Truncation error estimates of approximate differential operators of a particle method based on the Voronoi decomposition

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1. Introduction

Truncation errors are considered for approximate differential operators with a class of particle methods. Introducing sufficient conditions for the weight function and a regularity of the family of discrete parameters leads to truncation error estimates of approximate gradient and Laplace operators with a particle method based on the Voronoi decomposition. Moreover, some numerical results agree well with theoretical ones.

2. Formulation

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) \((d = 2, 3)\) and \( H \) a fixed positive number. For \( \Omega \) and \( H \), a domain \( \Omega_H \) is defined by
\[
\Omega_H := \left\{ x \in \mathbb{R}^d; \exists y \in \Omega \text{ s.t. } |x - y| < H \right\}.
\]
For \( H \) and \( N \in \mathbb{N} \), let \( X_{NH} \) be a set of points \( x_i \in \Omega_H \) \((i = 1, 2, \ldots, N)\) satisfying \( x_i \neq x_j \) \((i \neq j)\). We call \( x_i \) a particle and \( X_{NH} \) a particle distribution.

A set of weight functions \( \widetilde{W} \) is defined by
\[
\widetilde{W} := \left\{ w \in C^1(\mathbb{R}_+^d); \text{supp}(w) = [0, 1], \right\}
\]
\[
\int_{\mathbb{R}^d} w(|x|)dx = 1 \}
\]

For \( w \in \widetilde{W} \), we consider the following hypotheses:

Hypothesis 1 There exists \( k_0 \in \mathbb{N} \) such that, for all multi-indices \( \alpha \in \mathbb{N}_0^d \) satisfying \( 2 \leq |\alpha| \leq k_0 + 1, \)
\[
\int_{\mathbb{R}^d} e^{\omega} w(|x|)dx = 0.
\]
Hypothesis 2 Let a function \( \tilde{w} \) defined in \( \mathbb{R}^d_+ \) by
\[
\tilde{w}(r) = \begin{cases} 
\lim_{s \to 0} s^{-1}w(s), & r = 0, \\
\frac{1}{r^{d-1}}w(r), & r > 0.
\end{cases}
\]
Then, \( \tilde{w} \) belongs to \( C^1(\mathbb{R}^d_+) \).

Remark 3 An example of a weight function satisfying Hypothesis 1 with \( k_0 = 2 \) on \( d = 2 \) is shown in [4].

Let \( h \) be a positive number satisfying \( 0 < h < \min(1, H) \). We call \( h \) an influence radius. For \( w \in \tilde{W} \) and \( h \), set \( w_h(r) := h^{-d}w(r/h) \) (\( r \in \mathbb{R}^d_+ \)).

For \( x \in X_{NH} \), let \( \sigma_i \) be the Voronoi cell defined by
\[
\sigma_i := \{ x \in \Omega_H^\ast, |x_i - x| < |x_j - x|, \forall x_j \in X_{NH}(j \neq i) \},
\]
and \( \{ \sigma_i \}_{i=1}^N \) be the Voronoi decomposition. Set a particle volume \( V_i \) \((i = 1, 2, \ldots, N)\) by \( V_i := \text{meas}(\sigma_i) \).

For \( v \in C^1(\overline{\Omega_H}) \) and \( x \in \Omega_H \), an approximate gradient operator \( \nabla_h \) and an approximate Laplace operator \( \Delta_h \) are defined by
\[
\nabla_h v(x) := d \sum_{j \in A_0(x, h)} V_j \frac{v(x) - v(x_j)}{|x - x_j|} [x - x_j]^{-1} w_h(|x - x_j|), \quad (2)
\]
\[
\Delta_h v(x) := -2d \sum_{j \in A_0(x, h)} V_j \frac{v(x) - v(x_j)}{|x - x_j|^2} w_h(|x - x_j|), \quad (3)
\]
where \( A_0(x, h) := \{ i; i = 1, 2, \ldots, N, 0 < |x - x_i| < h \} \).

3. Truncation error estimates

Hereafter, we use \( c \) as a positive generic constant independent of \( N \) and \( h \). For \( X_{NH} \), we define \( r_v \) by
\[
r_v := \max_{i=1, 2, \ldots, N} \max_{x \in \sigma_i} |x_i - x|. \quad (4)
\]
As mentioned earlier, \( H \) is independent of \( N \) and \( h \). For \( m \geq 1 \), the family \( \{(X_{NH, h}, 0 < h < \min\{1, H\})\} \) is said to be regular with order \( m \) if there exists a positive constant \( c_0 > 0 \) such that for all elements of the family
\[
h^m \geq c_0 r_v. \quad (5)
\]
We introduce the spatial discrete spaces. For a set \( \omega \subset \mathbb{R}^d \), let \( \mathbb{F}^\omega(\omega) \) be the space of R-valued sequences \( v = \{v(x_i); x_i \in X_{NH} \cap \omega\} \) with its norm \( ||v||_{\mathbb{F}^\omega(\omega)} := \max \{||v(x_i)||; x_i \in X_{NH} \cap \omega\} \).

Theorem 4 For \( m \geq 1 \), \( \{(X_{NH, h}, 0 < h < \min\{1, H\})\} \) is regular with order \( m \). Set \( k \) by \( k_0 \) if \( w \) satisfies Hypothesis 1 with \( k_0 \) and by 0 otherwise. Then, there exists a positive constant \( c \) independent of \( N \) and \( h \) such that for all \( v \in C^{k+3}(\overline{\Omega_H}) \)
\[
||\nabla v - \nabla_h v||_{C^{k}(\Omega_H)} \leq c h^{k+2} ||v||_{C^{k+3}(\overline{\Omega_H})}, \quad (6)
\]
Moreover, assume \( m \geq 2 \) and \( w \) satisfies Hypothesis 2. Then, there exists a positive constant \( c \) independent of \( N \) and \( h \) such that for all \( v \in C^{k+4}(\overline{\Omega_H}) \)
\[
||\nabla v - \nabla_h v||_{C^{k}(\Omega_H)} \leq c h^{k+2} ||v||_{C^{k+4}(\overline{\Omega_H})}, \quad (7)
\]
Remark 5 Let \( w^S \) be a weight function used in SPH (see [1]) satisfying \( w^S \in \tilde{W} \cap C^2(\mathbb{R}^d_+) \) and \( dw^S/dr(r) < 0, \quad r \in (0, 1) \). By setting
\[
w(r) = -r \frac{d}{dr}w^S(r), \quad r \in \mathbb{R}^d_+, \quad (8)
\]
and (2) and (3) coincides with approximate differential operators of SPH [1] with \( m/\mu \) replaced by \( \text{meas}(\sigma) \). The weight function (8) does not satisfy Hypothesis 1, but satisfies \( w \in \tilde{W} \) and Hypothesis 2. Therefore, the convergence rates with respect to the influence radius are at least second order when \( m \geq 3 \) for the approximate gradient operator and \( m \geq 4 \) for the approximate Laplace operator.

4. Proof of Theorem 4

Before beginning the proof, we define operators \( \nabla_h \) and \( \Delta_h \) for \( v \in C^1(\overline{\Omega_H}) \) by
\[
\nabla_h v(x) := d \int_{\Omega_H} \frac{v(x) - v(y)}{|x - y|} w_h(|x - y|)dy,
\]
\[
\Delta_h v(x) := -2d \int_{\Omega_H} \frac{v(x) - v(y)}{|x - y|^2} w_h(|x - y|)dy
\]
for all \( x \in \Omega_H \), respectively.

Lemma 7 Set \( k \) by \( k_0 \) if \( w \) satisfies Hypothesis 1 with \( k_0 \) and by 0 otherwise. Then, there exists a positive constant \( c \) such that, for all \( v \in C^{k+3}(\overline{\Omega_H}) \),
\[
||\nabla v - \nabla_h v||_{C^{k}(\Omega_H)} \leq c h^{k+2} ||v||_{C^{k+3}(\overline{\Omega_H})}, \quad (9)
\]
and, for all \( v \in C^{k+4}(\overline{\Omega_H}) \),
\[
||\nabla v - \nabla_h v||_{C^{k}(\Omega_H)} \leq c h^{k+2} ||v||_{C^{k+4}(\overline{\Omega_H})}. \quad (10)
\]
Proof For \( y \in \mathbb{R}^d \) and \( r \in \mathbb{R}^+ \), let \( B_r(y) \) be an open ball in \( \mathbb{R}^d \) with the center \( y \) and the radius \( r \). We fix \( x \in \overline{\Omega} \). Since \( h < H \), we have \( B_h(x) \subset \Omega_H \). Then, for all \( v \in C^{k+1}(\overline{\Omega_H}) (n \in \mathbb{N}_0) \) and \( y \in B_h(x) \), we obtain the Taylor expansion of \( v \),
\[
v(y) = \sum_{|\alpha| \leq n} \frac{D^\alpha v(x)}{\alpha!} (y - x)^\alpha + R_n[v](x, y), \quad (11)
\]
where \( \alpha \in \mathbb{N}_0^d \) is a multi-index and \( R_n \) is the residual,
\[
R_n[v](x, y) = \sum_{|\alpha| = n+1} \frac{(y - x)^\alpha}{\alpha!} \times \int_0^1 (1 - t)^n D^\alpha v(ty + (1 - t)x)dt.
\]
Multiplying both sides of (11) with \( n = k + 2 \) by \( d(x - y)|x - y|^{-2}w_h(|x - y|) \) and integrating them over \( B_h(x) \)
with respect to $y$, we have

$$-\Delta_h v(x) = \sum_{j=1}^{d+3} I_j + d \int_{B_0(x)} \frac{R_{k+2}[v(x,y)r]}{|r|^2} w_h(|r|) dy,$$

where $r := x - y$. Since for multi-indices $\beta_1, \beta_2 \in \mathbb{N}_0^d (|\beta_1| = |\beta_2| = 1)$,

$$\int_{B_0(x)} \frac{(x-y)^{\beta_1+\beta_2}}{|y|^2} w_h(|y|) dy = \begin{cases} d^{-1}, & \beta_1 = \beta_2, \\
0, & \beta_1 \neq \beta_2. \end{cases} \tag{12}$$

we have $I_1[v](x) = -\nabla v(x)$. Considering a coordinate transformation to polar coordinates, we find that for all multi-indices $\alpha \in \mathbb{N}_0^d (|\alpha| \geq 2)$, there exists a multi-index $\beta \in \mathbb{N}_0^d (|\beta| = |\beta| = 2)$, and a constant $c_{\alpha, \beta}$ depending only on $\alpha$ and $\beta$ such that

$$\int_{B_0(x)} \frac{r^{\alpha}}{|r|^2} w_h(|r|) dy = c_{\alpha, \beta} \int_{\mathbb{R}^d} y^{\beta} w_h(|y|) dy. \tag{13}$$

Thus, we have $I_j[v](x) = 0$ ($j = 2, 3, \ldots, k + 2$) by Hypothesis 1. Therefore, we estimate

$$|\nabla v(x) - \Delta_h v(x)| = \left| \frac{d}{dx} \int_{B_0(x)} \frac{R_{k+3}[v(x,y)r]}{|r|^2} w_h(|r|) dy \right| \leq c \left| v \right|_{C^{k+3}(\Omega_H)} \int_{B_0(x)} |r|^{k+2} w_h(|r|) dy \leq c h^{k+2} \left| v \right|_{C^{k+3}(\Omega_H)}. \tag{14}$$

Then, we obtain (9).

Multiplying both sides of (11) with $n = k+3$ by $2d|x-y|/|x-y|^2 \int_{B_0(x)} w_h(|x-y|)$ and integrating them over $B_0(x)$ with respect to $y$, we have

$$\Delta_h v(x) = \sum_{j=1}^{k+3} J_j + 2d \int_{B_0(x)} \frac{R_{k+3}[v(x,y)r]}{|r|^2} w_h(|r|) dy,$$

$$J_j := (2d)^{-1} \int_{\mathbb{R}^d} \frac{r^{\alpha}}{|r|^2} w_h(|r|) dy. \tag{15}$$

Since $d = 2, 3$ and $w \in W$, we have $J_1[v](x) = 0$. From (12), we have $J_2[v](x) = \Delta v(x)$. Moreover, from (13), we have $J_j[v](x) = 0$ ($j = 3, 4, \ldots, k + 3$) by Hypothesis 1. Then, by a similar argument as in (14), we estimate

$$|\Delta v(x) - \Delta_h v(x)| \leq c h^{k+2} \left| v \right|_{C^{k+3}(\Omega_H)}. \tag{16}$$

Then, we obtain (10).

(QED)

**Proof of Theorem 4** First, we prove (6). For $i = 1, 2, \ldots, N$ and $y \in \Omega_H$, set $r_i(y) := x_i - y$. For $i = 1, 2, \ldots, N (i \neq j)$ and $y \in \Omega_H \setminus \{x_i\}$, set

$$\rho_{ij}(y) := \frac{v(x_i) - v(y)}{|r_i(y)|} - \frac{v(x_j) - v(y)}{|r_j(y)|},$$

$$\pi_{ij}(y) := w_h(|r_i(y)|) - w_h(|r_j(y)|),$$

$$\tau_{ij}(y) := \frac{v(x_i) - v(y)}{|r_i(y)|} - \frac{v(x_j) - v(y)}{|r_j(x_j)|}.$$

The left hand side of (6) can be split into the three terms as follows:

$$\| \nabla v - \nabla_h v \|_{C^0(\Omega_H)} \leq \| \nabla v - \Delta_h v \|_{C^0(\Omega_H)} + \| E_1 \|_{C^0(\Omega_H)} + \| E_2 \|_{C^0(\Omega_H)},$$

where $E_1$ and $E_2$ are defined by

$$E_1(x_i) := d \int_{\mathbb{R}^d} \rho_{ij}(y) w_h(|r_i(y)|) dy + \frac{v(x_i) - v(y)}{|r_i(y)|} \int_{\mathbb{R}^d} \pi_{ij}(y) w_h(|r_i(y)|) dy,$$

$$\text{and}$$

$$E_2(x_i) := d \int_{\mathbb{R}^d} \rho_{ij}(y) w_h(|r_i(y)|) \int_{\mathbb{R}^d} \pi_{ij}(y) dy,$$

respectively. We fix $x_i \in X_{\mathcal{F}_H} \cap \Omega$. By the Taylor expansion, for all $j = 1, 2, \ldots, N (j \neq i)$ and $y \in \sigma_j$, we have

$$|\pi_{ij}(y)| \leq \frac{v(x_j) - v(y)}{|r_i(y)|} \int_{\mathbb{R}^d} \left| \frac{1}{|r_i(y)|} - \frac{1}{|r_j(x_j)|} \right| dy \leq c r_j \frac{r_i(y)}{|r_i(y)|} \left| v \right|_{C^1(\Omega_H)}.$$

Then, by (16) and the Taylor expansion, we have

$$|E_1(x_i)| \leq c r_v \left| v \right|_{C^1(\Omega_H)} \left| \int_{\mathbb{R}^d} \frac{w_h(|r_i(y)|)}{|r_i(y)|} dy \right| \leq c r_v \frac{r_i(y)}{|r_i(y)|} \left| v \right|_{C^1(\Omega_H)}.$$

Therefore, owing to (5), we obtain

$$|E_1(x_i)| \leq c r^{-1} h^{-m-1} \left| v \right|_{C^1(\Omega_H)}.$$

Since $\sigma_j \subset B_{r_v}(x_i) (j = 1, 2, \ldots, N)$ and supp$w_h = [0, h]$, for all $j \not\in \Lambda(x_i, h + r_v)$, we have

$$\int_{\sigma_j} |\pi_{ij}(y)| dy = 0.$$

Then, from the definition of (4), we estimate

$$\sum_{j \not\in \Lambda_{x_i, h + r_v}} \int_{\sigma_j} |\pi_{ij}(y)| dy \leq c \frac{r_v}{h^{m+1}} \left| v \right|_{C^1(\Omega_H)} \sum_{j \not\in \Lambda(x_i, h + r_v)} \int_{\sigma_j} dy \leq c \left( 1 + 2r_v \right)^d \frac{r_v}{h^{m+1}} \left| v \right|_{C^1(\Omega_H)}.$$

Hence, owing to (5), we obtain

$$|E_2(x_i)| \leq c r^{-1} \left( 1 + 2c_0 h^{-m-1} \right) d h^{-m-1} \left| v \right|_{C^1(\Omega_H)}.$$
Next, we prove (7). By Hypothesis 2, we can take \( \tilde{w} \in C^1(\mathbb{R}_h^+ \mathbb{R}_h^+) \) defined in (1). For \( i, j = 1, 2, \ldots, N \) and \( y \in \Omega_H \), set \( \pi_{ij}(y) := \tilde{w}_h(r_i(y)) - \tilde{w}_h(r_j(x_j)) \). The left hand side of (7) can be split into the three terms as follows:

\[
\|\Delta v - \Delta_h v\|_{L^\infty(\Omega)} \leq \|\Delta v - \tilde{w}_h\|_{L^\infty(\Omega)} + \|E_3\|_{L^\infty(\Omega)} + \|E_4\|_{L^\infty(\Omega)},
\]

where \( E_3 \) and \( E_4 \) are defined by

\[
E_3(x_i) := \frac{2d}{h} \sum_{j \neq i} \int_{\sigma_j} \pi_{ij}(y) \tilde{w}_h(r_i(y)) dy - \frac{2d}{h} \int_{\sigma_i} v(x_i) - v(y) \frac{r(x_i)}{|r(y)|} \tilde{w}_h(r_i(y)) dy,
\]

\[
E_4(x_i) := \frac{2d}{h} \sum_{j \neq i} \pi_{ij}(y) \int_{\sigma_j} \tilde{w}_h(r_j(x_j)) dy,
\]

respectively. We fix \( x_i \in X_{NH} \cap \Omega \). By (15) and the Taylor expansion, we obtain

\[
|E_3(x_i)| \leq \frac{c}{h} |v|_{C^1(\mathbb{R}_h^+\mathbb{R}_h^+)} \sum_{i=1}^N \int_{\sigma_i} \frac{|\tilde{w}_h(r_i(y))|}{|r_i(y)|} dy \leq \frac{c}{h} |v|_{C^1(\mathbb{R}_h^+\mathbb{R}_h^+)}.
\]

Hence, owing to (5), we obtain

\[
|E_3(x_i)| \leq c_0^{-1}h^{m-2} |v|_{C^1(\mathbb{R}_h^+\mathbb{R}_h^+)}.
\]

By a similar argument as in (17), we estimate

\[
\sum_{j \neq i} \int_{\sigma_j} \pi_{ij}(y) dy \leq c \left( 1 + \frac{2c}{h} \right) \frac{d}{h} |v|_{C^1(\mathbb{R}_h^+\mathbb{R}_h^+)}.\]

Therefore, owing to (5), we obtain

\[
|E_4(x_i)| \leq c_0^{-1} \left( 1 + 2c_0^{-1}h^{m-1} \right) d h^{m-2} |v|_{C^1(\mathbb{R}_h^+\mathbb{R}_h^+)}.\]

By combining Lemma 7 and the estimates of \( E_3 \) and \( E_4 \), we conclude (7).

(QED)

5. Numerical results

We show some numerical results for Theorem 4. Set \( \Omega = (0, 1)^2 \). We consider a given function \( v(x, y) = \sin(2\pi(x + y)) \). Set \( H = 0.1 \) and

\[
X_{NH} = \{(i + \xi_{ij}^{(1)}\Delta x, (j + \xi_{ij}^{(2)}\Delta x) \in \Omega_H; i, j \in \mathbb{Z}\}.
\]

Here, \( \xi_{ij}^{(n)} (i, j \in \mathbb{Z}, n = 1, 2) \) denote random numbers in \((-0.25, 0.25) \) and \( \Delta x = 2^{-5}, 2^{-6}, \ldots, 2^{-12} \). In this case, \( N \approx 2^{10}, 2^{12}, \ldots, 2^{22} \) and \( 1.4\Delta x \leq \gamma_{x} \leq 1.8\Delta x \).

We consider \( w \) defined by (8), where \( w \) is the weight functions widely used in SPH, the cubic B-spline and the quintic B-spline [1]. In addition, in case of the approximate Laplace operators, we also consider \( w \) set by \( w(r) = r^2w^M(r), r \in \mathbb{R}_h^+ \), where \( w^M \) is the weight function used in MPS [2]. Set \( h \) by \( h = \{(3.14 \times 2^{-15})\Delta x \}^{1/4} \). Under the above setting, Theorem 4 is valid with \( O(h^2) \).

Fig. 1 shows graphs of the relative errors by \( L^\infty \) norm versus \( h \). The slopes of triangles show the theoretical convergence rates of \( \Theta(h^2) \) derived from Theorem 4. Ta-

![Fig. 1. Graphs of the relative errors versus h.](image)

Table 1. Numerical convergence rates between \( \Delta x = 2^{-10}, 2^{-11} \).

|                  | gradient | Laplacian |
|------------------|----------|-----------|
| SPH (cubic B-spline) | 2.00     | 2.00      |
| SPH (quintic B-spline) | 2.00    | 2.00      |
| MPS               | 2.07     |           |

Table 1 shows numerical convergence rates obtained from the slopes of the relative errors between \( \Delta x = 2^{-10} \) and \( 2^{-11} \). Fig. 1 and Table 1 show that numerical convergence rates agree well with theoretical ones.

6. Concluding remarks

We have estimated the truncation errors of the approximate gradient and Laplace operators obtained with a class of the particle methods, which coincide with approximate differential operators of SPH and MPS with their parameters replaced by ones based on the Voronoi volumes. Introducing sufficient conditions for the weight function and the regularity of the family of discrete parameters, we have established truncation error estimates of the approximate differential operators. In case of the approximate differential operators related to SPH and MPS, the convergence rates are at least second order with respect to the influence radius under these conditions. Moreover, we have confirmed that numerical convergence rates agree well with the theoretical ones.

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