Conjugate Codes and Applications to Cryptography

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Abstract—A conjugate code pair is defined as a pair of linear codes such that one contains the dual of the other. The conjugate code pair represents the essential structure of the corresponding Calderbank-Shor-Steane (CSS) quantum code. It is argued that conjugate code pairs are applicable to quantum cryptography in order to motivate studies on conjugate code pairs.

Index Terms—conjugate codes, quotient codes, cryptographic codes.

I. INTRODUCTION

Since the invention of the first algebraic quantum error-correcting code (QECC) by Shor [1] in 1995, the theory of QECCs has been developed rapidly. The first code was soon extended to a class of algebraic QECCs called Calderbank-Shor-Steane (CSS) codes [2] and then to a more general class of QECCs, which are called symplectic codes or stabilizer codes [3], [4], [5].

In this paper, we focus on CSS codes. It is well-known that this class of symplectic codes are useful for quantum key distribution (QKD), at least, in theory. In particular, Shor and Preskill [6] argued that the security of the famous Bennett-Brassard 1984 (BB84) QKD protocol could be proved by evaluating the fidelity of quantum error-correcting codes underlying the protocol.

The term ‘conjugate codes’ appearing in the title is almost a synonym for CSS codes if one forgets about quantum mechanical operations for encoding or decoding and pays attention only to what can be done in the coding theorists’ universe of finite fields. This term was coined here so that this issue would be more accessible to those unfamiliar with quantum information theory.

Recently, the present author [7] proved the existence of CSS codes that outperform those proved to exist in the literature [2], and quantified the security and reliability of CSS-code-based QKD schemes rigorously assuming ideal discrete quantum systems. Although we have treated QKD in [7], the CSS-code-based QKD scheme can be viewed as merely one application of conjugate codes (CSS codes). For example, the arguments in [7] also imply that conjugate codes can be used as cryptographic codes that directly encrypt secret data as well be elucidated in the sequel. Here, we remark that QKD means techniques for sharing a secret key between remote parties, and the shared key itself is not the secret message that the sender wishes to send. A typical scenario is that after sharing the key, the sender encrypts a secret data using the key and sends it to the receiver and the receiver decrypts the data using the shared key. The direct encryption is mightier, and can be used as QKD if one wishes. Turning back to the original motivation of (algebraic) QECCs, these codes deemed indispensable for quantum computing since quantum states are more vulnerable to errors or quantum noise, most notably to decoherence. Among QECCs, CSS codes are said to be suited for fault-tolerant quantum computing (e.g., [8] and references therein).

The aim of this work is to enhance motivation to study this class of codes. In particular, applications to cryptography, which allow direct encryption, are emphasized. We remark that a large portion of this paper is nearly a paraphrase of a part of [7] though our description is slightly more general in that we explicitly treat a general code pair \((C_1, C_2)\) satisfying a certain condition, which will be given shortly, whereas [7] describes the result for the case where \(C_1 = C_2\).

This paper is organized as follows. In Section II, conjugate codes are introduced, and in Section III CSS quantum codes are explained. In Section IV, it is argued that conjugate code pairs are applicable to quantum cryptography. General symplectic codes, and quotient codes, are explained in Sections V and VI respectively. Sections VII and VIII contain remarks and a summary, respectively.

II. CONJUGATE CODES

We write \(B \leq C\) if \(B\) is a subgroup of an additive group \(C\). We use a finite field \(F_q\) of \(q\) elements, and the dot product defined by

\[
(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i \quad (1)
\]

for vectors in \(F_q^n\). We let \(C^\perp\) denote \(\{ y \in F_q^n \mid \forall x \in C, \ x \cdot y = 0 \}\) for a subset \(C\) of \(F_q^n\).

We mean by an \([n, k]\) conjugate (complementary) code pair or CSS code pair over \(F_q\) a pair \((C_1, C_2)\) consisting of an
linear code $C_1$ and an $[n, k_2]$ linear code $C_2$ satisfying

$$C_2^⊥ \leq C_1,$$

which condition is equivalent to $C_1^⊥ \leq C_2$, and

$$k = k_1 + k_2 - n. \quad (3)$$

If $C_1$ and $C_2$ satisfy $A$, the quotient codes $C_1/C_2^⊥$ and $C_2/C_1^⊥$ are said to be conjugate. The notion of quotient codes was introduced in [9], and will be explained in Section VII.

The goal, in a long span, is to find a conjugate code pair $(C_1, C_2)$ such that both $C_1/C_2^⊥$ and $C_2/C_1^⊥$ have good performance. If the linear codes $C_1$ and $C_2$ both have good performance, so do $C_1/C_2^⊥$ and $C_2/C_1^⊥$. Hence, a conjugate code pair $(C_1, C_2)$ with good (not necessarily a technical term) $C_1$ and $C_2$ is also desirable.

### III. Calderbank-Shor-Steane Codes

The complex linear space of operators on a Hilbert space $H$ is denoted by $L(H)$. A quantum code usually means a pair $(Q, R)$ consisting of a subspace $Q$ of $H^\otimes n$ and a trace-preserving completely positive (TPCP) linear map $R$ on $L(H^\otimes n)$, called a recovery operator. The subspace $Q$ alone is also called a code. Symplectic codes have more structure: They are simultaneous eigenspaces of commuting operators on $H^\otimes n$. Once a set of commuting operators is specified, we have a collection of eigenspaces of them. A symplectic code refers to either such an eigenspace or a collection of eigenspaces, each possibly accompanied by a suitable recovery operator. In this section and the next, we assume $H$ is a Hilbert space of dimension $q$, and $q$ is a prime (but see Section VII). Then, $F_q = Z/qZ$. We fix an orthogonal basis $(|i\rangle)_{i=0}^{q-1}$ of $H$.

In constructing symplectic codes, the following basis of $L(H^\otimes n)$ is used. Let unitary operators $X, Z$ on $H$ be defined by

$$X|j\rangle = |j-1\rangle, \quad Z|j\rangle = \omega^j|j\rangle, \quad j \in F_q \quad (4)$$

with $\omega$ being a primitive $q$th root of unity (e.g., $e^{2\pi i/q}$). For $u = (u_1, \ldots, u_n) \in F_q^n$, let $X^u$ and $Z^u$ denote $X^{u_1} \otimes \cdots \otimes X^{u_n}$ and $Z^{u_1} \otimes \cdots \otimes Z^{u_n}$, respectively. The operators $X^u Z^w$, $u, w \in F_q^n$, form a basis of $L(H^\otimes n)$, which we call the Weyl (unitary) basis [15]. We have the commutation relation

$$(X^u Z^w)(X^{u'} Z^{w'}) = \omega^{u \cdot u'-w \cdot w'}(X^{u'} Z^{w'})(X^u Z^w), \quad (5)$$

for $u, u', w, w' \in F_q^n$, which follows from $XZ = ZX$. It is sometimes useful to rearrange the components of $(u, w)$ appearing in the operators $X^u Z^w$ in the Weyl basis as follows: For $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_n) \in F_q^n$, we denote by $[u, w]$ the rearranged one

$$([u_1, u_2], \ldots, [u_n, w_n]) \in X^n,$$

where $X = F_q \times F_q$. We occasionally use another symbol $N$ for the Weyl basis:

$$N_{[u, w]} = X^u Z^w,$$

$^1$When the number of elements of a code $C \subseteq F_q^n$ is $q^k$, it is called an $[n, k]$ code. Readers unfamiliar with coding theory are referred to [9, Section 2] or standard textbooks such as [10], [11], [12], [13], [14].

and

$$N_j = \{X_x \mid x \in J\}, \quad J \subseteq X^n.$$

An obvious but important consequence of $B$ is that $X^u Z^w$ and $X^{u'} Z^{w'}$ commute if and only if $u \cdot u' - w \cdot w' = 0$. The map

$$\{(u, w), (u', w')\} \mapsto u \cdot u' - w \cdot w' \quad (6)$$

is a symplectic bilinear form, which we refer to as standard.

A CSS code is specified by two classical linear codes (i.e., subspaces of $F_q^n$) $C_1$ and $C_2$ with $A$. Coset structures are exploited in construction of CSS codes. We fix some set of coset representatives of the factor group $F_q/C_1$, for which the letter $x$ is always used to refer to a coset representative in this section, that of $C_1/C_2^⊥$, for which $u$ is used, and that of $F_q/C_2$, for which $z$ is used. These may be written as

$$x \in F_q^n/C_1, \quad z \in F_q^n/C_2, \quad v \in C_1/C_2^⊥$$

where $i$ is in abuse as usual. Put $k_1 = \dim C_1$, $k_2 = \dim C_2$, and assume $g_1, \ldots, g_{n-k_1}$ form a basis of $C_2^⊥$, and $h_1, \ldots, h_{n-k_2}$ form a basis of $C_1^⊥$. We assume $k_1$ is larger than $n - k_2$.

The operators

$$Z^{h_1}, \ldots, Z^{h_{n-k_1}}, X^{g_1}, \ldots, X^{g_{n-k_2}}, \quad (7)$$

which generate the so-called stabilizer of the CSS code, commute with each other, so that we have simultaneous eigenspaces of these operators. Specifically, put

$$|\phi_{xzw}\rangle = \frac{1}{\sqrt{|C_2^⊥|}} \sum_{w \in C_2^⊥} \omega^{z \cdot w}|x + v + w\rangle \quad (8)$$

for coset representatives $x, z$ and $v$. Then, we have

$$Z^{h_j}|\phi_{xzw}\rangle = \omega^{x \cdot h_j}|\phi_{xzw}\rangle, \quad j = 1, \ldots, n - k_1$$

and

$$X^{g_j}|\phi_{xzw}\rangle = \omega^{z \cdot g_j}|\phi_{xzw}\rangle, \quad j = 1, \ldots, n - k_2.$$

It can be checked that $|\phi_{xzw}\rangle, x \in F_q^n/C_1, z \in F_q^n/C_2, v \in C_1/C_2^⊥$, form an orthonormal basis of $H^\otimes n$. In words, we have $g_1 + k_2 - n$-dimensional subspaces $Q_{xz}$ such that $\bigoplus_{x,z} Q_{xz} = H^\otimes n$ and $Q_{xz}$ is spanned by orthonormal vectors $|\phi_{xzw}\rangle$, $v \in C_1/C_2^⊥$, for each pair $(x, z) \in (F_q^n/C_1) \times (F_q^n/C_2)$. The subspaces $Q_{xz}, (x, z) \in (F_q^n/C_1) \times (F_q^n/C_2)$, are the simultaneous eigenspaces of the operators in $B$, and form a CSS code.

In [7], we have treated the case when $C_1 = C_2 = C^⊥$ with a code $C$. In this case, $C$ is necessarily self-orthogonal by $A$. We will consistently use $k$ to denote the logarithm of the dimension of $Q_{xz}$, viz.,

$$k = k_1 + k_2 - n = \log_q \dim C Q_{xz}. \quad (9)$$

$^2$Our code pair $(C_1, C_2)$ often appears as $(C_1, C_2^⊥)$ in the literature [2], [6], [9]. Our choice would be more acceptable to coding theorists because good (not necessarily a technical term) codes $C_1$ and $C_2$ result in a good CSS code while the performance of $C_2^⊥$ seemingly has no direct meaning.

$^3$A subspace $C$ with $C \subseteq C^⊥$, which is equivalent to $\forall x, y \in C, x \cdot y = 0$, is said to be self-orthogonal (with respect to the dot product).
Decoding or recovery operation for a CSS quantum code can be done as follows. If we choose a set $\Gamma_1$ of coset representatives of $\mathbb{F}_q^n/C_1$ ($i = 1, 2$), we can construct a recovery operation $R$ for $Q_{xz}$ so that the code $(Q_{xz}, R)$ is $N_{J(\Gamma_1, \Gamma_2)}$-correcting in the sense of [16], where $J(\cdot, \cdot)$ is defined by
\[ J(\Gamma_1, \Gamma_2) = \{ [x, z] | x \in \Gamma_1 \text{ and } z \in \Gamma_2 \} \tag{10} \]

In fact, $Q_{xz}$ is $N_{J(\Gamma_1', \Gamma_2')}$-correcting with
\[ \Gamma_1' = \Gamma_1 + C_{\perp}^2 \text{ and } \Gamma_2' = \Gamma_2 + C_{\perp}^1. \tag{11} \]

This directly follows from the general theory of symplectic codes [4], [5].[17, Proposition A.2] on noticing that the operators in the Weyl basis that commute with all of those in (7) are $X^u Z^w$, $u \in C_1$, $w \in C_2$ (see also Sections V and VI).

IV. CSS Codes as Cryptographic Codes

Schumacher [18, Section V-C], using the Holevo bound, argued that if a good quantum channel code is used as a cryptographic code, the amount of information leakage to the possible eavesdropper is small. In this section, we will apply Schumacher’s argument to CSS codes.

A. Quantum Codes and Quantum Cryptography

Suppose we send a $k$-digit secret information $V + C_{\perp}^2 \in C_1/C_2^2$ physically encoded into the state $|\phi_{xzv}\rangle \in Q_{xz}$, where we regard $X, Z$ as random variables, and assume $(X, Z)$ are randomly chosen according to some distribution $P_{xz}$. Once the eavesdropper, Eve, has done an eavesdropping, namely, a series of measurements, Eve’s measurement results form another random variable, say, $E$. We use the standard symbol $I$ to denote the mutual information (in information theory).

According to [18, Section V-C],
\[ I(V; E|X = x, Z = z) \leq S_{xz} \tag{12} \]

where $S_{xz}$ is the entropy exchange after the system suffers a channel noise $N$, Eve’s attack $E$, another channel noise $N'$, and the recovery operation $R = R_{xz}$ for $Q_{xz}$ at the receiver’s end. Let us denote by $F_{xz}$ the fidelity of the code $(Q_{xz}, R)$ employing the entanglement fidelity $F_e$ [18]. Specifically,
\[ F_{xz} = F_e(\pi_{Q_{xz}}, R N' E N') \]

where $\pi_{Q_{xz}}$ denotes the normalized projection operator onto $Q$, and $B A(\rho) = B(A(\rho))$ for two CP maps $A$ and $B$, etc. Then, by the quantum Fano inequality [18, Section VI], we have
\[ S_{xz} \leq h(F_{xz}) + (1 - F_{xz})2nR \tag{13} \]

where $h$ is the binary entropy function and $R = n^{-1} \log_q \dim Q_{xz}$. Combining (12) and (13) and taking the averages of the end sides, we obtain
\[ I(V; E|XZ) \leq Eh(F_{xz}) + (1 - EF_{xz})2nR \]
\[ \leq h(\mathbb{E}F_{xz}) + (1 - EF_{xz})2nR, \tag{14} \]

where $\mathbb{E}$ denotes the expectation operator with respect to $(X, Z)$. Hence, if $1 - \mathbb{E}F_{xz}$ goes to zero faster than $1/n$, then $I(V; E|XZ) \rightarrow 0$ as $n \rightarrow \infty$. We have seen in [7] that the convergence is, in fact, exponential for some good CSS codes, viz., $1 - \mathbb{E}F_{xz} \leq q^{-nE/o(n)}$ with some $E > 0$. This, together with (14), implies
\[ I(V; E|XZ) \leq 2q^{-nE+o(n)}[n(E + R) - o(n)], \tag{15} \]

where we used the upper bound $-2t \log t$ for $h(t)$, $0 \leq t \leq 1/2$, which can easily be shown by differentiating $\log t$ (or by Lemma 2.7 of [19]). Thus, we could safely send a secret data $v + C_{\perp}^2$ provided we could send the entangled state $|\phi_{xzv}\rangle$ in (8) and the noise level of the quantum channel including Eve’s action were tolerable by the quantum code.

In the above scheme, the legitimate sender, Alice, and receiver, Bob, should share the random variables $XZ$, say, by sending them through a public channel that is free from tampering of malicious parties (but see Section IVC). In particular, we assume Eve can possibly observe $XZ$ without tampering them as in the literature on quantum key distribution.

B. Reduction to Cryptographic Code

In [7], borrowing the idea of [6], we have reduced the above cryptographic scheme to the BB84 protocol. We now explain this problem, we use Shor and Preskill’s observation that the state (8) is entangled in general, and therefore the above scheme is not true. Thus, we can send secret data safely by the following scheme.

Conjugate-Code-Based Cryptographic Code. Alice sends a $k$-digit secret information $V + C_{\perp}^2 \in C_1/C_2^2$ physically encoded into the state in (16) where $X = x$ is chosen randomly according to some distribution $P_X$. 

A proof of (16) is given in Section VIII.
In this case, Alice and Bob should share \( X = x \). We remark the random variable \( Z \), the states \(|\psi_{xz}\rangle\) and the recovery operator \( R_{xz} \) are fictitious in that they do not appear in the above reduced cryptographic code, but they proved useful for demonstrating the security. These need only exist in theory, and need not be implemented in practice.

Up to now, we have fixed our attention on proving the security. However, we should ensure reliable transmission. Namely, the probability of disagreement between Alice’s data \( V \) and Bob’s data \( V' \), which should be the result of decoding the cryptographic code, must be reasonably small. For the conjugate-code-based (CSS-code-based) cryptographic code, we employ the following decoding principle. The receiver performs a decoding algorithm for the coset code \( x + C_1 \) that corrects errors in \( T_1 \), a set of coset representatives of \( \mathbb{F}_q^n/C_1 \). The algorithm is the obvious modification of a decoding algorithm for the linear code \( C_1 \). In the next subsection, we will see that a CSS quantum code with high fidelity results in a secure and reliable cryptographic code, where a reliable cryptographic code means that with small decoding error probability. Note that if \( f_X(x) = 1 \) for some \( x \), we do not have to send \( X \) (see the next subsection).

C. Evaluating Fidelity

Note that the underlying CSS codes (before the reduction) has the fidelity \( \mathbb{E}F_{XZ} \) which is bounded by
\[
1 - \mathbb{E}F_{XZ} \leq P_A(J(\Gamma_1', \Gamma_2')),
\]
where \( J, \Gamma_1' \) and \( \Gamma_2' \) are as in (10) and (11). The right-hand side of (17) can be written as \( \Pr\{\xi \notin \Gamma_1' \text{ or } \zeta \notin \Gamma_2'\} \), and hence we have \( 1 - \mathbb{E}F_{XZ} \leq \Pr\{\xi \notin \Gamma_1'\} + \Pr\{\zeta \notin \Gamma_2'\} \), where \( \xi \) and \( \zeta \) are random variables such that the distribution of \( [\xi, \zeta] \) is given by \( P_A \). The equality holds in (17) if \( |\Gamma_1'| = q^{n-k_1} \) and \( |\Gamma_2'| = q^{n-k_2} \) (namely, if they are complete sets of coset representatives). This follows from that the code is \( N_J(\Gamma_1', \Gamma_2') \) correcting and that the fidelity of an \( N_J \)-correcting symplectic code is \( 1 - P_A(J) \) for a channel \( A : L(H^{\otimes n}) \rightarrow L(H^{\otimes n}) \), where the probability distribution \( P_A \) is associated with \( A \) in the manner described in [7], [17] (see also Section V). In the present context, \( A = N'EN' \). An important fact is that the right-hand side of (17) is smaller than \( \Pr\{\xi \notin \Gamma_1'\} \), which is the decoding error probability when the quotient code \( C_1/C_2 \) is used as a conjugate-code-based cryptographic code. Hence, by bounding the fidelity of the underlying CSS quantum codes, we automatically obtain bounds on the security and reliability simultaneously.

There are subtleties on (17). This fidelity bound is true for a general TPCP map \( A : L(H^{\otimes n}) \rightarrow L(H^{\otimes n}) \) if \( F_{XZ} \) is uniform. This is because (17) is based on Corollary 4 to Theorem 3 in [17] or the alternative reasoning in [7, Appendix A] and any of these assumes the distribution of the syndrome \( (X, Z) \) is uniform. A desirable situation in the reduced cryptographic code is that the entropy of \( P_X \) is small. In particular, if \( P_X(x) = 1 \) for some \( x \), or (17) is true for such a random variable \( X \) for other reasons, we do not need the public channel to send \( X \). This is possible if the map \( A = N'EN' \) is known to the legitimate participants of the protocol as explained below.

The history of information theory suggests it would be reasonable to treat first the tractable case where \( E \) is known to the legitimate participants (and \( \mathcal{A} = \mathcal{E}^{\otimes n} \) ) to pursue the fundamentals of the issue of transmitting private data (cf. [20], [21]). Then, we can interpret the above argument as indicating the existence of a good cryptographic code (a kind of random coding proof). Namely, we can single out the best index \( \tilde{x} \) such that \( \mathbb{E}_ZF_{XZ} \geq \mathbb{E}_{\tilde{x}}F_{XZ} \), where \( \mathbb{E}_Y \) denotes the expectation operator with respect to a random variable \( Y \). Replacing the original random variable \( X \) with that whose probability concentrates on \( \tilde{x} \), we have a protocol that does not require transmission of information \( X \) through an auxiliary public channel.

Still, it would be desirable to remove the assumption that the legitimate participants know \( A \). That is, universal codes that do not depend on the channel characteristics are desirable. Regarding this issue, we make a small step forward. It seems difficult to construct universal cryptographic codes without transmission of auxiliary information for the completely general class of channels. However, this is possible with our conjugate-code-based cryptographic code if the class of \( A \) is restricted to those such that \( \mathbb{E}_ZF_{XZ} \) does not depend on \( x \). This situation occurs if, e.g., \( A \) has the form \( \rho \mapsto \sum_{u,w}F(u, w)X^uZ^w\rho(X^uZ^w)\dagger \) with a probability distribution \( P \) on \( (\mathbb{F}_q^n)^2 \). This condition is equivalent to that \( A \) is ‘Weyl-covariant’: \( N_xA = AN_x, \) where \( N_x : \rho \mapsto N_x\rho N_x^\dagger, x \in \mathbb{F}_q^n \) (see, e.g., [17, Section 2.5]). More generally, the situation occurs if \( A \) has the property
\[
A(X^u\rho X^{-u}) = X^uA(\rho)X^{-u}
\]
for any \( \rho \in L(H^{\otimes n}) \) and \( u \in \mathbb{F}_q^n \). This condition is equivalent to
\[
\langle l - u |A((i - u)(j - u))m - u \rangle = (l|A(|i\rangle \langle j|)m)
\]
for any \( i, j, l, m, u \in \mathbb{F}_q^n \), which reads ‘the channel looks the same if we translate the basis \(|i\rangle\) to \(|(i - u)\rangle\)’.

V. General Symplectic Codes

In this and next sections, the order \( q \) of the finite field \( \mathbb{F}_q \) is not necessarily a prime. In this section, we digress to explain how general symplectic codes are defined and how CSS codes are obtained from the general definition. The 2\( n \)-dimensional linear space \( \mathbb{F}_q^{2n} \) over \( \mathbb{F}_q \) equipped with the standard symplectic form
\[
f_{sp}((x_1, z_1, \ldots, x_n, z_n), (x'_1, z'_1, \ldots, x'_n, z'_n)) = \sum_i x_i z'_i - z_i x'_i
\]
which has already appeared in (6), plays a crucial role in algebraic QCPCs. We can define the dual \( L^{+\text{sp}} \) of \( L \) by \( L^{+\text{sp}} = \{ y \in \mathbb{F}_q^{2n} | \forall x \in L, f_{sp}(x, y) = 0 \} \). Let us call a subspace \( L \) with \( L^{+\text{sp}} \leq L \) an \( f_{sp}\)-dual-containing code or a dual-containing code (with respect to the symplectic form \( f_{sp} \)). Then, we have a quantum code whose performance is closely related to that of the classical code \( L \). The code is called a symplectic (quantum) code with parity check set \( (y_1, \ldots, y_{n-k}) \), where \( y_1, \ldots, y_{n-k} \in \mathbb{F}_q^{2n} \) form a basis of
$L^{\perp p}$, or a symplectic code with stabilizer $N_{L^{\perp p}}$. Here, $N : u \mapsto N_u$ is Weyl’s projective representation [15] of $\mathbb{F}_q^{2n}$ (the same as in Section III).

Suppose $A_{n,k}$ is the ensemble of $[2n, n+k]$ $f_{sp}$-dual-containing codes over $\mathbb{F}_q$. We can regard them $[n, (n+k)/2]$ additive codes over $\mathcal{X} = \mathbb{F}_q^2$ if we pair up the coordinates of any word $(x_1, z_1, \ldots, x_n, z_n)$ to have $((x_1, z_1), \ldots, (x_n, z_n)) \in \mathcal{X}^m$. We can associate with an $[n, (n+k)/2]$ $f_{sp}$-dual-containing code a set of $d^k$-dimensional subspaces of $H^{\otimes n}$, which can be used for quantum error correction [3], [4], [5]. Namely, we have the next lemma, which is a slight reformulation of the original one [3], [4].

**Lemma 1:** Suppose a subspace $L \in A_{n,k}$ and a set $J$ of representatives of cosets of $L$ in $\mathbb{F}_q^{2n}$ are given. Then, we have a $d^k$-dimensional subspace of $H^{\otimes n}$ that works as an $N_J$-correcting code with a suitable recovery operator, where $J = J + L^{\perp p} = \{x + y \mid x \in J, y \in L^{\perp p}\}$.

For a proof, see [4] or, e.g., [22], [17]. Roughly speaking, given a set of operators $\mathcal{F}$, a quantum code being $\mathcal{F}$-correcting or a code corrects ‘errors’ in $\mathcal{F}$ means that it recovers any state in the code subspace perfectly after the state suffers ‘errors’ belonging to $\mathcal{F}$ [16]. The precise definition of $\mathcal{F}$-correcting is not requisite for evaluating the performance of quantum codes. Indeed, the next fact is enough to treat symplectic codes [17]: If we properly define the performance measure of symplectic codes, it equals the probability $P_d(J)$. The performance measure is the entanglement fidelity averaged over the whole syndromes, which was already used in Section IV.

A CSS code is a symplectic code with stabilizer $N_{L^{\perp p}}$ such that $L^{\perp p}$ has the form $L^{\perp p} = \{[u, w] \mid u \in C_2, w \in C_1^\perp\}$ with some $C_1$ and $C_2$. In this case, $L = \{[u, w] \mid u \in C_1, w \in C_2\}$, so that $L^{\perp p} \leq L$ can be written as $C_2^\perp \leq C_1$, the requirement we have posed.

VI. QUOTIENT CODES

Now, we turn to the realm of finite fields or algebraic coding theory. In [9], the notion of quotient codes was introduced to explain QECCs. The aim of [9] was to exhibit the essence, at least, for algebraic coding theorists, of algebraic quantum coding. A **quotient code** of length $n$ over $\mathbb{F}_q$ is an additive quotient group $C/B$ with $B \leq C \leq \mathbb{F}_q^n$. In the scenario of quotient codes in [9], the encoder encodes a message into a channel. Clearly, if $C$ is a $J$-correcting in the ordinary sense, $C/B$ is $(J + B)$-correcting (since adding a word in $B$ to a code-coset does not change it). A conjugate-code-based cryptographic code effectively means a quotient code in this scenario. A conjugate-code-based cryptographic code may be said to be an error-correcting code that can protect information from eavesdroppers, and hence may be called a cryptographic error-correcting code.

**Lemma II** may read that if $L$ is a dual-containing code with respect to $f_{sp}$, and the quotient code $L/L^{\perp p}$ is $J$-correcting, then the corresponding symplectic quantum code is $N_J$-correcting. Turning our attention to the CSS code specified as above with $C_1, C_2$, the quotient code $L/L^{\perp p}$ has the form $C_1/C_2^\perp \oplus C_2/C_1^\perp$. Thus, the CSS code $Q_{x_2}$ in Section III is $N_{J(x_1, x_2)}$-correcting with $\Gamma_1 = C_1/C_2^\perp$ and $\Gamma_2 = C_2/C_1^\perp$. In particular, the CSS code has large fidelity if both $C_1/C_2^\perp$ and $C_2/C_1^\perp$ have small decoding error probabilities. This is the ground where the goal described in Section IV stems from. It might be said that the structure of quotient codes were inherent in quantum error-correcting codes and CSS-code-based cryptographic codes.

VII. REMARKS

A. Model of Eavesdropping

A measurement is modeled as a completely positive (CP) instrument whose measurement result belongs to a finite or countable set (e.g., [23], [24], [25], [26], [27]). The specific model employed in this work is the same as in [7] and as follows.

We assume a TPCP map $A : L(H^{\otimes n}) \to L(H^{\otimes n})$ represents the whole action of Eve (plus the other environment). This means that there exists a decomposition (CP instrument) $\{A_i\}_i$ such that $A = \sum_i A_i$, where $A_i$ are trace-nonincreasing CP maps, and when the initial state of the system of the whole sent digits is $\rho$, Eve obtains data $E = i$ with probability $\text{Tr}_E A_i(\rho)$ leaving the system in state $A_i(\rho)/\text{Tr}_E A_i(\rho)$. Here, the decomposition may depend on the other random variables available to Eve.

A minor comment follows. Let the random variable $E$ denotes Eve’s measurement result on the whole sent digits. Then, the random variable $E$ above mentioned has more information than $E'$ since $E$ includes the data relevant to the other environment. However, there is no harm in considering $E$ as Eve’s data for the purpose of proving the security.

B. Related Information Theoretic Problems

In [20], [21], information theoretic problems related to ours are treated. These and the present work or [7] share the goal of secure transmission of private data, but their specific purpose in [20], [21] is to establish coding theorems on the best asymptotically achievable rates. Our codes are linear codes while theirs lack such a helpful structure and are hard to conceive aimed at practical use.

The quantum theoretical models treated in the literature above mentioned can be regarded as generalizations of that of [28]. What are called conjugate-code-based cryptographic codes in the present work essentially fall in the class of coding systems in [28].

C. Wiesner’s Conjugate Coding

The term ‘conjugate coding’ appeared in the pioneering work on quantum cryptography [29], where the idea of encoding secret information into quantum states, more specifically, into conjugate bases, was proposed. This idea is still alive in CSS-code-based cryptographic codes or QKD schemes. However, this is a problem of modulation in the language of communication engineers. Thus, our meaning of ‘conjugate’ is different from, though related to, that of [29].
D. QKD Protocol

The BB84 QKD protocol as treated in [30], [6] or its variants is, roughly speaking, the CSS-code-based cryptographic code plus a scheme for estimating the noise level, where the noise includes the effect of eavesdropping. Mainly due to the scheme for noise estimation, the protocol needs public communication. We have used the dichotomy of cryptographic codes and estimation schemes in analysis of the QKD protocol [7], and have focused more on cryptographic codes in the present work.

E. Non-prime Alphabet

Let \( q = p^m \) with \( p \) prime. We have assumed \( m = 1 \) in Sections III and IV. When \( m > 1 \), a conjugate code pair \((C_1, C_2)\) over \( \mathbb{F}_q \) is still useful for quantum coding and cryptography. This is because elements of \( \mathbb{F}_q \) can be expanded into \( \mathbb{F}_p^m \) using dual bases in such a way that \( T_{\mathbb{F}_p^m} x y = \sum_i x_i d_i \), where \((x_1, \ldots, x_m)\) is the representation of \( x \) with respect to one basis and \((y_1, \ldots, y_m)\) is that of \( y \) with respect to the dual [31]. Applying these representations to \((C_1, C_2)\), we obtain a conjugate code pair over \( \mathbb{F}_p \). This follows easily from [32, Theorem 1], or [33, Theorem 1].

F. Proof of (16)

The left-hand side can be written as

\[
\frac{1}{|C_2|^2} \sum_{w, w' \in C_2^\perp} \sum_z \omega^z (w - w') |x + v + w\rangle \langle x + v + w'|
\]

and we see \( \sum_z \omega^z (w - w') \) vanishes whenever \( w \neq w' \).\(^6\) Hence, we have (16).

G. Other Comments

We take this opportunity to make corrections to related works of the present author [7], [34], [9]. (a) Ref. [7]: On p. 8313, line 5, \( T_n = \Gamma_n + 1^m \) should read \( T_n = \Gamma_n + \{0^n, 1^n\} \). (b) Ref. [9]: On p. 453, right column, 5th line from the bottom, ‘basis of \( L \)’ should read ‘basis of \( L^{\perp_{sp}} \).’ (c) Ref. [9]: On p. 453, right column, 4th line from the bottom, \( N_{L} \) should read \( N_{L^{\perp_{sp}}} \).’ (d) Ref. [34]: On p. 6, right column, line 16, the period should be removed, and ‘With’ in the subsequent line should be decapitalized.

VIII. Summary and Concluding Remarks

Conjugate codes were introduced without referring to Hilbert spaces so as to be more accessible to algebraic coding theorists. The bridge between the coding theorists’ universe, the vector space over a finite field, and quantum mechanical worlds that are represented by Hilbert spaces is Weyl’s projective representation \( N : \mathbb{F}_q^2 \cong \mathbb{C}^n \), \( N : \mathbb{C}^n \ni x \mapsto N_x \). Applicability of conjugate codes to cryptography was argued. A class of good conjugate code pairs will be given in future works [33], [35], [36].

\(^6\)This follows by an easy direct calculation, but may be seen as a basic property of characters (e.g., [12]): the map \( f : z \mapsto \omega^z (w - w') \) is a character, and \( f(z) \neq 1 \) for some \( z \) if \( w \neq w' \).

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