ON BLOW UP FOR THE ENERGY SUPER CRITICAL DEFOCUSING NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider the energy supercritical defocusing nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u - u|u|^{p-1} = 0 \]

in dimension \( d \geq 5 \). In a suitable range of energy supercritical parameters \((d,p)\), we prove the existence of \( C^\infty \) well localized spherically symmetric initial data such that the corresponding unique strong solution blows up in finite time. Unlike other known blow up mechanisms, the singularity formation does not occur by concentration of a soliton or through a self similar solution, which are unknown in the defocusing case, but via a front mechanism. Blow up is achieved by compression for the associated hydrodynamical flow which in turn produces a highly oscillatory singularity. The front blow up profile is chosen among the countable family of \( C^\infty \) spherically symmetric self similar solutions to the compressible Euler equation whose existence and properties in a suitable range of parameters are established in the companion paper [42].

1. Introduction

We consider the defocusing nonlinear Schrödinger equation

\[ (\text{NLS}) \quad i\partial_t u + \Delta u - u|u|^{p-1} = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \quad u(t,x) \in \mathbb{C}. \quad (1.1) \]

in dimension \( d \geq 3 \) for an integer nonlinearity \( p \in 2\mathbb{N}^* + 1 \) and address the problem of its global dynamics. We begin by giving a quick introduction to the problem and its development.

1.1. Cauchy theory and scaling. It is a very classical statement that smooth well localized initial data \( u_0 \) yield local in time, unique, smooth, strong solutions. For the global dynamics, two quantities conserved along the flow (1.1) are of the utmost importance:

\[
\begin{align*}
\text{mass:} & \quad M(u) = \int_{\mathbb{R}^d} |u(t,x)|^2 = \int_{\mathbb{R}^d} |u_0(x)|^2 \\
\text{energy:} & \quad E(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 + \frac{1}{p+1} \int_{\mathbb{R}^d} |u(t,x)|^{p+1} dx = E(u_0). \quad (1.2)
\end{align*}
\]

The scaling symmetry group

\[ u_\lambda(t,x) = \lambda^{\frac{4}{p-2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0 \]

acts on the space of solutions by leaving the critical norm invariant

\[ \int_{\mathbb{R}^d} |\nabla^s u_\lambda(t,x)|^2 = \int_{\mathbb{R}^d} |\nabla^s u(t,x)|^2 \quad \text{for} \quad s_c = \frac{d}{2} - \frac{2}{p-1}. \]

Accordingly, the problem (1.1) can be classified as energy subcritical, critical or supercritical depending on whether the critical Sobolev exponent \( s_c \) lies below, equal or above the energy exponent \( s = 1 \). This classification also reflects the (in)ability...
for the kinetic term in (1.2) to control the potential one via the Sobolev embedding $H^1 \hookrightarrow L^q$.

1.2. Classification of the dynamics. We review the main known dynamical results which rely on the scaling classification.

Energy subcritical case. In the energy subcritical case $s_c < 1$, the pioneering work of Ginibre-Velo [22] showed that for all $u_0 \in H^1$, there exists a unique strong solution $u \in C^0([0,T), H^1)$ to (1.1) and identified the blow up criterion

$$T < +\infty \implies \lim_{t \uparrow T} \|u(t)\|_{H^1} = +\infty.$$  \hspace{1cm}  (1.3)

Conservation of energy, which is positive definite and thus controls the energy norm $H^1$, then immediately implies that the solution is global, $T = +\infty$. In fact, it can be shown in addition that these solutions scatter as $t \to \pm \infty$, [23].

Energy critical problem. In the energy critical case $s_c = 1$, the criterion (1.3) fails and the energy density could concentrate. For the data with a small critical norm, Strichartz estimates allow one to rule out such a scenario, [10]. The large data critical problem has been an arena of an intensive and remarkable work in the last 20 years.

For large spherically symmetric data in dimensions $d = 3, 4$, the energy concentration mechanism was ruled out by Bourgain [7] and Grillakis [25] via a localized Morawetz estimate. In Bourgain’s work, a new induction on energy argument led to the statements of both the global existence and scattering. These results were extended to higher dimensions by Tao, [59].

The interaction Morawetz estimate, introduced in [11], led to a breakthrough on the global existence and scattering for general solutions without symmetry, first in $d = 3$, [11], then in $d = 4$, [54], and $d \geq 5$, [64].

A new approach was introduced in Kenig-Merle [29] in which, if there exists one global non-scattering solution, then using the concentration compactness profile decomposition [2, 45], one extracts a minimal blow up solution and proves that up to renormalization, such a minimal element must behave like a soliton. The existence of such objects is ruled out using the defocusing nature of the nonlinearity, which is directly related to the non existence of solitons for defocusing models.

In all of these large data arguments, the a priori bound on the critical norm provided by the conservation of energy plays a fundamental role. Let us note that in the energy critical focusing setting, the concentration of the critical norm is known to be possible via type II (non self similar) blow up with soliton profile, see e.g [33, 40, 52, 53, 51, 50].

Energy supercritical problem. In the energy supercritical range $s_c > 1$, local in time unique strong solutions can be constructed in the critical Sobolev space $H^{s_c}$, [10, 31]. Kenig-Merle’s approach, [30, 31], gives a blow up criterion

$$T < +\infty \implies \limsup_{t \uparrow T} \|u(t, \cdot)\|_{H^{s_c}} = +\infty,$$

but the question of whether this actually happens for any solution remained completely open. On the other hand, the main difficulty in proving that $T = \infty$ for all solutions is that there are no a priori bounds at the scaling level of regularity $H^{s_c}$.

1.3. Qualitative behavior for supercritical models. The question of global existence or blow up for energy supercritical models is a fundamental open problem in
many nonlinear settings, both focusing and defocusing. For focusing problems, the existence of finite energy type I (self similar) blow up solutions is known in various instances, see e.g [19, 37, 35, 15], and solitons have been proved to be admissible blow up profiles in certain type II (non self-similar) blow up regimes in all three settings of heat, wave and Schrödinger equations, see e.g. [28, 41, 14, 48, 38]. There are also several examples of supercritical problems with positive definite energy (wave maps, Yang-Mills) which admit smooth self-similar profiles and thus provide explicit blow up solutions, [57, 5, 18].

On the other hand, for defocusing problems, soliton-like solutions are known not to exist and admissible self similar solutions are expected not to exist. For a simple defocusing model like the scalar nonlinear defocusing heat equation, a direct application of the maximum principle ensures that bounded data yield uniformly bounded solutions which are global in time and in fact dissipate. We recall again that for the energy critical problems, blow up occurs in the focusing case, where solitons exist, and it does not in the defocusing case where solitons are known not to exist.

This collection of facts led to the belief, as explicitly conjectured by Bourgain in [6], that global existence and scattering should hold for the energy supercritical defocusing Schrödinger and wave equations. Indications of various qualitative behaviors supporting different conclusions have been provided (we give a highly incomplete list) in numerical simulations e.g. [12, 49], in model problems showing blow up e.g. [60, 61], in examples of global solutions e.g. [32, 4], in logarithmically supercritical problems e.g. [62, 58, 13], and in ill-posedness and norm inflation type results e.g. [24, 34, 1, 63].

The behavior of solutions in other supercritical models such as the ones arising in fluid and gas dynamics is extremely interesting and not yet well understood. We will not discuss it here.

1.4. Statement of the result. We assert that in dimensions $5 \leq d \leq 9$ the defocusing (NLS) model (1.1) admits finite time type II (non self similar) blow up solutions arising from $C^\infty$ well localized initial data. The singularity formation is based neither on soliton concentration nor self similar profiles, but on a new front scenario producing a highly oscillatory blow up profile.

Theorem 1.1 (Existence of energy supercritical type II defocusing blow up). Let

$$(d,p) \in \{(5,9), (6,5), (8,3), (9,3)\},$$

and let the critical blow up speed be

$$r^*(d,\ell) = \frac{\ell + d}{\ell + \sqrt{d}}, \quad \ell = \frac{4}{p-1}. \quad (1.5)$$

Then there exists a discrete sequence of blow up speeds $(r_k)_{k \geq 1}$ with

$$2 < r_k < r^*(d,\ell), \quad \lim_{k \to +\infty} r_k = r^*(d,\ell)$$

such that any all $k \geq 1$, there exists a finite co-dimensional manifold of smooth initial data $u_0 \in \cap_{m \geq 0} H^m(\mathbb{R}^d, \mathbb{C})$ with spherical symmetry such that the corresponding solution to (1.1) blows up in finite time $0 < T < +\infty$ at the center of symmetry with

$$\|u(t, \cdot)\|_{L^\infty} = \frac{cp_{p,r,d}(1+o_{t\to T}(1))}{(T-t)^{\frac{1}{p-1}(1+\frac{\ell^2}{r_k^2})}}, \quad cp_{p,r,d} > 0. \quad (1.6)$$
Comments on the result.

1. Hydrodynamical formulation. The heart of the proof of Theorem 1.1 is a study of (1.1) in its hydrodynamical formulation, i.e. with respect to its phase and modulus variables. The key to our analysis is the identification of an underlying compressible Euler dynamics. The latter arises as a leading order approximation of a "front" like renormalization of the original equation. In this process, the Laplace term applied to the modulus\(^1\) of the solution is treated perturbatively in the blow up regime. This is one of the key insights of the paper. The approximate Euler dynamics furnishes us with a self-similar solution, which requires very special properties and is constructed in the companion paper [42] and which, in turn, acts as a blow up profile for the original equation. The existence of these blow up profiles is directly related to the restriction on the parameters (1.4) which we discuss in comment 3 below. Let us recall that there is a long history of trying to use the hydrodynamical variables in (NLS) problems and exploit a connection with fluid mechanics, going back to Madelung’s original formulation of quantum mechanics in hydrodynamical variables, [36]. Geometric optics and the hydrodynamical formulation were used to address ill-posedness and norm inflation in the defocussing Schrödinger equations, [24, 1]. There is also a recent study of vortex filaments in [3] and its dynamical use of the Hasimoto transform. The scheme of proof of Theorem 1.1 will directly apply to produce the first complete description of singularity formation for the three dimensional compressible Navier-Stokes equation in the companion paper [43].

2. Blow up profile. The blow up profile of Theorem 1.1 is more easily described in terms of the hydrodynamical variables:

\[ u(t, x) = \rho_T(t, x)e^{i\phi(t, x)}. \]  

(1.7)

More precisely, we establish the decomposition

\[
\begin{align*}
\rho_T(t, x) &= \frac{1}{(T-t)^{\frac{1}{2}(1+\frac{2}{p-1})}}(\rho_P + \rho)(Z) \\
\phi(t, x) &= \frac{1}{(T-t)^{\frac{1}{2}}}(\Psi_P + \Psi)(Z), \\
Z &= \frac{x}{(T-t)^{\frac{1}{2}}}.
\end{align*}
\]

(1.8)

and prove the local asymptotic stability

\[
\lim_{t \to T} \|\Psi\|_{L^\infty(Z \leq 1)} + \|\rho\|_{L^\infty(Z \leq 1)} = 0.
\]

Here, the blow up profile \((\rho_P, \Psi_P)\) is, after a suitable transformation, picked among the family of spherically symmetric, smooth and decaying as \(Z \to +\infty\) self-similar solutions to the compressible Euler equations. The interest in self-similar solutions for the equations of gas dynamics goes back to the pioneering works of Guderley [26] and Sedov [56] (and references therein,) who in particular considered converging motion of a compressible gas towards the center of symmetry. However, the rich amount of literature produced since then is concerned with non-smooth self-similar solutions. This is partly due to the physical motivations, e.g. interests in solutions modeling implosion or detonation waves, where self-similar rarefaction or compression is followed by a shock wave (these are self-similar solutions which contain shock discontinuities already present in the data), and, partly due to the fact that, as it turns out, global solutions with the desired behavior at infinity and at the center of symmetry are generically not \(C^\infty\). This appears to be a fundamental feature of the self-similar Euler dynamics and, in the language of underlying acoustic geometry, means that generically such solutions are not smooth across the backward

\(^1\)But not to the phase!
light (acoustic) cone with the vertex at the singularity. The key of our analysis is to find those non-generic $C^\infty$ solutions and to discover that this regularity is an essential element in controlling suitable repulsivity properties of the associated linearized operator. This is at the heart of the control of the full blow up. A novel contribution of the companion paper [42] is the construction of $C^\infty$ spherically symmetric self-similar solutions to the compressible Euler equations with suitable behavior at infinity and at the center of symmetry for discrete values of the blow up speed parameter $r$ in the vicinity of the limiting blow up speed $r^*(d, \ell)$ given by (1.5).

3. Restriction on the parameters. There is nothing specific with the choice of parameters (1.4), and clearly the proof provides a full range of parameters. Two main constraints govern these restrictions. First of all, a fundamental restriction in order to make the Eulerian regime dominant is the constraint

$$r^*(d, \ell) > 2 \iff \ell < \ell_2(d) = d - 2\sqrt{d}$$

(1.9)

which provides a non empty set of nonlinearities iff

$$\ell_2(d) > 0 \iff d \geq 5.$$  

As a result, the case of dimensions $d = 3, 4$ is not amenable to our analysis at this point, and the existence of blow up solutions for $d = 3, 4$ remains open. The second restriction concerns the existence of $C^\infty$ smooth blow up profiles with suitable repulsivity properties of the associated linearized operator, as addressed in [42], see section 2.2 and remark 2.3 for detailed statements. In particular, a non degeneracy condition $S_\infty(d, \ell) \neq 0$ for an explicit convergent series required. An elementary numerical computation is performed in [42] to check the condition in the range (1.4).

4. Behavior of Sobolev norms. The conservation of mass and energy imply a uniform $H^1$ bound on the solution. This can also be checked directly on the leading order representation formulas (1.7), (1.8). For higher Sobolev norms, a computation, see Appendix D, shows that the blow up solutions of Theorem 1.1 break scaling, i.e., we can find

$$1 < \sigma < s_c = \frac{d}{2} - \frac{2}{p - 1}$$

such that

$$\lim_{t \to T} \|u(t)\|_{H^\sigma} = +\infty,$$

and the critical Sobolev norm $\|u(t, \cdot)\|_{H^{s_c}}$ blows up polynomially.

5. Stability of blow up. The blow up profiles of Theorem 1.1 have a finite number of instability directions, possibly none. Local asymptotic stability in the interior of the backward light cone (of the acoustical metric associated to the Euler profile) from the singularity relies on an abstract spectral argument for compact perturbations of maximal accretive operators. Related arguments have been used in the literature for the study of self-similar solutions both in focusing and defocusing regimes, for example [8, 21, 46, 44, 16] for parabolic and [19] for hyperbolic problems. The key to the control of the nonlinear flow in the exterior of the light cone is the propagation of certain weighted scale invariant norms. This generalizes a Lyapunov functional based approach developed in [41]. Counting the precise number of instability directions is an independent problem, disconnected to the nonlinear analysis of the blow up, and remains to be addressed.
6. Oscillatory behavior. The constructed solutions are smooth at the blow up time away from $x = 0$:

$$\forall R > 0, \lim_{t \to T} u(t, x) = u^*(x) \text{ in } H^k(|x| > R), \ k \in \mathbb{N}.$$  

As in the cases for blow up problems in the focusing setting, see e.g. [39], the profile outside the blow up point has a \textit{universal} behavior when approaching the singularity

$$u^*(x) = c_P (1 + o_{|x| \to 0}(1)) \left( \frac{|x|^{-2}}{|x|^{2r-2}} \right)^{1/2} e^{i(c_P x + \Psi(x))}, \ c_P \neq 0.$$  

What is unusual, and together with potential non-genericity perhaps responsible for difficulties in numerical detection of the blow up phenomena, is the \textit{highly oscillatory} behavior. This appears to be a deep consequence of the structure of the self-similar solution to the compressible Euler equation and the coupling of phase and modulus variable in the blow up regime, generating an \textit{anomalous Euler scaling}. The heart of our analysis is to show that after passing to the suitable renormalized variables provided by the front, the highly oscillatory behavior (1.11) becomes regular near the singularity and can be controlled with the \textit{monotonicity} estimates of energy type, without appealing to Fourier analysis.

7. Type I blow up. The existence of self similar solutions to the defocusing energy supercritical (NLS) decaying at infinity is an open problem. Such solutions are easily ruled out for the heat equation using the maximum principle, and we refer to [32] for further discussion in the case of the wave equation.

The paper is organized as follows. In section 2, we present the “front” renormalization of the flow which makes the Euler dynamics dominant, and recall all necessary facts about the corresponding self similar profile built in [42]. Theorem 1.1 reduces to building a global in time non vanishing solution to the renormalized flow (2.25) written in hydrodynamical variables. In section 2.4 we detail the strategy of the proof. In section 3, we introduce the functional setting related to maximal accretivity (modulo a compact perturbation) of the corresponding linear operator which leads to a statement of exponential decay in a neighborhood of the light cone for the space of solutions (modulo an a priori control of a finite dimensional manifold corresponding to the unstable directions.) In section 4, we describe our set of initial data and the set of bootstrap assumptions which govern the analysis. In sections 5, 6, 7, we close the control of weighted Sobolev norms and the associated pointwise bounds. In section 8, we close the exponential decay of low Sobolev norms by relying on spectral estimates and finite speed of propagation arguments.

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Notations. The bracket

$$\langle r \rangle = \sqrt{1 + r^2}.$$  

The weighted scalar product for a given measure $g$:

$$(u, v)_g = \int_{\mathbb{R}^d} u \overline{v} g dx.$$  

(1.12)
The integer part of $x \in \mathbb{R}$

$$x \leq [x] < x + 1, \quad [x] \in \mathbb{Z}.$$  

The infinitesimal generator of dilations

$$\Lambda = y \cdot \nabla.$$  

2. Front renormalization, blow up profile and strategy of the proof

In this section we introduce the hydrodynamical variables to study (1.1) and the associated renormalization procedure which makes the compressible Euler structure dominant. We collect from [42] the main facts about the existence of smooth spherically symmetric self-similar solutions to the compressible Euler equations which will serve as blow up profiles.

2.1. Hydrodynamical formulation and front renormalization. We renormalize the flow and, for non-vanishing solutions, write the equivalent hydrodynamical formulation in phase and modulus variables.

We begin with the standard self-similar renormalization

$$u(t, x) = \frac{1}{\lambda(t)^{p-1}} v(s, y) e^{i\gamma}, \quad y = \frac{x}{\lambda}$$

where we freeze the scaling parameter at the self-similar scale

$$\frac{d\tau}{dt} = \frac{1}{\lambda^2}, \quad y = \frac{x}{\lambda(t)}, \quad -\frac{\lambda}{\lambda^2} = \frac{1}{2},$$

then (1.1) becomes

$$i\partial_{\tau} v + \Delta v - \gamma_{\tau} v - i\frac{\lambda}{\lambda} \left( \frac{2}{p-1} v + \Lambda v \right) - v |v|^{p-1} = 0.$$  

(2.1)

In the defocusing case, (2.1) has no obvious type I self similar stationary solution, or type II soliton like solutions, [41], but, it turns out, that it admits approximate front like solutions. Their existence relies on a specific phase and modulus coupling and anomalous scaling. We introduce the parameters

$$|r| = \frac{2}{1-e}, \quad 0 < e < 1$$

$$\mu = \frac{1}{2}, \quad \ell = \frac{1-e}{2}$$

and claim:

**Lemma 2.1** (Front renormalization of the self similar flow). Define geometric parameters

$$-\frac{\lambda_{\tau}}{\lambda} = \frac{1}{2}, \quad \frac{b_{\tau}}{b} = -e, \quad \gamma_{\tau} = -\frac{1}{b}, \quad \frac{d\tau}{dt} = \frac{1}{\lambda^2}$$

(2.3)

and introduce the renormalization

$$u(t, x) = \frac{1}{\lambda(t)^{p-1}} v(s, y) e^{i\gamma}, \quad y = \frac{x}{\lambda}$$

with the phase and modulus

$$v = w e^{i\phi}$$

$$w(\tau, y) = \frac{1}{(\sqrt{b})^{p-1}} \rho_T(\tau, Z) \in \mathbb{R}^*$$

$$\phi(\tau, y) = \frac{1}{b} \Psi_T(\tau, Z)$$

$$Z = y \sqrt{b}$$
In these variables (1.1) becomes, on $[\tau_0, +\infty)$:
\[
\begin{align*}
\partial_\tau \rho_T &= -\rho_T \Delta \Psi_T - \frac{\mu(r-1)}{2} \rho_T - (2\partial_Z \Psi_T + \mu Z) \partial_Z \rho_T \\
\rho_T \partial_\tau \Psi_T &= b^2 \Delta \rho_T - \left[|\nabla \Psi_T|^2 + \mu(r-2)\Psi_T - 1 + \mu \Lambda \Psi_T + \rho_T^{p-1}\right] \rho_T.
\end{align*}
\] (2.4)

Proof. Starting from (2.1), we define a polar decomposition
\[ v = we^{i\phi} \]
so that
\[
\begin{align*}
v' &= (w' + i\phi' w)e^{i\phi}, \quad v'' = w'' - |\phi'|^2 w + 2i\phi' w' + i\phi'' w
\end{align*}
\]
and
\[
\begin{align*}
0 &= i\partial_\tau w + \Delta w + \left(-\partial_\tau \phi - |\nabla \phi|^2 - \gamma_\tau + \frac{\lambda_\tau}{\lambda} y \cdot \nabla \phi\right) w \\
&+ i \left(\Delta \phi - \frac{2}{p-1} \frac{\lambda_\tau}{\lambda}\right) w + i \left(2\nabla \phi - \frac{\lambda_\tau}{\lambda} y \cdot \nabla w - w|w|^{p-1} \right).
\end{align*}
\] (2.5)

Separating the real and imaginary parts yields the self-similar equations (2.1):
\[
\begin{align*}
\partial_\tau w = -\left(\Delta \phi + \frac{1}{p-1}\right) w - \left(2\frac{\partial \phi}{y} + \frac{1}{2}\right) \Lambda w \\
\partial_\tau \phi = \Delta w + \left(-|\nabla \phi|^2 - \gamma_\tau - \frac{1}{2} \Lambda \phi\right) w - w|w|^{p-1}
\end{align*}
\] (2.6)

We now renormalize according to
\[
\begin{align*}
w(\tau, y) &= \frac{1}{(\sqrt{b})^{p-1}} \rho_T(\tau, Z) \in \mathbb{R}_+, \quad \phi(\tau, y) = \frac{1}{b} \Psi_T(\tau, Z) \quad Z = y\sqrt{b}
\end{align*}
\]
with a fixed choice of parameters in the modulation equations
\[
\frac{b_\tau}{b} = -\epsilon, \quad \gamma_\tau = -\frac{1}{b}, \quad 0 < \epsilon < 1
\]
which transforms (2.6) into
\[
\begin{align*}
\partial_\tau \rho_T &= -\rho_T \Delta \Psi_T - \frac{\epsilon + 1}{p-1} \rho_T - (2\partial_Z \Psi_T + \frac{1}{p-1} Z) \partial_Z \rho_T \\
\rho_T \partial_\tau \Psi_T &= b^2 \Delta \rho_T - \left[|\nabla \Psi_T|^2 + e \Psi_T - 1 + \frac{p}{2}(1-e)\Lambda \Psi_T + \rho_T^{p-1}\right] \rho_T
\end{align*}
\]
We now compute from (2.2):
\[
\begin{align*}
\mu \frac{\tau(r-1)}{2} &= \frac{2}{p-1} (1 - \mu) = \frac{1+\epsilon}{p-1} \\
\mu(r-2) = 1 - (1 - \epsilon) = \epsilon
\end{align*}
\]
and (2.4) is proved. \hfill \qed

2.2. Blow up profile and Emden transform. A stationary solution $(\rho_P, \Psi_P)$ to (2.4) in the limiting Eulerian regime $b = 0$ satisfies the profile equation
\[
\begin{align*}
|\nabla \Psi_P|^2 + \rho_P^{p-1} + \mu(r-2)\Psi_P + \mu \Lambda \Psi_P = 1 \\
\Delta \Psi_P + \frac{\mu(r-1)}{2} (2\partial_Z \Psi_P + \mu Z) \frac{\partial \rho_P}{\rho_P} = 0
\end{align*}
\] (2.7)

We supplement it with the boundary conditions:
\[
\begin{align*}
\rho_P(0) &= 1, \quad \Psi_P(0) = 0, \\
\rho_P(x) &\to 0, \quad \Psi_P(x) \to \frac{1}{r} \quad \text{as} \quad x \to \infty
\end{align*}
\] (2.8)
We now show that the system (2.7), (2.8) is equivalent to the corresponding system of equations describing self-similar solutions of the Euler equations. We define the Emden variables:
\[ \phi = \frac{2}{3} \sqrt{\ell}, \quad p - 1 = \frac{4}{3}, \quad Q = \rho_p^{p-1} = \frac{1}{M^2}, \quad \frac{1}{M} = \phi Z \sigma, \quad x = \log Z, \] (2.9)
then (2.7) is mapped onto
\[
\begin{align*}
(w - 1)w' + \ell \sigma' + (w^2 - rw + \ell \sigma^2) &= 0 \\
\frac{\ell}{2} w' + (w - 1)\sigma' + \sigma \left[ w \left( \frac{d}{\ell} + 1 \right) - r \right] &= 0
\end{align*}
\] (2.10)
or equivalently
\[
\begin{align*}
a_1 w' + b_1 \sigma' + d_1 &= 0 \\
a_2 w' + b_2 \sigma' + d_2 &= 0
\end{align*}
\] (2.11)
The system (2.10) is exactly the one describing spherically symmetric self-similar solutions to the compressible Euler equation, [56] (and the references therein). For an explicit derivation see Appendix A. It is analyzed in [42], following pioneering work of Guderley, Sedov and others.

Let
\[ w_e = \frac{\ell(r - 1)}{d} \] (2.12)
and the determinants
\[
\begin{align*}
\Delta &= a_1 b_2 - b_1 a_2 = (w - 1)^2 - \sigma^2 \\
\Delta_1 &= -b_1 d_2 + b_2 d_1 = w(w - 1)(w - r) - d(w - w_e)\sigma^2 \\
\Delta_2 &= d_2 a_1 - d_1 a_2 = \ell \left[ (\ell + d - 1)w^2 - w(\ell + d + \ell r - r) + \ell r - \ell \sigma^2 \right]
\end{align*}
\] (2.13)
then
\[
\begin{align*}
w' &= -\frac{\Delta_1}{\Delta}, \quad \sigma' = -\frac{\Delta_2}{\Delta}, \quad \frac{dw}{d\sigma} = \frac{\Delta_1}{\Delta_2}.
\end{align*}
\] (2.14)
Solution curves \( w = w(\sigma) \) of the above system can be examined through its phase portrait in the \((\sigma, w)\) plane. The shape of the phase portrait depends crucially on the polynomials \( \Delta, \Delta_1, \Delta_2 \) and the parameters \((r, d, \ell)\). It is not hard to see that there is a unique solution with the normalization
\[ \rho_P(0) = 1, \quad \Psi_P(0) = 0, \] (2.15)
at \( x = 0 \), which is also \( C^\infty \) in the vicinity of \( x = 0 \), but the heart of the matter is the global behavior of this unique solution.

In particular, any such solution with the required asymptotics as \( x \to +\infty \) needs to pass through the point \( P_2 \) which lies on the so called sonic line\(^2\): \( \Delta = 0 \) but where also
\[ \Delta_1(P_2) = \Delta_2(P_2) \] (2.16)
(there are potentially two such points). It turns out that at \( P_2 \) the solution experiences an unavoidable discontinuity of high derivatives, except for discrete values of the speed \( r \). The following structural proposition on the blow up profile is proved in the companion paper [42].

\(^2\)Any point \( Z_0 \) on the sonic line corresponds to the acoustic cone described the equation \((\tau, Z = Z_0)\) for the acoustic metric defined by the profile passing through \( Z_0 \).
Theorem 2.2 (Existence and asymptotics of a $C^\infty$ profile, [42]). Let 
\[(d, p) \in \{(5, 9), (6, 5), (8, 3), (9, 3)\}\]
and recall (1.5). Then there exists a sequence \((r_k)_{k \geq 1}\) with
\[
\lim_{k \to \infty} r_k = r^*(d, \ell), \quad r_k < r^*(d, \ell)
\]
(2.17)
such that for all \(k \geq 1\), the following holds:

1. Existence of a smooth profile at the origin: the unique radially symmetric solution to (2.7) with Cauchy data at the origin (2.8) reaches in finite time \(Z_2 > 0\) the point \(P_2\).

2. Passing through \(P_2\): the solution passes through \(P_2\) with $C^\infty$ regularity.

3. Large \(Z\) asymptotic: the solution admits the asymptotics as \(Z \to +\infty\):
\[
\left| w(Z) = \frac{c_\sigma}{2r^2} \left(1 + O \left(\frac{1}{Z^r}\right)\right) \right|
\]
\[
\left| \frac{\sigma(Z)}{2r^2} = \frac{c_\sigma}{2r^2} \left(1 + O \left(\frac{1}{Z^r}\right)\right) \right|
\]
(2.18)
or equivalently
\[
\left| Q(Z) = \rho_P^{p-1}(Z) = \frac{c_p^{p-1}}{2r^{p-2}} \left(1 + O \left(\frac{1}{Z^r}\right)\right) \right|
\]
\[
\left| \Psi_P(Z) = \frac{1}{r} + \frac{c_p}{2r} \left(1 + O \left(\frac{1}{Z^r}\right)\right) \right|
\]
(2.19)
with non zero constants \(c_\sigma, c_P\). Similar asymptotics hold for all higher order derivatives.

4. Non vanishing: there holds
\[
\forall Z \geq 0, \quad \rho_P > 0.
\]

5. Repulsivity inside the light cone: let
\[
F = \sigma_P + \Lambda \sigma_P,
\]
then there exists \(c = c(d, \ell, r) > 0\) such that
\[
\forall 0 \leq Z \leq Z_2, \quad \left| (1 - w - \Lambda w)^2 - F^2 > c \right|
\]
\[
\left| 1 - w - \Lambda w - \frac{(1-w)F}{\sigma} > c. \right|
\]
(2.21)

6. Repulsivity outside the light cone:
\[
\exists c = c_{d, \ell, r} > 0, \quad \forall Z \geq Z_2, \quad \left| (1 - w - \Lambda w)^2 - F^2 > c \right|
\]
\[
\left| 1 - w - \Lambda w > c. \right|
\]
(2.22)

Remark 2.3 (Restriction on the parameters). The proof of Theorem 2.2 requires the non degeneracy of an explicit series \(S_\infty(d, \ell) \neq 0\) which is numerically checked in [42] in the range (1.4). The positivity properties (2.21), (2.22) are checked analytically in [42] and will be fundamental for the well-posedness of the linearized flow inside the light cone, and the control of global Sobolev norms outside the light cone. Let us insist that the restriction on parameters relies on the intersection of the conditions (1.9), \(S_\infty(d, \ell) \neq 0\) and (2.21), (2.22) hold. The range (1.4) is just an example where this holds, but a larger range of parameters can be directly extracted from [42], and the conclusion of Theorem 1.1 would follow. In particular and since we are working with non vanishing solutions, the fact that the non linearity is an odd integer can be relaxed as in [43].

From now on and for the rest of this paper, we assume (1.4). We observe from direct check that there holds:
\[
r^*(\ell) = \frac{d + \ell}{\ell + \sqrt{d}} > 2 \iff \ell < d - 2\sqrt{d} = \ell_2(d).
\]
Recalling (2.2), we may therefore assume from (2.17) that the blow speed \( r = r_k \) satisfies
\[
r > 2 \iff \epsilon = \frac{r - 2}{r} > 0.
\]

2.3. Linearization of the renormalized flow. We look for \( u \) solution to (1.1) and proceed to the decomposition of Lemma 2.1. We are left with finding a global, in self similar time \( \tau \in [\tau_0, +\infty) \), solution to (2.4):

\[
\begin{align*}
\partial_\tau \rho_T &= -\rho_T \Delta \Psi_T - \frac{\mu(r-1)}{2} \rho_T - (2\partial_\tau \Psi_T + \mu Z) \partial_\tau \rho_T \\
\rho_T \partial_\tau \Psi_T &= b^2 \Delta \rho_T - \left( |\nabla \Psi_T|^2 + \mu(r-2) \Psi_T - 1 + \mu \Lambda \Psi_T + \rho_p^{-1} \right) \rho_T
\end{align*}
\]

(2.23)

with non vanishing density \( \rho_T > 0 \). We define
\[
\begin{align*}
H_2 &= \mu + 2\frac{\mu}{\rho_p} - \mu(1-w) \\
H_1 &= -\left( \Delta \Psi_P + \frac{\mu(r-1)}{2} \right) = H_2 \frac{\Lambda p_p}{\rho_p} = \frac{\mu}{2} (1-w) \left[ 1 + \frac{\Lambda}{\sigma} \right]
\end{align*}
\]

(2.24)

We linearize \( \rho_T = \rho_P + \rho, \quad \Psi_T = \Psi_P + \Psi \) and compute, using the profile equation (2.7), for the first equation:

\[
\partial_\tau \rho = - (\rho_P + \rho) \Delta \Psi_P - \frac{\mu(r-1)}{2} (\rho_P + \rho) - (2\partial_\tau \Psi_P + \mu Z + 2\partial_\tau \Psi) (\partial_\tau \rho_P + \partial_\tau \rho)
\]

and for the second one:

\[
\begin{align*}
\rho_T \partial_\tau \Psi &= b^2 \Delta \rho_T - \rho_T \left\{ |\nabla \Psi_P|^2 + 2|\nabla \Psi_P \cdot \nabla \Psi + |\nabla \Psi|^2 \right\} \\
&- 1 + \mu(r-2) \Psi_P + \mu(r-2) \Psi + \mu(\Lambda \Psi_P + \Lambda \Psi) + (\rho_P + \rho)^{p-1}
\end{align*}
\]

with
\[
NL(\rho) = (\rho_P + \rho)^{p-1} - \rho_P^{p-1} - (p-1) \rho_P^{p-2} \rho.
\]

We arrive at the exact (nonlinear) linearized flow:

\[
\begin{align*}
\partial_\tau \rho &= H_1 \rho - H_2 \Lambda \rho - \rho_T \Delta \Psi - 2\partial_\tau \rho \cdot \nabla \Psi \\
\partial_\tau \Psi &= b^2 \frac{\Delta \rho_T}{\rho_T} - \left\{ H_1 \Lambda \Psi + \mu(r-2) \Psi + |\nabla \Psi|^2 + (p-1) \rho_P^{p-2} \rho + NL(\rho) \right\}
\end{align*}
\]

(2.25)

Theorem 1.1 is therefore equivalent to exhibiting a finite co-dimensional manifold of smooth well localized initial data leading to global, in renormalized \( \tau \)-time, solutions to (2.25).

2.4. Strategy of the proof. We now explain the strategy of the proof of Theorem 1.1.

**step 1** Wave equation and propagator estimate. After the change of variables \( \Phi = \rho_P \Psi \), we may schematically rewrite the linearized flow (2.25) in the form

\[
\partial_\tau X = MX + NL(X) - b^2 \begin{pmatrix} 0 \\ \Delta (\rho_P + \rho) \end{pmatrix}
\]

(2.26)

with

\[
X = \begin{pmatrix} \rho \\ \Phi \end{pmatrix}, \quad M = \begin{pmatrix} H_1 - H_2 \Lambda & -\Delta + H_3 \\ -\Delta + H_3 & -H_2 \Lambda - (p-1)Q - H_2 \Lambda \end{pmatrix}
\]

(2.27)
where \( Q, H_1, H_2, H_3 \) are explicit potentials generated by the profile \( \rho_P, \Psi_p \). During the first step the \( b^2 \Delta \) term is treated perturbatively. We commute the equation with the powers of the laplacian \( \Delta^k \) and obtain for \( X_k = \Delta^k X \)

\[
\partial_\tau X_k = M_k X + NL_k(X).
\]

(2.28)

We then show that, provided \( k \) is large enough, \( M_k \) is a finite rank perturbation of a maximally dissipative operator with a spectral gap \( \delta > 0 \). The topology in which maximal accretivity is established depends on the properties of the wave equation\(^3\) encoded in (2.28) and is based on weighted Sobolev norms with weights vanishing on the light cone corresponding to the point \( P2 \) of the profile. Indeed, the principal part of the wave equation is roughly of the form

\[
\partial^2_\tau \rho - D(Z) \partial^2_Z \rho
\]

where the weight \( D(Z) \) vanishes on the light cone \( Z = Z_2 \) corresponding to the \( P2 \) point. The corresponding propagation estimates for the wave equation produce an a priori control of the solution in the interior of the light cone \( Z < Z_2 \), modulo an a priori control of a finite number of directions corresponding to non positive eigenvalues of \( M_k \). An essential structural fact of this step is the \( C^\infty \) regularity of the profile. Indeed, we claim that for a generic non \( C^\infty \) solution at \( P2 \), the number of derivatives required to show accretivity of the linearized operator is always strictly greater than the regularity of the profile at \( P2 \). As a result such profiles may be completely unstable and are not amenable to our analysis. The \( C^\infty \) regularity obtained in [42] is therefore absolutely fundamental. The analytic properties leading to the maximality of the linearized operator will be consequences of (2.21), (2.22). We note that the coercivity constant in (2.21) degenerates as \( r \to r^\ast \), and the number of derivatives needed for accretivity is inversely proportional to this constant. This is a manifestation of a completely new nonlinear effect: the problem sees a scaling which depends on the chosen self similar profile.

**step 2** Extension slightly beyond the light cone. Exponential decay estimates provided in the first step yield control in the interior of the light cone \( Z < Z_2 \) only. It turns out that the analysis of the first step can be made more robust and extended\(^4\) slightly beyond the light cone, all the way to a spacelike hypersurface \( Z = Z_2 + a, \) \( 0 < a \ll 1 \), even though it is complicated by the dependence of the underlying wave equation on variable coefficients or, equivalently, on non constancy of the \( Q(Z) \) term in (2.27). We can revisit the first step by producing a new maximal accretivity structure for a norm which does not generate in the zone \( Z < Z_2 + a, \) \( 0 < a \ll 1 \). The argument relies on a new generalized monotonicity formula. The corresponding propagation estimates recovers exponential decay in the extended zone \( Z < Z_2 + a \).

Once decay has been obtained strictly beyond the light cone, a simple finite speed of propagation argument allows us to propagate decay to any compact set \( Z < Z_0, \) \( Z_0 \gg 1 \).

**step 3** Loss of derivatives. The decay obtained in step 2 relies on energy estimates compatible with the wave propagation and the Eulerian structure of approximation. The full evolution however is that of the Schrödinger equation and contains the \( b^2 \Delta \) term on the right hand side of (2.26). Such a term leads to an unavoidable loss of one derivative. However, this loss comes with a \( b^2 \) smallness in front. We then argue as follows. We pick a large enough regularity level \( k_m = k_m(r, d) \gg 2k_0 \), where \( k_0 \)

\(^3\)Reminiscent of the wave equation arising in a linearization of the compressible Euler equations.

\(^4\)Reminiscent of a non-characteristic energy estimate.
is the power of the laplacian used for commutation in step 2, and derive a global Schrödinger like energy identity on the full flow (2.25). The choice of phase and modulus as basic variables turns the equation quasilinear and makes this identity rather complicated and unfamiliar. An essential difficulty, which is deeply related to step 2, is that at the highest level of derivatives, the non trivial space dependence of the profile measured by $Q(Z) = \rho_p^{-1}(Z)$ in (2.27) produces a coupling term and a non trivial quadratic form. The condition (2.22) implies that the corresponding quadratic form is definite positive for $k_m$ large enough.

**step 4** Closing estimates. As explained above, we work with a linearized nonlinear equation, i.e., obtained after subtracting off the profile, written in terms of the phase and modulus unknowns $(\Psi, \rho)$, in renormalized self-similar variables $(\tau, Z)$, where the singularity corresponds to $(\tau = \infty, Z = 0)$, a special light cone is $(\tau, Z = Z_2)$ and where in the original variables $(t, r)$ the region $r \geq 1$ corresponds to $Z \geq e^{\mu \tau}$.

First, outside the singularity $r \geq 1$, we modify the profile by strengthening its decay to make it rapidly decaying and of finite energy. Relative to the self-similar variables this modification happens at $Z \sim e^{\mu \tau}$, far from the singularity, and as a result is harmless. Then, we run two sets of estimates. First, we employ wave propagation like estimates which go initially just slightly beyond the special light cone and then extend to any compact set in $Z$. These estimates are carried out at a sufficiently high level of regularity with $\sim 2k_0$ derivatives. The number $k_0$ emerges from the linear theory and is determined by the (conditional) positivity of a certain quadratic form responsible for maximal accretivity.

Then, we couple these estimates to global Schrödinger like estimates which take into account previously ignored $b^2 \Delta$ and take care of global control. These estimates are carried out at all levels of regularity up to $k_m$ derivatives with $k_m \gg k_0$. They are carefully designed weighted $L^2$ type estimates. The weights depend on the number of derivatives $k$: at first, their strength grows with $k$ but by the time we reach the highest level of regularity $k_m$ the weight function is identically $= 1$. The latter has to do with a well-known fact that even for a linear Schrödinger equation estimates, use of weights leads to a derivative loss ($\Delta$ is not self-adjoint on a weighted $L^2$ space.) Therefore, our highest derivative norm should correspond to an unweighted $L^2$ estimate. Of course, this last estimate also sees a positivity condition (2.22) responsible for the coercivity of an appearing quadratic form.

These global weighted $L^2$ bounds then allow us to prove pointwise bounds for the solution and its derivatives which, in turn, allow us to control nonlinear terms. The obtained sets of weighted $L^\infty$ bounds on derivatives recover in particular the non vanishing assumption required of the solution. We should note that while all the local (in $Z$) norms decay exponentially in $\tau$, the global norms are merely bounded. In the original $(t, r)$ variables this means that the perturbation decays inside and slightly beyond the backward light cone from the singular point but does not decay away from the singularity. This is, of course, entirely consistent with the global conservation of energy for NLS.

The whole proof proceeds via a bootstrap argument which also involves a Brouwer type argument to deal with unstable modes, if any, arising in linear theory of step 1. This is what produces a finite co-dimension manifold of admissible data.
3. Linear theory slightly beyond the light cone

Our aim in this section is to study the linearized problem \((2.25)\) for the exact Euler problem \(b = 0\). We in particular aim at setting up the suitable functional framework in order to apply classical propagator estimates which will yield exponential decay on compact sets \(Z \lesssim 1\) modulo the control of a finite number of unstable directions.

3.1. Linearized equations. Recall the exact linearized flow \((2.25)\) which we rewrite:

\[
\partial_t \rho = H_1 \rho - H_2 \Delta \rho - H_2 \Phi + \mu (r - 2) \Phi + \Lambda Q \rho
\]

with

\[
Q = \rho^{p-1}, \quad H_3 = \frac{\Delta \rho_p}{\rho_p}
\]

and the nonlinear terms:

\[
G_\rho = -\rho \Delta \Phi - 2 \nabla \rho \cdot \nabla \Phi
\]

\[
G_\Phi = -\rho \Delta \Phi + H_3 \Phi + \mu (r - 2) \Phi
\]

We transform \((3.2)\) into a wave equation for \(\Phi\) and compute:

\[
\partial_t^2 \Phi = -(p - 1) Q (H_1 \rho - H_2 A \rho - \Delta \Phi + H_3 \Phi + \mu (r - 2)) \partial_t \Phi + \mu (r - 2) \partial_t \Phi
\]

with

\[
A_1 = H_2 H_1 - H_2 A H_2 + H_2 (H_1 - \mu (r - 2)) + H_2 \frac{\Lambda Q}{Q}
\]

\[
A_2 = 2 H_1 - \mu (r - 2) + H_2 \frac{\Lambda Q}{Q}
\]

\[
A_3 = -(H_1 - \mu) H_1 + H_2 A H_1 - H_2 (H_1 - \mu (r - 2)) \frac{\Lambda Q}{Q} - (p - 1) Q H_3
\]

In this section we focus on deriving decay estimates for \((3.2)\).

Remark 3.1 (Null coordinates and red shift). We note that the principal symbol of the above wave equation is given by the second order operator

\[
\Box_Q := \partial_t^2 - ((p - 1) Q - H_2^2 Z^2) \partial_r^2 + 2 H_2 Z \partial_r^2
\]

In the variables of Emden transform this can be written equivalently as

\[
\Box_Q = \partial_t^2 - \mu^2 (\sigma^2 - (1 - w)^2) \partial_r^2 + 2 \mu (1 - w) \partial_r \partial_r
\]

The two principal null direction associated with the above equation are

\[
L = \partial_r + \mu [(1 - w) - \sigma] \partial_r, \quad L = \partial_r + \mu [(1 - w) + \sigma] \partial_r,
\]
so that
\[ \Box_Q = L \mathcal{L} \]
We observe that at \( P_2 \), we have \( L = \partial_\tau \) and the surface \( Z = Z_2 \) is a null cone. Moreover, the associated acoustical metric is
\[ g_Q = \mu^2 \Delta dx^2 - 2\mu(1 - w)d\tau dx + dx^2, \quad \Delta = (1 - w)^2 - \sigma^2 \]
for which \( \partial_\tau \) is a Killing field (generator of translation symmetry). Therefore, \( Z = Z_2 \) is a Killing horizon (generated by a null Killing field.) We can make it even more precise by transforming the metric \( g_Q \) into a slightly different form by defining the coordinate \( s \):
\[ s = \mu \tau - f(x), \quad f' = 1 - \frac{w}{\Delta}, \]
so that
\[ g_Q = \Delta (ds)^2 - \frac{\sigma^2}{\Delta} dx^2 \]
and then the coordinate \( x^* \):
\[ x^* = \int \frac{\sigma}{\Delta} dx, \]
so that
\[ g_Q = \Delta d(s + x^*) d(s - x^*) \]
and \( s + x^* \) and \( s - x^* \) are the null coordinates of \( g_Q \). The Killing horizon \( Z = Z_2 \) corresponds to \( x^* = -\infty \) and \( \Delta \sim e^{C x^*} \) for some positive constant \( C \). In this form, near \( Z_2 \) the metric \( g_Q \) resembles the \( 1 + 1 \)-quotient Schwarzschild metric near the black hole horizon.

The associated surface gravity \( \kappa \) which can be computed according to
\[ \kappa = \frac{\partial_x \Delta}{2 \Delta} |_{P_2} = \frac{\partial_x \Delta}{2 \sigma} |_{P_2} = \frac{-w'(1 - w) - \sigma' \sigma}{\sigma} |_{P_2} = (-w' - \sigma') |_{P_2} = 1 - w - \Lambda w - \frac{(1 - w)F}{\sigma}|_{P_2} > 0 \]
This is precisely the repulsive condition (2.21) (at \( P_2 \)). The positivity of surface gravity implies the presence of the red shift effect along \( Z = Z_2 \) both as an optical phenomenon for the acoustical metric \( g_Q \) and also as an indicator of local monotonicity estimates for solutions of the wave equation \( \Box_Q \phi = 0 \), [17]. The complication in the analysis below is the presence of lower order terms in the wave equation as well as the need for global in space estimates.

3.2. The linearized operator. Pick a small enough parameter
\[ 0 < a \ll 1 \]
and consider the new variable
\[ T = \partial_\tau \Phi + a H_2 \Lambda \Phi, \quad (3.5) \]
then
\[ \partial_\tau T = \partial^2 \Phi + a H_2 \Lambda \partial_\tau \Phi = \partial^2 \Phi + a H_2 \Lambda (T - a H_2 \Lambda \Phi) \]
\[ = \partial^2 \Phi + a H_2 \Lambda T - a^2 H_2 \Lambda H_2 \Lambda \Phi - a^2 H_2 ^2 \Lambda ^2 \Phi \]
which yields the \((T, \Phi)\) equation
\[ \partial_\tau \Phi = T - a H_2 \Lambda \Phi \]
and

$$\partial_r T = (p-1)Q \Delta \Phi - H^2_2 \Lambda^2 \Phi - 2H_2 \Lambda (T - aH_2 \Lambda \Phi) + A_1 \Lambda \Phi + A_2 (T - aH_2 \Lambda \Phi) + A_3 \Phi + aH_2 \Lambda T - a^2 H_2 \Lambda aH_2 \Lambda \Phi - a^2 H^2_2 \Lambda^2 \Phi + G_T$$

$$= (p-1)Q \Delta \Phi - (1-a)^2 H^2_2 \Lambda^2 \Phi + \tilde{A}_2 \Lambda \Phi + A_3 \Phi - (2-a)H_2 \Lambda T + A_2 T + G_T$$

with

$$G_T = \partial_r G \Phi - \left( H_1 + H_2 \frac{\Lambda Q}{Q} \right) G \Phi + H_2 \Lambda G \Phi - (p-1)QG \rho + \tilde{A}_2 \Lambda \Phi + A_3 \Phi - (2-a)^2 H_2 \Lambda T + A_2 T + G_T$$

and

$$\tilde{A}_2 = A_1 + (2a - a^2) H^2_2 \Lambda aH_2 \Lambda - aA_2 H_2.$$  

We rewrite these equations in vectorial form

$$\partial_r X = M X + G, X = \begin{bmatrix} \Phi \\ T \end{bmatrix}, G = \begin{bmatrix} 0 \\ G_T \end{bmatrix}$$  

with

$$M = \begin{bmatrix} -aH_2 \Lambda & 1 \\ (p-1)Q \Delta - (1-a)^2 H^2_2 \Lambda^2 + \tilde{A}_2 \Lambda + A_3 & -(2-a)H_2 \Lambda + A_2 \end{bmatrix}.$$  

3.3. Shifted measure. The fine structure of the operator (3.8) involves the understanding of the associated light cone.

**Lemma 3.2** (Shifted measure). Let

$$D_a = (1-a)^2 (w-1)^2 - \sigma^2$$

then for $0 < a < a^*$ small enough, there exists a $C^1$ map $a \mapsto Z_a$ with

$$Z_{a=0} = Z_2, \quad \frac{\partial Z_a}{\partial a} > 0$$

such that

$$D_a(Z_a) = 0, \quad -D_a(Z) > 0 \quad \text{on} \quad 0 \leq Z < Z_a$$

$$\lim_{Z \to 0} Z^2 (-D_a) > 0.$$  

**Proof of Lemma 3.2.** We recall the notations of the Emden transform:

$$x = \log Z, \quad \mu = \frac{1-w}{2}$$

$$F = \sigma + \Lambda \sigma$$

$$(p-1)Q = \mu^2 Z^2 \sigma^2, \quad \frac{\Lambda Q}{Q} = 2 + 2 \frac{\Lambda \sigma}{\sigma} = \frac{2F}{\sigma}$$

$$(p-1)\partial_Z Q = (p-1)\frac{\Lambda Q}{Q} = 2 \mu^2 Z \sigma^2 (1 + \frac{\Lambda \sigma}{\sigma}) = 2 \mu^2 Z \sigma F$$

$$H_2 = \frac{1-w}{2} + 2 \frac{\partial Z}{Z} F = \mu (1-w)$$

$$H_1 = H_2 \frac{\Lambda \sigma}{Z} = \frac{H_2}{2} \frac{\Lambda Q}{Q} = \frac{2F (1-w)}{(p-1)\sigma}$$

$$D = (w-1)^2 - \sigma^2.$$  

**step 1** Values of derivatives at $P_2$. Let

$$\Delta = (w-1)^2 - \sigma^2.$$  

Let the variables

$$w = w_2 + W, \quad \sigma = \sigma_2 + \Sigma,$$

then near $P_2$:

$$W = c_\Sigma + O(\Sigma^2).$$
Let
\[
\begin{align*}
c_1 &= \partial_W \Delta_1(P_2) \\
c_2 &= \partial_W \Delta_2(P_2) \\
c_3 &= \partial_\Sigma \Delta_1(P_2) \\
c_4 &= \partial_\Sigma \Delta_2(P_2) = -2\sigma_2^2
\end{align*}
\] (3.12)

Then, in our range of parameters,
\[
c_1 < 0, \; c_2 < 0, \; c_3 < 0, \; c_4 < 0,
\] (3.13)

and we have
\[
\begin{align*}
c_2c_- + c_4 &= \lambda_- \\
c_2c_+ + c_4 &= \lambda_- \\
c_\pm &= \frac{c_2c_- + c_4}{c_2c_+ + c_4}
\end{align*}
\] (3.14)

which imply
\[
c_1c_- + c_3 = c_-(c_2c_- + c_4) = c_-\lambda_-
\]
as well as
\[
-1 < c_- < 0 < c_+, \; \lambda_- < \lambda_+ < 0,
\] (3.15)

see Lemma 2.8 and Lemma 2.9 in [42].

We compute
\[
\begin{align*}
\Delta_1 &= c_1 W + c_3 \Sigma + O(W^2 + \Sigma^2) = (c_1 c_- + c_3) \Sigma + O(\Sigma^2) \\
\Delta_2 &= c_2 W + c_4 \Sigma + O(W^2 + \Sigma^2) = (c_2 c_- + c_4) \Sigma + O(\Sigma^2) \\
\Delta &= (1 - w_2 - W)^2 - (\sigma_2 + \Sigma)^2 = (\sigma_2 - W)^2 - (\sigma_2 + \Sigma)^2 = -2\sigma_2(c_- + 1) \Sigma + O(\Sigma^2)
\end{align*}
\]

This yields
\[
\begin{align*}
\frac{d\lambda_+}{d\Sigma} &= -\frac{\lambda_+}{\lambda_-} = -\frac{c_1 c_- + c_3 + O(\Sigma)}{2\sigma_2(1 + c_-) + O(\Sigma)} = \frac{|c_-||\lambda_+|}{2\sigma_2(1 + c_-) + O(\Sigma)} + O(\Sigma) \tag{3.16}
\end{align*}
\]

and hence
\[
\begin{align*}
Z_2 \frac{d\Delta}{dZ}(Z_2) &= \frac{d\Delta}{dx}(P_2) = -2(1 - w_2) \frac{dw}{dx}(P_2) - 2\sigma_2 \frac{d\sigma}{dx}(P_2) \\
&= -2\sigma_2 \left(\frac{|c_-||\lambda_+|}{2\sigma_2(1 + c_-)} - 2\sigma_2 \left(-\frac{|\lambda_+|}{2\sigma_2(1 + c_-)}\right) = \frac{|\lambda_+|}{1 + c_-} (1 - |c_-|) \right) \\
&= |\lambda_+| > 0 \tag{3.17}
\end{align*}
\]

**step 2** Computation of $Z_a$. Let $D_0(Z) = \Delta(Z)$, we have $D'_0(Z_2) > 0$ from (3.17) and hence by the implicit function theorem applied to the function $F(a, Z) = D_a(Z)$ at $(a, Z) = (0, Z_2)$ where $D_0(Z_2) = 0$, we infer for all $a$ small enough the existence of a locally unique solution $Z_a$ to
\[
D_a(Z_a) = 0 \tag{3.18}
\]

Furthermore, $Z_a$ is $C^1$ in a neighborhood of $a = 0$ and its derivative is given by
\[
\frac{\partial Z_a}{\partial a} \bigg|_{a=0} = -\left(\frac{\partial D_a(Z)}{\partial a} \bigg|_{a=0, Z=Z_2} \right) = \frac{2\sigma_2^2}{D'_0(Z_2)} > 0
\]

Thus
\[
\frac{\partial Z_a}{\partial a} \bigg|_{a=0} > 0, \; Z_a > Z_2 \text{ for } 0 < a \ll 1, \; D'_a(Z_a) > 0. \tag{3.19}
\]

We now observe
\[
D_a(Z) = ((1 - a)(1 - w) + \sigma)((1 - a)(1 - w) - \sigma)
\]
so that $D_a(Z)$ is of the sign of $(1 - a)(1 - w) - \sigma$ since $w < 1$ and $\sigma > 0$. Now from (3.16):

$$\frac{d}{dx}(1 - a)(1 - w) - \sigma = -(1 - a)\frac{|c_-|\lambda_+}{2\sigma_2(1 + c_-)} + \frac{|\lambda_+|}{2\sigma_2(1 + c_-)}$$

Thus, $(1 - a)(1 - w) - \sigma$ is increasing on $(0, Z_a]$ and vanishes at $Z = Z_a$ so that $D_a(Z) < 0$ on $(0, Z_a)$.

Moreover, we have in view of the behavior of $\sigma$ and $w$ as $Z \to 0^+$, see Lemma 3.1 in [42],

$$\lim_{Z \to 0^+} Z^2(-D_a(Z)) = \lim_{Z \to 0^+} Z^2\sigma^2 = 1 > 0.$$ This concludes the proof of (3.10). \hfill \Box

3.4. Commuting with derivatives. We define

$$T_k = \Delta^k T, \quad \Phi_k = \Delta^k \Phi.$$

**Lemma 3.3** (Commuting with derivatives). Let $k \in \mathbb{N}$. There exists a smooth measure $g$ defined for $Z \in [0, Z_a]$ such that the following holds. Let the elliptic operator

$$\mathcal{L}_g \Phi_k = \frac{\mu^2}{g(Z)} \partial_Z \left( Z^{d-1} Z^2 g(\partial_Z) \partial_Z \Phi_k \right),$$

then there holds

$$\Delta^k (MX) = \begin{aligned} m_k \bigg|_{T_k} + \tilde{m}_k X \end{aligned} \tag{3.20}$$

with

$$m_k \bigg|_{T_k} = \begin{cases} \frac{1}{2} \sum_{j=0}^{2k-1} |\partial_j^Z \Phi|, \\
\frac{1}{2} \sum_{j=0}^{2k-1} |\partial_j^Z \Phi| + \sum_{j=0}^{2k-1} |\partial_j^Z T|, \end{cases} \tag{3.21}$$

Moreover, $g > 0$ in $[0, Z_a)$ and admits the asymptotics:

$$\begin{align*}
g(Z) &= 1 + O(Z^2) \quad \text{as} \quad Z \to 0 \\
g(Z) &= c_{a,d,r} (Z_a - Z)^{c_\ast} [1 + O(Z - Z_a)] \quad \text{as} \quad Z \uparrow Z_a, \tag{3.22} \end{align*}$$

with

$$c_g > 0 \tag{3.23}$$

for all $k \geq k_1$ large enough and $0 < a < a^*$ small enough.

**Proof.** This is a direct computation.

**step 1** Proof of (3.20), (3.21). We recall (C.1):

$$[\Delta^k, V] \Phi - 2k \nabla V \cdot \nabla \Delta^{k-1} \Phi = \sum_{|\alpha| + |\beta| = 2k, |eta| \leq 2k-2} c_{k, \alpha, \beta} \partial^\alpha V \partial^\beta \Phi$$
which together with the commutator formulas
\[
\begin{align*}
[\Delta^k, \Lambda] &= 2k\Delta^k, \quad [\partial_Z, \Lambda] = \partial_Z \\
\Lambda^2 &= Z^2\Delta - (d-2)\Lambda \\
\partial_Z\Lambda &= \frac{\Lambda^2}{Z} = Z\Delta - (d-2)\partial_Z
\end{align*}
\]
\tag{3.24}
\]
yields
\[
\Delta^k(V\Lambda\Phi) = V\Delta^k(\Lambda\Phi) + 2k\nabla V \cdot \nabla \Delta^{k-1}\Lambda\Phi + \sum_{|\alpha|+|\beta|=2k,|\beta|\leq 2k-2} c_{k,\alpha,\beta} \partial^\alpha V \partial^\beta \Lambda\Phi.
\]
and
\[
2k\nabla V \cdot \nabla \Delta^{k-1}\Lambda\Phi = 2k\partial_Z V \partial_Z \left[ \Lambda \Delta^{k-1}\Phi + 2(k-1)\Phi_{k-1} \right]
\]
\[
= 2k\partial_Z V \left[ (Z\Delta - (d-2)\partial_Z)\Phi_{k-1} + 2(k-1)\partial_Z\Phi_{k-1} \right]
\]
\[
= 2k\Lambda V \Phi_k + 2k(2k-2-d+2)\partial_Z V \partial_Z\Phi_{k-1}
\]
from which for \(0 \leq Z \leq Z_a\):
\[
\Delta^k(V\Lambda\Phi) = V(2k+\Lambda)\Phi_k + 2k\Lambda V \Phi_k + O_{k,a} \left( \sum_{j=0}^{2k-1} |\partial_j^Z \Phi| \right).
\]
We then use
\[
[\Delta^k, \Lambda^2] = \Delta^k \Lambda^2 - \Lambda^2 \Delta^k = [\Delta^k, \Lambda] \Lambda + \Lambda \Delta^k \Lambda - \Lambda(-[\Delta^k, \Lambda] + \Delta^k \Lambda)
\]
\[
= 2k\Delta^k \Lambda + 2k\Lambda \Delta^k = 4k^2 \Delta^k + 4k\Lambda \Delta^k
\]
to compute similarly:
\[
\Delta^k(V\Lambda^2\Phi) = V\Delta^k(\Lambda^2\Phi) + 2k\nabla V \cdot \nabla \Delta^{k-1}\Lambda^2\Phi + \sum_{|\alpha|+|\beta|=2k,|\beta|\leq 2k-2} c_{k,\alpha,\beta} \partial^\alpha V \partial^\beta \Lambda\Phi
\]
\[
= V \left[ \Lambda^2 \Phi_k + 4k^2 \Phi_k + 4k\Lambda \Phi_k \right] + 2k\partial_Z V \partial_Z \Delta^{k-1}\Lambda^2\Phi + O \left( \sum_{j=0}^{2k} |\partial_j^Z \Phi| \right)
\]
and
\[
\partial_Z \Delta^{k-1}\Lambda^2\Phi = \partial_Z \left[ \Lambda^2 \Phi_{k-1} + 4(k-1)^2 \Phi_{k-1} + 4(k-1)\Lambda \Phi_{k-1} \right]
\]
\[
= \partial_Z \left( Z^2 \Phi_k - (d-2)\Lambda \Phi_{k-1} \right) + O \left( \sum_{j=0}^{2k} |\partial_j^Z \Phi| \right) = Z\Lambda \Phi_k + O \left( \sum_{j=0}^{2k} |\partial_j^Z \Phi| \right)
\]
and hence
\[
\Delta^k(V\Lambda^2\Phi) = V \left[ \Lambda^2 \Phi_k + 4k^2 \Phi_k + 4k\Lambda \Phi_k \right] + 2k\Lambda V \Lambda \Phi_k + O \left( \sum_{j=0}^{2k} |\partial_j^Z \Phi| \right).
\]
Recalling the definition of the operator (3.8), we obtain (3.20), (3.21) with
\[
M_k = \begin{vmatrix}
\Phi_k \\
T_k
\end{vmatrix} = \begin{vmatrix}
-aH_2\Lambda \Phi_k - 2ak(H_2 + \Lambda H_2)\Phi_k + T_k \\
(p-1)Q\Delta \Phi_k - (1-a)^2 H_2^2 \Lambda^2 \Phi_k + A_k \Lambda \Phi_k \\
-(2-a)H_2 \Lambda T_k - 2k(2-a)(H_2 + \Lambda H_2)T_k + A_2 T_k
\end{vmatrix}
\]
\[
A_k = 2k(p-1)\frac{\partial_Z Q}{Z} - (1-a)^2 4kH_2 [H_2 + \Lambda H_2] + \tilde{A}_2.
\]
**step 2** Equation for the measure. We compute using (3.11), (3.9):

\[
(p - 1)Q\Delta \Phi_k - (1 - a)^2 H_2^2 \Lambda^2 \Phi_k
\]

\[
= \mu^2 Z^2 \sigma^2 \left( \partial_Z^2 \Phi_k + \frac{d - 1}{Z} \partial_Z \Phi_k \right) - \mu^2 (1 - w)^2 (1 - a)^2 (Z^2 \partial_Z^2 \Phi_k + \Lambda \Phi_k)
\]

\[
= \mu^2 \left\{ -Z^2 D_a \partial_Z^2 \Phi_k + Z \partial_Z \Phi_k \left[ (d - 1) \sigma^2 - (1 - a)^2 (1 - w)^2 \right] \right\}
\]

and hence:

\[
A_k = 2k(p - 1) \frac{\partial_Z Q}{Z} - (1 - a)^2 4kh_2 [H_2 + \Lambda h_2] + \tilde{A}_2
\]

\[
= 4k\mu^2 \left[ \sigma F - (1 - a)^2 (1 - w) + (1 - a)^2 (1 - w) (w + \Lambda w) \right] + \tilde{A}_2
\]

and hence:

\[
(p - 1)Q\Delta \Phi_k - (1 - a)^2 H_2^2 \Lambda^2 \Phi_k + A_k \Lambda \Phi_k = -\mu^2 Z^2 D_a \partial_Z^2 \Phi_k
\]

\[
-\mu^2 \left[ (d - 1) \sigma^2 - (1 - a)^2 (1 - w)^2 \right]
\]

\[
= \mu^2 \left[ (d - 1) \sigma^2 - (1 - a)^2 (1 - w)^2 \right] + 4k\mu^2 \left[ \sigma F - (1 - a)^2 (1 - w) + (1 - a)^2 (1 - w) (w + \Lambda w) \right] + \tilde{A}_2.
\]

Equivalently:

\[
\left(-D_a\right) \frac{\Lambda g}{g} = -G
\]

(3.25)

with

\[
G = -(d - 1) \sigma^2 + (1 - w)^2 - (d + 1) D_a - \Lambda D_a
\]

\[
+ 4k \left[ (1 - a)^2 (1 - w) - \sigma F - (1 - a)^2 (1 - w) (w + \Lambda w) \right] - \frac{\tilde{A}_2}{\mu^2}.
\]

**step 3** Asymptotics of the measure. We now solve (3.25). Near the origin, the normalization (2.15) and (3.11) yield

\[
\sigma = \sqrt{p - 1} \mu Z \left[ 1 + O(Z^3) \right], \quad F = \sigma + \Lambda \sigma = O(Z), \quad -D_a = \sigma^2 + O(1).
\]

We compute

\[
\frac{\tilde{A}_2}{\mu^2} = O \left( \frac{|F|}{\sigma} + |\Lambda w| + |a| \right) = O(1)
\]

and hence

\[
G = -(d - 1) \sigma^2 - (d + 1)(-\sigma^2) - \Lambda(-\sigma^2) + O \left( 1 + |a| + \sigma |F| + |w| + |\Lambda w| \right)
\]

\[
= 2\sigma F + O(1) = O(1)
\]

which, recalling (3.10), yields:

\[
-\frac{G}{(-D_a)} = \frac{O(1)}{\sigma^2 + O(1)} = O(Z^2)
\]
and we may therefore choose explicitly:

\[ g = e^{\int_0^Z [-\frac{G}{-Da}] d\tau}. \]

To compute the behavior near \( Z_a \), recall from (3.18) (3.19) that we have

\[ D_a(Z_a) = 0, \quad D'_a(Z_a) > 0. \]

We infer in the neighborhood of \( Z = Z_a \)

\[ \partial_Z g \] \( \frac{g}{ZD_a} = \frac{G(Z_a) 1 + O(Z - Z_a)}{Z - Z_a} \]

\[ = \left( \frac{G(Z_2)}{\Lambda D_0(Z_2)} + O(|a|) \right) \frac{1 + O(Z - Z_a)}{Z - Z_a}. \] (3.27)

The fundamental computation is then at \( P_2 \) using (3.16):

\[ [(1 - w) - \sigma F - (1 - w)(w + \Lambda w)] = (1 - w_2)(1 - w_2 - \Lambda w) - \sigma_2(\sigma_2 + \Lambda \sigma) \]

\[ = \sigma_2(\sigma_2 - \Lambda w) - \sigma_2(\sigma_2 + \Lambda \sigma) = \sigma_2(-\Lambda w - \Lambda \sigma) \]

\[ = \sigma_2 \left[ -\frac{|c_-||\lambda_+|}{2\sigma_2(1 + c_-)} + \frac{|\lambda_+|}{2\sigma_2(1 + c_-)} \right] = \frac{|\lambda_+|}{2} > 0 \]

Hence from (3.26)

\[ G(Z_2) = 2k(|\lambda_+| + O(a)) + O(1). \]

and from (3.17)

\[ \frac{G(Z_2)}{\Lambda D_0(Z_2)} = \frac{2k(|\lambda_+| + O(a)) + O(1)}{|\lambda_+|} > k \]

for \( 0 < a < a^* \) small enough and \( k \geq k_1 \) large enough. Inserting this into (3.27) yields (3.22). \( \square \)

3.5. Hardy inequality and compactness. We let \( k \geq k_1 \) large enough so that (3.23) holds and extend the measure \( g \) by zero for \( Z \geq Z_a \). We let \( \chi \) be a smooth cut off function supported strictly inside the light cone \( |Z| < Z_2 \) with

\[ g \geq \frac{1}{2} \text{ on } \text{Supp} \chi. \]

Let

\[ \mathcal{D}_\Phi = \{ \Phi \in C^\infty([0, Z_2], \mathbb{C}) \text{ with spherical symmetry} \}, \]

be the space of test functions and

\[ \langle \langle \Phi, \hat{\Phi} \rangle \rangle = -(L_g \Phi, \hat{\Phi})_g + \int \chi \Phi \bar{\Phi} g Z^{d-1} dZ \] (3.28)

be a Hermitian scalar product, where we recall the notation (1.12). We let \( H_{\Phi} \) be the completion of \( \mathcal{D}_\Phi \) for the norm associated to (3.28). We claim the following compactness subcoercivity estimate:

**Lemma 3.4** (Subcoercivity estimate). For \( 0 < \nu \ll 1 \):

\[ \langle (\Phi, \bar{\Phi}) \rangle \geq \int \frac{|\Phi|^2}{Z_a - Z} g Z^{d-1} dZ + \sum_{m=0}^{2k} \int |\partial_Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1 - \nu}} Z^{d-1} dZ. \] (3.29)
Furthermore, there exists \( c > 0 \) and a sequence \( \mu_n > 0 \) with \( \lim_{n \to +\infty} \mu_n = +\infty \) and \( \Pi_n \in \mathbb{H}_\Phi, c_n > 0 \) such that \( \forall n \geq 0, \forall \Phi \in \mathbb{H}_\Phi, \)

\[
\langle \langle \Phi, \Phi \rangle \rangle \geq c \int \frac{|\Phi_k|^2}{Z_a - Z} g Z^{d-1} dZ + \mu_n \sum_{m=0}^{2k} \int |\partial_Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z^{d-1} dZ
- c_n \sum_{i=1}^n (\Phi, \Pi_i)^2.
\]

(3.30)

**Proof.** This is a classical Hardy and Sobolev based argument.

**step 1** Interior estimate. Let \( Z_0 < Z_a \) which will be chosen close enough to \( Z_a \) in step 2. Then, we have

\[
\int_0^{Z_0} \frac{|\Phi_k|^2}{Z_a - Z} g Z^{d-1} dZ + \sum_{m=0}^{2k} \int_0^{Z_0} |\partial_Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z^{d-1} dZ

\leq C_{Z_0} \|\Phi\|_{H^{2k}(0,Z_0)} \leq C_{Z_0} \left[ \int_0^{Z_0} |\partial_Z \Phi_k|^2 Z^{d-1} dZ + \int_0^{Z_0} \chi |\Phi(Z)|^2 Z^{d-1} dZ \right].
\]

Since \(-Z^2 D_a\) and \(g\) are smooth and satisfy \(-Z^2 D_a > 0\) and \(g > 0\) on \([0, Z_0]\), we infer

\[
\langle \langle \Phi, \Phi \rangle \rangle \geq c_{Z_0} \left[ \int_0^{Z_0} \frac{|\Phi_k|^2}{Z_a - Z} g Z^{d-1} dZ + \sum_{m=0}^{2k} \int_0^{Z_0} |\partial_Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z^{d-1} dZ \right]
\]

for some \( c_{Z_0} > 0 \). Thus, to prove (3.29), it remains to consider the region \((Z_0, Z_a)\). This will be done in step 2 and step 3.

**step 2** Hardy inequality with loss. Let \( 0 < \nu \ll 1 \), we claim the lossy Hardy bound for all \( \Phi \in \mathcal{D}_\Phi \):

\[
\sum_{m=0}^{2k} \int_{Z_0}^{Z_a} |\partial_Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z^{d-1} dZ \leq c_\nu \langle \langle \Phi, \Phi \rangle \rangle.
\]

(3.31)

Indeed, let \( Z_0 = Z_a - \delta \) with \( \delta > 0 \) small enough, we estimate by Taylor expansion for \( Z_0 \leq Z < Z_a \) for \( 0 \leq m \leq 2k \):

\[
|\partial_Z^m \Phi(Z)|^2 \leq C \left[ \sum_{j=m}^{2k} \left| \partial_Z^j \Phi(Z_0) \right|^2 + \left( \int_{Z_0}^Z \left| \partial_Z^{2k+1} \Phi(\tau) \right| d\tau \right)^2 \right]
\]

From Sobolev,

\[
\sum_{j=m}^{2k} \left| \partial_Z^j \Phi(Z_0) \right|^2 \leq C_{Z_0} \|\Phi\|_{H^{2k+1}(0,Z_0)}^2 \leq C_{Z_0} \langle \langle \Phi, \Phi \rangle \rangle
\]
and hence
\[
|\partial_Z^m \Phi(Z)|^2 \leq C_{Z_0} \langle \langle \Phi, \Phi \rangle \rangle + C \left( \int_{Z_0}^Z |\partial_Z \Delta^k \Phi(\tau)| d\tau \right)^2 + C \left( \int_{Z_0}^Z \sum_{j=1}^{2k} |\partial_Z^j \Phi| d\tau \right)^2
\]
\[
\leq C_{Z_0} \langle \langle \Phi, \Phi \rangle \rangle + C \left( \int_{Z_0}^Z |\partial_Z \Phi_k|^2 (Z_a - Z)^{1-\nu} Z dZ \right) \left( \int_{Z_0}^Z \frac{d\tau}{(Z_a - \tau)^{1-\nu}} \right) + C \sum_{j=1}^{2k} \left( \int_{Z_0}^Z \frac{|\partial_Z^j \Phi|^2}{(Z_a - Z)^{1-\nu}} d\tau \right)
\]
where we used the fact that 0 < \nu < 1 and Z_a - Z_0 = \delta. Using again 0 < \nu < 1 and Z_a - Z_0 = \delta, as well as Fubini and the fact that \partial_Z g < 0 on (Z_0, Z_a) for Z_0 close enough to Z_a so that g is decreasing on (Z_0, Z_a), we infer
\[
\sum_{m=0}^{2k} \int_{Z_0}^{Z_a} |\partial_Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z dZ
\]
\[
\leq C_{Z_0} \langle \langle \Phi, \Phi \rangle \rangle + C \delta^{\nu} \int_{Z_0}^{Z_a} \frac{g}{(Z_a - Z)^{1-\nu}} \left( \int_{Z_0}^Z |\partial_Z \Phi_k|^2 (Z_a - \tau)^{1-\nu} d\tau \right) Z dZ
\]
\[
+ C \delta^{\nu} \int_{Z_0}^{Z_a} \frac{g}{(Z_a - Z)^{1-\nu}} \left( \int_{Z_0}^Z \frac{|\partial_Z^j \Phi|^2}{(Z_a - \tau)^{1-\nu}} d\tau \right) Z dZ
\]
\[
\leq C_{Z_0} \langle \langle \Phi, \Phi \rangle \rangle + C \delta^{\nu} \int_{Z_0}^{Z_a} |\partial_Z \Phi_k|^2 g(\tau)(Z_a - \tau)^{1-\nu} \left( \int_{\tau}^{Z_a} \frac{dZ}{(Z_a - Z)^{1-\nu}} \right) \tau d\tau
\]
\[
+ C \delta^{\nu} \int_{Z_0}^{Z_a} \frac{|\partial_Z^j \Phi|^2 g(\tau)}{(Z_a - Z)^{1-\nu}} \left( \int_{\tau}^{Z_a} \frac{dZ}{(Z_a - Z)^{1-\nu}} \right) \tau d\tau
\]
\[
\leq C_{Z_0} \langle \langle \Phi, \Phi \rangle \rangle + C \delta^{\nu} \int_{Z_0}^{Z_a} |\partial_Z \Phi_k|^2 g(\tau)(Z_a - \tau)^{d-1} d\tau + C \delta^{\nu} \int_{\tau}^{Z_a} \frac{|\partial_Z^j \Phi|^2 g(\tau)}{(Z_a - \tau)^{1-\nu}} \tau d\tau.
\]
Letting \delta = \delta(\nu) small enough and estimating from (3.19)
\[
-(D_a)(Z) \geq c(Z_a - Z)
\]
yields (3.31).

**step 3** Sharp Hardy. We now claim the sharp Hardy inequality for f \in D_k:
\[
\langle \langle \Phi, \Phi \rangle \rangle \geq \int_{Z_0}^{Z_a} \frac{|\Phi_k|^2}{(Z_a - Z)} g Z dZ
\]
Indeed, recall (3.22), (3.23) near Z_a:
\[
g(Z) = c(Z_a - Z)^{\nu} [1 + O(Z - Z_0)],
\]
then integrating by parts:
\[
\int_{Z_0}^{Z_a} \frac{|\Phi_k|^2}{(Z_a - Z)} gZ^{d-1} dZ \lesssim \int_{Z_0}^{Z_a} |\Phi_k|^2 (Z_a - Z)^{c_g-1} dZ
\]
\[
= -\frac{1}{c_g} |\Phi_k|^2 (Z_a - Z)^{c_g} |Z_0^{Z_a} + \frac{1}{c_g} \int_{Z_0}^{Z_a} 2\Phi_k \partial Z \Phi_k (Z_a - Z)^{c_g} dZ
\]
\[
\lesssim |\Phi_k|^2 (Z_0) + \left( \int |\Phi_k|^2 (Z_a - Z)^{c_g-1} Z^{d-1} dZ \right)^{\frac{1}{2}} \left( \int |\partial Z \Phi_k|^2 (Z_a - Z)^{c_g+1} Z^{d-1} dZ \right)^{\frac{1}{2}}
\]
\[
\lesssim \langle \langle \Phi, \Phi \rangle \rangle + \left( \int_{Z_0}^{Z_a} \frac{|\Phi_k|^2}{(Z_a - Z)} gZ^{d-1} dZ \right)^{\frac{1}{2}} \left( \int |\partial Z \Phi_k|^2 g(-D_a) Z^{d-1} dZ \right)^{\frac{1}{2}}
\]
where we used (3.32). The bound (3.33) now follows using Hölder. Together with step 1 and step 2, this concludes the proof of (3.29).

**step 4** Compactness. We now turn to the proof of (3.30) which follows from a standard compactness argument. Let us consider \( T \in L_g^2 \). Then from (3.29), the antilinear form
\[
h \mapsto (T, h)_g,
\]
is continuous on \( \mathbb{H}_g \), and hence by Riesz, there exists a unique \( L(T) \in \mathbb{H}_g \) such that
\[
\forall h \in \mathbb{H}_g, \quad \langle \langle L(T), h \rangle \rangle = (T, h)_g, \quad (3.34)
\]
and the linear map \( L \) is bounded from \( L^2_g \) to \( \mathbb{H}_g \). For any \( 0 < \delta < Z_a \), we have in view of (3.29)
\[
\|h\|_{L^2_g} \leq \delta^{\frac{1}{2d}} \left( \left\| \frac{h}{(Z_a - Z)^{\frac{1}{2d}}} \right\|_{L^2_g} + \|h\|_{L^2_g(Z \leq Z_a - \delta)} \right) \lesssim \delta^{\frac{1}{2d}} \|h\|_{\mathbb{H}_g} + 1 \|h\|_{L^2_g(Z \leq Z_a - \delta)}.
\]
Relying on the smallness of \( \delta^{\frac{1}{2d}} \) for the first term, and Rellich for the second one, we easily infer that
\[
\mathbb{H}_g \text{ embeds compactly in } L^2_g. \quad (3.35)
\]
Since \( L \) is bounded from \( L^2_g \) to \( \mathbb{H}_g \), we infer that the map
\[
L : L^2_g \mapsto L^2_g
\]
is compact. Moreover, if \( \Phi_1 = L(T_1), \Phi_2 = L(T_2) \):
\[
(L(T_1), T_2)_g = (\Phi_1, T_2)_g = (\Phi_2, T_1)_g = \langle \langle LT_2, \Phi_1 \rangle \rangle = \langle \langle \Phi_1, \Phi_2 \rangle \rangle
\]
and hence interchanging the roles of \( T_1, T_2 \):
\[
(T_1, L(T_2))_g = (LT_2, T_1)_g = \langle \langle \Phi_2, \Phi_1 \rangle \rangle = \langle \langle \Phi_1, \Phi_2 \rangle \rangle = \langle \langle \Phi_1, \Phi_2 \rangle \rangle = \langle \langle L(T_1), T_2 \rangle \rangle
\]
and \( L \) is selfadjoint on \( L^2_g \). Since \( L \geq 0 \) from (3.34), we conclude that \( L \) is a diagonalizable with a non increasing sequences of eigenvalues \( \lambda_n > 0, \lim_{n \to +\infty} \lambda_n = 0 \), and let \( (\Pi_{n,i})_{1 \leq i \leq I(n)} \) be an \( L^2_g \) orthonormal basis for the eigenvalue \( \lambda_n \). The eigenvalue equation implies \( \Pi_{n,i} \in \mathbb{H}_g \).

Let then
\[
\mathcal{A}_n = \left\{ \Phi \in \mathbb{H}_g, \int |\Phi|^2 gZ^{d-1} dZ = 1, \ (\Phi, \Pi_{j,i})_g = 0, \ 1 \leq i \leq I(j), \ 1 \leq j \leq n \right\}
\]
and the minimization problem
\[
I_n = \inf_{\Phi \in \mathcal{A}_n} \langle \langle \Phi, \Phi \rangle \rangle,
\]
then the infimum is attained in view of (3.35) at \( \Phi \in \mathcal{A}_n \) and, by a standard Lagrange multiplier argument:

\[
\forall h \in \mathbb{H}_\Phi, \quad \langle \langle \Phi, h \rangle \rangle = \sum_{j=1}^{n} \sum_{i=1}^{I(j)} \alpha_{i,j}(\Pi_{j,i}, h)_g + \alpha(\Phi, h)_g.
\]

Letting \( h = \Pi_{i,j} \) implies \( \alpha_{i,j} = 0 \) and hence from (3.34):

\[
L(\Phi) = \frac{1}{\alpha} \Phi
\]

which together with our orthogonality conditions implies

\[
\frac{1}{\alpha} \leq \lambda_{n+1}
\]

and hence

\[
I_n = \langle \langle \Phi, \Phi \rangle \rangle = \alpha \langle \langle L(\Phi), \Phi \rangle \rangle = \alpha(\Phi, \Phi)_g = \alpha \geq \frac{1}{\lambda_{n+1}}. \tag{3.36}
\]

Also, for \( Z_0 = Z_a - \delta \) with \( \delta > 0 \) small enough, we estimate from (3.31)

\[
\sum_{m=0}^{2k} \int_{Z_0}^{Z_a} |\partial Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z^{d-1} dZ \leq c_\nu \delta^{\frac{\nu}{2}} \langle \langle \Phi, \Phi \rangle \rangle.
\]

On the other hand, from Rellich and an elementary compactness argument, for all \( Z_a > 0, \delta > 0, \epsilon > 0, k \geq 1 \), there exists \( c_{Z_a,\delta,\epsilon,k} > 0 \) such that

\[
\sum_{m=0}^{2k} \int_{Z \leq Z_a - \delta} |\partial Z^m \Phi(Z)|^2 Z^{d-1} dZ \leq \epsilon \int_{Z \leq Z_a - \delta} |\partial Z \Delta^k \Phi|^2 Z^{d-1} dZ + c_{Z_a,\delta,\epsilon,k} \int_{Z \leq Z_a - \delta} |\Phi|^2 Z^{d-1} dZ.
\]

Summing the two inequalities yields for all \( \delta > 0 \) small and \( \epsilon \) smaller still:

\[
\sum_{m=0}^{2k} \int_{Z=0}^{Z_a} |\partial Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z^{d-1} dZ \leq c_\nu \delta^{\frac{\nu}{2}} \langle \langle \Phi, \Phi \rangle \rangle + \tilde{c}_{Z_a,\delta,k} \int_0^{Z_a} |\Phi|^2 g Z^{d-1} dZ.
\]

Together with (3.36), this implies for any \( \Phi \) satisfying the orthogonality conditions \( (\Phi, \Pi_{j,i})_g = 0, \ 1 \leq i \leq I(j), \ 1 \leq j \leq n, \) and for any \( \delta > 0 \)

\[
\sum_{m=0}^{2k} \int_{Z=0}^{Z_a} |\partial Z^m \Phi(Z)|^2 \frac{g}{(Z_a - Z)^{1-\nu}} Z^{d-1} dZ \leq \left( c_\nu \delta^{\frac{\nu}{2}} + \tilde{c}_{Z_a,\delta,k} \lambda_{n+1} \right) \langle \langle \Phi, \Phi \rangle \rangle
\]

which yields (3.30).

\[\square\]

### 3.6. Acclativity

We now turn to the proof of the accretivity of the operator \( \mathcal{M} \).

Hilbert space. Recall (3.28). We define the space of test functions

\[\mathcal{D}_0 = \mathcal{D}_\Phi \times C_{radial}^\infty([0, Z_a], \mathbb{C}),\]

and let \( \mathbb{H}_{2k} \) be the completion of \( \mathcal{D}_0 \) for the scalar product:

\[
\langle X, \bar{X} \rangle = \langle \langle \Phi, \Phi \rangle \rangle + (T_k, \bar{T}_k)_g + \int \chi T \bar{T} Z^{d-1} dZ \tag{3.37}
\]

which is a coercive Hermitian form from (3.29).

Unbounded operator. Following (3.8), we define the operator

\[
\mathcal{M} = \left( \begin{array}{cc}
-a_2 \Lambda \\
(p - 1)_{Q\Delta} - (1 - a^2) H_2^2 \Lambda^2 + \tilde{A}_2 \Lambda + A_3 & 1 - (2 - a) H_2 \Lambda + A_2
\end{array} \right)
\]
with domain
\[ D(M) = \{ X \in \mathbb{H}_{2k}, \; MX \in \mathbb{H}_{2k} \} \] (3.38)
equipped with the domain norm. We then pick suitable directions \((X_i)_{1 \leq i \leq N} \in \mathbb{H}_{2k}\) and consider the finite rank projection operator
\[ A = \sum_{i=1}^{N} \langle \cdot, X_i \rangle X_i. \]
The aim of this section is to prove the following accretivity property:

**Proposition 3.5** (Maximal accretivity/dissipativity). Let
\[ \mu, r > 0. \]
There exist \(k^* \gg 1\) and \(0 < c^*, a^* \ll 1\) such that for all \(k \geq k^*\), \(\forall 0 < a < a^*\) small enough, there exist \(N = N(k,a)\) directions \((X_i)_{1 \leq i \leq N} \in \mathbb{H}_{2k}\) such that the modified unbounded operator
\[ \tilde{M} = M - A \]
is dissipative:\[ \forall X \in D(M), \; \Re(-\tilde{M}X,X) \geq c^*ak(X,X) \] (3.39)and maximal:
\[ \forall R > 0, \; \forall F \in \mathbb{H}_{2k}, \; \exists X \in D(M) \text{ such that } (-\tilde{M} + R)X = F. \] (3.40)

**Remark 3.6.** We recall that maximal dissipative operators are closed.

**Proof of Proposition 3.5.** Given \(R > R^*(k)\) large enough, we define the space of test functions
\[ D_R := \left\{ X = (\Phi, T), \; X \in C^{\chi\mathcal{R}}([0, Z_a]) \times C^{\chi\mathcal{R}}([0, Z_a]) \right\} \cap \left\{ X \mid (-\tilde{M} + R)X \in C^\infty([0, Z_a]) \times C^\infty([0, Z_a]) \right\}. \] (3.41)
In steps 1 to 3 below, we prove (3.39) for \(X \in D_R\) so that all integrations by parts in steps 1 to 3 are justified, and all boundary terms at \(Z = Z_a\) vanish due to the vanishing of \(g\) at \(Z = Z_a\). In steps 4 and 5, for any smooth \(F\) on \([0, Z_a]\), we show existence and uniqueness of a solution \(X \in \mathbb{H}_{2k}\) to \((-\tilde{M} + R)X = F\) for \(R > R^*(k)\) large enough. In step 6, we prove that \(D_R\) is dense in \(D(M)\). In step 7, we conclude the proof of (3.39) and (3.40).

**step 1** Main integration by parts. Let \(X \in D_R\) for \(R > R^*(k)\) large enough. We aim at proving (3.39) and split the computation in two:
\[
\begin{align*}
\langle X, \tilde{X} \rangle_1 &= -(\mathcal{L}_g \Phi_k, \Phi_k)_g + (T_k, \tilde{T}_k)_g, \\
\langle X, \tilde{X} \rangle_3 &= \int \chi \Phi \Phi + \int \chi T \tilde{T}.
\end{align*}
\]

5Equivalently, \(-\tilde{M}\) is accretive.
In step 1, we consider the principal part. We compute from (3.20):

$$- \Re (\mathcal{M}X, X) = \Re (\mathcal{L}_g \Delta^k (\mathcal{M}X)_\Phi, \Phi_k)_g - \Re (\Delta^k (\mathcal{M}X)_T, T_k)_g$$

$$= - \Re \left\{ \nabla \left[ -a H_2 M_k - 2ak(H_2 + \Lambda H_2) \Phi_k + T_k + (\hat{M}_k X)_\Phi \right] - \Re \Phi_k Z^2(-D_a)Z^{d-1} g dZ \right\}$$

$$= - \Re \left\{ \int \left[ \mathcal{L}_g \Phi_k - (2-a)H_2 \Lambda T_k - 2k(2-a)(H_2 + \Lambda H_2) T_k + A_2 T_k + (\hat{M}_k X)_\Phi \right] T_k dZ \right\}$$

$$= - \Re \left\{ \int \nabla \left[ -a H_2 M_k - 2ak(H_2 + \Lambda H_2) \Phi_k + (\hat{M}_k X)_\Phi \right] \cdot \nabla \Phi_k Z^2(-D_a)Z^{d-1} g dZ \right\}$$

$$= - \Re \left\{ \int [-(2-a)H_2 \Lambda T_k - 2k(2-a)(H_2 + \Lambda H_2) T_k + A_2 T_k + (\hat{M}_k X)_T] T_k dZ \right\}.$$ 

\(T_k\) terms. We use

$$\Re \left( \int f h \Xi h \right) = - \frac{1}{2} \int |h|^2 f \left( d + \frac{\Lambda f}{f} \right)$$

to compute

$$- \Re \left\{ \int [-(2-a)H_2 \Lambda T_k] T_k dZ \right\} = - \frac{2-a}{2} \int |T_k|^2 g H_2 \left( d + \frac{\Lambda H_2}{H_2} \right)$$

and hence

$$- \Re \left\{ \int [-(2-a)H_2 \Lambda T_k - 2k(2-a)(H_2 + \Lambda H_2) T_k + A_2 T_k] T_k dZ \right\} = (2-a) \int A_5 H_2 |T_k|^2 g dZ$$

with

$$A_5 := - \frac{1}{2} \left[ d + \frac{\Lambda g}{g} + \frac{\Lambda H_2}{H_2} \right] + 2k \left( 1 + \frac{\Lambda H_2}{H_2} \right) - \frac{A_2}{(2-a) H_2}. \quad (3.42)$$

\(\Phi_k\) terms. We first compute:

$$- \Re \left\{ \int \nabla \left[ -2ak(H_2 + \Lambda H_2) \Phi_k \right] \cdot \nabla \Phi_k Z^2(-D_a)Z^{d-1} g dZ \right\}$$

$$= 2ak \int (H_2 + \Lambda H_2) |\nabla \Phi_k|^2 Z^2(-D_a)Z^{d-1} g dZ$$

$$+ 2ak \Re \left\{ \int \Phi_k \nabla (H_2 + \Lambda H_2) \cdot \nabla \Phi_k Z^2(-D_a)Z^{d-1} g dZ \right\}$$

$$= 2ak \int (H_2 + \Lambda H_2) |\nabla \Phi_k|^2 Z^2(-D_a)Z^{d-1} g dZ$$

$$- ak \int |\Phi_k|^2 \nabla \cdot (Z^2(-D_a) \nabla (H_2 + \Lambda H_2) g) Z^{d-1} dZ.$$ 

For the second term:

$$- \Re \left\{ \int \nabla \left[ -a H_2 \Lambda \Phi_k \right] \cdot \nabla \Phi_k Z^2(-D_a)Z^{d-1} g dZ \right\} = - a \Re \left\{ \int \partial_Z (H_2 \Lambda \Phi_k) H_2 \bar{\Phi}_k \frac{D_a Z^d}{H_2} g dZ \right\}$$

$$= \frac{a}{2} \int |H_2 \Lambda \Phi_k|^2 \frac{D_a Z^d}{H_2} g \left( \frac{\partial Z D_a}{D_a} + \frac{d}{Z} - \frac{\partial Z H_2}{H_2} + \frac{\partial Z g}{g} \right) dZ$$

$$= - \frac{a}{2} \int |\partial_Z \Phi_k|^2 H_2 \left( \frac{\Lambda D_a}{D_a} + \frac{d}{Z} - \frac{\Lambda H_2}{H_2} + \frac{\Lambda g}{g} \right) (-D_a) g Z^2 Z^{d-1} dZ.$$
We have therefore obtained the formula:

\[
-\Re\langle MX, X\rangle_1 = (2-a) \int A_5 H_2 |T_k|^2 g + \mu^2 a \int |\nabla \Phi_k|^2 A_6 Z^2(-D_a)Z^{d-1}gdZ
- \mu^2 ak \int |\Phi_k|^2 \nabla \cdot (Z^2(-D_a)\nabla (H_2 + \Lambda H_2)g) Z^{d-1}dZ
- \mu^2 \Re \left\{ \int \nabla(\overline{M_k}X)_\Phi \cdot \nabla \Phi_k Z^2(-D_a)Z^{d-1}gdZ \right\} - \Re \left\{ \int (\overline{M_k}X)_T T_k gdZ \right\}
\]

where we have defined

\[A_6 = 2k(H_2 + \Lambda H_2) - \frac{H_2}{2} \left( \frac{\Lambda D_a}{D_a} + d - \frac{\Lambda H_2}{H_2} + \frac{\Lambda g}{g} \right).\]

We now claim the following lower bounds on \(A_5, A_6\): there exist universal constants \(k^* \gg 1, 0 < c^*, a^* \ll 1\) such that for all \(k \geq k^*\) and \(0 < a < a^*\),

\[
\forall 0 \leq Z \leq Z_1, \quad \left| A_5 \right| \geq \frac{c^*k}{Z_a - Z}, \quad \left| A_6 \right| \geq \frac{c^*k}{Z_a - Z}
\]

Proof of (3.44). Recall (3.25), (3.26):

\[
-\frac{\Lambda g}{g} = \frac{1}{(D_a)^{-1}} \left\{ - (d-1)\sigma^2 + (1-w)^2 - (d+1)D_a - \Lambda D_a \right\} + \frac{4k}{(D_a)^{-1}} \left(1 - \frac{\sigma F}{(w + \Lambda w)} - \frac{\Lambda}{\mu^2}\right)
\]

and hence from (3.42):

\[
A_5 = -\frac{1}{2} \left[ d + \frac{\Lambda g}{g} + \frac{\Lambda H_2}{H_2} \right] + 2k \left(1 + \frac{\Lambda H_2}{H_2}\right) - \frac{A_2}{(2-a)H_2} + 2k \left(1 - \frac{\Lambda w}{1-w} + O\left(\frac{1}{k}\right)\right)
\]

We now compute for \(Z \leq Z_2\)

\[
(1-w) - \sigma F - (1-w)(w + \Lambda w) + (-\Delta) \left(1 - \frac{\Lambda w}{1-w}\right)
\]

\[
= (1-w)(1-w - \Lambda w) - \sigma F + \left(\sigma^2 - (1-w)^2\right) \frac{1-w - \Lambda w}{1-w}
\]

\[
= \sigma^2 \frac{1-w - \Lambda w}{1-w} - \sigma F = \frac{\sigma^2}{1-w} \left[1-w - \Lambda w - \frac{1-w}{\sigma}F\right]
\]

\[
\geq c\sigma^2
\]

from the fundamental coercivity bound (2.21), and hence for \(Z \leq Z_a\) and \(a < a^*\) small enough:

\[
A_5 \geq \frac{k\sigma^2}{(D_a)^{-1}} \geq \frac{k\sigma^*}{Z_a - Z}
\]
for some $c^*$ independent of $k, a$. Similarly:

$$A_6 = \frac{2kH_2}{-D_a} \left\{ 1 + \Lambda H_2 + O \left( \frac{1}{k} \right) \right\} (-D_a) + (1 - w) - \sigma F - (1 - w)(w + \Lambda w) + O \left( a + \frac{1}{k} \right)$$

$$= \frac{2k\mu(1 - w)}{(-D_a)} \left\{ (1 - w) - \sigma F - (1 - w)(w + \Lambda w) + (-\Delta) \left( 1 - \frac{\Lambda w}{1 - w} \right) + O \left( a + \frac{1}{k} \right) \right\} \geq \frac{kc^*}{Z_a - Z}$$

arguing as above. This concludes the proof of (3.44).

**step 2** No derivatives term. We compute

$$-\Re \langle MX, X \rangle_3 = -\Re \left\{ \int \chi(MX)_\Phi^2 Z^{d-1} dZ + \int \chi(MX)_T Z^{d-1} dZ \right\}$$

$$= -\Re \left\{ \int \chi \left[ -aH_2 \Delta \Phi + T \right] \Phi Z^{d-1} dZ \right\}$$

$$- \Re \left\{ \int \chi \left[ (p - 1)Q \Delta \Phi - (1 - a)^2 H_2^2 \Delta^2 \Phi + \tilde{A}_2 \Delta \Phi + A_3 \Phi - (2 - a)H_2 \Lambda T + A_2 T \right] \Phi \right\}$$

$$= O \left( \int (\chi + |\Lambda \chi|) \left( |\Phi|^2 + |\partial_z \Phi|^2 + |\partial^2_z \Phi|^2 + |T|^2 \right) \right).$$

(3.45)

**step 3** Accretivity in $\mathcal{D}_0$. We compute from (3.45), (3.43):

$$-\Re \langle MX, X \rangle = -\Re \langle MX, X \rangle_1 - \Re \langle MX, X \rangle_3$$

$$= (2 - a) \int A_5 H_2 |T_k|^2 g + \mu^2 a \int |\nabla \Phi_k|^2 A_6 Z^{d-1} g dZ$$

$$- \mu^2 ak \int |\Phi_k|^2 \nabla \cdot (Z^{d-1} \nabla (H_2 + \Lambda H_2) g) Z^{d-1} dZ$$

$$- \mu^2 \Re \left\{ \int \nabla (\tilde{m}_k X)_\Phi \cdot \nabla \Phi_k Z^{d-1} g dZ \right\} - \Re \left\{ \int (\tilde{m}_k X)_T \Phi_k g dZ \right\}$$

$$+ O \left( \int (\chi + |\Lambda \chi|) \left( |\Phi|^2 + |\partial_z \Phi|^2 + |\partial^2_z \Phi|^2 + |T|^2 \right) \right).$$

We lower bound from (3.44):

$$(2 - a) \int A_5 H_2 |T_k|^2 g + \mu^2 a \int |\nabla \Phi_k|^2 A_6 Z^{d-1} g dZ$$

$$\geq c^* ak \left[ \int \frac{|T_k|^2}{Z_a - Z} + |\nabla \Phi_k|^2 Z^{d-1} g Z^{d-1} dZ \right]$$

The smoothness and boundedness of the profile together with (3.25), (3.26) ensure that

$$|\nabla \cdot (Z^2 (-D_a) \nabla (H_2 + \Lambda H_2) g)| \leq C_k \frac{Z^2 (-D_a) g}{Z_a - Z} \leq C_k g$$
and in view of (3.21),
\[ -\Re \left\{ \int \nabla (\overline{m_k}X) \cdot \nabla \Phi_k Z^2 (-D_a) Z^{d-1} g dZ \right\} - \Re \left\{ \int (\overline{m_k}X)_{\Gamma} \overline{\Phi_k} g dZ \right\} \]
\[ \leq C_k \left( \int |\nabla \Phi_k|^2 Z^2 (-D_a) g Z^{d-1} dZ \right)^{\frac{1}{2}} \left( \sum_{j=0}^{2k} \int |\partial^j_{\alpha} \Phi|^2 g Z^{d-1} dZ \right)^{\frac{1}{2}} + C_k \left( \int |T_k|^2 g Z^{d-1} dZ \right)^{\frac{1}{2}} \]
\[ + C_k \left( \int |T_k|^2 g Z^{d-1} dZ \right)^{\frac{1}{2}} \left( \sum_{j=0}^{2k-1} \int |\partial^j_{\alpha} \Phi|^2 g Z^{d-1} dZ \right)^{\frac{1}{2}} + \left( \sum_{j=0}^{2k} \int |\partial^j_{\alpha} \Phi|^2 g Z^{d-1} dZ \right)^{\frac{1}{2}} \]

The collection of above bounds yields:
\[ -\Re \langle MX, X \rangle \geq c^a ak \left[ \int \frac{|T_k|^2}{Z_a} g Z^{d-1} dZ + \int |\nabla \Phi_k|^2 \frac{Z^2 (-D_a) g Z^{d-1} dZ}{Z_a} \right] \]
\[ - C_k \left[ \sum_{j=0}^{2k} \int |\partial^j_{\alpha} \Phi|^2 g Z^{d-1} dZ + \sum_{j=0}^{2k-1} \int |\partial^j_{\alpha} T|^2 g Z^{d-1} dZ \right] . \]

We conclude using (3.30) with \( N = N(a, k) \) large enough and its analogue for \( T \):
\[ -\Re \langle MX, X \rangle \geq c^a ak \langle X, X \rangle - C_{a,k} \sum_{i=1}^{N} \left( \langle \Phi, \Pi_i \rangle \right)^2 + (T, \mathcal{I}_i)^2 \).

Therefore,
\[ -\Re \langle (M - \mathcal{A}) X, X \rangle \geq c^a ak \langle X, X \rangle + \sum_{i=1}^{N} \left( \langle X, X_{i,1} \rangle^2 + \langle X, X_{i,2} \rangle^2 - C_{a,k} \left[ \langle \Phi, \Pi_i \rangle \right]^2 + (T, \mathcal{I}_i)^2 \right) . \]

The linear from
\[ X = (\Phi, T) \mapsto \sqrt{C_{a,k} \langle \Phi, \Pi_i \rangle} \]
from \( (\mathbb{H}_{2k}, \langle \cdot, \cdot \rangle) \) into \( \mathbb{C} \) is continuous from Cauchy-Schwarz and (3.29), and hence by Riesz theorem, there exists \( X_i \in \mathbb{H}_{2k} \) such that
\[ \forall X \in \mathbb{H}_{2k}, \ \langle X, X_i \rangle = (\Phi, \Pi_i) \mathbb{I}_g , \]
and similarly for \( \mathcal{I}_i \), and (3.39) follows for \( X \in D_R \).

**step 4** ODE formulation of maximality. Our goal, in steps 4 to step 6, is to prove that for all \( R > 0 \) large enough,
\[ \forall F \in C^\infty([0, Z_a]), \ \exists ! X \in \mathbb{H}_{2k} \text{ such that } (-M + R)X = F . \] (3.46)

(3.46) corresponds to solving
\[ -[a H_2 \Delta] \Phi - T + R \Phi = F_\Phi , \]
\[ \left\{ \begin{array}{l}
(p - 1)Q \Delta - (1 - a) H_2^2 \Delta^2 + \tilde{A}_2 \Delta + A_3 \end{array} \right\} \Phi - (2 - a) H_2 \Delta^2 + A_2 \Delta \right\} + RT = F_T . \]

Solving for \( T \):
\[ T = (a H_2 \Delta + R) \Phi - F_\Phi , \] (3.47)
we look for \( \Phi \) - solution to the second order elliptic equation:
\[ \left\{ \begin{array}{l}
(p - 1)Q \Delta - (1 - a)^2 H_2^2 \Delta^2 + \tilde{A}_2 \Delta + A_3 \end{array} \right\} \Phi + \left\{ \begin{array}{l}
(2 - a) H_2 \Delta + A_2 \end{array} \right\} [a H_2 \Delta \Phi + R \Phi - F_\Phi] \]
\[ = -F_T + R (a H_2 \Delta \Phi + R \Phi - F_\Phi) \]
i.e.

\[(p - 1)Q \Delta \Phi - H_2^2 \Lambda^2 \Phi + \Lambda \Phi \left[ \tilde{A}_2 + aH_2A_2 - 2RH_2 - a(2 - a)H_2\Lambda H_2 \right] + (A_3 + RA_2 - \mu^2) \Phi \]

\[= -F_T - RF_\Phi + \left[ -(2 - a)H_2 \Lambda + A_2 \right] F_\Phi. \]

Now, we have

\[(p - 1)Q \Delta \Phi - H_2^2 \Lambda^2 \Phi = \left( (p - 1)Q - H_2^2 Z^2 \right) \partial_Z^2 \Phi + \left( \frac{(d - 1)(p - 1)Q}{Z} - H_2^2 Z \right) \partial_Z \Phi \]

and hence

\[\left( (p - 1)Q - H_2^2 Z^2 \right) \partial_Z^2 \Phi + \left( \frac{(d - 1)(p - 1)Q}{Z} - H_2^2 Z \right) \partial_Z \Phi + Z \left[ \tilde{A}_2 + aH_2A_2 - 2RH_2 - a(2 - a)H_2\Lambda H_2 \right] \partial_Z \Phi + (A_3 + RA_2 - \mu^2) \Phi \]

\[= -F_T - RF_\Phi + \left[ -(2 - a)H_2 \Lambda + A_2 \right] F_\Phi. \]

Since \((p - 1)Q = \mu^2 Z^2 \sigma^2\), we have

\[\left( Z^{-1} \partial_Z \left( Z^{d-1} \partial_Z \left( \mu^2 \sigma^2 - H_2^2 \right) Z^2 \partial_Z \Phi \right) \right) \]

with

\[\left( \frac{\partial_Z \omega}{\omega} + \frac{d - 1}{Z} \right) \left( \mu^2 \sigma^2 - H_2^2 \right) Z^2 + 2Z \left( \mu^2 \sigma^2 - H_2^2 \right) + \left( 2\mu^2 \sigma \partial_Z \sigma - 2H_2 \partial_Z H_2 \right) Z^2 \]

\[= (d - 1)\mu^2 Z \sigma^2 - H_2^2 Z + Z \left[ \tilde{A}_2 + aH_2A_2 - 2RH_2 - a(2 - a)H_2\Lambda H_2 \right], \]

i.e.

\[\frac{\partial_Z \omega}{\omega} = \frac{-2}{Z} \frac{2\mu^2 \sigma \partial_Z \sigma - 2H_2 \partial_Z H_2}{\mu^2 \sigma^2 - H_2^2} - \frac{2RH_2 - (d - 2)H_2^2 - \tilde{A}_2 - aH_2A_2 + a(2 - a)H_2\Lambda H_2}{\left( \mu^2 \sigma^2 - H_2^2 \right) Z}. \]

Recalling \(H_2 = \mu(1 - w)\) yields

\[\frac{\partial_Z \omega}{\omega} = \frac{-2}{Z} \frac{\partial_Z [\sigma^2 - (1 - w)^2]}{\sigma^2 - (1 - w)^2} - \frac{2(1 - w)R - (d - 2)(1 - w)^2 - \frac{\Lambda}{\mu^2} - a(1 - w)\frac{A_2}{\mu} + a(2 - a)(1 - w)\Lambda w}{Z \left( \sigma^2 - (1 - w)^2 \right)}. \]
We therefore define
\[\varpi(Z) = \begin{cases} 
e^{-F_-(Z)} & \text{for } 0 \leq Z \leq Z_2, \\ e^{-F_+(Z)} & \text{for } Z > Z_2. \end{cases}\] (3.48)

where\(^6\)
\[F_-(Z) = \int_{Z_2}^Z \frac{2(1-w)}{\mu} R - (d-2)(1-w)^2 - \frac{\Lambda_2}{\mu^2} - a(1-w)\frac{\Lambda_2}{\mu} - a(2-a)(1-w)\Lambda w}{Z'(\sigma^2 - (1-w)^2)} dZ' + C_-,\]
\[F_+(Z) = \int_{Z_2}^Z \frac{2(1-w)}{\mu} R - (d-2)(1-w)^2 - \frac{\Lambda_2}{\mu^2} - a(1-w)\frac{\Lambda_2}{\mu} - a(2-a)(1-w)\Lambda w}{Z'(\sigma^2 - (1-w)^2)} dZ' + C_+.\]

In view of the above, we have obtained the elliptic equation:
\[- \frac{1}{Z^{d-1}\varpi} \partial_Z \left( Z^{d-1} \varpi \left( \sigma^2 - (1-w)^2 \right) Z^2 \partial_Z \Phi \right) + \frac{1}{\mu^2} \left( R^2 - A_2 R - A_3 \right) \Phi = H,\]
\[H = \frac{1}{\mu^2} \left\{ F_T + R F_\Phi + \left( 2 - a \right) H_2 \Lambda - A_2 \right\} F_\Phi,\] (3.49)

with \(T\) recovered by (3.47). As \(Z \to Z_2\), we have from (3.17):
\[\Delta(Z) = \frac{|\lambda_+|}{Z_2} (Z - Z_2) + O((Z - Z_2)^2)\]
and hence
\[Z(\sigma^2 - (1-w)^2) = |\lambda_+|(Z_2 - Z) [1 + O(Z - Z_2)]\]
and hence
\[\frac{2(1-w)}{\mu} R - (d-2)(1-w)^2 - \frac{\Lambda_2}{\mu^2} - a(1-w)\frac{\Lambda_2}{\mu} - a(2-a)(1-w)\Lambda w}{Z'(\sigma^2 - (1-w)^2)} = \frac{2\sigma_2}{\mu |\lambda_+|} R \left[ 1 + O \left( \frac{1}{R} \right) \right] \]
\[\frac{2\sigma_2}{\mu |\lambda_+|} R \left( Z_2 - Z \right) \left[ 1 + O(Z - Z_2) \right]\]

Since the profile passes through \(P2\) in a \(C^\infty\) way, we obtain the development of the measure at \(P2\): for any \(M \geq 1,\)
\[\varpi(Z) = |Z_2 - Z|^{c_\varpi} \left[ 1 + \sum_{m=0}^M d_{\sigma,m,R}(Z_2 - Z)^m + O_M \left( |Z_2 - Z|^{M+1} \right) \right],\] (3.50)

where
\[c_\varpi = \frac{2\sigma_2}{\mu |\lambda_+|} R \left[ 1 + O \left( \frac{1}{R} \right) \right] \geq c^* R > 0\] (3.51)

for \(R > R^*\) large enough. Note that the above choice of \(C_\pm\) is made to fix the normalization constant in front of \(|Z_2 - Z|^{c_\varpi}\) to be equal to 1.

---

\(^6\)The choice of the lower limits \(\frac{2\sigma_2}{\mu}\) and \(2Z_2\) in the definition of \(F_\pm\) is arbitrary but dictate the choice of the constants \(C_\pm\) in such a way as to ensure that \(\lim_{Z \to Z_2} F_-(Z) - \lim_{Z \to Z_2} F_+(Z) = 0\). The additional degree of freedom in the choice of \(C_\pm\) is used to fix an overall normalization of \(\varpi\).
step 5 Solving (3.49). We analyze the singularity of (3.49) at $P2$ using a change of variables.

$0 \leq Z < Z_2$. We let

$$
\Phi(Z) = \Psi(Y), \quad Y = h(Z), \quad h(Z) = \int_{Z_2}^{Z} \frac{dZ'}{Z^{d-1-\sigma}Z'^{2}(\sigma^2 - (1 - w)^2)}
$$

which maps (3.49) onto:

$$
\begin{align*}
-\partial_Y^2 \Psi + \frac{1}{\mu}(R^2 - A_2R - A_3)Z^{2d-2}(\sigma^2 - (1 - w)^2)\Psi &= \tilde{H}, \\
\tilde{H} &= Z^{2d-2}(\sigma^2 - (1 - w)^2)H.
\end{align*}
$$

From (3.50),

$$
Y = h(Z) = \int_{Z_2}^{Z} \frac{dz}{Z^{d-1-\sigma}Z^2(\sigma^2 - (1 - w)^2)}
$$

$$
= \int_{Z_2}^{Z} \frac{dz}{Z^{d-1}Z^2|\lambda_+|(Z_2 - z)(Z_2 - z)^{\sigma} \left[ 1 + \sum_{m=0}^{M} d_{\sigma,m,R}(Z_2 - z)^m + O_M(|Z_2 - z|^{M+1}) \right]}
$$

where from (3.51) constant $C > 0$ is independent of $R$ and, choosing $M = \sqrt{R}$,

$$
\Gamma(Z) = \sum_{m=1}^{\sqrt{R}} \tilde{d}_{\sigma,m,R}(Z_2 - Z)^m + O((Z_2 - Z)^{\sqrt{R}+1})
$$

with similar estimates for derivatives. Hence the potential term in (3.52) can be expanded in $Y$ and estimated as $Y \to +\infty$ for $R$ large enough:

$$
\frac{1}{\mu^2}(R^2 - A_2R - A_3)Z^{2d-2}(\sigma^2 - (1 - w)^2) = C_R \left[ 1 + \sum_{j=1}^{\sqrt{R}} \frac{\tilde{d}_j}{Y^{j/R}} + O \left( \frac{1}{Y^{c_R(\sqrt{R}+1)}} \right) \right]
$$

for some universal constants $\tilde{d}_j$.

$$
C_R = C + O \left( \frac{1}{R} \right), \quad 0 < c_R = \frac{1}{c_\omega} \lesssim \frac{1}{R}
$$

where $C > 0$ is independent of $R$. Therefore, by an elementary fixed point argument, (3.52) with $\tilde{H} = 0$ admits a basis of solutions $\Psi_1^-$ and $\Psi_2^-$ with the following behavior as $Y \to +\infty$

$$
\begin{align*}
\Psi_1^- &= 1 + \sum_{j=1}^{\sqrt{R}} \frac{c_{j+1}}{Y^{j/R}} + O \left( \frac{1}{Y^{(\sqrt{R}+1)/R}} \right) \\
\Psi_2^- &= Y \left[ 1 + \sum_{j=1}^{\sqrt{R}} \frac{c_{j+1}}{Y^{j/R}} + O \left( \frac{1}{Y^{(\sqrt{R}+1)/R}} \right) \right]
\end{align*}
$$

with similar estimates for derivatives. The sequences $(c_{j,1})_{j=1,2}$ are uniquely determined inductively from (3.52) with $\tilde{H} = 0$ using the expansion of the potential (3.55).

$Z_2 < Z \leq Z_2$. To the right of $P2$, we let

$$
\Phi(Z) = \Psi(Y), \quad Y = h(Z), \quad h(Z) = \int_{2Z_2}^{Z} \frac{dz}{Z^{d-1-\sigma}Z^2(\sigma^2 - (1 - w)^2)} + \tilde{C}_+
$$
which sends$^7$ $Y \to +\infty$ as $Z \downarrow Z_2$. We construct a similar basis of homogenous solutions $\Psi_1^+$ and $\Psi_2^+$ as $Y \to +\infty$ with asymptotics given by:

$$\Psi_1^+ = 1 + \sum_{j=1}^{\sqrt{\pi}} \frac{c_{j,1}}{Y^{\sqrt{\pi}^j c_{j,1}}} + O\left(\frac{1}{Y^{(1+\sqrt{\pi})c_{j,1}}}\right), \quad \Psi_2^+ = Y \left[1 + \sum_{j=1}^{\sqrt{\pi}} \frac{c_{j,2}}{Y^{\sqrt{\pi}^j c_{j,2}}} + O\left(\frac{1}{Y^{(1+\sqrt{\pi})c_{j,2}}}\right)\right]$$

with the sequences $c_{j,1}, c_{j,2}$ the same as in (3.56).

Basis of fundamental solutions. The function $\Phi_1(Z) = \Psi_1^+(Y)$ for $Z < Z_2$ and $\Phi_1(Z) = \Psi_1^+(Y)$ for $Z > Z_2$, obtained by gluing $\Psi_1^+(Y)$ belongs to $C^{\sqrt{\pi}}([0, Z_2])$ and is a solution to the homogeneous equation (3.52). Let now $\Phi_{rad}(Z)$ be the radial solution to the homogeneous problem associated to (3.49) with $\Psi_{rad}(0) = 1$. Then the wronskian is given by

$$W = \partial_Z \Phi_1 \Phi_{rad} - \partial_Z \Phi_{rad} \Phi_1 = \frac{W_0}{Z^{d-1}Z^2(\sigma^2 - (1 - \omega)^2)}$$

where $W_0$ is a constant. We claim $W_0 \neq 0$. Indeed, otherwise $\Phi_{rad}$ is proportionate to $\Phi_1$ and hence is $C^{\sqrt{\pi}}$ on $[0, Z_2]$. In particular, if $T_{rad}$ is given by (3.47) with $F_\Phi = 0$, then $X_{rad} = (\Phi_{rad}, T_{rad})$ satisfies

$$(-M + R)X_{rad} = 0 \text{ on } (0, Z_a).$$

Since $X_{rad}$ is $C^{\sqrt{\pi}-1}([0, Z_2])$, we may apply the analysis in steps 1 to 4 for $R > R^*(k)$ large enough and (3.39) holds for $X_{rad}$, i.e.

$$0 = \Re\langle(-M + R)X_{rad}, X_{rad}\rangle = \Re\langle(-M + \A)X_{rad}, X_{rad}\rangle - \Re\langle\A X_{rad}, X_{rad}\rangle + R\|X_{rad}\|_{H_{2k}}^2 \geq R\|X_{rad}\|_{H_{2k}}^2 - \Re\langle\A X_{rad}, X_{rad}\rangle$$

so that for $R > R^*(k)$ sufficiently large

$$\frac{R}{2}\|X_{rad}\|_{H_{2k}}^2 \leq 0$$

and hence $X_{rad} = 0$ a contradiction. This concludes the proof of $W_0 \neq 0$.

Inner solution of the inhomogeneous problem. $(\Phi_{rad}, \Phi_1)$ is then a basis for the homogenous problem corresponding to (3.49). As a consequence, the only solution to (3.49) which is $o((Z_2 - Z)^{-\frac{1}{\sqrt{\pi}}})$ at $Z = Z_2$ is given by$^8$

$$\Phi(Z) = -\Phi_1(Z) \int_0^Z \frac{H(\tau)\Phi_{rad}(\tau)}{W(\tau)} d\tau - \Phi_{rad}(Z) \int_Z^{Z_2} \frac{H(\tau)\Phi_1(\tau)}{W(\tau)} d\tau.$$

For a smooth $H$, $\Phi$ is smooth on $[0, Z_2]$ and we study its regularity at $Z_2$. In $Y$ variables we obtain for some $Y_0$ large enough:

$$\Psi(Y) = c_{Y_0,H} \Psi_1^+(Y) - \Psi_1^+(Y) \int_{Y_0}^Y \hat{H}(\tau)\Psi_2^+(\tau) d\tau - \Psi_2^+(Y) \int_Y^{+\infty} \hat{H}(\tau)\Psi_1^+(\tau) d\tau.$$

We have from (3.50), (3.53):

$$(RY)^{cr} = \frac{1}{Z_2 - Z} \left(\sum_{m=0}^{\sqrt{\pi}} \beta_m (Z_2 - Z)^m + O(|Z_2 - Z|^{\sqrt{\pi} + 1})\right),$$

$^7$We add constant $\tilde{C}_1$ to match the asymptotic expansion of $Y$ in terms of $(Z_2 - Z)$. In principle, it is unnecessary as it influences the terms of order $R$ and higher while we only need the universality of the expansion up to the order $\sqrt{\pi}$.

$^8$Note that $\Phi_{rad}(Z) \sim (Z_2 - Z)^{-\frac{1}{\sqrt{\pi}}}$ as $Z \to Z_2$ in view of the behavior of $\Psi_2$ as $Y \to +\infty$. 

and hence
\[ Z_2 - Z = \sum_{m=1}^{\sqrt{\Re}} \frac{y_m}{Y^{mc_R}} + O \left( \frac{1}{Y^{(\sqrt{\Re}+1)c_R}} \right) \]
with similar estimates for derivatives. In particular, a smooth function \( H(Z) \) yields expansion for \( \tilde{H}(Z) \):
\[
\tilde{H} = (Z_2 - Z)^{1+2c_R^{-1}} \left( \sum_{m=0}^{\sqrt{\Re}} h_m(Z_2 - Z)^m + O \left( (Z_2 - Z)^{\sqrt{\Re}+1} \right) \right)
\]
\[
= \sum_{m=1}^{\sqrt{\Re}} \frac{g_m}{Y^{2+mc_R}} + O \left( \frac{1}{Y^{2+(\sqrt{\Re}+1)c_R}} \right).
\]
Conversely, an expansion of the form
\[
G = \sum_{m=0}^{\sqrt{\Re}-1} \frac{b_m}{Y^{mc_R}} + O \left( \frac{1}{Y^{\sqrt{\Re}c_R}} \right)
\]
defines a \( C^{\sqrt{\Re}} \) function \( G(Z) \) at \( Z = Z_2 \). Plugging in the asymptotic expansion for \( \Psi_1, \Psi_2 \) and \( H \) in (3.57) yields
\[
\Psi(Y) = c_{Y,H} \left( \sum_{m=0}^{\sqrt{\Re}} \frac{c_{m,1}}{Y^{mc_R}} + O \left( \frac{1}{Y^{(\sqrt{\Re}+1)c_R}} \right) \right) - \left( \sum_{m=0}^{\sqrt{\Re}} \frac{c_{m,1}}{Y^{mc_R}} + O \left( \frac{1}{Y^{(\sqrt{\Re}+1)c_R}} \right) \right) \int_{Y} \left( \sum_{m=0}^{\sqrt{\Re}} \frac{c_{m,2}}{Y^{mc_R}} + O \left( \frac{1}{Y^{(\sqrt{\Re}+1)c_R}} \right) \right) \tau^{1+jc_R} + O \left( \frac{1}{\tau^{1+(\sqrt{\Re}+1)c_R}} \right) \ d\tau
\]
We therefore have proved that for \( H \in C^\infty([0, Z_2]) \), there exists a unique solution \( \Phi \) to (3.49) on \([0, Z_2]\) which is \( o((Z_2 - Z)^{-\frac{1}{c_R}}) \) at \( Z = Z_2 \). Furthermore, this solution is smooth on \([0, Z_2]\), and is \( C^{\sqrt{\Re}} \) at \( Z = Z_2 \) where it admits an asymptotic expansion
\[
\Phi(Z) = \sum_{j=0}^{\sqrt{\Re}-1} c_j (Z_2 - Z)^j + O \left( (Z_2 - Z)^{\sqrt{\Re}} \right). \quad (3.58)
\]
Outer solution of the inhomogeneous problem. We argue similarly, considering the basis \( \Phi_1(Z) \) and \( \Phi_{rad}(Z) \) with \( \Phi_{rad}(Z_a) = 1 \), for \( Z_2 < Z \leq Z_a \) and construct \( \Phi \) solution to (3.49) on \([Z_2, Z_a]\) which is smooth on \([Z_2, Z_a]\), \( o((Z_2 - Z)^{-\frac{1}{c_R}}) \) at \( Z = Z_2 \) and \( C^{\sqrt{\Re}} \) at \( Z = Z_2 \). Furthermore, \( \Phi \) admits at \( Z = Z_2 \) the following asymptotic expansion analogous to (3.58)
\[
\Phi(Z) = \sum_{j=0}^{\sqrt{\Re}-1} \tilde{c_j} (Z_2 - Z)^j + O \left( (Z_2 - Z)^{\sqrt{\Re}} \right).
\]
The asymptotic expansion is uniquely determined from the equation (3.49) and the first coefficient $\tilde c_{0,\Phi}$. We now recall that the function $\Phi_1$ belongs to $C^{1/2}[0, Z_\alpha]$ and $\Phi_1(Z_2) = 1$. By adding $\Phi_1$ to the above expansion, we obtain another solution in which we can force the condition

$$\tilde c_{0,\Phi} = c_{0,\Phi}$$

with $c_{0,\Phi}$ appearing in (3.58). As a result, the asymptotic expansions of the inner and outer solutions are matched to order $\sqrt{R}$, so that the constructed solution is $C^{1/2}$ at $Z_2$. Finally, we have shown that given any smooth function $H$ on $[0, Z_\alpha]$, there exists a unique solution $\Phi$ to (3.49) on $[0, Z_\alpha]$ which is $o((Z_2 - Z)^{-1/4})$ at $Z = Z_2$. Furthermore, this solution is smooth for $Z \neq Z_2$ and $C^{1/2}$ at $Z = Z_2$.

In particular, with $T$ recovered by (3.47) and smooth for $Z \neq Z_2$ and $C^{1/2}$-1 at $Z = Z_2$, we have that $(\Phi, T) \in H_{2k}$ for $R > R(k)$ large enough. Also, since $(\Phi, T)$ with $\Phi \sim (Z_2 - Z)^{-1/4}$ near $Z = Z_2$ does not belong to $H_{2k}$, we have now proved that, in fact, there exists a unique solution $X = (\Phi, T)$ to $(-M + R)X = F$ on $[0, Z_\alpha]$ in $H_{2k}$, which concludes the proof of (3.46).

**step 6** Density of $D_R$. We now prove that $D_R$ given by (3.41) is dense in $D(M)$. Indeed, if $X \in D(M)$, then $X \in H_{2k}$ and $MX \in H_{2k}$ so that there exists a sequence $(Y_n)_{n \in \mathbb{N}} \in C^\infty([0, Z_\alpha], \mathbb{C}^2)$ with

$$\lim_{n \to +\infty} Y_n \to (-M + R)X \text{ in } H_{2k}.$$  

From step 5, for each integer $n$, there exist a unique $Z_n \in D_R$ solution to

$$(-M + R)Z_n = Y_n, \quad Z_n \in H_{2k},$$

and hence

$$(-M + R)Z_n \to (-M + R)X \text{ in } H_{2k}.$$  

Thus, to conclude, it remains to check that $Z_n$ converges to $X$ in $H_{2k}$. To this end, since $Z_n \in D_R$, (3.39) holds for $Z_n - Z_q$ and thus:

$$\Re\langle Y_n - Y_q, Z_n - Z_q \rangle = \Re\langle (-M + R)(Z_n - Z_q), Z_n - Z_q \rangle$$

$$= \Re\langle (-M + \mathcal{A})(Z_n - Z_q), Z_n - Z_q \rangle - \Re\langle \mathcal{A}(Z_n - Z_q), Z_n - Z_q \rangle$$

$$+ R\|Z_n - Z_q\|^2_{H_{2k}}$$

$$\geq R\|Z_n - Z_q\|^2_{H_{2k}} - \Re\langle \mathcal{A}(Z_n - Z_q), Z_n - Z_q \rangle$$

so that, since $\mathcal{A}$ is a bounded operator, we infer for $R$ sufficiently large

$$\frac{R}{2}\|Z_n - Z_q\|^2_{H_{2k}} \leq \|Y_n - Y_q\|^2_{H_{2k}}.$$  

In view of the convergence of $(Y_n)$ in $H_{2k}$, we deduce that $Z_n$ is a Cauchy sequence in $H_{2k}$ and hence converges, i.e.

$$\lim_{n \to +\infty} Z_n \to Z \text{ in } H_{2k}, \quad Z \in H_{2k}.$$  

Since $(-M + R)Z_n$ converges to $(-M + R)X$ in $H_{2k}$, we infer

$$(-M + R)(Z - X) = 0 \text{ in } D'(0, Z_\alpha), \quad Z - X \in H_{2k}.$$  

The uniqueness statement in (3.46) applied for $F = 0$ yields $Z = X$. Thus $Z_n \to X$ and $(-M + R)Z_n \to (-M + R)X$ in $H_{2k}$. Finally, we have obtained a sequence

$^9$Recall that $(c_R)^{-1} \geq R \gg 1$
\( Z_n \in \mathcal{D}_R \) such that \( Z_n \rightarrow X \) in \( D(M) \), and hence \( \mathcal{D}_R \) is dense in \( D(M) \) as claimed.

**step 7** Maximal accretivity. We have proved in steps 1 to 3 that (3.39) holds for \( X \in \mathcal{D}_R \), i.e.

\[
\forall X \in \mathcal{D}_R, \quad \Re \langle (-M + \mathcal{A})X, X \rangle \geq c^* a \langle X, X \rangle.
\]

Since \( \mathcal{D}_R \) is dense in \( D(M) \), in view of step 6, we have

\[
\forall X \in \mathcal{D}(M), \quad \Re \langle (-M + \mathcal{A})X, X \rangle \geq c^* a \langle X, X \rangle
\]

which concludes the proof of the accretivity property (3.39).

We now claim:

\[
\forall F \in \mathbb{H}^{2k}, \quad \exists X \in \mathcal{D}(M) \quad \text{such that} \quad (-M + R)X = F.	ag{3.59}
\]

Indeed, since \( F \in \mathbb{H}^{2k} \), by density, there exists

\[
\lim_{n \to +\infty} F_n \to F \quad \text{in} \quad \mathbb{H}^{2k}, \quad F_n \in C^\infty([0, Z_a])
\]

Since \( F_n \in C^\infty([0, Z_a]) \), by (3.46), there exists \( X_n \in \mathbb{H}^{2k} \) – solution to

\[
(-M + R)X_n = F_n.
\]

Using (3.39) and arguing as in step 6, we have for \( R \) sufficiently large

\[
\frac{R}{2} \| X_n - X_q \|_{\mathbb{H}^{2k}} \leq \| F_n - F_q \|_{\mathbb{H}^{2k}}.
\]

In view of the convergence of \((F_n)\) in \( \mathbb{H}^{2k} \), we deduce that \( X_n \) is a Cauchy sequence in \( \mathbb{H}^{2k} \) and hence converges, i.e.

\[
\lim_{n \to +\infty} X_n \rightarrow X \quad \text{in} \quad \mathbb{H}^{2k}, \quad X \in \mathbb{H}^{2k}.
\]

On the other hand, since \((-M + R)X_n = F_n\) convergence to \( F \) in \( \mathbb{H}^{2k} \), we infer

\[
(-M + R)X = F, \quad X \in \mathcal{D}(M)
\]

which concludes the proof of (3.59).

Finally, (3.39) and a classical and elementary induction argument ensures that the maximality property (3.40) is implied by:

\[
\exists R > 0, \quad \forall F \in \mathbb{H}^{2k}, \quad \exists X \in \mathcal{D}(M) \quad \text{such that} \quad (-\tilde{M} + R)X = F.
\]

Indeed, let \( R > 0 \) large enough and \( F \in \mathbb{H}^{2k} \). Since \( \mathcal{A} \) is a bounded operator, for \( R \) large enough, from (3.59) and (3.39),

\[
\Re \langle F, X \rangle = \Re \langle (-M + R)X, X \rangle = \Re \langle (-\tilde{M} - \mathcal{A} + R)X, X \rangle \geq \frac{R}{2} \| X \|_{\mathbb{H}^{2k}}^2.
\]

Therefore, for any \( F \in \mathbb{H}^{2k} \), solution \( X \) to (3.59) is unique. Therefore, \((-M + R)^{-1}\) is well defined on \( \mathbb{H}^{2k} \) with the bound

\[
\| (-M + R)^{-1} \|_{\mathcal{L}(\mathbb{H}^{2k}, \mathbb{H}^{2k})} \lesssim \frac{1}{R}.
\]

Hence

\[
-\tilde{M} + R = -M + \mathcal{A} + R = (-M + R) \left[ \text{Id} + (-M + R)^{-1} \mathcal{A} \right]
\]

is invertible on \( \mathbb{H}^{2k} \) for \( R \) large enough, which yields (3.40). This concludes the proof of Proposition 3.5. \( \Box \)
3.7. Growth bounds for dissipative operators. We conclude this section by recalling classical facts about unbounded operators and their semigroups. Let \((H, \langle \cdot, \cdot \rangle)\) be a hermitian Hilbert space and \(A\) be a closed operator with a dense domain \(D(A)\). We recall the definition of the adjoint operator \(A^*\): let 
\[ D(A^*) = \{ X \in H, \, \tilde{X} \in D(A) \mapsto \langle X, A\tilde{X} \rangle \text{ extends as a bounded functional on } H \}, \]
then \(A^*X\) is given by the Riesz theorem as the unique element of \(H\) such that 
\[ \forall \tilde{X} \in D(A), \, \langle A^*X, \tilde{X} \rangle = \langle X, A\tilde{X} \rangle. \]  
(3.60)

We recall the following classical lemma.

**Lemma 3.7** (Properties of maximal dissipative operators, [55] p.49). Let \(A\) be a maximal dissipative operator on a Hilbert space \(H\) with domain \(D(A)\), then:

(i) \(A\) is closed;

(ii) \(A^*\) is maximal dissipative;

(iii) \(\sigma(A) \subset \{ \lambda \in \mathbb{C}, \ |\Re(\lambda)\| \leq 0 \} ; \)

(iv) \(|\|(A - \lambda)^{-1}\| \| \leq |\Re(\lambda)|^{-1} \text{ for } \Re(\lambda) > 0. \)

We now recall from Hille-Yoshida’s theorem that a maximally dissipative operator \(A_0\) generates a strongly continuous semigroup \(T_0\) on \(H\), and so does \(A_0 + K\) for any bounded perturbation \(K\). Let us now recall the following classical properties of strongly continuous semigroup \(T(t)\). Let \(\sigma(A)\) denote the spectrum of \(A\), i.e., the complement of the resolvent set.

**Proposition 3.8** (Growth bound, [20] Cor 2.11 p.258). Let the growth bound of the semigroup be defined as 
\[ w_0 = \inf \{ w \in \mathbb{R}, \exists M_w \text{ such that } \forall t \geq 0, \, \|T(t)\| \leq M_we^{wt} \}. \]

Let \(w_{\text{ess}}\) denote the essential growth bound of the semigroup:
\[ w_{\text{ess}} = \inf \{ w \in \mathbb{R}, \exists M_w \text{ such that } \forall t \geq 0, \, \|T(t)\|_{\text{ess}} \leq M_we^{wt} \} \]
with 
\[ \|T(t)\|_{\text{ess}} = \inf_{K \in \mathcal{K}(H)} \|T(t) - K\|_{H \rightarrow H} \]
and \(\mathcal{K}(H)\) is the ideal of compact operators on \(H\); and let 
\[ s(A) = \sup \{ \Re(\lambda), \, \lambda \in \sigma(A) \}. \]

Then 
\[ w_0 = \max \{ w_{\text{ess}}, s(A) \} \]
and 
\[ \forall w > w_{\text{ess}}, \text{ the set } \Lambda_w(A) := \sigma(A) \cap \{ \Re(\lambda) > w \} \text{ is finite. } \]  
(3.61)

Moreover, each eigenvalue \(\lambda \in \Lambda_w(A)\) has finite algebraic multiplicity \(m^a_\lambda\): \(\exists k_\lambda \in \mathbb{Z}\) such that 
\[ \ker(A - \lambda I)^{k_\lambda} \neq \emptyset, \quad \ker(A - \lambda I)^{k_\lambda + 1} = \emptyset, \quad m^a_\lambda := \dim \ker(A - \lambda I)^{k_\lambda} \]

We note that the subspaces \(V_w(A) = \cup_{\lambda \in \Lambda_w(A)} \ker(A - \lambda I)^{k_\lambda}\) and \(V_w^+(A^*)\) are invariant for \(A\). In particular, \(A \left( D(A) \cap V_w^+(A^*) \right) \subset V_w^+(A^*)\). The invariance \(V_w(A)\) is immediate. To show that \(A \left( D(A) \cap V_w^+(A^*) \right) \subset V_w^+(A^*)\) we let \(X \in D(A) \cap V_w^+(A^*), \, Y \in V_w(A^*)\) and consider \(\langle AX, Y \rangle\). Since \(Y \in D(A^*)\) and \(V_w(A^*)\) is invariant for \(A^*\), 
\[ \langle AX, Y \rangle = \langle X, A^*Y \rangle = 0. \]

We claim the following corollary.
Lemma 3.9 (Perturbative exponential decay). Let $T_0$ be the strongly continuous semigroup generated by a maximal dissipative operator $A_0$, and $T$ be the strongly continuous semi group generated by $A = A_0 + K$ where $K$ is a compact operator on $H$. Then for any $\delta > 0$, the following holds:

(i) the set $\Delta_\delta(A) = \sigma(A) \cap \{ \lambda \in \mathbb{C}, \Re(\lambda) > \delta \}$ is finite, each eigenvalue $\lambda \in \Delta_\delta(A)$ has finite algebraic multiplicity $k_\lambda$. In particular, the subspace $V_\delta(A)$ is finite dimensional;

(ii) We have $\Delta_\delta(A) = \overline{\Delta_\delta(A^*)}$ and $\dim V_\delta(A^*) = \dim V_\delta(A)$. The direct sum decomposition

$$H = V_\delta(A) \bigoplus V_\delta^\perp (A^*)$$

is preserved by $T(t)$ and there holds:

$$\forall X \in V_\delta^\perp (A^*), \quad \|T(t)X\| \leq M_\delta e^{\delta t} \|X\|. \tag{3.63}$$

(iii) The restriction of $A$ to $V_\delta(A)$ is given by a direct sum of $(m_{\lambda} \times m_{\lambda})_{\lambda \in \Delta_\delta(A)}$ matrices each of which is the Jordan block associated to the eigenvalue $\lambda$ and the number of Jordan blocks corresponding to $\lambda$ is equal to the geometric multiplicity of $\lambda - m_{\lambda}^* = \dim \ker(A - \lambda I)$. In particular, $m_{\lambda}^* \leq m_{\lambda}^0 k_\lambda$. Each block corresponds to an invariant subspace $J_\lambda$ and the semigroup $T$ restricted to $J_\lambda$ is given by the nilpotent matrix

$$T(t)|_{J_\lambda} = \begin{pmatrix} e^{\lambda t} & e^{\lambda t} & \ldots & t^{m_{\lambda}-1} e^{\lambda t} \\
0 & e^{\lambda t} & \ldots & t^{m_{\lambda}-2} e^{\lambda t} \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & e^{\lambda t} \end{pmatrix}$$

Proof. This is a simple consequence of Proposition 3.8.

**step 1** Perturbative bound. First, since $A_0$ is maximally dissipative,

$$\forall t \geq 0, \quad \|T_0(t)\| \lesssim 1$$

implies $w_0(A_0) \leq 0$. By Proposition 3.8, $s(A_0) \leq 0$ and

$$w_{\text{ess}}(T_0) \leq 0.$$ 

On the other hand, from [20] Prop 2.12 p.258, compactness of $K$ implies

$$w_{\text{ess}}(T) = w_{\text{ess}}(T_0) \leq 0.$$ 

Let now $\lambda \in \sigma(A)$ with $\Re(\lambda) > 0$, then the formula

$$A - \lambda = A_0 + K - \lambda = (A_0 - \lambda)(\text{Id} + (A_0 - \lambda)^{-1}K)$$

and invertibility of $(A_0 - \lambda)$ imply that $\lambda$ belongs to the spectrum of the Fredholm operator $\text{Id} + (A_0 - \lambda)^{-1}K$. Therefore, $\lambda$ is an eigenvalue of $A$. On the other hand, $\Re(\lambda) > \delta$ implies $\Re(\lambda) > \delta > 0 \geq w_{\text{ess}}(T)$, and hence, by (3.61), there are finitely many eigenvalues with $\Re(\lambda) > \delta$. In fact, Proposition 3.8 also directly shows that each some $\lambda$ is an eigenvalue and implies the rest of (i).

Since $A^* = A_0^* + K^*$ and $A_0^*$ is maximally dissipative from Lemma 3.7, we can run the same argument as above for $A^*$. Moreover, $\sigma(A) = \sigma(A^*)$ ([55], prop. 2.7), (i) is proved.

The argument above, in fact, shows that $\{ \lambda \in \mathbb{C}, \ Re(\lambda) > \delta \} \cap \{ \lambda \in \sigma(A) \}$ is finite, since for every $\Re(\lambda) > 0$ and $\lambda \in \sigma(A)$, $\lambda$ is an eigenvalue of $A$.

**step 2** The first statement of (ii) is standard. We already explained that the subspaces $V_\delta(A)$ and $D(A) \cap V_\delta^\perp (A^*)$ are invariant for $A$. To prove the direct
We first argue that the subspace $V_{\delta}(A)$ is the image of $H$ under the spectral projection $P_{\delta}(A)$ associated to the set $\Lambda_{\delta}(A)$:

$$P_{\delta}(A) = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\lambda}{\lambda I - A}$$

where $\Gamma$ is an arbitrary contour containing the set $\Lambda_{\delta}(A)$. There is a direct decomposition

$$H = \text{Im} P_{\delta}(A) \bigoplus \ker P_{\delta}(A).$$

On the other hand, the adjoint

$$P_{\delta}^*(A) = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\lambda}{\lambda I - A^*} = P_{\delta}(A^*)$$

is the spectral projection of $A^*$ associated to the set $\overline{\Lambda_{\delta}(A)}$. The result is now immediate.

**Step 3** Semigroups generated by restriction and conclusion. Let $V = V_{\delta}(A)$, $U = V_{\delta}^\perp(A^*)$ and $P$ denote the projection on $V_{\delta}^\perp(A^*)$ in the direct decomposition $(3.62)$. Let $\tilde{A}$ denote the restriction of $A$ to $U$ with the domain $D(\tilde{A}) = U \cap D(A)$. By invariance

$$\forall X \in U \cap D(A), \quad \tilde{A}X = AX.$$

Let $\tilde{T}$ be the semigroup on $U$ generated by $\tilde{A} = A$. Then for all $X \in D(A) \cap U$, $\tilde{T}(t)X \in C^1([0, +\infty), D(\tilde{A}))$ is the unique strong solution to the ode

$$\frac{dX(t)}{dt} = AX(t), \quad X(0) = X.$$

This implies that $\tilde{T}(t)X = T(t)X$ for all $X \in D(A) \cap U$ and thus for all $X \in U$ by continuity of the semigroup. By Proposition 3.8 the growth bound of $\tilde{T}$ satisfies

$$w_0(\tilde{T}) \leq \max\{w_{\text{ess}}(\tilde{T}), s(\tilde{A})\}.$$

We first argue that

$$w_{\text{ess}}(\tilde{T}) \leq 0.$$

To prove that we note that we already established that $w_{\text{ess}}(T) \leq 0$. We then fix $\varepsilon > 0$ and, for any $t \geq 0$ choose a compact operator $K(t) \in \mathcal{K}(H)$ on $H$ such that,

$$\log \|T(t) - K(t)\|_{H \to H} < \varepsilon t + \log M$$

for some constant $M$ which may depend on $\varepsilon$. The restriction $\tilde{K}(t) = PK(t)$ of $K(t)$ to $U$ is a compact operator on $U$. Then, for any $t \geq 0$

$$\log \|\tilde{T}(t) - \tilde{K}(t)\|_{U \to U} = \log \|P(T(t) - K(t))\|_{U \to U} \leq \log \|C_P\|_{H \to H} \|T(t) - K(t)\|_{H \to H} < \log C_P + \log M + \varepsilon t,$$

where $C_P$ denotes the norm of the projector $P$. The desired conclusion follows.

To show that $s(\tilde{A}) \leq \delta$ we assume that $\lambda \in \sigma(\tilde{A})$ with $\Re(\lambda) > \delta$, then $\lambda$ is an eigenvalue of $\tilde{A}$ and, by invariance of $U$, $\lambda$ is an eigenvalue of $A$ with a non-trivial eigenvector $\psi \in U$. However, by construction, all such $\psi$ belong to the subspace $V = V_{\delta}(A)$, contradiction. Hence $s(\tilde{A}) \leq \delta$ and Proposition 3.8 yields $(3.63)$.

Finally, part (iii) is completely standard. \qed

We will use Lemma 3.9 in the following form.
Lemma 3.10 (Exponential decay modulo finitely many instabilities). Let $\delta > 0$ and let $T_0$ be the strongly continuous semigroup generated by a maximal dissipative operator $A_0$, and $T$ be the strongly continuous semigroup generated by $A = A_0 - \delta + K$ where $K$ is a compact operator on $H$. Let the (possibly empty) finite set
\[ \Lambda = \{ \lambda \in \mathbb{C}, \ \Re(\lambda) \geq 0 \} \cap \{ \lambda \text{ is an eigenvalue of } A \} = (\lambda_i)_{1 \leq i \leq N} \]
and let
\[ H = U \bigoplus V, \]
where $U$ and $V$ are invariant subspaces for $A$ and $V$ is the image of the spectral projection of $A$ associated to the set $\Lambda$. Then there exist $C, \delta > 0$ such that
\[ \forall X \in U, \quad \|T(t)X\| \leq Ce^{-\delta t}\|X\|. \quad (3.64) \]

Proof. We apply Lemma 3.9 to $\tilde{A} = A + \delta = A_0 + K$ with generates the semi group $\tilde{T}$. Hence the set
\[ \Lambda_\delta (\tilde{A}) = \{ \lambda \in \mathbb{C}, \ \Re(\lambda) > \frac{\delta}{4} \} \cap \{ \lambda \text{ is an eigenvalue of } \tilde{A} \} \]
is finite. Moreover
\[ AX = \lambda X \Leftrightarrow \tilde{A}X = (\lambda + \delta)X \]
and hence
\[ \Lambda \subset \Lambda_\delta \]
Let
\[ H = U_\delta \bigoplus V_\delta, \]
be the invariant decomposition of $\tilde{A}$ (and of $A$) associated to the set $\Lambda_\delta$. Clearly, $U_\delta \subset U$ and
\[ U = U_\delta \bigoplus O_\delta, \]
where $O_\delta$ is the image of the spectral projection of $A$ associated with the set $\Lambda_\delta \setminus \Lambda$. By Lemma 3.9,
\[ \forall X \in U_\delta, \quad \|\tilde{T}(t)X\| \leq M_\delta e^{\frac{\delta}{4}t}\|X\| \]
which implies
\[ \forall X \in U_\delta, \quad \|T(t)X\| = e^{-\delta t}\|\tilde{T}(t)X\| \leq M_\delta e^{-\frac{3\delta}{4}t}\|X\|. \quad (3.65) \]
Let now $X \in U$. Since $U_\delta$ is invariant by $T$ and (3.65) yields exponential decay on $U_\delta$, we assume $X \in O_\delta$. $O_\delta$ is an invariant subspace of $A$ generated by the eigenvalues $\lambda$ with the property that $-\frac{3}{4}\delta \leq \Re(\lambda) < 0$. Let $\delta_0 > 0$ be defined as
\[-\delta_0 := \sup\{\Re(\lambda) : -\frac{3}{4}\delta \leq \Re(\lambda) < 0\} \]
From part (iii) of Lemma 3.9,
\[ \|T(t)X\|_{O_\delta} \leq C \sup_{\Re(\lambda) < 0} e^{\lambda t} \|X\| \leq C^{-\frac{\delta_0}{2}t}\|X\| \]
This concludes the proof of Lemma 3.10. □

Our final result in this section is to set up a Brouwer type argument for the evolution of unstable modes.
Lemma 3.11. Let $A, \delta_g$ as in Lemma 3.10 with the decomposition

$$H = U \bigoplus V$$

into stable and unstable subspaces. Fix sufficiently large $t_0 > 0$ (dependent on $A$). Let $F(t)$ such that, $\forall t \geq t_0$, $F(t) \in V$ and

$$\|F(t)\| \leq e^{-2\delta_g t}$$

be given. Let $X(t)$ denote the solution to the ode

$$\begin{cases}
\frac{dX}{dt} = AX + F(t) \\
X(t_0) = x \in V.
\end{cases}$$

Then, for any $x$ in the ball $\|x\| \leq e^{-3\delta_g t_0}$, we have

$$\|X(t)\| \leq e^{-\delta_g t}, \quad t_0 \leq t \leq t_0 + \Gamma$$

for some large constant $\Gamma$ (which only depends on $A$ and $t_0$). Moreover, there exists $x^* \in V$ in the same ball as $x$ above such that $\forall t \geq t_0$,

$$\|X(t)\| \leq e^{-3\delta_g t_0}$$

Proof. According to Lemma 3.9 the subspace $V$ can be further decomposed into invariant subspaces on which $A$ is represented by Jordan blocks. We may therefore assume that $V$ is irreducible and corresponds to a Jordan block of $A$ of length $m_\lambda$ associated with an eigenvalue $\lambda$ with $\Re(\lambda) \geq 0$ and restrict $A$ to $V$. We decompose $A$ as

$$A = \lambda I + N,$$

where $N$ has the property that $N^{m_\lambda - 1} = 0$, and

$$e^{tN} = \begin{pmatrix}
1 & t & \ldots & t^{m_\lambda - 1} \\
0 & 1 & \ldots & t^{m_\lambda - 2} \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1
\end{pmatrix}$$

The claim (3.66) follows from the growth on the Jordan block:

$$\|X(t)\| = \left\|e^{(t-t_0)A}x + \int_{t_0}^{t} e^{(t-\tau)A}F(\tau)d\tau\right\| \leq CT^{m_\lambda - 1}e^{\Re(\lambda)\Gamma}e^{-\delta_g t_0} + \int_{t_0}^{t} C\{r - t_0\}^{m_\lambda - 1}e^{\Re(\lambda)(t-\tau)}e^{-2\delta_g \tau}d\tau \leq CT^{m_\lambda - 1}e^{\Re(\lambda)\Gamma}e^{-\delta_g t_0}$$

and hence the size of constant $\Gamma$ is determined from the inequality

$$CT^{m_\lambda - 1}e^{\Re(\lambda)\Gamma}e^{-\delta_g t_0} \leq e^{-\delta_g (t_0 + \Gamma)},$$

a sufficient condition being

$$\Gamma \leq \frac{t_0}{2} \left[ \frac{\delta_g}{10\Re(\lambda) + 5\delta_g} \right]$$

which can be made arbitrarily large by a choice of $t_0$. We now define a new variable

$$Y(t) = e^{-tN}e^{\frac{19\delta_g}{30}t}X(t).$$
Since $N$ and $A$ commute,
\[
\frac{dY}{dt} = \left( \lambda + \frac{19\delta_g}{30} \right) Y + \tilde{F}(t), \quad Y(t_0) = y
\]
where $\tilde{F}(t) = e^{-tN} e^{\frac{19\delta_g}{30} t} F(t)$ and
\[
\|\tilde{F}(t)\| \lesssim e^{-\frac{\delta_g}{\pi} t}.
\]
Since $t_0$ was chosen to be sufficiently large, we can assume that $\forall t \geq t_0$
\[
\|\tilde{F}(t)\| \lesssim e^{-\frac{\delta_g}{\pi} t}
\]
and $\epsilon < \Re(\lambda) + \frac{19\delta_g}{30}$. We now run a standard Brouwer type argument for $Y$. For any $y$ such that $\|y\| \leq 1$ we define the exit time $t^*$ to be the first time such that $\|Y(t^*)\| = 1$. If for some $y$, $t^* = \infty$, we are done. Otherwise, assume that for all $\|y\| \leq 1$, $t^* < \infty$ and define the map $\Phi : B \to S$ as $\Phi(y) = Y(t^*)$ mapping the unit ball to the unit sphere. Note that $\Phi$ is the identity map on the boundary of $B$. To prove continuity of $\Phi$ we compute
\[
\frac{d\|Y\|^2}{dt}(t^*) = 2\Re(\lambda) + \frac{19\delta_g}{15} + 2\Re(\tilde{F}(t^*)^*, Y(t^*)) \geq \frac{19\delta_g}{30} > 0.
\]
This is the outgoing condition which implies continuity. The Brouwer argument applies and shows that such $\Phi$ does not exist. We now reinterpret the result in terms of $X$. We have shown existence of $x$ such that the corresponding solution $X(t)$ has the property that $\forall t \geq t_0$,
\[
\|e^{-tN} X(t)\| \leq e^{-\frac{19\delta_g}{30} t}.
\]
Now $e^{-tN}$ is an invertible operator with the inverse given by $e^{tN}$ and its norm bounded by $Ct^{\alpha\lambda-1}$. The result follows immediately. We note that the resulting solution $X(t)$ has initial data $X(t_0)$ in the ball $\|X(t_0)\| \leq e^{-\frac{19\delta_g}{30} t_0}$.

4. Set up and the bootstrap

In this section we describe a set of smooth well localized initial data which lead to the conclusions of Theorem 1.1. The heart of the proof is a bootstrap argument coupled to the classical Brouwer topological argument of Lemma 2.11 to avoid finitely many unstable directions of the corresponding linear flow. Since our analysis relies essentially on the phase-modulus decomposition of solutions of the Schrödinger equation, our chosen data needs to give rise to nowhere vanishing solutions to (1.1) (at least for a sufficiently small time.)

4.1. Renormalized variables. Let $u(t,x) \in C([0,T],\cap_{k\geq0} H^k)$ be a solution to (1.1) such that $u(t,x)$ does not vanish at any $(t,x) \in [0,T] \times \mathbb{R}^d$. This will be a consequence of our choice of initial data and suitable bootstrap assumptions. We introduce for such a solution the decomposition of Lemma 2.1
\[
u(t,x) = \frac{1}{(\lambda \sqrt{b})^{\frac{2}{p-2}}} w(s,y) e^{i\gamma}, \quad w(\tau,y) = \rho_T(\tau,Z) e^{\frac{i}{b} \frac{\gamma_T}{p}}
\] (4.1)
with the renormalized space and times
\[
\begin{align*}
    Z &= y \sqrt{b} = Z^* x, \quad Z^* = e^{\rho_T}, \\
    \lambda(\tau) &= e^{-\frac{\tau}{b}}, \quad b(\tau) = e^{-\epsilon\tau}, \quad \gamma_T = -\frac{1}{b} = -e^{\epsilon\tau}, \\
    \tau &= -\log(T-t), \quad \tau_0 = -\log T.
\end{align*}
\] (4.2)
Here, $0 < e < 1$ is the fixed front speed such that

$$r = \frac{2}{1 - e} > 2.$$  

Up to a constant the phase can more explicitly be written in the form

$$\gamma(\tau) = -\frac{1}{eb}. \quad (4.3)$$

Our claim is that given $\tau_0 = -\log T$ large enough, we can construct a finite co-dimensional manifold of smooth well localized initial data $u_0$ such that the corresponding solution to the renormalized flow (2.23) is global in renormalized time $\tau \in [\tau_0, +\infty)$, bounded in a suitable topology and nowhere vanishing. Upon unfolding (4.1), this produces a solution to (1.1) blowing up at $T$ in the regime described by Theorem 1.1.

4.2. Stabilization and regularization of the profile outside the singularity.

The spherically symmetric profile solution $(\rho_P, \Psi_P)$ has an intrinsic slow decay as $Z \to +\infty$

$$\rho_P(Z) = \frac{c_P}{(Z)^{\frac{2r-1}{p-1}}} \left( 1 + O \left( \frac{1}{(Z)^r} \right) \right),$$

which needs to be regularized in order to produce finite energy non vanishing initial data.

1. Stabilization of the profile. Recall the asymptotics (2.19) and the choice of parameters (4.3), (4.2) which yield

$$\lambda^{2(r-2)} = b^r, \quad r = \frac{2}{1 - e}, \quad e = \frac{r - 2}{r}, \quad \mu = \frac{1 - e}{2}.$$  

For $Z = \frac{\sqrt{b}}{x} \gg 1$, i.e., outside the singularity:

$$u_P(t, x) = \frac{e^{i\gamma(\tau)}}{(\lambda \sqrt{b})^{\frac{p-1}{2}}} \rho_P(Z) e^{i\frac{\Psi_P}{b}} \rho_P(Z) e^{i\frac{\Psi_P}{b}}$$

$$= \frac{c_P e^{-\frac{1}{x^2}}}{(\lambda \sqrt{b})^{\frac{p-1}{2}}} e^{i \left[ \frac{1}{\lambda \sqrt{b}} \left( \frac{\sqrt{b}}{x} \right)^{\frac{2(r-1)}{p-1}} \right]} \left[ 1 + O \left( \frac{1}{(Z)^r} \right) \right]$$

$$= \frac{c_P}{x^{\frac{2r-1}{p-1}}} e^{i \frac{\Psi_P}{x^2}} \left[ 1 + O \left( \frac{1}{Z^r} \right) \right] \left[ 1 + O \left( \frac{1}{Z^r} \right) \right]. \quad (4.4)$$

We see that far away from the singularity the profile $u_P$ is stationary. It is precisely this property that will allow us to dampen the tail of the profile below and construct solutions arising from rapidly decaying (in particular, finite energy) initial data.

2. Dampening of the tail. We dampen the tail outside the singularity $x \geq 1$, i.e., $Z \geq Z^*$ as follows. Let

$$R_P(t, x) = \frac{1}{(\lambda \sqrt{b})^{\frac{p-1}{2}}} \rho_P(Z), \quad x = Ze^{-\mu \tau}, \quad (4.5)$$
then the asymptotics (4.4) imply the existence of a limiting profile for $x \geq 1$:

$$R_P(t, x) = -\frac{c_P}{x^{p-1}} \left(1 + O(e^{-\mu r})\right)$$

We then pick once and for all a large integer $n_P \gg 1$ and define a smooth non decreasing connection $\mathcal{K}(x)$

$$\mathcal{K}(x) = \begin{cases} 
0 & \text{for } |x| \leq 5 \\
 n_P - \frac{2(r-1)}{p-1} & \text{for } |x| \geq 10
\end{cases}$$  \hspace{1cm} (4.6)

for some large enough universal constant $n_P = n_P(d) \gg 1$.

We then define the dampened tail profile in original variables

$$R_D(t, x) = R_P(t, x) e^{-\int_0^x \frac{\mathcal{K}(x')}{x'} dx'}$$

and hence in renormalized variables:

$$\rho_D(\tau, Z) = (\lambda \sqrt{b})^{\frac{2}{p-1}} R_D(t, x)$$

Let

$$\zeta(x) = e^{-\int_0^x \frac{\mathcal{K}(x')}{x'} dx'},$$

we have the equivalent representation:

$$\rho_D(Z) = (\lambda \sqrt{b})^{\frac{2}{p-1}} R_D(\tau, x) \zeta(x) = \zeta \left(\frac{Z}{Z^*}\right) \rho_P(Z)$$ \hspace{1cm} (4.9)

Note that by construction for $j \in \mathbb{N}^*$:

$$- \frac{Z^j \partial_z^j \rho_D}{\rho_D} = \begin{cases} 
(-1)^{j-1} \left(\frac{2r-1}{p-1}\right)^j + O \left(\frac{1}{|Z|^r}\right) & \text{for } Z \leq 5Z^* \\
(-1)^{j-1} n_P^j + O \left(\frac{1}{|Z|^r}\right) & \text{for } Z \geq 10Z^*
\end{cases}$$ \hspace{1cm} (4.10)

and

$$\left| Z^j \partial_z^j \rho_D \right|_{L^\infty} \lesssim 1.$$

The obtained dampened profile for $Z \geq Z^*$ will be denoted

$$\left(\rho_D, \Psi_P\right), \quad Q_D = \rho_D^{p-1}.$$

4.3. Initial data. We now describe explicitly an open set of initial data which will be considered as perturbations of the profile $\left(\rho_D, \Psi_P\right)$ in a suitable topology. The conclusions of Theorem 1.1 will hold for a finite co-dimension set of such data.

We pick universal constants $0 < a \ll 1, Z_0 \gg 1$ which will be adjusted along the proof and depend only on $(d, \ell)$. We define two levels of regularity

$$\frac{d}{2} \ll k_0 \ll k_m$$

where $k_m$ denotes the maximum level of regularity required for the solution and $k_0$ is the level of regularity required for the linear spectral theory on the compact set $[0, Z_a]$. 

Variables and notations for derivatives. We define the variables
\[ \rho_T = \rho_T + \rho = \rho_D + \tilde{\rho}, \]
\[ \Psi_T = \Psi_T + \Psi = \rho_P. \]
\[ \Phi = \rho_P \Psi \]
and specify the data in the \((\tilde{\rho}, \Psi)\) variables. We will use the following notations for derivatives. Given \(k \in \mathbb{N}\), we note
\[ \partial^k = (\partial^1, ..., \partial^d), \]
\[ f^{(k)} := \partial^k f \]
the vector of \(k\)-th derivatives in each direction. The notation \(\partial^k_Z f\) is the \(k\)-th radial derivative. We let
\[ \tilde{\rho}_k = \Delta \tilde{\rho}, \]
\[ \Psi_k = \Delta \Psi. \]

1. Initializing the Brouwer argument. We define the variables adapted to the spectral analysis according to (3.1), (3.5):
\[ \Phi = \rho_P \Psi \]
\[ T = \partial_\tau \Phi + aH_2 \Lambda \Phi \]
\[ X = (\Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi
\]
and recall the scalar product (3.37). For \(0 < c_g, a \ll 1\) small enough, we choose \(k_0 \gg 1\) such that Proposition 3.5 applies in the Hilbert space \(H_{2k_0}\) with the spectral gap
\[ \forall X \in D(M), \quad \Re \langle (-M + \mathcal{A})X, X \rangle \geq c_g \langle X, X \rangle. \]
Hence
\[ M = (M - \mathcal{A} + c_g) - c_g + \mathcal{A} \]
and we may apply Lemma 3.10:
\[ \Lambda_0 = \{ \lambda \in \mathbb{C}, \quad \Re(\lambda) \geq 0 \} \cap \{ \lambda \text{ is an eigenvalue of } M \} = (\lambda_i)_{1 \leq i \leq N} \]
is a finite set corresponding to unstable eigenvalues, \(V\) is an associated (unstable) finite dimensional invariant set, \(U\) is the complementary (stable) invariant set
\[ H_{2k_0} = U \bigoplus V \]
and \(P\) is the associated projection on \(V\). We denote by \(N\) the nilpotent part of the matrix representing \(M\) on \(V:\)
\[ M|_V = N + \text{diag} \]
Then there exist \(C, \delta_g > 0\) such that (3.64) holds:
\[ \forall X \in U, \quad \| e^{\tau M} X \|_{H_{2k_0}} \leq C e^{\frac{\delta_g}{3} \tau} \| X \|_{H_{2k_0}}, \quad \forall \tau \geq \tau_0. \]
We now choose the data at \(\tau_0\) such that
\[ \| (I - P)X(\tau_0) \|_{H_{2k_0}} \leq e^{\frac{\delta_g}{3} \tau_0}, \quad \| PX(\tau_0) \|_{H_{2k_0}} \leq e^{\frac{3\delta_g}{3} \tau_0}. \]

2. Bounds on local low Sobolev norms. Let \(0 \leq m \leq 2k_0\) and
\[ \nu_0 = \frac{2(r - 1)}{p - 1} + \frac{\delta_g}{2p}, \]
let the weight function
\[ \chi_{\nu_0, m} = \frac{1}{(Z)^{d-2(r-1)+2(\nu_0-m)}} \zeta \left( \frac{Z}{Z^*} \right), \quad \zeta(Z) = \begin{cases} 1 & \text{for } Z \leq 2 \\ 0 & \text{for } Z \geq 3. \end{cases} \]


the smoothness of the nonlinearity since 0. Non vanishing and hydrodynamical variables.

Remark 4.1. and hence from our choice of initial data, the non vanishing of 

t for some large enough universal

generates a unique local solution

Then:
\[
\sum_{m=0}^{2k_0} \int (p - 1)Q(\partial^m \rho(\tau_0))^2 \chi_{\rho_{0,m}} + |\nabla \partial^m \Phi(\tau_0)|^2 \chi_{\rho_{0,m}} \leq e^{-\delta_{\rho_{\tau_0}}}.
\] (4.19)

4. Pointwise assumptions. We assume the following interior pointwise bounds

\[ \forall 0 \leq k \leq 2k_m, \left\| \frac{(Z)^k \partial^k \rho(\tau_0)}{\rho_D} \right\|_{L^\infty(Z \leq Z^*)} + \left\| (Z)^{r-2}(Z)^k \partial^k \Psi(\tau_0) \right\|_{L^\infty(Z \leq Z^*)} \leq b_0^C \] (4.20)

for some small enough universal constant \( c_0 \), and the exterior bounds:

\[ \forall 0 \leq k \leq 2k_m, \left\| \frac{Z^{k+1} \partial^k \rho(\tau_0)}{\rho_D} \right\|_{L^\infty(Z \geq Z^*)} + \left\| \frac{Z^{k+1} \partial^k \Psi(\tau_0)}{b_0} \right\|_{L^\infty(Z \geq Z^*)} \leq b_0^C \] (4.21)

for some large enough universal \( C_0(d, r, p) \). Note in particular that (4.20), (4.21) ensure for \( 0 < b_0 < b_0^* \ll 1 \) small enough:

\[ \left\| \frac{\partial^k(\tau_0)}{\rho_D} \right\|_{L^\infty} \leq \delta_0 \ll 1 \] (4.22)

and hence the data does not vanish.

5. Global rough bound for large Sobolev norms. We pick a large enough constant \( k_m(d, r, \ell) \) and consider the global Sobolev norm

\[ \| \partial^k \Psi \|^2_{k_m} := \sum_{j=0}^{k_m} \sum_{|u| = j} \int \frac{b^2 |\nabla \partial^u \partial^k \rho|^2 + (p - 1)\rho D \partial^{r-2} \rho T (\partial^u \rho)^2 + \rho T^2 |\nabla \partial^u \Psi|^2}{(Z)^{2(k_m - j)}}. \] (4.23)

then we require:

\[ \| \partial^k(\tau_0), \Psi(\tau_0) \|_{k_m} \leq \frac{1}{2}. \] (4.24)

The bound above is actually implied by the pointwise assumptions.

Remark 4.1. Note that we may without loss of generality assume \( u_0 \in \cap_{k \geq 0} H^k \).

4.4. Bootstrap bounds. We make the following bootstrap assumptions on the maximal interval \([\tau_0, \tau^*] \).

0. Non vanishing and hydrodynamical variables. From standard Cauchy theory and the smoothness of the nonlinearity since \( p \in 2N^* + 1 \), the smooth data \( u_0 \in \cap_{k \geq 0} H^k \) generates a unique local solution \( u \in C([0, T], \cap_{k \geq 0} H^k) \) with the blow up criterion

\[ T < +\infty \Rightarrow \lim_{t \to T} \| u(t, \cdot) \|_{H^\infty} = +\infty \] (4.25)

for some large enough \( k_\infty(d, p) \). To ensure non vanishing, we first note that since \( \inf_{|x| \leq \ell} |u_0(x)| > 0 \), the continuity of \( u \) in time ensures \( \inf_{|x| \leq \ell} |u(t, x)| > 0 \) for \( t \in [0, T], T > 0 \) small enough. For \( |x| \geq 10 \), we estimate from the flow

\[ |r^m |u(t, x)| - r^m |u_0|| \leq \int_0^t r^{np} \left| \Delta u - u |u|^{p-1} \right| dt \]

and hence from our choice of initial data, the non vanishing of \( u(t, x) \) follows on a time interval where

\[ T \| r^m (|\Delta u| + |u|^{p})\|_{L^\infty([0, T], |x| \geq 10)} \leq \delta \] (4.26)
for some sufficiently small universal constant $0 < \delta \ll 1$. Using spherical symmetry we can replace the above by
\[
T \left( \| \tilde{u}^{n+1} \|_{L^\infty(\mathbb{R};H^1)} + \| \rho^{2\nu} u^{p-1} \|_{L^\infty(\mathbb{R};L^{p+1}(\mathbb{R};H^1))} \right) \leq \delta
\]
for an arbitrarily small $\epsilon > 0$. Our initial data $u_0$ belongs to the space
\[
\cap_{k \geq 0} H^k \cap \| \tilde{u}^{n+1} \|_{L^2} \cap \| \tilde{u}^{n+1} \|_{L^2} \leq \| \tilde{u}^{n+1} \|_{L^2}.
\]
Existence of the desired time interval $[0, T)$ now follows from a local well-posedness for NLS in weighted Sobolev spaces which is (essentially) in \cite{27}.

We may therefore introduce the hydrodynamical variables \eqref{4.1} on such a small enough time interval and will bootstrap the smallness bound which ensures non vanishing:
\[
\left\| \frac{\tilde{\rho}}{p T} \right\|_{L^\infty} \leq \delta \tag{4.27}
\]
for some sufficiently small $0 < \delta = \delta(k_m) \ll 1$.

1. **Global weighted Sobolev norms.** Pick a small enough universal constant $0 < \nu < \nu^*(k_m) \ll 1$, we define
\[
\sigma_\nu = \nu + \frac{d}{2} - (r - 1) \quad m_0 = \frac{4 k_m}{p - 1} + 1 \quad \tilde{\nu} = \nu + \frac{2(r - 1)}{p - 1}
\]
and let the continuous function:
\[
\sigma(m) = \begin{cases} 
\sigma_\nu - m & \text{for } 0 \leq m \leq m_0 \\
-\alpha(k_m - m) & \text{for } m_0 \leq m \leq k_m
\end{cases} \tag{4.29}
\]
with the continuity requirement at $m_0$:
\[
\alpha(k_m - m_0) = m_0 - \sigma_\nu, \quad \alpha = \frac{m_0 - \sigma_\nu}{k_m - m_0} = \frac{4}{5} + O \left( \frac{1}{k_m} \right). \tag{4.30}
\]
In particular, $\alpha < 1$. We note that for all $1 \leq m \leq k_m$
\[
\sigma(m - 1) \geq \sigma(m) - \alpha. \tag{4.31}
\]
We also define the function
\[
\tilde{\sigma}(k) = \begin{cases} 
n_p - \frac{2(r - 1)}{p - 1} - (r - 2) + 2\tilde{\nu} & \text{for } 0 \leq k \leq \frac{2 k_m}{3} + 1 \\
\beta(k_m - k) & \text{for } \frac{2 k_m}{3} + 1 \leq k \leq k_m
\end{cases}
\]
\[
\leq n_p - \frac{2(r - 1)}{p - 1} - (r - 2) + 2\tilde{\nu} \tag{4.32}
\]
where $\beta$ is computed through the continuity requirement at $\frac{2 k_m}{3}$:
\[
\frac{k_m}{3} \beta = n_p - \frac{2(r - 1)}{p - 1} - (r - 2) + 2\tilde{\nu} \iff \beta = 3 \frac{n_p - \frac{2(r - 1)}{p - 1} - (r - 2) + 2\tilde{\nu}}{k_m}. \tag{4.33}
\]
We will choose $n_p \ll k_m$, e.g. $n_p = \frac{k_m}{30}$, so that in particular,
\[
\beta < \frac{1}{10}, \quad \alpha + \beta \leq 1.
\]
We also note that
\[
\tilde{\sigma}(m - 1) \leq \tilde{\sigma}(m) + \beta.
\]
We then define the weighted Sobolev norm:
\[
\left\| \tilde{\rho}, \Psi \right\|_{m,\sigma(m)}^2 = \sum_{k=0}^{m} \sum_{|\alpha|=k} \int \chi_{m,k,\sigma(m)} \left[ b^2 |\nabla \tilde{\rho}|^2 + \rho_D^{p-1} |\tilde{\rho}'|^2 + \rho_D^2 |\nabla \Psi|^2 \right]
\]
and for even \( m \)
\[
\left\| \tilde{\rho}, \Psi \right\|_{m,\sigma(m)}^2 = \sum_{k=0}^{m} \int \chi_{2k,\sigma(m)} \left[ b^2 |\nabla \tilde{\rho}|^2 + \rho_D^{p-1} |\tilde{\rho}'|^2 + \rho_D^2 |\nabla \Psi|^2 \right].
\]

Let us briefly sketch the proof. First, we note that the weight function \( \chi_{m,k,\sigma} \) can be replaced by a smooth function \( \tilde{\chi}_{m,k,\sigma} \) with similar properties. In particular,
\[
|\nabla^\alpha \tilde{\chi}_{m,k,\sigma}| \leq C_{\alpha,m,k,\sigma} \frac{\tilde{\chi}_{m,k,\sigma}}{|Z|^{\alpha}}
\]
The functions \( \rho_D^2 \tilde{\chi}_{m,k,\sigma} \) and \( \rho_D^{p-1} \tilde{\chi}_{m,k,\sigma} \) also obey the property above. We now consider the case \( m = 2 \), let \( \tilde{\chi} \) be a weight function obeying (4.37) and observe that
\[
\int \tilde{\chi} |\partial_1 \partial_2 f|^2 = \int \tilde{\chi} \partial_1^2 f \partial_2^2 f - \int \partial_1 \partial_2 f \partial_1 \partial_2 f + \int \partial_2 \partial_2 f \partial_1^2 f
\]
Therefore,
\[
\int \tilde{\chi} |\partial_1 \partial_2 f|^2 \lesssim \int \tilde{\chi} (|\partial_1^2 f|^2 + |\partial_2^2 f|^2) + \int \frac{\tilde{\chi}}{|Z|^2} (|\partial_1 f|^2 + |\partial_2 f|^2)
\]
Using this for \( f = \nabla \tilde{\rho}, \tilde{\rho}, \nabla \Psi \) and with any mixed derivative in place of \( \partial_1 \partial_2 \) immediately confirms the equivalence of the norms (4.33) and (4.35) for \( m = 2 \). The equivalence for higher derivatives can be proved by induction. The equivalence with (4.36) follows from a similar Bochner type identity
\[
\sum_{i,j=1}^{d} \int \tilde{\chi} |\partial_i \partial_j f|^2 = \int \tilde{\chi} |\Delta f|^2 - \sum_{i,j=1}^{d} \int \partial_i \tilde{\chi} \partial_j f \partial_i \partial_j f + \sum_{i,j=1}^{d} \int \partial_j \tilde{\chi} \partial_i f \partial_i \partial_i f
\]
implying
\[
\sum_{i,j=1}^{d} \int \tilde{\chi} |\partial_i \partial_j f|^2 \leq \int \tilde{\chi} |\Delta f|^2 + \int \frac{\tilde{\chi}}{|Z|^2} |\nabla f|^2 \leq \int \tilde{\chi} |\Delta f|^2 + \int \frac{\tilde{\chi}}{|Z|^2} |f|^2
\]
This gives the equivalence of (4.33) and (4.36) for \( m = 2 \). Once again, higher norms follow by induction.
Finally, note that the above norm equivalences are even independent of the assumption of spherical symmetry on \( \tilde{\rho}, \Psi \).
2. Global control of the highest Sobolev norm:
\[ \|\tilde{\rho}, \Psi\|_{k_m}^2 = \|\tilde{\rho}, \Psi\|_{k_m,\sigma(k_m)}^2 \leq 1. \] (4.38)

3. Local decay of low Sobolev norms: for any \(0 \leq k \leq 2k_0\), any large \(\hat{Z} \leq Z^*\) and universal constant \(C = C(k_0)\):
\[ \|\langle \tilde{\rho}, \Psi \rangle\|_{H^k(Z \leq \hat{Z})} \leq \hat{Z}^C e^{-\frac{3k}{\delta_g} \tau} \] (4.39)

4. Pointwise bounds:
\[ \forall 0 \leq k \leq \frac{2k_m}{3}, \quad \|Z^{n(k)}(\frac{b}{\rho})\|_{L^\infty} \leq 1 \]
\[ \forall 1 \leq k \leq \frac{2k_m}{3}, \quad \|Z^n(\frac{b}{\rho}) \|_{L^\infty(Z \leq Z^*)} + \|Z^n(\frac{b}{\rho}) \|_{L^\infty(Z \geq Z^*)} \leq 1 \] (4.40)
with
\[ n(k) = \begin{cases} k & \text{for } k \leq \frac{4k_m}{9}, \\ \frac{k_m}{4} & \text{for } \frac{4k_m}{9} < k \leq \frac{2k_m}{3}. \end{cases} \] (4.41)

**Remark 4.3.** Since \(b = e^{-\mu(r-2)\tau}\), (4.20) and (4.21) imply that the initial data verify the bootstrap inequalities (4.34), (4.38), (4.40) with the bound \(e^{-ct_0}\) for some small universal constant \(c\).

The heart of the proof of Theorem 1.1 is the following:

**Proposition 4.4** (Bootstrap). Assume (see (4.16)) that
\[ \|e^{\tau N} PX(\tau)\|_{H^2k_0} \leq e^{-\frac{19k_0}{3} \tau} \] (4.42)
for all \(\tau \in [\tau_0, \tau^*]\) and that the bounds (4.26), (4.34), (4.38), (4.39), (4.40), (4.27) hold on \([\tau_0, \tau^*]\) with \(\delta_1^{-1}, \tau_0\) large enough. Then the following holds:
1. Exit criterion. The bounds (4.26), (4.34), (4.38), (4.39), (4.40), (4.27) can be strictly improved on \([\tau_0, \tau^*]\). Equivalently, \(\tau^* < +\infty\) implies
\[ \|e^{\tau N} PX(\tau^*)\|_{H^2k_0} e^{\frac{19k_0}{3} \tau^*} = 1. \] (4.43)
2. Linear evolution. The right hand side \(G\) of the equation for \(X(\tau)\)
\[ \partial_\tau X = MX + G \]
satisfies
\[ \|G(\tau)\|_{H^2k_0} \leq e^{2k_0 \tau}, \quad \forall \tau \in [\tau_0, \tau^*] \] (4.44)

We will show in section 8.3 that Proposition 4.4 immediately implies Theorem 1.1.

**Remark 4.5.** We note that the assumption (4.42) implies that
\[ \|PX(\tau)\|_{H^2k_0} \leq e^{\frac{k_0}{3} \tau}, \quad \forall \tau \in [\tau_0, \tau^*] \] (4.45)
We will prove the bootstrap Proposition 4.4 under the weaker assumption (4.45). Specifically, we will define \([\tau_0, \tau^*]\) to be the maximal time interval on which (4.45) holds and will show that both the bounds (4.26), (4.34), (4.38), (4.39), (4.40), (4.27) can be improved and that \(G\) satisfies (4.44).

We now focus on the proof of Proposition 4.4 and work on a time interval \([0, \tau^*]\), \(\tau_0 < \tau^* \leq +\infty\) on which (4.26), (4.34), (4.38), (4.39), (4.40), (4.27) and (4.45) hold.
5. Control of high Sobolev norms

We first turn to the global in space control of high Sobolev norms. This is an essential step to control the \( b \) dependence of the flow and the dissipative structure which can neither be treated by spectral analysis nor perturbatively.

We claim an improvement of the bound (4.34), controlling all but the highest weighted Sobolev norm.

**Proposition 5.1.** The exists a universal constant \( c_{k_m} > 0 \) such that for all \( 0 \leq m \leq k_m - 1 \)

\[
\| \tilde{\rho}, \Psi \|_{m,\sigma(m)} \leq e^{-c_{k_m} \tau}.
\]  

(5.1)

The rest of this section is devoted to the proof of Proposition 5.1.

5.1. Algebraic energy identity. We derive the energy identity for high Sobolev norms which in the hydrodynamical formulation has a quasilinear structure.

**Step 1** Equation for \( \tilde{\rho}, \Psi \). Recall (2.23):

\[
\begin{align*}
\partial_t \rho_T &= -\rho_T \Delta \Psi_T - \frac{\mu(r-1)}{2} \rho_T - (2\partial_Z \Psi_T + \mu Z) \partial_Z \rho_T \\
\rho_T \partial_t \Psi_T &= b^2 \Delta \rho_T - [\nabla \Psi_T]^2 + \mu(r-2) \Psi_T - 1 + \mu \Delta \Psi_T + \rho_T^{-1} \rho_T
\end{align*}
\]

By construction

\[
\begin{align*}
|\nabla \Psi P|^2 + \rho_D^{p-1} + \mu(r-2) \Psi_P + \mu \Lambda \Psi_P - 1 &= \tilde{\Delta} \rho_P, \\
\partial_t \rho_D + \rho_D \left[ \Delta \Psi_P + \frac{\mu(r-1)}{2} + (2\partial_Z \Psi_P + \mu Z) \frac{\partial_Z \rho_D}{\rho_D} \right] &= \tilde{\Delta} \rho_P (\rho_D)
\end{align*}
\]

(5.2)

with \( \tilde{\Delta} \) supported in \( Z \geq 3Z^* \). The linearized flow is given by

\[
\begin{align*}
\partial_t \tilde{\rho} &= -\rho_T \Delta \Psi - 2\nabla \rho_T \cdot \nabla \Psi + H_1 \tilde{\rho} - H_2 \Lambda \tilde{\rho} - \tilde{\Delta} \rho_P, \\
\partial_t \Psi &= b^2 \Delta \rho_T - \left\{ H_2 \Lambda \Psi + \mu(r-2) \Psi + |\nabla \Psi|^2 + (p-1) \rho_D^{p-2} \tilde{\rho} + NL(\tilde{\rho}) \right\} - \tilde{\Delta} \rho_P
\end{align*}
\]

(5.3)

with the nonlinear term

\[
NL(\tilde{\rho}) = (\rho_D + \tilde{\rho})^{p-1} - \rho_D^{p-1} - (p-1) \rho_D^{p-2} \tilde{\rho}.
\]

Note that the potentials

\[
H_2 = \mu + 2 \frac{\Psi P}{Z}, \quad H_1 = - \left( \Delta \Psi_P + \frac{\mu(r-1)}{2} \right)
\]

remain the same in these equations: they are not affected by the profile localization introduced by passing from \( \rho_P \) to \( \rho_D \). We recall the Emden transform formulas (2.24):

\[
\begin{align*}
H_2 &= \mu(1 - \frac{\omega}{Z}), \\
H_1 &= \frac{\mu w}{Z} (1 - \frac{\omega}{Z})
\end{align*}
\]

(5.4)

which, using (2.18), yield the bounds:

\[
\begin{align*}
|H_2| + O \left( \frac{1}{Z^j} \right), & \quad H_1 = - \frac{2\mu(r-1)}{p-1} + O \left( \frac{1}{Z^j} \right), \\
|\langle Z \rangle^j \partial_Z^2 H_1| + |\langle Z \rangle^j \partial_Z^2 H_2| & \lesssim \frac{1}{(Z^j)}, \quad j \geq 1
\end{align*}
\]

(5.5)

Our main task is now to produce an energy identity for (5.3) which respects the quasilinear nature of (5.3) and does not lose derivatives.
\[ \partial^k := (\partial^{k}_1, \ldots, \partial^{k}_d) \]
\[ \tilde{\rho}^{(k)} = \partial^{k} \tilde{\rho}, \quad \Psi^{(k)} = \partial^{k} \Psi. \]

We use
\[ [\partial^k, \Lambda] = k\partial^k \]
to compute from (5.3):
\[
\partial_\tau \tilde{\rho}^{(k)} = (H_1 - kH_2)\tilde{\rho}^{(k)} - H_2 \Delta \tilde{\rho}^{(k)} - (\partial^k \rho_T)\Delta \Psi - k\partial_\rho \partial^{k-1} \Delta \Psi - \rho_T \Delta \Psi^{(k)} - 2\nabla(\partial^k \rho_T) \cdot \nabla \Psi - 2\nabla \rho_T \cdot \nabla \Psi^{(k)} + F_1 \]  
(5.6)

with
\[
F_1 = -\partial^k \tilde{\rho}_p + [\partial^k, H_1] \tilde{\rho} - [\partial^k, H_2] \Delta \tilde{\rho} \quad \quad (5.7)
\]

For the second equation:
\[
\partial_\tau \Psi^{(k)} = b^2 \left( \frac{\partial^{k} \Delta \rho_T}{\rho_T} - \frac{k\partial^{k-1} \Delta \rho_T \partial_\rho_T}{\rho_T^2} \right) - kH_2 \Psi^{(k)} - H_2 \Delta \Psi^{(k)} - \mu(r - 2) \Psi^{(k)} - 2\nabla \Psi \cdot \nabla \Psi^{(k)} - \left[ (p - 1)\rho_D^{p-2} \tilde{\rho}^{(k)} + k(p - 1)(p - 2)\rho_D^{p-3} \partial_\rho \partial^{k-1} \tilde{\rho} \right] + F_2 \]  
(5.8)

with
\[
F_2 = -\partial^k \tilde{\rho}_p \Psi + b^2 \left[ \partial^k \left( \frac{\Delta \rho_T}{\rho_T} \right) - \frac{\partial^{k} \Delta \rho_T}{\rho_T} + \frac{k\partial^{k-1} \Delta \rho_T \partial_\rho_T}{\rho_T^2} \right] - [\partial^k, H_2] \Delta \Psi - (p - 1) \left[ [\partial^k, \rho_D^{p-2}] \tilde{\rho} - k(p - 2)\rho_D^{p-3} \partial_\rho \partial^{k-1} \tilde{\rho} \right] \quad \quad (5.9)
\]

**Step 3** Algebraic energy identity. Let \( \chi \) be a smooth function. We compute:
\[
\frac{1}{2} \frac{d}{dt} \left\{ \int b^2 \chi |\nabla \tilde{\rho}^{(k)}|^2 + (p - 1) \int \chi \rho_D^{p-2} \partial^{(k)} \rho_T^2 + \int \chi \rho_T^2 |\nabla \Psi^{(k)}|^2 \right\} = \frac{1}{2} \int \partial_\tau \chi \left\{ b^2 |\nabla \tilde{\rho}^{(k)}|^2 + (p - 1) \rho_D^{p-2} \partial_T \tilde{\rho}^{(k)}^2 + \rho_T^2 |\nabla \Psi^{(k)}|^2 \right\} - eb^2 \int \chi |\nabla \tilde{\rho}^{(k)}|^2 + \int \partial_\rho \tilde{\rho}^{(k)} \left[ -b^2 \chi \Delta \tilde{\rho}^{(k)} - b^2 \nabla \chi \cdot \nabla \tilde{\rho}^{(k)} + (p - 1) \chi \rho_D^{p-2} \partial_T \tilde{\rho}^{(k)} \right] \right.

\[ + \quad \left. \frac{p - 1}{2} \int \chi (p - 2) \partial_\rho \rho_D^{p-3} \partial_T \tilde{\rho}^{(k)}^2 + \int \chi \partial_\rho \partial_T \left[ \frac{p - 1}{2} \rho_D^{p-2} \partial_T \tilde{\rho}^{(k)}^2 + \rho_T |\nabla \Psi^{(k)}|^2 \right] - \int \partial_\tau \Psi^{(k)} \left( 2\chi \rho_T \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T^2 \Delta \Psi^{(k)} + \rho_T^2 \nabla \chi \cdot \nabla \Psi^{(k)} \right) \right]. \]
We compute:

\[
\begin{align*}
\int \partial_\tau \bar{\rho}^{(k)} \left[ -b^2 \chi \Delta \bar{\rho}^{(k)} - b^2 \nabla \chi \cdot \nabla \bar{\rho}^{(k)} + (p - 1) \chi \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right] & \\
= \int F_1 \left[ -b^2 \nabla \cdot (\chi \nabla \bar{\rho}^{(k)}) + (p - 1) \chi \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right] \\
+ \int \left[ (H_1 - k H_2) \bar{\rho}^{(k)} - H_2 \Delta \bar{\rho}^{(k)} - (\partial^k \rho_T) \Delta \Psi - 2 \nabla (\partial^k \rho_T) \cdot \nabla \Psi \right] \\
\times \left[ -b^2 \chi \Delta \bar{\rho}^{(k)} - b^2 \nabla \chi \cdot \nabla \bar{\rho}^{(k)} + (p - 1) \chi \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right] \\
- \int k \partial \rho_T \partial^k \Delta \Psi \left[ -b^2 \chi \Delta \bar{\rho}^{(k)} - b^2 \nabla \chi \cdot \nabla \bar{\rho}^{(k)} + (p - 1) \chi \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right] \\
- \int (\rho_T \Delta \Psi^{(k)} + 2 \nabla \rho_T \cdot \nabla \Psi^{(k)}) \left[ -b^2 \chi \Delta \bar{\rho}^{(k)} - b^2 \nabla \chi \cdot \nabla \bar{\rho}^{(k)} + (p - 1) \chi \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right] \\
= b^2 \int \chi \nabla F_1 \cdot \nabla \bar{\rho}^{(k)} + (p - 1) \int \chi F_1 \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \\
+ \int \left[ (H_1 - k H_2) \bar{\rho}^{(k)} - H_2 \Delta \bar{\rho}^{(k)} - (\partial^k \rho_T) \Delta \Psi - 2 \nabla (\partial^k \rho_T) \cdot \nabla \Psi \right] \\
\times \left[ -b^2 \nabla \cdot (\chi \nabla \bar{\rho}^{(k)}) + (p - 1) \chi \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right] \\
- \int k \partial \rho_T \partial^k \Delta \Psi \left[ -b^2 \nabla \cdot (\chi \nabla \bar{\rho}^{(k)}) + (p - 1) \chi \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right] \\
+ b^2 \int \nabla \chi \cdot \nabla \bar{\rho}^{(k)} \left( \rho_T \Delta \Psi^{(k)} + 2 \nabla \rho_T \cdot \nabla \Psi^{(k)} \right) \\
- \int \chi (\rho_T \Delta \Psi^{(k)} + 2 \nabla \rho_T \cdot \nabla \Psi^{(k)}) \left[ -b^2 \Delta \bar{\rho}^{(k)} + (p - 1) \rho_D^{p-2} \rho_T \bar{\rho}^{(k)} \right]
\end{align*}
\]
Similarly:

\[- \int \partial_t \Psi^{(k)} \left[ 2 \chi \rho_T \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T^2 \Delta \Psi^{(k)} + \rho_T^2 \nabla \chi \cdot \nabla \Psi^{(k)} \right] = - \int F_2 \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \]

\[- \int \left\{ b^2 \left( \frac{\partial^k \Delta \rho_T}{\rho_T} - \frac{k \partial^{k-1} \Delta \rho_T \partial \rho_T}{\rho_T^2} \right) \right\} \left[ 2 \chi \rho_T \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T^2 \Delta \Psi^{(k)} + \rho_T^2 \nabla \chi \cdot \nabla \Psi^{(k)} \right] \]

\[- \int \left\{ - k H_2 \Psi^{(k)} - H_2 \Lambda \Psi^{(k)} - \mu (r - 2) \Psi^{(k)} - 2 \nabla \Psi \cdot \nabla \Psi^{(k)} \right\} \]

\[\left[ (p - 1) \rho_D^{p-2} \tilde{\rho}^{(k)} + k (p - 1) (p - 2) \rho_D^{p-3} \partial \rho_D \partial^{k-1} \tilde{\rho} \right] \]

\[\times \left[ 2 \chi \rho_T \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T^2 \Delta \Psi^{(k)} + \rho_T^2 \nabla \chi \cdot \nabla \Psi^{(k)} \right] \]

\[= \int \chi \rho_T^2 \nabla \Psi^{(k)} \cdot \nabla F_2 \]

\[- b^2 \int (\partial^k \Delta \rho_D + \Delta \tilde{\rho}^{(k)}) \left[ 2 \chi \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T \Delta \Psi^{(k)} + \rho_T \nabla \chi \cdot \nabla \Psi^{(k)} \right] \]

\[+ b^2 \int \frac{k \partial^{k-1} \Delta \rho_T \partial \rho_T}{\rho_T} \left[ 2 \chi \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T \Delta \Psi^{(k)} + \rho_T \nabla \chi \cdot \nabla \Psi^{(k)} \right] \]

\[- \int \left[ - k H_2 \Psi^{(k)} - H_2 \Lambda \Psi^{(k)} - \mu (r - 2) \Psi^{(k)} - 2 \nabla \Psi \cdot \nabla \Psi^{(k)} \right] \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \]

\[+ \int (p - 1) \rho_D^{p-2} \tilde{\rho}^{(k)} \left[ 2 \chi \rho_T \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T^2 \Delta \Psi^{(k)} + \rho_T^2 \nabla \chi \cdot \nabla \Psi^{(k)} \right] \]

\[= \int \chi \rho_T^2 \nabla \Psi^{(k)} \cdot \nabla F_2 - b^2 \int (\partial^k \Delta \rho_D) \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \]

\[+ \int \left( - b^2 \Delta \tilde{\rho}^{(k)} + (p - 1) \rho_D^{p-2} \partial \rho_D \partial^{k-1} \tilde{\rho} \right) \left[ 2 \chi \nabla \rho_T \cdot \nabla \Psi^{(k)} + \chi \rho_T \Delta \Psi^{(k)} + \rho_T \nabla \chi \cdot \nabla \Psi^{(k)} \right] \]

\[+ b^2 \int \frac{k \partial^{k-1} \Delta \rho_T \partial \rho_T}{\rho_T^2} \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \]

\[- \int \left[ - k H_2 \Psi^{(k)} - H_2 \Lambda \Psi^{(k)} - \mu (r - 2) \Psi^{(k)} - 2 \nabla \Psi \cdot \nabla \Psi^{(k)} \right] \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \]

\[+ \int (p - 1) (p - 2) \rho_D^{p-3} \partial \rho_D \partial^{k-1} \tilde{\rho} \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \].
This yields the algebraic energy identity:

\[
\frac{1}{2} \frac{d}{dt} \left\{ b^2 \chi |\nabla \rho(k)|^2 + (p - 1) \int \chi \rho_D^{p-2} \rho_T (\rho(k)) \rho_T (\rho(k)) \rho_D^{p-2} + \int \chi \rho_T^2 |\nabla \Psi(k)|^2 \right\} \\
= \frac{1}{2} \int \partial_t \chi \left\{ b^2 |\nabla \rho(k)|^2 + (p - 1) \rho_D^{p-2} \rho_T (\rho(k))^2 + \rho_T^2 |\nabla \Psi(k)|^2 \right\} \\
- b^2 \int (\partial_k \Delta \rho_D) \nabla \cdot (\chi \rho_T^2 \nabla \Psi(k)) \\
- e b^2 \int \chi |\nabla \rho(k)|^2 + \int \frac{\partial_T \rho_T}{\rho_T} \left[ \frac{p - 1}{2} \rho_D^{p-2} \rho_T (\rho(k))^2 + \rho_T^2 |\nabla \Psi(k)|^2 \right] \\
+ \frac{p - 1}{2} \int \chi (p - 2) \frac{\partial_T \rho_D}{\rho_D} \rho_D^{p-2} \rho_T (\rho(k))^2 \\
+ \int F_1 \chi (p - 1) \rho_D^{p-2} \rho_T \rho(k) + b^2 \int \chi \nabla F_1 \cdot \nabla \rho(k) + \int \chi \rho_T^2 \nabla F_2 \cdot \nabla \Psi(k) \\
+ \int \left[ (H_1 - k H_2) \rho(k) - H_2 \Delta \rho(k) - (\partial_k \rho_T) \Delta \Psi - 2 \nabla (\partial_k \rho_T) \cdot \nabla \Psi \right] \\
\times [-\rho_D^{p-2} \rho_T \rho(k)] \\
- \int \left[ -k H_2 \Psi(k) - H_2 \Delta \Psi(k) - \mu (\gamma - 2) \Psi(k) - 2 \nabla \Psi \cdot \nabla \Psi(k) \right] \nabla \cdot (\chi \rho_T^2 \nabla \Psi(k)) \\
- \int k \partial_k \partial_T \partial_{k-1} \Delta \Psi \left[ -b^2 \nabla \cdot (\chi \nabla \rho(k)) + (p - 1) \chi \rho_D^{p-2} \rho_T \rho(k) \right] \\
+ b^2 \int \frac{k \partial_k \Delta \rho_D \partial_T \rho_T \nabla \cdot (\chi \rho_T^2 \nabla \Psi(k))}{\rho_T^2} \\
+ \int k (p - 1) (p - 2) \rho_D^{p-3} \partial_D \partial_{k-1} \nabla \cdot (\chi \rho_T^2 \nabla \Psi(k)) \\
+ b^2 \int \nabla \chi \cdot \nabla \rho(k) \left( \rho_T \Delta \Psi(k) + 2 \nabla \rho_T \cdot \nabla \Psi(k) \right) \\
+ \int (-b^2 \Delta \rho(k) + (p - 1) \rho_D^{p-2} \rho_T \rho(k)) \left[ \rho_T \nabla \chi \cdot \nabla \Psi(k) \right].
\] (5.10)

5.2. **Weighted \(L^2\) bound for \(m \leq k_m - 1\).** Given \(\sigma \in \mathbb{R}\), we recall the notation

\[
\|\rho, \Psi\|_{k, \sigma}^2 = \sum_{k=0}^{m} \int \chi \rho_{k,m} [b^2 |\nabla \rho_m|^2 + (p - 1) \rho_D^{p-2} \rho_T^2 \rho_m + \rho_T^2 |\nabla \Psi_m|^2] \\
\chi_{k,m}(Z) = \frac{1}{(Z)^{2\sigma}} \xi_k \left( \frac{Z}{Z} \right)
\]

We let

\[
I_{k,\sigma} = \int \xi_k \left( \frac{Z}{Z} \right) \left[ b^2 |\nabla \rho(k)|^2 + (p - 1) \rho_D^{p-2} \rho_T (\rho(k))^2 + \rho_T^2 |\nabla \Psi(k)|^2 \right]. \quad \text{(5.11)}
\]

**Lemma 5.2 (Weighted \(L^2\) bound).** Recall the definition (4.28), (4.29) of \(\sigma(m)\) and let

\[
\sigma = \sigma(k) \\
\nu + \frac{2r-1}{p-1} = \tilde{\nu}
\]

then there exists \(c_{k_m} > 0\) such that for all \(0 < \nu < \tilde{\nu}(k_m) \ll 1\) and \(b_0 < b_0(k_m) \ll 1\), for all \(1 \leq k \leq k_m - 1\), \(I_k := I_{k,\sigma(k)}\) given by (5.11) satisfies the differential inequality

\[
\frac{dI_k}{dt} + 2 \mu \tilde{\nu} I_k \leq e^{-c_{k_m} \tau}. \quad \text{(5.13)}
\]

We claim that Lemma 5.2 implies Proposition 5.1.
Proof of Proposition 5.1. Integrating (5.13) on the interval \([\tau_0, \tau]\), with initial data prescribed at \(\tau_0\), we obtain
\[
I_k(\tau) \leq e^{-2\mu\bar{\nu}(\tau-\tau_0)} I_k(\tau_0) + \frac{1}{c_k} \left( e^{-2\mu\bar{\nu}(\tau-\tau_0) - c_k \tau_0} - e^{-c_k \tau} \right).
\]
We now recall, see Remark 4.3, that \(I_k(\tau_0) \leq e^{-c\tau_0}\). Choosing \(4\mu\bar{\nu} \leq \min\{c, c_k\}\) we obtain that
\[
I_k(\tau) \leq 2e^{-2\mu\bar{\nu}\tau}.
\] (5.14)

We now recall from (4.35) and (4.36) for even \(m\) that \(\|\tilde{\rho}, \Psi\|_{m, \sigma}\) controls all the corresponding Sobolev norms: let a multi-index \(\alpha = (\alpha_1, \ldots, \alpha_d)\) with
\[\alpha_1 + \cdots + \alpha_d = |\alpha|, \quad \nabla^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d},\]
then for all \(|\alpha| = k, \ 0 \leq k \leq m,
\[
\sum_\alpha b_2^2 \chi_{k,m,\sigma} |\nabla^\alpha \tilde{\rho}|^2 + (p - 1) \int \chi_{k,m,\sigma} \rho_D^{p-2} \rho_T |\nabla^\alpha \tilde{\rho}|^2 + \int \chi_{k,m,\sigma} \rho_D^{p-2} |\nabla^\alpha \Psi|^2 \lesssim \|\tilde{\rho}, \Psi\|_{k,\sigma}^2, (5.15)
\]
and similarly the norm \(\|\tilde{\rho}, \Psi\|_{k,\sigma}^2\) (with even \(k\)) is equivalent to the one where \(\partial^m\)
with \(1 \leq m \leq k\) derivatives are replaced by \(\Delta^m\) with \(1 \leq m \leq \frac{k}{2}\).

We now claim
\[
\|\tilde{\rho}, \Psi\|_{m, \sigma(m)}^2 \leq \sum_{k=0}^m I_{k, \sigma(k)}. (5.16)
\]
Combining this with (5.14) concludes the proof of (5.1) (with \(c_k = \mu\bar{\nu}\)).

Proof of (5.16). Indeed,
\[
\|\tilde{\rho}, \Psi\|_{m, \sigma(m)}^2 = \sum_{k=0}^m \int \chi_{k,m,\sigma(m)} \left[ b_2^2 |\nabla^\alpha \tilde{\rho}|^2 + (p - 1) \rho_D^{p-2} \rho_T |\nabla^\alpha \tilde{\rho}|^2 + \rho_D^{p-2} |\nabla^\alpha \Psi|^2 \right]
\]
\[
= \sum_{k=0}^m \int \frac{\langle Z \rangle^{2k}}{\langle Z \rangle^{2(m+\sigma(m))}} \xi_m(x) \left[ b_2^2 |\nabla^\alpha \tilde{\rho}|^2 + (p - 1) \rho_D^{p-2} \rho_T |\nabla^\alpha \tilde{\rho}|^2 + \rho_D^{p-2} |\nabla^\alpha \Psi|^2 \right]
\]
and
\[
\sum_{k=0}^m I_{k, \sigma(k)} = \sum_{k=0}^m \int \frac{\xi_k(x)}{\langle Z \rangle^{2\sigma(k)}} \left[ b_2^2 |\nabla^\alpha \tilde{\rho}|^2 + (p - 1) \rho_D^{p-2} \rho_T |\nabla^\alpha \tilde{\rho}|^2 + \rho_D^{p-2} |\nabla^\alpha \Psi|^2 \right]
\]
and hence (5.16) follows from \(\sigma(k) + k \leq \sigma(m) + m\) and \(\xi_k(x) \geq \xi_m(x)\) for \(0 \leq k \leq m\).

5.3. Proof of Lemma 5.2. This follows from the energy identity (5.10) coupled with the pointwise bound (4.40) to control the nonlinear term.

**step 1** Interpolation bounds. In what follows we use the convention \(\lesssim\) to denote any dependence on the universal constants, including \(k_m\). Constants \(c, c_k\) will stand for generic, universal small constants.

Our main technical tool below will be the following interpolation bound: for any \(0 \leq m \leq k - 1\) and \(\delta > 0\), there exists \(c_k, k_m > 0\) such that
\[
\|\tilde{\rho}, \Psi\|_{m, \sigma(m) + \delta}^2 \leq e^{-c_k, k_m \tau}. (5.17)
\]
Indeed, the claim follows by interpolating the local decay bootstrap bound (4.39) and the bound (4.38) for the highest Sobolev norm for \( Z \leq Z_c^* := (Z^*)^c \) and using the global weighted Sobolev bound for (4.34) for \( Z \geq Z_c^* \)

\[
\| \tilde{\rho}, \Psi \|^2_{m,\sigma(m)+\delta} \leq (Z^*)^C e^{-c km^\tau} + \frac{1}{(Z_c^*)^{2\tau}} \| \tilde{\rho}, \Psi \|^2_{m,\sigma(m)} \leq e^{-c \delta k m^\tau} \tag{5.18}
\]

We will also use the bound for the damped profile from (4.7), (4.8) and (4.9):

\[
|Z^k \partial^k_Z \rho_D| \lesssim \frac{1}{(Z)} \frac{1}{(Z^*)} Z \lesssim 1 + \frac{1}{(Z^*)} \frac{1}{(Z^*)^\beta} \frac{1}{\rho^\tau} Z \geq Z^*. \tag{5.19}
\]

We will also use the bound

\[
\chi_{k-1,k-1,\sigma(k-1)} \leq \langle Z \rangle^{2(\alpha + \beta)} \chi_{k,k,\sigma(k)} \leq \langle Z \rangle^2 \chi_{k,k,\sigma(k)}, \tag{5.20}
\]

which follows from

\[
\sigma(k - 1) + \alpha \geq \sigma(k), \quad \tilde{\sigma}(k - 1) \leq \tilde{\sigma}(k) + \beta \tag{5.21}
\]

and \( \alpha + \beta \leq 1 \).

**step 2** Energy identity. We run (5.10) with

\[
\chi = \frac{1}{(Z)^{2\sigma}} \xi_k \left( \frac{Z}{Z^*} \right), \quad \sigma = \sigma(k), \quad 1 \leq k \leq k_m - 1
\]

with \( \xi_k(x) = 1 \) for \( x \leq 1 \) and \( \xi_k(x) = x^{2\delta(k)} \) for \( x > 1 \), and estimate all terms. In our notations

\[
\chi = \chi_{k,k,\sigma(k)}.
\]

From (4.28), (4.29) and recalling \( m_0 = \frac{4km}{9} + 1 \):

\[
\sigma(k) + k = \sigma_\nu \text{ for } 0 \leq k \leq m_0
\]

\[
-\alpha(k_m - k) + k = (\alpha + 1)(k - m_0) + \sigma_\nu \text{ for } m_0 \leq k \leq k_m
\]

and

\[
\tilde{\sigma}(k) = \begin{cases} np - \frac{2(r-1)}{p-1} - (r-2) + 2\tilde{\nu} & \text{for } 0 \leq k \leq \frac{2km}{3} + 1 \\ \beta(k_m - k) & \text{for } \frac{2km}{3} + 1 \leq k \leq k_m \end{cases}
\]

\[
\leq np - \frac{2(r-1)}{p-1} - (r-2) + 2\tilde{\nu} \tag{5.23}
\]

which implies

\[
\chi = \frac{1}{(Z)^{2\sigma(k)}} \xi_k \left( \frac{Z}{Z^*} \right) \tag{5.24}
\]

\[
\leq \frac{1}{(Z)^{2\sigma(k)}} \left[ 1 + \left( \frac{Z}{Z^*} \right)^{2np - \frac{4(r-1)}{p-1} - 2(r-2) + 4\tilde{\nu}} \right] \chi_{Z \geq Z^*}
\]

\[
\leq \frac{1}{(Z)^{-2k+2} \left( \frac{4}{2} + \tilde{\nu} - \frac{2(r-1)}{p-1} \right) -(r-1)} + \left( \frac{Z}{Z^*} \right)^{2np - \frac{4(r-1)}{p-1} - 2(r-2) + 4\tilde{\nu}} \chi_{Z \geq Z^*}.
\]
which we will use below. The following additional inequality will be of particular significance ($b = (Z^*)^{2-r}$):

\[
\rho_T^2 \chi \lesssim \frac{1}{\langle Z \rangle^{-2k+2(\frac{d}{2}+\bar{r}-(r-1))}} \left[ 1_{Z \leq Z^*} + \left( \frac{Z}{Z^*} \right)^{4\delta-2(r-2)} 1_{Z \geq Z^*} \right]
\]

\[
= \frac{1}{\langle Z \rangle^{-2k+2(\frac{d}{2}+\bar{r}-(r-1))}} 1_{Z \leq Z^*} + \frac{1}{b^2 \frac{\bar{r}}{r-2}} \langle Z \rangle^{-2k+2(\frac{d}{2}-\bar{r}-1)} 1_{Z \geq Z^*} \tag{5.25}
\]

**step 3** Leading order terms. In what follows, we will systematically use the standard Pohozaev identity:

\[
\int \Delta g F \cdot \nabla g \, dx = \sum_{i,j=1}^{d} \int \partial_i^2 g F_j \partial_j g \, dx = - \sum_{i,j=1}^{d} \int \partial_i g (\partial_i F_j \partial_j g + F_j \partial_i^2 g)
\]

\[
= - \sum_{i,j=1}^{d} \int \partial_i F_j \partial_i g \partial_j g + \frac{1}{2} \int |\nabla g|^2 \nabla \cdot F \tag{5.26}
\]

which becomes in the case of spherically symmetric functions

\[
\int_{\mathbb{R}^d} f \Delta g \partial_r g \, dx = c_d \int_{\mathbb{R}^d} f \frac{\partial_r}{r^{d-1}} (r^{d-1} \partial_r g) r^{d-1} \partial_r g \, dx = - \frac{1}{2} \int_{\mathbb{R}^d} |\partial_r g|^2 \left[ f' - \frac{d-1}{r} f \right] \, dx
\]

**Cross terms.** We consider

\[
A_1 = b^2 k \left[ \int \partial_T \partial^{k-1} \Delta \Psi \nabla \cdot (\chi \nabla \rho^{(k)}) + \frac{\partial^{k-1} \Delta \rho_T \partial_T \rho_T}{\rho_T^2} \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \right].
\]

We compute:

\[
\partial_T \partial^{k-1} \Delta \Psi \nabla \cdot (\chi \nabla \rho^{(k)}) + \frac{\partial^{k-1} \Delta \rho_T \partial_T \rho_T}{\rho_T^2} \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)})
\]

\[
= \partial_T \partial^{k-1} \Delta \Psi \left[ \nabla \chi \cdot \nabla \rho^{(k)} + \chi \Delta \rho^{(k)} \right]
\]

\[
+ \partial^{k-1} \Delta \rho_T \partial_T \left[ \nabla \chi \cdot \nabla \Psi^{(k)} + 2 \chi \frac{\nabla \rho_T}{\rho_T} \cdot \nabla \Psi^{(k)} + \chi \Delta \Psi^{(k)} \right]
\]

\[
= \partial_T \partial^{k-1} \Delta \Psi \nabla \chi \cdot \nabla \rho^{(k)} + \partial^{k-1} \Delta \rho_T \partial_T \nabla \chi \cdot \nabla \Psi^{(k)} + 2 \partial^{k-1} \Delta \rho_T \partial_T \chi \frac{\nabla \rho_T}{\rho_T} \cdot \nabla \Psi^{(k)}
\]

\[
+ \chi \partial_T \partial^{k-1} \Delta \Psi \Delta \rho^{(k)} + \chi \partial^{k-1} \Delta \rho_T \partial_T \Delta \Psi^{(k)}.
\]
The last 2 terms require an integration by parts:
\[ b^2 k \left| \int \left[ \chi \partial_T \partial^{k-1} \Delta \Psi \Delta \tilde{\rho}^{(k)} + \chi \partial^{k-1} \Delta \rho_T \partial_T \Delta \Psi^{(k)} \right] \right| \]
\[ = b^2 k \left| \int \left[ -(\Delta \tilde{\rho}) \partial_T (\chi \partial_T \partial^{k-1} \Delta \Psi) + \chi \partial^{k-1} \Delta \rho_T \partial_T \Delta \Psi^{(k)} \right] \right| \]
\[ = b^2 k \left| \int \left[ -\partial^{k-1} \Delta \tilde{\rho} \left[ \chi \partial^2 \partial^{k-1} \Delta \Psi + \partial \chi \partial_T \partial^{k-1} \Delta \Psi \right] + \chi \partial^{k-1} \Delta \rho_T \partial_T \Delta \Psi^{(k)} \right] \right| \]
\[ \lesssim C_k b^2 \int \chi |\partial^{k-1} \Delta \Psi| \left[ |\partial^{k-1} \Delta \tilde{\rho} T| + \frac{|\partial^{k-1} \Delta \rho_T|}{\langle Z \rangle} + |\partial (\chi \partial^{k-1} \Delta \rho_T \partial_T) \right] \]
\[ \lesssim C_k b^2 \int \chi \rho_T |\partial^{k-1} \Delta \Psi| \left[ \frac{\rho_D}{\langle Z \rangle^{k+2}} + \frac{|\partial^{k-1} \Delta \tilde{\rho}|}{\langle Z \rangle} \right] \]
\[ \lesssim \sum_{|\alpha|=k} \frac{\chi \rho_D^2}{\langle Z \rangle^{2k+3}} |\nabla \partial^\alpha \tilde{\rho}|^2 + c_k b^4 \int \frac{\chi}{\langle Z \rangle} |\nabla \partial^\alpha \tilde{\rho}|^2 + b^4 \int \chi \frac{\rho_D^2}{\langle Z \rangle^{2k+3}}, \]
where in penultimate inequality we used the pointwise bound (4.40).

We now estimate the source term from (5.25):
\[ b^4 \int \frac{\chi \rho_D^2}{\langle Z \rangle^{2k+3}} \lesssim b^4 \int_{Z \leq Z^*} \frac{Z^{d-1} dZ}{\langle Z \rangle^{2k+3-2k+2(\frac{d}{2}+\varphi-(r-1))}} \]
\[ + b^4 \int_{Z \geq Z^*} \frac{\left( \frac{Z}{Z^*} \right)^{2\varphi} Z^{d-1} dZ}{\langle Z \rangle^{2k+3-2k+2(\frac{d}{2}+\varphi)}} \]
\[ \lesssim b^4 \int_{Z \leq Z^*} \frac{\langle Z \rangle^{2(r-2)-2-2\varphi}}{Z^{d-1}} dZ + b^4 \int_{Z \geq Z^*} \left( \frac{Z}{Z^*} \right)^{2\varphi} \langle Z \rangle^{-2-2\varphi} dZ \]
\[ \lesssim b^4 (Z^*)^{2(r-2)-1-2\varphi} \lesssim e^{-c \tau} \] (5.27)
and hence, using (5.18),
\[ b^2 k \left| \int \left[ \chi \partial_T \partial^{k-1} \Delta \Psi \Delta \tilde{\rho}^{(k)} + \chi \partial^{k-1} \Delta \rho_T \partial_T \Delta \Psi^{(k)} \right] \right| \lesssim e^{-c \tau} + \| \tilde{\rho}, \Psi \|_{k, \sigma+\frac{1}{2}}^2 \]
\[ \lesssim e^{-c_{km} \tau}. \]

We estimate similarly,
\[ k b^2 | \partial_T \partial^{k-1} \Delta \Psi \nabla \cdot \nabla \tilde{\rho}^{(k)} + \partial^{k-1} \Delta \rho_T \partial_T \nabla \cdot \nabla \Psi^{(k)} + 2 \partial^{k-1} \Delta \rho_T \partial_T \chi \frac{\nabla \rho_T}{\rho_T} \cdot \nabla \Psi^{(k)} | \]
\[ \lesssim \sum_{|\alpha|=k} \left[ b^4 \int \frac{\chi}{\langle Z \rangle} |\nabla \partial^\alpha \tilde{\rho}|^2 + \int \frac{\chi}{\langle Z \rangle} \rho_D^2 |\nabla \partial^\alpha \Psi|^2 \right] + b^4 \int \chi \frac{\rho_D^2}{\langle Z \rangle^{2k+3}} \]
\[ \lesssim e^{-c_{km} \tau} + \| \tilde{\rho}, \Psi \|_{k, \sigma+\frac{1}{2}}^2 \lesssim e^{-c_{km} \tau}. \]

The remaining cross terms are estimated as follows.
\[ k(p-1) \left| \int \chi \partial_T \partial^{k-1} \Delta \Psi \rho^{p-2}_D \rho_T \tilde{\rho}^{(k)} \right| \lesssim c_k \int \chi \frac{\rho_D^{p-1}}{\langle Z \rangle} \rho_T |\partial^{k-1} \Delta \Psi| |\tilde{\rho}^{(k)}| \]
\[ \lesssim \int \frac{\chi}{\langle Z \rangle} \rho_D^{p-1} |\tilde{\rho}^{(k)}|^2 + \int \frac{\chi}{\langle Z \rangle} \rho_T^2 |\nabla \partial^\alpha \Psi|^2 \lesssim \| \tilde{\rho}, \Psi \|_{k, \sigma+\frac{1}{2}}^2 \lesssim e^{-c_{km} \tau}, \]
where we used that \( p \geq 1 \) and a trivial bound \( |\rho_D| \leq 1 \). Similarly,
\[ \int |(p-1) \rho_D^{p-2} \tilde{\rho}^{(k)} \rho_T \nabla \cdot \nabla \Psi^{(k)} | \lesssim \int \frac{\chi}{\langle Z \rangle} \rho_D^{p-1} |\tilde{\rho}^{(k)}|^2 + \int \frac{\chi}{\langle Z \rangle} \rho_T^2 |\nabla \partial^\alpha \Psi|^2 \leq \| \tilde{\rho}, \Psi \|_{k, \sigma+\frac{1}{2}}^2 \lesssim e^{-c_{km} \tau}. \]
We use the pointwise bootstrap bound (4.40) we estimate after an integration by parts:

$$k(p-1)(p-2) \left| \int \nabla \cdot (\chi \rho_{\rho D}^2 \nabla \Psi^{(k)}) \rho_{\rho D}^{p-3} \partial \rho_D \partial^{k-1} \rho \right|$$

$$\lesssim \int \frac{\chi}{(Z)^2} \rho_{\rho D}^{p-1} \nabla \rho_{k-1}^2 \left. + \right. \int \frac{\chi}{(Z)^2} \rho_{\rho D}^{p-1} \rho_{k-1}^2 \left. + \right. \int \frac{\chi}{(Z)^2} \rho_{\rho D}^{2} \nabla \partial^\delta \Psi \right| ^2 \leq \| \rho, \Psi \| _{k, \sigma + \frac{1}{2}}^2$$

$$\lesssim e^{-\epsilon_{k\rho D} \tau}.$$ 

The other remaining cross term is estimated using an integration by parts:

$$\rho_k \text{ terms.}$$

We compute using (5.5):

$$\int \chi(H_2 - kH_2) \rho^{(k)}(-b^2 \Delta \rho^{(k)} + (p-1) \rho^{-2} \rho D \rho^{(k)}) - b^2 \int [H_1 - kH_2] \rho^{(k)} \nabla \chi \cdot \nabla \rho^{(k)}$$

$$= \int \chi(H_1 - kH_2) \left[ b^2 |\nabla \rho^{(k)}|^2 + (p-1) \rho^{-2} \rho D \rho^{(k)} \right] - \frac{b^2}{2} \int (\rho^{(k)})^2 \nabla \cdot [\chi \nabla (H_1 - kH_2)]$$

$$= - \mu \chi \left( k + \frac{2(r-1)}{p-1} + O \left( \frac{1}{(Z)^{\tau}} \right) \right) \left( b^2 |\nabla \rho^{(k)}|^2 + (p-1) \rho^{-2} \rho D \rho^{(k)} \right)$$

$$- \frac{b^2}{2} \int \rho^{(k)} \left| \frac{\Lambda \chi \Lambda (H_1 - kH_2)}{\chi Z^2} + \Delta (H_1 - kH_2) \right| ,$$

where we used the interpolation bound (5.18). Similarly, using that \( \chi_{k,k,\sigma} = (Z)^2 \chi_{k,k-1,\sigma} \) and \( |\rho_k| \leq |\nabla \rho_{k-1}| \) as well as (5.5), (5.18) gives

$$\frac{b^2}{2} \int \chi(\tilde{\rho}^{(k)})^2 \left[ \frac{\Lambda \chi \Lambda (H_1 - kH_2)}{\chi Z^2} + \Delta (H_1 - kH_2) \right] \lesssim \| \tilde{\rho}, \Psi \| _{k, \sigma (k) + \frac{1}{2}(1+r)} \lesssim e^{-\epsilon_{k\rho D} \tau}.$$ 

Next using

$$|\partial^\delta \rho_D| \lesssim \frac{\rho_D}{(Z)^k},$$

we estimate after an integration by parts:

$$b^2 \left[ \chi \Delta \rho^{(k)} + \nabla \chi \cdot \nabla \rho^{(k)} \right] \left[ (\partial^\delta \rho_D) \Delta \Psi + 2 \nabla (\partial^\delta \rho_D) \cdot \nabla \Psi \right]$$

$$\lesssim b^2 \int \chi |\nabla \rho^{(k)}| \left[ |\nabla (\partial^\delta \rho_D \Delta \Psi)| + |\nabla (\partial^\delta \rho_D \cdot \nabla \Psi)| + \frac{|(\partial^\delta \rho_D) \Delta \Psi + 2 \nabla (\partial^\delta \rho_D) \cdot \nabla \Psi|}{(Z)} \right]$$

$$\leq b^2 \int \chi \left| \frac{\nabla \rho^{(k)}}{(Z)} \right|^2 + b^2 \sum_{j=1}^3 \int \chi \left( \frac{\rho_D}{(Z)^{2k}} \right) \left( \frac{|\partial^j \Psi|}{(Z)^{3-\tau}} \right)^2$$

We use the pointwise bootstrap bound (4.40)

$$|(Z)^j \partial^j \Psi| \leq C_K \left[ \frac{1}{(Z)^{r-2}} + b \right] \lesssim \left[ \frac{1}{(Z)^{r-2}} + \frac{1}{(Z^*)^{r-2}} \right], \quad 1 \leq j \leq 3 \quad (5.28)$$
We recall that by definition of the norm:

\[ b^2 \sum_{j=1}^{3} \chi \langle Z \rangle \rho_D^2 \left( \frac{|\partial^j \Psi|}{\langle Z \rangle^{2k}} \right)^2 \leq b^2 C_K \int \frac{Z^{d-1}dZ}{\langle Z \rangle^{2\left(\frac{d}{2}-(r-1)\right)}} \frac{1}{\langle Z \rangle^5} \left( \left( \frac{1_{Z\leq Z^*}}{\langle Z \rangle^{2(r-2)}} + \frac{1_{Z> Z^*}}{\langle Z \rangle^{2(r-2)}} \right) \right) \]

Note that the last term in the case \( k = 0 \) should be treated with the help of the bound \( \rho \lesssim \rho_D \) and the estimate (5.27). For \( k \neq 0 \), we simply use \( |\rho_k| \leq |\nabla \rho_{k-1}| \).

We recall that by definition of the norm:

\[ \|\hat{\rho}, \Psi\|_{k,\sigma}^2 \gtrsim \sum_{m=0}^{k} \int \chi \frac{\rho_D^2 |\partial^m \nabla \Psi|^2}{\langle Z \rangle^{2(k-m)}} \gtrsim \sum_{m=1}^{k+1} \int \chi \frac{\rho_T^2 |\partial^m \Psi|^2}{\langle Z \rangle^{2(k+1-m)}}. \]

Hence, by the interpolation bound,

\[ \left| \int \chi \left( (\partial^k \rho_D) \Delta \Psi + 2\nabla (\partial^k \rho_D) \cdot \nabla \Psi \right) \right| (p-1) \rho_D^{p-2} \rho_T \rho^{(k)} \]

\[ \lesssim \int \chi \rho_D^{p-2} \rho_T \langle \hat{\rho}^{(k)} \rangle^2 \langle Z \rangle^{2k-1} + \int \chi \rho_T^{p-2} \rho_T^2 \left[ \frac{|\partial^2 \Psi|^2}{\langle Z \rangle^{2(k+1)-1}} + \frac{|\partial \Psi|^2}{\langle Z \rangle^{2(k+1)-1}} \right] \]

\[ \lesssim \|\hat{\rho}, \Psi\|_{k,\sigma+\frac{1}{2}}^2 \lesssim e^{-c_k \tau}. \]

For the nonlinear term, we integrate by parts and use (5.28):

\[ \left| \int \chi \left( \hat{\rho}^{(k)} \Delta \Psi + 2\nabla \hat{\rho}^{(k)} \cdot \nabla \Psi \right) \right| (p-1) \rho_D^{p-2} \rho_T \tilde{\rho}^{(k)} \]

\[ \lesssim \int \chi \frac{\rho_D^{p-2} \rho_T \langle \hat{\rho}^{(k)} \rangle^2}{\langle Z \rangle} \lesssim e^{-c_k \tau}. \]
From Pohozaev (5.26) and (5.5):
\[
- \int H_2 \chi H_2 \Lambda \tilde{\rho}(k) (-b^2 \Delta \tilde{\rho}(k)) + b^2 \int H_2 \Lambda \rho_k \nabla \chi \cdot \nabla \tilde{\rho}(k) + \frac{1}{2} \int \left\{ \int |\nabla \tilde{\rho}(k)|^2 \nabla \cdot \left( Z \chi H_2 \right) + \sum_{i,j=1}^{d} \int H_2 Z_j \partial_j \tilde{\rho}(k) \partial_i \chi \partial_i \tilde{\rho}(k) \right\} + \frac{1}{2} \int |\nabla \tilde{\rho}(k)|^2 \chi \left( H_2 + \frac{\Lambda \chi}{\chi} \right) \left( d + \frac{\Lambda \chi}{\chi} + O \left( \frac{1}{(Z)^{r-1}} \right) \right).
\]

Integrating by parts and using (5.5):
\[
- \int \chi H_2 \Lambda \tilde{\rho}(k) \left[ (p - 1) \rho_D^{p-2} \rho_T \tilde{\rho}(k) \right] + \frac{p-1}{2} \int \chi (p-2) \partial_T \rho_D \rho_D^{p-3} \rho_T \tilde{\rho}(k)^2 + \frac{p-1}{2} \int \chi \partial_T \rho_D \rho_D^{p-2} \tilde{\rho}(k)^2 = \frac{p-1}{2} \int \chi \partial_T \rho_T \rho_T^{p-2} \tilde{\rho}(k)^2 \left[ \left( \frac{\partial_T \rho_D + \mu \Lambda \rho_D}{\rho_D} \right) + \frac{\partial_T \rho_T + \mu \Lambda \rho_T}{\rho_T} + O \left( \frac{1}{(Z)^{r-1}} \right) \right] - 2 \frac{\mu (r-1)}{p-1} + O \left( \frac{1}{(Z)^r} \right).
\]

We now claim the fundamental behavior
\[
\frac{\partial_T \rho_D + \mu \Lambda \rho_D}{\rho_D} = -2 \frac{\mu (r-1)}{p-1} + O \left( \frac{1}{(Z)^r} \right) \quad (5.30)
\]
and
\[
\frac{\partial_T \rho_T + \mu \Lambda \rho_T}{\rho_T} = -2 \frac{\mu (r-1)}{p-1} + O \left( \frac{1}{(Z)^r} \right), \quad (5.31)
\]
Assume (5.30), (5.31), we obtain
\[
- \int \chi H_2 \Lambda \tilde{\rho}(k) \left[ (p - 1) \rho_D^{p-2} \rho_T \tilde{\rho}(k) \right] + \frac{p-1}{2} \int \chi (p-2) \partial_T \rho_D \rho_D^{p-3} \rho_T \tilde{\rho}(k)^2 + \frac{p-1}{2} \int \partial_T \rho_D \rho_D^{p-2} \rho_T \tilde{\rho}(k)^2
\]
\[
= \frac{p-1}{2} \int \chi \rho_D^{p-2} \rho_T \tilde{\rho}(k)^2 \left[ \frac{d + \frac{\Lambda \chi}{\chi}}{\chi} - 2 (r-1) \right] + O \left( \frac{1}{(Z)^r} \right) + O \left( e^{-c_{\chi} m} \right).
\]

Proof of (5.30). From (4.9):
\[
\partial_T \rho_D + \mu \Lambda \rho_D = -\mu \Lambda \zeta \left( \frac{Z}{Z^*} \right) \rho_P (Z) + \mu \Lambda \zeta \left( \frac{Z}{Z^*} \right) \rho_P (Z) + \mu \zeta \left( \frac{Z}{Z^*} \right) \Lambda \rho_P = \mu \zeta \left( \frac{Z}{Z^*} \right) \Lambda \rho_P
\]
\[
\frac{\partial_T \rho_D + \mu \Lambda \rho_D}{\rho_D} = \mu \frac{\Lambda \rho_P}{\rho_P} = -2 \frac{\mu (r-1)}{p-1} + O \left( \frac{1}{(Z)^r} \right) + O \left( \frac{1}{(Z)^r} \right).
\]
and (5.30) is proved.

Proof of (5.31). Recall (2.23)

\[ \partial_\tau \rho_T = -\rho_T \Delta \Psi_T - \frac{\mu \ell (r - 1)}{2} \rho_T - (2 \partial Z \Psi_T + \mu Z) \partial_Z \rho_T \]

which yields:

\[ \left| \frac{\partial_\tau \rho_T + \mu \Delta \rho_T}{\rho_T} + \frac{\mu (r - 1)}{2} \right| = \left| -\Delta \Psi_T - 2 \frac{\partial Z \Psi_T \partial_Z \rho_T}{\rho_T} \right| \]

and (5.31) follows from (5.28).

\( \Psi^{(k)} \) terms. Integrating by parts:

\[ b^2 \int \partial^k \Delta \rho_D \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) \leq b^2 \int \chi \rho_T^2 \frac{|\nabla \Psi^{(k)}|}{\langle Z \rangle^{k+3}} \]

\[ \lesssim \int \chi \rho_T^2 \frac{|\nabla \Psi^{(k)}|^2}{\langle Z \rangle} + b^4 \int \chi \rho_T^2 \frac{|\nabla \Psi^{(k)}|^2}{\langle Z \rangle^{2(k+3)-1}} \lesssim e^{-c_{km} \tau} \]

where we used (5.27).

Next:

\[ \mu (r - 2) \int \Psi^{(k)} \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) = -\mu (r - 2) \int \chi \rho_T^2 |\nabla \Psi^{(k)}|^2. \]

Similarly, using \( \partial_Z H_2 = O \left( \frac{1}{\langle Z \rangle^r} \right) \):

\[ k \int H_2 \Psi^{(k)} \nabla \cdot (\chi \rho_T^2 \nabla \Psi^{(k)}) = -k \left[ \int \chi \mu \left[ 1 + O \left( \frac{1}{\langle Z \rangle^{\frac{3}{r}}} \right) \right] \rho_T^2 |\nabla \Psi^{(k)}|^2 + O \left( \int \chi \rho_T^2 \frac{|\Psi^{(k)}|^2}{\langle Z \rangle^{2r - \frac{7}{2}}} \right) \right] \]

\[ = -k \mu \int \chi \rho_T^2 |\nabla \Psi^{(k)}|^2 + O \left( e^{-c_{km} \tau} \right), \]

where we also used that \( r > 2, k \neq 0 \) and

\[ \int \chi \kappa, \sigma(k) \rho_T^2 \frac{|\Psi^{(k)}|^2}{\langle Z \rangle^{2r - \frac{7}{2}}} \lesssim \int \chi \kappa, \sigma(k) \rho_T^2 \frac{|\nabla \Psi^{(k-1)}|^2}{\langle Z \rangle^{2r - \frac{7}{2}}} \lesssim \| \nabla, \Psi \|^2_{k, \sigma(k) + \frac{1}{2}} \]

Then using (5.28):

\[ \left| \int 2 \chi \rho_T^2 \nabla \cdot \nabla \Psi^{(k)} \left( 2 \frac{\partial \rho_T}{\rho_T} + \frac{\nabla \chi}{\chi} \right) \cdot \nabla \Psi^{(k)} \right| \lesssim \int \chi \rho_T^2 \frac{|\nabla \Psi^{(k)}|^2}{\langle Z \rangle^{2r - \frac{7}{2}}} \lesssim e^{-c_{km} \tau} \]

and from (5.26), (5.28):

\[ \left| \int 2 \chi \rho_T \nabla \Psi \cdot \nabla \Psi^{(k)} \right| \lesssim \int \chi \frac{|\nabla \Psi^{(k)}|^2}{\langle Z \rangle^{2r - \frac{7}{2}}} \lesssim e^{-c_{km} \tau} \]
We now carefully compute from \((5.26)\) again:

\[
\int \chi \rho T H_2 \Delta \Psi^{(k)} \left(2 \nabla \rho T \cdot \nabla \Psi^{(k)} + \rho T \Delta \Psi^{(k)}\right) + \int H_2 \Delta \Psi^{(k)} (\rho T^2 \nabla \chi \cdot \nabla \Psi^{(k)}) = 2 \sum_{i,j} \int \chi \rho T H_2 Z_j \partial_i \Psi^{(k)} \partial_i \rho T \partial_i \Psi^{(k)} - \sum_{i,j} \int \partial_i (\chi Z_j H_2 \rho T^2) \partial_i \Psi^{(k)} \partial_j \Psi^{(k)} \\
+ \frac{1}{2} \int \nabla \cdot (\chi Z H_2 \rho T^2) |\nabla \Psi^{(k)}|^2 + \sum_{i,j} H_2 \rho T^2 Z_j \partial_i \Psi^{(k)} \partial_i \chi \partial_i \Psi^{(k)}
\]

\[
= \sum_{i,j} H_2 \partial_j \Psi^{(k)} \partial_i \Psi^{(k)} \left[2 \chi \rho T \partial_i \rho T Z_j - \partial_i \chi Z_j \rho T^2 - \chi \delta_{ij} \rho T^2 - \chi Z_j \rho T \partial_i \rho T - \chi Z_j \frac{\partial_i Z_j H_2 \rho T^2 + Z_j \rho T \partial_i \chi}{H_2} \right] \\
+ \frac{1}{2} \int \chi H_2 \rho T^2 |\nabla \Psi^{(k)}|^2 \left[d + \frac{\Lambda \chi}{\chi} + \frac{\Lambda H_2}{H_2} + 2 \frac{\Lambda \rho T}{\rho T} \right] \\
= \frac{1}{2} \mu \int \chi \rho T^2 |\nabla \Psi^{(k)}|^2 \left[d - 2 + \frac{\Lambda \chi}{\chi} + 2 \frac{\Lambda \rho T}{\rho T} + O \left(\frac{1}{(Z)^r}\right)\right].
\]

Hence the final formula recalling \((5.31)\):

\[
\int \chi \rho T H_2 \Delta \Psi^{(k)} \left(2 \nabla \rho T \cdot \nabla \Psi^{(k)} + \rho T \Delta \Psi^{(k)}\right) + \int H_2 \Delta \Psi^{(k)} (\rho T^2 \nabla \chi \cdot \nabla \Psi^{(k)}) = \int \chi \partial_r \rho T \rho T |\nabla \Psi^{(k)}|^2
\]

\[
= \int \mu \chi \rho T^2 |\nabla \Psi^{(k)}|^2 \left[d - 2 + \frac{\Lambda \chi}{\chi} + \frac{\Lambda \rho T}{\rho T} + \frac{1}{\mu} \frac{\partial_i \rho T}{\rho T} + O \left(\frac{1}{(Z)^r}\right)\right] \\
= \int \mu \chi \rho T^2 |\nabla \Psi^{(k)}|^2 \left[d - 2 + \frac{\Lambda \chi}{\chi} - \frac{2(r-1)}{p-1} + O \left(\frac{1}{(Z)^r}\right)\right] \\
= \int \mu \chi \rho T^2 |\nabla \Psi^{(k)}|^2 \left[d - 2 + \frac{\Lambda \chi}{\chi} - \frac{2(r-1)}{p-1} + O(e^{-c_{km}})\right].
\]

Loss of derivatives terms. We integrate by parts the non linear term which must lose derivatives:

\[
b^2 \left[ \int \rho_T \nabla \cdot \nabla \Psi^{(k)} \Delta \hat{\rho}^{(k)} \right] \lesssim b^2 \left[ \int \rho_T \Delta \Psi^{(k)} \nabla \chi \cdot \nabla \hat{\rho}^{(k)} \right] + b^2 \int \chi \frac{\rho_T}{(Z)^2} |\nabla \Psi^{(k)}||\nabla \hat{\rho}^{(k)}|
\]

\[
\lesssim b^2 \int \chi |\nabla \hat{\rho}^{(k)}|^2 + b \left[ \int \chi \frac{\rho_T^2 |\Delta \Psi^{(k)}|^2}{(Z)^2} + \int \chi \frac{\rho_T^2 |\nabla \Psi^{(k)}|^2}{(Z)^4} \right]
\]

\[
\lesssim e^{-c_{km}} + b \int \chi \frac{\rho_T^2 |\Delta \Psi^{(k)}|^2}{(Z)^2}.
\]

We now use \((5.20)\) for \(0 \leq k \leq k_m - 1\) which implies

\[
\int \chi_{k,k,\sigma(k)} \frac{\rho_T^2 |\Delta \Psi^{(k)}|^2}{(Z)^2} \leq \int \chi_{k+1,k+1,\sigma(k+1)} \rho_T^2 |\Delta \Psi^{(k)}|^2 \lesssim \|\hat{\rho}, \Psi\|_{k+1,\sigma(k+1)}^2 \lesssim 1.
\]

Hence

\[
b^2 \left[ \int \rho_T \nabla \chi \cdot \nabla \Psi^{(k)} \Delta \hat{\rho}^{(k)} \right] + b^2 \left[ \int \rho_T \Delta \Psi^{(k)} \nabla \chi \cdot \nabla \hat{\rho}^{(k)} \right] + b^2 \left[ \int \nabla \chi \cdot \nabla \Psi^{(k)} \rho_T \cdot \nabla \Psi^{(k)} \right] \lesssim e^{-c_{km}}.
\]
Conclusion for linear terms. The collection of above bounds yields:

\[
\frac{1}{2} \frac{d}{dt} \left\{ \int b^2 \chi |\nabla \rho|^2 + (p - 1) \int \chi \rho_D^{-2} \rho_T \rho_D (\rho_P)^2 + \int \chi \rho_D^{2} |\nabla \Psi(k)|^2 \right\} \leq e^{-c_{m_\tau}}
\]

\[
+ \mu \int \chi \left[ -k + \frac{d}{2} - (r - 1) - \frac{2(r - 1)}{p - 1} + \frac{1}{2} \mu^{-1} \partial_\tau \chi + \Delta \chi \right]
\times \left[ b^2 |\nabla \rho|^2 + (p - 1) \rho_D^{-2} \rho_T \rho_D (\rho_P)^2 + \rho_D^{2} |\nabla \Psi(k)|^2 \right]
\]

\[
+ \int F_i \chi (p - 1) \rho_D^{-2} \rho_T \rho_P k + b^2 \int \chi \nabla F_i \cdot \nabla (\rho_P) + \int \chi \rho_D^2 \nabla F_2 \cdot \nabla \Psi(k).
\]

**step 4 F** terms. We recall (5.7) and claim the bound:

\[
(p - 1) \int \chi \rho_D^2 \nabla \rho \cdot b^2 \int \chi |\nabla F_1|^2 \leq e^{-c_{m_\tau}} \left[ 1 + \|\rho, \Psi\|_{k, \sigma}^2 \right]
\]

**Source term induced by localization.** Recall (5.2)

\[
\tilde{\varepsilon}_{P, \rho} = \partial_\tau \rho_D + \rho_D \left[ \Delta \Psi_P + \frac{\ell(r - 1)}{2} + (2 \partial_\tau \Psi_P + \mu Z) \frac{\partial_2 \rho_D}{\rho_D} \right]
\]

\[
\partial_\tau \rho_D + \mu \rho_D \Delta \Psi_P + \mu \ell(r - 1) \rho_D + \rho_D \Delta \Psi_P + 2 \partial_\tau \Psi_P \partial_\tau \rho_D.
\]

From the proof of (5.30)

\[
\rho_D = \zeta \left( \frac{Z}{Z^*} \right) \rho_P, \quad \partial_\tau \rho_D + \mu \rho_D \Delta \Psi_P = \mu \zeta \left( \frac{Z}{Z^*} \right) \Lambda \rho_P
\]

Therefore, using the profile equation for \( \rho_P \), we obtain

\[
\tilde{\varepsilon}_{P, \rho} = 2 \frac{\Psi'_P}{Z} \frac{Z}{Z^*} \zeta' \rho_P
\]

From (2.9) and (2.18) we then conclude that

\[
|\partial^k \tilde{\varepsilon}_{P, \rho}| \lesssim \frac{\rho_D}{(Z)^{k+\tau}} 1_{Z \geq Z^*}
\]

Hence, recalling (5.19) and (25):

\[
\int \chi \rho_D^{-2} \rho_T |\partial^k \tilde{\varepsilon}_{P, \rho}|^2 \lesssim \int_{Z \geq Z^*} \frac{Z^d dZ}{Z^{-2k+2} \left( \frac{d}{2} - (r - 1) \right)^{2(r - 1)}} \left( \frac{Z}{Z^*} \right)^{2k + 2(\tau - 1) + 2(r - 1) - 2} \lesssim e^{-c_{\tau}}
\]

Similarly, from (25):

\[
b^2 \int \chi |\nabla \partial^k \tilde{\varepsilon}_{P, \rho}|^2 \lesssim \int_{Z \geq Z^*} \frac{Z^d dZ}{Z^{2(1 - \nu) + 2 + 2\tau}} \lesssim (Z^*)^{-2r + 2\nu} \lesssim e^{-c_{\tau}}.
\]

**[\partial^k, H_1] term.** We use (5.5) to estimate:

\[
|\partial^k, H_1| \lesssim \sum_{j=0}^{k-1} |\partial^j \tilde{\rho} \partial^{k-j} H_1| \lesssim \sum_{j=0}^{k-1} \frac{|\partial^j \tilde{\rho}|}{(Z)^{r+k-j}}
\]

\[
|\nabla \partial^k, H_1| \lesssim \sum_{j=0}^{k} \frac{|\partial^j \tilde{\rho}|}{(Z)^{r+k+j}}
\]

(5.37)
Hence:

\[(p - 1) \int \chi \rho_D^{p-2} \rho_T (|D^k \hat{\rho}|^2) \lesssim \sum_{j=0}^{k-1} \int \chi \rho_D^{p-1} \frac{\rho^2}{|Z|^{2(r+k-j)}} \lesssim \|\hat{\rho}, \Psi\|_{k,\sigma(k)+r}^2 \lesssim e^{-c_{km} r},\]

and

\[b^2 \int \chi |\nabla (|D^k \hat{\rho}|^2) \lesssim b^2 \sum_{j=0}^{k} \int \chi \frac{|D^j \hat{\rho}|^2}{|Z|^{2(1+r+k-j)}} \lesssim e^{-c_{km} r},\]

where we used the bootstrap bound (4.40), the decay of $b^2$ and (5.25).

**$[\partial^k, H_2]$ term.** Similarly, from (5.5):

\[
\begin{align*}
|\partial^k, H_2| \Lambda \hat{\rho} & \lesssim \sum_{j=0}^{k-1} |\partial^j \Lambda \hat{\rho}| \partial^{k-j} H_2 \lesssim \sum_{j=1}^{k} \frac{\rho^2}{(Z)^{2(r+k+j-1)}}. \\
|\nabla [\partial^k, H_2] \Lambda \hat{\rho}| & \lesssim \sum_{j=1}^{k+1} \frac{\rho^2}{(Z)^{2(r+k+j)}}.
\end{align*}
\]

Hence, using $r > 1$:

\[(p - 1) \int \chi \rho_D^{p-2} \rho_T (|\partial^k, H_2| \Lambda \hat{\rho})^2 \lesssim \sum_{j=1}^{k} \int \chi \rho_D^{p-1} \frac{\rho^2}{|Z|^{2(r-1+k-j)}} \lesssim e^{-c_{km} r},\]

and

\[b^2 \int \chi |\nabla (|\partial^k, H_2| \Lambda \hat{\rho}|^2) \lesssim b^2 \sum_{j=1}^{k+1} \int \chi \frac{|\partial^j \hat{\rho}|^2}{|Z|^{2(r+k+j)}} \lesssim e^{-c_{km} r}.
\]

**Nonlinear term.** Changing indices, we need to estimate terms

\[N_{j_1, j_2} = \partial^{j_1} \rho_T \partial^{j_2} \nabla \Psi, \quad j_1 + j_2 = k + 1, \quad 2 \leq j_1, j_2 \leq k - 1.\]  

(5.39)

For the profile term:

\[|\partial^{j_1} \rho_D \partial^{j_2} \nabla \Psi| \lesssim \rho_D \frac{|\partial^{j_2} \nabla \Psi|}{|Z|^{j_1}} = \rho_D \frac{|\partial^{j_2} \nabla \Psi|}{|Z|^{k+1-j_2}}\]

and hence using from (5.19) the rough global bound:

\[\rho_D \lesssim \frac{1}{|Z|^{2(r-1)}}\]  

(5.40)

yields:

\[
\int (p - 1) \chi N_{j_1, j_2}^2 \rho_D^{p-2} \rho_T \lesssim \int \chi \frac{\rho_D^2 |\partial^{j_2} \nabla \Psi|^2}{|Z|^{2(k+1-j_2)+2(r-1)}} = \int \chi \frac{\rho_D^2 |\partial^{j_2} \nabla \Psi|^2}{|Z|^{2(k-j_2)+2r}} \lesssim e^{-c_{km} r}.
\]
Similarly, after taking a derivative:
\[ b^2 \int |\nabla N_{j_1,j_2}|^2 \lesssim b^2 \int \rho_D \frac{|\partial^{j_2} \nabla \Psi|^2}{\langle Z \rangle^{2(2j_2-j_1)}} \lesssim e^{-c_{km} \tau}. \]

We now turn to the control of the nonlinear term. If \( j_1 \leq \frac{4k_m}{9} \), then from (4.40):
\[ \int \rho_D^{p-1} |\partial^{j_1} \tilde{\rho} \partial^{j_2} \nabla \Psi|^2 \lesssim \int \rho_D^2 \frac{|\partial^{j_2} \nabla \Psi|^2}{\langle Z \rangle^{2(2j_2-j_1)+2(r-1)}} \lesssim e^{-c_{km} \tau}. \] (5.41)

If \( j_2 \leq \frac{4k_m}{9} \), then from (4.40) and \( b = \frac{1}{\langle Z \rangle^{2r-\tau}} \):
\[ \int \rho_D^{p-1} |\partial^{j_1} \tilde{\rho} \partial^{j_2} \nabla \Psi|^2 \lesssim \int_{Z \leq Z^*} \rho_D^{p-1} \frac{|\partial^{j_1} \tilde{\rho}|^2}{\langle Z \rangle^{2(k+1+(r-2)-j_1)}} + b^2 \int_{Z \geq Z^*} \rho_D^{p-1} \frac{|\partial^{j_1} \tilde{\rho}|^2}{\langle Z \rangle^{2(k+1-j_1)}} \lesssim e^{-c_{km} \tau}. \]

We may therefore assume \( j_1, j_2 \geq m_0 = \frac{4k_m}{9} + 1 \), which implies \( k \geq m_0 \) and \( j_1, j_2 \leq \frac{2k_m}{3} \). From (4.29):
\[ \sigma(k) = -\alpha(k_m - k) \geq -\alpha \left( k_m - \frac{4k_m}{9} \right) = -\frac{4}{5} \left( 1 - \frac{4}{9} \right) k_m + O_{k_m \to +\infty}(1) \]
\[ \geq -\frac{4k_m}{9} + O_{k_m \to +\infty}(1) \]  
(5.42)

From (4.41):
\[ \sigma(k) + n(j_1) + n(j_2) \geq -\frac{4k_m}{9} + \frac{k_m}{4} + n_{k_m \to +\infty}(1) \geq \frac{k_m}{20} \]  
(5.43)

and hence from (4.40) and interpolating on \( Z \leq Z^* \) with (4.39):
\[ \int \rho_D^{p-1} |\partial^{j_1} \tilde{\rho} \partial^{j_2} \nabla \Psi|^2 \lesssim e^{-c_{km} \tau} \]
\[ + \int_{Z \geq Z^*} \frac{Z_{d-1} dZ}{\langle Z \rangle^{\frac{k_m}{mp}}} \left[ 1 \mathbf{1}_{Z \leq Z^*} + \frac{Z}{Z^*} \mathbf{1}_{Z \geq Z^*} \right] \lesssim e^{-c_{km} \tau} \]

The \( b^2 \) derivative term and the other nonlinear term in (5.7) are estimated similarly.

We note that the relation
\[ k_m \gg n_P \gg 1 \]
ensures that the terms containing \( k_m \) are dominant and eliminates the need to track the dependence on \( n_P \).

**step 5** \( F_2 \) terms. We claim:
\[ \int \rho_D^2 |\nabla F_2|^2 \lesssim e^{-c_{km} \tau} \left[ 1 + \| \tilde{\rho}, \Psi \|_{k+1,\sigma(k+1)}^2 \right]. \] (5.44)

**Source term induced by localization.** Recall (5.2):
\[ \hat{e}_{P,\Psi} = |\nabla \Psi_P|^2 + \rho_D^{p-1} + e\Psi_P + \frac{1-e}{2} \Lambda \Psi_P - 1 = \rho_D^{p-1} - \rho_P^{p-1} \]
which yields the rough bound
\[ |\nabla \partial^k \hat{e}_{P,\Psi}| \lesssim \frac{1}{\langle Z \rangle^{k+1+2(r-1)}} \mathbf{1}_{Z \geq Z^*} \]
and hence, from (5.25),
\[
\int \chi \rho_T^2 \| \nabla \rho \tilde{\rho}_T \|^2 \lesssim \int_{Z \geq Z^*} \frac{\hat{Z}^{d-1}}{b(Z)^{-2k+2(\frac{d}{2}+\rho-1)}} (\frac{Z}{\hat{Z}})^{4\rho} \lesssim \hat{Z}^{2(r-2)-4(r-1)+4\rho} \lesssim e^{-c\tau}
\]

\[\Box[\varphi, \mathcal{H}_2] \Lambda \Psi \text{ term. From (5.5):}
\]
\[
\| \nabla (\Box[\varphi, \mathcal{H}_2] \Lambda \Psi) \| \lesssim \sum_{j=1}^{k+1} \frac{|\Box[\varphi]|}{(Z)^{r+k-j}} \lesssim \sum_{j=0}^{k} \frac{\| \nabla \Box[\varphi] \|}{(Z)^{r+k-j}}
\]

and hence:
\[
\int \chi \rho_T^2 \| \nabla (\Box[\varphi, \mathcal{H}_2] \Lambda \Psi) \|^2 \lesssim \sum_{j=0}^{k} \int \chi \rho_T^2 \frac{\| \nabla \Box[\varphi] \|^2}{(Z)^{2(r+1+k-j)}} \lesssim e^{-c_{km}\tau}.
\]

\[\Box[\varphi, \mathcal{H}_D^{p-2}] \varphi - k(p-2)\rho_D \Box[\varphi, \mathcal{H}_D^{k-1}] \varphi \text{ term. By Leibnitz:}
\]
\[
\left[\left[ \Box[\varphi, \mathcal{H}_D^{p-2}] \varphi - k(p-2)\rho_D \Box[\varphi, \mathcal{H}_D^{k-1}] \varphi \right] \right] \lesssim \sum_{j=0}^{k-1} \frac{|\Box[\varphi]|}{(Z)^{3(k-j)}} \rho_D^{-2}
\]
and, hence, taking a derivative:
\[
\int \chi \rho_T^2 \| \nabla \Box[\varphi, \mathcal{H}_D^{p-2}] \varphi - k(p-2)\rho_D \Box[\varphi, \mathcal{H}_D^{k-1}] \varphi \|^2 \lesssim \sum_{j=0}^{k-1} \rho_D^{2(p-2)+2} \frac{|\Box[\varphi]|}{(Z)^{3(k-j)+2}} \lesssim e^{-c_{km}\tau}.
\]

Nonlinear $\Psi$ term. Let
\[\partial N_{j_1,j_2} = \partial^{j_1} \nabla \Psi \partial^{j_2} \nabla \Psi, \quad j_1 + j_2 = k + 1, \quad j_1, j_2 \geq 1.
\]

If $j_1 \leq \frac{4k_m}{9}$, then from (4.40):
\[
\int \chi \rho_T^2 \| \nabla N_{j_1,j_2} \|^2 \lesssim \int \rho_T^2 \chi \frac{|\partial^{j_2} \nabla \Psi|^2}{(Z)^{2(k+1-j_2)+2}} \lesssim \| \tilde{\varphi} \|^2_{k,\sigma} \lesssim e^{-c_{km}\tau}.
\]

The expression being symmetric in $j_1,j_2$, we may assume $j_1,j_2 \geq m_0 = \frac{4k_m}{9} + 1$, $j_1,j_2 \leq \frac{2k}{3}$ and $k \geq m_0 = \frac{4k_m}{9} + 1$. Using (4.40), (5.43) and arguing as above $(k_m \gg n_p)$:
\[
\int \chi \rho_T^2 \| \nabla N_{j_1,j_2} \|^2 \lesssim e^{-c\tau} + \int_{Z \geq Z^*} \frac{dZ}{\langle Z \rangle^{\frac{4n_p+4\rho}{9}}} \left[ 1_{Z \leq Z^*} + \left( \frac{Z}{Z^*} \right)^{2n_p+4\rho} 1_{Z \geq Z^*} \right] \lesssim e^{-c_{km}\tau}.
\]

Quantum pressure term. We estimate from Leibniz:
\[
b^2 \left[ \Box[\varphi] \left( \frac{\Delta \rho_T}{\rho_T} \right) - \partial^{j_2} \frac{\Delta \rho_T}{\rho_T} \right] + k^{j_1} \Delta \rho_T \partial^{j_2} \frac{\rho_T}{\rho_T^2} \lesssim b^2 \sum_{j_1,j_2 = k,j_2 \geq 1} |\partial^{j_2} \frac{1}{\rho_T} | \left( \frac{1}{\rho_T} \right)
\]
and using the Faà-di Bruno formula:
\[
|\partial^{j_2} \frac{1}{\rho_T} | \lesssim \frac{1}{\rho_T^{j_2+1}} \sum_{q_1+2q_2+\ldots+2q_{j_2}+q_{j_2} = j_2} \Pi_{i=1}^{j_2} |(\partial^{j_i} \rho_T) |.
\]

We decompose $\rho_T = \rho_D + \tilde{\varphi}$ and control the $\rho_D$ term using the bound
\[
|\partial^{j_2} \rho_D | \lesssim \frac{\rho_D}{(Z)^j}.
\]
which yields

\[
\frac{1}{\rho_T^{j_2+1}} \sum_{m_1+2m_2+\cdots+j_2m_{j_2}=j_2} \Pi_{i=1}^{j_2} |(\partial^i \rho_D)^{m_i}| \lesssim \frac{1}{\rho_T(Z)^{j_2}} \tag{5.45}
\]

and hence the corresponding contribution to (5.44):

\[
b^4 \int \chi \rho_T^2 \left\{ \sum_{j_1+j_2=k, j_2 \geq 2} \frac{\rho_T^2(Z)^{2j_2}}{\rho_T^2(Z)^{2j_2+2}} \left[ \rho_T^2(Z)^{2j_2+2(j_1+3)} + \rho_T^2(Z)^{2j_2} \right] \right\}
\approx b^4 \sum_{j_1+j_2=k, j_2 \geq 2} \left[ \int \chi \rho_T^2 dZ \left( Z \right)^{2k+6} + b^4 \sum_{j_1=2}^k \int \chi \frac{\rho_T^2 dZ}{\rho_T^2(Z)^{2(k-j_1)+2}} \lesssim e^{-c_{km} \tau} \right.
\]

where we used (5.27) in the last step.

We now turn to the control of the nonlinear term and consider

\[
N_{j_1,j_2} = b^2 (\partial^{j_1+1} \Delta \rho_T) \frac{1}{\rho_T^{j_2+1}} \sum_{q_1+2q_2+\cdots+j_2q_{j_2}=j_2} \Pi_{i=1}^{j_2} (\partial^i \hat{\rho})^{q_i},
\]

where \( \hat{\rho} \) is either \( \rho_D \) or \( \tilde{\rho} \). In both cases we will use the weaker estimates (4.40). First assume that \( q_i = 0 \) whenever \( i \geq \frac{4km}{3} + 1 \), then from (4.40):

\[
|N_{j_1,j_2}| \lesssim b^2 |\partial^{j_1+1} \Delta \rho_T| \frac{1}{\rho_T^{j_2+1}} \sum_{q_1+2q_2+\cdots+j_2q_{j_2}=j_2} \Pi_{i=1}^{j_2} |(\partial^i \hat{\rho})^{q_i}| \lesssim b^2 |\partial^{j_1+1} \Delta \rho_T| \frac{1}{\rho_T(Z)^{j_2}}
\]

and the conclusion follows verbatim as above. Otherwise, there are at most two value \( \frac{4km}{3} \leq i_1 \leq i_2 \leq j_2 \) with \( q_{i_1}, q_{i_2} \neq 0 \) and \( q_{i_1} + q_{i_2} \leq 2 \). Hence from (4.40):

\[
\frac{1}{\rho_T^{j_2+1}} \Pi_{i=1}^{j_2} |(\partial^i \hat{\rho})^{q_i}| \lesssim \frac{1}{\rho_T^{j_2+1}} (\rho_D)^{q_{i_1}} |(\partial^{j_1+1} \Delta \rho_T)^{q_1} \Pi_{i=1}^{j_2} |(\partial^i \hat{\rho})^{q_i}| \lesssim \frac{1}{\rho_T(Z)^{j_2}}
\]

Assume first \( i_2 \geq \frac{2km}{3} + 1 \), then \( q_1 = 0, q_{i_2} = 1 \) and \( j_1 + 3 \leq \frac{4km}{3} \) from which:

\[
\int \chi \rho_T^2 |N_{j_1,j_2}| \lesssim b^4 \int \chi \rho_T^2 |\partial^{j_1+1} \Delta \rho_T|^2 \frac{1}{\rho_T^{j_2+1}} \sum_{q_1+2q_2+\cdots+j_2q_{j_2}=j_2} \Pi_{i=1}^{j_2} |(\partial^i \hat{\rho})^{q_i}| \lesssim b^4 \int \chi \frac{\rho_T^2 dZ}{\rho_T^2(Z)^{2(j_2-i_2)+2(j_1+3)}} \lesssim e^{-c_{km} \tau}.
\]

There remains the case \( \frac{4km}{3} + 1 \leq i_1 \leq i_2 \leq \frac{2km}{3} \) which imply \( j_1 + 3 \leq \frac{2km}{3} \), and we distinguish cases:

- **case** \( (m_{i_1}, m_{i_2}) = (0,1) \): if \( j_1 + 3 \leq \frac{4km}{3} \), we estimate

\[
\int \chi \rho_T^2 |N_{j_1,j_2}| \lesssim b^4 \int \chi \rho_T^2 |\partial^{j_1+1} \Delta \rho_T|^2 \frac{1}{\rho_T^{j_2+1}} \sum_{q_1+2q_2+\cdots+j_2q_{j_2}=j_2} \Pi_{i=1}^{j_2} |(\partial^i \hat{\rho})^{q_i}| \lesssim b^4 \int \chi \frac{\rho_T^2 dZ}{\rho_T^2(Z)^{2(j_2-i_2)+2(j_1+3)}} \lesssim e^{-c_{km} \tau}.
\]
Otherwise, \( \frac{4k_m}{9} + 1 \leq j_1 + 3 \leq \frac{4k_m}{3} \). Since \( j_2 \geq \frac{4k_m}{9} + 1 \), then necessarily \( j_2 \leq \frac{2k_m}{3} \).

Hence \( \frac{4k_m}{9} + 1 \leq j_1 + 3 \leq \frac{2k_m}{9}, \frac{4k_m}{3} + 1 \leq j_2 \leq \frac{2k_m}{3} \) and we estimate from (4.40):

\[
\int \chi \rho_T^2 |N_{j_1,j_2}|^2 \lesssim b^4 \int \frac{Z^{d-1}dZ}{\langle Z \rangle^{2(\sigma(k) + \frac{k_m}{2} + \frac{2}{3}j_2 - i_2)}} \left[ 1_{Z \leq Z^*} + \left( \frac{Z}{Z^*} \right)^{2n_P + 4\tilde{\nu}} 1_{Z \geq Z^*} \right] \lesssim b^4 \lesssim e^{-c_km^*},
\]

where we once again used that in this range of \( k \)

\[
\sigma(k) + \frac{k_m}{2} \geq \frac{k_m}{20}, \quad k_m \gg n_P \gg 1
\]

- case \( m_{i_1} + m_{i_2} = 2 \): we use (4.40) and estimate crudely:

\[
\int \chi \rho_T^2 |N_{j_1,j_2}|^2 \lesssim b^4 \int \chi |\partial^{j_1+1} \Delta \rho_T|^2 \left( \frac{1}{\langle Z \rangle^4} \right)^4 
\lesssim b^4 \int \frac{Z^{d-1}dZ}{\langle Z \rangle^{2(\sigma(k) + \frac{k_m}{2})}} \left[ 1_{Z \leq Z^*} + \left( \frac{Z}{Z^*} \right)^{2n_P + 4\tilde{\nu}} 1_{Z \geq Z^*} \right] \lesssim b^4 \lesssim e^{-c_km^*}.
\]

**NL(\( \tilde{\rho} \)) term.** We expand, using, according to our assumptions, that the power of the nonlinear term is an integer \( p \geq 3 \):

\[
\text{NL}(\tilde{\rho}) = (\rho_D + \tilde{\rho})^{p-1} - \rho_D^{p-1} - (p-1)\rho_D^{p-2} \tilde{\rho} = \sum_{q=2}^{p-1} c_q \rho_D^{p-1-q}
\]

and hence by Leibniz:

\[
\partial^k \text{NL}(\tilde{\rho}) = \sum_{q=2}^{p-1} \sum_{j_1+j_2=k} c_{q,j_1,j_2} \partial^{j_1}(\rho_D^q)\partial^{j_2}(\rho_D^{p-1-q})
\]

\[
= \sum_{q=2}^{p-1} \sum_{j_1+j_2=k} \sum_{\ell_1+\cdots+\ell_q=j_1} \partial^{\ell_1} \tilde{\rho} \cdots \partial^{\ell_q} \tilde{\rho} \partial^{j_2}(\rho_D^{p-1-q}).
\]

Let

\[
N_{\ell_1,\ldots,\ell_q,j_1,q} = \partial^{\ell_1} \tilde{\rho} \cdots \partial^{\ell_q} \tilde{\rho} \partial^{j_2}(\rho_D^{p-1-q}),
\]

\( \ell_1 \leq \cdots \leq \ell_q \), then

\[
|\nabla N_{\ell_1,\ldots,\ell_q,j_1,q}| \lesssim |N_{\ell_1,\ldots,\ell_q,j_1,q}^{(1)}| + |N_{\ell_1,\ldots,\ell_q,j_1,q}^{(2)}|
\]

with

\[
|N_{\ell_1,\ldots,\ell_q,j_1,q}^{(1)}| \lesssim |\partial^{m_1} \tilde{\rho} \cdots \partial^{m_q} \tilde{\rho}| \rho_D^{p-1-q} \frac{(Z)^{j_2}}{(Z)_{j_2}}, \quad 0 \leq m_1 \leq \cdots \leq m_q \leq k + 1
\]

\[
m_1 + \cdots + m_q = j_1 + 1.
\]

We estimate \( N_{\ell_1,\ldots,\ell_q,j_1,q}^{(1)} \), the other term being estimated similarly. We distinguish cases.

- case \( m_q \leq \frac{4k_m}{9} \), then from (4.40):

\[
|N_{m_1,\ldots,m_q,j_1,q}^{(1)}| \lesssim \frac{\tilde{\rho}}{\langle Z \rangle_{j_1+1}} \rho_D^{p-1-q} \frac{(Z)^{j_2}}{(Z)^{j_2+1}} \lesssim \rho_D^{p-1} \frac{(Z)^{k+1}}{(Z)^{k+1}}
\]
We now assume

On the other hand, if

which follows from

and the condition

and hence, from (5.19) and (5.24), the contribution of this term is given by

\[
\int \chi \rho_T^2 |N_{m_1, \ldots, m_j, j, q}|^2 \lesssim e^{-ct}\]

\[
+ \int_{Z \geq Z_c^*} \frac{(Z')^{2\sigma(k) + 2(k + 1)}}{(Z')^{4(r - 1) + 4(r - 1) - 2(r - 2) + 4\sigma}} \lesssim e^{-c m r}.
\]

We now assume \( m_q \geq \frac{4k_m}{9} + 1 \) and recall \( m_q \leq j_1 + 1 \leq k + 1 \leq k_m \).

- case \( m_{q-1} \leq \frac{4k_m}{9} \), then from (4.40):

\[
|N_{m_1, \ldots, m_j, j, q}| \lesssim \frac{\rho_{D}^{q-1}}{(Z)^{1 - m_q + 1}} |\partial^{m_q} \tilde{\rho}| \rho_{D}^{p_{j, q} - 1} \lesssim |\partial^{m_q} \tilde{\rho}| \rho_{D}^{p_{j, q} - 2} \frac{1}{(Z)^{k + 1 - m_q}}.
\]

If \( m_q \leq k \) then

\[
\int \chi \rho_T^2 |N_{m_1, \ldots, m_j, j, q}|^2 \lesssim \int \chi \rho_D^2 \frac{|\partial^{m_q} \tilde{\rho}|^2}{(Z)^{2(k - m_q + 1) + 4(r - 1)(p - 2) - p - 1}} \lesssim e^{-c m r}
\]

On the other hand, if \( m_q = k + 1 \), then, using (5.20)

\[
\int \chi_{k, k, \sigma(k)} \rho_T^2 |N_{m_1, \ldots, m_j, j, q}|^2 \lesssim \int \chi_{k, k, \sigma(k)} \rho_T^2 \rho_D^{2 - q} \rho_{D}^{p_{j, q} - 2} |\partial^{k + 1} \tilde{\rho}|^2
\]

\[
\lesssim \int_{Z < Z_c^*} \chi_{k + 1, k + 1, \sigma(k)} \rho_T^{p_{j, q} - 1} |\partial^{k + 1} \tilde{\rho}|^2 \lesssim e^{-c m r} \|hotilde\|_{k + 1, \sigma(k + 1)}^2
\]

where we used the following interpolation bound

\[
\|hotilde\|_{L^\infty(Z \leq Z_c^*)} \lesssim e^{-ct},
\]

the estimate

\[
\hat{\rho} \lesssim \rho_D \lesssim (Z)^{-\frac{2(r - 1)}{p - 1}}
\]

and the condition

\[-2 + 2(r - 1) > 0,\]

which follows from \( r > 2 \).

- case \( m_{q-1} \geq \frac{4k_m}{9} + 1 \), then necessarily \( m_{q-2} \leq \frac{4k_m}{9} + 1 \leq m_{q-1} \leq m_q < \frac{2k_m}{3} \)

and \( k \geq \frac{4k_m}{9} + 1 \). Hence:

\[
|N_{m_1, \ldots, m_j, j, q}| \lesssim \frac{\rho_D^{q}}{(Z)^{\frac{km}{6} + \frac{4km}{4}}} \rho_{D}^{p_{j, q} - 1} \lesssim \frac{\rho_{D}^{p_{j, q} - 1}}{(Z)^{\frac{km}{2}}},
\]

The integral for \( Z < Z_c^* \) is estimated as above, and we further estimate from (5.24) and (5.43), using that \( k_m \gg n_p \gg 1 \),

\[
\int_{Z \geq Z_c^*} \chi \rho_T^2 |N_{m_1, \ldots, m_j, j, q}|^2 \lesssim \int_{Z \geq Z_c^*} \frac{dZ}{(Z)^{\frac{km}{2}}} \lesssim e^{-c m r},
\]
Proof. We integrate (5.13) in time and obtain, by choosing $O$ where

\[ \frac{d}{dt} \left\{ \int b^2 \chi |\nabla \hat{\rho}(k)|^2 + (p-1) \int \chi \rho_D^{-2} \rho_T \hat{\rho}(k)^2 + \int \chi \rho_T^2 |\nabla \Psi(k)|^2 \right\} \]

\[ \leq \mu \int \chi \left[ -k \frac{d}{2} - (r-1) - \frac{2(r-1)}{p-1} + \frac{1}{2} \mu \frac{d}{2} - \frac{1}{2} \frac{d}{2} \right] + \frac{1}{2} \mu \frac{d}{2} \chi + \Lambda \chi \]

\[ \times \left[ b^2 \nabla \hat{\rho}(k)^2 + (p-1) \rho_D^{-2} \rho_T \hat{\rho}(k)^2 + \rho_T^2 |\nabla \Psi(k)|^2 \right] \]

+ $e^{-ck_m \tau}.$

We now compute, noting that $\partial_r Z^* = \mu Z^*$ and that $\xi_k$ only depends on $\tau$ through $Z^*$:

\[ \partial_r \chi + \mu \Lambda \chi \]

\[ = \frac{1}{\langle Z \rangle_2^{2r(k)}} \left[ \partial_r \xi_k \left( \frac{Z}{Z^*} \right) + \mu \Lambda \xi_k \left( \frac{Z}{Z^*} \right) \right] + \xi_k \left( \frac{Z}{Z^*} \right) \mu \Lambda \left( \frac{1}{\langle Z \rangle_2^{2r(k)}} \right) \]

\[ = \xi_k \left( \frac{Z}{Z^*} \right) \mu \Lambda \left( \frac{1}{\langle Z \rangle_2^{2r(k)}} \right). \]

Hence recalling (5.22):

\[ \frac{d}{2} + r - 1 + \frac{2(r-1)}{p-1} + \frac{1}{2} \mu \frac{d}{2} - \frac{1}{2} \frac{d}{2} = k + \sigma(k) - \frac{d}{2} + r - 1 + \frac{2(r-1)}{p-1} + O \left( \frac{1}{\langle Z \rangle} \right) \]

\[ \geq \sigma - \frac{d}{2} + r - 1 + \frac{2(r-1)}{p-1} + O \left( \frac{1}{\langle Z \rangle} \right) \geq \tilde{\nu} + O \left( \frac{1}{\langle Z \rangle} \right). \]

Using that $k \leq k_m - 1$ and the interpolation bound (5.17) we may absorb the $O \left( \frac{1}{\langle Z \rangle} \right)$ term and (5.13) is proved.

6. Pointwise bounds

We are now in position to close the control of the pointwise bounds (4.40). We start with inner bounds $|x| \lesssim 1$:

**Lemma 6.1** (Interior pointwise bounds). For all $0 \leq k \leq \frac{2k_m}{3}$:

\[
\begin{align*}
\forall 0 \leq k \leq \frac{2k_m}{3}, \quad & \| \frac{\langle Z \rangle^{n(k)} \rho_D}{\rho_T} \|_{L^\infty(Z \leq Z^*)} \leq \delta_0 \\
\forall 1 \leq k \leq \frac{2k_m}{3}, \quad & \| \langle Z \rangle^{n(k)} \langle Z \rangle^{-2} \partial_T^2 \Psi \|_{L^\infty(Z \leq Z^*)} \leq \delta_0 
\end{align*}
\]

(6.1)

where $\delta_0$ is a smallness constant depending on data.

**Proof.** We integrate (5.13) in time and obtain, by choosing $0 < \tilde{\nu} \ll c + c_k m$, $\forall 0 \leq m \leq k_m - 1$:

\[ I_m(\tau) \leq e^{-2\mu \tilde{\nu} (\tau - \tau_0)} I_m(0) + \frac{1}{c_k m - 2\mu \tilde{\nu}} \left( e^{-2\mu \tilde{\nu} - c_k m \tau_0} - e^{-c_k m \tau} \right) \]

\[ \leq e^{-2\mu \tilde{\nu} (\tau - \tau_0)} e^{-c \tau_0} + \frac{e^{-c_k m \tau_0}}{c_k m - 2\mu \tilde{\nu}} e^{-2\mu \tilde{\nu} \tau} \leq \delta_0 e^{-2\mu \tilde{\nu} \tau} \]

(6.2)

for some small constant $\delta_0$, which can be chosen to be arbitrarily small by increasing $\tau_0$. Below, we will adjust $\delta_0$ to remain small while absorbing any other universal constant.

Recalling (5.16):

\[ \forall 0 \leq m \leq k_m - 1, \quad \| \hat{\rho}, \nabla \Psi \|_{m, \sigma(m)} \leq \delta_0 e^{-\mu \tilde{\nu} \tau}. \]
This, in particular, already implies bounds on the Sobolev and pointwise norms of \((\tilde{\rho}, \Psi)\) on compact sets: for any \(Z_K < \infty\) and any \(k \leq k_m - d\)
\[
\|(\tilde{\rho}, \Psi)\|_{H^k(Z \leq Z_K)} \leq \delta_0 e^{-\mu \tau}, \quad \|(\partial^k \tilde{\rho}, \partial^k \Psi)\|_{L^\infty(Z \leq Z_K)} \leq \delta_0 e^{-\mu \tau} \quad (6.4)
\]
case \(m \leq \frac{4k_m}{9} + 1 = m_0\). Recall (4.29), then (6.2) implies: \(\forall 0 \leq m \leq m_0\),
\[
\left\| \left( Z \right)^{m - \frac{2}{3} + \frac{2(r-1)}{\mu + \nu - 1}} \partial_Z^m \tilde{\rho} \right\|_{L^2(Z \leq Z^*)}^2 + \left\| \left( Z \right)^{m - \frac{2}{3} + \frac{2(r-1)}{\mu + \nu - 1}} \partial_Z^{m-1} \Psi \right\|_{L^2(Z \leq Z^*)}^2 \leq \delta_0 e^{-2\mu \tau}. \quad (6.5)
\]
We now write for any spherically symmetric function \(u\) and \(\gamma > \frac{d}{2} - 1\):
\[
|u(Z)| \lesssim |u(1)| + \int_1^Z |\partial_Z u| \, d\sigma \lesssim |u(1)| + \left( \int_{1 \leq \sigma \leq Z} \frac{|\partial_Z u|^2 \tau^{d-1} \, d\tau}{\tau^{2\gamma}} \right)^{\frac{1}{2}} \left( \int_{1 \leq \sigma \leq Z} \frac{\tau^{2\gamma} \, d\tau}{\tau^{d-1}} \right)^{\frac{1}{2}} \quad (6.6)
\]
We pick \(1 \leq m \leq m_0\) and apply this to \(u = Z^{2\frac{r-1}{\mu + \nu - 1}} z^{m-1} \partial_Z^m \rho\), \(\gamma + 1 = \frac{d}{2} + \tilde{\nu}\) and obtain for \(Z \leq Z^*\) from (6.4) and (6.5):
\[
\left| Z^{m-1 + \frac{2(r-1)}{\mu + \nu - 1}} \partial_Z^m \rho \right| \lesssim e^{-c \tau} + \left( Z \right)^{\tilde{\nu}} \left\| \partial_Z (Z^{2\frac{r-1}{\mu + \nu - 1} + m-1} \partial_Z^m \rho) \right\|_{L^2(Z \leq Z^*)} \lesssim e^{-c \tau} + \left( Z \right)^{\tilde{\nu}} \lesssim \delta_0
\]
and hence
\[
\forall 0 \leq m \leq \frac{4k_m}{9}, \quad \left\| Z^{m} \partial_Z^m \rho \right\|_{L^\infty(Z \leq Z^*)} \leq \delta_0.
\]
We similarly pick \(1 \leq m \leq m_0\), apply (6.6) to \(u = (Z)^{r-2+m} \partial_Z^m \Psi\), \(\gamma + 1 = \frac{d}{2} + \tilde{\nu}\),
and obtain for \(Z \leq Z^*\) from (6.5):
\[
\left| (Z)^{r-2+m} \partial_Z^m \Psi \right| \lesssim e^{-c \tau} + \left( Z \right)^{\tilde{\nu}} \left\| \partial_Z (Z^{r-2+m} \partial_Z^m \Psi) \right\|_{L^2} + \left( Z \right)^{\tilde{\nu}} \left\| \partial_Z (Z^{r-2+m} \partial_Z^{m+1} \Psi) \right\|_{L^2(Z \leq Z^*)} \lesssim e^{-c \tau} + \delta_0 \left( Z \right)^{\tilde{\nu}} \lesssim \delta_0
\]
and hence
\[
\forall 1 \leq m \leq m_0 = \frac{4k_m}{9} + 1, \quad \left\| (Z)^{r-2+m} \partial_Z^m \Psi \right\|_{L^\infty(Z \leq Z^*)} \leq \delta_0.
\]

case \(m_0 \leq m \leq \frac{2k_m}{3} + 1\). Recall (5.22):
\[
\sigma(m) + m = -\alpha(k_m - m) + m = (\alpha + 1)(m - m_0) + \sigma_\nu
\]
and rewrite the norm:

\[
\|\tilde{\rho}, \Psi\|^2_{m,\sigma} \geq \sum_{k=0}^{m} \int_{Z \leq Z^*} \frac{1}{\langle Z \rangle^{2m-k+1} \langle \sigma \rangle^{m-k+1}} \left[ b^2 |\nabla \tilde{\rho}^{(k)}|^2 + (p-1) \tilde{\rho}_D^{p-2} \tilde{\rho}_T (\tilde{\rho}^{(k)})^2 + \rho_T^2 |\nabla \Psi^{(k)}|^2 \right]
\]

\[
= \sum_{k=0}^{m} \int_{Z \leq Z^*} \frac{(Z)^{2k}}{\langle Z \rangle^{2(m-k+\sigma(m))} + 2\sigma_c} \left[ b^2 |\nabla \tilde{\rho}^{(k)}|^2 + (p-1) \tilde{\rho}_D^{p-2} \tilde{\rho}_T (\tilde{\rho}^{(k)})^2 + \rho_T^2 |\nabla \Psi^{(k)}|^2 \right]
\]

We infer, using also (6.2):

\[
\int_{2Z \leq Z \leq Z^*} \left[ \frac{Z^{m-(\alpha+1)(m-m_0)} \partial_{Z}^m \rho}{\langle Z \rangle^{\frac{d}{2} - \frac{2(r-1)}{r-1} + \tilde{\nu}}} \right]^2 \left[ \frac{Z^{m-(\alpha+1)(m-m_0)} \langle Z \rangle^{r-1} \partial_{Z}^{m+1} \Psi}{\langle Z \rangle^{\frac{d}{2} + \tilde{\nu}}} \right]^2 \lesssim \|\rho, \Psi\|^2_{m,\sigma(m)} \leq d_0 e^{-2\tilde{\nu}T}.
\]  

(6.7)

Observe that for \( m_0 \leq m \leq \frac{2k_m}{3} + 1 \), from (4.30):

\[
m - (\alpha + 1)(m - m_0) = m_0(1 + \alpha) - \alpha m \geq m_0(1 + \alpha) - \alpha \left( \frac{2k_m}{3} + 1 \right) = k_m \left[ 4 \left( 1 + \frac{4}{5} \right) - \frac{24}{35} \right] + O_{k_m} \rightarrow \infty(1) = \frac{4k_m}{15} + O_{k_m} \rightarrow \infty(1) > \frac{k_m}{4} + 10.
\]  

(6.8)

We now apply (6.6), (6.7) to \( m_0 + 1 \leq m \leq \frac{2k_m}{3} + 1 \), \( u = \langle Z \rangle^{m + \frac{2(r-1)}{r-1} - (\alpha + 1) (m - m_0) - 1} \partial_{Z}^{m-1} \rho \), \( \gamma + 1 = \frac{d}{2} + \tilde{\nu} \) and obtain for \( Z \leq Z^* \):

\[
\|\langle Z \rangle^{m + \frac{2(r-1)}{r-1} - (\alpha + 1)(m - m_0) - 1} \partial_{Z}^{m-1} \rho \| \lesssim e^{-cT} + \langle Z \rangle^{\tilde{\nu}} \left\| \partial_{Z} (\langle Z \rangle^{m + \frac{2(r-1)}{r-1} - (\alpha + 1)(m-m_0) - 1} \partial_{Z}^{m-1} \rho) \right\|_{L^2(\mathbb{R})} \]

\[
\lesssim e^{-cT} + \langle Z \rangle^{\tilde{\nu}} \left\| \langle Z \rangle^{m + \frac{2(r-1)}{r-1} - (\alpha + 1)(m-m_0) - 1} \partial_{Z}^{m} \rho \right\|_{L^2(\mathbb{R})} \]

\[
+ \langle Z \rangle^{\tilde{\nu}} \left\| \langle Z \rangle^{m + \frac{2(r-1)}{r-1} - (\alpha + 1)(m-m_0) - 1} \partial_{Z}^{m-1} \rho \right\|_{L^2(\mathbb{R})} \lesssim d_0 \left[ 1 + \langle Z \rangle^{\tilde{\nu}} e^{-\tilde{\nu}T} \right]
\]

and hence using (6.8) for \( Z \leq Z^* \):

\[
\left\| \frac{Z^{\frac{2k_m}{3}} \partial_{Z}^{m-1} \rho}{\rho_D} \right\|_{L^\infty(\mathbb{R})} \lesssim \left\| Z^{m + \frac{2(r-1)}{r-1} - (\alpha + 1)(m-m_0) - 1} \partial_{Z}^{m-1} \rho \right\|_{L^\infty(\mathbb{R})} \lesssim d_0 \left[ 1 + \left( \frac{Z}{Z^*} \right)^{\tilde{\nu}} \right] \lesssim d_0
\]

and hence

\[
\forall \frac{4k_m}{9} + 1 \leq m \leq \frac{2k_m}{3}, \left\| \frac{Z^{\frac{2k_m}{3}} \partial_{Z}^{m} \rho}{\rho_D} \right\|_{L^\infty(\mathbb{R})} \leq d_0.
\]
For the phase, we apply (6.6), (6.7) to \( m_0 + 1 \leq m \leq \frac{2k_m}{3} + 1, \ \gamma + 1 = \frac{d}{2} + \tilde{\nu}, \)
\( u = \langle Z \rangle^{r-1+m-(\alpha+1)(m-m_0)\partial_Z^m \Psi} \) and obtain:
\[
\langle Z \rangle^{r-1+m-(\alpha+1)(m-m_0)}|\partial_Z^m \Psi| \lesssim e^{-\varepsilon \tau} + \langle Z \rangle^{\tilde{\nu}} \left| \partial_Z \left( \langle Z \rangle^{r-1+m-(\alpha+1)(m-m_0)} \partial_Z^m \Psi \right) \right| \left| \langle Z \rangle^{\frac{d}{2} + \tilde{\nu} - 1} \right|_{L^2(Z \leq Z^*)} \\
\lesssim e^{-\varepsilon \tau} + \langle Z \rangle^{\tilde{\nu}} \left| \partial_Z \left( \langle Z \rangle^{r-1+m-(\alpha+1)(m-m_0)} \partial_Z^m \Psi \right) \right| \left| \langle Z \rangle^{\frac{d}{2} + \tilde{\nu} - 1} \right|_{L^2(Z \leq Z^*)} \\
\quad + \langle Z \rangle^{\tilde{\nu}} \left| \partial_Z \left( \langle Z \rangle^{r-1+m-(\alpha+1)(m-m_0)} \partial_Z^m \Psi \right) \right| \left| \langle Z \rangle^{\frac{d}{2} + \tilde{\nu} - 1} \right|_{L^2(Z \leq Z^*)} \\
\quad \leq d_0 \left[ 1 + \langle Z \rangle^{\tilde{\nu}} e^{-\mu \tau} \right]
\]
and hence for \( m_0 + 1 \leq m \leq \frac{2k_m}{3} \) from (6.8) for \( Z \leq Z^* \):
\[
\langle Z \rangle^{r-2+k_m} \left| \partial_Z^m \Psi \right| \lesssim \langle Z \rangle^{r-1+m-(\alpha+1)(m-m_0)}|\partial_Z^m \Psi| \leq d_0
\]
which concludes the proof of (6.1).

\( \square \)

Similar to the above, we also have the following exterior bounds for \(|x| \geq 1:\)

**Lemma 6.2** (Exterior pointwise bounds). There holds:
\[
\begin{align*}
\forall 0 \leq k \leq \frac{2k_m}{3}, \quad & \left\| \langle Z \rangle^n \partial_Z^k \partial_Z^m \right\|_{L^2(Z \geq Z^*)} \leq d_0 \\
\forall 1 \leq k \leq \frac{2k_m}{3}, \quad & \left\| \langle Z \rangle^n \partial_Z^k \partial_Z^m \right\|_{L^2(Z \geq Z^*)} \leq d_0
\end{align*}
\]
where \( d_0 \) is a smallness constant depending on data.

**Proof.** We start with the case \( 0 \leq k \leq \frac{4k_m}{9} \). We have in that case
\[
I_{k, \sigma(k)} \geq \int_{Z \geq Z^*} \langle Z \rangle^{2k-2\sigma} \left( \frac{Z}{Z^*} \right)^{2n-p-4(r-1)p+2(r+2)+4\tilde{\nu}} \left[ b^2\left| \nabla \tilde{\rho}(k) \right|^2 + (p-1)\rho_D^{-1}(\tilde{\rho}(k))^2 + \rho_D^2 \left| \nabla \Psi(k) \right|^2 \right]
\]
We observe from (4.9), (4.28), (5.12) and \( b = Z^{2-r} \) that for \( Z \geq Z^* \)
\[
Z^{d-1} \langle Z \rangle^{2k-2\sigma} \left( \frac{Z}{Z^*} \right)^{2n-p-4(r-1)p+2(r+2)+4\tilde{\nu}} \left| \nabla \tilde{\rho}(k) \right|^2 \approx (Z)^{d-1+2k-2\sigma} \left( \frac{Z}{Z^*} \right)^{2n-p-4(r-1)p+2(r+2)+4\tilde{\nu}} \left| \nabla \tilde{\rho}(k) \right|^2 \left( \frac{Z^*}{Z} \right)^{-4\tilde{\nu} \rho_D^2}
\]
Similarly,
\[
Z^{d-1} \langle Z \rangle^{2k-2\sigma} \left( \frac{Z}{Z^*} \right)^{2n-p-4(r-1)p+2(r+2)+4\tilde{\nu}} \rho_D^2 \left| \nabla \Psi(k) \right|^2 \approx (Z)^{d-1+2k-2\sigma} \left( \frac{Z}{Z^*} \right)^{2n-p-4(r-1)p+2(r+2)+4\tilde{\nu}} \left| \nabla \Psi(k) \right|^2 \left( \frac{Z^*}{Z} \right)^{-4\tilde{\nu} \rho_D^2}
\]
Now, for a spherically symmetric function \( u, Z \geq Z^* \) and an arbitrary \( \lambda > 0 \)
\[
|u(Z)| = |u(Z^*) + \int_{Z^*} \partial_Z u| \leq |u(Z^*)| + \left( \int_{Z^*} \tau^{r+2\lambda} |\partial_Z u|^2 d\tau \right)^{\frac{1}{2}} \left( \int_{Z^*} \tau^{1-2\lambda} d\tau \right)^{\frac{1}{2}}
\]
\[
\leq |u(Z^*)| + (Z^*)^{-\lambda} \left( \int_{Z^*} \tau^{r+2\lambda} |\partial_Z u|^2 d\tau \right)^{\frac{1}{2}}
\]
We apply this to \( u = \left( \frac{\langle Z \rangle^k \hat{\rho}(k)}{(Z^*)^{2\nu} \rho_D} \right) \) for \( k \geq 1 \) and \( \lambda = \tilde{\nu} \):

\[
\left| \left( \frac{\langle Z \rangle^k \hat{\rho}(k)}{(Z^*)^{2\nu} \rho_D} \right) \langle Z \rangle \right| \leq \left| \left( \frac{\langle Z \rangle^k \hat{\rho}(k)}{(Z^*)^{2\nu} \rho_D} \right) (Z^*) \right| + (Z^*)^{-\tilde{\nu}} \left( \int_{Z^*} \tau^{1+2\tilde{\nu}} \left[ \left( \frac{\langle \tau \rangle^k |\nabla \hat{\rho}(k)|}{(Z^*)^{2\nu} \rho_D} \right)^2 + \left( \frac{\langle \tau \rangle^{k-1} \hat{\rho}(k)}{(Z^*)^{2\nu} \rho_D} \right)^2 \right] d\tau \right)^{\frac{1}{2}}
\]

\[
\lesssim (Z^*)^{-2\tilde{\nu}} \delta_0 + (Z^*)^{-\tilde{\nu}} (I_{k, \sigma(k)} + I_{k-1, \sigma(k-1)})^{\frac{1}{2}},
\]

where we used the already proved interior bounds (6.1). This, together with (6.2), immediately implies the exterior bound for \( \partial^2 \rho \) and \( 1 \leq k \leq \frac{4k_m}{3} + 1 \). The corresponding bound for \( \partial^2 \Psi \) is obtained similarly using \( u = \left( \frac{\langle Z \rangle^k \psi(k)}{(Z^*)^{2\nu} b} \right) \) and \( \lambda = \tilde{\nu} \). To prove the result for \( \rho \) in the case of \( k = 0 \) we note that the bootstrap assumptions imply that \( \hat{\rho} \to 0 \), so that, together with the above estimate for \( k \), we have, for \( Z \geq Z_* \),

\[
|\hat{\rho}(Z)| = \left| \int_{Z}^{+\infty} \partial_Z \hat{\rho} \right| \leq \delta_0 \int_{Z}^{+\infty} \frac{\rho_D(\tau)}{\tau} d\tau \leq \delta_0 \rho_D(Z)
\]
as desired.

Finally, we consider the regime \( \frac{4k_m}{9} + 1 \leq k \leq \frac{2k_m}{3} \). We have in that case

\[
I_{k, \sigma(k)} \geq \int_{Z \geq Z_*} \langle Z \rangle^{2\alpha(k_m-1)} \left( \frac{Z}{Z^*} \right)^{2n_p - \frac{4(r-1)}{p-1} - 2(r-2) + 4\tilde{\nu}} b^2 |\nabla \hat{\rho}(k)|^2 \approx \langle Z \rangle^\varpi(k) \left( \frac{\langle Z \rangle^{n(k)} |\nabla \hat{\rho}(k)|}{(Z^*)^{2\nu} \rho_D} \right)^2
\]

We observe, using \( n(k) = k_m/4 \) in that range, for \( Z \geq Z_* \)

\[
Z^{d-1} (Z^{2\alpha(k_m-1)} \left( \frac{Z}{Z^*} \right)^{2n_p - \frac{4(r-1)}{p-1} - 2(r-2) + 4\tilde{\nu}} b^2 |\nabla \hat{\rho}(k)|^2 \approx \langle Z \rangle^\varpi(k) \left( \frac{\langle Z \rangle^{n(k)} |\nabla \hat{\rho}(k)|}{(Z^*)^{2\nu} \rho_D} \right)^2
\]

and

\[
Z^{d-1} (Z^{2\alpha(k_m-1)} \left( \frac{Z}{Z^*} \right)^{2n_p - \frac{4(r-1)}{p-1} - 2(r-2) + 4\tilde{\nu}} b^2 |\nabla \hat{\rho}(k)|^2 \approx \langle Z \rangle^\varpi(k) \left( \frac{\langle Z \rangle^{n(k)} |\nabla \psi(k)|}{(Z^*)^{2\nu} b} \right)^2
\]

where

\[
\varpi(k) := 2\alpha(k_m-1) - k_m - \frac{k_m}{2} - 2(r-2) + 4\tilde{\nu} - \frac{4(r-1)}{p-1} + d - 1.
\]

Since \( k \leq \frac{2k_m}{3} \), and in view of the control of \( \alpha \) in (4.30), we have

\[
\varpi(k) \geq \left( \frac{8}{15} - \frac{1}{2} \right) k_m + O(1)_{k_m \to +\infty} = \frac{1}{30} + O(1)_{k_m \to +\infty} \geq \frac{k_m}{31}.
\]

In particular, we have \( \varpi(k) > 1 \), so that the proof for \( Z \geq Z_* \) in the case \( \frac{4k_m}{9} + 1 \leq k \leq \frac{2k_m}{3} \) is analogous to the case \( 0 \leq k \leq \frac{4k_m}{9} \). Details are left to the reader.

7. Highest Sobolev norm

In this section we improve the bootstrap bound (4.38) on the highest unweighted Sobolev norm of \( \langle \hat{\rho}, \psi \rangle \). Specifically, for (see (4.23))

\[
\| \hat{\rho}, \psi \|_{k_m}^{2} = \sum_{j=0}^{k_m} \sum_{|\alpha|=j} \int b^2 |\nabla^{\alpha} \hat{\rho}|^2 + (p-1) \rho^{-2} b^2 |\nabla^{\alpha} \psi|^2 \langle Z \rangle^{2(k_m-j)}
\]

(7.1)
we will establish the following

**Proposition 7.1** (Control of the highest Sobolev norm). For some small constant $d$ dependent on the data,

$$
\|\tilde{\rho}, \Psi\|_{k_m}^2 \leq \|(\tilde{\rho}, \Psi)(\tau_0)\|_{k_m}^2 + \delta.
$$

(7.2)

**Proof of Proposition 7.1.** This follows from the global *unweighted* quasilinear energy identity. We let

$$
k_m = 2K_m, \quad K_m \in \mathbb{N}
$$

and denote in this section

$$
k = k_m, \quad \tilde{\rho}^{(k)} = \Delta^{K_m} \tilde{\rho}, \quad \Psi^{(k)} = \Delta^{K_m} \Psi.
$$

We recall the notation (5.11)

$$
I_{k_m} = \int b^2|\nabla \partial^{K_m} \tilde{\rho}|^2 + (p - 1) \int \rho_D^{p-2} \rho_T |\partial^{K_m} \tilde{\rho}|^2 + \int \rho_T^2|\nabla \partial^{K_m} \Psi|^2.
$$

(7.3)

**step 1** Control of lower order terms. We recall the notation:

$$
\|\tilde{\rho}, \Psi\|_{k_m, \sigma(k_m)}^2 = \sum_{j=0}^{k_m} \chi_{j,k_m,\sigma(k_m)} b^2|\nabla \partial^j \tilde{\rho}|^2 + (p - 1) \int \chi_{j,k_m,\sigma(k_m)} \rho_D^{p-2} \rho_T |\partial^j \tilde{\rho}|^2 + \int \chi_{j,k_m,\sigma(k_m)} \rho_T^2|\nabla \partial^j \Psi|^2
$$

In view of (5.16) and (6.2), we have:

$$
\|\tilde{\rho}, \Psi\|_{k_m, \sigma(k_m)}^2 \leq \sum_{k=0}^{k_m} I_k \leq I_{k_m} + \delta_0
$$

(7.4)

By Remark 4.2 we can replace (up to the lower order terms controlled as above) $I_{k_m}$ with

$$
J_{k_m} := \int b^2|\nabla \Delta^{K_m} \tilde{\rho}|^2 + (p - 1) \int \rho_D^{p-2} \rho_T |\Delta^{K_m} \tilde{\rho}|^2 + \int \rho_T^2|\nabla \Delta^{K_m} \Psi|^2.
$$

(7.5)

We claim: there exist $k_m^*(d, r, p), c_{d,r,p} > 0$ such that for all $k_m > k_m^*(d, r, p)$, there holds:

$$
d \frac{d}{dt}[J_{k_m} [1 + O(\delta)])] + c_{d,r,k_m} J_{k_m} \leq \delta.
$$

(7.6)

Integrating the above in time, using (4.24), (7.4), yields (7.2).

**step 2** Energy identity. We revisit the computation of (5.6), (5.7), (5.8), (5.3) in order to extract all the coupling terms at the highest level of derivatives. Recall (5.3):

$$
\begin{align*}
\partial_t \tilde{\rho} &= -\rho_T \Delta \tilde{\rho} - 2\nabla \rho_T \cdot \nabla \Psi + H_1 \tilde{\rho} - H_2 \Delta \tilde{\rho} - \tilde{\rho}_P, \\
\partial_t \Psi &= b^2 \frac{\Delta \rho_T}{\rho_T} - \left\{ H_2 \Delta \Psi + \mu(r - 2) \Psi + |\nabla \Psi|^2 + (p - 1) \rho_D^{p-2} \tilde{\rho} + NL(\tilde{\rho}) \right\} - \tilde{\rho}_P.
\end{align*}
$$

We use

$$
[\Delta^{K_m}, \Lambda] = k_m \Delta^{K_m}
$$

and recall (C.1):

$$
[\Delta^k, V] \Phi - 2k \nabla V \cdot \nabla \Delta^{k-1} \Phi = \sum_{|\alpha| + |\beta| = 2k, |\beta| \leq 2k-2} c_{k,\alpha,\beta} \partial^\alpha \nabla \partial^\beta \Phi
$$

which gives:

$$
\Delta^{K_m}(H_2 \Delta \tilde{\rho}) = k_m (H_2 (H_2 + \Lambda H_2) \rho_k + H_2 \Delta \rho_k + \mathcal{A}_k(\tilde{\rho}))
$$
with

\[
\left| \mathcal{A}_k(\tilde{\rho}) \right| \lesssim c_k \sum_{j=1}^{k-1} \frac{\|\nabla^j \tilde{\rho}\|}{(2)^{k-m+j}} \quad (7.7)
\]

\[
\left| \nabla \mathcal{A}_k(\tilde{\rho}) \right| \lesssim c_k \sum_{j=1}^{k} \frac{\|\nabla^j \tilde{\rho}\|}{(2)^{k-m+j}}
\]

where \( \nabla^j = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \ j = \alpha_1 + \cdots + \alpha_d \) denotes a generic derivative of order \( j \).

Using (C.1) again:

\[
\begin{align*}
\partial_t \tilde{\rho}^{(k)} & = (H_1 - k(H_2 + \Lambda H_2))\tilde{\rho}_k - H_2 \hat{\rho}_k - (\Delta^{K_m} \rho_T) \Delta \Psi - k \nabla \rho_T \cdot \nabla \Psi^{(k)} - \rho_T \Delta \Psi_k \\
& \quad - 2\nabla(\Delta^{K_m} \rho_T) \cdot \nabla^2 \Psi - 2\nabla \rho_T \cdot \nabla \Psi_k + F_1 \quad (7.8)
\end{align*}
\]

with

\[
F_1 = -\Delta^{K_m} \tilde{\epsilon}_{p,\rho} + [\Delta^{K_m}, H_1] \tilde{\rho} - A_k(\tilde{\rho}) + \sum_{\substack{j_1 + j_2 = k \\ j_1 \geq 2, j_2 \geq 1}} c_{j_1,j_2} \nabla^{j_1} \rho_T \nabla^{j_2} \Delta \Psi - \sum_{\substack{j_1 + j_2 = k \\ j_1, j_2 \geq 1}} c_{j_1,j_2} \nabla^{j_1} \nabla \rho_T \cdot \nabla^{j_2} \nabla \Psi.
\]

For the second equation, we have similarly:

\[
\begin{align*}
\partial_t \Psi_k & = b^2 \left( \frac{\Delta^{K_m+1} \rho_T}{\rho_T} - k \nabla /Delta^{K_m+1} \rho_T \cdot \nabla \rho_T \right) \\
& \quad - k(H_2 + \Lambda H_2)\Psi_k - H_2 \Delta \Psi_k - \mu(r - 2) \nabla \Psi_k - 2\nabla \Psi \cdot \nabla \Psi_k \\
& \quad - (p - 1)\rho_p^{p-2} \tilde{\rho}_k + k(p - 1)(p - 2)\rho_D^{p-3} \nabla \rho_D \cdot \nabla \Delta^{K_m-1} \tilde{\rho} + F_2 \quad (7.10)
\end{align*}
\]

with

\[
\begin{align*}
F_2 & = -\partial^k \tilde{\epsilon}_{p,\rho} + b^2 \left[ \Delta^{K_m} \left( \frac{\Delta \rho_T}{\rho_T} \right) - \frac{\Delta^{K_m+1} \rho_T}{\rho_T} + \frac{k \nabla \Delta^{K_m} \rho_T \cdot \nabla \rho_T}{\rho_T^2} \right] \\
& \quad - (p - 1) \left[ \Delta^{K_m} \rho_D^{p-2} \tilde{\rho} - k(p - 2)\rho_D^{p-3} \nabla \rho_D \cdot \nabla \Delta^{K_m-1} \tilde{\rho} \right] \\
& \quad - \mathcal{A}_k(\Psi) - \sum_{\substack{j_1 + j_2 = k \\ j_1, j_2 \geq 1}} \nabla^{j_1} \nabla \Psi \cdot \nabla^{j_2} \nabla \Psi - \Delta^{K_m} \nabla \Psi \cdot \nabla \Psi - \Delta^{K_m} \nabla (\Psi) \quad (7.11)
\end{align*}
\]

and

\[
\left| \mathcal{A}_k(\Psi) \right| \lesssim \sum_{j=1}^{k-1} \frac{\|\nabla^j \Psi\|}{(2)^{k-m+j}} \quad (7.12)
\]

\[
\left| \nabla \mathcal{A}_k(\Psi) \right| \lesssim \sum_{j=1}^{k} \frac{\|\nabla^j \Psi\|}{(2)^{k-m+j}}
\]
We then run the global quasilinear energy identity similar to (5.10) with \( \chi = 1 \) and obtain:

\[
\frac{1}{2} \frac{d}{dt} \left\{ \int b^2 |\nabla \tilde{\rho}_k|^2 + (p - 1) \int \rho^p_D \rho_T \tilde{\rho}_k^2 + \int \rho_T^2 |\nabla \Psi_k|^2 \right\} 
= -\mu(r - 2)b^2 \int |\nabla \tilde{\rho}_k|^2 + \int \partial_\tau \partial_T \left[ \frac{p - 1}{2} \rho^p_D \rho_T \tilde{\rho}_k^2 + \rho_T |\nabla \Psi_k|^2 \right] + \frac{p - 1}{2} \int (p - 2) \partial_\tau \partial_D \rho^p_D \rho_T \tilde{\rho}_k^2 
+ \int F_1(p - 1) \rho^p_D \rho_T \tilde{\rho}_k + b^2 \int \nabla F_1 \cdot \nabla \tilde{\rho}_k + \int \rho_T^2 \nabla F_2 \cdot \nabla \Psi_k 
- \int k \nabla \rho_T \cdot \nabla \Psi_k (-b^2 \Delta \tilde{\rho}_k + (p - 1) \rho^p_D \rho_T \tilde{\rho}_k) + \int b^2 k \nabla \Delta^m \rho_T \cdot \nabla \rho_T (\rho_T \Delta \Psi_k + 2 \nabla \rho_T \cdot \nabla \Psi_k) 
\times \left[ -b^2 \Delta \tilde{\rho}_k + (p - 1) \rho^p_D \rho_T \tilde{\rho}_k \right] 
- \int \left[ b^2 \partial_\rho \Delta H^m D - k \rho_T (H_2 + \Lambda H_2) \Psi_k - \rho_T H_2 \Lambda \Psi_k - \mu(r - 2) \rho_T \Psi_k - 2 \rho_T \nabla \Psi \cdot \nabla \Psi_k \right] 
\times \left[ 2 \nabla \rho_T \cdot \nabla \Psi_k + \rho_D \Delta \Psi_k \right] 
+ \int k(p - 1)(p - 2) \partial_\rho \rho^p_D \rho_D \cdot \nabla \Delta^{m-1} \rho [2 \nabla \rho_T \cdot \nabla \Psi_k + \rho_T \Delta \Psi_k].
\]

We now estimate all terms in (7.13). The proof is similar to that one of Proposition 5.2 with two main differences: the absence of a cut-off function \( \chi \), and a priori control of lower order derivatives from (7.4). The challenge here is to avoid any loss of derivatives and to compute exactly the quadratic form at the highest level of derivatives. The latter will be shown to be positive on a compact set in \( Z \) provided \( k_m > k_m^*(d, r, p) \gg 1 \) has been chosen large enough.

In what follows, below, we will use \( \delta > 0 \) as a small universal constant and will assume that the pointwise bounds (6.1) obtained on the lower order derivatives of \( \tilde{\rho} \) and \( \Psi \) are dominated by \( \delta \). On the set \( Z \leq Z^* \), this will often be a source of smallness, while for \( Z \geq Z^* \), we may use the bootstrap bounds (4.40) and the \( \delta \)-smallness will be generated by extra powers of \( Z \). We also note that from (7.6) the quadratic form is expected to be proportionate to \( k_m I_{k_m} \). Choosing \( k_m \) large will allow us to dominate other quadratic terms without smallness but with the uniform dependence on \( k_m \). The notation \( \lesssim \) will allow dependence on \( k_m \), while \( O \) will indicate a bound independent of \( k_m \). As before, \( d_0 \) (as well as \( d \)) will denote small constants, dependent on the data (or, more precisely, on \( \tau_0 \)), that can be made arbitrarily small. In particular, we will use

\[
\| \tilde{\rho}, \Psi \|_{k_m - 1,\sigma(k_m - 1)} \leq d_0. \tag{7.14}
\]

The constants \( \delta \gg d_0 \) will be assumed to be smaller than any power of \( k_m \), so that our calculations will be unaffected by combinatorics generated by taking \( k_m \) derivatives of the equations.

**step 3 Leading order terms.**
\textit{Cross term.} Recall (5.26):

\[
\int \Delta g F \cdot \nabla g dx = \sum_{i,j=1}^{d} \int \partial_{ij}^{2} g F_{ij} \partial_{ij} g dx = - \sum_{i,j=1}^{d} \int \partial_{ij}(\partial_{i} F_{ij} \partial_{ij} g + F_{ij} \partial_{ij}^{2} g) \\
= - \sum_{i,j=1}^{d} \int \partial_{i} F_{ij} \partial_{ij} g + \frac{1}{2} \int |\nabla g|^{2} \nabla \cdot F.
\]

Letting \( g = g_{1} + g_{2} \) yields a bilinear off-diagonal Pohozhaev identity:

\[
\int \left[ \Delta g_{1} F \cdot \nabla g_{2} + \Delta g_{2} F \cdot \nabla g_{1} \right] dx = - \sum_{i,j=1}^{d} \int \partial_{i} F_{ij}(\partial_{i} g_{1} \partial_{ij} g_{2} + \partial_{i} g_{2} \partial_{ij} g_{1}) + \int \nabla g_{1} \cdot \nabla g_{2}(\nabla \cdot F).
\]

We may therefore integrate by parts the one term in (7.13) which has too many derivatives:

\[
b^{2} \int \left[ \nabla \rho_{T} \cdot \nabla \Psi_{k} \Delta \tilde{\rho}_{k} + \nabla \Delta^{K_{m}} \rho_{T} \cdot \nabla \rho_{T} \Delta \Psi_{k} \right] \\
= b^{2} \int \nabla \rho_{T} \cdot \nabla \Psi_{k} \Delta \tilde{\rho}_{k} + \nabla \rho_{T} \cdot \nabla \tilde{\rho}_{k} \Delta \Psi_{k} + \nabla \rho_{T} \cdot \nabla \Delta^{K_{m}} \rho_{D} \Delta \Psi_{k} \\
= b^{2} \left[ - \int \sum_{i,j=1}^{d} \partial_{i,j}^{2} \rho_{T}(\partial_{i} \tilde{\rho}_{k} \partial_{j} \Psi_{k} + \partial_{i} \Psi_{k} \partial_{j} \tilde{\rho}_{k}) + \int \nabla \tilde{\rho}_{k} \cdot \nabla \Psi_{k} \Delta \rho_{T} \right] - \int \nabla \Psi_{k} \cdot \nabla(\nabla \rho_{T} \cdot \nabla \Delta^{K_{m}} \rho_{D}) \\
\leq b^{2} \rho_{T} \left[ \frac{1}{\langle Z \rangle^{k+2}} + \frac{|\nabla \tilde{\rho}_{k}|}{\langle Z \rangle^{2}} \right] \leq \delta \int \rho_{T}^{2} |\nabla \Psi_{k}|^{2} + C_{4} b^{4} + C_{4} b^{4} \int |\nabla \tilde{\rho}_{k}|^{2} \\
\leq \delta J_{km} + C_{4} b^{4}.
\]

We estimate similarly:

\[
|k b^{2} \int \frac{\nabla \Delta^{K_{m}} \rho_{T} \cdot \nabla \rho_{T} \Delta \rho_{T} \cdot \nabla \Psi_{k}}{\rho_{T}}| \leq \delta \left[ b^{2} \int |\nabla \tilde{\rho}_{k}|^{2} + \int \rho_{T}^{2} |\nabla \Psi_{k}|^{2} \right] + C_{4} b^{4} \leq \delta J_{km} + C_{4} b^{4}.
\]

We use

\[
\frac{|\tilde{\rho}|}{\rho_{T}} + \frac{|\Delta \tilde{\rho}|}{\langle Z \rangle^{k} \rho_{T}} \leq \delta, \quad 0 < c \ll 1
\]

(7.15)

to compute the first coupling term:

\[
-k(p - 1) \int \nabla \rho_{T} \cdot \nabla \Psi_{k} \rho_{D}^{p-2} \rho_{T} \tilde{\rho}_{k} = -k \int \rho_{D} \rho_{D}^{p-1} \cdot \nabla \Psi_{k} \tilde{\rho}_{k} + O \left( \delta \int \frac{|\nabla \Psi_{k}| \rho_{D}^{p-1} \rho_{T} |\tilde{\rho}_{k}|}{\langle Z \rangle} \right) \\
= -k \int \rho_{D} \rho_{D}^{p-1} \cdot \nabla \Psi_{k} \tilde{\rho}_{k} + O \left( \delta J_{km} \right)
\]
The second coupling term is computed after an integration by parts using (7.15), the control of lower order terms (7.4) and the spherically symmetric assumption:

\[
\int (\rho T \Delta \Psi_k + 2 \nabla \rho T \cdot \nabla \Psi_k) k(p - 1)(p - 2) \rho T \rho D^{-3} \nabla \rho_D \cdot \nabla \Delta^{Km-1} \tilde{\rho}
\]

\[= k(p - 1)(p - 2) \int \nabla \cdot (\rho T^2 \nabla \Psi_k) \rho D^{-3} \nabla \rho_D \cdot \nabla \Delta^{Km-1} \tilde{\rho}
\]

\[= -k(p - 1)(p - 2) \int \rho T^2 \nabla \Psi_k \cdot \nabla \left( \rho D^{-3} \nabla \rho_D \cdot \nabla \Delta^{Km-1} \tilde{\rho} \right)
\]

\[= -k(p - 1)(p - 2) \int \rho T^2 \partial Z \Psi_k \partial Z \left( \rho D^{-3} \partial Z \rho_D \partial Z \Delta^{Km-1} \tilde{\rho} \right)
\]

\[= -k(p - 1)(p - 2) \int \rho T^2 \partial Z \rho_D \partial Z \partial Z \Delta^{Km-1} \tilde{\rho} + O \left( \int \rho T |\nabla \Psi_k| \rho T^{-1} \frac{|\nabla^{k-1} \tilde{\rho}|}{(Z)^2} \right)
\]

\[= -k(p - 2) \int \rho D \nabla \rho D^{-1} \cdot \nabla \Psi_k \tilde{\rho}_k + O (d_0 + \delta J_m),
\]

where in the last step we used that \( \frac{\rho_D k}{|Z|^2} \approx \frac{1}{(Z)^2} \).

**\( \rho_k \) terms.** We compute:

\[
\int (H_1 - k(H_2 + \Lambda H_2)) \tilde{\rho}_k(-b^2 \Delta \tilde{\rho}_k + (p - 1) \rho D^{-2} \rho T \tilde{\rho}_k)
\]

\[= \int (H_1 - k(H_2 + \Lambda H_2)) \left[ b^2 |\nabla \rho_k|^2 + (p - 1) \rho D^{-2} \rho T \tilde{\rho}_k^2 \right] - \frac{b^2}{2} \int \rho_k^2 \Delta (H_1 - k(H_2 + \Lambda H_2)).
\]

We now use the global lower bound, see properties (2.21) and (2.22) of the the profile \((w, \sigma)\),

\[H_2 + \Lambda H_2 = \mu(1 - w - \Lambda w) \geq c_{p,d,r}, \quad c_{p,d,r} > 0
\]

to estimate using (8.17), (7.14):

\[
\int (H_1 - k(H_2 + \Lambda H_2)) \tilde{\rho}_k(-b^2 \Delta \tilde{\rho}_k + (p - 1) \rho D^{-2} \rho T \tilde{\rho}_k)
\]

\[\leq -k \int \left[ 1 + O_{k_m \to +\infty} \left( \frac{1}{k_m} \right) \right] (H_2 + \Lambda H_2) \left[ b^2 |\nabla \rho_k|^2 + (p - 1) \rho D^{-2} \rho T \tilde{\rho}_k^2 \right] + C b^2 \int \frac{\rho_k^2}{(Z)^{2+r}}
\]

\[\leq -k \int (H_2 + \Lambda H_2) \left[ b^2 |\nabla \rho_k|^2 + (p - 1) \rho D^{-2} \rho T \tilde{\rho}_k^2 \right] + \delta_0.
\]

Next, using

\[|\partial^k \rho_D| \lesssim \frac{\rho_D}{(Z)^k},
\]
we estimate from (4.40):
\[ b^2 \int \Delta \tilde{\rho}_k \left[ (\Delta^{K_m} \rho_D) \Delta \Psi + 2 \nabla (\Delta^{K_m} \rho_D) \cdot \nabla \Psi \right] \]
\[ \lesssim b^2 \int |\nabla \tilde{\rho}_k| \left[ |\nabla (\Delta^{K_m} \rho_D)\Delta \Psi| + |\nabla (\Delta^{K_m} \rho_D) \cdot \nabla \Psi| \right] \]
\[ \leq b^2 \delta \int |\nabla \tilde{\rho}_k|^2 + \frac{b^2}{\delta} \sum_{j=1}^{3} \int \frac{\rho_D^2 |\partial \Psi|^2}{(Z)^{2(k+3-j)}} \leq \delta J_{km} + \delta_0 b^2. \]

For the nonlinear term, we use (6.1), (4.40), (5.26), (7.4) to estimate
\[ b^2 \left| \int \Delta \tilde{\rho}_k [\tilde{\rho}_k \Delta \Psi + 2 \nabla \tilde{\rho}_k \cdot \nabla \Psi] \right| \lesssim b^2 \left[ \int |\partial^2 \Psi||\nabla \tilde{\rho}_k|^2 + \int \frac{\tilde{\rho}_k^2}{(Z)^2} \right] \leq \delta J_{km} + \delta_0 b^2. \]

Next:
\[ \left| \int [(\Delta^{K_m} \rho_D) \Delta \Psi - 2 \nabla (\Delta^{K_m} \rho_D) \cdot \nabla \Psi] (p - 1) \rho_D^{p-2} \rho_T \tilde{\rho}_k \right| \]
\[ \leq \delta \int \rho_D^{p-2} \rho_T \tilde{\rho}_k^2 + C \delta \int \rho_D^{p-2} \rho_T \left[ \frac{|\partial^2 \Psi|^2}{(Z)^{2k}} + \frac{|\partial^2 \Psi|^2}{(Z)^{2(k+1)}} \right] \]
\[ \leq \delta J_{km} + \delta_0, \]

since we are assuming that \( \delta_0 \ll \delta \), and for the nonlinear term after an integration by parts:
\[ \left| \int [\tilde{\rho}_k \Delta \Psi - 2 \nabla \tilde{\rho}_k \cdot \nabla \Psi] (p - 1) \rho_D^{p-2} \rho_T \tilde{\rho}_k \right| \lesssim \delta \int \rho_D^{p-2} \rho_T \tilde{\rho}_k^2. \]

From Pohozaev (5.26):
\[ - \int H_2 \Delta \tilde{\rho}_k (-b^2 \Delta \tilde{\rho}_k) = b^2 \int \Delta \tilde{\rho}_k (ZH_2) \cdot \nabla \tilde{\rho}_k = O \left( b^2 \int |\nabla \tilde{\rho}_k|^2 \right). \]

Integrating by parts and using (8.17), (5.30), (5.31):
\[ - \int H_2 \Delta \tilde{\rho}_k \left[ (p - 1) \rho_D^{p-2} \rho_T \tilde{\rho}_k \right] + \frac{p-1}{2} \int (p - 2) \partial_T \rho_D \rho_D^{p-3} \rho_T \tilde{\rho}_k^2 + \frac{p-1}{2} \int \partial_T \rho_D \rho_D^{p-2} \tilde{\rho}_k^2 \]
\[ = \frac{p-1}{2} \int \rho_D^{p-2} \left[ \nabla \cdot (ZH_2 \rho_D^{p-2} \rho_T) + \partial_T (\rho_D^{p-2}) \rho_T + \partial_T \rho_D \rho_T \rho_D^{p-2} \right] = O \left( \int \rho_D^{p-2} \rho_T \tilde{\rho}_k^2 \right) \]

Note that the above two bounds, even though dependent on the highest order derivatives, contain no \( k \) dependence.

\( \Psi_k \) terms. After an integration by parts:
\[ \left| \int b^2 \Delta^{K_m+1} \rho_D [2 \nabla \rho_T \cdot \nabla \Psi_k + \rho_T \Delta \Psi_k] \right| \]
\[ \lesssim b^2 \int \rho_T \frac{|\nabla \Psi_k|^2}{(Z)^{k+3}} \leq \delta \int \rho_T^2 |\nabla \Psi_k|^2 + \delta_0. \]

Then
\[ \mu(r-2) \int \rho_T \Psi_k [2 \nabla \rho_T \cdot \nabla \Psi_k + \rho_T \Delta \Psi_k] \]
\[ = -\mu(r-2) \int \Psi_k^2 \nabla \cdot (\rho_T \nabla \rho_T) - \mu(r-2) \int \nabla \Psi_k \cdot \nabla (\rho_T^2 \Psi_k) \]
\[ = -\mu(r-2) \int \rho_T^2 |\nabla \Psi_k|^2 \]
and similarly, using (8.17), (7.4):

\[
\begin{align*}
k \int \rho_T (H_2 + \Lambda H_2) \Psi_k [2 \nabla \rho_T \cdot \nabla \Psi_k + \rho_T \Delta \Psi_k] &= k \int (H_2 + \Lambda H_2) \Psi_k \nabla \cdot (\rho_T^2 \nabla \Psi_k) \\
&= -k \left[ \int (H_2 + \Lambda H_2) \rho_T^2 |\nabla \Psi_k|^2 + \int \rho_T^2 \Psi_k^2 \left( \nabla \cdot \left( \frac{(\rho_T^2 \nabla (H_2 + \Lambda H_2))}{2\rho_T^2} \right) \right) \right] \\
&= -k \int (H_2 + \Lambda H_2) \rho_T^2 |\nabla \Psi_k|^2 + d_0,
\end{align*}
\]

where the $$\Psi_k^2$$ term is controlled, with the help of the bound

\[
\left| \nabla \cdot \left( \frac{\rho_T^2 \nabla (H_2 + \Lambda H_2)}{2\rho_T^2} \right) \right| \lesssim (Z)^{-2-r},
\]

by using the already bounded $$\| (\hat{\rho}, \Psi) \|_{k_m-1, \sigma(k_m-1)}$$-norm.

Then, from (6.1) and (6.9):

\[
\left| \int 2 \rho_T \nabla \Psi \cdot \nabla \Psi_k (2 \nabla \rho_T \cdot \nabla \Psi_k) \right| \lesssim d_0 \int \rho_T^2 |\nabla \Psi_k|^2
\]

and using (5.26):

\[
\left| \int 2 \rho_T \nabla \Psi \cdot \nabla \Psi_k (\rho_T \Delta \Psi_k) \right| \lesssim \int |\nabla \Psi_k|^2 \|\partial (\rho_T^2 \nabla \Psi)\| \lesssim d_0 \int \rho_T^2 |\nabla \Psi_k|^2.
\]

Arguing verbatim as in the proof of (5.32) produces the bound

\[
\left| \int \rho_T H_2 \Delta \Psi_k (2 \nabla \rho_T \cdot \nabla \Psi_k + \rho_T \Delta \Psi_k) \right| = O \left( \int \rho_T^2 |\nabla \Psi_k|^2 \right).
\]

**step 4** $$F_1$$ terms. We claim the bound:

\[
b^2 \int |\nabla F_1|^2 + (p - 1) \int \rho_D^{p-1} F_1^2 \lesssim \delta J_{k_m} + d_0. \tag{7.16}
\]

**Source term induced by localization.** From (5.36), for $$k_m$$ large enough:

\[
\int \rho_D^{p-2} \rho_T |\Delta^{K_m} \tilde{E}_{P,\rho}|^2 + b^2 \int |\nabla \Delta^{K_m} \tilde{E}_{P,\rho}|^2 \lesssim d_0.
\]

**$$[\Delta^{K_m}, H_1]$$ term.** We estimate from (5.37), (7.14)

\[
(p - 1) \int \rho_D^{p-1} |[\Delta^{K_m}, H_1]\hat{\rho}|^2 \lesssim \sum_{j=0}^{k-1} \int \rho_D^{p-1} \frac{|\partial_j \hat{\rho}|^2}{(Z)^{2(r+k-j)}} \leq d_0
\]

and

\[
b^2 \int |\nabla ( [\Delta^{K_m}, H_1]\hat{\rho} ) |^2 \lesssim b^2 \sum_{j=0}^{k} \int \frac{|\partial_j \hat{\rho}|^2}{(Z)^{2(1+r+k-j)}}
\]

\[
= b^2 \int \frac{\hat{\rho}^2 dZ}{(Z)^{2(1+r+k)}} + b^2 \sum_{j=0}^{k-1} \int \frac{|\partial_j \nabla \hat{\rho}|^2}{(Z)^{2(r+k+1-j)+2}} \lesssim b^2 + \| \hat{\rho}, \Psi \|_{k_m-1, \sigma(k_m-1)} \leq d_0.
\]

**$$\mathcal{A}_k(\hat{\rho})$$ term.** From (7.7), (7.14):

\[
(p - 1) \int \rho_D^{p-1} |\mathcal{A}_k(\hat{\rho})|^2 \lesssim \sum_{j=1}^{k-1} \int \rho_D^{p-1} \frac{|\nabla^j \hat{\rho}|^2}{(Z)^{2(r+k-j)}} \leq d_0
\]

\[
(p - 1) \int \rho_D^{p-1} |\Delta \Psi_k|^2 \lesssim \sum_{j=0}^{k-1} \int \rho_D^{p-1} \frac{|\nabla^j \Psi_k|^2}{(Z)^{2(r+k-j)}} \leq d_0
\]
and
\[ b^2 \int |\nabla (\mathcal{A}_k(\tilde{\rho}))|^2 \lesssim b^2 \sum_{j=0}^{k-1} \int \frac{|\nabla \nabla_j^i \tilde{\rho}|^2}{\langle Z \rangle^{2(r+k-3j)}} \leq d_0 \]
and (7.16) is proved for this term.

**Nonlinear term.** After changing indices, we need to estimate
\[ N_{j_1,j_2} = \nabla^{j_1} \rho_T \nabla^{j_2} \nabla \Psi, \quad j_1 + j_2 = k + 1, \quad 2 \leq j_1, j_2 \leq k - 1. \]
For the profile term:
\[ |\partial^{j_1} \rho_D \nabla^{j_2} \nabla \Psi| \lesssim \rho_D \frac{|\nabla^{j_2} \nabla \Psi|}{\langle Z \rangle^{j_1}} = \rho_D \frac{|\nabla^{j_2} \nabla \Psi|}{\langle Z \rangle^{k+1-j_2}} \]
and therefore, recalling (5.40), (7.14):
\[ \int (p-1)N_{j_1,j_2}^2 \rho_D^{p-1} \lesssim \int \frac{\rho_D^2 |\nabla^{j_2} \nabla \Psi|^2}{\langle Z \rangle^{2(k+1-j_2)+(p-1)(r-1)}} \leq d_0. \]
Similarly, after taking a derivative:
\[ b^2 \int |\nabla N_{j_1,j_2}|^2 \lesssim b^2 \int \rho_D^2 \frac{|\nabla^{j_2} \nabla \Psi|^2}{\langle Z \rangle^{2(k+1-j_2)}} + b^2 \int \rho_D^2 \frac{|\nabla^{j_2+1} \nabla \Psi|^2}{\langle Z \rangle^{2(k+1-j_2)}} \leq d_0 + \delta J_{km}. \]
The \( \delta J_{km} \) term above controls the case \( j_2 = k - 1 \).

We now turn to the control of the nonlinear term. If \( j_1 \leq \frac{4km}{9} \), then from (4.40), (7.4):
\[ \int \rho_D^{p-1} |\nabla^{j_1} \tilde{\rho} \nabla^{j_2} \nabla \Psi|^2 \lesssim \int \rho_D^2 \frac{|\nabla^{j_2} \nabla \Psi|^2}{\langle Z \rangle^{2(k+1-j_2)+(p-1)(r-1)}} \leq d_0. \]
If \( j_2 \leq \frac{4km}{9} \), then from (4.40) with \( b = \frac{1}{\langle Z \rangle^{r-j_2}} \):
\[ \int \rho_D^{p-1} |\nabla^{j_1} \tilde{\rho} \nabla^{j_2} \nabla \Psi|^2 \lesssim \int_{\langle Z \rangle \langle Z \rangle^{r-j_2}} \rho_D^{p-1} \frac{|\nabla^{j_1} \tilde{\rho}|^2}{\langle Z \rangle^{2(k+1-(r-2)-j_1)}} \leq d_0. \]
We may therefore assume \( j_1, j_2 \geq m_0 = \frac{4km}{9} + 1 \), which implies \( k \geq m_0 \) and \( j_1, j_2 \leq \frac{2km}{3} \) and hence from (4.40) and (6.1):
\[ \int \rho_D^{p-1} |\partial^{j_1} \tilde{\rho} \nabla^{j_2} \nabla \Psi|^2 \lesssim d_0 + \int_{\langle Z \rangle \langle Z \rangle^{r}} \frac{dZ}{\langle Z \rangle^{r+m}} \leq d_0. \]
The \( b^2 \) derivative contribution of the nonlinear term is estimated similarly.

**step 5** \( F_2 \) terms. We claim:
\[ \int \rho_T^2 |\nabla (F_2 + \Delta^K \text{NL}(\tilde{\rho}))|^2 \leq \delta J_{km} + d_0. \]
(7.17)
The nonlinear term \( \Delta^K \text{NL}(\tilde{\rho}) \) will be treated in the next step.

\( \mathcal{A}_k(\Psi) \) term. From (7.12)
\[ |\nabla \mathcal{A}_k(\Psi)| \lesssim \sum_{j=1}^{k} \frac{|\nabla^j \Psi|}{\langle Z \rangle^{r+k-j+1}} \]
and hence:  
\[ \int \rho_T^2 |\nabla \mathcal{A}_k(\Psi)|^2 \leq \sum_{j=0}^{k-1} \int \rho_T^2 \frac{\nabla \nabla^j \Psi}{(Z)^{2(r+k-j)}} \leq \delta_0. \]

[\Delta^{K_m}, \rho_D^{p-2}] term. From (C.1):

\[ |\Delta^{K_m}, \rho_D^{p-2}| \hat{\rho} - k(p-2)\rho_D^{p-3} \nabla \rho_D \cdot \nabla \Delta^{K_m-1} \hat{\rho} | \lesssim \sum_{j=0}^{k-2} \frac{|\nabla^j \hat{\rho}|}{(Z)^{2(k-j)}} \rho_D^{p-2} \]

After taking a derivative:

\[ \int \rho_T^2 |\nabla \left[ (\Delta^{K_m}, \rho_D^{p-2}) \hat{\rho} - k(p-2)\rho_D^{p-3} \nabla \rho_D \cdot \nabla \Delta^{K_m-1} \hat{\rho} \right] |^2 \]

\[ \lesssim \sum_{j=0}^{k-1} \rho_D^{2(p-2)+2} \frac{|\nabla^j \hat{\rho}|^2}{(Z)^{2(k-j)+2}} \leq \delta_0. \]

Nonlinear \( \Psi \) term. Let

\[ \partial N_{j_1,j_2} = \nabla^{j_1} \nabla \Psi \nabla^{j_2} \nabla \Psi, \quad j_1 + j_2 = k + 1, \quad j_1, j_2 \geq 1. \]

We first treat the highest derivative term using the \( L^\infty \) smallness of small derivatives: Using (4.40) and (6.1)

\[ \int \rho_T^2 |\nabla \nabla \Psi|^2 |\nabla^{k_m} \nabla \Psi|^2 \leq (\delta_0 + b^2) I_{k_m}. \]

We now assume \( j_1, j_2 \leq k_m - 1 \). If \( j_1 \leq \frac{4k_m}{9} \), then from (4.40), (7.4):

\[ \int \rho_T^2 |\nabla N_{j_1,j_2}|^2 \lesssim (\delta_0 + b^2) \int \rho_T^2 \frac{|\nabla^{j_2} \nabla \Psi|^2}{(Z)^{(k+1-j_2)}} \leq \delta_0. \]

The expression being symmetric in \( j_1, j_2 \), we may assume \( j_1, j_2 \geq m_0 = \frac{4k_m}{9} + 1, \quad j_1, j_2 \leq \frac{2k_m}{3} \), and using (4.40), (7.4):

\[ \int \rho_T^2 |\nabla N_{j_1,j_2}|^2 \lesssim \delta_0 \int_{Z \leq Z^*} \frac{dZ}{(Z)^{k_m/3}} + b^4 \int_{Z > Z^*} \frac{dZ}{(Z)^{k_m/3}} \leq \delta_0. \]

Quantum pressure term. We estimate from Leibniz and (C.1):

\[ b^2 \left| \Delta^{K_m} \left( \frac{\Delta \rho_T}{\rho_T} \right) \right| - \frac{\Delta^{K_m+1} \rho_T}{\rho_T} + \frac{k \nabla \Delta^{K_m} \rho_T \cdot \nabla \rho_T}{\rho_T^2} \lesssim_k b^2 \sum_{j_1+j_2=k, j_2 \geq 2} \nabla^{j_1} \Delta \rho_T \partial^{j_2} \left( \frac{1}{\rho_T} \right). \]

We use the Faa di Bruno formula:

\[ N_{j_1,j_2} = b^2 \nabla^{j_1+1} \Delta \rho_T \frac{1}{\rho_T^{j_2+1}} \sum_{m_1+2m_2+\cdots+j_2m_{j_2}=j_2} \Pi_{i=1}^{j_2} (\nabla^i \rho_T)^{m_i} \]
and \( m_1 + 2m_2 + \cdots + j_2m_{j_2} = j_2 \). We decompose \( \rho_T = \rho_D + \tilde{\rho} \) in the sum and estimate the \( \rho_D \) contribution:

\[
\begin{align*}
&b^4 \int \rho_T^2 \left\{ \sum_{j_1 + j_2 = k, j_2 \geq 2} \frac{|\nabla j_1 + 1 \Delta \rho_T|^2}{\rho_T^2 (Z)^{2j_2}} + \frac{|\nabla j_1 \Delta \rho_T|^2}{\rho_T^2 (Z)^{2j_2 + 2}} \right\} \\
&\lesssim b^4 \sum_{j_1 + j_2 = k, j_2 \geq 2} \left[ \int \frac{\rho_T^2}{(Z)^{2j_2 + 2(j_1 + 3)}} + \int \frac{|\nabla j_1 + 3 \tilde{\rho}|^2}{(Z)^{2j_2}} \right] \\
&\lesssim b^4 \left( 1 + \sum_{j_1 = 2}^k \int \frac{|\nabla j_1 \tilde{\rho}|^2}{(Z)^{2(k-j_1)+2}} \right) \leq \delta_0 + \delta J_{km}
\end{align*}
\]

In the general case, we replace \( (\nabla^i \rho_T)^{m_i} \) by \( (\nabla^i \tilde{\rho})^{m_i} \) where \( \tilde{\rho} \) is either \( \rho_D \) or \( \rho_T \). In both cases we will use the weaker estimates (4.40).

First, assume that \( m_i = 0 \) for \( i \geq \frac{4k_m}{9} + 1 \), then from (4.40):

\[
|N_{j_1, j_2}| \lesssim b^2 |\nabla j_1 + 1 \Delta \rho_T| \frac{1}{\rho_T^{j_2+1}} \sum_{m_1 + 2m_2 + \cdots + j_2m_{j_2} = j_2} \Pi_{i=0}^j |(\nabla^i \tilde{\rho})^{m_i}| \lesssim b^2 |\nabla j_1 + 1 \Delta \rho_T| \rho_T (Z)^{j_2}
\]

and the conclusion follows as above. Otherwise, there are at most two values \( \frac{4k_m}{9} \leq i_1 \leq i_2 \leq j_2 \) with \( m_{i_1}, m_{i_2} \neq 0 \) and \( m_{i_1} + m_{i_2} \leq 2 \). Hence from (4.40):

\[
\frac{1}{\rho_T^{j_2+1}} \Pi_{i=0}^j |(\nabla^i \tilde{\rho})^{m_i}| \lesssim \frac{1}{\rho_T^{j_2+1}} |\nabla^i \tilde{\rho}|^{m_{i_1}} |\nabla^i \tilde{\rho}|^{m_{i_2}} \Pi_{i=0}^j |(\nabla^i \tilde{\rho})^{m_i}| \frac{1}{\rho_D (Z)^{j_2-(m_{i_1}+m_{i_2})}}.
\]

Assume first \( i_2 \geq \frac{2k_m}{3} + 1 \), then \( m_{i_1} = 0, m_{i_2} = 1 \) and \( j_1 + 3 \leq \frac{4k_m}{9} \) from which:

\[
\frac{1}{\rho_T^{j_2+1}} |\nabla^i \tilde{\rho}|^{m_{i_1}} |\nabla^i \tilde{\rho}|^{m_{i_2}} \Pi_{i=0}^j |(\nabla^i \tilde{\rho})^{m_i}| \frac{1}{\rho_D (Z)^{2(j_2-i_2)+2}} \lesssim \int \frac{|\nabla^i \tilde{\rho}|^2}{(Z)^{2(j_2-i_2)+2}} \leq \delta_0
\]

There remains the case \( \frac{4k_m}{9} + 1 \leq i_1 \leq i_2 \leq \frac{2k_m}{3} \) which imply \( j_1 + 3 \leq \frac{2k_m}{3} \), and we distinguish cases:

- case \( (m_{i_1}, m_{i_2}) = (0, 1) \): if \( j_1 + 3 \leq \frac{4k_m}{9} \), we estimate

\[
\frac{1}{\rho_T^{j_2+1}} |\nabla^i \tilde{\rho}|^{m_{i_1}} |\nabla^i \tilde{\rho}|^{m_{i_2}} \Pi_{i=0}^j |(\nabla^i \tilde{\rho})^{m_i}| \frac{1}{\rho_D (Z)^{2(j_2-i_2)+2}} \lesssim \int \frac{|\nabla^i \tilde{\rho}|^2}{(Z)^{2(j_2-i_2)+2(j_2+1)}} \leq \delta_0
\]

Otherwise, \( \frac{4k_m}{9} + 1 \leq j_1 + 3 \leq \frac{2k_m}{3} \). Hence \( \frac{4k_m}{9} + 1 \leq j_1 + 3 \leq \frac{2k_m}{3}, \frac{4k_m}{9} + 1 \leq i_2 \leq \frac{2k_m}{3} \), and we estimate from (4.40), using \( k_m \) large:

\[
\int \frac{1}{\rho_T^{j_2+1}} |N_{j_1, j_2}|^2 \lesssim b^4 \int \frac{Z^{d-1}dZ}{(Z)^{2(k_m + \frac{4k_m}{9})}} \lesssim \delta_0.
\]

- case \( m_{i_1} + m_{i_2} = 2 \): we obtain from (4.40) and \( j_1 + 3 \leq \frac{2k_m}{3} \)

\[
\int \frac{1}{\rho_T^{j_2+1}} |N_{j_1, j_2}|^2 \lesssim b^4 \int \frac{1}{\rho_D^{j_2+1}} \frac{1}{(Z)^{2(j_2+1)}} \left( \frac{1}{(Z)^{k_m}} \right)^4 \leq \delta_0 \int \frac{dZ}{(Z)^{k_m}} \leq \delta_0.
\]
step 6 NL($\tilde{\rho}$) term. We need to estimate

$$\int \rho_T^2 \nabla \Delta K \mathcal{N} L(\tilde{\rho}) \cdot \nabla \Psi_k$$

which requires an integration by part in time for the highest order term. We expand using that, according to our assumptions, the nonlinearity is an integer:

$$\mathcal{N} L(\tilde{\rho}) = (\rho_D + \tilde{\rho})^{p-1} - \rho_D^{p-1} - (p-1)\rho_D^{p-2} \tilde{\rho} = \sum_{q=2}^{p-1} c_q \rho_D^{p-1-q}$$

and hence by Leibniz:

$$\Delta K^n \mathcal{N} L(\tilde{\rho}) = \sum_{q=2}^{p-1} c_q \rho_D^{p-1-q} \left( \Delta K^n \tilde{\rho} \right) + \sum_{q=2}^{p-1} \sum_{j_1 + j_2 = k} \left( \nabla^{\ell_1} \rho \ldots \nabla^{\ell_q} \rho \right) (\rho_D^{p-1-q})$$

Let

$$N_{\ell_1, \ldots, \ell_q;j_1,q} = \nabla^{\ell_1} \rho \ldots \nabla^{\ell_q} \rho (\rho_D^{p-1-q}), \quad \ell_1 \leq \cdots \leq \ell_q.$$

case $\ell_q \leq k_m - 2$: we estimate

$$|\nabla N_{\ell_1, \ldots, \ell_q,j_1,q}| \lesssim |\nabla^{m_1} \rho \ldots \nabla^{m_q} \rho| \rho_D^{p-1-q} \left( \frac{\rho_D}{\langle Z \rangle j_2} \right)^{j_1} \lesssim \frac{d_0}{\langle Z \rangle^{k_m}}$$

and hence the contribution of this term

$$\int \rho_T^2 |\nabla N_{\ell_1, \ldots, \ell_q,j_1,q}|^2 \lesssim d_0.$$

If $\frac{2k_m}{y} \leq m_q \leq \frac{2k_m}{y}$, then similarly, combining (6.9), (6.1):

$$|\nabla N_{\ell_1, \ldots, \ell_q,j_1,q}| \lesssim \frac{1}{\langle Z \rangle j_1 + 1} \frac{d_0}{\langle Z \rangle^{k_m}}$$

and the conclusion follows. If $m_q \geq \frac{2k_m}{y}$, then $m_{q-1} \leq \frac{4k_m}{y}$ from which:

$$|\nabla N_{\ell_1, \ldots, \ell_q,j_1,q}| \lesssim \frac{\rho_D^{p-1-q}}{\langle Z \rangle j_2} \frac{\rho_D^{p-1}}{\langle Z \rangle j_1 + 1 - \ell_q} |\nabla^{\ell_q} \rho| \lesssim \frac{\rho_D^{p-2} |\nabla^{\ell_q} \rho|}{\langle Z \rangle^{k_m - 1 - m_q}}$$

and hence the bound

$$\int \rho_T^2 |\nabla N_{\ell_1, \ldots, \ell_q,j_1,q}|^2 \lesssim \int \rho_T^{2(p-2)+2} \frac{|\nabla^{m_q} \rho|}{\langle Z \rangle^{2(k_m - m_q) + 2}} \lesssim \|\tilde{\rho}, \Psi\|_{k_m - 1, \sigma(k_m - 1)}^2 \leq d_0.$$

case $\ell_q = k_m - 1$: we compute $\nabla N_{\ell_1, \ldots, \ell_j,j_1,q}$. If the derivative falls on $\ell_j$, $j \leq q - 1$, we are back to the previous case, and we are therefore left with estimating

$$|\partial^{\ell_1} \rho \partial^{\ell_q \rho_D^{p-1-q} \rho D^{k_m \rho D^{p-1-q}} \left( \frac{\rho_D}{\langle Z \rangle j_2} \right)^{j_1 + \ell_j - 1 + k_m = j_1 + 1}}|$$
Using the extra decay in 

and the boundary term in time is small

Similarly, if \( j_1 = k_m \) then \( \ell_1 = \cdots = \ell_{q-2} = j_2 = 0 \) and \( \ell_{q-1} = 1 \).

And hence the corresponding contribution \((p \geq 3)\)

Similarly, after an integration by parts:

We treat this term by integration by parts in time using (7.8):

\[
\int \frac{\rho_{D}^{p-1}}{Z} |\partial k \rho | \cdot \nabla \Psi_k = \int \frac{\rho_{D}^{p-1}}{Z} \rho_{k} \cdot (\rho_{T} \nabla \Psi_k)
\]

We treat this term by integration by parts in time using (7.8):

The \( \partial_{t} \rho_{k} \) term is integrated by parts in time:

We then estimate:

\[
|\nabla^{\ell_{1}} \rho_{k} \cdots \nabla^{\ell_{q-1}} \rho_{k} \nabla k_{m} \tilde{\rho} \overline{p_{D}}^{p-1-q}(Z)_{j_{2}} | \lesssim |\tilde{\rho}^{q-1} \nabla k_{m} \tilde{\rho} \overline{p_{D}}^{p-1-q}(Z)_{j_{2}} | \lesssim \delta_{0} \rho_{D}^{p-2} |\nabla k_{m} \tilde{\rho}|
\]

and hence the corresponding contribution \((p \geq 3)\)

\[
\delta_{0} \int \frac{2(p-2)}{\rho_{D}^{2}} |\partial k \rho |^{2} \leq \delta J_{k_{m}}.
\]

Highest order term We are left with estimating the highest order term:

\[
N_{\ell_{1}, \ldots, \ell_{q}, j_{1}, q} = \tilde{\rho}^{q-1} \rho_{D}^{p-1-q} \Delta k_{m} \tilde{\rho}.
\]

We then estimate:

\[
\left| \int \frac{\rho_{D}^{p-1}}{Z} \rho_{k} \rho_{T}(H_{1} - k(H_{2} + \Lambda H_{2})) \rho_{k} \right| \lesssim k \delta \int \rho_{T}^{p-1} \rho_{k}^{2} \lesssim \delta J_{k_{m}}.
\]

Using the extra decay in \( Z \) and \( \| \Delta \Psi \|_{L^{\infty}} \leq \delta \ll 1: \)

\[
\left| \int \rho_{D}^{p-1-q} \rho_{k} \rho_{T}(\Delta k_{m} \rho_{T}) \Delta \Psi \right| \lesssim \delta_{0} \int \frac{dZ}{(Z)^{\frac{3-p}{2}}} + \int \rho_{T}^{p-1} \rho_{k}^{2} \Delta \Psi \leq \delta_{0} + \delta J_{k_{m}}.
\]

Similarly, after an integration by parts:

\[
\left| - \int \rho_{D}^{p-1-q} \rho_{k} \rho_{T} \nabla(\Delta k_{m} \rho_{T}) \cdot \nabla \Psi \right| \lesssim \delta_{0} + \left| \int \rho_{D}^{p-1-q} \rho_{T} \nabla(\rho_{k}^{2}) \cdot \nabla \Psi \right|
\]

\[
\leq \delta_{0} + \delta J_{k_{m}}.
\]

Similarly, after an integration by parts using (4.40):

\[
\left| - \int \rho_{D}^{p-1-q} \rho_{k} \rho_{T} H_{2} \Lambda \rho_{k} \right| \lesssim \delta_{0} + \left( \frac{\| \rho_{k} \|_{L^{\infty}}}{\rho_{T}} + \frac{\| Z \left| \nabla \rho_{k} \right| \rho_{T}^{2} \|_{L^{\infty}}}{\rho_{T}} \right) J_{k_{m}} \leq \delta_{0} + \delta J_{k_{m}}
\]
Proof of (7.20). The coupling term is lower order for which, after taking We compute the discriminant:
\[ \frac{\partial D}{\partial T} \{ J_{km}(1 + O(\delta)) \} \]
\[ \leq -k \left[ 1 + O \left( \frac{1}{k} \right) \right] \int (H_2 + \Lambda H_2) \left[ b^2 |\nabla \tilde{\rho}_k|^2 + (p - 1)\rho_D^{-2} \rho_T \tilde{\rho}_k^2 + \rho_T^2 |\nabla \Psi_k|^2 \right] \]
\[ - k \int (p - 1)\rho_D \partial_Z (\rho_D^{-1}) \tilde{\rho}_k \partial_Z \Psi_k + \delta. \]

We recall from (2.21), (2.22):
\[ H_2 + \Lambda H_2 = \mu(1 - w - \Lambda w) \geq c_{d,p} > 0 \quad (7.19) \]
and we now claim the pointwise coercivity of the coupled quadratic form: \( \exists c_{d,p} > 0 \) such that \( \forall Z \geq 0, \)
\[ (H_2 + \Lambda H_2) \left[ (p - 1)\rho_D^{-2} \rho_T \tilde{\rho}_k^2 + \rho_T^2 |\nabla \Psi_k|^2 \right] + (p - 1)\rho_D \partial_Z (\rho_D^{-1}) \tilde{\rho}_k \partial_Z \Psi_k \]
\[ \geq c_{d,p} \left[ (p - 1)\rho_D^{-2} \rho_T \tilde{\rho}_k^2 + \rho_T^2 |\nabla \Psi_k|^2 \right] \quad (7.20) \]
which, after taking \( k > k^*(d,p) \) large enough, concludes the proof of (7.6).

Proof of (7.20). The coupling term is lower order for \( Z \) large:
\[ |(p - 1)\rho_D \partial_Z (\rho_D^{-1}) \tilde{\rho}_k \partial_Z \Psi_k| \leq \frac{\rho_D^{-1}}{Z} \tilde{\rho}_k \rho_D \partial_Z \Psi_k \leq \delta \left[ (p - 1)\rho_D^{-2} \rho_T \tilde{\rho}_k^2 + \rho_T^2 |\nabla \Psi_k|^2 \right] \]
for \( Z > Z(\delta) \) large enough. On a compact set using the smallness (4.27), (7.20) is implied by:
\[ (H_2 + \Lambda H_2) \left[ (p - 1)Q \rho_T \tilde{\rho}_k^2 + \rho_T^2 |\nabla \Psi_k|^2 \right] + (p - 1)\rho_D \partial_Z Q \tilde{\rho}_k \partial_Z \Psi_k \]
\[ \geq c_{d,p} \left[ (p - 1)Q \rho_T \tilde{\rho}_k^2 + \rho_T^2 |\nabla \Psi_k|^2 \right] \quad (7.21) \]

We compute the discriminant:
\[ \text{Discr} = (p - 1)^2 \rho_D^2 (\partial_Z Q)^2 - 4\mu^2 (p - 1)\rho_D^2 Q (H_2 + \Lambda H_2)^2 \]
\[ = (p - 1)\rho_D^2 Q \left[ (p - 1)(\partial_Z Q)^2 - 4\mu^2 (1 - w - \Lambda w)^2 \right] \]
We compute from (2.9) recalling (2.20):
\[ (p - 1)\left( \frac{\partial Z Q}{Q} \right)^2 = (p - 1) \left( 2\partial Z \sqrt{Q} \right)^2 = (p - 1) \left( 1 - \frac{\epsilon}{2} \sqrt{\partial Z (\sigma_P Z)} \right)^2 = (1 - \epsilon)^2 (\partial Z (Z \sigma_P))^2 \]
\[ = \frac{4}{r^2} (\partial Z (Z \sigma_P))^2 = 4\mu^2 F^2 \]
and hence from (2.21), (2.22) the lower bound:
\[-\text{Discr} = 4\mu^2 (p - 1)^2 \rho_D^2 Q \left[ (1 - w - \Lambda w)^2 - F^2 \right] \geq c_{d,r} (p - 1)\rho_D^2 Q, \quad c_{d,r} > 0 \]
which together with (7.19) concludes the proof of (7.20).
8. Control of low Sobolev norms and proof of Theorem 1.1

Our aim in this section is to control weighted low Sobolev norms in the interior $r \leq 1$ ($Z \leq Z^*$). On our way we will conclude the proof of the bootstrap Proposition 4.4. Theorem 1.1 will then follow from a classical topological argument.

8.1. Exponential decay slightly beyond the light cone. We use the exponential decay estimate (3.64) for a linear problem to prove exponential decay for the nonlinear evolution in the region slightly past the light cone. We recall the notations of Section 3, in particular $Z_a$ of Lemma 3.2.

Lemma 8.1 (Exponential decay slightly past the light cone). Let

$$\tilde{Z}_a = \frac{Z_2 + Z_a}{2}.$$

$$\|\nabla \Phi\|_{H^{2k_0}(Z \leq \tilde{Z}_a)} + \|\rho\|_{H^{2k_0}(Z \leq \tilde{Z}_a)} \lesssim e^{-\frac{s_0}{2} \tau}. \quad (8.1)$$

Proof. The proof relies on the spectral theory beyond the light cone and an elementary finite speed propagation like argument in renormalized variables, related to [47].

**step 1** Semigroup decay in $X$ variables. Recall the definition (4.12) of $X = (\Phi, T)$

$$\begin{align*}
\Phi &= \rho \rho \Psi \\
T &= \partial_\tau \Phi + a H_2 \Delta \Phi = -(p-1)Q \rho + H_2 \Delta \Phi + (H_1 - \epsilon) \Phi + G_\Phi + a H_2 \Delta \Phi \quad (8.2)
\end{align*}$$

with $G_\Phi$ given by (3.4), the scalar product (3.37) and the definitions (4.14), (4.15):

$$\begin{align*}
\Lambda_0 &= \{ \lambda \in \mathbb{C}, \quad \Re(\lambda) \geq 0 \} \cap \{ \lambda \text{ is an eigenvalue of } M \} = (\lambda_i)_{1 \leq i \leq N}
\end{align*}$$

$$V = \bigcup_{1 \leq i \leq N} \ker(M - \lambda_i I)^{k_{\lambda_i}}$$

the projection $P$ associated with $V$, the decay estimate (3.64) on the range of $(I - P)$ and the results of Lemma 3.11. Relative to the $X$ variables our equations take the form

$$\partial_\tau X = M X + G,$$

which are considered on the time interval $\tau \geq \tau_0 \gg 1$ and the space interval $Z \in [0, Z_a]$ (no boundary conditions at $Z_a$). We consider evolution in the Hilbert space $H_{2k_0}$ with initial data such that

$$\| (I - P) X(\tau_0) \|_{H_{2k_0}} \leq e^{-\frac{s_0}{2} \tau_0}, \quad \| P X(\tau_0) \|_{H_{2k_0}} \leq e^{-\frac{3s_0}{4} \tau_0}. \quad (8.3)$$

According to the bootstrap assumption (4.45)

$$\| P X(\tau) \|_{H_{2k_0}} \leq e^{-\frac{s_0}{2} \tau}, \quad \forall \tau \in [\tau_0, \tau^*] \quad (8.4)$$

Lemma 3.11 shows that as long as

$$\| G \|_{H_{2k_0}} \leq e^{-\frac{2s_0}{4} \tau}, \quad \tau \geq \tau_0 \quad (8.5)$$

there exists $\Gamma$, which can be made as large as we want with a choice of $\tau_0$, such that

$$\| P X(\tau) \|_{H_{2k_0}} \leq e^{-\frac{s_0}{2} \tau}, \quad \tau_0 \leq \tau \leq \tau_0 + \Gamma. \quad (8.6)$$

This will allow us to show eventually that if we can verify (8.5), the bootstrap time $\tau^* \geq \tau_0 + \Gamma$. 

Moreover, as long as (8.5) holds, the decay estimate (3.64) implies that
\[
\| (I - P)X(\tau) \|_{H^{2k_0}} \lesssim e^{-\frac{\delta_2}{2}(\tau - \tau_0)} \| X(\tau_0) \|_{H^{2k_0}} + \int_{\tau_0}^{\tau} e^{-\frac{\delta_2}{2}(\tau - \sigma)} \| G(\sigma) \|_{H^{2k_0}} d\sigma
\]
\[
\lesssim e^{-\frac{\delta_2}{2}\tau} \left[ e^{\frac{\delta_2}{2}\tau_0} \| X(\tau_0) \|_{H^{2k_0}} + \int_{\tau_0}^{+\infty} e^{-\frac{\delta_2}{2}\tau} d\tau \right] \leq e^{-\frac{\delta_2}{2}\tau}.
\]
(8.7)

As a result,
\[
\| X(\tau) \|_{H^{2k_0}} \lesssim e^{-\frac{\delta_2}{2}\tau}, \quad \tau_0 \leq \tau \leq \tau^*.
\]
(8.8)

Below we will verify (8.5) \( \forall \tau \in [\tau_0, \tau^*] \) under the assumption (8.7), closing both. Once again, this will allow us to show eventually that the length of the bootstrap interval \( \tau^* - \tau_0 \geq \Gamma \) is sufficiently large.

Recall from (3.6), (3.7), (3.37):
\[
\| G \|_{H^{2k_0}}^2 \lesssim \int_{Z \leq Z_a} |\nabla \Delta^{k_0} G_T|^2 gZ^{d-1} dZ + \int_{Z \leq Z_a} G_T^2 Z^{d-1} dZ
\]
(8.9)

with
\[
G_T = \partial_\tau G_\Phi - \left( H_1 + H_2 \frac{\Lambda \Omega}{\Omega} \right) G_\Phi + H_2 \Lambda G_\Phi - (p - 1) QG_\rho,
\]
\[
G_\rho = -\rho \Delta \Psi - 2\nabla \rho \cdot \nabla \Psi,
\]
\[
G_\Phi = -\rho P(|\nabla \Psi|^2 + NL(\rho)) + \frac{\rho}{\rho_T} \Delta \rho_T.
\]

**Step 2** Semigroup decay for \((\rho, \Psi)\). We now translate the \( X \) bound to the bounds for \( \rho \) and \( \Psi \) and then verify (8.5). We recall (8.2) and obtain for any \( \tilde{Z} > Z_2 \)
\[
\| T \|_{H^{2k_0}(Z \leq \tilde{Z})} + \| \Phi \|_{H^{2k_0+1}(Z \leq \tilde{Z})} \lesssim \| \rho \|_{H^{2k_0}(Z \leq \tilde{Z})} + \| \Psi \|_{H^{2k_0+1}(Z \leq \tilde{Z})} + ||G_\Phi||_{H^{2k_0}(Z \leq \tilde{Z})}
\]
\[
\lesssim \| T \|_{H^{2k_0}(Z \leq \tilde{Z})} + \| \Phi \|_{H^{2k_0+1}(Z \leq \tilde{Z})} + ||G_\Phi||_{H^{2k_0}(Z \leq \tilde{Z})}
\]
and claim:
\[
||G_\Phi||_{H^{2k_0}(Z \leq \tilde{Z})} \lesssim ||\nabla \Psi||_{H^{2k_0}(Z \leq \tilde{Z})}^2 + ||\rho||_{H^{2k_0}(Z \leq \tilde{Z})}^2 + e^{-\delta_\rho \tau}.
\]
(8.10)

Indeed, since \( H^{2k_0}(Z \leq \tilde{Z}) \) is an algebra for \( k_0 \) large enough:
\[
||\rho P(|\nabla \Psi|^2 + NL(\rho)||_{H^{2k_0}(Z \leq \tilde{Z})} \lesssim ||\nabla \Psi||_{H^{2k_0}(Z \leq \tilde{Z})}^2 + ||\rho||_{H^{2k_0}(Z \leq \tilde{Z})}^2.
\]

The remaining quantum pressure term is treated using the pointwise bound (4.40) for small Sobolev norms and the smallness of \( b \) which imply:
\[
\left| \frac{b^2 \rho P \Delta \rho_T}{\rho_T} \right|_{H^{2k_0}(Z \leq \tilde{Z})} \lesssim C_K b^2 \leq e^{-\delta_\rho \tau}
\]
provided \( \delta_\rho > 0 \) has been chosen small enough, and (8.10) is proved. Choosing \( \tilde{Z} > Z_2 \), this implies from (8.2) and the initial bound (4.19):
\[
\| X(\tau_0) \|_{H^{2k_0}} \lesssim \| \Psi(\tau_0) \|_{H^{2k_0+1}(Z \leq \tilde{Z})} + ||\rho(\tau_0)||_{H^{2k_0}(Z \leq \tilde{Z})} + e^{-\delta_\rho \tau_0}
\]
\[
\lesssim e^{-\frac{\delta_\rho \tau_0}{2}}.
\]
(8.11)

This verifies (8.3). On the other hand, choosing \( \tilde{Z} = \tilde{Z}_a \) with
\[
\tilde{Z}_a = \frac{Z_2 + Z_a}{2},
\]
we also obtain from (8.8)
\[
\| \Psi(\tau) \|_{H^{2k_0+1}(Z \leq \tilde{Z}_a)} + ||\rho(\tau)||_{H^{2k_0}(Z \leq \tilde{Z}_a)} \lesssim \| X(\tau) \|_{H^{2k_0}} + e^{-\delta_\rho \tau} \lesssim e^{-\frac{\delta_\rho \tau}{2}}.
\]
(8.12)
The estimate (8.1) follows.

**step 3** Estimate for $G$. Proof of (8.5). We recall (8.9). On a fixed compact domain $Z \leq Z_0$ with $Z_0 > Z_2$, we can interpolate the bootstrap bound (4.39) with the global large Sobolev bound (4.38) and obtain for $k_m$ large enough and $b_0 < b_0(k_m)$ small enough:

$$
\|\rho\|_{H^{2k_0+10}(Z \leq Z_0)} + \|\Psi\|_{H^{2k_0+10}(Z \leq Z_0)} \leq C e^{-\frac{1}{4}k_m}\delta^r \leq e^{-\frac{1}{4}k_m}\delta^r \tag{8.13}
$$

and since $H^{2k_0}$ is an algebra and all terms are either quadratic or with a $b$ term, (8.13) implies

$$
\|G_T\|_{H^{2k_0+5}(Z \leq Z_0)} + \|G_\mu\|_{H^{2k_0+5}(Z \leq Z_0)} + \|G_\phi\|_{H^{2k_0+5}(Z \leq Z_0)} \leq e^{-\frac{1}{4}k_m}\delta^r \tag{8.14}
$$

which in particular using (8.9) implies (8.5). \hfill \square

### 8.2. Weighted decay for $m \leq 2k_0$ derivatives.

We recall the notation (3.1). We now transform the exponential decay (8.1) from just past the light cone into weighted decay estimate. It is essential for this argument that the decay (8.1) has been shown in the region strictly including the light cone $Z = Z_2$. The estimates in the lemma below close the remaining bootstrap bound (4.39).

**Lemma 8.2** (Weighted Sobolev bound for $m \leq 2k_0$). Let $m \leq 2k_0$ and $\nu_0 = \frac{\delta_p}{2m} - \frac{2(r-1)}{p-1}$, recall

$$
\lambda_{\nu_0,m} = \frac{1}{|Z|^{d-2(r-1)+2|\nu_0-m|}} \zeta \left( \frac{Z}{|Z|^*} \right), \quad \zeta(Z) = \begin{cases} 1 & \text{for } Z \leq 2 \\ 0 & \text{for } Z \geq 3, \end{cases}
$$

then:

$$
\sum_{m=0}^{2k_0} \sum_{i=1}^{d} \int (p-1)Q(\partial_i^m \rho)^2 \lambda_{\nu_0,m} + |\nabla \partial_i^m \Phi|^2 \lambda_{\nu_0,m} \leq C e^{-\frac{\delta_y}{\tau}}. \tag{8.15}
$$

**Proof of Lemma 8.2.** The proof relies on a sharp energy estimate with time dependent localization of $(\rho, \Phi)$. This is a renormalized version of the finite speed of propagation.

**step 1** $H^m$ localized energy identity. Pick a smooth well localized radially symmetric function $\chi(\tau, Z)$ and a coordinate $1 \leq i \leq d$ and note for $m$ integer

$$
\rho_m = \partial_i^m \rho, \quad \Phi_m = \partial_i^m \Phi,
$$

where we omit the $i$ dependence to simplify notations. We recall the Emden transform formulas (2.24):

$$
\begin{align*}
H_2 &= \mu(1-w) \\
H_1 &= \frac{\mu}{\lambda}(1-w) \left[ 1 + \frac{A_\sigma}{\sigma} \right] \\
H_3 &= \frac{\lambda_\sigma}{\rho_\sigma}
\end{align*}
$$

which yield the bounds using (2.18), (2.19):
and the commutator bounds:

\[
\begin{align*}
|[\partial_i^m, H_1]\rho| &\lesssim \sum_{j=0}^{m-1} \frac{|\partial_j^2\rho|}{(j!)^{m-j+1}} \\
|\nabla ([\partial_i^m, H_1]\rho)| &\lesssim \sum_{j=0}^{m} \frac{|\partial_j^2\rho|}{(j!)^{m-j+1}} \\
|([\partial_i^m, Q]\rho)| &\lesssim Q \sum_{j=0}^{m-1} \frac{|\partial_j^2\rho|}{(j!)^{m-j}} \\
|([\partial_i^m, H_2]\Lambda \rho)| &\lesssim \sum_{j=1}^{m} \frac{|\partial_j^2\rho|}{(j!)^{m-j}} \\
|\nabla ([\partial_i^m, H_2]\Lambda \Phi)| &\lesssim \sum_{j=1}^{m+1} \frac{|\partial_j^2\Phi|}{(j!)^{m-j+1}}.
\end{align*}
\]

(8.18)

Commuting (3.2) with \(\partial_i^m\):

\[
\begin{align*}
\partial_r \rho_m &= H_1 \rho_m - H_2(m + \Lambda) \rho_m - \Delta \Phi_m + \partial_i^m G_\rho + E_{m,\rho} \\
\partial_r \Phi_m &= -(p-1)Q \rho_m - H_2(m + \Lambda) \Phi_m + (H_1 - \mu) \Phi_m + \partial_i^m G_\Phi + E_{m,\Phi}
\end{align*}
\]

with the bounds

\[
\begin{align*}
|E_{m,\rho}| &\lesssim \sum_{j=0}^{m} \frac{|\partial_j^2\rho|}{(j!)^{m-j+2}} + \sum_{j=0}^{m} \frac{|\partial_j^2\Phi|}{(j!)^{m-j+2}} \\
|\nabla E_{m,\Phi}| &\lesssim Q \sum_{j=0}^{m} \frac{|\partial_j^2\rho|}{(j!)^{m-j+3}} + \sum_{j=0}^{m+1} \frac{|\partial_j^2\Phi|}{(j!)^{m-j}}.
\end{align*}
\]

Let \(\chi\) be an arbitrary smooth function. We derive the corresponding energy identity:

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left\{ \int (p-1)Q \rho_m^2 \chi + |\nabla \Phi_m|^2 \chi \right\} &= \frac{1}{2} \int \partial_r \chi \left[ (p-1)Q \rho_m^2 + |\nabla \Phi_m|^2 \right] \\
+ \int (p-1)Q \rho_m \chi [H_1 \rho_m - H_2(m + \Lambda) \rho_m - \Delta \Phi_m + \partial_i^m G_\rho + E_{m,\rho}] \\
+ \int \chi \nabla \Phi_m \cdot \nabla \left[ -(p-1)Q \rho_m - H_2(m + \Lambda \Phi_m) + (H_1 - \mu(r-2)) \Phi_m + \partial_i^m G_\Phi + E_{m,\Phi} \right] \\
= \frac{1}{2} \int \partial_r \chi \left[ (p-1)Q \rho_m^2 + |\nabla \Phi_m|^2 \right] \\
+ \int (p-1)Q \rho_m \chi [H_1 \rho_m - H_2(m + \Lambda) \rho_m + \partial_i^m G_\rho + E_{m,\rho}] + \int (p-1)Q \rho_m \chi \cdot \nabla \Phi_m \\
+ \int \chi \nabla \Phi_m \cdot \nabla \left[-H_2(m + \Lambda) \Phi_m + (H_1 - \mu(r-2)) \Phi_m + \partial_i^m G_\Phi + E_{m,\Phi} \right].
\end{align*}
\]

In what follows we will use \(\omega > 0\) as a small universal constant to denote the power of tails of the error terms. In most cases, the power is in fact \(r > 2\) which we do not need.

\(\rho_m\) terms. From the asymptotic behavior of \(Q\) (2.19) and (8.17):

\[
\begin{align*}
- \int (p-1)Q \rho_m \chi H_2 \Lambda \rho_m &= \frac{p-1}{2} \int \rho_m^2 \chi QH_2 \left[ d + \frac{\Lambda Q}{\mu} + \frac{\Lambda H_2}{\mu} + \frac{\Lambda \chi}{\mu} \right] \\
= \int \rho_m^2 (p-1) \chi Q \mu \left[ \frac{d}{2} - (r-1) + O \left( \frac{1}{(Z^\mu)^\omega} \right) \right] + \frac{1}{2} \int (p-1)Q H_2 \Lambda \chi \rho_m^2
\end{align*}
\]
\[ \Phi_m \text{ terms. We first estimate recalling (8.17):} \]
\[ \int \chi \nabla \Phi_m \cdot \nabla \left( (-mH_2 + H_1 - \mu (r-2))\Phi_m \right) \]
\[ = \int (-mH_2 + H_1 - \mu (r-2)) \chi |\nabla \Phi_m|^2 + O \left( \int \frac{\chi}{(Z)^{p-1}} |\nabla \Phi_m| |\Phi_m| \right) \]
\[ = - \left[ \mu (m + r - 2) + \frac{2\mu (r-1)}{p-1} \right] \int \chi |\nabla \Phi_m|^2 + O \left( \int \frac{\chi}{(Z)^{\omega}} \left[ |\nabla \Phi_m|^2 + \frac{\Phi_m^2}{(Z)^2} \right] \right) \]

From Pohozhaev identity (5.26) with \( F = \chi H_2(Z_1, \ldots, Z_d) \):
\[ - \int \chi |\nabla \Phi_m| \cdot \nabla (H_2 \Lambda \Phi_m) = \int H_2 \Lambda \Phi_m \left[ \chi |\nabla \Phi_m| + \nabla \chi \cdot \nabla \Phi_m \right] \]
\[ = - \sum_{i,j=1}^d \int \partial_i F \partial_i \Phi_m \rho \partial_j \Phi_m + \frac{1}{2} \int |\nabla \Phi_m|^2 \nabla \cdot F + \int H_2 \Lambda \Phi_m \nabla \chi \cdot \nabla \Phi_m \]
\[ = \sum_{i,j=1}^d \partial_i \Phi_m \partial_j \Phi_m \left[ -\partial_i (\chi H_2 Z_j) + H_2 Z_j \partial_i \chi \right] + \frac{1}{2} \int |\nabla \Phi_m|^2 \chi H_2 \left[ d + \frac{\Lambda \chi}{\chi} + \frac{\Lambda H_2}{H_2} \right] \]
\[ = \frac{\mu (d-2)}{2} \int \chi |\nabla \Phi_m|^2 + \frac{1}{2} \int H_2 \Lambda \chi |\nabla \Phi_m|^2 + O \left( \int \frac{\chi}{(Z)^{\omega}} |\nabla \Phi_m|^2 \right) \]

The collection of above bounds yields for some universal constant \( \omega > 0 \) the weighted energy identity:
\[ \frac{1}{2} \frac{d}{dt} \left\{ \int (p-1)Q \rho_m \chi + |\nabla \Phi_m|^2 \chi \right\} \]
\[ = - \int \chi \left[ (p-1)Q \rho_m^2 + |\nabla \Phi_m|^2 \right] \left[ \mu \left( m - \frac{d}{2} + r - 1 \right) + \frac{2\mu (r-1)}{p-1} + O \left( \frac{1}{(Z)^{\omega}} \right) \right] \]
\[ + \frac{1}{2} \int (p-1)Q \rho_m^2 \left[ \partial_r \chi + H_2 \Lambda \chi \right] + \frac{1}{2} \int |\nabla \Phi_m|^2 \left[ \partial_r \chi + H_2 \Lambda \chi \right] + \int (p-1)Q \rho_m \nabla \chi \cdot \nabla \Phi_m \]
\[ + O \left( \int \chi \left[ \sum_{j=0}^{m+1} \frac{|\partial_j^2 \Phi|^2}{(Z)^{2(m-j)+\omega}} + \sum_{j=0}^m \frac{Q |\partial_j \rho|^2}{(Z)^{2(m-j)+\omega}} \right] \right) \]
\[ + O \left( \int \chi |\nabla \Phi_m||\nabla \partial^m G_\phi| + \int \chi Q |\rho_m||\partial^m G_\rho| \right) \]

**step 2** Nonlinear and source terms. We claim the bound for \( \chi = \chi_{0,m} \):
\[ \sum_{m=0}^{2k_0} \sum_{i=1}^d \int \chi_{0,m} |\nabla \partial^m G_\phi|^2 + \int (p-1)Q \chi_{0,m} |\partial^m G_\rho|^2 \]
\[ \lesssim \left( \sum_{m=0}^{2k_0} \sum_{i=1}^d \int Q \rho_m^2 \chi_{0,m+1} |\nabla \Phi_m|^2 \chi_{0,m+1,m} + |\nabla \Phi_m|^2 \chi_{0,m+1,m} \right) + b^2. \]

**G_\rho term.** Recall (3.4)
\[ G_\rho = -\rho \Delta \Psi - 2\nabla \rho \cdot \nabla \Psi, \]
then by Leibniz:
\[ |\partial^m G_\rho|^2 \lesssim \sum_{j_1+j_2=m+2,j_2 \geq 1} |\partial^{j_1} \rho|^2 |\partial^{j_2} \Psi|^2. \]
We recall the pointwise bounds (4.40) for \( Z \leq 3Z^* \),
\[
|\partial^{j_1} \rho| \leq \frac{C_K}{\langle Z \rangle^{j_1 + \frac{2(r-1)}{r+1}}}, \quad |\partial^{j_2} \Psi| \leq \frac{C_K}{\langle Z \rangle^{j_2 + r - 2}}.
\]
This yields, recalling (8.33), for \( j_1 \leq 2k_0 \):
\[
\int \chi_{\nu_0,m} Q |\partial^{j_1} \rho|^2 |\partial^{j_2} \Psi|^2 \lesssim \int Q \zeta \left( \frac{Z}{Z^*} \right) \frac{|\partial^{j_1} \rho|^2}{Z^{2(r-1)+2(\nu_0+j_1)+2}} \lesssim \sum_{j=0}^{j_1} \int \chi_{\nu_0+1,j} Q |\partial^{j_2} \rho|^2
\]
\[
\lesssim \sum_{m=0}^{2k_0} \sum_{i=1}^{d} \int Q \rho_m^2 \chi_{\nu_0+1,m} + |\nabla \Phi_m|^2 \chi_{\nu_0+1,m}.
\]
For \( j_1 = m + 1 \), \( j_2 = 1 \), we use the other variable and the conclusion follows similarly.

\( G_\Phi \) term. Recall (3.4)
\[
G_\Phi = -\rho_P(|\nabla \Psi|^2 + \text{NL}(\rho)) + \frac{b^2 \rho_P}{\rho_T} \Delta \rho_T.
\]
We estimate using the pointwise bounds (4.40) for \( j_3 \leq 2k_0 \):
\[
|\nabla \partial^m (\rho_P |\nabla \Psi|^2)| \lesssim \sum_{j_1+j_2+j_3 = m+1,j_2 \leq j_3} \frac{\rho_P}{\langle Z \rangle^{j_1}} |\partial^{j_2+1} \Psi |^2 |\partial^{j_3+1} \Psi |
\]
\[
\lesssim \sum_{j_1+j_2+j_3 = m+1,j_2 \leq j_3} \frac{1}{\langle Z \rangle^{2(r-1)+j_1+r-2+j_2+1}} |\partial^{j_3+1} \Psi | \lesssim \sum_{j_3=0}^{2k_0} \frac{|\partial^{j_3+1} \Phi|}{\langle Z \rangle^{r+m-j_3}}.
\]
and since \( r > 1 \):
\[
\sum_{j_3=0}^{2k_0} \int \chi_{\nu_0,m} \frac{|\partial^{j_3+1} \Phi|^2}{\langle Z \rangle^{2(r-m-j_3)}} \lesssim \sum_{j_3=0}^{2k_0} \int \chi_{\nu_0+1,j_3} |\nabla \Phi_j|^2.
\]
For \( j_3 = 2k_0 + 1 \), we use the other variable and the conclusion follows similarly.

The quantum pressure term is estimated using the pointwise bounds (4.40):
\[
\int \chi_{\nu_0,m} \left| \nabla \partial^m \left( \frac{b^2 \rho_P}{\rho_T} \Delta \rho_T \right) \right|^2 \lesssim C_K b^4 \int Z \lesssim 3Z^* \frac{\chi_{\nu_0,m}}{\langle Z \rangle^{4(r-1)+2(m+3)}}
\]
\[
\lesssim C_K b^4 \int Z \lesssim 3Z^* \frac{Z^{d-1} dZ}{\langle Z \rangle^{d-2(r-1)+2(\nu_0+2)}+2(m+3)} \leq b^2.
\]
Recall that $Z$ propagate the bound (8.1) to the compact set and hence which concludes the proof of (8.23) and (8.15) for $\tilde{a}$.

step 2 Initialization and lower bound on the bootstrap time $\tau^*$. Fix a large enough $Z_0$ and pick a small enough universal constant $\omega_0$ such that
\[ \forall Z \geq 0, \ -\omega_0 + H_2 \geq \frac{\omega_0}{2} > 0 \] (8.21)
and let $\Gamma = \Gamma(Z_0)$ such that
\[ \frac{Z_0}{Z_a} e^{-\omega_0 \Gamma} = 1. \] (8.22)
We claim that provided $\tau_0$ has been chosen sufficiently large, the bootstrap time $\tau^*$ of Proposition 4.4 satisfies the lower bound
\[ \tau^* \geq \tau_0 + \Gamma. \] (8.23)
Indeed, in view of sections 5, 6, 7 there remains to control the bound (4.39) on $[\tau_0, \tau_0 + \Gamma]$. By (8.6) (8.7), the desired bounds already hold for $Z \leq \tilde{Z}_a$ on $[\tau_0, \tau_0 + \Gamma]$.

We now run the energy estimate (8.19) with $\chi = \chi_{\tau_0, m}$ and obtain from (8.19), (8.20) the rough bound on $[\tau_0, \tau^*]$
\[ \frac{d}{dt} \left\{ \int (p-1)Q\rho_m^2 \chi_{\tau_0, m} + |\nabla \Phi_m|^2 \chi_{\tau_0, m} \right\} \leq C \int (p-1)Q\rho_m^2 \chi_{\tau_0, m} + |\nabla \Phi_m|^2 \chi_{\tau_0, m} + b^2. \]
which yields using (4.19):
\[ \int (p-1)Q\rho_m^2 \chi_{\tau_0, m} + |\nabla \Phi_m|^2 \chi_{\tau_0, m} \leq e^{C(\tau - \tau_0)} \int (p-1)Q(\rho_m(0))^2 \chi_{\tau_0, m} + |\nabla \Phi_m(0)|^2 \chi_{\tau_0, m} + e^{C\tau} \int \int\]
and hence
\[ e^{\frac{4\delta_2}{\tau}} \left[ \int (p-1)Q\rho_m^2 \chi_{\tau_0, m} + |\nabla \Phi_m|^2 \chi_{\tau_0, m} \right] \]
\[ \leq e^{\frac{4\delta_2}{\tau_0}} e^{\frac{4\delta_2}{\tau_0}} \left[ \int (p-1)Q\rho_m^2 \chi_{\tau_0, m} + |\nabla \Phi_m|^2 \chi_{\tau_0, m} \int (p-1)Q\rho_m^2 \chi_{\tau_0, m} + |\nabla \Phi_m|^2 \chi_{\tau_0, m} \right] \]
\[ = 2e^{C\tau} C_0 e^{-\delta_2 \tau_0} e^{\frac{4\delta_2}{\tau_0}} \leq e^{2C\tau} e^{-\frac{\delta_2}{2\tau_0}} \leq 1 \]
which concludes the proof of (8.23) and (8.15) for $\tau \in [\tau_0, \tau_0 + \Gamma]$.

step 3 Finite speed of propagation. We now pick a time $\tau_f \in [\tau_0 + \Gamma, \tau^*]$ and propagate the bound (8.1) to the compact set $Z \leq \tilde{Z}_0$ using a finite speed of propagation argument. We claim:
\[ \|\rho\|^2_{H^{2\delta_0}(Z \leq \frac{Z_a}{2})} + \|\nabla \Phi_m\|^2_{H^{2\delta_0}(Z \leq \frac{Z_a}{2})} \leq C e^{-\delta_2 \tau_f} . \] (8.24)
Here the key is that (8.1) controls a norm on the set strictly including the light cone $Z \leq Z_2$.

Let
\[ \tilde{Z}_a = \frac{Z_a + Z_2}{2} \]
and note that we may, without loss of generality by taking $a > 0$ small enough, assume:
\[ \frac{\tilde{Z}_a}{Z_a} \leq 2. \] (8.25)
Recall that $\Gamma = \Gamma(Z_0)$ is parametrized by (8.22). We define
\[ \chi(\tau, Z) = \zeta \left( \frac{Z}{\nu(\tau)} \right), \quad \nu(\tau) = \frac{Z_0}{2Z_a} e^{-\omega_0 (\tau_f - \tau)} \]
with \( \omega_0 > 0 \) defined in (8.21), (8.22) and a fixed spherically symmetric non-increasing cut off function
\[
\zeta(Z) = \begin{cases} 1 & \text{for } 0 \leq Z \leq \tilde{Z}_a, \\ 0 & \text{for } Z \geq \tilde{Z}_a. \end{cases} \quad \zeta' \leq 0
\] (8.26)

We define
\[ \tau_T = \tau_f - \Gamma \]
so that from (8.22):
\[
\nu(\tau_T) = \frac{Z_0}{2Z_a} e^{-\omega_0 (\tau_f - \tau_T)} = \frac{Z_0}{2Z_a} e^{-\omega_0 \Gamma} = 1.
\] (8.27)

We pick
\[ 0 \leq m \leq 2k_0 \]
then (8.26), (8.27) ensure \( \text{Supp}(\chi(\tau_T, \cdot)) \subset \{ Z \leq \tilde{Z}_a \} \) and hence from (8.1):
\[
\left( \int (p - 1) Q \rho_m^2 \chi + \| \nabla \Phi_m \|^2 \chi \right)(\tau_T) \lesssim e^{-\delta g \tau_T}.
\] (8.28)

This estimate implies that we can integrate energy identity (8.19) only on the interval \([\tau_T, \tau_f]\). We now estimate all terms in (8.19).

**Boundary terms.** We compute the quadratic terms involving \( \Lambda \chi \) which should be thought of as boundary terms. First
\[
\partial_\tau \chi(\tau, Z) = -\frac{\partial_\tau \nu}{\nu} Z \partial_\tau \zeta \left( \frac{Z}{\nu} \right) = -\omega_0 \Lambda \chi.
\]

We now assume, recalling (8.16), that \( \omega_0 \) has been chosen small enough so that (8.21) holds, and hence the lower bound on the full boundary quadratic form using \( \Lambda \chi \leq 0 \):
\[
\frac{1}{2} \int (p - 1) Q \rho_m^2 [\partial_\tau \chi + H_2 \Lambda \chi] + \frac{1}{2} \int \| \nabla \Phi_m \|^2 \left[ \partial_\tau \chi + H_2 \Lambda \chi \right] + \int (p - 1) Q \rho_m \nabla \chi \cdot \nabla \Phi_m
= \int \left\{ \frac{1}{2} (p - 1) Q \rho_m^2 [-\omega_0 + H_2] + \frac{1}{2} \| \nabla \Phi_m \|^2 [-\omega_0 + H_2] + (p - 1) \frac{Q}{Z} \partial_\tau \Phi_m \rho_m \right\} \Lambda \chi.
\]

From (3.11), the discriminant of the above quadratic form is given by
\[
\left[ (p - 1) \frac{Q}{Z} \right]^2 - (-w_0 + H_2)^2 (p - 1) Q = (p - 1) Q \left[ (p - 1) \frac{Q}{Z^2} - (-w_0 + H_2)^2 \right]
= (p - 1) \mu^2 Q \left[ \sigma^2 - \left( -\frac{w_0}{\mu} + 1 - w \right)^2 \right] = (p - 1) \mu^2 Q \left[ -D(Z) + O(\omega_0) \right].
\]

We then observe by definition of \( \chi \) that for \( \tau \geq \tau_T \):
\[
Z \in \text{Supp} \Lambda \chi \Leftrightarrow \tilde{Z}_a \leq \frac{Z}{\nu(\tau)} \leq \tilde{Z}_a \Rightarrow Z \geq \nu(\tau) \tilde{Z}_a \geq \nu(\tau) \tilde{Z}_a = \tilde{Z}_a
\]
from which since \( \tilde{Z}_a > Z_2 \):
\[
Z \in \text{Supp} \Lambda \chi \Rightarrow -D(Z) + O(\omega_0) < 0
\]
provided \( 0 < \omega_0 \ll 1 \) has been chosen small enough.
Together with (8.21) and $\Lambda_\chi < 0$, this ensures: $\forall \tau \in [\tau_1, \tau^*],\;
0 < \frac{1}{2} \int (p - 1) Q \rho^2_m [\partial_\tau \chi + H_2 \Lambda \chi] + 1 \int |\nabla \Phi_m|^2 [\partial_\tau \chi + H_2 \Lambda \chi] + \int (p - 1) Q \rho_m \nabla \chi \cdot \nabla \Phi_m
(8.29)
\nonumber

Nonlinear terms. From (8.26), (8.25) for $\tau \leq \tau_f$:

$$\mbox{Supp} \chi \subset \{ Z \leq \nu(\tau) \tilde{Z}_a \} \subset \{ Z \leq \nu(\tau_f) \tilde{Z}_a \} = \left\{ Z \leq \frac{Z_0 \tilde{Z}_a}{2 \tilde{Z}_a} \right\} \subset \{ Z \leq Z_0 \},$$
and hence the bootstrap bounds (4.38) imply

$$\int \chi |\nabla \partial^m G \phi|^2 + \int (p-1) Q \chi |\partial^m G \rho|^2 \lesssim ||\nabla G \phi||^2_{H^{2k_0}(Z \leq Z_0)} + ||G \rho||^2_{H^{2k_0}(Z \leq Z_0)} \lesssim e^{-\frac{4g_2}{3} \tau}.$$

Conclusion. Injecting the collection of above bounds into (8.19) and summing over $m \in [0, 2k_0]$ yields the crude bound: $\forall \tau \in [\tau_1, \tau_f],$

$$\frac{d}{d\tau} \left\{ \sum_{m=0}^{2k_0} \int (p - 1) Q \rho^2_m \chi + |\nabla \Phi_m|^2 \chi \right\} \leq C \sum_{m=0}^{2k_0} \int (p - 1) Q \rho^2_m \chi + |\nabla \Phi_m|^2 \chi + e^{-\frac{4g_2}{3} \tau}.$$

We integrate the above on $[\tau_t, \tau_f]$ and conclude using

$$\chi(\tau_f, Z) = \zeta \left( \frac{Z}{\nu(\tau_f)} \right) = \zeta \left( \frac{Z}{Z_0} \right) = 1 \text{ for } Z \leq Z_0$$
and the initialization (8.28):

$$\left[ ||\rho||^2_{H^{2k_0}(Z \leq Z_0)} + ||\nabla \Psi||^2_{H^{2k_0}(Z \leq Z_0)} \right] (\tau_f) \lesssim e^{C(\tau_f - \tau_1)} e^{-\delta_\tau \tau_1} + \int_{\tau_1}^{\tau_f} e^{C(\tau_f - \sigma)} e^{-\frac{4g_2}{3} \sigma} d\sigma \lesssim C(\Gamma) e^{-\delta_\tau \tau_f} = C(\tau_f) e^{-\delta_\tau \tau_f}.$$

Since the time $\tau_f$ is arbitrary in $[\tau_0 + \Gamma, \tau^*]$, the bound (8.24) follows.

**step 4** Proof of (8.15). We run the energy identity (8.19) with $\chi_{m, m}$ and estimate each term.

terms $Z_0 \leq Z \leq \frac{Z_0}{2}$. In this zone, we have by construction

$$\rho = \bar{\rho}$$
and hence the bootstrap bounds (4.38) imply

$$||\rho||_{H^{k_0}(Z \leq \frac{Z_0}{2})} + ||\nabla \Psi||_{H^{k_0}(Z \leq \frac{Z_0}{2})} \lesssim 1$$
and hence interpolating with (8.24) for $k_0$ large enough:

$$||\rho||_{H^m(\frac{Z_0}{2} \leq Z \leq \frac{Z_0}{4})} \lesssim ||\rho||_{H^{k_0}(Z \leq \frac{Z_0}{2})} \lesssim e^{-\frac{4g_2}{10}}$$
$$\lesssim e^{-\frac{4g_2}{10} \left( 1 - \frac{m}{k_0} \right)} \leq e^{-\frac{4g_2}{10} \left( 1 - \frac{m}{k_0} \right)} \leq e^{-\frac{4g_2}{10} \left( 1 - \frac{m}{k_0} \right)}.$$  
(8.30)

and similarly for the phase

$$||\nabla \Psi||_{H^m(\frac{Z_0}{2} \leq Z \leq \frac{Z_0}{4})} \lesssim e^{-\frac{4g_2}{10} \left( 1 - \frac{m}{k_0} \right)} \leq e^{-\frac{4g_2}{10} \left( 1 - \frac{m}{k_0} \right)}.$$  
(8.31)
**Linear term.** We observe the cancellation using (8.17), (4.2):

\[
\partial_t \chi_{\nu_0,m} + H_2 \Lambda \chi_{\nu_0,m} = \frac{1}{(Z)^{d-2(r-1)+2(\nu_0-m)}} \left[ \mu \Lambda \zeta \left( \frac{Z}{Z^*} \right) \right] + \mu (1 - \omega) \frac{1}{(Z)^{d-2(r-1)+2(\nu_0-m)}} \Lambda \left( \frac{Z}{Z^*} \right) + \Lambda \left( \frac{1}{(Z)^{d-2(r-1)+2(\nu_0-m)+\omega}} \right) \zeta \left( \frac{Z}{Z^*} \right)
\]

for some universal constant \( \omega > 0 \). We now estimate the norm for \( 2Z^* \leq Z \leq 3Z^* \).

Using spherical symmetry for \( Z \geq 1 \) and \( m \geq 1 \):

\[
|Z^m \partial^m \rho| \lesssim \sum_{j=1}^m Z^m |\partial_j \rho| \lesssim \sum_{j=1}^m Z^j |\partial_j \rho|
\]

and hence using the outer \( L^\infty \) bound (4.40):

\[
\int_{2Z^* \leq Z \leq 3Z^*} \frac{(p-1)Q|\partial^m \rho|^2 + |\partial^m \nabla \Phi|^2}{(Z)^{d-2(r-1)+2(\nu_0-m)+\omega}} \leq \int_{2Z^* \leq Z \leq 3Z^*} \left[ \sum_{j=0}^m \frac{Z^j \partial_j \rho}{(Z)^{d+\nu_0+\nu}} \right] + \sum_{j=1}^{m+1} \frac{Z^j \partial_j \Phi}{(Z)^{d+\nu_0+(r-1)+1+\frac{\nu}{2}}} ^2 + \sum_{j=1}^{m+1} \frac{Z^j \partial_j \Psi}{(Z)^{d+\nu_0+(2r-1)+\frac{\nu}{2}}} ^2 \leq e^{-\delta_g \tau}
\]

using

\[
b(Z^*)^{-2} = e^{\tau(-\nu+\mu(2r-2))} = e^{\tau(-\nu+1+2\nu)} = 1
\]

and the explicit choice from (4.17):

\[
2\mu \left( \nu_0 + \frac{2(r-1)}{p-1} \right) = \delta_g
\]

**Conclusion** Injecting the above bounds into (8.19) yields:

\[
\int \frac{d}{dt} \left\{ \int (p-1)Q \rho^2 \chi_{\nu_0,m} + |\nabla \Phi_m|^2 \chi_{\nu_0,m} \right\} \leq -\int \chi_{\nu_0,m} \left( (p-1)Q \rho^2 + |\nabla \Phi_m|^2 \right) \left[ \mu \nu_0 + \frac{2\mu(r-1)}{p-1} \right] + O \left( \int Z_0 \leq Z \leq 2Z^* \chi_{\nu_0,m} \left[ \sum_{m=0}^{m+1} \frac{|\partial_j \Phi|^2}{(Z)^{2(m+1-j)+2\omega}} + \sum_{j=0}^m \frac{Q |\partial_j \rho|^2}{(Z)^{2(m-j)+2\omega}} \right] + e^{-\delta_g \tau} \right)
\]

and

\[
O \left( \int \chi_{\nu_0,m} \nabla \Phi_m ||\nabla^m G_\Phi| + \int \chi_{\nu_0,m} Q |\rho_m||\partial^m G_\rho| \right)
\]
and hence after summing over $m$:

\[
\frac{1}{2} \frac{d}{dt} \left\{ \sum_{m=0}^{2k_0} \int (p-1) Q \rho_m^2 \chi_{\nu_0,m} + |\nabla \Phi_m|^2 \chi_{\nu_0,m} \right\}
\]

\[
= -\mu \left[ \nu_0 + \frac{2(r-1)}{p-1} \right] \sum_{m=0}^{2k_0} \int \chi_{\nu_0,m} \left[ (p-1) Q \rho_m^2 + |\nabla \Phi_m|^2 \right]
\]

\[
+ O\left( e^{-4 \delta_g t} + \sum_{m=0}^{2k_0} \int (p-1) Q \rho_m^2 \chi_{\nu_0+\omega,m} + |\nabla \Phi_m|^2 \chi_{\nu_0+\omega,m} \right)
\]

\[
+ \sum_{m=0}^{2k_0} O\left( \int \chi_{\nu_0,m} |\nabla \Phi_m| \nabla \partial^m G \rho + \int |\chi_{\nu_0,m} Q |\rho_m||\partial^m G \rho| \right)
\]

Using (8.24) we conclude

\[
\frac{1}{2} \frac{d}{dt} \left\{ \sum_{m=0}^{2k_0} \int (p-1) Q \rho_m^2 \chi_{\nu_0,m} + |\nabla \Phi_m|^2 \chi_{\nu_0,m} \right\}
\]

\[
= -\mu \left[ \nu_0 + \frac{2(r-1)}{p-1} \right] \sum_{m=0}^{2k_0} \int \chi_{\nu_0,m} \left[ (p-1) Q \rho_m^2 + |\nabla \Phi_m|^2 \right]
\]

\[
+ O\left( e^{-4 \delta_g t} + \sum_{m=0}^{2k_0} \int \chi_{\nu_0,m} |\nabla \partial^m G \Phi|^2 + \int (p-1) Q \chi_{\nu_0,m} |\partial^m G \rho|^2 \right)
\]

Therefore, using also (8.20), for $Z_0$ large enough and universal and

\[
2\mu \left( \nu_0 + \frac{2(r-1)}{p-1} \right) = \delta_g,
\]

there holds

\[
\frac{d}{dt} \left\{ \sum_{m=0}^{2k_0} \int (p-1) Q \rho_m^2 \chi_{\nu_0,m} + |\nabla \Phi_m|^2 \chi_{\nu_0,m} \right\}
\]

\[
\leq -\frac{4 \delta_g}{5} \sum_{m=0}^{2k_0} \int \chi_{\nu_0,m} \left[ (p-1) Q \rho_m^2 + |\nabla \Phi_m|^2 \right] + Ce^{-4 \delta_g t}.
\]

Integrating in time and using (4.19) yields (8.15). \qed

8.3. Closing the bootstrap and proof of Theorem 1.1. We are now in position to prove the bootstrap Proposition 4.4 which immediately implies Theorem 1.1.

Proof of Proposition 4.4 and Theorem 1.1. Recall that the non vanishing of the solution is ensured by (4.27). It remains to close the bound (4.26). Indeed, from (4.1), (4.2), (4.8) for $Z \geq Z^*$:

\[
\frac{|\Delta u|}{\rho_D} \lesssim (Z^*)^2 \left[ |\Delta \rho_T| + \frac{|\partial_Z \rho_T| |\partial_Z \Psi_T|}{b} + \frac{\rho_T |\Delta \Psi_T|}{b} \right] \lesssim 1
\]

where we used (4.40) in the last step. The $|u|^p$ term is handled similarly, and (4.26) is improved for $b_0$ small enough\(^{10}\). Note also that the bounds (4.40) imply

\[
\|u(t)\|_{H^k} \leq C(t)
\]

\(^{10}\) The smallness of $b_0$ is responsible for the size of the time length between initial data and formation of a singularity.
for $\frac{d}{2} \ll k_c \ll k_m$ for times in the bootstrap interval and hence the bootstrap time is strictly smaller than the life time provided by standard Cauchy theory. We now conclude from a classical topological argument à la Brouwer. The bounds of sections 5,6,7,8 have been shown to hold for all initial data on the time interval $[\tau_0, \tau_0 + \Gamma]$ with $\Gamma$ large. Moreover, as explained in the proof of Lemma 8.1, they can be immediately propagated to any time $\tau^*$ after a choice of projection of initial data on the subspace of unstable modes $P_X(\tau_0)$. This choice is dictated by Lemma 3.11. A continuity argument implies $\tau^* = \infty$ for this data, and the conclusions of Theorem 1.1 follow. \[\Box\]

Appendix A. Comparison with compressible Euler dynamics

We consider the compressible Euler equations with a polytropic equation of state:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0, \\
\frac{\rho}{\partial t} u + \rho u \cdot \nabla u + \nabla P &= 0, \quad x \in \mathbb{R}^d. \\
P &= \frac{\gamma - 1}{\gamma} \rho^\gamma
\end{align*}
\] (A.1)

for $\gamma > 1$.

**step 1** Scaling and renormalization. The scaling symmetry\[^{11}\] is

\[
\lambda^{\frac{2}{\gamma+1}} \rho(\lambda^{\frac{2}{\gamma+1}} t, \lambda x), \quad \lambda^{\frac{\gamma-1}{\gamma+1}} u(\lambda^{\frac{2}{\gamma+1}} t, \lambda x).
\]

We renormalize self-similarly

\[
\frac{d\tau}{dt} = \frac{1}{\lambda^{\frac{2}{\gamma+1}}}, \quad \frac{\lambda}{\lambda} = \frac{1}{2}
\]

and obtain:

\[
\begin{align*}
\frac{\partial \rho}{\partial \tau} + \frac{1}{2} \left( \frac{2}{\gamma+1} \rho + y \cdot \nabla \rho \right) + \nabla \cdot (\rho u) &= 0, \\
\rho \frac{\partial u}{\partial \tau} + \frac{1}{2} \rho \left( \frac{\gamma-1}{\gamma+1} u + y \cdot \nabla u \right) + \rho u \cdot \nabla u + \nabla P &= 0. \\
\end{align*}
\] (A.2)

As above, we proceed with a front renormalization

\[
\rho \mapsto \frac{1}{b^{\frac{\gamma-1}{\gamma+1}}} \rho(y \sqrt{b}), \quad u \mapsto \frac{1}{b^{\frac{1}{2}}} u(y \sqrt{b})
\]

with

\[
\frac{b_{\tau}}{b} = -e
\]

and consider a potential spherically symmetric flow with $u = \nabla \Psi = \Psi'$. A direct computation in which we also integrate the second equation leads to

\[
\begin{align*}
-\frac{\partial \rho}{\partial \tau} + \Delta \Psi + \left( \frac{\gamma-1}{\gamma+1} \right) + \frac{\nabla \rho}{\rho} \cdot \left( \frac{1}{2} \right) Z + \nabla \Psi &= 0, \\
-\frac{\partial \Psi}{\partial \tau} = \frac{1}{2} |\nabla \Psi|^2 + \left(e - \frac{1}{\gamma+1}\right) \Psi - 1 + \left(\frac{1}{2}\right) Z \cdot \nabla \Psi + \rho^{\gamma-1}
\end{align*}
\]

A stationary solution of the above equation satisfies

\[
\begin{align*}
\frac{\Delta \Psi + \left( \frac{\gamma-1}{\gamma+1} \right) + \frac{\nabla \rho}{\rho} \cdot \left( \frac{1}{2} \right) Z + \nabla \Psi}{\frac{1}{2} |\nabla \Psi|^2 + \left(e - \frac{1}{\gamma+1}\right) \Psi + \left(\frac{1}{2}\right) Z \cdot \nabla \Psi + \rho^{\gamma-1}} &= 0
\end{align*}
\] (A.3)

\[^{11}\text{We choose a 1-parameter of scaling transformation, which is compatible with the Navier-Stokes equations, out of a larger 2-parameter family of possible transformations.}\]
**step 2** Emden transform. We introduce the variables

\[ V = \Psi', \quad S = \sqrt{2\rho^{\frac{2-\gamma}{\gamma}}}, \]

where \( S \) is the space dependent sound speed, so that equivalently taking the derivative of the second equation:

\[
\begin{align*}
V' + \frac{d-1}{Z} V + \left( e - \frac{1}{\gamma+1} \right) + 2 \frac{s'}{s} \left[ \left( \frac{1-e}{\gamma} \right) Z + V \right] &= 0 \\
VV' + \left( e - \frac{1}{\gamma+1} \right) V + \left( \frac{1-e}{\gamma} \right) (ZV' + V) + SS' &= 0.
\end{align*}
\]

Let \( x = \log Z, \quad V(Z) = v(x), \quad S(Z) = s(x), \quad Z \frac{d}{dZ} = \frac{d}{dX}. \)

First equation.

\[
\frac{v'}{Z} + \frac{(d-1)v}{Z} + \left( e - \frac{1}{\gamma+1} \right) + 2 \frac{s'}{s} \left[ \left( \frac{1-e}{\gamma} \right) Z + V \right] = 0
\]

and hence letting \( v(x) = e^x w, \quad s(x) = e^x \sigma \)

yields

\[
(w' + w) + (d-1)w + \left( e - \frac{1}{\gamma+1} \right) + 2 \frac{s'}{s} \left[ \left( \frac{1-e}{\gamma} \right) w + \frac{2\gamma}{\gamma^2-1} \right] = 0
\]

i.e.,

\[
\sigma w' + \frac{2}{\gamma-1} \left( \frac{1-e}{2} + w \right) \sigma' + \sigma \left[ \left( d + \frac{2}{\gamma-1} \right) w + \frac{2\gamma}{\gamma^2-1} \right] = 0
\]

Second equation. We get

\[
\frac{vv'}{Z} + \left( e - \frac{1}{\gamma+1} \right) v + \left( \frac{1-e}{2} \right) (v' + v) + \frac{ss'}{Z} = 0
\]

and hence

\[
w(w' + w) + \left( e - \frac{1}{\gamma+1} \right) w + \left( \frac{1-e}{2} \right) (w' + 2w) + \sigma (\sigma' + \sigma) = 0
\]

or equivalently

\[
\left( w + \frac{1-e}{2} \right) w' + \sigma \sigma' + \left( w^2 + \frac{\gamma}{\gamma+1} w + \sigma^2 \right) = 0
\]

We have obtained:

**Lemma A.1** (Emden transform). Let

\[
x = \log Z, \quad \Psi'(Z) = e^x w(x), \quad S(Z) = e^x \sigma(x), \quad S = \sqrt{2\rho^{\frac{2-\gamma}{\gamma}}},
\]

then

\[
\begin{align*}
(w + \frac{1-e}{2}) w' + \sigma \sigma' + \left( w^2 + \frac{\gamma}{\gamma+1} w + \sigma^2 \right) &= 0 \\
\sigma w' + \frac{2}{\gamma-1} \left( \frac{1-e}{2} + w \right) \sigma' + \sigma \left[ \left( d + \frac{2}{\gamma-1} \right) w + \frac{2\gamma}{\gamma^2-1} \right] &= 0
\end{align*}
\] (A.4)

**step 3** Renormalized form. We define

\[
\ell = \frac{2}{\gamma-1}, \quad r = \frac{2\gamma}{(1-e)(\gamma+1)}, \quad \phi^2 \left( \frac{2}{e-1} \right)^2 = \ell
\]

and the renormalized unknowns

\[
U = \frac{2}{e-1} w, \quad \Sigma = \frac{\sigma}{\phi}. \quad (A.5)
\]
The second equation becomes:

$$\sum \frac{e-1}{2} U' + \ell \frac{e-1}{2} (U - 1) \Sigma' + \sum \left(1 + \frac{d}{\ell} \right) \frac{e-1}{2} U + \frac{2\gamma}{(\gamma^2 - 1)\ell} = 0$$

i.e.,

$$\sigma \frac{\ell}{U} U' + (U - 1) \sigma' + \sigma \left(1 + \frac{d}{\ell} \right) U - \frac{2\gamma}{(1 - \epsilon)(\gamma + 1)} = 0.$$ 

The first equation becomes

$$\left( \frac{e-1}{2} \right)^2 U U' + \phi^2 \Sigma \Sigma' + \left( \frac{e-1}{2} \right)^2 U^2 + \frac{\gamma}{\gamma + 1} \frac{e-1}{2} U + \phi^2 \Sigma^2 = 0$$

and hence (A.5) yields:

$$(U - 1) U' + \ell \Sigma \Sigma' + \left[U^2 - \frac{2\gamma}{(1 - \epsilon)(\gamma + 1)} U + \ell \Sigma^2\right] = 0.$$ 

We arrive at the renormalized system

$$\left|\sum \frac{e-1}{2} U U' + (U - 1) \Sigma' + (U^2 - rU + \ell \Sigma^2) = 0 \right.$$ 

which is identical to the system (2.10) for the defocusing NLS but with the parameters

$$\ell = \frac{2}{\gamma - 1}, \quad r = \frac{2\gamma}{(1 - \epsilon)(\gamma + 1)}$$

in place of

$$\ell = \frac{4}{p - 1}, \quad r = \frac{2}{(1 - \epsilon)}$$

**Appendix B. Hardy inequality**

**Lemma B.1.** Assume $2\gamma \notin \mathbb{Z}$. Then, for all $u \in C^\infty_{rad}(r \geq 1)$ and $j \geq 1$:

$$\int_{r \geq 1} r^{2\gamma} u^2 dr \lesssim_{j, \gamma} \|u\|_{H^j(1 \leq r \leq 2)}^2 + \int_{r \geq 1} r^{2(\gamma + j)} |\partial_r^j u|^2 dr \quad (B.1)$$

**Proof.** Assume $2\gamma \neq -1$, We integrate by parts

$$\int_{r \geq 1} r^{2\gamma} u^2 dr = \frac{1}{2\gamma + 1} \left[ r^{2\gamma + 1} u^2 \right]_{1}^\infty - \frac{2}{2\gamma + 1} \int_{r \geq 1} r^{2\gamma + 1} u \partial_r u dr$$

$$\leq C\|u\|_{H^j(1 \leq r \leq 2)}^2 + C\left( \int_{r \geq 1} r^{2\gamma} u^2 dr \right)^\frac{1}{2} \left( \int_{r \geq 1} r^{2\gamma + 2} (\partial_r u)^2 dr \right)^\frac{1}{2}$$

where we used the one dimensional Sobolev embedding, and (B.1) for $j = 1$ follows by Hölder. For higher values of $j$, (B.1) now follows by induction. \qed

**Appendix C. Commutator for $\Delta^k$**

**Lemma C.1** (Commutator for $\Delta^k$). Let $k \geq 1$, then for any two smooth function $V, \Phi$, there holds:

$$[\Delta^k, V] \Phi - 2k \nabla V \cdot \nabla \Delta^{k-1} \Phi = \sum_{|\alpha| + |\beta| = 2k, |\beta| \leq 2k-2} c_{k, \alpha, \beta} \partial^\alpha V \partial^\beta \Phi. \quad (C.1)$$

where $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$. 


Proof. We argue by induction on $k$. For $k = 1$:

$$\Delta(\nabla \Phi) - \nabla \Delta \Phi = 2\nabla \nabla \cdot \nabla \Phi.$$ 

We assume (C.1) for $k$ and prove $k + 1$. Indeed,

$$\Delta^{k+1}(\nabla \Phi) = \Delta(\Delta^k \nabla \Phi + V \Delta^k \Phi)$$

$$= \Delta \left(2k \nabla \nabla \cdot \nabla \Delta^{k-1} \Phi + \sum_{|\alpha| + |\beta| = 2k, |\beta| \leq 2k-2} c_{k, \alpha, \beta} \partial^{\alpha} \nabla \partial^{\beta} \Phi + 2 \nabla \nabla \cdot \nabla \Delta^k \Phi \right)$$

$$+ \sum_{|\alpha| + |\beta| = 2k+2, |\alpha| \geq 2} c_{k+1, \alpha, \beta} \partial^{\alpha} \nabla \partial^{\beta} \Phi$$

$$= \nabla \Delta^{k+1} \Phi + 2(k + 1) \nabla \nabla \cdot \nabla \Delta^k \Phi + 2 \nabla \nabla \cdot \nabla \Delta^k \Phi$$

and (C.1) is proved. \hfill \square

Appendix D. Behaviour of Sobolev norms

We compute Sobolev norms assuming that the leading part of the solution is given by (1.8). Computations below are formal but could be justified as a consequence of the bootstrap estimates.

Dirichlet energy of the profile. We recall (1.7), (1.8) and compute:

$$||\nabla u||_{L^2}^2 \sim 1 + ||\nabla u||_{L^2(|x| \leq 1)}^2 = 1 + \int_{|x| \leq 1} |\nabla \rho|^2 dx + \int_{|x| \leq 1} \rho^2 |\nabla \phi|^2.$$ 

We compute for the first term:

$$\int_{|x| \leq 1} |\nabla \rho|^2 dx \sim \frac{1}{(T - t)^{\frac{4(r-1)}{p-1}}} \int_{|Z| \leq \frac{1}{(T - t)^{\frac{1}{p}}} |Z|^{\frac{4(r-1)}{p-1} + 2}} \frac{(T - t)^{\frac{d}{2}} |Z|^{d-1} dZ}{(T - t)^{\frac{d}{2}} \langle Z \rangle^{\frac{4(r-1)}{p-1} + 2}}$$

$$= \frac{1}{(T - t)^{\frac{d}{2}(1 - \sigma)}} \int_{|Z| \leq \frac{1}{(T - t)^{\frac{1}{p}}} \langle Z \rangle^{1 + 2(1 - \sigma)}} \frac{dZ}{\langle Z \rangle^1}$$

with

$$\sigma = s_c - \frac{2(r-2)}{p-1} > 1 \Leftrightarrow d - \frac{\ell}{2} - \frac{\ell}{2}(r-2) > 1 \Leftrightarrow d - \ell(r-1) > 2$$

$$\Leftrightarrow d - 2 > \ell \left(\frac{\ell + d}{\ell + \sqrt{d}} - 1\right) = \frac{\ell(d - \sqrt{d})}{\ell + \sqrt{d}} \Leftrightarrow (d - 2)\sqrt{d} + \ell(\sqrt{d} - 2) > 0$$

which holds and hence

$$\int_{|x| \leq 1} |\nabla \rho|^2 dx \lesssim 1.$$
Similarly:

\[
\int_{|z| \leq 1} \rho^2 |\nabla \phi|^2 = \frac{1}{(T-t)^{2(r-1)} + \frac{2(r-2)}{p-1}} \int_{|Z| \leq \frac{1}{(T-t)^{\frac{d}{r} Z}} (T-t)^{\frac{d}{r} Z} dZ = \frac{1}{(T-t)^{\frac{d}{r} Z}} \left( r + \frac{2(r-2)}{p-1} + \frac{2}{p-1 - \frac{d}{2}} \right) \int_{|Z| \leq \frac{1}{(T-t)^{\frac{d}{r} Z}}} dZ
\]

and at \( r^*(\ell) \):

\[
r - 2 + \frac{2(r-2)}{p-1} + 1 + \frac{2}{p-1 - \frac{d}{2}} \leq 0 \iff (r-1) \left( 1 + \frac{\ell}{2} \right) < \frac{d}{2}
\]

\[
\Leftrightarrow (2 + \ell)(d - \sqrt{d}) < d(\ell + \sqrt{d}) \Leftrightarrow d(\sqrt{d} - 2) + (\ell + 2)\sqrt{d} > 0
\]

which holds and hence

\[
\int_{|z| \leq 1} \rho^2 |\nabla \phi|^2 \lesssim 1.
\]

Blow up of large enough Sobolev norms below the scaling. We now unfold the change of variables

\[
\begin{align*}
|u(t, x)| &= \frac{1}{\lambda(t)^{\frac{d}{2}}} v(s, y) e^{i\gamma}, \quad y = \frac{x}{\lambda} \\
v(s, y) &= \frac{1}{(\sqrt{b})^{\frac{d}{2}}} \left( \rho_T e^{i\Psi_T} \right) (\tau, Z), \quad Z = y\sqrt{b}
\end{align*}
\]

which yields

\[
\|\nabla^s u\|_{L^2} = \frac{1}{\lambda^{5-sc}} \|\nabla^s v\|_{L^2} = \frac{1}{\lambda^{5-sc} (\sqrt{b})^{\frac{d}{2}}} (\sqrt{b})^{s-\frac{d}{2}} \|\nabla^s (\rho_T e^{i\Psi_T})\|_{L^2} \geq e^{\frac{s-\frac{d}{2} - \frac{d}{2}}{p-1}} \|\nabla^s (\rho_T e^{i\Psi_T})\|_{L^2(\|Z| \leq 1)} \geq e^{\frac{s-\frac{d}{2} - \frac{d}{2}}{p-1}} \frac{1}{b^\delta}
\]

which blows up as soon as

\[
s > \sigma = \frac{1}{1 + e} \left[ s_c - e \left( \frac{d}{2} + \frac{2}{p-1} \right) \right].
\]

We can check that at \( r^*(\ell) \):

\[
\sigma > 1 \Leftrightarrow \frac{d}{2} - \frac{2}{p-1} > 1 + e + e \left( \frac{d}{2} + \frac{2}{p-1} \right) \Leftrightarrow \frac{d}{2} (1 - e) > (1 + e) \left( 1 + \frac{\ell}{2} \right)
\]

\[
\Leftrightarrow \frac{d}{2} - \frac{2}{r} > \frac{1}{r} (r-1)(\ell + 2) \Leftrightarrow d > (\ell + 2) \left( \frac{d + \ell}{\ell + \sqrt{d}} - 1 \right) \Leftrightarrow d(\sqrt{d} - 2) + (\ell + 2)\sqrt{d} > 0
\]

The last inequality holds for our assumptions on \( d \geq 5 \) and \( \ell > 0 \).

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