On the Bands of the Schrödinger Operator with a Matrix Potential

O.A. Veliev
Dogus University, Istanbul, Turkey.
e-mail: oveliev@dogus.edu.tr

Abstract
In this article we consider the one-dimensional Schrödinger operator $L(Q)$ with a Hermitian periodic $m \times m$ matrix potential $Q$. We investigate the bands and gaps of the spectrum and prove that most of the positive real axis is overlapped by $m$ bands. Moreover, we find a condition on the potential $Q$ for which the number of gaps in the spectrum of $L(Q)$ is finite.

Key Words: Self-adjoint differential operator, Spectral bands, Periodic matrix potential.
AMS Mathematics Subject Classification: 34L05, 34L20.

1 Introduction and Preliminary Facts

Let $L(Q)$ be the differential operator generated in the space $L^{2}_{m}(-\infty, \infty)$ of the vector functions $y = (y_{1}, y_{2}, \ldots, y_{m})$ by the differential expression

$$-y'' + Qy,$$

(1)

where $y_{k} \in L^{2}_{2}(-\infty, \infty)$ for $k = 1, 2, \ldots, m$, $Q(x) = (q_{s,j}(x))$ is a $m \times m$ Hermitian matrix for all $x \in (-\infty, \infty)$, $q_{s,j}$ is the complex-valued locally square summable function and $Q(x+1) = Q(x)$. It is well-known that [2, Chap.XIII], [4, 10, 13] the spectrum $\sigma(L(Q))$ of the operator $L(Q)$ is the union of the spectra of the operators $L_{t}(Q)$ for $t \in (-\pi, \pi]$, where $L_{t}(Q)$ is the operator generated in $L^{2}_{m}[0, 1]$ by the differential expression (1) and the quasiperiodic conditions $y(1) = e^{it}y(0)$, $y'(1) = e^{it}y'(0)$. For $t \in (-\pi, \pi]$ the spectra $\sigma(L_{t}(Q))$ of the operators $L_{t}(Q)$ consist of the eigenvalues

$$\lambda_{1}(t) \leq \lambda_{2}(t) \leq \cdots$$

(2)

called the Bloch eigenvalues of $L(Q)$. The $n$-th band function $\lambda_{n}$ continuously depends on $t$ and its range

$$I_{n}(Q) = \{ \lambda_{n}(t) : t \in (-\pi, \pi]\}$$

(3)
is called the $n$-th band of the spectrum of $L$:

$$\sigma(L(Q)) = \bigcup_{n=1}^{\infty} I_n(Q).$$

The continuity of $\lambda_n$ in the case $m = 1$ was proved in [12]. The general case follows from the arguments of the perturbation theory described in [6] and [12]. In the Remark 1 of the next section, for the independence of this paper, we give a proof of this statement within the framework of this paper. The bands $I_n$ approach infinity as $n \to \infty$. The spaces between the bands $I_k$ and $I_{k+1}$ (if exist) for $k = 1, 2, ..., $ are called the gaps in the spectrum of $L(Q)$.

In this paper we investigate the set of the Bloch eigenvalues, bands and gaps of $L(Q)$. For this first we consider the set of the Bloch eigenvalues of the operators $L(O)$ and $L(C)$, where $O$ is the $m \times m$ zero matrix and $C = \int_{[0,1]} Q(x) \, dx$. (4)

It is clear that

$$\varphi_{k,1,t} = \begin{pmatrix} e_{k,t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \varphi_{k,2,t} = \begin{pmatrix} 0 \\ e_{k,t} \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad \varphi_{k,m,t} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e_{k,t} \end{pmatrix}$$

are the eigenfunctions of the operator $L(O)$ corresponding to the eigenvalue $(2\pi k + t)^2$, where $e_{k,t}(x) = e^{i(2\pi k + t)x}$. If $t \neq 0, \pi$ then the multiplicity of the eigenvalue $(2\pi k + t)^2$ is $m$ and the corresponding eigenspace is

$$E_k(t) = \operatorname{Span} \{\varphi_{k,1,t}, \varphi_{k,2,t}, \ldots, \varphi_{k,m,t}\}.$$

In the cases $t = 0$ and $t = \pi$ the multiplicities of the eigenvalues $(2\pi k)^2$ for $k \in (\mathbb{Z} \setminus \{0\})$ and $(2\pi k + \pi)^2$ for $k \in \mathbb{Z}$ are $2m$ and the corresponding eigenspaces are

$$E_k(0) = \operatorname{Span} \{\varphi_{n,j,0} : n = k, j = 1, 2, \ldots, m\}$$

and

$$E_k(\pi) = \operatorname{Span} \{\varphi_{n,j,\pi} : n = k, -(k+1), j = 1, 2, \ldots, m\}$$

respectively. Thus the points $(2\pi k)^2$ for $k \in (\mathbb{Z} \setminus \{0\})$ and $(2\pi k + \pi)^2$ for $k \in \mathbb{Z}$ are the exceptional Bloch eigenvalues of $L(O)$, in the sense that at these points the multiplicities of the eigenvalues are changed. It is clear that $(2\pi k + t)^2$ is an exceptional Bloch eigenvalue of $L(O)$ if and only if

$$(2\pi k + t)^2 = (2\pi n + t)^2$$

(5)
for some $n \neq k$. Since (5) holds only in the cases $t = 0, n = -k; k \neq 0$ and $t = \pi, n = -k - 1$, only the points $(2\pi k)^2$ for $k \in (\mathbb{Z} \setminus \{0\})$ and $(2\pi k + \pi)^2$ for $k \in \mathbb{Z}$ are the exceptional Bloch eigenvalues of $L(O)$.

To analyze the set of the Bloch eigenvalues and the spectrum of the operator $L(C)$ we introduce the following notations, where $C$ is defined in (4). Denote by $\mu_1 < \mu_2 < \ldots < \mu_p$ the distinct eigenvalues of the Hermitian matrix $C$. If the multiplicity of $\mu_j$ is $m_j$, then $m_1 + m_2 + \ldots + m_p = m$. Let $u_{j,1}, u_{j,2}, \ldots, u_{j,m_j}$ be the eigenvectors of the matrix $C$ corresponding to the eigenvalue $\mu_j$. It is not hard to see that

$$\Phi_{k,j,s,t}(x) = u_{j,s} e^{i(2\pi k + t)x}$$

for $s = 1, 2, \ldots, m_j$ are the eigenfunctions of $L_t(C)$ corresponding to the eigenvalue

$$\mu_{k,j}(t) = (2\pi k + t)^2 + \mu_j. \quad (6)$$

To consider the spectrum of $L(Q)$ we use the following result of [16].

Theorem 4(a) of [16]. All large eigenvalues of $L_t(Q)$ lie in $\varepsilon_k$ neighborhood

$$U_{\varepsilon_k}(\mu_{k,j}(t)) := (\mu_{k,j}(t) - \varepsilon_k, \mu_{k,j}(t) + \varepsilon_k)$$

of the eigenvalues $\mu_{k,j}(t)$ of $L_t(C)$, where $\varepsilon_k = c_1\left(\frac{1}{\ln|k|} + q_k\right)$,

$$q_k = \max\left\{\int_{[0,1]} q_{s,r}(x) e^{-2\pi inx} dx : s, r = 1, 2, \ldots, m; n = \pm 2k, \pm(2k + 1)\right\},$$

$c_1$ is a constant and does not depend on $t \in (-\pi, \pi)$. Moreover, for each large eigenvalue $\mu_{k,j}(t)$ of $L_t(C)$ there exists an eigenvalue of $L_t(Q)$ lying in $\varepsilon_k$ neighborhood of $\mu_{k,j}(t)$.

Now let’s explain a brief outline of this paper. Using Theorem 4(a) of [16], we first prove that most of the positive real axis is overlapped by $m$ bands and estimate the length of the gap between the bands (see Theorems 1 and 2). Then, in order to investigate the spectrum of $L(Q)$ in detail by using the asymptotic formulas and perturbation theory we consider the multiplicities of the eigenvalues of $L_t(C)$ and the large exceptional points of the spectrum of $L(C)$. We consider the operator $L(Q)$ as perturbation of $L(C)$ by $Q - C$ and prove that the perturbation $Q - C$ may generate the gaps in $\sigma(L(Q))$ only at the neighborhoods of the exceptional Bloch eigenvalues (see Theorem 3 and Corollary 2) of $L(C)$. In Theorem 4 we find a condition (see Condition 1) on the eigenvalues of the matrix $C$ for which the number of gaps in the spectrum of $L(Q)$ is finite. Note that in [16] we proved Theorem 4 under the assumption that the matrix $C$ has three simple eigenvalues $\mu_1, \mu_2, \mu_3$ satisfying Condition 1. These assumption simplifies the proof of Theorem 4. In this paper we prove Theorem 4 without any conditions on the multiplicity of these eigenvalues. Finally, note that in [8, 14, 15] we studied the non-self-adjoint operators with a periodic matrix potential. This paper can be considered as continuation of
the paper [16], in which the self-adjoint case was investigated. The self-adjoint case, was considered also in [1], where the main goal was to reformulate some spectral problems for the differential operator with periodic matrix coefficients as problems of conformal mapping theory.

Finally note that a great number of paper is devoted to the scalar case \( m = 1 \) (see for example the monographs [3] and [7] and the paper [9]). In this paper we consider the finite-zone potentials in the vectorial case. Therefore let us only stress the significant difference between the scalar and vectorial cases in the investigations of the finite-zone potentials. In case \( m = 1 \) the finite zone potentials are infinitely differentiable functions and have a special form expressed by Riemann \( \theta \) function (see [7, Chapters 8 and 9] and [5]), while in the vectorial case we guarantee finite number of gaps under simple algebraic condition on the eigenvalue of the matrix \( C \). Moreover, the method used in this paper for the investigation of the vectorial case is absolutely different from the methods used in the scalar case.

2 Main Results

One can easily verify that the set 

\[
\{(2\pi k + t)^2 : t \in (-\pi, \pi], \; k \in \mathbb{Z}\}
\]

of Bloch eigenvalues \((2\pi k + t)^2\) of \( L(O) \) overlap \( 2m \) times (counting the multiplicity) the half line \((0, \infty)\), since \((2\pi k + t)^2 = (-2\pi k - t)^2\) for \( t \in (0, \pi) \) and the multiplicity of the eigenvalues \((2\pi k)^2\) for \( k \in (\mathbb{Z} \setminus \{0\}) \) and \((2\pi k + \pi)^2\) for \( k \in \mathbb{Z} \) is \( 2m \). Since the both Bloch eigenvalues \((2\pi k + t)^2\) and \((-2\pi k - t)^2\) belong to the same band of \( L(O) \), any element of the half line \((0, \infty)\) is overlapped by \( m \) bands of \( L(O) \). To investigate this overlapping problem for the operator \( L(Q) \) let us note that the eigenvalues of \( L_t(Q) \) numbered in non-decreasing order (see (2)) continuously depend on \( t \in (-\pi, \pi] \).

Remark 1 Here we prove (within the framework of this paper) that the eigenvalues \( \lambda_n(t) \) defined in (2) continuously depend on \( t \). The eigenvalues of the operator \( L_t(Q) \) are the roots of the characteristic determinant

\[
\Delta(\lambda, t) = \det(U_v(Y_j))_{\nu=1}^2 = e^{i2mt} + f_1(\lambda)e^{i(2m-1)t} + f_2(\lambda)e^{i(2m-2)t} + \ldots + f_{2m-1}(\lambda)e^{it} + 1
\]

which is a polynomial of \( e^{it} \) with entire coefficients \( f_1(\lambda), f_2(\lambda), \ldots \), where

\[
U_v(Y_j) = Y_j^{(\nu-1)}(1, \lambda) - e^{it}Y_j^{(\nu-1)}(0, \lambda),
\]

\( Y_1(x, \lambda) \) and \( Y_2(x, \lambda) \) are the solutions of the matrix equation

\[-Y''(x) + Q(x)Y(x) = \lambda Y(x) \]
satisfying $Y_1(0, \lambda) = O$, $Y_1'(0, \lambda) = I$ and $Y_2(0, \lambda) = I$, $Y_2'(0, \lambda) = 0$ (see [11] Chapter 3).

Now using these statements we prove that for each $n$ the function $\lambda_n$ defined
in (3) is continuous at each point $t_0 \in (-\pi, \pi]$. Since $\lambda_n(t_0) \to \infty$ as $n \to \infty$, there
exist $k \leq n$ and $p \geq n$ such that $\lambda_{k-1}(t_0) < \lambda_k(t_0) = \lambda_{k+1}(t_0) = ... = \lambda_p(t_0) < \lambda_{p+1}(t_0)$ if $\lambda_n(t_0) > \lambda_1(t_0)$. Then the boundaries of the rectangles

$$R_1 = \{ c < x < d_1, |y| < 1 \} \text{ and } R_2 = \{ c < x < d_2, |y| < 1 \}$$

belong to the resolvent set of the operator $L_{t_0}(Q)$, where $\lambda_{k-1}(t_0) < d_1 < \lambda_k(t_0)$,
$\lambda_p(t_0) < d_2 < \lambda_{p+1}(t_0)$ and $c$ is a number for which $\sigma(L) \subset (c, \infty)$. It implies
that $\Delta(\lambda, t_0) \neq 0$ for each $\lambda \in \partial(R_1)$. Since $\Delta(\lambda, t_0)$ is a continuous function on
the compact $\partial(R_1)$, there exists $a > 0$ such that $|\Delta(\lambda, t_0)| > a$ for all $\lambda \in \partial(R_1)$.
Moreover, $\Delta(\lambda, t)$ is a polynomial of $e^{it}$ with entire coefficients. Therefore,
there exists $\delta_1 > 0$ such that $|\Delta(\lambda, t)| > a/2$ for all $t \in (t_0 - \delta_1, t_0 + \delta_1)$
and $\lambda \in \partial(R_1)$. It means that $\partial(R_1)$ belong to the resolvent set of $L_{t}(Q)$ for all
$\lambda \in (t_0 - \delta_1, t_0 + \delta_1)$. Moreover

$$(L_t - \lambda I)^{-1} f(x) = \int_0^1 G(x, \xi, \lambda, t) f(\xi) d\xi,$$

where $G(x, \xi, \lambda, t)$ is the Green’s function of $L_t - \lambda I$ defined by formula

$$G(x, \xi, \lambda, t) = g(x, \xi, \lambda) - \frac{1}{\Delta(\lambda, t)} \sum_{j,v=1}^2 Y_j(x, \lambda) V_{jv}(x, \lambda) U_v(g),$$

(see formula (8) of [7, p.117]). Here $g$ does not depend on $t$ and $V_{jv}$ is the transpose of that $n$th-order matrix consisting of the cofactor of the element
$U_v(Y_j)$ in the determinant $\det(U_v(Y_j))_{j,v=1}^2$. Hence the entries of the matrices
$V_{jv}(x, \lambda)$ and $U_v(g)$ either do not depend on $t$ or have the form $u(1, \lambda) - e^{it}u(0, \lambda)$
and $h(1, \xi, \lambda) - e^{it}h(0, \xi, \lambda)$ respectively, where the functions $u$ and $h$ do not depend on $t$. Therefore using these formulas and the last inequality for $|\Delta(\lambda, t)|$
one can easily verify that $(L_t - \lambda I)^{-1}$ continuously depend on $t \in (t_0 - \delta_1, t_0 + \delta_1)$
for $\lambda \in \partial(R_1)$. This implies that the operators $L_t$ for each $t \in (t_0 - \delta_1, t_0 + \delta_1)$
have $k - 1$ eigenvalues in $R_1$, since $L_{t_0}$ have $k - 1$ eigenvalues in $R_1$. It is clear that these eigenvalues are $\lambda_1(t), \lambda_2(t), ..., \lambda_{k-1}(t)$. In the same way we prove
that there exist $\delta_2 > 0$ such that the operators $L_t$ for $t \in (t_0 - \delta_2, t_0 + \delta_2)$
have $p$ eigenvalues in $R_2$ and they are $\lambda_1(t), \lambda_2(t), ..., \lambda_p(t)$. Thus the closed
rectangle $R = \{ d_1 \leq x \leq d_2, |y| \leq 1 \}$ contains $p - k + 1$ eigenvalues of $L_t$ for
t $\in (t_0 - \delta, t_0 + \delta)$ and they are $\lambda_k(t), \lambda_{k+1}(t), ..., \lambda_p(t)$, where $\delta = \min\{\delta_1, \delta_2\}$
and $n \in [k, p]$.

Now we are ready to prove that $\lambda_n$ is continuous at the point $t_0$ if $\lambda_n(t_0) > \lambda_1(t_0)$.
Consider any sequence $\{ (\lambda_n(t_k), t_k) : k \in \mathbb{N} \}$ such that $t_k \in (t_0 - \delta, t_0 + \delta)$
for all $k \in \mathbb{N}$ and $t_k \to t_0$ as $k \to \infty$. Let $(\lambda, t_0)$ be any limit point of the sequence
$\{ (\lambda_n(t_k), t_k) : k \in \mathbb{N} \}$. Since $\Delta$ is a continuous function with respect to the pair
$(\lambda, t)$ and $\Delta(\lambda_n(t_k), t_k) = 0$ for all $k$ we have $\Delta(\lambda, t_0) = 0$. It means that
\( \lambda \) is an eigenvalue of \( L_{t_0}(Q) \) lying in the rectangle \( R \), that is, \( \lambda = \lambda_k(t_0) = \lambda_{k+1}(t_0) = \ldots = \lambda_p(t_0) \), where \( n \in [k, p] \). Thus \( \lambda_n(t_k) \rightarrow \lambda_n(t_0) \) as \( k \rightarrow \infty \) for any sequence \( \{t_k : k \in \mathbb{N}\} \) converging to \( t_0 \) and \( \lambda_n \) is continuous at the point \( t_0 \). To prove the case \( \lambda_n(t_0) = \lambda_1(t_0) \) it is enough to consider only the rectangle \( R_2 = \{c < x < d_2, \ |y| < 1\} \), where \( \lambda_1(t_0) = \lambda_2(t_0) = \ldots = \lambda_p(t_0) < d_2 < \lambda_{p+1}(t_0) \).

First using this Remark and Theorem 4(a) of [16] we consider the overlapping problem for \( L(Q) \). For this in the following remark we explain Theorem 4(a) of [16] for the family of the operators \( L(C + \varepsilon(Q - C)) \) for \( \varepsilon \in [0, 1] \).

**Remark 2** To prove Theorem 4(a) we used the formula
\[
(\lambda_{k,j}(t) - \mu_{n,i}(t))(\Psi_{k,j,t}, \Phi_{n,i,t}) = (\varepsilon(Q - C)\Psi_{k,j,t}, \Phi_{n,i,t})
\] (7)
and proved that
\[
(\Psi_{k,j,t}(x), (Q(x) - C)\Phi_{n,i,t}(x)) = O\left(\frac{\ln |k|}{k}\right) + O(b_k).
\] (8)
Moreover, the last estimation does not depend on \( t \). If the potential \( Q \) is replaced by \( C + \varepsilon(Q - C) \) then the formula (7) has the form
\[
(\lambda_{k,j}(t) - \mu_{n,i}(t))(\Psi_{k,j,t}, \Phi_{n,i,t}) = (\varepsilon(Q(x) - C)\Psi_{k,j,t}, \Phi_{n,i,t})
\] (9)
Instead of (7) using (9) and repeating the proof of (8) we get
\[
(\Psi_{k,j,t}(x), \varepsilon(Q(x) - C)\Phi_{n,i,t}(x)) = O\left(\frac{\ln |k|}{k}\right) + O(\varepsilon q_k).
\]
Therefore, repeating the proof of Theorem 4(a) of [16] we obtain that all large eigenvalues of \( L_t(C + \varepsilon(Q - C)) \) for all \( \varepsilon \in [0, 1] \) lie in \( \varepsilon_k := c_1(\frac{\ln |k|}{k} + q_k) \) neighborhood of the eigenvalues \( \mu_{k,j}(t) \) for \( |k| \geq N \) and \( j = 1, 2, \ldots, p \), where the constants \( N \) and \( c_1 \) do not depend on \( t \) and \( \varepsilon \).

Now we are ready to prove the following theorem about the overlapping problem.

**Theorem 1** There exists a positive integer \( N_1 \) such that if \( s \geq N_1 \) then the intervals
\[
I(s) := \left( (s\pi)^2 + \mu_p + \varepsilon(s), (s\pi + \pi)^2 + \mu_1 - \varepsilon(s) \right)
\]
are contained in each of the bands
\[
I_{sm+1}(Q), I_{sm+2}(Q), \ldots, I_{sm+m}(Q),
\]
where \( \varepsilon(s) = \varepsilon_k \) if \( s \in \{2k, 2k + 1\} \) and \( \varepsilon_k \) is defined in Remark 2.
**Proof.** First consider the case $s = 2k$. One can easily verify that the number of the periodic Bloch eigenvalues of $L(C)$ (the eigenvalues of $L_0(C)$ counting the multiplicity) lying in the interval
\[ \left[ c, (s\pi)^2 + \mu_p + \varepsilon(s) \right] \]
is $sm + m$, where $c$ is a constant such that the spectra of the operators $L(C + \varepsilon(Q - C))$ are contained in $(c, \infty)$ for all $\varepsilon \in [0, 1]$. It follows from Remark 2 that there exist constants $N_1$ and $c$ such that if $s \geq N_1$, then the boundary of the rectangle
\[ R_1 = \left\{ c < x < (s\pi)^2 + \mu_1 + \varepsilon(s), \ |y| < 1 \right\} \]
belong to the resolvent set of the operators $L_0(C + \varepsilon(Q - C))$ for all $\varepsilon \in [0, 1]$. Hence, the projection of $L_0(C + \varepsilon(Q - C))$ defined by contour integration over the boundary of $R_1$ depends continuously on $\varepsilon$. It implies that the number of eigenvalues (counting the multiplicity) of $L_0(C + \varepsilon(Q - C))$ lying in $R_1$ are the same for all $\varepsilon \in [0, 1]$. Since $L_0(C)$ has $sm + m$ eigenvalues (counting the multiplicity) in $R_1$, the operator $L_0(Q)$ has also $sm + m$ eigenvalues.

In the same way we prove that the rectangle
\[ R_2 = \left\{ c < x < (s\pi)^2 + \mu_1 - \varepsilon(s), \ |y| < 1 \right\} \]
contains $sm - m$ eigenvalues of $L_0(Q)$. Therefore the interval
\[ \left( (s\pi)^2 + \mu_1 - \varepsilon(s), (s\pi)^2 + \mu_p + \varepsilon(s) \right) \] (10)
contains $2m$ periodic eigenvalues and they are
\[ \lambda_{sm-m+1}(0) \leq \lambda_{sm-m+2}(0) \leq \cdots \leq \lambda_{sm+m}(0). \]

In the similar way we prove that the interval
\[ \left( (s\pi + \pi)^2 + \mu_1 - \varepsilon(s), (s\pi + \pi)^2 + \mu_p + \varepsilon(s) \right) \] (11)
contains $2m$ antiperiodic eigenvalues and they are
\[ \lambda_{sm+1}(\pi) \leq \lambda_{sm+2}(\pi) \leq \cdots \leq \lambda_{sm+2m}(\pi) \]
Thus the bands $I_r$ for $r = sm + 1, sm + 2, \ldots, sm + m$ contain the point $\lambda_r(0)$ from interval (10) and the point $\lambda_r(\pi)$ from interval (11). Therefore the bands $I_{sm+1}, I_{sm+2}, \ldots , I_{sm+m}$ contains the interval $I(s)$. In the same way we prove the case $s = 2k + 1$. ■

**Theorem 1** immediately imply the following.

**Corollary 1.** Any spectral gap ($\alpha, \beta$) with $\alpha > (\pi N_1)^2$ (if exists) is contained in the intervals
\[ U(s) := ((\pi s)^2 + \mu_1 - \varepsilon(s - 1), (\pi s)^2 + \mu_p + \varepsilon(s)), \]
for $s > N_1$. Moreover, the spectral gap $(\alpha, \beta) \subset U(s)$ lies between the bands $I_{sm}(Q)$ and $I_{sm+1}(Q)$. 

7
Proof. By Theorem 1 the intervals \( I(s) \) for \( s \geq N_1 \) are the subsets of the spectrum. Therefore the spectral gap \((\alpha, \beta)\) with \( \alpha > (\pi N_1)^2 \) is contained between \( I(s-1) \) and \( I(s) \) for some \( s > N_1 \). It means that \((\alpha, \beta) \subset U(s)\). The bands \( I_{m+j}(Q) \) and \( I_{m+j+1}(Q) \) for \( j = 1, 2, ..., m - 1 \) have common intervals \( I(s) \) and hence there is not gaps between they. It means that the gap \((\alpha, \beta)\) is located between the bands \( I_{m_s} \) and \( I_{m_{s+1}} \). ■

Now using Corollary 1 and Theorem 4(a) of [16] we prove the following.

**Theorem 2** The length of the spectral gaps \((\alpha, \beta)\) lying in \( U(s) \) for \( s > N_1 \) is not greater than \( 2 \max \{\varepsilon(s-1), \varepsilon(s)\} \).

**Proof.** By Corollary 1 we have

\[
\left((\pi s)^2 + \mu_1 - \varepsilon(s-1)\right) \leq \alpha < \beta \leq (\pi s)^2 + \mu_p + \varepsilon(s).
\]

Now suppose on the contrary that the length of the gap \((\alpha, \beta)\) is greater than \( 2 \max \{\varepsilon(s-1), \varepsilon(s)\} \). Then it follows from last equalities that

\[
\frac{\alpha + \beta}{2} \in ((\pi s)^2 + \mu_1, (\pi s)^2 + \mu_p), \quad \frac{\beta - \alpha}{2} > \max \{\varepsilon(s-1), \varepsilon(s)\}
\]

Using (6) one can easily conclude that there exist \( t \in (-\pi, \pi] \) and \( j \in \{1, 2, ..., p\} \) such that the equality \( \mu_{s,j}(t) = \frac{\alpha + \beta}{2} \) holds. On the other hands, by Theorem 4(a) of [16] there exists an eigenvalue \( \lambda \) of \( L_t(Q) \) lying in \( \max \{\varepsilon(s-1), \varepsilon(s)\} \)

neighborhood of \( \frac{\alpha + \beta}{2} \). Therefore

\[
\lambda \in \left(\frac{\alpha + \beta}{2} - \frac{\beta - \alpha}{2}, \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2}\right) = (\alpha, \beta).
\]

Hence \((\alpha, \beta)\) is not a gap in the spectrum of \( L(Q) \). This contradiction imply the proof of the theorem. ■

To investigate the spectrum of \( L(Q) \) in detail by using the asymptotic formulas and perturbation theory we need to consider the multiplicities of the eigenvalues of \( L_t(C) \) and the exceptional points of the spectrum of \( L(C) \). The multiplicity of \( \mu_{n,j}(t) \) is \( m_j \) if \( \mu_{k,j}(t) \neq \mu_{n,i}(t) \) for all \( (n, i) \neq (k, j) \). The multiplicity of \( \mu_{k,j}(t) \) is changed, that is, \( \mu_{k,j}(t) \) is an exceptional point of the spectrum of \( L(C) \) if

\[
(2\pi k + t)^2 + \mu_j = (2\pi n + t)^2 + \mu_i \quad (12)
\]

for some \( (n, i) \neq (k, j) \). Since \((2\pi k + t)^2 = (-2\pi k - t)^2\), it is enough to study the equality (12) for \( t \in [0, \pi] \) and \( k \in \mathbb{Z} \). Moreover, we investigate the large exceptional Bloch eigenvalues \((2\pi k + t)^2 + \mu_j \) of \( \sigma(L(C)) \), because we are going to investigate the spectrum of the operator \( L(Q) \) by using the asymptotic formulas for the large eigenvalues. In other words, we need to consider (12) in the case when \( |k| \) is a large number. Then (12) has a solution \( t \in [0, \pi] \) only in the cases \( n = -k \) and \( n = -k - 1 \). In these cases (12) implies that the large eigenvalue \((2\pi k + t)^2 + \mu_j \) is an exceptional Bloch eigenvalue of \( L(C) \) if either

\[
\mu_{k,j}(t) - \mu_{-k,i}(t) = 8\pi kt + \mu_j - \mu_i = 0 \quad (13)
\]
or
\[ \mu_{k,j}(t) - \mu_{-k-1,i}(t) = 4\pi(2k + 1)(t - \pi) + \mu_{j} - \mu_{i} = 0 \] (14)
for some \( i = 1, 2, \ldots, p \). We denote by
\[ t(2k, j, i) = \frac{\mu_{i} - \mu_{j}}{4\pi(2k)} \]
and
\[ t(2k + 1, j, i) = \pi + \frac{\mu_{i} - \mu_{j}}{4\pi(2k + 1)} \]
the solutions of equations (13) and (14) lying in \([0, \pi]\). Thus eliminating the sets \( \{ t(2k, j, i) : i = 1, 2, \ldots, p \} \) and \( \{ t(2k + 1, j, i) : i = 1, 2, \ldots, p \} \) from \([0, \pi]\) we conclude that, if \( t \) belongs to the remaining part of \([0, \pi]\), then the multiplicity of the eigenvalue \( \mu_{k,j}(t) \) is \( m_{j} \), where \( k \) is a large number. In other word, \( \mu_{k,j}(t) \) is a non-exceptional Bloch eigenvalue of \( L(C) \). However, to investigate the perturbation of these non-exceptional Bloch eigenvalues by using the asymptotic formulas obtained in [16], we eliminate \( \delta_{k} \)-neighborhoods \( U_{\delta_{k}}(t(2k, j, i)) \) and \( U_{\delta_{k}}(t(2k + 1, j, i)) \) of \( t(2k, j, i) \) and \( t(2k + 1, j, i) \) from \([0, \pi]\), where \( \delta_{k} = o(k^{-1}) \).

Moreover, \( \delta_{k} \) can be chosen so that the remaining part of \([0, \pi]\) consists of the pairwise disjoint intervals \([a(k, j, s), b(k, j, s)]\) for \( s = 1, 2, \ldots, v \) :

\[ [0, \pi] \setminus \left( \bigcup_{i=1,2,\ldots,p} (U_{\delta_{k}}(t(2k, j, i)) \cup U_{\delta_{k}}(t(2k + 1, j, i))) \right) = \bigcup_{s=1}^{v} [a(k, j, s), b(k, j, s)] , \] (15)

where \( a(k, j, s) < b(k, j, s) < a(k, j, s+1) < b(k, j, s+1) \) for \( s = 1, 2, \ldots, v \). These intervals have the following property.

**Lemma 1** There exists \( N_{2} \) such that if \( |k| > N_{2} \),

\[ 4\pi(2|k| - 2)\delta_{k} = 2\max \{ \varepsilon_{k}, \varepsilon_{-k}, \varepsilon_{-k-1} \} , \] (16)

and the quasimomentum \( t \) belongs to the intervals \([a(k, j, s), b(k, j, s)]\) defined in (15), then the following statements hold.

(a) The inequality

\[ |\mu_{k,j}(t) - \mu_{n,i}(t)| \geq 4\pi(2|k| - 1)\delta_{k} \] (17)

holds for \( n = -k, -k - 1 \) and for all \( i = 1, 2, \ldots, p \).

(b) The closed interval

\[ U_{\varepsilon_{k}}(\mu_{k,j}(t)) = [\mu_{k,j}(t) - \varepsilon_{k}, \mu_{k,j}(t) + \varepsilon_{k}] \]

has no common points with \( U_{\varepsilon_{n}}(\mu_{n,i}(t)) \) for \( n \geq N_{2} \) and \( (n, i) \neq (k, j) \).
Proof. (a) Introduce the notations \( f(t) = \mu_{k,j}(t) - \mu_{-k,i}(t) \) and \( g(t) = \mu_{k,j}(t) - \mu_{-k-1,i}(t) \). By the definition of \( t(2k,j,i) \) and \( t(2k+1,j,i) \) we have \( f(t(2k,j,i)) = 0 \) and \( g(t(2k+1,j,i)) = 0 \) (see (13) and (14)). On the other hand, the derivatives of the functions \( f \) and \( g \) are \( 8\pi k \) and \( 4\pi (2k+1) \) respectively. Therefore if \( t \) does not belong to the \( \delta_k \)-neighborhood of \( t(2k,j,i) \) and \( t(2k+1,j,i) \) for \( i = 1,2,\ldots,p \) that is, if \( t \) belong to the intervals \([a(k,j),b(k,j,s)]\), then \( |f(t)| \geq 8\pi |k| \delta_k \) and \( |g(t)| \geq 4\pi |2k+1| \delta_k \). These inequalities imply (17).

(b) If (16) holds then it follows from (17) that the distance \( |\mu_{k,j}(t) - \mu_{n,i}(t)| \) between the centres of the intervals \( U_{\varepsilon_k}(\mu_{k,j}(t)) \) and \( U_{\varepsilon_n}(\mu_{n,i}(t)) \) is greater than the total sum \( \varepsilon_k + \varepsilon_n \) of the radii of these intervals for \( n = -k,-k-1 \). Therefore \( U_{\varepsilon_k}(\mu_{k,j}(t)) \) has no common points with the intervals \( U_{\varepsilon_n}(\mu_{n,i}(t)) \) for \( n = -k,-k-1 \). Similarly, if \( n \neq k,-k,-k-1 \), \( |k| > N_2 \) and \( t \in [0,\pi] \) then \( |\mu_{k,j}(t) - \mu_{n,i}(t)| \) is a large number and hence is greater than \( \varepsilon_k + \varepsilon_n \). If \( n = k \) and \( i \neq j \) then \( |\mu_{k,j}(t) - \mu_{n,i}(t)| = |\mu_j - \mu_i| > \varepsilon_k + \varepsilon_n \), since \( \varepsilon_k \to 0 \) as \( k \to \infty \). The lemma is proved.

Now using this lemma we consider the spectrum of \( L(Q) \).

**Theorem 3** Let \( j = 1,2,\ldots,p \) be fixed. For any interval \([a,b] := [a(k,j,s),b(k,j,s)]\) of (15) the following statements hold.

(a) If \( t \in [a,b] \), then the operator \( L_t(Q) \) has \( m_j \) eigenvalues (counting the multiplicity) lying in the interval \( U_{\varepsilon_k}(\mu_{k,j}(t)) = (\mu_{k,j}(t) - \varepsilon_k, \mu_{k,j}(t) + \varepsilon_k) \).

(b) There exists \( l \) such that the eigenvalues of \( L_t(Q) \) lying in \( U_{\varepsilon_k}(\mu_{k,j}(t)) \) are \( \lambda_{l+1}(t), \lambda_{l+2}(t), \ldots, \lambda_{l+m_j}(t) \) for all \( t \in [a,b] \).

(c) If \( k > 0 \) \((k < 0)\), then the interval \([\mu_{k,j}(a) + \varepsilon_k, \mu_{k,j}(b) - \varepsilon_k] \) \(([\mu_{k,j}(b) + \varepsilon_k, \mu_{k,j}(a) - \varepsilon_k])\) is a subset of the bands \( \Gamma_{l+1}, \Gamma_{l+2}, \ldots, \Gamma_{l+m_j} \) of the spectrum of \( L(Q) \).

**Proof.** (a) Theorem 4(a) of [16] implies that there exist \( n \) and \( N \) such that the eigenvalues \( \lambda_s(t) \) for \( s > n \) lie in \( U_{\varepsilon_k}(\mu_{k,i}(t)) \) for \( i = 1,2,\ldots,p \). On the other hand, by Lemma 1 the integer \( N \) can be chosen so that the closed interval \( U_{\varepsilon_k}(\mu_{k,i}(t)) \) for \( t \in [a,b] \) and \( |k| > N \) has no common points with the intervals \( U_{\varepsilon_n}(\mu_{n,i}(t)) \) for \( n \geq N \) and \((n,i) \neq (k,j)\). Therefore the circle

\[ \{ \lambda \in \mathbb{C} : |\lambda - \mu_{k,j}(t)| = \varepsilon_k \} \]

belong to the resolvent set of \( L_t(Q) \). Repeating the proof of the case \( \varepsilon = 1 \), one can easily verify that the circle (18) lies in the resolvent sets of \( L_t(C+\varepsilon(Q-C)) \) for all \( \varepsilon \in [0,1] \). Since \( L_t(C) \) has \( m_j \) eigenvalues (counting the multiplicity) in the circle (18), the operator \( L_t(Q) \) has also \( m_j \) eigenvalues.

(b) Let us denote the eigenvalues of \( L_t(Q) \) lying in \( U_{\varepsilon_k}(\mu_{k,j}(t)) \) by \( \lambda_{l(t)+1}(t), \lambda_{l(t)+2}(t), \ldots, \lambda_{l(t)+m_j}(t) \). We need to prove that \( t(t) \) does not depend on \( t \in [a,b] \). Since we numerate the eigenvalues of \( L_t(Q) \) in nondecreasing order (see (2)) \( \lambda_{l(t)}(t) \) and \( \lambda_{l(t)+m_j+1}(t) \) do not belong to the interval \( U_{\varepsilon_k}(\mu_{k,j}(t)) \) and

\[ \lambda_{l(t)}(t) < \lambda_{l(t)+s}(t) < \lambda_{l(t)+m_j+1}(t) \]

(19)
for all \( s = 1, 2, \ldots, m_j \). It follows from the continuity of the band functions and (19) that for each \( t \in [a, b] \) there exists a neighborhood \( U(t) \) of \( t \) such that

\[
\lambda_{l(t)}(y) < \lambda_{l(t)+s}(y) < \lambda_{l(t)+m_j+1}(y)
\]

for \( y \in U(t) \). In the other words, \( l(y) = l(t) \) for all \( y \in U(t) \). Thus we have

\[
\forall t \in [a, b], \exists U(t): l(y) = l(t), \forall y \in U(t). \tag{20}
\]

Let \( U(t_1), U(t_2), \ldots, U(t_\omega) \) be a finite subcover of the open cover \( \{U(t): t \in [a, b]\} \) of the compact \([a, b]\), where \( U(t) \) is the neighborhood of \( t \) satisfying (20). By (20), we have \( l(y) = l(t) \) for all \( y \in U(t_i) \). Clearly, if \( U(t_i) \cap U(t_j) \neq \emptyset \), then \( l(t_i) = l(z) = l(t_j) \), where \( z \in U(t_i) \cap U(t_j) \). Thus \( l(t_1) = l(t_2) = \ldots = l(t_\omega) \) and hence \( l(t) \) does not depend on \( t \in [a, b] \).

(c) We consider the case \( k > 0 \). The case \( k < 0 \) can be considered in the same way. Since

\[
\lambda_{l+s}(a) \in (\mu_{k,j}(a) - \varepsilon_k, \mu_{k,j}(a) + \varepsilon_k)
\]

and

\[
\lambda_{l+s}(b) \in (\mu_{k,j}(b) - \varepsilon_k, \mu_{k,j}(b) + \varepsilon_k)
\]

for \( s = 1, 2, \ldots, m_j \) the interval \([\mu_{k,j}(a) + \varepsilon_k, \mu_{k,j}(b) - \varepsilon_k]\) is a subset of \( \Gamma_{l+s} \) for \( s = 1, 2, \ldots, m_j \).

Using this theorem and the construction of the intervals (15) we prove the following consequence.

**Corollary 2** There exists \( N_3 > \max\{N, N_1, N_2\} \) and a sequence \( \{\gamma_k\} \to 0 \) such that \( \gamma_k \to 0 \) as \( k \to \infty \) and the spectral gap \( (\alpha, \beta) \) defined in Corollary 1 and lying in \( U(k) \) for \( k > N_3 \) is contained in the intersection of the sets \( S(1, k), S(2, k), \ldots, S(p, k) \), where

\[
S(j, k) = \bigcup_{i=1,2,\ldots,p} \left( (\pi k)^2 + \frac{\mu_i + \mu_j}{2} - \gamma_k, (\pi k)^2 + \frac{\mu_i + \mu_j}{2} + \gamma_k \right).
\]

**Proof.** We say that the intervals \([A + \varepsilon, B - \varepsilon]\) and \((A - \varepsilon, B + \varepsilon)\) are respectively the \( \varepsilon > 0 \) contraction and extension of the intervals \([A, B]\) and \((A, B)\). In Theorem 3 (c) we proved that the \( \varepsilon_k \) contraction of the images \( \mu_{k,j}([a, b]) \) of the intervals \([a, b]\) of (15) is a subset of the spectrum of \( L(Q) \). On the other hand, the intervals of (15) are obtained from \([0, \pi]\) by eliminating the open intervals

\[
(t(2k, j, i) - \delta_k, t(2k, j, i) + \delta_k), \ (t(2k + 1, j, i) - \delta_k, t(2k + 1, j, i) + \delta_k) \tag{21}
\]

for \( i = 1, 2, \ldots, p \). Therefore, it follows from (6) that the gaps in the spectrum are the subset of the \( \varepsilon_k \) extension of the images \( \mu_{k,j}((c, d)) \) of the intervals \((c, d)\) of (21). Since

\[
\mu_{k,j}(t(2k, j, i)) = (2\pi k)^2 + \frac{\mu_i + \mu_j}{2} + \left(\frac{\mu_i - \mu_j}{2\pi k}\right)^2
\]

11
and
\[\mu_{k,j}(t(2k + 1, j, i)) = (2\pi k + \pi)^2 + \frac{\mu_i + \mu_j}{2} + \left(\frac{\mu_i - \mu_j}{4\pi(2k + 1)}\right)^2,\]

using (6) and (16) we obtain that for any interval \((c, d)\) of (21) the interval 
\(\mu_{k,j}((c, d))\) and hence its \(\varepsilon_k\) extension are contained in \(S(j, k)\) for each \(j = 1, 2, ..., p\). The corollary is proved. ■

Now we find a condition on the eigenvalues of the matrix \(C\) for which the spectrum of \(L(Q)\) contains the interval \((H, \infty)\) for some constant \(H\). If the matrix \(C\) has only one eigenvalue \(\mu\) with multiplicity \(m\), then it is possible that the spectrum of \(L(Q)\) has infinitely many gaps. For example, if \(Q = qI,\) where \(q\) is not a finite zone scalar potential and \(I\) is the \(m \times m\) unit matrix then the spectrum of \(L(Q)\) has infinitely many gaps. If the matrix \(C\) has only two eigenvalues \(\mu_1\) and \(\mu_2\), then the sets \(S(1, k)\) and \(S(2, k)\) have a common interval
\[
\left((2\pi k)^2 + \frac{\mu_1 + \mu_2}{2} - \gamma_k, (2\pi k)^2 + \frac{\mu_1 + \mu_2}{2} + \gamma_k\right).
\]
Therefore, Corollary 2 does not imply that the number of the gaps in the spectrum of \(L(Q)\) is finite. However, we prove that if the number of different eigenvalues of the matrix \(C\) is greater than 2, then three sets \(S(j_1, k), S(j_2, k)\) and \(S(j_3, k)\) for the large values of \(k\) have no common intervals if the following condition holds.

**Condition 1** Suppose that there exists a triple \((j_1, j_2, j_3)\) such that
\[
\min_{i_1, i_2, i_3} \text{diam}\left(\{\mu_{j_1} + \mu_{i_1}, \mu_{j_2} + \mu_{i_2}, \mu_{j_3} + \mu_{i_3}\}\right) = d \neq 0,
\]
where minimum is taken under condition \(i_s \in \{1, 2, ..., p\}\) for \(s = 1, 2, 3\) and \(\text{diam}(E) = \sup_{x, y \in E} |x - y|\).

Let us first discuss why gaps in \(\sigma(L(Q))\) do not appear in the interval \((H, \infty)\) if \(H\) is a large number and Condition 1 holds. Then in Theorem 4 we give the mathematical proof of this statement. For each \(j \in \{1, 2, 3\}\) the set
\[\sigma_j(L(C)) = \left\{(2\pi k + t)^2 + \mu_j : k \in \mathbb{Z}, t \in (-\pi, \pi]\right\}
\]
(let us call it \(j\) spectrum) cover the interval \((H, \infty)\). The perturbation \(Q - C\) may generate a gap in \(\sigma_j(L(C))\) only at the neighborhood of the exceptional Bloch eigenvalues \((2\pi k + t)^2 + \mu_j\) (let us call it \(j\) exceptional Bloch eigenvalue). On the other hand, Condition 1 implies that the \(j_1, j_2\) and \(j_3\) exceptional Bloch eigenvalues have no common points. That is why, for each \(\lambda \in (H, \infty)\) there exists \(s \in \{1, 2, 3\}\) such that \(\lambda\) does not belong to the neighborhood of \(j_s\) exceptional Bloch eigenvalues. Hence the perturbation \(Q - C\) does not generate a gap in \(\sigma_{j_s}(L(C))\) at the neighborhood of \(\lambda\).

Now using Condition 1 and Corollary 2 we prove the following.
Theorem 4 If the matrix $C$ has three eigenvalues $\mu_{j_1}$, $\mu_{j_2}$ and $\mu_{j_3}$ satisfying Condition 1, then there exists a number $H$ such that $(H, \infty) \subset \sigma(L(Q))$, that is, the number of the gaps in the spectrum of $L(Q)$ is finite.

Proof. By Corollary 2 the gap $(\alpha, \beta)$ lying in $U(k)$ for $k > N_3$ belong to the set $S(j_s, k)$ for all $s \in \{1, 2, 3\}$ and for some $k > N_3$. Therefore it is enough to prove that

$$\bigcap_{s=1,2,3} S(j_s, k)$$

is an empty set for $k > N_3$. Since $\gamma_k \rightarrow 0$ the number $N_3$ can be chosen so that $4\gamma_k < d$ for $k > N_3$. If the set (22) contains an element $x$, then using the definitions of $S(j_s, k)$, we obtain that there exist $k > N_3$ and $i_s \in \{1, 2, \ldots, p\}$ such that

$$| x - (\pi k) |^2 - \frac{\mu_{j_s} + \mu_{i_s}}{2} | < \gamma_k$$

for all $s = 1, 2, 3$. This implies that

$$| (\mu_{j_u} + \mu_{i_u}) - (\mu_{j_v} + \mu_{i_v}) | < 4\gamma_k < d$$

for all $u, v \in \{1, 2, 3\}$, where $v \neq u$. This contradicts Condition 1. ■

References

[1] D. Chelkak and E. Korotyaev, Spectral estimates for Schrödinger operators with periodic matrix potentials on the real line, International Mathematics Research Notices, vol. 2006, Article ID 60314, 41 pages, 2006.

[2] N. Dunford and J.T. Schwartz, Linear Operators, Part II: Spectral Theory, Wiley-Interscience, New York, 1988.

[3] M. S. P. Eastham, The Spectral Theory of Periodic Differential Operators, Hafner, New York, 1974.

[4] I. M. Gelfand, Expansions in series of eigenfunctions of an equation with periodic coefficients, Sov. Math. Dokl. 73 (1950), 1117–1120.

[5] A. R. Its and V. B. Matveev, Hill operators with finitely many lacunae and multisoliton solutions of the Korteweg-de Vries equation, Trudy Mat. Fiz. 23 (1975), 51-67.

[6] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1980.

[7] B. M. Levitan, Inverse Sturm-Liouville Problems, Berlin, Boston: De Gruyter, 2018.

[8] F. G. Maksudov, O. A. Veliev, Spectral Analysis of Differential Operators with Periodic Matrix Coefficients, Differ. Equations 25 (1989), 271-277.
[9] V. A. Marchenko and I. V. Ostrovskii, *A characterization of the spectrum of the Hill operator*. Mat. USSR-Sb., **26** (1975), 493-554.

[10] D. C. McGarvey, *Differential operators with periodic coefficients in $L_p(-\infty, \infty)$*. Journal of Mathematical Analysis and Applications **11** (1965), 564-596.

[11] M. A. Naimark, *Linear Differential Operators*. London. George G. Harrap&Company 1967.

[12] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, Volume 4, Academic Press, New York, 1987.

[13] F. S. Rofe-Beketov, *The spectrum of non-self-adjoint differential operators with periodic coefficients*, Soviet Math. Dokl. **4** (1963), 1563-1566.

[14] O. A. Veliev, *On the non-self-adjoint Sturm-Liouville operators with matrix potentials*, Mathematical Notes **81** (2007), 440-448.

[15] O. A. Veliev, *Uniform convergence of the spectral expansion for a differential operator with periodic matrix coefficients*, Boundary Value Problems **ID 628973** (2008), 22 pp.

[16] O. A. Veliev, *On the Hill’s operator with a matrix potential*, Mathematische Nachrichten **281** (2008), 1341-1350.