Maximally connected and super arc-connected Bi-Cayley digraphs *

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Abstract Let $X = (V, E)$ be a digraph. $X$ is maximally connected, if $\kappa(X) = \delta(X)$. $X$ is maximally arc-connected, if $\lambda(X) = \delta(X)$. And $X$ is super arc-connected, if every minimum arc-cut of $X$ is either the set of inarcs of some vertex or the set of outarcs of some vertex. In this paper, we will prove that the strongly connected Bi-Cayley digraphs are maximally connected and maximally arc-connected, and the most of strongly connected Bi-Cayley digraphs are super arc-connected.

Keywords: Bi-Cayley digraph; atom; $\lambda$−atom; $\lambda$−superatom

1 Introduction

A digraph is a pair $X = (V, E)$, where $V$ is a finite set and $E$ is an irreflexive relation on $V$. Thus $E$ is a set of ordered pairs $(u, v) \in V \times V$ such that $u \neq v$. The elements of $V$ are called the vertices or nodes of $X$ and the elements of $E$ are called the arcs of $X$. Arc $(u, v)$ is said to be an inarc of $v$ and an outarc of $u$; we also say that $(u, v)$ originates at $u$ and terminates at $v$. If $u$ is a vertex of $X$, then the outdegree of $u$ in $X$ is the number $d^+_X(u)$ of arcs of $X$ originating at $u$ and the indegree of $u$ in $X$ is the number $d^-_X(u)$ of arcs of $X$ terminating at $u$. The minimum outdegree of $X$ is $\delta^+(X) = \min\{d^+_X(u) \mid u \in V\}$ and the minimum indegree of $X$ is $\delta^-(X) = \min\{d^-_X(u) \mid u \in V\}$. We denote by $\delta(X)$ the minimum of $\delta^+(X)$ and $\delta^-(X)$.

The reverse digraph of digraph $X = (V, E)$ is the digraph $X^{(r)} =
Digraph $X = (V, E)$ is symmetric if $E = E^r$ and is antisymmetric if $E \cap E^r = \emptyset$. An (undirected) graph is a pair $X = (V, E)$ where $V$ is a finite set and $E$ is a collection of two-element subsets of $V$. We will in general identify an undirected graph $X = (V, E)$ with the symmetric digraph $X_\text{s} = (V, E_\text{s})$ where $E_\text{s} = \{(u, v) | (u, v) \in E\} \cup \{(v, u) | (u, v) \in E\}$. A digraph with exactly one vertex (and therefore no arcs) is called a trivial digraph. We denote by $K_n^\text{s}$ the digraph with vertices the integers from 1 to $n$ and arcs all pairs $(i, j)$ of such integers with $i \neq j$. A digraph isomorphic to $K_n^\text{s}$ is said to be a complete symmetric digraph.

For a digraph $X = (V, E)$ and a subset $A$ of $V$, we can get a subdigraph $X[A]$ of $X$ whose vertex set is $A$ and whose arc-set consists of all arcs of $X$ which have both ends in $A$. And we call the subdigraph $X[A]$ is an induced subdigraph of $X$.

**Definition 1.1.** Let $G$ be a group and $T_0, T_1 \subseteq G$. We define the Bi-Cayley digraph $X = BD(G, T_0, T_1)$ to be the bipartite digraph with vertex set $G \times \{0, 1\}$ and arc set $\{(g, 0), (to, 1)\}, \{(t_1, g, 1), (g, 0)\} \mid g \in G, t_0 \in T_0, t_1 \in T_1$.

By definition we observe that $d_X^+(g, 0) = |T_0|, d_X^+(g, 1) = |T_1|, d_X^+(g, 1)) = |T_1|, d_X^+(g, 1)) = |T_0|$, for any $g \in G$.

In this paper, we always denote $X_0 = G \times \{0\}$ and $X_1 = G \times \{1\}$. Some new results on the Bi-Cayley graph are referred to [2, 6]. In this paper, we will consider Bi-Cayley digraphs. Denote $R(G) = \{R(a) | R(a) : (g, i) \rightarrow (ga, i), for a, g \in G and i = 0, 1\}$. We will get the following proposition.

**Proposition 1.2.** Let $X = BD(G, T_0, T_1)$, then

1. $R(G) \leq Aut(X)$, furthermore $Aut(X)$ acts transitively both on $X_0$ and $X_1$.

2. $X$ is strongly connected if and only if $|T_0| \geq 1$, $|T_1| \geq 1$ and $G \triangleleft \frac{T_1^{-1} T_0^T}.$

**Proof.** (1) For any $R(a) \in R(G)$ and $((g_1, 0), (g_2, 1)) \in E(X)$, there exists some $t_0 \in T_0$ such that $g_2 = tog_1$, then $g_2 a = t_0 g_1 a$. Thus $((g_1, 0), (g_2, 1))^{R(a)} = ((g_1 a, 0), (g_2 a, 1)) \in E(X)$. Similarly, if $((g_1, 1), (g_2, 0)) \in E(X)$, then $((g_2, 1), (g_1, 0))^{R(a)} \in E(X)$. So $R(a)$ is an automorphism of the Bi-Cayley digraph $X$, thus $R(G) \leq Aut(X)$. Since $(g_1, i)^{R(g_1^{-1} g_2)} = (g_2, i)$ for any $g_1, g_2 \in G$, $Aut(X)$ acts transitively both on $X_0$ and $X_1$.

(2) If $X$ is strongly connected, then $|T_0| \geq 1$, $|T_1| \geq 1$ and there exists a directed path from $(1_G, 0)$ to $(g, 0)$ for any $g \in G$. Thus there exists an integer $n$, $t_0^{(i)} \in T_0$ and $t_1^{(i)} \in T_1 (1 \leq i \leq n)$ such that $1_G \rightarrow t_0^{(1)} \rightarrow (t_1^{(1)})^{-1} t_0^{(1)} \rightarrow \ldots \rightarrow$.
\[ (t^{(n)})^{-1}t^{(n)} \cdot (t^{(2)})^{-1}t^{(2)} (t^{(1)})^{-1}t^{(1)} = g, \text{ that is } G = < T^{-1}T_0 >. \]

On the other hand, for any \( h, g \in G \), \( h^{-1}g \) is in \( G = < T^{-1}T_0 > \) if and only if it can be written as a product of elements of \( T^{-1}T_0 \cup (T^{-1}T_0)^{-1} \). Thus we can easily know there exists a path from \( (g, i) \) to \( (h, i) \). And since \( |T_0| \geq 1 \) and \( |T_1| \geq 1 \), \( (g, 0) \) has both outarcs and inarcs for any \( g \in G \). So \( X \) is strongly connected.

\[ \square \]

2 Connectivity

Let \( X = (V, E) \) be a strongly connected digraph. An arc disconnecting set of \( X \) is a subset \( W \) of \( E \) such that \( X \setminus W = (V, E \setminus W) \) is not strongly connected. An arc disconnecting set is minimal if no proper subset of \( W \) is an arc disconnecting set of \( X \) and is a minimum arc disconnecting set if no other arc disconnecting set has smaller cardinality than \( W \). The arc connectivity \( \lambda(X) \) of a nontrivial digraph \( X \) is the cardinality of a minimum arc disconnecting set of \( X \).

The positive arc neighborhood of a subset \( A \) of \( V \) is the set \( \omega^+(X)(A) \) of all arcs which initiate at a vertex of \( A \) and terminate at a vertex of \( V \setminus A \). The negative neighborhood of subset \( A \) of \( V \) is the set \( \omega^-(X)(A) \) of all arcs which initiate in \( V \setminus A \) and terminate in \( A \). Thus \( \omega^+_X(A) = \omega^-_X(V \setminus A) \). Arc neighborhoods of proper, nonempty subsets of \( V \), often called arc-cuts, are clearly arc disconnecting sets. Thus for any proper, nonempty subset \( A \) of \( V \), \( |\omega^+(A)| \geq \lambda(X) \). If we consider the cases where \( A \) consists of a single vertex or the complement of a single vertex, we easily see that \( \lambda(X) \leq \delta(X) \).

A nonempty subset \( A \) of \( V \) is called a positive(respectively, negative) arc fragment of \( X \) if \( |\omega^+(A)| = \lambda(X) \) (respectively, \( |\omega^-(A)| = \lambda(X) \)). An arc fragment \( A \) with \( 2 \leq |A| \leq |V(X)| - 2 \) is called a strict arc fragment of \( X \). An arc fragment of minimum cardinality is called \( \lambda \)-atom of \( X \) and a strict arc fragment of least possible cardinality is called a \( \lambda \)-superatom of \( X \). Note that a \( \lambda \)-atom(respectively, \( \lambda \)-superatom) may be either a positive arc fragment or a negative arc fragment or both. A \( \lambda \)-atom which is a positive(respectively, negative) arc fragment is called a positive(respectively, negative) \( \lambda \)-atom and a \( \lambda \)-superatom which is a positive(respectively, negative) arc fragment is called a positive(respectively, negative) \( \lambda \)-superatom.

A vertex disconnecting set of \( X \) is a subset \( F \) of \( V(X) \) such that \( X \setminus F \) is either trivial or is not strongly connected. We often call \( F \) a vertex-cut. The connectivity \( \kappa(X) \) of a nontrivial digraph \( X \) is the cardinality of a minimum vertex disconnecting set of \( X \).
The positive neighborhood of a subset \( F \) of \( V \) is the set \( N^+(F) \) of all vertices of \( V \setminus F \) which are targets of arcs initiating at a vertex of \( F \). The positive closure \( C^+(F) \) of \( F \) is the union of \( F \) and \( N^+(F) \). The negative neighborhood of subset \( F \) of \( V \) is the set \( N^-(F) \) of all vertices of \( V \setminus F \) which are the initial vertices of arcs which terminate at a vertex of \( F \). The negative closure \( C^-(F) \) of \( F \) is the union of \( F \) and \( N^-(F) \).

If \( F \) is a nonempty subset of \( V \) with \( C^+(F) \neq V \), then the positive neighborhood of \( F \) is clearly a vertex disconnecting set for \( X \). Thus for each such set \( F \), \( |N^+(F)| \geq \kappa(X) \). If we consider the cases where \( F \) consists of a single vertex or the complement of a single vertex, we easily see that \( \kappa(X) \leq \delta(X) \).

A nonempty subset \( F \) of \( V \) is called a positive (respectively, negative) fragment of \( X \) if \( |N^+(F)| = \kappa(X) \) and \( C^+(F) \neq V \) (respectively, \( |N^-(F)| = \kappa(X) \) and \( C^-(F) \neq V \)). A fragment of minimum cardinality is called atom.

Note that an atom may be either a positive fragment or a negative fragment or both. A atom which is a positive (respectively, negative) fragment is called a positive (respectively, negative) atom.

A digraph \( X \) is maximally arc connected (respectively, maximally connected), or more simply, max-\( \lambda \) (respectively, max-\( \kappa \)), if \( \lambda(X) = \delta(X) \) (respectively, \( \kappa(X) = \delta(X) \)). And \( X \) is super arc connected, or more simply, super-\( \lambda \) (respectively, super-\( \kappa \)), if every minimum arc-cut of \( X \) is either the set of inarcs of some vertex or the set of outarcs of some vertex. The relationship of \( \lambda(X) \) and \( \kappa(X) \) is well known: \( \kappa(X) \leq \lambda(X) \leq \delta(X) \). So if \( \kappa(X) = \delta(X) \), then \( \lambda(X) = \delta(X) \). In the following of this section we will try to prove that \( \kappa(X) = \delta(X) \) for Bi-Cayley digraphs.

A desirable property one wishes any type of atom to have is that, if nontrivial, they form imprimitive blocks for the automorphism group of the digraph. To be precise, an imprimitive block for a group \( \Phi \) of permutations of a set \( T \) is a proper, nontrivial subset \( A \) of \( T \) such that if \( \varphi \in \Phi \) then either \( \varphi(A) = A \) or \( \varphi(A) \cap A = \emptyset \). In the following proposition Hamidoune has proved that the positive (respectively, negative) atoms of \( X \) are imprimitive blocks of \( X \). The following proposition indicates why imprimitivity is so useful.

**Proposition 2.1.** ([4]) Let \( X = (V, E) \) be a graph or digraph and let \( Y \) be the subgraph or subdigraph induced by an imprimitive block \( A \) of \( X \). Then
1. If \( X \) is vertex-transitive then so is \( Y \);
2. If \( X \) is a strongly connected arc-transitive digraph or a connected edge-transitive graph and \( A \) is a proper subset of \( V \), then \( A \) is an independent subset of \( X \).
Proposition 2.2. Let $X = (V, E)$ be a strongly connected digraph which is not a complete symmetric digraph and let $A$ be a positive (respectively, negative) atom of $X$. If $B$ is a positive (respectively, negative) fragment of $X$ with $A \cap B \neq \emptyset$, then $A \subset B$.

Proposition 2.3. Let $X$ be a strongly digraph with $\kappa(X) < \delta(X)$, and $A$ be an atom of $X$, then $X[A]$ is strongly connected.

Clearly if $X = BD(G, T_0, T_1)$ is a strongly connected Bi-Cayley digraph with $\kappa(X) < \delta(X)$ and $A$ an atom of $X$, then $A_i = A \cap X_i \neq \emptyset$ for $i = 0, 1$.

Lemma 2.4. Let $X = BD(G, T_0, T_1)$ be a strongly connected Bi-Cayley digraph with $\kappa(X) < \delta(X)$. If $A$ is an atom of $X$, then
(1) $V(X)$ is a disjoint union of distinct positive (or, negative) atoms of $X$;
(2) Let $Y = X[A]$, then $Aut(Y)$ acts transitively both on $A_0$ and $A_1$;
(3) If $(1, i) \in A_i = H_i \times \{i\}$, then $H_i$ is the subgroup of $G$ for $i=0,1$;
(4) $|A_0| = |A_1|$.

Proof. (1) and (2) follow from the results that the distinct positive (negative) atoms are disjoint and $Aut(X)$ acts transitively both on $X_0$ and $X_1$. (3) For any $g \in H_0$, $Ag$ is also a positive atom since $R(g) \in Aut(X)$. And $g \in A \cap Ag$, then we get that $A = Ag$, thus $A_0g = A_0$ and $A_1g = A_1$. The former equality means that $H_0$ is a subgroup of $G$.

(4) From proposition 1.2(1) and proposition 2.2, we can get $V(X) = \cup_{i=1}^{k} \phi_i(A)$ where $\phi_i \in Aut(X)$ such that $\phi_i(A) \cap \phi_j(A) = \emptyset$ if $i \neq j$, then $X_i = \cup_{i=1}^{k} \phi_i(A_i)$. Since $|X_0| = |X_1|$, we have $|A_0| = |A_1|$. \hfill \Box

From the proof of lemma 2.4, $Y = X[A]$ has the property that $d^+_Y((g_i, 0)) = d^+_Y((g_j, 0))$ and $d^-_Y((g_i, 0)) = d^-_Y((g_j, 0))$ for any vertices $(g_i, 0), (g_j, 0) \in A_0$. And if $(1, 0) \in A_0$, then $A_1g = A_1$ is right for any $g \in H_0$, so $H_1H_0 = H_1$. It means $H_1$ is a left coset of $H_0$ since $|H_0| = |H_1|$. We have the following lemma.

Lemma 2.5. Let $X = BD(G, T_0, T_1)$ be a strongly connected Bi-Cayley digraph with $\kappa(X) < \delta(X)$, and $A$ be a positive atom. Let $A_0 = \{g_1, g_2, ..., g_m\} \times \{0\} = H_0 \times \{0\}$ and $A_1 = \{g_1, g_2, ..., g_m\} \times \{1\} = H_1 \times \{1\}$. Then
(1) If $t_ig_j \in H_1$ for some $t_i \in T_1$ ($i=0, 1$) and some some $j (1 \leq j \leq m)$, then $t_ig_k \in H_1$ for any $k (1 \leq k \leq m)$;
(2) If $t^{-1}_ig_j \in H_0$ for some $t_i \in T_1$ and some $j (1 \leq j \leq m)$, then $t^{-1}_ig_k \in H_0$ for any $k (1 \leq k \leq m)$.

Proof. (1) Assume $(1, 0) \in A_0$, then $H_1H_0 = H_1$. If $t_ig_j \in H_1$, then $t_ig_jH_0 = t_igH_1$. It means $t_ig_k \in H_1$ for any $k (1 \leq k \leq m)$. 

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(2) Similarly, assume \((1,0) \in A_0\), then \(H_1H_0 = H_1\). If \(t_1^{-1}g'_j \in H_0\), then \(g'_j \in t_1H_0\). So \(H_1 = t_1H_0\). It means that \(t_1^{-1}g'_k \in H_0\) for any \(k(1 \leq k \leq m)\).

**Theorem 2.6.** Let \(X = BD(G, T_0, T_1)\) be a strongly connected Bi-Cayley digraph, then \(\kappa(X) = \delta(X)\)

**Proof.** Suppose \(X\) is not max-\(\kappa\). Without loss of generality, assume that \(A = A_0 \cup A_1\) is a positive atom. Denote \(A_0 = H_0 \times \{0\}\) and \(A_1 = H_1 \times \{1\}\). If \(|N^+(A_0) \setminus A_1| \neq 0\), then by lemma 2.5 we have \(|N^+(A_0) \setminus A_1| \geq |H_0|\). Thus \(|N^+(A)| = |N^+(A_0) \setminus A_1| + |N^+(A_1) \setminus A_0| = |N^+(A_0) \setminus A_1| + |T_1^{-1}H_1 \setminus H_0| \times \{0\}| \geq |H_0| + |T_1^{-1}H_0| \geq |T_1^{-1}| \geq \delta(X)\), a contradiction. If \(|N^+(A_1) \setminus A_0| \neq 0\), then by lemma 2.5 we have \(|N^+(A_1) \setminus A_0| \geq |H_1|\). Thus \(|N^+(A)| = |N^+(A_0) \setminus A_1| + |N^+(A_1) \setminus A_0| = |T_0H_0 \setminus H_1| \times \{1\}| + |N^+(A_0) \setminus A_1| \geq |T_0H_1| + |H_1| \geq |T_0| \geq \delta(X)\), a contradiction. Therefore \(N^+(A) = \emptyset\), it is a contradiction.

**Corollary 2.7.** Let \(X = BD(G, T_0, T_1)\) be a strongly connected Bi-Cayley digraph, then \(\kappa(X) = \lambda(X) = \delta(X)\).

### 3 Super arc-connectivity

A **weak path** of a digraph \(X\) is a sequence \(u_0, ..., u_r\) of distinct vertices such that for \(i = 1, ..., r\), either \((u_{i-1}, u_i)\) or \((u_i, u_{i-1})\) is an arc of \(X\). A directed graph is **weakly connected** if any two vertices can be joined by a weak path.

**Proposition 3.1.** Let \(X = BD(G, T_0, T_1)\) be a strongly connected Bi-Cayley digraph and \(A\) be a \(\lambda\)--superatom, then

1. \(Y = X[A]\) is weakly connected;
2. \(|A| \geq \delta(X)|.

**Proof.** Suppose \(A\) is a positive \(\lambda\)--superatom.

1. If \(|A| = 2\), then we obtain that \(A\) is not an independent set since \(|N^+(A)| = \delta(X)\) and \(N^+(u) \neq 0\) for any \(u \in V(X)\). Now assume \(|A| \geq 3\). If \(Y = X[A]\) is not weakly connected, we can get a \(\lambda\)--superatom with cardinality less than \(A\), a contradiction.

2. \(\lambda(X) = |\omega_X^+(A)| \geq |A|(|\delta(X) - (|A| - 1)|) = |A|(|\delta(X) - |A| + 1|)

we can verify that \(\lambda(X) > \delta(X)\) when \(2 \leq |A| < \delta(X)\), a contradiction.

Any digraph with \(d^+(x) = d^-(x)\) for every vertex \(x\) of \(X\) is said to be a **balanced** digraph.
Proposition 3.2. [4] Let $X = (V, E)$ be a strongly connected, balanced digraph and let $A$ and $B$ be arc fragments of $X$ such that $A \nsubseteq B$ and $B \nsubseteq A$. If $A \cap B \neq \emptyset$ and $A \cup B \neq V$, then each of the sets $A \cap B$, $A \cup B$, $A \setminus B$ and $B \setminus A$ is an arc fragment of $X$.

Theorem 3.3. [4] Let $X = (V, E)$ be a strongly connected balanced digraph which is not a symmetric cycle, is not super arc-connected and has $\delta(X) \geq 2$. If $\delta(X) > 2$ or $X$ is vertex-transitive, then distinct $\lambda-$superatoms of $X$ are vertex disjoint.

Similarly, we can also achieve the analogous results.

Proposition 3.4. Let $X = (V, E)$ be a strongly connected digraph and let $A$ and $B$ be positive(respectively, negative) arc fragments of $X$ such that $A \nsubseteq B$ and $B \nsubseteq A$. If $A \cap B \neq \emptyset$ and $A \cup B \neq V$, then each of the sets $A \cap B$, $A \cup B$, $A \setminus B$ and $B \setminus A$ is a positive(respectively, negative) arc fragment of $X$.

Theorem 3.5. Let $X = (V, E)$ be a strongly connected digraph which is not a symmetric cycle, is not super arc-connected and has $\delta(X) \geq 2$. If $\delta(X) > 2$ or $X$ is vertex-transitive, then distinct positive(respectively, negative) $\lambda-$superatoms of $X$ are vertex disjoint.

Lemma 3.6. Let $X = BD(G, T_0, T_1)$ be strongly connected but not super-$\lambda$. If $X$ is neither a directed cycle nor a symmetric cycle, then distinct positive(respectively, negative) $\lambda-$superatoms of $X$ are vertex disjoint.

Proof. Suppose to the contrary that there are distinct positive $\lambda-$superatoms $A$, $B$ of $X$ with $A \cap B \neq \emptyset$. By proposition 3.4, each of $A \cap B$, $A \cup B$, $A \setminus B$, $B \setminus A$ is a positive arc fragment which is a proper subset of a $\lambda-$superatom. Therefore, each of these sets must have cardinality 1 so that we may assume $A = \{u, v\}$, $B = \{v, w\}$ with $u \neq w$. Thus we have $d^+_X[A](u) = d^-_X[A](v) \leq 1$, $d^-_X[A](u) = d^+_X[A](v) \leq 1$, $d^+_X[B](v) = d^-_X[B](w) \leq 1$ and $d^-_X[B](v) = d^+_X[B](w) \leq 1$.

Case 1 $\delta(X) = 1$.

$d^+_X(u) = d^+_X(v) = d^+_X(w) = 1$, so $|T_0| = 1$, $|T_1| = 1$. And because $X$ is a strongly connected digraph, we can get $X$ is a directed cycle, a contradiction.

Case 2 $\delta(X) = 2$.

$d^+_X(u) = d^+_X(v) = d^+_X(w) = 2$. Because $X[A]$ and $X[B]$ are weakly connected and $A$, $B$ and $A \cup B$ are arc fragments, we can deduce

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$|T_0| = |T_1| = 2$ and $T_0 = T_1$.

Because $X$ is strongly connected, $X$ is a cycle, a contradiction.

**Case 3** $\delta(X) \geq 3$.

It is true by Theorem 3.5. \hfill \Box

For the rest of the paper we set $A_i = A \cap X_i = H_i \times i$, $i = 0, 1$. Similarly to Lemma 2.4, we can derive the following theorem.

**Lemma 3.7.** Let $X = BD(G, T_0, T_1)$, which is neither a directed cycle nor a symmetric cycle, be strongly connected but not super-$\lambda$. Let $A$ be a $\lambda$-superatom of $X$, then

1. $V(X)$ is a disjoint union of distinct positive(negative) $\lambda$-superatoms;
2. Let $Y = X[A]$, then $\text{Aut}(Y)$ acts transitively both on $A_0$ and $A_1$;
3. If $A_i$ contains $(1,i)(i = 0, 1)$, then $H_i$ is a subgroup of $G$;
4. $|A_0| = |A_1|$.

Similarly as lemma 2.4, we also have $H_1H_0 = H_1$ if $(1,0) \in A_0$ and $H_0H_1 = H_0$ if $(1,1) \in A_1$. The following proposition is easy to get.

By a similar argument as lemma 2.5, the following lemma is obtained.

**Lemma 3.8.** Let $X = BD(G, T_0, T_1)$, which is neither a directed cycle nor a symmetric cycle, be strongly connected but not super-$\lambda$. Let $A$ be a $\lambda$-superatom of $X$ and set $A_0 = \{g_1, g_2, g_3, \ldots, g_m\} \times \{0\} = H_0 \times \{0\}$ and $A_1 = \{g_1, g_2, g_3, \ldots, g_m\} \times \{1\} = H_1 \times \{1\}$. Then

1. If $t_i g_j \in H_1$ for some $t_i \in T_i$ ($i=0, 1$) and some some $j(1 \leq j \leq m)$, then $t_i g_k \in H_1$ for any $k(1 \leq k \leq m)$;
2. If $t_i^{-1} g_j \in H_0$ for some $t_i \in T_1$ and some $j(1 \leq j \leq m)$, then $t_i^{-1} g_k \in H_0$ for any $k(1 \leq k \leq m)$.

**Theorem 3.9.** Let $X = BD(G, T_0, T_1)$ be strongly connected. If $X$ is neither a directed cycle nor a symmetric cycle, then $X$ is not super-$\lambda$ if and only if $X$ satisfies one of the following conditions:

1. There exists a subgroup $H \leq G$ and there distinct elements $t_0', t_1', t_0'' \in T_0$ such that

$|H| = \delta(X), T_1^{-1} t_0 \subseteq H, t_0^{-1}(T_0 \setminus \{t_0'\}) \subset H$ and $t_0^{-1} t_0' \notin H.$

or

$|H| = \delta(X)/2, T_1^{-1} t_0 \subseteq H, t_0^{-1}(T_0 \setminus \{t_0', t_0''\}) \subset H$ and $t_0^{-1} t_0', t_0^{-1} t_0'' \notin H.$

Where $t_0' \neq t_0$ and $t_0'' \neq t_0$

2. There exists a subgroup $H \leq G$ and two distinct elements $t_1', t_1'' \in T_1$ and some element $t_0 \in T_0$ such that

$|H| = \delta(X), t_0^{-1} T_0 \subseteq H, (T_1 \setminus \{t_1'\})^{-1} t_0 \subset H$ and $t_1^{-1} t_0 \notin H.$
\[|H| = \delta(X)/2, \ t_0^{-1}T_0 \subset H, (T_1 \setminus \{t_1, t'_1\})^{-1}t_0 \subset H \text{ and } t_1^{-1}t_0, t_1^{-1}t_0 \notin H.\]

(3) There exists a subgroup \( H \leq G \) and two distinct elements \( t_0, t'_0 \in T_0 \) and some element \( t_1 \in T_1 \) such that \( |H| = \delta(X)/2, \ t_0^{-1}(T_0 \setminus \{t'_0\}) \subset H, \ t_0^{-1}t'_0 \notin H, (T_1 \setminus \{t_1\})^{-1}t_0 \subset H \text{ and } t_1^{-1}t_0 \notin H, \) where \( t_0 \neq t_0'. \)

**Proof.** Necessity. Without loss of generality assume \( A \) is a positive \( \lambda \)-superatom of \( X \) and \( (1, 0) \in A \). From lemma 3.7, \( H_0 \) is a subgroup of \( G \) and \( Y = X[A] \) is a bipartite digraph with \( d^+_Y((g_k, i)) = d_Y^-(A(i=0, 1)) \) and \( d^-_Y((g_k, i)) = d^-_Y((g_k, 1)) \) for any vertices \( (g_k, i), (g_k, i) \in A(i=0, 1) \). Furthermore, \( d^+_Y((g_k, 0)) = d^-_Y((g_k, 1)) \) and \( d^-_Y((g_i, 0)) = d^+_Y((g_k, 1)) \). Denote \( d_Y^+((g_k, 0)) = d_Y^-((g_k, 1)) = p, \ d_Y^-(Y, 0)) = d^-_Y((g_k, 1)) = q \). Let \( H = H_0 \).

**Claim:** There exist at least an element \( t_0 \in T_0 \) such that \( H_1 = t_0H_0 \) if \((1, 0) \in A \).

The **proof of the Claim:** If \( p = 0 \), then \( \delta(X) = \lambda(X) = |\omega^+_X(A)| = |A_0|(|T_0| - p) + |A_1|(|T_1| - q) = |A_0|T_0 + |A_1|(|T_1| - q) \geq |T_0| + |A_1|(|T_1| - q) \geq |T_0| \geq \delta(X) \). So \( |A_0| = |A_1| = 1 \) and \( |T_1| = q = 0 \). So \( \omega^+(A) = \omega^+(A_0) \), thus \( X \) is super-\( \lambda \). By a similar argument we can prove \( X \) is super-\( \lambda \) when \( q = 0 \). A contradiction. So \( pq \neq 0 \).

If \((1, 0) \in A_0 \), then \( H_1 = H_1H_0 \). And because \( p \neq 0 \), there exist at least an element \( t_0 \in T_0 \) such that \( t_0 \in H_1 \). Thus \( H_1 = t_0H_0 \). So the Claim is true.

Since \( \delta(X) = \lambda(X) = |\omega^+_X(A)| = |A_0|(|T_0| - p) + |A_1|(|T_1| - q), |A| \geq \delta(X) \) and \( |A_0| = |A_1| \), we have \( |A_0| = |A_1| \geq \delta(X)/2 \) and \( |T_0| - p + |T_1| - q \leq 2 \). Now we consider fine cases.

**Case 1** \( |T_0| - p = 1 \) and \( |T_1| - q = 0 \).

(i) \( \lambda(X) = |\omega^+_X(A)| = |A_0| = |H_0| = |H| = \delta(X) \), since \( |T_0| - p = 1 \) and \( |T_1| - q = 0 \).

(ii) Since \( |T_0| - p = 1 \), there exists an element \( t'_0 \in T_0 \) such that \( (T_0 \setminus \{t'_0\})H_0 \subset H_1 \) and \( t_0^{-1}H_0 \cap H_1 = \emptyset \). It means \( (T_0 \setminus \{t'_0\})H_0 \subset t_0H_0 \) and \( t_0^{-1}t'_0 \notin H_0 \).

(iii) since \( |T_1| - q = 0 \), we have that \( T_1^{-1}H_1 \subset H_0 \). It means \( T_1^{-1}t_0H_0 \subset H_0 \), so \( T_1^{-1}t_0 \notin H_0 \).

**Case 2** \( |T_0| - p = 0 \) and \( |T_1| - q = 1 \).

(i) \( \lambda(X) = |\omega^+_X(A)| = |A_1| = |H_1| = |H| = \delta(X) \), since \( |T_0| - p = 0 \) and \( |T_1| - q = 1 \).

(ii) Since \( |T_0| - p = 0 \), we have that \( T_0H_0 \subset H_1 \). It means \( T_0H_0 \subset t_0H_0 \), so \( t_0^{-1}t_0 \notin H_0 \).

(iii) since \( |T_1| - q = 1 \), there exists an element \( t_1 \in T_1 \) such that \( (T_1 \setminus \{t_1\})^{-1}H_1 \subset H_0 \) and \( t_1^{-1}H_1 \cap H_0 = \emptyset \). It means \( (T_1 \setminus \{t_1\})^{-1}t_0H_0 \subset H_0 \).
and $t_1^{-1}t_0H_0 \cap H_0 = \emptyset$, so $(T_1 \setminus \{t_1\})^{-1}t_0 \subset H_0$ and $t_1^{-1}t_0 \notin H_0$.

**Case 3** $|T_0| - p = 2$ and $|T_1| - q = 0$.

It is similar to Case 1, we have

(i) $|H| = |H_0| = \delta(X)/2$.

(ii) $t_0^{-1}(T_0 \setminus \{t_0, t_0''\}) \subset H_0$ and $t_0^{-1}t_0, t_0^{-1}t_0'' \notin H_0$ for some $t_0, t_0'' \in T_0$.

(iii) $T_1^{-1}t_0 \subseteq H_0$.

**Case 4** $|T_0| - p = 0$ and $|T_1| - q = 2$.

It is similar to Case 2, we have

(i) $|H| = |H_1| = \delta(X)/2$.

(ii) $t_0^{-1}T_0 \subset H_0$.

(iii) $(T_1 \setminus \{t_1, t_1''\})^{-1}t_0 \subset H_0$ and $t_1^{-1}t_0, t_1''^{-1}t_0 \notin H_0$ for some $t_1, t_1'' \in T_1$.

**Case 5** $|T_0| - p = 1$ and $|T_1| - q = 1$.

(i) $\lambda(X) = |\omega^1_X(A)| = |A_0| = |H_0| = |H| = \delta(X)/2$, since $|T_0| - p = 1$, and $|T_1| - q = 1$.

(ii) since $|T_0| - p = 1$, then $t_0^{-1}(T_0 \setminus \{t_0\}) \subset H_0$ and $t_0^{-1}t_0 \notin H_0$ for some element $t_0 \in T_0$ and $t_0 \neq t_0$.

(iii) since $|T_1| - q = 1$, $(T_1^{-1}\{t_1^{-1}\})t_0 \subset H_0$ and $t_1^{-1}t_0 \notin H_0$ for some $t_1 \in T_1$.

**Sufficiency.** Set $A = H \times \{0\} \cup \langle t_0H \rangle \times \{1\}$. Thus $(1, 0) \in A$, $H_0 = H$ and $H_1 = t_0H_0$.

(1) If $t_0^{-1}(T_0 \setminus \{t_0\}) \subset H$ and $t_0^{-1}t_0 \notin H$, then $t_0^{-1}(T_0 \setminus \{t_0\})H = H$ and $t_0 \notin t_0H_0$, it is $(T_0 \setminus \{t_0\})H = t_0H = H_1$ and $t_0H_0 \cap H_1 = \emptyset$. So $|T_0| - p = 1$. And if $T_1^{-1}t_0 \subseteq H$, then $T_1^{-1}t_0H \subseteq H$. It is $T_1^{-1}H \subseteq H$. So $|T_1| - q = 0$. Associate with the condition $|H| = \delta(X)$, we have $\lambda(X) = |\omega^1_X(A)| = |A_0| = |H| = \delta(X)$. So A is a $\lambda$–superatom of X.

Similarly, If $|H| = \delta(X)/2$, $T_1^{-1}t_0 \subseteq H$, $t_0^{-1}(T_0 \setminus \{t_0, t_0''\}) \subset H$ and $t_0^{-1}t_0, t_0''^{-1}t_0 \notin H$, we can prove A is a $\lambda$–superatom of X.

(2) If $(T_1 \setminus \{t_1^{-1}\})^{-1}t_0 \subset H$ and $t_1^{-1}t_0 \notin H$, then $(T_1 \setminus \{t_1^{-1}\})^{-1}t_0H \subset H$ and $t_0 \notin t_1H$, it is $(T_1 \setminus \{t_1^{-1}\})^{-1}H_1 \subset H$ and $t_0H \cap t_1H = \emptyset$. So $|T_1| - q = 1$. And if $t_0^{-1}T_0 \subset H$, then $t_0^{-1}T_0H \subset H$, it is $T_0H \subset t_0H$. So $|T_0| - p = 0$. Associate with the condition $|H| = \delta(X)$, we get $\lambda(X) = |\omega^1_X(A)| = |A_1| = |H_1| = |H_0| = \delta(X)$. So A is a $\lambda$–superatom of X.

Similarly, If $|H| = \delta(X)/2, t_0^{-1}t_0 \subset H, (T_1 \setminus \{t_1, t_1^{-1}\})^{-1}t_0 \subset H$ and $t_1^{-1}t_0, t_1^{-1}t_0 \notin H$, we can prove A is a $\lambda$–superatom of X.

(3) If $t_0^{-1}(T_0 \setminus \{t_0\}) \subset H, (T_1 \setminus \{t_1\})^{-1}t_0 \subset H$ and $t_0^{-1}t_0, t_0^{-1}t_0 \notin H$ then $t_0^{-1}(T_0 \setminus \{t_0\})H \subset H, (T_1 \setminus \{t_1\})^{-1}t_0H \subset H, t_0 \notin t_0H$ and $t_0 \notin t_1H$, it is $(T_0 \setminus \{t_0\})H \subset t_0H = H_1, (T_1 \setminus \{t_1\})^{-1}H_1 \subset H_0 \subset t_0H \cap t_1H = \emptyset$ and $t_0H \cap t_1H = \emptyset$. So $|T_0| - p = 1$ and $|T_1| - q = 1$. Thus associate with the condition $|H| = \delta(X)/2$, we have $\lambda(X) = |\omega^1_X(A)| = |A_0| + |A_1| = |H_1| = |H_0| + |H_1| = \delta(X)$. So A is a $\lambda$–superatom of X. □
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