Rudnick and Soundarajan’s Theorem over Prime Polynomials for the Rational Function Field

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ABSTRACT: In this paper, we use the methods of Andrade, Rudnick and Soundarajan to prove a Theorem about Lower bounds of moments of quadratic Dirichlet L-functions associated to monic irreducible polynomials over function fields.

1 Introduction

A fundamental problem in Analytic Number Theory is to understand the asymptotic behaviour of moments of families of L-functions. For example, in the case of the Riemann-zeta function, a problem is to establish an asymptotic formula for

\[ M_k(T) = \int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} \, dt, \]  

(1.1)

where \( k \) is a positive integer and \( T \to \infty \). Asymptotic formulas for the first two moments have been explicitly calculated, the first by Hardy and Littlewood \([10]\), in which they proved that

\[ M_1(T) \sim T \log T \]

and Ingham \([11]\) proved that

\[ M_2(T) \sim \frac{1}{2\pi^2} T \log^4 T. \]

Although no higher moments have been explicitly calculated, it is conjectured that

\[ M_k(T) \sim c_k T (\log T)^{k^2} \]

where, due to Conrey and Ghosh \([6]\), the constant \( c_k \) assumes a more explicit form, namely

\[ c_k = \frac{a_k g_k}{\Gamma(k^2 + 1)}, \]
where
\[ a_k = \prod_p \left[ \left( 1 - \frac{1}{p^2} \right) k^2 \sum_{m \geq 0} \frac{d_k(m)^2}{p^m} \right], \]
g_k is an integer when k is an integer and \( d_k(m) \) is the number of ways to represent \( m \) as a product of \( k \) factors. Ramachandra [14] obtained a lower bound for moments of the Riemann-zeta function for positive integers \( k \). In particular he showed that
\[ M_k(T) \gg T (\log T)^{k^2}. \]

For the family of Dirichlet L-functions, \( L(s, \chi_d) \) associated to the quadratic character \( \chi_d \), a problem is to understand the asymptotic behaviour of
\[ S_k(X) = \sum_{|d| \leq X} L \left( \frac{1}{2}, \chi_d \right)^k, \]
where the sum is over fundamental discriminants \( d \) as \( X \to \infty \). Jutila [12], proved that
\[ S_1(X) \sim C_1 X \log X \]
and
\[ S_2(X) \sim C_2 X \log^3 X, \]
where \( C_1 \) and \( C_2 \) are positive constants. Restricting \( d \) to be odd, square-free and positive, so that \( \chi_{8d} \) are real, primitive characters with conductor \( 8d \) and with \( \chi_{8d}(-1) = 1 \), Soundarajan [18] proved
\[ \sum_{|d| \leq X} L \left( \frac{1}{2}, \chi_{8d} \right)^3 \sim C_3 X \log^6 X, \]
for some positive constant \( C_3 \). It is conjectured, by Keating and Snaith [13] that
\[ S_k(X) \sim C_k X (\log X)^{\frac{k(k+1)}{2}}, \]
for some positive constant \( C_k \). Rudnick and Soundarajan [17] proved a result for the lower bounds for moments of these Dirichlet L-functions, when \( k \) is an even integer. In particular, they showed that
\[ S_k(X) \gg X (\log X)^{\frac{k(k+1)}{2}}. \]

In the Function Field setting, the analogue problem of (1.2) is to understand the asymptotic behaviour of
\[ I_k(g) = \sum_{D \in \mathcal{H}_{2g+1}} L \left( \frac{1}{2}, \chi_D \right)^k \]
as \( |D| = q^{\deg(D)} \to \infty \), where \( \mathcal{H}_{2g+1} \) denotes the space of monic, square-free polynomials of degree \( 2g + 1 \) over \( \mathbb{F}_q[T] \). In the setting of fixing \( q \), where in particular \( q \equiv 1(\mod 4) \), and letting \( g \to \infty \), Andrade and Keating [2] and Florea [7] proved that
\[ I_1(g) \sim \tilde{C}_1 |D| \log_q |D| \]

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for some positive constant $\tilde{C}_1$. Andrade and Keating \cite{1} conjectured that

$$I_k(g) \sim \tilde{C}_k |D| (\log_q |D|)^{\frac{k(k+1)}{2}}$$

for some positive constant $\tilde{C}_k$. The second, third and fourth moments were explicitly calculated by Florea \cite{8, 9}, where the computed constants agreed with the constants conjectured by Andrade and Keating. Similar to the calculations done by Rudnick and Soundarajan in the number field setting, Andrade \cite{1} proved a result for the lower bounds of (1.3) when $k$ is an even integer. In particular, he proved that

$$I_k(g) \gg |D| (\log_q |D|)^{\frac{k(k+1)}{2}}.$$

Another problem in Function Fields is to understand the asymptotic behaviour of

$$\sum_{P \in \mathcal{P}_{2g+1}} L \left( \frac{1}{2}, \chi_P \right)^k$$

where $\mathcal{P}_{2g+1}$ denotes the spaces of monic, irreducible polynomials of degree $2g+1$ over $\mathbb{F}_q[T]$. Andrade and Keating \cite{3} proved

$$\sum_{P \in \mathcal{P}_{2g+1}} (\log_q |P|) L \left( \frac{1}{2}, \chi_P \right) \sim |P| \log_q |P|,$$

while, along with Bui and Florea \cite{15}, in the same paper computed the second moment

$$\sum_{P \in \mathcal{P}_{2g+1}} L \left( \frac{1}{2}, \chi_P \right)^2 \sim |P| (\log_q |P|)^2.$$

In this paper, we use similar methods to that of Andrade, Rudnick and Soundarajan to obtain a lower bound for (1.4). The main result is the following.

**Theorem 1.1.** For every even natural number $k$ and $n = 2g + 1$ or $n = 2g + 2$ we have,

$$\frac{1}{|\mathcal{P}_n|} \sum_{P \in \mathcal{P}_n} L \left( \frac{1}{2}, \chi_P \right)^k \gg_k (\log_q |P|)^{\frac{k(k+1)}{2}}.$$

## 2 Background and Preliminaries

Before we prove Theorem 1.1, we state some facts which can generally be found in \cite{15}. Fix a finite field $\mathbb{F}_q$, where $q \equiv 1 \pmod{4}$ and let $\mathbb{A} = \mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$ and let $k = \mathbb{F}_q(T)$ be the rational function field over $\mathbb{F}_q$. We use the notation $\mathbb{A}^+, \mathbb{A}_n^+$ and $\mathbb{A}_{2n}^+$ to denote the set of all monic polynomials in $\mathbb{A}$, the set of all monic polynomials of degree $n$ in $\mathbb{A}$ and the set of all monic polynomials of degree less than or equal to $n$ in $\mathbb{A}$ respectively. The zeta function associated to $\mathbb{A}$ is defined by the infinite series

$$\zeta_{\mathbb{A}}(s) = \sum_{f \in \mathbb{A}^+} \frac{1}{|f|^s},$$

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where $|f| = q^{\deg(f)}$ if $f \neq 0$ and $|f| = 0$ if $f = 0$. There are $q^n$ monic polynomials of degree $n$, therefore we have

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}}.$$ 

Let $P$ be a monic irreducible polynomial in $A$ of odd degree. We denote by $\chi_P$ the quadratic character defined in terms of the quadratic residue symbol for $\mathbb{F}_q[T]$:

$$\chi_P(f) = \left(\frac{P}{f}\right)$$

where $f \in \mathbb{A}$. For more details see [13], chapter 3. Thus, if $Q \in \mathbb{A}$ is a monic, irreducible polynomial, we have

$$\chi_P(Q) = \begin{cases} 0 & \text{if } Q | P \\ 1 & \text{if } Q \nmid P \text{ and } P \text{ is a square mod } P \\ -1 & \text{if } Q \nmid P \text{ and } P \text{ is not a square mod } P. \end{cases}$$

Thus, the corresponding Dirichlet L-function associated with the quadratic character $\chi_P$ is defined as

$$L(s, \chi_P) = \sum_{f \in \mathbb{A}^*} \frac{\chi_P(f)}{|f|^s}. $$

Let

$$\mathcal{P}_n = \{P \in \mathbb{A}, \text{monic, irreducible, } \deg(P) = n\}. $$

If $P \in \mathcal{P}_{2g+1}$, then by the argument given in [3], $L(s, \chi_P)$ is a polynomial in $q^{-s}$ of degree $2g$ given by

$$L(s, \chi_P) = \sum_{n=0}^{2g} \sum_{f \in \mathbb{A}^*_n} \chi_P(f)q^{-ns}$$

and satisfies the functional equation

$$L(s, \chi_P) = (q^{1-2s})^{g}L(1-s, \chi_P)$$

and the Riemann Hypothesis for curves proved by Weil [19] tells us that all the zeros of $L(s, \chi_P)$ have real part equal to $\frac{1}{2}$. The next results will be fundamental in proving Theorem 1.1.

**Theorem 2.1** (Prime Polynomial Theorem). We have that

$$|\mathcal{P}_n| = \frac{q^n}{n} + O\left(\frac{q^{\frac{2}{3}}}{n}\right).$$

**Proof.** See [13], Theorem 2.2. \qed

**Proposition 2.2.** If $f \in \mathbb{A}$, monic, $\deg(f) > 0$ and $f$ is not a perfect square, then

$$\left|\sum_{P \in \mathcal{P}_n} \chi_P(f)\right| \ll \frac{\deg(f)}{n}q^{\frac{n}{2}}.$$
Proof. See [16], section 2. □

Lemma 2.3 (Approximate Functional Equation). For $P \in \mathcal{P}_{2g+1}$, we have

$$L\left(\frac{1}{2}, \chi_P\right) = \sum_{f_1 \in \mathbb{A}_{2g}} \frac{\chi_P(f_1)}{\sqrt{|f_1|}} + \sum_{f_2 \in \mathbb{A}_{2g-1}} \frac{\chi_P(f_2)}{\sqrt{|f_2|}}. \quad (2.1)$$

Proof. The proof is similar to that given in [2], Lemma 3.3. □

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let $k$ be a given even number and let $x = \frac{2(2g)}{15^k}$. We define

$$A(P) = \sum_{n \in \mathbb{A}_{\frac{2g}{x}}} \frac{\chi_P(n)}{\sqrt{|n|}} \quad (3.1)$$

and let

$$S_1 = \sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right) A(P)^{k-1} \quad (3.2)$$

and

$$S_2 = \sum_{P \in \mathcal{P}_{2g+1}} A(P)^k. \quad (3.3)$$

An application of Triangle Inequality followed by Hölder’s inequality gives us that

$$\left| \sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right) A(P)^{k-1} \right| \leq \sum_{P \in \mathcal{P}_{2g+1}} |L\left(\frac{1}{2}, \chi_P\right)||A(P)|^{k-1} \leq \left( \sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right)^k \right) \left( \sum_{P \in \mathcal{P}_{2g+1}} A(P)^k \right)^{k-1}.$$

Rearranging gives

$$\sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right)^k \geq \frac{\left( \sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right) A(P)^{k-1}\right)^k}{\left( \sum_{P \in \mathcal{P}_{2g+1}} A(P)^k \right)^{k-1}} = \frac{S_1^k}{S_2^{k-1}}.$$

Thus, to prove Theorem 1.1, we only need to give satisfactory estimates for $S_1$ and $S_2$.

3.1 Evaluating $S_2$

We have that

$$A(P)^k = \sum_{n_j \in \mathbb{A}_{\frac{2g}{x}} \atop j=1,\ldots,k} \frac{\chi_P(n_1 \cdots n_k)}{\sqrt{|n_1| \cdots |n_k|}}.$$
Therefore we can conclude that

\[ S_2 = \sum_{\substack{\eta_j \in A_{n_k}^+ \\ j = 1, \ldots, k}} \sum_{\substack{p \in \mathbb{P}_{2g+1} \\ P \in \mathbb{P}_{2g+1}}} \chi_P(n_1 \ldots n_k) \]

Using similar methods to that given in [1], we see that

Using Theorem 2.1 and Proposition 2.2, we have

\[ S_2 = \frac{|P|}{\log |P|} \sum_{\substack{\eta_j \in A_{n_k}^+ \\ j = 1, \ldots, k \ n_1 \ldots n_k = \square}} \frac{1}{\sqrt{|n_1| \ldots |n_k|}} + \sum_{\substack{\eta_j \in A_{n_k}^+ \\ j = 1, \ldots, k \ n_1 \ldots n_k = \square}} \frac{1}{\sqrt{|n_1| \ldots |n_k|}} O\left(\frac{|P|^{\frac{1}{2}}}{\log |P| \deg(n_1 \ldots n_k)}\right). \]

Using the choice of \( x \) given and after some manipulation with the \( O \)-terms, we get that

\[ S_2 = \frac{|P|}{\log |P|} \sum_{\substack{\eta_j \in A_{n_k}^+ \\ j = 1, \ldots, k \ n_1 \ldots n_k = \square}} \frac{1}{\sqrt{|n_1| \ldots |n_k|}} + O(|P|^{\frac{1}{8}}). \]

Writing \( n_1 \ldots n_k = m^2 \) we see that

\[ \sum_{m \in \mathbb{A}_{n_k}^+} \frac{d_k(m^2)}{|m|} \leq \sum_{\eta_j \in \mathbb{A}_{n_k}^+ \ j = 1, \ldots, k} \frac{1}{\sqrt{|n_1| \ldots |n_k|}} \leq \sum_{m \in \mathbb{A}_{n_k}^+} \frac{d_k(m^2)}{|m|}. \]

Using similar methods to that given in [1], we see that

\[ \sum_{m \in \mathbb{A}_{n_k}^+} \frac{d_k(m^2)}{|m|} \sim C(k) z^{\frac{k(k+1)}{2}}. \quad (3.4) \]

Therefore we can conclude that

\[ S_2 \ll |P| (\log |P|)^{\frac{k(k+1)}{2} - 1}. \quad (3.5) \]

### 3.2 Evaluating \( S_1 \)

Using Lemma 2.3, we have that

\[ S_1 = \sum_{\substack{\eta_j \in \mathbb{A}_{n_k}^+ \\ j = 1, \ldots, k - 1}} \frac{1}{\sqrt{|f_1| \ldots |n_1| \ldots |n_k-1|}} \sum_{\substack{p \in \mathbb{P}_{2g+1} \\ P \in \mathbb{P}_{2g+1}}} \chi_P(f_1 n_1 \ldots n_{k-1}) \]

\[ + \sum_{\substack{\eta_j \in \mathbb{A}_{n_k}^+ \\ j = 1, \ldots, k - 1}} \frac{1}{\sqrt{|f_2| \ldots |n_1| \ldots |n_k-1|}} \sum_{\substack{p \in \mathbb{P}_{2g+1} \\ P \in \mathbb{P}_{2g+1}}} \chi_P(f_2 n_1 \ldots n_{k-1}). \]
The two sums for $S_1$ are the same apart from the size of the sums. So we will only estimate the first sum, as the second follows from replacing $g$ with $g - 1$. Thus we have that

$$
\sum_{f \in A_g} \frac{1}{\sqrt{|f||n_1| \ldots |n_{k-1}|}} \sum_{p \in \mathcal{P}_{2g+1}} \chi_P(fn_1 \ldots n_{k-1})
= \sum_{f \in A_g} \frac{1}{\sqrt{|f||n_1| \ldots |n_{k-1}|}} \sum_{p \in \mathcal{P}_{2g+1}} \chi_P(fn_1 \ldots n_{k-1}).
$$

Using Theorem 2.1 and Proposition 2.2, we have that

$$
\sum_{f \in A_g} \frac{1}{\sqrt{|f||n_1| \ldots |n_{k-1}|}} \sum_{p \in \mathcal{P}_{2g+1}} \chi_P(fn_1 \ldots n_{k-1})
= \frac{|P|}{\log_q |P|} \sum_{f \in A_g} \frac{1}{\sqrt{|f||n_1| \ldots |n_{k-1}|}} + O\left(\frac{|P|^{\frac{1}{2}}}{\log_q |P|}\right)
$$

Using the choice of $x$ given and after some manipulation with the O-terms, we get that

$$
\sum_{f \in A_g} \frac{1}{\sqrt{|f||n_1| \ldots |n_{k-1}|}} \sum_{p \in \mathcal{P}_{2g+1}} \chi_P(fn_1 \ldots n_{k-1})
= \frac{|P|}{\log_q |P|} \sum_{f \in A_g} \frac{1}{\sqrt{|f||n_1| \ldots |n_{k-1}|}} + O\left(|P|^{\frac{1}{2}}\right)
$$

(3.6)

For the main term we let $n_1 \ldots n_{k-1} = rh^2$ and $f = r l^2$ and thus we get

$$
\sum_{f \in A_g} \frac{1}{\sqrt{|f||n_1| \ldots |n_{k-1}|}} = \sum_{n \in A_g} \frac{1}{\sqrt{|n||n_1| \ldots |n_{k-1}|}}.
$$

We see that

$$
\sum_{l \in A_g} \frac{1}{|l|^j} \sim C(r, h)(\log_q |P|).
$$
for some positive constant $C(r, h)$. Therefore it follows that the main term in (3.6) is

$$\gg \frac{|P|}{\log_q |P|} \sum_{n_j \in \mathbb{Z}_t} \frac{1}{|r h|} \gg \frac{|P|}{\log_q |P|} \sum_{r, h \text{monic}} \frac{d_{k-1}(r h^2)}{|r h|} \gg |P| \left( \log_q |P| \right)^{\frac{k(k+1)}{2}}.$$ 

where the last bound follows from (3.4) but replacing $k$ with $k - 1$. Therefore we have that

$$S_1 \gg |P| \left( \log_q |P| \right)^{\frac{k(k+1)}{2}}.$$  \hspace{1cm} (3.7)

Combining (3.5) and (3.7) and using Theorem 2.1 proves Theorem 1.1.

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