PORT-HAMILTONIAN SYSTEMS AND MONOTONICITY

M.K. CAMLIBEL∗ AND A.J. VAN DER SCHAFT†

Abstract. The relationships between port-Hamiltonian systems modeling and the notion of monotonicity are explored. The earlier introduced notion of incrementally port-Hamiltonian systems is extended to maximal cyclically monotone relations, together with their generating functions. This gives rise to new classes of incrementally port-Hamiltonian systems, with examples stemming from physical systems modeling as well as from convex optimization. An in-depth treatment is given of the composition of maximal monotone and maximal cyclically monotone relations, where in the latter case the resulting maximal cyclically monotone relation is shown to be computable through the use of generating functions. Furthermore, connections are discussed with incremental versions of passivity, and it is shown how incrementally port-Hamiltonian systems with strictly convex Hamiltonians are (maximal) equilibrium independent passive. Finally, the results on compositionality of monotone relations are employed for a convex optimization approach to the computation of the equilibrium of interconnected incrementally port-Hamiltonian systems.

1. Introduction. Port-based modeling of physical systems leads to their description as port-Hamiltonian systems. Such models have turned out to be powerful for purposes of analysis, simulation and control, see e.g. [1, 2, 13]. On the other hand, during the last decades the concept of monotonicity has attracted much attention from multiple points of view. In relation with the current paper the following two aspects of monotonicity are most relevant. First, monotonicity has been a key notion in the study of nonlinear electrical circuits and general nonlinear network dynamics; see e.g. the recent paper [3] for a historical context and references. From a systems and control point of view this view on monotonicity is strongly related to notions of incremental passivity [4] and contraction [5]. Second, monotonicity has evolved as a key concept in convex optimization (see e.g. [6] and the references therein), as well as in nonlinear analysis (see e.g. [7, 8]).

The present paper takes a closer look at the connections between port-Hamiltonian systems and monotonicity, and explores overarching notions. Already in our paper [9], inspired by [10], we defined a new class of dynamical systems, coined as incrementally port-Hamiltonian systems. This was done by replacing the composition of the Dirac structure and the energy-dissipating relation in the standard definition of port-Hamiltonian systems by a general (maximal) monotone relation. Furthermore, in [9] it was shown how monotone relations share the same compositionality property as Dirac structures, and sufficient conditions for the composition of two maximally monotone relations to be again maximally monotone were given. Moreover, the connections between incrementally port-Hamiltonian systems and the notions of incremental and differential passivity were briefly discussed. In the current paper this line of research is continued by developing a full-fledged theory of composition of maximally monotone relations. Furthermore, we take an in-depth look at (maximal) cyclically monotone relations in the context of port-Hamiltonian systems modeling. In particular, we show how, under mild technical conditions, the composition of two maximal cyclically monotone relations is again a maximal cyclically monotone relation. Maximal cyclically monotone relations are of special interest because they correspond to extended convex functions, in the sense that any maximal cyclically monotone rela-

∗Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, The Netherlands, (m.k.camilbel@rug.nl)
†Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, The Netherlands, (a.j.van.der.schaft@rug.nl).
tion is given as the subdifferential of a convex function, called the generating function of the relation. We show how the composition of two maximal cyclically monotone relations can be directly computed via their generating functions. Also, we define an appealing class of incrementally port-Hamiltonian systems which are defined by convex functions of the state and input. Obviously, this connects the theory of incrementally port-Hamiltonian systems to convex optimization. In fact, simple examples are continuous-time gradient algorithms for convex functions, and primal-dual gradient algorithms in case of minimization under affine constraints. Furthermore, we show how the equilibrium of interconnections of incrementally port-Hamiltonian systems defined by maximal cyclically monotone relations can be computed by convex optimization; thereby extending the innovative work of [11]. Finally, another connection with convex analysis appears if we assume the Hamiltonian function of the incrementally port-Hamiltonian system to be convex. This leads to shifted passivity [1] of steady states, and in particular to (maximal) equilibrium independent passivity [11,12].

The organization of the paper is as follows. In Section 2, we quickly review the concepts of Dirac structures and standard port-Hamiltonian systems. This is followed by the definition of incrementally port-Hamiltonian systems in Section 3, and the discussion of a number of examples. In Section 4, we prove that under mild technical conditions the composition of two (maximal) (cyclically) monotone relations is (maximal) cyclically monotone, and thus the power-conserving interconnection of incrementally port-Hamiltonian systems is again an incrementally port-Hamiltonian system. In Section 5 we investigate the various relationships and differences between incrementally port-Hamiltonian and standard port-Hamiltonian systems. As a direct application of the established compositionality theory we study in Section 6 the structure of the set of steady states of incrementally port-Hamiltonian systems, and the computation of the equilibrium of interconnected maximal cyclically monotone port-Hamiltonian systems via convex optimization. Furthermore, assuming convexity of the Hamiltonian, the relations with (maximal) equilibrium independent passivity are investigated. Finally in Section 7 the relations of incrementally port-Hamiltonian systems with incremental passivity and differential passivity are discussed, continuing upon the explorations in [9]. The conclusions are in Section 8.

2. Review of port-Hamiltonian systems on linear state spaces. In order to motivate the definition of incrementally port-Hamiltonian systems we first review the definition of ‘ordinary’ port-Hamiltonian systems; cf. [1,2,13] for more details and ramifications.

Underlying the definition of a port-Hamiltonian system is the geometric notion of a Dirac structure, which relates the power variables of the constituting elements of the system in a power-conserving manner. Since incrementally port-Hamiltonian systems will be defined on linear state spaces we restrict as well attention to port-Hamiltonian systems on linear state spaces, and correspondingly to constant Dirac structures on linear spaces.

Power variables (such as voltages and currents, and forces and velocities), appear in conjugated pairs, whose products have dimension of power. They take values in dual linear spaces, with product meaning duality product. In particular, let $\mathcal{F}$ be a finite-dimensional linear space and $\mathcal{E} := \mathcal{F}^*$ be its dual space. We call $\mathcal{F}$ the space of flow variables, and $\mathcal{E}$ the space of effort variables. The duality product for the pair

\footnote{For the extension to port-Hamiltonian systems on manifolds, and the corresponding notions of Dirac on manifolds we refer to e.g. [1,14,15].}
\((\mathcal{E}, \mathcal{F})\), denoted by \(\langle \cdot \mid \cdot \rangle\), is given as
\[
\langle e \mid f \rangle = e^T f \in \mathbb{R}
\]
for \(e \in \mathcal{E}\) and \(f \in \mathcal{F}\), and is the power associated to the pair \((f, e)\). Furthermore on \(\mathcal{F} \times \mathcal{E}\) an indefinite bilinear form is defined as
\[
\langle ((f_1, e_1), (f_2, e_2)) \rangle = \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle,
\]
where \((f_i, e_i) \in \mathcal{F} \times \mathcal{E}\) with \(i \in \{1, 2\}\). For any subspace \(S \subset \mathcal{F} \times \mathcal{E}\), we denote its orthogonal companion with respect to this indefinite bilinear form by \(S^\perp\).

Throughout the paper, we will work with various spaces of flow/effort variables. By convention, if \(F\) denotes a certain space of flow variables then \(E := F^*\) will denote the corresponding space of effort variables.

**Definition 2.1.** Let \(\mathcal{F}\) be a linear space. A subspace \(D \subset \mathcal{F} \times \mathcal{E}\) is a constant Dirac structure on \(\mathcal{F}\) if \(D = D^\perp\).

From now on in this paper a Dirac structure will simply refer to a constant Dirac structure on a linear space.

**Remark 2.2.** An equivalent definition is the following [13–15]. A Dirac structure is any subspace \(D\) with the property
\[
\langle e \mid f \rangle = 0 \text{ for all } (f, e) \in D,
\]
which is **maximal** with respect to this property. (That is, there does not exist a subspace \(D' \supset D\) such that \(\langle e \mid f \rangle = 0\) for all \((f, e) \in D'\).)

In the finite-dimensional case (as will be the case throughout this paper) the maximal dimension of any subspace \(D\) satisfying (2.1) equals \(\dim \mathcal{F} = \dim \mathcal{E}\). Thus, equivalently, a Dirac structure is any subspace \(D\) satisfying (2.1) together with
\[
\dim D = \dim \mathcal{F}.
\]

The definition of a **port-Hamiltonian system** on a linear space contains the following ingredients (see e.g. [1, 2, 14, 16]). First a Dirac structure \(D\) defined on the space of all flow variables, that is,
\[
D \subset \mathcal{F}_x \times \mathcal{F}_P \times \mathcal{F}_R \times \mathcal{E}_x \times \mathcal{E}_P \times \mathcal{E}_R
\]
Here \((f_x, e_x) \in \mathcal{F}_x \times \mathcal{E}_x\) are the flow and effort variables linking to the energy-storing elements, \((f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R\) are the flow and effort variables linking to energy-dissipating elements, and finally \((f_P, e_P) \in \mathcal{F}_P \times \mathcal{E}_P\) are the flow and effort port variables. The port-Hamiltonian system is defined by specifying, next to its Dirac structure \(D\), the constitutive relations of the energy-dissipating elements, and of the energy-storing elements. An **energy-dissipating relation** is any subset \(R \subset \mathcal{F}_R \times \mathcal{E}_R\) with the property
\[
\langle e_R \mid f_R \rangle \geq 0 \text{ for all } (f_R, e_R) \in R.
\]
The constitutive relations of the energy-storing elements are specified by a Hamiltonian \(H : \mathcal{X} \to \mathbb{R}\), where \(\mathcal{X} = \mathcal{F}_x\). Thus the total energy while at state \(x\) is given as \(H(x)\). This defines the following constitutive relations between the state variables \(x\) and the flow and effort vectors \((f_x, e_x)\) of the energy-storing elements\(^2\)
\[
\dot{x} = -f_x \quad \text{and} \quad e_x = \frac{\partial H}{\partial x}(x).
\]
\(^2\)Throughout this paper the vector \(\frac{\partial H}{\partial x}(x)\) denotes the column vector of partial derivatives; the corresponding row vector denoted as \(\frac{\partial H}{\partial x}^T(x)\).
Definition 2.3. Consider a Dirac structure (2.2), a Hamiltonian $H : X \to \mathbb{R}$, and an energy-dissipating relation $R \subset F_R \times E$ as above. Then the dynamics of the corresponding port-Hamiltonian system on $X$ is given as

\begin{align}
(2.5a) \quad & \left( -\dot{x}(t), f_P(t), -f_R(t), \frac{\partial H}{\partial x}(x(t)), e_P(t), e_R(t) \right) \in D, \\
(2.5b) \quad & (f_R(t), e_R(t)) \in R
\end{align}

at (almost) all time instants $t$.

Equation (2.4) immediately implies the energy balance $\frac{d}{dt} H = \frac{\partial H}{\partial x}(x) \dot{x} = -\langle e_x \mid f_x \rangle$.

Furthermore, the composition $D \rightleftharpoons R := \{(f_x, f_P, e_x, e_P) \in F_x \times F_P \times E_x \times E_P \mid \exists (f_R, e_R) \in R \text{ s.t. } (f_x, f_P, -f_R, e_x, e_P, e_R) \in D\}$ satisfies by the power-conserving property of the Dirac structure and (2.3)

$$\langle e_x^T f_x + e_P^T f_P = e_R^T f_R \geq 0 \rangle$$

for all $(f_x, f_P, e_x, e_P) \in D \rightleftharpoons R$. Taken together this implies that

$$\frac{d}{dt} H(x(t)) \leq e_P^T(t) f_P(t),$$

showing cyclo-passivity of any port-Hamiltonian system, and passivity if $H : X \to \mathbb{R}_+$.

3. Incrementally port-Hamiltonian systems. The basic idea in the definition of an incrementally port-Hamiltonian system, as first introduced in [9], is to replace the composition $D \rightleftharpoons R$ of a Dirac structure $D$ and an energy-dissipating relation $R$ by a monotone relation $\mathcal{M}$. To do so, we begin with a quick review of monotone relations.

Definition 3.1. A relation $\mathcal{M} \subset F \times E$ is said to be
- monotone if
  $$\langle e_1 - e_2 \mid f_1 - f_2 \rangle \geq 0$$
  for all $(f_i, e_i) \in \mathcal{M}$ with $i \in \{1, 2\}$.
- cyclically monotone if
  $$\langle e_0 \mid f_0 - f_1 \rangle + \langle e_1 \mid f_1 - f_2 \rangle + \cdots + \langle e_{m-1} \mid f_{m-1} - f_m \rangle + \langle e_m \mid f_m - f_0 \rangle \geq 0$$
  for all $m \geq 1$ and $(f_i, e_i) \in \mathcal{M}$ with $i \in \{0, 1, \ldots, m\}$.

Since $\langle e_0 \mid f_0 - f_1 \rangle + \langle e_1 \mid f_1 - f_0 \rangle = \langle e_0 - e_1 \mid f_0 - f_1 \rangle$ for all $e_0, f_0, e_1, f_1$, every cyclically monotone relations is automatically monotone.

A simple example of a monotone relation $\mathcal{M} \subset \mathbb{R} \times \mathbb{R}$ is the graph of a monotone (i.e., non-decreasing), possibly discontinuous, function. For example, the graph of the discontinuous function $\theta : \mathbb{R} \to \mathbb{R}$ given by

$$\theta(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
1 & \text{if } x \geq 0
\end{cases}$$

is a monotone relation. This example already motivates the strengthened definition of a maximal monotone relation.
**Definition 3.2.** A relation \( M \subset F \times E \) is called maximal (cyclically) monotone if it is (cyclically) monotone and the implication

\[
M' \text{ is (cyclically) monotone and } M \subset M' \implies M = M'
\]

holds.

The graph of the discontinuous function \( \theta \) in (3.1) is monotone, but not maximal monotone. In fact, its graph can be enlarged so as to obtain the following maximal monotone relation

\[
(3.2) \quad M = \begin{cases} (x,y) \mid y \in \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases} \end{cases}.
\]

Note that the function \( \theta \) in (3.1) can be regarded as the description of a relay, while its closure given by the maximal monotone relation \( M \) defined in (3.2) defines for example an ideal Coulomb friction characteristic.

A few well-known facts are noteworthy. For continuous functions, monotonicity of the graph implies maximal monotonicity (see e.g. [17]). Also, every maximal monotone relation on \( R \times R \) is maximal cyclically monotone. (Hence the above Coulomb friction characteristic in (3.2) is maximal cyclically monotone.) In higher dimensions, however, not every maximal monotone relation enjoys the cyclical monotonicity property. Indeed, for example the relation given by

\[
\{ \left( \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} -y \\ x \end{bmatrix} \right) \mid x, y \in \mathbb{R} \} \subset \mathbb{R}^2 \times \mathbb{R}^2
\]

is maximal monotone but not cyclically monotone. More generally, later on (Proposition 5.4) we will see that Dirac structures are maximal monotone, but not cyclically monotone if they are the graph of a non-zero map.

The importance of maximal cyclically monotone relations \( M \) lies in the fact that they correspond to extended real-valued convex functions. This will be briefly review next, for more details we refer to [17]. Let \( \phi : F \to (-\infty, +\infty] \) be a proper convex function. Its **effective domain** is defined by

\[
\text{dom } \phi := \{ f \in F \mid \phi(f) < +\infty \},
\]

its **subdifferential** of \( \phi \) at \( f \) by

\[
\partial \phi(f) := \begin{cases} \{ e \in E \mid \phi(f) \geq \phi(f) + \langle e, \tilde{f} - f \rangle \quad \forall \tilde{f} \in F \} & \text{if } f \in \text{dom}(\phi) \\ \emptyset & \text{otherwise} \end{cases},
\]

and its **conjugate** \( \phi^* : E \to (-\infty, +\infty] \) by

\[
\phi^*(e) := \sup \{ \langle e, f \rangle - \phi(f) \mid f \in F \}.
\]

If, in addition, \( \phi \) is lower semicontinuous, then \( \phi = (\phi^*)^* \) and

\[
(3.3) \quad e \in \partial \phi(f) \iff f \in \partial \phi^*(e).
\]

It turns out (see [17, Thm. 12.25]) that a relation \( M \subset F \times E \) is maximal cyclically monotone if and only if there exists a proper lower semicontinuous convex function \( \phi \) such that

\[
M = \{ (f, e) \mid e \in \partial \phi(f) \} = \{ (f, e) \mid f \in \partial \phi^*(e) \}.
\]
In this case, we say that $\mathcal{M}$ is generated by $\phi$, or that $\phi$ is a generating function of $\mathcal{M}$. Note that $\phi$ is determined by $\mathcal{M}$ uniquely up to an additive constant.

As an example, consider the relation given by (3.2). One easily verifies that $\mathcal{M}$ is generated by the convex function $\phi(x)$ given by $x \mapsto |x|$. Furthermore,

$$
\phi^*(y) = \begin{cases} 
0 & \text{if } y \in [-1, 1] \\
+\infty & \text{if } y \notin [-1, 1].
\end{cases}
$$

The definition of an incrementally port-Hamiltonian system as introduced in [9] is now extended as follows.

**Definition 3.3.** Consider a (maximal) (cyclically) monotone relation

$$
\mathcal{M} \subset \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_p \times \mathcal{E}_p
$$

and a Hamiltonian $H : \mathcal{F}_x \to \mathbb{R}$. Then the dynamics of the corresponding (maximal) (cyclically) monotone port-Hamiltonian system, is given by the requirement

$$
\left( -\dot{x}(t), f_p(t), \frac{\partial H}{\partial x}(x(t)), e_p(t) \right) \in \mathcal{M}
$$

for all time instants $t$.

**Remark 3.4.** Throughout the terminology incrementally port-Hamiltonian system will be used as shorthand for all systems defined with respect to monotone relations $\mathcal{M}$. Whenever we need to be more precise about the properties of the monotone relation $\mathcal{M}$ we will refer to the system as a (maximal) (cyclically) monotone port-Hamiltonian system.

It follows that the dynamics of any incrementally port-Hamiltonian system satisfies the incremental dissipation inequality

$$
\langle \frac{\partial H}{\partial x}(x_1) - \frac{\partial H}{\partial x}(x_2) \mid \dot{x}_1 - \dot{x}_2 \rangle \leq \langle e^1_p - e^2_p \mid f^1_p - f^2_p \rangle
$$

for all quadruples $(x_i, \dot{x}_i, f^i_p, e^i_p), i = 1, 2$, satisfying $\left( -\dot{x}_i, f^i_p, \frac{\partial H}{\partial x}(x_i), e^i_p \right) \in \mathcal{M}$ with $i \in \{1, 2\}$. The consequences of this dynamical inequality, and especially the relation with incremental and differential passivity, will be discussed in Section 7.

Incrementally port-Hamiltonian systems are ubiquitous in physical systems modeling, as already illustrated by the following examples.

**Example 3.5 (Mechanical systems with friction).** Consider a mechanical system subject to friction. The friction characteristic is given by a relation between $f_R, e_R$.

In the case of a scalar friction characteristic of the form $e_R = R(f_R)$ the system is port-Hamiltonian if the graph of the function $R : \mathbb{R} \to \mathbb{R}$ is in the first and third quadrant. On the other hand, it is maximal cyclically monotone port-Hamiltonian if the function $R$ is a monotonically non-decreasing and moreover continuous, or otherwise the graph of $R$ is extended by the interval between the left- and right limit values at its discontinuities. (A typical example of the latter is Coulomb friction as mentioned above.)

**Example 3.6 (Systems with constant sources).** Physical systems containing non-zero internal constant sources are not port-Hamiltonian but can be incrementally port-Hamiltonian. Consider for example any LC-circuit with passive resistors/conductors and constant voltage and/or current sources. The same holds for an arbitrary mechanical system with constant actuation: incrementally port-Hamiltonian but not port-Hamiltonian for nonzero constant actuation.
**Example 3.7 (Van der Pol oscillator).** Consider an electrical LC-circuit (with possibly nonlinear capacitors and inductors), together with a single conductor with current \( f_R = I \) and voltage \( e_R = V \). In case of a linear conductor \( I = G V, G > 0 \), the system is both port-Hamiltonian and maximal monotone port-Hamiltonian. For a nonlinear conductor \( I = G(V) \) the system is port-Hamiltonian if and only if the graph of the function \( G \) is in the first and third quadrant and maximal monotone port-Hamiltonian if \( G \) is monotonically non-decreasing and continuous, or otherwise the graph of \( G \) is extended by the interval between the left- and right limit values at its discontinuities. For example, the conductor characteristic \( I = \Phi(V) \) where \( \Phi(z) = \gamma z^3 - \alpha z, \alpha, \gamma > 0 \), defines a system which is port-Hamiltonian but not monotone port-Hamiltonian, since the function \( \Phi \) is not monotone. On the other hand, by adding a constant source voltage \( V_0 \) and constant source current \( I_0 \) in such a way that the tunnel diode characteristic

\[ I = \Phi(V - V_0) + I_0, \]

passes through the origin the resulting system (the Van der Pol oscillator) is not port-Hamiltonian, since close to the origin the characteristic is in the second and fourth quadrant, while neither is it incrementally port-Hamiltonian.

An appealing class of maximal cyclically monotone port-Hamiltonian systems is defined as follows. Consider any Hamiltonian \( H : X \to \mathbb{R} \), and any convex function \( K : X^* \times U \to \mathbb{R} \). Then the system

\[
\begin{align*}
\dot{x} &= -\frac{\partial K}{\partial e} (\nabla H(x), u), \quad e = \nabla H(x) := \frac{\partial H}{\partial x}(x) \\
y &= \frac{\partial K}{\partial u}(\nabla H(x), u)
\end{align*}
\]

is a maximal cyclically monotone port-Hamiltonian system with maximal cyclically monotone relation \( M = \text{graph} (\partial K) \). Special case occurs if the convex function \( K(e, u) \) is of the form

\[
K(e, u) = P(e) + e^T Bu,
\]

with \( P \) a convex function of \( e \), and \( B \) an \( n \times m \) matrix. This yields the restricted system class

\[
\begin{align*}
\dot{x} &= -\frac{\partial P}{\partial e}(\nabla H(x)) - Bu \\
y &= B^T \nabla H(x)
\end{align*}
\]

A physical example of the form (3.6) is the following.

**Example 3.8 (Nonlinear RC electrical circuit).** Consider an RC electrical circuit, with nonlinear conductors at the edges and grounded nonlinear capacitors at part of the nodes, while the remaining nodes are the boundary nodes (terminals). Let the circuit graph be defined by an incidence matrix \( D \), split according to the splitting of the capacitor and boundary nodes as

\[
D = \begin{bmatrix} D_c \\ D_b \end{bmatrix}
\]

Furthermore, let the conductors at the edges be given as \( I_j = G_j(V_j) \), where \( I_j, V_j \) are the current through, respectively, voltage, across the \( j \)-th edge, \( j = 1, \ldots, m \). Assume that the conductance functions \( G_j \) are all monotone (however not necessarily in the
first and third quadrant). This means that there exist convex functions $\hat{K}_j$ such that $G_j(V_j) = \frac{d\hat{K}_j}{dV_j}(V_j)$ (if for simplicity we assume that the functions $G_j$ are continuous, and $\hat{K}_j$ are differentiable). Define the convex functions

\[(3.10) \quad \hat{K}(V_1, \ldots, V_m) := \sum_{j=1}^{m} \hat{K}_j(V_j), \quad K(\psi) := \hat{K}(D^T \psi),\]

where $\psi$ is the vector of node voltage potentials. (Recall that by Kirchhoff’s voltage law $V = D^T \psi$.) It is immediately checked that $\frac{\partial K}{\partial \psi} = D \frac{\partial \hat{K}}{\partial V}(D^T \psi)$. Denote the vector of charges of the grounded capacitors by $Q$. It follows by Kirchhoff’s current laws that the dynamics of the nonlinear RC circuit is given by

\[(3.11) \quad \dot{Q} = -D_e \frac{\partial \hat{K}}{\partial V}(D^T \psi), \quad I_e = D_e \frac{\partial \hat{K}}{\partial V}(D^T \psi),\]

where $I_e$ is the vector of injected currents at the boundary nodes of the electrical circuit. According to the splitting of the nodes in capacitor and boundary nodes write $\psi = \begin{bmatrix} \psi_c \\ \psi_e \end{bmatrix}$. Then by specifying the nonlinear grounded capacitors by a Hamiltonian function $H(Q)$ it follows that $\psi_c = \frac{\partial H}{\partial Q}(Q)$.

The system (3.11) is a maximal cyclically monotone port-Hamiltonian system of the form (3.6), with inputs $\psi_e$, state $Q$ (dimension equal to the number of capacitor nodes), and outputs $I_e$. The generating function of its maximal cyclically monotone relation is given by the convex function $K(\psi)$. Finally note that the system equations can be also written in terms of the alternative state vector $\psi_c$ (under the assumption that the map $Q \mapsto \frac{\partial H}{\partial Q}(Q)$ is invertible), by substituting $\dot{\psi}_c = \frac{\partial^2 H}{\partial Q^2}(Q) \dot{Q}$.

An example of a maximal cyclically monotone port-Hamiltonian system of the form (3.6) that is not stemming from physical systems modeling, but instead from optimization, is the following.

**Example 3.9** (Gradient algorithm in continuous time). Consider the problem of minimizing a convex function $P : \mathbb{R}^n \to \mathbb{R}$. The gradient algorithm in continuous time is given as

\[(3.12) \quad \tau \dot{q} = -\frac{\partial P}{\partial q}(q) - Bu, \quad y = B^T q,\]

where $\tau$ is a positive definite matrix determining the time-scales of the algorithm. Here, an input vector $u \in \mathbb{R}^n$ is added in order to represent possible interaction with other algorithms or dynamics (e.g., if the gradient algorithm is carried out in a distributed fashion), defining a conjugate output vector as $y = B^T q \in \mathbb{R}^n$.

This defines a maximal cyclically monotone port-Hamiltonian system with state vector $x := \tau q$, quadratic Hamiltonian $H(x) = \frac{1}{2} x^T \tau^{-1} x$, and maximal cyclically monotone relation

\[(3.13) \quad \mathcal{M} = \{(f, e, s, y, u) \mid -f_s = \frac{\partial P}{\partial q}(q) + Bu, y = B^T e_s\},\]

where $e_s = \frac{\partial H}{\partial x}(x) = \tau^{-1} x$. 

An extended class of maximal monotone port-Hamiltonian systems is defined as

\begin{align}
\dot{x} &= J \nabla H(x) - \frac{\partial P}{\partial e}(\nabla H(x)) - Bu \\
y &= B^T \nabla H(x)
\end{align}

where \( J \) is a skew-symmetric matrix, and \( P \) a convex function as above. It will follow from Proposition 5.4 that if \( J \neq 0 \) then the underlying maximal monotone relation is not derivable from a convex function, and the system is not cyclically monotone port-Hamiltonian anymore. An example within this class is the following.

Example 3.10 (Primal-dual gradient algorithm [18]). Consider the constrained optimization problem

\begin{align}
\begin{align}
\min_{q:Aq=b} P(q),
\end{align}
\end{align}

where \( P: \mathbb{R}^n \to \mathbb{R} \) is a convex function, and \( Aq = b \) are affine constraints for some \( k \times n \) matrix \( A \) and vector \( b \in \mathbb{R}^k \). The resulting Lagrangian function is defined as

\begin{align}
L(q, \lambda) := P(q) + \lambda^T (Aq - b), \quad \lambda \in \mathbb{R}^k,
\end{align}

which is convex in \( q \) and concave in \( \lambda \). The primal-dual gradient algorithm for solving the optimization problem in continuous time is given as

\begin{align}
\begin{align}
\tau_q \dot{q} &= -\frac{\partial L}{\partial q}(q, \lambda) = -\frac{\partial P}{\partial q}(q) - A^T \lambda + u \\
\tau_\lambda \dot{\lambda} &= \frac{\partial L}{\partial \lambda}(q, \lambda) = Aq - b \\
y &= q,
\end{align}
\end{align}

where \( \tau_q, \tau_\lambda \) are positive-definite matrices determining the time-scales of the algorithm. Again, an input vector \( Bu \in \mathbb{R}^n \) is added in order to represent possible interaction with other algorithms or dynamics, defining a conjugated output vector \( y = G^T q \in \mathbb{R}^n \).

This defines a maximal monotone port-Hamiltonian system with state vector \( x = (x_q, x_\lambda) := (\tau_q q, \tau_\lambda \lambda) \), quadratic Hamiltonian

\begin{align}
H(x) = \frac{1}{2} x_q^T \tau_q^{-1} x_q + \frac{1}{2} x_\lambda^T \tau_\lambda^{-1} x_\lambda,
\end{align}

and maximal monotone relation

\begin{align}
\mathcal{M} = \{(f_S, e_S, y, u) \mid -f_S = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} e_S - \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u, y = \begin{bmatrix} 0 & B^T \end{bmatrix} e_S \},
\end{align}

where

\begin{align}
e_S = \nabla H(x) = \begin{bmatrix} \tau_q^{-1} x_q \\ \tau_\lambda^{-1} x_\lambda \end{bmatrix} = \begin{bmatrix} q' \\ \lambda \end{bmatrix}
\end{align}

See for an application the optimization of social welfare in a dynamic pricing algorithm for power networks [19].
4. Composition of monotone relations. A cornerstone of port-Hamiltonian systems theory is the fact that the power-conserving interconnection of port-Hamiltonian systems defines again a port-Hamiltonian system. This in turn is based on the fact that the composition of Dirac structures is again a Dirac structure. In this section we will show that the same property holds for incrementally port-Hamiltonian systems. This follows from the corresponding compositionality property of (maximal) (cyclically) monotone relations.

Let us start by considering two monotone relations \( \mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{F} \times \mathcal{E}_a \times \mathcal{E} \) and \( \mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F} \times \mathcal{E}_b \times \mathcal{E} \). Define the composition of \( \mathcal{M}_a \) and \( \mathcal{M}_b \), denoted as \( \mathcal{M}_a \rightrightarrows \mathcal{M}_b \), as before, by

\[
\mathcal{M}_a \rightrightarrows \mathcal{M}_b := \{(f_a, f_b, e_a, e_b) \in \mathcal{F}_a \times \mathcal{F}_b \times \mathcal{E}_a \times \mathcal{E}_b \mid \exists (f, e) \in \mathcal{F} \times \mathcal{E} \text{ s.t. } (f_a, f, e_a, e) \in \mathcal{M}_a, (f_b, -f, e_b, e) \in \mathcal{M}_b\}.
\]

(4.1) Thus the composition of \( \mathcal{M}_a \) and \( \mathcal{M}_b \) is obtained by imposing the interconnection constraints

\[
f_1 = -f_2, \quad e_1 = e_2,
\]

on the vectors \((f_a, f_1, e_a, e_1) \in \mathcal{M}_a \) and \((f_b, f_2, e_b, e_2) \in \mathcal{M}_b \) and looking at the resulting vectors \((f_a, f_b, e_a, e_b) \in \mathcal{F}_a \times \mathcal{F}_b \times \mathcal{E}_a \times \mathcal{E}_b \).

Whenever interconnection flow and effort spaces \( \mathcal{F} \) and \( \mathcal{E} \) are clear from the context, we will simply write \( \mathcal{M}_a \rightrightarrows \mathcal{M}_b \). The following result is straightforward.

**Proposition 4.1.** Let \( \mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{F} \times \mathcal{E}_a \times \mathcal{E} \) and \( \mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F} \times \mathcal{E}_b \times \mathcal{E} \) be (cyclically) monotone relations. Then, \( \mathcal{M}_a \rightrightarrows \mathcal{M}_b \subset \mathcal{F}_a \times \mathcal{F}_b \times \mathcal{E}_a \times \mathcal{E}_b \) is (cyclically) monotone.

**Proof.** Suppose that both \( \mathcal{M}_a \) and \( \mathcal{M}_b \) are monotone relations. Let

\[
(f_a, f_b, e_a, e_b), (\tilde{f}_a, \tilde{f}_b, \tilde{e}_a, \tilde{e}_b) \in \mathcal{M}_a \rightrightarrows \mathcal{M}_b.
\]

Then, there exist \((f, e), (\tilde{f}, \tilde{e}) \in \mathcal{F} \times \mathcal{E} \) such that \((f_a, f, e_a, e) \in \mathcal{M}_a \) and \((f_b, -f, e_b, e) \in \mathcal{M}_b \). From monotonicity of \( \mathcal{M}_a \) and \( \mathcal{M}_b \), we have

\[
\begin{bmatrix} e_a - \tilde{e}_a \\ e - \tilde{e} \end{bmatrix} \begin{bmatrix} f_a - \tilde{f}_a \\ f - \tilde{f} \end{bmatrix} \succeq 0 \quad \text{and} \quad \begin{bmatrix} e_b - \tilde{e}_b \\ e - \tilde{e} \end{bmatrix} \begin{bmatrix} f_b - \tilde{f}_b \\ -f + f \end{bmatrix} \succeq 0.
\]

By adding these left hand sides of these inequalities, we obtain

\[
\begin{bmatrix} e_a - \tilde{e}_a \\ e_b - \tilde{e}_b \end{bmatrix} \begin{bmatrix} f_a - \tilde{f}_a \\ f_b - \tilde{f}_b \end{bmatrix} \succeq 0.
\]

This means that \( \mathcal{M}_a \rightrightarrows \mathcal{M}_b \) is monotone. The cyclical monotone case follows in a similar fashion. \( \blacksquare \)

Also the composition of two maximal monotone relations turns out to be maximal monotone; provided certain (mild) regularity conditions are met. To elaborate on this, we first introduce some nomenclature and review some known facts about maximal monotone relations.

For a set \( S \subset \mathcal{F} \), \( \text{cl} S \) denotes its closure. The relative interior of a convex set \( C \subset \mathcal{F} \) is denoted by \( \text{rint} C \). A set \( S \subset \mathcal{F} \) is said to be nearly convex if there exists
a convex set $C \subseteq \mathcal{F}$ such that $C \subseteq S \subseteq \text{cl}C$. For a nearly convex set $S$, in general, there can be multiple convex sets $C$ satisfying $C \subseteq S \subseteq \text{cl}C$. For any such set $C$, however, we have that $\text{cl}C = \text{cl}S$. As such, $\text{cl}S$ is convex if $S$ is nearly convex. Based on this observation, one can extend the notion of relative interior to nearly convex sets by defining $\text{rint}S = \text{rint}(\text{cl}S)$.

Let $S \subseteq \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{E}_1 \times \mathcal{E}_2$. The projection of $S$ on $\mathcal{F}_1 \times \mathcal{F}_2$, denoted by $\Pi(S, \mathcal{F}_1 \times \mathcal{F}_2)$, is defined as

$$\Pi(S, \mathcal{F}_1 \times \mathcal{F}_2) := \{(f_1, f_2) \mid \exists (e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \text{ s.t. } (f_1, f_2, e_1, e_2) \in S\}.$$ 

We define projections of $S$ on $\mathcal{E}_1 \times \mathcal{E}_2$, $\mathcal{F}_i \times \mathcal{E}_j$, $\mathcal{F}_i$, and $\mathcal{E}_j$ in a similar fashion.

Let $S \subseteq \mathcal{F} \times \mathcal{G}$ be a nearly convex set. Then, both $\Pi(S, \mathcal{F})$ and $\Pi(S, \mathcal{G})$ are nearly convex sets. Furthermore, one can show that

$$\text{rint}S = \{(f, g) \mid f \in \text{rint} \Pi(S, \mathcal{F}) \text{ and } g \in \text{rint} \Pi(S \cap \{(f) \times \mathcal{G}), \mathcal{G})\}. \tag{4.3}$$

Let $\mathcal{M} \subset \mathcal{F} \times \mathcal{E}$ be a maximal monotone relation. Then, the projections $\Pi(\mathcal{M}, \mathcal{F})$ and $\Pi(\mathcal{M}, \mathcal{E})$ are nearly convex sets [17, Thm. 12.41].

Let $L : \mathcal{G} \to \mathcal{H}$ be a linear map and $L^* : \mathcal{H}^* \to \mathcal{G}^*$ denote its adjoint. For maximal monotone relations $\mathcal{M} \subseteq \mathcal{H} \times \mathcal{H}^*$ and $\mathcal{N} \subseteq \mathcal{G} \times \mathcal{G}^*$, define $\mathcal{M}_L \subseteq \mathcal{G} \times \mathcal{G}^*$ and $L\mathcal{N} \subseteq \mathcal{H} \times \mathcal{H}^*$ by

$$\mathcal{M}_L = \{(g, L^*h^*) \mid (Lg, h^*) \in \mathcal{M}\}$$

$$L\mathcal{N} = \{(Lg, h^*) \mid (g, L^*h^*) \in \mathcal{N}\}.$$ 

From [17, Thm. 12.43], we know that $\mathcal{M}_L$ is maximal monotone if

$$\text{im}L \cap \text{rint} \Pi(\mathcal{M}, \mathcal{H}) \neq \emptyset \tag{4.4}$$

and $L\mathcal{N}$ is maximal monotone if

$$\text{im}L^* \cap \text{rint} \Pi(\mathcal{N}, \mathcal{G}) \neq \emptyset. \tag{4.5}$$

Furthermore, if $\mathcal{M}$ is generated by $\phi : \mathcal{H} \to (-\infty, +\infty]$ and

$$\text{im}L \cap \text{rint} \text{dom}(\phi) \neq \emptyset, \tag{4.6}$$

then $\mathcal{M}_L$ is generated by $\phi \circ L$ given by $h \mapsto \phi(Lh)$. Dually, if $\mathcal{N}$ is generated by $\psi : \mathcal{G} \to (-\infty, +\infty]$ and

$$\text{im}L^* \cap \text{rint} \text{dom}(\psi^*) \neq \emptyset, \tag{4.7}$$

then $L\mathcal{N}$ is generated by the function $(\psi^* \circ L^*)^*$.

**Theorem 4.2.** Let $\mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{E}_a \times \mathcal{E}$ and $\mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F} \times \mathcal{E}_b \times \mathcal{E}$ be maximal monotone relations. Let

$$C_f = \{(f_1, f_2) \mid f_1 \in \Pi(\mathcal{M}_a, \mathcal{F}) \text{ and } f_2 \in \Pi(\mathcal{M}_b, \mathcal{F})\}.$$ 

and

$$C_e = \{(e_1, e_2) \mid \exists f \text{ s.t. } (f, e_1) \in \Pi(\mathcal{M}_a, \mathcal{F} \times \mathcal{E}) \text{ and } (-f, e_2) \in \Pi(\mathcal{M}_b, \mathcal{F} \times \mathcal{E})\}.$$ 

Suppose that there exists $((\bar{f}, \bar{e})) \in \mathcal{F} \times \mathcal{E}$ such that
Proof. First, we give an alternative characterization of \( M_a =: M_b \). Let \( M =: (F,a) \times F \times F \times F \times F \times F \) be defined by

\[
M := \{ (f_a, f_1, f_2, e_a, e_1, e_2) \mid (f_a, f_1, e_a, e_1) \in M_a \text{ and } (f_2, e_b, e_2) \in M_b \}.
\]

Since \( M_a \) and \( M_b \) are both maximal monotone, so is \( M \). Let \( A : F_a \times F \times F_b \to F_a \times F \times F_b \times F \) be the linear map given by

\[
(f_a, f, f_b) \mapsto (f_a, f, f_b, -f)
\]

and \( B : F_a \times F \times F_b \to F_a \times F_b \) be the linear map given by

\[
(f_a, f, f_b) \mapsto (f_a, f_b).
\]

Note that \( A^* : E_a \times E \times E_b \times E \to E_a \times E \times E_b \) is given by

\[
(e_a, e_1, e_2) \mapsto (e_a, e_1 - e_2, e_b)
\]

and \( B^* : E_a \times E_b \to E_a \times E \times E_b \) given by

\[
(e_a, e_b) \mapsto (e_a, 0, e_b).
\]

Now, we claim that \( M_a =: M_b =: B(M_A) \). To see this, note that

\[
M_A := \{ (f_a, f_b, e_a, e_1 - e_2, e_b) \mid (f_a, f, f_b, -f, e_a, e_1, e_2) \in M \}
\]

and

\[
B(M_A) := \{ (f_a, f_b, e_a, e_b) \mid \exists f \in F \text{ s.t. } (f_a, f, f_b, e_a, 0, e_b) \in M_a \}
\]

\[
= \{ (f_a, f_b, e_a, e_b) \mid \exists (f, e) \in F \times E \text{ s.t. } (f_a, f, e_a, e) \in M_a \text{ and } (f_b, -f, e_b, e) \in M_b \}
\]

\[(4.8)\]

where \( A : F_a \times V_a \times F_b \times V_b \to F_a \times V_a \times V_c \times V_d \times F_b \times V_b \) is the linear map given by

\[
(f_a, v_a, f_b, v_b) \mapsto (f_a, v_a, v_b, f_b, v_b)
\]

Since \( M \) is maximal monotone, we see from (4.4) that \( M_A \) is maximal monotone if

\[
(4.9) \quad \text{im } A \cap \text{rint } \Pi(M, F_a \times F \times F_b \times F) \neq \emptyset.
\]

From (4.3), it follows that

\[
\text{rint } S_A = \{ (f_a, f_1, f_2) \mid (f_1, f_2) \in \text{rint } \Pi(S_A, F \times F) \}
\]

and \( (f_a, f_b) \in \text{rint } \Pi(S_A \cap (F_a \times \{f_1\} \times F_b \times \{f_2\}), F_a \times F_b) \).
where $S_A = \Pi(M, F_a \times F \times F_b \times F)$. By observing that $\Pi(S_A, F \times F) = \Pi(M, F \times F) = C_f$, we see that the condition (4.9) is equivalent to the existence of $\tilde{f} \in F$ such that $(\tilde{f}, \tilde{e}) \in \text{rint} \ C_f$. Therefore, $M_A$ is maximal monotone due to (i). As such, it follows from (4.5) that $B(M_A)$, and thus $M_a \preceq M_b$, is maximal monotone if

\begin{equation}
(4.10) \quad \text{im} B^* \cap \text{rint} \Pi(M_A, E_a \times E \times E_b) \neq \emptyset.
\end{equation}

To verify this condition, let $S_B = \Pi(M_A, E_a \times E \times E_b)$ and note that

$$
\text{rint} \ S_B = \left\{ (e_a, e_e) \mid e \in \text{rint} \Pi(S_B, E) \right\}
$$

and $(e_a, e_e) \in \text{rint} \Pi(S_B \cap (E_a \times \{e\} \times E_b))$.

Therefore, (4.10) holds if and only if $0 \in \text{rint} \Pi(S_B, E)$. Note that

$$
\Pi(S_B, E) = \left\{ e \mid \exists (e_1, e_2) \text{ s.t. } e = e_1 - e_2 \right\}.
$$

As such, (ii) is equivalent to $0 \in \text{rint} \Pi(S_B, E)$ and hence (4.10). Consequently, $M_a \preceq M_b$ is maximal monotone.

Furthermore, maximal cyclical monotonicity is also preserved under composition as stated in the following theorem.

**Theorem 4.3.** Let $M_a \subset F_a \times F \times E_a \times E$ and $M_b \subset F_b \times F \times E_b \times E$ be maximal cyclical monotone relations that are generated by proper lower semicontinuous convex functions $\phi_a : F_a \times F \rightarrow (-\infty, +\infty]$ and $\phi_b : F_b \times F \rightarrow (-\infty, +\infty]$, respectively. Let

\begin{equation}
C_f = \left\{ (f_1, f_2) \mid f_1 \in \Pi(\text{dom} \phi_a, F) \text{ and } f_2 \in \Pi(\text{dom} \phi_a, F) \right\}.
\end{equation}

and

\begin{equation}
C_e = \left\{ (e_1, e_2) \mid \exists f \text{ s.t. } (f, e_1) \in \text{dom} \phi_a \text{ and } (-f, e_2) \in \text{dom} \phi_b \right\}.
\end{equation}

Suppose that there exists $(\tilde{f}, \tilde{e}) \in F \times E$ such that

(i) $(\tilde{f}, -\tilde{f}) \in \text{rint} C_f$ and

(ii) $(\tilde{e}, -\tilde{e}) \in \text{rint} C_e$.

Then, $M_a \preceq M_b \subset F_a \times F_b \times E_a \times E_b$ is a maximal cyclical monotone relation that is generated by $\theta^* : F_a \times F_b \rightarrow (-\infty, +\infty]$ where $\theta : E_a \times E_b \rightarrow (-\infty, +\infty]$ is given by

$$
\theta(e_a, e_b) = \phi^*(e_a, 0, e_b)
$$

and $\phi : F_a \times F \times F_b \rightarrow (-\infty, +\infty]$ is given by

$$
\phi(f_a, f, f_b) = \phi_a(f_a, f) + \phi_b(f_b, -f).
$$

**Proof.** Let $M, A, B, M_A, \text{ and } B(M_A)$ be as in the proof of Theorem 4.2. Note that $M$ is generated by the proper lower semicontinuous convex function $\phi_{ab} : F_a \times F \times F_b \times F \rightarrow (-\infty, +\infty]$ given by

$$
\phi_{ab}(f_a, f_1, f_2, f_b) = \phi_a(f_a, f_1) + \phi_b(f_b, f_2).
$$

For a proper convex function $\Psi : G \rightarrow (-\infty, +\infty]$ and a linear map $L : H \rightarrow G$, let $\psi \circ L : H \rightarrow (-\infty, +\infty]$ denote the function given by $h \mapsto \psi(Lh)$. It follows from the definition of $M_A$ that $(f_a, f, f_b, e_a, e, e_b)$ if and only if

\begin{equation}
(4.11) \quad (e_a, e, e_b) \in A^* \partial \phi_{ab}(A(f_a, f, f_b)).
\end{equation}
Similar arguments as employed in the proof Theorem 4.2 show that (i) is equivalent to
\[ \text{im } A \cap \text{rint dom } \phi_{ab} \neq \emptyset. \]
Then, it follows from [20, Prop. 5.4.5] that \( A^* \partial \phi_{ab}(Ax) = \partial(\phi_{ab} \circ A)(x) \) for all \( x \in F_a \times F \times F_b \). Since \( \phi_{ab} \) is lower semicontinuous, so is \( \phi_{ab} \circ A \). As such, we see from (4.11) that \( \mathcal{M}_A \) is maximal cyclically monotone and generated by \( \phi = \phi_{ab} \circ A \). Now, it follows from (4.8), the definition of \( B(M_A) \), and (3.3) that \( (f_a, f_b, e_a, e_b) \in \mathcal{M}_a \Rightarrow \mathcal{M}_b \)
if and only if
\[ (f_a, f_b) \in B \partial \phi^* (B^*(e_a, e_b)) \]
One can show that (ii) is equivalent to
\[ \text{im } B^* \cap \text{rint dom } \phi^* \neq \emptyset \]
by employing similar arguments to those in the proof of Theorem 4.2. Then, it follows from [20, Prop. 5.4.5] that \( B \partial \phi^* (B^*y) = \partial(\phi^* \circ B^*)(y) \) for all \( y \in F_a \times F_b \). Since \( \phi^* \) is lower semicontinuous, so is \( \phi^* \circ B^* \). Consequently, (4.12) and (3.3) imply that \( \mathcal{M}_a \Rightarrow \mathcal{M}_b \) is maximal cyclically monotone and generated by \( (\phi^* \circ B^*)^\circ \). Since \( \theta = \phi^* \circ B^* \), this concludes the proof.

The following adaptation of Theorem 4.3 applies to the alternative interconnection \( e_2 = f_3, e_3 = f_2 \). Consider two maximal cyclically monotone relations \( \mathcal{M}_a, \mathcal{M}_b \) with generating convex functions \( g(e_1, e_2) \) and \( h(e_3, e_4) \):
\[ \mathcal{M}_a = \{(e_1, e_2, f_1 = \partial e_1 g, f_2 = \partial e_2 g)\}, \quad \mathcal{M}_b = \{(e_3, e_4, f_3 = \partial e_3 h, f_4 = \partial e_4 h)\} \]
Assume \( \dim e_2 = \dim e_3 \), and consider the positive feedback interconnection \( e_2 = f_3, e_3 = f_2 \). This yields the relation
\[ \mathcal{M} := \{(e_1, e_4, f_1, f_4) | \exists e_2 = f_3, e_3 = f_2 \text{ s.t. } (e_1, e_2, f_1, f_2) \in \mathcal{M}_a, (e_3, e_4, f_3, f_4) \in \mathcal{M}_b \}. \]

**Proposition 4.4.** The relation \( \mathcal{M} \) is maximal cyclically monotone, with generating function
\[ (e_1, e_4) \mapsto \inf_{e_2, e_3} (g(e_1, e_2) + h(e_3, e_4) - e_2^T e_3) \]

**Proof.** The proof can be directly based on the proof of Theorem 4.3 formulated for the canonical interconnection \( f_2 = -f_3, e_2 = e_3 \). Define \( h^* \) as the partial convex conjugate of \( h \) with respect to \( e_3 \), i.e.,
\[ h^*(f_3, e_4) = \sup_{e_3} [e_3^T f_3 - h(e_3, e_4)] \]
Note that by the definition of the partial convex conjugate
\[ \partial_{f_3} h^* = e_3, \quad \partial_{e_3} h^* = -\partial_{e_4} h \]
Hence \( \mathcal{M} \) as obtained from \( \mathcal{M}_a, \mathcal{M}_b \) via the interconnection equations \( e_2 = f_3, e_3 = f_2 \), can be also understood as the canonical interconnection of \( \mathcal{M}_a \) and \( \mathcal{M}_b^* \), where
\[ \mathcal{M}_b^* = \{(f_3, e_4) | \exists e_3 = \partial_{f_3} h^*, f_4 = -\partial_{e_4} h^*)\} \]
Thus in view of Theorem 4.3 \( M \) has the generating function
\[
\inf_x [g(e_1, x) - h^*(x, e_4)]
\]
Substitution of the expression of \( h^* \) is immediately seen to result in (4.14). 

An application of Proposition 4.4 is the following. Consider two maximal cyclically monotone port-Hamiltonian systems of the form as given in (3.6), that is
\[
\begin{align*}
\dot{x}_i &= -\frac{\partial K_i}{\partial e_i}(\nabla H_i(x_i), u_i), \\
y_i &= \frac{\partial K_i}{\partial u_i}(\nabla H_i(x_i), u_i), \quad i = 1, 2
\end{align*}
\]
Now interconnect both systems by the positive feedback \( u_1 = y_2, u_2 = y_1 \). By Proposition 4.4 this leads to the maximal cyclically monotone port-Hamiltonian system
\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -\begin{bmatrix} \frac{\partial K_1}{\partial x_1} \\ \frac{\partial K_2}{\partial x_2} \end{bmatrix} (\nabla H_1(x_1), \nabla H_2(x_2)),
\]
where the convex function \( K(e_1, e_2) \) is given as
\[
K(e_1, e_2) = \inf_{u_1, u_2} (K_1(e_1, u_2) + K_2(e_2, u_2) - u_1^T u_2)
\]

**Example 4.5.** Consider two gradient algorithms in continuous time, i.e.,
\[
\begin{align*}
\tau_i \dot{q}_i &= -\frac{\partial P_i}{\partial q_i}(q_i) - B_i u_i \\
y_i &= B_i^T q_i, \quad i = 1, 2
\end{align*}
\]
which converge for \( u_i = 0 \) to the minimum of the convex functions \( P_i(q_i) \). Now consider the coupled gradient algorithm that is resulting from the interconnection \( u_1 = y_2, u_2 = y_1 \). This leads to a maximal cyclically monotone port-Hamiltonian system with respect to the convex function determined as
\[
\begin{align*}
\inf_{u_1, u_2} (P_1(q_1) + P_1(q_2) + q_1^T B_1 u_1 + q_2^T B_2 u_2 - u_1^T u_2)
\end{align*}
\]
Clearly the minimum is attained for \( u_1 = B_1^T q_2, u_2 = B_2^T q_1 \), leading to the convex function
\[
P(q_1, q_2) := P_1(q_1) + P_2(q_2) + q_1^T B_1 B_2^T q_2
\]
Hence the coupling of the two gradient algorithms computes the minimum of \( P(q_1, q_2) \).

**5. When are port-Hamiltonian systems incrementally port-Hamiltonian, and conversely.** Let us first relate (maximal) (cyclically) monotone relations to the notions of Dirac structures and energy-dissipating relations as used in the definition of port-Hamiltonian systems.

We begin with showing that every Dirac structure is a maximal monotone relation, and maximal cyclically monotone if and only if it belongs to a special subclass of Dirac structures. This special subclass is defined and characterized as follows [21].

**Definition 5.1.** A Dirac structure \( D \subset F \times E \) is separable if
\[
<e_a | f_b> = 0, \quad \text{for all } (f_a, e_a), (f_b, e_b) \in D
\]
Separable Dirac structures have the following simple geometric characterization [21].

**Proposition 5.2.** Any separable Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ can be written as

$$
\mathcal{D} = \mathcal{K} \times \mathcal{K}^\perp
$$

for some subspace $\mathcal{K} \subset \mathcal{F}$, where $\mathcal{K}^\perp = \{ e \in \mathcal{E} \mid \langle e \mid f \rangle = 0, \forall f \in \mathcal{K} \}$. Conversely, any subspace $\mathcal{D}$ as in (5.2) for some $\mathcal{K} \subset \mathcal{F}$ is a separable Dirac structure.

**Remark 5.3.** A typical example of a separable Dirac structure is provided by Kirchhoff’s current and voltage laws of an electrical circuit. Indeed, take e.g. $\mathcal{F}$ to be the space of currents, $\mathcal{K}$ the space of currents satisfying Kirchhoff’s current laws. Then $\mathcal{E} = \mathcal{F}^\ast$ is the space of voltages, and $\mathcal{K}^\perp$ defines Kirchhoff’s voltage laws. Moreover, $\langle e \mid f \rangle = 0$ for all $(f_a, e_a), (f_b, e_b) \in \mathcal{K} \times \mathcal{K}^\perp$ expresses Tellegen’s law.

**Proposition 5.4.** Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is maximal monotone. It is maximal cyclically monotone if and only if $\mathcal{D}$ is separable. If $\mathcal{D}$ is the graph of a mapping $J : \mathcal{E} \to \mathcal{F}$ or $J : \mathcal{F} \to \mathcal{E}$ then $\mathcal{D}$ is cyclically monotone if and only if $J = 0$.

**Proof.** Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ be a Dirac structure. Let $(f_i, e_i) \in \mathcal{D}$ with $i = 1, 2$. Since $\langle e \mid f \rangle = 0$ for all $(f, e) \in \mathcal{D}$ due to Remark 2.2, we obtain by linearity

$$
\langle e_1 - e_2 \mid f_1 - f_2 \rangle = 0.
$$

Therefore, $\mathcal{D}$ is monotone on $\mathcal{F} \times \mathcal{E}$. Let $\mathcal{D}'$ be a monotone relation on $\mathcal{F} \times \mathcal{E}$ such that $\mathcal{D} \subseteq \mathcal{D}'$. Let $(f', e') \in \mathcal{D}'$ and $(f, e) \in \mathcal{D}$. Since $\mathcal{D}'$ is monotone, $\mathcal{D}$ is a subspace, and $\mathcal{D} \subseteq \mathcal{D}'$, we have

$$
0 \leq \langle e' - \alpha e \mid f' - \alpha f \rangle = \langle e' \mid f' \rangle - \alpha (\langle e' \mid f \rangle + \langle e \mid f' \rangle)
$$

for any $\alpha \in \mathbb{R}$. This means that $\langle e' \mid f \rangle + \langle e \mid f' \rangle = 0$, and hence $(f', e') \in \mathcal{D}^\perp = \mathcal{D}$. Therefore, we see that $\mathcal{D}' \subseteq \mathcal{D}$, and thus $\mathcal{D}' = \mathcal{D}$. Consequently, $\mathcal{D}$ is maximal monotone.

Next, let $\mathcal{D}$ be separable, i.e. $\langle e_a \mid f_b \rangle = 0$ for all $(f_a, e_a), (f_b, e_b) \in \mathcal{D}$. Then it immediately follows from Definition 3.1 that $\mathcal{D}$ is cyclically monotone. Conversely, let $\mathcal{D}$ be cyclically monotone. Then take any $(f_i, e_i) \in \mathcal{D}$ with $i \in \{0, 1, 2\}$. It follows from Definition 3.1 that

$$
\langle e_0 \mid f_0 - f_1 \rangle + \langle e_1 \mid f_1 - f_2 \rangle + \langle e_2 \mid f_2 - f_0 \rangle \geq 0.
$$

Since $\langle e \mid f \rangle = 0$ for all $(f, e) \in \mathcal{D}$ due to Remark 2.2, we see that

$$
\langle e_0 \mid -f_1 \rangle + \langle e_1 \mid -f_2 \rangle + \langle e_2 \mid -f_0 \rangle \geq 0.
$$

As $\mathcal{D}$ is a subspace, $(-f_0, -e_0) \in \mathcal{D}$. Therefore, we see from (5.3) that

$$
\langle -e_0 \mid -f_1 \rangle + \langle e_1 \mid -f_2 \rangle + \langle e_2 \mid f_0 \rangle \geq 0.
$$

By summing this inequality and (5.3), we obtain $\langle e_1 \mid -f_2 \rangle \geq 0$. By using the fact that $\mathcal{D}$ is a subspace, we see that $\langle e_1 \mid f_2 \rangle = 0$, and thus $\mathcal{D}$ is separable.

Finally, let $\mathcal{D}$ be the graph of a mapping $J : \mathcal{E} \to \mathcal{F}$. Since $\mathcal{D}$ is a Dirac structure necessarily $J$ is skew-symmetric. Take again any $(f_i, e_i) \in \mathcal{D}$ with $i \in \{0, 1, 2\}$, where now $f_1 = Je_i$. Then if $\mathcal{D}$ is cyclically monotone

$$
\langle e_0 \mid J(e_0 - e_1) \rangle + \langle e_1 \mid J(e_1 - e_2) \rangle + \langle e_2 \mid J(e_2 - e_0) \rangle \geq 0
$$

Using $\langle e_i \mid Je_i \rangle = 0$ by skew-symmetry of $J$ this yields $\langle e_1 \mid Je_2 \rangle \geq 0$ for all $e_1, e_2$, which clearly implies $J = 0$. Similarly for $J : \mathcal{F} \to \mathcal{E}$. ■
Remark 5.5. As noticed before, a typical example of a separable Dirac structure is given by Kirchhoff’s current and voltage laws. In particular, it follows that for any electrical circuit there exists a convex function specifying Kirchhoff’s current and voltage laws. Indeed, consider an electrical circuit whose circuit graph is given by the incidence matrix $D$. Identify as above $F$ with the set of currents $f = I$ through the edges, and $E = F^*$ with the set of voltages $e = V$ across the edges. Then Kirchhoff’s current laws are given as $DI = 0$ and Kirchhoff’s voltage laws as $V \in \text{im} D^T$. The convex function generating the resulting separable Dirac structure is given by (see the proof lines of Lemma 5.6)

\[
\phi(f) = \begin{cases} 
0 & \text{if } f \in \ker D \\
\infty & \text{otherwise}
\end{cases}
\]

An arbitrary energy-dissipating relation need not be a (maximal) monotone relation; as was also demonstrated by some of the examples in the previous section. A special type of energy-dissipating relation that is a maximal cyclically monotone relation is that of a linear energy-dissipating relation which is of maximal dimension. Such an energy-dissipating relation in the port-variables $(f, e) \in F \times E$ can be represented as a subspace

\[
R = \{(f, e) \in F \times E \mid R_f f - R_e e = 0\},
\]

where the matrices $R_f, R_e$ satisfy the property

\[
R_f R_e^T = R_e R_f^T \succeq 0,
\]

together with the dimensionality condition

\[
\text{rank } [R_f \quad R_e] = \dim F.
\]

First of all, this is seen to define an energy-dissipating relation as follows. By the dimensionality condition (5.7) and the equality in (5.6) we can equivalently rewrite the kernel representation (5.5) as an image representation

\[
f = R_e^T \lambda, \quad e = R_f^T \lambda.
\]

That is, any pair $(f, e)$ satisfying (5.5) also satisfies (5.8) for some $\lambda$, and conversely, every $(f, e)$ satisfying (5.8) for some $\lambda$ also satisfies (5.5). Hence by (5.6) for all $(f, e)$ satisfying (5.5)

\[
e^T f = (R_f^T \lambda)^T R_e^T \lambda = \lambda^T R_f R_e^T \lambda \geq 0
\]

A subspace $R \subset F \times E$ as in (5.5) where $R_f, R_e$ satisfy (5.6) and (5.7) is called a linear resistive structure. A linear resistive structure can be regarded as a geometric object having properties which are opposite to those of a Dirac structure, in the sense that a Dirac structure can be regarded as the generalization of a skew-symmetric map, while a linear resistive relation as the generalization of a positive semi-definite symmetric map. (Geometrically $R$ defines a Lagrangian subspace of the linear space $F \times E$.)

It turns out that every linear resistive structure $R \subset F \times E$ is maximal cyclically monotone. To elaborate further, note that there exists $R = R^T \succeq 0$ such that

\[
R_e R R_e^T = R_e R_f^T
\]
due to [22, Thm. 2.5]. In general, $R$ is not unique but the matrix $RR^T$ does not depend on the choice of $R = R^T \succeq 0$ satisfying (5.10). Now, define the extended real-valued convex function 

\begin{equation}
\phi_R(f) = \begin{cases}
\frac{1}{2} f^T R f & \text{if } f \in \text{im} R^T_e \\
+\infty & \text{otherwise}
\end{cases}
\end{equation}

With these preparations, we can state the following characterization for linear resistive structures.

**Lemma 5.6.** Let $\mathcal{R} \subset \mathcal{F} \times \mathcal{E}$ be a linear resistive structure and $\mathcal{D} \subset \mathcal{F}' \times \mathcal{F} \times \mathcal{E}' \times \mathcal{E}$ be a Dirac structure. Then, the following statements hold:

(a) $\mathcal{R}$ is generated by $\phi_R$ and hence is maximal cyclically monotone.

(b) The composition $\mathcal{D} \sqsubset \mathcal{R}$ is maximal monotone.

(c) Any port-Hamiltonian system with Dirac structure and linear resistive structure is maximal monotone port-Hamiltonian.

**Proof.** (a): Clearly, $\phi_R$ is a proper lower semicontinuous function with $\text{dom} \phi = \text{im} R^T_e$ and

\begin{equation}
\partial \phi_R(f) = \begin{cases}
R f + \ker R_e & \text{if } f \in \text{im} R^T_e \\
\emptyset & \text{otherwise}
\end{cases}
\end{equation}

Now, we claim that $\mathcal{R}$ is generated by $\phi_R$ and hence maximal cyclically monotone. To verify this claim, one needs to show that

\begin{equation}
\mathcal{R} = \{(f, e) \mid e \in \partial \phi_R(f)\}.
\end{equation}

To see this, first let $(f, e) \in \mathcal{R}$. Then, we see from (5.8) that $f = R^T_e \lambda$ and $e = R^T_f \lambda$ for some $\lambda$. Note that

\begin{equation}
\text{im}(RR^T_e - R^T_f) \subseteq \ker R_e
\end{equation}
due to (5.10). As such there must exist $\mu \in \ker R_e$ such that $R^T_f \lambda = RR^T_e \lambda + \mu$. Therefore, it follows from (5.12) that $e \in \partial \phi_R(f)$. This proves that

\begin{equation}
\mathcal{R} \subseteq \{(f, e) \mid e \in \partial \phi_R(f)\}.
\end{equation}

To see that the reverse inclusion also holds, let $(f, e)$ be such that $e \in \partial \phi_R(f)$. From (5.12), we see that there exist $\lambda$ and $\mu \in \ker R_e$ such that $f = R^T_e \lambda$ and $e = RR^T_e \lambda + \mu$. Since $\ker R_e \subseteq R^T_f \ker R^T_e$ due to (5.5) and (5.8), it follows from (5.14) that $e = R^T_f (\lambda + \theta)$ where $\theta \in \ker R^T_e$. Note that $f = R^T_e \lambda = R^T_e (\lambda + \theta)$. Consequently, we see that

\begin{equation}
\{(f, e) \mid e \in \partial \phi_R(f)\} \subseteq \mathcal{R}
\end{equation}

which, together with (5.15), proves (5.13).

(b): Note first that $\mathcal{R}$ is clearly maximal monotone. Since both $\mathcal{D}$ and $\mathcal{R}$ are subspaces, the sets $\Pi(\mathcal{D}, \mathcal{F})$, $\Pi(\mathcal{R}, \mathcal{F})$, $\Pi(\mathcal{D}, \mathcal{F} \times \mathcal{E})$, and $\Pi(\mathcal{R}, \mathcal{F} \times \mathcal{E}) = \mathcal{R}$ are all subspaces. As such, the conditions (i) and (ii) of Theorem 4.2 are trivially satisfied by the choices $\bar{f} = 0 = \bar{e}$. Consequently, the composition $\mathcal{D} \sqsubset \mathcal{R}$ is maximal monotone.

(c): This immediately follows from the fact that the definition of a port-Hamiltonian system entails the composition $\mathcal{D} \sqsubset \mathcal{R}$ of $\mathcal{D}$ and $\mathcal{R}$. \[\blacksquare\]
Furthermore, if we replace in the definition of a port-Hamiltonian system the energy-dissipating relation \( R \) by a relation \( R' \) such that \(-R'\) is monotone, then also \( D \Rightarrow R' \) is monotone, and thus we obtain a monotone port-Hamiltonian system (which is however not necessarily port-Hamiltonian).

6. Steady state analysis of incrementally port-Hamiltonian systems.
In this section we utilize the theory from the previous section to analyze the set of steady states (for non-zero constant inputs) of an incrementally port-Hamiltonian system and of interconnections of incrementally port-Hamiltonian systems. For simplicity of exposition, we will denote throughout this section \( Y := F_p, U := E_p \), and correspondingly set \( y = f_p, u = e_p \).

6.1. The steady-state input-output relation. First recall the notion of steady-state input-output relation. Consider an input-state-output system \( \Sigma \) given as

\[
\Sigma : \begin{align*}
\dot{x} &= f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\
y &= h(x, u), \quad y \in \mathbb{R}^m
\end{align*}
\]

Consider any constant input vector \( \bar{u} \) for which there exists an \( \bar{x} \in \mathbb{R}^n \) with \( 0 = f(\bar{x}, \bar{u}) \), and denote \( \bar{y} = h(\bar{x}, \bar{u}) \). Then the set of all such pairs \((\bar{y}, \bar{u})\), i.e.,

\[
\mathcal{G} = \{(\bar{y}, \bar{u}) | \exists \bar{x}, 0 = f(\bar{x}, \bar{u}), \bar{y} = h(\bar{x}, \bar{u})\}
\]

is called the steady-state input-output relation of \( \Sigma \).

In the case of incrementally port-Hamiltonian systems more can be said about the structure of steady-state input-output relations. First we note the following direct applications of Theorem 4.2 and Theorem 4.3.

**Corollary 6.1.** Consider an incrementally port-Hamiltonian system with underlying maximal monotone relation \( M \subset F_S \times E_S \times Y \times U \). Assume \( M \) satisfies

\[
0 \in \text{rint} \Pi(M, F_S)
\]

and

\[
\text{there exists } \bar{e} \text{ such that } \bar{e} \in \text{rint}\{e_S | (0, e_S) \in \Pi(M, F_S \times E_S)\}
\]

Then

\[
M_s = \{(y, u) | \exists e_S \text{ such that } (0, e_S, y, u) \in M\}
\]

is also a maximal monotone relation.

Furthermore, in case the maximal monotone relation is cyclically monotone, and thus is given as the subdifferential of some convex function \( K(e_S, u) \), then

\[
M_s = \text{graph} (\partial K_s),
\]

where the convex function \( K_s : U \rightarrow \mathbb{R} \) is given as

\[
K_s(u) = K^*(0, u)
\]

with \( K^*(f_S, u) \) the partial convex conjugate of \( K \) with respect to \( e_S \).
Proof. First note that \( \mathcal{M}_s \) is the composition of \( \mathcal{M} \) with the trivial maximally monotone relation \( \{(f_S = 0, e_S) \in F_S \times E_S\} \). Thus in order to apply Theorem 4.2 we need to show that there exists \((\bar{f}_S, \bar{e}_S) \in F_S \times E_S\) such that (following the notation of Theorem 4.2)

\[
\begin{align*}
(i) \, (\bar{f}_S, -\bar{f}_S) & \in \text{rint } D_f \\
(ii) \, (\bar{e}_S, \bar{e}_S) & \in \text{rint } D_e
\end{align*}
\]

where \( D_f = \Pi(\mathcal{M}, F_S) \times 0 \) and \( D_e = \{(e_1, e_2) \mid (0, e_1) \in \Pi(\mathcal{M}, F_S \times E_S)\} \). It is easily seen that conditions (i), (ii) reduce to (6.3) and (6.4).

The rest of the proof follows from Theorem 4.3.

From now on we will throughout assume that the conditions (6.3), (6.4) are satisfied, implying that \( \mathcal{M}_s \) is maximal monotone.

It is directly seen that the steady-state input-output relation \( \mathcal{G} \) of the maximal monotone port-Hamiltonian system with maximal monotone relation \( \mathcal{M} \) is contained in the maximal monotone relation \( \mathcal{M}_s \). Indeed, if \( \bar{u}, \bar{x}, \bar{y} \) is such that \( (0, \frac{\partial H}{\partial x}(\bar{x}), \bar{y}, \bar{u}) \in \mathcal{M} \) (and thus \( (\bar{y}, \bar{u}) \in \mathcal{G} \)) then clearly \( (0, e_S, \bar{y}, e_S) \in \mathcal{M} \), where \( e_S = \frac{\partial H}{\partial x}(\bar{x}) \). Consequently, \( \mathcal{G} \) is at least monotone.

However \( \mathcal{M}_s \) may be larger than \( \mathcal{G} \) since there may not exist for every \( e_S \) such that \( (0, e_S, \bar{y}, \bar{u}) \in \mathcal{M} \) an \( \bar{x} \) such that \( e_S = \frac{\partial H}{\partial x}(\bar{x}) \). In fact, this non-existence of \( \bar{x} \) may be due to two reasons. First, the Hamiltonian \( H \) may be such that there does not exist for any \( e_S \) an \( x \) (not necessarily a steady state) such that \( e_S = \frac{\partial H}{\partial x}(x) \). Secondly, if such an \( x \) exists it may not be a steady state. A simple example illustrating the second reason is provided by the nonlinear integrator

\[
\dot{x} = u, \quad y = \frac{\partial H}{\partial x}(x),
\]

where we assume that \( H \) is a strictly convex function such that the mapping \( x \mapsto \frac{\partial H}{\partial x}(x) \) is surjective. This defines an incrementally port-Hamiltonian system with \n
\[
\mathcal{M} = \{(f_S, e_S, y, u) \mid f_S = -u, y = e_S\}
\]

Clearly, \( \mathcal{M}_s = \{(y, u) \mid u = 0\} \). However the set of steady states \( \bar{x} \) is either empty or is given as the singleton \( \{\bar{x} \mid \frac{\partial H}{\partial x}(\bar{x}) = 0\} \), implying that also \( \mathcal{G} \) is a singleton, and hence not equal to \( \mathcal{M}_s \).

6.2. Equilibrium independent passivity. Up to now, no conditions were imposed on the Hamiltonian \( H \) in the definition of an incrementally port-Hamiltonian system. If additionally \( H \) is strictly convex, as well as differentiable, then for every \( \bar{x} \) the function \( S_\bar{x} : X \to \mathbb{R} \) defined as

\[
H_\bar{x}(x) := H(x) - \frac{\partial H}{\partial x}(\bar{x})(x - \bar{x}) - H(\bar{x})
\]

(as a function of \( x \) and \( \bar{x} \) also called the Bregman divergence of \( H \) [23], or as a function of \( x \) alone for fixed \( \bar{x} \) the shifted Hamiltonian [1]) has a strict minimum at \( \bar{x} \), and is again strictly convex. Furthermore,

\[
\frac{\partial H_\bar{x}}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(\bar{x})
\]
Hence for any \((\bar{u}, \bar{y})\) in the steady-state input-output relation of an incrementally port-Hamiltonian system one computes
\[
\frac{d}{dt} H_x = \frac{\partial S}{\partial x^T}(x) \dot{x} = \left( \frac{\partial H}{\partial x^T}(x) - \frac{\partial H}{\partial x^T}(\bar{x}) \right) (\dot{x} - 0) \leq (y - \bar{y})^T (u - \bar{u}),
\]
implying passivity with respect to the shifted passivity supply rate \((y - \bar{y})^T (u - \bar{u})\).
This was called shifted passivity in [1], while the property that this holds for any steady state values \((\bar{u}, \bar{x}, \bar{y})\) was coined as equilibrium independent passivity in [12].

Summarizing

**Proposition 6.2.** Consider a maximal monotone port-Hamiltonian system with respect to the maximal monotone relation
\[
\mathcal{M} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,
\]
with a strictly convex differentiable Hamiltonian \(H : \mathbb{R}^n \to \mathbb{R}\). Then the system is equilibrium independent passive, with static input-output relation given by the monotone relation \(\mathcal{G} \subset \mathcal{G}_{\mathcal{M}}\), and with storage functions \(H_x\) having a strict minimum at \(\bar{x}\). If additionally \(H\) is such that for every \(\bar{e}_x\) there exists an \(\bar{x}\) with \(\bar{e}_x = \frac{\partial H}{\partial x}\) then \(\mathcal{G} = \mathcal{G}_{\mathcal{M}}\).

The case \(\mathcal{G} = \mathcal{G}_{\mathcal{M}}\) was called maximal equilibrium independent passivity in [11]. (Maximal) equilibrium independent passivity is a desirable property for showing (asymptotic) stability of the steady state values of a port-Hamiltonian system for different constant input values, since by (6.8) the shifted Hamiltonians can be employed as Lyapunov functions for \(u = \bar{u}\).

### 6.3. Determination of the steady state of the interconnection of incrementally port-Hamiltonian systems.

In this subsection we analyze how the steady-state of the interconnection of multiple maximal cyclically monotone port-Hamiltonian systems can be computed, under additional assumptions, by solving a convex optimization problem. This subsection is motivated by some of the developments in [11].

Consider \(k\) maximal monotone port-Hamiltonian systems with input and output vectors \(u_i \in \mathbb{R}^{m_i}, y_i \in \mathbb{R}^{m_i}\), and maximal monotone relations \(\mathcal{M}_i, i = 1, \cdots, k\). Let, as before, \(\mathcal{M}_i^* \subset \mathbb{R}^{m_i} \times \mathbb{R}^{m_i}, i = 1, \cdots, k\) be maximal monotone relations. Additionally, assume that \(\mathcal{M}_i^*, i = 1, \cdots, k\) are maximal cyclically monotone, and thus the graphs of subdifferentials of convex functions \(K_i(u_i), i = 1, \cdots, k\).

Consider now an interconnection of the following general type. For any subset \(\pi \subset \{1, \cdots, k\}\) define
\[
\begin{align*}
    f_i & := u_i, \quad i \in \pi, \quad f_i := y_i, \quad i \notin \pi \\
    e_i & := y_i, \quad i \in \pi, \quad e_i := u_i, \quad i \notin \pi
\end{align*}
\]
Furthermore, consider any subspace \(\mathcal{C}\) of the linear space of variables \((f_1, \cdots, f_k) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}\), and define interconnection constraints
\[
(f_1, \cdots, f_k) \in \mathcal{C}, \quad (e_1, \cdots, e_k) \in \mathcal{C}^\perp
\]
The main message of this subsection is that finding a steady state of the interconnected system can be performed by solving a convex minimization problem. Define the convex function \(K : \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} \to \mathbb{R} \cup \{\infty\}\) given as
\[
K(f_1, \cdots, f_k) := \sum_{i \in \pi} K_i(u_i) + \sum_{i \notin \pi} K_i^*(y_i)
\]
Now consider the minimization

\[
(6.12) \quad \min_{(f_1, \ldots, f_k) \in C} K(f_1, \ldots, f_k)
\]

and write \( C = \ker C \) for some constraint matrix \( C = \col(C_1, \ldots, C_k) \). Then the minimization is equivalent to the \textit{unconstrained} minimization

\[
(6.13) \quad \min_{(f_1, \ldots, f_k, \lambda)} K(f_1, \ldots, f_k) - \sum_{i} \lambda_i^T C_i f_i
\]

where \( \lambda \) is a corresponding vector of Lagrange multipliers. This yields the first-order optimality conditions

\[
(6.14) \quad 0 \in \partial K_i(u_i) - C_i^T \lambda, \quad i \in \pi
\]

\[
0 \in \partial K_i^*(y_i) - C_i^T \lambda, \quad i \notin \pi
\]

Consider a solution \((\bar{f}_1, \ldots, \bar{f}_k) \in C\) of these first-order optimality conditions. Hence there exist \( \bar{e}_i = \bar{y}_i \in \partial K_i(u_i), i \in \pi \), and \( \bar{e}_i = \bar{u}_i \in \partial K_i^*(y_i), i \notin \pi \), such that \( \bar{e} \in \im C^T \), which is nothing else than \( \bar{e} \in C^\perp \).

This yields the following theorem regarding the equilibrium of the interconnection of maximal cyclically monotone port-Hamiltonian systems.

**Theorem 6.3.** Consider \( k \) maximal cyclically monotone port-Hamiltonian systems with input and output variables \( u_1, \ldots, u_k, y_1, \ldots, y_k \) where \( u_i \in \mathbb{R}^{m_i}, y_i \in \mathbb{R}^{m_i}, i = 1, \ldots, k \). Assume that the maximal monotone relations \( \mathcal{G}_{Ai} \subset \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \) are given as the graph of subdifferentials \( \partial K_i \) for convex functions \( K_i, i = 1, \ldots, k \). Furthermore, let \( \pi \subset \{1, \ldots, k\} \) be an index set and consider any constraint subspace \( C \subset \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} \) leading to the interconnection

\[
(f_1, \ldots, f_k) \in C, (e_1, \ldots, e_k) \in C^\perp
\]

Then if \((\bar{f}_1, \ldots, \bar{f}_k) \in C\) is a solution of the minimization

\[
(6.15) \quad \min_{(f_1, \ldots, f_k) \in C} K(f_1, \ldots, f_k)
\]

then there exists \((\bar{e}_1, \ldots, \bar{e}_k) \in C^\perp\).

Note that once we have computed \((\bar{e}_1, \ldots, \bar{e}_k)\) and there exists \((\bar{x}_1, \ldots, \bar{x}_k)\) such that

\[
\frac{\partial H_i}{\partial x_i}(\bar{x}_i) = \bar{e}_i, i = 1, \ldots, k,
\]

then this means that \((\bar{x}_1, \ldots, \bar{x}_k)\) is an equilibrium of the interconnected system. Furthermore, if we additionally assume that the Hamiltonians \( H_i \) are strictly convex, it follows that this equilibrium is \textit{stable}.

Finally, note that the interconnection constraints can be equivalently formulated as the solution of the \textit{dual} minimization problem

\[
(6.16) \quad K^*(\bar{e}_1, \ldots, \bar{e}_k) := \sum_{i \in \pi} K_i^*(y_i) + \sum_{i \notin \pi} K_i(u_i)
\]
7. Connections with other passivity notions. In the previous section we already observed that the notion of incrementally port-Hamiltonian systems is closely related to shifted passivity and equilibrium independent passivity. In this section we will discuss how it is closely related to incremental passivity and differential passivity as well; at least in case the Hamiltonian is quadratic-affine. Thus let $H(x) = \frac{1}{2}x^TQx + Ax + c$ for some symmetric positive semi-definite matrix $Q$, matrix $A$ and constant $c$. In this case, the inequality (3.5) reduces to

$$\langle Q(x_1 - x_2) \mid \dot{x}_1 - \dot{x}_2 \rangle \leq \langle e_1^T, e_2^T \mid f_1^T - f_2^T \rangle$$

which is equivalent to

$$\frac{1}{2} \frac{d}{dt} (x_1(t) - x_2(t))^T Q(x_1(t) - x_2(t)) \leq (e_1^T(t) - e_2^T(t))^T (f_1^T(t) - f_2^T(t))$$

Recall [4, 24, 25] that a system $\dot{x} = f(x, u), y = h(x, u)$ with $x \in \mathbb{R}^n, u, y \in \mathbb{R}^m$ is called incrementally passive if there exists a nonnegative function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{d}{dt} V(x_1, x_2) \leq (u_1 - u_2)^T (y_1 - y_2)$$

for all $(x_i, u_i, y_i), i = 1, 2$ satisfying $\dot{x} = f(x, u), y = h(x, u)$. We immediately obtain the following result.

**Proposition 7.1.** Any incrementally port-Hamiltonian system with quadratic-affine Hamiltonian $H(x) = \frac{1}{2}x^TQx + Ax + c$ with $Q \succeq 0$ is incrementally passive.

**Proof.** The function $V(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^T Q(x_1 - x_2)$ is immediately seen to define an incremental storage function for incremental passivity. \qed

Recall furthermore from [26–28] the following definition of differential passivity.

**Definition 7.2.** Consider a nonlinear control system $\Sigma$ with state space $X$, affine in the inputs $u$, and with an equal number of outputs $y$, given as

$$\Sigma : \begin{align*}
\dot{x} &= f(x) + \sum_{j=1}^{m} u_j g_j(x), \\
y_j &= H_j(x), & j = 1, \ldots, m,
\end{align*}$$

The variational system along any input-state-output trajectory

$$t \in [0, T] \mapsto (x(t), u(t), y(t))$$

is given by the following time-varying system, cf. [29]

$$\dot{x}(t) = \frac{\partial f}{\partial x}(x(t)) \delta x + \sum_{j=1}^{m} u_j \frac{\partial g_j}{\partial x}(x(t)) \delta x(t) + \sum_{j=1}^{m} \delta u_j \hat{g}_j(x(t))$$

with state $\delta x \in \mathbb{R}$, where $\delta u = (\delta u_1, \ldots, \delta u_m)$, $\delta y = (\delta y_1, \ldots, \delta y_m)$ denote the inputs and the outputs of the variational system. Then $\Sigma$ is called differentially passive if the system together with all its variational systems is dissipative with respect to the supply
rate $\delta u^T \delta y$, that is, if there exists a function $P : TX \to \mathbb{R}^+$ (called the differential storage function) satisfying

\begin{equation}
\frac{d}{dt} P \leq \delta u^T \delta y
\end{equation}

for all $x, u, \delta u$.

Similar to incremental passivity we obtain

**Proposition 7.3.** A monotone port-Hamiltonian system with quadratic-affine Hamiltonian $H(x) = \frac{1}{2} x^T Q x + Ax + c$ with $Q \geq 0$ is differentially passive.

**Proof.** Consider the infinitesimal version of (3.5). In fact, let $(f^1_P, e^1_P, x_1)$ and $(f^2_P, e^2_P, x_2)$ be two triples of system trajectories arbitrarily near each other. Taking the limit we deduce from (3.5)

\begin{equation}
\delta x^T \frac{\partial^2 H}{\partial x^2} (x) \delta \dot{x} \leq \delta e^T_P \delta f_P
\end{equation}

where $\delta x$ denotes the variational state, and $\partial f_P, \partial e_P$ the variational inputs and outputs). If the Hamiltonian $H$ is a quadratic function $H(x) = \frac{1}{2} x^T Q x + Ax + c$ then the left-hand side of the inequality (7.7) is equal to $\frac{1}{2} \delta x^T Q \delta x$, and hence amounts to the differential dissipativity inequality

\begin{equation}
\frac{d}{dt} \frac{1}{2} \delta x^T Q \delta x \leq \delta e^T_P \delta f_P,
\end{equation}

implying that the monotone port-Hamiltonian system is differentially passive, with differential storage function $\frac{1}{2} \delta x^T Q \delta x$. \hfill \blacksquare

Of course, the assumption of a quadratic-affine Hamiltonian $H(x) = \frac{1}{2} x^T Q x + Ax + c$ in order to let the monotone port-Hamiltonian system be incrementally passive and differentially passive is restrictive. On the other hand, it is known from the literature [3, 30] that for ‘unconditional’ incremental properties such an assumption may be necessary as well. For example we can formulate the following simple result.

Consider a scalar nonlinear integrator system

\begin{equation}
\dot{x} = u, \quad y = \frac{dH}{dx}(x)
\end{equation}

In order to evaluate its incremental properties consider two copies

\begin{equation}
\dot{x}_1 = u_1, \dot{x}_2 = u_2, \quad y_1 = \frac{dH}{dx_1}(x_1), y_2 = \frac{dH}{dx_2}(x_2)
\end{equation}

Then the system (7.9) is incrementally passive iff there exists $S(x_1, x_2) \geq 0$ satisfying

\begin{equation}
\frac{\partial S}{\partial x_1} (x_1) u_1 + \frac{\partial S}{\partial x_2} (x_2) u_2 \leq (u_1 - u_2) \left( \frac{dH}{dx_1}(x_1) - \frac{dH}{dx_2}(x_2) \right)
\end{equation}

for all $x_1, x_2, u_1, u_2$ related by (7.10). This is equivalent to

\begin{equation}
\frac{\partial S}{\partial x_1} (x_1, x_2) = \frac{dH}{dx_1}(x_1) - \frac{dH}{dx_2}(x_2) = - \frac{\partial S}{\partial x_2} (x_1, x_2)
\end{equation}
for all $x_1, x_2$. Differentiation of the first equality with respect to $x_2$, and of the second equality with respect to $x_1$, yields

$$
-\frac{d^2 H}{d x_2^2}(x_2) = \frac{\partial^2 S}{\partial x_1 \partial x_2}(x_1, x_2) = -\frac{d^2 H}{d x_1^2}(x_1),
$$

implying that $\frac{d^2 H}{d x_2^2}(x)$ is a constant; i.e., $H(x)$ must be a quadratic-affine function $H(x) = \frac{1}{2}q x^2 + ax + c$, for some constants $q, a, c$. Hence the (7.9) is incrementally passive \textit{if and only if} $H$ is quadratic-affine (in which case the integrator is actually linear). This example is easily extendable to more general situations, basically implying that unconditional incremental passivity implies a quadratic-affine storage function.

8. Conclusions. The notion of an incrementally port-Hamiltonian system was first introduced in [9]; basically replacing the composition of a Dirac structure and an energy-dissipation relation in a standard port-Hamiltonian system by a general monotone relation. The present paper discusses the properties of incrementally port-Hamiltonian systems in much more detail; including a wealth of examples and the formulation of specific system subclasses. In particular, the current paper studies the class of maximal cyclically monotone port-Hamiltonian systems and its connection to convex generating functions. From a mathematical point of view a key contribution of the present paper is a detailed treatment of composition of maximal (cyclically) monotone relations, and its implications for the interconnection of incrementally port-Hamiltonian systems. Indeed, it is shown that under mild technical conditions the composition of maximal (cyclically) monotone relations defines a maximal (cyclically) monotone relation.

Apart from the abundance of physical examples, this relates incrementally port-Hamiltonian systems to convex optimization as well. Such relations are multi-faceted; from the formulation of gradient and primal-dual gradient algorithms in continuous time as incrementally port-Hamiltonian systems to the computation of the equilibrium of interconnected incrementally port-Hamiltonian systems via convex optimization. Furthermore, apart from the convex generating functions of maximal cyclically monotone relations, another use of convexity in this incrementally port-Hamiltonian framework is the consideration of convex Hamiltonian functions. The use of the Bregman divergence of a convex function already turns out to be natural in assessing the stability of steady states of (interconnected) incrementally port-Hamiltonian systems, but much more connections between port-Hamiltonian theory and convex analysis are still to be explored.

The precise dynamical properties of incrementally port-Hamiltonian systems still remain somewhat illusive. The dynamical implications of the key inequality 3.5 are only fully clear if the Hamiltonian $H$ is a quadratic-affine function in suitable coordinates. Indeed, in this case the incrementally port-Hamiltonian system is incrementally and differentially passive. On the other hand, as shown in [30] and in the example of a scalar integrator discussed at the end of the previous section, unconditional incremental properties are typically very demanding (see also the theory of contractive systems [5]), and one could argue that the notion of incrementally port-Hamiltonian systems is less restrictive (although yet less clear from a dynamical perspective).

REFERENCES

[1] A. J. van der Schaft, \textit{L}_2-Gain and Passivity Techniques in Nonlinear Control, Communic-
tions and Control Engineering. Springer International Publishing, Cham, Switzerland, 3rd edition edition, 2017.

[2] A.J. van der Schaft and D. Jeltsema, Port-Hamiltonian Systems Theory: An Introductory Overview, Foundations and Trends in Systems and Control. now Publishers, 2014.

[3] T. Chaffey and R. Sepulchre, “Monotone one-port circuits,” https://arxiv.org/pdf/2111.15407.pdf, 2021.

[4] C.A. Desoer and M. Vidyasagar, Feedback Systems: Input-Output Properties, Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, USA, 1975.

[5] W. Lohmiller and J.E. Slotine, “On contraction analysis for non-linear systems,” Automatica, vol. 34, pp. 683–696, 1998.

[6] N. Parikh and S. Boyd, Proximal Algorithms, Foundations and Trends in Systems and Control. now Publishers, 2013.

[7] H. Brézis, Opérateurs Maximaux Monotones, North-Holland, 1973.

[8] H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, CMS Books in Mathematics. Springer, 2011.

[9] M. K. Camlibel and A. J. van der Schaft, “Incrementally port-Hamiltonian systems,” in 52th IEEE Conference on Decision and Control, Florence, Italy, 2013.

[10] M. K. Camlibel and J. M. Schumacher, “Linear passive systems and maximal monotone mappings,” Mathematical Programming, vol. 157, no. 2, pp. 397–420, 2016.

[11] M. Burger, D. Zelazo, and F. Allgower, “Duality and network theory in passivity-based cooperative control,” Automatica, vol. 47, no. 9, pp. 1949–1956, 2011.

[12] G. H. Hines, M. Arcak, and A. K. Packard, “Equilibrium-independent passivity: A new definition and numerical certification,” Automatica, vol. 47, no. 9, pp. 1949–1956, 2011.

[13] V. Duindam, A. Macchelli, and S. Stramigioli, Eds., Modeling and Control of Complex Physical Systems: The Port-Hamiltonian Approach, Springer London, Limited, 2009.

[14] M. K. Camlibel and A. J. van der Schaft, “Incrementally port-Hamiltonian systems,” in 9th IFAC Symposium on Nonlinear Control Systems, Florence, Italy, 2013.

[15] T. J. Courant, “Dirac manifolds,” Trans. Amer. Math. Soc., vol. 319, no. 2, pp. 631–661, 1990.

[16] A.J. van der Schaft and B.M. Maschke, “The Hamiltonian formulation of energy conserving physical systems with external ports,” Archive für Elektronik und Übertragungstechnik, vol. 49, pp. 362–371, 1995.

[17] R. T. Rockafellar and J. B. Wets, Variational Analysis, A Series of Comprehensive Studies in Mathematics 317, Springer, 1998.

[18] K. J. Arrow, L. Hurwicz, and H. Uzawa, Studies in Linear and Non-linear Programming, vol. 2 of Stanford Mathematical Studies in the Social Sciences, Stanford University Press, 1958.

[19] T. W. Stegink, C. De Persis, and A. J. van der Schaft, “A unifying energy-based approach to stability of power grids with market dynamics,” IEEE Transactions Automatic Control, vol. 62, no. 6, pp. 2612–2622, 2017.

[20] D.P. Bertsekas, Convex Optimization Theory, Athena Scientific, 2009.

[21] A.J. van der Schaft and B. Maschke, “Port-Hamiltonian systems on graphs,” SIAM J. Control Optim., vol. 51, no. 2, pp. 906–937, 2013.

[22] C.G. Khatri and S.K. Mitra, “Hermitian and nonnegative definite solutions of linear matrix equations,” SIAM J. Appl. Math., vol. 31, pp. 579–585, 1976.

[23] L. M. Bregman, “The relaxation method of finding the common points of convex sets and its application to the solution of problems in convex programming,” USSR Computational Mathematics and Mathematical Physics, vol. 7, no. 3, pp. 200–217, 1967.

[24] A. Pavlov and L. Marconi, “Incremental passivity and output regulation,” Systems and Control Letters, vol. 57, no. 5, pp. 400–409, 2008.

[25] D. Angeli, “A Lyapunov approach to incremental stability properties,” IEEE Transactions Automatic Control, vol. 47, no. 2, pp. 410–421, 2000.

[26] F. Forni and R. Sepulchre, “On differentially dissipative dynamical systems,” in 9th IFAC Symposium on Nonlinear Control Systems, Toulouse, 2013, pp. 4–6.

[27] A.J. van der Schaft, “On differential passivity,” in 9th IFAC Symposium on Nonlinear Control Systems, Toulouse, 2013, pp. 21–25.

[28] F. Forni, R. Sepulchre, and A.J. van der Schaft, “On differential passivity of physical systems,” in 52nd IEEE Conference on Decision and Control, Florence, Italy, 2013, pp. 6580–6585.

[29] P.E. Crouch and A.J. van der Schaft, Variational and Hamiltonian control systems, Lectures Notes in Control and Inf. Sciences 101. Springer-Verlag, New York, 1987.

[30] V. V. Kulkarni and M. G. Safonov, “Incremental positivity non-preservation by stability multipliers,” in 40th IEEE Conf. on Decision and Control, Orlando, USA, 2001, pp. 33–38.