A PROPOSAL FOR THE QUANTUM THEORY OF GRAVITY

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1. Introduction.

The purpose of this paper is to give a proposal for a candidate for the quantum theory of general relativity in 3+1 dimensions. The proposal is easily generalized to a large family of similar theories which could be interpreted as including various matter fields. The idea is to modify the 4d topological quantum field theory (TQFT) constructed in [1] in such a way as to preserve diffeomorphism invariance, but to introduce propagating modes. The generalizations would amount to replacing the tensor category at the heart of the construction in [1] (or equivalently, the quantum group) by a larger one.

Since the construction in [1] is believed to be a discrete construction of the $B \wedge F$ theory, while the lagrangian of general relativity can be written as $B \wedge F$ plus a lagrange multiplier term [2], it is not implausible to try to quantize general relativity by augmenting the discrete state sum in [1] with an extra insertion. We will not try to motivate the insertion by a heuristic argument involving the two lagrangians, but rather to guess it from the category-theoretical character of the state sum.

The proposal is, at this stage, a guess. Demonstrating that it is in fact related to general relativity will require extensive computation within the proposed model.

This proposal is, in a reasonable sense, a completion of the program of the paper “Clock and Category”[3]. We will make the argument that a certain structure in the braided monoidal categories has the effect of introducing a clock into the TQFT of [1]. This proposal differs from the proposal of [3], however, in using an actual modification of the state sum formula of [1], rather than a simple reinterpretation of it, or a more complex TQFT. For this change of direction, I am indebted to Fotini Marcopoulo and Lee Smolin [4], who recently reproduced the theory in [1] in a different form, with the suggestion that the terms of the state sum should be modified, to include the distinction of the timelike directions.

It was with some reluctance that I refrained from entitling this paper “what makes a category tick?” The suggested clock operator is a very natural piece of the structure of the family of categories from which TQFTs can be constructed. In this sense, this proposal attempts to bring quantized general relativity into the sphere of “categorical physics”.

It should be remembered that the type of tensor categories utilized in [1] were first introduced into physics in [5] as part of an attempt to construct string theories. They are “stringy” categories in the sense that their categorical structure is related to operations on the world sheet of the string. It is possible to regard the model in this paper as a construction of a sort of string field
theory. The generalizations mentioned above correspond to various choices of string theories. Whether the anomaly cancellations in string theory will play a part in the development of this model remains to be seen.

We warn the reader that the treatment in this announcement is not completely self contained, but requires a knowledge of [1] and [3], although we do summarize briefly.

One issue we do not address carefully in this paper is the “Wick rotation” into lorentzian signature spacetime. We believe that this can be done rather straightforwardly by raising the amplitudes to the power i, since $e^{i\theta} = (e^\theta)^i$. Something along these lines does indeed work in dimension 3 [6].

It is necessary, but very difficult, to address the relationship between this proposal and the idea of spin network states [6] for the Ashtekar variables. Spin networks can be dualized to represent labellings on triangulated surfaces, so the states in that picture can be related to terms in the state sum we are using. In our picture, labels that do not sit on surfaces we choose to treat as observers can be summed over, so that states on two triangulations coincide if one is a refinement of the other. If this identification is not used, the spin network states in the Rovelli-Smolin picture become problematically multiple.

As far as the law of propagation is concerned, we have made a guess, based on mathematical elegance. All the other proposals really amount to guesses too, related to Hamiltonian operators at boundaries, clock fields, and various other physical approximations. We may well be in a period like the origin of quantum mechanics, where apparently very different proposals turn out to be equivalent, much like wave mechanics and matrix mechanics. The proposal we are considering has the advantage of mathematical rigor and computability.

2. State Sums and Tensor Categories.

Let us recall the formula with which we constructed a 4d TQFT in [1]. The procedure in that paper is to first label each 2-simplex of a triangulated oriented 4-manifold with a tilting module of the quantum group $U_q(sl(2))$ for $q$ a suitable root of unity. More simply, the labels are half integers up to some finite bound, which should be thought of as quantum spins. We then put in tensor operators connecting the four spins at each tetrahedron, and combine the spins and operators around each 4-simplex into a number called a q-15j symbol. The 15 comes from the fact that a 4-valent tensor operator in this category is given by a quantum spin, so the expression has 10+5=15 spins in it. We then form the expression:

$$\sum N^{\text{#vertices}-\text{#edges}} \prod_{\text{faces}} \dim_q(j) \prod_{\text{tetrahedra}} \dim_{-1}(p) \prod_{\text{4-simplices}} 15J_q \ (*)$$

where $\dim_q$ is the quantum dimension, and N is the sum of the quantum dimensions of the irreducible representations. This expression is independent
of the triangulation, and can be used to construct a 4d TQFT. The expression should be thought of as the discrete analog of a path integral.

We can represent this procedure graphically by embedding a certain surface in the boundary of each 4-simplex. The surface is a union of tubes dual to each 2-simplex and one per 3-simplex in the boundary of the 4-simplex, joined together to make a closed surface. (The summation in each 4-simplex is then equivalent to a sum over a basis for the space of conformal blocks for the surface in the sense of [5].) The surfaces in the boundaries of adjacent 4-simplices share the tubes corresponding to shared faces and tetrahedra, so the summations for different 4-simplices overlap.

The tubular diagrams we use can also be thought of a ribbon graphs [7]. Remember that a closed ribbon graph can be evaluated to give a number, but that an open ribbon graph is an intertwining map between the tensor product of the representations on the open ends at the top and the same at the bottom. In [1] we used only the numbers on closed graphs, below, we use the intertwining maps as well.

There is a natural generalization, stated in [1], which uses a more complex tensor category with analogous structure to the representation category of the quantum group. The whole proof is very categorical, in that the fundamental structures of the category translate nicely into the ingredients of the proof of topological invariance.

This construction can be thought of as filling up the spacetime with a dense collection of tubes related to the triangulation. We are then summing over all the spaces of conformal blocks of the tubes, if we think of the tensor category as the modular tensor category of a rational conformal field theory as in [5]. It is then a sort of algebraic miracle that the propagation is independent of the triangulation.

In [1] we considered as spaces of states only formal sums of labels for representations. This is because all the operators used to construct the TQFT are intertwining operators, hence acting as a scalar on each irreducible representation. The operator we introduce below as a clock does not have this property, so we will be considering larger spaces of states corresponding to vectors in the tensor product of the actual representations themselves. Since our proof of invariance under change of triangulation in [1] is a proof of the invariance of an operator, we can speak of topological propagation for this new and larger class of states as well.

A formal heuristic argument can be made which suggests that this theory is a discrete construction of $B \wedge F$ theory [2]. An important point of the heuristic is that the representations in the diagram come from summing over a group element which corresponds to the holonomy of the connection along some path. Thus in some sense, our state sum has the right variables to be a discretized quantum theory of gravity. We do not alter the variables below when we add the ticks.

The space of physical states of this theory in a closed 3-manifold is one
dimensional, but it is possible to obtain larger spaces of states associated to 3-manifolds with boundary. In the program of [3], these are interpreted as spaces of states which an observer on the boundary surface would observe. Thus it is possible to find large spaces of states in this theory. Unfortunately, in a certain sense, they are “nonpropagating.” If we take the cartesian product of the 3 manifold with boundary by an interval, we get a 4-manifold with corners, which we can think of as a process of time passing in our theory; this process gives us the identity operator on all our states.

This is the fundamental difference between TQFT and quantum gravity. The finite dimensional hilbert spaces in a TQFT can easily be combined into an infinite dimensional limit space [9], but the non-propagation is a basic problem. On the other hand, diffeomorphism invariance is a common feature which we would like to preserve. Finally, the TQFT we have outlined has no timelike directions.

So can we modify the construction in [3] to solve all these problems?

3. The tick of time

At this point, we explain the modification of the state sum in [1] which could transform the resulting theory into general relativity. Our discussion as we have warned the reader, is not self contained, but presupposes a knowledge of [1]. We also do not supply the proof of isotopy invariance, which is not difficult.

We propose to place insertions of certain operators on the category of representations we used in [1] inside the diagrams corresponding to the triangulated 4-manifold in such a way as to introduce a clock into the theory. The operators to be inserted are the canonical grouplike element of the quantum group $U_q(SU(2))$ for which the labels on the faces in the TQFT of [1] are representations. This hopf algebra has a canonical grouplike element and all other grouplikes are powers of it; the larger tensor categories would have different possible choices for a grouplike on them. For a discussion of the grouplike element and its properties see [10,11].

The grouplike in $U_q(SU(2))$ is easy to understand. It is $q^H$, where $H$ is the $q$ analog of the diagonal matrix $J_z$. The grouplike property is related to the fact that $J_z$ remains a good quantum number for $q$-deformed spins. ($J_x$ and $J_y$ do not.)

A grouplike element in a hopf algebra obeys the equation $\Delta g = g \otimes g$, $\epsilon(g) = 1$, $s(g) = g^{-1}$. The first equation means that when applied to a representation which then has a tensor operator taking it to a product of two representations, it can be “passed through” the vertex, to act on each of the two representations below the vertex. The second implies that when we change a downward tube to an upward one in a diagram the grouplike switches to its inverse. These properties allow us to use the grouplike element to put an insertion into the state sum on a hypersurface in such a way that the result is unaffected if the hypersurface is isotoped. The hypersurface cannot be isotoped to the boundary.
We refer to a hypersurface on which the tubes are decorated as a “tick,” which we think of as a space slice where the clock time advances by one Planck time. The model would be to calculate the state sum for the TQFT with many ticks.

In order to define the tick precisely, it is necessary to pick an ordering of the set of tetrahedra in the triangulation. This gives a direction on each tube in the model (which is dual to a face in the boundary of a particular 4-simplex). We next pick a hypersurface in general position with respect to the triangulation, which traverses the cone on each tetrahedron with apex the center of each 4-simplex which the tetrahedron bounds regularly and once only or not at all. We further require the hypersurface to be oriented with a collar neighborhood.

For the nonmathematician, we are requiring the hypersurface to divide the spacetime into two sides, a past and a future.

The exact prescription is now to decorate the tubes dual to the faces of each tetrahedron with copies of the grouplike if the tetrahedron is ordered lower than the one on the other side of the face and the center of the 4-simplex is to the past of the tetrahedron, reversing to the inverse of the grouplike if the tetrahedron is ordered higher than its opposite or if the center of the 4-simplex is in the future, and labelling with the grouplike if the opposite tetrahedron is lower and the center is in the future of the tetrahedron. It is equivalent to do the labellings only on faces which bound between traversed and nontraversed tetrahedra (always in the boundary of a particular 4-simplex), because of cancellations.

Basically, this prescription works because a spin depicted as a down line is expanded in a dual basis from an up line, so that the grouplike is transformed to its inverse.

A mathematical paper with the details and proof of isotopy invariance of this prescription will follow [12]. It is also possible, but unenlightening to include “wigglier” hypersurfaces.

The idea now is to put a long sequence of “ticks” in a 4-manifold with corners, and to see how the states on neighboring surfaces interact. It is not necessary for the string of hypersurfaces to be parallel, they can also spiral or have various topological configurations.

Let us mention that the 4d state sum we are modifying, and hence the TQFT and also the ticking model we construct from it, are actually more flexible than our description shows. The “blob lemma” of [1] shows that we can replace the triangulation by any polyhedral decomposition. Furthermore, the work of Roberts [13], shows that we can replace the triangulation with a complicated weave in the boundary of a 4-ball. The tick is equally well defined in these situations.

Thus we can represent our ticking model as a weave passing through a long series of ticking surfaces. This may make calculations easier.

We believe that the procedure we outline here has several properties which are suggestive of general relativity. First, the independence of the tick location from the triangulation is reminiscent of the difference between clock time and
coordinate time. Second is the manifest diffeomorphism invariance of the constructed theory. Third is the natural introduction of a fundamental length scale into the theory, while still being able to consider any triangulation. Fourth is the identification of the states in this theory with the spin network states derived from the loop variable picture [4].

None of this shows that the “ticking” theory we construct actually resembles general relativity in any way. That can only be addressed by actual calculations in the ticking model. It is possible to say, at least, that the model is well defined; so that a number of calculational investigations are possible. Let us list some natural things to investigate: 1. the analog of the Wheeler -De Witt equation at a tick; 2. the behavior of sectional curvatures around an edge near a tick; 3. the propagation of states through a sequence of ticks.

4. Towards an investigation of the model.

At this point we wish to make some preliminary remarks concerning the physical interpretation of the model.

First, we state without proof some facts about the effect of insertion of the grouplike in a diagram in the category.

Insertion of a grouplike on a single strand of a closed diagram has the effect of multiplying the evaluation of the diagram by the quantum dimension of the representation on the edge where the grouplike is inserted. Insertion of grouplikes at more than one place, however, has a more complex effect, since the operations are correlated. If two edges are joined to a third at a vertex, the joint effect of grouplikes on both of them is equal to the effect of a grouplike on the third. This means that states on the pair which are strongly correlated (ie add up to a larger representation in the tensor product) are weighted more heavily by the effect of the insertions of the grouplikes.

The definition of a “tick” we made above reduces to putting grouplikes on the tubes dual to the faces of one tetrahedron at a time.

Since each set of four such tubes is connected at the center of the tetrahedron in our model, the effect of a tick will be to cause the states attached to the faces of a tetrahedron to become correlated. We believe that this has a classical limit which reproduces the form of Einstein’s equation in Regge calculus. If this is so, then our model will genuinely have general relativity as a classical limit. The motivation for this idea is to think of the minimal example, in which the representations are q-deformed spins, and the quantum dimension is the q-analog of angular momentum. The spin states should be thought of as generalized fourier coefficients of sectional curvatures. The correlation between them would correspond to some weighted sum of the sectional curvatures at a tetrahedron, which is a discetized analog of the Ricci curvature. A more refined analysis of this argument, with especial care taken as to the effects of Wick rotating into the lorentzian regime will be necessary to see if there is really any validity to it.
In general, since each tick is producing correlations between neighboring observers, we expect that our model will show propagation. Also, different vectors in the same irreducible representation will now have fundamentally different evolutions because of the tick.

Now we can understand the problem of writing down propagating modes. The eigenstates of $J_z$ are multiplied by appropriate powers of $q$ as we pass a tick. This is analogous to the behavior of eigenstates of the momentum operator for a quantum mechanical free particle. In order to see that quantum mechanical particles propagate, we need to form wave packets.

We can imagine joining together all the tubes labelled on a tick to a single tube, which would end up labelled by a very large representation which would be the tensor product of those on the individual tubes. The $J_z$ operator for the combined representation would be like a Hamiltonian for the total system. The fact that $J_z$ is a grouplike allows us to break it up into local pieces.

The problem of actually constructing a propagating mode will use this local decomposition of the tick/Hamiltonian. Interpretation is made more subtle by the diffeomorphism invariance of the construction, which must be somehow broken in a semiclassical picture. The necessary ingredients all seem to be present, however.

In order to actually demonstrate propagating modes, we would need to pick a background spacetime geometry (determined by choosing the irreducible representations at the 2-simplices rather than summing over them) and studying how particular vectors in it propagate in the state sum with ticks. It should be possible to construct wave packets using the different phases different vectors pick up at the ticks. This will be a formidable computation, however.

The passage to our larger notion of states, mentioned above, is crucial to allow this. The states of the TQFT, which do not distinguish vectors inside the representations, do not propagate under the tick.

Assuming this proposal goes through, the propagating states would live on 2-simplices of the triangulation. This suggests that they would look a lot like gravitons, i.e. have a tensorial character.

5 Strings and things.

It is natural to ask at this point what connections this picture may have with string theory. The most salient connection is the fact that both the 4d-TQFTs we modify here and rational conformal field theories are constructed from the same family of tensor categories [5]. The same categories are also used to construct the state spaces on surfaces. (We think of surfaces in spacelike 3-manifolds as the boundaries where observers observe. See [3].)

It is worth recalling that the program of string theory faltered through its inability to construct a “string field theory” to which the string amplitudes gave perturbative terms. The construction we are proposing amounts to filling the spacetime with a network of Planck scale strings, obtaining a propagation law
unaltered by any further refinement of the network. This circumvents the nasty mathematical problems of constructing a string field theory on loop space.

Also, the stringy states on the tubes which meet the past and future boundaries of our spacetime are combined to produce the vector spaces on surfaces which we interpret as relative states for observers. Perhaps, since the state space of an observer is dominated by states which almost turn it into a black hole, the strings really come into the theory as boundaries for quantum observers which are rather like black holes. In other words, maybe strings are things.

The most difficult part of our model to interpret in stringy terms is the grouplike in the hopf algebra. Here we note that Moore and Seiberg in [5] actually regarded a group as a “classical” rational conformal field theory. In that analogy, elements of the group are “grouplike” in the same sense as the grouplike element of the quantum group. The grouplike represents a remaining symmetry unbroken in the deformation producing the quantum group.

6. Conclusions.

At this point our proposal is really only the beginning of a program of investigation. It is certainly natural to ask if any other modifications of the state sums of [1] exist which might be worth studying.

One proposal would be to modify the 15j symbols by cutting them along the hypersurface and inserting some carefully chosen spin diagrams. One possibility would be to connect adjacent tubes with crosstubes labelled with the basic representation. This is strongly reminiscent of the hamiltonian proposed for quantum gravity in [14].

A drawback of such a procedure is the difficulty of finding one with any sort of invariance property. This could be a simple lack of imagination, but the elegance of the behavior of the grouplike is seductive. Also, any such proposal would not distinguish vectors inside a representation.

A drawback of this proposal is the absence of any heuristic argument connecting the laplace multiplier term in the lagrangian of GR to our picture. Perhaps more careful thought would suggest a different insertion from that quarter.

At any rate we have proposed a model with the following features:

1. A natural distance scale
2. Variables which are reasonable for a discretized version of general relativity
3. A perturbation of $B \wedge F$ theory
4. Diffeomorphism invariance
5. A clock
6. (plausibly) propagating modes resembling gravitons.

Clearly, this model deserves further study.
Let us remind ourselves of another feature of string theory, namely the cancellation of conformal anomalies and the related choice of a special class of models. This phenomenon has a counterpart in the construction of 3d TQFTs from modular tensor categories: the amplitudes for propagation of topological states has a phase ambiguity unless we choose a model with “central charge” a multiple of 24 [15].

We have not yet computed the propagation of states on surfaces carefully enough for the ticking model, but it is very likely that a phenomenon of this type will recur. If so, there will be a topological reason to prefer the same models which appeared as string motivated GUT theories during the heyday of the string. If this does turn out to be the case, the physical applicability of the ticking models will be more direct.

As we mentioned in the introduction, there is a natural proposal for Wick rotation for this model, namely raising each term in a state sum to the power i. It needs to be carefully studied if the ticking together with this prescription makes the timelike directions tend to align.

Another interesting feature of this model is that a tick on a closed contractible hypersurface can be contracted away, leaving no result. It is tempting to try to use this feature of the ticking model to shed some light on the difficult questions of time evolution of states which fall into black holes. The suggestion is that classical states with event horizons may occur in quantum evolutions which do not possess them. What this may tell us about information loss remains to be explored.

In this paper we have been constructing the “stringy” tensor categories as representations of quantum groups. They also appear as representations of loop algebras with central extensions. If we could find a natural choice of grouplike for the universal enveloping algebras of the Kac Moody algebras, then we could repeat our construction, with genuinely stringy state spaces on surfaces. The grouplike would have to behave well with respect to the tensor product of Moore and Seiberg [5]. This would lead to rather different problems of physical interpretation.

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ADDENDUM

We wish to note that the definition of the tick above is actually more flexible than we realized. It is possible to modify the construction so that the grouplike appears in the same places as in I, but raised to the power of the absolute value of the casimir of the representation on the tube it is attached to. This will have
the effect that if a vector in a representation on some face of the triangulation is followed through the tubular diagrams in the 4-manifold to another face, the contributions along various paths will pick up different phases, corresponding to the total “lengths” of the paths. This seems much like a discrete analog of the path integral expression for a propagator for a massless field in a gravitational background.

We believe that with this modification the model has every indication of possessing propagating modes, although demonstrating them will still be a formidable calculational problem.

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