COMPREHENSIVE ANALYSIS OF INTEGER-ORDER, CAPUTO-FABRIZIO (CF) AND ATANGANA-BALEANU (ABC) FRACTIONAL TIME DERIVATIVE FOR MHD OLDROYD-B FLUID WITH SLIP EFFECT AND TIME DEPENDENT BOUNDARY CONDITION

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ABSTRACT. This article is focused on the slip effect in the unsteady flow of MHD Oldroyd-B fluid over a moving vertical plate with velocity $U_0 f(t)$. The Laplace transformation and inversion algorithm are used to evaluate the expression for fluid velocity and shear stress. Fractional time derivatives are used to analyze the impact of fractional parameters (memory effect) on the dynamics of the fluid. While making a comparison, it is observed that the fractional-order model is best to explain the memory effect as compared to the classical model. The behavior of slip condition as well as no-slip condition is discussed with all physical quantities. The influence of dimensionless physical parameters like magnetic force $M$, retardation time $\lambda_r$, fractional parameter $\alpha$, and relaxation time $\lambda$ on fluid velocity has been discussed through graphical illustration. Our results suggest that the velocity field decreases by increasing the value of the magnetic field. In the absence of a slip parameter, the strength of the magnetic field is maximum. Furthermore, it is noted that the Atangana-Baleanu derivative in Caputo sense (ABC) is the best to highlight the dynamics of the fluid.

1. Introduction. The theory of Newtonian and non-Newtonian fluids describes the mechanical behaviour of different real fluids. A few subdivisions of non-Newtonian fluid are plastics, greases, toothpaste, and foodstuff. There exist many mathematical models to efficiently anticipate the key features of different non-Newtonian fluids. The attributes of fluid flow trace the diversity of physical structure for non-Newtonian fluid flow. In such fluid, stress and rate of strain have a nonlinear relationship. Oldroyd-B fluids have become a significant model of rate type fluid. It describes the relaxation and retardation phenomena of viscoelastic
Table 1. Nomenclature

| Symbol | Quantity                                      |
|--------|----------------------------------------------|
| $u$    | Fluid velocity                               |
| $B_0$  | Magnitude of applied magnetic field          |
| $q$    | Laplace transforms parameter                 |
| $S$    | Extra stress tensor                          |
| $A$    | Rivlin Ericken                               |
| $L$    | Velocity gradient                            |
| $R$    | Reynold number                               |
| $\rho$ | Fluid density                                |
| $\lambda$ | Relaxation time                             |
| $\lambda_r$ | Retardation time                         |
| $\mu$  | Dynamic viscosity                            |
| $\nu$  | Kinematic viscosity                          |
| $\beta$| slip parameter                               |
| $\nabla$| gradient operator                            |
| $\tau$ | shear stress                                  |

In industry, there are many applications to the effect of the slip parameter. The roughness of the surface may generate slip condition and rarefaction of the fluid and the velocity on the surface. Many researchers investigated the application of MHD flow with the help of a magnetic force. This model converts to Maxwell fluid by $\lambda = 0$, second grade fluid by $\lambda = 0$ and $\lambda_r = \alpha_1$ and reduce to Newtonian fluid by $\lambda = \lambda_r = 0$.

During the last few years, fractional calculus has played a significant role in viscoelastic models. The derivative of fractional order can be achieved by constitutive equations of well-known models through time ordinary derivative. Recently, many fractional time derivative problems have been studied [4]-[5]. Different real-life problems have been investigated through Caputo-Fabrizio ($CF$) and Atangana-Baleanu ($ABC$) time derivative [16]-[1]. These modern approaches have been presented without a singular kernel [8]. Hepatitis-B and Hepatitis-E diseases were analyzed through the $CF$ approach. The solution and its uniqueness can be obtained through a numerical approach, which is the suitable application of these definitions in research [27]-[15]. Viscous fluid over the isothermal plate was discussed by Shah et al. [21]. Vieru et al. [29] analyzed time-fractional flow with free convection Newtonian heating near the vertical plate. Riaz et al. [2] investigated the solution of a Maxwell fluid using Caputo-Fabrizio derivative with slip condition. The solution for fractional second order with the help of the Laplace and Zakians algorithm was explored by Tassaddiq [25].
The technique of fractional calculus has been used to formulate mathematical modeling in various technological development, engineering applications, and industrial sciences. Different valuable work has been discussed for modeling fluid dynamics, signal processing, viscoelasticity, electrochemistry, and biological structure through fractional time derivatives. This fractional differential operator found useful conclusions for experts to treat cancer cells with a suitable amount of heat source and have compared the results to see the memory effect of temperature function. As compared to classical models, the memory effect is much stronger in fractional derivatives. From the past to the present, modeling of the different processes is handled through various types of fractional derivatives and fractal-fractional differential operators, such that Caputo (Power law), Atangana-Baleanu (Mittag-Leffler law), Caputo-Fabrizio (exponential law), Riemann-Liouville, modified Riemann-Liouville (Power law with boundaries) and few others [7]-[9]. A numerical approach controls the convergence and its stability. Mittag-Leffler nonsingular kernel is considered as a new fractional operator that provides a bounded solution and stabilizing point.

The main objective of this paper is to study MHD Oldroyd-B fluid with a definition of integer and non-integer order derivatives. The solutions of fluid velocity and shear stress are obtained via Integer and non-integer order derivatives with the help of Laplace transformation and numerical inversion algorithm. The study of this article proceeds as follows. The statement of the problem and helps to drive the governing partial differential equations in section 2. The solution of velocity profile and shear stress achieve through the classical model, $CF$, and $ABC$ fractional models with the help of Laplace transformation and inversion algorithm in section 3. In section 4, the influence of physical parameters discusses through graphically using MATHCAD software with the different conditions on the motion of the plate. Finally, the conclusion present in the end.

2. Statement of the problem. Let us assume an incompressible MHD Oldroyd-B fluid is suited in cartesian plane with upward positive y-direction. Initially, fluid and plate are at rest. The fluid starts the motion along x-axis. The velocity of the fluid is $u_w(t) = U_0 f(t)$, where $U_0$ is constant and $f(t)$ is piecewise continuous function defined on $(0, \infty)$ and $f(0) = 0$. The velocity is given as $v = (u(y,t), 0, 0)$. The mathematical modeling of the problem is given below [2]:

$$\frac{\partial u(y,t)}{\partial t} + \lambda \frac{\partial u^2(y,t)}{\partial t^2} = \left( v + \nu \lambda r \frac{\partial}{\partial t} \right) \frac{\partial^2 u(y,t)}{\partial y^2} - \frac{\sigma B_0^2}{\rho} \left( 1 + \lambda \frac{\partial}{\partial t} \right) u(y,t), \quad (1)$$

$$\left( 1 + \lambda \frac{\partial}{\partial t} \right) \tau(y,t) = \mu \left( 1 + \lambda r \frac{\partial}{\partial t} \right) \frac{\partial u(y,t)}{\partial y}, \quad (y,t) \in (0,1) \times (0, \infty). \quad (2)$$

The set of suitable boundary conditions under slip effect as well as initial conditions are:

$$u(0,t) - \beta \frac{\partial u(y,t)}{\partial y} |_{y=0} = U_0 f(t), \quad t \geq 0,$$

$$u(h,t) + \beta \frac{\partial u(y,t)}{\partial y} |_{h=0} = 0, \quad t > 0,$$

$$u(y,0) = \frac{\partial u(y,t)}{\partial t} |_{t=0} = \tau(y,0) = 0. \quad (3)$$
Equation (1)-(3) can be non-dimensionalized by given below dimensionless variables:

\[ y^* = \frac{y}{h}, \ t^* = \frac{t}{T}, \ u^* = \frac{u}{V_0}, \ \beta^* = \frac{\beta}{h}, \ \lambda^* = \frac{\lambda}{T}, \ \lambda_r^* = \frac{\lambda_r}{T}, \ \tau^* = \frac{\tau}{\frac{T}{f}(hV_0)}. \]  

(4)

Finally, the set of governing equations with initial and boundary condition becomes:

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial u(y,t)}{\partial t} = \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial^2 u(y,t)}{\partial y^2} - M^2 \left(1 + \lambda \frac{\partial}{\partial t}\right) u(y,t),
\]

(5)

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) \tau(y,t) = \frac{1}{R} \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial u(y,t)}{\partial y},
\]

(6)

\[
u(0, t) - \beta \frac{\partial u(y, t)}{\partial y}|_{y=0} = g(t), \ t \geq 0,
\]

(7)

\[
u(1, t) + \beta \frac{\partial u(y, t)}{\partial y}|_{y=1} = 0, \ t > 0,
\]

(8)

\[
u(y, 0) = \frac{\partial u(y, t)}{\partial t}|_{t=0} = \tau(y, 0) = 0,
\]

(9)

where \( M^2 = \frac{\sigma h^2 T}{\rho} \), Reynold number \( R = \frac{h^2}{\nu}, \ 0 < \beta < 1 \) and \( g(t^*) = f(Tt^*) \).

3. Solution of the problem.

3.1. Integer-Order derivative.

3.1.1. Velocity field and shear stress. Taking Laplace transform (LT) to Eq. (5) with given conditions (7)-(8),

\[
(q + \lambda q^2) \pi(y, q) = \frac{1}{R} \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial^2 \pi(y, q)}{\partial y^2} - M^2 \pi(y, q) - M^2 \lambda q \pi(y, q),
\]

(10)

\[
\frac{\partial^2 \pi}{\partial y^2} - a(q) \pi = 0,
\]

(11)
More simplified form of above Eq. (16) is:

\[
\ddot{u}(0, q) - \beta \frac{\partial \ddot{u}(0, q)}{\partial y} = G(q), \quad q > 0,
\]

\[
\ddot{u}(1, q) + \beta \frac{\partial \ddot{u}(1, q)}{\partial y} = 0, \quad q > 0,
\]

(12)

where \( a(q) = \frac{R(q) + M^2(1 + \lambda q)}{1 + \lambda q} \) and \( \mathcal{L}(g(t)) = G(q) \).

The solution of Eq. (11) is,

\[
\pi(y, q) = C_1 \cosh(\sqrt{a(q)} y) + C_2 \sinh(\sqrt{a(q)} y),
\]

(13)

to get the constant values \( C_1 \) and \( C_2 \) using (12),

\[
C_1 = \frac{\left( \beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + \sinh(\sqrt{a(q)} y) \right) N(q)}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)}
\]

(14)

\[
C_2 = -\frac{\left( \beta \sqrt{a(q)} \sinh(\sqrt{a(q)} y) + \cosh(\sqrt{a(q)} y) \right) G(q)}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)}
\]

(15)

Putting \( C_1 \) and \( C_2 \) into (13). We have:

\[
\pi(y, q) = \frac{\left( \beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + \sinh(\sqrt{a(q)} y) \right) G(q)}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)} \times \cosh(\sqrt{a(q)} y)
\]

\[
-\frac{\left( \beta \sqrt{a(q)} \sinh(\sqrt{a(q)} y) + \cosh(\sqrt{a(q)} y) \right) G(q)}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)} \times \sinh(\sqrt{a(q)} y).
\]

(16)

It can be written as:

\[
\pi(y, q) = A_1(q) \times G(q) - A_2(q) \times G(q),
\]

(17)

where

\[
A_1(q) = \frac{\left( \beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + \sinh(\sqrt{a(q)} y) \right) \cosh(\sqrt{a(q)} y)}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)},
\]

\[
A_2(q) = -\frac{\left( \beta \sqrt{a(q)} \sinh(\sqrt{a(q)} y) + \cosh(\sqrt{a(q)} y) \right) \sinh(\sqrt{a(q)} y)}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)}.
\]

More simplified form of above Eq. (16) is:

\[
\pi(y, q) = \frac{\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} (1 - y)) + \sinh(\sqrt{a(q)} (1 - y))}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)} \times G(q).
\]

(18)

The solution of Eq. (6) by using Eq. (16), we have

\[
\tau(y, q) + \lambda q \tau(y, q) = \frac{1}{R} \left( 1 + \lambda q \right) \frac{\partial \tau}{\partial y},
\]

(19)

\[
\tau(y, q) = \frac{1}{R} \left( 1 + \lambda q \right) \frac{\partial \tau}{\partial y}.
\]

(20)

\[
\tau(y, q) = \frac{1}{R} \left( 1 + \lambda q \right) \frac{\beta a(q) \sinh(\sqrt{a(q)} (1 - y)) + \sqrt{a(q)} \cosh(\sqrt{a(q)} (1 - y))}{2\beta \sqrt{a(q)} \cosh(\sqrt{a(q)} y) + (1 + \beta^2 a(q)) \sinh(\sqrt{a(q)} y)} \times G(q).
\]

(21)
We achieved some results from literature by taking assumptions on different parameters.

• By taking retardation time \( \lambda_r = 0 \) and magnetic field \( M = 0 \), we get same results obtained by Azhar et al (Eq. (30) in [28]). Those results help us to confirmation of our results.

• By taking magnetic field \( M = 0 \), we get identical results obtained by Shakeel et al. (Eq. (15) in [22]).

3.2. Caputo-Fabrizio fractional time derivative.

3.2.1. Velocity field and shear stress. We utilize integral transformation to acquire the solutions of fluid velocity and shear stress given by Eqs. (5)-(6) by using (7)-(9). In order to construct the Oldroyd-B model fluid, exchange the partial derivative with fractional derivative of order \( \alpha \), and Eqs.(5)-(6) can be written as:

\[
(1 + \lambda CF^{\alpha}) \frac{\partial u(y,t)}{\partial t} = \frac{1}{R} \left( 1 + \lambda CF^{\alpha} \right) \frac{\partial^2 u(y,t)}{\partial y^2} - M^2 \left( 1 + \lambda CF^{\alpha} \right) u(y,t),
\]

\[
(1 + \lambda CF^{\alpha}) \tau(y,t) = \frac{1}{R} \left( 1 + \lambda CF^{\alpha} \right) \frac{\partial u(y,t)}{\partial y},
\]

where \( CF^{\alpha} \) is known as \((CF)\) time fractional operator and it is defined as:

\[
CF^{\alpha} f(y,t) = \frac{1}{1 - \alpha} \int_0^t \exp \left( -\frac{\alpha(t-\tau)}{1 - \alpha} \right) \frac{\partial f(y,\tau)}{\partial \tau} d\tau, \quad 0 < \alpha < 1.
\]

The Laplace transform of Eq. (24) is:

\[
\mathcal{L} \left( CF^{\alpha} f(y,t) \right) = \frac{s\mathcal{L} \left( f(y,t) \right) - f(y,0)}{(1-\alpha)s + \alpha}.
\]

Apply the definition of \((CF)\) and its Laplace transform given by (24)-(25) to the Eq.(5) and Eq.(6). We get:

\[
\left( 1 + \frac{\lambda q}{(1-\alpha)q + \alpha} \right) \bar{u}_{(CF)}(y,q) = \frac{1}{R} \left( 1 + \frac{\lambda_r q}{(1-\gamma)q + \gamma} \right) \frac{\partial^2 \bar{u}_{(CF)}(y,q)}{\partial y^2} - M^2 \left( 1 + \frac{\lambda q}{(1-\alpha)q + \alpha} \right) \bar{u}_{(CF)}(y,q),
\]

\[
\left( 1 + \frac{\lambda q}{(1-\alpha)q + \alpha} \right) \bar{\tau}_{(CF)}(y,q) = \frac{1}{R} \left( 1 + \frac{\lambda_r q}{(1-\gamma)q + \gamma} \right) \frac{\partial \bar{u}_{(CF)}(y,q)}{\partial y},
\]

with appropriate transformed boundary conditions are given as:

\[
\bar{u}(0,q) - \beta \frac{\partial \bar{u}(y,q)}{\partial y} \bigg|_{y=0} = G(q), \quad q > 0,
\]

\[
\bar{u}(1,q) + \beta \frac{\partial \bar{u}(y,q)}{\partial y} \bigg|_{y=1} = 0, \quad q > 0.
\]

To get the solution of homogenous part of Eqs. (26), we get:

\[
\bar{u}_{(CF)}(y,q) = C_1 \cosh \left( \sqrt{R(a(q))} y \right) + C_2 \sinh \left( \sqrt{R(a(q))} y \right),
\]
using Eq.\((28)\) to get the constant values \(C_1\) and \(C_2\) for the velocity field. We have
\[
C_1 = \frac{\beta \left( \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)} \right) + \sinh \left( \sqrt{Ra(q)} \right) \right)}{2\beta \left( \sqrt{Ra(q)} \right) \cosh \left( \sqrt{Ra(q)} \right) + (1 + \beta^2 Ra(q)) \sinh \left( \sqrt{Ra(q)} \right)} G(q),
\]
\[
C_2 = -\frac{\beta \left( \sqrt{Ra(q)} \right) \sinh \left( \sqrt{Ra(q)} \right) + \cosh \left( \sqrt{Ra(q)} \right)}{2\beta \left( \sqrt{Ra(q)} \right) \cosh \left( \sqrt{Ra(q)} \right) + (1 + \beta^2 Ra(q)) \sinh \left( \sqrt{Ra(q)} \right)} G(q).
\]
Substitute the Eqs. \((30-31)\) into Eq. \((29)\). Finally,
\[
\bar{u}_{(CF)}(y, q) = \frac{\beta \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)} \right) + \sinh \left( \sqrt{Ra(q)} \right) \cosh \left( \sqrt{Ra(q)} \right) y}{2\beta \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)} \right) + (1 + \beta^2 Ra(q)) \sinh \left( \sqrt{Ra(q)} \right)} \times G(q)
\]
\[
-\frac{\beta \sqrt{Ra(q)} \sinh \left( \sqrt{Ra(q)} \right) + \cosh \left( \sqrt{Ra(q)} \right) \cosh \left( \sqrt{Ra(q)} \right) y}{2\beta \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)} \right) + (1 + \beta^2 Ra(q)) \sinh \left( \sqrt{Ra(q)} \right)} \times G(q).
\]
The comprehensive form of Eq. \((32)\) is given below:
\[
\bar{u}_{(CF)}(y, q) = \frac{\beta \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)}(1 - y) \right) + \sinh \left( \sqrt{Ra(q)}(1 - y) \right)}{2\beta \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)} \right) + (1 + \beta^2 Ra(q)) \sinh \left( \sqrt{Ra(q)} \right)} G(q).
\]
The solution of Eq. \((27)\) using \((28)\), we have
\[
\bar{\tau}_{(CF)}(y, q) = \frac{1}{R} \left( \frac{((1 - \gamma + \lambda_r)q + \gamma) \times ((1 - \alpha)q + \alpha)}{((1 - \alpha + \lambda)q + \alpha) \times ((1 - \gamma)q + \gamma)} \right) \frac{\partial \bar{u}_{(CF)}(y, q)}{\partial y}.
\]
Substitute the value of \((33)\) into \((34)\). We get:
\[
\bar{\tau}_{(CF)}(y, q) = -\frac{1}{R} \left( \frac{((1 - \gamma + \lambda_r)q + \gamma) \times ((1 - \alpha)q + \alpha)}{((1 - \alpha + \lambda)q + \alpha) \times ((1 - \gamma)q + \gamma)} \right) \times \frac{\beta Ra(q) \sinh \left( \sqrt{Ra(q)}(1 - y) \right) + \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)}(1 - y) \right)}{2\beta \sqrt{Ra(q)} \cosh \left( \sqrt{Ra(q)} \right) + (1 + \beta^2 Ra(q)) \sinh \left( \sqrt{Ra(q)} \right)} \times G(q),
\]
where
\[
a(q) = (q + M^2) \times \left( \frac{((1 - \gamma + \lambda_r)q + \gamma) \times ((1 - \alpha)q + \alpha)}{((1 - \alpha + \lambda)q + \alpha) \times ((1 - \gamma)q + \gamma)} \right).
\]
### 3.3. Atangana-Baleanu fractional time derivative.

#### 3.3.1. Velocity field and shear stress.
We have to calculate the velocity field and shear stress by the definition of Atangana-Baleanu fractional derivative with Caputo sense \((ABC)\) and their Laplace transform are defined respectively:
\[
^{ABC}D_t^\alpha f(y, t) = \frac{1}{1 - \alpha} \int_0^t E_\alpha \left( -\frac{\alpha(t - \tau)^\alpha}{1 - \alpha} \right) \frac{\partial f(y, \tau)}{\partial \tau} d\tau,
\]
\[
\mathcal{L} \left( ^{ABC}D_t^\alpha f(y, t) \right) = \frac{s^\alpha \mathcal{L} (f(y, t)) - s^{\alpha-1} f(y, 0)}{(1 - \alpha)s^\alpha + \alpha}.
\]
Apply Eqs. (36)-(37) to the Eqs. (5) and (6), we get:
\[
\left(1 + \frac{\lambda q^\alpha}{(1 - \alpha) q^\alpha + \alpha}\right) \tilde{u}_{(ABC)}(y, q) = \frac{1}{R} \left(1 + \frac{\lambda \gamma q^\gamma}{(1 - \gamma) q^\gamma + \gamma}\right) \frac{\partial^2 \tilde{u}_{(ABC)}(y, q)}{\partial y^2}
- M^2 \left(1 + \frac{\lambda q^\alpha}{(1 - \alpha) q^\alpha + \alpha}\right) \tilde{u}_{ABC}(y, q),
\]
(38)
\[
\left(1 + \frac{\lambda q^\alpha}{(1 - \alpha) q^\alpha + \alpha}\right) \tilde{\tau}_{(ABC)}(y, q) = \frac{1}{R} \left(1 + \frac{\lambda \gamma q^\gamma}{(1 - \gamma) q^\gamma + \gamma}\right) \frac{\partial \tilde{u}_{(ABC)}(y, q)}{\partial y}.
\]
(39)
The solution of the Eq. (38) using Eq. (28) is given below:
\[
u_{(ABC)}(y, q) = C_1 \cosh \left(\sqrt{Ra_1(q)}\right) y + C_2 \sinh \left(\sqrt{Ra_1(q)}\right) y,
\]
(40)
where
\[
C_1 = \frac{\beta \sqrt{Ra_1(q)} \cosh \sqrt{Ra_1(q)} + \sinh \sqrt{Ra_1(q)}}{2\beta \sqrt{Ra_1(q)} \cosh \sqrt{Ra_1(q)} + (1 + \beta^2 Ra_1(q)) \sinh \sqrt{Ra_1(q)}} \times G(q),
\]
(41)
and
\[
C_2 = -\frac{\beta \sqrt{Ra_1(q)} \sinh \sqrt{Ra_1(q)} + \cosh \sqrt{Ra_1(q)}}{2\beta \sqrt{Ra_1(q)} \cosh \sqrt{Ra_1(q)} + (1 + \beta^2 Ra_1(q)) \sinh \sqrt{Ra_1(q)}} \times G(q).
\]
(42)
Putting the constant $C_1$ and $C_2$ into Eq. (40). The final form of (40) after simplification, we have:
\[
\tilde{u}_{(ABC)}(y, q) = \frac{\beta \sqrt{Ra_1(q)} \cosh \left(\sqrt{Ra_1(q)}(1 - y)\right) + \sinh \left(\sqrt{Ra_1(q)}(1 - y)\right)}{2\beta \sqrt{Ra_1(q)} \cosh \sqrt{Ra_1(q)} + (1 + \beta^2 Ra_1(q)) \sinh \sqrt{Ra_1(q)}} \times G(q).
\]
(43)
To determine the solution of shear stress given by Eq. (39) using (36)-(37). We have:
\[
\tilde{\tau}_{(ABC)}(y, q) = \frac{1}{R} \left(\frac{(1 - \gamma + \lambda \gamma) q^\gamma + \gamma}{(1 - \alpha + \lambda) q^\alpha + \alpha} \times (1 - \gamma) q^\gamma + \gamma\right) \frac{\partial \tilde{u}_{(ABC)}(y, q)}{\partial y}.
\]
(44)
By substitute the value of (43) into (44). We get the final form of shear stress.
\[
\tilde{\tau}_{(ABC)}(y, q) = -\frac{1}{R} \times \left(\frac{(1 - \gamma + \lambda \gamma) q^\gamma + \gamma}{(1 - \alpha + \lambda) q^\alpha + \alpha} \times (1 - \gamma) q^\gamma + \gamma\right) \times \frac{\beta Ra_1(q) \sinh \left(\sqrt{Ra_1(q)}(1 - y)\right) + \sqrt{Ra_1(q)} \cosh \left(\sqrt{Ra_1(q)}(1 - y)\right)}{2\beta \sqrt{Ra_1(q)} \cosh \sqrt{Ra_1(q)} + (1 + \beta^2 Ra_1(q)) \sinh \sqrt{Ra_1(q)}} \times G(q),
\]
(45)
where
\[a_1 = (q + M^2) \left(\frac{(1 - \gamma + \lambda \gamma) q^\gamma + \gamma}{(1 - \alpha + \lambda) q^\alpha + \alpha} \times (1 - \gamma) q^\gamma + \gamma\right)\].

As $\alpha \to 1$, the non-integer fractional models are reduce into classical model. Further, if we neglect magnetic field ($M = 0$) and ($\lambda_r = 0$) in Eq. (5), the results are identical which obtained by Riaz et al (Eqs.(16) and (18) in [2]). Moreover, if ($\lambda_r = 0$) and ($M = 0$) the resultant result becomes identity to Riaz et al. (Eqs. (22) and (29) in [20]).
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In our fluid models, we use classical computational technique to solve the fluid models using definitions of fractional derivatives. There are many numerical algorithms which are used to calculate their inverses like Stehfest’s and Tzou’s algorithms for semi-analytical solutions [24]-[26]. Recently, Riaz et al. [11] and madeeha et al. [17] analyzed the numerical Laplace method to show the accuracy of inversion algorithms by solving fractional differential equation in a effective and reliable way. Tzou’s calculation for our numerical inverse Laplace

\[ v(r, t) = e^{4.7t} \left[ \frac{1}{2} \bar{v}(r, \frac{4.7t}{t}) + \text{Re} \left\{ \sum_{k=1}^{N_1} (-1)^k \bar{v}(r, \frac{4.7t+k\pi i}{t}) \right\} \right], \]

where \( \text{Re}(.) \) is the real part, \( i \) is the imaginary unit and \( N_1 \) is a natural number [26].

4. Results and discussion. The channel flow of MHD Oldroyd-B fluid on a flat plate has been analyzed in this article with consideration of the slip effect. The solution of the dimensionless governing equation has been obtained by Laplace transform with semi-analytical techniques. The integer order derivative and non-integer order derivative based on singular, nonsingular, and nonlocal kernels. The graphical representation of MHD Oldroyd-B fluid with slip condition using integer order, Caputo-Fabrizio (\( CF \)) and Atangana-Baleanu (\( ABC \)) are studied. In the end, we studied the influence of all physical parameters on the fluid flow velocity with the relation of slip condition. Here we consider three different cases of motion on the plate, e.g., \( g(t) = \sin(t) \), \( g(t) = H(t) \), and \( g(t) = t \). These three cases are discussed through a graphical presentation using MATHCAD software.

Figures (2) – (10) are plotted to see the influence of \( M \) on the profile of fluid velocity versus \( t \) is analyzed through classical model, (\( CF \)) and (\( ABC \)) with three cases of motion on flat plate, i.e. \( g(t) = \sin(t) \), \( g(t) = H(t) \) and \( g(t) = t \) respectively. It is noted that the velocity decreases in all cases for the classical model, (\( ABC \)), and (\( CF \)) models by enlarging the value of \( M \) due to Lorenz forces. It is a type of resistive force that helps to decrease the velocity by increase the magnetic force. The maximum strength of \( M \) considers at \( \beta = 0 \). Furthermore, when \( \beta \) increase, the resultant velocity decrease. The velocity for the nonslip condition is larger than the slip condition by enhancing the value of \( M \). The role of slip parameter (\( \beta \)) on the magnetic field makes a special impact on the velocity profile.

Figures (11) – (19) are plotted to discuss the influence of relaxation time (\( \lambda \)) on the flow of fluid velocity, by increasing the value of \( \lambda \), the resultant velocity decrease. Furthermore, it is also compared with a small and large value of \( \beta \). The velocity reduces by increasing the value of the slip effect. When \( \beta = 0 \), the velocity is maximum as compared to other values of \( \beta \).

Figures (20) – (28) are depicted to show the behavior of retardation time. The influence of \( \lambda_r \) is opposite to \( \lambda \) on the velocity field. To increase the value of \( \lambda_r \), the resultant velocity increase with a small and immense value of \( \beta \) the decrease in \( \beta \) as a result of fluid increases. The behavior of \( \lambda_r \) and \( \alpha \) are the same in fluid velocity.

The reaction of fractional parameter (\( \alpha \)) on fluid velocity discussed in figure (29 – 34). It is perceived that with an increase of \( \alpha \), the velocity increases in (\( ABC \)) and (\( CF \)) models. The graphical relations between \( \alpha \) and \( \beta \) on fluid velocity are opposite to each other.

In general, we conclude that the velocity without slip effect is much considerable as compared to the slip effect because the velocity reduces by enhancing the value...
of the slip parameter. The fractional model is useful to explain the memory effect and flow behavior of the fluid with reference to the classical model. Furthermore, the ABC model is good for the dynamics of the fluid. Atangana-Balean fractional derivative is excellent in exhibiting the memory effect in fluid flow problems. The kernel of ABC is a Mittag-Leffler function, which is a superset of the kernel of CF. Mittag-Leffler (nonsingular & nonlocal) kernel is considered as a new fractional operator that provides a bounded solution and stabilizing point. Mittag-Leffler function recovers full memory effect in comparison to the exponential decaying function.

Figure 2. Velocity profile of Classical model with variation of $M$ and $\beta$ for $g(t) = \sin(t)$.

5. Conclusions. The analysis of MHD Oldroyd-B fluid has been discussed using integer-order and non-integer order models for three different types of motion on the fluid. The inversion algorithm and Laplace transform are used to find the velocity profile and shear stress. The graphical results are plotted for fluid velocity with the effect of the slip parameter. The influence of different parameters on velocity has been analyzed for the classical model, (CF), and (ABC). Some significant remarks for this problem are:
1. The velocity field increases for increasing the value of $\alpha$.
2. The velocity parameter decreases by increasing the value of $\beta$.
3. The relation between $\alpha$ and $\beta$ is opposite.
4. Velocity decrease by increase in relaxation time $\lambda$.
5. The velocity decrease by magnifying the value of the magnetic field.
6. The behaviors of velocity profile is opposite for $\lambda$ and $\lambda_r$.
7. The velocity increases for small values of slip parameter.
8. ABC model is suitable as compared to CF for different physical parameters.
Figure 3. Atangana-Baleanu velocity profile for $g(t) = \sin(t)$ with variation of $M$ and $\beta$.

Figure 4. Caputo-Fabrizio velocity profile for $g(t) = \sin(t)$ with variation of $M$ and $\beta$.

Acknowledgments. The authors are highly thankful and grateful to their respective departments and Universities for supporting the research work.
Figure 5. Velocity profile of Classical model with variation of $M$ and $\beta$ for $g(t) = H(t)$.

Figure 6. Atangana-Baleanu velocity profile for $g(t) = H(t)$ with variation of $M$ and $\beta$. 
Figure 7. Caputo-Fabrizio velocity profile for $g(t) = H(t)$ with variation of $M$ and $\beta$.

Figure 8. Velocity profile of Classical model with variation of $M$ and $\beta$ for $g(t) = t$. 
Figure 9. Atangana-Baleanu velocity profile for $g(t) = t$ with variation of $M$ and $\beta$.

Figure 10. Caputo-Fabrizio velocity profile for $g(t) = t$ with variation of $M$ and $\beta$. 
Figure 11. Velocity profile of Classical model for $\lambda$ variation with $g(t) = \sin(t)$.

Figure 12. Velocity profile of Atangana-Baleanu for $\lambda$ variation with $g(t) = \sin(t)$. 
Figure 13. Velocity profile of Caputo-Fabrizio for $\lambda$ with $g(t) = \sin(t)$.

Figure 14. Velocity profile of Classical model for $\lambda$ variation with $g(t) = H(t)$.
Figure 15. Velocity profile of Atangana-Baleanu for $\lambda$ variation with $g(t) = H(t)$.

Figure 16. Velocity profile of Caputo-Fabrizio for $\lambda$ with $g(t) = H(t)$. 
Figure 17. Velocity profile of Classical model for $\lambda$ variation with $g(t) = t$.

Figure 18. Velocity profile of Atangana-Baleanu for $\lambda$ variation with $g(t) = t$. 
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Figure 19. Velocity profile of Caputo-Fabrizio for $\lambda$ with $g(t) = t$.

Figure 20. Integer order velocity profile for $\lambda_r$ variation with $g(t) = \sin(t)$. 
Figure 21. Atangana-Baleanu velocity profile for $\lambda_r$ variation with $g(t) = \sin(t)$.

Figure 22. Caputo-Fabrizio velocity profile for $\lambda_r$ variation with $g(t) = \sin(t)$.
Figure 23. Integer order velocity profile for $\lambda_r$ variation with $g(t) = H(t)$.

Figure 24. Atangana-Baleanu velocity profile for $\lambda_r$ variation with $g(t) = H(t)$.
Figure 25. Caputo-Fabrizio velocity profile for $\lambda_r$ variation with $g(t) = H(t)$.

Figure 26. Integer order velocity profile for $\lambda_r$ variation with $g(t) = t$. 
Figure 27. Atangana-Baleanu velocity profile for $\lambda_r$ variation with $g(t) = t$.

Figure 28. Caputo-Fabrizio velocity profile for $\lambda_r$ variation with $g(t) = t$. 
Figure 29. Atangana-Baleanu velocity profile with $\alpha$ variation for $g(t) = \sin(t)$.

Figure 30. Caputo-Fabrizio velocity profile with $\alpha$ variation for $g(t) = \sin(t)$. 
Figure 31. Atangana-Baleanu velocity profile with $\alpha$ variation for $g(t) = H(t)$.

Figure 32. Caputo-Fabrizio velocity profile with $\alpha$ variation for $g(t) = H(t)$.
Figure 33. Atangana-Baleanu velocity profile with $\alpha$ variation for $g(t) = t$.

Figure 34. Caputo-Fabrizio velocity profile with $\alpha$ variation for $g(t) = t$. 
REFERENCES

[1] T. Abdeljawad, Different type kernel h-fractional differences and their fractional h-sums, *Chaos, Solitons and Fractals*, **116** (2018), 146–156.

[2] A. Asif, Z. Hammouch and M. B. Riaz, Analytical solution of a Maxwell fluid with slip effects in view of the Caputo-Fabrizio derivative, *Eur. Phys. J. Plus*, **133** (2018), 272.

[3] A. Atangana, On the new fractional derivative and application to nonlinear fishers reaction-diffusion equation, *Appl. Math. Comput.*, **273** (2016), 948–956.

[4] D. Avc, M. Yavuz and N. Zdemir, Fundamental Solutions to the Cauchy and Dirichlet Problems for a Heat Conduction Equation Equipped with the Caputo-Fabrizio Differentiation. *Heat Conduction: Methods, Applications and Research*, Nova Science Publishers, (eds: Jordan Hristov, Rachid Bennacer) Book Chapter, ISBN: 978-1-53614-673-8 2019, 95–107.

[5] D. Avc, M. Yavuz and N. Zdemir, Fractional Optimal Control of a Diffusive Transport Acting on a Spherical Region, *Methods of Mathematical Modelling: Fractional Differential Equations*, Taylor and Francis Group Publishing, (eds: Harendra Singh, Devendra Kumar, Dumitru Baleanu) Book Chapter, ISBN: 978-0-367-22008-2 (2019), 63–82.

[6] D. Baleanu, A. Jajarmi, S. S. Sajjadi and D. Mozyrska, A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **29** (2019), 083127, 15 pp.

[7] M. A. Dokuyucu, E. Celik, H. Bulut and H. M. Baskonus, Cancer treatment model with the caputo-fabrizio fractional derivative, *Eur. Phys. J. Plus*, **133** (2018), 93.

[8] C. Fetecau, D. Vieru, C. Fetecau and S. Akhter, General solutions for magnetohydrodynamic natural convection flow with radiative heat transfer and slip condition over a moving plate, *Z Naturforsch. A*, **68** (2013), 659–667.

[9] M. Hajipour, A. Jajarmi and D. Baleanu, On the accurate discretization of a highly nonlinear boundary value problem, *Numerical Algoritham*, **79** (2018), 679–695.

[10] T. Hayat, A. M. Siddiqui and S. Asghar, Some simple flows of an Oldroyd-B fluid, *International Journal of Engineering Science*, **39** (2001), 135–147.

[11] M. A. Imran, M. B. Riaz, N. A. Shah and A. A. Zafar, Boundary layer flow of MHD generalized Maxwell fluid over an exponentially accelerated infinite vertical surface with slip and Newtonian heating at the boundary, *Results in physics*, **8** (2018), 1061–1067.

[12] A. Jajarmi, D. Baleanu, S. S. Sajjadi and J. H. Asad, A new feature of the fractional Euler-Lagrange equations for a coupled oscillator using a nonsingular operator approach, *Frontiers in Physics*, **196** (2019), 7.

[13] A. Jajarmi, S. Arshad and D. Baleanu, A new fractional modelling and control strategy for the outbreak of dengue fever, *Physica A: Statistical Mechanics and its Applications*, **535** (2019), 122524.

[14] S. Jamshad, T. Gul, S. Islam, M. A. Khan, R. A. Shah, S. Nasir and H. Rasheed, Flow of unsteady second grade fluid between two vertical plates when one of the plate oscillating and other is stationary, *J. Appl. Environ. Biol. Sci.*, **4** (2014), 41–52.

[15] M. A. Khan, Z. Hammouch and D. Baleanu, Modeling the dynamics of hepatitius-E via the Caputo-Fabrizio derivative, *Math. Model. Nat. Phenom.*, **14** (2019), 311.

[16] Z. Li, Z. Liu and M. A. Khan, Fractional investigation of bank data with fractal-fractional Caputo derivative, *Chaos, Solitons and Fractals*, (2020), 109528.

[17] T. Madeeha, M. A. Imran, N. Raza, M. Abdullah and A. Maryam, Wall slip and noninteger order derivative effects on the heat transfer flow of Maxwell fluid over an oscillating vertical plate with new definition of fractional Caputo-Fabrizio derivatives, *Results in physics*, **7** (2017), 1887–1898.

[18] M. Navier, Memoire sur les lois du mouvement des fluides, *Mem. LAcad. Sci. LInst. France*, **6** (1823), 389–440.

[19] K. R. Rajagopal and R. K. Bhatnagar, Exact solutions for some simple flows of an Oldroyd-B fluid, *Acta Mechanica*, **113** (1995), 233–239.

[20] M. B. Riaz, N. A. Asif, A. Atangana and M. I. Asjad, Couette flows of a viscous fluid with slip effects and non-integer order derivative without singular kernal, *Discrete and Continuous Dynamical System*, **12** (2019), 645–664.

[21] N. A. Shah, M. A. Imran and F. Miraj, Exact solutions of time fractional free convection flows of viscous fluid over an isothermal vertical plate with caputo and caputo-fabrizio derivatives, *J. Prime Res. Math.*, **13** (2017), 56–74.
[22] A. Shakeel, S. Ahmad, H. Khan, N. A. Shah and S. U. Haq, Flows with slip of Oldroyd-B fluids over a moving plate, *Advances in Mathematical Physics*, (2016), Art. ID 8619634, 9 pp.

[23] A. M. Siddiqui, T. Haroon, M. Zahid and A. Shahzad, Effect of slip condition on unsteady flows of an Oldroyd-B fluid between parallel plates, *World Applied Sciences Journal*, 13 (2011), 2282–2287.

[24] H. Stehfest Algorithm, Numerical inversion of Laplace transforms, *Commun. ACM*, 13 (1970), 9–47.

[25] A. Tassaddiq, MHD flow of a fractional second grade fluid over an inclined heated plate, *Chaos, Solitons and Fractals*, 123 (2019), 341–346.

[26] DY. Tzou, Macro to Microscale Heat Transfer: The Lagging Behaviour, *Washington: Taylor and Francis*, (1970).

[27] S. Ullah, M. A. Khan and M. Farooq, A new fractional model for the dynamics of the hepatitis-B virus using the Caputo-Fabrizio, *Eur. Phys. J. Plus.*, 133 (2018), 237.

[28] D. Vieru and A. A. Zafar, Some Couette flows of a Maxwell fluid with wall slip condition, *Appl. Math. Inf. Sci.*, 7 (2013), 209–219.

[29] D. Vieru, C. Fetecau and C. Fetecau, Time-fractional free convection flow near a vertical plate with newtonian heating and mass diffusion, *Therm. Sci.*, 19 (2015), 85–98.

[30] H. Zaman, Z. Ahmad and M. Ayub, A note on the unsteady incompressible MHD fluid flow with slip conditions and porous walls, *ISRN Mathematical Physics*, 1 (2013), 1–10.

[31] Z. Zhang, C. Fu and W. Tan, Linear and nonlinear stability analyses of thermal convection for Oldroyd-B fluids in porous media heated from below, *Phys. Fluids*, 20 (2008), 84–103.

Received November 2019; 1st revision December 2019; 2nd revision January 2020.

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