Exploration Enhancement of Nature-Inspired Swarm-based Optimization Algorithms
Kwok Pui Choi, Enzio Hai Hong Kam, Tze Leung Lai, Xin T. Tong, and Weng Kee Wong

Abstract
Nature-inspired swarm-based algorithms have been widely applied to tackle high-dimensional and complex optimization problems across many disciplines. They are general purpose optimization algorithms, easy to use and implement, flexible and assumption-free. A common drawback of these algorithms is premature convergence and the solution found is not a global optimum. We provide sufficient conditions for an algorithm to converge almost surely (a.s.) to a global optimum. We then propose a general, simple and effective strategy, called Perturbation-Projection (PP), to enhance an algorithm’s exploration capability so that our convergence conditions are guaranteed to hold. We illustrate this approach using three widely used nature-inspired swarm-based optimization algorithms: particle swarm optimization (PSO), bat algorithm (BAT) and competitive swarm optimizer (CSO). Extensive numerical experiments show that each of the three algorithms with the enhanced PP strategy outperforms the original version in a number of notable ways.

Index Terms
Bat algorithm, Competitive swarm optimizer, Convergence to the global optimum, Exploration, Exploitation, Particle swarm optimization.

I. INTRODUCTION
OVER the past decades, Swarm Intelligence has and continues to inspire a steadily rising numbers of nature-inspired swarm-based algorithms in optimizing high-dimensional complex cost functions, including those that do not have analytic forms. Examples of swarm-based algorithms include particle swarm optimization (PSO), competitive swarm optimizer (CSO), ant colony optimization, bat algorithm (BAT) etc. Swarm-based optimization algorithms are mainly motivated by nature or animal behavior and then thoughtfully formulated into an algorithm that iterates to the optimum according to a couple of equations. Generally, these algorithms are easy to code and implement, and do not need gradient information or other sophisticated mathematical techniques to work well. Computer codes are widely and freely available, which have undoubtedly helped fuel numerous and various applications of these algorithms to tackle many different types of complex real-world optimization problems. Documentation of their effectiveness is widespread resulting in their meteoric applications in an increasing number of sectors in industry and in academia [1, 2].

There are known shortcomings of nature-inspired swarm-based optimization algorithms. We highlight two: (1) They require effective tuning of their parameters to achieve optimal performance [3] and (2) they often suffer from premature convergence to a local optimum of the cost function, especially when the cost function is high-dimensional and multi-modal. Recently, [4] proposed a general strategy to tackle the first shortcoming using statistical tools.

In this paper, we propose an innovative and simple strategy to address the latter problem by modifying a nature-inspired swarm-based algorithm to ensure its almost sure (a.s) convergence to a global optimum. The proposed methodology is general, as long as certain unrestricted technical conditions are met, and we demonstrate the techniques using three popular swarm-based algorithms mentioned above. Theorem II.1 in Section II shows that if a swarm-based algorithm satisfy conditions (C1)–(C3), the algorithm is guaranteed to converge a.s. to a global solution. We also provide comments on possible extension of the proposed methodology to evolutionary algorithms, like differential evolution.

We next propose a general, yet simple and effective strategy, Perturbation-Projection (PP), to modify an algorithm, if necessary, to ensure conditions (C1) and (C2) to hold. In Section III, we demonstrate how the PP strategy is applied to the original PSO, BAT and CSO algorithms with appropriate modifications to meet the required conditions for almost sure convergence to the global optimum. We denote these modified algorithms of their counterparts by mPSO, mBAT and mCSO. When it is necessary to further improve the exploitation ability of an algorithm but not at the expense of its exploration ability, we introduce a Heterogeneous Perturbation-Projection (HPP) strategy in Section IV to satisfy conditions (C1)–(C2). In Section V, we conduct extensive numerical experiments using many commonly used test functions and show that the modified algorithms remain very competitive with the original algorithms. We also conduct tuning parameter analysis for the PP and HPP strategies. The paper ends with conclusion and related future work.

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Manuscript received ???, ???; revised ???, ???. This work is supported in part by Singapore MOE Academic Research Fund R-155-000-222-114.
In the literature, there are many different modifications of the original algorithms or new techniques to address the premature convergence issue for swarm-based algorithms; see, for example [5, 6, 7, 8]. These methods tend to either apply to only one specific algorithm, e.g. PSO, instead of a class of algorithms, or, require additional detection or tuning mechanisms that can be complicated to implement. A commonality among these works is that they do not have rigorous mathematical theory to support the modifications for improved convergence to the global optimum. The distinctive features of our proposed modifications are that they are supported by mathematical theory, simple to implement and applicable to any swarm-based algorithm.

II. STOCHASTIC ENHANCEMENT OF AN ALGORITHM’S EXPLORATION

In this section, we introduce a simple and effective strategy to modify a nature-inspired swarm-based algorithm to ensure the modified algorithm converges almost surely (a.s.) to a global optimum. The cost function can be a high-dimensional, non-differentiable, non-separable or non-convex with multiple local or global optima. The key idea behind the proposed algorithms is simple; we incorporate an additional stochastic component with an appropriate noise level to a nature-inspired swarm-based algorithm to enable it to escape from a local minimum. While this idea may not be entirely new, the distinguishing feature of proposed modification is its generality and the modified algorithm is guaranteed to converge a.s. to a global optimum.

Without loss of generality, we assume that we have a minimization problem as \( \min_{x \in D} f(x) = -\max_{x \in D} \{-f(x)\} \). Here \( f(x) \) is a user-specified real-valued cost function defined on a given compact subset \( D \) of \( \mathbb{R}^d \). For many real-world applications, there are physical and budgetary constraints and we assume that these constraints have been appropriately translated into the compact search space. For many other problems, like those in machine learning, regularization terms, sometimes denoted \( \|x\|_1 \) or \( \|x\|_2^2 \) are often added to the nonnegative loss/cost function \( f(x) \) to prevent over-fitting, where the regularization parameter \( \rho \) can often be selected through various cross validation methods [9, 10]. In such situations, we minimize the function \( F(x) = f(x) + \rho \|x\|_1 \) (or \( f(x) + \rho \|x\|_2^2 \)) over \( D \), which is the closed \( L_1 \)-ball (or \( L_2 \)-ball) centered at the origin with radius \( f(0)/\rho \). This is because for \( x \notin D \), \( F(x) > F(0) \) so the global minima are in \( D \).

To solve the optimization problem, we resort to nature-inspired swarm-based algorithms, which are increasingly used to solve complex and high-dimensional optimization problems. They generically employ an evolutionary-like or swarm-based strategy to search for an optimum by first randomly generating a user-specified set of candidate solutions. Depending on the particular swarm-based algorithm, candidate solutions are called agents or particles. These algorithms embrace common principles that include good exploration and exploitation abilities and they have stochastic components and tuning parameters. For example, for a swarm-based type of algorithm, it can be typically described as follows.

At time or iteration \( t = 1, 2, \ldots \), let \( x_i(t) \) represent the \( i \)-th particle’s position or candidate solution of the optimization problem. Let \( \{x_i(t) : 1 \leq i \leq n\} \) denote the collection of candidate solutions of the swarm of size \( n \). In addition, there is auxiliary information from each member of the swarm and we denote them collectively by \( \{v_i(t) : 1 \leq i \leq n\} \). In example, in PSO, the auxiliary information of each particle is its velocity with which it flies to the current position. We denote them by \( v_i(t) = (v_{1i}(t), \ldots, v_{ni}(t)) \).

Suppose the algorithm has a stochastic update rule of the following form: for agent \( i \)

\[
x_i(t+1) = \Phi_i(x_i(t), v_i(t), f),
\]

\[
v_i(t+1) = \Psi_i(x_i(t), v_i(t), f),
\]

for \( 1 \leq i \leq n \), where \( \Phi_i, \Psi_i \) are some functions of which the outputs can be random. At either a pre-selected deterministic time or a stopping time \( T \), the algorithm is terminated and the best solution \( \min_{1 \leq i \leq n} f(x_i(T)) \) is output.

A key desirable property of a global optimization algorithm is its ability to identify a global optimum eventually. To achieve convergence to the global optimum, Törn and Žilinskas [11] remark that the algorithm has to be able to explore densely in the search space. Inspired by their remark, we formulate the following conditions on the algorithm’s agents, \( x_1(t), \ldots, x_n(t) \).

(C1) At any iteration, all agents stay in \( D \), i.e., \( x_i(t) \in D \) for \( 1 \leq i \leq n \) and \( t = 0, 1, \ldots, T \).

(C2) Conditional on the information up to the current iteration \( t \), there exists an agent at the next iteration, \( t+1 \), that has a positive probability to explore a region of improvement in \( D \). Specifically, there exists an \( \alpha > 0 \), independent of \( t \), such that for any \( d \)-dimensional ball \( B \) in \( D \) satisfying

\[
f(y) \leq \min_{1 \leq i \leq n} f(x_i(t)), \quad \text{for all } y \in B,
\]

the following holds

\[
\max_{1 \leq i \leq n} \mathbb{P}_t(x_i(t+1) \in B) \geq \alpha |B|,
\]

with \( |B| \) denoting the volume of \( B \) and \( \mathbb{P}_t \) the conditional probability with information at the \( t \)-th iteration.

(C3) The algorithm is time improving, i.e.,

\[
\min_{1 \leq i \leq n} f(x_i(t+1)) \leq \min_{1 \leq i \leq n} f(x_i(t)), \quad \text{a.s.}
\]
Suppose $f$ is a continuous function defined on a given compact subset $D$ of $\mathbb{R}^d$ with at least one minimizer in the interior of $D$. Any swarm-based algorithm that satisfies (C1)–(C3) will converge to the global minimum almost surely (a.s.). In other words, if the swarm-based algorithm has $n$ agents, $\lim_{t \to \infty} \min_{1 \leq i \leq n} f(x_i(t)) = \min_{x \in D} f(x)$ a.s.

**Proof.** Let an interior point $x_* \in D$ denote a minimizer of $f$. Without loss of generality, we assume $f(x_*) = 0$. By (C3), $m(t) := \min_{1 \leq i \leq n} f(x_i(t))$ is a non-increasing sequence, thus it suffices to show that $m(t) \xrightarrow{P} 0$ as $t \to \infty$. Given a fixed $\varepsilon > 0$, by continuity of $f$, there exists $\delta > 0$ such that $0 \leq f(x) \leq \varepsilon$ for $x \in B(x_*, \delta) := \{x : \|x - x_*\| \leq \delta\}$. Since $x_*$ is an interior point, $B(x_*, \delta) \subset D$ for a sufficiently small $\delta$. By (C2), at each iteration, there is an agent $x_i$ such that either $f(x_i(t)) \leq \varepsilon$ or

$$\Pr \left( x_{i(t+1)}(t+1) \in B(x_*, \delta) \right) \geq \alpha|B(x_*, \delta)| =: \alpha(\varepsilon).$$

If $f(x_i(t)) \leq \varepsilon$, then $m(t) \leq \varepsilon$. If $x_{i(t+1)}(t+1) \in B(x^*, \delta)$, then $m(t+1) \leq f(x_{i(t+1)}(t+1)) \leq \varepsilon$. By the tower property of conditional expectation,

\[
\begin{align*}
\Pr(m(t) > \varepsilon) & \leq \Pr(x_{i(s+1)}(s+1) \notin B(x^*, \delta), \text{ for } s = 1, \ldots, t-1) \\
& = \mathbb{E} \prod_{s=1}^{t-1} \Pr_{x_i} (x_{i(s+1)}(s+1) \notin B(x^*, \delta)) \\
& < \mathbb{E} \prod_{s=1}^{t-1} (1 - \alpha(\varepsilon)) = (1 - \alpha(\varepsilon))^{t-1}.
\end{align*}
\]

Therefore, $m(t) \xrightarrow{P} 0$ as $t \to \infty$. \hfill \square

Theorem II.1 shows that any swarm-based algorithm satisfying (C1)–(C3) is guaranteed to converge to a global optimal solution eventually. Many swarm-based algorithms satisfy (C3), but not all swarm-based algorithms stay within the search space $D$ in their iterations thus violating condition (C1) or they are exploratory in the sense of condition (C2). However, Section II-A below shows that, with slight modifications of an algorithm, we can ensure the modified algorithm satisfies (C1)–(C2).

### A. Perturbation-projection strategy

A first step is to ensure all agents stay in $D$. For this purpose, we introduce the projection onto $D$ ensuring the agents stay inside the search space, $D$, a compact subset in $\mathbb{R}^d$:

$$\chi_D(x) = \arg \min_{y \in D} \|y - x\|_2,$$

where $\|y - x\|_2$ denotes the Euclidean distance between $x$ and $y$. In other words, $\chi_D(x)$ is a point in $D$ that is closest to $x$. When ties occur, they can be broken arbitrarily. By definition, $\chi_D(x) = x$ if $x \in D$. In general, adding the projection to an algorithm’s iterations reduces the agents’ distances to the optimal solution [10]. In particular, if $x_* \in D$ and $D$ is convex, it can be shown that $\|\chi_D(x) - x_*\| \leq \|x - x_*\|$. In many optimization problems, the search space, $D$, is usually a $d$-dimensional hyper-rectangle, it follows that $D$ is convex.

A second step is to enhance an algorithm’s exploration. We achieve this by injecting noise into its dynamics. In stochastic optimization algorithms, such as the perturbed gradient descent and the Langevin algorithm, Gaussian noise is added to the classical gradient descent algorithm. This injection of noise effectively helps the algorithm to get out of saddle points or local minimums efficiently [12, 13]. We adopt the same strategy in this article due to its simplicity. Other form of noise injection also exists. For example, in multi-arm bandit problem, one can enforce exploration in the classical greedy algorithm by allocating a small proportion of trials on random arms.

By combining these two steps, the modified algorithm can be formulated as

$$x_i(t+1) = \chi_D(\chi_D(x'_i(t+1)) + w_i(t)), \quad (1)$$

where $x'_i(t+1)$ is the position of the $i$th agent after applying the algorithm’s update rule at $t+1$, and $w_i(t)$ is an independent draw from a density $p_i$ such that there is a constant $\alpha > 0$

$$p_i(y - x) > \alpha, \quad \text{for } x, y \in D. \quad (2)$$

The noise density $p_i$ in general can be adaptive, depending on the evolution of the algorithm up to time $t$. Here we only need it to be strictly positive and bounded away from zero. We call this method of modification in (1) a perturbation-projection (PP) modification or strategy. The proposition below shows that with PP strategy, the modified algorithm satisfies (C1) and (C2).

**Proposition II.2.** Suppose $A$ is a swarm-based algorithm and at each iteration, every agent is projected onto $D$. If the perturbation-projection modification (1) is applied to at least one of the agents, then (C1) and (C2) hold.
Proof. Since \( \chi_D \) is applied to all agents, (C1) clearly holds. At the \((t+1)\)th iteration, denote the agent to which the modification is applied by \( i = i(t+1) \). Let \( y = \chi_D(x_i^*(t+1)) \).

Then, for any \( d \)-dimensional ball \( B \) in \( D \),

\[
\max_{1 \leq k \leq n} \mathbb{P}_t(x_k(t+1) \in B) \\
\geq \mathbb{P}_t(x_{i(t+1)}(t+1) \in B) \\
\geq \mathbb{P}_t(y + w_i(t) \in B) \\
= \int_B \mathbb{P}_t(y + w_i(t) \in d\mathbf{x}) \\
= \int_B \mathbb{P}_t(w_i(t) \in d(\mathbf{x} - y)) \\
= \int_B p_i(y - \mathbf{x})d\mathbf{x} \geq \alpha|B|,
\]

showing (C2) holds. \(\square\)

III. APPLICATIONS

In this section, we illustrate how to implement our PP strategy to three nature-inspired swarm-based optimization algorithms, namely, particle swarm optimization (PSO), bat algorithm (BAT) and competitive swarm optimization (CSO).

A. PSO and mPSO

Kennedy and Eberhart [14] proposed the particle swarm optimization (PSO) algorithm. The algorithm has stimulated many refinements and inspired many variants, and these algorithms find wide applications in many fields. A recent search of “particle swarm optimization” in the Web of Science generated more than 32,000 articles and almost 600 review articles. A small sampling of early examples of how PSO has been studied and modified over the years are [15, 16, 17, 18, 19, 20, 19, 21, 22, 23]. PSO probably has the most modified versions among nature-inspired metaheuristic algorithms and they continue to this day.

PSO algorithm models after the movement of a flock of birds looking for food collectively. To align with our terminology in this article, we think of birds in PSO as agents, and food as a global minimizer of an objective function. At the next iteration, \( t + 1 \), each agent veers towards the best location it has found (cognitive/memory component) and the best location known to the flock up to iteration \( t \) (social component). To describe the memory effect, we allocate a personal “memory” agent \( x_i^*(t) \) to record the best location agent \( i \) has found up to iteration \( t \), and a global “memory” agent \( x^*(t) \) to record the best location found by all agents collectively so far. Recall the basic PSO algorithm runs the following iterations iteratively:

1) Generate two independent random vector \( U_1 \) and \( U_2 \) from the uniform distribution on \([0,1]^d\) and let

\[
v_i(t+1) = wv_i(t) + c_1U_1 \circ (x_i^*(t) - x_i(t)) + c_2U_2 \circ (x^*(t) - x_i(t)),
\]

where \( w, c_1, c_2 \) are given constants, and \( \circ \) denotes the Hadamard product.

2) Update \( x_i(t+1) = x_i(t) + v_i(t+1) \).

3) Update personal best

\[
x_i^*(t+1) = \begin{cases} 
x_i(t+1), & \text{if } f(x_i^*(t)) > f(x_i(t+1)); \\
x_i^*(t), & \text{otherwise}.
\end{cases}
\]

4) Repeat steps 1–3 for all agents, then update the global best agent

\[
x^*(t+1) = \begin{cases} 
x_i^*(t+1), & \text{if condition H holds,} \\
x^*(t), & \text{otherwise,}
\end{cases}
\]

where \( j = \arg\min_{1 \leq i \leq n} f(x_i^*(t+1)) \), \( m(t) = \min_{1 \leq i \leq n} f(x_i^*(t)) \) and condition H: \( m(t+1) < m(t) \).

Our modified PSO, denoted by mPSO, is to apply the PP strategy in step 2:

2’) Update

\[
x_i(t+1) = \chi_D (\chi_D(x_i(t) + v_i(t+1)) + w_i(t)) \tag{3}
\]

where \( w_i \)'s are independent multivariate normal distributed with mean vector \( 0 \) and covariance matrix \( \sigma\mathbf{I} \).

Since its introduction in 1995, there have been numerous attempts to analyze the basic PSO convergence behavior. In our opinion, Yuan and Yin [24] provide the first rigorous proof of the weak convergence of PSO to a global optimum, without overly restrictive assumptions. Moreover, using stochastic approximation technique, they provide the rate of convergence. In another direction, Tong et al. [25] introduce two smoothed versions of PSO and prove that they converge a.s. to a cost function’s global optimum.
The next proposition shows that if we modify the basic PSO algorithm by replacing the update step 2 by the new noise enhanced update step 2', the modified mPSO algorithm converges to a global minimum of the cost function a.s.

**Proposition III.1.** Let $f$ be a continuous function defined on a compact subset $\mathcal{D}$ of $\mathbb{R}^d$ and one of its minimizers is in the interior of $\mathcal{D}$. Then the algorithm mPSO converges to $\min_{x \in \mathcal{D}} f(x)$ a.s.

**Proof.** By Theorem II.1, it suffices to verify that mPSO satisfies (C1)–(C3). Note that $x_i(t+1) \in \mathcal{D}$ by projection. As $x_i^*(t+1)$ takes either value $x_i^*(t)$ or $x_i(t+1)$, therefore $x_i^*(t+1) \in \mathcal{D}$ if $x_i^*(t) \in \mathcal{D}$. Similar argument applies to $x^*(t+1)$, so (C1) holds by induction. Since we add noise to all agents except the $x_i^*(1 \leq i \leq n)$ and $x^*$ agents, Proposition II.2 implies (C2). Consider the global memory agent, by step 4, $f(x^*(t+1)) \leq f(x^*(t))$, so (C3) is verified. This completes the proof. \qed

**B. BAT and mBAT**

The success of bats, dolphins, shrews and other animals in applying echolocation technique to hunt and navigate inspired Yang [26] to propose the Bat Algorithm (BAT) in 2010. BAT meets with rising popularity in applications. According to a recent search of the Web of Science, over 1,500 articles, nearly 800 proceedings papers and 48 review papers have been written on this algorithm and its variants in just ten years. We refer interested readers in BAT and its wide ranging applications to the review articles in [27, 28] and the references therein. Recent applications of the BAT algorithm include [29, 30]. As with other metaheuristic algorithms, BAT has also many modified or hybridized versions for improved performance in various ways; some examples of the modified versions can be found in [31, 32, 33, 34, 35] and some examples of BAT algorithm hybridized with other metaheuristic algorithm are [36, 37, 38].

Let $x_i(t), 1 \leq i \leq n$, denotes the position of the $i$-th bat at $t$-th iteration, and $x^*(t)$ tracking the best position found by the cauldron of $n$ bats. Recall BAT runs the following iterations iteratively:

1) Generate $n$ independent random number $U_i$ from the uniform distribution on $[f_{\min}, f_{\max}]$,
   $$v_i(t+1) = v_i(t) + U_i(x_i(t) - x^*(t)).$$

2) Let $r_0$ (pulse rate) denote a user-specified threshold. Generate $n$ independent random numbers $r_i$ from the uniform distribution on $[0, 1]$, $1 \leq i \leq n$.
   Update according to (a): $r_i < r_0$ or (b)$r_i \geq r_0$:
   (a) If $r_i < r_0$, $x_i(t+1) = x_i(t) + v_i(t+1)$;
   (b) If $r_i \geq r_0$, $x_i(t+1) = x^*(t) + \epsilon_i(t)$, where the components of $\epsilon_i(t)$ are independent mean 0 normal distribution with standard deviation 0.001.

3) Generate $n$ independent random numbers $r_i$ from the uniform distribution on $[0, 1]$. If $r_i$ is less than a threshold $r_A$ (loudness), or $f(\chi_\mathcal{D}(x_i(t))) < f(x^*(t+1))$ update $x_i(t+1) = \chi_\mathcal{D}(x_i(t))$.

4) Update global best
   $$x^*(t+1) = x_{i^*},(t+1),$$
   where $i^* = \arg\min f_i(x_i(t+1)).$

In the procedure above, $f_{\min}, f_{\max}$ are given constants which describe the minimal and maximal frequencies. The threshold probabilities $r_0$ and $r_A$ describe the pulse and emission rates. The random noise $\epsilon_i(t)$ describes the loudness of each bat. Yang [26] also suggests using time varying $\epsilon_i(t)$ and $r_0$ for improved performance. Here we fix them as in the standard values suggested by the MATLAB package provided by the same author.

Our modified BAT, denoted by mBAT, is to apply the PP strategy by an additional step after step 3

3’ Update
   $$x_i(t+1) = \chi_\mathcal{D}(\chi_\mathcal{D}(x_i(t) + v_i(t+1)) + w_i(t+1)).$$

**Proposition III.2.** Let $f$ be a continuous function defined on $\mathcal{D}$, a compact subset of $\mathbb{R}^d$. Suppose further that a minimizer of $f$ lies in the interior of $\mathcal{D}$, then the algorithm mBAT converges to $\min_{x \in \mathcal{D}} f(x)$ a.s.

**Proof.** By Theorem II.1, it suffices to verify mBAT satisfies (C1)–(C3). Note that $x_i(t+1) \in \mathcal{D}$ by projection. As $x^*(t+1)$ takes either value $x^*(t)$ or $x_i(t+1)$ for some $i$, therefore it is in $\mathcal{D}$ if $x^*_i(t) \in \mathcal{D}$. So (C1) holds by induction.

To verify (C2), we denote the value of $x_i(t+1)$ after step 3’ (4) as $y_i(t)$. Following the proof of Proposition II.2, we can show that for any d-ball $B \subset \mathcal{D}$,
   $$P(y_i(t) \in B) \geq \alpha|B|.$$ 

Then if all $y \in B$ satisfies $f(y) < f(x_i(t))$, we find that
   $$P(x_i(t+1) \in B) \geq (1 - r_A)P(y_i(t) \in B) \geq \alpha(1 - r_A)|B|.$$ 

So (C2) holds.

By step 4, $f(x^*(t+1)) \leq f(x^*(t))$, so (C3) is verified. This completes the proof of Proposition III.2. \qed
C. CSO and mCSO

Inspired by the particle swarm optimization, Cheng and Jin [39] in 2014 proposed the Competitive Swarm Optimizer (CSO) for "large scale optimization problems and is able to effectively solve problems of dimensionality up to 5000". It is generally deemed to outperform PSO and it is also continually improved in various ways such as [40, 41]. Most recently, [42] applied CSO to optimize a multitasking problem and Zhang et al. [43] modified CSO to find optimal experiment designs to estimate interesting parameters in a nonlinear regression model with several interacting factors.

We shall call the particles in CSO agents. In each CSO iteration, agents are randomly paired and their functional values are compared. In each pair, agent with smaller functional value is declared winner and its configuration in the next iteration will move closer to the average configuration of all the agents, see steps 4 and 5 below. Specifically, CSO repeats steps 1 to 5 until termination criterion is met.

1) Randomly pair the agents. Repeat the steps 2 to 5 below for each pair.
2) Consider a pair of agents indexed as $i$ and $j$. Define $w = i, \ell = j$ if $f(x_i(t)) < f(x_j(t))$. Otherwise $w = j, \ell = i$.
3) Let $x_w(t + 1) = x_w(t), v_w(t + 1) = v_w(t)$.
4) Generate three independent random vectors $U_1, U_2, U_3$ from the uniform distribution on $[0, 1]^d$ and let
   $$v_t(t + 1) = U_1 \circ v_t(t) + U_2 \circ (x_w(t) - x_t(t))$$
   $$+ \phi U_3 \circ (\bar{x}(t) - x_t(t)),$$
   where $\circ$ denotes element-wise product, $\phi$ a given constant, and $\bar{x}(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t)/n$.
5) Update $x_t(t + 1) = x_t(t) + v_t(t + 1)$.

Our modified CSO, denoted by mCSO, is to apply the PP strategy in step 5 only to the loser agent: 5') Update
   $$x_w(t + 1) = \chi_D (\chi_D (x_t(t) + v_t(t + 1)) + w_t(t)).$$

It is not necessary to apply projection on $x_w(t + 1)$ because $x_w(t + 1) = x_w(t) \in D$.

Proposition III.3. Let $f$ be a continuous function over $D$, a compact subset of $\mathbb{R}^d$. Suppose further that a minimizer of $f$ lies in the interior of $D$, then the algorithm mCSO converges to $\min_{x \in D} f(x)$ a.s.

Proof. By Theorem II.1 and Proposition II.2, our claim will be proved if we can show that mCSO satisfies (C1)--(C3). Since $x_t(t + 1) \in D$ by projection, and $x_w(t + 1) = x_w(t)$, (C1) holds. As we add noise to the loser agents, (C2) holds by Proposition II.2. Let $i_* = \arg \min_{1 \leq i \leq n} f(x_t(t))$. In the next iteration, $x_{i_*}(t)$ will be a winner no matter whom he is paired with. So $x_{i_*}(t + 1) = x_{i_*}(t)$; and $\min_i f(x_i(t + 1)) \leq f(x_{i_*}(t + 1)) = \min_i f(x_i(t))$. This proves that (C3) is satisfied and hence the proof of the proposition.

IV. HETEROGENEOUS PP

In applications, the strength of stochastic perturbation in the PP strategy often needs proper tuning. In general, a stronger perturbation enhances exploration at the expense of exploitation. During the final stage of optimization, the agents will have difficulties in narrowing down the optimal solution with strong perturbations. There can be different ways to solve this issue such as choosing a diminishing perturbation as in the original BAT algorithm; using adaptive adjustment mechanism; or using reinforcement learning tools to find the optimal parameter. However, these approaches lead to further tuning parameters or additional computational complexity. Instead, we propose a simpler strategy, which is nature-inspired, that involves role heterogeneity of members in the swarm.

In swarms in nature, it is common to observe they adoption division of labors. Agents specializing in different tasks collaborate to improve collective effectiveness. In the same vein, we set some agents in the swarm to specialize in exploration while others in exploitation. In the context of PP strategy, we only perturb the exploration agents so they help exploring the search space while we do not perturb exploitation agents so as not to hamper their convergence capability. We call this heterogeneous perturbation-projection, abbreviated to HPP, strategy. In contrast with the other ways to find optimal perturbation strength, HPP strategy is very easy to implement. For example, if we want to have half exploration agents and half exploitation agents, we will apply PP strategy only to the first half of the agents while the other agents we only apply the projection modification. This idea can be applied directly to mPSO and mBAT. As for mCSO, because pairs are randomly formed, we will apply PP strategy only to the loser agents in the first half of pairs.

Our HPP strategy can be considered as a special case of cooperative or collaborative learning, which suggests that mixing agents of different settings can lead to improved overall performance. Existing works on cooperative swarm-based algorithms can be found in [20, 44, 45]. In comparison, our HPP strategy focuses more on the cooperation between exploration agents and exploitation agents. Recently, [46] also applies this idea to local optimization algorithms such as gradient descent and Langevin algorithm.
Note that Proposition II.2 only requires the perturbation strategy applied to at least one agent, so this proposition is still applicable to algorithms with HPP, which perturbs half of their agents (or loser agents in CSO). Consequently, Theorem II.1 guarantees H-PP modified algorithms converge a.s to a global optimal solution. We denote the modified version of an algorithm \( A \) using H-PP strategy by \( hmA \).

V. NUMERICAL EXPERIMENTS

In this section, we conduct extensive numerical experiments to investigate whether our modified versions of \( A \) (\( A = PSO, BAT \) or CSO), \( mA \) and \( hmA \), outperform or at least perform on par with \( A \) given a computing budget for a large class of very different types of cost functions. We use two measures to evaluate the performance of the modified algorithms relative to the original algorithm. They measure how likely and by how much the modified algorithm obtains better results than the original algorithm.

A. Test functions and details of experiments

We use a total of 30 test functions of varying dimensions listed in [47, Table 2] for our experiments. The list of test functions was compiled from various sources by the authors in [47]. They include commonly used test functions such as Ackley, Griewank, Powell, Rastrigin, Sphere etc; and are of different kinds, unimodal, multi-modal, separable and inseparable. Majority of the functions in their table can be defined for arbitrary dimension \( d \). For such functions, we consider \( d = 5, 10, 20 \) and \( 40 \) to investigate the effect of dimension on the optimization algorithms. The other test functions (except one which can only be defined for dimension 4, which we exclude in our experiments) can only be defined for \( d = 2 \). In total, we have a collection of 70 test functions, see Appendix A, denoted by \( \mathcal{C} \). We shall partition \( \mathcal{C} \) into different collections according to \( d = 2, 5, 10, 20 \) and 40 in reporting our results.

For each \( f \in \mathcal{C} \), we conduct 100 runs of algorithms \( A, mA \) and \( hmA \) for 10,000 iterations where \( A = PSO, BAT, \) or CSO. To gauge the progress of each run of an algorithm, we output the algorithm best functional value found at \( t \)-th iteration, where \( t = 50, 100, 200, 400, 1000, 3000 \) and 10000.

Throughout the experiments, the number of agents/particles used is \( n = 32 \). For \( mA \) or \( hmA \) (\( A = PSO, BAT \) or CSO), the stochastic perturbation used is independent normal distributions with mean 0 and standard deviation 0.005.

We follow common parameter settings for PSO, BAT and CSO:

- PSO: \( w = 0.729 \) (inertia weight), \( c_1 = c_2 = 1.5 \) (acceleration constants);
- BAT: \( Q_{\text{min}} = 0 \) (minimum frequency), \( Q_{\text{max}} = 100 \) (maximum frequency), \( r_0 = 0.5 \) (pulse rate), \( r_A = 0.5 \) (loudness);
- CSO: \( \phi = 0 \).

B. Comparison and results

This subsection compares the performance of an algorithm \( A \) versus one of its modification \( B \) using several test functions taken from the set \( \mathcal{F} \). The functions in the set are commonly used to compare performance of an algorithm in the engineering literature. To study how the performance on a problem depends on the dimension of the problem, each \( \mathcal{F} \) here is the subset of \( \mathcal{C} \) that contains functions of the same dimension \( d = 2, 5, 10, 20 \) or 40. We are interested in the following two questions: How likely does \( B \) outperform \( A \)? How much does \( B \) exceed the best possible value from using \( A \) and \( B \)? Here \( B = mA \) or \( hmA \).

(a) How likely will \( B \) outperform \( A \)?

Let \( A_I(t) \) and \( B_I(t) \) denote the (theoretical) best functional values from algorithms \( A \) and \( B \) respectively for the test function \( f \) up to the \( t \)-th iteration. Similarly, let \( A_r(f) \) and \( B_r(f) \) denote the best functional values output by algorithms \( A \) and \( B \) respectively up to the \( t \)-th iteration at the \( r \)-th run. The probability that algorithm \( B \) outperforms algorithm \( A \) for \( f \) at the \( t \)-th iteration is \( P(B_I(t) < A_I(t)) \), and it can be estimated by \( \sum_{r=1}^{100} I(B_r(f) < A_r(f))/100 \). The average of these estimates over \( \mathcal{F} \), denoted by \( P_{B \succ A}(t) \), can be thought of as an average winning proportion of \( B \) over \( A \) (or simply, it’s winning proportion). Specifically,

\[
P_{B \succ A}(t) := \frac{1}{100|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{r=1}^{100} I(B_r(f) < A_r(f)).
\]

Similarly, interchanging \( A \) and \( B \) above, we define the winning proportion of \( A \) over \( B \), \( P_{A \succ B}(t) \). Since \( P_{A \succ B}(t) + P_{B \succ A}(t) = 1 \), it suffices to track \( P_{B \succ A}(t) \). Loosely speaking, if \( P_{B \succ A}(t) > 1/2 \), \( B \) is more likely to outperform \( A \) for \( f \in \mathcal{F} \). Indeed, the larger the \( P_{B \succ A}(t) \), the more likely \( B \) outperforms \( A \).

The numerical results are plotted in Figure 1. Based on these plots, we observe have the followings. When \( t \) is fixed, the winning proportion of PP/HP modified algorithm outperforming its counterpart increases as the dimension of the test function increases. Both mBAT and hmBAT are comparable in their performance against BAT, and the winning proportion increases
Fig. 1: Plots of winning proportion of $mA$ against $A$ (top row), and that of $hmA$ against $A$ (bottom row) where $A =$ PSO, BAT and CSO categorized according to the dimension of the test functions.

steady from about 0.5 to 0.7 for dimension at least 5. Interestingly, the HPP strategy has more significant enhancement effect on PSO and CSO. For test functions with dimension 40, winning proportion of hmCSO is nearly 1 after 3000 iterations.

(b) How much does $B$ (or $A$) outperform $A$ (or $B$)?

For simplicity, we suppress the dependence of a notation on other parameters/notations when the context is clear. To this end, let $m_*$ and $m^*$ denote respectively, the minimum (i.e., the best) and maximum (i.e., the worst) of the outputs from 100 runs of $A$ and 100 runs of $B$ for $f$ at the $t$-th iteration. We view $m_*$ as a proxy of the best possible output by $A$ and $B$; and let $R = m^* - m_*$ denote the range of the 200 outputs of $A$ and $B$. Define the relative error of $A$ with respect to $A$ and $B$, $RE_{A,A,B,f}(t)$, to be the average of $(A_{t,f}(t) - m_*)/R$ over 100 runs; and the relative error of $B$ relative to $A$ and $B$, $RE_{B,A,B,f}(t)$ is similarly defined. We consider the relative instead of absolute error in order to cancel out the effect due to a scale change of $f$ (i.e., if we consider $cf$ instead of $f$). For brevity, we simply call this quantity the relative error. Note that the relative errors lie between 0 and 1. Small relative error of $A$ (i.e., $RE_{A,A,B,f}(t)$) indicates the results from algorithm $A$ are closer to $m_*$ than those by $B$’s. A similar interpretation can be extended over a class of functions $f \in \mathcal{C}$ if we define $RE(A, A \cdot B)(t) := \frac{1}{|\mathcal{C}|} \sum_{f \in \mathcal{C}} RE_{A,A,B,f}(t)$ and call this the overshoot of $A$ relative to $A$ and $B$.

Figure 2 plots the relative errors of $A$ (dashed blue curve) and $mA$ (solid blue curve) with respect to one another; and the relative errors of $A$ (dashed red curve) and $hmA$ (solid red curve) with respect to one another. We only display the plots for $d = 10$ and $d = 40$. We highlight some observations from these plots in Figure 2. (i) hmBAT and mBAT outperform BAT by a significant margin. (ii) At $d = 40$, both mPSO and hmPSO are nearly 0 from 1000 iterations onwards. This implies mPSO
and hmPSO significantly outperform the basic PSO from 1000 iterations. At $d = 10$, only hmPSO (resp., hmCSO) improves the basic PSO (resp., CSO) from 400 iterations.

Combining the observations in (a) and (b), Figures 1 and 2 convincingly demonstrate that the HPP strategy significantly improves BAT, CSO and PSO particularly when the dimension of a test function is large. Improvement is measured in two complementary aspects, “how likely” and “how much” one algorithm outperforms the other.

C. Tuning parameter analysis for PP and HPP strategies

As demonstrated in Section V-B, additional stochastic component in a modified swarm-based algorithm enhances the exploration ability of the algorithm under consideration. The stochastic component chosen is Gaussian with mean 0 and standard deviation, $\sigma = 0.005$. Expectedly the noise level, in terms of $\sigma$ or a heavier-tailed distribution, affects the efficiency of the modified algorithm since strong level of noise interferes the algorithm’s exploitation ability. We conduct further numerical experiments with the same set-up as above to investigate the effects of $\sigma$ in the Gaussian noise and $t$-distributions, which have heavier tails than the Gaussian, on the performance of $mA$ and $hmA$.

(i) On the standard deviation of the Gaussian noise We tried $\sigma = 0.005, 0.01, 0.02$ and 0.05. The plots of $P_{B < A}(t)$ (analogous to Figure 1) and $RE(A, A \cdot B)(t)$ (analogous to Figure 2) are very similar for $hmA$ ($A =$ PSO, BAT and CSO) for all the $\sigma$s; and $\sigma = 0.005$ and 0.01. Hence, we recommend using $\sigma$ in $[0.005, 0.01]$. Interested readers can refer to Figures S4 and S5 in Appendix B.

(ii) On the choice of distribution We examine the effect on the performance of the modified algorithms if we change the Gaussian distribution with a heavier-tailed distribution. We choose distributions $0.01 \sqrt{(m - 2)/m} t_m$ for $m = 5, 10, 30$ and 60 where $t_m$ denotes a $t$-distribution with $m$ degrees of freedom. The scaling factor $0.01 \sqrt{(m - 2)/m}$ is to ensure the scaled $t$-distribution has the same standard deviation as that of the Gaussian noise in (i) above. Note also that $t_m$ converges to a standard normal. For $m$ large, we do not expect essential difference between the $t$-distribution and the Gaussian noise.

The analogous plots of $P_{B < A}(t)$ and $RE(A, A \cdot B)(t)$ using $t$-distributions for the choices of degrees of freedom (df) look almost the same as the corresponding plots using Gaussian noise. Interested readers can view these plots as Figure S6-S7 in Appendix B.

VI. Conclusion

In this article, we delineate sufficient conditions for a swarm-based algorithm to converge almost surely to an optimum of an objective function. This assumes that there is no constraint on the computation budget and the algorithm is allowed to run indefinitely. The class of objective functions for which Theorem II.1 holds is very large. We proposed two strategies, perturbation-project (PP) and heterogeneous perturbation-projection (HPP), to modify a given swarm-based algorithm so that the sufficient conditions (C1)-(C3) hold, and hence Theorem II.1 guarantees the modified algorithm converges almost surely to the global optimum. The proposed strategies are simple to implement and not algorithm-specific. We demonstrated how these strategies are applied to PSO, Bat Algorithm and CSO. Extensive numerical experiments were conducted and the results show that the modified algorithm either out-performs or performs on par with the original algorithm over a finite computational budget, and especially so for HPP modified algorithm. Our simulation results suggest that applying the HPP strategy with mean 0 and standard deviation 0.005 Gaussian distribution for the additional stochastic component is likely to be effective in improving a swarm-based algorithm.

To conclude this article, we report two interesting observations from our experiments. The first observation is that when $A$ outperforms $mA$ or $hmA$, $A$ frequently produces a marginally better result; however, when $mA$ or $hmA$ outperforms $A$, the modified version quite often outperforms $A$ by relatively larger margin. We offer a heuristic argument. When both $A$ and $mA$ are exploring in the same neighbourhood of a local minimum, the stochastic element hinders the exploitation by a small amount and hence $A$ is more likely to produce only marginally better solution. However, when the occasion for $mA$ to make a big jump leaving the local minimum’s neighbourhood arises, $mA$ has the potential to explore a better region and hence produces noticeably better solution than the original algorithm does. The same heuristic reasoning carries over to the comparison of $A$ and $hmA$. Moreover, $hmA$ version due to the built in diversity of agents, better balances the conflicting demands of exploitation and exploration, and expectedly amplifies this observation.

The second observation comes from our application of PP and HPP strategies to Differential Evolution (DE), an evolutionary algorithm. The same numerical experiments were conducted to compare DE with mDE, and DE with hmDE. We found that DE outperforms mDE and hmDE; the plots can be found in Appendix C as Figures S8-S9. A reason may be that DE already performs very well in the optimization problem over majority of the test functions in the experiment and so any additional stochastic component will either not help or even worsen the search solution. Moreover, the random perturbation we apply here mimic the random local exploration of an ant or other animal. Genetic mutation in general cannot describe such dynamics and require some other more structured mechanism.
ACKNOWLEDGMENT

Choi’s research was supported by the Singapore MOE Academic Research Funds R-155-000-222-114. Lai’s research was supported by the National Science Foundation under DMS-1811818. Tong’s research was supported by MOE Academic Research Funds R-146-000-292-114. Wong’s research was supported by the National Institutes of Health under R01GM107639.

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# APPENDIX A

**Test functions used in the numerical simulations**

| Label | Name                  | min \( f(x) \) | Unimodal | Separable |
|-------|-----------------------|----------------|----------|-----------|
| F1    | Ackley                | 0              | No       | No        |
|       | \(-20 \exp(-0.2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2}) - \exp(\frac{1}{n} \sum_{i=1}^{n} \cos(2\pi x_i)) + 20 + e\) |
| F2    | Bohachevsky2          | 0              | No       | No        |
|       | \(\sum_{i=1}^{d-1} [x_i^2 + 2x_{i+1} - 0.3 \cos(3\pi x_i) \cos(4\pi x_{i+1}) + 0.3]\) |
| F3    | Bohachevsky3          | 0              | No       | No        |
|       | \(\sum_{i=1}^{d-1} [x_{i+1}^2 + 2x_{i+2}^2 - 0.3 \cos(3\pi x_{i+1}) + 4\pi x_{i+1} + 0.3]\) |
| F4    | Bukin6                | 0              | No       | No        |
|       | \(100\sqrt{|x_2 - 0.01x_1^2| + 0.01|x_1 + 10|}\) |
| F5    | DropWave              | -1             | No       | No        |
|       | \(\frac{1+\cos(12\sqrt{\frac{x_1^2+x_2^2}{2}})}{0.01(x_1^2+x_2^2)}\) |
| F6    | Eggholder             | -959.6407      | No       | No        |
|       | \(-25.1667 \sin(\sqrt{\frac{x_1^2 + \frac{4\pi}{9} x_2^2} + 4.7}) - x_1 \sin(\sqrt{x_1^2 + (x_2 + 4.7)^2})\) |
| F7    | GoldsteinPrice        | 3              | No       | No        |
|       | \[
\left[1 + (x_1 + 2x_2 + 1)^2(19 + 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)\right] \\
\times\left[30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 4x_2 - 36x_1x_2 + 27x_2^2)\right]
\]
| F8    | Griewank              | 0              | No       | No        |
|       | \(\frac{1 + \sum_{i=1}^{d} x_i^2}{4000} - \prod_{i=1}^{d} \cos\left(x_i\frac{\pi}{2}\right)\) |
| F9    | McCormick             | -1.9133        | No       | No        |
|       | \(\sin(x_1 + x_2) + (x_1 - x_2)^2 - 1.5x_1 + 2.5x_2 + 1\) |
| F10   | Schaffer2             | 0              | No       | No        |
|       | \(0.5 + \frac{\sin^2(x_1^2 - x_2^2)}{(1+0.001(x_1^2 + x_2^2))^2}\) |
| F11   | Schaffer4             | 0.292579       | No       | No        |
|       | \(0.5 + \frac{\cos^2(\sin(x_1^2 - x_2^2))}{(1+0.001(x_1^2 + x_2^2))^2}\) |
| F12   | Bohachevsky1          | 0              | No       | Yes       |
|       | \(\sum_{i=1}^{d-1} x_i^2 + 2x_{i+1} - 0.3 \cos(3\pi x_i) - 0.4 \cos(4\pi x_{i+1}) + 0.7\) |
| F13   | Booth                 | 0              | No       | Yes       |
|       | \((x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2\) |
| F14   | Bralin                | 0.397887       | No       | Yes       |
|       | \(\frac{x_2 - 0.1x_1}{\sqrt{x_1^2 + x_2^2}} - \frac{x_1}{\sqrt{x_1^2 + x_2^2}} - 6)^2 + 10(1 - \frac{1}{8\pi} \cos x_1 + 10)\) |
| F15   | Michalewicz5          | -4.687658      | No       | Yes       |
|       | \(-\sum_{i=1}^{5} \sin(x_i) \sin^2(\frac{x_i^2}{\pi})\) |
| F16   | Rastrigin             | 0              | No       | Yes       |
|       | \(10d + \sum_{i=1}^{d} \left|x_i^2 - 10\cos(2\pi x_i)\right|\) |
| F17   | Shubert               | -186.73        | No       | Yes       |
|       | \(\prod_{j=1}^{\sqrt{n}} i \cos((i + 1)x_j + i)\) |
| F18   | Beale                 | 0              | Yes      | No        |
|       | \((1.5 - x_1 + x_1x_2)^2 + (2.25 - x_1 + x_1x_2^2)^2 + (2.625 - x_1 + x_1x_2^2)^2\) |
| F19   | DixonPrice            | 0              | Yes      | No        |
|       | \((x_1 - 1)^2 + \sum_{i=2}^{d} (2x_i - x_{i-1})^2\) |
| F20   | Easom                 | -1             | Yes      | No        |
|       | \(-\cos(x_1) \cos(x_2) \exp(-(x_1 - \pi)^2 - (x_2 - \pi)^2)\) |
| F21   | Matyas                | 0              | Yes      | No        |
|       | \(0.26(x_1^2 + x_2^2) - 0.48x_1x_2\) |
| Function | Name | Known Min | Unimodal | Separable |
|----------|------|------------|----------|-----------|
| F22      | Powell | 0          | Yes      | No        |
|          | $\sum_{i=1}^{d}\left( (x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-2} - 2x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4 \right)$ |
| F23      | Rosenbrock | 0        | Yes      | No        |
|          | $\sum_{i=1}^{d-1}\left[ 100(x_{i+1} - x_i)^2 + (x_i - 1)^2 \right]$ |
| F24      | Schwefel | -418.9829d | Yes      | No        |
|          | $418.9829d - \sum_{i=1}^{d} x_i \sin(\sqrt{|x_i|})$ |
| F25      | Trid6 | -d(d+4)(d-1)/6 | Yes    | No        |
|          | $\sum_{i=1}^{d} (x_i - 1)^2 - \sum_{i=1}^{d} x_i x_{i-1}$ |
| F26      | Zakharov | 0          | Yes      | No        |
|          | $\sum_{i=1}^{d} x_i^2 + (\sum_{i=1}^{d} 0.5ix_i)^2 + (\sum_{i=1}^{d} 0.5ix_i)^4$ |
| F27      | Sphere | 0          | Yes      | Yes       |
|          | $\sum_{i=1}^{d} x_i^2$ |
| F28      | Sumsquare | 0          | Yes      | Yes       |
|          | $\sum_{i=1}^{d} i x_i^2$ |

**TABLE S1:** Table of test functions containing their function label, name, known minimum value, unimodality and separability. Functions F4, 5, 6, 7, 9, 10, 11, 13, 14, 17, 18, 20, 21 have dimension 2, and F15 is dimension 5. All other functions were tested at dimensions 5, 10, 20 and 40.
APPENDIX B
ADDITIONAL NUMERICAL RESULTS

Here we report the simulation results of dimensions (Figure S3) and noise setups (Figures S4-S7).

Fig. S3: Plots of relative error of $A$ relative to $A$ and $mA$ (dashed blue curve), and relative error of $mA$ relative to $A$ and $mA$ (solid blue curves). Plots of relative error of $A$ relative to $A$ and $mA$ (dashed red curves), and relative error of $mA$ relative to $A$ and $mA$ (solid red curves).
Fig. S4: Plots of winning proportion of $mA$ against $A$ where $A = \text{PSO, BAT and CSO}$ categorized according to the dimension of the test functions. Each row shows the result for one value of sigma $\sigma$, where $\sigma = 0.005, 0.01, 0.02, 0.05$. 
Fig. S5: Plots of winning proportion of \( hmA \) against \( A \) where \( A = \text{PSO}, \text{BAT} \) and \( \text{CSO} \) categorized according to the dimension of the test functions. Each row shows the result for one value of sigma \( \sigma \), where \( \sigma = 0.005, 0.01, 0.02, 0.05 \).
Fig. S6: Plots of winning proportion of \( mA-t \) against \( A \) where \( A = \) PSO, BAT and CSO categorized according to the dimension of the test functions. Each row shows the result for one value of degrees of freedom \( df \), where \( df = 0.005, 0.01, 0.02, 0.05 \).
Fig. S7: Plots of winning proportion of $hmA - t$ against $A$ where $A =$ PSO, BAT and CSO categorized according to the dimension of the test functions. Each row shows the result for one value of degrees of freedom $df$, where $df = 0.005, 0.01, 0.02, 0.05$. 

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$hmPSO-t df=5$

$hmBAT-t df=5$

$hmCSO-t df=5$

$hmPSO-t df=10$

$hmBAT-t df=10$

$hmCSO-t df=10$

$hmPSO-t df=30$

$hmBAT-t df=30$

$hmCSO-t df=30$

$hmPSO-t df=60$

$hmBAT-t df=60$

$hmCSO-t df=60$
Here we review the DE algorithm and call the particles in DE as agents. In each DE iteration, each agent considers a new proposed location and move to the new location if it provides a better function value. The proposed movement is obtained by truncating the difference between two other random agents at a random component. More specifically, DE repeats steps 1 to 6 until the user-specified termination criterion is met.

1) Repeat the following step for each agent $x_i(t)$
2) Randomly find two other agents $x_j(t)$ and $x_k(t)$.
3) Compute $y_i(t) = x_i(t) + F(x_j(t) - x_k(t))$, where $F \in [0, 2]$ is a constant known as differential weight and its default value is 0.8.
4) Randomly find a component index $\ell \in \{1, \ldots, d\}$.
5) For component $m \in \{1, \ldots, d\} \setminus \{\ell\}$, replace the $m$th component of $y_i(t)$ by the corresponding component of $x_i(t)$ with probability $1 - \omega$, where $\omega \in [0, 1]$ is a constant known as crossover probability, and its default value is 0.9.
6) If $f(x_i(t)) > f(y_i(t))$, let $x_i(t + 1) = y_i(t)$; otherwise let $x_i(t + 1) = x_i(t)$.

Our modified DE, denoted by mDE, is to add an additional step for step 6

6'. Let $y'_i(t) = \chi_D(\chi_D(y_i(t)) + \omega_\ell(t))$ and

$$x_i(t + 1) = \begin{cases} y'_i(t), & \text{if } f(y'_i(t)) < f(x_i(t)); \\ x_i(t), & \text{if } f(y'_i(t)) \geq f(x_i(t)). \end{cases}$$

(6)

The simulations use the same setups as the ones for PSO. The results are plotted in Figures S8 and S9. The modification actually worsen the optimization output.
Fig. S9: Plots of relative error of $DE$ relative to $DE$ and $mDE$ (dashed blue curve), and relative error of $mDE$ relative to $DE$ and $mDE$ (solid blue curve). Plots of relative error of $DE$ relative to $DE$ and $hmDE$ (dashed red curves), and relative error of $hmDE$ relative to $DE$ and $hmDE$ (solid red curve).