Making tight contact manifolds non-fillable

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Abstract

For any \(n \geq 2\), we prove that the \((2n+1)\)-dimensional sphere admits a tight non-fillable contact structure that is homotopically standard. By taking connected sums, we deduce that any \((2n+1)\)-dimensional manifold that admits a strongly fillable contact structure with torsion first Chern class also admits a tight but not strongly fillable contact structure in the same almost contact class. Lastly, we also obtain infinitely many such structures on Weinstein fillable contact manifolds of dimension at least 11.

1 Introduction

In contact topology, there is an important dichotomy, stating that contact structures can either be overtwisted or tight. The first ones appear in abundance in any odd dimension [Eli89, BEM15]. However, tight contact structures are much harder to construct, and let alone classify. Another fundamental question in the field is that of symplectic fillability, which consists in determining which contact manifolds are fillable, i.e. which ones arise as (convex) boundaries of symplectic manifolds.

The first step in the direction of classification is to simplify the standard classical topology as much as possible, i.e. focus on the case of (odd-dimensional) spheres. On these manifolds, there is always a standard contact structure, i.e. the boundary of a standard symplectic ball, and so fillable. Moreover, an early fundamental result in the field, obtained by Eliashberg in [Eli92], is the fact that the standard structure is the unique tight contact structure on \(S^3\). In higher dimensions, as it turns out, the history of the problem of understanding tight contact structures on spheres is much richer:

1. Eliashberg showed in [Eli91] that \(S^{2n+1}\) admits non-standard fillable contact structures for every \(n \geq 2\). This appeals to a previous result of Eliashberg–Floer–Mcduff [McD91], which states that any such filling of the standard tight contact structure on the \((2n+1)\)–sphere is diffeomorphic to the \((2n+2)\)–ball.

2. Ustilovsky [Ust99] proved that, for \(n \geq 2\), the sphere \(S^{4n+1}\) admits infinitely many non-isomorphic contact structures in each formal homotopy class\(^1\). These are all fillable, as they may be viewed as suitable Brieskorn manifolds (see [KvK16] for a nice survey on the role of Brieskorn manifolds in contact topology).

3. In [DG04], Ding–Geiges have shown that every odd-dimensional sphere admits non-standard, but homotopically standard contact structures.

4. Uebele [Ueb16] constructed infinitely many non-isomorphic contact structures in each formal homotopy class on \(S^7, S^{11}\) and \(S^{15}\), all of which are again Brieskorn manifolds, and hence fillable.

\(^1\)The obstruction to the existence of a contact structure is the existence of an almost contact structure, which is purely topological, i.e. the structure group of the tangent bundle should admit a reduction to \(U(n) \times \mathbb{I}\). In fact, almost contact manifolds are indeed contact [BEM15]. In every sphere, by homotopically standard, we mean that the almost contact class is that induced by the standard contact structure.
5. In [Laz20], Lazarev gave infinitely many examples of homotopically standard contact structures on spheres which admit flexible fillings, in dimension at least 5.

6. Recently, Alves–Meiwes [AM19] also gave examples of contact structures on spheres of dimension at least 7 with non-standard dynamical properties, i.e. for which all Reeb flows have positive topological entropy; these examples are also all fillable.

In this paper, we contribute new examples to the cabinet of oddities. Namely, based on the study of Bourgeois manifolds carried out in [BGM22] as well as symplectic cohomology computations as in [Zho21, Zho22a], we prove the following:

**Theorem A.** For every \( n \geq 2 \), the sphere \( S^{2n+1} \) admits a tight, non-fillable and homotopically standard contact structure.

**Remark 1.** As weak fillability is equivalent to strong fillability for the case of a sphere according to [MNW13, Proposition 6], these contact structures are in fact not weakly fillable.

**Geometric construction.** In order to obtain one example \((S^{2n+1}, \xi_{ex})\) as in Theorem A in the case \( n \geq 3 \), one applies subcritical surgery to Bourgeois contact structures on \( S^{2n-1} \times T^2 \) that arise from Milnor open books on \( S^{2n-1} \) which are dynamically convex (cf. Section 5). The resulting contact structure on \( S^{2n-1} \times T^2 \) then inherit a weaker form of convexity, namely they are asymptotically dynamically convex (ADC) as explained in Section 2.2. The desired obstruction to fillability can then be obtained either from an adaptation of [BGM22, Theorem B] (see Theorem 32) or from symplectic cohomology (see Theorems 34 and 36). To ensure tightness, we use a stronger form of the ADC-condition to be able to appeal to algebraic tightness, which implies tightness, and is well-behaved under cobordisms (see Proposition 25 and Example 26). The 5-dimensional case is slightly different as here one also needs to consider critical surgeries and the obstruction is then given by Theorem 37.

**Remark 2.** The examples described above arise via contact surgery on a contact structure which is in fact not weakly fillable. In particular, this shows that weak fillability is not preserved under contact surgery.

By taking a contact connected sum with the geometrically constructed non-standard sphere \((S^{2n+1}, \xi_{ex})\) of Theorem A, and using the obstructions to fillability based on symplectic cohomology, we obtain the following result:

**Theorem B.** If \((M^{2n+1}, \xi)\) with \( n \geq 2 \) is strongly fillable with torsion first Chern class \( c_1(\xi) \), then \( M \) admits a tight but not strongly fillable contact structure in the same almost contact class.

We now specialize to the case of almost contact manifolds that occur as the boundary of almost Weinstein domains. More precisely we consider an almost Weinstein filling of an almost contact manifold \((M, \zeta)\), which is given by the data \((W, J, \varphi)\) of an even-dimensional manifold \( W \) with boundary \( M = \partial W \), an almost complex structure \( J \) so that \( TM \cap JT M = \zeta \), and a Morse function \( \varphi : W \to \mathbb{R} \) with no critical points of index greater than \( \dim(W)/2 \). By taking further contact connected sums with judiciously chosen Weinstein fillable contact structures on the sphere, we further obtain a version of Theorem A for general almost symplectic manifolds of dimension at least 11, which yields an infinitude of non-standard contact structures:

**Theorem C.** Let \( n \geq 5 \), and \((M^{2n+1}, \zeta)\) be an almost contact manifold admitting an almost Weinstein filling and with \( c_1(\zeta) \) torsion. Then, \( M \) admits infinitely many non-isomorphic, tight and not strongly fillable contact structures in the same almost contact class.

In fact, we conjecture the following:
**Conjecture 1.** A connected sum of an arbitrary tight contact manifold with the non-standard sphere \((S^{2n+1}, \xi_{ex})\) of Theorem A is tight and admits no strong filling.

This would yield higher-dimensional fillability obstructions coming from non-standard contact spheres, which one could think of as analogues of Giroux torsion but on a manifold with trivial topology. In particular, these could be implemented on any contact manifold.

**Outline of Paper.** Section 2 contains important preliminaries and background, including a description of the Bourgeois construction together with a result concerning dynamical convexity of its Reeb vector field, as well as statements about the existence of strong cobordisms from contact manifolds, obtained from subcritical (resp. flexible) surgeries on \(S^1\)-equivariant ones, to subcritically (resp. flexibly) fillable ones.

In Section 3, we describe the notion of algebraic tightness, together with relevant properties, and prove that the Bourgeois construction is a rich source of contact manifolds satisfying this property.

Section 4 contains results on obstructions to strong fillability. There are two different techniques used to give such obstructions, one topological in nature, and the other involving symplectic cohomology. The second route gives stronger obstructions, with the most general statement concerning contact manifolds which are connected sums of a strongly fillable one with a flexibly fillable one.

Finally, the remaining two sections are devoted to the proofs of the results stated above. Namely, Section 5 contains the proof of Theorem A, and Section 6 those of Theorems B and C. The last section further contains a discussion of a speculative strategy for extending the last theorem to dimensions 7 and 9, assuming that the set consisting of all linearized non-equivariant contact homologies/positive symplectic cohomologies for all possible augmentations is a well-defined invariant of contact structures.

**Outlook: classification of symplectic fillings in higher dimensions.** Theorems 34, 36 and 37, which in this paper we will use as some of the obstructions to fillability, should be understood within the lens of the classification of fillings of special families of contact manifolds in higher dimensions. Taking the celebrated Eliashberg–Floer–McDuff theorem [McD91] as a starting point (i.e. the case of spheres), many contact manifolds are expected to have unique exact/symplectically aspherical fillings, at least up to diffeomorphism.

Up until now, results in this direction mainly apply to the contact boundary of a split manifold of the form \(V \times \mathbb{D}^2\), up to subcritical/flexible surgeries; we refer to [OV12, BGZ19, GKZ0, Zho21, Zho22a, BGM22].

If one considers the more general class of strong fillings, these are no longer unique, as one can take a blow-up of any given filling without affecting the boundary. However, it is plausible that all the ambiguity comes from such birational surgeries, which will only increase the complexity of the topology of the filling; see [Zho21, §8] for supporting evidences. Incarnations of such expectations at the level of homology lead to our obstructions to strong fillings.

On the other hand, Eliashberg [Eli90] showed in dimension 3, that if \(W\) is a symplectic filling of \(Y \# Y'\), then \(W\) is obtained from attaching a 1-handle to a symplectic filling \(W'\) of \(Y_1 \cup Y_2\) (which might be connected, cf. [Bow12]). Although the higher dimensional analogue has not been established, partial generalizations were obtained by Ghiggini–Niederkrüger–Wendl [GNW16]. In view of this, the obstructions to strong fillings described above should be inherited by connected sums. In this direction, Theorems 36 and 37 constitute further “homological” evidence of these two broader expectations.

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2 Bourgeois contact manifolds

In this section, we recall the Bourgeois construction [Bou02b], following the presentation in [BGM22].

Consider a closed, oriented, connected smooth manifold $M^{2n-1}$ and an open book decomposition that we denote $(B, \theta)$, together with a defining map $\Phi: V \to \mathbb{R}^2$ so that each $z \in \text{int}(\mathbb{D}^2)$ is a regular value. Here, $B \subset V$ is a closed codimension-2 submanifold, $\theta: M \setminus B \to S^1$ is a fiber bundle, and $\Phi$ is such that $\Phi^{-1}(0) = B$ and $\theta = \Phi/|\Phi|$. Let us also denote by $\rho$ the norm $|\Phi|$.

Recall that a 1-form $\alpha$ on $M$ is said to be adapted to $\Phi$ if it induces a contact structure on the regular fibers of $\Phi$ and if $d\alpha$ is symplectic on the fibers of $\theta = \Phi/|\Phi|$. In particular, if $\xi$ is a contact structure on $M$ supported by $(B, \theta)$, in the sense of [Gir02], then (by definition) there is such a pair $(\alpha, \Phi)$ with $\alpha$ defining $\xi$.

**Theorem 3** (Bourgeois [Bou02b]). Let $(B, \theta)$ denote an open book decomposition of $M^{2n-1}$, represented by a map $\Phi = (\Phi_1, \Phi_2): V \to \mathbb{R}^2$ as above, and let $\alpha$ be a 1-form adapted to $\Phi$. Then, $\alpha_{BO} := \alpha + \Phi_1 dq_1 - \Phi_2 dq_2$ is a contact form on $M \times \mathbb{T}^2$, where $(q_1, q_2)$ are coordinates on $\mathbb{T}^2$.

The contact form $\alpha_{BO}$ on $M \times \mathbb{T}^2$ will be called the Bourgeois form associated to $(\alpha, \Phi)$ in what follows. It will be useful to also think in terms of abstract open books, i.e. in terms of the Liouville page $\Sigma = (\Sigma, d\lambda)$ and the compactly supported symplectic monodromy $\psi$. In this case we write $OB(\Sigma, \psi)$ for the contact manifold obtained via the Thurston-Winkelnkemper construction, and we denote $BO(\Sigma, \psi) = (M \times \mathbb{T}^2, \ker \alpha_{BO})$. We invite the reader to consult [LMN19] for details on the correspondence between Bourgeois contact structures for geometric and abstract open books.

**Remark 4.** The almost contact structure underlying the Bourgeois contact manifold is (up to homotopy) just the sum

$$(\xi, d\alpha) \oplus (TT^2, \Omega),$$

where $\Omega$ is an area form on $\mathbb{T}^2$. This can be seen via an explicit homotopy; see for instance [Gir19, Lemma 4.1.(1)]. In particular if $c_1(\xi)$ is torsion, then the same is true for the Bourgeois contact structure, independently of the auxiliary choice of open book.

2.1 Reeb dynamics of Bourgeois contact structures

In this somewhat technical section, we consider the Reeb dynamics of the Bourgeois contact forms more closely. First of all applying a lemma of Giroux (see e.g. [DGZ14, Section 3]), on a neighborhood of the form $\mathcal{N} = B \times \mathbb{D}^2 \times \mathbb{T}^2$ of $B$ where $\rho = |\Phi|$ coincides with the radial coordinate of $\mathbb{D}^2$, the Bourgeois contact form looks like

$$\alpha_{BO} = \alpha + \rho(r)(\cos(\theta)dq_1 - \sin(\theta)dq_2)$$

$$= h_1(r)\alpha_B + h_2(r)d\theta + r(\cos(\theta)dq_1 - \sin(\theta)dq_2).$$

Here, $\alpha_B = \alpha|_B$ is the contact form on the binding, $(r, \theta)$ are polar coordinates on $\mathbb{D}^2$, and the functions $h_1, h_2$ satisfy the following conditions:

i. $h_1(0) > 0$ and $h_1(r) = h_1(0) + O(r^2)$ for $r \to 0$;

ii. $h_2(r) \sim r^2$ for $r \to 0$;

iii. $\frac{h_1^{n-1}}{r} (h_1 h_2' - h_2 h_1') > 0$ for $r \geq 0$ (contact condition);

iv. $h_1'(r) < 0$ for $r > 0$ (symplectic condition on the pages).
We also point out that there is a natural (orientation preserving) diffeomorphism
\[ \mathcal{N} = B \times \mathbb{D}^2 \times \mathbb{T}^2 \to B \times D^*\mathbb{T}^2 \]
\[ (b, p_1, p_2, q_1, q_2) \mapsto (b, p_1, q_1, -p_2, q_2). \]  

We then have the following global description of the Reeb vector field (c.f. [Bou02b, Gir20a]):

**Lemma 5.** The Reeb vector field of the contact form \( \alpha_{BO} \) is given by
\[ R_{BO} = \mu(r)R_B + \nu(r)[\sin(\theta)\partial_{q_1} - \cos(\theta)\partial_{q_2}], \]  
where \( R_B \) is the Reeb vector field of the restriction \( \alpha_B = \alpha|_B \), and the coefficients are
\[ \mu = \begin{cases} \frac{\nu}{\sqrt{p_1^2 - \rho_1^2}} & \text{in } \mathcal{N}, \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad \nu = \begin{cases} -\frac{h_1'}{\rho_1} & \text{in } \mathcal{N}, \\ 1 & \text{elsewhere} \end{cases}. \]

The computation in [Bou02b] shows that, in order for \( \alpha_{BO} \) to be a contact form in the neighborhood \( \mathcal{N} \cong B \times D^*\mathbb{T}^2 \) of \( B \times \mathbb{T}^2 \) where \( \alpha = h_1\alpha_B + h_2d\theta \), it is in fact enough that \( h_1(h_1 - h_1') > 0 \), a condition which only depends on \( h_1 \). In particular, one can homotope the pair \((h_1, h_2)\) in a compactly supported way among pairs of functions satisfying this condition, and this will result in a homotopy of contact forms (hence isotopy by Gray’s stability) on \( M \times \mathbb{T}^2 \), independently of the fact that the resulting \( \alpha \) on \( M \) might not be adapted to the open book or even a contact form. Moreover, the explicit formula in Lemma 5 still holds for the homotoped contact form, as the explicit computation does not use any specific property of the pair of functions \((h_1, h_2)\) listed above. In particular, up to homotopy one can achieve the following form, which will be useful below:

**Lemma 6.** For \( \delta > 0 \) sufficiently small, up to a deformation among contact structures on \( M \times \mathbb{T}^2 \) supported in the neighbourhood of radius \( 2\delta \) of \( B \times \mathbb{T}^2 \), one can assume that
\begin{itemize}
  \item \( h_1(r) = 1 \) for \( r \leq \delta \),
  \item \( h_1(h_1 - h_1') > 0 \) everywhere,
  \item \( h_2(r) = 0 \) for \( r \leq 3\delta/2 \).
\end{itemize}
In particular, under the diffeomorphism in Equation (1), \( \alpha_{BO} \) coincides with \( \alpha_B + \lambda_{std} \) for \( r = \sqrt{p_1^2 + p_2^2} \leq \delta \), where \( \lambda_{std} = p_1dq_1 + p_2dq_2 \) is the standard Liouville form on \( D^*\mathbb{T}^2 \). Moreover, Equation (2) also holds for the Reeb vector field of the deformed contact form; in particular, it coincides with \( R_B \) for \( r \leq \delta \).

### 2.2 ADC Bourgeois contact manifolds

In this section, we will assume that the first Chern class \( c_1(\xi) \) of all contact structures are torsion. In this case there is a well-defined Conley-Zehnder index \( \mu_{CZ}(\gamma) \) for any non-degenerate contractible periodic Reeb orbit \( \gamma \). If the contact manifold is of dimension \( 2n - 1 \), then the SFT degree of \( \gamma \) is defined as \( |\gamma| := n - 3 + \mu_{CZ}(\gamma) \). Notice that this does not correspond to the degree in symplectic cohomology of \( \gamma \) when viewed as a periodic Hamiltonian orbit, which is just given by \( n \) minus the Conley-Zehnder index.

We now recall the following definition from [Zho21], that generalizes the one from [Laz20]:

**Definition 7.** A contact structure \((M, \xi)\) is called \( k \) asymptotically dynamically convex (\( k \)-ADC) if there is a sequence of contact forms \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \cdots \) so that all contractible periodic orbits of \( \alpha_i \)-action \( \leq D_i \) are non-degenerate and have degree \( > k \), with \( D_i \to \infty \).
In particular, 0-ADC as in [Zho21] is just ADC as in [Laz20]. An important special case is when there is a contact form $\alpha$ that is \textit{index-positive}, meaning, following [CO18, Section 9.5], that all periodic orbits are non-degenerate and have positive degree for $\alpha \equiv \alpha$ the given contact form. A very useful aspect of the ADC condition is that it is preserved under flexible surgery [Laz20], whereas index-positivity is not.

**Example 8** (Brieskorn Manifolds). Consider the $2n-1$-dimensional Brieskorn spheres $\Sigma_n(k,2,\cdots,2)$ given as the link of the $A_{k-1}$-singularity, i.e.

$$\Sigma_n(k,2,\cdots,2) = \{z_0^k + z_1^2 + \cdots + z_n^2 = 0\} \cap \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}.$$  

This is a contact manifold which is index-positive. The degrees of the generators are at least $2n-4$, provided that $n \geq 3$. This follows from [Ust99] in case $n = 2m + 1$ is odd, and from [vK08, Section 3] in general.

More generally, one can consider Brieskorn spheres $\Sigma_n(p_1,p_2,\cdots,p_{n-2},2,2,2)$ for odd primes $p_i$, which according to [vK08, Section 4.1], satisfy the same properties in terms of their Conley-Zehnder indices. This also follows from a deep theorem of McLean [McL16, Theorem 1.1], which implies that the Brieskorn manifold $\Sigma_n(p_1,p_2,\cdots,p_{n+1})$ is ADC if the singularity is terminal.

**Example 9** (3-dimensional case). For the case of 3-dimensional Brieskorn manifolds, the computations are more involved as the manifolds are not simply connected, but one can still prove that links of $A_{k-1}$-singularities admit contact forms whose contractible periodic orbits all have Conley-Zehnder indices at least 3; see [AHKNS17].

From another perspective, one can view links of $A_{k-1}$-singularities as quotients of the standard contact 3-sphere so that contractible periodic orbits on the links can be lifted to the 3-sphere, which then have Conley-Zehnder indices at least 3. However, when we construct a 7-dimensional Bourgeois contact structure $\mathbb{S}^5 \times \mathbb{T}^2$ with the 3-dimensional $A_{k-1}$ singularity link as the binding, these non-contractible orbits in the binding become contractible in the Bourgeois contact manifold. Since $H_1(\Sigma(k,2,2); \mathbb{Z})$ is torsion, all Reeb orbits, including those non-contractible orbits, can be assigned with well-defined rational Conley-Zehnder indices [McL16, §4], which are at least 1 by the computation in [MR18, §2.1].

We next observe that the $k$-ADC condition is stable under the Bourgeois construction:

**Lemma 10.** If the binding of an open book $OB(\Sigma, \psi)$ is simply connected and $k$-ADC, then the Bourgeois contact manifold $BO(\Sigma, \psi)$ is $(k+2)$-ADC.

**Proof.** In view of Lemma 6, we can assume that, for some fixed $\delta > 0$ sufficiently small, on a neighborhood $B \times \mathbb{D}^2_{\delta} \times \mathbb{T}^2$ of the (stabilized) binding of the open book, the Bourgeois contact form is

$$\alpha_{BO} = h_1(r)\alpha_B + h_2(r)d\theta + \lambda_{std}$$

where $\lambda_{std}$ is the standard Liouville form on the portion $\mathbb{D}^2_{\delta} \times \mathbb{T}^2 \simeq D^*_{\delta} \mathbb{T}^2$ of unit cotangent bundle made of vectors of norm less than $2\delta$, and where the functions $h_1, h_2$ are such that the following conditions hold:

- $h_1(r) \equiv 1$ for $r \leq \delta$ and $h_1'(r) < 0$ otherwise;
- $h_2(r) \equiv 0$ for $r \leq 3\delta/2$ and $h_2'(r) > 0$ otherwise.

Consider then the Hamiltonian $H(q_1,p_1,q_2,p_2) = \eta(r)(f(q_1,q_2) + g(r))$, where $(q_i, p_i)$ are coordinates on $D^*_{\delta} \mathbb{T}^2$, $r = \sqrt{p_1^2 + p_2^2}$, $\eta$ is a bump cut-off function, $f: \mathbb{T}^2 \rightarrow \mathbb{R}$ is a Morse-function on $\mathbb{T}^2$, and $g: [0,1] \rightarrow \mathbb{R}$ is Morse with a unique maximum at $r = 0$. Following Bourgeois [Bou02a] we consider

$$\alpha_\epsilon = (h_1 + \epsilon H)\alpha_B + \lambda_{std}$$
on the 2δ-neighbourhood $B \times \mathbb{D}_{2\delta}^2 \times \mathbb{T}^2$. Then one computes that the Reeb vector field is of the form

$$\frac{1}{h_1 + \varepsilon H - \lambda_{std}(X_{h_1} + \varepsilon X_H)} [R_B - X_{h_1} - \varepsilon X_H],$$

where we use the notation $X_h$ for the Hamiltonian vector field of a function $h : D^*\mathbb{T}^2 \to \mathbb{R}$ with respect to $\omega_{std} = d\lambda_{std}$. Note that the flow generated by the vector field $X_{h_1}$ is just a rescaling of the geodesic flow on $D^*\mathbb{T}^2$ and is stationary precisely where $h_1 = 0$.

Now for a fixed action threshold $D$, taking $\varepsilon$ small, we see that the only contractible periodic orbits up to action $D$ must lie in the region where $h_1$ is constant, i.e. $B \times \mathbb{D}_\delta^2 \times \mathbb{T}^2$. These orbits are of the form $\gamma_q = \gamma_B \times \{(q, r = 0)\}$ with $q$ a critical point of $f$ and $\gamma_B$ a contractible Reeb orbit in the binding. Moreover, one can compute, using the properties of the Conley-Zehnder index (c.f. [Sal99, Section 2.4]) that

$$\mu_{CZ}(\gamma_q) = \mu_{CZ}(\gamma_B) - 2 + \text{ind}_q(H) \geq \mu_{CZ}(\gamma_B),$$

as $\text{ind}_q(H) = \text{ind}_q(f) + 2 \geq 2$. Thus the degree on $M^{2n-1} \times \mathbb{T}^2$ is

$$|\gamma_q| = (n + 1) - 3 + \mu_{CZ}(\gamma_B) \geq (n - 1) - 3 + \mu_{CZ}(\gamma_B) + 2 = |\gamma_B| + 2,$$

where $|\gamma_B|_B$ denotes the degree of $\gamma_B$ as an orbit in $B$. Consider a decreasing sequence $\alpha_{B,i}$ on the binding whose contractible orbits have degrees at least $k$ as in the definition of $k$-ADC, i.e. up to some action threshold $D_i \to \infty$. By multiplying by appropriate constants we can assume that this sequence is strictly decreasing. We then multiply the adapted contact form on $OB(\Sigma, \psi)$ by an appropriate function $f_i < 1$ so that the form on the binding $\alpha_{B,i}$ is replaced by $\alpha_{B,i}'$ to obtain a decreasing sequence of forms $\alpha_{B,i}'$ for the Bourgeois contact structure whose contractible orbits also have degrees at least $k + 2$ up to the action threshold $D_i$. Then perturbing as above to obtain non-degeneracy near the binding, we obtain contact forms $\alpha_{B,i}'$ so that the action of the above contractible orbit $\gamma_q$ is $(1 + \varepsilon_i H(q, 0))$ times its $\alpha_{B,i}'$-action and $\varepsilon_i \ll 1$ small.

Strictly speaking these contact forms determine distinct contact structures, as their kernels may differ, but applying Gray stability this can be remedied by an isotopy, which can be made arbitrarily close to the identity by shrinking $\varepsilon_i$ as needed. We deduce that the Bourgeois contact structure is $(k + 2)$-ADC if the contact structure on $B$ is $k$-ADC.

**Remark 11.** When $B$ is not simply connected, there might be Reeb orbits of $B$ which are non-contractible in $B$ but contractible in the ambient open book $OB(\Sigma, \psi)$. These orbits should then be included in the verification of the ADC condition for the open book, even though they are not relevant for that of $B$. In our applications, this will only happen in settings where $H_1(B; \mathbb{Z})$ is torsion. In this case, if the contact structure on $B$ has torsion first Chern class, every Reeb orbit can be assigned with a well-defined rational Conley-Zehnder index [McL16, §4]. One can then consider the notion of $k$-ADC manifolds but for all Reeb orbits. The computation in Lemma 10 is still valid for the rational Conley-Zehnder index.

We have the following examples that will be used in the proof of Theorem A.

**Example 12.** Consider the Bourgeois contact manifold associated to the Milnor open book on $\mathbb{S}^{2n-1}$ coming from the $A_k$-singularity, whose binding is a Brieskorn sphere $B = \Sigma_{n-1}(k, 2, \cdots, 2)$ and whose page is the Milnor fiber $V_{n-1}(k, 2, \cdots, 2) = \{z_0^k + z_1^2 + \cdots + z_{n-1}^2 = \epsilon\} \cap \mathbb{D}^n$, for small $\epsilon > 0$. When $n \geq 4$, $B$ is a 1-ADC contact manifold according to Example 8 and simply connected. We thus obtain from Lemma 10 that the corresponding Bourgeois contact manifolds on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ are 1-ADC. When $n = 3$, the corresponding Bourgeois contact manifolds on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ are also 1-ADC by Example 9 and Remark 11.
2.3 $S^1$–invariant contact structures and convex decompositions

Bourgeois contact structures are $T^2$-equivariant, and therefore in particular $S^1$-equivariant. As discussed in [DG12, Section 6] any $S^1$–invariant contact structure on a manifold $V^{2n} \times S^1$ induces a decomposition of the first factor into so called ideal Liouville domains, as defined in [Gir20b]. In particular, we have a topological decomposition

$$V \times S^1 = V_+ \times S^1 \cup \overline{V}_- \times S^1.$$  

Any (constant) section $V \times \{pt\}$ is a convex hypersurface and the decomposition $V = V_+ \cup \overline{V}_-$ is a convex decomposition. In the case of the Bourgeois contact manifolds the ideal Liouville pieces are given as products of the page of the initial open book with an annulus, i.e. $V_\pm = \Sigma \times D^* S^1$. For details related to the notion of ideal Liouville domains and to the Bourgeois case we refer to [DG12, MNW13, Gir20b]. The pieces in this decomposition are (round) contactizations of ideal Liouville domains and are referred to as Giroux domains following [MNW13].

2.3.1 Capping Giroux domains

In [BGM22, Theorem B] certain homological obstructions to the symplectic fillability Bourgeois contact manifolds were obtained (see also Section 4.1 below). The argument of the proof uses a capping construction of Massot-Niederkrüger-Wendl [MNW13]. More precisely, one observes that there is a natural blow down construction given an embedded Giroux domain $G = V \times S^1$. Namely one removes the interior $B$ and collapses all circle fibers at the remaining boundary to points. By [MNW13, Section 5.1] the resulting smooth manifold $M_{bd}$ (where $bd$ stands for blown down) carries a contact structure that is well-defined (up to isotopy). The following is a special case of [MNW13, Theorem 6.1].

**Theorem 13.** Let $M$ be a contact manifold containing a Giroux domain $G = V \times S^1$. Then there is a strong symplectic cobordism from $M$ to the contact manifold $M_{bd}$ obtained by blow down of $G$.

Topologically, this cobordism is obtained by attaching $V \times D^2$ on top of $V \times S^1$ (and smoothing corners), where the latter is seen as the positive boundary of $[0,1] \times V \times S^1$. We emphasize that this cobordism is not Liouville, as there are only local Liouville vector fields near the boundary in general.

**Example 14.** It is perhaps instructive to describe the above cobordism explicitly in the case where $M$ itself is given as the contact boundary of a product of Liouville domains $Y \times D^* S^1$. In this case, the cobordism is given by embedding the annulus factor into the 2-disc and the local Liouville vector fields given by the sum of the Liouville vector field on the factors $Y$ together with, respectively, that of $D^* S^1$ at the negative end and the radial expanding one on $D^2$ at the positive end.

In the case where we begin with an $S^1$-invariant contact structure and apply this blow down cobordism to Giroux domain $V_- \times S^1$, we end up with a convex boundary $M_{bd}$ which is just the contact open book $OB(V_+, \text{Id})$ with page $V_+$ and trivial monodromy. Equivalently this is then the contact boundary of $V_+ \times D^2$. In other words, we have the following:

**Corollary 15.** Let $M = V \times S^1$ be a contact manifold be a contact manifold with an $S^1$-invariant contact structure and convex decomposition

$$V \times S^1 = V_+ \times S^1 \cup \overline{V}_- \times S^1.$$  

Then there is a strong symplectic cobordism from $M$ to the contact boundary of $V_+ \times D^2$. Furthermore, if $c_1(\xi) = 0$ on $M$ then the same is true for the cobordism.
In particular, for surgeries of index 1

Remark 20 (Low index surgeries). In many situations the condition of being binding compatible is vacuous. In particular, for surgeries of index 1, 2 this is the case since framings are in 1-1 correspondence with elements...
of the homotopy groups $\pi_i(U(n-i))$ for $U(n-i)$ the unitary group. For $i = 2$ these groups are trivial and for $i = 1$ we have $\pi_1(U(n-i)) \cong \pi_1(U(1)) = \mathbb{Z}$ so that all framings are homotopic to ones that are compatible with the splitting at the binding, since they can be obtained from a fixed trivialisation by twisting in the $D^2$ factor.

Now, since $V_+$ is Weinstein, it has the homotopy type of its middle-dimensional skeleton. In particular, when performing the subcritical contact surgeries of interest on $V_+ \times S^1$, seen as sitting inside $OB(V_+ , \text{Id})$, one can smoothly push all the surgery spheres into the binding of the open book. What’s more, we claim that one can in fact push them via a contact isotopy, using the $h$-principles for isotropic (non Legendrian) submanifolds and for immersed Legendrians [EM02]. This can be done as follows.

Denote the smooth isotopy of submanifolds with $(S_t)_{t \in [0,1]}$, so that $S_0$ is the attaching sphere of the contact surgery and $S_1$ is contained in the binding. Using the $h$-principle for immersed Legendrians, $S_1$ can be homotoped (among immersions) to a Legendrian $L$ in the binding. Moreover, this homotopy of immersions can be perturbed (relative endpoints) to a homotopy of embeddings in a neighborhood of the binding inside the ambient manifold. In other words, we have a homotopy of smooth embeddings from the isotropic $S_0$ to the isotropic (and Legendrian in the binding) $L$. This can be perturbed, relative endpoints, to the desired isotopy of isotropic submanifolds from $S_0$ to $L$ using the $h$-principle for isotropic submanifolds, thus proving the claim.

The hypothesis on the framing now comes into play as well. As the attaching spheres now sit in the binding, which is just the boundary of a page of the open book, thanks to the compatibility assumption on the framing one can then consider the corresponding handle attachments done on a single page $V_+$; denote by $V'_+$ the resulting Weinstein domain.

With these preliminaries, we can now state the promised more precise version of Proposition 18 in the subcritical case:

**Proposition 21 (Subcritical Case).** Consider an $S^1$-equivariant contact structure on $M = V \times S^1$, with induced convex splitting $V = V_+ \cup V_-$ and assume that $V_+$ is Weinstein. Let $(M',\xi')$ be obtained from the former via a sequence of subcritical surgeries on isotropic spheres contained in the complement of $V_- \times S^1$, with contact framings which are compatible with the splitting at the binding (if the surgeries are seen as done on $V_+ \times S^1 \subset OB(V_+ , \text{Id})$).
Then there is a strong symplectic cobordism from \((M', \xi')\) to the contact manifold \(M'_+=OB(V'_+, \text{Id})\) obtained from \(OB(V_+, \text{Id})\) via the corresponding sequence of contact surgeries (pushed to the binding via a contact isotopy).

**Proof.** The proof is analogous to the above one, using the fact that, after pushing the attaching spheres to the binding, thanks to the hypothesis on the framings, one can identify \(M'_+\) with \(OB(V'_+, \text{Id})\) via the corresponding sequence of contact surgeries (pushed to the binding via a contact isotopy).

**Remark 22** (Choice of Framings). The condition on the framings in Proposition 21 is *a priori* restrictive. However, in most applications the precise (contact) framings will not play an important role so that one can just choose them so that the proposition applies.

### 3 Tightness and fillability from an algebraic perspective

Among all versions of SFT invariants of a contact manifold \(Y\), the simplest one is the contact homology algebra \(\text{CHA}(Y)\), which at the level of homology is defined rigorously by Pardon [Par19]. By [BvK10], the contact homology algebra vanishes for overtwisted contact manifolds. Motivated from this, Bourgeois and Niederkrüger introduced the notion of algebraically overtwisted manifolds [BN10]. In the following, we introduce the analogous notions for tight and fillable manifolds. In this section, we do not assume \(c_1\) of the contact manifold is torsion.

**Definition 23.** Let \(Y\) be a contact manifold and \(\text{CHA}(Y; \Lambda)\) the contact homology algebra of \(Y\) over the Novikov field \(\Lambda\). We say

1. \(Y\) is algebraically overtwisted if \(\text{CHA}(Y; \Lambda) = 0\) [BN10];
2. \(Y\) is algebraically tight if \(\text{CHA}(Y; \Lambda) \neq 0\);
3. \(Y\) is algebraically non-fillable if there is no differential graded algebra (DGA) augmentation at the chain level of \(\text{CHA}(Y; \Lambda)\);
4. \(Y\) is algebraically fillable if there is a DGA augmentation at the chain level of \(\text{CHA}(Y; \Lambda)\).

**Remark 24.** Although the contact homology algebra is established as a contact invariant, the homotopy type of the underlying chain level DGA is not. This is the main reason why the computationally simpler linearized contact homology is not well-defined yet. However, even using the chain level structure, the notion of algebraic (non-)fillability above is well-defined. To see this, given any pairs of contact forms \(\alpha_1, \alpha_2\) (together with the needed auxiliary choices), consider the trivial exact cobordism from \((Y, \alpha_1)\) to \((Y, k\alpha_2)\) for \(k > 0\) (and vice versa). The functoriality in [Par19] then implies that there is a DGA augmentation at the chain level of \(\text{CHA}(Y)\) using \(\alpha_1\) (and associated auxiliary choices) if and only if there is a DGA augmentation at the chain level using \(\alpha_2\) (and associated auxiliary choices).

**Proposition 25.** We have the following properties.

1. If \(Y\) is overtwisted, then \(Y\) is algebraically overtwisted. Hence algebraically tight contact manifolds are tight.
2. \(Y\) is algebraically overtwisted if and only if one connected component of \(Y\) is algebraically overtwisted.
3. Let \(W\) be an exact cobordism with convex boundary \(\partial_+ W\) and concave boundary \(\partial_- W\). If \(\partial_+ W\) is algebraically overtwisted then so is \(\partial_- W\).
4. Algebraically fillable contact manifolds are algebraically tight.

5. The following contact manifolds are algebraically tight.
   
   (a) Hypertight contact manifolds;
   (b) Contact manifolds with vanishing rational first Chern class and a contact form $\alpha$, such that there is no contractible Reeb orbit of SFT degree 1;
   (c) 1-ADC contact manifolds.

6. If $Y$ is algebraically non-fillable, then $Y$ has no strong filling.

Proof. The first claim follows from [BvK10] and the main result of [CMP19]. Although [BvK10] uses $\mathbb{Q}$-coefficients, the equivalence with using $\Lambda$-coefficients can be found in [MZ20, Proposition 3.12]. The second claim follows from the fact that $\text{CHA}(Y_1 \cup Y_2) = \text{CHA}(Y_1) \otimes \text{CHA}(Y_2)$. The third claim follows from the functoriality of contact homology algebra, i.e. we have an unital algebra map $\text{CHA}(\partial_+ W) \to \text{CHA}(\partial_- W)$. The fourth claim follows from the fact that $\text{CHA}(Y; \Lambda) = 0$ is an obstruction to the existence of augmentations, see e.g. [MZ20, Proposition 2.11]. Conditions in (a) and (b) of the fifth claim mean that $1 \neq 0 \in \text{CHA}(Y)$, which implies algebraic tightness by homotopy class or degree reasons. Roughly speaking, (c) is a special case of (b). (Recall that $k$–ADC contact structures have torsion $c_1$ by definition.) However, to cope with the asymptotic property in the definition of 1-ADC, we can argue as follows. If $\text{CHA}(Y) = 0$, then for any fixed contact form $\alpha$, there exists a positive number $A$, such that $1 = 0 \in \text{CHA}^{<A}(Y, \alpha)$, i.e. the homology of the sub-complex generated by $\alpha$-orbits with contact action at most $A$. By the 1-ADC condition, we can find a contact form $\alpha_0 < \alpha$, such that all $\alpha_0$-Reeb orbits with action smaller than $A$ have SFT degree at least 2. Then the functoriality $\text{CHA}^{<A}(Y, \alpha) \to \text{CHA}^{<A}(Y, \alpha_0)$ implies that $1 = 0 \in \text{CHA}^{<A}(Y, \alpha_0)$, which is impossible due to degree reasons. The sixth claim follows from [MZ20, Proposition 3.13].

The advantage of obtaining algebraic tightness instead of tightness is that any contact surgery of an algebraically tight manifold remains algebraically tight by Proposition 25. The following example illustrates how one may use these properties.

Example 26 (1-ADC Bourgeois manifolds). Starting with an open book whose binding is 1-ADC and simply connected, according to Lemma 10 the associated Bourgeois contact structure is also 1-ADC (in fact 3-ADC), hence algebraically tight. Then after contact surgeries, the contact manifold remains algebraically tight, hence tight.

Remark 27 (Tightness without CH). In fact, proving that 1-ADC implies tightness can be done directly without needing to pass through contact homology (cf. [MNW13, Lemma 3.2]). Namely, assuming that the contact structure is a negative stabilisation, which is equivalent to overtwistedness by [CMP19], Bourgeois-van Koert [BvK10] showed the existence of contact form whose symplectization contains a Fredholm-regular finite-energy plane of index 1 that is unique up to $\mathbb{R}$-translation. Then one can deform this symplectic form so that the negative end has only orbits of degree at least 2 for arbitrarily large action (leaving the positive end fixed). As a portion of the $\mathbb{R}$–family of planes survives near the positive end, one gets a resulting moduli space of planes, whose compactification must have SFT-type breaking. This forces the existence of an orbit of degree at most 1 at the lower end, contradicting the 1-ADC condition.

Note that the same argument also applies to the positive end of any Liouville cobordism whose negative end is 1-ADC.

Remark 28 (Algebraic vs. geometric). One of the most fundamental questions in symplectic and contact topology is whether the boundary between rigidity and flexibility is captured by pseudo-holomorphic curves. In this framework, it was shown by Avdek [Avd20] that algebraic overtwistedness in dimension 3 does not
imply overtwistedness. Avdek used another tightness criterion from Heegaard-Floer homology, which is still based on pseudo-holomorphic curves. Unlike the situation in dimension 3, in higher dimensions, it seems that algebraic tightness is the only currently available criterion to ensure tightness. It is an important question to develop further criteria. There are potentially further obstructions to the existence of DGA augmentations, other than the vanishing of the contact homology algebra. For example, as augmentations are solutions to a family of algebraic equations, one could exploit the algebraic non-closedness of $\Lambda$. Hence we do not expect the converse of the fourth point of Proposition 25 to hold. Moreover, it is possible that $Y$ is algebraically fillable, but $Y$ has no strong filling, as there is a hierarchy of obstructions to strong fillability coming from the full SFT [LW11] or the rational SFT [MZ20].

In dimension 5, we can reinterpret the proof of [BGM22, Theorem A] to get the following.

**Theorem 29.** 5-dimensional Bourgeois contact structures $BO(\Sigma, \psi)$ are algebraically tight if $\Sigma$ is non-sporadic (i.e. not a sphere with 3 or less punctures).

**Proof.** The hypothesis of page non-sporadic gives that there is a strong symplectic cobordisms whose negative end consists of hypertight contact manifolds [BGM22, Theorem 3.1 and Corollary 4.1]. Now the symplectic form on this cobordism is exact (hence no sphere bubbling can occur), however the given global primitive 1-form is not compatible with the contact structure on the negative end (this cobordism is called pseudo-Liouville in [BGM22]). In terms of functoriality in SFT, a priori this leads to deformations of the algebraic structure on the negative boundary by counting holomorphic caps in the cobordism. However, symplectic caps for this specific cobordism can be ruled out as shown in the proof of [BGM22, Lemma 5.1]. Hence the usual functoriality for contact homology holds (over Novikov coefficients), and the positive boundary is also algebraically tight, as hypertight contact manifolds are algebraically tight.

By point 3. of Proposition 25, we then have the following useful consequence:

**Corollary 30.** Any contact manifold obtained via contact surgery on a 5-dimensional Bourgeois contact structure $BO(\Sigma, \psi)$ for non-sporadic $\Sigma$ is algebraically tight, and hence in particular tight.

In fact, this result can be proved directly without appealing to Proposition 25, i.e. to the functoriality of contact homology. Indeed, the argument in the proof above used to rule out holomorphic caps applies directly to any cobordism obtained from that used in the proof of Theorem 29 by adding a Liouville cobordism on top.

### 4 Obstructions to strong fillings

#### 4.1 Fillability of $\mathbb{S}^1$-equivariant contact manifolds and subcritical surgery

In this section, we recall the fillability obstruction [BGM22, Theorem B] for $\mathbb{S}^1$-equivariant contact manifolds, and prove in Theorem 32 that this also holds for manifolds obtained from these via certain subcritical surgeries. In the following section, we give alternative proofs, based on symplectic cohomology.

We use the notation as introduced in Section 2 so that we have the following obstruction to strong fillability for $\mathbb{S}^1$-equivariant contact manifolds:

**Theorem 31 ([BGM22, Theorem B]).** Suppose that $M = V^{2n} \times \mathbb{S}^1$ admits an $\mathbb{S}^1$-equivariant contact structure with induced convex splitting $V = V_+ \cup V_-$ and let $N := \partial V_+ = \partial V_-$ be the dividing set, which we assume to be connected. Let also $(W^{2n+2}, \omega)$ be a strong filling of $V \times \mathbb{S}^1$, and consider the natural inclusions and induced maps on homology:

$$N \hookrightarrow V_\pm \hookrightarrow W \quad \text{and} \quad H_*(N, \mathbb{Q}) \xrightarrow{I_\pm} H_*(V_\pm, \mathbb{Q}) \longrightarrow H_*(W, \mathbb{Q}).$$
Then the second inclusion induces an injection on the image $\text{Im} (I_\pm)$ in rational homology.

The argument of the proof uses a capping construction of Massot-Niederkrüger-Wendl (cf. Theorem 13) to a split stable Hamiltonian structure on $N \times S^2$, and the second factor gives a moduli space of holomorphic spheres which is then used to obtain the above result.

We now wish to state a generalisation of Theorem 31 to a contact manifold $M'$ obtained from an $S^1$-equivariant contact structure via subcritical surgery in one of the convex regions, e.g. $V_+ \times S^1$. Again we adopt the notation of Section 2, and assume that $V_+$ is Weinstein. As explained in the preamble and proof of Proposition 21, we then have a cobordism to a subcritically fillable contact manifold $OB(V_+, \text{Id})$ where $V'_+ \supseteq V_+$ is the Weinstein manifold obtained by pushing all surgeries into the binding of the open book $OB(V_+, \text{Id})$ obtained after blowing down. There is then a well defined embedding of $V_+ \hookrightarrow M'$ (up to smooth isotopy), since all attaching spheres can be made disjoint from a slice $V_+ \times \{pt\}$ via a smooth isotopy.

**Theorem 32.** Consider an $S^1$-equivariant contact structure on $M = V^{2n} \times S^1$, with induced convex splitting $V = V_+ \cup V_-$. Suppose now that $V_+$ is Weinstein, and that $(M', \xi')$ is obtained from $M$ via subcritical surgery on isotropic spheres contained in the interior of $V_+ \times S^1$, with framings satisfying the compatibility condition of Definition 19. Let also $(W^{2n+2}, \omega)$ be a strong filling of $M'$, and consider the natural inclusions and induced maps on homology:

$$V'_+ \hookrightarrow V_+ \hookrightarrow W \quad \text{and} \quad H_q(V'_+, \mathbb{Q}) \xrightarrow{i'} H_q(V_+, \mathbb{Q}) \xrightarrow{i_w} H_q(W, \mathbb{Q}).$$

Then $\text{Ker}(i') = \text{Ker}(i_W)$ for all $q \leq n - 1$. In particular, since the inclusion factors as $V_+ \hookrightarrow M' \hookrightarrow W$, we deduce $\dim H_q(M', \mathbb{Q}) \geq \dim \text{Im}(i')$.

**Proof.** We will follow the proof of Theorem 31 in [BGM22] more or less word for word. More precisely, we first apply Proposition 21 to cap $V_- \times S^1$ and obtain a cobordism to $OB(V'_+, \text{Id})$. We then cap again by attaching $V'_+ \times D^2$ yielding a strong cobordism denoted $W_{\text{cap}}$ whose positive end is $\partial V'_+ \times S^2$ with a split Hamiltonian structure. We then deduce as in [BGM22, Section 6.2] that the inclusion $\partial V'_+ \hookrightarrow W_{\text{cap}}$ must be injective in rational homology for those classes in $\partial V'_+$ which are non-trivial in $H_q$. We then have the following diagram of induced maps:

\[
\begin{array}{ccc}
H_q(\partial V'_+; \mathbb{Q}) & \xrightarrow{\cong} & H_q(W_{\text{cap}}; \mathbb{Q}) \\
\uparrow & & \uparrow \\
H_q(V'_+; \mathbb{Q}) & \xrightarrow{i'} & H_q(W; \mathbb{Q}) \\
\downarrow i_w & & \downarrow i_w \\
H_q(V_+; \mathbb{Q}) & \rightarrow & H_q(M'; \mathbb{Q})
\end{array}
\]

Here, the top left vertical arrow is an isomorphism for $q \leq n - 1$ since $V'_+ \times S^1$ is Weinstein, and we have proven above that the top horizontal map is injective. Now, the vertical top right arrow is injective on $\text{Im}(i_W)$, as can be seen by comparing the long exact sequence for the pair $(W_{\text{cap}}, W)$ to that of the handle $(V'_+ \times \{pt\}) \times D^2$ relative to its boundary, using the excision and the fact that the inclusion of $V'_+$ into the handle gives an injection on homology. Then, by commutativity we have that the kernels $\text{Ker}(i') = \text{Ker}(i_W)$ for all $q \leq n - 1$, thus concluding the proof.

\[\square\]

### 4.2 Obstructions to fillability from symplectic cohomology

The proof of Theorem 32 uses a capping construction and holomorphic spheres to derive information on homology. In the following, we will prove a similar result without capping that is based on symplectic
cohomology. We point out that the strategy given above for Theorem 32 requires the use of polyfold theory developed by Hofer-Wysocki-Zehnder [HWZ17] to deal with sphere bubbling in strong fillings. On the other hand, the symplectic cohomology approach relies on understanding the existence and non-existence of certain holomorphic curves which are cut out transversely, and formal properties of the package of symplectic cohomology. To guarantee the existence of symplectic cohomology (with coefficients in the Novikov field $\Lambda$ over $\mathbb{Q}$) as well as its formal properties for general strong fillings, we need to deploy Pardon’s VFC package [Par16] for Hamiltonian-Floer cohomology.

4.2.1 Properties of Symplectic Cohomology

We now recall briefly some important properties of symplectic cohomology, a more detailed account of the theory can be found e.g. in [CO18]. Let $W$ be a strong filling with strict contact boundary $(Y, \alpha)$. Unless otherwise specified, we will take coefficients in the Novikov field $\Lambda$ over $\mathbb{Q}$.

1. For any $D > 0$, which is not the period of some Reeb orbits of $\alpha$, we have a $\mathbb{Z}/2$-graded (with grading given by $\dim W - \mu_{\text{CZ}}$) filtered positive symplectic cohomology $SH^*_{> D}(W)$. Its underlying cochain complex is generated by (hat and check orbits of) Reeb orbits with period up to $D$. The existence of positive symplectic cohomology for strong fillings is a consequence of the asymptotic behaviour lemma [CO18, Lemma 2.3]. We also have the filtered symplectic cohomology $SH^*_{> D}(W)$, whose cochain complex is generated by the above orbits and a Morse cochain complex of $W$. Moreover, they fit into a tautological long exact sequence

$$
\ldots \to H^* (W) \to SH^*_{> D}(W) \to SH^*_{> D} (W) \xrightarrow{\delta} H^* (W)[1] \to \ldots
$$

The filtered symplectic cohomology can be computed using any Hamiltonian of slope $D$.

2. We define $\delta_\mathcal{B} : SH^*_{> D} (W) \to H^* (Y)[1]$ to be the composition of the connecting map $\delta$ and the restriction map in cohomology associated to the inclusion $Y \hookrightarrow W$. Then, according to [Zho21, §3.1], $\delta_\mathcal{B}$ can be defined directly by counting rigid configurations consisting of a holomorphic plane in $W$ and a tail of gradient flow line in $Y$ for any chosen Morse function on $Y$. When $\delta_\mathcal{B}(x)$ has a non-trivial component in $H^d (Y)$, then we have

$$
d - \frac{1}{2} \dim W + \mu_{\text{CZ}}(x) - 1 = 0,
$$

where $\mu_{\text{CZ}}(x)$ is the Conley-Zehnder index computed w.r.t. the trivialization induced by the holomorphic plane which contributes to the non-trivial component in $H^d (Y)$.

3. We will repeatedly apply neck-stretching [BEH+03] to moduli spaces in symplectic cohomology. Roughly speaking, this has two uses:

(a) exclude certain moduli spaces, which in the exact case follow simply by action/index considerations;

(b) establish the existence of certain moduli spaces by comparing them to the situation in the “standard filling”.

Both uses will yield a simpler presentation of differentials as well as other structural maps (e.g. $\delta_\mathcal{B}$) in symplectic cohomology. Here we present a very brief account and refer readers to [CO18] (see also [Zho21, Zho22a]) for details on applying neck-stretching to moduli spaces in symplectic cohomology.

Let $(Y, \alpha)$ be a contact type hypersurface in a symplectic filling $W$. Assume that $Y$ divides $W$ into the union of a cobordism $X$ from $Y$ to $\partial W$ and a filling $W'$ of $Y$. For any almost complex structure
$J$, which is compatible with the contact structure near $Y$, we can find a $[0, 1]$–family of compatible almost complex structures on $\tilde{W}$ starting at $J_0 = J$ and converging, for $t \to 1$, to almost complex structures on the completions $\tilde{X}$, $\tilde{W}'$ and $Y \times \mathbb{R}_+$ by stretching the almost complex structure in the Liouville vector direction along $Y$. Holomorphic curves and/or Floer cylinders for this family of almost complex structures $J_t$, will then converge to SFT buildings for $t \to 1$, in such a way that the top-level curve, i.e. the curve contained in $\tilde{X}$ will develop negative punctures asymptotic to Reeb orbits on $Y$. Let $\Gamma$ be the set of Reeb orbits to which these negative punctures are asymptotic. There are then mainly two constraints on the top-level curve:

(a) Action constraint:

$$A_H(\text{negative end}) - A_H(\text{positive end}) - \sum_{\gamma \in \Gamma} \int_{\gamma} \gamma^* \alpha > 0$$

where $A_H$ is the symplectic action [Zho21, (2.1)] for the Hamiltonian $H$, which is well-defined if $\tilde{X}$ is exact.

(b) Index constraint: the top-level curve must have a non-negative expected dimension. More precisely, when $c_1(X)$ is torsion, then

$$m - \sum_{\gamma \in \Gamma} \left( \mu_{CZ}(\gamma) + \frac{\dim W}{2} - 3 \right) \geq 0,$$

where $m$ is the expected dimension of the moduli space without negative punctures and $\mu_{CZ}(\gamma) + \frac{\dim W}{2} - 3$ is the SFT degree of $\gamma$. Note that when we speak of expected dimension we will always consider the unparametrised moduli space where we quotient out any translation symmetries.

4. This last point is irrelevant to our proofs, but would be helpful if one wants to compare the results in this paper to those in [Zho22a]. If $W$ is exact or symplectically aspherical, one can define $SH^{*,<D}(W; \mathbb{Z})$, $SH^{*,<D}(W; \mathbb{Z})$ over $\mathbb{Z}$. Moreover, if $SH^{*,<D}(W; \mathbb{Z}) \to H^*(W; \mathbb{Z})[1] \to H^0(W; \mathbb{Z})$ is non-zero, then $\delta : SH^{*,<D}(W; \mathbb{Z}) \to H^*(W; \mathbb{Z})[1]$ is surjective. This is a key difference between strong fillings and exact fillings, which allows to obtain stronger restrictions on cohomology for exact fillings in [Zho22a].

Remark 33 (Foundations of symplectic cohomology). Going beyond semi-positive symplectic fillings, we will need to apply virtual techniques (e.g. polyfolds, VFC) to define symplectic cohomology for general strong fillings. The proof of results in this paper relies on the first three properties above only and that a virtual count/perturbed count can be arranged to be the geometric count when transversality holds, e.g. by [Par19, Proposition 4.33]. In this paper, we refer foundations of symplectic cohomology for strong fillings to the construction of Hamiltonian-Floer cohomology for general symplectic manifolds in [Par16]. Note that we do not need anything beyond differentials and continuation maps (i.e. no product structure, or higher structures) for the first three properties above, hence we do not need to go beyond what is established explicitly in [Par16].

4.2.2 Controlled dynamics on $\partial(V \times \mathbb{D}^2)$

Let $V$ be a $2n$–dimensional Liouville domain. We choose a Morse function on $V$ such that $\partial_c f > 0$ near $\partial V$, where $\partial_c$ is the locally defined outwards Liouville vector field. As done in [Zho22a, §2.1], one can define using $f$ a contact form $\alpha$ on $\partial(V \times \mathbb{D}^2) = OB(V, \text{Id})$ with the following properties:
1. The Reeb orbits of period at most 2 are precisely the simple Reeb orbits $\gamma_p$ wrapping around $\partial \mathbb{D}^2$ over critical points $p$ of $f$.

2. For $p, q$ in the set $\text{Crit}(f)$ of critical points of $f$, $\gamma_p \neq \gamma_q$ if and only if $f(p) < f(q)$. Moreover, there is $0 < \epsilon \ll 1$ so that, for all $p \in \text{Crit}(f)$, $1 < \gamma_p \alpha < 1 + \epsilon$.

3. For all $p \in \text{Crit}(f)$, the Conley-Zehnder index $\mu_{CZ}(\gamma_p)$ of $\gamma_p$, defined using the obvious bounding disk in the $\mathbb{D}^2$ direction, is $n + 2 - \text{ind}(p)$. This is moreover a canonical/global Conley-Zehnder index if $c_1(V)$ is torsion. In particular, the SFT degrees $\mu_{CZ} + (n + 1) - 3$ are positive. In the setting of Hamiltonian-Floer cohomology, one gets two non-degenerate Hamiltonian orbits $\gamma_p, \hat{\gamma}_p$ (or two critical points on $\text{im}(\gamma_p)$ in the cascades model) with $\mu_{CZ}(\gamma_p) = \mu_{CZ}(\hat{\gamma}_p) - 1$.

4.2.3 Theorem 32 from symplectic cohomology

We first point out that the moduli space of holomorphic spheres in the proof of Theorem 32 in fact gives Theorem 34 below, that will serve as a model for increasingly more general statements below and whose proof is already contained in [Zho22a].

Assume that $W$ is a strong filling of $Y = \partial(V \times \mathbb{D}^2)$, where $V$ is Liouville domain (not necessarily Weinstein). Then the composition $H_a(V \times \{pt\}; \mathbb{Q}) \to H_a(Y; \mathbb{Q}) \to H_a(W; \mathbb{Q})$ is injective.

Before giving a formal proof, we first give an outline of the argument for the reader's convenience. We are going to prove the dual statement that $H^*(W; \mathbb{Q}) \to H^*(Y; \mathbb{Q}) \xrightarrow{I^V} H^*(V \times \{pt\}; \mathbb{Q})$ is surjective. Note that, for any $D > 0$, $\delta_\delta : SH^*_< (W) \to H^*(Y)[1]$ factors through the first restriction map by definition, so it is enough to prove that $I^V \circ \delta_\delta$ is surjective for any strong filling. For this, we will consider $\delta_\delta$ on the filtered positive symplectic cohomology $SH^*_< (W)$, which is generated by Reeb orbits of period up to 2, i.e. all the simple orbits above. For the standard filling $V \times \mathbb{D}^2$, one can compute that the following composition is surjective:

$$I^V \circ \delta_\delta : SH^*_< (V \times \mathbb{D}^2) \to H^*(V \times \mathbb{D}^2)[1] \to H^*(Y)[1] \to H^*(V \times \{pt\})[1].$$

Next, we consider a hypothetical filling $W$ of $Y$. One then argues, using neck-stretching, that the moduli spaces involved in (the surjectivity of) the analogous map

$$I^V \circ \delta_\delta : SH^*_< (W) \to H^*(W)[1] \to H^*(Y)[1] \to H^*(V \times \{pt\})[1]$$

are in fact independent of the choice of the filling, even in the non-aspherical case. We point out that, although such moduli spaces are cut out transversely for any strong filling, the use of Pardon’s virtual fundamental cycles [Par19] is inevitable to guarantee that such moduli spaces encode the surjectivity of $H^*(W) \to H^*(V \times \{pt\})$ by the package of symplectic cohomology (Property 2 in §4.2.1). This then gives Theorem 34 below, that will serve as a model for increasingly more general statements below and whose proof is already contained in [Zho22a].

For simplicity we will assume that $c_1(V)$ is torsion, so that computations of the expected dimension of moduli spaces contained in the symplectization of the end with only depends on asymptotic orbits. The necessary modifications for the general case are discussed after the proof.

Proof of Theorem 34. By [Zho22a, Proposition 2.7, 2.8, 2.11], for the standard filling we have an isomorphism

$$SH^*_{<1+\epsilon} (V \times \mathbb{D}^2) \cong H^*(V)[1] \oplus H^*(V)[2]$$
and the connecting map in the tautological long exact sequence is the projection to the first component

\[ SH^*_{\pm; <1+\varepsilon}(V \times \mathbb{D}^2) \overset{\delta}{\to} H^*(V \times \mathbb{D}^2)[1] \cong H^*(V \times \{pt\})[1]. \]

The underlying cochain complex is generated by \( \tilde{\gamma}_p \) and \( \tilde{\gamma}_q \) with differentials corresponding to Morse differentials of \( f \). In particular, in the splitting above, the component \( H^*(V)[1] \) is generated by the \( \tilde{\gamma}_p \)'s and the component \( H^*(V)[2] \) by the \( \tilde{\gamma}_p \)'s. Moreover, the isomorphism \( SH^*_{\pm; <1+\varepsilon}(W) \cong H^*(V)[1] \oplus H^*(V)[2] \) also holds for any strong filling \( W \) by neck-stretching as there is no action room for the differential to depend on \( W \).

Now let \( h \) be a Morse function on \( Y \). Then [Zho22a, Proposition 3.2] asserts the following:

- \( \delta_{\varepsilon}(\tilde{\gamma}_p) = a + b \), where \( a, b \) are Morse cocycles of \( h \), such that their Morse indices satisfy \( \text{ind}(a) = \text{ind}(p) < \text{ind}(b) \), and with \( a \) independent of the particular strong filling.

To prove this one applies neck-stretching to the moduli space contributing to the coefficient \( \langle \delta_{\varepsilon}(\tilde{\gamma}_p), \alpha \rangle \) for a critical point \( \alpha \) of \( h \). Strictly speaking, we stretch along a slightly pushed-in copy of the contact boundary so that the Morse flow tail in the definition \( \delta_{\varepsilon} \) is contained in the top level. Either the moduli space is completely contained outside of the filling in the completed filling, or the top-level curves develop some negative punctures asymptotic to some \( \gamma_q \) with \( q \in \text{Crit}(f) \). The expected dimension of the top-level curves is (using our assumption that \( c_1(V) \), and hence \( c_1(Y) \), is torsion)

\[ \text{ind}(\alpha) - (n + 1 - \mu_{CZ}(\gamma_p)) - 1 - (\mu_{CZ}(\gamma) + n + 1 - 3) = \text{ind}(\alpha) - \text{ind}(p) - (\mu_{CZ}(\gamma) + n - 2). \]  

Since \( \mu_{CZ}(\gamma_q) + n - 2 = 2n - \text{ind}(q) \geq 1 \), we know that the top-level curve will have negative expected dimension when \( \text{ind}(a) \leq \text{ind}(p) \). In particular, in this case, we see that the relevant curves must be contained outside the compact part of the completed filling for a sufficiently stretched almost complex structure on any strong filling \( W \). As genericity can be achieved in that region, we can thus assume that all the curves satisfying this index constraint are regular, i.e. that transversality holds. In particular, even though the computation of \( \delta_{\varepsilon}(\tilde{\gamma}_p) \) would need virtual counts using Pardon’s VFC techniques, by [Par19, Proposition 4.33] these virtual counts agree with geometric counts when there is transversality. Thus we know that at least a part of the counts necessary for \( \delta_{\varepsilon}(\tilde{\gamma}_p) \), namely those where the index inequality is satisfied in the determination of the \( a \) component, can actually be determined using an honest geometric count. We deduce that the projection of \( \delta_{\varepsilon}(\tilde{\gamma}_p) \) to \( H^k(Y) \) is well-defined and independent of the filling for all \( k \leq \text{ind}(p) \).

On the other hand, as we are assuming that \( c_1(V \times \mathbb{D}^2) \) is torsion there is a global grading on \( SH^*(V \times \mathbb{D}^2) \). Thus we know that \( \langle \delta_{\varepsilon}(\tilde{\gamma}_p), \alpha \rangle \neq 0 \) is possible only for \( \text{ind}(\alpha) = \text{ind}(p) \) in the case of the standard filling. Therefore, by comparing to the standard filling, we deduce the claimed property for \( \delta_{\varepsilon} \) for every strong filling.

Similarly, we have

- \( \delta_{\varepsilon}(\tilde{\gamma}_p) = b \) for \( \text{ind}(p) - 1 < \text{ind}(b) \).

This follows from the fact that \( \delta_{\varepsilon}(\tilde{\gamma}_p) = 0 \) for the standard filling, which uses a \( S^1 \)-transversality argument as in [Zho22a, Proposition 3.2], i.e. the relevant holomorphic curves are never rigid because one can rotate the Floer cylinder\(^2\). Since \( \mu_{CZ}(\tilde{\gamma}_p) = \mu_{CZ}(\gamma_p) + 1 \), the dimension computation as before yields the claim.

Now let \( a \in H^q(Y) \) be such that \( I^V(a) \neq 0 \). Then for the standard filling \( V \times \mathbb{D}^2 \), we have a closed cochain \( x = \sum a_i \tilde{\gamma}_p_i + \sum b_i \tilde{\gamma}_q \), with \( \text{ind}(p_i) = \text{ind}(q_i) - 1 = d \), such that \( \delta_{\varepsilon}(x) = a \) in cohomology. Then we know that \( x \) is a closed cochain for any strong filling \( W \) and \( \delta_{\varepsilon}(x) = a + b \) on cohomology, with \( \text{ind}(b) > d \). This implies that \( I^V(a) + I^V(b) \) is in the image of \( I^V \circ \delta_{\varepsilon} \) for \( W \), hence in the image of \( H^*(W) \to H^*(V \times \{pt\}) \). Therefore \( I^V(a) \) is in the image of \( H^*(W) \to H^*(V \times \{pt\}) \) and the theorem follows. \( \square \)

\(^2\)Although one should expect that \( \delta_{\varepsilon}(\tilde{\gamma}_p) = 0 \) for any strong filling, the \( S^1 \)-equivariant transversality is not as easy as in the exact case.
Remark 35 (Non torsion first Chern class). If $c_1(V)$ is not torsion, the expected dimension in (3) also depends on the relative homology class of the curve. In this setting, action and compactness arguments were used in [Zho22a, Proposition 3.1] to argue the relative homology class must be trivial. Similarly, the above proof can be adapted to the non-torsion case. This being said, in the more general statements below, it is unclear how to generalize the action and compactness arguments, hence we preferred to give the proof above only in the case of $c_1(V)$ torsion, and will restrict to this case in the following subsections.

4.2.4 A more general obstruction

As mentioned in the introduction, motivated by Eliashberg’s result on contact connected sums [Eli90], the obstruction of strong fillings in Theorem 34 should persist under contact connected sums. More precisely, we have the following, which is crucial for Theorem B.

Theorem 36. Let $V$ be a Liouville domain such that $c_1(V)$ is torsion. Assume $(M, \xi)$ is a strongly fillable contact manifold, such that $c_1(\xi)$ is torsion. Let $W$ be a strong filling of $Y = \partial(V \times \mathbb{D}^2)\#M$. Then the following composition is injective

$$H_\ast(V \times \{pt\}; \mathbb{Q}) \rightarrow H_\ast(Y; \mathbb{Q}) \rightarrow H_\ast(W; \mathbb{Q}).$$

The strategy to prove Theorem 36 is similar to Theorem 34. Let $F$ be a strong filling of $M$ and denote the boundary connected sum. We can then compute $SH^\ast,\ast(\{V \times \mathbb{D}^2\}_\#F) \rightarrow H^\ast(V \times \{pt\})[1]$ if we equip $M$ with an extremely large contact form, as the Reeb orbits on $M$ do not contribute to the computation because of the action filtration. We then understand the image of such map through the kernel of $H^\ast((V \times \mathbb{D}^2)\#F) \rightarrow SH^\ast,\ast(\{V \times \mathbb{D}^2\}_\#F)$. For this, we use the Hamiltonians in [Cie02, Fau20] on the 1-handle to simplify the generators in the 1-handle itself. Next we switch the Hamiltonian to a $C^2$-small one on the filling $(V \times \mathbb{D}^2)\#F$ (for the sake of positive symplectic cohomology), and argue by neck-stretching that the surjectivity of $SH^\ast,\ast(W) \rightarrow H^\ast(V \times \{pt\})[1]$ on $\bigoplus_{i \geq 0} H^i(V)$ is independent of filling $W$ of $Y$. Here follows a detailed proof.

Proof of Theorem 36. We will prove the dual statement on cohomology, namely that the composition

$$H^\ast(W; \mathbb{Q}) \rightarrow H^\ast(Y; \mathbb{Q}) \rightarrow H^\ast(V \times \{pt\}; \mathbb{Q})$$

is surjective. Let $F$ be a strong filling of $M$ and consider the boundary connected sum $X := (V \times \mathbb{D}^2)\#F$, with boundary $Y$. We take a contact form $\alpha$ on $Y = \partial X$, such that 1-handle is very thin (so that the simple Reeb orbit in the co-core has small period) and $F$ is very fat in the following sense: all Reeb orbits on $Y$ with period smaller than 2 are either those simple orbits winding around $V$ once, or multiple covers of $\gamma_1$, the simple orbit in the co-core winding around the core of the 1-handle in $X$. The Conley-Zehnder index of the $k$-th multiple cover $\gamma_1^k$ is given by $n - 1 + 2k$, where $2n = \dim V$ [Laz20, Theorem 3.15]. Following [Cie02, Fau20], we can build a Hamiltonian $H$ of slope $1 + \epsilon$ on the completion $\hat{X}$ (which is not $C^2$-small on the handle), such that the 1-periodic orbits of $H$ are one of the following types:

1. constant orbits (Morse critical points) on $V \times \mathbb{D}^2$,
2. Hamiltonian orbits perturbed from simple Reeb orbits in $V \times \partial \mathbb{D}^2$ that wind around $\partial \mathbb{D}^2$ once,
3. constant orbits on $F$,
4. a constant orbit $\gamma_1^N$ on the handle with Conley-Zehnder index $n - 1 + 2N + 1 = n + 2N$ for $N \gg 0$, i.e. the same as that of $\gamma_1^N$ for $N = [(1 + \epsilon)/\int \gamma_1^N \alpha]$.
Note that $H$ is not $C^2$-small on the handle, and the orbit $\gamma_h$ may be viewed as the only survivor after the massacre between $\gamma_1, \ldots, \gamma_N^k$ and the constant Morse critical point on the handle using a $C^2$-small Hamiltonian in a homotopy connecting those two Hamiltonians. Moreover, orbits in (1) and (2) are orbits of an extension $H_1$ to the whole $\tilde{V} \times \mathbb{D}^2$ of the truncation of $H$ on $V \times \mathbb{D}^2$. Their Conley-Zehnder indices are well-defined (using any bounding disk in $V \times \mathbb{D}^2$) as we assume $c_1(V)$ is torsion and they furthermore have positive SFT degrees which are bounded. Similarly, orbits in (3) are orbits of the Hamiltonian $H_2$ on $\tilde{F}$ obtained by completing the truncation of $H$ on $F$. The symplectic action of constant orbits is approximately zero and the symplectic action of non-constant orbits is approximately the negative period of the corresponding Reeb orbit, i.e. approximately $-1$.

**Claim 1:** There is no differential to or from $\gamma_h$ and hence we have a tautological splitting:

$$HF^*(H) = HF^*(H_1) \oplus H^*(H_2) \oplus \langle \gamma_h \rangle.$$  

**Proof.** Assume by contradiction that there are curves between $\gamma_h$ and orbits in (1) or (3). Then by neck-stretching along the boundary of the disjoint union $(V \times \mathbb{D}^2) \sqcup F$, the top component, which must have punctures, will have negative energy, as Reeb orbits on $\partial((V \times \mathbb{D}^2) \sqcup F)$ have period at least 1. This is a contradiction.

Similarly, if by contradiction there are differentials from $\gamma_h$ to orbits in (2), we can again apply neck-stretching along $\partial((V \times \mathbb{D}^2) \sqcup F)$, and the top component must also have negative energy, a contradiction.

Finally if by contradiction there are differentials from orbits in (2) to $\gamma_h$, after neck-stretching along $\partial((V \times \mathbb{D}^2) \sqcup F)$, we know that the top component has no puncture asymptotic to orbits on $\partial F$ as they have large period. Note that the expected dimension of the top curve is the difference of Conley-Zehnder indices of the orbit in (2) and $\gamma_h$, minus the sum of SFT degrees of Reeb orbits on the negative punctures, minus 1. Since orbits on $\partial(V \times \mathbb{D}^2)$ with period smaller than 2 all have positive SFT degrees and bounded Conley-Zehnder indices, the top component must have negative virtual dimension when $N \gg 0$, hence generically does not exist, giving a contradiction. 

As the filtered symplectic cohomology only depends on the slope of the Hamiltonian, we have the equality of vector spaces $HF^*(H) = SH_{+}^{*,<1+\epsilon}(X)$. From the splitting, we see that the map

$$H^*(V \times \mathbb{D}^2) \to H^*(V \times \mathbb{D}^2) \oplus H^*(F) \to HF^*(H) = SH_+^{*,<1+\epsilon}(X)$$

is zero as $H^*(V \times \mathbb{D}^2) \to HF^*(H_1)$ is zero. As a consequence of the tautological exact sequence, we have

$$SH_+^{*,<1+\epsilon}(X) \to H^*(X)[1] \to H^*(V \times \mathbb{D}^2)[1] \cong H^*(V \times \{pt\})[1]$$  \hspace{1cm} (4)$$

is surjective in positive degrees. Note that $1 \in H^*(V \times \{pt\})$ is not in the image of the map in Equation (4), since $(1,1) \in H^*(V \times \mathbb{D}^2) \oplus H^*(F)$ maps to a non-zero element in $HF^*(H)$.

We now claim that $SH_+^{*,<1+\epsilon}(W) \to H^*(V \times \{pt\})[1]$ is surjective in positive degrees for any strong filling $W$ of $Y$, i.e. the above property is independent of the filling. On $W$, we will of course not use $H$, which is not defined there, but a new Hamiltonian $\tilde{H}$, which is zero on $W$ and linear with slope $1 + \epsilon$ on the cylindrical end, and computes the filtered positive symplectic cohomology with action bound $1 + \epsilon$. Following §4.2.2, take a Morse function $f$ on $V$ with a unique minimum $m$. Then the generators of the co-chain complex $SC_+(\tilde{H})$ computing $SH_+^{*,<1+\epsilon}(W)$ are given by $\gamma_p, \hat{\gamma}_p$ for critical points $p$ of $f$, and $\gamma_1, \hat{\gamma}_1$ where $1 \leq k \leq N = [(1 + \epsilon)/\sqrt{\alpha}]$. By choosing the period of $\gamma_1$ suitably, we may assume the period of $\gamma_1^k$ for $k \leq N$ is strictly smaller than that of $\gamma_p$. As a consequence, there is no differential from $\gamma_1^k$ to $\gamma_p$ by neck-stretching.

**Claim 2:** There is no differential from $\gamma_p$ to $\gamma_1^k$, unless $p = m$ and $k = 1$.

\footnote{For a general strong filling, the symplectic action is not well-defined but requires a bounding disk. Here we use the obvious bounding disk in $V \times \mathbb{D}^2$, then the symplectic action is the same as the one using the Liouville form on $V \times \mathbb{D}^2$.}

\footnote{There is one such differential in this case.}
Proof. If there is such a differential from \( x \in \{ \tilde{\gamma}_p, \tilde{\gamma}_p' \} \) to \( y \in \{ \gamma_k^1, \gamma_k^2 \} \), we apply neck-stretching along \( \partial (V \times \mathbb{D}^2) \cap M \). Then the top level can have at most one negative puncture, which is asymptotic to some \( \gamma_q \) on \( \partial (V \times \mathbb{D}^2) \). The expected dimension is
\[
\mu_{CZ}(x) - \mu_{CZ}(y) - 1, \quad \text{or} \quad \mu_{CZ}(x) - \mu_{CZ}(y) - (\mu_{CZ}(\gamma_q) + n - 2) - 1.
\]
In either case, since \( \mu_{CZ}(\gamma) = \mu_{CZ}(\tilde{\gamma}) = \mu_{CZ}(\tilde{\gamma}) - 1 \), it is not larger than
\[
\mu_{CZ}(x) - \mu_{CZ}(y) - 1 \leq \mu_{CZ}(\gamma_p) - \mu_{CZ}(\gamma_1^k) = n + 2 - \text{ind}(p) - (n - 1 + 2k) = 3 - 2k - \text{ind}(p).
\]
Therefore the expected dimension is negative unless one of the following holds.
1. \( p = m \) and \( k = 1 \);
2. \( x = \tilde{\gamma}_p \) for \( \text{ind}(p) = 1 \) and \( y = \tilde{\gamma}_1^1 \).

It remains to rule out the second case. Note that for the moduli space of differentials from (simple orbit) \( \tilde{\gamma}_p \) to \( \tilde{\gamma}_1^1 \), we can use an almost complex structure that is independent of \( S^1 \) in the cascades model. There is no rigid cascade from a hat orbit to a check orbit as slightly rotating the Floer cylinder gives again a cascade. Therefore there is no differential in the second case even if we choose an \( S^1 \)-dependent almost complex structure that is close to the above \( S^1 \)-independent almost complex structure.

As a consequence, \( \tilde{\gamma}_p, \tilde{\gamma}_p' \) for \( p \neq m \) generate a subcomplex of \( SC_+ (\hat{H}) \). We write \( \Phi \) for the composition
\[
\langle \tilde{\gamma}_p, \tilde{\gamma}_p' \rangle_{p \neq m} \to SC_+ (\hat{H}) \to C^*(h)
\]
where \( C^*(h) \) is the Morse cochain complex for a Morse function \( h \) on the boundary \( Y \).

**Claim 3:** On the standard filling \( X \), we have
1. \( \Phi(\tilde{\gamma}_p) = 0 \);
2. \( \Phi(\tilde{\gamma}_p) = a \) where \( \text{ind}(a) = \text{ind}(p) \).

On the cohomology level, we have \( I^V \circ \Phi \) is surjective onto \( \bigoplus_{* > 0} H^*(V) \), where \( I^V \) is induced from the inclusion \( V \times \{ pt \} \subset Y \).

**Proof.** We apply neck-stretching along \( M \) to the curves in the moduli spaces defining \( \Phi \). The top-level component of the limit configuration cannot have negative punctures asymptotic to Reeb orbits on \( M \) for action reasons. Hence all curves are contained outside \( F \), where the symplectic manifold is exact with torsion first Chern class. The first property holds when we use an \( S^1 \)-independent almost complex structure and a cascades model as in [Zho22a, Proposition 3.2]. The fact that \( \Phi(\tilde{\gamma}_m) = 0 \) follows by the same reason. The second property follows from the global grading on the complement of \( F \). It is then left to prove the cohomological part of the claim.

We consider a Morse function \( g \) on \( X \), which is assembled from Morse functions \( g_1, g_2 \) on \( V \times \mathbb{D}^2 \) and \( F \) and a Morse function \( g_3 \) on the one handle. We consider the connecting map
\[
\delta : SC_+ (\hat{H}) \to C^*(g) = C^*(g_1) \oplus C^*(g_2) \oplus C^*(g_3) = C_1^* \oplus C_2^* \oplus C_3^*,
\]
where the latter is not a direct sum of cochain complexes, as there are differentials from \( C_1^*, C_2^* \) to \( C_3^* \). We write \( \delta_i \) for the corresponding components of \( \delta \).

We then claim that the first component \( \delta_1(\tilde{\gamma}_1^1) \) is trivial. To see this, we assume by contradiction that such curves exist. Then we apply neck-stretching along \( \partial (V \times \mathbb{D}) \cap M \), as the curve must develop negative punctures, we have a contradiction by action reasons.
We now further claim that $\delta_1(\bar{\gamma}_m)$ has no components in positive degrees. Indeed, we can apply neck-stretching along $M$ to see the curves involved in the definition of this projection are contained outside of $F$. Since the complement of $F$ has torsion first Chern class, then it follows that $\delta_1(\bar{\gamma}_m)$ has no components with positive degrees.

Therefore the surjectivity follows the surjectivity of (4) in positive degrees, thus concluding. \hfill \Box

Claim 4: The cochain complex $\langle \bar{\gamma}_p, \bar{\gamma}_q \rangle_{p \neq m}$ is independent of the choice of strong filling.

Proof. For $x, y$ orbits in this cochain complex, we apply neck-stretching to the moduli space of differentials from $x$ to $y$ as usual. The top curve cannot have negative punctures asymptotic to $\gamma_p$ by action reasons. Assume then that there are negative punctures that are asymptotic to $\gamma_m$. Asymptotic orbits of a negative puncture in the neck-stretching have positive SFT degree.

This is proved exactly as in Theorem 34 using Claim 3 above, as all Reeb orbits that could be asymptotic to $\gamma_m$ are non-degenerate and have Conley-Zehnder indices at least 1, hence for a Hamiltonian $H(\bar{x}_1, \ldots, \bar{x}_n)$ for $m \geq 1$. The expected dimension of the associated moduli space is then

$$\mu_{CZ}(x) - \mu_{CZ}(y) - \sum_{i=1}^{m}((n-1+2k_i) + n + 1 - 3) - 1 \leq \mu_{CZ}(x) - \mu_{CZ}(y) - 2n.$$  

Since $\mu_{CZ}(x), \mu_{CZ}(y) \in [3-n, n+2]$, the expected dimension above is negative. As a consequence, all differentials can be assumed to be contained outside the compact part of the completed filling, and are hence independent of the particular choice of filling. \hfill \Box

Claim 5: On a hypothetical strong filling $W$ we have

1. $\Phi(\bar{\gamma}_p) = b$, with $\text{ind}(p) - 1 < \text{ind}(b)$;
2. $\Phi(\bar{\gamma}_p) = a + b$ where $\text{ind}(a) = \text{ind}(p)$ and $\text{ind}(b) > \text{ind}(p)$. Moreover, $a$ is independent of $W$.

Proof. This is proved exactly as in Theorem 34 using Claim 3 above, as all Reeb orbits that could be asymptotic orbits of a negative puncture in the neck-stretching have positive SFT degree. \hfill \Box

We can now conclude the proof exactly as in the proof of Theorem 34, where the surjectivity in degree 0 is automatic. \hfill \Box

4.2.5 Obstructions from flexibly fillable manifolds

In the 5-dimensional case, Theorem 36 can’t be applied directly as the exotic sphere requires applying critical surgeries. Therefore we need a more flexible version of Theorem 36 to prove Theorem A in dimension 5.

Theorem 37. Let $W_0$ be a 2n-dimensional flexible Weinstein domain with $c_1(W_0)$ torsion and $(M, \xi)$ a strongly fillable contact manifold with $c_1(\xi)$ torsion. Then for any strong filling $W$ of $Y := \partial W_0 \# M$, we have that $H^*(W; \mathbb{Q}) \to H^*(Y; \mathbb{Q})$ is surjective onto the image of $H^*(W_0; \mathbb{Q}) \to H^*(\partial W_0; \mathbb{Q}) \subset H^*(Y; \mathbb{Q}) = H^*(\partial W_0; \mathbb{Q}) \otimes H^*(M; \mathbb{Q})$ for $1 \leq * \leq n$ when $n$ is odd and for $2 \leq * \leq n$ when $n$ is even.

Proof. By [BEE12], we have that $SH^+(W_0) \to H^*(W_0)[1]$ is an isomorphism for the standard flexible filling $W_0$. Moreover, according to [Laz20], there exists a contact form $\alpha$ on $\partial W_0$ such that all Reeb orbits of period $\leq D$ are non-degenerate and have Conley-Zehnder indices at least 1, hence for a Hamiltonian $H_1$ of slope $D$ the connecting map $H^*(SC_+^+(H_1)) \to H^*(W_0)[1]$ is surjective.

Let $F$ be a strong filling of $M$. As in the proof of Theorem 36, we have a Hamiltonian $H$ on $\overline{W_0 \# F}$ built from $H_1$, such that $H^*(SC_+^{<D}(H)) \to H^*(W_0 \# F)[1]$ maps surjectively onto $H^*(W_0) \subset H^*(W_0 \# F)$ for $* > 0$. In particular, $H^*(SC_+^{<D}(H)) \to H^*(Y)[1]$ maps surjectively onto the image of $H^*(W_0) \to H^*(\partial W_0) \subset H^*(Y)$ for $* > 0$.  

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As in the proof of Theorem 36, we use the Hamiltonian $\tilde{H}$ on the completion of our given strong filling $\tilde{W}$ to compute the same Floer cohomology as $\tilde{H}$. The non-constant orbits of $\tilde{H}$ are those of $H_1$ (corresponding to Reeb orbits on $(\partial W_0, \alpha)$ up to a certain period threshold), and multiple covers of the short orbit from the 1-handle. All of these orbits have canonically defined boundary Conley-Zehnder indices (given by a trivialization of $\det C \xi^N$), which are all at least 1.

Now let $x$ be the closed cochain of pure degree in the positive cochain complex of $\tilde{H}$ on $\tilde{W} \mathbb{Z}^F$ that is mapped to $\beta \in H^*(\partial W_0)$, with $* \geq 1$, by the connecting map $\delta_2$. We claim $x$ is also closed in the complex $SC^+_{\mathcal{D}}(W, \tilde{H})$ as long as $n$ is odd or $\deg(\beta) > 1$. This is no longer obvious as in the subcritical case where action based arguments were sufficient, but can still be argued as follows.

For simplicity, we assume that $x$ consists of a single orbit. Now, since $x$ is an orbit that maps to $\beta$ under the connecting homomorphism for $W_0$, it must have (boundary) Conley-Zehnder index $n + 1 - |\beta|$. Suppose that there is a differential from $x$ to $y$ in $SC^+_{\mathcal{D}}(W, \tilde{H})$. Then we apply neck-stretching and assume that there are some SFT degenerations. The top level curve of any resulting building will be a punctured infinite cylinder, with the cylindrical ends asymptotic to $x$ and $y$ and the punctures negatively asymptotic to Reeb orbits $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$. Now, $\Gamma$ can be assumed to consist of Reeb orbits on $\partial W_0$ and on the handle, since we can take $M$ to be “large” as in Theorem 36. The expected dimension of the associated moduli space, after dividing out the $\mathbb{R}$-translation (notice the punctures are allowed to move), is then

$$
(\mu_{\text{CZ}}(x) - \mu_{\text{CZ}}(y) - 1) - \sum_{\gamma \in \Gamma} (\mu_{\text{CZ}}(\gamma) + n - 3) = (n + 1 - |\beta| - \mu_{\text{CZ}}(y) - 1) - \sum_{\gamma \in \Gamma} (\mu_{\text{CZ}}(\gamma) + n - 3) \leq 1 - |\beta|,
$$

where we used the fact that $\mu_{\text{CZ}}(y) \geq 1$ in view of the assumption of flexible fillability. Therefore when $|\beta| > 1$, we cannot have such curves. When $|\beta| = 1$ and $n$ is odd, $\mu_{\text{CZ}}(y)$ must be even due to the $\mathbb{Z}/2$-grading, hence $\mu_{\text{CZ}}(y) \geq 2$. Hence we cannot have such a curve either. In particular, as argued in the previous proofs, this also shows that the closedness of $x$ is independent of fillings for such a cochain and we thus obtain an element denoted $x_W \in SH^+_{\mathcal{D}}(W)$.

Recall now that $x$ maps to $\beta \in H^*(\partial W_0) = H^*(\partial W)$ via the connecting homomorphism. This may not hold for $x_W$ a priori, but by a neck-stretching argument and using the ADC condition as in the proof of Theorem 34, we can argue that $x_W$ is mapped to $\beta$ plus terms with higher degrees under $\delta_2$, i.e. under the composition of the connecting homomorphism and restriction to the boundary:

$$
SH^+,<\mathcal{D}(W) \to H^*(W)[1] \to H^*(\partial W)[1].
$$

This then gives the claimed surjectivity of the second map, since maps on ordinary cohomology do respect gradings.

\[ \square \]

Remark 38 (Semi-positivity). We point out that in dimension $2n = 6$ semi-positivity holds so that one could choose a generic almost complex structure to define symplectic homology in this case. In particular, this allows for a proof of Theorem 37 in this case that does not need to appeal to the VFC package.

Remark 39. The main technical difference between Theorem 36 and Theorem 37 is that in the case of the latter one has more positivity in SFT degrees, whereas the positivity of SFT degrees on $V \times \mathbb{D}^2$ is minimal when $V$ has a codimension-1 skeleton. This being said, the Reeb dynamics on $V \times \mathbb{D}^2$ is simpler and we use the information of periods in that setting, which allows us to work under the assumption of $V$ Liouville instead of the more restrictive Weinstein situation.
5 Exotic contact structures: a geometric construction

We first construct one tight and non-fillable contact structure on \( S^{2n+1} \), with \( n \geq 2 \), via a geometric construction to prove Theorem A. The 5-dimensional case requires a slightly different argument. We will then use this exotic sphere to obtain Theorem B. In the next section, we will upgrade this to give infinitely many examples as in Theorem C, i.e. under a further dimensional restriction, namely \( n \geq 5 \).

**Proof of Theorem A.** We first prove the theorem in the case \( n \geq 3 \). We consider the Bourgeois contact structure coming from a Milnor open book on a sphere \( S^{2n-1} \). For instance, one can take the open book coming from the \( A_1 \)-singularity, with binding \( B = \Sigma_{n-1}(2,2,\cdots,2) \cong S^2 \times S^{n-1} \). This is just the positive stabilisation \( S^{2n-1} = OB(D^2S^{n-1}, \tau) \) of the standard open book \( OB(D^2S^{n-1}, \text{Id}) \), where \( \tau \) is the Dehn-Seidel twist. Topologically, the associated Bourgeois manifold is of the form \( S^{2n-1} \times \mathbb{T}^2 \).

We then perform subcritical surgeries to topologically obtain a sphere \( S^{2n+1} \). More precisely, first observe that by the \( h \)-principle for isotropic embeddings one can always realise surgeries of smooth surgeries of index 1 and 2 via contact surgeries. In view of this we first perform two index 1 surgeries that kill the generators of the fundamental group in the torus factor, to obtain a simply connected manifold with the homology of \( S^{2n-1} \times S^2 \). Notice that, since these generators are non-trivial in homology, the resulting cobordism will have trivial \( c_1 \). An index 2 surgery then kills the generator in degree 2 (which is indeed spherical by the Hurewicz theorem). In order to ensure that this surgery is possible one needs to make sure that the normal bundle is trivial: however, this is ensured by the assumption that \( c_1 \) vanishes, since the obstruction to the normal bundle being trivial is given by the second Stiefel-Whitney class \( w_2 \), to which \( c_1 \) reduces modulo 2. Finally, the corresponding handle attachments applied to the boundary of the null-cobordism \( \mathbb{D}^2 \times \mathbb{T}^2 \) yield a simply connected homology ball, which is then smoothly a ball by the \( h \)-cobordism theorem, giving that the manifold obtained by these surgeries is indeed a smoothly standard sphere. Interpreting these surgeries as Weinstein handle attachment we thus obtain the desired contact structure \( \xi_{ex} \) on \( S^{2n-1} \).

Now, by Example 12 we see that the Bourgeois contact structure is 1-ADC, and thus algebraically tight in view of Proposition 25. Since algebraic tightness is preserved under contact surgery and implies tightness it follows that \( (S^{2n+1}, \xi_{ex}) \) is tight. What is more, the almost contact structure underlying \( \xi_{ex} \) is the standard one on \( S^{2n+1} \), since it extends as an almost complex structure on the smooth filling which is just a ball, on which any two almost complex structures are homotopic.

To obstruct fillability, we remark that the surgeries can all be done on one side of the \( S^1 \)-equivariant decomposition for the Bourgeois contact structure described in Section 4.1, i.e. away from the dividing region \( N \times S^1 \). Note that the initial Milnor page \( \Sigma = V_{n-1}(2,2,\cdots,2) \) has nontrivial homology in degree \( n-1 \). Moreover, this class can be pushed into the boundary of \( V_\pm = D^2S^{n-1} \times D^1 = D^*(S^{n-1} \times S^1) \), since this is then a trivial bundle topologically. In the case that \( n \geq 4 \), performing the index 1 and 2 surgeries on \( V_+ \) corresponding to the handle attachments on \( M \) (as explained in the preamble to Proposition 21) does not affect this class and hence we obtain a non-trivial class in \( H_{n-1}(\partial V'_+) \) that is homologous to a class in \( V_+ \). In the case \( n = 3 \), the same is true since the index 2 surgery kills a class that is homologically non-trivial in the manifold, whereas the zero section already dies in the \( S^{2n-1} \) factor. Note that the framing condition in Theorem 32 is automatic for surgeries of index 1 and 2 (c.f. Remark 20) and hence we deduce that the inclusion \( V_+ \to W \) is non-trivial on degree \( n-1 \) homology. Since the inclusion \( V_+ \to W \) factors through the inclusion \( V_+ \to S^{2n+1} = \partial W \), we obtain a contradiction since \( H_{n-1}(S^{2n+1}) = 0 \). We thus deduce that the contact structure is not fillable.

It is then left to prove the 5-dimensional case. We start with any open book \( OB(\Sigma, \psi) \) for the standard contact structure on \( S^5 \), whose page \( \Sigma \) has homology \( H_1(\Sigma) \) of rank at least 3 – for example take repeated positive stabilisations of the trivial open book. Then, applying surgery to the Bourgeois contact manifold \( BO(\Sigma, \psi) \) as done above, we obtain a contact structure on \( S^5 \). The Bourgeois contact structure is algebraically tight by Corollary 30, hence the contact structure on \( S^5 \) obtained by flexible surgeries on it will also be
algebraically tight, hence tight. Furthermore, by our assumptions, the resulting flexible contact manifold giving by first capping then doing surgery as in Proposition 18 has non-trivial $H_1$ rationally. We can apply Proposition 18 and Theorem 37 (setting $(M, \xi) = (S^5, \xi_{ex})$) to deduce non-fillability. Lastly, the proof of the fact that the obtained exotic structure on $S^5$ is homotopically standard is analogous to the case of dimensions at least 7.

Remark 40 (Freedom of choice of Open Book). For concreteness we chose to take the open book determined by the $A_1$-singularity in the proof above. This said, essentially any Milnor open book whose binding satisfies some sort of index-positivity condition would suffice. Distinguishing the resulting contact structures is however more subtle.

Proof of Theorem B. Assume first $n \geq 3$. Let $(S^{2n+1}, \xi_{ex})$ be the exotic sphere of Theorem A, that was obtained via subcritical surgery on $BO(D*S^{n-1}, \tau)$ away from the dividing set. By Proposition 25, the contact manifolds $(M, \xi)$, $(S^{2n+1}, \xi_{ex})$ and $Y := (M \# S^{2n+1}, \xi \# \xi_{ex}) = (M, \xi')$ are all algebraically tight, and hence tight. Notice also that $\xi'$ is in the same almost contact class as $\xi$ on $Y \approx M$, as $\xi_{ex}$ on $S^{2n+1}$ is homotopically standard. As in the proof of Theorem 32, using the convex decomposition of $BO(D*S^{n-1}, \tau)$ described in Section 4.1, we have a strong cobordism $W$ from $Y$ to $M \# (\partial V'_+ \times D^2)$, where $V'_+$ is obtained from $V_+ = D^*S^{n-1} \times D^*S^1$ by subcritical surgeries. Just as in the proof of Theorem A, there is then a non-trivial class $H_{n-1}(\partial V'_+) \neq 0$ that is mapped to zero in $H_{n-1}(W)$, as it factors through a map

$$H_{n-1}(V_+) \rightarrow H_{n-1}(S^{2n+1} \setminus D^{2n+1}) = 0,$$

where we consider the first factor of the connect sum $Y = (M \# S^{2n+1}, \xi \# \xi_{ex})$ with some small ball removed. Therefore Theorem 36 implies that $Y$ has no strong filling.

The case $n = 2$ is proved similarly. More precisely, in this case recall that the exotic sphere $(S^5, \xi_{ex})$ of Theorem A can be obtained by contact surgeries from a Bourgeois contact structure associated to an open book on $S^3$ with page $\Sigma$ having large $b_1(\Sigma) \gg 3$. Consider now $Y := (M \# S^5, \xi \# \xi_{ex}) = (M, \xi')$ as above. We then use Proposition 18 to obtain a strong cobordism from $(S^5, \xi_{ex})$ to a flexibly filled contact manifold $M'$. This then gives a strong cobordism from $Y$ to $M' \# (M, \xi)$. Using the properties of $M'$ from Proposition 18, one can then use Theorem 37 to obstruct fillability. Tightness follows from algebraic tightness exactly in view of Corollary 30.

6 Exotic contact structures in abundance

We now upgrade the construction from the previous section to obtain infinitely many exotic structures in sufficiently high dimensions. We let $(S^{2n+1}, \xi_{ex})$ denote the tight non-fillable contact structure that we constructed in the previous section. This was obtained (for dimension at least 7) via subcritical surgery on the Bourgeois manifold $BO(D*S^{n-1}, \tau)$, which, as originally observed by Sam Lisi, has an associated spinal open book decomposition (SOBD); c.f. [BGM22, Section 2] for details. Namely, we have a decomposition

$$BO(D*S^{n-1}, \tau) = S^*S^{n-1} \times D^*T^2 \bigcup_{S^*S^{n-1} \times T^3} \text{Map}(D*S^{n-1}, \tau) \times T^2,$$

where $\text{Map}(D*S^{n-1}, \tau)$ denotes the mapping torus of the Dehn-Seidel twist $\tau : D*S^{n-1} \rightarrow D*S^{n-1}$.

Lemma 41. When $n \geq 3$, the contact manifold $(S^{2n+1}, \xi_{ex})$ is $(n - 3)$-ADC.

Proof. Lemma 10 gives the indices of contractible Reeb orbits on $BO(D*S^{n-1}, \tau)$. On the other hand, the subcritical surgeries kill part of the topology of $BO(D*S^{n-1}, \tau)$ making non-contractible orbits of $BO(D*S^{n-1}, \tau)$ contractible in $(S^{2n+1}, \xi_{ex})$. We then need to compute the Conley-Zehnder indices of them as well.
For simplicity we pick a global trivialisation of the almost contact structure underlying the Bourgeois contact structure, which is split as an almost contact structure according to Remark 4, by simply taking a trivialisation of \((S^{2n-1}, \xi_{st})\) and trivialising the tangent bundles of the \(T^2\)-fibers in an equivariant manner. The Reeb orbits on \(BO(D^*S^{n-1}, \tau)\) then have Conley-Zehnder indices as follows:

1. On \(B \times D^2\), where \(B = S^*S^{n-1}\), the computation of the Conley-Zehnder index is exactly that as done in Lemma 10. Namely,

\[
\mu_{CZ}(\gamma_q) = \mu_{CZ}(\gamma_B) - 2 + \text{ind}_q(H) \geq \mu_{CZ}(\gamma_B).
\]

Here, \(\gamma_q = \gamma_B \times \{(q, r = 0)\}\) is the contractible orbit on the binding corresponding to the critical point \(q \in T^2 \times \{0\}\) of a Morse function \(H\) on \(D^2\). From the computations in [KvK16, p. 37] for the \(A_1\)-singularity (i.e. \(k = 1\), and with \(n\) there replaced by \(n - 1\)), we obtain that \(\mu_{CZ}(\gamma_B) \geq n - 2\), and so

\[
\mu_{CZ}(\gamma_q) \geq n - 2,
\]

for every orbit in this region.

2. On the product \(\text{Map}(D^*S^{n-1}, \tau) \times T^2\), the orbits stay tangent to the pages of the open book. Moreover, for a fixed angle \(\theta \in S^1\), the closed Reeb orbits tangent to the \(\theta\)-page are natural parametrized by the page \(D^*S^{n-1}\) itself, namely they are flat geodesics on \(T^2\) with rational slope \(\theta\) (recall Formula (2)), and so they form an \(S^1\) Morse–Bott family when viewed as Reeb orbits on \(T^3\). As in Lemma 10, we locally perturb the contact form with a Morse function with a unique local maximum on \(D^*S^{n-1}\), with two critical points \(q_1, q_2\) lying in the zero section, respectively of index \(2n - 2\) and \(n - 1\). Moreover, the minimal Conley-Zehnder index of the orbits on \(T^3\) is zero after perturbing with a Morse function on \(S^1\) with critical points \(p_1, p_2\). By standard properties of the Conley-Zehnder index, if we denote by \(\gamma_{p_j}\) the Reeb orbit on \(T^3\) corresponding to \(p_j\), then the non-degenerate orbit \(\gamma_{q_i,p_j}\) corresponding to the pair \((q_i, p_j)\) has Conley-Zehnder index

\[
\mu_{CZ}(\gamma_{q_i,p_j}) = \text{ind}(q_i) - \frac{1}{2} \dim(D^*S^{n-1}) + \mu_{CZ}(\gamma_{p_j})
\]

\[
= \text{ind}(q_i) - n + 1 + \mu_{CZ}(\gamma_{p_j})
\]

\[
\geq n - 1 - n + 1 + 0 = 0
\]

Therefore the Conley-Zehnder index of every orbit is non-negative along this region, and their SFT degrees are at least \(n - 2\).

3. Along the interface region \(S^*S^{n-1} \times T^3\), the situation is modelled as a smoothed corner of the product \(D^*S^{n-1} \times D^2\). Computations in this setting have already been done in detail in [Zho21, Proposition 6.18 (3)], and the output is that the minimal Conley-Zehnder index is the sum of minimal Conley-Zehnder indices on \(S^*S^{n-1}\) (i.e. \(n - 2\)) and \(S^2\) (i.e. 0).\(^5\) Hence we have \(\mu_{CZ}(\gamma) \geq n - 2\) for every orbit \(\gamma\) on this region.

4. The attaching of two 1-handles and a 2-handles create Reeb orbits of index at least \(n - 2\). When we attach the two 1-handles to kill the fundamental group of \(BO(D^*S^{n-1}, \tau)\), to maintain the correctness of Conley-Zehnder indices computed above, we need to choose the framing so that the induced trivialization of the contact structure (used in cases 2 and 3 above) along the isotropic circle extends to a trivialization of the core disk in the handle. In fact since the \(T^2\)-fibers over the binding of the

\(^5\)Note that [Zho21] considered the \(S^1\)-family of Reeb orbits, whose generalized Conley-Zehnder index has an extra 1/2. But here we consider non-degenerate Reeb orbits after a small perturbation of the Morse-Bott family, hence the minimal Conley-Zehnder index adds.
open book in the base are isotropic, these give natural choices of isotropic circles for surgery (of any slope). One then simply takes the framing of the normal bundle induced from the trivialisation on the symplectic normal induced from the projection to the base.

In total we see that after surgery all periodic orbits have degree at least \( n + 1 - 3 = n - 2 \) (up to a certain action threshold). Thus the resulting contact structure is \((n - 3)\)-ADC.

As proved in [Laz20], the ADC condition is sufficient to obtain well-defined invariants of Liouville-fillable contact manifolds using positive symplectic cohomology (generated by contractible orbits) of any of its Liouville fillings (with \( q_1 = 0 \)). Under a slightly stronger assumption, namely that there is a non-degenerate Reeb vector field all of whose orbits have SFT degree at least equal to 2, Cieliebak-Oancea [CO18, Section 9.5] described a version of symplectic homology on the trivial symplectic cobordism \( M \times [0, 1] \) obtained with \( (M, \xi) \) simply by

\[
\text{Proposition 42.} \quad \text{When } n \geq 5, \text{ there exists a fillable } (S^{2n+1}, \xi') \text{ realised the standard almost contact structure, which is } 1 \text{-ADC, and } 0 < N := \dim SH^{2n-3}(S^{2n+1}, \xi') < \infty.
\]

Proof. We consider the Brieskorn sphere \( Y := \Sigma(n, \ldots, n, n+1, p) \), where \( p \) is a prime number with \( p \gg n \).

By [KvK16, (14)] or [Zho22b, (5.12)], the minimal Conley-Zehnder index of a small perturbation of the standard contact form on \( Y \) is \( 4 - n \) (obtained with \( |I| = n + 2, |I_T| = n, N = 1, T = n \) in [Zho22b, (5.12)]).

The SFT degree of the orbit is 2. The minimal Conley-Zehnder indices of other Morse-Bott families of Reeb orbits of the natural contact form is \( 6 - n \) (given by \( N = 2, T = n \) in [Zho22b, (5.12)]) as \( p \gg n \) and so they have SFT degree 4. As a consequence, we have that \( Y \) is 1-ADC.

Moreover, by the Morse-Bott spectral sequence and the index gap above, we have \( \dim SH^{(n+1)-(4-n)}(Y) = \dim SH^{2n-3}(Y) = 1 \), solely contributed to by the family with the minimal Conley-Zehnder index. Now, by [KvK16, Proposition 3.6], \( Y \) is homeomorphic to a sphere. Then there exists \( N \), such that the \( (N-1) \)-th iterated self connected sum \( Y \#^N \) is diffeomorphic to the standard sphere. Since \( \dim Y \geq 11 \), the contact sum preserves the 1-ADC property and \( \dim SH^{2n-3}(Y \#^N) = \dim \bigoplus^N \dim SH^{2n-3}(Y) = N \) by [CO18, Theorem 7.1, Proposition 9.19]. To obtain the standard almost contact structure, we choose \( Y' \) to be the flexibly fillable sphere with the opposite homotopy class of almost contact structures as that of \( Y \#^N \), as in [Laz20, Corollary 1.6]. (To be precise, here “opposite” is to be interpreted as the inverse w.r.t. the natural group structure on the space of almost contact structures on the sphere given by the connected sum operation.) Then \( Y' \#^N Y \) is a standard sphere with the standard almost contact structure, and it is 1-ADC. Moreover, \( \dim SH^{2n-3}(Y') = 0 \) when \( n \geq 5 \). Hence we may take \( Y' \#^N Y \) as the desired contact sphere.

\[\text{Corollary 43.} \quad \text{For } n \geq 5, \text{ there exists infinitely many different contact structures on } S^{2n+1} \text{ that are tight, non-fillable and homotopically standard.}\]

Proof. If \( (S^{2n+1}, \xi_{\infty}) \) is the exotic example from Theorem A and \( (S^{2n+1}, \xi') \) the one from Proposition 42, then the iterated connected sums \( (S^{2n+1}, \xi_{\infty}) \#^i (S^{2n+1}, \xi') \) give the desired infinite family of examples, for \( i \in \mathbb{N} \). Indeed, they are tight and non-fillable by the same arguments as in the proof of Theorem B. Moreover, since \( SH^+(S^{2n+1}, \xi_{\infty}) \) is supported in degrees at most \( n + 1 \) because Conley-Zehnder indices are non-negative in that case, then if \( 2n - 1 > n + 1 \), i.e. \( n \geq 5 \), we have \( \dim SH^+(S^{2n+1}, \xi_{\infty}) \#^i (S^{2n+1}, \xi') = N \cdot i \), where \( N = \dim SH^{2n-3}(S^{2n+1}, \xi') \). As a consequence, the contact structures given by the contact sums above are pairwise distinguished by \( SH^*, \) and are hence pairwise distinct contact structures.
We now have all the needed ingredients to prove the last result stated in the Introduction:

**Proof of Theorem C.** One first realises the given almost contact structure as the contact boundary \((M, \xi_{flex})\) of a flexible Weinstein domain using Eliashberg’s h-principle [CE12]. As in the proof of Corollary 43, one then considers, for \(i \in \mathbb{N}\), the iterated connected sums \((M, \xi_{flex}) \#^i (\mathbb{S}^{2n+1}, \xi')\) with the contact sphere \((\mathbb{S}^{2n+1}, \xi')\) from Proposition 42. Notice that \(M\) is 1-ADC, as it is flexibly fillable [Laz20], and that \(SH^*_+ (M)\) is supported in degree smaller than \(n+1\), in view of the vanishing of symplectic homology for flexible Weinstein domains. Then, the same argument as in Corollary 43 allows to prove that the contact structures given by the connected sums are tight, non-fillable, all homotopic to \(\eta\), and pairwise distinct.

**Discussion about lower dimensions.** When \(n = 3\), \((\mathbb{S}^{2n+1}, \xi_{flex})\) is not 1-ADC, which costs us the well-definedness of the positive symplectic cohomology. When \(n = 4\), it is not easy to find a degree such that positive symplectic cohomology is different for varying summands in the connected sum. However, we can appeal to the symplectic cohomology of a DGA augmentation. Namely, given a contact manifold \(Y\), we can consider a Hamiltonian on \(Y \times (0, \infty)\) which is zero on \(Y \times (0, 1)\) and quadratic on \(Y \times (1, \infty)\). By counting solutions to the Hamiltonian Floer equation, possibly with negative punctures capped off by the given algebraic augmentation \(\epsilon\) of the contact DGA of \(Y\), we have a well-defined cochain complex and the cohomology is denoted by \(SH^*(Y; \epsilon)\). The equivariant analogue is sketched in [MZ20, §4]. (In the case of \(Y\) being 1-ADC and \(\epsilon\) being the trivial augmentation, this is the symplectic cohomology of the boundary used above [CO18], and it is a contact invariant itself.)

**Definition 44.** Assume \((Y, \xi)\) is contact manifold with \(c_1(\xi)\) torsion. In the following, we consider the contact DGA generated by contractible orbits, which is canonically \(\mathbb{Z}\)-graded. We define

\[
SH^+_*(Y) := \{SH^+_*(Y, \epsilon) \mid \epsilon \text{ is a } \mathbb{Z}\text{-graded DGA augmentation of the contact homology algebra of } Y \}
\]

where the symplectic cohomology is also generated by contractible orbits only.

It is natural to expect that \(SH^+_*(Y)\) as a set is a contact invariant. However, one would need to settle the homotopy issues to prove that \(SH^+_*(Y)\) is independent of contact forms and other auxiliary choices. Assuming the existence of such a contact invariant, one can then extend Theorem C to the problematic dimensions:

**Corollary 45.** Assume Definition 44 is well-defined. Then, for any \(n \geq 3\), there exists infinitely many non-isomorphic contact structures on \(\mathbb{S}^{2n+1}\) that are homotopically standard, tight and not strongly fillable. The analogue of Theorem C also holds.

**Proof.** We consider a Brieskorn sphere \(Y = \Sigma(a_0, \ldots, a_{n+1})\) with \(a_i \gg 0\). Then the Conley-Zehnder indices are all \(\ll 0\). Assume that the maximal index is \(N \ll 0\), only realised by \(\gamma\). Note that \(SH^+_{n-N}(Y) = \mathbb{Q}\) is composed from the (Hamiltonian) orbits with Conley-Zehnder index \(N + 1\), which are solely contributed by the hat orbit \(\hat{\gamma}\) (there are two cancelling differentials from \(\hat{\gamma}\)). We claim that \(SH^+_{n-N}(\mathbb{S}^{2n+1}, \xi_{flex}) \#^i Y\) are different for different \(i\).

As the augmentations are \(\mathbb{Z}\)-graded, we only need to consider contractible orbits with SFT degree 0, which are solely contributed by \((\mathbb{S}^{2n+1}, \xi_{flex})\) since the augmentation evaluates to zero on other generators. On the other hand, \(SH^+_{n-N}(\mathbb{S}^{2n+1}, \xi_{flex}) \#^i Y\) is solely contributed by the hat orbit \(\hat{\gamma}\) of each copy of \(Y\). By choosing the contact form on \((\mathbb{S}^{2n+1}, \xi_{flex})\) much larger than that of \(Y\), we see the differential from \(\hat{\gamma}\) does not depend on the augmentation by action reasons. One the other hand, there is no differential to \(\hat{\gamma}\) by degree reasons. Therefore we have \(SH^+_{n-N}(\mathbb{S}^{2n+1}, \xi_{flex}) \#^i Y\) = \(\{\mathbb{Q}\}\), as there is one \(\hat{\gamma}\) for each copy of \(Y\). We then use a flexibly fillable contact sphere to correct the almost contact structure as before. This will not affect \(SH^+_{n-N}\) for \(N \ll 0\) as Conley-Zehnder indices on the flexibly contact manifolds are (asymptotically) positive. Tightness and non-fillability follow again from the same argument as that in the proof of Theorem B.  

\[28\]
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