The rational parts of one-loop QCD amplitudes I: The general formalism

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\textbf{Abstract}

A general formalism for computing only the rational parts of one-loop QCD amplitudes is developed. Starting from the Feynman integral representation of the one-loop amplitude, we use tensor reduction and recursive relations to compute the rational parts directly. Explicit formulas for the rational parts are given for all bubble and triangle integrals. Formulas are also given for box integrals up to two-mass-hard boxes which are the needed ingredients to compute up to 6-gluon QCD amplitudes. We use this method to compute explicitly the rational parts of the 5- and 6-gluon QCD amplitudes in two accompanying papers.

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1 Introduction

The forthcoming experimental program at CERN’s Large Hadron Collider (LHC) requires many computations at the next-to-leading order (NLO) or one-loop with many particles as final states [1]. However the analytic computation of the matrix elements is very difficult. Only for special helicity configurations [2, 3, 4] or special models [5], some analytic results are known for higher point amplitudes. The current state of the art in NLO computation is 5-point for QCD processes and 6-point for electroweak processes [6, 7, 8]. The recent development in tackling the multi-leg amplitudes by semi-numerical/analytic methods shows promise for improving traditional capabilities [9, 10, 11, 12, 13, 14, 15, 16]. All helicity configurations for the 6-gluon amplitude are evaluated for a single-space point [9]. These results are used to check the available analytic results [17].

Following Witten’s twistor string theory [18], the CSW approach [19] and the use of maximally-helicity-violating (MHV)1 vertices [21, 22], there has been spectacular progress [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] in the perturbative QCD computations in the last two years or so, by using the unitarity cut method of Bern, Dunbar, Dixon and Kosower [3, 48, 50] and the spinor-helicity formalism [51, 52] (see [53] for a review). In particular, Bedford, Brandhuber, Spence and Travaglini [28, 31] applied the MHV vertices to one-loop calculations. Britto, Buchbinder, Cachazo, Feng and Mastrolia [38, 39, 40] developed an efficient technique for evaluating the rational coefficients in an expansion of the one-loop amplitude in terms of scalar box, triangle and bubble integrals (the cut-constructible part, see below and Sect. 4). By using their technique, it is much easier to calculate the coefficients of box integrals without doing any integration. Recently, Britto, Feng and Mastrolia completed the computation of the cut-constructible terms for all the 6-gluon helicity amplitudes [40].

In order to complete the QCD calculation for the 6-gluon amplitude, the remaining challenge is to compute the rational part of the amplitude with scalars circulating in the loop, commonly called the \( N = 0 \) case in a supersymmetric decomposition of QCD amplitudes:

\[
A^{QCD} = A^{N=4} - 4A^{N=1 \text{ chiral}} + A^{N=0 \text{ or scalar}}. \tag{1}
\]

\(^1\)The 2 dimensional origin of the MHV amplitudes in gauge theory was first given in [20]
The above strategy of splitting the computation of the QCD amplitude into various supersymmetric parts plus a scalar part is quite fruitful. By using a theorem of Bern, Dunbar, Dixon and Kosower [50], the supersymmetric parts are cut-constructible, meaning that these amplitudes can be determined completely by using 4-dimensional unitarity. Even for the scalar part which is not cut-constructible, we can still split it into two parts: a cut-constructible part and a rational part. As we said before, the recent development inspired by twistor string theory has lead to very efficient techniques to compute the cut-constructible part [38, 39, 40]. To complete the program it is quite important to have efficient and powerful methods to compute the rational part.

There are various ways to compute the rational part. The first approach [48] is to use the factorization properties by trial and error. This is a quite effective method if the final result is simple enough. The difficulty with this approach is that we do not know how to automate the method to make effective use of the advances in computer industry. The correctness of the obtained result is almost guaranteed by checking factorizations for all channels. For higher point amplitudes, the complexity of the analytic results makes this method impractical.

The second approach uses the unitarity relation. In principle the rational part can be constructed by using the $D$-dimensional unitarity method [49, 50, 54, 42]. The problem with this approach is that too much information is kept and tree amplitudes in $D$-dimension are even more difficult [55]. In fact this approach loses the simplicity of 4-dimensional helicity amplitudes as given by the MHV formula.

The third approach is the bootstrap recursive approach of Bern, Dixon and Kosower [34]. This approach is quite promising and powerful. It is a streamlined approach of the first one by adding the insights of the recent tree-level recursive method of Britto, Cachazo, Feng and Witten [32]. This approach has already produced a wealth of general results for special helicity configurations, notably the “split-helicity” configurations [34, 46, 17]. It can also be used to compute one-loop QCD amplitudes with general helicities as outlined in [17].

Given the complexity of the results for the cut-constructible part of the 6-gluon amplitude [40] and its important applications to LHC related experiments, it is quite worthy and even mandatory to have other methods to compute the rational part of the QCD amplitude. In particular one would
like to bypass the need of using the cut-constructible part and have an independent method to compute the rational part. Of course, the testing ground for any method is a complete computation of the 6-gluon QCD amplitude where only partial results for some helicity configurations exist. The present status for the marching to one loop 6-gluon QCD amplitude were summarized in [55, 56]. For more recent developments, we refer the reader to [57, 58].

In this paper we will study the problem of computing the rational parts of one-loop amplitudes directly from Feynman integral representations. These integrals can be written down directly by drawing all Feynman diagrams and by using the Feynman rules. With the present technology these can be done quite effectively by using the various packages like GRACE [62], FeynArts [63] and Qgraf [64] et. al. (See [65, 62] for reviews.) Fortunately these powerful methods are not needed to compute up to the 6-gluon amplitude. For computing higher point amplitudes or 6-parton amplitudes they may be a necessity.

It is easy to imagine that the rational part is already contained in the integral representation of the amplitude. If one could obtain the complete rational coefficients by doing tensor reduction to scalar box, triangle and bubble integrals, one can simply get the rational part by making an expansion with the dimension $D$ around 4 ($D = 4 - 2\epsilon$ is the parameter of dimensional regularization). This is extraordinarily difficult because of the complexity of tensor reduction for $N \geq 5$. However if one only needs to compute the rational part, it is not necessary to know the complete coefficients from tensor reductions. By the BDDK theorem [3], we know that many terms simply do not contribute to the rational part. Following this path of thought, the remaining problem is: is there an efficient way to compute these rational parts in one-loop QCD amplitudes directly from Feynman integrals?

In this paper we show that there is actually a quite efficient and powerful method to compute the rational part directly from Feynman integrals. Because we are concerned only with the rational part of the amplitude, there is no need for tensor reduction all the way down to scalar integrals. We only need tensor reduction to reduce the degree of the numerator by 2. So the original complexity of tensor reduction is bypassed in the computation of the rational part.

In our approach of computing the rational part, we will exploit the theorem of Bern, Dunbar, Dixon and Kosower [3] and directly extract the rational part from the one-loop Feynman integrals. We also use the simple tensor re-
duction by using spinors as developed in [48, 66, 67]. We point out that
the tensor reduction formulas used in our computations are actually quite
simple, as one can see from eqs. (14) and (16) in Sect. 3.

As we will demonstrate in this paper, the computation of the rational
part is reduced to tree-level like calculations. As our method also applies to
massive theory and theories with fermions, we envisage wider applications of
our method in the computation of one-loop amplitudes, in combination with
the $D = 4$ unitarity method [3, 48] and the efficient technique for computing
generic unitarity cuts [38]. The once most difficult part of the one-loop
amplitude can actually be attacked by the traditional technique.

In this paper and the accompanying two papers [68, 69], we will develop
our method and apply it to the computation of the rational parts of the
one-loop 5- and 6-gluon QCD amplitudes. This paper mainly deals with the
general theoretical formalism of the method. In [68], we show the efficiency
of the method by computing the rational parts for the 5-gluon amplitudes
for the two MHV helicity configurations by giving most of the intermediate
steps. In [69], we will present the results for the rational parts of the 6-gluon
amplitudes for the two MHV and two NMHV helicity configurations. The
rational parts of the 6-gluon amplitudes for the "split helicity" configurations
are known already [34, 17]. Recently all one-loop maximally helicity violat-
ing gluonic amplitudes were computed by Berger, Bern, Dixon, Forde and
Kosower [70]. We refer the reader to [70, 69] for details about the explicit
analytic results and comparisons.

This paper is organized as follows: in Sect. 2 we set up our notation for
spinor products and composite currents for sewing trees to the loop. Some
simple tensor reduction formulas are given in Sect. 3. Starting from Sect. 4,
we begin to develop the method of extracting the rational parts of Feynman
integrals. We use the recursive approach to compute any Feynman integral
as developed in [60]. In Sect. 6 and 7 we give explicit results for triangle
and box integrals. In Sect. 8 we compute the correction terms to the naive
$D = 4$ tensor reduction of box and triangle integrals, which arise from the
ultra-violet divergent part of the box and triangle amplitudes.
2 Notation

We mainly follow the notation of BDK [34] and the QCD-literature convention for the square bracket \([i j]\). By abusing of notation the product between 2 holomorphic spinors or 2 anti-holomorphic spinors is formed by a round bracket:

\[
(\lambda_i, \lambda_j) = \langle i j \rangle, \quad (\bar{\lambda}_i, \bar{\lambda}_j) = [i j].
\]

(2)

The scalar product between 2 vectors (written either in 4d vector notation or in 2 spinor notation) is also denoted by a round bracket. We have

\[
(\lambda_i, \bar{\lambda}_j \cdot k_j) = (\lambda_i, \bar{\lambda}_j) = \langle i j \rangle [j i].
\]

(3)

For spinor strings, we simply use \(\langle \lambda_i | (k_a + k_b) | \lambda_j \rangle\) or \(\langle i | (k_a + k_b) | j \rangle\) to denote \(\langle i^- | (a + b) | j^- \rangle\):

\[
\langle \lambda_i | (k_a + k_b) | \lambda_j \rangle = \langle i | (k_a + k_b) | j \rangle = \langle i | (a + b) | j \rangle = \langle i^- | (a + b) | j^- \rangle = \langle i a | a j \rangle + \langle i b | b j \rangle.
\]

(5)

We do not use gamma matrix traces. Instead we use bra and ket notation with multiple insertions of momenta:

\[
\langle i | k_1 k_2 \cdots k_n | j \rangle = \begin{cases} 
\langle i 1 \rangle [1 2] \cdots [n j], & n = \text{odd}, \\
\langle i 1 \rangle [1 2] \cdots \langle n j \rangle, & n = \text{even}.
\end{cases}
\]

(6)

Sometimes we also write \(\langle i | K | j \rangle\) for \(\langle i | K | j \rangle\), with the understanding that sometimes the last \(j\) should actually stand for \(\bar{\lambda}_j\) and with the bracket \]. For example we have

\[
\langle i | k_1 k_2 k_3 | j \rangle = \langle i 1 \rangle [1 2] \langle 2 3 \rangle [3 j].
\]

(7)

For \(i = j\) the above notation is just the gamma matrix trace. For simplicity we will not write the slash:

\[
\langle i | k_1 k_2 k_3 | i \rangle = \text{tr}_- (k_i k_1 k_2 k_3).
\]

(8)

We note that the above notation only happens for an odd number of momenta inserted between 2 spinors (one holomorphic and one anti-holomorphic). Of
course the momentum can be either massless or a sum of several massless momenta.

The sums of cyclicly consecutive external momenta are denoted generically by $K_i$ in a Feynman diagram. In our explicit computation we use $k_{12} = k_1 + k_2$ and $k_{234} = k_2 + k_3 + k_4$, etc. The kinematic variables are denoted as $s_{12} = (k_1 + k_2)^2$ and $s_{123} = (k_1 + k_2 + k_3)^2$ in a self explaining notation. For 6-gluon case we also have $s_{123} = (k_4 + k_5 + k_6)^2$ by momentum conservation.

![Figure 1: The composition of 3 external particles in tree amplitudes.](image1)

![Figure 2: The composition of 4 external particles in tree amplitudes. The blob denotes an expansion as given in Fig. 1. The explicit expression of $\epsilon_{i(i+1)(i+2)(i+3)}$ will not be given.](image2)

For sewing trees to the loop, we define the following composite currents or polarization vectors:

$$
\epsilon_{i(i+1)} = P(\epsilon_i, k_i; \epsilon_{i+1}, k_{i+1}) \equiv \frac{1}{(k_i + k_{i+1})^2} ((\epsilon_i, k_{i+1}) \epsilon_{i+1} - (\epsilon_{i+1}, k_i) \epsilon_i + \frac{1}{2} (\epsilon_i, \epsilon_{i+1}) (k_i - k_{i+1})) \tag{9}
$$
\[ \epsilon_{i(i+1)(i+2)} = P(\epsilon_{i(i+1)}, k_{i(i+1)}; \epsilon_{i+2}, k_{i+2}) + P(\epsilon_{i+1}, k_{i+1}; \epsilon_{i+2}, k_{i+1}) \]
\[
+ \frac{1}{s_{i(i+1)(i+2)}} \left( (\epsilon_{i}, \epsilon_{i+2}) \epsilon_{i+1} - \frac{1}{2} (\epsilon_{i}, \epsilon_{i+1}) \epsilon_{i+2} - \frac{1}{2} (\epsilon_{i+1}, \epsilon_{i+2}) \epsilon_{i} \right) ,
\]

where \( s_{i(i+1)(i+2)} = (k_{i} + k_{i+1} + k_{i+2})^2 \). The above procedure is a simplified version of the general recursive calculation of the tree-level \( n \)-gluon amplitudes [71]. We note that \( \epsilon_{i(i+1)} \) is anti-symmetric and \( \epsilon_{i(i+1)(i+2)} \) is symmetric under the reversing of the order of the particles. This generalizes to composite currents with more legs, which we have not written down explicitly. The diagrammatic representations of \( \epsilon_{i(i+1)(i+2)} \) and \( \epsilon_{i(i+1)(i+2)(i+3)} \) are given in Figs. 1 and 2.

In the formalism of [52] (see [53, 22] for reviews), the gluon polarization vectors are defined as
\[
\epsilon_\mu^{(+)}(k; q) = \frac{\langle q^-|\gamma_\mu|k^- \rangle}{\sqrt{2} \langle q^-|k^- \rangle}, \quad \epsilon_\mu^{(-)}(k; q) = \frac{\langle q^+|\gamma_\mu|k^+ \rangle}{\sqrt{2} \langle k^+|q^- \rangle} ,
\]

where \( k \) is the momentum of the polarization null vector and \( q \) is the reference null vector. In terms of the holomorphic and anti-holomorphic spinors ( \( k = \lambda \tilde{\lambda} \) and \( q = \eta \tilde{\eta} \)), these polarization vectors can be recast as
\[
\epsilon_{\alpha \beta}^{(+)}(k; q) = \frac{\sqrt{2} \eta_{\alpha} \lambda_{\beta}}{\langle \eta \lambda \rangle}, \quad \epsilon_{\alpha \beta}^{(-)}(k; q) = \frac{\sqrt{2} \lambda_{\alpha} \tilde{\eta}_{\beta}}{[\lambda \eta]}.
\]

In our notation, we will use \( \epsilon \) to denote the polarization vectors, and the relations between \( \epsilon \) and \( \epsilon \) is \( \epsilon = \sqrt{2} \epsilon \). The troublesome \( \sqrt{2} \) will be absorbed in the overall coefficient of the amplitude. Given this notation, the bracket product of polarizations is more natural than the dot product:
\[
(\epsilon_j^\dagger(k_j; q_j), \epsilon_i^\dagger(k_i; q_i)) = \epsilon_j^\dagger(k_j; q_j) \cdot \epsilon_i^\dagger(k_i; q_i) = \frac{\langle j | l \rangle [q_j | q_i]}{[j | q_j] [l | q_i]}.
\]

In our calculations, it is also convenient to omit the denominators in the definition of the polarizations, and reinstate them in the last step.

### 3 Tensor reduction of the one-loop amplitudes

There is a vast literature on this subject. The original Passrino-Veltman approach [59] is quite general but it is not quite practical to obtain compact
analytic results. In fact the tensor reduction relations we will use for our calculations of the 5- and 6-gluon amplitudes are quite simple. It is based on the BDK trick [48] of multiplying and dividing by spinor square roots. To make more effective use of this trick, we have purposely chosen the reference momenta to make the tensor reduction simple. See [69] for further details about the specific choices of the reference momenta and the tensor reductions involved in the computation of (the rational parts of) the 6-gluon amplitudes.

Figure 3: The tensor reduction for two adjacent same helicities generally gives three terms.

There are basically only two different cases to consider. As shown in Fig. 3, the two polarization vectors have the same helicity. If we choose the same reference momentum (denoted by the spinor $\eta$), we have

\[
(\eta \tilde{\lambda}_1, p) (\eta \tilde{\lambda}_2, p) = -\frac{(\eta \tilde{\lambda}_{k_{12}}^{(0)}, p + k_1)}{\langle \frac{1}{2} \rangle I^{(2)}}
+ \frac{\langle 1 \rangle (\eta \tilde{\lambda}_1, p)}{\langle \frac{1}{2} \rangle} I^{(3)} + \frac{\langle 2 \rangle (\eta \tilde{\lambda}_2, p)}{\langle \frac{1}{2} \rangle} I^{(1)},
\]

(14)

where $I^{(1)} = (p + k_1)^2$, $I^{(2)} = p^2$ and $I^{(3)} = (p - k_2)^2$ are various inverse propagators. The above tensor reduction formula is shown diagrammatically in Fig. 3, omitting the relevant factors.

In deriving eq. (14), we assumed that $p$ is a four-dimensional vector. Because pentagon and higher point one-loop amplitudes are ultra-violet convergent, the use of the above formula in tensor reduction is correct in dimensional regularization up to infinitesimal terms. However one must be quite careful to apply the above formula to the tensor reduction of the box and triangle tensor integrals because the difference is a finite rational part. Some correction terms must be included for tensor reduction with box and triangle tensor integrals. This also applies to the following tensor reduction formulas
given later in this section. We will compute these correction terms later in Sect. 8.

\[
\begin{align*}
&+ \frac{1}{2} \quad + \frac{1}{2}
\end{align*}
\]

Figure 4: For two adjacent same helicities, the tensor reduction for the combination of two diagrams is even simpler by a judicious choice of the reference momenta.

An even simpler version of the above tensor reduction relation is to consider a combination of two diagrams together as shown in Fig. 4. The reduction formula is:

\[
\frac{(\epsilon_1, p + k_1)(\epsilon_2, p)}{(p + k_1)^2 p^2 (p - k_2)^2} + \frac{(\epsilon_{12}, p + k_1) - (\epsilon_1, \epsilon_2)/2}{(p + k_1)^2 (p - k_2)^2}
\]

\[
= -\frac{1}{p^2} + \frac{1/2}{(p + k_1)^2} + \frac{1/2}{(p - k_2)^2}.
\]

(16)

for \( \epsilon_1 = \lambda_1 \tilde{\lambda}_2 \) and \( \epsilon_2 = \lambda_2 \tilde{\lambda}_1 \). All the factors appearing on the left-hand side of the above equations are read off directly from Feynman rules. These tensor reduction relations show quite clearly the simplicity of the diagrams when there are adjacent particles with the same helicity.

For three adjacent particles with the same helicity gluons, we choose the following polarization vectors (omitting an overall factor for each polarization vector):

\[
\begin{align*}
\epsilon_4 &= \lambda_5 \tilde{\lambda}_4, & \epsilon_5 &= \eta \tilde{\lambda}_5, & \epsilon_6 &= \lambda_5 \tilde{\lambda}_6.
\end{align*}
\]

(17)

Then we have

\[
\begin{align*}
\epsilon_{45} &= -\eta \tilde{\lambda}_4 + \frac{1}{2} \frac{\langle \eta 5 \rangle}{\langle 4 5 \rangle} k_{45}, & \epsilon_{56} &= \eta \tilde{\lambda}_6 - \frac{1}{2} \frac{\langle \eta 5 \rangle}{\langle 6 5 \rangle} k_{56}.
\end{align*}
\]

(18)

(19)
Figure 5: For three adjacent same helicities, the tensor reduction for the combination of these four diagrams is also quite simple if we choose the reference momenta appropriately.

\[
\begin{align*}
A_{456} &= (\epsilon_4, p - k_{45}) (\epsilon_5, p - k_4) (\epsilon_6, p - k_4) \\
&+ \left( (\epsilon_4, p - k_{45}) - \frac{1}{2} (\epsilon_4, \epsilon_5) \right) (\epsilon_6, p - k_4) I^{(5)} \\
&+ (\epsilon_4, p - k_4) (\epsilon_{56}, p - k_4) - \frac{1}{2} (\epsilon_5, \epsilon_6) \right) I^{(6)} \\
&= ((\lambda_5 \tilde{\lambda}_4, p - k_4) (\eta \tilde{\lambda}_6, p - k_4) - (\eta \tilde{\lambda}_4, p - k_4) (\lambda_5 \tilde{\lambda}_6, p - k_4)) I^{(6)} \\
&+ \frac{1}{2} \langle \eta 5 \rangle \left( \lambda_5 \tilde{\lambda}_6, p - k_4 \right) (I^{(4)} - I^{(6)}) I^{(5)} \\
&- \frac{1}{2} \langle \eta 5 \rangle \left( \lambda_5 \tilde{\lambda}_4, p - k_4 \right) (I^{(5)} - I^{(*)}) I^{(6)},
\end{align*}
\]

by doing tensor reduction with \( k_{45} \) for the first term. This can be further simplified by expressing \( \eta \) in terms of a linear combination of \( \lambda_{4,5} \). The final

\[
\begin{align*}
(\epsilon_{45}, p - k_{45}) - \frac{1}{2} (\epsilon_4, \epsilon_5) &= - (\eta \tilde{\lambda}_4, p - k_{45}) + \frac{1}{2} \langle \eta 5 \rangle (I^{(4)} - I^{(6)}) , \quad (20) \\
(\epsilon_{56}, p - k_4) - \frac{1}{2} (\epsilon_5, \epsilon_6) &= (\eta \tilde{\lambda}_6, p - k_4) - \frac{1}{2} \langle \eta 5 \rangle (I^{(5)} - I^{(*)}) , \quad (21)
\end{align*}
\]

where \( I^{(4)} = p^2, I^{(5)} = (p - k_4)^2, I^{(6)} = (p - k_{45})^2 \) and \( I^{(*)} = (p - k_{456})^2 \) are inverse propagators.

Now considering the three terms coming from the first 3 Feynman diagrams shown in Fig. 5, we have:

\[
A_{45} = (\epsilon_4, p - k_4) (\epsilon_5, p - k_4) (\epsilon_6, p - k_4) \\
+ ((\epsilon_{45}, p - k_{45}) - \frac{1}{2} (\epsilon_4, \epsilon_5)) (\epsilon_6, p - k_4) I^{(5)} \\
+ (\epsilon_4, p - k_4) (\epsilon_{56}, p - k_4) - \frac{1}{2} (\epsilon_5, \epsilon_6) \right) I^{(6)} \\
= ((\lambda_5 \tilde{\lambda}_4, p - k_4) (\eta \tilde{\lambda}_6, p - k_4) - (\eta \tilde{\lambda}_4, p - k_4) (\lambda_5 \tilde{\lambda}_6, p - k_4)) I^{(6)} \\
+ \frac{1}{2} \langle \eta 5 \rangle \left( \lambda_5 \tilde{\lambda}_6, p - k_4 \right) (I^{(4)} - I^{(6)}) I^{(5)} \\
- \frac{1}{2} \langle \eta 5 \rangle \left( \lambda_5 \tilde{\lambda}_4, p - k_4 \right) (I^{(5)} - I^{(*)}) I^{(6)},
\]
result is:

\[ A_{456} = \frac{1}{2} \langle \eta | 5 \rangle \left( \lambda_5 \tilde{\lambda}_6, p - k_4 \right) I^{(4)} I^{(5)} + \frac{1}{2} \langle \eta | 5 \rangle \left( \lambda_5 \tilde{\lambda}_4, p - k_4 \right) I^{(*)} I^{(6)} + I^{(5)} I^{(6)} \langle \eta | 5 \rangle \left[ [6, 4] - \frac{1}{2} \lambda_5 \left( \frac{\tilde{\lambda}_4}{6} \right) + \frac{\tilde{\lambda}_6}{6} \right] \]

which has a nice symmetric property under the flipping operation \( 4 \leftrightarrow 6 \). The last term in eq. (23) actually cancels the contribution from the last Feynman diagram in Fig. 5.

For the case of different neighboring helicities, we can use the following reduction formulas:

\[ (\lambda_1 \tilde{\eta}, p) (\lambda_1 \tilde{\lambda}, p) = \frac{\langle \lambda_1 | p k_2 K k_1 | p \tilde{\eta} \rangle}{\langle 2 | K | 1 \rangle} \]

\[ = \frac{1}{\langle 2 | K_4 | 1 \rangle} \left( I^{(1)} (\lambda_1 \tilde{\lambda}, p) \langle 2 | (K_4 + k_1) | \tilde{\eta} \rangle + I^{(2)} \langle 1 | p - k_2 (K_4 k_1 - k_2 K_4) | \tilde{\eta} \rangle - I^{(3)} \langle 1 | p K_4 | 1 \rangle | \tilde{\eta} \rangle - I^{(4)} (\lambda_1 k_2, p) \langle 2 1 | [1 \tilde{\eta}] + (K_4 + k_1)^2 (\lambda_1 \tilde{\lambda}, p) \langle 2 1 | [1 \tilde{\eta}] \rangle \right) \]

\[ (\lambda_1 \tilde{\lambda}, p) (\eta \tilde{\lambda}, p) = \frac{\langle \eta | p k_2 K k_1 | \tilde{\lambda} \rangle}{\langle 2 | K | 1 \rangle} \]

\[ = \frac{1}{\langle 2 | K_3 | 1 \rangle} \left( I^{(1)} \langle \eta | k_2 K_3 p | 2 \rangle + I^{(2)} \langle \eta (K_3 k_1 - k_2 K_3) (p + k_1) | 2 \rangle - I^{(3)} \langle \eta | (k_2 + K_3) | 1 \rangle (\lambda_1 \tilde{\lambda}, p) + I^{(*)} (\lambda_1 \tilde{\lambda}, p) \langle \eta 2 | 2 1 \rangle - (k_2 + K_3)^2 (\lambda_1 \tilde{\lambda}, p) \langle \eta 2 | 2 1 \rangle \right) \]

where \( I \)'s are the various inverse propagators:

\[ I^{(1)} = (p + k_1)^2, \quad I^{(2)} = p^2, \quad I^{(3)} = (p - k_2)^2, \quad I^{(*)} = (p - k_2 - K_3). \]

In eqs. (24) and (25), the momentum \( K \) can be chosen as one of the nearby momenta to avoid the spurious pole associated with a composite momen-
tum. For two-mass-hard box, \( K \) can only be chosen as one of the composite momenta.

Because of the complexity of the above tensor reduction formula for different neighboring helicities, it is better not to use them directly. Luckily we are able to avoid using them directly for tensor reduction with 5- and 6-point diagrams by a judicious choice of reference momenta for all the polarization vectors. The details will be given in [69]. For two-mass-hard box integrals, we must use the above general tensor reduction in order to obtain comparatively compact analytic expressions. We will use a slightly different reduction formula in Sect. 7.4 to compute the rational part of the two-mass-hard box integral.

The above is just the first step for the tensor reduction. Of course this procedure can be applied recursively. By a rough examination of this recursive method, one immediately finds two problems: 1) the association of the resulting polarization vector with an external (composite) momentum may not satisfy the physical conditions; 2) the above formula is no-longer applicable if one has a massive external momentum (a composite one arising from the reduction or a pinched line by sewing the tree to the loop). There are other methods to do further tensor reduction, some involving Gram determinants. Exactly because of these problems, tensor reduction is usually the most difficult part and the bottleneck for directly computing the one loop amplitude. Quite elaborate methods are developed to tackle these problems. See, for example, [8, 61].

Because of the complexity of doing further tensor reduction, we immediately see the problem why directly computing the amplitude by using Feynman rules is an extraordinarily difficult task for higher point amplitudes. There are mainly two difficulties to overcome: too many diagrams and the complexity for tensor reduction (especially at a later stage of tensor reduction as mentioned in the above). By using computers it is not too difficult to manage the numerous Feynman diagrams (of the order 1000). But the complexity of tensor reduction with the appearance of spurious poles is actually the bottleneck for analytic computation. Tensor reduction is also the bottleneck for doing numerical calculations. See, for example, [62].

In contrast we also see that why it is possible to compute the rational part by using the conventional Feynman integrals. First by using the supersymmetric decomposition, the number of Feynman diagrams is about 50 (1 hexagon, 6 pentagons, 15 boxes, 20 triangles and 15 bubbles) by only com-
puting the scalar loop contributions. Second by computing only the rational part, it is not necessary to do tensor reduction all the way down to scalar integrals. One needs only to do tensor reduction to reduce the degree by 2 (see next section). So tensor reduction does not complicate the analytic expressions significantly. In fact for special helicity assignments, there is an almost mutual cancellation between higher point diagrams and lower point diagrams, as we demonstrated in eqs. (14) and (16). This is a manifestation of gauge invariance. This property can be used to check and to organize the results of our calculation. It is very important to have some “local” cancellations before adding all the results together in order to obtain relatively compact analytic results for the rational parts of QCD amplitudes.

4 The BDDK theorem and the structure of one-loop amplitudes

\[
\begin{array}{c}
\begin{array}{c}
\text{\textbf{K}_1} \\
\text{\textbf{K}_2} \\
\cdots \\
\text{\textbf{K}_m} \\
p-K_1 \\
p \\
p+K_m \\
K_1 \\
K_m
\end{array}
\end{array}
\]

Figure 6: A generic one-loop diagram with external momenta $K_1, \cdots, K_m$. $p$ is the internal momentum between the external lines $K_1$ and $K_m$.

A generic $m$-point one-loop Feynman diagram shown in Fig. 6 and its integral is given as follows (by using Feynman rules):

\[
I^D_m[f(p)] = \int \frac{d^D p}{i \pi^{D/2}} \frac{f(p)}{p^2(p-K_1)^2 \cdots (p+K_m)^2},
\]

where $f(p)$ is a polynomial function of the internal momentum $p$. For phenomenologically interesting models and by choosing a suitable gauge, the
degree of \( f(p) \) is always not greater than \( m \). \( f(p) \) also depends on the external momenta \( k \) (\( K_i \)'s are sums of cyclicly consecutive external momenta \( k \)'s) and the polarization vectors \( \epsilon_i \). For \( f(p) = 1 \) it is called the scalar integral. The strategy of computing \( I^D_m[f(p)] \) is to reduce it recursively into lower degree polynomials and/or lower point integrals. It is a well-known result that \( I^D_m[f] \) can be generically written as:

\[
I^D_m[f] = \sum_i c_{4,i}(\epsilon, k; D) I^{D(i)}_4[1] + \sum_i c_{3,i}(\epsilon, k; D) I^{D(i)}_3[1] + \sum_i c_{2,i}(\epsilon, k; D) I^{D(i)}_2[1],
\]

(29)

up to infinitesimal terms arising from tensor reduction from pentagon or higher point diagrams. Here \( c_{j,i} \)'s are rational functions of the external momenta when all polarization vectors are written in terms of spinor products. We note that these coefficients also depend on the (arbitrary) space-time dimension \( D \) in dimensional regularization (in the FDH scheme \([72]\)).

A brute-force computation of these coefficients from Feynman integrals is an impossible task for 6 or higher point amplitudes. The 5-point case was computed by string-inspired method by using a table for all Feynman parameter integrals (see below) \([7]\). However the string-inspired method is still not powerful enough to compute even the 6-gluon amplitude due to the complexity of the Feynman integrals and the intermediate expressions.

For physical application what we actually need is an expansion in \( \epsilon \) of the above formula for \( D = 4 - 2\epsilon \), up to finite terms. If we forget the infrared divergence for the moment, there are only simple pole (\( 1/\epsilon \)) terms in the scalar integrals \( I^D_{4,3,2}[1] \) with rational coefficients. So we can write the \( n \)-gluon amplitude (addition of all contributing \( I^D_m[f(p)] \) from all Feynman diagrams) as follows:

\[
A_n = \sum_i c_{4,i}(\epsilon, k; 4) I^{D(i)}_4[1] + \sum_i c_{3,i}(\epsilon, k; 4) I^{D(i)}_3[1] + \sum_i c_{2,i}(\epsilon, k; 4) I^{D(i)}_2[1] + \text{(rational function)} + O(\epsilon).
\]

(30)

For supersymmetric theories, Bern, Dixon, Dunbar and Kosower \([3]\) proved a theorem which states that the rational function is exactly zero. This is due to the better ultra-violet behaviour of one-loop amplitudes in supersymmetric theories. So we need only to compute the rational coefficients (called the
cut-constructible part hereafter) exactly at $D = 4$. What they proved is actually a more general theorem: if $f(p)$ is a polynomial (in $p$) of degree $m - 2$ or less, the rational part for $I_m^{D}[f(p)]$ arising by expanding in $\epsilon$ is exactly zero. Generally speaking, the rational part is non-vanishing for degree $m$ and $m - 1$ polynomials.

For a non-supersymmetric theory like QCD we also need to compute the rational function (called the rational part hereafter). In a series of papers [3, 48, 50], Bern, Dunbar, Dixon and Kosower have developed a method of computing the cut-constructible part, i.e., the coefficients $c_{j,i}(\epsilon, k; 4)$, from 4-dimensional unitarity. The nicety of 4-dimensional unitarity is that all the ingredients in the unitarity relation are on-shell quantities. However 4-dimensional unitarity loses all information about the rational function part in eq. (30). One must use some other methods to compute the rational part. In [48], they use the factorization properties with trial and error. The difficulty of computing the rational function prevents the wider application of the unitarity method to calculate more general amplitudes. As we mentioned in the introduction, the rational function part could be computed by going to $D$-dimensional unitarity [50, 42].

In our papers, the precise definition of the rational part $R_n$ of the QCD amplitude is as follows. The $n$-point one loop color-ordered partial amplitude $A_{n;1}^{[0]}$ defined in eq. (2.10) in the second paper of [34] can be decomposed as

$$A_{n;1}^{[0]} = \frac{(4\pi)^{\epsilon}}{16\pi^2} \left( \sum_i c_{4,i}(\epsilon, k; 4) I_4^{D(i)}[1] + \sum_i c_{3,i}(\epsilon, k; 4) I_3^{D(i)}[1] + \sum_i c_{2,i}(\epsilon, k; 4) I_2^{D(i)}[1] + 2R_n + O(\epsilon) \right),$$

(31)

where $R_n$ is the rational part for one real scalar circulating in the loop. We will use notations like $R(+-+-++)$ to denote $R_n$ in different cases in [68] and [69].

In the following sections we will exploit the BKK theorem to compute the rational part directly from the Feynman integrals. By using the recursive relations satisfied by the tensor integrals we will derive the recursive relations for the rational parts by making an expansion in $\epsilon$. Our integration method of computing the rational part may also be used to compute the rational part by using $D$-dimensional unitarity. We use then recursive relations to derive explicit formulas for the rational parts of all bubble and triangle integrals.
in Sect. 6. In Sect. 7, we derive the formulas for box integrals up to two-mass-hard boxes. Formulas for 3-mass and 4-mass box integrals can also be derived. They are much more complicated than the two-mass-box formulas. Fortunately they are not needed in the computation of 6-gluon amplitudes and will not be given here. We note that the recursive relations for tensor integrals and the rational parts can also be derived for massive internal loop and/or external fermion lines. For simplicity all formulas are given only for the cases with vanishing internal masses.

5 The recursive relations of one-loop amplitudes

In this section we study the recursive relations of one-loop amplitudes [60, 61, 8]. By using Feynman parametrization we have

\[
I_n^D \equiv \int \frac{d^D p}{i \pi^{D/2}} \frac{1}{p^2 (p-k_1)^2 \cdots (p+k_n)^2} = (-1)^n \Gamma(n-D/2) \int d^n a \frac{\delta(1-\sum a_i)}{(a \cdot S \cdot a)^{n-D/2}}, \quad (32)
\]

where

\[
a \cdot S \cdot a = \sum_{i,j=1}^n a_i a_j S_{ij}
\]

and the matrix \(S\),

\[
S = -\frac{1}{2} \begin{pmatrix} 0 & k_1^2 & (k_1+k_2)^2 & \cdots & (k_1+k_2+\cdots+k_{n-1})^2 \\ * & 0 & k_2^2 & \cdots & (k_2+k_3+\cdots+k_{n-1})^2 \\ : & : & : & \vdots & : \\ * & * & * & 0 & k_n^2 \\ * & * & * & * & 0 \end{pmatrix}, \quad (33)
\]

is an \(n \times n\) symmetric matrix of external kinematic variables (extension to massive loop is straightforward). For tensor integral \(I_n^D[f(p)]\) it is given by the Feynman parameter integral with an extra polynomial of \(a\) in the
numerator:

\[ \hat{I}_n^D [g(a)] = \Gamma(n - D/2) \int d^n a \frac{\delta(1 - \sum_i a_i)}{(a \cdot S \cdot a)^{n-D/2}}. \]  

(34)

The degree of \( g(a) \) is the same as the degree of \( f(p) \) in \( p \).

As explained in [60], a set of recursive relations for these tensor integrals can be derived by performing the following integration:

\[
F = \Gamma(\alpha) \int_0^1 da_m \int_0^{1-a_m} \cdots \int_0^{1-a_1-a_2-\cdots-a_m-\cdots-a_{n-1}} da_m \cdots \\
\times \frac{\partial}{\partial a_m} \left[ \frac{f(a)}{(a \cdot S \cdot a)^\alpha} \right]_{a_n=1-a_1-\cdots-a_{n-1}},
\]

(35)

in two ways: one by partial integration and one by direct differentiation. Then we obtain the following recursive relation:

\[
-2\Gamma(\alpha + 1) \int_0^1 d^n a \delta(1 - a) \frac{f(a)((S \cdot a)_m - (S \cdot a)_n)}{(a \cdot S \cdot a)^{\alpha+1}}
\]

\[= \Gamma(\alpha) \int_0^1 d^n a \delta(1 - a) \frac{f(a)}{(a \cdot S \cdot a)^\alpha} \bigg|_{a_n=0}
\]

\[-\Gamma(\alpha) \int_0^1 d^n a \delta(1 - a) \frac{f(a)}{(a \cdot S \cdot a)^\alpha} \bigg|_{a_m=0}
\]

\[= \frac{\partial_m f(a) - \partial_n f(a)}{(a \cdot S \cdot a)^\alpha}.
\]

(36)

By carefully examining the composition of \( a \cdot S \cdot a \) for \( a_m = 0 \) one recognizes that this corresponds to a pinching limit of the original Feynman diagram: \( k_{m-1}, \ k_m \rightarrow k_{m-1} + k_m \). By setting \( \alpha = n - 1 - \frac{D}{2} \) and using the definition for \( \hat{I}_n^D [f] \), the above recursive relation translates into the following form:

\[
\hat{I}_n^D \left[ -2f(a)((S \cdot a)_m - (S \cdot a)_n) \right]
\]

\[= \hat{I}_n^{D(n)} [f(a)] - \hat{I}_n^{D(m)} [f(a)] - \hat{I}_n^{D+2}[\partial_m f(a)] + \hat{I}_n^{D+2}[\partial_n f(a)].
\]

(37)

\(^2\text{We note that the difference between Feynman parameter integral } \hat{I}_n[g(a)] \text{ and } I_n[f(p)] \text{ is only the } (-1)^n \text{ factor. For even } n \text{ and } f(p) = g(a) = 1, \text{ they are the same and we will not distinguish them in this case.}\)
Here $\hat{I}_{n-1}^{D(i)}[f(a)]$ denotes the $(n-1)$-point one-loop Feynman integral obtained by pinching $k_{i-1}$ and $k_i$. Coupled with one more equation from the delta function:

$$\sum_{i=1}^{n} \hat{I}_{n}^{D}[f(a) a_i] = \hat{I}_{N}^{D}[f(a)], \quad (38)$$

we can solve $\hat{I}_{n}^{D}[f(a) a_i]$ in terms of $\hat{I}_{n-1}^{D(m)}[f(a)]$, $\hat{I}_{n}^{D+2}[\partial_m f(a)]$ and $\hat{I}_{n}^{D}[f(a)]:$

$$\hat{I}_{n}^{D}[f(a) a_i] = c^{(n)}_{ij} \hat{I}_{n-1}^{D(j)}[f(a)] + d^{(n)}_{ij} \hat{I}_{n}^{D+2}[\partial_j f(a)] + e^{(n)}_{ij} \hat{I}_{n}^{D}[f(a)]. \quad (39)$$

The above reasoning goes through so long as we can invert the relevant matrix. This matrix is

$$\begin{bmatrix} -2(S_{ij} - S_{nj}) & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix} \quad (40)$$

and it is singular for $n \geq 6$. For our purpose of computing the rational part we will only need these recursive relations for $n \leq 4$. Higher point tensor integrals are reduced directly in $D = 4$ as we have done in the last section. For further discussions about the tensor reduction and its close relation with the above recursive relation, we refer the reader to [61, 8].

The above recursive relation is not symmetric for all $a_i$. The symmetric recursive relation can be obtained by firstly solving $\hat{I}_{n}^{D}[f(a) (S \cdot a)_n]$ and substituting it back to the system of equations. By taking $f = gl(a) a_m$ ($gl(a)$ is a homogeneous polynomial of degree $l$ in $a$), we can solve $\hat{I}_{n}^{D}[gl(a) (S \cdot a)_n]$ to get:

$$2\hat{I}_{n}^{D}[gl(a) (S \cdot a)_n] = \hat{I}_{n-1}^{D(n)}[gl(a)] + (n - 1 - l - D) \hat{I}_{n}^{D+2}[gl(a)] + \hat{I}_{n}^{D+2}[\partial_n gl(a)]. \quad (41)$$

Substituting this back into eq. (37) and multiplying both sides of the equation by $S_{im}^{-1}$ and then summing over $m$, we have [61]:

$$\hat{I}_{n}^{D}[gl(a) a_i] = \frac{1}{2} (n - 1 - l - D) \gamma_i \hat{I}_{n}^{D+2}[gl(a)]$$

$$+ \frac{1}{2} \sum_{j} S_{ij}^{-1} \hat{I}_{n-1}^{D(j)}[gl(a)] + \frac{1}{2} \sum_{j} S_{ij}^{-1} \hat{I}_{n}^{D+2}[\partial_j gl(a)], \quad (42)$$

where

$$\gamma_i = \sum_{j} S_{ij}^{-1}. \quad (43)$$
One can check that the above recursive relation is equivalent to the following recursive relation:

\[ \hat{I}_{D}^{n}[a_{i}f(a)] = P_{ij} \left( \hat{I}_{D}^{n}(j)[f(a)] + \hat{I}_{D}^{n+2}[\partial_{j}f(a)] \right) + \frac{\gamma_{i}}{\Delta} \hat{I}_{D}^{n}[f(a)], \tag{44} \]

\[ P_{ij} = \frac{1}{2} \left( S_{ij}^{-1} - \frac{\gamma_{i} \gamma_{j}}{\Delta} \right), \quad \Delta = \sum \gamma_{i}. \tag{45} \]

The good point of this recursive relation is that all coefficients have no explicit dependence on the space-time dimension \( D \) and so it is well suited to compute the rational part.

We note that the above reasoning would go through if \( S \) is invertible. In the two-mass and one-mass triangle cases, \( S \) is not invertible. Nevertheless we can still use the above formulas by taking a limit from the general 3-mass triangle case. The point is that all we need is to have a well-behaved limit for the quantity \( P_{ij} \) and \( \frac{\Delta}{2} \). We have verified that directly taking the massless limit gives the same result as one would obtained by solving the non-singular system of equations (37) and (38).

From eq. (44) we see that higher dimensional integrals also appear in the recursive relations. These higher dimensional tensor integrals can be reduced to even higher and/or lower point tensor integrals by using these recursive relations repeatedly. At the end only scalar integrals are left. These higher dimensional scalar integrals can be reduced to lower dimensional and lower point scalar integrals by using the following recursive relation:

\[ \hat{I}_{D}^{n+2}[1] = \frac{1}{(n-1-D)\Delta} \left[ 2 \hat{I}_{D}^{n}[1] - \sum_{j} \gamma_{j} \hat{I}_{D}^{n+2}[1] \right]. \tag{46} \]

This equation is derived from eq. (42) by setting \( g_{l}(a) = 1 \) and summing over \( i \). The explicit dependence on the space-time dimension \( D \) in eq. (46) is very important. Otherwise all the rational coefficients in eq. (29) would have no explicit dependence on \( D \) and there would be no rational part.

To be more specific, we give some explicit examples for bubble and triangle integrals in what follows.

For bubble integral we have

\[ I_{2}^{D}[f(p)] = \int \frac{d^{D}p}{(-p)^{D/2}} \frac{f(p)}{p^{2}(p+K)^{2}}, \tag{47} \]
where $K$ is the sum of momenta on one side of the bubble diagram. For $K^2 = 0$ this integral is 0 in dimensional regularization. So we will assume $K^2 \neq 0$ hereafter. By direct computation we have the following results ($D = 4 - 2\epsilon$):

$$I^D_2[1] = \frac{r_{r\Gamma}}{\epsilon(1-2\epsilon)} (-K^2)^{-\epsilon}, \quad r_{r\Gamma} = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)},$$

$$I^D_2[p^\mu] = -\frac{K^\mu}{2} I^D_2[1],$$

$$I^D_2[a_1^2] = I^D_2[a_2^2] = \frac{2 - \epsilon}{2(3-2\epsilon)} I^D_2[1].$$

The recursive relation (46) becomes:

$$I^D_2^{D+2}[1] = \frac{K^2}{2(D-1)} I^D_2[1].$$

This can be applied recursively to compute arbitrarily higher dimensional bubble integrals.

![Triangle Diagram](image)

Figure 7: A generic three-mass triangle diagram. The 2-mass triangle diagram is obtained by setting one of momentum to be massless, for example $K_1 = k_1$ and $k_i^2 = 0$.

A generic triangle diagram is shown in Fig. 7 and the integral is

$$I^D_3[f(p)] \equiv \int \frac{d^Dp}{i \pi^{D/2}} \frac{f(p)}{p^2 (p - K_1)^2 (p + K_3)^2}.$$  

$^3$A minus sign is not included in the definition of $I^D_3[f(p)]$. 

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This integral is finite (free of ultra-violet (for degree 3 or less polynomial \(f(p)\) and infrared divergences) for generic external momenta (i.e., \(K_i^2 \neq 0,\ i = 1, 2, 3\)). The explicit formula can be found in [3]. For degenerate cases we have

\[
I_D^{[1]} = \frac{1}{\epsilon^2} \frac{\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \frac{(-K_2^2)^{-\epsilon} - (-K_3^2)^{-\epsilon}}{(-K_2^2) - (-K_3^2)}, \quad K_1^2 = 0, \quad K_1^2 = \frac{K_2^2}{K_1^2}, \quad K_1^2 = K_2^2 = 0. \tag{53}
\]

In the next section we will use these results and the more general recursive relations to derive the rational parts of the bubble, triangle and box integrals by making an expansion in the space-time parameter \(\epsilon\).

6 The rational parts of the triangle integrals

6.1 \(\epsilon\)-expansion and the rational parts: the bubble integrals

The \(\epsilon\)-expansion of the scalar bubble integral is:

\[
I_D^{[1]} = \frac{1}{\epsilon} + O(1). \tag{55}
\]

By using this result in eq. (51) we have (\(D = 4 - 2\epsilon\)):

\[
I_D^{D+2}[1] = \frac{K_2^2}{6} I_D^{[1]} + \frac{K_2^2}{9} + O(\epsilon), \tag{56}
\]

\[
I_D^{D+4}[1] = \frac{(K_2^2)^2}{60} I_D^{[1]} + \frac{4 (K_2^2)^2}{225} + O(\epsilon). \tag{57}
\]

We note that the first term on the right-hand of the above two equations still depends on \(D\) through the scalar integral \(I_D^{[1]}\). What is important is the second term which is a pure rational function and does not depend on the space-time dimension \(D\). This is the rational function part we need to keep track of later.

By making an expansion in \(\epsilon\) for eq. (50), we have:

\[
\hat{I}_D^{[a_1^2]} = \hat{I}_D^{[a_2^2]} = \frac{1}{3} I_D^{[1]} + \frac{1}{18} + O(\epsilon). \tag{58}
\]
From this equation we read off the rational part of $\hat{I}_2^D[a_1^2]$ and $\hat{I}_2^D[a_2^2]$ as $\frac{1}{18}$.

Translating back to the momentum integral the result is:

$$I_2^D[(\epsilon_1, p) (\epsilon_2, p)] = \left( \frac{(\epsilon_1, K) (\epsilon_2, K)}{3} - \frac{K^2 (\epsilon_1, \epsilon_2)}{6} \right) I_2^D[1]$$

$$+ \frac{1}{18} ((\epsilon_1, K) (\epsilon_2, K) - 2 K^2 (\epsilon_1, \epsilon_2)). \quad (59)$$

This result has already appeared in [50] (eq. (31) on p. 133). We interpret the second term in the above equation as the rational part. If we discard the first term we can simply write:

$$I_2[(\epsilon_1, p) (\epsilon_2, p)] = \frac{1}{18} ((\epsilon_1, K) (\epsilon_2, K) - 2 K^2 (\epsilon_1, \epsilon_2)), \quad (60)$$

by dropping also the explicit dependence of $I_2$ on $D$. However we still retain this dependence for higher dimensional Feynman integrals and simply drop all the cut-constructible parts. Explicitly we have:

$$I_2[1] = 0, \quad (61)$$

$$\hat{I}_2[a_1^2] = \hat{I}_2[a_3^2] = -\hat{I}_2[a_1 a_2] = \frac{1}{18}, \quad (62)$$

$$I_2^{D+2}[1] = \frac{K^2}{9}. \quad (63)$$

### 6.2 The rational parts of the higher-dimensional scalar integrals

For higher dimensional scalar integrals we can use the recursive relation eq. (46) to derive their rational parts.

For three-mass triangle the explicit recursive relation is:

$$\hat{I}_3^{3m(D+2)}[1] = \frac{1}{(2 - D) \Delta} \left[ 2 s_1 s_2 s_3 \hat{I}_3^{3m} + s_2 (s_1 + s_3 - s_2) I_2^{(1)} \right.$$

$$+ s_3 (s_1 + s_2 - s_3) I_2^{(2)} + s_1 (s_2 + s_3 - s_1) I_2^{(3)} \bigg], \quad (64)$$

$$\Delta = s_1^2 + s_2^2 + s_3^2 - 2(s_1 s_2 + s_1 s_3 + s_2 s_3), \quad (65)$$

where $s_i = K_i^2$, $i = 1, 2, 3$. 

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For 3-mass triangle integral $I_3^{D}[1]$ is regular as $D \to 4$. By using the $\epsilon$-expansion of $I_2^{D}$ we have:

$$
\hat{I}_3^{3m(D+2)}[1] = \frac{-1}{2\Delta} \left[ 2 s_1 s_2 s_3 \hat{I}_3^{3m} + s_2 (s_1 + s_3 - s_2) I_2^{(1)} + s_3 (s_1 + s_2 - s_3) I_2^{(2)} + s_1 (s_2 + s_3 - s_1) I_2^{(3)} \right] + \frac{1}{2} + O(\epsilon). \quad (66)
$$

So the rational part of (the Feynman parameter integral) $\hat{I}_3^{D+2}[1]$ is $\frac{1}{2}$. This result also applies to the two-mass and one-mass triangle, although there are intricacies of infrared divergences. This can be explicitly checked by using the following explicit recursive relations for the two-mass and one-mass triangle integrals:

$$
\hat{I}_3^{2m(D+2)}[1] = \frac{1}{D-2} \left( \frac{s_1}{s_1 - s_2} I_2^{(3)} - \frac{s_2}{s_1 - s_2} I_2^{(1)} \right), \quad (67)
$$

$$
\hat{I}_3^{1m(D+2)}[1] = \frac{1}{D-2} I_2^{(3)}, \quad (68)
$$

and the explicit formulas for $I_2^{D(i)}$'s. We note that the above recursive relations can be derived either by taking the limit $s_1 = K_1^2 \to 0$ (2-mass) or $s_{1,2} = K_{1,2}^2 \to 0$ (1-mass) or by solving eqs. (37) and (38). In above eqs. (67) and (68), the terms with infrared divergences really do not appear. This is because the coefficients of the infrared divergent terms in eq. (65) are automatically zero. The infrared divergences also do not contribute to the rational parts in the four point cases, which can be checked by explicit calculations. A heuristic argument is that since higher dimensional integrals are free of infrared divergences, we expect that there should not be infrared divergence in the square bracket on the right hand side of eq. (46) like in eq. (67) and (68), and hence the infrared divergences do not contribute to the rational part.

The end result of the above analysis is

$$
\hat{I}_3^{D+2}[1] = \frac{1}{2} + c(s) \hat{I}_3^D[1] + \sum_i c_i(s) I_2^{D(i)}[1] + O(\epsilon), \quad (69)
$$

where $c(s)$ and $c_i(s)$'s are (rational) functions of the external kinematic variables $s_i$ and the polarization vectors. This formula applies to all possible
triangle integrals. This shows explicitly that the rational part of the higher
dimensional scalar integral $\hat{I}_{3}^{D+2}[1]$ is a purely ultra-violet effect.

Setting the cut-constructible part to 0, we can effectively write the fol-
lowing formula for the rational part:

$$\hat{I}_{3}^{D+2}[1] = \frac{1}{2}. \quad (70)$$

We have also studied in detail the box integrals by using the explicit
formulas of [3]. We checked all the degenerate cases. The results can be
simply stated as follows:

$$I_{4}^{D+2}[1] = \hat{I}_{4}^{D+2}[a] = 0, \quad (71)$$

$$I_{4}^{D+4}[1] = \frac{5}{18}, \quad (72)$$

by dropping all the cut-constructible part. For 5-point integrals we have

$$\hat{I}_{5}^{D+2}[1] = \hat{I}_{5}^{D+2}[a] = \hat{I}_{5}^{D+2}[a_{i} a_{j}] = 0, \quad (73)$$

$$\hat{I}_{5}^{D+4}[1] = \hat{I}_{5}^{D+4}[a] = 0, \quad (74)$$

$$\hat{I}_{5}^{D+6}[1] = \frac{13}{144}, \quad (75)$$

although they are not used in the computation of the QCD amplitudes.

The rational part for the tensor integral is computed by first transforming
it into Feynman parameter integrals and then using the recursive relations.
The recursive relation for the rational part is exactly the same as for the
complete Feynman integral. However all lower degree ($m - 2$ or less for $m$-
point) Feynman integrals can be set to zero by BDDK theorem. Effectively
the recursive relations are truncated. In the next few sections we compute
explicitly the rational parts for triangle and box integrals. From hereafter
all Feynman integrals will mean their rational parts, except explicitly stated
otherwise.

6.3 The triangle integrals: the general case

For triangle integrals we will consider 2 cases: the two-mass triangle and
the three-mass triangle, although the two-mass case can be obtained simply
by setting one of the masses to be 0. The reason for doing this is that
the formulas simplify greatly for the two-mass triangle integral. For some computations one may only need the simplified formulas (for example the 5-gluon amplitudes in [68]). We will first give the formulas for the general three-mass triangle integrals.

The three-mass triangle diagram is shown in Fig. 7 and the external momenta are denoted by \( K_i \) (\( i = 1, 2, 3 \)). We use the convention of denoting a light-like momentum by a lower case \( k \), i.e. \( k^2 = 0 \).

We will give the rational parts for both the degree 3 and degree 2 polynomials\(^4\). First we make the following definitions:

\[
I_3(\epsilon_1, \epsilon_2, \epsilon_3) \equiv \int \frac{d^D p}{i\pi^{D/2}} \frac{\epsilon_1(p) (\epsilon_2, p - K_1)(\epsilon_3, p + K_3)}{p^2 (p - K_1)^2 (p + K_3)^2}, \tag{76}
\]

\[
I_3(\epsilon_1, \epsilon_2) \equiv \int \frac{d^D p}{i\pi^{D/2}} \frac{\epsilon_1(p) (\epsilon_2, p)}{p^2 (p - K_1)^2 (p + K_3)^2}. \tag{77}
\]

The actual computation of the rational part is done by using Feynman parametrization and we have:

\[
I_3(\epsilon_1, \epsilon_2, \epsilon_3) = -\hat{I}_3[(\epsilon_1, a)(\epsilon_2, a)(\epsilon_3, a)] + \sum_{i=1}^3 (\epsilon_i, \epsilon_{i+1}) \hat{I}_3^{D+2}[(\epsilon_{i+2}, a)], \tag{78}
\]

\[
I_3(\epsilon_1, \epsilon_2) = -\hat{I}_3[(\epsilon_1, a)(\epsilon_2, a)] + \frac{1}{2} (\epsilon_1, \epsilon_2), \tag{79}
\]

by using the previous result for \( \hat{I}_3^{D+2}[1] \). In the above we have used the following shortened notation:

\[
(\epsilon_1, a) = (\epsilon_1, K_1) a_2 - (\epsilon_1, K_3) a_3, \tag{80}
\]

\[
(\epsilon_2, a) = (\epsilon_2, K_2) a_3 - (\epsilon_2, K_1) a_1, \tag{81}
\]

\[
(\epsilon_3, a) = (\epsilon_3, K_3) a_1 - (\epsilon_3, K_2) a_2, \tag{82}
\]

where \( a_i \)’s are Feynman parameters as used in Sect. 5. In order to give compact formulas for the various quantities appearing in eqs. (78) and (79), we first define the following functions:

\[
F_0(s_1, s_2, s_3) \equiv -\hat{I}_3[a_1 a_2 a_3]
\]

\(^4\)It is also possible to derive the rational parts for higher degree polynomials but they have no practical usage in application to computations in the electroweak theory and QCD.
determinant” for the triangle diagram.

where $s_i = K_i^2$ and $\Delta = s_1^2 + s_2^2 + s_3^2 - 2(s_1s_2 + s_2s_3 + s_3s_1)$ is the “Gram determinant” for the triangle diagram.

By using these functions we have

\[
\hat{I}_3[(\epsilon_1,a)(\epsilon_2,a)(\epsilon_3,a)] = F_0(s_1,s_2,s_3)((\epsilon_1,K_1)(\epsilon_2,K_1)(\epsilon_3,K_2)
+ (\epsilon_1,K_3)(\epsilon_2,K_2)(\epsilon_3,K_2) + (\epsilon_1,K_3)(\epsilon_2,K_1)(\epsilon_3,K_3)
+ (\epsilon_1,K_3)(\epsilon_2,K_1)(\epsilon_3,K_2) - (\epsilon_1,K_1)(\epsilon_2,K_2)(\epsilon_3,K_3))
+ \sum_{i=1}^3 (\epsilon_1,K_i)(\epsilon_2,K_i)(\epsilon_3,K_i) F_i(s_1,s_2,s_3)
+ \frac{1}{2\Delta}((s_1 - s_2 - s_3)(\epsilon_1,K_1)(\epsilon_2,K_1)(\epsilon_3,K_3)
+ (s_2 - s_3 - s_1)(\epsilon_1,K_1)(\epsilon_2,K_2)(\epsilon_3,K_2)
+ (s_3 - s_1 - s_2)(\epsilon_1,K_3)(\epsilon_2,K_2)(\epsilon_3,K_3)),
\]  

\[
\hat{I}_3^{D+2}[(\epsilon_1,a)] = (\epsilon_1,K_1 - K_3) \left( \frac{7}{36} + \frac{s_2(s_3 + s_1 - s_2)}{12\Delta} \right)
+ (\epsilon_1,K_2) \frac{(s_1 - s_3)(s_3 + s_1 - s_2)}{12\Delta},
\]

\[
\hat{I}_3^{D+2}[(\epsilon_2,a)] = (\epsilon_2,K_2 - K_1) \left( \frac{7}{36} + \frac{s_3(s_1 + s_2 - s_3)}{12\Delta} \right)
+ (\epsilon_2,K_3) \frac{(s_2 - s_1)(s_1 + s_2 - s_3)}{12\Delta},
\]
\[ I^{D+2}_3[(\epsilon_3, a)] = (\epsilon_3, K_3 - K_2) \left( \frac{7}{36} + \frac{s_1(s_2 + s_3 - s_1)}{12\Delta} \right) \]
\[ + (\epsilon_3, K_1) \frac{(s_3 - s_2)(s_2 + s_3 - s_1)}{12\Delta}, \]  
(90)

and
\[ I_3[(\epsilon_i, a) (\epsilon_j, a)] = \frac{1}{2\Delta} \left( s_1 ((\epsilon_i, K_2) (\epsilon_j, K_3) + (\epsilon_i, K_3) (\epsilon_j, K_2)) \right. \]
\[ + s_2 ((\epsilon_i, K_3) (\epsilon_j, K_1) + (\epsilon_i, K_1) (\epsilon_j, K_3)) \]
\[ + s_3 ((\epsilon_i, K_1) (\epsilon_j, K_2) + (\epsilon_i, K_2) (\epsilon_j, K_1)). \]  
(91)

We note that some formulas (and also the function \( F_i \)) in the above are related by permutations. We purposely wrote down the complete formulas in order to see the pattern.

### 6.4 The triangle integrals: the two-mass case

For two-mass triangle we set \( K_1 = k_1 \). Then we can derive the simplified formulas in the two-mass triangle case by setting \( s_1 = 0 \) in the above formulas. Also we redefine \( (\epsilon_3, a) \) to be \( (\epsilon_3, k_1) a_2 - (\epsilon_3, K_3) a_3 \). This corresponds to the change of \( (\epsilon_3, p + K_3) \) to \( (\epsilon_3, p) \). This gives a more symmetric form for the result, as one can see from the Feynman integral representation by doing the transformation 2 ↔ 3. We list all the explicit formulas here because they are used heavily in the actual computation of the 6 particle amplitudes. This is necessary to obtain compact analytic formulas for the rational part of the amplitude. We have:

\[ \hat{I}_3[(\epsilon_1, a) (\epsilon_2, a) (\epsilon_3, a)] = \frac{(s_2 + s_3)}{6(s_2 - s_3)^2} (\epsilon_1, K_2) (\epsilon_2, k_1) (\epsilon_3, k_1) \]
\[ + \frac{(\epsilon_1, K_2)}{6(s_2 - s_3)} ((\epsilon_2, k_1) (\epsilon_3, K_3) - (\epsilon_2, K_2) (\epsilon_3, k_1)). \]  
(92)

\[ \hat{I}^{D+2}_3[(\epsilon_1, a)] = \frac{1}{9} (\epsilon_1, K_2), \]  
(93)

\[ \hat{I}^{D+2}_3[(\epsilon_2, a)] = -\frac{7}{36} (\epsilon_2, k_1) + \frac{1}{9} (\epsilon_2, K_2) - \frac{(s_2 + s_3)(\epsilon_2, k_1)}{12(s_2 - s_3)}, \]  
(94)

\[ \hat{I}^{D+2}_3[(\epsilon_3, a)] = \frac{7}{36} (\epsilon_3, k_1) - \frac{1}{9} (\epsilon_3, K_3) - \frac{(s_2 + s_3)(\epsilon_3, k_1)}{12(s_2 - s_3)}. \]  
(95)
\[ \hat{I}_3[(\epsilon_i, a) (\epsilon_j, a)] = -\frac{(s_2 + s_3)}{2(s_2 - s_3)^2} (\epsilon_i, k_1) (\epsilon_j, k_1) \]
\[ -\frac{((\epsilon_i, k_1) (\epsilon_j, K_2 - K_3) + (\epsilon_j, k_1) (\epsilon_i, K_2 - K_3))}{4(s_2 - s_3)}. \tag{96} \]

We note that in the last formula the double pole term is absent if one of the polarization vectors is associated with the first momentum \(k_1\) and satisfies the physical condition \((\epsilon, k_1) = 0\).

By using the above results we have

\[ I_3(\epsilon_1, \epsilon_2) = \int \frac{d^D p}{i \pi^{D/2}} \frac{(\epsilon_1, p) (\epsilon_2, p)}{p^2 (p - k_1)^2 (p + K_3)^2} \]
\[ = \frac{1}{2} (\epsilon_1, \epsilon_2) + \frac{(K_2^2 + K_3^2)}{2(K_2^2 - K_3^2)^2} (\epsilon_1, k_1) (\epsilon_2, k_1) \]
\[ + \frac{((\epsilon_1, K_2) (\epsilon_2, k_1) - (\epsilon_1, k_1) (\epsilon_2, K_3))}{2(K_2^2 - K_3^2)}, \tag{97} \]

\[ I_3(\epsilon_1, \epsilon_2) = \frac{1}{2} (\epsilon_1, \epsilon_2) + \frac{(\epsilon_1, K_2) (\epsilon_2, k_1)}{2(K_2^2 - K_3^2)}, \quad (\epsilon_1, k_1) = 0, \tag{98} \]

and

\[ I_3(\epsilon_i) = \int \frac{d^D p}{i \pi^{D/2}} \frac{(\epsilon_1, p) (\epsilon_2, p - k_1) (\epsilon_3, p)}{p^2 (p - k_1)^2 (p + K_3)^2} \]
\[ = \frac{1}{36} \left( (\epsilon_2, 4K_2 - 7k_1) (\epsilon_1, \epsilon_3) - (2 \leftrightarrow 3) + 4(\epsilon_1, K_2) (\epsilon_2, \epsilon_3) \right) \]
\[ - \frac{(K_2^2 + K_3^2)}{6(K_2^2 - K_3^2)^2} (\epsilon_1, K_2) (\epsilon_2, k_1) (\epsilon_3, k_1) \]
\[ - \frac{(\epsilon_1, K_2) ((\epsilon_2, k_1) (\epsilon_3, K_3) - (\epsilon_2, K_2) (\epsilon_3, k_1))}{6(K_2^2 - K_3^2)} \]
\[ - \frac{(K_2^2 + K_3^2)}{12(K_2^2 - K_3^2)} ( (\epsilon_1, \epsilon_2) (\epsilon_3, k_1) + (\epsilon_1, \epsilon_3) (\epsilon_2, k_1) ). \tag{99} \]

The above formula is anti-symmetric under the exchange \(2 \leftrightarrow 3\) by noting \((\epsilon_1, K_2) = -(\epsilon_1, K_3)\). For later application we denote the two-mass rational part of \(I_3\) by \(I_3^{2m(i)}\) where \(i\) denote the massless external line. To distinguish the two possible two-mass triangle diagrams with the same massless external line in the 6-gluon amplitude case, we put a tilde on \(I_3^{2m}\) for
one of the two-mass triangle integrals with 1, 3 and 2 external momenta in a clockwise direction. Referring to Fig. 7, \( I^{2m(i)}_2 \) has external momenta \( \{k_i, k_{i+1} + k_{i+2}, k_{i+3} + k_{i+4} + k_{i+5}\} \), whereas \( I^{2m(i)}_3 \) has external momenta \( \{k_i, k_{i+1} + k_{i+2} + k_{i+3}, k_{i+4} + k_{i+5}\} \).

7 The rational parts of the box integrals

A generic box diagram is shown in Fig. 8. The kinematic variables \( s \) and \( t \) are defined by following the standard notation:

\[
s = (K_1 + K_2)^2 = (K_3 + K_4)^2, \quad t = (K_2 + K_3)^2 = (K_4 + K_1)^2. \tag{100}
\]

For our purpose of computing up to 6-gluon amplitudes, we need to consider up to two-mass boxes. There are two kinds of two-mass boxes: the two-mass-hard box and two-mass-easy box. By a judicious choice of reference momenta the two-mass-hard box does not show up in the computation of MHV amplitudes. So we will discuss only the two-mass case here and set \( K_1 = k_1 \), etc. by following the convention of writing the light-like momenta in lower case \( k \). The one-mass box case is obtained as a special case of the two-mass-easy box case by setting further \( K_2 = k_2 \). We will list the explicit formula for one-mass box for quick reference and the ease of use.

Figure 8: A generic box diagram. \( p \) is the internal momenta between \( K_4 \) and \( K_1 \). Other internal momenta are also shown explicitly.
7.1 The box integrals of degree 3 polynomials: the two-mass-easy case

Generally we need to compute the rational part of the following box integral:

$$I_4(\epsilon_1, \epsilon_2, \epsilon_3) \equiv \int \frac{d^D}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p)(\epsilon_3, p)}{p^2(p-K_1)^2(p-K_1-K_2)^2(p+K_4)^2}. \quad (101)$$

The complete rational part in the general case is quite complicated and should be avoided. From our experience it is always the case that at least one of the polarization vectors satisfies the physical condition for one of the massless external momenta $k_1$ or $k_3$. In fact one can always expand an arbitrary 4-dimensional vector in terms of the 2 independent spinors of the two external massless momenta. So we can assume that $\epsilon_1$ satisfies the physical condition: $(\epsilon_1, k_1) = 0$. To be specific we can take $\epsilon_1 = \eta \tilde{\lambda}_1$. The negative helicity case can be obtained from this case (the positive helicity one) by conjugation. If one of the polarizations satisfies the physical condition for $k_3$ we can rotate the two-mass-easy box diagram by $\pi$ and relabel $k_3$ as $k_1$.

By explicit computation, we found that if the reference momentum of $\epsilon_1$ is $k_3$, the rational part becomes quite simple and is given as follows:

$$I_4(\lambda_3 \tilde{\lambda}_1, \epsilon_2, \epsilon_3) = \frac{\langle 3 | K_2 | 1 \rangle}{2} \left[ \frac{(\epsilon_2, k_3)(\epsilon_3, k_3)}{(K_2^2 - t)(K_4^2 - s)} - \frac{(\epsilon_2, k_1)(\epsilon_3, k_1)}{(K_2^2 - s)(K_4^2 - t)} \right]. \quad (102)$$

By using this result, the computation of the rational part of the degree 3 polynomial can be proceeded by changing the reference momentum of $\epsilon_1$ to $k_3$. This is equivalent to expanding the spinor in terms of $\lambda_{1,3}$:

$$\eta = \frac{\langle \eta 3 \rangle}{\langle 13 \rangle} \lambda_1 + \frac{\langle \eta 1 \rangle}{\langle 31 \rangle} \lambda_3. \quad (103)$$

In so doing we also generate 2 triangle diagrams which have been computed in last subsection. Explicitly we have:

$$I_4(\eta \tilde{\lambda}_1, \epsilon_2, \epsilon_3) = \frac{\langle \eta 1 \rangle}{\langle 31 \rangle} \frac{\langle 3 | K_2 | 1 \rangle}{2} \left[ \frac{(\epsilon_2, k_3)(\epsilon_3, k_3)}{(K_2^2 - t)(K_4^2 - s)} - \frac{(\epsilon_2, k_1)(\epsilon_3, k_1)}{(K_2^2 - s)(K_4^2 - t)} \right]$$

$$+ \frac{\langle \eta 3 \rangle}{\langle 13 \rangle} \left[ \tilde{I}_2^m(\epsilon_2, \epsilon_3) - I_3^m(\epsilon_2, \epsilon_3) \right], \quad (104)$$
\[ I_3^{2m}(\epsilon_2, \epsilon_3) = \frac{1}{2} (\epsilon_2, \epsilon_3) + \frac{(K_2^2 + t)}{2(K_2^2 - t)^2} (\epsilon_2, k_3)(\epsilon_3, k_3) \]
\[ + \frac{((\epsilon_2, k_3)(\epsilon_3, K_2) - (\epsilon_3, k_3)(\epsilon_2, K_4 + k_1))}{2(K_2^2 - t)}. \]
\[ I_3^{2m}(\epsilon_2, \epsilon_3) = \frac{1}{2} (\epsilon_2, \epsilon_3) + \frac{(K_4^2 + s)}{2(K_4^2 - s)^2} (\epsilon_2, k_3)(\epsilon_3, k_3) \]
\[ + \frac{((\epsilon_2, k_3)(\epsilon_3, K_4) - (\epsilon_3, k_3)(\epsilon_2, k_1 + K_2))}{2(K_4^2 - s)}. \] (105)

Taking into account the anti-symmetric property of the product \((\lambda_3 \tilde{\lambda}_1, K_2) = -(\lambda_3 \tilde{\lambda}_1, K_4)\), the above formulas are actually symmetric (which must be the case) under the interchange \(1 \leftrightarrow 3\) and \(2 \leftrightarrow 4\) (the polarization vectors are kept fixed because they should be invariant under the permutation by themselves).

### 7.2 The box integrals of degree 3 polynomials: the two-mass-hard case

For the two-mass-hard box case, we follow the same strategy. For \(\epsilon_1 = \lambda_1 \tilde{\lambda}_2\) or \(\lambda_2 \tilde{\lambda}_1\), we have

\[ I_4^{2mh}(\lambda_1 \tilde{\lambda}_2, \epsilon_2, \epsilon_3) = \frac{\langle 1|K_3|2 \rangle}{4 \delta} I_4(\epsilon_2, \epsilon_3), \] (106)

\[ I_4^{2mh}(\lambda_2 \tilde{\lambda}_1, \epsilon_2, \epsilon_3) = \frac{\langle 2|K_3|1 \rangle}{4 \delta} I_4(\epsilon_2, \epsilon_3), \] (107)

where

\[ I_4(\epsilon_2, \epsilon_3) = (\epsilon_2, k_1)(\epsilon_3, K_4) + (\epsilon_2, K_4)(\epsilon_3, k_1) + (\epsilon_2, k_2)(\epsilon_3, K_3) \]
\[ + (\epsilon_2, K_3)(\epsilon_3, k_2) - \frac{1}{\Delta} \left[ 2(K_3^2K_4^2 - t^2 + \delta)(\epsilon_2, k_12)(\epsilon_3, k_12) \right. \]
\[ + (K_3^2 + K_4^2 - s - 2t)((K_3^2 - K_4^2 + s)(\epsilon_2, K_4)(\epsilon_3, K_4) \]
\[ + (K_4^2 - K_3^2 + s)(\epsilon_2, K_3)(\epsilon_3, K_3)) \right] \]
\[ + \frac{K_3^2 + t}{K_4^2 - t} (\epsilon_2, k_1)(\epsilon_3, k_1) + \frac{K_3^2 + t}{K_3^2 - t} (\epsilon_2, k_2)(\epsilon_3, k_2). \] (108)

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\[
\delta = K_3^2 K_4^2 - (K_3^2 + K_4^2) t + (s + t) t,
\]
\[
\Delta = \Delta(k_{12}^2, K_3^2, K_4^2),
\]
\[
\Delta(s_1, s_2, s_3) = s_1^2 + s_2^2 + s_3^2 - 2(s_1 s_2 + s_2 s_3 + s_3 s_1),
\]
are functions of the external momentum invariants. In particular \( \Delta \) is the Gram determinant of the three-mass triangle integral arising from the tensor reduction of the two-mass-hard box integral.

The general case is obtained by changing the reference momentum:

\[
I_4(\eta \lambda_1, \epsilon_2, \epsilon_3) = \frac{\langle \eta \rangle}{\langle \epsilon_2 \rangle} I_4(\lambda_2 \lambda_1, \epsilon_2, \epsilon_3)
\]
\[
+ \frac{\langle \eta \rangle}{\langle \epsilon_2 \rangle} \left[ I_3^{2m}(\epsilon_2, \epsilon_3) - I_3^{3m}(\epsilon_2, \epsilon_3) \right],
\]
\[
I_3^{2m}(\epsilon_2, \epsilon_3) = \frac{1}{2} (\epsilon_2, \epsilon_3) + \frac{(K_3^2 + t)}{2(K_3^2 - t)} (\epsilon_2, k_2) (\epsilon_3, k_2)
\]
\[
+ \frac{((\epsilon_2, k_2) (\epsilon_3, K_3) - (\epsilon_3, k_2) (\epsilon_2, K_4 + k_1))}{2(K_3^2 - t)},
\]
and \( I_3^{3m}(\epsilon_2, \epsilon_3) \) is the three-mass triangle integral with external momenta \( \{k_{12}, K_3, K_4\} \). There are two triangle diagrams from this reduction and one is a three-mass triangle diagram.

### 7.3 The box integrals of degree 4 polynomials: the two-mass-easy case

Now we discuss the computation of the rational parts for degree 4 polynomials. First we define

\[
I_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \equiv \int \frac{d^D p}{i \pi^{D/2}} \frac{(\epsilon_1, p) (\epsilon_2, p - K_1) (\epsilon_3, p - K_{12}) (\epsilon_4, p + K_4)}{p^2(p - K_1)^2(p - K_{12})^2(p + K_4)^2},
\]
where \( K_{12} = K_1 + K_2 \). By using Feynman parametrization we have

\[
I_4(\epsilon_i) = \hat{I}_4[(\epsilon_1, a)(\epsilon_2, a)(\epsilon_3, a)(\epsilon_4, a)] - \sum_{i < j} (\epsilon_i, \epsilon_j) \hat{I}_4^{D+2}[(\epsilon_k, a)(\epsilon_l, a)]
\]
\[
+ ((\epsilon_1, \epsilon_2)(\epsilon_3, \epsilon_4) + (\epsilon_1, \epsilon_3)(\epsilon_2, \epsilon_4) + (\epsilon_1, \epsilon_4)(\epsilon_2, \epsilon_3)) I_4^{D+4}[1].
\]
We also assume that $\epsilon_{1,3}$ satisfy the physical conditions in the two-mass-easy box integral and $\epsilon_{1,2}$ satisfy the physical conditions for the 2-mass-hard box integral. As we said before this can always be done by expanding all polarization vectors in terms of the 2 spinors from the 2 massless momenta.

The direct calculation of the rational part of the two-mass-easy box integral gives rather complicated formulas for generic polarization vectors (even after using the physical condition for $\epsilon_{1,3}$). By choosing appropriate reference momenta for $\epsilon_{1,3}$, the result simplifies greatly. In particular the reference momentum of $\epsilon_1$ should be $k_3$ and the reference momentum of $\epsilon_3$ should be $k_1$. In some sense this is equivalent to the tensor reduction with the two factors $(\epsilon_1, p)(\epsilon_3, p - k_1 - K_2)$. The explicit results are given as follows:

$$I_4(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4) = -\frac{1}{4} \left( \frac{K_2^2 + s + K_3^2 + t}{K_2^2 - s + K_3^2 - t} \right) (\epsilon_2, k_1)(\epsilon_4, k_1)$$

$$-\frac{1}{4} \left( \frac{K_2^2 + t + K_3^2 + s}{K_2^2 - t + K_3^2 - s} \right) (\epsilon_2, k_3)(\epsilon_4, k_3) - \frac{5}{9} (k_1, k_3)(\epsilon_2, \epsilon_4)$$

$$+ \frac{4}{9} \left( (\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1) \right),$$

$$I_4(\lambda_1\tilde{\lambda}_3, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4) = \frac{5}{9} \left( 1|\epsilon_2[3] \langle 1|\epsilon_4|3 \right)$$

$$+ \frac{1|K_2|3^2}{3} \left[ \frac{(\epsilon_2, k_1)(\epsilon_4, k_1)}{(K_2^2 - s)(K_3^2 - t)} + \frac{(\epsilon_2, k_3)(\epsilon_4, k_3)}{(K_2^2 - t)(K_3^2 - s)} \right].$$

Other cases can be either obtained by conjugation or by relabelling $k_{1,3}$. In fact $I_4(\lambda_1\tilde{\lambda}_3, \epsilon_2, \lambda_3\tilde{\lambda}_1, \epsilon_4) = I_4(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4)$ as it is invariant under conjugation.

For the general case, we use the reduction formula by changing the reference appropriately. For example we have:

$$I_4(\eta_{1}\tilde{\lambda}_1, \epsilon_2, \eta_3\tilde{\lambda}_3, \epsilon_4) = \frac{\langle \eta_{1} \rangle^{3}}{1|3} I_4(k_1, \epsilon_2, \epsilon_3, \epsilon_4) + \frac{\langle \eta_{3} \rangle 1}{3|1} I_4(\epsilon_1, \epsilon_2, k_3, \epsilon_4)$$

$$- \frac{\langle \eta_{1} \rangle 1}{1|3} \frac{\langle \eta_{3} \rangle 3}{3|1} I_4(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4)$$

$$+ \frac{\langle \eta_{1} \rangle 3}{1|3} \frac{\langle \eta_{3} \rangle 1}{3|1} I_4(k_1, \epsilon_2, k_3, \epsilon_4).$$

(118)
By using the explicit result of the 2-mass triangle integral we have

\[
I_4(\epsilon_1, \epsilon_2, k_3, \epsilon_4) = \frac{K_2^2 + s}{6(K_2^2 - s)^2} (\epsilon_1, K_2)(\epsilon_2, k_1)(\epsilon_4, k_1)
+ \frac{K_4^2 + t}{6(K_4^2 - t)^2} (\epsilon_1, K_4)(\epsilon_2, k_1)(\epsilon_4, k_1)
+ \frac{1}{12} \left( \frac{K_2^2 + s}{K_2^2 - s} + \frac{K_4^2 + t}{K_4^2 - t} \right) ((\epsilon_1, \epsilon_2)(\epsilon_4, k_1) + (\epsilon_1, \epsilon_4)(\epsilon_2, k_1))
+ \frac{(\epsilon_1, K_2)}{6(K_2^2 - s)} (\epsilon_2, k_1)(\epsilon_4, k_3) + \frac{(\epsilon_1, K_4)}{6(K_4^2 - t)} (\epsilon_2, k_3)(\epsilon_4, k_1)
+ \left[ \frac{(\epsilon_1, K_2)}{6(K_2^2 - s)} - \frac{(\epsilon_1, K_4)}{6(K_4^2 - t)} \right] ((\epsilon_2, k_1)(\epsilon_4, K_4) - (\epsilon_2, K_2)(\epsilon_4, k_1))
+ \frac{1}{9} ((\epsilon_1, \epsilon_2)(\epsilon_4, k_3) + (\epsilon_1, \epsilon_4)(\epsilon_2, k_3) + (\epsilon_2, \epsilon_4)(\epsilon_1, k_3))
+ \frac{1}{2} (\epsilon_2, k_1)(\epsilon_4, K_4) \left[ \frac{(\epsilon_1, K_4)}{K_4^2 - t} - \frac{(\epsilon_1, K_2)}{K_2^2 - s} \right].
\] (119)

Except for the last term, this formula is invariant under the interchange 2 \leftrightarrow 4 (s \leftrightarrow t). By setting \( \epsilon_1 = k_1 \) we get

\[
I_4(k_1, \epsilon_2, k_3, \epsilon_4) = \frac{1}{18} (2(k_1, k_3)(\epsilon_2, \epsilon_4) - ((\epsilon_2, k_1)(\epsilon_2, k_3) + (\epsilon_2, k_3)(\epsilon_2, k_1))).
\] (120)

This agrees with the result by direct computation by first doing the tensor reduction and then using the result for bubble integrals. These formulas are quite useful for obtaining compact analytic formulas for QCD amplitudes.

For easy reference we also give here the relevant formulas in terms of Feynman parameters:

\[
\hat{I}_4[(\bar{\epsilon}_1, a)(\epsilon_2, a)(\bar{\epsilon}_3, a)(\epsilon_4, a)]
= \frac{(\bar{\epsilon}_1, K_2)(\epsilon_3, K_2)}{3} \left( \frac{(\epsilon_2, k_1)(\epsilon_4, k_1)}{(K_2^2 - s)(K_4^2 - t)} + \frac{(\epsilon_2, k_3)(\epsilon_4, k_3)}{(K_2^2 - t)(K_4^2 - s)} \right)
\]

\[
\hat{I}_4^{\rho+2}[(\epsilon_2, a)(\epsilon_4, a)] = \frac{1}{6(K_2^2 + K_4^2 - s - t)} [(\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1)]
- (K_2^2 K_4^2 - s t) \left( \frac{(\epsilon_2, k_1)(\epsilon_4, k_1)}{(K_2^2 - s)(K_4^2 - t)} + \frac{(\epsilon_2, k_3)(\epsilon_4, k_3)}{(K_2^2 - t)(K_4^2 - s)} \right).\] (121)
All other $I^D_{4+2}[(\epsilon_i, a)(\epsilon_j, a)]'s$ are identically zero. In the above formulas $\tilde{\epsilon}_{1,3}$ satisfy 2 conditions: $(\tilde{\epsilon}_i, k_{1,3}) = 0$, i.e. the reference momentum of $\tilde{\epsilon}_1$ is $k_3$ and the reference momentum of $\tilde{\epsilon}_3$ is $k_1$. We also note when $(\tilde{\epsilon}_1, \tilde{\epsilon}_3)$ is not equal to 0, it cancels with the factor $(K_2^2 + K_4^2 - s - t) = (k_1, k_3) = -(\tilde{\epsilon}_1, \tilde{\epsilon}_3)$.

### 7.4 The box integrals of degree 4 polynomials: the two-mass-hard case

As before, the direct computation of the rational part of the two-mass-hard box gives very complicated algebraic expressions. In fact it gives the most complicated formula up to now. Changing the reference momenta simplifies a little bit, but the resulting formula is still not intelligible. The main complication comes from the presence of the 3-mass triangle integrals. To organize the final result, we proceed to do tensor reduction one more time.

To begin with let us define the integral we want to compute:

$$I^2_{4}^mh(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4; c_3, c_4) 
\equiv I_4[(\epsilon_1, p)(\epsilon_2, p - k_1)((\epsilon_3, p + K_4 + c_3)((\epsilon_4, p + K_4 + c_4))]
= \int \frac{d^Dp}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p - k_1)((\epsilon_3, p + K_4 + c_3)((\epsilon_4, p + K_4 + c_4))}{p^2 (p - k_1)^2 (p - k_{12})^2 (p + K_4)^2}.
\tag{122}$$

The external momenta are $k_1, k_2, K_3$ and $K_4$ as shown in Fig. 8 by setting $K_1 = k_1$ and $K_2 = k_2$. $k_{1,2}$ are the two massless external legs and $K_{3,4}$ are the two massive external legs. We set $t = (k_2 + K_3)^2 = (K_4 + k_1)^2$. We also require that $\epsilon_{1,2}$ satisfy the physical condition, i.e. $(\epsilon_1, k_1) = 0$ and $(\epsilon_2, k_2) = 0$. The formula for generic $\epsilon_{1,2}$ is beyond the scope of this paper. ($\epsilon_{3,4}$ are arbitrary 4-dimensional polarization vectors.)

There are 4 possible cases for $\epsilon_{1,2}$. In the same helicity cases, the box integrals can be easily reduced to triangle integrals by using the reduction formulas eq. (14) and (16) in Sect. 3. So we will only consider the difficult cases where $\epsilon_{1,2}$ have different helicities. To be definite we set $\epsilon_1 = \lambda_1 \bar{\eta}_1$ and $\epsilon_2 = \eta_2 \lambda_2$. The opposite case can be obtained from this one simply by conjugation. We will give the explicit formula for this case at the end of this subsection.

To do tensor reduction for the factors associated with the two massless
external legs, we have:

\[ T^{++} = (\epsilon_1, p) (\epsilon_2, p - k_1) = \frac{\langle \eta_2 | (p - k_1) k_2 K_3 k_1 (p - k_1) | \tilde{\eta}_1 \rangle}{(2 | K_3 | 1)} \] (123)

The middle factor in the above can be decomposed by moving the first factor of \((p - k_1)\) towards the second \((p - k_1)\). Explicitly we have:

\[
\begin{align*}
(p - k_1) k_2 K_3 k_1 (p - k_1) &= T_1 + T_2, \\
T_1 &= \frac{1}{2} (p - k_1) k_2 K_3 + I^{(2)} (K_3 k_1 - k_2 K_3) p + I^{(3)} K_4 k_1 p, \\
T_2 &= p^2 - \frac{1}{2} k_2 K_3 k_1 + (I^{(4)} - t) k_2 k_1 p \\
&+ I^{(1)} (-I^{(2)} K_3 + I^{(3)} k_2 + K_3) - (I^{(4)} - t) k_2).
\end{align*}
\] (124)

Omitting the factor \(\langle 2 | K_3 | 1 \rangle\), we now compute the various terms. We have:

\[
\begin{align*}
\langle \eta_2 | T_2 | \tilde{\eta}_1 \rangle &= p^2 - \frac{1}{2} \langle \eta_2 | k_2 K_3 k_1 | \tilde{\eta}_1 \rangle \\
&+ (I^{(4)} - t) \langle \eta_2 | 2 \rangle [\tilde{\eta}_1 | 1 \rangle (\lambda_1 \tilde{\lambda}_2, p) - I^{(2)} \langle \eta_2 | k_2 | \tilde{\eta}_1 \rangle \\
&+ I^{(1)} (I^{(3)} \langle \eta_2 | (k_2 + K_3) | \tilde{\eta}_1 \rangle - I^{(2)} \langle \eta_2 | K_3 | \tilde{\eta}_1 \rangle).
\end{align*}
\] (127)

This gives the following rational terms:

\[
\begin{align*}
A_2 &= -\frac{1}{6} \langle \eta_2 | k_2 K_3 k_1 | \tilde{\eta}_1 \rangle (\epsilon_3, \epsilon_4) - t \langle \eta_2 | 2 \rangle [\tilde{\eta}_1 | 1 \rangle \left( \frac{1}{2} \langle 1 | \epsilon_3 | 2 \rangle c_4 + \langle 1 | \epsilon_4 | 2 \rangle c_3 \\
&+ \frac{1}{18} (\langle 1 | \epsilon_3 | 2 \rangle \epsilon_4 + \langle 1 | \epsilon_4 | 2 \rangle \epsilon_3, 7 k_1 + 2 k_2 + 9 K_4) \right) \\
&+ \frac{1}{18} \langle \eta_2 | 2 \rangle [\tilde{\eta}_1 | 2 \rangle ((\epsilon_3, k_1) (\epsilon_4, k_1) - 2 s_{12} (\epsilon_3, \epsilon_4)) \\
&+ \frac{1}{18} \langle \eta_2 | (k_2 + K_3) | \tilde{\eta}_1 \rangle ((\epsilon_3, k_2 + K_3)(\epsilon_4, k_2 + K_3) - 2 t (\epsilon_3, \epsilon_4)) \\
&- \frac{1}{18} \langle \eta_2 | K_3 | \tilde{\eta}_1 \rangle ((\epsilon_3, K_3)(\epsilon_4, K_3) - 2 K_3^2 (\epsilon_3, \epsilon_4)).
\end{align*}
\] (128)

The explicit formulas for the other 3 terms in \(T_1\) are not quite illuminating and we refrain from writing the explicit results here. We can write the result in terms of the rational parts for the three-mass and two-mass triangle
integrals arising from the tensor reduction. We have:

\[ A_1 = \langle 2|K_3|\tilde{\eta}_1 \rangle (I_{3}^{2m}(\eta_2\tilde{\lambda}_2, \epsilon_3, \epsilon_4) + I_{3}^{m}(\eta_2\tilde{\lambda}_2, (c_3 - (\epsilon_3, K_3))\epsilon_4 + (c_4 + (\epsilon_4, K_4 + k_1))\epsilon_3) + I_{3}^{3m}(v, \epsilon_3, \epsilon_4) + I_{3}^{3m}(v, (c_3 - (\epsilon_3, K_3))\epsilon_4 + c_4\epsilon_3), \quad (129) \]

where

\[ v = \langle \eta_2|K_3|1 \rangle \lambda_1\tilde{\eta}_1 + \langle \eta_2|K_3|2 \rangle \lambda_2\tilde{\eta}_1 - (k_2, K_3)\eta_2\tilde{\eta}_1. \quad (131) \]

Combining the above results together we have:

\[ I_{4}^{2m}(\lambda_1\tilde{\eta}_1, \eta_2\tilde{\lambda}_2, \epsilon_3, \epsilon_4; c_3, c_4) = \frac{A_1 + A_2}{\langle 2|K_3|1 \rangle}. \quad (132) \]

For the opposite helicity case the formula is:

\[ I_{4}^{2m}(\eta_1\tilde{\lambda}_1, \lambda_2\tilde{\eta}_2, \epsilon_3, \epsilon_4; c_3, c_4) = \frac{1}{\langle 1|K_3|2 \rangle} \left[ -\frac{1}{6} \langle \eta_1|k_1K_3k_2|\tilde{\eta}_2 \rangle (\epsilon_3, \epsilon_4) \right.

\[ - t \langle \eta_1 1 | [\tilde{\eta}_2 2 | I_{4}^{2m}(\lambda_2\tilde{\lambda}_1, \epsilon_3, \epsilon_4) + t \langle \eta_1 |k_2|\tilde{\eta}_2 \rangle I_{3}^{m}(\epsilon_3, \epsilon_4)

\[ + \langle \eta_1 1 | [\tilde{\eta}_2 2 | \left( \frac{1}{2} \langle 2|\epsilon_3|1 \rangle c_4 + (2|\epsilon_4|1 \rangle c_3

\[ + \frac{1}{18} (2|\epsilon_3|1 \rangle \epsilon_4 + (2|\epsilon_4|1 \rangle \epsilon_3, 7k_1 + 2k_2 + 9K_4) \right)

\[ + \frac{1}{18} \langle \eta_1 2 | [\tilde{\eta}_2 2 | ((\epsilon_3, k_12) (\epsilon_4, k_12) - 2 s_{12} (\epsilon_3, \epsilon_4)

\[ + \frac{1}{18} \langle \eta_1 |(k_2 + K_3)|\tilde{\eta}_2 \rangle (\epsilon_3, k_2 + K_3)(\epsilon_4, k_2 + K_3) - 2 t (\epsilon_3, \epsilon_4)

\[ - \frac{1}{18} \langle \eta_1|K_3|\tilde{\eta}_2 \rangle ((\epsilon_3, K_3)(\epsilon_4, K_3) - 2K_3^2(\epsilon_3, \epsilon_4))

\[ + \langle \eta_1|K_3|2 |I_{3}^{2m}(\lambda_2\tilde{\eta}_2, \epsilon_3, \epsilon_4)

\[ + I_{3}^{m}(\lambda_2\tilde{\eta}_2, (c_3 - (\epsilon_3, K_3))\epsilon_4 + (c_4 + (\epsilon_4, K_4 + k_1))\epsilon_3) + I_{3}^{3m}(\tilde{v}, \epsilon_3, \epsilon_4) + I_{3}^{3m}(\tilde{v}, (c_3 - (\epsilon_3, K_3))\epsilon_4 + c_4\epsilon_3), \quad (133) \]

where

\[ \tilde{v} = \langle 1|K_3|\tilde{\eta}_2 \rangle \eta_1\tilde{\lambda}_1 + \langle 2|K_3|\tilde{\eta}_2 \rangle \eta_1\tilde{\lambda}_2 - (k_2, K_3)\eta_1\tilde{\eta}_2. \quad (135) \]
8 Extra terms for box and triangle tensor reduction

8.1 Extra terms for box tensor reduction

In order to get a comparatively compact analytic formula for the two-mass-hard box integral, it is necessary to do tensor reduction because a direct computation of the rational part by the recursive method gives a very complicated formula. It would be a better idea to organize it into lower degree and lower point integrals. A naive tensor reduction directly in $D = 4$ would give an incorrect result because the box integral is ultra-violet divergent for a degree 4 polynomial of momenta in the numerator. It turns out that the difference between the $D = 4$ tensor reduction and the correct tensor reduction is just a rational function. In this subsection we will compute this extra rational function explicitly.

For a degree 2 polynomial $g(\epsilon, k, p)$ in the internal momentum $p$, the general form of the $D = 4$ tensor reduction we used is as follows:

$$g(\epsilon, k, p) = \tilde{g}(\epsilon, k, p) - p^2 f(\epsilon, k).$$  \hspace{1cm} (136)

We assume that all polarization vectors $\epsilon$ and momenta $k$ are 4-dimensional and the above relation is derived by assuming $p$ is also a 4-dimensional momentum. We also assume that $g$ and $\tilde{g}$ depend on the momentum $p$ through scalar products $(\epsilon, p)$ and/or $(k, p)$. Because we use FDH regularization [72], $p$ is actually promoted to be in $D = 4 - 2\epsilon$ dimensions in our later calculations of the rational part. This affects only the last term in eq. (136). For arbitrary dimensional internal momentum $p$, the above formula is still valid if we make the substitution:

$$p^2 \rightarrow p^2 - p^2_{D-4}.  \hspace{1cm} (137)$$

That is, the extra dimensional part $(D - 4 = -2\epsilon)$ of the momentum $p$ must be subtracted from $p^2$ which actually stands for the scalar product in $D$ dimensions in the subsequent computation in the four-dimensional helicity regularization scheme. Because pentagon and hexagon diagrams are ultra-violet convergent we can safely discard this term in the tensor reduction for 5- or 6-point diagrams. This term does give a non-vanishing contribution to the rational parts for box and triangle integrals.
For box integral we have

\[
A_{\text{box}} = I_4^{D=4-2\epsilon}[p^2_{-2\epsilon} (\epsilon_3, p) (\epsilon_4, p)]
\]
\[
= \int_0^1 d^4a_i \delta(1 - \sum_i a_i) \int_0^\infty dT T^3 \times \int \frac{d^D p}{i\pi^{D/2}} p^2_{-2\epsilon} (\epsilon_3, p) (\epsilon_4, p) e^{p^2 T - T \cdot S \cdot a},
\]

(138)

by transforming it into a Feynman parameter integral and omitting terms which are vanishing in the limit \( \epsilon \to 0 \).

The integration over the momentum \( p \) can be done easily by splitting it into a \( D = 4 \) part and a \((-2\epsilon)\)-dimensional part. We have then

\[
A_{\text{box}} = \epsilon (- (\epsilon_3, \epsilon_4)) I_4^{D+4}[1] = -\frac{1}{6} (\epsilon_3, \epsilon_4) + O(\epsilon).
\]

(139)

By using this result the correct formula for computing the rational part of the box tensor integral is:

\[
I_4[g(\epsilon, k, p) ((\epsilon_3, p) + c_3) ((\epsilon_4, p) + c_4)] = I_4[(\tilde{g}(\epsilon, k, p) - p^2 f(\epsilon, k))
\times((\epsilon_3, p) + c_3) ((\epsilon_3, p) + c_4)] - \frac{1}{6} f(\epsilon, k) (\epsilon_3, \epsilon_4).
\]

(140)

### 8.2 Extra terms in triangle tensor reduction

For triangle integrals, using the same reduction formula as given in eq. (136) also gives incorrect results. The extra terms are given by the following formula (\( p \) is the internal momentum between the external legs \( K_1 \) and \( K_2 \)):

\[
I_3[g(\epsilon, K, p) ((\epsilon_3, p) + c_3)] = I_3[(\tilde{g}(\epsilon, K, p) - p^2 f(\epsilon, k))
\times((\epsilon_3, p) + c_3)| - f(\epsilon, k) \left[ \frac{1}{2} c_3 + \frac{1}{6} (\epsilon_3, K_1 - K_3) \right].
\]

(141)

We note that the rational part \( I_3[g(\epsilon, K, p) ((\epsilon_3, p) + c_3)] \) is defined by the Feynman integral without the usual minus sign, just as we did before in eq. (52).
9 Conclusion

The calculation of multi-leg one-loop amplitudes in QCD is highly motivated through the need of precise predictions for multi-particle scattering at TeV colliders like the Tevatron and the LHC. Whereas unitarity based methods are very successful to provide the cut-constructible parts of one-loop amplitudes, the evaluation of rational terms which are not defined by the cuts alone, is a much harder question. Recently the bootstrap recursive proposal of Bern, Dixon and Kosower [34] has been made for their evaluation. (It was used to do real QCD calculations [34, 70].) In this paper we have developed a general formalism targeting only the computation of the rational terms by using Feynman diagrammatic methods.

In comparing with the previous use of diagrammatic methods in the computation of multi-leg one-loop amplitudes, the computation of the rational terms only is greatly simplified by using the BDDK theorem [3]. By using this theorem and a thorough analysis of the recursive relations we derived recursive relations also for the rational terms. Quite explicit results are given for all bubble and triangle integrals, and box integrals up to two-mass cases. These results are all the ingredients for computing up to the 6-gluon one-loop amplitudes in QCD. We will compute the (rational parts of the) 5-gluon and 6-gluon amplitudes in [68] and [69] respectively.

By using the recursive relations it is straightforward to derive the rational terms of the 3-mass and 4-mass (tensor) box integrals. The formulas obtained are quite complicated. It may only be useful for implementing numerically.

The method developed in this paper can be extended to the cases with massive internal particles and internal (massive) quarks. Our method complements the twistor-inspired approach quite nicely. We envisage that an automatic implementation of both approaches could push the present limit of 6-gluon to about 8-gluon or higher.

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References

[1] G. P. Salam, “Developments in perturbative QCD,” Invited talk at 22nd International Symposium on Lepton-Photon Interactions at High Energy (LP 2005), Uppsala, Sweden, 30 Jun - 5 Jul 2005, hep-ph/0510090.

[2] Z. Bern, L. Dixon and D. A. Kosower, “New QCD Results from String Theory,” talk presented by Z. Bern at Strings 1993, May 24-29, Berkeley CA, USA, hep-th/9311026; Z. Bern, G. Chalmers, L. Dixon and D. A. Kosower, “One-Loop N Gluon Amplitudes with Maximal Helicity Violation via Collinear Limits,” Phys. Rev. Lett. 72 (1994) 2134-2137, hep-ph/9312333.

[3] Z. Bern, L. Dixon, D. C. Dunbar and D.A. Kosower, “One-Loop n-Point Gauge Theory Amplitudes, Unitarity and Collinear Limits,” Nucl. Phys. B425 (1994) 217-260, hep-ph/9403226; “Fusing Gauge Theory Tree Amplitudes Into Loop Amplitudes,” Nucl. Phys. B435 (1995) 59-101, hep-ph/9409265.
[4] G. Mahlon, “One Loop Multiphoton Helicity Amplitudes,” Phys. Rev. D49 (1994) 2197-2210, hep-ph/9311213; “Multigluon Helicity Amplitudes Involving a Quark Loop,” Phys. Rev. D49 (1994) 4438-4453, hep-ph/9312276.

[5] T. Binoth, J. P. Guillet, G. Heinrich and C. Schubert, “Calculation of 1-loop Hexagon Amplitudes in the Yukawa Model,” Nucl. Phys. B615 (2001) 385-401, hep-ph/0106243.

[6] For a summary of the present status, see, for example, C. Buttar, S. Dittmaier, et. al., “Les Houches Physics at TeV Colliders 2005, Standard Model and Higgs working group: Summary report,” hep-ph/0604120

[7] Z. Bern, L. J. Dixon and D. A. Kosower, “One Loop Corrections to Five Gluon Amplitudes,” Phys. Rev. Lett. 70 (1993) 2677-2680; Z. Bern, “String based perturbative methods for gauge theories,” TASI 92 lectures, hep-ph/9304249.

[8] A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, “Complete electroweak $O(\alpha)$ corrections to charged-current $e^+e^- \rightarrow 4$ fermion processes,” Phys. Lett. B612 (2005) 223-232, hep-ph/0502063; “Electroweak corrections to charged-current $e^+e^- \rightarrow 4$ fermion processes - technical details and further results,” Nucl.Phys. B724 (2005) 247-294, hep-ph/0505042; A. Denner and S. Dittmaier, “Reduction schemes for one-loop tensor integrals,” Nucl. Phys. B734, B734 (2006) 62-115, hep-ph/0509141.

[9] R. K. Ellis, W. T. Giele and G. Zanderighi, “The one-loop amplitude for six-gluon scattering,” JHEP 0605 (2006) 027, hep-ph/0602185; “Semi-Numerical Evaluation of One-Loop Corrections,” Phys. Rev. D72, 054018 (2005), hep-ph/0503083; W. T. Giele and E. W. N Glover, “A calculational formalism for one-loop integrals,” JHEP 0404 (2004) 029, hep-th/0402152.

[10] T. Binoth, J. Ph. Guillet, G. Heinrich, E. Pilon and C. Schubert, “An algebraic/numerical formalism for one-loop multi-leg amplitudes,” JHEP 0510 (2005) 015, hep-ph/0504267;
T. Binoth, G. Heinrich, and N. Kauer, “A numerical evaluation of the scalar hexagon integral in the physical region,” Nucl. Phys. B654 (2003) 277-300, hep-ph/0210023;

T. Binoth, M. Ciccolini and G. Heinrich, “Towards LHC phenomenology at the loop level: A new method for one-loop amplitudes,” hep-ph/0601254.

[12] F. del Aguila and R. Pittau, “Recursive numerical calculus of one-loop tensor integrals,” JHEP 0407 (2004) 017, hep-ph/0404120.

[13] A. Ferroglia, M. Passera, G. Passarino, and S. Uccirati, “All-Purpose Numerical Evaluation of One-Loop Multi-Leg Feynman Diagrams,” Nucl. Phys. B650 (2003) 162–228, hep-ph/0209219.

[14] Z. Nagy and D. E. Soper, “General subtraction method for numerical calculation of one-loop QCD matrix elements,” JHEP 0309 (2003) 055, hep-ph/0308127.

[15] C. Anastasiou and A. Daleo, “Numerical evaluation of loop integrals,” hep-ph/0511176.

[16] M. Kramer and D. E. Soper, “Next-to-leading order numerical calculations in Coulomb gauge,” Phys. Rev. D66 (2002) 054017, hep-ph/0204113;

M. Czakon, “Automatized analytic continuation of Mellin-Barnes integrals,” hep-ph/0511200.

[17] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, “Bootstrapping One-Loop QCD Amplitudes with General Helicities,” hep-ph/0604195.

[18] E. Witten, “Perturbative gauge theory as a string theory in twistor space,” Commun. Math. Phys. 252 (2004) 189-258, hep-th/0312171.

[19] F. Cachazo, P. Svrček and E. Witten, “MHV vertices and tree amplitudes in gauge theory,” JHEP 0409, (2004) 006, hep-th/0403047.

[20] V. Nair, “A Current Algebra For Some Gauge Theory Amplitudes,” Phys. Lett. B78 (1978) 464.
[21] S. J. Parke and T. R. Taylor, “An amplitude for N gluon scattering,” Phys. Rev. Lett. 56 (1986) 2459; F. A. Berends and W. T. Giele, “Recursive Calculations For Processes With N Gluons,” Nucl. Phys. B306 (1988) 759.

[22] M. Mangano and S. J. Parke, “Multiparton Amplitudes In Gauge Theories,” Phys. Rep. 200 (1991) 301, hep-th/0509223.

[23] C. -J. Zhu, “The googly amplitude in gauge theory,” JHEP 0404 (2004) 032, hep-th/0403115.

[24] G. Georgiou and V. V. Khoze, “Tree Amplitudes in Gauge Theory as Scalar MHV Diagrams,” JHEP 0405 (2004) 070, hep-th/0404072;
G. Georgiou, E. W. N. Glover and V. V. Khoze, “Non-MHV Tree Amplitudes in Gauge Theory,” JHEP 0407 (2004) 048, hep-th/0407027;
V. V. Khoze, “Gauge Theory Amplitudes, Scalar Graphs and Twistor Space,” in “From Fields to Strings: Circumnavigating Theoretical Physics”, in memory of Ian Kogan, vol. 1, 622-657, hep-th/0408233;
S. D. Badger, E. W. N. Glover and V. V. Khoze, “MHV Rules for Higgs Plus Multi-Parton Amplitudes,” JHEP 0503 (2005) 023, hep-th/0412275;
S.D. Badger, E. W. N. Glover, V. V. Khoze and P. Svrcek, “Recursion Relations for Gauge Theory Amplitudes with Massive Particles,” JHEP 0507 (2005) 025, hep-th/0504159;
S. D. Badger, E. W. N. Glover and V. V. Khoze, “Recursion Relations for Gauge Theory Amplitudes with Massive Vector Bosons and Fermions,” JHEP 0601 (2006) 066, hep-th/0507161;
V. V. Khoze, “Amplitudes in the beta-deformed Conformal Yang-Mills, JHEP 0602 (2006) 040, hep-th/0512194.

[25] J.-B. Wu and C.-J. Zhu, “MHV Vertices And Scattering Amplitudes In Gauge Theory,” JHEP 0407, 032 (2004), hep-th/0406085; “MHV vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos,” JHEP 0409, 063 (2004), hep-th/0406146.
[26] I. Bena, Z. Bern and D. A. Kosower, “Twistor-Space Recursive Formulation of Gauge-Theory Amplitudes,” Phys. Rev. D71 (2005) 045008, hep-th/0406133;
D. A. Kosower, “Next-to-Maximal Helicity Violating Amplitudes in Gauge Theory,” Phys.Rev. D71 (2005) 04500768, hep-th/0406175;
I. Bena, Z. Bern, D. A. Kosower and R. Roiban, “Loops in Twistor Space,” Phys. Rev. D71 (2005) 106010, hep-th/0410054;
Z. Bern, V. Del Duca, L. J. Dixon and D. A. Kosower, “All Non-Maximally-Helicity-Violating One-Loop Seven-Gluon Amplitudes in N=4 Super-Yang-Mills Theory,” Phys. Rev. D71 (2005) 045006, hep-th/0410224;
S. J. Bidder, N. E. J. Bjerrum-Bohr, L. J. Dixon and D. C. Dunbar, “N=1 Supersymmetric One-loop Amplitudes and the Holomorphic Anomaly of Unitarity Cuts,” Phys. Lett. B606 (2005) 189-201, hep-th/0410296;
Z. Bern, D. Forde, D. A. Kosower and P. Mastrolia, “Twistor-Inspired Construction of Electroweak Vector Boson Currents,” Phys.Rev. D72 (2005) 025006, hep-ph/0412167;
Z. Bern, N. E. J. Bjerrum-Bohr and D. C. Dunbar, “Inherited Twistor-Space Structure of Gravity Loop Amplitudes,” JHEP 0505 (2005) 056, hep-th/0501137;
Z. Bern, L. J. Dixon and D. A. Kosower, “All Next-to-Maximally-Helicity-Violating One-Loop Gluon Amplitudes in N=4 Super-Yang-Mills Theory,” Phys. Rev. D72 (2005) 045014, hep-th/0412210.

[27] F. Cachazo, P. Svrcek and E. Witten, “Twistor Space Structure Of One-Loop Amplitudes In Gauge Theory,” JHEP 0410 (2004) 074, hep-th/0406177.

[28] A. Brandhuber, B. Spence and G. Travaglini, “One-Loop Gauge Theory Amplitudes In N = 4 Super Yang-Mills From MHV Vertices,” Nucl. Phys. B 706 (2005) 150-180, hep-th/0407214; “From Trees to Loops and Back,” JHEP 0601 (2006) 142, hep-th/0510253.

[29] F. Cachazo, “Holomorphic Anomaly Of Unitarity Cuts And One-Loop Gauge Theory Amplitudes,” hep-th/0410077.
[30] C. Quigley and M. Rozali, “One-Loop MHV Amplitudes in Supersymmetric Gauge Theories,” JHEP 0501 (2005) 053, hep-th/0410278.

[31] J. Bedford, A. Brandhuber, B. Spence and G. Travaglini, “A Twistor Approach to One-Loop Amplitudes in N=1 Supersymmetric Yang-Mills Theory,” Nucl. Phys. B706 (2005) 100-126, hep-th/0410280; “Non-Supersymmetric Loop Amplitudes and MHV Vertices,” Nucl. Phys. B712 (2005) 59-85, hep-th/0412108; “A recursion relation for gravity amplitudes,” Nucl. Phys. B721 (2005) 98-110, hep-th/0502146.

[32] R. Britto, F. Cachazo and B. Feng, “New Recursion Relations for Tree Amplitudes of Gluons,” Nucl. Phys. B715 (2005) 499-522, hep-th/0412308;
R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct Proof Of Tree-Level Recursion Relation In Yang-Mills Theory,” Phys. Rev. Lett. 94 (2005) 181602, hep-th/0501052.

[33] S. J. Bidder, N.E.J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, “Twistor Space Structure of the Box Coefficients of N=1 One-loop Amplitudes,” Phys. Lett. B608 (2005) 151-163, hep-th/0412023;
S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, “One-Loop Gluon Scattering Amplitudes in Theories with $N < 4$ Supersymmetries,” Phys. Lett. B612 (2005) 75-88, hep-th/0502028;
N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, “Six-Point One-Loop N=8 Supergravity NMHV Amplitudes and their IR behaviour,” Phys. Lett. B621 (2005) 183-194, hep-th/0503102.

[34] Z. Bern, L. J. Dixon and D. A. Kosower, “On-Shell Recurrence Relations for One-Loop QCD Amplitudes,” Phys. Rev. D71 (2005) 105013, hep-th/0501240. “Bootstrapping Multi-Parton Loop Amplitudes in QCD,” Phys. Rev. D73 (2006) 065013, hep-ph/0507005; “The Last of the Finite Loop Amplitudes in QCD,” Phys. Rev. D72 (2005) 125003, hep-ph/0505055.

[35] Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, “Recursive Calculation of One-Loop QCD Integral Coefficients,” JHEP 0511 (2005) 027, hep-ph/0507019.
[36] Z. Bern, L. J. Dixon and V. A. Smirnov, “Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond,” Phys. Rev. D72 (2005) 085001, hep-th/0505205.

[37] M. -X. Luo and C. -K. Wen, “One-Loop Maximal Helicity Violating Amplitudes in N=4 Super Yang-Mills Theories,” JHEP 0411 (2004) 004, hep-th/0410045; “Systematics of One-Loop Scattering Amplitudes in N=4 Super Yang-Mills Theories,” Phys. Lett. B609 (2005) 86-94, hep-th/0410118; “Recursion Relations for Tree Amplitudes in Super Gauge Theories,” JHEP 0503 (2005) 004, hep-th/0501121; “Compact Formulas for Tree Amplitudes of Six Partons,” Phys. Rev. D71 (2005) 091501, hep-th/0502009.

[38] R. Britto, F. Cachazo and B. Feng, “Generalized Unitarity and One-Loop Amplitudes in N=4 Super-Yang-Mills,” Nucl. Phys. B725 (2005) 275-305, hep-th/0412103;
R. Britto, F. Cachazo and Bo Feng, “Coplanarity In Twistor Space Of N=4 Next-To-MHV One-Loop Amplitude Coefficients,” Phys. Lett. B611 (2005) 167-172, hep-th/0411107;
R. Britto, B. Feng, R. Roiban, M. Spradlin and A. Volovich, “All Split Helicity Tree-Level Gluon Amplitudes,” Phys. Rev. D71 (2005) 105017, hep-th/0503198.

[39] R. Britto, E. Buchbinder, F. Cachazo, B. Feng, “One-Loop Amplitudes Of Gluons In SQCD,” Phys. Rev. D72 (2005) 065012, hep-ph/0503132.

[40] R. Britto, B. Feng and P. Mastrolia, “The Cut-Constructible Part of QCD Amplitudes,” Phys. Rev. D73 (2006) 105004, hep-ph/0602178.

[41] F. Cachazo and P. Svrcek, “Lectures on Twistor Strings and Perturbative Yang-Mills Theory,” PoS RTN2005 (2005) 004, hep-th/0504194.

[42] A. Brandhuber, S. McNamara, B. Spence and G. Travaglini, “Loop Amplitudes in Pure Yang-Mills from Generalised Unitarity,” JHEP 0510 (2005) 011, hep-th/0506068.

[43] X. Su and J. -B. Wu, “Six-Quark Amplitudes from Fermionic MHV Vertices,” Mod. Phys. Lett. A20 (2005) 1065-1076, hep-th/0409228.
[44] Y. Abe, V. P. Nair and M. Park, “Multigluon amplitudes, $\mathcal{N} = 4$ constraints and the WZW model,” Phys. Rev. D71 (2005) 025002, hep-th/0408191.

[45] C. Schwinn and S. Weinzierl, “Born amplitudes in QCD from scalar diagrams,” hep-th/0510054; “SUSY Ward identities for multi-gluon helicity amplitudes with massive quarks,” JHEP 0603 (2006) 030, hep-th/0602012; “Scalar diagrammatic rules for Born amplitudes in QCD,” hep-th/0503015.

[46] D. Forde and D. A. Kosower, “All-Multiplicity One-Loop Corrections to MHV Amplitudes in QCD,” Phys. Rev. D73 (2006) 061701, hep-ph/0509358.

[47] Yu-tin Huang, “$\mathcal{N}=4$ SYM NMHV Loop Amplitude in Superspace,” Phys. Lett. B631 (2005) 177-186, hep-th/0507117;
K. Risager, “A direct proof of the CSW rules,” JHEP 0512 (2005) 003, hep-th/0508206;
C. Quigley and M. Rozali, “Recursion relations, Helicity Amplitudes and Dimensional Regularization,” JHEP 0603 (2006) 004, hep-ph/0510148;
A. Gorsky and A. Rosly, “From Yang-Mills Lagrangian to MHV Diagrams,” JHEP 0601 (2006) 101, hep-th/0510111;
P. D. Draggiotis, R. H. P. Kleiss, A. Lazopoulos, C. G. Papadopoulos, “Diagrammatic proof of the BCFW recursion relation for gluon amplitudes in QCD,” hep-ph/0511288;
H. Kunitomo, “One-Loop Amplitudes in Supersymmetric QCD from MHV Vertices,” hep-th/0604210;

[48] Z. Bern, L. Dixon and D. A. Kosower, “One-loop amplitudes for $e^+e^-$ to four partons,” Nucl. Phys. B513 (1998) 3–86, hep-ph/9708239.

[49] Z. Bern and A. G. Morgan, “Massive Loop Amplitudes from Unitarity,” Nucl. Phys. B467 (1996) 479-509, hep-ph/9511336.

[50] For review see: Z. Bern, L. Dixon and D. A. Kosower, “Progress in One-Loop QCD Computations,” Ann. Rev. Nucl. Part. Sci. 46 (1996)
109–148, hep-ph/9602280; “Unitarity-based Techniques for One-Loop Calculations in QCD,” Nucl. Phys. Proc. Suppl. 51C (1996) 243–249, hep-ph/9606378.

[51] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans and T. T. Wu, “Single Bremsstrahlung Processes In Gauge Theories,” Phys. Lett. B103 (1981) 124; P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, “Multiple Bremsstrahlung In Gauge Theories At High-Energies. 1. General Formalism For Quantum Electrodynamics,” Nucl. Phys. B206 (1982) 53; R. Kleiss and W. J. Stirling, “Spinor Techniques For Calculating P Anti-P → W± / Z0 + Jets,” Nucl. Phys. B262 (1985) 235; R. Gastmans and T. T. Wu, The Ubiquitous Photon: Helicity Method For QED And QCD Clarendon Press, 1990.

[52] Z. Xu, D.-H. Zhang and L. Chang, “Helicity Amplitudes For Multiple Bremsstrahlung In Massless Nonabelian Theories,” Nucl. Phys. B291 (1987) 392.

[53] L. J. Dixon, “Calculating Scattering Amplitudes Efficiently,” hep-ph/9601359.

[54] Z. Bern, L. Dixon, D. C. Dunbar and D.A. Kosower, “One-Loop Self-Dual and N=4 Super Yang-Mills ,” Phys. Lett. B394 (1997) 105–115, hep-th/9611127.

[55] L. Dixon, “On-Shell Recursion Relations for QCD Loop Amplitudes,” presentation at “From Twistors to Amplitudes” at QMUL, 3-5 Nov. 2005.

[56] Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, “Recursive Approach to One-loop QCD Matrix Elements,” hep-ph/0603187.

[57] Z. Bern, “A bootstrap approach to loop amplitudes,” presentation at “8th DESY Workshop on Elementary Particle Theory – Loops and Legs in Quantum Field Theory”, Eisenach, 23-28 April, 2006.

[58] D. A. Kosower, “Complete QCD amplitudes,” presentation at “8th DESY Workshop on Elementary Particle Theory – Loops and Legs in Quantum Field Theory”, Eisenach, 23-28 April, 2006.
G. Passarino and M. J. G. Veltman, “One-loop Corrections for $e^+e^-$
anihilation into $\mu^+\mu^-$ in the Weinberg modes,” Nucl. Phys. B160 (1979) 151-207.

Z. Bern, L. Dixon and D. A. Kosower, “Dimensionally Regulated One-Loop Integrals,” Phys.Lett. B302 (1993) 299-308, Erratum-ibid. B318 (1993) 649, hep-ph/9212308; “Dimensionally Regulated Pentagon Integrals,” Nucl. Phys. B412 (1994) 751-816, hep-ph/9306240.

T. Binoth, J.Ph. Guillet, G. Heinrich, “Reduction formalism for dimensionally regulated one-loop N-point integrals,” Nucl.Phys. B572 (2000) 361-386, hep-ph/9911342.

G. Belanger, F. Boudjema, J. Fujimoto, T. Ishikawa, T. Kaneko, K. Kato and Y. Shimizu, “Grace at One-Loop: Automatic calculation of 1-loop diagrams in the electroweak theory with gauge parameter independence checks,” hep-ph/0308080.

R. Mertig, M. Böhm and A. Denner, Comp. Phys. Commun. 64 (1991) 345;
T. Hahn and M. Perez-Victoria, “Automatized One-Loop Calculations in 4 and D dimensions,” Comp. Phys. Commun. 118 (1999) 153-165, hep-ph/9807565;
T. Hahn, “Generating Feynman Diagrams and Amplitudes with FeynArts 3,” Comput. Phys. Commun. 140 (2001) 418-431, hep-ph/0012260.

P. Noguerira, J. Compt. Phys. 105 (1993) 279.

R. Harlander and M. Steinhauser, “Automatic computation of Feynman diagrams,” Prog. Part. Nucl. Phys. 43 (1999) 167-228.

R. Pittau, “A simple method for multi-leg loop calculations,” Comput. Phys. Commun. 104 (1997) 23–36, hep-ph/9607309; “A simple method for multi-leg loop calculations 2: a general algorithm,” Comput. Phys. Commun. 111 (1998) 48-52, hep-ph/9712418.
[67] S. Weinzierl, “Reduction of multi-leg loop integrals,” Phys. Lett. B450 (1999) 234–240, hep-ph/9811365; “The art of computing loop integrals,” hep-ph/0604068.

[68] X. Su, Z. -G. Xiao, G. Yang and C. -J. Zhu, “The rational parts of one-loop QCD amplitudes II: The 5-Gluon case,” hep-ph/0607016.

[69] Z. -G. Xiao, G. Yang and C. -J. Zhu, “The rational parts of one-loop QCD amplitudes III: The 6-Gluon case,” hep-ph/0607017.

[70] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, “All One-loop Maximally Helicity Violating Gluonic Amplitudes in QCD,” hep-ph/0607014.

[71] A. Berends and W. T. Giele, “Recursive Calculations For Processes With N Gluons,” Nucl. Phys. B306 (1988) 759.

[72] Z. Bern, A. De Freitas, L. Dixon and H. L. Wong “Supersymmetric Regularization, Two-Loop QCD Amplitudes and Coupling Shifts,” Phys. Rev. D66 (2002) 085002, hep-ph/0202271.