MEAN-FIELD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS ON MARKOV CHAINS

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Abstract. In this paper, we deal with a class of mean-field backward stochastic differential equations (BSDEs) related to finite state, continuous time Markov chains. We obtain the existence and uniqueness theorem and a comparison theorem for solutions of one-dimensional mean-field BSDEs under Lipschitz condition.

1. Introduction

The general (nonlinear) backward stochastic differential equations (BSDE in short) were firstly introduced by Pardoux and Peng [22] in 1990. Since then, BSDEs have been studied with great interest, and they have gradually become an important mathematical tool in many fields such as financial mathematics, stochastic games and optimal control, etc, see for example, Peng [23], Hamadène and Lepeltier [13] and El Karoui et al. [11].

McKean-Vlasov stochastic differential equation of the form

\[ dX(t) = b(X(t), \mu(t))dt + dW(t), \quad t \in [0, T], \quad X(0) = x, \]

was suggested by Kac [14] as a stochastic toy model for the Vlasov kinetic equation of plasma. The study of which was initiated by Mckean [21]. Since then, many authors made contributions on McKean-Vlasov type SDEs and applications,

\[ b(X(t), \mu(t)) = \int_{\Omega} b(X(t, \omega), X(t; \omega')) P(\omega') = E[b(\xi, X(t))|\xi = X(t)], \]

where \( b : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R} \) being a (locally) bounded Borel measurable function and \( \mu(t; \cdot) \) being the probability distribution of the unknown process \( X(t) \), was suggested by Kac [14] as a stochastic toy model for the Vlasov kinetic equation of plasma. The study of which was initiated by Mckean [21]. Since then, many authors made contributions on McKean-Vlasov type SDEs and applications,
Mathematical mean-field approaches have been used in many fields, not only in physics and chemistry, but also recently in economics, finance and game theory, see for example, Lasry and Lions [17], they have studied mean-field limits for problems in economics and finance, and also for the theory of stochastic differential games.

Motivated by the above works, the present paper deal with a class of mean-field BSDEs on Markov Chains of the form

\[ Y_t = \xi + \int_t^T E'[f(s, Y^1_{s-}, Z^1_{s-}, Y^2_s, Z^2_s)] ds - \int_t^T Z_s dM_s, \]

where \((Y^1, Z^1)\) is a copy of \((Y, Z)\). To the best of our knowledge, so far little is known about this new kind of BSDEs. Our aim is to find a pair of adapted
processes \((Y, Z)\) in an appropriate space such that (3) hold. We also present a comparison theorem for the solutions of BSDEs (3). We remark that our BSDE (3) includes BSDE (2) as a special case.

The paper is organized as follows. In Section 2, we introduce some preliminaries. Section 3 is devoted to the proof of the existence and uniqueness of the solutions to mean-field BSDEs on Markov chains. In Section 4, we give a comparison theorem for the solutions of mean-field BSDEs.

2. Preliminaries

Let \(T > 0\) be fixed throughout this paper. Let \(X = \{X_t, t \in [0, T]\}\) be a continuous time finite state Markov chain. The state space of \(X\) can be identified with the set of unite column vectors \(\{e_1, e_2, \ldots, e_N\}\) in \(\mathbb{R}^N\), where \(e_i = (0, \ldots, 1, \ldots, 0)^*\) with 1 in the \(i\)-th position, \(N\) is the number of states and \([:\]^*\) denotes vector/matrix transposition.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space. We denote by \(\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}\) the natural filtration generated by \(X = \{X_t, t \in [0, T]\}\) and augmented by all \(\mathbb{P}\)-null sets, i.e.,

\[\mathcal{F}_t = \sigma\{X_u, 0 \leq u \leq t\} \vee \mathcal{N}_P,\]

where \(\mathcal{N}_P\) is the set of all \(\mathbb{P}\)-null subsets.

Let \(A_t\) be the rate matrix for the chain \(X\) at time \(t\), then this chain has the representation (see Appendix B of Elliott et al. [12])

\[X_t = X_0 + \int_0^t A_u X_u - du + M_t,\]

where \(M_t\) is a martingale related to the chain \(X = \{X_t, t \in [0, T]\}\). The optional quadratic variation of \(M_t\) is given by the matrix process

\[\langle M, M \rangle_t = \sum_{0 < u \leq t} \Delta M_u \Delta M_u^*\]

and

\[\langle M, M \rangle_t = \int_{[0, t]} [\text{diag}(A_u X_u - ) - \text{diag}(X_u - ) A_u^* - A_u \text{diag}(X_u - )] du.\]

Let \(\Phi_t\) be the nonnegative definite matrix

\[\Phi_t := \text{diag}(A_t X_t - ) - \text{diag}(X_t - ) A_t^* - A_t \text{diag}(X_t - )\]

and

\[\|Z\|_{X_{\cdot-}} := \sqrt{\text{Tr}(Z \Phi_t Z^*)}.\]

Then \(\| \cdot \|_{X_{\cdot-}}\) defines a (stochastic) seminorm, with the property that

\[\text{Tr}(Z \text{d}(M, M) \cdot Z_t^*) = \|Z\|^2_{X_{\cdot-}} dt.\]

Now, we provide some spaces and notations used in the sequel.

- \(L^p(\Omega, \mathcal{F}_T, \mathbb{P}) := \{\xi : \text{real valued } \mathcal{F}_T\text{-measurable random variable } E[\xi]^p < +\infty, p \geq 1\};\)
Definition 1. A solution to the mean-filed BSDE (4) is a couple \((Y, Z)\) such that \((Y, Z) \in S^2(\mathbb{R}) \times H^2_{X,Y}(\mathbb{R}^N)\).
3. Existence and uniqueness of solutions

In this section, we aim to derive the existence and uniqueness result for the solutions of mean-field BSDEs on Markov chains.

Before stating our main theorem, we recall an existence and uniqueness result in Cohen and Elliott [7].

**Lemma 3.1.** Given \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \). Suppose assumptions (A1) and (A2) hold. Then BSDE (2) has a unique solution \((Y, Z) \in S^2_\mathcal{F}(\mathbb{R}) \times H^2_{X,P}(\mathbb{R}^N)\), and the solution is the unique such solution, up to indistinguishability for \( Y \) and equality \( d\langle M, M \rangle_t \times P \)-a.s. for \( Z \).

For the solutions of mean-field BSDE (4), we first establish the following unique result.

**Lemma 3.2.** Given \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \). Suppose assumptions (A1) and (A2) hold. Then mean-field BSDE (4) has at most one solution \((Y, Z) \in S^2_\mathcal{F}(\mathbb{R}) \times H^2_{X,P}(\mathbb{R}^N)\).

**Proof.** Let \((Y^i, Z^i) \in S^2_\mathcal{F}(\mathbb{R}) \times H^2_{X,P}(\mathbb{R}^N), i = 1, 2\) be two solutions of mean-field BSDE (4). Define \( \hat{Y} = Y^1 - Y^2, \hat{Z} = Z^1 - Z^2 \), we then have

\[
\hat{Y}(t) = \int_t^T E'[\hat{f}(s)]ds - \int_t^T \hat{Z}_s dM_s,
\]

where \( \hat{f}(s) = f(s, Y^1_{s-}^{1'}, Z^1_{s-}^{1'}, Z^1_{s-}^{1'}, Y^1_{s-}^{2'}, Z^2_{s-}^{2'}, Y^2_{s-}^{2'}, Z^2_{s-}^{2'}) - f(s, Y^2_{s-}^{1'}, Z^2_{s-}^{1'}, Y^2_{s-}^{1'}, Z^2_{s-}^{2'}, Z^2_{s-}^{2'}) \).

Using the Stieltjes chain rule for products, we get

\[
|\hat{Y}_t|^2 = |\hat{Y}_0|^2 - 2 \int_0^T \hat{Y}_s E'[\hat{f}(s)]ds + 2 \int_0^T \hat{Y}_s \hat{Z}_s dM_s + \sum_{0 < s \leq t} |\Delta Y^1_s - \Delta Y^2_s|^2.
\]

Taking expectation on both sides of (5) and evaluating at \( t = T \), we obtain

\[
E|\hat{Y}_T|^2 = 2 \int_t^T E[\hat{Y}_s E'[\hat{f}(s)]|ds - E \sum_{t < s \leq T} |\Delta Y^1_s - \Delta Y^2_s|^2
\]
\[= 2 \int_t^T E[\hat{Y}_s E'[\hat{f}(s)]|ds - E \sum_{t < s \leq T} |(Z^1_s - Z^2_s)\Delta M_s|^2
\]
\[= 2 \int_t^T E[\hat{Y}_s E'[\hat{f}(s)]|ds - \int_t^T E\|\hat{Z}_s\|_{X,P}^2|ds.
\]

On the other hand, by (A1) and Young’s inequality \( 2ab \leq \frac{1}{\rho}a^2 + \rho b^2 \), for any \( \rho > 0 \), it holds that

\[
2 \int_t^T E[\hat{Y}_s E'[\hat{f}(s)]|ds
\]
We have the following existence and uniqueness result.

Let $\rho = 3C$, we obtain

$$2 \int_t^T E[\hat{Y}_s - E'[\hat{f}(s)]] ds \leq (6C^2 + 4C) \int_t^T E[\hat{Y}_s - E'[\hat{f}(s)]] ds + \frac{2}{3} \int_t^T E[\hat{Z}_s] ds.$$

This together with (6) implies

$$E[\hat{Y}_t] = 0, \quad E[\hat{Z}_t]|_{X_t} = 0,$$

i.e., $Y_t^1 = Y_t^2$ and $Z_t^1 = Z_t^2$ P.a.s. for each $t$. The proof is complete. \qed

Next, let’s consider a simplified version of mean-field BSDEs (4) as follows

$$Y_t = \xi + \int_t^T E'[f(s,Y_s',Y_s,Z_s)] ds - \int_t^T Z_s dM_s.$$

We have the following existence and uniqueness result.

**Lemma 3.3.** Given $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$. Suppose assumptions (A1) and (A2) hold. Then mean-field BSDE (7) has a unique solution $(Y_t,Z_s) \in S^2_\mathbb{P}(\mathbb{R}) \times H^{2}_{X,X}([0,T])$.

**Proof.** Let $Y_t^0 = 0$, $t \in [0,T]$, we consider the following mean-field BSDE:

$$Y_t^{n+1} = \xi + \int_t^T E'[f(s,Y_s',Y_s,Z_s^{n+1})] ds - \int_t^T Z_s^{n+1} dM_s.$$

According to Lemma 3.1, we can define recursively $(Y_t^{n+1},Z_s^{n+1})$ be the solution of BSDE (8). For $t \in [0,T]$, we have

$$Y_t^{n+1} - Y_t^n = \int_t^T E'[f(s,Y_s',Y_s,Z_s^{n+1}) - f(s,Y_s',Y_s^{n-1},Z_s^n)] ds - \int_t^T (Z_s^{n+1} - Z_s^n) dM_s$$

$$= Y_0^{n+1} - Y_0^n - \int_0^t E'[f(s,Y_s',Y_s^{n+1},Z_s^{n+1}) - f(s,Y_s',Y_s^{n-1},Z_s^n)] ds - \int_0^t (Z_s^{n+1} - Z_s^n) dM_s.$$
Using the Stieltjes chain rule for products, we have
\[ |Y_t^{n+1} - Y_t^n|^2 \]
\[ = |Y_0^{n+1} - Y_0^n|^2 - 2 \int_0^t (Y_s^{n+1} - Y_s^n)E'[f(s,Y^{n_t},Y^n_s,Z_s^{n+1})] \]
\[ - f(s,Y_s^{n-1},Y_s^n,Z_s^n)]ds + 2 \int_0^t (Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n)dM_s \]
\[ + \sum_{0 < s \leq t} |\Delta Y_s^{n+1} - \Delta Y_s^n|^2. \]

Taking expectation and evaluating at \( t = T \), we obtain
\[ E|Y_T^{n+1} - Y_T^n|^2 \]
\[ = 2 \int_t^T E\{(Y_s^{n+1} - Y_s^n)E'[f(s,Y^{n_t},Y^n_s,Z_s^{n+1})] \]
\[ - f(s,Y_s^{n-1},Y_s^n,Z_s^n)]ds - \int_t^T E\|Z_s^{n+1} - Z_s^n\|_X^n \]
\[ \leq C \int_t^T E\{(Y_s^{n+1} - Y_s^n)E'[Y_s^{n_t} - Y_s^{n-1}] \]
\[ + |Y_s^n - Y_s^{n-1}| + \|Z_s^{n+1} - Z_s^n\|_X^n \}ds \]
\[ \leq \frac{3C}{\rho} \int_t^T E\{Y_s^{n+1} - Y_s^n\}^2ds + 2\rho C \int_t^T E\|Y_s^n - Y_s^{n-1}\|^2 \]
\[ + \rho C \int_t^T E\|Z_s^{n+1} - Z_s^n\|_X^n \]

Choosing \( \rho = \frac{1}{2C^2} \), combining (10) and (11), we then have
\[ E|Y_T^{n+1} - Y_T^n|^2 \]
\[ \leq c \int_t^T E\|Y_s^n - Y_s^{n-1}\|^2 \]

where \( c = \max\{6C^2,1\} \). Let \( u^n(t) = \int_t^T E\|Y_s^n - Y_s^{n-1}\|^2 \)
\[ du^{n+1}(t) - cu^{n+1}(t) \leq cu^n(t), \quad u^{n+1}(T) = 0. \]

Integration gives
\[ u^{n+1}(t) \leq c \int_t^T e^{c(s-t)}u^n(s)ds. \]
Iterating above inequality, we obtain
\[ u^{n+1}(0) \leq \left( \frac{ce}{n!} \right) u^1(0). \]
This implies that \( \{Y^n\} \) is a Cauchy sequence in \( S^2_F(\mathbb{R}) \). Then by (12), \( \{Z^n\} \) is a Cauchy sequence in \( H^2_{X,\mathbb{R}}(\mathbb{R}^N) \).

Passing to the limit on both sides of (8), by (A2) and the dominated convergence theorem, it follows that
\[ Y := \lim_{n \to \infty} Y^n, \quad Z := \lim_{n \to \infty} Z^n \]
solves BSDE (7). The uniqueness is a direct consequence of Lemma 3.2. The proof is complete. \( \square \)

The main result of this section is the following theorem.

**Theorem 3.4.** Assume that (A1) and (A2) hold true. Then for any given terminal conditions \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), the mean-field BSDE (4) has a unique solution \( (Y, Z) \in S^2_F(\mathbb{R}) \times H^2_{X,\mathbb{R}}(\mathbb{R}^N) \).

**Proof.** According to Lemma 3.2, all we need to prove is the existence of solution for mean-field BSDE (4).

Let \( Z^0_t = 0, \ t \in [0, T] \), in virtue of Lemma 3.3, we can define recursively the pair of processes \( (Y^{n+1}_t, Z^{n+1}_t) \) be the unique solution of the following mean-field BSDE:

\[ Y^{n+1}_t = \xi + \int_t^T E'[f(s, Y^{n+1}_{s-}, Z^{n+1}_{s-}, Y_s^{n+1} - Y_s^n, Z_s^{n+1} - Z_s^n)] ds - \int_t^T Z^{n+1}_s dM_s. \]

Using the same procedure as above, we get
\[ E[Y^{n+1}_t - Y^n_t] \]
\[ = 2 \int_t^T E\{ (Y^{n+1}_s - Y^n_s) E'[f(s, Y^{n+1}_{s-}, Z^{n+1}_{s-}, Y_s^{n+1} - Y_s^n, Z_s^{n+1} - Z_s^n)] \}
\[ - f(s, Y^{n+1}_{s-}, Z^{n+1}_{s-}, Y_s^n, Z_s^n) ds - \int_t^T E\|Z^{n+1}_s - Z_s^n\|_X^2 ds \]
\[ \leq 2C \int_t^T E\{ (Y^{n+1}_s - Y^n_s) E'[|Y^{n+1}_{s-} - Y^n_{s-}| + |Y^{n+1}_s - Y^n_s| \]
\[ + \|Z^{n+1}_s - Z^{n+1}_{s-}\|_X + \|Z^{n+1}_s - Z^n_s\|_X] ds \]
\[ - \int_t^T E\|Z^{n+1}_s - Z_s^n\|_X^2 ds. \]

With the help of (A1) and Young’s inequality, for any \( \rho > 0 \), we have
\[ E[Y^{n+1}_t - Y^n_t] \]
\[ \leq 2C \int_t^T E\{ (Y^{n+1}_s - Y^n_s) E'[|Y^{n+1}_{s-} - Y^n_{s-}| + |Y^{n+1}_s - Y^n_s| \]
\[ + \|Z^{n+1}_s - Z^{n+1}_{s-}\|_X + \|Z^{n+1}_s - Z^n_s\|_X] ds \]
\[ - \int_t^T E\|Z^{n+1}_s - Z_s^n\|_X^2 ds. \]
Choosing \( \rho \) (14), Iterating above inequality implies that

\[
\{ Y, Z \}_{\text{dimensional mean-field BSDEs on Markov chains.}}
\]

unique solution of mean-field BSDE (4).

\[\square\]

By (14), we know that

\[
\int_t^T E[|Y_s|^{n+1} - Y_s^n|^2]ds + \rho C \int_t^T E[Z_s^{n+1} - Z_s^n|^2]ds
\]

\[+ (\rho C - 1) \int_t^T E[Z_u^{n+1} - Z_u^n|^2]du\]

\[(14)\]

Choosing \( \rho = \frac{1}{2C} \), we get

\[
\int_t^T E[Z_s^{n+1} - Z_s^n|^2]ds + ke^{-kt} \int_t^T e^{ks} \int_s^T E[Z_u^{n+1} - Z_u^n|^2]du ds
\]

\[\leq \frac{1}{2} \left( \int_t^T E[Z_s^n - Z_s^{n-1}|^2]ds + ke^{-kt} \int_t^T e^{ks} \int_s^T E[Z_u^n - Z_u^{n-1}|^2]du ds \right).\]

Iterating above inequality implies that \( \{ Z^n \} \) is a Cauchy sequence in \( H^2_{X,F}(\mathbb{R}^N) \) under the equivalent norm.

By (14), we know that \( \{ Y^n \} \) is a Cauchy sequence in \( H^2_F(\mathbb{R}) \). We denote their limits by \( Y \) and \( Z \) respectively. By (A2) and the dominated convergence theorem, for any \( t \in [0,T] \), we have

\[
\int_t^T E[f(s, Y_{s-}^{n+1}, Z_{s-}^{n+1}, Y_{s}^n, Z_{s}^n) - f(s, Y_{s-}, Z_{s-}, Z_s)]ds \to 0, n \to \infty.
\]

We now pass to the limit on both sides of (13), it follows that \( (Y,Z) \) is the unique solution of mean-field BSDE (4).

4. A comparison theorem

In this section, we discuss a comparison theorem for the solutions of one-dimensional mean-field BSDEs on Markov chains.

Let \( (Y^1, Z^1) \) and \( (Y^2, Z^2) \) be respectively the solutions for the following two mean-field BSDEs

\[
Y^i_t = \xi^i + \int_t^T E[f_i(s, Y^i_s, Y^i_s, Z^i_s, Z_s)]ds - \int_t^T Z^i_sdM_s,
\]

\[(15)\]
where $i = 1, 2$.

**Theorem 4.1.** Assume that $f_1, f_2$ satisfy (A1) and (A2), $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, P)$. Moreover, we suppose:

(i) $\xi^1 \geq \xi^2$, $P$-a.s.;

(ii) for any $t \in [0, T], \quad f_1(\omega', \omega, t, Y^2_t, Z^2_t, Y^1_t, Z^1_t) \geq f_2(\omega', \omega, t, Y^2_t, Z^2_t, Y^1_t, Z^1_t), \bar{P}$-a.s.

It is then true that $Y^1 \geq Y^2$ on $[0, T], P$-a.s.

**Proof.** We omit the $\omega', \omega$ and $s$ for clarity. By assumption (i), $(\xi^2 - \xi^1)^+ = 0$, a.s.. Since for $t \in [0, T], (Y^2_t - Y^1_t)^+ = \frac{1}{2}[|Y^2_t - Y^1_t| + (Y^2_t - Y^1_t)]$, then by the Stieltjes chain rule for products, we have

$$ ((Y^2_t - Y^1_t)^+)^2 $$

$$ = -2 \int_t^T (Y^2_s - Y^1_s)^+ d(Y^2_s - Y^1_s)^+ - \sum_{t < s \leq T} \Delta(Y^2_s - Y^1_s)^+ \Delta(Y^2_s - Y^1_s)^+ $$

$$ = - \int_t^T (Y^2_s - Y^1_s)^+ d[|Y^2_s - Y^1_s| + (Y^2_s - Y^1_s)] $$

$$ - \sum_{t < s \leq T} \Delta(Y^2_s - Y^1_s)^+ \Delta(Y^2_s - Y^1_s)^+ $$

$$ = - \int_t^T (Y^2_s - Y^1_s)^+ d|Y^2_s - Y^1_s| - \int_t^T (Y^2_s - Y^1_s)^+ d(Y^2_s - Y^1_s) $$

$$ - \sum_{t < s \leq T} \Delta(Y^2_s - Y^1_s)^+ \Delta(Y^2_s - Y^1_s)^+ $$

$$ = -2 \int_t^T I_{[Y^2 > Y^1]} (Y^2_s - Y^1_s) d(Y^2_s - Y^1_s) $$

$$ - \sum_{t < s \leq T} I_{[Y^2 > Y^1]} \Delta(Y^2_s - Y^1_s) \Delta(Y^2_s - Y^1_s) $$

$$ = -2 \int_t^T I_{[Y^2 > Y^1]} (Y^2_s - Y^1_s) d(Y^2_s - Y^1_s) $$

$$ - \sum_{t < s \leq T} I_{[Y^2 > Y^1]} [(Z^2_s - Z^1_s) \Delta M_s]^2. $$

For $t \in [0, T], \text{by assumption (ii), (A1) and Young's inequality, for any } \rho > 0, \text{we have}$

$$ E((Y^2_t - Y^1_t)^+)^2 + E \int_t^T I_{[Y^2 > Y^1]} \|[Z^2_s - Z^1_s]\|^2_{\bar{X}_s} ds $$

$$ = 2 \int_t^T E I_{[Y^2 > Y^1]} (Y^2_s - Y^1_s) E[|f_2(Y^2_s, Z^2_s, Y^1_s, Z^1_s) - f_1(Y^2_s, Z^2_s, Y^1_s, Z^1_s)|] ds $$
\[
\leq 2 \int_t^T E\{I_{\{Y^2_s > Y^1_s\}}(Y^2_s - Y^1_s)E'[f_1(Y^2_s, Z^2_s, Y^2_s, Z^2_s) - f_1(Y^1_s, Z^1_s, Y^1_s, Z^1_s)]\} ds
\]
\[
\leq 2C \int_t^T E\{I_{\{Y^2_s > Y^1_s\}}(Y^2_s - Y^1_s)^2 + \|Z^2_s - Z^1_s\|_{X_{\infty}}\} ds
\]
\[
+ 2(2 + \rho) C \int_t^T E[(Y^2_s - Y^1_s)^2 ds
\]
\[
+ 2\rho CE \int_t^T I_{\{Y^2_s > Y^1_s\}} \|Z^2_s - Z^1_s\|^2_{X_{\infty}} ds
\]
\[
\leq (4C + 2\rho C) \int_t^T E[(Y^2_s - Y^1_s)^2 ds
\]
\[
+ 2\rho CE \int_t^T I_{\{Y^2_s > Y^1_s\}} \|Z^2_s - Z^1_s\|^2_{X_{\infty}} ds.
\]
Choosing \(\rho = \frac{1}{4C}\), it follows from Gronwall's inequality that \(E[(Y^2_s - Y^1_s)^2] = 0, t \in [0, T]\). It is then true that \(Y^1_s \geq Y^2_s\) on \([0, T]\), P-a.s. The proof is complete. \(\square\)

**Remark 4.2.** Compared to the comparison results in Cohen and Elliott [8], our assumptions on coefficients \(f_1\) and \(f_2\) are natural. Moreover, we don’t make restrictions on the two solutions, hence it’s easier to use.

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