ON THE COMMUTATIVITY OF CLOSED
SYMMETRIC OPERATORS

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Abstract. In this article, we give conditions guaranteeing the
commutativity of a bounded self-adjoint operator with an unbounded
closed symmetric operator.

1. Essential background

All operators considered here are linear but not necessarily bounded.
If an operator is bounded and everywhere defined, then it belongs to
$B(H)$ which is the algebra of all bounded linear operators on $H$ (see
[19] for its fundamental properties).

Most unbounded operators that we encounter are defined on a sub-
space (called domain) of a Hilbert space. If the domain is dense, then
we say that the operator is densely defined. In such case, the adjoint
exists and is unique.

Let us recall a few basic definitions about non-necessarily bounded
operators. If $S$ and $T$ are two linear operators with domains $D(S)$ and
$D(T)$ respectively, then $T$ is said to be an extension of $S$, written as
$S \subset T$, if $D(S) \subset D(T)$ and $S$ and $T$ coincide on $D(S)$.

An operator $T$ is called closed if its graph is closed in $H \oplus H$. It is
called closable if it has a closed extension. The smallest closed exten-
sion of $T$ is called its closure and it is denoted by $\overline{T}$ (a standard result
states that a densely defined $T$ is closable iff $T^*$ has a dense domain,
and in which case $\overline{T} = T^{**}$). If $T$ is closable, then

$S \subset T \Rightarrow \overline{S} \subset \overline{T}$.

If $T$ is densely defined, we say that $T$ is self-adjoint when $T = T^*$;
symmetric if $T \subset T^*$; normal if $T$ is closed and $TT^* = T^*T$.

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The product $ST$ and the sum $S + T$ of two operators $S$ and $T$ are defined in the usual fashion on the natural domains:

$$D(ST) = \{x \in D(T) : Tx \in D(S)\}$$

and

$$D(S + T) = D(S) \cap D(T).$$

In the event that $S$, $T$ and $ST$ are densely defined, then

$$T^* S^* \subset (ST)^*,$$

with the equality occurring when $S \in B(H)$. If $S+T$ is densely defined, then

$$S^* + T^* \subset (S + T)^*$$

with the equality occurring when $S \in B(H)$.

Let $T$ be a linear operator (possibly unbounded) with domain $D(T)$ and let $B \in B(H)$. Say that $B$ commutes with $T$ if

$$BT \subset TB.$$ 

In other words, this means that $D(T) \subset D(TB)$ and

$$BTx = TBx, \forall x \in D(T).$$

If a symmetric operator $T$ is such that $\langle Tx, x \rangle \geq 0$ for all $x \in D(T)$, then we say that $T$ is positive, and we write $T \geq 0$. When $T$ is self-adjoint and $T \geq 0$, then we can define its unique positive self-adjoint square root, which we denote by $\sqrt{T}$.

If $T$ is densely defined and closed, then $T^* T$ (and $TT^*$) is self-adjoint and positive (a celebrated result due to von-Neumann, see e.g. [26]). So, when $T$ is closed then $T^* T$ is self-adjoint and positive whereby it is legitimate to define its square root. The unique positive self-adjoint square root of $T^* T$ is denoted by $|T|$. It is customary to call it the absolute value or modulus of $T$. If $T$ is closed, then (see e.g. Lemma 7.1 in [26])

$$D(T) = D(|T|) \text{ and } \|Tx\| = \||T|x\|, \forall x \in D(T).$$

Next, we recall some definitions of unbounded non-normal operators. A densely defined operator $A$ with domain $D(A)$ is called hyponormal if

$$D(A) \subset D(A^*) \text{ and } \|A^* x\| \leq \|Ax\|, \forall x \in D(A).$$

A densely defined linear operator $A$ with domain $D(A) \subset H$, is said to be subnormal when there are a Hilbert space $K$ with $H \subset K$, and a normal operator $N$ with $D(N) \subset K$ such that

$$D(A) \subset D(N) \text{ and } Ax = Nx \text{ for all } x \in D(A).$$
2. SOME APPLICATIONS TO THE COMMUTATIVITY OF SELF-ADJOINT OPERATORS

In [1], [6], [10], [11], [13], [14], [16], [17], [18], [22], [21], and [25], the question of the self-adjointness of the normal product of two self-adjoint operators was tackled in different settings (cf. [2]). In all cases, the commutativity of the operators was reached.

Here, we deal with the similar question where the unbounded (operator) factor is closed and symmetric which, and it is known, is weaker than self-adjointness. More precisely, we show that if $B \in B(H)$ is self-adjoint and $A$ is densely defined, closed and symmetric, then $BA \subseteq AB$ given that $AB$ or $BA$ is e.g. normal.

As in e.g. [16], the use of the Fuglede-Putnam theorem is primordial but just in the beginning of the proof. The desired conclusions do not come as straightforwardly as when the operator was self-adjoint. Thankfully, these technical difficulties have been overcome.

Now, recall the well-known Fuglede-Putnam theorem:

**Theorem 2.1.** ([8], [24]) If $A \in B(H)$ and if $M$ and $N$ are normal (non-necessarily bounded) operators, then

$$AN \subseteq MA \implies AN^* \subseteq M^*A.$$

There have been many generalizations of the Fuglede-Putnam theorem since Fuglede’s paper. However, most generalizations were devoted to relaxing the normality assumption. Apparently, the first generalization of the Fuglede theorem to an unbounded $A$ was established in [23]. Then the first generalization involving unbounded operators of the Fuglede-Putnam theorem is:

**Theorem 2.2.** If $A$ is a closed and symmetric operator and if $N$ is an unbounded normal operator, then

$$AN \subseteq N^*A \implies AN^* \subseteq NA$$

whenever $D(N) \subseteq D(A)$.

The previous result was established in [16] under the assumption of the self-adjointness of $A$. However, and by scrutinizing its proof in [16] or [17], it is seen that only the closedness and the symmetricity of $A$ were needed. This key observation is all what one needs to start the proof of some of the results below.

Let us also recall some perhaps known auxiliary results (cf. Lemmata 2.1 & 2.2 in [9]). See also [15] for the case of normality.

**Lemma 2.3.** ([14]) Let $A$ and $B$ be self-adjoint operators. Assume that $B \in B(H)$ and $BA \subseteq AB$. Then the following assertions hold:
We shall also have need for the following result:

**Lemma 2.4.** Let $B \in B(H)$ be self-adjoint. If $BA \subset AB$ where $A$ is closed, then $f(B)A \subset Af(B)$ for any continuous function $f$ on $\sigma(B)$. In particular, and if $B$ is positive, then $\sqrt{BA} \subset A\sqrt{B}$.

**Remark.** In fact, the previous lemma was shown in ([16], Proposition 1) under the assumption "$A$ being unbounded and self-adjoint", but by looking closely at its proof, we see that only the closedness of $A$ was needed (cf. [3] and [12]).

We are now ready to state and prove the first result of this section.

**Theorem 2.5.** Let $A$ be an unbounded closed and symmetric operator with domain $D(A)$, and let $B \in B(H)$ be positive. If $AB$ is normal, then $BA \subset AB$, and so $AB$ is self-adjoint. Also, $BA$ is self-adjoint.

Besides, $|A|B \subset |B|A$, and so $|A|B$ is self-adjoint and positive. Moreover, $|A|B = \overline{B|A|}$.

**Proof.** Since $B \in B(H)$ is self-adjoint, we have $(BA)^* = A^*B$ and $BA^* \subset (AB)^*$. Now, write $B(AB) = BAB \subset BA^*B \subset (AB)^*B$.

Since $AB$ and $(AB)^*$ are both normal, the Fuglede-Putnam theorem applies and gives

$$B(AB)^* \subset (AB)^**B = \overline{AB}B = AB^2,$$

i.e.

$$B^2A \subset B^2A^* \subset B(AB)^* \subset AB^2.$$

Since $A$ is closed and $B \in B(H)$ is positive, Lemma 2.4 gives $BA \subset AB$.

To show that $AB$ is self-adjoint, we proceed as follows: Observe that $BA \subset BA^* \subset (AB)^*$.

Since we also have $BA \subset AB$, we now know that $BAx = ABx = (AB)^*x$ for all $x \in D(A)$. This says that $AB$ and $(AB)^*$ coincide on $D(A)$. Denoting the restrictions of the latter operators to $D(A)$ by $T$ and $S$ respectively, it is seen that $T - S \subset 0$, $T \subset AB$, and $S \subset (AB)^*$. 
Hence
\[(AB)^* - AB \subset T^* - S^* \subset (T - S)^* = 0.\]
Since \(D(AB) = D[(AB)^*]\) thanks to the normality of \(AB\), it ensues that \(AB = (AB)^*\), that is, \(AB\) is self-adjoint.

Now, we show that \(BA\) is self-adjoint. First, we show that \(BA\) is normal. Clearly \(BA^* \subset A^*B\) for we already know that \(BA \subset AB\). Hence
\[BA^*A \subset A^*BA \subset A^*AB.\]
Therefore
\[BA(BA)^* = ABA^*B \subset AA^*B^2\]
and
\[(BA)^*BA = A^*BAB \subset A^*AB^2.\]
By Lemma 2.3, it is seen that both of \(AA^*B^2\) and \(AA^*B^2\) are self-adjoint. By the maximality of self-adjoint operators, it ensues that
\[BA(BA)^* = AA^*B^2\]
and \((BA)^*BA = A^*AB^2\). Therefore, we have shown that
\[(BA)^*BA = BA(BA)^*.\]
In other words, \(BA\) is normal.

To infer that \(BA\) is self-adjoint, observe that \(BA \subset AB\) gives \(BA \subset AB\), but because normal operators are maximally normal, we obtain \(BA = AB\), from which we derive the self-adjointness of \(BA\).

To show the last claim of the theorem, consider again \(BA^*A \subset A^*AB\). So, \(B|A| \subset |A|B\) by the spectral theorem say. Since \(B \geq 0\), Lemma 2.3 gives the self-adjointness and the positivity of \(|A|B\), as well as \(|A|B = B|A|\). This completes the proof. \(\square\)

**Remark.** Under the assumptions of the preceding theorem (by consulting [3]), we have:
\[|AB| = |BA| = |A|B = |B|A.\]

**Corollary 2.6.** Let \(A\) be an unbounded closed and symmetric operator and let \(B \in B(H)\) be positive. Suppose that \(AB\) is normal. Then
\[BA\] is closed \(\iff\) \(A\) is self-adjoint.
In particular, if \(B\) is invertible, then \(A\) is self-adjoint.
Proof. By Theorem 2.5, \( BA \) is self-adjoint and \( BA = AB \). Hence
\[
BA^* \subset (AB)^* = (BA)^* = BA.
\]
So, when \( BA \) is closed, \( BA^* \subset BA \). Therefore, \( D(A^*) \subset D(A) \), and so \( D(A) = D(A^*) \). Thus, \( A \) is self-adjoint, as required. \( \square \)

Corollary 2.7. Let \( A \) be an unbounded closed and symmetric operator with domain \( D(A) \), and let \( B \in B(H) \) be positive. If \( BA^* \) is normal, then \( BA \subset AB \), and so \( BA^* \) is self-adjoint.

Proof. Since \( BA^* \) is normal, so is \( (BA^*)^* = AB \). To obtain the desired conclusion, one just need to apply Theorem 2.5. \( \square \)

The case of the normality of \( BA \) was unexpectedly trickier. After a few attempts, we have been able to show the result.

Theorem 2.8. Let \( A \) be an unbounded closed and symmetric operator with domain \( D(A) \), and let \( B \in B(H) \) be self-adjoint. Assume \( BA \) is normal. Then \( A \) is necessarily self-adjoint.

If we further assume that \( B \) is positive, then \( BA \) becomes self-adjoint and \( \overline{BA} = BA \).

We are now ready to show Theorem 2.8.

Proof. First, recall that since \( BA \) is normal, \( BA \) is closed and \( D(BA) = D((BA)^*) \).

Write
\[
A(BA) \subset A^*BA = (BA)^*A.
\]
Since \( BA \) is normal and \( D(BA) = D(A) \), Theorem 2.2 is applicable and it gives
\[
A(BA)^* \subset (BA)^*A = BAA = BA^2,
\]
i.e. \( AA^*B \subset BA^2 \). Since \( A \) is symmetric, we may push the previous inclusion to further obtain \( AA^*B \subset BAA^* \), that is, \( |A^*|^2B \subset B|A^*|^2 \).

Next, we claim that \( B|A^*| \) is closed too. To see that, observe that as \( B \in B(H) \), then \( (BA)^* = A^*B \). Hence \( BA = (A^*B)^* \) or \( BA = (A^*B)^* \) because \( BA \) is already closed. By Lemma 11 in [5], the last equation is equivalent to \( (|A^*|^2B) \) which gives the closedness of \( B|A^*| \) as needed.

Now, we have
\[
B|A^*|(B|A^*|)^* = B|A^*|^2B \subset B^2|A^*|^2.
\]
It then follows by Corollary 1 in [7] that
\[
B|A^*|(B|A^*|)^* = B^2|A^*|^2
\]
for $B|A^*(B|A^*)|^* \in B$, and $|A^*|^2$ are all self-adjoint. The self-adjointness of $B|A^*(B|A^*)|^*$ also implies that $B^2|A^*|^2$ is self-adjoint as well, i.e.
\[
B^2|A^*|^2 = (B^2|A^*|^2)^* = |A^*|^2 B^2.
\]
In particular, $B^2|A^*|^2$ is closed. So, Proposition 3.7 in [6] implies that $B|A^*|^2$ is closed.

The next step is to show that $B|A^*|^2$ is normal. As $|A^*|^2 B \subset B|A^*|^2$, it ensues that
\[
B|A^*|^2 (B|A^*|^2)^* = B|A^*|^4 B \subset B^2|A^*|^4
\]
and
\[
(B|A^*|^2)^* B|A^*|^2 = |A^*|^2 B^2|A^*|^2 \subset B^2|A^*|^4.
\]
Since $B|A^*|^2 (B|A^*|^2)^*$, $(B|A^*|^2)^* B|A^*|^2$, $B^2$, and $|A^*|^2$ are all self-adjoint, Corollary 1 in [7] yields
\[
B|A^*|^2 (B|A^*|^2)^* = (B|A^*|^2)^* B|A^*|^2 \subset B^2|A^*|^4.
\]
Therefore, $B|A^*|^2$ is normal. So, since $B \in B(H)$ is self-adjoint and $|A^*|^2$ is self-adjoint and positive, it follows by Theorem 1.1 in [11] that $B|A^*|^2$ is self-adjoint and $B|A^*|^2 = |A^*|^2 B$.

By applying Theorem 10 in [3], it is seen that
\[
B|A^*| = |A^*| B
\]
due to the self-adjointness and the positivity of $|A^*|$.

We now have all the necessary tools to establish the self-adjointness of $A$. Indeed,
\[
D(A^*) = D(|A^*|) = D(B|A^*|) = D(|A^*|B) = D(A^*B) = D((BA)^*) = D(BA) = D(A).
\]
Thus, $A$ is self-adjoint as it is already symmetric.

Finally, when $B \in B(H)$ is positive and since $A$ is self-adjoint, $(BA)^* = AB$ is normal. By Theorem 2.5, $AB$ is self-adjoint or $(BA)^*$ is self-adjoint. In other words,
\[
BA = (BA)^* = AB,
\]
and this marks the end of the proof.}

Generalizations to weaker classes than normality vary. Notice in passing that in [3], the self-adjointness of $BA$ was established for a positive $B \in B(H)$ and an unbounded self-adjoint $A$ such that $BA$ is hyponormal and $\sigma(BA) \neq \mathbb{C}$. The next result is of the same kind.
**Proposition 2.9.** Let $B \in B(H)$ be positive and let $A$ be a densely defined closed symmetric operator. If $(AB)^* \text{ is subnormal or if } BA^* \text{ is closed and subnormal, then } BA \subset AB$.

Moreover, if $A$ is self-adjoint, then $AB$ is self-adjoint. Besides, $AB = BA$.

**Proof.** The proof relies on a version of the Fuglede-Putnam theorem obtained by J. Stochel in [27]. Write

$$B[(AB)^*]^* = B(AB)^* = BAB \subset BA^*B \subset (AB)^*B.$$  

Since $(AB)^*$ is subnormal, Theorem 4.2 in [27] yields

$$B^2A \subset B^2A^* \subset B(AB)^* \subset (AB)^*B = AB^2.$$  

The same inclusion is obtained in the event of the subnormality of $BA^*$. Indeed, write

$$B(AB^*)^* = BAB \subset BA^*B.$$  

Applying again Theorem 4.2 in [27] gives

$$B(AB^*) \subset (BA^*)^*B = AB^2.$$  

Therefore, and as above, we obtain $B^2A \subset AB^2$.

Now, since $B \geq 0$ and $A$ is closed, it follows that $BA \subset AB$.

Finally, when $A$ is self-adjoint, Lemma 2.3 implies that $AB$ is self-adjoint and $AB = BA$, as needed. □

There are still more cases to investigate. As is known, if $N \in B(H)$ is such that $N^2$ is normal, then $N$ need not be normal (cf. [20]). The same applies for the class of self-adjoint operators.

The first attempted generalization is the following: Let $A,B$ be two self-adjoint operators, where $B$ is positive, and such that $(AB)^n$ is normal for some $n \in \mathbb{N}$ such that $n \geq 2$. Does it follow that $AB$ is self-adjoint?

The answer is negative even when $A$ and $B$ are $2 \times 2$ matrices. This is seen next:

**Example 2.10.** Take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

Then $A$ is self-adjoint and $B$ is positive (it is even an orthogonal projection). Also,

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{whilst} \quad (AB)^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
for all \( n \geq 2 \). In other words, \( AB \) is not self-adjoint while all \( (AB)^n \), \( n \geq 2 \), are patently self-adjoint.

Let us pass to other possible generalizations.

**Proposition 2.11.** Let \( B \in B(H) \) be positive and let \( A \) be a closed and symmetric operator. Assume \( AB^n \) is normal for a certain positive integer \( n \in \mathbb{N} \). Then

1. \( BA \subset AB \) (hence \( BA \) is symmetric).
2. If it is further assumed that \( B \) is invertible, then \( A \) is self-adjoint. Besides, all of \( AB^{1/n} \) and \( B^{1/n}A \) are self-adjoint for all \( n \geq 1 \).

**Proof.**

1. Since \( B^n \) is positive for all \( n \) and \( AB^n \) is normal, it follows by Theorem 2.5 that \( AB^n \) is self-adjoint and \( B^nA \subset AB^n \). By Lemma 2.4, it is seen that \( BA \subset AB \).
2. Since \( AB^n \) is normal and \( B^nA \) is closed (as \( B^n \) is invertible), Corollary 2.6 yields the self-adjointness of \( A \).

Finally, since \( BA \subset AB \) and \( B \) is positive, it follows that \( B^{1/n}A \subset AB^{1/n} \), from which we derive the self-adjointness of \( AB^{1/n} \) and \( B^{1/n}A = B^{1/n}A \), as suggested.

\( \square \)

Similarly, we have:

**Proposition 2.12.** Let \( B \in B(H) \) be positive and let \( A \) be a closed and symmetric operator. Assume that \( B^nA \) is normal for some positive integer \( n \in \mathbb{N} \). Then \( A \) and \( BA \) are self-adjoint, and \( BA = AB \).

One of the tools to prove this result is:

**Lemma 2.13.** (Cf. Proposition 3.7 in [6]) Let \( B \in B(H) \) and let \( A \) be an arbitrary operator such that \( B^nA \) is closed for some integer \( n \geq 2 \). Suppose further that \( BA \) is closable. Then \( BA \) is closed.

**Proof.** Let \( (x_p) \) be in \( D(B^nA) \) and such that \( x_p \to x \) and \( BA x_p \to y \). Since \( B^{n-1} \in B(H) \), \( B^nA x_p \to B^{n-1}y \). Since \( B^nA \) is closed, we obtain \( x \in D(B^nA) = D(A) \). Since \( BA \subset \overline{BA} \) and \( x \in D(BA) \), we have

\[
BAx = \overline{BAx} = \lim_{p \to \infty} BA x_p = y
\]

by the definition of the closure of an operator. We have therefore shown that \( BA \) is closed, as wished.

\( \square \)

Now, we show Proposition 2.12.
Proof. Since $B^n$ is positive, Theorem 2.8 gives both the self-adjointness of $A$ and $B^n A$. Moreover, $B^n A = AB^n$. Using Lemma 2.4 or else, we get $BA \subset AB$ (only the inclusion suffices to finish the proof). The equation $B^n A = AB^n$ contains the closedness of $B^n A$ which, by a glance at Lemma 2.13 yields $BA = AB$ by consulting Lemma 2.3. □

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