The Canonical Foliation On Null Hypersurfaces in Low Regularity

Stefan Czimek · Olivier Graf

Received: 4 February 2020 / Accepted: 5 April 2022
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022

Abstract
Let $\mathcal{H}$ denote the future outgoing null hypersurface emanating from a spacelike 2-sphere $S$ in a vacuum spacetime $(\mathcal{M}, g)$. In this paper we study the so-called canonical foliation on $\mathcal{H}$ introduced in [13, 22] and show that the corresponding geometry is controlled locally only in terms of the initial geometry on $S$ and the $L^2$ curvature flux through $\mathcal{H}$. In particular, we show that the ingoing and outgoing null expansions $\text{tr}\chi$ and $\text{tr}\chi^\ast$ are both locally uniformly bounded. The proof of our estimates relies on a generalisation of the methods of [15–17] and [1, 2, 26, 32] where the geodesic foliation on null hypersurfaces $\mathcal{H}$ is studied. The results of this paper, while of independent interest, are essential for the proof of the spacelike-characteristic bounded $L^2$ curvature theorem [12].

1 Introduction

1.1 Einstein vacuum equations

A Lorentzian 4-manifold $(\mathcal{M}, g)$ is called a vacuum spacetime if it solves the Einstein vacuum equations

$$\text{Ric} = 0,$$

where $\text{Ric}$ denotes the Ricci tensor of the Lorentzian metric $g$. Expressed in general coordinates, (1.1) is a non-linear coupled system of partial differential equations of order 2 for $g$. In so-called wave coordinates, it can be shown that (1.1) is a system of...
nonlinear wave equations. It therefore admits an initial value formulation. Moreover, the characteristic hypersurfaces of these equations are the null hypersurfaces of the spacetime $(\mathcal{M}, g)$.

### 1.2 The weak cosmic censorship conjecture and the bounded $L^2$ curvature theorem

The global behaviour of solutions to (1.1) is subject to the celebrated conjecture of weak cosmic censorship formulated by Penrose [24].

**Conjecture 1.1** (Weak cosmic censorship conjecture, [24]) For generic initial data, all singularities forming in gravitational collapse are hidden in a black hole region.

In the seminal work [6], it is shown that the conjecture holds true in the case of spherical symmetry for Einstein equations coupled with a scalar field. The result relies on the sharp breakdown criterion and local existence result proved in [5] at the level of data with bounded variation, which is adapted to the $(1 + 1)$-setting of spherical symmetry. In the case of Einstein vacuum equations (1.1) without symmetry, local existence results are naturally formulated in terms of $L^2$-based function spaces (see the discussion in the introduction of [18]). In this context, the sharpest known local existence result in terms of regularity of the initial data is the celebrated bounded $L^2$ curvature theorem (see [18] and the companion papers [27–31]). The following is a rough statement of that result.

**Theorem 1.2** (Bounded $L^2$ curvature theorem, [18]) For initial data to the Einstein equations (1.1) on a spacelike hypersurface $\Sigma$ such that the spacetime curvature tensor $R$ is bounded in $L^2(\Sigma)$, there exists a local Cauchy development that satisfies Einstein equations (1.1).

In the proof [6] of the weak cosmic censorship conjecture in spherical symmetry, it is crucial that the local existence result in [5] is formulated on null hypersurfaces, in order to highlight a trapped surface formation mechanism (see [4, 6] and also [19] for further discussion).

The aim of the present paper is to initiate the proof of a local existence result for initial data on null hypersurfaces with no symmetry assumption, assuming only finite $L^2$ curvature. Together with the companion paper [12], this will amount to a proof of a spacelike-characteristic bounded $L^2$ curvature theorem that generalises the bounded $L^2$ curvature Theorem 1.2 to the case of initial data posed on a characteristic hypersurface instead of a spacelike hypersurface (see Section 1.6 for further discussion).

### 1.3 Null hypersurfaces, foliations and geometry

In various problems, foliating vacuum spacetimes by null hypersurfaces is a powerful tool to capture the propagation features of the Einstein equations. We refer the reader for example to [8, 18, 27–30] where the spacetimes are foliated by a mixed spacelike-null foliation (one family of null hypersurfaces and one family of spacelike hypersurfaces), and for example to [7, 13, 21] where the spacetimes are foliated by a double null foliation (two families of transversely intersecting null hypersurfaces).
When using mixed spacelike-null foliations, the family of null hypersurfaces is typically determined by prescribing the corresponding induced foliation on an initial spacelike hypersurface. This is equivalent to prescribing the values of an optical function $u$, whose level sets are the null hypersurfaces, on the initial spacelike hypersurface. In particular, the regularity of the induced foliation on the initial spacelike hypersurface determines the regularity of the corresponding foliation by null hypersurfaces and hence needs to be carefully picked depending on the situation (see the different constructions of the optical functions in [8] and in [27] for example).

The family of spacelike hypersurfaces is itself typically determined by defining them to be maximal hypersurfaces and fixing their asymptotics towards spacelike infinity or prescribing their finite boundary. In case of spacelike-null foliations these boundaries can be naturally prescribed by choosing the induced foliation on an initial null hypersurface, see [3]. Similarly, in double null foliations, the two families of null hypersurfaces are entirely determined by the foliation they induce on two transversely intersecting initial null hypersurfaces.

A standard choice of foliation on initial null hypersurfaces is the geodesic foliation (see below for a definition and [7, 20, 21] for examples where this foliation is used as an initial foliation). In more specific situations, other foliations have to be considered: the so-called canonical foliation on null hypersurfaces in [13] and [22] (see also Definition 1.4) for its additional regularity features, the so-called constant expansion and constant mass aspect function foliations in [25] to obtain monotonicity properties for the Hawking mass (see also [23]).

In this paper, we consider foliations on an outgoing truncated regular null hypersurface $H$ emanating from a spacelike 2-sphere $S$, given by the level sets $S_v$ of a scalar function $v \in [1, 2]$ and we assume that the first leaf of this foliation coincides with $S$, i.e. $S = S_{v=1}$. Given a null geodesic generator $L$ of $H$, we define the null lapse $\Omega$ of the foliation $(S_v)$ to be

$$\Omega := Lv.$$  

The geodesic foliation corresponds to $\Omega = 1$ and we call its parameter $s$. Note that it depends on the choice of $L$, which we assume to be fixed, here and in the rest of the paper.

We denote by $L$ the null vector field orthogonal to $S_v$ and transverse to $H$ such that $g(L, L) = -2$. The geometry of the foliation $(S_v)$ on $H$ is described by the induced metric $g$ on the 2-spheres $S_v$ by $g$, the intrinsic null second fundamental form $\chi$, the torsion $\zeta$ and the extrinsic null second fundamental form $\bar{\chi}$, respectively defined by

$$\chi(X, Y) := g(D_X L, Y), \quad \zeta(X) := \frac{1}{2} g(D_X L, L), \quad \bar{\chi}(X, Y) := g(D_X L, Y),$$

where $X, Y$ are $S_v$-tangent vectors and $D$ denotes the covariant derivative on $(\mathcal{M}, g)$. The quantities $\chi$, $\zeta$ and $\bar{\chi}$ are called the null connection coefficients. In the following, we often split up $\chi$ and $\bar{\chi}$ into their trace and tracefree parts,

---

1 The (portion of the) null hypersurface $H$ is regular if there exists a smooth non-vanishing null geodesic vector field $L$ with integral curves threading (the portion of) $H$. 

---

Springer
\[ \text{tr} \chi := g^{AB} \chi_{AB}, \quad \hat{\chi}_{AB} := \chi_{AB} - \frac{1}{2} \text{tr} g^{AB}, \]

Geometrically, the intrinsic second fundamental form \( \chi \) measures how the spheres \( S_v \) and their first fundamental form \( g \) change along \( \mathcal{H} \) and the extrinsic second fundamental form \( \hat{\chi} \) measures how the 2-spheres \( S_v \) change in the null direction transverse to \( \mathcal{H} \) given by \( L \). In particular, we see that the geometry of hypersurfaces emanating from the 2-spheres \( S_v \) transversely to \( \mathcal{H} \) must critically depend on the regularity of \( \chi \) on \( S_v \).

We also have the following decomposition of the spacetime curvature tensor \( R \) into the null curvature components relative to \( L \) and \( L_\perp \).

\[
\alpha(X, Y) := R(X, L, Y, L), \quad \beta(X) := \frac{1}{2} R(X, L, L, L), \quad \rho := \frac{1}{4} R(L, L, L, L),
\]

\[
\sigma := \frac{1}{4} \ast R(L, L, L, L), \quad \beta(X) := \frac{1}{2} R(X, L, L, L), \quad \varpi(X, Y) := R(L, X, L, Y),
\]

where \( X \) and \( Y \) are \( S_v \)-tangent vectors. We define the \( L^2 \) curvature flux through \( \mathcal{H} \) by

\[
\mathcal{R}_\mathcal{H} := \left( \| \alpha \|^2_{L^2(\mathcal{H})} + \| \beta \|^2_{L^2(\mathcal{H})} + \| \rho \|^2_{L^2(\mathcal{H})} + \| \sigma \|^2_{L^2(\mathcal{H})} + \| \varpi \|^2_{L^2(\mathcal{H})} \right)^{1/2}. \tag{1.2}
\]

### 1.4 Regularity of the foliation on \( \mathcal{H} \)

Our goal in this paper is to provide a local initial foliation \((S_v)\) on \( \mathcal{H} \) such that the geometry of a family of hypersurfaces transverse to \( \mathcal{H} \) emanating from the 2-spheres \( S_v \) can be locally controlled under the assumption of finite \( L^2 \) curvature flux. One needs to control the intrinsic and extrinsic geometry of the foliation \((S_v)\), that is, to provide bounds on \( \mathcal{H} \) for the null lapse \( \Omega \), the induced metric \( g \) and the null connection coefficients \( \chi, \zeta \) and \( \hat{\chi} \) of the foliation \((S_v)\) assuming only a control on the \( L^2 \) curvature flux.

Here and in the rest of the paper, all quantities specific to the geodesic foliation will be noted with a prime. In [15], the following groundbreaking result is proved for the geodesic foliation.

**Theorem 1.3** (Control of the geodesic foliation, [15]) Let \((\mathcal{M}, g)\) be a vacuum spacetime. Let \( \mathcal{H} \) be an outgoing null hypersurface emanating from a spacelike 2-sphere \((S, \hat{g})\) foliated by the geodesic foliation associated to the affine parameter \( s \) going from \( s|_S = 1 \) to \( s = 2 \). Let \( I'_S \) denote low regularity norms on \( \chi', \zeta' \) and \( \hat{\chi}' \) on \( S \) (see Section 2.11 for a definition). Assume that \( I'_S \) and the \( L^2 \)-curvature flux \( \mathcal{R}'_\mathcal{H} \) are sufficiently small. Then

\[
\frac{3}{s} \| \text{tr} \chi' - \frac{2}{s} \|_{L^\infty(\mathcal{H})} \leq I'_S + \mathcal{R}'_\mathcal{H},
\]
\[
\|\tilde{\chi}'\|_{L^2(\mathcal{H})} + \|\nabla'\tilde{\chi}'\|_{L^2(\mathcal{H})} \lesssim I_S' + R_H', \\
\|\zeta'\|_{L^2(\mathcal{H})} + \|\nabla'\zeta'\|_{L^2(\mathcal{H})} \lesssim I_S' + R_H', \\
\|\operatorname{tr}\chi' + \frac{2}{s}\|_{L^2(\mathcal{H})} + \|\tilde{\chi}'\|_{L^2(\mathcal{H})} \lesssim I_S' + R_H',
\]

where \(\nabla' \in \{\nabla', \nabla'_L\}\) (see Definition 2.5). This holds together with additional, more specific estimates for \(\chi', \zeta'\) and \(\chi'\) (see Section 2.11 for further estimates which are used in this paper, and see [15] for the full estimates).

**Remarks**

1. The smallness assumption on \(I_S'\) implies that the 2-sphere \(S\) is close to the Euclidean 2-sphere of radius 1 in a weak sense, and the smallness assumption on \(R_H'\) implies that the null curvature components are close to their (trivial) value in Minkowski spacetime in \(L^2\) on \(\mathcal{H}\).

2. The proof of Theorem 1.3 is obtained by analysing the so-called null structure equations and null Bianchi equations which are consequences of geometric constraints and the Einstein equations (1.1) and relate \(g', \chi', \zeta', \chi'\) to \(\alpha', \beta', \rho', \sigma', \beta',\) see Sections 2.3 and 2.4 and [8] and [15] for details.

3. Sharp bilinear and trace estimates and geometric Littlewood-Paley theory are used in the proof of Theorem 1.3, see also [16, 17] and the subsequent [1, 2, 26–30, 32].

4. This theorem gives a local control of the geodesic foliation in terms of the \(L^2\) curvature flux. In [1] and [2], a global control on the geodesic foliation was obtained provided that the (weighted) \(L^2\) curvature flux is sufficiently close to Schwarzschild data. In [32], a local control result was obtained when \(\mathcal{H}\) is the null cone emanating from a point.

5. The control of the extrinsic coefficients \(\operatorname{tr}\chi'\) and \(\tilde{\chi}'\) in Theorem 1.3 is significantly weaker than the control of \(\operatorname{tr}\chi'\) and \(\tilde{\chi}'\) on \(\mathcal{H}\).

One needs bounds for \(\operatorname{tr}\chi\) and \(\tilde{\chi}\) comparable to the ones for \(\operatorname{tr}\chi'\) and \(\tilde{\chi}'\) in order to control transversely emanating hypersurfaces. As outlined above, one does not obtain such bounds using the techniques from [15–17] in the case of the geodesic foliation. In the next section we turn to the study of the canonical foliation on \(\mathcal{H}\), which was defined in [13] and [22] for its improved regularity features for \(\operatorname{tr}\chi\) and \(\tilde{\chi}\) (see the discussion in [14]). Relying on the same ideas, we can obtain a similar regularity improvement for \(\operatorname{tr}\chi\) and \(\tilde{\chi}\) which is sufficient for the proof of the spacelike-characteristic bounded \(L^2\) curvature theorem in our companion paper [12] (see also Section 1.6 for more details).

1.5 **The canonical foliation and a first version of the main result**

In this section, we introduce the canonical foliation on \(\mathcal{H}\). Notation and more precise definitions are given in Section 2.
Definition 1.4 (Canonical foliation) A foliation \((S_v)\) on \(\mathcal{H}\) is called canonical foliation, if the null lapse \(\Omega\) satisfies

\[
\Delta (\log \Omega) = - \text{div} \xi + \rho - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \bar{\rho} + \frac{1}{2} \hat{\chi} \cdot \hat{\chi},
\]

\[
\int_{S_v} \log \Omega = 0.
\]

where \(\Delta\) denotes the induced Laplace-Beltrami operator on \(S_v\), and \(\text{div}\) the divergence operator acting on \(S_v\)-tangent vector fields, \(\bar{\rho}\) and \(\hat{\chi} \cdot \hat{\chi}\) denote the average of \(\rho\) and \(\hat{\chi} \cdot \hat{\chi}\) on \(S_v\) respectively.

Remark 1.5 In this paper, we use the definitions for the connection coefficients from [15], for this choice makes the null lapse \(\Omega\) disappear from the null structure equations (see Section 2.3). This accounts for the apparent discrepancy with the original canonical foliation definition (see Definition 3.3.2 in [13]).

The following is a first version of the main result of this paper, see Theorem 2.37 for the precise version.

Theorem 1.6 (Existence and control of the canonical foliation, version 1) Let \((\mathcal{M}, g)\) be a vacuum spacetime. Let \(\mathcal{H}\) be an outgoing null hypersurface emanating from a spacelike 2-sphere \((S, g)\) and foliated by a smooth geodesic foliation associated to the affine parameter \(s\) taking values between \(s\big|_S = 1\) and \(s = 5/2\). Assume that the initial data norm \(\mathcal{I}'_S\) at \(S_1\) and the \(L^2\) curvature flux \(\mathcal{R}'_H\) (with respect to the geodesic foliation) are sufficiently small. Then:

1. **\(L^2\)-regularity.** The canonical foliation \((S_v)\) on \(\mathcal{H}\) is well-defined from \(v = 1\) to \(2\) and

\[
\left\| \text{tr} \chi - \frac{2}{v} \right\|_{L^\infty(\mathcal{H})} + \left\| \text{tr} \hat{\chi} + \frac{2}{v} \right\|_{L^\infty(\mathcal{H})} \lesssim \mathcal{I}'_S + \mathcal{R}'_H,
\]

\[
\left\| \hat{\chi} \right\|_{L^2(\mathcal{H})} + \left\| \nabla \hat{\chi} \right\|_{L^2(\mathcal{H})} \lesssim \mathcal{I}'_S + \mathcal{R}'_H,
\]

\[
\left\| \xi \right\|_{L^2(\mathcal{H})} + \left\| \nabla \xi \right\|_{L^2(\mathcal{H})} \lesssim \mathcal{I}'_S + \mathcal{R}'_H,
\]

\[
\left\| \hat{\chi} \right\|_{L^2(\mathcal{H})} + \left\| \nabla \hat{\chi} \right\|_{L^2(\mathcal{H})} \lesssim \mathcal{I}'_S + \mathcal{R}'_H,
\]

\[
\left\| \Omega - 1 \right\|_{L^\infty(\mathcal{H})} + \left\| \nabla \Omega \right\|_{L^2(\mathcal{H})} + \left\| \nabla \Omega \right\|_{L^2(\mathcal{H})} \lesssim \mathcal{I}'_S + \mathcal{R}'_H,
\]

where \(\nabla \in \{ \nabla, \nabla_L \} \) (see Definition 2.5). Additional, more specific estimates hold for \(\chi, \xi, \Omega\) and \(\hat{\chi}\) (see Theorem 2.37).

2. **Higher regularity.** The smoothness of the geodesic foliation implies smoothness of the canonical foliation.

Remarks (1) In Theorem 1.6, the regularity of \(\text{tr} \chi\) and \(\hat{\chi}\) is improved compared to Theorem 1.3. In particular, the regularity of \(\hat{\chi}\) is sufficient for the spacelike-characteristic bounded \(L^2\) curvature theorem in the companion paper [12] (see also Section 1.6).
(2) The canonical foliation displays better regularity features for $\chi$ than the geodesic foliation because of a simplified transport equation for $\text{tr}_\chi$. More precisely, while in the geodesic foliation it holds that

$$L(\text{tr}_\chi') + \frac{1}{2} \text{tr}_\chi' \text{tr}_\chi' = -2 \text{div}_\chi + 2 \left( \rho' - \frac{1}{2} \hat{\chi}' \cdot \hat{\chi}' \right) + 2 |\zeta'|^2$$

where a low regularity curvature term is present on the right-hand side, in the canonical foliation we have

$$L(\text{tr}_\chi) + \frac{1}{2} \text{tr}_\chi \text{tr}_\chi = 2 \rho - \hat{\chi} \cdot \hat{\chi} + 2 |\nabla \Omega - \zeta|^2,$$

where the right-hand side has improved tangential regularity (see Lemma 2.22). This allows for an improved control of $\text{tr}_\chi$ and subsequently $\hat{\chi}$ on $\mathcal{H}$.

(3) The methods in the proof of Theorem 1.6 are reminiscent of [15–17] and the subsequent [1, 2, 26, 32] where the geodesic foliation is studied (see also [27–31]). A new difficulty that arises in our analysis is that, in contrast to the geodesic foliation where $\Omega \equiv 1$, the null lapse $\Omega$ has only low regularity and hence must be treated with care (see Sections 2, 4, 5 and 6).

(4) The functional calculus tools are listed in Section 3 and are mostly taken from [26], which is the latest version of the ideas from the groundbreaking [15, 16] and [17] (see also [32] and [27–31]).

(5) We use that the geodesic connection coefficients are controlled by Theorem 1.3 and that a small change of foliation leaves the null curvature components, the second fundamental form $\chi$ and some geometric norms essentially invariant (see Sections 2, 4, 5 and 6).

(6) The proof of Theorem 1.6 implies in particular that if a given canonical foliation on $\mathcal{H}$ has small initial norm $I_S$ at $S$ and small $L^2$ curvature flux $\mathcal{R}_\mathcal{H}$ with respect to the canonical foliation on $\mathcal{H}$, then the foliation geometry on $\mathcal{H}$ is controlled as stated in Theorem 1.6 with $I_S$ and $\mathcal{R}_\mathcal{H}$ on the right-hand side. Since such a formulation would require that the canonical foliation a priori exists, we prefered to state the smallness assumptions with respect to the geodesic foliation.

1.6 The spacelike-characteristic Cauchy problem of general relativity in low regularity

Our motivation for the main Theorem 1.6 in this paper is its application to the authors’ spacelike-characteristic bounded $L^2$ curvature theorem [12]. First we define the volume radius of a Riemannian 3-manifold.

**Definition 1.7** (Volume radius) Let $(\Sigma, g)$ be a Riemannian 3-manifold with boundary, and let $r > 0$ be a real number. The *volume radius of $\Sigma$ at scale $r$* is defined by
\[
 r_{\text{vol}}(\Sigma, r) := \inf_{p \in \Sigma} \inf_{0 < r' < r} \frac{\text{vol}_g(B_g(p, r'))}{(r')^3},
\]
where \(B_g(p, r')\) denotes the geodesic ball of radius \(r'\) centred at \(p \in \Sigma\).

**Theorem 1.8** (The spacelike-characteristic bounded \(L^2\) curvature theorem, [12])
Consider smooth initial data for the Einstein vacuum equations posed on a maximal spacelike hypersurface \(\Sigma \cong \overline{B} \subset \mathbb{R}^3\) and the outgoing null hypersurface \(\mathcal{H}\) emanating from \(S := \partial \Sigma\). Assume that for some \(\varepsilon > 0\),

\[
 T'_S + R'_H \leq \varepsilon, \quad \|\text{Ric}\|_{L^2(\Sigma)} \leq \varepsilon, \quad \|k\|_{L^2(\Sigma)} + \|
abla k\|_{L^2(\Sigma)} \leq \varepsilon, \quad r_{\text{vol}}(\Sigma, 1/2) \geq 1/4, \quad \text{vol}_g(\Sigma) < \infty,
\]

where the initial foliation geometry \(T'_S\) and the \(L^2\) curvature flux \(R'_H\) are the same as in Theorem 1.6, and \(\text{Ric}\) and \(k\) denote the Ricci tensor and second fundamental form of \(\Sigma \subset M\). Then:

1. **\(L^2\)-regularity.** There is a universal constant \(\varepsilon_0 > 0\) such that if \(0 < \varepsilon < \varepsilon_0\), then the maximal globally hyperbolic development of \((M, g)\) contains a future region of \(\Sigma \cup H\) which is foliated by maximal spacelike hypersurfaces \(\Sigma_t\) given as level sets of a time function \(t\) such that \(\Sigma_1 = \Sigma\) and

\[
\partial \Sigma_t = S_t \text{ on } \mathcal{H},
\]

where \((S_t)_{1 \leq t \leq 2}\) is the canonical foliation on \(\mathcal{H}\), and the following control holds for \(1 \leq t \leq 2\),

\[
\|\text{Ric}\|_{L^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \|k\|_{L^\infty L^2(\Sigma_t)} + \|
abla k\|_{L^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \inf_{1 \leq t \leq 2} r_{\text{vol}}(\Sigma_t, 1/2) \geq \frac{1}{8}, \quad \text{vol}_g(\Sigma_t) < \infty.
\]

2. **Higher regularity.** Smoothness is propagated from initial data into \(\mathcal{M}\) up to \(t = 2\).

**Remarks**
1. In the proof of Theorem 1.8, the boundary regularity of the hypersurfaces \(\Sigma_t\) is directly related to the regularity of the canonical foliation \((S_t)\) on \(\mathcal{H}\). More specifically, for the control of \(k\) on \(\partial \Sigma_t\), it is necessary to control both \(\chi\) and \(\chi\) in sufficiently high regularity, which is achieved by the main result of this paper, Theorem 1.6.
2. In addition to the estimates of this paper, the proof of Theorem 1.8 relies on the bounded \(L^2\) curvature theorem [18], the extension procedure for the constraint equations [9], Cheeger-Gromov convergence theory on manifolds (with boundary) in low regularity [10, 11], and global estimates for maximal spacelike hypersurfaces.
2 Geometric setup and main results

In this section, we introduce the geometric setup of this paper, give a precise statement of our main result (see Section 2.12) as well as a proof, assuming that the results from Sections 3, 4, 5, 6 can be obtained.

2.1 Foliations on null hypersurfaces

In this section, we set up foliations on null hypersurfaces following the notations (and normalisations) of [15]. Let \((M, g)\) be a Lorentzian 4-manifold and let \(S \subset M\) be a spacelike 2-sphere. Let \(\mathcal{H}\) denote the outgoing null hypersurface emanating from \(S\). In the following, we restrict the null hypersurface \(\mathcal{H}\) to a regular truncation of itself, \textit{i.e.} such that there exists a smooth non-vanishing null geodesic vector field \(L\) with integral curves threading it.

**Definition 2.1** (Geodesic foliation on \(\mathcal{H}\)) Let \(L\) be a null generator of \(\mathcal{H}\). We say that \(s\) is the \textit{associated affine parameter} to \(L\) on \(\mathcal{H}\) if

\[
Ls = 1 \text{ on } \mathcal{H}, \quad s|_S = 1.
\]

Denote the level sets of \(s\) by

\[
S'_{s_0} = \{s = s_0\}
\]

and denote the geodesic foliation by \((S'_{s})\). In the following, we will only consider \textit{2-sphere geodesic foliations, i.e.} such that all the level sets \(S'_{s}\) are 2-spheres.

**Definition 2.2** (General 2-sphere foliations on \(\mathcal{H}\)) Let \(v\) be a given scalar function on \(\mathcal{H}\) with \(dv \neq 0\) such that its level sets \(S_v = \{v = v_0\}\), are 2-spheres. We denote by \((S_v)\) the foliation of \(\mathcal{H}\) by the level sets \(S_v\). We define the \textit{null lapse} \(\Omega\) of \((S_v)\) on \(\mathcal{H}\) by

\[
\Omega := Lv.
\]  

**Definition 2.3** (Orthonormal null frame) Let \((S_v)\) be a foliation on \(\mathcal{H}\). Let \(L\) be the unique null vector field on \(\mathcal{H}\) orthogonal to the 2-spheres \(S_v\) and such that \(g(L, L) = -2\). The pair \((L, L)\) is called a \textit{null pair for the foliation} \((S_v)\). Let \((e_1, e_2)\) be an orthonormal frame tangential to the 2-spheres \(S_v\). The frame \((L, L, e_1, e_2)\) is called an \textit{orthonormal null frame for the foliation} \((S_v)\).

**Remark 2.4** Let \((S_v)\) be a foliation on \(\mathcal{H}\), \((L, L)\) an associated null pair and \(X\) an \(\mathcal{H}\)-tangent vector field. Decomposing \(X\) onto an orthonormal null frame for the foliation \((S_v)\) and using (2.1) we have
\[ Xv = -\frac{1}{2} \Omega g(L, X). \quad (2.2) \]

**Definition 2.5** Let \((S_v)\) be a foliation on \(\mathcal{H}\). We denote by \(g\) and \(\nabla\) the induced Riemannian metric and covariant derivative on the 2-spheres \(S_v\) and define for any \(S_v\)-tangential \(k\)-tensor \(T\) the derivative \(\nabla L T\) by

\[
\nabla L T_{A_1 \ldots A_k} := \Pi_{A_1}^{\beta_1} \cdots \Pi_{A_k}^{\beta_k} D_L T_{\beta_1 \ldots \beta_k},
\]

where \(\Pi\) denotes the projection operator onto the tangent space of \(S\), \(D\) is the covariant derivative on \((\mathcal{M}, g)\) and we tacitly use, as in the rest of this paper, the Einstein summation convention.

Here and in the following, indices \(A, B, C, D, E \in \{1, 2\}\) denote evaluation of \(S_v\)-tangent tensors on the components \((e_1, e_2)\) of an orthonormal frame \((L, L, e_1, e_2)\) for the foliation \((S_v)\). In the following, we recall definitions from Section 1 and make new additional definitions.

**Definition 2.6** (Null connection coefficients) We define the null connection coefficients, to be the \(S_v\)-tangent tensors such that

\[
\begin{align*}
\chi_{AB} := g(D_A L, e_B), & \quad \chi_{\bar{A} \bar{B}} := g(D_A \bar{L}, e_B), \\
\zeta_A := \frac{1}{2} g(D_A L, \bar{L}), & \quad \eta_A := \frac{1}{2} g(D_L \bar{L}, e_A),
\end{align*}
\]

(2.3)

**Lemma 2.7** The connection coefficients \(\eta\) and \(\zeta\) and the null lapse \(\Omega\) verify

\[ \eta = -\zeta - \nabla (\log \Omega). \quad (2.4) \]

**Proof** Using equations (2.1) and (2.2), we have

\[
\begin{align*}
\eta_A &= -\frac{1}{2} g(L, D_L e_A) \\
&= -\frac{1}{2} g(L, D_A L) - \frac{1}{2} g(L, [L, e_A]) \\
&= -\zeta_A + \Omega^{-1}[L, e_A](v) \\
&= -\zeta_A - \nabla_A (\log \Omega),
\end{align*}
\]

as desired. \(\square\)

We have the following relations between covariant derivatives and null connection coefficients (see [8]),

\[
\begin{align*}
D_L L &= 0, & D_L \bar{L} &= 2\eta_A e_A, \\
D_A L &= \chi_{AB} e_B - \zeta_A L, & D_A \bar{L} &= \chi_{\bar{A} \bar{B}} e_B + \zeta_A \bar{L}, \\
D_L e_A &= \nabla_L e_A + \eta_A L, & D_A e_B &= \nabla_A e_B + \frac{1}{2} \chi_{AB} L + \frac{1}{2} \chi_{\bar{A} \bar{B}} \bar{L}. 
\end{align*}
\]

\(\odot\) Springer
If the orthonormal null frame is such that $\nabla L e_A = 0$, we call it \textit{Fermi propagated}. We have the following decomposition of $\chi$ and $\bar{\chi}$ into their trace and tracefree parts

$$\tr \chi := g^{AB} \chi_{AB}, \quad \bar{\chi}_{AB} := \chi_{AB} - \frac{1}{2} \tr \chi g^{AB},$$
$$\tr \bar{\chi} := g^{AB} \bar{\chi}_{AB}, \quad \bar{\chi}_{AB} := \bar{\chi}_{AB} - \frac{1}{2} \tr \bar{\chi} g^{AB}.$$  

\textbf{Definition 2.8} (Null curvature components) We define the null curvature components to be the $S_v$-tangent tensors such that

$$\alpha_{AB} := R(L, e_A, L, e_B), \quad \beta_A := \frac{1}{2} R(e_A, L, L, L),$$
$$\rho := \frac{1}{4} R(L, L, L, L), \quad \sigma := \frac{1}{4} *R(L, L, L, L),$$
$$\bar{\beta}_A := \frac{1}{2} R(e_A, L, L, L), \quad \bar{\alpha}_{AB} := R(L, e_A, L, e_B),$$

where $*R$ denotes the Hodge dual of $R$, given by $*R_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} R^\mu\nu_{\gamma\delta}$, with $\epsilon$ the volume form associated to the metric $g$.  

\subsection*{2.2 Tensor calculus on 2-spheres}

We introduce the following notation.

\textbf{Definition 2.9} (Hodge duals) For a $S_v$-tangent 1-tensor $\phi$, we define its left Hodge dual by

$$*\phi_A := \ell_{AB} \phi_B,$$

where $\ell_{AB} := \epsilon_{AB LL}$. Similarly, for a $S_v$-tangent symmetric 2-tensor $\phi$, let

$$*\phi_{AB} := \ell_{AC} \phi_{CB}.$$  

\textbf{Definition 2.10} ($S_v$-tangent tensor calculus) For $S_v$-tangent $r$-tensors $\phi$, $\phi^{(1)}$ and $\phi^{(2)}$, we define

$$\phi^{(1)} \cdot \phi^{(2)} := g^{A_1 B_1} \ldots g^{A_r B_r} \phi^{(1)}_{A_1 \ldots A_r} \phi^{(2)}_{B_1 \ldots B_r}, \quad |\phi|^2 = \phi \cdot \phi,$$

and

$$d \nabla \phi_{A_2 \ldots A_r} := g^{AB} \nabla_A \phi_{BA_2 \ldots A_r}, \quad \nabla \phi_{A_2 \ldots A_r} = \ell_{AB} \nabla_A \phi_{BA_2 \ldots A_r}.$$  

For a 1-form $\phi$, we define

$$(\nabla \hat{\otimes} \phi)_{AB} := \nabla_A \phi_B + \nabla_B \phi_A - d \nabla \phi g_{AB}.$$
For 1-forms \( \phi^{(1)} \) and \( \phi^{(2)} \) we define

\[
(\phi^{(1)} \mathring{\otimes} \phi^{(2)})_{AB} := \phi^{(1)}_A \phi^{(2)}_B + \phi^{(1)}_B \phi^{(2)}_A - g_{AB} \phi^{(1)} \cdot \phi^{(2)},
\]

\[
\phi^{(1)} \wedge \phi^{(2)} := \varepsilon^{AB} \phi^{(1)}_A \phi^{(2)}_B.
\]

For symmetric 2-tensors \( \phi^{(1)} \) and \( \phi^{(2)} \) we define the wedge product,

\[
(\phi^{(1)} \wedge \phi^{(2)}) := \varepsilon^{AB} g^{CD} \phi^{(1)}_{AC} \phi^{(2)}_{BD}.
\]

### 2.3 Null structure equations on \( \mathcal{H} \)

The Einstein vacuum equations (1.1) induce the following null structure equations on \( \mathcal{H} \), see ([8], pp. 168-170). These equations hold for any general 2-sphere foliation. We have the first variation equation,

\[
\mathcal{L}_L g = 2\chi,
\]

the null transport equations,

\[
\nabla_L \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 = -|\mathring{\chi}|^2,
\]

\[
\nabla_L \mathring{\chi} + \text{tr} \mathring{\chi} = -\alpha,
\]

\[
\nabla_L \text{tr} \chi + \frac{1}{2} \text{tr} \chi \text{tr} \chi = 2 \text{div} \eta + 2\rho - \mathring{\chi} \cdot \mathring{\chi} + 2|\eta|^2,
\]

\[
\nabla_L \mathring{\chi} + \frac{1}{2} \text{tr} \chi \mathring{\chi} = (\nabla \mathring{\otimes} \eta) - \frac{1}{2} \text{tr} \mathring{\chi} + (\eta \mathring{\otimes} \eta),
\]

\[
\nabla_L \zeta + \frac{1}{2} \text{tr} \chi \zeta = \frac{1}{2} \text{tr} \chi \eta - \mathring{\chi} \cdot (\zeta - \eta) - \beta,
\]

the torsion equation,

\[
\text{curl} \eta = -\text{curl} \zeta = -\sigma + \frac{1}{2} \mathring{\chi} \wedge \mathring{\chi},
\]

and the Gauss-Codazzi equations,

\[
K = -\frac{1}{4} \text{tr} \chi \text{tr} \chi - \rho + \frac{1}{2} \mathring{\chi} \cdot \mathring{\chi},
\]

\[
\text{div} \mathring{\chi} - \frac{1}{2} \nabla \text{tr} \chi = -\zeta \cdot \mathring{\chi} + \frac{1}{2} \zeta \text{tr} \chi - \beta,
\]

\[
\text{div} \mathring{\chi} - \frac{1}{2} \nabla \text{tr} \chi = \zeta \cdot \mathring{\chi} - \frac{1}{2} \zeta \text{tr} \chi + \beta,
\]

where \( K \) denotes the Gauss curvature of \( S_v \).
Remark 2.11 Only the trace and the symmetrised traceless part of the transport equation for $\chi$ are stated in [8]. By rederiving the equation, or simply using the null transport equation for the traced and the symmetrised traceless tensor together with the torsion equation, one has more generally the following transport equation for the full tensor $\chi$

$$\nabla_L \chi_{AB} + \chi_{AC} \chi_{CB} = 2\nabla_A \eta_B + 2\eta_A \eta_B + \rho \xi_{AB} + \sigma \xi_{AB}.$$ 

Remark 2.12 Similarly, only the divergence part of Codazzi equations are stated in [8]. By rederiving the equation, or simply using the Gauss-Codazzi equation for $\text{div} \hat{\chi}$, more generally it holds that

$$\text{curl} \chi = -\zeta \cdot \hat{\chi} + \frac{1}{2} \text{tr} \chi^* \zeta - \beta.$$

2.4 Null Bianchi identities on $\mathcal{H}$

The Einstein vacuum equations (1.1) further yield the following null Bianchi identities on $\mathcal{H}$, see ([8], p. 161).

$$\nabla_L \chi + \frac{1}{2} \text{tr} \chi = -(\nabla \otimes \beta) - 3\hat{\chi} \rho + 3\hat{\chi} \sigma + ((\zeta - 4\eta) \otimes \beta),$$

$$\nabla_L \beta + \text{tr} \chi \beta = -\nabla \rho + \ast \nabla \sigma + 2\hat{\chi} \cdot \beta - 3\rho \sigma + 3\ast \eta \sigma,$$

$$\nabla_L \rho + \frac{3}{2} \text{tr} \chi = \text{div} \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2\eta \cdot \beta,$$

$$\nabla_L \sigma + \frac{3}{2} \text{tr} \chi \sigma = -\text{curl} \beta + \frac{1}{2} \hat{\chi} \wedge \alpha - \zeta \wedge \beta - 2\eta \wedge \beta,$$

$$\nabla_L \beta + 2\text{tr} \chi \beta = \text{div} \alpha + (2\zeta + \eta) \cdot \alpha.$$ 

We have the following renormalisations of the curvature components, which have the advantage of eliminating $\hat{\chi}$ from the source terms of the null transport equations. See also [15].

Definition 2.13 (Renormalised null curvature components) Let the renormalised curvature components $\tilde{\rho}, \tilde{\sigma}, \tilde{\beta}$ be

$$\tilde{\rho} := \rho - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}, \quad \tilde{\sigma} := \sigma - \frac{1}{2} \hat{\chi} \wedge \hat{\chi}, \quad \tilde{\beta} := \beta + 2\hat{\chi} \cdot \zeta.$$ 

We have the following transport equations for $\tilde{\rho}, \tilde{\sigma}$ and $\tilde{\beta}$.

Lemma 2.14 The renormalised null curvature component $\tilde{\rho}, \tilde{\sigma}$ and $\tilde{\beta}$ satisfy

$$\nabla_L \tilde{\rho} + \frac{3}{2} \text{tr} \chi \tilde{\rho} = \text{div} \beta + \zeta \cdot \beta + 2\eta \cdot \beta - \frac{1}{2} \nabla \otimes \eta \cdot \hat{\chi} + \frac{1}{4} \text{tr} \chi^2 - \frac{1}{2} (\eta \otimes \eta) \cdot \hat{\chi}.$$ 

(2.9a)
\[ V_L \ddot{\sigma} + \frac{3}{2} \text{tr} \dot{\sigma} = - \text{curl} \beta - \zeta \wedge \beta - 2 \eta \wedge \beta - \frac{1}{2} \dot{\chi} \wedge (V_\eta \eta) - \frac{1}{2} \ddot{\chi} \wedge (\eta \otimes \eta) \]  
(2.9b)

\[ V_L \ddot{\beta} + \text{tr} \dot{\beta} = - V_\rho + V_\sigma + 2(V_\eta \eta) \cdot \zeta - 3 \eta \rho + 3^* \eta \sigma \]
\[- \text{tr} \chi \zeta \cdot \ddot{\chi} + \text{tr} \chi \eta \cdot \ddot{\chi} + 2 \zeta \cdot (\eta \otimes \eta) - 2 \ddot{\chi} \cdot (\zeta - \eta). \]  
(2.9c)

**Proof** From p. 14 in [15], we have the first two equations. Using Bianchi equation (2.7b) and the null structure equation for \( \ddot {\chi} \) and \( \zeta \) from Section 2.3, we have

\[ V_L \ddot{\beta} + \text{tr} \dot{\beta} = V_L \beta + \text{tr} \chi \beta + 2 \left( V_L \dot{\chi} + \frac{1}{2} \text{tr} \chi \chi \right) \cdot \zeta + 2 \ddot{\chi} \cdot \left( V_L \zeta + \frac{1}{2} \text{tr} \chi \zeta \right) \]
\[ = - V_\rho + V_\sigma + 2 \ddot{\chi} \cdot \beta - 3 \eta \rho + 3^* \eta \sigma \]
\[ + 2 \zeta \cdot \left( V_\eta \eta - \frac{1}{2} \text{tr} \chi \chi + \eta \otimes \eta \right) \]
\[ + 2 \ddot{\chi} \cdot \left( \frac{1}{2} \text{tr} \chi \eta - \ddot{\chi} \cdot (\zeta - \eta) - \beta \right) \]
\[ = - V_\rho + V_\sigma + 2(V_\eta \eta) \cdot \zeta - 3 \eta \rho + 3^* \eta \sigma \]
\[- \text{tr} \chi \zeta \cdot \ddot{\chi} + \text{tr} \chi \eta \cdot \ddot{\chi} + 2 \zeta \cdot (\eta \otimes \eta) - 2 \ddot{\chi} \cdot (\zeta - \eta), \]

as desired. \( \square \)

### 2.5 Commutation formulas on \( \mathcal{H} \)

The next proposition follows from p. 159 in [8].

**Proposition 2.15** *(Commutation formulas)* For an \( S_v \)-tangent \( r \)-tensor \( \phi \), it holds that

\[ [V_L, V_B] \phi_{A_1 \ldots A_r} = - \frac{1}{2} \text{tr} \chi V_B \phi_{A_1 \ldots A_r} - \ddot{\chi}_{BC} V_B C \phi_{A_1 \ldots A_r} + (\eta_B + \zeta_B) V_L \phi_{A_1 \ldots A_r} \]
\[ + \sum_{i=1}^{r} (\chi_{A_i B} \eta_C - \ddot{\chi}_{BC} \eta_{A_i} + \xi_{A_i C} \beta_B) \phi_{A_1 \ldots C \ldots A_r}. \]  
(2.10)

For an \( S_v \)-tangent 1-form \( \phi \), it holds that

\[ [V_L, \text{d} \psi] \phi = - \frac{1}{2} \text{tr} \chi \text{d} \psi \phi - \ddot{\chi} \cdot \text{d} \phi \phi + (\eta + \zeta) \cdot V_L \phi \]
\[ + \text{tr} \chi \eta \cdot \phi - \eta_A \phi_C \chi_{AC} + \beta \cdot \phi, \]  
(2.11)

\[ [V_A, \phi] \phi_B = - 3 K V_A \phi_B + 2 g_{AB} K \ddot{\chi} \phi + 2 \xi_{AB} K \text{curl} \phi \]
\[ + g_{AB} \phi \cdot K - \phi_A V_B K. \]  
(2.12)

where \( K \) denotes the Gauss curvature of \( S_v \). For a scalar function \( \phi \), it holds that

\[ [V_L, V_B] \phi = - \frac{1}{2} \text{tr} \chi V_B \phi - \ddot{\chi}_{BC} V_B C \phi + (\eta_B + \zeta_B) L \phi. \]  
(2.13)
\[ [\nabla_L, \phi] = - \text{tr} \chi \phi - 2 \hat{\chi} \cdot \nabla \phi - (\hat{\eta} + \zeta) \cdot (\nabla \nabla_L + \nabla_L \nabla) \phi \\
+ (\text{tr} \chi \eta - \text{div} \chi) \cdot \nabla \phi - \eta_A \chi_{AB} \nabla B \phi \\
+ (\text{div} \eta + \text{div} \zeta) \nabla_L \phi + \beta \cdot \nabla \phi, \quad (2.14) \]

\[ [\nabla_A, \phi] = - K \nabla_A \phi. \quad (2.15) \]

**Proposition 2.16** For any scalar function \( f \) on \( \mathcal{H} \), it holds that

\[
\Omega^{-1} L (\bar{f}) = \Omega^{-1} L f + \Omega^{-1} \text{tr} \chi f - \Omega^{-1} \text{tr} \chi \cdot \bar{f},
\]

where \( \bar{f} \) denotes the mean value of \( f \) on \( S_v \).

**Proof** Using the null structure equations from Section 2.3, we have

\[
\Omega^{-1} L \left( \int_{S_v} f \right) = \frac{d}{dv} \left( \int_{S_v} f \right) = \int_{S_v} \Omega^{-1} (Lf + \text{tr} \chi f) .
\]

We therefore deduce

\[
\Omega^{-1} L (\bar{f}) = - \Omega^{-1} L (\log (|S_v|)) \bar{f} + \Omega^{-1} (Lf + \text{tr} \chi f)
\]

\[
= - \Omega^{-1} \text{tr} \chi \cdot \bar{f} + \Omega^{-1} (Lf + \text{tr} \chi f),
\]

where \( |S_v| \) denotes the area of \( S_v \). This proves the desired result. \( \square \)

### 2.6 The mass aspect function on \( \mathcal{H} \)

**Definition 2.17** Let the mass aspect function \( \mu \) on \( \mathcal{H} \) be defined by

\[
\mu := - \tilde{\rho} - \text{div} \zeta . \quad (2.16)
\]

We have the following transport equation for \( \mu \).

**Lemma 2.18** The mass aspect function \( \mu \) verifies

\[
L(\mu) + \text{tr} \chi \mu = \frac{1}{2} \text{tr} \chi \tilde{\rho} - \frac{1}{2} \text{tr} \chi \text{div} \eta - 2 \zeta \cdot \beta + (\zeta - \eta) \cdot \nabla \text{tr} \chi + 2 \hat{\chi} \cdot \nabla \zeta
\]

\[
- \frac{1}{2} \hat{\chi} \cdot \nabla \eta + \text{tr} \chi \left( |\zeta|^2 - \zeta \cdot \eta - \frac{1}{2} |\eta|^2 \right) - \frac{1}{4} \text{tr} \chi |\hat{\chi}|^2
\]

\[
+ 2 \hat{\chi} \cdot \zeta \cdot \eta - \frac{1}{2} \hat{\chi} \cdot \eta \cdot \eta .
\]
Proof Using the transport equation for $\zeta$ from the null structure equations from Section 2.3 and commutation formula (2.11), we have

\[
L(d\psi \zeta) = d\psi \nabla_L \zeta + [\nabla_L, d\psi] \zeta
\]

\[
= d\psi \left( -\frac{1}{2} \text{tr} \chi \zeta + \frac{1}{2} \text{tr} \chi \eta - \hat{\chi} \cdot (\zeta - \eta) - \beta \right)
- \frac{1}{2} \text{tr} \chi d\psi \zeta - \hat{\chi} \cdot \nabla \zeta + (\eta + \zeta) \cdot \nabla_L \zeta + \text{tr} \chi \eta \cdot \zeta - \eta \cdot \zeta \cdot \chi + \beta \cdot \zeta
\]

\[
= -\text{tr} \chi d\psi \zeta + \frac{1}{2} \text{tr} \chi d\psi \eta - d\psi \beta + F_1,
\]

where

\[
F_1 := -\frac{1}{2} \zeta \cdot \nabla \text{tr} \chi + \frac{1}{2} \eta \cdot \nabla \text{tr} \chi - d\psi \hat{\chi} \cdot (\zeta - \eta) - \hat{\chi} \cdot (\nabla \zeta - \nabla \eta)
- \hat{\chi} \cdot \nabla \zeta + (\eta + \zeta) \cdot \left( -\frac{1}{2} \text{tr} \chi \zeta + \frac{1}{2} \text{tr} \chi \eta - \hat{\chi} \cdot (\zeta - \eta) - \beta \right)
+ \text{tr} \chi \eta \cdot \zeta - \eta \cdot \zeta \cdot \chi + \beta \cdot \zeta.
\]

From the above and transport equation (2.9a) for $\hat{\rho}$, we have

\[
L(\mu) + \text{tr} \chi \mu = -L(\hat{\rho}) - \text{tr} \chi \hat{\rho} - L(d\psi \zeta) - \text{tr} \chi d\psi \zeta
\]

\[
= \frac{3}{2} \text{tr} \chi \hat{\rho} - d\psi \beta + F_2 - \text{tr} \chi \hat{\rho}
- \frac{1}{2} \text{tr} \chi d\psi \eta + d\psi \beta - F_1 = \frac{1}{2} \text{tr} \chi \hat{\rho} - \frac{1}{2} \text{tr} \chi d\psi \eta + F_2 - F_1,
\]

where the nonlinear term $F_2$ is given by

\[
F_2 := -\zeta \cdot \beta - 2\eta \cdot \beta + \frac{1}{2} (\nabla \eta) \cdot \hat{\chi} - \frac{1}{4} \text{tr} \chi |\hat{\chi}|^2 + \frac{1}{2} \eta \cdot \hat{\chi}.
\]

Rearranging the nonlinear term $F_2 - F_1$ then gives the desired result. \(\square\)

2.7 The canonical foliation on $\mathcal{H}$

Definition 2.19 (Canonical foliation) Let $\mathcal{H}$ be a regular outgoing null hypersurface emanating from a spacelike 2-sphere $S$. A 2-sphere foliation $(S_v)$ of $\mathcal{H}$ (see Definition 2.2) is called a canonical foliation if the null lapse $\Omega$ satisfies the following elliptic equation on the leaves $S_v$.
\[ \Delta (\log \Omega) = -d\hat{\nabla} \zeta + \tilde{\rho} - \bar{\rho}, \]  
\[ \int_{S_v} \log \Omega = 0, \quad (2.17) \]

where \( \bar{\rho} \) is the average of \( \tilde{\rho} \) on the 2-sphere \( S_v \). In the following, we will moreover consider canonical foliations with \( v|_S = 1 \).

**Remark 2.20** Using (2.16), the elliptic equation (2.17) can also be rewritten

\[ \Delta (\log \Omega) = -2d\hat{\nabla} \zeta - \mu + \bar{\mu}, \]
\[ = 2(\tilde{\rho} - \bar{\rho}) + \mu - \bar{\mu}. \quad (2.18) \]

**Notation.** From now on, primed quantities on \( \mathcal{H} \) will correspond to the geodesic foliation of \( \mathcal{H} \), while unprimed quantities correspond to the canonical foliation. Moreover, we call \( S_1 = S = S'_1 \).

As a first consequence of Definition 2.19, we note that in a canonical foliation, the quantities \( \eta \) and \( \text{tr} \chi \) satisfy the following equations.

**Lemma 2.21** In a canonical foliation, \( \eta \) satisfies the following equation

\[ d\hat{\nabla} \eta = -\tilde{\rho} + \bar{\rho}. \quad (2.19) \]

**Proof** Using relation (2.4) and the elliptic equation (2.17), we have

\[ d\hat{\nabla} \eta = -d\hat{\nabla} \zeta - \Delta (\log \Omega) \]
\[ = -\tilde{\rho} + \bar{\rho}, \]

as desired. \[ \square \]

**Lemma 2.22** In a canonical foliation, \( \text{tr} \chi \) satisfies the null transport equation

\[ L(\text{tr} \chi) + \frac{1}{2} \text{tr} \chi \text{tr} \chi = 2\tilde{\rho} + 2|\eta|^2. \quad (2.20) \]

**Proof** Using the transport equation for \( \text{tr} \chi \) from Section 2.3 and relation from Lemma 2.21, we have

\[ L(\text{tr} \chi) + \frac{1}{2} \text{tr} \chi \text{tr} \chi = 2d\hat{\nabla} \eta + 2\tilde{\rho} + 2|\eta|^2 \]
\[ = 2\tilde{\rho} + 2|\eta|^2, \]

as desired. \[ \square \]
Lemma 2.23 In a canonical foliation, the transport equation for the mass aspect function can be written in the following form

\[ L(\mu) + \text{tr}_\mathcal{X} \mu = \text{tr}_\mathcal{X} \tilde{\rho} - \frac{1}{2} \text{tr}_\mathcal{X} \tilde{\rho} - 2 \zeta \cdot \beta + (\zeta - \eta) \cdot \nabla \text{tr}_\mathcal{X} + 2 \tilde{\chi} \cdot \nabla \zeta \]
\[ - \frac{1}{2} \tilde{\chi} \cdot \nabla \eta + \text{tr}_\mathcal{X} \left( |\zeta|^2 - \zeta \cdot \eta - \frac{1}{2} |\eta|^2 \right) - \frac{1}{4} \text{tr}_\mathcal{X} |\tilde{\chi}|^2 \] \tag{2.21}
\[ + 2 \tilde{\chi} \cdot \zeta \cdot \eta - \frac{1}{2} \tilde{\chi} \cdot \eta \cdot \eta. \]

For convenience, we summarise the full null structure equations in a canonical foliation

\[ \mathcal{L}_L g^\mu = 2 \chi, \] \tag{2.22a}
\[ \nabla_L \text{tr}_\mathcal{X} + \frac{1}{2} (\text{tr}_\mathcal{X})^2 = -|\tilde{\chi}|^2, \] \tag{2.22b}
\[ \nabla_L \tilde{\chi} + \text{tr}_\mathcal{X} \tilde{\chi} = -\alpha, \] \tag{2.22c}
\[ \nabla_L \tilde{\chi} + \chi \cdot \tilde{\chi} = 2 \nabla \eta + 2 \eta \eta + \rho g + \sigma \ne, \] \tag{2.22d}
\[ \nabla_L \text{tr}_\mathcal{X} + \frac{1}{2} \text{tr}_\mathcal{X} \text{tr}_\mathcal{X} = 2 \tilde{\rho} + 2 |\eta|^2, \] \tag{2.22e}
\[ \nabla_L \tilde{\chi} + \frac{1}{2} \text{tr}_\mathcal{X} \tilde{\chi} = (\nabla \tilde{\chi} \eta) - \frac{1}{2} \text{tr}_\mathcal{X} \tilde{\chi} + (\eta \tilde{\chi} \eta), \] \tag{2.22f}
\[ \nabla_L \zeta + \frac{1}{2} \text{tr}_\mathcal{X} \zeta = \frac{1}{2} \text{tr}_\mathcal{X} \eta + \tilde{\chi} \cdot (\eta - \zeta) - \beta, \] \tag{2.22g}
\[ \text{curl} \eta = -\text{curl} \zeta = -\tilde{\sigma}, \] \tag{2.22h}
\[ K = -\frac{1}{4} \text{tr}_\mathcal{X} \text{tr}_\mathcal{X} - \tilde{\rho}, \] \tag{2.22i}
\[ \text{d} \text{div} \tilde{\chi} - \frac{1}{2} \nabla \text{tr}_\mathcal{X} = -\zeta \cdot \tilde{\chi} + \frac{1}{2} \zeta \text{tr}_\mathcal{X} - \beta, \] \tag{2.22j}
\[ \text{d} \text{div} \tilde{\chi} - \frac{1}{2} \nabla \text{tr}_\mathcal{X} = \zeta \cdot \tilde{\chi} - \frac{1}{2} \zeta \text{tr}_\mathcal{X} + \beta, \] \tag{2.22k}
\[ \text{div} \eta = -\tilde{\rho} + \tilde{\rho}, \] \tag{2.22l}
\[ \text{curl} \mathcal{X} = -\zeta \cdot \star \tilde{\chi} + \frac{1}{2} \text{tr}_\mathcal{X} \star \zeta - \star \beta, \] \tag{2.22m}
\[ \Delta (\log \Omega) = - \text{d} \text{div} \zeta + \tilde{\rho} - \tilde{\rho}. \] \tag{2.22n}

2.8 Comparison of foliations

In this section, we derive equations that are used to compare a geodesic foliation and a canonical foliation starting from a common sphere \( S \).

We first introduce the derivative of the geodesic parameter \( s \) in the canonical foliation.
Definition 2.24 We define the $S_v$-tangent 1-form $\Upsilon$ to be

$$\Upsilon := \nabla_{\mathcal{F}}.$$ 

We have the following proposition.

Proposition 2.25 (Null frame comparison) Let $(e'_A)_{A=1,2}$ and $(e_A)_{A=1,2}$ be Fermi propagated null frames respectively for the geodesic and the canonical foliation, such that $e'_A = e_A$ on $S$. For $A = 1, 2$, it holds that

$$e'_A = e_A - \Upsilon_A L,$$  \hspace{1cm} (2.23)

and

$$L' = L - 2\Upsilon_A e_A + |\Upsilon|^2 L.$$  \hspace{1cm} (2.24)

Proof It is straight-forward to verify that the vectors $e_A - \Upsilon_A L$, $A = 1, 2$, are $S'_s$-tangent vector fields that coincide with $e'_A$ on $S$. Moreover, for any $X \in TS'_s$, we have

$$g(X, D_L(e_A - \Upsilon_A L)) = g(X, \eta_A L - L(\Upsilon_A)L) = 0.$$ 

Thus, the vectors $e_A - \Upsilon_A L$ are Fermi propagated with respect to the geodesic foliation and we deduce that they coincide with $e'_A$ on $\mathcal{H}$. One then directly checks that the vector field

$$Z := L - 2\Upsilon_A e_A + |\Upsilon|^2 L$$ satisfies $g(Z, e'_A) = g(Z, Z) = 0$ and $g(Z, L) = -2$. \hfill $\square$

We have the following definition of the projection of tensors from one foliation to another, see also Section 2.2 in [1].

Definition 2.26 Let $\phi'$ be a $S'_s$-tangent $r$-tensor. We define the projection $(\phi')^{\dagger}$ to be the $S_v$-tangent $r$-tensor defined by

$$(\phi')^{\dagger}_{A_1 \cdots A_r} = (\phi')^{\dagger}(e_{A_1}, \cdots, e_{A_r}) := \phi'(e'_{A_1}, \cdots, e'_{A_r}) = \phi'_{A_1 \cdots A_r}.$$ \hspace{1cm} (2.25)

Reciprocally, for $\phi$ a $S_v$-tangent $r$-tensor, we define the projection $(\phi)^\sharp$ to be the $S'_s$-tangent $r$-tensor defined by

$$(\phi)^\sharp_{A_1 \cdots A_r} = (\phi)^\sharp(e_{A_1}, \cdots, e_{A_r}) := \phi(e_{A_1}, \cdots, e_{A_r}) = \phi_{A_1 \cdots A_r}.$$ \hspace{1cm} (2.26)

Remark 2.27 The projection of tensors from one foliation to another does not depend on the choice of Fermi propagated frames.
We introduce the following projection of $\Upsilon$.

**Definition 2.28** We define the $S'_s$-tangent 1-form $\Upsilon'$ to be

$$\Upsilon' := -(\Upsilon)^\dagger.$$ 

We have the following relation between $\Upsilon'$ and the derivative of $v$ in the geodesic foliation.

**Lemma 2.29** We have

$$\Upsilon' = \Omega^{-1} \nabla v.$$ 

**Proof** We have

$$\nabla v = (e_A - \Upsilon_A L)v = -\Omega \Upsilon_A = \Omega \Upsilon'_A,$$

as desired. \[\square\]

We have the following correspondences for $S_v$-tangential derivatives of projected tensors.

**Proposition 2.30** (Projection calculus) Let $r \geq 1$ be an integer. Let $\phi'$ be an $S'_s$-tangent $r$-tensor. Then it holds that

$$\nabla_L (\phi')_{A_1 \ldots A_r} = (\nabla_L (\phi'))_{A_1 \ldots A_r},$$

$$\nabla_A (\phi')_{A_1 \ldots A_r} = (\nabla_A (\phi'))_{A_1 \ldots A_r} + \Upsilon_A (\nabla_L (\phi'))_{A_1 \ldots A_r}$$

$$+ \chi_{AA_i} \Upsilon_B (\phi')_{A_1 \ldots B \ldots A_r} - \chi_{AB} \Upsilon_A (\phi')_{A_1 \ldots B \ldots A_r}.$$ (2.27)

Similarly, for a given $S_v$-tangent $r$-tensor $\phi$, it holds that

$$\nabla_L (\phi')_{A_1 \ldots A_r} = (\nabla_L (\phi'))_{A_1 \ldots A_r},$$

$$\nabla_A (\phi')_{A_1 \ldots A_r} = (\nabla_A (\phi'))_{A_1 \ldots A_r} + \gamma'_{AA_i} (\nabla_L (\phi'))_{A_1 \ldots A_r}$$

$$- \chi_{AB} \gamma'_{BA_1} (\phi')_{A_1 \ldots B \ldots A_r}.$$ (2.28)

**Proof** First, using that both frames $(e_1, e_2)$ and $(e'_1, e'_2)$ are Fermi propagated, we have

$$\nabla_L (\phi')_{A_1 \ldots A_r} = L((\phi')^\dagger (e_{A_1}, \ldots, e_{A_r}))$$

$$= L(\phi'(e'_{A_1}, \ldots, e'_{A_r}))$$

$$= \nabla_L (\phi'(e'_{A_1}, \ldots, e'_{A_r}))$$

$$= (\nabla'_L (\phi'))_{A_1 \ldots A_r}.$$
Second,
\[ \nabla_A (\phi')_{A_1...A_r}^\dagger = e_A (\phi'_{A_1...A_r}) - (\phi')^\dagger (e_{A_1}, \ldots, \nabla_A e_{A_r}, \ldots, e_{A_r}), \]
and
\[ e_A (\phi'_{A_1...A_r}) = e'_A (\phi'_{A_1...A_r}) + \gamma_A L (\phi'_{A_1...A_r}) \]
\[ = \nabla^\prime_A \phi'_{A_1...A_r} + \phi' (e'_{A_1}, \ldots, \nabla_A e'_{A_1}, \ldots, e'_{A_r}) + \gamma_A \nabla'_L \phi'_{A_1...A_r}. \]

Therefore, we deduce
\[ \nabla_A (\phi')_{A_1...A_r}^\dagger = (\nabla' \phi')_{A_1...A_r}^\dagger + \gamma_A (\nabla'_L \phi')_{A_1...A_r}^\dagger + \phi' (e'_{A_1}, \ldots, \nabla_A e'_{A_1}, \ldots, e'_{A_r}). \] (2.29)

To compute the third term on the right-hand side of (2.29), write
\[ g(\nabla_A e_A, e'_B) = g(D_A e_A, \frac{1}{2} \chi_{AA_i} L - \frac{1}{2} \chi_{AA_i} L, e'_B) = g(D_A e_A, e'_B) - \chi_{AA_i} \gamma_B \]
\[ = g(D_A e_A, \gamma'_A L (e'_{A_1} + \gamma_A L) e'_B) = \gamma_A, g(D_A e'_A, e'_B) - \chi_{AA_i} \gamma_B \]
\[ = g(D_A e'_A, e'_B) + \gamma_A, \chi_{AB} - \chi_{AA_i} \gamma_B. \]

where we used (2.5), (2.24) and the fact that both frames are Fermi propagated. Moreover, it follows from (2.23) that \( \chi'_{AA_i} = \chi_{AA_i} \), and therefore
\[ \phi' (e'_{A_1}, \ldots, \nabla_A e'_{A_1} - g(\nabla_A e_A, e'_B) e'_B, \ldots, e'_{A_r}) \]
\[ = - \gamma_A, \chi_{AB} (\phi')_{A_1...A_r}^\dagger + \chi_{AA_i} \gamma_B (\phi')_{A_1...A_r}^\dagger. \] (2.30)

Plugging (2.30) into (2.29) concludes the proof of (2.27). In view of (2.23) and Lemma 2.29, the proof of (2.28) follows by replacing \( \gamma_A \) by \( \gamma'_{A} \). This finishes the proof of Proposition 2.30. \( \square \)

We have the following transport equation for \( \gamma \).

**Lemma 2.31** We have
\[ \nabla_L \gamma = - \nabla (\log \Omega) - \chi \cdot \gamma. \] (2.31)

**Proof** Using commutation formula (2.10), relation (2.4) and \( Ls = 1 \), we have
\[ \nabla_L \gamma_A = \nabla_L \nabla s \]
\[ = \nabla L(s) + [\nabla_L, \nabla] A_s \]

Springer
\[-\nabla_A (\log \Omega) - \chi_{AB} \nabla_B s,\]
as desired. \qed

We have the following comparison between null curvature components and connection coefficients and projected null curvature components and connections coefficients. The proofs are postponed to Appendix A.

**Proposition 2.32** (Null curvature component comparison) The following relations hold.

\[
\alpha_{AB} = (\alpha')^\dagger_{AB}, \quad \beta_A = (\beta')^\dagger_A + \Upsilon_B (\alpha')^\dagger, \\
\rho = \rho' + 2 \Upsilon \cdot (\beta')^\dagger + \Upsilon \cdot (\gamma')^\dagger, \\
\sigma = \sigma' - 2 \Upsilon \cdot (\gamma')^\dagger - \Upsilon \cdot \gamma \cdot (\alpha')^\dagger, \\
\rho = \rho' + 2 \Upsilon \cdot (\beta')^\dagger, \\
\beta_A = (\beta')^\dagger_A - 3 \rho' \Upsilon A + 3 \sigma' \Upsilon A - 2 \left( \Upsilon \cdot (\beta')^\dagger \right) \Upsilon A - 4 \left( \Upsilon \cdot (\gamma')^\dagger \right) \Upsilon A + |\Upsilon|^2 \beta_A' - 2 \left( \Upsilon \cdot \gamma \cdot (\alpha')^\dagger \right) \Upsilon A + |\Upsilon|^2 \gamma \cdot (\alpha')^\dagger_A. \tag{2.32}
\]

**Proposition 2.33** (Null connection coefficients comparison) The following relations hold.

\[
\chi_{AB} = (\chi')^\dagger_{AB}, \\
\zeta_A = (\zeta')^\dagger_A + (\chi')^\dagger_{AB} \Upsilon_B, \\
\eta_A = (\eta')^\dagger_A + \nabla L \Upsilon A, \\
\chi_{AB} = (\chi')^\dagger_{AB} + 2 \Upsilon A (\eta')^\dagger_B - 2 \Upsilon B (\zeta')^\dagger_A + 2 \nabla A \Upsilon B - |\Upsilon|^2 \chi_A'. \tag{2.33}
\]

**2.9 Norms on \(H\)**

In this section, we define norms on \(H\). Throughout this section, we denote by \((\tilde{S}_v)_{1 \leq v \leq v^*}\) either the geodesic foliation \((S'_v)\) or the canonical foliation \((S_v)\).

**Definition 2.34** (\(\tilde{S}_v\)-mixed norms) Let \(v^* \geq 1\). Let \(F\) be an \(\tilde{S}_v\)-tangent tensor. We define the mixed norms on \(H\) with respect to the foliation \((\tilde{S}_v)_{1 \leq v \leq v^*}\),

\[
\| F \|_{L^p_{\tilde{S}_v}([1,v^*]) L^q} := \left( \int_1^{v^*} \| F \|_{L^p_{\tilde{S}_v}(\tilde{S}_v) \, d\tilde{v}} \right)^{\frac{1}{p}}, \\
\| F \|_{L^p L^q_{\tilde{S}_v}([1,v^*])} := \left\| \left( \int_1^{v^*} |F|^p \, d\tilde{v} \right)^{\frac{1}{p}} \right\|_{L^q(S_1)}. 
\]
Definition 2.35 Let $v^* \geq 1$ and let $F$ be an $\tilde{S}_v$-tangent tensor. Define

$$N^v_{1, v^*}(F) := \| F \|_{H^{1/2}(S_1)} + \| F \|_{L^2_v((1, v^*))^2} + \| \tilde{\nabla} F \|_{L^2_v((1, v^*))^2} + \| \tilde{\nabla}_L F \|_{L^2_v((1, v^*))^2},$$

where $\tilde{\nabla}$ and $\tilde{\nabla}_L$ denote the induced covariant derivatives on $\tilde{S}_v$. Moreover, for $m \geq 1$, define

$$N^v_{m, [1, v^*]}(F) := \sum_{k \leq m-1} \| \tilde{\nabla}^k F \|_{H^{1/2}(S_1)} + \sum_{k \leq m} \| \tilde{\nabla}^k F \|_{L^2_v((1, v^*))^2},$$

where $\tilde{\nabla} \in \{\tilde{\nabla}, \tilde{\nabla}_L\}$. We refer to Section 3.2 for a precise definition of the space of tensors $H^{1/2}(S_1)$.

2.10 Weak regularity of 2-spheres

In this section, we define the weak regularity assumption on $S_1$, see [26] Section 2.4.

Definition 2.36 Let $N \geq 1$ be an integer and $c > 0$ a real number. A Riemaniann 2-sphere $(S, g)$ is weakly regular with constants $N, c$ if it can be covered by $N$ coordinate patches $(x^1, x^2)$ with a partition of unity $\eta$ adapted to the coordinate patches and with functions $0 \leq \tilde{\eta} \leq 1$ that are compactly supported in the patches and equal to 1 on the support of $\eta$, and if on each patch there exists an orthonormal frame $(e_1, e_2)$ such that for $a, b = 1, 2$ and $A = 1, 2$,

$$c^{-1} \leq \sqrt{\det g} \leq c,$$

$$c^{-1} \left( (\xi^1)^2 + (\xi^2)^2 \right) \leq g_{ab} \xi^a \xi^b \leq c \left( (\xi^1)^2 + (\xi^2)^2 \right), \quad \forall (\xi^1, \xi^2) \in \mathbb{R}^2,$$

$$|\partial_{x^a} \eta| + |\partial_{x^a} \partial_{x^b} \eta| + |\partial_{x^a} \tilde{\eta}| \leq c,$$

$$\|\nabla \partial_{x^a} \|_{L^2(S)} + \|\nabla e_A\|_{L^4(S)} \leq c.$$

2.11 Norms for the geodesic and canonical foliation geometry

In this section, we introduce norms to measure the geometry of the geodesic foliation and the canonical foliation on $\mathcal{H}$ at the level of bounded $L^2$ curvature. The definitions of the Besov spaces $B^0(S)$ and fractional Sobolev space $H^{1/2}(S)$ are postponed to Section 3.2.

Norms for null connection coefficients of the geodesic foliation on $S_1$.

$$I'_{S_1} := \| \text{tr} \chi' - 2 \|_{L^\infty(S_1)} + \| \nabla \text{tr} \chi' \|_{B^0(S_1)} + \| \text{tr} \chi' + 2 \|_{L^\infty(S_1)} + \| \nabla \text{tr} \chi' \|_{L^2(S_1)} + \| \mu' \|_{B^0(S_1)} + \| \varsigma' \|_{H^{1/2}(S_1)} + \| \hat{\chi}' \|_{H^{1/2}(S_1)} + \| \hat{\chi}' \|_{H^{1/2}(S_1)}.$$

Springer
Norms for null connection coefficients of the canonical foliation on $S_1$.

$$\mathcal{I}_{S_1} := \| \text{tr} \chi - 2 \|_{L^\infty(S_1)} + \| \text{tr} \chi + 2 \|_{L^\infty(S_1)} + \| \nabla \text{tr} \chi \|_{B_0^0(S_1)} + \| \nabla \text{tr} \chi \|_{L^2(S_1)}$$

$$+ \| \mu \|_{B_0^0(S_1)} + \| \xi \|_{H^{1/2}(S_1)} + \| \nabla \xi \|_{H^{1/2}(S_1)} + \| \nabla \xi \|_{H^{1/2}(S_1)}$$

$$+ \| \nabla \log \Omega \|_{H^{1/2}(S_1)} + \| \nabla \log \Omega \|_{L^2(S_1)}$$

$$+ \| \Omega - 1 \|_{L^\infty(S_1)} + \| \mu \|_{L^2(S_1)} .$$

Norms for null connection coefficients of the geodesic foliation on $\mathcal{H}$.

$$\mathcal{O}_{[1,s^*]} := \left\| \text{tr} \chi' - \frac{2}{s} \right\|_{L^\infty(S_1)} + \| v \|_{L^\infty(S_1)} + \| v \|_{L^2(S_1)}$$

$$+ \| \nabla \text{tr} \chi' \|_{B_0^0(S_1)} + \| \nabla \text{tr} \chi' \|_{L^2(S_1)}$$

$$+ \| \nabla \log \Omega \|_{H^{1/2}(S_1)} + \| \nabla \log \Omega \|_{L^2(S_1)}$$

$$+ \| \Omega - 1 \|_{L^\infty(S_1)} + \| \mu \|_{L^2(S_1)} .$$

Norms for null connection coefficients of the canonical foliation on $\mathcal{H}$.

$$\mathcal{O}_{[1,v^*]} := \left\| \text{tr} \chi - \frac{2}{v} \right\|_{L^\infty(S_1)} + \| v \|_{L^\infty(S_1)} + \| v \|_{L^2(S_1)}$$

$$+ \| \nabla \text{tr} \chi' \|_{B_0^0(S_1)} + \| \nabla \text{tr} \chi' \|_{L^2(S_1)}$$

$$+ \| \nabla \log \Omega \|_{H^{1/2}(S_1)} + \| \nabla \log \Omega \|_{L^2(S_1)}$$

$$+ \| \Omega - 1 \|_{L^\infty(S_1)} + \| \mu \|_{L^2(S_1)} .$$

Norms for null curvature components of the geodesic foliation on $\mathcal{H}$.

$$\mathcal{R}'_{[1,s^*]} := \| \alpha' \|_{L^2(S_1)} + \| \beta' \|_{L^2(S_1)} + \| \rho' \|_{L^2(S_1)}$$

$$+ \| \sigma' \|_{L^2(S_1)} + \| \beta' \|_{L^2(S_1)}$$

Norms for null curvature components of the canonical foliation on $\mathcal{H}$.

$$\mathcal{R}_{[1,v^*]} := \| \alpha \|_{L^2(S_1)} + \| \beta \|_{L^2(S_1)} + \| \rho \|_{L^2(S_1)}$$

$$+ \| \sigma \|_{L^2(S_1)} + \| \beta \|_{L^2(S_1)}$$
2.12 Main results

The following is the main result of this paper.

**Theorem 2.37 (Existence and control of the canonical foliation, version 2)** Let \((M, g)\) be a smooth spacetime and \(\mathcal{H}\) be a smooth regular null hypersurface emanating from a smooth spacelike 2-sphere \(S\). Assume that there exists a 2-sphere geodesic foliation \((S^{'s})\) starting at \(S\) with \(s = 1\), well-defined and smooth up to \(s = 5/2\). Let \(N \geq 1\) be an integer and \(c > 0\) be a real number and assume that \(S\) is weakly regular with constants \(N, c\). Assume moreover that for some \(\varepsilon > 0\),

\[
\mathcal{I}'_{S^{'1}} + \mathcal{R}'_{[1,5/2]} + \mathcal{O}'_{[1,5/2]} \leq \varepsilon. \tag{2.38}
\]

Then, there is a universal constant \(\varepsilon_0 > 0\) such that if \(0 < \varepsilon < \varepsilon_0\), the following holds.

(1) **Existence and uniqueness of the canonical foliation.** There exists a unique canonical foliation (see Definition 2.17), well-defined from \(v = 1\) to \(v = 2\), and we have the following comparison estimate with respect to the geodesic foliation,

\[
\|\Omega - 1\|_{L^\infty(\mathcal{I}_{[1,2]}^\infty L^\infty)} \lesssim \varepsilon, \quad \|\Upsilon\|_{L^\infty(\mathcal{I}_{[1,2]}^\infty L^\infty)} \lesssim \varepsilon. \tag{2.39}
\]

(2) **\(L^2\)-regularity.** There is a constant \(C = C(N, c) > 0\) such that the canonical foliation is uniformly weakly regular with constants \(N, C\) for \(v = 1\) to \(v = 2\), and moreover,

\[
\mathcal{I}_{S^{'1}} + \mathcal{R}_{[1,2]} + \mathcal{O}_{[1,2]} \lesssim \varepsilon. \tag{2.40}
\]

(3) **Smoothness.** The canonical foliation is smooth up to \(v = 2\).

**Remarks**

(1) The comparison estimate (2.39) implies in particular that \(|s - v| \lesssim \varepsilon\), so that the foliations remain close to each other.

(2) Using the conclusions of [15] (see also Theorem 1.3), we have that under the assumption \(\mathcal{I}'_{S^{'1}} + \mathcal{R}'_{[1,5/2]} \leq \varepsilon\), control of the geodesic connection coefficient norm \(\mathcal{O}'_{[1,5/2]} \lesssim \varepsilon\) can be obtained. The smallness hypothesis (2.38) can therefore be replaced by \(\mathcal{I}'_{S^{'1}} + \mathcal{R}'_{[1,5/2]} \leq \varepsilon\) and involves only \(L^2\)-norms of curvature components on \(\mathcal{H}\) and low regularity connection coefficients bounds on \(S_1\).

(3) Using the a priori estimates from the previous remark, together with a topological assumption on the hypersurface \(\mathcal{H}\) could lead to existence and non-degeneracy for the geodesic foliation from \(s = 1\) to \(s = 5/2\). For simplicity, we rather make the existence and smoothness of the geodesic foliation from \(s = 1\) to \(s = 5/2\) an assumption.

(4) The null curvature components are essentially invariant by a change of foliation (see Proposition 2.32) and so is the smallness assumption \(\mathcal{R} \leq \varepsilon\). Regarding the previous remarks, it is consistent however to assume it for the geodesic foliation, since we rely on its existence and on the control on the connection coefficient norm.
\( \mathcal{O}'_{[1,5/2]} \lesssim \varepsilon \) to obtain existence for the canonical foliation together with bounds for the corresponding canonical foliation connection coefficients.

(5) The equations for the canonical foliation reduce to a system of coupled quasilinear elliptic and transport equations on \( \mathcal{H} \) (see equations (2.22b)-(2.22n)), having curvature components as source terms, which are essentially invariant under a change of foliation. Thus, Theorem 2.37 can be seen as a small data time 1 existence result for which the smallness is measured in terms of \( L^2(\mathcal{H}) \)-norms of the null curvature components.

(6) The desired bound for \( \text{tr} \chi \) is obtained as part of the estimates (2.40), which are all needed to obtain existence and control of the canonical foliation on the interval \( 1 \leq v \leq 2 \).

(7) Here and in the rest of the paper, smooth means \( C^\infty \) with respect to the \( C^\infty \)-topology of the manifolds \( \mathcal{M}, \mathcal{H}, \) etc.

(8) The smoothness of the canonical foliation is a consequence of the smoothness of the geodesic foliation and is obtained by higher regularity comparison estimates (see Step 3 in Section 2.13 and Section 5). Since we only work with smooth foliations, we did not seek any sharpness in these higher regularity estimates. For example in the proof of Proposition 2.41, we assume \( C^k \)-regularity with \( k \) arbitrarily large of the geodesic foliation to prove \( C^{k'} \)-regularity with \( k' \ll k \) of the canonical foliation.

### 2.13 Proof of Theorem 2.37

The proof of Theorem 2.37 relies on a bootstrap argument which we set up and prove in this section, assuming for the moment the local existence result and estimates that will be proved in Sections 4, 5 and 6.

Let \( D > 0 \) be a fixed (large) constant and let \( v^* \in [1, 2] \). We say that a foliation \( (S_v)_{1 \leq v \leq v^*} \) satisfies the bootstrap assumptions \( \text{BA} \mathcal{D} \varepsilon, [1, v^*] \), if

\[
\| \Omega - 1 \|_{L^\infty([1, v^*])L^\infty} + \| \nabla \|_{L^\infty([1, v^*])L^\infty} + \mathcal{O}_{[1, v^*]} \leq D \varepsilon.
\]

We say that a foliation \( (S_v)_{1 \leq v \leq v^*} \) is regular if \( v \) is a \( C^1 \)-function and if

\[
\sum_{l \leq 5} \left( \| \nabla^l \Omega \|_{L^\infty([1, v^*])L^2} + \| \nabla^l (\Omega - 1) \|_{L^\infty([1, v^*])L^2} \right) < \infty.
\]

We define \( V \in [1, 2] \) as

\[
V := \sup_{v^* \in [1, 2]} \left\{ \text{There exists a regular function } v \text{ on } \mathcal{H} \text{ taking values from } 1 \text{ to } v^* \text{ and defining a canonical foliation such that the assumptions } \text{BA} \mathcal{D} \varepsilon, [1, v^*] \text{ are satisfied} \right\},
\]

and we show in the rest of this section that \( V = 2 \).
Step 1 It holds that $V > 1$. Indeed, this follows by the next local existence result.

Theorem 2.38 (Local existence and continuation for the canonical foliation) Let $(\mathcal{M}, g)$ be a smooth vacuum spacetime and $\mathcal{H} \subset \mathcal{M}$ a smooth null hypersurface foliated by a well-defined and smooth geodesic foliation $(S'_s)_{1 \leq s \leq s^*}$. Let $v^* \in [1, 2]$ be a real number. Assume there exists a $C^1$-function $v$ on $\mathcal{H}$ taking values from 1 to $v^*$ and defining a canonical foliation. Assume moreover that

$$
\|s\|_{H^5(S_{v^*})} < \infty, \quad \|\Omega - 1\|_{L^\infty(S_{v^*})} < \frac{1}{100}
$$

and that $S_{v^*}$ is close to the Euclidean 2-sphere in a weak sense (see Definition 3.14). There exists $\delta > 0$ and a unique $C^1$-function $\tilde{v}$ taking values from 1 to $v^* + \delta$, coinciding with $v$ on $\{1 \leq v \leq v^*\}$ and such that $(S_{\tilde{v}})_{1 \leq \tilde{v} \leq v^* + \delta}$ is a canonical foliation.

Remarks on Theorem 2.38

1) In Theorem 2.38, the time of existence $\delta$ depends on $\sum_{l \leq 5} \|\nabla^l (s - v^*)\|_{L^2(S_{v^*})}$ and $\|g'\|_{C^5(\mathcal{H})}$ (see Section 6).

2) The proof of Theorem 2.38 is made by a fixed-point argument for a more general system of coupled quasilinear elliptic and transport equations and is detailed in Section 6. The required assumption $|\Omega - 1| < 1/100$ at the initial sphere $v = s = 1$ and its weak sphericality are consequences of the low regularity bounds (2.42) proved in Proposition 2.39 (see also Section 4.1 and Remark 4.2).

3) A local existence result for canonical foliations is proved in [22] but under the stronger smallness assumption

$$
\mathcal{R}'_2 \leq \varepsilon,
$$

where $\mathcal{R}'_2$ contain $L^\infty$-norms of curvature components on $\mathcal{H}$. Similarly, a local existence result was proved in [25] for general foliations under $L^\infty(\mathcal{H})$-smallness assumptions on the curvature. As such smallness conditions can not be assumed in our low regularity setting, we give a new proof of a stronger local existence result.

Step 2 For $v^* \in [1, 2]$ we can improve $BA_{D, [1, v^*]}$ to $BA_{D', [1, v^*]}$ for a real number $D' < D$. We first show that the assumptions of Theorem 2.37 imply that the canonical connection coefficients are controlled on $S_1$.

Proposition 2.39 (Connection coefficients bounds on $S_1$) Assume that for some real $\varepsilon > 0$,

$$
\mathcal{I}'_{S_1} \leq \varepsilon. \quad (2.41)
$$

There exists $\varepsilon_0 > 0$ small such that if $0 < \varepsilon < \varepsilon_0$, then we have

$$
\mathcal{I}_{S_1} \lesssim \varepsilon. \quad (2.42)
$$
The proof of Proposition 2.39 is carried out in Section 4.1 and goes by direct comparison between the geodesic and the canonical connection coefficients. Namely most of the coefficients are identical since the first 2-spheres of the two foliations coincide,

\[ S_{s=1} = S = S_{v=1}. \]

Then the next proposition shows that we can improve the bootstrap assumptions.

**Proposition 2.40** *(Low regularity estimates)* Assume that for some real number \( \varepsilon > 0 \)

\[
R'_{[1,5/2]} + O'_{[1,5/2]} \leq \varepsilon, \quad \mathcal{I}_S \leq \varepsilon.
\]

Let \( 1 < v^* \leq 2 \) and assume that the canonical foliation \((S_v)_{1 \leq v \leq v^*}\) is regular and satisfies the bootstrap assumptions \( BA_{D, [1,v^*]} \) with \( D > 0 \) a fixed constant. There exists \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \), then the canonical foliation satisfies the bootstrap assumptions \( BA_{D', [1,v^*]} \) for a real number \( D' < D \).

Proposition 2.40 is proved in Section 4. The first step is to show that under the bootstrap assumptions \( BA_{D, [1,v^*]} \), the null curvature components in the canonical foliation are comparable to the geodesic null curvature components

\[
R_{[1,v^*]} \lesssim R'_{[1,5/2]}.
\]

Then, under the bounds (2.43) and weak regularity of \( S_1 \) with constants \( N, c \), we can show that the foliation \((S_v)_{1 \leq v \leq v^*}\) is uniformly weakly regular and spherical with constants depending only on \( N, c, \varepsilon \) (see Definitions 3.1 and 3.14). At this level of regularity, calculus inequalities can be derived on \( \mathcal{H} \) with constants depending only on \( N, c, \varepsilon \). Using these inequalities together with the null structure equations (2.22b)-(2.22n), the bounds obtained on \( S_1 \) (2.42) and the obtained bounds for the null curvature components (2.43), it follows that there exists \( \varepsilon_0 > 0 \) small enough such that if \( 0 < \varepsilon < \varepsilon_0 \), then the bootstrap assumptions \( BA_{D, [1,v^*]} \) can be improved to \( BA_{D', [1,v^*]} \).

**Step 3** The canonical foliation is regular on \([1, V]\). Indeed, we show more generally the following proposition.

**Proposition 2.41** *(Higher regularity comparison estimates)* Assume that the geodesic foliation \((S'_s)_{1 \leq s \leq 5/2}\) is smooth and well-defined from \( 1 \leq s \leq 5/2 \), and assume that for \( 1 < v^* < 2 \) and for some real number \( \varepsilon > 0 \) the canonical foliation \((S_v)_{1 \leq v \leq v^*}\) is regular and satisfies the bootstrap assumptions \( BA_{[1,v^*]} \). There exists \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \) then, we have for all integers \( m \geq 0 \)

\[
\sum_{l+k \leq m} \left\| \nabla^l \nabla^k (s - v) \right\|_{L^2([1,v^*]) L^2} \lesssim C \left( \| g' \|_{C^{m+2}(\mathcal{H})}, m \right).
\]

The proof of Proposition 2.41 goes by standard Grönwall argument and is carried out in Section 5. In particular, for \( m = 6 \), this gives the desired regularity result in our continuity argument.

By continuity, we therefore deduce that the canonical foliation is regular on the full interval \([1, V]\). Additionally, using these higher regularity estimates, one can deduce the smoothness of the canonical foliation.
Step 4: The foliation can be continued past \( V \) for \( V < 2 \). Using the estimates from Step 2 and 3, the assumptions of the local existence and continuation Theorem 2.38 are satisfied and therefore we deduce that the canonical foliation can be extended past \( V \) which therefore implies that \( V \geq 2 \).

2.14 Organisation of the paper

The rest of the paper is organised as follows.

- In Section 3, we state the calculus prerequisites that hold under weak regularity conditions for a foliation on \( \mathcal{H} \).
- Section 4 is dedicated to the proof of the low regularity bounds for the canonical connection coefficients on the sphere \( S_1 \) and the improvement of the bootstrap assumptions.
- Section 5 is dedicated to the proof of higher regularity bounds for the canonical foliation.
- Section 6 is dedicated to the proof of the local existence Theorem 2.38.

3 Calculus prerequisites

In this section, we state the necessary calculus prerequisites for Sections 4, 5 and 6. The results are based on the pioneering works [15, 16] and [17] (see also [32]), with further improvements and simplifications taken from [26], whose presentation we shall follow and whose calculus results we shall use as a black box.

3.1 Uniform weak regularity of foliations

We now state the definition of uniform weak regularity for a foliation which allows to develop uniform calculus on the leaves of the foliation (see [26] Sections 3.3 and 4.3).

Definition 3.1 Let \( N \geq 1 \) be an integer and \( C > 0 \) a real number. Let \( v^* > 1 \) be a real number. We say that a foliation \( (S_v)_{1 \leq v \leq v^*} \) on \( \mathcal{H} \) is uniformly weakly regular with constants \( N, C \) if the 2-sphere \( S_1 \) is weakly regular with constants \( N, C \) in the sense of Definition 2.36, the following bounds are satisfied

\[
\| \Omega - 1 \|_{L^\infty} \leq 1/10, \quad \| \text{tr} \chi \|_{L^\infty} \leq C, \quad \| \tilde{\chi} \|_{L^\infty} \leq C, \quad \mathcal{N}_1(\chi) + \mathcal{N}_1(\Omega) \leq C,
\]

and there exists a \( S_v \)-tangent 3-tensor \( \Psi \) satisfying

\[
\nabla_L \Psi_{ABC} = \Omega \nabla_A (\Omega^{-1} \chi_{BC}) - \Omega \nabla_C (\Omega^{-1} \chi_{BA}),
\]
such that
\[ \| \Psi \|_{L^4 L^\infty_v} \leq C. \] (3.3)

**Remarks**

1. For simplicity these assumptions are stronger than necessary and imply in particular assumptions (F2) of Section 4.5 in [26] with constants \( N, C \) and \( B \equiv C \), since the tensor \( k \) in [26] reads in the present paper \( k \equiv \Omega^{-1} \chi \).
2. Under the assumptions (3.1) (3.2) (3.3), one can deduce that each 2-sphere \( S_v \) is weakly regular in the sense of Definition 2.36 with uniform constants \( N, C' \), where \( C'(N, C) > 0 \) (see Proposition 4.13 in [26]).
3. The assumption (3.3) is designed so that using the local frames \( (e_A)_A=1,2 \) defined on the first 2-sphere \( S_1 \), the regularity of their associated Fermi propagated frames on \( \mathcal{H} \) is transported, i.e. \( \| \nabla e_A \|_{L^4 L^\infty_v} \leq C \). This regularity is then sufficient in [26] for running a scalarisation procedure for tensorial estimates and then comparing geometric Besov norms for scalars to coordinate-based Besov norms (see Sections 4,5 and Appendix A in that paper).
4. From the weak regularity of each 2-sphere \( S_v \), we deduce in particular that
\[ C^{-1} \lesssim \sqrt{\det(g)} \lesssim C, \]
uniformly on \( \mathcal{H} \). As a consequence, for all \( S_v \)-tangent tensor \( F \) and for all \( 1 \leq p \leq q \leq \infty \), we have
\[ \| F \|_{L^q L^p_v} \lesssim \| F \|_{L^p_v L^q}, \]
where the constant depends only on \( N, C \).

**Notations.** Here and in the rest of this section, we take out any reference to \( v^* \) in the \( L^p L^q \)-norms for ease of notation and we moreover denote \( \mathcal{N}_m := \mathcal{N}_m^u \).

### 3.2 Littlewood-Paley theory and Besov spaces

In this section, we define Littlewood-Paley projections and Besov spaces on Riemannian 2-spheres \((S, g)\).

Let \( \Delta \) denote the Laplacian on \((S, g)\). Interpreting \(-\Delta\) as a positive self-adjoint unbounded operator acting on tensors in \( L^2(S) \), we have the spectral decomposition (see [26] for details)
\[ -\Delta = \int_0^\infty \lambda dE_\lambda. \]

We define the corresponding Littlewood-Paley operators as follows.

- Let \( \phi \in C^\infty(\mathbb{R}) \) be a function such that \( \text{supp} \phi \subset \{1/2 \leq |\xi| \leq 2 \} \) and
\[ \sum_{k \in \mathbb{Z}} \phi(2^{-2k} \xi) = 1 \text{ for all } \xi \in \mathbb{R} \setminus \{0\}. \]
For each $k \in \mathbb{Z}$, define the Littlewood-Paley operator acting on tensors in $L^2(S)$ by

$$
P_k = \phi(-2^{-2k} \triangle), \quad P_- = \delta_{\{0\}}(-\triangle),$$

where $\delta_{\{0\}}(-\triangle)$ denotes the $L^2$-projection onto the kernel of $-\triangle$.

For $k \in \mathbb{Z}$, define the aggregated operators

$$
P_{<k} = P_- + \sum_{l < k} P_l,$$

where the summation is in the strong operator topology. In particular,

$$
P_{<0} + \sum_{k \geq 0} P_k = 1. \quad (3.4)$$

Using the Littlewood-Paley operators, we next define Besov spaces.

**Definition 3.2** (Geometric tensorial Besov space) For a $S$-tangent tensor $F$ we define the norms

$$
\| F \|_{B^0(S)} := \sum_{k \geq 0} \| P_k F \|_{L^2(S)} + \| P_{<0} F \|_{L^2(S)},
$$

We have moreover the following $v$-integrated Besov spaces.

**Definition 3.3** (Geometric tensorial $v$-integrated Besov spaces) Define for a $S_v$-tangent tensor $F$ on $\mathcal{H}$,

$$
\| F \|_{P^0_v} := \sum_{k \geq 0} \| P_k F \|_{L^2_v L^2} + \| P_{<0} F \|_{L^2_v L^2},
$$

$$
\| F \|_{Q^{1/2}_v} := \left( \sum_{k \geq 0} 2^k \| P_k F \|_{L^\infty_v L^2}^2 + \| P_{<0} F \|_{L^\infty_v L^2}^2 \right)^{1/2}.
$$

**Remark 3.4** The space $B^0(S)$ corresponds to the $L^2(S)$-based Besov space on $S$ with parameters $s = 0$ and $a = 1$ in [26].

The $v$-integrated spaces $P^0_v$ and $Q^{1/2}_v$ correspond to the $L^2(S)$-based $v$-integrated Besov space with parameters respectively $s = 0$, $a = 1$ and $p = 2$ and $s = 1/2$, $a = 2$ and $p = \infty$ in [26].

Finally, set for real numbers $s \in \mathbb{R}$ and $S$-tangent tensors $F$,

$$
\| F \|_{H^s(S)} := \| (I - \Delta)^{s/2} F \|_{L^2(S)},
$$
where the fractional Laplace operator is defined as in [26]. With this definition, we have (see [26, p. 836])

$$\| F \|_{H^1(S)} \simeq \| \nabla F \|_{L^2(S)} + \| F \|_{L^2(S)} .$$

The next lemma is proved in Appendix B.

**Lemma 3.5** For an S-tangent tensor $F$, we have

$$\| \nabla F \|_{H^{-1/2}(S)} \lesssim \| F \|_{H^{1/2}(S)} .$$

### 3.3 Sobolev inequalities on 2-spheres

The next lemma is proved in Section 2.5 in [26].

**Lemma 3.6** (Classical Sobolev inequalities on $S$) Let $(S, g)$ be weakly regular 2-sphere with constants $N, C$. Then for a scalar function $f$ and for an S-tangent tensor $F$, we have

$$\| F \|_{L^4(S)} \lesssim \| \nabla F \|_{L^2(S)}^{1/2} \| F \|_{L^2(S)}^{1/2} + \| F \|_{L^2(S)},$$

$$\| F \|_{L^\infty(S)} \lesssim \| \nabla^2 F \|_{L^2(S)}^{1/2} \| F \|_{L^2(S)}^{1/2} + \| F \|_{L^2(S)},$$

$$\| F \|_{L^\infty(S)} \lesssim \| \nabla F \|_{L^4(S)} + \| F \|_{L^4(S)},$$

where the constants depend only on $N, C$.

The next lemma follows from Proposition 3.3 in [26].

**Lemma 3.7** (Besov-Sobolev inequalities on $S$) Let $(S, g)$ be a weakly regular 2-sphere with constants $N, C$. Then for an S-tangent tensor $F$, we have

$$\| F \|_{L^4(S)} \lesssim \| \nabla F \|_{L^4(S)},$$

$$\| F \|_{L^\infty(S)} \lesssim \| \nabla F \|_{L^4(S)} + \| F \|_{L^4(S)},$$

where the constant depends only on $N, C$.

### 3.4 Sobolev inequalities on $\mathcal{H}$

Let $\mathcal{H}$ be a null hypersurface.

**Lemma 3.8** (Classical Sobolev inequalities on $\mathcal{H}$) Let $(S_v)$ be a uniformly weakly regular foliation on $\mathcal{H}$ with constants $N, C$. Then for an $S_v$-tangent tensor $F$ on $\mathcal{H}$,

$$\| F \|_{L^4(S_v)} \lesssim \mathcal{N}_1(F),$$

$$\| F \|_{L^\infty(S_v)} \lesssim \mathcal{N}_1(F),$$

$$\| F \|_{L^5_vL^5_v} \lesssim \mathcal{N}_1(F),$$

$$\| F \|_{L^\infty_vL^\infty_v} \lesssim \mathcal{N}_1(\nabla F) + \mathcal{N}_1(F),$$

where the constants depend only on $N, C$.
\[ \| F \|_{L^2_1 L^1} \lesssim \| \nabla F \|_{L^2_1 L^2} \frac{1}{2} \| F \|_{L^2_1 L^2} + \| F \|_{L^2_1 L^2}. \] (3.9)

All constants in these estimates depend only on \( N, C. \)

**Proof** Estimate (3.5) is Proposition 4.15 in [26]. Estimate (3.5) is a direct consequence of (3.6). Estimate (3.7) is obtained in the proof of Proposition 4.15 in [26]. Estimates (3.8) and (3.9) are direct consequences of the estimates of Lemma 3.6. \( \square \)

We have the following Besov-Sobolev estimate, see Proposition 5.3 in [26].

**Lemma 3.9** *(Besov-Sobolev inequalities on \( \mathcal{H} *)* Let \( (S_v) \) be a uniformly weakly regular foliation on \( \mathcal{H} \) with constants \( N, C. \) Let \( F \) be a \( S_v \)-tangent tensor. We have

\[ \| F \|_{Q^1_1} \lesssim N_1(F), \]

where the constant depends only on \( N, C. \)

We have the following product estimate in Besov spaces, see Theorem 3.6 in [26].

**Lemma 3.10** *(Besov product estimates)* Let \( (S_v) \) be a uniformly weakly regular foliation on \( \mathcal{H} \) with constants \( N, C. \) Let \( F \) and \( G \) be two \( S_v \)-tangent tensors. We have

\[ \| FG \|_{P^0_v} \lesssim N_1(F)(\| G \|_{L^2_1 L^2} + \| \nabla G \|_{L^2_1 L^2}), \]

where the constant depends only on \( N, C. \)

### 3.5 Null transport equations on \( \mathcal{H} \)

We have the following \( L^p L^\infty_1 \)-estimates for solutions of null transport equations.

**Lemma 3.11** *(\( L^p L^\infty_1 \)-estimates for transport equations)* Let \( (S_v) \) be a uniformly weakly regular foliation of \( \mathcal{H} \) with constants \( N, C. \) Let \( \kappa \) be a real number. For \( F \) an \( S_v \)-tangent tensor satisfying on \( \mathcal{H} \)

\[ \nabla F + \kappa \text{tr} \chi F = W, \]

and for all \( 1 \leq p \leq \infty \), we have

\[ \| F \|_{L^p L^\infty_1} \lesssim \| F \|_{L^p(S_1)} + \| W \|_{L^p L^1_1}, \]

where the constant depends only on \( N, C, p, \kappa. \)

**Proof** The proof follows by applying Proposition 4.6 in [26] to the transport equation obtained for the renormalised quantity \( \exp \left( \int_1^{v_1} \Omega^{-1} \text{tr} \chi \, dv' \right) F \) and using the uniform weak regularity bounds (3.1) for \( \Omega \) and \( \text{tr} \chi \). Details are left to the reader. \( \square \)

**Remark 3.12** Proposition 4.6 in [26] does not require assumption (3.3). This will be used when proving that assumption (3.3) holds.
3.6 $L^\infty L^2_v$ trace estimate

By Theorem 5.7 in [26], we have the next trace estimate.

**Lemma 3.13** (Trace estimate) Let $(S_v)$ be a uniformly weakly regular foliation with constants $N, C$ and let $F$ be a $S_v$-tangent tensor such that

$$\nabla F = \nabla L P + E.$$

Then

$$\|F\|_{L^\infty L^2_v} \lesssim \mathcal{N}_1(P) + \|E\|_{P^0} + \mathcal{N}_1(F),$$

where the constant only depends on $N, C$.

**Proof** This is Theorem 5.7 in [26] together with the bounds (3.1) of Definition 3.1 and Sobolev Lemma 3.8. \qed

3.7 Uniform weak sphericality

In order to have uniform estimates for Hodge systems on 2-spheres $S_v$, we introduce the following definition of uniform weak sphericality (see Section 6.1 in [26]).

**Definition 3.14** Let $N \geq 1$ be an integer and $C, D_{sph} > 0$ be reals. We say that a 2-sphere $S$ is **weakly spherical** with constants $N, C, D_{sph}$ and radius $R$ if it is weakly regular in the sense of Definition 2.36 with constants $N, C$ and if the Gauss curvature $K$ of $S$ can be written as

$$K - \frac{1}{R^2} = \text{div} \Psi + \Theta,$$

where

$$\|\Psi\|_{H^{1/2}(S)} + \|\Theta\|_{L^2(S)} \leq D_{sph}.$$

We say that a foliation $(S_v)$ of $\mathcal{H}$ is **uniformly weakly spherical** with constants $N, C, D_{sph}$ if it is uniformly weakly regular in the sense of 3.1 with constants $N, C$ and such that the Gauss curvature $K$ of $S_v$ can be written as

$$K - \frac{1}{v^2} = \text{div} \Psi + \Theta,$$

where

$$\|\Psi\|_{Q^{1/2}_v} + \|\Theta\|_{L^\infty L^2_v} \leq D_{sph}.$$

**Remark 3.15** From the Definition 3.3 of the Besov space $Q^{1/2}_v$, it is clear that every 2-sphere $S_v$ of a uniformly weakly spherical foliation is weakly spherical with radius $v$ and uniform constants.
3.8 Bochner identities on 2-spheres and consequences

We first recall the Bochner identity on spheres (see [15], p. 483 and p. 488).

Lemma 3.16 (Bochner identities) Let $(S, g)$ be a Riemannian 2-sphere. For scalar functions $f$ on $S$, we have

$$
\int_S |\nabla^2 f|^2 = \int_S |\Delta f|^2 - \int_S K|\nabla f|^2,
$$

where $K$ denotes the Gauss curvature of $S$. For an $S$-tangent 1-form $F$, we have

$$
\int_S |\nabla^2 F|^2 = \int_S |\Delta F|^2 - 2\int_S K|\nabla F|^2 + \int_S K|\text{div} F|^2 + \int_S K^2|F|^2.
$$

The Bochner identities of Lemma 3.16 imply the next estimates, see Section 6 in [26].

Lemma 3.17 (Bochner estimates) For a weakly spherical 2-sphere $S$ of radius 1 with constants $N, C, D_{sph}$, there exists a universal constant $D_0 > 0$ such that if $D_{sph} < D_0$, then the following holds.

1. For scalar function $f$ on $S$, we have

$$
\left\| \nabla^2 f \right\|_{L^2(S)} + \| \nabla f \|_{L^2(S)} \lesssim \| \Delta f \|_{L^2(S)}.
$$

2. For an $S$-tangent 1-form $F$, we have

$$
\left\| \nabla^2 F \right\|_{L^2(S)} \lesssim \| \Delta F \|_{L^2(S)} + \| \nabla F \|_{L^2(S)} + \| F \|_{L^2(S)}.
$$

Moreover, for a uniformly weakly spherical foliation $(S_v)$ of $\mathcal{H}$ with constants $N, C, D_{sph}$, with $D_{sph} < D_0$, the following holds.

1. For scalar functions $f$ on $\mathcal{H}$, we have

$$
\left\| \nabla^2 f \right\|_{L^2_v(L^2)} + \| \nabla f \|_{L^2_v(L^2)} \lesssim \| \Delta f \|_{L^2_v(L^2)}.
$$

2. For a $S_v$-tangent 1-form $F$, we have

$$
\left\| \nabla^2 F \right\|_{L^2_v(L^2)} \lesssim \| \Delta F \|_{L^2_v(L^2)} + \| \nabla F \|_{L^2_v(L^2)} + \| F \|_{L^2_v(L^2)}.
$$

3.9 Hodge systems on 2-spheres

In this section, we recall standard Hodge theory on Riemannian 2-spheres, see for example [8].
**Definition 3.18** Let \((S, g)\) be a Riemannian 2-sphere. We define the Hodge operators \(\mathcal{D}_1\) and \(\mathcal{D}_2\) that act respectively on \(S\)-tangent 1-forms \(\phi\) and on \(S\)-tangent traceless symmetric 2-tensors \(\psi\) by

\[
\mathcal{D}_1 \phi := (\text{div}\phi, \text{curl}\phi), \\
(\mathcal{D}_2 \psi)_A := \text{div}\psi_A.
\]

We denote by \(*\mathcal{D}_1\) and \(*\mathcal{D}_2\) their \(L^2\)-adjoint. For scalar functions \(f, h\) on \(S\) and for a \(S\)-tangent 1-form \(\phi\), we have

\[
*\mathcal{D}_1 (f, h) = -\nabla_A f + *\nabla_A h, \\
*\mathcal{D}_2 \phi = -\frac{1}{2} \nabla \hat{\Box} \phi.
\]

We have the following identities (see [8])

\[
\mathcal{D}_1 *\mathcal{D}_1 = -\Delta, \\
*\mathcal{D}_1 \mathcal{D}_1 = -\Delta + K, \\
\mathcal{D}_2 *\mathcal{D}_2 = -\frac{1}{2} \Delta - \frac{1}{2} K, \\
*\mathcal{D}_2 \mathcal{D}_2 = -\frac{1}{2} \Delta + K.
\]

We have the following invertibility properties and \(L^2\)-estimates for Hodge systems.

**Lemma 3.19 (Estimate for Hodge systems)** For a weakly spherical 2-sphere \(S\) of radius 1 with constants \(N, C, \mathcal{D}_{sph}\), there exists \(D_0 > 0\) such that if \(\mathcal{D}_{sph} < D_0\), the following holds.

- The operator \(\mathcal{D}_1\) is a bijection from the space of smooth vector fields onto the space of pairs of smooth functions with vanishing means.
- The operator \(\mathcal{D}_2\) is one-to-one from the space of smooth symmetric tracefree 2-tensors into the space of smooth vector fields.
- The operator \(*\mathcal{D}_1\) is a bijection from the space of pairs of smooth functions with vanishing means onto the space of smooth vector fields.
- For an \(S\)-tangent tensor of appropriate type \(F\), we have

\[
\left\| \nabla \mathcal{D}^{-1} F \right\|_{L^2(S)} + \left\| \mathcal{D}^{-1} F \right\|_{L^2(S)} \lesssim \left\| F \right\|_{L^2(S)}, \\
\left\| \nabla (*\mathcal{D}_1)^{-1} F \right\|_{L^2(S)} + \left\| (*\mathcal{D}_1)^{-1} F \right\|_{L^2(S)} \lesssim \left\| F \right\|_{L^2(S)},
\]

where \(\mathcal{D}^{-1} \in \{\mathcal{D}_1^{-1}, \mathcal{D}_2^{-1}\}\) with \(\mathcal{D}_1^{-1}\) and \(\mathcal{D}_2^{-1}\) denoting the inverses of \(\mathcal{D}_1\) and \(\mathcal{D}_2\) composed with the projections onto their respective domain, and the constants depend only on \(N, C\).
- For a uniformly weakly spherical foliation \((S_v)\) of \(\mathcal{H}\) with constants \(N, C, \mathcal{D}_{sph}\), there exists \(D_0 > 0\) such that if \(\mathcal{D}_{sph} < D_0\), the following holds. For an \(S_v\)-tangent
tensor of appropriate type $F$, we have

$$
\left\| \nabla \mathcal{P}^{-1} F \right\|_{L^2_v L^2} + \left\| \mathcal{P}^{-1} F \right\|_{L^2_v L^2} \lesssim \| F \|_{L^2_v L^2},
$$

$$
\left\| \nabla \left( \mathcal{P}_1 \right)^{-1} F \right\|_{L^2_v L^2} + \left\| \left( \mathcal{P}_1 \right)^{-1} F \right\|_{L^2_v L^2} \lesssim \| F \|_{L^2_v L^2},
$$

where $\mathcal{P}^{-1} \in \{ \mathcal{P}^{-1}_1, \mathcal{P}^{-1}_2 \}$ and the constants depend only on $N, C$.

**Proof** The fact that $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_1^*$ are one-to-one respectively on the space of smooth vector fields, smooth symmetric traceless 2-tensors and smooth pairs of functions with vanishing means is a direct consequence of Proposition 6.4 in [26]. That $\mathcal{P}_1, \mathcal{P}_1^*$ are onto follows from adjoint properties, see the discussion in [26, p. 875]. The results in the rest of the lemma are taken from Proposition 6.5 in [26].

We have the following elliptic estimates. The proof is postponed to Appendix B.

**Lemma 3.20** For a weakly spherical 2-sphere $S$ of radius 1 with constants $N, C, D_{sph}$, there exists a universal constant $D_0 > 0$ such that if $D_{sph} < D_0$, then the following holds.

Assume that $f$ satisfies the equation

$$
\Delta f = \text{div} P + h,
$$

then

$$
\| \nabla f \|_{L^2(S)} + \| f - \bar{f} \|_{L^2(S)} \lesssim \| P \|_{L^2(S)} + \| h \|_{L^{4/3}(S)}.
$$

Moreover, for a uniformly weakly spherical foliation $(S_v)$ of $\mathcal{H}$ with constants $N, C, D_{sph}$, with $D_{sph} < D_0$, the following holds. Assume that $f$ satisfies (3.11) then,

$$
\| \nabla f \|_{L^2_v L^2} + \| f - \bar{f} \|_{L^2_v L^2} \lesssim \| P \|_{L^2_v L^2} + \| h \|_{L^{4/3}_v L^2},
$$

where the constants only depend on $N, C$.

The next lemma follows from Theorem 6.8 in [26]. Its proof goes along the same lines as the proof of estimate (6.18) in Proposition 6.10 of [26].

**Lemma 3.21** (Elliptic estimates in fractional Sobolev spaces) For a weakly spherical 2-sphere $S$ with constants $N, C, D_{sph}$, there exists a constant $D_0 > 0$ such that if $D_{sph} < D_0$, then the following holds.

(1) Let $f$ be a scalar function on $S$ and $X$ a $S$-tangent 1-form satisfying

$$
\Delta f = \text{div} X, \quad \int_S f = 0.
$$
Then,
\[ \| \nabla f \|_{H^{1/2}(S)} + \| f \|_{L^2(S)} \lesssim \| X \|_{H^{1/2}(S)}. \]

(2) For an $S$-tangent 1-form $F$, we have
\[ \left\| \ast \vec{\Phi}_1^{-1} F \right\|_{H^{1/2}(S)} \lesssim \| F \|_{H^{-1/2}(S)}. \]

### 4 Low regularity estimates

This section is dedicated to the proofs of Propositions 2.39 and 2.40 (see Step 2 in Section 2.13). Let $(S'_s)_{1 \leq s \leq 5/2}$ denote the geodesic foliation on $\mathcal{H}$, and assume that for some $\varepsilon > 0$,

\[ \mathcal{I}'_{S_1} + \mathcal{R}'_{[1,5/2]} + \mathcal{O}'_{[1,5/2]} \leq \varepsilon. \]  \tag{4.1}

- In Section 4.1, we show that, for $\varepsilon$ small enough, we have $\mathcal{I}_{S_1} \lesssim \varepsilon$. This proves Proposition 2.39.
- From Section 4.2 on, we assume that $1 < v^* < 2$ is a real number and that the canonical foliation $(S_v)_{1 \leq v \leq v^*}$ is regular. We suppose further that for a fixed large constant $D$, for $1 \leq v \leq v^*$,

\[ \| \Omega - 1 \|_{L^\infty([1,v^*])L^\infty} + \| \Upsilon \|_{L^\infty([1,v^*])L^\infty} + \mathcal{O}_{[1,v^*]} \leq D\varepsilon. \]  \tag{4.2}

- We prove in Sections 4.2 to 4.7 that for $\varepsilon > 0$ sufficiently small, we can improve (4.2), i.e. we show that,

\[ \| \Omega - 1 \|_{L^\infty([1,v^*])L^\infty} + \| \Upsilon \|_{L^\infty([1,v^*])L^\infty} + \mathcal{O}_{[1,v^*]} + \mathcal{R}_{[1,v^*]} \leq D'\varepsilon \]  \tag{4.3}

for a constant $0 < D' < D$. This proves Proposition 2.40.

**Remark 4.1** To prove the improved bootstrap bound (4.3), we will show in Sections 4.2 to 4.7 that for $\varepsilon > 0$ sufficiently small, each norm $\| F \|$ composing the left-hand side of (4.3) is bounded by

\[ \| F \| \leq \hat{D}_F \varepsilon, \]

where $\hat{D}_F$ is a constant, independent of $D$. Defining

\[ D' := \sum_F \hat{D}_F, \]

we will then obtain the desired (4.3), provided that $D$ was chosen such that $D > D'$. In the following we will omit the mention of the quantities $F$ in the constants $\hat{D}$ for simplicity.
Notation. To ease the notations, we suppress all references to \(v^*\) in the following and we denote by \(\mathcal{N}_m := N_m^u\) and by \(\mathcal{N}_m^v := N_m^{s,5/2}\).

4.1 Bounds for the connection coefficients on \(S_1\)

This section is dedicated to the proof of Proposition 2.39.

By Proposition 2.33 and the fact that \(\Upsilon = 0\) on \(S_1\) since \(s = 1\), we have

\[
\chi = (\chi')^\dagger, \quad \chi = (\chi')^\dagger, \quad \zeta = (\zeta')^\dagger,
\]

which by using (2.16) also gives \(\mu = \mu'\).

Using the elliptic equation for \(\log \Omega\) (2.18), using the hypotheses estimates (4.1) and applying Lemma 3.17 and Lemma 3.21, we have

\[
\|\nabla \log \Omega\|_{H^{1/2}(S_1)} + \|\log \Omega\|_{L^2(S_1)} \lesssim \|\zeta\|_{H^{1/2}(S_1)} + \|\mu - \mathcal{M}\|_{L^2(S_1)} \lesssim \varepsilon. \tag{4.4}
\]

From this and relation (2.4), we deduce

\[
\|\eta\|_{H^{1/2}(S_1)} \lesssim \|\zeta\|_{H^{1/2}(S_1)} + \|\nabla \log \Omega\|_{H^{1/2}(S_1)} \lesssim \varepsilon.
\]

Moreover, using Sobolev Lemmas 3.6 and 3.7, together with the previous estimates, we have

\[
\|\log \Omega\|_{L^\infty(S_1)} \lesssim \|\nabla \log \Omega\|_{H^{1/2}(S_1)} + \|\log \Omega\|_{L^2(S_1)} \lesssim \varepsilon,
\]

which implies

\[
\|\Omega - 1\|_{L^\infty(S_1)} \lesssim \varepsilon.
\]

This finishes the proof of the bound (2.42) and of Proposition 2.39.

Remark 4.2 From the Gauss equation (2.22i) and the definition of the mass aspect function (2.16), we have on \(S_1\)

\[
K - 1 = \Theta + \text{d}\bar{\nu}\zeta,
\]

where

\[
\Theta := \frac{1}{2} (\text{tr} \chi - 2) - \frac{1}{2} (\text{tr} \chi + 2) - \frac{1}{4} (\text{tr} \chi - 2) (\text{tr} \chi + 2) + \mu.
\]

Using estimate (2.42), this implies that the 2-sphere \(S_1\) is weakly spherical with radius 1 and constants \(N, C, C'\varepsilon\) in the sense of Definition 3.14, where \(C' > 0\) is a universal constant. This assumption is required to apply Theorem 2.38 in Step 1 of Section 2.13.
4.2 Uniform weak regularity and sphericality of the canonical foliation

In this section, we show that under the bootstrap assumptions (4.2), the regularity of $S_1$ is propagated to the canonical foliation. More specifically, we show that the canonical foliation is uniformly weakly regular in Lemma 4.3 and uniformly weakly spherical in Lemma 4.4.

Lemma 4.3 (Uniform weak regularity) Let $S_1$ be a weakly regular 2-sphere with constants $N, c$. There exists $\varepsilon_0 > 0$ and $C(N, c) > 0$ such that for $\varepsilon < \varepsilon_0$ the canonical foliation is uniformly weakly regular with constants $N, C$.

Proof The bounds (3.1) directly follow from the bootstrap assumptions (4.2). Using Gauss-Codazzi equation (2.22m) for curl $\chi$, we deduce that for $\Psi$ a 3-tensor verifying equation (3.2), we have

$$\nabla_L \Psi_{ABC} = \Omega \nabla_A (\Omega^{-1} \chi_{BC}) - \Omega \nabla_C (\Omega^{-1} \chi_{BA})$$

$$= - \Omega^{-1} (\nabla_A \Omega \chi_{BC} - \nabla_C \Omega \chi_{BA})$$

$$+ \hat{\xi}_{AC} \text{curl} \chi_B$$

$$= - \Omega^{-1} (\nabla_A \Omega \chi_{BC} - \nabla_C \Omega \chi_{BA})$$

$$+ \hat{\xi}_{AC} \left( - \xi D^* \hat{\chi}_{DB} + \frac{1}{2} \text{tr} \chi^* \xi_B - \hat{\chi}_{BD} \eta_D \right).$$

Using the transport equation (2.22g) for $\zeta$, we have

$$\ast \beta_B = \frac{1}{2} \text{tr} \chi (- \ast \zeta_B + \ast \eta_B) + \ast \hat{\chi}_{BD} (\eta_D - \xi_D) - \nabla_L \ast \xi_B.$$

Therefore, we have

$$\nabla_L (\Psi_{ABC} - \hat{\xi}_{AC} \ast \xi_B) = E_{ABC},$$

with

$$E_{ABC} := - \Omega^{-1} (\nabla_A \Omega \chi_{BC} - \nabla_C \Omega \chi_{BA})$$

$$+ \hat{\xi}_{AC} \left( \text{tr} \chi^* \xi_B - \frac{1}{2} \text{tr} \chi^* \eta_B - \ast \hat{\chi}_{BD} \eta_D \right).$$

Using the transport Lemma 3.11 (see also Remark 3.12), the bootstrap assumptions (4.2) and choosing $\Psi_{ABC} := \hat{\xi}_{AC} \ast \xi_B$ on $S_1$, we have

$$\| \Psi - \hat{\xi}^* \xi \|_{L^4 L^\infty_v} \lesssim \| E \|_{L^4 L^1_v}$$

$$\lesssim \| \Omega^{-1} \|_{L^\infty_v L^1_v} \| \chi \|_{L^\infty_v L^1_v} \| \nabla \Omega \|_{L^2_v L^2_v}$$

$$+ \| \text{tr} \chi \|_{L^\infty_v L^\infty} \left( \| \xi \|_{L^4_v L^1_v} + \| \eta \|_{L^4_v L^1_v} \right).$$

\(\Box\) Springer
Using the bootstrap assumptions (4.2), we deduce
\[
\|\Psi\|_{L^4_v L^\infty_v} \lesssim \|\zeta\|_{L^4_v L^\infty_v} + (D\varepsilon)^2
\lesssim D\varepsilon + (D\varepsilon)^2
\leq C.
\]
This finishes the proof of Lemma 4.3. \(\square\)

**Lemma 4.4** (Uniform weak sphericality) There exists \(\varepsilon_0 > 0\), such that for \(\varepsilon < \varepsilon_0\), the canonical foliation is uniformly weakly spherical with constants \(N, C, D_{sph}\) and we have \(D_{sph} < D_0\), where \(D_0\) is the constant from Lemmas 3.17, 3.19, 3.20, and 3.21.

**Proof** We have by the Gauss equation (2.22i) and the definition of the mass aspect function (2.16)
\[
K - \frac{1}{v^2} = \Theta + d\nu \xi,
\]
where
\[
\Theta := \frac{1}{2} v^{-1} \left( \text{tr} \chi - \frac{2}{v} \right) - \frac{1}{2} v^{-1} \left( \text{tr} \chi + \frac{2}{v} \right) - \frac{1}{4} \left( \text{tr} \chi - \frac{2}{v} \right) \left( \text{tr} \chi + \frac{2}{v} \right) + \mu.
\]
Using bootstrap assumptions (4.2), we have
\[
\|\Theta\|_{L^\infty_v L^2_v} \lesssim \left\| \text{tr} \chi - \frac{2}{v} \right\|_{L^\infty_v L^\infty_v} + \left\| \text{tr} \chi + \frac{2}{v} \right\|_{L^\infty_v L^\infty_v} + \left\| \mu \right\|_{L^\infty_v L^2_v}
\lesssim D\varepsilon + (D\varepsilon)^2.
\]
And moreover, using bootstrap assumptions (4.2) and Lemma 3.9, we have
\[
\|\xi\|_{Q^{1/2}_v} \lesssim N_1(\xi) \lesssim D\varepsilon.
\]
Therefore
\[
\|\Theta\|_{L^\infty_v L^2_v} + \|\xi\|_{Q^{1/2}_v} \lesssim (D\varepsilon) + (D\varepsilon)^2 \leq D_0,
\]
for \(\varepsilon > 0\) small enough. This finishes the proof of Lemma 4.4. \(\square\)
The next lemma will allow us to compare norms for the geodesic and the canonical foliation components.

**Lemma 4.5 (Integral comparison)** For all \(1 \leq p \leq \infty\), and for all \(S_v\)-tangent tensors \(F\), we have

\[
\|F\|_{L^p_v([1,v^*])L^p} \lesssim \|F^\sharp\|_{L^p_v([1,5/2])L^p},
\]

where we extend \(F^\sharp\) by 0 on the domain of the geodesic foliation \((S'_s)_{1 \leq s \leq 5/2}\) not covered by the canonical foliation \((S_v)_{1 \leq v \leq v^*}\). To stress this foliation independence, we shall replace all \(L^p_v L^p\) and \(L^p_v L^p_s\) norms by \(L^p(H)\) in the rest of the paper.

**Proof** Let \((x^1, x^2)\) be local coordinates on \(S_1\). Extend \((x^1, x^2)\) on \(H\) by \(L(x^a) = 0\) on \(H\) for \(a = 1, 2\).

The triplets \((s, x^1, x^2)\) and \((v, x^1, x^2)\) are local coordinates on \(H\). Let \((\partial_s, \partial'_x, \partial'_x)\) and \((\partial_v, \partial'_x, \partial'_x)\) be the respective corresponding coordinate vector fields. Then the following relations hold,

\[
\partial_v = \Omega^{-1} \partial_s, \\
\partial'_x = \partial'_x - \Omega^{-1} (\partial'_x v) \partial_s \text{ for } a = 1, 2.
\]

In particular,

\[
g'^{ab} = g(\partial'_x a, \partial'_x b) = g(\partial_x a, \partial_x b) = g^{ab} \text{ for } a, b = 1, 2,
\]

and

\[
\gamma := \sqrt{\det(g'^{ab})} = \sqrt{\det(g^{ab})},
\]

where the indices \(a, b\) with \(a, b = 1, 2\) correspond to evaluation with respect to the coordinate vector fields \(\partial_x a, \partial_x b\). Performing a change of variable in the integrals, using the previous relations, and the bootstrap assumption (4.2) on \(\Omega\), we therefore deduce

\[
\|F\|_{L^p_v([1,v^*])L^p} = \int_v \int_{S_v} |F|^p \, dv \\
= \int_s \int_{S'_s} |F^\sharp|^p \Omega \, ds \\
\lesssim \int_s \int_{S'_s} |F^\sharp|^p \, ds \\
\lesssim \|F^\sharp\|_{L^p_v([1,5/2])L^p},
\]

as desired. \(\square\)
4.3 Bounds for the null curvature components of the canonical foliation on $\mathcal{H}$

In this section, we estimate the canonical curvature components $(\alpha, \beta, \rho, \sigma, \beta)$ and $(\tilde{\rho}, \tilde{\sigma}, \tilde{\beta})$ on $\mathcal{H}$ by comparing them to the geodesic components.

Lemma 4.6 For $\varepsilon > 0$, sufficiently small, it holds that

$$\|\alpha\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} + \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} \leq \hat{D}\varepsilon. \quad (4.5)$$

Proof By Proposition 2.32, it holds schematically that

$$R = R' + (\Upsilon) R' + (\Upsilon)^2 R' + (\Upsilon)^3 R',$$

where

$$R \in \{\alpha, \beta, \rho, \sigma, \beta\} \text{ and } R' \in \{\alpha', \beta', \rho', \sigma', \beta'\}.$$

Using the above and the bootstrap assumptions (4.2), we have

$$\|R\|_{L^2(\mathcal{H})} \lesssim \|R'\|_{L^2(\mathcal{H})} + \|\Upsilon\|_{L^{\infty}(\mathcal{H})} \|R'\|_{L^2(\mathcal{H})} + \|\Upsilon\|_{L^{\infty}(\mathcal{H})}^2 \|R'\|_{L^2(\mathcal{H})}$$

$$+ \|\Upsilon\|_{L^{\infty}(\mathcal{H})}^3 \|R'\|_{L^2(\mathcal{H})} \lesssim \varepsilon + (D\varepsilon)\varepsilon + (D\varepsilon)^2 \varepsilon + (D\varepsilon)^3 \varepsilon \leq \hat{D}\varepsilon.$$

This finishes the proof of Lemma 4.6.

Moreover, we have the following estimates for the renormalised canonical curvature components.

Lemma 4.7 We have,

$$\|\tilde{\rho}\|_{L^2(\mathcal{H})} + \|\tilde{\sigma}\|_{L^2(\mathcal{H})} + \|\tilde{\beta}\|_{L^2(\mathcal{H})} \leq \hat{D}\varepsilon. \quad (4.6)$$

Proof Using equation (2.8), the bootstrap assumptions (4.2) and the previous estimate (4.5), we have

$$\|\tilde{\rho}\|_{L^2(\mathcal{H})} \lesssim \|\rho\|_{L^2(\mathcal{H})} + \|\hat{\chi}\|_{L^{\infty}L^4} \|\hat{\chi}\|_{L^{\infty}L^4} \lesssim \varepsilon + (D\varepsilon)^2 \leq \hat{D}\varepsilon.$$

The estimates for $\tilde{\sigma}$ and $\tilde{\beta}$ follow analogously and are left to the reader.
4.4 Schematic notation for null connection coefficients and null curvature components

For ease of presentation, in the following sections, we employ the next schematic notation for null connection coefficients and null curvature components.

**Notation.** Let

\[ A \in \{ \text{tr} \chi - \frac{2}{v}, \hat{\chi}, \zeta, \eta, \nabla \log \Omega, \text{tr} \chi + \frac{2}{v} \} \cup \{ \Omega - 1, \log \Omega, \Omega^{-1} - 1 \}, \]

\[ \hat{A} \in A \cup \{ \hat{\chi} \}, \]

\[ R \in \{ \alpha, \beta, \rho, \sigma, \hat{\beta}, \hat{\rho}, \hat{\sigma}, \hat{\beta} \}. \]

Let moreover

\[ \nabla \in \{ \nabla, \nabla_L \}. \]

Using the above notation, the bootstrap assumptions (4.2), the bounds on \( S_1 \) (2.42), the improved curvature bounds (4.5) and (4.6), can be written as follows.

**Lemma 4.8** It holds that

\[ \| A \|_{L^{1/2}(S)} \lesssim \epsilon, \]

\[ \| R \|_{L^2(H)} \leq \hat{D} \epsilon, \]

\[ \| A \|_{L^\infty L^2 v} + \mathcal{N}_1(A) + \| A \|_{L^4 L^\infty v} \lesssim D \epsilon. \]

**Remark 4.9** To improve the bootstrap assumption for \( \mathcal{N}_1(A) \), it is enough to improve only \( \| \nabla L A \|_{L^2(H)} + \| \nabla A \|_{L^2(H)} + \| A \|_{L^2(H)} \), since the \( H^{1/2}(S) \) norms are already controlled by Lemma 3.9.

4.5 Improvement of the estimates for \( \Omega \) and \( A \)

In this section, we improve the bounds for \( \Omega \) and \( A \) using the previously improved bounds for the null curvature components.

**Lemma 4.10** For \( \epsilon > 0 \) sufficiently small, it holds that

\[ \| \mu \|_{L^2 L^\infty} \leq \hat{D} \epsilon. \]  

(4.7)

**Proof** We rewrite schematically the transport equation (2.21) under the form

\[ L(\mu) + \text{tr} \chi \mu = \hat{\rho} - \frac{1}{2} \hat{\rho} + AR + A \nabla A + A^2 + A^3. \]
Using Lemma 3.11, the bootstrap assumptions (4.2), the bounds on $S_1$ (2.42), and the renormalised curvature bounds (4.5) and (4.6), we have

$$\|\mu\|_{L^\infty L^\infty_v} \lesssim \|\mu\|_{L^2(S_1)} + \|\tilde{\rho}\|_{L^2 L^1_v}$$

$$+ \|AR\|_{L^2 L^1_v} + \|A\nabla A\|_{L^2 L^1_v} + \|A^2\|_{L^2 L^1_v} + \|A^3\|_{L^2 L^1_v}$$

$$\lesssim \|\mu\|_{L^2(S_1)} + \|\tilde{\rho}\|_{L^2(H)}$$

$$+ \|A\|_{L^\infty L^2_v} (\|R\|_{L^2(H)} + \|\nabla A\|_{L^2(H)} + \|A\|_{L^2(H)} + \|A\|^2_{L^4(H)})$$

$$\lesssim \varepsilon + (D\varepsilon)^2 + (D\varepsilon)^3$$

$$\leq \tilde{D}\varepsilon,$$

as desired.

\[\square\]

**Remark 4.11** From now on, we include $\mu$ in the schematic notation $R$.

**Lemma 4.12** For $\varepsilon > 0$ sufficiently small, it holds that

$$\|\nabla^2 \log \Omega\|_{L^2(H)} + \|\nabla \log \Omega\|_{L^2(H)} + \|\log \Omega\|_{L^2(H)} \leq \tilde{D}\varepsilon. \quad (4.8)$$

**Proof** Using Lemma 3.20 on the elliptic equation (2.18) and the improved estimates (4.7) and (4.6), we get that

$$\|\nabla \log \Omega\|_{L^2(H)} + \|\log \Omega\|_{L^2(H)} \lesssim \|\mu\|_{L^2 L^{4/3}} + \|\tilde{\rho}\|_{L_v^2 L^{4/3}}$$

$$\lesssim \|\mu\|_{L^2 L^\infty} + \|\tilde{\rho}\|_{L^2(H)}$$

$$\leq \tilde{D}\varepsilon.$$

Using Lemma 3.17 on the same elliptic equation (2.18) gives

$$\|\nabla^2 \log \Omega\|_{L^2(H)} + \|\nabla \log \Omega\|_{L^2(H)} \lesssim \|\tilde{\rho}\|_{L^2(H)} + \|\mu\|_{L^2(H)}$$

$$\leq \tilde{D}\varepsilon,$$

as desired.

\[\square\]

**Lemma 4.13** For $\varepsilon > 0$ sufficiently small, it holds that

$$\|\nabla \log \Omega\|_{L^2(H)} + \|\log \Omega\|_{L^2(H)} \leq \tilde{D}\varepsilon, \quad (4.9)$$

$$\|\nabla L \log \Omega\|_{L^2(H)} \leq \tilde{D}\varepsilon. \quad (4.10)$$

**Proof** Consider first (4.9). We want to derive an equation for $L \log \Omega$ and apply Lemma 3.20. Commuting equation (2.18) with $L$ gives the following elliptic equation
\[ \Delta (L(\log \Omega)) = F \]

with the source term \( F \)

\[ F := L(\mu) + 2L(\tilde{\rho}) - [L, \Delta] \log \Omega - 2L(\tilde{\rho}) - L(\bar{\rho}). \]

Using commutation formula (2.14), we have

\[ [L, \Delta] \log \Omega = - \text{tr} \chi \Delta (\log \Omega) - 2\tilde{\chi} \cdot \nabla^2 \log \Omega \]
\[ + (\zeta + \eta) (\nabla (\log \Omega) + \nabla L \nabla \log \Omega) \]
\[ + (\text{tr} \chi \eta - \text{div} \tilde{\chi}) \cdot \nabla \log \Omega - \eta \cdot \chi \cdot \nabla \log \Omega \]
\[ + (\text{div} \chi \zeta + \text{div} \eta \tilde{\eta}) L(\log \Omega) + \beta \cdot \nabla \log \Omega \]
\[ = - 2v^{-1} \Delta \log \Omega \]
\[ - \left( \text{tr} \chi - \frac{2}{v} \right) \Delta \log \Omega - 2\tilde{\chi} \cdot \nabla^2 \log \Omega \]
\[ + (\zeta + \eta) (\nabla (\log \Omega) + \nabla L \nabla \log \Omega) \]
\[ + (\text{tr} \chi \eta - \text{div} \tilde{\chi}) \cdot \nabla \log \Omega - \eta \cdot \chi \cdot \nabla \log \Omega \]
\[ + (\text{div} \chi \zeta + \text{div} \eta \tilde{\eta}) L(\log \Omega) + \beta \cdot \nabla \log \Omega. \quad (4.11) \]

Using Bianchi equation (2.9a) for \( \tilde{\rho} \), and the transport equation (2.21) for \( \mu \), we obtain the formula

\[ L(\mu) + 2L(\tilde{\rho}) = - 2v^{-1} \mu - 4v^{-1} \tilde{\rho} - v^{-1} \tilde{\rho} + 2 \text{div} \beta \]
\[ - \left( \text{tr} \chi - \frac{2}{v} \right) \mu - 2 \left( \text{tr} \chi - \frac{2}{v} \right) \tilde{\rho} - \frac{1}{2} \left( \text{tr} \chi - \frac{2}{v} \right) \tilde{\rho} \]
\[ + 4\eta \cdot \beta + (\zeta - \eta) \cdot \nabla \text{tr} \chi + \tilde{\chi} \cdot \nabla \zeta \]
\[ - \frac{1}{2} \tilde{\chi} \cdot \nabla \eta + \text{tr} \chi \left| \zeta \right|^2 - \zeta \cdot \eta - \frac{1}{2} \left| \eta \right|^2 \]
\[ + \frac{1}{4} \text{tr} \chi \left| \tilde{\chi} \right|^2 + 2\tilde{\chi} \cdot \zeta \cdot \eta - \frac{3}{2} \tilde{\chi} \cdot \eta \cdot \eta. \quad (4.12) \]

Using Proposition 2.16 and Remark 4.11, we can moreover schematically write

\[ L(\bar{\rho}) + 2L(\tilde{\rho}) = L(\mu) + 2L(\tilde{\rho}) + AR. \quad (4.13) \]

Using the three equations (4.11), (4.12) and (4.13) we deduce that \( F \) can be rewritten in the following schematic form
\[ F = F^L + F^{NL}, \]

with the linear source terms \( F^L \)
\[ F^L = -2v^{-1} \Delta \log \Omega - 2v^{-1}(\mu - \bar{\mu}) - 4v^{-1}(\tilde{\rho} - \bar{\rho}) + 2 \text{div} \beta, \]

and the non-linear source terms \( F^{NL} \) being of the form
\[ F^{NL} = \nabla (AL \log \Omega) + (A + A^2)L(\log \Omega) + A\nabla L \log \Omega + AR + A\nabla A + A^2 + A^3. \]

On the one hand, it holds that
\[ F^L = \text{div} P^L + W^L, \]
with
\[ P^L = 2\beta - 2v^{-1} \nabla \log \Omega, \]
\[ W^L = -4v^{-1}(\tilde{\rho} - \bar{\rho}) - 2v^{-1}(\mu - \bar{\mu}), \]

and using the already improved bounds (4.5), (4.6), (4.7) and (4.8), we get
\[
\left\| P^L \right\|_{L^2(\mathcal{H})} + \left\| W^L \right\|_{L^2 L^{4/3}} \lesssim \| \beta \|_{L^2(\mathcal{H})} + \| \nabla \log \Omega \|_{L^2(\mathcal{H})} + \| \tilde{\rho} \|_{L^2(\mathcal{H})} + \| \mu \|_{L^2 L^\infty} \leq \hat{D}\varepsilon. \tag{4.14}
\]

On the other hand, we have
\[ F^{NL} = \mathcal{V} P^{NL} + W^{NL}, \]
with
\[ P^{NL} = AL(\log \Omega) \]
\[ W^{NL} = (A + A^2)L(\log \Omega) + A\nabla L \log \Omega + AR + A\nabla A + A^2 + A^3, \]

and using the bootstrap assumptions (4.2), the estimates (4.5) and the Sobolev estimates of Lemmas 3.6 and 3.8, we get
\[
\left\| P^{NL} \right\|_{L^2(\mathcal{H})} + \left\| W^{NL} \right\|_{L^2 L^{4/3}} \lesssim \| A \|_{L^\infty L^4} \| L(\log \Omega) \|_{L^2 L^4} + \| R \|_{L^2(\mathcal{H})} + \| \nabla A \|_{L^2(\mathcal{H})} + \| A \|_{L^2(\mathcal{H})} + \| A \|_{L^2(\mathcal{H})}^2 \]
\[
\lesssim (D\varepsilon)^2 + (D\varepsilon)^3 \leq \hat{D}\varepsilon. \tag{4.15}
\]
Applying Lemma 3.20, using the bounds (4.14) and (4.15), we have
\[
\|\nabla L (\log \Omega)\|_{L^2(\mathcal{H})} + \|L (\log \Omega) - \overline{L}(\log \Omega)\|_{L^2(\mathcal{H})} \\
\lesssim \|P_L\|_{L^2(\mathcal{H})} + \|W_L\|_{L^2 L^{4/3}} + \|P^{NL}_L\|_{L^2(\mathcal{H})} + \|W^{NL}_L\|_{L^2 L^{4/3}} \tag{4.16}
\]
\[
\lesssim \|\Omega^{-1}\|_{L^\infty(\mathcal{H})} \|\log \Omega\|_{L^2(\mathcal{H})} + \|\Omega^{-1}\tr \chi \log \Omega + \Omega^{-1}\tr \chi \cdot \log \Omega + (1 - \Omega^{-1})\|L (\log \Omega)\|_{L^2(\mathcal{H})}
\]
Using Proposition 2.16 and the equation (2.17), we have
\[
L (\log \Omega) = \Omega^{-1} L (\log \Omega) + (1 - \Omega^{-1}) L (\log \Omega) \\
= \Omega^{-1} L (\log \Omega) - \Omega^{-1}\tr \chi \log \Omega + \Omega^{-1}\tr \chi \cdot \log \Omega + (1 - \Omega^{-1}) L (\log \Omega)
\]
Using the improved bound (4.8) and the bootstrap assumptions (4.2) we therefore deduce
\[
\|\nabla L (\log \Omega)\|_{L^2(\mathcal{H})} \lesssim \|\log \Omega\|_{L^2(\mathcal{H})} \|\Omega^{-1}\|_{L^\infty(\mathcal{H})} \|\tr \chi\|_{L^\infty(\mathcal{H})} \\
+ \|\Omega - 1\|_{L^\infty(\mathcal{H})} \|\Omega^{-1}\|_{L^\infty(\mathcal{H})} \|L (\log \Omega)\|_{L^2(\mathcal{H})} \lesssim \varepsilon + D\varepsilon \|L (\log \Omega)\|_{L^2(\mathcal{H})}
\]
Using (4.16) the above estimate and an absorption argument, we finally get
\[
\|\nabla L (\log \Omega)\|_{L^2(\mathcal{H})} + \|L (\log \Omega)\|_{L^2(\mathcal{H})} \lesssim \varepsilon + D\varepsilon \|L (\log \Omega)\|_{L^2(\mathcal{H})} \leq \hat{\mathcal{D}}\varepsilon.
\]
By the above bounds, commutation formula (2.10), the bootstrap assumptions (4.2) and the Sobolev estimates of Lemma 3.8 we also get that
\[
\|\nabla L \nabla \log \Omega\|_{L^2(\mathcal{H})} \lesssim \|\nabla L (\log \Omega)\|_{L^2(\mathcal{H})} + \|\tr \chi\|_{L^\infty(\mathcal{H})} \|\log \Omega\|_{L^2(\mathcal{H})} \\
+ \|\nabla \log \Omega\|_{L^\infty L^4} \|\log \Omega\|_{L^2 L^4} \\
+ \left(\|\xi\|_{L^\infty L^4} + \|\eta\|_{L^\infty L_4}\right) \|L (\log \Omega)\|_{L^2 L^4} \lesssim \varepsilon + (D\varepsilon)^2 \leq \hat{\mathcal{D}}\varepsilon.
\]
This finishes the proof of Lemma 4.13. \qed

Lemma 4.14 For \(\varepsilon > 0\) sufficiently small, we have
\[
\mathcal{N}_1 (\log \Omega) + \mathcal{N}_1 (\nabla \log \Omega) \leq \hat{\mathcal{D}}\varepsilon, \tag{4.17}
\]
\[
\| \log \Omega \|_{L^\infty(\mathcal{H})} + \| \Omega - 1 \|_{L^\infty(\mathcal{H})} + \| L(\log \Omega) \|_{L^2 L^4} \leq \hat{D} \epsilon. \tag{4.18}
\]

**Proof** Estimate (4.17) is a recapitulation of estimates (4.4) (4.8) (4.9) and (4.10). Estimate (4.18) is a consequence of estimate (4.17) and Sobolev embeddings from Lemma 3.8. \(\square\)

**Lemma 4.15** For \(\epsilon > 0\) sufficiently small, it holds that

\[
\mathcal{N}_1(\zeta) + \mathcal{N}_1(\eta) \leq \hat{D} \epsilon. \tag{4.19}
\]

**Proof** By equations (2.16) and (2.22h), \(\zeta\) satisfies the Hodge system

\[
\mathcal{P}_1(\zeta) = (-\mu - \tilde{\rho}, \tilde{\sigma}).
\]

Using Lemma 3.19, the improved bounds (4.6) and (4.7), we get that

\[
\| \zeta \|_{L^2(\mathcal{H})} + \| \nabla \zeta \|_{L^2(\mathcal{H})} \lesssim \| \mu \|_{L^2(\mathcal{H})} + \| \tilde{\rho} \|_{L^2(\mathcal{H})} + \| \tilde{\sigma} \|_{L^2(\mathcal{H})} \leq \hat{D} \epsilon.
\]

By relation (2.4), the improved bounds (4.8) and the above improvement, we directly deduce

\[
\| \eta \|_{L^2(\mathcal{H})} + \| \nabla \eta \|_{L^2(\mathcal{H})} \leq \hat{D} \epsilon.
\]

Using equation (2.22g), the bootstrap assumptions (4.2), the curvature bounds (4.5) and the just obtained improved estimate for \(\zeta\) and \(\eta\), we have

\[
\| \nabla L \zeta \|_{L^2(\mathcal{H})} \lesssim \| \text{tr} \chi \|_{L^\infty} (\| \zeta \|_{L^2(\mathcal{H})} + \| \eta \|_{L^2(\mathcal{H})} ) \\
+ \| \tilde{\Phi} \|_{L^\infty L^4} (\| \zeta \|_{L^\infty L^4} + \| \eta \|_{L^\infty L^4} ) + \| \beta \|_{L^2(\mathcal{H})} \\
\lesssim \epsilon + (D \epsilon)^2 \\
\leq \hat{D} \epsilon.
\]

By relation (2.4) and the improved bounds (4.10), we therefore also deduce

\[
\| \nabla L \eta \|_{L^2(\mathcal{H})} \leq \hat{D} \epsilon,
\]

and this finishes the proof of Lemma 4.15. \(\square\)

**Lemma 4.16** For \(\epsilon > 0\) sufficiently small, it holds that

\[
\| \nabla \text{tr} \chi \|_{L^2 L^\infty} \leq \hat{D} \epsilon, \tag{4.20}
\]
\[
\| \text{tr} \chi - \frac{2}{v} \|_{L^\infty(\mathcal{H})} \leq \hat{D} \varepsilon, \tag{4.21}
\]

\[
\mathcal{N}_1 \left( \text{tr} \chi - \frac{2}{v} \right) + \mathcal{N}_1 (\hat{\chi}) \leq \hat{D} \varepsilon. \tag{4.22}
\]

**Proof** Consider first (4.20). Commuting equation (2.22b) with \( \nabla / \), we get

\[\nabla / L \left( \nabla / \text{tr} \chi \right) + \frac{3}{2} \text{tr} \chi \nabla / \text{tr} \chi = G,\]

with

\[G = -2 \hat{\chi} \cdot \nabla / \hat{\chi} - \hat{\chi} \cdot \nabla / \text{tr} \chi + (\zeta + \eta) \left( \frac{1}{2} (\text{tr} \chi)^2 + |\hat{\chi}|^2 \right).\]

By the improved bounds (4.19) for \( \zeta \) and \( \eta \) and the bootstrap assumptions (4.1), we have

\[
\| G \|_{L^2 L_v^1} \lesssim \| \hat{\chi} \|_{L^\infty L_v^2} \| \nabla / \hat{\chi} \|_{L^2(\mathcal{H})} + \| \hat{\chi} \|_{L^\infty L_v^2} \| \nabla / \text{tr} \chi \|_{L^2(\mathcal{H})}
+ (\| \zeta \|_{L^2(\mathcal{H})} + \| \eta \|_{L^2(\mathcal{H})}) \| \text{tr} \chi \|_{L^\infty(\mathcal{H})}^2
+ (\| \zeta \|_{L^\infty L_v^2} + \| \eta \|_{L^\infty L_v^2}) \| \hat{\chi} \|_{L^4(\mathcal{H})}^2
\lesssim \varepsilon + (D \varepsilon)^2
\leq \hat{D} \varepsilon.
\]

Therefore, we deduce from Lemma 3.11 with (4.1) that (4.20) holds.

Next, we consider (4.21). The transport equation for \( \text{tr} \chi \) (2.22b) can be rewritten

\[
L \left( \text{tr} \chi - \frac{2}{v} \right) + \text{tr} \chi \left( \text{tr} \chi - \frac{2}{v} \right) = 2 v^{-2} (\Omega - 1) - |\hat{\chi}|^2 + \frac{1}{2} \left( \text{tr} \chi - \frac{2}{v} \right)^2. \tag{4.23}
\]

Using Lemma 3.11, the bootstrap assumptions (4.2), the bounds on \( S_1 \) (2.42), and the improved bound (4.18), we have

\[
\| \text{tr} \chi - \frac{2}{v} \|_{L^\infty(\mathcal{H})} \lesssim \| \text{tr} \chi - 2 \|_{L^\infty(S_1)} + \| \Omega - 1 \|_{L^\infty(\mathcal{H})}
+ \| \hat{\chi} \|_{L^\infty L_v^2}^2 + \| \text{tr} \chi - \frac{2}{v} \|_{L^\infty(\mathcal{H})}^2
\lesssim \varepsilon + (D \varepsilon)^2
\leq \hat{D} \varepsilon,
\]

which proves (4.21).
It remains to prove (4.22). Using transport equation (4.23) for $\tr \chi$, we deduce that
\[
\left\| L \left( \tr \chi - \frac{2}{v} \right) \right\|_{L^2(\mathcal{H})} \leq \hat{D} \varepsilon,
\]
and therefore
\[
\mathcal{N}_1 \left( \tr \chi - \frac{2}{v} \right) \leq \hat{D} \varepsilon.
\]

Applying Hodge Lemma 3.19 to the Codazzi equation on $\hat{\chi}$ (2.22j), with the curvature bounds (4.5), the improved bound (4.19), and the bound just proven for $\nabla \tr \chi$ gives
\[
\begin{align*}
\| \nabla \hat{\chi} \|_{L^2(\mathcal{H})} + \| \hat{\chi} \|_{L^2(\mathcal{H})} & \lesssim \| \nabla \tr \chi \|_{L^2 L^\infty} + \| \xi \|_{L^4 L^\infty} \| \hat{\chi} \|_{L^4 L^\infty} \\
& \quad + \| \tr \chi \|_{L^\infty(\mathcal{H})} \| \xi \|_{L^2(\mathcal{H})} + \| \beta \|_{L^2(\mathcal{H})} \\
& \lesssim \varepsilon + (D \varepsilon)^2 \\
& \leq \hat{D} \varepsilon.
\end{align*}
\]
Taking directly the $L^2(\mathcal{H})$-norm in the transport equation for $\hat{\chi}$ (2.22c), we finally obtain
\[
\| \nabla L \hat{\chi} \|_{L^2(\mathcal{H})} \lesssim \varepsilon
\]
and this finishes the proof of Lemma 4.16.

**Lemma 4.17** For $\varepsilon > 0$ sufficiently small, it holds that
\[
\begin{align*}
\left\| \nabla \tr \chi \right\|_{L^2 L^\infty(\mathcal{H})} & \leq \hat{D} \varepsilon, & (4.24) \\
\left\| \tr \chi + \frac{2}{v} \right\|_{L^\infty(\mathcal{H})} & \leq \hat{D} \varepsilon, & (4.25) \\
\mathcal{N}_1 \left( \tr \chi + \frac{2}{v} \right) + \mathcal{N}_1 (\hat{\chi}) & \leq \hat{D} \varepsilon. & (4.26)
\end{align*}
\]

**Proof** Consider (4.24). Commuting the transport equation for $\tr \chi$ (2.22e) with $\nabla$, we get
\[
\nabla L \nabla \tr \chi + \tr \chi \nabla \tr \chi = G,
\]
with
\[
G = 4 \eta \cdot \nabla \eta - \hat{\chi} \cdot \nabla \tr \chi + (\xi + \eta) \left( -\frac{1}{2} \tr \chi \tr \chi + 4 \beta + 4 |\eta|^2 \right).
\]
which can be rewritten in the schematic form

\[ G = 2v^{-2}(\zeta + \eta) + A(\nabla A + A + A^2 + R). \]

Using Lemma 3.11, the bootstrap assumptions (4.2), the initial bounds (2.42) and the improved bounds (4.19), we have

\[
\| \nabla \chi \|_{L^2 H^\infty} \lesssim \| \nabla \chi \|_{L^2(S_1)} + \| G \|_{L^2 L^1_v} \\
\lesssim \varepsilon + \| \zeta \|_{L^2(H)} + \| \eta \|_{L^2(H)} \\
+ \| A \|_{L^\infty L^1_v} (\| \nabla A \|_{L^2(H)} + \| A \|_{L^2(H)} + \| A \|_{L^4(H)}^2 + \| R \|_{L^2(H)}) \\
\lesssim \varepsilon + (D\varepsilon)^2 + (D\varepsilon)^3 \\
\leq \hat{D}\varepsilon.
\]

We turn to estimate (4.25). The transport equation for \( \text{tr} \chi (2.22e) \) can be rewritten in the following form

\[
L\left( \text{tr} \chi + \frac{2}{v} \right) + \frac{1}{2} \text{tr} \chi \left( \text{tr} \chi + \frac{2}{v} \right) \\
= -2v^{-2}(\Omega - 1) + v^{-1} \left( \text{tr} \chi - \frac{2}{v} \right) + 2\bar{\rho} + 2|\eta|^2.
\]

Using Lemma 3.11, the bootstrap assumptions (4.2), the bounds on \( S_1 (2.42) \) and the improved bounds (4.21)(4.18), we have

\[
\left\| \text{tr} \chi + \frac{2}{v} \right\|_{L^\infty(H)} \lesssim \left\| \text{tr} \chi + 2 \right\|_{L^\infty(S_1)} + \| \Omega - 1 \|_{L^\infty(H)} \\
+ \left\| \text{tr} \chi - \frac{2}{v} \right\|_{L^\infty(H)} + \| \bar{\rho} \|_{L^2 L^1_v} + \| \eta \|_{L^2 L^1_v}^2 \\
\lesssim \varepsilon + (D\varepsilon)^2 \\
\leq \hat{D}\varepsilon.
\]

To prove estimate (4.26), we apply Hodge Lemma 3.19 to the Codazzi equation for \( \hat{\chi} \) and since \( \nabla \text{tr} \chi \) and \( \text{tr} \chi \) have already been estimated the \( \nabla \)-control of \( \text{tr} \chi + \frac{2}{v} \) and \( \hat{\chi} \) follows. The estimates for \( L(\text{tr} \chi + \frac{2}{v}) \) and \( \nabla L \hat{\chi} \) are obtained by taking directly the \( L^2(H) \) norm in the transport equations for \( \text{tr} \chi \) and \( \hat{\chi} (2.22e) \) and (2.22f) since all linear source terms have already been estimated. This concludes the proof of Lemma 4.17. \( \square \)
4.6 Improvement of $\Upsilon$

In this section, we improve the estimate for $\|\Upsilon\|_{L^\infty(H)}$ which is the key quantity to compare the geodesic and canonical foliations. Using the estimates proved in the previous sections, we can first improve the $L^\infty L^2_v$ estimate for $\eta$.

**Lemma 4.18** For $\varepsilon > 0$ sufficiently small, it holds that

$$\|\eta\|_{L^\infty L^2_v} \leq \hat{D}\varepsilon. \quad (4.27)$$

**Proof** Our goal is to apply the trace estimate of Lemma 3.13. By the improved estimates (4.19) for $\eta$, it suffices to prove that there exist $P$ and $E$ such that

$$\nabla \eta = \nabla_L P + E,$$

with

$$\mathcal{N}_1(P) \leq \hat{D}\varepsilon, \quad \|E\|_{P^0_v} \leq \hat{D}\varepsilon.$$

From the transport equation for $\chi$ (2.22d), we have

$$\nabla_A \eta_B = \frac{1}{2} \nabla_L \chi_{AB} - \frac{1}{2} \rho^A \chi_{AB} - \frac{1}{2} \sigma^A \chi_{AB} + \frac{1}{2} \chi_{AC} \chi_{CB} - \eta_A \eta_B$$

$$= \frac{1}{2} \nabla_L \left( \chi_{AB} + \frac{2}{v} \chi_{AB} \right) - \frac{1}{2} \rho^A \chi_{AB} - \frac{1}{2} \sigma^A \chi_{AB} + E_{AB},$$

where

$$E_{AB} := \left( \frac{1}{2} \text{tr} \chi \left( \text{tr} \chi - \frac{2}{v} \right) + 2v^{-2}(\Omega - 1) + \frac{1}{2} \text{tr} \chi \left( \text{tr} \chi + \frac{2}{v} \right) \right) g_{AB}$$

$$+ \frac{1}{4} \text{tr} \chi \chi_{AB} + \frac{1}{4} \text{tr} \chi \chi_{AB} + \frac{1}{2} \chi_{AC} \chi_{CB} - \eta_A \eta_B.$$

First, using the results of Lemma 4.17, we have

$$\mathcal{N}_1 \left( \chi + \frac{2}{v} \chi \right) \leq \hat{D}\varepsilon.$$

Second, using Lemma 3.10 and the improved bounds for $\Omega, \chi, \chi$ and $\eta$ (4.18), (4.22), (4.26), (4.19) we have

$$\|E\|_{P^0_v} \lesssim \left( \mathcal{N}_1(\Omega - 1) + \mathcal{N}_1 \left( \chi - \frac{2}{v} \chi \right) + \mathcal{N}_1 \left( \chi + \frac{2}{v} \chi \right) + \mathcal{N}_1(\eta) \right)$$
\[ \times \left( 1 + \mathcal{N}_1(\Omega - 1) + \mathcal{N}_1 \left( \chi - \frac{2}{v} \bar{g} \right) + \mathcal{N}_1 \left( \chi + \frac{2}{v} \bar{g} \right) + \mathcal{N}_1(\eta) \right) \]
\[ \leq \hat{D} \varepsilon. \]

Third, we define \((\phi, \psi)\) to be the solution of the transport equation

\[ \begin{align*}
L \phi &= \rho, \\
L \psi &= \sigma,
\end{align*} \tag{4.28} \]

Using the curvature bounds (4.5), we have directly

\[ \| L \phi \|_{L^2(\mathcal{H})} + \| L \psi \|_{L^2(\mathcal{H})} \leq \hat{D} \varepsilon \tag{4.29} \]

Using the definition of \(\tilde{\beta}\) (2.8) and the Codazzi equation for \(\hat{\chi}\) (2.22k), we have schematically

\[ \tilde{\beta} = \text{div} \hat{\chi} - \frac{1}{2} \nabla \text{tr} \chi + A + AA. \]

Therefore, using Lemma 3.5 and the bounds on \(S_1\) (2.42), we have

\[ \| \tilde{\beta} \|_{H^{-1/2}(S_1)} \lesssim \| \hat{\chi} \|_{H^{1/2}(S_1)} + \left\| \text{tr} \chi + \frac{2}{v} \right\|_{H^{1/2}(S_1)} + \| A \|_{L^2(S_1)} + \| A A \|_{L^2(S_1)} \]
\[ \lesssim \| \hat{\chi} \|_{H^{1/2}(S_1)} + \left\| \text{tr} \chi + \frac{2}{v} \right\|_{H^{1/2}(S_1)} + \| A \|_{L^2(S_1)} + \| A \|_{H^{1/2}(S_1)} \| A \|_{H^{1/2}(S_1)} \]
\[ \lesssim \varepsilon. \]

Thus, using Lemma 3.21, we have

\[ \| (\phi, \psi) \|_{H^{1/2}(S_1)} \lesssim \| \mathcal{P}_1^{-1} \tilde{\beta} \|_{H^{1/2}(S_1)} \lesssim \| \tilde{\beta} \|_{H^{-1/2}(S_1)} \lesssim \varepsilon. \tag{4.30} \]

Using the transport Lemma 3.11 with these bounds, we deduce

\[ \| (\phi, \psi) \|_{L^2L^\infty} \lesssim \| L(\phi, \psi) \|_{L^2(\mathcal{H})} + \| (\phi, \psi) \|_{H^{1/2}(S_1)} \leq \hat{D} \varepsilon. \tag{4.31} \]

Commuting the transport equation (4.28) by \(*\mathcal{P}_1\), using Bianchi equation (2.9c) for \(\tilde{\beta}\) and commutation formula (2.13) gives
\[ \nabla^L \varphi_1(\phi, \psi) = \nabla^L (\varphi_1 + [\varphi, \varphi]) \]

Using Lemma 3.11 (with \( \kappa = 1/2 \)), the bootstrap assumptions (4.2), the curvature bounds (4.6) and the condition (4.28) on \( S_1 \) for \((\phi, \psi)\), we have

\[ \left\| \nabla^L \varphi_1(\phi, \psi) - \tilde{\beta} \right\|_{L^2} \lesssim \left\| \nabla^L \varphi_1(\phi, \psi) - \tilde{\beta} \right\|_{L^2} + \left\| \text{tr} \chi \right\|_{L^\infty} \left\| \tilde{\beta} \right\|_{L^2} \]

\[ + \| A \|_{L^\infty} \left( \| R \|_{L^2} + \| \nabla A \|_{L^2} + \| A \|_{L^2} + \| A \|_{L^4} \right) \]

\[ + \| A \|_{L^\infty} \left( \| \nabla (\phi, \psi) \|_{L^2} \right) \]

\[ \lesssim \epsilon + (D\epsilon)^2 + (D\epsilon) \mathcal{N}_1(\phi, \psi). \]

Using Hodge Lemma 3.19 and the curvature bound (4.6), we deduce from the above that

\[ \left\| \nabla (\phi, \psi) \right\|_{L^2} \lesssim \left\| \nabla^L \varphi_1(\phi, \psi) \right\|_{L^2} \]

\[ \lesssim \left\| \tilde{\beta} \right\|_{L^2} + \left\| \nabla^L \varphi_1(\phi, \psi) - \tilde{\beta} \right\|_{L^\infty} \]

\[ \lesssim \epsilon + (D\epsilon) \mathcal{N}_1(\phi, \psi). \]

For \( \epsilon > 0 \) sufficiently small, we therefore have by (4.29), (4.30), (4.31), (4.32) and an absorption argument that

\[ \mathcal{N}_1(\phi, \psi) = \left\| (\phi, \psi) \right\|_{L^2} + \left\| (\phi, \psi) \right\|_{H^{1/2}} \]

\[ \lesssim \epsilon + (D\epsilon) \mathcal{N}_1(\phi, \psi) \]

\[ \leq \mathcal{D}\epsilon. \]

This finishes the proof of the lemma. \( \square \)

**Lemma 4.19** We have the improved bound

\[ \left\| \Upsilon ' \right\|_{L^\infty} + \left\| \Upsilon \right\|_{L^\infty} \leq \mathcal{D}\epsilon. \]

**Proof** From Proposition 2.33, we have

\[ \nabla^L \Upsilon_A' = \eta_A' - (\eta_A')^\perp \]

\[ = \eta_A' - \eta(e_A). \]

Integrating in \( s \) and since \( \Upsilon ' = 0 \) on \( S_1 \), we deduce

\[ \Upsilon_A' = \int_1^s L(\Upsilon_A') \, ds' \]
\[
\int_1^s \left( \eta_A' - \eta(e_A) \right) \, ds' \\
= \int_1^s \eta_A' \, ds' - \int_{v'=1}^{v(s)} \eta_A \Omega^{-1} \, dv'.
\]

Therefore, using the assumed bound \((4.1)\) on the geodesic connection coefficient \(\eta' = -\zeta'\) and the improved bound for \(\eta\) \((4.27)\), we obtain

\[
\left\| Y' \right\|_{L^\infty(H)} \lesssim \left\| \eta' \right\|_{L^\infty L^2_v} + \left\| \Omega^{-1} \right\|_{L^\infty(H)} \left\| \eta \right\|_{L^\infty L^2_v} \leq \hat{D}\varepsilon,
\]

which, together with Definition 2.28 proves \((4.33)\). \(\square\)

### 4.7 Improvement of \(L^\infty L^2_v\) estimates for \(A\)

In section 4.6, we proved \(L^\infty L^2_v\) estimate for \(\eta\). In this section, we prove the remaining \(L^\infty L^2_v\) estimates for \(\hat{\chi}, \zeta, \nabla \log \Omega\) by comparing the canonical foliation to the geodesic foliation on \(H\). This concludes the improvement of the bootstrap assumptions \((4.2)\), thus finishes the proof of Proposition 2.40.

**Lemma 4.20** For \(\varepsilon > 0\) sufficiently small, we have

\[
\left\| \hat{\chi} \right\|_{L^\infty L^2_v} \leq \hat{D}\varepsilon, \quad (4.34)
\]

\[
\left\| \xi \right\|_{L^\infty L^2_v} \leq \hat{D}\varepsilon, \quad (4.35)
\]

\[
\left\| \nabla \log \Omega \right\|_{L^\infty L^2_v} \leq \hat{D}\varepsilon. \quad (4.36)
\]

**Proof** From Proposition 2.33, we have

\[
\hat{\chi}_{AB} = \hat{\chi}_A^A, \quad \hat{\xi}_A = \zeta_A' - 2 \gamma_B^\prime \chi_{AB}',
\]

Thus, the estimate \((4.34)\) is a direct consequence of the bounds \((4.1)\) and \((4.33)\). Further, by the improved bound on \(Y'\) \((4.33)\) and the assumed bound \((2.38)\) for the geodesic connection coefficients, we get

\[
\left\| \xi \right\|_{L^\infty L^2_v} \lesssim \left\| \zeta' \right\|_{L^\infty L^2_v} + \left\| Y' \right\|_{L^\infty(H)} \left\| \chi' \right\|_{L^\infty L^2_v} \leq \hat{D}\varepsilon.
\]

Estimate \((4.36)\) then follows directly using relation \((2.4)\). This finishes the proof of Lemma 4.20. \(\square\)

### 4.8 Additional bounds for \(Y\)

In Section 5, we will use the following additional estimates.
Lemma 4.21  For $\varepsilon > 0$ sufficiently small, we have

$$
\| \nabla L \chi \|_{L^2(H)} + \| \nabla \chi \|_{L^2 L^\infty_v} \leq \hat{D} \varepsilon. \quad (4.37)
$$

Proof  Taking the $L^2(H)$-norm in the transport equation (2.31) for $\chi$, using the improved bound (4.8) for $\Omega$ and the improved bound on $\chi$ (4.33), we have

$$
\| \nabla L \chi \|_{L^2(H)} \lesssim \| \text{tr} X \|_{L^\infty(H)} \| \chi \|_{L^\infty(H)} + \| \hat{\nabla} \|_{L^2 L^\infty_v} \| \chi \|_{L^\infty(H)} + \| \nabla \log \Omega \|_{L^2(H)} \\
\lesssim \varepsilon + \varepsilon^2 \\
\leq \hat{D} \varepsilon.
$$

To obtain the other bound, we make the additional bootstrap assumption $\| \nabla \chi \|_{L^2 L^\infty_v} \leq D \varepsilon$. We commute the transport equation (2.31) by $\nabla$

$$
\nabla L \nabla_A \chi_B = -\frac{1}{2} \text{tr} X \nabla_A \chi_B - \frac{1}{2} \nabla_A \chi \text{tr} X \chi_B + \nabla_A \chi_B \log \Omega \\
- \nabla_A \hat{\chi}_{BC} \chi_C - \hat{\chi}_{BC} \nabla_A \chi_C + [\nabla L, \nabla] A \chi_B, \quad (4.38)
$$

where by using formula (2.10) and (2.31) we have

$$
[\nabla L, \nabla] A \chi_B = -\frac{1}{2} \text{tr} X \nabla_A \chi_B - \hat{\chi}_{AC} \nabla_X \chi_C - \nabla_A (\log \Omega) \nabla_B (\log \Omega) \\
- (\nabla \log \Omega) A \hat{\chi}_{BC} \chi_C \\
+ \chi_{AB} \eta_C \chi_C - \chi_{AC} \eta_B \chi_C - * \beta_A * \chi_B.
$$

Therefore, applying Lemma 3.11, using that $\chi = 0$ on $S_1$, the improved bounds and the additional bootstrap assumption, we obtain

$$
\| \nabla \chi \|_{L^2 L^\infty_v} \lesssim \| \nabla^2 \log \Omega \|_{L^2(H)} + \| \nabla \log \Omega \|_{L^2 L^4}^2 \\
+ \| \chi \|_{L^\infty(H)} (\| \nabla X \|_{L^2(H)} \\
+ \| X \|_{L^2 L^\infty_v} (\| \nabla \log \Omega \|_{L^2(H)} + \| \eta \|_{L^2(H)} ) + \| \beta \|_{L^2(H)} ) \\
+ \| \nabla \chi \|_{L^2 L^\infty_v} \| \hat{\nabla} \|_{L^\infty(H)}^2 \\
\lesssim \varepsilon + (D \varepsilon)^2 \\
\leq \hat{D} \varepsilon,
$$

which improves the additional bootstrap assumption and hence finishes the proof of Lemma 4.21.  \(\square\)
5 Higher regularity estimates

This section is dedicated to the proof of Proposition 2.41 and completes Step 3 in Section 2.13. We assume that \((\mathcal{M}, g)\) is a smooth spacetime and \(\mathcal{H}\) a smooth null hypersurface foliated by a smooth geodesic foliation. We assume moreover that the following bounds hold on \([1, v^*]_v\),

\[
\|\Omega - 1\|_{L^\infty_v([1,v^*])} + \|\nabla_v \log \Omega\|_{L^\infty_v([1,v^*])} \lesssim \varepsilon, \\
\|\Upsilon\|_{L^\infty_v([1,v^*])} \lesssim \varepsilon.
\]  

(5.1)

For all \(m \geq 0\), we will prove the following estimates

\[
\sum_{l \leq m} \left( \|\nabla^l (\Omega - 1)\|_{L^\infty_v([1,v^*])} + \|\nabla^l \Upsilon\|_{L^\infty_v([1,v^*])} \right) \leq C \left( \|g'\|_{C^m(\mathcal{H})}, m \right) .
\]

(5.2)

Moreover, we will also have the following estimates on the \(L\)-derivatives

\[
\sum_{l' \leq m} \left( \|\nabla^l L^k (\Omega)\|_{L^\infty_v([1,v^*])} \right) \leq C \left( \|g'\|_{C^{m+k+2}(\mathcal{H})}, m + k \right) .
\]

(5.3)

for all \(m \geq 0\) and all \(k \geq 0\). This will complete the proof of Proposition 2.41.

Before turning to the proof of (5.2) and (5.3), we prove the following lemma that is a rewriting of equations (2.31) and (2.17). This will also be used in the proof of the local existence Theorem 2.38.

Lemma 5.1 We have

\[
\nabla \log \Omega = -(\chi')^{\dagger} \cdot \Upsilon - \nabla \log \Omega,
\]

\[
\log \Omega = \Delta^{-1} \left( F_1' + (F_2')^{\dagger} \cdot \Upsilon + (F_3')^{\dagger} \cdot \Upsilon \cdot \Upsilon + (F_4')^{\dagger} \cdot \nabla \Upsilon \right),
\]

(5.4)

(5.5)

where \(F_1', F_2', F_3', F_4'\) are (contractions of) geodesic quantities,\(^2\) and \(u := \Delta^{-1} f\) denotes the solution of the following elliptic equation

\[
\Delta u = f - \int_{S_v} f, \\
\int_{S_v} u = 0.
\]

Remark 5.2 As far as higher regularity is concerned, we are not interested in proving sharp estimates. Thus, the specific structure of the terms \((F_i')^{\dagger}\) is not needed.

\(^2\) Throughout this section, we call “geodesic quantities” arbitrary contractions of arbitrary numbers of derivatives of null connection coefficients and null curvature components associated to the geodesic foliation.
\textbf{Proof} Equation (5.4) is a rewriting of (2.31). Equation (5.5) is a rewriting of (2.22n). Namely, we have using the relations from Proposition \ref{prop:2.33} and the derivatives relation from Proposition \ref{prop:2.30}

\[
dj\vec{v}\xi = dj\vec{v}\left((\xi')^\dagger + (\chi')^\dagger \cdot \gamma\right)
= dj\vec{v}\xi' + (\nabla_L^\dagger \xi')^\dagger \cdot \gamma + (\text{tr} \chi')^\dagger (\xi')^\dagger \cdot \gamma
- (\chi')^\dagger \cdot (\xi')^\dagger \cdot \gamma + (\chi')^\dagger \cdot (\nabla \gamma) + (dj\vec{v} \chi'')^\dagger \cdot \gamma
+ \nabla_L^\dagger \chi' \cdot \gamma \cdot \gamma + (\text{tr} \chi')^\dagger (\chi')^\dagger \cdot \gamma \cdot \gamma - 2(\chi')^\dagger \cdot (\chi')^\dagger \cdot \gamma \cdot \gamma.
\]

Using the relations from Proposition \ref{prop:2.32}, we have

\[
\rho = \rho' + (\beta')^\dagger \cdot \gamma + (\alpha')^\dagger \cdot \gamma \cdot \gamma.
\]

Using the relations from Proposition \ref{prop:2.33}, we have

\[
-\frac{1}{2} \hat{\chi} \cdot \hat{\chi} = -\frac{1}{2} (\tilde{\xi}')^\dagger \cdot (\tilde{\xi}')^\dagger - 4(\xi')^\dagger \cdot \gamma + 2\nabla \gamma - |\gamma|^2(\chi')^\dagger.
\]

Therefore, using the definition of \(\rho (2.8)\) and defining

\[
F_1' := -dj\vec{v} \xi' + \rho' - \frac{1}{2} \hat{\chi}' \cdot \hat{\chi}',
F_2' := -\nabla_L^\dagger \xi' - \text{tr} \chi' \xi' + \chi' \cdot \xi' - (dj\vec{v} \chi'')^\dagger + \beta' + 2\hat{\chi}' \cdot \xi',
F_3' := + \alpha' + \frac{1}{2} \hat{\chi} \cdot \chi' - \frac{1}{2} |\hat{\chi}'|^2 - (\text{tr} \chi')^\dagger (\chi')^\dagger + 2(\chi')^\dagger \cdot (\chi')^\dagger,
F_4' := - \text{tr} \chi' \gamma' - 2\hat{\chi}',
\]

we have

\[
\log \Omega = \Delta^{-1} \left( F_1' + (F_2')^\dagger \cdot \gamma + (F_3')^\dagger \cdot \gamma \cdot \gamma + (F_4')^\dagger \cdot \nabla \gamma \right).
\]

This finishes the proof of Lemma \ref{lem:5.1}.

\[\square\]

\textbf{Proof of (5.2) and (5.3)} The proof of (5.2) goes by induction on \(m\). The cases \(m = 0\) and \(m = 1\) were already obtained in Section \ref{sec:4}. We prove the case \(m = 2\) and the cases \(m \geq 3\) are proved similarly and are left to the reader. In what follows, we use that the quantities \(F_1', F_2', F_3', F_4'\) appearing in Lemma \ref{lem:5.1} are smooth in the geodesic foliation. More precisely, we are going to obtain bounds in terms of

\[
\sum_{k \geq 2} \left( \left\| (\nabla')^k F_1' \right\|_{L^\infty(H)} + \left\| (\nabla')^k F_2' \right\|_{L^\infty(H)} + \left\| (\nabla')^k F_3' \right\|_{L^\infty(H)} + \left\| (\nabla')^k F_4' \right\|_{L^\infty(H)} \right).
\]

For simplicity, we do not write the exact bound and this quantity shall be always implicitly included in the constants \(C\) appearing in the following.
First, applying Lemma 3.17 to elliptic equation (5.5), we have
\[
\|\nabla^2 \log \Omega\|_{L^\infty_v L^2} \lesssim \|F'_1\|_{L^\infty(H)} + \|F'_2\|_{L^\infty(H)} \|\gamma\|_{L^\infty(H)}
+ \|F'_3\|_{L^\infty(H)} \|\gamma\|_{L^\infty(H)}^2 + \|F'_4\|_{L^\infty(H)} \|\nabla\gamma\|_{L^\infty_v L^2}
\lesssim C(\varepsilon).
\]

Second, commuting equation (5.4) by \(\nabla^2\), we have schematically
\[
\nabla^2 \nabla^2 \gamma = -(\chi')^\dagger \cdot \nabla^2 \gamma + G'(\gamma) \cdot (\nabla \gamma + \nabla \log \Omega) + \nabla^3 (\log \Omega) + [\nabla \gamma, \nabla^2] \gamma,
\]
where \(G'(\gamma)\) denotes geodesic quantities and (arbitrary number of) contractions of geodesic quantities with \(\gamma\).

Using commutation formula (2.10), the commutator can be schematically rewritten
\[
[\nabla \gamma, \nabla^2 \gamma] = [\nabla \gamma, \nabla \gamma] + [\nabla \gamma, \nabla^2 \log \Omega]
= (\chi')^\dagger \cdot \nabla^2 \gamma + \nabla \log \Omega \cdot \nabla^2 \log \Omega.
\]

We therefore obtain the following schematic formula
\[
\nabla \gamma [\nabla \gamma, \nabla^2] \gamma = (\chi')^\dagger \cdot \nabla^2 \gamma + \nabla^3 (\log \Omega) + \nabla \log \Omega \cdot \nabla^2 \log \Omega
\]

Using Lemma 3.11, the assumptions (5.1), and the above formulas, we therefore get
\[
\|\nabla^2 \gamma\|_{L^\infty_v L^2} \lesssim \|\chi\|_{L^\infty(H)} \|\nabla^2 \gamma\|_{L^1_v L^2} + \|\nabla^3 \log \Omega\|_{L^1_v L^2}
+ \|\nabla \log \Omega\|_{L^\infty_v L^2} \|\nabla^2 \log \Omega\|_{L^2(H)}
+ \|G'(\gamma)\|_{L^\infty(H)} \left(\|\nabla \gamma\|_{L^\infty_v L^2} + \|\nabla \log \Omega\|_{L^\infty_v L^2}\right)
\lesssim C + C \int_v \left(\|\nabla^2 \gamma\|_{L^2(S_v)} + \|\nabla^3 \log \Omega\|_{L^2(S_v)}\right) dv.
\]

On the other hand, commuting elliptic equation (5.5), with \(\nabla\), we obtain schematically
\[
\Delta \nabla \log \Omega = G'(\gamma) + G'(\gamma) \cdot \nabla \gamma + (F'_4)^\dagger \cdot \nabla^2 \gamma + [\Delta, \nabla] \log \Omega,
\]
where using formula (2.15) and Propositions 2.32 and 2.33, the commutator can be rewritten
\[
[\Delta, \nabla] \log \Omega = -K \nabla \log \Omega
\]
\[
= - \left( -\frac{1}{4} \text{tr}\chi \text{tr}\chi + \frac{1}{2} \tilde{\chi} \cdot \tilde{\chi} - \rho \right) \nabla \log \Omega \\
= (G'(\Upsilon) + G'(\Upsilon) \cdot \nabla \Upsilon) \nabla \log \Omega.
\]

We therefore obtain the following formula
\[
\Delta \nabla \log \Omega = G'(\Upsilon) + G'(\Upsilon) \cdot (\nabla \Upsilon + \nabla \log \Omega) \\
+ G'(\Upsilon) \cdot \nabla \Upsilon \cdot \nabla \log \Omega + (F_4') \cdot \nabla^2 \Upsilon.
\]

Therefore, using Lemma 3.17, assumptions (5.1) and Sobolev Lemma 3.8, we have
\[
\begin{align*}
\left\| \nabla^3 \log \Omega \right\|_{L^2(S_v)} & \lesssim \left\| G'(\Upsilon) \right\|_{L^\infty(H)} \\
& + \left\| G'(\Upsilon) \right\|_{L^\infty(H)} \left( \left\| \nabla \Upsilon \right\|_{L^\infty L^2} + \left\| \nabla \log \Omega \right\|_{L^\infty L^2} \right) \\
& + \left\| F_4' \right\|_{L^\infty(H)} \left\| \nabla^2 \Upsilon \right\|_{L^2(S_v)} \\
& \lesssim C \left( 1 + \left\| \nabla^2 \Upsilon \right\|_{L^2(S_v)} \right).
\end{align*}
\]

Plugging this estimate into (5.6), we obtain
\[
\left\| \nabla^2 \Upsilon \right\|_{L^\infty L^2} \lesssim C + C \int_v \left\| \nabla^2 \Upsilon \right\|_{L^2(S_v)} \, dv.
\]

By a Grönwall argument, we deduce
\[
\left\| \nabla^2 \Upsilon \right\|_{L^\infty L^2} \lesssim C.
\]

This finishes the proof of the Lemma in the case \( m = 2 \).

To prove estimates (5.3), we commute elliptic equation (5.5) for \( \log \Omega \) with \( L \) and using the formula (5.4), the right-hand side can be expressed in terms of lower order derivatives of \( \log \Omega \). Details are left to the reader. \( \square \)

### 6 Proof of local existence

In this section, we prove Theorem 2.38 by showing a more general local existence theorem for equations of the type (5.4)-(5.5), where the unknown is the function
\[
(v, \omega) \in [1, 2] \times S^2 \mapsto s(v, \omega).
\]

This strategy is similar to writing a foliation by geometric flows as family of graphs and was already used in [22] and [25].
6.1 Geometric setup and theorem

Let $v_0 \in [1, 2)$, and define

$$C := \{ (v, \omega) \in [v_0, 2] \times S^2 \}, \quad S_v := \{ v \} \times S^2 \subset C.$$ 

Similarly, let

$$C' := \{ (s, \omega) \in [1, 5/2] \times S^2 \}, \quad S'_s := \{ s \} \times S^2 \subset C'.$$

Let $g$ be a smooth degenerate metric on $C'$ such that the induced metric on $S'_s$ is Riemannian. Let $F'_1, F'_2, F'_3, F'_4$ be respectively a fixed scalar field, a fixed 1, 2 and 2 $S'_s$-tangent tensor.

For a function $s : C \to [1, 5/2]$, we define

$$\Phi_1(s) : C \to C', \quad (v, \omega) \mapsto (s(v, \omega), \omega).$$

For a function $s : C \to [1, 5/2]$, we define $g(s)$ to be the induced Riemannian metric on $S_v$ by $\Phi(s)^* g$, and $F_i(s) := (\Phi(s)^* F'_i)^\dagger$, where $^\dagger$ denotes the projection of $C$-tangent tensors to $S_v$-tangent tensors defined in Definition 2.26.

Remark 6.1 By the degeneracy of the metric $g$ and the definition of $\Phi$, the metric $g(s)$ depends only on $s$ and not on derivatives of $s$. Similarly, since the tensors $F'_i$ are $S'_s$-tangent and by the definition of $\Phi$, the tensors $F_i(s)$ only depend on $s$.

In what follows, our goal is to prove local existence for the system of quasilinear elliptic transport equations in $C$

$$\log \Omega = \Delta^{-1} \left( F_1(s) + F_2(s) \cdot \nabla s + F_3(s) \cdot \nabla s \cdot \nabla s + F_4(s) \cdot \nabla^2 s \right), \quad \partial_{v, s} \Omega = \Omega^{-1}, \quad \text{(6.1)}$$

where $\nabla$ and $\Delta$ are respectively the covariant derivative and the Laplacian associated to $g(s)$ and where for a Riemannian 2-sphere $(S, g)$, $u := \Delta^{-1} f$ is the solution of

$$\Delta u = f - \int_S f, \quad \int_S u = 0,$$

with integrals taken with respect to the metric $g$.

For ease of notation, we shall define $F(s, \nabla s, \nabla^2 s)$ to denote schematically

$$F(s, \nabla s, \nabla^2 s) := F_1(s) + F_2(s) \cdot \nabla s + F_3(s) \cdot \nabla s \cdot \nabla s + F_4(s) \cdot \nabla^2 s.$$
Let \( s_0 \) be a function \( S_{v_0} \rightarrow [1, 5/2] \) and extend it to \( C \) by requiring that \( \partial_v s_0 = 0 \). Define \( (0)\nabla \) and \( (0)\Delta \) to be respectively the covariant derivative and Laplacian on all spheres \( S_v \) associated to \( g(s_0) \). Define \( \log \Omega_0 \) on all spheres \( S_v \) by

\[
\log \Omega_0 := (0)\Delta^{-1} \left( F(s_0, (0)\nabla s_0, (0)\nabla^2 s_0) \right).
\]

We have the following result.

**Theorem 6.2** Assume that \( s_0 \in H^5(S_{v_0}) \) and that \(|\log \Omega_0| \leq 1/100\). Assume moreover that \((S_{v_0}, g(s_0))\) is a weakly spherical 2-sphere of radius \( v_0 \) (see Definition 3.14) with constants such that the Bochner and Hodge estimates from Lemmas 3.17, 3.19, 3.20 hold true. There exists

\[
\delta \left( \left\| (s_0 - 5/2)^{-1} \right\|_{L^\infty(S_{v_0})}, \left\| s_0 \right\|_{H^5(S_{v_0})}, \left\| F'_i \right\|_C^3 \right) > 0,
\]

and a unique\(^3\)

\[
s \in \mathcal{C}^0([v_0, v_0 + \delta], H^5) \cap \mathcal{C}^1([v_0, v_0 + \delta], H^4),
\]

such that on \( C \) the system of equations (6.1) is satisfied for \( s \), with the initial condition \( s|_{S_{v_0}} = s_0 \). Moreover, we have

\[
|\log \Omega| < 1/10.
\]

\[6.2\] Proof of Theorem 6.2

Recall that the goal of Theorem 6.2 is to find a function \( s \) (together with its associated function \( \Omega \)) solutions to the general system of transport-elliptic equations (6.1). The proof goes by a classical Banach-Picard iteration. The general scheme is to define a sequence of functions \((s_n, \Omega_n)\) approximate solutions to (6.1) and to show that the time of existence \( \delta \) can be chosen sufficiently small so that the sequence is bounded in the desired functional space, i.e.

\[
\left\| s_n \right\|_{\mathcal{C}^0([v_0, v_0 + \delta], H^5(S_v))} + \left\| \Omega_n \right\|_{\mathcal{C}^0([v_0, v_0 + \delta], H^4(S_v))} \lesssim 1,
\]

and is a contraction, i.e.

\[
\left\| s_{n+1} - s_n \right\|_{\mathcal{C}^0([v_0, v_0 + \delta], H^5(S_v))} + \left\| \Omega_{n+1} - \Omega_n \right\|_{\mathcal{C}^0([v_0, v_0 + \delta], H^4(S_v))} \leq \kappa \left( \left\| s_n - s_{n-1} \right\|_{\mathcal{C}^0([v_0, v_0 + \delta], H^5(S_v))} + \left\| \Omega_n - \Omega_{n-1} \right\|_{\mathcal{C}^0([v_0, v_0 + \delta], H^4(S_v))} \right),
\]

with \( \kappa < 1 \). From a classical argument, the sequence \((s_n, \Omega_n)\) then converges and the limit \((s, \Omega)\) is the desired solution of (6.1).

\(^3\) The uniqueness of \( s \) follows from re-running the contraction argument of the proof of Theorem 6.2 in Section 6.2. Details are left to the reader.
Definition of the iteration

As defined previously, we have $s_0(v, \omega) := s_0(\omega)$ and $\Omega_0(v, \omega) := \Omega_0(\omega)$. For all $n \geq 0$, we define $s_{n+1}$ and $\log \Omega_{n+1}$ on $C$ by

$$s_{n+1}(v, \omega) := s_0(\omega) + \int_{v_0}^v \Omega_n^{-1}(v', \omega) \, dv', \quad (6.3)$$

$$\log(\Omega_{n+1}) := (n+1) \Delta^{-1} \left( F \left( s_{n+1}, (n+1) \nabla s_{n+1}, (n+1) \nabla^2 s_{n+1} \right) \right). \quad (6.4)$$

We define

$$M_0 := \sum_{l \leq 5} \left\| (0) \nabla^l s_0 \right\|_{L^2(S_{v_0})},$$

and

$$M_\delta := \sup_{v_0 \leq v \leq v_0 + \delta} \left( \sum_{l \leq 5} \left\| (0) \nabla^l (s_n(v) - s_0) \right\|_{L^2(S_{v_0})} + \sum_{l \leq 5} \left\| (0) \nabla^l \log \Omega_n(v) \right\|_{L^2(S_{v_0})} \right).$$

Boundedness of the iteration

In this section, we show that if

$$\delta < \delta \left( \left\| (s_0 - 5/2)^{-1} \right\|_{L^\infty(S_{v_0})}, M_0, \| F \|_{C^3} \right),$$

then, defining $M := 2M_0$, we have for all $n \geq 0$

$$M_\delta \leq M, \quad (6.5)$$

$$\sup_{v_0 \leq v \leq v_0 + \delta} s_n(v) < 5/2, \quad (6.6)$$

$$\sup_{v_0 \leq v \leq v_0 + \delta} \left\| \log \Omega_n \right\|_{L^\infty(S_v)} \leq 1/10. \quad (6.7)$$

We argue by induction and assume that these assumptions hold for an arbitrary $n \in \mathbb{N}$. First, using the transport equation (6.3) and estimate (6.7), if $\delta$ is small enough depending on $\left\| (s_0 - 5/2)^{-1} \right\|_{L^\infty(S_{v_0})}$, we have

$$\sup_{v_0 \leq v \leq v_0 + \delta} s_{n+1}(v) < 5/2,$$

and therefore (6.6) is proved for $n + 1$.

Second, using estimate (6.5) and (6.7) at step $n$, we obtain that

$$\sup_{v_0 \leq v \leq v_0 + \delta} \sum_{l \leq 5} \left\| (0) \nabla^l (\Omega_n^{-1}) \right\|_{L^2(S_{v_0})} \leq C(M),$$
where here and in the following \( C (\cdot) \) are smooth universal positive increasing functions with \( C(0) = 0 \). These functions \textit{a priori} differ from one line to another but we chose to keep them under the same notation for simplicity. Therefore, differentiating and estimating equation (6.3), we obtain

\[
\sup_{v_0 \leq v \leq v_0 + \delta} \sum_{l \leq 5} \left\| (0) \mathcal{W}^{l} (s_{n+1} (v) - s_{0}) \right\|_{L^{2}(S_{v})} \leq \delta C(M). \tag{6.8}
\]

Third, we can rewrite equation (6.4)

\[
(0) \Delta (\log \Omega_{n+1} - \log \Omega_{0}) = \left( (0) \Delta - \left( n + 1 \right) \Delta \right) \left( \log \Omega_{n+1} - \log \Omega_{0} \right) + E_{n+1}, \tag{6.9}
\]

with

\[
E_{n+1} := \left( (0) \Delta - \left( n + 1 \right) \Delta \right) \left( \log \Omega_{0} \right) + F (s_{n+1}, \left( n + 1 \right) \nabla s_{n+1}, \left( n + 1 \right) \nabla^{2} s_{n+1}) - F (s_{0}, (0) \nabla s_{0}, (0) \nabla^{2} s_{0}).
\]

From the weak sphericity assumption from Theorem 6.2, we can apply the elliptic estimate from Lemma 3.20 and the Bochner estimate for scalar function of Lemma 3.17, which add together yield

\[
\left\| (0) \nabla^{2} \left( \log \Omega_{n+1} - \log \Omega_{0} \right) \right\|_{L^{2}(S_{v})} + \left\| (0) \nabla \left( \log \Omega_{n+1} - \log \Omega_{0} \right) \right\|_{L^{2}(S_{v})} + \left\| \log \Omega_{n+1} - \log \Omega_{0} - \log \Omega_{n+1} - \log \Omega_{0} \right\|_{L^{2}(S_{v})} \leq \delta C(M) \left\| \log \Omega_{n+1} - \log \Omega_{0} \right\|_{L^{2}(S_{v})} + \left\| E_{n+1} \right\|_{L^{2}(S_{v})}, \tag{6.10}
\]

where, on the left-hand side, the average is taken with respect to the metric \( \bar{g}(s_{0}) \). Using Remark 6.1 and the fact that by definition \( \log \Omega_{n+1} \) and \( \log \Omega_{0} \) have vanishing means on respectively the Riemannian spheres \( (S_{v}, \bar{g}(s_{n+1})) \) and \( (S_{v}, \bar{g}(s_{0})) \), we have

\[
\int_{(S_{v}, \bar{g}(s_{0}))} \left( \log \Omega_{n+1} - \log \Omega_{0} \right) = \int_{(S_{v}, \bar{g}(s_{0}))} \log \Omega_{n+1} - \int_{(S_{v}, \bar{g}(s_{n+1}))} \log \Omega_{n+1} \leq \int_{(S_{v}, \bar{g}(s_{0}))} \left| \log \Omega_{n+1} \right| \left| \sqrt{\bar{g}_{(s_{n+1})}} - \sqrt{\bar{g}_{(s_{0})}} \right| \leq C \left( \| s_{n+1} - s_{0} \|_{L^{\infty}(S_{v})} \right) \left\| \log \Omega_{n+1} \right\|_{L^{2}(S_{v})} \leq C \left( \sum_{l=0}^{5} \left\| (0) \mathcal{W}^{l} (s_{n+1} - s_{0}) \right\|_{L^{2}(S_{v})} \right) \times \left\| \log \Omega_{n+1} \right\|_{L^{2}(S_{v})} \leq \delta C(M) \left\| \log \Omega_{n+1} - \log \Omega_{0} \right\|_{L^{2}(S_{v})} + \delta C(M), \tag{6.11}
\]
where the integral of the second line is to be understood as being in coordinate patches and where $\sqrt{|g(s)|}$ is the determinant of the Riemannian metric $g(s)$ in these coordinate patches and where from the second to the third line we used a standard Sobolev estimate. Moreover, using that in coordinates

$$(0) \Delta - (n+1) \Delta = \frac{1}{\sqrt{|g(s_0)|}} \partial_i \left( \sqrt{|g(s_0)|} g^{ij}(s_0) \partial_j \right)$$

$$- \frac{1}{\sqrt{|g(s_{n+1})|}} \partial_i \left( \sqrt{|g(s_{n+1})|} g^{ij}(s_{n+1}) \partial_j \right)$$

$$= \sum_{l=0}^{2} f_l^0(s_0, s_{n+1}, (0) \nabla s_0, (0) \nabla s_{n+1})(s_{n+1} - s_0), (0) \nabla l$$

$$+ \sum_{l=0}^{1} f_l^1(s_0, s_{n+1}, (0) \nabla s_0, (0) \nabla s_{n+1})(0) \nabla s_{n+1} - (0) \nabla s_0), (0) \nabla l,$$

with $f_l^0$, $f_l^1$ smooth, together with standard Sobolev estimates, we have

$$\left\| \left( (0) \Delta - (n+1) \Delta \right) (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)}$$

$$\leq \left( \sum_{l=0}^{1} \left\| f_l^0(s_0, s_{n+1}, (0) \nabla s_0, (0) \nabla s_{n+1}) \right\|_{L^\infty(S_v)} \right)$$

$$\times \left( \|s_{n+1} - s_0\|_{L^\infty(S_v)} + \left\| (0) \nabla (s_{n+1} - s_0) \right\|_{L^\infty(S_v)} \right)$$

$$\times \left( \sum_{l=2}^{l \leq 2} \left\| (0) \nabla l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \right)$$

$$\leq C \left( \sum_{l=5}^{l \leq 5} \left\| (0) \nabla l s_0 \right\|_{L^2(S_v)} \cdot \sum_{l=5}^{l \leq 5} \left\| (0) \nabla l s_{n+1} \right\|_{L^2(S_v)} \right)$$

$$\times C \left( \sum_{l=5}^{l \leq 5} \left\| (0) \nabla l (s_{n+1} - s_0) \right\|_{L^2(S_v)} \right)$$

$$\times \left( \sum_{l=2}^{l \leq 2} \left\| (0) \nabla l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \right)$$

$$\leq \delta C(M) \left( \sum_{l=2}^{l \leq 2} \left\| (0) \nabla l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \right),$$

and similarly

$$\|E_{n+1}\|_{L^2(S_v)} \leq \left\| \left( (0) \Delta - (n+1) \Delta \right) (\log \Omega_0) \right\|_{L^2(S_v)}$$
The elliptic estimate (6.10) combined with estimates (6.11), (6.12) and (6.13) gives

\[
2 \sum_{l=0}^{2} \left\| (0) \partial^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \\
\leq \left\| (0) \Delta - (n+1) \Delta \right\|_{L^2(S_v)} \left( \log \Omega_{n+1} - \log \Omega_0 \right) + E_{n+1} + \left\| \log \Omega_{n+1} - \log \Omega_0 \right\|_{L^2(S_v)} \\
\leq \delta C(M) + \delta C(M) \sum_{l=0}^{2} \left\| (0) \partial^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \leq \delta C(M),
\]

where the last line is obtained by absorption provided that \( \delta C(M) < 1 \). We now prove that we have the following bounds for the remaining higher order derivatives

\[
5 \sum_{l=0}^{5} \left\| (0) \partial^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \leq \delta C(M). \tag{6.14}
\]

The proof goes by induction on \( l \) for \( l = 2 \) to \( l = 5 \). We only do the case \( l = 5 \) assuming that the bounds for \( l \leq 4 \) have been obtained, since it will be clear that the proof for the other cases is almost identical. Commuting equation (6.9) with \((0) \partial^3\) gives

\[
(0) \Delta (0) \partial^3 (\log \Omega_{n+1} - \log \Omega_0) = [(0) \Delta, (0) \partial^3] (\log \Omega_{n+1} - \log \Omega_0) \\
+ (0) \partial^3 \left( \left( (0) \Delta - (n+1) \Delta \right) (\log \Omega_{n+1} - \log \Omega_0) \right) \\
+ (0) \partial^3 (E_{n+1}). \tag{6.15}
\]
The commutator \([\mathcal{A}, \mathcal{V}^3]\) is a fourth-order linear differential operator and we have

\[
\begin{aligned}
\left\| \left[ \mathcal{A}, \mathcal{V}^3 \right] (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_e)} &\leq C \left( 1 + \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l s_0 \right\| \right) \\
\times \left( \sum_{l=0}^{4} \left\| (0)^l \mathcal{V}^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_e)} \right) \\
&\leq \delta C(M),
\end{aligned}
\]

(6.16)

where we implicitly used standard Sobolev estimates and where we used that by the induction hypothesis the bounds for \(l \leq 4\) have been obtained. Similarly, we have

\[
\begin{aligned}
\left\| \mathcal{V}^3 E_{n+1} \right\|_{L^2(S_e)} &\leq \left\| \mathcal{V}^3 \left[ \left( \mathcal{A} - (n+1) \mathcal{A} \right) (\log \Omega_0) \right] \right\|_{L^2(S_e)} \\
&+ \left\| \mathcal{V}^3 \left( F(s_{n+1}, (n+1) \mathcal{V}s_{n+1}, (n+1) \mathcal{V}^2 s_{n+1}) - F(s_0, (0)^0 \mathcal{V}s_0, (0)^2 \mathcal{V}s_0) \right) \right\|_{L^2(S_e)} \\
&\leq C \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l s_0 \right\|_{L^2(S_e)} \right) \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l \log \Omega_0 \right\|_{L^2(S_e)} \right) \\
&+ C \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l \right\|_{L^2(S_e)} \right) \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l s_{n+1} \right\|_{L^2(S_e)} \right) \\
&\times \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l (s_{n+1} - s_0) \right\|_{L^2(S_e)} \right) \leq \delta C(M),
\end{aligned}
\]

(6.17)

and

\[
\begin{aligned}
\left\| (0)^0 \mathcal{V}^3 \left[ \left( \mathcal{A} - (n+1) \mathcal{A} \right) (\log \Omega_{n+1} - \log \Omega_0) \right] \right\|_{L^2(S_e)} &\leq C \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l s_0 \right\|_{L^2(S_e)} \right) \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l s_{n+1} \right\|_{L^2(S_e)} \right) \\
&\times \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l (s_{n+1} - s_0) \right\|_{L^2(S_e)} \right) \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_e)} \right) \\
&\leq \delta C(M) \left( \sum_{l=0}^{5} \left\| (0)^l \mathcal{V}^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_e)} \right).
\end{aligned}
\]

(6.18)
Using a Bochner estimate similar to Lemma 3.17 for tensors of arbitrary type applied to equation (6.15) together with estimates (6.16), (6.17) (6.18) and the induction hypothesis, we deduce that

\[ \left\| (0) \bar{\nabla}^5 (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_c)} \overset{\text{absorption}}{\leq} \left\| (0) \bar{\nabla}^3 \left( (0) \Delta - (n+1) \Delta \right) (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_c)} + \left\| (0) \bar{\nabla}^3 E_{n+1} \right\|_{L^2(S_c)} + \sum_{l=0}^{4} \left\| (0) \bar{\nabla}^{1} (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_c)} \leq \delta C(M) + \delta C(M) \left\| (0) \bar{\nabla}^5 (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_c)} \leq \delta C(M), \]

where the last line is obtained by absorption, provided that \( \delta C(M) < 1 \). This finishes the proof of the desired bound (6.14).

Combining (6.8) and (6.14), we have proved that

\[ M^\delta_{n+1} \leq M_0 + \delta C(M) \leq M, \]

provided that \( \delta > 0 \) was chosen such that \( \delta C(M) < M_0 \).

Moreover, by Sobolev embedding, we deduce that

\[ \| \log \Omega_{n+1} - \log \Omega_0 \|_{L^\infty(S_c)} \overset{\text{absorption}}{\leq} \sum_{l\leq 5} \left\| (0) \bar{\nabla}^{l} (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_c)} \leq \delta C(M). \]

Therefore, for \( \delta \) such that \( \delta C(M) < 1/100 \), and using the assumption \( |\log \Omega_0| \leq 1/100 \), this proves the bound (6.7) for \( n+1 \). This finishes the proof of the boundedness of the sequence \( s_{n+1} \).

**Contraction of the iteration** We define

\[ \Delta^\delta_{n+1} := \sup_{v_0 \leq v \leq v_0 + \delta} \left( \sum_{l\leq 5} \left\| (0) \bar{\nabla}^{l} (s_{n+1} - s_n) \right\|_{L^2(S_c)} + \sum_{l\leq 5} \left\| (0) \bar{\nabla}^{l} (\log \Omega_{n+1} - \log \Omega_n) \right\|_{L^2(S_c)} \right), \]

and we show, provided that \( \delta \) has been chose small enough, that we have

\[ \Delta^\delta_{n+1} \leq \kappa \Delta^\delta_n, \tag{6.19} \]

with \( \kappa < 1 \). The proof follows the lines of the proof of the boundedness. First, we have

\[ \sum_{l\leq 5} \left\| (0) \bar{\nabla}^{l} \left( \Omega^{-1}_{\Delta n+1} - \Omega^{-1}_{\Delta n} \right) \right\|_{L^2(S_c)} \leq C(M) \left( \sum_{l\leq 5} \left\| (0) \bar{\nabla}^{l} (\log \Omega_n - \log \Omega_{n-1}) \right\|_{L^2(S_c)} \right). \]
Therefore we deduce using equation (6.3) that
\[ \sum_{l \leq 5} \left\| (0) \nabla^l (s_{n+1} - s_n) \right\|_{L^2(S_v)} \leq \delta C(M) \Delta_n^\delta, \]
which using the boundedness of the Banach-Picard iteration also gives
\[ \sum_{l \leq 5} \left\| (n) \nabla^l (s_{n+1} - s_n) \right\|_{L^2(S_v)} \leq \delta C(M) \Delta_n^\delta. \]  \hspace{1cm} (6.20)

Performing the exact same elliptic estimates as in the proof of the boundedness of the iteration – replacing 0 in that proof by \( n \) and replacing the use of estimate (6.8) by the use of (6.20) –, we obtain that
\[ \sum_{l \leq 5} \left\| (n) \nabla^l \left( \log \Omega_{n+1} - \log \Omega_n \right) \right\|_{L^2(S_v)} \leq \delta C(M) \Delta_n^\delta, \]
from which it follows that
\[ \sum_{l \leq 5} \left\| (0) \nabla^l \left( \log \Omega_{n+1} - \log \Omega_n \right) \right\|_{L^2(S_v)} \leq \delta C(M) \Delta_n^\delta. \]

Thus, for \( \delta \) such that \( \kappa := \delta C(M) < 1 \), we deduce the desired (6.19) and this finishes the proof of the contraction and of Theorem 6.2.

### 6.3 Proof of Theorem 2.38

In this section, we show how Theorem 2.38 follows from Theorem 6.2.

We define \( v_0 := v^* \) and \( s_0 := s^* = s|_{v^*} \), and \( F'_1, F'_2, F'_3, F'_4 \) to be the tensors defined in Lemma 5.1. By assumptions and since the \( F'_i \) have been defined as in Lemma 5.1, the quantity \( \log \Omega_0 \) defined in Section 6.1 coincides with \( \log \Omega|_{S_{v_0}} \).

By assumption of Theorem 2.38, we have \( s_0 \in H^5(S_{v_0}), |\log \Omega_0| \leq 1/100 \) and that \( (S_{v_0}, g(s_0)) \) is a weakly spherical 2-sphere of radius \( v_0 \).

Applying Theorem 6.2, there exists \( \delta > 0 \) and a function \( s \in C^1([v_0, v_0 + \delta], S^2) \), satisfying the system of equations (6.1). Since by estimate (6.2) we have \( |\partial_v s - 1| < 1/5 \), the map \( \Phi(s) : [v_0, v_0 + \delta] \times S^2 \to [1, 5/2] \times S^2 \) admits a \( C^1 \)-inverse by the global inverse theorem. This defines a \( C^1 \)-function \( v \) in geodesic coordinates, and therefore on \( \mathcal{H} \), taking values from \( v^* = v_0 \) to \( v^* + \delta \), which, using the conclusion of Theorem 6.2, is regular. Since equations (6.1) are satisfied and since the \( F'_i \) have been defined as in Lemma 5.1, we deduce that \( (S_v)_{v^* \leq v \leq v^* + \delta} \) is a canonical foliation. In case \( (S_v)_{1 \leq v \leq v^*} \) was a regular canonical foliation, using equations (6.1), one deduces that \( (S_v)_{v^* \leq v \leq v^* + \delta} \) is a regular extension thereof. This finishes the proof of Theorem 2.38.

**Acknowledgements** Both authors are very grateful to Jérémie Szeftel for many interesting and stimulating discussions. The second author is supported by the ERC grant ERC-2016 CoG 725589 EPGR.

\( \S \) Springer
Appendix A. Proof of Propositions 2.32 and 2.33

In this section we prove the formulas from Proposition 2.32 and 2.33. The following computations are standard and can be found in various forms in [1, 22, 25] for instance. In what follows, we use the formulas from [8] pp. 149-150. We have

\[ \alpha_{AB} = R(L, e_A, L, e_B) = R(L, e'_A + \gamma_A L, L, e'_B + \gamma_B L) \]
\[ = R(L, e'_A, L, e'_B) = (\alpha')^\dagger_{AB}, \]

and

\[ \beta_A = \frac{1}{2} R(e_A, L, L, L) \]
\[ = \frac{1}{2} R(e'_A + \gamma_A L, L, L' + 2\gamma_B e'_B + |\gamma|^2 L, L) \]
\[ = \frac{1}{2} R(e'_A, L, L', L) + \gamma_B R(e'_A, L, e'_B, L) \]
\[ = \beta'_A + \gamma_B \alpha'_{AB} \]
\[ = (\beta')^\dagger_A + \gamma_B (\alpha')^\dagger_{AB}, \]

and

\[ \rho = \frac{1}{4} R(L, L, L, L) \]
\[ = \frac{1}{4} R(L', 2\gamma_A e'_A + |\gamma| L, L' + 2\gamma_B e'_B + |\gamma|^2 L, L) \]
\[ = \frac{1}{4} R(L', L, L', L) + \frac{1}{2} \gamma_A R(e'_A, L, L', L) \]
\[ + \frac{1}{2} \gamma_B R(L', L, e'_B, L) + \gamma_A \gamma_B R(e'_A, L, e'_B, L) \]
\[ = \rho' + (\beta')^\dagger \cdot \gamma + (\alpha')^\dagger \cdot \gamma \cdot \gamma. \]

Following the previous computation for \( \rho \) we also obtain

\[ \sigma = \frac{1}{4} * R(L, L, L, L) = \sigma' + \gamma_A * R(e'_A, L, L', L) + \gamma_A \gamma_B * R(e'_A, L, e'_B, L) \]
\[ = \sigma' - (\beta')^\dagger \cdot (\gamma) - (\alpha')^\dagger \cdot \gamma \cdot \gamma. \]

We have

\[ \beta_A = \frac{1}{2} R(e_A, L, L, L) = \frac{1}{2} R(e'_A + \gamma_A L, L' + 2\gamma_B e'_B + |\gamma|^2 L, L' + 2\gamma_C e'_C + |\gamma|^2 L, L) \]
\[ = \frac{1}{2} R(e'_A, L', L', L) + \frac{1}{2} \gamma_A R(L, L', L', L) \]
This finishes the proof of Proposition 2.32. We turn to the connection coefficients. We have immediately $\chi_{AB} = \chi'_{AB}$. We also have

\[
\zeta_A = \frac{1}{2} g(D_A L, L)
\]

\[
= \frac{1}{2} g(D e'_A + \gamma_A L, L' + 2 \gamma_B e'_B + |\gamma|^2 L)
\]

\[
= \frac{1}{2} g(D e'_A, L') + \gamma_B g(D e'_A, e'_B)
\]

\[
= (\zeta')^\dagger_A + \gamma \cdot \chi_A,
\]

and

\[
\eta_A = \frac{1}{2} g(D_L L, e_A)
\]

\[
= \frac{1}{2} g(D_L L' + 2 \gamma_B e'_B + |\gamma|^2 L, e'_A + \gamma_A L)
\]

\[
= \frac{1}{2} g(D_L L', e'_A) + g(D_L (\gamma_B e'_B), e'_A)
\]

\[
= (\eta')^\dagger_A + \gamma L \gamma_A.
\]

Finally, we have

\[
\chi_{AB} = g(D_A L, e_B)
\]

\[
= g(D e_A (L' + 2 \gamma_C e_C - |\gamma|^2 L), e_B)
\]

\[
= 2 \gamma_A \gamma_B + g(D e_A (L' - |\gamma|^2 L), e_B)
\]

\[
= 2 \gamma_A \gamma_B + g(D e_A (L' - |\gamma|^2 L), e'_B + \gamma_B L)
\]

\[
= 2 \gamma_A \gamma_B + g(D e_A (L' + \gamma_A L), e'_B) + \gamma_A g(D_L L', e'_B)
\]

\[
+ \gamma_B g(D e'_A L', L) - |\gamma|^2 g(D e'_A L', e'_B)
\]

\[
= 2 \gamma_A \gamma_B + (\chi')^\dagger_{AB} + 2 \gamma_A (\eta')^\dagger_B - 2 \gamma_B (\zeta')^\dagger_A - |\gamma|^2 (\chi')^\dagger_{AB}.
\]

This finishes the proof of Proposition 2.33.
Appendix B. Proof of Lemmas 3.5 and 3.20

B.1. Proof of Lemma 3.5

This section is dedicated to the proof of Lemma 3.5. In fact, we prove the following more general estimate

\[ \| \nabla F \|_{H^s(S)} \lesssim \| F \|_{H^{s+1}(S)}, \quad \text{(B.1)} \]

for \(-1 < s < 0\).

**Remark B.1** As it will be clear from what follows, the proof below does not work for other ranges of exponents \(s\), and would require additional regularity assumptions on the 2-sphere \(S\).

From Proposition 2.3 in [26], we have the following characterisation of \(H^s(S)\) using the Littlewood-Paley projectors defined in Section 3.2

\[ \| F \|_{H^s(S)}^2 \simeq \sum_{k \geq 0} 2^{2sk} \| P_k F \|_{L^2(S)}^2 + \| P_{<0} F \|_{L^2(S)}^2. \quad \text{(B.2)} \]

From Section 2.2 and Proposition 2.1 in [26], we recall the following properties of the Littlewood-Paley projection operators defined in Section 3.2. For all \(k \in \mathbb{Z}\), we have

\[ P_k = P_k P_{k-1} + P_k P_k + P_k P_{k+1}, \quad \text{(B.3)} \]

and for \(F\) an \(S\)-tangent tensor and for all \(k \in \mathbb{Z}\), we have

\[ \| P_k F \|_{L^2(S)} \lesssim \| F \|_{L^2(S)}, \quad \| P_{<0} F \|_{L^2(S)} \lesssim \| F \|_{L^2(S)} \quad \text{(B.4)} \]

and,

\[ \| P_k \nabla F \|_{L^2(S)} \lesssim 2^k \| F \|_{L^2(S)}, \quad \| \nabla P_k F \|_{L^2(S)} \lesssim 2^k \| F \|_{L^2(S)}, \quad \text{(B.5)} \]

We turn to the proof of estimate (B.1). Using (3.4), (B.2) and (B.5), we have

\[ \| \nabla F \|_{H^s(S)}^2 \lesssim \sum_{k \geq 0} 2^{2sk} \| P_k \nabla F \|_{L^2(S)}^2 + \| P_{<0} \nabla F \|_{L^2(S)}^2 \]

\[ \lesssim \sum_{k \geq 0} 2^{2sk} \| P_k \nabla P_{>k} F \|_{L^2(S)}^2 + \sum_{k \geq 0} 2^{2sk} \| P_k \nabla P_{\leq k} F \|_{L^2(S)}^2 + \| F \|_{L^2(S)}^2. \quad \text{(B.6)} \]

The first term in the right-hand side of (B.6) can be estimated using (B.3), (B.5) and that \(-1 < s < 0\).
\[
\sum_{k \geq 0} 2^{2sk} \| P_k \nabla P_{>k} F \|_{L^2(S)}^2 \lesssim \sum_{k \geq 0} 2^{2(s+1)k} \| P_{>k} F \|_{L^2(S)}^2 \\
\lesssim \sum_{k \geq 0} \sum_{l > k} 2^{2(s+1)k} \| P_l F \|_{L^2(S)}^2 \\
\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \| P_l F \|_{L^2(S)}^2 \sum_{k = 0}^{l-1} 2^{2(s+1)(k-l)} \\
\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \| P_l F \|_{L^2(S)}^2 .
\]

For the second term in the right-hand side of (B.6), using (B.4), we first write the following decomposition

\[
\| \nabla P_{\leq k} F\|_{L^2(S)}^2 = \sum_{0 \leq l \leq k} \sum_{0 \leq l' \leq k} \int_S \nabla P_l F \cdot \nabla P_{l'} F + 2 \sum_{0 \leq l \leq k} \int_S \nabla P_l F \cdot \nabla P_{<0} F \\
+ \int_S \nabla P_{<0} F \cdot \nabla P_{<0} F.
\]

The first term can be estimated, using that \( \Delta \) preserves the support of the projectors \( P_k \) (see also Section 2.2 in [26]) and (3.4),

\[
\sum_{0 \leq l \leq k} \sum_{0 \leq l' \leq k} \int_S \nabla P_l F \cdot \nabla P_{l'} F = - \sum_{0 \leq l \leq k} \sum_{0 \leq l' \leq k} \int_S P_l F \Delta P_{l'} F \\
= - \sum_{0 \leq l \leq k} \sum_{l' = l-1}^{l+1} \int_S P_l F \Delta P_{l'} F \\
\lesssim \sum_{0 \leq l \leq k} \sum_{l' = l-1}^{l+1} 2^{2l} \| P_l F \|_{L^2(S)} \| P_{l'} F \|_{L^2(S)} \\
\lesssim \sum_{0 \leq l \leq k+1} 2^{2l} \| P_l F \|_{L^2(S)}^2 ,
\]

and similarly, we deduce for the last two terms, using (B.4) and (B.5)

\[
2 \sum_{0 \leq l \leq k} \int_S \nabla P_l F \cdot \nabla P_{<0} F + \int_S \nabla P_{<0} F \cdot \nabla P_{<0} F \lesssim \| F \|_{L^2(S)}^2 .
\]

Using this, (B.4) and that \( -1 < s < 0 \), we therefore deduce that for the second term of (B.6) we have

\( \square \) Springer
\[
\sum_{k \geq 0} 2^{2sk} \| P_k \nabla P_{\leq k} F \|_{L^2(S)}^2 \lesssim \sum_{k \geq 0} 2^{2sk} \| \nabla P_{\leq k} F \|_{L^2(S)}^2 \\
\lesssim \sum_{k \geq 0} \sum_{0 \leq l \leq k+1} 2^{2sk} 2^{2l} \| P_l F \|_{L^2(S)}^2 + \sum_{k \geq 0} 2^{2sk} \| F \|_{L^2(S)}^2 \\
\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \| P_l F \|_{L^2(S)}^2 \left( \sum_{k \geq l-1} 2^{2sk} \right) + \| F \|_{L^2(S)}^2 \\
\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \| P_l F \|_{L^2(S)}^2 + \| F \|_{L^2(S)}^2.
\]

Finally, plugging the above estimates into (B.6) and using (B.2), we obtain
\[
\| \nabla / F \|_{H^s(S)}^2 \lesssim \sum_{l \geq 0} 2^{2(s+1)l} \| P_l F \|_{L^2(S)}^2 + \| F \|_{L^2(S)}^2 \\
\lesssim \| F \|_{H^{s+1}(S)}^2.
\]

This finishes the proof of Lemma 3.5.

**B.2. Proof of Lemma 3.20**

This section is dedicated to the proof of Lemma 3.20. We assume that \( f \) is a scalar function satisfying the elliptic equation (3.11)
\[
\Box f = d^\dagger v P + h.
\]

Multiplying equation (3.11) by \( f - \bar{f} \) and integrating by part, we have
\[
\| \nabla f \|_{L^2(S)}^2 \leq \| P \|_{L^2(S)} \| \nabla f \|_{L^2(S)} + \| h \|_{L^{4/3}(S)} \| f - \bar{f} \|_{L^4(S)}.
\]

Using Lemma 3.19, we have the following Poincaré inequality
\[
\| f - \bar{f} \|_{L^2(S)} = \| (f - \bar{f}, 0) \|_{L^2(S)} = \| (^* \mathcal{P}_1)^{-1} (\nabla f) \|_{L^2(S)} \lesssim \| \nabla f \|_{L^2(S)}.
\]

Therefore, using Sobolev Lemma 3.8, we have
\[
\| \nabla f \|_{L^2(S)}^2 + \| f - \bar{f} \|_{L^2(S)}^2 \lesssim \| P \|_{L^2(S)} \| \nabla f \|_{L^2(S)} \\
+ \| h \|_{L^{4/3}(S)} \left( \| \nabla f \|_{L^2(S)} + \| f - \bar{f} \|_{L^2(S)} \right)
\]

and the bound holds by a standard absorption argument. The bound on \( \mathcal{H} \) follows by integration in \( v \). This finishes the proof of Lemma 3.20.
References

1. Alexakis, S., Shao, A.: Bounds on the Bondi energy by a flux of curvature. J. Eur. Math. Soc. (JEMS) 18(9), 2045–2106 (2016)
2. Alexakis, S., Shao, A.: On the geometry of null cones to infinity under curvature flux bounds. Class. Quantum Gravity 31(19), 62 (2014)
3. Bartnik, R.: Existence of maximal surfaces in asymptotically flat spacetimes. Commun. Math. Phys. 94(2), 155–175 (1984)
4. Christodoulou, D.: The formation of black holes and singularities in spherically symmetric gravitational collapse. Commun. Pure Appl. Math. 44(3), 339–373 (1991)
5. Christodoulou, D.: Bounded variation solutions of the spherically symmetric Einstein-scalar field equations. Commun. Pure Appl. Math. 46, 1131–1220 (1993)
6. Christodoulou, D.: The instability of naked singularities in the gravitational collapse of a scalar field. Ann. Math. 149, 183–217 (1999)
7. Christodoulou, D.: The Formation of Black Holes in General Relativity. EMS Monographs in Mathematics (2009)
8. Christodoulou, D., Klainerman, S.: The global nonlinear stability of the Minkowski space. Princeton Mathematical Series, 41. Princeton University Press, Princeton, NJ, (1993). x+514 pp
9. Czimek, S.: An extension procedure for the constraint equations. Ann. PDE 4, no. 1, Art. 2, 122, (2018)
10. Czimek, S.: Boundary harmonic coordinates on manifolds with boundary in low regularity. Commun. Math. Phys. (2019). https://doi.org/10.1007/s00220-019-03430-7
11. Czimek, S.: The localised bounded $L^2$ curvature theorem. Commun. Math. Phys. (2019). https://doi.org/10.1007/s00220-019-03458-9
12. Czimek, S., Graf, O.: The spacelike-characteristic Cauchy problem of general relativity in low regularity. arXiv, (2019), 90
13. Klainerman, S., Nicolò, F.: The evolution problem in general relativity. Progress in Mathematical Physics, 25. Birkhäuser Boston, Inc., Boston, MA, (2003). xiv + 385 pages
14. Klainerman, S., Nicolò, F.: On local and global aspects of the Cauchy problem in general relativity. Class. Quantum Grav. 16, R73 (1999)
15. Klainerman, S., Rodnianski, I.: Causal geometry of Einstein-vacuum spacetimes with finite curvature flux. Invent. Math. 159(3), 437–529 (2005)
16. Klainerman, S., Rodnianski, I.: A geometric approach to the Littlewood-Paley theory. Geom. Funct. Anal. 16(1), 126–163 (2006)
17. Klainerman, S., Rodnianski, I.: Sharp trace theorems for null hypersurfaces on Einstein metrics with finite curvature flux. Geom. Funct. Anal. 16(1), 164–229 (2006)
18. Klainerman, S., Rodnianski, I., Szefel, J.: The bounded $L^2$ curvature conjecture. Invent. Math. 202(1), 91–216 (2015)
19. Liu, J., Li, J.: A robust proof of the instability of naked singularities of a scalar field in spherical symmetry. Commun. Math. Phys. 363(2), 561–578 (2018)
20. Luk, J.: On the local existence for the characteristic initial value problem in general relativity. Int. Math. Res. Not. IMRN 20, 4625–4678 (2012)
21. Luk, J., Rodnianski, I.: Local propagation of impulsive gravitational waves. Commun. Pure Appl. Math. 68(4), 511–624 (2015)
22. Nicolò, F.: Canonical foliation on a null hypersurface. J. Hyperbolic Differ. Equ. 1(3), 367–428 (2004)
23. Roesch, H.: Proof of a Null Penrose Conjecture Using a New Quasi-local Mass. Ph.D Thesis, Duke University (2017)
24. Penrose, R.: Gravitational Collapse: the Role of General Relativity. Rivista del Nuovo Cimento, Numero Speciale I, 252–276 (1969)
25. Sauter, J.: Foliations of null hypersurfaces and the Penrose inequality. Ph.D. Thesis, ETH Zurich (2008)
26. Shao, A.: New tensorial estimates in Besov spaces for time-dependent (2+1)-dimensional problems. J. Hyperbolic Differ. Equ. 11(4), 821–908 (2014)
27. Szefel, J.: Parametrix for wave equations on a rough background I: regularity of the phase at initial time. arXiv:1204.1768, (2012), 145
28. Szefel, J.: Parametrix for wave equations on a rough background II: construction and control at initial time. arXiv:1204.1769, (2012), 84
29. Szefel, J.: Parametrix for wave equations on a rough background III: space-time regularity of the phase. Astérisque 401, 321 (2018)
30. Szeftel, J.: Parametrix for wave equations on a rough background IV: control of the error term. arXiv:1204.1771, (2012), 284
31. Szeftel, J.: Sharp Strichartz estimates for the wave equation on a rough background. Ann. Sci. Ecol. Norm. Supérieure 49(6), 1279–1309 (2016)
32. Wang, Q.: On the geometry of null cones in Einstein-vacuum spacetimes. Ann. Inst. H. Poincaré Anal. Non Linéaire 26(1), 285–328 (2009)

Publisher's Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.