Sharp Bounds for the Signless Laplacian Spectral Radius in Terms of Clique Number*

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Dedicated to Professors Abraham Berman, Moshe Goldberg, and Raphael Loewy in recognition of their important contributions to linear algebra and the linear algebra community.

Abstract

In this paper, we present a sharp upper and lower bounds for the signless Laplacian spectral radius of graphs in terms of clique number. Moreover, the extremal graphs which attain the upper and lower bounds are characterized. In addition, these results disprove the two conjectures on the signless Laplacian spectral radius in [P. Hansen and C. Lucas, Bounds and conjectures for the signless Laplacian index of graphs, Linear Algebra Appl., 432(2010) 3319-3336].

Key words: Signless Laplacian spectral radius; clique number, Turán graph.
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1 Introduction

Throughout this paper, we only consider simple and undirected graphs. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set

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Let \( A(G) = (a_{ij}) \) be the \((0,1)\) adjacency matrix of \( G \) with \( a_{ij} = 1 \) for \( v_i \) adjacent to \( v_j \) and 0 otherwise. Moreover, let \( D(G) = \text{diag}(d(u), u \in V) \) be the diagonal matrix of vertex degrees \( d(u) \) of \( G \). Then \( Q(G) = D(G) + A(G) \) is called the signless Laplacian matrix of \( G \). The signless Laplacian matrix \( Q(G) \) can be viewed as an operator on the space of functions \( f : V(G) \to \mathbb{R} \) which satisfies

\[
Q(G)f(u) = \sum_{v \sim u} (f(u) + f(v)),
\]

where " \( \sim \) " stands for the adjacency relation. The largest eigenvalue of \( Q(G) \) is called the signless Laplacian spectral radius of \( G \) and denoted by \( q_1(G) \), or for short \( q_1 \).

Recently, the signless Laplacian matrix of a graph has received increasing attention. For example, Desai and Rao [5] used the smallest eigenvalue of the signless Laplacian matrix of a connected graph to serve as a measure of how much a graph is close to bipartite, since 0 is the smallest eigenvalue of \( Q(G) \) if and only if \( G \) is bipartite. Liu and Liu [9] presented lower and/or upper bounds for the clique number and the independence number in terms of the signless Laplacian eigenvalues. Oliveira et. al. [16] gave several upper and lower bounds for the signless Laplacian spectral radius. Zhang [20, 21] investigated the largest signless Laplacian spectral radius for a given degree graphic sequence. Cvetović and Simić [2]-[4] surveyed spectral graph theory based on the signless Laplacian matrix of a graph. Recently, Hansen and Lucas [8] proposed some conjectures on the signless Laplacian spectral radius.

**Conjecture 1.1** ([8]) Let \( G \) be a connected graph on \( n \geq 4 \) vertices with signless Laplacian spectral radius \( q_1 \) and clique number \( \omega \). Then

\[
q_1 - \omega \leq \frac{3}{2}n - 4, \text{ if } n \text{ is even,}
\]

\[
\frac{q_1}{\omega} \leq \frac{n}{2}.
\]

The bound for (1) is attained by and only by the complement of a perfect matching when \( n \geq 6 \) is even. Moreover, when \( n \geq 9 \) is odd, \( q_1 - \omega \) is maximum for and only for the complement of a perfect matching on \( n - 3 \) vertices and a triangle on three remaining vertices. The bound for (2) is attained by and only by the complete bipartite graph \( K_{p,q} \).

**Conjecture 1.2** ([8]) Let \( G \) be a connected graph on \( n \geq 6 \) vertices with signless Laplacian spectral radius \( q_1 \) and chromatic number \( \chi \). Then

\[
q_1 - \chi \leq \frac{3}{2}n - 4, \text{ if } n \text{ is even.}
\]
The bound is attained by and only by the complement of a perfect matching when
\( n \) is even. Moreover, when \( n \geq 9 \) is odd, \( q_1 - \chi \) is maximum for and only for
the complement of a perfect matching on \( n - 3 \) vertices and a triangle on the three
remaining vertices.

Hansen and Lucas [7] obtained some results on signless Laplacian related to clique
number and chromatic number. For other results, see [2]-[4] and the references therein.

On the other hand, there are many Turán-type extremal problems, i.e., given a
forbidden graph \( H \), determine the maximal number of edges in a graph on \( n \) vertices
that does not contain a copy of \( H \). It states that among \( n \)-vertex graphs not containing
a clique of size \( t+1 \), the complete \( t \)-partite graph \( T_{n,t} \) with (almost) equal parts, which
is called Turán graph, has the maximum number of edges. Spectral graph theory has
similar Turán extremal problems which determine the largest (or smallest) eigenvalue
of a graph not containing a subgraph \( H \).

Nikiforov [10] proved a spectral extremal Turán theorem: let \( \lambda(G) \) be the largest
eigenvalues of the adjacency matrix of \( G \) not containing complete graph \( K_t \) of order
\( t \) as a subgraph, then \( \lambda(G) \leq \lambda(T_{n,t-1}) \) with equality if and only if \( G = T_{n,t-1} \);
the same result has been proved previously by Guiduli [6], but his proof was made
public only after the publication of [10]. Further, Nikiforov explicitly advocated the
study of general Turán problems in many publications (see [10]-[15]). For instance, he
determined [13] the maximum spectral radius of graphs without paths of given length
and presented [15] a comprehensive survey on these topics. In addition, Sudakov et al.
[17] presented a generalization of Turán Theorem in terms of Laplacian eigenvalues.

Motivated by these conjectures and Turán-type extremal problems, we investigate
in this paper the extremal graphs with maximal or minimal signless Laplacian spectral
radius among all graphs of order \( n \) with given clique number, which may be regarded
as a part of spectral extremal theory. The main result of this paper reads:

**Theorem 1.3** Let \( G \) be a connected graph of order \( n \) with clique number \( \omega \geq 2 \).
Then

\[
q_1(G) \leq \frac{(3\omega - 4)k + 3r - 2 + \sqrt{k^2\omega^2 + [(2r + 4)\omega - 8r]k + (r - 2)^2}}{2},
\]

where \( n = k\omega + r, 0 \leq r < \omega \). Moreover, equality holds in (4) if and only if \( G \) is
complete bipartite graph for \( \omega = 2 \) and Turán graph \( T_{n,\omega} \) for \( \omega \geq 3 \).
2 Proof of Theorem 1.3

In order to prove Theorem 1.3, we need more notation and preliminary results. Two vertices in a graph $G$ are called duplicate if they have precisely the same neighbors. In the operation of duplication of a vertex $u$ to a vertex $v$, all edges incident to vertex $u$ are deleted and then all edges between $u$ and vertices adjacent to $v$ are added. Clearly, after this duplication, $u$ and $v$ are duplicate, and duplication in a graph not containing $K_{t+1}$ preserves this property. The complement $\overline{G}$ of a simple graph $G = (V(G), E(G))$ is the simple graph with vertex set $V(G)$, two vertices being adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. For two disjoint graphs $G$ and $H$, the union $G + H$ of $G$ and $H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$; the join $G \vartriangle H$ of $G$ and $H$ is the graph obtained from the union $G + H$ by joining each vertex of $G$ to each vertex of $H$.

Let $f$ be a nonnegative function on vertex set $V(G)$ of $G$. Then $f$ is also regarded as a nonnegative vector corresponding to vertex set $V(G)$. For any vertex $u$, the weight $w_G(u)$ of vertex $u$ with respect to $f$ in $G$ is defined as

$$w_G(u) = Q(G)f(u) = \sum_{v \sim u} (f(u) + f(v))$$

if $u$ is not isolated vertex and 0 otherwise, where ”$\sim$” stands for adjacency relation.

We adapt a variation of Zykov’s proof of Turán’s theorem, which was also used in [6].

Lemma 2.1 Let $G$ be a simple graph of order $n$ not containing $K_{t+1}$. Then there exists a simple graph $G_1$ such that $G_1 = \overline{K_p} \vartriangle H$ and $q_1(G) \leq q_1(G_1)$, where $H$ is a simple graph not containing $K_t$ and $1 \leq p \leq n - 1$. Moreover, equality holds if and only if $G = G_1$.

Proof. Without loss of generality, we assume that $G$ is connected, since $q_1(G)$ is an increasing function with respect to adding edges. Let $f$ be the unit positive eigenvector on $V(G)$ corresponding to the eigenvalue $q_1(G)$ of $G$ such that $q_1(G) = \langle f, Q(G)f \rangle$ and $\langle f, f \rangle = 1$. Let $w_G(u) = \max \{w_G(v) : v \in V(G)\}$. Denote by $V_1$ the set of all neighbors of vertex $u$ and $V_2 = V(G) \setminus (V_1 \cup \{u\})$ with $|V_2| = p - 1$. Now we construct a new graph $G_1$ obtained from $G$ by a duplication of each vertex $v \in V_2$ to vertex $u$. Then $G_1$ still does not contain $K_{t+1}$ and the induced subgraph $G[V_1]$ does not contain $K_t$. Hence $G_1$ can be written as $G_1 = \overline{K_p} \vartriangle H$, where $H = G[V_1]$ does not contain $K_t$.
On the other hand, for any \( v \in V_1 \), any vertex adjacent to \( v \) in \( G \) must be adjacent to \( v \) in \( G_1 \). Then

\[
w_G(v) = \sum_{xv \in E(G)} (f(v) + f(x)) \leq \sum_{xv \in E(G_1)} (f(v) + f(x)) = w_{G_1}(v)
\]

with equality holding if and only if \( v \) is adjacent to each vertex in \( V_2 \) in \( G \). For any vertex \( v \in V_2 \),

\[
q_1(G)f(u) = Q(G)f(u) = w_G(u) \geq w_G(v) = Q(G)f(v) = q_1(G)f(v).
\]

Hence \( f(u) \geq f(v) \). In addition,

\[
w_G(u) = q_1(G)f(u) = \sum_{xu \in E(G)} (f(u) + f(x)) = d_G(u)f(u) + \sum_{x \in V_1} f(x)
\]

and \( q_1(G) > d_G(v) \) for any \( v \in V(G) \) (see [16]). Therefore, for any \( v \in V_2 \),

\[
w_{G_1}(v) = \sum_{xv \in E(G_1)} (f(v) + f(x)) = d_{G_1}(v)f(v) + \sum_{x \in V_1} f(x)
\]

\[
= d_G(u)f(v) + w_G(u) - d_G(u)f(u)
\]

\[
= w_G(v) + (w_G(u) - w_G(v)) - d_G(u)(f(u) - f(v))
\]

\[
= w_G(v) + (q_1(G) - d_G(v))(f(u) - f(v))
\]

\[
\geq w_G(v).
\]

Moreover, \( w_{G_1}(u) = w_G(u) \). Then

\[
\langle f, Q(G_1)f \rangle = \sum_{v \in V(G_1)} f(v)w_{G_1}(v) \geq \sum_{v \in V(G)} f(v)w_G(v) = q_1(G).
\]

Hence by Rayleigh quotient, \( q_1(G_1) \geq q_1(G) \) with equality holding if and only if \( w_{G_1}(v) = w_G(v) \) for all \( v \in V(G) = V(G_1) \), which implies \( G = G_1 \). \( \blacksquare \)

**Lemma 2.2** Let \( G \) be a simple graph not containing \( K_{t+1} \) and \( R \) be another simple graph. Then there exists a simple graph \( G_1 \) such that \( G_1 = \overline{K_p} \triangle H \) and \( q_1(R \triangle G) \leq q_1(R \triangle G_1) = q_1(R \triangle \overline{K_p} \triangle H) \), where \( H \) is a simple graph not containing \( K_t \) and \( 1 \leq p \leq n - 1 \).

**Proof.** Let \( f \) be unit positive eigenvector of \( Q(R \triangle G) \) corresponding to \( q_1(R \triangle G) \). Let

\[
w_{R \triangle G}(v) = Q(R \triangle G)f(v) = \sum_{xv \in E(R \triangle G)} (f(v) + f(x)) = q_1(R \triangle G)f(v)
\]
for any \( v \in V(R \triangle G) \), and let \( w_{R \triangle G}(u) = \max \{ w_{R \triangle G}(v) : v \in V(G) \} \). Therefore, as in \( \square \),

\[
q_1(R \triangle G)f(u) = \sum_{x \in E(R \triangle G)} (f(u) + f(x)) = w_{R \triangle G}(u) \geq w_{R \triangle G}(v) = q_1(R \triangle G)f(v)
\]

which implies \( f(u) \geq f(v) \) for any \( v \in V(G) \). Denote by \( V_1 \) the set of all neighbors of vertex \( u \) in \( V(G) \) and \( V_2 = V(G) \setminus (V_1 \cup \{ u \}) \) with \( |V_2| = p - 1 \). Now we construct a new graph \( G_1 \) obtained from \( G \) by a duplicatation of each vertex \( v \in V_2 \) to vertex \( u \). Then \( G_1 \) still does not contain \( K_{t+1} \) and the induced subgraph \( G[V_1] \) does not contain \( K_t \). Hence \( G_1 \) can be written as \( G_1 = \overline{K_p} \setminus H \), where \( H = G[V_1] \) does not contain \( K_t \).

Clearly, for any vertex \( v \in V(R) \), \( w_{R \triangle G_1}(v) = w_{R \triangle G}(v) \). For any vertex \( v \in V_1 \), any neighbor of \( u \) in \( R \triangle G \) is also its neighbor in \( R \triangle G_1 \). So for any \( v \in V_1 \), \( w_{R \triangle G_1}(v) \geq w_{R \triangle G}(v) \) with equality if and only if \( v \) is adjacent to each vertex in \( V_2 \) in \( G \). For any vertex \( v \in V_2 \), by simple calculations, as in the previous proof, we have

\[
w_{R \triangle G_1}(v) = \sum_{x \in E(G_1)} (f(v) + f(x)) + \sum_{x \in V(R)} (f(v) + f(x))
\]

\[
= \sum_{x \in V_1} ((f(u) + f(x)) - (f(u) - f(v))) + \sum_{x \in V(R)} ((f(u) + f(x)) - (f(u) - f(v)))
\]

\[
= \sum_{x \in V_1} (f(u) + f(x)) + \sum_{x \in V(R)} (f(u) + f(x)) - (|V_1| + |V(R)|)(f(u) - f(v))
\]

\[
= w_{R \triangle G}(u) - (|V(R)| + d_G(u))(f(u) - f(v))
\]

\[
= w_{R \triangle G}(v) + (w_{R \triangle G}(u) - w_{R \triangle G}(v)) - (|V(R)| + d_G(u))(f(u) - f(v))
\]

\[
= w_{R \triangle G}(v) + (q_1(R \triangle G)f(u) - q_1(R \triangle G)f(v)) - (|V(R)| + d_G(u))(f(u) - f(v))
\]

\[
= w_{R \triangle G}(v) + (q_1(R \triangle G) - (|V(R)| + d_G(u)))(f(u) - f(v))
\]

\[
\geq w_{R \triangle G}(v),
\]

since \( q_1(R \triangle G) > |V(R)| + d_G(u) \) by Perron-Frobenius theorem. Moreover, \( w_{R \triangle G_1}(u) = w_{R \triangle G}(u) \). Therefore

\[
q_1(R \triangle G) = \langle f, Q(R \triangle G)f \rangle = \sum_{v \in V(R \triangle G)} f(v)w_{R \triangle G}(v)
\]

\[
\leq \sum_{v \in V(R \triangle G_1)} f(v)w_{R \triangle G_1}(v) = \langle f, Q(R \triangle G_1)f \rangle
\]

\[
\leq q_1(R \triangle G_1).
\]
Moreover, equality holds if and only if \( f \) is an eigenvector of \( Q(R \vee G_1) \), which implies \( w_{R \vee G}(v) = w_{R \vee G_1}(v) \) for all \( v \in V(R \vee G) \). So equality holds if and only if \( G = G_1 \). Hence the assertion holds.

The following lemma on the signless Laplacian spectral radius of a complete \( t \)-partite graph is well-known (see, for example, [1] or [19]).

**Lemma 2.3** ([1], [19]) Let \( G \) be a complete \( t \)-partite graph on \( n \) vertices. If \( n = tk + r, 0 \leq r < t \), then \( q_1(G) = n \) for \( t = 2 \), and

\[
q_1(G) \leq \frac{(3t - 4)k + 3r - 2 + \sqrt{t^2k^2 + [(2r + 4)t - 8r]k + (r - 2)^2}}{2}
\]  

for \( t \geq 3 \). Moreover, equality holds if and only if \( G \) is Turán graph \( T_{n,t} \) which is the complete \( t \)-partite graph on \( n \) vertices in which the partite sets are of size \( k \) or \( k+1 \).

**Corollary 2.4** For any \( 2 \leq t \leq n - 1 \),

\[
q_1(T_{n,t}) < q_1(T_{n,t+1}).
\]

**Proof.** It follows directly from Lemma 2.3 with a simple calculation.

Now we are ready to present a proof of Theorem 1.3.

**Proof.** Let \( \omega(G) = t \) and \( f \) be the positive eigenvector of \( Q(G) \) corresponding to \( q_1(G) \) with \( \langle f, f \rangle = 1 \). We consider the following two cases.

**Case 1:** \( t = 2 \). Then by [3], \( G \) is bipartite and \( q_1(G) \leq n \) with equality if and only if \( G \) is complete bipartite graph. Hence [11] holds with equality if and only if \( G \) is complete bipartite graph.

**Case 2:** \( t \geq 3 \). Then \( G \) does not contain \( K_{t+1} \) as a subgraph. By Lemma 2.1 there exists a graph \( G_1 = \overline{K_n} \neq H_1 \) such that \( H_1 \) does not contain \( K_3 \) and \( q_1(G_1) \leq q_1(G) \). Moreover, equality holds if and only if \( G = G_1 \). Since \( \overline{K_n} \) is a simple graph and \( H_1 \) does not contain \( K_3 \), by Lemma 2.2 there exists a graph \( G_2 = \overline{K_{n_1}} \vee \overline{K_{n_2}} \vee H_2 \) such that \( H_2 \) does not contain \( K_{t-1} \) and \( q_1(G_1) \leq q_2(G_2) \). Moreover, equality holds if and only if \( G_2 = G_1 \). If \( H_2 \) does not contain any edges, then \( H_2 = \overline{K_{n_3}} \). Hence \( G_2 = \overline{K_{n_1}} \vee \overline{K_{n_2}} \vee \overline{K_{n_3}} \) is complete 3-partite graph and \( q_1(G) \leq q_1(G_1) \leq q_1(G_2) = q_1(\overline{K_{n_1}} \vee \overline{K_{n_2}} \vee \overline{K_{n_3}}) \). If \( H_2 \) contains at least one edge, then by Lemma 2.2, there exists a graph \( G_3 \) such that \( G_3 = \overline{K_{n_1}} \vee \overline{K_{n_2}} \vee \overline{K_{n_3}} \vee H_3 \) and \( q_1(G_2) \leq q_1(G_3) \) with \( H_3 \) not containing \( K_{t-2} \), since \( H_2 \) does not contain \( K_{t-1} \) and \( \overline{K_{n_1}} \vee \overline{K_{n_2}} \) is a simple graph. By repeated use of Lemma 2.2 there exists a series of graphs \( G_1, \ldots, G_s \) such that
\( G_i = \overline{K_{n_1}} \triangledown \overline{K_{n_2}} \triangledown \cdots \triangledown \overline{K_{n_i}} \triangledown H_i \) and \( q_1(G) \leq q_1(G_1) \leq \cdots \leq q_1(G_i) \) with equality if and only if \( G_i = G \), where \( H_i \) does not contain \( K_{t+1} \) and \( i = 1, \cdots, s \leq t \). Moreover, \( H_s = \overline{K_{n_s}} \). Therefore, \( G_s \) is a complete \( s \)-partite graph. Further, by Lemma 2.3 and Corollary 2.4

\[
q_1(G) \leq q_1(G_1) \leq \cdots \leq q_1(G_s) \leq q_1(T_{n,s}) \leq q_1(T_{n,t}) = (3t - 4)k + 3r - 2 + \sqrt{t^2k^2 + [(2r + 4)t - 8r]k + (r - 2)^2} / 2
\]

with equality if and only if \( G = G_1 = \cdots = G_s = T_{n,s} = T_{n,t} \). This completes the proof. \( \blacksquare \)

**Remark:** From Theorem 1.3, we are able to deduce Turán theorem for \( t \geq 3 \).

**Corollary 2.5** Let \( G \) be a connected graph of order \( n \) not containing \( K_{t+1} \). If \( t \geq 3 \) and \( n = kt + r, 0 \leq r < t \), then

\[
|E(G)| \leq |E(T_{n,t})| = \frac{t^2 - t}{2}k^2 + (t - 1)rk + \frac{r(r - 1)}{2}
\]

Moreover, equality holds if and only if \( G = T_{n,t} \).

**Proof.** Clearly \( |E(T_{n,t})| = \frac{t^2 - t}{2}k^2 + (t - 1)rk + \frac{r(r - 1)}{2} \). By Rayleigh’s quotient, it is easy to see that \( q_1(T_{n,t}) \geq \frac{4|E(T_{n,t})|}{n} \). Then \( \frac{nq_1(T_{n,t})}{4} - |E(T_{n,t})| \geq 0 \) and

\[
\varepsilon \equiv \frac{nq_1(T_{n,t})}{4} - |E(T_{n,t})| = -\frac{1}{8} \left\{ t^2k^2 + [(2r + 2)t - 4r]k + r^2 - 2r - (tk + r) \sqrt{t^2k^2 + [(2r + 4)t - 8r]k + (r - 2)^2} \right\} \geq 0.
\]

On the other hand, let

\[
\varphi(x) \equiv 4x^2 + \left\{ t^2k^2 + [(2r + 2)t - 4r]k + r^2 - 2r \right\}x + (r - t)rk^2 + (r - t)rk.
\]

Then \( \varphi(x) = 0 \) has a root

\[
x_1 = \frac{1}{8} \left\{ t^2k^2 + [(2r + 2)t - 4r]k + r^2 - 2r - (tk + r) \sqrt{t^2k^2 + [(2r + 4)t - 8r]k + (r - 2)^2} \right\} = \varepsilon,
\]

i.e., \( \varepsilon \) is a nonnegative root of \( \varphi(x) = 0 \). Further, since

\[
\varphi(0) = (r - t)rk^2 + (r - t)rk \leq 0
\]
and
\[
\varphi(1) = (t^2 - rt + r^2)k^2 + [(r + 2)t + r^2 - 4r]k + (r - 1)^2 > 0,
\]
equation \(\varphi(x) = 0\) has only two roots: one is negative and the other is nonnegative which lies in [0, 1). Then \(0 \leq \varepsilon < 1\), i.e., \(|\varepsilon| = 0\), where \(|a|\) is the largest integer not greater than \(a\). Hence \(\left\lfloor \frac{\nu_1(T_{n,t})}{4} \right\rfloor = |E(T_{n,t})|\). Therefore, by Theorem 1.3,
\[
|E(G)| \leq \left\lfloor \frac{\nu_1(G)}{4} \right\rfloor \leq \left\lfloor \frac{\nu_1(T_{n,t})}{4} \right\rfloor = |E(T_{n,t})|.
\]
So the assertion holds. \(\blacksquare\)

The following result has been proved in [1] and [19].

**Corollary 2.6** ([1], [19]) Let \(G\) be a connected simple graph of order \(n\) with chromatic number \(\chi \geq 3\). If \(n = k\chi + r, 0 \leq r < \chi\), then
\[
q_1(G) \leq \frac{(3\chi - 4)k + 3r - 2 + \sqrt{k^2\chi^2 + [(2r + 4)\chi - 8r]k + (r - 2)^2}}{2}
\]
with equality if and only if \(G\) is \(T_{n,\chi}\).

**Proof.** The assertion follows from \(\omega(G) \leq \chi(G)\) and Theorem 1.3. \(\blacksquare\)

The following corollary gives some conditions under which Conjectures 1.1 and 1.2 hold, or need not hold.

**Corollary 2.7** Let \(G\) be a simple connected graph of order \(n \geq 10\) with clique number \(\omega\).

(i). If \(\omega \leq 4\) or \(\omega \geq \lceil \frac{n}{2} \rceil\), then \(q_1(G) \leq \frac{3n}{2} + \omega - 4\). Moreover, equality holds if and only if \(G\) is Turán graph \(T_{n,4}\) and \(n = 4k\) or Turán graph \(T_{n,k}\) and \(n = 2k\). In other words, Conjectures 1.1 and 1.2 hold.

(ii). If \(5 \leq \omega < \lceil \frac{n}{2} \rceil\), then \(q_1(T_{n,\omega}) > \frac{3n}{2} + \omega - 4\). In other words, Conjectures 1.1 and 1.2 generally do not hold.

(iii). \(q_1(G) \leq \frac{n}{2}\).

**Proof.** If \(\omega = 2\), then \(q_1(G) \leq n\) and \(q_1(G) \leq n \leq \frac{3n}{2} + 2 - 4\). So (i) holds.

If \(\omega = 3\) and \(n = 3k + r\) with \(0 \leq r < 3\), then by (4),
\[
q_1(G) \leq \frac{5k + 3r - 2 + \sqrt{9k^2 + (12 - 2r)k + (r - 2)^2}}{2}
\]
\[
\leq \frac{5k + 3r - 2 + 3k + 2}{2}
\]
\[
\leq \frac{9k + 3r - 2}{2} = \frac{3n}{2} + \omega - 4.
\]
So (i) holds.

If \( \omega = 4 \), by (4), we have

\[
q_1(G) \leq \frac{8k + 3r - 2 + \sqrt{16k^2 + 16k + (r - 2)^2}}{2} \leq \frac{8k + 3r - 2 + (4k + 2)}{2} = \frac{3n}{2} + \omega - 4
\]

with equality if and only if \( n = 4k \). Hence (i) holds.

Now assume that \( \omega \geq \lceil \frac{n}{2} \rceil \). If \( n = 2\omega \), then by (4),

\[
q_1(G) \leq \frac{2(3\omega - 4) - 2 + \sqrt{4\omega^2 + 8\omega + 4}}{2} = 4\omega - 4 = \frac{3n}{2} + \omega - 4.
\]

Moreover, equality if and only if \( n = 2\omega \). If \( n \neq 2\omega \), then \( n = \omega + r \) with \( r < \omega \) and

\[
q_1(G) \leq \frac{3\omega - 4 + (3r - 2) + \sqrt{\omega^2 + (2r + 4)\omega - 8r + (r - 2)^2}}{2} < \frac{3\omega - 4 + (3r - 2) + 2\omega - 2}{2} = \frac{3n}{2} + \omega - 4.
\]

Hence (i) holds.

(ii). Suppose that \( 5 \leq \omega < \lceil \frac{n}{2} \rceil \). Let \( n = k\omega + r \), \( 0 \leq r < \omega \). Then \( r \neq 0 \) or \( k > 2 \). Hence by Theorem 1.3 and some calculations, it is easy to verify that

\[
q_1(T_{n,\omega}) = \frac{(3\omega - 4)k + 3r - 2 + \sqrt{\omega^2k^2 + (2r + 4)\omega - 8r + (r - 2)^2}}{2} \geq \frac{(3\omega - 4)k + 3r - 2 + (k\omega + 2)}{2} = \frac{3n}{2} + \frac{(\omega - 4)k}{2} \geq \frac{3n}{2} + \omega - 4.
\]

Moreover, if \( q_1(T_{n,\omega}) = \frac{3n}{2} + \omega - 4 \), then \( k = 2 \) and \( r = 0 \) which is impossible. So (ii) holds.

(iii). If \( \omega = 1 \), then \( q_1(G) = 0 \). If \( \omega \geq 2 \), then by (3), \( q_1(G) \leq n = \frac{nw}{2} \). If \( \omega = 3 \), the assertion follows from (1). If \( \omega \geq 4 \), then by (3), \( q_1(G) \leq 2n - 2 < \frac{nw}{2} \). Hence (iii) holds. ■

**Remark:** In fact, Theorem 1.3 also confirms the following conjecture of Hansen and Lucas [7].

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Corollary 2.8 Let $G$ be a graph of order $n$ with clique number $\omega$. Then $q_1(G) \leq 2n(1 - \frac{1}{\omega})$. Moreover, if $\omega \neq 2$, the upper bound is sharp if and only if $G$ is a complete regular $\omega$-partite graph.

**Proof.** If $\omega = 1$, then $q_1(G) = 0$ and the assertion holds. Now assume that $\omega \geq 2$ and $n = k\omega + r$ with $0 \leq r < \omega$. Then $(r - 2)^2 \leq (r + 2 - \frac{4r}{\omega})^2$ with equality if and only if $r = 0$. Hence by (4),

\[
q_1(G) \leq \frac{(3\omega - 4)k + 3r - 2 + \sqrt{k^2\omega^2 + (2r + 4)\omega - 8r}k + (r - 2)^2}{2} \\
\leq \frac{(3\omega - 4)k + 3r - 2 + \sqrt{k^2\omega^2 + 2k\omega(r + 2 - \frac{4r}{\omega}) + (r + 2 - \frac{4r}{\omega})^2}}{2} \\
= \frac{(3\omega - 4)k + 3r - 2 + k\omega + r + 2 - \frac{4r}{\omega}}{2} \\
= \frac{4k\omega + 4r - 4k - \frac{4r}{\omega}}{2} \\
= 2n(1 - \frac{1}{\omega}).
\]

So the assertion holds. ■

### 3 Further Results

In this section, we begin with a lower bound for the signless Laplacian spectral radius of a graph in terms of clique number.

**Theorem 3.1** Let $G$ be a connected graph of order $n$ with clique number $\omega$.

(i). If $\omega = 2$, then $q_1(G) \geq 2 + 2\cos \frac{\pi}{n}$ with equality if and only if $G$ is a path of order $n$.

(ii). If $\omega \geq 3$, then

\[
q_1(G) \geq q_1(K_{i_{n,\omega}}),
\]

where $K_{i_{n,\omega}}$ is the kite graph of order $n$ which is obtained by joining one vertex of a complete graph $K_\omega$ to one end vertex of a path $P_{n-\omega}$ with a bridge. Moreover, equality holds if and only if $G$ is the kite $K_{i_{n,\omega}}$.

**Proof.** If $\omega = 2$, (i) follows from Lemma 1 in [8]. Now assume that $\omega \geq 3$. Since $q_1(G)$ is an strictly increasing function with respect to adding edges for a connected graph, there exists a connected graph $G_1$ of order $n$ obtained from $G$ by deleting
some edges such that \( \omega(G_1) = \omega \) and any proper subgraph of \( G_1 \) is disconnected or the clique number fewer than \( \omega \). Hence \( q_1(G) \geq q_1(G_1) \) with equality if and only if \( G_1 = G \). By repeated use of Theorem 2.2 in [3], \( q_1(G_1) \geq q_1(K_{\omega,n}) \) with equality if and only if \( G_1 = K_{\omega,n} \). Hence the assertion holds.

**Corollary 3.2** Let \( G \) be a connected graph of order \( n \) with clique number \( \omega \geq 3 \). Then

\[
q_1(G) \geq 2\omega - 1 + \frac{\sqrt{4\omega^2 - 12\omega + 17}}{2}
\]

with equality if and only if \( G = K_{n,n-1} \) and \( \omega = n-1 \).

**Proof.** By Theorem 3.1, \( q_1(G) \geq q_1(K_{\omega,n}) \). Since the line graph of \( K_{\omega+1,n} \) is the induced subgraph of the line graph of \( K_{\omega,n} \), it is easy to see that \( q_1(K_{\omega,n}) \geq q_1(K_{\omega+1,n}) \) with equality if and only if \( \omega = n-1 \). Hence by [8], we have

\[
q_1(K_{\omega,n}) \geq q_1(K_{\omega+1,n}) = \frac{2\omega - 1 + \sqrt{4\omega^2 - 12\omega + 17}}{2}.
\]

Hence the assertion holds.

**Remark:** Oliveira et.al. [16] presented several sharp upper bounds for the signless Laplacian spectral radius of a graph in terms of vertex degrees and 2-average degree. Let \( G \) be a graph with degree sequence \( d_1 \geq d_2 \geq \cdots \geq d_n \) and \( m_i = \sum_{v_j \notin E(G)} d_j \) for \( i = 1, \ldots, n \). Part results can be stated as follows:

\[
q_1(G) \leq \max_{1 \leq i \leq n} \{ d_i + \sqrt{d_i m_i} \} \tag{11}
\]

and

\[
q_1(G) \leq \max_{1 \leq i \leq n} \left\{ d_i + \sqrt{d_i^2 + 8d_i m_i} \right\}. \tag{12}
\]

Liu and Liu [9] gave an upper bound in terms of the largest degree and clique number, i.e.,

\[
q_1(G) \leq n + d_1 - \frac{n}{\omega(G)}. \tag{13}
\]

Yu et. al. [18] obtained the following upper bound in terms of degree sequence, i.e.,

\[
q_1(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_i + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\}. \tag{14}
\]

In general, these bounds (11), (12), (13), (14) are not comparable. We present two examples to illustrate that our bounds are best in some cases.
Example 3.3 Let $G_1$ be Turán graph $T_{10,3}$ of order 10 and $G_2$ be a graph of order 7 as follows:

![Graph](image)

Then we have following results

|       | $q_1(G)$ | $\{4\}$  | $\{1\}$  | $\{1\}$  | $\{1\}$  | $\{1\}$  |
|-------|----------|-----------|-----------|-----------|-----------|-----------|
| $T_{10,3}$ | 13.2915  | 13.2915  | 13.7082   | 13.6119   | 13.6667   | 13.5826   |
| $G_2$    | 8.7417   | 9.2749   | 9.5826    | 9.4462    | 9.6667    | 8.8284    |

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