MODE DEPENDENT FIELD RENORMALIZATION AND TRIVIALITY

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ABSTRACT

We critically analyze the introduction of an independent zero momentum mode field renormalization for $\phi^4$. It leads to an infrared divergent effective action. It does not achieve its purpose: triviality still gives massless particles in the broken phase in the continuum limit. It leads to an effective potential which is not the low energy limit of the effective action.

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Lattice $\phi^4$ is a theory which depends on three parameters: two which characterize the action, $m$ and $\lambda$, and the lattice spacing $a$. For particle physics only the scaling region is of interest. It is characterized by a correlation length $\xi$ which is much larger than the lattice spacing, $\xi >> a$. Physics, as long as $E << a^{-1}$, is then insensitive to the lattice spacing, and we are doing essentially continuum physics. In the scaling region, the only one we will be interested in, and at not too high energies, lattice $\phi^4$ is a two parameter field theory. It is convenient to parametrize the theory with low energy parameters, defined in terms of renormalized Green functions at low or zero energy. They are the renormalized mass, $m_R$, and the renormalized coupling, $\lambda_R$. The critical theory is characterized by an infinitive correlation length. The two parameters of the action are not independent anymore, $m = m_c(\lambda)$. It is, at not too high energies, a one parameter theory. Its renormalized mass vanishes, $m_R = 0$.

Continuum $\phi^4$ is a two parameter theory, a priori at least. It is defined by approaching the critical theory, and taking at the same time the lattice spacing to zero. In doing so $a$ is traded for some external length, $L$. Defining $\hat{m}(\lambda, ma) \equiv m_R a$, and recalling that for the critical theory $\hat{m}(\lambda, m_c a) = 0$, one takes the continuum limit in such a way that the continuum renormalized parameters stay finite, and in general nonvanishing,

\[
\lambda_r \equiv \lim_{a \to 0} m \to m_c \lambda_R(\lambda, ma) \quad (1) \\
\]

\[
m_r \equiv \lim_{a \to 0} m \to m_c \frac{\hat{m}(\lambda, ma)}{a} \\
\]

This finetuned limit requires in principle three renormalizations: field, mass and coupling. The field renormalization takes care of the multiplicative renormalization of the Green functions. It is normalized by the relation between Green functions and amplitudes. The mass and coupling renormalizations are normalized by the two low energy parameters of the
theory, $m_R$ and $\lambda_R$. But, as the theory is almost surely trivial [1], there is no interaction eventually and the renormalized continuum coupling vanishes, $\lambda_r = 0$. In other words, the renormalized coupling $\lambda_R$ does depend in a trivial theory in an essential way on the UV cutoff $a$ when the continuum limit is taken. There does not exist a scaling region for interacting physics when one takes the continuum limit. The theory is thus a one parameter theory in the continuum limit. The parameter is, in the symmetric phase, $m_r$.

The same scenario very likely holds in the broken phase [2]. Since the theory is now broken its renormalized vacuum expectation value should be nonvanishing, $v_R \neq 0$. But as the theory still only has at not too high energies two independent parameters as no new counterterms are required, there has to be a relation between $v_R$, $m_R$ and $\lambda_R$. It is conveniently given (up to a finite renormalization) by

$$\frac{3m_R^2}{v_R^2} = \lambda_R$$

(2)

because this relation holds at tree level and because the UV divergent parts of the counterterms satisfy it. Eq. 2 is often taken as a definition of $\lambda_R$. In taking the continuum limit triviality transforms it into

$$\frac{m_r^2}{v_r^2} = 0$$

(3)

As renormalized continuum parameters should be finite eq. 3 implies, as $v_r \neq 0$, that $m_r = 0$. Triviality leads to a massless particle theory in the continuum broken phase. The steps from eq. 2 to eq. 3 have important consequences for the high energy structure of the minimal standard model, leading to a bound of the Higgs mass which, very conservatively, reads $m_H < 1\text{TeV}$ [3].

There have been attempts of finding a non-trivial, or, later, a trivial massive theory in the broken phase. One of these started from a gaussian effective potential which,
after renormalization, was non-trivial [4]. It was soon realized, however, that one could not extend the renormalization to the effective action, or, in other words, the effective action was UV divergent [5], and that effective potential studies were too limited for allowing a thorough understanding of triviality issues [6]. This is because an unconventional field renormalization has to be checked against the UV behaviour of the kinetic energy of the effective action, even if it leads to a finite effective potential. These difficulties were bypassed by introducing two field renormalizations, one for non-zero momentum modes and one for the zero momentum mode [7]. The first one is relevant to the effective kinetic energy, the second to the effective potential. This conjecture soon led to the prediction of a 2 TeV Higgs [8], its analytical finite volume study [9], the proposal of its lattice test [10], its first lattice results [11] and a recollection of the main ideas which lie behind a 2 TeV Higgs [12].

This attempt has produced a substantial amount of publications on O(N) extensions, finite temperature analysis, postgaussian corrections, perturbative approaches, renormalization group studies, etc. to which we do not refer. It now hinges on the assumption of performing two independent field renormalizations, which then would lead to a trivial two parameter continuum theory, massive in the broken phase, with \( v_r \) and \( m_r \) nonvanishing, contrary to the generally accepted and very solidly founded understanding of triviality [1, 2, 3]. Then, determining \( v_r \) from Fermi’s constant and relating \( m_r \) to \( v_r \) with a further assumption, one obtains the unconventional Higgs mass prediction.

Now, although it is enough to check an unconventional field renormalization for non-zero momentum modes with the effective action, it is not enough to check an unconventional zero momentum mode field renormalization with the effective potential, as there is one primitively UV divergent zero momentum Green function missing from it, because it is not proper: the connected one point function. In other words, if there is a new renor-
malization it comes from new UV divergences, and these have to show up in the one point function. In the standard picture of symmetry breaking the one point function divergences are not independent, but determined by the ones of the symmetric phase. Surprisingly in none of the many publications on this subject the unconventional zero-momentum renormalization has been checked against the one point function to see whether it remains finite after renormalization. It does not if the nonvanishing momentum modes are massive. This would then lead to massless nonvanishing momentum modes, and thus massless particles, and massive vanishing momentum modes, unrelated to any particles. We will come to this conclusion, not by performing a specific computation, but proving it from the general structure of trivial quantum field theory with two field renormalizations. Thus two independent field renormalizations do not live up to their expectations; they are useless and lead to pathologies.

One could of course dismiss such an undertaking offhand on two grounds: first, one of the renormalizations, being only a zero momentum mode renormalization, will lead to an IR divergent effective action; and second, by renormalizing zero momentum modes differently from nonvanishing momentum modes the renormalized Green functions will be discontinuous at vanishing momenta, which makes their zero momentum values physically irrelevant. This is because the bare Green functions cannot have a new UV divergence at zero momentum; any new zero momentum divergence can only be IR in nature. There are two reasons why we feel it is nevertheless worth showing that two field renormalizations are not possible: first, the still ongoing work on the attempt started at [4]; and second, the irony of the fact that it is precisely the masslessness of continuum, broken, trivial $\phi^4$ which allows to start the whole attempt, as we will show.

In a QFT all the renormalizations are dictated by the UV divergences of the Green functions, which are given by the theory itself. In a trivial theory, because there are so
many UV zeroes (all the three and more point Green functions) and specially because in the broken phase the zero momentum two point function vanishes in the continuum limit, one could think of an independent renormalization of the zero momentum mode, which would make the zero momentum Green functions finite: it does not absorb UV divergences but cancels UV zeroes, except for the one point function, where it absorbs UV divergences.

In order to show how this idea is implemented, and why it fails, let us construct the effective action in the standard way for a $\phi^4$ field theory with two noncompeting sources, one coupled to the zero momentum mode and the other coupled to all the other modes. No mode should couple to both sources, as otherwise its renormalization would be ambiguous. With euclidean metric and working in momentum space, the generating functional is defined by

$$e^{W[J,j]} = N \int D\tilde{\phi} \exp[-S[\tilde{\phi}] + j\tilde{\phi}(0) + \int dk \tilde{J}(-k)\tilde{\phi}(k)]$$ (4)

with the Lorentz invariant constraint

$$\tilde{J}(0) = 0$$ (5)

and the normalizing factor $N$ such that $W[0,0] = 0$. $\tilde{\phi}(k)$ and $\tilde{J}(k)$ are the Fourier transformed $\phi(x)$ and $J(x)$. An UV cutoff $a$ is in place. We assume Lorentz invariance as we are in the scaling region and neglect scaling violations. The thermodynamic or infinite volume limit is understood. The theory, $\phi^4$, is symmetric, $S[\tilde{\phi}] = S[-\tilde{\phi}]$, and $dk$ is the four-dimensional measure divided by $(2\pi)^4$. The variable source $\tilde{J}(k)$ is a smooth function which tends to zero for large $k$, well below $a^{-1}$. The advantage of working in momentum space is clear from eq. 5.

We are interested in the broken phase. The sign of $j$ determines which of the
two equivalent SSB vacua is chosen by the theory. The standard approach starts from
\[ W[\tilde{J}] = \lim_{j \to 0} W[\tilde{J}, j], \] with \( \tilde{J} \) unconstrained.

The effective field is given by
\[
\frac{\delta W}{\delta \tilde{J}(-k)} \equiv \tilde{\Phi}(k), \quad \lim_{\tilde{J} \to 0} \tilde{\Phi}(k) = 0, \quad k \neq 0
\tag{6}
\]
and by
\[
\frac{dW}{dj} \equiv \tilde{\Phi}(0) \equiv \tilde{\delta}(0)v, \quad \lim_{\tilde{J} \to 0} v = v_j
\tag{7}
\]
where the extensive character of the ground state is made explicit by the zero mode in form
of the IR divergent volume factor \( \tilde{\delta}(0) \). Actually \( v = v_j \) as \( \tilde{J} \) only produces IR subleading
differences. SSB means that
\[
\lim_{j \to 0} v_j = v_o \neq 0
\tag{8}
\]
The effective field \( \tilde{\Phi}(k) \) is a smooth function of \( k \), except at \( k = 0 \), which vanishes for large
\( k \).

The effective action is given by a double Legendre transform,
\[
\Gamma[\tilde{\Phi}] \equiv W[\tilde{J}, j] - \int dk \tilde{\Phi}(k)\tilde{J}(-k) - \tilde{\Phi}(0)j
\tag{9}
\]
so that \( \Gamma[\tilde{\delta}(k)v_o] = 0 \). Also
\[
\frac{\delta \Gamma}{d\tilde{\Phi}(k)} = -\tilde{J}(-k), \quad \lim_{\tilde{\Phi} \to 0} \tilde{J}(-k) = 0, \quad k \neq 0
\tag{10}
\]
and

\[
7
\]
\[ \frac{d\Gamma}{d\Phi(0)} = -j, \quad \lim_{\tilde{\delta}(k) \rightarrow \tilde{\delta}(0)} \tilde{\delta}(k)v_0 j = 0 \]  

(11)

A Taylor expansion around \( \tilde{\Phi}(k) = \tilde{\delta}(k)v_o \) shows its character as a generating functional of proper (one-particle irreducible, truncated, tadpole reducible) Green functions:

\[ \Gamma[\tilde{\Phi}] = \sum_{n=2}^{\infty} \frac{1}{n!} \Pi_{i=1}^{n} \left( \int dk_i (\tilde{\Phi}(k_i) - \tilde{\delta}(k_i)v_o) \tilde{\delta}(k_1 + k_2 + \cdots k_n) \tilde{\Gamma}^{(n)}(k_1, k_2, \cdots k_n) \right) \]  

(12)

This is the standard expression, except that \( \tilde{\Phi}(k) - \tilde{\delta}(k)v_o \) is now discontinuous at \( k = 0 \), the discontinuity being \( (v - v_o)\tilde{\delta}(k) \). One can rewrite eq. 12 as

\[ \Gamma[\tilde{\Phi}] = \sum_{n=2}^{\infty} \frac{1}{n!} \Pi_{i=1}^{n} \left( \int dk_i (\tilde{\Phi}(k_i) - \tilde{\delta}(k_i)v) \tilde{\delta}(k_1 + \cdots k_n) \tilde{\Gamma}^{(n)}(k_1, \cdots k_n) \right) + \cdots + \tilde{\delta}(0) \sum_{n=2}^{\infty} \frac{1}{n!} (v - v_o)^n \tilde{\Gamma}^{(n)}(0, 0, \cdots 0) \]  

(13)

where only the first term, which only depends on non-zero modes, and the last term, which only depends on zero modes, have been written out explicitly. The zero-momentum Green functions could be IR divergent, but this should be of no relevance to the issue at hand, which refers to the UV structure of the theory.

Since the only \( x \)-independent source \( J(x) \) which satisfies eq. 5 is \( J(x) = 0 \), the effective potential is defined as

\[ V(v_j) \equiv -\frac{1}{\delta(0)} \tilde{\Gamma}[v_j \tilde{\delta}(k)] = - \sum_{n=2}^{\infty} \frac{1}{n!} (v_j - v_o)^n \tilde{\Gamma}^{(n)}(0, \cdots 0) \]  

(14)

Notice that the last term of the effective action eq. 13 is IR divergent; this is of course due to the constant source \( j \). This is why sources have to decay for large \( x \), and when they are taken constant, as when one defines the effective potential, the IR divergent volume factor is divided out.
In fact, and as \( v = v_j \), the effective action actually contains the effective potential for a constant field. This never happens in the standard formalism with one source, as there the effective potential is obtained from the effective action for an \( x \)-independent effective field incompatible with a source which decays for large \( x \).

Up to here the theory was bare and regularized. The bare Green functions depend on \( \tau \equiv |1 - \frac{m}{m_c}|, \lambda \) and \( a \). They are renormalized multiplicatively

\[
\tilde{\Gamma}_R^{(n)}(k_1 \cdots k_n) = Z_R^{(n)}(\Phi) \tilde{\Gamma}^{(n)}(k_1 \cdots k_n)
\]

(15)

The renormalization of the nonvanishing momentum modes thus is

\[
\tilde{\Phi}_R(k) = Z_R^{-1/2} \Phi(k), \quad k \neq 0
\]

(16)

Suppose now that the theory is trivial in the known sense \([2, 13]\), i.e.

\[
m_R \sim \tau^{1/2} |\ln \tau|^{-1/6} a^{-1}
\]

\[
\lambda_R \sim \frac{32\pi^2}{3} |\ln \tau|^{-1}
\]

\[
Z_R \sim 1
\]

\[
v_R \sim C \tau^{1/2} |\ln \tau|^{1/3} a^{-1}
\]

(17)

for small \( \tau \), which satisfies eq. 2 for a conveniently chosen constant \( C \). Notice that this scaling behaviour is very solidly founded, because renormalization group improved perturbation theory is, at low energies, and because of triviality, very reliable. Let us now take the continuum limit in such a way that
\[ a \sim \tau^{1/2} |\ln \tau|^{1/12} L \]  

so that in the finetuned continuum limit one finds

\[ m_R \sim |\ln \tau|^{-1/4} L^{-1} \]
\[ \lambda_R \sim \frac{32\pi^2}{3} |\ln \tau|^{-1} \]
\[ Z_R \sim 1 \]
\[ v_R \sim C |\ln \tau|^{1/4} L^{-1} \]

and \( m_r = 0, \lambda_r = 0 \) and \( v_r \) diverges. The continuum limit shown in eq. 19 can be generalized to all Green functions:

\[ \tilde{\Gamma}^{(n>2)}_R(k_1 \cdots k_n) \sim Z_A^{n/2} A^{(n)}(k_1 \cdots k_n) \]
\[ \tilde{\Gamma}^{(2)}_R(k) \sim k^2 + Z_A A^{(2)} \]
\[ v_R \sim Z_A^{-1/2} A \]

where \( A^{(n)}(k_1 \cdots k_n) \) is nonvanishing, \( A^{(n)} \equiv A^{(n)}(0,0,\cdots,0) \) and

\[ Z_A \sim |\ln \tau|^{-1/2} \]

Now comes the crucial observation: in eq. 20 all the zero-momentum proper Green functions have an UV zero, and the connected one point function has an UV divergence in precisely such a way that a further zero momentum field renormalization makes all of them finite. It is given by

\[ \tilde{\Phi}_A(0) = Z_R^{-1/2} Z_A^{1/2} \tilde{\Phi}(0) \]
This is what was (unconsciously) discovered in [4] and is the conjecture on which the prediction of a 2 TeV Higgs hinges. The effective action then becomes in the continuum limit

\[
\Gamma[\tilde{\Phi}_R, \tilde{\Phi}_A] = -\frac{1}{2} \int dk k^2 \tilde{\Phi}_R(k) \tilde{\Phi}_R(-k) + \delta(0) \sum_{n=2} (A - A_o)^n \frac{n!}{n!} A^{(n)} \tag{23}
\]

where, from eqs. 20 and 22,

\[
A = \lim_{a \to 0} Z_R^{-1/2} Z_A^{1/2} v \\
A_o = \lim_{a \to 0} Z_R^{-1/2} Z_A^{1/2} v_o \tag{24}
\]

Notice that only the two terms written out in eq. 13 survive the continuum limit. Eq. 23 is our main result. It contains a finite, interacting effective potential, and only trivial nonvanishing momentum physics, as put forward in references [7-12]. But it has three features of relevance, missed in these references: First, the effective action is still IR divergent. Sources cannot be constant. Second, the particles of eq. 23 are still massless, the new renormalization has not changed this. Third, the massive, interacting effective potential of eq. 23 is not the low energy limit of the effective action. Renormalized Green functions are discontinuous at zero momentum, and thus is the discontinuity devoid of physics. These results show that two independent field renormalizations do not lead to a physically meaningful effective action, and in any case to no new physics.
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