Constructive Matrix Theory for Higher Order Interaction II: Hermitian and Real Symmetric Cases

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Abstract. This paper provides the constructive loop vertex expansion for stable matrix models with (single trace) interactions of arbitrarily high even order in the Hermitian and real symmetric cases. It relies on a new and simpler method which can also be applied in the previously treated complex case. We prove analyticity in the coupling constant of the free energy for such models in a domain uniform in the size $N$ of the matrix.

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1. Introduction

In this sequel to [1], we extend the analyticity results on $N$ by $N$ complex matrix models to the case of Hermitian or real symmetric matrices with a higher than quartic positive even interaction\(^1\). Such models are interesting for many areas of theoretical physics, in particular in the context of two-dimensional quantum gravity. Notice that Feynman graphs of Hermitian matrix models pave orientable Riemann surfaces of arbitrary genus and in the real symmetric case also pave non-orientable surfaces.

Since this paper is a sequel to [1], we refer to the latter introduction for further explanation on our program and motivation. But we would like to stress that the improved method introduced in this paper is both simpler and more powerful. The basic formalism is still the Loop Vertex Representation or LVR\(^2\) first introduced in [5], joined to Cauchy holomorphic matrix calculus as in [1]. But when [1] used contour integral parameters attached to every vertex

\(^1\)Presumably our method extends also to the ten Altland-Zirnbauer discrete symmetry classes [2].

\(^2\)This LVR representation is itself a generalization of the Loop Vertex Expansion [3]. The latter is well-adapted only to quartic interactions (see [4] for a recent review).
of the loop representation, this paper introduces more contour integrals, one for each loop vertex corner. This results in simpler bounds for the norm of the corner operators.

In the scalar case $N = 1$ [3], the corresponding analytic contour integrals and bounds reduce to some “poor man” particular case of the general theory of resurgent calculus of Jean Écalle and followers [6,7]. See in this respect [8] for a recent reference on scalar partition functions in zero dimension. However, the emphasis in this paper as in [1] is on obtaining uniform bounds for the matrix free energies as $N \to \infty$.

Let us in this respect also emphasize that the LVE expresses the partition function of a quantum field theory as a sum over a weighted combinatorial species of decorated forests, which has the same advantage than the traditional expansion in terms of Feynman graphs, namely its logarithm or free energy is given by the same sum but restricted to the corresponding connected species of trees, but has in addition the great advantage of being a convergent sum.

2. Hermitian Case

Let $d\mu(H)$ be the standard normalized GUE measure with iid covariance $1/N$ between matrix elements so that

$$d\mu(H) = \frac{1}{\pi^{N^2}} e^{-\frac{N}{2} \text{Tr} H^2} dH,$$

(2.1)

where $dH = \prod_i dH_{ii} \prod_{i<j} dH_{ij} d\bar{H}_{ij}$. We consider the Hermitian matrix model with stable interaction of order $2p$ with $p \geq 2$

$$S_0(\lambda, H) := \lambda^{p-1} \text{Tr} H^{2p},$$

(2.2)

where $\lambda$ is the coupling constant. Remark that the case $p = 2$ is much simpler than the general case $p \geq 3$, and has been first treated with the help of the intermediate field representation in [3]. The partition function and free energy of the model are given by

$$Z(\lambda, N) := \int d\mu(H) e^{-NS_0(\lambda, H)},$$

(2.3)

$$F(\lambda, N) := \frac{1}{N^2} \log Z(\lambda, N).$$

(2.4)

We perform the one-to-one change of variables (not singular for $\lambda$ real positive)

$$K := H \sqrt{1 + \lambda^{p-1} H^{2p-2}}, \quad K^2 = H^2 + \lambda^{p-1} H^{2p},$$

(2.5)

and put $T_p := \frac{H^2}{K^2}$. The corresponding Fuss–Catalan equation is [1]:

$$z T_p^p(z) - T_p(z) + 1 = 0,$$

(2.6)

with $z := -\lambda^{p-1} K^{2p-2}$. The change of variables inverts to $H(K) := K \sqrt{T_p(z)}$. We keep implicit that $H(K)$ also depends on $\lambda$ and also write simply $H$ for
$H(K)$ and so on when no confusion is expected. We also define the corresponding scalar functions

$$
\begin{align*}
  f_\lambda(u) &:= \sqrt{T_p(-\lambda^{p-1}u^{2p-2})}, \\
  h_\lambda(u) &:= uf_\lambda(u), \\
  k_\lambda(v) &:= v\sqrt{1+\lambda^{p-1}v^{2p-2}}.
\end{align*}
$$

(2.7)

They will be used below to express $H$ in terms of $K$ and $K$ in terms of $H$, as $h_\lambda$ and $k_\lambda$ are inverse of each other

$$
h_\lambda \circ k_\lambda(z) = k_\lambda \circ h_\lambda(z) = z
$$

(2.8)
in the cut complex plane which is the natural domain of the square root and Fuss–Catalan functions [3]. The Jacobian of the change of variables (2.5) produces a new non-polynomial interaction. According to [1] it writes

$$
\left| \frac{\delta H}{\delta K} \right| = \left| \det \frac{H \otimes 1 - 1 \otimes H}{K \otimes 1 - 1 \otimes K} \right|.
$$

(2.9)

In the following, we do not take the absolute of $\frac{\delta H}{\delta K}$, since it is positive for $\lambda > 0$ and can be extended to all other $\lambda$'s from the pacman domain by means of the analytical continuation. We prove the positivity of $\frac{\delta H}{\delta K}$ in Appendix B. Applying to (2.9) the trace-log formula of [1], we obtain the following expression for the partition function

$$
Z(\lambda, N) = \int d\mu(K) \exp\{S(\lambda, K)\},
$$

(2.10)

$$
S(\lambda, K) = \text{Tr}_\otimes \log \frac{\partial H}{\partial K} = \text{Tr}_\otimes \log \frac{H \otimes 1 - 1 \otimes H}{K \otimes 1 - 1 \otimes K}.
$$

(2.11)

The application of the LVE machinery goes along the same line as for complex matrices and allows one to express the free energy of the Hermitian matrix model as a sum over trees.

We first expand the partition function as

$$
Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu(K) \prod_{i=1}^{n} S(\lambda, K_i).
$$

(2.12)

Then we apply the BKAR formula [9,10] as in [1]. It replaces the covariance $C_{ij} = N^{-1}$ by $C_{ij}(x) = N^{-1}x_{ij}$ ($x_{ij} = x_{ji}$) evaluated at $x_{ij} = 1$ for $i \neq j$ and $C_{ii}(x) = N^{-1} \forall i$ and expands according to the BKAR forest Taylor formula. The result is a sum over the set $F_n$ of forests $\mathcal{F}$ on $n$ labeled vertices

$$
Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F} \in F_n} \int dw_{\mathcal{F}} \partial_{\mathcal{F}} Z_n \bigg|_{x_{ij} = x_{ij}^{\mathcal{F}}(w)}
$$

(2.13)

$$
\int dw_{\mathcal{F}} := \prod_{(i,j) \in \mathcal{F}} \int_{0}^{1} dw_{ij}, \quad \partial_{\mathcal{F}} := \prod_{(i,j) \in \mathcal{F}} \frac{\partial}{\partial x_{ij}},
$$

(2.14)

$$
Z_n := \int d\mu_C(x)\{K\} \prod_{i=1}^{n} S(\lambda, K_i)
$$

(2.15)
\[ x_{ij}^F := \begin{cases} \inf_{(k,l) \in P_{i \leftrightarrow j}^F} w_{kl} & \text{if } P_{i \leftrightarrow j}^F \text{ exists}, \\ 0 & \text{if } P_{i \leftrightarrow j}^F \text{ does not exist}. \end{cases} \quad (2.16) \]

In this formula, \( w_{ij} \) is the weakening parameter of the edge \((i, j)\) of the forest, and \( P_{i \leftrightarrow j}^F \) is the unique path in \( F \) joining \( i \) and \( j \) when it exists [9,10].

The differentiation with respect to \( x_{ij} \) in (2.14) results in
\[
\frac{\partial}{\partial x_{ij}} \left( \int d\mu_{C(x)}(\{K\}) f(K) \right) = \frac{1}{N} \int d\mu_{C(x)}(\{K\}) \text{Tr} \left[ \frac{\partial}{\partial K_i} \frac{\partial}{\partial K_j} \right] f(K). \quad (2.17) \]

The operator \( \text{Tr} \left[ \frac{\partial}{\partial K_i} \frac{\partial}{\partial K_j} \right] \) acts on two distinct loop vertices \((i \text{ and } j)\) and connects them by an edge. Introducing the condensed notations
\[
\partial^K_F = \prod_{(i,j) \in F} \text{Tr} \left[ \frac{\partial}{\partial K_i} \frac{\partial}{\partial K_j} \right], \quad S_n = \prod_{i=1}^{n} S(\lambda, K_i), \quad (2.18) \]
we obtain
\[
Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F} \in \mathcal{S}_n} N^{-|\mathcal{F}|} \int dw_{\mathcal{F}} \int d\mu_{C(x)}(\{K\}) \partial^K_F S_n \bigg|_{x_{ij} = x_{ij}^F(w)}. \quad (2.19) \]

As usual, since the right hand side of (2.19) is now factorized over the connected components of the forest \( F \), which are spanning trees, its logarithm, which selects only the connected parts, is expressed by exactly the same formula but summed over trees. For a tree on \( n \) vertices, we have \(|T| = n - 1\), and taking into account the \( N^{-2} \) factor in the normalization of \( F \) in (2.4), we obtain the expansion of the free energy as (remark the sum which starts now at \( n = 1 \) instead of \( n = 0 \))
\[
F(\lambda, N) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{T \in \mathcal{T}_n} A_T, \quad (2.20) \]
\[
A_T := N^{-n-1} \int dw_T \int d\mu_{C(x)}(\{K\}) \partial^K_T S_n \bigg|_{x_{ij} = x_{ij}^T(w)}, \quad (2.21) \]
where \( \mathcal{T}_n \) is the set of spanning trees over \( n \geq 1 \) labeled vertices.

From now on let us write \( O(1) \) for a generic constant (independent of \( N \)) which however may depend on \( \epsilon \). Our main result for the Hermitian matrix model is given by the following theorem, similar to the complex case of [1]

**Theorem 2.1.** For any \( \epsilon > 0 \), there exists \( \eta > 0 \) small enough such that the expansion (2.20) is absolutely convergent and defines an analytic function of \( \lambda \), uniformly bounded in \( N \), in the uniform in \( N \) “pacman domain”
\[
P(\epsilon, \eta) := \left\{ 0 < |\lambda| < \eta, |\arg \lambda| < \frac{\pi}{2} + \frac{\pi}{p - 1} - \epsilon \right\}. \quad (2.22) \]
More precisely, for fixed $\epsilon$ and $\eta$ as above there exists a constant $O(1)$, independent of $N$ such that for $\lambda \in P(\epsilon, \eta)$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{T \in \mathcal{T}_n} |A_T| \leq O(1) < \infty.$$  

(2.23)

Before we prove this result in the following section, let us make some comments.

First, we actually prove analyticity in sectors $D_\alpha$

$$D_\alpha = \left\{ 2\alpha - \frac{\pi}{p-1} < \arg \lambda < 2\alpha + \frac{\pi}{p-1}, |z| < \eta \right\}$$  

(2.24)

for each $\alpha$ such that $-\pi/4 < \alpha < \pi/4$, by rotations in the contour integration over the matrices $K_i$ by an angle $-\alpha$. This results in a convergent sum over trees of analytic functions of $\lambda \in D_\alpha$. These functions are analytic continuations of the functions defined in sectors containing the positive axis so that they define an analytic function on the whole pacman domain for $p > 2$. For $p = 2$, the pacman domain covers the negative axis which defines a branch cut.

Some further remarks on the Borel summability of the series defined by (2.20) are in order. Let us put $\lambda_p = \lambda^{p-1}$, so that $S_0(\lambda, H)$ is redefined as $S_0'(\lambda_p, H) := \lambda_p \text{Tr} H^{2p}$. The partition function is naturally defined as a perturbative series in $\lambda_p$, since it rewrites as $Z'(\lambda_p, N) := \int d\mu(H) e^{-NS_0'(\lambda_p, H)}$.

For $N = 1$, the Taylor series around the origin is

$$Z'(\lambda_p, 1) = a_n(-\lambda_p)^n, a_n = \frac{(2np)!}{2^{np}(np)!n!} \sim O(1)n^{n(p-1)}, n \geq 0,$$  

(2.25)
and the growth of the Taylor coefficients at the origin indicates that this series can be at best Borel-Leroy summable of order $p - 1$ in $\lambda_p$, e.g., $Z(\lambda, 1)$ can be at best Borel summable with respect to $\lambda$.

In fact we conjecture that $F(\lambda, N)$ is indeed Borel summable in $\lambda$ uniformly in $N$. We prove in this current paper the analyticity part of that conjecture since a pacman domain contains certainly a disk tangent to the imaginary axis. By the usual road of the Nevanlinna-Sokal criterion, to prove Borel summability just now requires some Taylor remainder estimate of the type $O(1)|\lambda|^q$ for the Taylor remainder at order $q$ in a disk tangent to the imaginary axis. We think that the estimates of the resolvent of section 6.2 page 115 of [11] can be adapted, but with some effort, to this task of bounding our resolvent in this paper, e.g., $[1 \otimes (1 + \Sigma_\lambda(K))]^{-1}$ in (3.2).

3. Proof of Theorem 2.1

We need first to compute $\partial T K S_n$ assuming $n \geq 2$ (as usual the special case $n = 1$ requires an additional integration by parts). Since trees have arbitrary coordination numbers, we need a formula for the action on a vertex factor $S(\lambda, K_i)$ of a certain number $r_i \geq 1$ of derivatives $\frac{\partial}{\partial K_i}$ with $\sum_i r_i = 2n - 2$.

Let us fix a given loop vertex and forget for a moment to write the vertex index $i$. We need to develop a formula for the action of a product of $r_\sqcup$ symbols as in [1] to indicate the $r_\sqcup$ pairs of external indices of the $r_\sqcup$ derivatives. The final tree amplitude will be obtained later by gluing these $\sqcup$ symbols along the edges of the trees.

The first $\frac{\partial}{\partial K}$ derivative is a bit special as it destroys forever the logarithm in $S$ and gives

$$\left[ \frac{\partial}{\partial K} \right] \text{Tr} \otimes \log [1 \otimes \Sigma] = [1 \otimes \Sigma]^{-1} \frac{\partial \Sigma}{\partial K}. \quad (3.1)$$

We can use holomorphic functional matrix calculus as in [1] to write

$$\frac{K \otimes 1 - 1 \otimes K}{H \otimes 1 - 1 \otimes H}, = [1 \otimes \Sigma_\lambda(K)]^{-1} = \frac{\delta K}{\delta H} \quad (3.2)$$

where we have defined

$$\Sigma_\lambda(K) := \oint_{\Gamma} du \left[ h_\lambda(u) - u \right] \frac{1}{u - K} \otimes \frac{1}{u - K} \quad (3.3)$$

with a contour $\Gamma$ enclosing the spectrum of $K$. This spectrum lies on the real axis so the contour has to enclose this real axis, avoiding any singularity of the function $h_\lambda$.

Recall that the function $h_\lambda$ is defined by

$$h_\lambda(u) = \sqrt{\frac{T_p(-\lambda p - 1 u^{2p - 2})}{u - K} \otimes \frac{1}{u - K}}. \quad (3.4)$$
where the Fuss–Catalan function is analytic on $\mathbb{C}$ except for a cut on $[C_p, +\infty]$, with $C_p := \frac{(p-1)}{p^{p/(p-1)}}$. Therefore for arbitrary complex $\lambda$, $h_\lambda(u)$ is analytic on the complex plane with $2p - 2$ cuts at

$$|u| \geq \sqrt{\frac{C_p}{|\lambda|}}, \quad \text{arg} \, u = -\frac{\text{arg} \, \lambda}{2} + \frac{\pi}{2(p-1)} + \frac{k\pi}{p-1}$$

(3.5)

with $k = -p + 1, -p + 2, \ldots, -1, 0, 1, \ldots, p - 2$.

There are plenty of possible choices for $\Gamma$ to avoid the cuts, but a key condition is that the spectrum of $z = [\lambda K^2]$ stays within the region that avoid each one of the $2p - 2$ cuts. This condition is implied, for instance, if our contour stays within

$$\left\{ u \in \mathbb{C} \text{ such that } |u| \leq \frac{1}{2} \sqrt{C_p/\eta} \text{ or } \frac{-\pi}{2p - 2} + \epsilon \leq \text{arg} \, u \leq \frac{\pi}{2p - 2} - \epsilon \right\},$$

(3.6)

for some $\epsilon > 0$. One of the simplest contour satisfying the condition (3.6), inspired by [1], is to choose, for $\lambda > 0$, for $\Gamma$ the finite symmetric “keyhole” $\Gamma_{R,r,\psi}$ parametrized by $R, r$ and $\psi$ as in Fig. 2, with $R$ large (e.g. larger than $2\|K\|$), $r = \frac{1}{2} \sqrt{C_p/\eta}$ and $\psi$ small as $\epsilon \to 0$ (e.g., $\psi = \frac{\epsilon}{4}$).

The singularities of $h_\lambda(u)$ imply that for hermitian $K$ the function $\Sigma_\lambda(K)$ is only defined for $|\text{arg} \, \lambda| < \frac{\pi}{p-1}$. To remedy this situation, we simultaneous rotate the contour $\Gamma$ and the integration contour in the matrices $K$ by an angle $-\alpha$, see Fig. 3, thus replacing $u \to e^{-i\alpha}u$ and $K \to e^{-i\alpha}K$. The function we
integrate over $u$ being analytic with suitable decrease at infinity, both contours yield the same integral. The dependence on $e^{-i\alpha}$ drops, except in $\lambda$ and $K$,

$$
\oint_{e^{-i\alpha} \Gamma} du \left[ h_{\lambda}(u) - u \right] \frac{1}{u - e^{-i\alpha} K} \otimes \frac{1}{u - e^{-i\alpha} K}
= \oint_{\Gamma} du \left[ h_{e^{-2i\alpha}\lambda}(u) - u \right] \frac{1}{u - K} \otimes \frac{1}{u - K}. \tag{3.7}
$$

Moreover, the matrix integral over the space of Hermitian matrices $H_N$ in (2.21) is unaffected by the contour integration since it only involves analytic
functions, considering the \( N^2 \) real entries of \( K \) as independent complex variables. Therefore, for any matrix \( K \) and analytic function \( F \) with suitable behavior at infinity,

\[
\int_{H_N} d\mu_C(K) F(K) = \int_{e^{-i\alpha} H_N} d\mu_C(K) F(K) = \int_{H_N} d\mu_{e^{2i\alpha} C}(K) F(e^{-i\alpha} K)
\]

which is well defined provided the real part of \( C \) remains positive definite, i.e., \( \frac{\pi}{4} < \alpha < \frac{\pi}{4} \). Additionally, there is a factor of \( e^{2(n-1)i\alpha} \) for each tree amplitude with \( n \) vertices since it comes with \( n - 1 \) covariances. Altogether, equation (2.21) is modified into

\[
A_{T,\alpha}(\lambda) := e^{2(n-1)i\alpha} N^{-n-1} \int dw_T \int d\mu_{C(x)}(\{K\}) \partial_T^K S_{n,\alpha} \big|_{x_{ij}=x_T^w(u)},
\]

where the effective action \( S_{n,\alpha} \) involves the coupling \( e^{-2i\alpha} \lambda \).

By construction, \( A_{T,\alpha}(\lambda) \) is an analytic function of \( \lambda \) in the sector \( D_\alpha \). Moreover, on a connected intersection of two sectors \( D_\alpha \cap D_\beta \) where \( A_{T,\alpha}(\lambda) \) and \( A_{T,\beta}(\lambda) \) are both defined, they agree since one can pass from one to the other by a change of integration variable. This intersection is always connected, except possibly for \( p = 2 \) when it covers the negative axis, in which case the function is multivalued. With this in mind, for any \( \epsilon > 0 \) it defines a single analytic function on the pacman domain

\[
\bigcup_{-\frac{\pi}{4} + \frac{\epsilon}{2} < \alpha < \frac{\pi}{4} - \frac{\epsilon}{2}} D_\alpha = \left\{ -\frac{\pi}{2} - \frac{\pi}{p-1} + \epsilon < \arg \lambda < \frac{\pi}{2} + \frac{\pi}{p-1} - \epsilon \ | |\lambda| < \eta \right\} = P(\epsilon, \eta),
\]

with \( \eta \) small enough so that the contour \( \Gamma \) can go around the origin in the \( u \) plane without meeting a branch point of \( h_{e^{-i\alpha} \lambda} \).

In the remaining part of this section, we establish bounds on \( A_{T,\alpha}(\lambda) \) that will ensure the convergence of the tree expansion, thus proving the analyticity in a pacman domain, adapting the value of \( \eta' \). In order to do so, we have to bound the integrand in (2.21), where the rotation angle only enters through \( h_{e^{-2i\alpha} \lambda} \). Therefore, it is sufficient to bound \( A_T(\lambda, \alpha = 0) = A_T(\lambda) \) for \( |\arg \lambda| < \frac{\pi}{p-1} - \epsilon \), the result extending immediately to \( A_{T,\alpha}(\lambda) \) for \( |\arg \lambda - 2\alpha| < \frac{\pi}{p-1} - \epsilon \).

Noticing that \( g_\lambda(u) = [h_\lambda(u) - u] \) vanishes at \( \lambda = 0 \), it can be written as

\[
g_\lambda(u) = \int_0^\lambda dt \partial_t g_t(u) = -\frac{p-1}{2} \int_0^\lambda dt \ t^{p-2} u^{2p-1} e_t(u) f_t(u),
\]

where we define \( e_t(u) := \frac{T_p^\epsilon}{T_p} (-t^{p-1} u^{2p-2}) \).

From now on we use \( O(1) \) as a generic name for any inessential numerical constant (which may depend on the parameters \( \epsilon \) and \( \eta \) of our fixed pacman domain).

**Lemma 3.1.** On the contour \( \Gamma \), we have the bound

\[
|g(\lambda)| \leq O(1)|\lambda|^{\frac{p-1}{4p^2}} |u|^{1 + \frac{2p}{2p-1} - \frac{1}{2p^2}}.
\]
Proof. We can use the rather standard estimates on \( T_p \) and \( E_p(z) = \frac{T'_{p}}{T_p}(z) \) proven in section III of [5] (see Lemma III.1). In particular it is proven there that in a domain avoiding a small angular opening \( \epsilon \) around the cut of \( T_p \) we have

\[
T_p(z) \leq (1 + |z|)^{-\frac{1}{p}}, \quad E_p(z) \leq \frac{O(1)}{(1 + |z|)}. \tag{3.14}
\]

In our case, this means that on our contour \( \Gamma \), for any \( 0 < \delta < 1 \) there is a constant \( C_\delta \) such that

\[
|e_t(u)f_t(u)| \leq \frac{C_\delta}{(1 + |u|^{2p-2})^{1+\frac{1}{2p}(1-\delta)}}, \tag{3.15}
\]

Choosing \( \delta = \frac{1}{2p} \) give

\[
|e_t(u)f_t(u)| \leq \frac{C_\delta}{(1 + |u|^{2p-2})^{1+\frac{1}{4p^2}}}. \tag{3.16}
\]

Taking into account, (3.12) give

\[
|g_\lambda(u)| \leq (p-1)C_\delta \int_0^{\lambda} \frac{t^{p-2}u^{2p-1}}{(1 + |u|^{2p-2})^{1+\frac{1}{4p^2}}} dt, \tag{3.17}
\]

\[
\leq O(1)|u|^{1+\frac{1}{2p} - \frac{1}{4p^2}} \int_0^\lambda t^{\frac{p-1}{4p^2}} \frac{dt}{t}, \tag{3.18}
\]

This gives (3.13). □

From now on and when there is no risk of ambiguity, we write simply \( \Sigma \) for \( \Sigma_\lambda(K) \). It remains to compute \( \frac{\partial \Sigma}{\partial K} \) in (3.1). The \( \partial \) derivative can act on the left or right side of the tensor product so that

\[
\frac{\partial \Sigma}{\partial K} = \oint_{\Gamma} du g_\lambda(u) \left[ \frac{1}{u-K} \boxplus \frac{1}{u-K} \otimes \frac{1}{u-K} \right] + \frac{1}{u-K} \otimes \frac{1}{u-K} \boxplus \frac{1}{u-K}. \tag{3.19}
\]

Then, the next derivatives iterate in a similar pattern. Each \( \frac{\partial \Sigma}{\partial K} \) derivative

- either derives a \( [1 + \Sigma]^{-1} \) factor and creates a new \( \frac{\partial \Sigma}{\partial K} \) through the resolvent formula (easily checked algebraically)

\[
\frac{\partial}{\partial K} [1 + \Sigma]^{-1} = -[1 + \Sigma]^{-1} \frac{\partial \Sigma}{\partial K} [1 + \Sigma]^{-1}. \tag{3.20}
\]

To this \( \frac{\partial \Sigma}{\partial K} \) is associated a new integration contour through (3.19).

- or derives again an existing \( \Sigma \). In this case, it results in no new contour but in a new \( \boxplus \) and a multiplication by a new factor \( \frac{1}{u-K} \otimes 1 + 1 \otimes \frac{1}{u-K} \).

The combinatorics to sum over the choices is the usual one relying on the Faà di Bruno formula. Since it is similar to the one explained in detail of the complex case [1], we would not discuss it further here. The result for a loop vertex of degree \( r \), hence with \( r \) “corners” \( c \) between half edges, is a sum over sequences of \( 1 \leq m \leq r \) contour-corner operators \( O^c \) and \( r - m \)
derivative-corner operators $\hat{O}^c$. Each such operator is sandwiched between two \ $\sqcup$ insertions. The tree amplitude $A_T$, as also detailed in [1], is obtained by identifying the two ends of each pair of \ $\sqcup$ symbols along each edge of $T$. This pairing of the $2n - 2 \ \sqcup$ symbols then exactly glue the $2n$ traces of the tensor products present in the $n$ vertices into $n + 1$ traces.

However, we have not yet given the exact formula for the contour-corner operators $O^c$ and the derivative-corner operators $\hat{O}^c$. The derivative-corner operators are simply defined as

$$\hat{O}^c(u_{k_c}) := \frac{1}{u_{k_c} - K} \otimes 1 + 1 \otimes \frac{1}{u_{k_c} - K}. \quad (3.21)$$

where $k_c \in [1, m]$ is the index of its “parent” contour-corner. For contour-corner operators the formula is more interesting and contains a subtlety, pictured in Fig. 4. To each contour corner $c_k, k \in [1, m]$ is associated a contour integral $\int du_k g_{\lambda}(u_k)[1 + \Sigma]^{-1}$ factor. But since the two $\frac{1}{u_k - K}$ operators of the same side of the $\otimes$ tensor product in (3.19) are separated by a $\sqcup$ they do not belong to the same corner. Therefore we must attribute one of them to the next corner, in a cyclic way around the vertex. As a consequence, the formula for a contour-corner operator $O^{c_k}, k \in [1, m]$ contains both $u_k$ and $u_{k+1}$ (with the cyclic convention $u_{m+1} := u_1$). Taking out in front of the loop vertex the global contour integral $\prod_{k=1}^{m} \oint_{\Gamma} du_k g_{\lambda}(u_k)$, we define the $k$-th contour-corner operator therefore as

$$O^{c_k}(u_k, u_{k+1}) := [1 \otimes + \Sigma]^{-1} \left[ \frac{1}{u_k - K} \frac{1}{u_{k+1} - K} \otimes \frac{1}{u_k - K} \right.$$ \left. + \frac{1}{u_k - K} \otimes \frac{1}{u_k - K} \frac{1}{u_{k+1} - K} \right]. \quad (3.22)$$

Remark that this operator contains therefore both $u_k$ and $u_{k+1}$.

To bound the amplitude $A_T$, we first have to bound the corner operators. The bound on a derivative-corner operator is rather trivial. Since the contour $\Gamma$ is never closer than $r \sin \psi$ to the spectrum of $K$ (see Fig. 2), we have

$$\|\hat{O}^c(u_{k_c})\| \leq 2(r \sin \psi)^{-1}. \quad (3.23)$$

For the contour-corner operators, the bound is more delicate. We remark that all corner operators of a given loop vertex commute as they involve to the same replica field $K$. In fact they are diagonalized by the tensor basis $e_i \otimes e_j$ where $e_i$ is the basis diagonalizing $K$. Let us call $\mu_i$ the eigenvalue of $K$ on $e_i$.

The operator $O^{c_k}(u_k, u_{k+1})$ is diagonal on the basis $e_i \otimes e_j$, with eigenvalues

$$O^{c_k}_{ij}(u_k, u_{k+1}) = [1 \otimes + \Sigma]^{-1}_{ij} \left[ \frac{1}{u_k - \mu_i} \frac{1}{u_{k+1} - \mu_i} \otimes \frac{1}{u_k - \mu_j} + \frac{1}{u_k - \mu_i} \otimes \frac{1}{u_k - \mu_j} \frac{1}{u_{k+1} - \mu_j} \right]. \quad (3.24)$$
Figure 4. A vertex with some of its corner operators. The label \( k \) indicates the corresponding contour variable. The upper left corner between the two half-edges \( \sqcup \) symbols contains three \( (u-K)^{-1} \) operators with indices \( k \), \( k \) and \( k+1 \).

**Lemma 3.2.** For complex \( \lambda \) such that \(|\arg \lambda| < \frac{\pi}{p-1} - \epsilon\) there exists some constant \( O(1) \) such that

\[
|1 \otimes (1 + \Sigma)^{-1} | \leq O(1) \Lambda_{ij} \tag{3.25}
\]

\[
\Lambda_{ij} := \sup \{ 1, |\lambda|^{\frac{p-1}{2p}}, |\mu_i|^{1 - \frac{1}{p}}, |\lambda|^{\frac{p-1}{2p}}, |\mu_j|^{1 - \frac{1}{p}} \}. \tag{3.26}
\]

**Proof.** Calling \( \nu_i = h_\lambda(\mu_i) \), (3.2) means that

\[
(1 \otimes (1 + \Sigma)^{-1} = \frac{k_\lambda(\nu_i) - k_\lambda(\nu_j)}{\nu_i - \nu_j}. \tag{3.27}
\]

Hence it is bounded by \( \sup_{\nu \in [\nu_i, \nu_j]} |k_\lambda'(\nu)| \) where the sup is taken along the \([\nu_i, \nu_j]\) segment in the complex plane. \( k_\lambda' \) can be explicitly computed (see (2.7)) and from the large \( z \) behavior of the function \( T(z) \sim z^{-1/p} \) derived from its functional equation (2.6) the bound follows easily on the pacman domain. \( \square \)

The next step is to compensate the growth of this bound as \( \mu_i \) or \( \mu_j \) becomes large with the decay hidden in the \( \frac{1}{u-\mu} \) factors. The contour \( \Gamma \) in Fig. 2 has been chosen so that everywhere along the contour

\[
\left| \frac{1}{u-\mu} \right| \leq O(1) \inf \left( \frac{1}{1+|u|}, \frac{1}{1+|\mu|} \right), \tag{3.28}
\]

for some other constant \( O(1) \).
Lemma 3.3. For complex $\lambda$ such that $|\arg \lambda| < \frac{\pi}{p-1} - \epsilon$

$$\|O^e(u_k, u_{k+1})\| \leq O(1) \frac{1}{(1 + |u_k|)^{1+\frac{1}{p}}} \frac{1}{1 + |u_{k+1}|}. \quad (3.29)$$

Proof. Suppose $eg \Lambda_{ij} = |\lambda|^{\frac{p-1}{2p}} |\mu_i|^{\frac{1}{p}}$. Using (3.28), we bound the $\frac{1}{u_k - \mu_i}$ factor in (3.24) as

$$\left| \frac{1}{u_k - \mu_i} \right| \leq \left[ \frac{1}{1 + |u_k|} \right]^{1/p} \left[ \frac{1}{1 + |\mu_i|} \right]^{-1/p}. \quad (3.30)$$

Combining with (3.25) leads to

$$|O^e_{ij}(u_k, u_{k+1})| \leq O(1) \left( \frac{1}{1 + |u_k|} \right)^{1+1/p} \frac{1}{1 + |u_{k+1}|}. \quad (3.31)$$

The other cases $\Lambda_{ij} = |\lambda|^{\frac{p-1}{2p}} |\mu_j|^{\frac{1}{p}}$ or $\Lambda_{ij} = 1$ are obviously similar.

Since the bound (3.31) is independent of $i$ and $j$, it implies (3.29). \qed 

Still keeping the integral over the contour parameters for later we now glue the $\sqcup$ operators and perform all traces. We obtain

Lemma 3.4. There exists some constant $O(1)$ such that

$$|A_T| \leq [O(1)]^n \prod_v \prod_{k=1}^{m(v)} \int_{\Gamma} |g_{\lambda}(u_k)| \left[ \frac{1}{1 + |u_k|} \right]^{2+\frac{1}{p}} du_k. \quad (3.32)$$

Proof. We bound recursively all tree traces. The simplest way to understand how it works is to start from a leaf $f$, which has $r = m = 1$. The associated operator is therefore a single contour-corner operator $O^c$ whose norm, by (3.29), is bounded by $O(1)\left( \frac{1}{1 + |u_k|} \right)^{2+\frac{1}{p}}$. The amplitude for $A_T$ contains a partial trace on one $H$ factor of the tensor product $H \otimes H$ of the leaf vertex, leading to a simpler operator on $H$ only, with norm bounded by $NO(1)\left( \frac{1}{1 + |u_k|} \right)^{2+\frac{1}{p}}$. After gluing this factor between the two appropriate corners in the parent vertex $v(f)$, we can find a new leaf and iterate. This leads to the bound. Indeed this induction collects exactly $n + 1$ factors $N$ (since the last vertex of the tree brings two such factors). This exactly compensates with the $N^{-n-1}$ factor in (2.21). Finally, the $\int dw_T \int d\mu_C(x)\{K\}$ integrals are normalized so do not add anything to the bounds. \qed

To complete the bound on $A_T$, it remains only to perform all contour integrals. From the choice of our contour

Lemma 3.5. There exists some constant $O(1)$ such that

$$\int_{\Gamma} |g_{\lambda}(u)| \left[ \frac{1}{1 + |u|} \right]^{2+\frac{1}{p}} du \leq O(1)|\lambda|^{rac{p-1}{2p^2}}. \quad (3.33)$$

Proof. Inserting (3.13) proves (3.33) since the integral $\int_{\Gamma} |u|^{1+\frac{1}{p}} |\lambda|^{rac{p-1}{2p^2}} du$ is absolutely convergent and bounded by a constant at fixed $p$. \qed
Finally since each vertex has at least one contour operator, the number of \(|\lambda|^{p-1}\) factors in the bound is at least \(n\). Taking into account that the number of (labeled) trees is bounded by \(K^n n!\) for some constant \(K\) completes the proof of (2.23), hence of Theorem 2.1.

4. The (not-so-)trivial \(n = 1\) Tree

This section is devoted to establish a not-so trivial bound on the trivial tree amplitude with a single vertex, namely

\[
A_{T_0} = N^{-2} \int d\mu \ S(\lambda, K), \quad S(\lambda, K) = \text{Tr}_\otimes \log \frac{\partial H}{\partial K}. \tag{4.1}
\]

More precisely, it is devoted to prove

**Lemma 4.1.** We have

\[
|A_{T_0}| \leq O(1)|\lambda|^{1/4}. \tag{4.2}
\]

**Proof.** Let us rewrite the Jacobian matrix as

\[
\frac{\partial H}{\partial K} = 1_\otimes + \Sigma_\lambda(K) = 1_\otimes f_\lambda(K) + K \frac{\partial f_\lambda(K)}{\partial K} \tag{4.3}
\]

\[
= 1_\otimes f_\lambda(K) + \oint_{\Gamma} du f_\lambda(u) \frac{K}{u - K} \otimes \frac{1}{u - K}, \tag{4.4}
\]

so that \(A_{T_0} = A_1 + A_2\) with

\[
A_1 := \frac{1}{2N} \int d\mu \ \text{Tr} \log T_p(-\lambda^{p-1} K^{2p-2}) \tag{4.5}
\]

\[
= -\frac{p - 1}{2N} \int d\mu \ \text{Tr} \int_0^\lambda dt \oint_{\Gamma} u^{2p-3t^{p-2}} e_t(u) \frac{K}{u - K} du, \tag{4.6}
\]

and

\[
A_2 := N^{-2} \int d\mu \ \text{Tr}_\otimes \log [1_\otimes + V_\lambda(K)], \tag{4.7}
\]

\[
1_\otimes + V_\lambda(K) = [1_\otimes f_\lambda(K)]^{-1} [1_\otimes + \Sigma_\lambda(K)]. \tag{4.8}
\]

We can write \(V_\lambda(K)\) as a double contour integral

\[
V_\lambda(K) := \oint_{\Gamma} du \oint_{\Gamma'} dv \phi(\lambda, u, v) \frac{K}{u - K} \otimes \frac{1}{v - K}, \tag{4.9}
\]

\[
\phi(\lambda, u, v) := \frac{1}{u - v} f_\lambda(u), \tag{4.10}
\]

where \(\Gamma'\) is another keyhole contour similar to \(\Gamma\) surrounding the spectrum of \(K\) but inside \(\Gamma\) and with half its opening angle.

The \(A_1\) part is easy to bound and to prove in addition that it tends to zero as \(\lambda \to 0\). We simply integrate by parts the \(K\) numerator in (4.6) to get

\[
A_1 = -\frac{p - 1}{2N^2} \int d\mu \ \text{Tr}_\otimes \int_0^\lambda dt \oint_{\Gamma} du u^{2p-3t^{p-2}} e_t(u) \frac{1}{u - K} \otimes \frac{1}{u - K}. \tag{4.11}
\]
and recalling (3.14) one can use $|e_t(u)| \leq |t|^{p-1}|u|^{2p-2}^{-1+2p-2}$ to conclude easily that $A_1$ is $O(1)|\lambda|^{1/2}$.

Turning to $A_2$, we remark that $V$ vanishes at $\lambda = 0$, hence we can rewrite it as $\int_0^\lambda dt \partial_t V_t$ with

$$\partial_t V_t(K) = \oint_{\Gamma} \int_{\Gamma'} du \int_{\Gamma'} dv \partial_t \phi(t, u, v) \frac{K}{u - K} \otimes \frac{1}{v - K}, \quad (4.12)$$

with $\partial_t \phi$ easily computed as

$$\partial_t \phi(t, u, v) = \frac{p}{2(u - v)} f_t(u) \left[ v^{2p-2} t^{p-2} e_t(v) - u^{2p-2} t^{p-2} e_t(u) \right], \quad (4.13)$$

so that

$$A_2 := N^{-2} \int d\mu \int_0^\lambda dt \oint_{\Gamma} \int_{\Gamma'} du \int_{\Gamma'} dv \partial_t \phi(t, u, v) \quad (4.14)$$

$$\text{Tr}_\otimes \left[ 1 \otimes + V(K) \right]^{-1} \frac{K}{u - K} \otimes \frac{1}{v - K} \quad (4.15)$$

On the contours, we can easily bound $f_t(u)/f_t(v)$ by $1 + |v|^{1-\frac{1}{2p}}$, hence $1/|u-v| f_t(u)$ by $O(1)[1 + |u| + |v|]^{-\frac{1}{2p}}$, $|u|^{2p-2} e_t(u)$ by $|t|^{-1+\frac{1}{2p}} |u|^{\frac{1}{2p}}$ and similarly $|v|^{2p-2} e_t(v)$ by $|t|^{-1+\frac{1}{2p}} |v|^{\frac{1}{2p}}$, so that finally

$$|\partial_t \phi(t, u, v)| \leq O(1) \frac{|t|^{-1+\frac{1}{2p}}}{[1 + |u| + |v|]^{\frac{1}{2p}}}. \quad (4.16)$$

Then, we integrate by part the $K$ numerator in (4.12). We get five terms, two of which are “triple trace” and three of which “single trace.” Since we remain in the commutative algebra generated by $K$, we can diagonalize all tensor products and compute all traces. We define the resolvent $R_t(K) := \left[ 1 \otimes + V_t(K) \right]^{-1}$ and write simply $V$ for $V_t(K)$, $R$ for $R_t(K)$ and so on. Remember that from (4.8) we have

$$\left[ 1 \otimes + V \right]^{-1} = \left[ 1 \otimes f \right] \left[ 1 \otimes + \Sigma \right]^{-1}. \quad (4.17)$$

$R$ is diagonal on the basis $e_i \otimes e_j$ with eigenvalue $R_{ij}$. Since $1 \otimes f$ is also diagonal on that basis $e_i \otimes e_j$, with eigenvalue decaying as $|t|^{p-1} |\mu|^{2p-2}^{-1/2p}$, from (3.25)–(3.26) we get easily

$$|R_{ij}| \leq O(1) \sup \left\{ 1, |t|^{p-1} |\mu|^{1-\frac{1}{2p}} \right\} \quad (4.18)$$

We define $R_{\text{diag}}$ and $(1 \otimes \Sigma)^{-1}_{\text{diag}}$ as the diagonal “single thread” $N$ by $N$ matrix with eigenvalue $R_{\text{diag}} := R_{ii}$ or $(1 \otimes \Sigma)^{-1}_{ii}$ on $e_i$, and perform a careful analysis of the tensor threads involved, hopefully helped by Fig. 5. It gives

$$A_2 = N^{-3} \int d\mu \int_0^\lambda dt \oint_{\Gamma} \int_{\Gamma'} du \int_{\Gamma'} dv \partial_t \phi(t, u, v) \left[ B_1 + B_2 + B_3 + B_4 + B_5 \right], \quad (4.19)$$
Figure 5. The five terms $B_1$, $B_2$, $B_3$, $B_4$ and $B_5$. The arrow indicates the action of the $\partial_K$ matrix derivative

where the first two terms are obtained when $\partial_K$ hits $\left[\frac{1}{u-K} \otimes \frac{1}{v-K}\right]$, giving

$$B_1 = \text{Tr}_{\otimes 3} \left[ [R] \otimes 1 \right] \left[ \frac{1}{u-K} \otimes \frac{1}{v-K} \otimes \frac{1}{u-K} \right],$$

$$B_2 = \text{Tr} \ R_{\text{diag}} \frac{1}{(u-K)(v-K)^2},$$

The last three terms $B_3$, $B_4$ and $B_5$ come from $\partial_K$ hitting $R$

$$\partial_K R = \partial_K (1 \otimes f)(1 + \Sigma)^{-1}$$

$$= -R [\partial_K \Sigma] (1 + \Sigma)^{-1} + [1 \otimes \partial_K f](1 + \Sigma)^{-1}. \quad (4.23)$$

We can recompute $\partial_K \Sigma$, as

$$\frac{\partial \Sigma}{\partial K} = \oint \Gamma dw \ w f_t(w) \left[ \frac{1}{w-K} \otimes \frac{1}{w-K} \otimes \frac{1}{w-K} \right].$$
recalling (3.19) but modifying it slightly. We also compute easily
\[
\partial_K f = \oint_{\Gamma} dw f_\lambda(w) \frac{1}{w-K} \otimes \frac{1}{w-K},
\]
We obtain
\[
B_3 = \oint_{\Gamma} dw Tr_3 \left[ [R \otimes 1][1 \otimes (1 + \Sigma)^{-1}] \right] \frac{w f_i(w)}{(u-K)(w-K) \otimes (v-K)(w-K) \otimes 1},
\]
\[
B_4 = \oint_{\Gamma} dw Tr (1 + \Sigma)^{-1}_{\text{diag}} R_{\text{diag}} \frac{w f_i(w)}{(u-K)(v-K)(w-K)^3},
\]
\[
B_5 = \oint_{\Gamma} dw Tr (1 + \Sigma)^{-1}_{\text{diag}} \frac{f_i(w)}{(u-K)(v-K)(w-K)^2}.
\]

**Lemma 4.2.** We have
\[
\sup_{i=1}^{5} \{ |B_i| \} \leq O(1) \frac{1}{1 + |u|} \frac{1}{1 + |v|}.
\]

**Proof.** We remark that the decay of $f_i(w)$ at large $w$ means that
\[
|f_i(w)| \leq O(1) \inf\{ |w|, |t|^{-(p-1)/2p} |w|^{-1+1/p} \},
\]
\[
|w f_i(w)| \leq O(1) \inf\{ |w|, |t|^{-(p-1)/2p} |w|^{1/p} \}
\]
• For $B_1$, it is essential to remark that (4.18) implies that the growth of $R$ can occur only in its left eigenvalue. It can therefore be bounded by $O(1)$ using the corresponding left $\frac{1}{u-K}$ factor. There remains a $\| \frac{1}{u-K} \frac{1}{v-K} \|$ factor.
• For $B_2$ the growth of $R_{\text{diag}}$ can be bounded by $O(1)$ using one $\frac{1}{v-K}$ factor. There remains a $\| \frac{1}{u-K} \frac{1}{v-K} \|$ factor.
• For $B_3$, the growth of $[R \otimes 1][1 \otimes (1 + \Sigma)^{-1}]$ can be compensated by the decay of the $\frac{1}{w-K} \otimes 3$ factor. This is again subtle and true only because that factor decays separately on each of the three threads in the tensor product, and by (4.18) the possible growth of $R$ occurs only on the first thread and the potential growth of $(1 + \Sigma)^{-1}$ occurs only on the second or third thread. Hence, they never conspire on the same thread. Using this fact, we find the bound
\[
\| [R \otimes 1][1 \otimes (1 + \Sigma)^{-1}] \| \leq O(1) \sup \left\{ \frac{1}{[1 + |w|]^3}, \frac{|t|^{\frac{p-1}{2p}}}{[1 + |w|]^{2+\frac{2}{p}}}, \frac{|t|^{\frac{p-1}{2p}}}{[1 + |w|]^{1+\frac{2}{p}}} \right\}.
\]
Adding the $wf$ factor combined with bound (4.31), we get an overall bound on the $B_3$ $w$-integrand

\[
O(1) \inf \left\{ \frac{1}{|1 + |w||^2}, \frac{1}{1 + |w||1 + \frac{1}{p}} \right\} \leq O(1) \frac{1}{|1 + |w||^{1 + \frac{1}{p}}}
\]

(4.33)

which is integrable on $\Gamma$, and we still have a $\| \frac{1}{u-K} v - K \|$ factor left.

- For $B_4$, the bound is the same as for $B_3$; just easier because there is no subtle discussion of the threads.

- For $B_5$, we can simply bound the $f_i(w)$ factor by $O(1)$. The growth of $(1 + \Sigma)_{\text{diag}}^{-1}$ is bounded by one $\| \frac{1}{w-K} \|^{1 - \frac{2}{p}}$ factor. There remains therefore a factor $[1 + |w|]^{-1 - \frac{2}{p}}$ to integrate over $w$, and after that is done, there remains a $\| \frac{1}{u-K} v - K \| \leq O(1) \frac{1}{1 + |u|} \frac{1}{1 + |v|}$ factor.

□

Combining Lemma 4.2 with the bound (4.16) on $\partial_t \phi$, the five $B$ terms are all given by absolutely convergent integrals on $u$ and $v$ and the result is that $|A_2| \leq O(1) |\lambda|^{\frac{1}{p}}$. Combining with the better bound $O(1) |\lambda|^{\frac{1}{2}}$ on $A_1$ completes the proof of (4.2). □

Remark that $B_2$, $B_4$ and $B_5$ are proportional to $N^{-2}$, hence subleading at large $N$. □

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A Appendix: The $O(N)$ Real Symmetric Case

In the core of this article, we focused on Hermitian matrix models for simplicity. However, the same techniques can be applied to real symmetric and quaternionic Hermitian matrices, with only a few minor changes. In this appendix, we outline how our techniques can be extended to these cases.

First, recall that we work with Hermitian matrices $H \in M_N(\mathbb{H})$, $H_{ij} = H^*_{ji}$, with $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$. In the real case, these are just real symmetric matrices, while in the quaternionic case, diagonal elements are real numbers and off-diagonals one form pairs of conjugate quaternions. These are, respectively, invariant under the groups $O(N)$, $U(N)$ and USp($N$) (unitary matrices with quaternionic entries).
Using the symmetries, the covariance in the normalized Gaussian case is shown to be
\[
\int dH \, H_{ij} H_{kl} \exp \left(-\frac{\beta N}{4} \text{Tr} \, H^2\right) = \frac{1}{N} \delta_{il} \delta_{jk} + \frac{2-\beta}{N \beta} \delta_{ik} \delta_{jl},
\]
(A.1)
for complex (\(\beta = 2\)), real (\(\beta = 1\)) or quaternionic (\(\beta = 4\)). The first term is conveniently represented by a ribbon and the second one by a twisted ribbon.

In order to derive the general formula for the change of variables, it is convenient to first diagonalize the matrices, \(H = U \text{diag}(\mu_1, \ldots, \mu_N) U^\dagger\), with \(\mu_i\) real and \(U\) an element of the corresponding unitary group. Let us denote by \(\mathcal{V}_N(K)\) the volume of this unitary group after division by diagonal matrices and permutations. Then, the partition function can be written as
\[
Z = \int dH \exp \left(-\frac{\beta N}{2} \left\{\frac{1}{2} \text{Tr} \, H^2 + \frac{\lambda^{p-1}}{p} \text{Tr} \, H^{2p}\right\}\right) = \mathcal{V}_N \int \prod_i d\mu_i \prod_{i < j} |\mu_i - \mu_j|^{\beta} \exp \left(-\frac{\beta N}{2} \sum_i \left\{\frac{1}{2} \mu_i^2 + \frac{\lambda^{p-1}}{p} \mu_i^{2p}\right\}\right). \tag{A.3}
\]

Next, we perform the same change of variable \(\mu_i = h_\lambda(\nu_i)\) and rewrite the result in terms of a matrix integral over \(K\), whose eigenvalues are the \(\nu_i\)'s,
\[
Z = \mathcal{V}_N \int \prod_i d\nu_i \left|h'_\lambda(\nu_i)\right| \times \prod_{i < j} \left|h_\lambda(\nu_i) - h_\lambda(\nu_j)\right|^{\beta} \exp \left(-\frac{\beta N}{4} \sum_i \nu_i^2\right) \tag{A.4}
\]
with the new effective action
\[
S(K) = \left(1 - \frac{\beta}{2}\right) \text{Tr} \log h'_\lambda(K) + \frac{\beta}{2} \text{Tr}_\otimes \log \left(\frac{h_\lambda(K) \otimes 1 - 1 \otimes h_\lambda(K)}{K \otimes 1 - 1 \otimes K}\right). \tag{A.6}
\]

The main difference with the complex Hermitian case (see (2.11)) is the occurrence of the single trace term that involves the derivative. Note the formal similarity between the single and the double trace terms: the former can be obtained from the latter in the limit of coinciding eigenvalues. Therefore, we can apply the previous techniques with only minor modifications, as we sketch below.

In order to apply the LVE formalism, we have to derive the effective action with respect to \(K\). As in the previous section, we use the holomorphic functional calculus to introduce resolvents, so that the first derivative is
\[
\frac{\partial}{\partial K} \text{Tr} \log h'_\lambda(K) = \left[1 + \tilde{\Sigma}\right]^{-1} \frac{\partial \tilde{\Sigma}}{\partial K}. \tag{A.7}
\]
As before, higher derivatives with respect to $K$ act either on $[1 + \tilde{\Sigma}]^{-1}$ or on $\frac{\partial \tilde{\Sigma}}{\partial K}$. The net result is a product of derivatives of $\tilde{\Sigma}$ with respect to $K$, separated by insertions of $[1 + \tilde{\Sigma}]^{-1}$.

The latter factor is nothing but the inverse of $h_\lambda'(K)$. Since the functions $h_\lambda$ and $k_\lambda$ are inverses one of the other,

$$[1 + \tilde{\Sigma}]^{-1} = \int_{\Gamma_u} du k_\lambda'(u) \frac{1}{u - H}.$$  \hspace{1cm} (A.8)

In a basis in which $K$ and therefore also $H$ are diagonal, it obeys the bound (3.25) with $R_i = \sup(1, |\mu_i|)$.

The second term is obtained by deriving $1 + \tilde{\Sigma}$ with respect to $K$. It simply corresponds to the one obtained in the previous section, except that tensor products are replaced by ordinary products. Explicitly, it reads (see (3.19) for comparison)

$$\frac{\partial \tilde{\Sigma}}{\partial K} = \int_{\Gamma_v} dv g_\lambda(v) \left( \frac{1}{(v - K)^2} \sqcup \frac{1}{v - K} + \frac{1}{v - K} \sqcup \frac{1}{(v - K)^2} \right),$$ \hspace{1cm} (A.9)

where $\sqcup$ stands for an insertion an insertion of the two indices of the derivative, as before. Higher order derivatives create new resolvents, separated by insertions $\sqcup$.

As a consequence, we obtain the same expression as in the complex case, with the following changes:

- there is a single eigenvalue index $i$, so that the limit $\nu_j \to v_j$ has to be taken;
- tensor products are replaced by ordinary matrix products;
- double trace vertices are multiplied by $\frac{\beta}{2}$ and single trace ones by $1 - \frac{\beta}{2}$;
- tree edges can be twisted or untwisted, with a weight given by (A.1).

Therefore, all the bounds on the corner and derivative operators remain valid for single trace operators, up to multiplicative factors that do not depend on $\epsilon$. Let us also note that the contribution of any single trace vertex is suppressed by a power of $1/N$ in the bounds since it involves one instead of two eigenvalues.

Moreover, on a tree all twisted edges can be untwisted, so that we conclude that the proof we detailed in the previous section for the complex Hermitian case remains valid in the more general case of real symmetric or quaternionic Hermitian matrices, albeit with modified constants.

B Appendix: Positivity of the Jacobian

In this section, we prove the positivity of the Jacobian for $\lambda > 0$ in case of Hermitian matrices.

**Lemma B.1.** \textit{For all $\lambda > 0$, the transformation $\frac{\partial H}{\partial K} > 0$.}
Proof. The eigenvalues $s_i$ of $K$ are real and in the corresponding eigen-basis the Jacobian (2.9) can be written as

$$J_H = \exp \left( \sum_{i,j} \log \frac{s_i \sqrt{T_p(-\lambda^p s_i^{2p-2})} - s_j \sqrt{T_p(-\lambda^p s_j^{2p-2})}}{s_i - s_j} \right). \quad (B.1)$$

When eigenvalues $s_i$ and $s_j$ have different signs, the expression under the logarithm is positive due to the positivity of the Fuss–Catalan function for $\lambda > 0$ [1] and it produces the positive contribution (as a multiplier) to the total Jacobian $J_H$. When $s_i$ and $s_j$ have the same sign, we decompose the corresponding contributions to the Jacobian as

$$\exp \left( \sum_{i,j} \log \left[ \frac{s_i^2 T_p(-\lambda^p s_i^{2p-2}) - s_j^2 T_p(-\lambda^p s_j^{2p-2})}{s_i^2 - s_j^2} \right] \right) + \log \left[ \frac{s_i + s_j}{s_i \sqrt{T_p(-\lambda^p s_i^{2p-2})} + s_j \sqrt{T_p(-\lambda^p s_j^{2p-2})}} \right]. \quad (B.2)$$

Here, the argument of the last logarithm under the exponent is positive again due to $T_p(-\lambda^p s_i^{2p-2}) > 0$ for $\lambda > 0$. Using the functional equation (2.6), we rewrite the argument of the first logarithm in (B.2) as

$$\left( 1 + \lambda^p \frac{s_i^2 T_p(-\lambda^p s_i^{2p-2}) - s_j^2 T_p(-\lambda^p s_j^{2p-2})}{s_i^2 T_p(-\lambda^p s_i^{2p-2}) - s_j^2 T_p(-\lambda^p s_j^{2p-2})} \right)^{-1}$$

$$= \left( 1 + \lambda^p \sum_{k=0}^{p-1} (s_i^2 T_p(-\lambda^p s_i^{2p-2}))^k (s_j^2 T_p(-\lambda^p s_j^{2p-2}))^{p-1-k} \right)^{-1} > 0. \quad (B.3)$$

The positivity of (B.3) follows from $\lambda > 0$ and $s_i^2 T_p(-\lambda^p s_i^{2p-2}) > 0$. Since all multipliers in the Jacobian are positive, we have $J_H > 0$. □

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