An Overview of Nominal-Typing versus Structural-Typing in Object-Oriented Programming
(with code examples)

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Abstract

NOOP [4, 6, 7, 8] is a mathematical model of nominally-typed OOP that proves the identification of inheritance and subtyping in mainstream nominally-typed OO programming languages and the validity of this identification [11]. This report gives an overview of the main notions in OOP relevant to constructing a mathematical model of OOP such as NOOP. The emphasis in this report is on defining nominality, nominal typing and nominal subtyping of mainstream nominally-typed OO languages, and on contrasting the three notions with their counterparts in structurally-typed OO languages, i.e., with structurality, structural typing and structural subtyping, respectively. An additional appendix demonstrates these notions and other related notions, and the differences between them, using some simple code examples. A detailed, more technical comparison between nominal typing and structural typing in OOP is presented in other publications (e.g., [5]).

1 Main Notions in OOP

1.1 Objects, Fields and Methods

An object in object-oriented programming can be viewed as a “service provider”, i.e., as ‘an entity that provides a service’. An object provides its service by providing a set of object fields, as the inactive component of the object, and a set of object methods, as its active component.

A field of an object is a binding of a name to some other object. The field name is used to access and interact with the object bound to it. A method of an object is a binding of a name to a function that performs some computation when the method is invoked (on zero or more objects, as method arguments).
and returns an object as a result of the computation. The computation done by a method involves accessing fields of some objects and/or calling their methods. The object containing a method is always available as an implicit argument to that method (e.g., under the special name \texttt{this} or \texttt{self}), thereby providing access to the fields and methods of the object inside the code for the method.

Collectively, the fields and methods of an object are called \textit{members} of the object. The set of members of an object is finite. In statically-typed OO languages this set is also fixed, and thus cannot change at runtime. An object responds to a method call by performing the computation associated with the method name, as implemented by the code of the method, and it returns the result object. Collectively, the response of an object to field accesses and method calls and the logical relation between this response and the method arguments define the behavior of the object. The behavior of an object defines the service the object provides.

1.2 Encapsulation

Object-oriented programming is defined by two features: encapsulation and inheritance. Because an object is viewed as a whole integrated service provider, its members are not a collection of unrelated, independent members. Members of an object usually mutually depend on each other. Their interplay collectively provides the service of the object. Methods of an object often call one another, where an object can recursively invoke its own methods (via the special argument \texttt{self/this}). Methods of an object access its fields to obtain information on the “state” of the object. The high dependency and coupling between fields and methods of an object necessitates the two components (i.e., fields and methods) to be bound together (“embedded”) inside the object. The embedding and binding together of the active and inactive members of an object inside the object is expressed by stating that objects \textit{encapsulate} their members. Further, encapsulation in OOP not only refers to the bundling together of data and code that accesses this data inside objects but also to allowing some implementation details of an object (e.g., the code of its methods) and some details of its representation and structure (i.e., some of its fields and methods) to be hidden from other objects. This aspect of OOP encapsulation is sometimes also called information hiding. Information hiding helps maintain invariants of objects. It also allows implementation and representation details of objects that are hidden (i.e., not expressed in object interfaces, discussed below) to change, without the changes impacting client code.

1.3 Contracts, Classes, Class Names and Nominality

Many objects in an OO program share similar behavior, have similar properties, and provide the same service, only with some little variations (e.g., these objects may only differ in the exact values/objects bound to their fields). A \textit{contract} is a set of formal or informal statements that express common behavior and common properties of objects shared by a set of similar objects. As an expression of
the common service provided by the set of similar objects, the statements of a contract are assumed to hold true for all objects of the set. A class is a syntactic construct that OO languages offer for the specification of the behavior of objects that provide more or less the same service and that abide by and maintain the same contract. A class is used as a template for the construction of these similar objects. Objects produced using a certain class are called instances of that class.

A class has a name, called its class name. Class names are required to be unique in an OO program (using fully-qualified class names, if necessary). A class name is always associated, even if informally, with the common contract maintained by instances of the class. To relate objects in class-based OOP to the semantic class contracts the objects maintain, instances of a class have the name of the class as well as class names of its superclasses as part of their identity. That is, in class-based OOP class names inside objects are part of what it means to be an object. Class name information inside an object is called its nominal information. The association of class names with contracts and the availability of nominal information at runtime enables OO developers to design their software and control its behavior based on the contracts of objects in their software.

To emphasize the fact that objects in class-based OOP have class names as part of their meaning they are sometimes called nominal objects. A nominal object is always tied to the class (and superclasses) from which it was produced, via the class name information (i.e., nominal information) embedded inside the object. Having class names as part of the meaning of objects is called nominality. An OO language with nominal objects is a nominal OO language. Examples of nominal OO languages include Java, C#, Smalltalk, C++, Scala, and X10.

An OO language that does not embed class names in objects is called a structural OO language. The term ‘structural’ comes from the fact that an object in such a language is simply viewed as a record containing fields and methods but with no class name information, and thus no mention of the contracts maintained by the object. The view of objects as records thus reflects only their structure. Examples of structural OO languages include Strongtalk, Moby, PolyToil, and OCaml.

1.4 Shapes, Object Interfaces and Nominal Typing

We call the set of names of members of an object the shape of the object. Object shapes are sets of labels, i.e., are sets of names of members of the objects. Given that the shape of all instances of a class in statically-typed mainstream OO languages is fixed (i.e., is an invariant of instances of the class), we also talk

\footnote{Using, for example, tests on class names, such as the \texttt{instanceof} check in Java and the \texttt{isMemberOf} operation in Smalltalk. Nominally-typed OO languages also do runtime checks akin to \texttt{instanceof} when passing arguments to called methods and when returning objects returned by the methods, based on method signatures.}
about shapes of classes. The notion of encapsulation (as information hiding) in OOP motivates the notion of object interfaces. An object interface is an informal notion that specifies, possibly incompletely, how an object is viewed and should be interacted with by “the outside world”, i.e., by other objects of an OO program. It is hard to find a universally-accepted definition of the notion of object interfaces. An object interface typically contains information on the names of fields and methods of objects (their shapes), and the interfaces of these members themselves. We adopt a nominal view of object interfaces. In the nominal view of interfaces we adopt, an object interface further includes the class name information of the class from which the object is produced. Accordingly, the contracts associated with names of classes are part of the public interface of the object; these contracts are an essential part of how the object should be viewed and interacted with by other objects. Class name information inside an object interface can be used to derive other information about an object, like its full class inheritance information. This inheritance information is also part of how an object should be viewed by other objects.

In class-based OOP, object interfaces are the basis for the formal definition of class signatures and, in statically-typed OO languages, are also the basis for the formal definition of class types. The nominality of object interfaces in mainstream OOP causes class signatures, and class types, to be nominal notions as well. A statically-typed nominal OO language where objects are further associated with class types is thus called a nominally-typed OO language.

Nominally-typed OO languages allow readily expressing circular (i.e., mutually-dependent) class definitions. The wide need for circular class definitions and the ease by which recursive typing can be expressed in nominally-typed OO languages is one of the main advantages of nominally-typed OOP. According to Benjamin Pierce [17], “The fact that recursive types come essentially for free in nominal systems is a decided benefit [of nominally-typed OO languages].”

OO languages where types of objects are structural types (i.e., expressed as record type expressions that denote record types, with no class name information) are called structurally-typed OO languages. See the appendix for examples of both nominal and structural OO type expressions.

1.5 Inheritance and Nominal Subtyping

Inheritance is the second defining feature of OOP, where it is the main mechanism for sharing and reuse. Inheritance is a syntactic relation explicitly expressed and defined between classes of an OO program. It allows the code of...
a class (including its field and method definitions) to be defined in terms of
the code of other classes. The defined class is said to inherit from (or, extend)
the other classes. Inheritance is sometimes also called subclassing, where the
inheriting class is called a subclass while a class that is inherited from is called
a superclass.

Two levels at which inheritance takes place need to be distinguished. At the
level of values (i.e., objects), inheritance involves the sharing of code of object
methods and fields (implementation inheritance). At the type level, inheritance
involves the sharing of interfaces and contracts of objects. Given our goal of
analyzing and improving type systems of mainstream OO languages, our main
focus is on type-level inheritance. Unless otherwise noted, all references to
inheritance refer to type-level inheritance rather than object-level inheritance.

Inheritance in mainstream OOP is a nominal relation. Inheritance is spec-
ified between classes using class names in mainstream OO languages, asserting
that a subclass also inherits the class contracts associated with the names of its
superclasses. Because of being explicitly specified, inheritance in class-based
OOP is always an intended relation between classes and is never an implicit,
accidental or “spurious” relation. Most importantly, in nominally-typed main-
stream OOP, a subclass that explicitly inherits from some superclass is explicitly
declaring that its instances maintain the same contract associated with the su-
perclass.

2 Types and Typing in OOP

In computer programming, object-oriented or otherwise, typing (i.e., the use of
types and type systems) is mainly a means to disallow improper use of data
values (“well-typed programs can’t go wrong” [15]). The type soundness (also
called type safety) of a programming language is a statement about the language
stating that “well-typed” programs written using this language do not exhibit
certain language-specific undesirable program behaviors (i.e., programs that
have no type errors do not “go wrong”). Type-soundness is a desirable property
of programming languages. Much of PL research has been focused on the study
of type systems of programming languages to prove their type safety.

The most common interpretation of types in programming languages is that
a type is a set of similar values. Under this set-theoretic interpretation, types
can be large sets of values that satisfy weak similarity constraints, or be small
sets of values that satisfy strong similarity constraints, with other types filling
the spectrum in-between these two extremes.

The most desirable criteria for judging the similarity of software values and
data is to judge the similarity of their behavior and of how these values should

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4Similar to OO languages such as Java and C#, we allow multiple inheritance of
classes/interfaces. The names of classes and interfaces, alike, are always associated
with class contracts in mainstream OO languages. For example, as part of its “class” con-
tract, interface Comparable in Java requires a total ordering on instances of any inheriting
class.
be used. Defining types according to the similarity of behavior and usage of
data values is called semantic typing.

Judging the similarity of behavior and usage of data values is known to be
computationally undecidable, generally-speaking. The intractability of semantic
typing motivates using syntactic typing. Syntactic typing dictates that program
values are assigned syntactic expressions, called type expressions, that denote the
(syntactic) type of these values and phrases. Type expressions usually express
properties of values that can be easily checked, thereby making syntactic typing
tractable. The type corresponding to (or, denoted by) a given type expression
is the set of values that can be assigned the given type expression as their
(programmatic) type. In syntactic typing, the similarity of the values of a type
comes from the fact that all the values have the same property expressed by the
type expression denoting the type.

2.1 Class Types and Nominal Typing

In the context of OOP, typing is a means to disallow improper use of objects.
Examples of improper use of objects include: (1) attempting to access a non-
eexisting field or to call a non-existing method of an object, (2) allowing a field
access or a method call for a field or method not expressed in an object interface
(i.e., breaching the abstraction/information hiding offered by object interfaces),
or (3) not maintaining the contract associated with the class of an object or the
contracts associated with its superclasses.

Nominally-typed OOP languages use syntactic object features such as inher-
ance relations and object interfaces to decide the similarity of objects and to
translate their types. As mentioned earlier, given that class names are part
of object interfaces (i.e., are part of the public view of objects) and inheritance
relations are explicitly specified between class names, type expressions for ob-
jects in nominally-typed OOP are nominal type expressions. The sets of objects
these type expressions denote are called class types. Class types are nominal
notions, since class types with different class names are different class types that
defer different sets of objects. In structurally-typed OOP, on the other hand,
object type expressions express a structural view of object interfaces that does
not include class names nor inheritance relation information. As such, type
expressions of objects in structurally-typed OOP are the same as record type
expressions. The sets of objects these type expressions denote are the same as
record types well-known in the world of functional programming and among PL
researchers.

In many mainstream OO languages such as Java and C#, types of objects
are class types not record types, i.e., they are nominally-typed not structurally-
typed. Reflecting the nominality of class types in nominally-typed OO lan-
guages, class names are used also as names for class types in nominally-typed
OOP. This fact demonstrates the central role played by class names in the type
systems of nominally-typed mainstream OO languages. A crucial advantage of
the nominality of class types is that objects of a class type are not only simi-
lar structurally but the objects of the class type (given the association of class
names with contracts) all maintain the same contract associated with the name of the class type. This property, which gets typing in nominally-typed mainstream OOP closer to semantic typing, is a main drive behind the adoption of nominal typing among OO developers and language designers.

2.2 Subtyping and Substitutability

Given that objects in OOP can have different criteria for judging similarity, and varying degrees of similarity can be used, the same object can belong to multiple “sets of similar objects”, according to the degree of similarity required of elements of the sets and the criteria used to judge their similarity. If these sets of similar objects are denotable by type expressions of the language, the sets are types of the language (programmatic types). If values of a programming language can be assigned multiple types the language is said to support type polymorphism. When some type is a subset of (i.e., its elements are included in the elements of) another type, the first type is said to be a subtype of the second type, which in turn is called a supertype of the first type. Because of subsumption, a value assigned a subtype automatically polymorphically is assigned any supertype of the subtype. This form of type polymorphism is called subtyping polymorphism, or just subtyping. Subtyping has an intuitive counterpart in everyday lives, with plenty of everyday examples—sometimes under the name of ‘inclusion’ or ‘containment’—that get paralleled in OO software.

All statically-typed OO languages support subtyping polymorphism. In nominally-typed OOP, inheritance relations and object interfaces (upon which the definition of type expressions of objects is based) allow varying degrees of similarity between objects to be defined in a natural way. When concrete expressions of object interfaces (e.g., class signatures) are interpreted as type expressions of objects, object interfaces of classes that have more superclasses and more object members are interpreted as requiring more object-similarity constraints. These richer object interfaces thus denote smaller sets of objects (i.e., smaller class types).

2.3 The Liskov Substitution Principle and Nominal Subtyping

Just as for typing, semantic subtyping is more desirable but less tractable than syntactic subtyping. Semantic subtyping is commonly expressed as the Liskov Substitution Principle (LSP), which is familiar to many OO developers. The LSP states that in a computer program, a type S is a subtype of a type T if and only if objects of type T may be replaced with (i.e., substituted by) objects of type S, without altering any of the main behavioral properties of that program.

To achieve more precise subtyping, the inheritance relation between classes and the subtyping relation between class types are completely identified (are in one-to-one correspondence) in nominally-typed OOP [11]. Accordingly, an OO language that puts nominal information, and thus class contracts, in consideration while deciding the subtyping relation is a nominally-subtyped OO language
(all known nominally-typed OO languages are also nominally-subtyped). An
OO language that does not use nominal information while deciding the subtyp-
ing relation is a \textit{structurally-subtyped} OO language. In nominally-typed main-
stream OO languages, given the association of class names with class contracts,
the inclusion of class names in deciding the subtyping relation makes the sub-
typing relation semantically more precise (since it incorporates more behavioral
properties) than structural subtyping.

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In this appendix we present code examples that concretely demonstrate the concepts and notions discussed in this report. Unless otherwise noted, code examples in this appendix use the syntax of Java-like OO languages. To simplify the inheritance and subtyping relations, in the examples we do not assume that all classes have to inherit from a single superclass (e.g., class `Object`).

A Classes

As a first example, we assume a declaration of class `Object` as in Figure 1.
class Object {
    // Classes with no explicit constructors are always
    // assumed to have a default constructor that simply
    // initializes the fields of the constructed object.

    Boolean equals(Object o){
        return (o is Object);
        // Equivalent to: 'o.getClass() == Object.class'
    }
}

Figure 1: Class Object

In class Object, and in other classes declared below, we make use of the
standard class Boolean. We assume class Boolean has values true and false
(or equivalents) as its instances, and that it supports (via its methods) standard
boolean operations on boolean values.

Next, we will make use of classes A, B, C, D and E, whose declarations are as
presented in Figure 2. These five simple classes are not quite realistic and they
serve no purpose except demonstrating the concepts and notions we discuss.

class A { // no superclasses
    }

class B extends A {
    // add no members
}

class C extends B {
    D foo(D d) { return d; }
}

class D { // no superclasses
    A bar() { return new A(); }
}

class E extends D {
    A meth() { return new A(); }
}

Figure 2: Classes A, B, C, D and E

As more complex example, we also assume the declaration of class Pair as
presented in Figure 3.
class Pair extends Object {
    Object first;
    Object second;

    Boolean fstEqSnd(){
        return first.equals(second);
    }
    Boolean equalTo(Pair p){
        return first.equals(p.first) &&
        second.equals(p.second);
    }
    Boolean equals(Object p){
        if(p instanceof Pair)
            return equalTo((Pair)p);
        return false;
    }
    Pair setFirst(Object fst){
        return new Pair(fst, second);
    }
    Pair setSecond(Object snd){
        return new Pair(first, snd);
    }
    Pair swap(){
        return new Pair(second, first);
    }
}

Figure 3: Class Pair

B Shapes

Assuming a single namespace for fields and methods, the shape of (instances of) class Object, as declared in Figure 1, is the set

\{equals\}.

The shapes of classes A, B, C, D and E, as declared in Figure 2, are, respectively, the sets

\{\}, \{\}, \{foo\}, \{bar\}, \{bar, meth\}

(Note that the shapes of instances of A and B are the same set, namely the empty set \(\phi=\{\}\).) The shape of class E is a supershape of (i.e., a superset of)
the shape of class D, which in turn, is a supershape of the shape of classes A and B.

The shape of class Pair, as declared in Figure 3, is the set

\{equals, first, second, fstEqSnd, equalTo, setFirst, setSecond, swap\}.

The shape of class Pair is a supershape of the shape of class Object\(^5\).

C Record Type Expressions

A record type expression of class Object, expressing a structural view of the object interface for instances of Object, is

\[
\text{OSOI} \triangleq \text{record_type} \mu O. \{ \\
\text{B equals}(O) \\
\} \text{ and } \mu B. \{... \text{ member interfaces of class Boolean...}\}
\]

where \(\mu\) is the recursive types operator, and \text{and} defines mutually-recursive types. (See the discussions in [17, 6, 5] on circularity in OOP.)

Record type expressions of classes A, B, C, D and E, respectively, are

\[
\begin{align*}
\text{ASOI} & \triangleq \text{record_type} \{} \\
\text{BSOI} & \triangleq \text{record_type} \{} \\
\text{DSOI} & \triangleq \text{record_type} \{ \\
\text{BSOI} & \text{ bar()} \\
\} \text{ Note the need to include the full record type expression.} \\
\text{ESOI} & \triangleq \text{record_type} \{ \\
\text{ASOI} & \text{ bar()}, \\
\text{ASOI} & \text{ meth()} \\
\} \text{ Because of structurality, BSOI or the equivalent} \\
\text{unfolded expression ‘record_type {}’ could be used} \\
\text{in place of ASOI everywhere ASOI is used.}
\end{align*}
\]

\(^5\)Note that because class Object has no fields, all its instances are mathematically-equivalent. Mathematically-speaking, thus, class Object has only one instance.
A record type expression expressing a structural view of the object interface for instances of class Pair is

\[
\text{PSOI} \triangleq \text{record_type } \mu P. \{ \\
\quad \text{B equals}(O), \quad \text{with no rebinding from O to P} \\
\quad \text{O first, O second,} \\
\quad \text{B fstEqSnd()}, \\
\quad \text{B equalTo}(P), \\
\quad \text{P setFirst}(O), \\
\quad \text{P setSecond}(O), \\
\quad \text{P swap()} \\
\} \quad \text{and } \mu O. \{ \ldots \text{member interfaces of class Object} \ldots \} \\
\quad \text{and } \mu B. \{ \ldots \text{member interfaces of class Boolean} \ldots \}
\]

D Structural Subtyping

The following record type expressions, from Section C, are in the structural subtyping relation, \(<:\).

\[
\begin{align*}
\text{BSOI} & <\: \text{ASOI} \quad \text{(and spuriously ASOI} & <\: \text{BSOI, because ASOI} = \text{BSOI)} \\
\text{CSOI} & <\: \text{BSOI} \quad \text{(a genuine "is-a")}
\end{align*}
\]

\[
\begin{align*}
\text{DSOI} & <\: \text{BSOI} \quad \text{(spurious subtyping. unwarranted "is-a")} \\
\text{ESOI} & <\: \text{DSOI} \quad \text{(a genuine "is-a")}
\end{align*}
\]

\[
\begin{align*}
\text{OSOI} & <\: \text{BSOI} \quad \text{(spurious subtyping. unwarranted "is-a")} \\
\text{PSOI} & <\: \text{OSOI} \quad \text{(a genuine "is-a")}
\end{align*}
\]

Note that a pair in the structural subtyping relation could express a genuine “is-a” relation or an unwarranted, accidental relation between instances of classes. For example,

\[
\text{record_type } \{ \} <\: \text{record_type } \{ \}
\]

which we expressed above, in disguise, as BSOI < ASOI (and ASOI < BSOI), intuitively holds true when in reference to objects of class B being also objects of class A. This is something the developer (of class B) intended. It is thus a genuine is-a relation. The same relation does not hold true, however, when in it refers to objects of class A being objects of class B. Viewing objects of A as objects of B may not have been intended by the developer of class A. It is an accidental (“spurious”) is-a relation. It is only a result of the fact that structural subtyping does not capture the full intention of class developers.\(^6\)

\(^6\)Another example is the example of sets and multisets. Mathematically, every set is a multiset, but a multiset that has repeated elements is not a set. Classes modeling sets and multiset may have their structural types (i.e., record type expressions) not reflecting the contracts associated with the names ‘set’ and ‘multiset’ (of disallowing and allowing repetitions, respectively). Structural OO implementations of sets and multisets may thus allow instances of the two classes modeling sets and multisets to be mixed, in particular allowing multisets to be incorrectly viewed and interacted with as sets!
The subtyping pairs \( \text{CSOI} <: \text{BSOI} \), \( \text{ESOI} <: \text{DSOI} \) and \( \text{PSOI} <: \text{OSOI} \) express genuine is-a relations when referencing objects of classes \( C \) being \( B \)'s, \( E \)'s being \( D \)'s, and \( \text{Pair} \)'s being \( \text{Object} \)'s, respectively. The pairs \( \text{DSOI} <: \text{BSOI} \) and \( \text{OSOI} <: \text{BSOI} \) express an unwarranted is-a relation when referencing objects of \( D \) being \( B \)'s, and of \( \text{Object} \) being \( B \)'s.

E  Signatures and Subsigning

Based on declarations in Section A, the signature of class \( \text{Object} \), expressing a nominal view of the interface of \( \text{Object} \), is

\[
\text{Obj} \triangleq \text{sig } \text{Object} \{
\quad \text{equals: } \text{Object} \rightarrow \text{Boolean}
\}
\]

and the signatures of classes \( A \), \( B \), \( C \), \( D \) and \( E \), expressing nominal views of the interfaces of these five classes, are

\[
\begin{align*}
A \triangleq & \text{sig } A \{ \\
B \triangleq & \text{sig B ext } A \{ \\
C \triangleq & \text{sig C ext } B \{ \\
& \quad \text{foo: } D \rightarrow D \\
D \triangleq & \text{sig D } \\
& \quad \text{bar: } () \rightarrow A \\
E \triangleq & \text{sig E ext } D \{ \\
& \quad \text{bar: } () \rightarrow A, \\
& \quad \text{meth: } () \rightarrow A
\}
\end{align*}
\]

Also, the signature of class \( \text{Pair} \) is

\[
\text{Pair} \triangleq \text{sig Pair ext } \text{Object} \{
\quad \text{equals: } \text{Object} \rightarrow \text{Boolean}, \\
\quad \text{first: } \text{Object}, \\
\quad \text{second: } \text{Object}, \\
\quad \text{fstEqSnd: } () \rightarrow \text{Boolean}, \\
\quad \text{equalTo: } \text{Pair} \rightarrow \text{Boolean}, \\
\quad \text{setFirst: } \text{Object} \rightarrow \text{Pair}, \\
\quad \text{setSecond: } \text{Object} \rightarrow \text{Pair}, \\
\quad \text{swap: } () \rightarrow \text{Pair}
\}
\]

(It should be noted that the syntax used to present examples of class signatures above is different from the syntax generated by the more mathematically-oriented abstract syntax rules for signatures that are presented with NOOP [7].)
Even though equally informative, the syntax of signatures we used here is closer to the concrete syntax of classes that most mainstream OO developers are familiar with.)

Next, we define the following signature environments and signature closures

\[
\begin{align*}
ObjectSE &= \{ \text{Obj}, \text{Bool} \} \\
&\quad // \text{where Bool is the class signature of class Boolean} \\
Ase &= \{ \text{A} \} \\
Bse &= \{ \text{A}, \text{B} \} \\
Cse &= \{ \text{A}, \text{B}, \text{C} \} \\
Dse &= \{ \text{A}, \text{D} \} \\
Ese &= \{ \text{A}, \text{D}, \text{E} \} \\
PairSE &= \{ \text{Obj}, \text{Bool}, \text{Pair} \}
\end{align*}
\]

\[
\begin{align*}
ObjSC &= (\text{Object}, ObjectSE) \\
Asc &= ( \text{A, Ase} ), Bsc &= ( \text{B, Bse} ), Csc &= ( \text{C, Cse} ) \\
Dsc &= ( \text{D, Dse} ), Esc &= ( \text{E, Ese} ) \\
PairSC &= ( \text{Pair, PairSE} )
\end{align*}
\]

By the rules for signature environment extension we have

\[
\begin{align*}
Bse &\preceq Ase \quad (\text{but } Ase \npreceq Bse) \\
Cse &\preceq Bse \\
Ese &\preceq Dse \\
PairSE &\preceq ObjSE \\
\end{align*}
\]

and by the rules of subsigning (See, for example, \cite{7, Sec. 4.5}) we have

\[
\begin{align*}
Bsc &\preceq Asc \quad (\text{but } Asc \npreceq Bsc) \\
Csc &\preceq Bsc \\
Esc &\preceq Dsc \\
PairSC &\preceq ObjSC \\
\end{align*}
\]

Note that pairs in subsigning relation only express genuine “is-A” relations. In particular, unlike we had for structural subtyping (e.g., in Section D), for subsigning we do not have

\[
\begin{align*}
Dsc &\preceq Bsc \quad (\text{unwarranted by rules of subsigning,} \\
&\quad \text{since } Dse \npreceq Bse) \\
Dsc &\preceq Asc \quad (\text{unwarranted by rules of subsigning, even though} \\
&\quad Dse \npreceq Ase, \text{ since } A \notin \text{super_sigs}(D) = \{\}) \\
ObjSC &\preceq Bsc \quad (\text{unwarranted by rules of subsigning,} \\
&\quad \text{since } ObjSE \npreceq Bse)
\end{align*}
\]

Using the class declarations presented in Section A, the reader is invited to construct more examples of signature closures in and outside the subsigning
relation. Unlike structural subtyping (<;), the examples for subsigning demonstrate that subsigning, and thus nominal subtyping, fully captures the intention of class developers.