CENTRAL INVARIANTS REVISITED

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ABSTRACT. We give a new proof of the statement of Dubrovin-Liu-Zhang that the Miura-equivalence classes of the deformations of semi-simple bi-Hamiltonian structures of hydrodynamic type are parametrized by the so-called central invariants.

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1. INTRODUCTION

In 2004, in a series of papers, Dubrovin, Liu, and Zhang considered the problem of classification of deformations of semi-simple pencils of local Poisson brackets with respect to a dispersive parameter [8, 5]. This type of structures is universally important in the theory of integrable systems since this gives a way to construct and study integrable systems of evolutionary equations with one spatial variable. In particular, in the case of $N$ dependent variables, they proved that the Miura equivalence class of deformations of a given semi-simple pencil of local Poisson brackets of hydrodynamic type is specified by a choice of $N$ functions of one variable. They called these functions central invariants, and conjectured that for any choice of central invariants the corresponding Miura equivalence class is non-empty. This conjecture was proved in [4].

As any deformation theory of this type, its space of infinitesimal deformations as well as the space of obstructions for the extensions of infinitesimal deformations are controlled by some cohomology groups. In this case these are the so-called bi-Hamiltonian cohomology in cohomological degrees 2 and 3, and one should also consider there the degree with respect to the total $\partial_x$-derivative, where $x$ is the spatial variable. So, central invariants span the second bi-Hamiltonian cohomology group in $\partial_x$-degree 3, and the second bi-Hamiltonian cohomology groups in $\partial_{\xi}\partial_x$-degrees 2 and $\geqslant 4$ are equal to zero.

The computation of bi-Hamiltonian cohomology is a delicate issue. It is defined on the space of local stationary polyvectorfields on the loop space of an
open $n$-dimensional ball $M$. A useful tool for this undertaking is the so-called $\theta$-formalism [7]. The main technical difficulty is that we can’t immediately work with the space of densities, since there is a necessary factorization by the kernel of the integral along the loop. For the central invariants it is done in [5] essentially by hand.

In [9], Liu and Zhang came up with an important new idea: they invented a way to lift the computation of the bi-Hamiltonian cohomology from the space of local polyvector fields to the space of their densities. The latter can also be considered as the functions on the infinite jet space of the loop space of the shifted tangent bundle $T_M[-1]$, independent of the loop variable $x$. Their approach was used intensively in a number of papers; it has been applied to show that the deformation of the dispersionless KdV brackets is unobstructed [9] and to compute the higher cohomology in this case as well [2]. More generally, this approach allowed a complete computation of the bi-Hamiltonian cohomology in the scalar ($N = 1$) case [3]. Finally, it was used to show that the deformation theory for any semi-simple Poisson pencil is unobstructed [4].

At the moment, it is not completely clear yet how widely this approach can be applied to the computation of the bi-Hamiltonian cohomology. In the case of $N > 1$ the full bi-Hamiltonian cohomology is not known, and moreover, as the computation in the case $N = 1$ shows, the full answer should depend on the formulas for the original hydrodynamic Poisson brackets. So far the computational techniques worked well only for the groups of relatively high cohomological grading and/or grading with respect to the total $\partial_x$-derivative degree. In particular, the most fundamental result of this whole theory, the fact that the infinitesimal deformations are controlled by the central invariants, was out of reach of this technique until now.

In this paper, we extend the computational techniques of [4] further and give a new proof of the theorem of Dubrovin-Liu-Zhang that the space of the Miura classes of the infinitesimal deformations of a semi-simple Poisson pencil is isomorphic to the space of $N$ functions of one variable. An advantage of our approach is that we use only the general shape of the differential induced on the jet space of $T_M[-1]$, and, for instance, the Ferapontov equations for compatible Poisson brackets of hydrodynamic type [6] enter the computation only through the fact that the differential squares to zero. A disadvantage is that in the cohomological approach of Liu-Zhang it is not possible to reproduce the explicit formula for the central invariants of a given deformation as in [5, Equation 1.49].

1.1. Organization of the paper. The outline of the article is as follows. In section 2 we recall some standard notations and formulate our main results, based on the computation of some of the cohomology of a certain complex $(\hat{A}[\lambda], D_1)$ in the rest of the paper. In section 3 we give a streamlined version of the proof [4] of the vanishing theorem for the cohomology of $(\hat{A}[\lambda], D_1)$. In the next sections we proceed to compute other parts of this cohomology that will lead us in particular to the identification of the parameters of the infinitesimal deformations. In section 4 we compute the full cohomology of the complex $(\hat{d}_i(\hat{C}_i), D_i)$. In section 5 we compute the cohomology of the subcomplex $(\hat{C}[\lambda], \Delta_{0,1})$ given by degrees $p = d$. In section 6 we prove a vanishing result in degrees $(p, d) = (3, 2)$, which is essential to complete the reconstruction of the second bi-Hamiltonian cohomology group. In section 7 we collect the results of the previous sections and, using standard spectral sequences arguments, we compute new parts of the cohomology $H(\hat{A}[\lambda], D_1)$. 
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2. Recollections and main results

Let $N$ be the number of dependent variables. We consider a hypercube $U$ in $\mathbb{R}^N$ outside the diagonals. Let $u^1, \ldots, u^N$ be the coordinate functions of $\mathbb{R}^N$ restricted to $U$. We denote the corresponding basis of sections of $T_U[-1]$ by $\theta^0_1, \ldots, \theta^0_N$. We denote by $\mathcal{A}$ the space of functions on the jet space of the loop space of $T_U[-1]$ that do not depend on the loop variables $x$, that is,

$$\mathcal{A} := C^\infty(U) \left[ \{u^{id} \mid d=1, \ldots, N \}, \left\{ \theta^d_\gamma \mid d=1, \ldots, N \right\} \right]$$

Sometimes it is convenient to denote the coordinate functions $u^d$ by $u^i$, $i = 1, \ldots, N$. The standard derivation, that is, the total derivative with respect to the variable $x$, is given by

$$\partial_x := \sum_{d=0}^\infty \left( u^{i,d+1} \frac{\partial}{\partial u^d} + \theta^{d+1} \frac{\partial}{\partial \theta^d} \right).$$

(we assume the summation over the repeated indices).

A semi-simple pencil of hydrodynamic Poisson brackets is determined by $N$ non-vanishing functions $f^1, \ldots, f^N$, subject to the following equations derived by Ferapontov [6]. Let $H_i := (f^i)^{-1/2}$, $i = 1, \ldots, N$, be Lamé coefficients and $\gamma_{ij} := (H_i)^{-1} \partial_i H_j$, $i \neq j$, be rotation coefficients for the metric determined by $f^1, \ldots, f^N$. Here we denote by $\partial_i$ the derivative $\partial/\partial u^i$. Then we have:

$$\partial_i \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \quad i \neq j \neq k \neq i; \quad (1)$$

$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i,j} \gamma_{ik} \gamma_{kj} = 0, \quad i \neq j; \quad (2)$$

$$u^i \partial_i \gamma_{ij} + u^j \partial_j \gamma_{ji} + \sum_{k \neq i,j} u^k \gamma_{ik} \gamma_{kj} + \frac{1}{2} (\gamma_{ij} + \gamma_{ji}) = 0, \quad i \neq j. \quad (3)$$

One can associate with the pencil given by the functions $f^1, \ldots, f^N$, a differential operator $D_1$ defined on the space $\mathcal{A}[\lambda]$, where $\lambda$ is a formal variable. The operator $D_1$ is then defined as $D_1 := D(u^1 f^1, \ldots, u^N f^N) - AD(f^1, \ldots, f^N)$, where

$$D(g^1, \ldots, g^N) = \sum_{s \geq 0} \partial^s \left( g^i \frac{\partial}{\partial u^i} \right) \frac{\partial}{\partial u^s}.$$
Note that both spaces \( \hat{\mathcal{A}} \) and \( \hat{\mathcal{F}} \) have two gradations: the standard gradation that we also call the \( \partial_\tau \)-degree in the introduction, given by \( \deg u^i = \deg \theta^i = s \), \( i = 1, \ldots, N \), \( s \geq 0 \), and the super gradation that we also call the cohomological or the \( \theta \)-degree, given by \( \deg u^i = 1 \), \( i = 1, \ldots, N \), \( s \geq 0 \). We denote by \( \hat{\mathcal{A}}_d^s \) (respectively, \( \hat{\mathcal{F}}_d^p \)) the subspace of \( \hat{\mathcal{A}} \) (respectively, \( \hat{\mathcal{F}} \)) of \( \partial_\tau \)-degree \( d \) and \( \theta \)-degree \( p \).

Consider a semi-simple pencil of hydrodynamic Poisson brackets defined by the functions \( f^1, \ldots, f^N \), as above. One can associate with it a concept of bi-Hamiltonian cohomology, \( BH \), that also has standard and super gradations. We denote by \( BH_d^p \) the subspace of \( BH \) of \( \partial_\tau \)-degree \( d \) and cohomological degree \( p \).

The space of the infinitesimal dispersive deformations of this pencil of brackets is given by \( \oplus_{d \geq 2} BH_d^2 \). We refer to \( [8, 5, 9] \) for the definition of \( BH \).

The key lemma of \( [9] \), see also \( [1] \), implies that for \( d \geq 2 \) we have \( BH_d^p \cong H^p_\lambda(\hat{\mathcal{F}}[\lambda], d_1) \). Another idea of Liu and Zhang \( [9] \) is that in order to compute the cohomology of \( (\hat{\mathcal{F}}[\lambda], d_1) \) one might use the long exact sequence in the cohomology since they are always standard and super gradations. We give a streamlined proof of this theorem in the next section. This theorem is a theorem of Dubrovin-Liu-Zhang \([9] \) and \( \partial_\tau \)-degree \( d \) and cohomological degree \( p \).

We want to derive this theorem from the exact sequence given by Equation (4).

**Theorem 2.1.** We have: \( H^3_\lambda(\hat{\mathcal{F}}[\lambda]) \) (and, therefore, \( BH_3^3 \)) is equal to 0 for \( d = 2 \) and \( d \geq 4 \). In the case \( d = 3 \), \( H^2_\lambda(\hat{\mathcal{F}}[\lambda]) \) (and, therefore, \( BH_2^1 \)) is isomorphic to \( \oplus_{i=1}^N C^\infty(u^i) \).

We want to derive this theorem from the exact sequence given by Equation (4). For that, let us recall that in \([1] \) the following vanishing theorem for the cohomology of the complex \( (\hat{\mathcal{A}}[\lambda], D_\lambda) \) was proved.

**Theorem 2.2.** The cohomology \( H^p_d(\hat{\mathcal{A}}[\lambda]) \) vanishes for all bi-degrees \( (p, d) \), unless \( (p, d) = (d + k, d) \) with

\[
k = 0, \ldots, N - 1, \quad d = 0, \ldots, N + 2 \quad \text{or} \quad k = N, \quad d = 0, \ldots, N.
\]

We give a streamlined proof of this theorem in the next section. This theorem implies that the exact sequence (4) is equal to

\[
0 \to 0 \to H^0_d(\hat{\mathcal{F}}[\lambda]) \to 0 \to 0 \quad (5)
\]

in the case \( d \geq 4 \). So we immediately have the vanishing of \( H^2_\lambda(\hat{\mathcal{F}}[\lambda]) \), and, therefore, \( BH_2^3 \) for \( d \geq 4 \). In the case \( d = 3 \) we have:

\[
H^2_\lambda(\hat{\mathcal{A}}[\lambda]) \to 0 \to H^3_\lambda(\hat{\mathcal{F}}[\lambda]) \to H^3_\lambda(\hat{\mathcal{A}}[\lambda]) \to 0, \quad (6)
\]

and in the case \( d = 2 \) we have

\[
H^1_\lambda(\hat{\mathcal{A}}[\lambda]) \to H^2_\lambda(\hat{\mathcal{A}}[\lambda]) \to H^2_\lambda(\hat{\mathcal{F}}[\lambda]) \to H^3_\lambda(\hat{\mathcal{A}}[\lambda]) \to H^3_\lambda(\hat{\mathcal{A}}[\lambda]) \quad (7)
\]
The main contribution of this paper is the following result about the cohomology of $\hat{A}[\lambda]$.

**Theorem 2.3.** We have: $H^3_2(\hat{A}[\lambda]) = 0$, $H^3_2(\hat{A}[\lambda]) = 0$, $H^3_3(\hat{A}[\lambda]) \cong \bigoplus_{i=1}^N C^\infty(u')$.

This theorem follows from a bit more general statements that we derive in the rest of the paper and collect in section 7. It allows us to prove immediately theorem 2.1.

**Proof of theorem 2.1.** The statement for $d > 4$ follows from the exact sequence (5) and it was already derived this way in [4]. For $d = 3$ the exact sequence (6) implies that $H^3_2(\hat{F}[\lambda]) \cong H^3_3(\hat{A}[\lambda]),$ and $H^3_3(\hat{A}) \cong \bigoplus_{i=1}^N C^\infty(u')$ by theorem 2.3.

For $d = 2$ the second and the forth terms in the exact sequence (7) are equal to zero by theorem 2.3, which implies that $H^2_2(\hat{F}[\lambda]) = 0.$ □

We conclude this section with one more piece of notation that we use throughout the rest of the paper: for a multi-index $I = \{i_1, \ldots, i_s\}$, we write $f_I = \prod_{i \in I} f_i$, $\theta_I = \theta_{i_1} \cdots \theta_{i_s}$, etc.

3. The vanishing theorem

In this section we give a streamlined proof of Theorem 2.2 with the purpose of recalling some objects that will be used later.

3.1. Let $\deg_u$ be the degree on $\hat{A}$ defined by assigning

$$\deg_u u^{i_s} = 1, \quad s > 0$$

and zero on the other generators. The operator $D_A$ splits in the sum of its homogeneous components

$$D_A = \Delta_{-1} + \Delta_0 + \ldots,$$

where $\deg_u \Delta_k = k$.

To the degree $\deg_u + \deg_0$ we associate a decreasing filtration of $\hat{A}[\lambda]$. Let us denote by $^1E$ the associated spectral sequence. The zero page $^1E_0$ is simply given by $\hat{A}[\lambda]$ with differential $\Delta_{-1}$:

$$(^1E_0, ^1d_0) = (\hat{A}[\lambda], \Delta_{-1}).$$

To find the first page $^1E_1$, we have to compute the cohomology of this complex.

3.2. Let us compute the cohomology of the complex $(\hat{A}[\lambda], \Delta_{-1})$. The differential can be written as

$$\Delta_{-1} = \sum_i (\lambda + u') f_i \hat{d}_i$$

where $\hat{d}_i$ is the de Rham-like differential

$$\hat{d}_i = \sum_{s > 1} \theta_i^{s+1} \frac{\partial}{\partial u^{i_s}}.$$

It is convenient to split $\hat{A}$ in a direct sum

$$\hat{A} = \hat{C} \oplus \left( \bigoplus_{i=1}^N \hat{C}_i \right) \oplus \hat{M}.$$

Here

$$\hat{C} = C^\infty(U)[\theta^0_1, \ldots, \theta^0_N, \theta^1_1, \ldots, \theta^1_N].$$
and
\[ \hat{C}_i = \hat{C}[[u^{i,s}, \theta_i^{s+1} \mid s \geq 1]], \]
while \( \hat{C}^\text{nt}_i \) denotes the subspace of \( \hat{C}_i \) spanned by nontrivial monomials, i.e., all monomials that contain at least one of the variables \( u^{i,s}, \theta_i^{s+1} \) for \( s \geq 1 \). By \( \hat{M} \) we denote the subspace of \( \hat{A} \) spanned by monomials which contain at least one of the mixed quadratic expressions
\[ u^{i,s}u^{j,s}, \quad u^{i,s}\theta_j^{s+1}, \quad \theta_i^{s+1}\theta_j^{s+1} \]
for some \( s, t \geq 1 \) and \( i \neq j \).

**Lemma 3.1.** The differential \( \Delta_{-1} \) leaves invariant each direct summand in
\[ \hat{A}[\lambda] = \hat{C}[\lambda] \oplus \bigoplus_{i=1}^{N} \hat{C}^\text{nt}_i[\lambda] \oplus \hat{M}[\lambda], \]
and in particular maps \( \hat{C}[\lambda] \) to zero.

**Proof.** It is easy to check that
\[ \hat{d}_i(\hat{C}) = 0, \quad \hat{d}_i(\hat{M}) \subseteq \hat{M}, \]
\[ \hat{d}_i(\hat{C}^\text{nt}_i) \subseteq \hat{C}^\text{nt}_i, \quad \hat{d}_i(\hat{C}^\text{nt}_j) = 0 \quad i \neq j, \]
from which the lemma follows immediately. \( \square \)

The cohomology of \( \hat{A}[\lambda] \) is therefore the direct sum of the cohomologies of the summands in the direct sum (8), and in particular
\[ H(\hat{C}[\lambda], \Delta_{-1}) = \hat{C}[\lambda]. \]

Let us first observe that the cohomology of the de Rham complex \((\hat{C}_i, \hat{d}_i)\) is trivial in positive degree.

**Lemma 3.2.**
\[ H(\hat{C}_i, \hat{d}_i) = \hat{C}. \]

**Proof.** The proof is completely analogous to the standard proof of the Poincaré lemma. \( \square \)

In particular we have that
\[ H(\hat{C}^\text{nt}_i, \hat{d}_i) = 0, \]
therefore the kernel of \( \hat{d}_i \) in \( \hat{C}^\text{nt}_i \) coincides with \( \hat{d}_i(\hat{C}_i) \).

**Lemma 3.3.**
\[ H(\hat{C}^\text{nt}_i[\lambda], \Delta_{-1}) = \frac{\hat{d}_i(\hat{C}_i)[\lambda]}{(-\lambda + u')d_i(\hat{C}_i)[\lambda]}. \]

**Proof.** On \( \hat{C}^\text{nt}_i[\lambda] \) the differential \( \Delta_{-1} \) is equal to \((-\lambda + u')f'd_i \). Its kernel coincides with the kernel of \( \hat{d}_i \) on \( \hat{C}^\text{nt}_i[\lambda] \), which is \( d_i(\hat{C}_i)[\lambda] \). Its image is \((-\lambda + u')d_i(\hat{C}_i)[\lambda] \).

Finally we prove that the complex \((\hat{M}[\lambda], \Delta_{-1})\) is acyclic.

**Lemma 3.4.**
\[ H(\hat{M}[\lambda], \Delta_{-1}) = 0. \]
Proof. This lemma can be proved by induction on $N$. Denote, for convenience, the corresponding space and the differential by $\hat{\mathcal{M}}[\lambda]_N$ and $\Delta_{-1}(N)$. We also use in the proof the notation $\hat{\mathcal{A}}(N)$ and $\hat{\mathcal{C}}(N)$.

The differential $\Delta_{-1}$ is naturally the sum of two commuting differentials,

$$\Delta_{-1}(N) = \Delta_{-1}(N-1) + (\lambda + u^N)f^N \hat{d}_N.$$  

The cohomology of $(-\lambda + u^N)f^N \hat{d}_N$ on $\hat{\mathcal{M}}[\lambda]_N$ is equal to the direct sum of two subcomplexes, $\hat{\mathcal{C}}(N) \otimes \hat{\mathcal{C}}_{N-1}(N)$ and

$$\hat{d}_N(\hat{C}_N^\text{nat}) \otimes \hat{\mathcal{C}}(N) \left( \bigoplus_{i=1}^{N-1} \hat{C}_i^\text{nat}[\lambda] \right) \otimes \hat{\mathcal{C}}(N) \otimes \hat{\mathcal{C}}_{N-1}(N) \hat{\mathcal{M}}[\lambda]_{(N-1)}^2.$$  

On the first component the induced differential is equal to $\Delta_{-1}(N-1)$, so we can use the induction assumption. On the second component the induced differential is equal to

$$(\Delta_{-1}(N-1))_{\lambda + u^N},$$  

so, up to rescaling by non-vanishing functions, it is a de Rham-like differential acting only on the second factor of the tensor product. This second factor can be identified with $\hat{\mathcal{C}}(N) \otimes \hat{\mathcal{C}}_{N-1}(N)/(\hat{\mathcal{C}}(N-1)/\hat{\mathcal{C}}(N))$, so the possible non-trivial cohomology is quotiented out (cf. the standard proof of the Poincaré lemma).

This completes the computation of the cohomology of the complex $(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$:

**Proposition 3.5.**

$$H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}) = \hat{\mathcal{C}}[\lambda] \oplus \left( \bigoplus_{i=1}^{N} \frac{\hat{d}_i(\hat{C}_i)[\lambda]}{(-\lambda + u^N)\hat{d}_i(\hat{C}_i)[\lambda]} \right)$$  

(9)

3.3. The first page $^1E_1$ of the first spectral sequence is given by the cohomology of the complex $H(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$ with the differential induced by the operator $\Delta_0$:

$$(^1E_1, ^1d_1) = (H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}), \Delta_0).$$  

We remind the formula for the operator $\Delta_0$ in the appendix. To get the second page $^1E_2$ of the first spectral sequence we have to compute the cohomology of this complex.

Let $\deg_\varrho$ be the degree on $\hat{\mathcal{A}}$ defined by setting

$$\deg_\varrho \theta_i^1 = 1 \quad i = 1, \ldots, N$$

and zero on the other generators. The operator $\Delta_0$ splits in its homogeneous components

$$\Delta_0 = \Delta_{0,1} + \Delta_{0,0} + \Delta_{0,-1}$$

where $\deg_\varrho \Delta_{0,k} = k$.

To the degree $\deg_\varrho$ we associate a decreasing filtration of $H(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$, and denote by $^2E$ the associated spectral sequence. The zero page $^2E_0$ is given by $H(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$ with the differential induced by $\Delta_{0,1}$:

$$(^2E_0, ^2d_0) = (H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}), \Delta_{0,1}).$$  

The first page $^2E_1$ is given by the cohomology of this complex.
3.4. To obtain a simple expression for the action of $\Delta_{0,1}$ on the cohomology $H$, it is convenient to perform a change of basis in $\mathcal{A}$. Let $\Psi$ be the invertible operator that rescales the generators of $\mathcal{A}$ as follows

$$u^{i,s} \mapsto (f^i)^{\frac{s}{2}} u^{i,s}, \quad \theta^i \mapsto (f^i)^{\frac{s+1}{2}} \theta^i.$$

The operator $\Delta_{0,1}$ has a simpler form when conjugated with $\Psi$, and since $\Psi$ leaves invariant all the subspaces that we consider, such conjugation does not affect the computation of the cohomology.

**Lemma 3.6.** The operator $\Delta_{0,1}$ acts on the cohomology $H$ as the operator $\Psi \tilde{\Delta}_{0,1} \Psi^{-1}$ where

$$\tilde{\Delta}_{0,1} = \sum_{i} (\lambda + u^i) \frac{\partial}{\partial u^i} + \sum_{i,j} (\lambda + u^i)(\gamma_{ij} \theta^j) \frac{\partial}{\partial \theta^j} + \theta^i \mathcal{E}_i,$$

and leaves invariant each of the summands in $H$. Here $\mathcal{E}_i$ is the Euler operator that multiplies any monomial $m$ by its weight $w_i(m)$ defined by

$$w_i(u^{i,s}) = \frac{s}{2} + 1, \quad w_i(\theta^i s) = \frac{s}{2} - 1 \quad s \geq 1$$

and zero on the other generators.

**Proof.** Recall that $\Delta_{0,1}$ is the deg$_u = 0$ and deg$_{\theta^i} = 1$ homogeneous component of the differential $D$. An explicit expression can be found in [4]. By a straightforward computation, we have that $\Psi^{-1} \Delta_{0,1} \Psi$ is equal to $\tilde{\Delta}_{0,1}$ plus two extra terms

$$- \sum_{i,j} (\lambda + u^i) \left( \frac{f^j}{f^i} \right)^{\frac{s}{2}} \left( (s+2) \gamma_{ij} \theta^j + sy_{ij} \theta^j \right) u^{i,s} \frac{\partial}{\partial u^{i,s}}$$

$$+ \sum_{i,j} (\lambda + u^i) \left( \frac{f^j}{f^i} \right)^{\frac{s}{2}} \left( (1-s) \gamma_{ij} \theta^j - (1+s) \gamma_{ij} \theta^j \right) \theta^j \frac{\partial}{\partial \theta^j}.$$ 

The following formulas are useful in the computation of the conjugated operator:

$$\Psi^{-1} \frac{\partial}{\partial u^i} \Psi = (f^i)^{\frac{s}{2}} \frac{\partial}{\partial u^i}, \quad \Psi^{-1} u^{i,s} \Psi = (f^i)^{\frac{s}{2}} u^{i,s},$$

$$\Psi^{-1} \frac{\partial}{\partial \theta^i} \Psi = (f^i)^{\frac{s+1}{2}} \frac{\partial}{\partial \theta^i}, \quad \Psi^{-1} \theta^i \Psi = (f^i)^{-\frac{s+1}{2}} \theta^i,$$

$$\Psi^{-1} \frac{\partial}{\partial u^i} \Psi = \frac{\partial}{\partial u^i} + \sum_j \frac{\partial \log f^j}{\partial u^i} \sum_{s \geq 0} \left( \frac{s}{2} u^{j,s} \frac{\partial}{\partial u^{j,s}} + \frac{s+1}{2} \theta^j \frac{\partial}{\partial \theta^j} \right).$$

By construction the operator $\Delta_{0,1}$ induces a map on the cohomology $H$, and so does the conjugated operator $\Psi^{-1} \Delta_{0,1} \Psi$.

Let us make a few easy to check observations in order to simplify this operator:

1. $\tilde{\Delta}_{0,1}$ maps $C[\lambda]$ to itself, while the two extra terms send it to zero;
2. the two extra terms, when $j \neq i$, send $d_i(C_j)[\lambda]$ to $M_i[\lambda]$ which is trivial in cohomology;
3. both $\tilde{\Delta}_{0,1}$ and the extra terms for $j = i$ map $d_i(C_i)[\lambda]$ to $\tilde{C}^n_i[\lambda]$, and, because they need to act on cohomology, they actually send it to $d_i(C_i)[\lambda]$;
4. terms in $d_i(C_i)[\lambda]$ which are proportional to $\lambda - u^i$ actually vanish in cohomology, so we can set $\lambda$ equal to $u^i$; this in particular kills the $i = j$ part of the extra terms.
3.6. We have the following vanishing result for the cohomology of \( \hat{C}[\lambda] \) in the proposition.

\[
\frac{\hat{d}_i(\hat{C})[\lambda]}{(-\lambda + u')\hat{d}_i(\hat{C})[\lambda]} \approx \hat{d}_i(\hat{C})
\]

(10)

by setting \( \lambda \) equal to \( u' \). Let \( D_i \) be the operator induced by \( \Delta_{0,1} \) on \( \hat{d}_i(\hat{C}) \) by this identification. Let us give its explicit form.

**Corollary 3.7.** The operator \( D_i \) on \( \hat{d}_i(\hat{C}) \) is given by \( D_i = \Psi \hat{D}_i \Psi^{-1} \) with

\[
\hat{D}_i = \sum_k \theta_k \left[ (u^k - u') \left( \frac{\partial}{\partial u^k} + \sum_j \gamma_{jk} \theta_j \frac{\partial}{\partial \theta_j} \right) + \sum_j (u' - u') \gamma_{jk} \theta_j \frac{\partial}{\partial \theta_j} + E_k \right].
\]

The first page of the second spectral sequence is therefore given by the following direct sum

\[
2^E_1 \approx H(\hat{C}[\lambda], \Delta_{0,1}) \oplus \left( \bigoplus_{i=1}^N H(\hat{d}_i(\hat{C}), D_i) \right).
\]

3.5. A vanishing result for the cohomology of \( \hat{C}[\lambda] \) is obtained by a simple degree counting argument.

**Proposition 3.8.** The cohomology \( H^p_d(\hat{C}[\lambda], \Delta_{0,1}) \) vanishes for all \((p, d)\), unless

\[
d = 0, \ldots, N, \quad p = d, \ldots, d + N.
\]

**Proof.** The possible bi-degrees of the elements of \( \hat{C} \) are precisely those excluded in the proposition. \( \square \)

3.6. We have the following vanishing result for the cohomology of \( (\hat{d}_i(\hat{C}), D_i) \).

**Proposition 3.9.** The cohomology \( H^p_d(\hat{d}_i(\hat{C}), D_i) \) vanishes for all \((p, d)\), unless

\[
d = 2, \ldots, N + 2, \quad p = d, \ldots, d + N - 1.
\]

**Proof.** To prove this result let us introduce a third spectral sequence. For fixed \( i \), let \( \deg_{\theta_i} \) be the degree that assigns degree one to \( \theta_i \) and degree zero to the remaining generators. Consider the decreasing filtration associated to the degree \( \deg_{\theta_i} - \deg_{\theta_i} \).

Let \( 3^E \) be the associated spectral sequence. Let \( D_{i,1} \) be the homogeneous component of \( D_i \) with \( \deg_{\theta_i} = 1 \), i.e., \( D_{i,1} = \Psi \hat{D}_{i,1} \Psi^{-1} \) with

\[
\hat{D}_{i,1} = \theta_i \left[ \sum_j (u' - u') \gamma_{jk} \theta_j \frac{\partial}{\partial \theta_j} + E_i \right].
\]

The zero page \( 3^E_0 \) is given by \( \hat{d}_i(\hat{C}) \) with differential \( D_{i,1} \):

\[
(3^E_0, 3^d_0) = (\hat{d}_i(\hat{C}), D_{i,1}).
\]

To prove the proposition it is sufficient to prove the vanishing of the cohomology of this complex in the same degrees, which we will do in the next lemma. \( \square \)

**Lemma 3.10.** The cohomology \( H^p_d(\hat{d}_i(\hat{C}), D_{i,1}) \) vanishes for all \((p, d)\), unless

\[
d = 2, \ldots, N + 2, \quad p = d, \ldots, d + N - 1.
\]
Proof. As before let us work with the operator $\tilde{D}_{i,1}$. Let $m$ be a monomial in the variables $u^j$, $\theta^i$ for $s \geq 1$. For $g \in \tilde{C}$, we have
\[
\tilde{D}_{i,1} \left( g \tilde{d}_i(m) \right) = \theta_i^j \left( \sum_j (u^j - u^j ) \psi_j \frac{\partial}{\partial \theta^j} g + (w_i(g) + w_i(m) - 1)g \right) \tilde{d}_i(m),
\]
where $w_i$ is the weight defined in Lemma 3.6. Therefore $\tilde{D}_{i,1}$ leaves $\tilde{C} \tilde{d}_i(m)$ invariant for each monomial $m$. We will now prove that the cohomology of the subcomplex $\tilde{C} \tilde{d}_i(m)$ vanishes for all monomials $m$, except for the case $m = u^{i,1}$, therefore the cohomology of $\tilde{d}_i(\tilde{C})$ is just given by the cohomology of $\tilde{C} \tilde{d}_i(u^{i,1})$. Notice that $\tilde{d}_i(\tilde{C})$ is nonzero only for $w_i(m) \geq \frac{3}{2}$, and the case $w_i(m) = \frac{3}{2}$ corresponds to $m = u^{i,1}$ and $\tilde{d}_i(m) = \theta_i^2$.

Let us split $\tilde{C} = \tilde{C}' \oplus \theta_i^2 \tilde{C}'_0$, where $\tilde{C}'_0$ is the subspace spanned by monomials that do not contain $\theta_i^2$. Given $g \in \tilde{C}'_0$ we have
\[
\tilde{D}_{i,1} \left( g \tilde{d}_i(m) \right) = \theta_i^j (w_i(m) - 1)g \tilde{d}_i(m).
\]
Notice that the coefficient $w_i(m) - 1$ is non-vanishing, therefore $\tilde{D}_{i,1}$ is acyclic on the subcomplex $\tilde{C}'_0 \tilde{d}_i(m)$. For $g \in \theta_i^2 \tilde{C}'_0$, the differential $\tilde{D}_{i,1}$ maps $g \tilde{d}_i(m)$ to $\theta_i^j (w_i(m) - \frac{3}{2})g \tilde{d}_i(m) \in \theta_i^2 \tilde{C}'_0 \tilde{d}_i(m)$ plus an element in $\tilde{C}'_0 \tilde{d}_i(m)$.

It is well-known that when a complex $(C, d)$ contains an acyclic subcomplex $C'$, its cohomology is given by the cohomology of a subspace $C''$ complementary to $C'$ with differential given by the restriction and projection of $d$ to $C''$.

In the present case this implies that the cohomology of $\tilde{C} \tilde{d}_i(m)$ is equivalent to the cohomology of $\theta_i^2 \tilde{C}'_0$ with differential given by the operator of multiplication for the element $\theta_i^j (w_i(m) - \frac{3}{2})$. Such complex is acyclic as long as $w_i(m) \neq \frac{3}{2}$. The only nontrivial case is when $m = u^{i,1}$, and in such case the cohomology is given by $\theta_i^2 \tilde{C}'_0 \tilde{d}_i(u^{i,1}) = \tilde{C}'_0 \theta_i^2 \theta_i^2$.

Counting the degrees of the possible elements in this space we obtain the vanishing result above. $\square$

3.7. From the previous two propositions it follows that $^2E_1$ is zero if the bi-degree $(\rho, d)$ is not in one of the two specified ranges, i.e., in their union given in theorem 2.2. Clearly the vanishing of $^2E_1$ in certain degrees implies the vanishing of $^1E_2$ and consequently of $H(\mathcal{A}[\lambda], D_4)$ in the same degrees. This concludes the proof of theorem 2.2.

4. The cohomology of $(\tilde{d}_i(\tilde{C}), \tilde{D}_i)$

In this section we extend the vanishing result of 3.6 to a computation of the full cohomology of the complex $(\tilde{d}_i(\tilde{C}), \tilde{D}_i)$.

First, we can represent the space $\tilde{d}_i(\tilde{C})$ as a direct sum
\[
\tilde{d}_i(\tilde{C}) = \tilde{C}_0 \theta_i^2 \oplus \tilde{C}_0 \theta_i^2 \oplus \tilde{C} \oplus \tilde{d}_i(V_i)
\]
where, as before in 3.6, we denote by $\tilde{C}_0$ the subspace of $\tilde{C}$ spanned by monomials that do not contain $\theta_i^2$. We denote by $V_i$ the space of polynomials in $u^{i,1}$, $\theta_i^2$ of standard degree $\geq 2$. 

Lemma 4.1. The differential $D_\ell$ leaves invariant the spaces $\hat{C}_\ell^i\theta_2^2$ and $\hat{C} \otimes \hat{d}(V_i)$, while
\[D_\ell(\hat{C}_\ell^i\theta_2^2) \subset \hat{C}_\ell^i\theta_2^2 = \hat{C}^i_\ell\theta_2^2 \oplus \hat{C}_\ell^i\theta_2^2.\]
Proof. As before we can equivalently work with $\tilde{D}_\ell$. The statement is a simple check, noticing that $[\tilde{D}_\ell, \hat{d}] = -\delta_\ell i \hat{d}_\ell$.

As we know from §3.5 the cohomology is a subquotient of $\hat{C}_\ell^i\theta_2^2$. Therefore the subcomplexes $\hat{C}_\ell^i\theta_2^2$ and $\hat{d}(V_i)$ are acyclic and the cohomology is given by
\[H(\hat{d}(\hat{C}_\ell), D_\ell) = H(\hat{C}_\ell^i\theta_2^2, D_\ell'),\]
where $D_\ell'$ is the restriction and projection of $D_\ell$ to $\hat{C}_\ell^i\theta_2^2$. Explicitly $D_\ell' = \Psi \hat{D}' \Psi^{-1}$ is given by removing the terms in $\hat{D}_\ell$ that decrease the degree in $\theta_2^2$, which gives
\[
\tilde{D}_\ell' = \sum_{k=1}^{\infty} \theta_k \left[ (u^k - u^k) \frac{\partial}{\partial u^k} + \sum_{j \neq k} \gamma_{jk} \theta_j \frac{\partial}{\partial \theta_j} + \sum_j (u^k - u^k) \gamma_{jk} \theta_j \frac{\partial}{\partial \theta_j} + E_k \right].
\]
Notice that $E_i$ maps $\hat{C}^i\theta_2^2$ to zero, since both $\theta_1^i$ and $\theta_0^j\theta_2^2$ have degree $w_j$ equal to zero. We can now split $\hat{C}^i\theta_2^2$ in the direct sum $\hat{C}^i_{0,1}\theta_2^2 \oplus \hat{C}^i_{0,1}\theta_2^2$ where $\hat{C}^i_{0,1}$ is the subspace of $\hat{C}^i$ spanned by monomials that do not depend on $\theta_1^i$. Since $\tilde{D}_\ell'$ does not act on $\theta_1^i\theta_2^2$ or $\theta_0^j\theta_2^2$, we can reduce our problem to computing the cohomology of the complex $(\hat{C}^i_{0,1}, D_\ell')$. Let us denote by $\delta_k^i$ the coefficient of $\theta_k^i$ in $\tilde{D}_\ell'$, i.e.,
\[
\tilde{D}_\ell' = \sum_{k=1}^{\infty} \delta_k^i \theta_k^i.
\]

Lemma 4.2. The cohomology $H^*(\hat{C}^i_{0,1}, D_\ell')$ is nontrivial only in degrees $d = 0$ and $p = 0, \ldots, N - 1$. In degree $(d = 0, p)$ it is isomorphic to $C^\infty(u^i) \otimes \wedge^p \mathbb{R}^{N-1}$ and is represented by an element
\[F = \sum_{j \in [n] \cap [i]} F_j(u^1, \ldots, u^N) \theta_j \in \bigcap_{k=1}^{\infty} \ker \delta_k^i,
\]
which depends on a single function of the variable $u^i$.

Proof. We represent the space of coefficients $\hat{C}^i_{0,1}$ as a direct sum $\bigoplus_{\ell, \ell=0}^{n-1} K^\ell$, where an element of $K^\ell$ can be written down as
\[
\sum_{I \subseteq [n] \cap [i]} f_I \theta_I \cdot \sum_{j \in [n] \cap [i]} \delta_j^I F_{I,j}(u^1, \ldots, u^n).
\]
The action of $D_\ell'$ can be described, in both cases, as a map $K^\ell \to K^{\ell+1}$ given by the following formula on the components of the corresponding vectors: $F_{I,j} \mapsto G_{S,T}$, where
\[
G_{S,T} = \sum_{s \in S} \frac{\partial}{\partial u^s} F_{I,[s],T} + (A_{s,t})_I^j F_{I,[s],T},
\]
where the coefficients of the matrices $(A_{s,t})_I^j$ can easily be reconstructed from the formula for the operator $\tilde{D}_\ell'$. So, this way we can describe each of the subcomplexes $K^\ell \theta_1^i\theta_2^2$, $K^\ell \theta_0^j\theta_2^2$, $\ell = 0, \ldots, n - 1$, as a tensor product of the de Rham
complex of smooth functions in \( n - 1 \) variable \( u^k, k \neq i \), with a vector space whose basis is indexed by monomials of degree \( t \) in \( \theta_q^i, q \neq i \). The differential (the restriction of \( \mathcal{D}_F \) to this subcomplex) is equal to the de Rham differential \( \sum_{p \neq i} \theta_p \frac{\partial}{\partial u^p} \) twisted by a linear map:

\[
\sum_{p \neq i} \theta_p \left( \frac{\partial}{\partial u^p} + A_{p,i} \right).
\]

(the coefficients of \( A_{p,i} \) depend on whether we consider the case of \( K^\bullet(t^0, \theta_i^i) \) or \( K^\bullet(t^0, \theta_j^i) \), but the shape of the differential is the same in both cases).

The cohomology of the differential \( \mathcal{H} \) is isomorphic to the cohomology of the de Rham differential \( \sum_{p \neq i} \theta_p \frac{\partial}{\partial u^p} \). It is represented by the differential forms of order 0, that is, it is non-trivial only for \( \ell = 0 \), whose vector of coefficients \( F_{0,\ell} \) solves the differential equations

\[
\frac{\partial F_{0,\ell}}{\partial u^p} + (A_{p,i})_F F_{0,\ell} = 0
\]

for \( p \neq i \). The solution of this equation is uniquely determined by the restriction \( F_{0,0,\ell,0, p \neq i} \), that is, by a single function of \( u^l \). So, finally, we obtain the statement of the lemma.

Taking into account the action of \( \Psi \) we obtain the cohomology the complex \((\hat{d}_i, \mathcal{D}_i)\).

**Proposition 4.3.** The cohomology of \((\hat{d}_i, \mathcal{D}_i)\) is nontrivial only in the degrees \((p, d) = (2, 2), \ldots, (N + 1, 2)\) and \((p, d) = (3, 3), \ldots, (N + 2, 3)\). In the degrees \((2 + t, 2)\) and \((3 + t, 3)\) it is isomorphic to \( C^\text{\infty}(u^l) \otimes \bigwedge^t \mathbb{R}^{N-1}, t = 0, \ldots, N - 1 \). More precisely, representatives of cohomology classes in degrees \((2 + t, 2)\) and \((3 + t, 3)\) are given respectively by elements of the form

\[
F \cdot (f^t)^{1/2 + \ell} \partial^1 \theta_i^1, \quad G \cdot (f^t)^{1/2 + \ell} \partial^1 \theta_i^2
\]

for \( F, G \) representatives of \( H^1_0(\hat{\mathcal{C}}_{0,1}, \hat{\mathcal{D}}_1) \) as given in the previous lemma.

5. The cohomology of \((\hat{\mathcal{C}}[\lambda], \Delta_{0,1})\) at \( p = d \)

In this section we extend the result of §3.3 by computing the cohomology of the subcomplex of \((\hat{\mathcal{C}}[\lambda], \Delta_{0,1})\) defined by setting \( p = d \).

From proposition 3.8 we already know that the complex \((\hat{\mathcal{C}}[\lambda], \Delta_{0,1})\) is non-trivial only for \( d \in \{0, \ldots, n\} \) and \( p \in \{d, \ldots, d+n\} \). As usual, being the differential of bidegree \((p, d) = (1, 1)\), it splits in subcomplexes of constant \( p - d \). Here we consider the case \( p = d \).

**Proposition 5.1.** For \( p = d \) the cohomology of the complex \((\hat{\mathcal{C}}[\lambda], \Delta_{0,1})\) is given by

\[
H^p_0(\hat{\mathcal{C}}[\lambda], \Delta_{0,1}) = \begin{cases} \mathbb{R}[\lambda] \oplus \bigoplus_{i=1}^{N} C^\infty(u^l) \theta_i^1 & p = 0, \\ C^\infty(U)[\theta_1^1, \ldots, \theta_N^1] & p = 1, \\ 0 & \text{else.} \end{cases}
\]

**Proof.** For \( p = d \) the complex \( \hat{\mathcal{C}}[\lambda] \) is equal to

\[ C^\infty(U)[\theta_1^1, \ldots, \theta_N^1]. \]
Let us compute the cohomology of $\tilde{\Delta}_{0,1}$. Because there is no dependence on $\theta_0^1$ and the degree $w_k$ of $\theta_k^1$ is zero, the differential simplifies to

$$\tilde{\Delta}_{0,1} = \sum_i \delta_i, \quad \delta_i = (-\lambda + u^i)\theta_k^1 \frac{\partial}{\partial u^i}.$$ 

We will let $J \subseteq \{1, \ldots, N\}$ denote a multi-index and write $\theta_j^1$ for the lexicographically ordered product $\prod_{j \in J} \theta_j^1$. For each of the $\theta_1^1, \theta_2^1, \ldots, \theta_N^1$, we can define a degree $\deg_{\theta_j^1} - \deg_{\theta_k^1}$, which again induces a decreasing filtration. The filtration associated to $\theta_j^1$ has $\delta_i$ as differential on the zeroth page of the spectral sequence. Considering all these filtrations, we get the following picture:

![Spectral sequence diagram]

So the complex can be visualised as an $N$-dimensional hypercube with a term in every corner.

On the first page of the $\theta_1^1$-spectral sequence, the differential is $\sum_{j \neq 1} \delta_j$, and we can use the $\theta_j^1$-filtration to get another spectral sequence. This procedure can be repeated inductively.

Consider an element in $\mathcal{C}^\infty(U)[\lambda]\theta_1^1$. Clearly it is in $\ker \delta_1$ if $J$ contains 1 or if it does not depend on $u^1$:

$$\ker \delta_1 = \bigoplus_{J \not= \{1\}} \mathcal{C}^\infty(U)[\lambda]\theta_1^1 \oplus \bigoplus_{J \not= \{1\}} \mathcal{C}^\infty(u^2, \ldots, u^N)[\lambda]\theta_1^1,$$

where $\mathcal{C}^\infty(u^2, \ldots, u^N)$ denotes the functions in $\mathcal{C}^\infty(U)$ which are constant in $u^1$.

On the other hand, we clearly have

$$\im \delta_1 = \bigoplus_{J \not= \{1\}} (u^1 - \lambda)\mathcal{C}^\infty(U)[\lambda]\theta_1^1,$$

therefore the first page of the spectral sequence is

$$H(\hat{C}[\lambda], \delta_1) = \bigoplus_{J \not= \{1\}} \frac{\mathcal{C}^\infty(U)[\lambda]}{(u^1 - \lambda)\mathcal{C}^\infty(U)[\lambda]}\theta_1^1 \oplus \bigoplus_{J \not= \{1\}} \mathcal{C}^\infty(u^2, \ldots, u^N)[\lambda]\theta_1^1.$$ 

As these arguments do not depend on the $\theta_i^1$ for $i \neq 1$ in any way, on the first page of the spectral sequence we can use the $\theta_1^1$ filtration and use the same arguments to find the first page of its spectral sequence. Completing the induction, we get the
following result for the $\tilde{\Delta}_{0,1}$-cohomology on $\ tilde{\mathcal{C}}[\lambda]$:

$$
\bigoplus_{J \subseteq \{1, \ldots, N\}} \frac{C^\infty((u^J)_J)[\lambda]}{\sum_{j \in J} (\lambda - \theta^j)}
$$

where the sum in the denominator is an ideal sum. If $|J| \geq 2$, this ideal sum contains the invertible element $u^j - u^j = (u^j - \lambda) - (u^j - \lambda)$ for $i, j \in J$, so the cohomology is zero. The cohomology of $\tilde{\Delta}_{0,1}$ is therefore nontrivial only in degree zero, where it equals $\mathbb{R}[\lambda]$, and in degree one, where it is given by the sum $\bigoplus_{i=1}^N C^\infty(u^i)\theta^i_1$. To find the cohomology of $\Delta_{0,1}$ we need to take into account the action of the operator $\Psi$. Hence the cohomology of $\Delta_{0,1}$ in degree one is $\bigoplus_{i=1}^N C^\infty(u^i)f^i(\theta^i_1)$.

The proposition is proved.

\[\square\]

6. A vanishing result for $1E_2$ at $(p, d) = (3, 2)$

We now go back to the first spectral sequence $1E$ associated with $\ deg_u$ in §3.1 and prove a vanishing result for its second page.

Proposition 6.1. The cohomology of the complex $(H(\tilde{\mathcal{A}}[\lambda], \Delta_{-1}), \Delta_0)$ vanishes in degree $(p, d) = (3, 2)$.

Proof. In §3.3 the vanishing result for $1E_2$ is proved by introducing a filtration in the degree $\ deg_\theta$. In order to extend the vanishing to the case $(p, d) = (3, 2)$, we split the differential $\Delta_0$ in a different way. Recall that the operator $\Delta_0$ is by definition the homogeneous component of $D_1$ of degree $\ deg_u$ equal to zero. It induces a differential on the first page $1E_1$ of the first spectral sequence, that is on the cohomology $H(\tilde{\mathcal{A}}[\lambda], \Delta_{-1})$ given by $\Theta$.

From Proposition 4.3 we know that the cohomology of this complex is vanishing for $\ deg_u$ positive. We can therefore limit our attention to the subcomplex with $\ deg_u$ equal to zero

$$
1E_1^0 = \tilde{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^N \frac{\tilde{\mathcal{C}}[[\theta^{\geq 2}_i]]^{nt}[\lambda]}{(\lambda - u^i)\tilde{\mathcal{C}}[[\theta^{\geq 2}_i]]^{nt}[\lambda]},
$$

where the superscript in $\tilde{\mathcal{C}}[[\theta^{\geq 2}_i]]^{nt}$ indicates that every monomial should include at least one $\theta^{\geq 2}_i$.

Let us denote by $\ deg_\theta$ the degree that counts the number of $\theta^i_j$, $j = 1, \ldots, N$, and split $\Delta_0$ its homogeneous components

$$
\Delta_0 = \Delta^1_0 + \Delta^0_0,
$$

where $\ deg_\theta \Delta^k_0 = k$.

The decreasing filtration on $1E_1^0$ associated to the degree $\ deg_\theta - \ deg_\theta$ induces a spectral sequence $4E$, whose zero page is clearly $1E_1^0$, with differential $4d_0 = \Delta^0_0$. The first page $4E_1$ is given by the cohomology of $(1E_1^0, \Delta^1_0)$ which we now consider.

The form of $\Delta^1_0$ can be easily derived from the explicit expression of $\Delta_0$, see appendix A. When acting on $1E_1^0$ it simplifies to the following operator, which for simplicity we still denote $\Delta^1_0$:

$$
\Delta^1_0 = \frac{1}{2} \sum_j \theta_j \sum_{s \geq 1} \theta_j^{s+1} \partial \theta_i^s.
$$
with
\[ \tilde{\partial}_0^i := f^i \theta_0^i + \sum_{j \neq i} (u^j - u^i) f^j \frac{\partial f^i}{\partial f^j}. \]

We consider now the spectral sequence on \( E_1^0 \) induced by the degree \( \deg_{\theta^{(2)}} \), which assigns degree one to all \( \theta_i \) with \( s > 2 \). Let \( \Delta_0^1 = \Delta_0^{1,0} + \Delta_0^{1,1} \) where
\[ \Delta_0^{1,0} = \frac{1}{2} \sum_i \tilde{\partial}_0^i \sum_{s \neq 2} \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s}, \quad \Delta_0^{1,1} = \frac{1}{2} \sum_i \theta_i^2 \theta_i \frac{\partial}{\partial \theta_i^2}, \]
are of degree \( \deg_{\theta^{(2)}} \). Let \( \Delta_0^{1,k} = k \).

We can rewrite our complex as
\[ \tilde{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^N \bigoplus_{k \geq 1} \frac{\tilde{\mathcal{C}}[\theta_i^{(2)}][k][\lambda]}{(\lambda - u^i)\tilde{\mathcal{C}}[\theta_i^{(2)}][k][\lambda]}, \]
where \( \tilde{\mathcal{C}}[\theta_i^{(2)}][k] \) denotes the homogeneous polynomials with \( \deg_{\theta^{(2)}} \) equal to \( k \).

Each of the summands is invariant under \( \Delta_0^{1,0} \), so it forms a subcomplex whose cohomology we can compute independently. Notice that the differential vanishes on \( \tilde{\mathcal{C}}[\lambda] \), while it acts like multiplication by \( \tilde{\partial}_0^i \) on the \( k = 1 \) subcomplex
\[ \tilde{\mathcal{C}} \theta_i^2 \to \tilde{\mathcal{C}} \theta_i^3 \to \tilde{\mathcal{C}} \theta_i^4 \to \cdots, \]
which is therefore acyclic except for the first term, where the cohomology is given by the kernel of the multiplication map, i.e., the ideal of \( \tilde{\partial}_0^i \) in \( \tilde{\mathcal{C}} \) multiplied by \( \theta_i^2 \).

The first page of the spectral sequence is therefore given by
\[ \tilde{\mathcal{C}}[\lambda] \oplus \bigoplus_{i=1}^N \bigoplus_{k \geq 1} \frac{\tilde{\mathcal{C}} \theta_i^2[\lambda]}{(\lambda - u^i)\tilde{\mathcal{C}} \theta_i^2[\lambda]} \oplus \bigoplus_{k \geq 2} \bigoplus_{l \geq 1} H(\tilde{\mathcal{C}}[\theta_i^{(2)}][k], \Delta_0^{1,0}). \] (12)

While it is not difficult to compute the cohomology groups appearing in the third summand, it can be easily seen that they give no contribution to \( E_1^2 \). Indeed, we know from proposition 4.3 that the cohomology with standard degree \( d \geq 4 \) is a subquotient of \( \tilde{\mathcal{C}}[\lambda] \), but the minimal degree of elements in the third summand above is \( d = 5 \).

On this page the differential is induced by \( \Delta_0^{1,1} \), which has \( \deg_{\theta^{(2)}} \) equal to one. When acting on the second summand \( \tilde{\partial}_0^i \theta_i^2 \), it vanishes, since it produces a mixed term \( \theta_i^2 \theta_i^2 \) which cannot be in \( \tilde{\mathcal{C}}[\theta_i^{(2)}][k] \) for \( k > 2 \). Therefore the cohomology of the first two summands is determined by the kernel and the image of the map
\[ \Delta_0^{1,1} : \tilde{\mathcal{C}}[\lambda] \to \bigoplus_i \frac{\tilde{\mathcal{C}} \theta_i^2[\lambda]}{(\lambda - u^i)\tilde{\mathcal{C}} \theta_i^2[\lambda]}. \]

The image can be computed in the following way: first of all, it is clear that an element in the image is a linear combination of \( \theta_i^2, i = 1, \ldots, N \), where the coefficient of each \( \theta_i^2 \) does not depend on \( \theta_i^1 \) and is in the ideal generated by \( \tilde{\partial}_0^i \) in \( \tilde{\mathcal{C}} \). Therefore the image is a subspace of
\[ \bigoplus_{i=1}^N \frac{\tilde{\mathcal{C}} \theta_i^2[\lambda]}{(\lambda - u^i)}. \] (13)
where $\hat{C}_1$ is the subspace of $\hat{C}$ generated by monomials that do not depend on $\theta^1_i$.

Second, it is sufficient to consider the fact that the image of the ideal $\prod_{j \neq i} (-\lambda + u^j)\hat{C}[[\lambda]]$ under $\Delta_{0,1}^{1,1}$ is

$$\frac{\hat{C}_1 \cdot \partial^i \theta^j_1 [\lambda]}{(\lambda - u^i)}$$

to conclude that the image of $\Delta_{0,1}^{1,1}$ is the whole space (13).

So, the cohomology of $\Delta_{0,1}^{1,1}$ on the second term in (12) is

$$\bigoplus_{i=1}^N \frac{\hat{C}_1 \cdot \partial^i \theta^j_1 [\lambda]}{(\lambda - u^i)}.$$

In particular, we see that it cannot give any contribution to the cohomology of degree $(p, d) = (3, 2)$.

The second page of the spectral sequence associated to $\deg_{\theta^2}$ is

$$\text{Ker} \Delta_{0,0}^{1,1} \subset \hat{C}[[\lambda]] \oplus \bigoplus_i \frac{\hat{C}_i \cdot \partial^i \theta^j_1 [\lambda]}{(\lambda - u^i)} \oplus \bigoplus_{k \geq 2} \bigoplus_i \bigoplus_{\lambda \geq 0} H(\hat{C} \cdot \theta^2_1 [\lambda]) \oplus \Delta_{0,0}^{1,1},$$

where, as discussed before, the third summand does not give any contribution to $^4E_2$, and can therefore be ignored here. Since $\Delta_{0,0}^{1,1}$ vanishes on this page, (14) gives the cohomology of $(^4E_1, \Delta_{0,0}^{1,1})$ which coincides with the first page $^4E_1$ of the spectral sequence $^4E$.

The differential $^4d_1$ on $^4E_1$ is the one induced by $\Delta_{0,0}^{1,1}$, the degree $\deg_{\theta^2}$ zero part of $\Delta_0$. The three summands in (14) are invariant under the action of the differential $\Delta_{0,0}^{1,1}$, which in particular vanishes on the second term. To see this observe that since the standard degree of the second term is $d = 3$ and that of the third term is $d \geq 5$, there can be no terms mapped between these two spaces by $\Delta_{0,0}^{1,1}$, nor from the second space to itself. The third term cannot map to the first one, since $\Delta_{0,0}^{1,1}$ cannot remove more than one $\theta^2$.

The operator $\Delta_{0,0}^{1,1}$ has to increase the standard degree and the $\theta$ degree by one, while keeping $\deg_{\theta^2}$ unchanged. This can only be achieved on $\hat{C}[[\lambda]]$ by increasing $\deg_{\theta^2}$ by one, therefore $\Delta_{0,0}^{1,1} = \Delta_{0,1,1}$, which is given in Lemma 5.6. Explicitly:

$$\Delta_{0,0}^{1,1} = (u - \lambda) f^l j^j_k \frac{\partial}{\partial u^l} - (\partial_i \log f^k) \theta^i_k \frac{\partial}{\partial \theta^i_k} - \frac{1}{2} (\partial_i \log f^k) \theta^i_k \frac{\partial}{\partial \theta^i_k}$$

$$- \frac{1}{2} (u - \lambda) \partial_i f^j \theta^j_k \frac{\partial}{\partial \theta^i_k} + \frac{1}{2} f^i \theta^j_k \theta^j_k \frac{\partial}{\partial \theta^i_k} + (u - \lambda) f^j \partial_j \frac{\partial}{\partial \theta^i_k}.$$
a sum over all subsets \( I \subset \{1, \ldots, n\}, |I| = t \), of the elements of the form

\[
\sum_{j=1}^{n} F_j(u, \lambda) \partial^0_j \cdot \prod_{i \in I} (-\lambda + u^i) \partial^1_i + \sum_{i \in I} G(u) \partial^0_i \cdot \prod_{j \neq i} (-\lambda + u^j) \partial^1_j.
\]

This representation naturally splits the kernel of \( \Delta^{1,1}_{0,1} \) into two summands, let us call them \( F \) and \( G \).

Observe that the splitting of the \( p = d + 1 \) part of the kernel of \( \Delta^{1,1}_{0,1} \) on \( \tilde{C}[\lambda] \) into the direct sum \( F \oplus G \) defines a filtration for the operator \( \Delta^0_{0,1} \). We can see this by using the base change \( \Psi \). First, define

\[
\tilde{\partial}^0_i := \Psi^{-1} \partial^0_i = \partial^0_i + 2(u^i - u^j) \gamma_{ij} \partial^1_i
\]

From the formula above for \( \Delta^0_{0,1} \) we have that we can write \( \Delta^0_{0,1} = \Psi \tilde{\Delta} \Psi^{-1} \), for

\[
\tilde{\Delta} = (u^i - \lambda) \partial^1_i \frac{\partial}{\partial u^i} + (u^i - \lambda) \gamma_{ij} \partial^1_i \frac{\partial}{\partial u^j} - (u^i - \lambda) \gamma_{ij} \partial^1_i \frac{\partial}{\partial u^j} + \frac{1}{2} \partial^0_i \partial^1_i \frac{\partial}{\partial \partial^0_i}.
\]

The first three terms preserve \( F = \Psi^{-1} F \), while the last sends \( F \) to \( G := \Psi^{-1} G \). Moreover, the entire operator preserves \( G \). Furthermore, the parts \( F \to F \) and \( G \to G \) form deformed de Rham differentials \( d + A \). Therefore, the only possible cohomology is in the lowest degree in \( \theta^i \), which is zero for \( F \) and 1 for \( G \). So, only nontrivial cohomology in the case \( p = d+1 \) is possible in the degree \((t+1, t) = (1, 0)\) and \((t+1, t) = (2, 1)\). This implies the the cohomology of degree \((3, 2)\) is equal to zero.

**Remark 6.2.** Note that it is not clear from the definitions that \( \tilde{\Delta} G \subset \tilde{G} \). But it must be there because of the restriction of the degree in \( \lambda \) and the fact that \( \Delta^0_{0,1} \) must preserve the kernel of \( \Delta^{1,1}_{0,1} \). The more direct proof requires the flatness equations for \( f^i \). We give this calculation in appendix \( \mathbb{A} \).

### 7. Main result: the cohomology of \((\tilde{A}[\lambda], D_{\lambda})\)

Let us combine the results of the previous sections to compute the cohomology of the complex \((\tilde{A}[\lambda], D_{\lambda})\). Our main result is the following:

**Theorem 7.1.** For \( p = d \), the cohomology of \( \tilde{A}[\lambda] \) is given by:

\[
H^p_{\theta}(\tilde{A}[\lambda], D_{\lambda}) \cong \begin{cases} \mathbb{R}[\lambda] & p = 0, \\ \bigoplus_{i=1}^{N} C^\infty(u^i) \partial^1_i \theta^1_i & p = 3, \\ 0 & \text{else}. \end{cases}
\]

**Proof.** As observed in \( \S 3.4 \) the first page \( \tilde{2}E_1 \) is given by the direct sum \( \bigoplus \). From propositions \( 4.3 \) and \( 5.1 \) we get

\[
\tilde{2}E_1^p \cong \begin{cases} \mathbb{R}[\lambda] & p = 0, \\ \bigoplus_{i=1}^{N} C^\infty(u^i) \partial^1_i & p = 1, \\ \bigoplus_{i=1}^{N} C^\infty(u^i) \partial^1_i \theta^1_i & p = 2, \\ \bigoplus_{i=1}^{N} C^\infty(u^i) \partial^1_i \theta^1_i \theta^2_i & p = 3, \\ 0 & \text{else}. \end{cases}
\]
On this first page, the differential $^2d_1$ must lower the spectral sequence degree $\text{deg}_{E_0}$ by one, in other words, since the differential must still be of bidegree $(1, 1)$, it must leave the degree $\text{deg}_{E_0}$ unchanged, which is impossible on this subcomplex. Hence, the differential $^2d_1$ is equal to zero, and $(^2E_2)_p^i \cong (^2E_1)_p^i$.

On the second page, the differential $^2d_i$ must lower the spectral sequence degree by two, i.e., it must be of degree $\text{deg}_{E_0}$ equal to $-1$. Therefore, on this subcomplex the differential can only be non-trivial between $p = 1$ and $p = 2$. Looking back at the formula for $\Delta_0$, one can easily identify the terms of degree $\text{deg}_{E_0} = -1$, which give

$$\Delta_{0,-1} = \sum_i \frac{1}{2} \left[ \sum_j (u^j - \lambda)(\partial_i f^j \partial_i^2 + f^j \partial_i^j f^i - (\partial_i \partial_i^2 f^i - \partial_i^i \partial_i^j f^j)) \right] \frac{\partial}{\partial \theta_i^j}.$$ 

$\Delta_{0,-1}$ induces an operator on $H(\hat{\mathcal{A}}(\lambda), \Delta_{-1})$. Since we are interested only in the differential at degree $p = 1$, we need to consider just the action of such operator on $\hat{\mathcal{C}}(\lambda)$, which is, taking into account the identification $\theta_0 = f^i i^2$, the only surviving term is $\frac{1}{2} f^i \partial_i^j f^i \theta_i^j \frac{\partial}{\partial \theta_i^j}$, which gives an isomorphism $^2d_2 : (^2E_2)^1 \rightarrow (^2E_2)^2$.

The differential is therefore zero on $(^2E_2)_0^p$ for $p \neq 1$ and an isomorphism for $p = 1$, so $(^2E_2)_0^p$ is zero unless $p = 0$ or $p = 3$, when it is equal to $(^2E_2)_1^p$. This spectral sequence has no other non-trivial differentials, so $(^2E_2)_p^i$ has the same form. As $^2\mathcal{E} \Rightarrow 1\mathcal{E}$, we get that $(1E_2)_p^i$ is of this form as well. Because all differentials must have $(p, d)$-bidegree $(1, 1)$, there can be no higher non-trivial differentials on this part of the first spectral sequence. Now, $1\mathcal{E} \Rightarrow H(\hat{\mathcal{A}}(\lambda),\Delta_{-1})$, yielding the result.

Notice that can also extend our vanishing result: the vanishing at degrees $d = N, N + 1$ follows from proposition 4.3, and the vanishing at $(3, 2)$ follows from proposition 6.1.

**Proposition 7.2.** The cohomology $H_p(\hat{\mathcal{A}}(\lambda),\Delta_{-1})$ vanishes for

$$\begin{align*}
\text{if } p < d, & \quad d > 0, \\
\text{if } p > d + N, & \quad d > 0, \\
\text{if } d < p \leq d + N, & \quad d > \max(3, N), \\
\text{if } p = 3, & \quad d = 2.
\end{align*}$$

**Remark 7.3.** Observe that the cohomology is still unknown on the subcomplexes $p = d + 1, \ldots, N$ for $d < N$, unless $(p, d) = (3, 2)$. 
Appendix A. Formula for and calculations with $\Delta_0$

We recall from \cite{4} the formula for the degree $\deg_u$ zero part of the operator $D_u$.

$$\Delta_0 = (-\lambda + u')f^i\theta_i^1 \frac{\partial}{\partial u^i}$$

$$+ \sum_{s=a+b} (-\lambda + u') \left( \frac{s}{b} \right) \theta_j f^i u^j \theta_i^{1+b} \frac{\partial}{\partial u^i} + \sum_{s=c+d} \left( \frac{s}{b} \right) f^i u^d \theta_i^{1+c} \frac{\partial}{\partial u^i}$$

$$\frac{1}{2} \sum_{s=a+b} (-\lambda + u') \left( \frac{s}{b} \right) \theta_j f^i \frac{\partial f^j}{\partial u^i} \theta_i^{1+b} \frac{\partial}{\partial u^i} + \frac{1}{2} \sum_{s=c+d} \left( \frac{s}{b} \right) f^i u^d \theta_i^{1+c} \frac{\partial}{\partial u^i}$$

$$\frac{1}{2} \sum_{s=a+b} (-\lambda + u') \left( s \right) \frac{\partial f^j}{\partial u^i} \theta_j f^i \theta_i^{1+b} \frac{\partial}{\partial u^i} - \frac{1}{2} \sum_{s=c+d} \left( s \right) f^i \theta_i^{1+c} \frac{\partial}{\partial u^i}$$

The proof that $\Delta \mathcal{G} \subset \mathcal{G}$ in proposition \cite{6.1} goes as follows:

**Lemma A.1.** The operator $\Delta$ preserves $\mathcal{G}$, where

$$\Delta = (u' - \lambda) \frac{\partial}{\partial u^i} + (u' - \lambda) \gamma_{ij} \theta_i^1 \theta_j^1 \frac{\partial}{\partial \theta_i^1} - (u' - \lambda) \gamma_{ij} \theta_i^1 \theta_j^1 \frac{\partial}{\partial \theta_i^1} + \frac{1}{2} \theta_i^1 \frac{\partial}{\partial \theta_i^1}$$

and

$$\mathcal{G} = \bigoplus_{i=1}^N C^\infty(U) \left\{ |(u' - \lambda)\theta_i^1| \right\}_{i=1}^N \theta_i^1$$

**Proof.** When calculating the action of $\Delta$ on an element of the form $G(u) \theta_i^1 \theta_i^1 \in \mathcal{G}$, we get the following (where $i$ is a fixed index, and $k$, $l$, and $m$ are summed over)

$$\Delta G(u) \theta_i^1 \theta_i^1 = \frac{\partial}{\partial u^i} (G(\theta_i^1 + 2(u' - \lambda) \gamma_{ij} \theta_j^1) \theta_i^1 (u' - \lambda) \theta_k^1$$

$$+ G \gamma_{ik} \theta_k^1 \frac{\partial}{\partial \theta_i^1} (\theta_i^1 + 2(u' - \lambda) \gamma_{ij} \theta_j^1) \theta_i^1 (u' - \lambda) \theta_k^1$$

$$- G \gamma_{ik} \theta_k^1 \frac{\partial}{\partial \theta_i^1} (\theta_i^1 + 2(u' - \lambda) \gamma_{ij} \theta_j^1) \theta_i^1 (u' - \lambda) \theta_k^1$$

$$+ \frac{1}{2} G \frac{\partial}{\partial \theta_i^1} (\theta_i^1 + 2(u' - \lambda) \gamma_{ij} \theta_j^1) \theta_i^1 \theta_i^1$$
Multiplying with a factor

By equation (3), we get

conclude that the part of the third term adds up to the sixth term.

\[ G(\theta_{i}^{j} \theta_{k}^{j}) = \frac{\partial G}{\partial \theta_{i}^{j}} \theta_{k}^{j} (u^{k} - \lambda) \theta_{k}^{j} + 2G_{\gamma \kappa l} \theta_{k}^{j} \theta_{l}^{j} (u^{k} - \lambda) \theta_{k}^{j} + 2G(u^{k} - u^{l}) G_{\gamma \kappa l} \theta_{k}^{j} \theta_{l}^{j} (u^{k} - \lambda) \theta_{k}^{j} + G(u^{l} - \lambda) \gamma_{\kappa l} \theta_{k}^{j} \theta_{l}^{j} \]

Using equation (1) for the third term if \( i, k, l \) distinct, that part of the third term adds up to the sixth term.

\[ \Delta G(\theta_{i}^{j} \theta_{k}^{j}) = \frac{\partial G}{\partial \theta_{i}^{j}} \theta_{k}^{j} (u^{k} - \lambda) \theta_{k}^{j} + 2G_{\gamma \kappa l} \theta_{k}^{j} \theta_{l}^{j} (u^{k} - \lambda) \theta_{k}^{j} + 2G(u^{k} - u^{l}) G_{\gamma \kappa l} \theta_{k}^{j} \theta_{l}^{j} (u^{k} - \lambda) \theta_{k}^{j} + G(u^{l} - \lambda) \gamma_{\kappa l} \theta_{k}^{j} \theta_{l}^{j} \]

By the definition of \( \theta_{k}^{j} \), the last two terms drop out against half of the second term. So we get

\[ \Delta G(\theta_{i}^{j} \theta_{k}^{j}) = \frac{\partial G}{\partial \theta_{i}^{j}} \theta_{k}^{j} (u^{k} - \lambda) \theta_{k}^{j} + G_{\gamma \kappa l} \theta_{k}^{j} \theta_{l}^{j} (u^{k} - \lambda) \theta_{k}^{j} + 2G_{\gamma \kappa l} \theta_{k}^{j} \theta_{l}^{j} (u^{k} - \lambda) \theta_{k}^{j} + G(u^{l} - \lambda) \gamma_{\kappa l} \theta_{k}^{j} \theta_{l}^{j} \]

By equation (3), we get

\[ \Delta G(\theta_{i}^{j} \theta_{k}^{j}) = \frac{\partial G}{\partial \theta_{i}^{j}} \theta_{k}^{j} (u^{k} - \lambda) \theta_{k}^{j} - Gu^{l} \frac{\partial G}{\partial \theta_{k}^{j}} \theta_{k}^{j} (u^{k} - \lambda) \theta_{k}^{j} - G_{\gamma \kappa l} \theta_{k}^{j} \theta_{l}^{j} (u^{k} - \lambda) \theta_{k}^{j} - G(u^{l} - \lambda) \gamma_{\kappa l} \theta_{k}^{j} \theta_{l}^{j} \]

Applying equation (2) gives

\[ \Delta G(\theta_{i}^{j} \theta_{k}^{j}) = \frac{\partial G}{\partial \theta_{i}^{j}} \theta_{k}^{j} (u^{k} - \lambda) \theta_{k}^{j} - G_{\gamma \kappa l} (u^{l} - \lambda) \theta_{k}^{j} \theta_{l}^{j} \theta_{k}^{j} \]

Multiplying with a factor \( \prod_{\lambda \neq j} (u^{j} - \lambda) \theta_{j}^{j} \) does not change the calculation, so we can extend this calculation to all of \( G \), showing that \( \Delta \) does indeed preserve this space.

\( \square \)

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