Global dynamics of a SD oscillator

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Abstract In this paper, we derive the global bifurcation diagrams of a SD oscillator which exhibits both smooth and discontinuous dynamics depending on the value of a parameter $a$. We research all possible bifurcations of this system, including Pitchfork bifurcation, degenerate Hopf bifurcation, homoclinic bifurcation, double limit cycle bifurcation, Bautin bifurcation and Bogdanov–Takens bifurcation. Besides, we show that the system has five limit cycles, including four small limit cycles and one large limit cycle. At last, we give all numerical phase portraits to illustrate our results.

Keywords SD oscillator · Homoclinic loop · Limit cycle · Hopf bifurcation · Bogdanov–Takens bifurcation · Averaging method

Mathematics Subject Classification 34C29 · 34C25 · 47H11

1 Introduction and main results

In recent years, SD (smooth and discontinuous, for short) oscillator was proposed and investigated for studying the transition from smooth to discontinuous dynamics, see, for instance [2–5,15]. In [5,15], the van der Pol damped SD oscillator is given by

$$\ddot{x} + \xi (b + x^2) \dot{x} + x \left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) = 0$$

for studying this transition, where $a \geq 0$, $b$ and $\xi$ can take arbitrary real values. More precisely, the smooth dynamics appears when $a > 0$, while the discontinuous dynamic behavior occurs at $a = 0$. The global dynamics was completely studied in [5] when $a = 0$, and in [15] when $|a - 1| < \varepsilon$, $|\xi| < \varepsilon$ and $\varepsilon$ are sufficiently small. Besides, Cao et al. in [2–4] studied the following linear damped SD oscillator with periodic excitation

$$\ddot{x} + 2\xi \dot{x} + x \left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) = f_0 \cos(\omega t)$$

using the fact that when $f_0 = 0$ this system admits a Hamiltonian formulation, and they analyze the dynamics of the perturbed Hamiltonian system by Melnikov methods when $f_0$ is small. Moreover, they gave many numerical results. Clearly, the dynamics of periodic system (2) is different from the dynamics here studied, the one of system (1).

Clearly, system (1) can be rewritten as the 2-dimensional differential system

\[\ddot{x} + \xi (b + x^2) \dot{x} + x \left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) = f_0 \cos(\omega t)\]
\( \dot{x} = y - \hat{\xi} \left( b x + x^3 \right) =: y - F(x), \)
\( \dot{y} = -x \left( 1 - \frac{1}{\sqrt{x^2 + a^2}} \right) =: -g(x), \)
\( \hat{\xi} = \xi / 3, \hat{b} = 3b, \) and for simplicity in what follows we still denote \( \xi \) and \( b \) by \( \hat{\xi} \) and \( \hat{b} \), respectively. Note that system (3) is invariant by the transformation \((y, t, \xi) \mapsto (-y, -t, -\hat{\xi}). \) Therefore, we only need to consider the set of parameters

\[ G := \{(a, b, \xi) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \}, \]

where \( \mathbb{R}^+ = [0, +\infty). \)

In this paper, we shall describe the dynamics of system (3). Thus, the following theorem is our main result.

**Theorem 1** System (3) has three equilibria \( E_L = (-\sqrt{1 - a^2}, -(1 - a^2 + b)\xi \sqrt{1 - a^2}), E_0 = (0, 0) \) and \( E_R = (\sqrt{1 - a^2}, (1 - a^2 + b)\xi \sqrt{1 - a^2}) \) if \( a < 1, \) and only \( E_0 \) if \( a \geq 1. \) The global bifurcation diagram of system (3) consists of the following bifurcation surfaces:

i. Pitchfork surface \( P := \{(a, b, \xi) \in G : a = 1\}; \)

ii. Hopf surfaces

\[ H_1 := \{(a, b, \xi) \in G : a > 1, b = 0\}, \]
\[ H_2 := \{(a, b, \xi) \in G : b = 3a^2 - 3, 1/\sqrt{3} < a < 1\} \text{ and} \]
\[ H_3 := \{(a, b, \xi) \in G : b = 3a^2 - 3, 0 < a < 1/\sqrt{3}\}; \]

iii. Homoclinic surface \( \text{HL} := \{(a, b, \xi) \in G : b = \varphi(a, \xi), 0 < a < 1\}; \)

iv. Double limit cycle surfaces

\[ \text{DL}_1 := \{(a, b, \xi) \in G : b = \phi_1(a, \xi), 0 < a < 1\} \text{ and} \]
\[ \text{DL}_2 := \{(a, b, \xi) \in G : b = \phi_2(a, \xi), 0 < a < 1/\sqrt{3}\}; \]

v. Codimension 2 Bogdanov–Takens bifurcation with symmetry curve:

\[ BT := \{(a, b, \xi) \in G : a = 1, b = 0\}; \]

vi. Bautin bifurcation curve: \( B_0 := \{(a, b, \xi) \in G : a = 1/\sqrt{3}, b = -2\}, \)

where

\[ \varphi(a, \xi) < \phi_1(a, \xi) < a^2 - 1, \]
\[ \varphi(1, \xi) = \phi_1(1, \xi) = 0, \]
\[ 2\sqrt{3}a - 4 < \phi_2(a, \xi) < \min\{\varphi(a, \xi), 3a^2 - 3\}, \]
\[ \phi_2(1/\sqrt{3}, \xi) = -2, \text{ and} \]
\[ \varphi(0, \xi) = \phi_1(0, \xi) = \phi_2(0, \xi), \]

as shown in Fig. 1. Consequently, the complete classification of the phase portraits of system (3) is given in Fig. 2, where

![Figure 1](image-url)
Fig. 2 Phase portraits of system (3) on the Poincaré disk.
Table 1 Limit cycles and homoclinic loops of system (3)

| Subsets of $G$ | Large limit cycles surrounding all $E_R, E_0, E_L$ | Small limit cycles only surrounding $E_0, E_L, E_R$, respectively | Homoclinic loops |
|----------------|-------------------------------------------------|---------------------------------------------------------------|-----------------|
| $1, 1I, H_1, H_3, P_1, BT$ | 0 | 0; 0; 0 | 0 |
| $1I, P_2$ | 0 | 1 stable; 0; 0 | 0 |
| $IV, H_2, B_0$ | 1 stable | 0; 2 the inner one is stable, the outer one is unstable; 2 the inner one is stable, the outer one is unstable | 0 |
| $V$ | 1 stable | 0; 1 stable; 1 stable | 0 |
| $VI$ | 2 the inner one is unstable, the outer one is stable; | 0; 1 stable; 1 stable | 0 |
| $VII$ | 0 | 0; 0; 0 | 0 |
| $VIII, H_3_2$ | 2 the inner one is unstable, the outer one is stable; | 0; 1 unstable; 1 unstable | 0 |
| $IX, H_3_3$ | 1 stable | 0; 1 stable; 1 stable | 0 |
| $DL_{11}$ | 1 semistable | 0; 0; 0 | 0 |
| $DL_{12}$ | 1 semistable | 0; 1 semistable; 1 semistable | 0 |
| $DL_2$ | 1 stable | 0; 1 stable; 1 stable | Figure-eight loop, unstable |
| $HL_1$ | 1 stable | 0; 1 stable; 1 stable | Figure-eight loop, unstable |
| $HL_2$ | 1 stable | 0; 0; 0 | Figure-eight loop, unstable |

Moreover, all results about limit cycles and homoclinic loops are listed in Table 1.

In this paper, we call large limit cycles to the ones surrounding all three equilibria and small limit cycles the ones surrounding a single equilibrium. For the notions and definitions which appear in the statement of Theorem 1, see its proof.

The paper is organized as follows. In Sect. 2, we analyze the local bifurcations, namely pitchfork bifurcation, Hopf bifurcation, Bautin bifurcation and codimension 2 Bogdanov–Takens bifurcation with symmetry. In Sect. 3, we estimate the number of limit cycles in difference parameter regions and curves. In Sect. 4, we study the global bifurcations, namely the different kinds of homoclinic connections and the double limit cycles. In Sect. 5, we give the numerical phase portraits in different parameter regions.

2 Local bifurcations

Computing the Jacobian matrix at equilibrium $E_0$, we have

$$J_0 := \begin{pmatrix} -b \xi & 1 \\ \frac{1}{a} - 1 & 0 \end{pmatrix}.$$ 

Then, at $E_0$ the determinant $\det(J_0) = 1 - 1/a$ and the trace $\text{tr}(J_0) = -b \xi$, implying that $E_0$ is a saddle if $a < 1$, stable focus or node if $a > 1$ and $b > 0$, and unstable focus or node if $a > 1$ and $b < 0$. When $a = 1$, equilibrium $E_0$ is a stable node if $b > 0$, and unstable node if $b < 0$ by applying [8, Theorem 2.19]. By the symmetry of vector field (3), equilibrium $E_L$ is of the same type as $E_R$. The Jacobian matrix at $E_R$ is

$$J_R := \begin{pmatrix} -\xi(b + 3 - 3a^2) & 1 \\ -(1 - a^2) & 0 \end{pmatrix}.$$ 

Calculate $\det(J_R) = 1 - a^2$ and $\text{tr}(J_R) = -\xi(b + 3 - 3a^2)$. Hence, $E_R$ is a stable focus or node if $a < 1$ and
\[ b + 3 - 3a^2 > 0, \text{ and unstable focus or node if } a < 1 \]
\[ b + 3 - 3a^2 < 0. \]

2.1 Pitchfork bifurcation

From the expressions of the equilibria \( E_L, E_0 \) and \( E_R \), it follows immediately that a pitchfork bifurcation occurs at the origin of coordinates when \( a = 1 \); i.e., for \( a \geq 1 \) we have a unique antisaddle, while for \( 0 < a < 1 \) from the previous antisaddle it bifurcates at \( a = 1 \) a saddle and two antisaddles. For more details on this kind of bifurcation, see [7, 10].

2.2 Hopf bifurcations

There are two kinds of Hopf bifurcations, one at the equilibrium \( E_0 \) and the other at the equilibria \( E_L \) and \( E_R \), which is essentially the same bifurcation in both, because due to the invariance of system (3) with respect to the symmetry \( (x, y) \mapsto (-x, -y) \), what occurs at the equilibrium point \( E_L \) occurs to its symmetric \( E_R \).

2.2.1 Hopf bifurcation at \( E_0 \)

The next result characterizes the Hopf bifurcation at the equilibrium point \( E_0 \), it is proved using the averaging theory, in this way we also can estimated the shape of the limit cycle bifurcating from \( E_0 \), and we avoid the computation of the Liapunov constant.

**Proposition 2** The following statements hold for differential system (3).

a. If \( a > 1 \), \( b = 0 \) and \( \xi > 0 \), then a Hopf bifurcation takes place at the equilibrium point located at the origin of coordinates, and the limit cycle \( \gamma \) bifurcating from this equilibrium exists for \( b < 0 \) sufficiently small.

b. For \( \varepsilon > 0 \) sufficiently small if \( b = -\beta\varepsilon^2 < 0 \), then the limit cycle \( \gamma \) passes through the point

\[ \left( 2\sqrt[3]{\beta} \varepsilon + O(\varepsilon^3), 0 \right) . \]

Moreover, this limit cycle is stable.

**Proof** Assume that \( a > 1, \xi > 0 \) and \( b = -\beta\varepsilon^2 < 0 \) where \( \varepsilon \) is a sufficiently small parameter. Writing differential system (3) in polar coordinates, we get

\[ \dot{r} = \frac{r \cos \theta \sin \theta}{\sqrt{a^2 + r^2 \cos^2 \theta}} - \xi r^3 \cos^4 \theta + \varepsilon^2 \xi \beta \cos \theta, \]
\[ \dot{\theta} = -1 + \frac{\cos^2 \theta}{\sqrt{a^2 + r^2 \cos^2 \theta}} + \xi r^2 \sin \theta \cos^3 \theta - \varepsilon^2 \xi \beta \cos \theta \sin \theta . \]

(4)

Since we want to study the Hopf bifurcation at the origin of coordinates, we blow up the origin doing the scaling \( r = \varepsilon R, \) differential system (4) taking as new independent variable the \( \theta \) becomes

\[ \frac{dR}{d\theta} = \frac{R \sin \theta \cos \theta}{\cos^2 \theta - a} + \varepsilon^2 \]
\[ \times \frac{R \left( 2\xi a^2 (a-1)(R^2 - 2\beta R^2 \cos(2\theta)) + R^2 \sin(2\theta) \right) \cos^2 \theta}{4a(\cos^2 \theta - a)^2} + O(\varepsilon^4). \]

(5)

In order to apply the averaging theory described in appendix, we need that differential equation (5) starts at least with order \( \varepsilon \). So we do the change of variables \( R \mapsto \rho \) defined by

\[ R = \frac{2(1-a)}{1 - 2a + \cos(2\theta)} \rho . \]

Then, differential equation (5) in the new variable \( \rho \) writes

\[ \frac{d\rho}{d\theta} = -\frac{2(a-1)\rho \cos^2 \theta}{a(1 - 2a + \cos(2\theta))^3} \]
\[ \times \left( 2\xi a^2 (a-1)\rho^2 - 2a\beta + \beta \right) \]
\[ + \frac{2\xi a^2 (a-1)\rho^2 + \beta}{\cos^2 \theta + \rho^2 \sin(2\theta)} + O(\varepsilon^4) \]
\[ = \varepsilon^2 F_1 (\theta, \rho) + O(\varepsilon^4) . \]

(6)

Differential equation (6) is written into normal form (34) for applying the averaging theory summarized in appendix, using the notation of appendix we only need to take \( n = 1, x = \rho, t = \theta, \mu = \varepsilon^2, F_1 (t, \mu) = F_1 (\theta, \rho) \) and \( T = 2\pi \), and all the necessary assumptions for applying the averaging theory described in appendix are satisfied. Then, we compute

\[ f_1 (\rho) = \frac{1}{2\pi} \int_0^{2\pi} F_1 (\theta, \rho) d\theta \]
\[ = \frac{\xi}{8\sqrt{a - 1}} \rho \left( 3\rho^2 - 4\beta \right) . \]

Since \( \beta > 0 \) the averaged function \( f_1 (\rho) \) has a unique positive zero, \( \rho = 2\sqrt[3]{\beta/3} \) which satisfies the condition
In fact, this last expression is positive because \(a > 1\) and \(\xi > 0\), and consequently, by the results described in appendix differential equation (6) has a periodic solution \(\rho(\theta, \varepsilon)\) satisfying that

\[
\rho(0, 0) = 2\sqrt{\frac{\beta}{3}} + O(\varepsilon^2).
\]

Moreover, this periodic solution \(\rho(\theta, \varepsilon)\) is unstable because derivative (7) is positive.

Now we will go back through the changes of variables for obtaining the periodic solution bifurcating from the equilibrium at the origin of coordinates of differential system (3). Thus, the periodic solution \(\rho(\theta, \varepsilon)\) satisfying initial condition (8) becomes the periodic solution

\[
\rho(\theta, \varepsilon) = \sqrt{\frac{2(1-a)}{1-2a+\cos(2\theta)}} \rho(\theta(\varepsilon), \varepsilon) + O(\varepsilon^3),
\]

satisfying

\[
R(0, 0) = 2\sqrt{\frac{\beta}{3}} + O(\varepsilon^2).
\]

This periodic solution in differential system (4) becomes \((r(t, \varepsilon), \theta(t, \varepsilon))\) with

\[
r(t, \varepsilon) = \sqrt{\frac{2(1-a)}{1-2a+\cos(2\theta(t, \varepsilon))}} \rho(\theta(t, \varepsilon), \varepsilon) + O(\varepsilon^3),
\]

and it pass through point

\[
\left(2\sqrt{\frac{\beta}{3}} + O(\varepsilon^3), 0\right)
\]

\[
\text{in the coordinates } (r, \theta). \text{ Finally, this periodic solution in the coordinates of system (3) is the periodic solution } (x(t, \varepsilon), y(t, \varepsilon)) \text{ given by }
\]

\[
\varepsilon \sqrt{\frac{2(1-a)}{1-2a+\cos(2\theta(t, \varepsilon))}} \rho(\theta(t, \varepsilon), \varepsilon)
\]

\[
\left(\cos(\theta(t, \varepsilon)), \sin(\theta(t, \varepsilon))\right) + O(\varepsilon^3),
\]

passing through point (9) now in coordinates \((x, y)\). Therefore, when \(\varepsilon \to 0\) such periodic solution tends to the origin, it is a periodic solution of a Hopf bifurcation.

We remark that the periodic solution \(\rho(\theta, \varepsilon)\) was an unstable limit cycle, but due to the fact that \(\dot{\theta}\) is negative in a neighborhood of the origin, when we pass the unstable limit cycle \(R(\theta, \varepsilon)\) to the periodic solution \((r(t, \varepsilon), \theta(t, \varepsilon))\) it changes to a stable limit cycle. \(\square\)

In short, Proposition 2 shows the existence of the Hopf bifurcation surface \(H_1\).

### 2.2.2 Hopf bifurcation at \(E_L\) and \(E_R\)

**Proposition 3** The following statements hold for differential system (3).

a. If \(0 < a < 1, a \neq 1/\sqrt{3}, b = 3(a^2 - 1)\) and \(\xi > 0\), then one limit cycle bifurcates from each of the equilibria \(E_L\) and \(E_R\).

b. For \(\varepsilon > 0\) sufficiently small if \(b = 3a^2 - 3 + \beta \varepsilon^2\), then those two limit cycles exist if \(\beta(1-3a^2) < 0\), and they pass through the points

\[
\pm \left(\sqrt{1-a^2} + \sqrt{\frac{\beta(1-a^2)}{3(a^2-1)}} \varepsilon \right)
\]

\[
+ O(\varepsilon^3), \xi \sqrt{1-a^2} - 2\left(2a^2 - 2 + \beta \varepsilon^2\right)
\]

Moreover, these limit cycles are stable if \(\beta < 0\) and unstable if \(\beta > 0\).

**Proof** Assume \(0 < a < 1, a \neq 1/\sqrt{3}, \xi > 0\) and \(b = 3(a^2 - 1) + \beta \varepsilon^2\), where \(\varepsilon\) is a sufficiently small parameter. We shall prove the proposition studying the Hopf bifurcation at the equilibrium point \(E_R\).

We translate the equilibrium point \(E_R\) to the origin of coordinates doing the change

\[
(x, y) = (X + \sqrt{1-a^2}, Y + \xi \sqrt{1-a^2} - 2a^2 - 2 + \beta \varepsilon^2),
\]

and system (3) is transformed into

\[
\dot{X} = -\varepsilon \xi \beta X + Y - 3\xi \sqrt{1-a^2} X^2 - \xi X^3,
\]

\[
\dot{Y} = -X - \sqrt{1-a^2} \times \left(1 - \frac{1}{\sqrt{X^2 + 2\sqrt{1-a^2} X + 1}}\right).
\]

Writing differential system (10) in polar coordinates, we get

\[
\dot{r} = -r \cos \theta \left(\xi r^2 \cos^3 \theta + 3\xi \sqrt{1-a^2} r \cos^2 \theta - \sin \theta\right)
\]

\[
- \left(\cos \theta + \sqrt{1-a^2}\right)
\]

\[
\times \left(1 - \frac{1}{\sqrt{r^2 \cos^2 \theta + 2\sqrt{1-a^2} r \cos \theta + 1}}\right) \sin \theta
\]

\[
- \varepsilon^2 \xi \beta r \cos^2 \theta.
\]
\[ \dot{\theta} = -1 + \xi \left( r^2 \cos^2 \theta + 3\sqrt{1 - a^2} r \cos \theta \right) \sin \theta \cos \theta + \frac{2(\sqrt{1 - a^2} r \cos \theta + \sqrt{1 - a^2} r \sin \theta) \sqrt{r^2 \cos^2 \theta + 2\sqrt{1 - a^2} r \cos \theta + 1}}{r \sqrt{1 - a^2} r \cos \theta} \].  

Again, since we want to study the Hopf bifurcation now at the origin of coordinates, we blow up the origin doing the scaling \( r = \varepsilon R \); then, differential system (11) taking as new independent variable the \( \theta \) writes

\[ \frac{dR}{d\theta} = \frac{a^2 R \cos \theta \sin \theta}{a^2 \cos^2 \theta - 1} - \frac{3\sqrt{1 - a^2} R \cos^2 \theta (2\xi (a^2 - 1) \cos \theta - a^2 \sin \theta)}{2(a^2 \cos^2 \theta - 1)^2} + \varepsilon^2 \frac{R \cos^2 \theta}{32(a^2 \cos^2 \theta - 1)} \left( 54\xi a^6 R^2 + 2a^6 R^2 \sin(2\theta) \right)
+ a^6 \sin(4\theta) - 102\xi a^4 R^2 - 16\xi a^4 \left( 36\xi a^2 R^2 \sin(4\theta) + a^4 R^2 \sin(4\theta) + 64\xi a^2 R^2 + 48\xi a^2 + 2\xi a^2 \left( 9a^2 - 29a^2 + 20 \right) R^2 \cos(4\theta) \right)
+ 144\xi^2 a^2 R^2 \sin(2\theta) + 32a^2 R^2 \sin(2\theta)
+ 72\xi a^2 R^2 \sin(2\theta) - 16\xi R^2 - 32\xi \beta + 8\xi (a^2 - 1)
\times \left( 9R^2 a^4 - a^2 \left( 11R^2 + 2\beta \right) + 2R^2 \right) \cos(2\theta)
- 72\xi R^2 \sin(2\theta) - 36\xi^2 R^2 \sin(4\theta) + O(\varepsilon^3), \quad (12) \]

Again for applying the averaging theory of appendix, we need that differential equation (12) starts at least with order \( \varepsilon \). Hence, we do the change of variables \( R \mapsto \rho \) defined by

\[ R = \sqrt{\frac{2(1 - a^2)}{2 - a^2 + a^2 \cos(2\theta)}} \rho. \]

Then, differential equation (12) in the new variable \( \rho \) writes

\[ \frac{d\rho}{d\theta} = \varepsilon F_1(\theta, \rho) + \varepsilon^2 F_2(\theta, \rho) + O(\varepsilon^3), \quad (13) \]

where

\[ F_1(\theta, \rho) = \frac{3(a^2 - 1) \rho \cos^2 \theta (2\xi (a^2 - 1) \cos \theta - a^2 \sin \theta)}{\sqrt{2(a^2 \cos^2 \theta - 1)^2 \sqrt{2 - a^2 - a^2 \cos(2\theta)}}}, \]

\[ F_2(\theta, \rho) = \frac{(a^2 - 1) \rho \cos^2 \theta}{2(\cos(2\theta)a^2 + a^2 - 2)^2} \left( 54\rho^2 - 2 \rho^2 \sin(2\theta) \right)
+ 2\rho^2 \sin(2\theta) a^6 + \rho^2 \sin(4\theta) a^6 - 102\rho^2 a^4
- 12\rho a^4 - 72\rho^2 \sin(2\theta) a^4 - 38\rho^2 \sin(2\theta) a^4 - 36\rho^2 \sin(4\theta) a^4 + \rho^2 \sin(4\theta) a^4
+ 64\rho^2 a^2 + 32\rho a^2
+ 2\rho \left( 9\rho^2 a^4 - (29\rho^2 + 2\beta) a^2 + 20\rho^2 \right) \cos(4\theta) a^2
+ 144\rho^2 a^2 \sin(2\theta) a^2 + 32\rho^2 a^2 \sin(2\theta) a^2
+ 72\rho^2 \sin(4\theta) a^2 - 16\rho a^2 - 32\rho + 8\rho \rho^2 a^6
- 2 \left( 10\rho^2 + \beta \right) a^4 + (13\rho^2 + 4\beta) a^2 - 2\rho^2 \cos(2\theta)
- 72\rho^2 \sin(2\theta) - 36\rho^2 \sin(4\theta). \]

Differential equation (13) is already into normal form (34) for applying the averaging theory of appendix. Again using the notation of appendix, we take \( n = 1, \lambda = \rho, t = \theta, \mu = \varepsilon, F_1(t, x) = F_1(\theta, \rho) \) and \( T = 2\pi \), and all the necessary hypotheses for applying the averaging theory of appendix hold. Then, we compute

\[ f_1(\rho) = \frac{1}{2\pi} \int_0^{2\pi} F_1(\theta, \rho) d\theta \equiv 0. \]

Since the first averaged function \( f_1(\rho) \) is identically zero, we must compute the second one \( f_2(\rho) \). We start calculating

\[ \int_0^{\theta} F_1(\theta, \rho) d\theta = \frac{\rho^2 N(\theta)}{2\sqrt{2 - 2a^2 (2 - a^2 - a^2 \cos(2\theta))^{3/2}}}, \]

where

\[ N(\theta) = 3a^2 (1 - a^2)^{3/2} \cos \theta
- a^2 \sqrt{2 (2 - a^2 - a^2 \cos(2\theta))^{3/2}}
+ \sqrt{2 - a^2 (a^2 - a^4) \cos(3\theta)}
+ 2\xi (3(a^2 - 3) \sin \theta + (3a^2 - 1) \sin(3\theta)). \]

Then, the second averaged function

\[ f_2(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \left[ D_\rho F_1(\theta, \rho)
- \int_0^{\theta} F_1(s, \rho) d\theta + F_2(\theta, \rho) \right] d\theta = \frac{\xi \rho (3 - 3a^2) \rho^2 + 4\beta}{8\sqrt{1 - a^2}} \]

has a unique positive zero, \( \rho = 2\sqrt{\beta/(3(3a^2 - 1))} \); recall that \( a \neq 1/\sqrt{3} \). This zero satisfies the condition.
by assumptions. Consequently, by the results described in appendix differential equation (13) has a periodic solution \( \rho(\theta, \varepsilon) \) satisfying initial condition (15) in the variables of differential system (12) becomes the periodic solution

\[
\rho(0, 0) = 2 \sqrt{\frac{\beta}{3(\alpha^2 - 1)}} + O(\varepsilon). \tag{15}
\]

Moreover, from (14) this periodic solution \( \rho(\theta, \varepsilon) \) is unstable if \( \beta < 0 \) and stable if \( \beta > 0 \).

Now we will go back through the changes of variables for obtaining the periodic solution bifurcating from the equilibrium \( E_R \) of differential system (3). Thus, the periodic solution \( \rho(\theta, \varepsilon) \) satisfying initial condition (15) in the variables of differential system (12) becomes the periodic solution

\[
R(\theta, \varepsilon) = \sqrt{\frac{2(1 - a^2)}{2 - a^2 + a^2 \cos(2\theta)}} \rho(\theta, \varepsilon),
\]

satisfying

\[
R(0, 0) = 2 \sqrt{\frac{\beta(1 - a^2)}{3(\alpha^2 - 1)}} + O(\varepsilon).
\]

This periodic solution in differential system (11) becomes \((r(t, \varepsilon), \theta(t, \varepsilon))\) with

\[
r(t, \varepsilon) = \sqrt{\frac{2(1 - a^2)}{2 - a^2 + a^2 \cos(2\theta(t, \varepsilon))}} \rho(\theta(t, \varepsilon), \varepsilon) + O(\varepsilon^2),
\]

and it pass through the point

\[
\left(2 \sqrt{\frac{\beta(1 - a^2)}{3(\alpha^2 - 1)}} \varepsilon + O(\varepsilon^2), 0\right) \tag{16}
\]

in the coordinates \((r, \theta)\). This periodic solution in the coordinates of system (10) is the periodic solution \((X(t, \varepsilon), Y(t, \varepsilon))\) given by

\[
\varepsilon \left( \frac{2(1 - a^2)}{2 - a^2 + a^2 \cos(2\theta(t, \varepsilon))} \rho(\theta(t, \varepsilon), \varepsilon) \right) (\cos(\theta(t, \varepsilon)), \sin(\theta(t, \varepsilon))) + O(\varepsilon^2),
\]

passing through point (16) now in coordinates \((X, Y)\). Finally, we get the periodic solution \((x(t, \varepsilon), y(t, \varepsilon))\) given by

\[
(x(t, \varepsilon), y(t, \varepsilon)) = (X(t, \varepsilon) + \sqrt{1 - a^2}, Y(t, \varepsilon) + \xi \sqrt{1 - a^2} (2a^2 - 2 + \beta \varepsilon^2)),
\]

and passing through the point

\[
\left(\sqrt{1 - a^2} + 2 \sqrt{\frac{\beta(1 - a^2)}{3(\alpha^2 - 1)}} \varepsilon + O(\varepsilon^2), \xi \sqrt{1 - a^2} (2a^2 - 2 + \beta \varepsilon^2)\right)
\]

in coordinates \((x, y)\). So, when \( \varepsilon \to 0 \) such periodic solution tends to the equilibrium \( E_R \), so it is a periodic solution of a Hopf bifurcation.

Again we note that the limit cycle \( \rho(\theta, \varepsilon) \) was unstable if \( \beta < 0 \), and stable if \( \beta > 0 \), but due to the fact that \( \dot{\theta} \) is negative in a neighborhood of the origin, when we pass the limit cycle \( R(\theta, \varepsilon) \) to the cycle \((r(t, \varepsilon), \theta(t, \varepsilon))\) it changes its type of stability. \( \square \)

In particular, Proposition 3 shows the existence of the Hopf bifurcation surfaces \( H_2 \) and \( H_3 \).

### 2.2.3 The Bautin bifurcation curve \( B_0 \)

The standard or classical Hopf bifurcation in a 2-dimensional differential system, i.e., that a limit cycle bifurcates from an equilibrium point, takes place in an equilibrium point with purely imaginary eigenvalues which is not a center because the first Liapunov constant at that equilibrium is not zero. These are the Hopf bifurcations studied in Sect. 2.2.2. But when the first Liapunov constant is zero, and also can bifurcate a limit cycle of the equilibrium point if the second Liapunov constant is not zero, such more degenerate Hopf bifurcation is called for some authors a Bautin bifurcation. See for more details about these Hopf bifurcations [11, Chapter 8].

When \( b = 3(\alpha^2 - 1) \), with the change of variables \( x_1 = x - \sqrt{1 - a^2}, y_1 = y/\sqrt{1 - a^2} - \xi (b + 1 - a^2) \) and \( \tau = \sqrt{1 - a^2} t \) in a small neighborhood of \( E_R \), system (3) can be written as

\[
\begin{align*}
\frac{dx}{dr} &= y - 3 \xi x^2 - \frac{\xi}{\sqrt{1 - a^2}} x^3 =: y + \hat{f}(x), \\
\frac{dy}{dr} &= -x - \frac{3a^2 x^2}{2\sqrt{1 - a^2}} + \frac{(4a^2 - 5a^4) x^3}{2 - 2a^2} + O(x^4) =: -x + \hat{h}(x),
\end{align*}
\tag{17}
\]
where, for simplicity, we still use the variables \((x, y, t)\) instead of the new ones \((x_1, y_1, \tau)\). From [10, p. 156], we compute the first Liapunov constant at the origin of system (17) and we get \(\hat{g}_1 = 3\xi(3a^2 - 1)/(8\sqrt{1 - a^2})\). From the expressions of \(\hat{g}_1\), we can confirm that the classical Hopf bifurcation happens when \(a \neq 1/\sqrt{3}\) and \(b = 3a^2 - 3\). Clearly, when \(a = 1/\sqrt{3}\) and \(b = -2\) we have \(\hat{g}_1 = 0\). By [11, Chapter 8], we can obtain that the second Liapunov constant \(\hat{g}_2 = 5\sqrt{6}\xi/32 > 0\) at those values of \(a\) and \(b\).

In short, system (3) exhibits a Bautin bifurcation at \(a = 1/\sqrt{3}\) and \(b = -2\) by [11, Chapter 8], i.e., at the intersection point of the surfaces \(H_2, H_3\) and \(DL_2\), in particular \(\phi_2(1/\sqrt{3}, \xi) = -2\).

### 2.3 Codimension 2 Bogdanov–Takens bifurcation with symmetry

From [15] a codimension 2 Bogdanov–Takens bifurcation with symmetry happens in system (3) in a neighborhood of the curve \(a = 1\) and \(b = 0\). So, in a neighborhood of the intersection point of \(P, H_1, H_2, HL\) and \(DL_1\) of the bifurcation diagram in Fig. 1, i.e., \(\varphi(1, \xi) = \phi_1(1, \xi) = 0\).

### 2.4 The dynamics near infinity

In this subsection, we will discuss the qualitative properties of the equilibria at infinity, which describe the behavior of the orbits of system (3) when \(x\) and \(y\) are sufficiently large.

**Proposition 4** As shown in Fig. 3, differential system (3) with \(\xi > 0\) has four equilibria at infinity \(I^+_A, I^+_B, I^-_A, I^-_B\), where \(I^+_A\) are the two endpoints of the \(x\)-axis, and \(I^-_B\) are the two endpoints of the \(y\)-axis. The equilibria \(I^+_A\) are unstable star nodes, and the equilibria \(I^-_B\) are degenerate equilibria, formed by the union of two hyperbolic sectors.

**Proof** Doing the Poincaré transformation \(x = 1/z, y = u/z\), system (3) becomes

\[
\frac{du}{d\tau} = \xi u + b\xi uz^2 - u^2z^2 - z^2 + \frac{|z|^3}{\sqrt{1 + a^2z^2}},
\]

\[
\frac{dz}{d\tau} = \xi z + b\xi z^3 - uz^3,
\]

where \(dt = z^2d\tau\). Obviously, this system has a unique equilibrium \(A(0, 0)\) on the \(u\)-axis (the infinity), where \(A\) is an unstable star node.

Doing the other Poincaré transformation \(x = v/z, y = 1/z\), system (3) writes as

\[
\frac{dv}{d\tau} = z^2 - b\xi vz^2 - \xi v^3 + v^2z^2 - \frac{v^2|z|^3}{\sqrt{v^2 + a^2z^2}}.
\]

\[
\frac{dz}{d\tau} = vz^3 - \frac{vz^3|z|}{\sqrt{v^2 + a^2z^2}} = \Psi_2(v, z),
\]

where \(dt = z^2d\tau\). In this local chart, we only need to study the equilibrium \(B(0, 0)\) of system (18), which corresponds to two equilibria \(I^+_B\) and \(I^-_B\) at infinity of system (3) at the endpoints of the positive and negative \(y\)-semiaxes, respectively. By Lemmas 1 and 3 of [13, Chapter 2], we only need to discuss the orbits along characteristic directions of system (18) at \(B\).

Applying the polar coordinate changes \(x = r\cos\theta, y = r\sin\theta\), system (18) can be written

\[
\frac{1}{r}\frac{dr}{d\theta} = \frac{H(\theta) + o(1)}{G(\theta) + o(1)}, \quad \text{as} \quad r \to 0,
\]

where

\[
G(\theta) = -\sin^3\theta \sqrt{\cos^2\theta + a^2\sin^2\theta},
\]

\[
H(\theta) = \sin^2\theta \cos\theta \sqrt{\cos^2\theta + a^2\sin^2\theta}.
\]

Hence, a necessary condition for \(\theta = \theta_0\) to be an characteristic direction is \(G(\theta_0) = 0\), which has exactly
two roots 0 and $\pi$. Except these two directions, there are no directions along which system (18) has orbits connecting $B$.

Notice that vector field (18) is symmetric with respect to the $v$-axis. Thus, we only need to discuss the orbits connecting the origin $B$ of (18) in the half-plane $z \geq 0$. We will construct some related open quasi-sectors to determine how many orbits of (18) connect $B$ in the first and the second quadrants.

Observing that system (18) has four horizontal isoclines: the $v$-axis, the $z$-axis and

$\mathcal{H}^\pm := \{(u, z) \in \mathbb{R}^2 : v = \pm \sqrt{1 - a^2z},\ 0 < r < \ell,\ a < 1\},$

where $\ell > 0$ is a sufficiently small constant. Set

$V^\pm := \{(v, z) \in \mathbb{R}^2 : z = 0,\ \pm v > 0,\ 0 < r < \ell\}.$

The possible vertical isocline is

$\mathcal{V} := \{(v, z) \in \mathbb{R}^2 : v = \xi^{-\frac{1}{2}}z^{\frac{3}{2}} + o(z^{\frac{3}{2}}),\ 0 < r < \ell\}.$

Obviously, the isocline $\mathcal{V}$ is tangent to the $v$-axis at the origin. Set

$L^\pm := \{(v, z) \in \mathbb{R}^2 : z = \pm \sigma v,\ 0 < r < \ell,\ a = 1\},$

where $\sigma > 0$ is a small constant. Hence, if there exist orbits of system (18) connecting $B$ along the direction of the $v$-axis in the first and the second quadrants, then near the origin must lie in the sector regions $\Delta V^+ B\mathcal{H}^+$ or $\Delta V^- B\mathcal{H}^-$ if $a < 1$, and $\Delta V^+ B\mathcal{L}^+$ or $\Delta V^- B\mathcal{L}^-$ if $a = 1$. The directions of vector field of (18), i.e., the directions of arrows, and the positions of the isoclines are shown in Fig. 4.

Fig. 4 Vector field of system (18)

Firstly, we consider the case $a < 1$. We can check that $\dot{v} > 0$ and $\dot{z} > 0$ in $\Delta V^+ B\mathcal{H}^+$; $\dot{v} < 0$ and $\dot{z} > 0$ in $\Delta V^- B\mathcal{V}$; and $\dot{v} > 0$ and $\dot{z} < 0$ in $\Delta V^- B\mathcal{H}^-$. Lemma 4 in [14] guarantees that no orbits connect $B$ in $\Delta V^+ B\mathcal{V}$. There are also no orbits connecting $B$ in the interior of $\Delta V^- B\mathcal{H}^-$, because $\Psi_2(v, z)/\Psi_1(v, z)$ is not equal to the slopes of the curves tangent to the $v$-axis. On the other hand, we compute that $(\partial/\partial v)(\Psi_1(v, z)/\Psi_2(v, z)) < 0$ in the generalized normal sector $\Delta \mathcal{V} B\mathcal{H}^+$ of class II, i.e., $r > 0$ in $\Delta \mathcal{V} B\mathcal{H}^+$ and all positive semi-orbits starting from the curves $B\mathcal{V}$ and $B\mathcal{H}^+$ go into $\Delta \mathcal{V} B\mathcal{H}^-$. The definition of generalized normal sectors can be seen in [14, Section 2]. Therefore, there exists a unique orbit leaving from $B$ in $\Delta \mathcal{V} B\mathcal{H}^+$ by Lemmas 2 and 5 in [14].

Similarly, in case $a = 1$, we can also prove that exactly one orbit connects $B$ along the $v$-axis, which lies in $\Delta \mathcal{V} B\mathcal{L}^+$. \hfill $\square$

3 Limit cycles

Lemma 5 Assume that $a \geq 1$. System (3) has no limit cycles if $b \geq 0$ and a unique limit cycle if $b < 0$.

Proof When $b \geq 0$, since the divergence of system (3) is $f(x) = F'(x) = \xi(b + 3x^2) \geq 0$, by the Bendixson criterion (see, for instance, [8, Theorem 7.10]), system (3) has no limit cycles.

When $b < 0$, the following conditions are satisfied.

i. $g(x)$ is an odd function and $xg(x) > 0$ if $x \neq 0$;
ii. $F(x)$ is an odd function, $F(x) < 0$ if $0 < x < \sqrt{-b}$, and $F(x) \geq 0$ if $x \geq \sqrt{-b}$;
iii. $\int_0^\infty f(x)\,dx = \int_0^\infty g(x)\,dx = +\infty$;
iv. $f(x)$ and $g(x)$ satisfy the Lipschitz condition in any bounded interval.

Then, by Theorem 4.1 of [17, Chapter 4], system (3) has a unique limit cycle, which is stable. \hfill $\square$
Note that the phase portraits (b) and (c) in Fig. 2 in Theorem 1 are obtained from Lemma 5 and from the properties of equilibria.

Since the phase portrait of system (3) is symmetric with respect to the point \( E \), the small limit cycles surrounding \( E_L \) are of the same type as that surrounding \( E_R \). Hence, in what follows we only consider the small limit cycles around \( E_R \).

**Lemma 6** If \( 0 < a < 1 \) and \( b \geq a^2 - 1 \), then system (3) has no limit cycles.

Since the proof is similar to Lemma 4 of [5], we omit it.

Lemma 6 shows that \( \phi_1(a, \xi) < a^2 - 1 \) and the phase portrait (a) in Fig. 2 in Theorem 1 is obtained. Consider equation

\[
\frac{dz}{dy} = y - \hat{F}(z), \quad 0 \leq z \leq z_0,
\]

where both \( \hat{F}(z) \) and \( \hat{F}'(z) \) are continuous in \([0, z_0] \), and \( \hat{F}(0) = 0 \). Let \( L_J \) be the integral curve of (19) passing through the point \( P(z_J, \hat{F}(z_J)) \) on the curve \( y = \hat{F}(z) \). Also, let \( y = \varphi_J(z) \) and \( y = \tilde{\varphi}_J(z) \) represent the orbit segments of \( L_J \) below and above the curve \( y = \hat{F}(z) \). When \( 0 < z < z_J \), we clearly have \( \varphi_J(z) < \hat{F}(z) < \tilde{\varphi}_J(z) \) and \( \varphi_j'(z) > 0 \geq \tilde{\varphi}_j'(z) \). Moreover, we introduce the symbol

\[
V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z)) = \frac{\hat{F}'(z)}{\hat{F}(z) - \varphi_j(z)} + \frac{\hat{F}'(z)}{\tilde{\varphi}_J(z) - \hat{F}(z)}.
\]

Then, we have

\[
\int_{a_0}^{z_J} \hat{F}'(z) dy = \int_{a_0}^{z_J} V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z)) dz
\]

for some \( a_0 \).

**Lemma 7** (Lemma 4.5 of [17, Chapter 4]) For equation (19), suppose there is \( a_0 \geq 0 \) with \( \hat{F}(a_0) = 0 \), and \( \hat{F}(z) > 0 \), \( \hat{F}(z) \hat{F}'(z) \) is nondecreasing for \( z > a_0 \). Then,

\[
\int_{a_0}^{z_J} V(\hat{F}(z), \varphi_Q(z), \tilde{\varphi}_Q(z)) dz
\]

\[
\leq \int_{a_0}^{z_J} V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z)) dz, \quad \text{for } a_0 < z_Q < z_J.
\]

Lemma 7 will be applied in the following lemma.

**Lemma 8** If \( 0 < a < 1 \) and \( b < a^2 - 1 \), then system (3) has at most two large limit cycles.

Proof Assume that system (3) has at least two large limit cycles surrounding the three equilibria \( E_L, E_0 \) and \( E_R \), and that \( L_1 \) and \( L_2 \) are the most external limit cycles, where \( L_2 \) denotes the outer one.

We first consider \( 3a^2 - 3 \leq b < a^2 - 1 \). The corresponding phase portrait is shown in Fig. 5a. By the Bendixson criterion, each \( L_i \) has two intersection points, denoted by \( B_i \) and \( C_i \) (i = 1, 2) with the straight line \( x = x_0 \), where \( x_0 \) is the abscissa of the equilibrium \( E_R \), as shown in Fig. 5a. By the symmetry of the phase portrait.
for $i = 1, 2$. On the arcs $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$, let $y = y_1(x)$ and $y = y_2(x)$, respectively. In fact, for each $i = 1, 2$, we have

$$
2 \int_{\widehat{A_iB_i}} f(x)dt = \int_{L_i} f(x)dt \\
= - \int_{L_i} \text{div}(y - F(x), -g(x))dt,
$$

we obtain

$$(y_1(x_0) - y_2(x_0))F(x_0) \quad y_1(x_0)(y_2(x_0) - F(x_0)) > 0, \quad \int_0^{x_0} \tilde{F}(x)dx > 0.
$$

Thus,

$$
\int_{\widehat{A_2B_2}} f(x)dt > \int_{\widehat{A_1B_1}} f(x)dt.
$$

Similarly, we obtain that

$$
\int_{\widehat{C_1F_1}} f(x)dt - \int_{\widehat{C_1F_2}} f(x)dt > 0.
$$

Setting $z = f_0^{-1} g(s)ds$, from (20), we get

$$
\int_{B_1G_1\widehat{C}_1} F'(x)dx = \int_{B_1G_1\widehat{C}_1} \tilde{F}'(z)dz \\
= \int_{x_0}^{z_{G_2}} V(\tilde{F}(z), \varphi G_1(z), \tilde{\varphi} G_1(z))dz.
$$

By Lemma 7, in order to prove the inequality

$$
\int_{z(x_0)}^{z_{G_2}} V(\tilde{F}(z), \varphi G_1(z), \tilde{\varphi} G_1(z))dz \\
> \int_{z(x_0)}^{z_{G_1}} V(\tilde{F}(z), \varphi G_1(z), \tilde{\varphi} G_1(z))dz,
$$

where $\tilde{F}(z) = F(x) - F_0(x)$, we only need to prove that $\tilde{F}(z(x_0)) = 0$, $\tilde{F}(z) > 0$ and $\tilde{F}(z) \tilde{F}'(z)$ is non-decreasing for $z > z(x_0)$. Clearly, we have $\tilde{F}(z(x_0)) = 0$ and $\tilde{F}(z) > 0$ for $z > z(x_0)$. Note that

$$
\tilde{F}(z) \tilde{F}'(z) = [F(x) - F(\sqrt{1 - a^2})]f(x)/g(x).
$$

For $x > \sqrt{1 - a^2}$, we have

$$
\frac{F(x) - F(\sqrt{1 - a^2})}{g(x)} f(x) \\
= \frac{\xi^2}{\xi^2 - 1} \left[ b + x^2 + (1 - a^2 + (1 - a^2) \sqrt{1 - a^2}) \right] (b + 3x^2) \\
= \frac{\xi^2}{\xi^2 - 1} \left[ b + x^2 + (1 - a^2 + (1 - a^2) \sqrt{1 - a^2}) \right] (b + 3x^2) (\sqrt{x^2 + a^2} + x^2 + a^2) \\
= \frac{\xi^2}{\xi^2 - 1} \left[ b + x^2 + (1 - a^2 + (1 - a^2) \sqrt{1 - a^2}) \right] (b + 3x^2) (\sqrt{x^2 + a^2} + x^2 + a^2),
$$

where the three factors of the last line are positive and increasing. Therefore, $[F(x) - F(\sqrt{1 - a^2})]f(x)/g(x)$ is positive and increasing. By Lemma 7, we have that

$$
\int_{B_2G_2\widehat{C}_2} f(x)dt - \int_{B_1G_1\widehat{C}_1} f(x)dt > 0.
$$

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Therefore,
\[
\int_{L_1} \text{div}(y - F(x), -g(x))dt > \int_{L_2} \text{div}(y - F(x), -g(x))dt.
\]
(21)

Now we consider \( b < 3a^2 - 3 \). The corresponding phase portrait is shown in Fig. 5b. By the Bendixson criterion again, each \( L_i \) has two intersection points with the straight line \( x = x_0 \), being \( x_0 \) the unique positive zero of \( F'(x) \) when \( x > 0 \), and the intersection points are denoted by \( B_i \) and \( C_i \) \((i = 1, 2)\). Then, inequality (21) can be proved in a similar way to the case \( 3a^2 - 3 < b < a^2 - 1 \). Since \( f(x) < 0 \) for \( 0 < x < x_0 \) and \( y_i(x) - F(x) > 0 \) \((i = 1, 2)\), we have
\[
\int_{A_2B_2} f(x)dt - \int_{A_1B_1} f(x)dt = \int_{\partial K} f(x)d\sigma = 0,
\]
\[
\int_{C_2F_2} f(x)dt - \int_{C^1F_1} f(x)dt > 0.
\]
Similarly, we obtain that
\[
\int_{C_2F_2} f(x)dt - \int_{C^1F_1} f(x)dt > 0.
\]

Now, using again Lemma 7, we only need to prove that \( \bar{F}(z(x_0)) = 0 \), and \( \bar{F}(z) > 0, \bar{F}(z) \bar{F}'(z) \) is nondecreasing for \( z > z(x_0) \), where \( \bar{F}(z) = F(x) - F(x_0). \) Clearly, we have \( \bar{F}(z(x_0)) = 0 \) and \( \bar{F}(z) > 0 \) for \( z > z(x_0) \). Note that \( \bar{F}(z) \bar{F}'(z) = [F(x) - F(\sqrt{-b/3})]f(x)/g(x). \) For \( x > \sqrt{-b/3} \), we have
\[
\frac{[F(x) - F(\sqrt{-b/3})]f(x)}{g(x)} = \frac{x^2 + x^3 - x\sqrt{-b/3} - (-b/3)\sqrt{-b/3}}{x(1 - 1/\sqrt{x^2 + x^2})} = \frac{x^2 + x^3 + \sqrt{-b/3}}{x(1 - 1/\sqrt{x^2 + x^2})} = \frac{2b/3 - \sqrt{1 - a^2}}{x(\sqrt{-b/3} - x\sqrt{1 - a^2})} + x\sqrt{-b/3} - \sqrt{1 - a^2} = \left( \frac{\sqrt{x^2 + a^2} + x^2 + a^2}{x - \sqrt{1 - a^2}} \right) \left( \frac{1 + \frac{1 - a^2 - \sqrt{-b/3}}{x - \sqrt{1 - a^2}}}{x + \sqrt{x^2 + a^2} + x^2 + a^2} \right).
\]

We consider a generic Liénard system
\[
\dot{x} = y - \bar{F}(x), \quad \dot{y} = -\bar{g}(x), \tag{22}
\]
where \( \bar{F} \) is \( C^2, \bar{g} \) is \( C^1 \) and \( x \in (\alpha, \beta) \)(\( \alpha, \beta \) can be \( \pm \infty \)). The following proposition partly improves the result of [9, Theorem 2.1].

**Proposition 9** Consider system (22), which satisfies (i)–(iv), where

i. \( \bar{f}(x) = \bar{F}'(x) \) has a unique zero and \( \bar{f}(x) < 0 \) (resp. \( > 0 \)) as \( \alpha < x < 0 \) (resp. \( 0 < x < \beta \));

ii. \( \bar{F}(0) = 0 \);

iii. \( x\bar{g}(x) > 0 \) for \( x \neq 0 \), \( x \in (\alpha, \beta) \);

iv. the system
\[
\bar{F}(x_1) = \bar{F}(x_2), \quad \bar{\lambda}(x_1) = \bar{\lambda}(x_2)
\]
has at most one solution \( x_2 < 0 < x_1, \) where \( \bar{\lambda} := \bar{g}(x)/\bar{f}(x) \). Moreover, when \( \bar{F}(x_1) = \bar{F}(x_2), \bar{\lambda}(x_1) > \bar{\lambda}(x_2) \) (resp. \( \bar{F}(x_1) = \bar{F}(x_2), \bar{\lambda}(x_1) < \bar{\lambda}(x_2) \)) as \( \|x_1\|, \|x_2\| \) are small, system (22) satisfies either (v) or (v'), where

v. the function \( \bar{F}(x)\bar{f}(x)/\bar{g}(x) \) is decreasing (resp. increasing) for \( \alpha < x < 0 \);
Clearly, $-\bar{y}$ as shown in Fig. 6a. In the following, we will ascertain which deduces

$$\frac{d}{dt} \bar{F}(x) = \lim_{x \to a^+} \bar{F}(x).$$

Then, system (22) has at most one closed orbit in the region $\{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta\}$. The closed orbit is simple and unstable (resp. stable) if it exists.

**Proof** Assume that system (22) exists a limit cycle $\gamma$, as shown in Fig. 6a. In the following, we will ascertain the sign of

$$\oint_y \bar{f}(x) \, dt = \oint_y \text{div}(y - \bar{F}(x), -\bar{g}(x)) \, dt.$$  

Clearly, $w = \bar{F}(x)$ has two inverse functions, $x_1(w)$ (respectively, $x_2(w)$), on the right (resp. left) side of the origin. The functions $\bar{\lambda}(x_i(w))$ will be denoted simply by $\lambda_i(w)$.

By $w = \bar{F}(x)$, we rewrite system (22) into

$$\dot{w} = \bar{f}(x_i(w))(y - w), \quad \dot{y} = -\bar{g}(x_i(w)),$$  

which deduces

$$\frac{dy}{dw} = \frac{\lambda_1(w)}{w - y}. \quad (25)$$

Let $y_1(w)$ and $y_2(w)$ (resp. $z_1(w)$ and $z_2(w)$) be functions determined by the orbits of (24) below (resp. above) the line $y = w$, which correspond the parts of the trajectories of system (22) below (resp. above) the curve $y = \bar{F}(x)$ and depend whether they are to the left or right of the origin.

From condition (iv), $\lambda_1(w) = \lambda_2(w)$ has at most one root. When the equation $\lambda_1(w) = \lambda_2(w)$ has no roots, by the comparison theorem we obtain either $z_1(w) > z_2(w)$ and $y_1(w) < y_2(w)$ or $z_1(w) < z_2(w)$ and $y_1(w) > y_2(w)$. By Green formula, we have

$$\oint_y \bar{g}(x) \, dx + (y - \bar{F}(x)) \, dy = -\int_{\text{Int}_x} \bar{f}(x) \, dx \, dy$$

$$= \int_{\text{Int}_y, x < 0} \, dw \, dy - \int_{\text{Int}_y, x > 0} \, dw \, dy \neq 0,$$  

which contradicts $\oint_y \bar{g}(x) \, dx + (y - \bar{F}(x)) \, dy = 0$. When the equation $\lambda_1(w) = \lambda_2(w)$ has a unique root and the equation $z_1(w) = z_2(w)$ has no roots, we can similarly prove that (22) has no limit cycles. When the equation $\lambda_1(w) = \lambda_2(w)$ has a unique root, applying again the comparison theorem the equation $z_1(w) = z_2(w)$ has at most one root.

Now we only need to discuss the system $\lambda_1(w) = \lambda_2(w)$ and $z_1(w) = z_2(w)$ having a unique root. Here we consider that condition (v) holds. By $(w, y) \to (\mu w, \mu y)$ with $\mu := w_2^*/w_1^* > 1$, where $w_i^*$ is the intersection of $z_i(w)$ and the line $y = w$, system (25) deduces

$$\frac{dy}{dw} = \frac{\lambda_2(\mu w)}{\mu(w - y)}.$$  

First, we consider that $\lambda_1(w) > \lambda_2(w)$ as $w \to 0$. Moreover, the limit cycle $\gamma$ of (22) with $x > 0$ corresponds to the broken curve in Fig. 6b and $y_2(w)$, $z_2(w)$ are changed into two features, denoted $\hat{y}_2(w)$ and $\hat{z}_2(w)$, respectively. It is easy to check that $y_2(w) = \mu \hat{y}_2(w/\mu)$ and $z_2(w) = \mu \hat{z}_2(w/\mu)$. By
\( \mu > 1 \) and the increase of \( \lambda(w)/w \), we get \( \lambda(\mu w)/\mu > \lambda(w) \). Furthermore, using again the comparison theorem to (25) and (26), we obtain \( y_1(w) < \tilde{y}_2(w) \) and \( z_1(w) > \tilde{z}_2(w) \). Thus,

\[
\oint \tilde{f}(x) \, dt = \oint \frac{\tilde{f}(x)}{\tilde{F}(x)} \, dx = \int_0^{u_2^*} \frac{dw}{y_2(w) - w} - \int_0^{u_2^*} \frac{dw}{z_2(w) - w} + \int_0^{u_1^*} \frac{dw}{y_1(w) - w} - \int_0^{u_1^*} \frac{dw}{\lambda(\mu w) - \mu w} + \int_0^{u_1^*} \frac{dw}{z_1(w) - w} - \int_0^{u_1^*} \frac{dw}{y_1(w) - w} \]

\[
= \int_0^{u_1^*} \frac{(y_1(w) - w)(\tilde{y}_2(w) - w)}{(z_1(w) - w)(\tilde{z}_2(w) - w)} < 0.
\]

So in the region \( \{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta\} \) \( \gamma \) is unstable and simple if it exists. Moreover, it is impossible to have two attracting (repelling) limit cycles surrounding the same equilibrium adjacent one to the other. Therefore, the uniqueness has also been proved.

For the case \( \lambda_1(w) < \lambda_2(w) \) as \( w \) is small, we can prove that \( \int f(x) \, dt > 0 \) in a similar way to the case \( \lambda_1(w) > \lambda_2(w) \). So, in the region \( \{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta\} \), the limit cycle \( \gamma \) is stable and simple if it exists. Therefore, we have completed this proof. \( \square \)

The proof of Proposition 9 gives the following corollary directly.

**Corollary 10** Assume that system (22) satisfies conditions (i)–(iii) of Proposition 9. If there is no solutions to (23), then system (22) has no closed orbits.

Under the preparations of Proposition 9 and Corollary 10, we obtain the existence of small limit cycles on the parameter surfaces \( H_2 \) and \( H_3 \) as follows.

**Lemma 11** Assume that \( 0 < a < 1 \) and \( b = 3a^2 - 3 \). Then, system (3) has no small limit cycles if \( 1/\sqrt{3} \leq a < 1 \) and at most two small limit cycles if \( 0 < a < 1/\sqrt{3} \).

**Proof** By the transformation \( (x, y) \to (x + \sqrt{1 - a^2}, y + F(\sqrt{1 - a^2})) \), system (3) can be rewritten as

\[
\dot{x} = y - F(x + \sqrt{1 - a^2}) + F(\sqrt{1 - a^2}),
\]

\[
\dot{y} = -g(x + \sqrt{1 - a^2}).
\]

It is easy to show that system (27) satisfies conditions (i)–(iii) of Proposition 9 when \( x \in (-\sqrt{1 - a^2}, +\infty) \). Condition (iv) of Proposition 9 is equivalent to the fact that system

\[
F(\tilde{x}_1 + \sqrt{1 - a^2}) = F(\tilde{x}_2 + \sqrt{1 - a^2}),
\]

\[
g(\tilde{x}_1 + \sqrt{1 - a^2}) = g(\tilde{x}_2 + \sqrt{1 - a^2}),
\]

\[
f(\tilde{x}_1 + \sqrt{1 - a^2}) = f(\tilde{x}_2 + \sqrt{1 - a^2})
\]

has at most one solution, where \( -\sqrt{1 - a^2} < \tilde{x}_1 < 0 < \tilde{x}_2 \). Clearly (28) is equivalent to the fact that the system

\[
F(x_1) = F(x_2), \quad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)}
\]

has a unique solution when \( 0 < x_1 < \sqrt{1 - a^2} < x_2 \) and \( x_j = \tilde{x}_j + \sqrt{1 - a^2}, \) \( j = 1, 2 \). Let \( s := x_1 + x_2 \). From \( F(x_1) = F(x_2) \), we have \( x_1x_2 = 3(a^2 - 1) + s^2 \). Since \( 0 < x_1 < \sqrt{1 - a^2} < x_2 \), we obtain \( \sqrt{3} - 3a^2 < s < 2\sqrt{1 - a^2} \). Note that

\[
g(x) \quad \frac{f(x)}{f(x)} = \frac{x}{3s(\sqrt{x^2 + a^2} + x^2 + a^2)}.
\]

From the second equality of (29) and (30), we have

\[
s^2 + 2a^2 - 3 - \frac{a^2s}{x_2\sqrt{x_1^2 + a^2} + x_2\sqrt{x_2^2 + a^2}} = 0.
\]

And consequently

\[
\left( x_2\sqrt{x_1^2 + a^2} + x_1\sqrt{x_2^2 + a^2} \right)^2
\]

\[
= 2x_1^2x_2^2 + a^2(x_1^2 + x_2^2)
\]

\[
+ 2x_1x_2\sqrt{x_1^2x_2^2 + a^2(x_1^2 + x_2^2) + a^4}
\]

\[
= 2s^4 + (11a^2 - 12)s^2 + 6(1 - a^2)(3 - 2a^2)
\]

\[
+ 2(s^2 + 3a^2 - 3)\sqrt{s^4 + (5a^2 - 6)s^2 + (2a^2 - 3)^2}.
\]

Let \( \eta := s^2 \). Then, for \( 3(1 - a^2) < \eta < 4(1 - a^2) \), we have
Thus, if \( h(\eta) := \eta + 2a^2 - 3 - a^2 \sqrt{\eta}/[x_2\sqrt{x_1^2 + a^2} + x_1\sqrt{x_2^2 + a^2}] \), then the function \( h(\eta) \) is increasing in \( 3(1-a^2) < \eta < 4(1-a^2) \). On the other hand, we have \( h(3-3a^2) = -a-a^2 < 0 \) and \( h(4-4a^2) = 1-3a^2 \).

Therefore, when \( \sqrt{3}/3 \leq a < 1 \), the function \( h(\eta) \) has no solutions for \( 3(1-a^2) < \eta < 4(1-a^2) \).

By Corollary 10, system (3) has no limit cycles in the region \( x > 0 \). When \( 0 < a < \sqrt{3}/3 \), the function \( h(\eta) \) has a unique root in \( 3(1-a^2) < \eta < 4(1-a^2) \). Now we only need to verify the condition \( (\gamma') \) of Proposition 9.

We have

\[
(F(x) - F(\sqrt{1-a^2}) f(x))
\]

\[
g(x) = 3\xi^2 \frac{\sqrt{x^2 + a^2} + x^2 + a^2}{x} \left((3a^2 - 3)x + 2(1-a^2)^{3/2} + x^3\right)
\]

\[
= 3\xi^2 \left(3a^2 - 3 + \frac{2(1-a^2)^{3/2}}{x} + x^2\right)
\]

\[
= (\sqrt{x^2 + a^2} + x^2 + a^2).
\]

Then, we obtain

\[
\frac{d}{dx} \left[(F(x) - F(\sqrt{1-a^2}) f(x))/(3\xi^2 g(x))\right]
\]

\[
= \left(-2(1-a^2)^{3/2}/x^2 + 2x\right)(\sqrt{x^2 + a^2} + x^2 + a^2)
\]

\[
+ \left(3a^2 - 3 + \frac{2(1-a^2)^{3/2}}{x} + x^2\right)
\]

\[
\times \left(\frac{x}{\sqrt{x^2 + a^2}} + 2x\right) > 0,
\]

for \( 0 < a < \sqrt{3}/3 \) and \( x > \sqrt{1-a^2} \), because

\[
-2(1-a^2)^{3/2}/x^2 + 2x > 0
\]

\[
3a^2 - 3 + \frac{2(1-a^2)^{3/2}}{x} + x^2 = \frac{(x - \sqrt{1-a^2})(x^2 + \sqrt{1-a^2}x - 2(1-a^2))}{x} > 0.
\]

So condition \((\gamma')\) of Proposition 9 holds. Therefore, when \( 0 < a < \sqrt{3}/3 \), system (3) has at most one limit cycle which lies in the region \( x > 0 \) by Proposition 9.

**Lemma 12** When \( 1/\sqrt{3} \leq a < 1 \) and \( b = 3a^2 - 3 \), system (3) has a unique limit cycle surrounding all equilibria.

**Proof** If \( 1/\sqrt{3} \leq a < 1 \) and \( b = 3a^2 - 3 \), i.e., on the parameter curve \( H_2 \), we obtain Fig. 7 by Proposition 4 and Lemma 11, which shows the existence of a Poincaré–Bendixson annulus, i.e., any trajectory starting at a point of the boundary curves of the annulus enters (or leaves) the annulus, and inside the annulus there is no equilibrium point. So, the existence of some large limit cycles is obtained. Assume that system (3) has two large limit cycles. Let the outer limit cycle and inner one denoted by \( \gamma_2 \) and \( \gamma_1 \) Therefore, \( \int_{\gamma_1} \text{div}(y - F(x), -g(x)) dt \leq 0 \) for \( i = 1, 2 \).

By (21), \( \int_{\gamma_2} \text{div}(y - F(x), -g(x)) dt > \int_{\gamma_2} \text{div}(y - F(x), -g(x)) dt \). However, it is impossible to have...
two attracting (repelling) limit cycles surrounding the same equilibrium (equilibria) adjacent one to the other. Therefore,
\[
\begin{align*}
\oint_{\gamma_1} \text{div}(y - F(x), -g(x)) \, dt &= 0 \\
\oint_{\gamma_2} \text{div}(y - F(x), -g(x)) \, dt \\
\end{align*}
\]

which contradicts \( \oint_{\gamma_1} \text{div}(y - F(x), -g(x)) \, dt \) \( > \oint_{\gamma_2} \text{div}(y - F(x), -g(x)) \, dt. \) Therefore, system (3) has at most one large limit cycle. Thus, the uniqueness of limit cycle is proved and the phase portrait (d) in Fig. 2 in Theorem 1 is obtained. □

Moreover, we can get the phase portraits (i), (e) and (n) in Fig. 2 in Theorem 1 by Proposition 3, Lemmas 8, 11, 12 and the continuity of the vector fields.

From [5], it follows that system (3) has no limit cycles for \( a \rightarrow 0 \) and \( b = 3a^2 - 3. \)

Proposition 13 System (3) has no small limit cycles for the values of the parameters in \( G_1 := \{ (a, b, \xi) \in G : \sqrt{3}/3 \leq a < 1, b < 3a^2 - 3 \} \) and \( G_2 := \{ (a, b, \xi) \in G : 0 < a < \sqrt{3}/3, b \leq 2\sqrt{3}a - 4 \}. \)

Proof In the case \( x > 0, \) via the change of variables \( w = \sqrt{x^2 + a^2} \) and the time scaling \( dt = \sqrt{1 + a^2/x^2} \, d\tau, \) system (3) becomes
\[
\begin{align*}
\dot{w} &= y - \xi \{ b\sqrt{(w+1)^2 - a^2} + [(w+1)^2 - a^2]^{3/2} \} \\
&= y - \hat{F}(w), \\
\dot{y} &= -w. \\
\end{align*}
\]

It is obvious that \( w \in (a - 1, +\infty). \) Further, we can compute that
\[
\hat{F}(w) - \hat{F}(-w) = \xi \left( \sqrt{(w+1)^2 - a^2} - \sqrt{1 - (w-1)^2 - a^2} \right) \kappa(s),
\]
where \( \kappa(s) = b + 4 + 2s + \sqrt{s^2 - 4a^2} \) and \( s = w^2 - 1 - a^2. \) By \( w \in (a - 1, 0), \) it follows that \( s \in (-1 - a^2, -2a). \) We compute that
\[
\kappa'(s) = 2 + \frac{s}{\sqrt{s^2 - 4a^2}},
\]

\[\sup_{s \in (-1 - a^2, -2a)} \kappa'(s) = \kappa(-1 - a^2) = \frac{1 - 3a^2}{1 - a^2}.\]

Hence, when \( \sqrt{3}/3 \leq a < 1, \)
\[
\sup_{s \in (-1 - a^2, -2a)} \kappa(s) = \kappa(-1 - a^2) = b + 3 - 3a^2.
\]

In other words, in \( G_1 \) we have that \( \hat{F}(w) > \hat{F}(-w) \) by (33) for \( w \in (a - 1, 0). \) When \( 0 < a < \sqrt{3}/3, \) it is clear that \( \kappa'(s) > 0 \) for \( s \in (-1 - a^2, -4a/\sqrt{3}) \) and \( \kappa'(s) < 0 \) for \( s \in (-4a/\sqrt{3}, -2a). \) Hence, we obtain
\[
\sup_{s \in (-1 - a^2, -2a)} \kappa(s) = \kappa(-4a/\sqrt{3}) = b + 4 - 2\sqrt{3}a,
\]
indicating that \( \hat{F}(w) \geq \hat{F}(-w) \) and \( \hat{F}(w) \neq \hat{F}(-w) \) for \( w \in (a - 1, 0) \) in \( G_2. \) By Proposition 2.1 of [6], system (3) has no small limit cycles in \( G_1 \) and \( G_2. \) The proof is completed. □

4 Global bifurcation

The aim of this section is to show that the global bifurcation surfaces \( HL, DL_1, DL_2 \) for homoclinic loops and double limit cycles in Fig. 1 exist and how they are located in the bifurcation diagram of system (3).

In the following results, we discuss the existence and location of large limit cycles and figure-eight homoclinic loops of system (3).

Proposition 14 When \( 0 < a < 1 \) and \( b < a^2 - 1, \) there exist two continuous functions \( \varphi(a, \xi) \) and \( \phi_1(a, \xi) \) such that \( \varphi(0, \xi) = \phi_1(0, \xi), \) \( \varphi(1, \xi) = \phi_1(1, \xi) = 0, \)
\[
\varphi(a, \xi) < \phi_1(a, \xi) < a^2 - 1
\]

\[\]
Fig. 8 Separatrices of the saddle at the origin

i. when $f_1(a, \xi) < b < a^2 - 1$, system (3) has no large limit cycles;

ii. when $b = \phi_1(a, \xi)$, system (3) has exactly one large limit cycle, which is semistable;

iii. when $\varphi(a, \xi) < b < \phi_1(a, \xi)$, system (3) has exactly two large limit cycles, where the outer one is stable and the inner one is unstable;

iv. when $b = \varphi(a, \xi)$, system (3) has exactly one stable large limit cycle and one unstable figure-eight homoclinic loop.

Proof We consider equivalent system (31) of system (3). When $b = a^2 - 1$, system (31) has no limit cycles by Lemma 6 and equilibria $E_R, E_L$ are stable by the beginning part of Sect. 2. Thus, the separatrices of the saddle at the origin of (31) are shown in Fig. 8a when $b = a^2 - 1$. When $b = 3a^2 - 3$ and $\sqrt{3}/3 < a < 1$, system (31) has exactly one large limit cycle; by Lemma 12 and equilibria, $E_R, E_L$ are unstable by Sect. 2.2.3, yielding that the separatrices of the saddle at the origin of (31) are shown in Fig. 8b. Moreover, $(a, b, \xi) = (1, 0, \xi^*)$ is a Bogdanov–Takens bifurcation point of codimension 2 by [15] and $(a, b, \xi) = (0, b^*, \xi^*)$ is a grazing bifurcation point by [5], where $\xi^* > 0$ is an arbitrary fixed value and $b^* < -3$ is a certain value. As it was proved in [5], the separatrices of the saddle move monotonically when $a, \xi$ are arbitrarily fixed and $b$ increases. Hence, when $a$ and $\xi$ are fixed, there exists a unique value $b = \varphi(a, \xi)$ such that system (31) has exactly one figure-eight loop. Note that $\text{div}(y, -g(x) - f(x)y)|_{(x,y)=(0,0)} = -b\xi > 0$ for $b < 0$. By [7, Chapter 3], the figure-eight loop is unstable. Moreover, from Fig. 3 and Bendixson annulus Theorem, system (31) has at least one large limit cycle if $b = \varphi(a, \xi)$. Using an analogous argument of Lemma 12, we can show that system (31) has at most one large limit cycle in this case. Therefore, statement (iv) is proved.

When $b = \varphi(a, \xi) + \epsilon$ for small $\epsilon > 0$, a homoclinic bifurcation occurs, i.e., an unstable large limit cycle appears in a small neighborhood of the figure-eight loop. Hence, by Lemma 8 system (31) has exactly two large limit cycles when $b = \varphi(a, \xi) + \epsilon$.

For arbitrarily fixed values of $a$ and $\xi$, system (31) is a rotational family of vector fields (see [17] for definitions and properties) with respect to the parameter $b$, implying that unstable limit cycles increase and stable ones decrease in size when $b$ increases. Furthermore, the double limit cycle which is stable in its outside part splits into a pair of limit cycles. Hence, when $a$ and $\xi$ are arbitrarily fixed, there exists a unique value $b = \phi_1(a, \xi)$ satisfying that system (31) has exactly one large limit cycle, which is semistable. Thus, statements (ii) and (iii) are proved. By properties of rotated system (31) in $b$, statement (i) can be proved by a similar way as case (iii).

Lemma 15 The surface $DL_2$ lies in the region $\mathcal{G}_3 := \{(a, b, \xi) \in \mathcal{G} : 0 < a < \sqrt{3}/3, 2\sqrt{3}a - 4 < \phi_2(a, \xi) < \min\{\varphi(a, \xi), 3a^2 - 3\}\}$.

Proof By Lemma 11, system (3) has no small limit cycles if $1/\sqrt{3} \leq a < 1$ and $b = 3a^2 - 3$, and at most one small limit cycle around equilibria $E_R$ or $E_L$ if $0 < a < 1/\sqrt{3}$ and $b = 3a^2 - 3$. Therefore, $DL_2$ cannot intersect with the curve $b = 3a^2 - 3$ except when $a = 1/\sqrt{3}$. Computing the trace at $E_0$,
Fig. 9 Knots of $DL_2$

Fig. 10 Simulations with a single equilibrium

we obtain $\text{tr}(J_0) = -b\xi > 0$, because $b < 0$ and $\xi > 0$. So the homoclinic loops have to be unstable if they exist by [7, Chapter 3, Theorem 3.3]. Assume that system (3) exhibits at least two small limit cycles surrounding $E_R$ for $(a, b, \xi) \in \mathcal{G}_4 := \{(a, b, \xi) \in \mathcal{G} : 0 < a < 1, \ 3a^2 - 3 < b < a^2 - 1\}$. For fixed $a$ and $\xi$, we obtain that system (3) has a small semistable limit cycle $\Gamma_0$ when $b = b_0$ from the rotational properties of system (3) with respect to $b$. Now, given $b, \xi$ and a perturbation $a \to a + \varepsilon$, there exists a solution $\tilde{\phi}(t, x_0, y_0)$ for $t \in (0, T)$ which lies in the small neighborhood of $\Gamma_0$ from the continuous depen-
Fig. 11 Simulations with three equilibria

Remark 1 The existence of surface $DL_2$ of double small limit cycles can be guaranteed by the Bautin bifurcation, as seen in Sect. 2.2.3. If the graph of $DL_2$ can be expressed as a function $b = \phi_2(a, \xi)$ and has not any knot, there are at most two small limit cycles around the equilibria $E_R$ or $E_L$ for system (3), as shown in Fig. 2. If the graph of $DL_2$ has knots, system (3) will have more than two small limit cycles in the parameter region $V$ in Fig. 1. For example, system (3) has exactly two small limit cycles for special parameter value, where the two ones are quadruple, as shown in Fig. 9.

From [5], we can obtain that a pair of grazing loop is stable for $a = 0$ if they exist. However, the pair of homoclinic loops have to be unstable if they exist when $a \neq 0$. In fact, $HL, DL_1, DL_2$ have a common intersection point for the limit value $a = 0$, i.e., $\varphi(0, \xi) = \phi_1(0, \xi) = \phi_1(0, \xi) < -3$. Since sys-
system (31) is a rotational vector field with respect to the parameter $b$, the manifolds of $E_0$ move monotonically as $a, \xi$ are fixed and $b$ increases, see [5,6]. Therefore, it is worthwhile to note that $HL$, $DL_1$ and $DL_2$ have no intersection points except at endpoints. Summarizing the previous results, we can obtain Theorem 1, as shown in Fig. 1.

5 Numerical examples

In this section, we give several numerical examples of previous results.

Example 1 Let $a = \sqrt{2}$ and $\xi = 1$. When $b = 1$, the system has a unique equilibrium $(0, 0)$, which is a sink, and no limit cycles, as shown in Fig. 10a.

However, when $b = -1$ the system has a unique equilibrium, the origin $(0, 0)$ which is a source. Furthermore, from Lemma 5, there is a unique limit cycle, which is stable, as shown in Fig. 10b.

Example 2 Let $\xi = 1$. When $a = \sqrt{2}/2$ and $b = -2$, the system has three equilibria and exactly one large limit cycle, as shown in Fig. 11a.

When $a = 0.3$ and $b = -2.77$, the system has three equilibria and exactly two small limit cycles, as shown in Fig. 11b.

When $a = \sqrt{2}/2$ and $b = -0.5$, the system has three equilibria and no limit cycles, as shown in Fig. 11c.

Example 3 Let $\xi = 0.1$. When $a = 0.9$ and $b = -0.46$, the system has three equilibria and exactly two large limit cycles, as shown in Fig. 12a.

When $a = 0.9$ and $b = -0.555$, the system has three equilibria, exactly two small limit cycles and one large limit cycle, as shown in Fig. 12b.
When \( a = 3\sqrt{2}/10 \) and \( b = -2.461 \), the system has three equilibria, exactly two small limit cycles and two large limit cycles, as shown in Fig. 12c.

When \( a = \sqrt{5}/5 \) and \( b = -2.41 \), the system has three equilibria, exactly four small limit cycles and one large limit cycle, as shown in Fig. 12d.

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Appendix: Averaging theory of first and second order

The averaging theory of second order for studying specifically periodic orbits can be found in [12] for \( S^3 \) differential systems and in [1] for Lipschitz differential systems, see also Chapter 11 of [16]. Here we present a brief summary with the results that we need for studying the Hopf bifurcation of the SD oscillator systems.

Consider the differential system
\[
\dot{x}(t) = \mu F_1(t, x) + \mu^2 F_2(t, x) + \mu^3 R(t, x, \mu), \quad (34)
\]
where \( F_1, F_2 : \mathbb{R} \times D \to \mathbb{R} \), \( R : \mathbb{R} \times D \times (-\mu_0, \mu_0) \to \mathbb{R} \) are continuous functions, \( T \)-periodic in the first variable and \( D \) is an open subset of \( \mathbb{R}^n \). Assume:

i. \( F_1(t, \cdot) \in S^2(D), F_2(t, \cdot) \in S^1(D) \) for all \( t \in \mathbb{R}, F_1, F_2, R, D^2_F F_1, D_x F_2 \) are locally Lipschitz with respect to \( x \), and \( R \) is twice differentiable with respect to \( \mu \).

We define the functions \( F_{k0} : D \to \mathbb{R} \) for \( k = 1, 2 \) as follows
\[
F_1(x) = \frac{1}{T} \int_0^T F_1(s, x) ds, \\
F_2(x) = \frac{1}{T} \int_0^T \left[ D_x F_1(s, x) \int_0^s F_1(t, x) dt + F_2(s, x) \right] ds.
\]

ii. For \( V \subset D \) an open and bounded set and for each \( \mu \in (-\mu_0, \mu_0) \setminus \{0\} \), suppose that either \( f_1(x) \neq 0 \), there exists \( a \in V \) such that \( f_1(a) = 0 \) and the Jacobian \( \det D_x f_1(a) \neq 0 \), or \( f_1(x) \equiv 0 \), there exists \( a \in V \) such that \( f_2(a) = 0 \) and the Jacobian \( \det D_x f_2(a) \neq 0 \).

Then, for \( |\mu| > 0 \) sufficiently small there exists a \( T \)-periodic solution \( x(t, \mu) \) of system (34) such that \( x(0, \mu) \mapsto a \) when \( \mu \mapsto 0 \).

If for the \( i \) for which \( f_i(a) = 0 \) the real part of all the eigenvalues of the Jacobian matrix \( D_x f_i(a) \) are negative, then the periodic solution \( x(t, \mu) \) is asymptotically stable, if some eigenvalue has a positive real part then it is unstable.

The averaging theory of first order takes place when \( f_1(x) \neq 0 \). If \( f_1(x) \equiv 0 \) and \( f_2(x) \neq 0 \), we say that that we work with the averaging theory of second order.

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