EXACT COMPUTATION OF THE CUMULATIVE DISTRIBUTION FUNCTION OF THE EUCLIDEAN DISTANCE
BETWEEN A POINT AND A RANDOM VARIABLE UNIFORMLY DISTRIBUTED IN DISKS, BALLS, OR POLYHEDRONS AND APPLICATION TO PROBABILISTIC SEISMIC HAZARD ANALYSIS

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Abstract. We consider a random variable expressed as the Euclidean distance between an arbitrary point and a random variable uniformly distributed in a closed and bounded set of a three-dimensional Euclidean space. Four cases are considered for this set: a union of disjoint disks, a union of disjoint balls, a union of disjoint line segments, and the boundary of a polyhedron. In the first three cases, we provide closed-form expressions of the cumulative distribution function and the density. In the last case, we propose an algorithm with complexity $O(n \ln n)$, $n$ being the number of edges of the polyhedron, that computes exactly the cumulative distribution function. An application of these results to probabilistic seismic hazard analysis and extensions are discussed.

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1. Introduction

Consider a closed and bounded set $S \subset \mathbb{R}^3$ and a random variable $X : \Omega \to S$ uniformly distributed in $S$. Given an arbitrary point $P \in \mathbb{R}^3$, we study the distribution of the Euclidean distance $D : \Omega \to \mathbb{R}^+$ between $P$ and $X$ defined by $D(\omega) = \|\overrightarrow{PX(\omega)}\|_2$ for any $\omega \in \Omega$.

Denoting respectively the density and the cumulative distribution function (CDF) of $D$ by $f_D(\cdot)$ and $F_D(\cdot)$, we have $f_D(d) = F_D(d) = 0$ if $d < \min_{Q \in S} \|\overrightarrow{PQ}\|_2$ while $f_D(d) = 0$ and $F_D(d) = 1$ if $d > \max_{Q \in S} \|\overrightarrow{PQ}\|_2$. For $\min_{Q \in S} \|\overrightarrow{PQ}\|_2 \leq d \leq \max_{Q \in S} \|\overrightarrow{PQ}\|_2$, we have

$$F_D(d) = \mathbb{P}(D \leq d) = \frac{\mu(B(P, d) \cap S)}{\mu(S)}$$

where $\mu(A)$ is the Lebesgue measure of the set $A$ and $B(P, d)$ is the ball of center $P$ and radius $d$. As a result, the computation of the CDF of $D$ amounts to computing the Lebesgue measures of $S$, and of $B(P, d) \cap S$ for any $d \in \mathbb{R}_+$. We consider four cases for $S$, represented in Figure 1 and denoted by (A), (B), (C), and (D) in this figure: (A) a disk, (B) a ball, (C) a line segment, and (D) the boundary of a polyhedron. The cases where $S$ is a union of disks, a union of balls,

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Figure 1. Different supports \( S \) for random variable \( X \).

or a union of line segments are straightforward extensions of cases (A), (B), and (C).

The study of these four cases is useful for Probabilistic Seismic Hazard Analysis (PSHA) to obtain the distribution of the distance between a given location on earth and the epicenter of an earthquake which, in a given seismic zone, is usually assumed to have a uniform distribution in that zone modelled as a disk, a ball, a line segment, or a union of flat polyhedrons (the boundary of a polyhedron in \( \mathbb{R}^3 \)). This application, which motivated this study, is described in Section 2 following the lines of the seminal papers [3], [6], which paved the way for PSHA.

In this context, the outline of the paper is as follows. In Section 2, the four basic steps of PSHA are introduced and the application of our results to this problem is discussed. In Section 2 and Subsection 5.1 we respectively consider case (A), the case where \( S \) is a disk. In Section 4 and Subsection 5.1 we respectively consider case (B), where \( S \) is a ball and case (C), where \( S \) is a line segment. In these three cases (A), (B), and (C), we obtain closed-form expressions for the CDF and the density of \( D \). The main mathematical contribution of this paper is Subsection 5.2 which provides for case (D), i.e., the case where \( S \) is the boundary of a polyhedron, an algorithm with complexity \( O(n \ln n) \) where \( n \) is the number of edges of the polyhedron, that computes exactly the CDF of \( D \). An approximate density for \( D \) can then be obtained.

We are not aware of other papers with these results. However, particular cases have been discussed: in [2], cases (A) and (C) are considered taking for \( P \) respectively the center of the disk and a point on the perpendicular bisector of the line segment. In the recent paper [9], as a particular case of (D), a rectangle is considered for \( S \) while \( P \) is the center of the rectangle. In the case where \( S \) is the boundary of a polyhedron, to our knowledge, the current versions of the most popular softwares for PSHA (OPENQUACK [11], CRISIS 2012 [12]) do not compute exactly the CDF of \( D \). For instance, CRISIS 2012 uses an approximate algorithm
that performs a spatial integration subdividing the boundary of the polyhedron into small triangles.

Extensions of our results, in particular to handle the case of a general polyhedron and the case where the $\ell_2$-norm is replaced by either the $\ell_1$-norm or the $\ell_\infty$-norm, are discussed in the last Section 6.

Throughout the paper, we use the following notation. If $A$ is an $m_1 \times n$ matrix and $B$ is an $m_2 \times n$ matrix then $(A; B)$ is the $(m_1 + m_2) \times n$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. For a point $A$ in $\mathbb{R}^3$, we denote its coordinates with respect to a given Cartesian coordinate system by $x_A$, $y_A$, and $z_A$. For two points $A, B \in \mathbb{R}^3$, $\overrightarrow{AB}$ is the line segment joining points $A$ and $B$, i.e., $\overrightarrow{AB} = \{tA + (1-t)B : t \in [0, 1]\}$ and $\overrightarrow{AB}$ is the vector whose coordinates are $(x_B - x_A, y_B - y_A, z_B - z_A)$. Given two vectors $x, y \in \mathbb{R}^3$, we denote the usual scalar product of $x$ and $y$ in $\mathbb{R}^3$ by $\langle x, y \rangle = x^\top y$. For $P \in \mathbb{R}^2$, we denote the circle and the disk of center $P$ and radius $d$ by respectively $C(P, d)$ and $D(P, d)$.

2. Overview of the Four Steps of PSHA

An important problem in civil engineering is to determine the level of ground shaking a given structure can withstand. In regions with high levels of seismic activity, it makes sense to invest in structures able to resist high levels of ground shaking. On the contrary, in regions without seismic activity during the structure lifetime, we should not invest in such structures. More precisely, it would be reasonable to design structures able to resist up to a Peak Ground Acceleration $A^* m/s^2$ that is very rarely exceeded, say with a small probability $\varepsilon$, over a given time window. This approach is used in PSHA: the confidence level $\varepsilon$ and the time window being fixed (say of $t$ years), the main task of PSHA is to estimate at a given location $P$, the Peak Ground Acceleration (PGA) $A^*$ such that the probability of the event

\[
E_t(A^*, P) = \{\text{There is at least an earthquake causing a PGA greater than } A^* \text{ at } P \text{ in the next } t \text{ years}\}
\]

is $\varepsilon$. We present the approach introduced by [3], [6], to model and solve this problem. In this approach, we consider the seismic zones that could have an impact on the Peak Ground Acceleration at $P$ (see Figure 2 for an example of 4 zones with $P$ belonging to one of these zones). These zones are bounded sets that do not overlap: typically disks, line segments, or flat polyhedrons. The number of earthquakes provoking Peak Ground Accelerations at $P$ greater than $A^*$ over the next $t$ years depends on the frequency of earthquakes in each zone. As for the ground acceleration at $P$ provoked by the earthquakes of a given zone, it will depend on the magnitudes of these earthquakes, which are random, and the locations of their epicenters, which are random too. To take these factors into account, PSHA uses a four-step process (see Figure 2):

(i) in zone $i$, the process of earthquake arrivals is modelled as a Poisson process with rate $\lambda_i$. We will assume that the earthquake arrival processes in the different zones are independent.

(ii) In zone $i$, the magnitude of earthquakes is modelled as a random variable $M_i$ with density $f_{M_i}(\cdot)$.

(iii) The distance between $P$ and the epicenter of the earthquakes of zone $i$ is modelled as a random variable $D_i$ with density $f_{D_i}(\cdot)$.
(iv) A ground motion prediction model is chosen expressed as a regression of the ground acceleration on magnitude, distance, and possibly other factors. We now detail these steps and explain how to combine them to achieve the main task of PSHA: compute the probability of event (2.1) for any $A^*$. The ability to compute this probability for any $A^*$ makes possible the estimation, by dichotomy, of an acceleration $A^*$ satisfying $\mathbb{P}(E(A^*, P)) = \varepsilon$.

From (i), we obtain that the distribution of the number of earthquakes $N_{ti}$ in zone $i$ on a time window of $t$ time units is given by

$$
\mathbb{P}(N_{ti} = k) = e^{-\lambda_i t} \left(\frac{(\lambda_i t)^k}{k!}\right), \quad k \in \mathbb{N},
$$

where the rate $\lambda_i$ represents the mean number of earthquakes in zone $i$ per time unit, say per year. From now on, we fix an acceleration $A^*$ and introduce the event

$$
E(A^*, P, i) = \{ \text{An earthquake from zone } i \text{ causes a PGA greater than } A^* \text{ at } P \}
$$

with its probability $p_i = \mathbb{P}(E(A^*, P, i))$. For each earthquake in zone $i$, either event $E(A^*, P, i)$ occurs for this earthquake, i.e., this earthquake causes a PGA greater than $A^*$ at $P$, or not. As a result, we can define two new counting processes for zone $i$: the process $\tilde{N}_{ti}$ counting the earthquakes causing $PGA > A^*$ at $P$ (events represented by black balls in Figure 3) and the process counting the earthquakes causing $PGA \leq A^*$ at $P$. To proceed, we need the following well-known lemma:

**Lemma 2.1.** Consider a Poisson process $N_t$ with arrival rate $\lambda$. Assume that arrivals are of two types I and II: type I with probability $p$ and type II with probability $1 - p$. We also assume that the arrival types are independent. Then the process $\tilde{N}_t$ of type I arrivals is a Poisson process with rate $\lambda p$. 

![Figure 2. Seismic zones around a given point P.](image-url)
Poisson process for the earthquakes of zone $i$ causing $PGA \leq A^*$ at $P$, rate $\lambda_i(1 - p_i)$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Splitting of the process of earthquake arrivals in zone $i$ into a process of earthquakes causing $PGA > A^*$ at $P$ (arrivals represented by black balls) and a process of earthquakes causing $PGA \leq A^*$ at $P$.}
\end{figure}

Proof. We compute for every $k \in \mathbb{N}$,
\[ \mathbb{P}(\tilde{N}_t = k) = \sum_{j=k}^{+\infty} \mathbb{P}(\tilde{N}_t = k | N_t = j) \mathbb{P}(N_t = j) \] [Total Probability Theorem]
\[ = \sum_{j=k}^{+\infty} C_j^k p^k (1 - p)^{j-k} e^{-\lambda_t (\lambda t)^j} \frac{j!}{j!} \]
\[ = e^{-\lambda_t (\lambda pt)^k} \sum_{j=0}^{+\infty} \frac{[\lambda (1 - p) t]^j}{j!} = e^{-\lambda pt} \frac{(\lambda pt)^k}{k!}, \]
which shows that $\tilde{N}_t$ is a Poisson random variable with parameter $\lambda pt$. We conclude using the independence of the arrival types on disjoint time windows. □

This lemma shows that the process $(\tilde{N}_{ti})_t$ is a Poisson process with rate $\lambda_i p_i$. Denoting by $N$ the number of zones, it follows that the probability to have $k$ earthquakes causing a PGA greater than $A^*$ at $P$ over the next time window of $t$ years is
\[ \mathbb{P}\left( \sum_{i=1}^{N} \tilde{N}_{ti} = k \right) = \sum_{x_1 + \ldots + x_N = k} \mathbb{P}\left( \tilde{N}_{t1} = x_1; \ldots; \tilde{N}_{tN} = x_N \right) \]
\[ = \sum_{x_1 + \ldots + x_N = k} \prod_{i=1}^{N} \mathbb{P}(\tilde{N}_{ti} = x_i) \]
\[ = \sum_{x_1 + \ldots + x_N = k} \prod_{i=1}^{N} e^{-\lambda_i p_i t} \frac{(\lambda_i p_i t)^{x_i}}{x_i!}, \]
where for the second equality we have used the independence of $\tilde{N}_{t1}, \ldots, \tilde{N}_{tN}$. Taking $k = 0$ in the above relation, we obtain
\[ 1 - \mathbb{P}(E_t(A^*, P)) = \mathbb{P}(\overline{E_t(A^*, P)}) = e^{-(\sum_{i=1}^{N} \lambda_i p_i)t}. \]
Setting $\tilde{N}_t = \sum_{i=1}^{N} \tilde{N}_{ti}$, the expectation of $\tilde{N}_t$ which is the mean number of earthquakes causing a PGA greater than $A^*$ at $P$ over the next $t$ years, can be expressed
as
\[
\lambda_i(A^*, P) = \mathbb{E} \left[ (\ln N_i) \right] = \sum_{i=1}^{N_\Lambda} \mathbb{E} \left[ (\ln t_i) \right] = \left( \sum_{i=1}^{N_\Lambda} \lambda_i p_i \right)t.
\]

Using this relation and \(2.3\), the probability of event \(E_t(A^*, P)\) can be rewritten
\[
P(E_t(A^*, P)) = 1 - e^{-\lambda_i(A^*, P)}
\]
with \(\lambda_i(A^*, P)\) given by \(2.4\).

It remains to explain how the probability \(p_i\) of event \(2.2\) is computed. This computation is based on a ground motion prediction model (step (iv) above) which is a regression equation representing the Peak Ground Acceleration induced by an earthquake of magnitude \(M\) at distance \(D\) of its epicenter. This relation takes the form
\[
\ln PGA = \ln PGA(M, D, \theta) + \sigma(M, D, \theta)\varepsilon.
\]
In this relation, \(\ln PGA(M, D, \theta)\) (resp. \(\sigma(M, D, \theta)\)) is the conditional mean (resp. standard deviation) of \(\ln PGA\) given the magnitude \(M\) and distance \(D\) to the epicenter while \(\varepsilon\) is a standard Gaussian random variable. We see that the Peak Ground Acceleration depends on the magnitude, the distance to the epicenter and other parameters, generally referred to as \(\theta\) (such as the ground conditions). More precisely, the mean \(\ln PGA(M, D, \theta)\) should increase with \(M\) (the higher the magnitude, the higher the PGA) and decrease with \(D\) (the larger the distance, the lower the PGA). As an example, the ground motion prediction model in \[3\] is of the form
\[
\ln PGA = 0.152 + 0.859M - 1.803\ln(D + 25) + 0.57\varepsilon.
\]
which amounts to take \(\ln PGA(M, D, \theta) = 0.152 + 0.859M - 1.803\ln(D + 25)\) and \(\sigma(M, D, \theta) = 0.57\).

The density \(f_{M_i}(\cdot)\) used for the distribution of the magnitude of the earthquakes of zone \(i\) depends on the history of the magnitudes of the earthquakes of that zone. For a large number of seismic zones, the density proposed by Gutenberg and Richter \[5\] has shown appropriate. It is of the form
\[
f_{M_i}(m) = \beta_i e^{-\beta_i(m-M_{\text{min}}(i))} \left( 1 - e^{-\beta_i(M_{\text{max}}(i) - M_{\text{min}}(i))} \right)
\]
for some parameter \(\beta_i > 0\) where the support of \(M_i\) is \([M_{\text{min}}(i), M_{\text{max}}(i)]\).

In each zone, the epicenter has a uniform distribution in that zone. The seismic zones usually considered in PSHA are disks, balls, line segments, or the boundary of a polyhedron. As a result, the determination of the density \(f_{D_i}(\cdot)\) of the distance \(D_i\) between \(P\) and the epicenter in zone \(i\) can be determined analytically or approximately using Sections 3, 4, 5.1, and 5.2.

Gathering the previous ingredients, assuming that \(D_i\) and \(M_i\) are independent, and using the Total Probability Theorem, we obtain
\[
p_i = \int_{m_i=M_{\text{min}}(i)}^{M_{\text{max}}(i)} \int_{x_i=0}^{\infty} \mathbb{P} \left( PGA > A^* | M_i = m_i; D_i = x_i \right) f_{M_i}(m_i)f_{D_i}(x_i) dm_i dx_i
\]
where \(\mathbb{P} \left( PGA > A^* | M_i = m_i; D_i = x_i \right)\) is given by the ground motion prediction model \(2.5\). For implementation purposes, the above integral is generally estimated discretizing the continuous distributions of magnitude \(M_i, i = 1, \ldots, N\), and distance \(D_i, i = 1, \ldots, N\).
Finally, we mention the existence of an alternative, zoneless approach to PSHA introduced by [4] and [10].

3. DISTANCE TO A RANDOM VARIABLE UNIFORMLY DISTRIBUTED IN A DISK

Let $S = \mathcal{D}(S_0, R_0)$ be a disk of center $S_0$ and radius $R_0 > 0$ and let $P$ be a point in the plane containing $S$ at Euclidean distance $R_1$ of $S_0$. We first consider the case where $R_1 = 0$. If $0 \leq d \leq R_0$, we get $F_D(d) = \frac{\pi d^2}{2R_0^2} = (d/R_0)^2$ and $f_D(d) = 2\frac{\pi d}{R_0^2}$ if $d > R_0$ we have $F_D(d) = 1$ and $f_D(d) = 0$ while if $d < 0$ we have $F_D(d) = f_D(d) = 0$. Let us now consider the case where $R_1 \geq R_0$. If $d > R_1 + R_0$ we have $F_D(d) = 1$ and $f_D(d) = 0$ while if $d < R_1 - R_0$ we have $F_D(d) = f_D(d) = 0$. Let us now take $R_1 - R_0 \leq d \leq R_1 + R_0$. The intersection of the disks $\mathcal{D}(S, R_0)$ and $\mathcal{D}(P, d)$ is the union of two lenses having a line segment $AB$ in common (see Figures 4 and 5). Without loss of generality, assume that $(S_0 P)$ is the $x$-axis and that the equations of the boundaries of the disks are given by $x^2 + y^2 = R_0^2$ and $(x - R_1)^2 + y^2 = d^2$. From these equations, we obtain that the abscissa of the intersection points $A$ and $B$ of the boundaries of the disks is $x^* = \frac{R_0^2 + R_1^2 - d^2}{2R_1}$.

Note that $A = B$ if and only if $d = R_1 \pm R_0$. In Figure 4, we represented a situation where $x^* \geq 0$ and a situation where $x^* < 0$. In both cases, $\mathcal{D}(S_0, R_0) \cap \mathcal{D}(P, d)$ is the union of a lens of height $h_1(d)$ in a disk of radius $d$ (the disk $\mathcal{D}(P, d)$) and of a lens of height $h_2(d)$ in a disk of radius $R_0$ (the disk $\mathcal{D}(S_0, R_0)$) where

\[
\begin{align*}
    h_1(d) &= d - R_1 + x^* = d - R_1 + \frac{R_0^2 + R_1^2 - d^2}{2R_1} \\
    h_2(d) &= R_0 - x^* = R_0 - \frac{R_0^2 + R_1^2 - d^2}{2R_1}.
\end{align*}
\]

Recall that the area $A(R, h)$ of a lens of height $h$ contained in a disk of radius $R$ (see Figure 4) is $A(R, h) = R^2 \frac{A}{2} - R^2 \sin(\frac{A}{2}) \cos(\frac{A}{2})$ with $\cos(\frac{A}{2}) = \frac{R-h}{R}$, i.e.,

\[
A(R, h) = R^2 \arccos \left( \frac{R-h}{R} \right) - (R-h)\sqrt{R^2 - (R-h)^2}.
\]

In the sequel, we will denote by $A(S)$ the area of a surface $S$. With this notation, it follows that

\[
A(\mathcal{D}(S_0, R_0) \cap \mathcal{D}(P, d)) = A(h_1(d)) + A(h_2(d)).
\]
Figure 5. Random variable $X$ uniformly distributed in a ball of radius $R_0$ and center $S_0$. Case where $R_1 \geq R_0 > 0$.

where

\begin{equation}
A(d, h_1(d)) = d^2 \arccos \left( \frac{d^2 + R_1^2 - R_0^2}{2 R_1 d} \right) - \frac{d^2 + R_1^2 - R_0^2}{2 R_1} \sqrt{d^2 - \left( \frac{d^2 + R_1^2 - R_0^2}{2 R_1} \right)^2} \tag{3.9}
\end{equation}

and

\begin{equation}
A(R_0, h_2(d)) = R_0^2 \arccos \left( \frac{R_0^2 + R_1^2 - d^2}{2 R_0 R_1} \right) - \frac{R_0^2 + R_1^2 - d^2}{2 R_1} \sqrt{R_0^2 - \left( \frac{R_0^2 + R_1^2 - d^2}{2 R_1} \right)^2}. \tag{3.10}
\end{equation}

For $R_1 - R_0 \leq d \leq R_1 + R_0$, we obtain $F_D(d) = \frac{\hat{A}(d, h_1(d)) + \hat{A}(R_0, h_2(d))}{\pi R_0^2}$ where $\hat{A}(d, h_1(d))$ and $\hat{A}(R_0, h_2(d))$ are given by (3.9) and (3.10). The density is

\begin{equation}
f_D(d) = \frac{1}{\pi R_0^2} \left[ h_2'(d) \frac{\partial \hat{A}(R_0, h_2(d))}{\partial h} + \frac{\partial \hat{A}(d, h_1(d))}{\partial R} + h_1'(d) \frac{\partial \hat{A}(d, h_1(d))}{\partial h} \right] \tag{3.11}
\end{equation}

where $h_1'(d) = -\frac{d}{R_1}$, $h_2'(d) = \frac{d}{R_1}$, and

\begin{align}
\frac{\partial \hat{A}(R, h)}{\partial R} &= 2R \arccos \left( 1 - \frac{h}{R} \right) - 2\sqrt{h(2R-h)}, \tag{3.12} \\
\frac{\partial \hat{A}(R, h)}{\partial h} &= 2 \sqrt{h(2R-h)}.
\end{align}

We now consider the case where $R_1 > 0$ and $R_1 < R_0$ (see Figure 6). If $0 \leq d \leq R_1$,
be two points of the boundary of the disk such that \( \vec{P} \) independent. We introduce the projection matrix \( \begin{bmatrix} A \end{bmatrix} \) then the matrix \( A \) onto \( \vec{P} \) in the plane \( \vec{P} \) it is a disk of center \( \vec{P} \). Since vectors \( \vec{D} \) are linearly independent, if \( \vec{0} \parallel \vec{A} \) and \( \vec{0} \parallel \vec{A} \) both in the case where the abscissa \( x^* \) of the intersection points between the boundaries of \( \mathcal{D}(S_0, R_0) \) and \( \mathcal{D}(P, d) \) is positive and negative, we check (see Figure 8) that the area of \( \mathcal{D}(S_0, R_0) \cap \mathcal{D}(P, d) \) is still given by (3.8) with \( h(d, h_1(d)) \) and \( h(R_0, h_2(d)) \) given respectively by (3.9) and (3.10). Summarizing, if \( 0 < R_1 < R_0 \) then if \( R_1 + R_0 \geq d > R_0 - R_1 \), the density of \( D \) at \( d \) is given by (3.11) and if \( 0 \leq d \leq R_0 - R_1 \), we have \( f_D(d) = \frac{d}{R_0^2} \). The density of \( D \) when \( X \) is uniformly distributed in a disk is given for some examples in Figure 7.

Finally, we consider the case where \( S \) is a disk \( \mathcal{D} \) and \( P \in \mathbb{R}^3 \) is not contained in the plane \( \mathcal{P} \) containing this disk. Let \( S_0 \) be the center of \( S \) and let \( S_1, S_2 \) be two points of the boundary of the disk such that \( \vec{S_0S_1} \) and \( \vec{S_0S_2} \) are linearly independent. We introduce the projection \( P_0 = \pi_{\mathcal{D}}[P] = \arg\min_{Q \in \mathcal{P}} \| \vec{Q} \| \) of \( P \) onto \( \mathcal{P} \). Since vectors \( \vec{S_0S_1} \) and \( \vec{S_0S_2} \) are linearly independent, if \( A \) is the (3, 2) matrix \( [\vec{S_0S_1}, \vec{S_0S_2}] \) whose first column is \( \vec{S_0S_1} \) and whose second column is \( \vec{S_0S_2} \), then the matrix \( A^\top A \) is invertible. It follows that the projection \( P_0 = \pi_{\mathcal{D}}[P] \) of \( P \) onto \( \mathcal{P} \) can be expressed as \( \vec{S_0P_0} = A^\top A)^{-1}A^\top \vec{S_0P} \). With this notation, the intersection of \( \mathcal{P} \) and the ball \( B(P, d) \) of center \( P \) and radius \( d \) is either empty or it is a disk of center \( P_0 \) and radius

(3.13) \[ R(d) = \sqrt{d^2 - \| \vec{P} \|^2} \]

(see Figure 8). In the latter case, denoting this disk by \( \mathcal{D}(P_0, R(d)) \) and using the fact that \( \mathcal{D} = \mathcal{D} \cap \mathcal{P} \) (recall that \( \mathcal{D} \subset \mathcal{P} \)), we obtain

\[ \mathcal{D} \cap B(P, d) = \mathcal{D} \cap \mathcal{P} \cap B(P, d) = \mathcal{D} \cap \mathcal{D}(P_0, R(d)). \]

Since \( \mathcal{D} \) and \( \mathcal{D}(P_0, R(d)) \) are disks contained in the plane \( \mathcal{P} \), setting \( R_0 = \| \vec{S_0S_1} \| \) and \( R_1 = \| \vec{S_0P_0} \| \), the previous results provide the area of their intersection and the following CDFs and densities for \( D \):

**Figure 6.** Random variable \( X \) uniformly distributed in a ball of radius \( R_0 \) and center \( S_0 \). Case where \( 0 < R_1 < R_0 \).

\[
\begin{align*}
R_1 + d &\leq R_0 & R_1 + d &> R_0 \text{ and } x^* \geq 0 & R_1 + d &> R_0 \text{ and } x^* < 0
\end{align*}
\]
Figure 7. Density of $D$ where $X$ is uniformly distributed in a disk of radius $R_0 = 1$: some examples. Top left: $R_1 = 0$, top right: $R_1 = 0.5$, bottom left: $R_1 = 0.75$, bottom right: $R_1 = 6$.

Case where $P_0 = S_0$: The CDF and density of $D$ are given by

$$
\begin{align*}
F_D(d) &= f_D(d) = 0 & \text{if } d < \|S_0\|_2, \\
F_D(d) &= \frac{R(d)^2}{\|S_0S_1\|_2^2} = \frac{d^2 - \|\overrightarrow{S_0P}\|_2^2}{\|S_0S_1\|_2^2} & \text{if } \|\overrightarrow{S_0P}\|_2 \leq d \leq \sqrt{\|\overrightarrow{S_0P}\|_2^2 + \|S_0S_1\|_2^2}, \\
f_D(d) &= 1 & \text{if } d > \sqrt{\|\overrightarrow{S_0P}\|_2^2 + \|S_0S_1\|_2^2},
\end{align*}
$$
Case where $0 < \|S_0\vec{P}_0\|_2 < \|\vec{S}_0\vec{S}_1\|_2$: Setting

\[
d_{\text{min}} = \sqrt{\|\vec{P}_0\|_2^2 + (\|\vec{S}_0\vec{S}_1\|_2 - \|\vec{S}_0\vec{P}_0\|_2)^2} \quad \text{and}
\]
\[
d_{\text{max}} = \sqrt{\|\vec{P}_0\|_2^2 + (\|\vec{S}_0\vec{S}_1\|_2 + \|\vec{S}_0\vec{P}_0\|_2)^2},
\]
the CDF of $D$ is given by

\[
F_D(d) = \begin{cases} 
\frac{d^2 - \|\vec{P}_0\|_2^2}{\|\vec{S}_0\vec{S}_1\|_2^2} & \text{if } d < \|\vec{P}_0\|_2, \\
\frac{\|\vec{S}_0\vec{P}_0\|_2^2 + \|\vec{P}_0\|_2^2 - d^2}{2\|\vec{S}_0\vec{P}_0\|_2} & \text{if } \|\vec{P}_0\|_2 \leq d \leq d_{\text{min}}, \\
\frac{\|\vec{S}_0\vec{S}_1\|_2^2 + \|\vec{S}_0\vec{P}_0\|_2^2 + \|\vec{P}_0\|_2^2 - d^2}{2\|\vec{S}_0\vec{P}_0\|_2} & \text{if } d_{\text{min}} \leq d \leq d_{\text{max}}, \\
1 & \text{if } d > d_{\text{max}},
\end{cases}
\]

where the expression of $A$ is given by (3.7) and, where, using the expressions of $h_1$ and $h_2$ and recalling that $R_0 = \|\vec{S}_0\vec{S}_1\|_2$ and $R_1 = \|\vec{S}_0\vec{P}_0\|_2$,

\[
h_1(R(d)) = \sqrt{d^2 - \|\vec{P}_0\|_2^2 - \|\vec{S}_0\vec{P}_0\|_2^2} + \frac{\|\vec{S}_0\vec{P}_0\|_2^2 + \|\vec{P}_0\|_2^2 - d^2}{2\|\vec{S}_0\vec{P}_0\|_2},
\]
\[
h_2(R(d)) = \frac{\|\vec{S}_0\vec{P}_0\|_2^2 + \|\vec{P}_0\|_2^2 - d^2}{2\|\vec{S}_0\vec{P}_0\|_2}.
\]

It follows that $F_D(d) = 0$ if $d < \|\vec{P}_0\|_2$ or $d > d_{\text{max}}$ while $F_D(d) = \frac{2d}{\|\vec{S}_0\vec{S}_1\|_2}$ if $\|\vec{P}_0\|_2 \leq d \leq d_{\text{min}}$. Finally, if $d_{\text{min}} \leq d \leq d_{\text{max}}$, we have

\[
f_D(d) = \frac{1}{\pi\|\vec{S}_0\vec{S}_1\|_2^2} \left[ \frac{d}{\|\vec{S}_0\vec{P}_0\|_2} \frac{\partial h(R(d),h_1(R(d)))}{\partial h} + \frac{d}{\sqrt{d^2 - \|\vec{P}_0\|_2^2}} \frac{\partial h(R(d),h_2(R(d)))}{\partial h} \right]
\]

where the expressions of $\frac{\partial h(R,h)}{\partial h}$ and $\frac{\partial h(R,h)}{\partial \vec{P}_0}$ are given by (3.12).

Case where $\|\vec{S}_0\vec{P}_0\|_2 \geq \|\vec{S}_0\vec{S}_1\|_2$: With the definitions (3.14) of $d_{\text{min}}$ and $d_{\text{max}}$, if $d < d_{\text{min}}$ then $F_D(d) = f_D(d) = 0$, if $d > d_{\text{max}}$ then $F_D(d) = 1$ and $f_D(d) = 0$, while if $d_{\text{min}} \leq d \leq d_{\text{max}}$, $f_D(d)$ is given by (3.17) and $F_D(d)$ is given by (3.15)-(c) with $h_1(R(d))$ and $h_2(R(d))$ given by (3.16).

4. Distance to a Random Variable Uniformly Distributed in a Ball

Let $S = B(S_0, R_0)$ be a ball of radius $R_0 > 0$ and center $S_0$ in $\mathbb{R}^3$ and let $P$ be at Euclidean distance $R_1$ of $S_0$. The computations are identical to those of the previous section replacing two dimensional lenses and disks by three dimensional caps and balls. If $R_1 = 0$ then if $d > R_0$, we have $f_D(d) = 0$ and $F_D(d) = 1$, if $d < 0$, we have $f_D(d) = F_D(d) = 0$ while if $0 \leq d \leq R_0$, we obtain $F_D(d) = \frac{(d/|S_0\vec{S}_1|)^3}{(4/3)\pi R_0^3}$, i.e., $f_D(d) = 3\frac{d^2}{R_0^3}$ (see Figure 0). If $0 < R_1 < R_0$, then if $d > R_0 + R_1$, we have $F_D(d) = 1$ and $f_D(d) = 0$, if $d < 0$, we have $F_D(d) = f_D(d) = 0$ while if $0 \leq d \leq R_0 - R_1$, we have $F_D(d) = \frac{(d/|S_0\vec{S}_1|)^3}{(4/3)\pi R_0^3}$, i.e., $f_D(d) = 3\frac{d^2}{R_0^3}$ (see Figure 0). If $R_1 \geq R_0$ then if $d > R_0 + R_1$, we have $f_D(d) = 0$ and $F_D(d) = 1$ and if $d < R_1 - R_0$, we have $f_D(d) = F_D(d) = 0$. If $0 < R_1 < R_0$ and $R_0 - R_1 < d \leq R_0 + R_1$ or if $R_1 \geq R_0$ and $R_1 - R_0 \leq d \leq R_1 + R_0$, then $B(S_0, R_0) \cap B(P,d)$ is the union of a spherical cap.
of height $h_1(d)$ contained in a ball of radius $d$ (the ball $B(P, d)$) and of a spherical cap of height $h_2(d)$ contained in a ball of radius $R_0$ (the ball $B(S_0, R_0)$) where the expressions (3.6) for $h_1(d)$ and $h_2(d)$ are still valid. Now recall that the volume of a spherical cap (see Figure 4 for a cut of this cap) of height $h$ contained in a ball of radius $R$ in $\mathbb{R}^3$ is

\begin{equation}
V(R, h) = \int_{x=R-h}^{R} \pi r^2(x)dx = \int_{x=R-h}^{R} \pi[R^2 - x^2]dx = \frac{\pi h^2}{3}(3R - h).
\end{equation}

It follows that if $0 < R_1 < R_0$ and $R_0 - R_1 < d \leq R_0 + R_1$ or if $R_1 \geq R_0$ and $R_1 - R_0 \leq d \leq R_1 + R_0$, we have

\begin{align*}
F_D(d) &= \frac{3}{4\pi R_0^3} \left[ V(d, h_1(d)) + V(R_0, h_2(d)) \right] \\
&= \frac{1}{4\pi R_0^3} \left[ h_1^2(d)(3d - h_1(d)) + h_2^2(d)(3R_0 - h_2(d)) \right]
\end{align*}

where we recall that $h_1(d)$ and $h_2(d)$ are given by (3.6) and the density is

\begin{align*}
f_D(d) &= \frac{3}{4\pi R_0^3} \left[ \frac{\partial V}{\partial R}(d, h_1(d)) + h_1'(d)\frac{\partial V}{\partial h}(d, h_1(d)) + h_2'(d)\frac{\partial V}{\partial h}(R_0, h_2(d)) \right]
\end{align*}

where

\begin{align*}
\frac{\partial V}{\partial R}(R, h) &= \pi h^2 \\
\frac{\partial V}{\partial h}(R, h) &= \pi h(2R - h).
\end{align*}

The density of $D$ when $X$ is uniformly distributed in a ball is given for some examples in Figure 9.

5. Distance to a random variable uniformly distributed in a polyhedron

5.1. Distance to a random variable uniformly distributed on a line segment. Let $S = \overrightarrow{AB}$ be a line segment in $\mathbb{R}^3$ with $A \neq B$ and let $P \in \mathbb{R}^3$. We introduce the projection $P_0$ of $P$ onto line $(AB)$:

\[ P_0 = A + \frac{\langle \overrightarrow{AB}, \overrightarrow{P} \rangle}{\|\overrightarrow{AB}\|^2} \overrightarrow{AB}. \]

This projection $P_0$ belongs to line segment $\overrightarrow{AB}$ if and only if $\langle P_0 \overrightarrow{A}, \overrightarrow{P} \overrightarrow{B} \rangle \leq 0$ (see Figure 11). In this case, setting $d_{\min} = \min(\|P \overrightarrow{A}\|_2, \|P \overrightarrow{B}\|_2)$ and $d_{\max} = \max(\|P \overrightarrow{A}\|_2, \|P \overrightarrow{B}\|_2)$, we obtain the following CDF for $D$ (see Figure 11):

\begin{align*}
F_D(d) &= \begin{cases} 
0 & \text{if } d < \|P \overrightarrow{P_0}\|_2, \\
\frac{2d}{\|\overrightarrow{AB}\|^2} - \frac{\sqrt{d^2 - \|P \overrightarrow{P_0}\|_2^2}}{\|\overrightarrow{AB}\|^2} & \text{if } \|P \overrightarrow{P_0}\|_2 \leq d \leq d_{\min}, \\
\min(\|P_0 \overrightarrow{A}\|_2, \|P_0 \overrightarrow{B}\|_2) + R(d) & \text{if } d_{\min} \leq d \leq d_{\max}, \\
\min(\|P_0 \overrightarrow{A}\|_2, \|P_0 \overrightarrow{B}\|_2) + \frac{\sqrt{d^2 - \|P \overrightarrow{P_0}\|_2^2}}{\|\overrightarrow{AB}\|^2} & \text{if } d_{\max} \leq d \leq d_{\max}, \\
1 & \text{if } d > d_{\max}.
\end{cases}
\end{align*}
If $P_0$ does not belong to $AB$, i.e., if $\langle \overrightarrow{P_0A}, \overrightarrow{P_0B} \rangle > 0$, we obtain the following CDF for $D$ (see Figure 10):

\[
F_D(d) = \begin{cases} 
0 & \text{if } d < d_{\text{min}}, \\
\frac{R(d) - R(d_{\text{min}})}{\|AB\|_2} & \text{if } d_{\text{min}} \leq d \leq d_{\text{max}}, \\
1 & \text{if } d > d_{\text{max}}.
\end{cases}
\]
An analytic expression of the density can be obtained deriving the above CDF. The density of $D$ when $X$ is uniformly distributed in a line segment is given for two examples in Figure 11.

5.2. **General non-self-intersecting polyhedron.** Let $S$ be a non-self-intersecting polyhedron contained in a plane given by its extremal points $\{S_1, S_2, \ldots, S_n\}$ where the boundary of $S$ is $\cup_{i=1}^{n} S_i S_{i+1}$ with the convention that $S_{n+1} = S_1$ and where $S_i \neq S_j$ for $i \neq j$. We assume that when travelling on the boundary of $S$ from $S_1$ to $S_2$, then from $S_2$ to $S_3$ and so on until the last line segment $S_n S_1$, one always has the interior of $S$ to the left (see Figure 12). Let $P$ be a point in the plane $\mathcal{P}$ containing $S$. The value $F_D(d)$ of the CDF of $D$ evaluated at $d$ is the area of the intersection of $S$ and the disk $\mathcal{D}(P, d)$ of center $P$ and radius $d$ divided by the area of $S$. These areas will be computed making use of a special case of Green’s theorem: if $\mathcal{D}$ is a closed and bounded region in the plane then the area $\mathcal{A}(\mathcal{D})$ of $\mathcal{D}$
can be expressed as a line integral over the boundary \( \partial D \) of \( D \):

\[
A(\mathcal{D}) = \frac{1}{2} \oint_{\partial D} [x \, dy - y \, dx].
\]

Since the boundary of \( S \) is a union of line segments and the boundary of \( S \cap D(P, d) \) is made of line segments and arcs, we need to compute \( \int_{\mathcal{C}} [x \, dy - y \, dx] \) with \( C \) a line segment or an arc. If \( C = AB \) is a line segment, denoting respectively the coordinates of \( A \) and \( B \) by \((x_A, y_A)\) and \((x_B, y_B)\), we obtain

\[
I_{AB} := \int_{AB} [x \, dy - y \, dx] = y_B x_A - y_A x_B.
\]

Now let \( C = \hat{AB}_{R_0, P} \) be an arc starting at \( A = (x_A, y_A) \) and ending at \( B = (x_B, y_B) \) with \( A \) and \( B \) belonging to the circle of center \( P = (x_P, y_P) \) and radius \( R_0 > 0 \). We assume that when travelling along the arc from \( A \) to \( B \), the interior of the disk is to the left. If \( \theta(A, B) \) is the angle \( \angle APB \), using (5.21) we obtain

\[
\frac{R_0^2 \theta(A, B)}{2} = \frac{1}{2} (I_{\hat{AB}_{R_0, P}} + I_{PA} + I_{PB})
\]

where \( I_{\hat{AB}_{R_0, P}} := \int_{\hat{AB}_{R_0, P}} [x \, dy - y \, dx] \). Using (5.22), the above relation can be written

\[
I_{\hat{AB}_{R_0, P}} = R_0^2 \theta(A, B) + x_P(y_B - y_A) - y_P(x_B - x_A).
\]

We introduce the function \( \text{Angle} \) defined on the boundary of \( \mathcal{D}(P, d) \) taking values in \([0, 2\pi]\) and given by

\[
\text{Angle}(x, y) = \text{Arccos} \left( \frac{x - x_P}{R_0} \right) \text{ if } y \geq y_P \text{ and } \\
\text{Angle}(x, y) = 2\pi - \text{Arccos} \left( \frac{x - x_P}{R_0} \right) \text{ if } y < y_P.
\]

This function associates to a point of the boundary of \( \mathcal{D}(P, d) \) its angle. With this notation, for two points \( A = (x_A, y_A) \) and \( B = (x_B, y_B) \) of the boundary of \( \mathcal{D}(P, d) \), we have

\[
\theta(A, B) = \text{Angle}(x_B, y_B) - \text{Angle}(x_A, y_A) \text{ if } \text{Angle}(x_A, y_A) \leq \text{Angle}(x_B, y_B) \\
\theta(A, B) = 2\pi + \text{Angle}(x_B, y_B) - \text{Angle}(x_A, y_A) \text{ otherwise}
\]

and formula (5.25) can be written

\[
\begin{cases}
I_{\hat{AB}_{R_0, P}} = R_0^2 (\text{Angle}(x_B, y_B) - \text{Angle}(x_A, y_A)) + x_P(y_B - y_A) - y_P(x_B - x_A) \\
\quad \text{if } \text{Angle}(x_A, y_A) \leq \text{Angle}(x_B, y_B) \text{ and } \\
I_{\hat{AB}_{R_0, P}} = R_0^2 (2\pi + \text{Angle}(x_B, y_B) - \text{Angle}(x_A, y_A)) + x_P(y_B - y_A) - y_P(x_B - x_A) \\
\quad \text{if } \text{Angle}(x_A, y_A) > \text{Angle}(x_B, y_B).
\end{cases}
\]

To compute the area of the intersection \( S \cap \mathcal{D}(P, d) \), we need to determine the intersections between the boundary of \( S \) and the circle \( C(P, d) \) of center \( P \) and radius \( d \). This will be done using Algorithm 1 which computes the intersection between a given line segment \( \overline{AB} \) with \( A \neq B \) and the sphere of center \( P \) and radius \( d \) in \( \mathbb{R}^3 \). When this intersection is nonempty, let \( I_1(d) \) and \( I_2(d) \) be the intersection points (eventually \( I_1(d) = I_2(d) \)). Writing \( I_i(d) \) as

\[
I_i(d) = A + t_i \overline{AB},
\]

(5.26)
$t_i$ solves $\|\overrightarrow{PA} + t_i \overrightarrow{AB}\|^2 = d^2$. Introducing

$\Delta = \langle \overrightarrow{PA}, \overrightarrow{AB}\rangle^2 - \|\overrightarrow{AB}\|^2 (\|\overrightarrow{PA}\|^2 - d^2),$

if $\Delta < 0$ then the boundary of $S$ and $C(P, d)$ have an empty intersection while if $\Delta \geq 0$ the intersections $I_1(d)$ and $I_2(d)$ are given by (5.26) where

$$t_i = -\frac{\langle \overrightarrow{PA}, \overrightarrow{AB}\rangle \pm \sqrt{\Delta}}{\|\overrightarrow{AB}\|^2}.$$  

We are now in a position to write Algorithm 1, observing that $I_i(d) \in (AB)$ belongs to line segment $\overrightarrow{AB}$ if and only if $\langle \overrightarrow{IA}, \overrightarrow{IB}\rangle \leq 0$.

Algorithm 1: Computation of the intersection points between line segment $\overrightarrow{AB}$ with $A \neq B$ and the sphere of center $P$ and radius $d$ in $\mathbb{R}^3$.

**Inputs:** $A, B, P, d$.

**Initialization:** \(N=0; \) //Will store the number of intersections (0, 1, or 2).
\(\text{List}_\text{Intersections}=\text{Null}; \) //Will store the intersection points.

//Check if line $(AB)$ and the sphere have an empty intersection or not
Compute $\Delta = \langle \overrightarrow{PA}, \overrightarrow{AB}\rangle^2 - \|\overrightarrow{AB}\|^2 (\|\overrightarrow{PA}\|^2 - d^2)$.
If $\Delta \geq 0$ then //if $\Delta < 0$ the intersection is empty.
  If $\Delta = 0$ then //the intersection of $(AB)$ and the sphere is a singleton \{I\}
    Compute $I = A + t\overrightarrow{AB}$ where $t = -\langle \overrightarrow{PA}, \overrightarrow{AB}\rangle / \|\overrightarrow{AB}\|^2$ (see (5.26), (5.28)) and
    check if $I$ belongs to $\overrightarrow{AB}$:
    If $\langle \overrightarrow{IA}, \overrightarrow{IB}\rangle \leq 0$, then //I belongs to $\overrightarrow{AB}$
      $\text{List}_\text{Intersections}=\{I\}, N=1$.
  End If
Else
  Compute the intersections $I_1(d)$ and $I_2(d)$ of $(AB)$ and the sphere given by (5.26), (5.28).
  If $\langle I_1(d)A, I_1(d)B\rangle \leq 0$ then //$I_1(d)$ belongs to $\overrightarrow{AB}$
    If $\langle I_2(d)A, I_2(d)B\rangle \leq 0$ then //$I_2(d)$ belongs to $\overrightarrow{AB}$
      //$I_1(d)$ and $I_2(d)$ belong to $\overrightarrow{AB}$
      $\text{List}_\text{Intersections}=\{I_1(d), I_2(d)\}, N=2$.
    Else //Only $I_1(d)$ belongs to the intersection
      $\text{List}_\text{Intersections}=\{I_1(d)\}, N=1$.
  End If
Else
  If $\langle I_2(d)A, I_2(d)B\rangle \leq 0$ then //$I_2(d)$ belongs to $\overrightarrow{AB}$
    $\text{List}_\text{Intersections}=\{I_2(d)\}, N=1$.
End If
End If
End If
End If
Figure 13. Increase in the crossing number when the ray passes through an extremal point of the polyhedron or when an edge of the polyhedron is contained in the ray.

Outputs: N, List_Intersection.

Algorithm 4 which computes the CDF of $D$ will also make use of Algorithm 2 that (i) computes the minimal distance $d_{\text{min}}$ and maximal distance $d_{\text{max}}$ between $P$ and the boundary of $S$, (ii) computes the area of $S$, and (iii) determines if $P$ belongs to the interior of $S$ or not. The computation of the area of $S$ will be done using formula (5.21). To know if $P$ belongs to the interior of $S$ or not, we compute the crossing number (stored in variable Crossing Number of Algorithm 2) for point $P$ and polyhedron $S$. Let $R$ be the ray starting at $P$ and parallel to the positive $x$-axis. The crossing number counts the number of times ray $R$ crosses the boundary of $S$ going either from the inside to the outside of $S$ or from the outside to the inside of $S$. If the crossing number is odd then $P$ belongs to the interior of $S$. Otherwise, the crossing number is even and $P$ is on the boundary of $S$ or outside $S$.

Though the computation of the crossing number (the value of variable Crossing Number in the end of Algorithm 2) is known (see for instance [8]), we recall it here for the sake of self-completeness. For each edge $S_iS_{i+1}$ of the polyhedron, we consider its intersection with $R$. Each time a single intersection point is found that belongs to the interior of an edge, Crossing Number increases by one. If the intersection between the edge and the ray is nonempty but is not a single point from the interior of the edge, then either this intersection is an extremal point or it is the whole edge. There are 8 possibilities, denoted by A-H in Figure 13. This figure also provides the increase in the crossing number in each case. To deal with these cases, the following (known) rules are used in Algorithm 2: (a) horizontal edges (edges $S_iS_{i+1}$ with $y_{S_i} = y_{S_{i+1}}$) are not considered, (b) for upward edges (edges $S_iS_{i+1}$ with $y_{S_i} < y_{S_{i+1}}$), only the final vertex is counted as an intersection, and (c) for downward edges (edges $S_iS_{i+1}$ with $y_{S_i} > y_{S_{i+1}}$), only the starting
vertex is counted as an intersection. The increase in the crossing number using these rules is reported for cases A–H in Figure 13. Comparing with the expected increase in the crossing number in each case, we see that the variable $\text{Crossing Number}$ that is updated using these rules in Algorithm 2, will be even if and only if $P$ is on the boundary of the polyhedron or outside the polyhedron, as expected.

Algorithm 2: Given a polyhedron $S$ contained in a plane and a point $P$ in that plane, the algorithm computes the area of $S$, the crossing number, and the minimal and maximal distances from $P$ to the boundary of $S$.

Inputs: $P$ and the vertices $S_1, S_2, \ldots, S_n$ of a polyhedron contained in a plane.

Initialization: $L = 0$. //Will store line integral $I_{AB}$, $I_{AT}$ is given by (5.22).

$d_{\text{min}} = +\infty$. //Will store the crossing number.

$d_{\text{max}} = 0$. //Will store the minimal distance from $P$ to the boundary of $S$.

For $i = 1, \ldots, n$,

$L = L + \frac{1}{2} I_{S_i S_{i+1}}$ where for a line segment $AB$, $I_{AT}$ is given by (5.22).

//Computation of the crossing number

If $y_{S_i} < y_p \leq y_{S_{i+1}}$ or $y_{S_{i+1}} < y_p \leq y_{S_i}$ then

//Compute the abscissa $x_I$ of the intersection $I$ of the line $y = y_p$ and line segment $S_i S_{i+1}$:

$$x_I = x_{S_i} + \frac{x_{S_{i+1}} - x_{S_i}}{y_{S_{i+1}} - y_{S_i}} (y_p - y_{S_i}).$$

If $x_I > x_P$ then

$$\text{Crossing Number} = \text{Crossing Number} + 1$$

End If

End If

//Computation of the maximal distance from $P$ to the boundary of $S$

$d_{\text{max}} = \max(d_{\text{max}}, \|\overrightarrow{PS_i}\|_2)$

//Computation of the minimal distance from $P$ to the boundary of $S$

Compute the projection $P_0$ of $P$ onto line $(S_i, S_{i+1})$:

$$P_0 = S_i + \frac{(S_i \overrightarrow{P}, \overrightarrow{S_{i+1}})}{\|S_i S_{i+1}\|_2^2} S_i S_{i+1}.$$ 

If $\langle P_0 S_i, \overrightarrow{P_0 S_{i+1}} \rangle \leq 0$ then

//$P_0$ belongs to $AB$

$d_{\text{min}} = \min(d_{\text{min}}, \|\overrightarrow{PP_0}\|_2)$. Else

$$d_{\text{min}} = \min(d_{\text{min}}, \|\overrightarrow{PS_i}\|_2, \|\overrightarrow{PS_{i+1}}\|_2).$$

End If

\footnote{Alternatively, we can of course count only the starting vertices of upward edges and the final vertices of downward edges.}
End For

**Outputs:** Crossing Number, $\mathcal{L}, d_{\text{min}}, d_{\text{max}}$.

The outputs of Algorithm 2 allow us to know if $P$ belongs to $\mathcal{S}$ or not. Indeed, $P$ belongs to $\mathcal{S}$ if and only if $P$ belongs to the interior of $\mathcal{S}$, which occurs if and only if the crossing number is odd, or if $P$ is on the boundary of $\mathcal{S}$, which occurs if and only if $d_{\text{min}} = 0$. As a result, $P$ belongs to $\mathcal{S}$ if and only if Crossing Number is odd or $d_{\text{min}} = 0$.

**Remark 5.1.** The crossing number computed replacing the condition $x_I > x_P$ by $x_I \geq x_P$ in Algorithm 2 will not necessarily be odd if $P$ belongs to the boundary of $\mathcal{S}$. For instance, if $\mathcal{S}$ is the rectangle $\mathcal{S} = \{(x, y) : x_1 \leq x \leq x_2, y_1 \leq y \leq y_2\}$ then if the condition $x_I > x_P$ is replaced by $x_I \geq x_P$ in Algorithm 2, if we take $P = (x_1 + x_2)/2, (y_1 + y_2)/2$ this variable will be odd. However, both points belong to the boundary of $\mathcal{S}$.

Let us now comment on Algorithm 4 that computes the cumulative distribution function of $\mathcal{D}$ using Algorithms 1 and 2.

We first explain the different steps of Algorithm 4 when there is at least an edge of $\mathcal{S}$ that has a nonempty intersection with both the interior of $\mathcal{S}$ and the complement of $\mathcal{S}$. In other words, we exclude for the moment the cases $\mathcal{D}(P, d) \subset \mathcal{S}$, $\mathcal{S} \subset \mathcal{D}(P, d)$, and $\mathcal{D}(P, d) \cap \mathcal{S} = \emptyset$.

In this case, at the end of Algorithm 4, $\ell$ stores line integral \(5.21\) with $\mathcal{D} = \mathcal{S} \cap \mathcal{D}(P, d)$, i.e., the area of $\mathcal{S} \cap \mathcal{D}(P, d)$.

In the first **For** loop of Algorithm 4, starting from $\ell = 0$, we update $\ell$ travelling along the edges of $\mathcal{S}$ always leaving the interior of $\mathcal{S}$ to the left. In the end of this loop, $\ell$ is the sum of line integrals \(5.22\) computed for all the line segments belonging to the boundary of $\mathcal{S} \cap \mathcal{D}(P, d)$. More precisely, at iteration $i$ of this loop, we consider edge $S_iS_{i+1}$. For this edge, 6 cases can happen:

(i) $S_i$ belongs to $\mathcal{D}(P, d)$ and $S_{i+1}$ belongs to the interior of $\mathcal{D}(P, d)$. In this case, the whole segment $S_iS_{i+1}$ belongs to the boundary of $\mathcal{S} \cap \mathcal{D}(P, d)$ and

Figure 14. Cases where $S_{i+1}$ is not on the boundary of $\mathcal{D}(P, d)$. 
\[ \ell \leftarrow \ell + \frac{1}{2} T_{S_i S_{i+1}} \]  
This corresponds to subcases A1 (where \( S_i \) is on the boundary of \( D(P, d) \)) and A2 (where \( S_i \) belongs to the interior of \( D(P, d) \)) in Figure 14.

(ii) \( S_i \) belongs to \( D(P, d) \) and \( S_{i+1} \) does not belong to \( D(P, d) \). In this situation, either \( S_i \) belongs to the boundary of \( D(P, d) \) (subcases B1 and B3 in Figure 14) or \( S_i \) belongs to the interior of \( D(P, d) \) (subcase B2 in Figure 14). If \( S_i S_{i+1} \) and \( C(P, d) \) have an intersection point \( I \) that is different from \( S_i \) then \( S_i I_i \) belongs to the boundary of \( S \cap D(P, d) \) and \( \ell \leftarrow \ell + \frac{1}{2} T_{I_i S_i} \).

(iii) \( S_i \) belongs to \( D(P, d) \) and \( S_{i+1} \) is on the boundary of \( D(P, d) \). As in (i), the whole segment \( S_i S_{i+1} \) belongs to the boundary of \( S \cap D(P, d) \) and \( \ell \leftarrow \ell + \frac{1}{2} T_{S_i S_{i+1}} \).

(iv) \( S_i \) does not belong to \( D(P, d) \) and \( S_{i+1} \) belongs to the interior of \( D(P, d) \) (case C in Figure 14). In this case, \( S_i S_{i+1} \) and \( C(P, d) \) have a single intersection point \( I \); \( I_i S_{i+1} \) belongs to the boundary of \( S \cap D(P, d) \), and \( \ell \leftarrow \ell + \frac{1}{2} T_{I_i S_{i+1}} \).

(v) Both \( S_i \) and \( S_{i+1} \) are outside \( D(P, d) \). There are three subcases: \( S_i S_{i+1} \) and \( C(P, d) \) have two intersection points \( I_{1i} \) and \( I_{2i} \) (case D1 in Figure 14); \( S_i S_{i+1} \) and \( C(P, d) \) have a single intersection point (case D2 in Figure 14); or \( S_i S_{i+1} \) and \( C(P, d) \) have an empty intersection (case D3 in Figure 14). In case D2, \( I_{1i} I_{2i} \) belongs to the boundary of \( S \cap D(P, d) \) and \( \ell \leftarrow \ell + \frac{1}{2} T_{I_{1i} I_{2i}} \).

(vi) \( S_i \) does not belong to \( D(P, d) \) and \( S_{i+1} \) is on the boundary of \( D(P, d) \). If \( S_i S_{i+1} \) and \( C(P, d) \) have two intersection points \( I_i \) and \( S_{i+1} \) then \( I_i S_{i+1} \) belongs to the boundary of \( S \cap D(P, d) \) and \( \ell \leftarrow \ell + \frac{1}{2} T_{I_i S_{i+1}} \).

We also have to determine the arcs that belong to the boundary of \( S \cap D(P, d) \). A simple way to do this would be as follows:

(a) store all the intersections between the edges of the polyhedron and the boundary of \( D(P, d) \).

(b) Sort these intersection points \((x_i, y_i)\) in ascending order of their angles \( \text{Angle}(x_i, y_i) \).

(c) To know if a given arc belongs to \( S \cap D(P, d) \), take the middle \( M \) of this arc and compute the crossing number and \( d_{\text{min}} \) for \( S \) and \( M \) using Algorithm 2. The corresponding arc belongs to \( S \cap D(P, d) \) if and only if the crossing number is odd or \( d_{\text{min}} = 0 \).

The complexity of this algorithm is \( O(n^2) \) where \( n \) is the number of edges. Algorithm 4 which has complexity \( O(n \ln n) \) selects the appropriate arcs in a more efficient manner. In this algorithm, the extremities of these arcs are stored, without repetitions, in the list \texttt{Intersections} which is updated along the iterations of the first \texttt{For} loop of Algorithm 4: \texttt{Intersections(i)} will be the \( i \)-th "relevant" (see below) intersection point found. To know the arcs that belong to \( S \cap D(P, d) \), a second list \texttt{Arcs} is used: the \( i \)-th element of list \texttt{Arcs} is 1 if and only if the arc from the boundary of \( D(P, d) \) obtained starting at \texttt{Intersections(i)} and ending at the next element from list \texttt{Intersections} found travelling counter clockwise on the boundary of \( D(P, d) \) belongs to \( S \cap D(P, d) \). To produce this information, when an intersection between \( S \) and \( C(P, d) \) is found we need to know the type of this intersection, knowing that there are three types of intersections:

- \( T_1 \): the intersection is not "relevant", i.e., there is no arc from \( S \cap C(P, d) \) starting or ending at this point;
Now let us go back to the 6 cases (i)-(vi) discussed above and considered in the first For loop of Algorithm 4. It remains to explain how to determine in each of these cases the intersection type when an intersection is found.

First, since vertices belonging to the boundary of \( \mathcal{D}(P,d) \) are starting vertices of an edge and ending vertices of another edge, to avoid counting them twice, we do not consider the intersection points that are starting vertices of an edge. With this convention, in case (i), i.e., subcases \( A_1 \) and \( A_2 \) in Figure 14, we do not need to store intersection points, even if \( S_i \) belongs to \( \mathcal{D}(P,d) \).

In case (ii), corresponding to subcases \( B_1, B_2, \) and \( B_3 \) in Figure 14 if \( S_iS_{i+1} \) and \( \mathcal{C}(P,d) \) have an intersection point that is different from \( S_i \) then this intersection point is stored in list \textbf{Intersections} and it is of type \( T_2 \): the corresponding entry in \textbf{Arcs} is one (these type \( T_2 \) intersections are represented by red balls in Figure 14).

In case (iv), corresponding to case \( C \) in Figure 14 there is a single intersection point between \( S_iS_{i+1} \) and \( \mathcal{C}(P,d) \) and it is of type \( T_3 \): the corresponding entry in \textbf{Arcs} is zero (these type \( T_3 \) intersections are represented by red circles in Figure 14).

Case (v) corresponds to cases \( D_1, D_2, \) and \( D_3 \) in Figure 14. In subcase \( D_1 \), i.e., when \( S_iS_{i+1} \) and \( \mathcal{C}(P,d) \) have two intersections, the first one encountered when travelling from \( S_i \) to \( S_{i+1} \) is of type \( T_3 \) while the second one is of type \( T_2 \). In subcase \( D_2, \) \( S_iS_{i+1} \) and \( \mathcal{C}(P,d) \) have a single intersection which is of type \( T_1 \).

Let us now consider cases (iii) and (vi), the cases where \( S_{i+1} \) is on the boundary of \( \mathcal{D}(P,d) \). We want to determine the intersection type for \( S_{i+1} \). This is done using an auxiliary algorithm, Algorithm 3, that takes as entries \( P \) and \( d \), the center and radius of \( \mathcal{C}(P,d) \) and three successive vertices \( S_i, S_{i+1}, \) and \( S_{i+2} \) of \( \mathcal{S} \), knowing that \( S_{i+1} \) is on the boundary of \( \mathcal{D}(P,d) \). The output variable \textbf{Arc} of this algorithm is one (resp. zero) if and only if \( S_{i+1} \) is of type \( T_2 \) or \( T_3 \) (resp. type \( T_1 \)). What matters to determine the intersection type for \( S_{i+1} \) is whether \( S_iS_{i+1} \) is contained in the complement of the interior of \( \mathcal{D}(P,d) \) or not. An additional input variable of Algorithm 3, variable \textbf{In}, takes the value zero in the former case and the value one in the latter case.

To explain this algorithm, it is convenient to introduce two half spaces \( \mathcal{H}_{\text{Right}} \) and \( \mathcal{H}_P \) and a line \( L_1 \). These half spaces and lines depend on the entries of Algorithm 3, i.e., \( P \) and \( d \), the center and radius of \( \mathcal{C}(P,d) \), and three successive vertices \( S_i, S_{i+1}, \) and \( S_{i+2} \) of \( \mathcal{S} \). Line \( L_1 \) is the line that contains line segment \( S_iS_{i+1} \). The open half space \( \mathcal{H}_{\text{Right}} \) is the set of points that are to the right of line \( L_1 \) when travelling on this line in the direction \( S_i \rightarrow S_{i+1} \). Denoting by \( L_2 \) the line that is tangent to the circle \( \mathcal{C}(P,d) \) at \( S_{i+1} \) (recall that \( S_{i+1} \) belongs to \( \mathcal{C}(P,d) \)), the closed half space \( \mathcal{H}_P \) is the set of points that are on the side of line \( L_2 \) that does not contain \( P \), including \( L_2 \). The definitions of these sets follow.
To know the intersection type for $D$ of (5.30) we obtain:

$$\begin{cases}
\text{if } x_{Si+1} = x_S \text{ and } y_{Si+1} > y_S \text{ then} \\
L_1 = \{(x, y) : x = x_S\}, \\
H_{\text{right}} = \{(x, y) : x > x_S\}.
\end{cases}$$

If $x_{Si+1} = x_S$ and $y_{Si+1} < y_S$ then

$$\begin{cases}
L_1 = \{(x, y) : x = x_S\}, \\
H_{\text{right}} = \{(x, y) : x < x_S\}.
\end{cases}$$

(5.29)

If $x_{Si+1} > x_S$ then

$$\begin{cases}
L_1 = \{(x, y) : y = y_S + \frac{y_{Si+1} - y_S}{x_{Si+1} - x_S}(x - x_S)\}, \\
H_{\text{right}} = \{(x, y) : y < y_S + \frac{y_{Si+1} - y_S}{x_{Si+1} - x_S}(x - x_S)\}.
\end{cases}$$

If $x_{Si+1} < x_S$ then

$$\begin{cases}
L_1 = \{(x, y) : y = y_S + \frac{y_{Si+1} - y_S}{x_{Si+1} - x_S}(x - x_S)\}, \\
H_{\text{right}} = \{(x, y) : y > y_S + \frac{y_{Si+1} - y_S}{x_{Si+1} - x_S}(x - x_S)\}.
\end{cases}$$

For $H_P$, we obtain:

$$\begin{cases}
\text{if } y_{Si+1} = y_P \text{ and } x_{Si+1} > x_P \text{ then} \\
H_P = \{(x, y) : x \geq x_{Si+1}\}.
\end{cases}$$

If $y_{Si+1} = y_P$ and $x_{Si+1} < x_P$ then

$$\begin{cases}
H_P = \{(x, y) : x \leq x_{Si+1}\}.
\end{cases}$$

If $y_{Si+1} \neq y_P$ and $y_P > y_{Si+1} + \frac{x_{Si+1} - x_P}{y_{Si+1} - y_P}(x - x_{Si+1})$ then

$$\begin{cases}
H_P = \{(x, y) : y \leq y_{Si+1} + \frac{x_{Si+1} - x_P}{y_{Si+1} - y_P}(x - x_{Si+1})\}.
\end{cases}$$

If $y_{Si+1} \neq y_P$ and $y_P < y_{Si+1} + \frac{x_{Si+1} - x_P}{y_{Si+1} - y_P}(x - x_{Si+1})$ then

$$\begin{cases}
H_P = \{(x, y) : y \geq y_{Si+1} + \frac{x_{Si+1} - x_P}{y_{Si+1} - y_P}(x - x_{Si+1})\}.
\end{cases}$$

To know the intersection type for $S_{i+1}$, note that if $S_{i+1}$ belongs to the boundary of $D(P, d)$ and if we take a ball of center $S_{i+1}$ and radius $\varepsilon$ sufficiently small then four situations, represented in Figure 15, can happen:

1. The angle $\angle S_i S_{i+1} S_{i+2}$ is acute and there is an arc starting or ending at $S_{i+1}$ belonging to $S \cap \mathcal{C}(P, d)$. In this case, represented in the top left figure of Figure 15, $S_{i+1}$ is a type $T_2$ or $T_3$ intersection.
2. The angle $\angle S_i S_{i+1} S_{i+2}$ is acute and there is no arc from $S \cap \mathcal{C}(P, d)$ that starts or ends at $S_{i+1}$. In this case, represented in the top right figure of Figure 15, $S_{i+1}$ is a type $T_1$ intersection.
3. The angle $\angle S_i S_{i+1} S_{i+2}$ is obtuse and there is an arc starting or ending at $S_{i+1}$ belonging to $S \cap \mathcal{C}(P, d)$. In this case, represented in the bottom left figure of Figure 15, $S_{i+1}$ is a type $T_2$ or $T_3$ intersection.
4. The angle $\angle S_i S_{i+1} S_{i+2}$ is obtuse and there is no arc from $S \cap \mathcal{C}(P, d)$ starting or ending at $S_{i+1}$. In this case, represented in the bottom right figure of Figure 15, $S_{i+1}$ is a type $T_1$ intersection.

Let us first consider the case where input variable $\text{In}$ of Algorithm 3 is one, i.e., the case where $S_i S_{i+1}$ is not contained in the complement of the interior of $D(P, d)$.

In this case, the edge $S_{i+1} S_{i+2}$ can belong to three different regions, denoted by $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ in Figure 16 and respectively represented in pink at the top left, in green at the top right, and in yellow in the middle left figures of Figure 16. In this Figure 16, type $T_2$ intersections are represented by red balls while type $T_3$ intersections are represented by red circles. Regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are given by
(see Figure 10):

\[ R_1 = \overline{H_P} \cap H_{\text{right}} \cup L_1, \quad R_2 = H_P, \quad \text{and} \quad R_3 = \overline{H_P} \cap H_{\text{right}}. \]

If \( S_{i+2} \) belongs to \( R_1 \), we are in the situation of the top right figure of Figure 10 and \( S_{i+1} \) is a type \( T_1 \) intersection. If \( S_{i+2} \) belongs to \( R_2 \) (resp. \( R_3 \)), we are in the situation of the bottom left or top left (resp. bottom right) figure of Figure 15 and \( S_{i+1} \) is a type \( T_2 \) (resp. type \( T_1 \)) intersection.

We now consider the case where input variable \( \text{In} \) of Algorithm 3 is zero, i.e., the case where \( S_i S_{i+1} \) is contained in the complement of the interior of \( D(P, d) \). In this case, \( S_{i+2} \) can belong to three different regions, denoted by \( R_4, R_5, \) and \( R_6 \) in Figure 10 and respectively represented in pink in the middle right, in green in the bottom left, and in yellow in the bottom right figures of Figure 10.

Regions \( R_4, R_5, \) and \( R_6 \) are given by (see Figure 10):

\[ R_4 = \overline{H_P} \cap H_{\text{right}} \cup L_1, \quad R_5 = H_P, \quad \text{and} \quad R_6 = \overline{H_P} \cap H_{\text{right}}. \]

If \( S_{i+2} \) belongs to \( R_4 \), we are in the situation of the top right figure of Figure 10 and \( S_{i+1} \) is a type \( T_1 \) intersection. If \( S_{i+2} \) belongs to \( R_5 \) (resp. \( R_6 \)), we are in the situation of the bottom left or top left (resp. bottom right) figure of Figure 15 and \( S_{i+1} \) is a type \( T_3 \) (resp. type \( T_1 \)) intersection.

Summarizing our observations, if \( S_{i+1} \) belongs to the boundary of \( D(P, d) \), this intersection is stored as a “relevant” intersection (it is not a type \( T_1 \) intersection) if and only if \( \text{In}=1 \) and \( S_{i+2} \in R_2 \) (in this case, it is a type \( T_2 \) intersection) or \( \text{In}=0 \) and \( S_{i+2} \in R_5 \) (in this case, it is a type \( T_3 \) intersection).

Algorithm 3: Given three successive vertices \( S_i, S_{i+1}, \) and \( S_{i+2} \) of a non-self intersecting polyhedron \( S \) and a circle of center \( P \) and radius \( d > 0 \) with \( S_{i+1} \) belonging to this circle, the algorithm determines if \( S_{i+1} \)
Figure 16. The six cases where an endpoint $S_{i+1}$ of an edge is on the boundary of $D(P,d)$. 

is or is not a starting or ending point of an arc from the boundary of $D(P,d) \cap S$.

Inputs: $P, d, S_i, S_{i+1}, S_{i+2}, \text{In}$.

Initialization: $\text{Arc}=0$.

If $\text{In}$ and $y_P \neq y_{S_{i+1}}$ then

If $y_P > y_{S_{i+1}} \pm \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P}(x_P - x_{S_{i+1}})$ and

$y_{S_{i+2}} \leq y_{S_{i+1}} \pm \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P}(x_{S_{i+2}} - x_{S_{i+1}})$ then $\text{Arc} = 1$.

Else if $y_P < y_{S_{i+1}} \pm \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P}(x_P - x_{S_{i+1}})$ and

$y_{S_{i+2}} \geq y_{S_{i+1}} \pm \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P}(x_{S_{i+2}} - x_{S_{i+1}})$ then $\text{Arc} = 1$.

End if.

Else if $\overline{\text{In}}$ and $y_P \neq y_{S_{i+1}}$ then
If $y_P > y_{S_{i+1}} + \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P} (x_P - x_{S_{i+1}})$ and $y_{S_{i+2}} > y_{S_{i+1}} + \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P} (x_{S_{i+2}} - x_{S_{i+1}})$ then $\text{Arc} = 1.$

Else if $y_P < y_{S_{i+1}} + \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P} (x_{S_{i+2}} - x_{S_{i+1}})$ and $y_{S_{i+2}} < y_{S_{i+1}} + \frac{x_P - x_{S_{i+1}}}{y_{S_{i+1}} - y_P} (x_{S_{i+2}} - x_{S_{i+1}})$ then $\text{Arc} = 1.$

End If

Else if $y_P = y_{S_{i+1}}$ then

If $x_P < x_{S_{i+1}}$ and $x_{S_{i+1}} \leq x_{S_{i+1}}$ or $x_P > x_{S_{i+1}}$ and $x_{S_{i+1}} \leq x_{S_{i+1}}$ then $\text{Arc} = 1.$

End If

Else if $y_P = y_{S_{i+1}}$ then

If $x_P < x_{S_{i+1}}$ and $x_{S_{i+1}} < x_{S_{i+1}}$ or $x_P > x_{S_{i+1}}$ and $x_{S_{i+1}} > x_{S_{i+1}}$ then $\text{Arc} = 1.$

End If

End if

Output: $\text{Arc}.$

In the end of the first For loop of Algorithm 4, the “relevant” intersections points $(x_i, y_i)$ of $S$ and $C(P, d)$ are stored in list $\text{Intersections}$. We then sort these intersections in ascending order of their angles $\text{Angle}(x_i, y_i)$ where we recall that $\text{Angle}$ is defined in (5.24). The values in list $\text{Arcs}$ are sorted correspondingly. For $\text{Nb\_Intersections}$ intersection points, this defines $\text{Nb\_Intersections}$ arcs on the circle. At $i$-th iteration of the second For loop of Algorithm 4, the $i$-th arc is considered. If this arc belongs to $S \cap D(P, d)$, i.e., if $\text{Arcs}(i) = 1$, the corresponding line integral (5.25) is computed. The sum of these line integrals makes up the last part of line integral (5.21) for $D = D(P, d) \cap S$.

It remains to check that the algorithm correctly computes $F_D(d)$ when the value of variable $\text{Nb\_Intersections}$ in the end of Algorithm 4 is null. This can occur in three different manners reported in Figure 17: (i) $D(P, d) \cap S = \emptyset$, (ii) the polyhedron $S$ is contained in $D(P, d)$, and (iii) the disk $D(P, d)$ is contained in $S$. Case (ii) corresponds to $\ell = \mathcal{L}$ and in this case $F_D(d) = 1$. If $\ell \neq \mathcal{L}$, case (i) occurs when $P$ is outside $S$ and case (iii) when $P$ belongs to the interior of $S$. To know if case (i) or case (iii) occurs, we use the crossing number computed by Algorithm 2. If the crossing number is odd then $P$ is inside $S$ and $F_D(d) = \pi d^2 / \mathcal{L}$. Otherwise, the crossing number is even, $P$ is outside $S$ (case (i)) and $F_D(d) = 0$.

Algorithm 4: Computation of the value $F_D(d)$ of the cumulative distribution function of $D$ at $d$ when $X$ is uniformly distributed in a polyhedron contained in a plane with $P$ in that plane.

**Inputs:** $P$, the vertices $S_1, \ldots, S_n$, of polyhedron $S$, $\text{Crossing\_Number}$, $\mathcal{L}$, $d$.

**Initialization:** $\ell = 0$ //Will store line integral (5.21) taking $D = S \cap D(P, d)$, //i.e., will compute the area of $D = S \cap D(P, d)$. 
For $i = 1, \ldots, n$,

// Check if $S_i$ belongs to $\mathcal{D}(P, d)$ or not:

If $\|S_i P\|_2 \leq d$ then

If $\|S_{i+1} P\|_2 < d$ then // Cases A1 and A2 in Figure 14

$\ell \leftarrow \ell + \frac{1}{2} T_{S_i S_{i+1}}$ where for a line segment $AB$, $T_{AB}$ is given by (5.22).

Else If $\|S_{i+1} P\|_2 > d$ // Cases B1, B2, and B3 in Figure 14

Call Algorithm 1 to compute the intersections between the circle of center $P$ and radius $d$ with the line segment $S_i S_{i+1}$.

If there is an intersection point different from $S_i$ then

Let $I_i$ be this intersection point.

$\ell \leftarrow \ell + \frac{1}{2} T_{S_i I_i}$ where for a line segment $AB$, $T_{AB}$ is given by (5.22).

$\text{Nb\_Intersections} \leftarrow \text{Nb\_Intersections} + 1$.

$\text{Intersections}[\text{Nb\_Intersections}] = I_i$.

$\text{Arcs}[\text{Nb\_Intersections}] = 1$.

End If

Else

$\ell \leftarrow \ell + \frac{1}{2} T_{S_i S_{i+1}}$ where for a line segment $AB$, $T_{AB}$ is given by (5.22).

Call Algorithm 3 with input variables $P, d, S_i, S_{i+1}, S_{i+2}$ and with variable $\text{In}$ set to 1.

If the variable $\text{Arc}$ returned by this algorithm is 1 then

$\text{Nb\_Intersections} \leftarrow \text{Nb\_Intersections} + 1$.

$\text{Intersections}[\text{Nb\_Intersections}] = S_{i+1}$.

$\text{Arcs}[\text{Nb\_Intersections}] = 1$.

End If
End If
Else
If $\|\overrightarrow{S_iS_{i+1}}\| < d$ then //Case C in Figure 14
Call Algorithm 1 to compute the intersection $I_i$ between the circle of center $P$ and radius $d$ with the line segment $S_iS_{i+1}$ (note that the intersection is a single point).
$\ell \leftarrow \ell + \frac{1}{2}T_{I_iS_{i+1}}$, where for a line segment $\overline{AB}$, $T_{\overline{AB}}$ is given by (5.22).
Nb_Intersections $\leftarrow$ Nb_Intersections + 1.
Intersections[Nb_Intersections] = $I_i$.
End If
Else If $\|\overrightarrow{S_iS_{i+1}}\| > d$ then //Cases $D_1$, $D_2$, and $D_3$ in Figure 14
Call Algorithm 1 to compute the intersection between the circle of center $P$ and radius $d$ with the line segment $S_iS_{i+1}$.
If there are two intersection points then
Let $I_{i1}$ and $I_{i2}$ be these intersection points where $I_{i1}$ and $I_{i2}$ satisfy
$x_{S_{i+1}} - x_{S_i} \leq x_{S_{i+1}} - x_{S_i}$ if $x_{S_{i+1}} \neq x_{S_i}$ and $\frac{y_{S_{i+1}} - y_{S_i}}{y_{S_{i+1}} - y_{S_i}} \leq \frac{y_{S_{i+1}} - y_{S_i}}{y_{S_{i+1}} - y_{S_i}}$ if $x_{S_{i+1}} = x_{S_i}$.
$\ell \leftarrow \ell + \frac{1}{2}T_{I_{i1}I_{i2}}$, where for a line segment $\overline{AB}$, $T_{\overline{AB}}$ is given by (5.22).
Nb_Intersections $\leftarrow$ Nb_Intersections + 2.
Intersections[Nb_Intersections - 1] = $I_{i1}$.
Intersections[Nb_Intersections] = $I_{i2}$.
Arcs[Nb_Intersections - 1] = 0.
Arcs[Nb_Intersections] = 1.
End If
Else
If there are one intersection then
Call Algorithm 3 with input variables $P, d, S_i, S_{i+1}, S_{i+2}$ and with variable $In$ set to 0.
If the variable Arc returned by this algorithm is 1 then
Nb_Intersections $\leftarrow$ Nb_Intersections + 1.
Intersections[Nb_Intersections] = $S_{i+1}$.
Arcs[Nb_Intersections] = 0.
End If
Else If there are two intersections $I_i$ and $S_{i+1}$ then
$\ell \leftarrow \ell + \frac{1}{2}T_{I_iS_{i+1}}$, where for a line segment $\overline{AB}$, $T_{\overline{AB}}$ is given by (5.22).
Nb_Intersections $\leftarrow$ Nb_Intersections + 1.
Intersections[Nb_Intersections] = $I_i$.
Arcs[Nb_Intersections] = 0.
Call Algorithm 3 with input variables $P, d, S_i, S_{i+1}, S_{i+2}$ and with variable $In$ set to 1.
If the variable Arc returned by this algorithm is 1 then
Nb_Intersections $\leftarrow$ Nb_Intersections + 1.
Intersections[Nb_Intersections] = $S_{i+1}$.
Arcs[Nb_Intersections] = 1.
End If
End If
End If
End If
End For

If Nb_Intersections=0 then
If \( \ell = \mathcal{L} \) then
\[ F_D(d) = 1 \]
Else if variable Crossing_Number is odd then
\(/\!\!/D(P, d)\) is inside the polyhedron
\[ F_D(d) = \frac{\pi d^2}{\mathcal{L}} \]
Else
\(/\!\!/D(P, d)\) has no intersection with the polyhedron
\[ F_D(d) = 0 \]
End If
Else
Sort (call Quicksort algorithm) the elements \((x_i, y_i)\) of list Intersections by ascending order of their angles \(\text{Angle}(x_i, y_i)\) and sort the elements of list Arcs correspondingly.
Let again Intersections and Arcs be the corresponding sorted lists.
For \(i = 1, \ldots, \text{Nb}_\text{Intersections}\)
If \(\text{Arcs}[i]=1\) then
\[ \ell \leftarrow \ell + \frac{1}{2} I_{\bar{A}B_d, p} \text{ where } A = \text{Intersections}[i], \]
\[ B = \begin{cases} \text{Intersections}[i+1] & \text{if } i < \text{Nb}_\text{Intersections}, \\ \text{Intersections}[1] & \text{if } i = \text{Nb}_\text{Intersections}, \end{cases} \]
and where \(I_{\bar{A}B_d, p}\) is obtained substituting \(R_0\) by \(d\) in \([5, 25]\).
End If
End For
\[ F_D(d) = \ell / \mathcal{L}. \]
End If

Output: \(F_D(d)\).

After calling Algorithm 2, if the crossing number is odd, we know that \(P\) belongs to the interior of \(\mathcal{S}\) and for \(0 \leq d \leq d_{\min}\), we have \(f_D(d) = \frac{2\pi d}{\mathcal{L}}\). For \(d \geq d_{\max}\) or \(d \leq 0\), the density is null. If the crossing number is even, \(f_D(d)\) is null for \(0 \leq d \leq d_{\min}\). For \(d_{\min} \leq d \leq d_{\max}\), Algorithm 5 provides approximations of the density at points \(d_i, i = 1, \ldots, N - 1\).

**Algorithm 5: Computation of the approximate density of \(D\) (distance from \(P\) to a random variable uniformly distributed in a polyhedron) in the range \([d_{\min}, d_{\max}]\).**

**Inputs:** The vertices \(S_1, \ldots, S_n\) of a polyhedron contained in a plane, a point \(P\) in this plane, and the number \(N\) of discretization points.

**Initialization:** Call Algorithm 2 to compute \(d_{\min}, d_{\max}\), the crossing number,
and the area $\mathcal{L}$ of $\mathcal{S}$.

If the crossing number is odd then
\[
F_{\text{old}} = \frac{d_{\text{min}}^2}{\mathcal{L}}
\]
Else
\[
F_{\text{old}} = 0.
\]
End If

For $i = 1, \ldots, N - 1$,
Compute $d_i = d_{\text{min}} + \frac{(d_{\text{max}} - d_{\text{min}})i}{N}$.
Call Algorithm 4 with input variables the crossing number, $\mathcal{L}$, $d_{\text{min}}$, $d_{\text{max}}$, and $d = d_i$ to compute $F_D(d_i)$.
Compute $\hat{f}_D(d_i) = \frac{\sum_{i=1}^{N} [F_D(d_i) - F_{\text{old}}]}{d_{\text{max}} - d_{\text{min}}}$ and set $F_{\text{old}} = F_D(d_i)$.
End For

Outputs: $\hat{f}_D(d_i), i = 1, \ldots, N - 1$.

Algorithm 5 was implemented and tested to obtain approximations of the density of $D$ when $X$ is uniformly distributed in some polyhedrons. The results are reported in Figures 13 and 19. Three polyhedrons represented in these figures were considered: a triangle, a rectangle and an arbitrary polyhedron. In each case, the algorithm was tested taking a point $P$ inside the polyhedron and a point $P$ outside.

Finally, we consider the case where the polyhedron is contained in a plane $\mathcal{P}$ and $P$ is not contained in that plane. In this situation, referring to arguments from Section 3, we can use the previous results reparametrizing the problem and replacing $P$ and $d$ respectively by $P_0$, the projection of $P$ onto $\mathcal{P}$, and $R(d) = \sqrt{d^2 - \|PP_0\|^2}$. Indeed, since $\mathcal{S} \subset \mathcal{P}$, we have
\[
\mathcal{S} \cap \mathcal{B}(P, d) = \mathcal{S} \cap \mathcal{P} \cap \mathcal{B}(P, d) = \mathcal{S} \cap \mathcal{D}(P_0, R(d))
\]
where $\mathcal{D}(P_0, R(d))$ is the disk of center $P_0$ and radius $R(d)$ contained in the plane $\mathcal{P}$ (see Figure 8). Since $S_1, S_2$, and $S_3$ are consecutive extremal points of $\mathcal{S}$, the vectors $\overrightarrow{S_2S_1}$ and $\overrightarrow{S_2S_3}$ are linearly independent. Using Gram-Schmidt orthonormalization process, we obtain two points $S'_1$ and $S'_2$ of the plane $\mathcal{P}$ such that the vectors $\overrightarrow{S_2S'_1}$ and $\overrightarrow{S_2S'_2}$ are orthonormal and for any point $Q$ in plane $\mathcal{P}$, the vector $\overrightarrow{S_2Q}$ can be uniquely written as a linear combination of these vectors. Vectors $\overrightarrow{S_2S'_1}$ and $\overrightarrow{S_2S'_2}$ are given by
\[
\overrightarrow{S_2S'_1} = \frac{\overrightarrow{S_2S_1}}{\|\overrightarrow{S_2S_1}\|_2} \quad \text{and} \quad \overrightarrow{S_2S'_2} = \frac{\overrightarrow{S_2S_3} - \langle \overrightarrow{S_2S_3}, \overrightarrow{S_2S'_1} \rangle \overrightarrow{S_2S'_1}}{\|\overrightarrow{S_2S_3} - \langle \overrightarrow{S_2S_3}, \overrightarrow{S_2S'_1} \rangle \overrightarrow{S_2S'_1}\|_2}
\]
It follows that if $A$ is the $(3, 2)$ matrix $[\overrightarrow{S_2S'_1}, \overrightarrow{S_2S'_2}]$ whose first column is $\overrightarrow{S_2S'_1}$ and whose second column is $\overrightarrow{S_2S'_2}$, then the matrix $A^\top A$ is invertible and the projection $P_0 = \pi_{\mathcal{P}}[P]$ of $P$ on $\mathcal{P}$ can be expressed as
\[
(5.31) \quad \overrightarrow{S_2P_0} = A(A^\top A)^{-1} A^\top \overrightarrow{S_2P}.
\]
Before calling Algorithms 2 and 4, we need to reparametrize the problem: we write $\overrightarrow{S_2P_0} = x_{P_0} \overrightarrow{S_2S'_1} + y_{P_0} \overrightarrow{S_2S'_2} = A(x_{P_0}, y_{P_0})$ and $\overrightarrow{S_2S_i} = A(x_i, y_i)$ for $i = 1, \ldots, n$. In
particular, we have \((x_1, y_1) = (\|S_2S_1\|_2, 0)\) and \((x_2, y_2) = (0, 0)\). Since \(A\) has rank 2, eventually after re-ordering the lines of \(A\), we can assume that \(A\) is of the form \(A = [A_0; a_0]\) where \(A_0\) is a \((2, 2)\) invertible matrix with \(A_0(1, 1) \neq 0\). Using Gaussian
elimination, the system \( \overrightarrow{S_2 P_0} = A(x_{P_0}, y_{P_0}) \) can be written \[
\begin{bmatrix}
U_0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_{P_0} \\
y_{P_0}
\end{bmatrix}
= 
\begin{bmatrix}
b \\
0
\end{bmatrix}
\]
for some two-dimensional vector \( b \) and an invertible upper triangular matrix \( U_0 = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \). Another by-product of Gaussian elimination is the lower triangular matrix \( L_0 = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \) such that \( A = L_0 U_0 \) is the \( LU \) decomposition of \( A_0 \). We obtain
\[
(5.32) \quad x_{P_0} = \frac{S_2 P_0(1)}{U_{11}} [1 + \frac{U_{12} L_{21}}{U_{22}}] - \frac{U_{12}}{U_{11}} S_2 P_0(2), \quad y_{P_0} = \frac{\overrightarrow{S_2 P_0(2)} - L_{21} S_2 P_0(1)}{U_{22}},
\]
and for \( i \geq 3 \),
\[
(5.33) \quad x_i = \frac{S_2 S_i(1)}{U_{11}} [1 + \frac{U_{12} L_{21}}{U_{22}}] - \frac{U_{12}}{U_{11}} S_2 S_i(2), \quad y_i = \frac{\overrightarrow{S_2 S_i(2)} - L_{21} S_2 S_i(1)}{U_{22}}.
\]

Algorithms 2, 3, and 4 can now be used with \( P \) replaced by \( (x_{P_0}, y_{P_0}) \) and where the coordinates of the extremal points of the polyhedron are \( (x_i, y_i), i = 1, \ldots, n \). First, Algorithm 2 is called to compute the area \( \mathcal{L} \) of \( S \), the crossing number for \( P_0 \) and \( S \), and the minimal and maximal distances from \( P_0 \) to the boundary of \( S \), respectively denoted by \( d_{\text{min}} \) and \( d_{\text{max}} \). Recalling the definition (5.31) of \( P_0 \), we introduce
\[
(5.34) \quad d_m = \sqrt{d_{\text{min}}^2 + \| PP_0 \|^2_2} \quad \text{and} \quad d_M = \sqrt{d_{\text{max}}^2 + \| PP_0 \|^2_2}.
\]
With this notation, for $d \geq d_M$ or $d \leq 0$, the density is null and if the crossing number is odd, i.e., if $P_0$ belongs to the interior of $\mathcal{S}$, then for $0 \leq d \leq d_m$, we have $f_D(d) = \frac{2\pi d}{L}$. Otherwise, if the crossing number is even, $f_D(d)$ is null for $0 \leq d \leq d_m$. For $d_m \leq d \leq d_M$, Algorithm 6 provides approximations $\hat{f}_D(d_i)$ of the value of the density at points $d_i, i = 1, \ldots, N - 1$.

### Algorithm 6: Computation of the approximate density of $D$ (distance from $P$ to a random variable uniformly distributed in a polyhedron) in the range $[d_m, d_M]$.

**Inputs:** The vertices $S_1, \ldots, S_n$ of a polyhedron contained in a plane, the point $P$, and the number $N$ of discretization points.

**Initialization:** Call Algorithm 2 with $P$ replaced by $(x_{P_0}, y_{P_0})$ (see equation (5.32)) and where the coordinates of the extremal points of the polyhedron are $(x_i, y_i), i = 1, \ldots, n$, given by (5.33). This will compute the area $L$ of $\mathcal{S}$, the crossing number for $P_0$ and $\mathcal{S}$, and the minimal and maximal distances from $P_0$ to the boundary of $\mathcal{S}$, respectively denoted by $d_{\text{min}}$ and $d_{\text{max}}$.

If the crossing number is odd then

$$F_0d = \frac{\pi d_{\text{min}}}{L}$$

Else

$$F_0d = 0.$$ 

End If

Compute $d_m$ and $d_M$ given by (5.34).

For $i = 1, \ldots, N - 1$,

- Compute $d_i = d_m + \left(\frac{d_M - d_m}{N}\right)i$.
- Call Algorithm 4 with input variables the crossing number, $L$, $d_{\text{min}}$, $d_{\text{max}}$, and $d = \sqrt{d_i^2 - \|P_0P\|^2}$ to compute $F_D(d_i)$.
- Compute $\hat{f}_D(d_i) = \frac{N [F_D(d_i) - F_0d]}{d_{\text{max}} - d_{\text{min}}}$ and set $F_0d = F_D(d_i)$.

End For

**Outputs:** $\hat{f}_D(d_i), i = 1, \ldots, N - 1$.

### 6. Application to PSHA and Extensions

The results of Sections 3, 4, and 5 can be used to determine for the application presented in Section 2 the distribution of the distance between the epicenter in $\mathcal{S}$ and an arbitrary point $P$ when $\mathcal{S}$ is a union of disks, a union of balls, or a union of flat polyhedrons (the boundary of a polyhedron in $\mathbb{R}^3$). For this application, the coordinates of $P$, the centers of the disks and of two points on the boundaries of these disks, of the centers of the balls, and of the vertices $S_1, \ldots, S_n$ of the polyhedron are given providing for each point its latitude, its longitude, and its depth measured from the surface of the earth. To apply the computations of the previous sections, we need to choose a Cartesian coordinate system and use the corresponding Cartesian coordinates of these points. These coordinates are given
as follows. We take for the positive $x$-axis the ray $OA$ where $O$ is the center of the earth and $A$ is the point on the surface of the earth with longitude 0 and latitude 0. We take for the positive $z$-axis the ray $OB$ where $O$ is the center of the earth and $B$ is the north pole. The positive $y$-axis is chosen correspondingly and corresponds to ray $OC$ where $C$ is the point on the surface of the earth with longitude 0 and latitude $90^\circ$ East. Let $P$ be a point at depth $d$ from the surface of the earth with latitude $\varphi \in [0,90^\circ]$ (North or South) and longitude $\lambda \in [0,180^\circ]$ (East or West). If the latitude is $\varphi$ North (resp. $\varphi$ South), we use the notation $\varphi N$ (resp. $\varphi S$) while if the longitude is $\lambda$ East (resp. $\lambda$ West), we use the notation $\lambda E$ (resp. $\lambda W$). Denoting by $R$ the earth radius, the Cartesian coordinates of $P$ in the chosen Cartesian coordinate system are

\[
\begin{align*}
(R-d) \cos \varphi \cos \lambda, (R-d) \cos \varphi \sin \lambda, (R-d) \sin \varphi \quad &\text{if } P = (R-d, \lambda E, \varphi N), \\
(R-d) \cos \varphi \cos \lambda, (R-d) \cos \varphi \sin \lambda, -(R-d) \sin \varphi \quad &\text{if } P = (R-d, \lambda E, \varphi S), \\
(R-d) \cos \varphi \cos \lambda, -(R-d) \cos \varphi \sin \lambda, (R-d) \sin \varphi \quad &\text{if } P = (R-d, \lambda W, \varphi N), \\
(R-d) \cos \varphi \cos \lambda, -(R-d) \cos \varphi \sin \lambda, -(R-d) \sin \varphi \quad &\text{if } P = (R-d, \lambda W, \varphi S).
\end{align*}
\]

In the case where the $\ell_2$-norm is replaced by either the $\ell_1$-norm or the $\ell_\infty$-norm and when $S$ is a union of disks contained in a plane with $P$ in that plane, we can use the results of Section 5. Indeed, since the level curves of the $\ell_1$-norm and the $\ell_\infty$-norm in the plane are squares, to compute the CDF of $D$ at a given point in these cases we need to determine the area of the intersection of a square (a particular polyhedron) with disks. It also possible to extend Algorithm 5 to the case where the $\ell_2$-norm is replaced by either the $\ell_1$-norm or the $\ell_\infty$-norm and $S$ is a union of flat polyhedrons.

Another extension of interest is the case where $S$ is an arbitrary polyhedron in $\mathbb{R}^3$. In this case, the CDF and density of the corresponding random variable $D$ given by $D(\omega) = \|PX(\omega)\|_2$ for any $\omega \in \Omega$ can be approximated using Monte Carlo methods. This is possible if we have at hand a black box able to decide if a given point in $\mathbb{R}^3$ belongs to polyhedron $S$ or not.

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