(SEMI)SIMPLE EXERCISES IN QUANTUM COHOMOLOGY

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Abstract. The paper is dedicated to the study of algebraic manifolds whose quantum cohomology or a part of it is a semisimple Frobenius manifold. Theorem 1.8.1 says, roughly speaking, that the sum of \((p,p)\)–cohomology spaces is a maximal Frobenius submanifold that has chances to be semisimple. Theorem 1.8.3 provides a version of the Reconstruction theorem, assuming semisimplicity but not \(H^2\)–generation. Theorem 3.6.1 establishes the semisimplicity for all del Pezzo surfaces, providing an evidence for the conjecture that semisimplicity is related to the existence of a full system of exceptional sheaves of the appropriate length. Finally, in \(\S\)2 we calculate special coordinates for three families of Fano threefolds with minimal cohomology.

\(\S\)0. Introduction

0.1. Role of semisimplicity. This paper is a contribution to the study of algebraic manifolds whose quantum cohomology (or at least its appropriate subspace) is a generically semisimple Frobenius manifold. This class of manifolds is important in many respects. For example, one expects that the higher genus correlators and the correlators with gravitational descendants for such manifolds can be calculated entirely in terms of the Frobenius structure, when it is semisimple (see [DuZh], [G2], and the references therein). Unfolding spaces of the hypersurface singularities (Landau–Ginzburg models) are generically semisimple as well, therefore any mirror isomorphism between such an unfolding space and (a part of) quantum cohomology must involve semisimple cohomology. Finally, to establish such an isomorphism, we have to compute for both Frobenius manifolds only finitely many constants (monodromy data, or special coordinates) whereas one generally needs infinitely many numbers in order to describe a non–semisimple manifold (like the quantum cohomology of a quintic).

In this paper, we focus on the values of Dubrovin’s canonical coordinates \((u_i)\) at the points of \(H^2\) that is, parameter space of the small quantum cohomology. As soon as their identification at a tame semisimple points is made, the remaining part of the mirror picture requires a choice of the flat metric compatible with multiplication. Unless it is explicitly given, we have to recourse to one of the following equivalent data describing it in an implicit way:

(i) Values of the diagonal coefficients of the flat metric \(\sum_i \eta_i (du^i)^2\) at a tame semisimple point completed by the values of its first derivatives \(\eta_{ij}\), that is, initial data for the second structure connection (cf. [Ma2], II.3).
Monodromy data for the first structure connection and oscillating integrals for the deformed flat coordinates (cf. [G1], [Du] and the references therein).

(iii) Choice of one of K. Saito’s primitive forms.

(iv) Choice of a filtration on the cohomology space of the Milnor fiber (M. Saito, cf. [He2] and the references therein).

(v) Use of the semi–infinite Hodge structure (this is a far-reaching refinement of (iii), cf. [Bar]).

In this paper we discuss only (i).

0.2. General notation for quantum cohomology. Let $V$ be a smooth projective manifold. We choose a homogeneous basis of its cohomology space $(\Delta_a), \Delta_a \in H^{[\Delta_a]}(V)$ where $a$ runs over a set of indices such that $\Delta_0$ always denotes the dual class of $[V]$. The dual coordinates on $H^*(V)$ are denoted $(x_a)$. The Poincaré form $g_{ab} = (\Delta_a, \Delta_b)$ allows us to raise indices, e. g. $\Delta^a := \sum_b g^{ab} \Delta_b$. The letter $\beta$ generally denotes a variable algebraic element of the group $H_2(V, \mathbb{Z})/(\text{tors})$, and $q^\beta$ is the respective element of the Novikov ring. We write $k$ or $k(V)$ for $c_1(V)$ and $k(\beta)$ for $(c_1(V), \beta)$.

The genus zero Gromov–Witten invariants are written as correlators, that is polynlinear functions indexed by $\beta$:

$$H^*(V, \mathcal{Q}) \otimes^n \to \mathcal{Q}: \Delta_{a_1} \otimes \cdots \otimes \Delta_{a_n} \mapsto \langle \Delta_{a_1} \ldots \Delta_{a_n} \rangle_{\beta}.$$ 

For $n = 0$ we write the respective correlator as $\langle \emptyset \rangle_{\beta}$. If $\dim V \geq 3$, we have $\langle \emptyset \rangle_{\beta} = 0$.

Sometimes it is convenient to consider the total correlators with values in the completed Novikov ring: $\langle \Delta_{a_1}, \ldots, \Delta_{a_n} \rangle := \sum_\beta \langle \Delta_{a_1}, \ldots, \Delta_{a_n} \rangle_{\beta} q^\beta$ and extend them to the completed tensor algebra of $H^*(V)$. The potential of quantum cohomology is the formal series in $(x_a)$ over the Novikov ring which can be compactly written in this notation as $\Phi := e^{\sum_a x_a \Delta_a}$. The quantum multiplication table is given by $\Delta_a \circ \Delta_b = \sum_c \Phi_{abc} \Delta^c$ where $\Phi_{abc} = \partial_a \partial_b \partial_c \Phi$, $\partial_a = \partial / \partial x_a$. Using the standard properties of the Gromov–Witten invariants (symmetry, divisor axiom and identity axiom) we can rewrite this in the following form convenient for further calculations:

$$\sum_{\beta \neq 0} \sum_{c \neq 0} \langle \Delta_a \Delta_b \Delta_c \rangle_{\beta} e^{\sum_{k: |\Delta_k| > 2} x_k \Delta_k} \beta e^{\sum_{k: |\Delta_k| = 2} x_k \Delta_k, \beta} q^\beta. \quad (0.1)$$

Since $\beta \mapsto e^{\sum_{k: |\Delta_k| = 2} x_k \Delta_k, \beta} q^\beta$ is a generic character of $H_2(V, \mathbb{Z})/(\text{tors})$, as well as $\beta \mapsto q^\beta$, we will allow ourselves in the future to use (0.1) omitting $e^{\sum_{k: |\Delta_k| = 2} x_k \Delta_k, \beta}$ and instead interpreting $q^\beta$ as a monomial function on the algebraic torus $T_V$ with the character group $H^2(V, \mathbb{Z})/(\text{tors})$.

0.3. Plan of the paper. We start §1 with a review of general properties of semisimple Frobenius manifolds. Actually, a natural context for this discussion is
the realm of $F$–manifolds, that is, "Frobenius manifolds without metric", introduced in [HeMa]. We recall their definition and the main properties related to the semisimplicity in 1.1–1.7.

In 1.8 we then prove the first main result of this paper, Theorem 1.8.1. Roughly, it says that the most natural candidate for semisimple quantum cohomology is not the whole cohomology of a Fano manifold, but only its $(p,p)$–part. The second result, Theorem 1.8.3, is a new version of the First Reconstruction Theorem in [KoMa]. Finally, we discuss the special coordinates for Fano manifolds with minimal $(p,p)$–cohomology (rank one in every dimension).

In §2 we provide explicit canonical coordinates and the initial data for the second structure connection for three families of Fano threefolds with minimal $(p,p)$–cohomology. They can be useful for finding Landau–Ginzburg superpotentials and primitive forms for them. Since these threefolds are not toric, no clear–cut prescription is available for doing this.

The whole §3 is dedicated to the quantum cohomology of the del Pezzo surfaces. This subject was already treated in several papers, see [GP] and the references therein. We systematically use the symmetry properties of the quantum cohomology with respect to the Weyl groups, established in [GP]. This allows us to encode a large amount of numerical data in rather compact tables and, more important, to perform calculations in quantum cohomology in terms of the geometry of root systems and reflections. Since the expected mirror picture involving the Lagrangian vanishing cycles is also naturally formulated in this way, we feel that this is the right language.

The principal result of §3 is the Theorem 3.6, establishing generic semisimplicity of the quantum cohomology of all del Pezzo surfaces $V_r$. For $r \leq 3$ when $V_r$ are toric, the point $q = 1$ turns out to be tame semisimple. This ceases to be true for $r \geq 5$, and instead we study the behavior of quantum cohomology near the boundary of the partial compactification of the torus $T_{V_r}$, where the exponentiated coordinate corresponding to an exceptional curve vanishes.

For this proof, the calculations of 3.3–3.5 are not needed, and the reader may prefer to skip them. On the other hand, they might be useful for finding the Landau–Ginzburg superpotentials in the non–toric cases.

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§1. Semisimplicity and $F$–manifolds

1.1. Definition. An $F$–manifold is a triple $(M, ◦, e)$, where $M$ is a manifold and $◦$ is an associative supercommutative $O_M$–bilinear multiplication $T_M × T_M → T_M$ with identity vector field $e$, satisfying the following axiom: for any (local) vector fields $X, Y, Z, W$ we have

$$[X ◦ Y, Z ◦ W] − [X ◦ Y, Z] ◦ W − Z ◦ [X ◦ Y, W]$$
$$− X ◦ [Y, Z ◦ W] + X ◦ [Y, Z] ◦ W + X ◦ Z ◦ [Y, W]$$
$$− Y ◦ [X, Z ◦ W] + Y ◦ [X, Z] ◦ W + Y ◦ Z ◦ [X, W] = 0 \quad (1.1)$$

This notion makes sense in any of the standard categories of manifolds, and also supermanifolds, if appropriate signs are introduced in the structural identity, cf. [HeMa]. Below we will assume that $M$ is a complex manifold.

A shorter formulation of (1.1) is

$$\text{Lie}_{X ◦ Y}(◦) = X ◦ \text{Lie}_Y(◦) + Y ◦ \text{Lie}_X(◦) \quad (1.2)$$

where $X, Y$ are any two local vector fields, and $◦$ is considered as a tensor.

1.1.1. Question. Does the quantum multiplication in $K$–theory (whose construction is sketched by Givental in [G3]) satisfy (1.1)–(1.2)?

Returning to $F$–manifolds, we will say that $x ∈ M$ is a semisimple point of the $F$–manifold $M$, if the following equivalent conditions are satisfied:

(i) $T_x M$ endowed with the multiplication $◦$ is a (commutative) semisimple algebra, that is, isomorphic to $\mathbb{C}^n$, $n = \dim M$, with componentwise multiplication.

(ii) In a neighborhood of $x$, there exists a basis of vector fields $e_i, i = 1, \ldots, n,$ such that $e_i ◦ e_j = \delta_{ij} e_i$ and $[e_i, e_j] = 0$ for all $i, j$.

$(M, ◦, e)$ is called generically semisimple, if its set of semisimple points is open and dense.

From (ii) it follows that in a neighborhood of any semisimple point, $M$ admits a system of Dubrovin’s canonical coordinates $(u_1, \ldots, u_n)$. They have the following property: $e_i := \partial/\partial u_i$ form a complete system of pairwise orthogonal idempotents with respect to $◦$. Any two local systems of canonical coordinates differ by a constant shift $(u_j + c_j)$ followed by a permutation.

$F$–manifolds admit a natural product operation:

$$(M_1, ◦_1, e_1) × (M_2, ◦_2, e_2) := (M_1 × M_2, ◦_1 ◦_2, e_1 ⊕ e_2).$$

The main structural identity (1.1) can be used as an integrability condition, in order to show the following result ([He1], Theorem 4.2): for any $x ∈ M$, the canonical
decomposition of $(T_xM, \circ)$ into a product of local algebras uniquely extends to the direct decomposition of the germ of $(M, x)$ into the product of irreducible germs of $F$–manifolds.

1.2. Euler fields. A semisimple point $x$ endowed with local canonical coordinates $u_i$ is called tame semisimple, if $u_i(x) \neq u_j(x)$ for all $i \neq j$. There are two basic setups in which the eventual constant shifts of canonical coordinates can be eliminated and the notion of tame semisimplicity made coordinate–independent.

First, it might happen that the fundamental group of the submanifold of semisimple points acts transitively on the elements of a local idempotent system $(e_i)$. Then $(u_i)$ can be normalized up to a common shift $(u_i + c)$ and a permutation, and tameness or otherwise with respect to such coordinates is defined unambiguously.

Second, it might happen that the $F$–manifold $(M, \circ, e)$ is endowed with a structure Euler vector field $E$ (this is the case of quantum cohomology). In the context of $F$–manifolds, a local vector field $E$ is called an Euler field of (constant) weight $d_0$ if $\text{Lie}_E(\circ) = d_0 \circ$ that is, for all local vector fields $X, Y$ we have

$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = d_0 X \circ Y.$$

(1.3)

Euler fields form a sheaf of Lie algebras, and weight is a linear function on this sheaf. The commutator of two Euler fields has weight zero.

It is easy to describe all Euler fields in a neighborhood of a semisimple point endowed with canonical coordinates $(u_i)$. Namely, writing (1.3) for $X = e_i, Y = e_j$, we see that any local Euler field of weight $d_0$ is of the form

$$E = d_0 \sum_i u_i e_i + \sum_j c_j e_j,$$

(1.4)

where $c_j$ are arbitrary constants. Reversing this argument, we see that given an Euler field $E$ of non–zero weight, we can define uniquely up to permutation canonical coordinates $(u_i)$ by the condition $E = d_0 \sum_i u_i e_i$. Moreover, functions $(d_0 u_i)$ can be then invariantly described as the spectrum of the operator $E \circ$ acting upon local vector fields. Notice that in quantum cohomology $E$ is normalized in such a way that $d_0 = 1$, and we will adopt this convention.

1.3. The spectral cover. Let $(M, \circ, e)$ be an $F$–manifold. By definition, its spectral cover is the relative analytic spectrum $L := \text{Specan} (T_M, \circ)$, together with two structure maps $\pi : L \to M$ and $\sigma : T_M \to \pi_*(\mathcal{O}_L)$. Clearly, $\sigma$ is an isomorphism of sheaves of $\mathcal{O}_M$–algebras. A point $x \in M$ is semisimple iff $\pi$ is étale at $x$.

Moreover, $L$ is endowed with a canonical closed embedding $i : L \to T^*M$ to the complex analytic cotangent bundle of $M$. This embedding is induced by the map of sheaves of algebras $S(T_M) \to (T_M, \circ)$ identical on $T_M$. 

The definition of $L$, as well as the definition of a semisimple point, does not require $\circ$ to satisfy (1.1). Generic semisimplicity is equivalent to the fact that $L$ is reduced (has no nilpotents in the structure sheaf). If this condition is satisfied, (1.1) becomes equivalent to the condition that $L$ is Lagrangian.

1.4. K. Saito’s frameworks and Landau–Ginzburg models. A large class of generically semisimple $F$–manifolds is furnished by the construction due to K. Saito in the theory of singularities and axiomatized in [Ma1], [Ma2] (for the richer Frobenius structure) under the name of K. Saito’s frameworks. Briefly, let $p : N \to M$ be a submersion of complex manifolds, generally non–compact. Then $N$ carries the complex of sheaves of relative holomorphic forms $(\Omega^*(N/M), d_{N/M})$. Let $\Psi$ be a closed relative 1–form, and $C \subset N$ the closed analytic subspace $\Psi = 0$ of $N$. We can define a map $s : T_M \to p_*(\mathcal{O}_C)$ in the following way. Let $X$ be a local vector field on $M$, defined in a sufficiently small open subset $U$. Cover $p^{-1}(U)$ by small open subsets $U_j$ in such a way that on each $U_j$ there exists a lift $X_j$ of $X$. Then the maps $X \mapsto i_{X_j}(\Psi)$ on $U_j$ glue together and produce a well defined map of sheaves $s : T_M \to p_*(\mathcal{O}_C)$. (In [Ma1], [Ma2], only the case $\Psi = d_{N/M}F$ was considered, which is not a restriction locally on $N$ but overlooks more global effects).

Assume that $s$ is an isomorphism of $\mathcal{O}_M$–modules, in particular, $p$ restricted to $C$ is flat. Then $s$ induces a multiplication $\circ$ on $T_M$ coming from $p_*(\mathcal{O}_C)$ and endowed with the identity vector field $e = s^{-1}(1)$. If $p : C \to M$ is generically étale, $(M, \circ, e)$ is an $F$–manifold. Its spectral cover $L$ is then canonically identified with $C$, and $s$ with $\sigma$. When these conditions are satisfied, we will call $(p : N \to M, \Psi)$ a K. Saito’s framework (in the context of $F$–manifolds).

If an $F$–manifold $(M, \circ, e)$ is initially defined by another construction, e. g. is the quantum cohomology of a manifold $V$, then any isomorphism of it with the base of some Saito’s framework $(p : N \to M, \Psi)$ is called a Landau–Ginzburg representation of it. When $\Psi = d_{N/M}F$ for some local function $F$ on $M$, $F$ is called the respective LG–superpotential of $(M, \circ, e)$. Canonical coordinates on a simply connected $U \subset M$ over which $C$ is étale, are precisely the critical values of $F$, that is, restrictions of $F$ on various connected components of $p^{-1}(U)$.

If $V$ is a Fano manifold with generically semisimple quantum cohomology, any Landau–Ginzburg representation of it is traditionally considered as a mirror construction.

1.5. Example: unfolding of isolated hypersurface singularities. Let $f = f(z_1, \ldots, z_m)$ be the germ of an analytic function with an isolated singularity at zero. Let $g_1, \ldots, g_n$ be germs of analytic functions at zero whose classes modulo the Jacobian ideal $J := (\partial f/\partial z_i)$ form a basis of the Milnor ring of $f$. Put $F = F(z, t) = f(z) + \sum_{j=1}^n t_j g_j(z)$. Consider $F$ as a function on the germ $N$ at zero of the analytic space with coordinates $(z, t)$ endowed with the projection $(z, t) \mapsto (t)$ to the germ $M$ of the $(t)$–space. Then $(N \to M, p, d_{N/M}F)$ is a K. Saito’s framework. It produces a
generically semisimple $F$–structure on the base of miniversal deformation $M$ whose spectral cover $C$ is smooth. Moreover, this (germ of) $F$–manifold is irreducible, that is, cannot be represented as a product of $F$–manifolds, because its tangent space at zero is a local algebra.

Conversely, any irreducible germ of a generically semisimple $F$–manifold with smooth spectral cover $L$ is isomorphic to the miniversal unfolding base of an isolated hypersurface singularity, and this singularity is unique up to stable right equivalence, that is, adding squares of extra $z$–coordinates and performing invertible coordinate changes. This beautiful theorem was proved in [He1] (see Theorem 16.6), using earlier results by V. Arnold and A. Givental.

It gives a very neat intrinsic description of irreducible germs of $F$–manifolds allowing the Landau–Ginzburg representations of the well studied type.

If we only know that an $F$–manifold $(M, \circ, e)$ has a smooth spectral cover $L$, then Hertling’s local decomposition theorem together with the result above produces a rather detailed local information about possible $LG$–potentials, but neither proves its existence, nor characterizes it uniquely.

1.6. Example: $LG$–superpotentials of the quantum cohomology of projective spaces. Quantum cohomology of $H^\ast(P^r)$ is semisimple at zero. The canonical coordinates of this point defined as the eigenvalues of the multiplication by the standard Euler field $E$ are simply

$$u_k = \zeta^k(r + 1), \quad \zeta = e^{\frac{2\pi i}{r+1}}. \quad (1.5)$$

The respective K. Saito’s framework which is traditionally considered as the mirror of $P^r$ is constructed as follows (cf. [G1], [Bar]). Denote by $M$ the germ of $C^{r+1}$ at zero, with local coordinates $t_0, \ldots, t_r$. Denote by $N$ the direct product of $M$ and the torus $z_0 \cdots z_r = 1$, and by $p$ its projection on $M$. Put $f(z) := z_0 + \cdots + z_r$. Finally, consider the function

$$F(z; t) := f(z) + \sum_{j=0}^{r} t_j f(z)^j. \quad (1.6)$$

An easy calculation shows that at $t = 0$, the critical points of $F(z; 0) = f(z)$ are $z_0 = \cdots = z_r = \zeta^k$, $k = 0, \ldots, r + 1$, with the respective critical values $\zeta^k(r + 1)$. Therefore, there exists a unique isomorphism of the $F$–manifold $(M, \circ, e)$ produced by this framework with the germ of the quantum cohomology of $H^\ast(P^r)$ extending the above identification of the canonical coordinates at zero.

The deformation of $f(z)$ explicitly described by (1.6) is formally similar to the usual construction of the unfolding space of the isolated singularity: in fact, the first $r + 1$ powers of $f(z)$ generate a basis of the Milnor ring of $f$ (which is semisimple
rather than local this time). However, theoretical reasons for choosing this deformation are less compelling than in the classical case: we are dealing with a global but non-compact deformation problem.

For a thorough discussion of a class of such problems, see [Sa] and the references therein.

1.7. Metric and flat coordinates. Let again \((M, \circ, e)\) be an \(F\)-manifold with semisimple base point. Choose also an Euler field \(E\) whose eigenvalues constitute a normalized system of canonical coordinates \(u\) on \(M\). An additional structure which turns \(M\) into a Frobenius manifold is the choice of a (complex analytic) flat metric \(g\) such that \(g(X \circ Y, Z) = g(X, Y \circ Z)\) for all local vector fields \(X, Y, Z\). Dubrovin has shown that in the canonical coordinates such a metric can be expressed as \(\sum_i e_i \eta (du_i)^2\) where \(\eta\) is a local function defined up to adding a constant. For quantum cohomology, \(g\) is the Poincaré form, and one can take for \(\eta\) the flat coordinate dual to the cohomology class of a point. For the latter, cf. [Ma2], Ch. I, Proposition 3.5 c).

If the base point of \(M\) is tame semisimple, then the metric can be uniquely reconstructed from the values of \(\eta_i = e_i \eta\) and \(\eta_{ij} = e_i e_j \eta\) at this point. For a proof, see [Ma2], I.3.

The essential (and difficult) problem in this case consists in studying the global analytic properties of flat coordinates near the boundary of the tame semisimple domain. An appropriate language for such a study is the language of monodromy data for the first and the second structure connections of \((M, \circ, g)\) for which we refer to [Du], [Guz], and references therein (first connection and the Stokes data), and [Ma2], [He2] (second connection, Schlesinger’s equations).

1.8. Semisimple quantum cohomology. We now turn to the main subject of this section, \(F\)-manifolds and Frobenius manifolds of quantum cohomology.

First of all, semisimple Frobenius manifolds cannot have odd coordinates. Hence we should restrict ourselves from the start by the even–dimensional quantum cohomology. It is well known that it is endowed with the induced \(F\)-manifold structure, Euler field, flat invariant metric and flat identity.

1.8.1. Theorem. a) Let \(V\) be a projective manifold whose even–dimensional quantum cohomology is generically semisimple as a formal Frobenius manifold over the Novikov ring. Then

\[
h^{p,q}(V) = 0 \text{ for all } p \neq q, \ p + q \equiv 0 \mod 2. \tag{1.7}
\]

b) Generally, for an arbitrary projective manifold \(V\), the formal completion at of \(\oplus_{p \in \mathbb{Z}} H^{p,p}(V)\) at the point of the classical limit is endowed with an induced structure of Frobenius submanifold with flat identity and Euler field.

Recall that at the point of the classical limit vanish all the flat coordinates outside \(H^2\) and exponentiated coordinates on \(H^2\). More precisely, the \(H^2\) component of
this point lies in the partial compactification of the respective torus corresponding to the Kähler cone.

1.8.2. Dubrovin’s conjecture and a generalization. Classical calculations show that there exist Fano complete intersections in projective spaces for which (1.7) fails. For example, $2m$–dimensional intersections of three quadrics have

$$h^{m-1,m+1} = h^{m+1,m-1} = m(m+1)/2.$$ 

Hence their even–dimensional quantum cohomology cannot be semisimple.

Thus, the best result that one can expect is that $\bigoplus_{p\in\mathbb{Z}} H^{p,p}(V)$ is semisimple for an interesting class of Fano manifolds.

B. Dubrovin ([Du], 4.2.2) conjectured that the even quantum cohomology of $V$ is semisimple if and only if $V$ is Fano and $\text{Der}^b(\text{Coh} V)$ admits a full system of exceptional objects whose length equals $\dim H^*(V)$. This agrees with Theorem 1.8.1, because on such a manifold all cohomology classes are algebraic, and in particular, $H^{odd}(V) = 0$ and $h^{p,q} = 0$ for $p \neq q$. Theorem 1.8.1 suggests the following strengthening of this conjecture: $\bigoplus_{p\in\mathbb{Z}} H^{p,p}(V)$ (with quantum multiplication) is semisimple if and only if $V$ is Fano and $\text{Der}^b(\text{Coh} V)$ admits a full system of exceptional objects whose length equals $\sum_p h^{p,p}(V)$.

For some positive results on the semisimplicity in a more narrow sense, see [Beau], [TX], and the references in these papers. In particular, in [TX] it is shown that for certain Fano complete intersections the operator of the quantum multiplication by the canonical class is generically semisimple on the classical subring of $\bigoplus_{p\in\mathbb{Z}} H^{p,p}(V)$ generated by $H^{1,1}(V)$. However, we do not know, whether this subring is stable with respect to the quantum multiplication.

Proof of the Theorem 1.8.1. a) It is based on the consideration of Euler fields, discussed in 1.2. Namely, from (1.4) one easily deduces that if on a generically semisimple $F$–manifold two Euler fields of non–zero weights commute, they must be proportional.

Now let us turn to the quantum cohomology. As Sheldon Katz remarked, it admits generally at least two commuting Euler fields. In order to write them down explicitly, choose a bigraded homogeneous basis $(\Delta_a)$ of $H = H^{*,*}(V,\mathbb{C})$ considered as the space of flat vector fields, and let $(x_a)$ be the dual flat coordinates vanishing at zero. Let $\Delta_a \in H^{p_a,q_a}(V)$. Put $-K_V = \sum_{p_a+q_a=2} r_b \Delta_b$. Then

$$E_1 := \sum_a (1 - p_a)x_a \Delta_a + \sum_b r_b \Delta_b, \quad (1.8)$$

$$E_2 := \sum_a (1 - q_a)x_a \Delta_a + \sum_b r_b \Delta_b \quad (1.9)$$
are Euler of weight 1. To check this, one can repeat the calculations made in [Ma2], III.5.4 with obvious changes. In the final count, the reason for this is that Gromov–Witten correspondences are algebraic, and therefore are represented by the cohomology classes of types \((p, p)\).

Assuming now that \(h^{p,q}(V) \neq 0\) for some \(p \neq q\), \(p + q \equiv 0 \mod 2\), we see that the restrictions of (1.8) and (1.9) to the even–dimensional cohomology cannot be proportional. This contradiction concludes the proof of a).

b) With the same notation, a correlator \(\langle \Delta_{a_1} \cdots \Delta_{a_n} \rangle_\beta\) vanishes unless \(\sum_i p_{a_i} = \sum_i q_{a_i}\) because the Gromov–Witten correspondences are represented by algebraic cycles. Let \(\Phi\) be the potential of the quantum cohomology. Assume that \(p_a = q_a\) and \(p_b = q_b\), but \(p_c \neq q_c\) for some \(a, b, c\). From the preceding remark it follows that \(\Phi_{ab}^c\) vanishes after restriction to the subspace \(\oplus_p H^{p,p}(V)\) given by the equations \(x_e = 0\) for all \(e\) such that \(p_e \neq q_e\). Hence the \(\circ\) product of two vector fields tangent to this submanifold is again tangent to it. It is also clear that the Poincaré form restricts to a non–degenerate flat metric, and the identity field and the Euler field are tangent to it as well.

We will now prove a theorem which establishes a result of the same type as the First Reconstruction Theorem of [KoMa], but under the different assumptions.

**1.8.3. Reconstruction Theorem.** Assume that the \((p, p)–\)part of the quantum cohomology of \(V\) is generically semisimple and, moreover, admits a tame semisimple point lying in the subspace \(H^2(V)\) (parameter space of the small quantum deformation).

In this case all genus zero Gromov–Witten invariants of \(\oplus_p H^{p,p}(V)\) can be reconstructed from the correlators \(\langle \gamma_1 \otimes \cdots \otimes \gamma_n \rangle_\beta\) with \(n \leq 4\), \(\gamma_i \in H^{a_i,a_i}(V)\), and \(\beta \in H_2(V, \mathbb{Z})\) which can be non–vanishing only if

\[
k(\beta) := (c_1(V), \beta) = \sum_{i=1}^n (a_i - 1) + 3 - \dim V.
\]

In particular, if \(\pm c_1(V)\) is numerically effective, finitely many correlators suffice for the complete reconstruction.

**Proof.** As was explained in 1.7, the whole germ of the Frobenius manifold at such tame semisimple point of \(H^2\) is determined by its canonical coordinates \((u_i^0)\), the values \(\eta^0\) at this point of the coefficients \(\eta_i\) of the metric (Poincaré form) \(\sum \eta_i (du_i)^2\), and the values \(\eta^0_{ij}\) of their first derivatives \(\eta_{ij} = e_i \eta_{ij}\).

One easily sees that in order to calculate these data, it suffices to know the expressions of \((u_i)\) through some flat coordinates modulo \(J^2\) where \(J\) is the ideal of equations of \(H^2\) inside \(\oplus H^{p,p}\). Now, \((u_i)\) are the eigenvalues of the operator \(E\circ\) acting on vector fields. To calculate them modulo \(J^2\), it suffices to know the potential \(\Phi\) of quantum cohomology modulo \(J^5\) because the structure constants of
the \(\circ\)-multiplication are the third derivatives of the potential. With this precision, \(\Phi\) is determined by all four-point correlators. Of course, (1.10) is a special case of the Dimension Axiom.

1.9. Fano manifolds with minimal \((p,p)\)-cohomology. The finite family of numbers \((u^0_i, \eta^0_i, \eta^0_{ij})\) essentially coincides with what was called special coordinates of the tame semisimple germ of Frobenius manifold, cf. [Ma2], III.7.1.1. In this subsection, we will show how to calculate them for the \(\oplus H^{p,p}(V)\)-part of the quantum cohomology of those Fano manifolds, for which \(\dim H^{p,p}(V) = 1\) for all \(1 \leq p \leq \dim V := r\). This generalizes our old computation for projective spaces.

We will work with a homogeneous basis \(\Delta_p \in H^{p,p}(V)\) consisting of rational cohomology classes satisfying the following conditions: \(\Delta_0 = \text{the dual class of } [V]\), \(\Delta_1 = c_1(V)/\rho\) is the ample generator of \(\text{Pic } V\), and \(\rho\) is called the index of \(V\). Furthermore, \(\Delta_{r-p}\) is dual to \(\Delta_p\), that is \((\Delta_p, \Delta_{r-p}) = 1\), \(\Delta_r = \text{the dual class of a point}\). The dual coordinates are denoted \(x_0, \ldots, x_r\). The nonvanishing \(\beta = 0\) correlators are coefficients of the cubic self-intersection form

\[
(x_0 \Delta_0 + \cdots + x_r \Delta_r)^3.
\]

We put

\[
[d; a_1, \ldots, a_k] := \langle \Delta_{a_1} \cdots \Delta_{a_k} \rangle_{d \Delta_{r-1}}.
\]

These symbols satisfy the following relations:

(i) If \(r \geq 3\), \([d; a_1, \ldots, a_k] \neq 0\) and \(d > 0\), then necessarily

\[
k > 0, \ a_i > 0 \text{ for all } i, \ \text{and } dp = \sum_{i=1}^{k} (a_i - 1) + 3 - r \tag{1.11}
\]

(see (1.10)).

(ii) \([d; a_1, \ldots, a_k]\) is symmetric with respect to the permutations of \(a_1, \ldots, a_k\).

(iii) \([d; 1, a_2, \ldots, a_k] = d [d; a_2, \ldots, a_k]\).

(iv) Associativity relations, expressing the associativity of the multiplication (0.1).

The multiplication table in the first infinitesimal neighborhood of \(H^2(V)\) involves only up to four-point correlators and looks as follows:

\[
\Delta_a \circ \Delta_b = \Delta_a \cup \Delta_b + \\
\sum_{d \geq 1} \sum_{c \geq 1} \left( [d; a, b, c] + \sum_{f \geq 2} [d; a, b, c, f] x_f \right) \Delta_{r-c} q^d. \tag{1.12}
\]
Finally, the (restricted) Euler field of weight 1 is

$$E = \sum_{p=0}^{r} (1-p) x_p \Delta_p + \rho \Delta_1$$  \hspace{1cm} (1.13)

As we already remarked, under the assumptions of the Theorem 1.8.3, the eigenvalues of $E \circ$ at the generic point of $H^2$ are pairwise distinct and determine the canonical coordinates of this point. We will have to calculate in the first infinitesimal neighborhood of $H^2$ and therefore we will consider all the relevant quantities as consisting of two summands: restriction to $H^2$ and the linear (in $x_a$) correction term, in particular

$$u_i := u_i^{(0)} + u_i^{(1)}.$$  

The remaining special coordinates are given by the following formulas.

1.9.1. Theorem. Put

$$e_i := \frac{\prod_{j \neq i} (E - u_j)}{\prod_{j \neq i} (u_i - u_j)} = e_i^{(0)} + e_i^{(1)}.  \hspace{1cm} (1.14)$$

where the multiplication is understood in the sense of quantum cohomology with the coefficient ring extended by $(u_i)$ and $(u_i - u_j)^{-1}$.

Then we have on $H^2$:

$$\eta_i = e_i^{(0)}(x_r), \quad \eta_{ij} = e_i^{(0)} e_j^{(0)} (x_r)$$  \hspace{1cm} (1.15)

where in (1.15) $e_i$ are considered as vector fields acting upon coordinates via $\Delta_a = \partial / \partial x_a$.

Proof. The elements $e_i$ are the basic pairwise orthogonal idempotents in the quantum cohomology ring at the considered point satisfying $E \circ e_i = u_i e_i$. The metric potential $\eta$ is $x_r$.

Here is an efficient way of computing $e_i^{(1)}$. First, compute $\omega_i$ defined by the identity in the first neighborhood:

$$e_i^{(0)} \circ e_i^{(0)} = e_i^{(0)} + \omega_i.$$  \hspace{1cm} (1.16)

Then we have

$$e_i^{(1)} = -\frac{\omega_i}{2 e_i^{(0)} - 1} = \omega_i \circ (1 - 2 e_i^{(0)})$$  \hspace{1cm} (1.17)

where the division resp. multiplication is again made in the first neighborhood.

In fact, this follows from

$$(e_i^{(0)} + e_i^{(1)})^2 = e_i^{(0)} + e_i^{(1)}$$

and (1.16).
§2. Fano threefolds with minimal cohomology

2.1. Notation. Let $V$ be a Fano threefold. We keep the general notation of 1.3, but now consider only the case $r = 3$. Besides the index $\rho$, we consider the degree $\delta := (c_1(V)^3)/\rho^3$ of $V$.

There exist four families of Fano threefolds $V = V_5$ with cohomology $H^{p,q}(V, \mathbb{Z}) \cong \mathbb{Z}$ for $p = 0, \ldots, 3$ and $H^{p,q}(V, \mathbb{Z}) = 0$ for $p \neq q$. Besides $V_1 = \mathbb{P}^3$ and the quadric $V_2 = Q$, they are $V_5$ and $V_{22}$, with degree as subscript; their indices are, respectively, $4, 3, 2, 1$. One can get a $V_5$ by considering a generic codimension three linear section of the Grassmannian of lines in $\mathbb{P}^5$ embedded in $\mathbb{P}^9$.

The nonvanishing $\beta = 0$ correlators are coefficients of the cubic self-intersection form

$$(x_0\Delta_0 + \cdots + x_3\Delta_3)^3 = \delta x_1^3 + 3x_0^2x_3 + x_0x_1x_2.$$ 

In this section, we will deal only with $Q$, $V_5$ and $V_{22}$, since projective spaces of any dimension were treated by various methods earlier: see [Ma3] for special coordinates, [Du], 4.2.1 and [Guz] for monodromy data, [Bar] for semiinfinite Hodge structures.

2.2. Tables of correlators. The following tables provide the coefficients of the multiplication table (1.12).

It suffices to tabulate the primitive correlators, where primitivity means that $a_i > 1$ and $a_i \leq a_{i+1}$. The symmetry and the divisor identities furnish the remaining correlators.

**Manifold $Q$:**

| $[a; b, c]$ | $[1; 2,3]$ | $[1; 2,2,2]$ | $[2; 3,3,3]$ | $[2; 2,2,3,3]$ |
|-----------|-----------|-------------|-------------|-------------|
| $\delta$  | 1         | 1           | 1           | 1           |

**Manifold $V_5$:**

| $[a; b, c]$ | $[1; 3]$ | $[1; 2,2]$ | $[2; 3,3]$ | $[2; 2,2,3]$ | $[3; 3,3,3]$ | $[2; 2,2,2,2]$ | $[3; 2,2,3,3]$ | $[4; 3,3,3,3]$ |
|-----------|----------|-----------|-------------|-------------|-------------|--------------|--------------|-------------|
| $\delta$  | 3        | 1         | 1           | 1           | 1           | 2            | 3            | 2           |

**Manifold $V_{22}:$**

| $[a; b, c]$ | $[1; 2]$ | $[2; 3]$ | $[2; 2,2]$ | $[3; 2,3]$ | $[4; 3,3]$ | $[3; 2,2,2]$ | $[4; 2,2,3]$ | $[5; 2,3,3]$ |
|-----------|----------|-----------|-------------|-------------|-------------|--------------|--------------|-------------|
| $\delta$  | 2        | 6         | 1           | 3           | 10          | 1            | 4            | 16          |
|           | 65       | 2         | 9           | 41          | 186         | 840          |              |             |

The tables were compiled in the following way. First, (1.10) furnishes the list of all primitive correlators that might be (and actually are) non–vanishing. Second, several correlators corresponding to the smallest values of $n$ in (1.10) must be
computed geometrically: $n = 2$ for $Q$, $n = 1, 2$ for $V_5$, and $V_{22}$. These values were computed by A. Bondal, D. Kuznetsov and D. Orlov. Third, the associativity equations uniquely determine all the remaining correlators, in the spirit of the First Reconstruction Theorem of [KoMa].

2.3. Canonical coordinates. The canonical coordinates on $H^2 \cap \{x_0 = 0\}$ expressed in terms of the flat coordinates are the roots $u_0, \ldots, u_3$ of the following characteristic equations of the operator $E_0$:

$$Q : \quad u^4 - 108 q u = 0,$$

$$V_5 : \quad u^4 - 44 q u^2 - 16 q^2 = 0,$$

$$V_{22} : \quad (u + 4 q) (u^3 - 8 q u^2 - 56 q^2 u - 76 q^3) = 0.$$

(2.1) (2.2) (2.3)

If $x_0 \neq 0$, one must simply add $x_0$ to the values above.

2.3.1. Question. Find “natural” functions $f(z)$ whose critical values at 0 are roots of (2.1)–(2.3) and whose unfolding space carries an appropriate flat metric.

2.4. Multiplication tables, idempotents, and metric coefficients. The remaining special coordinates $\eta_i, \eta_{ijk}$ were calculated using the multiplication tables in the first neighborhood of $H^2$ obtained by specializing (1.12). We calculated $e^{(0)}_i$ by determining the eigenvectors of $\text{ad} E$; then we used equation (1.17) to get $e^{(1)}_i$.

Manifold $Q$:

$$\Delta_1^2 = 2 \Delta_2 + q \Delta_1 x_3 + q \Delta_0 x_2,$$

$$\Delta_1 \Delta_2 = \Delta_3 + q \Delta_0 + q \Delta_2 x_3 + q \Delta_1 x_2,$$

$$\Delta_1 \Delta_3 = q \Delta_1 + q \Delta_2 x_2 + 2q^2 \Delta_0 x_3,$$

$$\Delta_2^2 = q \Delta_1 + q \Delta_2 x_2 + q^2 \Delta_0 x_3,$$

$$\Delta_3 \Delta_3 = q \Delta_2 + q^2 \Delta_0 x_2 + q^2 \Delta_1 x_3,$$

$$\Delta_3^2 = q^2 \Delta_0 + q^2 \Delta_1 x_2 + 2q^2 \Delta_2 x_3.$$

Let $\xi_i, i = 1, \ldots, 3$ be the three roots of $\xi^3 = 4q$. Then the respective idempotents have the following form:

$$e_0 = \frac{1}{2} \Delta_0 - \frac{1}{2q} \Delta_3 + \frac{x_2}{4} \Delta_1 + \frac{x_3}{2} \Delta_2$$

and, for $i = 1, \ldots, 3$,

$$e_i = \frac{1}{6} \Delta_0 + \frac{\xi_i^2}{12q} \Delta_1 + \frac{\xi_i}{6q} \Delta_2 + \frac{1}{6q} \Delta_3 - \frac{\xi_i x_2}{36} \Delta_0$$
\[-\left(\frac{x_2}{12} + \frac{\xi_i x_3}{12}\right) \Delta_1 - \left(\frac{\xi_i^2 x_2}{18q} + \frac{x_3}{6}\right) \Delta_2 - \left(\frac{\xi_i x_2}{27q} + \frac{\xi_i^2 x_3}{12q}\right) \Delta_3.\]

Probably the most direct and efficient test of our computations is to simply compute the pairwise products of these idempotents using the multiplication table above. This verifies the formulas of the \(e_i\) while checking at the same time that our multiplication table yields an associative product.

Note that \(\Delta_i = \frac{\partial}{\partial x_i}\) and \(\frac{\partial}{\partial x_i} q = q\) since we identify \(q\) with \(e^{x_1}\) to get a proper (non–formal) Frobenius manifold. So we get as special coordinates (where \(i, j \in \{1, 2, 3\}\)):

\[\eta_0 = -\frac{1}{2q}, \quad \eta_i = \frac{1}{6q}, \quad \eta_{00} = 0,\]

\[\eta_{0i} = \frac{\xi_i^2}{12q} \Delta_1 \left(\frac{-1}{2q}\right) = \frac{\xi_i^2}{24q^2}, \quad \eta_{0i} = -\frac{1}{2q} \Delta_3 \left(-\frac{\xi_i^2 x_3}{12q}\right) = \frac{\xi_i^2}{24q^2},\]

\[\eta_{ij} = \frac{\xi_i^2}{12q} \Delta_1 \left(\frac{1}{6q}\right) - \frac{\xi_i}{6q} \Delta_2 \left(\frac{\xi_i x_2}{27q}\right) - \frac{1}{6q} \Delta_3 \left(\frac{\xi_i^2 x_3}{12q}\right) = -\frac{\xi_i^2}{72q^2} - \frac{\xi_i \xi_j}{162q^2} - \frac{\xi_i^2}{72q^2}.\]

The symmetry of \(\eta_{ij}\) is just an additional check of our computations, as we know in general that \(e_i\) are commuting vector fields.

**Manifold \(V_5\):**

\[\Delta_1 = 5\Delta_2 + 3q \Delta_0 + 3q \Delta_2 x_3 + q \Delta_1 x_2 + 4q^2 \Delta_0 x_3,\]

\[\Delta_1 \Delta_2 = \Delta_3 + q \Delta_1 + q \Delta_2 x_2 + 2q^2 \Delta_1 x_3 + 2q^2 \Delta_0 x_2,\]

\[\Delta_1 \Delta_3 = 3q \Delta_2 + 2q^2 \Delta_0 + 4q^2 \Delta_2 x_3 + 2q^2 \Delta_1 x_2 + 3q^3 \Delta_0 x_3,\]

\[\Delta_2 = q \Delta_2 + q^2 \Delta_0 + 2q^2 \Delta_2 x_3 + q^2 \Delta_1 x_2 + 2q^3 \Delta_0 x_3,\]

\[\Delta_2 \Delta_3 = q^2 \Delta_1 + 2q^2 \Delta_2 x_2 + 2q^3 \Delta_1 x_3 + 2q^3 \Delta_0 x_2,\]

\[\Delta_3 = 2q^2 \Delta_2 + q^3 \Delta_0 + 3q^3 \Delta_2 x_3 + 2q^3 \Delta_1 x_2 + 3q^4 \Delta_0 x_3.\]

Let \(u_i\) be the roots of \(u^4 - 44q u^2 - 16q^2\). The idempotents are given by

\[4000q^3 e_i = 1440q^3 \Delta_0 - 20q^2 u_i^2 \Delta_0 + 70q u_i^3 \Delta_1 - 3040q^2 u_i \Delta_1\]

\[-880q^2 \Delta_2 + 40 q u_i \Delta_2 + 4920 q u_i \Delta_3 - 110 u_i^3 \Delta_3\]

\[-1968 u_i q^3 x_2 \Delta_0 + 44 u_i^3 q^2 x_2 \Delta_0 + 352 q^4 x_3 \Delta_0 - 16 u_i^2 q^2 x_3 \Delta_0\]

\[+ 176 q^2 x_2 \Delta_1 - 8 q^2 u_i^2 \Delta_1 x_2 - 5412 u_i q^3 \Delta_1 x_3 + 121 u_i^2 q^2 \Delta_1 x_3\]

\[- 2864 u_i q^2 x_2 \Delta_2 + 62 q u_i^3 x_2 \Delta_2 + 1056 q^3 \Delta_2 x_3 - 48 x_3 u_i^2 q^2 \Delta_2\]
We calculated

\[-16qu_1^2x_2\Delta_3 + 6036q^2u_i x_3\Delta_3 + 352q^2x_2\Delta_3 - 138qu_1^3x_3\Delta_3.\]

The special coordinates \(\eta_{ii}\) are

\[\eta_{ii} = \frac{-964q + 21u_i^2}{800q^3} = \frac{-251 \pm 105\sqrt{5}}{400q^2}.\]

Now since the Galois group of \(u^4 - 44qu^2 - 16q^2\) obviously does not act transitively on the pairs of roots, we have to distinguish two cases in determining the \(\eta_{ij}\). So we fix a root \(u_1\); the other roots are given by \(u_2 = -u_1\) and \(u_{3,4} = 11q + u_1^2 = 0\). We calculated

\[\eta_{12} = \eta_{21} = \frac{932q - 21u_1^2}{800q^3} = \frac{47 \pm 21\sqrt{5}}{80q^2}\]

and

\[\eta_{13} = \frac{-3u_3u_1 + 4q}{200q^3}\]

All other coordinates are obtained from these via Galois permutations.

**Manifold \(V_{22}\):**

\[
\begin{align*}
\Delta_1^2 &= 22\Delta_2 + 2q\Delta_1 + 24q^2\Delta_0 \\
+2q\Delta_2x_2 + 48q^2\Delta_2x_3 + 4q^2\Delta_1x_2 + 27q^3\Delta_1x_3 + 27q^3\Delta_0x_2 + 160q^4\Delta_0x_3, \\
\Delta_1\Delta_2 &= \Delta_3 + 2q\Delta_2 + 2q^2\Delta_1 + 9q^3\Delta_0 \\
+4q^2\Delta_2x_2 + 27q^3\Delta_2x_3 + 3q^3\Delta_1x_2 + 16q^4\Delta_1x_3 + 16q^4\Delta_0x_2 + 80q^5\Delta_0x_3, \\
\Delta_1\Delta_3 &= 24q^2\Delta_2 + 9q^3\Delta_1 + 40q^4\Delta_0 \\
+27q^3\Delta_2x_2 + 160q^4\Delta_2x_3 + 16q^4\Delta_1x_2 + 80q^5\Delta_1x_3 + 80q^5\Delta_0x_2 + 390q^6\Delta_0x_3, \\
\Delta_2^2 &= 2q^2\Delta_2 + q^3\Delta_1 + 4q^4\Delta_0 \\
+3q^3\Delta_2x_2 + 16q^4\Delta_2x_3 + 2q^4\Delta_1x_2 + 9q^5\Delta_1x_3 + 9q^5\Delta_0x_2 + 41q^6\Delta_0x_3, \\
\Delta_2\Delta_3 &= 9q^3\Delta_2 + 4q^4\Delta_1 + 16q^5\Delta_0 \\
+16q^4\Delta_2x_2 + 80q^5\Delta_2x_3 + 9q^5\Delta_1x_2 + 41q^6\Delta_1x_3 + 41q^6\Delta_0x_2 + 186q^7\Delta_0x_3, \\
\Delta_3^2 &= 40q^4\Delta_2 + 16q^5\Delta_1 + 65q^6\Delta_0 \\
+80q^5\Delta_2x_2 + 390q^6\Delta_2x_3 + 41q^6\Delta_1x_2 + 186q^7\Delta_1x_3 + 186q^7\Delta_0x_2 + 840q^8\Delta_0x_3.
\end{align*}
\]

The first idempotent is given by

\[e_0 = \frac{\Delta_0^2}{2} + \frac{2\Delta_2}{q^2} - \frac{\Delta_3}{2q^3} + qx_2\Delta_0 + 2q^2x_3\Delta_0\]
Now let $u_i, i = 1, \ldots, 3$ be the roots of $u^3 - 8qu^2 - 56q^2u - 76q^3$. Then the respective idempotents are given by

$$5324q^4e_i =$$

$$(-71742q^4 - 24552q^3u_i + 2354q^2u_i^2) \Delta_0 + (-30272q^3 - 10186q^2u_i + 979qu_i^2) \Delta_1$$

$$+(-118712q^2 - 38126qu_i + 3696u_i^2) \Delta_2 + \left(49346q + 16192u_i - 1562\frac{u_i^2}{q}\right) \Delta_3$$

$$+(-130876q^5 - 43283u_iq^4 + 4168u_i^2q^3) x_2 \Delta_0$$

$$+(-483464q^6 - 161648u_iq^5 + 15528u_i^2q^4) x_3 \Delta_0$$

$$+(-38977q^4 - 12940u_iq^3 + 1245u_i^2q^2) x_2 \Delta_1$$

$$+(-145898q^5 - 48889u_iq^4 + 4694u_i^2q^3) x_3 \Delta_1$$

$$+(-143992q^3 - 46818u_iq^2 + 4522u_i^2q) x_2 \Delta_2$$

$$+(-491334q^4 - 164832u_iq^3 + 15822u_i^2q^2) x_3 \Delta_2$$

$$+(-35042q^2 + 11348u_iq - 1098u_i^2) x_2 \Delta_3$$

$$+(-112272q^3 + 37824u_iq^2 - 3633u_i^2q) x_3 \Delta_3.$$ 

From this we compute the special coordinates

$$\eta_0 = -\frac{1}{2q^3}, \quad \eta_i = 49346q + 16192u_i - 1562\frac{u_i^2}{q},$$

$$\eta_{00} = -\frac{1}{q^4}, \quad \eta_{0i} = \eta_{0i} = -\frac{-2536q^2 - 688u_iq + 69u_i^2}{968q^6},$$

$$\eta_{ii} = \frac{-3412q^2 - 260u_iq + 41u_i^2}{484q^6}$$

$$\eta_{ij} = \frac{404}{121q^4} - \frac{34}{121}\frac{u_i + u_j}{q^5} + \frac{13}{968}\frac{u_ju_i}{q^6}.$$
§3. Del Pezzo surfaces

3.1. Notation. Let $V$ be a surface. For surfaces, we will denote the dual class of a point by $\Delta_2$, and the dual coordinate by $z$. Assuming $h^{2,0}(V) = 0$, we may and will identify the group $H_2(V, \mathbb{Z})/(\text{tors})$ with $H^{1,1}(V, \mathbb{Z})$.

We start again with calculating the structure constants of the quantum multiplication of classes in $H^{p,p}$ restricted to the first infinitesimal neighborhood of $H^{1,1}$. Essentially, this means that we must calculate the third derivatives of the potential modulo $z^2$. The canonical coordinates of this “very small quantum cohomology” are the eigenvalues of the operator $k_0$, $k = c_1(V)$, calculated at the points of $H^{1,1}$ so that only triple correlators are involved in their calculation.

3.2. Quantum multiplication table. Since $\Delta_0$ is the identity with respect to $\circ$, we need only the following formulas, which follow from (0.1). Let $D_1, D_2 \in H^2(V)$, then

$$D_1 \circ D_2 = (D_1, D_2) \Delta_2 + \sum_{\beta \neq 0} (e^z \Delta_2)_{\beta}(D_1, \beta) (D_2, \beta) q^\beta + \left( \sum_{\beta \neq 0} \langle \Delta_2 e^z \Delta_2 \rangle_{\beta}(D_1, \beta) (D_2, \beta) q^\beta \right) \Delta_0. \quad (3.1)$$

Similarly, for $D \in H^2(V)$,

$$D \circ \Delta_2 = \sum_{\beta \neq 0} \langle \Delta_2 e^z \Delta_2 \rangle_{\beta}(D, \beta) q^\beta + \left( \sum_{\beta \neq 0} \langle \Delta_2^2 e^z \Delta_2 \rangle_{\beta}(D, \beta) q^\beta \right) \Delta_0. \quad (3.2)$$

Finally,

$$\Delta_2 \circ \Delta_2 = \sum_{\beta \neq 0} \langle \Delta_2^2 e^z \Delta_2 \rangle_{\beta} q^\beta + \left( \sum_{\beta \neq 0} \langle \Delta_2^3 e^z \Delta_2 \rangle_{\beta} q^\beta \right) \Delta_0. \quad (3.3)$$

If we want only terms modulo $z^2$, (1.5) can be used in order to restrict the summation over $\beta$:

$$D_1 \circ D_2 = (D_1, D_2) \Delta_2 + \sum_{\beta: k(\beta) = 1} \langle \emptyset \rangle_{\beta}(D_1, \beta) (D_2, \beta) q^\beta + \left( \sum_{\beta: k(\beta) = 2} \langle \Delta_2 \rangle_{\beta}(D_1, \beta) (D_2, \beta) q^\beta \right) \Delta_0 + z \sum_{\beta: k(\beta) = 2} \langle \Delta_2^2 \rangle_{\beta}(D_1, \beta) (D_2, \beta) q^\beta + z \left( \sum_{\beta: k(\beta) = 3} \langle \Delta_2^3 \rangle_{\beta}(D_1, \beta) (D_2, \beta) q^\beta \right) \Delta_0 + O(z^2), \quad (3.4)$$
\[ D \circ \Delta_2 = \sum_{\beta : k(\beta) = 2} \langle \Delta_2 \rangle_{\beta} (D, \beta) q^\beta \beta + \left( \sum_{\beta : k(\beta) = 3} \langle \Delta_2^2 \rangle_{\beta} (D, \beta) q^\beta \right) \Delta_0 + \]
\[ z \sum_{\beta : k(\beta) = 3} \langle \Delta_2^2 \rangle_{\beta} (D, \beta) q^\beta \beta + z \left( \sum_{\beta : k(\beta) = 4} \langle \Delta_2^3 \rangle_{\beta} (D, \beta) q^\beta \right) \Delta_0 + O(z^2). \quad (3.5) \]

In the case when one of the divisor classes is \( k \), (3.4) and (3.5) further simplify:

\[ k \circ D = k(D) \Delta_2 + \]
\[ \sum_{\beta : k(\beta) = 1} \langle \emptyset \rangle_{\beta} (D, \beta) q^\beta \beta + 2 \left( \sum_{\beta : k(\beta) = 2} \langle \Delta_2 \rangle_{\beta} (D, \beta) q^\beta \right) \Delta_0 + \]
\[ 2z \sum_{\beta : k(\beta) = 2} \langle \Delta_2 \rangle_{\beta} (D, \beta) q^\beta \beta + 3z \left( \sum_{\beta : k(\beta) = 3} \langle \Delta_2^2 \rangle_{\beta} (D, \beta) q^\beta \right) \Delta_0 + O(z^2). \quad (3.6) \]

In particular,

\[ k \circ k = (k, k) \Delta_2 + \sum_{\beta : k(\beta) = 1} \langle \emptyset \rangle_{\beta} q^\beta \beta + 4 \left( \sum_{\beta : k(\beta) = 2} \langle \Delta_2 \rangle_{\beta} q^\beta \right) \Delta_0 + \]
\[ 4z \sum_{\beta : k(\beta) = 2} \langle \Delta_2 \rangle_{\beta} q^\beta \beta + 9z \left( \sum_{\beta : k(\beta) = 3} \langle \Delta_2^2 \rangle_{\beta} q^\beta \right) \Delta_0 + O(z^2). \quad (3.7) \]

Furthermore,

\[ k \circ \Delta_2 = 2 \sum_{\beta : k(\beta) = 2} \langle \Delta_2 \rangle_{\beta} q^\beta \beta + 3 \left( \sum_{\beta : k(\beta) = 3} \langle \Delta_2^2 \rangle_{\beta} q^\beta \right) \Delta_0 + \]
\[ 3z \sum_{\beta : k(\beta) = 3} \langle \Delta_2^2 \rangle_{\beta} q^\beta \beta + 4z \left( \sum_{\beta : k(\beta) = 4} \langle \Delta_2^3 \rangle_{\beta} q^\beta \right) \Delta_0 + O(z^2). \quad (3.8) \]

### 3.3. Del Pezzo surfaces

Let \( V = V_r \) be obtained from \( \mathbb{P}^2 \) by blowing up \( r \) points in general position. We get del Pezzo surfaces (characterized by the ampleness of \( k \)) for \( r \leq 8 \). Sums in the right hand sides of (3.4)–(3.8) for larger ranks \( r + 1 \) of \( H^{1,1}(V) \) are long and for \( r \geq 9 \) are infinite. However, for del Pezzo surfaces (and to a certain degree, for more general rational surfaces) the situation
becomes simpler because a Weyl group acts upon $\beta \in H^{1,1}(V)$ leaving $k(\beta)$, $(\beta, \beta)$ and the correlators $\langle \Delta_2^{k(\beta)-1} \rangle_\beta$ invariant. For $k(\beta) \leq 4$ the number of orbits is quite small, and the resulting expressions become manageable. We start with some classical results on the structure of $H^2(V, \mathbb{Z})$ (see e.g. [Ma3]).

Fix a representation of $V_r$ as blow up of $\mathbb{P}^2$. Then $k$ and the classes of the resulting exceptional curves $l_1, \ldots, l_r$ generate a sublattice of index 3 in the Néron–Severi group coinciding with $H^2(V, \mathbb{Z})$. The whole lattice $N_r$ is generated by the orthonormal basis $(l_0; l_1, \ldots, l_r)$ where $l_0$ is the class of a line lifted to $V$. We identify $\beta = a l_0 - \sum_i b_i l_i$ with the vector $(a; b_1, \ldots, b_r)$ so that $a = a(\beta) = (\beta, l_0)$ and $b_i = b_i(\beta) = (\beta, l_i)$. We have $k = k_r = (3; 1, \ldots, 1)$ in this notation.

Denote by $R_r$ the set $\{ \beta \mid k(\beta) = 0, (\beta, \beta) = -2 \}$. It is a system of roots, lying entirely in the orthogonal complement $O_r$ to $k_r$. Reflections with respect to the roots generate the Weyl group $W_r$ fixing $k_r$. In our representation, this group for $r \geq 3$ is generated by all permutations of $l_r$'s and by the Cremona transformation

$$(a; b_1, \ldots, b_r) \mapsto (a + \delta; b_1 + \delta, b_2 + \delta, b_3 + \delta, b_4, \ldots, b_r), \quad \delta := a - b_1 - b_2 - b_3. \quad (3.9)$$

The function $\beta \mapsto \langle \Delta_2^{k(\beta)-1} \rangle_\beta$ is constant on $W_r$–orbits. This is well known. One argument is given in [GP], 5.1. It involves geometric Cremona transformations (blowing up and down), which generate $W_r$, together with permutations of $(l_1, \ldots, l_r)$. Alternatively, one can invoke Hirschowitz's theorem ([Hirsch]) which shows that the action of $W_r$ on $N_r$ is induced by the action of $W_r$ on a generic del Pezzo surface and its definition field.

In fact, a related reasoning readily extends to a more general situation. Consider a family of algebraic manifolds over a base whose fundamental group acts on the fiber cohomology in a nontrivial way. A typical situation is the existence of several Lefschetz type transforms associated with various cusps of the moduli space. If they are reflections, a version of the following proposition will hold.

**3.3.1. Proposition.** *Let $\rho$ be a root in $R_r$, $s_\rho : \gamma \mapsto \gamma + (\gamma, \rho)\rho$ the respective reflection. It induces the involution of the algebraic torus $T_{V_r}$ with the character group $H^2(V_r, \mathbb{Z})$. Let $T_\rho$ be the codimension one subtorus consisting of fixed points of this involution: $q^{s_\rho(\beta)} = q^\beta$ on $T_\rho$ for all $\beta$. Then we have, after restriction on $T_\rho$:*

a) *For any $D \in H^2(V)$ orthogonal to $\rho$*

$$D \circ \rho = \mu_{D, \rho} \rho, \quad \mu_{D, \rho} = -\frac{1}{2} \sum_\beta \langle e^{z \Delta_2} \rangle_\beta (\rho, \beta)^2 (D, \beta) q^\beta. \quad (3.11)$$

*Similarly,*

$$\Delta_2 \circ \rho = \nu_\rho \rho, \quad \nu_\rho = -\frac{1}{2} \sum_\beta \langle e^{z \Delta_2} \rangle_\beta (\rho, \beta)^2 q^\beta. \quad (3.12)$$
b) If $D_1, D_2$ are orthogonal to $\rho$, then $D_1 \circ D_2$ is orthogonal to $\rho$ on $T_\rho$; the same is true for $\rho \circ \rho$.

**Proof.** We have the following general identity. Let $S$ be a $W_r$–orbit, $D_1, \ldots, D_n$ orthogonal to $\rho$, and $m$ an integer. Then we have on $T_\rho$

$$\sum_{\beta \in S} (D_1, \beta) \ldots (D_n, \beta) (\rho, \beta)^{2m+1} q^\beta = 0. \quad (3.13)$$

To see this, replace $s$ by $s_\rho(\beta)$ in the summands of (3.13). The whole sum will remain the same, whereas each summand will change sign, because of the factor $(\rho, \beta)^{2m+1}$.

Consider now (3.1) for $D_1 = D, D_2 = \rho$. In view of (3.13), the coefficient of $\Delta_0$ vanishes on $T_\rho$. Take the scalar product with any $D'$ orthogonal to $\rho$. It will vanish, again in view of (3.13). Hence our vector is proportional to $\rho$. To calculate the coefficient $\mu_\rho$, it suffices to take the scalar product with $\rho$. In order to prove (3.12), we treat similarly (3.2) with $D = \rho$.

To establish b), look at (3.1) with both $D_i$ orthogonal to $\rho$, resp. with $D_1 = D_2 = \rho$. (Notice that in these cases the coefficient of $\Delta_0$ does not vanish, but of course, $\Delta_0$ is still orthogonal to $\rho$ in the Poincaré metric). This ends the proof.

The intersection of all subtori $T_\rho$ is the one–dimensional subgroup $q^\beta = e^{tk_r(\beta)}$, $t$ arbitrary. On this subgroup, and in particular at $t = 0$ (which we write as $q = 1$), the statements of the Proposition 3.3.1 are valid simultaneously for all roots $\rho$. Hence we get the following corollary.

**3.3.2. Corollary.** On the subgroup defined above, $k_r \circ, \Delta_2 \circ$ act on the orthogonal complement $O_r$ to $k_r$ as multiplication by the scalars (independent on the choice of $\rho \in R_r$) denoted respectively

$$\mu = -\frac{1}{2} \sum_{\beta} \langle e^{z\Delta_2} \rangle_\beta (\rho, \beta)^2 k_r(\beta) e^{tk_r(\beta)}, \quad (3.14)$$

$$\nu = -\frac{1}{2} \sum_{\beta} \langle \Delta_2 e^{z\Delta_2} \rangle_\beta (\rho, \beta)^2 e^{tk_r(\beta)}, \quad (3.15)$$

In fact, $O_r$ is spanned by all roots in $R_r$.

**3.4. Structure of the sets** $k(\beta) = a$. For each $3 \leq r \leq 8$ we denote by $I_r, F_r, G_r, H_r$ the following subsets of $N_r$, each constituting one orbit of $W_r$. The statements accompanying their definitions are proved, for example, in [Ma3], Ch. 4.

$$I_r := \{ \beta | k(\beta) = (\beta, \beta) = -1 \} = W_r(0; -1; 0, \ldots, 0). \quad (3.16)$$
Elements of $I_r$ are exactly classes of exceptional curves on a general del Pezzo surface of degree $9 - r$. (This is also true for $r = 9$ but false for larger values of $r$: cf. [Hirsch], 3.4).

$$F_r := W_r(l_0 - l_1) = W_r(1; 1, 0, \ldots, 0).$$  \hspace{1cm} (3.17)

$$G_r := W_r l_0 = W_r(1; 0, 0, \ldots, 0).$$  \hspace{1cm} (3.18)

$$H_r := W_r(l_1 + l_2) = W_r(0; 0, -1, 0, \ldots, 0).$$  \hspace{1cm} (3.19)

Elements of $H_r$ for $r \geq 4$ are exactly classes of cycles $\lambda + \mu$ where $\lambda$ and $\mu$ are two disjoint exceptional curves. For $r = 3$, however, there are two $W_3$–orbits of such cycles: blowing down two disjoint exceptional curves can produce either $\mathbf{P}^2$ blown up at a single point or $\mathbf{P}^1 \times \mathbf{P}^1$. The orbit $H_3$ corresponds to the first case.

3.4.1. Proposition. The total support and nonzero values of the function $\beta \mapsto \langle \Delta_2^{k(\beta)-1} \rangle_{\mu}$ for $k(\beta) \leq 3$ are given in the following list:

- a) $\langle \emptyset \rangle_{\beta} = 1$ on $I_r$ for all $3 \leq r \leq 8$.
  - In $N_8$ in addition $\langle \emptyset \rangle_{k_8} = 12$ (recall that $k_r$ is $W_r$–invariant for any $r$).
- b) $\langle \Delta_2 \rangle_{\beta} = 1$ on $F_r$ for all $3 \leq r \leq 8$.
  - In $N_7$ in addition $\langle \Delta_2 \rangle_{k_7} = 12$.
  - In $N_8$ in addition $\langle \Delta_2 \rangle_{\beta} = 12$ on $k_8 + I_8$ and $\langle \Delta_2 \rangle_{2k_8} = 90$.
- c) $\langle \Delta_2^2 \rangle_{\beta} = 1$ on $G_r$ for all $3 \leq r \leq 8$.
  - In $N_6$ in addition $\langle \Delta_2^2 \rangle_{k_6} = 12$.
  - In $N_7$ in addition $\langle \Delta_2^2 \rangle_{\beta} = 12$ on $k_7 + I_7$.
  - In $N_8$ there are four additional orbits with the following values: 12 on $k_8 + H_8$; 96 on $k_8 + F_8$; 576 on $2k_8 + I_8$; 2880 on $3k_8$.

Proof. The argument depends on compiling and studying a table, fortunately rather short one.

It lists all non–zero 9–uples $\beta = (a; b_1, \ldots, b_8)$ in $\mathbf{Z}^9$ with the following properties:

$$1 \leq k(\beta) = 3a - b_1 - \cdots - b_9 \leq 3, \ b_1 \geq b_2 \geq \cdots \geq b_8 \geq 0, \ a \geq b_1 + b_2 + b_3. \hspace{1cm} (3.20)$$

In addition, it includes $(0; 0, -1, 0, \ldots, 0)$ and contains 10 entries altogether. In the notation omitting final zeros and rendering multiplicities of $b_i$’s by superscripts (cf. [GP], p. 87), they are

$$(0; -1), \ (1; \cdot), \ (1; 1), \ (3; 1^6), \ (3; 1^7), \ (3; 1^8), \ (4; 2, 1^7), \ (6; 2^7, 1), \ (6; 2^8), \ (9; 3^8). \hspace{1cm} (3.21)$$

According to [GP], sections 3 and 5, any $\beta \in N_8$ with $\langle \Delta_2^{k(\beta)-1} \rangle_{\beta} \neq 0$ and $k(\beta) \leq 3$ is $W_8$–equivalent to one of the entries of this list. (Actually, to a single entry, because they all are pairwise distinguished by the values of pairs $k(\beta), (\beta, \beta)$). Moreover, if an entry ends with $\geq s$ zeroes, then for $s \geq 3$ its $W_{8-s}$–orbit contains all $\beta \in N_{8-s}$ with non–zero correlators and respective value of $k(\beta)$. 

In fact, for any such $\beta$ we can first permute $b_i$’s to make them decreasing. If we get then $\delta := a - b_1 - b_2 - b_3 < 0$, we can decrease $a$ by applying the respective Cremona transformation (3.9) or, which is the same, by reflecting $\beta$ with respect of one of the simple roots in $N_{8-s}$. If after several steps of this kind we get a vector with $a > 0$ and some negative $b_i$, or with $a = 0$ but not with exactly one $b_i = -1$ and zeros on the remaining places, then $\langle \Delta_2^{k(\beta)-1} \rangle_\beta = 0$ according to [GP]. Hence we have to consider only the 10 classes listed above.

The values of $\langle \Delta_2^{k(\beta)-1} \rangle_\beta$ for these classes can be read off from the tables on p. 88 of [GP]. One exception is $(9; 3, \ldots, 3)$ which must be calculated using recursion; its value was communicated to us by L. Göttscbe.

Finally, we can directly identify the relevant $W_r$-orbits with the sets listed in the Proposition. Notice however that adding zeros changes the invariant description of the vector. For example, $(3; 1^6)$ produces $k_6$ in $N_6$, but $k_7 + l_7$ in $N_7$, and for the respective orbits we have $W_6k_6 = \{k_6\}$, $W_7(k_7 + l_7) = k_7 + l_7$.

The complete list of $F_8$ (calculated for other purposes) in this format can be found in [MaTschi] on p. 329. Similarly, the complete list of $G_8$ can be extracted from the table on p. 330 in the following way: replace the misprinted entry $(10; 5^2, 4^3, 3^2, 2)$ by the correct one $(10; 5^2, 3^5, 2)$ and delete the following six entries belonging to another orbit (with vanishing Gromov–Witten invariant): $(5; 3^2, 1^6), (7; 3^5, 1^3), (9; 5, 3^6, 1), (11; 5^3, 3^5), (13; 5^6, 3^2), (15; 7, 5^7)$.

3.5. The point $q = 1, z = 0$. We will state some simple lemmas which use the $W_r$-invariance and will simplify the calculation of the quantum cup product; at the end of this section we will combine the obtained information to describe the quantum cohomology algebra at $q = 1, z = 0$ in the case $r \geq 5$.

First, note the following

3.5.1 Fact. If $r \geq 5$ and $\rho$ is a root, then the orthogonal subspace $\rho^\perp$ inside the root space $O_r$ is generated by the roots orthogonal to $\rho$.

Indeed, if $r = 5$ and $\rho = (0; -1, 1, 0, 0, 0)$, then $(1; 1, 1, 1, 0, 0), (1; 0, 0, 1, 1, 1), (0; 0, 0, -1, 1, 10)$ and $(0; 0, 0, 0, -1, 1)$ are linearly independent roots and all orthogonal to $\rho$. This argument can easily be extended to higher $r$.

This fact will actually make a difference in several computations, which is why the Proposition 3.5.5 would be false for $r \leq 4$.

3.5.2. Lemma. Let $S$ be a $W_r$ orbit in $N_r$, $k_r(S) := k_r(\beta)$ for an arbitrary $\beta \in S$, and let $\rho \perp \rho'$ be two orthogonal roots. Then we have:

\begin{align*}
a) \sum_{\beta \in S} \beta &= k_r(S) \frac{|S|}{9-r} k_r. \\
b) \sum_{\beta \in S} (\rho, \beta) \beta &= -\frac{1}{2} \sum_{\beta \in S} (\rho, \beta)^2 \rho. \\
c) \sum_{\beta \in S} (\rho, \beta)^m (\rho', \beta) &= \sum_{\beta \in S} (\rho, \beta)^{2m+1} = 0. \end{align*}
d) If \( r \geq 5 \), then \( \sum_{\beta \in S} (\rho, \beta)^2 \beta = \frac{k_r(S)}{9 - r} \sum_{\beta \in S} (\rho, \beta)^2 k_r. \)

e) \( \sum_{\beta \in S} (\rho, \beta)(\rho', \beta)\beta = 0. \)

**Proof.** a) Since \( \sum_{\beta \in S} \beta \) is a \( W_r \)-invariant element of \( N_r \), it must be proportional to \( k_r. \) To calculate the proportionality coefficient, it remains to take the intersection index with \( k_r. \)

b) This follows from symmetrizing over \( \beta \) and \( s_\rho(\beta) = \beta + (\rho, \beta)\rho: \)

\[
\sum_{\beta \in S} (\rho, \beta)\beta = \frac{1}{2} \sum_{\beta \in S} ((\rho, \beta) + (\rho, s_\rho(\beta))s_\rho(\beta)) = \frac{1}{2} \sum_{\beta \in S} ((\rho, \beta) - (\rho, \beta)(\beta + (\rho, \beta)\rho)).
\]

c) These are special cases of (3.13).

d) Since we assume \( r \geq 5 \), this formula can be checked by taking the intersection index of the right- and left-hand side with \( k_r \), with \( \rho \) and with an arbitrary root \( \rho' \perp \rho \), respectively. For \( \rho, \rho' \) this follows from c), and for \( k_r \) it is obvious.

e) As in b), we symmetrize over \( \beta \) and \( s_\rho(\beta) \), from which we see that the sum is equal to \( -\frac{1}{2} \sum_{\beta \in S} (\rho, \beta)^2 (\rho', \beta)\rho; \) this is zero according to c).

### 3.5.3. Cardinalities.

The following table lists the cardinalities of the sets introduced above:

| \( r \) | 3 | 4 | 5 | 6 | 7 | 8 |
| --- | --- | --- | --- | --- | --- | --- |
| \( |W_r| \) | \( 2^3 \) | \( 2^3 \cdot 5 \) | \( 2^7 \cdot 5 \) | \( 2^7 \cdot 5 \cdot 3 \) | \( 2^{10} \cdot 5 \cdot 7 \) | \( 2^{14} \cdot 5 \cdot 27 \) |
| \( |R_r| \) | \( 2^3 \) | \( 2^3 \cdot 5 \) | \( 2^3 \cdot 5 \) | \( 2^3 \cdot 2 \) | \( 2 \cdot 3^2 \) | \( 2 \cdot 3 \cdot 5 \) |
| \( |I_r| \) | \( 2 \cdot 3 \) | \( 2 \cdot 5 \) | \( 2^4 \) | \( 3^3 \) | \( 2^3 \cdot 7 \) | \( 2^3 \cdot 5 \) |
| \( |F_r| \) | \( 3 \) | \( 5 \) | \( 2 \cdot 5 \) | \( 3^3 \) | \( 2 \cdot 3 \cdot 7 \) | \( 2^3 \cdot 3 \cdot 5 \) |
| \( |G_r| \) | \( 2 \) | \( 5 \) | \( 2^4 \) | \( 2^3 \cdot 2 \) | \( 2^6 \cdot 3 \cdot 2 \) | \( 2^7 \cdot 3 \cdot 5 \) |
| \( |H_r| \) | \( 2 \cdot 3 \) | \( 2 \cdot 3 \cdot 5 \) | \( 2^4 \cdot 5 \) | \( 2^3 \cdot 3 \) | \( 2^2 \cdot 3 \cdot 7 \) | \( 2^6 \cdot 3 \cdot 5 \cdot 7 \) |

The orders of \( W_r, R_r \) and \( I_r \) are well known. The remaining entries can be calculated directly for \( r = 3 \) and then inductively in the following way. There are exactly \( r \) exceptional classes in \( N_r \) having zero intersection index with \( (1; 0^r) \), and they are pairwise orthogonal. Hence \( |G_r| \) contains as many elements as there are maximal contractible \( r \)-tuples of exceptional classes in \( N_r \), that is \( |I_r| |G_{r-1}| / r. \) Similarly, \( |F_r| \) counts non-maximal contractible \( (r - 1) \)-tuples so that \( |F_r| = |I_r| |F_{r-1}| / 2(r - 1). \) Finally, \( |H_r| \) counts unordered contractible pairs, contained in maximal contractible \( r \)-tuples so that \( |H_r| = |I_r| |I_{r-1}| / 2. \)

### 3.5.4. Proposition.

At \( q = 1, z = 0 \), the operator \( k_r \circ \) admits two complementary orthogonal invariant subspaces: one spanned by \( \Delta_0, k_r, \Delta_2, \) the other being \( O_r. \)

On the first subspace, the characteristic polynomial \( \det (\lambda \text{id} - k_r \circ) \) is

\[
R(\lambda) = \lambda^3 - B_r \lambda^2 - D_r \lambda - 36 (9 - r) C_r \tag{3.22}
\]
where
\[ B_r = \frac{|I_r|}{9 - r}, \quad C_r = \frac{|G_r|}{12} + \delta_{r,6} + 56 \delta_{r,7} + 35760 \delta_{r,8}, \]
\[ D_r = 8 (|F_r| + 12 \delta_{r,7} + 12 \delta_{r,8}|I_8| + 90 \delta_{r,8}) \quad (3.23) \]
(\(\delta_{r,a}\) being the Kronecker symbol).

On the second subspace, \(k_r \circ \) is multiplication by a constant \(\mu_r\) if \(r \geq 4\). This constant is given by the following table:

| \(r\) | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|
| \(\mu_r\) | -3 | -4 | -6 | -12 | -60 |

**Proof.** Formulas (3.7) and (3.8), combined with Proposition 3.4.1 and Lemma 3.5.1, lead to the following action of \(k_r \circ \) at \(q = 1, z = 0\) on the first subspace:

\[ k_r \circ \Delta_0 = k_r \text{ and } k_r \circ \Delta_2 = 2 \left( \frac{2|F_r|}{9 - r} + 12 \delta_{r,7} + 24 \delta_{r,8}|I_8| + 180 \delta_{r,8} \right) k_r + \left(\frac{2|F_r|}{9 - r} + 12 \delta_{r,7} + 24 \delta_{r,8}|I_8| + 180 \delta_{r,8} \right) \Delta_0. \quad (3.24) \]

The direct calculation of the characteristic polynomial then gives (3.22) and (3.23).

To treat the orthogonal complement \(O_r\) of \(k_r\) in \(H^2(V)\), notice first of all that it is spanned by roots \(\rho \in R_r\). If \(r \geq 4\), they form a single \(W_r\)-orbit. We calculate \(k_r \circ \rho\) at \(q = 1, z = 0\) using (3.6). The coefficient of \(\Delta_0\) vanishes according to 3.5.2 c), while the first sum in (3.6) is equal to \(-\frac{1}{2} \sum_{\beta} \langle \emptyset \rangle_{\beta}(\rho, \beta)^2 \rho =: \mu_r\) by lemma 3.5.2 b). In view of \(W_r\)-invariance, the coefficient cannot depend on the root \(\rho\) if \(r \geq 4\).

The value of \(\mu_r\) can be computed explicitly by using a complete list of the \(W_r\)-orbit \(I_r\) as given in [Ma3]; see the relevant remark in the proof of the Proposition 3.5.5 for a more elegant way to compute it.

**3.5.5. Proposition.** If \(r \geq 5\), then the quantum cohomology algebra at the point \(q = 1, z = 0\) is isomorphic to \(\mathbb{C} \oplus \mathbb{C}[X_1, \ldots, X_r] / (X_1^2 = X_2^2, X_i X_j = 0)\). To establish this isomorphism, we can send \(X_1, \ldots, X_r\) to an arbitrary orthonormal basis of \(O_r\), in which case \(X_1^2 = \cdots = X_r^2\) gets sent to

\[ \Gamma := \Delta_2 - \frac{1}{2(9 - r)} \sum_{\beta : k_r(\beta) = 1} \langle \emptyset \rangle_{\beta}(\rho, \beta)^2 k_r - \frac{1}{2} \sum_{\beta : k_r(\beta) = 2} \langle \Delta_2 \rangle_{\beta}(\rho, \beta)^2 \Delta_0. \quad (3.26) \]
**Proof.** Let \( \rho \) be a root. Using lemma 3.5.2 d) we compute

\[
\rho \circ \rho = -2\Delta_2 + \sum_{\beta : k_r(\beta) = 1} \langle \emptyset \rangle_\beta (\rho, \beta)^2 \beta + \sum_{\beta : k_r(\beta) = 2} \langle \Delta_2 \rangle_\beta (\rho, \beta)^2 \Delta_0 = -2\Delta_2 + \frac{1}{9 - r} \sum_{\beta : k_r(\beta) = 1} \langle \emptyset \rangle_\beta (\rho, \beta)^2 k_r + \sum_{\beta : k_r(\beta) = 2} \langle \Delta_2 \rangle_\beta (\rho, \beta)^2 \Delta_0 = -2\Gamma.
\]

Now let \( \rho' \) be a root orthogonal to \( \rho \). By 3.5.2 c) and e) we see that

\[
\rho \circ \rho' = \sum_{\beta : k_r(\beta) = 1} \langle \emptyset \rangle_\beta (\rho, \beta)(\rho', \beta) + \sum_{\beta : k_r(\beta) = 2} \langle \Delta_2 \rangle_\beta (\rho, \beta)(\rho', \beta) \Delta_0 = 0.
\]

Since \( \rho \) and the roots \( \rho' \) orthogonal to \( \rho \) span \( O_r \), we can conclude that for any \( D' \) which lies in this space, we have \( \rho \circ D' = (\rho, D') \Gamma \). Since the roots span \( O_r \), the formula \( D \circ D' = (D, D') \Gamma \) holds for all \( D, D' \in O_r \).

Note that \( \Gamma \circ \rho = \rho' \circ \rho = 0 \) and hence \( \Gamma \circ \Gamma = 0 = \rho^{\circ 3} \).

By associativity, \( \Gamma \) must be an eigenvector of \( k_r \circ \) to the eigenvalue \( \mu_r \). So \( \mu_r \) must be a root of the characteristic polynomial \( R(\lambda) \) defined in (3.22). One can check that it is actually a double root. In fact, if it was a simple root, then the eigenspace of \( k_r \circ \) to the eigenvalue \( \mu_r \) would consist only of nilpotent elements, which is impossible since it is a direct summand of an algebra with identity. (This fact gives the most direct way to determine \( \mu_r \): as the double root of \( R(\lambda) \).) Since \( R(0) < 0 \), it can’t be a triple root.

Thus we have an \((r + 2)\)–dimensional eigenspace of \( k_r \circ \) and a splitting of \( H^*(V) \) into a one–dimensional and an \((r + 2)\)–dimensional subalgebra. The latter one has a basis consisting of its identity, \( \Gamma \) and an arbitrary orthonormal basis of \( k_r^\perp \); in this basis the structure is evidently as described in the proposition.

**3.6. Semisimplicity of the quantum cohomology of del Pezzo surfaces.**

We have seen that if \( r \geq 5 \), the quantum cohomology is far from being semisimple at the symmetric point \( q = 1 \); also proposition 3.5.4 shows that this cannot be a tame semisimple point if \( r \geq 3 \). However, we will prove:

**3.6.1. Theorem.** The Frobenius manifold associated to the quantum cohomology of a del Pezzo surface is generically semisimple on \( H^2 \).

Our proof is based on extending the Frobenius manifold associated to \( V_r \) to a boundary; on the boundary, it ceases to be a Frobenius manifold, yet the boundary itself will be the Frobenius manifold associated to \( V_{r-1} \).

First, we introduce appropriate coordinates: We write the generic character \( q^\beta \) on \( H^2 \) as

\[
q^\beta = e^{\sum_{i=0}^r x_i(l_i, \beta)},
\]
where the $l_i$ are as defined in the beginning of section 3.3. We define $q_i := e^{x_i}$ and thus can write

$$q^\beta = \prod_{i=0}^r q_i^{(l_i, \beta)}.$$  

As mentioned in the beginning of section 3.3, all the sums appearing in equations (3.4) to (3.6) are finite, and each summand is a monomial in the variables $q_0, \ldots, q_r$ and $q_1^{-1}, \ldots, q_r^{-1}$. Hence, the spectral cover map restricted to $TV$ is the morphism of affine schemes

$$\text{Spec } H^*(V_r, \mathbb{Q})[q_0, q_1, \ldots, q_r, q_1^{-1}, \ldots, q_r^{-1}] \to \text{Spec } \mathbb{Q}[q_0, q_1, \ldots, q_r, q_1^{-1}, \ldots, q_r^{-1}]$$

(3.27)

where the left hand side is the small quantum cohomology algebra.

**3.6.2. Lemma.** The map (3.27) can be extended in a flat way to the boundary $q_r = 0$. Moreover, the fiber over $q_r = 0$ is isomorphic to a disjoint sum of the identity map and the respective map (3.27) for the case $r - 1$.

**Proof.** As we already mentioned, from the properties (P1) and (P2) of Theorem 4.1 in [GP] it follows that the only $\beta \in H_2(V, \mathbb{Z})$ with non–vanishing Gromov–Witten invariant $\langle \Delta_2, (\beta, l_r) \rangle < 0$ is $l_r$ itself, for which we have $\langle \emptyset \rangle_{l_r} = 1$. This will be the key fact used in our proof.

Let $F$ be the free module over $\mathbb{Q}[q_0, \ldots, q_r, q_1, \ldots, q_r^{-1}]$ with basis $\Delta_0, \Delta_2, l_0, \ldots, l_{r-1}, q_r l_r$.

Let $D$ be an arbitrary element in $H^2(V_r)$ orthogonal to $l_r$, i. e. $D \in H^2(V_{r-1})$, and let $T_1$ and $T_2$ be two arbitrary elements in $H^*(V_r)$ orthogonal to $l_r$, so $T_1, T_2 \in H^*(V_{r-1})$. We will denote their product by $T_1 \circ_{V_r} T_2$ indicating in which quantum cohomology this product is to be taken.

We claim that:

$$q_r l_r \circ q_r l_r = q_r l_r + O(q_r^2)$$

$$q_r l_r \circ D = O(q_r^2)$$

$$q_r l_r \circ \Delta_2 = O(q_r^2)$$

$$T_1 \circ_{V_r} T_2 = T_1 \circ_{V_{r-1}} T_2 + O(q_r)$$

(3.27)

All these formulas follow from (3.4)–(3.6) and the fact mentioned in the beginning of our proof. To give an example of the computations, we treat the case of $q_r l_r \circ D$:

$$l_r \circ D = \sum_{\beta: k_r(\beta) = 1} \langle \emptyset \rangle_{\beta} q_0^{(l_0, \beta)} \cdots q_r^{(l_r, \beta)} (D, (\beta, l_r, \beta)) \beta$$
\[ + \sum_{\beta: k_r(\beta) = 2} \langle \Delta_2 \rangle_{\beta} q_0^{(l_r, \beta)} \ldots q_r^{(l_r, \beta)} (D, \beta)(l_r, \beta) \Delta_0 \]

The only term that could contribute a negative power of \( q_r \) is \( \beta = l_r \); however, this term is cancelled by the factor \((D, \beta) = 0\). The terms with \( q_r^0 \) as factor do not contribute to the sum either since \((l_r, \beta) = 0\). Hence we get the above formula.

The other cases are dealt with by very similar arguments. In the case of \( l_r \circ l_r \), the summand with \( \beta = l_r \) gives \( \frac{1}{q_r l_r} \) as the only term containing a negative power of \( q_r \), which yields our formula. In the product of \( T_1, T_2 \), the summand \( \beta = l_r \) either does not appear at all if \( T_1 \) or \( T_2 \) is \( \Delta_2 \) because of \( k_r(l_r) = 1 \); otherwise, if both \( T_1, T_2 \in H^2(V_r) \), this summand is cancelled by the factor \((T_1, \beta)\). Now if we set \( q_r = 0 \), in the relevant sums computing \( T_1 \circ V_r T_2 \) only summands with \((\beta, l_r) = 0\) are left, i.e. with \( \beta \in H_2(V_{r-1}) \). Since the Gromov–Witten invariants of such an \( \beta \) are the same, whether they are computed in \( V_r \) or in \( V_{r-1} \), we get the last equation of (3.27).

We have thus seen that the product extends to \( F \). Since it obviously remains associative, \( F \) is an algebra over \( \mathbb{Q}[q_0, \ldots, q_r, q_1^{-1}, \ldots, q_{r-1}^{-1}] \). So we have proved the first assertion of our lemma.

We want to examine the fiber \( F_0 := F/q_r F \). Clearly, \( q_r l_r \) is an idempotent, so we get a splitting of \( F_0 \) into the image and the kernel \( K \) of multiplication with \( q_r l_r \). The kernel \( K \) is spanned by \( H^2(V_{r-1}), \Delta_2 \) and \( \Delta_0 - q_r l_r \), the image is the span of \( q_r l_r \).

The kernel \( K \) is isomorphic to \( F_0/q_r l_r \). From the last equation in the list (3.27) we see that \( F_0/q_r l_r \) is isomorphic to the small quantum cohomology algebra of \( V_{r-1} \), which proves the second statement of the lemma.

**Proof of Theorem 3.6.1.** As mentioned already in 1.6, it is well known that the quantum cohomology of \( \mathbb{P}^r \) is generically semisimple. We proceed by induction.

So assume we know that the quantum cohomology of \( V_{r-1} \) is generically semisimple on \( H^2 \). According to the lemma, this implies that

\[ \text{Spec } F \to \text{Spec } \mathbb{Q}[q_0, \ldots, q_r, q_1^{-1}, \ldots, q_{r-1}^{-1}] \]

is generically semisimple on the fiber \( q_r = 0 \). Since semisimplicity is a Zariski–open condition, this map is generically semisimple; restricting to \( q_r \neq 0 \), we see that the quantum cohomology of \( V_r \) is generically semisimple on \( H^2(V_r) \).

**3.6.3. Remark.** It is well known that there exists a full collection of exceptional sheaves for del Pezzo surfaces, see [KuOr]. Hence the Theorem 3.6.1 is a special case of Dubrovin’s conjecture mentioned in 1.8.2.
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