ANALOGS OF GENERALIZED RESOLVENTS AND EIGENFUNCTION EXPANSIONS OF RELATIONS GENERATED BY PAIR OF DIFFERENTIAL OPERATOR EXPRESSIONS ONE OF WHICH DEPENDS ON SPECTRAL PARAMETER IN NONLINEAR MANNER

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Abstract. For the relations generated by pair of differential operator expressions one of which depends on the spectral parameter in the Nevanlinna manner we construct analogs of the generalized resolvents which are integro-differential operators. The expansions in eigenfunctions of these relations are obtained.

Introduction

We consider either on finite or infinite interval operator differential equation of arbitrary order

\[ l_\lambda[y] = m[f], \quad t \in \mathcal{I}, \quad \mathcal{I} = (a, b) \subseteq \mathbb{R}^1 \]

in the space of vector-functions with values in the separable Hilbert space \( \mathcal{H} \), where

\[ l_\lambda[y] = l[y] - \lambda m[y] - n_\lambda[y]. \]

\( l[y], m[y] \) are symmetric operator differential expression. The order of \( l_\lambda[y] \) is equal to \( r > 0 \). For the expression \( m[y] \) the subintegral quadratic form \( m\{y, y\} \) of the Dirichlet integral

\[ m\{y, y\} = \int_{\mathcal{I}} m[y, y] dt \]

is nonnegative for \( t \in \mathcal{I} \). The leading coefficient of the expression \( m[y] \) may lack the inverse from \( B(\mathcal{H}) \) for any \( t \in \mathcal{I} \) and even it may vanish on some intervals. For the operator differential expression \( n_\lambda[y] \) the form \( n_\lambda\{y, y\} \) depends on \( \lambda \) in the Nevanlinna manner for \( t \in \mathcal{I} \). Therefore the order \( s \geq 0 \) of \( m[y] \) is even and \( \leq r \).

In the Hilbert space \( L^2_{m}(\mathcal{I}) \) with metrics generated by the form \( m[y, y] \) for equation (1)-(2) we construct analogs \( R(\lambda) \) of the generalized resolvents which in general are non-injective and which possess the following representation:

\[ R(\lambda) = \int_{\mathbb{R}^1} \frac{dE_\mu}{\mu - \lambda} \]

where \( E_\mu \) is a generalized spectral family for which \( E_\infty \) is less or equal to the identity operator. (Abstract operators which possess such representation were studied in \[14\].)

This construction is based on a special reduction of the equation

\[ l[y] = m[f] \]

to the first order system with weight. Here \( l \) and \( m \) are operator differential expressions which are not necessary symmetric (in contrast to \[2\]). For construction of \( R(\lambda) \) we also introduce

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that are taken in the norm any point of $\Delta$ means that $f(\Delta)$ ends belonging to $f(\Delta)$ indexes if it is necessary.

The notation $f(\Delta)$ was constructed in [20].

Further in the work we consider the boundary value problem obtained by adding to equation (1)-(2) the dissipative boundary conditions depending on a spectral parameter. We prove that for some boundary conditions solutions of such problems are generated by the operators $R(\lambda)$ if, in contrast to the case $s = 0$, $n_\lambda[y] = H_\lambda(t)y$, the boundary conditions contain the derivatives of vector-function $f(t)$ that are taken on the ends of the interval.

In the work we calculate $E_\Delta$ and derive an inequality of Bessel type. In the case when the expression $n_\lambda[y]$ submits in a special way to the expression $m[y]$ we obtain the transformation formulae and the Parseval equality. The general results obtained in the work are illustrated on the example of equation (1) with coefficients which are periodic on the semi-axes. We remark that in the case $n_\lambda[y] \equiv 0$ it follows from [17, 18, 19, 23] that if $I = \mathbb{R}^1$, $r > s$ and $\dim H < \infty$ then $E_\mu$ for equation (1) with periodic coefficients on the axis have no jumps. (For $r = s$ in the described case $E_\mu$ may have jump (see, e.g., [23])). We show that in contrast to the case $n_\lambda[y] \equiv 0$ if $r = 1$, $\dim H = 2$ then $E_\mu$ for equation (1) with periodic coefficients on the axis may have jump.

In the case $n_\lambda[y] \equiv 0$ the results listed above are known [23], and $R(\lambda)$ is the generalized resolvent of the minimal relation generated by the pair of expressions $l[y]$ and $m[y]$. For this case we show in the work that in the regular case all generalized resolvents are exhausted by the operators $R(\lambda)$, and thereby by virtue of [21] their full description with the help of boundary conditions is given. A review of other results for the case $n_\lambda[y] \equiv 0$ is in the work [22].

In the works [8, 9] the question of the conditions for holomorphy and continuous reversibility of the restrictions of maximal relations generated by $l[y]$ (2) with $m[y] \equiv 0$, $n_\lambda[y] = H_\lambda(t)y$ in $L^2_{3\lambda_0}(\Delta)$ and also by the integral equation with the Nevanlinna matrix measure was studied (using some of the results from [21]). We remark that the relations inverse to those ones considered in [8, 9] do not possess the representation (3). Also we note that the resolvent equation (1)-(2) does not reduced to the equations considered in [8, 9].

Many question, that concern differential operators and relations in the space of vector-functions, are considered in the monographs [1, 3, 4, 15, 25, 26, 31, 32] containing an extensive literature. The method of studying of these operators and relations based on use of the abstract Weyl function and its generalization (Weyl family) was proposed in [11, 12, 13].

We denote by $(,)$ and $\|\cdot\|$ the scalar product and the norm in various spaces with special indexes if it is necessary.

Let an interval $\Delta \subseteq \mathbb{R}^1$, $f(t)$ ($t \in \Delta$) be a function with values in some Banach space $B$. The notation $f(t) \in C^k(\Delta, B)$, $k = 0, 1, \ldots$ (we omit the index $k$ if $k = 0$) means, that in any point of $\Delta$ $f(t)$ has continuous in the norm $\|\cdot\|_B$ derivatives of order up to and including $l$ that are taken in the norm $\|\cdot\|_B$; if $\Delta$ is either semi-open or closed interval then on its ends belonging to $\Delta$ the one-side continuous derivatives exist. The notation $f(t) \in C^0(\Delta, B)$ means that $f(t) \in C^k(\Delta, B)$ and $f(t) = 0$ in the neighbourhoods of the ends of $\Delta$. 

$$l_\lambda[y] = -\frac{(3\lambda_0)[f]}{3\lambda}, \ t \in \bar{I}.$$ (5)
1. The reduction of equation (1) to the first order system of canonical type with weight. The Green formula

We consider in the separable Hilbert space \( \mathcal{H} \) equation (1), where \( l[y] \) and \( m[f] \) are differential expressions (that are not necessary symmetric) with sufficiently smooth coefficients from \( B(\mathcal{H}) \) and of orders \( r > 0 \) and \( s \) correspondingly. Here \( r \geq s \geq 0 \), \( s \) is even and these expressions are presented in the divergent form. Namely:

\[
l[y] = \sum_{k=0}^{r} i^k l_k[y],
\]

where \( l_{2j} = D^j p_j(t) \ D^j, \ l_{2j-1} = \frac{1}{2} D^{j-1} \{ D q_j(t) + s_j(t) D \} D^{j-1}, \ p_j(t), q_j(t), s_j(t) \in C^j (\mathcal{T}, B(\mathcal{H})), \ D = d/dt; \ m[f] \) is defined in a similar way with \( s \) instead of \( r \) and \( \tilde{p}_j(t), \ \tilde{q}_j(t), \ \tilde{s}_j(t) \in B(\mathcal{H}) \) instead of \( p_j(t), \ q_j(t), \ s_j(t) \).

In the case of even \( r = 2n \geq s \), \( p_n^{-1} \in B(\mathcal{H}) \) we denote

\[
Q(t,l) = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix} = \frac{J}{t}, \ S(t,l) = Q(t,l), \ H(t,l) = \| h_{\alpha\beta} \|^2_{\alpha, \beta = 1}, \ h_{\alpha\beta} \in B(\mathcal{H}^n),
\]

where \( I_n \) is the identity operator in \( B(\mathcal{H}^n) \); \( h_{11} \) is a three diagonal operator matrix whose elements under the main diagonal are equal to \( \left( \frac{1}{2} q_1, \ldots, \frac{1}{2} q_{n-1} \right) \), the elements over the main diagonal are equal to \( \left( -\frac{1}{2} s_1, \ldots, -\frac{1}{2} s_{n-1} \right) \), the elements on the main diagonal are equal to \( \left( -p_0, \ldots, -p_{n-2}, \frac{1}{2} s_p q_n - p_{n-1} \right) \); \( h_{12} \) is an operator matrix with the identity operators \( I_1 \) under the main diagonal, the elements on the main diagonal are equal to \( \left( 0, \ldots, 0, -\frac{1}{2} s_p q_n \right) \), the rest elements are equal to zero; \( h_{21} \) is an operator matrix with identity operators \( I_1 \) over the main diagonal, the elements on the main diagonal are equal to \( \left( 0, \ldots, 0, \frac{1}{2} p_n^{-1} q_n \right) \), the rest elements are equal to zero; \( h_{22} = \text{diag} \left( 0, \ldots, 0, p_n^{-1} \right) \).

Also in this case we denote

\[
W(t,l,m) = C^{s-1}(t,l) \left\{ \| m_{\alpha\beta} \|^2_{\alpha, \beta = 1} \right\} C^{-1}(t,l), m_{\alpha\beta} \in B(\mathcal{H}^n),
\]

where \( m_{11} \) is a tree diagonal operator matrix whose elements under the main diagonal are equal to \( \left( -\frac{1}{2} \tilde{q}_1, \ldots, -\frac{1}{2} \tilde{q}_{n-1} \right) \), the elements over the main diagonal are equal to \( \left( \frac{1}{2} \tilde{s}_1, \ldots, \frac{1}{2} \tilde{s}_{n-1} \right) \), the elements on the main diagonal are equal to \( \left( \tilde{p}_0, \ldots, \tilde{p}_{n-1} \right) \); \( m_{12} = \text{diag} \left( 0, \ldots, 0, \frac{1}{2} \tilde{s}_n \right) \), \( m_{21} = \text{diag} \left( 0, \ldots, 0, -\frac{1}{2} \tilde{q}_n \right) \), \( m_{22} = \text{diag} \left( 0, \ldots, 0, \tilde{p}_n \right) \).

Operator matrix \( C(t,l) \) is defined by the condition

\[
C(t,l) \left\{ f(t), f'(t), \ldots, f^{(n-1)}(t), f^{(2n-1)}(t), \ldots, f^{(n)}(t) \right\} =
\]

\[
= \text{col} \left\{ f^{[0]}(t|l), f^{[1]}(t|l), \ldots, f^{[n-1]}(t|l), f^{[2n-1]}(t|l), \ldots, f^{[n]}(t|l) \right\},
\]

where \( f^{[k]}(t|L) \) are quasi-derivatives of vector-function \( f(t) \) that correspond to differential expression \( L \).

\textsuperscript{1}W(t,l,m) \) is given for the case \( s = 2n \). If \( s < 2n \) one have set the corresponding elements of operator matrices \( m_{\alpha\beta} \) be equal to zero. In particular if \( s < 2n \) then \( m_{12} = m_{21} = m_{22} = 0 \) and therefore \( W(t,l,m) = \text{diag} (m_{11}, 0) \) in view of (14).
The quasi-derivatives corresponding to $l$ are equal (cf. [30]) to
\begin{align}
y^{[j]} (t | l) &= y^{(j)} (t), \quad j = 0, \ldots, \left\lfloor \frac{r}{2} \right\rfloor - 1, \quad (11) \\
y^{[n]} (t | l) &= \begin{cases} 
p_n y^{(n)} - \frac{i}{2} q_{n-1} y^{(n-1)}, & r = 2n \\
-\frac{i}{2} q_{n+1} y^{(n+1)}, & r = 2n + 1 \end{cases}, \quad (12) \\
y^{[r-j]} (t | l) &= -D y^{[r-j-1]} (t | l) + p_j y^{(j)} + \frac{i}{2} \left[ s_{j+1} y^{(j+1)} - q_j y^{(j-1)} \right], \quad j = 0, \ldots, \left\lfloor \frac{r-1}{2} \right\rfloor, \quad q_0 \equiv 0. \quad (13)
\end{align}

At that $l [y] = y^r (t | l)$. The quasi-derivatives $y^{[k]} (t | m)$ corresponding to $m$ are defined in the same way with even $s$ instead of $r$ and $\tilde{p}_j, \tilde{q}_j, \tilde{s}_j$ instead of $p_j, q_j, s_j$.

It is easy to see that
\begin{equation}
C (t, l) = \left( \begin{array} {ccc}
I & 0 \\
C_{21} & C_{22} \end{array} \right), \quad C_{\alpha\beta} \in B (\mathcal{H}^n), \quad (14)
\end{equation}

$C_{21}, C_{22}$ are upper triangular operator matrices with diagonal elements $(-\frac{i}{2} q_1, \ldots, -\frac{i}{2} q_n)$ and $((-1)^n - 1 p_n, (-1)^{n-2} p_n, \ldots, p_n)$ correspondingly.

In the case of odd $r = 2n + 1 > s$ we denote
\begin{align}
Q (t, l) &= \begin{cases} \frac{J}{q_1} \oplus q_{n+1}, & q_1 \\
q_1, & n > 0 \end{cases}, \quad S (t, l) = \begin{cases} \frac{J}{s_1} \oplus s_{n+1}, & n > 0 \\
\frac{s_1}{s_1}, & n = 0 \end{cases}, \quad (15) \\
H (t, l) &= \begin{cases} \|h_{\alpha, \beta}\|^2_{\alpha, \beta=1}, & n > 0 \\
\|p_0\|, & n = 0 \end{cases}, \quad (16)
\end{align}

where $B (\mathcal{H}^n) \ni h_{11}$ is a three-diagonal operator matrix whose elements under the main diagonal are equal to $\left( \frac{i}{2} q_1, \ldots, \frac{i}{2} q_{n-1} \right)$, the elements over the main diagonal are equal to $(-\frac{i}{2} s_1, \ldots, -\frac{i}{2} s_{n-1})$, the elements on the main diagonal are equal to $(-p_0, \ldots, -p_n)$, the rest elements are equal to zero. $B \left( \mathcal{H}^{n+1}, \mathcal{H}^n \right) \ni h_{12}$ is an operator matrix whose elements with numbers $j, j-1$ are equal to $I_1, j = 2, \ldots, n$, the element with number $n, n+1$ is equal to $\frac{1}{2} s_n$, the rest elements are equal to zero. $B \left( \mathcal{H}^n, \mathcal{H}^{n+1} \right) \ni h_{21}$ is an operator matrix whose elements with numbers $j-1, j$ are equal to $I_1, j = 2, \ldots, n$, the element with number $n+1, n$ is equal to $\frac{1}{2} q_n$, the rest elements are equal to zero. $B \left( \mathcal{H}^{n+1}, \mathcal{H}^n \right) \ni h_{22}$ is an operator matrix whose last row is equal to $(0, \ldots, 0, -i I_1, -p_n)$, last column is equal to $col (0, \ldots, 0, i I_1, -p_n)$, the rest elements are equal to zero.

Also in this case we denote\footnote{See the previous footnote}
\begin{equation}
W (t, l, m) = \|m_{\alpha\beta}\|^2_{\alpha, \beta=1}, \quad (17)
\end{equation}

where $m_{11}$ is defined in the same way as $m_{11}$ \footnote{See the previous footnote}. $B \left( \mathcal{H}^{n+1}, \mathcal{H}^n \right) \ni m_{12}$ is an operator matrix whose element with number $n, n+1$ is equal to $-\frac{1}{2} s_n$, the rest elements are equal to zero. $B \left( \mathcal{H}^n, \mathcal{H}^{n+1} \right) \ni m_{21}$ is an operator matrix whose element with number $n+1, n$ is equal to $-\frac{1}{2} q_n$, the rest elements are equal to zero. $B \left( \mathcal{H}^{n+1}, \mathcal{H}^n \right) \ni m_{22} = \text{diag} (0, \ldots, 0, \tilde{p}_n)$.

Obviously for $H (t, l)$ \footnote{See the previous footnote} and $W (t, l, m)$ \footnote{See the previous footnote}, one has
\begin{equation}
H^* (t, l) = H (t, l^*), W^* (t, l, m) = W (t, l, m^*). \quad (18)
\end{equation}
Lemma 1.1. Let the order of $\mathcal{H}$ be even. Then
\begin{equation}
\mathcal{H} (t, l) = W (t, l, -\mathcal{H}) = W (t, l^*, -\mathcal{H}) .
\tag{19}
\end{equation}

Proof. Let us prove the first equality in (19) for even $r = 2n$. Let us represent $H (t, l)$ in the form
\begin{equation}
H (t, l) = A (t, l) + B (t, l) ,
\tag{20}
\end{equation}
where
\begin{equation}
B (t, l) = \| B_{jk} \|^2_{j, k = 1} , \quad B_{jk} \in B (\mathcal{H}^n) ,
\tag{21}
\end{equation}
\begin{equation}
B_{11} = \text{diag} \left( 0, ..., 0, s_n p_n^{-1} q_n / 4 \right) , \quad B_{12} = \text{diag} \left( 0, ..., 0, -i s_n p_n^{-1} / 2 \right) ,
\tag{22}
\end{equation}
\begin{equation}
B_{21} = \text{diag} \left( 0, ..., 0, i p_n^{-1} q_n / 2 \right) , \quad B_{22} = \text{diag} \left( 0, ..., 0, p_n^{-1} \right) .
\tag{23}
\end{equation}

In view of (14), (21) - (23) one has
\begin{equation}
u_{n2n} = -i s_n / 2 , \quad u_{2n} 2n = I_1 , \quad \text{rest} \quad u_{jk} = 0.
\end{equation}
Hence
\begin{equation}
C^* (t, l) B (t, l) C (t, l) = \| v_{jk} \|^2_{j, k = 1} , \quad v_{jk} \in B (\mathcal{H}) ,
\end{equation}
\begin{equation}
v_{2n} = -i \left( s_n - q_n^* \right) , \quad v_{2n} 2n = p_n^* , \quad \text{rest} \quad v_{jk} = 0.
\end{equation}
Hence $C^* (t, l) \mathcal{H} (t, l) C (t, l) = C^* (t, l) W (t, l, -\mathcal{H}) C (t, l)$ in view of (8), (9), (10), (20).

The first equality in (19) for even $r$ is proved. Its proof for odd $r$ follows from (16), (17).

One can see from the proof that
\begin{equation}
W (t, l, \mathcal{H}) = -\mathcal{H} (t, l) .
\tag{25}
\end{equation}

The second equality in (19) is a corollary of (25) and (18). Lemma 1.1 is proved.

For sufficiently smooth vector-function $f (t)$ by corresponding capital letter we denote (if $f (t)$ has a subscript then we add the same subscript to $F$)
\begin{equation}
\mathcal{H} \ni F (t, l, m) =
\begin{cases}
\sum_{j=0}^{s / 2} \oplus f^{(j)} (t) \oplus 0 \oplus ... \oplus 0 , & r = 2n , \quad r = 2n + 1 > 1 , \quad s < 2n , \\
\sum_{j=0}^{n-1} \oplus f^{(j)} (t) \oplus 0 \oplus ... \oplus 0 \oplus \left( -i f^{(n)} (t) \right) , & r = 2n + 1 > 1 , \quad s = 2n , \\
f (t) , & r = 1 , \\
\sum_{j=0}^{n-1} \oplus f^{(j)} (t) \oplus \left( \sum_{j=1}^{n} \oplus f^{[r-j]} (t l) \right) , & r = s = 2n
\end{cases}
\tag{26}
\end{equation}

From now on in equation (14)
\begin{equation}
p_n^{-1} (t) \in B (\mathcal{H}) \quad (r = 2n) ; \quad (q_{n+1} (t) + s_{n+1} (t))^{-1} \in B (\mathcal{H}) \quad (r = 2n + 1).
\end{equation}

Theorem 1.1. Equation (14) is equivalent to the following first order system
\begin{equation}
\frac{i}{2} \left( \left( Q (t) \tilde{y} \right) ' + S (t) \tilde{y} ' \right) + H (t) \tilde{y} = W (t) F (t)
\tag{27}
\end{equation}
sufficiently smooth vector-function $f$, weight $W(t) = \sum_{j=0}^{n-1} y^{(j)}(t)$, and with $F(t) = F(t, l^*, m)$ that are obtained from (4), (17) and (26) correspondingly with $l^*$ instead of $l$. Namely if $y(t)$ a solution of equation (4) then

$$\bar{y}(t) = \bar{y}(t, l, m, f) = \begin{cases} \sum_{j=0}^{n-1} y^{(j)}(t) + \sum_{j=1}^{n} \left( y^{[r-j]}(t \mid l) - f^{[s-j]}(t \mid m) \right), & r = 2n \\ \sum_{j=0}^{n-1} y^{(j)}(t) + \sum_{j=1}^{n} \left( y^{[r-j]}(t \mid l) - f^{[s-j]}(t \mid m) \right) \oplus (-iy^{(r)}(t)), & r = 2n + 1 > 1 \\ y(t), & r = 1 \end{cases} \quad (28)$$

is a solution of (27) with the coefficients, weight and $F(t)$ mentioned above. Any solution of equation (27) with such coefficients, weight and $F(t)$ is equal to (28), where $y(t)$ is some solution of equation (4).

Let us notice that different vector-functions $f(t)$ can generate different RHSs of equation (27) but unique RHS of equation (4).

Proof. We need the following three lemmas.

**Lemma 1.2.** Let $L_\alpha$ be a differential expression of $l$ type and of order $\alpha$. Let us add to $L_\alpha$ the expressions of $t^k$ type, where $k = \alpha + 1, \ldots, \beta$, with coefficients equal to zero. We obtain the expressions $L_\beta$ which formally has the order $\beta$, but in fact $L_\beta$ and $L_\alpha$ coincide. Then for sufficiently smooth vector-function $f(t)$

$$f^{[\beta-j]}(t \mid L_\beta) = \begin{cases} f^{[\alpha-j]}(t \mid L_\alpha), & j = 0, \ldots, \left[ \frac{\alpha+1}{2} \right] \\ 0, & j = \left[ \frac{\alpha+1}{2} \right] + 1, \ldots, \left[ \frac{\beta}{2} \right] \end{cases} \quad (29)$$

(here $f^{[0]}(t \mid L_1)$ is defined by (28) with $r = 1$).

Proof. Proof of Lemma 1.2 follows from formulae (12) – (13) for quasi-derivatives.

**Lemma 1.3.** Let $f(t) \in C^s((\alpha, \beta], \mathcal{H})$, $y(t)$ is a solution of corresponding equation (4). Then the sequence $f_k(t) \in C^s((\alpha, \beta], \mathcal{H})$ and solutions $y_k(t)$ of equation (4) with $f(t) = f_k(t)$ exist such that

$$f_k(t), \quad y_k(t) \quad (30)$$

This is trivial consequence of Weierstrass theorem for vector-functions (33) and formula (1.21) from [10].
Lemma 1.4. Let vector-function $f(t) \in C^s(I, H)$. Then

\[
W(t, l^*, m) F(t, l^*, m) =
\begin{cases}
\left( \sum_{j=0}^{s/2-1} \oplus \left( f^{[s-j]}(t|m) + (f^{[s-j]}(t|m))' \right) + \\
\oplus f^{[s/2]}(t|m) \oplus 0 \oplus \ldots \oplus 0, \quad r = 2n + 1, r = 2n, 0 < s < 2n \\
\left( \sum_{j=0}^{s/2-1} \oplus \left( f^{[s-j]}(t|m) + (f^{[s-j]}(t|m))' \right) + \\
\oplus 0 \oplus \ldots \oplus 0 \right) + H(t,l)(0 \oplus \ldots \oplus 0 \oplus f^{[n]}(t|m)), \quad r = 2n + 1, s = 2n > 0 \\
\end{cases}
\]

(29)

Let us notice that $W(t, l^*, m) F(t, l^*, m)$ does not change if the null-components in $F(t, l^*, m)$ we change by any $H$-valued vector-functions.

Proof. Let us prove Lemma 1.4 for $r = s = 2n$. It is sufficient to verify that

\[
\left( \|m_{\alpha\beta}(t)\|^2_{\alpha,\beta=1} \right)_{\text{col}} \left\{ f(t), f'(t), \ldots, f^{(n-1)}(t), f^{(2n-1)}(t), \ldots, f^{(n)}(t) \right\} = \left. C^s(t,l^*) \left\{ \left( \sum_{j=0}^{n-1} \left( f^{[r-j]}(t|m) + (f^{[r-j]}(t|m))' \right) + 0 + \ldots + 0 \right) + \\
+ H(t,l)(0 \oplus 0 \oplus \ldots \oplus 0 \oplus f^{[n]}(t|m)) \right\}. \right. 
\]

(30)

But in view of (11), (12), (13) the left side of equality (30) is equal to

\[
\left( \sum_{j=0}^{n-1} \left( f^{[r-j]}(t|m) + (f^{[r-j]}(t|m))' \right) + 0 + \ldots + 0 \oplus f^{[n]}(t|m) \right).
\]

And hence equality (30) is true since $C(t,l^*)[..] = [..]$ and the last column of $C^s(t,l^*) H(t,l)$ is equal to $col(0,\ldots,0,I_1)$ in view of (8), (11).

The proof for $r = 2n + 1, s = 2n$ is carried out via direct calculation using (17), (12), (13).

The proof for $s < 2n$ follows from the case $s = 2n$ consiered above, Lemmas 1.2, 1.3 and fact that elements $u_{j,k} \in B(H)$ of matrix $W(t,l^*,m)$ are equal to zero if $s < 2n$ and $i > s/2$ or $j > s/2$. Lemma 1.4 is proved.

Let us return to the proof of Theorem 1.1. Let $y(t)$ is a solution of equation (4). Then

\[
\frac{i}{2} \left\{ \left( Q(t,l) \tilde{y}(t,l,m,0) \right)' + S(t,l) \tilde{y}'(t,l,m,0) \right\} - H(t,l) \tilde{y}(t,l,m,0) = \left. \text{diag} \left( y^{[r]}(t,l) , 0, \ldots, 0 \right) \right. 
\]

(31)

in view of formulae that are analogues to formulae (4.10), (4.11), (4.24), (4.25) from [24]. Using (31) and Lemma 1.4 we show via direct calculations that $\tilde{y}(t,l,m,f)$ (28) is a solution...
of (27) for \( r = s = 2n, \ r = 2n + 1, \ s = 2n \). Therefore in view of Lemmas [1.2, 1.3] \( \bar{y}(t, l, m, f) \) is a solution of (27) for \( s < 2n \).

Conversely let \( \tilde{y}(t) = \text{col} (y_1, \ldots, y_r) \) is a solution of (27). Let \( y(t) \) is a solution of Cauchy problem that is obtained by adding the initial condition \( \tilde{y}(0, l, m, f) = \tilde{y}(0) \) to equation (4). Then \( \tilde{y}(t) = \bar{y}(t, l, m, f) \) in view of existence and uniqueness theorem. Theorem 1.1 is proved

Let us notice that Theorem 1.1 remains valid if null-components of \( F(t, l^*, m) \) we change by any \( \mathcal{H} \)-valued vector-functions.

For differential expression \( L = \sum_{k=0}^{R} j^k L_k \), where \( L_{2j} = D^j P_j(t) D^j \).

\( L_{2j-1} = \frac{1}{2} D^{j-1} \{ DQ_j(t) + S_j(t) D \} D^{j-1} \), we denote by

\[
L [f, g] = \int_I L \{ f, g \} dt,
\]

the bilinear form which corresponds to Dirichlet integral for this expression. Here

\[
L \{ f, g \} = \sum_{j=0}^{[R/2]} \left( P_j(t) f^{(j)}(t), g^{(j)}(t) \right) + \frac{j}{2} \sum_{j=1}^{[R+1]} \left( S_j(t) f^{(j)}(t), g^{(j-1)}(t) \right) - \left( Q_j(t) f^{(j-1)}(t), g^{(j)}(t) \right) \label{33}
\]

**Theorem 1.2** (On the relationships between bilinear forms). Let \( f(t), y(t), f_k(t), y_k(t) (k = 1, 2) \) be sufficiently smooth vector-function. Starting from these functions by the formulae (26), (28) we construct \( F(t, l, m), F_k(t, l, m), \bar{y}(t, l, m, f), \bar{y}_k(t, l, m, f_k) \). Then:

1. \( (W(t, l, m) F_1(t, l, m), F_2(t, l, m)) = m \{ f_1, f_2 \} \).

2. a) If the order of \( \exists l \) is even, then

\[
(W(t, l, -\exists l) \bar{y}(t, l, m, f), \bar{y}(t, l, m, f)) - \exists (W(t, l^*, m^*) \bar{y}(t, l, m, f)), F(t, l^*, m) = - (\exists l) \{ y, f \} - \exists (m^* \{ y, f \}). \label{35}
\]

b) \( m \{ y_1, f_2 \} - m \{ f_1, y_2 \} = (W(t, l, m) \bar{y}_1(t, l, m, f_1), F_2(t, l, m)) - (W(t, l^*, m) F_1(t, l^*, m), \bar{y}_2(t, l^*, m^*, f_2)) \) \label{36}

although for \( r = s \) the corresponding terms in the right-and left-hand side of \( \ref{35} \) and \( \ref{36} \) do not coincide.

**Proof.** 1. follows from [31]. (17). [26], [33].

2. Let \( r = s = 2n \). For convenience when using notations of [26] type we omit the argument \( m \). For example by \( F(t, l^*) \) we denote \( F(t, l^*, m) \).

a) We denote

\[
F(t, m) = \text{col} \left\{ 0, \ldots, 0, f^{[2n-1]}(t | m), \ldots, f^{[n]}(t | m) \right\} \in \mathcal{H}^r. \label{37}
\]
One has
\[
(W(t,l, -3l) \tilde{g}(t,l,m,f), \tilde{g}(t,l,m,f)) = (W(t,l, -3l) Y(t,l), Y(t,l)) - \\
- (W(t,l, -3l) Y(t,l), \mathcal{F}(t,m)) - (W(t,l, -3l) \mathcal{F}(t,m), Y(t,l)) + \\
+ (W(t,l, -3l) \mathcal{F}(t,m), \mathcal{F}(t,m)) = -(3l)[y,y] + \\
+ 2\Re\left(p^{-1}_{r}f^{[n]}(t|m)\right) + 3\left(p^{-1}_{r}f^{[n]}(t|m), f^{[n]}(t|m)\right).
\]
Here the last equality follows from (18), (34), (29), (8). On the other hand we have
\[
\Re\left(W(t,l,m)\tilde{g}(t,l,m,f), F(t,l)\right) = \Re\left(W(t,l^*, m^*) Y(t,l^*), F(t,l^*)\right) + \\
+ \Re\left(W(t,l^*, m^*) (Y(t,l) - Y(t,l^*)) - \mathcal{F}(t,m), F(t,l^*)\right) = \Re\left(m^*\{y,f\}\right) + \\
+ 2\Re\left(p^{-1}_{r}f^{[n]}(t|m)\right) + 3\left(p^{-1}_{r}f^{[n]}(t|m), f^{[n]}(t|m)\right).
\]
Here the last equality is proved similarly to (38) taking into account that \(y^{[n]}(t|l) - y^{[n]}(t|l^*) = 2iy^{[n]}(t|3l)\). Comparing (38), (39) we obtain (35).

b) In view of (28), (34), (18) and Lemma 1.4 we have
\[
(W(t,l,m)\tilde{g}_{1}(t,l,m,f), F_{2}(t,l)) = m\{y_{1}(t,l,m,f), f_{2}\} - \\
- \left(\mathcal{F}_{1}(t,m), H(t,l^*)\right)\col(0, ..., 0, f_{2}^{[n]}(t|m^*))\right) = \\
= m\{y_{1}, f_{2}\} - \left(p^{-1}_{n}f_{1}^{[n]}(t|m), f_{2}^{[n]}(t|m^*)\right).
\]
Similarly
\[
(W(t,l^*, m) F_{1}(t,l^*), \tilde{g}_{2}(t,l^*, m^*, f_{2})) = m\{f_{1}, y_{2}\} - \left(p^{-1}_{n}f_{1}^{[n]}(t|m), f_{2}^{[n]}(t|m^*)\right)
\]
Comparing (40), (41) we obtain (39).

For \(r = 2n + 1, s = 2n \) or \(r = 2n + 1 \vee 2n, s = 2n\), the corresponding terms in (35), (36) coincide in view of (9), (17), (26), (28), (34). For example in these cases
\[
(W(t,l, -3l) \tilde{g}(t,l,m,f), \tilde{g}(t,l,m,f)) = (W(t,l, -3l)) Y(t,l), Y(t,l)) = -(3l)\{y,y\}
\]
Theorem 1.2 is proved. □

Let us notice that Theorem 1.2 remains valid if null-components in \(F_{k}(t,l,m), F(t,l^*, m), F_{1}(t,l^*, m)\) we change by any \(H\)-valued vector-functions.

**Theorem 1.3** (The Green formula). Let \(l_{k}, m_{k} (k = 1, 2)\) are differential expressions of \(l\) (9), \(m\) type correspondingly. The orders of \(l_{k}\) are equal to \(r\), the orders \(m_{k}\) are different in general, even and are equal to \(s_{k} \leq r\). Let \(y_{k}(t) \in C^{r}([\alpha, \beta], H), f_{k}(t) \in C^{s_{k}}([\alpha, \beta], H), \) and \(l_{k}[y_{k}] = m_{k}[f_{k}], k = 1, 2\). Then
\[
\int_{\alpha}^{\beta} m_{1}\{f_{1}, y_{2}\} dt - \int_{\alpha}^{\beta} m_{2}\{y_{1}, f_{2}\} dt - \int_{\alpha}^{\beta} (l_{1} - l_{2})\{y_{1}, y_{2}\} dt = \\
= \left(\frac{i}{2}(Q(t,l_{1}) + Q^{*}(t,l_{2})) \tilde{g}_{1}(t, l_{1}, m_{1}, f_{1}), \tilde{g}_{2}(t, l_{2}, m_{2}, f_{2})\right)\bigg|_{\alpha}^{\beta},
\]
where \(Q(t,l_{k}), \tilde{g}_{k}(t,l_{k}, m_{k}, f_{k})\) correspond to equations \(l_{k}[y] = m_{k}[f]\) by formulae (7), (15), (28) with \(l_{k}, m_{k}, y_{k}, f_{k}\) instead of \(l, m, y, f\) correspondingly.

**Proof.** We need the following
Lemma 1.5. For sufficiently smooth vector-function \( g_1(t), g_2(t) \) one has
\[
\left( (H(t, l_1) - H(t, l_2)) \tilde{g}_1(t, l_1, m_1, 0), \tilde{g}_2(t, l_2, m_2, 0) \right) = \begin{cases} 
-(l_1 - l_2^r) \{g_1, g_2\}, & r = 2n \\
-(l_1 - l_2^r) \{g_1, g_2\} + \left( l_{2n+1}^l - l_{2n+1}^r \right) \{g_1, g_2\}, & r = 2n + 1,
\end{cases}
\]
where \( l_{2n+1}^r \) are the analogs of \( l_{2n+1} \).

Proof. Let \( r = 2n \). Then in view of \([20]-[21], [28], [10], [18]\) we have
\[
\left( (H(t, l_1) - H(t, l_2^r)) \tilde{g}_1(t, l_1, m_1, 0), \tilde{g}_2(t, l_2, m_2, 0) \right) =
\left( (A(t, l_1) - A(t, l_2^r)) \tilde{g}_1(t, l_1, m_1, 0), \tilde{g}_2(t, l_2, m_2, 0) \right) + 
\left( C^*(t, l_2) B(t, l_1) C(t, l_1) \text{col} \left\{ g_1, g_1', ..., g_1^{(n-1)}, g_1^{(2n-1)}, ..., g_1^{(n)} \right\}, \right.
\text{col} \left\{ g_2, g_2', ..., g_2^{(n-1)}, g_2^{(2n-1)}, ..., g_2^{(n)} \right\} \left\} \right.
- \left( \text{col} \left\{ g_1, g_1', ..., g_1^{(2n-1)}, ..., g_1^{(n)} \right\}, C^*(t, l_1) B(t, l_2) C(t, l_2) \text{col} \left\{ g_2, g_2', ..., g_2^{(n-1)}, g_2^{(2n-1)}, ..., g_2^{(n)} \right\} \right) =
- \left( l_{2n+1}^l \left\{ f, g \right\} \right).
\]

The proof of \((43)\) for \( r = 2n + 1 \) follows directly from \([16], [28]\). Lemma 1.5 is proved. \( \square \)

Now Green formula \((42)\) we obtain from the following Green formula for the equation \([27]\) that correspond to equations \( l_k [g] = m_k [f] \)
\[
\int_\alpha^\beta \left( W(t, l_1^r, m_1) F_1(t, l_1^r, m_1), \tilde{g}_2(t, l_2, m_2, f_2) \right) dt -
- \int_\alpha^\beta \left( W(t, l_2^r, m_2^r) \tilde{g}_1(t, l_1, m_1, f_1), F_2(t, l_2^r, m_2) \right) dt + 
+ \int_\alpha^\beta \left( (H(t, l_1) - H(t, l_2^r)) \tilde{g}_1(t, l_1, m_1, f_1), \tilde{g}_2(t, l_2, m_2, f_2) \right) dt - 
- \int_\alpha^\beta \left\{ \left( (S(t, l_1) - Q^*(t, l_2)) \tilde{g}_1'(t, l_1, m_1, f_1), \tilde{g}_2(t, l_2, m_2, f_2) \right) - 
- \left( (Q(t, l_1) - S^*(t, l_2)) \tilde{g}_1(t, l_1, m_1, f_1), \tilde{g}_2'(t, l_2, m_2, f_2) \right) \right\} dt =
= \left( \frac{i}{2} (Q(t, l_1) + Q^*(t, l_2)) \tilde{g}_1(t, l_1, m_1, f_1), \tilde{g}_2(t, l_2, m_2, f_2) \right) \bigg|_\alpha^\beta. \quad (44)
\]

Let \( r = s_k = 2n \). For convenience by \( F_k(t, l_k^r), Y_k(t, l_k) \) we denote \( F_k(t, l_k^r, m_k), Y_k(t, l_k, m_k) \) correspondingly. Then in view of \([8], [28], [29], [34], [13]\) one has:
\[
(W(t, l_1^r, m_1) F_1(t, l_1^r), \tilde{g}_2(t, l_2, m_2, f_2)) = m_1 \{ f_1, y_2 \} + 
+ \left( H(t, l_1) \text{col} \left\{ 0, ..., 0, f_1^{[n]}(t | m_1) \right\}, \text{col} \left\{ 0, ..., 0, y_2^{[n]}(t | l_2) - y_2^{[n]}(t | l_1^r) - f_2^{[n]}(t | m_2) \right\} \right) ; \quad (45)
\]
\[
(W(t, l_2^r, m_2^r) \tilde{g}_1(t, l_1, m_1, f_1), F_2(t, l_2^r)) = m_2 \{ y_1, f_2 \} + 
+ \left( H(t, l_2^r) \text{col} \left\{ 0, ..., 0, y_1^{[n]}(t | l_1) - y_1^{[n]}(t | l_2) - f_1^{[n]}(t | m_1) \right\}, \text{col} \left\{ 0, ..., 0, f_2^{[n]}(t | m_2) \right\} \right) ; \quad (46)
\]
Hence the proof of (42) for Remark footnote 1.

\[ ((H(t, l_1) - H(t, l_2^*)) \tilde{y}_1(t, l_1, m_1, f_1), \tilde{y}_2(t, l_2, m_2, f_2)) = -(l_1 - l_2^*) \{y_1, y_2\} - \\
- ((H(t, l_1) - H(t, l_2)) Y_1(t, l_1), \mathcal{F}_2(t, m_2)) - ((H(t, l_1) - H(t, l_2^*)) \mathcal{F}_1(t, m_1), Y_2(t, l_2)) + \\
+ \left( (H(t, l_1) - H(t, l_2^*)) \text{col} \left\{ 0, \ldots, 0, f_1^{[n]}(t | m_1) \right\}, \text{col} \left\{ 0, \ldots, 0, f_2^{[n]}(t | m_2) \right\} \right). \quad (47) \]

where \( \mathcal{F}_k(t, m_k) \) are the analogs of (47).

Let us denote by \( p_k^j, q_j^k, s_j^k \) the coefficients of \( l_j \). Then in view of (48)

\[ \begin{align*}
(H(t, l_1) \text{col} \left\{ 0, \ldots, 0, f_1^{[n]}(t | m_1) \right\}, \text{col} \left\{ 0, \ldots, 0, y_1^{[n]}(t | l_2) - y_2^{[n]}(t | l_1^*) \right\}) &= \\
= \left( (p_n^1)^{-1} f_1^{[n]}(t | m_1), y_2^{[n]}(t | l_2) - y_2^{[n]}(t | l_1^*) \right), \quad (48) \end{align*} \]

and

\[ \begin{align*}
(\text{col} \left\{ 0, \ldots, 0, y_1^{[n]}(t | l_1) - y_1^{[n]}(t | l_2^*) \right\}, H(t, l_2) \text{col} \left\{ 0, \ldots, 0, f_2^{[n]}(t | m_2) \right\}) &= \\
= \left( y_1^{[n]}(t | l_1) - y_1^{[n]}(t | l_2^*) \right), (p_n^2)^{-1} f_2^{[n]}(t | m_2)). \quad (49) \end{align*} \]

On the another hand in view of (48), (49) we have

\[ \begin{align*}
- ((H(t, l_1) - H(t, l_2^*)) Y_1(t, l_1), \mathcal{F}_2(t, m_2)) &= \\
= - \left( i(p_n^1)^{-1} q_n^1/2 - i(p_n^2)^{-1} r_n^2/2 \right) y_1^{(n-1)} + \left( (p_n^1)^{-1} - (p_n^2)^{-1} \right) y_1^{[n]}(t | l_1), f_2^{[n]}(t | m_2) \right) &= \\
= \left( (p_n^2)^{-1} \left( y_1^{[n]}(t | l_1) - y_1^{[n]}(t | l_2^*) \right), f_2^{[n]}(t | m_2) \right), \quad (50) \end{align*} \]

where the last equality is a corollary of (12) and its following modification:

\[ (p_n^1)^{-1} y_1^{[n]}(t | l_1) = y_1^{(n)} - i/2 (p_n^1)^{-1} q_n^1 y_1^{(n-1)} \]

Analogously it can be proved that

\[ ((H(t, l_1) - H(t, l_2^*)) \mathcal{F}_1(t, m_1), Y_2(t, l_2)) = \]

\[ \left( f_1^{[n]}(t | m_1), (p_n^1)^{-1} \left( y_2^{[n]}(t | l_2) - y_2^{[n]}(t | l_1^*) \right) \right). \quad (51) \]

Comparing (43) - (51) we get (12) since the last \( f_\alpha^\beta \) in the left-hand-side of (14) is equal to zero if \( r = 2n \) in view of (7).

For \( s < r = 2n \) the proof of (12) easy follows from (26), (28), (31), (34), (41) in view of footnote 11.

Now let \( r = 2n + 1 \). Then the last \( f_\alpha^\beta \) in the left-hand-side of (14) is equal to \( f_\alpha^\beta \left( l_{2n+1}^1 - l_{2n+1}^2 \right) \{y_1, y_2\} \text{dt} \). Hence the proof of (12) for \( s \leq 2n < r = 2n + 1 \) follows from (17), (26), (28), (31), (34), (41). Theorem 1.3 is proved.

\[ \square \]

Remark 1.1. In view of Lemmas 1.2, 1.3 all results of this item are valid if the condition of evenness of \( s \) is changed by the condition \( s \leq 2 \left[ \frac{r}{2} \right] \).

2. Characteristic operator

We consider an operator differential equation in separable Hilbert space \( \mathcal{H}_1 \):

\[ \frac{i}{2} \left( (Q(t) x(t))' + Q^*(t) x'(t) \right) - H_\lambda(t) x(t) = W_\lambda(t) F(t), \quad t \in \mathcal{I}, \quad (52) \]
where \( Q(t), [\Re Q(t)]^{-1}, H_\lambda (t) \in B (\mathcal{H}_1), Q(t) \in C^1 (\bar{I}, B (\mathcal{H}_1)) \); the operator function \( H_\lambda (t) \) is continuous in \( t \) and is Nevanlinna’s in \( \lambda \). Namely the following condition holds:

(A) The set \( \mathcal{A} \supset C \setminus \mathbb{R}^1 \) exists, any its point have a neighbourhood independent of \( t \in \bar{I} \), in this neighbourhood \( H_\lambda (t) \) analytic \( \forall t \in \bar{I} \); \( \forall \lambda \in \mathcal{A} H_\lambda (t) = H_\lambda ^*(t) \in C (\bar{I}, B (\mathcal{H}_1)) \); the weight \( W_\lambda (t) = 3H_\lambda (t)/3\lambda \geq 0 (3\lambda \neq 0) \).

In view of \( [21] \) \( \forall \mu \in \mathcal{A} \cap \mathbb{R}^1 : W_\mu (t) = \partial H_\lambda (t)/\partial \lambda|_{\lambda=\mu} \) is Bochner locally integrable in the uniform operator topology.

For convenience we suppose that \( 0 \in \bar{I} \) and we denote \( \Re Q(0) = G \).

Let \( X_\lambda (t) \) be the operator solution of homogeneous equation \( [52] \) satisfying the initial condition \( X_\lambda (0) = I \), where \( I \) is an identity operator in \( \mathcal{H}_1 \). Since \( H_\lambda (t) = H_\lambda ^*(t) \) then

\[
X_\lambda ^*(t)[\Re Q(t)]X_\lambda (t) = G, \; \lambda \in \mathcal{A}.
\]

For any \( \alpha, \beta \in \bar{I}, \alpha \leq \beta \) we denote \( \Delta_\lambda (\alpha, \beta) = \int^\beta_\alpha X_\lambda ^*(t) W_\lambda (t) X_\lambda (t) \, dt, \quad N = \{ h \in \mathcal{H}_1 | h \in \text{Ker} \Delta_\lambda (\alpha, \beta) \forall \alpha, \beta \}, P \) is the ortho-projection onto \( N ^\perp \). \( N \) is independent of \( \lambda \in \mathcal{A} \) [21].

For \( x(t) \in \mathcal{H}_1 \) or \( x(t) \in B (\mathcal{H}_1) \) we denote \( U [x(t)] = ([\Re Q(t)]x(t), x(t)) \) or \( U [x(t)] = x^*(t) [\Re Q(t)] x(t) \) respectively.

As in \( [20, 21] \) we introduce the following

**Definition 2.1.** An analytic operator-function \( M(\lambda) = M^* (\bar{\lambda}) \in B (\mathcal{H}_1) \) of non-real \( \lambda \) is called a characteristic operator of equation \( [52] \) on \( \mathcal{I} \) (or, simply, c.o.), if for \( 3\lambda \neq 0 \) and for any \( \mathcal{H}_1 \) - valued vector-function \( F(t) \in L^2_{W_\lambda} (\mathcal{I}) \) with compact support the corresponding solution \( x_\lambda (t) \) of equation \( [52] \) of the form

\[
x_\lambda (t, F) = \mathcal{R}_\lambda F = \int_{\mathcal{I}} X_\lambda (t) \left\{ M(\lambda) - \frac{1}{2} \text{sgn} (s-t) (iG)^{-1} \right\} X^*_\lambda (s) W_\lambda (s) F(s) \, ds
\]

satisfies the condition

\[
(3\lambda) \lim_{(\alpha, \beta) \uparrow \mathcal{I}} (U [x_\lambda (\beta, F)] - U [x_\lambda (\alpha, F)]) \leq 0 \; \; (3\lambda \neq 0).
\]

Let us note that in \( [21] \) c.o. was defined if \( Q(t) = Q^*(t) \). Our case is equivalent to this one since equation \( [52] \) coincides with equation of \( [52] \) type with \( \Re Q(t) \) instead of \( Q(t) \) and with \( H_\lambda (t) - \frac{1}{2} \Re Q(t) \) instead of \( H_\lambda (t) \).

The properties of c.o. and sufficient conditions of the c.o.’s existence are obtained in \( [20, 21] \).

In the case \( \dim \mathcal{H}_1 < \infty, Q(t) = \mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}, -\infty < a = c \) the description of c.o.’s was obtained in \( [28] \) (the results of \( [28] \) were specified and supplemented in \( [22] \)). In the case \( \dim \mathcal{H}_1 = \infty \) and \( \mathcal{I} \) is finite the description of c.o.’s was obtained in \( [21] \). These descriptions are obtained under the condition that

\[
\exists \lambda_0 \in \mathcal{A}, \; [\alpha, \beta] \subseteq \mathcal{I}: \; \Delta_{\lambda_0} (\alpha, \beta) \gg 0.
\]

**Definition 2.2.** \( [20, 21] \) Let \( M(\lambda) \) be the c.o. of equation \( [52] \) on \( \mathcal{I} \). We say that the corresponding condition \( [55] \) is separated for nonreal \( \lambda = \mu_0 \) if for any \( \mathcal{H}_1 \)-valued vector function \( f(t) \in L^2_{W_{\mu_0} (t)} (\mathcal{I}) \) with compact support the following inequalities holds simultaneously for the solution \( x_{\mu_0} (t) \) \( [54] \) of equation \( [52] \):

\[
\lim_{\alpha \uparrow \mathcal{I}} \exists \mu_0 U [x_{\mu_0} (\alpha)] \geq 0, \; \lim_{\beta \uparrow \mathcal{I}} \exists \mu_0 U [x_{\mu_0} (\beta)] \leq 0.
\]
Theorem 2.1. [20] [21] (see also [31]) Let $M(\lambda)$ be the c.m. of equation (52). We represent $M(\lambda)$ in the form

$$M(\lambda) = (\mathcal{P}(\lambda) - \frac{1}{2}I)(iG)^{-1}. \quad (58)$$

Then the condition (55) corresponding to $M(\lambda)$ is separated for $\lambda = \mu_0$ if and only if the operator $\mathcal{P}(\mu_0)$ is the projection, i.e.

$$\mathcal{P}(\mu_0) = \mathcal{P}^2(\mu_0). \quad (59)$$

Definition 2.3. [20] [21] If the operator-function $M(\lambda)$ of the form (58) is the c.o. of equation (52) on $\mathcal{I}$ and, moreover, $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, then $\mathcal{P}(\lambda)$ is called a characteristic projection (c.p) of equation (52) on $\mathcal{I}$ (or, simply, c.p).

The properties of c.p.'s and sufficient conditions for their existence are obtained in [21]. Also [21] contains the description of c.p.'s and abstract an analogue of Theorem 2.1.

The following statement gives necessary and sufficient conditions for existence of c.o., which corresponds to such separated boundary conditions that corresponding boundary condition in regular point is self-adjoint. This statement follows from Theorem 2.1.

Let us denote $\mathcal{H}_+$ ($\mathcal{H}_-$) the invariant subspace of operator $G$, which corresponds to positive (negative) part of $\sigma(G)$.

Theorem 2.2. Let $-\infty < a$. If $P = I$ then for existence of c.o. $M(\lambda)$ of equation (52) on $(a, b)$ such that

$$\exists \mu_0 \in \mathbb{C} \setminus \mathbb{R}^1: U[x_{\mu_0}(a, F)] = U[x_{\bar{\mu}_0}(a, F)] = 0 \quad (60)$$

(and therefore condition (55) is separated on $\lambda = \mu_0$, $\lambda = \bar{\mu}_0$) it is necessary and sufficient that

$$\dim \mathcal{H}_+ = \dim \mathcal{H}_- \quad (61)$$

(in (60) $x_\lambda(t, F)$ is a solution (54) of (52) which corresponds to c.o. $M(\lambda)$, $L^2_{\omega, \theta(t)}(a, b) \supset F = F(t)$ is any $\mathcal{H}_1$-valued vector-function with compact support). If condition (60) holds then condition (61) is also sufficient for the existence of such c.o.

Proof. Necessity. Since $P = I$ we obtain

$$U[X_{\mu_0}(a)(I - \mathcal{P}(\mu_0))] = U[X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0))] = 0 \quad (62)$$

in view of the proof of n°2° of Theorem 1.1 from [21].

Let for definiteness $\Im \mu_0 > 0$. Then in view of Theorem 2.4 and formula (1.69) from [21], (59), (62) and the fact that

$$\exists \lambda (X_{\lambda}(a)\mathcal{R}(a)X_\lambda(a) - G) \leq 0, \lambda \in \mathcal{A} \quad (63)$$

we conclude that $X_{\mu_0}(a)(I - \mathcal{P}(\mu_0)) \mathcal{H}_1$ and $X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0)) \mathcal{H}_1$ are correspondingly maximal $\mathcal{R}(a)$-nonnegative and maximal $\mathcal{R}(a)$-nonpositive subspaces which are $\mathcal{R}(a)$-neutral and which are $\mathcal{R}(a)$-orthogonal in view of Remark 3.2 from [21], Theorem 2.1 and (53). Hence

$$(X_{\mu_0}(a)(I - \mathcal{P}(\mu_0)) \mathcal{H}_1)^{[1]} = X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0)) \mathcal{H}_1$$

in view of [2] p.73 (here by $[1]$ we denote $\mathcal{R}(a)$-orthogonal complement). Therefore $X_{\mu_0}(a)(I - \mathcal{P}(\mu_0)) \mathcal{H}_1$ is hypermaximal $\mathcal{R}(a)$-neutral subspace in view of [2] p.43. Thus we obtain that in view of [3] p.42 that $\dim \mathcal{H}_+(a) = \dim \mathcal{H}_-(a)$, where $\mathcal{H}_+(a)$ are analogs of $\mathcal{H}_+$ for $\mathcal{R}(a)$. In view of (63) $X_{\mu_0}^{-1}(a) \mathcal{H}_+(a)$ and $X_{\bar{\mu}_0}^{-1}(a) \mathcal{H}_-(a)$ are correspondingly maximally uniformly $G$-positive and maximal uniformly $G$-negative subspaces. Therefore $\mathcal{H}_1$ is equal to the direct
and $G$-orthogonal sum of these subspaces in view of \textbf{[53]} and \textbf{[2]} p.75. Hence we obtain \textbf{[61]} in view of the law of inertia \textbf{[2]} p.54.

Sufficiency follows from Theorem 4.4. from \textbf{[21]}. Theorem is proved. \hfill $\square$ 

It is obvious that in Theorem 2.2 the point $a$ can be replaced by the point $b$ if $b < \infty$, but cannot be replaced by the point $b$ if $b = \infty$ as the example of operator $id/dt$ on the semi-axis shows. Also this example shows that condition \textbf{[60]} is not necessary for the fulfilment of the condition $U[x_{\mu_0}(a, F)] = 0$ only.

In the case of self-adjoint boundary conditions the analogue of Theorem 2.2 for regular differential operators in space of vector-functions was proved in \textbf{[29]} (see also \textbf{[31]}). For finite canonical systems depending on spectral parameter in a linear manner such analogue was proved in \textbf{[27]}. These analogs were obtained in a different way comparing with Theorem 2.2.

Let us consider operator differential expression $l_\lambda$ of \textbf{[61]} type with coefficients $p_j = p_j(t, \lambda), q_j = q_j(t, \lambda), s_j = s_j(t, \lambda)$ and of order $r$. Let $-l_\lambda$ depends on $\lambda$ in Nevanlinna manner. Namely, from now on the following condition holds:

\textbf{(B)} The set $B \supseteq \mathbb{C} \setminus \mathbb{R}^1$ exists, any its points have a neighbourhood independent on $t \in \mathbb{I}$, in this neighbourhood coefficients $p_j(t, \lambda), q_j(t, \lambda), s_j(t, \lambda)$ of the expression $l_\lambda$ are analytic $\forall t \in \mathbb{I}; \forall \lambda \in B, p_j(t, \lambda), q_j(t, \lambda), s_j(t, \lambda) \in C^j(\mathbb{I}, B(\mathcal{H}))$ and

\begin{equation}
\left.\frac{p_n^{-1}(t, \lambda)}{B(\mathcal{H})} \right|_{r = 2n}, (q_{n+1}(t, \lambda) + s_{n+1}(t, \lambda))^{-1} \in B(\mathcal{H}) |_{r = 2n + 1}, t \in \mathbb{I}; \quad (64)
\end{equation}

these coefficients satisfy the following conditions

\begin{equation}
l_j(t, \lambda) = p_j(t, \lambda), q_j(t, \lambda) = s_j(t, \lambda), \lambda \in B
\end{equation}

\begin{equation}
\lambda = l_\lambda \Leftrightarrow \lambda = l^{**}_\lambda \quad \text{in view of \textbf{[11]}}
\end{equation}

\forall h_0, \ldots, h_{[\frac{r}{2}]} \in \mathcal{H}:

\begin{equation}
\Im \left( \sum_{j=0}^{[\frac{r}{2}]} (p_j(t, \lambda) h_j, h_j) + \frac{r+1}{2} \sum_{j=1}^{[\frac{r+1}{2}]} \left\{ (s_j(t, \lambda) h_j, h_{j-1}) - (q_j(t, \lambda) h_{j-1}, h_j) \right\} \right) \leq 0,
\end{equation}

$t \in \mathbb{I}, \quad \Im \lambda \neq 0. \quad (66)$

Therefore the order of expression \Im $l_\lambda$ is even and therefore if $r = 2n + 1$ is odd, then $q_{m+1}, s_{m+1}$ are independent on $\lambda$ and $s_{n+1} = q^{*}_{n+1}$.

Condition \textbf{[66]} is equivalent to the condition: \Im \{(f, f) / 3 \lambda \leq 0, t \in \mathbb{I}, \Im \lambda \neq 0 \}.

Hence $W(t, \mu, -\Im l_\lambda) = \Im H(t, l_\lambda) / 3 \lambda \geq 0, t \in \mathbb{I}, \Im \lambda \neq 0$ due to Lemma \textbf{[11]} and Theorem \textbf{[12]} and therefore $H(t, l_\lambda)$ satisfy condition \textbf{(A)} with $A = B$. Therefore $\forall \mu \in B \cap \mathbb{R}^1$ \Im $W(t, \mu, -\Im l_\lambda) = \Im H(t, l_\lambda) / 3 \lambda \mid_{\lambda = \mu}$ is Bochner locally integrable in uniform operator topology. Here in view of \textbf{[8]}, \textbf{[16]} \forall $\mu \in B \cap \mathbb{R}^1$ \Im $\frac{\Im l_{\mu}}{3 \lambda} \equiv \Im \frac{\Im l_{\mu + \alpha} \mid_{\lambda = \mu}}{3 \lambda} = \frac{\partial l_{\mu}}{\partial \lambda} \mid_{\lambda = \mu}$, where the coefficients $\frac{\partial l_{\mu}}{\partial \lambda}, \frac{\partial l_{\mu}}{\partial \lambda}$ of expression $\partial_{\mu} / \partial_{\mu}$ are Bochner locally integrable in the uniform operator topology.

Let us consider in $\mathcal{H}_1 = \mathcal{H}^\tau$ the equation

\begin{equation}
\frac{i}{2} \left( (Q(t, l_\lambda) \tilde{\gamma}(t)') + Q^*(t, l_\lambda) \tilde{\gamma}'(t) \right) - H(t, l_\lambda) \tilde{\gamma}(t) = W(t, l_\lambda, -\Im l_\lambda) F(t).
\end{equation}
Lemma 2.1. Let $F$ be a c.o. of equation (5) as the fulfillment of (71) imply its fulfillment with $\delta(\lambda) > 0$ instead of $\delta$ for all $\lambda \in \mathcal{B}$.

**Lemma 2.1.** Let $M(\lambda)$ be a c.o. of equation (5), for which condition (71) holds with $P = I_r$, if $\mathcal{I}$ is infinite. Let $\exists \lambda \neq 0, \mathcal{H}'$-valued $F(t) \in L^2_{W(t, l_\lambda, m)}(\mathcal{I})$ (in particular one can set $F(t) = F(t, l_\lambda, m)$, where $f(t) \in C^s(\mathcal{I}, \mathcal{H}), m[f, f] < \infty$). Then the solution

$$x_\lambda(t, F) = R_\lambda F = \int_{\mathcal{I}} X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} \text{sgn} (s - t) (i G)^{-1} \right\} X_\lambda^*(s) W(s, l_\lambda, m) F(s) ds$$

(73)
of equation (70) with \( F(t) \) instead \( F(t, l, m) \), satisfies the following inequality
\[
\| R_\lambda F \|_{L^2_w(t, l, m)}^2 (\mathcal{I}) \leq 3 \| R_\lambda F \|_{L^2_w(t, l, m)} (\mathcal{I}) / 3 \lambda, \quad 3 \lambda \neq 0,
\]
where \( X_\lambda(t) \) is the operator solution of homogeneous equation (70) such that \( X_\lambda(0) = I_r \), \( G = RQ(0, l_\lambda) \); integral (73) converges strongly if \( \mathcal{I} \) is infinite.

**Proof.** Let us denote
\[
K(t, s, \lambda) = X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} \text{sgn}(s - t)(iG)^{-1} \right\} X_\lambda^*(s)
\]

If (71) holds with \( P = I_r \) if \( \mathcal{I} \) is infinite, then in view of (69) and [21, p.166] there exists a locally bounded on \( s \) and on \( \lambda \) constant \( k(s, \lambda) \) such that
\[
\forall h \in \mathcal{H} : \quad \|K(t, s, \lambda)h\|_{L^2_w(t, l, m)} (\mathcal{I}) \leq k(s, \lambda) \|h\|.
\]
Hence integral (73) converges strongly if \( \mathcal{I} \) is in finite.

Let \( F(t) \) have compact support and \( \text{supp}F(t) \subseteq [\alpha, \beta] \). Then in view of (12)
\[
\int_\alpha^\beta \left( W(t, l, m) - \frac{3l}{3}\lambda \right) R_\lambda F, R_\lambda F \right) dt - \frac{3}{3\lambda} \int_\alpha^\beta \left( W(t, l, m) R_\lambda F, R_\lambda F \right) dt \]
\[
= \frac{1}{2} \left( \frac{\text{R}(t, l, m) R_\lambda F, R_\lambda F}{\lambda} \right) \right|_\alpha^\beta \leq 0
\]
where the last inequality is a corollary of \( n^2 \). Theorem 1.1 from [21, p.162] and the following

**Lemma 2.2.** Let \( F_\lambda \) is the set of \( \mathcal{H}^r \)-valued function from \( L^2_w(t, l, m) (\alpha, \beta) \),
\[
I_\lambda(\alpha, \beta) F = \int_\alpha^\beta X_\lambda^*(t) W(t, l, m) F(t) dt, \quad F(t) \in F_\lambda
\]
Then
\[
I_\lambda(\alpha, \beta) F \in \left\{ \text{Ker} \int_\alpha^\beta X_\lambda^*(t) W(t, l, m) X_\lambda(t) dt \right\} \subseteq N^\perp.
\]

**Proof.** Let \( h \in \text{Ker} \int_\alpha^\beta X_\lambda^*(t) W(t, l, m) X_\lambda(t) dt \Rightarrow W(t, l, m) X_\lambda(t) h = 0 \Rightarrow I_\lambda(\alpha, \beta) F \perp h \). The second enclosure in (78) is a corollary of condition (69). Lemma 2.2 and inequality (76) are proved.

Thus Lemma 2.1 is proved if \( \mathcal{I} \) is finite. Let us prove it for infinite \( \mathcal{I} \). Let finite intervals \( (\alpha_n, \beta_n) \uparrow \mathcal{I}, \quad F_n = \chi_n F \), where \( \chi_n \) - is a characteristic function of \( (\alpha_n, \beta_n) \). If \( (\alpha, \beta) \subseteq (\alpha_n, \beta_n) \) then
\[
\| R_\lambda F_n \|_{L^2_w(t, l, m)} (\mathcal{I}) \leq \frac{\|F\|_{L^2_w(t, l, m)} (\mathcal{I})}{|3\lambda|}
\]
in view of (76), (69). But local uniformly on \( t \): \( (R_\lambda F_n)(t) \rightarrow (R_\lambda F)(t) \), in view of (75). Hence
\[
\| R_\lambda F \|_{L^2_w(t, l, m)} (\mathcal{I}) \leq \frac{\|F\|_{L^2_w(t, l, m)} (\mathcal{I})}{|3\lambda|}.
\]
for any finite \((\alpha, \beta)\). Hence \((79)\) holds with \(I\) instead of \((\alpha, \beta)\). In view of last fact \(R_\lambda F_n \to R_\lambda F\) in \(L^2_W(t, l, \cdot, -\frac{\alpha}{\omega}) (I)\). Hence \((74)\) is proved since it is proved for \(F_n\). Lemma 2.1 is proved.

Let us notice that in view of \((21)\) \(PM(\lambda) P\) is a c.o. of equation \((5)\), if \(M(\lambda)\) is its c.o. Obviously the closures of operators \(R_\lambda\) corresponding to c.o.s \(M(\lambda)\) and \(PM(\lambda) P\) are equal in \(B \left( L^2_W(t, l, \cdot, m) (I), L^2_W(t, l, \cdot, -\frac{\alpha}{\omega}) (I) \right)\).

Let us notice what in view of \((68)\) \(l\) can be a solution of equation \((70)\) for any finite \(n\). Let us denote by \(\lambda\) of Proposition 3.1.

Proof. Let us notice that in view of \((21)\) \(PM(\lambda) P\) is a c.o. of equation \((5)\), if \(M(\lambda)\) is its c.o. Obviously the closures of operators \(R_\lambda\) corresponding to c.o.s \(M(\lambda)\) and \(PM(\lambda) P\) are equal in \(B \left( L^2_W(t, l, \cdot, m) (I), L^2_W(t, l, \cdot, -\frac{\alpha}{\omega}) (I) \right)\).

Let us notice what in view of \((68)\) \(l\) can be a represented in form \((2)\) where

\[ l = \Re l_i, n_\lambda = l_\lambda - l - \lambda m; \exists n_\lambda \{ f, f \} / \exists \lambda \geq 0, t \in \bar{I}, \exists \lambda \neq 0. \quad (80) \]

From now on we suppose that \(l_\lambda\) has a representation in \((2), (80)\) and therefore the order of \(n_\lambda\) is even.

3. Main results

We consider pre-Hilbert spaces \(H\) and \(H\) of vector-functions \(y(t) \in C^0(\bar{I}, \mathcal{H})\) and \(y(t) \in C^s(\bar{I}, \mathcal{H}), m[y(t), y(t)] < \infty\) correspondingly with a scalar product

\[ (f(t), g(t))_m = m[f(t), g(t)], \]

where \(m[f, g]\) is defined by \((32)\) with expression \(m\) from condition \((68)\) instead of \(L\).

The null-elements of \(H\) is given by

Proposition 3.1. Let \(f(t) \in H\). Then

\[ m[f, f] = 0 \iff m[f] = f^{[s]}(t) = \ldots = f^{[s/2]}(t) = 0, \quad t \in \bar{I}. \]

Proof. Let us denote by \(m(t) \in B(\mathcal{H}^{n+1})\) the operator matrix corresponding to the quadratic form in left side of \((68)\). Since \(m(t) \geq 0\) one has

\[ m[f, f] = 0 \iff m(t) \text{col} \left\{ f(t), \ldots, f^{[s/2]}, 0, \ldots, 0 \right\} = 0 \iff f^{[s]}(t) = \ldots = f^{[s/2]} = 0 \]

\(\Box\)

Example 3.1. Let \(\dim \mathcal{H} = 1, s = 2, \tilde{p}_1(t) > 0, |\tilde{q}_1(t)|^2 = 4\tilde{p}_1(t) \tilde{p}_0(t)\). Then for expression \(m\) the first inequality \((68)\) holds and \(m\{f_0, f_0\} \equiv 0\) for \(f_0(t) = \exp \left( \frac{1}{2} \int_0^t \tilde{q}_1/\tilde{p}_1 \, dt \right) \neq 0\) in view of Proposition 3.1.

By \(L^2_m(I)\) and \(L^2_m(I)\) we denote the completions of spaces \(\hat{H}\) and \(H\) in the norm \(||\cdot||_m = \sqrt{(\cdot, \cdot)_m}\) correspondingly. By \(\hat{H}\) we denote the orthoprojection in \(L^2_m(I)\) onto \(L^2_m(I)\).

Theorem 3.1. Let \(M(\lambda)\) is a c.o. of equation \((5)\), for which the condition \((71)\) with \(P = I_r\) holds if \(I\) is infinite. Let \(\exists \lambda \neq 0, f(t) \in H\) and

\[ \text{col} \left\{ y_j(t, \lambda, f) \right\} = \int_I X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} \text{sgn} (s - t) (iG)^{-1} \right\} X_\lambda^*(s) W(s, l, m) F(s, l, m) \, ds, \quad y_j \in \mathcal{H}. \quad (81) \]

be a solution of equation \((70)\), that corresponds to equation \((1)\), where \(X_\lambda(t)\) is the operator solution of homogeneous equation \((70)\) such that \(X_\lambda(0) = I_r; G = \text{Re} Q(0, l, \lambda)\) (if \(I\) is infinite
integral (81) converges strongly). Then the first component of vector function (81) is a solution of equation (1). It defines densely defined in $L^2_m (I)$ integro-differential operator

$$R (\lambda) f = y_1(t, \lambda, f), \ f \in H$$

which has the following properties after closing $f^\circ$

$$R^* (\lambda) = R (\bar{\lambda}), \ \exists \lambda \neq 0$$

(83)

$^\circ$

$R (\lambda)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}^1$

(84)

$$\| R (\lambda) f \|^2_{L^2_m (I)} \leq \frac{\exists (R (\lambda) f, f)_{L^2_m (I)}}{\exists \lambda}, \ \exists \lambda \neq 0, \ f \in L^2_m (I)$$

(85)

Let us notice that the definition of the operator $R (\lambda)$ is correct. Indeed if $f (t) \in H$, $m [f, f] = 0$, then $R (\lambda) f \equiv 0$ since $W (t, l_\lambda, m) F(t, l_\lambda, m) \equiv 0$ due to (41), (69).

**Proof.** In view of Lemma 2.1 integral (81) converges strongly if $I$ is infinite. In view of Theorem 1.1 $y_1 (t, \lambda, f) (82)$ is a solution of equation (1).

In view of (68), (35)

$$|\mathcal{I} (R (\lambda) f, f)_{L^2_m (I)} | \leq (\lambda) f, f)_{L^2_\alpha, \beta} (\alpha, \beta) - \frac{\exists (R (\lambda) f, f)_{L^2_m (\alpha, \beta)}}{\exists \lambda} =$$

$$= \| R (t, l_\lambda, m) \|_{L^2 W(t, l_\lambda, m)} (\alpha, \beta) - \frac{\exists (\mathcal{R}_\lambda F(t, l_\lambda, m), F(t, l_\lambda, m))_{L^2 W(t, l_\lambda, m)} (\alpha, \beta)}{\exists \lambda}.$$ (86)

In view of Lemma 2.1 a nonnegative limit of the right-hand-side of (86) exists, when $(\alpha, \beta) \uparrow I$. Hence (85) is proved.

Let $H^\circ$-valued $F(t) \in L^2 W(t, l_\lambda, m) (I)$. Then in view of (69), Lemma 2.1 (19) one has

$$\| \mathcal{R}_\lambda F \|^2_{L^2 W(t, l_\lambda, m)} (I) \leq \| \mathcal{R}_\lambda F \|^2_{L^2 W(t, l_\lambda, -\frac{\lambda}{2}) (I)} \leq \frac{\exists (\mathcal{R}_\lambda F, F)_{L^2 W(t, l_\lambda, m)} (I)}{\exists \lambda},$$ (87)

$$\| \mathcal{R}_\lambda F \|^2_{L^2 W(t, l_\lambda, m)} (I) \leq \| \mathcal{R}_\lambda F \|^2_{L^2 W(t, l_\lambda, -\frac{\lambda}{2}) (I)} = \| \mathcal{R}_\lambda F \|^2_{L^2 W(t, l_\lambda, -\frac{\lambda}{2}) (I)}.$$ (88)

In view of (87), (88) we have

$$\| \mathcal{R}_\lambda F \|^2_{L^2 W(t, l_\lambda, m)} (I) \leq \| F \|^2_{L^2 W(t, l_\lambda, m)} (I) / \| \exists \lambda |,$$ (89)

$$\| \mathcal{R}_\lambda F \|^2_{L^2 W(t, l_\lambda, m)} (I) \leq \| F \|^2_{L^2 W(t, l_\lambda, m)} (I) / \| \exists \lambda |.$$ (90)

Let $F(t) \in L^2 W(t, l_\lambda, m) (I), G(t) \in L^2 W(t, l_\lambda, m) (I)$ are $H^\circ$-valued functions with compact support. We have

$$\langle \mathcal{R}_\lambda F, G \rangle_{L^2 W(t, l_\lambda, m)} (I) = \langle F, \mathcal{R}_\lambda G \rangle_{L^2 W(t, l_\lambda, m)} (I).$$ (91)
since \( M(\lambda) = M^*(\bar{\lambda}) \). Due to inequalities (92), (93) equality (88) is valid for \( F(t), G(t) \) with non-compact support.

Now it follows from, (87), (91) that \( \forall f(t), g(t) \in H \)

\[
m[R(\lambda)f, g] - m[f, R(\bar{\lambda})g] = (\mathcal{R}_\lambda F(t, l, m), G(t, l, m))_{L^2_w(t, l, m)} - (F(t, l, m), \mathcal{R}_\lambda G(t, l, m))_{L^2_w(t, l, m)} = 0
\]

Thus the closure of the operator \( R(\lambda)f \) in \( L^2_m(I) \) possesses property (84).

Since in view of (85) for any \( f(t), g(t) \in H \)

\[
(R(\lambda)f, g)_{L^2_m(\alpha, \beta)} \rightarrow (R(\lambda)f, g)_{L^2_m(I)} \quad \text{as} \quad (\alpha, \beta) \uparrow I
\]

uniformly in \( \lambda \) from any compact set from \( \mathbb{C} \setminus \mathbb{R}^1 \), we see that, in view of the analyticity of the operator function \( M(\lambda) \) and vector-function \( W(t, l, m)F(t, l, \lambda) \) (see (29) with \( l = l_\lambda \)) the operator \( R(\lambda) \) depends analytically on the non-real \( \lambda \) in view of [16] p. 195). Theorem 3.1 is proved. \( \square \)

For \( r = 1, n_\lambda[y] = H_\lambda(t) y \) Theorem 3.1 is known [20].

Let us notice that if \( L^2_m(I) = L^2_m(\bar{I}) \) then Theorem 3.1 is valid with \( f(t) \in H^0 \) instead of \( f(t) \in H \) and without condition (71) with \( P = I_e \) for infinite \( I \).

The following theorem establishes a relationship between the resolvents \( R(\lambda) \) that are given by Theorem 3.1 and the boundary value problems for equation (1), (2) with boundary conditions depending on the spectral parameter. Similarly to the case \( n_\lambda[y] = 0 \) [23] we see that the pair \( \{y, f\} \) satisfies the boundary conditions that contain both \( y \) derivatives and \( f \) derivatives of corresponding orders at the ends of the interval.

**Theorem 3.2.** Let the interval \( I = (a, b) \) be finite and condition (71) with \( P = I_e \) holds.

Let the operator-functions \( \mathcal{M}_\lambda, \mathcal{N}_\lambda \in B(\mathcal{H}^r) \) depend analytically on the non-real \( \lambda \),

\[
\mathcal{M}_\lambda^* [\mathcal{R}Q(a, l_\lambda)] \mathcal{M}_\lambda = \mathcal{N}_\lambda^* [\mathcal{R}Q(b, l_\lambda)] \mathcal{N}_\lambda \quad (\exists \lambda \neq 0),
\]

where \( Q(t, l_\lambda) \) is the coefficient of equation (70) corresponding by Theorem 1.1 to equation (1),

\[
\| \mathcal{M}_\lambda^* h \| + \| \mathcal{N}_\lambda^* h \| > 0 \quad (0 \neq h \in \mathcal{H}^r, \exists \lambda \neq 0),
\]

the linear \( \{\mathcal{M}_\lambda h \oplus \mathcal{N}_\lambda h \} \in \mathcal{H}^r \) is a maximal \( \mathcal{Q} \)-nonnegative subspace if \( \exists \lambda \neq 0 \), where \( \mathcal{Q} = (\exists \lambda) \text{diag } (\mathcal{R}Q(a, l_\lambda), -\mathcal{R}Q(b, l_\lambda)) \) (and therefore

\[
\exists \lambda (\mathcal{N}_\lambda^* [\mathcal{R}Q(a, l_\lambda)] \mathcal{N}_\lambda - \mathcal{M}_\lambda^* [\mathcal{R}Q(a, l_\lambda)] \mathcal{M}_\lambda) \leq 0 \quad (\exists \lambda \neq 0).
\]

Then

1°. For any \( f(t) \in H \) the boundary problem that is obtained by adding the boundary conditions

\[
h = h(\lambda, f) \in \mathcal{H}^r: \quad \bar{g}(a, \lambda, m, f) = \mathcal{M}_\lambda h, \quad \bar{g}(b, \lambda, m, f) = \mathcal{N}_\lambda h
\]

to the equation (1), where \( \bar{g}(t, \lambda, f) \) is defined by (28), has the unique solution \( R(\lambda)f \) in \( C^r(I, \mathcal{H}) \) as \( \exists \lambda \neq 0 \). It is generated by the resolvent \( R(\lambda) \) that is constructed, as in Theorem 3.1 using the c.o.

\[
M(\lambda) = -\frac{1}{2} \left( X^{-1}_\lambda(a) \mathcal{M}_\lambda + X^{-1}_\lambda(b) \mathcal{N}_\lambda \right) \left( X^{-1}_\lambda(a) \mathcal{M}_\lambda - X^{-1}_\lambda(b) \mathcal{N}_\lambda \right)^{-1} (iG)^{-1},
\]

where

\[
\left( X^{-1}_\lambda(a) \mathcal{M}_\lambda - X^{-1}_\lambda(b) \mathcal{N}_\lambda \right)^{-1} \in B(\mathcal{H}^r) \quad (\exists \lambda \neq 0),
\]
$X_\lambda(t)$ is an operator solution of the homogeneous equation (70) such that $X_\lambda(0) = I_r$.

For any operator $R(\lambda)$ from Theorem 3.1 vector-function $R(\lambda)f$ ($f \in H$) is a solution of some boundary problem as in $I^\circ$.

Let us notice that if $f(t) = g(t)$ then in boundary conditions (95): $\vec{y}(t, l, m, f) = \vec{y}(t, l, m, g)$ in view of (28) and Proposition 3.1.

Proof. The proof of Theorem 3.2 follows from Theorems 1.1, 3.1 and from [21, Remark 1.1]. □

For the case $n_\lambda [y] \equiv 0$, Theorem 3.2 is known [21].

The example below show that the following is possible: for some resolvent $R(\lambda)$ from Theorem 3.1 $\exists f_0(t) \neq 0$ such that $m[f_0] = 0$ and therefore the "resolvent" equation (11) for $R(\lambda)f_0$ is homogeneous but $R(\lambda)f_0 \neq 0, \exists \lambda \neq 0$.

Example 3.2. Let $m$ in (11) be such expression that equation $m[f] = 0$ has a solution $f_0(t) \neq 0$.

Let in Theorem 3.2 $M_\lambda = \begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $N_\lambda = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}$, $R(\lambda)$ is the corresponding resolvent. Then $R(\lambda)f_0 \neq 0, \exists \lambda \neq 0$, while if $M_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ I_n & 0 & 0 \end{pmatrix}$, $N_\lambda = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}$ then for the corresponding resolvent $R(\lambda)f_0 \neq 0, \exists \lambda \neq 0$ (and therefore in view of [14, p. 87] $E_\infty f_0 = 0$ for generalized spectral family $E_\mu$, which corresponds to $R(\lambda)$ by (3)).

It is known [14, p.86] that the operator-function $R(\lambda)$ (83)-(85) can be represented in the form

$$R(\lambda) = (T(\lambda) - \lambda)^{-1},$$

where $T(\lambda)$ is such linear relation that

$$\exists T(\lambda) \leq 0 \ (\text{max}), \ T(\lambda) = T^*(\lambda), \ \lambda \in \mathbb{C}_+,$$

the Cayley transform $C_\mu(T(\lambda))$ defines a holomorphic function in $\lambda \in \mathbb{C}_+$ for some (and hence for all) $\mu \in \mathbb{C}_+$. Applications of abstract relations of $T(\lambda)$ type (Nevanlinna families) to the theories of boundary relations and of generalized resolvents are proposed in [12, 13].

The description of $T(\lambda)$ corresponding to $R(\lambda)$ from Theorem 3.1 in regular case gives
Corollary 3.1. Let \( \mathcal{I} \) is finite and condition (71) with \( P = I_r \) holds. Let us consider the relation \( T(\lambda) = T'(\lambda) \) as \( \exists \lambda \neq 0 \), where

\[
T'(\lambda) = \left\{ \begin{array}{l}
\{ \tilde{y}(t), \tilde{f}(t) \} | \tilde{y}(t) \mid \mathcal{L}^s_{m}(\mathcal{I}) y(t) \in C^r(\mathcal{I}), \tilde{f}(t) \mid \mathcal{L}^s_{m}(\mathcal{I}) f(t) \in H, (l-n\lambda)[y] = m[f], \\
\tilde{y}(t,l-n\lambda,m,f) \) satisfy boundary condition
\end{array} \right.
\]

\( \exists h = h(\lambda,f) \in \mathcal{H}^r : \tilde{y}(a,l-n\lambda,m,f) = M_\lambda h, \tilde{y}(b,l-n\lambda,m,f) = N_\lambda h, \) where operators \( M_\lambda, N_\lambda \) satisfy the conditions of Theorem 3.2.

\[
\tilde{y}(t,l-n\lambda,m,f) \text{def} \tilde{y}(t,l,m,f) |_{m=0} = \begin{cases}
\left( \sum_{j=0}^{n-1} \oplus y^{(j)}(t) \right) \oplus \left( \sum_{j=1}^{n} \oplus (y^{[s'-j]}(t) | l-n\lambda) - f^{[s-j]}(t | m) \right), & r = 2n \\
\left( \sum_{j=0}^{n-1} \oplus y^{(j)}(t) \right) \oplus \left( \sum_{j=1}^{n} \oplus (y^{[r-j]}(t) | l-n\lambda) - f^{[s-j]}(t | m) \right) \oplus (-iy^{(n)}(t)), & r = 2n + 1 > 1
\end{cases}
\]

(98)

Then

\( ^* \). \( (T(\lambda) - \lambda)^{-1} \) is equal to resolvent \( R(\lambda) \) (81), (82) from Theorem 3.1 corresponding to c.o. \( M(\lambda) \) (99).

\( ^2 \). Let \( R(\lambda) \) is resolvent (81), (82) from Theorem 3.1. Then \( R(\lambda) = (T(\lambda) - \lambda)^{-1} \), where \( T(\lambda) \) is some relation as in item \( ^* \).

Proof. The proof follows from (28), Lemma 1.2, Theorem 3.2 and Remark 1.1 from [21]. \( \Box \)

Let in (1), (2) \( n\lambda[y] \equiv 0 \) i.e. \( l\lambda = l - \lambda m \), where \( l = l^* \), \( m = m^* \) and coefficients of expressions \( m \) satisfy condition (68).

We consider in \( \mathcal{L}^s_{m}(\mathcal{I}) \) the linear relation

\[
\mathcal{L}'_0 = \left\{ \{ \tilde{y}(t), \tilde{g}(t) \} | \tilde{y}(t) \mid \mathcal{L}^s_{m}(\mathcal{I}) y(t), \tilde{g}(t) \mid \mathcal{L}^s_{m}(\mathcal{I}) g(t), y(t) \in C^r(\mathcal{I}, \mathcal{H}), g(t) \in H, l[y] = m[g], \tilde{g}(t,l,m,g) \right\}
\]

is equal to zero in the edge of \( \mathcal{I} \) if this edge is finite and \( \tilde{g}(t,l,m,g) \) is equal to zero in the some neighbourhood of the edge of \( \mathcal{I} \) if this edge is infinite

(99)

where \( \tilde{g}(t,l,m,g) \) is defined by (89) with \( l\lambda = l - \lambda m, f = g \).

Below we assume that relation \( \mathcal{L}'_0 \) consists of the pairs of \( \{ y, g \} \) type.

The relation \( \mathcal{L}'_0 \) is symmetric due to the following Green formula with \( \lambda_k = 0 \):

\footnote{Let us notice that vector-function \( g(t) \) in (99) may be non-equal to zero in the finite edge or in the some neighbourhood of infinite edge of \( \mathcal{I} \).}
Let \( y_k (t) \in C^r ([\alpha, \beta], \mathcal{H}) \), \( f_k (t) \in C^s ([\alpha, \beta], \mathcal{H}) \), \( \lambda_k \in \mathbb{C} \), \( l \ [y_k] - \lambda_k m \ [y_k] = m \ [f_k] \), \( k = 1, 2 \). Then
\[
\int_\alpha^\beta m \{f_1, y_2\} \, dt = \int_\alpha^\beta m \{y_1, f_2\} \, dt + (\lambda_1 - \bar{\lambda}_2) \int_\alpha^\beta m \{y_1, y_2\} \, dt = \]
\[
= i (\mathbb{R}Q(t, l_\lambda) \bar{y}_1(t, l_{\lambda_1}, m, f_1), \bar{y}_2(t, l_{\lambda_2}, m, f_2)) |^{\bar{\beta}}_\alpha, \quad (100)
\]
where \( \bar{y}_k(t, l_{\lambda_k}, m, f_k) \) for \( \lambda_k \in \mathbb{R}^1 \) is defined by (98) with \( l_{\lambda} = l - \lambda m \).

This formula is a corollary of Theorem 3.3 if \( \Im \lambda \neq 0 \). For its proof for example in the case \( \lambda_1 \in \mathbb{R}^1 \) we need to modify (12) for equation \( l \ [y_1] - (\lambda_1 + i \varepsilon) m \ [y] = m \ [f_1 - i \varepsilon y_1] \) and then to pass to limit in (12) as \( \varepsilon \to +0 \).

In general the relation \( L_0^0 \) is not closed. We denote \( L_0 = \mathcal{L}_0^0 \).

**Theorem 3.3.** Let \( l_{\lambda} = l - \lambda m \) and the conditions of Theorem 3.1 hold. Then the operator \( R(\lambda) \) from Theorem 3.1 is the generalized resolvent of the relation \( L_0 \). Let \( I \) be finite and additionally the condition (71) hold. Then every such generalized resolvent can be constructed as the operator \( R(\lambda) \).

**Proof.** In view of (14) and taking into account properties (83)-(85) of the operator \( R(\lambda) \) it is sufficient to prove that \( R(\lambda) \ (L_0 - \lambda) \subseteq I \), where \( I \) is a graph of the identical operator in \( L_0^2 (\mathcal{I}) \). But this proposition is proved similarly to (12) taking into account (100) and the fact that in view of (101) \( (\bar{y} - y)(t, l - \lambda m, m, 0) = \bar{y}(t, l - \lambda m, m, g - \lambda y) - \bar{y}(t, l, m, g) \) if \( y, g \in L_0^0, \bar{y} = R(\lambda)(g - \lambda y) \).

Conversely let \( I \) be finite, \( R_\lambda \) a generalized resolvent of relation \( L_0 \). We denote \( N_\lambda = \{ y(t) \in C^r (\mathcal{I}, \mathcal{H}), \lambda \in \mathcal{B}, l \ [y] - \lambda m \ [y] = 0 \} \). We need the following

**Lemma 3.1.** Let condition (71) hold. Then the lineal \( N_\lambda \) is closed in \( L_0^2 (\mathcal{I}) \).

**Proof.** The proof of Lemma 3.1 follows from (34). \( \square \)

**Lemma 3.2.** Let \( \lambda \in \mathcal{B} \). Then \( R (L_0^0 - \lambda) = N_\lambda^+ \).

**Proof.** Let \( x(t) \in N_\lambda \), \( f(t) \in H \). Then a solution of the following Cauchy problem:
\[
l \ [y] - \lambda m \ [y] = m \ [f] , \quad \bar{y}(a, l_{\lambda}, m, f) = 0.
\]
Then
\[
m \ [f, x] = i (\mathbb{R}Q(b, l_{\lambda}) \bar{y}(b, l_{\lambda}, m, f), \bar{x}(b, l_{\lambda}, m, 0)) \quad (102)
\]
in view of Green formula (100). Therefore \( R (L_0^0 - \lambda) \subseteq N_\lambda^+ \).

Let \( g(t) \in N_\lambda^+ \). Then \( \exists H \ni g_n \overset{L_0^2 (\mathcal{I})}{\rightarrow} g, \ n = x_n \oplus f_n, \ x_n \in N_\lambda, \ f_n \in N_\lambda^+ \Rightarrow f_n \in H \). Let \( y_n \) be a solution of problem (101) with \( f_n \) instead of \( f \). In view of (102) with \( f = f_n \), one has:
\[
\bar{y}_n(b, l_{\lambda}, m, f_n) = 0 \Rightarrow f_n \in R (L_0^0 - \lambda). \quad \text{But} \quad f_n \overset{L_0^2 (\mathcal{I})}{\rightarrow} f. \quad \text{Therefore} \quad R (L_0^0 - \lambda) \subseteq N_\lambda^+. \quad \text{Lemma 3.2 is proved.} \quad \square
\]

**Lemma 3.3.** Let the condition (71) hold, \( \lambda \in \mathcal{B} \). Let \( \{ \bar{y}, \bar{f} \} \in L_0^* - \lambda, \quad \bar{f} \overset{L_0^2 (\mathcal{I})}{\rightarrow} f \in H \). Then \( \bar{y} \overset{L_0^2 (\mathcal{I})}{\rightarrow} y \in C^r (\mathcal{I}, \mathcal{H}) \) and \( y(t) \) satisfies the equation (11).

**Proof.** Let \( C^r (\mathcal{I}, \mathcal{H}) \ni y_0 \) be a solution of (11). Let \( \{ \varphi, \psi \} \in L_0^* - \lambda \). Then \( \varphi (a, l_{\lambda}, m, \psi) = \bar{\varphi}(b, l_{\lambda}, m, \psi) = 0 \) in view of (38), (99). Hence \( m \ [\varphi, f] = m \ [\psi, y_0] \) due to Green formula (100). But \( m \ [\varphi, f] = (\psi, \bar{y})_{L_0^2 (\mathcal{I})} \) in view of the definition of the adjoint relation. Hence
where $\Delta$ is the linear way. Therefore $\tilde{y} - y_0 \overset{L^2_n (\mathcal{I})}{=} y - y_0 \in N_\lambda$ in view of Lemmas 3.1, 3.2. Hence $\tilde{y} \overset{L^2_n (\mathcal{I})}{=} y \in C^r (\mathcal{I}, \mathcal{H})$ and $y$ is a solution of (1). Lemma 3.3 is proved. \hfill $\square$

We return to the proof of Theorem 3.3.

Let $f \in H$. Then in view of Lemma 3.3 $R_{\lambda} f \overset{L^2_n (\mathcal{I})}{=} y \in C^r (\mathcal{I}, \mathcal{H})$ and $y$ satisfies equation (1). Therefore taking into account Theorem 1.1, [10, p.148] and (53) we have

$$y (t) = [X_\lambda (t)]_1 \left\{ h - \frac{1}{2} (iG)^{-1} \left( \int_a^b \text{sn}(s - t) X_\lambda^* (s) W (s, l_\lambda, m) F (s, l_\lambda, m) ds \right) \right\}, \quad (103)$$

where $[X_\lambda (t)]_1 \in B (\mathcal{H}^r, \mathcal{H})$ is the first row of operator solution $X_\lambda (t)$ from Theorem 3.1 that is written in the matrix form, $h = h_\lambda (f) \in N_+^\perp$ is defined in the unique way in view of (34) and condition (71).

Let us prove that $h$ depends on $I_\lambda f \overset{\text{def}}{=} \int_a^b X_\lambda^* (s) W (s, l_\lambda, m) F (s, l_\lambda, m) ds$ in unique way. Operator $(I_\lambda : H \to N^\perp)$ in view of Lemma 2.2 Moreover $I_\lambda N^\perp = N^\perp$ i.e. $\forall h_0 \in N^\perp \exists f_0 \in H : h_0 = I_\lambda f_0$. For example we can set

$$f_0 = f_0 (t, \lambda) = [X_\lambda (t)]_1 \{ \Delta_\lambda (\mathcal{I}) |_{N_+^\perp} \}^{-1} h_0 \quad (104)$$

and to utilize the equality.

$$W (s, l_\lambda, m) F_0 (s, l_\lambda, m) = W (s, l_\lambda, m) X_\lambda (s) \{ \ldots \}^{-1} h_0$$

If $f (t), g (t) \in H$ are such functions that $I_\lambda f = I_\lambda g$, then in view of (103)

$$\exists \lambda \left( (\mathcal{R} Q (t, l_\lambda)) \Delta_\lambda (t, \lambda, m, f - g), \Delta_\lambda (t, \lambda, m, f - g) \right)^{\beta}_{\alpha} = \exists \lambda (\mathcal{R} Q (t, l_\lambda)) X_\lambda (t) (h_\lambda (f) - h_\lambda (g)), X_\lambda (t) (h_\lambda (f) - h_\lambda (g)) \right)^{\beta}_{\alpha}, \quad (105)$$

where $\Delta y = R_{\lambda} [f - g]$. But in view of (103) the left hand side of (105) is nonpositive since $R_{\lambda}$ has property of (85) type. The right hand of (103) is nonnegative in view of (42). Hence $h_\lambda (f) = h_\lambda (g)$ in view of (12), (71). Thus $h$ depends on $I_\lambda f$ in unique way and obviously in the linear way. Therefore

$$h = M (\lambda) I_\lambda f, \quad (106)$$

where $M (\lambda) : N_+^\perp \to N_+^\perp$ is a linear operator and so $R_{\lambda} f (f \in H)$ can be represented in the form (32).

Further, for definiteness, we will consider the most complicated case $r = s = 2n$.

Let us prove that $M (\lambda) \in B (N^\perp)$, $\exists \lambda \neq 0$. Let $h_0 \in N^\perp$, $y = R_{\lambda} f_0$, where $f_0 = f_0 (t, \lambda)$ see (104). Then in view of (103) and Theorem 1.2 we have

$$X_\lambda (t) M (\lambda) h_0 = Y (t, l_\lambda, m) - F_0 (t, m) - \frac{1}{2} X_\lambda (t) (iG)^{-1} (I_\lambda (a, \tau) - I_\lambda (b, \tau)) F_0, \quad (107)$$

where $Y (t, l_\lambda, m)$, $F_0 = F_0 (t, l_\lambda, m)$, $F_0 (t, m)$ are defined by (20), (37) correspondingly with $y$ and $f_0$ correspondingly instead of $f$, $I_\lambda (0, t) F_0$ is defined by (77). Therefore

$$\Delta_\lambda (a, b) M (\lambda) h_0 = \left. I_\lambda y - I_\lambda (a, b) \left( F_0 (t, m) + \frac{1}{2} X_\lambda (t) (iG)^{-1} (I_\lambda (a, \tau) - I_\lambda (b, \tau)) F_0 \right) \right|_{a,b}, \quad (108)$$
where \( I_{\bar{\lambda}} y, \ I_{\bar{\lambda}} (a, b) (\ldots) \in N^\perp \) in view of \((78)\). But

\[
\forall g \in \mathcal{H}^r : \ |(I_{\bar{\lambda}} y, g)| \leq \max_{t \in \mathcal{I}} \|X_{\lambda} (t)\| \left\{ \int_{\mathcal{I}} \|W (t, I_{\lambda}, m)\| \ dt \right\}^{1/2} \|R_{\lambda} f_0\|_{L_{m}^2 (\mathcal{I})} \|g\|
\]

in view of Cauchy inequality and \((54)\). Therefore

\[
\exists \text{ constant } \ c (\lambda) : \ |(I_{\bar{\lambda}} y, g)| \leq c (\lambda) \|g\| \|g\| \ (109)
\]

since

\[
\|R_{\lambda} f_0\|_{L_{m}^2 (\mathcal{I})} \leq \|\Delta_{\bar{\lambda}} (a, b)\|^{1/2} \|\left(\Delta_{\bar{\lambda}} (a, b) |_{N^\perp}\right)^{-1}\| \|h_0\| / |\Im \lambda| \n\]

in view of \((54)\), \((109)\) and inequality: \(\|R_{\lambda} f_0\|_{L_{m}^2 (\mathcal{I})} \leq \|f_0\|_{L_{m}^2 (\mathcal{I})} / |\Im \lambda|\).

Obviously \(|(I_{\bar{\lambda}} (a, b) (\ldots), g)|\) satisfies the estimate of type \((109)\). Therefore \(M (\lambda) \in B (N^\perp)\).

Now we have to prove that \(M (\lambda)\) is a c.o. e equation \((67)\).

Let us prove that \(M (\lambda)\) is strongly continuous for nonreal \(\lambda\). To prove this fact it is enough to verify it for \(\Delta_{\bar{\lambda}} (a, b) M (\lambda)\); while the last one obviously follows from strongly continuity of vector-function \(I_{\bar{\lambda}} R_{\lambda} f_0 (t, \lambda)\).

In view of \((54)\) we have \(\forall g \in \mathcal{H}^r\)

\[
(I_{\bar{\lambda}} R_{\lambda} f_0 (t, \lambda) - I_{\bar{\mu}} R_{\lambda} f_0 (t, \mu), g) = m [R_{\lambda} f_0 (t, \lambda), [X_{\lambda} (t)]_1 g] - m [R_{\mu} f_0 (t, \mu), [X_{\mu} (t)]_1 g].
\]

Then the required statement can be derived from the equality

\[
m \{[X_{\lambda} (t) - X_{\mu} (t)]_1 g, [X_{\lambda} (t) - X_{\mu} (t)]_1 g\} = (W (t, l_{\lambda}, m) ([X_{\lambda} (t) - X_{\mu} (t)] g + (\lambda - \mu) F(t, m)), (X_{\lambda} (t) - X_{\mu} (t)) g + (\lambda - \mu) F(t, m)),
\]

where \(F(t, m)\) is defined by \((37)\) with \(f (t) = [X_{\mu} (t)]_1 g, \|X_{\lambda} (t) - X_{\mu} (t)\| \to 0\) uniformly in \(t \in [a, b]\), and from the analogous equality for \(m \{f_0 (t, \lambda) - f_0 (t, \mu), f_0 (t, \lambda) - f_0 (t, \mu)\}\).

Let us prove that \(M (\lambda)\) is analytic for nonreal \(\lambda\). To prove this fact it is enough in view of strongly continuity of \(M (\lambda)\) to prove the analyticity in \(\lambda\) of \((I_{\bar{\lambda}} M (\lambda) I_{\bar{\lambda}} f, g)\), where \(f (t) \in C^{\infty} (\mathcal{I}, \mathcal{H}), \ g \in \mathcal{H}^r, \ (\Im \lambda)(3\mu) > 0,\)

\[
I_{\bar{\lambda}} M (\lambda) I_{\bar{\lambda}} f, g) = m [R_{\lambda} f, [X_{\mu} (t)]_1 g] + (\lambda - \mu) \int_a^b \left( (R_{\lambda} f)^{[n]} (t | m, g^{(n)} (t) \right) dt + \text{ terms independent on } R_{\lambda} f \text{ and analytic in } \lambda, \ (110)
\]

where \(g^{(n)} (t) \overset{\text{def}}{=} (p_{m} - \bar{\mu} p_{m})^{-1} ([X_{\mu} (t)]_1 g)^{[n]} (t | m)\).

For scalar or vector-function \(F (\lambda)\) let us denote

\[
\Delta_{km} F (\lambda) = \frac{F (\lambda + \Delta_k \lambda) - F (\lambda)}{\Delta_k \lambda} - \frac{F (\lambda + \Delta_m \lambda) - F (\lambda)}{\Delta_m \lambda}.
\]

Let us denote

\[
R_n (\lambda) = \int_a^b \left( \bar{p}_n (R_{\lambda} f)^{[n]} (t | m, g^{(n)} (t) \right) dt.
\]
In view of (12) we have

$$|\Delta_{km} R_n (\lambda)| \leq (m [\Delta_{km} R_{\lambda} f, \Delta_{km} R_{\lambda} f])^{1/2} \left( \int_a^b (p_n g^{(n)}, g^{(n)}) dt \right)^{1/2}$$  \hspace{1cm} (111)

Therefore \( R_n (\lambda) \) depends analytically on nonreal \( \lambda \) in view of analyticity of \( R_{\lambda} \) and so analyticity of \( M (\lambda) \) is proved in view of (111).

Let us consider the solution \( x_{\lambda} (t, F) = R_{\lambda} F \) of equation (67). Let us prove that \( x_{\lambda} (t, F) \) satisfies the condition (82). Let us denote \( y (t, \lambda, f) = R_{\lambda} f \). Then in view of Green formula (12)

$$m [y, y] - \frac{3m [y, f]}{3\lambda} = \frac{1}{2} (\Re (Q (t, l_{\lambda}) \check{g}(t, l_{\lambda}, m, f), \check{g}(t, l_{\lambda}, m, f))) \bigg|_a^b /3\lambda$$  \hspace{1cm} (112)

But the left hand side of (112) is \( \leq 0 \) since \( R_{\lambda} f \) is generalized resolvent. So

$$\forall f \in H : \Re (Q (t, l_{\lambda}) \check{g}(t, l_{\lambda}, m, f), \check{g}(t, l_{\lambda}, m, f))) \bigg|_a^b /3\lambda < 0.$$  \hspace{1cm} (113)

But for every \( H^s \)-valued \( F (t) \in L^2_{W(t, l_{\lambda}, m)} (I) \) there exists such vector-function \( f (t) \in H \) that \( x_{\lambda} (a, F) = \check{g}(a, l_{\lambda}, m, f), x_{\lambda} (b, F) = \check{g}(b, l_{\lambda} m, f) \). So \( (55) \) is proved in view of (113).

To prove that \( M (\lambda) \) is a c.o. of equation (67) it remains to show that \( M (\lambda) = M^* (\lambda) \).

Let us consider the following operator \( \check{M} (\lambda) = M (\lambda) \), \( \check{M} (\bar{\lambda}) = M^* (\lambda) \), \( 3\lambda > 0 \).

This operator is a c.o. of equation (67) in view of (21). This c.o. generate by Theorem 3.1 the operator \( R (\lambda) \) (82).

But \( R (\lambda) = R_{\lambda}, 3\lambda > 0 \Rightarrow R (\bar{\lambda}) = R^* (\lambda) = R^*_{\lambda} = R_{\bar{\lambda}}, 3\lambda > 0 \Rightarrow \Rightarrow \forall f \in H:

\[
\left\| [X_{\lambda} (t)]_1 (M^* (\lambda) - M (\bar{\lambda})) \int_a^b X^*_\lambda (s) W (s, l_{\lambda}, m) F (s, l_{\lambda}, m) ds \right\|_m = 0.
\]

\( \Rightarrow \forall h \in N^\perp : \Delta_{\lambda} (a, b) (M (\bar{\lambda}) - M^* (\lambda)) h = 0 \Rightarrow \check{M} (\bar{\lambda}) = M^* (\lambda) .

Theorem 3.3 is proved. \( \Box \)

For generalized resolvents of differential operators a representation of (82) type was obtained in (35) for the scalar case and in (5) for the case of operator coefficients. For generalized resolvents for (10), (2) with \( s = 0, n_{\lambda} [y] \equiv 0 \) the representation of such a type was obtained in (6) [7 ] [19].

Let \( I_k, k = 1, 2 \) be finite intervals, \( I_1 \subset I_2 \). Then, in spite of the fact that \( f (t) \in C^s (I_2, H) \) but \( \chi_{I_2} f (t) \notin C^s (I_2, H) \), where \( \chi_{I_i} \) is the characteristic function of \( I_i \), one has.

**Corollary 3.2.** Let \( 0 \in I_1 \) and the condition (111) with \( \mathcal{I} = I_2 \) holds. Let \( R_{\lambda} \) be the generalized resolvent of the relation \( L_0 \) in \( L^2_m (I) \) with \( \mathcal{I} = I_2 \). Then by Theorems 3.1, 3.3 there exists c.o. \( M (\lambda) \) of equation (5) such that \( R_{\lambda} f = y_1 (t, \lambda, f) \) (31), \( t \in I = I_2, f \in H (= H (I_2)) \). Let us define the operator \( y_1 (t, \lambda, f) = R_{\lambda}^1 f, t \in I = I_1, f \in H (= H (I_1)) \) by the same formula (31) as operator \( R_{\lambda} f \), but with \( \mathcal{I} = I_1 \) instead of \( \mathcal{I} = I_2 \). Then this operator is (after closing) the generalized resolvent of the relation \( L_0 \) in \( L^2_m (I) \) with \( \mathcal{I} = I_1 \).

It is known [11] that (33) - (35) implies (3), where \( E_{\mu} \in B (L^2_m (I)) \), \( E_{\mu} = E_{\mu - 0} \),

$$0 \leq E_{\mu_1} \leq E_{\mu_2} \leq I, \mu_1 < \mu_2; E_{-\infty} = 0.$$  \hspace{1cm} (114)

Here \( I \) is the identity operator in \( L^2_m (I) \). We denote \( E_{\alpha \beta} = \frac{1}{2} [E_{\beta + 0} + E_{\beta} - E_{\alpha + 0} - E_{\alpha}] \).
Theorem 3.4. Let $M (\lambda)$ be the characteristic operator of equation (5) (and therefore by [21, p.162] $\exists \operatorname{PM} (\lambda) P/\exists \lambda \geq 0$ as $\exists \lambda \neq 0$) and $\sigma (\mu) = w - \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{0}^{\mu} \exists \operatorname{PM} (\mu + i \varepsilon) Pd\mu$ be the spectral operator-function that corresponds to $\operatorname{PM} (\lambda) P$.

Let for equation (5) the condition (114) be valid. Let $E_{\mu}$ be generalized spectral family (114) corresponding by (3) to the resolvent $R (\lambda)$ from Theorem 3.4 which is constructed with the help of c.o. $M (\lambda)$. Then for any $[\alpha, \beta] \subset \mathcal{B}$ the equalities

$$
P \varphi (\mu, f) = \begin{cases} \int_{\mathcal{I}} (|X_\mu (t)|^2) m [f] dt \text{ if } f (t) \in \mathcal{H} \text{, } \mathcal{I} \text{ is infinite,} \\ \int_{\mathcal{I}} (|X_\mu (t)|^2) W (t, t, m) F (t, l, m) dt \text{ if } f(t) \in \mathcal{H} \text{ or } f (t) \in H, \mathcal{I} \text{ is finite} \end{cases},
$$

for $\mathcal{I}$ in $L^2_m (\mathcal{I})$, where $|X_\lambda (t)| \in B (\mathcal{H}, \mathcal{H})$ is the first row of the operator solution $X_\lambda (t)$ of homogeneous equation (7) with coefficients (7), (8), (15), (16) that is written in the matrix form and such that $X_\lambda (0) = I_r$,

$$
\varphi (\mu, f) = \begin{cases} \int_{\mathcal{I}} (|X_\mu (t)|^2) m [f] dt \text{ if } f (t) \in \mathcal{H} \text{, } \mathcal{I} \text{ is infinite,} \\ \int_{\mathcal{I}} (|X_\mu (t)|^2) W (t, t, m) F (t, l, m) dt \text{ if } f(t) \in \mathcal{H} \text{ or } f (t) \in H, \mathcal{I} \text{ is finite} \end{cases},
$$

are valid in $L^2_m (\mathcal{I})$, where $m \in [\alpha, \beta]$.

Moreover, if vector-function $f (t)$ satisfy the following conditions

$$
P E_{\infty} f = f, \text{ } P \int_{\mathcal{I}} \mathcal{E} dE_{\mu} f = 0 \text{ if } f \in \mathcal{H}, \mathcal{I} \text{ is infinite,}$$

$$E_{\infty} f = f, \text{ } \int_{\mathcal{I}} \mathcal{E} dE_{\mu} f = 0 \text{ if } f \in H, \mathcal{I} \text{ is finite}$$

then the inverse formulae in $L^2_m (\mathcal{I})$

$$
f (t) = P \int_{\mathcal{I}} \mathcal{E} dE_{\mu} f = 0 \text{ if } f \in \mathcal{H}, \mathcal{I} \text{ is infinite,}$$

$$f (t) = \int_{\mathcal{I}} \mathcal{E} dE_{\mu} f = 0 \text{ if } f \in H, \mathcal{I} \text{ is finite}$$

and Parcevel’s equality

$$m [f, g] = \int_{\mathcal{I}} (d \sigma (\mu) \varphi (\mu, f), \varphi (\mu, g)),$$

are valid, where $g (t) \in \mathcal{H}$ if $\mathcal{I}$ is infinite or $g (t) \in H$, if $\mathcal{I}$ is finite.

In general case for $f (t), g (t) \in \mathcal{H}$ if $\mathcal{I}$ is infinite or $f (t), g (t) \in H$ if $\mathcal{I}$ is finite, the inequality of Bessel type

$$m [f (t), g (t)] \leq \int_{\mathcal{I}} (d \sigma (\mu) \varphi (\mu, f), \varphi (\mu, g))$$

is valid.

Let us notice that $\mathcal{B} = \cup_{k} (a_k, b_k), (a_j, b_j) \cap (a_k, b_k) = \emptyset, k \neq j$ since $\mathcal{B}$ is an open set. In (118) $P \int_{\mathcal{B}} = \sum_{k} \lim_{\alpha_{k} \downarrow a_{k} \subset \alpha_{k} \to b_{k}} \int_{\alpha_{k}}^{\beta_{k}} \mathcal{E} dE_{\alpha_{k}}$. In (118)–(120) we understand $\int_{\mathcal{B}}$ similarly.

Proof. Let for definiteness $r = s = 2n, \mathcal{I}$ is infinite (for another cases the proof becomes simpler). Let the vector-functions $f (t), g (t) \in \mathcal{H}$, $\lambda = \mu + i \varepsilon, G_\lambda (t, \lambda, m)$ be defined by (26).
with \( g(t) \) instead of \( f(t) \). In view of the Stieltjes inversion formula, we have

\[
(E_{\alpha,\beta} f, g)_{L^2_{m}(\mathcal{I})} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\alpha} \left( y_1(t, \lambda, f) - y_1(t, \lambda, f) \right)_m d\mu = \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\alpha} \left[ \left( \overline{g}(t, l_\lambda, m, f), G(t, l_\lambda, m) \right)_{L^2_{W(t, l_\lambda, m)}(\mathcal{I})} \right] d\mu = \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\alpha} \left[ \left( \overline{g}(t, l_\lambda, m, f), G(t, l_\lambda, m) \right)_{L^2_{W(t, l_\lambda, m)}(\mathcal{I})} \right] + \\
+ 2i \int_{\mathcal{I}} \left( (3\overline{p}_m)(t, \lambda) \right) f^{[n]}(t|m), g^{[n]}(t|m) dt \right) d\mu = \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\alpha} \left[ \left( M(\lambda) \int_{\mathcal{I}} X_\lambda^+(t) W(t, l_\lambda, m) F(t, l_\lambda, m) dt, \int_{\mathcal{I}} X_\lambda^+(t) W(t, l_\lambda, m) G(t, l_\lambda, m) dt \right) - \\
- \left( M^*(\lambda) \int_{\mathcal{I}} X_\lambda^-(t) W(t, l_\lambda, m) F(t, l_\lambda, m) dt, \int_{\mathcal{I}} X_\lambda^-(t) W(t, l_\lambda, m) G(t, l_\lambda, m) dt \right) \right] d\mu = \\
= \int_{\alpha} \left( d\sigma(\mu) \int_{\mathcal{I}} X_\mu^+(t) W(t, l_\mu, m) F(t, l_\mu, m) dt, \int_{\mathcal{I}} X_\mu^+(t) W(t, l_\mu, m) G(t, l_\mu, m) dt \right), (121)
\]

where the second equality is a corollary of (110), the next to last is a corollary of (81) and the last one follows from the well-known generalization of the Stieltjes inversion formula [35, Proposition (B), p. 803], [5] Lemma, p. 952. (In the case of finite \( \mathcal{I} \) we have to substitute in (121) \( M(\lambda) \) by \( PM(\lambda)P \) and then when passing to the next to the last equality in (121) we have to use the remark after the proof of Lemma 2.1) But for \( \lambda \in \mathcal{B} \)

\[
\int_{\mathcal{I}} X_\lambda^+(t) W(t, l_\lambda, m) F(t, l_\lambda) dt = \int_{\mathcal{I}} \left( [X_\lambda(t)]^+_1 \right) m[f] dt, (122)
\]

because, in view of (54),

\[
\forall h \in \mathcal{H}^r : \left( \int_{\mathcal{I}} X_\lambda^+(t) W(t, l_\lambda, m) F(t, l_\lambda) dt, h \right) = \\
= \int_{\mathcal{I}} (W(t, l_\lambda, m) F(t, l_\lambda), X_\lambda(t) h) dt = \left( \int_{\mathcal{I}} \left( [X_\lambda(t)]^+_1 \right) m[f], h \right) dt.
\]

Due to (121), (122), (116)

\[
(E_{\alpha,\beta} f, g)_{L^2_{m}(\mathcal{I})} = \int_{\alpha} (d\sigma(\mu) \varphi(\mu, f), \varphi(\mu, g)) . (123)
\]

The equality (119) and inequality (120) are the corollaries of (123).

Representing \( \varphi(\mu, g) \) in (123) by the second variant of (116), changing in (123) the order of integration and replacing \( \int_{\alpha}^{\beta} \) by integral sum and using (34) we obtain that

\[
(E_{\alpha,\beta} f, g)_{L^2_{m}(\mathcal{I})} = \left( \int_{\alpha}^{\beta} [X_\mu(t)]_1 d\sigma(\mu) \varphi(\mu, f), g(t) \right)_{L^2_{m}(\mathcal{I})} = \\
= \int_{\alpha}^{\beta} [X_\mu(t)]_1 d\sigma(\mu) \varphi(\mu, f), g(t) \right)_{L^2_{m}(\mathcal{I})} \quad (124)
\]

and (115) is proved since \( g(t) \in \mathcal{H}^2 \). Equalities (118) are the corollary of (115), (117). Theorem 3.4 is proved.

Let us notice that if \( L^2_{m}(\mathcal{I}) = \mathcal{I}^2 \) then Theorem 3.4 is valid without condition \( \mathcal{I}^2 \) with \( P = I_r \) if \( \mathcal{I} \) is infinite. \( \square \)
It is known (see for example [17] or Example [3,2]) that just in the case \( n_{\lambda} [y] \equiv 0 \) in (1), (2) there is such \( E_{\mu} \) satisfying (3), (81) – (85), (111) that \( E_{\infty} \neq I \).

On the other hand if \( n_{\lambda} [y] \equiv 0 \) then \( \forall f \in D (L_0) E_{\infty} f = f \) in view of [17], [19].

Let expression \( n_{\lambda} \) in representation (2), (80) have a divergent form with coefficients \( \tilde{p}_j = \tilde{q}_j, \tilde{r}_j = \tilde{s}_j \) at \( t, \lambda \).

We denote \( m (t) \) three-diagonal \((n + 1) \times (n + 1)\) operator matrix, whose elements under main diagonal are equal to \((-\frac{i}{2}, \ldots, -\frac{i}{2})\), the elements over the main diagonal are equal to \((\frac{i}{2}, \ldots, \frac{i}{2})\), the elements on the main diagonal are equal to \((\tilde{p}_0, \ldots, \tilde{p}_n)\), where \( \tilde{p}_j, \tilde{q}_j, \tilde{s}_j = \tilde{q}_j^* \) are the coefficients of expressions \( m \). (Here either \( 2n \) or \( 2n + 1 \) is equal to the order \( r \) of \( l_{\lambda} \). If order of \( n_{\lambda} \) is less or equal to \( 2n \), we denote \( n (t, \lambda) \) the analogues \((n + 1) \times (n + 1)\) operator matrix with \( \tilde{p}_j, \tilde{q}_j, \tilde{s}_j \) instead of \( \tilde{p}_j, \tilde{q}_j, \tilde{s}_j \). If order \( m \) or order \( n_{\lambda} \) is less than \( 2n \), we set the correspondent elements of \( m (t) \) or \( n (t, \lambda) \) be equal to zero.

**Theorem 3.5.** Let in (1), (2) the order of the expression \( n_{\lambda} \) is less or equal to the order of the expression \( l - \lambda m \) (and therefore in view of (80) the order of \( l - \lambda m \) is equal to \( r \); so \( Q (t, l_{\lambda}) = Q (t, l - \lambda m) \)). Let \( y = R_{\lambda} f, \ f \in H \) be the generalized resolvent of the relation \( L_0 \) and \( y \) satisfy equation (1). Let \( y_1 = R (\lambda) f, \ f \in H \) be the operator (81), (82) from Theorem 3.7.

Let the following conditions hold for \( \tau > 0 \) large enough:

\[
\lim_{\alpha \downarrow a, \beta \uparrow b} \frac{\Re Q (t, l_{\lambda}) (\tilde{y}_1 (t, l_{\lambda}, m, f) - \tilde{y} (t, l - \lambda m, m, f)) \} (\tilde{y}_1 (t, l_{\lambda}, m, f) - \tilde{y} (t, l - \lambda m, m, f))}{\Re l_{\lambda}} |_{\alpha}^{\beta} \leq 0, \ \lambda = i \tau \tag{125}
\]

\[
\Re n (t, \lambda) \leq c (t, \tau) m (t), \ t \in \mathbb{I}, \ \lambda = i \tau, \tag{126}
\]

where the scalar function \( c (t, \tau) \) satisfies the following condition:

\[
\sup_{t \in \mathbb{I}} c (t, \tau) = o (\tau), \ \tau \to +\infty. \tag{127}
\]

Then for generalized spectral family \( E_{\mu} \) (114) corresponding by (3) to the resolvent \( R (\lambda) \) (81), (82) from Theorem 3.7 and for generalized spectral family \( E_{\mu} \) corresponding to the generalized resolvent \( R_{\lambda} \) one has \( E_{\infty} = E_{\infty} \).

Let us notice that in view of (126) the coefficient at the highest derivative in the expression \( l - \lambda m \) has inverse from \( B (\mathcal{H}) \) if \( t \in \mathbb{I}, \ \Re \lambda \neq 0 \).

**Proof.** Let \( f (t) \in H, \ y_1 = R (\lambda) f, \ y = R_{\lambda} f \). Then \( z = y_1 - y \) satisfies the following equation

\[
l [z] - \lambda m [z] = n_{\lambda} [y_1]. \tag{128}
\]

Applying to the equation (128) the Green formula (42), one has

\[
\int_{\alpha}^{\beta} \Re (n_{\lambda} \{ y_1, z \}) \ dt + \int_{\alpha}^{\beta} m \{ z, z \} \ dt = \frac{1}{2} \Re (Q (t, l_{\lambda}) z, z) |_{\alpha}^{\beta},
\]
where \( \vec{z} = \vec{z}(t, l - \lambda m, n, y_1) = \vec{y}_1(t, l_\lambda, m, f) - \vec{y}(t, l - \lambda m, m, f) \) in view of (28) and Lemma 1.2.

Hence for \( \tau > 0 \) large enough

\[
m[z, z] \leq - \int_I \Re (n[\lambda, y_1, z]) \, dt / \tau \leq \int_I \left| \left( n(t, \lambda) \cos \left\{ y_1, y'_1, \ldots, y^{(n)}_1 \right\}, \cos \left\{ z, z', \ldots, z^{(n)} \right\} \right) \right| \, dt / \tau, \quad \lambda = i\tau \quad (129)
\]

in view of (125). But due to the inequality of the Cauchy type for dissipative operators [55, p. 199] and (126), (127): subintegral function in (129) is less or equal to \( (m \{ z, z \})^{1/2} (m \{ y_1, y_1 \})^{1/2} o(1) \) with \( \lambda = i\tau, \tau \to +\infty \). Therefore \( \| z \| \leq o(1/\tau) \| f \|_m \) since \( \| R_{x} \| \leq 1/|\Im \lambda| \). Hence

\[
R(\lambda) - R_{x} \leq o(1/\tau), \quad \lambda = i\tau, \quad \tau \to +\infty
\]

To complete the proof of the theorem it remains to prove the following

**Lemma 3.4.** Let \( R_k(\lambda) = \int_{R^1} \frac{dE_k}{\mu - \lambda} \), \( k = 1, 2 \), where \( E_k(\mu) \) are the generalized spectral families the type \( (114) \) in Hilbert space \( \mathbf{H} \). If \( \| R_1(\lambda) - R_2(\lambda) \| \leq o(1/\tau), \lambda = i\tau, \tau \to +\infty \), then \( E_1 = E_{\infty} \).

**Proof.** Let \( f \in \mathbf{H} \) \( \sigma(\mu) = (E_1^\lambda - E_2^\lambda) \, f, f \). One has

\[
\| (R_1(\lambda) - R_2(\lambda)) f, f \| =
\]

\[
= \frac{1}{\tau} - \int_{\Delta} d\sigma(\mu) + \int_{\Delta} \frac{\mu d\sigma(\mu)}{\mu - \lambda} + \int_{R^1 \setminus \Delta} \frac{\lambda d\sigma(\mu)}{\mu - \lambda} \leq o(1/\tau) \| f \|, \lambda = i\tau, \tau \to +\infty
\]

Therefore

\[
\left| - \int_{\Delta} d\sigma(\mu) + \int_{\Delta} \frac{\mu d\sigma(\mu)}{\mu - \lambda} + \int_{R^1 \setminus \Delta} \frac{\lambda d\sigma(\mu)}{\mu - \lambda} \right| \leq o(1), \lambda = i\tau, \tau \to +\infty. \quad (130)
\]

For an arbitrarily small \( \varepsilon > 0 \) we choose such finite interval \( \Delta(\varepsilon) \) that for any finite interval \( \Delta \supset \Delta(\varepsilon) \) for any finite interval \( \Delta \supset \Delta(\varepsilon) \) \( \exists N = N(\Delta) \):

\[
\forall \tau > N : \left| \int_{\Delta} \frac{\mu d\sigma(\mu)}{\mu - \lambda} < \frac{\varepsilon}{2}, \lambda = i\tau \right. \text{. But for any finite interval } \Delta \supset \Delta(\varepsilon) \exists N = N(\Delta) : \forall \tau > N : \left| \int_{\Delta} \frac{\mu d\sigma(\mu)}{\mu - \lambda} < \frac{\varepsilon}{2}, \lambda = i\tau \right. \text{. Therefore } \forall \varepsilon > 0, \Delta \supset \Delta(\varepsilon) : |\int_{\Delta} d\sigma(\mu)| < \varepsilon \text{ in view of (130). Hence } \forall f \in \mathbf{H}: (E_1^\lambda f, f) = (E_{\infty}^\lambda f, f) \Rightarrow E_1^\lambda = E_{\infty}^\lambda.
\]

Lemma 3.3 and Theorem 3.5 are proved. \( \square \)

**Corollary 3.3.** Let the conditions of Theorems 3.4, 5 hold. Then for generalized spectral family \( E_\mu \) from Theorem 3.4 \( \forall f \in \mathbf{D} \) \( E_\infty f = f \).

**Remark 3.1.** If \( L_m^2(\mathcal{I}) = L_m^2(\mathcal{I}) \), then it is sufficient to verify condition (125) in Theorem 3.5 for \( f \in \mathbf{H} \).

**Proposition 3.2.** Let the order of expression \( n_\lambda \) be less or equal to the order of expression \( l - \lambda m \) and the coefficient of \( l - \lambda m \) at the highest derivative has the inverse from \( \mathcal{B}(\mathcal{H}) \) for \( t \in \mathcal{I}, \lambda \in \mathcal{B}(l - \lambda m) \), where \( \mathcal{B}(l - \lambda m) \) is an analogue of the set \( \mathcal{B} = \mathcal{B}(l_\lambda) \). Let interval \( \mathcal{I} \) be finite and for equation (11), (12) with \( n_\lambda[y] \equiv 0 \) condition (11) holds with \( P = I_r \). Then for equation (11), (12) this condition also holds with \( P = I_r \) and resolvents \( y = R_{\lambda f}, y_1 = R(\lambda) f \) from Theorem 3.5 satisfies condition (125) for \( \forall \lambda \neq 0 \) if they are the solutions of boundary
value problems for equations \((1), (2)\), \((n_\lambda [y] \equiv 0)\) and \((1), (2)\) with boundary conditions from Theorem 3.2 with the same operators \(\mathcal{M}_\lambda, \mathcal{N}_\lambda\).

**Proof.** In view of Theorem 3.2 it is sufficient to prove only proposition about condition \((71)\).

Let for definiteness order \(l = order\ m = order\ n_\lambda = 2n.\)

Let for equation \((1), (2)\), \((n_\lambda [y] \equiv 0)\) condition \((71)\) with \(P = I_r\) hold, but for equation \((1), (2)\) that is not true. Then in view of \([21]\) the solutions \(y_k(t)\) of equation \((1), (2)\) with \(\lambda = i\) exist for which

\[
\int_\alpha^\beta (m + 3ni) \{y_k, y_k\} dt \to 0, \quad \vec{y}_k(0, l, m, 0) = f_k, \quad ||f_k|| = 1, \quad (131)
\]

where \(i3n_i = n_i\) in view of \((80)\). Hence in view of \((34)\)

\[
\int_\alpha^\beta (W_i(t, l + im, n_i)Y_k(t, l + im, n_i), Y_k(t, l + im, n_i)) dt = \int_\alpha^\beta ni \{y_k, y_k\} dt \to 0. \quad (132)
\]

On the other hand

\[
X_i(t) f_k = \vec{X}_i(t) f_k + \vec{X}_i(t) \int_0^t \vec{X}_i^{-1}(s) J^{-1} W(s, l + im, n_i)Y_k(s, l + im, n_i) ds. \quad (133)
\]

in view of Theorem 1.1 and the fact that \(\vec{y}_k(t, l - im, n_i, y_k) = \vec{y}_k(t, l, m, 0)\), where \(\vec{X}_i(t)\) is an analogue of \(X_\lambda(t)\) for the case \(n_\lambda [y] \equiv 0.\)

Comparing \((132), (133)\) we see that

\[
\left| X_i(t) f_k - \vec{X}_i(t) f_k \right| \to 0 \quad (134)
\]

uniformly in \(t \in [\alpha, \beta].\)

In view of \((131)\) subsequence \(y_{kq}\) exist such that

\[
m \{y_{kq}, y_{kq}\} \to 0, \quad n_i \{y_{kq}, y_{kq}\} \to 0. \quad (135)
\]

Due to second proposition \((134)\) and the arguments as in the proof of Proposition 3.1 one has

\[
y^{[j]}_k(t | n_i) \to 0 \quad j = n, \ldots, 2n. \quad (136)
\]

Let us denote \(\vec{y}_k(t) = \vec{X}_i(t) f_k.\) In view of Theorem 1.1 and \((134)\)

\[
\left| y^{(j)}_k(t) - \vec{y}^{(j)}_k(t) \right| \to 0, \quad j = 1, \ldots, n - 1, \quad (137)
\]

\[
\left| (p_n(t) - i\vec{p}_n(t)) \left[ y^{(n)}_{kq}(t) - \vec{y}^{(n)}_{kq}(t) \right] \right. - \left. \frac{i}{2} (q_n(t) - i\vec{q}_n(t)) \left[ y^{(n-1)}_{kq}(t) - \vec{y}^{(n-1)}_{kq}(t) \right] - y^{[n]}_k(t | l_j) \right| = \quad (138)
\]

uniformly in \(t \in [\alpha, \beta].\) Comparing \((135), (136), (138)\) and using \((p_n(t) - i\vec{p}_n(t))^{-1} \in B(\mathcal{H})\) we have

\[
\left( \vec{p}_n(t) \vec{y}^{(n)}_{kq}(t), \vec{y}^{(n)}_{kq}(t) \right) \to 0. \quad (139)
\]

In view of \((137), (139), (135)\)

\[
m \{\vec{y}_{kq}, \vec{y}_{kq}\} \to 0, \quad \vec{y}_{kq}(0, l - im, m, 0) = f_{kq}, \quad (140)
\]

that contradicts to the condition \((71)\) with \(P = I_r\) for equation \((1), (2)\) with \(n_\lambda [y] \equiv 0.\) Proposition 3.2 is proved. \(\square\)
In the next theorem \( I = \mathbb{R}^1 \) and condition (71) hold with \( P = I_r \) both on the negative semi-axis \( R_- \) (i.e. as \( I = R_- \)) and on the positive semi-axis \( R_+ \) (i.e. as \( I = R_+ \)).

**Theorem 3.6.** Let \( I = \mathbb{R}^1 \), the coefficient of the expression \( l_\lambda \) be periodic on each of the semi-axes \( R_- \) and \( R_+ \) with periods \( T_- > 0 \) and \( T_+ > 0 \) correspondingly. Then the spectrums of the monodromy operators \( X_\lambda (\pm T_\pm) \) (\( X_\lambda (t) \) is from Theorem 3.1) do not intersect the unit circle as \( \Im \lambda \neq 0 \), the c.o. \( M (\lambda) \) of the equation (5) is unique and equal to

\[
M (\lambda) = \left( \mathcal{P} (\lambda) - \frac{1}{2} I_r \right) (iG)^{-1} \quad (\Im \lambda \neq 0),
\]

where the projection \( \mathcal{P} (\lambda) = P_+ (\lambda) (P_+ (\lambda) + P_- (\lambda))^{-1} \), \( P_{\pm} (\lambda) \) are Riesz projections of the monodromy operators \( X_\lambda (\pm T_\pm) \) that correspond to their spectrums lying inside the unit circle, \( (P_+ (\lambda) + P_- (\lambda))^{-1} \in B (\mathcal{H}^\nu) \) as \( \Im \lambda \neq 0 \).

Also let \( \dim \mathcal{H} < \infty \), a finite interval \( \Delta \subset \mathcal{B} \). Then in Theorem 3.4 \( \sigma (\mu) = d \sigma_{ac} (\mu) + d \sigma_d (\mu) \), \( \mu \in \Delta \). Here \( \sigma_{ac} (\mu) \in AC (\Delta) \) and, for \( \mu \in \Delta \),

\[
\sigma'_{ac} (\mu) = \frac{1}{2 \pi} G^{-1} (Q^*_+ (\mu) G Q^- (\mu) - Q^*_+ (\mu) G Q^+ (\mu)) G^{-1},
\]

where the projections \( Q_{\pm} (\mu) = q_{\pm} (\mu) (P_+ (\mu) + P_- (\mu))^{-1} \), \( q_{\pm} (\mu) \) are Riesz projections of the monodromy matrices \( X_\mu (\pm T_\pm) \) corresponding to the multiplicators belonging to the unit circle and such that they are shifted inside the unit circle as \( \mu \) is shifted to the upper half plane, \( P_{\pm} (\mu) = P_{\pm} (\mu + i0) \); \( \sigma_d (\mu) \) is a step function.

Let us notice that the sets on which \( q_{\pm} (\mu), P_{\pm} (\mu), (P_+ (\mu) + P_- (\mu))^{-1} \) are not infinitely differentiable do not have finite limit points \( \in \mathcal{B} \) as well as the set of points of increase of \( \sigma_d (\mu) \).

**Proof.** The proof of Theorem 3.6 is similar to that on in the case \( n_\lambda [y] \equiv 0 \). \( \square \)

The following examples demonstrate effects that are the results of appearance in \( l_\lambda \) of perturbation \( n_\lambda \) depending nonlinearly on \( \lambda \).

In Examples 3.3, 3.4 nonlinear in \( \lambda \) perturbation does not change the type of the spectrum.

In this examples \( \dim \mathcal{H} = 1 \), \( m [y] = -y'' + y \). \( L_0^2 (I) = L_0^2 (I) = W^1_2 (\mathbb{R}^1) \). In Example 3.5 such perturbation implies an appearance of spectral gap with "eigenvalue" in this gap.

**Example 3.3.** Let

\[
l_\lambda [y] = iy' - \lambda (-y'' + y) - \left( \frac{h}{\lambda} y \right) \quad (h \geq 0).
\]

Here \( \mathcal{B} = \mathbb{C} \setminus 0 \), \( E_0 = E_{+0} \), spectral matrix \( \sigma (\mu) \in AC_{loc} \).

\[
\sigma' (\mu) = \frac{1}{2 \pi} \begin{pmatrix} \sqrt{4h + 1 - 4 \mu^2} & 0 \\ 0 & \frac{1}{2} \sqrt{4h + 1 - 4 \mu^2} \end{pmatrix}, \text{ as } |\mu| < \sqrt{h + 1/4},
\]

\[
\sigma' (\mu) = 0, \text{ as } |\mu| > \sqrt{h + 1/4}.
\]

In Example 3.3 nonlinear in \( \lambda \) perturbation change edges of spectral band.

**Example 3.4.** Let

\[
l_\lambda [y] = y^{(IV)} - \lambda (-y'' + y) - \left( \frac{h}{\lambda} y \right) \quad (h \geq 0).
\]
Here $B = \begin{cases} \mathbb{C} \setminus \{0\}, & h \neq 0 \\ \mathbb{C}, & h = 0 \end{cases}$, $E_0 = E_{+0}$, spectral matrix $\sigma(\mu) \in AC_{loc}$,

\[
\sigma' (\mu) = \begin{pmatrix}
\frac{1}{2\pi} \sqrt{\frac{\lambda + \sqrt{D}}{D}} & 0 & 0 & -1 \\
0 & 1 & -\frac{\lambda + \sqrt{D}}{2} & 0 \\
0 & -\frac{\lambda + \sqrt{D}}{2} & \left(\frac{\lambda - \sqrt{D}}{2}\right)^2 & 0 \\
-1 & 0 & 0 & \frac{\lambda + \sqrt{D}}{2}
\end{pmatrix}, \quad \text{as } -\sqrt{h} < \mu < 0, \mu > \sqrt{h},
\]

where $D = \lambda^2 - 4q$, $q = h/\lambda - \lambda$, $\mu^* = \mu^*(h)$ - nonnegative root of equation $D = 0$. $\sigma'(\mu) = 0$, as $\mu \notin [-\sqrt{h}, 0] \cup [\mu^*, \infty)$.

In Example 3.4 nonlinear in $\lambda$ perturbation implies an appearance of additional spectral band $[-h, 0]$, variation of edge of semi in finite spectral band and appearance of interval $(\mu^*, \sqrt{h})$ of fourfold spectrum.

Example 3.5. Let $\text{dim} \mathcal{H} = 2$,

\[
i_\lambda [y] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y' - \lambda y - \begin{pmatrix} -h/\lambda & 0 \\ 0 & 0 \end{pmatrix} y, \quad h \geq 0.
\]

Here $B = \begin{cases} \mathbb{C} \setminus \{0\}, & h \neq 0 \\ \mathbb{C}, & h = 0 \end{cases}$, spectral matrix $\sigma(\mu) = \sigma_{ac}(\mu) + \sigma_d(\mu)$, $\sigma_{ac}(\mu) \in AC_{loc}$, $\sigma_{ac}'(\mu) \neq 0$, as $|\mu| > \sqrt{h}$, $\sigma_{ac}'(\mu) = 0$, as $|\mu| < \sqrt{h}$, step-function $\sigma_d(\mu)$ has only one jump $\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{h}/2 \end{pmatrix}$ in point $\mu = 0$ (inside of spectral gap). In this point

\[
(E_{+0} - E_0) f = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{h}/2 \end{pmatrix} \int_{-\infty}^{\infty} e^{-\sqrt{h}|t-s|} f(s) ds, \quad f(t) \in L^2(\mathbb{R}),
\]

Let us notice that in view of Floquet theorem conditions of Theorem 3.5 ([125] with account of Remark 3.1) hold for all Examples 3.3-3.5.

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