The behavior of vortices in dirty type-II superconductors has been a subject of intense studies in the last decade \cite{1,2}. Aside from the obvious technological significance of vortex pinning, understanding the physics of such interacting many-body systems in the presence of quenched disorder is a central theme of modern condensed matter physics. Similarities between the randomly-pinned vortex system and the more familiar mesoscopic electronic systems \cite{2} are highlighted by a recent experimental study of a planar vortex array threaded through a thin crystal of 2H-NbSe\textsubscript{2} by Bolle \textit{et al.} \cite{2}. Interesting behaviors, including the sample-dependent magnetic responses known as “finger prints”, have been observed for such a mesoscopic vortex system.

The disordered planar vortex array is well studied theoretically \cite{3,4,5,6,7}. It is one of the few disorder-dominated systems for which quantitative predictions can be made, including a finite-temperature “vortex glass” phase \cite{3}, characterized by anomalous vortex displacements \cite{4}, and universal variation of magnetic susceptibility \cite{5}. However, until the work of Bolle \textit{et al.}, there were hardly any experimental studies of this system, with difficulties stemming partly from the weak magnetic signals in such 2d systems. Also, numerical simulations have been limited by the slow glassy dynamics \cite{6}, although the availability of special optimization algorithms did lead to the elucidation of the zero-temperature problem in recent years \cite{7}. In this letter, we describe numerical studies of the thermodynamics of the vortex glass via a mapping to a discrete dimer model with quenched disorder. A new polynomial algorithm for the dimer problem circumvents the glassy dynamics and enables us to study large systems at finite temperatures. Our results obtained in the dilute (single-flux-pinning) regime compare well with the experiment by Bolle \textit{et al.} \cite{2}, while those obtained in the collective-pinning regime strongly support the renormalization-group theory of the vortex glass, including its prediction of universal susceptibility variation \cite{6}.

\textbf{The Model:} The dimer model consists of all complete dimer coverings \{\textit{D}\} on a square lattice \textit{L} as illustrated in Fig. 1(a). The partition function is

\[ Z = \sum_{\{\textit{D}\}} \exp \left[ -\sum_{\langle ij \rangle \in \textit{D}} \epsilon_{ij} / T_d \right], \]  

where the sum in the exponential is over all dimers of a given covering, and \( T_d \) is the dimer temperature. Quenched disorder is introduced via random bond energies \( \epsilon_{ij} \), chosen independently and uniformly in the interval \( (-\frac{1}{2}, \frac{1}{2}) \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.pdf}
\caption{(a) Snapshot of a dimer covering (thick bonds) together with the associated “height” values \( h(\textbf{r}) \) for a lattice of size \( L = 8 \) at temperature \( T_d = 1.0 \). (b) Dimer covering of the fixed reference. (c) Vortex line configuration (thick lines) obtained as the difference between (a) and (b).}
\end{figure}

The dimer model is related to the planar vortex-line array via the well-known mapping to the solid-on-solid (SOS) model (see Figs. 1): Take the centers of the square cells of \textit{L} to form the dual square lattice \( \hat{\textit{L}} \). Orient all bonds of \( \hat{\textit{L}} \) such that the elementary squares of \( \hat{\textit{L}} \) that enclose the sites of the chosen sublattice of \textit{L} (indicated by the solid dots in Fig. 1(a)) are circled counterclockwise. It is now possible to assign a single-valued “height” function \( h(\textbf{r}) \) on the lattice points \( \textbf{r} \) of \( \hat{\textit{L}} \), such that the difference of every pair of neighboring heights across the oriented bonds is \(-3\) if a dimer is crossed and \(+1\) otherwise. For the dimer covering of Fig. 1(a), the values of the associated height function are shown at their respective positions. In terms of the height configuration \( h(\textbf{r}) \), the partition function \( \langle \rangle \) can be written alternatively as

\[ Z = \sum_{\{h(\textbf{r})\}} e^{-\beta H(h)}, \]  

where the SOS Hamiltonian takes the following form in the continuum limit,
Here, $K$ is an effective stiffness caused by the inability of a tilted surface to take as much advantage of the low weight bonds as a flatter surface, and $f(r)$ is a random local tilt bias. The periodicity of the cosine potential in (1) is given by $G = 2\pi/4$ since the smallest “step” of this height profile is four. In the present context of a randomly pinned vortex array, $h(r)$ describes the coarse-grained displacement field of the vortex array with respect to a uniform reference state at the same vortex line density; see Fig. 1(c) and Refs. [5,8]. This mapping between the dimer model and the planar vortex array allows us to compute statistical properties of the vortex array by monitoring appropriate quantities of the dimer model. The latter can be accomplished using a polynomial algorithm described below.

The Algorithm: Computing the partition function of complete dimer coverings on an arbitrary weighted lattice is likely to be algorithmically intractable [12]. Weights here refer to the Boltzmann factors $w_{ij} \equiv \exp(-e_{ij}/T_d)$ on the bonds. A weighted planar lattice $G$ can however be oriented, denote by $\tilde{G}$, so that the square root of the determinant of the weighted adjacency matrix defined on $\tilde{G}$ yields the partition function (2) exactly [3]. Computation of the determinant can be achieved by various row or column reduction schemes in time that grows polynomially with the matrix size.

Recently, Propp and coworkers [4] furnished the above algebraic reduction with an elegant graphical interpretation for the case of bipartite planar lattices [3]. The partition function $Z_L$ on a lattice of linear size $L$ is related to $Z_{L-1}$ as $Z_L(\{w\}) = C(\{w\})Z_{L-1}(\{w'\})$ after a simple weight transformation $\{w\} \rightarrow \{w'\}$. The prefactor $C(\{w\})$ is independent of dimer coverings. The partition function is obtained in a “deflation” process in which the above recursive procedure is carried out down to $L = 0$ with $Z_0 = 1$. This deflation process can also be reversed in an “inflation” process where a dimer covering at size $L-1$ can be used to stochastically generate a dimer covering at size $L$ according to $Z_L$ already obtained. Repeating the inflation process thus generates uncorrelated “importance samplings” of the dimer configurations, or equivalently the equilibrium height configurations, without the need to run the slow relaxational dynamics. The ensuing numerical results are obtained by taking various measurements of the height configurations generated this way. A somewhat inconvenient feature of this approach is that the algorithm requires open boundary condition on the dimer model; this in turn fixes the total number of vortex lines, e.g., to $L/2$ on a $L \times L$ lattice (see Fig. 1).

Numerical Results: Since the temperature of the vortex array, given by $K^{-1}$ in (3), is generally different from the dimer temperature $T_d$, we first need to calibrate the temperature scale. To do so, we exploit a statistical rotational symmetry [17] of the system (2) which guarantees that at large scales, the effective $K$ is the same as that of the pure system. In particular, $K$ can be obtained by measuring the disorder-averaged thermal fluctuation of the displacement field $\tilde{W}_T^2 \equiv \left\langle h_T^2(r) \right\rangle \equiv \left\langle h_T^2(r) - \left\langle h_T^2(r) \right\rangle \right\rangle = (2\pi K)^{-1}\ln(L)$, where $h_T(r) \equiv h(r) - \left\langle h(r) \right\rangle$ measures the thermal distortion superposed on the distorted “background” $h_D(r) \equiv \left\langle h(r) \right\rangle$ by the quenched disorder. We used $<...>,\{\}$, and overline to denote spatial, thermal, and disorder averages respectively. With our polynomial algorithm, we were able to perform thorough disorder averages for equilibrated systems of sizes up to $512 \times 512$. To reduce boundary effects, we focus on the central $L/2 \times L/2$ piece of the system and compute its displacement fluctuations. Fig. 2(a) illustrates the dependence of $\tilde{W}_T^2(L)$ for various dimer temperatures $T_d$. The linear dependence on $\ln(L)$ is apparent. Identifying the proportionality constant with $(2\pi K)^{-1}$, we obtain the empirical relation between $K^{-1}$ and $T_d$ shown in Fig. 2(b). Note that our result recovers the exact relation $K^{-1}(T_d \rightarrow \infty) = 16/\pi$ for the dimer model without disorder [14]. Since the glass transition of the system (3) is expected to occur at temperature $K^{-1} \sim 4\pi/G^2 = 16/\pi$, our system is glassy for the entire range of dimer temperatures. Fig. 2(b) shows that a linear relation holds between $K^{-1}$ and $T_d$.

A striking feature of the vortex glass phase is the anomalous fluctuation of the disorder-induced displacement, $\tilde{W}_D^2 \equiv \left\langle h_D^2(r) \right\rangle - \left\langle h_T^2(r) \right\rangle = 2\pi K^{-1}\ln(L)$, where $h_D(r) \equiv h(r) - \left\langle h(r) \right\rangle$ measures the disorder-induced distortion superposed on the distorted “background” $h_D(r) \equiv \left\langle h(r) \right\rangle$ by the quenched disorder. We used $<...>,\{\}$, and overline to denote spatial, thermal, and disorder averages respectively. With our polynomial algorithm, we were able to perform thorough disorder averages for equilibrated systems of sizes up to $512 \times 512$. To reduce boundary effects, we focus on the central $L/2 \times L/2$ piece of the system and compute its displacement fluctuations. Fig. 2(a) illustrates the dependence of $\tilde{W}_T^2(L)$ for various dimer temperatures $T_d$. The linear dependence on $\ln(L)$ is apparent. Identifying the proportionality constant with $(2\pi K)^{-1}$, we obtain the empirical relation between $K^{-1}$ and $T_d$ shown in Fig. 2(b). Note that our result recovers the exact relation $K^{-1}(T_d \rightarrow \infty) = 16/\pi$ for the dimer model without disorder [14]. Since the glass transition of the system (3) is expected to occur at temperature $K^{-1} \sim 4\pi/G^2 = 16/\pi$, our system is glassy for the entire range of dimer temperatures.

\[ \beta \mathcal{H} = \int d^2r \left[ \frac{K}{2}(\nabla h)^2 - f(r) \cdot \nabla h + g \cos(\Theta h) \right]. \] (2)
To probe the RG prediction further, we examine the temperature dependence of the quadratic coefficient $C_2$. Plotting $C_2$ extracted from the quadratic fit in Fig. 3(a) vs. the reduced temperature $\tau = K_g/K - 1$. The best power law fit yields an exponent of 2.04(9).

Another interesting feature of the vortex glass phase is the sample-to-sample variation of the magnetic susceptibility $\chi$. It was predicted that the fractional variance,

$$\text{var}[\chi]/\chi^2 \xrightarrow{L \to \infty} D|\tau| \quad \text{for} \quad 0 < -\tau < 1, \quad (3)$$

is universal, with $D$ being a computable, size-independent constant of order unity. Due to the constraint of fixed vortex density when using the dimer representation, it is not easy to probe the magnetic susceptibility directly by varying vortex density. Instead, we use the fluctuation-dissipation relation, e.g., $\chi \propto (\langle \partial_x h \rangle)^2 - (\langle \partial_x h \rangle)^2$. To circumvent lattice effects, we take as definition $\chi(L) \equiv \mathcal{Q}^2 (\langle \hat{h}_q \rangle \langle \hat{h}_{-q} \rangle - \langle \hat{h}_q \rangle \langle \hat{h}_{-q} \rangle)$, where $\hat{h}_q$ is the Fourier transform of $h(r)$, and the right-hand side is evaluated at $(q_x, q_y) = (1/L, 0)$. As shown in Fig. 4, the fractional variance is indeed size-independent over the range $L = 32$ to $L = 256$ examined. Its temperature dependence can again be deduced from the relation between $K^{-1}$ and $T_d$ given in Fig. 2(b); it is well described by the linear form close to the glass transition with $D = 0.454(5)$.

The universal variation of magnetic susceptibility is reminiscent of the phenomena of universal conductance fluctuation much studied in mesoscopic electronic systems. This mesoscopic vortex system in fact exhibits interesting fluctuations even deep into the dilute regime, i.e., close to the threshold field $H_{c1}$. But to study the vortex behavior there, we must relax the constraint of fixed vortex number. To do so, we focus on a $L_x \times L_y$ located at the center of the $L \times L$ lattice. Random bond energies are assigned only to this “inner” region. While the total number of vortex lines must be fixed, the number of lines in the inner region can be changed by adding an extra bond energy $\delta v_{ij} = (-1)^{i+j} v$ in a staggered fashion to all vertical bonds of the staggered reference dimer pattern (Fig. 1(b)) in this region. This has the effect of increasing the “self-energy” of the vortex lines such that when $v$ is sufficiently large, i.e., for $v \geq v_c$, all vortex lines will be expelled from the inner region to the outer region; see Fig. 5. A dilute regime in the inner region is then achieved for small $\delta v = v_c - v$. This setup could be interpreted as embedding the dirty type-II superconductor of focus within another pure type-II superconductor which has a much smaller $H_{c1}$. In the limit $L \gg L_y$, the outer region can be regarded as an infinite reservoir in which the vortex density change is negligible. The response we measure will thus come essentially from the inner region.

![Fig. 4](image-url) Fractional variance as a function of the reduced temperature, $\tau$, for $L = 32, 64, 128$, and 256 averaged over $10^3$ disorder realizations. The solid line is a least-square linear fit for $|\tau| < 0.15$.

![Fig. 5](image-url) Snapshots of vortex line configurations in the inner region at different repulsion strength $v$ for a given disorder realization and $T_d = 0.01$. Quenched disorder exists only in the inner region of size $L_x = L_y = L/2 = 64$.

In Fig. 6(a), we plot the thermal averaged number of vortex lines $\langle n \rangle$ in a thin strip of size $L_x = 16$, $L_y = 256$ embedded in a $512 \times 512$ square lattice. We vary the repulsion strength $v$, holding the temperature fixed at $T_d = 0.01$ (corresponding to only $\sim 10\%$ of the glass transition temperature). The solid squares are data taken from a given realization of disorder. We see a disordered
stair-case structure in the dependence of $\langle n \rangle$ on $\delta v$, similar to experimental magnetic response curves observed by Bolle et al [3]. The step width dispersion results from a delicate energetic balance between the line-line repulsion and the random pinning; it can be regarded as a finger print of this disordered sample. By averaging such stair-case-response curves over 100 realizations, we obtain results indicated by the open squares in Fig. 6(a). We can extract a linear dependence, i.e., $\langle n \rangle \sim \delta v$ (except for finite-size rounding very close to the threshold). This finding agrees well with the experimental result of Bolle et al [3]. It should be contrasted with the very different behavior when the disorder is absent. The latter is shown in Fig. 6(b) for a system which is twice larger in all linear dimensions. The data are well described by a square-root singularity, $\langle n \rangle \sim (\delta v)^{1/2}$, as expected from the theory of Pokrovsky and Talapov [18]. The origin of the anomalous linear dependence of $\langle n \rangle$ on $\delta v$ in the random case has been discussed previously [19,8]; it can be readily understood from the properties of a single vortex line pinned in a two-dimensional random potential.

The numerical findings in the dilute regime are also in agreement with recent experiment on a micron-sized single crystal of 2H-NbSe$_2$. It is hoped that the present study will stimulate further experimental investigations of the fascinating physics of mesoscopic vortex systems.

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FIG. 6. (a) Average number of vortex lines $\langle n \rangle$ vs. different repulsion strength $v$ at temperature $T = 0.01$ in a narrow $L_x \times L_y$ stripe. The solid squares are for a given disorder realization and the open squares denote results from average over 100 realizations. The line is a fit to a linear dependence in the vicinity of the threshold $v_c$. Slight rounding near $v_c$ is due to the small but still finite ratio $L_x/L_y$. (b) $\langle n \rangle$ vs. $v$ in the absence of disorder. The line is a fit to the square-root form expected in the vicinity of $v_c$.

Conclusion. Finite temperature simulations were performed on a disordered dimer model to study various equilibrium properties of a planar array of vortex lines. Numerical results on the anomalous displacement fluctuation strongly support the renormalization-group theory of vortex glass. Universal susceptibility variations in the collective pinning regime were observed in accordance with the theory; critical behaviors near the glass transition compared well with analytic predictions. Suitable modification of the dimer model allowed a direct study of the dilute regime where a few vortex lines penetrate the system. Disordered stair-case-like magnetic finger prints were obtained; the statistics are in agreement with theoretical predictions, and drastically different from the well-known behavior of the pure system.

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