Quantizing Billiards with Arbitrary Trajectories

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The structure of the semiclassical trace formula can be used to construct a quasi-classical evolution operator whose spectrum has a one-to-one correspondence with the semiclassical quantum spectrum. We illustrate this for marginally unstable integrable and non-integrable billiards and demonstrate its utility by quantizing them using arbitrary non-periodic trajectories.

Periodic orbits form the skeleton around which generic classical motion is built, and not surprisingly, many dynamical quantities of interest can be computed quite accurately in terms of these invariant trajectories [1]. They also form the basis of modern semiclassical quantization schemes which express the trace of the energy dependent quantum propagator in terms of lengths, stabilities and focal points of periodic orbits [2]. Thus resonances in open systems and the energy eigenspectra of closed systems can be determined semiclassically even when the dynamics is chaotic.

A similar relationship also exists for the trace of the classical propagator, $L^t$, though this is known explicitly only when the system is hyperbolic [3]. Here $L^t$ governs the evolution of densities

$$ L^t \circ \phi(x) = \int dy \, \delta(x - f^t(y)) \, \phi(y) $$

where $x = (q, p)$ and $f^t$ refers to the flow in the full phase space. We denote by $\Lambda_n(t)$ the eigenvalue corresponding to an eigenfunction $\phi_n(x)$ such that $L^t \circ \phi_n(x) = \Lambda_n(t) \phi_n(x)$. It then follows that the autocorrelation of any analytic function, $A(x(t))$ can be expressed as $\sum_n a_n \Lambda_n(t)$ where the coefficients $\{a_n\}$ are determined by $A(x)$ and the eigenfunctions $\{\phi_n(x)\}$. The classical spectrum, $\{\Lambda_n(t)\}$, is thus crucial to understanding the evolution of correlations and the first step towards determining this is to evaluate the trace of $L^t$. For hyperbolic systems [3]

$$ \text{Tr} \, L^t = \sum_n \Lambda_n(t) = \sum_p \sum_{r=1}^{\infty} \frac{T_p \delta(t - rT_p)}{|\det(1 - J_p^T)|} \tag{2} $$

where the summation over $p$ refers to all primitive periodic orbits, $T_p$ is the time period and $J_p$ the Jacobian matrix evaluated on the orbit. Note that the semi-group property, $L^{t_1} \circ L^{t_2} = L^{t_1 + t_2}$, for continuous time implies that the eigenvalues $\Lambda_n(t)$ are of the form $e^{\lambda_n t}$. The poles in the Laplace transform of $\text{Tr} \, L^t$ thus occur at $\lambda_n$ much in the same way as the trace of the energy-dependent quantum propagator has poles at the eigenenergies, $\{E_n\}$, of the system. There are important differences though between the traces of the quantum and classical evolution operators. The quantum case involves the square root of $|\det(1 - J_p^T)|$ and in addition has Maslov phases so that the classical spectrum in general bears no relationship to the quantum spectrum [4].

Our intention here is to construct quasi-classical propagators similar to Eq. (1), but based on the quantum trace formula so that $\{\lambda_n\}$ is directly related to $\{E_n\}$. This can be used to our advantage by evolving classical functions and extracting the semiclassical spectrum from the peak positions in its power spectrum. Importantly, the recipe does not involve periodic orbits which often have a highly non-trivial topological organisation due to which a systematic computation is difficult. To keep matters simple however, we shall carry out this exercise for both integrable and non-integrable systems that are marginally unstable. Compared to chaotic systems, these are simpler to handle by far but nevertheless serve to illustrate the essential idea.

Quasi-classical propagators have been studied before by Cvitanović and Vattay [5] though from a different standpoint. They consider an extended dynamical space to evolve the quasi-classical wavefunction and ultimately derive a

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trace formula that is distinct from the usual Gutzwiller trace formula for chaotic systems, and, whose corresponding Fredholm determinant is entire. Thus, they do not consider the question of quantization using arbitrary trajectories. A resemblance to the method employed here can however be found in the approach of Heller and Tomsovic [9] in that they work in the time domain and use the fourier transform of the correlation function to determine the spectrum, \( \{ E_n \} \). There are basic differences though, primarily concerning the nature of the evolution. While we use here the classical propagator and its quasi-classical adaptation, Heller and Tomsovic [9] use a semiclassical approximation of the quantum propagator. As a consequence, the quantities that need to be computed differ.

We shall, without much loss of generality, confine ourselves to billiards [11]. The ones that are marginally unstable include the integrable circle billiard and rational polygonal billiards. An important characteristic of these systems is the existence of two independent constants of motion due to which the invariant surface is two dimensional. In case of integrable motion, this is topologically equivalent to a torus while in the general case, the invariant surface has the topology of a sphere with \( g \) holes. Examples of integrable billiards are few and generic rational polygonal billiards are pseudo-integrable [11] with \( g > 2 \).

Before considering the question of quasi-classical quantization, we first introduce the appropriate classical evolution operator. For integrable systems, this is easily defined as

\[
L^t \circ \phi(\theta_1, \theta_2) = \int d\theta'_1 d\theta'_2 \delta(\theta_1 - \theta'_1)\delta(\theta_2 - \theta'_2) \phi(\theta'_1, \theta'_2)
\]

(3)

where \( \theta_1 \) and \( \theta_2 \) are the angular coordinates on the torus and evolve in time as \( \theta_i^t = \omega_i(I_1, I_2)t + \theta_i \) with \( \omega_i = \partial H(I_1, I_2)/\partial I_i \) and \( I_i = \frac{1}{2\pi} \oint_{\Gamma_i} \mathbf{p} \cdot dq \). Here \( \Gamma_i, i = 1, 2 \) refer to the two irreducible circuits on the torus and \( \mathbf{p} \) is the momentum conjugate to the coordinate \( q \).

It is easy to see that the eigenfunctions \( \{ \phi_n(\theta_1, \theta_2) \} \) are such that \( \phi_n(\theta_1, \theta_2) = \Lambda_n(t) \phi_n(\theta_1, \theta_2) \) where \( \Lambda_n(t) = e^{i\alpha_n t} \). On demanding that \( \phi_n(\theta_1, \theta_2) \) be a single valued function of \( (\theta_1, \theta_2) \), it follows that \( \phi_n(\theta_1, \theta_2) = e^{i(n_1 \theta_1 + n_2 \theta_2)} \) where \( \mathbf{n} = (n_1, n_2) \) is a point on the integer lattice. Thus the eigenvalue, \( \Lambda_n(t) = \exp\{it(n_1 \omega_1 + n_2 \omega_2)\} \). We shall evaluate this explicitly for the rectangular billiard while discussing the trace of \( L^t \) and we now introduce the appropriate evolution operator that applies also to rational polygonal billiards which are generically non-integrable.

For both integrable and pseudo-integrable polygonal billiards, the dynamics in phase space can be viewed in a singly connected region by executing 2\( g \) cuts in the invariant surface and identifying edges appropriately.

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FIG. 1. The singly connected region for an L-shaped billiard consists of four copies with edges appropriately identified. A trajectory originating near the 3\( \pi/2 \) vertex in 1 is plotted in configuration space using bold lines and the corresponding unfolded trajectory is also shown. The latter consists of parallel segments and the trajectory can be parametrized by the angle \( \varphi \) that it makes for example with the \( q_1 \) axis.
At a given energy, the motion is parametrised by the angle, $\varphi$, that a trajectory makes with respect to one of the edges. As a trivial example, consider the rectangular billiard. The singly connected region is a larger rectangle consisting of four copies corresponding to the four directions that a trajectory can have and these can be glued appropriately to form a torus $\mathbb{T}^2$. As a non-trivial example (see Fig 1), consider the L-shaped billiard which is pseudo-integrable with its invariant surface having, $g = 2$. Alternately, the surface can be represented by a singly connected region in the plane and consists of four copies corresponding to the four possible directions an orbit can have and these are glued appropriately. A trajectory in phase space thus consists of parallel segments at an angle $\varphi$ measured for example with respect to one of the sides. It will be useful to note at this point that the same trajectory can also be represented by parallel segments at angles $\pi - \varphi$, $\pi + \varphi$ and $2\pi - \varphi$. In general, the number of directions for representing a trajectory equals the number of copies, $N$, that constitute the invariant surface.

The classical propagator on an invariant surface parametrised by $\varphi$ is thus

$$L^t(\varphi)\phi(q) = \int dq' \delta(q - q'(\varphi)) \phi(q')$$

where $q$ refers to the position in the singly connected region and $q'(\varphi)$ is the time evolution parametrised by $\varphi$ as described above. We denote by $\{\Lambda_n(t; \varphi)\}$, the eigenvalues of $L^t(\varphi)$ and from its multiplicative nature, it follows that $\Lambda_n(t; \varphi) = e^{\lambda_n(\varphi)t}$.

The trace of $L^t$ takes into account all possible invariant surfaces that exist and hence involves an additional integration over $\varphi$. Thus

$$Tr L^t = \int d\varphi L^t(\varphi) = \int d\varphi \sum \lambda_n(\varphi)t = \int d\varphi \int dq \delta(q - q^t(\varphi))$$

Clearly the only orbits that contribute are the ones that are periodic. Further, the $q$ integrations are simpler if we transform to a local coordinate system with one component parallel to the trajectory and the other perpendicular. Thus $\delta_{||}(q_l - q^t_l) = \sum_p \sum_r \int_0^T \delta(t - rT_p)$ where $v$ is the velocity, $T_p$ is the period of the orbit and $r$ is the repetition number. Similarly, for an orbit of period $rT_p$ parametrised by the angle $\varphi_p$, $\delta_{\perp}(q_{\perp} - q^t_{\perp}) = \delta(\varphi - \varphi_p)/[\partial q_{\perp}/\partial \varphi]_{\varphi = \varphi_p}$ where $[\partial q_{\perp}/\partial \varphi]_{\varphi = \varphi_p} = rT_p$ for marginally unstable billiards. Putting these results together and noting that each periodic orbit occurs in general at $N$ different values of $\varphi$, we finally have

$$Tr L^t = N \sum_p \sum_r \frac{a_p}{rT_p} \delta(l - rT_p)$$

where $l = tv$ and the summation over $p$ refers to all primitive periodic orbit families with length $l_p$ and occupying an area $a_p$. We thus have the analogue of Eq. (9) for integrable and polygonal billiards.

In some cases, it is possible to interpret the periodic orbit sum in Eq. (6) starting with the appropriate quantum trace formula

$$\sum_n \delta(E - E_n) = da_v(E) + \frac{1}{\sqrt{8\pi^l}} \sum_p \sum_{r=1}^{\infty} \frac{a_p}{\sqrt{krT_p}} \cos(krl_p - \frac{\pi}{4} - \pi r n_p - r \nu_p \frac{\pi}{2})$$

Here $da_v(E)$ refers to the average density of quantal eigenstates, $k = \sqrt{E}$, $l_p$ is the length of a primitive periodic orbit family, $n_p$ the number of bounces that it suffers at the boundary and $\nu_p$ the number of caustics encountered by the orbit. Note that in the Neumann case, $n_p = 0$ since there is no phase loss on reflection while for polygonal billiards, $\nu_p = 0$. For convenience, we have set $\hbar = 1$ and the mass $m = 1/2$. Starting with the function

$$\sum_n f(\sqrt{E_n})e^{-\beta E_n} = \int_0^{\infty} dE f(\sqrt{E})e^{-\beta E} \sum_n \delta(E - E_n)$$

where $f(x) = \sqrt{\frac{\pi}{x^2}} \cos(x - \pi/4)$ and $0 < \epsilon < E_0$, it is possible to show using Eq. (7) that for polygonal billiards

$$\sum_p \sum_{r=1}^{\infty} \frac{a_p}{\sqrt{krT_p}} \delta(l - rT_p) = 2\pi b_0 + 2\pi \sum_n f(\sqrt{E_n})l$$

for $\beta \rightarrow 0^+$. In the above, $b_0 = \sum_p \sum_{r=1}^{\infty} \frac{a_p}{4\pi} \int_0^{\infty} dE f(\sqrt{E})f(\sqrt{E}rT_p)$ and is a constant. It follows from Eqs. (8) and (9) and the fact that $h(x)\delta(x - a) = h(a)\delta(x - a)$ that
where \( \{ E_n \} \) are the Neumann eigenvalues of the system. Thus \( \lambda_n(\varphi) = i\sqrt{E_n} \sin(\varphi) \) [18].

For integrable polygonal billiards, Eq. (10) can in fact be derived directly from the expression for \( \Lambda_n(t) \) given earlier. To illustrate this, we consider a rectangular billiard where the Hamiltonian expressed in terms of the actions, \( I_1, I_2 \) is \( H(I_1, I_2) = \pi^2(L_1^2/L_1^2 + L_2^2/L_2^2) \) where \( L_1, L_2 \) are the lengths of the two sides. With \( I_1 = \sqrt{E}L_1 \cos(\varphi)/\pi \) and \( I_2 = \sqrt{E}L_2 \sin(\varphi)/\pi \), it is easy to see that at a given energy, \( E \), each torus is parametrised by a particular value of \( \varphi \). Thus \( \Lambda_n(t; \varphi) = e^{i2\pi\nu \sqrt{E}(n_1 \cos(\varphi)/L_1 + n_2 \sin(\varphi)/L_2)} \) and the spectrum is continuous, parametrized by \( \varphi \). The trace of the classical propagator is thus

\[
\text{Tr} \ L^t = \sum_n \int_{-\pi}^{\pi-n} d\varphi \ e^{i\sqrt{E_n} \sin(\varphi+\pi-n)} = 2\pi \sum_n J_0(\sqrt{E_n}l) \tag{11}
\]

where \( l = 2t\sqrt{E} \), \( \tan(\mu_n) = n_1 L_2/(n_2 L_1) \) and \( E_n = \pi^2(n_1^2/L_1^2 + n_2^2/L_2^2) \). On separating out \( n = (0,0) \) from the rest, restricting the summation to the first quadrant of the integer lattice and noting that for a rectangle \( b_0 = 1/4 \), Eq. (11) follows [18,19].

The classical evolution operator thus serves to determine the Neumann spectrum in polygonal billiards. We have confirmed this numerically for the rectangular and equilateral triangle billiards and an outline of the method used can be found towards the end of this paper. Appropriate modifications however need to be made for the Dirichlet spectrum and for systems that have caustics (the circle billiard is an example) and the construction of the evolution operator is then guided by the nature of the quantum trace formula.

The quasi-classical evolution operator linking the classical and semiclassical eigenvalues can be defined as

\[
L_{qc}^t(\varphi)\circ\phi(q) = \int dq' \delta(q - q'^t(\varphi))e^{-in(t)\pi-n(t)\frac{\varphi}{2}} \phi(q') \tag{12}
\]

where \( \nu(t) = \nu(q^{-t}(\varphi)) \) and \( n(t) = n(q^{-t}(\varphi)) \) count respectively the number of caustics and (phase altering) reflections encountered by the trajectory \( q^{-t}(\varphi) \) in time \( t \). As before, the multiplicative nature of \( L_{qc}^t(\varphi) \) implies that its spectrum is of the form \( \{ e^{\lambda_n(\varphi)t} \} \) and we shall now show that for the quasi-classical operator defined in Eq. (12), \( \{ \lambda_n \} \) has a one-to-one correspondence with the appropriate quantum spectrum.

The trace of \( L_{qc} \) is given by

\[
\text{Tr} \ L_{qc}^t = \int d\varphi \sum_n e^{\lambda_n(\varphi)t} = \int d\varphi \int dq \delta(q - q^t(\varphi)) e^{-in(t)\pi-n(t)\frac{\varphi}{2}} \tag{13}
\]

and the contributions to it are due to orbits that are periodic. Since they occur in families, the integrand is independent of \( q \) as shown earlier and we have

\[
\text{Tr} \ L_{qc}^t = N \sum_p \int_{1}^{\infty} \frac{dp}{r_p} \delta(l - rl_p) \int d\varphi \delta(\varphi - \varphi_p)e^{-in(q^{-1}(\varphi)\pi-n(q^{-1}(\varphi))\frac{\varphi}{2}} \tag{14}
\]

Starting with the function \( \sum_n g(\sqrt{E_n}t) \exp(-\beta E_n) \), it follows from Eq. (10) that for \( \beta \to 0^+ \),

\[
\text{Tr} \ L_{qc}^t = \sum_n \Lambda_n(t) = 2\pi NC + 2\pi N \sum_n g(\sqrt{E_n}l) \tag{16}
\]

where \( \{ E_n \} \) now refers to the desired quantum spectrum, \( g(x) = \sqrt{\frac{1}{4\pi}} \exp(-ix + \pi/4) \) and \( C \) is a constant. It follows from

\[
g(x) \simeq \frac{1}{2\pi} \int_{0}^{2\pi} e^{-i\sqrt{E_n}l \sin(\varphi/2)} d\varphi \tag{17}
\]

that
\[ \lambda_n(\phi) = i \sqrt{E_n} \sin(\phi/2) \]  

Eq. (18) forms the central result of this paper.

In order to numerically demonstrate our results, we first note that

\[ L^t_{qc}(\phi) \circ \phi(q) = \phi(q^{-t}(\phi))e^{-in(q^{-t}(\phi))\pi-iv(q^{-t}(\phi))} = \sum_n c_n e^{i\sqrt{E_n} \sin(\phi/2)} \phi_n(q) \]  

where \( \{\phi_n\} \) are the eigenfunctions of \( L^t_{qc} \) and \( \{c_n\} \) are the expansion coefficients for the function \( \phi(q) \). Thus on evolving \( \psi(t; \phi) = \phi(q^{-t}(\phi))e^{-in(q^{-t}(\phi))\pi-iv(q^{-t}(\phi))} \) and averaging this over \( \phi \), we can extract the quantum eigenvalues from the power spectrum of \( \langle \psi(t) \rangle \) where \( \langle \ldots \rangle \) denotes averaging over \( \phi \). In practice though, we use the function \( \langle \psi(t) \rangle e^{-\beta t^2} \) to compensate for the finite data set with \( \beta > 0 \).

The first application of the quasi-classical evolution operator is provided for a circle billiard of unit radius in Fig. (2). The function \( \phi(q) \) is chosen to be a Gaussian inside a cell centred at \( q_c \) and 0 outside [20]. The power spectrum, \( S(k) \) is displayed in Fig. (2) where the arrows mark the positions of the Dirichlet EBK eigenvalues [21].

![FIG. 2. Power spectrum of \( \langle \psi(t) \rangle \) for a circle billiard of unit radius. The arrows mark the position of the Dirichlet EBK eigenvalues.](image)

A total of 1000 arbitrary trajectories with different values of angular momentum have been used and the the power spectrum is averaged over 4 cells. We have also generated the Neumann spectrum of the circle billiard (this is distinct from the Dirichlet spectrum) by setting \( n(t) = 0 \).

Finally, we provide results for a non-integrable triangular billiard with base angles \( (\pi/5, 3\pi/10) \) in Fig. (3). The function \( \langle \psi(t) \rangle \) has been generated using 300 arbitrary trajectories and averaging over 150 cells while the power spectrum has been obtained with \( \beta = 0.006 \). We find convergence to be rapid with the number of trajectories used. The width of individual peaks depends on \( \sqrt{\beta} \) and the choice of \( \beta \) is dictated by several factors which include the degree of smoothening attained by averaging over different cells and the length of trajectories. In the present context, the value of \( \beta \) is also influenced by the height of the shortest peak (at 18.5) and is chosen such that it is distinct from the noisy baseline. As in the circle billiard [21], peak positions in the non-integrable case do not coincide with the exact quantum spectrum. The average error is about 8 percent of the mean level spacing in the triangular billiard and we find this to be of the same magnitude as in the eigenvalues obtained from Bogomolny’s [22] transfer operator (see Fig. (3)).
FIG. 3. Power spectrum of $\langle \psi(t) \rangle$ for a non-integrable triangular billiard with base angles $(\pi/5, 3\pi/10)$. The upper set of arrows mark the positions of the exact Dirichlet eigenvalues while the lower set is obtained from Bogomolny’s transfer operator.

The neglect of higher order corrections thus introduces errors as in all semiclassical schemes and it is not immediately obvious how effects such as diffraction can be taken into account in the quantization recipe developed here. We wish to point out however that direct applications of periodic orbits have met with little success in determining individual quantum levels of pseudointegrable systems [23] and the results provided here are thus of some significance.

In addition to the cases presented here, we have also generated the semiclassical spectrum of a circle billiard with a single magnetic flux line passing through the centre as well as other non-integrable triangular billiards using arbitrary trajectories [10]. Also, it is possible to generate excited states by suitably choosing the function $\phi(q)$ such that it has nodes. We have thus been able to generate the first 100 levels of an arbitrary triangle [10].

We provide below a summary of our results:

- (i) we have introduced the classical evolution operator for marginally unstable systems and shown that for polygonal billiards, the classical spectrum on an invariant surface labelled by $\varphi$ is $\{ e^{i\sqrt{E_n}\sin(\varphi)t} \}$ where $\{E_n\}$ is the (quantum) Neumann spectrum.

- (ii) we have also shown how the classical evolution operator can be appropriately modified to construct a quasilocal adaptation whose spectrum has a one-to-one correspondence with the desired semiclassical quantum spectrum.

- this yields a quantization procedure using arbitrary trajectories and its utility is demonstrated by determining individual levels in a class of systems where direct applications of periodic orbits have met with little success so far.

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Details of this work together with a study of systems other than billiards will be published elsewhere.

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For the circle billiard, the angular momentum is conserved and $\varphi$ is then a measure of the impact parameter, $R_0 = R \sin(\varphi)$ where $R$ is the radius of the circle.

We have neglected here the contribution to the trace from isolated periodic orbits. As in the quantum case [14], we expect them to be less important.

The respective contributions of isolated and diffractive periodic orbits are $O(k^{-1})$ and $O(k^{-1-\nu/2})$ where $\nu$ is the total number of (singular) vertex connections in a diffractive periodic orbit.

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Numerical verification of this for a generic polygon can be found in D. Biswas, Phys. Rev. E54, R1044 (1996).

The function $f(x)$ is the asymptotic expansion of $J_0(x)$.

Note that the term $2\pi b_0 N$ in Eq. (11) reflects the fact that there is a unit eigenvalue at each $\varphi$.

The results are better for functions $\psi(q)$ that are discontinuous possibly because all modes are excited. We have obtained the same results with function other than a Gaussian. In particular, $\psi(q)$ can be defined to be 1 inside the cell and 0 outside and $\langle \psi(t) \rangle$ is then a weighted recurrence fraction of trajectories.

The exact quantum eigenvalues are different from the EBK eigenvalues for the circle billiard [2].

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