Domination games played on line graphs of complete multipartite graphs

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Abstract

The domination game on a graph $G$ (introduced by B. Brešar, S. Klavžar, D.F. Rall) consists of two players, Dominator and Staller, who take turns choosing a vertex from $G$ such that whenever a vertex is chosen by either player, at least one additional vertex is dominated. Dominator wishes to dominate the graph in as few steps as possible, and Staller wishes to delay this process as much as possible. The game domination number $\gamma_g(G)$ is the number of vertices chosen when Dominator starts the game; when Staller starts, it is denoted by $\gamma'_g(G)$.

In this paper, the domination game on line graph $L(K_m^n)$ of complete multipartite graph $K_m^n$ $(m \equiv (m_1, ..., m_n) \in \mathbb{N}^n)$ is considered, the exact values for game domination numbers are obtained and optimal strategy for both players is described. Particularly, it is proved that for $m_1 \leq m_2 \leq ... \leq m_n$ both $\gamma_g(L(K_m^n)) = \min \left\{ \left\lceil \frac{n}{2} \right\rceil |V(K_m^n)| \right\}$, $2 \max \left\{ \left\lceil \frac{n}{2} (m_1 + ... + m_{n-1}) \right\rceil, m_{n-1} \right\}$ $= 1$ when $n \geq 2$ and $\gamma'_g(L(K_m^n)) = \min \left\{ \left\lceil \frac{n}{2} (|V(K_m^n)| - 2) \right\rceil, 2 \max \left\{ \left\lceil \frac{n}{2} (m_1 + ... + m_{n-1} - 1) \right\rceil, m_{n-1} \right\} \right\}$ when $n \geq 4$.

Keywords. domination game; game domination number; line graph; complete multipartite graph; optimal strategy

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1 Introduction

We consider only finite undirected graphs without loops and multi-edges. The set of vertices of a graph $G$ is denoted by $V(G)$, and the set of edges of $G$, by $E(G)$. For a vertex $v \in V(G)$ the closed vertex neighborhood is denoted by $N[v] = \{ u \in V(G) : (u, v) \in E(G) \} \cup \{ v \}$ and for an edge $e \in E(G)$, the closed edge neighborhood by $N[e] = \{ e' \in E(G) : e \neq e', e$ and $e'$ are adjacent in $G \} \cup \{ e \}$. The line graph of a graph $G$, denoted by $L(G)$, is the graph with vertex set $E(G)$ in which two vertices are adjacent if and only if the respective edges of $G$ have a vertex in common, i.e. $V(L(G)) = E(G)$ and $E(L(G)) = \{ (e_1, e_2) : e_1 \in E(G), e_2 \in N[e_1], e_1 \neq e_2 \}$. A complete graph on $m$ vertices is denoted by $K_m$, and a complete $n$-partite ($n \geq 2$) graph with partite classes $V_1, V_2, ..., V_n$ of order $m_1, m_2, ..., m_n$ respectively is denoted by $K_{m_1, ..., m_n}$. Non-defined concepts can be found in [4].

According to the terminology of [1]-[3], we describe two vertex domination games and their edge-analogs played on a finite graph $G$. In Game $\mathcal{D}_v$, two players, Dominator and Staller, alternate taking turns choosing a vertex from $G$, with Dominator going first. Let $S$ denote the sequence of vertices $s_1, s_2, ..., s_i$ chosen by the players. These vertices must be chosen in such a way that whenever a vertex is chosen by either player, at least one additional vertex of the graph $G$ is dominated that was not dominated by the vertices previously chosen. That is, for each $i$:

$$N[s_i] \cup_{j=1}^{i-1} N[s_j] \neq \emptyset \quad (1 < i \leq |S|).$$

In Game $\mathcal{D}_e$, the players alternate choosing vertices satisfying to condition (1) as in Game $\mathcal{D}_v$, except that Staller begins. Since the graph $G$ is finite, each of the defined games will end in some finite number of moves regardless of how the vertices are chosen. In each of the games, Dominator chooses vertices using a strategy that will force the game to end in the fewest number of moves, and Staller uses a strategy that will prolong the game as long as possible. Following [1], we define the vertex game domination number of $G$, denoted by $\gamma_g(G)$, and the Staller-start vertex game domination number of $G$, denoted by $\gamma'_g(G)$, to be the total number of vertices chosen when they play respectively Game $\mathcal{D}_v$ and Game $\mathcal{D}_e$ on graph $G$ using optimal strategies.

In the Dominator-start edge domination game, denoted by Game $\mathcal{D}_e$, and in the Staller-start edge domination game, denoted by Game $\mathcal{D}_e'$, Dominator and Staller are taking edges instead, under the condition (1) where $S = s_1, s_2, ..., s_{|S|}$ is a sequence of chosen edges. Analogously, the edge game domination number of $G$, denoted by $\gamma_{e,g}(G)$, and the Staller-start edge domination number of $G$, denoted by $\gamma'_{e,g}(G)$, are the total numbers of edges chosen when they play respectively Game $\mathcal{D}_e$ and Game $\mathcal{D}_e'$ on graph $G$ using optimal strategies.

Remark 1. From definitions it immediately follows that $\gamma_g(L(G)) = \gamma_{e,g}(G)$ and $\gamma'_g(L(G)) = \gamma'_{e,g}(G)$ for every graph $G$.

A set of covered vertices, denoted by $C_{S,i}$, at step $i$ ($1 \leq i \leq |S|$) in an instance $S = s_1, s_2, ..., s_{|S|}$ of Game $\mathcal{D}_e$ played on a graph $G$ is defined as a union of endpoints of chosen edges $s_1, s_2, ..., s_i$. A vertex $v \in V(G)$ is called uncovered in $S$ at step $i$ ($1 \leq i \leq |S|$) if $v \notin C_{S,i}$. For short, put $C_S = C_{S,|S|}$ and $C_{S,0} = \Theta$. 

In Section 2, helper properties for edge domination games are given. In Section 3, the game domination number when at the end of the game at most one uncovered vertex remains is obtained, and as a corollary, an exact value of \( \gamma_g(L(K_m)) \) is calculated. In Section 4, an semi-greedy strategy for Staller for edge domination game played on complete multipartite graph is introduced. Through that strategy, the lower bound for domination number, when at the end of the game at least two uncovered vertices are left, is determined. Then from the equality of the obtained upper and lower bounds, by using result from Section 5, game domination number \( \gamma_g(L(K_m)) \) is obtained, and the optimality of semi-greedy strategy for Staller is shown. In Section 5, Staller-start game domination number \( \gamma_g'(L(K_m)) \) is determined.

## 2 Preliminaries and Basic Properties

Following [2], we use the following definitions. Let \( G \) be a graph on which several turns of the edge domination game have already been taken. We say that a edge \( e \) of \( G \) is dominated if some edge within \( N[e] \) has been played. A partially edge dominated graph \( G_A \) is a graph \( G \) in which we suppose that some edges \( A \subseteq E(G) \) have already been dominated, i.e. some moves have already been made, although we are concerned with which edges have thus far been dominated, rather than which have been chosen. If \( G_A \) is a partially edge dominated graph, then let \( \gamma_{e,g}(G_A) \) denote the number of turns remaining in the game if Dominator has the next move. Similarly, let \( \gamma'_{e,g}(G_A) \) denote the number of turns remaining if Staller has the next move.

On the basis of Remark 1, the Continuation Principle (see [2], Lemma 2.1) can be verbatim rewritten for partially edge dominated graphs.

**Proposition 1 (Continuation Principle).** Let \( G \) be a graph and \( A \subseteq B \subseteq E(G) \). If \( G_A \) and \( G_B \) are the partially edge dominated graphs corresponding to \( G \), with \( A \) dominated and with \( B \) dominated respectively, then \( \gamma_{e,g}(G_A) \geq \gamma_{e,g}(G_B) \) and \( \gamma'_{e,g}(G_A) \geq \gamma'_{e,g}(G_B) \).

**Proposition 2.** Let \( S \) be an instance of Game \( D_e \) played on a graph \( G \). Then the vertices of the set \( V(G) \setminus C_{S,i} \) \((1 \leq i \leq |S|)\) are independent in \( G \) if and only if game \( S \) is over, i.e. \( i = |S| \).

**Proof.** If \( V(G) \setminus C_{S,i} \) \((1 \leq i \leq |S|)\) is independent in \( G \) then all edges of \( G \) are dominated and game \( S \) is over, i.e. \( i = |S| \).

If \( v_1, v_2 \in V(G) \setminus C_{S,i} \) \((1 \leq i \leq |S|)\) and \( (v_1, v_2) \in E(G) \) then, since at step \( i \) there are no chosen edges adjacent to either \( v_1 \) or \( v_2 \), edge \((v_1, v_2)\) is not dominated at step \( i \). So, \( i < |S| \). Thus, \( V(G) \setminus C_{S,i} \leq |S| \) is independent in \( G \).

**Proposition 3.** For every graph \( G \) there exists an optimal strategy \( S \) for Game \( D_e \) played on \( G \) such that at each step Dominator chooses an edge which covers exactly two new vertices, i.e. for an arbitrary instance \( S \) of Game \( D_e \) played on \( G \) with strategy \( S \) and for each odd \( i \) \((1 \leq i \leq |S|)\), \(|C_{S,i} \setminus C_{S,i-1}| = 2 \).

**Proof.** Let at step \( i \) \((1 \leq i \leq \gamma_{e,g}(G))\) edges \( E_i \subseteq E(G) \) are dominated and Dominator by playing with an optimal strategy on move \( i \) chooses edge \( s_i \) which (by definition) dominating at least one new edge \( s'_i \). If edge \( s_i \) covers new vertices then strategy \( S \) will choose \( s_i \), otherwise Dominator will choose edge \( s'_i \) instead of edge \( s_i \), and since in that case \( E_i \cup N[s_i] \subseteq E_i \cup N[s'_i] \), due to the Continuation Principle (see Propositions 1), \( S \) is also optimal strategy.

Let \( dist(v, u) \) be the distance between vertices \( v, u \in V(G) \). The vertex-edge diameter of a connected graph \( G \) (with \( E \neq \emptyset \)) denoted by \( diam(G) \) is defined as:

\[
\text{diam}(G) = \max_{(v, u) \in E(G)} \min_{w \in V(G)} \{\text{dist}(v, w), \text{dist}(w, u)\}.
\] (2)

A strategy \( S \) for Game \( D_e \) is called a 2-1-strategy if on each move, Dominator covers exactly two new vertices and Staller covers exactly one, i.e. for an arbitrary instance \( S \) of Game \( D_e \) played on \( G \) with strategy \( S \) and for each \( i \) \((1 \leq i \leq |S|)\) both \(|C_{S,i} \setminus C_{S,i-1}| = 2 \) when \( i \) is odd and \(|C_{S,i} \setminus C_{S,i-1}| = 1 \) when \( i \) is even.

**Proposition 4.** For every connected graph \( G \) if \( diam(G) = 1 \) then there exists an optimal 2-1-strategy \( S \) for Game \( D_e \) played on \( G \).

**Proof.** Let Dominator plays with strategy \( S \) as described in proof of Propositions 5. Let \( E_i \subseteq E(G) \) be dominated edges at step \( i \) \((1 \leq i \leq \gamma_{e,g}(G))\), let \( v \) be a previously covered vertex and let Staller by playing with an optimal strategy on move \( i \) chooses edge \( s_i = (u, v) \). Since \( diam(G) = 1 \) either \((v, u) \in E(G) \) or \((v, w) \in E(G) \). If \( s_i \) covers one new vertex then in strategy \( S \) Staller will also choose \( s_i \), otherwise Staller will choose edge \( s'_i \) (either \( s'_i = (v, u) \) if \((v, u) \in E(G) \) or \( s'_i = (v, w) \) if \((v, w) \in E(G) \)) instead of edge \( s_i \), and since \( E_i \cup N[s'_i] \subseteq E_i \cup N[s_i] \), due to Continuation Principle (see Proposition 1), \( S \) is also optimal strategy.
Proposition 5. Let \( \gamma_{c,g}(G) \) be even. If there is an instance \( S \) of Game \( D_c \) played on a graph \( G \) with a 2-1-strategy then at the end of the game the number of uncovered vertices \( V(G)/C_S \) is not less than 1, i.e. \( |V(G)| - |C_S| \geq 1 \).

Proof. Since \( S \) played with a 2-1-strategy and the last move was made by Staller (because of \( \gamma_{c,g}(G) \) is even), then on the last move exactly one new vertex is covered, i.e. \( |C_{S,\gamma_{c,g}(G)}| = |C_{S,\gamma_{c,g}(G)-1}| + 1 \). Since Game \( D_c \) is not over at step \( \gamma_{c,g}(G) - 1 \), then due to Proposition 2 \( |C_{S,\gamma_{c,g}(G)}| \leq |V(G)| - 2 \). Thus, \( |C_{S,\gamma_{c,g}(G)}| \leq |V(G)| - 1 \). ☐

Proposition 6. Let \( S \) be an instance of Game \( D_c \) played on a graph \( G \) with an optimal 2-1-strategy and let \( S' \) be an instance played on \( G \) with a 2-1-strategy such that Dominator plays optimally. Then

\[
|C_{S'}| \leq |C_S|.
\] (3)

Proof. Since Dominator plays optimally in games \( S \) and \( S' \), and Staller plays optimally in game \( S \), it immediately follows that \( |S'| \leq |S| \). Since both \( S \) and \( S' \) are played with 2-1-strategies, then (3). ☐

Proposition 7. Let \( S \) and \( S' \) be instances of Game \( D_c \) played on graph \( G \) with 2-1-strategies. If \( |V(G)\setminus C_S| \leq 1 \) and \( |V(G)\setminus C_{S'}| \leq 1 \) then \( |S| = |S'| \).

Proof. Let \( |S| \neq |S'| \). Since \( |V(G)\setminus C_S| \leq 1 \) and \( |V(G)\setminus C_{S'}| \leq 1 \) then \( ||S_S| - |C_{S'}|| \leq 1 \). On the other hand, by Proposition 5 \( \max \{|S_S|, |S'_{S'}|\} \) is odd, as \( \min \{|V(G)\setminus C_S|, |V(G)\setminus C_{S'}|\} = 0 \). So, \( ||S_S| - |C_{S'}|| \geq 2 \) as \( S \) and \( S' \) are 2-1-strategies. Hence, the obtained contradiction proves the proposition. ☐

Proposition 8. If \( U \subset V(G) \) is an independent set in a connected graph \( G \) and \( M \subset E(G) \) is a matching in induced subgraph \( G[V(G)\setminus U] \), then

\[
\gamma_{c,g}(G) \leq 2(|V(G)\setminus U| - |M|) - 1.
\] (4)

Proof. Since Dominator at most with \( |M| + |V(G)\setminus U| - 2|M| \) steps dominates all edges of graph \( G \), then upper bound (4) holds immediately. ☐

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Lemma 1. If there is an instance \( S \) of Game \( D_c \) played on a graph \( G \) with an optimal 2-1-strategy such that \( |V(G)| - |C_S| \leq 1 \) then

\[
\gamma_{c,g}(G) = \left\lceil \frac{2}{3} |V(G)| \right\rceil - 1.
\]

Proof. Consider the following three cases.

Case 1. \( \gamma_{c,g}(G) \) is even.

From Proposition 3 it follows that \( |C_S| = |V(G)| - 1 \). Since \( S \) played with an optimal 2-1-strategy and \( \gamma_{c,g}(G) \) is even, then \( |C_S| = \frac{2}{3} \gamma_{c,g}(G) \). Hence, \( \gamma_{c,g}(G) = \frac{2}{3} (|V(G)| - 1) \). Accordingly, \( |V(G)| = 1 \) (mod 3).

Case 2. \( \gamma_{c,g}(G) \) is odd and \( |C_S| = |V(G)| - 1 \).

Since \( \gamma_{c,g}(G) \) is odd, \( |C_S| = \frac{1}{3} (|V(G)| - 1) + 2 \). Hence, \( \gamma_{c,g}(G) = \frac{2}{3} (|V(G)| - 2) + 1 \). So, \( |V(G)| = 2 \) (mod 3).

Case 3. \( \gamma_{c,g}(G) \) is odd and \( |C_S| = |V(G)| \).

Analogously, \( \gamma_{c,g}(G) = \frac{2}{3} |V(G)| - 1 \). So, \( |V(G)| = 0 \) (mod 3).

Therefore, (a) if \( |V(G)| = 0 \) (mod 3) then \( \gamma_{c,g}(G) = \frac{2}{3} |V(G)| - 1 \); (b) if \( |V(G)| = 1 \) (mod 3) then \( \gamma_{c,g}(G) = \frac{2}{3} |V(G)| - \frac{2}{3} \); and (c) if \( |V(G)| = 2 \) (mod 3) then \( \gamma_{c,g}(G) = \frac{2}{3} |V(G)| - \frac{1}{3} \).

Theorem 1. Let \( m \in \mathbb{N} \). Then

\[
\gamma_{c,g}(K_m) = \left\lceil \frac{2}{3} |V(K_m)| \right\rceil - 1.
\] (5)

Proof. Let \( m \geq 3 \). Since \( \text{diam}(K_m) = 1 \), from Proposition 4 it follows that there is an instance \( S \) for Game \( D_c \) played on \( K_m \) with an optimal 2-1-strategy. Then, due to Proposition 2 \( |C_S| \geq |V(K_m)| - 1 \). Hence, formula (5) immediately follows from Lemma 1. ☐
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Proposition 9. The vertex-edge diameter $\text{diam}(G)$, defined in formula (3), of a connected graph $G$ with $|E(G)| \geq 2$ is equal to 1 if and only if $G$ is a complete multipartite graph.

Proof. If $\overline{m} \in \mathbb{N}^n$ and $|E(K_m^n)| \geq 2$ then $\text{diam}(K_m^n) = 1$. Thus, the sufficiency is proved.

Let $\text{diam}(G) = 1$. A binary relationship $\alpha$ on $V(G)$ is defined as follows:

$$v \alpha u \Leftrightarrow (v, u) \notin E(G) \quad \forall v, u \in V(G).$$

It is trivial that $\alpha$ is reflexive (for every $v \in V(G)$) and symmetric ($v \alpha u \Leftrightarrow u \alpha v$ for every $v, u \in V(G)$). Assume $(v, u) \notin E(G)$ and $(u, w) \notin E(G)$ then $(v, w) \notin E(G)$. Otherwise, if $(v, w) \in E(G)$, then due to $\text{diam}(G) = 1$, either $(v, u) \in E(G)$ or $(u, w) \in E(G)$. Hence $\alpha$ is transitive. Thus, $\alpha$ is a relationship of equivalence. So, $V(G)$ can be partitioned into disjoint sets $U_1, ..., U_r$, such that $U_i \ (1 \leq i \leq r)$ is an independent set in $G$. Therefore, $G$ is isomorphic to $K_{(|U_1|, ..., |U_r|)}$. Thus, the necessity is proved. ■

Definition 1. Let $S$ be an instance of Game $D_e$ played on graph $K_m^n$ ($\overline{m} \in \mathbb{N}^n$, $n \geq 2$) and let for each $i \ (1 \leq i \leq |S|)$ (partition classes $V_1, ..., V_n$ of graph $K_m^n$ be renumbered as $V_{i_1}, ..., V_{i_n}$) to satisfy condition $|V_{i_1} \cap K_{S,i-1}| \leq \ldots \leq |V_{i_{n-1}} \cap K_{S,i-1}| \leq |V_{i_n} \cap K_{S,i-1}|$. Then say that Staller plays $S$ with a semi-greedy strategy if for each even $i \ (1 \leq i \leq |S|)$ Staller chooses an edge which covers exactly one new vertex $c_i$ which satisfies to following conditions:

$$c_i \in V_{i_1}, \quad \text{when } |V_{i_1} \cap K_{S,i-1}| > |V_{i_2} \cap K_{S,i-1}|,$$

$$c_i \in V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_n}, \quad \text{when } |V_{i_1} \cap K_{S,i-1}| = |V_{i_2} \cap K_{S,i-1}| \quad \text{and } V(K_m^n) \cap (V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_n}) \cap K_{S,i-1} \neq \varnothing,$$

$$c_i \in V_{i_1} \cup V_{i_2}, \quad \text{when } |V_{i_1} \cap K_{S,i-1}| = |V_{i_2} \cap K_{S,i-1}| \quad \text{and } V(K_m^n) \cap (V_{i_1} \cup V_{i_2}) \cap K_{S,i-1} = \varnothing.$$

Proposition 10 (Lower Bound). Let $n \geq 2$, $\overline{m} \in \mathbb{N}^n$, $m_1 \leq m_2 \leq \ldots \leq m_n$ and let $S$ be an instance of Game $D_e$ played on graph $K_m^n$ with a 2-1-strategy such that (a) Dominator plays with an optimal strategy and (b) Staller plays with a semi-greedy strategy. If at the end of the game the number of uncovered vertices $V(K_m^n) \setminus C_S$ is not less than 2 then

$$\gamma_{e,g}(K_m^n) \geq |S| \geq 2 \max \left\{ \left\lfloor \frac{1}{2} \sum_{j=1}^{n-1} m_j \right\rfloor, m_{n-1} \right\} - 1. \quad (6)$$

Proof. Since Proposition [2] there is partite class $V_i$ such that $V(K_m^n) \setminus C_S \subseteq V_i$.

Claim 1. The number $|S|$ is odd.

Proof. Let $|S|$ be even. Then at last step exactly one new vertex $w \in V(K_m^n) \setminus C_{S,|S|-1}$ is covered, as $S$ is played with a 2-1-strategy. By Proposition [2] $w \notin V_i$, i.e. there is an index $i'$ such that $i' \neq i$ and $w \in V_{i'}$. Since $|V_{i'} \setminus C_{S,|S|-1}| = 1$ and $|V_i \setminus C_{S,|S|-1}| \geq 2$, from Staller’s strategy it follows that in $S$ at last step must be chosen vertex from $V_i$ and game will not be over at step $|S|$. Thus, the obtained contradiction proves Claim 1. ■

Claim 2. For $i = 1, 3, ..., |S|$

$$|V_i \setminus C_{S,i}| > |V_k \setminus C_{S,i}| \quad k = 1, 2, ..., n; k \neq i. \quad (7)$$

Proof. Since $V(K_m^n) \setminus C_S \subseteq V_i$, (7) holds when $|S| = 1$. Hence, assume $|S| > 1$. Claim [2] when $|S| > 1$ will be proved by a contrary assumption. It is assumed there exist some even $p \ (1 < p < |S|)$ and partite class $V_p$ such that

$$|V_i \setminus C_{S,i}| > |V_k \setminus C_{S,i}| \quad k = 1, 2, ..., n; k \neq i; i = p + 1, p + 3, ..., |S|, \quad (8)$$

and

$$|V_i \setminus C_{S,p-1}| \leq |V_p \setminus C_{S,p-1}|. \quad (9)$$

From inequalities (6), due to Staller’s strategy, follows that

$$c_i \in V_i \quad i = p + 2, p + 4, ..., |S| - 1. \quad (10)$$

Let $f$ be the number of remaining moves for Staller to complete the game after $p^{th}$ move, i.e. $f = \frac{1}{2}(|S| - 1 - p)$, and since $|S|$ is odd, Dominator needs $f + 1$ moves to complete the game. On the strength of Staller’s strategy, consider the following three cases.
Case 1. \(|V_l \setminus C_{S,p-1}| < |V_l \setminus C_{S,p-1}|\).

Since (a) at each step Dominator can cover at most one vertex from the independent set \(V_l\), (b) at step \(p\) Staller can cover at most one vertex from \(V_l\) and (c) in remaining \(f\) moves Staller covers vertices only from \(V_l\) (see \((10)\)), then

\[ |V_l \setminus C_s| \geq |V_l \setminus C_{S,p-1}| - (f + 1) - 1. \quad (11) \]

and

\[ |V_l \setminus C_s| \leq |V_l \setminus C_{S,p-1}| - f. \quad (12) \]

Since \((11)\) and \((12)\),

\[ |V_l \setminus C_s| \geq |V_l \setminus C_{S,p-1}| - f - 2 \geq |V_l \setminus C_{S,p-1}| + 1 - f - 2 \geq |V_l \setminus C_s| - 1 \geq 1, \]

which contradicts to \(|V_l \setminus C_s| = 0\) (since \(V(K_m) \setminus C_s \subseteq V_l\)). Thus, case 1 is impossible.

Case 2. \(|V_l \setminus C_{S,p-1}| = |V_l \setminus C_{S,p-1}|\) and \(c_p \notin V_l\).

From \((10)\) it follows that \((12)\) holds. Since at each remaining step Dominator can cover at most one vertex from the independent set \(V_l\), if \((10)\) is taken into account, then

\[ |V_l \setminus C_s| \geq |V_l \setminus C_{S,p-1}| - (f + 1). \quad (13) \]

From inequalities \((12)\) and \((13)\) it follows that

\[ 0 = |V_l \setminus C_s| \geq |V_l \setminus C_{S,p-1}| - f - 1 \geq |V_l \setminus C_{S,p-1}| - f - 1 \geq |V_l \setminus C_s| - 1 \geq 1, \]

which is contradictory. Thus, case 2 is also impossible.

Case 3. \(|V_l \setminus C_{S,p-1}| = |V_l \setminus C_{S,p-1}|\) and \(c_p \in V_l\).

Since Staller’s strategy \(V(K_m) (V_l \cup V_l) \setminus C_{S,p-1} = \emptyset\), so at each step Dominator covers one vertex from both independent set \(V_l\) and independent set \(V_l\), if \((10)\) is taken into account, then

\[ |V_l \setminus C_s| = |V_l \setminus C_{S,p-1}| - f - (f + 1). \quad (14) \]

and \((11)\) holds. Inequalities \((11)\) and \((14)\) yield contradictory \(0 = |V_l \setminus C_s| \geq |V_l \setminus C_{S,p-1}| - f - 1 = |V_l \setminus C_{S,p-1}| - f - 2 = |V_l \setminus C_s| + f - 1 = 1 + f \geq 1. \)

Thus, case 3 is impossible as well.

Thus, the obtained contradictions prove Claim 2.

Claim 3. \(|V_l| = |V_n|\).

Proof. Since Claim 2, either \(n = l\) or \(|V_l \setminus C_{S,l}| > |V_n \setminus C_{S,l}|\). So, \(|V_l| = |V_n|\) as \(|V_n \cap C_{S,l}| \leq 1.\)

In virtue of Claim 3, assume that \(l = n\). So, from Claims 2 and 3 it follows that in \(S\) Stellar does not cover vertices from \(V(K_m) \setminus V_n\). Hence, on the one hand, since Dominator needs at least \(m_{n-1}\) steps to cover all vertices of independent set \(V_{n-1}\) to complete the game, \(\gamma_{e,g} (K_m) \geq |S| \geq 2m_{n-1} - 1. \)

On the other hand, as Dominator covers exactly two new vertices at each step, Dominator needs at least \( \left\lceil \frac{1}{2} |V(K_m) \setminus V_n| \right\rceil \) steps to cover all vertices of \(V(K_m) \setminus V_n\). So, \(\gamma_{e,g} (K_m) \geq |S| \geq 2 \left\lceil \frac{1}{2} |V(K_m) \setminus V_n| \right\rceil - 1. \)

Thus, lower bound \((13)\) holds.

Example 1. Let \(G\) be a graph with vertexes \(\{v_1, ..., v_7\}\) and edges \(\{(v_1, v_3), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_5, v_7)\}\).

Although, \(S = (v_3, v_4)(v_4, v_5)\) is an instance of Game \(D_s\) played on \(G\) with an optimal 2-1-strategy such that \(|V(G) \setminus C_s| = 4\) but \(|S|\) is even. So, Claim 1 does not work for \(S\) because of dima \((G) \neq 1\) and Staller could not play with semi-greedy strategy.

Proposition 11 (Upper Bound). Let \(n \geq 2, \sigma_m \in \mathbb{N}^n, m_1 \leq m_2 \leq ... \leq m_n.\) Then

\[ \gamma_{e,g} (K_m) \leq 2 \max \left\{ \left\lceil \frac{1}{2} \sum_{j=1}^{n-1} m_j \right\rceil, m_{n-1} \right\} - 1. \quad (15) \]

Proof. Put \(\sigma_0 \equiv 0\) and \(\sigma_k \equiv m_1 + ... + m_k\) for \(k = 1, ..., n\). On the one hand, if \(m_{n-1} \leq \sigma_{n-2}\) then in subgraph \(K_m (m_1, ..., m_{n-1})\) of \(K_m\) there is a matching with \(\left\lceil \frac{1}{2} \sigma_{n-1} \right\rceil\) edges (see 8). Hence, from Proposition 8 it follows that \(\gamma_{e,g} (K_m) \leq 2 \left( |V(K_m) \setminus V_n| - \left\lceil \frac{1}{2} \sigma_{n-1} \right\rceil \right) - 1 = 2 \left\lceil \frac{1}{2} \sigma_{n-1} \right\rceil - 1\) when \(m_{n-1} \leq \sigma_{n-2}\). On the other hand, if \(m_{n-1} > \sigma_{n-2}\) then in subgraph \(K_m (m_1, ..., m_{n-1})\) of \(K_m\) there is a matching with \(\sigma_{n-2}\) edges. Hence, from Proposition 8 it follows that \(\gamma_{e,g} (K_m) \leq 2 \left( |V(K_m) \setminus V_n| - \sigma_{n-2} \right) - 1 = 2m_{n-1} - 1\) when \(m_{n-1} > \sigma_{n-2}\). Thus, \(\gamma_{e,g} (K_m) \leq 2 \max \left\{ \left\lceil \frac{1}{2} \sigma_{n-1} \right\rceil, m_{n-1} \right\} - 1\) in both cases.
Theorem 2. Let $n \geq 2, \overline{m} \in \mathbb{N}^n$ and $m_1 \leq m_2 \leq \ldots \leq m_n$. Then

$$
\gamma_{e,g}(K_{\overline{m}}) = \min \left\{ \left[ \frac{2}{3} |V(K_{\overline{m}})| \right], 2 \max \left\{ \left[ \frac{n-1}{2} m_j \right], m_{n-1} \right\} \right\} - 1.
$$

Proof. Let $S$ be an instance of Game $D_e$ played on graph $K_{\overline{m}}$ with an optimal 2-1-strategy.

Case 1. $|V(K_{\overline{m}})\setminus C_S| \geq 2$.

Since $\text{diam}(K_{\overline{m}}) = 1$, there exists an instance $S'$ of Game $D_e$ played on graph $K_{\overline{m}}$ with a 2-1-strategy such that Dominator plays with an optimal strategy and Staller plays with a semi-greedy strategy. From Proposition 6 it follows that $|V(K_{\overline{m}})\setminus C_{S'}| \geq |V(K_{\overline{m}})\setminus C_S| \geq 2$. Hence, from Propositions 10 and 11 it follows that

$$
\gamma_{e,g}(K_{\overline{m}}) = 2 \max \left\{ \left[ \frac{n-1}{2} m_j \right], m_{n-1} \right\} - 1.
$$

Since $|S'|$ is odd (see Claim 1 inside Proposition 10), $\frac{2}{3}(|S'| - 1) + 2 = |C_{S'}| \leq |V(K_{\overline{m}})| - 2$. So, $\gamma_{e,g}(K_{\overline{m}}) = |S'| \leq \left[ \frac{2}{3} |V(K_{\overline{m}})| \right] - 1$.

Case 2. $|V(K_{\overline{m}})\setminus C_S| \leq 1$.

From Lemma 11 it follows that $\gamma_{e,g}(K_{\overline{m}}) = \left[ \frac{2}{3} |V(K_{\overline{m}})| \right] - 1$. Hence, from Proposition 10 follows that

$$
\left[ \frac{2}{3} |V(K_{\overline{m}})| \right] - 1 = \gamma_{e,g}(K_{\overline{m}}) \leq 2 \max \left\{ \left[ \frac{n-1}{2} m_j \right], m_{n-1} \right\} - 1.
$$

Thus, the proof is completed. ■

Corollary 1. Let $S$ be an instance of Game $D_e$ played on graph $K_{\overline{m}}$ ($n \geq 2$) with 2-1-strategy such that (a) Dominator plays with an optimal strategy and (b) Staller plays with a semi-greedy strategy. Then $|S| = \gamma_{e,g}(K_{\overline{m}})$.

Proof. From Propositions 10 and 11 it follows that $|S| = \gamma_{e,g}(K_{\overline{m}})$ when $|V(K_{\overline{m}})\setminus C_S| \geq 2$. Let $S'$ be an instance of Game $D_e$ played on graph $K_{\overline{m}}$ with an optimal 2-1-strategy. Hence, from Proposition 7 it follows that $|S| = |S'| = \gamma_{e,g}(K_{\overline{m}})$ when $|V(K_{\overline{m}})\setminus C_S| \leq 1$, as $|V(K_{\overline{m}})\setminus C_{S'}| \leq |V(K_{\overline{m}})\setminus C_S|$ due to Proposition 6. Thus, semi-greedy strategy is an optimal strategy for Staller for Game $D_e$ played on graph $K_{\overline{m}}$. ■

Example 2. Since Theorem 2 $\gamma_{e,g}(K_{2,2,6,0}) = 10$, $\gamma_{e,g}(K_{2,2,4,5}) = 7$ and $\gamma_{e,g}(K_{1,2,5,5}) = 8$. Hence, just ”greedy” strategy for Staller, when at each step Staller choose edge to cover vertex from some maximum independent set, is not optimal and it is expedient to use semi-greedy strategy instead.

Corollary 2. Let $n \geq 2, \overline{m} \in \mathbb{N}^n$, $m_1 \leq m_2 \leq \ldots \leq m_n$, let $M$ be a maximal matching in induced subgraph $K_{\overline{m}}[V(K_{\overline{m}})\setminus V_n]$ and let $S = s_1 \ldots s_{|S|}$ be an instance of Game $D_e$ played on graph $K_{\overline{m}}$ with 2-1-strategy such that Staller plays with an optimal strategy. If Dominator at each step $i$ ($i \leq |S|$) chooses edge from $M$ when $M \setminus \{s_1, \ldots, s_{i-1}\} \neq \emptyset$ then $|S| = \gamma_{e,g}(K_{\overline{m}})$.

Proof. Let $S'$ be an instance of Game $D_e$ played on graph $K_{\overline{m}}$ with an optimal 2-1-strategy. From Propositions 10 and 11 it follows that $|S'| = |S| = \gamma_{e,g}(K_{\overline{m}})$ when $|V(K_{\overline{m}})\setminus C_{S'}| \geq 2$. On the other hand, from Proposition 7 it follows that $|S| = |S'| = \gamma_{e,g}(K_{\overline{m}})$ when $|V(K_{\overline{m}})\setminus C_{S'}| \leq 1$, as $|V(K_{\overline{m}})\setminus C_S| \leq |V(K_{\overline{m}})\setminus C_{S'}|$ due to Proposition 6. ■

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Proposition 12. Let $n \geq 3, \overline{m} \in \mathbb{N}^n$ and $m_1 \leq m_2 \leq \ldots \leq m_n$. Then

$$
\gamma'_{e,g}(K_{\overline{m}}) = \min \left\{ \left[ \frac{2}{3} \left( |V(K_{\overline{m}})| - 2 \right) \right], 2 \max \left\{ \left[ \frac{1}{2} \left( \sum_{j=1}^{n-1} m_j - 1 - \eta \right) \right], m_{n-1} - \eta \right\} \right\},
$$

where $\mu$ equals 1 when $n = 3$ and $m_{n-1} = m_n$, and otherwise $\mu$ equals 0; and $\eta$ equals 1 when $m_{n-1} = m_n$, and otherwise $\eta$ equals 0.
Proof. For \( r = 1, \ldots, n \) put \( \overline{m}^{(r)} \equiv (m_1, \ldots, m_{r-1}, m_r - 1, m_{r+1}, \ldots, m_n) \). Since \( \gamma_{e,g}'(K_m) = \max_{1 \leq r < t \leq n} \left\{ \gamma_{e,g}(K_m^{(r,t)}) \right\} + 1 \), from Theorem 2 and from the equality
\[
\max_{1 \leq z \leq q} \left\{ \min \{a, \max \{b_z, c_z\}\} \right\} = \min \left\{ a, \max \left\{ \max_{1 \leq z \leq q} \{b_z\}, \max_{1 \leq z \leq q} \{c_z\}\right\} \right\} \quad \forall q \in \mathbb{N}; a, b_1, \ldots, b_q, c_1, \ldots, c_q \in \mathbb{R};
\]
it follows that (16) holds. \( \blacksquare \)

Theorem 3. Let \( n \geq 4 \), \( m \in \mathbb{N}^n \) and \( m_1 \leq m_2 \leq \ldots \leq m_n \). Then
\[
\gamma_{e,g}'(K_m) = \min \left\{ \left\lfloor \frac{2}{3} (|V(K_m)| - 2) \right\rfloor, 2 \max \left\{ \left\lceil \frac{1}{2} \sum_{j=1}^{n-1} m_j - 1 \right\rceil, m_{n-1} \right\} \right\}. \quad (17)
\]
Proof. From Proposition 12 it immediately follows that (17) holds when \( m_{n-1} \neq m_n \). Put \( \sigma_k \equiv m_1 + \ldots + m_k \) for \( k = 1, \ldots, n \). From \( m_{n-1} = m_n \) it follows that (a) if \( \frac{1}{2} (\sigma_{n-1} - 2) \leq m_{n-1} \) then \( \frac{2}{3} (\sigma_{n-1} + m_n - 2) \leq \frac{2}{3} (2m_{n-1} + 2 + m_n - 2) = 2m_{n-1} \) and (b) if \( \frac{1}{2} (\sigma_{n-1} - 2) > m_{n-1} \) then \( \frac{2}{3} (\sigma_{n-1} + m_n - 2) < \frac{2}{3} (\sigma_{n-1} + \frac{1}{2} (\sigma_{n-1} - 2) - 2) \leq 2 \left\lceil \frac{1}{2} (\sigma_{n-1} - 2) \right\rceil \). Thus, formulas (16) and (17) are equivalent when \( m_{n-1} = m_n \). \( \blacksquare \)

Remark 2. If \( m_2 \geq 1 \) then \( \gamma_{e,g}'(K_{m_1, m_2}) = 1 \) and if \( m_2 \geq m_1 \geq 2 \) then \( \gamma_{e,g}'(K_{m_1, m_2}) = \gamma_{e,g}(K_{m_1, 1, m_2 - 1}) + 1 \).

Remark 3. Since Proposition 12 \( \gamma_{e,g}'(K_{m_1, m_2, m_3}) = \min \left\{ \left\lfloor \frac{2}{3} (|V(K_{m_1, m_2, m_3})| - 2) \right\rfloor, 2(m_2 - 1) \right\} \) when \( m_1 \leq m_2 = m_3 \), and \( \gamma_{e,g}'(K_{m_1, m_2, m_3}) = \min \left\{ \left\lfloor \frac{2}{3} (|V(K_{m_1, m_2, m_3})| - 2) \right\rfloor, 2m_2 \right\} \) when \( m_1 \leq m_2 < m_3 \).

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References

[1] B. Brešar, S. Klavžar, D.F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979-991.

[2] B.Kinnersley, D. B. West, R. Zamani, Extremal problems for game domination number, SIAM J. Discrete Math., 27 (2013), 2090–2107.

[3] B. Brešar, S. Klavžar, D.F. Rall, Domination game played on trees and spanning subgraphs, Discrete Math. 313 (2013), 915–923.

[4] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.

[5] D. Sitton, Maximum matchings in complete multipartite graphs, Electronic Journal of Undergraduate Math. (1996) 6-16.