AFFINE MOTION OF 2D INCOMPRESSIBLE FLUIDS
SURROUNDED BY VACUUM AND FLOWS IN SL(2, R)

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Abstract. The affine motion of two-dimensional (2d) incompressible fluids surrounded by vacuum can be reduced to a completely integrable and globally solvable Hamiltonian system of ordinary differential equations for the deformation gradient in SL(2, R). In the case of perfect fluids, the motion is given by geodesic flow in SL(2, R) with the Euclidean metric, while for magnetically conducting fluids (MHD), the motion is governed by a harmonic oscillator in SL(2, R). A complete classification of the dynamics is given including rigid motions, rotating eddies with stable and unstable manifolds, and solutions with vanishing pressure. For perfect fluids, the displacement generically becomes unbounded, as \( t \to \pm \infty \). For MHD, solutions are bounded and generically quasi-periodic and recurrent.

Contents

0. Introduction 2
1. Matrix inner product space and groups 3
2. The geometry of \( \text{SL}(2, \mathbb{R}) \) 5
3. The equations of affine motion 7
4. Existence of global solutions 10
5. Hamiltonian formalism 11
6. Reduction to the phase plane in the case \( X_2 \neq 0 \) 14
7. Reduction to the phase plane in the case \( X_2 = 0 \) 17
8. The Lagrange multiplier 20
9. Rigid solutions 22
10. Asymptotic behavior for MHD, \( \kappa > 0 \) 24
11. Asymptotic behavior for perfect fluids, \( \kappa = 0 \) 29
12. The picture in the tangent space 32
Appendix A. Glossary of notation 36
References 37

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0. Introduction

The equations of motion for the velocity $u$, the magnetic field $b$, and the pressure $p$ of a 2d incompressible magnetically conducting fluid are

\begin{align}
D_t u &= -\nabla p + b \cdot \nabla b \\
D_t b &= b \cdot \nabla u \\
\nabla \cdot u &= \nabla \cdot b = 0.
\end{align}

Here, $D_t = \partial_t + u \cdot \nabla$ is the material time derivative. The system \((0.1)\) is to be solved in a space-time domain of the form $\{(x,t) \in \mathbb{R}^2 \times I : x \in \Omega(t)\}$, where $I \subset \mathbb{R}$ is some time interval. The equations are supplemented by the vacuum free boundary conditions

\begin{align}
0.2 \quad p = 0 \text{ and } b \cdot n = 0 \text{ on } \partial \Omega(t),
\end{align}

where dot “·” denotes the inner product on $\mathbb{R}^2$. The free boundary $\partial \Omega(t)$ is also assumed to move with the fluid. When the magnetic field $b$ vanishes identically, the system reduces to the incompressible Euler equations, which will treated as a distinct case.

For initial data satisfying the Rayleigh-Taylor sign condition, local well-posedness for the incompressible free boundary Euler equations with bulk vorticity was established in [2], [6], [3], [7], [11], [4] and for the incompressible free boundary MHD problem in [5], [10].

In this article we explore the affine motion of incompressible planar fluids surrounded by vacuum. An affine motion is one whose deformation and velocity gradients depend only on time. The use of affine deformations is a well-established tool in continuum mechanics, first introduced in the context of the vacuum free boundary incompressible Euler system in [8], [9]. Under this assumption, the fluid equations \((0.1), (0.2)\) reduce to a globally solvable constrained Lagrangian system of ordinary differential equations \((3.1)\) for the deformation gradient in the special linear group $\text{SL}(2, \mathbb{R})$. The natural phase space is the 6-dimensional tangent bundle of $\text{SL}(2, \mathbb{R})$, which we regard as being embedded in $\mathbb{R}^8$ with the Euclidean metric. This system possesses three integrals of motion corresponding to conservation of energy and invariance under the left and right action of the special orthogonal group $\text{SO}(2, \mathbb{R})$.

Using a suitable of parameterization of $\text{SL}(2, \mathbb{R})$ followed by a Legendre transformation, we obtain an equivalent completely integrable Hamiltonian system \((5.3)\). We shall then provide a complete description of all such motions in terms of the values of the invariants in both cases: MHD and Euler.

Taking the unit disk as the reference domain, the time-dependent fluid domains arising from incompressible affine motion are ellipses of constant area. The principle axes of these fluid ellipses are determined by the eigendirections and eigenvalues of the stretch tensor. Incompressible affine motion allows for compression along one axis and expansion along the other, combined with rigid rotation of the reference and fluid domains.

Once the deformation gradient and fluid domains are known, the pressure, velocity, and magnetic field, if present, are recovered through explicit formulae, taking into account the boundary conditions. The sign of the pressure is preserved by the motion. There exist special solutions whose pressure vanishes identically. In this case, the equations of motion are linear, and solutions may be found explicitly.

The affine motion of incompressible perfect fluids in the plane is described by geodesic flow for the deformation gradient in $\text{SL}(2, \mathbb{R})$, echoing the classic result of Arnold on geodesic flow in the space of volume preserving diffeomorphims [1]. By energy conservation, the material
velocity is bounded. However, the fluid domain becomes unbounded, generically, as $|t| \to \infty$, and its diameter grows linearly in time approaching infinite eccentricity. Additionally, there is an invariant manifold of initial data leading to rigidly rotating fluid disks (eddies) of arbitrary angular velocity. These solutions are represented by curves in $\text{SO}(2, \mathbb{R})$. The manifold of rotational solutions is hyperbolic, possessing both stable and unstable invariant manifolds. Solutions on these manifolds are semi-bounded, and they decay exponentially to a rotating disk, as $t \to \infty$ in the stable case and as $t \to -\infty$ in the unstable case. Unbounded solutions are asymptotic to straight lines in the space of $2 \times 2$ matrices. In the special case of vanishing pressure, solutions coincide with geodesic lines in $\text{SL}(2, \mathbb{R})$.

The affine motion of incompressible magnetically conducting planar fluids can be viewed as a simple harmonic oscillator, constrained to $\text{SL}(2, \mathbb{R})$, through the addition of a one-parameter restoring force to the equations of geodesic motion. Here, all solutions are bounded and, generically, quasi-periodic and recurrent. There exist rigid solutions with fluid ellipses of arbitrary eccentricity. Included among these are rotating disks of arbitrary angular velocity. For sufficiently large angular velocities, the rotating disk solutions possess a homoclinic invariant manifold. Solutions on this manifold are asymptotic to a pair of rotational solutions, with an exponential convergence rate, as $t \to \pm \infty$.

The main results concerning the asymptotic behavior of solutions for MHD and perfect fluids are given in Sections 10 and 11 respectively. Up to that point, the exposition for the two cases is presented in parallel. The initial sections are devoted to the algebraic and geometric properties of the phase space for the system of ordinary equations describing the motion of the deformation gradient. In Section 3 the primary equations (3.1) are derived from the fundamental fluid equations (0.1). The global existence theorem is presented in the next section along with a discussion of the integral invariants of this system. We then devote considerable time developing the Hamiltonian structure of the equations of motion. In Section 5, we derive an equivalent completely integrable Hamiltonian system (5.3). The qualitative behavior of this system is decisively influenced by the values of the invariants, and in particular, the presence or absence of solutions which take values in $\text{SO}(2, \mathbb{R})$. These two cases are examined in Sections 6 and 7 respectively. The results given in Theorems 6.12 and 7.5 hinge on the fact that the systems (6.1) and (7.1) each partially uncouple, allowing for a complete phase plane analysis. The remainder of the paper is devoted to cataloging the rich behavior of solutions for all admissible initial conditions. For example, we find solutions with vanishing pressure, rigid solutions, and invariant manifolds for rigid rotations, and in Section 12 we classify the initial velocities in the tangent space of the initial position which produce these distinguished solutions. For the convenience of the reader, a glossary of notation appears at the end of the paper in Appendix A.

1. Matrix inner product space and groups

Definition 1.1. By $\mathbb{M}^2$, we denote the set of $2 \times 2$ matrices over $\mathbb{R}$ with the Euclidean inner product

$$\langle A, B \rangle = \sum_{ij} A_{ij} B_{ij} = \text{tr} A^\top B$$

and norm

$$|A| = \langle A, A \rangle^{1/2}.$$
Lemma 1.2. For all $A, B, C \in \mathbb{M}^2$,\[
\langle AB, C \rangle = \langle B, A^\top C \rangle = \langle A, CB^\top \rangle.
\]

Definition 1.3. Define the following vectors in $\mathbb{M}^2$:
\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Lemma 1.4. The vectors $I, Z, K, M$ are an orthogonal basis for $\mathbb{M}^2$ satisfying the relations
\[
Z^2 = -I, \quad K^2 = M^2 = I, \quad ZK = M, \quad MK = Z, \quad MZ = K.
\]

Definition 1.5. The special linear group is given by
\[
\text{SL}(2, \mathbb{R}) = \{ A \in \mathbb{M}^2 : \det A = 1 \}
\]
and the special orthogonal group is
\[
\text{SO}(2, \mathbb{R}) = \{ U \in \text{SL}(2, \mathbb{R}) : U^{-1} = U^\top \}.
\]

Lemma 1.6. For all $A \in \mathbb{M}^2$ and $U, V \in \text{SO}(2, \mathbb{R})$,
\[
|UAV| = |A|.
\]
The left and right actions of $\text{SO}(2, \mathbb{R})$ on $\mathbb{M}^2$ and on $\text{SL}(2, \mathbb{R})$ are isometries.

The subgroup $\text{SO}(2, \mathbb{R})$ will play a special role in the sequel. Here is the first of several characterizations that we shall repeatedly use.

Lemma 1.7. Elements of $\text{SO}(2, \mathbb{R})$ are norm minimizers in $\text{SL}(2, \mathbb{R})$. There holds
\[
\min \{ |A|^2 : A \in \text{SL}(2, \mathbb{R}) \} = 2
\]
and
\[
\text{SO}(2, \mathbb{R}) = \{ A \in \text{SL}(2, \mathbb{R}) : |A|^2 = 2 \}.
\]

Proof. Given $A \in \text{SL}(2, \mathbb{R})$, we have that
\[
\det A^\top A = 1,
\]
and so the eigenvalues of the positive definite symmetric matrix $A^\top A$ satisfy
\[
\lambda_1 \geq \lambda_2 > 0 \quad \text{and} \quad \lambda_1 \lambda_2 = 1.
\]
Therefore,
\[
|A|^2 = \text{tr} A^\top A = \lambda_1 + \lambda_2 \geq 2,
\]
with equality if and only if $A^\top A = I$. Using the polar decomposition, there exists $U \in \text{SO}(2, \mathbb{R})$ such that
\[
A = U(A^\top A)^{1/2}.
\]
So by Lemma 1.6, $|A|^2 = 2$ if and only if $A = U$. \hfill \Box

Definition 1.8. Define the one-parameter family of rotations
\[
U(\theta) = \exp(\theta Z) = \cos \theta I + \sin \theta Z = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \theta \in \mathbb{R}.
\]

Lemma 1.9. With this definition, we have $\text{SO}(2, \mathbb{R}) = \{ U(\theta) : \theta \in \mathbb{R} \}$. 

Lemma 1.10. The cofactor map \( \text{cof} : \mathbb{M}^2 \to \mathbb{M}^2 \) satisfies
\[
\text{cof} A = ZAZ^\top = Z^\top AZ.
\]
It is linear, symmetric, and orthogonal.

Proof. The identity is easily verified. By Lemma 1.2, we have that
\[
\langle \text{cof} A, B \rangle = \langle ZAZ^\top, B \rangle = \langle A, Z^\top BZ \rangle = \langle A, \text{cof} B \rangle.
\]
and
\[
\langle \text{cof} A, \text{cof} B \rangle = \langle A, \text{cof} \text{cof} B \rangle = \langle A, B \rangle.
\]
\[\square\]

Remark 1.11. The linearity of the cofactor map on \( \mathbb{M}^2 \) is a distinguishing feature of the planar case, and it plays a critical role in our analysis.

Lemma 1.12. For any \( A \in \mathbb{M}^2 \), we have
\[
\frac{1}{2} \det A = \langle A, \text{cof} A \rangle.
\]

Lemma 1.13. The determinant map \( \det : \mathbb{M} \to \mathbb{R} \) is \( C^\infty \) and
\[
\frac{\partial}{\partial A} \det A = \text{cof} A.
\]

Lemma 1.14. For all \( A, B \in \mathbb{M}^2 \), there holds
\[
\langle A, B \rangle \langle \text{cof} A, B \rangle + \langle ZA, B \rangle \langle AZ, B \rangle = \frac{1}{2} \langle \text{cof} A, A \rangle |B|^2 + \frac{1}{2} \langle \text{cof} B, B \rangle |A|^2.
\]
Proof. If we expand the vectors \( A, B \) in the basis of Definition 1.3
\[
A = \frac{1}{\sqrt{2}}(a_1 I + a_2 Z + a_3 K + a_4 M) \quad \text{and} \quad B = \frac{1}{\sqrt{2}}(b_1 I + b_2 Z + b_3 K + b_4 M),
\]
then both sides are equal to \( (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_3^2 + a_4^2)(b_3^2 + b_4^2) \).
\[\square\]

2. The geometry of \( SL(2, \mathbb{R}) \)

Lemma 2.1. The group \( SL(2, \mathbb{R}) \) is a smooth three-dimensional embedded submanifold of \( \mathbb{M}^2 \). The vector \( \text{cof} A \) is normal to \( SL(2, \mathbb{R}) \) at the point \( A \in SL(2, \mathbb{R}) \), and the tangent space to \( SL(2, \mathbb{R}) \) at \( A \) is
\[
T_A SL(2, \mathbb{R}) = \{ B \in \mathbb{M}^2 : \langle B, \text{cof} A \rangle = \text{tr} BA^{-1} = 0 \}.
\]
For each \( A \in SL(2, \mathbb{R}) \), \( T_A SL(2, \mathbb{R}) \) is a three-dimensional subspace in \( \mathbb{M}^2 \).

Proof. This follows from Definition 1.3 and Lemma 1.13 \[\square\]

Definition 2.2. Define the tangent bundle
\[
\mathcal{D} = \{(A, B) \in \mathbb{M}^2 \times \mathbb{M}^2 : A \in SL(2, \mathbb{R}), B \in T_A SL(2, \mathbb{R})\}.
\]
We also define
\[
\mathcal{D}_0 = \{(A, B) \in \mathcal{D} : A \in SO(2, \mathbb{R})\}.
\]

Lemma 2.3. \( \mathcal{D} \) is a smooth 6-dimensional embedded submanifold of \( \mathbb{M}^2 \times \mathbb{M}^2 \).
**Definition 2.4.** Define the smooth map \( \varphi : \mathbb{R}^3 \to \mathbb{M}^2 \) by
\[
\varphi(x) = \frac{1}{\sqrt{2}} (\rho(x) \cos x_3 I + \rho(x) \sin x_3 Z + x_1 K + x_2 M),
\]
with
\[
\rho(x) = (2 + x_1^2 + x_2^2)^{1/2}.
\]

**Remark 2.5.** We shall occasionally find it convenient to identify \( \mathbb{R}^3 \) with \( \mathbb{R}^2 \times \mathbb{R} \) by writing \( x = (x_1, x_2, x_3) = (\bar{x}, x_3) \). Thus, for example, we have \( \rho(x) = (1 + |\bar{x}|^2)^{1/2} \).

**Lemma 2.6.** The map \( \varphi \) is an immersion from \( \mathbb{R}^3 \) onto \( \text{SL}(2, \mathbb{R}) \),
\[
|\varphi(x)|^2 = \rho(x)^2 + |\bar{x}|^2 = 2(1 + |\bar{x}|^2),
\]
and \( \varphi(x) \in \text{SO}(2, \mathbb{R}) \) if and only if \( \bar{x} = (x_1, x_2) = 0 \).

**Proof.** Let \( A \in \mathbb{M}^2 \) and set
\[
\bar{A} = A - \frac{1}{\sqrt{2}} (x_1 K + x_2 M), \quad \text{with} \quad x_1 = \frac{1}{\sqrt{2}} \langle A, K \rangle \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}} \langle A, M \rangle.
\]
Then \( \langle \bar{A}, K \rangle = \langle \bar{A}, M \rangle = 0 \), and so \( \bar{A} \in \text{span}\{I, Z\} \). Thus, we can find \( \rho \geq 0 \) and \( x_3 \in \mathbb{R} \) such that
\[
\bar{A} = \frac{1}{\sqrt{2}} (\rho \cos x_3 I + \rho \sin x_3 Z).
\]
We now have
\[
A = \frac{1}{\sqrt{2}} (\rho \cos x_3 I + \rho \sin x_3 Z + x_1 K + x_2 M).
\]
It follows from Lemma 1.12 that
\[
\det A = \frac{1}{2} \langle A, \text{cof} A \rangle = \frac{1}{2} (\rho^2 - |\bar{x}|^2).
\]
Therefore, \( A \in \text{SL}(2, \mathbb{R}) \) if and only if \( \rho^2 = 2 + |\bar{x}|^2 \). This proves that \( \varphi : \mathbb{R}^3 \to \text{SL}(2, \mathbb{R}) \) is surjective.

Since \( |\varphi(x)|^2 = \rho(x)^2 + |\bar{x}|^2 = 2(1 + |\bar{x}|^2) \), Lemma 1.7 implies that \( \varphi(x) \in \text{SO}(2, \mathbb{R}) \) if and only if \( \bar{x} = 0 \).

For any \( x \in \mathbb{R}^3 \), it is straightforward to verify that the linear map \( D_x \varphi(x) : \mathbb{R}^3 \to \mathbb{M}^2 \) satisfies
\[
\ker D_x \varphi(x) = \{0\} \quad \text{and} \quad \text{Im} D_x \varphi(x) = (\text{span cof} \varphi(x))^\perp.
\]
By Lemma 2.1, this says that \( D_x \varphi(x) \) is a bijection from \( \mathbb{R}^3 \) onto \( T_{\varphi(x)} \text{SL}(2, \mathbb{R}) \), thereby proving that \( \varphi \) is an immersion.

**Lemma 2.7.** The map \( \varphi \) defines a local coordinate chart for \( \text{SL}(2, \mathbb{R}) \) in a neighborhood of any point \( x \in \mathbb{R}^3 \). The metric on \( T_{\varphi(x)} \text{SL}(2, \mathbb{R}) \) in these coordinates is given by
\[
g(x) = \begin{bmatrix} 1 + x_1^2 / \rho(x)^2 & 0 & 0 \\ x_1 x_2 / \rho(x)^2 & 1 + x_2^2 / \rho(x)^2 & 0 \\ 0 & 0 & \rho(x)^2 \end{bmatrix}.
\]

**Proof.** That \( \varphi \) defines local coordinate charts follows from Lemma 2.6. The form of the metric is \( g(x) = D_x \varphi(x)^T D_x \varphi(x) \), the components of which are \( g_{ij}(x) = \langle D_{x_i} \varphi(x), D_{x_j} \varphi(x) \rangle \).

**Remark 2.8.** For each \( r^2 > 1 \), the sphere \( \{ A \in \text{SL}(2, \mathbb{R}) : |A|^2 = r^2 \} \) corresponds to the torus \( \{ \varphi \in \mathbb{R}^4 : \varphi_1^2 + \varphi_2^2 = \frac{1}{r^2} r^2 + 1, \varphi_3^2 + \varphi_4^2 = \frac{1}{r^2} r^2 - 1 \} \).
Definition 2.9. Define the map \( \Phi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{M}^2 \times \mathbb{M}^2 \) by
\[
\Phi(x, y) = (\varphi(x), D_x\varphi(x)y).
\]

Lemma 2.10. \( \Phi \) maps \( \mathbb{R}^3 \times \mathbb{R}^3 \) onto \( \mathcal{D} \).

Proof. This follows from Lemma 2.6. \( \square \)

Remark 2.11. In this context, we can think of \( \mathbb{R}^3 \times \mathbb{R}^3 \) as the tangent bundle of \( \mathbb{R}^3 \), the linear map \( D_x\varphi(x) \) as the push-forward from \( T_x\mathbb{R}^3 \) to \( T_{\varphi(x)}\text{SL}(2, \mathbb{R}) \subset \mathbb{M}^2 \), and the linear map \( D_x\varphi(x)^\top \) as the pull-back from \( T_{\varphi(x)}\text{SL}(2, \mathbb{R}) \) to \( T_x\mathbb{R}^3 \).

3. The equations of affine motion

Definition 3.1. An incompressible affine motion defined on the unit disk \( \mathcal{B} \subset \mathbb{R}^2 \) is a one-parameter family of volume preserving diffeomorphisms of the form
\[
x(t, y) = A(t)y, \quad y \in \mathcal{B}, \quad t \in \mathcal{I},
\]
on some interval \( \mathcal{I} \subset \mathbb{R} \), with
\[
A \in C^0(\mathcal{I}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathcal{I}, \mathbb{M}^2).
\]

Here, \( \mathcal{B} \) is the reference domain, and the domain occupied by the material (fluid) at time \( t \) is the image
\[
\Omega(t) = A(t)\mathcal{B} = \{ x \in \mathbb{R}^2 : |A(t)^{-1}x|^2 \leq 1 \}.
\]
For affine motion, \( \Omega(t) \) is an ellipse centered at the origin with principal axes determined by the eigendirections and eigenvalues of the positive definite symmetric (stretch) matrix \( (A(t)A(t)^\top)^{1/2} \), for all \( t \in \mathbb{R} \).

The spatial velocity field associated to an affine motion is defined by
\[
u(t, x(t, y)) = \partial_t x(t, y) = \dot{A}(t)y, \quad y \in \mathcal{B},
\]
or equivalently
\[

u(t, x) = \dot{A}(t)A(t)^{-1}x, \quad x \in \Omega(t).
\]

Definition 3.2. Define the special linear Lie algebra
\[
\mathfrak{sl}(2, \mathbb{R}) = T_1\text{SL}(2, \mathbb{R}) = \{ L \in \mathbb{M}^2 : \text{tr} L = 0 \} = \text{span}\{ K, M, Z \}.
\]

Lemma 3.3. If \( A \in C^0(\mathcal{I}, \text{SL}(2, \mathbb{R})) \cap C^1(\mathcal{I}, \mathbb{M}^2) \) for some interval \( \mathcal{I} \), then \( (A, \dot{A}) \in C^0(\mathcal{I}, \mathcal{D}) \).

In particular, \( \dot{A}A^{-1} \in C^0(\mathcal{I}, \mathfrak{sl}(2, \mathbb{R})) \).

This leads to the following definition.

Definition 3.4. Define the mapping \( L : \mathcal{D} \rightarrow \mathfrak{sl}(2, \mathbb{R}) \) by
\[
L(A, B) = BA^{-1},
\]
so that the spatial velocity gradient of an affine motion \( x(t, y) = A(t)y \) is given by \( \nabla u(t, x) = L(A(t), \dot{A}(t)) \).
Lemma 3.3 suggests that the tangent bundle $\mathcal{D}$ is the natural phase space for affine incompressible motion.

Consider a solution of (0.1), (0.2). Let us assume that the velocity $u(t, x)$ and the fluid domains $\Omega(t)$ arise from an incompressible affine motion $x(t, y) = A(t)y$, as described above. By Lemma 3.3 the velocity field is divergence free:

$$\nabla \cdot u(t, x) = \text{tr} \tilde{A}(t)A(t)^{-1} = 0, \quad t \in \mathbb{R}.$$ 

Let us assume that

$$b(t, x) = \beta(t)A(t)^{-1}x \quad \text{and} \quad -\nabla p(t, x) = A(t)^{-T} \varpi(t)A(t)^{-1}x,$$

with $\beta, \varpi \in C^2(\mathbb{R}, M^2)$. We use the notation $A^{-T} = (A^{-1})^T$. (As motivation, note that if we assume that the unknowns $b$ and $p$ are also spatially homogeneous, then the PDEs imply that $\nabla p(t, x)$ and $b(t, x)$ should be homogeneous of degree one in the variable $x$.)

The equation for the magnetic field implies that

$$\dot{\beta} = \dot{A}A^{-1}\beta$$

from which it follows that

$$\beta(t) = A(t)A_0^{-1}\beta_0, \quad \text{where} \quad A_0 = A(0), \quad \beta_0 = \beta(0).$$

The normal vector to $\partial \Omega(t)$ at a point $x \in \partial \Omega(t)$ has the direction of the vector $A(t)^{-T}A(t)^{-1}x$. So the boundary condition implies that for all $|y| = 1$,

$$0 = b(t, x(t, y)) \cdot n(t, x(t, y)) = \beta(t)y \cdot A(t)^{-T}y = A_0^{-1}\beta_0 y \cdot y.$$

It follows that $A_0^{-1}\beta_0$ is anti-symmetric, and so there exists a constant $c_0$ such that

$$A_0^{-1}\beta_0 = c_0Z.$$

Thus, we have shown that

$$b(t, x) = c_0A(t)ZA(t)^{-1}x.$$

As a consequence,

$$\nabla \cdot b(t, x) = c_0 \text{tr} A(t)ZA(t)^{-1} = c_0 \text{tr} Z = 0,$$

so that $b$ is divergence free.

Since $A(t)^{-T} \varpi(t)A(t)^{-1}x$ is a gradient, the matrix $A(t)^{-T} \varpi(t)A(t)^{-1}$ must be symmetric. Thus,

$$\nabla p(t, x) = -\frac{1}{2} \nabla [A(t)^{-T} \varpi(t)A(t)^{-1}x \cdot x].$$

We find that

$$p(t, x) = \frac{1}{2} \left[ \lambda(t) - \varpi(t)A(t)^{-1}x \cdot A(t)^{-1}x \right],$$

for some scalar function $\lambda(t)$. The other boundary condition implies that

$$0 = p(t, x(t, y)) = \frac{1}{2} [\lambda(t) - \varpi(t)y \cdot y],$$

for all $|y| = 1$. This forces

$$\varpi(t) = \lambda(t)I,$$

and so

$$p(t, x) = \frac{1}{2} \lambda(t)[1 - |A(t)^{-1}x|^2].$$

Finally, from the velocity equation, we derive

$$\dot{A}(t) = \lambda(t)A(t)^{-T} + A(t)(c_0Z)^2 = \lambda(t)A(t)^{-T} - c_0^2 A(t).$$

Since $A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R}))$, we may write $A^{-T} = \text{cof} A(t)$. Thus, we have proven
Lemma 3.5. Suppose that
\[ A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2), \quad \lambda \in C^0(\mathbb{R}, \mathbb{R}), \quad c_0 \in \mathbb{R}, \quad \text{and} \quad \kappa = c_0^2. \]
Define
\[ u(t, x) = \dot{A}(t)A(t)^{-1}x, \]
\[ b(t, x) = c_0 A(t)ZA(t)^{-1}x, \]
\[ p(t, x) = \frac{1}{2} \lambda(t) \left[ 1 - |A(t)^{-1}x|^2 \right], \]
and
\[ \Omega(t) = A(t)B. \]
Then \( u(t, x), b(t, x), p(t, x) \) solve the MHD system (0.1), (0.2) in \( \Omega(t) \) if and only if
(3.1) \[ \dot{A}(t) + \kappa A(t) = \lambda(t) \text{cof } A(t). \]

Remark 3.6. Note that \( 1 - |A(t)^{-1}x|^2 > 0 \), for \( x \in \Omega(t) \), and so the sign of \( p(t, x) \) is determined by the sign of \( \lambda(t) \). Thus, an affine solution satisfies the Taylor sign condition if and only if \( \lambda(t) > 0 \). We shall see later on (in Corollary 8.3) that the sign of this function is preserved under the motion.

Remark 3.7. The equations of motion (3.1) are the Euler-Lagrange equations associated to the Lagrangian \( L_0 : \mathbb{M}^2 \times \mathbb{M}^2 \times \mathbb{R} \to \mathbb{R} \) given by
\[ L_0(A, \dot{A}, \lambda) = \frac{1}{2} |\dot{A}|^2 - \frac{\kappa}{2} |A|^2 + \lambda(\det A - 1). \]
The scalar function \( \lambda(t) \) in (3.1) is a Lagrange multiplier which will now be identified.

Definition 3.8. Given a parameter value \( \kappa \geq 0 \), define the Lagrange multiplier map \( \Lambda : \mathcal{D} \to \mathbb{R} \) by
\[ \Lambda(A, B) = \frac{2(\kappa - \det B)}{|A|^2}. \]
The dependence of \( \Lambda \) on \( \kappa \) will be suppressed.

Lemma 3.9. Fix \( \kappa \geq 0 \). Suppose that \( A \in C^2(\mathcal{J}, \mathbb{M}^2) \) satisfies (3.1) on some interval \( \mathcal{J} \subset \mathbb{R} \).
Then \( A \in C^0(\mathcal{I}, \text{SL}(2, \mathbb{R})) \) if and only if \( (A(t_0), \dot{A}(t_0)) \in \mathcal{D} \) for some \( t_0 \in \mathcal{J} \) and
(3.2) \[ \lambda(t) = \Lambda(A(t), \dot{A}(t)), \quad \text{for} \quad t \in \mathcal{J}. \]

Proof. Suppose that \( A \in C^2(\mathcal{J}, \mathbb{M}^2) \) satisfies (3.1). By Lemma 1.12, we have
\[ J(t) = \det A(t) = \frac{1}{2} \langle \text{cof } A(t), A(t) \rangle. \]
Recall that the map \( \text{cof} : \mathbb{M}^2 \to \mathbb{M}^2 \) is linear, symmetric, and orthogonal, by Lemma 1.10.
Using this and Lemma 1.12, we have
(3.3) \[ \dot{J} = \langle \text{cof } A, \dot{A} \rangle \]
and
(3.4) \[ \ddot{J} = \langle \text{cof } A, \ddot{A} \rangle + \langle \text{cof } \dot{A}, \dot{A} \rangle \]
\[ = \langle \text{cof } A, -\kappa A + \lambda \text{cof } A \rangle + 2 \det \dot{A} \]
\[ = -2\kappa J + \lambda |A|^2 + 2 \det \dot{A}. \]
If $A \in C^0(J, \text{SL}(2, \mathbb{R}))$, then $J(t) = 1$, $t \in J$, and by \[3.4\], we obtain \[3.2\]. Moreover, by Lemma \[3.3\] we have $(A(t), \dot{A}(t)) \in D$, for all $t \in J$.

On the other hand, if \[3.2\] holds, then we see that
\[
\ddot{J}(t) + 2\kappa J(t) = 0, \quad t \in J.
\]
If $(A(t_0), \dot{A}(t_0)) \in D$, then $(J(t_0), \ddot{J}(t_0)) = (1, 0)$ by \[3.3\], and so by uniqueness, we see that $J(t) = 1$, $t \in J$. Thus, $A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R}))$.

\[Remark\ 3.10\] Since this result depends on the linearity of the cofactor map on $\mathbb{M}^2$, it does not carry over to incompressible flows in 3d. A dimension independent formula for $\lambda$ was given in \[9\].

\section{Existence of global solutions}

\[Definition\ 4.1\] Fix $\kappa \geq 0$. For $i = 1, 2, 3$, we define the maps $X_i : \mathbb{M}^2 \times \mathbb{M}^2 \to \mathbb{R}$ by
\[
X_1(A, B) = \frac{1}{2}|B|^2 + \frac{\kappa}{2}|A|^2
\]
\[
X_2(A, B) = \frac{1}{2} \langle ZA - AZ, B \rangle
\]
\[
X_3(A, B) = \frac{1}{2} \langle ZA + AZ, B \rangle.
\]

We suppress the dependence of $X_1$ on the fixed parameter $\kappa \geq 0$.

\[Theorem\ 4.2\] If $A \in C^0(I, \text{SL}(2, \mathbb{R})) \cap C^2(I, \mathbb{M}^2)$ solves \[3.1\] on an interval $I$, then the quantities $X_i(A(t), \dot{A}(t))$, $i = 1, 2, 3$, are invariant.

\[Proof\] Suppose that $A \in C^0(I, \text{SL}(2, \mathbb{R})) \cap C^2(I, \mathbb{M}^2)$ solves \[3.1\] on $I$. By Lemma \[3.3\] $(A, \dot{A}) \in C^0(I, D)$, so that $\langle \dot{A}(t), \text{cof} A(t) \rangle = 0$ on $I$. From this we obtain
\[
\frac{d}{dt} X_1(A, \dot{A}) = \langle \dot{A} + \kappa A, \dot{A} \rangle = \lambda \langle \text{cof} A, \dot{A} \rangle = 0,
\]
which proves the invariance of $X_1(A, \dot{A})$. Next we compute
\[
\frac{d}{dt} \langle ZA, \dot{A} \rangle = \langle Z\dot{A}, \dot{A} \rangle + \langle ZA, -\kappa A + \lambda \text{cof} A \rangle.
\]
Each term on the right-hand side vanishes because $\langle Z\dot{A}, \dot{A} \rangle = 0$ for any $\dot{A} \in \mathbb{M}^2$. To see that the last term vanishes in this way we use Lemmas \[1.10\] and \[1.2\]. As a result, the quantity $\langle ZA, \dot{A} \rangle$ is invariant. The same holds for $\langle AZ, \dot{A} \rangle$. Therefore, the statements for $i = 2, 3$ are now clear.

\[Remark\ 4.3\] The invariants $X_2 \pm X_3$ correspond to the invariance of the Lagrangian $\mathcal{L}_0$ under the left and right actions of $\text{SO}(2, \mathbb{R})$.

\[Remark\ 4.4\] Note that by Lemma \[1.7\], $X_1(A, B) \geq \frac{\kappa}{2}|A|^2 \geq \kappa$, for all $(A, B) \in D$.

A version of the next result (in 3d and with $\kappa = 0$) appeared in Lemma 4 of \[9\].

\[Theorem\ 4.5\] Given a parameter value $\kappa \geq 0$ and initial data $(A_0, B_0) \in D$, the initial value problem
\[
\begin{align*}
\dot{A}(t) + \kappa A(t) &= \Lambda(A(t), \dot{A}(t)) \text{cof} A(t), \\
(A(0), \dot{A}(0)) &= (A_0, B_0)
\end{align*}
\]
has a unique global solution \( A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2) \). Additionally, there holds \((A, \dot{A}) \in C^0(\mathbb{R}, \mathcal{D})\).

**Proof.** Express the equation as an equivalent first order system

\[
\dot{A} = B \\
\dot{B} = -\kappa A + \Lambda(A, B) \text{cof}(A).
\]

The vector field on the right-hand side is a smooth function of \((A, B) \in M^2 \times M^2\) provided \(A \neq 0\), by Lemma [1.13]. Since the initial data satisfies \(A_0 \in SL(2, \mathbb{R})\), we have \(A_0 \neq 0\). Therefore, by the Picard existence and uniqueness theorem, the problem has a unique solution \((A, B) \in C^1(\mathcal{I}, M^2)\) defined on some maximal interval \(\mathcal{I}\), and since \(\dot{A} = B\), we have \(A \in C^2(\mathcal{I}, M^2)\). So by Lemma [3.9] the solution satisfies \(A \in C^0(\mathcal{I}, SL(2, \mathbb{R}))\), and by Lemma [3.3] \((A, \dot{A}) \in C^0(\mathcal{I}, \mathcal{D})\). By Lemma [1.7] \(|A(t)|^2 \geq 2\), on \(\mathcal{I}\), and so \(|A(t)|\) is bounded away from 0. By Theorem [4.2] the quantity \(X_1(A(t), B(t))\) is conserved. If \(\kappa > 0\), then the norm of the solution is uniformly bounded in \(M^2 \times M^2\), and we conclude that the solution is global. If \(\kappa = 0\), then the norm of \(B\) is uniformly bounded in \(M^2\). But since \(\dot{A} = B\), we see that the norm of \(A\) can not blow up in finite time, so again we conclude that the solution is global.

**Corollary 4.6.** A curve \(A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2)\) is a geodesic in \(SL(2, \mathbb{R})\) with the (induced) Euclidean metric if and only if it satisfies (3.1) with \(\kappa = 0\).

**Remark 4.7.** We include constant solutions as geodesics.

**Proof.** A geodesic curve is one for which \(\dot{A}(t)\) is parallel along \(A(t)\). That is

\[
\frac{D}{dt} \dot{A}(t) = 0,
\]

in which the covariant derivative of \(\dot{A}(t)\) along \(A(t)\) is is the projection of \(\dot{A}(t)\) onto \(T_{A(t)}SL(2, \mathbb{R})\). Thus, the curve \(A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2)\) is a geodesic if and only if \(\dot{A}(t) \in \text{span cof}(A(t))\). This is equivalent to (3.1). \(\square\)

## 5. Hamiltonian Formalism

We now proceed to reduce the constrained problem for (3.1) first to an unconstrained Lagrangian system using the map \(\Phi\) and then to a completely integrable Hamiltonian system using the Legendre transformation.

**Definition 5.1.** Given \(\kappa \geq 0\), define a Lagrangian density \(\mathcal{L} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\) by

\[
\mathcal{L}(x, y) = \frac{1}{2} |D_x \varphi(x)y|^2 - \frac{\kappa}{2} |\varphi(x)|^2 = \frac{1}{2} (g(x)y, y) - \frac{\kappa}{2} |\varphi(x)|^2.
\]

**Lemma 5.2.** If \(x \in C^2(\mathcal{I}, \mathbb{R}^3)\) is a solution of the Lagrangian equations of motion for \(\mathcal{L}:

\[
\frac{d}{dt} D_y \mathcal{L}(x, \dot{x}) - D_x \mathcal{L}(x, \dot{x}) = 0,
\]

on some time interval \(\mathcal{I}\), then \(A = \varphi \circ x \in C^0(\mathcal{I}, SL(2, \mathbb{R})) \cap C^2(\mathcal{I}, M^2)\) is a solution of (3.1).

**Proof.** The system (5.1) implies

\[
D_x \varphi \circ x^\top [\dot{A} + \kappa A] = 0,
\]
or in other words, by (2.1)

\[ \hat{A} + \kappa A \in \ker D_x \varphi \circ x^\top = (\text{Im} D_x \varphi \circ x)^\perp = \text{span} \, \text{cof} \, \varphi \circ x = \text{span} \, \text{cof} \, A. \]

This shows that \( A \) satisfies (3.1).

\[ \square \]

Lemma 5.3. The Hamiltonian\(^1\) associated to \( \mathcal{L} \) is

\[ H(x, p) = \frac{1}{2} \langle g^{-1}(x)p, p \rangle + \frac{\kappa}{2} |\varphi(x)|^2, \quad (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3, \]

with

\[ g^{-1}(x) = \begin{bmatrix} 1 - x_1^2/|\varphi(x)|^2 & -x_1 x_2/|\varphi(x)|^2 & 0 \\ -x_1 x_2/|\varphi(x)|^2 & 1 - x_2^2/|\varphi(x)|^2 & 0 \\ 0 & 0 & 1/\rho(x)^2 \end{bmatrix}. \]

Proof. The Legendre transformation associated to \( \mathcal{L} \) is

\[ P(x, y) = D_y \mathcal{L}(x, y) = g(x)y, \quad (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3. \]

The inverse transformation is

\[ Y(x, p) = g^{-1}(x)p, \quad (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3, \]

and the Hamiltonian associated to \( \mathcal{L} \) is

\[ H(x, p) = \langle Y(x, p), p \rangle - \mathcal{L}(x, Y(x, p)) = \frac{1}{2} \langle g^{-1}(x)p, p \rangle + \frac{\kappa}{2} |\varphi(x)|^2. \]

The formula (5.2) is a direct computation from (2.7).

Here we can think of \((x, p) \in \mathbb{R}^3 \times \mathbb{R}^3\) as a point in the cotangent bundle of \( \mathbb{R}^3 \), with the Legendre transformation \( y \mapsto p = g(x)y \) taking \( T_x \mathbb{R}^3 \) to \( T^*_x \mathbb{R}^3 \).

Using the facts that

\[ \rho(x)^2 = 2 + |\bar{x}|^2, \quad |\varphi(x)|^2 = \rho(x)^2 + |\bar{x}|^2, \quad |\bar{x}|^2 |\bar{p}|^2 = \langle \bar{x}, \bar{p} \rangle^2 + \langle Z \bar{x}, \bar{p} \rangle^2, \]

we can write

\[ \langle g^{-1}(x)p, p \rangle = |\bar{p}|^2 - \frac{\langle \bar{x}, \bar{p} \rangle^2}{|\varphi(x)|^2} + \frac{p_3^2}{\rho(x)^2} = \frac{\rho(x)^2 |\bar{p}|^2 + \langle Z \bar{x}, \bar{p} \rangle}{|\varphi(x)|^2} + \frac{p_3^2}{\rho(x)^2}, \]

and thus,

\[ H(x, p) = \frac{1}{2} \left( \frac{\rho(x)^2 |\bar{p}|^2 + \langle Z \bar{x}, \bar{p} \rangle}{|\varphi(x)|^2} + \frac{p_3^2}{\rho(x)^2} \right) + \frac{\kappa}{2} |\varphi(x)|^2. \]

The Hamiltonian system

\[ \dot{x} = D_p H(x, p), \quad \dot{p} = -D_x H(x, p) \]

\(^1\)In this section, \( p \in \mathbb{R}^3 \) denotes the generalized momentum, and not the pressure.
takes the explicit form
\[\begin{align*}
\dot{x} &= \frac{\rho(x)^2}{|\varphi(x)|^2} \ddot{p} + \frac{\langle Z\bar{x}, \ddot{p} \rangle}{|\varphi(x)|^2} Z\bar{x} \\
\dot{x}_3 &= \frac{p_3}{\rho(x)^2} \\
\dot{p} &= \left( \frac{2|\ddot{p}|^2 + 2 \langle Z\bar{x}, \ddot{p} \rangle^2}{|\varphi(x)|^4} + \frac{p_3^2}{\rho(x)^4} - 2\kappa \right) \ddot{x} + \frac{\langle Z\bar{x}, \ddot{p} \rangle}{|\varphi(x)|^2} Z\ddot{p} \\
\dot{p}_3 &= 0.
\end{align*}\] (5.3)

**Lemma 5.4.** If \((x,p) \in C^1(J, \mathbb{R}^3 \times \mathbb{R}^3)\) is a solution of (5.3) on \(J\), then \(A = \Phi \circ x\) is a solution of (3.1) in \(C^0(J, \text{SL}(2, \mathbb{R})) \cap C^2(J, M^2)\).

**Proof.** Since \(H\) is smooth, it follows from (5.3) that \(x \in C^2(J, \mathbb{R}^3)\), and thus, by Lemma 5.2, \(\varphi \circ x \in C^2(J, \text{SL}(2, \mathbb{R}))\).

That \(x\) solves the associated Lagrangian system (5.1) is a standard fact, and by Lemma 5.6 we have that \(\varphi \circ x\) solves (3.1). \(\square\)

**Lemma 5.5.** The map 
\[\Gamma(x,p) = (x, g^{-1}(x)p)\]
is a bijection on \(\mathbb{R}^3 \times \mathbb{R}^3\), and the map \(\Phi \circ \Gamma : \mathbb{R}^3 \times \mathbb{R}^3 \to D\) is surjective.

**Proof.** The first statement is obvious, and the second follows from Lemma 2.10. \(\square\)

**Lemma 5.6.** If \((A, B) = \Phi \circ \Gamma(x, p)\), for some \((x,p) \in \mathbb{R}^3 \times \mathbb{R}^3\), then
\[\begin{align*}
X_1(A, B) &= H(x, p), \\
X_2(A, B) &= x_1p_2 - x_2p_1 = \langle Z\bar{x}, \ddot{p} \rangle, \\
X_3(A, B) &= p_3.
\end{align*}\]

**Proof.** This lemma is a direct computation based on the definitions of \(X_i, \Phi,\) and \(\Gamma\). \(\square\)

The appearance the invariants \(X_2\) and \(X_3\) in (5.3) will eventually allow us to uncouple the system.

**Theorem 5.7.** If \((x,p) \in C^1(J, \mathbb{R}^3 \times \mathbb{R}^3)\) is a solution of (5.3) on \(J\), then the quantities
\[X_i \circ \Phi \circ \Gamma(x(t), p(t)), \quad i = 1, 2, 3\]
are invariant. The three invariants Poisson commute, and so the system (5.3) is completely integrable.

**Proof.** By Lemma 5.4 we know that \(A = \varphi \circ x\) is a solution of (3.1) in \(C^0(J, \text{SL}(2, \mathbb{R})) \cap C^2(J, M^2)\). Since \(\dot{x} = D_p H(x, p) = g^{-1}(x)p\), we see that
\[\Phi \circ \Gamma(x,p) = (\varphi(x), D_x \varphi(x)g^{-1}(x)p) = (\varphi(x), D_x \varphi(x)\dot{x}) = (A, \dot{A}).\]
This implies that
\[X_i \circ \Phi \circ \Gamma(x,p) = X_i(A, \dot{A}),\]
so it follows from Lemma 1.2 that these quantities are conserved. \(\square\)

**Theorem 5.8.** For any \(\kappa \geq 0\) and any initial data \((x(0), p(0)) \in \mathbb{R}^3 \times \mathbb{R}^3\), the system (5.3) has global solutions.
Proof. By Theorem 5.7, $X_1(x(t), p(t))$ is conserved along any solution $(x(t), p(t))$. If $\kappa > 0$, this implies that all solutions remain bounded, and thus, global existence holds. If $\kappa = 0$, then $|p(t)|$ is bounded. It then follows from (5.3) that $|\dot{x}(t)|$ is uniformly bounded, which prevents blow up in finite time. □

Lemma 5.9. If $(A, B) \in D_0$, then $X_2(A, B) = 0$. If $(A, B) \in D$ and $X_2(A, B) \neq 0$, then $A \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$.

Proof. If $(A, B) \in D_0$, then $A \in SO(2, \mathbb{R})$. By Lemma 1.9, $ZA - AZ = 0$, and so we have $X_2(A, B) = 0$. If $(A, B) \in D$ and $X_2(A, B) \neq 0$, then $(A, B) \in D \setminus D_0$, so $A \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$. □

Corollary 5.10. Let $(x, p) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)$ be a (necessarily global) solution of (5.3), and set

$$X_2(t) = x_1(t)p_2(t) - x_2(t)p_1(t).$$

If $X_2(t_0) \neq 0$ for a single time $t_0 \in \mathbb{R}$, then $\bar{x}(t) \neq 0$ and $\varphi \circ x(t) \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$, for all $t \in \mathbb{R}$. If $\bar{x}(t_0) = 0$ for a single time $t_0 \in \mathbb{R}$, then $X_2(t) = 0$, for all $t \in \mathbb{R}$.

Proof. By Theorem 5.7, the quantity $X_2(t)$ is invariant. Therefore, either $X_2(t) = 0$ for all $t \in \mathbb{R}$, or $X_2(t) \neq 0$, for all $t \in \mathbb{R}$. So if $X_2(t)$ vanishes for a single time, then it vanishes identically. Otherwise, $X_2(t) \neq 0$ on $\mathbb{R}$ which forces $\bar{x}(t) \neq 0$ on $\mathbb{R}$, and by Lemma 2.6 $\varphi \circ x(t) \in SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$ on $\mathbb{R}$. □

The behavior of the system (5.3) is topologically different depending on whether the invariant $X_2$ is nonzero or not. We shall now consider each case in turn.

6. Reduction to the phase plane in the case $X_2 \neq 0$

According to Corollary 5.10, the property $\bar{x}(t_0) \neq 0$ is preserved by the flow of (5.3). Therefore, it is natural to introduce polar coordinates for $\bar{x}$ in this case.

Definition 6.1. Define the map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\psi(q) = \begin{bmatrix} q_1 \cos q_2 \\ q_1 \sin q_2 \\ q_3 \end{bmatrix}.$$

Remark 6.2. We shall mostly consider $\psi$ on the restricted domain $\mathbb{R}^3_+ = \{ q \in \mathbb{R}^3 : q_1 > 0 \}$.

Lemma 6.3. The map $\varphi \circ \psi : \mathbb{R}^3 \to SL(2, \mathbb{R})$ is surjective, and $\varphi \circ \psi(q) \in SO(2, \mathbb{R})$ if and only if $q_1 = 0$.

The map $\varphi \circ \psi$ restricted to $\mathbb{R}^3_+$ is an immersion from $\mathbb{R}^3_+$ onto $SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$. The restricted map $\varphi \circ \psi$ defines a local coordinate chart for $SL(2, \mathbb{R}) \setminus SO(2, \mathbb{R})$ in a neighborhood of any point $q \in \mathbb{R}^3_+$.

Proof. The map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ is surjective, and $\psi_1(q) = \psi_2(q) = 0$ if and only if $q_1 = 0$, so the first statement follows by Lemma 2.6.

For each $q \in \mathbb{R}^3_+$, $D_q \psi(q)$ is a bijection on $\mathbb{R}^3$, and so by Lemma 2.6,

$$D_q[\varphi \circ \psi(q)] : \mathbb{R}^3 \to T_{\varphi \circ \psi(q)} SL(2, \mathbb{R})$$
is a bijection for each \( q \in \mathbb{R}^3_+ \). This shows that \( \varphi \circ \psi \) is an immersion, and it follows that \( \varphi \circ \psi \) defines a local coordinate chart near any point \( q \in \mathbb{R}^3_+ \).

\[ \square \]

**Lemma 6.4.** If \( A \in \text{SL}(2, \mathbb{R}) \) and \( A = \varphi \circ \psi(q) \), for some \( q \in \mathbb{R}^3 \), then \( \frac{1}{2} |A|^2 = 1 + q_1^2 \) and

\[
A = \frac{1}{\sqrt{2}} U \left( \frac{1}{2}(q_3 + q_2) \right) (\rho I + q_1 K) U \left( \frac{1}{2}(q_3 - q_2) \right), \quad \text{with} \quad \rho = (2 + q_1^2)^{1/2}.
\]

The image of the unit disk under \( A \) is an ellipse with principal axes of lengths

\[
\left[ \left( \frac{1}{2} |A|^2 + 1 \right)^{1/2} \pm \left( \frac{1}{2} |A|^2 - 1 \right)^{1/2} \right] \sqrt{2}.
\]

**Proof.** By definition, we have

\[
A = \varphi \circ \psi(q) = \frac{1}{\sqrt{2}} \left[ \rho (\cos q_3 I + \sin q_3 Z) + q_1 (\cos q_2 K + \sin q_2 M) \right],
\]

with \( \rho = (2 + q_1^2)^{1/2} \). Recalling definition [1.8] we set \( U(\theta) = \cos \theta I + \sin \theta Z \). Since \( M = ZK = -KZ \), we have

\[
\cos \theta K + \sin \theta M = (\cos \theta I + \sin \theta Z) K = U(\theta) K = KU(-\theta).
\]

Therefore, we obtain

\[
\sqrt{2} A = \rho U \left( \frac{1}{2}(q_3 + q_2) \right) + q_1 U \left( \frac{1}{2}(q_3 - q_2) \right) K
\]

\[
= U \left( \frac{1}{2}(q_3 + q_2) \right) (\rho I + q_1 K) U \left( \frac{1}{2}(q_3 - q_2) \right) K,
\]

\[
= U \left( \frac{1}{2}(q_3 + q_2) \right) (\rho I + q_1 K) U \left( \frac{1}{2}(q_3 - q_2) \right).
\]

Since \( \rho I + q_1 K = \text{diag} \left[ \rho + q_1, \rho - q_1 \right] \), the formula shows that the image of the unit disk under \( A \) is an ellipse with principal axes of lengths \( (\rho \pm q_1)/\sqrt{2} \), giving (6.2). \( \square \)

**Definition 6.5.** Set \( \mathbb{R}^3_1 = \{(0,0,q_3) \in \mathbb{R}^3 : q_3 \in \mathbb{R} \} \), and define the map \( \Psi : \mathbb{R}^3_+ \times \mathbb{R}^3 \to (\mathbb{R}^3 \setminus \mathbb{R}^3_1) \times \mathbb{R}^3 \) by

\[
\Psi(q, \xi) = (\psi(q), D\psi(q)^{-\top} \xi).
\]

\( \Psi \) is well-defined since \( D\psi(q) \) is invertible when \( q \in \mathbb{R}^3_+ \).

**Lemma 6.6.** The transformation \( \Psi \) is canonical.

**Lemma 6.7.** The composition \( \Phi \circ \Gamma \circ \Psi \) maps \( \mathbb{R}^3_+ \times \mathbb{R}^3 \) onto \( \mathcal{D} \setminus \mathcal{D}_0 \). The induced metric on \( T_{\varphi \circ \psi(q)} \text{SL}(2, \mathbb{R}) \) is

\[
h(q) = \text{diag} \left[ \frac{2 + q_1^2}{2(1 + q_1^2)} \quad \frac{1}{q_1} \quad \frac{1}{2 + q_1^2} \right].
\]

**Proof.** The first statement follows from Lemmas [2.10] 5.5 and 6.3. Explicitly, we have

\[
\Phi \circ \Gamma \circ \Psi(q, \xi) = (\varphi \circ \psi(q), T(q) \xi)
\]

with

\[
T(q) = D_x \varphi \circ \psi(q) g^{-1} \circ \psi(q) D_q \psi(q)^{-1} : \mathbb{R}^3_+ \to T_{\varphi \circ \psi(q)} \text{SL}(2, \mathbb{R}).
\]

The metric \( h(q) \) can be found by computing

\[
h(q) = T(q)^\top T(q) = D_q \psi^{-1}(q) g^{-1} \circ \psi(q) D_q \psi(q)^{-\top}.
\]

\( \square \)
Definition 6.8. Define the Hamiltonian

\[ \bar{H}(q, \xi) = H \circ \Psi(q, \xi) = \frac{1}{2} \left( \langle h(q)\xi, \xi \rangle + \frac{2}{\epsilon} |\varphi \circ \psi(q)|^2 \right) \]

\[ = \frac{1}{2} \left( \frac{(2 + q_1^2)\xi_1^2}{2(1 + q_1^2)} + \frac{\xi_2^2}{q_1} + \frac{\xi_3^2}{2(2 + q_1^2)} + \kappa(1 + q_1^2) \right), \]

for \((q, \xi) \in \mathbb{R}^3_+ \times \mathbb{R}^3\).

The corresponding system is

\[ \begin{align*}
\dot{q}_1 &= \frac{(2 + q_1^2)\xi_1}{2(1 + q_1^2)} \\
\dot{\xi}_1 &= \left( \frac{\xi_1^2}{2(1 + q_1^2)} + \frac{\xi_2^2}{q_1} + \frac{\xi_3^2}{(2 + q_1^2)^2} - 2\kappa \right)q_1 \\
\dot{q}_2 &= \frac{\xi_2}{q_1} \\
\dot{\xi}_2 &= 0 \\
\dot{q}_3 &= \frac{\xi_3}{2 + q_1^2} \\
\dot{\xi}_3 &= 0.
\end{align*} \] (6.1)

Notice that our choice of polar coordinates for \(\bar{x}\) has created a singularity at \(q_1 = |\bar{x}| = 0\), corresponding to \(SO(2, \mathbb{R})\).

Lemma 6.9. If \((q, \xi) \in C^1(\mathcal{J}, \mathbb{R}^3_+ \times \mathbb{R}^3)\) is a solution of (6.1) on \(\mathcal{J}\), then \((x, p) = \Psi(q, \xi)\) solves (5.3) on \(\mathcal{J}\).

Proof. The statement holds because the transformation \(\Psi\) is canonical. \(\square\)

Lemma 6.10. If \((A, B) = \Phi \circ \Gamma \circ \Psi(q, \xi)\), for some \((q, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3\), then

\[ X_1(A, B) = \bar{H}(q, \xi), \quad X_i(A, B) = \xi_i, \quad i = 2, 3, \]

and these quantities are invariant along solutions of (6.1).

Lemma 6.11. If \((q, \xi) \in C^1(\mathcal{J}, \mathbb{R}^3_+ \times \mathbb{R}^3)\) is a solution of (6.1) on \(\mathcal{J}\) with \(\xi_2 \neq 0\), then \(q_1(t)\) is bounded away from zero:

\[ q_1(t)^2 \geq \frac{\xi_2^2}{2\bar{H}(q, \xi)} = \frac{X_2^2}{2X_1} > 0, \]

and \((q(t), \xi(t))\) is defined for all \(t \in \mathbb{R}\).

Proof. The lower bound follows from Definition 6.8 and Lemma 6.10. Global existence is now a consequence of the invariance of \(X_1\). \(\square\)

Theorem 6.12. Fix \(\kappa \geq 0\). Let \((A_0, B_0) \in \mathcal{D} \setminus \mathcal{D}_0\) with \(X_2(A_0, B_0) \neq 0\). Choose \((q(0), \xi(0)) \in \mathbb{R}^3_+ \times \mathbb{R}^3\) such that \(\Phi \circ \Gamma \circ \Psi(q(0), \xi(0)) = (A_0, B_0)\). Let \((q, \xi) \in C^1(\mathbb{R}, \mathbb{R}^3_+ \times \mathbb{R}^3)\) be the global solution of (6.1) with initial data \((q(0), \xi(0))\).

Then

\[ A(t) = \varphi \circ \psi \circ q(t), \quad t \in \mathbb{R}, \]

is the solution of (3.1) in \(C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \text{M}^2)\) with initial data \((A_0, B_0)\). Explicitly, we have

\[ A(t) = \frac{1}{\sqrt{2}} U \left( \frac{1}{2}(q_3(t) + q_2(t)) \right) (\rho(t)I + q_1(t)K) U \left( \frac{1}{2}(q_3(t) - q_2(t)) \right), \]

with \(\rho(t) = (2 + q_1(t)^2)^{1/2}\). There holds

\[ \frac{1}{2} |A(t)|^2 = 1 + q_1(t)^2, \]
and the fluid domain $\Omega(t) = A(t)B$ is an ellipse with principal axes of lengths
\[
(6.2) \quad \frac{1}{\sqrt{2}} \left[ \left( \frac{1}{2} |A(t)|^2 + 1 \right)^{1/2} \pm \left( \frac{1}{2} |A(t)|^2 - 1 \right)^{1/2} \right],
\]
for all $t \in \mathbb{R}$.

Proof. That $A$ is a global solution of (3.1) follows from Lemmas 6.11, 6.9, and 5. The remaining statements follow from Lemma 6.4.

The Hamiltonian flow (6.1) is easily understood. Since $\xi_i = X_i(A_0, B_0)$, $i = 2, 3$, by Lemma 6.10, we see from (6.1) that the equations for the pair $(q_1, \xi_1)$ uncouple from the others. The corresponding orbits are simply the level curves of $\tilde{H}$ in the half plane $\{(q_1, \xi_1) : q_1 > 0\}$ with the known fixed values of $(\xi_2, \xi_3)$. These level curves are described in Theorem 6.13 and illustrated in Figures 1 and 2. Given $q_1$, the other coordinates $q_2, q_3$ are found by integration of the remaining equations in (6.1).

**Theorem 6.13.** For each fixed $(\xi_2, \xi_3) \in \mathbb{R}^2$ with $\xi_2 \neq 0$, $\tilde{H}$ is a strictly convex function of $(q_1, \xi_1)$ on the set $\{(q_1, \xi_1) : q_1 > 0\}$. Its level sets are symmetric with respect to the $\xi_1$ axis. If $\kappa > 0$, $\tilde{H}$ has a minimum value at a unique point $(q_1(\xi_2, \xi_3), 0)$, and all other level sets are smooth simple closed curves. See Fig. 1. If $\kappa = 0$, the level sets of $\tilde{H}$ are smooth curves, bounded in $\xi_1$ and unbounded in $q_1$. See Fig. 2.

**Figure 1.** Typical level curves of $\tilde{H}$, in the case $\kappa > 0$.

7. Reduction to the phase plane in the case $X_2 = 0$

When $X_2 = 0$, we shall rely on the fact that for any fixed unit vector $\bar{v} \in \mathbb{R}^2$, the set
\[
\{(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle \bar{x}, \bar{v} \rangle = \langle \bar{p}, \bar{v} \rangle = 0\}
\]
is invariant under the flow of (5.3).

**Definition 7.1.** Define $\Psi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ by $\Psi_0(q, \xi) = (\psi(q), \psi(\xi))$, where $\psi$ was given in Definition 6.1.
**Figure 2.** Typical level curves of $\tilde{H}$, in the case $\kappa = 0$. 

**Definition 7.2.** Define the Hamiltonian

$$H_0(q, \xi) = H \circ \Psi_0(q, \xi)|_{q_2=\xi_2} = \frac{1}{2} \left( \frac{(2 + q_1^2)\xi_1^2}{2(1 + q_1^2)} + \frac{\xi_3^2}{2 + q_1^2} \right) + \kappa(1 + q_1^2), \quad (q, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

**Lemma 7.3.** If $(A, B) \in \mathcal{D}$ and $X_2(A, B) = 0$, then there exists $(q, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $q_2 = \xi_2$ such that

$$(A, B) = \Phi \circ \Gamma \circ \Psi_0(q, \xi).$$

In this case, we have

$$X_1(A, B) = H_0(q, \xi), \quad X_2(A, B) = 0, \quad \text{and} \quad X_3(A, B) = \xi_3.$$

**Proof.** Let $(A, B) \in \mathcal{D}$ with $X_2(A, B) = 0$. By Lemma 5.5, we can choose $(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\Phi \circ \Gamma(x, p) = (A, B).$$

By Lemma 5.6, we have $x_1p_2 - x_2p_1 = 0$. This says that the vectors $\bar{x}$ and $\bar{p}$ are dependent. Therefore, we can find a single unit vector $\bar{\omega} \in \mathbb{R}^2$ and scalars $q_1, \xi_1 \in \mathbb{R}$ such that

$$\bar{x} = q_1\bar{\omega} \quad \text{and} \quad \bar{p} = \xi_1\bar{\omega}.$$

We can express $\bar{\omega}$ in the form

$$\bar{\omega} = (\cos q_2, \sin q_2),$$

for some $q_2 \in \mathbb{R}$. Now, if we define

$$q = (q_1, q_2, x_3) \quad \text{and} \quad \xi = (\xi_1, q_2, p_3),$$

then $\Psi_0(q, \xi) = (x, p)$ and $q_2 = \xi_2$. It follows that $(A, B) = \Phi \circ \Gamma \circ \Psi_0(q, \xi)$.

The form of the nonzero invariants follows from Lemma 5.6 and Definition 7.2. □
Although $H_0$ is independent of $q_2, q_3, \xi_2$, we shall still regard it as a function on $\mathbb{R}^3 \times \mathbb{R}^3$. As such, the corresponding Hamiltonian system takes the form

$$
\begin{align*}
\dot{q}_1 &= \frac{(2 + q_1^2) \xi_1}{2(1 + q_1^2)} \\
\dot{q}_2 &= 0 \\
\dot{q}_3 &= \frac{\xi_3}{2 + q_1^2}
\end{align*}
(7.1)
$$

Notice that (7.1) is formally obtained from (6.1) by deleting the terms which are singular at $q_1 = 0$, although here we have $q_1 \in \mathbb{R}$ rather than $q_1 > 0$.

**Lemma 7.4.** For any $\kappa \geq 0$ and any initial data $(q(0), \xi(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$, the system (7.1) has a unique global solution $(q, \xi) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)$.

**Proof.** The follows by the conservation of the Hamiltonian $H$ along the flow of (7.1). □

**Theorem 7.5.** Fix $\kappa \geq 0$. Let $(A_0, B_0) \in \mathcal{D}$ with $X_2(A_0, B_0) = 0$. Choose $(q(0), \xi(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $q_2(0) = \xi_2(0)$ such that $\Phi \circ \Gamma \circ \Psi_0(q(0), \xi(0)) = (A_0, B_0)$. Let $(q, \xi) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)$ be the global solution of (7.1) with initial data $(q(0), \xi(0))$. Then

$$
A(t) = \varphi \circ \psi \circ q(t), \quad t \in \mathbb{R}
$$

is the solution of (3.1) in $C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ with initial data $(A_0, B_0)$. Explicitly, we have

$$
A(t) = \frac{1}{\sqrt{2}} U \left( \frac{1}{2} \langle q_3(t) + q_2(t) \rangle \right) (\rho(t) I + q_1(t) K) U \left( \frac{1}{2} \langle q_3(t) - q_2(t) \rangle \right),
$$

with $\rho(t) = (2 + q_1(t)^2)^{1/2}$. There holds

$$
\frac{1}{2} |A(t)|^2 = 1 + q_1(t)^2,
$$

and the fluid domain $\Omega(t) = A(t) \mathcal{B}$ is an ellipse with principal axes of lengths given in (6.2).

**Proof.** Since $q_2(0) = \xi_2(0)$, the solution $(q, \xi)$ of (7.1) satisfies

$$
q_2(t) = q_2(0) = \xi_2(0) = \xi_2(t), \quad \text{for all} \quad t \in \mathbb{R}.
$$

Let $\bar{\omega} = (\cos q_2(0), \sin q_2(0))$. Define $(x, p) = \Psi_0(q, \xi)$. Then $(x, p) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)$ and

$$
\begin{align*}
x(t) &= (\bar{x}(t), x_3(t)) = (q_1(t) \bar{\omega}, q_3(t)), \\
p(t) &= (\bar{p}(t), p_3(t)) = (\xi_1(t) \bar{\omega}, \xi_3(t)),
\end{align*}
$$

for all $t \in \mathbb{R}$. According to our definitions, we find

$$
\langle Z \bar{x}, \bar{p} \rangle = 0, \quad \rho(x)^2 = 2 + q_1^2, \quad |\varphi(x)|^2 = 2(1 + q_1^2), \quad \text{and} \quad |\bar{p}|^2 = \xi_1^2,
$$

where for notational convenience we have suppressed the time argument.

It is now straightforward to verify that $(x, p)$ is a global solution of (5.3) with initial data $\Psi_0(q(0), \xi(0))$. It follows from Lemma 5.4 that

$$
A = \varphi \circ x = \varphi \circ \psi \circ q \in C(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)
$$

is a solution of (3.1).

The remaining statements follow from Lemma 6.4 □

Again, we can understand the flow of (7.1) by looking at the level sets of $H_0$ with $\xi_3$ fixed. These are described in Theorem 7.6 and illustrated in Figures 3, 4, 5, and 6.
Theorem 7.6. Fix $\kappa \geq 0$ and $\xi_3 \in \mathbb{R}$. As a function of $(q_1, \xi_1) \in \mathbb{R}^2$, the level curves of $H_0$ are symmetric with respect to the coordinate axes. They have the following structure.

If $0 < \kappa < \frac{1}{8} \xi_3^2$, then $H_0$ has three nondegenerate critical points: a saddle point at $(0, 0)$ and two centers $(\pm q_1(\xi_3), 0)$ where $H_0$ achieves its minimum value. Each level set when $\min H_0 < H_0 < H_0(0, 0)$ corresponds to a pair of closed orbits. Each level set when $H_0 > H_0(0, 0)$ corresponds to single closed orbit. The level set when $H_0 = H_0(0, 0)$ corresponds to the union of an equilibrium point at $(0, 0)$ and two homoclinic orbits. See Fig. 3.

If $\kappa > 0$ and $\kappa \geq \frac{1}{8} \xi_3^2$, then $H_0$ has a single critical point at $(0, 0)$ where it achieves its minimum value. Each level set when $H_0 > \min H_0$ corresponds to a closed orbit. See Fig. 4.

If $\kappa = 0$ and $\xi_3 \neq 0$, then $H_0$ has a saddle point at $(0, 0)$. The level set when $H = H_0(0, 0)$ corresponds to an equilibrium solution and four orbits, bounded in $\xi_1$ and unbounded in $q_1$, comprising the stable and unstable manifolds of the equilibrium. Each level set for $H_0 \neq H_0(0, 0)$ corresponds to a pair of orbits, bounded in $\xi_1$ and unbounded in $q_1$. See Fig. 5.

If $\kappa = 0$ and $\xi_3 = 0$, then $\min H_0 = 0$, and the minimum level set corresponds to a line of equilibria $\xi_1 = 0$. Each level set for $H_0 > 0$ corresponds to a pair of orbits, bounded in $\xi_1$ and unbounded in $q_1$. See Fig. 6.

8. The Lagrange Multiplier

We now obtain an alternate expression for $\Lambda(A, B)$.

Lemma 8.1. If $(A, B) \in \mathcal{D}$, then

$$\Lambda(A, B) = \frac{4X_1(A, B) + 2X_2(A, B)^2 - 2X_3(A, B)^2}{|A|^4}.$$ 

Proof. If $(A, B) \in \mathcal{D}$, then $\det A = \frac{1}{2} \langle \text{cof} A, A \rangle = 1$ and $\langle \text{cof} A, B \rangle = 0$, and so Lemma 1.14 gives

$$\langle ZA, B \rangle \langle AZ, B \rangle = |B|^2 + \det B |A|^2.$$
Figure 4. Typical level curves of $H_0$, in the case $\kappa > 0$, $\kappa \geq \frac{1}{8} X_3^2$.

Figure 5. Typical level curves of $H_0$, in the case $\kappa = 0$, $\xi_3 \neq 0$.

By Definition 4.1, this is equivalent to

$$-X_2(A, B)^2 + X_3(A, B)^2 = 2X_1(A, B) - \kappa |A|^2 + \det B |A|^2.$$ 

The lemma now follows from Definition 3.8.

Theorem 8.2. Let $\kappa \geq 0$. If $A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2)$ satisfies (3.1) for some $\lambda \in C^0(\mathbb{R}, \mathbb{R})$, then

$$\lambda(t) = \Lambda(A(t), \dot{A}(t)) = \frac{4X_1 + 2X_2^2 - 2X_3^2}{|A(t)|^4},$$

with

$$X_i = X_i(A(0), \dot{A}(0)), \quad i = 1, 2, 3.$$ 

Proof. This follows by Lemma 3.9, Lemma 8.1 and Theorem 4.2.
Corollary 8.3. The sign of the pressure is preserved under affine motion.

Proof. This is a consequence of Lemma 3.5 (see Remark 3.6) and Theorem 8.2.

We now consider the case of vanishing pressure.

Lemma 8.4. Let $A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be the solution of (3.1) with initial data $(A_0, B_0) \in \mathcal{D}$ such that

$$2X_1(A_0, B_0) + X_2(A_0, B_0)^2 - X_3(A_0, B_0)^2 = 0.$$

If $\kappa = 0$, then

$$A(t) = B_0 t + A_0,$$

and if $\kappa > 0$, then

$$A(t) = (\cos \sqrt{\kappa} t)A_0 + \frac{1}{\sqrt{\kappa}} (\sin \sqrt{\kappa} t)B_0.$$

Proof. By Theorem 8.2 and the assumption on the invariants, we have $\lambda(t) = 0$. Thus, (3.1), reduces to the linear equation $\ddot{A} + \kappa A = 0$ whose solutions are as stated above.

Remark 8.5. When $\kappa > 0$, the solution $A(t)$, and hence the motion $x(t,y)$, is $2T$-periodic with $T = \pi/\sqrt{\kappa}$, while $|A(t)|$ and the fluid domain $\Omega(t)$ is $T$-periodic.

Remark 8.6. When $\kappa = 0$, we obtain geodesics in SL(2, $\mathbb{R}$) and in $\mathbb{M}^2$.

9. Rigid solutions

Definition 9.1. A solution $A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ of (3.1) is called rigid if $|A(t)|$ is constant for all $t \in \mathbb{R}$. A solution $A$ is called a rigid rotation if $A(t) \in \text{SO}(2, \mathbb{R})$, for all $t \in \mathbb{R}$.

Recall that by Lemma 1.7, $A(t) \in \text{SO}(2, \mathbb{R})$ if and only if $|A(t)|^2 = 2$, so a rigid rotation is a rigid solution, according to the definitions. Any equilibrium solution is rigid.
As noted in Theorems 6.12 and 7.5, the fluid domains $\Omega(t)$ of a solution $A(t)$ form a family of ellipses whose dimensions depend only on $|A(t)|$. So for rigid motion, the $\Omega(t)$ are constant up to rotation, and $\Omega(t) = B$ for a rigid rotation.

**Theorem 9.2.** Fix $\kappa \geq 0$. Let $A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2)$ be a solution of (3.1) with initial data $(A_0, B_0) \in D$, and set $X_i = X_i(A_0, B_0)$, $i = 1, 2, 3$.

i. $A$ is an equilibrium if and only if $X_1 = \kappa$.

ii. $A$ is a rigid rotation if and only if $A_0 \in SO(2, \mathbb{R})$ and $X_1 = \kappa + \frac{1}{4} X_3^2$. In this case, $A(t) = U(\frac{1}{2}X_3 t) A_0$.

iii. $A$ is rigid with $|A_0|^2 > 2$ if and only if

$$
X_1 = \frac{1}{2} \left( \frac{X_2^2}{\left(\frac{1}{2}|A_0|^2 - 1\right)^2} + \frac{X_3^2}{\left(\frac{1}{2}|A_0|^2 + 1\right)^2} \right) + \frac{\kappa}{2} |A_0|^2
$$

and

$$
X_2^2 \left(\frac{1}{2}|A_0|^2 - 1\right)^2 + X_3^2 \left(\frac{1}{2}|A_0|^2 + 1\right)^2 = 2\kappa.
$$

**Proof.** If $A$ is an equilibrium, then $\dot{A} = 0$, and so in particular, $B_0 = 0$. This implies that

$$
X_1 = \frac{\kappa}{2} |A_0|^2 \quad \text{and} \quad X_i = 0, \quad i = 2, 3.
$$

If $\kappa = 0$, then we have verified that $X_1 = \kappa$. So we may assume that $\kappa > 0$. By Lemma 8.1 and (9.3), we see that

$$
\Lambda(A_0, B_0) = 4X_1/|A_0|^4 = \kappa^2/X_1.
$$

Now $A$ solves (4.1), so setting $\dot{A} = 0$ and then $t = 0$, we get

$$
\kappa A_0 = \Lambda(A_0, B_0) \operatorname{cof} A_0 = (\kappa^2/X_1) \operatorname{cof} A_0.
$$

Since the cofactor map is norm-preserving, we find

$$
\kappa = \kappa^2/X_1.
$$

The assumption $\kappa > 0$ implies that $X_1 = \kappa$.

Conversely, if $X_1 = \kappa$, then

$$
\frac{1}{2} |\dot{A}|^2 = X_1(A, \dot{A}) - \frac{\kappa}{2} |A|^2 = X_1 - \frac{\kappa}{2} |A|^2 = \frac{\kappa}{2} (2 - |A|^2) \leq 0,
$$

by Lemma 1.7. This shows that $\dot{A} = 0$, so $A$ is an equilibrium. This completes the proof of statement i.

Let $A_0 \in SO(2, \mathbb{R})$. Then $(A_0, B_0) \in D_0$. By Lemma 5.9, $X_2 = 0$. By Lemma 7.3, we may choose $(q(0), \xi(0)) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$
(A_0, B_0) = \Phi \circ \Gamma \circ \Psi(q(0), \xi(0)), \quad X_1 = H_0(q(0), \xi(0)), \quad \xi_2(0) = 0, \quad X_3 = \xi_3(0).
$$

Since $A_0 \in SO(2, \mathbb{R})$, we also have $q_1(0) = 1/2 |A_0|^2 - 1 = 0$. Thus,

$$
q(0) = (0, q_2(0), q_3(0)) \quad \text{and} \quad \xi(0) = (\xi_1(0), 0, X_3).
$$

Let $(q(t), \xi(t))$ be the solution of (7.1) with initial data $(q(0), \xi(0))$. Then by Theorem 7.5

$$
A(t) = \frac{1}{\sqrt{2}} U \left( \frac{1}{2} (q_3(t) + q_2(0)) \right) (\rho(t) I + q_1(t) K) U \left( \frac{1}{2} (q_3(t) - q_2(0)) \right).
$$

We also note that since $q_1(0) = 0$

$$
X_1 = H_0(q(0), \xi(0)) = \frac{1}{4} (\xi_1(0)^2 + X_3^2) + \kappa.
$$
Given that \((9.4)\) and \((9.6)\) hold, we can now assert that the following statements are equivalent:

- \(X_1 = \kappa + \frac{1}{4}X_3^2\),
- \(\xi_1(0) = 0\),
- \(q(t) = (0, q_2(0), \frac{1}{2}X_3t + q_3(0))\) and \(\xi(t) = (0, 0, X_3)\),
- \(q_1(t) = 0\) for all \(t \in \mathbb{R}\),
- \(|A(t)|^2 = 2\) for all \(t \in \mathbb{R}\),
- \(A(t) \in SO(2, \mathbb{R})\) for all \(t \in \mathbb{R}\).

Finally, we note that if \(A(t) \in SO(2, \mathbb{R})\), for all \(t \in \mathbb{R}\), then \(q_1(t) = 0, \rho(t) = \sqrt{2}\), and the formula \((9.5)\) reduces to

\[
\begin{align*}
A(0, B_0) &= \Phi \circ \Gamma \circ \Psi(q(0), \xi(0)),
\end{align*}
\]

and by and Lemma \(6.10\)

\[
(9.7) \quad X_1 = \tilde{H}(q(0), \xi(0)), \quad X_2 = \xi_2(0), \quad X_3 = \xi_3(0).
\]

Let \((q, \xi) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)\) be the solution of \((6.1)\) with initial data \((q(0), \xi(0))\). By Theorem \(6.12\) we have

\[
A(t) = \varphi \circ \psi \circ q(t).
\]

Since \(q_1(0)^2 = \frac{1}{2}|A_0|^2 - 1\), we see from \((9.7)\) and Definition \(6.8\) that \((9.1)\) is equivalent to \(\xi_1(0) = 0\). By \((6.1)\), we also note that \((9.1)\) and \((9.2)\) are equivalent to \(\xi_1(0) = 0\). Therefore, if \((9.1)\), \((9.2)\) hold, then the unique solution of \((6.1)\) is

\[
q(t) = \left( q_1(0), \frac{X_2}{q_1(0)^2}t + q_2(0), \frac{X_3}{2 + q_1(0)^2}t + q_3(0) \right), \quad \xi(t) = (0, X_2, X_3).
\]

Since \(q_1(t)\) is constant, \(A\) is rigid.

Conversely, if \(A\) is rigid with \(|A(t)|^2 = |A_0|^2 > 2\), then \(q_1(t) = q_1(0) > 0\) and \(\dot{q}_1(t) = 0\), for all \(t \in \mathbb{R}\). It follows from \((6.1)\) that \(\xi_1(t) = 0\), for all \(t \in \mathbb{R}\). As noted above, this implies that \((9.1)\) holds. We also see that \(\dot{\xi}_1(t) = 0\), for all \(t \in \mathbb{R}\), and \(q_1(t) = q_1(0) > 0\), \((6.1)\) implies that \((9.2)\) holds.

If \(X_2 = 0\), then the result follows from the analogous argument using Lemma \(7.3\) and Theorem \(7.5\).

10. Asymptotic behavior for MHD, \(\kappa > 0\)

**Theorem 10.1.** Fix \(\kappa > 0\). Let \(A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2)\) be a solution of \((3.1)\). Then either

- \(A\) is rigid,
- \(|A|\) is nonconstant and \(T\)-periodic with \(T > 0\),
- or \(|A| > 2\) and \(|A(t)| \searrow 2\), as \(t \to \pm \infty\).

**Proof.** Since \(\frac{1}{2}|A|^2 = 1 + q_1^2\) by Theorems \(6.12\) and \(7.5\), this summarizes Theorems \(6.13\) and \(7.6\) when \(\kappa > 0\). □

The next result provides a detailed analysis of the third option in Theorem 10.1.
Theorem 10.2. Fix $\kappa > 0$. If $X_1 = \kappa + \frac{1}{4}X_3^2$ with $X_3^2 > 8\kappa$, then the set
\begin{equation}
W(X_3) = \{(A, B) \in \mathcal{D} : X_i(A, B) = X_i, \quad i = 1, 3, \quad X_2(A, B) = 0\}
\end{equation}
is nonempty and invariant under the flow of \eqref{3.1} in $C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$.

Let $0 < \mu < \frac{1}{2}(X_3^2 - 8\kappa)^{1/2}$. If $A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ is a solution of \eqref{3.1} with initial data $(A_0, B_0) \in W(X_3)$, then there exist a time $T > 0$ and phases $\theta_{\pm}$ such that
\begin{align*}
\left| \left( \frac{d}{dt} \right)^j \left[ A(t) - U \left( \frac{1}{2}X_3t + \theta_{+} \right) \right] \right| &\lesssim e^{-\mu t}, \quad j = 0, 1, \quad \left\langle A(t), \dot{A}(t) \right\rangle < 0, \quad t > T, \\
\left| \left( \frac{d}{dt} \right)^j \left[ A(t) - U \left( \frac{1}{2}X_3t + \theta_{-} \right) \right] \right| &\lesssim e^{\mu t}, \quad j = 0, 1, \quad \left\langle A(t), \dot{A}(t) \right\rangle > 0, \quad t < -T.
\end{align*}

Proof. As outlined in Theorem 7.6, the assumptions on the invariants imply that the set
\[ \mathcal{C}(X_3) = \{(q, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 : \xi_2 = 0, \quad \xi_3 = X_3, \quad H_0(q_1, \xi_1, X_3) = H_0(0, 0, X_3)\} \]
is the union of an equilibrium point at $(0, 0)$ and two homoclinic orbits. The image of $\mathcal{C}(X_3)$ under $\Phi \circ \Gamma \circ \Psi_0$ is equal to $W(X_3)$, and so $W(X_3) \neq \emptyset$.

The invariance of $W(X_3)$ follows by virtue of the invariance of the quantities $X_i(A, \dot{A})$, according to Theorem 4.2.

Fix $\kappa > 0, \quad X_3 > 8\kappa$ and let $X_1 = \kappa + \frac{1}{2}X_3^2$. Suppose that $(q, \xi) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3)$ is a solution of \eqref{7.1} such that $(q_1(t), \xi_1(t))$ parameterizes the homoclinic orbit in $\mathcal{C}(X_3)$ with $q_1(t) > 0$. From \eqref{7.1}, we see that this orbit satisfies
\[ q_1(t) \searrow 0, \quad \xi_1(t) \nearrow 0, \quad \text{as } t \to \infty, \]
and
\[ q_1(t) \searrow 0, \quad \xi_1(t) \nearrow 0, \quad \text{as } t \to -\infty. \]

Let $\alpha^2 = \frac{1}{4}X_3^2 - 2\kappa$. Then for any $\varepsilon > 0$, we can find a $T \gg 1$ such that
\begin{equation}
0 < (|q_1(t)| + |\xi_1(t)|)^3 < \varepsilon(\alpha q_1(t) - \xi_1(t)), \quad t > T,
\end{equation}
and
\begin{equation}
0 < (|q_1(t)| + |\xi_1(t)|)^3 < \varepsilon(\alpha q_1(t) + \xi_1(t)), \quad t < -T.
\end{equation}

From \eqref{7.1}, we see that
\[ \dot{q}_1 - \dot{\xi}_1 = O(|q_1|^3 + |\xi_1|^3), \quad \dot{\xi}_1 - \alpha^2 q_1 = O(|q_1|^3 + |\xi_1|^3). \]

Taking linear combinations of these two equations yields
\[ \frac{d}{dt}(\alpha q_1 - \xi_1) + \alpha(\alpha q_1 - \xi_1) = O(|q_1| + |\xi_1|)^3 \]
and
\[ \frac{d}{dt}(\alpha q_1 + \xi_1) - \alpha(\alpha q_1 + \xi_1) = O(|q_1| + |\xi_1|)^3. \]

Choose $0 < \mu < \alpha$. Using \eqref{10.2} and \eqref{10.3}, we can find a $T > 0$ such that
\[ \frac{d}{dt}(\alpha q_1 - \xi_1) + \alpha(\alpha q_1 - \xi_1) < (\alpha - \mu)(\alpha q_1 - \xi_1), \quad t > T \]
From here, we obtain the bounds

\begin{equation}
0 < \alpha q_1(t) - \xi_1(t) \lesssim e^{-\mu t}, \quad t > T
\end{equation}

and

\begin{equation}
0 < \alpha q_1(t) + \xi_1(t) \lesssim e^{\mu t}, \quad t < -T.
\end{equation}

Going back to (7.1), we can now write

\[ q_3(t) = \frac{1}{2} X_3 t + \theta_+ + \frac{1}{2} X_3 \int_t^\infty \frac{q_1(s)^2}{2 + q_1(s)^2} ds, \quad \theta_+ = q_1(0) - \frac{1}{2} X_3 \int_0^\infty \frac{q_1(s)^2}{2 + q_1(s)^2} ds \]

and

\[ q_3(t) = \frac{1}{2} X_3 t + \theta_- - \frac{1}{2} X_3 \int_{-\infty}^t \frac{q_1(s)^2}{2 + q_1(s)^2} ds, \quad \theta_- = q_1(0) + \frac{1}{2} X_3 \int_{-\infty}^0 \frac{q_1(s)^2}{2 + q_1(s)^2} ds. \]

The integrals converge, thanks to (10.4), (10.5). Moreover, we have

\begin{equation}
|q_3(t) - (\frac{1}{2} X_3 t + \theta_+)| \lesssim e^{-\mu t}, \quad t > T
\end{equation}

and

\begin{equation}
|q_3(t) - (\frac{1}{2} X_3 t + \theta_-)| \lesssim e^{\mu t}, \quad t < -T.
\end{equation}

Recycling the estimates (10.4), (10.5), (10.6), (10.7) in (7.1) yields identical bounds for the derivatives.

By Theorem 7.5, we can write

\[ A(t) - U(\frac{1}{2} X_3 t + \theta_+) \]

\[ = U\left(\frac{1}{2}(q_3(t) + q_2(0))\right) \left((\rho(t) - 1)I + q_3(t)K\right) U\left(\frac{1}{2}(q_3(t) - q_2(0))\right)
\]

\[ + \left(U(q_3(t) - \frac{1}{2} X_3 t - \theta_+) - I\right) U\left(\frac{1}{2} X_3 t + \theta_+\right). \]

Thus, applying the bounds (10.4) and (10.6), we obtain

\[ |A(t) - U(\frac{1}{2} X_3 t + \theta_+)| \leq |(\rho(t) - 1)I + q_3(t)K| + |U(q_3(t) - \frac{1}{2} X_3 t - \theta_+) - I| \lesssim e^{-\mu t}, \]

for \( t > T \). The corresponding estimate for \( t < -T \) is proven in the same way using (10.5) and (10.7).

We note that

\[ \langle A(t), \dot{A}(t) \rangle = \frac{d}{dt} \frac{1}{2} |A(t)|^2 = \frac{d}{dt} (1 + q_1(t)^2) = q_1(t) \dot{q}_1(t). \]

By (7.1), we see that

\[ \text{sign} \langle A(t), \dot{A}(t) \rangle = \text{sign} \xi_1(t). \]

The argument for the case \( q_1(t) < 0 \) is symmetric.

\[ \square \]

**Remark 10.3.** The total phase shift is given by the expression

\[ \theta_+ - \theta_- = -\frac{1}{2} X_3 \int_{-\infty}^\infty \frac{q_1(s)^2}{2 + q_1(s)^2} ds. \]
Corollary 10.4. Fix $\kappa > 0$. If $X_1 = \kappa + \frac{1}{4} X_2^2$ with $X_2^2 > 8\kappa$, then the set
\begin{equation}
\mathcal{R}(X_3) = \{ (A, B) \in W(X_3) : A \in \text{SO}(2, \mathbb{R}) \}
\end{equation}
corresponds to the orbit of the rigid rotation $U \left( \frac{1}{2} X_3 t \right)$. The set $W(X_3) \setminus \mathcal{R}(X_3)$ is a stable and unstable manifold for $\mathcal{R}(X_3)$. Every solution orbit $(A(t), A'(t))$ in $W(X_3) \setminus \mathcal{R}(X_3)$ is homoclinic to $\mathcal{R}(X_1)$, that is,
\[ \lim_{|t| \to \infty} e^{\mu |t|} \text{dist}[(A(t), A'(t)), \mathcal{R}(X_3)] = 0, \]
for some $\mu > 0$.

Proof. The function $A(t) = U \left( \frac{1}{2} X_3 t \right)$ is a rigid rotation, by Theorem 9.2. Its orbit is $\{ (A, \frac{1}{2} X_3 Z A) : A \in \text{SO}(2, \mathbb{R}) \}$, and it is easily verified that this set coincides with $\mathcal{R}(X_3)$.

The final statement is a consequence of Theorem 10.2.

Remark 10.5. The corollary shows that the rigid rotational solutions are unstable within the class of affine motions when $X_3^2 > 8\kappa$.

Theorem 10.6. If $A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ is a solution of (3.1) such that $|A|$ is nonconstant and $T$-periodic for some $T > 0$, then the solution has the form
\[ A(t) = U \left( \frac{1}{2} (\omega_1 + \omega_2) t \right) \hat{A}(t) U \left( \frac{1}{2} (\omega_1 - \omega_2) t \right), \]
where $\hat{A}(t)$ is $T$-periodic if $A(t) \notin \text{SO}(2, \mathbb{R})$, for all $t \in \mathbb{R}$, and $2T$-periodic if $A(t) \in \text{SO}(2, \mathbb{R})$, for some $t \in \mathbb{R}$. The frequencies are defined by
\[ \omega_1 = \frac{1}{T} \int_0^T \frac{X_3}{\frac{1}{2} |A(t)|^2 + 1} \, dt \]
and
\[ \omega_2 = \begin{cases} 
0, & X_2 = 0 \\
\frac{1}{T} \int_0^T \frac{X_2}{\frac{1}{2} |A(t)|^2 - 1} \, dt, & X_2 \neq 0,
\end{cases} \]
where $X_i = X_i(A, \hat{A})$, $i = 2, 3$.

Proof. By Theorems 6.12 and 7.5 we can write
\begin{equation}
A(t) = \frac{1}{\sqrt{2}} U \left( \frac{1}{2} (q_3(t) + q_2(t)) \right) (\rho(t) I + q_1(t) K) U \left( \frac{1}{2} (q_3(t) - q_2(t)) \right),
\end{equation}
where the $q_i(t)$ are obtained by solving (6.1), if $X_2 \neq 0$, or (7.1), if $X_2 = 0$, with appropriate initial data.

We have that $q_1^2 = \frac{1}{2} |A|^2 - 1$ is nonconstant and $T$-periodic with $T > 0$. Note $q_1(t) = 0$ if and only if $A(t) \in \text{SO}(2, \mathbb{R})$. By Theorems 6.13 and 7.6 we see that either $q_1(t) \neq 0$, for all $t \in \mathbb{R}$, whence $q_1$ is also $T$-periodic, or $q_1(t) = 0$ for a sequence of times $t_0 + kT$, $k \in \mathbb{Z}$, whence by symmetry, $q_1$ is $2T$-periodic. Thus, we have that
\[ \rho(t) I + q_1(t) K \]
is either $T$- or $2T$-periodic, depending on whether $A$ passes through $\text{SO}(2, \mathbb{R})$ or not.

Notice that for each $i = 2, 3$, the quantity $\omega_i$ defined above is the mean of $\dot{q}_i$ over one period. Thus, we have that
\[ q_i(t) - \omega_i t \]
is $T$-periodic. So the now result follows from (10.9) by writing
\[ U\left(\frac{1}{2}(q_3(t) \pm q_2(t))\right) = U\left(\frac{1}{2}(\omega_1 \pm \omega_2)t\right) U\left(\frac{1}{2}(q_3(t) - \omega_1 t \pm q_2(t) \mp \omega_2 t)\right) \]
\[ \square \]

Remark 10.7. Note that the result holds for rigid solutions. In this case, the quantity $|A(t)|$ is constant and thus $T$-periodic for all $T \geq 0$. Any value of $T > 0$ can be used in computing the frequencies. Thus, solutions are generically quasiperiodic.

**Theorem 10.8.** Let $A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{R}^2)$ be a solution of (3.1) such that the quantity $|A(t)|$ is nonconstant and $T$-periodic for some $T > 0$.

For every $N \in \mathbb{N}$, there exists $\ell \in \{1, \ldots, N^2\}$ depending on $N$ such that
\[ |A(t + 2\pi T) - A(t)| \leq 8\pi|A(t)|/N, \quad \text{for all } t \in \mathbb{R}. \]
If $A(t)$ is rigid, then either $A(t)$ is periodic or the range of $A(t)$ is dense in the sphere of radius $|A(0)|$ in $SL(2, \mathbb{R})$.

**Proof.** By Theorem 10.6 we may write
\[ A(t) = U(\omega_1 t)\hat{A}(t)U(\omega_2 t), \]
in which $\hat{A}(t)$ is $2T$-periodic. (If $A(t)$ does not pass through $SO(2, \mathbb{R})$, then we know that $\hat{A}(t)$ is $T$-periodic.)

For every $x \in \mathbb{R}$, there is a unique $k \in \mathbb{Z}$ such that
\[ \{x\} \equiv x - 2\pi k \in [0, 2\pi). \]
Consider the set of $N^2 + 1$ ordered pairs
\[ \{ (\omega_1 jT, \omega_2 jT) : j = 0, 1, \ldots, N^2 \} \]
contained in the square $[0, 2\pi) \times [0, 2\pi)$. Partition this square into $N^2$ congruent subsquares of side $2\pi/N$. By the pigeonhole principle, two of these ordered pairs belong to the same subsquare. It follows that there exist $k, \ell \in \mathbb{Z}$ such that $0 \leq k < k + \ell \leq N^2$ and
\[ |\{\omega_1 kT\} - \{\omega_2 (k + \ell)T\}| \leq 2\pi/N, \quad i = 1, 2. \]
Thus, there exist $m_i \in \mathbb{Z}$ such that
\[ |\omega_i 2\ell T + 2\pi m_i| \leq 2\pi/N, \quad i = 1, 2. \]
Define
\[ \tau_i = \omega_i 2\ell T + 2\pi m_i. \]

For $i = 1, 2$ and $t \in \mathbb{R}$, we have using Definition 1.8 and the mean value theorem
\[ |U(\omega_i 2\ell T + t) - U(t)| = |U(\omega_i 2\ell T) - I| \]
\[ = |U(\tau_i) - I| \]
\[ = \sqrt{2}[(\cos \tau_i - 1)^2 + \sin^2 \tau_i]^{1/2} \]
\[ = 2(1 - \cos \tau_i)^{1/2} \]
\[ \leq 2|\tau_i| \]
\[ \leq 4\pi/N. \]
For any \( t \in \mathbb{R} \), we have
\[
A(2\ell T + t) = U(\omega_1(2\ell T + t))A(2\ell T + t)U(\omega_2(2\ell T + t)) = U(\omega_12\ell T)U(\omega_1 t)\hat{A}(t)U(\omega_22\ell T) = U(\tau_1)A(t)U(\tau_2).
\]
We now estimate as follows
\[
|A(2\ell T + t) - A(t)| = |U(\tau_1)A(t)U(\tau_2) - A(t)| = ||U(\tau_1) - I||A(t)U(\tau_2) + A(t)[U(\tau_2) - I]| \leq |U(\tau_1) - I||A(t)||U(\tau_2)| + |A(t)||U(\tau_2) - I| \leq 2(4\pi/N)|A(t)|.
\]
This proves the first statement.

If \( A(t) \) is rigid, then \( |A(t)| = |A(0)| \), and so by Theorem 10.6
\[
A(t) = U(\omega_1 t)A(0)U(\omega_2 t) = U(\{\omega_1 t\})A(0)U(\{\omega_2 t\}).
\]
If \( \omega_1 \) and \( \omega_2 \) are rationally dependent, then \( A(t) \) is periodic. The curve
\[
t \mapsto (\{\omega_1 t\}, \{\omega_2 t\})
\]
represents linear flow on the torus. If \( \omega_1 \) and \( \omega_2 \) are rationally independent, then it is well-known that the image of the curve is dense in the square \([0, 2\pi) \times [0, 2\pi)\). The set
\[
\{UA(0) \in SO(2, \mathbb{R}) : U, V \in \text{SO}(2, \mathbb{R})\}
\]
coinsides with the sphere of radius \( |A(0)| \) in \( \text{SL}(2, \mathbb{R}) \). Thus, the range of \( A(t) \) is dense in this sphere. \(\square\)

Remark 10.9. The only solutions \( A(t) \) for which \( |A(t)| \) is not periodic are those which are homoclinic to a rigid rotation. Thus, the result shows that, generically, solutions are recurrent.

Remark 10.10. Since
\[
|A(t)| \leq \left[ \frac{2}{\pi}X_1(A(t), \hat{A}(t)) \right]^{1/2}
\]
and the energy is conserved, Theorem 10.8 shows that
\[
|A(2\ell T + t) - A(t)| \lesssim 1/N, \quad \text{for all} \quad t \in \mathbb{R}.
\]

11. Asymptotic behavior for perfect fluids, \( \kappa = 0 \)

Theorem 11.1. Let \( \kappa = 0 \), and suppose that \( A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2) \) is a solution of (3.1). Then either
- \( A \) is rigid,
- \( |A(t)| \nearrow \infty \), as \( |t| \to \infty \),
- \[A(t), \hat{A}(t)\] > \( 0, t \in \mathbb{R} \), \( |A(t)| \nearrow \infty \), as \( t \to \infty \), and \( |A(t)| \searrow 2 \), as \( t \to -\infty \), or
- \[A(t), \hat{A}(t)\] < \( 0, t \in \mathbb{R} \), \( |A(t)| \nearrow \infty \), as \( t \to -\infty \), and \( |A(t)| \searrow 2 \), as \( t \to \infty \).
Proof. By Theorems 6.12, 7.5 and Lemma 6.4, we have \( \frac{1}{2} |A|^2 = 1 + q_1^2 \), where \( q_1 \) is obtained by solving (11.1) or (11.4). Thus, we find that

\[
\langle A, \dot{A} \rangle = \frac{d}{dt} \frac{1}{2} |A|^2 = 2q_1 \dot{q}_1,
\]

and so

\[
\text{sign} \langle A, \dot{A} \rangle = \text{sign} q_1 \xi_1.
\]

Thus, the result summarizes Theorems 6.13 and 7.6 when \( \kappa = 0 \). \( \square \)

In analogy with Theorem 10.2, we have

**Theorem 11.2.** Fix \( \kappa = 0 \). If \( X_1 = \frac{1}{4} X_3^2 > 0 \), then the sets

\[
W_s(X_3) = \{ (A, B) \in D : X_i(A, B) = X_i, \ i = 1, 3, \ X_2(A, B) = 0, \langle A, B \rangle < 0 \} 
\]

and

\[
W_u(X_3) = \{ (A, B) \in D : X_i(A, B) = X_i, \ i = 1, 3, \ X_2(A, B) = 0, \langle A, B \rangle > 0 \}
\]

are nonempty and invariant under the flow of \( (3.1) \) in \( C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2) \).

Let \( 0 < \mu < \frac{1}{4} |X_3| \). If \( A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2) \) is a solution of (3.1) with initial data \( (A_0, B_0) \in W_s(X_3) \), then there exist a time \( T > 0 \) and a phase \( \theta_+ \) such that

\[
\left| \begin{pmatrix} d \\ dt \end{pmatrix}^j [A(t) - U(\frac{1}{2} X_3 t + \theta_+)] \right| < e^{-\mu t}, \quad j = 0, 1, \quad t > T.
\]

If \( (A_0, B_0) \in W_u(X_3) \), then there exist a time \( T > 0 \) and a phase \( \theta_− \) such that

\[
\left| \begin{pmatrix} d \\ dt \end{pmatrix}^j [A(t) - U(\frac{1}{2} X_3 t + \theta_−)] \right| < e^{\mu t}, \quad j = 0, 1, \quad t < -T.
\]

**Proof.** As outlined in Theorem 7.6, the assumptions on the invariants imply that the sets

\[
C_s(X_3) = \{ (q, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 : q_1 \xi_1 < 0, \ \xi_2 = 0, \ \xi_3 = X_3, \ H_0(q_1, \xi_1, X_3) = H_0(0, 0, X_3) \}
\]

and

\[
C_u(X_3) = \{ (q, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 : q_1 \xi_1 > 0, \ \xi_2 = 0, \ \xi_3 = X_3, \ H_0(q_1, \xi_1, X_3) = H_0(0, 0, X_3) \}
\]

are each the union of two semi-bounded orbits of the system for \( (q_1, \xi_1) \). The images of \( C_s(X_3) \) and \( C_u(X_3) \) under \( \Phi \circ \Gamma \circ \Psi_0 \) are equal to \( W_s(X_3) \) and \( W_u(X_3) \), respectively.

The invariance of \( W_s(X_3) \) and \( W_u(X_3) \) follows by virtue of the invariance of the quantities \( X_i(A, \dot{A}) \), by Theorem 4.2, together with (11.1).

The remainder of the proof is similar to that of Theorem 10.2 and so we shall omit it. \( \square \)

**Theorem 11.3.** Let \( A \in C^0(\mathbb{R}, SL(2, \mathbb{R})) \cap C^2(\mathbb{R}, M^2) \) be a solution of (3.1) with initial data \( (A_0, B_0) \in D \). If \( |A(t)|^2 \nrightarrow \infty \) as \( t \to \infty \), then there exist \( A_\infty, B_\infty \in M^2 \) with \( B_\infty \neq 0 \) such that for \( t > 0, j = 0, 1, 2, \)

\[
(11.4) \quad \left| \begin{pmatrix} d \\ dt \end{pmatrix}^j [A(t) - (B_\infty t + A_\infty)] \right| \lesssim (1 + t)^{-1-j}.
\]

If \( A_\infty, B_\infty \in M^2 \) is any pair such that

\[
(11.5) \quad \lim_{t \to \infty} |A(t) - (B_\infty t + A_\infty)| = 0,
\]

and so

\[
(11.1) \quad \text{sign} \langle A, \dot{A} \rangle = \text{sign} q_1 \xi_1.
\]
then \((A_\infty, B_\infty) = (A_\infty, B_\infty)\).

The vectors \(A_\infty, B_\infty\) satisfy

\[
X_i(A_\infty, B_\infty) = X_i(A_0, B_0) \equiv X_i, \quad i = 1, 2, 3,
\]

and

\[
(B_\infty, \text{cof } A_\infty) = \det B_\infty = 0, \quad \det A_\infty = \frac{X_2^2 - X_2^2}{2X_1^2}.
\]

If \(2X_1 + X_2^2 - X_3^2 = 0\), then \((A_\infty, B_\infty) = (A_0, B_0) \in \mathcal{D}\) and

\[
A(t) = B_0t + A_0.
\]

Proof. By Theorems 6.12 and 7.5, we can write \(A(t)\) in terms of functions \(q_i(t), i = 1, 2, 3\), obtained by solving (6.1), if \(X_2 \neq 0\), or (7.1), if \(X_2 = 0\), with appropriate initial data. Since \(q_1^2(t) = \frac{1}{2}|A(t)|^2 - 1 \nearrow \infty\) as \(t \to \infty\), using Lemmas 6.10 and 7.3, we can write

\[
X_1 = \frac{1}{2} \left( \frac{(2 + q_1(t)^2)x_1(t)^2}{2(1 + q_1(t)^2)} + \frac{X_2^2}{q_1(t)^2} + \frac{X_3^2}{2 + q_1(t)^2} \right),
\]

for \(t \gg 1\). Thus, there exists a time \(t_0\) such that \(\xi_1(t)^2 > 2X_1\), for all \(t > t_0\). From (6.1), (7.1), it follows that

\[
|\dot{q}_1(t)| \geq |\xi_1(t)|/2 \geq (X_1/2)^{1/2}, \quad t \geq t_0.
\]

This implies the lower bound

\[
|A(t)|^2 \geq 2(1 + q_1(t)^2) \gtrsim (1 + t)^2, \quad t \geq 0.
\]

Since \(A \in C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)\) solves (3.1), we obtain from Lemma 8.2 that

\[
|\dot{A}(t)| \lesssim |A(t)|^{-3} \lesssim (1 + t)^{-3}, \quad t \geq 0.
\]

Thus, by Lemma 6 of [9], we can write

\[
A(t) = B_\infty t + A_\infty + A_1(t),
\]

with

\[
B_\infty = B_0 + \int_0^\infty \dot{A}(s)ds,
\]

\[
A_\infty = A_0 - \int_0^\infty \int_s^\infty \dot{A}(\sigma)d\sigma ds,
\]

\[
A_1(t) = \int_t^\infty \int_s^\infty \dot{A}(\sigma)d\sigma ds.
\]

Note that our estimate for \(|\dot{A}(t)|\) implies that

\[
\left| \left( \frac{d}{dt} \right)^j A_1(t) \right| \lesssim (1 + t)^{-1-j}, \quad t \geq 0, \quad j = 0, 1, 2,
\]

thereby proving (11.4).

If (11.5) holds, then using (11.4), we find that

\[
\lim_{t \to \infty} |(B_\infty - B_\infty)t + (A_\infty - \dot{A}_\infty)| = 0,
\]

and uniqueness of the states \((A_\infty, B_\infty)\) follows from this.
Applying (11.4), we find

\[ X_1 = \frac{1}{2} |\dot{A}(t)|^2 = \frac{1}{2} |B_\infty + \dot{A}_1(t)|^2 = \frac{1}{2} |B_\infty|^2 + O(t^{-1}), \quad t > 0. \]

Sending \( t \to \infty \) shows that \( X_1 = \frac{1}{2} |B_\infty|^2 \).

For the other invariants, we have for \( i = 2, 3 \),

(11.8) \[ X_i = X_i(A(t), \dot{A}(t)) = X_i(B_\infty t + A_\infty + A_1(t), B_\infty + \dot{A}_1(t)) = tX_i(B_\infty, B_\infty) + X_i(A_\infty, B_\infty) + O(t^{-1}). \]

By Lemmas 1.2 and 1.4 we see that \( X_i(B_\infty, B_\infty) = 0, i = 2, 3 \), and so letting \( t \to \infty \) we obtain \( X_i = X_i(A_\infty, B_\infty), i = 2, 3 \).

Since \( A(t) \in \text{SL}(2, \mathbb{R}) \), we get from Lemma 1.12

\[ 2 = 2 \det A(t) = \langle A(t), \text{cof } A(t) \rangle = t^2 \langle B_\infty, \text{cof } B_\infty \rangle + 2t \langle A_\infty, \text{cof } B_\infty \rangle + O(1) = 2t^2 \det B_\infty + 2t \langle A_\infty, \text{cof } B_\infty \rangle + O(1). \]

This implies that

\[ \det B_\infty = 0 \quad \text{and} \quad \langle A_\infty, \text{cof } B_\infty \rangle = 0. \]

So by Lemma 1.14 we get

\[ -X_2^2 + X_3^2 = \langle Z A_\infty, B_\infty \rangle \langle A_\infty Z, B_\infty \rangle = \frac{1}{2} \langle \text{cof } A_\infty, A_\infty \rangle |B_\infty|^2 = \det A_\infty \cdot 2X_1. \]

This verifies the statements (11.6).

If \( 2X_1 + X_2^2 - X_3^2 = 0 \), then \( A(t) = 0 \), by Theorem 8.2. By (11.7), we obtain \( A_1(t) = 0 \) and \( (A_\infty, B_\infty) = (A_0, B_0) \), so that \( A(t) = B_\infty t + A_\infty \). \( \square \)

Remark 11.4. In Theorem 11.5 if \( 2X_1 + X_2^2 - X_3^2 \neq 0 \), then \( A_\infty \notin \text{SL}(2, \mathbb{R}) \). Hence \((A_\infty, B_\infty) \notin \mathcal{D} \), and \( B_\infty t + A_\infty \notin C^0(\mathbb{R}, \text{SL}(2, \mathbb{R})) \).

Remark 11.5. An analogous result holds when \( |A(t)|^2 \nearrow \infty \), as \( t \to -\infty \).

12. The picture in the tangent space

In this final section, we examine the following question: For a given initial position \( A \in \text{SL}(2, \mathbb{R}) \), which initial velocities \( B \in T_A \text{SL}(2, \mathbb{R}) \) launch to a solution with vanishing pressure, a rigid solution, or a solution on an invariant manifold of the rigid rotations? We shall translate the conditions involving the invariants \( X_i(A, B) \) and the magnitude \( |A| \) given in Theorem 8.2, Theorem 9.2 and Theorems 10.2, 11.2 respectively, into local coordinates in \( T_A \text{SL}(2, \mathbb{R}) \).

We first suppose that \( A \in \text{SL}(2, \mathbb{R}) \setminus \text{SO}(2, \mathbb{R}) \) so that \((A, B) \in \mathcal{D} \setminus \mathcal{D}_0 \). Lemma 6.7 says that there exists \((q, \xi) \in \mathbb{R}^2_+ \times \mathbb{R}^3 \) such that

\[ (A, B) = \Phi \circ \Gamma \circ \Psi(q, \xi) = (\varphi \circ \psi(q), T(q)\xi). \]

Thus, the columns of \( T(q) \) span \( T_A \text{SL}(2, \mathbb{R}) \), and these vectors are orthogonal since the metric \( h(q) \) in these coordinates is diagonal. Let us normalize the coordinates by setting

\[ \xi = h(q)^{1/2} \xi. \]
If we fix $A$ by

\[
\text{(and hence also the pressure) vanishes if and only if}
\]
degenerates, and the only rigid solution is an equilibrium. This represents an ellipse in the $\tau$ plane. When $\kappa > 0$, we obtain a pair of lines

\[
\frac{\hat{\xi}_1^2}{1 + q_1^2} + \frac{\hat{\xi}_2^2}{2 + q_1^2} - \frac{\hat{\xi}_3^2}{2 + q_1^2} = 0.
\]

This relation describes a two-sheeted hyperboloid when $\kappa > 0$ and a cone when $\kappa = 0$. Note that the region of positive pressure is

\[
\frac{\hat{\xi}_1^2}{1 + q_1^2} + \frac{\hat{\xi}_2^2}{2 + q_1^2} - \frac{\hat{\xi}_3^2}{2 + q_1^2} > 0.
\]

**Rigid solutions.** By Theorem 9.2, the rigid solutions in $\text{SL}(2, \mathbb{R}) \setminus \text{SO}(2, \mathbb{R})$ are characterized by the relations (9.1) and (9.2) which reduce to

\[
\hat{\xi}_1 = 0, \quad \frac{\hat{\xi}_2^2}{q_1^2} + \frac{\hat{\xi}_3^2}{2 + q_1^2} = 2\kappa = 0.
\]

This represents an ellipse in the $\tau_2, \tau_3$ plane when $\kappa > 0$. When $\kappa = 0$, the condition degenerates, and the only rigid solution is an equilibrium.

**Stable and unstable manifolds of the rigid rotations.** When $\kappa > 0$, the collection of rigid rotations $\bigcup \{\mathcal{R}(X_3) : X_3^2 > 8\kappa\}$ have an invariant manifold $\bigcup \{\mathcal{W}(X_3) : X_3^2 > 8\kappa\}$, where $\mathcal{W}(X_3)$ and $\mathcal{R}(X_3)$ were defined in (10.1) and (10.8), respectively. In local coordinates, the defining conditions for this invariant manifold are

\[
\frac{1}{2} \hat{\xi}_1^2 + \kappa q_1^2 = \frac{1}{4} q_1^2 \hat{\xi}_3^2, \quad \hat{\xi}_2 = 0.
\]

If we fix $A \in \text{SL}(2, \mathbb{R}) \setminus \text{SL}(2, \mathbb{R})$, then $q_1 > 0$ and this defines a hyperbola in the $\tau_1, \tau_3$ plane. When $\kappa = 0$, we obtain a pair of lines

\[
|\hat{\xi}_1| = (q_1/\sqrt{2})|\hat{\xi}_3|, \quad \hat{\xi}_2 = 0.
\]

The interested reader can verify that $\text{sign} \langle A, B \rangle = \text{sign} \hat{\xi}_1$, so that the stable directions $B \in T_A \text{SL}(2, \mathbb{R})$ with $\langle A, B \rangle \in \mathcal{W}_u(X_3)$ correspond to points with $\xi_1 < 0$ on these lines while the unstable directions $B \in T_A \text{SL}(2, \mathbb{R})$ with $\langle A, B \rangle \in \mathcal{W}_u(X_3)$ correspond to points with $\xi_1 > 0$, (see (11.2), (11.3)).

These sets are illustrated in Figure 7 when $\kappa > 0$ and in Figure 8 when $\kappa = 0$.

The situation for $A \in \text{SO}(2, \mathbb{R})$ is fairly simple. We can use the orthonormal basis

\[
\tau_1(A) = \frac{1}{\sqrt{2}} AK, \quad \tau_2(A) = \frac{1}{\sqrt{2}} AM, \quad \tau_3(A) = \frac{1}{\sqrt{2}} AZ,
\]

where the columns of $\hat{T}(q)$ are orthonormal. Denote this orthonormal basis for $T_A \text{SL}(2, \mathbb{R})$ by $\{\tau_i(A)\}_{i=1}^3$. In these coordinates, we have according to Lemma 6.10

\[
X_1 = \frac{1}{2} \xi_1^2 + \kappa (1 + q_1^2), \quad X_2 = q_1 \hat{\xi}_2, \quad X_3 = (2 + q_1^2)^{1/2} \hat{\xi}_3,
\]

where $q_1^2 = \frac{1}{2} |A|^2 - 1 > 0$ is fixed.

**Solutions with vanishing pressure.** Theorem 8.2 shows that the Lagrange multiplier (and hence also the pressure) vanishes if and only if

\[
2X_1 + X_2^2 - X_3^2 = 0,
\]

which is equivalent to the equation

\[
\frac{\hat{\xi}_2^2}{q_1^2} + \frac{\hat{\xi}_3^2}{2 + q_1^2} - \frac{\hat{\xi}_1^2}{2 + q_1^2} = 0.
\]
Figure 7. Distinguished directions in $T_A\text{SL}(2, \mathbb{R})$ for a fixed $A \in \text{SL}(2, \mathbb{R}) \setminus \text{SO}(2, \mathbb{R})$, with $\kappa > 0$. The branch of pressureless directions in the half space $\xi_3 < 0$ is not shown.

Figure 8. Distinguished directions in $T_A\text{SL}(2, \mathbb{R})$ for a fixed $A \in \text{SL}(2, \mathbb{R}) \setminus \text{SO}(2, \mathbb{R})$, with $\kappa = 0$. The cone of pressureless directions in the half space $\xi_3 < 0$ is not shown.

in $T_A\text{SL}(2, \mathbb{R})$. In this case, if

$$B = \hat{\xi}_i \tau_i(A) \in T_A\text{SL}(2, \mathbb{R}),$$
then

\[ X_1 = \frac{1}{2} \mid \xi \mid^2 + \kappa, \quad X_2 = 0, \quad X_3 = \sqrt{2} \xi_3. \]

The solutions with vanishing pressure are

\[ \xi_1^2 + \xi_2^2 - \xi_3^2 - 2\kappa = 0, \]

consistent with sending \( q_1 \to 0 \) above. There are no directions \( B \in T_{A SL(2,\mathbb{R})} \) so that \((A, B)\) launches from \( D_0 \) to an invariant manifold in \( D \setminus D_0 \). Any \( B \in \text{span} \tau_3(A) \) leads to a rigid rotation.
# Appendix A. Glossary of notation

| Symbol | Reference | Description |
|--------|-----------|-------------|
| $\mathbb{M}^2$ | Def 1.1 | vector space of $2 \times 2$ matrices over $\mathbb{R}$ |
| $\langle \cdot, \cdot \rangle$ | Def 1.1 | Euclidean inner product on $\mathbb{M}^2$ or on $\mathbb{R}^n$ |
| $I, Z, K, M$ | Def 1.3 | orthogonal basis vectors in $\mathbb{M}^2$ |
| $\text{SL}(2, \mathbb{R})$ | Def 1.5 | special linear group |
| $\text{SO}(2, \mathbb{R})$ | Def 1.5 | special orthogonal group |
| $U(\theta)$ | Def 1.8 | parameterization of SO(2, $\mathbb{R}$) |
| $\text{cof}$ | Lem 1.10 | cofactor map |
| $T_{\text{ASL}(2, \mathbb{R})}$ | Lem 2.1 | tangent space at $A \in \text{SL}(2, \mathbb{R})$ |
| $\mathcal{D}$ | Def 2.2 | tangent bundle / phase space |
| $\mathcal{D}_0$ | Def 2.2 | subset of $\mathcal{D}$ |
| $\varphi(x)$ | Def 2.4 | immersion |
| $g(x)$ | Lem 2.7 | metric |
| $\Phi(x, y)$ | Def 2.9 | tangent bundle map |
| $\mathfrak{sl}(2, \mathbb{R})$ | Def 3.2 | special linear Lie algebra |
| $L(A, B)$ | Def 3.4 | velocity gradient map |
| $\Lambda(A, B)$ | Def 3.8 | Lagrange multiplier |
| $X_i(A, B)$ | Def 4.1 | invariant quantities |
| $H(x, p)$ | Lem 5.3 | Hamiltonian |
| $\Gamma(x, p)$ | Lem 5.5 | Legendre transformation |
| $\psi(q)$ | Def 6.1 | immersion |
| $\mathbb{R}_+^3$ | Rem 6.2 | half space |
| $\Psi(x, p)$ | Def 6.5 | canonical transformation |
| $\mathbb{R}_+^1$ | Def 6.5 | one-dimensional subspace |
| $h(q)$ | Lem 6.7 | metric |
| $\tilde{H}(q, \xi)$ | Def 6.8 | Hamiltonian |
| $H_0(q, \xi)$ | Def 7.2 | Hamiltonian |
| $\mathcal{W}(X_3)$ | Eq (10.1) | invariant manifold |
| $\mathcal{R}(X_3)$ | Eq (10.8) | rotational invariant manifold |
| $\mathcal{W}_s(X_3)$ | Eq (11.2) | stable manifold |
| $\mathcal{W}_u(X_3)$ | Eq (11.3) | unstable manifold |
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