Efficient Identification of Additive Link Metrics: Theorem Proof and Evaluations

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I. INTRODUCTION

As described in [1], algorithm STPC (Spanning Tree-based Path Construction) builds three independent spanning trees (\(T_1, T_2\) and \(T_3\)) wrt node \(r\) in the 3-vertex-connected graph \(G^*_e\) (\(G^*_e\) is obtained by adding virtual nodes and virtual links to the original graph \(G\)) for constructing a sufficient number of simple paths such that all links in \(G\) can be identified. Based on these three independent spanning trees, STPC constructs 3 internally vertex disjoint paths wrt each node in \(G\), thus resulting in \(3|V|\) paths totally, where \(V\) is the set of nodes in \(G\). However, there exists redundant information in these \(3|V|\) paths since some of them are exactly the same. The main objective of this report is to prove that the number of distinct paths constructed by STPC in fact equals \(|L|\), where \(L\) is the set of links in \(G\). By linear algebra, we know that the minimum number of the required path measurements for full link identifications is \(|L|\) and all these links must be linearly independent. Therefore, STPC is characterized by high efficiency in that all constructed distinct paths are linearly independent, i.e., all link metrics in \(G\) can be identified without conducting any unnecessary path measurements. In the end, we show some additional simulation results to verify the superior efficiency of STPC and STLTI. The terms and notations used in this report are defined in Table I

II. INDEPENDENT SPANNING TREES

With respect to \(G^*_e\), let \(P_{STPC}\) denote the path set obtained by running STPC algorithm. As discussed before, the goal is to prove that the number of distinct paths in \(P_{STPC}\) is the number of links in \(G\), i.e., \(|P_{STPC}| = |L|\). For this purpose, we first briefly discuss how to construct the 3 independent spanning trees, some unique features of which will be used in later proofs.

A. Three Independent Spanning Trees

In this section, we consider how to construct three independent spanning trees wrt node \(r\) in \(G^*_e\).

Definition 1. (3) Nonseparating ear decomposition: Nonseparating ear decomposition of \(G\) is a decomposition \(V(G) = V(E_1 \cup E_2 \cup \cdots \cup E_{n_e})\) (each \(E_i\) is called an ear, \(n_e\) ears in total) such that

1) \(E_i\) is an induced cycle,
2) \(E_i\) (\(2 \leq i \leq n_e\)) is an induced simple path with only its end-points in common with \(E_1 \cup \cdots \cup E_{i-1}\),
3) Let \(\overline{G} = G \setminus (E_1 \cup E_2 \cup \cdots \cup E_{i-1})\). For each \(E_i\), \(\overline{G}\) is connected and each internal node of \(E_i\) has a neighbor in \(\overline{G}\), and
4) \(|E_{n_e}| = 3\) and the internal node of \(E_{n_e}\) does not appear in other ears.

Definition 2. (4) s-t numbering: s-t numbering of \(G\) is a one-one function \(f : V \rightarrow [1 \cdots n] (n = |V|)\) such that

1) \(f(s) = 1\), \(f(t) = n\), and
2) For each vertex \(v\) in \(G \setminus \{s, t\}\), it has a neighbor \(u\) with \(f(u) < f(v)\) and a neighbor \(w\) with \(f(w) > f(v)\).

Suppose \(E_j\) is the first ear with node \(v\) on it, then \(j\) is called the ear level of \(v\), denoted by \(g(v) = j\). It is proved in [3] that a 3-vertex-connected graph \(G^*_e\) has a nonseparation ear decomposition, by which each node in \(G^*_e\) can be assigned an ear level and an s-t number. With the knowledge of the ear level and the s-t number of each node, three independent spanning trees can be constructed accordingly. Without loss of generality, in the sequel, we assume \(\mu_1\) is a non-cutvertex monitor in \(G\) and virtual node \(r\) connects to \(\mu_2\), i.e., in the definition [1] of \(G^*_e\), \(\mu_1\) equals \(\mu_2\). Due to the special structure of \(G^*_e\), the complicated algorithm [3] of finding \(E_1\) can be avoided, i.e., simply choose \(r\mu_1\mu_2\) as \(E_1\). Then

1. A non-cutvertex monitor can be found since it is impossible that all monitors are cutvertices in an identifiable network according to the minimum monitor placement algorithm MMP [5].
the rest ears can be found by the ear decomposition algorithm described in the proof of Theorem 1. To construct 3 independent spanning trees, one rule for selecting the last ear  can be obtained by selecting the last ear  from  for each  and  are sequentially numbered from  to .

By running Algorithm 1 the final s-t numbers for  and  are  and  respectively. With the computed ear level and s-t numbers of a given 3-vertex-connected graph  it is easy to show that each node  in  has three neighbors  such that:

1) ear level 2) 3) 

The above three properties are also the three rules to construct the corresponding independent spanning trees:

- : First, add link to . Next, for each node  in  involves link where  is a neighbor node of .
- : First, add link to . Next, for each node  in  involves link where  is a neighbor node of .
- : First, add link to . Next, for each node  in  involves link where  is a neighbor node of .

Accordingly, three independent spanning trees  are constructed in .

III. NUMBER OF DISTINCT PATHS CONSTRUCTED BY STPC

With the constructed three independent spanning trees, let . Then each node in  (virtual nodes  and  are also included) has three internally vertex disjoint paths (denoted by  and ) to , each along the corresponding spanning tree. Combining any two of these paths, we get three cycles, i.e.,  and  and . Let  be the set of all these cycles. To count the number of distinct paths constructed by STPC, we first count the number of distinct cycles in . Next, removing all virtual links and the resulting isolated nodes in each cycle , of one, one monitor-to-monitor simple path  is formed, thus generating a path set . We then study the number of linearly independent paths in  and yield path set by removing the linearly dependent paths in .

Finally, we show that for each path  with more than 2 monitors while retaining the property of full tree link identification in . We can prove path set has paths  is the set of virtual links in , all of which are linearly independent and can cover all paths constructed by STPC for tree link identification in . Therefore, for tree link identification, the number of distinct paths constructed by STPC is exactly the number of tree links. In the end, we show that the auxiliary algorithm of STPC constructs one path for each non-tree link in , thus completing the proof of Theorem IV.2 in as a link in  is either a tree or non-tree link.

A. Number of Distinct Cycles

Let ,  and  denote three internally vertex disjoint paths from  ( are the number of edges in ). The number of distinct cycles in is obtained by counting the number of distinct cycles in . For illustrative purpose, we color the links on as blue, as red in the sequel. Since the independent paths are constructed wrt each link in , , and has a natural direction toward  when selecting (i.e., 3). Therefore, for the simplicity of the following proofs, each link in , , and is assigned a direction toward . Note this direction is for and only for the theorem proof and only exists on , , and , meaning the assumption that the original graph is undirected still holds. With this notation, links  and  on  are different colors and different directions might correspond to the same link in .

Lemma III.1. The number of distinct cycles in  is the number of links (both real and virtual) in , i.e., . Furthermore, all these distinct cycles are linearly independent.

Proof: In , let  be the number of links appearing only in  (colored as blue), and  be the number of links appearing in two trees: one is  (colored as blue), the other one is either  (colored as green) or  (colored as red). Then we have:



since is a spanning graph of . Consider ear . Let be the number of newly added nodes to by . Let  denote the nodes in , where  are end-points already existed in previous ears. Then we can derive the relationship among the number of links, the number of nodes and the number of ears in . Suppose there are  ears in total, then  have  new nodes has  links since .
is a cycle, whereas $\mathcal{E}_i$ ($2 \leq i \leq n$) which is a path with $\delta_i$ new nodes has $\delta_i + 1$ new links. In addition to these links, there exist $b_1$ links only appearing in the blue tree and not involving in any ears. Hence, we have

$$||G_m|| = b_1 + \delta_1 + \sum_{i=2}^{n}(\delta_i + 1)$$

$$= b_1 + \delta_1 + \sum_{i=2}^{n}\delta_i + \sum_{i=2}^{n}1$$

$$= b_1 + ||G_m'|| + n - 1.$$  \hspace{1cm} (2)

To count distinct cycles in $C$, we consider merging the cycles obtained from each pair of independent spanning trees.

(i) We first consider the cycles obtained by combining the red and green paths. For ear $\mathcal{E}_i$, each newly added node in $\mathcal{E}_i$ corresponds to a cycle obtained by combining the red and green paths. However, as Fig. 1 shows, each link between $v_1$ and $v_{\delta_i}$ corresponds to two colors, red and green, with opposite directions. Therefore, cycles formed by combining the red and green paths wrt $v_1 \cdots \delta_{i+1}$ are identical. Thus, ear $\mathcal{E}_i$ only contributes one distinct cycle when combining the red and green paths for any of the newly added nodes $v_1 \cdots \delta_i$.

(ii) Next, we consider merging the cycles generated by combining the red and the blue paths. For the $\delta_i$ new nodes in ear $\mathcal{E}_i$, each path formed by combining the red and the blue paths are distinct from each other. However, as Fig. 2 displays, the $red+blue$ cycle wrt $v_1$ might be identical with the $red+blue$ cycle wrt $v_0$ (when $v_0$ was first added in the previous operations). Thus, the same cycle might be counted twice. Let $\varepsilon_i$ indicate if link $v_0v_1$ is colored as blue, i.e., $\varepsilon_i = 1$ if true, and $\varepsilon_i = 0$ otherwise. With this notation, the number of distinct cycles formed by combining the red and the blue paths is $\delta_i - \varepsilon_i$ for ear $\mathcal{E}_i$ ($2 \leq i \leq n$), and $\delta_i - 1 - \varepsilon_i$ for ear $\mathcal{E}_1$.

(iii) Finally, we consider merging the cycles formed by combining the green and the blue paths. Following the same argument in (ii), each path formed by combining the green and the blue paths are distinct from the $\delta_i$ new nodes in ear $\mathcal{E}_i$. However, the cycles wrt to $v_k$ and $v_{\delta_i+1}$ might be identical (see Fig. 2). Let $\varepsilon_i'$ indicate if link $v_{\delta_i}v_{\delta_i+1}$ is colored blue, i.e., $\varepsilon_i' = 1$ if true, and $\varepsilon_i' = 0$ otherwise. Accordingly, the number of distinct cycles formed by combining the green and the blue paths is $\delta_i - \varepsilon_i'$ for ear $\mathcal{E}_i$ ($2 \leq i \leq n$), and $\delta_i - 1 - \varepsilon_i'$ for ear $\mathcal{E}_1$.

Therefore, the number of distinct cycles, denoted by $Q_i$, contributed by ear $\mathcal{E}_i$ is:

$$Q_1 = 1 + \delta_1 - 1 - \varepsilon_1 + \delta_1 - 1 - \varepsilon_1'$$

$$Q_i = 1 + \delta_i - \varepsilon_i + \delta_i - \varepsilon_i' \ (2 \leq i \leq n).$$  \hspace{1cm} (3)

Thus, the number of distinct cycles in $C$ is

$$|C| = Q_1 + \sum_{i=2}^{n}Q_i$$

$$= 2\delta_1 - \varepsilon_1 - \varepsilon_1' + 1 + 2\sum_{i=2}^{n}\delta_i + \sum_{i=2}^{n}1 - \sum_{i=2}^{n}(\varepsilon_i + \varepsilon_i')$$

$$= 2|G_m'| + n - 2 - \sum_{i=1}^{n}(\varepsilon_i + \varepsilon_i')$$

$$= 2|G_m'| + n - 2 - b_2.$$

Subtracting (3) from (4) and utilizing (1), we get

$$|C| = ||G_m'||.$$  \hspace{1cm} (5)

Let $G'_m := G_m$ except all virtual links/nodes in $G_m$ are real links/nodes in $G'_m$. Suppose cycle measurement is allowed and $r$ is the only monitor in $G'_m$; then all cycle measurements associated with $C$ is sufficient to identify all links in $G'_m$ (following the same method in STLI, see [1]). Moreover, we have $||G_m'|| = ||G'_m||$; therefore, all cycles in $C$ are linearly independent.

B. Number of Linearly Independent Paths After Removing All Virtual Links

In section [122], we get a cycle set $C$ with $||G_m'||$ linearly independent cycles. For any cycle in $C$, removing all involved virtual links and the resulting isolated nodes, we can obtain a monitor-to-monitor simple path since each cycle in $C$ is obtained by combining two internally vertex disjoint paths ($X^{(i)}$ and $X^{(j)}$) ($i,j \in \{1,2,3\}, i \neq j$) along two independent spanning trees and virtual links only appear at the end of $X^{(i)}$ and $X^{(j)}$. Therefore, following the same operation to each cycle in $C$ iteratively, we can generate a new set $Y$ with each element $\gamma_i$ representing a monitor-to-monitor simple paths. In this section, we investigate on the number of linearly independent paths in $Y$. Note that some paths in $Y$ might contain more than 2 monitors. We will show how to shorten these paths in the next section.
Let $V$ denote the set of virtual links in $G_m$. The goal is to prove the number of linearly independent paths in $Y$ is $|V_m| - |V|$, and these linearly independent paths are sufficient to identify all tree links in the original graph $G$.

For all redundant paths generated by mapping from $C$ to $Y$, they can be divided into 3 categories, i.e., 1) trivial topology, 2) linearly dependent paths, and 3) duplicate paths. Note in Fig. 3-Fig. 9 for all paths (or path segments) colored as blue/green/red, they represent virtual links if they are outside $G$; and real simple paths otherwise.

1) Trivial topology (a graph with no nodes or links). As shown in Fig. 7 combining the red and green paths wrt $\mu_1$ ($G_m$ involves $r\mu'_1\mu_1\mu_2^r\mu$), cycle $r\mu'_1\mu_1\mu_2^r\mu$ is formed. However, the corresponding path of $r\mu'_1\mu_1\mu_2^r\mu$ after removing the virtual links and the resulting isolated nodes is empty. Therefore, this trivial path has no contribution for identifying real links in $G$.

2) Linearly dependent paths. The following 3 cases can generate redundant linearly dependent paths.

(i) Consider the three cycles constructed wrt $\mu_2$ (shown in Fig. 8). The paths associated with blue + red cycle and blue + green cycle are $\mu_1 e_1 \mu_2$ and $\mu_2 e_2 \mu_j$, respectively. However, the path associated red + green cycle, $\mu_1 e_1 \mu_2 e_2 \mu_j$, is the sum of previously constructed paths $\mu_1 e_1 \mu_2$ and $\mu_2 e_2 \mu_j$. Therefore, redundant path $\mu_1 e_1 \mu_2 e_2 \mu_j$ is linearly dependent with the other two.

(ii) Now consider the case shown in Fig. 9. For $G_m$, there exists one and only one monitor $\mu$ with link $\mu_2 \mu_a$ colored as blue in the $\mu_2 \rightarrow \mu_a$ direction, according to the rules for blue tree construction. Now consider $\mu_a$. According to the s-t numbering rule and the processing of ear decomposition, we have $f(\mu_a') < f(\mu_a)$ and $g(\mu_a') < g(\mu_a)$; therefore, $\mu'_2 \mu_a$ is colored as green in the other direction. Then $\mu_2 \epsilon_1 \mu_a$ and $\mu_2 \epsilon_2 \mu_a$ are two paths obtained from green + blue cycle and green + red cycle wrt $\mu_a$, respectively. However, the path generated by the blue + red cycle is the sum of $\mu_2 \epsilon_1 \mu_a$ and $\mu_1 e_2 \mu_a$, thus redundant in $Y$.

(iii) Analogously, there exists one and only one monitor $\mu_b$ with link $\mu'_1 \mu_b$ colored as blue in the $\mu'_1 \rightarrow \mu_b$ direction (see Fig. 6). Based on the nonseparating ear decomposition, we have $f(\mu'_1) > f(\mu_b)$ and $g(\mu'_1) < g(\mu_b)$; therefore, $\mu'_1 \mu_b$ is colored as red in the other direction. Then $\mu_1 e_1 \mu_2$ and $\mu_2 e_2 \mu_j$ are two paths obtained from red + blue cycle and red + green cycle wrt $\mu_b$, respectively. However, the path generated by the blue + green cycle is the sum of $\mu_1 e_1 \mu_2$ and $\mu_2 e_2 \mu_j$, thus not providing new information for link identifications in $G$.

3) Duplicate paths. Each of the following 3 cases can generate the same path twice.

(i) We have considered combining the red and green paths wrt $\mu_1$. Now consider the paths associated with the blue + red cycle and blue + green cycle. As Fig. 7 displays, the paths associated with these two cycles are exactly the same, i.e., path $\mu_1 e_1 \mu_2$ is generated twice.

(ii) Following the similar argument, the paths associated with the blue + red cycle and blue + green cycle wrt $\mu'_2$ are also the same, i.e., path $\mu_2 e_2 \mu_a$ (shown in Fig. 7) is generated twice when mapping from $C$ to $Y$. Note $\mu_a$ cannot be the same as $\mu_2$ since $\mu_2$ only appears in the last ear. Therefore, path $\mu_2 e_2 \mu_a$ is non-trivial.

Note $\mu_a$ might have more than one neighbor $w$ with $f(w) < f(\mu_a)$ and $g(w) < g(\mu_a)$, in which case we can still choose $\mu'_2 \mu_a$ to color it as green.

Note $\mu_b$ might have more than one neighbor $u$ with $f(u) > f(\mu_b)$ and $g(u) < g(\mu_b)$, in which case we can still choose $\mu'_1 \mu_b$ to color it as red.
which is linearly dependent with the other two. To prove this claim, consider $\mu_{g1}$ as an example. Two paths $\mu_2, \mu_{g1}$ and $\mu_c, \mu_2, \mu_{g1}$ (see Fig. 9) can be obtained from the green + blue cycle and green + red cycle wrt $\mu_{g1}$. However, the path associated with the blue + red cycle wrt $\mu_{g1}$ is the sum of $\mu_2, \mu_{g1}$ and $\mu_c, \mu_2, \mu_{g1}$, thus linearly dependent with the other two. The same argument can be applied to other nodes in $\{\mu_{g1}, \ldots, \mu_{c3}, \mu_{c4}, \ldots, \mu_{c6}\}$. Therefore, we can identify another $N_1 + N_2$ linearly dependent paths in $Y$ regarding these $N_1 + N_2$ virtual links.

In sum, 1)–4) cover all the possible cases of redundant paths when mapping from $C$ to $Y$ and the number of these redundant paths is $7 + N_1 + N_2$, which is also the number of virtual links (denoted by $|V|$) in $G_m$. Therefore, removing these redundant paths, the cardinality of the resulting path set $Y'$ is $|G_m| - (7 + N_1 + N_2) = |G_m| - |V|$.

Now we can explore if $Y'$ is sufficient to identify all tree links in $G^*_{ex}$. Recall that each path (for identifying tree links) constructed by STPC consists of one or two path segments, each path segment terminating at the first monitor it encounters. If we let each segment terminate at the last monitor before traversing a virtual link and combine any two segments wrt real node $v$ ($v$ can be either monitor or non-monitor), then a new measurement path set $J$ is formed. It is easy to show (using STLI) that this new path set $J$ is sufficient to identify all tree links in $G^*_{ex}$. Recall the mapping process from cycle set $C$ to path set $Y$. Each path in $Y$ is obtained by removing the virtual links and all resulting isolated nodes of the corresponding cycle in $C$. Then each path $P_j$ in $J$ can be obtained by applying the same operation to the corresponding cycle $C_j (C_j \supset P_j)$ in $C$; therefore, each path in $J$ must be involved as one element in $Y$. Accordingly, $Y$ is sufficient to identify all tree links in $G^*_{ex}$. Since $Y'$ is obtained by removing redundant paths in the procedure of mapping $C$ to $Y$, then $Y'$ is also sufficient to identify all tree links in $G^*_{ex}$, i.e., the measurement matrices associated with $J$, $Y$, and $Y'$ have the same column rank.

In sum, for graph $G_m$ with $|G_m| - |V|$ real links, the property of $Y'$ is that it contains exactly $|G_m| - |V|$ paths, with each path containing only tree links in $G^*_{ex}$. Therefore, the measurement matrix associated with $Y'$ is a square matrix with full rank.

### C. Number of Distinct Paths by STPC

**Theorem III.2.** The number of distinct paths constructed by STPC equals $n$, the number of links in $G$.

**Proof:** We have proved that the corresponding measurement matrix of $Y'$ is square and sufficient to identify all tree links in $G^*_{ex}$. In this section, we prove that all paths with more than 2 monitors in $Y'$ can be shortened to form a new path set $Z$ which can cover all the paths obtained by STPC for tree link identification, i.e., paths obtained by line 1–11 in STPC algorithm.

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5Note $\mu_c$ can be any node in $\{\mu_{r1}, \ldots, \mu_{r6}, \mu_b\}$ in Fig. 9.
1. First, we consider the number of generated paths wrt to node \( v \) (line 5–9 in STPC).

(i) If \( v \) is a monitor in \( \{p_2, \mu_a, \mu_b, \mu_g, \mu_{g1}, \ldots, \mu_{gN1}, \mu_{r1}, \ldots, \mu_{rN2}\} \) (see Fig. 2), then one path in \( \{P_{v1}, P_{v2}, P_{v3}\} \) contains no links, resulting to be an invalid path and discarded without appending to \( \mathbf{R} \) in line 10. In this case, only two paths are generated in line 6.

(ii) If \( v = \mu_1 \), then line 6 of STPC only generates one path, since the other two paths contain only one node, i.e., \( \mu_1 \) itself. In this case, we have also shown that only path \( \{\mu_1, \mu_2, \mu_3\} \) is the combination of the other two paths wrt the same node, has been removed, thus not existed in \( Y' \).

Therefore, in \( Y' \), there are also two paths wrt to nodes belonging to \( \{p_2, \mu_a, \mu_b, \mu_g, \mu_{g1}, \ldots, \mu_{gN1}, \mu_{r1}, \ldots, \mu_{rN2}\} \).

(3) Let \( \mu' \) correspond to 3 measurement paths both in \( Y' \) and \( \mathcal{P}_{STPC} \).

Therefore, when constructing path segment \( S_a \) wrt \( v \), there is no need to terminate at \( \mu_{last} \). \( S_a \) can also terminate at \( \mu_{first} \). Since \( W_{S_a} \) can be known from other path measurements directly (there are only two monitors on \( S_a \)) or indirectly (in Fig. 10) \( W_{\mu_{first}\rightarrow\mu_{last}} \) and \( W_{\mu_{last}\rightarrow\mu_{first}} \) are obtained from paths constructed wrt \( \mu_i \) and \( \mu_{first} \), respectively. Using this operation, each path with more than 2 monitors in \( Y' \) can be shortened.

3. Let \( Y'' \) denote the path set obtained from \( Y' \) after the above shortening process. We can show in set \( Y'' \), for a monitor \( \mu' \) which is not in \( \{\mu_2, \mu_a, \mu_b, \mu_g, \mu_{g1}, \ldots, \mu_{gN1}, \mu_{r1}, \ldots, \mu_{rN2}\} \), the corresponding measurement paths can be further shortened. Let \( S'_1, S'_2 \) and \( S'_3 \) denote the corresponding shortened path segments wrt \( \mu' \), then the associated measurement paths in \( Y'' \) are \( S'_1 \cup S'_2 \), \( S'_2 \cup S'_3 \) and \( S'_3 \cup S'_1 \), which can be further shortened to \( S''_1, S''_2 \) and \( S''_3 \), respectively, forming the same 3 paths as those obtained by STPC. Suppose the newly formed set from \( Y'' \) by path shortening is \( Z \), then \( Z \) is also sufficient to identify all tree links in \( \mathcal{G}_m \). This is because, from the perspective of linear algebra, this shortening operation means that one part of the original linear equation can be eliminated by subtracting the linear combinations of some other rows; therefore, the resulting matrix rank is the same as the original matrix. Since we have proved that the measurement matrix associated with \( Y'' \) has a full rank, the measurement matrix associated with \( Z \) also has a full rank.

To identify tree links in \( \mathcal{G}_{ex}^* \), based on the above three arguments, we know that the number of paths constructed wrt \( v \) are the same in both \( Y'' \) and \( \mathcal{P}_{STPC} \). Moreover, wrt the same node and along the same independent spanning tree(s), the paths obtained by these two methods are exactly the same after the path shortening operation. Thus, the newly generated path set \( Z \) can cover all the paths selected by STPC for tree link identification. The number of paths in \( Z \) is the number of tree links in \( \mathcal{G}_{ex}^* \); therefore, the number of distinct paths constructed by line 1–11 in STPC is exactly the number of tree links in \( \mathcal{G}_{ex}^* \).

Finally, we consider the non-tree links of the original graph \( \mathcal{G} \). As \( \mathcal{G}_{in} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \), all non-tree links are involved in set \( L(\mathcal{G} \setminus \mathcal{G}_{in}) \). With the knowledge of tree link metrics, based on the auxiliary algorithm of STPC for non-tree link identification, in line 12–15, each non-tree link \( l \) (with link metric \( W_l \)) corresponds to one new path measurement involving \( W_l \) as the only unknown link metric. Therefore, the number of newly added paths by line 12–15 equals the number of non-tree links in \( \mathcal{G} \) and they are linearly independent with all other selected paths. A link in \( \mathcal{G} \) is either a tree or non-tree link; therefore, the number of distinct paths constructed by STPC equals the number of links in \( \mathcal{G} \).

IV. PERFORMANCE EVALUATIONS ON RANDOM GRAPHS

Besides the main simulation results in [1], we also simulate random graphs with a different number of links and observe similar results in Table II.

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| graph | m   | κ  | rSUCC | τ   | tSTPC (s) | tRWPC (s) | tSTLI (ms) | tRWLI (ms) | hSTPC | hRWPC |
|-------|-----|----|-------|-----|-----------|-----------|------------|------------|-------|-------|
| ER    | 150 | 9.39 | 99.00% | 99.76% | 10.21 | 38.73 | 5.27 | 17.8 | 17.94 | 14.2 |
| RG    | 150 | 14.96 | 4.00% | 93.91% | 10.62 | 109.21 | 5.10 | 23.34 | 22.48 | 14.43 |
| BA    | 150 | 3   | 72.00% | 99.70% | 8.41  | 90.48  | 5.39 | 20.45 | 15.08 | 8.85  |

TABLE II
Sparsely-Connected Random Graphs (ER: p = 0.0390, RG: dc = 0.11943, BA: p = 3, \( I_{\text{max}} \times n \))