ON THE NODAL STRUCTURES OF RANDOM FIELDS - A DECADE OF RESULTS

IGOR WIGMAN

ABSTRACT. We survey a decade worth of work pertaining to the nodal structures of random fields, with emphasis on the transformative techniques that shaped the field.

1. Introduction

In 2011 I published the survey [70] of results pertaining to the distribution of local functionals of some Gaussian ensembles of random fields, a subject that, in hindsight, was only in its early stage. Some 10 years later, one has, what I believe, a complete understanding of the said subject, and it seems like a good time for another survey of this research direction, to be written. While this manuscript does not mean to contain an exhaustive list of papers on the subject, it aims to describe the transformative or principal results that shaped the field, and outline the main techniques of proofs. We will outline the essence of the arguments behind the proofs of selected results, and emphasize the subtleties of the problems.

1.1. Euclidean vs. Riemannian setting. For $n \geq 1$ let $F : \mathbb{R}^n \to \mathbb{R}$ be a random field, that, unless specified otherwise, will be assumed stationary Gaussian, a.s. $C^\infty$-smooth. One is interested in the geometry of $F$, and, importantly, the structure of the (random) nodal set $\mathcal{N}_F := F^{-1}(0) \subseteq \mathbb{R}^n$. It is usual to restrict $F|_{B(R)}$ to a large centred ball $B(R) \subseteq \mathbb{R}^n$, and study the geometry of the restricted field in the limit $R \to \infty$; the corresponding restricted nodal set is $\mathcal{N}_F \cap B(R)$. Associated to $F|_{B(R)}$ one distinguishes a class of quantitative local properties, that could be described as additive functionals of the domain. For example, for $n = 1$ the number of zeros of $F$ is a local property, since if $I_1 = (a, b)$ and $I_2(c, d)$ are two disjoint intervals, then the number of zeros on $I_1 \cup I_2$ is the sum of zeros of $F$ on $I_1$ and on $I_2$. Similarly, for $n = 2$ (resp. $n \geq 2$), the nodal length (resp. $(n-1)$-volume) of $F|_{B(R)}$, i.e. the length of the smooth curve $F^{-1}(0)$ could be thought of as the functional $\mathcal{L}_F : \mathcal{D} \mapsto \mathcal{L}_F \mathcal{D} \in \mathbb{R}_{\geq 0}$, where $\mathcal{L}_F \mathcal{D}$ is the nodal length of $F$ restricted to a domain $\mathcal{D} \subseteq \mathbb{R}^2$.

Any quantitative property of the geometry of $F$ that fails to be local is said to be nonlocal. Examples of important nonlocal properties of $F$ include the number of nodal components of $F|_{B(R)}$ (i.e. the connected components of $F^{-1}(0) \cap B(R)$), or the percolation properties of excursion sets $F^{-1}(0, +\infty)$ on rectangles $[a, b] \times [c, d] \subseteq \mathbb{R}^2$ and their dilates. The number of nodal components of $F|_{B(R)}$ satisfies an intermediate property of semi-locality [46, 47]. That is, as $R \to \infty$, with high probability, most nodal components of $F|_{B(R)}$ lie in some radius-$r$ ball, with $r > 0$ growing slowly with $R$. In this survey we will only focus our attention to the local properties of ensembles of random fields in a number of scenarios, especially their nodal volumes, though we will also mention the critical values, and nodal intersections against smooth curves.

A particularly important planar random field is Berry’s Random Wave Model (RWM), that is the centred Gaussian random field $u : \mathbb{R}^2 \to \mathbb{R}$ of monochromatic isotropic waves, uniquely prescribed by the covariance function

$$r_{RW}(x, y) = r_{RW}(x - y) := \mathbb{E}[u(x) \cdot u(y)] = J_0(|x - y|),$$

where $J_0$ is the zeroth order Bessel function of the first kind.
with $J_0$ Bessel’s J function of order 0. The said random field, used as a model for ocean waves by Longuet-Higgins, is believed [6] to represent the (deterministic) Laplace eigenfunctions on generic chaotic surfaces, in the high energy limit.

Equivalently, one may define $u$ via its spectral measure: It is the arc length measure of the unit circle $S^1 \subseteq \mathbb{R}^2$, easily extended to higher dimensions. In general, the spectral measure $\rho = \rho_F$ of a stationary random field $F : \mathbb{R}^n \to \mathbb{R}$ is the Fourier transform of its covariance function

$$r_F(x, y) = r_F(y - x) := \mathbb{E}[F(x) \cdot F(y)],$$

$x, y \in \mathbb{R}^n$, thought of as of a function of a single variable $r_F(z)$. That $\rho$ is indeed a probability measure is the statement of Bochner’s theorem. The spectral measure of $F$ uniquely prescribes its law, and oftentimes, rather than imposing some constraints on the covariance function $r_F$, one imposes conditions on the law of $F$ via $\rho_F$. For example, the support of $\rho_F$ having a point in its interior is a strong non-degeneracy condition on the law of $F$, implying, in particular, that the distribution of $(F(0), \nabla F(0)) \in \mathbb{R}^{n+1}$ is non-degenerate, and, by the stationarity of $F$, so is the distribution of $(F(x), \nabla F(x)) \in \mathbb{R}^{n+1}$ for every $x \in \mathbb{R}^n$.

Another than the asymptotic geometry of a fixed random field over expanding subdomains of $\mathbb{R}^n$, one is interested in ensembles of Gaussian random fields defined on a Riemannian manifold $(\mathcal{M}, g)$: $f_n : \mathcal{M} \to \mathbb{R}$, where $n$ belongs to a discrete set of indexes. In many interesting cases, $f_n$ admits scaling limits, defined via the tangent plane, with stationary or even isotropic limit Euclidean random field. One in particular important such ensemble with a scaling random field is that of band-limited functions (see e.g. [60]) on a smooth compact Riemannian manifold $\mathcal{M}$, defined in the following section.

1.2. Random band-limited functions. Let $\Delta$ be the Laplace-Beltrami operator on $\mathcal{M}$ with Dirichlet boundary conditions (say), known to have a purely discrete spectrum, and denote the corresponding (negative) Laplace eigenvalues $\{\lambda_i^2\}_{i \geq 0}$ with associate orthonormal basis of $L^2(\mathcal{M})$ consisting of Laplace eigenfunctions $\{\varphi_i\}_{i \geq 0}$, i.e.

$$\Delta \varphi_i + \lambda_i^2 \varphi_i = 0.$$

Fix a number $\alpha \in [0, 1]$, and for the spectral parameter $\lambda \gg 0$ define the random field of random $\alpha$-band limited functions

$$f_\lambda(x) = f_{\mathcal{M}, \alpha; \lambda}(x) = \frac{1}{V(\lambda)} \sum_{\alpha \cdot \lambda \leq \lambda_i \leq \lambda} c_i \varphi_i(x),$$

indexed by $x \in \mathcal{M}$, where $c_j$ are i.i.d. standard Gaussian, and $V(\lambda) > 0$ is a convenience pre-factor chosen so that to make $f_\lambda(\cdot)$ asymptotically univariate. For $\alpha = 1$, the range of the summation on the r.h.s. of (1.2) should be understood as

$$\lambda - \eta(\lambda) \leq \lambda_i \leq \lambda,$$

with $\eta(\lambda) = o_{\lambda \to \infty}(\lambda)$ but $\|\eta(\lambda) \to \infty$ - these are the “monochromatic” waves on $\mathcal{M}$.

One may identify the covariance function of $f_\lambda(\cdot)$, uniquely prescribing the law of $f_\lambda$, as the spectral projector

$$r_\lambda(x, y) = r_{\mathcal{M}, \alpha; \lambda}(x, y) := \frac{1}{V(\lambda)} \sum_{\alpha \cdot \lambda \leq \lambda_i \leq \lambda} \varphi_i(x) \cdot \varphi_i(y).$$

It is well known in the microlocal analysis literature that $r_\lambda(x, y)$, and thereby $f_\lambda(\cdot)$, scale around every reference point $x_0 \in \mathcal{M}$, in the following sense. Let $R > 0$ be a number, and assuming that $R/\lambda$ is less than the injectivity radius of $x_0$, the (scaled) exponential map

$$\text{exp}_{x_0}(\cdot/\lambda) : B(R) \to \mathcal{M}$$

1In fact, $\eta(\lambda)$ does not need to grow, under some more restrictive assumptions on $\mathcal{M}$ of geometric nature, e.g. if the measure of geodesic loop directions through $x \in \mathcal{M}$ is 0 for a.a. $x \in \mathcal{M}$, see a discussion in [15] and the references therein, in particular [59]. The case $\eta \equiv 1$ is the most interesting, but also the most difficult one from the microlocal analytic point of view.
assigning $z \mapsto \exp_{x_0}(z/\lambda) \in \mathcal{M}$ is a bijection between the (Euclidean) ball $B(R) \subseteq T_{x_0} \mathcal{M} \cong \mathbb{R}^n$ and a neighbourhood of $x_0$ in $\mathcal{M}$. We define the scaled Gaussian random field on $B(R)$:

\begin{equation}
\lambda \mapsto f_\lambda(\exp_{x_0}(z/\lambda)),
\end{equation}

whose covariance function

\begin{equation}
\widetilde{r}_\lambda(z, w) = \frac{1}{V(\lambda)} \sum_{\alpha \lambda \leq \lambda, \leq \lambda} \varphi_\lambda(\exp_{x_0}(z/\lambda)) \cdot \varphi_\lambda(\exp_{x_0}(w/\lambda))
\end{equation}

converges, uniformly on $B(R) \times B(R)$, together with any finite number of its mixed derivatives, to the Fourier transform of the characteristic function of the annulus $\{\xi \in \mathbb{R}^n : \alpha \leq \|\xi\| \leq 1\}$:

\begin{equation}
\widetilde{r}_\lambda(z, w) \rightarrow \widetilde{r}_{\alpha;\infty} := \int_{\alpha \leq \|\xi\| \leq 1} \exp(2\pi i(z - w, \xi))d\xi,
\end{equation}

with effective control over the error term in terms of power decay in $\lambda$. For $\alpha = 1$, the said characteristic function of the annulus is understood in the limit sense as the hypersurface volume measure of the unit hypersphere $\|\xi\| = 1$.

The above means that, for every $R > 0$, as $\lambda \rightarrow \infty$, the (not stationary) random field $g_\lambda(\cdot)$ converges to the stationary isotropic Gaussian random field $g_{\alpha;\infty}$ on $B(R)$ whose covariance function is $\widetilde{r}_{\alpha;\infty}$. For $\alpha = 1$, the random field $g_{\alpha;\infty}$ is identified as Berry’s RWM with $n = 2$, and, more generally, for $\alpha = 1$, $n \geq 2$ these are the monochromatic waves on $\mathbb{R}^n$. These scaling properties allow for inferring results on $f_\lambda$ from the corresponding results on $g_\infty$, in Planck scale balls (i.e., geodesic balls of radius $\approx \frac{1}{\lambda}$), by a straightforward application of the Continuous Mapping theorem (see §1.7), or, since the asymptotics (1.4) is valid for all $R > 0$, tour de force slightly above it, i.e. geodesic balls of radii $R/\lambda$ with $R \rightarrow \infty$ sufficiently slowly.

In 1d the arising Gaussian ensemble is that of trigonometric polynomials: It is the Gaussian ensemble of stationary random processes

\begin{equation}
X_N(x) = \sum_{n=1}^{N}(a_n \sin(nt) + b_n \cos(nt)),
\end{equation}

with $a_n, b_n \sim \mathcal{N}(0, 1)$ standard i.i.d. Gaussian random variables, $t \in [0, 2\pi]$, and $N \geq 1$. The analogous scaling limit process is the Paley-Wiener process on $\mathbb{R}$, with the covariance $\frac{\sin x}{x}$. In this manuscript we will avoid dealing with the 1-dimensional case that was resolved [29], and instead only survey the high dimensional literature.

1.3. Random spherical harmonics and Arithmetic Random Waves. The sphere $S^2 \subseteq \mathbb{R}^3$ admits high spectral degeneracy, and, as a result, here, rather than superimposing eigenfunctions belonging to different eigenspaces (1.2), it allows for the introduction of the ensemble of random spherical harmonics, that is a particular instance of monochromatic random waves (i.e. band-limited functions (1.2) with $\alpha = 1$). The Laplace eigenvalues of the sphere are all the numbers

\begin{equation}
\lambda_\ell^2 := \ell(\ell + 1),
\end{equation}

parameterized by $\ell \in \mathbb{Z}_{\geq 0}$ nonnegative integer numbers. Given $\ell \geq 0$, the eigenspace corresponding to $\lambda_\ell^2$ is the space of degree-$\ell$ spherical harmonics, of dimension $2\ell + 1$, and let

\begin{equation}
\{\eta_1 = \eta_{\ell,1}, \ldots \eta_{2\ell+1} = \eta_{\ell,2\ell+1}\}
\end{equation}

be its arbitrary $L^2$-orthonormal basis. The degree-$\ell$ random spherical harmonic is

\begin{equation}
T_\ell(x) = \frac{1}{\sqrt{2\ell + 1}} \sum_{k=1}^{2\ell+1} a_k \eta_k(x),
\end{equation}

with $a_k$ i.i.d. standard Gaussian, and the convenience pre-factor $\frac{1}{\sqrt{2\ell + 1}}$ making $T_\ell$ univariate. It is easy to generalize the definition of $T_\ell(\cdot)$ for higher dimensional spheres. Since the standard multivariate
Gaussian distribution is invariant w.r.t. orthogonal transformations, the law of \( T_\ell \) is invariant w.r.t. the choice of the \( L^2 \)-orthonormal basis.

Alternatively (and equivalently) \( T_\ell (\cdot) \) is the Gaussian random field uniquely prescribed by the covariance function

\[
(1.7) \quad r_\ell (x, y) = P_\ell (\cos d(x, y)),
\]

where \( P_\ell \) are the Legendre polynomials parameterized by their degree \( \ell \geq 1 \), and \( d(\cdot, \cdot) \) is the spherical distance between \( x, y \in S^2 \). For every \( \ell \geq 1 \), the random field \( T_\ell (\cdot) \) is isotropic, in the sense that for every rotation \( g \) of \( S^2 \), the law of \( T(g \cdot) \) is identical to the law of \( T(\cdot) \).

As all spherical harmonics are either even or odd (w.r.t. to the transformation \( x \mapsto -x \) on \( S^2 \)), depending on whether \( \ell \) is even or odd respectively, so a.s. is \( T_\ell (\cdot) \), defined in (1.6). This means that the Gaussian vector \((T_\ell (x), T_\ell (-x))\) is fully degenerate, hence the Kac-Rice formula (2.4) below is not applicable on \( S^2 \). On the other hand, it also means that the nodal set of \( T_\ell \) is invariant w.r.t. \( x \mapsto -x \), hence, to recover all the information on the nodal set of \( T_\ell \), it is sufficient to restrict \( T_\ell \) to a hemisphere, where it is applicable.

As \( \ell \to \infty \), the asymptotic behaviour of \( P_\ell \) is described by Hilb’s approximation [64, Formula (8.21.17) on p. 197]

\[
(1.8) \quad P_\ell (\cos \phi) = \left( \frac{\phi}{\sin \phi} \right)^{1/2} J_0 ((\ell + 1/2) \phi) + \delta(\phi),
\]

where \( \delta(\cdot) = \delta_\ell (\cdot) \) is a small error term. As we fix a reference point \( x_0 \in S^2 \) (e.g. \( x_0 = N \), the northern pole), rescale the variables around \( x_0 \) and flatten the sphere in a small vicinity of \( x_0 \), the covariance function (1.7) is approximately

\[
(1.9) \quad r_\ell (x, y) \approx \left( \frac{\psi/\ell}{\sin (\psi/\ell)} \right)^{1/2} J_0 (\psi),
\]

where \( \psi = \psi(x, y) = (\ell + 1/2) \phi = (\ell + 1/2) d(x, y) \in [0, (\ell + 1/2) \pi] \), and \( x, y \in S^2 \), both in the vicinity of \( x_0 \). We identify the factor \( J_0 (\psi) \) in (1.9) as the covariance function (1.1) of Berry’s RWM; the extra factor \( \left( \frac{\psi/\ell}{\sin (\psi/\ell)} \right)^{1/2} \), reminiscent of the geometry of the sphere and the rescaled variables, is only close to 1 if \( d(x, y) \to 0 \), consistent to the general monochromatic waves (1.2) (with \( \alpha = 1 \)), though on a bigger neighbourhood, and faster effective convergence rate. Hence the scaled random field \( T_\ell \), restricted to a small, vanishing, neighbourhood of a reference point (and only restricted one) converges, in appropriate sense, to the RWM. Mind that the sphere with the geodesic flow is a completely integrable dynamical system, hence the RWM is not expected to (nor does it) represent the global behaviour of the individual spherical harmonics.

It is known that every square-summable isotropic random field \( G \) on \( S^2 \) can be decomposed in \( L^2 \) as

\[
G(x) \overset{L^2(S^2)}{\Rightarrow} \sum_{\ell=1}^{\infty} C_\ell T_\ell (x)
\]

in the sense that

\[
\mathbb{E} \left[ \left\| G - \sum_{\ell=1}^{L} C_\ell T_\ell \right\|_{L^2(S^2)}^2 \right] \overset{\ell \to \infty}{\to} 0.
\]

The collection of the nonnegative numbers \( \{C_\ell\}_{\ell \geq 1} \) is called the power spectrum of \( G \). That is, \( \{T_\ell\}_{\ell \geq 1} \) are the Fourier components of every nice Gaussian isotropic field. For example, in cosmology these represent the Cosmic Microwave Background (CMB) radiation measurements under the Gaussianity assumption, and the high energy limit \( \ell \to \infty \) stands for the high precision of those measurements.
Another surface admitting high spectral degeneracy is the standard 2-dimensional torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$. Oravecz-Rudnick-Wigman [50] introduced the ensemble of random toral eigenfunctions, usually referred to as “Arithmetic Random Waves” (ARW). Let 
\[ S = \{a^2 + b^2 : a, b \in \mathbb{Z} \} \subseteq \mathbb{Z} \]
be the set of all positive integers expressible as a sum of two squares, and for $n \in S$ we let 
\[ \Lambda_n := \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \|\lambda\|^2 = n \} = \mathbb{Z}^2 \cap \sqrt{n}S^1 \]
be the collection of all lattice points of squared Euclidean norm $n$, or equivalently, lattice points lying on the centred circle of radius $\sqrt{n}$ (alternatively, the nonempty set of representations of $n$ as a sum of two squares). Denote $N_n = r_2(n) := |\Lambda_n|$ to be the number of lattice points lying on the said circle (also the number of representations of $n$ as sum of two squares). It is well-known that the toral Laplace eigenvalues are all the number of the form \(4\pi^2n : n \in S\), and the eigenspace corresponding to \(4\pi^2n\) is the collection of all (complex) linear combinations of the plane waves $e((\lambda, x))$, where $x = (x_1, x_2) \in T^2$,
\[ \langle \lambda, x \rangle = \lambda_1 x_1 + \lambda_2 x_2 \]
is the standard Euclidean inner product, and $e(\cdot) = e^{2\pi i \cdot}$.

The Arithmetic Random Waves is the Gaussian ensemble\footnote{Note the abuse of notation, as in this context $n$ denotes the energy rather than the manifold dimension.} $f_n : T^2 \to \mathbb{R}$ of the random linear combinations of the plane waves
\[
(1.10) \quad f_n(x) = \frac{1}{\sqrt{2N_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e(\langle \lambda, x \rangle),
\]
where the coefficients $\{a_\lambda \|\lambda\|^2 = n\}$ are standard complex Gaussian i.i.d., save for the relation $a_{-\lambda} = \overline{a_\lambda}$, making $f_n$ real-valued. As for the spherical harmonics, the convenience pre-factor $\frac{1}{\sqrt{2N_n}}$ was introduced to make $f_n$ univariate. Equivalently, the $f_n$ are stationary Gaussian random waves indexed by $T^2$, with the covariance function
\[
(1.11) \quad r_n(x) = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi \langle \lambda, x \rangle),
\]
x $\in T^2$. Finally, $f_n$ could be uniquely prescribed via (the scaled version of) its spectral measure supported on $S^1 \subseteq \mathbb{R}^2$:
\[ \mu_n := \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}}, \]
with $\delta_x$ the Dirac delta function supported on $x$ (cf. the spectral measure of Berry’s RWM described above).

1.4. Nodal intersections against smooth curves and hypersurfaces. Let $M$ be a surface and $C \subseteq M$ be a smooth curve. One is interested in the following (deterministic) questions: 1. Is the number of the intersections $M_{C,i}$ of the nodal line of Laplace eigenfunction $\varphi_i$ (as in §1.2) against $C$ finite for $i$ sufficiently large? 2. If the number of nodal intersections is finite, what is its asymptotic behaviour as $i \to \infty$? The natural scaling of the problem suggests that, provided that, given $C$, the answer to (1) is “yes”, then the number of nodal intersections should be asymptotic to
\[ c_C \cdot \sqrt{\lambda_i}, \]
with an appropriate $c_C > 0$. Alternatively, at least optimal, up to a constant, upper and lower bounds
\[ c_C \cdot \lambda_i \leq M_{C,i} \leq C_C \cdot \lambda_i, \]
$0 < c_C < C_C$ should be pursued. Of course, same questions could be posed for manifolds of arbitrarily high dimension, and $\Sigma$, a hypersurface of arbitrary codimension, in place of $C$ (depending on the dimensions of $M$ and $\Sigma$, these might not be discrete zeros, whence the relevant quantity is the intersection volume).
These questions were addressed by Toth-Zelditch for complex analytic manifolds, by methods of complexification, i.e. the number of nodal intersections is bounded by the number of (complex) zeros of the extension of $\varphi_i$ in a complex neighbourhood of $\mathcal{M}$ around $\mathcal{C}$. An optimal upper bound was proved [67] for $\mathcal{C}$ satisfying a “goodness” condition, which, in practice, might be quite difficult to validate for a given $\mathcal{C}$, see also the references within [67]. A particularly important scenario is when $\mathcal{M}$ is a billiard (a planar surface with a boundary), and $\mathcal{C}$ is its boundary, addressed in [68], where an optimal upper bound was proved in a number of scenarios. A precise result for the expected number of intersections of the band-limited functions against the boundary of a generic billiard was established in [69].

The same questions for the particular case of (deterministic) toral eigenfunctions of arbitrary dimension, susceptible to the number theoretic methods, was addressed by Bourgain-Rudnick. They proved [14, 13] the finiteness and the optimal upper bound for the intersection number (or volume) for generic curves, and a nearly optimal lower bound, along with some other interesting results of relevant nature.

2. Main Ideas: Kac-Rice Formulae and Their Applications

2.1. Zero Density. Let $F : \mathcal{D} \to \mathbb{R}$ indexed by the domain $\mathcal{D} \subseteq \mathbb{R}^n$, be a sufficiently smooth Gaussian random field so that for every $x \in \mathcal{D}$, $F(x)$ is non-degenerate, and, with no loss of generality, one may assume that $F$ is centred, i.e. for every $x \in \mathcal{D}$, $E[F(x)] = 0$. Assuming that for every $x \in \mathcal{D}$, $F(x)$ is nondegenerate Gaussian r.v., one may define the zero density (also called the first intensity) of $F$ as

\begin{equation}
K_1(x) = K_{F;1}(x) := \phi_{F(x)}(0) \cdot E[\|\nabla F(x)\| \mid F(x) = 0],
\end{equation}

$x \in \mathcal{D}$, where $\phi_{F(x)}(0) = \frac{1}{\sqrt{2\pi \sqrt{\text{Var}(F(x))}}}$ is the probability density function of $F(x)$ evaluated at 0, and $E[\|\nabla F(x)\| \mid F(x) = 0]$ is the Gaussian expectation of the norm of $\nabla F(x)$ conditioned on $F(x) = 0$. Then, under very mild assumptions on $F$, one may evaluate the expected nodal volume of $F$ on compact $\mathcal{D}$ as

\begin{equation}
E[\text{Vol}_{n-1}(F^{-1}(0))] = \int_{\mathcal{D}} K_1(x) dx.
\end{equation}

To the best knowledge of the author, the mildest known sufficient condition for (2.2) is [2] Theorem 6.8], applicable under the only extra assumption that $F$ has a.s. no degenerate zeros, for which, in turn, a generalization of Bulinskaya’s Lemma [2 Proposition 6.12] gives a sufficient condition in terms of the boundedness of the density of $(F(x), \nabla F(x))$ at $(0, 0)$, uniform w.r.t. $x \in \mathcal{D}$. It is easy to modify the definition of the zero density (2.1) corresponding to other local quantities: For example, at level $t \in \mathbb{R}$, it assumes the shape

\begin{equation}
K_{F;1,t}(x) := \phi_{F(x)}(t) \cdot E[\|\nabla F(x)\| \mid F(x) = t],
\end{equation}

and it is also an easy (though a bit more complicated) variation to adapt [20, 17] for the critical points or critical points whose values are restricted to lie in a window. In the Riemannian case the above holds true, except that the meaning of $\nabla F(x)$ is naturally adapted.

If, as we are accustomed, $F$ is stationary, then the density $K_1 \equiv K_1(x)$ does not depend on $x$, and the formula (2.2) assumes a particularly simple shape

$$E[\text{Vol}_{n-1}(F^{-1}(0))] = \text{Vol}(\mathcal{D}) \cdot K_1,$$

i.e. the expected number of the nodal volume of a stationary random field restricted to a domain is proportional to its volume. In this case, the number $K_1$ is expressed as a Gaussian integral (2.1), whose parameters are easily computed in terms of the covariance $r_F$ and its mixed derivatives up to 2nd order, all evaluated at the origin.
2.2. The 2-point correlation function. Further, if for every \( x, y \in \mathcal{D} \) so that \( x \neq y \), the bivariate Gaussian vector \((F(x), F(y))\) is non-degenerate, one can define the 2-point correlation function (2nd intensity) of the nodal set as

\[
K_2(x, y) = K_{F;2}(x, y) := \phi_{F(x),F(y)}(0,0) \cdot \mathbb{E}[(\|\nabla F(x)\| \cdot \|\nabla F(y)\| | F(x) = F(y) = 0)],
\]

\( x, y \in \mathbb{R}^n, x \neq y \), where \( \phi_{F(x),F(y)}(\cdot, \cdot) \) is the Gaussian density of \((F(x), F(y))\), and

\[
\mathbb{E}[\|\nabla F(x)\| \cdot \|\nabla F(y)\| | F(x) = F(y) = 0]
\]

is the Gaussian expectation of the product \( \|\nabla F(x)\| \cdot \|\nabla F(y)\| \) of norms conditioned on \( F(x) = F(y) = 0 \). For \( F \) sufficiently nice, for \( n \geq 2 \), one may express the 2nd moment of the nodal volume of \( F \) via the double integral

\[
\mathbb{E}[\text{Vol}_{n-1}(F^{-1}(0))^2] = \int_{\mathcal{D} \times \mathcal{D}} K_2(x, y) dx dy,
\]

which, for \( n = 1 \) yields the factorial 2nd moment\(^3\)

\[
\mathbb{E}[(\#(F^{-1}(0))^2) - \#(F^{-1}(0))] = \int_{[a,b]^2} K_2(x, y) dx dy
\]

of the number of the discrete zeros of \( F \) on a compact interval \([a, b]\). To the best knowledge of the author, the mildest sufficient condition for \((2.4)\) and \((2.5)\) is given by \([2, \text{ theorems } 6.8-6.9]\), a little more than the non-degeneracy of \((F(x), F(y))\) for every \( x \neq y \). Unless specified otherwise, we will be willing to assume for a while that \( n \geq 2 \).

The definition \((2.3)\) of the 2-point correlation function gives for \( x \neq y \) the value of \( K_2(x, y) \) in terms of a certain Gaussian integral depending on the covariance function \( r_F \) and its mixed derivatives up to 2nd order, evaluated at \((x, y)\) and the diagonal points \((x, x)\) and \((y, y)\); in 1d one may evaluate that integral explicitly to express it as an elementary function of the said derivatives. If \( F \) is stationary, then one may normalize it to be univariate: for every \( x \in \mathcal{D}, \text{Var}(F(x)) = 1 \). Then (assuming \( n \geq 2 \)), the 2-point correlation function depends on \( z := y - x \) only, and it simplifies to\(^4\)

\[
K_2(z) = \frac{1}{2\pi \sqrt{1 - r_F(z)^2}} \mathbb{E}[\|\nabla F(0)\| \cdot \|\nabla F(z)\| | F(0) = F(z) = 0],
\]

whereas \((2.4)\) reads

\[
\mathbb{E}[\text{Vol}_{n-1}(F^{-1}(0))^2] = \int_{\mathcal{D} \times \mathcal{D}} K_2(x - y) dx dy,
\]

with an obvious adaptation for \( n = 1 \). When \( F \) is defined on a manifold possessing some symmetries w.r.t. which the law of \( F \) is invariant (such as the standard torus with respect to translations, or the sphere with respect to rotations), the double integral on the r.h.s. of \((2.6)\) is expressible in terms of a simple integral.

Typically, for \( F \) stationary, defined on the whole of \( \mathbb{R}^n \), the asymptotic behaviour of \( K_2(z) \) as \( |z| \to \infty \) determines the asymptotic law for the variance of the nodal volume \( \text{Vol}_{n-1}(F^{-1}(0) \cap B(R)) \) of \( F \) restricted to increasing balls \( B(R), R \to \infty \); in the Riemannian setting with scaling, the relevant asymptotic behaviour is for the scaled variables diverging. However, the 2-point correlation function carries so much more information than merely a precise evaluation of the 2nd moment \( \mathbb{E}[\text{Vol}_{n-1}(F^{-1}(0) \cap \mathcal{D})^2] \) or the variance \( \text{Var}(\text{Vol}_{n-1}(F^{-1}(0))) \), or the asymptotic nodal volume variance restricted to increasing balls. First, it gives the variance on arbitrary shapes or their homotheties with no re-calculation, and in the Riemannian setting, the variance of the nodal volume of \( f_k \) restricted to subdomains, possibly shrinking slower than the scaling (see the applications given in \( \S 4.2 \)).

\(^3\)This is the contribution of the so-called “diagonal” - tuples of the discrete zeros \((z_i, z_i)\) of \( F \), see e.g. the argument given by Cramer-Leadbetter \([23]\).

\(^4\)We tacitly assume that \( 0 \in \mathcal{D} \), otherwise shift the random field.
and §3.2.2. It also endows the notions of zero attraction or repulsion with a proper meaning: we say that if zeros repel if $K_2(x, y)$ vanishes as $\|x - y\| \to 0$, and attract if $K_2(x, y)$ grows to infinity as $\|x - y\| \to 0$.

When in addition, $F$ is isotropic, then $K_2(x, y) = K_2(\|x - y\|)$ depends on the distance between $x$ and $y$ only. Hence, in this case, one encodes the attraction of repulsion of zeros in terms of the value of $K_2(0)$ at the origin, understood in the limit sense; for the former (attraction) $K_2(0) = +\infty$, whereas for the latter, $K(0) = 0$. In the isotropic case, one may also entertain an intermediate notion of zero attraction or repulsion, in the situation when $K_2(0) > 0$ is a finite, strictly positive number. For example, one may compare [9, 10] this value at the origin of the 2-point correlation function corresponding to the critical points of $F$ to that of the Poisson point process of the same intensity.

Analogously to the above, one may introduce the $k$-point correlation function, $k \geq 3$, to relate to the higher moments of the nodal volume. It is easy to modify the definition of the zero density (2.1) and the other correlation functions for other local quantities: For example, at level $t \in \mathbb{R}$, the density function assumes the shape

$$K_{F,1,t}(x) := \phi_{F(x)}(t) \cdot \mathbb{E}[\|\nabla F(x)\| | F(x) = t],$$

and it is also an easy (though a bit more complicated) variation to adapt [20, 17] for the critical points or critical points whose values are restricted to lie in a window. In the Riemannian case all of the above holds true, except that the meaning of $\nabla F(x)$ is naturally adapted.

### 2.3. Approximate and Mixed Kac-Rice formulae.

Provided that the said sufficient conditions on $F$ are satisfied, the Kac-Rice integral (2.4) on $\mathbb{D}$ computes the second moment (or the second factorial moment in 1d) on the whole of $\mathbb{D}$, and hence its variance,

$$\text{Var} \left( \text{Vol}_{n-1}(F^{-1}(0)) \right) = \int_{\mathbb{D} \times \mathbb{D}} K_2(x, y) dx dy - (\mathbb{E}[\text{Vol}_{n-1}(F^{-1}(0))])^2$$

(2.7)

$$= \int_{\mathbb{D} \times \mathbb{D}} (K_2(x, y) - K_1(x)K_1(y)) dx dy,$$

$n \geq 2$. If $\mathbb{D}_1, \mathbb{D}_2 \subseteq \mathbb{D}$ are two domains, then one may obtain the covariance of the nodal volume of $F$ restricted to $\mathbb{D}_1$ and $\mathbb{D}_2$ respectively by restricting the range of the integral on the r.h.s. of (2.7):

$$\text{Cov} \left( \text{Vol}_{n-1}(F^{-1}(0) \cap \mathbb{D}_1), \text{Vol}_{n-1}(F^{-1}(0) \cap \mathbb{D}_2) \right)$$

$$= \int_{\mathbb{D}_1 \times \mathbb{D}_2} K_2(x, y) dx dy - \mathbb{E}[\text{Vol}_{n-1}(F^{-1}(0) \cap \mathbb{D}_1)] \cdot \mathbb{E}[\text{Vol}_{n-1}(F^{-1}(0) \cap \mathbb{D}_2)]$$

(2.8)

$$= \int_{\mathbb{D}_1 \times \mathbb{D}_2} (K_2(x, y) - K_1(x)K_1(y)) dx dy.$$

The upshot is that, in some important cases, when (2.7) either does not hold, or it is technically demanding to validate its sufficient conditions, (2.8) may still be verifiable for a wide family of carefully chosen pairs of non-intersecting subdomains $\mathbb{D}_1, \mathbb{D}_2 \subseteq \mathbb{D}$. This way it might still be possible to justify that (2.7) (“Approximate Kac-Rice” [57, Proposition 1.3], see §4.3 below), or, in some cases, further approximation of (2.7), holds asymptotically [58] in some regime, also allowing for the asymptotic analysis of the variance $\text{Var} \left( \text{Vol}_{n-1}(F^{-1}(0)) \right)$ of the total nodal volume.

Let $F, G : \mathbb{D} \to \mathbb{R}$ be two Gaussian random fields defined on the same probability space. In analogy to the above, under suitable conditions on $F, G$, one may define the cross-correlation function (see e.g. [29] p. 37 or [40] p. 7-8); it is contained in

---

5For $n = 1$ obvious adjustments are due.

6For $n = 1$ some adjustment is required. Otherwise, it is applicable away from the diagonal unimpaired.
the 1970’s literature in the 1d case)
\[ K_2(x, y) = K_{2,F,G}(x, y) := \phi_{F(x),G(y)}(0, 0) \cdot \mathbb{E}[\|\nabla F(x)\| \cdot \|\nabla F(y)\| | F(x) = F(y) = 0], \]
expressible in terms of the cross-covariance
\[ \overline{r}_{F,G}(x, y) = \mathbb{E}[F(x) \cdot G(y)] \]
and its mixed derivatives up to 2nd order. Under suitable conditions, the corresponding (“mixed”) Kac-Rice integral computes the covariance between the zeros of \( F \) and \( G \) on \( \mathcal{D} \):

\[ \text{Cov}(\text{Vol}_{n-1}(F^{-1}(0)), \text{Vol}_{n-1}(G^{-1}(0))) = \int_{\mathcal{D} \times \mathcal{D}} \left( \overline{K}_2(x, y) - K_{1,F}(x) \cdot K_{1,G}(y) \right) dx dy. \tag{2.9} \]

The Mixed Kac-Rice formula is in particular useful when either \( F \) represents an ensemble \( \{f_k\} \) of Gaussian random fields converging in law to a limit Gaussian random field \( G \), or when \( G \) is a mollified version of \( F \); for example, \( G \) could be stationary or even isotropic, whereas \( F \) is only asymptotically such, or \( G \) could be \( M \)-dependent\(^7\) for \( M \) sufficiently slowly growing parameter. This way, from quantitative convergence of \( r_F(\cdot, \cdot) \) and its mixed derivatives to \( r_G(\cdot, \cdot) \) and its derivatives respectively, one may infer various properties on \( F \) from the analogue properties on \( G \): for example, one may infer the variance and the limit law of the total number of zeros of random trigonometric polynomials \((1.5)\) from the Paley-Wiener process on \( \mathbb{R} \). The proximity alone of \( r_F \) to \( r_G \) does not guarantee the success of this procedure; the key step is constructing a coupling of \( F \) and \( G \) so that the cross-covariance \( \overline{r}_{F,G} \) is quantitatively close, in the suitable norm, to \( r_F \) (and \( r_G \)). Combining the two ideas of Approximate and Mixed Kac-Rice to express covariances of nodal volume on different domains can give extra boost to these powerful methods. As another example, for \( n \geq 2 \), one may reproduce this argument to infer the variance and the limit law of the nodal volume of band-limited functions \((1.2)\) from the corresponding limit random field, in somewhat more restrictive scenario, only in geodesic balls of radius\(^8\) logarithm power higher than the Plack scale.

Rather than comparing the nodal volume of two different random fields, as in \((2.9)\), one may derive the cross-correlation formula that compares between different properties of the same random field, or different random fields, and integrate it to obtain the covariance between these. For example, this way one may compute the covariance between the nodal volume and the number of critical points of the same random field \( F \) on a domain \( \mathcal{D} \) or, for instance, the covariance of the volumes of two nonzero levels \( \text{Cov}(\text{Vol}_{n-1}(F^{-1}(t_1)), \text{Vol}_{n-1}(F^{-1}(t_2))) \), or, more generally, the covariance of any local properties of two random fields restricted to arbitrary domains.

2.4. Auxiliary function for linear statistics, or Euclidean random fields. In a situation when \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a stationary isotropic Gaussian process, for example, Berry’s RWM (alternatively, \( F : \mathcal{S}^2 \rightarrow \mathbb{R} \) is invariant w.r.t. rotations), the 2-point correlation function
\[ K_{F,2}(x, y) = K_2(||x - y||) \]
is a function of the distance \( ||x - y|| \) (resp. of the spherical distance \( d(x, y) \)). One may then use \((2.4)\) with Fubini to rewrite
\[ \mathbb{E}[\text{Vol}_{n-1}(F^{-1}(0))^2] = \int_0^{\text{diam}(\mathcal{D})} K_2(t) W_\mathcal{D}(t) dt, \]

\(^7\)Similarly to above, one may restrict the range of the integral to \( \mathcal{D}_1 \times \mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{D} \), resulting in the covariance between the nodal volume of \( F \) and \( G \) restricted to \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) respectively.

\(^8\)That is, \( r_{FG}(x, y) = 0 \) for \( ||x - y|| > M \).

\(^9\)This is a by-product of the fact that the convergence of \( r_{F_1} \) to the limit covariance is restricted only slightly above Plack scale if \( M \) has no conjugate points; otherwise, other than Zoll surfaces or rational tori, no general result above Plack scale is known.
where \( \text{diam}(\mathcal{D}) \) is the diameter of \( \mathcal{D} \), and \( W_\mathcal{D}(\cdot) \) is the auxiliary function \( \text{[72]} \).

\[
W_\mathcal{D}(t) = \text{meas}\{(x, y) \in \mathbb{D}^2 : \|x - y\| = t\} = \int_\mathcal{D} \text{len}\{(y \in \mathcal{D} : \|x - y\| = t\}) \, dx,
\]

independent of \( F \). Given \( \mathcal{D} \) finding the precise values of \( W_\mathcal{D}(\cdot) \) might be quite technically demanding or not have a closed formula, even in some elementary cases, e.g. \( \mathcal{D} \) is a disc. However, in many cases, such as the situation of the random spherical harmonics, the bulk of the variance \( \text{(2.4)} \) is concentrated around the diagonal, hence what matters is only the asymptotics of \( W_\mathcal{D}(\cdot) \) near the origin.

More generally, one may consider the linear statistics

\[
\text{Vol}_\mathcal{D}(F) := \int_{F^{-1}(0)} \varphi(x) \, dx,
\]

where \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) (or \( \varphi : \mathcal{M} \to \mathbb{R} \)) is some test function, e.g. it could be a smooth function or the characteristic function \( \varphi = 1_\mathcal{D} \) of a domain \( \mathcal{D} \subseteq \mathbb{R}^2 \) or \( \mathcal{D} \subseteq \mathcal{M} \). Then one has \( \text{[72]} \)

\[
\mathbb{E}[(\text{Vol}_\mathcal{D}(F))^2] = \int_0 W_\varphi(t) \, dt
\]

with

\[
W_\varphi(t) = \int_{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|x - y\| = t} \varphi(x) \varphi(y) \, dy = \int_{\mathbb{R}^2} dx \int_{\partial B_t} \varphi(y) \, dy.
\]

The function \( W_\varphi \) has a smoothing effect on \( \varphi \) in a way similar to convolving functions; in fact, it is possible to express it as a convolution of certain by-products of \( \varphi \). It gives a powerful tool for evaluating the variance of linear statistics (in particular, of the nodal volume restricted to a subdomain of an ambient manifold) of the nodal lines, assuming that the asymptotic behaviour of \( K_2(\cdot) \) is understood, with no need for re-computations or solving the difficult geometric problem \( \text{(2.10)} \), that is independent of \( F \).

2.5. Wiener Chaos expansion of Kac-Rice integrals. For a “nice” Gaussian random field \( F : \mathbb{R}^n \to \mathbb{R} \) (alternatively, \( F : B(R) \to \mathbb{R} \) or \( F : \mathcal{M} \to \mathbb{R} \) etc.), defined on the sample space \( \Omega \), let \( \mathcal{H} = \mathcal{H}_F \subseteq L^2(\Omega) \) be the (complex) closed Gaussian Hilbert space generated by the vectors \( \{F(x) : x \in \mathbb{R}^n\} \), and \( \mathcal{H} = \mathcal{H}_F \subseteq L^2(\Omega) \) be the Hilbert space of random variables in \( L^2(\Omega) \), measurable w.r.t. the sub-\( \sigma \)-algebra generated by \( \mathcal{H} \). For \( q \geq 0 \) the \( q \)‘th Wiener chaos space is the closed span \( \mathcal{H}_q \) of

\[
\{H_\alpha(\xi_1, \ldots, \xi_k) : k \geq 1, \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_\geq^k, \alpha_1 + \ldots + \alpha_k = q\},
\]

where the \( H_\alpha(\cdot) \) is the \( k \)-variate Hermite polynomial of multi-index \( \alpha \). For example, \( \mathcal{H}_0 \) consists of all a.s. constant random variables. One has the Wiener chaos decomposition

\[
\mathcal{H} = \bigoplus_{q=0}^\infty \mathcal{H}_q,
\]

orthogonal w.r.t. the inner product

\[
\langle \xi, \xi' \rangle = \mathbb{E}[\xi \cdot \bar{\xi'}],
\]

that is, if \( \xi \in \mathcal{H}_q, \xi' \in \mathcal{H}_{q'} \) with \( q \neq q' \), then \( \xi, \eta \) are uncorrelated.

\footnote{It is easy to combine it with the ideas presented towards the end of \( \S 2.3 \) to express
\[
\text{Cov}(\text{Vol}_1^\varphi(F), \text{Vol}_1^\varphi(G))
\]
for test functions \( \varphi_1, \varphi_2 \), and stationary isotropic Gaussian random fields \( F, G \). It is also easy to generalize it for non-isotropic random fields.}
Using Federer’s Co-Area formula \[27\], one may formally express the nodal volume of a sufficiently smooth function \( f : \mathcal{M} \to \mathbb{R} \) as

\[
\text{Vol}_{n-1}(f^{-1}(0)) = \int_{\mathcal{M}} \delta_0(f(x)) \cdot \|\nabla f(x)\| dx,
\]

where \( \delta \) is the Dirac delta function. More generally, if \( g : \mathcal{M} \to \mathbb{R}^k \), where \( \mathcal{M} \) is a smooth \( n \)-manifold, and \( k \leq n \), then the nodal \((n-k)\)-volume of \( g \) is given by

\[
\text{Vol}_{n-k}(g^{-1}(0)) = \int_{\mathcal{M}} \delta_0(f(x)) \cdot \|J_g(x)\| dx,
\]

where \( J_g(\cdot) \) is the volume of the parallelogram spanned by the row vectors of the Jacobi matrix of \( g \), e.g., if \( n = k \), then \( J_g(\cdot) \) is the Jacobian of \( g \). One can infer from the expression (2.12), that, given a random field \( F : \mathcal{M} \to \mathbb{R} \) (or, for example \( F : B(R) \to \mathbb{R} \) with the Euclidean ball \( B(R) \subseteq \mathbb{R}^n \)) the random variable \( \mathcal{V} := \text{Vol}_{n-1}(f^{-1}(0)) \) is measurable w.r.t. the random variables generated by \( \mathcal{F} \), hence it satisfies \( \mathcal{V} \in \mathcal{H} \), if only it has finite variance, whence one may project it \( \mathcal{V}_q := \text{Proj}_{\mathcal{H}_q}(\mathcal{V}) \) onto \( \mathcal{H}_q \) for \( q \geq 0 \), and thus decompose it w.r.t. the decomposition (2.11). For example, the 0’th Wiener chaotic component of \( \mathcal{V} \)

\[
\mathcal{V}_0 = \mathbb{E}[\mathcal{V}]
\]

is its expectation, as one has for an arbitrary \( X \in \mathcal{H} \).

This way, rather than study the distribution of the stochastic integral of the type (2.12), one may decompose it into the components \( \mathcal{V}_q \), as above, and study their distributions separately, before combining the individual results for the purpose of obtaining a limit law of the nodal volume \( \mathcal{V} \). This was first done by Kratz-León [34] for Euclidean random fields \( F : \mathbb{R}^n \to \mathbb{R} \) over expanding regions, such as \( B(R) \), \( R \to \infty \). This approach was used in [44, 42, 43] (and then elsewhere) in order to study the generic functionals of the random spherical harmonics (1.6), using a somewhat more sophisticated modern language developed in the mean time, that saves one from approximating the given random fields by \( M \)-dependent ones (see §3.2.2 below for some details).

3. Survey of results: Band-limited functions, random spherical harmonics and their Euclidean limits

3.1. Berry’s random wave model. Recall that Berry’s random wave model is the centred isotropic Gaussian random field \( u : \mathbb{R}^2 \to \mathbb{R} \) whose covariance is

\[
r_u(x, y) = r_u(x - y) = J_0(||x - y||).
\]

By the standard asymptotics of the Bessel functions, as \( \psi \to \infty \), \( J_0(\psi) \) admits the 2-term asymptotics

\[
J_0(\psi) = \sqrt{\frac{2}{\pi}} \sin(\psi + \pi/4) - \frac{1}{8} \sqrt{\frac{2}{\pi}} \cos(\psi + \pi/4) + O\left(\frac{1}{\psi^{5/2}}\right),
\]

yielding the asymptotics as \( ||x - y|| \to \infty \) for \( r_u(x, y) \), and one may also derive. Alternatively, one may derive the asymptotic expansion of \( J_0(\psi) \) by writing its expression as the Fourier transform of the characteristic function of the unit circle, and using the stationary phase method.

Since \( u \) is stationary (ditto isotropic), its zero density is constant on \( \mathbb{R}^2 \), easily evaluated to be \( K_1(x) = K_{u;1}(x) = \frac{1}{28/2} \).

\[1\] In reality, one replaces \( \delta(\cdot) \) with \( \frac{1}{\pi} \chi_{\mathcal{C} \setminus \mathcal{C}}(\cdot) \), performs all the steps below, and takes \( \epsilon \downarrow 0 \).

\[2\] A full asymptotic expansion is well-known, 2 terms are sufficient for the presented results.

\[3\] Though [27] did not state it in this precise form, it is easy to extract it from Berry’s manuscript [7].
hence the expected nodal length of $u$ restricted to $B(R)$ equals precisely to
\[
\mathbb{E} \left[ \text{len}(u^{-1}(0) \cap B(R)) \right] = \frac{1}{232/3} \cdot \pi R^2.
\]

Berry then exploited the asymptotics \((3.1)\) of the covariance function and its mixed derivative to yield a 2-term asymptotic expansion for the 2-point correlation function at infinity: Since $u$ is isotropic, $K_{u,2}(x, y) = K_2(||x - y||) = K_2(\psi)$ depends only on the distance $\psi := ||x - y||$ between the two points, and he asserted that
\[
K_2(\psi) = \frac{1}{8} + \frac{\sin(2\psi)}{4\pi \psi} + \frac{1}{512\pi^2} \frac{1}{\psi^2} + E(\psi),
\]
where $E(\psi)$ contains purely oscillatory terms of order of magnitude $\frac{1}{\psi^2}$ as well as a term of order of magnitude $O\left(\frac{1}{\psi}\right)$.

Berry’s asymptotics \((3.3)\) is quite remarkable, in more than one way. First, it gives a precise asymptotic law for the variance of the nodal volume of $u$ restricted to $B(R)$ as $R \to \infty$: only the regime $\|x - y\| \to \infty$ contributes to the integral on the r.h.s. of \((2.7)\), the constant term $\frac{1}{8}$ in \((3.3)\) perfectly cancels out with the squared expectation via \((3.2)\), since at infinity (i.e. $\|x - y\| \to \infty$), $(u(x), \nabla u(x))$ and $(u(y), \nabla u(y))$ are stochastically independent. Further, one needs to integrate the two remaining terms on the r.h.s. of \((3.3)\) on $B(R) \times B(R)$ (recall that $\psi := \|x - y\|$), on dismissing the error term $E(\cdot)$. Berry argued that, given a “generic” $x \in B(R)$ and $\psi \in (0, R)$, for the purpose of deriving the leading asymptotics of the double integral, one may assume that $y$ is unrestricted, lying on the circle of radius $\psi$ centred at $x$. This simplified model makes the integral into simple, 1-dimensional integral, for it disregards the geometric factor that is naturally induced on us when we are to evaluate the measure of the set of tuples
\[
\{(x, y) \in B(R) \times B(R) : \|x - y\| = \psi\}
\]
at distance $\psi$, that, to the author’s best knowledge, does not have a simple or elegant answer (cf. \S\S\S 2.4). The other remarkable fact about Berry’s asymptotics \((3.3)\) is that the remaining leading term $\frac{\sin(2\psi)}{4\pi \psi}$ is purely oscillatory, and will not contribute to the integral \((2.7)\). Instead, the bulk of the variance comes from the term $\frac{1}{512\pi^2} \frac{1}{\psi^2}$, which gives a logarithmic contribution, and hence the variance is
\[
\text{Var}(u^{-1}(0) \cap B(R)) \sim \frac{1}{128} R^2 \log R,
\]
of lower order than $\sim R^3$, that would be expected by the natural scaling considerations of the problem. This remarkable phenomenon was named \[72\] “Berry’s cancellation”, that is highly acclaimed in the recent literature. It has many different, easily identifiable, appearances in other contexts, see e.g. \S\S 3.2 or \S\S 4 below. Though in his original manuscript \[7\], Berry did not pursue a complete rigorous proof for the asymptotics \((3.3)\) for the nodal length variance, it could be fully validated by adding some details: Bound for the contribution of the origin and the intermediate regimes, more detailed derivation of \((3.3)\), and the “symmetrization” of the domain; all of this was performed for the random spherical harmonics (cf. \S\S 3.2), with the understanding that it could have been fully performed by Berry had he pursued a rigorous proof for \((3.4)\). Interestingly, Berry’s cancellation does not persist \[70\] for nonzero levels, see an explanation in \S\S 2.4.\footnote{In \[70\] it concerned the same situations for random spherical harmonics \[1.6\]. But, as explained in \S\S 3.2 below, given the result for random spherical harmonics, the corresponding result for Berry’s RWM is automatic.}

Berry’s original elegant derivation of the asymptotics \((3.3)\) capitalized on the particular structure of the Gaussian integral \((2.3)\). Since the covariance function $r_u$ (and all of its derivatives) decay at infinity, the covariance matrix of $(u(x), u(y), \nabla u(x), \nabla u(y))$ (since $u$ is isotropic, $K_2$ is invariant w.r.t. translations and rotations in $\mathbb{R}^2$, so, accordingly, we may assume that $x = (0, 0)$, $y = (\psi, 0)$), properly normalized, is a perturbation of the identity matrix, hence the resulting Gaussian integral approximates one with separation of variables, whose value cancels the squared expectation. In general (for example, when treating the Gaussian integral corresponding to counting critical points rather than...
the nodal volume \([20]\), one invokes perturbation theory\(^{15}\) (i.e. study the eigenvalues and eigenfunctions of the perturbed covariance matrix) to derive an asymptotic formula for \(K_2(x, y)\) as \(|x-y| \to \infty\).

This approach also shows that, if one is interested in an upper bound for the variance only, then one may control it via the \(L^2\)-norm of the covariance function and its derivatives, properly normalized, for example, via a bound of the type

\[
(3.5) \quad \text{Var}(\text{len}(u^{-1}(0) \cap B(R))) \ll \|r_u\|_{L^2(B(R) \times B(R))}^2 + \|\partial_x r_u\|_{L^2(B(R) \times B(R))}^2 + \|\partial_y r_u\|_{L^2(B(R) \times B(R))}^2 + \|\partial_{xx} r_u\|_{L^2(B(R) \times B(R))}^2 + \|\partial_{xy} r_u\|_{L^2(B(R) \times B(R))}^2 + \|\partial_{yy} r_u\|_{L^2(B(R) \times B(R))}^2.
\]

3.2. Random spherical harmonics: nodal volume and generic functionals.

3.2.1. Nodal volume. Let us now consider the analogue problem for the random spherical harmonics \((1.6)\). Here both the distribution of the total nodal length, and the nodal length of the function restricted to subdomains, possibly shrinking as \(\ell \to \infty\), makes sense. The asymptotics \((1.9)\) shows that \(T_\ell\), restricted to spherical caps of shrinking radii (e.g. radius \(1/\log \ell\)), and rescaled by a factor of \(\ell\) (rather, \(\ell+1/2\)), defined on the flattened coordinates of the sphere, converges to Berry’s RWM. Since the law of \(T_\ell(\cdot)\) is invariant w.r.t. rotations of the sphere, it follows that the zero density is independent of \(x \in S^2\), hence only depends on \(\ell\). An easy and straightforward computations shows that

\[
(3.6) \quad K_{\ell, 1}(x) \equiv \frac{1}{2^{4/2}} \sqrt{\lambda_\ell},
\]

in particular, the expected nodal length of \(T_\ell\) is given precisely by

\[
(3.7) \quad \mathbb{E}[\text{len}(T_\ell^{-1}(0))] = \sqrt{2\pi} \cdot \sqrt{\lambda_\ell},
\]

established by Béard \([5]\) by different methods. Neuheisel \([48]\) gave the useful upper bound

\[
\text{Var}(\text{len}(T_\ell^{-1}(0))) = O\left(\ell^{2-1/7}\right),
\]

showing that, with high probability, the nodal length is concentrated around the mean. Neuheisel’s bound was improved \([71]\) to

\[
\text{Var}(\text{len}(T_\ell^{-1}(0))) = O\left(\ell^{2-1/2}\right).
\]

The question of asymptotic variance was finally resolved in \([72]\). It was shown that, up to an admissible error term depending on \(\ell\), an analogue of \((3.3)\) holds in this case, namely, that\(^{14}\) for \(x, y\) belonging to a (fixed) hemisphere

\[
(3.8) \quad K_{2, \ell}(x, y) = \ell \cdot \left(1 + \frac{1}{4\pi \ell} \cdot \frac{\sin(2\psi)}{\sin(\psi/(\ell + 1/2))} + \frac{1}{512\pi^2 \ell} \cdot \frac{\sin(2\psi)}{\psi \cdot \sin(\psi/(\ell + 1/2))} + E_\ell(\psi)\right),
\]

with, as above, \(\psi = (\ell + 1/2)d(x, y)\). Integrating \((3.8)\) on a hemisphere yields the precise asymptotics \([72\text{ Theorem 1.1}]\)

\[
(3.9) \quad \text{Var}(\text{len}(T_\ell^{-1}(0))) = \frac{1}{32} \log \ell + O(1)
\]

for the nodal length variance of \(T_\ell\).

A few observations are due. In accordance with the locality of the convergence of the scaled and flattened \(T_\ell(\cdot)\) to Berry’s RWM, so is the convergence of \((3.8)\) to \((3.3)\), e.g. not holding for \(\psi \approx \ell\). However, since the bulk of the variance is contained within a small neighbourhood of the diagonal, e.g. inside

\[
\{(x, y) : d(x, y) < 1/\log \ell\},
\]

where, in addition, the sphere is close to be flat, the asymptotic law of the variance \((3.9)\) is the same as \((3.4)\), properly scaled and understood. One may however notice that, if one scales the sphere by \(\ell\), then there is a discrepancy of factor 2 between the leading constants of \((3.9)\) compared to \((3.4)\) per

\(^{15}\)It is also the Taylor expansion of the Gaussian integral as a function of the (perturbed) covariance matrix, about the (unperturbed) identity matrix, that we get at infinity.

\(^{16}\)The normalization in \([72]\) was slightly different, and here we present the result so that to compare it to \((3.3)\).
unit area. This is a by-product of the fact that the nodal line of $T_\ell$ is fully determined by its restriction to a hemisphere, hence the nodal length is twice the nodal length of $T_\ell$ restricted to a hemisphere.

In the same paper [72], the result (3.9) was extended in two directions. Let $\varphi : S^2 \to \mathbb{R}$ be a $C^1$-smooth test function, and consider the linear statistics

$$\mathcal{V}_\ell^\varphi = \mathcal{V}^\varphi(T_\ell) := \int_{T_\ell^{-1}(0)} \varphi(x) \, dx$$

of the nodal line of $T_\ell$. For example, if $\varphi = 1_A$ is the characteristic function of a nice subdomain $A \subseteq S^2$, then $\mathcal{V}_\ell^\varphi$ is identified as simply the nodal length of $T_\ell$ restricted to $A$ (though, in this case, $\varphi$ is not smooth). To avoid tautologies, one may assume w.l.o.g. that $\varphi$ is an even, not identically vanishing, function. Using (3.6) it is easy to evaluate the expectation

$$\mathbb{E}[\mathcal{V}_\ell^\varphi] = c_0(\varphi) \cdot \sqrt{\lambda_\ell}.$$  

One may further exploit (3.8) with the auxiliary function $W^\varphi$ is in §2.4 to obtain the asymptotic law for the variance [72, Theorem 1.4]

$$\mathbb{E}[\mathcal{V}_\ell^\varphi] = c_1(\varphi) \cdot \log \ell + O(1),$$

with some $c_1(\varphi) > 0$. Only the behaviour of $W^\varphi$ at the origin will determine the variance (3.10).

Since the characteristic function of a subdomain of $S^2$ is not continuous, let alone $C^1$, the result (3.10) does not apply in this case. Instead, the result (3.10) was also proved [72, Theorem 1.5] for a different class of functions, namely for bounded functions of bounded variation that do contain the characteristic functions of nice subdomains of the sphere. To prove it, a density argument is given, with the use of (3.10) for smooth $\varphi$ approximating the given characteristic function, and effective control over the dependence of the error term on $\varphi$ as the basis.

Importantly, only the nodal case, invariant w.r.t. multiplication of $T_\ell$ by any real number, is susceptible to Berry’s cancellation, whereas the nonzero levels are not. Let $t \in \mathbb{R} \setminus \{0\}$, and consider the level curve length $\text{len}(T_\ell^{-1}(t))$. Then it is easy to evaluate the density in this case to yield the expectation

$$\mathbb{E}[\text{len}(T_\ell^{-1}(t))] = e^{-t^2/2} \cdot \sqrt{2\pi} \cdot \sqrt{\lambda_\ell},$$

cf. (3.7). For the variance one has [70, Formula (18)]

$$\text{Var}(\text{len}(T_\ell^{-1}(t))) = c \cdot t^4 e^{-t^2} \ell + O(\log \ell)$$

with some absolute number $c > 0$, in contrast to (3.9). Moreover, for every $t_1, t_2 \neq 0$, the lengths of the corresponding level curves asymptotically fully correlate [70, Formula (19)], in the sense that

$$\text{Corr}(\text{len}(T_\ell^{-1}(t_1)), \text{len}(T_\ell^{-1}(t_2))) = 1 - o_{\ell \to \infty}(1),$$

where, as usual, for two random, non-degenerate, variables $X, Y$,

$$\text{Corr}(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}} \in [-1, 1].$$

To prove either (3.11) or (3.12) one asymptotically expands the appropriate the 2-point correlation function (resp. cross-correlation function) above Planck scale (see §2.3). No Berry’s cancellation in the non-nodal case means that our job here is easier as compared to (3.8), as the leading term will give the bulk of contribution to the variance (resp. covariance), so there is no need for 2-term expansion.

Finally, asymptotically, as $\ell \to \infty$, there is no correlation between the nodal length and any level curve at nonzero level $t \neq 0$,

$$\text{Corr}(\text{len}(T_\ell^{-1}(t)), \text{len}(T_\ell^{-1}(0))) = o_{\ell \to \infty}(1).$$

While it is easy to check that fact via the cross-correlation route, it is intuitively clear from the corresponding result for the excursion sets [42]. Because of that it was also later observed [41] that the projection of the non-nodal length at level $t \neq 0$ to the 2nd Wiener chaos, dominating the fluctuations
of the level curve length, is, up to the factor $\sqrt{e^{-t^2/2}t^2}$, equal to the squared norm of $T_\ell$. This is in contrast to the nodal case, that is invariant to products by a constant, hence the projection onto the 2nd Wiener chaos vanishes precisely, and the fluctuations of the nodal length are dominated by its projection onto the 4th Wiener chaos. Knowing that, one may also re-prove [41] Theorem 1.3 both (3.12) and (3.13).

3.2.2. Generic nonlinear functionals of $T_\ell$ and CLT for nodal length. Let $H_q(\cdot): \mathbb{R} \to \mathbb{R}$, $q \geq 0$ be the $q$'th degree Hermite polynomials, and consider the centred random variables

$$h_{q,\ell} := \int_{S^2} H_q(T_\ell(x)) dx.$$  

For example, $h_{4,\ell}$ is usually referred to as “sample trispectrum” of $T_\ell$ [40]. The random variable $h_{q,\ell}$ belongs to the $q$'th Wiener chaos generated by the Gaussian measure induced by $T_\ell(\cdot)$. More generally, let $G: \mathbb{R} \to \mathbb{R}$ be a function, and, assuming that

$$\mathbb{E}[G(T_\ell(\cdot))^2] < +\infty,$$

define [43] the random variable (“nonlinear functional”)

$$\mathcal{G}(T_\ell) := \int_{S^2} G(T_\ell(x)) dx.$$  

One may Hermite expand

$$G(T_\ell) \overset{L^2(S^2)}{=} \sum_{q=0}^{\infty} \frac{J_q(G)}{q!} H_q(T_\ell),$$

where the coefficients are

$$J_q(G) := \mathbb{E}[G(Z) \cdot H_q(Z)]$$

with $Z$ standard Gaussian, and integrate (3.15) on $S^2$ to obtain the Wiener chaos expansion

$$\mathcal{G}(T_\ell) = \sum_{q=0}^{\infty} \frac{J_q(G)}{q!} h_{q,\ell}$$

for $\mathcal{G}(T_\ell)$. Hence the $h_{q,\ell}$ are the building blocks of all square summable functionals of $T_\ell$.

If both $q$ and $\ell$ is odd, then, since in this case, $T_\ell(\cdot)$ is odd, $h_{q,\ell} \equiv 0$, and, for that reason, in what follows we assume that $\ell$ is even. It is easy to extend our analysis for either $\ell$ odd, $q$ even, or, otherwise, pass to a subdomain of $S^2$ to avoid tautologies (e.g. restrict $T_\ell$ to a hemisphere, as we did above). It is evident from the properties of Hermite polynomials, that the variance of $h_{q,\ell}$ is, up to an explicit constant, precisely the $q$'th moment of the Legendre polynomial $P_\ell$ on $[0, 1]$:

$$\text{Var}(h_{q,\ell}) = q!(4\pi)^2 \int_0^1 P_\ell(t)^2 dt = (4\pi^2)^q \int_0^{\pi/2} P_\ell(\cos \theta)^q \sin \theta d\theta = q! \cdot \int_{S^2 \times S^2} \langle x, y \rangle dx dy,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^3$.

By using Hilb’s asymptotics [18] (and special functions for $q = 2, 3$), it is possible to evaluate the asymptotics for the moments of the Legendre polynomials, and, therefore, for the variance of $h_{q,\ell}$ as $\ell \to \infty$ along even integers:

$$\text{Var}(h_{q,\ell}) = \begin{cases} \frac{1}{\ell} (1 + o_{\ell \to \infty}(1)) & q = 2 \\ \frac{\log \ell}{\ell^2} (1 + o_{\ell \to \infty}(1)) & q = 4 \\ (4\pi^2)^q! \frac{\log \ell}{\ell^2} (1 + o_{\ell \to \infty}(1)) & q = 3, q \geq 5 \end{cases}.$$
Here
\[ c_q = \int_0^\infty \psi J_0(\psi)^q d\psi \geq 0 \]
with the integral conditionally convergent for \( q = 3 \), and absolutely convergent for \( q \geq 5 \). It is then clear that, for every \( q \geq 4 \) even, the a priori nonnegative numbers \( c_q \) are actually positive, and also \[43\] Lemma 3.1] \( c_3 > 0 \) by evaluating it explicitly via special functions; it is not known to date for which other odd \( q \), \( c_q > 0 \), though it is believed for all \( q \geq 3 \).

Further, for every \( q \geq 2 \) even, \( q = 3 \), and every \( q \geq 5 \) so that \( c_q > 0 \) a quantitative Central Limit Theorem (q-CLT) was established for \( h_{q,\ell} \) as \( \ell \to \infty \) along even \( \ell \). Using these results, one may establish the asymptotics of the variance for generic functionals of type (3.14), depending on the coefficients of their Hermite expansion (3.15), e.g. if \( c_q > 0 \) for at least one \( q \) such that \( J_q(G) \neq 0 \). This approach also implies a q-CLT for generic functionals \( \mathcal{G}(T_\ell) \) of the said type.\(^\text{17}\)

One in particular important functional of this type is the so-called defect of \( T_\ell \), appearing in the physics literature (see e.g. \[8\]): It is the area of the positive excursion set minus the area of the negative excursion set:
\[ D_\ell = D(T_\ell) = \int_{S^2} \mathcal{H}(T_\ell(x)) dx, \]
where
\[ \mathcal{H}(t) = \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases} \]
is the Heaviside function.

The expansion (3.16) for the defect only contains odd summands \( q \geq 3 \), all whose variance is of order of magnitude \( \approx \frac{1}{q} \), and there is no single term dominating the rest of the summands. The asymptotic law of the variance of \( D_\ell \) was previously established to be \[44\]
\[ \text{Var}(D_\ell) \sim \frac{C}{\ell^2}, \]
with \( C > 0 \) given by the conditionally converging integral
\[ C = 32\pi \int_0^\infty \psi (\arcsin(J_0(\psi)) - J_0(\psi)) d\psi, \]
incorporating the odd moments \( q \geq 3 \) of the Legendre polynomials (the Bessel function in the limit). Using the q-CLT for the building blocks \[44] h_{q,\ell}, \) a q-CLT for \( D_\ell \), via a standard route \[43\].

The functionals (3.14) do not formally include the nodal length \( \text{len}(T_\ell^{-1}) \) of \( T_\ell \), since the integral in (3.14) is only allowed to depend on the values of \( T_\ell \) and not its derivatives, cf. (2.12). Nonetheless, it was observed that \( h_{4,\ell} \) asymptotically fully correlates \[40\] Theorem 1.2] with \( \text{len}(T_\ell^{-1}) \) as \( \ell \to \infty \), hence a (quantitative) Central Limit Theorem for \( h_{4,\ell} \) will also yield \[40\] Corollary 1.3] a (quantitative) Central Limit Theorem for \( \text{len}(T_\ell^{-1}) \). This concluded a question opened up a few decades ago.

It is worth mentioning that the projection of \( \text{len}(T_\ell^{-1}(0)) \) to all the odd Wiener chaos spaces vanish, and, in this scenario, the projection of \( \text{len}(T_\ell^{-1}(0)) \) to the 2nd chaos vanishes precisely. The precise vanishing of the projection of the nodal length (and other quantities) is not inherent to Berry’s cancellation, as in some other situations it will only be majorized by the projection onto the 4th Wiener chaos, such as, for example, the same setting of nodal length of \( T_\ell \), only this time restricted to some general position subdomain of \( S^2 \) (also see the discussion below on RWM). The precise vanishing of the projection onto the 2nd chaos is merely a manifestation of the symmetries of \( S^2 \). This indicates that the asymptotics at infinity of the 2-point correlation function, as it was originally done in \[7\, 72\],

\(^{17}\) At first, claim these for finite combinations (3.16). Then use a standard limit procedure, for sufficiently rapidly coefficients.

\(^{18}\) At least those with \( c_q > 0 \); the others have smaller fluctuations, hence, knowing that \( c_3 > 0 \), these may be neglected.
is a more natural viewpoint on Berry’s cancellation phenomenon than the Wiener chaos expansion language.

A. Vidotto [69] showed that, as suggested, the above indicated proof yielding the CLT for the nodal length of $T_\ell$ also yields the CLT for Berry’s RWM restricted to expanding discs $B(R)$ as $R \to \infty$, proved via a somewhat different argument [51], also involving the Wiener chaos expansion. A. P. Todino [65] extended the said results (logarithmic variance, and CLT for nodal length) for nodal length restricted to a spherical length of radius above Planck scale, i.e. for any sequence of radii $r_\ell$, possibly shrinking, so that $r_\ell \cdot \ell \to \infty$. This is a uniform result, optimal w.r.t. the radius, as if $r \ll \frac{1}{\ell}$, then one does not expect the distribution of nodal length to exhibit a limit law of the said type, because of the comparison, after scaling to the RWM, whence the domain is not expanding.

3.3. Critical points and nodal intersections of independent copies of random spherical harmonics. Higher dimensional spherical harmonics and monochromatic waves. Berry proposed [7, Formula (2)] to model the complex random waves by two independent copies, $u_1$, $u_2$, of the RWM $u$ in §1.1 so their a.s. discrete set of zeros is an intersection of the nodal lines of two independent plane waves. As in case of “real” plane wave, Berry found that the leading term in the expansion at infinity of the 2-point correlation of the zeros of the complex plane wave (“nodal number”) is purely oscillatory, and does not contribute to the variance. The bulk of the contribution comes from the non-oscillatory part of the 2nd leading term, and the nodal number variance restricted to $B(R)$ is asymptotic to

$$\text{Var}(\# \{ x \in \mathbb{R}^2 : u_1(x) = u_2(x) = 0 \}) \sim c_0 R^2 \log R$$

with some $c_0 > 0$.

This way, we may also consider the number of points in the intersection of the nodal lines of two independent copies of $T_\ell$, or, more generally, independently drawn $T_{\ell_1}$ and $T_{\ell_2}$ with some $\ell_1 \leq \ell_2$. Given all the hard work we did to establish (3.8) (and (3.9)), evaluating the zero density precisely, the 2-point correlation function at infinity after scaling, and the Wiener chaos expansion for the zeros of

$$x \in S^2 \mapsto (T_{\ell_1}(x), T_{\ell_2}(x)),$$

for the regime $\ell_1 \to \infty$, $\ell_2 = \ell_2(\ell_1) \geq \ell_1$, amounts in essence to evaluating the asymptotics of the determinant of a block diagonal perturbation of the unit matrix, given the determinants of the individual blocks. By this method, it is easy to evaluate the expected number of points in $T_{\ell_1}^{-1}(0) \cap T_{\ell_2}^{-1}(0) \subseteq S^2$ to be proportional to $\ell_1 \cdot \ell_2$ (more precisely, $\sqrt{\lambda_{\ell_1} \cdot \lambda_{\ell_2}}$, in line with (3.21)).

Further, in the said regime, under further assumption that $\ell_2$ is not much bigger than $\ell_1$, e.g. $\ell_2 \leq C \cdot \ell_1$ for some $C > 0$ (or beyond it), one may derive the asymptotics for the variance of the type

$$\text{Var}(\#T_{\ell_1}^{-1}(0) \cap T_{\ell_2}^{-1}(0)) = c_1(\ell_1^2 + \ell_2^2) \log \ell_1 + c_2 \ell_1 \ell_2 \log \left( \frac{\ell_1}{\ell_2 + 1 - \ell_1} \right),$$

for some explicitly computable constants $c_1, c_2 > 0$; in particular,

$$\text{Var}(\#T_{\ell_1}^{-1}(0) \cap T_{2^{-1}}^{-1}(0)) \sim c_0 \ell^2 \log \ell;$$

the purely oscillatory nature of the leading term for $K_2$ indicates that Berry’s cancellation occurs in this case again. If, on the other hand, $\ell_2$ is bigger by order of magnitude than $\ell_1$, then, at some point, a phase transition occurs, which is easy to explicate by analyzing the resulting Kac-Rice integral. To derive an asymptotic law for $\#T_{\ell_1}^{-1}(0) \cap T_{\ell_2}^{-1}(0)$ one uses a Wiener chaos expansion, that, in light of the above, is, in the said regime, most certainly dominated by the 4th chaos projection, analysis of which will yield the Central Limit Theorem for the said quantity.

What is more technically demanding than the analysis of intersection of the nodal lines of random independent copies of $T_\ell$ is the analysis of the number of critical points and critical values (i.e. the values of $T_\ell(x_k)$ where $\{x_k\} \subseteq S^2$ are the critical points of $T_\ell$) of $T_\ell$, since this amounts to the intersection of nodal lines of two dependent random Gaussian functions, namely, the derivatives of

---

19This model of complex random functions is very different from the models considered by Sodin-Tsirelson (see e.g. [63] in the pure mathematics literature.
be neglected, and do impact the asymptotic distribution of the number of critical points (and their values). It is easy to evaluate the expected total number of critical points to be proportional to $\lambda_1 \sim \ell^2$, and, by expanding the 2-point correlation function at infinity, their variance is proportional \(^{20}\) to $\ell^2 \log \ell$, with Berry’s cancellation. For critical values on $\mathbb{R}$, a non-universal limit distribution has been derived \[^{17}\] as a point-process, and a variance for the number of critical values belonging to a generic interval $I = [a, b] \subseteq \mathbb{R}$ has been established \[^{17}\] to be proportional to $\ell^2$; for non-generic $I$ the variance is of lower order, most likely (this special class of $I$ is reflecting their symmetry, for example, as it was mentioned, $I = \mathbb{R}$ has a lower order variance of $\ell^2 \log \ell$, as it is expected from other intervals in this class). Attraction and repulsion of critical points (also finer, maxima, minima and saddles) were studied for Berry’s RWM \[^{9}\], and generic Gaussian isotropic processes \[^{10, 1}\].

### 3.4. Generalizations.

All the above results generalize for $n$-dimensional sphere, $n \geq 2$ and any number $k \leq n$ of independent spherical harmonics of either equal or non-equal degrees, or monochromatic waves on $\mathbb{R}^n$, see e.g. \[^{21, 25}\], whence one is interested in the corresponding $(n - k)$-volume (number of discrete zeros for $k = n$); every configuration $(n, k)$ is susceptible to Berry’s cancellation. Alternatively, one may consider intersections of level sets: The analogous results to \[^{3.2.1}\] (especially towards the end) will hold in this more general setting, with the only outstanding question, being somewhat vague, is that of the number of points $\#T_{1, \ell}^k(0) \cap T_{2, \ell}^k(t)$ in the intersection of the nodal line of $T_\ell$ and level $t \neq 0$ curve of its independent copy. Without performing the necessary (straightforward) computation, it seems that this case will exhibit some features of nonzero level curve length of a single spherical harmonic and than that of the nodal length (in terms of Berry’s cancellation and the dominating chaos projection).

### 3.5. Nodal intersections against smooth curves.

Let $\gamma : [0, L] \to \mathbb{R}^2$ be a (fixed) smooth curve, arc length parameterized. In this context, one is interested in the number of the intersections of Berry’s RWM $u(\cdot)$ with $\gamma$, or, rather, of the wavenumber-$k$ plane waves $u(k\cdot)$, $k \to \infty$. Equivalently, this is the number of the a.s. discrete zeros of the function

$$f_k(t) := u(k\gamma(t)) : [0, L] \to \mathbb{R}.$$  

Note that the random Gaussian process is not stationary. However, it is still quite easy to evaluate the zero density $K_1(t) = K_{1, f_k}(t)$ in this case, and exploit the univariate nature of $f_k$ to find that it is universally given by

$$K_1(t) \equiv \frac{k}{\sqrt{2\pi}},$$

independent of the geometry of $\gamma$, so that the expected number of nodal intersections in this case is given precisely by

$$\mathbb{E}[\# f_k^{-1}(0)] = \frac{k}{\sqrt{2\pi}} \cdot L.$$  

Evaluating the variance of the nodal intersections number is much more technically demanding and subtle. Nevertheless it was successfully performed to show that, as $k \to \infty$,

$$\text{Var}(\# f_k^{-1}(0)) \sim \frac{L}{2\pi^2 k} k \log k.$$  

A few observations are due. Somewhat surprisingly, the leading asymptotics of the variance is universally independent of the geometry of $\gamma$. This is especially unexpected, or even shocking, due to the existence of the so-called “static” curves in case of Arithmetic Random Waves (see \[^{4.3}\] below) that exhibit lower variance than the generic curves, and exhibit non-central limit law. Despite the existence of logarithmic term in the variance, there is no Berry’s cancellation here, and the bulk of the contribution to the variance arises from the leading term of the expansion of $K_2(t_1, t_2)$ away from the diagonal, i.e. $|t_2 - t_1| \gg 1/k$. This indicates that, most certainly, the projection to the 2nd Wiener chaos of $\# f_k^{-1}(0)$ will dominate its fluctuations, and will yield the quantitative CLT via the standard

\[^{20}\]It seems that $C^2$-smoothness should be sufficient
route. It is highly likely that same techniques yield the small scale analogues of these all the way to the Planck scale, cf. [29, Theorem 1.2]. Finally, same results are, with no doubt, also applicable for the nodal intersections of the random spherical harmonics $T_i$ against a fixed smooth spherical curves. Higher dimensional analogues should also be treatable via similar methods.

3.6. Berry’s cancellation - generic Gaussian random fields on Euclidean space. Let $F$ be a stationary random Gaussian random field on $\mathbb{R}^2$, and let $\rho$ be its spectral measure, i.e. $\rho$ is the Fourier transform of the covariance function of $F$. Assuming that $F$ is monochromatic, i.e. $\rho$ is supported on $S^1 \subseteq \mathbb{R}^2$, the covariance function of $F$ is given by the inverse Fourier transform

$$r_F(x) = \int_{S^1} e^{i(x, \vartheta)} d\rho(\vartheta),$$

where $x = (x_1, x_2), \vartheta = (\cos \theta, \sin \theta) \in S^1, \theta \in [0, 2\pi]$, $(x, \vartheta) = x_1 \cos \theta + x_2 \sin \theta$, and we may further assume w.l.o.g. that $\rho$ is invariant w.r.t. rotation by $\pi$. It is then straightforward to obtain a precise expression for the expected nodal length of $F$ restricted to $B(R)$ to be $c(\rho) \cdot \pi R^2$, where $c(\rho)$ is explicitly computable in terms of a couple of power moments of $\rho$. The question put forward is, again, that of the asymptotic variance (and the asymptotic law), as $R \to \infty$, the intrigue being whether Berry’s cancellation occurs, and in what generality if it does.

One may obtain an asymptotic expansion of $r_F(x)$ and its derivatives at infinity, i.e. when $|x| \to \infty$, in analogy to (3.1) for the isotropic case (1.1) of Berry’s RWM, and use these to obtain an asymptotic expression at infinity for $K_{F,2}(\cdot)$, the 2-point correlation function (2.3). First, we assume that $\rho$ has a density on $S^1$: $d\rho(\theta) = g(\theta) d\theta$ for some nonnegative “nice” function $g : [0, 2\pi) \to \mathbb{R}_{\geq 0}$ ($C^2$ should be sufficient). In this case, it is possible to use the stationary phase method on (3.17) to find that, after cancelling out the squared expectation, the leading term in the expansion of $K_{F,2}(x)$ at infinity is of purely oscillatory nature, and, using the polar representation $x = |x|e^{i\vartheta}$, the next term is proportional to

$$g(\vartheta)^4 \cdot \frac{1}{|x|^2}.$$

Hence this model is susceptible to Berry’s cancellation. By integrating (2.4) the latter asymptotics, and subtracting the squared expectation, we may then deduce that, as $R \to \infty$,

$$\text{Var}(F^{-1}(0) \cap B(R)) \sim c(g) \cdot R^2 \log R,$$

with

$$c(g) = c_0 \cdot \int_{S^1} g(\theta)^4 d\theta,$$

and $c_0 > 0$ an absolute constant. The leading constant (3.18) is reminiscent of the previously discovered one for the Arithmetic Random Waves (see (4.6) in §4 below), although it bears the 4th Wiener component of the spectral measure, as opposed to the 4th moment (3.18) of its density. Note that, interestingly, the expression $g(\theta)^4$ is minimized by the uniform probability measure, due to Jensen’s inequality; somewhat surprisingly, the leading constant (3.18) does not bear the derivatives of $g$, despite the contrary for the intermediate expressions (various cancellations yielding a very neat and compact expression for the leading non-oscillatory asymptotics of the 2-point correlation, and the nodal length variance).

There is no doubt that, using the Wiener chaos expansion for $\text{Var}(F^{-1}(0) \cap B(R))$ reveals that, in this case, the projection onto the 4th Wiener chaos is dominating the projection onto the 2nd one (which asymptotically vanishes, due to the oscillatory nature of the 2 leading term of $K_{F,2}$), and also the higher Wiener chaos spaces. A Central Limit Theorem for $\text{Var}(F^{-1}(0) \cap B(R))$ also follows from the Central Limit Theorem for the projection onto the 4th Wiener chaos, via the standard route indicated above. If, on the other hand, $\rho$ has jump discontinuities, or $\rho$ is either purely atomic, or $\rho$ is a superposition of atoms and an absolutely continuous part, then the atoms would dominate the asymptotics at infinity of the covariance function (3.17), and the nodal length variance, though
some notion of Berry’s cancellation will be holding in some generality (for example, in case of jump discontinuities only, the variance will be logarithmic, with leading constant depending on the jumps). Since if the support of \( \rho \) has a point of interior, \( \tau_F(\cdot) \) and its derivatives have rapid decay in infinity, the nodal length variance will be of smaller order of magnitude \( R^2 \), with no Berry’s cancellation.

In this light we may formulate the following principle or meta-theorem: **Under appropriate assumptions on \( F \) (or \( \rho \)), it is susceptible to Berry’s cancellation, if and only if \( F \) is monochromatic.** All of the above (including the results stated in previous sections) could be generalized to any dimensions \( n \geq 2 \), and intersections of \( k \leq n \) nodal sets of independent copies of \( F \) (or different functions), or other related quantities associated with \( F \), such as, for example the number of critical points.

Finally, recall that the ensemble \( T_\ell(\cdot) \) scales like Berry’s RWM around every reference point \( x \in S^2 \). It would be interesting, given a monochromatic random field \( F : \mathbb{R}^2 \to \mathbb{R} \) of the said type, to construct an ensemble of Gaussian functions \( T_{F,\ell}(\cdot) \), supported on the space of degree \( \ell \) spherical harmonics, that scales like \( F \) around every (or, at least, almost every) reference point in \( S^2 \).

### 3.7. Band-limited functions - general case.

Let \( M \) be a smooth closed \( n \)-manifold with \( n \geq 2 \), i.e. \( M \) is a smooth compact Riemannian manifold with empty boundary. Under further assumption\footnote{This condition is satisfied for “most” of the manifolds in many different senses.} that the measure of the geodesic loops through a point with the

\[
  \text{the condition is satisfied for “most” of the manifolds in many different senses.}
\]

\[ c(3.19) \]

in several ranges, to yield the handy bound [15, Theorem 1] normalized or, equivalently, to show that the restriction, take

\[
  f_\lambda(x) = \sum_{\lambda \in [\lambda, \lambda+1]} c_i \varphi_i(x),
\]

with the \( c_i \) i.i.d. standard Gaussian r.v., i.e. \( f_\lambda(\cdot) \) are the random Gaussian monochromatic waves on \( M \), corresponding to the shortest possible energy window. Alternatively, without the geometric restriction, take

\[
  f_\lambda(x) = \sum_{\lambda \in [\lambda-\eta(\lambda), \lambda]} c_i \varphi_i(x),
\]

with some \( \eta \to \infty \) (but \( \eta = o(\lambda) \)).

Zelditch [73] showed that the total expected nodal \((n-1)\)-volume of \( f_\lambda \) is asymptotic to

\[
  \mathbb{E}[\text{Vol}_{n-1}(f_\lambda^{-1}(0))] = c_M \cdot \lambda,
\]

with \( c_M > 0 \) explicit, proportional to the Riemannian volume of \( M \). Canzani-Hanin [15] employed the 2-point correlation function (2.3) to give an upper bound for the variance \( \text{Var}(\text{Vol}_{n-1}(f_\lambda^{-1}(0))) \). They developed a sophisticated partition argument, carefully bounding the contribution of the \( K_2(\cdot, \cdot) \) in several ranges, to yield the handy bound [15, Theorem 1]

\[
  \text{Var}(\text{Vol}_{n-1}(f_\lambda^{-1}(0))) = O(\lambda^{2-(n-1)/2}),
\]

or, equivalently, to show that the **normalized** variance

\[
  \text{Var} \left( \frac{\text{Vol}_{n-1}(f_\lambda^{-1}(0))}{\lambda} \right) = O \left( \lambda^{-(n-1)/2} \right)
\]

vanishes.

It seems that, by employing the *Approximate* Kac-Rice technique as in §2.3 in place of the (precise) Kac-Rice, one can simplify or shorten Canzani-Hanin’s argument validating the sufficient Gaussian non-degeneracy conditions for the Kac-Rice to hold. Instead, it is sufficient to prove the same for most \((x, y) \in M \times M \). Further, by writing the Kac-Rice formula and using the upper bound in terms of the \( L^2 \)-energy of the covariance function and its derivatives of the type (3.5), it is expected to yield a sharper upper bound than (3.19) in terms of the power of \( \lambda \), not unlikely

\[
  \text{Var}(\text{Vol}_{n-1}(f_\lambda^{-1}(0))) = O(\lambda).
\]

Unfortunately, all of these shed no light on the true behaviour of the nodal volume variance, nor give a nontrivial *lower* bound for the said variance, ditto of the same order of magnitude as the upper bound, cf. (3.9). That the *local* convergence of the covariance (even on a logarithmic scale above the Planck
scale \( r \approx 1/\lambda \) and its derivatives to those of the RWM does not indicate the same for the total nodal length follows from the toral application, see \[4.2\].

Let \( x_0 \in \mathcal{M} \) be a reference point, and consider the function \( f_\lambda \) restricted to the geodesic ball \( B_{x_0}(1/\lambda) \). It follows that the scaled random field \((1.3)\), defined on \( B(R) \), converges to the \((n\text{-dimensional})\) RWM \( g_\infty \). One may then use the Continuous Mapping Theorem \[15\] (see also the proof in \[11\] p. 70), to infer that, up to the scaling, the distribution of the nodal volume of \( f_\lambda \) restricted to \( B(R/\lambda) \) converges in law to the distribution of the nodal volume of \( g_\infty \) on \( B(R) \) (with no information what this distribution is). That the sufficient conditions for the application of the Continuous Mapping Theorem are satisfied for the nodal volume of \( g_\lambda \) (i.e. the nodal volume is a.s. continuous w.r.t. \( g_\lambda \)) follows from an application of the well-known Bulinskaya’s Lemma. Since the said result holds for every \( R > 0 \), one may also \textit{tour de force} the Continuous Mapping Theorem while using Berry’s asymptotics \[3.4\] and CLT for the nodal length variance of the RWM in \( R > 0 \), to infer the analogous result for the nodal length of \( f_\lambda \) restricted to geodesic balls in \( \mathcal{M} \) of radius \( R/\lambda \), with \( R \) slowly growing (with no control how slow the growth of \( R \) should be).

Finally, by using an extension due to Keeler \[32\] of the scaling property \[1.4\] for \( \alpha = 1 \), holding for a further restricted (but still generic) class of manifold, a logarithm power above the Planck scale, one may pushforward \[52\] the asymptotic variance and the CLT logarithmically above the Planck scale, for the said class of manifolds. As the Arithmetic Random Waves of \[4.1\] is an ensemble of random Gaussian band-limited functions that locally scales like the RWM for a generic sequence of toral energies, though the asymptotic distribution of the total nodal length is very different, we conclude that the above is a local phenomenon, generally not indicative of the global nature of the nodal line.

4. Survey of results: Arithmetic random waves

4.1. Lattice points: angular distribution, spectral correlations, quasi-correlations, and semi-correlations. Recall that \( f_n \) are the Arithmetic Random Waves (ARW) given by \((1.10)\), with covariance function given by \((1.11)\). Unlike the spherical harmonics, unfortunately, \( r_n(\cdot) \) (resp. the mixed derivatives of \( r_n \)) does not admit an asymptotic law (after scaling), valid on the whole of \( \mathbb{T}^2 \), cf. \((1.7)\) and \((1.8)\). As a substitute, one can work \[33\] with the \textit{moments} of \( r_n(\cdot) \) (resp. its derivatives) on \( \mathbb{T}^2 \). For an integer number \( k \geq 2 \) one has

\[
(4.1) \quad \int_{\mathbb{T}^2} r_n(x)^k dx = \frac{1}{N_n} \#R_k(n),
\]

where

\[
(4.2) \quad \#R_k(n) := \{(\lambda_1, \ldots, \lambda_k) \in \Lambda_n^k : \lambda_1 + \ldots + \lambda_k = 0\}
\]

is the length-\( k \) spectral correlation set.

The sets \( R_k(n) \) (equivalently, the moments of \( r_n(\cdot) \)) play a crucial role in the analysis of the nodal length of \( f_n \). For \( k = 2 \), trivially

\[
\mathcal{R}_2(n) = \{ (\lambda, -\lambda) : \lambda \in \Lambda_n \}.
\]

The analogue of this holds as well for \( k = 4 \), thanks to Zygmund’s trick \[74\] observing that given \( \lambda_1, \lambda_2 \in \Lambda_n \), the \( \lambda_3, \lambda_4 \) \( \in \Lambda_n \) are determined, up to a sign and a permutation. That is, the length-4 spectral correlation set equals to the diagonal set, i.e. it is the set

\[
\mathcal{R}_4(n) = \{ \pi(\lambda, -\lambda, \lambda', -\lambda') : \lambda, \lambda' \in \Lambda_n, \pi \in S_4 \}
\]

of permutations of those 4-tuples cancelling out in pairs, hence its cardinality is

\[
\#R_4(n) = 3N_n(N_n - 1).
\]

For our purposes we need a nontrivial bound for \( \#R_6(n) \). As above, given \( \lambda_1, \ldots, \lambda_4 \in \Lambda_n \), it determines, up to a sign and a permutation, \( \lambda_5, \lambda_6 \in \Lambda_n \). Therefore, it follows that \( \#R_4(n) = O(N_n^4) \).
For our needs, any improvement of the type
\[(4.3) \quad \#\mathcal{R}_6(n) = o_{N_n \to \infty}(N_n^4)\]
is required. The result (4.3) was established by Bourgain, published as part of [33]. Since then, a better bound of
\[\#\mathcal{R}_6(n) = O(N_n^{7/2})\]
was proven [12], using the power of the ring of Gaussian integers \(\mathbb{Z}[i] \cong \mathbb{Z}^2\). Some better bounds of
\[\#\mathcal{R}_6(n) = O(N_n^3)\]
were also asserted [12], albeit either conditional, or valid unconditionally for a generic subsequence of \(S\).

Let us say a few words about the dimension \(N_n = r_2(n)\) of the eigenspaces of toral eigenfunctions. It fluctuates in an erratic way. First, it is easy to show that \(N_n = O(n^{o(1)})\), i.e., that for every \(\varepsilon > 0\), \(N_n = O(n^\varepsilon)\). On the one hand, \(N_n\) could grow as a power of \(\log n\), and its normal order is \(N_n \approx \log n \log 2 / 2\), that is, for every \(\varepsilon > 0\),
\[\log n^{\log 2 / 2 - \varepsilon} \leq N_n \leq \log n^{\log 2 / 2 + \varepsilon}\]
for a density-1 sequence \(\{n\} \subseteq S\). On the other hand, \(N_n\) is as small as \(N_n = 8\) for an infinite sequence of primes \(p \equiv 1 \mod 4\). We do assume throughout that \(N_n \to \infty\).

To measure the angular distribution
\[\{\lambda / \sqrt{n} : \lambda \in \Lambda_n\} \subseteq S^1\]
of the lattice points \(\Lambda_n\) lying on the circle \(\sqrt{n} S^1\), one introduces [33] the atomic probability measure on \(S^1\)
\[(4.4) \quad \mu_n := \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda / \sqrt{n}}\]
supported on the angles corresponding to these lattice points. It is well-known [31, 26] that, for a density-1 sequence, the angles of \(\Lambda_n\) become equidistributed in a quantitative sense, so that, in particular,
\[(4.5) \quad \mu_n \Rightarrow \frac{d\vartheta}{2\pi}\]
This also means that the covariance function, \(r_n\) as in (1.11), of the ARW, tends, locally at Planck scale, to the covariance function (1.1) of Berry’s RWM, together with their respective derivatives. Finer than that, the quantitative equidistribution asserts that the same is true at radii a power of logarithm above Planck scale.

On the other hand, other limit measures in (4.5) are possible: Cilleruelo [22] constructed a “Cilleruelo” sequence \(\{n\} \subseteq S\) so that
\[\mu_n \Rightarrow \frac{1}{4} (\delta_{\pm 1} + \delta_{\pm i})\]
where we think of \(S^1 \subseteq \mathbb{C}\). The rich structure of all possible measure “attainable” as a limit of a subsequence of \(\mu_n\) was studied [35, 62].

Another subject, naturally occurring when studying the eigenfunctions on a small disc of radius above Planck scale is that of quasi-correlations. For \(k \geq 2\), and \(\varepsilon > 0\), define the quasi-correlation set as
\[Q_k(n, n^{1/2 - \varepsilon}) = \left\{ (\lambda_1, \ldots, \lambda_k) \in \Lambda_n^k : 0 < \left\| \sum_{j=1}^k \lambda_j \right\| < n^{1/2 - \varepsilon} \right\},\]
i.e. those \( k \)-tuples whose sum has cancellations of order of magnitude \( n' \). For example, for \( k = 2 \), the quasi-correlations are identified as the close-by pairs, and it was asserted \([30]\) that for a generic \( n \in S \),

\[
\mathcal{Q}_2(n, n^{1/2-\varepsilon}) = \emptyset,
\]

and the same was inferred \([55\text{, Lemma 5.2}]\) for \( k = 4 \). Additionally, the notion of semi-correlations was introduced \([19\text{, Eq. (3.6.3)}]\) i.e. those \( k \)-tuples \((\lambda_1, \ldots, \lambda_k) \in \Lambda_n^k\), with the 1st coordinate of \( \lambda_1 + \ldots + \lambda_k \) vanishing, for the study of the boundary-adapted ARW, i.e. random Laplace eigenfunctions on the unit square.

### 4.2. Nodal length of Arithmetic Random Waves.

It is easy to compute the expected nodal length of \( f_n \) by invoking the stationarity of \( f_n \), so that the zero density \( K_1(\cdot) \) depends only on \( n \). Hence, an explicit computation of the involved Gaussian integral yields a precise expression for the expected nodal length for every \( n \in S \) to be \([56]\)

\[
\mathbb{E}[\text{len}(f_n^{-1}(0))] = \frac{\pi}{\sqrt{2}} \sqrt{n}.
\]

A handy upper bound was also given for the variance

\[
\text{Var}(\text{len}(f_n^{-1}(0))) = O\left(\frac{n}{\sqrt{N_n}}\right),
\]

as \( N_n \to \infty \). A precise asymptotic law \([33]\) of the form

\[
\text{Var}(\text{len}(f_n^{-1}(0))) = (1 + \mu_n(4)^2) \frac{\pi^2}{128 N_n^2} (1 + o_{N_n \to \infty}(1))
\]

was finally derived for the variance. To control the error term in the asymptotics \((4.6)\), a crucial role has been played by a nontrivial upper bound

\[
\# \mathcal{R}_6(n) = o_{N_n \to \infty}(N_n^4)
\]

due to J. Bourgain.

A few observations on \((4.6)\) are due. Remarkably, \((4.6)\) shows that, assuming \( N_n \to \infty \), the corresponding sequence of variances \( \text{Var}(\text{len}(f_n^{-1}(0))) \) fluctuates depending on the angular distribution of the lattice points \( \Lambda_n \). Thus, to exhibit an asymptotic law, it is essential to pass to a subsequence of \( S \) so that the corresponding \( \Lambda_n \) would observe a limit angular distribution, i.e. \( \mu_n \Rightarrow \mu \) for some probability measure \( \mu \) on \( S^1 \). The variance \((4.6)\) is of a smaller order of magnitude as compared to what was the expected order of magnitude \([56]\) of \( \frac{n}{N_n} \). This is due to another unexpected cancellation ("arithmetic Berry’s cancellation"), of a similar nature, but different appearance, to Berry’s cancellation \((3.4)\) in the context of RWM, or random spherical harmonics. Similar to random spherical harmonics, Berry’s cancellation does not occur for level curves at nonzero level: For every \( t \neq 0 \),

\[
\text{Var}(\text{len}(f_n^{-1}(t))) \sim c_0 t^4 e^{-t} \frac{n}{N_n} \to \infty \text{, with some absolute constant } c_0 > 0 \text{ (cf. } (3.11)\text{). Further, the full correlation phenomenon } (3.12) \text{ is observed in this context too: for every } t_1, t_2 \neq 0:\n\]

\[
\text{Corr}(\text{len}(f_n^{-1}(t_1)), \text{len}(f_n^{-1}(t_2))) = 1 - o_{N_n \to \infty}(1).\]

To study the fluctuations of \( \text{len}(f_n^{-1}(0)) \) in more details, the Wiener chaos expansion technique was applied. In accordance to the analogous situation for the spherical harmonics (see \((3.2.2)\), the 2nd Wiener chaos component of \( \text{len}(f_n^{-1}(0)) \) vanishes precisely, whence the bulk of the contribution to the fluctuations of \( \text{len}(f_n^{-1}(0)) \) comes from its projection to the 4th Wiener chaos component; bounding the contribution of the higher Wiener chaos spaces amounts to bounding the size of the spectral correlations \( \mathcal{R}_k(n) \) in \((4.2)\), \( k \geq 6, n \to \infty \), an approach that could also be used re-prove the

\footnote{This is an opportunity to correct an informal statement made in that manuscript. There are two negative biases of different nature. The perpendicular intersection of the nodal line with the boundary has only a local effect on the density in its neighbourhood, and is unrelated to the negative excess term infinitely many wavelengths away from the boundary.}
variance estimate \( (4.6) \), implemented within \([54]\). A non-universal non-Central Limit theorem was established \([45]\) for

\[
\widetilde{\mathcal{L}}_n := \frac{\text{len}(f_n^{-1}(0)) - \mathbb{E}[\text{len}(f_n^{-1}(0))]}{\sqrt{\text{len}(f_n^{-1}(0))}}
\]

for the regime \( N_n \to \infty \), where to exhibit a limit law, it is essential to separate the sequence \( S \) according to the angular distribution of the lattice points \( \Lambda_n \), in the following, explicit, manner. For \( \eta \in [-1,1] \) denote the random variable

\[
M_\eta := \frac{1}{2\sqrt{1+\eta^2}} \left( 2 - (1+\eta)X_1^2 - (1-\eta)X_2^2 \right),
\]

where \( X_1, X_2 \) are i.i.d. standard Gaussian r.v. Then, for \( \eta \in [-1,1] \), if \( \{n\} \subseteq S \) is a subsequence of energies so that \( N_n \to \infty \) and

\[
\widehat{\mu}_n(4) \to \eta,
\]

one has

\[
(4.7)
\]

convergence in distribution (and also in some other senses).

As the variance and the limit law of the total nodal length of \( f_n \) are completely understood, the natural question is that of the local analogues of these, above Plack scale, i.e. the nodal length of \( f_n \) restricted to a disc of radius \( r = r(n) \) so that \( r_n \sqrt{n} \to \infty \); by the stationarity of \( f_n(\cdot) \), one may only consider the centred discs \( B(r_n) = B_0(r_n) \). As in the total length case, it is also naturally invoking the moments of the covariance \( r_n \) in \((1.11)\) and its derivatives, this time restricted to \( B(r_n) \) (or, rather, to \( B(r_n) \times B(r_n) \)). Fix a small number \( \epsilon > 0 \), and consider \( r_n := n^{-1/2+\epsilon} \). The analogue equality to \((4.1)\) holds, this time, up to an error term that is expressed in terms of the size of the quasi-correlations set, with \( \delta = n^{1/2-\epsilon} \). It was shown \([4]\) Theorem 1.4] that, for a “generic” (density-1) sequence, for given a number \( k \geq 2 \), the spectral quasi-correlation set of length \( m = 2, \ldots, k \) is empty, and, in addition, such a generic sequence \( \{n\} \) could be selected sufficiently rich for the accumulation set of \( \{\widehat{\mu}_n(4)\} \) to contain the interval \([0,1]\).

Let \( \mathcal{L}_n \) be the total nodal length of \( f_n, \epsilon > 0, r_n = n^{-1/2+\epsilon} \), and \( \mathcal{L}_{n,r_n} \) be the nodal length of \( f_n \) restricted to \( B(r_n) \). It was shown that if \( \{n\} \subseteq S \) is a sequence of energies, with the quasi-correlation sets of both lengths 2 and 6 sufficiently small\(^{23}\) then

\[
(4.8)
\]

\text{Corr}(\mathcal{L}_n, \mathcal{L}_{n,r_n}) = 1 - o_{n \to \infty}(1).
\]

In particular, the local analogues of \((4.6)\) and \((4.7)\) are satisfied just above the Planck scale, for a generic sequence of energy levels, with \( \eta \) (rather, \( \eta^2 \)) unrestricted. The full correlation \((4.8)\) was proved by using an appropriate Mixed Kac-Rice formula, to evaluate the relevant covariance, and upon using \((4.6)\), and its local analogue that is the bulk of the proof. One reuses the result on the 2-point correlation function that was instrumental for establishing \((4.6)\), to make a 2-line proof of \((4.8)\) given all the developed technology. The alternative for demonstrating this result is to use the Wiener chaos expansion approach, that, up to an admissible error and explicit factors, identifies the Wiener chaos expansion of \( \mathcal{L}_{n,r_n} \) with that of \( \mathcal{L}_n \). This also shows that the bulk of the fluctuations of \( \mathcal{L}_{n,r_n} \) is its projection onto the 4th Wiener chaos space, though, unlike \( \mathcal{L}_n \), its projection onto the 2nd Wiener chaos does not vanish precisely (consistent to the sphere).

Sartori \([61]\) Theorem 1.3] refined the latter result for \((4.8)\) (and thus the local variance and limit law) to hold a logarithm power above the Planck scale, i.e. for \( r_n = \frac{\log^A n}{n^B} \) with \( A > 0 \) sufficiently large, instead of a power of \( n \) above the Planck scale. On the other hand, the angular equidistribution result \((4.5)\) holding for generic \( n \) means that, for a density-1 sequence \( \{n\} \subseteq S \), the restriction \( f_n|_{B(r_n)} \) to \( B(r_n) \) the random field \( f_n|_{B(r_n)} \) with \( r_n = \frac{\log^A n}{n^B}, \delta > 0 \) sufficiently small, converge to Berry’s RWM. By constructing an appropriate coupling of \( f_n \) with Berry’s RWM, it is therefore possible \([61]\) Theorem 1.4] to infer that, for this choice of \( r_n, \mathcal{L}_{n,r_n} \) has Berry’s logarithmic variance and converges

\(^{23}\): This, in particular, is satisfied when the quasi-correlations sets are empty. 2. In particular, this implies \( N_n \to \infty \).
to the Gaussian distribution (see also [52]). Therefore, the Arithmetic Random Waves is one example, when the local distribution of the nodal length is not indicative of the total nodal length (see the discussion at the end of §3.7).

Benatar and Maffucci [3] considered the 3-dimensional Arithmetic Random Waves, i.e. the 3-dimensional random toral eigenfunctions. On one hand, for the 3d torus, the angles of the lattice points lying on spheres of growing radii equidistribute [28, 24] for the full sequence of energy levels (i.e. the numbers representable as sum of 3 squares), e.g. in the sense of convergence of the analogues of the probability measures (4.4) to the unit volume Lebesgue measure on the sphere. On the other hand, the subtlety of the 3-dimensional case is that, to pursue the nodal hypersurface volume, one encounters the correlations of the lattice points $\mathbb{Z}^3$ (as opposed to $\mathbb{Z}^2$), the difficulty being that the $\mathbb{Z}^3$ does not conveniently correspond to a ring of integers or a number field, like $\mathbb{Z}^2 \cong \mathbb{Z}[i]$. Most specifically, if one wants to analyse the nodal volume variance, then the analogue result to Zygmund’s trick (4.1) on the length-4 correlations, and analogue on (4.3) on the length-6 spectral correlations are required. Both of these 3d analogues were successfully resolved in [3], by special methods, and, as a result, an analogue of (4.6) was established. Building on their results on the correlations of lattice points, an analogue non-central limit theorem to (4.7) was established [16] in this case (again, converging to a single limit for the full sequence of energies).

The defect distribution of the ARW and deterministic toral eigenfunctions was addressed in [36]. As for the spherical harmonics (§3.3), all the mentioned results are applicable for the intersection of 2 independent ARW in $\mathbb{S}^d$ [53], or 2 or 3 independent ARW in 3d [49], and some other variations of the basic problem, e.g. [18]. However, for dimensions higher than 3, the results on spectral correlations are not known (and, in some cases, decisively fail), hence, save for an upper bound [21], the problem of fluctuations of the nodal volume in dimensions $\geq 4$ is open.

### 4.3. Nodal intersections against smooth curves and approximate Kac-Rice.

Let $C$ be a simple smooth curve, and $\gamma : [0, L] \to \mathbb{T}^2$ its unit speed parametrization, where we are interested in the number of nodal intersections $\mathcal{Z}_{C,n} = \mathcal{Z}_C(f_n)$ of the ARW $f_n$ against $C$, i.e. the number of zeros of

$$f_n \circ \gamma = f_n(\gamma(t)),$$

$t \in [0, L]$. Note that, unless $C$ is a straight line segment, $f_n \circ \gamma$ is not stationary. However, since it is univariate regardless, it is easy to establish the expected number of the nodal intersections to be universally given by [57]

$$\mathbb{E}[\mathcal{Z}_{C,n}] = \sqrt{2n} \cdot L.$$

The question of the variance of $\mathcal{Z}_{C,n}$ is far more subtle, and, in fact, depends both on the geometry of $C$ and the angular distribution of the lattice points $\Lambda_n$. First, it turns out that one needs to separate the two most interesting, extreme, scenarios: $C$ is a straight segment, and $C$ has nowhere vanishing curvature, whereas the intermediate cases could be analysed by combining both techniques. The former case of $C$ straight segment has been dealt with by Maffucci [37], whence various upper bounds were asserted, both conditionally and unconditionally, but no precise estimates for the variance was given for a single case.

For the other extreme case, when $C$ is “generic” simple smooth curve with nowhere vanishing curvature, one may evaluate the variance of $\mathcal{Z}_{\gamma,n}$, asymptotically precisely, as follows. Let $B_C(\Lambda_n)$ be the quantity

$$B_C(\Lambda_n) := \int_{\mathcal{C} \times \mathcal{C}} \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \left( \frac{\lambda}{\sqrt{N}} \cdot \hat{\gamma}(t_1) \right)^2 \left( \frac{\lambda}{\sqrt{N}} \cdot \hat{\gamma}(t_2) \right)^2 \, dt_1 \, dt_2.$$

$$= \int_0^L \int_0^L \int_{\mathbb{S}^1} \langle \theta, \hat{\gamma}(t_1) \rangle \cdot \langle \theta, \hat{\gamma}(t_2) \rangle \, d\mu_n(\theta) \, dt_1 \, dt_2,$$
with \( \mu_n \) as in (4.4), easily generalized for an arbitrary probability measure on \( S^1 \). Observe that the expression

\[
(4B_C(\Lambda_n) - L^2)
\]

is in general fluctuating, depending on both \( \gamma \) and the angular distribution of \( \Lambda \), and is easy to show that for every \( C, \Lambda_n \) sufficiently symmetric set (equivalently, probability measure on \( S^1 \)),

\[
\frac{1}{4}L^2 \leq B_C(\Lambda_n) \leq \frac{1}{2}L^2.
\]

Under the above notation, and assuming that \( C \) has nowhere vanishing curvature, it was shown [57, Theorem 1.2], that

\[
\text{Var}(\mathcal{Z}_{C,n}) = (4B_C(\Lambda_n) - L^2) \cdot \frac{n}{N_n} + O\left(\frac{n}{N_n^{3/2}}\right).
\]

In light of the above, we have

\[
(4B_C(\Lambda_n) - L^2) \geq 0
\]

always, and it does not ever vanish for a “generic” \( C \). However, the vanishing of \( (4B_C(\Lambda_n) - L^2) \) genuinely occurs, in two scenarios of a very different nature (still, under the nowhere vanishing curvature assumption) [57] Proposition 7.1, Corollary 7.2: 1. Given a curve \( C \), \( (4B_C(\Lambda_n) - L^2) \) vanishes identically for every symmetric set \( \Lambda_n \) (in a more generalized sense, for every probability measure arising from a symmetric set). In this case \( C \) is called “static”. An example of a static curve is a circle or a semi-circle. 2. Given a curve \( C \), \( (4B_C(\Lambda_n) - L^2) \) vanishes for precisely either the “Cilleruelo” measure

\[
\frac{1}{4}(\delta_{\pm 1} + \delta_{\pm i})
\]

or its tilt by \( \pi/4 \) tilted Cilleruelo. Otherwise, if both (1) or (2) do not hold, then the number \( (4B_C(\Lambda_n) - L^2) \) is bounded away from 0.

Rossi and Wigman [55] studied the distribution of \( \mathcal{Z}_{C,n} \) in more details, via the Wiener chaos expansion. They proved the CLT in a generic scenario: If, for some sequence \( \{n\} \subseteq S \), the numbers \( 4B_C(\Lambda_n) - L^2 \) are bounded away from 0, then [55, Theorem 1.1]

\[
\frac{\text{Var}(\mathcal{Z}_{C,n}) - \mathbb{E}[\mathcal{Z}_{C,n}]}{\sqrt{\text{Var}(\mathcal{Z}_{C,n})}}
\]

converges, in distribution, to the standard Gaussian. For instance, the CLT applies in scenario (2) above, so long as one excludes the Cilleruelo and tilted Cilleruelo sequences. In this case the bulk of the fluctuations of \( \mathcal{Z}_{C,n} \) “lives” in the projection onto the 2nd Wiener chaos space, with no Berry’s cancellation.

On the other hand, the static curves observe a very different behaviour, both in terms of the asymptotic variance and the limit theorem. They proved [55, Theorem 1.3] the following results, assuming that \( C \) is static, and under an extra assumption on the sequence \( \{n\} \subseteq S \), that is generic. Namely, it was shown that [55, Theorem 1.3(1)], if \( C \) is static, the variance of \( \mathcal{Z}_{C,n} \) is of order in \( \frac{1}{N_n} \), with the leading constant depending on both the angular distribution of \( \Lambda_n \) and the geometry of \( C \). Further, an explicit non-Central Limit Theorem [55, Theorem 1.3(2)] was asserted for

\[
\frac{\text{Var}(\mathcal{Z}_{C,n}) - \mathbb{E}[\mathcal{Z}_{C,n}]}{\sqrt{\text{Var}(\mathcal{Z}_{C,n})}}.
\]

In this case Berry’s cancellation occurs: The bulk of the fluctuations of \( \mathcal{Z}_{C,n} \) “lives” in the projection onto the 4th Wiener chaos space, with the projection onto the 2nd Wiener chaos space vanishing precisely (by construction). Surprisingly enough, no analogue of static curves exist for this problem for either the random spherical harmonics or the RWM, see [33, 55]. Finally, the analogue 3d problems were studied: Nodal intersections against curves [58], straight segments [38], and surfaces [39].

Note that, depending on whether \( n \) is odd or even, \( f_n \) is either odd w.r.t. to the involution

\[
\tau_1 : \cdot \mapsto \cdot + (1/2, 1/2)
\]
or w.r.t. the involution $\tau_2\cdot \mapsto \cdot + (1/2,0)$ respectively. Hence if there are sub-curves of $C$ whose image, under $\tau_1$ or $\tau_2$, is lying in $C$, then the sufficient conditions (i.e. non-degeneracy of the Gaussian distribution of $(f_n(\gamma(t_1)), f_n(\gamma(t_2))), t_1 \neq t_2$) for the applicability of the Kac-Rice formula \((2.5)\) fail. We do not know whether or not such smooth curves with nowhere vanishing curvature (i.e. curves of this type that are partially invariant w.r.t. either of the $\tau_i$) actually exist, but, in this light, we were not able to validate whether the Kac-Rice formula \((2.5)\) holds precisely in this case, i.e. whether the r.h.s. of \((2.5)\) computes the 2nd factorial moment of $\mathcal{D}_{C;n}$. However, that it does give a good estimate for the variance, up to an admissible error term, follows from the Approximate Kac-Rice formula [57, Proposition 1.3], especially developed for these needs, whose implementation, after much effort analysing the integral on the r.h.s. of \((2.5)\), eventually yields the asymptotic variance \((4.9)\). This approach also interprets the factorial (i.e. linear) part in \((2.5)\) as the length of the projection onto the 1st component of the diagonal

$$D := \{(t,t) : t \in [0,L]\} \subseteq [0,L]^2,$$

corresponding to those Gaussian vectors $(f_n(\gamma(t_1)), f_n(\gamma(t_2)))$ that fail the non-degeneracy condition, and also gives an answer to what the r.h.s. of \((2.5)\) actually computes in case that there are degeneracies outside of $D$ that occur\(^{24}\).

\section*{References}

[1] Azaïs, J.M. and Delmas, C. Mean number and correlation function of critical points of isotropic Gaussian fields and some results on GOE random matrices. Stochastic Processes and their Applications (2022).
[2] Azaïs, J.M. and Wschebor, M. Level sets and extrema of random processes and fields. John Wiley & Sons (2009).
[3] Benatar, J. and Maffucci, R.W. Random waves on: nodal area variance and lattice point correlations. International Mathematics Research Notices, 2019(10), pp.3032–3075 (2019).
[4] Benatar, J., Marinucci, D. and Wigman, I. Planck-scale distribution of nodal length of arithmetic random waves. Journal d’Analyse Mathématique, 141(2), pp.707–749 (2020).
[5] Béard, P. Volume des ensembles nodaux des fonctions propres du laplacien. Séminaire de théorie spectrale et géométrie, 3, pp.1–9 (1985).
[6] Berry, M.V. Regular and irregular semiclassical wavefunctions. Journal of Physics A: Mathematical and General, 10(12), p.2083 (1977).
[7] Berry, M.V. Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. Journal of Physics A: Mathematical and General, 35(13), p.3025 (2002).
[8] Blum G., Gnutzmann, S., and Smilansky, U. Nodal domain statistics: a criterion for quantum chaos Phys. Rev. Lett. 88 114101 (2002)
[9] Beliaev, D., Cammarota, V. and Wigman, I. Two point function for critical points of a random plane wave. International Mathematics Research Notices, 2019(9), pp.2661–2689 (2019).
[10] Beliaev, D., Cammarota, V. and Wigman, I. No repulsion between critical points for planar Gaussian random fields. Electronic Communications in Probability, 25, pp.1–13 (2020).
[11] Billingsley, P. Convergence of probability measures. John Wiley & Sons (2013).
[12] Bombieri, E. and Bourgain, J. A problem on sums of two squares. International Mathematics Research Notices, 2015(11), pp.3343–3407 (2015).
[13] Bourgain, J. and Rudnick, Z. Restriction of toral eigenfunctions to hypersurfaces and nodal sets. Geometric and Functional Analysis, 22(4), pp.878–937 (2012).
[14] Bourgain, J. and Rudnick, Z. On the nodal sets of toral eigenfunctions. Inventiones mathematicae, 185(1), pp.199–237 (2011).
[15] Canzani, Y. and Hanin, B. Local universality for zeros and critical points of monochromatic random waves. Communications in Mathematical Physics, 378(3), pp.1677–1712 (2020).
[16] Cammarota, V. Nodal area distribution for arithmetic random waves. Transactions of the American Mathematical Society, 372(5), pp.3539–3564 (2019).
[17] Cammarota, V., Marinucci, D. and Wigman, I. On the distribution of the critical values of random spherical harmonics. The Journal of Geometric Analysis, 26(4), pp.3252–3324 (2016).
[18] Cammarota, V., Marinucci, D. and Rossi, M. Lipschitz-Killing curvatures for arithmetic random waves. arXiv preprint arXiv:2010.14165 (2020).

\(^{24}\)Such situation will occur, for example, for $C$ smooth with nowhere vanishing curvature, partially invariant w.r.t. either of $\tau_i$, when the answer is given in terms of a special “degeneracy index” of $C$. The author was informed that this was already known to Pavel Bleher in a different context.
[19] Cammarota, V., Klurman, O. and Wigman, I. Boundary effect on the nodal length for arithmetic random waves, and spectral semi-correlations. Communications in Mathematical Physics, 376(2), pp.1261–1310 (2020).

[20] Cammarota, V. and Wigman, I. Fluctuations of the total number of critical points of random spherical harmonics. Stochastic Processes and their Applications, 127(12), pp.3825–3869 (2017).

[21] Cherubini, G. and Laaksonen, N. On the variance of the nodal volume of arithmetic random waves. Forum Mathematicum, Vol. 34, No. 2, pp. 279–292, De Gruyter (2022).

[22] Cilleruelo, J. The distribution of the lattice points on circles. Journal of Number theory, 43(2), pp.198–202 (1993).

[23] Cramér, H. and Leadbetter, M.R. Stationary and related stochastic processes: Sample function properties and their applications. Courier Corporation (2013).

[24] Duke, W. Hyperbolic distribution problems and half-integral weight Maass forms, Invent. Math. 92, no. 1, p. 73–90 (1988).

[25] Dalmao, F., Estrade, A. and León, J. On 3-dimensional Berry’s model. ALEA. Lat. Am. J. Probab. Math. Stat. (2021).

[26] Erdős, P. and Hall, R.R. On the angular distribution of Gaussian integers with fixed norm. Discrete mathematics, 200(1-3), pp.87–94 (1999).

[27] Federer, H. Geometric measure theory, Springer (2014).

[28] Golubeva, E. P. and Fomenko, O. M. Asymptotic distribution of lattice points on the three-dimensional sphere, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 160, p. 54–71 (1987).

[29] Granville, A. and Wigman, I. The distribution of the zeros of random trigonometric polynomials. American journal of mathematics, 133(2), pp.295–357 (2011).

[30] Granville, A. and Wigman, I. Planck-scale mass equidistribution of toral Laplace eigenfunctions. Communications in Mathematical Physics, 355(2), pp.767–802 (2017).

[31] I. Kátai and I. Környei. On the distribution of lattice points on circles. Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 19:87–91 (1977).

[32] Keeler, B. A logarithmic improvement in the two-point Weyl law for manifolds without conjugate points. arXiv preprint arXiv:1905.05136 (2019).

[33] Krishnapur, M., Kurlberg, P. and Wigman, I. Nodal length fluctuations for arithmetic random waves. Annals of Mathematics, pp.699–737 (2013).

[34] Kratz, M.F. and León, J.R., 2001. Central limit theorems for level functionals of stationary Gaussian processes and fields. Journal of Theoretical Probability, 14(3), pp.639-672.

[35] Kurlberg, P. and Wigman, I. On probability measures arising from lattice points on circles. Mathematische Annalen, 367(3), pp.1057–1098 (2017).

[36] Kurlberg, P., Wigman, I. and Yesha, N. The defect of toral Laplace eigenfunctions and arithmetic random waves. Nonlinearity, 34(9), p.6651 (2021).

[37] Maffucci, R.W. Nodal intersections of random eigenfunctions against a segment on the 2-dimensional torus. Monatshefte für Mathematik, 183(2), pp.311–328 (2017).

[38] Maffucci, R.W. Nodal intersections for random waves against a segment on the 3-dimensional torus. Journal of Functional Analysis, 272(12), pp.5218–5254 (2017).

[39] Maffucci, R.W. and Rossi, M. Asymptotic distribution of Nodal Intersections for ARW against a Surface. arXiv preprint arXiv:2110.08218 (2021).

[40] Marinucci, D., Rossi, M. and Wigman, I. The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. In Annales de l’Institut Henri Poincaré, Probabilités et Statistiques (Vol. 56, No. 1, pp. 374–390). Institut Henri Poincaré (2020).

[41] Marinucci, D. and Rossi, M. On the correlation between nodal and nonzero level sets for random spherical harmonics. In Annales Henri Poincaré Vol. 22, No. 1, pp.275–307. Springer International Publishing (2021).

[42] Marinucci, D. and Wigman, I. On the area of excursion sets of spherical Gaussian eigenfunctions. Journal of Mathematical Physics, 52(9), p.093301 (2011).

[43] Marinucci, D. and Wigman, I. On nonlinear functionals of random spherical eigenfunctions. Communications in Mathematical Physics, 327(3), pp.849–872 (2014).

[44] Marinucci, D. and Wigman, I. The defect variance of random spherical harmonics. Journal of Physics A: Mathematical and Theoretical, 44(35), pp.355206 (2011).

[45] Marinucci, D., Peccati, G., Rossi, M. and Wigman, I. Non-universality of nodal length distribution for arithmetic random waves. Geometric and Functional Analysis, 26(3), pp.926–960 (2016).

[46] Nazarov, F. and Sodin, M. On the number of nodal domains of random spherical harmonics. American Journal of Mathematics, 131(5), pp.1337–1357 (2009).

[47] Nazarov, F. and Sodin, M. Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. Journal of Mathematical Physics, Analysis, Geometry (Kharkov) v.12 no. 3, 205–278 (2016).

[48] J. Neuheisel, The asymptotic distribution of nodal sets on spheres, Johns Hopkins Ph.D. thesis (2000).
Notarnicola, M. Fluctuations of nodal sets on the 3-torus and general cancellation phenomena. ALEA, 18, pp.1127–1194 (2021).

Oravecz, F., Rudnick, Z. and Wigman, I. The Leray measure of nodal sets for random eigenfunctions on the torus. In Annales de l'institut Fourier Vol. 58, No. 1, pp. 299–335 (2008).

Nourdin, I., Peccati, G. and Rossi, M. Nodal statistics of planar random waves. Communications in Mathematical Physics, 369(1), pp.99–151 (2019).

Dierickx, G., Nourdin, I., Peccati, G. and Rossi, M. Small scale CLTs for the nodal length of monochromatic waves. arXiv preprint arXiv:2005.06577 (2020).

Dalmao, F., Nourdin, I., Peccati, G. and Rossi, M. Phase singularities in complex arithmetic random waves. Electronic Journal of Probability, 24, pp.1–45 (2019).

Peccati, G. and Rossi, M. Quantitative limit theorems for local functionals of arithmetic random waves. In The Abel Symposium pp. 659–689. Springer, Cham. (2016).

Rossi, M. and Wigman, I. Asymptotic distribution of nodal intersections for arithmetic random waves. Nonlinearity, 31(10), p.4472 (2018).

Rudnick, Z. and Wigman, I. On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. In Annales Henri Poincare (Vol. 9, No. 1, pp. 109–130). SP Birkhäuser Verlag Basel (2008).

Rudnick, Z. and Wigman, I. Nodal intersections for random eigenfunctions on the torus. American Journal of Mathematics, 138(6), pp.1605–1644 (2016).

Rudnick, Z., Wigman, I. and Yehsa, N. Nodal intersections for random waves on the 3-dimensional torus. In Annales de l’Institut Fourier, Vol. 66, No. 6, pp. 2455–2484 (2016).

Y. G. Safarov, Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition, Funktsional. Anal. i Prilozhen., 22, pp. 53—65, 96 (1988).

Sarnak, P. and Wigman, I. Topologies of nodal sets of random band-limited functions. Communications on pure and applied mathematics, 72(2), pp.275–342 (2019).

Sartori, A. Spectral Quasi Correlations and Phase Transitions for the Nodal Length of Arithmetic Random Waves. International Mathematics Research Notices, 11, pp.8472–8507 (2022).

Sartori, A. On the fractal structure of attainable probability measures. Bulletin Polish Acad. Sci. Math., 66, pp.123–133 (2018).

Sodin, M. and Tsirelson, B. Random complex zeroes, I. Asymptotic normality. Israel Journal of Mathematics, 144(1), pp.125–149 (2004).

Szego, G. Orthogonal polynomials. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I. (1975).

Todino, A.P. Nodal lengths in shrinking domains for random eigenfunctions on \(S^2\). Bernoulli, 26(4), pp.3081–3110 (2020).

Toth, J.A. and Wigman, I. Counting open nodal lines of random waves on planar domains. International Mathematics Research Notices, 2009(18), pp.3337–3365 (2009).

Toth, J.A. and Zelditch, S. Nodal intersections and geometric control. Journal of Differential Geometry, 117(2), pp.345–393 (2021).

Toth, J.A. and Zelditch, S. Counting nodal lines which touch the boundary of an analytic domain. Journal of Differential Geometry, 81(3), pp.649–686 (2009).

Vidotto, A. A note on the reduction principle for the nodal length of planar random waves. Statistics & Probability Letters, 174, p.109090 (2021).

Wigman, I. On the nodal lines of random and deterministic Laplace eigenfunctions. Spectral geometry, 84, pp.285–297 (2012).

Wigman, I. On the distribution of the nodal sets of random spherical harmonics. Journal of mathematical physics, 50(1), p.013521 (2009).

Wigman, I. Fluctuations of the nodal length of random spherical harmonics. Communications in Mathematical Physics, 298(3), pp.787–831 (2010).

Zelditch, S. Real and complex zeros of Riemannian random waves. Contemporary Mathematics, 14, p.321 (2009).

A. Zygmund, On Fourier coefficients and transforms of functions of two variables, Studia Math. 50, pp. 189—201 (1974).