Mixed inequalities for commutators with multilinear symbol

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Abstract
We prove mixed inequalities for commutators of Calderón–Zygmund operators (CZO) with multilinear symbols. Concretely, let $m \in \mathbb{N}$ and $b = (b_1, b_2, \ldots, b_m)$ be a vectorial symbol such that each component $b_i \in \text{Osc}_{\exp L^r}$, with $r_i \geq 1$. If $u \in A_1$ and $v \in A_\infty(u)$ we prove that the inequality

$$uv \left( \left\{ x \in \mathbb{R}^n : \frac{|T_b(fv)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} \Phi \left( \|b\| \frac{|f(x)|}{t} \right) u(x)v(x) \, dx$$

holds for every $t > 0$, where $\Phi(t) = t(1 + \log^+ t)^r$, with $1/r = \sum_{i=1}^m 1/r_i$. We also consider operators of convolution type with kernels satisfying less regularity properties than CZO. In this setting, we give a Coifman type inequality for the associated commutators with multilinear symbol. This result allows us to deduce the $L^p(w)$-boundedness of these operators when $1 < p < \infty$ and $w \in A_p$. As a consequence, we can obtain the desired mixed inequality in this context.

Keywords Multilinear symbol · Commutators · Young functions · Muckenhoupt weights

Mathematics Subject Classification 42B20 · 42B25

1 Introduction
The problem of characterizing the nonnegative functions $w$ for which the Hardy–Littlewood maximal operator $M$ is bounded in $L^p(w)$, for $1 < p < \infty$, was first solved by Muckenhoupt in [11] and established the emergence of the well-known $A_p$ classes.

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Some years later, looking for an alternative proof of Muckenhoupt theorem, Sawyer showed in [17] that the inequality
\[ uv \left( \left\{ x \in \mathbb{R} : \frac{M(fv)(x)}{v(x)} > t \right\} \right) \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)|u(x)v(x) \, dx \]
holds if \( u \) and \( v \) are \( A_1 \) weights, for every positive \( t \). Notice that this inequality can be seen as a generalization of the weak (1, 1) type of \( M \), which corresponds to the case \( v = 1 \), characterized by \( A_1 \) weights. Thus, Muckenhoupt theorem can be obtained by combining the inequality above with interpolation techniques and the Jones factorization theorem. We shall refer to these type of estimates as mixed inequalities because they include two different weights.

A further extension of this inequality was established and proved in [4]. More precisely, it was shown that if \( u \) and \( v \) satisfy either \( u, v \in A_1 \) or \( u \in A_1 \) and \( v \in A_\infty(u) \), then the inequality
\[ uv \left( \left\{ x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t \right\} \right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|u(x)v(x) \, dx \]
holds for every positive \( t \) where \( T \) is either \( M \) or a Calderón–Zygmund operator (CZO). More general results were proved in [9] and in [2] by considering weaker conditions on the weights given by \( u \in A_1 \) and \( v \in A_\infty \). The former involves the operator \( M \) and the latter, the generalized maximal operator \( M_b \) associated to a Young function \( \Phi \) with additional properties.

In the next sections we state our main results concerning to Calderón–Zygmund and Hörmander type operators, separately. We shall be dealing with a linear operator \( T \), bounded on \( L^2(\mathbb{R}^n) \) and such that for \( f \in L^2 \) with compact support we have the representation
\[ Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y) \, dy, \quad x \notin \text{supp}(f), \quad (1.1) \]
where \( K \) is a measurable function defined away from the origin. Note that we are assuming this integral representation only for \( x \notin \text{supp}(f) \).

### 1.1 Commutators of CZO

Given a locally integrable function \( b \) and an operator \( T \) as in (1.1), the commutator of \( T \) is denoted by \( T_b \) or \([b, T] \) and defined by the expression
\[ [b, T]f(x) = bTf(x) - T(bf)(x). \quad (1.2) \]
For \( m \in \mathbb{N} \), the higher order commutator of \( T \) is given recursively by
\[ T_b^m f = [b, T_b^{m-1} f]. \]
In [3] we proved mixed inequalities for higher order commutators of Calderón–Zygmund operators. More precisely, given \( u \in A_1, v \in A_\infty(u), b \in \text{BMO}, m \in \mathbb{N} \) and a CZO \( T \), the inequality
holds for every positive $t$, where $\Phi(t) = t(1 + \log^+ t)^n$. Notice that condition $v \in A_\infty(u)$ implies that both $uv$ and $v$ belong to $A_\infty$. In this case many classical tools of Harmonic Analysis, like Calderón–Zygmund decomposition, can be used to achieve the desired estimates. On the other hand, when $v = 1$ inequality (1.3) was proved by Pérez in [12].

In this paper we study mixed inequalities for commutators associated to a multilinear symbol $b = (b_1, b_2, \ldots, b_m)$, where each $b_i$ is a locally integrable function. These operators are defined by

$$
T_{b'} = \left[b_1, T_{\overline{b}}f\right],
$$

(1.4)

where $\overline{b_i} = (b_2, \ldots, b_m)$ (see Sect. 3 for details). In fact, we shall prove that we can define $T_b$ by commuting any component $b_i$ with the corresponding operator $T_{\overline{b_i}}$ (see Corollary 11 below). If $b = (b_1)$ we simply write $T_b = T_{b_1}$ as in (1.2).

We describe now the space of functions considered for the components of $b$. Given $r \geq 1$ and $b \in L^1_{\text{loc}}$ we say that $b \in \text{Osc}_\text{exp}L^r$ if

$$
\|b\|_{\text{Osc}_\text{exp}L^r} = \sup_Q \|b - b_Q\|_{\text{exp}L^r, Q} < \infty
$$

(1.5)

where $Q$ denotes any cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $b_Q$ is the average of $b$ over $Q$ and $\|g\|_{\varphi,Q}$ denotes a $\varphi$-Luxemburg type average (see section below for details). Given $b = (b_1, \ldots, b_m)$ where $b_i \in \text{Osc}_\text{exp}L^r$, and $r_i \geq 1$ for every $1 \leq i \leq m$, we will denote

$$
\|b\| = \prod_{i=1}^m \|b_i\|_{\text{Osc}_\text{exp}L^{r_i}}.
$$

We are now in a position to state our main result concerning to mixed inequalities for $T_{b'}$.

**Theorem 1** Let $m \in \mathbb{N}$, $r_i \geq 1$ for $1 \leq i \leq m$ and $1/r = \sum_{i=1}^m 1/r_i$. Let $\Phi(t) = t(1 + \log^+ t)^{1/r}$ and $b = (b_1, b_2, \ldots, b_m)$ a multilinear symbol such that each $b_i \in \text{Osc}_\text{exp}L^r$. If $u \in A_1$, $v \in A_\infty(u)$ and $T$ is a CZO, then there exists a positive constant $C$ such that the inequality

$$
uv\left(\left\{x \in \mathbb{R}^n : \frac{|T_{b'}(f^v)(x)|}{v(x)} > t\right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{\|b\|_{\text{Osc}_\text{exp}L^r} \|f^v(x)\|}{t}\right)u(x)v(x) \, dx,
$$

holds for every $t > 0$.

Observe that if we consider $b_i = b \in \text{Osc}_\text{exp}L^r$ for every $1 \leq i \leq m$, then we get (1.3), since $\text{Osc}_\text{exp}L^r = \text{BMO}$.

### 1.2 Commutators of operators of Hörmander type

Let $T$ be a linear operator as in (1.1). We shall consider kernels $K$ with less regularity properties associated to a Young function $\varphi$, that will be called $H_{\varphi,m}$, where $m \in \mathbb{N}$ and $H_{\varphi,0} = H_\varphi$ (for further details see Sect. 2).
In this section we state mixed inequalities involving the commutators of these type of operators with multilinear symbol \( b = (b_1, \ldots, b_m) \). Our main result is contained in the following theorem.

**Theorem 2** Let \( 1 < q < \infty \) and \( q^2/(2q-1) < \beta < q \). Assume that \( u \in A_1 \cap RH_s \) for some \( s > 1 \) and \( v^d \in A_{\alpha/(2q)}(u) \), with \( \alpha = \beta(q-1)/(q-\beta) \). Let \( m \in \mathbb{N}, r_i \geq 1 \) for every \( 1 \leq i \leq m \), \( 1/r = \sum_{i=1}^{m} 1/r_i \) and \( b = (b_1, b_2, \ldots, b_m) \) where \( b_i \in \text{Osc}_{exp} L^\infty \), for \( 1 \leq i \leq m \). If \( T \) is an operator with kernel \( K \in H_{\phi,m} \), where \( \tilde{\eta}^{-1}(t)\phi^{-1}(t)(\log t)^{1/r} \lesssim t \) for \( t \geq e \) and \( \tilde{\eta} \in B_\rho \) for every \( \rho \geq \min(\beta, s) \), then there exists \( C > 0 \) such that the inequality

\[
uv \left( \left\{ x \in \mathbb{R}^n : \left| \frac{T_b(fv)(x)}{v(x)} \right| > t \right\} \right) \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\|b\|\|f(x)\|}{t} \right) u(x)v(x) \, dx
\]

holds for every positive \( t \), and where \( \Phi(t) = t(1 + \log^+ t)^{1/r} \).

**Remark 1** Notice that the hypothesis \( K \in H_{\phi,m} \) implies that \( K \in H_{\eta,m} \), since we are assuming the relation

\[
\tilde{\eta}^{-1}(t)\phi^{-1}(t)(\log t)^{1/r} \lesssim t \quad \text{for} \quad t \geq e.
\]

When we consider a single symbol \( b \in \text{BMO} \) the result above was proved by the authors in [3]. The proof involves the Calderón–Zygmund decomposition combined with the boundedness of \( T_b^m \) on \( L^p(w) \) for \( 1 < p < \infty \) and \( w \in A_p \). As far as we know, this last result is unknown for the commutators of \( T \) when \( b \) is a multilinear symbol with components belonging to the spaces \( \text{Osc}_{exp} L^r \), \( r \geq 1 \). We prove this result in the next theorem.

**Theorem 3** Let \( m \in \mathbb{N}, r_i \geq 1 \) for every \( 1 \leq i \leq m \) and \( 1/r = \sum_{i=1}^{m} 1/r_i \). Let \( 1 \leq \beta < p \) and \( \eta, \phi \) be two Young functions verifying \( \tilde{\eta}^{-1}(t)\phi^{-1}(t)(\log t)^{1/r} \lesssim t \), for every \( t \geq e \). Let \( T \) be an operator with kernel \( K \in H_{\phi,m} \) and \( b = (b_1, b_2, \ldots, b_m) \) where \( b_i \in \text{Osc}_{exp} L^\infty \), for \( 1 \leq i \leq m \). If \( \tilde{\eta} \in B_\rho \) for \( \rho > \beta \) and \( w^d \in A_{p/\beta} \), then there exists a positive constant \( C \) such that

\[
\| (T_b^m f)w \|_{L^p} \leq C \| b \| \| f \|_{L^p},
\]

provided the left-hand side is finite.

When \( b \) is a single symbol belonging to \( \text{BMO} \) this result was proved in [10]. On the other hand, if \( T \) is a CZO and \( b \) as above an analogous result is contained in [15]. The theorem above is a consequence of a Coifman type estimate that we shall state in Sect. 5.

The article is organized as follows. In Sect. 2 we give the previous definitions and basic results. In Sect. 3 we prove some technical facts concerning commutators with a multilinear symbol \( b \). Finally, in Sects. 4 and 5 we prove the results for CZO and Hörmander type operators, respectively.

**2 Preliminaries and basic definitions**

In this section we provide the concepts and basic results required for the reading of this work.
A weight \( w \) is a locally integrable function defined on \( \mathbb{R}^n \), such that \( 0 < w(x) < \infty \) in a.e. \( x \in \mathbb{R}^n \). For \( 1 < p < \infty \) the Muckenhoupt \( A_p \) class is defined as the set of all weights \( w \) for which there exists a positive constant \( C \) such that the inequality

\[
\left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} \leq C
\]

holds for every cube \( Q \subset \mathbb{R}^n \) with sides parallel to the coordinate axes. For \( p = 1 \) we say that \( w \in A_1 \) if there exists a positive constant \( C \) such that

\[
\frac{1}{|Q|} \int_Q w \leq C \inf_Q w(x),
\]

for every cube \( Q \subset \mathbb{R}^n \). The smallest constant \( C \) for which the Muckenhoupt condition holds is called the characteristic \( A_p \)-constant of \( w \), and denoted by \( [w]_{A_p} \). The \( A_{\infty} \) class is defined by the collection of all the \( A_p \) classes. It is easy to see that if \( p < q \) then \( A_p \subseteq A_q \).

Given weights \( u \) and \( w \) we say \( w \in A_p(u) \), \( 1 \leq p \leq \infty \), if the above inequalities hold with the Lebesgue measure replaced by \( d\mu(x) = u(x) \, dx \). The corresponding characteristic constants are denoted by \( [w]_{A_p(u)} \). The following result about \( A_p(u) \) classes was proved in [4].

**Lemma 4** If \( 1 \leq p \leq \infty \), \( u \in A_1 \) and \( v \in A_p(u) \), then \( uv \in A_p \).

The following result will be key for some estimates in our main results. The proof can be found in [3, p. 538].

**Lemma 5** Let \( u \in A_1 \) and \( v \) such that \( uv \in A_\infty \). Then there exists a positive constant \( C \) such that for every cube \( Q \)

\[
\frac{uv(Q)}{v(Q)} \leq C \inf_Q u.
\]

An important property of Muckenhoupt weights is the reverse Hölder condition. Given \( 1 < s < \infty \) we say that \( w \in RH_s \) if there exists a positive constant \( C \) such that for every cube \( Q \)

\[
\left( \frac{1}{|Q|} \int_Q w^s(x) \, dx \right)^{1/s} \leq C \frac{1}{|Q|} \int_Q w(x) \, dx.
\]

We say \( w \) belongs to \( RH_\infty \) if there exists a positive constant \( C \) such that

\[
\sup_Q w \leq C \frac{1}{|Q|} \int_Q w,
\]

for every \( Q \subset \mathbb{R}^n \). We denote by \( [w]_{RH} \) the smallest constant \( C \) for which the conditions above hold. Notice that \( RH_\infty \subseteq RH_s \subseteq RH_q \) for every \( 1 < q < s \).

We say that \( \varphi : [0, \infty) \to [0, \infty] \) is a Young function if it is strictly increasing, convex, \( \varphi(0) = 0 \) and \( \varphi(t) \to \infty \) when \( t \to \infty \). Given a Young function \( \varphi \) and a Muckenhoupt weight \( w \), the generalized maximal operator \( M_{\varphi,w} \) is defined by...
where \( \|f\|_{\phi,Q,w} \) denotes the weighted Luxemburg average over \( Q \) defined by
\[
\|f\|_{\phi,Q,w} := \inf \left\{ \lambda > 0 : \frac{1}{w(Q)} \int_Q \phi \left( \frac{|f|}{\lambda} \right) w \leq 1 \right\}.
\]
(2.1)

When \( w = 1 \) we simply denote \( \|f\|_{\phi,Q,w} = \|f\|_{\phi,Q} \). We use \( \overline{\phi} \) to denote the complementary function associated to \( \phi \), defined for \( t \geq 0 \) by
\[
\overline{\phi}(t) = \sup \{ ts - \phi(s) : s \geq 0 \}.
\]

It is well known that \( \overline{\phi} \) satisfies
\[
t \leq \phi^{-1}(t) \overline{\phi}^{-1}(t) \leq 2t, \quad \forall t > 0,
\]
where \( \phi^{-1} \) denotes the generalized inverse of \( \phi \), defined by
\[
\phi^{-1}(t) = \inf \{ s \geq 0 : \phi(s) \geq t \},
\]
where we understand \( \inf \emptyset = \infty \).

We say that \( \phi \) belongs to \( B_p \), \( 1 < p < \infty \), if there exists \( c > 0 \) satisfying
\[
\int_c^\infty \frac{\phi(t)}{t^p} \frac{dt}{t} < \infty.
\]
(2.2)

These classes were introduced in [13] and it was shown that \( \phi \in B_p \) if and only if \( M_\phi \) is bounded on \( L^p(\mathbb{R}^m) \). Observe that if \( 1 < p \leq q < \infty \), then \( B_p \subseteq B_q \).

The following lemma contains a useful fact about \( M_\phi \) and \( A_1 \) weights.

**Lemma 6** ([3], Lemma 15) Let \( s > 1 \), \( u \in A_1 \cap RH_s \) and \( \varphi \) a Young function in \( B_s \). Then there exists a positive constant \( C \) such that \( M_\varphi u(x) \leq Cu(x) \), for almost every \( x \).

We will be dealing with functions of the type \( \Phi(t) = t(1 + \log^+ t)^{\delta} \), with \( \delta > 0 \). It is well-known that
\[
\Phi^{-1}(t) \approx \frac{t}{(1 + \log^+ t)^{\delta}} \quad \text{and} \quad \overline{\Phi}(t) \approx \left( e^{t^{1/\delta}} - e \right) X_{(1,\infty)}(t).
\]
(2.3)

When \( \delta = 1/r \), \( r \geq 1 \), we shall refer to the average \( \|f \cdot \|_{\phi,Q,w} \) as \( \|f \cdot \|_{\exp L^r,Q,w} \).

The generalized Hölder inequality
\[
\frac{1}{w(Q)} \int_Q |fg| w \lesssim \|f\|_{\phi,Q,w}\|g\|_{\phi,Q,w}
\]
holds for every Muckenhoupt weight \( w \) and every Young function \( \varphi \). More generally, if \( \varphi \), \( \phi \) and \( \eta \) are Young functions such that the inequality
\[
\phi^{-1}(t) \eta^{-1}(t) \leq C \varphi^{-1}(t)
\]
holds for every \( t \geq t_0 > 0 \), then

\[\text{ Springer}\]
\[
\|fg\|_{\varphi, Q, w} \leq C \|f\|_{\varphi, Q, w} \|g\|_{\eta, Q, w}.
\]

We can also replace the cube \( Q \) above with any bounded measurable set \( E \).

The next lemma is a variant of Lemma 11 in [3], where it was shown for the case \( r = 1 \). Its proof can be achieved with minor modifications and we will omit it.

**Lemma 7** Let \( r \geq 1 \) and \( w \in RH_s \), for some \( s > 1 \). Then there exists \( C > 0 \) such that

\[
\|f\|_{\exp L^r, Q, w} \leq C \|f\|_{\exp L^r, Q}
\]

for every cube \( Q \).

The following fact is a well-known property satisfied for \( BMO \) symbols. A proof for the \( \Oscexp L^r \) case can be found in [15].

**Lemma 8** Let \( r \geq 1 \) and \( b \in \Oscexp L^r \). For every \( k \in \mathbb{N} \) and every cube \( Q \) we have that

\[
|b_Q - b_{2^kQ}| \leq Ck \|b\|_{\Oscexp L^r}.
\]

The next is a purely technical result, which will be useful in some of our estimates. The proof can be performed by induction and we shall omit it.

We shall introduce some useful notation before state this result. Given \( m \in \mathbb{N} \), we denote with \( S_m \) the

\[
S_m = \{0, 1\}^m = \{ \sigma \in \mathbb{R}^m : \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m), \text{ where } \sigma_i = 0 \text{ or } \sigma_i = 1 \}.
\]

Notice that \( S_m \) has exactly \( 2^m \) elements. We shall write this as \( \#S_m = 2^m \). For \( \sigma \in S_m \) we define \( |\sigma| = \sum_{i=1}^m \sigma_i \) and \( \tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_m) \), where

\[
\tilde{\sigma}_i = \begin{cases} 0 & \text{if } \sigma_i = 1, \\ 1 & \text{if } \sigma_i = 0. \end{cases}
\]

**Lemma 9** Let \( a_i \) and \( b_i \) real numbers for \( 1 \leq i \leq m \). Then

\[
\prod_{i=1}^m (a_i + b_i) = \sum_{\sigma \in S_m} \prod_{i=1}^m a_i^{\sigma_i} b_i^{\tilde{\sigma}_i}.
\]

Let \( T \) be as in (1.1). Recall that \( T \) is a CZO if \( K \) is a standard kernel, which means that \( K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) satisfies a size condition given by

\[
|K(x)| \lesssim \frac{1}{|x|^n},
\]

and the smoothness conditions, usually called Lipschitz conditions,

\[
|K(x-y) - K(x-z)| \lesssim \frac{|x-z|}{|x-y|^{n+1}}, \quad \text{if } |x-y| > 2|y-z|,
\]

(2.5)

We write \( f(t) \lesssim g(t) \) when there exists a positive constant \( C \) such that \( f(t) \leq Cg(t) \) for every \( t \) in the domain. Throughout the paper, the constant \( C \) may change on each occurrence.
We shall also deal with kernels with less regularity properties than (2.5). Given a Young function \( \varphi \), we denote
\[
\|f\|_{\varphi,[|x|^{-s}]} = \|f \tilde{\mathcal{X}}_{[|x|^{-s}]}\|_{\varphi,B(0,2s)},
\]
where \( |x| \sim s \) means \( s < |x| \leq 2s \).

We say that \( K \) satisfies the \( L^p \)–Hörmander condition and we denote it by \( K \in H_\varphi \) if there exist constants \( c \geq 1 \) and \( C_\varphi > 0 \) such that, for every \( y \in \mathbb{R}^n \) and \( R > c|y| \)
\[
\sum_{k=1}^{\infty} (2^k R)^n \|K(\cdot - y) - K(\cdot)\|_{\varphi,[|x|^{-2^k R}]} \leq C_\varphi. \tag{2.6}
\]
We also say that \( K \in H_{\varphi,m} \) for \( m \in \mathbb{N} \) if there exist constants \( c \geq 1 \) and \( C_{\varphi,m} > 0 \) such that the inequality
\[
\sum_{k=1}^{\infty} (2^k R)^m \|K(\cdot - y) - K(\cdot)\|_{\varphi,[|x|^{-2^k R}]} \leq C_{\varphi,m}. \tag{2.7}
\]
holds for every \( y \in \mathbb{R}^n \) and \( R > c|y| \). It is immediate from the definition that \( H_{\varphi,m} \subset H_{\varphi,1} \) for every \( 0 \leq i \leq m \).

3 Some useful facts about \( T_b \)

We devote this section to prove some technical results concerning the representation of commutators with multilinear symbols. Although the most part of them do not rely on specific properties of \( T \), we shall assume that \( T \) is a linear operator.

If \( b = (b_1, \ldots, b_m) \) and \( \sigma \in S_m \) we denote by \( b_\sigma \) the symbol with \( |\sigma| \) components corresponding to those indices \( i \) for which \( \sigma_i = 1 \). For example, for \( m = 4 \), \( b = (b_1, b_2, b_3, b_4) \) and \( \sigma = (1, 0, 1, 0) \), we have \( b_\sigma = (b_1, b_3) \). When we need to emphasize that \( b_\sigma \) has length \( k \) we will write \( b^k_\sigma \). Also, we shall use \( \tilde{b}_i \) to denote the multilinear symbol containing all the components of \( b \) except \( b_i \), this is
\[
\tilde{b}_i = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_m).
\]

**Proposition 10** Let \( b = (b_1, b_2, \ldots, b_m) \) and \( \lambda_i \) be fixed constants for \( 1 \leq i \leq m \). Then
\[
T_b f(x) = \sum_{\sigma \in S_m} (-1)^{|\sigma|-|\lambda|} \left( \prod_{i=1}^{m} [b_i(x) - \lambda_i]^{\sigma_i} \right) T \left( \prod_{i=1}^{m} [b_i - \lambda_i]^{\sigma_i} f \right) (x). \tag{3.1}
\]

**Proof** We proceed by induction on \( m \). If \( m = 1 \) then \( b = (b_1) \). In this case, the right-hand side of (3.1) becomes
\[
(b_1(x) - \lambda_1) T f(x) - T \left( (b_1 - \lambda_1) f \right) (x) = b_1(x) T f(x) - T(b_1 f)(x) = T_b f(x),
\]
since \( T \) is linear.

Assume now that the representation holds for every multilinear symbol with \( k \) components and let \( b = (b_1, b_2, \ldots, b_{k+1}) \). The right-hand side of (3.1) can be split in two terms \( A + B \), where
By writing these sums over elements of $S_{n+1}$ we have that
\[
A = \sum_{\sigma \in S_{n+1} : \sigma_1 = 1} (-1)^{k+1-|\sigma|} \left( b_1(x) - \lambda_1 \right) \left( \prod_{i=2}^{k+1} [b_i(x) - \lambda_i]^{\sigma_i} \right) T \left( \prod_{i=2}^{k+1} [b_i - \lambda_i]^{\sigma_i} \right) f(x)
\]

and
\[
B = \sum_{\sigma \in S_{n+1} : \sigma_1 = 0} (-1)^{k+1-|\sigma|} \left( \prod_{i=2}^{k+1} [b_i(x) - \lambda_i]^{\sigma_i} \right) T \left( \prod_{i=2}^{k+1} [b_i - \lambda_i]^{\sigma_i} \right) f(x).
\]

Now, by using the inductive hypothesis, we have that
\[
A = \left( b_1(x) - \lambda_1 \right) \left[ \sum_{\theta \in S_k} (-1)^{k-|\theta|} \left( \prod_{i=2}^{k+1} [b_i(x) - \lambda_i]^{\theta_i} \right) T \left( \prod_{i=2}^{k+1} [b_i - \lambda_i]^{\theta_i} \right) f \right] = (b_1(x) - \lambda_1) T_{b_{\theta}} f(x).
\]

On the other hand, since $T$ is linear
\[
B = T_{b_{\theta}} f(x) + \lambda_1 T_{b_{\theta}} f(x).
\]

Finally, we obtain that
\[
A + B = b_1(x) T_{b_{\theta}} f(x) - T_{b_{\theta}}(b_1 f)(x) = \left[ b_1, T_{b_{\theta}} f \right](x) = T_{b_{\theta}} f(x),
\]

which completes the proof. \(\square\)

**Remark 2** If we take $\lambda_i = 0$ for every $i$ in the result above, we obtain
\[
T_{b_{\theta}} f(x) = \sum_{\sigma \in S_m} (-1)^{m-|\sigma|} \left( \prod_{i=1}^{m} [b_i(x)]^{\sigma_i} \right) T \left( \prod_{i=1}^{m} [b_i]^{\sigma_i} \right) f(x).
\]

(3.2)

**Corollary 11** Given $1 \leq \ell \leq m$ and $b = (b_1, b_2, \ldots, b_m)$ we have that
\[
T_{b_{\ell}} f(x) = \left[ b_{\ell}, T_{b_{\theta}} f \right](x).
\]

**Proof** By virtue of (3.2) we have that
\[
T_{b_{\ell}} f(x) = \sum_{\sigma \in S_m} (-1)^{m-|\sigma|} \left( \prod_{i=1}^{m} [b_i(x)]^{\sigma_i} \right) T \left( \prod_{i=1}^{m} [b_i]^{\sigma_i} \right) f(x) = \sum_{\sigma \in S_m : \sigma_\ell = 0} + \sum_{\sigma \in S_m : \sigma_\ell = 1}
\]

By writing these sums over elements of $S_{m-1}$ we get
We shall deal with linear operators \( T \) such that \( Tf(x) \) has an integral representation for adequate values of \( x \). The following proposition states that its commutators have the same property. The proof is straightforward and we shall omit it.

**Proposition 12** Let \( T \) be linear, \( m \in \mathbb{N} \) and \( \mathbf{b} = (b_1, \ldots, b_m) \) be a multilinear symbol. Then

(a) \( T_{\mathbf{b}} \) is a linear operator;
(b) if \( T \) has the representation

\[
Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y) \, dy
\]

for every \( x \notin \text{supp}(f) \), then

\[
T_{\mathbf{b}}f(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} (b_i(x) - b_i(y)) K(x-y)f(y) \, dy
\]

for these \( x \).

The following result states a representation for \( T_{\mathbf{b}} \) by means of lower order commutators.

**Proposition 13** Let \( T \) be an operator with the representation given by (3.3) for \( x \notin \text{supp}(f) \) and \( \mathbf{b} = (b_1, \ldots, b_m) \). If \( \lambda_i \) are fixed constants for every \( 1 \leq i \leq m \) and \( x \notin \text{supp}(f) \), then

\[
T_{\mathbf{b}}f(x) = \left( \prod_{i=1}^{m} (b_i(x) - \lambda_i) \right) T f(x) - T \left( \left( \prod_{i=1}^{m} (b_i - \lambda_i) \right) f \right)(x)
\]

\[
- \sum_{\sigma \in S_m, 0 < |\sigma| < m} T_{\mathbf{b}_\sigma} \left( \left( \prod_{i=1}^{m} (b_i - \lambda_i)^{\sigma_i} \right) f \right)(x).
\]

**Proof** By induction on \( m \), if \( \mathbf{b} = (b_1) \) by applying Proposition 10 we have that

\[
T_{\mathbf{b}}f(x) = (b_1(x) - \lambda_1) Tf(x) - T((b_1 - \lambda_1)f)(x),
\]

which implies the thesis since the set

\[
\{ \sigma \in S_1 : 0 < |\sigma| < 1 \}
\]

is empty. Assume now that the equality holds for every multilinear symbol with \( k \) components and let \( \mathbf{b} = (b_1, \ldots, b_k, b_{k+1}) \). From Corollary 11 and Proposition 12 we can write

\[
T_{\mathbf{b}}f(x) = [b_{k+1}, T_{\mathbf{b}_{k+1}} f](x) = (b_{k+1}(x) - \lambda_{k+1}) T_{\mathbf{b}_{k+1}} f(x) - T_{\mathbf{b}_{k+1}} ((b_{k+1} - \lambda_{k+1}) f)(x),
\]

where \( \mathbf{b}_{k+1} = (b_1, \ldots, b_k) \). By the inductive hypothesis we write the first term as
(b_{k+1}(x) - \lambda_{k+1})T_{b_{k+1}}f(x) = \left( \prod_{i=1}^{k+1} (b_i(x) - \lambda_i) \right) Tf(x)
- (b_{k+1}(x) - \lambda_{k+1})T\left( \prod_{i=1}^{k} (b_i(x) - \lambda_i) \right)f(x)
- \sum_{\theta \in S_k, 0 < |\theta| < k} (b_{k+1}(x) - \lambda_{k+1})T_{b_{\theta}}\left( \prod_{i=1}^{k} (b_i(x) - \lambda_i) \right)f(x)
= A + B + C.

We shall use Proposition 12 to estimate some of the previous terms. We have
\[
B = - \int_{\mathbb{R}^n} (b_{k+1}(x) - b_{k+1}(y) + b_{k+1}(y) - \lambda_{k+1})\left( \prod_{i=1}^{k} (b_i(y) - \lambda_i) \right)K(x-y)f(y) \, dy
= -T_{b_{k+1}}\left( \prod_{i=1}^{k} (b_i(x) - \lambda_i) \right)f(x) - T\left( \prod_{i=1}^{k+1} (b_i(x) - \lambda_i) \right)f(x).
\]

On the other hand, if we set
\[
G_{k+1} = \left\{ \sigma \in S_{k+1} : \sigma_{k+1} = 1 \text{ and } 0 < \sum_{i=1}^{k} \sigma_i < k \right\}
\]
and
\[
H_{k+1} = \left\{ \sigma \in S_{k+1} : \sigma_{k+1} = 0 \text{ and } 0 < \sum_{i=1}^{k} \sigma_i < k \right\},
\]
we can write
\[
C = - \sum_{\theta \in S_k, 0 < |\theta| < k} \int_{\mathbb{R}^n} (b_{k+1}(x) - \lambda_{k+1})\left( \prod_{i=1}^{k} (b_i(x) - b_i(y))^{\theta_i} \right)\left( \prod_{i=1}^{k} (b_i(y) - \lambda_i)^{\theta_i} \right)K(x-y)f(y) \, dy
= - \sum_{\theta \in S_k, 0 < |\theta| < k} \int_{\mathbb{R}^n} (b_{k+1}(x) - b_{k+1}(y))\left( \prod_{i=1}^{k} (b_i(x) - b_i(y))^{\theta_i} \right)\left( \prod_{i=1}^{k} (b_i(y) - \lambda_i)^{\theta_i} \right)K(x-y)f(y) \, dy
- \sum_{\theta \in S_k, 0 < |\theta| < k} \int_{\mathbb{R}^n} (b_{k+1}(y) - \lambda_{k+1})\left( \prod_{i=1}^{k} (b_i(x) - b_i(y))^{\theta_i} \right)\left( \prod_{i=1}^{k} (b_i(y) - \lambda_i)^{\theta_i} \right)K(x-y)f(y) \, dy
= - \sum_{\sigma \in G_{k+1}} T_{b_{\sigma}}\left( \prod_{i=1}^{k+1} (b_i - \lambda_i)^{\delta_{\sigma}} f \right)(x) - \sum_{\sigma \in H_{k+1}} T_{b_{\sigma}}\left( \prod_{i=1}^{k+1} (b_i - \lambda_i)^{\delta_{\sigma}} f \right)(x).
\]
This yields the thesis since
\[ G_{k+1} \cup H_{k+1} \cup \{(0,0,\ldots,0,1),(1,1,\ldots,1,0)\} = \{\sigma \in S_{k+1} : 0 < |\sigma| < k + 1\}. \]

Proposition 14 If \( b = (b_1, \ldots, b_m) \) and \( \lambda_i \) are fixed constants for every \( 1 \leq i \leq m \), then

\[
T_b f = (-1)^m T \left( \prod_{i=1}^{m} (b_i - \lambda_i) \right) f + \sum_{\sigma \in S_m, |\sigma| < m} (-1)^{m-|\sigma|} \left( \prod_{i=1}^{m} (b_i - \lambda_i)^{\delta_i} \right) T_{b_\sigma} f.
\]

Proof We proceed by induction. Recall that \( T_{b_0} = T \) when each component of \( \sigma \) is zero, so that the result trivially holds when \( m = 1 \). Now assume that it is true for every multilinear symbol with \( k \) components and let us prove for \( b = (b_1, b_2, \ldots, b_{k+1}) \). For simplicity, we will denote \( b = b_{k+1} = (b_1, b_2, \ldots, b_k) \). Then, from Corollary 11 we have that

\[
T_b f(x) = [b_{k+1}, T_b] f(x)
\]

\[
= (b_{k+1} - \lambda_{k+1}) T_b f(x) - T_b((b_{k+1} - \lambda_{k+1}) f)(x)
\]

\[
= (b_{k+1} - \lambda_{k+1}) T_b f(x) - (-1)^k T \left( \prod_{i=1}^{k+1} (b_i - \lambda_i) \right) f(x) +
\]

\[
+ \sum_{\theta \in S_k, |\theta| < k} (-1)^{k-|\theta|} \left( \prod_{i=1}^{k} (b_i - \lambda_i)^{\delta_i} \right) T_{b_\theta}((b_{k+1} - \lambda_{k+1}) f)(x).
\]

By defining the sets

\[ H_k = \left\{ \sigma \in S_k : \sigma_k = 1 \text{ and } \prod_{i=1}^{k-1} \sigma_i = 0 \right\} \quad \text{and} \quad G_k = \left\{ \sigma \in S_k : \sigma_k = 0 \text{ and } \prod_{i=1}^{k-1} \sigma_i = 0 \right\} \]

we get

\[ \{\theta \in S_k, |\theta| < k\} = H_k \cup G_k \cup \{\eta^k\}, \]

where \( \eta^k = (1, 1, \ldots, 1, 0) \in S_k \). Since \#(H_k \cup G_k) = #H_{k+1} \), for every \( \sigma \in H_{k+1} \) we can associate a vector \( \theta = \theta(\sigma) \in S_k \), with at least one of its components equal to 0 such that \( \sigma_i = \theta_i \) for each \( 1 \leq i \leq k \). By using the relation

\[
T_b f(x) = [b_{k+1}, T_b] f(x)
\]

\[
= (b_{k+1} - \lambda_{k+1}) T_b f(x) - T_b((b_{k+1} - \lambda_{k+1}) f)(x).
\]

we can write the sum

\[
\sum_{\theta \in S_k, |\theta| < k} (-1)^{k-|\theta|} \left( \prod_{i=1}^{k} (b_i - \lambda_i)^{\delta_i} \right) T_{b_\theta}((b_{k+1} - \lambda_{k+1}) f)(x)
\]

as

\[ \square \]
\[
\sum_{\sigma \in H_{k+1}} (-1)^{k-|\sigma|+1} \left( \prod_{i=1}^{k} (b_i - \lambda_i)^{\theta_i} \right) ((b_{k+1} - \lambda_{k+1}) T_{b_{i+1}k} f(x) - T_{b_{i}k} f(x)) = I + II,
\]
where
\[
I = \sum_{\sigma \in G_{k+1}} (-1)^{k-|\sigma|} \left( \prod_{i=1}^{k+1} (b_i - \lambda_i)^{\theta_i} \right) T_{b_{i+1}k} f(x)
\]
and
\[
II = \sum_{\sigma \in H_{k+1}} (-1)^{k-|\sigma|} \left( \prod_{i=1}^{k} (b_i - \lambda_i)^{\theta_i} \right) ((b_{k+1} - \lambda_{k+1}) T_{b_{i}k} f(x)).
\]
Finally, the equality
\[
\{ \sigma \in S_{k+1}, |\sigma| < k + 1 \} = H_{k+1} \cup G_{k+1} \cup \{ \eta^{k+1} \},
\]
where \( \eta^{k+1} = (1, 1, 1, \ldots, 1, 0) \in S_{k+1} \), implies that the sum
\[
(b_{k+1}(x) - \lambda_{k+1}) T_{b_{i}k} f(x) + \sum_{\theta \in S_{k+1}, |\theta| < k} (-1)^{k-|\theta|} \left( \prod_{i=1}^{k} (b_i - \lambda_i)^{\theta_i} \right) T_{b_{i}k} ((b_{k+1} - \lambda_{k+1}) f)(x)
\]
is equal to
\[
\sum_{\sigma \in S_{k+1}, |\sigma| < k+1} (-1)^{k-|\sigma|} \left( \prod_{i=1}^{k+1} (b_i - \lambda_i)^{\theta_i} \right) T_{b_{i}k} f(x).
\]
This concludes the proof. \( \square \)

4 Mixed inequalities for commutators of CZO

We shall use a mixed weak type estimate for \( T \) given below. This result was set and proved in [4]. Also, the proof can be performed in an alternative way by following the ideas involved in (1.3) in [3].

**Theorem 15 ([4])** Let \( T \) be a Calderón–Zygmund operator. If \( u, v \in A_1 \) or \( u \in A_1 \) and \( v \in A_{\infty}(u) \), then there exists a positive constant \( C \) such that the inequality
holds for every $t > 0$ and every bounded function $f$ with compact support.

**Proof of Theorem 1** We shall proceed by induction on $m$. The case corresponding to $m = 1$ can be achieved by following similar lines as in the proof of (1.3) by substituting BMO condition by $\text{Osc}_{\text{exp}L^r}$, with $r \geq 1$. For more clarity, we shall provide the details for the case $m = 2$.

Let us assume, without loss of generality, that $f$ is nonnegative, bounded and has compact support. From (3.2) we obtain that

$$uv \left( \left\{ x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t \right\} \right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|u(x)v(x) \, dx,$$

where $\tilde{T} = (b_1/\|b_1\|_{\text{Osc}_{\text{exp}L^1}}, b_2/\|b_2\|_{\text{Osc}_{\text{exp}L^2}})$. Therefore, it will be enough to achieve the estimate for the case in which $\|b_i\|_{\text{Osc}_{\text{exp}L^i}} = 1$ for each $i$.

The hypothesis on $v$ ensures that it belongs to $A_\infty$. We fix $t > 0$ and perform the Calderón–Zygmund decomposition of $f$ at level $t$, with respect to the measure given by $d\mu(x) = v(x) \, dx$. This yields a disjoint collection of dyadic cubes $\{Q_j\}_{j=1}^\infty$ that verify

$$t < \frac{1}{v(Q_j)} \int_{Q_j} fv \leq C_n t,$$

for every $j$. We also split $f = g + h$, where

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^n \setminus \Omega; \\ f_{Q_j}^v & \text{if } x \in Q_j, \end{cases}$$

where $f_{Q_j}^v$ denotes the average of $f$ over $Q_j$ with respect to the measure $\mu$ and $\Omega = \bigcup_j Q_j$. Also $h(x) = \sum_j h_j(x)$, where

$$h_j(x) = (f(x) - f_{Q_j}^v) \chi_{Q_j}(x).$$

That is, every function $h_j$ is supported in $Q_j$ and

$$\int_{Q_j} h_j v = 0 \quad (4.1)$$

for every $j$. If we set $Q_j^* = 3Q_j$ and $\Omega^* = \bigcup_j Q_j^*$, then we obtain
\[ uv \left( \left\{ x : \frac{|T_b(gv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) = uv \left( \left\{ x : \frac{|T_b(gv)(x)|}{v(x)} > t \right\} \right) \\
+ uv \left( \left\{ x : \frac{|T_b(hv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \\
\leq uv \left( \left\{ x : \frac{|T_b(gv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) + uv(\Omega^*) \\
+ uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \frac{|T_b(hv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \\
= I + II + III. \]

We begin with the estimate of \( I. \) Since \( v \in A_\infty(u), \) there exists \( p > 1 \) such that \( v \in A_p(u). \) This implies that \( v^{1/p'} \in A_{p'}(u) \) and by Lemma 4 we get \( uv^{1-p'} \in A_{p'}. \) By applying Tchebychev inequality and the strong type estimate for \( T_b \) proved in [15] we have

\[ I \leq \frac{1}{p'} \int_{\mathbb{R}^n} |T_b(gv)|^{p'} uv^{1-p'} \]
\[ \leq \frac{C}{p'} \int_{\mathbb{R}^n} g^{p'} uv \]
\[ \leq \frac{C}{t} \left( \int_{\mathbb{R}^n} fuv + \sum_{j} \int_{Q_j} f_{Q_j}^\# uv \right) \]
\[ \leq \frac{C}{t} \left( \int_{\mathbb{R}^n} fuv + \sum_{j} \frac{uv(Q_j)}{v(Q_j)} \int_{Q_j} fv \right) \]
\[ \leq \frac{C}{t} \left( \int_{\mathbb{R}^n} fuv + C \sum_{j} \int_{Q_j} fu \right) \]
\[ \leq \frac{C}{t} \int_{\mathbb{R}^n} fuv \]

by virtue of Lemma 5. We turn now to the estimate of \( II. \) Since \( uv \in A_\infty, \) it is a doubling weight. Then

\[ II = uv(\Omega^*) \leq C \sum_{j} \frac{uv(Q_j)}{v(Q_j)} \frac{1}{t} \int_{Q_j} fv \]
\[ \leq C \sum_{j} \frac{1}{t} \int_{Q_j} fu \leq C \int_{\mathbb{R}^n} fuv. \]

For \( III, \) we fix constants \( \lambda_{i,j} \) for \( i = 1, 2 \) and every \( j \) that will be chosen later. By applying Proposition 10 we write

\[ T_b(hv)(x) = \sum_{j} T_b(h_jv)(x) \]
\[ = \sum_{j} \left( T((b_1 - \lambda_{1,j})(b_2 - \lambda_{2,j})h_jv) - (b_2(x) - \lambda_{2,j})T((b_1 - \lambda_{1,j})h_jv)(x) \right) \\
- (b_1(x) - \lambda_{1,j})T((b_2 - \lambda_{2,j})h_jv)(x) + (b_1(x) - \lambda_{1,j})(b_2(x) - \lambda_{2,j})T(h_jv)(x) \right). \]
If \( x \in \mathbb{R}^n \setminus \Omega^* \) we can use the integral representation of \( T \) to write
\[
(b_2(x) - \lambda_{2j})T((b_1 - \lambda_{1j})h_jv)(x)
\]
\[
= \sum_j \int_{\mathbb{R}^n \setminus \Omega^*} (b_2(x) - \lambda_{2j})(b_1(y) - \lambda_{1j})h_j(y)K(x - y)v(y)\,dy
\]
\[
= \sum_j \int_{\mathbb{R}^n \setminus \Omega^*} (b_2(x) - b_2(y))(b_1(y) - \lambda_{1j})h_j(y)K(x - y)v(y)\,dy
\]
\[
+ \sum_j \int_{\mathbb{R}^n \setminus \Omega^*} (b_2(y) - \lambda_{2j})(b_1(y) - \lambda_{1j})h_j(y)K(x - y)v(y)\,dy
\]
\[
= \sum_j T_{b_2}((b_1 - \lambda_{1j})h_jv)(x) + \sum_j T((b_1 - \lambda_{1j})(b_2 - \lambda_{2j})h_jv)(x).
\]

Similarly,
\[
(b_1 - \lambda_{1j})T((b_2 - \lambda_{2j})h_jv) = \sum_j T_{b_1}((b_2 - \lambda_{2j})h_jv)
\]
\[
+ \sum_j T((b_1 - \lambda_{1j})(b_2 - \lambda_{2j})h_jv).
\]

Therefore, if \( x \in \mathbb{R}^n \setminus \Omega^* \), we have
\[
T_b(hv)(x) = - \sum_j \left( T((b_1 - \lambda_{1j})(b_2 - \lambda_{2j})h_jv) + (b_1(x) - \lambda_{1j})(b_2(x) - \lambda_{2j})T(h_jv)(x) \right)
\]
\[
- \sum_j T_{b_1}((b_2 - \lambda_{2j})h_jv(x)) - \sum_j T_{b_2}((b_1 - \lambda_{1j})h_jv(x)),
\]

which gives
\[
III \leq uv\left( \sum_j \frac{|T((b_1 - \lambda_{1j})(b_2 - \lambda_{2j})h_jv)(x)|}{v(x)} > \frac{t}{8} \right)
\]
\[
+ uv\left( \sum_j \frac{|(b_1(x) - \lambda_{1j})(b_2(x) - \lambda_{2j})T(h_jv)(x)|}{v(x)} > \frac{t}{8} \right)
\]
\[
+ uv\left( \sum_j \frac{|T_{b_1}(\sum_j (b_2 - \lambda_{2j})h_jv)(x)|}{v(x)} > \frac{t}{8} \right)
\]
\[
+ uv\left( \sum_j \frac{|T_{b_2}(\sum_j (b_1 - \lambda_{1j})h_jv)(x)|}{v(x)} > \frac{t}{8} \right)
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]

Let us first estimate \( I_1 \). Let us fix \( \lambda_{ij} = b_{ij} \), where as usual
$$b_{i,j} = \frac{1}{|Q_j|} \int_{Q_j} b_i(x) \, dx \quad \text{for } i = 1, 2 \text{ and every } j.$$  

By Theorem 15 we have

$$I_1 \leq C \int_{\mathbb{R}^n} \sum_j |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| h_j(x) |u(x)v(x)| \, dx$$

$$\leq C \int_{\mathbb{R}^n} \sum_j |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| f(x) |u(x)v(x)| \, dx$$

$$+ C \int_{\mathbb{R}^n} \left( \int_{Q_j} |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| u(x)v(x) \, dx \right) \left( \frac{1}{v(Q_j)} \int_{Q_j} f(y)v(y) \, dy \right)$$

$$= A + B.$$  

We start with $A.$ Let $\varphi_i(t) = e^{rt} - 1,$ for $i = 1, 2.$ Observe that

$$\varphi_1^{-1}(t)\varphi_2^{-1}(t)\Phi^{-1}(t) \lesssim (\log t)^{1/r_1 + 1/r_2} \frac{t}{(\log t)^{1/r}} = t,$$  

if $t \geq e.$ It is well-known that for a Young function $\Phi$ and an $A_\infty$ weight $w$ we have

$$\|f\|_{\Phi,E,w} \approx \inf_{\tau > 0} \left\{ \tau + \frac{\tau}{w(E)} \int_E \Phi \left( \frac{|f(x)|}{\tau} \right) w(x) \, dx \right\},$$  

(4.2)

for every measurable set $E$ with $0 < |E| < \infty$ (see, for example, [16] or [8] for the case $w = 1;$ the proof for $w \in A_\infty$ can be achieved by adapting the argument).

Since $uv \in A_\infty$ we apply generalized Hölder inequality with the functions $\varphi_1, \varphi_2$ and $\Phi,$ with respect to the measure $d\nu(x) = u(x)v(x) \, dx.$ We also combine Lemmas 7 and 5 with (4.2) to get

$$A \leq C \int_{\mathbb{R}^n} \sum_j (uv)(Q_j) \|b_1 - b_{1,j}\|_{\exp L^1, Q_j, uv} \|b_2 - b_{2,j}\|_{\exp L^2, Q_j, uv} \|f\|_{\Phi, Q_j, uv}$$

$$\leq C \int_{\mathbb{R}^n} \sum_j (uv)(Q_j) \left( t + \frac{t}{uv(Q_j)} \int_{Q_j} \Phi \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx \right)$$

$$\leq C \int_{\mathbb{R}^n} \sum_j \left( \frac{uv(Q_j)}{v(Q_j)} \int_{Q_j} f(x)v(x) \, dx + C \sum_j \int_{Q_j} \Phi \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx \right)$$

$$\leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx$$

$$\leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx.$$  

For the term $B$ we apply the same Hölder inequality as above and Lemmas 7 and 5. This yields
\[ B \leq \sum_{j} \frac{(uv)(Q_j)}{v(Q_j)} \| b_1 - b_{1,j} \|_{\exp L^1, Q_j, uv} \| b_2 - b_{2,j} \|_{\exp L^2, Q_j, uv} \left( \int_{Q_j} f y \right) \]

\[ \leq C \sum_{j} \frac{(uv)(Q_j)}{v(Q_j)} \| b_1 - b_{1,j} \|_{\exp L^1, Q_j} \| b_2 - b_{2,j} \|_{\exp L^2, Q_j} \left( \int_{Q_j} f y \right) \]

\[ \leq C \sum_{j} \int_{Q_j} f(x)u(x)v(x) \, dx \]

\[ \leq C \int_{\mathbb{R}^n} f(x)u(x)v(x) \, dx. \]

We turn now to the estimation of \( I_2 \). Let \( x_{Q_j} \) be the centre of the cube \( Q_j \), for every \( j \). By applying Tchebychev inequality, Tonelli theorem and (4.1) we have that

\[ I_2 \leq \frac{8}{t} \sum_{j} \int_{\mathbb{R}^n \setminus Q_j} |(b_1(x) - b_{1,j})(b_2(x) - b_{2,j})T(h_j y)(x)|u(x) \, dx \]

\[ \leq \frac{8}{t} \sum_{j} \int_{\mathbb{R}^n \setminus Q_j} |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| \left| \int_{Q_j} h_j(y)v(y)(K(x - y) - K(x - x_{Q_j})) \, dy \right| u(x) \, dx \]

\[ \leq \frac{8}{t} \sum_{j} \int_{Q_j} |h_j(y)| v(y) \left( \int_{\mathbb{R}^n \setminus Q_j} |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| |K(x - y) - K(x - x_{Q_j})| u(x) \, dx \right) \, dy \]

\[ = \frac{8}{t} \sum_{j} \int_{Q_j} |h_j(y)| v(y) F_j(y) \, dy. \]

We shall prove that there exists \( C > 0 \) that satisfies

\[ F_j(y) \leq Cu(y), \text{ for } y \in Q_j \text{ and every } j. \] (4.3)

Fix \( j \) and \( y \in Q_j \). Let \( \ell_j = \ell(Q_j)/2 \), where \( \ell(Q_j) \) is the length of the sides of \( Q_j \), and \( A_{j,k} = \{ x : 2^k \ell_j \leq |x - x_{Q_j}| < 2^{k+1} \ell_j \} \). By using the smoothness condition of the kernel \( K \), we have

\[ F_j(y) \leq \sum_{k=1}^{\infty} \int_{A_{j,k}} |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| \frac{|y - x_{Q_j}|}{|x - x_{Q_j}|^{n+1}} u(x) \, dx \]

\[ \leq C \sum_{k=1}^{\infty} \left( \frac{\ell_j}{(2^k \ell_j)^{n+1}} \right) \int_{2^{k+1} Q_j} |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| u(x) \, dx \]

\[ \leq C \sum_{k=1}^{\infty} \frac{2^{-k}}{|2^{k+1} Q_j|} \int_{2^{k+1} Q_j} |b_1(x) - b_{1,j}| |b_2(x) - b_{2,j}| u(x) \, dx. \]

Let \( b_{i,j}^k = |2^k Q_j|^{-1} \int_{2^k Q_j} b_i \) for \( i = 1, 2 \). Thus,
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\[ |b_1(x) - b_1(y)| |b_2(x) - b_2(y)| \leq |b_1(x) - b_1^{k+1}(x) - b_2(x) - b_2^{k+1}(x)| + |b_1(x) - b_1^{k+1}(x) - b_2(x) - b_2^{k+1}(x)| + |b_1(x) - b_1^{k+1}(x) - b_2(x) - b_2^{k+1}(x)|
\]
\[ \leq |b_1(x) - b_1^{k+1}(x) - b_2(x) - b_2^{k+1}(x)| + C(k+1)\|b_2\|_{Osc_c}\|b_1(x) - b_1^{k+1}(x) - b_2(x) - b_2^{k+1}(x)| + C(k+1)^2\|b\|.
\]

by virtue of Lemma 8. Then

\[
F_j(y) \leq C \sum_{k=1}^{\infty} \frac{2^{-k}}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j} |b_1(x) - b_1^{k+1}(x) - b_2(x) - b_2^{k+1}(x)| u(x) \, dx
\]
\[ + C \sum_{k=1}^{\infty} \frac{2^{-k}(k+1)}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j} b_1(x) - b_1^{k+1}(x) - b_2(x) - b_2^{k+1}(x)| u(x) \, dx
\]
\[ + C \sum_{k=1}^{\infty} \frac{2^{-k}(k+1)}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j} b_2(x) - b_2^{k+1}(x)| u(x) \, dx
\]
\[ + C \sum_{k=1}^{\infty} \frac{2^{-k}(k+1)^2}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j} u(x) \, dx.
\]

We now apply a generalized Hölder inequality to each term above, with respect to \(d\zeta(x) = u(x) \, dx\) and with the functions \(\varphi_i(t) = e^{\epsilon i} - 1\), for \(i = 1, 2\). This yields

\[ F_j(y) \leq C \left( \inf_{2^{k+1}Q_j} u \right) \sum_{k=1}^{\infty} 2^{-k}(k+1)^2 \leq Cu(y),
\]

which proves (4.3). By combining this estimate with Lemma 5 we get

\[ I_2 \leq C \sum_{k=1}^{\infty} \int_{Q_j} f(y) u(y) v(y) \, dy + \frac{C}{t} \sum_{Q_j} uv(Q_j) \int_{Q_j} f(y) v(y) \, dy
\]
\[ \leq C \int_{\mathbb{R}^n} f(y) u(y) v(y) \, dy.
\]

Let us conclude with the estimates of \(I_3\) and \(I_4\). We need to use here the case \(m = 1\), which can be proved by following the ideas for the BMO case (see [3], Theorem 1). Then

\[ I_3 \leq uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| T_{b_1} \left( \sum_j (b_2 - \lambda_2 j) v \chi_{Q_j} \right) (x) \right| / v(x) > t/8 \right\} \right)
\]
\[ + uv \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| T_{b_1} \left( \sum_j (b_2 - \lambda_2 j) v \chi_{Q_j} \right) (x) \right| / v(x) > t/8 \right\} \right)
\]
\[ = I_{3,1} + I_{3,2}.
\]

By taking \(\Phi_1(t) = t(1 + \log^+ t)^{1/4}\) and \(\lambda_2 j = b_2 j\), we have

\[ \text{Springer} \]
\[
I_{3,1} \leq C \int_{\mathbb{R}^n} \Phi_1 \left( \sum_j |b_2(x) - b_{2,j}| \frac{f(x) \chi_{Q_j}(x)}{t} \right) u(x)v(x) \, dx
\]
\[
\leq C \sum_j \int_{Q_j} \Phi_1 \left( \frac{|b_2(x) - b_{2,j}| f(x)}{t} \right) u(x)v(x) \, dx.
\]

Since \( \Phi^{-1}(t) \varphi_2^{-1}(t) \lesssim \Phi_1^{-1}(t) \) for \( t \geq e \), we obtain that
\[
\Phi_1(st) \leq \Phi(s) + \varphi_2(t).
\]

Notice that \( 1 = \|b_2\|_{\text{Osc,exp} L^2} \geq \|b_2 - b_{2,j}\|_{\text{exp} L^2, Q_j} \geq C \|b_2 - b_{2,j}\|_{\exp L^2, Q_j, uv} \). This implies that
\[
I_{3,1} \leq C \sum_j \int_{Q_j} \left( \Phi \left( \frac{f(x)}{t} \right) + \varphi_2(|b_2 - b_{2,j}|) \right) u(x)v(x) \, dx
\]
\[
\leq \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx + C \sum_j uv(Q_j)
\]
\[
\leq \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx + C \left( \sum_j \frac{uv(Q_j)}{v(Q_j)} \right) \int_{Q_j} f(x)v(x) \, dx
\]
\[
\leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{t} \right) u(x)v(x) \, dx,
\]
where we have used Lemma 5. On the other hand, since \( f_{Q_j}^v/t \leq C \) we have that
\[
I_{3,2} \leq C \int_{\mathbb{R}^n} \Phi_1 \left( \sum_j |b_2(x) - b_{2,j}| \frac{f_{Q_j}^v \chi_{Q_j}(x)}{t} \right) u(x)v(x) \, dx
\]
\[
\leq C \sum_j \int_{Q_j} \Phi_1 \left( C|b_2(x) - b_{2,j}| \right) u(x)v(x) \, dx
\]
\[
\leq C \sum_j \int_{Q_j} \left( \Phi(C) + \varphi_2(|b_2(x) - b_{2,j}|) \right) u(x)v(x) \, dx,
\]
and the estimate follows similarly as above. The term \( I_4 \) can be bounded by interchanging the roles of the components \( b_1 \) and \( b_2 \). The proof for \( m = 2 \) is complete.

Suppose now that the result is true for every symbol with \( k \) components and consider \( b = (b_1, b_2, \ldots, b_{k+1}) \). Let us assume again that \( \|b_i\|_{\text{Osc,exp} L^2} = 1 \) for every \( i \). Fixed \( t > 0 \), we perform the Calderón–Zygmund decomposition of \( f \) with respect to the measure \( d\mu = v(x) \, dx \) at level \( t \), obtaining a disjoint collection of dyadic cubes \( \{Q_j\}_j \). We also decompose \( f = g + h \). If \( Q_j^* = 3Q_j \) and \( \Omega^* = \bigcup_j Q_j^* \) we have that
\[
uv\left( \left\{ x : \frac{|T_b(fv)(x)|}{v(x)} > t \right\} \right) = uv\left( \left\{ x : \frac{|T_b(gv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \\
+ uv\left( \left\{ x : \frac{|T_b(hv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \\
\leq uv\left( \left\{ x : \frac{|T_b(gv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) + uv(\Omega^*) \\
+ uv\left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \frac{|T_b(hv)(x)|}{v(x)} > \frac{t}{2} \right\} \right) \\
= I + II + III.
\]

The estimate of \( I \) follows from the strong \((p, p)\) type of \( T_b \) for \( A_p \) weights proved in [15], and for \( II \) we proceed as in the case \( m = 2 \).

For \( III \) we use Proposition 13. If \( \lambda_{i,j} \) are constants to be chosen for \( 1 \leq i \leq k + 1 \) and \( j \in \mathbb{N} \), then

\[
T_b(hv)(x) = \sum_j T_b(h_jv)(x) \\
= \sum_j \left( \prod_{i=1}^{k+1} (b_i(x) - \lambda_{i,j}) \right) T(h_jv)(x) - \sum_j T\left( \left( \prod_{i=1}^{k+1} (b_i - \lambda_{i,j}) \right) h_jv \right)(x) \\
- \sum_{\sigma \in S_{k+1}, 0 < |\sigma| < k+1} T_{b_{\sigma}} \left( \sum_j \left( \prod_{i=1}^{k+1} (b_i - \lambda_{i,j})_{\sigma_i} \right) h_jv \right)(x),
\]

which implies that

\[
III \leq uv\left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \sum_j \left( \prod_{i=1}^{k+1} |b_i(x) - \lambda_{i,j}| \right) \frac{|T(h_jv)(x)|}{v(x)} > \frac{t}{2k+2} \right\} \right) \\
+ uv\left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \sum_j \left| \frac{T\left( \left( \prod_{i=1}^{k+1} (b_i - \lambda_{i,j}) \right) h_jv \right)(x)}{v(x)} \right| > \frac{t}{2k+2} \right\} \right) \\
+ \sum_{\sigma \in S_{k+1}, 0 < |\sigma| < k+1} uv\left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \frac{|T_{b_{\sigma}} \left( \sum_j \left( \prod_{i=1}^{k+1} (b_i - \lambda_{i,j})_{\sigma_i} \right) h_jv \right)(x)}{v(x)} > \frac{t}{2k+2} \right\} \right) \\
= I_1 + I_2 + I_3.
\]

Let us first estimate \( I_1 \). Take \( \lambda_{i,j} = b_{i,j} \) for every \( j \) and every \( 1 \leq i \leq k + 1 \). By applying Tchebychev inequality we have that
We shall prove that \( F_{j,k}(y) \leq C u(y) \), for \( y \in Q_j \) and \( C \) independent of \( j \). Indeed, by using the smoothness condition of \( K (2.5) \) we get

\[
F_{j,k}(y) \leq \sum_{\ell=0}^\infty \int_{\Lambda_{\ell, j}} \left( \frac{|y - x_{Q_j}|}{|x - x_{Q_j}|^{n+1}} \left( \prod_{i=1}^{k+1} |b_i(x) - b_{i,j}| \right) u(x) \right) dx 
\]

\[
\leq C \sum_{\ell=0}^\infty \frac{2^{-\ell}}{|2^{\ell+1} Q_j|} \int_{2^{\ell+1} Q_j} \left( \prod_{i=1}^{k+1} |b_i(x) - b_{i,j}| \right) u(x) \right) dx. 
\]

Let \( b_{i,j}^\ell = [2^\ell Q_j]^{-1} \int_{2^\ell Q_j} b_i \). By virtue of Lemmas 8 and 9 we obtain

\[
\prod_{i=1}^{k+1} |b_i(x) - b_{i,j}| = \sum_{\sigma \in S_{k+1}} \prod_{i=1}^{k+1} |b_i - b_{i,j}^\ell| \sigma_i |b_i^\ell + 1 - b_{i,j}| \sigma_i 
\]

\[
\leq C(\ell + 1)^{k+1} \prod_{i=1}^{k+1} |b_i - b_{i,j}^\ell| \sigma_i. 
\]

We apply now generalized Hölder inequality with functions \( \varphi_i(t) = e^{r_i t} - 1, 1 \leq i \leq k + 1 \), with respect to the measure \( d\zeta = u(x) \) dx. This yields

\[
F_{j,k}(y) \leq C \sum_{\ell=0}^\infty \frac{u(2^{\ell+1} Q_j)}{|2^{\ell+1} Q_j|} 2^{-\ell}(\ell + 1)^{k+1} 
\]

\[
\times \sum_{\sigma \in S_{k+1}} \left( \prod_{i : \sigma_i = 1} \|b_i - b_{i,j}^\ell\| \varphi_i,2^{\ell+1} Q_j, u \right) \left( \prod_{i : \sigma_i = 0} \|X_{2^{\ell+1} Q_j}\| \varphi_i,2^{\ell+1} Q_j, u \right) 
\]

\[
\leq C[u]_{A_j} \sum_{\ell=0}^\infty 2^{-\ell}(\ell + 1)^{k+1} u(y) 
\]

\[
= C u(y). 
\]

From here the estimate follows in the same way as in page 17.

Let us focus now on \( I_2 \). Considering again \( \lambda_{i,j} = b_{i,j} \), Theorem 15 allows us to get
\[ I_2 \leq \frac{C}{t} \sum_j \int_{Q_j} \left( \prod_{i=1}^{k+1} |b_i(x) - b_{i,j}| \right) |h_j(x)| u(x) v(x) \, dx \]

\[ \leq \frac{C}{t} \sum_j \int_{Q_j} \left( \prod_{i=1}^{k+1} |b_i(x) - b_{i,j}| \right) f(x) u(x) v(x) \, dx \]

\[ + \frac{C}{t} \sum_j \int_{Q_j} \left( \prod_{i=1}^{k+1} |b_i(x) - b_{i,j}| \right) f_{Q_j}^u u(x) v(x) \, dx \]

\[ = I_2^1 + I_2^2. \]

For \( I_2^1 \), observe that

\[ \left( \prod_{i=1}^{k+1} \phi_i^{-1}(t) \right) \Phi^{-1}(t) \lesssim t, \]

for \( t \geq e \). By applying generalized Hölder inequality with respect to \( uv \), Lemma 7 and (4.2) we obtain that

\[ I_2^1 \leq \frac{C}{t} \sum_j (uv)(Q_j) \left( \prod_{i=1}^{k+1} \|b_i - b_{i,j}\|_{\phi_i,Q_{ij,u,v}} \right) \left\| f \right\|_{\Phi,Q_{ij,u,v}} \]

\[ \leq \frac{C}{t} \sum_j (uv)(Q_j) \left( \prod_{i=1}^{k+1} \|b_i - b_{i,j}\|_{\phi_i,Q_{ij,u,v}} \right) \left\{ t + \frac{t}{(uv)(Q_j)} \int_{Q_j} \Phi \left( \frac{f(x)}{t} \right) u(x) v(x) \, dx \right\} \]

\[ \leq C \sum_j \left( \frac{uv(Q_j)}{v(Q_j)} \right) \frac{1}{t} \int_{Q_j} f(x) v(x) \, dx + \int_{Q_j} \Phi \left( \frac{f(x)}{t} \right) u(x) v(x) \, dx \]

\[ \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f(x)}{t} \right) u(x) v(x) \, dx, \]

by virtue of Lemma 5. For \( I_2^2 \) we apply again Hölder inequality to get

\[ I_2^2 \leq \frac{C}{t} \sum_j (uv)(Q_j) \left( \int_{Q_j} f v \right) \left( \prod_{i=1}^{k+1} \|b_i - b_{i,j}\|_{\phi_i,Q_{ij,u,v}} \right) \left\| \mathcal{X}_{Q_j} \right\|_{\Phi,Q_{ij,u,v}} \]

\[ \leq \sum_j \frac{C}{t} \left( \int_{Q_j} f u v \right) \left( \prod_{i=1}^{k+1} \|b_i - b_{i,j}\|_{\phi_i,Q_{ij}} \right) \]

\[ \leq \frac{C}{t} \left( \int_{\mathbb{R}^n} f u v \right). \]

Finally, we estimate \( I_3 \). Fix \( \sigma \in S_{k+1} \) such that \( 0 < |\sigma| < k + 1 \). Let \( \Phi_{\sigma}(t) = t(1 + \log^+ t)^{1/r_\sigma} \), where

\[ \frac{1}{r_\sigma} = \sum_{i : \sigma_i = 1} \frac{1}{r_i}. \]

By using the inductive hypothesis we conclude
\[
uv \left( \left\{ \frac{T_{b_i} \left( \sum_{j} \left( \prod_{i=1}^{k+1} (b_i - \lambda_{ij})^{\phi_i} \right) h_j v \right)(x) }{v(x)} > \frac{t}{2^{k+1}} \right\} \right) \leq I_3^1(\sigma) + I_3^2(\sigma),
\]

where

\[
I_3^1(\sigma) = C \sum_j \int_{Q_j} \Phi_{\sigma} \left( \frac{\left( \prod_{i=1}^{k+1} |b_i - \lambda_{ij}|^{\phi_i} \right) f_j}{t} \right) uv
\]

and

\[
I_3^2(\sigma) = C \sum_j \int_{Q_j} \Phi_{\sigma} \left( \frac{\left( \prod_{i=1}^{k+1} |b_i - \lambda_{ij}|^{\phi_i} \right) f_j^v}{t} \right) uv.
\]

To estimate \( I_3^1(\sigma) \) observe that

\[
\Phi^{-1}(t) \left( \prod_{i: \sigma_i = 0} \varphi_i^{-1}(t) \right) \approx \frac{t}{(1 + \log^+ t)^{1/r}} \left( 1 + \log^+ t \right)^{1/r - 1/r_\sigma} = \frac{t}{(1 + \log^+ t)^{1/r_\sigma}} = \Phi^{-1}(t),
\]

which implies that

\[
\Phi_{\sigma} \left( \frac{\left( \prod_{i=1}^{k+1} |b_i - \lambda_{ij}|^{\phi_i} \right) f_j}{t} \right) \leq \Phi \left( \frac{f_j}{t} \right) + \sum_{i: \sigma_i = 0} \varphi_i(|b_i - b_{ij}|).
\]

Since \( 1 = \|b_i\|_{\text{osc}_{L^r}} \geq \|b_i - b_{ij}\|_{\exp L^{r':Q_j}} \geq C \|b_i - b_{ij}\|_{\exp L^{r':Q_j,uv}} \) for every \( i \) such that \( \sigma_i = 0 \), we have

\[
I_3^1(\sigma) \leq C \sum_j \int_{Q_j} \Phi \left( \frac{f_j}{t} \right) uv + C \sum_j \sum_{i: \sigma_i = 0} \int_{Q_j} \varphi_i(|b_i - b_{ij}|) uv
\]

\[
\leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f_j}{t} \right) uv + C \sum_j \frac{(uv(Q_j))}{v(Q_j)} \int_{Q_j} \frac{f_j v}{t}
\]

\[
\leq C \int_{\mathbb{R}^n} \Phi \left( \frac{f_j}{t} \right) uv,
\]

by Lemma 5. On the other hand, since \( f_j^v/t \leq C \)
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Therefore the result holds for every symbol \( \mathbf{b} \) with \( k + 1 \) components. The proof of Theorem 1 is complete. \( \square \)

5 Operators with kernel of Hörmander type

This section will be devoted to prove the strong \((p, p)\) type of \( T_{\mathbf{b}} \) on the Hörmander setting as well as a mixed inequality for these operators. When \( b \) is a single symbol belonging to BMO and \( T = K * f \), the first result follows by combining a Coifman type inequality proved in [10] with the weighted strong \((p, p)\) type of \( M_{\bar{\eta}} \) obtained in [1], where \( \varphi \) is a Young function with certain properties. One of the keys for the proof of the aforementioned Coifman inequality is a pointwise relation between the lower order commutators of \( T_{m} b \) and the sharp-\( \delta \) maximal function \( M^{\#}_{\delta} f(x) = (M^{\#}_{\delta} f^{\#}(x))^{1/\delta} \), where

\[
M^{\#}_{\delta} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_{Q} |f(y) - f_{Q}| dy
\]

and \( f_{Q} = |Q|^{-1} \int_{Q} f \). It is well-known (see, for example, [5]) that

\[
M^{\#}_{\delta} f(x) \approx \sup_{Q} \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_{Q} |f(y) - a| dy
\]  

and that for every \( 0 < p < \infty \) and \( w \in A_{m} \), the inequality

\[
\int_{\mathbb{R}^{n}} (Mf(x))^{p} w(x) dx \leq C \int_{\mathbb{R}^{n}} (M^{\#}_{\delta} f(x))^{p} w(x) dx,
\]

holds whenever the left-hand side is finite.

The following proposition gives the pointwise relation involving \( M^{\#}_{\delta} \) on the multilinear context.

**Proposition 16** Let \( m \in \mathbb{N}, r_{i} \geq 1 \) for \( 1 \leq i \leq m \) and \( 1/r = \sum_{i=1}^{m} 1/r_{i} \). Let \( \eta, \phi \) be Young functions that verify \( \bar{\eta}^{-1}(t)\phi^{-1}(t)(\log t)^{r_{i}} \leq t, \) for \( t \geq e \). Let \( T \) be an operator with kernel \( K \in H_{\phi, n} \) and \( \mathbf{b} = (b_{1}, b_{2}, \ldots, b_{m}) \), where \( b_{i} \in \text{Osc}_{exp \mathcal{L}^{r}_{m}} \) for \( 1 \leq i \leq m \). Then the inequality

\[
M^{\#}_{\delta}(T_{\mathbf{b}} f)(x) \leq C \|\mathbf{b}\| M_{\delta}^{\#} f(x) + C \sum_{\sigma \in S_{m}, |\sigma| < m} \|b_{\sigma}\| M_{\varepsilon}(T_{\mathbf{b}} f)(x)
\]

holds for every \( 0 < \delta < \varepsilon < 1 \).

**Proof** Fix \( 0 < \delta < \varepsilon < 1 \). By virtue of Proposition 14 we can write
\[ T_b f(x) = (-1)^m T \left( \prod_{i=1}^{m} (b_i - \lambda_i) f \right)(x) + \sum_{\sigma \in S_m, |\sigma| < m} (-1)^{m-1-|\sigma|} \prod_{i=1}^{m} (b_i - \lambda_i) \delta \quad \text{for} \quad 1 \leq i \leq m. \]

where \( \lambda_i \) are constants to be chosen later for \( 1 \leq i \leq m \).

By homogeneity we can assume, without loss of generality, that \( \| b_i \|_{\text{Osc}_{L^q}} = 1 \) for every \( 1 \leq i \leq m \). Fix \( x \in \mathbb{R}^n \) and \( B = B(x_B, R) \) a ball that contains \( x \). By using (5.1), we decompose \( f = f\chi_{2B} + f\chi_{(2B)^c} = f_1 + f_2 \) and take a constant \( a_B \) that we shall choose properly. Thus

\[
\left( \frac{1}{|B|} \int_B |T_b f(y)|^\delta - |a_B|^\delta \right)^{1/\delta} \leq \left( \frac{1}{|B|} \int_B |T_b f(y) - a_B|^\delta \right)^{1/\delta}
\]

\[
\leq C_\delta \sum_{\sigma \in S_m, |\sigma| < m} \left( \frac{1}{|B|} \int_B \left( \prod_{i=1}^{m} |b_i - \lambda_i| \delta \right) |T_b f| \right)^{1/\delta}
\]

\[
+ C_\delta \left( \frac{1}{|B|} \int_B T \left( \prod_{i=1}^{m} (b_i - \lambda_i) f_1 \right) \delta \right)^{1/\delta}
\]

\[
+ C_\delta \left( \frac{1}{|B|} \int_B T \left( \prod_{i=1}^{m} (b_i - \lambda_i) f_2 \right) \delta \right)^{1/\delta}
\]

\[
= I + II + III.
\]

Let us first estimate \( I \). If \( q = \varepsilon/\delta > 1 \), then for every \( \sigma \in S_m \) with at least one component equal to 0 we have that for \( t \geq t_0 \)

\[
t^{1/q} \prod_{i: \sigma_i = 0} (\log t)^{1/r_i} \leq t^{1/q}(\log t)^{1/r_i} \leq t.
\]

By taking \( \lambda_i = (b_i)_{2B} \) and applying generalized Hölder inequality, we get

\[
I \leq \sum_{\sigma \in S_m, |\sigma| < m} \left( \prod_{i: \sigma_i = 0} \| b_i - (b_i)_{2B} \|_{\text{exp} L^q} \right)^{1/(\delta q)} \left( \frac{1}{|B|} \int_B |T_b f(y)|^\delta \right)^{1/(\delta q)}
\]

\[
\leq \sum_{\sigma \in S_m, |\sigma| < m} \left( \prod_{i: \sigma_i = 0} \| b_i - (b_i)_{2B} \|_{\text{exp} L^q} \right) M_{\delta} (T_b f)(x)
\]

\[
\leq C \sum_{\sigma \in S_m, |\sigma| < m} M_{\delta} (T_b f)(x).
\]

We shall now estimate \( II \). Since \( H_\varphi \subset H_1 \), \( T \) is of weak \((1, 1)\) type. By applying Kolmogorov inequality we have that
$II \leq \frac{C}{|B|} \int_B \left( \prod_{i=1}^m |b_i(y) - (b_i)_{2B}| \right) |f(y)| \, dy$

$\leq \frac{C}{|2B|} \int_{2B} \left( \prod_{i=1}^m |b_i(y) - (b_i)_{2B}| \right) |f(y)| \, dy$

$\leq C \left( \prod_{i=1}^m \|b_i - (b_i)_{2B}\|_{\exp L^{r_i, 2B}} \right) \|X_{2B}\|_{\phi, 2B M_\varphi f(x)}$

$\leq CM_\varphi f(x),$ where we have used generalized Hölder inequality with functions $\varphi_i(t) = e^{i/r_i} - 1$, $1 \leq i \leq m$, $\phi(t)$ and $\bar{\eta}$, since

$\bar{\eta}^{-1}(t)\varphi^{-1}(t)(\log t)^{1/r} \leq t$

for $t \geq e$.

Finally, we estimate $III$. Pick $a_B = T(\prod_{i=1}^m (b_i - (b_i)_{2B}) f_2)(x_B).$ Jensen inequality yields

$III \leq \frac{1}{|B|} \int_B \left| T\left( \left( \prod_{i=1}^m (b_i - (b_i)_{2B}) \right) f_2 \right)(y) - T\left( \left( \prod_{i=1}^m (b_i - (b_i)_{2B}) \right) f_2 \right)(x_B) \right| \, dy$

$\leq \frac{1}{|B|} \int_B F(y) \, dy.$

We define, for $j \geq 1$, the sets

$A_j = \{ y : 2^j R \leq |x_B - y| < 2^{j+1} R \}.$

By using the integral representation of $T$, we have

$F(y) = \int_{\mathbb{R}^n} \prod_{i=1}^m |b_i(z) - (b_i)_{2B}| |f_2(z)| |K(y-z) - K(x_B - z)| \, dz.$

Lemmas 8 and 9 imply

$\prod_{i=1}^m |b_i(z) - (b_i)_{2B}| \leq \prod_{i=1}^m \left( |b_i(z) - (b_i)_{2^{i+1}B}| + |(b_i)_{2^{i+1}B} - (b_i)_{2B}| \right)$

$\leq \prod_{i=1}^m \left( |b_i(z) - (b_i)_{2^{i+1}B}| + Cj \right)$

$= \sum_{\sigma \in S_m} \prod_{i=1}^m |b_i(z) - (b_i)_{2^{i+1}B}|^{\sigma_i} (Cj)^{\sigma_i}$

$\leq C \sum_{\sigma \in S_m} \prod_{i=1}^m |b_i(z) - (b_i)_{2^{i+1}B}|^{\sigma_i}.$

Thus, by applying again Hölder inequality
\[ F(y) \leq \sum_{\sigma \in S_m} \sum_{j=1}^{\infty} f^{j_m-|\sigma|} \left( \frac{2^j R^n}{|2^{j+1}B|} \int_{2^{j+1}B} \left( \prod_{i=1}^{\infty} |b_i(z) - (b_i)_{2^{j+i+1}B}|^\sigma \right) \times |K(y-z) - K(x_B - z)| \left| \mathcal{X}_\lambda(z) f(z) \right| dz \right) \leq CM_B f(x), \]

since \( K \in H_{\phi,m} \) implies \( K \in H_{\phi,m-|\sigma|} \). By combining these estimates we get

\[ M_\phi(T_b f)(x) \leq CM_B f(x) + C \sum_{\sigma \in S_m, |\sigma| < m} M_\sigma(T_b f)(x). \]

\[ \square \]

The following result contains a Coifman type estimate involving \( T_b \). The proof can be performed by using the proposition above and following similar arguments as in Theorem 3.3 in [10]. We shall omit the details.

**Theorem 17** Let \( m \in \mathbb{N}, r_i \geq 1 \) for every \( 1 \leq i \leq m \) and \( 1/r = \sum_{i=1}^{m} 1/r_i \). Let \( \eta \) and \( \phi \) be Young functions that verify \( \tilde{\eta}^{-1}(t)\phi^{-1}(t)(\log t)^{1/r} \leq t \), for \( t \geq e \). Let \( T \) be an operator with kernel \( K \in H_{\phi,m} \) and \( \mathbf{b} = (b_1, b_2, \ldots, b_m) \) where \( b_i \in \text{Osc}_{\exp L^s} \), for \( 1 \leq i \leq m \). Then for every \( 0 < p < \infty \) and \( w \in A_{\infty} \) there exists a positive constant \( C \) such that the inequality

\[ \int_{\mathbb{R}^n} |T_b f(x)|^p w(x) \, dx \leq C \|\mathbf{b}\|_p \int_{\mathbb{R}^n} (M_\phi f(x))^p w(x) \, dx \]

holds for every bounded function \( f \) with compact support, provided the left-hand side is finite.

When we consider a single symbol in BMO, the result above was proved in [10]. The corresponding result for commutators of CZO was obtained in [14].

Now we are in a position to give the proof of the strong \( (p, p) \) type for \( T_b \).

**Proof of Theorem 3** We start again by assuming that \( \|b_i\|_{\text{Osc}_{\exp L^s}} = 1 \) for each \( 1 \leq i \leq m \). It is well known under the hypotheses of this theorem (see [1], Theorem 2.5) we have that

\[ \|(M_\phi f)w\|_{L^p} \leq C \|fw\|_{L^p}. \]

The conclusion follows immediately from Theorem 17 applied with \( w^\alpha \). \( \square \)

We close this section with the proof of mixed inequalities for \( T_b \). We shall need a mixed inequality involving the operator \( T \), which was set and proved in [3].

**Theorem 18** Let \( 1 < q < \infty \) and \( q^2/(2q - 1) < \beta < q \). Assume that \( w \in A_1 \cap RH_s \) for some \( s > 1 \) and \( v^\alpha \in A_{(q/\beta)\alpha}(a) \), where \( \alpha = \beta(q - 1)/(q - \beta) \). Let \( T \) be an operator with kernel...
$K \in H_\phi$, where $\phi$ is a Young function that verifies $\phi \in B_\rho$ for every $\rho \geq \min\{\beta, s\}$. Then there exists $C > 0$ such that the inequality

$$uv\left( \left\{ x \in \mathbb{R}^n : \left| \frac{T(fv)(x)}{v(x)} \right| > t \right\} \right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|u(x)v(x) \, dx$$

holds for every $t > 0$.

**Proof of Theorem 2** We proceed by induction on $m$. When $m = 1$ we have a symbol with a single component; the proof can be achieved by following the corresponding version of the BMO case ([3], Theorem 4). Assume now that the result holds for any symbol with $k$ components and let us prove it for $b = (b_1, b_2, \ldots, b_{k+1})$. Also assume without loss of generality that $f$ is bounded with compact support and that $\|b_i\|_{\text{One}_{\text{exp}L^q_t}} = 1$ for each $1 \leq i \leq k + 1$. Since $\alpha > 1$ we have $v \in A_{(q/\beta)p}^\alpha(u) \subset A_{\infty}(u)$, and this implies that $d\mu(x) = v(x) \, dx$ is doubling. We perform the Calderón–Zygmund decomposition of $f$ at height $t > 0$, with respect to $v$, obtaining a disjoint collection of dyadic cubes $\{Q_j\}_j$ that verify

$$t < f_{Q_j}^\alpha \leq Ct,$$

for each $j$. We also split $f = g + h$. Let $Q_j^* = 2c_\delta nQ_j$, where $c = \min\{c_\delta, c_\phi\}$ and these constants are the parameters appearing in the conditions $H_{\phi,m}$ and $H_\phi$ for $K$. If $\Omega^* = \bigcup_j Q_j^*$, then

$$uv\left( \left\{ x : \left| \frac{T_b(fv)(x)}{v(x)} \right| > t \right\} \right) \leq uv\left( \left\{ x : \left| \frac{T_b(gv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right) + uv(\Omega^*)$$

$$+ uv\left( \left\{ x \in \mathbb{R}^n \setminus \Omega^* : \left| \frac{T_b(hv)(x)}{v(x)} \right| > \frac{t}{2} \right\} \right)$$

$$= I + II + III.$$

Notice that the hypothesis on $v$ yields $v^{\alpha(1-(q/\beta))} \in A_{q/p}(u)$, that is $v^{1-q} \in A_{q/\beta}(u)$ and therefore $w^q = uv^{1-q} \in A_{q/\beta}$. By applying Tchebychev inequality with $q$ and Theorem 3 we get

$$I \leq \frac{C}{t^q} \int_{\mathbb{R}^n} |T_b(gv)(x)|^q w(x)^q \, dx$$

$$\leq \frac{C}{t^q} \int_{\mathbb{R}^n} |g(x)v(x)|^q w(x)^q \, dx$$

$$= \frac{C}{t^q} \int_{\mathbb{R}^n} |g|^q u(x)v(x) \, dx,$$

provided $\|T_b(gv)w\|_{L^q_t}$ is finite. From this point the estimate follows exactly as in page 13.

Let us prove that $\|T_b(gv)w\|_{L^q_t}$ is indeed finite. As a first step assume that $w^q$ and every component function of $b$ are in $L^\infty$. From (3.2) we have that

$$|T_b(gv)(x)| \leq \sum_{\sigma \in S_{k+1}} \left( \prod_{i=1}^{k+1} b_i^{\sigma_i}(x) \right) |T\left( \prod_{i=1}^{k+1} b_i^{\sigma_i} \right)(gv)(x)|,$$

and using the strong $(q, q)$ type of $T$ we obtain that
\[
\int_{\mathbb{R}^n} |T_b(gv)(x)|^q w^\varphi(x) \, dx \leq C \|w^\varphi\|_L^q \left( \prod_{i=1}^{k+1} \|b_i\|_{L^\infty}^q \right) \int_{\mathbb{R}^n} |g(x)|^q v^\varphi(x) \, dx \\
\leq C \|w^\varphi\|_L^q \left( \prod_{i=1}^{k+1} \|b_i\|_{L^\infty}^q \right) \|g\|_{L^\infty}^q \int_{\text{supp}(g)} v^\varphi(x) \, dx \\
< \infty,
\]

since condition \( \beta > q^2 / (2q - 1) \) implies that \( v^\varphi \) is a weight and, consequently, locally integrable. The additional assumption that \( w^\varphi \) and every \( b_i \) are in \( L^\infty \) can be removed by following standard arguments similarly as in Theorem 3.3 in [10].

The estimate of II follows exactly as we did in the proof of Theorem 1. For III we proceed as in the proof of Theorem 1 in page 19 to get

\[
III \leq I_1 + I_2 + I_3.
\]

For \( I_1 \), as we did in page 19, we get

\[
I_1 \leq \sum_j \int_{Q_j} |b_j(y)| v(y) F_{j,k}(y) \, dy.
\]

We shall prove that \( F_{j,k}(y) \leq Cu(y) \), for every \( y \in Q_j \) and with \( C \) independent of \( j \). By setting \( b_{ij}^{\ell} = [2^\ell Q_j]^{-1} \int_{2^\ell Q_j} b_i \). By Lemmas 8 and 9 we can conclude that

\[
\prod_{i=1}^{k+1} |b_i(x) - b_{ij}| = \sum_{\sigma \in S_{k+1}} \prod_{i=1}^{k+1} |b_i - b_{ij}^{\ell}|^{\sigma_i} |b_{ij}^{\ell+1} - b_{ij}|^{\delta_i} \\
\leq C (\ell + 1)^{k+1} + \sum_{\sigma \in S_{k+1}, [\sigma] > 0} (\ell + 1)^{k+1-[\sigma]} \prod_{i: \sigma_i = 1} |b_i - b_{ij}^{\ell+1}|.
\]

Let \( A_{j,\ell} = \{ x : 2^{\ell-1} r_j < |x - x_{Q_j}| \leq 2^\ell r_j \} \), where \( r_j = c \sqrt{n\ell}(Q_j) \). Then we have that

\[
F_{j,k}(y) \leq C \sum_{\ell=0}^{\infty} (\ell + 1)^{k+1} \int_{A_{j,\ell}} |K(x - y) - K(x - x_{Q_j})|u(x) \, dx \\
+ \sum_{\sigma \in S_{k+1}, [\sigma] > 0} \sum_{\ell=0}^{\infty} (\ell + 1)^{k+1-[\sigma]} \int_{A_{j,\ell}} \prod_{i: \sigma_i = 1} |b_i(x) - b_{ij}^{\ell+1}| |K(x - y) - K(x - x_{Q_j})|u(x) \, dx \\
= F_{j,k}(y) + \sum_{\sigma \in S_{k+1}, [\sigma] > 0} F_{j,k}^\sigma(y).
\]

Let \( B_{j,\ell} = B(x_{Q_j}, 2^\ell r_j) \). Recall that from Remark we have that \( K \in H_{\eta,k+1} \). Then we apply generalized Hölder inequality with functions \( \eta \) and \( \tilde{\eta} \) to get
\[
F_{j,k}^1(y) \leq C \sum_{\ell=0}^{\infty} \epsilon^{k+1+2\ell} r_j^\alpha \| (K(\cdot - y) - K(\cdot - x_{Q_j})) \mathcal{X}_{j,\ell} \|_{\eta,B'_j} \| u \|_{\eta,B'_j}
\]

\[
= C \sum_{\ell=0}^{\infty} \epsilon^{k+1+2\ell} r_j^\alpha \| K(\cdot - y) - K(\cdot - x_{Q_j}) \|_{w_{\alpha,1}-2\ell r_j} \| u \|_{\eta,B'_j}
\]

\[
\leq C \mu M_\eta u(y)
\]

by virtue of Lemma 6.

On the other hand, \( K \in H_{\phi,k+1} \) implies \( K \in H_{\phi,i} \) for every \( 0 \leq l \leq k + 1 \). Fixed \( \sigma \in S_{k+1} \), we apply generalized Hölder inequality with functions \( \phi_i(t) = e^{t/\sigma} - 1, 1 \leq i \leq k + 1, \phi \) and \( \bar{\eta} \), since

\[
\bar{\eta}^{-1}(t)\phi_i^{-1}(t) \prod_{i: \sigma_i = 1} (\log t)^{1/r_i} = \bar{\eta}^{-1}(t)\phi_i^{-1}(t)(\log t)^{1/\sigma} \leq \bar{\eta}^{-1}(t)\phi_i^{-1}(t)(\log t)^{1/\tau} \leq t.
\]

Therefore,

\[
F_{j,k}^\eta(y) \leq C \sum_{\ell=0}^{\infty} \epsilon^{k+1-\sigma} (2\ell r_j)^\alpha \left( \prod_{i: \sigma_i = 1} \| b_i - b_i^{\ell+1} \|_{\exp L^1,B_j} \right) \| K(\cdot - y) - K(\cdot - x_{Q_j}) \|_{\phi,1-2\ell r_j} \| u \|_{\eta,B'_j}.
\]

Notice that \( B'_j \subset 2^{\ell+1} Q_j \). Let \( n_0 \) be the smallest integer that verifies \( 2^{n_0} \geq 2\sqrt{nc} \), then we have that \( 2^{\ell+1} Q_j \subset 2^{n_0+\ell} Q_j \). By virtue of Lemma 8 we get

\[
\prod_{i: \sigma_i = 1} \| b_i - b_i^{\ell+1} \|_{\exp L^1,B_j} \leq \prod_{i: \sigma_i = 1} \| b_i - b_i^{\ell+1} \|_{\exp L^1,2^{n_0+1} Q_j}
\]

\[
\leq \prod_{i: \sigma_i = 1} \left( \| b_i - b_i^{\ell+1} \|_{\exp L^1,2^{n_0+1} Q_j} + \| b_j^{\ell+1} - b_i^{\ell+1} \|_{\exp L^1,2^{n_0+1} Q_j} \right)
\]

\[
\leq \prod_{i: \sigma_i = 1} \left( \| b_i - b_i^{\ell+1} \|_{\exp L^1,2^{n_0+1} Q_j} + Cn_0 \right)
\]

\[
\leq C.
\]

Consequently,

\[
F_{j,k}^\eta(y) \leq C \sum_{\ell=0}^{\infty} \epsilon^{k+1-\sigma} (2\ell r_j)^\alpha \| K(\cdot - y) - K(\cdot - x_{Q_j}) \|_{\phi,1-2\ell r_j} \| u \|_{\eta,B'_j}
\]

\[
\leq CM_\eta u(y) \sum_{\ell=0}^{\infty} \epsilon^{k+1-\sigma} (2\ell r_j)^\alpha \| K(\cdot - y) - K(\cdot - x_{Q_j}) \|_{\phi,1-2\ell r_j}
\]

\[
\leq C \mu M_\eta u(y)
\]

by Lemma 6 again. From these two estimates it follows that \( F_{j,k}(y) \leq Cu(y) \). Then
\[ I_1 \leq C \sum_{j} \int_{Q_j} |h_j(y)|u(y)v(y) \, dy, \]

and from this point we can continue the estimate in the same way as in page 17.

We can achieve the estimate of \( I_2 \) by applying Theorem 18 and proceeding as in page 20.

We conclude with the estimate of \( I_3 \). Since \( K \in H_{\phi} \cap H_{n,k+1} \) we have that \( K \in H_{\phi} \cap H_{n,l} \) for each \( 1 \leq l \leq k+1 \). Fix \( \sigma \in S_{k+1} \) such that \( 0 < |\sigma| < k + 1 \). Then, by applying the inductive hypothesis we get

\[
\left\{ x \in \mathbb{R}^n \setminus \Omega^*: \frac{\left| T_{b_x} \left( \sum_j \prod_{i=1}^{k+1} (b_i - \lambda_{ij})^\theta \right) h_jv \right|(x)}{v(x)} > \frac{t}{2^{k+2}} \right\}
\]

is bounded by

\[
C \int_{\mathbb{R}^n} \Phi_\sigma \left( \sum_j \prod_{i=1}^{k+1} \left| b_j(x) - b_{ij} \right|^\theta \left| h_j(x) \right| \right) u(x)v(x) \, dx,
\]

where \( \Phi_\sigma \) is as in the proof of Theorem 1. Since \( h_j \) is supported in \( Q_j \), this last expression can be written as

\[
I_3^1(\sigma) + I_3^2(\sigma) = C \sum_j \int_{Q_j} \Phi_\sigma \left( \sum_j \prod_{i=1}^{k+1} \left| b_j(x) - b_{ij} \right|^\theta \left| f(x) \right| \right) u(x)v(x) \, dx
\]

\[
+ \sum_j \int_{Q_j} \Phi_\sigma \left( \sum_j \prod_{i=1}^{k+1} \left| b_j(x) - b_{ij} \right|^\theta \right| f_{\lambda_{ij}}(x) \right) u(x)v(x) \, dx.
\]

These two quantities can be estimated as we did in the proof of Theorem 1. This completes the proof. \( \square \)

**Remark 3** We want to point out that similar results than those contained in this article can be achieved by considering non necessarily convolution operators with the obvious changes in the hypothesis of the kernels (see for example the conditions on \( K \) given in [7]).

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