Cosmic crystallography with a pullback

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Abstract

We present a modified version of the cosmic crystallography method, especially useful for testing closed models of negative spatial curvature. The images of clusters of galaxies in simulated catalogs are “pulled back” to the fundamental domain before the set of distances is calculated.

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1 Introduction

The method of cosmic crystallography was devised by Lehoucq, Lachièze-Rey, and Luminet [1] to investigate the global spatial topology of the universe. See also [2].

Succinctly (see [3] for details), while the spatial sections of a Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model are usually taken to be one of the simply connected spaces of constant curvature (spherical space $S^3$, Euclidean space $E^3$, and hyperbolic space $H^3$), they can more generally be represented by a quotient manifold $M = \bar{M}/\Gamma$, where $\bar{M}$ is one of the mentioned spaces and $\Gamma$ is a discrete group of isometries (or rigid motions) acting freely and properly discontinuously on $\bar{M}$. In practice $M$ is described by a Dirichlet domain or fundamental polyhedron (FP) in $\bar{M}$, with faces pairwise identified through the action of the elements of $\Gamma$; the latter is said to tesselate $\bar{M}$ into cells which are replicas of FP, so that $\bar{M} = \Gamma(FP)$. Mathematically $\bar{M}$ is the universal covering space of $M$, while physically it is the locus of repeated images of sources in $M$, one for each cell.

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The method of \cite{1} consists of plotting the distances between cosmic images of clusters of galaxies vs. the frequency of occurrence of each of these distances. If real space turns out to be such a manifold, and $\Gamma$ contains Clifford translations (cf. \cite{4}, \cite{5}), then we may expect to see neat peaks or spikes in this plot, and their pattern would be related to the topology of cosmic space.

In the case of hyperbolic space, the only Clifford translation in $\Gamma$ is the trivial motion \cite{6}. Therefore the original crystallography method may reveal very little, or nothing, of the global topology; cf. \cite{4}, \cite{5}, \cite{7}. Some modified forms of the method have been proposed for application to hyperbolic models; see \cite{7}, \cite{4}, \cite{5}, \cite{8}. All of these, as well as more general proposals based on other principles - see, for example, \cite{8}, \cite{10} and references there - are dependent on much expected, needed improvements on the observational side, and will certainly be found to be complementary to each other. Here we present still another variant of the crystallography idea, which may be particularly useful if it becomes known that space is a hyperbolic manifold. Namely, we assume space to be a definite hyperbolic manifold, with a fixed $FP$ in a fixed orientation in astronomical space. The position of the observer inside $FP$ does not influence the result, so one may work as if he or she were located at the center (or basepoint) of $FP$.

The original crystallography scheme relied on the elements of $\Gamma$ bringing a source’s position in $FP$ to the positions of its repeated images in other cells. For hyperbolic manifolds a given action $\gamma \in \Gamma$ on points $p \in FP$ depends on $p$, and thus we do not get the wealth of equal distances that make up the neat peaks in the case of Clifford translations.

In the present variation of the method, we pull each image back to its pre-image’s position in $FP$. If space is really $H^3/\Gamma$, with the observer’s position at the center of $FP$, then the distribution of distances between the sources and the pulled back images will strongly peak at zero distance - more precisely, near zero, because of little known facts like evolution and peculiar velocities, and data inaccuracies. If real space is rotated with respect to the assumed manifold, $FP$ will also be rotated, and the pullback operation does not bring an image to its real pre-image’s position. So the neat peak at zero distance is destroyed; if the rotation is small ($\lesssim 5^\circ$) a less sharp peak is still visible near zero distance, but it quickly disappears as the angle of rotation increases. Therefore one might have to check about a thousand orientations of $FP$, for each candidate manifold in order to find the significant peak near zero distance.
2 The simulated catalogs

We shall be working with the spacetime metric of FLRW hyperbolic model,

\[ ds^2 = a^2(\eta)(d\eta^2 - d\lambda^2), \]

where \( a(\eta) \) is the expansion factor or curvature radius, and

\[ d\lambda^2 = d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \]

is the standard or normalized metric of \( H^3 \) - cf.\[11\]. We assume for the cosmological parameters the values \( \Omega_0 = 0.3, H_0 = 65 \text{ km s}^{-1}\text{Mpc}^{-1} \), and \( \Lambda = 0 \). The present value of the curvature radius is then \( a(\eta_0) = 5512.62 \text{ Mpc} \).

Our computer simulated catalogs are similar to those in \[7\]. There is an improvement in the making of tables of pseudo-random points: The volume element in hyperbolic space is \( dV = \sinh^2 \chi \sin \theta d\chi d\theta d\phi \); if we define new coordinates \( u(\chi) = (\sinh \chi \cosh \chi - \chi)/2 \), \( v(\theta) = \cos^{-1} \theta \), and \( \phi = \phi \), we get \( dV = du dv d\phi \), so that the probability density of points in \((u,v,\phi)\)-space is uniform, and we need not weight the distribution of random values for these coordinates.

We did the simulations for two compact hyperbolic models, both with a regular icosahedron as \( FP \) but different groups \( \Gamma \). They are the first and second manifolds in Table 1 in \[12\], and appear in the ‘closed census’ of \( \text{SnapPea} \) \[13\] as \( v2051(+3, 2) \) and \( v2293(+3, 2) \), respectively. The results are quite similar for both manifolds; here we will only report those for \( v2293 \). We took from \[7\] the set of 92 neighboring copies of \( FP \). These 93 cells (including the original \( FP \)) completely cover a ball of radius \( \chi = 2.33947 \) in \( H^3 \). If we take \( Z = 1300 \) for the redshift of the last scattering surface (SLS) we get for the latter’s normalized radius \( \chi_{\text{SLS}} = 2.33520 < \chi \), hence the presently observable universe fits within the described 93-cell region.

To build simulated catalogs for the compact models we first created 100 random points inside the ball with the radius \( \chi_{\text{out}} = 1.38257 \) of the sphere circumscribing the icosahedron. We then excluded those points outside \( FP \), and took the 29 points remaining inside as sources. From the latter a datafile of 897 potential images within the \( \chi_{\text{SLS}} \)-ball (including the sources themselves) was created, using the 93 elements of \( \Gamma \) that cover this ball.

However, since we intended to displace the observer to position \((0.1, 0, 0)\) in Klein coordinates, which is at a distance \( \delta = \tanh^{-1} 0.1 \) from the center, we took the catalogs’ radii to be \( \chi_{\text{max}} = \chi_{\text{SLS}} - \delta = 2.23486 \).
Two catalogs were then prepared from the $\chi_{SLS}$-ball datafile, both with radius $\chi_{\text{max}}$ and the orientation of the SnapPea’s coordinates, centered on the observer at (0,0,0) in one catalog and at (0.1, 0, 0) in the other.

### 3 The pullback crystallography

#### 3.1 Observer at basepoint

Assuming that our $\nu 2293$ manifold represents cosmic space, with the same orientation and basepoint as given by SnapPea, our first simulation places the observer at the basepoint, which is the center of the icosahedron.

For all images $q$ in the catalog with radius $\chi_{\text{max}}$, we find their pre-images in $FP$, that is, we find $p = \gamma^{-1} q$, $\gamma \in \Gamma$. The computer procedure for this process was essentially written by Weeks [14], in the context of SnapPea; it is based on the very definition of a Dirichlet domain.

Let $S$ be the set of sources in $FP$, and $n(p)$ be the number of images in the catalog with pre-image $p \in S$. The pullback process takes all of them to $p$. When the distances between the sources and the pulled back images are calculated, there will be $\sum_{p \in S} n(p)$ null distances, hence a strong peak at zero. The counting for other distances follows a pattern similar to those of the failed attempts to do the standard crystallography with hyperbolic manifolds - cf. [7] and references there. The result is shown if Fig. 1a, with percentage of occurrences plotted versus distances, the latter in bins of 100 Mpc.

In Fig. 1b we show the result of pulling back the images of a random distribution of sources in the open FLRW model. A catalog with the same radius was assumed, and the sources were pulled back with the operation of the same group $\Gamma$ as above. Fig. 1c shows the differences between the previous two plots.

#### 3.2 Observer displaced from basepoint

Moving the observer from (0,0,0) to (0.1, 0, 0), we built a new catalog, with images from the $\chi_{SLS}$-ball with distances up to $\chi_{\text{max}}$ from the new center. Then we proceeded as above, and obtained practically the same results. This is as expected, since the set $S$ of sources and the pulled back images are the same, only with a new set of frequencies $n'(p)$, whose sum is approximately the same as before.
Figure 1: The results of crystallography with pullback for a simulated catalog of clusters of galaxies. Figure a corresponds to the multiply connected model, with the observer at the center of the fundamental region, b to a simply connected universe with the same physical parameters as a, and c to the difference between a and b.
3.3 Manifold in a different orientation

Our next step is to rotate \( FP \). Suppose our manifold with its sources and repeated images represents the real distribution of cosmic images, with the observer at \((0,0,0)\). Also suppose we do not know that the orientation of this manifold in space is that implied by the coordinates used in the SnapPea census. We want to try the pullback process with several rotated versions of \( FP \), to see if the characteristic peaks near zero distance are still present.

If we represent \( H^3 \) as the upper branch of hyperboloid \( X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1 \) in Minkowski space, then the rigid motions in \( H^3 \), like the elements \( \gamma \in \Gamma \), are represented by \( 4 \times 4 \) Lorentz transformation matrices (see, for example, \([13]\)). A rotation with Euler angles \( \phi, \theta, \psi \) corresponds to a matrix \( R_{\mu\nu} \) (\( \mu, \nu = 0 - 3 \)), where \( R_{00} = R_{\nu0} = \delta_{\nu0} \) and \( R_{ij} = R_{ij}(\phi, \theta, \psi) \) is the \( 3 \times 3 \) rotation matrix, as given in \([14]\). Let the face-pairing generators of \( \Gamma \) in SnapPea be \( \gamma_k \), \( k = 1 - 20 \); then the generators will be \( \gamma'_k = R\gamma_k R^{-1} \) for the rotated manifold. The result is that if the images are pulled into the rotated \( FP \), using the new generators, they no longer coincide with their pre-images in \( FP \), with the exception of eventual images of our own Galaxy.

Because of this the peak near zero distance tends to quickly disappear with increasing angle of rotation. In Figs. 2a-c we show the difference plots for angles \( (\phi, \theta, \psi) = (2^\circ, 0, 0) \), \( (0, 5^\circ, 0) \), and \( (150^\circ, 100^\circ, 60^\circ) \), respectively. The plot for a pulled back random distribution has been subtracted out, as in Fig. 1-c. The peak near zero is quite visible in the first, is less sharp in the second, and does not appear at all in the third case.

On the other hand, away from zero distance, these plots resemble the difference plot in \([7]\), whose topological significance is still uncertain - cf. \([17]\).

4 Conclusion

As suggested in Sec. 1, this method would be most useful if we previously knew that space could be a given manifold. This emphasizes our belief that the search for the true cosmic topology will not be easy or straightforward, given the usual inaccuracies and fragmentary nature of observational data. The various proposals that have appeared (cf. \([10]\)) and will continue to appear should all contribute to this much desired discovery, its confirmation and development, and its role in the building of a new, richer, and truer cosmological picture.
Figure 2: The difference between the plots for three models with the same fundamental polyhedron in different orientations in space, and that for the simply connected universe of Fig. 2b.
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