Symmetry Protected Topological Order by Folding a One-Dimensional Spin-1/2 Chain

Pejman Jouzdani

1Department of Physics, University of Central Florida, Orlando, Florida 32816, USA

(Dated: March 10, 2015)

We present a toy model with a Hamiltonian $H_T^{(2)}$ on a folded one-dimensional spin chain. The non-trivial ground states of $H_T^{(2)}$ are separated by a gap from the excited states. By analyzing the symmetries in the model, we find that the topological order is protected by a $\mathbb{Z}_2$ global symmetry. However, by using perturbation series and excluding thermal effects, we show that the $\mathbb{Z}_2$ symmetry is stable in comparison to a standard nearest-neighbor Ising model with a Hamiltonian $H_I$. We find that $H_T^{(2)}$ is a member of a family of Hamiltonians that are adiabatically connected to $H_I$. Furthermore, the generalizations of this class of Hamiltonians, their adiabatic connection to $H_I$, and the relation to quantum error-correcting codes are discussed. Finally, we show the correspondence between the two ground states of $H_T^{(2)}$ and the unpaired Majorana modes, and provide numerical examples.

*Intro.*—A large effort has been made to construct robust protocols for quantum computation by exploiting topological properties of many-body systems [1, 2]. Models such as the toric and surface codes [3–5] are proposed. A particular attention has been given to detect and employ exotic non-Abelian excitations in quantum computation [6, 7], with a special attention to the Majorana fermions [8, 9]. In this order, a crucial step is to detect and characterize the topological orders. It has been shown by Levin and Wen [10] that the ground state of quantum many-body systems with non-trivial orders can be seen as a condensate of fluctuating string-like objects. In particular, quantum phases are commonly studied by the projective symmetry groups (PSG) tool [11] which has been used to identify symmetry protected topological orders (SPT) [12, 13], and with focus to quantum computation and quantum error correction [14–18].

Consider a one-dimensional spin-1/2 Ising model with the Hamiltonian

$$H_I = -J \sum_{i=1}^{N-1} \sigma^z_i \sigma^z_{i+1},$$

on a chain with $N$ spins ($J > 0$) where $\sigma^\nu_i$ is the $\nu$-th component of the Pauli matrices acting on site $i$. Ignoring thermal excitation for a moment, a longitudinal field perturbation $U_x = \sum h_i \sigma^x_i$ lifts the degeneracy of the ground states for any non-zero $h_i$. However, in the absence of a longitudinal field, the degeneracy is topologically protected against a perturbation $U_x = \sum h_i \sigma^x_i$.

In comparison to the toric code [19] where the system is defined on a $L \times L$ lattice and the protection against perturbation is of the order of $L$, we could think of the one-dimensional Ising chain as a $1 \times N$ two-dimensional lattice. The lattice has a width of only “one” lattice site. Therefore, any (longitudinal) single-spin perturbation already reaches the size of the lattice. Thus, the Ising chain has a topological phase, but the phase can not be realized due to the short width of the lattice. Although in different words, this claim was originally expressed in a footnote in Ref. [20].

In this Letter we introduce an adiabatic transformation that does not change the topological characteristic of the Ising Hamiltonian $H_I$, but effectively folds the spin chain to a $2 \times \frac{N}{2}$ lattice. As a result, we obtain a Hamiltonian $H_T^{(2)}$ and the topological protection of the one-dimensional Ising chain tremendously improves, unexpectedly. In order to do so, the steps bellow are followed.

First, a Hamiltonian $H_T^{(2)}$ on an open chain of spin-1/2 is defined. $H_T^{(2)}$ is a sum of four-spin operators. The set of these operators form a group that is denoted by $\mathcal{S}^{(2)}$. In addition, we find two symmetry groups $\mathcal{S}_1^{(2)}$ and $\mathcal{S}_2^{(2)}$ that commute with all the elements of $\mathcal{S}^{(2)}$. We find that $(\mathcal{S}_1^{(2)} \otimes \mathcal{S}_2^{(2)})/\mathcal{S}^{(2)} = \mathbb{Z}_2$ which indicates the non-trivial order of the model is protected as long as the global symmetry $\mathbb{Z}_2$ is preserved. Next, we show through a degenerate perturbation analysis that the global symmetry $\mathbb{Z}_2$ is robust. Then, the extension to models with wider width $H_T^{(\text{width})}$ and with symmetry groups $\{\mathcal{S}_1^{(\text{width})}\}$ is discussed. Finally, we show the Hamiltonian $H_T^{(2)}$ is adiabatically connected to $H_I$ in Eq. (1),

$$H_T^{(2)} = R(\pi) H_I R^\dagger(\pi),$$

where $R(\alpha)$ is the unitary transformation

$$R(\alpha) = e^{i\alpha V}.$$  

Here $V$ is a sum of two-spin interactions and $\alpha$ is a scalar parameter. The relation between Majorana modes of the Kitaev toy model [21] and the two ground states of $H_T^{(2)}$ are explained as well.

*The Model.*—Consider a one-dimensional spin-1/2 system of a length $N = 4m + 2$, with a positive and integer...
FIG. 1. (a) The standard one-dimensional Ising model with nearest-neighbor interactions. (b) The reshaped Ising model into a ladder after the unitary transformation $R$ in spin space. The diagonal dashed and dash-dot lines show the two-spin terms $\sigma_i^x \sigma_j^x$ used in the transformation. (c) The Hamiltonian $H_{T}^{(2)}$ resulting from the transformation has four-spin interactions, associated to plaquettes. The bulk has two different types of plaquettes, $A$ and $B$. There are three operators of type $A$ and one operator of type $B$. The two left and right plaquettes on the boundaries are named $R$ and $L$, respectively. There are two operators acting on each of these boundary plaquettes.

number $m$. We define $H_{T}^{(2)}$ as

$$H_{T}^{(2)} = \sum_s O_s$$

$$= -J \sum_{i=1}^{m-1} \left[ A_i^{(1)} + A_i^{(2)} + A_i^{(3)} \right] - J \sum_{i=0}^{m-1} B_i$$

$$- J \left( L_1 + R_1 + L_2 + R_2 \right),$$

(4)

where we have introduced the following operators (the stabilizers $\{O_s\}$):

$$A_i^{(1)} = \sigma_i^x \sigma_{i+1}^y \sigma_{i+2}^y \sigma_{i+3}^y$$

(5a)

$$A_i^{(2)} = \sigma_i^y \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x$$

(5b)

$$A_i^{(3)} = \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x$$

(5c)

$$B_j = \sigma_j^y \sigma_{j+1}^x \sigma_{j+2}^x \sigma_{j+3}^x \sigma_{j+4}^y \sigma_{j+5}^y$$

(5d)

$$L_1 = \sigma_1^x \sigma_2^y \sigma_3^y \sigma_4^y$$

(5e)

$$R_1 = \sigma_3^y \sigma_4^y \sigma_5^y \sigma_6^y$$

(5f)

$$L_2 = \sigma_2^x \sigma_3^x \sigma_4^x \sigma_5^x$$

(5g)

$$R_2 = \sigma_4^x \sigma_5^x \sigma_6^x \sigma_7^x$$

(5h)

with $i = 1, \ldots, m-1$ and $j = 0, \ldots, m-1$.

All the terms on the r.h.s. of Eq. (4) commute with each other. As shown in Fig. 1c, one can identify $A_i^{(k)}$ and $B_j$ as bulk plaquettes operators, while $R_{1,2}$ and $L_{1,2}$ act as boundary plaquette operators. For $N = 4m + 2$ there are $3(m-1)$ plaquette operators of type $A$, $m$ plaquette operators of type $B$, and $2$ plaquette operators of types $R$ and $L$ each. Therefore, there are overall $4m + 1$ plaquette operators - that is $N - 1$. Without proof, the set of the stabilizers defined in Eq. (4) generates a group that we denote by $S_{\alpha}^{(2)}$.

Separately, consider the set of the operators $\{\sigma_i^x \sigma_j^x\}$, on pairs $\{(i,j)\}$. Unless otherwise mentioned, we use the notation $(i,j)$ for the two sites $i$ and $j$ that are connected with a dashed or a dot-dash line in Fig. 1b. We define the group generated by these operators as $S_{\alpha}^{(2)} = (\sigma_1^x \sigma_2^x, \ldots, \sigma_N^x \sigma_{N-1}^x)$. All elements in $S_{\alpha}^{(2)}$ commute with all elements in $S_{\alpha}^{(2)}$.

Furthermore, consider the set of operators $\{\sigma_i^x \sigma_j^y\}$ on pairs $\{(i,j)\}$. We define the group generated by these operators as $S_{\beta}^{(2)} = (\sigma_1^x \sigma_2^y, \ldots, \sigma_N^x \sigma_{N-1}^y)$. All elements in $S_{\beta}^{(2)}$ commute with all elements in $S_{\alpha}^{(2)}$ and all elements in $S_{\alpha}^{(2)}$. Especially, we find $(S_{\alpha}^{(2)} \otimes S_{\beta}^{(2)})/S_{\alpha}^{(2)} = \mathbb{Z}_2$ (there are $N/2$ generators in $S_{\alpha}^{(2)}$, $N/2$ generators in $S_{\beta}^{(2)}$, and $N - 1$ generators in $S_{\alpha}^{(2)}$; $2^N/2^{N-1} = 2$). A comprehensive classification of SPT orders in one-dimensional spin systems is given in Ref. [10].

The physical implication of the above statements is the following. If we manage to have a fixed “gauge” $S_{\alpha}^{(2)}(G) = +|G\rangle$ for the ground state subspace $|G\rangle$ of $H_{T}^{(2)}$, as long as the $\mathbb{Z}_2$ symmetry is not broken, the ground state is doubly degenerate and we can have $|G_{+}\rangle$ such that $\gamma_{G_{+}} = e^{i\alpha/2}|G_{-}\rangle$ for any generator $\gamma$ of $S_{\alpha}^{(2)}$.

The phase $e^{i\pi/2}$ is a global phase and it is equal to $e^{i\pi/2}$ for $H_{T}^{(2)}$ defined in Eq. (4).

To see this, consider the states $|\uparrow\rangle_z = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_N$ and $|\downarrow\rangle_z = |\downarrow\rangle_1 \otimes \cdots \otimes |\downarrow\rangle_N$ where by definition $\sigma_i^z |\uparrow\rangle_i = + |\uparrow\rangle_i$ and $\sigma_i^z |\downarrow\rangle_i = - |\downarrow\rangle_i$. Next, for every element $s_k \in S_{\alpha}^{(2)}$ we have $s_k |\uparrow\rangle_{z} = + |\uparrow\rangle_{z}$ (since $s_k |\downarrow\rangle_{z} = + |\downarrow\rangle_{z}$), and thus the gauge $S_{\alpha}^{(2)}$ is set to $+1$. Then, by applying the group elements of $S_{\alpha}^{(2)}$ on each of the states $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ we obtain

$$|G_{+}\rangle = \frac{1}{\sqrt{4^m}} \prod_s [1 + O_s] |\uparrow\rangle_z$$

(6)

and

$$|G_{-}\rangle = \frac{1}{\sqrt{4^m}} \prod_s [1 + O_s] |\downarrow\rangle_z,$$

(7)

where the product is over the stabilizers $O_s$. Notice that $2m = N - 1$ is the number of distinguishable plaquettes in Fig. 1b. Thus, $4^m$ is the number of “loops” that can be constructed on the folded chain using the plaquettes as the unit blocks. The basis states appear as condensates of string-like configurations [12]. Interestingly, for a generator $\gamma = \sigma_i^x \sigma_j^y$ of $S_{\alpha}^{(2)}$

$$\gamma|G_{+}\rangle = i|G_{-}\rangle,$$

(8)

since $\sigma^y |\uparrow\rangle = +i |\downarrow\rangle$. The equations (6), (7), and the property in Eq. (8) can be examined in the examples given at the end of this Letter.
In fact, the property in Eq. (8) is not a coincidence if one remembers that by a Jordan-Wigner transformation the Ising model maps to the unpaired Majorana problem [23]. Thus, the generators in the group $S^{(2)}_2$ act equivalently as a logical operation for the ground states of $H^{(2)}_T$.

Stability of the $\mathbb{Z}_2$ symmetry.– In a one-dimensional Ising spin on an open chain a non-zero longitudinal field opens a gap between $|\uparrow\rangle$ and $|\downarrow\rangle$. The gap stimulates topological excitations (a propagating domain wall) from a false vacuum to a true vacuum and eventually destroys the symmetry [21].

Similar to $H_I$, the Hamiltonian $H^{(2)}_T$ has a discrete energy spectrum and at low temperature ($k_BT \ll J$) thermal excitations are energetically costly and can be considered forbidden. By a degenerate perturbation series approach we see that a transverse field perturbation such as $U_x = \sum_i h_i \sigma_i^x$ (or $U_y = \sum_i h_i \sigma_i^y$) has vanishing matrix elements $\langle G|U_x^i|G\rangle$ for all the powers $l < N$, in the ground states subspace $\langle G\rangle \equiv \{G_+\langle G_+| + |G_-\rangle\langle G_-|\}$. Therefore, we have a topological protection with respect to this transversal field. Especially, the only non-vanishing term is $\langle G|X = \prod_{l=1}^N \sigma_x^i|G\rangle$.

Considering the bases $\{G_+\} \pm i \{G_-\}$, we find $X = \prod_{l=1}^N \sigma_x^i$ and any generator $\gamma \in S^{(2)}_2$ as the bit-flip and phase-flip logical operators, respectively.

In contrast to $H_I$, in the presence of two transversal fields $(H^{(2)}_T + U_x + U_y)$, the first non-vanishing term $\langle G|U_x U_y|G\rangle \propto \langle G|\sigma_i^x \sigma_j^y|G\rangle$ appears at the second order of the perturbation series and only on pairs $(i,j)$. This means that, out of $\binom{N}{2}$ number of possible terms in the second order, only $\frac{N^2}{2}$ of them are non-zero. Thus, by increasing the length, the second order non-vanishing terms are suppressed by a factor of $O(\frac{1}{N})$. This is opposite to the $H_I + U_x$ case where in the second order of perturbation there are $\binom{N}{2} - (N-1)$ non-vanishing terms ($O(N)$). This pattern continues in all the orders. The odd orders vanish. In the fourth order there is a suppression factor of $O(\frac{1}{N^2})$, etc.

Thus, as long as the perturbation (noise) affects single spins (no correlated noise) we should expect that the interplay between multiplicity and energy cost in the perturbation series (the statistical ground for a phase transition) to be substantially suppressed in our model with $H^{(2)}_T$ Hamiltonian. Therefore, we expect a stable global symmetry $\mathbb{Z}_2$. Whether the $\mathbb{Z}_2$ symmetry will still be destroyed through other mechanisms is a question to be answered. For a ladder of the toric code this has been studied [23].

Translational symmetry breaking and generalization.– By a close look at Fig. 8k and Fig. 8l, we notice that the periodicity changes as we move from the Ising chain in Fig. 8k to $H^{(2)}_T$ in Fig. 8l. The unit cell in $H^{(2)}_T$ is the two-plaques $AB$ and the periodicity goes as $\cdots ABAB\cdots$ in the bulk, while in the Ising chain it goes as $\cdots ZZZZ\cdots$. The emergence of non-trivial topological orders in one dimensional “organic” polymers by breaking translational symmetry has an old history [25].

Also, notice that the width of the folded chain in Fig. 9 contains two sites. It should be now clear to the reader why we chose $N = 4 \times m + 2$. There are 4 sites ($2 \times \text{width}$) in each unit cell and the last 2 sites are added to keep the inversion symmetry and have $(S^{(2)}_1 \otimes S^{(2)}_2)/S^{(2)}_2 = \mathbb{Z}_2$. Although, it is not clear whether the inversion symmetry is necessary.

Similarly, we can construct a Hamiltonian $H^{(3)}_T$ with a width of “three” sites and define stabilizers with “six” operators [26]. In this case, the length of the chain is chosen to be $N = 6m$ with a positive integer $m$. That is, $m-1$ unit cells in the bulk. There are five different $A$-type operators, defined on $A$ plaquettes, and one $B$-type stabilizer (in a hexagonal shape), with two boundary stabilizers on each edge. Therefore, there are $N-1$ number of stabilizers. The non-trivial part is to show that there are two independent symmetry groups $S^{(3)}_1$ and $S^{(3)}_2$ that commute with the group of stabilizers of $H^{(3)}_T, S^{(3)}$. That is, they satisfy $(S^{(3)}_1 \otimes S^{(3)}_2)/S^{(3)} = \mathbb{Z}_2$.

$H^{(3)}_T$ is the first member of the quantum error-correcting codes that can be constructed by folding a line and has a code distance of $O(1)$; That is roughly half of the width (3) of the folded configuration. In principle, it should be possible to construct quantum error-correcting codes with longer code distance and larger stabilizers with Hamiltonian $H^{\text{width}_T}$. As we will see for the case of $H^{(2)}_T$ below, there is an adiabatic connection between this class of Hamiltonians and $H_I$ in Eq. (8).

The Hamiltonian $H^{(2)}_T$ is adiabatically connected to the one-dimensional Ising Hamiltonian $H_I$.– We begin with the standard nearest-neighbor Ising Hamiltonian in Eq. 1 with $J > 0$. Next, the unitary transformation $R$ defined in Eq. (4) with

$$V = \sum_{(i,j)} \sigma_i^x \sigma_j^y$$

is used to map $H_I$ to $H^{(2)}_T$ according to Eq. (5). The result is

$$H(\alpha) = R(\alpha) H I R^{\dagger}(\alpha)$$

$$= \cos^2 \left(\frac{\alpha}{2}\right) H_I + \cos \left(\frac{\alpha}{2}\right) \sin \left(\frac{\alpha}{2}\right) H_I + \sin^2 \left(\frac{\alpha}{2}\right) H^{(2)}_T,$$  

where $H_I$ and $H^{(2)}_T$ involve three-body and four-body interaction terms, respectively. For $\alpha = \pi$, the contribution of $H_I$ to the total Hamiltonian drops out and we obtain $H(\pi) = H^{(2)}_T$.

Since we obtain $H^{(2)}_T$ by a unitary transformation from $H_I$, the energy spectrum must stay unchanged and the
ground state subspace of $H_T^{(2)}$ must be two-fold degenerate. However, if two gapped states are connected by a set of local unitary transformations they belong to the same phase \([27]\). This can be seen by applying $R(\pi)$ on a state $|\uparrow\rangle$. One obtains

$$|\psi\rangle = R(\pi)|\uparrow\rangle = \prod_{(i,j)} (|\uparrow\rangle_i |\uparrow\rangle_j - i |\downarrow\rangle_i |\downarrow\rangle_j),$$  

which is a product state. As it can be checked, clearly a second degenerate state is not accessible by using Eq. \([6]\). That is, for a generator $\gamma$ of the group $S_2^2$, we have $\gamma |\psi\rangle = - |\psi\rangle$ while $N = 4m + 2$. This means $|\psi\rangle = \frac{1}{\sqrt{2}} (|G_+\rangle - i |G_-\rangle)$. This is not surprising if one thinks of the Majorana counterpart of the problem where the two unpaired Majorana modes ($|G_+\rangle$ and $|G_-\rangle$) are paired up and experimentally undetectable.

Then, how can we observe the phase that corresponds to $|G_+\rangle$ (or $|G_-\rangle$)? One quick answer is to find a way to implement logical quantum gates. We are seeking an operation such that $M^{(\pm)} |\psi\rangle = |G_\pm\rangle$. Since $\gamma$ is a logical phase-flip operation and $X = \prod_{i=1}^N \sigma_i^x$ is the logical bit-flip operation, we should be able to decompose $M^{(-)}$ and have

$$M^{(-)} = \frac{1}{\sqrt{2}} [\gamma X - i \gamma].$$  

Notice that the logical operation $M^{(-)}$ is unitary but non-local. Implementing $M^{(-)}$, if ever possible experimentally, would change the quantum phase from a product state in Eq. \((\textbf{11})\) to a Majorana mode \((\textbf{6})\). This can be checked for a chain with $N = 6$ in the examples bellow.

Notice that we can obtain $\gamma$ and $X$, corresponding to the logical operations of $H_T^{(2)}$, by transforming the $X = \prod_{i=1}^N \sigma_i^x$ and the order parameter $\sigma_i^x$ corresponding to $H_I$ under $R$. In general, to obtain the logical operations corresponding to $H_T^{(\text{width})}$ one needs to know the adiabatic transformation from $H_I$ and the knowledge of the symmetry groups of the Hamiltonian becomes irrelevant.

**Examples and numerical results.**—Let us consider two examples. The first example is a chain of $N = 6$. Following Eq. \((\textbf{11})\), we have (see Fig. \(\textbf{2b}\))

$$H_T^{(2)}_{N=6} = -J (L_1 + L_2 + B_0 + R_1 + R_2).$$  

By exact diagonalization we obtain the two basis states

$$|G_{+, N=6}\rangle = \frac{1}{\sqrt{4}} [|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\downarrow\downarrow\rangle - |\uparrow\uparrow\uparrow\downarrow\downarrow\rangle]$$

$$|G_{-, N=6}\rangle = \frac{1}{\sqrt{4}} [|\downarrow\downarrow\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\downarrow\downarrow\rangle - |\downarrow\uparrow\uparrow\uparrow\uparrow\rangle]$$  

The second example is a chain with $N = 4$. It does not exactly follow the prescription defined in Eq. \((\textbf{6})\). However, we can define (see Fig. \(\textbf{3a}\))

$$H_{T, N=4}^{(2)} = -J (\sigma_i^x \sigma_j^x \sigma_j^x \sigma_i^x + \sigma_i^y \sigma_j^y \sigma_j^y \sigma_i^y + \sigma_i^z \sigma_j^z \sigma_j^z \sigma_i^z),$$  

and the inversion symmetry in this case is still preserved. It has the degenerate basis states

$$|G_{+, N=4}\rangle = \frac{1}{\sqrt{2}} [|\uparrow\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle]$$

$$|G_{-, N=4}\rangle = \frac{1}{\sqrt{2}} [|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle].$$  

Notice that by applying $\sigma_i^x \sigma_j^y \in S_2^2$ on $|G_{+}^{(4)}\rangle (|G_{+}^{(6)}\rangle)$ one obtains $i(G_{+}^{(4)} (iG_{+}^{(6)}))$.

We use the dependence of the splitting of the ground states, $\Delta(\alpha) = E_{+}(\alpha) - E_{-}(\alpha)$, on the global external magnetic field as a criterion to numerically verify the enhanced protection in $H_T$. The numerical calculation of the splitting for $N = 4$ at the points $\alpha = 0 (H_I)$ and $\alpha = \pi (H_T)$ is shown in Figs. \(\textbf{3}\) and \(\textbf{4}\) as a function of $h_z$ (longitudinal) and $h_y$ (transverse) external magnetic fields, respectively. One can see that the dependence of the splitting $\Delta_g$ goes from linear for $H_I$ to quadratic for $H_T$, indicating increased protection. Figure \(\textbf{3b}\) shows the gap $\Delta_g$ which behaves topologically protected for both $H_I$ and $H_T$, as expected.

**Summary.**—The standard one-dimensional Ising chain with a Hamiltonian $H_I$ could theoretically be in a topological phase if the global $Z_2$ symmetry were stable. The symmetry is not stable since the order parameter of the system is just a single spin. We showed how to adiabatically obtain a Hamiltonian $H_T^{(2)}$ by a local unitary transformation with only two-spin interactions from $H_I$. We showed that the protection against single-spin errors in the transformed Hamiltonian $H_T^{(2)}$ scales with the length
FIG. 3. The energy splitting between the two low-lying states at $\alpha = 0$ ($H_I$, circles) and $\alpha = 1$ ($H_T$, squares) as a function of external field for a $N = 4$ spin chain. (a) The splitting as a function of a longitudinal field $h_z$ is shown for the two $H_I$ and $H_T^{(2)}$. For any non-zero longitudinal magnetic field $h_z$, a gap opens linearly for $H_I$. The situation is visibly different (quadratic) for $H_T^{(2)}$ as discussed by perturbation analysis. (b) The splitting as function of a transverse field $h_y$ is the same for both $H_I$ and $H_T^{(2)}$.

of the chain. We discussed a family of Hamiltonians that are adiabatically connected to $H_I$.

Acknowledgments.— The author thanks Eduardo Mucciolo for his advise and support and Alioscia Hamma for his critical comments. This work was supported in part by the National Science Foundation grant CCF-1117241.

[1] S. D. Sarma, M. Freedman, and C. Nayak, Phys. Today 59, 32 (2006).
[2] A. Kitaev, Ann. Phys. 321, 2 (2006).
[3] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. 43, 4452 (2002).
[4] S. B. Bravyi and A. Y. Kitaev, arXiv:quant-ph/9811052.
[5] A. Yu. Kitaev, Ann. Phys. 303, 2 (2003).
[6] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[7] J. Alicea, Phys. Rev. B 81, 125318, (2010).
[8] L. Fu and C. L. Kane, Phys. Rev. Lett. 102, 216403 (2009).
[9] E. Ginossar and E. Grosfeld, Nat. Comm. 5, 4772 (2014).
[10] A. Bühlner, N. Lang, C. V. Kraus, G. Möller, S. D. Huber and H. P. Büchler, Nat. Comm. 5, 4504 (2014).
[11] C. V. Kraus et al., New J. Phys. 14 113036, (2013).
[12] M. A. Levin and X. G. Wen, Phys. Rev. B 71, 045110 (2005).
[13] X. G. Wen, Phys. Rev. B 65, 165113 (2002).
[14] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa Phys. Rev. B 85, 075125 (2012).
[15] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. 59, 799 (1987).
[16] X. Chen, Z.-C. Gu, X.-G. Wen, Phys. Rev. B 83, 035107 (2011).
[17] X. Chen, Y. Lu, and A. Vishwanath, Nat. Comm. 5, 3507 (2014).
[18] W. Son, L. Amico, R. Fazio, A. Hamma, S. Pascazio, and V. Vedral, Europhys. Lett. 95, 50001 (2011).
[19] A. Miyake, Phys. Rev. Lett. 105, 040501 (2010).
[20] D. V. Else, I. Schwarz, S. D. Bartlett, and A. C. Doherty, Phys. Rev. Lett. 108, 240505 (2012).
[21] A. Altland and B. D. Simons, Condensed Matter Field Theory (Cambridge University Press, 2008), Ch. 3.
[22] Notice that $S^{(2)}_i \subset S^{(2)}$. Thus, only one generator is enough to represent the whole subgroup.
[23] A. Y. Kitaev, Phys.-Usp. 44, 131 (2001).
[24] V. Karimipour, L. Memarzadeh, and P. Zarkeshian, Phys. Rev. A 87, 032322 (2013).
[25] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).
[26] The details will be published elsewhere.
[27] X. Chen, Z. C. Gu, and X. G. Wen, Phys. Rev. B 82, 155138 (2010).