Non-Induced Representations of Finite Cyclic Groups

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Abstract

Let $K$ be an algebraically closed field of characteristic 0 and let $G$ be a finite cyclic group of order $n$. In this note we prove, using induction on the number of prime divisors of $n$, that $R_K(G)/I \cong Z[X]/\Phi_n(X)$ where $R_K(G)$ denotes the ring of $K$-representations of $G$ and $I$ is the sum of ideals $\text{Ind}_I^G(R_K(H))$ of $R_K(G)$ as $H$ varies over all proper subgroups of $G$. This gives us an idea of how many representations of $G$ are not induced from representations of a proper subgroup.

1 Introduction

Let $K$ be a field and let $G$ be an Abelian group. By a $K$-representation of $G$ we mean a pair $(\rho, V)$ where $V$ is a $K$-vector space and $\rho : G \to \text{GL}(V)$ is a group homomorphism. Moreover, $(\rho, V)$ is said to be finite dimensional if $V$ is a finite dimensional $K$-vector space. Two representations $(\rho_1, V_1)$ and $(\rho_2, V_2)$ are said to be isomorphic if there exists a $K$-linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi \circ \rho_1 = \rho_2 \circ \varphi$. In fact if $\dim_K V = n$ and if we choose a basis for $V$, then saying that the two representations $\rho_1$ and $\rho_2$ are isomorphic is the same as saying that the matrices corresponding to $\rho_1$ and $\rho_2$ are conjugate over $\text{GL}_n(K)$.

Let $F_K(G) = \bigoplus_{(\rho, V)} Z[(\rho, V)]$ be the free Abelian group generated by the isomorphism classes of finite dimensional irreducible $K$-representations of $G$, and let $S_K(G) \subseteq F_K(G)$ be the subgroup generated by the elements $[(\psi_1, V_1)] + [(\psi_2, V_2)] = [(\psi_1 \oplus \psi_2, V_1 \oplus V_2)]$ as $\psi_1$ and $\psi_2$ vary over all finite dimensional irreducible $K$-representations of $G$. Then the representation ring $R_K(G)$ of $G$ is defined by

$$R_K(G) := \frac{F_K(G)}{S_K(G)}.$$ 

Although a priori $R_K(G)$ is just an Abelian group, we can equip it with a product $\times$: for two isomorphism classes $[(\rho_1, V_1)]$ and $[(\rho_2, V_2)]$, define $[(\rho_1, V_1)] \times [(\rho_2, V_2)] = [(\rho_1 \otimes \rho_2, V_1 \otimes V_2)]$, and extend this product $\mathbb{Z}$-linearly to $R_K(G)$. (Here and in the sequel, $\otimes$ denotes tensor product over $K$.) That this product is well defined in $R_K(G)$ follows from the fact that tensor product distributes over direct sum. From the same reason it also follows that equipped with this product, $R_K(G)$ forms a commutative ring with identity $[(1_G, K)]$, the class of the trivial representation of $G$.

We recall that given a $K$-representation $(\rho, V)$ of $G$, we may view $V$ as a module over the commutative algebra $K[G]$. Indeed, any $g \in G$ acts on a vector $v \in V$ as $v \mapsto \rho(g)(v)$. Conversely, given a $K[G]$-module $V$ we obtain a representation $\rho : G \to \text{GL}(V)$ by defining $\rho(g)(v) = g \cdot v$. Thus in what follows, we interchangeably use $\rho$ or $V$ to denote the representation $(\rho, V)$, where we view $V$ as a $K[G]$-module via the action of $\rho$.

Given a subgroup $H$ of $G$, we view $K[H]$ as a $K$-subalgebra of $K[G]$ via the natural inclusion $H \hookrightarrow G$. Thus given a representation $V$ of $G$, we may view $V$ as a $K[H]$-module, and therefore as a representation of $H$. We denote this representation of $H$ by $\text{Res}_H^G(V)$. Conversely to a representation $W$ of $H$ we may associate a representation of $G$ by defining

$$\text{Ind}_H^G(W) := W \otimes_{K[H]} K[G].$$

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In other words, \( \text{Ind}_H^G(W) \) is obtained by base change of the \( K[H] \)-module \( W \) to \( K[G] \). Since \( \text{Ind}_H^G(W_1 \oplus W_2) \cong \text{Ind}_H^G(W_1) \oplus \text{Ind}_H^G(W_2) \), we obtain a well-defined map at the level of representation rings (which, by abuse of notation, we also denote by \( \text{Ind}_H^G \)) by extending \( \text{Ind}_H^G \) \( \mathbb{Z} \)-linearly, \( \text{Ind}_H^G : R_K(H) \to R_K(G) \). It is easily seen that \( \text{Ind}_H^G \) is a homomorphism of Abelian groups. Furthermore, for any representations \( V \) of \( G \) and \( W \) of \( H \), we have the following \( K[G] \)-module isomorphisms.

\[
V \otimes \text{Ind}_H^G(W) = V \otimes (W \otimes_{K[H]} K[G]) \\
\cong (V \otimes W) \otimes_{K[H]} K[G] \\
\cong (\text{Res}_H^G(V) \otimes W) \otimes_{K[H]} K[G] \\
= \text{Ind}_H^G(\text{Res}_H^G(V) \otimes W).
\]

This shows that the image \( \text{Ind}_H^G(R_K(H)) \) is in fact an ideal in \( R_K(G) \). Thus we may form the quotient ring \( R_K(G)/\text{Ind}_H^G(R_K(H)) \), which can be thought of as a measure of the representations of \( G \) which are \textit{not} induced from \( H \).

In this note we wish to get a sense of how many representations of a finite cyclic group are not induced from representations of a proper subgroup. In other words, we wish to understand the ring

\[
\frac{R_K(G)}{\sum_{H<G} \text{Ind}_H^G(R_K(H))}
\]

where \( G \) is a finite cyclic group and as \( H \) ranges over all proper subgroups of \( G \). To that end we have the following result.

**Theorem 1.1.** Let \( G \) be a finite cyclic group of order \( n \), and let \( K \) be an algebraically closed field of characteristic 0. Then

\[
\frac{R_K(G)}{\sum_{H<G} \text{Ind}_H^G(R_K(H))} \cong \frac{\mathbb{Z}[X]}{(\Phi_n(X))}
\]

where \( \Phi_n \) denotes the \( n \)th cyclotomic polynomial.

**Organisation.** In Section 2 we compute the representation ring of a finite cyclic group \( G \) in terms of a primitive character \( \chi \) of \( G \). More specifically, we show that \( R_K(G) \cong \mathbb{Z}[\chi] \). We then use this isomorphism in Section 3 to compute the sum of ideals \( \text{Ind}_H^G(R_K(H)) \) as \( H \) ranges over all proper subgroups of \( G \). This allows us to reformulate Theorem 1.1 in the form of Theorem 3.1, which is precisely what we prove in Section 5 after establishing a few properties of the cyclotomic polynomials in Section 4.

In passing, we remark that Theorem 3.1 is interesting in its own right. For any positive integer \( n \) and a divisor \( d \) of \( n \), it tells us that the ideal generated by polynomials of the type \( P_{n,d} \) (see Section 3 for the definition of \( P_{n,d} \)) in \( \mathbb{Z}[X] \) is in fact a principal ideal, generated by \( \Phi_n(X) \). To look at it in another way, Theorem 3.1 says that the ideal generated by \( \Phi_n(X) \) (the coefficients of which are somewhat mysterious) in \( \mathbb{Z}[X] \) can be generated by the polynomials \( P_{n,d} \), which are just sums of powers of \( X \). Readers who wish to see the proof of Theorem 3.1 can skip directly to Section 4.

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## 2 Representation Ring of Finite Cyclic Groups

In this section we determine the representation ring of finite cyclic groups. Since any finite dimensional representation can be written as a sum of irreducible representations, it suffices to classify the irreducible representations.
Lemma 2.1. Let $K$ be an algebraically closed field and $G$ an Abelian group. Let $\rho : G \to GL(V)$ be a finite dimensional irreducible $K$-representation of $G$. Then for every $g \in G$, $\rho(g)$ acts on $V$ as $\lambda_g \cdot 1_V$ for some $\lambda_g \in K^\times$.

Proof. Fix $g \in G$. Since $K$ is assumed to be algebraically closed, the characteristic polynomial of $\rho(g)$ has a root, say, $\lambda_g \in K^\times$, so that $\{0\} \subseteq \ker(\rho(g) - \lambda_g \cdot 1_V)$. Moreover, as $G$ is assumed to be Abelian, $\ker(\rho(g) - \lambda_g \cdot 1_V)$ is in fact a non-zero $K[G]$-submodule of $V$. However, $\rho$ is irreducible which implies that $V$ is a simple $K[G]$-module. Therefore $\ker(\rho(g) - \lambda_g \cdot 1_V) = V$, or $\rho(g) = \lambda_g \cdot 1_V$. \hfill \Box

Lemma 2.1 says that if $(\rho, V)$ is a finite dimensional irreducible representation of $G$, then $\rho(g)$ acts as a scalar on $V$ for every $g \in G$, so that any subspace of $V$ will be stable under the action of $G$. Since $\rho$ is assumed to be irreducible, we conclude that dim$_K V = 1$. Thus we have proved the following lemma.

Lemma 2.2. Let $K$ be an algebraically closed field and $G$ an Abelian group. Then any irreducible $K$-representation of $G$ is one-dimensional.

The results in this section so far hold for any irreducible $K$-representation of an Abelian group $G$, as long as $K$ is algebraically closed. We now specialize to finite cyclic groups. Let $G$ be a finite cyclic group of order $n$ with a generator $\sigma$. Then by Lemma 2.2, any irreducible representation of $G$ is a character $\chi : G \to K^\times$. Fix a primitive $n$th root of unity $\zeta_n$ in $K$. Then the representation $\chi_G$ defined on the generator $\sigma$ of $G$ by $\chi_G(\sigma) = \zeta_n$ is an irreducible representation, since it is one-dimensional. We shall refer to $\chi_G$ as the primitive character of $G$. The following proposition shows that any other irreducible representation of $G$ is just a power of $\chi_G$.

Proposition 2.1. Any irreducible representation of $G$ is of the form $\chi_G^k$ where $0 \leq k < n$. (Here $\chi_G^k$ denotes the $k$-fold tensor product of $\chi_G$, $\chi_G^k$.)

Proof. Indeed, since the order of $G$ is $n$, we see for any character $\rho$ that $\rho(\sigma)$ must be a $n$th root of unity. Writing $\rho(\sigma) = \zeta_n^k$ for some $0 \leq k < n$, it follows that $\rho = \chi_G^k$. \hfill \Box

The above proposition provides us with a clean expression for the representation ring of $G \cong \mathbb{Z}/n\mathbb{Z}$:

$$R_K(G) \cong \mathbb{Z}[\chi_G] \cong \frac{\mathbb{Z}[X]}{(X^n - 1)}$$  \hfill (2)

where the second isomorphism is defined by mapping $\chi_G$ to $X$.

3 Reformatulation of Theorem 1.1

As before, let $G$ be a cyclic group of order $n$ generated by $\sigma$ and let $\chi_G$ denote the primitive character of $G$, i.e., $\chi_G(\sigma) = \zeta_n$. For a positive divisor $d$ of $n$, let $H = \langle \sigma^d \rangle$ be the unique cyclic subgroup of $G$ of order $n/d$. Then the restriction $\chi_G|_H$ is the primitive character $\chi_H$ of $H$. Indeed, $\chi_H(\sigma^d) = \zeta_n^d = \zeta_n^{n/d}$. It then follows that every irreducible representation of $H$ (which is a character by Lemma 2.2 and therefore a power of $\chi_H$ by Proposition 2.1) is the restriction of a representation of $G$. Since any finite dimensional representation is a sum of irreducible representations, we conclude that any finite dimensional representation of $H$ arises as the restriction of some representation of $G$.

Next we would like to determine the ideal $\text{Ind}_H^G(R_K(H))$. To that end, let $(V, \rho)$ be a representation of $H$. By the discussion in the previous paragraph, $(V, \rho)$ is a restriction of a representation $(W, \psi)$ of $G$: $\rho = \psi|_H$. Let $1_H : H \to K^\times$ denote the trivial representation of $H$. Then

$$\text{Ind}_H^G(V) = \text{Ind}_H^G(\text{Res}_H^G(W))$$

$$\cong W \otimes_{K[H]} K[G]$$

$$\cong (W \otimes K) \otimes_{K[H]} K[G]$$

$$\cong W \otimes (K \otimes_{K[H]} K[G])$$

$$\cong W \otimes \text{Ind}_H^G(1_H)$$
where in the above $K[G]$-module isomorphisms we view $K$ as a $K[H]$-module where $H$ acts trivially on $K$. (That is, $K$ is the trivial representation of $H$.) This shows that the ideal $\text{Ind}_H^G(1_H)$ is the principal ideal generated by $\text{Ind}_H^G(1_H)$. Therefore in order to determine $\text{Ind}_H^G(1_H)$, it suffices to determine $\text{Ind}_H^G(1_H)$. This is precisely the content of the following proposition.

**Proposition 3.1.** With the above notations,

$$\text{Ind}_H^G(1_H) \cong \bigoplus_{k=0}^{d-1} \chi_G^{kn/d}. \tag{3}$$

**Proof.** By [Ser, Cor. 2, Thm. 4, Ch. 2] it suffices to show that the characters of $\rho := \text{Ind}_H^G(1_H)$ and $\psi := \bigoplus_{k=0}^{d-1} \chi_G^{kn/d}$ agree. First, the character $\chi_{\psi}$ of $\psi$ is just $\chi_{\psi} = \sum_{k=0}^{d-1} \chi_G^{kn/d}$. Let $u \in G$. If $u \in H$, then $\chi_{\psi}(u) = \sum_{k=0}^{d-1} \chi_G^{kn/d}(u) = d$. On the other hand suppose $u \notin H$. Then we may write $u = \sigma^m$ for some positive integer $m$. Hence

$$\chi_{\psi}(u) = \sum_{k=0}^{d-1} \chi_G^{kn/d}(u) = \sum_{k=0}^{d-1} \chi_G^{kn/d}(\sigma^m) = \sum_{k=0}^{d-1} \chi_G^{kn/d}(u) = d.$$

Thus,

$$\chi_{\psi}(u) = \begin{cases} 0 & \text{if } u \notin H, \\ d & \text{if } u \in H. \end{cases}$$

For the character of $\rho$ we have

$$\chi_\rho(u) = \sum_{t \in G/H, t^{-1}ut \in H} 1_H(t^{-1}ut) = \sum_{t \in G/H, u \in H} 1_H(u) = \begin{cases} 0 & \text{if } u \notin H, \\ d & \text{if } u \in H \end{cases}$$

where the first equality follows from [Ser, Thm. 12, Ch. 3]. So $\chi_{\psi} = \chi_\rho$. \hfill \Box

Let $\varphi : R_K(G) = \mathbb{Z}[\chi_G] \to \mathbb{Z}[X]/\langle X^n - 1 \rangle$ denote the second isomorphism from Equation 2. Then by the above proposition,

$$\varphi \left( \text{Ind}_H^G(1_H) \right) = \varphi \left( \bigoplus_{k=0}^{d-1} \chi_G^{kn/d} \right) = \bigoplus_{k=0}^{d-1} \chi_G^{kn/d}.$$

Furthermore, if $P_{n,d} \in \mathbb{Z}[X]$ denotes the polynomial $\sum_{k=0}^{d-1} \chi_G^{kn/d}$, then

$$\varphi \left( \sum_{H \leq G} \text{Ind}_H^G(R_K(H)) \right) = \langle P_{n,d} : d > 1, d \mid n \rangle$$

as ideals in $\mathbb{Z}[X]$. Thus we may restate Theorem 1.1 as follows.

**Theorem 3.1.** Let $n$ be a positive integer. Then

$$\langle P_{n,d} : d > 1, d \mid n \rangle = \langle \Phi_n(X) \rangle \tag{4}$$

as ideals in $\mathbb{Z}[X]$. 

4
4 Preliminary Results

In this section we establish a few properties of the cyclotomic polynomials which will help us in proving Theorem 3.1.

Lemma 4.1. Let $p$ be a prime and $r$ a positive integer such that $\gcd(r, p) = 1$.

- For any integer $m \geq 1$,
  \[ \Phi_{rp^m}(X) = \Phi_{rp}(X^{p^{m-1}}). \]

- We have
  \[ \Phi_{rp}(X) = \frac{\Phi_r(X^p)}{\Phi_r(X)}. \]

Proof. Since
  \[ X^n - 1 = \prod_{d|n} \Phi_d(X), \]
we obtain by the M"obius inversion formula
  \[ \Phi_n(X) = \prod_{d|n} (X^d - 1)^{\mu(n/d)} \]  \hspace{1cm} (5)
where $\mu : \mathbb{Z} \to \{-1, 0, 1\}$ is the M"obius function defined as
  \[
  \mu(n) = \begin{cases} 
  0 & \text{if } n \text{ is not squarefree,} \\
  1 & \text{if } n \text{ is squarefree and has an even number of prime factors,} \\
  -1 & \text{if } n \text{ is squarefree and has an odd number of prime factors.}
  \end{cases}
  \]
We now prove the first part of the lemma. From Equation (5) we have
  \[ \Phi_{rp^m}(X) = \prod_{d|r p^m} (X^d - 1)^{\mu(rp^m/d)}. \]  \hspace{1cm} (6)
However if $\text{ord}_p(d) \leq m - 2$ then $rp^m/d$ would be divisible by $p^2$ and consequently $\mu(rp^m/d) = 0$. Thus the product in Equation (6) ranges over all divisors $d$ of $rp^m$ such that $\text{ord}_p(d) = m$ or $m - 1$. Writing $f = dp^{-(m-1)},$
  \[
  \Phi_{rp^m}(X) = \prod_{f|rp^m} (X^{f p^{m-1}} - 1)^{\mu(rp^m/f p^{m-1})} \\
  = \prod_{f|rp} (X^{f p^{m-1}} - 1)^{\mu(rp/f)} \\
  = \Phi_{rp}(X^{p^{m-1}}).
  \]
For the second part, note that any divisor of $pr$ is either a divisor of $r$ or $p$ times a divisor of $r$. Thus
  \[ \Phi_{rp}(X) = \prod_{d|r} (X^d - 1)^{\mu(rp/d)} \\
  = \prod_{d|r} (X^d - 1)^{\mu(rp/d)} \cdot \prod_{d|r} (X^{pd} - 1)^{\mu(rp/dp)}. \]
Furthermore, since $\gcd(p, r) = 1$, we see that $\mu(rp/d) = \mu(p)\mu(r/d) = -\mu(r/d)$. Hence
  \[ \Phi_{rp}(X) = \prod_{d|r} (X^d - 1)^{-\mu(r/d)} \cdot \prod_{d|r} (X^{dp} - 1)^{\mu(r/dp)} \\
  = \prod_{d|r} (X^d - 1)^{-\mu(r/d)} \cdot \prod_{d|r} (X^{dp} - 1)^{-\mu(r/d)} \\
  = \Phi_r(X^p) \frac{\Phi_r(X)}{\Phi_r(X)}. \]
**Proposition 4.1.** Let \( n \) be a positive multiple of \( p \) and let \( a = \text{ord}_p(n) \geq 1 \). Then

\[
\Phi_n(X) = \frac{\Phi_{np^{a}}(X^{p^a})}{\Phi_{np^{a}}(X^{p^{a-1}})}.
\]

**Proof.** From the first part of Lemma 4.1,

\[
\Phi_n(X) = \Phi_{p^{a}(np^{a-1})}(X) = \Phi_{p(np^{a-1})}(X)
\]

and from the second part of the same lemma,

\[
\Phi_{p(np^{a-1})}(X) = \frac{\Phi_{np^{a-1}}(X^{p^a})}{\Phi_{np^{a-1}}(X^{p^{a-1}})}
\]

and the result follows. \( \square \)

**Proposition 4.2.** Let \( n \) be a positive multiple of \( p \) and \( a = \text{ord}_p(n) \geq 1 \). Then for a prime divisor \( q \) of \( n \) other than \( p \),

\[
\frac{X^{n/p} - 1}{(X^{n/pq} - 1) \cdot \Phi_{np^{a}}(X^{p^{a-1}})} \in \mathbb{Z}[X]. \tag{7}
\]

**Proof.** Let \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \) where \( p_1, p_2, \ldots, p_k \) are the distinct prime divisors of \( n \) and \( a_1, a_2, \ldots, a_k \) are positive integers. Without loss of generality, we prove the lemma when \( p = p_k \) and \( q = p_{k-1} \). Writing \( m = p_1 p_2 \cdots p_{k-1} \),

\[
\prod_{d|m} \Phi_d(X) = X^m - 1, \tag{8}
\]

and since \( p_1 p_2 \cdots p_{k-2} \) divides \( m \),

\[
\prod_{d|m} \Phi_d(X) = P(X) \cdot \left[ \prod_{d|p_1 p_2 \cdots p_{k-2}} \Phi_d(X) \right] \cdot \Phi_m(X)
\]

for some polynomial \( P \in \mathbb{Z}[X] \). So from Equation 8 we conclude that

\[
\frac{X^{p_1 p_2 \cdots p_{k-1}} - 1}{(X^{p_1 p_2 \cdots p_{k-2}} - 1) \cdot \Phi_{p_1 p_2 \cdots p_{k-1}}(X)} = P(X) \in \mathbb{Z}[X].
\]

Substituting \( X^{p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}} \) for \( X \) in the above expression,

\[
\frac{X^{n/p_k} - 1}{(X^{n/p_{k-1} p_k} - 1) \cdot \Phi_{p_1 p_2 \cdots p_{k-1}}(X^{p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}})} \in \mathbb{Z}[X]. \tag{9}
\]

By a repeated application of the first part of Lemma 4.1,

\[
\Phi_{n p_k^{-a_k}}(X^{p_k^{a_k-1}}) = \Phi_{p_1^{a_1} p_2^{a_2} \cdots p_{k-1}^{a_{k-1}}}(X^{p_k^{a_k-1}}) = \Phi_{p_1 p_2 \cdots p_{k-1}}(X^{p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}})
\]

and so substituting this into Equation 9 we find

\[
\frac{X^{n/p_k} - 1}{(X^{n/p_{k-1} p_k} - 1) \cdot \Phi_{n p_k^{-a_k}}(X^{p_k^{a_k-1}})} \in \mathbb{Z}[X]
\]

which proves the lemma. \( \square \)
**Lemma 4.2.** For positive integers \(a\) and \(b\) with \(d = \gcd(a,b)\),
\[
(X^d - 1) = (X^a - 1, X^b - 1)
\]
as ideals in \(\mathbb{Z}[X]\).

**Proof.** The inclusion \((X^a - 1, X^b - 1) \subseteq (X^d - 1)\) is clear. We may assume without loss of generality that \(a \geq b\). Writing \(r_0 = a, r_1 = b\), we obtain from the Euclidean algorithm
\[
\begin{align*}
    r_0 &= q_1r_1 + r_2 \\
    r_1 &= q_2r_2 + r_3 \\
    &\vdots \\
    r_{\ell-1} &= q_\ell r_\ell
\end{align*}
\]
where \(r_\ell = d\), and \(0 \leq r_{i+1} < r_i\) for all \(i = 1, 2, \ldots, \ell\). Then for \(i = 1, \ldots, \ell - 1\) the polynomial
\[
f_i(X) = \frac{X^{n/r_i} - 1}{X^{r_i} - 1} \cdot X^{r_{i+1}}
\]
has integer coefficients, and
\[
(X^{r_i-1} - f_i(X)) (X^{r_i} - 1) = X^{r_{i+1}} - 1.
\]
Therefore \((X^{r_i} - 1, X^{r_{i+1}} - 1) \subseteq (X^{r_i-1} - 1, X^{r_i} - 1)\) for all \(0 \leq i < \ell\), and it follows that
\[
\begin{align*}
    (X^d - 1) &= (X^{r_\ell} - 1) \\
    &\subseteq (X^{r_\ell-1} - 1, X^{r_\ell} - 1) \\
    &\subseteq (X^a - 1, X^b - 1)
\end{align*}
\]
which completes the proof of the lemma. \(\square\)

# 5 Proof of Theorem 3.1

First let us show that \(\langle P_{n,d} : d > 1, d \mid n \rangle \subseteq \langle \Phi_n(X) \rangle\). We note that if \(\zeta_n\) is a primitive \(n\)th root of unity, then \(\zeta_n^{n/d}\) is a primitive \(d\)th root of unity and thus
\[
P_{n,d}(\zeta_n) = \sum_{k=0}^{d-1} \zeta_n^{kn/d} = 0.
\]
So \(P_{n,d} = Q_d \cdot \Phi_n\) for some polynomial \(Q_d \in \mathbb{Q}[X]\). But since both \(P_{n,d}\) and \(\Phi_n\) are monic polynomials in \(\mathbb{Z}[X]\), it follows that \(Q_d \in \mathbb{Z}[X]\) and therefore \(Q_d \in \mathbb{Z}[X]\) for all \(d > 1, d \mid n\). This shows the inclusion \(\langle P_{n,d} : d > 1, d \mid n \rangle \subseteq \langle \Phi_n(X) \rangle\).

The following proposition establishes the inclusion in the other direction.

**Proposition 5.1.** Let \(n\) be a positive integer with prime divisors \(p_1, p_2, \ldots, p_t\). Then there exist polynomials \(f_1, f_2, \ldots, f_t \in \mathbb{Z}[X]\) such that
\[
\sum_{i=1}^{t} \frac{X^n - 1}{X^{n/p_i} - 1} \cdot f_i(X) = \Phi_n(X).
\]

**Proof.** We use induction on the number of prime divisors \(t\) of \(n\). The case \(t = 1\) simply rephrases Lemma 4.1. Indeed, if \(n = p^m\) for a prime \(p\) and a positive integer \(m\) then
\[
\frac{X^n - 1}{X^{n/p} - 1} = \frac{X^{p^m} - 1}{X^{p^{m-1}} - 1} = \Phi_p(X)^{p^{m-1}} = \Phi_{p^m}(X) = \Phi_n(X)
\]
where the second equality follows from Lemma 4.1.

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So assume that the proposition holds for all positive integers having at most \( k - 1 \) distinct prime divisors, and let \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \). Then the positive integer \( m = np_k^{a_k} \) has at most \( k - 1 \) distinct prime factors, so by our hypothesis

\[
\sum_{i=1}^{k-1} \frac{X^{m - 1}}{X^{m/p_i} - 1} \cdot f_i(X) = \Phi_m(X)
\]

for some polynomials \( f_1, f_2, \ldots, f_{k-1} \in \mathbb{Z}[X] \). Substituting \( X^{p_k^{a_k}} \) for \( X \),

\[
\sum_{i=1}^{k-1} \frac{X^{mp_k^{a_k} - 1}}{X^{mp_k^{a_k}/p_i} - 1} \cdot f_i \left( X^{p_k^{a_k}} \right) = \Phi_m \left( X^{p_k^{a_k}} \right)
\]
or

\[
\sum_{i=1}^{k-1} \frac{X^{n - 1}}{X^{n/p_i} - 1} \cdot f_i(X) = \Phi_{np_k^{a_k}} \left( X^{p_k^{a_k}} \right)
\]

(10)

where \( F_i(X) = f_i \left( X^{p_k^{a_k}} \right) \in \mathbb{Z}[X] \). Now for \( 1 \leq i \leq k - 1 \), define the polynomials \( g_i \) by

\[
g_i(X) = \frac{X^{n/p_k} - 1}{\left( X^{n/p_k} - 1 \right) \Phi_{np_k^{a_k}} \left( X^{p_k^{a_k}} \right)} \cdot F_i(X).
\]

Then it follows by Proposition 4.2 that \( g_i \in \mathbb{Z}[X] \) for all \( i \). So by dividing Equation 10 by \( \Phi_{np_k^{a_k}} \left( X^{p_k^{a_k}} \right) \),

\[
\sum_{i=1}^{k-1} \frac{(X^n - 1) (X^{n/p_i} - 1)}{(X^n/p_i - 1) (X^{n/p_i} - 1)} \cdot g_i(X) = \frac{\Phi_{np_k^{a_k}} \left( X^{p_k^{a_k}} \right)}{\Phi_{np_k^{a_k}} \left( X^{p_k^{a_k}} \cdot 1 \right)}.
\]

(11)

For each \( 1 \leq i \leq k - 1 \) there exist polynomials \( a_i, b_i \in \mathbb{Z}[X] \) by Lemma 4.2 such that

\[
(X^{n/p_i} - 1)a_i(X) + (X^{n/p_k} - 1)b_i(X) = (X^{n/p_k} - 1).
\]

Substituting for \( (X^{n/p_k} - 1) \) in Equation 11 and using Proposition 4.1 on the right hand side of the equation,

\[
\sum_{i=1}^{k-1} (X^n - 1) \cdot \left( \frac{a_i(X)}{X^{n/p_k} - 1} + \frac{b_i(X)}{X^{n/p_i} - 1} \right) \cdot g_i(X) = \Phi_n(X).
\]

Upon rearranging the sum we obtain

\[
\sum_{i=1}^{k} \frac{X^n - 1}{X^{n/p_i} - 1} \cdot h_i(X) = \Phi_n(X)
\]

where for \( 1 \leq i \leq k - 1 \),

\[
h_i(X) = b_i(X)g_i(X) \in \mathbb{Z}[X]
\]

and

\[
h_k(X) = \sum_{i=1}^{k-1} a_i(X)g_i(X) \in \mathbb{Z}[X].
\]

This completes the proof of the proposition, and therefore also of Theorem 1.1.

\[\square\]

References

[Ser] J.-P. Serre. Linear Representations of Finite Groups, Springer-Verlag, New York Inc., 1977.