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Short–time dynamics of the three–dimensional O(4) model

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Abstract. We study the short-time dynamics of the O(4) spin model in three dimensions. Following the heat-bath time evolution algorithm, we obtain the dynamic critical exponents \( y \) and \( z \) related to the second moment of the magnetization and to the time correlation length, respectively. We also compute the ratio between the static critical exponents \( \beta \) and \( \nu \). Our results are compatible with the ones obtained from other approaches.

1. Introduction
After Janssen et al. \cite{1} and Huse \cite{2} more attention has been drawn to the behavior of the critical dynamics of spin model observables, in particular for the Ising model \cite{3,4}, majority vote \cite{5}, damage spreading \cite{6}, Potts Model \cite{7} and percolation theory \cite{8} (see \cite{9,10} for reviews). It is possible to find in the literature numerical measurements of dynamic critical exponents in discrete spin models, but there are no such studies for continuous spin models — except for the three-dimensional Heisenberg model \cite{11} — and for phase transitions in short-time dynamics. In this paper we investigate the short-time phase transition for a continuous spin model. The main idea of this work is to obtain some dynamic critical exponents for the continuous O(4) spin model by using a heat-bath time evolution algorithm. The concept of using the heat-bath method is to update the whole lattice, site by site, changing each spin to a new value without considering the previous spin configuration. The new spin state is selected taking into account its Boltzmann weight. After some iterations all spins in the lattice will obey a Boltzmann distribution and form the heat-bath.

2. Continuous-spin models
O(N) spin models (or nonlinear sigma models) are defined by

\[
\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - \vec{H} \cdot \sum_i \vec{S}_i ,
\]

where \( i \) and \( j \) are the nearest-neighbors sites on a \( d \)-dimensional lattice, \( \vec{S}_k = (S^1_k, S^2_k, ..., S^N_k) \) is an \( N \)-component unit vector at site \( k \) and \( \vec{H} \) is an external magnetic field. For \( N = 1 \) we have the Ising model. The case \( N = 2 \) describes the liquid Helium superfluid transition and \( N = 3 \) is
the classical ferromagnetic Heisenberg model. In this paper we are interested in the $N = 4$ case because it is expected that the phase transition of two-flavor full-QCD would be on the same universality class as the O(4) models [12]. Therefore, the study of short-time phase transition can contribute to the understanding of the full-QCD phase transition.

We consider the short-time dynamics of the three-dimensional O(4) model without external magnetic field ($\vec{H} = 0$). The total magnetization is given by

$$\mathcal{M} = \sqrt{M_1^2 + M_2^2 + M_3^2 + M_4^2}$$

where $M_\alpha = \frac{1}{V} \sum_i S_i^\alpha$ ($\alpha = 1, 2, 3, 4$) (2)

and the order parameter is the fourth coordinate ($M_4$). $V = L^3$ is the number of lattice sites (volume).

The auto-correlation function is written as

$$A(t) = \frac{1}{V} \left\langle \sum_i \vec{S}_i(0) \cdot \vec{S}_i(t) \right\rangle$$

where $\vec{S}_i(0)$ is the initial spin configuration.

3. Short-time scaling

Using renormalization-group theory, it can be shown (see Janssen et al. [1]) that the early time evolution of the order parameter (the magnetization fourth coordinate, $M_4$, from here on represented just by $M$) already displays universal critical behavior, given by the $k-th$ moment the order parameter

$$M^{(k)}(t, \epsilon, L, m_0) = b^{-k\beta/\nu} \mathcal{M}^{(k)}(t b^{-z}, \epsilon b^{1/\nu}, L b^{-1}, m_0 b^{\nu})$$

where $m_0$ is the initial magnetization, $\epsilon \equiv (T - T_c)/T_c$, $\mathcal{M}$ is a universal function and $b$ is a scale factor, which can be taken equal to $t^{1/z}$. We thus expect, for $T = T_c$, $k = 1$ and small $m_0$, a power-law behavior at early times

$$M(t)_{t \to 0} \sim m_0 t^\theta,$$

with $\theta \equiv (x_0 - \beta/\nu)/z$, where $\beta$ and $\nu$ are static critical exponents, $x_0$ and $z$ are dynamic critical exponents.

When $m_0 = 0$ the auto-correlation function [Eq. (3)] obeys the power law $A(t, t' = 0) \sim t^{-\lambda}$ and defines the dynamic critical exponent $\lambda \equiv (d/z) - \theta$, where $d = 3$ is the lattice dimension. The scaling behavior of the second moment of the magnetization can also be described by Eq. (4). Indeed, for $m_0 = 0$, $\epsilon = 0$, and $k = 2$ we have

$$M^{(2)}(t, L) = t^{-2\beta/\nu} \mathcal{M}^{(2)}(t^{-1/z}L)$$

which behaves as $M^{(2)}(t) \sim t^y$,

where the dynamic critical exponent $y$ is defined as $y \equiv (d - 2\beta/\nu)/z$.

The order parameter presents an anomalous time evolution behavior. At the beginning it evolves following an increasing power law [Eq. (5) with $\theta > 0$]. After reaching the maximum value it decreases in two steps: first, following $M \sim t^{-\lambda}$ and then, after a long time, the system reaches the equilibrium value. Then, the spin correlation appears and the magnetization goes to $M \sim \exp(-t/\xi_t)$ for a finite system at $T = T_c$, where $\xi_t$ is the time correlation length, given by $\xi_t \approx L^{z}$.

For a direct measurement of the critical exponent $z$ one can use the method of Ref. [13]

$$F_2(t) = \frac{M^{(2)}(t)_{m_0 = 0}}{[M(t)]^2_{m_0 = 1}} \sim \frac{t^{(d-2\beta/\nu)/z}}{t^{-2\beta/\nu z}} = t^{d/z}.$$

(7)
Figure 1: (a) Short-time behavior of the order parameter $M(t)$ [Eq. (5)] for $L = 60$ with different initial conditions ($m_0 = 0.01$, 0.02, and 0.03). Values were averaged over 5000 samples. The inset plot shows the exponent $\theta$ as a function of the initial magnetization $m_0$. The best linear fit gives $\theta = 0.152(3)$ for $m_0 = 0$. (b) Time correlation function [Eq. (8)] for $L = 30$ and $m_0 = 0$ (not sharp). In this case, we obtained $\theta = 0.153(2)$ with $\chi^2/d.o.f. = 0.16$. The solid line corresponds to the best fit.

In order to avoid the extrapolation and the cumulative error when calculating $\theta$, one can use the time correlation function [14]

$$Q(t) \equiv \frac{1}{V} \left\langle \sum_{i=1}^{V} \sum_{j=1}^{V} \vec{S}_i(t) \cdot \vec{S}_j(0) \right\rangle \sim t^\theta,$$

starting from an initial state with zero correlation length and zero (not-sharp) magnetization. This technique was used to study short-time dynamics in the Glauber model, majority vote, “extreme” model [14] and Ising models [15].

The results of our numerical simulations are presented in the following section. For a consistency check, we computed the ratio $\beta/\nu$ from our dynamic critical exponents and compared with the results obtained through other approaches. Some final discussion and conclusions are presented in Section 5.

4. Results
We have studied the three-dimensional O(4) model, performing Monte Carlo simulations with 5,000 seeds (samples or histories) and 100 time steps for each seed. To measure the dynamic critical exponent $\theta$, the simulations were performed at the critical temperature $T_c = 1.0683$ [16] for different values of the initial magnetization ($m_0 = 0.01$, 0.02, and 0.03). Starting from a configuration with null magnetization, we randomly flipped some spins to obtain the desired $m_0$.

4.1. Dynamic critical exponent $\theta$
The exponent $\theta$ was calculated by extrapolating $m_0$ to zero. For each $m_0$ we perform Monte Carlo simulations for two lattice sizes, $L = 30$ and 60, corresponding respectively to volumes $V = 30^3$ and $60^3$. We have used the heat-bath algorithm [17] with 5,000 seeds and 100 time steps for each seed.

We found a fine power-law behavior of the order parameter in every initial magnetization for both volumes studied, as shown in Fig. 1 (a). The dynamic critical exponent $\theta$, for different values of $m_0 = 0$, is listed in Tab. 1. We have extrapolated $m_0 \to 0$ (last column in the table) for both volumes to obtain $\theta = 0.152(3)$ for the larger size.
Table 1: Dynamic critical exponent $\theta$ of the O(4) model. The value for $m_0 = 0$ was obtained by extrapolation.

| $V$  | $m_0 = 0.03$ | $m_0 = 0.02$ | $m_0 = 0.01$ | $m_0 = 0.00$ |
|------|-------------|-------------|-------------|-------------|
| $30^3$ | 0.147(1)    | 0.149(1)    | 0.150(1)    | 0.151(6)    |
| $60^3$ | 0.148(4)    | 0.149(3)    | 0.151(1)    | 0.152(3)    |

Figure 2: (a) Measurement of dynamic critical exponent $z = 2.176(7)$, $\chi^2/d.o.f. = 0.08$ using the method of [Eq. (7)] with $V = 30^3$ and 5,000 samples. The best fit in the time interval [15 : 100] is represented by the solid line. (b) Behavior of the second moment of magnetization [Eq. (6)] for $V = 30^3$ and 5,000 samples. The best fit in the interval [10 : 90] results in $y = 0.908(3)$, $\chi^2/d.o.f. = 0.01$.

The anomalous behavior of the expected order parameter increases with $t^\theta$ [Eq. (5)] for very short times and then decreases as $t^{-b\nu/z}$ [Eq. (3)]. On the other hand, although we have a history for each simulation it is not convenient to consider the initial time values in the fit. The power-law of the time evolution just appears after the so called microscopic time ($t_{mic}$) [4]. In our simulation $t_{mic}$ is around 3 time steps for both adopted volumes. It is the usual value for heat-bath algorithms. The $\chi^2/d.o.f.$ is smaller than 1 for every $\theta$ fits. Indeed, in our fits we have not used magnetization values with $t > 72$ when $m_0 = 0.01$, with $t > 51$ when $m_0 = 0.02$ and with $t > 42$ to $m_0 = 0.03$ in both volumes $30^3$ and $60^3$.

As an alternative to avoid the extrapolation to $m_0 \to 0$ when measuring $\theta$, one can use the time correlation function [Eq. (8)]. We see in Fig. 1 (b) the behavior of the time correlation function for $V = 30^3$, with 5,000 histories and $m_0 = 0$. The data fit results $\theta = 0.153(2)$ with $\chi^2/d.o.f. = 0.16$.

4.2. Dynamic critical exponents $z$ and $y$

By using Eq. (7) we calculate the dynamic critical exponent $z$ using $V = 30^3$ and 5,000 samples. The fit results in $z = 2.176(7)$, with $\chi^2/d.o.f. = 0.08$. In this fit we use the time interval [25 : 100]. Both $F_2$ and the fit results are showed in Fig. 2 (a).

The dynamic exponent of the second moment of the magnetization ($y$) can be calculated from $M^{(2)}(t)|_{m_0=0}$. Indeed, this quantity follows a power law [Eq. (6)] for $m_0 = 0$ and $\epsilon = 0$. Then, using the numerical simulation for $F_2$, it is possible to measure $y$. Our data shows $y = 0.908(3)$, with $\chi^2/d.o.f. = 0.01$, in the time interval [10 : 90]. The values of $M^{(2)}(t)$ and the best fit can be seen in Fig. 2 (b).
5. Conclusions
We investigated the order parameter short-time behavior of the O(4) continuous spin model. Our results showed a power-law behavior of this observable, with the critical exponent $\theta = 0.152(3)$. For a more precise measurement of the dynamic critical exponent it is advisable to avoid numerical extrapolation. Indeed, it is possible to use the time correlation function. Proceeding this way, we obtained $\theta = 0.153(2)$. Note that, by means of the time correlation function, the final result was improved even using the smaller lattice. Furthermore, the extrapolation (which causes a cumulative error in the final result) was avoided. From the second moment of the magnetization [Eq. (6)] we measured the dynamic critical exponent $y = 0.908(3)$ and, following the method of Ref. [13], we calculated $z = 2.176(7)$.

The short-time power-law behavior of the order parameter, the time correlation function, and the second moment of the magnetization showed evidence of a second order phase transition of the O(4) spin model [9]. The dynamic critical exponents $\theta$, $z$, and $y$ were obtained with $\chi^2$/d.o.f. < 1. As a consistency check, we have evaluated the ratio $\beta/\nu$ by using the exponents $z$ and $y$ (obtained from our simulations) through the equation $\beta/\nu = (d - zy)/2$. Our estimate is $\beta/\nu = 0.512(5)$. This result agrees with the one obtained through perturbation theory approach, $\beta/\nu = 0.52 \pm 0.02$ [18], and also with the result from Monte Carlo simulations with the single-cluster algorithm technique, $\beta/\nu = 0.5129(11)$ [16].

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