Cognition of the circle in ancient India

S.G. Dani

Next only to the rectilinear figures such as triangles and rectangles, the circle is the simplest geometrical figure that would have touched human life even in the primitive stages, especially after the advent of the wheel. Apart from the everyday secular aspects of life, the circle gained significance in ritual and spiritual respects. Considerable understanding was acquired over a period in the ancient times, concerning various geometric features of the circle. Progress in understanding the circle may be readily correlated with progress of human civilization in general. Our overall knowledge of history across ancient cultures has many limitations, in terms of source material and means of interpretation. Nevertheless, in the Indian context we are endowed with information from various sources such as Śulvasūtras, the Jaina compositions, works from the mathematical astronomy tradition starting with Āryabhaṭa, and finally the Kerala school of mathematics, from different periods in history, that give an interesting perspective on how ideas developed on the issue.

The Śulvasūtra period

The Śulvasūtras are compositions concerned with construction of altars (Vedi) and fire platforms (citi) for the Vedic rituals\footnote{Performance of yajnas, fire rituals, in pursuance of material and/or spiritual benefits is one of the dominant features of the Vedic civilisation. It involved both the royalty as well as laity from the priestly class of the time. There are detailed prescriptions, about specific yajnas to be performed for various objectives, as well as the procedures to be followed.}. Apart from elaborate instruction on laying of bricks (of simple rectilinear shapes, such as squares, triangles etc.) to achieve approximations to various elaborate shapes such as falcons, tortoise, wheel, etc., the compositions also include enunciation of various geometric principles, geometric constructions with practical or theoretical import, etc., thus providing us a glimpse of the mathematical knowledge at that time. There were many Śulvasūtras, of which Baudhāyana, Āpastamba, Mānava and Kātyāyana Śulvasūtras are especially noted for their significance from a mathematical point of view. The period of the Śulvasūtras is somewhat uncertain, as there are no internal clues in the compositions other than their style and language, but there now seems to be a general consensus among scholars that they belong to the period from about 800 BCE to 200 BCE, Baudhāyana being the oldest and Kātyāyana the latest. For further general details the reader is referred to [13], [12], [1], and
the references cited there - here we shall focus on the specific theme at hand, concerning the circle.

One of the simplest questions that one can think of about the circle is the ratio of its circumference to the diameter. As in other ancient cultures, in ancient India also this ratio was believed to be 3. In the context of the Vedic tradition this is reflected in an indirect reference in the Baudhāyana Śulvasūtra in the statement “The pits for the sacrificial posts are 1 pada in diameter, 3 padas in circumference.” (Baudhāyana Śulvasūtra 4.15, see [18]); pada, which literally means foot, was a measure of length equivalent to about 28 cm. The second part of the statement is evidently meant as an elaboration/clarification of the first part, but provides us a clue that they considered the circumference to be 3 times the diameter.

The choice of the value 3 for the ratio would today seem quite surprising, as one would expect that many everyday experiences could have suggested the value to be a little more. The following seems to me to be a plausible explanation for this (which does not seem to have appeared in literature before): the idea of the ratio being 3 dates back to the time when mankind was yet to think in terms of fractions (except perhaps for “half”, which may have meant a substantial portion of the whole, rather than its precise value as we understand it) and developed into a belief (perhaps linked with religion). The ratio was assigned the value 3 in the sense that it is not, say 2 or 4, or even “three hand a half”. The belief, once it was rooted deeply, was not reviewed for a long time, even after fractions became part of human thought process. While our encounter with the circle, especially in the context of wheels, is over 5000 years old, fractions seem to have appeared on the scene in a serious way, in Indian as well as Egyptian cultures, substantially later, possibly only in the first millennium BCE. The difference between the actual ratio and 3 is small enough not to come in to serious conflict with everyday experience to warrant doubting an accepted proposition, which furthermore may have the backing of religious authority, and an appeal on account of universal significance associated with the number three. Also, while for a first-time determination of an entity one typically avails of prevalent techniques of any given time, a belief often remains untested until coming in conflict with another idea or experience.

The Mānava Śulvasūtra however breaks out of the mould, and we encounter the following:

\[
\text{viṣkambhahpañcabhāgaśca viṣkambhastrigunaśca yaḥ} \\
\text{sa maṇḍalaparikṣeṇa na vālamatiricyate} \\
\text{(Mānava Śulvasūtra 10.3.2.13)}
\]

A fifth of the diameter and three times the diameter, that is the cir-
cumference of the circle, not even a hair-breadth remains.

Apparently over the years it was recognised that the ratio is indeed a little larger than 3. Mānava seems to have taken a leaf out of this and came up with a better estimate. The exultation over it is striking!

Unlike the circumference, the area of the circle is seen to have been of direct interest to the authors of the Śulvasūtras. There is no indication in the Śulvasūtras of their being aware of the ratio of the circumference to the diameter being the same as the ratio of the area to the square of the radius; no occasion seems to have presented itself that would inspire a comparison of the two ratios. The issue of area, which was involved in the construction of altars, is treated independently. There were citis (fire platforms) constructed in the shape of a chariot wheel, a circular trough etc. with stipulated areas, which motivated the issue of how to transform a square into a circle having the same area.

**Transforming a square into a circle**

Baudhāyana describes a procedure to produce a circle with the same area as a given square, which goes as follows: take a string with length half the diagonal of the square, and stretch from the centre across a side of the square, viz. PS as in Figure 1, and draw the circle including a third of the extra part stretching outside the square, viz. PR as in the figure with \( QR = \frac{1}{3} PS \), as the radius.

\[ \text{Circling the square, Baudhāyana Śulvasūtra} \]

For a square with side \( 2a \) this radius works out to be
\[
\left( a + \frac{1}{3}(\sqrt{2} - 1)a \right) = (2 + \sqrt{2})a/3.
\]

For the unit square the area of this circle works out to 1.01725...
about 1.7% more than the correct value 1. If one is to compute the value of \( \pi \) with 
\[
(2 + \sqrt{2})a/3
\]
as the radius of the circle corresponding to a square with side 2a, it works out to be 3.0883..., in place of 3.1415.... It should be borne in mind that what they had was a procedure for producing the circle and not a numerical value for \( \pi \); the latter had not emerged as a concept, and they were not trying to compute such a ratio. The comparison, here and in similar contexts below, is only for facilitation in overall comprehension of the relative values.

The Āpastamba Śulvasūtra gives the same construction for the circle, as Baudhāyana. Mānava Śulvasūtra is seen to provide another construction for the circle with the area of the given square. The following interpretation of a verse in Mānava Śulvasūtra was introduced in \( \Pi \) by this author. For convenience I shall discuss the verse and background around it, after first describing the procedure, according to the interpretation. The steps involved are illustrated by Figure 2.

\[\text{Circling the square, Mānava Śulvasūtra}\]

Draw the lines dividing the square into 3 equal strips. Produce one of these lines to meet the circle passing through the vertices of the square. On the segment of the line that is outside the square and inside the circle, viz. QS as in Figure 2, take the point at a distance \( \frac{1}{5} \)th of the length of the segment, from the square, viz. the point R in the figure, with QR=\( \frac{1}{5} \)QS. The circle with PR as the radius, where P is the centre of the square, is given as the desired circle, with area equal to that of the original square.
For a square of side length 2 the length of the segment between the square and the outer circle is seen to be \( \left( \frac{\sqrt{17}}{3} - 1 \right) \), so the radius \( r \) of the circle is given by

\[
r^2 = \left\{ 1 + \frac{1}{5} \left( \frac{\sqrt{17}}{3} - 1 \right) \right\}^2 + \frac{1}{9}.
\]

For the unit square the area of this circle works out to 0.9946..., a substantially more accurate value compared to the earlier one, the error involved being just about \( \frac{1}{2} \% \) (which is now on the other side). The value of \( \pi \) in this case works out to 3.1583....

The verse in question, from Mānava Śulvasūtra is:

\[
\begin{align*}
caturaśraṃ navadāḥ kuryāddhanuḥkoṭyastraḥdātraḥdāḥ | 
\text{utsedhātpaṃcamaṃ lumpetpurīṣeṇeḥa tāvatsamaṃ} || 
\end{align*}
\]
(Mānava Śulvasūtra 11.15)

There seems to be considerable confusion about, and discomfort with, this verse in literature. In [18] it is suggested that possibly “squares are drawn without any mathematical significance”. In [12] there is an interpretation, the conclusion of which manifestly wrong. There is another interpretation in [7], concluding the value of \( \pi \) according to the sūtra to be \( \frac{25}{8} \), but it may be seen that in the interpretation the first line of the verse plays no role at all, while that of the second is quite ad hoc. Appropriate transliteration seems to be at the heart of the issue.

I propose the following translation:

Divide the square into nine parts, (by) dividing the sides into three (equal) parts each. Mark a fifth of the part jutting out (of the square) and cover (the corresponding circle with centre at the origin) with loose earth.

The meaning of this (according to my interpretation) is described above, with the help of Figure 2. It would be out of place to go into the linguistic details with regard to the interpretation. I shall instead focus on highlighting two reasons for which the present interpretation ought to be appropriate. Firstly, as noted above, it leads to a significantly improved result; this could not be a mere coincidence. Secondly if one is to try to read their mind on how they might have attempted
to remedy a perceived discrepancy in a known result, the construction seems to arise as a natural development: In the first place, it is reasonable to suppose in this respect that over a period it had been realised that the circle produced by the Baudhāyana construction was slightly larger than it should be. Since taking a point on the bisector of the square along a side did not seem to work, they chose to consider trisectors of square. So far there is no divergence in various interpretations. The crucial, and distinctive point in the above interpretation is that they picked a point on the trisecting line, which is actually natural in the context of the comparison with the Baudhāyana construction, but does not seem to have been taken note of by the earlier authors. Furthermore in analogy with the earlier construction a point had to be picked up on the segment of the trisector jutting out from the square. In the earlier construction one third of the jutting out part was added to get the radius of the desired circle, and it may be noted here that though the circle through the vertices of the square finds no mention in the verse, it would be lurking in the their minds, in the context of the Baudhāyana construction. The fraction $\frac{1}{5}$th of the extra part is then likely to have been based on an ad hoc observation that the resulting line segment for the radius is slightly smaller than the Baudhāyana construction, as was desired.

Clearly, the Mānava construction is the result of keen attempts to improve upon the original Baudhāyana construction, through refinement of the overall scheme. How the specific details were conceived and how, and to what extent, it was confirmed to be more accurate, remains unclear.

It may be mentioned here that there were also other constructions adopted, in the broader Vedic community; while indeed the Vedic civilization shared a certain common body of knowledge, there are many variations in the individual Śulvasūtras adopted by different sub-communities. A lesser known Śulvasūtra by the name Maitrāyanīya, which is akin to Mānava Śulvasūtra (in that it belongs to the same saṃhita), gives a construction for circling the square which involves taking the radius of the desired circle to be $\frac{9}{16}$ times the side of the square; see [9]. For a unit square the area of the resulting circle turns out to be 0.9940..., comparable to the one above, with 3.1604... as the corresponding value of $\pi$. It may be recalled here that the Egyptians took the area of a circle of radius $r$ to be $(\frac{16r}{9})^2$, to which the above value, for the reverse process, corresponds exactly.

**Squaring the circle**

The converse problem of “squaring the circle”, viz. finding a square with the same area as a given circle, is also considered in the Śulvasūtras. Baudhāyana

---

2This should not be confused with the problem in Greek geometry of finding a square with
gives the following expression for the ratio of the side of the square to the diameter of the circle (the original description is in words):

\[
\frac{7}{8} + \frac{1}{8 \times 29} - \frac{1}{8 \times 29 \times 6} + \frac{1}{8 \times 29 \times 6 \times 8}.
\] (1)

For the circle with unit radius, the area according to this works out to 3.0883..., a little more than 98.3% of the actual value. It may be noted that in this case also the error is about 1.7 percent, in the opposite direction. It could not be a coincidence (as has been noted also by earlier authors - see [19], [17]), that the errors in the two prescriptions, corresponding to mutually opposite operations, while substantial, are quite matching in their order and opposite in the orientation. It suggests that for want of a geometrical procedure in the reverse direction (unlike for transforming a square into a circle) they obtained it through inversion of the previous ratio, in some way which is not entirely clear so far (see below). To get the inverse of \((2 + \sqrt{2})/3\) they sought out a value of \(\sqrt{2}\), as a familiar fraction.

The square root of 2

Three of the four Śulvasūtras, Baudhāyana, Āpastamba and Kātyāyana, give the following expression for \(\sqrt{2}\) (in words):

\[
1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}.
\] (2)

In decimal expansion the value of the expression is 1.4142157... This is remarkably close to the actual value 1.4142136..., and this fact has been a subject of much laudatory comment in literature. It may be recalled in this context that Babylonians also had a value, about a thousand years earlier, describing \(\sqrt{2}\) in the sexagesimal system, which works out to 1.4142129... (see [4], for instance).

Various aspects including the presentation of the number as above and the substantial relative difference of the values (including the side of the error), rule out any organic link between the values. There has been considerable speculation and discussion on how the Śulvasūtra value of \(\sqrt{2}\) may have been arrived at; we shall however not digress to these details here (see for instance [1] and other references cited there).

Precisely the same area through a ruler and compass construction. The context of the sūtrakāras was entirely different, and their objective would have been only to find a square with the area of the circle, within the levels of accuracy they were used to, or desired. They may have liked to find a geometric construction, like by ruler and compass, but that was not the thrust. The problem in their perspective involved finding such a square by whatever available means.
As noted above the motivation for finding a value for $\sqrt{2}$ would have come from the problem of computing the inverse of $(2 + \sqrt{2})/3$, viz. for obtaining the formula (1). This numerical value of $\sqrt{2}$ is not involved elsewhere in the Śulvasūtras. In other contexts they are seen to use only the geometric form of $\sqrt{2}$ as the diagonal of the unit square, which in fact went by the special name dvikaraṇī.

How the inversion would have been effected, using the value of $\sqrt{2}$ as above, has been discussed by Thibaut [19] and also other later authors. The older explanations, however, presuppose considerable dexterity on the part of the sūtrakāras in dealing with fractions, for which there is no corroborative evidence, and are thus unsatisfactory. In a recent paper Kichenassamy [11] has proposed a resolution of the issue which is more in tune with the Śulvasūtras ethos; the paper also discusses at length the inadequacies of the earlier arguments.

It would also be worthwhile to note here another Śulvasūtra construction which relates in a way to properties of the circle. Baudhāyana Śulvasūtra describes a construction of a square which involves drawing a perpendicular to a given line, say L, at a point P on L, by drawing circles with centres at points on either side of P on L at equal distance, with radii larger than the distance, and joining the points of intersection of the two circles (in very much the same way as taught in schools today). Underlying the construction is the realisation that the line joining the two points of intersection of two circles meets the line joining their centres orthogonally; though the construction involves the principle for circles of equal radius, it seems reasonable to assume that they were aware of this “orthogonality principle” in that generality. In most constructions requiring perpendiculars, they were however produced using the converse of the Pythagoras theorem rather than the construction as above (implemented in a certain way, the former turns out to be simpler than the latter; see [1]).

Let me conclude this section on the Śulvasūtras with the following comment. There has been a tendency with regard to Śulvasūtras to assume that the sūtrakāras lay great store on accuracy. While the value of $\sqrt{2}$ does seem like an example of this, a careful reading of the Śulvasūtras shows that high degree of accuracy was not seen as a primary objective. In many contexts, alternate values or constructions are described, which are of a crude variety, alongside some

---

The theorem named after Pythagoras has been known in India at least since the time of the earliest Baudhāyana Śulvasūtra (ca. 800 BCE), where an explicit statement of it is found. The converse of the theorem, namely that a triangle in which the square of one of the sides equals the sum of squares of the other two sides, was used extensively for producing perpendiculars (see [1] for details).
relatively accurate ones, which shows that in their overall conception, the benefits meant to accrue from the ritual performances would not be seriously affected if approximate procedures were adopted. Where accuracy was pursued, it seems to be the result of keen academic enquiry, rather than an imperative arising from practical issues of the time, or the philosophical framework involved. On the other hand a supposition as above does them a disservice in the context of the less accurate values such as in circling a square. Mathematics of ancient cultures needs to be understood and appreciated in their specific context, and not judged through generalised abstract tests. The issue of circling the square arose for instance from the desire of having a fire platform with the same area, that would not bring with it an intrinsic demand for high degree of accuracy, and it is incorrect to wonder why their value of the area is not accurate - it was not meant to be.

The Jaina tradition

Apart from the Vedic religion (if one may call it that) Jainism and Buddhism flourished during the first millennium BCE (and later during certain periods). There was a long tradition among the Jainas of engagement with mathematics, as is evidenced from various compositions that have come down to us. As for Buddhism, though certain constructions involved in Buddhist pursuit, called Mandalas involve intricate designs which seem mathematically significant, no textual composition involving mathematical concepts has come down to us.

The motivation of the Jainas for mathematics did not come, per se, from any rituals, which they indeed abhorred, but from contemplation of the cosmos, of which they had evolved an elaborate and unique conception of their own. In the Jaina cosmography the world is supposed to be an infinite flat plane, with concentric annular regions surrounding an innermost circular region with a diameter of 100000 yojana\(^4\) known as the Jambudvīpa (island of Jambu, that corresponded to the Earth), and the annular regions alternately consist of water and land, and the width of each successive ring being twice that of the previous one. The geometry of the circle played an important role in the overall discourse, even when the scholars engaged in it were more of philosophers than mathematicians. Unfortunately, many historical and chronological details of the Jaina tradition are uncertain (even more so than the Hindu tradition) and have been a subject of speculation. Definitive references to the properties of the circle known in the Jaina tradition can be found in the work in fourth or fifth century (\cite{14}, page 59). It has however been mentioned by Datta in \cite{2} that they are found in Tattvārthādhigama-sūtra-bhāṣya,

\(^4\)yojana was a measure of length, of the order of 15 to 20 kilometres, with local variations.
a philosophical work of Umāsvāti, who is supposed to have lived around 150 BCE according to the Śvetāmbara tradition and in the second century CE according to the Digambara tradition. Datta suggests that Umāsvāti was probably not the discoverer of the formulae and they would have been known centuries before him, and discusses some evidence in this respect. Saraswati Amma attributes the basic formulae to Sūryaprajñāpti, a composition which is believed to be from the fifth century BCE.

In the Jaina tradition the departure from old belief of 3 as the ratio of the circumference to the diameter is quite pronounced; Sūryaprajñāpti records the then traditional value 3 for it, and discards it in favour of \( \sqrt{10} \). The Jainas also knew the ratio of the area of the circle to the square of the radius to be the same number as the ratio of the circumference to the diameter. In fact they had the formula directly relating the area to the circumference, that the former is a fourth of the product of the circumference and the diameter of the circle, which in particular readily implies the equality of the two ratios as above. Incidentally, \( \sqrt{10} \) which is about 3.16227... , may be seen to be a better approximation for \( \pi \) compared to the Baudhāyana construction, involving an error of only about \( \frac{2}{3} \) per cent.

The value was very convenient to the Jaina theologian mathematicians, in their computations. For example, in Jambudvīpa prajñāpti the value the circumference of Jambudvīpa, a circle of diameter 100,000 yojana, is computed, with \( \sqrt{10} \) as the value for the ratio of the circumference to the diameter, by computing the square-root of 10^{11}.

This value for \( \pi \) was used for over a thousand years, even after better values were known; indeed so routine was its use in the Jaina texts that it is often known as the Jaina value for \( \pi \). The value was also adopted in Pañcasiddhāntika, in the Siddhānta tradition sometime during 1st to 6th century, and by Brahmagupta in the 7th century.

There has been some speculation on the origin of \( \sqrt{10} \) as a value for \( \pi \). One explanation, attributed to Hunrath, goes as follows (15, page 65): The square of the side of a regular 12-sided polygon inscribed in a circle of unit radius is seen to be \( (1 - \frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \) and with the choice of \( \frac{5}{3} \) as an approximation for \( \sqrt{3} \) it works out to \( \frac{\sqrt{10}}{12} \); thus the perimeter of a regular 12-gon is about \( \frac{\sqrt{10}}{12} \) times its principal diagonals, but a little less. This may have inspired the formula for the circumference of the circle. The explanation however involves many assumptions.

\[ \text{\textsuperscript{5}Śvetāmbara and Digambara are two ancient branches of Jainism, with certain differences both in terms of their philosophy as also practices in everyday life.} \]
about which there is little evidence. An interested reader may also consult \cite{6} for various other explanations.

Virasena, a Jaina mathematician from the 8th century states:

\begin{verbatim}
vyāsāṃ śoḍaśaguṇītaṃ śoḍaśa sahitāṃ trirūparūpairbhaktām |
vyāsāṃ triguṇītaṃ sūkṣmādapi tadbhavet sūkṣmām |
\end{verbatim}

\textit{(Ṣaṭkhaṇḍāgama, Vol. IV. page 42)}

A routine translation of this would be as follows (slightly simplified from \cite{17}):

Sixteen times the diameter, together with 16, divided by 113 and thrice the diameter is a very fine value (of the circumference).

There is something strange about the formula (with the interpretation as above), that it prescribes “together with 16” - surely it was known to the author that the circumference is proportional to the diameter and that adding 16, independent of the diameter, would not be consistent with this. It seems reasonable however to suppose that the author meant $3 + \frac{16}{113} = \frac{355}{113}$ to be the factor by which to multiply the diameter to get the circumference, which is indeed a good approximation, as the author stresses with the phrase “sūkṣmādapi sūkṣhamām” (finest of the finest!).

In China this approximation for $\pi$ was given by Chong-Zhi (429-500). Its value is 3.1415929... in place of 3.1415926..., accurate to 6 decimals.

In \textit{Trilokasūra}, which is another account of the Jaina scholarship, composed by Nemicandra, who lived around 980 CE, also one finds another value for the ratio $\pi$, apart from $\sqrt{10}$: it is the value $(\frac{16}{9})^2$, that we saw from the \textit{Maitrāyaniya Śulvasūtra} (shared also with the Egyptians). This may suggest a relation with the Hindu tradition, but the time gap is rather intriguing.

Apart from the circle as a whole, the Jaina mathematicians were also interested in the interrelation between the arcs of a circle and the corresponding chords. This is related to their conception of the geography of \textit{Jambudvīpa}, including various

\footnote{There does not seem to be much of scope for attributing the issue to corruption in the course of transmission at some level; there are however issues of grammar and interpretation involved, and this accepted translation may be flawed; it is possible, for instance, that \textit{śoḍaśa sahitāṃ} (“together with sixteen”), which is the culprit, has the role of emphasising that while dividing by 113, one is to divide the previous product, which involved 16 - thus “together with 16” is not about adding 16, but reiterating that the following division by 113 is to be subjected to the output together with the earlier 16. While this indeed does seem odd in what is intended to be a formula, it need not be ruled out, given that the direct interpretation is odd anyway, and the author obviously would not have meant it. In part, the somewhat curious presentation may have been the result of needs of versification.}
regions, mountains etc. Umāsvāti notes various relations between the length $c$ of a chord, the height $h$ of the corresponding “arrow” (viz. the segment joining the midpoint of the chord to the midpoint of the arc) and the diameter $d$ of the circle. One of the relations noted is

$$c = \sqrt{4h(d - h)};$$

various other forms which are equivalent to this one algebraically, from a modern point of view, are also presented. An interested reader may consult [17], [2] and also [8] for further discussion on this issue; the last two references have some details also of analogous formulae from other ancient cultures.

There is also an interesting formula for the length of the minor arc (the smaller of the two arcs cut out by the chord), say $a$, as

$$a = \sqrt{6h^2 + c^2}$$

(with notation as above). As can be seen, such a relation does not actually hold exactly. It may be noted that in the special case when the chord is a diameter, so that the arc is a semicircle, the equation corresponds to the ratio of the length of the semicircle to the radius being $\sqrt{10}$; so the relation holds with their value for $\pi$. As we go to small arcs however the assertion goes quite off the mark. Surprisingly however, the formula continued to be part of Jaina literature all the way, including the famous mathematical work Gaṇita sāra saṅgraha of Mahāvīra in 850 (Ch. VII, verse 73\textsuperscript{1/2}; see [14], page 469). The formula also appears in Trilokasāra of Nemicandra (see [3]), who was mentioned above.

Given a chord of a circle, apart from the length of the arc segment one may also ask about the area cut out by the chord (with the minor arc). Gaṇita sāra sangraha gives the value of the area to be $\frac{1}{4}\sqrt{10}ch$, where $c$ is the length of the chord and $h$ is the height over the chord (length of the arrow). The formula is also found in Trilokasāra of Nemicandra. This formula also holds strictly only for a semicircular segment with $\pi$ in place of $\sqrt{10}$ but diverges from the actual value for smaller arcs.

A different formula for the area of the segment cut out by the chord is given in Triśatika of Śridhara (ca. 750\textsuperscript{7}) and also quoted in some later works, including Bhāskara (see below for more about him). It may be worth mentioning here that Śridhara does not quite seem to fit in the astronomer mathematicians tradition -

\footnote{Though until some time there had been an argument over when he lived and his background, there is now a general consensus that Śridhara is from the 8th century, and was Jaina, at least during the time of his writings - his mathematical work is seen to be consistent with this.}

12
his known works deal exclusively with mathematics, and he is well-known for his
procedure for solving a quadratic equation. According to Śridhara the area \( A \) of
the segment between a chord and the corresponding arc is given by

\[
A = \frac{\sqrt{10}}{3} \left( \frac{h(c+h)}{2} \right);
\]

Clearly \( \sqrt{10} \) here is meant to be for the ratio \( \pi \).

It would seem that many formulae for arc segments cut out by chords were
written down by extrapolating relations that were noted for the case of the semi-
circle to a general arc segment; if they had some (heuristic) reasoning for it, it is
not found recorded. From a historical point of view this highlights the difficulties
faced by the ancient mathematicians in grasping the lengths of arcs and the areas
bounded by them, and their endeavour to get around the difficulties, before the
ideas of trigonometry, and then calculus emerged.

**Āryabhaṭa and the astronomical tradition**

Āryabhaṭa, born in 476 CE (as has been indicated by the author in his work
Āryabhaṭīya), was the pioneer of what is termed as the siddhānta tradition, of as-
tronomer mathematicians in India that flourished for almost eight hundred years,
until Bhāskarāchārya in the 12th century, and even beyond, and in turn led to
the Kerala school of mathematics. While the tradition has some manifest linkages
with the older Hellenistic mathematical astronomy, after the early influences it
seems to have charted a course of its own. Many new mathematical ideas were
developed, both in response to the theoretical demands in the study of astron-
omy, and also in pursuit of pure mathematical thought. In particular a deeper
understanding of the circle evolved, both in terms of geometry and trigonometry.
In his work Āryabhaṭīya we find the following:

\[
\text{Caturadhikam šatamaṣṭagunam dvāṣaṣṭistathā sahasrāṇam |}
\text{ayutadvayaviskambhasyāsanno vrṛtapariṇahah ||}
\]

(Gaṇitapāda 10, in Āryabhaṭīya)

The circumference of a circle with diameter twenty thousand is ap-
proximately a hundred and four times eight, and sixty-two thousand
[viz. 62832].

This gives the value of \( \pi \) as approximately 3.1416, which indeed coincides with the
correct value of \( \pi \) truncated at 4 decimal places. It may be recalled that in Greek
astronomy, Ptolemy had the value, in sexagesimal expression, which corresponds to 3.14166... There is no direct information on how Āryabhata arrived at the value. One may anticipate that, like in similar instances in other cultures, the value was obtained through repeated application of the formula

\[ S_{2n} = \sqrt{\frac{S_n^2}{4} + \left(1 - \sqrt{\frac{4r^2 - S_n^2}{4}}\right)^2}, \]

where \( S_n \) is the side of the regular \( n \)-gon inscribed in a circle of unit radius. The formula follows from the “Pythagoras theorem”, which, as noted earlier, had been known in India since the Baudhāyana Śulvasūtra (8th century BCE) and is also stated in Āryabhatīya (in Gañitapāda - 17). It is suggested by Ganeṣa, a sixteenth century commentator of Āryabhaṭīya, that an inscribed polygon with 384 sides was used as an approximation for the circle, and the above formula was used, starting with a hexagon (for which the side coincides with the radius of the circle), until reaching the polygon with the number of sides \( 384 = 6 \times 2^6 \). The choice of 20,000 as the measure for diameter is readily seen to facilitate computation of square-roots in integral values; the values would have been rounded, up or down, to integer values at various stages of application of the above formula, and the square root computed using the well-known procedure for the purpose, that is attributed to Āryabhaṭa. It may be noted that the value of \( \pi \) as above is slightly greater than the actual value, despite its representing the perimeter of an inscribed regular polygon, due to rounding up at some stages.

In Āryabhatīya we also have the trigonometric sine functions. Āryabhaṭīya (499 CE) provides a sine table, in a verse, for angles up to 90° that are multiples of 3°45′ (24th part of the right angle): taking the circle whose circumference is 21600 (equal to the total measure of the circumference in minutes), the differences between the values of half-chords corresponding to angles that are successive multiples of 3°45′ are recounted sequentially; the radius of the circle, which features as the total of the differences recounted, is 3438. A similar table also appears in Panca-siddhāntikā an older composition from the early centuries of CE in which the value of the radius involved is 120. Once such tables were available, the lengths of circular arcs could be calculated using the sine table (for the specific values), without recourse to any special formula as in Jaina mathematics. There were also interpolation methods for dealing with intermediate angles.

While the Greeks did trigonometry with chords, it was in India that the trigonometry in terms of the half-chords originated.
Apart from the sine tables there was also a curious approximate formula for
the sine function in vogue in the Siddhānta tradition. It is generally attributed to
Bhaskara I (7th century CE), being part of his Mahābhāskarīya, but is also
found independently in the contemporaneous work Brāhmaśputa siddhānta of
Brahmagupta. In the modern notation the formula may be stated as

$$\sin \theta = \frac{4\theta(180 - \theta)}{40500 - \theta(180 - \theta)},$$

where $\theta$ is the angle measured in degrees. The formula is seen to be remarkably
accurate, involving an error of less than 1%, except for very small angles. It is
unclear how such a formula was derived. (see [5] and [20] for further details in
this respect).

Knowledge of various properties of the circle and trigonometry gradually be-
came crucial part of learning in the Siddhānta tradition, being a prerequisite for
pursuing mathematical astronomy. The tradition sustained itself, though perhaps
somewhat feebly during certain periods than others, and individual exponents
made fresh contributions to knowledge, apart from carrying forward the body
of knowledge that was getting built. We shall not go into the finer historical
details in this respect here. Bhāskara II, from the 12th century, (also known
as Bhāskara the teacher) is considered the last major exponent
from the tradition. Apart from mastering the knowledge flowing in the tradition,
Bhāskara made substantial contributions of his own in various respects. By his
time, the attendant mathematics, especially arithmetic and geometry, that went
with mathematical astronomy, had acquired a wider appeal, and applicability, in
the society. Bhāskarācārya composed a comprehensive work, Siddhānta Śiromaṇi
which, in the tradition of Siddhānta works, had a chapter devoted the mathe-
natical topics as above, called Līlavati. The latter however acquired a life of
its own, and a reputation as a mathematical work, with large number of copies
being produced. It served as a textbook of mathematics for several centuries, in
a large part of India. Specifically with regard to the circle I will only recall the
following (approximate) formula from Līlavati for the length of an arc of a circle;
the formula itself may be seen to be related to Bhāskara I’s formula for the sine
function, when expressed in radians:

$$a = \frac{p}{2} - \sqrt{\frac{p^2}{4} - \frac{5p^2c}{4(c + 4d)}} = \frac{p}{2} \left(1 - \sqrt{\frac{1 - \frac{5c}{c + 4d}}{}}\right),$$

where $p$ denotes the circumference (perimeter) of the circle, and the other notation
is as above, namely $a$ is the length of the arc, $c$ is the length of the chord, and $d$
is the diameter of the circle; the first expression as above is akin to the way it is given in the original verse and the second is a simplification. A more integral view is seen to have evolved with regard to geometry of the circle and trigonometry.

The Kerala School

We conclude this article with a few observations on the Kerala school in the context of the above theme. The school originated with the work of Madhava in the second half of the 14th century, and flourished, as a teacher-student continuity, with multiple names involved during some periods, for about 250 years. They took remarkable strides towards calculus, introducing techniques involving infinitesimals, and in particular had obtained Gregory-Leibnitz series for the arctan function and the Newton series for the sine function (over two centuries before their European counterparts). We shall not go into a detailed discussion on the mathematics from the Kerala school, which has been a subject of much study in recent years. The interested reader is referred to [10], [14], and [16].

Determining accurate values for \( \pi \), which is something that concerns our theme here, seems to have been a passion for the school. In particular the following remarkably close approximation to \( \pi \) is credited to Madhava, by Śankara Vāriar (1556), in Kriyākramakarī (cf. [14]): the measure of the circumference in a circle of diameter 900,000,000,000 is 2,827,433,388,233. Thus

\[
\pi = \frac{2,827,433,388,233}{900,000,000,000} = 3.141592653592 \ldots,
\]

in place of 3.141592653589\ldots, accurate to 11 decimals, when rounded. As the series expansion

\[
\text{Circumference} = 4 \text{diameter} \left(1 - \frac{1}{3} + \frac{1}{5} + \ldots\right)
\]

that they had obtained converges very slowly, and hence not useful in getting good approximations for \( \pi \). Madhava had introduced an ingenious device to get over this difficulty, called antya samskāra, “the end correction”. With \( S_n \) as the sum of the series truncated at the \( n \)th term, he introduced sequences \( a_n \) such that the sequence \( S_n + a_n \) converges faster. The third, the final one that was recorded, produces the sequence

\[
S_n + (-1)^{n-1} \frac{n^2 + 1}{4n^3 + 5n}.
\]

The 50th term of this is accurate to 11 decimals.
References

[1] S.G. Dani, Geometry in the Śulvasūtras, in Studies in History of Mathematics, Proceedings of Chennai Seminar, Ed. C.S. Seshadri, Hindustan Book Agency, New Delhi, 2010.

[2] Bibhutibhushan Datta, The Jaina School of Mathematics, Bull. Cal. Math. Soc. 21 (1929), 115 -145.

[3] Bibhutibhushan Datta, Mathematics of Nemicandra, The Jaina Antiquary, 1 (1935),

[4] D. Fowler and E. Robson, Square Root Approximations in Old Babylonian Mathematics: YBC 7289 in Context, Historia Math. 25 (1998) 366 - 378,

[5] R.C. Gupta, Bhāskara I’s approximation to sine, Indian J. History Sci. 2 (1967), 121 - 136.

[6] R.C. Gupta, Mādhavacandra’s and other octagonal derivations of the Jaina value \( \pi = \sqrt{10} \), Indian J. Hist. Sci. 21 (1986), no. 2, 131 - 139.

[7] R.C. Gupta, New Indian values of \( \pi \) from the Mānava śulba sūtra, Centaurus 31 (1988), no. 2, 114 - 125.

[8] R.C. Gupta, Area of a bow-figure in India, Studies in the history of the exact sciences in honour of David Pingree, 517 - 532, Islam. Philos. Theol. Sci., LIV, Brill, Leiden, 2004.

[9] R.C. Gupta, Śulvasūtras: earliest studies and a newly published manual, Indian J. Hist. Sci. 41 (2006), 317 - 320.

[10] George Gheverghese Joseph, A passage to infinity, Medieval Indian mathematics from Kerala and its impact, Sage Publications, Los Angeles, CA, USA, 2009.

[11] S. Kichenassamy, Baudhāyana’s rule for the quadrature of the circle, Historia Mathematica 33 (2006), 149-183.

[12] Raghunath P. Kulkarni, Chār Śulbasūtra (in Hindi), Maharshi Sandipani Rashtriya Vedavidya Pratishthana, Ujjain, 2000.

[13] Padmavathamma, Śrī Mahāvīrācarya’s Gañitasaṅgraha, Publ.: Sri Siddhāntakṛiti Granthamāla, Sri Hombuja Jain Math, Hombuja, Shimoga Dist., Karnataka - 577436, India, 2000.

[14] Kim Plofker, Mathematics in India : 500 BCE - 1800 CE, Princeton University Press, NJ, USA, 2008.
[15] T.A. Saraswati Amma, Geometry in Ancient and Medieval India, Motilal Banarsidas, Delhi, 1979.

[16] K. Ramasubramanian and M.D. Srinivas, Development of calculus in India, *Studies in the history of Indian mathematics*, 201 - 286, Cult. Hist. Math., 5, Hindustan Book Agency, New Delhi, 2010, 201-286.

[17] A. Seidenberg, The ritual origin of geometry, Archive for History of Exact Sciences, 1 (1962), 488 - 527.

[18] S.N. Sen and A.K. Bag, The Śulvasūtras of Baudhāyana, Āpastamba, Kātyāyana, and Mānava, Indian National Science Academy, 1983.

[19] G. Thibaut, The Śulvasūtras, The Journal, Asiatic Society of Bengal, Part I, 1875, Printed by C.B. Lewis, Baptist Mission Press, Calcutta, 1875.

[20] Glen Van Brummelen, The mathematics of the heavens and the earth; The early history of trigonometry, Princeton University Press, Princeton, NJ, USA, 2009.

**Note:** The papers of R.C. Gupta cited here are also available in the compilation of *Ganitānanda*, edited by K. Ramasubramanian, Published by the Indian Society for History of Mathematics (ISHM), 2015.

Department of Mathematics
Indian Institute of Technology
Powai, Mumbai, 400005
E-mail: sdani@math.iitb.ac.in