ASYMPTOTIC BEHAVIOUR OF THE RELATIVISTIC BOLTZMANN EQUATION IN THE
ROBERTSON-WALKER SPACETIME

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Abstract. In this paper, we study the relativistic Boltzmann equation in the spatially flat
Robertson-Walker spacetime. For a certain class of scattering kernels, global existence of classical
solutions is proved. We use the standard method of Illner and Shinbrot for the global existence
and apply the splitting technique of Guo and Strain for the regularity of solutions. The main interest
of this paper is to study the evolution of matter distribution, rather than the evolution of spacetime.
We obtain the asymptotic behaviour of solutions and will understand how the expansion of the
universe affects the evolution of matter distribution.

1. Introduction

One of the simplest ways to describe an expanding universe is to consider the spatially homoge-
neous and isotropic spacetimes with suitable matter models. This kind of spacetimes are called the
Robertson-Walker (RW) spacetimes, and in this paper we are interested in a kinetic matter model.
One may consider the Vlasov equation as a kinetic matter model to get the Einstein-Vlasov system,
and this system of equations has been extensively studied for last several decades. In this case, more
general spacetime models such as the spacetimes of Bianchi types can be considered, and many in-
teresting results can be found in the literature, for instance see [1,14,21,22,23,24,25,26,27,28].
In this paper, we only consider the spatially flat RW spacetime with the metric
\[ ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2), \]
and as a matter model we will consider the Boltzmann equation. Here, the scale factor \( R \) will be
assumed to be given. The main purpose of this paper is to study the evolution of matter distribution,
rather than the evolution of spacetime, and in this sense we study the asymptotic behaviour of
solutions of the Boltzmann equation in the RW spacetime.

In the kinetic theory, matter is treated as a collection of particles, and the Boltzmann equation
takes into account the effect of collisions between particles. It is well known that collisions between
particles lead to an interesting phenomenon, which is not observed in the Vlasov case, such that
almost every solution of the Boltzmann equation converges to an equilibrium state. This can be
proved in a mathematically rigorous way, and indeed in the Newtonian and the Minkowski cases
innumerable references can be found on this subject. We only refer to [4,7] and their references
for the Newtonian Boltzmann equation and [12,30,31,32] for recent results on the relativistic
Boltzmann equation in the Minkowski spacetime. In this paper, we will consider the relativistic
Boltzmann equation in the RW spacetime. The main interest of this paper is to investigate how the
expansion of the universe affects the asymptotic behaviour of solutions of the Boltzmann equation.

Recently in [15], this question was answered in the context of Newtonian cosmology. The main result
of [15] can be stated as follows: depending on the rate of growth of the scale factor, solutions of the
Boltzmann equation may or may not approach equilibrium states. In this paper, we will answer this
question in the framework of general relativity.

The Boltzmann equation in general relativity has not been studied much. Local existence was
proved by Bancel and Choquet-Bruhat many years ago [2,3], and Noutchegueme and his colleagues
have studied the Boltzmann equation in some cosmological settings with strong assumptions on
scattering kernels [18,19,20,33]. A hard potential case has been recently studied in [17] in the
RW spacetime. In this paper, we will also study the Boltzmann equation in the RW spacetime, but we will consider a different type of scattering kernels with a different type of existence proof. In the following two subsections, we will review basic information on the Boltzmann equation in the RW spacetime. In Section 2 we introduce a change of variables to write the Boltzmann equation in a simple form, and then we make the main assumptions on the transformed equation. After establishing several pointwise estimates, we prove global existence of solutions in Section 3.

1.1. The Boltzmann equation. The relativistic Boltzmann equation in the RW spacetime can be written as follows:

\[ \partial_t f - 2\frac{\dot{R}}{R} \sum_{i=1}^{3} p^i \partial_{p^i} f = Q(f, f), \]

where \( f = f(t, p) \) is the distribution function, \( \dot{R} \) denotes the time derivative of \( R \), and \( Q \) is called the Boltzmann collision operator. It can be written as

\[ Q(f, f) = R^3 \int \int_{\mathbb{S}^2} v_\phi \sigma(g, \omega) \left( f(p') f(q') - f(p) f(q) \right) \, d\omega \, dq, \quad v_\phi = \frac{g \sqrt{s}}{p^0 q^0}, \]

where \( v_\phi \) and \( \sigma \) are called the Møller velocity and the scattering kernel respectively, and the scalar quantities \( g \) and \( s \) are defined by

\[ g = \sqrt{(p_\alpha - q_\alpha)(p^\alpha - q^\alpha)} \quad \text{and} \quad s = -(p_\alpha + q_\alpha)(p^\alpha + q^\alpha). \]

Here, \( p^\alpha \) and \( q^\alpha \) denote four-dimensional momentum variables, and \( p \) and \( q \) in the equation (1.1) denote the spatial components of \( p^\alpha \) and \( q^\alpha \), i.e.

\[ p^\alpha = (p^0, p^1, p^2, p^3) \quad \text{and} \quad p = (p^0, p^1, p^2, p^3). \]

Latin indices will be assumed to run from 1 to 3 as in (1.1), while Greek indices run from 0 to 3 as in (1.3), and the Einstein summation convention will be assumed. The indices are lowered through the metric considered, i.e. \( p_\alpha = g_{\alpha\beta} p^\beta \), and in the RW case we have \( p_0 = -p^0 \) and \( p_k = R^2 p^k \).

Assuming that all the particles have the same mass, we get the mass shell condition

\[ p_\alpha p^\alpha = -m^2 = -1, \]

where we assumed \( m = 1 \) for simplicity, and in the RW case it can be written as

\[ p^0 = \sqrt{1 + R^2 |p|^2}, \]

where \( | \cdot | \) denotes the modulus of \( p \) in \( \mathbb{R}^3 \). The momentum variables with primes like \( p' \) and \( q' \) in (1.2) denote post-collisional momenta for given pre-collisional momenta \( p \) and \( q \). In the following section, we briefly review several different types of representations of post-collisional momenta. For more details on the relativistic kinetic equations, we refer to [1] [5] [6] [16] [22] [29].

1.2. Post-collisional momenta. Suppose that two particles having momenta \( p^\alpha \) and \( q^\alpha \) collide, and let \( p'^\alpha \) and \( q'^\alpha \) be their momenta after the collision. In the collision process it is assumed that their total energy and momentum are conserved, which can be written by

\[ p'^\alpha + q'^\alpha = p^\alpha + q^\alpha. \]

Due to this energy-momentum conservation, the post-collisional momenta can be parametrized by \( p^\alpha \) and \( q^\alpha \) with some additional parameters. In the nonrelativistic case, the following two different types of representations of post-collisional momenta are easily found. For given three dimensional vectors \( \xi \) and \( \xi_* \), we have

\[ \xi' = \frac{\xi + \xi_*}{2} + \frac{|\xi - \xi_*|}{2} \sigma, \quad \xi'_* = \frac{\xi + \xi_*}{2} - \frac{|\xi - \xi_*|}{2} \sigma, \quad \sigma \in \mathbb{S}^2, \]

or

\[ \xi' = \xi - ((\xi - \xi_*) \cdot \omega) \omega, \quad \xi'_* = \xi_* + ((\xi - \xi_*) \cdot \omega) \omega, \quad \omega \in \mathbb{S}^2, \]

where \( \xi' \) and \( \xi'_* \) denote post-collisional momenta for given pre-collisional momenta \( \xi \) and \( \xi_* \).
where $\cdot$ and $|\cdot|$ are the usual inner product and the corresponding norm in $\mathbb{R}^3$, and they are sometimes called the $\sigma$-representation and the $\omega$-representation respectively (pages 126–127 of [34]). To find a relativistic analogue of (1.5), we make an assumption that the post-collisional momenta have the form of

$$p^{\alpha} = \frac{p^{\alpha} + q^{\alpha}}{2} + \frac{g}{2} \Omega^{\alpha}, \quad q^{\alpha} = \frac{p^{\alpha} + q^{\alpha}}{2} - \frac{g}{2} \Omega^{\alpha},$$

for some four-vector $\Omega^{\alpha}$. In a similar way, we may write

$$p^{\alpha} = p^\beta - \left((p_\beta - q_\beta)\Omega^\beta\right)\Omega^{\alpha}, \quad q^{\alpha} = q^\beta + \left((p_\beta - q_\beta)\Omega^\beta\right)\Omega^{\alpha},$$

for some different four-vector $\Omega^{\alpha}$. Then, the mass shell condition (1.9) gives two constraints on the parameter $\Omega^{\alpha}$. In the case of (1.7), we have

$$-1 = \frac{s}{4} + \frac{g^2}{4} n_\alpha \Omega^\alpha + \frac{g^2}{4} \alpha \Omega^\alpha,$$

where $n^{\alpha} = p^{\alpha} + q^{\alpha}$, and equivalently we get two constraint equations:

$$n_\alpha \Omega^\alpha = 0 \quad \text{and} \quad \Omega_\alpha \Omega^\alpha = 1.$$

In the case of (1.8), the same constraint equations are obtained. Hence, the parameter $\Omega^{\alpha}$ reduces to a four-vector which can be represented by two independent variables, and we take $\omega \in S^2$ as usual. The simplest way to find $\Omega^{\alpha}$ is to construct a vector $t^{\alpha}$ which is orthogonal to $n^{\alpha}$ and then normalize it. Using the parameter $\omega \in S^2$, we find $t^{\alpha} = (n^\alpha \omega, -n_0 \omega)$ and normalize it to get

$$\Omega^\alpha = \frac{t^\alpha}{\sqrt{t_\beta t^\beta}} \quad \text{with} \quad t^\alpha = (n^\alpha \omega, -n_0 \omega).$$

It is easy to see that $n_\alpha t^{\alpha} = 0$ and $\Omega_\alpha \Omega^\alpha = 1$. Plugging (1.10) into (1.7) or (1.8), we obtain two parametrizations of $p^{\alpha}$ and $q^{\alpha}$ in terms of $p^{\alpha}$, $q^{\alpha}$, and $\omega$. We note that the parameter $\Omega^{\alpha}$ is basically the same with the one that was previously found by Glassey and Strauss [9, 10] in the Minkowski case. It is written in the Minkowski space as

$$\Omega^\alpha = \frac{t^\alpha}{\sqrt{t_\beta t^\beta}} \quad \text{with} \quad t^\alpha = (n^\alpha \omega, n^\alpha \omega),$$

where $n^\cdot \omega$ is the usual inner product of $n$ and $\omega$ in $\mathbb{R}^3$. We combine (1.8) and (1.11) to get the parametrization of [9, 10].

In the Minkowski case we can find a different form of the parameter $\Omega^{\alpha}$. To find a vector orthogonal to $n^{\alpha}$, we have constructed $t^{\alpha}$ in (1.10) and then normalized it. However, we may decompose $\omega = \omega_1 + \omega_2$ such that $n \cdot \omega_1 = 0$ and $n \cdot \omega_2 = n \cdot \omega$, for instance

$$\omega_1 = \omega - \frac{(n \cdot \omega)n}{|n|^2} \quad \text{and} \quad \omega_2 = \frac{(n \cdot \omega)n}{|n|^2}.$$

Then, $t^\alpha = t^\alpha(\omega)$, as a linear function of $\omega$, can also be decomposed as

$$t^\alpha = t^\alpha(\omega) = t^\alpha(\omega_1) + t^\alpha(\omega_2) = t_1^\alpha + t_2^\alpha.$$

We can observe

$$n_\alpha t_1^\alpha = n_\alpha t_2^\alpha = t_{1\alpha} t_{1\beta} t_{2\gamma} = 0,$$

and

$$t_{1\alpha} t_2^\alpha = (n^0)^2 |\omega_1|^2, \quad t_{2\alpha} t_{2\beta} = s |\omega_2|^2.$$

Since $|\omega_1|^2 + |\omega_2|^2 = 1$, we find a different form of $\Omega^{\alpha}$ as

$$\Omega^\alpha = \frac{1}{n^0} t_1^\alpha + \frac{1}{s} t_2^\alpha \quad \text{with} \quad t_k^\alpha = (n \cdot \omega_k, n^0 \omega_k), \quad k = 1, 2.$$

It is easy to see that (1.13) satisfies the constraints (1.9). We remark that the representation combining (1.7) and (1.13) is basically the one that Strain has derived and used in a series of his
papers, for instance [12, 30, 31, 32]. It has turned out that using the parameter (1.13) is crucial in the proof of existence of regular solutions to the relativistic Boltzmann equation. We refer to [12, 32] for more detailed explanation of the parametrization of this type.

In this paper, we consider the RW spacetime. In this case the parameter $\Omega^\alpha$ in (1.10) reduces to the following:

$$\Omega^\alpha_R = \frac{t^\alpha}{\sqrt{t^\beta t^\beta}} \quad \text{with} \quad t^\alpha = (R^2 n \cdot \omega, n^0 \omega).$$

On the other hand, the parameter $\Omega^\alpha_R$ in (1.13) is only valid in the Minkowski case, hence we generalize it to the RW case as follows:

$$\Omega^\alpha_{RS} = \frac{1}{R n^\alpha t^\alpha_1} + \frac{1}{R \sqrt{s} q^2} \quad \text{for} \quad t^\alpha_k = (R^2 n \cdot \omega_k, n^0 \omega_k), \quad k = 1, 2,$$

where $\omega_1$ and $\omega_2$ are given by (1.12). In the present paper we only use the representation (1.14), but for the parameter $\Omega^\alpha$ we use both of (1.14) and (1.15) as in [12]. Since we are interested in classical solutions, we need to control derivatives of post-collisional momenta, and the splitting technique and interplay between (1.14) and (1.15) proposed in [12] will efficiently control them.

2. Preliminaries

In this section, we first introduce a change of variables so that the Boltzmann equation in the RW spacetime is written in a simple form, and then we make the main assumptions on the transformed equation. At the end of the section, we collect elementary lemmas which can be easily proved by direct calculations.

2.1. Change of variables. In this paper we will consider the RW spacetimes, and in this case the Boltzmann equation is written in a simple form if we use covariant variables. To be explicit, the distribution function $f$ will be considered as a function of $t$ and

$$p_k = g_{k\beta} p^\beta = R^2 p^k, \quad k = 1, 2, 3.$$  

This argument was previously used in [17], but in this paper for simplicity, instead of using lower indices, we introduce a new variable $v$ such that

$$(2.1) \quad v = (v^1, v^2, v^3), \quad v^k = R^2 p^k, \quad v^0 = \sqrt{1 + R^{-2} |v|^2} = p^0,$$

where $|v|$ is the modulus of $v$ in $\mathbb{R}^3$.

The post-collisional momentum $p^\alpha$ in the representation of (1.7) and (1.14) is written as follows:

$$p^0 = \frac{p^0 + q^0}{2} + \frac{g}{2} \frac{R^2 n \cdot \omega}{\sqrt{2 R^2 (n^0)^2 - R^4 (n \cdot \omega)^2}} = \frac{v^0 + u^0}{2} + \frac{g}{2 R} \frac{\tilde{n} \cdot \omega}{\sqrt{(n^0)^2 - R^{-2} (n \cdot \omega)^2}},$$

$$p^k = \frac{p^k + q^k}{2} + \frac{g}{2} \frac{n^0 \omega_k}{\sqrt{2 R^2 (n^0)^2 - R^4 (n \cdot \omega)^2}} = \frac{v^k + u^k}{2 R^2} + \frac{g}{2 R} \frac{\tilde{n}^0 \omega_k}{\sqrt{(n^0)^2 - R^{-2} (n \cdot \omega)^2}},$$

where $\tilde{n} = v + u$ and $\tilde{n}^0 = v^0 + u^0$, and we may write

$$(2.2) \quad v^k = R^2 p^k, \quad u^k = R^2 q^k, \quad v^0 = p^0, \quad u^0 = q^0.$$ 

In a similar way the post-collisional momentum in (1.7) and (1.15) is written as

$$p^0 = \frac{p^0 + q^0}{2} + \frac{R g}{2} \frac{1}{\sqrt{s}} (n \cdot \omega_2) = \frac{v^0 + u^0}{2} + \frac{g}{2 R} \frac{1}{\sqrt{s}} (\tilde{n} \cdot \omega),$$

$$p^k = \frac{p^k + q^k}{2} + \frac{g}{2 R} \frac{\omega_1 + n^0 \omega_2}{\sqrt{s}} = \frac{v^k + u^k}{2 R^2} + \frac{g}{2 R} \left( \frac{(\tilde{n} \cdot \omega) \tilde{n}}{|\tilde{n}|^2} + \frac{\tilde{n}^0 (\tilde{n} \cdot \omega) \tilde{n}}{|\tilde{n}|^2} \right),$$

where $\omega_1 = \sqrt{s} (n \cdot \omega_2)$.
and we write $v'$ and $u'$ as in (2.4). Throughout the paper we will only consider the new variables $v$ and $u$, hence we may drop the tildes to define

$$(2.3) \quad n^0 := v^0 + u^0 \quad \text{and} \quad n := v + u,$$

and in the representation of (1.7) and (1.14) we have

$$(2.4) \quad \begin{pmatrix} v^0 \\ v^k \end{pmatrix} = \begin{pmatrix} \frac{v^0 + u^0}{2} + \frac{g}{2R} \sqrt{(n^0)^2 - R^{-2}(n \cdot \omega)^2} \\ \frac{v^k + u^k}{2} + \frac{Rg}{2} \sqrt{(n^0)^2 - R^{-2}(n \cdot \omega)^2} \end{pmatrix}, \quad \omega \in \mathbb{S}^2,$$

while in the representation of (1.7) and (1.15) we have

$$(2.5) \quad \begin{pmatrix} v^0 \\ v^k \end{pmatrix} = \begin{pmatrix} \frac{v^0 + u^0}{2} + \frac{g}{2R} \sqrt{(n \cdot \omega)} \\ \frac{v^k + u^k}{2} + \frac{Rg}{2} \left( (\omega^k - (n \cdot \omega)n^k) + \frac{n^0 (n \cdot \omega)n^k}{\sqrt{s}} \right) \end{pmatrix}, \quad \hat{\omega} \in \mathbb{S}^2,$$

where $\omega$ and $\hat{\omega}$ denote unit vectors in $\mathbb{R}^3$.

We note that $v'$ is written in (2.4) and (2.5) with different unit vectors $\omega$ and $\hat{\omega}$ on $\mathbb{S}^2$. For a fixed $v'$, we may compare (2.5) and (2.6) to find a relation between $\omega$ and $\hat{\omega}$. Let $A$ and $B^k$ be the quantities in (2.4) such that

$$v^0 = \frac{v^0 + u^0}{2} + \frac{g}{2R} A, \quad v^k = \frac{v^k + u^k}{2} + \frac{Rg}{2} B^k,$$

and then $\hat{\omega}$ is determined by comparing (2.4) and (2.5) such that

$$\frac{1}{\sqrt{s}} (n \cdot \hat{\omega}) = A, \quad \left( \hat{\omega}^k - \frac{(n \cdot \hat{\omega})n^k}{|n|^2} \right) + \frac{n^0 (n \cdot \hat{\omega})n^k}{\sqrt{s} |n|^2} = B^k.$$

To be explicit, we plug the first one to the second one to get

$$(2.6) \quad \hat{\omega}^k = B^k + \frac{\sqrt{s} A n^k}{|n|^2} - \frac{n^0 A n^k}{|n|^2}$$

$$= \frac{1}{\sqrt{(n^0)^2 - R^{-2}(n \cdot \omega)^2}} \left( n^0 \omega^k + \frac{\sqrt{s} (n \cdot \omega)n^k}{|n|^2} - \frac{n^0 (n \cdot \omega)n^k}{|n|^2} \right).$$

We observe that $\omega = \hat{\omega}$ in two special cases. Multiplying $\omega$ to (2.6), we obtain

$$\omega \cdot \hat{\omega} = \frac{1}{\sqrt{(n^0)^2 - R^{-2}(n \cdot \omega)^2}} \left( n^0 + \frac{\sqrt{s} (n \cdot \omega)^2}{|n|^2} - \frac{n^0 (n \cdot \omega)^2}{|n|^2} \right),$$

and this quantity is unity when $n \cdot \omega = 0$ or $(n \cdot \omega)^2 = |n|^2$. In the other cases the unit vectors $\omega$ and $\hat{\omega}$ will in general have different values. For instance, we may consider from (2.6)

$$n \cdot \hat{\omega} = \frac{\sqrt{s}}{\sqrt{(n^0)^2 - R^{-2}(n \cdot \omega)^2}} (n \cdot \omega),$$

which shows that $n \cdot \omega$ and $n \cdot \hat{\omega}$ have the same sign and satisfy

$$(2.7) \quad |n \cdot \omega| \geq |n \cdot \hat{\omega}|,$$

where we used (2.19) of Lemma 2.1. With these variables the Boltzmann equation is written as

$$(2.8) \quad \partial_t f = R^{-3} \int v_\phi \sigma(g, \omega) \left( f(v') f(u') - f(v) f(u) \right) d\omega du, \quad v_\phi = \frac{g \sqrt{s}}{v_0^3 u_0}.$$
where $v'$ and $u'$ are parametrized by (2.4) or (2.5). The scalar quantities $g$ and $s$ are understood as functions of $v$ and $u$, i.e.

$$(2.9) \quad g = \sqrt{-(v^0 - u^0)^2 + R^{-2}|v - u|^2} = \sqrt{-2 - 2v^0u^0 + 2R^{-2}(v \cdot u)}, \quad s = 4 + g^2.$$  

Note that the equation (2.8) is equivalent to (1.1) for classical solutions. In this paper the Boltzmann equation will refer to the equation (2.8).

2.2. Main assumptions. To prove existence of solutions of the equation (2.8), we will follow the standard argument of Illner and Shinbrot [13], where existence of small solutions of the Boltzmann equation was proved in the Newtonian case. The original idea has been applied to several different physical settings, for instance see [31] [34], where the Vlasov-Poisson-Boltzmann system and the relativistic Boltzmann equation in the Minkowski spacetime have been studied. The followings are our main assumptions in this paper. We first assume that the scale factor is given as an increasing function, and then choose a suitable weight function. We also make assumptions on the scattering kernel, and the relevance of the assumption will be discussed.

Scale factor. We assume that the scale factor $R$ satisfies

$$(2.10) \quad R(0) = 1 \quad \text{and} \quad R'(t) \geq 0 \quad \text{with} \quad \lim_{t \to \infty} R(t) = \infty.$$  

Weight function. We choose the weight function as $e^{\nu^2}$ and define

$$(2.11) \quad \|f(t)\| := \sup \left\{ e^{\nu^2} \partial_{ij} f(\tau, v) : 0 \leq \tau \leq t, \ v \in \mathbb{R}^3, \ j = 0, 1, \ k = 1, 2, 3 \right\}.$$  

Scattering kernel. We assume that the scattering kernel satisfies the following conditions: for some positive constant $A$ and $0 \leq b < 3$,

$$(2.12) \quad 0 \leq \sigma(g, \omega) \leq A(1 + g^{-b})\sigma_0(\omega) \quad \text{and} \quad |\partial_2 \sigma(g, \omega)| \leq Ag^{-b-1}\sigma_0(\omega),$$  

where $\sigma_0$ is bounded and supported in $S^2_R$ such that

$$(2.13) \quad 0 \leq \sigma_0(\omega) \leq \sigma_1 1_{S^2_R}(\omega)$$  

for some positive constant $\sigma_1$. Here, $1_{S^2_R}(\omega)$ is the indicator function of the set $S^2_R$, which is defined as follows: for some positive constant $B$, we define

$${S^2_R} := \left\{ \omega \in S^2 : \frac{|v - u|^2|n \times \omega|^2}{2R^2s} \leq B \right\},$$  

where $n$ and $s$ are defined by (2.23) and (2.24) respectively.

We introduced a cutoff set $S^2_R$ on the angular part of the scattering kernel. It depends on $v$, $u$, and $t$, but we can see that for each $v$ and $u$ the cutoff set $S^2_R$ converges to $S^2$, since $s$ has a lower bound as in (2.13) and $R$ is an increasing function of $t$. More explicitly, we can find a finite $t_0$ such that we have $S^2_R = S^2$ for $t \geq t_0$, hence the restriction disappears for large $t$. A similar restriction on the scattering kernel can be found in a different physical situation as in [30], where the author studied the Newtonian limit of the Boltzmann equation and the speed of light $c$ was treated as a parameter. To first prove global existence of solutions, the author introduced a cutoff set $S^2_R$ and showed that the cutoff set $S^2_R$ converges to $S^2$ for large $c$ (see Lemma 3.1 of [30]). This restriction to $B_1$ was crucial in the proof of the existence theorem of [30], and in the present paper the cutoff set $S^2_R$ will play the role similar to $B_1$. In Section 5.1 we will see that the weight function $e^{\nu^2}$ works well in our case under the restriction of (2.13).

In this paper, we will use two different representations of post-collisional momenta, and then for a given $v'$ two different unit vectors $\omega$ and $\hat{\omega}$ will be used in the representations (2.9) and (2.10), respectively. Therefore, the scattering kernel $\sigma$ in principle should be given in different forms, for instance $\sigma(g, \omega)$ and $\tilde{\sigma}(g, \hat{\omega})$ respectively. However, in this paper we will assume that both $\sigma$ and $\tilde{\sigma}$ satisfy (2.12) and (2.13) and will not distinguish $\sigma$ and $\tilde{\sigma}$. Note that $\omega$ and $\hat{\omega}$ are related to each
Lemma 2.1. The following estimates hold for the quantities defined in the previous sections:

\begin{align}
(2.14) \\
& s = 4 + g, \
& 2 \leq \sqrt{s}, \quad g \leq \sqrt{s}, \\
(2.15) \\
& g \leq \sqrt{s} \leq 2\sqrt{v^0 u^0}, \\
(2.16) \\
& \frac{|v - u|}{\sqrt{v^0 u^0}} \leq R g \leq |v - u|, \\
(2.17) \\
& |v| \leq R v^0, \quad v^0 = \sqrt{1 + R^{-2} |v|^2} \leq \sqrt{1 + |v|^2}, \\
(2.18) \\
& R g = |v - u| \sqrt{1 - \frac{|v + u|^2 \cos^2 \theta_0}{R^2 (v^0 + w^0)^2}}, \\
(2.19) \\
& \sqrt{s} \leq \sqrt{(v^0)^2 - R^{-2} (n \cdot \omega)^2}, \\
(2.20) \\
& \sqrt{s} \geq \max \left\{ \sqrt{\frac{v^0}{w^0}}, \sqrt{\frac{u^0}{v^0}} \right\},
\end{align}

where \( \theta_0 \) is the angle between \( v + u \) and \( v - u \).

Proof. Since \( s = -(p_\alpha + q_\alpha)(p^\alpha + q^\alpha) \) and \( g = \sqrt{(p_\alpha - q_\alpha)(p^\alpha - q^\alpha)} \), we have

\[ s = 2 - 2p_\alpha q^\alpha = 4 - 2 - 2p_\alpha q^\alpha = 4 + g^2. \]

The inequalities \( 2 \leq \sqrt{s} \) and \( g \leq \sqrt{s} \) are now clear. The second inequality of \( (2.16) \) is obtained by

\[ s = 2 - 2p_\alpha q^\alpha = 2 + 2p^0 q^0 - 2R^2 (p \cdot q) \]

\[ \leq 2p^0 q^0 + 2 \sqrt{1 + R^2 |p|^2 + R^2 |q|^2 + R^4 |p|^2 |q|^2} = 4p^0 q^0 = 4v^0 u^0. \]

For \( (2.16) \), we observe that

\[
\begin{align*}
(p^0)^2 (q^0)^2 - (1 + R^2 (p \cdot q))^2 \\
& = 1 + R^2 |p|^2 + R^2 |q|^2 + R^4 |p|^2 |q|^2 - 1 - 2R^2 (p \cdot q) - R^4 (p \cdot q)^2 \\
& \geq R^2 |p - q|^2,
\end{align*}
\]

and this implies

\[
\begin{align*}
g^2 \\
& = 2p^0 q^0 - 2 \left( 1 + R^2 (p \cdot q) \right) = 2 \frac{(p^0)^2 (q^0)^2 - (1 + R^2 (p \cdot q))^2}{p^0 q^0 + 1 + R^2 (p \cdot q)} \\
& \geq \frac{R^2 |p - q|^2}{p^0 q^0} = \frac{1}{R^2} \frac{|v - u|^2}{v^0 u^0}.
\end{align*}
\]

The second inequality of \( (2.16) \) comes from \( (2.18) \), and this proves \( (2.16) \). We have assumed that \( R = R(t) \) is an increasing function with \( R(0) = 1 \), hence \( (2.17) \) is clear. For \( (2.18) \), we first note that

\[
R^2 (v^0)^2 - R^2 (u^0)^2 = |v|^2 - |u|^2 = (v - u) \cdot (v + u).
\]
Then, by a direct calculation we have
\[ R^2 y^2 = -R^2 (p^0 - q^0)^2 + R^4 |p - q|^2 = |v - u|^2 - R^2 (v^0 - u^0)^2 \]
\[ = |v - u|^2 - \left( \frac{(v - u) \cdot (v + u)}{R(v^0 + u^0)} \right)^2 \]
\[ = |v - u|^2 \left( 1 - \frac{|v + u|^2 \cos^2 \theta_0}{R^2(v^0 + u^0)^2} \right), \]
where \( \theta_0 \) is the angle between \( v + u \) and \( v - u \). This proves (2.18). For (2.19), we note that
\[ s = (p^0 + q^0)^2 - R^2 |p + q|^2 = (v^0 + u^0)^2 - R^{-2} |v + u|^2 \]
\[ = (n^0)^2 - R^{-2} |n|^2 \geq (n^0)^2 - R^{-2} (n \cdot \omega)^2, \]
and this proves (2.19). For the last inequality (2.20), we note that
\[ s = (n^0)^2 - R^{-2} |n|^2 = (v^0)^2 + 2v^0 u^0 + (u^0)^2 - R^{-2} |v|^2 - 2R^{-2} (v \cdot u) - R^{-2} |u|^2 \]
\[ \geq 2 + 2\sqrt{1 - |v|^2} \sqrt{1 + R^{-2} |u|^2} - 2R^{-2} |v||u| \]
\[ = 2 + 2 \frac{1 + R^{-2} |v|^2 + R^{-2} |u|^2}{\sqrt{1 + R^{-2} |v|^2} \sqrt{1 + R^{-2} |u|^2}} \]
\[ \geq 2 + \frac{1 + R^{-2} |v|^2 + R^{-2} |u|^2}{v^0 u^0} \geq \frac{v^0}{u^0} + \frac{u^0}{v^0}, \]
and this completes the proof of the lemma.

**Lemma 2.2.** The following estimates hold for the quantities defined in the previous sections:

\( \partial_{v^0} v^0 = \frac{v^i}{R^2 v^0}, \quad |\partial_{v^0} v^0| \leq \frac{1}{R} \)

\( \partial_{v^i} g = \frac{u^0}{Rg} \left( \frac{v^i}{R v^0} - \frac{u^i}{R u^0} \right), \quad |\partial_{v^i} g| \leq \frac{2u^0}{Rg} \)

\( \partial_{v^i} \sqrt{s} = \frac{u^0}{R \sqrt{s}} \left( \frac{v^i}{R v^0} - \frac{u^i}{R u^0} \right), \quad |\partial_{v^i} \sqrt{s}| \leq \frac{2u^0}{R \sqrt{s}} \)

\( \partial_{v^i} \sqrt{(n^0)^2 - R^{-2} (n \cdot \omega)^2} = \frac{v^0}{R \sqrt{(n^0)^2 - R^{-2} (n \cdot \omega)^2}} \left( \frac{v^i}{R v^0} - \frac{(u^0) \omega^i}{R u^0} \right) \)
\[ + \frac{v^0}{R \sqrt{(n^0)^2 - R^{-2} (n \cdot \omega)^2}} \left( \frac{(v \cdot \omega) \omega^i}{R u^0} \right) \]
\[ \leq \frac{2u^0 + 2v^0}{R \sqrt{(n^0)^2 - R^{-2} (n \cdot \omega)^2}} \]

\( |\partial_{v^i} g| \leq \frac{u^0 \sqrt{v^0 u^0}}{R}, \quad |\partial_{v^i} \sqrt{s}| \leq \frac{u^0 \sqrt{v^0 u^0}}{R} \)

\( \partial_{v^i} \left[ \frac{(n \cdot \omega)n^k}{|n|^2} \right] = \frac{\omega^i n^k}{|n|^2} + \frac{(n \cdot \omega) \delta^i k}{|n|^2} - 2 \frac{(n \cdot \omega)n^i n^k}{|n|^4}, \quad |\partial_{v^i} \left[ \frac{(n \cdot \omega)n^k}{|n|^2} \right] | \leq \frac{3}{|n|} \)

Note that (2.23) reduces to (2.22) in the case of \((n \cdot \omega)^2 = |n|^2\).

**Proof.** The first identities of (2.21)–(2.24) are direct calculations from the definitions of \( v^0, g, \) and \( s \). The inequality of (2.21) is clear. The inequalities of (2.22), (2.23), and (2.25) come from (2.21). For the inequalities (2.26), we note that
\[ \left| \frac{R^{-1} v^i}{\sqrt{1 + R^{-2} |v|^2}} - \frac{R^{-1} u^i}{\sqrt{1 + R^{-2} |u|^2}} \right| \leq R^{-1} |v - u| \leq g \sqrt{v^0 u^0}, \]
and applying it to (2.4) and (2.5) we obtain (2.6). The estimate (2.7) is an elementary calculation, and this completes the proof of the lemma.

**Lemma 2.3.** The following integral for $0 \leq \alpha < 3$ is estimated as
\[
\int_{\mathbb{R}^3} |v - u|^{-\alpha} e^{-|u|^2} \, du \leq C_\alpha (1 + |v|^2)^{-\frac{\alpha}{2}},
\]
where $C_\alpha$ is a positive constant depending on $\alpha$.

**Proof.** The quantity $|v - u|^{-\alpha}$ is integrable near $v \approx u$ for $\alpha < 3$, hence the above integral is bounded for small $v$. Then, for large $v$ we separate the domain as
\[
\int_{\mathbb{R}^3} \frac{1}{|v - u|^\alpha} e^{-|u|^2} \, du \leq \int_{\{|v - u| > \frac{|u|}{2}\}} \cdots \, du + \int_{\{|v - u| \leq \frac{|u|}{2}\}} \cdots \, du,
\]
and note that $|u| \geq \frac{1}{3}|v|$ in the second case. We estimate them as follows:
\[
\int_{\mathbb{R}^3} \frac{1}{|v - u|^\alpha} e^{-|u|^2} \, du \leq C_\alpha |v|^{-\alpha} + e^{-\frac{1}{3}|v|^2} \int_{\{|v - u| \leq \frac{|u|}{2}\}} \frac{1}{|v - u|^\alpha} \, du
\leq C_\alpha |v|^{-\alpha} + C_\alpha |v|^{3-\alpha} e^{-\frac{1}{3}|v|^2} \leq C_\alpha |v|^{-\alpha}.
\]
Combining this estimate with the case of small $v$, we obtain the desired result. □

### 3. Global existence

In this section, we prove global existence of classical solutions to the Boltzmann equation (2.8). We first make some remark about the weight function and then establish several pointwise estimates. The global existence of classical solutions will be proved in Section 3.3 and some discussions on the result will be given in Section 3.4.

#### 3.1. Weight function.

In this part we make some remark about the weight function that we have chosen in (2.1). Note that the energy conservation between colliding particles can be written as
\[
E^0 + u^0 = v^0 + u^0,
\]
and one may try to estimate the equation (2.8) as
\[
e^{v^0} \partial_t f = R^{-3} \int \cdots e^{-u^0} \left( e^{v^0} f(v') e^{u^0} f(u') - e^{v^0} f(v) e^{u^0} f(u) \right) \, d\omega \, du
\leq CR^{-3} \|f(t)\|^2 \int e^{-u^0} \, du.
\]
However, the integral in the last inequality gives a factor $R^3$ because $u^0$ is defined by (2.1), and then the right hand side will only be bounded by $\|f(t)\|^2$. Hence, it is not easy to apply the argument of Illner and Shinbrot [13], because integrability is not guaranteed. Instead, if we use the weight function $e^{v^2}$, then the loss term is easily estimated as
\[
e^{v^2} \partial_t f = (\text{gain term}) - R^{-3} \int \cdots e^{-u^2} \left( e^{v^2} f(v) e^{u^2} f(u) \right) \, d\omega \, du
\leq (\text{gain term}) + CR^{-3} \|f(t)\|^2 \int e^{-u^2} \, du,
\]
and we can apply the method of [13] in the case that $R$ grows fast enough such that $R^{-3}$ is integrable. On the other hand, as for the gain term, the energy conservation (3.1) does not apply, thus we need the restriction on the scattering kernel given in (2.13). Detailed calculations will be given in the following sections.

Another motivation for the weight function is that the representation of post-collisional momenta (2.23) or (2.24) converges to that of the Newtonian case as $t$ tends to infinity. From the definitions of
We first consider the case of (2.4). The quadratic quantity in (3.4) can be written as
\begin{equation}
(3.5) \quad \omega
\end{equation}
where \( \omega \) is a unit vector.

The restriction (2.13) is enough to control the gain term.

On the other hand, since \( R(0) = 1 \), we have the following:
\begin{equation}
(3.3) \quad \sqrt{1 + |v|^2} + \sqrt{1 + |u'|^2} = \sqrt{1 + |v|^2} + \sqrt{1 + |u|^2} \quad \text{at} \quad t = 0.
\end{equation}

We may now compare (3.2) and (3.3) to conjecture that the energy conservation (3.4) has some structure which becomes close to the Newtonian case rather than the Minkowski case after a large time. In this sense we choose the weight function as \( e^{\|v\|^2} \). Since (3.2) does not apply at a finite time, we need the restriction (2.13) on the scattering kernel, and the following lemma shows that the restriction (2.13) is enough to control the gain term.

**Lemma 3.1.** Let \( v \) and \( u \) be given. Suppose that \( v' \) and \( u' \) are parametrized by (2.4) or (2.5) with a unit vector \( \omega \) on \( S^2_R \). Then, we have the following estimate:
\begin{equation}
(3.4) \quad |v|^2 + |u|^2 - |v'|^2 - |u'|^2 \leq B,
\end{equation}
where \( B \) is the constant given in (2.13).

**Proof.** We first consider the case of (2.4). The quadratic quantity in (3.4) can be written as
\begin{equation}
(3.5) \quad |v|^2 + |u|^2 - |v'|^2 - |u'|^2 = 2R^2(v^0u^0 - v^0u^0),
\end{equation}
where we used the energy conservation \( v^0 + u^0 = v^0 + u^0 \). We apply (2.4) to get
\[ v^0u^0 = \frac{(v^0)^2}{4} + \frac{(u^0)^2}{4} + \frac{v^0u^0}{2} - \frac{g^2(n \cdot \omega)^2}{4R^2((n^0)^2 - R^{-2}(n \cdot \omega)^2)}, \]
and then
\[ 2R^2(v^0u^0 - v^0u^0) = 2R^2 \left( \frac{(v^0 - u^0)^2}{4} - \frac{g^2(n \cdot \omega)^2}{4R^2((n^0)^2 - R^{-2}(n \cdot \omega)^2)} \right) = \frac{1}{2} (Rv^0 - Ru^0)^2 - \frac{(Rg)^2(n \cdot \omega)^2}{2((Rn^0)^2 - (n \cdot \omega)^2)}. \]

The first quantity above is written as
\[ \frac{1}{2} (Rv^0 - Ru^0)^2 = \frac{1}{2} \left( \frac{|v|^2 - |u|^2}{Rv^0 + Ru^0} \right)^2 = \frac{1}{2} \left( \frac{(v + u) \cdot (v - u)}{R^2(n^0)^2} \right)^2 \leq \frac{|v - u|^2 |n|^2 \cos^2 \theta_0}{2R^2(n^0)^2}, \]
and the second quantity is written by
\[ \frac{(Rg)^2(n \cdot \omega)^2}{2((Rn^0)^2 - (n \cdot \omega)^2)} = \frac{|v - u|^2}{2} \left( 1 - \frac{|n|^2 \cos^2 \theta_0}{R^2(n^0)^2} \right) \left( \frac{n \cdot \omega^2}{(Rn^0)^2 - (n \cdot \omega)^2} \right), \]
where we used (2.13). Then, (3.5) is estimated as
\[ |v|^2 + |u|^2 - |v'|^2 - |u'|^2 \leq \frac{|v - u|^2}{2} \left( \frac{|n|^2 \cos^2 \theta_0}{R^2(n^0)^2} \left( 1 + \frac{(n \cdot \omega)^2}{(Rn^0)^2 - (n \cdot \omega)^2} \right) - \frac{(n \cdot \omega)^2}{(Rn^0)^2 - (n \cdot \omega)^2} \right), \]
where
\[ |v - u|^2 \left( \frac{|n|^2 \cos^2 \theta_0}{(Rn^0)^2 - (n \cdot \omega)^2} - \frac{(n \cdot \omega)^2}{(Rn^0)^2 - (n \cdot \omega)^2} \right) \leq \frac{|v - u|^2 |n \times \omega|^2}{2R^2s} \leq B, \]
where we used the assumption (2.13). In the second case, we apply (2.5) to the quadratic quantity in (3.5) to obtain
\[
2R^2(v^0 u^0 - v^0 u^0) = 2R^2 \left( \frac{(v^0 - u^0)^2}{4} - \frac{g^2(n \cdot \hat{\omega})^2}{4R^2 s} \right) \\
= \frac{1}{2}(Rv^0 - Ru^0)^2 - \frac{(Rg)^2(n \cdot \hat{\omega})^2}{2R^2 s}.
\]
The first quantity is the same as above,
\[
\frac{1}{2}(Rv^0 - Ru^0)^2 = \frac{|v - u|^2|n|^2 \cos \theta_0}{R^2(n^0)^2},
\]
while the second quantity is estimated as
\[
\frac{(Rg)^2(n \cdot \hat{\omega})^2}{2R^2 s} = \frac{|v - u|^2}{2} \frac{(n^2 \cos^2 \theta_0 - (n \cdot \hat{\omega})^2)}{R^2(n^0)^2}.
\]
Then, (3.5) is estimated as
\[
|v|^2 + |u|^2 - |v'|^2 - |u'|^2 \leq \frac{|v - u|^2}{2} \frac{(|n|^2 \cos^2 \theta_0 - (n \cdot \hat{\omega})^2)}{R^2(n^0)^2} \leq \frac{|v - u|^2}{2} \frac{|n \times \hat{\omega}|^2}{R^2(n^0)^2} \leq B.
\]
This completes the proof of the lemma. \(\square\)

3.2. Pointwise estimates. In this part, we collect several pointwise estimates which will be used in the following section for the global existence theorem. The following lemma is trivial from the assumption on the angular part of the scattering kernel.

**Lemma 3.2.** Suppose that \(\sigma_0(\omega)\) satisfies (2.13). Then, we have
\[
\int \int \sigma_0(\omega)e^{-|\omega|^2}d\omega \leq C.
\]
**Proof.** Since \(\sigma_0(\omega)\) is bounded, the lemma is clear. \(\square\)

**Lemma 3.3.** For \(0 \leq \beta < 4\), we have the following estimate:
\[
\int_{\mathbb{R}^3} v_\phi g^{-\beta}e^{-|u|^2}du \leq \begin{cases}
C & \text{for } 0 \leq \beta \leq 1, \\
CR^{\beta-1} & \text{for } 1 \leq \beta < 4,
\end{cases}
\]
where \(C\) is a positive constant depending on \(\beta\).

**Proof.** Since the lower and upper bounds for \(g\) are different (2.16), we separate the cases as follows. Note that \(R(t)\) is an increasing function.

Case 1. \((0 \leq \beta \leq 1)\) By the definition of \(v_\phi\) and (2.15), we obtain
\[
\int v_\phi g^{-\beta}e^{-|u|^2}du = \int g^{1-\beta}e^{-|v|^2}dv_\phi e^{-|u|^2}du \leq C \int \frac{|v^0 u^0|}{v^0 u^0} e^{-|u|^2}du \\
\leq C \int (v^0 u^0)^{-\frac{2}{\alpha}}e^{-|u|^2}du \leq C.
\]
Case 2. (1 \leq \beta \leq 2) We use the lower bound for g in (2.10) with Lemma 2.3 to estimate
\[ \int v_0 g^{-\beta} c^{-|u|^2} du = \int \frac{\sqrt{s}}{\sqrt{v_0 u_0}} \frac{1}{g^{\beta-1}} e^{-|u|^2} du \leq C \int \frac{1}{\sqrt{v_0 u_0}} R^{\beta-1}(v_0 u_0)^{\frac{\beta-1}{2}} e^{-|u|^2} du \]
\[ \leq CR^{\beta-1} \int \frac{1}{|v-u|^\beta (v_0 u_0)^{\frac{\beta-1}{2}}} e^{-|u|^2} du \]
\[ \leq CR^{\beta-1} \int \frac{1}{|v-u|^\beta} e^{-|u|^2} du \leq C R^{\beta-1}(1 + |v|^2)^{-\frac{\beta-1}{2}}, \]
where we used Lemma 2.3.

Case 3. (2 \leq \beta < 4) We use (2.17) and Lemma 2.3 to estimate
\[ \int v_0 g^{-\beta} c^{-|u|^2} du = \int \frac{\sqrt{s}}{\sqrt{v_0 u_0}} \frac{1}{g^{\beta-1}} e^{-|u|^2} du \leq C \int \frac{1}{\sqrt{v_0 u_0}} R^{\beta-1}(v_0 u_0)^{\frac{\beta-1}{2}} e^{-|u|^2} du \]
\[ \leq CR^{\beta-1} \int \frac{1}{|v-u|^\beta (v_0 u_0)^{\frac{\beta-1}{2}}} e^{-|u|^2} du \]
\[ \leq CR^{\beta-1} \int \frac{(1 + |v|^2)^{\frac{\beta-2}{2}} (1 + |u|^2)^{\frac{\beta-2}{2}}}{|v-u|^\beta} e^{-|u|^2} du \]
\[ \leq C R^{\beta-1}(1 + |v|^2)^{-\frac{\beta}{2}} \leq C R^{\beta-1}(1 + |v|^2)^{-\frac{\beta}{4}}. \]
We combine the above estimates to complete the proof of the lemma.

Lemma 3.4. Consider the representation for \( v' \) in (2.4). We have the following estimate:
\[ |\partial_{v^i} v^k| \leq C v_0(u_0)^4, \]
where the constant \( C \) does not depend on \( R \).

Proof. For simplicity, let \( r \) denote
\[ r = \sqrt{(n^0)^2 - R^{-2}(n \cdot \omega)^2}. \]
Then, we take \( v^i \)-derivative on \( v^k \) to have
\[ \partial_{v^i} v^k = \frac{\delta^i_k}{2} + \partial_{v^i} \left[ R g \frac{n^0 \omega^k}{r} \right] \]
\[ = \frac{\delta^i_k}{2} + \frac{R(\partial_{v^i} g) n^0 \omega^k}{2 r} + \frac{R g (\partial_{v^i} n^0 \omega^k)}{2 r} - \frac{R g n^0 \omega^k}{r^2} (\partial_{v^i} r), \]
where \( \delta^i_k \) is the Kronecker delta. We now collect (2.25), (2.19), and (2.20) to estimate
\[ \left| \frac{R(\partial_{v^i} g) n^0 \omega^k}{2 r} \right| \leq \frac{R u_0 \sqrt{v_0 u_0}}{2} \sqrt{(v_0^0 + u_0^0)} \sqrt{\frac{u_0^0}{v_0^0}} \leq C \left( v_0(v_0^0)^2 + (u_0^0)^3 \right), \]
and (2.15), (2.21), (2.19), and (2.20) to estimate
\[ \left| \frac{R g (\partial_{v^i} n^0 \omega^k)}{2 r} \right| \leq \frac{R}{2} 2 \sqrt{v_0 u_0} \frac{1}{R} \sqrt{\frac{u_0^0}{v_0^0}} \leq C u_0^0, \]
and (2.15), (2.19), (2.20), and (2.25) to estimate
\[ \left| \frac{R g n^0 \omega^k}{2 r^2} (\partial_{v^i} r) \right| \leq \frac{R}{2} 2 \sqrt{v_0 u_0} (v_0^0 + u_0^0) \left( \frac{u_0^0}{v_0^0} \right) \frac{2(v_0^0 + u_0^0)}{R} \sqrt{\frac{u_0^0}{v_0^0}} \leq C \left( v_0(v_0^0)^2 + (u_0^0)^3 + \frac{(u_0^0)^4}{v_0^0} \right). \]
Since \( v_0 \geq 1 \) and \( u_0 \geq 1 \), we combine the above estimates to obtain the desired result. \( \square \)
Lemma 3.5. Consider the representation for \( v' \) in (2.20). We have the following estimate:

\[
|\partial_{v'}v^k| \leq C \left( \frac{R_0}{|v-u|} + \frac{R_0}{|v+u|} + \frac{R^2(v^0)^2}{|v-u|^2} \right) (u^0)^3,
\]

where the constant \( C \) does not depend on \( R \).

Proof. The proof of this lemma is also a direct calculation as in Lemma 3.4. We take \( v' \)-derivative on \( v^k \) in the representation of (2.20) to get

\[
\partial_{v'}v^k = \frac{\delta^k}{2} + \partial_{v'} \left[ \frac{Rg}{2} \left( \frac{(n \cdot \omega)n^k}{|n|^2} + \frac{n^0(n \cdot \bar{\omega})n^k}{\sqrt{s}|n|^2} \right) \right]
\]

and by (2.14), (2.23), (2.15), and (2.16), we obtain

\[
|\partial_{v'} v^k| \leq \frac{R}{2} \frac{(n \cdot \omega)n^k}{|n|^2} + \frac{R}{2} \frac{\partial_{v'} (n \cdot \bar{\omega})n^k}{|n|^2} + R \frac{n^0(n \cdot \bar{\omega})n^k}{\sqrt{s}|n|^2}
\]

As in the proof of Lemma 3.4, we separate the above quantities. We first apply (2.22) and (2.16) to the second quantity above to get

\[
\left| \frac{R}{2} (\partial_{v'} g) \frac{n^0(n \cdot \bar{\omega})n^k}{|n|^2} \right| \leq \frac{R u_0^0}{|v-u|} \leq \frac{R u_0^0 \sqrt{v^0 u_0}}{|v-u|},
\]

and similarly

\[
\left| \frac{R}{2} (\partial_{v'} g) \frac{(n \cdot \omega)n^k}{|n|^2} \right| \leq \frac{R u_0^0 \sqrt{v^0 u_0}}{|v-u|}.
\]

The inequalities in (2.27) and (2.15) give

\[
\left| \frac{Rg}{2} \frac{\partial_{v'} (n \cdot \bar{\omega})n^k}{|n|^2} \right| \leq C \frac{Rg}{|n|} \leq C \frac{R \sqrt{v^0 u_0}}{|v+u|}.
\]

For the fifth quantity, we apply (2.22), (2.16), and (2.10) to get

\[
\left| \frac{R}{2} (\partial_{v'} g) \frac{n^0(n \cdot \bar{\omega})n^k}{|n|^2} \right| \leq \frac{R u_0^0 (v^0 + u_0)}{2 \sqrt{s} |n|^2} \leq C \frac{R^2 v^0 (u_0)^2 (v^0 + u_0)}{|v-u|^2}.
\]

Applying (2.15), (2.16), and (2.21), we have

\[
\left| \frac{Rg (\partial_{v'} v^0)(n \cdot \bar{\omega})n^k}{2 \sqrt{s} |n|^2} \right| \leq \frac{R v_0^0 (v^0 + u_0)}{2 \sqrt{s} |n|^2} \leq \frac{R v_0^0 u_0}{|v-u|},
\]

and by (2.14), (2.22), (2.15), and (2.10), we obtain

\[
\left| \frac{Rg n^0}{2 \sqrt{s} (\partial_{v'} \sqrt{s}) (n \cdot \bar{\omega})n^k}{|n|^2} \right| \leq \frac{C u_0^0 (v^0 + u_0)}{s} \leq \frac{C R^2 v^0 (u_0)^2 (v^0 + u_0)}{|v-u|^2}.
\]

The last quantity is estimated by (2.14) and (2.27) as

\[
\left| \frac{Rg n^0}{2 \sqrt{s} \partial_{v'} \frac{(n \cdot \bar{\omega})n^k}{|n|^2}} \right| \leq C \frac{R(v^0 + u_0)}{|v+u|}.
\]

Since \( v^0 \geq 1 \) and \( u^0 \geq 1 \), we combine the above estimates to obtain the desired result. \( \Box \)

The above estimates in Lemma 3.4 and 3.5 will be crucially used in the proof of existence of classical solutions. To estimate the collision operator, we will decompose the integration domain into three different cases. We fix a finite time \( t \) and separate the cases as follows:

- **Case 1**: \( |v| \leq R \),
- **Case 2**: \( |v| \geq R \) and \( |v| \leq 2|u| \),
- **Case 3**: \( |v| \geq R \) and \( |v| \geq 2|u| \).
In the first and second cases, the estimate of Lemma 3.4 is further estimated as follows:

(3.6) (Case 1) \[ |\partial_v v^k| \leq C \sqrt{1 + R^{-2}|v|^2(u^0)}^4 \leq C(u^0)^4, \]

(3.7) (Case 2) \[ |\partial_v v^k| \leq C \sqrt{1 + R^{-2}|v|^2(u^0)}^4 \leq C(u^0)^5. \]

In the third case, we note that \(|v \pm u| \geq \frac{1}{2}|v|\) and use Lemma 3.5 to estimate

(3.8) (Case 3) \[ |\partial_v v^k| \leq C \left( \frac{R\sigma^0}{|v|} + \frac{(R\sigma^0)^2}{|v|^2} \right) (u^0)^3 \leq C(u^0)^3. \]

We observe that the estimates (3.6)–(3.8) do not produce a growth in \(v\). The main idea is to use the representation (2.5) together with (2.4) and decompose the integration domain into the three cases as above. This argument was originally suggested by Guo and Strain in [12], and the existence of classical solutions of the relativistic Vlasov-Maxwell-Boltzmann system was proved. In the following section, we will use the estimates (3.6)–(3.8) to prove the existence of classical solutions to the Boltzmann equation in the RW spacetime.

### 3.3. Global existence

We now prove the global existence theorem.

**Lemma 3.6.** We have the following estimate for \(f\):

\[
|e^{|v|^2} f(t, v)| \leq \|f_0\| + C\|f(t)\|^2 \int_0^t R^{-3}(s) + R^{b-4}(s) \, ds,
\]

where \(C\) is a positive constant depending on \(b\).

**Proof.** Multiplying the weight function to the Boltzmann equation (2.8) as in Section 3.1, we obtain

\[
\partial_t \left[ e^{|v|^2} f(t, v) \right] = R^{-3} \int v \phi \sigma(g, \omega) e^{|v|^2 + |u|^2 - |v'|^2 - |u'|^2} \left( e^{|v|^2} f(v') e^{u|v|^2} f(u') \right) e^{-|u|^2} \, d\omega \, du
\]

\[ - R^{-3} \int v \phi \sigma(g, \omega) \left( e^{|v|^2} f(v) e^{u|v|^2} f(u) \right) e^{-|u|^2} \, d\omega \, du \]

\[ =: I_1 + I_2,
\]

and \(I_2\) is easily estimated as

\[
|I_2| \leq R^{-3} \|f(t)\|^2 \int v \phi \sigma(g, \omega) e^{-|u|^2} \, d\omega \, du
\]

\[ \leq CR^{-3} \|f(t)\|^2 \left( \int v \phi e^{-|u|^2} \, du + \int v \phi g^{-b} e^{-|u|^2} \, du \right)
\]

\[ \leq C \left( R^{-3}(t) + R^{b-4}(t) \right) \|f(t)\|^2,
\]

where we used Lemma 3.3. For \(I_1\), we use Lemma 3.1 to estimate

\[
|I_1| \leq R^{-3} \|f(t)\|^2 \int v \phi \sigma(g, \omega) e^{|v|^2 + |u|^2 - |v'|^2 - |u'|^2} e^{-|u|^2} \, d\omega \, du
\]

\[ \leq CR^{-3} \|f(t)\|^2 \int v \phi \sigma(g, \omega) e^{-|u|^2} \, d\omega \, du
\]

\[ \leq C \left( R^{-3}(t) + R^{b-4}(t) \right) \|f(t)\|^2.
\]

Then, we obtain a differential inequality from (3.9)–(3.11). Integrating it from 0 to \(t\), we have

\[
e^{|v|^2} f(t, v) \leq \|f_0\| + C\|f(t)\|^2 \int_0^t R^{-3}(s) + R^{b-4}(s) \, ds,
\]

and this completes the proof. \(\square\)
Lemma 3.7. We have the following estimate for $\partial_v f$:

$$|e^{\|v\|^2} \partial_v f(t, v)| \leq \|f_0\| + C\|f(t)\|^2 \int_0^t R^{-3}(s) + R^{b-4}(s) \, ds, \quad i = 1, 2, 3,$$

where $C$ is a positive constant depending on $b$.

Proof. In this lemma, we make use of (3.6)–(3.8). We take $v^i$-derivative on the Boltzmann equation (2.8) and multiply the weight function to have the following equation for $\partial_v f$:

$$\partial_t \left[ e^{\|v\|^2} \partial_v f \right] = R^{-3} \int \partial_v \left[ v^j \sigma(g, \omega) \right] e^{\|v\|^2} \left( f(v')f(u') - f(v)f(u) \right) \, d\omega \, du$$

$$+ R^{-3} \int v^j \sigma(g, \omega) e^{\|v\|^2} \partial_v \left[ f(v')f(u') \right] \, d\omega \, du$$

$$- R^{-3} \int v^j \sigma(g, \omega) e^{\|v\|^2} (\partial_v f)(v)f(u) \, d\omega \, du$$

$$=: J_1 + J_2 + J_3.$$

For $J_1$, we note that

$$\partial_v \left[ v^j \sigma(g, \omega) \right] = \left( \partial_v g \right) \frac{\nabla_s}{v^0 u^0} \sigma(g, \omega) + \left( \partial_v \sqrt{s} \right) g \frac{\nabla_{s \omega}}{v^0 u^0 \sigma(g, \omega)}$$

$$- \left( \partial_v v^j \right) g \frac{\sqrt{s}}{v^0 u^0} \sigma(g, \omega) + g \frac{\nabla_{s \omega}}{v^0 u^0} (\partial_v g) \partial_\theta \sigma(g, \omega).$$

Applying (2.26), (2.21), and (2.15) to the above, we estimate

$$\left| \partial_v \left[ v^j \sigma(g, \omega) \right] \right| \leq \frac{u^0 \sqrt{u^0 v^0}}{R} \frac{\nabla_s}{v^0 u^0} \sigma(g, \omega) + \frac{u^0 \sqrt{u^0 v^0}}{R} \frac{g}{v^0 u^0} \sigma(g, \omega)$$

$$+ \frac{1}{R} \frac{g}{(v^0)^2 u^0} \sigma(g, \omega) + \frac{u^0 \sqrt{u^0 v^0}}{R} \frac{\nabla_{s \omega}}{v^0 u^0} (\partial_v g) \partial_\theta \sigma(g, \omega)$$

$$\leq \frac{Cu^0}{R} \left( \sigma(g, \omega) + g \partial_\theta \sigma(g, \omega) \right).$$

By the assumption on the scattering kernel (2.12), we get

$$\left| \partial_v \left[ v^j \sigma(g, \omega) \right] \right| \leq CR^{-1} u^0 (1 + g^{-b}) \sigma_0(\omega).$$

Since $u^0 \leq \sqrt{1 + |u|^2}$, we estimate $J_1$ by the same argument as in Lemma 3.6

$$\left| J_1 \right| \leq C \left( R^{-4}(t) + R^{b-5}(t) \right) \|f(t)\|^2.$$

The estimate of $J_3$ is exactly the same with Lemma 3.6. We obtain

$$\left| J_3 \right| \leq C \left( R^{-3}(t) + R^{b-4}(t) \right) \|f(t)\|^2.$$

For $J_2$, we write

$$J_2 = R^{-3} \int v^j \sigma(g, \omega) e^{\|v\|^2} \left( \partial_v v^j (\partial_v f)(v') f(u') + f(v') (\partial_v u^j (\partial_v f))(u') \right) \, d\omega \, du,$$

and separate the cases as in (3.6)–(3.8). We fix a momentum $v$ and note that $R(t)$ is an increasing function with $R(0) = 1$. Then, we can find a finite time $t_0$ such that

$$t \geq t_0 \iff |v| \leq R(t).$$
Note that $t_0$ can be zero for small $v$. We first consider the case of $t \geq t_0$. In this case, we apply \eqref{4.6}, and the quantities $\partial_v v'$ and $\partial_v u'$ are bounded by $(u^0)^3$. This quantity can be controlled by the weight function as in \eqref{5.14}, we obtain the following estimate:

\begin{equation}
(3.15) \quad t \geq t_0 \implies |J_2| \leq C \left(R^{-3}(t) + R^{b-4}(t)\right)\|f(t)\|^2.
\end{equation}

In the case of $t \leq t_0$, we decompose the integration domain as in \eqref{3.7} and \eqref{3.8}. We write $J_2$ as

$$J_2 = R^{-3} \int_{|v| \leq |u|} \cdots d\omega du + R^{-3} \int_{|v| \geq 2|u|} \cdots d\omega du =: J_{21} + J_{22},$$

and apply \eqref{3.7} to $J_{21}$. Then, the quantities $\partial_v v'$ and $\partial_v u'$ are bounded by $(u^0)^5$. In the case of $J_{22}$, we consider the post-collisional momenta $v'$ and $u'$ in the representation of \eqref{2.5}. Applying \eqref{3.8} to $J_{22}$, we control $\partial_v v'$ and $\partial_v u'$ to be bounded by $(u^0)^3$. By the same argument as in the proof of Lemma \ref{5.6} we now obtain the following estimate:

\begin{equation}
(3.16) \quad t \leq t_0 \implies |J_2| \leq C \left(R^{-3}(t) + R^{b-4}(t)\right)\|f(t)\|^2.
\end{equation}

We combine the estimates \eqref{3.12}–\eqref{3.16} and apply them to \eqref{3.12} to obtain

$$|e^{|v|^2} \partial_v f(t, v)| \leq \|f_0\| + C\|f(t)\|^2 \int_0^t R^{-3}(s) + R^{b-4}(s) \, ds,$$

and this completes the proof. \hfill \Box

We are now ready to prove global existence of solutions to the Boltzmann equation in the RW spacetime. With Lemma \ref{3.6} and \ref{3.7} it is easy to apply the method of \cite{13}. The estimates of those lemmas give the following estimate for $f$:

$$\|f(t)\| \leq \|f_0\| + C\|f(t)\|^2 \int_0^t R^{-3}(s) + R^{b-4}(s) \, ds$$

for some positive constant $C$. In the case that the scale factor $R$ grows fast enough such that the integral above converges, we obtain the following inequality:

$$\|f(t)\| \leq \|f_0\| + C\|f(t)\|^2.$$

The above inequality shows that if initial data is sufficiently small, then global existence of solutions is guaranteed. For detailed arguments of the proof of this framework, we refer to \cite{7}. By following the proofs of \cite{8} \cite{11} \cite{13} \cite{50}, we finally obtain the following theorem.

**Theorem 3.1.** Consider the relativistic Boltzmann equation in the spatially flat Robertson-Walker spacetime in the form of \eqref{2.8}. Suppose that the scattering kernel satisfies \eqref{2.12} and \eqref{2.13}, and let the scale factor $R$ be given and satisfy \eqref{2.10} together with the following condition:

\begin{equation}
(3.17) \quad \int_0^\infty R^{-3}(t) + R^{b-4}(t) \, dt < \infty,
\end{equation}

where $b$ is the constant given in \eqref{2.12}. Let $f_0$ be an initial data such that it is differentiable and satisfies $\|f_0\| < \varepsilon$ for some positive constant $\varepsilon$. If the constant $\varepsilon$ is sufficiently small, then there exists a unique nonnegative classical solution of the Boltzmann equation \eqref{2.8} such that

\begin{equation}
(3.18) \quad \sup_{0 \leq t < \infty} \|f(t)\| \leq C\varepsilon,
\end{equation}

where $C\varepsilon$ is some positive constant depending on $\varepsilon$. 

3.4. Discussions. In Theorem 3.1 we have obtained global existence of classical solutions to the Boltzmann equation \((2.8)\). Note that the equation \((2.8)\) is written in the transformed variable \(v\), hence we write it back in the original variable \(p\), and then (3.18) can be written as
\[
(3.19) \quad f(t, v) \leq C e^{-|v|^2} \quad \text{or} \quad f(t, p) \leq C e^{-R^4(t)|p|^2}.
\]
We may compare this result with the Vlasov case. The Vlasov equation in the RW spacetime is obtained by simply ignoring the right hand side of the Boltzmann equation \((2.8)\), i.e. \(\partial_t f = 0\) in the transformed variable, and we get \(f(t, v) = f_0(v)\). It is usual in the Vlasov case to assume that initial data has a compact support such that \(f_0(v) = 0\) for \(|v| \geq C\), hence we obtain \(f(t, v) = 0\) for \(|v| \geq C\), or \(f(t, p) = 0\) for \(|p| \geq CR^{-2}(t)\).

This kind of estimates for the Vlasov equation has already been obtained in more general cases. In [14], the author studied the Einstein-Vlasov system with a positive cosmological constant in the spacetimes of all Bianchi types except IX. She showed that the distribution function \(f\) satisfies the above estimates with \(R(t) \approx e^\text{ct}\) (see Theorem 3.8 of [14]). A similar result has been obtained in [21], where the Einstein-Vlasov system was studied with zero cosmological constant. We also remark that the integrability condition \((3.17)\) does not seem to be a strong restriction. The condition \((3.17)\) is indeed satisfied by several special solutions of the Einstein equations, for instance the Einstein-de Sitter model \(R(t) = t^{2/3}\) and the de Sitter spacetime \(R(t) = e^{Ht}\) (see [22] for more details). We expect that the condition \((3.17)\) holds in more general spacetimes.

To summarize, we have studied the relativistic Boltzmann equation in a given RW spacetime. By applying the argument of Guo and Strain [12], we could prove the global existence of classical solutions together with the asymptotic behaviour \((3.19)\). In this paper, we assumed that the scale factor \(R\) is given, but the result of this paper can be easily applied to the Einstein-Boltzmann case. The Boltzmann equation is coupled to the Einstein equations through the energy-momentum tensor, and the energy-momentum tensor of the Boltzmann equation has the same form with that of the Vlasov equation. Hence, existence of solutions will be proved in a way similar to the Einstein-Vlasov cases. We also hope that this work can be applied to more general cases.

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