end{bmatrix} $$

(4)

### 2.3 Solution of the coupled region

The displacement field of the CCM-part of the system is obtained by solving the differential equation of a rod of length $l_{ccm}$ for a fixed right end and a displacement $u_0$ on the left hand side oscillating with an excitation frequency $\Omega$.

$$ u(x, t) = u_0 \left[ \cos \left( \frac{\Omega x}{c} \right) - \sin \left( \frac{\Omega x}{c} \right) \cos \left( \frac{\Omega l_{ccm}}{c} \right) \right] \cos(\Omega t) , \quad c = \sqrt{\frac{E}{\rho}} $$

(5)

The virtual work of the interior forces $W_{int}$ and inertia forces $W_{dyn}$ in the classical system is obtained by a Galerkin approach and a test-function $v(x)$ which have the same shape as $u(x)$.

$$ W_{int,ccm} = \int_0^{l_{ccm}} E u''(x,t) v(x,t) \, dx \quad , \quad W_{dyn,ccm} = \int_0^{l_{ccm}} \rho \ddot{u}(x,t) v(x,t) \, dx $$

(6)

### 2.4 Arlequin method

The Arlequin method yields a framework to find displacement fields $u_{peri}$ and $u_{ccm}$ for a model consisting of two subregions as well as a Lagrange multiplier field $\lambda$ in the coupling region $S_g$ [2]. Since it is based on the principle of virtual work, the virtual test fields $v_{peri}, v_{ccm}$ and a virtual lagrange multiplier field $\mu$ has to be introduced.

$$ W_{dyn}(w_1, w_2, v_1, v_2) + W_{int}(w_1, w_2, v_1, v_2) + W_{arl}(\lambda, w_1, w_2) = W_{ext}(w_1) \quad (on \ S_{peri}, S_{ccm}) $$

$$ W_{arl}(\mu, w_1, w_2) = 0 \quad (on \ S_g) $$

(7)

### 3 Numerical Example

We consider a single span rod with the following properties: \( l_{peri} = 1.0 \ m \), \( l_{ccm} = 1.5 \ m \), \( A = 0.001 \ m^2 \). Young’s Modulus \( E = 2100000 \ kN/m^2 \), density \( \rho = 7850 \ k g/m^3 \). Simulation parameters: \( \delta = 3 \cdot h \). Length of coupling region \( S_g \): \( l_{S_g} = 1 \cdot h \). Investigated are two cases, differing in the excitation frequency and the point of excitation. Both coupled simulations are compared to a classical finite element analysis.

![Fig. 1: \( F_0 \cos(\Omega \cdot t) \) applied at \( x = 0.75 \ m \), \( \Omega = 1 \cdot \text{rad/s} \) \( u_{peri}, u_{FEM} \)](image)

![Fig. 2: \( F_0 \cos(\Omega \cdot t) \) applied at \( x = 0.5 \ m \), \( \Omega = 34000 \cdot \text{rad/s} \) \( u_{peri}, u_{FEM} \)](image)

### 4 Conclusion

Both Fig.1 and Fig.2 show good congruence though the peridynamic solution leads to slightly higher amplitudes. Due to the peridynamic horizon, the influence of the single load extends to the three adjacent elements to the left and right.

### References

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