A CORRELATION INEQUALITY FOR THE
EXPECTATIONS OF NORMS OF STABLE VECTORS.

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Abstract. For $0 < q \leq 2$, $1 \leq k < n$, let $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ be symmetric $q$-stable random vectors so that the joint distributions of $X_1, ..., X_k$ and $X_{k+1}, ..., X_n$ are equal to the joint distributions of $Y_1, ..., Y_k$ and $Y_{k+1}, ..., Y_n$, respectively, but $Y_i$ and $Y_j$ are independent for every $1 \leq i \leq k$, $k+1 \leq j \leq n$. We prove that $E(f(X)) \geq E(f(Y))$ where $f$ is any continuous, positive, homogeneous of the order $p \in (-n, 0)$ function on $\mathbb{R}^n\setminus\{0\}$ such that $f$ is a positive definite distribution in $\mathbb{R}^n$, and $f(u, v) = f(u, -v)$ for every $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$. As a particular case, we show that $E(\max_{i=1}^n |X_i|)^p \geq E(\max_{i=1}^n |Y_i|)^p$ for every $p \in (-n, -n + 1)$. The latter inequality is related to Slepian’s Lemma and to the Gaussian correlation problem.

1. Introduction

Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be symmetric jointly Gaussian random variables. The well-known Slepian’s Lemma [20], [9] states that if $E X_i^2 = E Y_i^2$ and $E(X_i X_j) \geq E(Y_i Y_j)$ for every $i, j = 1, ..., n$, then

$$E (\max_{i=1}^n X_i) \leq E (\max_{i=1}^n Y_i),$$

and, even more, for every $t \in \mathbb{R}$ one has

$$P(\max_{i=1}^n X_i > t) \leq P(\max_{i=1}^n Y_i > t),$$

These inequalities mean that the maximum of Gaussians tends to be larger when they are less correlated.

One may ask a question of whether the absolute values of Gaussians behave in a similar way, namely, given a number $k \in \mathbb{N}$, $1 \leq k < n$ and fixed joint distributions of $X_1, ..., X_k$ and of $X_{k+1}, ..., X_n$ is it true that the quantities $E (\max_{i=1}^n |X_i|)$ and $P(\max_{i=1}^n |X_i| > t)$ are maximal when the random variables $X_i$ and $X_j$ are independent for every $1 \leq i \leq k$, $k+1 \leq j \leq n$? However, this question

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for $P(\max_{i=1,\ldots,n} |X_i| > t)$ appears to be equivalent to the famous correlation problem for Gaussian measures of symmetric convex sets: Is it true that $\mu(A \cap B) \geq \mu(A)\mu(B)$ for any symmetric convex sets $A, B$ in $\mathbb{R}^d$ where $\mu$ is the standard symmetric Gaussian measure in $\mathbb{R}^d$? Pitt [16] has proved that the answer to the correlation problem is positive in the case $d = 2$, and, therefore, confirmed the proper behaviour of the Gaussian random variables generated by a two-dimensional Gaussian vector. For $d > 2$ the problem remains open (see [19] for the history of the problem and partial results).

This note provides some evidence supporting the conjecture on the behaviour of the absolute values of Gaussians. Let $0 < q \leq 2$, $1 \leq k < n$. Throughout the paper, $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are symmetric $q$-stable random vectors such that the joint distributions of $X_1, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$ are equal to the joint distributions of $Y_1, \ldots, Y_k$ and $Y_{k+1}, \ldots, Y_n$, respectively, but $Y_i$ and $Y_j$ are independent for every $1 \leq i \leq k$, $k + 1 \leq j \leq n$. We prove that in this situation

\begin{equation}
\mathbb{E} \left( \max_{i=1,\ldots,n} |X_i| \right)^p \geq \mathbb{E} \left( \max_{i=1,\ldots,n} |Y_i| \right)^p,
\end{equation}

for every $p \in (-n, -n + 1)$.

We show this result as a particular case of the following more general inequality:

**Theorem 1.** Let $q, k, X, Y$ be as above, and let $-n < p < 0$ and $f$ be a continuous, positive, homogeneous of the order $p$ function on $\mathbb{R}^n \setminus \{0\}$ such that $f$ is a positive definite distribution in $\mathbb{R}^n$, and $f(u, v) = f(u, -v)$ for every $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$. Then $\mathbb{E}(f(X)) \geq \mathbb{E}(f(Y))$.

The inequality (1) will follow from Theorem 1 and a simple fact that, for $-n < p < -n + 1$, every positive, continuous, homogeneous of the order $p$ function on $\mathbb{R}^n \setminus \{0\}$ is also a positive definite distribution.

We refer the reader to [7, 17, 18] for other results related to the Slepian Lemma.

### 2. Expectations of positive powers of norms

We start with an inequality for the expectations of positive powers of certain norms. The techniques used in this case are standard, but the positive case makes more clear what happens later in the case of negative powers.

We need a few simple inequalities for the $L_q$-norms which follow from Clarkson’s inequality (see [2]). For the reader’s convenience we include the proof. We denote by $\| \cdot \|_q$ the norm of the space $L_q([0, 1])$.

**Lemma 1.** Let $x, y \in L_q([0, 1])$, $0 < q \leq 2$. Then

\begin{equation}
\exp(-\|x + y\|_q^q) + \exp(-\|x - y\|_q^q) \geq 2 \exp(-\|x\|_q^q - \|y\|_q^q).
\end{equation}
Also for every $0 < p \leq q$

$$\|x + y\|^p_q + \|x - y\|^p_q \leq 2(\|x\|^q_q + \|y\|^q_q)^{p/q}. \tag{3}$$

Finally, for $q = 2$ and $p > 2$ the inequality (3) goes in the opposite direction.

Proof. First, note that for any $0 < q \leq 2$

$$\|x + y\|^q_q + \|x - y\|^q_q \leq 2(\|x\|^q_q + \|y\|^q_q), \tag{4}$$

and this is a simple consequence of the same inequality for real numbers. Now to get (2) apply the relation between the arithmetic and geometric means and then use (4). The inequality (3) also follows from (4):

$$\left(\frac{\|x + y\|^p_q + \|x - y\|^p_q}{2}\right)^{1/p} \leq \left(\frac{\|x + y\|^q_q + \|x - y\|^q_q}{2}\right)^{1/q} \leq (\|x\|^q_q + \|y\|^q_q)^{1/q}.$$

Finally, if $q = 2$ the latter calculation works for $p > 2$ where the first inequality goes in the opposite direction, and the second inequality turns into an equality. □

For $0 < q \leq 2$, $1 \leq k < n$, let $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$ be the symmetric $q$-stable random vectors defined in Introduction.

The characteristic function of the vector $X$ has the form

$$\phi(\xi) = \exp(-\|\sum_{i=1}^n \xi_i s_i\|^q_q), \quad \xi \in \mathbb{R}^n,$$  \tag{5}

where $s_1, \ldots, s_n \in L_q([0, 1])$.

Then the characteristic function of $Y$ is equal to

$$\phi_0(\xi) = \exp(-\|\sum_{i=1}^k \xi_i s_i\|^q_q - \|\sum_{i=k+1}^n \xi_i s_i\|^q_q).$$

Let $(\mathbb{R}^n, \| \cdot \|)$ be an $n$-dimensional subspace of $L_p([0, 1])$, $p > 0$. A well-known easy fact due to P.Levy [13] is that an $n$-dimensional space is isometric to a subspace of $L_p([0, 1])$ if and only if its norm admits the following Levy representation:

$$\|x\|^p = \int_S |(x, \xi)|^p \, d\gamma(\xi) \tag{6}$$

for every $x \in \mathbb{R}^n$, where $S$ is the unit sphere in $\mathbb{R}^n$, $(x, \xi)$ stands for the scalar product, and $\gamma$ is a finite Borel (non-negative) measure on $S$.

**Proposition 1.** Let $q, k, X, Y$ be as in the Introduction, $0 < p \leq q$ and $(\mathbb{R}^n, \| \cdot \|)$ is a subspace of $L_p$ with the norm satisfying $\|(u, v)\| = \|(u, -v)\|$ for every $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$. Then $E(\|X\|^p) \leq E(\|Y\|^p)$. Also if $q = 2$ and $p > 2$ the inequality goes in the opposite direction.

Proof. A basic property of the stable vector with the characteristic function (5) is that, for any vector $\xi \in \mathbb{R}^n$, the random variable $(X, \xi)$ has the same distribution as...
\[ \| \sum_{i=1}^{n} \xi_i s_i \|_q Z, \] where \( Z \) is the standard one-dimensional \( q \)-stable random variable. Therefore, if \( p < q \) then

\[ \mathbb{E} |(X, \xi)|^p = c_{p,q} \| \sum_{i=1}^{n} \xi_i s_i \|_q^p, \]

where \( c_{p,q} \) is the \( p \)-th moment of \( Z \) (which exists only for \( p < q \) if \( q < 2 \), and it exists for every \( p > 0 \) if \( q = 2 \); see [22] for a formula for \( c_{p,q} \)). Similarly, we get

\[ \mathbb{E} |(X_-, \xi)|^p = c_{p,q} \| \sum_{i=1}^{k} \xi_i s_i - \sum_{i=k+1}^{n} \xi_i s_i \|_q^p, \]

where \( X_- = (X_1, \ldots, X_k, -X_{k+1}, \ldots, -X_n) \). Also,

\[ \mathbb{E} |(Y, \xi)|^p = c_{p,q}(\| \sum_{i=1}^{k} \xi_i s_i \|_q^p + \| \sum_{i=k+1}^{n} \xi_i s_i \|_q^p)^{p/q}. \]

Since \( (\mathbb{R}^n, \| \cdot \|) \) is a subspace of \( L_p([0, 1]) \), we can use the Levy representation (6) and after that the formula (7) to get

\[ \mathbb{E} (\| X \|)^p = \int_{S} \mathbb{E} (| (X, \xi)|^p) \ d\gamma(\xi) = c_{p,q} \int_{S} \| \sum_{i=1}^{n} \xi_i s_i \|_q^p d\gamma(\xi). \]

Similarly,

\[ \mathbb{E} (\| Y \|)^p = c_{p,q} \int_{S} (\| \sum_{i=1}^{k} \xi_i s_i \|_q^p + \| \sum_{i=k+1}^{n} \xi_i s_i \|_q^p)^{p/q} d\gamma(\xi), \]

\[ \mathbb{E} (\| X_- \|)^p = c_{p,q} \int_{S} \| \sum_{i=1}^{k} \xi_i s_i - \sum_{i=k+1}^{n} \xi_i s_i \|_q^p d\gamma(\xi). \]

Since \( 0 < p \leq q \), the equalities (8), (9), (10) in conjunction with (3) imply

\[ \mathbb{E} (\| X \|)^p + \mathbb{E} (\| X_- \|)^p \leq 2 \mathbb{E} (\| Y \|)^p, \]

and now the result follows from the property of the norm that \( \| X \| = \| X_- \| \). In the case \( q = 2, \ p > 2 \) we use the corresponding part of Lemma 1. \( \square \)

**Remarks.**

(i) For \( p > q, \ q < 2 \) the expectation of \( \| X \|_p \) does not exist so the statement of Proposition 1 does not make sense in that case.

(ii) In view of Proposition 1, it is natural to ask how can one check whether a given space is isometric to a subspace of \( L_p \). This question is the matter of an old problem raised by P.Levy [13]. In the same paper P.Levy showed that an \( n \)-dimensional space is isometric to a subspace of \( L_p \) if and only if its norm admits...
the representation (6). Since then, a few criteria involving the Fourier transform have appeared. Bretagnolle, Dacunha-Castelle and Krivine [1] proved that, for $0 < p \leq 2$, a Banach space is isometric to a subspace of $L_p$ if and only if the function $\exp(-\|x\|^p)$ is positive definite, and, in particular, showed that the spaces $L_q$ embed isometrically into $L_p$ if $0 < p < q \leq 2$. Another Fourier transform criterion was given in [10], [11]: for any $p \in (0, \infty) \setminus \{\text{even integers}\}$, an $n$-dimensional space is isometric to a subspace of $L_p$ if and only if the restriction of the Fourier transform of $\|x\|^p \Gamma(-p/2)$ to the unit sphere $S$ in $\mathbb{R}^n$ is a finite Borel measure on $S$ (the Fourier transform is considered in the sense of distributions). Recently, two criteria were shown that were in terms of the derivatives of the norm: Zastavny [21] proved that a three-dimensional space is not isometric to a subspace of $L_p$ with $0 < p \leq 2$ if there exists a basis $e_1, e_2, e_3$ so that the function $(y, z) \mapsto \|xe_1 + ye_2 + ze_3\|_x (1, y, z)/\|e_1 + ye_2 + ze_3\|$, $y, z \in \mathbb{R}$ belongs to the space $L_1(\mathbb{R}^2)$. By inverting the representation (6), it was shown in [12] that an $n$-dimensional space is isometric to a subspace of $L_p$ with $n + [p]$ being an even integer if $(-1)^{(n+\lfloor p\rfloor)/2} \Delta^{(n+\lfloor p\rfloor)/2} \|x\|^p$ is a positive continuous function on the sphere $S$ ($\Delta$ is the Laplace operator, $p$ is not an even integer; in the case where $n + [p]$ is odd the formula must be slightly modified.)

(iii) Misiewicz [15] proved that the spaces $\ell_\infty^n$, $n > 2$ do not embed in any of the spaces $L_p$, $p > 0$, therefore Proposition 1 does not tell anything about the behaviour of $\max(|X_i|)$ (except for the case $n = 2$ where one can use the well-known fact due to Herz [8], Ferguson [3], Lindenstrauss [14] that any two-dimensional Banach space embeds isometrically in each one of the spaces $L_p$ with $0 < p \leq 1$.)

### 3. Expectations of Negative Powers of Norms

As one can see from Remarks (ii) and (iii) the condition of Proposition 1 that the norm embeds isometrically in $L_p$ is quite restricting and is not easy to check. In this section we replace this condition by an equivalent one, and that allows us to extend the result of Proposition 1 to the case of negative powers $p$ and, more important, to a much larger class of norms.

To formulate the equivalent condition, we need some notation. As usual, we denote by $S(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions in $\mathbb{R}^n$, and by $S'(\mathbb{R}^n)$ the space of distributions over $S(\mathbb{R}^n)$. Recall that the Fourier transform of a distribution $f \in S'(\mathbb{R}^n)$ is defined by $(\hat{f}, \phi) = (f, \hat{\phi})$ for every $\phi \in S(\mathbb{R}^n)$. We say that a distribution $f \in S'(\mathbb{R}^n)$ is positive definite in a domain $D \subset \mathbb{R}^n$ if the Fourier transform of $f$ is a positive distribution in $D$, namely, $(\hat{f}, \phi) \geq 0$ for every non-negative function $\phi \in S(\mathbb{R}^n)$ supported in $D$. We need the following.
Lemma 2. Let $p \in (-1, \infty)$, $p$ is not an even integer. Let $\phi$ be a function from the space $S(\mathbb{R}^n)$. Then for every $\xi \in \mathbb{R}^n$, $\xi \neq 0$,

$$
\int_{\mathbb{R}^n} |(x, \xi)|^p \hat{\phi}(x) \, dx = (2\pi)^{n-1} c_p(|z|^{-1-p}, \phi(z\xi))
$$

where $c_p = (2^{p+1} \Gamma((p+1)/2))/\Gamma(-p/2)$, and $(|z|^{-1-p}, \phi(z\xi))$ is the value of the one-dimensional distribution $|z|^{-1-p}$ at the test function $z \to \phi(z\xi)$, $z \in \mathbb{R}$.

Proof. By the Fubini theorem

(11) $\int_{\mathbb{R}^n} |(x, \xi)|^p \hat{\phi}(x) \, dx = \int_{\mathbb{R}} |t|^p \left( \int_{(x, \xi)=t} \hat{\phi}(x) \, dx \right) \, dz = (|t|^p, \int_{(x, \xi)=t} \hat{\phi}(x) \, dx)$.

It is well-known that the Fourier transform of the function $t \to |t|^p$, $t \in \mathbb{R}$ is equal to $(|t|^p)^\wedge(z) = c_p |z|^{-1-p}$, $z \neq 0$, for every $p \in (-1, \infty)$ which is not an even integer (see [4]). Also the function $z \to (2\pi)^n \hat{\phi}(-z\xi)$ is the Fourier transform of the function

$$
t \to \int_{(x, \xi)=t} \hat{\phi}(x) \, dx
$$

(this is the connection between the Fourier transform and the Radon transform, see [6]). Passing to the Fourier transforms in the equality (11) we get

$$
(|t|^p, \int_{(x, \xi)=t} \hat{\phi}(x) \, dx) = (1/2\pi)(c_p |z|^{-1-p}, (2\pi)^n \phi(z\xi)). \quad \Box
$$

Now we are able to show the equivalent condition mentioned above:

Lemma 3. Let $p$ be a positive number which is not an even integer. A space $(\mathbb{R}^n, \| \cdot \|)$ is isometric to a subspace of $L_p([0,1])$ if and only if there exists a finite Borel (non-negative) measure $\gamma$ on the unit sphere $S$ in $\mathbb{R}^n$ so that, for every $\phi \in S(\mathbb{R}^n)$,

(12) $((|x|^p)^\wedge, \phi) = c_p \int_S (|z|^{-1-p}, \phi(z\xi)) \, d\gamma(\xi)$.

Proof. A simple fact going back to P.Levy [13] is that a space $(\mathbb{R}^n, \| \cdot \|)$ is isometric to a subspace of $L_p([0,1])$ if and only if the norm admits the Levy representation (6) with a measure $\gamma$ on the sphere $S$. By (6) and Lemma 2, the space embeds into $L_p([0,1])$ if and only if, for every $\phi \in S(\mathbb{R}^n)$,

$$
((|x|^p)^\wedge, \phi) = (|x|^p, \hat{\phi}) = \int_{\mathbb{R}^n} |x|^p \hat{\phi}(x) \, dx = 
$$

$$
\int_S d\gamma(\xi) \left( \int_{\mathbb{R}^n} |(x, \xi)|^p \hat{\phi}(x) \, dx \right) = c_p \int_S (|z|^{-1-p}, \phi(z\xi)) \, d\gamma(\xi). \quad \Box
$$
If the function $\phi$ in (12) is supported in $\mathbb{R}^n \setminus \{0\}$, we have $\langle |z|^{-1-p}, \phi(z\xi) \rangle = \int_{\mathbb{R}} |z|^{-1-p}\phi(z\xi)$ which is non-negative if the function $\phi$ is non-negative. Therefore,

**Corollary 1.** If a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p([0, 1])$ with $p > 0$, $p \neq 2k$, $k \in \mathbb{N}$, then the distribution $\|x\|^p \Gamma(-p/2)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$.

Now we are able to prove an analog of Proposition 1 for negative powers $p$ replacing the embedding in $L_p$ by positive definiteness of $\|x\|^p$ (note that for negative $p$ the numbers $\Gamma(-p/2)$ are always positive). Also in the case of negative $p$ we will be able to replace the norm to the power $p$ by any positive, continuous, homogeneous of the order $p$ function. (The latter means that $f(tx) = |t|^p f(x)$ for every $t \in \mathbb{R}$, $t \neq 0, x \in \mathbb{R}^n \setminus \{0\}$.)

**Theorem 1.** Let $q, k, X, Y$ be as in the Introduction, and let $-n < p < 0$ and $f$ be a continuous, positive, homogeneous of the order $p$ function on $\mathbb{R}^n \setminus \{0\}$ such that $f$ is a positive definite distribution in $\mathbb{R}^n$, and $f(u, v) = f(u, -v)$ for every $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$. Then $\mathbb{E}(f(X)) \geq \mathbb{E}(f(Y))$.

**Proof.** We use the following generalization of Bochner’s theorem (see [5]): if $f$ is a positive definite distribution in $\mathbb{R}^n$ (over $\mathcal{S}(\mathbb{R}^n)$) then $f$ is the Fourier transform (in the sense of distributions) of a tempered measure $\mu$ in $\mathbb{R}^n$. (Recall that a measure is called tempered if $\int_{\mathbb{R}^n} (1 + \|x\|^2)^\alpha \, d\mu(x) < \infty$ for some $\alpha < 0$.) Let $\mu$ be the tempered measure whose Fourier transform is equal to $f$.

Let $P_X$ be the $q$-stable measure in $\mathbb{R}^n$ according to which the random vector $X$ is distributed. Applying the Parseval equality and the expression (5) for the characteristic function of $X$ we get

$$\mathbb{E}(f(X)) = \int_{\mathbb{R}^n} f(x) \, dP_X(x) = \int_{\mathbb{R}^n} \widehat{P_X}(\xi) \, d\mu(\xi) = \int_{\mathbb{R}^n} \exp(-\|\sum_{i=1}^n \xi_i s_i\|_q^q) \, d\mu(\xi).$$

Note that the function $f$ is locally integrable in $\mathbb{R}^n$ because $-n < p < 0$. Similarly,

$$\mathbb{E}(f(X_-)) = \int_{\mathbb{R}^n} \exp(-\|\sum_{i=1}^k \xi_i s_i - \sum_{i=k+1}^n \xi_i s_i\|_q^q) \, d\mu(\xi),$$

where $X_- = (X_1, ..., X_k, -X_{k+1}, ..., -X_n)$, and

$$\mathbb{E}(f(Y)) = \int_{\mathbb{R}^n} \exp(-\|\sum_{i=1}^k \xi_i s_i\|_q^q - \|\sum_{i=k+1}^n \xi_i s_i\|_q^q) \, d\mu(\xi).$$

Now by the inequality (2) from Lemma 1 and taking in account that $\mu$ is a positive measure, we get

$$\mathbb{E}(f(X)) + \mathbb{E}(f(X_-)) \geq 2\mathbb{E}(f(Y)),$$

and the result follows from the property of the function $f$ that $f(X) = f(X_-)$. \(\square\)
Let us show that the norm of every subspace of the spaces $L_r$, $0 < r \leq 2$ has all the properties of the function $f$ in Theorem 1.

**Proposition 2.** Let $(\mathbb{R}^n, \| \cdot \|)$ be a subspace of $L_r([0, 1])$ with $0 < r \leq 2$. Then, for any $p \in (-n, 0)$ the function $\|x\|^p$ is a positive definite distribution on $\mathbb{R}^n$, and, therefore, $E(\|X\|^p) \geq E(\|Y\|^p)$.

**Proof.** By the result of Bretagnolle, Dacunha-Castelle and Krivine [1], the function $\exp(-\|x\|^r)$ is a positive definite function in $\mathbb{R}^n$. It is easy to see that

$$\|x\|^p = \frac{r}{\Gamma(-p/r)} \int_0^\infty |t|^{-1-p} \exp(-|t|^r \|x\|^r) \, dt.$$ 

The integral in the right-hand side converges because $p < 0$. Also that integral represents a positive definite function of $x$, since $\exp(-|t|^r \|x\|^r)$ is a positive definite function of $x$ for every $t > 0$. The inequality for the expectations follows from Theorem 1. □

We are going to show now that the number of spaces for which $\|x\|^p$ is a positive definite distribution in $\mathbb{R}^n$ becomes quite large when $p \to -\infty$. For example, the norm of every $n$-dimensional Banach space has this property if $-n < p < -n + 1$.

**Proposition 3.** Let $-n < p < -n + 1$. Then every even, continuous, positive, homogeneous of the order $p$ function $f$ on $\mathbb{R}^n \setminus \{0\}$ is a positive definite distribution in $\mathbb{R}^n$.

**Proof.** Let $\phi$ be any non-negative function from $\mathcal{S}(\mathbb{R}^n)$. Writing the integral in the spherical coordinates we get

$$(\hat{f}, \phi) = \int_{\mathbb{R}^n} f(x) \hat{\phi} \, dx = (1/2) \int_\mathbb{R} \int_\mathbb{R} f(r\theta) |r|^{n-1} \hat{\phi}(r\theta) \, dr \, d\theta =$$

$$(1/2) \int_\mathbb{R} f(\theta) \, d\theta \left( \int_\mathbb{R} |r|^{n+p-1} \hat{\phi}(r\theta) \, dr \right).$$

The integral over $\mathbb{R}$ in (13) converges because $n+p-1 \in (-1, 0)$. The Fourier transform of the distribution $|r|^{n+p-1}$ is equal to $(|r|^{n+p-1})^\wedge(t) = c_{n+p-1}|t|^{-n-p}$, $t \in \mathbb{R}$ where $c_{n+p-1} = 2^{n+p} \sqrt{\pi} \Gamma((n+p)/2)/\Gamma((-n-p+1)/2)$ is a positive constant. On the other hand, by the connection between the Fourier transform and the Radon transform, the function $r \to \hat{\phi}(r\theta)$ is the one-dimensional Fourier transform of the function $t \to \int_{(x,\theta) = t} \phi(x) \, dx$ (the latter function is the Radon transform of $\phi$ in the direction of $\theta$.) Therefore, for any non-negative function $\phi \in \mathcal{S}(\mathbb{R}^n)$, switching to the Fourier transforms we get

$$\int |r|^{n+p-1} \hat{\phi}(r\theta) \, dr = (|r|^{n+p-1}, \hat{\phi}(r\theta)) =$$

$$= (|r|^{n+p-1})^\wedge(t) \int_\mathbb{R} \hat{\phi}(s \cdot r\theta) \, ds \leq$$

$$\leq \int_\mathbb{R} \hat{\phi}(s \cdot r\theta) \, ds = \int_\mathbb{R} \hat{\phi}(s) \, ds.$$
\[ c_{n+p-1}(t) = c_{n+p-1} \int_{\mathbb{R}} |t|^{-n-p} \left( \int_{(x,\theta)=t} \phi(x) \, dx \right) \, dt \geq 0, \]

where the last integral converges since \(-n-p \in (-1,0)\). We conclude that the integral (13) is non-negative, which means that \(\hat{f}\) is a positive distribution on \(\mathbb{R}^n\). \(\square\)

An immediate consequence of Theorem 1 and Proposition 3 is the following

**Corollary 2.** Let \(-n < p < -n+1\) and \(f\) be any even, continuous, positive, homogeneous of the order \(p\) function in \(\mathbb{R}^n \setminus \{0\}\) such that \(f(u,v) = f(u,-v)\) for every \(u \in \mathbb{R}^k, v \in \mathbb{R}^{n-k}\). Then \(E(f(X)) \geq E(f(Y))\).

Putting \(f(x) = \max_{i=1,\ldots,n} |x_i|^p, p \in (-n,-n+1)\) in Corollary 2 we get the inequality (1):

**Corollary 3.** For any \(p \in (-n,-n+1)\) and \(q,k,X,Y\) as in the Introduction, we have

\[ E \left( \max_{i=1,\ldots,n} |X_i|^p \right) \geq E \left( \max_{i=1,\ldots,n} |Y_i|^p \right). \]

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