MANKIEWICZ’S THEOREM AND THE MAZUR–ULAM PROPERTY FOR C*-ALGEBRAS

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Abstract. We prove that every unital C*-algebra $A$, possibly except for the 2 by 2 matrix algebra, has the Mazur–Ulam property. Namely, every surjective isometry from the unit sphere $S_A$ of $A$ onto the unit sphere $S_Y$ of another normed space $Y$ extends to a real linear map. This extends the result of A. M. Peralta and F. J. Fernández-Polo who have proved the same under the additional assumption that both $A$ and $Y$ are von Neumann algebras. In the course of the proof, we strengthen Mankiewicz’s theorem and prove that every surjective isometry from a closed unit ball with enough extreme points onto an arbitrary convex subset of a normed space is necessarily affine.

1. Introduction

The celebrated Mazur–Ulam theorem ([MU]) asserts that every surjective isometry between normed spaces $X$ and $Y$ is necessarily affine. This was extended by P. Mankiewicz ([Ma]) to surjective isometries between the closed unit balls $B_X$ and $B_Y$. Motivated by these results, D. Tingley ([Ti]) has posed the problem in 1987: Does every surjective isometry $T: S_X \to S_Y$ between the unit spheres of normed spaces $X$ and $Y$ extend to a real linear isometry between $X$ and $Y$? Currently, no counterexample to Tingley’s problem is known. A Banach space $X$ is said to have the Mazur–Ulam property ([CD]) if Tingley’s problem has an affirmative answer for an arbitrary target $Y$. The main result of the present paper is the following.

Theorem 1. The following Banach spaces have the Mazur–Ulam property.

1. A unital complex C*-algebra $A \not\cong M_2(\mathbb{C})$, as a real Banach space.
2. A real von Neumann algebra $A \not\cong M_2(\mathbb{R}), M_2(\mathbb{C}), M_2(\mathbb{H})$.

Tingley’s problem between von Neumann algebras has been solved earlier in [PFP] (see also [Mo, Pe, Tan]). The Mazur–Ulam property for commutative C*-algebras has been proved in [Di, FW, Liu, PCA]. For more examples of Banach spaces with the Mazur–Ulam property, see [THL] for example. The starting point of the present and many other works on Tingley’s problem for operator algebras is R. Tanaka’s observation ([Tan]) that a surjective isometry from the unit sphere $S_A$ of a C*-algebra $A$ onto another unit sphere $S_Y$ maps closed faces onto closed faces.

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The unit sphere of a C*-algebra has many faces that are approximable by closed unit balls of C*-algebras. To exploit this, we revisit Mankiewicz's theorem. Let $K$ be a convex subset in a normed space $X$. It is called a convex body if it has non-empty interior in $X$. P. Mankiewicz ([Ma]) has proved that any isometry between convex bodies is necessarily affine. We say $K$ has the strong Mankiewicz property if every surjective isometry $T$ from $K$ onto an arbitrary convex subset $L$ in a normed space $Y$ is affine.

Every convex subset of a strictly convex normed space has this property because it is uniquely geodesic (cf. Lemma 6.1 in [BFGM]), but some convex subset of $L^1[0,1]$ does not ([Sc]), see Example 5. Every normed space also has this property by Figiel's theorem ([Fi]). This probably suggests that the same is true for every convex body, but this is not clear to the authors (since the range $L$ is not assumed to have non-empty interior).

Theorem 2. Let $X$ be a Banach space such that the closed convex hull of the extreme points $\text{ext} B_X$ of the closed unit ball $B_X$ has non-empty interior in $X$. Then, every convex body $K \subset X$ has the strong Mankiewicz property.

Corollary 3. Let $A$ be a unital complex C*-algebra or a real von Neumann algebra. Then, the closed unit ball $B_A$ has the strong Mankiewicz property.

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Notations and Remarks on real C*-algebras. Throughout this paper, $X$ and $Y$ are real normed (Banach) spaces and $A$ is a real or complex C*-algebra. The unit sphere and the closed unit ball of $X$ are denoted respectively by $S_X$ and $B_X$. For any projection $p$ in a unital C*-algebra, we write $p^p := 1 - p$. By definition, a real C*-algebra $A$ is the real part of a complex C*-algebra $A_C$ with respect to a conjugate-linear *-automorphism $J$ such that $J^2 = \text{id}$. Many of the standard operations in complex C*-algebras work equally well for real C*-algebras. For example, the modulus $|a|$ of an element $a$ in $A$ is firstly considered in the complexification $A_C$ and by uniqueness one sees that $|a|$ belongs to the real part $A$. For any complex continuous function $f$ and any normal element $a$ in $A_C$, one has $J(f(a)) = \bar{f}(J(a^*))$. A projection $p$ in a real von Neumann algebra $A$ is minimal if and only if $pAp = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, in which case $p$ has rank at most 2 in the complexification $A_C$. See [Li] for more on real operator algebras.

2. ON THE STRONG MANKIEWICZ PROPERTY

Lemma 4. If $B_X$ has the strong Mankiewicz property, then every convex body $K \subset X$ has the strong Mankiewicz property.

Proof. The assumption implies that every interior point $x$ in $K$ has a neighborhood on which $T$ is affine. By continuation (see Proof of Theorem 2 in [Ma]), one sees that $T$ is affine everywhere. □
Proof of Theorem [2]. By the above lemma, it suffices to show every surjective isometry $T: B_X \to L \subset Y$ is affine. We may assume that $T(0) = 0$, which implies that $\|T(x)\| = \|x\|$ for all $x \in B_X$. Let $a \in \text{ext} B_X$. Since the line segment $[-a, a]$ is the unique geodesic path between $-a$ and $a$, the map $T$ is affine (linear) on $[-a, a]$. We claim that if $x \in X$ and $\lambda \in \mathbb{R}$ are such that $\|x\| \leq \frac{1}{2}$ and $\|x\| + |\lambda| \leq 1$, then

$$\|T(x + \lambda a) - (T(x) + \lambda T(a))\| \leq 4\|x\||\lambda|.$$ 

For this, we may assume that $\lambda \geq 0$ as $T(-a) = -T(a)$. Since $T$ is affine on $[x, x + (1 - \|x\|)a]$ by the similar reason as above, one has

$$T(x + \lambda a) - T(x) = \frac{\lambda}{1 - \|x\|} (T(x + (1 - \|x\|)a) - T(x)) \approx \|x\| \lambda T(a).$$

Let $T_n : nB_X \to nB_Y$ be the map defined by $T_n(x) = nT(\frac{1}{n}x)$. By the previous inequality, for any $a_k \in \text{ext} B_X$ and $\lambda_k \in \mathbb{R}$ such that $C := \sum_k |\lambda_k|$, one has

$$\|T_n(\sum_k \lambda_k a_k) - \sum_k \lambda_k T(a_k)\| \leq 4n(\sum_k |\lambda_k| n)^2 \leq 4C^2 n$$

for all $n \geq 2C$. We consider the Banach space $Z_\infty := \ell_\infty(N; Z)/c_0(N; Z)$ for $Z = X$ or $Y$ and define $\hat{T} : X_\infty \to Y_\infty$ by $\hat{T}([x_n]) = [T_n(x_n)]$. Here $[x_n]$ denotes the element in $X_\infty$ represented by $(x_n) \in \ell_\infty(N; X)$. Observe that $\hat{T}$ is a well-defined isometry which is moreover linear by $(\ast)$ and the assumption on $X$. We claim that

$$\delta B_{\hat{T}(X_\infty)} = \hat{T}(\delta B_{X_\infty}) \subset L_\infty := \{[y_n] \in Y_\infty : y_n \in L\} \subset \hat{T}(X_\infty).$$

Here $\delta > 0$ is such that $\delta B_{X_\infty}$ is contained in the closed convex hull of ext $B_X$. Thus the first inclusion follows from $(\ast)$ and the fact that $\sum_k \lambda_k T(a_k) \in L$ for every $a_k \in \text{ext} B_X$ and $\lambda_k \geq 0$ with $\sum_k \lambda_k = 1$. The second follows from the fact that if $y := [y_n] \in L_\infty$, then for $x_n := nT^{-1}(\frac{1}{n}y_n)$, one has $[y_n] = \hat{T}(x_n) \in \hat{T}(X_\infty)$. This claim implies that $L$ has non-empty interior in its linear span. Indeed, if $y \in L$ and $\lambda \in \mathbb{R}$ are such that $\|\lambda y\| \leq \delta$, then the constant sequence $y$ belongs to $L_\infty$ and so is $\lambda y$, which means that there is a sequence $(z_n)_n$ in $L$ such that $\|\lambda y - z_n\| \to 0$ and so $\lambda y \in L$. Therefore, we can apply Mankiewicz’s theorem (Ma) to $T$ and conclude that $T$ is affine.

Proof of Corollary [3]. In the unital complex case, this follows from the Russo–Dye theorem (Theorem I.8.4 in [Da]). Let $A$ be a real von Neumann algebra and $x \in B_A$ be arbitrary. By polar decomposition $x = v|x|$ and by maximality argument, one can find a partial isometry $w \in A$ such that $x = w|x|$ and $(1 - w^*)A(1 - w^*w) = 0$. It is well-known (Theorem I.10.2 in [Tak]) that any partial isometry $u$ such that $(1 - uu^*)A(1 - u^*u) = 0$ is an extreme point of $B_A$. Since $|x| \in B_A$ belongs to a closed convex hull of the self-adjoint unitary elements in $A$, one sees that ext $B_A$ has a dense convex hull in $B_A$. □
The following beautiful example without the strong Mankiewicz property is provided for us by G. Schechtman (Sc).

**Example 5.** Consider the set $K_0$ of all continuous strictly increasing functions $f$ from $[0,1]$ onto $[0,1]$ and put $K := \overline{K_0} \subset L^1[0,1]$. Then, $K$ is compact and in particular it has empty interior. Let $T_0 : K_0 \to K_0$ be the map that sends $f$ to its inverse. Since the $L^1$ distance between two functions is the area enclosed by their graphs, $T$ is an isometry by the Fubini theorem. The continuous extension $T : K \to K$ of $T_0$ is a surjective isometry which is not affine. Hence, the compact convex subset $K \subset L^1[0,1]$ does not have the strong Mankiewicz property.

3. **Proof of Theorem 4 for $B(H)$**

Since the proof of Theorem 4 for the type I factor $B(H)$ is much simpler than the general $C^*$-case, we give it here as an appetizer.

**Lemma 6** (cf. Lemma 2.1 in [FW]). Let $T : S_X \to S_Y$ be a surjective isometry. Assume that there are $\{\varphi_i\}_i \subset B_{X^*}$ and $\{\psi_i\}_i \subset B_{Y^*}$ such that $\varphi_i = \psi_i \circ T$ and that the family $\{\varphi_i\}_i$ is norming for $X$. Then, $T$ extends to a linear isometry.

**Proof.** By assumption, the linear map $U : X \to \ell_\infty$, defined by $(Ux)_i = \varphi_i(x)$, is isometric. Since $\{\psi_i\}_i$ is also norming, the same holds true for $V : Y \to \ell_\infty$, which satisfies $U|_{S_X} = V \circ T$. Thus $T$ extends to a linear isometry (which is $V^{-1} \circ U$).

We note that any real linear functional $\varphi$ on a complex Banach space $X_C$ has the complexification $\varphi_C$ on $X_C$, which satisfies that $\varphi = \Re \varphi_C$ and $\|\varphi\| = \|\varphi_C\|$ (see Lemma III.6.3 in [Cd]).

**Lemma 7.** Let $\varphi$ be a norm-one real or complex linear functional on a $C^*$-algebra $A$. If $a \in B_A$ is such that $\varphi(a) = 1$, then $\varphi(x) = \varphi(aa^*xa^*)$ for all $x \in A$.

**Proof.** This is a simple consequence of the Arens trick. Since

$$\|(1 - aa^*)x + a\| = \|[(1 - aa^*) a] [x] \| \leq \|[1 - aa^*] a\| \|[x]\| \leq \sqrt{1 + \|x\|^2},$$

one has

$$|\lambda \varphi((1 - aa^*)x + a)|^2 = |\varphi(\lambda(1 - aa^*)x + a)|^2 \leq 1 + |\lambda|^2\|x\|^2$$

for all $\lambda$. This is possible only if $\varphi((1 - aa^*)x) = 0$. The proof of the other side is similar.

We call a closed face $F \subset S_X$ an intersection face if

$$F = \bigcap \{E : E \subset S_X \text{ a maximal face containing } F\}.$$

By Corollary 3.4 in [Tan], every norm-closed face of the unit sphere of a $C^*$-algebra is an intersection face.

**Lemma 8.** Let $T : S_X \to S_Y$ be a surjective isometry and $F$ be an intersection face. Then, $T(F)$ is an intersection face such that $T(-F) = -T(F)$. 

Proof. Recall from [CD, Ti] and Proposition 2.3 in [Mo] that for any maximal face \( E \subset S_X \), its image \( T(E) \subset S_Y \) is a maximal face such that \( T(-E) = -T(E) \). Thus \( T(F) = T(\bigcap E) = \bigcap T(E) \) is an intersection face such that \( T(-F) = T(\bigcap -E) = -\bigcap T(E) = -T(F) \).

Proof of the Mazur–Ulam property for \( \mathbb{B}(\mathcal{H}) \), \( \dim \mathcal{H} > 2 \). Let \( \xi, \eta \in \mathcal{H} \) be unit vectors and \( \varphi(\cdot) = \langle \cdot, \xi, \eta \rangle \) be the corresponding linear functional on \( A := \mathbb{B}(\mathcal{H}) \). We consider the maximal face

\[
E_\varphi := \{ x \in S_A : \varphi(x) = 1 \} = \{ x \in S_A : \mathbf{1} \xi = \mathbf{1} \eta \}.
\]

By Lemma 8, there is \( \psi \in B_{Y^*} \) such that \( \psi = 1 \) on the face \( T(E_\varphi) \). We will show \( \mathcal{R} \varphi = \psi \circ T \), which along with Lemma 7 proves Theorem 1 for \( A = \mathbb{B}(\mathcal{H}) \).

Let \( v \in A \) denote the rank-one partial isometry such that \( v \xi = \mathbf{1} \eta \). Let \( u \in S_A \) be an arbitrary unitary element. Since \( \dim \mathcal{H} > 2 \), there is a sub-partial isometry \( w \) of \( u \) such that \( w \perp v \), for example, pick a unit vector \( \zeta \in \{ \xi, u^* \eta \} \perp \) and set \( w = up \zeta \), where \( p \zeta \) is the rank-one projection corresponding to \( \zeta \). Put

\[
F(w) := \{ x \in S_A : xw^*w = w \} = \{ w + y : y \in B_{(1-w^*w)A(1-w^*w)} \}
\]

the corresponding closed face, which contains \( u \). Since \( T|_{F(w)} \) is affine by Lemma 8 and Corollary 3, there is \( \theta \in B_{A^*} \) such that \( (\psi \circ T)(w + y) = (\psi \circ T)(w) + \mathcal{R} \theta(y) \) for \( y \in B_{(1-w^*w)A(1-w^*w)} \). Since \( w \perp v \in F(w) \cap (\pm E_\varphi) \), one has \( \pm 1 = (\psi \circ T)(w \pm v) = (\psi \circ T)(w) \pm \mathcal{R} \theta(v) \). This implies that \( (\psi \circ T)(w) = 0 \) and \( \theta(v) = 1 \), which means \( \theta = \varphi \) by Lemma 7 and the fact that \( vv^*v = \varphi(x)v \) for all \( x \in A \). It follows that \( (\psi \circ T)(u) = \mathcal{R} \varphi(u) \) for every unitary element \( u \). Now let \( w' \) be an arbitrary rank-one partial isometry. Since \( T|_{F(w')} \) is affine and \( F(w') \) is the closed convex hull of the unitary elements in \( F(w') \), the previous result implies that \( \psi \circ T = \mathcal{R} \varphi \) on \( F(w') \). Since \( \bigcup w' F(w') \) is dense in \( S_A \), we conclude by continuity that \( \psi \circ T = \mathcal{R} \varphi \).

4. Convex combinations of a face and its opposite

For any face \( E \subset S_X \) and \( \lambda \in [-1, 1] \), put

\[
E_\lambda := \{ x \in S_X : \text{dist}(x, E) \leq 1 - \lambda \text{ and } \text{dist}(x, -E) \leq 1 + \lambda \}.
\]

Since \( \text{dist}(E, -E) = 2 \) for any convex subset \( E \subset S_X \), the inequalities in the RHS are actually equalities. We upgrade Lemma 8 as follows.

**Lemma 9.** Let \( T : S_X \to S_Y \) be a surjective isometry and \( F \subset S_X \) be an intersection face. Then, \( T(F) \) is an intersection face such that \( T(F_\lambda) = T(F)|_{S_X} \) for every \( \lambda \in [-1, 1] \).

**Lemma 10.** For any face \( E \subset S_X \), \( \lambda_1, \lambda_2 \in [-1, 1] \), and \( \alpha \in [0, 1] \), one has

\[
(\alpha E_\lambda_1 + (1-\alpha) E_\lambda_2) \cap S_X \subset E_{\alpha \lambda_1 + (1-\alpha) \lambda_2}.
\]

**Proof.** For given \( x_i \in E_\lambda_i \), put \( x_3 := \alpha x_1 + (1-\alpha) x_2 \) and \( \lambda_3 := \alpha \lambda_1 + (1-\alpha) \lambda_2 \). For any \( \varepsilon > 0 \), take \( y_i \in E \) such that \( \| y_i - x_i \| \approx \varepsilon 1 - \lambda_i \). Then, \( \alpha y_1 + (1-\alpha) y_2 \in E \) and

\[
\text{dist}(x_3, E) \leq \alpha \| x_1 - y_1 \| + (1-\alpha) \| x_2 - y_2 \| \approx \varepsilon 1 - \lambda_3.
\]
Since \( \varepsilon > 0 \) was arbitrary, one has \( \text{dist}(x_3, E) \leq 1 - \lambda_3 \). The proof of the other inequality is similar and one has \( x_3 \in E_{\lambda_3} \).

Combining Lemmas 9 and 10 we obtain the following.

**Lemma 11.** Let \( T: S_X \to S_Y \) be a surjective isometry and \( E \subset S_X \) be an intersection face. Then, for every \( \lambda_1, \lambda_2 \in [-1, 1] \) and \( \alpha \in [0, 1] \), one has
\[
(\alpha T(E_{\lambda_1}) + (1 - \alpha) T(E_{\lambda_2})) \cap S_Y \subset T(E_{\alpha \lambda_1 + (1 - \alpha) \lambda_2}).
\]

5. **Facial structure of a C*-algebra**

The following first three results are about Kadison’s transitivity theorem. They are all well-known in the complex case.

**Lemma 12** (Kadison’s transitivity theorem). Let \( A \) be a C*-algebra and \( p \in A^{**} \) be a finite rank projection. Then, for any norm-one (resp. self-adjoint) element \( x \in pA^{**}p \), there is a norm-one (resp. self-adjoint) element \( a \in A \) such that \( ap = x = pa \). For any unitary element \( x \) in \( \pm U_0(pA^{**}p) \), the connected component of \( \pm 1 \) is a unitary element in \( A \) (or the unitization of, if \( A \) is not unital) such that \( ap = x = pa \).

**Proof.** In the complex case, this follows from Theorem II.4.15 in [Tak]. We deal with the case of a real C*-algebra \( A \). Let \( (A_C, \mathcal{J}) \) be its complexification. Then for every norm-one (resp. self-adjoint) element \( x \in pA^{**}p \), there is a norm-one (resp. self-adjoint) element \( a' \in A_C \) such that \( a'p = x = pa' \). Thus \( a := \frac{1}{2}(a' + \mathcal{J}(a')) \in A \) satisfies the desired condition. Now let \( x \in \pm U_0(pA^{**}p) \) be a unitary element. Since \( x = \pm x_1 \cdots x_n \) for some \( x_k \in U_0(pA^{**}p) \) with \( \|x_k - p\| < 2 \), we may assume that \(-1\) is not in the spectrum of \( x \). Let \( \log(\sqrt{-1} \lambda) = \sqrt{-1} \lambda \) for \( \lambda \in (-\pi, \pi) \). Then, \( h := \frac{1}{\sqrt{-1}} \log(x + p^\perp) \in A^{**}_C \) is a self-adjoint element such that \( hp = ph \) and \( \mathcal{J}(h) = \frac{1}{\sqrt{-1}} \log(x^* + p^\perp) = -h \). Similarly as above, there is a self-adjoint element \( b \in A_C \) such that \( hp = php = pb \) and \( \mathcal{J}(b) = -b \). It follows that \( a = \exp(\sqrt{-1} b) \) is a unitary element in (the unitization of) \( A \) such that \( ap = x = pa \). \( \square \)

The assumption on the unitary element \( x \) cannot be removed in general: E.g., there is no unitary element \( f \) in \( \{f \in C([0, 1], M_2(\mathbb{R})): f(0) \in \mathbb{R}1\} \) such that \( f(1) = [\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}] \). However, this is a rather special case. Since \( [\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}] \) is connected to \( [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}] \) via \( [\begin{smallmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{smallmatrix}] \), unless there is a nonzero central projection \( z \) in \( A^{**} \) such that \( zp = z \), any unitary element \( x \in pA^{**}p \) can be inflated to a unitary element in \( U_0(p_0A^{**}p_0) \) for some finite rank projection \( p_0 \geq p \).

Recall that a linear functional \( \varphi \) on a C*-algebra \( A \) is called a state if it is positive and has norm one. It is said to be pure if it is an extreme point of the state space.

**Lemma 13.** Let \( \varphi \) be a pure state on a C*-algebra \( A \). Then, \( p := \text{supp}(\varphi) \) is a minimal projection in \( A^{**} \). Let \( L := \{a \in A : \varphi(a^* a) = 0\} \) be the corresponding left ideal and \( (e_n)_n \) be an approximate unit for the C*-subalgebra \( L \cap L^* \). Then, \( e_n \to p^\perp \) ultrastrongly. In particular, any \( \theta \in B_{A^*} \) such that \( \lim_n \theta(1 - e_n) = 1 \) coincides with \( \varphi \).
Proof. We view $\varphi$ as a normal state on the second dual $A^{**}$. Recall that $p := \text{supp}(\varphi)$ is the smallest projection in $A^{**}$ such that $\varphi(p) = 1$. In the complex case, it is well-known that $p$ has rank one. In the real case, the set $\Omega$ of states $\varphi_0$ on $A_C$ such that $\mathbb{R}\varphi_0 = \varphi$ on $A$ is a weak$^*$-closed face of the state space of $A_C$ and any pure state $\varphi_0 \in \partial \Omega$ satisfies $\varphi_C = \frac{1}{2}(\varphi_0 + \bar{\varphi}_0)$ on $A_C$, where $\bar{\varphi}_0(a) = \varphi_0(\overline{\mathcal{J}(a)})$ and $\varphi_C$ is the complexification of $\varphi$, which is the unique state extension of $\varphi$ on $A_C$ such that $\varphi_C = \bar{\varphi}_C$. It follows that $p = \text{supp}(\varphi_C)$ has rank at most two in $A_C^{**}$. Hence, by compactness, for any nonzero projection $q \leq p$ there is $\varepsilon > 0$ such that $\varepsilon\varphi(q \cdot q) \leq \varphi(p \cdot p)$. Since $\varphi(p \cdot p) = \varphi(\cdot)$ is a pure state, this implies $q = p$, proving that the projection $p$ is minimal. We note that there is only one state on $pA^{**}p \cong \mathbb{R}, \mathbb{C}, \mathbb{H}$. Since $\varepsilon_n\varepsilon_m \to \varepsilon_m$ as $n \to \infty$, the ultrastrong limit $e := \sup e_n \in A^{**}$ is a projection such that $e^* \geq p$. By Kadison’s transitivity theorem (Lemma 12), one has

$$\{x \in A : pxp = 0\} = L + L^* \subset \{x \in A : e^*xe^* = 0\}.$$

Indeed, for any $x \in A$ such that $pxp = 0$, there is a finite rank projection $q \geq p$ in $A^{**}$ such that $px = pxp$ and there is $a \in A$ such that $aq = pxq = qa$. Since $ap = pxp = 0$ and $p(x-a) = pxq - paq = 0$, one has $a \in L$ and $x - a \in L^*$. The right inclusion is obvious. Hence the map $pAp \ni pxp \mapsto e^*xe^*$ is a well-defined continuous linear map. Since $p$ is a pure state, this map is ultraweakly continuous on $pA^{**}p$ and so $e^* \leq p$. This proves $e = p^\perp$. If $\theta \in B_{A^*}$ is such that $\lim_n \theta(1 - e_n) = 1$, then $\theta(e^*) = 1$, which implies that $\theta$ is a state and $\theta = \varphi$ by Lemma 7 and the uniqueness of the state on $e^*A^{**}e^*$.

Recall that for any maximal face $E \subset S_X$ there is $\varphi \in \text{ext} B_{X^*}$ such that

$$E = E_{\varphi} := \{x \in S_X : \varphi(x) = 1\}.$$

Lemma 14. Let $A$ be a C*-algebra and $\varphi \in \text{ext} B_{A^*}$. Then, $|\varphi|$ is a pure state and there is a unitary element $u$ in the unitization of $A$ such that $\varphi(\cdot) = |\varphi|(u^* \cdot)$ and $E_{\varphi} = uE_{|\varphi|}$.

Proof. Every $\varphi \in S_{A^*}$ has a polar decomposition $\varphi(\cdot) = |\varphi|(v^* \cdot)$ (see Section III.4 in [Tak]), where $v$ is a partial isometry in $A^{**}$. By Lemma 7 one sees that the state $|\varphi|$ is pure if (and only if) $\varphi \in \text{ext} B_{A^*}$. Hence $p := v^*v$ and $q := vv^*$ are minimal projections in $A^*$. By Lemma 7, it suffices to show that there is a unitary element $u$ in (the unitization of) $A$ such that $up = v$.

If $p = q$, then $v$ is a unitary element in $pA^{**}p \cong \mathbb{R}, \mathbb{C}, \mathbb{H}$ and so there is a unitary element $u$ such that $up = v$ by Kadison’s transitivity theorem (Lemma 12). From now on, we assume that $p \neq q$. It follows that $p \vee q - p \sim q - p \wedge q = q$ (Proposition V.1.6 in [Tak]) is a minimal projection and $(p \vee q - p)v(p \vee q - q)v^*(p \vee q - p) = \alpha(p \vee q - p)$ for some $\alpha > 0$. Hence $w := \alpha^{-1/2}(p \vee q - q)v^*(p \vee q - p)$ is a minimal partial isometry in $(p \vee q)A^{**}(p \vee q)$ such that $w \perp v$. We claim that at least one of the two unitary elements $v \pm w$ in $(p \vee q)A^{**}(p \vee q)$ does not have both $\pm 1$ in its spectrum. If this was not the case, then $v + w = e - e^\perp$ and $v - w = f - f^\perp$ for some minimal projections $e$ and $f$, but since $v = e + f - 1$ is a partial isometry, this implies $e = f$ or $e \perp f$, which is absurd. Hence, by Kadison’s transitivity theorem (see Proof of Lemma 12), there is a unitary element $u$ in (the unitization of) $A$ such that $up = v$. \qed
Lemma 15. Let $A$ be a (real or complex) von Neumann algebra and $E \subset S_A$ be a maximal face. Then, there is a net $(\Theta_n)_n$ of affine contractions from $E$ onto ultraweakly-closed face $E_n \subset E$ such that $\Theta_n \to \text{id}_E$ in the point-norm topology and each $E_n$ is affinely isometrically isomorphic to the closed unit ball of a real von Neumann algebra. In particular, any surjective isometry $T: S_A \to S_Y$ is affine on $E$.

Proof. By Lemma 14, we may assume that $E = E_\varphi$ for some pure state $\varphi$ on $A$. Let $L := \{a \in A : \varphi(a^*a) = 0\}$ and take an approximate unit $(e_n)_n$ for $L \cap L^*$. By enlarging the index set if necessary, we may find $\varepsilon(n) > 0$ such that $\varepsilon(n) \to 0$ and put $q_n := \chi_{(\varepsilon(n),1]}(e_n) \in L \cap L^*$. Then any $x \in E$ satisfies $1 - x \in L \cap L^*$ and $q_n(1 - x) \approx 1 - x \approx (1 - x)q_n$. This implies $(1 - q_n)x \approx 1 - q_n \approx x(1 - q_n)$ and $q_nx \approx xq_n$. Therefore $\Theta_n(x) := q_n + q_nxq_n \in q_n + B_{q_nAq_n} \subset E_\varphi$ satisfies $\Theta_n(x) \to x$ in norm. That $T|_E$ is affine follows from Lemma 8 and Corollary 3. □

This is enough for the proof of Theorem 1 for von Neumann algebras. More technical results below are needed to deal with $C^*$-algebras.

Let $A$ be a $C^*$-algebra and $A^{**}$ be its second dual. A projection $p$ in $A^{**}$ is said to be compact if it is the ultrastrong limit of an decreasing net of norm-one positive elements in $A$. (See [AP] for more detail.) For any nonzero compact projection $p \in A^{**}$, denote the corresponding face by

$$F(p) := \{x \in S_A : xp = p = px\} = A \cap \{p + y : y \in B_{p^{+}A^{*}p^{+}}\}$$

and put for $\lambda \in [-1, 1]$,

$$F(p, \lambda) := \{x \in S_A : xp = \lambda p = px\} = S_A \cap \{\lambda p + y : y \in B_{p^{+}A^{*}p^{+}}\}.$$

Lemma 16. One has $\overline{F(p)}^{(A^{**}, A^r)} = \{p + y : y \in B_{p^{+}A^{*}p^{+}}\}$.

Proof. The inclusion $\subset$ is clear. For the other inclusion, take $y \in B_{p^{+}A^{*}p^{+}}$ arbitrary. By Kaplansky’s density theorem, there is a net $y_n \in B_A$ such that $y_n \to y$ ultrastrongly. Since $p$ is compact, there is a net $p_n \in A$ such that $0 \leq p_n \leq 1$ and $p_n \searrow p$. (We may assume that these nets are indexed by the same directed set.) Then, $p_np = p = pp_n$ and hence the net

$$z_n := p_n + (1 - p_n)1/2y_n(1 - p_n)1/2 = \sqrt{1/2} \begin{bmatrix} p_n & 0 \\ 0 & y_n \end{bmatrix} \sqrt{1/2}$$

belongs to $F(p)$ (the latter expression shows $\|z_n\| \leq 1$). Since it converges ultrastrongly to $p + y$, we are done. □

Lemma 17. One has $F(p)_\lambda = F(p, \lambda)$.

Proof. We view $A^{**} \subset \mathcal{B}(H)$. Let $x \in F(p)_\lambda$ be given and take a unit vector $\xi \in pH$. For every $\varepsilon > 0$, there are $y \in F(p)$ and $z \in -F(p)$ such that $\|y - x\| \approx_\varepsilon 1 - \lambda$ and $\|z - x\| \approx_\varepsilon 1 + \lambda$. Since one has $y\xi = \xi = -z\xi$ and

$$2 \leq \|\xi - x\xi\| + \|\xi + x\xi\| \leq \|y - x\| + \|z + x\| \approx_\varepsilon 2,$$
Lemma 18. Let \( A \) be a unital complex C*-algebra and \( E \subset S_A \) be a maximal face. Then, \( E \) coincides with the closed convex hull of unitary elements in \( E \).

Proof. By Lemma 14 we may assume that \( A = E_\varphi \) for some pure state \( \varphi \) on \( A \). Let \( p := \text{supp}(\varphi) \) and view elements in \( A \) as operator valued \( 2 \times 2 \) matrix in accordance with \( p \oplus p^\perp \). Thus, \( E = \{ [1,0] \} \). Let \( x \in E \) and \( \varepsilon > 0 \) be given. We set \( v_0 := 1 \) and choose \( u_k,v_k \in E \cap \mathcal{U}(A) \) inductively as follows. Fix \( \delta > 0 \) very small and let \( (1+\delta)v_{k-1} + x = w_k[(1+\delta)v_{k-1} + x] \) be the polar decomposition. Note that \( w_k \in E \cap \mathcal{U}(A) \), as it is easily seen from the operator valued matrix viewpoint. We define \( u_k,v_k \in E \cap \mathcal{U}(A) \) to be 

\[
  u_k + v_k = w_k|v_{k-1} + x| \approx_{\delta'} w_k|(1+\delta)v_{k-1} + x| = (1+\delta)v_{k-1} + x \approx_{\delta} v_{k-1} + x,
\]

where \( \delta' > 0 \) depends only on \( \delta \) and converges to 0 as \( \delta \) converges to 0. Thus, by choosing \( \delta > 0 \) small enough, one has \( v_{k-1} + x \approx_{\varepsilon/3} u_k + v_k \). It follows that 

\[
  v_0 + nx \approx_{\varepsilon/3} u_1 + v_1 + (n-1)x \approx_{\varepsilon/3} \cdots \approx_{\varepsilon/3} u_1 + \cdots + u_n + v_n
\]

and 

\[
\| x - \frac{1}{n} \sum_{k=1}^n u_k \| < \varepsilon \quad \text{for} \quad n \geq 3/\varepsilon. \]

We will need the following ad hoc result.

Lemma 19. Let \( A \) be a unital complex C*-algebra and \( \psi : A \to \mathbb{C} \) be a nonzero multiplicative linear functional. Then the closed unit ball of the real C*-algebra \( A^\psi_\mathbb{R} := \psi^{-1}(\mathbb{R}) \) has the strong Mankiewicz property.

Proof. Let \( \bar{\bar{A}} = \{ \bar{\bar{a}} : a \in A \} \) denote the complex conjugate C*-algebra of \( A \). Then, \( A^\psi_\mathbb{R} \) is the real part of the complex C*-algebra \( A^\psi := \{ a \oplus \bar{\bar{b}} : a \oplus \bar{\bar{b}} \in A \oplus \bar{\bar{A}} : \psi(a) = \bar{\bar{\psi(b)}} \} \) with respect to the conjugate linear automorphism \( J(a \oplus \bar{\bar{b}}) = b \oplus \bar{\bar{a}} \). By Theorem 2 it suffices to show that \( A^\psi_\mathbb{R} \) coincides with the closed convex hull of unitary elements in \( A^\psi_\mathbb{R} \). Let \( x \in B_{A^\psi_\mathbb{R}} \) and \( \varepsilon > 0 \) be given. We consider the second dual von Neumann algebra \( A^{**} \) and the ultraweakly continuous extension \( \psi : A^{**} \to \mathbb{C} \). Then, \( M := \ker \psi \) is a von Neumann subalgebra such that \( A^{**} = \mathbb{C} \oplus M \) as a von Neumann algebra. Hence there are unitary elements \( u_k \in (A^{**})^\psi_\mathbb{R} = \mathbb{R} \oplus M \) such that \( x \approx_\varepsilon \frac{1}{n} \sum_{k=1}^n u_k \). Let \( h_k := \frac{1}{\sqrt{-1}} \log u_k \in (A^{**})_{a.a.} \), where \( \log \exp(\sqrt{-1} \lambda) = \sqrt{-1} \lambda \) for \( \lambda \in (-\pi, \pi) \). Since \( x \approx_\varepsilon \frac{1}{n} \sum_{k=1}^n u_k \),
\( \psi(u_k) \in \{1, -1\} \), one has \( \psi(h_k) = \frac{1}{\sqrt{n}} \log \psi(u_k) \in \{0, -\pi\} \). By Kaplansky’s density theorem, there are bounded nets \((h_{k,i})_i\) in \( A_{s.a.} \) such that \( h_{k,i} \to h_k \) ultrastrongly. We may assume that \( \psi(h_{k,i}) \in \{0, -\pi\} \).

By Lemma 14, we may assume that \( K \) is affinely isometrically isomorphic to the closed unit ball of a real \( C^* \)-algebra \( C_n \), and each \( T(K_n) \) is convex. In the case \( A \) is a unital complex \( C^* \)-algebra, the above \( C_n \) can be taken so that \( B_{C_n} \) has the strong Mankiewicz property and so \( T|_{E} \) is affine.

**Proposition 20.** Let \( A \) be a \( C^* \)-algebra, \( T : S_A \to S_Y \) be a surjective isometry, and \( E \subset S_A \) be a maximal face. Then, there is a net \((\Theta_n)_n\) of affine contractions from \( E \) into closed convex subsets \( K_n \subset E \) such that \( \Theta_n \to \text{id}_E \) in the point-norm topology, each \( K_n \) is affinely isometrically isomorphic to the closed unit ball of a real \( C^* \)-algebra \( C_n \), and each \( T(K_n) \) is convex. In the case \( A \) is a unital complex \( C^* \)-algebra, the above \( C_n \) can be taken so that \( B_{C_n} \) has the strong Mankiewicz property and so \( T|_{E} \) is affine.

**Proof.** By Lemma 14 we may assume that \( E = E_\varphi \) for some pure state \( \varphi \) on \( A \). We denote by \( \tilde{A} \) the unitization of \( A \) if \( A \) is not unital, else \( \tilde{A} = A \). We consider \( \varphi \) a pure state on \( \tilde{A} \) and consider \( L := \{ a \in \tilde{A} : \varphi(a^*a) = 0 \} \).

One can skip this paragraph if \( A \) is unital. In case \( A \) is not unital, let \( \sigma \) denote the character on \( \tilde{A} \) corresponding to the unitization. We claim that there is an approximate unit \((e_n)_n\) for \( L \cap L^* \) such that \( \sigma(e_n) = 1 \). Let \((e_n)_n\) be any approximate unit. Then, by Lemma 13 we may assume \( \lambda := \inf \sigma(e_n) > 0 \). We consider the continuous function \( h(t) = \min(\lambda^{-1}t, 1) \). Then, \( h(e_n) \) is an approximate unit such that \( \sigma(h(e_n)) = h(\sigma(e_n)) = 1 \) for all \( n \).

Let \((e_n)_n\) be an approximate unit for \( L \cap L^* \) such that \( 1 - e_n \in A \) for all \( n \) (which is equivalent to \( \sigma(e_n) = 1 \)). By perturbation using functional calculus, we may assume that there is \( f_n \in L \cap L^* \) such that \( 0 \leq e_n \leq f_n \leq 1 \) and \( e_n f_n = e_n = f_n e_n \). (Although \((e_n)_n\) may not be increasing anymore, this does not matter for the following.) Since \( 1 - x \in L \cap L^* \) for every \( x \in E_\varphi \), one has \( e_n (1 - x) \approx 1 - x \approx (1 - x) e_n \). This implies \( (1 - e_n)x \approx 1 - e_n \approx x(1 - e_n) \) and \( e_n x \approx x e_n \). Therefore
\[
 x = ((1 - e_n) + e_n)x \approx (1 - e_n) + e_n^{1/2} x e_n^{1/2} =: \Theta_n(x) \in E_\varphi.
\]
See Proof of Lemma 16 for the proof that \( \Theta_n \) is contractive and \( \|\Theta_n(x)\| \leq 1 \) for every \( x \in E_\varphi \).

To ease notation, we fix \( n \) and write \( f := f_n \). We consider \( s := 1 - f \in E_\varphi \), its support projection \( p := \chi_{[0,1]}(s) \in A^{**} \), and the closed face
\[
 F := \{ x \in S_A : xs = s = sx \} = \{ x \in S_A : xp = p = px \} \subset E_\varphi
\]
(although \( p \) is not a compact projection). Since \( (1 - e_n)s = s = s(1 - e_n) \), one has \( \Theta_n(E_\varphi) \subset F \). If \( p \in A \), then the face \( K_n := F + B_{p^\perp Ap^\perp} \) satisfies the desired property with \( C_n = p^\perp Ap^\perp \). Let’s assume \( p \notin A \) and consider the \( C^* \)-subalgebra
\[
 C := \{ a \in A : ap = \gamma p = pa \text{ for some scalar } \gamma \}.
\]
For the following, it is probably easier to digest if one views elements in $A$ as operator valued $2 \times 2$ matrices in accordance with $p \oplus p^\perp$. Thus $s = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (slightly abusing the notation), $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $F = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \}$, and $C = \{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$. Since $p \notin A$, one has $\|px\| \leq \|p^\perp x\|$ for all $x \in C$. Hence, there is a norm-one (multiplicative) linear functional $\psi$ on $p^\perp C$ such that $\psi(p^\perp x)p = px$ for all $x \in C$, or equivalently $x = [\psi(y) 0]_y$ for all $x \in C$ and $y := p^\perp x$. Hence, $\|x\| = \|(1 - s)x\|$ for all $x \in C$ and one has

$$F \subset K_n := s + (1 - s)B_{C^\alpha_k} = S_A \cap \{ [s + \gamma(1 - s) 0]_0 \gamma \in [-1, 1] \} \subset E_\varphi.$$  

By Lemma [19], the closed unit ball $B_{C^\alpha_k}$ satisfies the strong Mankiewicz property provided that $A$ (and hence $C$) is a unital complex $C^*$-algebra. It is left to show that $T(K_n)$ is convex. For $\gamma \in [-1, 1]$, put $h_\gamma(\lambda) := \lambda + \gamma(1 - \lambda)$. For $i = 1, 2$, put

$$G^i_m(\gamma) := E_\varphi \cap \bigcap_k F(\chi_{[2k-2+\varepsilon, 2k+1]}(s), h_\gamma(\frac{k}{m})) \text{ and } H^i_m(\gamma) := E_\varphi \cap \mathcal{N}_\delta(m)(G^i_m(\gamma)),$$

where the intersection is over $k = 1, 2, \ldots, m$ for which $\chi_{[2k-2+\varepsilon, 2k+1]}(s) \neq 0$ and $\mathcal{N}_\delta$ means the $\delta$-neighborhood in $A$. Note that $\chi_{[2k-2+\varepsilon, 2k+1]}(s) \leq p$ for all $i, m$, and $k$. By Lemmas [17] and [10] one has

$$\alpha G^i_m(\gamma_1) + (1 - \alpha)G^i_m(\gamma_2) \subset G^i_m(\gamma_3) \text{ and } \alpha H^i_m(\gamma_1) + (1 - \alpha)H^i_m(\gamma_2) \subset H^i_m(\gamma_3)$$

for every $\gamma_1, \gamma_2 \in [-1, 1]$, $\alpha \in [0, 1]$, and $\gamma_3 := \alpha \gamma_1 + (1 - \alpha)\gamma_2$. We claim that

$$K(\gamma) := \{ x \in E_\varphi : px = h_\gamma(s) = xp \} = \bigcap_{m \in \mathbb{N}} (H^1_m(\gamma) \cap H^2_m(\gamma)).$$

To prove the inclusion $\subset$, we define $g_m$ to be the continuous function such that $g_m(0) = \gamma$, $g_m(\lambda) = h_\gamma(\frac{k}{m})$ for $\lambda \in [\frac{2k-1}{2m}, \frac{2k}{2m}]$, and linear on $[\frac{2k-1}{2m}, \frac{2k}{2m}]$ for $k = 1, \ldots, m$. Then, $\|g_m - h_\gamma\| \leq \frac{1}{m}$ and $(g_m - h_\gamma)(0) = 0$. It follows that $(g_m - h_\gamma)(s) \in A \oplus pAp$ and for any $x \in K(\gamma)$, one has $x + (g_m - h_\gamma)(s) \in G^1_m(\gamma)$. This proves $x \in H^1_m(\gamma)$. The proof of $x \in H^2_m(\gamma)$ is similar. For the converse inclusion, take $x$ from the RHS of the claimed equality. Since $x \in H^1_m(\gamma) \cap H^2_m(\gamma)$, there are $y^i_m \in G^i_m(\gamma)$ such that $\|x - y^i_m\| \leq \frac{1}{m}$. For the projection $p^i_m := \sum_{k=1}^{m} \chi_{[2k-2+\varepsilon, 2k+1]}(s)$ in $A^{**}$, one has $\|h_\gamma(s)p^i_m - y^i_mp^i_m\| \leq \frac{1}{m}$. Hence, $\|(h_\gamma(s) - x)(p^1_m \vee p^2_m)\| \leq \frac{2}{m}$. Since $p^1_m \vee p^2_m \to p$ ultrastrongly, one sees $h_\gamma(s) = xp$. The proof of $h_\gamma(s) = px$ is similar. Now, since $K_n = \bigcup_{\gamma \in [-1, 1]} K(\gamma)$ and

$$T^{-1}(\alpha T(K(\gamma_1)) + (1 - \alpha)T(K(\gamma_2))) \subset \bigcap_{m \in \mathbb{N}} (H^1_m(\gamma_3) \cap H^2_m(\gamma_3)) = K(\gamma_3)$$

by Lemma [14], one concludes that $T(K_n)$ is convex. 

6. Proof of Theorem [1]

Proof of Theorem [1]. We will show that for any surjective isometry $T : S_A \to S_Y$ and any pure state $\varphi$ on $A$, there is $\psi \in B_{Y^{**}}$ such that $\mathbb{R} \varphi = \psi \circ T$. This yields the assertion by Lemmas [6] and [14]. Let $p := \text{supp}(\varphi) \in A^{**}$ and consider the maximal face

$$E_\varphi := \{ x \in S_A : \varphi(x) = 1 \} = \{ x \in S_A : xp = p = px \}.$$
By Lemma [8] there is $\psi \in B_{Y^*}$ such that $\psi = 1$ on the face $T(E_\varphi)$. We will show $\Re \varphi = \psi \circ T$.

If $A$ is a type $I_1$ factor $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, then $A$ is a real Hilbert space and the Mazur–Ulam property is already known (see [CD]). Hence we may assume that $A$ is not a type $I_k$ factor with $k \leq 2$. Let $u \in S_A$ be an arbitrary unitary element. By the previous remark, there is a minimal projection $q \in A^{**}$ such that $q \perp p \vee u^*pu$. We consider the face

$$F := \{ x \in S_A : xq = uq \} = S_A \cap (uq + B_{(uqu^*)^*} Aq^*).$$

Since $T|_F$ is affine by Proposition 20 or Lemma 15, there are $\gamma \in \mathbb{R}$ and $\theta \in B_{A^*}$ such that $(\psi \circ T)(x) = \gamma + \Re \theta(x)$ for $x \in F$. By Kadison's transitivity theorem (Lemma 12), there is $x_0 \in B_A$ such that $x_0q = uq$ and $x_0p = px_0$. Then, $x_0 \in F \cap L \cap L^*$, where $L := \{ a \in A : \varphi(a^*a) = 0 \}$. Let $(e_n)_n$ be an approximate unit for the $C^*$-subalgebra $L \cap L^*$ such that $e_n \geq |x_0|$ for all $n$. Since $e_n \geq |x_0|$, one has $e_nq = q$ and $x_0 \pm (1 - e_n) \in F \cap (E \varphi)$ for all $n$. Thus

$$\pm 1 = (\psi \circ T)(x_0 \pm (1 - e_n)) = (\psi \circ T)(x_0) \pm \theta(1 - e_n).$$

This implies that $(\psi \circ T)(x_0) = 0$ and $\theta(1 - e_n) = 1$ for all $n$. By Lemma 13, one sees $\theta = \varphi$ and so $(\psi \circ T)(u) = \Re \varphi(u)$. Now $\Re \varphi = \psi \circ T$ follows from Lemma 18 or that the unitary elements have norm-dense convex hull in the closed unit ball of any von Neumann algebra (see Proof of Corollary 3).

**Remark 21.** A similar proof yields the Mazur–Ulam property for the self-adjoint part of a unital complex $C^*$-algebra (and a real von Neumann algebra) which is not isomorphic to the $2 \times 2$ matrix algebra.

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