Boundary Structure and Module Decomposition of the Bosonic $Z_2$ Orbifold Models with $R^2 = 1/2k$.

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Abstract

The $Z_2$ bosonic orbifold models with compactification radius $R^2 = 1/2k$ are examined in the presence of boundaries. Demanding the extended algebra characters to have definite conformal dimension and to consist of an integer sum of Virasoro characters, we arrive at the right splitting of the partition function. This is used to derive a free field representation of a complete, consistent set of boundary states, without resorting to a basis of the extended algebra Ishibashi states. Finally the modules of the extended symmetry algebra that correspond to the finitely many characters are identified inside the direct sum of Fock modules that constitute the space of states of the theory.
1 Introduction

The boundary structure of the $Z_2$ bosonic orbifold models with $R^2 = 1/2k$ is examined. It was discovered that for these particular values of the radius the unorbifolded torus models admit a complete consistent set of Newmann boundary states \[^1\]. To derive the complete set of boundary states for the orbifold it is necessary to find the correct splitting of the partition function of the theories, and from this to read off the characters. This is achieved by requiring each character to have definite conformal dimension and to split into a positive integer sum of Virasoro characters. The guiding principle is the identification of the $k = 1$ orbifold model with the $k = 4$ torus model, which happens at the self-dual point in the Ginsparg classification \[^3\].

The boundary states need to satisfy two consistency conditions. The first is the Ishibashi condition \[^3\], which is necessary for the introduction of boundaries in a conformal field theory. More precisely, it arises from the requirement that the variation of the correlation functions under a conformal transformation produces the correct Ward identities. Of course if the symmetry algebra is larger than the Virasoro algebra, then the Virasoro Ishibashi condition gets modified. The second is the Cardy condition \[^4\], which stems from the identification of the one loop open string amplitude with certain consistent boundary conditions, to the closed string amplitude between two boundary states that correspond to the open string boundary conditions. A solution to the Cardy condition for a RCFT was derived in \[^4\], by making use of the Verlinde formula \[^9\], in terms of the S matrix for the characters. Nevertheless this solution makes use of a complete set of extended algebra Ishibashi states, a free field representation of which was lacking.

Our approach to construct a free field representation of the boundary states consists of writing the boundary states as linear combinations of the Virasoro Ishibashi states (which are infinite), and then, instead of trying to find the extended algebra Ishibashi states, apply directly the Cardy condition. This permits the determination of the free field representation of the boundary states of our theories. Finally, the consistency of the boundary states is checked explicitly, rederiving in this way the fusion rules.

Next we study the extended algebra module structure of the space of states of the theory. First the vacuum character is used to find a set of generators of the algebra. The generating functions of these generators include chiral bilocal fields in the sense of \[^5\]. Next a set of primary states is identified. The above generators, acting on these states produce a set of modules of the extended symmetry algebra. It is shown that the space of states of the theory splits into a direct sum of these modules. Furthermore, traces over these modules produce the correct characters (in the right splitting) of the theory.

2 Decomposition of the Orbifold Partition Function

We consider the $Z_2$ orbifold conformal field theory of the free bosonic field theory compactified on a circle of radius $R = 1/\sqrt{2k}$. For this choice of radius the unorbifolded theory has been shown to admit a consistent set of Newmann boundary states \[^1\]. The action of our theory is

$$ S = \frac{1}{2\pi} \int \partial X \bar{\partial} X $$

(1)

where $0 \leq \sigma \leq \beta$ and is invariant under the $Z_2$ symmetry $g : X \rightarrow -X$. This corresponds to $\alpha_n \rightarrow -\alpha_n$, $\bar{\alpha}_n \rightarrow -\bar{\alpha}_n$, $\hat{P} \rightarrow -\hat{P}$ and $\hat{W} \rightarrow -\hat{W}$, where $\hat{P}$ and $\hat{W}$ are the momentum and
winding operators respectively. Their eigenvalues are \( m/R \) and \( nR \). This symmetry permits us to have two kinds of boundary conditions, the untwisted one corresponding to \( X(\sigma + \beta, t) = X(\sigma, t) \) and the twisted one corresponding to \( X(\sigma + \beta, t) = -X(\sigma, t) \).

The space of states associated to the untwisted sector is

\[
H^U = \bigoplus_{m,n \in \mathbb{Z}} H^U_{m,n} \oplus_{n \geq 0} H^U_{0n}
\]

where

\[
H^U_{mn} = \{ \alpha_{-n_1} \cdots \alpha_{-n_j} \bar{\alpha}_{-n_j+1} \cdots \bar{\alpha}_{-n_{2l}} (|m, n > + | -m, -n >) \}
\]

\[
\oplus \{ \alpha_{-n_1} \cdots \alpha_{-n_j} \bar{\alpha}_{-n_j+1} \cdots \bar{\alpha}_{-n_{2l+1}} (|m, n > - | -m, -n >) \}
\]

and \( n_i \in \mathbb{Z}^+ \). This is the space of states that are invariant under the \( Z_2 \) symmetry. The Hilbert space corresponding to the twisted sector is

\[
H^T = \{ \alpha_{-n_1} \cdots \alpha_{-n_j} \bar{\alpha}_{-n_j+1} \cdots \bar{\alpha}_{-n_{2l}} |1/16, 1/16 >_0 \}
\]

\[
\oplus \{ \alpha_{-n_1} \cdots \alpha_{-n_j} \bar{\alpha}_{-n_j+1} \cdots \bar{\alpha}_{-n_{2l+1}} |1/16, 1/16 >_{\pi R} \}
\]

where \( n_i \in (Z + 1/2)^+ \). The states \( |1/16, 1/16 >_0 \) and \( |1/16, 1/16 >_{\pi R} \) are primary states of conformal weight \( (1/16, 1/16) \) corresponding to the fixed points of the symmetry. Note that the twisted boundary condition forces the momentum and winding eigenvalues to be zero, while the only option left for the position operator is to take values at the orbifold points \( 0, \pi R \).

The untwisted Virasoro generators are

\[
L^U_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{m-n} \alpha_n :
\]

giving rise to the Hamiltonian

\[
H^U_\beta = \frac{2\pi}{\beta} \left( L^U_0 + L^U_0 - \frac{1}{12} \right),
\]

while the twisted ones are

\[
L^T_m = \frac{1}{2} \sum_{n \in \mathbb{Z}+1/2} : \alpha_{m-n} \alpha_n : + \frac{1}{16} \delta_{m,0}
\]

giving rise to the Hamiltonian

\[
H^T_\beta = \frac{2\pi}{\beta} \left( L^T_0 + L^T_0 - \frac{1}{12} \right).
\]

The radius \( R \) orbifold partition function has the well known form

\[
\mathbb{Z}^R_{orb}(q) = \frac{1}{2} \left( \mathbb{Z}^R_{circ}(q) + \frac{\mathbb{Z}^R_{circ}(q) \theta_3(q) \theta_4(q)}{\eta^2(q)} + \frac{\mathbb{Z}^R_{circ}(q) \theta_3(q) \theta_4(q)}{\eta^2(q)} \right) + \frac{\mathbb{Z}^R_{circ}(q) \theta_3(q) \theta_4(q)}{\eta^2(q)}.
\]

Here, \( q = e^{2\pi i \tau} = e^{-2\pi \beta} \) where we have taken \( \tau \) to be pure imaginary, as in [4]. The precise definitions of the theta and eta functions are given in Appendix A. In the particular case of \( R^2 = 1/2k, k \in \mathbb{Z}^+ \), the \( \mathbb{Z}^R_{circ}(q) \) simplifies considerably [1] and takes the form

\[
\mathbb{Z}^k_{circ}(q) = \sum_{s=0}^{2k-1} \chi_s(q) \chi_s(q)
\]
where
\[ \chi_s(q) = \sum_{n \in \mathbb{Z}} \frac{q^{k(n+\frac{1}{16})^2}}{\eta(q)}. \] (11)

It is convenient to define the following extra chiral components:
\[
\begin{align*}
\chi_a(q) &= \frac{1}{2} \frac{\sqrt{\theta_2(q)^4}}{\eta(q)} = \frac{1}{2\sqrt{2}} \frac{q^{1/48}}{\prod_{n=1}^\infty (1-q^n)} = \frac{1}{2\sqrt{2}} \frac{\sum_{n \in \mathbb{Z}} q^{(n+1/2)^2}}{\eta(q)}, \\
\chi_b(q) &= \frac{1}{2} \frac{\sqrt{\theta_2(q)^4}}{\eta(q)} = \frac{1}{2\sqrt{2}} \frac{q^{1/48}}{\prod_{n=1}^\infty (1+q^n)} = \frac{1}{2\sqrt{2}} \frac{\sum_{n \in \mathbb{Z}} q^{(n+1/2)^2} e^{i\pi(n+1/2)}}{\eta(q)}, \\
\chi_c(q) &= \frac{\sqrt{\theta_2(q)^4}}{\eta(q)} = \frac{q^{-1/24}}{\prod_{n=1}^\infty (1+q^n)} = \sum_{n \in \mathbb{Z}} q^{n^2(-1)^n}.
\end{align*}
\]

Using this notation the orbifold partition function becomes
\[ Z_{orb}^k = \frac{1}{2} \sum_{s=0}^{2k-1} (\chi_s(q) + 2\chi_a(q)\chi_a(q) + 2\chi_b(q)\chi_b(q) + \frac{1}{2}\chi_c(q)\chi_c(q)). \] (13)

Noticing that \( \chi_s(q) = \chi_{2k-s}(q) \) we can rewrite the above sum as
\[ Z_{orb}^k = \frac{1}{2} \chi_0(q)\chi_0(q) + \sum_{s=0}^{k-1} \chi_s(q)\chi_s(q) + \frac{1}{2} \chi_k(q)\chi_k(q) \] (14)
\[ + 2\chi_a(q)\chi_a(q) + 2\chi_b(q)\chi_b(q) + \frac{1}{2}\chi_c(q)\chi_c(q). \]

One problem in this decomposition of the partition function is that \( \chi_a(q) \) and \( \chi_b(q) \) do not have a definite conformal dimension. To eliminate this problem we rewrite \( 2\chi_a(q)\chi_a(q) + 2\chi_b(q)\chi_b(q) \) in the form \( 2\chi_+(q)\chi_+(q) + 2\chi_-(q)\chi_-(q) \) where
\[
\begin{align*}
\chi_+(q) &= \frac{1}{\sqrt{2}} (\chi_a(q) + \chi_b(q)) = \sum_{n \in \mathbb{Z}} \frac{q^{4(n+1/8)^2}}{\eta(q)}, \\
\chi_-(q) &= \frac{1}{\sqrt{2}} (\chi_a(q) - \chi_b(q)) = \sum_{n \in \mathbb{Z}} \frac{q^{4(n+3/8)^2}}{\eta(q)}.
\end{align*}
\]

The components (15) satisfy
\[
\begin{align*}
\chi_+(e^{2\pi i q}) &= e^{2\pi i \frac{1}{16}} \chi_+(q), \\
\chi_-(e^{2\pi i q}) &= e^{2\pi i \frac{3}{16}} \chi_-(q),
\end{align*}
\]
so their conformal dimensions are 1/16 and 9/16 respectively.

Another problem with this decomposition is that not all components are an integer sum of Virasoro characters. This is necessary since we are going to search for a larger algebra that groups the infinitely many Virasoro characters into finitely many groups which are to be interpreted as the characters of the larger algebra. This is because we are searching for an algebra that is going to contain the Virasoro algebra as a subalgebra of its universal enveloping algebra. This suggests that we should change again the splitting of the partition function. To
see which is the right splitting it is instructive to look more closely at the \( k = 1 \) model. This is none other than the self-dual point in the Ginsparg’s classification \[2\]. This model should be the same as the \( k = 4 \) torus model, so we should be able to identify the components of the partition functions. For the \( k = 4 \) torus model we have the characters

\[
\chi^{k=4}_s(q) = \sum_{n \in \mathbb{Z}} \frac{q^{4(n+\frac{1}{2})^2}}{\eta(q)}, \quad 0 \leq s \leq 7
\]

while for the \( k = 1 \) orbifold model we have the components

\[
\chi^{k=1}_s(q) = \sum_{n \in \mathbb{Z}} \frac{q^{(n+\frac{1}{2})^2}}{\eta(q)}, \quad 0 \leq s \leq 1
\]

as well as \( \chi_+(q), \chi_-(q) \) and \( \chi_c(q) \). The identification proceeds as follows:

\[
\frac{1}{2} \left( \chi^{k=1}_0(q) + \chi_c(q) \right) = \chi^{k=4}_0(q),
\]

\[
\frac{1}{2} \left( \chi^{k=1}_0(q) - \chi_c(q) \right) = \chi^{k=4}_4(q),
\]

\[
\frac{1}{2} \chi^{k=1}_1(q) = \chi^{k=4}_2(q) = \chi^{k=4}_6(q),
\]

\[
\chi_+(q) = \chi^{k=4}_1(q) = \chi^{k=4}_7(q),
\]

\[
\chi_-(q) = \chi^{k=4}_3(q) = \chi^{k=4}_5(q).
\]

Under this identification the two partition functions are equal. So in the case of \( k = 1 \) we have the following splitting of the orbifold partition function into characters:

\[
Z^{k=1}_{\text{orb}} = \chi^*_+\chi^*_+(q) + \chi^*_-\chi^*_-(q) + 2\chi^*_1\chi^*_1(q) + 2\chi_+(q)\chi_+(q) + 2\chi_-(q)\chi_-(q)
\]

where

\[
\chi^\pm_{k=1}(q) = \frac{1}{2} \left( \chi_0(q) \mp \chi_c(q) \right),
\]

\[
\chi^*_1(q) = \frac{1}{2} \chi_1(q) = \sum_{n \in \mathbb{Z}} \frac{q^{4k(n+1/4)^2}}{\eta(q)} = \sum_{n=0}^{\infty} \frac{q^{k(n+1/2)^2}}{\eta(q)}.
\]

This seems to suggest that for general \( k \) the right splitting is

\[
Z^k_{\text{orb}} = \chi^*_+\chi^*_+(q) + \chi^*_-\chi^*_-(q) + 2\chi^*_k\chi^*_k(q) + \sum_{s=1}^{k-1} \chi_s(q)\chi_s(q)
\]

\[
+ \ 2\chi_+(q)\chi_+(q) + 2\chi_-(q)\chi_-(q).
\]

Now we have the following lemma:

**Lemma 1** The extended algebra characters \( \chi^*_\pm(q), \chi^*_k(q), \chi_s(q) \) and \( \chi_\pm(q) \) split into an integer sum of Virasoro characters.

The proof of this lemma is in appendix A. It is worth noting that the splitting into Virasoro characters is different in the cases \( k \neq l^2 \) and \( k = l^2 \). This suggests different behavior of the theories in the two cases. Indeed the models with \( k = l^2 \) can be identified with the \( D_l \) dihedral group orbifolds of the \( SU(2) \times SU(2) \) level one theory, which corresponds to the \( k = 1 \) \((R = 1/\sqrt{2})\) torus model.
3 Boundary States

There are two conditions that the boundary states have to satisfy. One is the Ishibashi condition and the other is the Cardy condition.

The Ishibashi condition appears when we restrict the conformal field theory to the upper half plane. The variation of the correlation functions under $z \rightarrow z + a(z)$ is given by

$$\delta < \phi_h(z_1, \bar{z}_1) \cdots \phi_h(z_N, \bar{z}_N) > = \frac{1}{2\pi i} \oint_C a(z) < T(z) \phi_h(z_1, \bar{z}_1) \cdots \phi_h(z_N, \bar{z}_N) > dz \quad (23)$$

$$- \frac{1}{2\pi i} \oint_C a(\bar{z}) < \bar{T}(\bar{z}) \phi_h(z_1, \bar{z}_1) \cdots \phi_h(z_N, \bar{z}_N) > d\bar{z}$$

where the contour $C$ contains all the points $z_i$. This can be deformed to a contour on a large semicircle and a contour on the real line. The integral over the real line has to vanish for the Ward identities to be valid, and this implies that $T(z) = \bar{T}(\bar{z})$ on the real line. Since the real line is the boundary of the upper half plane this translates (in the closed string picture) into the condition

$$(L_n - \bar{L}_{-n}) |B> = 0 \quad (24)$$

for the boundary states $|B>$. It is worth mentioning that if we have a larger symmetry algebra of generators $(W_n^{(r)}, \bar{W}_{\bar{n}}^{(r)})$ then the corresponding Ishibashi condition becomes

$$(W_n^{(r)} - (-1)^s \bar{W}_{\bar{n}}^{(r)}) |B> = 0 \quad (25)$$

where $s$ is the spin of the generators of the algebra.

The solution of the Virasoro Ishibashi condition in the case of a free field with untwisted boundary conditions gives two kinds of states. One corresponding to Dirichlet and the other to Newmann boundary conditions in the target space:

$$|i^{D}_{m/2R}> = e^{\sum_{n=1}^{\infty} \frac{a_{-n}a_n}{n+1}} |m, 0 >,$$

$$|i^{N}_{nR}> = e^{\sum_{n=1}^{\infty} \frac{a_{-n}a_n}{n}} |0, n >.$$

Upon orbifolding only the linear combinations invariant under $Z_2$ survive so we get

$$|i^{UD}_{m/2R}> = e^{\sum_{n=1}^{\infty} \frac{a_{-n}a_n}{n+1/2}} \frac{1}{\sqrt{2}} (|m, 0 > + |m, 0 >),$$

$$|i^{UN}_{nR}> = e^{-\sum_{n=1}^{\infty} \frac{a_{-n}a_n}{n-1/2}} \frac{1}{\sqrt{2}} (|0, n > + |0, n >).$$

In the case of the twisted boundary conditions we get

$$|i^{TD}_{0,\pi R}> = e^{\sum_{n=1}^{\infty} \frac{a_{-n+1/2}a_{-n+1/2}}{n+1/2}} |1/16, 1/16 >_{0,\pi R},$$

$$|i^{TN}_{0}> = e^{\sum_{n=1}^{\infty} \frac{a_{-n+1/2}a_{-n+1/2}}{n-1/2}} \frac{1}{\sqrt{2}} (|1/16, 1/16 >_0 + |1/16, 1/16 >_{\pi R}),$$

$$|i^{TN}_{\pi/2R}> = e^{-\sum_{n=1}^{\infty} \frac{a_{-n+1/2}a_{-n+1/2}}{n-1/2}} \frac{1}{\sqrt{2}} (|1/16, 1/16 >_0 - |1/16, 1/16 >_{\pi R}).$$

The Cardy condition comes from the identification of the partition function of the closed string between two boundary states, with the one loop partition function of the open string.
with the boundary conditions corresponding to the boundary states. Here we are going to view the above cylinder as a strip of length $\beta$ and width $1/2$. The two ends corresponding to length 0 and $\beta$ are identified. The length corresponds to the open string time, while the width corresponds to the closed string time. In the open string picture the partition function (for a general CFT) takes the form

$$Z_{ab} = \sum_i n_{ab}^i \chi_i(q)$$  \hspace{1cm} (29)

where $n_{ab}^i$ counts how many copies of the channel $i$ exist in the open string partition function with boundary conditions $a$ and $b$. In the closed string picture we have

$$Z_{ab} = \sum_j <a|j><j|b> \chi_j(\tilde{q})$$  \hspace{1cm} (30)

for the boundary states corresponding to the open string boundary conditions. Note that the time in the open string picture corresponds to space in the closed string one, and this explains why in the closed string partition function we have $\tilde{q} = e^{-2\pi/\beta}$ instead of $q = e^{-2\pi \beta}$. Here $|j>$ is a complete set of states satisfying the extended algebra Ishibashi condition. They are normalized according to

$$<j|j'> = \delta_{jj'} S_{0j}$$  \hspace{1cm} (31)

and satisfy

$$<j'|\tilde{q}^{(L_0+\bar{L}_0-\frac{c}{2})}|j> = \delta_{jj'} \chi_j(\tilde{q}) \hspace{1cm} (32)$$

These are known to be in one to one correspondence with the characters of the theory $[8]$. The boundary states are constructed as linear combinations of these states.

When we identify (29) with (30) we get

$$\sum_i S_{ij}^0 n_{ab}^i = <a|j><j|b>$$  \hspace{1cm} (33)

where $S_{ij}$ is the S matrix for the characters. This is the Cardy condition. A solution to this condition, once one knows the set of characters and the complete basis of Ishibashi states, is obtained by using the Verlinde formula $[9]$. This formula tells us that

$$\sum_i S_{ij}^i N_{kl}^i = S_{kj} S_{ij}^l / S_{ij}^0$$  \hspace{1cm} (34)

where $N_{kl}^i$ represent the fusion rules. This suggests the following solution for the Cardy condition $[4]$: 

$$|l> = \sum_j \frac{S_{lj}^j}{(S_{ij}^j)^{1/2}} |j>,$$

$$|l'\rangle = \sum_j \frac{(S_{lj}^j)^*}{(S_{ij}^j)^{1/2}} |j>,$$

$$n_{k,l}^i = N_{kl}^i.$$
Note that among the boundary states there is a special state, the vacuum state
\[ |0> = \sum_j (S_0^j)^{1/2} |j> \]  
which satisfies
\[ |0^\vee> = |0>, \quad n_{00}^i = N_{00}^i = \delta_0^i, \quad n_{01}^i = N_{01}^i = \delta_1^i. \]
This means that
\[ Z_{00} = \chi_0(q), \quad Z_{01} = \chi_1(q). \]
A set of boundary states is consistent if it contains a vacuum boundary state satisfying (38) and for different combinations of boundary states the partition function is an integer sum of characters. At this point we should remark that if we knew what the states \( |j> \) were, the construction of the boundary states would be easy, since we could read the S matrix from the characters that appear in the right diagonalization of the torus partition function. But to construct these states in the way presented in [3], we need to understand the representation theory of the extended symmetry algebra. An alternative way to proceed is to construct directly the boundary states as (infinite) linear combinations of the free field solutions to the Virasoro Ishibashi condition. After all, this free field representation of the boundary states makes calculations a lot more tractable.

In the sequel we need the inner products of the Virasoro Ishibashi states for the orbifold. In the untwisted sector these are given by
\begin{align*}
< i_{n_R}^{UN} | e^{-H^U_\beta / 2} | i_{m_R}^{UN} > &= \frac{\tilde{q}^{\frac{s_R^2}{2}}}{\eta(q)} \delta_{n,m}, \\
< i_{n/2_R}^{UD} | e^{-H^U_\beta / 2} | i_{m/2_R}^{UD} > &= \frac{\tilde{q}^{\frac{s_{R^2}}{4}}}{\eta(q)} \delta_{n,m}, \\
< i_{n_R}^{UN} | e^{-H^U_\beta / 2} | i_{m_R}^{UD} > &= \tilde{q}^{-\frac{1}{2}} \prod_{l=1}^{\infty} \frac{1}{1 + q^l} \delta_{n,0} \delta_{m,0} = \frac{\sum_{l \in Z} (-1)^l q^{l^2}}{\eta(q)} \delta_{n,0} \delta_{m,0} = \chi_c(\tilde{q}) = \sqrt{2}(\chi_+(q) + \chi_-(q))
\end{align*}
while in the twisted one they are
\begin{align*}
< i_0^{TN} | e^{-H^T_\beta / 2} | i_0^{TN} > &= < i_0^{TN} | e^{-H^T_\beta / 2} | i_0^{TN} > = < i_{\pi/2_R}^{TN} | e^{-H^T_\beta / 2} | i_{\pi/2_R}^{TN} > = \tilde{q}^{\frac{1}{48}} \prod_{l=1}^{\infty} \frac{1}{1 - q^{l^2/2}} = \frac{1}{2} \sum_{l \in Z} q^{l^2} \frac{e^{\frac{1}{2}l(l-1)/2}}{\eta(q)} \\
&= \sqrt{2} \chi_a(\tilde{q}) = \frac{1}{\sqrt{2}}(\chi_+(q) - \chi_-(q)), \\
< i_0^{TD} | e^{-H^T_\beta / 2} | i_0^{TN} > &= < i_0^{TD} | e^{-H^T_\beta / 2} | i_0^{TN} > = < i_{\pi/2_R}^{TD} | e^{-H^T_\beta / 2} | i_{\pi/2_R}^{TN} > = \frac{1}{2} \sum_{l \in Z} q^{l^2} \frac{e^{\frac{1}{2}l(l-1)/2}}{\eta(q)} \\
&= \chi_b(\tilde{q}) = \frac{1}{\sqrt{2}}(\chi_+(q) - \chi_-(q)), \\
< i_{\pi_R}^{TD} | e^{-H^T_\beta / 2} | i_{\pi_R}^{TN} > &= -\frac{1}{2} \sum_{l \in Z} q^{l^2} \frac{e^{\frac{1}{2}l(l-1)/2}}{\eta(q)} = -\frac{1}{\sqrt{2}}(\chi_+(q) - \chi_-(q)).
\end{align*}
Here we have used the identity
\[
\sum_{n \in \mathbb{Z}} \frac{\tilde{q}^{A(n+b)^2}}{\eta(\tilde{q})} = \frac{1}{\sqrt{2A}} \sum_{n \in \mathbb{Z}} \frac{q^{n^2} e^{2\pi inb}}{\eta(q)}
\]
which comes from the Poisson resummation formula.

The next step is to search for the vacuum boundary state in the case \( R^2 = 1/2k \). This has to be a linear combination of the twisted and the untwisted Virasoro Ishibashi states and it has to give \( Z_{00}(q) = \chi_+(q) \). The reason is that the character \( \chi_+(q) \) is the only one that admits the interpretation of a vacuum character since it has the right conformal dimension (while the character \( \chi^-(q) \) has conformal dimension 1). Using the inner products (39), (40) it is not very difficult to construct such a state:
\[
|X_0^{N+}\rangle = \frac{1}{\sqrt{2\sqrt{2}k}} \sum_{n \in \mathbb{Z}} |i_{UN}^{\sqrt{2k}}\rangle + \frac{1}{\sqrt{2}} |i_{TN}^{\sqrt{k}}\rangle.
\]  

Next we need to construct boundary states whose partition function with the vacuum boundary state gives all the characters. A set of such states is
\[
|X_0^{N-}\rangle = \frac{1}{\sqrt{2\sqrt{2}k}} \sum_{n \in \mathbb{Z}} |i_{UN}^{\sqrt{2k}}\rangle - \frac{1}{\sqrt{2}} |i_{TN}^{\sqrt{k}}\rangle,
\]
\[
|X_m^N\rangle = \frac{\sqrt{2}}{\sqrt{2}k^{2k-1}} \sum_{l=0}^{2k-1} e^{2\pi i ml} \sum_{n \in \mathbb{Z}} |i_{UN}^{\sqrt{2k(n+\frac{l}{m})}}\rangle, \quad 1 \leq m \leq k - 1
\]
\[
|X_k^{N\pm}\rangle = \frac{1}{\sqrt{2\sqrt{2}k}} \sum_{n \in \mathbb{Z}} (-1)^n |i_{UN}^{\sqrt{2k(n+\frac{k}{m})}}\rangle \pm \frac{1}{\sqrt{2}} |i_{TN}^{\sqrt{k}}\rangle,
\]
\[
|X_0^{D\pm}\rangle = \frac{1}{\sqrt{2\sqrt{2}k}} \sum_{n \in \mathbb{Z}} |i_{UD}^{\sqrt{2}n\sqrt{3}}\rangle \pm \frac{1}{\sqrt{2}} |i_{TD}^{\sqrt{k}}\rangle,
\]
\[
|X_k^{D\pm}\rangle = \frac{1}{\sqrt{2\sqrt{2}k}} \sum_{n \in \mathbb{Z}} (-1)^n |i_{UD}^{\sqrt{2}n\sqrt{3}}\rangle \pm \frac{1}{\sqrt{2}} |i_{TR}^{\sqrt{k}}\rangle.
\]

These give that
\[
Z_{00}^{N+,N+}(q) = Z_{00}(q) = \chi_+(q),
\]
\[
Z_{00}^{N+,N-}(q) = \chi^-(q),
\]
\[
Z_{0m}^{N+,N}(q) = \chi_m(q), \quad 1 \leq m \leq k - 1
\]
\[
Z_{0k}^{N+,N+}(q) = Z_{0k}^{N+,N-}(q) = \chi_+(q),
\]
\[
Z_{0k}^{N+,D+}(q) = Z_{0k}^{N+,D-}(q) = \chi+(q),
\]
\[
Z_{0k}^{N+,D-}(q) = Z_{0k}^{N+,D+}(q) = \chi-(q).
\]

Of course there is no a priori guarantee that all other partition functions are going to be integer combinations of characters, but there is not much choice left for the states. In fact the full set of partition functions, each of which is an integer sum of characters is contained in Appendix B. Note that we were forced (by the twisted sector) to include Dirichlet boundary states at the orbifold fixed points which correspond to \( m = 0, m = k \), while in the case of the torus models the space of boundary states contained pure Neumann states. This is because the orbifold fixed points correspond to the possible eigenvalues of the position operator in the expansion of the free field.
4 Module Decomposition

In this section we study the extended algebra module decomposition of the space of states for the $c = 1$ $Z_2$ orbifold with $R^2 = 1/2k$.

The vacuum character: The first question is what the extended symmetry algebra is. One way to get this algebra is to study the vacuum character. We are going to find which subspace $H_0$ of the known overall space of states $H^U$ gives

$$\text{Tr}_{H_0}(q^{-L_{0,0} - 1/24} q^{L_{0,0} - 1/24}) = \chi^*_+(q) \chi^*_-(q)$$

$$(45)$$

Next we are going to determine a set of operators that generates $H_0$ from the vacuum state. The algebra of these operators is the extended symmetry algebra.

To this end we need the following lemma [3]:

Lemma 2

$$q^{-1/12} \text{Tr}_{H^U_{mn}}(q^{L_{0,0}} q^{L_{0,0}}) = \frac{1}{\eta^2(q)} q^{\frac{1}{2}((m - \frac{c}{24})^2 + \frac{1}{2} \frac{q^{-1/12} \delta_{mn} \delta_{0,0}}{2 \prod_{n=1}^{\infty} (1 + q^n)^2}}$$

$$(46)$$

The traces that can contribute to the vacuum character are the ones for which $m = \rho$ and $n = k\rho'$ where $\rho - \rho' = 0 \mod 2$. Otherwise we get powers of $q$ that correspond to noninteger conformal dimension. If $\rho - \rho' = 2s'$ and $\rho + \rho' = 2s$ then we get that the following traces can contribute

$$q^{-1/12} \text{Tr}_{H_{s,s',k(s-s')}}(q^{L_{0,0}} q^{L_{0,0}}) = \frac{1}{\eta^2(q)} q^{ks^2} q^{ks'^2} + \frac{1}{2 \prod_{n=1}^{\infty} (1 + q^n)^2}$$

$$(47)$$

where $s, s' \in \mathbb{Z}$. Of course we really need half of these spaces because there is also the product $\chi^*_+(q) \chi^*_-(q)$ that should be generated by traces of the same $H^U_{mn}$. Thus we define the following subspaces of $H^U_{s,s',k(s-s')}$, $H^U_{s,ks}$, $H^U_{s',-ks'}$ (with $s, s' > 0$), and of $H^U_{0,0}$:

$$H^U_{s,s',k(s-s')} = \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j+1}} \cdots \bar{\alpha}_{-n_{2l}} (|s + s', k(s - s') >$$

$$(48)$$

$+ | - (s + s'), -k(s - s') > + |s - s', k(s + s') > + | - (s - s'), -k(s + s') > \}$$

$+ \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j}} \cdots \bar{\alpha}_{-n_{2l+1}} (|s + s', k(s - s') >$

$- | - (s + s'), -k(s - s') > - |s - s', k(s + s') > - | - (s - s'), -k(s + s') > \}$$

$+ \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j}} \cdots \bar{\alpha}_{-n_{2l+1}} (|s + s', k(s - s') >$

$- | - (s + s'), -k(s - s') > - |s - s', k(s + s') > + | - (s - s'), -k(s + s') > \},$

$$H^U_{s,ks} = \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j+1}} \cdots \bar{\alpha}_{-n_{2l}} (|s, ks > + | - s, -ks > \}$$

$+ \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j}} \cdots \bar{\alpha}_{-n_{2l+1}} (|s, ks > - | - s, -ks > \},$

$$H^U_{s',-ks'} = \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j+1}} \cdots \bar{\alpha}_{-n_{2l}} (|s', -ks' > + | - s', ks' > \}$$

$+ \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j+1}} \cdots \bar{\alpha}_{-n_{2l+1}} (|s', -ks' > - | - s', ks' > \},$

$$H^U_{0,0} = \{ \alpha_{n_1} \cdots \alpha_{n_{2j-1}} \bar{\alpha}_{-n_{2j+1}} \cdots \bar{\alpha}_{-n_{2l}} |0, 0 > \}.$$

The traces over the subspaces (48) are summarized in the following lemma:
Lemma 3

\[ q^{-1/12} \text{Tr}_{H_{s+s',k(s-s')}}(q^{L_0}q^{\bar{L}_0}) = \frac{1}{\eta^2(q)} q^{ks_2}q^{\bar{s}_2^{s'}}, \quad s, s' > 0 \]  

(49)

\[ q^{-1/12} \text{Tr}_{H_{s,k(s)}}(q^{L_0}q^{\bar{L}_0}) = \frac{1}{\eta^2(q)} q^{ks_2} \left( \sum_{n \geq 0} q^{n^2(-1)^n} \right), \quad s > 0 \]

\[ q^{-1/12} \text{Tr}_{H_{s',-k(s')}}(q^{L_0}q^{\bar{L}_0}) = \frac{1}{\eta^2(q)} \left( \sum_{n \geq 0} q^{n^2(-1)^n} \right) q^{ks_2}, \quad s' > 0 \]

\[ q^{-1/12} \text{Tr}_{H_{0,0}^v}(q^{L_0}q^{\bar{L}_0}) = \frac{1}{\eta^2(q)} \left( \sum_{n \geq 0} q^{n^2(-1)^n} \right)^2 \]  

For the proof of this lemma see Appendix C.

Now we are in a position to define our space \( H_0 \):

\[ H_0 = \sum_{s, s' > 0} H_{s+s',k(s-s')}^v \oplus \sum_{s > 0} H_{s,k(s)}^v \oplus \sum_{s' > 0} H_{s',-k(s')}^v \oplus H_{0,0}^v. \]  

(50)

The trace over \( H_0 \) is

\[ q^{-1/12} \text{Tr}_{H_0}(q^{L_0}q^{\bar{L}_0}) = \frac{1}{\eta^2(q)} \left[ \left( \sum_{s > 0} q^{ks_2} \right) \left( \sum_{s' > 0} q^{\bar{s}_2^{s'}} \right) + \left( \sum_{s > 0} q^{s_2} \right) \left( \sum_{n \geq 0} q^{n^2(-1)^n} \right) \right] \]

\[ + \left( \sum_{n \geq 0} q^{n^2(-1)^n} \right) \left( \sum_{s' > 0} q^{\bar{s}_2^{s'}} \right) + \left( \sum_{n \geq 0} q^{n^2(-1)^n} \right)^2 \]

\[ = \chi^+_{s}(q)\chi^+_{s'}(q). \]  

Having identified the vacuum module, it is interesting to try to read what the symmetry algebra is. Recall that in the unorbifolded theory we had three generators of the algebra \( \mathfrak{g} \), \( J^0(z) = \frac{1}{\sqrt{2k}} \partial \phi(z) \) and \( J^\pm(z) = e^{\pm i\sqrt{2k}\phi(z)} \). Now we have to select elements of the algebra generated by the modes that remain invariant under the \( Z_2 \) symmetry. Such elements are the following:

\[ S = \{ a_n, a_n^\dagger, J^+_k, J^-_k, a_n(J^+_k - J^-_k), \bar{a}_n, \bar{a}_n^\dagger, \bar{J}^+_k, \bar{J}^-_k, \bar{a}_n(\bar{J}^+_k - \bar{J}^-_k) \}. \]  

(52)

Of course these are not the only ones. Now we have the following proposition:

**Proposition 1** Starting from a vacuum \(|0,0>\), \( S \) generates precisely the module \( H_0 \).

For the proof of this proposition see Appendix D. So the symmetry algebra of the theory is the algebra generated by the elements of \( S \).

The character \( \chi^+(q) \): The space \( H_1 \) that generates the product \( \chi^+(q)\chi^+(q) \) can be constructed similarly. Define

\[ H_{s+s',k(s-s')}^{U_b} = \{ \alpha_{-n_1} \cdots \alpha_{-n_{2j-1}} \bar{\alpha}_{n_{2j}} \cdots \bar{\alpha}_{-n_{2l}}(|s+s', k(s-s') > \]

\[ + | - (s+s'), -k(s-s') > + | s-s', k(s+s') > + | - (s-s'), -k(s+s') > \}

\[ \oplus \{ \alpha_{-n_1} \cdots \alpha_{-n_{2j}} \bar{\alpha}_{n_{2j+1}} \cdots \bar{\alpha}_{-n_{2l}}(|s+s', k(s-s') > \]

11
The proof of this lemma is similar to the proof of lemma 3. So finally we have for the trace

| Proposition 2 |

Now we have the following proposition:

\[ H^*_1 = \sum_{s,s'>0} H^*_s H^*_{s',k(s-s')} + \sum_{s>0} H^*_s H^*_{s',k(s-s')} + \sum_{s'>0} H^*_s H^*_{s',k(s-s')} + H^*_{0,0}. \] (54)

The associated traces are computed in the following lemma:

**Lemma 4**

\[ q^{-1/12}Tr_{H^*_s} (q^{L_0}q^{L_0}) = \frac{1}{\eta(q)} q^{ks^2} q^{ks'q^2}, \quad s, s' > 0 \] (55)

\[ q^{-1/12}Tr_{H^*_s} (q^{L_0}q^{L_0}) = \frac{1}{\eta(q)} q^{ks^2} \left( \sum_{n \geq 1} q^{n^2(-1)^{n-1}} \right), \quad s > 0 \]

\[ q^{-1/12}Tr_{H^*_s} (q^{L_0}q^{L_0}) = \frac{1}{\eta(q)} \left( \sum_{n \geq 1} q^{n^2(-1)^{n-1}} \right) q^{ks^2}, \quad s' > 0 \]

\[ q^{-1/12}Tr_{H^*_0} (q^{L_0}q^{L_0}) = \frac{1}{\eta(q)} \left( \sum_{n \geq 1} q^{n^2(-1)^{n-1}} \right) \left( \sum_{n \geq 1} q^{n^2(-1)^{n-1}} \right). \]

The proof of this lemma is similar to the proof of lemma 3. So finally we have for the trace over \( H_1 \)

\[ q^{-1/12}Tr_{H_1} (q^{L_0}q^{L_0}) = \frac{1}{\eta(q)} \left( \sum_{s>0} q^{ks^2} \right) \left( \sum_{s'>0} q^{ks'q^2} \right) + \left( \sum_{s>0} q^{ks^2} \right) \left( \sum_{n \geq 1} q^{n^2(-1)^{n-1}} \right) \]

\[ + \left( \sum_{n \geq 1} q^{n^2(-1)^{n-1}} \right) \left( \sum_{s'>0} q^{ks'q^2} \right) + \left( \sum_{n \geq 1} q^{n^2(-1)^{n-1}} \right) \left( \sum_{s>0} q^{ks^2} \right) \]

\[ = \chi^*_s(q) \chi^*_s(q). \] (56)

Now we have the following proposition:

**Proposition 2** Starting from the state \( a_{-1}a_{-1}|0,0> \), \( S \) generates precisely the module \( H_1 \).
Next we ask what the direct sum \( H_0 \oplus H_1 \) is. To find this we need the identities

\[
\begin{align*}
H_{s+s', k(s-s')}^U \oplus H_{s+s', k(s-s')}^U &= H_{s+s', k(s-s')}^U \oplus H_{s-s', k(s+s')}^U, \quad s, s' > 0 \\
H_{s, k}^U \oplus H_{s, k}^U &= H_{s, k}^U, \quad s > 0 \\
H_{s', -k s'}^U \oplus H_{s', -k s'}^U &= H_{s', -k s'}^U, \quad s' > 0 \\
H_{0, 0}^U \oplus H_{0, 0}^U &= H_{0, 0}^U
\end{align*}
\]

where it is understood that \( H_{s-s', k(s+s')}^U = H_{s-s', -k(s+s')}^U \) if \( s-s' < 0 \). Summing over \( s, s' \geq 0 \) we get

\[
H_0 \oplus H_1 = \bigoplus_{s, s' \geq 0} (H_{s+s', k(s-s')}^U \oplus H_{s-s', k(s+s')}^U) \oplus_{s > 0} H_{s, k}^U \oplus_{s' > 0} H_{s', -k s'}^U \oplus H_{0, 0}^U
\]

(58)

where the condition \( s + s' > 0 \) corresponds to the condition \( m > 0 \) (if \( m = 0, n \geq 0 \)) in the original Hilbert space.

The characters \( \chi_l(q) \): The next question is what module \( H_{l^2/4k} \) gives rise to the product \( \chi_l(q) \chi_l(q) \) for \( 0 < l < k \). The spaces that can contribute to this product have to produce the correct powers of \( q \) in the trace. This requires

\[
\frac{1}{2} \left( \frac{m}{2} + nR \right) \frac{(mk + n)^2}{4k} = \frac{l^2}{4k} \mod 2.
\]

(59)

Equation (59) is satisfied provided that \( m = \rho \) and \( n = l + k \rho' \) where \( \rho - \rho' = 0 \mod 2 \). Defining again \( 2s = \rho + \rho' \) and \( 2s' = \rho - \rho' \) we get that only the modules \( H_{s+s', k(s-s')}^U \) can contribute. It is natural to guess that \( H_{l^2/4k} = \bigoplus_{s, s' \in Z} H_{s+s', k(s-s')}^U \). The modules \( H_{s+s', k(s-s')}^U \) with \( s + s' < 0 \) make sense through the identity \( H_{s+s', k(s-s')}^U = H_{s+s', (s-s')+1}^U = H_{-(s+s'), -k(s-s')-2+2k-l}^U \). So now we have

\[
H_{l^2/4k} = \bigoplus_{s, s' \geq 0} H_{s+s', k(s-s')}^U \bigoplus_{s > 0} H_{s, k}^U \bigoplus_{s' > 0} H_{s', -k s'}^U
\]

(60)

Summing over \( l \) for \( 1 \leq l \leq k - 1 \) we get

\[
\bigoplus_{l=1}^{k-1} H_{l^2/4k} = \bigoplus_{l=1}^{k-1} \bigoplus_{s, s' \geq 0} H_{s+s', k(s-s')}^U \bigoplus_{s > 0} H_{s, k}^U \bigoplus_{s' > 0} H_{s', -k s'}^U
\]

(61)

Now we have the following lemma:

**Lemma 5**

\[
q^{-1/12} Tr_{H_{l^2/4k}} (q^{L_0} q^{L_0}) = q^{-1/12} \sum_{s, s' \in Z} Tr_{H_{s+s', k(s-s')}^U} (q^{L_0} q^{L_0}) = \chi_l(q) \chi_l(q).
\]

(62)
The traces over these spaces are computed in the following lemma:

\[ q^{-1/12} Tr_{H_{s+s',k(s-s')}^U} (q^{L_0} q^{L_0}) = \frac{1}{\eta^2(q)} q^{k(s+1/2k)^2} q^{k(s'-l/2k)^2}, \quad s, s' \in Z \]  

(63)

which comes from lemma 2. Now we have the following proposition:

**Proposition 3** Starting from the state \( |0, l > + |0, -l > \), \( S \) generates precisely the module \( H_{l/4} \).

The character \( \chi^*_k(q) \): Now we need the modules \( H_{k/4}^v, H_{k/4}^b \) that give rise to the product \( 2\chi^*_k(q) \). The modules that can contribute to this product are \( H_{\rho,k\rho'}^U \) where \( \rho - \rho' = 1 \mod 2 \). So we introduce the following spaces:

\[
H_{s+s'+1,k(s-s')}^U = \{ \alpha_{-n_1} \cdots \alpha_{-n_2} \cdots \alpha_{-n_{2j+1}} \cdots \alpha_{-n_{2k}} (|s+s'+1, k(s-s') > \\
+ |(-s+s'+1), -k(s-s') > + |s-s', k(s+s'+1) > + |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > - |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > + |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > - |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > + |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > - |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >). \\
\]  

(64)

\[
H_{s+s'+1,k(s-s')}^b = \{ \alpha_{-n_1} \cdots \alpha_{-n_2} \cdots \alpha_{-n_{2j+1}} \cdots \alpha_{-n_{2k}} (|s+s'+1, k(s-s') > \\
+ |(-s+s'+1), -k(s-s') > + |s-s', k(s+s'+1) > + |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > - |s-s', k(s+s'+1) > + |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > + |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > - |s-s', k(s+s'+1) > + |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > + |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >) \\
+ |(-s+s'+1), -k(s-s') > - |s-s', k(s+s'+1) > - |-(s-s'), -k(s+s'+1) >). \\
\]  

(65)

These definitions lead to

\[
H_{k/4}^v = \bigoplus_{s, s' \geq 0} H_{s+s'+1,k(s-s')}^U, \\
H_{k/4}^b = \bigoplus_{s, s' \geq 0} H_{s+s'+1,k(s-s')}^b. 
\]  

The traces over these spaces are computed in the following lemma:

**Lemma 6**

\[
q^{-1/12} Tr_{H_{k/4}^v} (q^{L_0} q^{L_0}) = q^{-1/12} Tr_{H_{k/4}^b} (q^{L_0} q^{L_0}) = \chi^*_k(q) \chi^*_k(q). 
\]  

(66)

Note that

\[
\bigoplus_{s, s' \geq 0} H_{s+s'+1,k(s-s')}^U \oplus H_{s+s'+1,k(s-s')}^b = \bigoplus_{s, s' \geq 0} H_{0,k(2s+1)}. 
\]  

(67)

Furthermore we have the additional proposition:

**Proposition 4** Starting from the states \( |1, 0 > + |-1, 0 > + |0, k > + |0, -k > \), \( S \) generates precisely the modules \( H_{l/4}^v \) and \( H_{l/4}^b \).
So we have identified completely the extended symmetry algebra to be the algebra of products of elements of S. The generating functions that give rise to the elements of S are nonlocal as expected, since the orbifold symmetry is nonlocal, and are:

\[ J^0(z)J^0(w), \quad J^+(z) + J^-(z), \quad J^0(z)(J^+(w) - J^-(w)) \quad \text{and c. c.} \quad (68) \]

Again this algebra splits into a chiral and an antichiral part. Since

\[ H_0 \oplus H_1 \oplus \bigoplus_{l=1}^{k-1} H_{l^2/4k} \oplus H_{l^2/4} \oplus H_{l^2/4} = H^U \quad (69) \]

we come to the conclusion that the periodic part of the Hilbert space of the theory splits into a sum of modules of the extended symmetry algebra, each module giving rise to a product of extended characters that appear in the partition function of the theory.

The characters \( \chi_+(q), \chi_-(q) \): Now consider the antiperiodic sector. Upon changing the boundary conditions the only elements of S that survive are \( a_{n_1}a_{n_2} \) and \( \bar{a}_{n_1}\bar{a}_{n_2} \) where now \( n_1 \) and \( n_2 \) become half integers. The currents disappear since in the antiperiodic sector there is no charge changing operator. Now we define the modules

\[ H_{0,\pi R}^{Te} = \alpha_{-n_1} \cdots \alpha_{-n_{2j}} \bar{\alpha}_{-n_{2j+1}} \cdots \bar{\alpha}_{-n_{2l}} |1/16, 1/16 >_{0,\pi R}, \]

\[ H_{0,\pi R}^{To} = \alpha_{-n_1} \cdots \alpha_{-n_{2j-1}} \bar{\alpha}_{-n_{2j}} \cdots \bar{\alpha}_{-n_{2l}} |1/16, 1/16 >_{0,\pi R}. \quad (70) \]

We have the following lemma:

**Lemma 7**

\[ q^{-1/12} \text{Tr}_{H_{0,\pi R}^{Te}}(q^{L_0}q^{L_0}) = \chi_+(q)\chi_+(q) \quad (71) \]

\[ q^{-1/12} \text{Tr}_{H_{0,\pi R}^{To}}(q^{L_0}q^{L_0}) = \chi_-(q)\chi_-(q) \]

Finally we have the proposition:

**Proposition 5** Starting from the states \( |1/16, 1/16 >_{0,\pi R} \) and \( a_{-1/2}a_{-1/2} |1/16, 1/16 > \), the surviving elements of S generate precisely the modules \( H_{0,\pi R}^{Te} \) and \( H_{0,\pi R}^{To} \).

It is worth mentioning that the new definitions of modules simply give another decomposition of the original Hilbert space into modules of the orbifold algebra. So the direct sum of the modules that give rise to the products of characters gives the original Hilbert space.

## 5 Conclusions

We studied the boundary state structure of the \( Z_2 \) orbifold models with \( R^2 = 1/2k \). We found a complete set of boundary states that satisfies both the Virasoro Ishibashi condition and the Cardy condition. It is interesting to emphasize that in the case of the \( R^2 = 1/2k \) torus models all the boundary states were Newmann states while in the orbifold case we were forced to introduce four Dirichlet states, one pair corresponding to each orbifold fixed point. Another point of interest is that the decomposition of the extended algebra characters into a sum of Virasoro characters changes nature in the case \( k = l^2 \). The models with this k are identified to be the \( D_l \) dihedral group orbifolds of the \( SU(2) \times SU(2) \) level one theory, which corresponds
to the $k = 1$ ($R = 1/\sqrt{2}$) torus model. Note that the T-duality $R \rightarrow 1/2R$ just exchange Newmann and Dirichlet states preserving the essential structure of the theory.

Finally, the space of states of the theory admits a decomposition into highest weight modules of the extended symmetry algebra. It is worth noting that two of the generating functions of our set of generators are nonlocal fields (they are bilocal), but they respect chirality. This is of course expected, since the $Z_2$ symmetry is a global symmetry of the theory.

Appendix A

The elliptic theta functions and the Dedekind eta function are:

$$\theta_1(w, q) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} w^{n+\frac{1}{2}},$$  
(A.1)

$$\theta_2(w, q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} w^n,$$

$$\theta_3(w, q) = \sum_{n=-\infty}^{\infty} q^{n^2} w^n,$$

$$\theta_4(w, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} w^n,$$

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Proof of Lemma 1: The characters of the Virasoro algebra with $c = 1$ are

$$\chi_{h}^{Vir}(q) = \frac{q^h}{\eta(q)}, \quad h \neq n^2/4$$  
(A.2)

$$\chi_{n^2}^{Vir}(q) = \frac{q^{n^2} - q^{(n+1)^2}}{\eta(q)},$$

$$\chi_{(n+1/2)^2}^{Vir}(q) = \frac{q^{(n+1/2)^2} - q^{n^2}}{\eta(q)}.$$
where the splitting of the components into a sum of Virasoro characters is exhibited. This splitting is no longer valid for all components if \( k = l^2 \). This is justified by the fact that some powers of \( q \) are of the form \( q^{n^2/4} \) and \( q^{n^2/4}/\eta(q) \) is not a Virasoro character. The characters that contain such powers are \( \chi^*_\pm(q) \), \( \chi^*_k(q) \) and \( \chi_s(q) \) for \( s = pl, 1 \leq p \leq l - 1 \). To deduce the splitting in this case we consider the sum

\[
\sum_{n \geq 0} \frac{q^{k(n+s/2k)^2}}{\eta(q)} = \sum_{n \geq 0} \frac{q^{(2n+l+p)^2/4}}{\eta(q)}
\]

(A.4)

where

\[
1/\eta(q) \left((q^{p^2/4} - q^{(p+2)^2/4}) + (q^{(p+2)^2/4} - q^{(p+4)^2/4}) + \ldots + (q^{(p+2l-2)^2/4} - q^{(p+2l)^2/4})
\]

+ \(2(q^{(p+2l)^2/4} - q^{(p+2l+2)^2/4}) + \ldots + 2(q^{(p+4l-2)^2/4} - q^{(p+4l)^2/4})
\]

+ \(3(q^{(p+4l)^2/4} - q^{(p+4l+2)^2/4}) + \ldots + 3(q^{(p+6l-2)^2/4} - q^{(p+6l)^2/4})
\]

+ \(\ldots\)

= \(\sum_{n=0}^{l-1} \sum_{m=0}^{l-1} (n+1) \chi^*_{(p/2+nl+m)^2}(q)\).

Note that

\[
\chi_{pl}(q) = \sum_{n \geq 0} \frac{q^{(2n+l+p)^2/4}}{\eta(q)} + \sum_{n \geq 0} \frac{q^{(2n+l+2l-p)^2/4}}{\eta(q)}.
\]

(A.5)

Now the splitting of the problematic components takes the form

\[
\chi^*_+(q) = \sum_{n=1}^{l-1} \sum_{m=0}^{l-1} n \chi^*_{(nl+m)^2}(q) + \sum_{n=0}^{\infty} \chi^*_{(2n)^2}(q),
\]

(A.6)

\[
\chi^*_-(q) = \sum_{n=1}^{l-1} \sum_{m=0}^{l-1} n \chi^*_{(nl+m)^2}(q) + \sum_{n=0}^{\infty} \chi^*_{(2n+1)^2}(q),
\]

\[
\chi^*_k(q) = \sum_{n=0}^{l-1} \sum_{m=0}^{l-1} (n+1) \chi^*_{(l/2+nl+m)^2}(q),
\]

\[
\chi_s(q) = \sum_{n=0}^{l-1} \sum_{m=0}^{l-1} (n+1) \chi^*_{(p/2+nl+m)^2}(q) + \sum_{n=1}^{l-1} \sum_{m=0}^{l-1} n \chi^*_{(p/2+nl+m)^2}(q),
\]

while the rest of the components preserves its splitting.

**Appendix B**

The partition functions corresponding to combinations of the boundary states that do not involve the vacuum boundary state are:

\[
Z_{00}^{N-,N-}(q) = \chi^*_+(q),
\]

(B.1)

\[
Z_{0m}^{N-,N-}(q) = \chi_m(q),
\]

\[
Z_{0k}^{N-,N\pm}(q) = \chi^*_k(q),
\]

\[
Z_{00}^{N-,D\pm}(q) = \chi^*_\pm(q),
\]
Proof of Lemma 3: To prove the first equation we need to realize that all the states
\[ \pm |s+s', k(s-s') > \pm | -(s+s'), -k(s-s') > \pm |s-s', k(s+s') > \pm | -(s+s'), -k(s-s') > \]
have the same \( L_0 \) and \( \bar{L}_0 \) eigenvalues, \( k s^2 \) and \( k s'^2 \), and so
\[ q^{-1/12} Tr_{H^{u,v}_{s+s',k(s-s')}}(q^{L_0}q^{\bar{L}_0}) = q^{-1/12} Tr_{H^{\bar{u},\bar{v}}_{s,s'}}(q^{L_0}q^{\bar{L}_0}) \] (C.2)
where
\[ H^{\bar{u}}_{s,s'} = a_{-n_1} \cdots a_{-n_j} \bar{a}_{-n_{j+1}} \cdots \bar{a}_{-n_l} |v_{s,s'} > \] (C.3)
and \( L_0 |v_{s,s'}\rangle = k s^2 |v_{s,s'}\rangle \). Now we have that

\[
q^{-1/12} \text{Tr}_{H^0_{s,s'}}(q^{L_0} q^{L_0}) = \left( q^{-1/24} \text{Tr}_{H^0_{s,s'}}(q^{L_0}) \right) \left( q^{-1/24} \text{Tr}_{H^0_{s,s'}}(q^{L_0}) \right)
\]

(C.4)

where

\[
H^0_{s,s'} = a_{-n_1} \cdots a_{-n_j} |v_{s,s'}\rangle
\]

and \( H^0_{s,s'} \) is the conjugate of \( H^0_{s,s'} \). But

\[
q^{-1/24} \text{Tr}_{H^0_{s,s'}}(q^{L_0}) = q^{-1/24} q^{ks^2} \sum_{n=0}^{\infty} P(n) q^n = \frac{q^{ks^2}}{\eta(q)}
\]

(C.6)

where \( P(n) \) is the number of partitions of \( n \). So we finally have

\[
q^{-1/12} \text{Tr}_{H^0_{s,s'}}(q^{L_0} q^{L_0}) = \frac{q^{ks^2}}{\eta(q)} \frac{q^{ks^2}}{\eta(q)}.
\]

(C.7)

To prove the remaining equations, we need to introduce the projection operators \( (1 + g_c)/2 \), \( (1 - g_c)/2 \), \( (1 + g_a)/2 \) and \( (1 - g_a)/2 \). Here \( g_c \) maps \( a_n \) to \( -a_n \) and \( g_a \) maps \( \bar{a}_n \) to \( -\bar{a}_n \). The traces in the presence of the projection operators become:

\[
q^{-1/24} \text{Tr}_{H^0_{s,0}} \left( \frac{1 + g_a}{2} q^{L_0} \right) = \frac{1}{2} q^{-1/24} \text{Tr}_{H^0_{s,0}} \left( q^{L_0} \right) + \frac{1}{2} q^{-1/24} \text{Tr}_{H^0_{s,0}} \left( g_a q^{L_0} \right)
\]

(C.8)

\[
= \frac{1}{2} q^{-1/24} \sum_{n=0}^{\infty} P(n) q^n + \frac{1}{2} q^{-1/24} \prod_{n=1}^{\infty} (1 + q^n) = \frac{1}{2} \frac{\eta(q) \sum_{n \geq 1} (-1)^n q^{n^2}}{\eta(q)}
\]

Similarly we get

\[
q^{-1/24} \text{Tr}_{H^0_{s,0}} \left( \frac{1 - g_a}{2} q^{L_0} \right) = \sum_{n \geq 1} \frac{(-1)^{n-1} q^{n^2}}{\eta(q)}
\]

(C.9)

and the conjugate formulas.

Now we have that

\[
q^{-1/12} \text{Tr}_{H^{0c}_{s,k}} \left( q^{L_0} q^{L_0} \right) = q^{-1/24} \text{Tr}_{H^{0c}_{s,0}} \left( q^{L_0} \right) \cdot q^{-1/24} \text{Tr}_{H^{0c}_{s,0}} \left( \frac{1 + g_a}{2} q^{L_0} \right)
\]

(C.10)

\[
= \frac{q^{ks^2}}{\eta(q)} \sum_{n \geq 0} q^{n^2} (-1)^n \eta(q)
\]

\[
q^{-1/12} \text{Tr}_{H^{0c}_{s',-k}} \left( q^{L_0} q^{L_0} \right) = q^{-1/24} \text{Tr}_{H^{0c}_{s,0}} \left( \frac{1 + g_a}{2} q^{L_0} \right) \cdot q^{-1/24} \text{Tr}_{H^{0c}_{s,0}} \left( q^{L_0} \right)
\]

(C.11)

\[
= \sum_{n \geq 0} \frac{q^{n^2} (-1)^n q^{ks^2}}{\eta(q)} \eta(q)
\]
\[ q^{-1/12} \text{Tr}_{H_{0,0}}(q^{L_0} q^{\hat{L}_0}) = q^{-1/24} \text{Tr}_{H_{0,0}} \left( \frac{1 + g_c}{2} q^{L_0} \right) \cdot q^{-1/24} \text{Tr}_{H_{0,0}} \left( \frac{1 + g_a}{2} q^{\hat{L}_0} \right) \] (C.12)

\[
= \sum_{n \geq 0} q^{n^2} (-1)^n \sum_{n \geq 0} q^{n^2} (-1)^n \eta(q).
\]

The proof of the rest of the lemmas proceeds similarly.

**Appendix D**

To proceed with the proofs of the propositions we need first to derive the way the currents act on the lowest level states in our modules. It is convenient to change the notation of our states in order to keep track of the charge. So we rename the states as follows:

\[
|s + s', k(s - s') + l \rangle \rightarrow |s\sqrt{2k} + \frac{l}{\sqrt{2k}}, s'\sqrt{2k} - \frac{l}{\sqrt{2k}} \rangle.
\] (D.1)

The free field representation of the currents gives

\[
\sum_{n \in \mathbb{Z}} J_{-n} z^{-n} \langle s\sqrt{2k} + \frac{l}{\sqrt{2k}}, s'\sqrt{2k} - \frac{l}{\sqrt{2k}} | (s \pm 1)\sqrt{2k} + \frac{l}{\sqrt{2k}}, s'\sqrt{2k} - \frac{l}{\sqrt{2k}} \rangle >, (D.2)
\]

Equating the lowest powers of \(z\) we get

\[
J_{-(2s+1)k-l}^+ |s\sqrt{2k} + \frac{l}{\sqrt{2k}}, s'\sqrt{2k} - \frac{l}{\sqrt{2k}} \rangle = |(s + 1)\sqrt{2k} + \frac{l}{\sqrt{2k}}, s'\sqrt{2k} - \frac{l}{\sqrt{2k}} \rangle, (D.3)
\]

\[
J_{-(2s-1)k+l}^- |s\sqrt{2k} - \frac{l}{\sqrt{2k}}, s'\sqrt{2k} + \frac{l}{\sqrt{2k}} \rangle = |-(s + 1)\sqrt{2k} + \frac{l}{\sqrt{2k}}, s'\sqrt{2k} + \frac{l}{\sqrt{2k}} \rangle, (D.4)
\]

while for the other powers of \(z\) we need to expand the exponentials in powers of \(z\). This can be achieved conveniently using the Schur polynomials which are defined by the relation

\[
e^\sum_{n=1}^\infty t_n z^n = \sum_{N=0}^\infty z^N S_N(t_1, t_2, \cdots, t_N) \] (D.4)

and turn out to be

\[
S_N(t_1, t_2, \cdots, t_N) = \sum_{i_1, i_2, \cdots, i_N}^{N=0} t_1^{i_1} \cdots t_N^{i_N} \frac{n_1! \cdots n_N!}{i_1! \cdots i_N!}.
\] (D.5)
Using definition (D.3) we get the following relations:

\[
J_{-(2s+1)k-l}^+ - s\sqrt{2k} - \frac{l}{\sqrt{2k}}, s\sqrt{2k} + \frac{l}{\sqrt{2k}} >
\]

\[
= S_{4ks+2f}(\sqrt{2k}\frac{a-1}{2}, \sqrt{2k}\frac{a-2}{2}, \cdots, \sqrt{2k}\frac{a-(4ks+2l)}{(4ks+2l)!}) | - (s - 1)\sqrt{2k} - \frac{l}{\sqrt{2k}}, s\sqrt{2k} + \frac{l}{\sqrt{2k}} >,
\]

\[
J_{-(2s+1)k-l}^- s\sqrt{2k} + \frac{l}{\sqrt{2k}}, s\sqrt{2k} - \frac{l}{\sqrt{2k}} >
\]

\[
= S_{4ks+2f}(-\sqrt{2k}\frac{a-1}{2}, -\sqrt{2k}\frac{a-2}{2}, \cdots, -\sqrt{2k}\frac{a-(4ks+2l)}{(4ks+2l)!}) | (s - 1)\sqrt{2k} + \frac{l}{\sqrt{2k}}, s\sqrt{2k} - \frac{l}{\sqrt{2k}} > .
\]

The r.h.s. of the above equations vanishes unless \(2ks + l > 0\).

Proof of Proposition 1: Starting from the vacuum state \(|0,0>\) and acting on it with bilinears of the form \(a_{-n_1}a_{-n_2}\) the space \(H_{0,0}^{U_v}\) is generated. To construct the remaining spaces we need to change the charge of the vacuum state. This can be accomplished by acting with the currents. Consider

\[
(J^+_k + J^-_k)|0,0> = |\sqrt{2k},0> + |\sqrt{2k},0>,
\]

\[
a_{-n}(J^+_k - J^-_k)|0,0> = a_{-n}(|\sqrt{2k},0>- |\sqrt{2k},0>).
\]

The bilinears \(a_{-n_1}a_{-n_2}\) and their conjugates acting on these states are sufficient to generate the module \(H_{1,k}^{U_v}\). Proceeding inductively, suppose we have generated the spaces \(H_{s-1,k(s-1)}^{U_v}\). Acting on the state \(|s\sqrt{2k},0> + |- s\sqrt{2k},0>\) with \(J^+_{-(2s+1)k} + J^-_{-(2s+1)k}\) and \(a_{-n}(J^+_{-(2s+1)k} - J^-_{-(2s+1)k})\) we get

\[
(J^+_{-(2s+1)k} + J^-_{-(2s+1)k})(|s\sqrt{2k},0> + |- s\sqrt{2k},0>)
\]

\[
= |(s+1)\sqrt{2k},0> + |- (s+1)\sqrt{2k},0> + S_{4ks}(\sqrt{2k}\frac{a-1}{2}, \sqrt{2k}\frac{a-2}{2}, \cdots, \sqrt{2k}\frac{a-(4ks)}{(4ks)!}) | -(s - 1)\sqrt{2k},0>
\]

\[
+ S_{4ks}(-\sqrt{2k}\frac{a-1}{2}, -\sqrt{2k}\frac{a-2}{2}, \cdots, -\sqrt{2k}\frac{a-(4ks)}{(4ks)!})(s - 1)\sqrt{2k},0>
\]

\[
= |(s+1)\sqrt{2k},0> + |- (s+1)\sqrt{2k},0> + S_{4ks}^o(\sqrt{2k}\frac{a-1}{2}, \sqrt{2k}\frac{a-2}{2}, \cdots, \sqrt{2k}\frac{a-(4ks)}{(4ks)!}) (|(s - 1)\sqrt{2k},0> + |- (s - 1)\sqrt{2k},0>)
\]

\[
= (|(s+1)\sqrt{2k},0> + |- (s+1)\sqrt{2k},0>) \bmod H_{s-1,k(s-1)}^{U_v},
\]

\[
a_{-n}(J^+_{-(2s+1)k} - J^-_{-(2s+1)k})(|s\sqrt{2k},0> + |- s\sqrt{2k},0>)
\]

\[
= a_{-n}(|(s+1)\sqrt{2k},0> - |(s+1)\sqrt{2k},0>) + a_{-n}S_{4ks}(\sqrt{2k}\frac{a-1}{2}, \sqrt{2k}\frac{a-2}{2}, \cdots, \sqrt{2k}\frac{a-(4ks)}{(4ks)!}) | -(s - 1)\sqrt{2k},0>
\]

\[
- a_{-n}S_{4ks}(-\sqrt{2k}\frac{a-1}{2}, -\sqrt{2k}\frac{a-2}{2}, \cdots, -\sqrt{2k}\frac{a-(4ks)}{(4ks)!})(s - 1)\sqrt{2k},0>
\]

\[
= a_{-n}(|(s+1)\sqrt{2k},0> - |(s+1)\sqrt{2k},0>)
\]
\[ a_{-n} S_{4k}^s (\sqrt{2k^{a-1}}/1, \sqrt{2k^{a-2}}/2, \cdots, \sqrt{2k^{a-4k_s}}/4k_s!) |((s-1)\sqrt{2k}, 0 > + |-(s-1)\sqrt{2k}, 0 >) \]
\[ a_{-n} S_{4k}^s (\sqrt{2k^{a-1}}/1, \sqrt{2k^{a-2}}/2, \cdots, \sqrt{2k^{a-4k_s}}/4k_s!) |((s-1)\sqrt{2k}, 0 > - |-(s-1)\sqrt{2k}, 0 >) \]
\[ = a_{-n}( |(s+1)\sqrt{2k}, 0 > - |-(s+1)\sqrt{2k}, 0 >) \mod H_{s+1,k(s-1)}^{U_v} \]

where \( S_{4k}^s \) and \( S_{4k}^s \) are the parts of the Schur polynomials that contain products of even or odd numbers of oscillator modes respectively. In this way we have generated the lowest level states of the module \( H_{s+1,k(s-1)}^{U_v} \). Acting on these states with the bilinears of modes we get the whole \( H_{s+1,k(s-1)}^{U_v} \). Thus the induction step is complete and we have generated all the modules \( H_{s,k}^{U_v} \) starting from the vacuum state.

Similarly we get \( H_{s',k(s'-1)}^{U_v} \). Before embarking on induction to get the spaces \( H_{s+s',k(s-1)}^{U_v} \) we need to construct explicitly the space \( H_{1+1,k(1-1)}^{U_v} = H_{2,0}^{U_v} \). This can be constructed from the bilinears of modes and the relations:

\[
(J^+_k + J^-_k)(J^+_k + J^-_k)|0,0 > = \sqrt{2k}, \sqrt{2k} > + |\sqrt{2k}, -\sqrt{2k} >
\]
\[ a_{-n_1}(J^+_k - J^-_k)\tilde{a}_{n_2}(J^-_k - J^-_k)|0,0 > = a_{-n_1}\tilde{a}_{n_2}( |\sqrt{2k}, \sqrt{2k} > - |\sqrt{2k}, -\sqrt{2k} > - | - \sqrt{2k}, \sqrt{2k} > - | - \sqrt{2k}, -\sqrt{2k} > )
\]
\[ a_{-n}(J^+_k - J^-_k)(J^+_k + J^-_k)|0,0 > = a_{-n}( |\sqrt{2k}, \sqrt{2k} > + |\sqrt{2k}, -\sqrt{2k} > - | - \sqrt{2k}, \sqrt{2k} > - | - \sqrt{2k}, -\sqrt{2k} > )
\]
\[ (J^+_k + J^-_k)\tilde{a}_{n}(J^-_k - J^-_k)|0,0 > = \tilde{a}_{n}( |\sqrt{2k}, \sqrt{2k} > - |\sqrt{2k}, -\sqrt{2k} > - | - \sqrt{2k}, \sqrt{2k} > - | - \sqrt{2k}, -\sqrt{2k} > ).
\]

The inductive step is as follows: Assuming that we have constructed all the spaces \( H_{s+s',k(s-1)}^{U_v} \) for \( s, s' \leq p \) where \( p \geq 1 \) we will prove that we can construct all the spaces \( H_{s+s',k(s-1)}^{U_v} \) with \( s, s' \leq p+1 \). To show this we need to construct the spaces \( H_{(p+1)+s',k((p+1)-s')}^{U_v} \) and \( H_{2p+2,0}^{U_v} \). To build up the first spaces we need the relations:

\[
(J^+_{(p+1)k} + J^-_{(p+1)k})(|p\sqrt{2k}, s'\sqrt{2k} > + | - p\sqrt{2k}, s'\sqrt{2k} >
\]
\[ a_{-n}(J^+_{(p+1)k} - J^-_{(p+1)k})(|p\sqrt{2k}, s'\sqrt{2k} > + | - p\sqrt{2k}, s'\sqrt{2k} >
\]
\[ a_{-n}(|p\sqrt{2k}, s'\sqrt{2k} > + | - p\sqrt{2k}, s'\sqrt{2k} >
\]

The inductive step is as follows: Assuming that we have constructed all the spaces \( H_{s+s',k(s-1)}^{U_v} \) for \( s, s' \leq p \) where \( p \geq 1 \) we will prove that we can construct all the spaces \( H_{s+s',k(s-1)}^{U_v} \) with \( s, s' \leq p+1 \). To show this we need to construct the spaces \( H_{(p+1)+s',k((p+1)-s')}^{U_v} \) and \( H_{2p+2,0}^{U_v} \). To build up the first spaces we need the relations:
The eigenspace of $g$ can be constructed from Appendix C, by making them send the above relations acting on it by the operators in $S$. Extend now the definition of the operators $H_S$. Since the elements of $S$ change the conformal dimension of the states by integers, the space $H_{p-1} \equiv H_{p-1}^{U_T}$ our space is a subspace of $H_{p-1}^{U}$, as the eigenspace of $c$ with eigenvalue +1 (also the eigenspace of $c'$ resp. of $c$) is generated from the vacuum state and acting on it by the operators in $S$. Extend now the definition of the operators $g_c, g_a$ defined in Appendix C, by making them send $s$ to $-s$ and $s'$ to $-s'$ respectively:

$$g_c(|s\sqrt{2k}, s'\sqrt{2k}>= | - s\sqrt{2k}, s'\sqrt{2k}>, \quad g_a(|s\sqrt{2k}, s'\sqrt{2k}>= | s\sqrt{2k}, -s'\sqrt{2k}>) .$$

It is an easy exercise to check that these operators commute with the action of the elements of $S$. Since the elements of $C$ change the conformal dimension of the states by integers, the space they generate starting from the vacuum is a subspace of $H_0 \oplus H_1$. But the space $H_0$ is the eigenspace of $g_c$ with eigenvalue $+1$ (also the eigenspace of $g_a$ with eigenvalue $+1$) while $H_1$ is the eigenspace of $g_c$ (resp. of $g_a$) with eigenvalue $-1$ (resp. $-1$). Since the generators of $S$ commute with $g_c$ (resp. $g_a$) our space is a subspace of $H_0$. But since $H_0$ is generated from the vacuum, the action of $S$ on the vacuum generates precisely $H_0$.

**Proof of Proposition 2:** The proof of proposition 2 proceeds along the same lines as proposition 1 with the only difference that equations (D.3) are replaced by

$$J_{-(2s+1)k}^+ a_{-s-1} \bar{a}_{-1} |s\sqrt{2k}, s'\sqrt{2k}> = -\sqrt{2k} a_{-1} |s+1\sqrt{2k}, s'\sqrt{2k}>,$$

and equations (D.6) are replaced by

$$J_{-(2s+1)k}^+ a_{-s-1} \bar{a}_{-1} |s\sqrt{2k}, s'\sqrt{2k}> = \left[-\sqrt{2k} S_{4ks} \frac{a_{-1}}{1}, \frac{a_{-2}}{2}, \ldots, \sqrt{2k} \frac{a_{-4ks}}{(4ks)!}\right] a_{-1} |s+1\sqrt{2k}, s'\sqrt{2k}>.$$
Now we have

\[ a_n(J^+_{k+t} - J^-_{k+t})(J^+_{k+t} + J^-_{k+t})(\frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}}) > +| - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >= a_n(| \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} > -| - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >) \] (D.16)

\[-a_n(J^+_{k+t} - J^-_{k+t})(J^+_{k+t} + J^-_{k+t})(\frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}}) > +| - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >= a_n(| \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} > -| \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >)\]

\[-a_n^2(J^+_{k+t} - J^-_{k+t})a_n(J^+_{k+t} - J^-_{k+t})(J^+_{k+t} + J^-_{k+t})(| \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} > +| - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >) = a_n^2 a_n(| \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} > +| - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >)\]

These states together with the bilinears of the modes are sufficient to construct \( H^U_{0,0} \). Also we have

\[(J^+_{k+t} + J^-_{k+t})(| \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} > -| - \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} >) \] (D.17)

\[-a_n(J^+_{k+t} - J^-_{k+t})(| \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} > -| - \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} >) = a_n(| -\sqrt{2k} + \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} > - |\sqrt{2k} - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >),\]

\[(J^+_{k+t} + J^-_{k+t})a_n(| \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} > -| - \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} >) = a_n(| -\sqrt{2k} + \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} > - |\sqrt{2k} - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >),\]

\[-a_n^2(J^+_{k+t} - J^-_{k+t})a_n^2(| \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} > -| - \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} >) = a_n^2 a_n(| -\sqrt{2k} + \frac{l}{\sqrt{2k}}, -\frac{l}{\sqrt{2k}} > + |\sqrt{2k} - \frac{l}{\sqrt{2k}}, \frac{l}{\sqrt{2k}} >)\]

and, together with the bilinears of the modes, we are able to construct \( H^U_{-1,0} \). Similarly we can construct the spaces \( H^U_{1,-1} \) and \( H^U_{0,-1} \).

Now we have to proceed inductively on \( s \). Recall that the spaces \( H^U_{1,0}, H^U_{1,-1}, H^U_{0,-1} \) and \( H^U_{0,0} \) have been constructed. Assuming that the spaces \( H^U_{s,s'} \) for \(-1 \leq s, s' \leq p\) have been constructed we will construct the spaces \( H^U_{p+1,s'}, H^U_{s,p+1} \) and \( H^U_{p+1,p+1} \). For the spaces \( H^U_{p+1,s'} \) we need the following relations:

\[(J^+_{(2p+1)k+t} + J^-_{(2p+1)k-t})(|p\sqrt{2k} + \frac{l}{\sqrt{2k}}, \sqrt{2k} > -| -p\sqrt{2k} - \frac{l}{\sqrt{2k}}, \sqrt{2k} >) \] (D.18)

\[= (|p+1)\sqrt{2k} + \frac{l}{\sqrt{2k}}, \sqrt{2k} > +| -p+1)\sqrt{2k} - \frac{l}{\sqrt{2k}}, \sqrt{2k} > \text{mod} H^U_{p+1,s'},\]

\[a_n(J^+_{(2p+1)k-t} - J^-_{(2p+1)k-t})(|p\sqrt{2k} + \frac{l}{\sqrt{2k}}, \sqrt{2k} > -| -p\sqrt{2k} - \frac{l}{\sqrt{2k}}, \sqrt{2k} >) \]

\[= a_n(|p+1)\sqrt{2k} + \frac{l}{\sqrt{2k}}, \sqrt{2k} > -| -p+1)\sqrt{2k} - \frac{l}{\sqrt{2k}}, \sqrt{2k} > \text{mod} H^U_{p+1,s'},\]

\[(J^+_{(2p+1)k-t} + J^-_{(2p+1)k-t})a_n(|p\sqrt{2k} + \frac{l}{\sqrt{2k}}, \sqrt{2k} > -| -p\sqrt{2k} - \frac{l}{\sqrt{2k}}, \sqrt{2k} >) \]
In the new notation the relevant states become: construct and the bilinears of the modes. The second and third spaces are constructed similarly. So we need the relations
\[ a_n\bigl((p+1)\sqrt{2}k + \frac{1}{\sqrt{2}k}, s'\sqrt{2}k - \frac{1}{\sqrt{2}k}\bigr) > -| - (p+1)\sqrt{2}k - \frac{1}{\sqrt{2}k}, -s'\sqrt{2}k + \frac{1}{\sqrt{2}k}\bigr)modH_{p-1}^{U_l} \]

Together with the bilinears of modes we can construct the spaces \( H_{s,s'}^{U_l} \). Similarly we can construct the spaces \( H_{s,s'+1}^{U_l} \), while the space \( H_{s+1,s'}^{U_l} \) can be constructed from the spaces \( H_{s,s'+1}^{U_l} \) and \( H_{s+1,s'}^{U_l} \) using the above relations. So finally we have constructed all the spaces \( H_{s,s'}^{U_l} \) with \( s, s' \geq -1 \).

To proceed further we need induction on the negative values of \( s, s' \). Assuming again that all the spaces \( H_{s,s'}^{U_l} \) for \( s, s' \geq -p \) have been constructed we will show that we can construct the spaces \( H_{s,s'}^{U_l} \) for \( s, s' \geq -p \).

For the spaces \( H_{s,s'}^{U_l} \) we need the relations
\[ a_n\bigl((p+1)\sqrt{2}k + \frac{1}{\sqrt{2}k}, s'\sqrt{2}k - \frac{1}{\sqrt{2}k}\bigr) > +| + (p+1)\sqrt{2}k - \frac{1}{\sqrt{2}k}, -s'\sqrt{2}k + \frac{1}{\sqrt{2}k}\bigr)modH_{p+1,1}^{U_l} \]

and the bilinears of the modes. The second and third spaces are constructed similarly. So we have shown that starting from the state \( |l/\sqrt{2}k, -l/\sqrt{2}k > + | - l/\sqrt{2}k, l/\sqrt{2}k > \) we can construct \( H_{s+1,k}^{U_l} = \bigoplus_{s,s' \in \mathbb{Z}} H_{s,s'}^{U_l} \).

**Proof of Proposition 4** Here we will denote \( H_{s,s'}^{U_{vk}} \equiv H_{s+s'+1,k(s-s')}^{U_{vk}} \) and \( H_{s,s'}^{U_{bk}} \equiv H_{s+s'+1,k(s-s')}^{U_{bk}} \).

In the new notation the relevant states become:
\[ |s + s' + 1, k(s-s') > = |s + (s'+1), k(s-(s'+1)) + k > \]
\[ \rightarrow |s\sqrt{2}k + \frac{k}{\sqrt{2}k}, (s'+1)\sqrt{2}k - \frac{k}{\sqrt{2}k} > = |(s + 1/2)\sqrt{2}k, (s'+1/2)\sqrt{2}k >, \]
\[ | - (s + s' + 1), -k(s-s') > = | - (s + (s'+1)), -k(s-(s'+1)) - k > \]
\[ \rightarrow | - s\sqrt{2}k - \frac{k}{\sqrt{2}k}, -(s'+1)\sqrt{2}k + \frac{k}{\sqrt{2}k} > = | - (s + 1/2)\sqrt{2}k, -(s'+1/2)\sqrt{2}k >, \]
\[ |s - s', k(s+s'+1) > = |s - s', k(s+s') + k > \]
\[ \rightarrow |s\sqrt{2}k + \frac{k}{\sqrt{2}k}, -s'\sqrt{2}k - \frac{k}{\sqrt{2}k} > = |(s + 1/2)\sqrt{2}k, -(s'+1/2)\sqrt{2}k >, \]
\[ | - (s - s'), -k(s+s'+1) > = | - (s - s'), -k(s+s') - k > \]
\[ \rightarrow | - s\sqrt{2}k - \frac{k}{\sqrt{2}k}, -s'\sqrt{2}k + \frac{k}{\sqrt{2}k} > = | - (s + 1/2)\sqrt{2}k, -(s'+1/2)\sqrt{2}k >. \]
We have to prove that starting from the states
\[
\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} >, \quad \text{(D.21)}
\]
we can generate the modules \(H_{k/4}^a\) and \(H_{k/4}^b\) respectively. We show only the construction of \(H_{k/4}^v\) since the construction of \(H_{k/4}^b\) proceeds similarly. To generate \(H_{0,0}^{U_{vk}}\) we need the following relations:
\[
ap_{-n}(J_{0}^+ - J_{0}^-)\left(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > \right)
\]
\[
= a_{-n}(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} >),
\]
\[
a_{-n}(J_{0}^+ - J_{0}^-)\left(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > \right)
\]
\[
= a_{-n}(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} >),
\]
\[
a_{-n}(J_{0}^+ - J_{0}^-)\left(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > \right)
\]
\[
= a_{-n}(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} >).
\]

To generate \(H_{1,0}^{U_{vk}}\) we need the following relations:
\[
(J_{-2k}^+ + J_{-2k}^-)\left(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > \right)
\]
\[
= \left(\text{(1 + \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - (1 + \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| (1 + \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - (1 + \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > modH_{0,0}^{U_{vk}} \right).
\]

Similarly we can generate the spaces \(H_{0,1}^{U_{vk}}\) and \(H_{1,1}^{U_{vk}}\). We are now in position to proceed with induction. For the induction we will assume that we have generated the spaces \(H_{s,s'}^{U_{vk}}\) where \(0 \leq s, s' \leq p\) and we will generate the spaces \(H_{p+1,s'}^{U_{vk}}, H_{s,p+1}^{U_{vk}}\) and \(H_{p+1,p+1}^{U_{vk}}\). To generate the first of these spaces we need the relations
\[
(J_{-2k}^+ + J_{-2k}^-) a_{-n}(\frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} >)
\]
\[
= \left(\text{(1 + \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > +| - (1 + \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| (1 + \frac{1}{2}\sqrt{2k}, -\frac{1}{2}\sqrt{2k} > +| - (1 + \frac{1}{2}\sqrt{2k}, \frac{1}{2}\sqrt{2k} > modH_{0,0}^{U_{vk}} \right).
\]

Similarly we can generate the spaces \(H_{0,1}^{U_{vk}}\) and \(H_{1,1}^{U_{vk}}\). We are now in position to proceed with induction. For the induction we will assume that we have generated the spaces \(H_{s,s'}^{U_{vk}}\) where \(0 \leq s, s' \leq p\) and we will generate the spaces \(H_{p+1,s'}^{U_{vk}}, H_{s,p+1}^{U_{vk}}\) and \(H_{p+1,p+1}^{U_{vk}}\). To generate the first of these spaces we need the relations
\[
(J_{-(2p+2)}^+ + J_{-(2p+2)}^-) a_{-n}(J_{-(2p+2)}^+ - J_{-(2p+2)}^-)
\]
\[
= \left(\text{(p + \frac{1}{2}\sqrt{2k}, (s' + \frac{1}{2}\sqrt{2k} > +| - (p + \frac{1}{2}\sqrt{2k}, (s' + \frac{1}{2}\sqrt{2k} > modH_{p+1,s'}^{U_{vk}} \right).
\]

Similarly we can generate the spaces \(H_{0,1}^{U_{vk}}\) and \(H_{1,1}^{U_{vk}}\). We are now in position to proceed with induction. For the induction we will assume that we have generated the spaces \(H_{s,s'}^{U_{vk}}\) where \(0 \leq s, s' \leq p\) and we will generate the spaces \(H_{p+1,s'}^{U_{vk}}, H_{s,p+1}^{U_{vk}}\) and \(H_{p+1,p+1}^{U_{vk}}\). To generate the first of these spaces we need the relations
\[
(J_{-(2p+2)}^+ + J_{-(2p+2)}^-) a_{-n}(J_{-(2p+2)}^+ - J_{-(2p+2)}^-)
\]
\[
= \left(\text{(p + \frac{1}{2}\sqrt{2k}, (s' + \frac{1}{2}\sqrt{2k} > +| - (p + \frac{1}{2}\sqrt{2k}, (s' + \frac{1}{2}\sqrt{2k} > modH_{p+1,s'}^{U_{vk}} \right).
\]

Similarly we can generate the spaces \(H_{0,1}^{U_{vk}}\) and \(H_{1,1}^{U_{vk}}\). We are now in position to proceed with induction. For the induction we will assume that we have generated the spaces \(H_{s,s'}^{U_{vk}}\) where \(0 \leq s, s' \leq p\) and we will generate the spaces \(H_{p+1,s'}^{U_{vk}}, H_{s,p+1}^{U_{vk}}\) and \(H_{p+1,p+1}^{U_{vk}}\). To generate the first of these spaces we need the relations
\[
(J_{-(2p+2)}^+ + J_{-(2p+2)}^-) a_{-n}(J_{-(2p+2)}^+ - J_{-(2p+2)}^-)
\]
\[
= \left(\text{(p + \frac{1}{2}\sqrt{2k}, (s' + \frac{1}{2}\sqrt{2k} > +| - (p + \frac{1}{2}\sqrt{2k}, (s' + \frac{1}{2}\sqrt{2k} > modH_{p+1,s'}^{U_{vk}} \right).
\]
Similarly we generate the other two spaces. So finally we have generated the whole $H^u_{k/4}$.

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