ZETA FUNCTIONS OF INFINITE GRAPH BUNDLES

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Abstract. We compute the equivariant zeta function for bundles over infinite graphs and for infinite covers. In particular, we give a “transfer formula” for the zeta function of infinite graph covers. Also, when the infinite cover is given as a limit of finite covers, we give a formula for the limit of the zeta functions.

1. Introduction

The Ihara zeta function of a finite graph reflects combinatorial and spectral properties of that graph ([13], [2], [18]). Originally, Ihara defined the zeta function on finite graphs imitating the classical definition of the zeta function:

\[ \zeta_X(z) = \prod [C] (1 - z^{\ell(C)})^{-1}, \]

where the product is over all equivalence classes of primitive closed loops \( C \) in \( X \) and \( \ell(C) \) denotes the length of \( C \). In [2], it was shown that, for a finite graph \( X \):

\[ \zeta_X(z)^{-1} = (1 - z^2)^{\epsilon - \nu} \text{det}(I - zA + z^2Q), \]

where \( \epsilon \) is the number of edges, \( \nu \) is the number of vertices, \( A \) is the adjacency matrix of \( X \), and \( Q \) is the diagonal matrix with entries \( \deg(v) - 1 \), for each \( v \in V(X) \). In [8], the definition of Ihara zeta function was extended to infinite graphs that are limits of sequences of finite graphs. In particular, it was shown in [8], using the results in [17], that the sequence of the zeta functions of the finite graphs converges. In [3], [4], [10], [11], [12], the expression of the zeta function as a rational function was extended to infinite graphs that admit an action of a discrete group \( \Gamma \) with finite quotient. The determinant in the finite case is replaced by the determinant in a von Neumann algebra \( \mathcal{N}_0(X, \Gamma) \) of all the bounded operators on \( L^2(V(X)) \). In [6] the zeta function of finite graph bundles over finite was computed generalizing the results on graph coverings that appear [18], [19], [20]. Their results can be described as transfer results for the Ihara zeta function.

We combine the results on infinite graphs and bundles to derive a transfer formula for infinite bundles and coverings. Let \( \phi \) be an \( \text{Aut}(F) \)-assignment on \( X \). Let \( (\Gamma, \Delta) \) be a pair of groups that act on \( X \) and \( F \) in such a way that the actions are \( \phi \)-compatible and by finite co-volume.

∗Partially supported by an NSF REU grant.
**Partially supported by Canisius College Summer Grant and an NSF REU grant.
Theorem (Main Theorem 1). With the above assumptions, the equivariant zeta function is given by:

\[
\zeta_{X \times F, \Gamma \times \Delta}(z)^{-1} = (1 - z^2)^{-\chi^{(2)}(X \times F)} \det_{\Gamma \times \Delta} \left( I - \sum_{\gamma \in \text{Aut}(F)} (A_{X,\gamma}^{-1} \otimes P_\gamma + I_X \otimes A_F)z + Qz^2 \right),
\]

where \(A_{X,\gamma}^{-1}\) is the adjacency matrix of the directed graph spanned by the edges in \(\psi^{-1}(\gamma)\), \(P_\gamma\) is the permutation matrix induced by the action of \(\gamma\) on \(V(F)\), \(\chi^{(2)}\) is the Euler characteristic of the quotient \(X_1 \times \phi^1 F\), \(\det_{\Gamma \times \Delta}\) is the determinant defined on the von Neumann algebra of \(\Gamma \times \Delta\), and \(Q\) is the diagonal operator such that \(Q(y,i) = \deg(y) + \deg(i) - 1\).

Using similar methods, we prove a decomposition formula of the Ihara zeta function for infinite covers. Let \(p : Y \to X\) be a cover with \(X\) finite. Let \(\Gamma = \text{Cov}(p)\).

Theorem (Main Theorem 2). With the above notation,

\[
\zeta_{Y, \Gamma}(z)^{-1} = (1 - z^2)^{-\chi^{(2)}(Y)} \det_{\Gamma} \left( I - \sum_{\gamma \in \Gamma} A_{Y,\gamma}^{-1} \otimes P_\gamma \right) z + Qz^2\right),
\]

where \(Q\) is the diagonal operator with \((x, \gamma)\)-entry \(\deg(x) - 1\).

We apply the above calculations to sequences of strongly convergent graphs. In particular, a sequence \(\{(X_n, w_n)\}_{n \in \mathbb{N}}\) is strongly convergent to \((X, w)\) if it is a covering sequence of regular graphs converging to \(X\) in such a way that \(X\) covers compatibly each element of the sequence. Such sequences appear when we consider the Cayley graphs of finite quotients of a groups converging to the Cayley graph of the group.

2. Preliminaries

We now define a number of terms that we will use later on.

Definition 2.1. Let \(G\) be any locally finite graph. Then we define the adjacency operator \(A_G\) as follows: for any \(u, v \in V(G)\),

\[
A_G(u, v) = \begin{cases} 
1, & \text{if } u \sim v \\
0, & \text{otherwise}.
\end{cases}
\]

The definition makes sense even if the graph is directed. If \(G\) is undirected, the \(A_G\) is symmetric.

Definition 2.2. Let \(\bar{G}\) and \(G\) be locally finite graphs. We say that

\(p : \bar{G} \to G\)

is a graph covering if the following two conditions hold:

1. If \(x \sim_{\bar{G}} y\), then \(p(x) \sim_G p(y)\).
(2) For any \( x \in \tilde{G}, \ p : N(x) \to N(p(x)) \) is a bijection.

The first condition in the definition means that \( p \) is a graph map. The second condition is a local triviality condition.

Graph bundles are defined in [15]. They generalize the graph coverings in the sense that the “fiber graph” is allowed to have a non-empty set of edges. We will concentrate on bundles with finite fibers. For a graph \( X \), we denote by \( E(X) \) the set of ordered edges—i.e., each edge of \( X \) appears twice, each with opposite orientation.

**Definition 2.3.** Let \( G \) be any locally finite graph (possibly infinite), let \( F \) be a finite graph. We define an Aut\((F)\)-voltage assignment on \( G \) by

\[
\phi : E(\tilde{G}) \to \text{Aut}(F), \quad \phi(uv) = \phi(vu)^{-1}.
\]

**Definition 2.4.** Let \( G \) be a locally finite graph, \( F \) a finite graph, and \( \phi \) an Aut\((F)\)-voltage assignment on \( G \). We define a graph bundle \( G \times_\phi F \) to be the graph with vertex set \( V(G) \times V(F) \), with two vertices \((u,i),(v,j)\) in \( G \times_\phi F \) adjacent if either one of the following two conditions hold:

1. \( u \sim v \) and \( j = i\phi(uv) \)
2. \( u = v \) and \( i \sim j \).

Let \( \phi \) be a Aut\((F)\)-voltage assignment on \( G \). Let \( \gamma \in \text{Aut}(F) \).

1. Let \( \tilde{G}_{(\phi,\gamma)} \) denote the spanning subgraph of the digraph \( \tilde{G} \) whose directed edge set is \( \phi^{-1}(\gamma) \).
2. We define the permutation operator \( P_\gamma \) by the following formula: for any two vertices \( i,j \) in \( V(F) \),

\[
P_\gamma(i,j) = \begin{cases} 
1, & \text{if } j = i^\gamma \\
0, & \text{otherwise}.
\end{cases}
\]

**Remark 2.5.** When the graphs are infinite, the matrices defined above are operators on the Hilbert space with basis the vertex set of the graph. More precisely, if \( G \) is any locally finite graph, set \( L^2(G) \) to be the Hilbert space:

\[
L^2(G) = \left\{ f : V(G) \to \mathbb{C} : \sum_{u \in V(G)} |f(u)|^2 < \infty \right\}.
\]

Then the adjacency operator is given by

\[
A(f)(u) = \sum_{u \sim v} f(v).
\]

With the above notation,

\[
P_\gamma(f)(i) = f(\gamma(i)).
\]

The following combines covering maps and bundles.
**Theorem 2.6.** Let $F$ and $X$ be locally finite graphs. Let $X$ be equipped with an $\text{Aut}(F)$-voltage assignment $\phi$. Let $p : Y \to X$ be a covering map and $\psi$ the $\text{Aut}(F)$-voltage assignment

\[ \psi : E(Y) \to \text{Aut}(F), \quad \psi(xy) = \phi(p(x)p(y)). \]

Define a graph map

\[ \tilde{p} : Y \times \psi F \to X \times \phi F, \quad \tilde{p}(x,i) = (p(x), i). \]

Then $\tilde{p}$ is a covering map.

**Proof.** First we will show prove that $\tilde{p}$ is a graph map i.e., that is preserves adjacency. Let $(x, i) \sim (y, j)$ in $Y \times \psi F$. There are two cases to consider:

1. Suppose $x \sim y$ in $Y$. Then $p(x) \sim p(y)$, and $j = \psi(xy) = \psi(p(x)p(y))$. Thus, by definition,
   \[ \tilde{p}(x, i) = (p(x), i) \sim (p(y), j) = \tilde{p}(y, j), \quad \text{in } X \times \phi F. \]

2. Suppose $x = y$. Then $i \sim j$ in $F$, and clearly $p(x) = p(y)$. Thus, by definition, $\tilde{p}(x, i) \sim \tilde{p}(y, j)$.

Thus $\tilde{p}$ preserves adjacency.

Now we must show that $\tilde{p}|_{N(x,i)}$ is a bijection.

$\tilde{p}|_{N(x,i)}$ is an injection. Let $(y_1, j_1), (y_2, j_2) \in N(x, i)$ with $\tilde{p}(y_1, j_1) = \tilde{p}(y_2, j_2)$. Then we know that $p(y_1) = p(y_2)$ and $j_1 = j_2$. Now there are two cases to consider:

1. Suppose $y_1 = x$ and $i \sim j_1$. Then $i \sim j_2$, and since $(x, i) \sim (y_2, j_2)$, we must have $x = y_2$.
   
   Thus $y_1 = y_2$, so $(y_1, j_1) = (y_2, j_2)$. The same argument works if $y_2 = x$.

2. Suppose $y_1 \sim x$, and $i = j_1 \psi(y_1x) = j_1 \phi(p(y_1)p(x))$. Since $p(y_1) = p(y_2)$, we see that $i = j_2 \phi(p(y_2)p(x))$. Thus, since $(y_2, j_2) \sim (x, i)$, we must have $y_2 \sim x$. Now, since $p$ is a graph covering map, $p|_{N(x)}$ is a bijection. But $y_1, y_2 \in N(x)$ and $p(y_1) = p(y_2)$; thus, $y_2 = y_1$, so $(y_1, j_1) = (y_2, j_2)$.

$\tilde{p}|_{N(x,i)}$ is a surjection. Let $(u, k) \in N((p(x), i))$. Again there are two cases to consider:

1. Suppose $u = p(x)$ and $i \sim k$. Then $\tilde{p}(x, k) = (u, k)$, and by definition, $(x, k) \in N(x, i)$.

2. Suppose $u \sim p(x)$ and $i = k \phi(up(x))$. Since $p|_{N(x)}$ is a surjection, there exists some $y \in N(x)$ such that $p(y) = u$. Then $y \sim x$ and $i = k \phi(p(y)p(x)) = k \psi(yx)$. Thus, $(y, k) \in N(x, i)$ and $\tilde{p}(y, k) = (u, k)$.

Therefore, $\tilde{p}|_{N(x,i)}$ is a bijection. This completes the proof. \qed

The vertex set of a bundle over $G$ is $V(G) \times V(F)$. Then

\[ L^2(G \times \phi F) = L^2(G) \otimes L^2(F) \]
where the tensor product takes place in the category of Hilbert spaces. More precisely, it is the completion of the algebraic tensor product. The following theorem (proved in [15] for the finite case) provides a decomposition for the adjacency operator of any graph bundle.

**Theorem 2.7.** Let \( \phi \) be an \( \text{Aut}(F) \)-voltage assignment on a locally finite graph \( G \), with \( F \) locally finite. Then

\[
A_{G \times \phi F} = \bigoplus_{\gamma \in \text{Aut}(F)} A_{\overline{G}(\phi, \gamma)} \otimes P_{\gamma} + I_{G} \otimes A_{F}.
\]

**Proof.** It is enough to prove the result for functions of the form \( f \otimes g \), where \( f \in L^2(G) \) and \( g \in L^2(F) \). Let \((u, i) \in V(G \times \phi F)\). Then

\[
A_{G \times \phi F}(f \otimes g)(u, i) = \sum_{(u, i) \sim (v, j)} f(v)g(j).
\]

The right hand side is given by:

\[
\bigoplus_{\gamma \in \text{Aut}(F)} A_{\overline{G}(\phi, \gamma)} P_{\gamma}(f \otimes g)(u, j) + I_{G} \otimes A_{F}(f \otimes g)(u, j) = \bigoplus_{\gamma \in \text{Aut}(F), u \in \overline{G}(\phi, \gamma)} A_{\overline{G}(\phi, \gamma)}(f\mid)(u)P_{\gamma}(g)(i) + f(u)A_{F}(g)(i)
\]

There are two possibilities for \((u, i) \sim (v, j)\):

1. \( u \sim v \) and \( i = j^{\phi(uv)} \). Then the right hand side becomes:

\[
A_{\overline{G}(\phi, \phi(uv))}(f\mid)(u)P_{\gamma}(g)(i) = f(v)g(j).
\]

2. \( u = v \) and \( i \sim j \). In the right hand side, only the last summand is non-zero and it is equal to \( f(u)g(j) \).

Finally, it is clear that if neither \( u \sim v \) nor \( u = v \), then the sum on the right hand side is zero. This completes the proof. \( \square \)

By a **marked graph**, we mean a pair \((X, w)\) with \( X \) a graph and \( w \) a distinguished vertex.

**Definition 2.8.** On the space of marked graphs there is a metric \( \text{dist} \) defined as follow:

\[
\text{dist}\left((X_1, w_1), (X_2, w_2)\right) = \inf \left\{ \frac{1}{n+1}; B_{X_1}(w_1, n) \text{ is isometric to } B_{X_2}(w_2, n) \right\},
\]

where \( B_X(w, n) \) is the combinatorial ball of radius \( n \) in \( X \) centered on \( v \).

For a sequence of marked graphs \( \{(X_n, w_n)\}_{n \in \mathbb{N}} \), we say that \((X, w)\) is the limit graph if

\[
\lim_{n \to \infty} \text{dist}((X, w), (X_n, w_n)) = 0.
\]

For a finite graph \( X \), let \( c_r(X) \) denote the number of closed paths in \( X \) of length \( r \). Let

\[
(X, w) = \lim_{n \to \infty} (X_n, w_n),
\]
where \( \{(X_n, w_n)\}_{n \in \mathbb{N}} \) is a covering sequence of \( k \)-regular marked graph. In [8], the definition of the number \( c_r \) is extended for the graph \( X \) as follows:

\[
\tilde{c}_r = \lim_{n \to \infty} \frac{c_r(X_n)}{|X_n|}.
\]

In [8], it was shown that the limit exists. The zeta function \( \zeta(X, w) \) of the marked graph \( X \), with respect to the sequence \( \{(X_n, w_n)\}_{n \in \mathbb{N}} \), is defined by

\[
\ln \zeta_{X, w}(z) = \lim_{n \to \infty} \frac{1}{|X_n|} \ln \zeta_{X_n}(z) = \sum_{r=1}^{\infty} \frac{\tilde{c}_rz^r}{r}, \quad |z| < \frac{1}{k-1}.
\]

The proof that the series has a non-trivial radius of convergence is given in [8] and depends on results from [17].

Let \( X \) be a graph such that the degrees of vertices is bounded. Let \( \Gamma \) be a group of graph automorphisms of the graph \( X \) that acts on \( X \) without inversions and satisfying the following properties:

1. For each \( v \in V(X) \), the stabilizer \( \Gamma_v = \{ \gamma \in \Gamma : \gamma v = v \} \) is finite.
2. If \( \mathcal{F}_0 \subset V(X) \) is a complete set of orbit representatives of the action of \( \Gamma \) on \( V(X) \), then

\[
\text{vol}(X/\Gamma) = \sum_{v \in \mathcal{F}_0} \frac{1}{|\Gamma_v|} < \infty.
\]

In particular, if the action of \( \Gamma \) on \( V(X) \) is free, the second condition is equivalent to the condition that the orbit space \( V(X)/\Gamma \) is finite. In this case, the Ihara zeta function is defined as

\[
\zeta_{X, \Gamma}(z) = \prod_{C \in \mathcal{P}/\Gamma} \left( 1 - z^{\ell(C)} \right)^{|\Gamma_C|}.
\]

where:

- \( \mathcal{P} \) are equivalence classes of closed, primitive, tail-less edge-paths without backtracking. Two such circuits are equivalent if they differ only by a shift. \( \mathcal{P}/\Gamma \) denotes the orbit space of \( \mathcal{P} \) under the \( \Gamma \) action.
- For each class \( C \in \mathcal{P}/\Gamma \), \( \ell(C) \) denotes the length of \( C \) i.e., the number of edges in \( C \).
- \( \Gamma_C \) denotes the isotropy group of \( C \).

This formula generalizes the classical zeta function on finite graphs.

We will describe the analogue of Bass’ formula for \( \zeta_{X, \Gamma}(z) \) Let \( L^2(X) \) be the Hilbert space of functions on \( V(X) \). A unitary representation is given by:

\[
\lambda_0 : \Gamma \to U(L^2(X)), \quad (\lambda_0(\gamma)f)(v) = f(\gamma^{-1}v), \quad \gamma \in \Gamma, \ f \in L^2(X), \ v \in V(X).
\]

Then the von Neumann algebra of all bounded operators on \( L^2(X) \) that commute with the \( \Gamma \) action is defined as:

\[
\mathcal{N}_0(X, \Gamma) = \{ \lambda_0(\gamma) : \gamma \in \Gamma \}'.
\]
The algebra $\mathcal{N}_0(X, \Gamma)$ inherits a trace given by:

$$\text{Tr}_\Gamma(A) = \sum_{v \in \mathcal{F}_0} \frac{1}{|\Gamma_v|} A(v, v), \quad A \in \mathcal{N}_0(X, \Gamma).$$

With this setting, the Bass formula for the Ihara zeta function has the form ([3], [4], [10], [11], [12]):

$$\zeta_{X,\Gamma}^{-1}(z) = (1 - z^2)^{-\chi(2)(x)} \det_\Gamma(\Delta_{X,z}),$$

where

- $\det_\Gamma = \exp \circ \text{Tr}_\Gamma \circ \ln$ is the determinant in the von Neumann algebra $\mathcal{N}_0(X, \Gamma)$.
- $\Delta_{X,z} = I - Az + Qz^2$, with $A$ the adjacency operator on $X$, and $Q$ is the operator on $L^2(X)$ given by:

$$Q(f)(v) = (\deg(v) - 1)f(v), \quad \text{for each } v \in V(X).$$

Remark 2.9.

1. In [3], [4], [10], [11], [12], it was shown that the function $\zeta_{X,\Gamma}$ is defined for sufficiently small $|u|$. More precisely, if $k$ is the maximum degree of $X$, $\zeta_{X,\Gamma}(u)$ is a holomorphic function for all $|u| < \frac{1}{d - 1}$.
2. $\chi(2)(X)$ is the Euler characteristic defined in [3]. In most applications, it is equal to $\chi(X/\Gamma)$, the Euler characteristic of the orbit space.
3. Let $X$ be a $k$-regular graph and $q = k - 1$. Using the determinant formula, the zeta function can be extended to a holomorphic function in the open set ([3], [10]):

$$\Omega_q = \mathbb{R}^2 \setminus \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \frac{1}{q}\right\} \cup \left\{(x, 0) \in \mathbb{R}^2 : \frac{1}{q} \leq |x| \leq 1\right\}.$$

4. In the above references there is an interpretation of the Bass formula over the determinant on $\mathcal{N}_1(X, \Gamma)$, the von Neumann algebra on the set of edges of $X$.

Notation. There are three types of zeta functions used in this paper.

1. We write $\zeta_X(z)$ for the classical zeta function defined for a finite graph $X$.
2. We write $\zeta_{X,\Gamma}(z)$ for the equivariant zeta function defined on an infinite graph $X$ equipped with an action of a group $\Gamma$ with finite co-volume.
3. We write $\zeta_{(X,w)}(u)$ for the zeta function that it is the limit of $\zeta_{X_n}(z)^{1/|V(X_n)|}$, where $\{(X_n, w_n)\}_{n \in \mathbb{N}}$ is a covering sequence of finite regular graphs converging to $(X, w)$.

Definition 2.10. The sequence $\{X_n, w_n\}_{n \in \mathbb{N}}$ strongly converges to $(X, w)$ if:
(1) \( \{X_n, w_n\}_{n \in \mathbb{N}} \) is a covering sequence of marked \( k \)-regular graphs with
\[
p_{m-1} : X_m \to X_{m-1}
\]
the covering map.
(2) \( X \) is \( k \)-regular.
(3) There are covering maps
\[
\rho_n : X \to X_n
\]
such that:
(a) \( \rho_1(u) = p_1 p_2 \ldots p_{n-1}(\rho_n(u)) \), for all \( n \).
(b) For each \( n \), the isometry between \( B_X(w, s_n) \) and \( B_{X_n}(w_n, s_n) \) is given by the restriction of \( \rho_n \).

Remark 2.11. Cayley graphs of groups give sequences of graphs that strongly converge. Let \( \Gamma \) be a group, \( S \) a symmetric set of generators and \( \{K_n\}_{n \in \mathbb{N}} \) a sequence of normal subgroups of finite index such that:
\[
K_1 \supset K_2 \supset K_3 \ldots, \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} K_n = \{K\}.
\]
Then the sequence of the marked Schreier graphs \( \{(S(\Gamma, K_n, S), 1)\}_{n \in \mathbb{N}} \) strongly converges to \( (S(\Gamma, K, S), 1) \).

Let \( \{X_n, w_n\}_{n \in \mathbb{N}} \) strongly converge to \( (X, w) \) and \( \Gamma_n = \text{Cov}(X, X_n) \). The next results gives a connection between the different types of zeta functions.

**Theorem 2.12.** Assume that all the graphs in the sequence are \( k \)-regular finite graphs converging to a \( k \)-regular graph \( (X, w) \). Then
\[
\lim_{n \to \infty} \zeta_{X_n}(z)^{|V(X_n)|} = \zeta_{X, \Gamma_1}(z) = \zeta_{(X, w)}(z)^{|V(X_1)|}.
\]

**Proof.** By [4], Theorem 2.1,
\[
\lim_{n \to \infty} \zeta_{X_n}(z)^{|N_n|} = \zeta_{X, \Gamma_1}(z)
\]
where \( N_n = [\Gamma_n, \Gamma_1] = |V(X_n)|/|V(X_1)| \). The result follows from the definition of \( \zeta_{(X, w)}(z) \). \( \square \)

The following is the main part of the proof of Theorem 2.1 in [4].

**Corollary 2.13.** With the above notation,
\[
\det_{\Gamma_1}(\Delta_{X, z}) = \lim_{n \to \infty} (\Delta_{X_n, z})^{\frac{|V(X_1)|}{|V(X_n)|}}
\]

Let \( F \) be a locally finite graph and \( \phi_1 \) an \( \text{Aut}(F) \)-voltage assignment on \( X_1 \). Inductively, define an \( \text{Aut}(F) \)-voltage assignment on \( X_n \) by:
\[
\phi_n(uv) = \phi_{n-1}(p_{n-1}(u)p_{n-1}(v)).
\]
Also, define an $\text{Aut}(F)$-voltage assignment on $X$ by:

$$\phi(uv) = \phi_1(\rho_1(u)\rho_1(v)).$$

The details are presented in the following diagram.

Now, by Theorem 2.6, we know that, for any finite, $d$-regular graph $F$, the sequence \{\(X_n \times^{\phi_n} F\)\}_{n \in \mathbb{N}} is a $kd$-regular covering sequence; thus, by [8], it converges. We will show that in fact it converges to the graph $X \times^{\phi} F$. To do this will need the following:

**Lemma 2.14.** Assume that \{(\(X_n, w_n\))\}_{n \in \mathbb{N}} strongly converges to \((X, w)\). Then

$$\tilde{\rho}_n : B_{X \times^{\phi} F}((w, i), s_n) \rightarrow B_{X_n \times^{\phi_n} F}((w_n, i), s_n), \quad \tilde{\rho}_n(u, i) = ((\rho_n(u), i),$$

is an isometry, for any $i \in V(F)$ and for all $n \in \mathbb{N}$. 
Proof. Since $\rho_n$ is a bijection, it is clear that $\tilde{\rho}_n$ is a bijection; thus, we must show that $\tilde{\rho}_n$ preserves adjacency. To this end, assume $(u, i) \sim (v, j)$, for $(u, i), (v, j) \in B_{X \times \phi F}(w, i)$. Then there are two cases:

1. $u \sim v$ and $j = i^{\phi(uv)}$
2. $u = v$ and $i \sim j$.

In the case where $u = v$, since clearly $\rho_n(u) = \rho_n(v)$, we must have $\tilde{\rho}_n(u, i) \sim \tilde{\rho}_n(v, j)$. In the case where $u \sim v$, we must have $\rho_n(u) \sim \rho_n(v)$. Thus we must show that

$$j = i^{\phi_n(\rho_n(u)\rho_n(v))}.$$ 

Now, by the definition of $\phi$,

$$\phi(uv) = \phi_1(\rho_1(u)\rho_1(v)),$$

and by assumption,

$$\phi_1(\rho_1(u)\rho_1(v)) = \phi_1(p_{n-1}...p_1(\rho_n(u)), p_{n-1}...p_1(\rho_n(v))).$$

But by the definition of $\phi_n$,

$$\phi_n(\rho_n(u)\rho_n(v)) = \phi_1(p_{n-1}...p_1(\rho_n(u)), p_{n-1}...p_1(\rho_n(v))).$$

This shows that

$$\phi_n(\rho_n(u)\rho_n(v)) = \phi(uv),$$

and thus

$$j = i^{\phi(uv)} \implies j = i^{\phi_n(\rho_n(u)\rho_n(v))}.$$ 

This shows that $\tilde{\rho}_n$ preserves adjacency, and thus is an isometry. This completes the proof. \hfill \Box

As a corollary, we have the following theorem.

**Theorem 2.15.** For each $i \in F$, the covering sequence $\{(X_n \times^{\phi_n} F, (w_n, i))\}_{n \in \mathbb{N}}$ strongly converges to $(X \times^{\phi} F, (w, i))$.

**Proof.** Theorem 2.6 implies that the covering conditions of the strong convergence are satisfied. The rest of the proof follows from Lemma 2.14 and [8]. \hfill \Box

3. Zeta Functions for Bundles and Coverings

In this section we will use our previous result and [3, 4, 10, 11, 12], to generalize the results of [6] to infinite graph bundles.

**Definition 3.1.** Let $X$ be a graph equipped with an $\text{Aut}(F)$-voltage assignment $\phi$.

1. An action of a group $\Gamma$ on $X$ without edge inversions is called $(F, \phi)$-compatible if

$$\phi(\gamma(u)\gamma(v)) = \phi(uv), \text{ for all } u, v \in V(X), \gamma \in \Gamma.$$
(2) An action of a group $\Delta$ without inversions on $F$ is called $(X, \phi)$-compatible if $\text{Im}(\phi) \subset C_{\text{Aut}(F)}(\Delta)$ i.e., the image of $\phi$ centralizes $\Delta$.

(3) The pair of groups $(\Gamma, \Delta)$ as before is called $\phi$-compatible if $\Gamma$ is $(F, \phi)$-compatible and $\Delta$ is $(X, \phi)$-compatible.

Lemma 3.2. With the above notation, if the pair $(\Gamma, \Delta)$ is $\phi$-compatible, then the product action:

$$(\gamma, \delta)(x, i) = (\gamma x, \delta i), \quad (\gamma, \delta) \in \Gamma \times \Delta, \quad (x, i) \in V(X \times F),$$

is an action by graph automorphisms on $X \times F$. Furthermore, if the action of $\Gamma$ and $\Delta$ are of finite co-volume, so is the action of $\Gamma \times \Delta$ on $X \times F$.

Proof. The proof follows from the definitions. □

Theorem 3.3. Assume that $(\Gamma, \Delta)$ is a pair of $\phi$-compatible actions. Also, assume that the actions are of finite co-volume. Then

$$\zeta_{X \times F, \Gamma \times \Delta}(z)^{-1} = (1 - z^2)^{-\chi^{(2)}(X \times F)} \det_{\Gamma \times \Delta} \left( I - \sum_{\gamma \in \text{Aut}(F)} (A^{X, (\phi, \gamma)}_\Delta \otimes P_\gamma + I_X \otimes A_F) z + Qz^2 \right),$$

$\chi^{(2)}$ is the Euler characteristic and $Q$ is the diagonal operator with $(x, i)$-entry $\text{deg}(x) + \text{deg}(i) - 1$. Furthermore, the zeta function is holomorphic for $|z| < \frac{1}{k+d-1}$. If $X$ is $k$ regular and $F$ $d$-regular, then $\zeta_{X \times F, \Gamma}(z)$ can be extended to a holomorphic function on $\Omega_{k+d-1}$.

Proof. From [10], [11], we have

$$\zeta_{X \times F, \Gamma \times \Delta}(z)^{-1} = (1 - z^2)^{-\chi^{(2)}(X \times F)} \det_{\Gamma \times \Delta} \left( I - zA_{X \times F} + z^2Q_{X \times F} \right).$$

The theorem now follows immediately from Theorem 2.7. □

We will now use Theorem 3.3 to provide a decomposition for the zeta function of any infinite cover. Let $p : Y \to X$ be a cover with $X$ finite and $Y$ locally finite. Let $\text{Cov}(p) = \Gamma$. Now we define the function

$$\phi : E(X) \to \Gamma.$$ 

For this we write $X = \{x_1, \ldots, x_n\}$. For each $i$, choose $v_i \in Y$ such that $p(v_i) = x_i$. Now, since $p : N(v_i) \to N(x_i)$ is a bijection, for each $x_j \in N(x_i)$ there exists a unique $u_j \in N(v_i)$ such that $p(u_j) = x_j$. So, since $p(v_j) = x_j = p(u_j)$, there exists some $\gamma \in \Gamma$ such that $\gamma v_j = u_j$. Thus, define

$$\phi : E(X) \to \Gamma, \quad \phi(x_i x_j) = \gamma.$$ 

We then have the following:

Lemma 3.4. Let $\phi$ be defined as above. Then

1. The $\Gamma$ action on $Y$ is of finite co-volume and it is $\phi$-compatible.
The map
\[ \alpha : Y \to X \times \phi \Gamma, \quad \alpha(u) = (p(u), \beta) \]
is an isomorphism, where \( \beta u = v_i, \ p(u) = x_i = p(v_i) \).

**Proof.** The proof is folklore. \( \square \)

Now, in order to prove an analogue of 2.7 for \( Y \simeq X \times \phi \Gamma_1 \), we need to define the following operator: for \( \gamma_1, \gamma_2, \gamma \in \Gamma_1 \),

\[ P_\gamma(\gamma_1, \gamma_2) = \begin{cases} 1, & \text{if } \gamma_2 = \gamma_1 \gamma \\ 0, & \text{otherwise} \end{cases} \]

**Lemma 3.5.** Let \( \phi \) be defined as above. Then
\[ A_{X \times \phi \Gamma} = \sum_{\gamma \in \Gamma} A_{X_{(\phi, \gamma)}} \otimes P_\gamma, \]

**Proof.** The proof is analogous to that of 2.7. \( \square \)

The following theorem provides a decomposition for the zeta function of any infinite cover.

**Theorem 3.6.** With the above notation,
\[ \zeta_{Y, \Gamma}(z)^{-1} = (1 - z^2)^{-\chi(X)} \det_\Gamma \left( I - \sum_{\gamma \in \Gamma} A_{X_{(\phi, \gamma)}} \otimes P_\gamma \right) z + Qz^2 \]
where \( Q \) is the diagonal operator with \( (x, \gamma) \)-entry \( \deg(x) - 1 \). The function is holomorphic for \( |z| < \frac{1}{k-1} \). If \( Y \) is \( k \)-regular then \( \zeta_{Y, \Gamma}(z) \) is holomorphic on \( \Omega_{k-1} \).

**Proof.** This follows immediately from 3.3 and 3.4. \( \square \)

Let \( \{(X_n, w_n)_{n \in \mathbb{N}} \) be a sequence of finite \( k \)-regular marked graphs that strongly converges to the \( k \)-regular marked graph \((X, w) \). Let \( F \) be a finite \( d \)-regular graph. With the notation as in Theorem 2.15 we know that

We write \( a_n = |V(X_n)| \) and \( f = |F| \).

**Corollary 3.7.** With the above notation, for \( |z| < \frac{1}{k+d-1} \),
\[ \zeta_{(X \times \phi \Gamma, (w, i))}(z)^{-1} = \]

\[ = \lim_{n \to \infty} \left[ (1 - z^2)^{-\chi(X_n \times \phi \Gamma, F)} \det \left( I - \sum_{\gamma \in \text{Aut}(F)} (A_{X_n_{(\phi, \gamma)}} \otimes P_\gamma + I_X \otimes A_F)z + Q_nz^2 \right) \right]^{\frac{1}{f_{\text{aut}}}} \]
\[ = \left[ (1 - z^2)^{-\chi(X \times \phi \Gamma, F)} \det_\Gamma \left( I - \sum_{\gamma \in \text{Aut}(F)} (A_{X_{(\phi, \gamma)}} \otimes P_\gamma + I_X \otimes A_F)z + Qz^2 \right) \right]^{\frac{1}{f_{\text{aut}}}} \]
\section*{4. APPLICATION}

Let \( p : Y \to X \) be a cover with \( \text{Cov}(p) = \Gamma \) and \( X \) a finite graph. Let \( F \) be a finite \( d \)-regular graph with \( n \) such that \( \text{Aut}(F) \) contains a the dihedral group \( D_{2n} \) of order \( 2n \). Let \( \phi \) be an \( \text{Aut}(F) \)-voltage assignment on \( X \) whose image is contained into \( D_{2n} \) and \( \psi \) the induced \( \text{Aut}(F) \)-voltage assignment on \( Y \) (Theorem 2.6). By Theorem 2.6, the induced map

\[
\bar{p} : Y \times \psi F \to X \times \phi F, \quad \bar{p}(x, i) = (p(x), i)
\]

is a covering map. Also, \( \text{Cov}(\bar{p}) = \Gamma \).

The following is the setup (for the finite case this is the same as in [14] and [6]): set \( V(F) = \{1, 2, \ldots, n\} \) and \( S_n \) the symmetric group on \( V(F) \). Let \( a = (1\ 2\ \ldots\ n-1\ n) \) be an \( n \)-cycle and let

\[
b = \begin{cases} 
(1\ n)(2\ n-1)\ldots\left(\frac{n-1}{2} \frac{n+3}{2}\right)\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd}, \\
(1\ n)(2\ n-1)\ldots\left(\frac{n}{2} \frac{n+2}{2}\right) & \text{if } n \text{ is even}
\end{cases}
\]

be a permutation in \( S_n \). The permutations \( a \) and \( b \) generate the dihedral subgroup \( D_n \) of \( S_n \):

\[
D_n = \langle a, b \mid a^n = b^2 = 1,\ bab = a^{-1} \rangle.
\]

Let \( \mu = \exp(2\pi i/n) \) and \( x_k = (1, \mu^k, \mu^{2k}, \ldots, \mu^{(n-1)k})^T \) be the column vector in \( \mathbb{C}^n \). Then 1, \( \mu \), \( \ldots \), \( \mu^{n-1} \) are the distinct eigenvalues of the permutation matrix \( P(a) \) and \( x_k \) is the eigenvector corresponding to the eigenvalue \( \mu^k \). Let \( P(b) \) be the permutation matrix of \( b \) and

\[
M = \begin{cases} 
[x_0 \ P(b)x_1 \ x_2 \ P(b)x_2 \ldots \ x_{(n-1)/2} \ P(b)x_{(n-1)/2}] & \text{if } n \text{ is odd} \\
[x_0 \ P(b)x_1 \ x_2 \ P(b)x_2 \ldots \ x_{(n-2)/2} \ P(b)x_{(n-2)/2} \ x_{n/2}] & \text{if } n \text{ is even}
\end{cases}
\]

In [14], (also [6]) it was shown that \( P(b)x_k \) is an eigenvector of \( P(a) \) associated with the eigenvalue \( \mu^{n-k} \). Thus \( M \) is invertible. Also, \( P(a) \) and \( A_F \) commute and thus they are simultaneously diagonalizable. Also, \( x_k \) and \( P(b)x_k \) (\( 1 \leq k \leq (n-1)/2 \) when \( n \) is odd and \( 1 \leq k \leq (n-2)/2 \) when \( n \) is even) are eigenvectors of \( A_F \) with the same eigenvalue of \( P(b) \), denoted \( \lambda_{(F,k)} \). Also, \( x_0 \) is
the eigenvector of $A_F$ corresponding to the eigenvalue $d$ and, for $n$ even, $\lambda_{(F,n/2)}$ is the eigenvalue associated to the eigenvector $x_2$. Then as in [14], using Theorem 2.7, we get that

$$(I_Y \otimes M)^{-1} A_{Y^\psi} (I_Y \otimes M) =$$

$$= \begin{cases} (A_Y + dI_Y) \oplus \left( \bigoplus_{i=1}^{(n-1)/2} (A_t + \lambda_{(F,t)} (I_Y \oplus I_Y)) \right) & \text{if } n \text{ is odd} \\ (A_Y + dI_Y) \oplus \left( \bigoplus_{i=1}^{(n-2)/2} (A_t + \lambda_{(F,t)} (I_Y \oplus I_Y)) \oplus (B + \lambda_{(F,n/2)} I_Y) \right) & \text{if } n \text{ is even} \end{cases}$$

where

$$B = \sum_{k=0}^{n-1} \left( (-1)^k A(Y_{(\psi,a^k)}) + (-1)^{k+1} A(Y_{(\psi,a^k,b)}) \right),$$

and

$$A_t = \sum_{k=0}^{n-1} \begin{pmatrix} \mu^t A(Y_{(\psi,a^k)}) & \mu^t A(Y_{(\psi,a^k,b)}) \\ \mu^{(n-t)k} A(Y_{(\psi,a^k)}) & \mu^{(n-t)k} A(Y_{(\psi,a^k)}) \end{pmatrix}$$

Also, let $L_Y = (Q_Y + dI_Y) \otimes I_2$. Then the calculation in section 4 in [6] can be carried through in our setting and we get the following:

**Theorem 4.1.** Let $p : Y \to X$ be as above. Then

$$\zeta_{Y^\psi,X,F,\Gamma}(z)^{-1} = (1 - z^2)^{-\chi^{(2)(Y^\psi,F)}} f_{Y,F}(z) \prod_{i=1}^{(n-1)/2} g_{Y,F,i}(z)$$

when $n$ is odd, and

$$\zeta_{Y^\psi,X,F,\Gamma}(z)^{-1} = (1 - z^2)^{-\chi^{(2)(Y^\psi,F)}} f_{X,F}(z) h_{Y,F}(z) \prod_{i=1}^{(n-2)/2} g_{Y,F,i}(z)$$

when $n$ is even, where

1. $g_{Y,F,i}(z) = \det \left( Y \oplus I_Y - (A_t + \lambda_{(F,t)} (I_Y \oplus I_Y)) z + L_Y z^2 \right)$
2. $h_{Y,F}(z) = \det \left( Y - (B + \lambda_{(F,n/2)} I_Y) z + (Q_Y + dI_Y) z^2 \right)$
3. $f_{X,F}(z) = \det \left( Y - (A_Y + dI_Y) z + (Q_Y + dI_Y) z^2 \right)$

**Proof.** This follows by simple calculation from [6] with $\Delta = \{1\}$, and Theorem 9 of [4].

Let $\{(X_m, w_m)\}_{m \in \mathbb{N}}$ be a sequence of finite regular graphs that strongly converges to $(X, w)$. Let $F$ be a finite $d$-regular with $n$ vertices such that $\mathrm{Aut}(F)$ contains $D_{2n}$. Let $\phi$ be an $\mathrm{Aut}(F)$-voltage assignment on $X_1$ whose image is contained in $D_{2n}$. Let $\phi_n$ be the induced $\mathrm{Aut}(F)$-voltage on $X_m$ and $\psi$ be the induced $\mathrm{Aut}(F)$-voltage assignment on $X$. Set

$$\Gamma = \mathrm{Cov}(X \to X_1), \quad \Delta_m = \mathrm{Cov}(X_m \to X_1), \quad m \in \mathbb{N}.$$

**Corollary 4.2.** Let $a_m = |V(X_m)|$. With the above notation,
(1) If $n$ is odd:

$$
\zeta_{X \times \psi F, \Gamma}(z)^{-1} = \zeta_{(X \times \psi F, (w,i))}(z)^{-a_1} = \lim_{m \to \infty} \left( (1 - z^2)^{-\chi(X_m \times \psi F)} f_{X_m, F}(z) \prod_{t=1}^{(n-1)/2} g_{X_m, F, t}(z) \right) \frac{a_1}{a_m}.
$$

(2) If $n$ is even

$$
\zeta_{X \times \psi F, \Gamma}(z)^{-1} = \zeta_{(X \times \psi F, (w,i))}(z)^{-a_1} = \lim_{m \to \infty} \left( (1 - z^2)^{-\chi(X_m \times \psi F)} f_{X_m, F}(z) h_{X_m, F}(z) \prod_{t=1}^{(n-1)/2} g_{X_m, F, t}(z) \right) \frac{a_1}{a_m}.
$$

Proof. It follows from Theorem 2.12, Theorem 2.15 and Theorem 4.1. □

In some cases, we can get a better description of the functions appearing in the expression for the zeta function of the limit. Assume that all the graphs $X_m, m \in \mathbb{N}$, and $X$ are $p$-regular. Following [8], for each $m \in \mathbb{N}$ set:

$$
\sigma_m = \sum \frac{\delta_{\lambda_i(X_m)}}{a_m}
$$

where $\lambda_i(X_m)$ are the eigenvalues of the Markov operator $(1/k)A_{X_m}$ on $X_m$ and $\delta_x$ is the Dirac function. The sequence $\{\sigma_m\}_{m \in \mathbb{N}}$ weakly converges to the spectral measure $\sigma$ associated to the Markov operator $(1/k)A_X$. Using the calculation of Section 5 in [8] and Corollary 2.13, we get:

$$
\ln f_{X,F}(z) = \ln \det (I_Y - (A_Y + dI_Y) z + (Q_Y + dI_Y) z^2)
$$

$$
= \lim_{m \to \infty} \frac{1}{a_m} \ln \det (I_{X_m} - (A_{X_m} + dI_{X_m}) z + (Q_{X_m} + dI_{X_m}) z^2)
$$

$$
= \lim_{m \to \infty} \int_{-1}^{1} \ln (1 - (p\lambda + d)z + (p - 1 + d)z^2) \, d\sigma_m(\lambda)
$$

$$
= \int_{-1}^{1} \ln (1 - (p\lambda + d)z + (p - 1 + d)z^2) \, d\sigma(\lambda)
$$

where $\sigma$ is the spectral measure associated to $(1/p)A_X$.

Summarizing:

**Corollary 4.3.** With the above notation,

$$
\ln f_{X,F}(z) = \int_{-1}^{1} \ln (1 - (p\lambda + d)z + (p - 1 + d)z^2) \, d\sigma(\lambda), \text{ for } |z| < \frac{1}{p + d - 1},
$$

where $\sigma$ is the spectral measure associated to the regular random walk on $X$.

We give a specific example. The same method works for any group for which the spectral measure is known. Let $\Gamma$ be the Grigorchuk group ([1], [5], [7], [8]). Then $\Gamma$ can be represented as a subgroup of automorphisms of the rooted binary tree. Let $P = \text{St}(1^\infty)$ be the stabilizer of the
infinite sequence of 1’s. Let $P_m$ be the stabilizer of all the elements that start with $m$ 1’s and it has finite index in $\Gamma$. Then

$$P = \bigcup_{m=1}^{\infty} P_m.$$ 

If $S = \{a, b, c, d\}$ be the standard set of generators of $\Gamma$, then the Schreier graphs $\{S_m = S(\Gamma, P_m, S)\}_{m \in \mathbb{N}}$ converge to $S = S(\Gamma, P, S)$. All the graphs have as the base point the identity coset. Then in $S$, Corollary 9.2, we have that

$$\ln \zeta_{S,P}(z) = -3 \ln(1 - z^2) - \int_{-1/2}^{1/2} \frac{(1 - 8xz + 7z^2) |1 - 4x|}{2\pi \sqrt{x(2x - 1)(2x + 1)(1 - x)}} dx$$

Let $F$ be a $d$-regular graph as in the beginning of the section and $\phi$ an Aut($F$)-voltage assignment whose image lies into $D_{2n}$. Combining Theorem 4.1 and Corollary 4.3 we get.

**Theorem 4.4.** With the above notation, let $\psi$ be the Aut($F$)-voltage assignment on $S$ and $R = P_1/P$. Then

$$\ln \zeta_{S \times \psi F, (P, i)}(z) = \frac{1}{da_1} \left[ -\chi(2)(S \times \phi) \ln(1 - z^2) + \text{Indet}_R \left( I - \sum_{\gamma \in \text{Aut}(F)} (A(S(\psi, \gamma)) \otimes P + I_S \otimes A_F) z + Qz^2 \right) \right]$$

Furthermore,

$$\zeta_{S \times \psi F, (P, i)}(z) = (1 - z^2)^{-\chi(2)(S \times \psi F)} f_{S, F}(z) \prod_{t=1}^{(n-1)/2} g_{S, F, t}(z)$$

when $n$ is odd, and

$$\zeta_{S \times \psi F, (P, i)}(z) = (1 - z^2)^{-\chi(2)(S \times \psi F)} f_{X, F}(z) h_{S, F}(z) \prod_{t=1}^{(n-2)/2} g_{S, F, t}(z)$$

when $n$ is even, where $g$ and $h$ are as in Theorem 4.1 and

$$\ln f_{S, F}(z) = \int_{-1/2}^{0} \frac{\ln(1 - (8x + d)z + (7 + d)z^2) |1 - 4x|}{2\pi \sqrt{x(2x - 1)(2x + 1)(1 - x)}} dx + \int_{1/2}^{1} \frac{\ln(1 - (8x + d)z + (7 + d)z^2) |1 - 4x|}{2\pi \sqrt{x(2x - 1)(2x + 1)(1 - x)}} dx$$

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