SPECTRUM OF PARTIAL INTEGRAL OPERATORS WITH DEGENERATE KERNEL

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Abstract. In the paper we consider self-adjoint partial integral operators of Fredholm type $T$ with a degenerate kernel on the space $L^2([a, b] \times [c, d])$. Essential and discrete spectra of $T$ are described.

1. Introduction

Linear equations and operators involving partial integrals appear in elasticity theory [1, 2, 3], continuum mechanics [2, 4, 5, 6], aerodynamics [7] and in PDE theory [8, 9]. Self-adjoint partial integral operators arise in the theory of Schrödinger operators [10, 11, 12, 13]. Spectrum of a discrete Schrödinger operator $H$ are tightly connected (see [13, 14]) with that of partial integral operators which participate in the presentation of the operator $H$.

Let $\Omega_1$ and $\Omega_2$ be closed boundary subsets in $\mathbb{R}^{\nu_1}$ and $\mathbb{R}^{\nu_2}$, respectively. Partial integral operator (PIO) of Fredholm type in the space $L^p(\Omega_1 \times \Omega_2)$, $p \geq 1$ is an operator of the form [15]:

$$T = T_0 + T_1 + T_2 + K,$$

where operators $T_0$, $T_1$, $T_2$ and $K$ are defined by the following formulas:

$$T_0 f(x, y) = k_0(x, y) f(x, y),$$

$$T_1 f(x, y) = \int_{\Omega_1} k_1(x, s, y) f(s, y) ds,$$

$$T_2 f(x, y) = \int_{\Omega_2} k_2(x, t, y) f(x, t) dt,$$

$$K f(x, y) = \int_{\Omega_1} \int_{\Omega_2} k(x, y; s, t) f(s, t) ds dt.$$

Here $k_0, k_1, k_2$ and $k$ are given measurable functions on $\Omega_1 \times \Omega_2$, $\Omega_1^2 \times \Omega_2^2$, $\Omega_1 \times \Omega_2$ and $(\Omega_1 \times \Omega_2)^2$, respectively, and all integrals have to be understood in the Lebesgue sense, where $ds = d\mu_1(s)$, $dt = d\mu_2(t)$, $\mu_k(\cdot)$ is the Lebesgue measure on the $\sigma$-algebra of subsets $\Omega_k$, $k = 1, 2$.

In 1975, Likhtarnikov and Vitova [16] spectral properties of partial integral operators are studied. In [16], the following restrictions were imposed: $k_1(x, s) \in L^2(\Omega_1 \times \Omega_1)$, $k_2(y, t) \in L^2(\Omega_2 \times \Omega_2)$ and $T_0 = K = 0$. In [17], spectral properties of PIO with positive kernels were studied (under restriction $T_0 = K = 0$). In Kalitvin and Zabrejko [18] spectral properties of PIO with kernels of two variables

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in $L_p, p \geq 1$ are studied. In [19, 20, 21, 22, 23] for the more general equations with continuous kernels or kernels in $C(L_1)$ spectral properties of the PIO and solvability of partial integral equations in the space $C([a, b] \times [c, d])$ were studied.

Self-adjoint PIO with $T_0 \neq 0$ were first studied in [10], where theorem above essential spectrum is proved. Finiteness and infiniteness of a discrete spectrum of self-adjoint PIO, arising in the theory of Schrödinger operators, are investigated in [11, 12, 14]. In [24] applications of partial integral equations and operators to problems of continuous mechanics, elasticity problems and other problems were considered. Still, some important spectral properties of PIO in the space $L_2$ were left open. The present paper is dedicated to the mentioned problem for PIO with degenerate kernels from the class $L_2$.

Let $T_1$ be a linear integral operator in the space $L_2([a, b] \times [c, d])$ given by the formula

$$
(T_1f)(x, y) = \int_a^b k(x, s, y)f(s, y)ds.
$$

Here $k(x, s, y)$ is a measurable function on $[a, b]^2 \times [c, d]$.

The kernel $k(x, s, y)$ of the integral operator $T_1$ usually satisfies the condition

$$
\int_a^b k(x, s, y)f(s, y)ds \in L_2([a, b] \times [c, d]), \quad \forall f \in L_2([a, b] \times [c, d]).
$$

Consequently, the operator $T_1$ is a linear bounded operator on $L_2([a, b] \times [c, d])$. If, in addition, the kernel $k(x, s, y)$ satisfies the condition:

$$
k(x, s, y) = \bar{k}(s, x, y), \quad \text{for almost all } y \in [c, d],
$$

then the operator $T_1$ is a self-adjoint operator on the Hilbert space $L_2([a, b] \times [c, d])$.

Let $\{\varphi_k(x)\}_{k=1}^n$ be a orthonormal system of functions from the $L_2([a, b], d\nu)$, and let $\{h_k(y)\}_{k=1}^n$ be a system of essential bounded real functions on $[c, d]$.

We define a measurable function $k_1(x, s, y)$ on $[a, b]^2 \times [c, d]$ by the following rule:

$$
k_1(x, s, y) = \sum_{k=1}^n \varphi_k(x)\varphi_k(s)h_k(y).
$$

Then the PIO $T_1$ with the kernel $k_1(x, s, y)$ is a self-adjoint bounded linear operator on $L_2([a, b] \times [c, d])$.

Let $\{\psi_k(y)\}_{k=1}^m$ be a some orthonormal system of functions from the $L_2([c, d], d\nu)$, $\{p_k(x)\}_{k=1}^m$ be a system of essential bounded real functions on $[a, b]$. We define the measurable function $k_2(x, t, y)$ on $[a, b] \times [c, d]^2$ by the following rule:

$$
k_2(x, t, y) = \sum_{j=1}^m p_j(x)\psi_j(y)\psi_j(t).
$$

Then the PIO $T_2$ with the kernel $k_2(x, t, y)$:

$$
T_2f(x, y) = \int_c^d k_2(x, t, y)f(x, t)d\mu_2(t)
$$

is a linear bounded self-adjoint operator on $L_2([a, b] \times [c, d])$.

For an essential bounded function $\varphi \geq 0$ on the measurable set $\Omega \subset \mathbb{R}^\nu$, we define

$$
\text{esssup}_\Omega(\varphi) = \inf\{C : \mu(\{\xi \in \Omega : \varphi(\xi) > C\}) = 0\},
$$
where $\mu(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. For a measurable function $\varphi$ on the set $\Omega \subset \mathbb{R}^n$, the number $\lambda \in \mathbb{R}$ is called an essential value of the function $\varphi$ if
\[ \mu(\{\xi \in \Omega : \lambda - \varepsilon < \varphi(\xi) < \lambda + \varepsilon\}) > 0 \]
for all $\varepsilon > 0$. We denote by $\text{Essran}(\varphi)$ the set of all essential values of the function $\varphi$.

In this paper, we study essential and discrete spectra of PIO of the form $T_1 + T_2$ with the degenerate kernels. The resolvent set, spectrum, essential spectrum and discrete spectrum are denoted by $\rho$, $\sigma$, $\sigma_{\text{ess}}$ and $\sigma_{\text{disc}}$, respectively (see [25]).

2. Spectral Property of PIO $T_1$ and $T_2$

In this section, we study spectra of PIO $T_1$ and $T_2$.

**Proposition 2.1.** Zero is an eigenvalue of $T_1$ of infinite multiplicity. A number $\lambda_0 \neq 0$ is an eigenvalue of $T_1$ if and only if there exists $1 \leq j_0 \leq n$ such that
\[ \mu_2(h_{j_0}^{-1}(\{\lambda_0\})) > 0 \]
We denote by $M$ the subspace of a Hilbert space $L_2[a,b]$ constructed by the orthogonal system $\{\varphi_1, \ldots, \varphi_n\}$. Then $\dim M = n$, and for the subspace $\mathcal{H} = L_2[a,b] \cap M$, we have $\dim \mathcal{H} = \infty$.

Let $\{g_k\}_{k \in \mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, and $\psi \in L_2[c,d]$, $\|\psi\| = 1$. Then $f_k(x,y) = g_k(x)\psi(y) \in L_2(a,b) \times [c,d)$, $k \in \mathbb{N}$, and the system $\{f_k\}_{k \in \mathbb{N}}$ is orthonormal in $L_2([a,b] \times [c,d])$. Clearly,
\[ T_1 f_k = \sum_{i=1}^{n} \int_{a}^{b} \varphi_i(x) \overline{\varphi_i(s)} f_k(s,y) d\mu_1(s) = 0, \quad k \in \mathbb{N}, \]
i.e. zero is an eigenvalue of PIO $T_1$ of infinite multiplicity.

“if” part. Let $\lambda_0 \in \mathbb{C} \setminus \{0\}$ be an eigenvalue of the PIO $T_1$. Then there exists $f \in L_2([a,b] \times [c,d])$, $\|f\| = 1$ such that
\[ T_1 f = \lambda_0 f. \]
We define compact self-adjoint integral operators $K_\omega$ in $L_2[a,b]$ as follows:
\[ K_\omega \varphi(x) = \int_{a}^{b} k_1(x, s, w) \varphi(s) d\mu_1(s), \quad \omega \in \Omega_0, \]
where
\[ \Omega_0 = \{\omega \in [c,d] : p_\omega(x, s) = k_1(x, s, \omega) \in L_2([a,b]^2)\}. \]
We have $\mu_2([c,d] \setminus \Omega_0) = 0$. Put
\[ \mathcal{M}_1 = \{\omega \in \Omega_0 : f_\omega(x) = f(x, \omega) \in L_2[a,b]\}. \]
Then $\mu_2(\mathcal{M}_1) > 0$.

We define measurable subsets:
\[ D_k = \{\omega \in [c,d] : h_k(\omega) = \lambda_0\}, \quad k \in \{1, \ldots, n\}. \]

Put
\[ D_0 = \bigcup_{k=1}^{n} D_k. \]
Let $\omega \in \mathcal{M}_1$. Then $K_\omega f_\omega = \lambda_0 f_\omega$, i.e. the number $\lambda_0$ is an eigenvalue of operators $K_\omega, \omega \in \mathcal{M}_1$. We define by $\{\alpha_1^{(\omega)}, ..., \alpha_n^{(\omega)}\}$ the set of eigenvalues of the operator $K_\omega$ which are different from zero. Then

$$\lambda_0 \in \{\alpha_1^{(\omega)}, ..., \alpha_n^{(\omega)}\} \subset \{h_1(\omega), ..., h_n(\omega)\},$$

i.e. there exists $j_0 \in \{1, ..., n\}$ such that $h_{j_0}(\omega) = \lambda_0$.

Consequently, we have $\omega \in \mathcal{D}_0$. Thus, $\mathcal{M}_1 \subset \mathcal{D}_0$. It means that $\mu_2(\mathcal{D}_0) > 0$. Then there exists $j_0 \in \{1, ..., n\}$ such that

$$\mu_2(\{\omega \in [c, d] : h_{j_0}(\omega) = \lambda_0\}) > 0.$$ 

“only if” part. Let for a $j_0 \in \{1, ..., n\}$ we have $\mu_2(h_{j_0}^{-1}(\{\lambda_0\}) > 0$. Put $\mathcal{D} = h_{j_0}^{-1}(\{\lambda_0\})$. We define the function $\psi \in L_2[c, d]$ in the following way:

$$\psi(y) = \frac{\chi_\mathcal{D}(y)}{\sqrt{\mu_2(\mathcal{D})}}, \ y \in [c, d],$$

where $\chi_\mathcal{D}(\cdot)$ is the characteristic function of a set $\mathcal{D}$. Obviously, $||\psi|| = 1$. Let $f(x, y) = \varphi_{j_0}(x)\psi(y)$. Then $f \in L_2([a, b] \times [c, d])$ and $||f|| = 1$. On the other hand,

$$T_1f(x, y) = \sum_{k=1}^{n} \varphi_k(x) \int_{a}^{b} \varphi_k(s)h_k(y)\varphi_{j_0}(s)\psi(y)d\mu_1(s) = \varphi_{j_0}(x)h_{j_0}(y)\psi(y) = \lambda_0 f(x, y),$$

i.e. the number $\lambda_0$ is an eigenvalue of $T_1$.

We consider the following projectors $P_k$ in the space $L_2([a, b] \times [c, d])$:

$$P_kf(x, y) = \int_{a}^{b} \varphi_k(x)\varphi_k(s)f(s, y)d\mu_1(s), \ k \in \{1, ..., n\}.$$

Let $P = P_1 + ... + P_n$ and $P_0 = E - P$, where $E$ is the identical operator. Then projectors $P_i$ and $P_j$ ($i \neq j$) are orthogonal.

**Proposition 2.2.** If $\lambda \neq 0$ and $\lambda \in \bigcup_{k=1}^{n} \text{Essran}(h_k)$, then the operator $T_1 - \lambda E$ is invertible in $L_2([a, b] \times [c, d])$, and the operator $(T_1 - \lambda E)^{-1}$ is bounded in $L_2([a, b] \times [c, d])$, moreover

$$(T_1 - \lambda E)^{-1}f(x, y) = -\frac{1}{\lambda} \left( f(x, y) - \sum_{k=1}^{n} \frac{h_k(y)}{h_k(y) - \lambda} P_kf(x, y) \right).$$

**Proof.** Let $\lambda \neq 0$ and $\lambda \in \bigcup_{k=1}^{n} \text{Essran}(h_k)$. We define the operator $B_\lambda$ on the $L_2([a, b] \times [c, d])$ by the formula:

$$B_\lambda f(x, y) = \sum_{k=1}^{n} \frac{1}{h_k(y) - \lambda} P_kf(x, y) - \frac{1}{\lambda} P_0f(x, y).$$

It is clear,

$$\lambda \in \text{Essran}(h_k) \cup \{0\}, \text{ the operator }$$

$$A_kf(x, y) = \frac{1}{h_k(y) - \lambda} P_kf(x, y), \ f \in L_2([a, b] \times [c, d])$$

(5) $$\lambda E)B_\lambda = B_\lambda(T_1 - \lambda E) = E.$$ 

For all $\lambda \in \text{Essran}(h_k) \cup \{0\}$, the operator
is a bounded operator in $L_2([a,b] \times [c,d])$. Then the operator $B_\lambda$ is a bounded operator in $L_2([a,b] \times [c,d])$. For each $\lambda \in \{0\} \cup \left( \bigcup_{k=1}^n \text{Essran}(h_k) \right)$, (5) implies 

$$(T - \lambda E)^{-1} = B_\lambda.$$ 

Hence, we have

$$B_\lambda f(x,y) = -\frac{1}{\lambda} \left( f(x,y) - \sum_{k=1}^n \frac{h_k(y)}{h_k(y) - \lambda} P_k f(x,y) \right).$$

**Theorem 2.3.** For the spectra $\sigma(T_1)$ of the PIO $T_1$ with a degenerate kernel (3), the following formula holds:

$$\sigma(T_1) = \{0\} \cup \left( \bigcup_{k=1}^n \text{Essran}(h_k) \right).$$

**Proof.** By proposition 2.2, we obtain

$$\sigma(T_1) \subset \{0\} \cup \left( \bigcup_{k=1}^n \text{Essran}(h_k) \right).$$

However, by proposition 2.1 we have $0 \in \sigma(T_1)$. Now we prove

$$\bigcup_{k=1}^n \text{Essran}(h_k) \subset \sigma(T_1).$$

Let $\lambda_0 \in \text{Essran}(h_{j_0})$, $\lambda_0 \neq 0$ and $t_0$ be arbitrary point from the subset $h_{j_0}^{-1}(\{\lambda_0\})$.

Put

$$V_i = \left\{ t \in [c,d] : \frac{1}{i+1} < |t_0 - t| < \frac{1}{i} \right\}, \quad i \in \mathbb{N}.$$ 

Then there exists $n_0 \in \mathbb{N}$ such that $\mu_2(V_i) > 0$ for all $i \geq n_0$. We consider the following sequence of orthonormal functions $\chi_p(y) \in L_2[c,d] :$

$$\chi_p(y) = \begin{cases} \frac{1}{\sqrt{\mu_2(V_p)}} & y \in V_p, \\ 0 & y \not\in V_p, \end{cases}$$

where $p \geq n_0$. We define by $f_p(x,y) \in L_2([a,b] \times [c,d])$ the orthonormal system of functions: $f_p(x,y) = \varphi_{j_0}(x)\chi_p(y)$, $p \geq n_0$. Then we have

$$(T_1 - \lambda_0 E)f_p(x,y) = (h_{j_0}(y) - \lambda_0) f_p(x,y).$$

Hence,

$$\| (T_1 - \lambda_0 E)f_p \| \leq \text{esssup}_{V_p} (h_{j_0}(y) - \lambda_0)^2, \quad p \geq n_0.$$ 

Since zero is an essential value of the function $h_{j_0}(y) - \lambda_0$, then for large $n_1 \geq n_0$, there exists a small number $\delta_{n_1}$ such that

$$| h_{j_0}(y) - \lambda_0 | < \delta_{n_1} \quad \text{for almost all } y \in V_p, \quad p \geq n_1.$$ 

Therefore

$$\| (T_1 - \lambda_0 E)f_p \| < \delta_{n_1}, \quad p \geq n_1.$$
i.e. \( \lim_{n \to \infty} \| (T_1 - \lambda_0 E) f_n \| = 0 \). This and the Weyl criterion for an essential spectrum of self-adjoint operators \(^{25}\) imply \( \lambda_0 \in \sigma_{ess}(T_1) \subset \sigma(T_1) \).

**Proposition 2.4.** Any eigenvalue of the PIO \( T_1 \) is of infinite multiplicity.

**Proof.** Let \( \lambda \in \mathbb{R} \setminus \{0\} \) be an eigenvalue of \( T_1 \). Then there exists \( f_0 \in L_2([a, b] \times [c, d]), \| f_0 \| = 1 \) such that \( T_1 f_0 = \lambda f_0 \). We define the measurable subset \( \Omega_0 \subset [c, d] : \)

\[
\Omega_0 = \left\{ y \in [c, d] : \int_a^b |f_0(x, y)|^2 d\mu_1(x) \neq 0 \right\}.
\]

Obviously, \( \mu_2(\Omega_0) > 0 \). Define the function:

\[
f_0(x, y) = \begin{cases} \frac{f_0(x, y)}{\sqrt{\int_a^b |f_0(s, y)|^2 d\mu_1(s)}}, & x \in [a, b], y \in \Omega_0 \\ 0, & x \in [a, b], y \notin \Omega_0, \end{cases}
\]

Then \( f_0 \in L_2([a, b] \times [c, d]), \) and \( f_0 \neq 0 \).

Let \( \{\tilde{\psi}_k\}_k \subset \mathbb{N} \) be a system of orthonormal functions from \( L_2(\Omega_0) \). Consider the sequence of functions from \( L_2([a, b] \times [c, d]) : \)

\[
f_k(x, y) = f_0(x, y)\tilde{\psi}_k(y), \quad k \in \mathbb{N},
\]

where

\[
\tilde{\psi}_k(y) = \begin{cases} \tilde{\psi}_k(y), & y \in \Omega_0 \\ 0, & y \notin \Omega_0. \end{cases}
\]

Then

\[
\int_a^b \int_c^d |f_n(x, y)|^2 d\mu_1(x) d\mu_2(y) = \int_{\Omega_0} |\tilde{\psi}_n(y)|^2 d\mu_2(y) = 1,
\]

and

\[
(f_i, f_j) = \int_{\Omega_0} \tilde{\psi}_i(y) \tilde{\psi}_j(y) d\mu_2(y) = 0
\]

for \( i \neq j \).

Clearly

\[
T_1 f_k(x, y) = \lambda f_k(x, y), \quad k \in \mathbb{N},
\]

i.e. the number \( \lambda \) is an eigenvalue of the PIO \( T_1 \) of infinite multiplicity.

**Corollary 2.5.** A discrete spectrum of the PIO \( T_1 \) with a degenerate kernel \(^{3}\) is absent.

**Corollary 2.6.** If every function \( h_k, k \in \{1, ..., n\} \) is continuous and strictly monotone on \([c, d]\), then there is not an eigenvalue of the PIO \( T_1 \) different from zero.

Consider the following projectors \( Q_j \) in the space \( L_2([a, b] \times [c, d]) : \)

\[
Q_j f(x, y) = \int_c^d \psi_j(y) \overline{\psi_j(t)} f(x, t) d\mu_2(t), \quad j \in \{1, ..., m\}.
\]
Proposition 2.7. If \( \lambda \neq 0 \) and \( \lambda \not\in \bigcup_{j=1}^{m} \text{Essran}(p_j) \), then the operator \( T_2 - \lambda E \) is invertible on \( L_2([a, b] \times [c, d]) \), and the operator \( (T_2 - \lambda E)^{-1} \) is bounded in \( L_2([a, b] \times [c, d]) \), moreover

\[
(T_2 - \lambda E)^{-1} f(x, y) = -\frac{1}{\lambda} \left( f(x, y) - \sum_{j=1}^{m} \frac{p_j(x)}{p_j(x) - \lambda} Q_j f(x, y) \right).
\]

Theorem 2.8. For the spectrum \( \sigma(T_2) \) of the PIO \( T_2 \) with a degenerate kernel \( \lambda \), the following formula is hold:

\[
\sigma(T_2) = \{0\} \cup \left( \bigcup_{j=1}^{m} \text{Essran}(p_j) \right).
\]

3. Solvability of partial integral equations

We consider the Fredholm partial integral equation (PIE) of the second kind

\[(6) \quad f(x, y) - \tau(T_1 + T_2)f(x, y) = g(x, y)\]

in the Hilbert space \( L_2([a, b] \times [c, d]) \), where \( f \) is an unknown function from \( L_2([a, b] \times [c, d]) \), \( g \in L_2([a, b] \times [c, d]) \) is a given function, and \( \tau \in \mathbb{C} \) is the parameter of the equation.

The homogeneous PIE corresponding to (6) has the following form:

\[
f(x, y) - \tau(T_1 + T_2)f(x, y) = 0.
\]

In this section, we reduce to some necessary results for PIE of the second kind. Assume that \( \tau \neq 0 \) and \( \tau^{-1} \in \rho(T_1) \). Then the operator \( E - \tau T_1 \) is invertible in \( L_2([a, b] \times [c, d]) \), and the operator \( (E - \tau T_1)^{-1} \) is bounded on \( L_2([a, b] \times [c, d]) \), moreover, by proposition 2.2 we have

\[
(E - \tau T_1)^{-1} f(x, y) = f(x, y) + \tau \sum_{k=1}^{m} \frac{h_k(y)}{1 - \tau h_k(y)} P_k f(x, y).
\]

Analogously, if \( \tau \neq 0 \) and \( \tau^{-1} \in \rho(T_2) \), then the operator \( E - \tau T_2 \) is invertible in \( L_2([a, b] \times [c, d]) \), and the operator \( (E - \tau T_2)^{-1} \) is bounded on \( L_2([a, b] \times [c, d]) \), moreover, by proposition 2.7

\[
(E - \tau T_2)^{-1} f(x, y) = f(x, y) + \tau \sum_{j=1}^{m} \frac{p_j(x)}{1 - \tau p_j(x)} Q_j f(x, y).
\]

Let \( \tau \neq 0 \) and \( \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \). We define compact operators \( W_1(\tau) \) and \( W_2(\tau) \) by the following formulas:

\[
W_1(\tau) = (E - \tau T_2)^{-1} S_1(\tau) T_2, \quad W_2(\tau) = (E - \tau T_1)^{-1} S_2(\tau) T_1,
\]

where

\[
S_1(\tau) f(x, y) = \sum_{k=1}^{n} \frac{h_k(y)}{1 - \tau h_k(y)} (P_k f)(x, y), \quad S_2(\tau) f(x, y) = \sum_{i=1}^{m} \frac{p_i(x)}{1 - \tau p_i(x)} (Q_i f)(x, y).
\]
Lemma 3.1. Let $\tau \neq 0$ and $\tau^{-1} \in \rho(T_1) \cap \rho(T_2)$. Then the following three homogeneous Fredholm PIE of the second kind are equivalent:

\begin{align}
(7) & \quad f - \tau(T_1 + T_2)f = 0, \\
(8) & \quad f - \tau^2 W_1(\tau)f = 0, \\
(9) & \quad f - \tau^2 W_2(\tau)f = 0.
\end{align}

In \cite{19, 20, 21, 22, 23} lemmas similar to the lemma 3.1 for the case of general PIE’s in $C([a,b] \times [c,d])$ with continuous kernels or kernels in $C(L_1)$ were proved. The scheme for the proof of lemma 3.1 can be seen from these works.

Assume that $\tau \neq 0$ and $\tau^{-1} \in \rho(T_1) \cap \rho(T_2)$. We denote by $\Delta_1(\tau)$ and $\Delta_2(\tau)$ the Fredholm determinants of the operators $E - \tau^2 W_1(\tau)$ and $E - \tau^2 W_2(\tau)$, respectively. Define in $\mathbb{C}$ the following subsets

\begin{align*}
R_1 &= \{ \tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \text{ and } \Delta_1(\tau) \neq 0 \}, \\
R_2 &= \{ \tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \text{ and } \Delta_2(\tau) \neq 0 \}, \\
D_1 &= \{ \tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \text{ and } \Delta_1(\tau) = 0 \}, \\
D_2 &= \{ \tau \in \mathbb{C} \setminus \{0\} : \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \text{ and } \Delta_2(\tau) = 0 \}.
\end{align*}

It follows from lemma 3.1 that $R_1 = R_2$ and $D_1 = D_2$. Put

$$R = R(T) = R_1 \text{ and } D = D(T) = D_1.$$ 

Then we obtain

\begin{equation}
R \cup D = \{ \tau \in \mathbb{C} : \tau \in \mathbb{C} \setminus \{0\} \text{ and } \tau^{-1} \in \rho(T_1) \cap \rho(T_2) \}
\end{equation}

and $R \cap D = \emptyset$.

Lemma 3.1 implies the following

Theorem 3.2. Let $\tau \in R \cup D$. Homogenous PIE (7) has a non-trivial solution if and only if $\tau \in D$.

Theorem 3.3. Let $\tau \in R$. Then the Fredholm PIE of the second kind

\begin{equation}
f - \tau(T_1 + T_2)f = g
\end{equation}

has the unique solution $f_0 \in L_2([a,b] \times [c,d])$ for any $g \in L_2([a,b] \times [c,d])$.

4. Spectrum of the PIO $T_1 + T_2$

We put

$$D_0 = D_0(T) = \left\{ \xi : \xi = \frac{1}{\tau}, \quad \tau \in D(T) \right\}.$$ 

Lemma 4.1. For each $\lambda \in \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2) \cup D_0(T))$, the resolvent $R_\lambda(T)$ of the PIO $T = T_1 + T_2$ exists and is bounded on $L_2([a,b] \times [c,d])$. 

Proof. Let $\lambda \in \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2) \cup \mathcal{D}_0(T))$. Then $\lambda \neq 0$, and $\lambda \in \rho(T_1) \cap \rho(T_2)$. Then operators $E - \frac{1}{\lambda} T_1$ and $E - \frac{1}{\lambda} T_2$ are injective and
\[
\left( E - \frac{1}{\lambda} T_1 \right)^{-1} = E + \frac{1}{\lambda} S_1 \left( \frac{1}{\lambda} \right), \quad \left( E - \frac{1}{\lambda} T_2 \right)^{-1} = E + \frac{1}{\lambda} S_2 \left( \frac{1}{\lambda} \right).
\]
However, from $\lambda \not\in \mathcal{D}_0(T)$ and by (111), we obtain $\Delta_1 \left( \frac{1}{\lambda} \right) \neq 0$. Consequently, the operator $E - \frac{1}{\lambda^2} W_1 \left( \frac{1}{\lambda} \right)$ is injective. By the other hand, we have
\[
T_1 + T_2 - \lambda E = (T_1 - \lambda E)(E + (T_1 - \lambda E)^{-1} T_2) =
\]
\[
(T_1 - \lambda E) \left[ E - \frac{1}{\lambda} \left( E + \frac{1}{\lambda} S_1 \left( \frac{1}{\lambda} \right) \right) T_2 \right] =
\]
\[
= (T_1 - \lambda E) \left( E - \frac{1}{\lambda^2} \left( E - \frac{1}{\lambda} T_2 \right)^{-1} S_1 \left( \frac{1}{\lambda} \right) T_2 \right) =
\]
\[
= -\lambda \left( E - \frac{1}{\lambda} T_1 \right) \left( E - \frac{1}{\lambda} T_2 \right) \left( E - \frac{1}{\lambda^2} W_1 \left( \frac{1}{\lambda} \right) \right).
\]
Hence,
\[
R_\lambda(T) = -\frac{1}{\lambda} \left( E - \frac{1}{\lambda^2} W_1 \left( \frac{1}{\lambda} \right) \right)^{-1} \left( E - \frac{1}{\lambda} T_2 \right)^{-1} \left( E - \frac{1}{\lambda} T_1 \right)^{-1}.
\]
Boundedness of the operator $R_\lambda(T)$ follows from the last equality.

Corollary 4.2. For the resolvent operator $R_\lambda(T)$ of the PIO $T = T_1 + T_2$, the formula
\[
R_\lambda(T) = -\frac{1}{\lambda} \left( E - \frac{1}{\lambda^2} W_1 \left( \frac{1}{\lambda} \right) \right)^{-1} \left( E - \frac{1}{\lambda} T_2 \right)^{-1} \left( E - \frac{1}{\lambda} T_1 \right)^{-1}
\]
holds for each $\lambda \in \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2) \cup \mathcal{D}_0(T))$.

Proposition 4.3. Zero is an eigenvalue of infinite multiplicity of the PIO $T = T_1 + T_2$.

Proof. Define by $\mathcal{L}$ the subspace of a Hilbert space $L_2([a, b] \times [c, d])$ constructed by the orthogonal system \{\( \varphi_i(x)\psi_j(y) \)\}_{i=1}^{m} \times j=1^{m}. Then $\dim \mathcal{L} = m \times n$ and for subspace $\mathcal{H}_0 = L_2([a, b] \times [c, d]) \ominus \mathcal{L}$, we have $\dim \mathcal{H}_0 = \infty$.

Let $\{f_k\}_{k \in \mathbb{N}}$ be an orthonormal basis in $\mathcal{H}_0$. It is obvious that
\[
T_1 f_p(x, y) = \sum_{i=1}^{n} \int_{a}^{b} \varphi_i(x) \overline{\varphi_i(s)} h_i(y) f_k(s, y) d\mu_1(s) = 0, \quad p \in \mathbb{N},
\]
\[
T_2 f_p(x, y) = \sum_{j=1}^{m} \int_{c}^{d} p_j(x) \psi_j(y) \overline{\psi_j(t)} f_k(x, t) d\mu_2(t) = 0, \quad p \in \mathbb{N},
\]
i.e.
\[
(T_1 + T_2) f_k(x, y) = 0, \quad k \in \mathbb{N}.
\]

Proposition 4.4. The inclusion $\sigma(T_1) \cup \sigma(T_2) \subset \sigma_{ess}(T_1 + T_2)$ holds.
Proof. We show that \( \sigma(T_1) \subset \sigma_{ess}(T_1 + T_2) \) (the inclusion \( \sigma(T_2) \subset \sigma_{ess}(T_1 + T_2) \) can be proved analogously). We have

\[
\sigma(T_1) = \{0\} \cup \left( \bigcup_{i=1}^{n} \text{Essran}(h_i) \right).
\]

By proposition 4.3 \( 0 \in \sigma_{ess}(T_1 + T_2) \). Assume that \( \lambda_0 \in \text{Essran}(h_{i_0}), \lambda_0 \neq 0, \) where \( i_0 \in \{1, 2, ..., n\} \) and \( y_0 \) is arbitrary point of the set \( h_{i_0}^{-1}(\{\lambda_0\}) \). Put

\[
V_k = \left\{ y \in [c, d] : \frac{1}{k+1} < |y_0 - y| < \frac{1}{k}, \ k \in \mathbb{N} \right\}.
\]

Then there exists \( k_0 \in \mathbb{N} \) such that \( \mu_2(V_k) > 0 \) for all \( k \geq k_0 \), and \( \lim_{k \to \infty} \mu_2(V_k) = 0 \). Consider the sequence of orthonormal functions \( \Phi_k(y) \in L_2[c, d] : \)

\[
\Phi_k(y) = \frac{\chi_{V_k}(y)}{\sqrt{\mu_2(V_k)}}, \ k \in \mathbb{N}, \ k \geq k_0.
\]

Put

\[
f_k(x, y) = \varphi_{i_0}(x)\Phi_k(y), \ k \geq k_0.
\]

Then the system of functions \( \{f_k(x, y)\}_{k \geq k_0} \) of \( L_2([a, b] \times [c, d]) \) is an orthonormal system.

Now we prove that \( \lim_{k \to \infty} \|(T_1 + T_2 - \lambda_0 E)f_k\| = 0 \). However, \( \lim_{k \to \infty} \|(T_1 - \lambda_0 E)f_k\| = 0 \) (see the proof of theorem 2.3). We show that \( \lim_{k \to \infty} \|T_2f_k\| = 0 \).

We define operators \( A_i, i = 1, ..., m \) in the following way:

\[
A_if(x, y) = \int_{c}^{d} p_i(x)\sqrt{\psi_i(y)}|f(x, t)|d\mu_2(t), \ f \in L_2([a, b] \times [c, d]).
\]

Then

\[
\|A_if_k\|^2 \leq \int_{a}^{b} \int_{c}^{d} \left( \int_{c}^{d} |p_i(x)| \cdot |\psi_i(t)| \cdot |\psi_i(y)| \cdot |\varphi_{i_0}(x)| \cdot |\Phi_k(t)| d\mu_2(t) \right)^2 d\mu_1(x) \cdot d\mu_2(y) \leq C_i^2 \left( \int_{c}^{d} |\psi_i(t)| \cdot |\Phi_k(t)| d\mu_2(t) \right)^2 \leq C_i^2 \cdot \int_{V_k} |\psi_i(t)|^2 d\mu(t), \ i \in \{1, ..., n\},
\]

where \( C_i = \text{esssup}_{[a, b]} |p_i(x)| \). Since Lebesgue integrals are absolute continuous, we obtain

\[
\lim_{k \to \infty} \int_{V_k} |\psi_i(t)|^2 d\mu_2(t) = 0
\]
at \( \lim \mu_2(V_k) = 0 \) and \( \|\psi_i\| = 1 \). Thus, we get \( \lim_{k \to \infty} \|A_i f_k\| = 0, i \in \{1, ..., m\} \).

However,

\[
\|T_2f_k\| \leq \sum_{i=1}^{m} \|A_i f_k\|,
\]
what follows \( \lim_{k \to \infty} \|T_2 f_k\| = 0 \). Hence, \( \lim_{k \to \infty} \|(T_1 + T_2 - \lambda_0 E) f_k\| = 0 \). Finally, by the Weyl criterion for an essential spectrum of self-adjoint operators \([25]\), \( \lambda_0 \in \sigma_{\text{ess}}(T_1 + T_2) \).

**Proposition 4.5.** Each \( \lambda \in \mathcal{D}_0(T) \) is an eigenvalue of finite multiplicity of the PIO \( T = T_1 + T_2 \).

*Proof.* Let \( \lambda \in \mathcal{D}_0(T) \). Then \( \lambda \neq 0 \) and \( \Delta_1 \left( \frac{1}{\lambda} \right) = 0 \), where \( \Delta_1(\tau) \) is the Fredholm determinant of the operator \( E - \tau^2 W_1(\tau) \). It means that the number 1 is an eigenvalue of the compact integral operator \( \frac{1}{\lambda^2} W_1 \left( \frac{1}{\lambda} \right) \). By lemma \([3.3]\) the number \( \lambda \) is an eigenvalue of the PIO \( T_1 + T_2 \). Since the following integral equations

\[
    f - \frac{1}{\lambda}(T_1 + T_2)f = 0
\]

and

\[
    f - \frac{1}{\lambda^2} W_1 \left( \frac{1}{\lambda} \right) f = 0
\]

are equivalent, the number \( \lambda \) is an eigenvalue of finite multiplicity of \( T_1 + T_2 \) because of every eigenvalue \( \alpha \neq 0 \) of compact operators is of finite multiplicity.

The next theorem follows from lemma \([4.1]\) and propositions \([4.3, 4.5]\).

**Theorem 4.6.** For the spectrum \( \sigma(T) \) of the PIO \( T = T_1 + T_2 \) with a degenerate kernels, the following formula

\[
    \sigma(T_1 + T_2) = \{0\} \cup \left( \bigcup_{i=1}^n \text{Essran}(h_i) \right) \cup \left( \bigcup_{j=1}^m \text{Essran}(p_j) \right) \cup \mathcal{D}_0(T)
\]

holds.

5. **Discrete spectrum of the PIO \( T_1 + T_2 \)**

Put \( G = \mathbb{C} \setminus (\sigma(T_1) \cup \sigma(T_2)) \). It is well-known that spectra of a linear bounded self-adjoint operators are compact set in the set of all real numbers. Consequently, the set \( \sigma(T_1) \cup \sigma(T_2) \) is a compact subset in \( \mathbb{R} \). Therefore the set \( G \) is an open subset in \( \mathbb{C} \) and \( G \) is unbounded domain in \( \mathbb{C} \).

For each \( \lambda \in G \), we consider the kernel of the compact integral operator \( W_1 \left( \frac{1}{\lambda} \right) \) given as follows:

\[
    W_1 \left( \frac{1}{\lambda} \right) = \left( E - \frac{1}{\lambda} T_2 \right)^{-1} S_1 \left( \frac{1}{\lambda} \right) T_2.
\]

**Proposition 5.1.** For the kernel \( K(x, y; s, t|\lambda) \ (\lambda \in G) \) of the Fredholm integral operator \( W_1 \left( \frac{1}{\lambda} \right) \), the equality

\[
    K(x, y; s, t|\lambda) = \lambda \sum_{j=1}^n \sum_{k=1}^m F_{k,j}(x, y; \lambda) B_{k,j}(s, t),
\]

is valid, where

\[
    F_{k,j}(x, y; \lambda) = \varphi_j(x) \left( \frac{\psi_k(y) h_j(y)}{\lambda - h_j(y)} + \sum_{i=1}^m \frac{p_i(x) \psi_i(y)}{\lambda - p_i(x)} \int_{\mathbb{R}} \frac{h_j(\xi)}{\lambda - h_j(\xi)} \psi_k(\xi) \psi_i(\xi) d\mu_2(\xi) \right),
\]
\[ B_{k,j}(s, t) = p_k(s)\varphi_j(s) \cdot \psi_k(t). \]

Proof. Let \( \lambda \in G \). Then \( \lambda \in \rho(T_1) \cap \rho(T_2) \), and we get
\[
\left( E - \frac{1}{\lambda} T_2 \right)^{-1} = E + \frac{1}{\lambda} S_2 \left( \frac{1}{\lambda} \right).
\]
However,
\[
W_1 \left( \frac{1}{\lambda} \right) = S_1 \left( \frac{1}{\lambda} \right) T_2 + \frac{1}{\lambda} S_2 \left( \frac{1}{\lambda} \right) S_1 \left( \frac{1}{\lambda} \right) T_2.
\]

For each \( f \in L_2([a, b] \times [c, d]) \), using representations of operators \( S_1(\tau) \) and \( S_2(\tau) \), we obtain
\[
S_1 \left( \frac{1}{\lambda} \right) T_2 f(x, y) = \sum_{j=1}^{n} \sum_{k=1}^{m} \int_{c}^{d} K_{k,j}(x, y; \lambda) B_{k,j}(s, t) f(s, t) d\mu_1(s) d\mu_2(t),
\]
\[
S_2 \left( \frac{1}{\lambda} \right) S_1 \left( \frac{1}{\lambda} \right) T_2 f(x, y) = \sum_{j=1}^{n} \sum_{k=1}^{m} \int_{c}^{d} G_{k,j}(x, y; \lambda) B_{k,j}(s, t) f(s, t) d\mu_1(s) d\mu_2(t),
\]
where
\[
K_{k,j}(x, y; \lambda) = \frac{\lambda \varphi_j(x) \psi_k(y) h_j(y)}{\lambda - h_j(y)},
\]
\[
G_{k,j}(x, y; \lambda) = \lambda^2 \sum_{i=1}^{m} \frac{p_i(x) \psi_k(y)}{\lambda - p_i(x)} \int_{c}^{d} \frac{h_j(\xi) \psi_k(\xi) \psi_i(\xi)}{\lambda - h_j(\xi)} d\mu_2(\xi).
\]
Hence, we obtain equality (12) for the kernel \( K(x, y; \tau, s|\lambda) \) of the integral operator \( W_1 \left( \frac{1}{\lambda} \right) \).

Set
\[
\Gamma_1 = \{1, 2, ..., m\}, \quad \Gamma_2 = \{1, 2, ..., n\} \quad \text{and} \quad \Gamma = \Gamma_1 \times \Gamma_2.
\]
We can introduce the relation of partial order in the set \( \Gamma \) by the following way: for elements \( \omega = (k_1, j_1) \in \Gamma \) and \( \omega' = (k_2, j_2) \in \Gamma \), we write \( \omega \leq \omega' \) if \( k_1 < k_2 \) or \( k_1 = k_2, j_1 \leq j_2 \). As the set \( \Gamma \) is finite, the set \( \Gamma \) is linear complete ordered, i.e. for arbitrary \( \omega, \omega' \in \Gamma \), we have \( \omega \leq \omega' \) or \( \omega' \leq \omega \). Thus, we can give elements of \( \Gamma \) in the increase order:
\[
\Gamma = \{\omega_1, \omega_2, ..., \omega_{m(n-1)}, \omega_{m.n}\},
\]
moreover
\[
\omega_1 = (1, 1) < \omega_2 = (1, 2) < ... < \omega_n = (1, n) < \omega_{n+1} = (2, 1) < ... < \omega_{m.n} = (m, n).
\]

Let \( \lambda \in G \) be fixed. For every \( \omega = (k, j) \in \Gamma \), we define the function \( F_\omega(x, y; \lambda) \) on \([a, b] \times [c, d]\) by the following formula:
\[
F_\omega(x, y; \lambda) = F_{k,j}(x, y; \lambda).
\]
Consider the homogenous Fredholm integral equation
\[
(14) \quad f(x, y) - \frac{1}{\lambda^2} W_1 \left( \frac{1}{\lambda} \right) f(x, y) = 0, \quad f \in L_2([a, b] \times [c, d]).
\]
Set
\[
\int_a^b \int_c^d f_\omega(x, y; \lambda)f(x, y)d\mu(x)d\mu(y) = A_\omega(\lambda), \quad i \in \{1, \ldots, m \cdot n\}.
\]

Then the homogenous equation (14) turns into the equation
\[
f(x, y) = \frac{1}{\lambda} \sum_{i=1}^{m-n} A_\omega(\lambda)F_\omega(x, y; \lambda).
\]

Let
\[
\int_a^b \int_c^d f_\omega(x, y; \lambda)B_\omega(x, y)d\mu_1(x)d\mu_2(y) = \Pi_{i,l}(\lambda), \quad i, l \in \{1, \ldots, m \cdot n\},
\]

where
\[
B_\omega(x, y) = B_{k_l,j_l}(x, y), \quad \omega_l = (k_l, j_l).
\]

Then we obtain a system of homogenous linear algebraic equations for unknown numbers \(A_\omega(\lambda)\):
\[
(15) \quad A_\omega(\lambda) - \frac{1}{\lambda} \sum_{i=1}^{m-n} \Pi_{i,l}(\lambda)A_\omega(\lambda) = 0, \quad i \in \{1, \ldots, m \cdot n\}.
\]

**Lemma 5.2.** Let \(\lambda \in G\). The homogenous integral equation (14) has a nontrivial solution if and only if \(\Delta(\lambda) = 0\), where
\[
\Delta(\lambda) = \begin{vmatrix}
\Pi_{1,1}(\lambda) - \lambda & \Pi_{1,2}(\lambda) & \ldots & \Pi_{1,m-n}(\lambda) \\
\Pi_{2,1}(\lambda) & \Pi_{2,2}(\lambda) - \lambda & \ldots & \Pi_{2,m-n}(\lambda) \\
\ldots & \ldots & \ldots & \ldots \\
\Pi_{m-n,1}(\lambda) & \Pi_{m-n,2}(\lambda) & \ldots & \Pi_{m-n,m-n}(\lambda) - \lambda
\end{vmatrix}.
\]

**Proof.** Let \(\lambda \in G\). Then equivalence of the Fredholm integral equation of the second kind (14) and the system of linear algebraic homogeneous equations (15) is clear. The determinant \(\tilde{\Delta}(\lambda)\) of the system of equations (15) has the following form:
\[
\tilde{\Delta}(\lambda) = \begin{vmatrix}
1 - \frac{\Pi_{1,1}(\lambda)}{\lambda} & -\frac{\Pi_{1,2}(\lambda)}{\lambda} & \ldots & -\frac{\Pi_{1,m-n}(\lambda)}{\lambda} \\
-\frac{\Pi_{2,1}(\lambda)}{\lambda} & 1 - \frac{\Pi_{2,2}(\lambda)}{\lambda} & \ldots & -\frac{\Pi_{2,m-n}(\lambda)}{\lambda} \\
\ldots & \ldots & \ldots & \ldots \\
-\frac{\Pi_{m-n,1}(\lambda)}{\lambda} & -\frac{\Pi_{m-n,2}(\lambda)}{\lambda} & \ldots & 1 - \frac{\Pi_{m-n,m-n}(\lambda)}{\lambda}
\end{vmatrix},
\]

and
\[
\tilde{\Delta}(\lambda) = \left(\frac{-1}{\lambda}\right)^{m-n} \Delta(\lambda).
\]
It is well-known, the system of linear homogeneous equations (13) has nontrivial solution if and only if \( \Delta(\lambda) = 0 \), i.e. \( \Delta(\lambda) = 0 \). However, we obtain that the homogeneous Fredholm equation (14) has nontrivial solution if and only if \( \Delta(\lambda) = 0 \).

**Lemma 5.3.** The function \( \Delta(z) \) is holomorphic in the domain \( G \).

**Proof.** Let \( \omega \in \Gamma \). It is known, the function \( F_\omega(x, y; \lambda) \) is holomorphic by \( \lambda \) in the domain \( G \) for almost all \((x, y) \in [a, b] \times [c, d]\), and for every \( \lambda \in G \) the integral

\[
\int_a^b \int_c^d F_\omega(x, y; \lambda) B_{\omega'}(x, y) d\mu_1(x) d\mu_2(y), \quad \omega, \omega' \in \Gamma
\]

exists and is finite. Then for every \( \omega = (i, l) \in \Gamma \), the function \( \Pi_{i,l}(z) \) is a holomorphic function in \( G \). Consequently, the function \( \Delta(z) \) is a sum of holomorphic functions \( F_{\omega_1}(z), F_{\omega_2}(z), \ldots, F_{\omega_{m-n}}(z) \), i.e. \( \Delta(z) \) is holomorphic in \( G \).

**Remark 5.4.** An analogue of Lemma 5.3 can be proved for the general PIE.

**Theorem 5.5.** The discrete spectrum of the PIO \( T = T_1 + T_2 \) coincides with the set \( D_0(T) \).

**Proof.** Lemmas 3.1 and 5.2 imply

\[
D_0(T) = \{ \lambda \in G : \Delta(\lambda) = 0 \}.
\]

By proposition 4.4, we have

\[
\sigma(T_1) \cup \sigma(T_2) \subset \sigma_{\text{ess}}(T_1 + T_2).
\]

By theorem 4.6, we obtain

\[
\sigma_{\text{disc}}(T) \subset D_0(T).
\]

Let \( \lambda_0 \in D_0(T) \) be arbitrary. Then by proposition 4.5, the number \( \lambda_0 \) is an eigenvalue of finite multiplicity of the operator \( T_1 + T_2 \). Since the function \( \Delta(z) \) is holomorphic in the \( G \), arbitrary point \( \lambda \) form the \( D_0(T) \) is isolate in \( D_0(T) \). Then the point \( \lambda_0 \) is isolate in the spectrum \( \sigma(T_1) \cup \sigma(T_2) \cup D_0(T) \) of the operator \( T \) since \( (\sigma(T_1) \cup \sigma(T_2)) \cap D_0(T) = \emptyset \). Thus, by definition of a discrete spectrum, we obtain \( \lambda_0 \in \sigma_{\text{disc}}(T) \), i.e. \( D_0(T) \subset \sigma_{\text{disc}}(T) \).

Theorems 4.6 and 5.3 implies

**Theorem 5.6.**

\[
\sigma_{\text{ess}}(T_1 + T_2) = \{0\} \cup \left( \bigcup_{i=1}^n \text{Essran}(h_i) \right) \cup \left( \bigcup_{j=1}^m \text{Essran}(p_j) \right).
\]

**Example 5.7.** Let \( h_i(y) \equiv a_i \in \mathbb{R} \setminus \{0\}, \ i \in \{1, \ldots, n\} \) and \( p_j(x) \equiv b_j \in \mathbb{R} \setminus \{0\}, \ j \in \{1, \ldots, m\} \) for the kernels \( k_1(x, s, y) \) and \( k_2(x, t, y) \) of PIO \( T_1 \) and \( T_2 \).

Then by theorem 2.3 and 2.8, we obtain

\[
\sigma(T_1) = \{0, a_1, \ldots, a_n\}, \quad \sigma(T_2) = \{0, b_1, \ldots, b_m\}.
\]

By theorem 5.6, we have

\[
\sigma_{\text{ess}}(T_1 + T_2) = \{0, a_1, \ldots, a_n, b_1, \ldots, b_m\}.
\]
We obtain from (13):

\[ F_{k,j}(x,y;\lambda) = \frac{\lambda a_j}{(a_j - \lambda)(b_k - \lambda)} \varphi_j(x)\psi_k(y), \quad \lambda \in \sigma_{\text{ess}}(T_1 + T_2), \]

and

\[ B_{k,j}(s,t) = b_k \varphi_j(s) \cdot \psi_k(t). \]

Then the homogeneous Fredholm integral equation (14) becomes the following form (16)

\[ f(x,y) - \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{a_j b_k}{(a_j - \lambda)(b_k - \lambda)} \int_{a}^{b} \int_{c}^{d} \varphi_j(x)\psi_k(y) \varphi_j(s) \cdot \psi_k(t) d\mu_1(s) d\mu_2(t) = 0. \]

However, using the property of Fredholm integral equations with a degenerate kernel, we obtain

\[ \frac{a_j b_k}{(a_j - \lambda)(b_k - \lambda)} = 1, \quad j \in \{1, \ldots, n\}, \quad k \in \{1, \ldots, m\}, \quad \lambda \in \sigma_{\text{ess}}(T_1 + T_2). \]

It means that

\[ a_j b_k = (a_j - \lambda)(b_k - \lambda), \quad \lambda \in \{0\} \cup \{a_1\} \cup \{b_1\}, \]

i.e. the integral equation (16) has nontrivial solution if and only if

\[ \lambda = a_j + b_k \in \{0\} \cup \{a_1\} \cup \{b_1\}. \]

Set

\[ \Lambda = \{ \lambda : \lambda = a_j + b_k \in \{0, a_1, \ldots, a_n, b_1, \ldots, b_m\}, \quad j = 1, n, \quad k = 1, m \}. \]

Therefore according to theorem 4.6 we obtain \( \sigma_{\text{disc}}(T_1 + T_2) = \Lambda \) and theorem 4.6 implies

\[ \sigma(T_1 + T_2) = \{ \lambda : \lambda = a + b : a \in \sigma(T_1), \quad b \in \sigma(T_2) \}. \]

It should be noted, the equality (17) was proved for the PIO \( T \) in \( L_p, p \geq 1 \) with more general kernels \( k_1(x,s,y) = k_1(x,s), k_2(x,t,y) = k_2(t,y) \) in the paper [13], [24].

Remark 5.8. It is known, that the discrete spectrum \( \sigma_{\text{disc}}(K) \) for each self-adjoint Fredholm integral operator \( K \) with a degenerate kernel is finite (since \( \sigma_{\text{disc}}(K) \) is the set of all eigenvalues of \( K \) different from zero). The following question is arisen: Does this property hold for the PIO \( T = T_1 + T_2 \) with a degenerate kernels (3) and (4) This question is still an open problem.

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