Lie point symmetry reductions of Bondi’s radiating metric

S. Dimas, D. Tsoubelis and P. Xenitidis
University of Patras, Department of Mathematics
Rio 26 500 - Greece
E-mail: spawn@math.upatras.gr, tsoubeli@math.upatras.gr, xeniti@math.upatras.gr

Abstract. The Lie point symmetries of the Einstein vacuum equations corresponding to the Bondi form of the line element are presented. Using these symmetries, we study reductions of the field equations, which might lead to new asymptotically flat solutions, representing gravitational waves emitted by an isolated source.

1. Introduction
Most of the exact solutions of the Einstein field equations constructed so far depend on two independent variables, only. This is the case, in particular, with the whole class of stationary and axially symmetric metrics, which cover the physically most realistic solutions. However, in order to arrive at a general relativistic description of dynamic gravitational phenomena, we must free ourselves from ... the two-dimensional chains.

The evolution of a compact object, such as a star, is the prime example of the above kind of phenomena. It is a process expected to be accompanied by the emission of gravitational radiation and, as a result, it is of special interest in connection of the intense current effort, on an international scale, to detect gravitational waves.

The foundations for constructing models of gravitational waves emitted by a compact source were laid down by the pioneering work of Sir Herman Bondi and his collaborators in the 1960s [1]: In a nutshell, the results of their investigations can be described by the following set of requirements:

(i) In the axially symmetric case, the space-time region outside the radiating object is described by the line element

\[ ds^2 = \left( \frac{e^{2\beta}V}{r} - r^2 e^{2\gamma}U^2 \right) du^2 + 2e^{2\beta} dudr + 2r^2 e^{2\gamma}Udud\theta - r^2 \left( e^{2\gamma}d\theta^2 + \sin^2 \theta e^{-2\gamma}d\phi^2 \right) . \]

This is referred to as Bondi’s radiating metric, and \( \beta, \gamma, U \) and \( V \) represent functions depending only on \( u, r, \) and \( \theta \).
(ii) The latter must comply to the boundary conditions
\[ \beta \to 0, \quad \gamma \to 0, \quad r \, U \to 0, \quad \frac{V}{r} \to 1, \quad (r \to \infty) \]
which insure that the space-time is asymptotically flat. In other words, as \( r \to \infty \), the line element must approach
\[ ds^2 = du^2 + 2dudr - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]
which is but the Minkowski line element
\[ ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \]
in the retarded, spherical coordinates \((u, r, \theta, \phi)\) defined by
\[ t = u + r, \quad z = r \cos \theta, \quad x + y = r \sin \theta e^\phi. \]

(iii) For \( u_0 \leq u \leq u_1, \quad r_0 \leq r \leq \infty, \quad 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi \), one can expand all metric components in powers of \( r^{-1} \) with at most a finite pole at \( r = \infty \), and the resulting series can be added, multiplied, differentiated, etc., freely.

(iv) As \( \sin \theta \to 0, \beta, V, U/\sin \theta \) and \( \gamma/\sin^2 \theta \) should be regular functions of \( \cos \theta \). This regularity at the axis condition is imposed by the fact that the axis of rotational symmetry is free of matter.

Unfortunately, the vacuum field equations corresponding to Bondi’s radiating metric are so complicated that no exact solution of really dynamic character has been obtained yet. Hoping that symmetry analysis could help in breaking out of this impasse, we have started a systematic investigation of the symmetries admitted by the above system. In the following sections, we present the first results of our effort: The Lie point symmetries of the Einstein vacuum equations, the corresponding reductions, and some known solutions that can be obtained from the reduced system.

2. Einstein’s vacuum field equations
In the case of the Bondi metric, the vacuum field equations split into the following two subsystems:

(a) **The four main equations**
\[
2\beta_{,r} - r\gamma_{,r}^2 = 0 \\
\rho^2 U_{,r}(4 - 2r\beta_{,r} + 2r\gamma_{,r}) + r^3 U_{,rr} + e^{2(\beta - \gamma)} \left\{ 2\beta_{,\theta} + r[2(\cot \theta - \gamma_{,\theta})\gamma_{,r} - \beta_{,\theta r} + \gamma_{,r\theta}] \right\} = 0 \\
e^{2(\beta - \gamma)}[-1 + \beta_{,\theta}^2 + \beta_{,\theta}(\cot \theta - 2\gamma_{,\theta}) - 3 \cot \theta \gamma_{,\theta} + 2\gamma_{,\theta}^2 + \beta_{,\theta \theta} - \gamma_{,\theta \theta}] + \\
e^{-2(\beta - \gamma)}r^3 U_{,r}^2 + 4V_{,r} - 2r[4 \cot \theta U + 4U_{,\theta} + r(\cot \theta U_{,r} + U_{,r\theta})] = 0 \\
e^{2(\beta - \gamma)}[-1 + 2\beta_{,\theta}(\cot \theta - \gamma_{,\theta}) - 3 \cot \theta \gamma_{,\theta} + 2\gamma_{,\theta}^2 - \gamma_{,\theta \theta}] + V_{,r} - V_{,r} + \\
r U_{,\theta}(-1 + r\gamma_{,r}) + rU[2\gamma_{,\theta} + \cot \theta(-3 + r\gamma_{,r}) + 2r\gamma_{,r\theta}] - r[r(\cot \theta - \gamma_{,\theta})U_{,r} + \\
V_{,r}\gamma_{,r} + V_{,rr} - 2(\gamma_{,u} + r\gamma_{,ur})] = 0
\]
(b) The two supplementary equations

\[ U U^2 r^6 - e^{2(\beta - \gamma)} \csc \theta \{ r \cos \theta [r U_x + 2U(r \gamma_r + 1)] + 4U^2 (\gamma_\theta + r \gamma_r \theta) r + \
\sin \theta \{2U_x (U_\theta - \beta_{uu} + \gamma_{u}) + U_{ur}] r^2 + V[2U_x (r \beta_r - r \gamma_r - 2) - r U_{rr}] + \\
U \{r[-2r(\beta_{u} - 2\gamma_{u})]U_r - 2V_x \gamma_r + 2U_\theta (r \gamma_r + 3) + 3r U_{r \theta} - 2V \gamma_{rr} + 4\gamma_{u} + 4r \gamma_{ur} - \}
\]
\[2(V_x + V \gamma_r)\} \} r^2 + e^{4(\beta - \gamma)} \{V_{\beta}(2r \beta_r - 2r \gamma_r - 1) + \\
\}
\]
\[r \{V_{r \theta} + 2r \{U[\beta_{u} (\cot \theta + 2 \beta_{u} - 2 \gamma_{u}) + \beta_{u \theta}] + 2(\cot \theta - \gamma_{u}) \gamma_{u} - \beta_{u \theta} + \gamma_{u \theta}\} \} = 0 \\
\]
\[-e^{2(\gamma - \beta)} V U_x^2 r^4 + 2e^{2(\gamma - \beta)} U^3 [(r \gamma_r + 1) \cot \theta + 2 \gamma_{u} + 2r \gamma_{u \theta}] r^4 + \\
U^2 \{ - e^{4(\gamma - \beta)} U_{x}^2 r^4 - 2[2 \beta_{u} (\beta_\theta - \gamma_{u}) + \cot \theta \gamma_{u} + \gamma_{u \theta}] + \\
2e^{2(\beta - \gamma)} \{ - V_x - V \gamma_r + r \{r \cos \theta \beta - 3 \gamma_{u} U_x - \\
V_x \gamma_r + U_\theta (r \gamma_r + 3) - V \gamma_{rr} + 2\gamma_{u} + 2r (U_{r \theta} + \gamma_{u})\} \} r^3 + \\
e^{2(\gamma - \beta)} \{ 2r^2 \{[2U_x (U_{r} - \beta_{uu} + \gamma_{u}) + U_{ur}] r^2 + V[2U_x (r \beta_r - r \gamma_r - 2) - r U_{rr}] \} - \\
e^{2(\beta - \gamma)} \{ - 4V_{\beta} + V[4 \beta_{\theta} + \cot \theta (2r \beta_r - 1) + 4r \beta_{r \theta}] + \\
r \{2U_\theta (\cot \theta - 2 \beta_{u} + 2 \gamma_{u}) + \cot \theta V_x + \\
4V_\theta (\beta_r - \gamma_r) + 2r \{U_{r \theta} + 2[\cot \theta \beta_{u} - \beta_{u \theta} + \gamma_{u \theta}]\} \} \} r + \\
e^{2(\gamma - \beta)} [(V_\theta + 2V \beta_\theta) \{ \cot \theta + 2 \beta_{\theta} - 2 \gamma_{u} \} + V_{\theta \theta} + 2V \beta_{u \theta}] - \\
\csc \theta \{ 2 \cos \theta \beta_\theta U_{x}^3 + \sin \theta ( - 2( \beta_r + r \beta_{r \theta}) V^2 + \\
r [2r \beta_{u} U_{r} - 2V_x \beta_r + U_\theta (2r \beta_r - 1) - V \gamma_{rr} + 4 \beta_{u} + 4r \beta_{ur}] V + \\
r \{2r U_{r \theta}^2 + (V_x - 4r \beta_{u} + 4r \gamma_{u}) U_\theta + 4r \gamma_{u}^2 - V \theta U_x + 2r U_{u \theta} - 2V \theta \} \} = 0 \\
\]

The first objective of the symmetry analysis is to determine the point transformations

\[(x^a, g_{ab}) \rightarrow (\tilde{x}^a, \tilde{g}_{ab})\]

that leave the above set of equations invariant. A short outline of the way this group of transformations can be determined and exploited in constructing exact solutions is presented in the following section.

3. Symmetries of systems of PDEs

Let \((x, u)\) denote an arbitrary point of the space \(\mathbb{R}^n \times \mathbb{R}^m\). In other words, let

\[(x, u) := (x^1, \ldots, x^n, u^1, \ldots, u^m).\]

Then the vector field

\[X = \xi^i(x, u) \partial_{x^i} + \eta^\mu(x, u) \partial_{u^\mu}, \quad i = 1, 2, \ldots, n; \quad \mu = 1, 2, \ldots, m,\]

where

\[\partial_{x^i} := \frac{\partial}{\partial x^i}, \quad \partial_{u^\mu} := \frac{\partial}{\partial u^\mu},\]

determines an infinitesimal transformation of the following form:

\[(\tilde{x})^i = x^i + \varepsilon \xi^i(x, u) + \cdots, \quad (\tilde{u})^\mu = u^\mu + \varepsilon \eta^\mu(x, u) + \cdots.\]
When the $u^\mu$ are considered to be functions of the coordinates $x^i$, the above transformation in the space $\mathbb{R}^n \times \mathbb{R}^m$ (of the independent and dependent variables) can be extended to the partial derivatives of the $u^\mu$ with respect to the $x^i$. Specifically,

$$\tilde{u}^\mu_{ij} = u^\mu_{ij} + \varepsilon \eta^\mu_i(x, u, \partial u) + \cdots$$

where

$$\eta^\mu_i(x, u, \partial u) = D_i \eta^\mu - u^\mu_j D_i \xi_j^i.$$  

In the last expression $D_i$ denotes the total derivative operator. Thus, from the fact that $\eta^\mu = \eta^\mu(x, u)$ it follows that

$$D_i \eta^\mu \equiv \frac{D \eta^\mu}{D x^i} = \partial_{x^i} \eta^\mu + (\partial_{u^\nu} \eta^\mu) u^\nu_j.$$  

This extension process can be generalized easily to the derivatives of all higher orders. For example,

$$\tilde{u}^\mu_{ij} = u^\mu_{ij} + \varepsilon \eta^\mu_{ij}(x, u, \partial u, \partial^2 u) + \cdots,$$

where

$$\eta^\mu_{ij}(x, u, \partial u, \partial^2 u) = D_j \eta^\mu_i - u^\mu_k D_j \xi^k.$$  

Now $\eta^\mu_i = \eta^\mu_i(x, u, \partial u)$, i.e. the coefficients $\eta^\mu_i$ depend, in general, on the first derivatives of the functions $u^\nu$. Therefore,

$$D_j \eta^\mu_i \equiv \frac{D \eta^\mu_i}{D x^j} = \frac{\partial \eta^\mu_i}{\partial x^j} + \frac{\partial \eta^\mu_i \partial u^\nu}{\partial x^j} + \frac{\partial \eta^\mu_i \partial u^\nu_k}{\partial x^j}.$$  

Equivalently,

$$D_j \eta^\mu_i = \partial_{x^j} \eta^\mu_i + (\partial_{u^\nu} \eta^\mu_i) u^\nu_j + (\partial_{u^\nu_k} \eta^\mu_i) u^\nu_{k,j}.$$  

A system of $S$ partial differential equations (PDEs) for the functions $u^\mu$ can be written as

$$H_A(x, u, \partial u, \partial^2 u, \cdots, \partial^p u) = 0, \quad A = 1, 2, \ldots, S.$$  

It is said to be invariant under the one-parameter group of transformations

$$(x, u) \rightarrow (\tilde{x}, \tilde{u}) = G(x, u, \varepsilon), \quad G(x, u, 0) = (x, u)$$

if

$$H_A(\tilde{x}, \tilde{u}, \partial \tilde{u}, \partial^2 \tilde{u}, \cdots, \partial^p \tilde{u}) = 0.$$  

Then, the above transformation is called a **Lie point symmetry** of the given system of PDEs.

At the infinitesimal level, the condition

$$X H_A = 0 \quad \text{(mod } H_A = 0)$$

is necessary and sufficient for the transformation $(x, u) \rightarrow (\tilde{x}, \tilde{u})$ to be a symmetry of the system $H_A = 0$. Here, $X$ denotes the extension of the vector field determined by $(x, u) \rightarrow (\tilde{x}, \tilde{u})$ to the order $p$ of the system (the order of the highest derivative of $u^\mu$ that appears in the system), i.e.

$$X = \xi^i(x, u) \partial_{x^i} + \eta^\mu_i(x, u) \partial_{u^\mu} + \eta^\mu_j(x, u, \partial u) \partial_{u^\nu_j} + \cdots + \eta^\mu_{ij \cdots p}(x, u, \partial u, \partial^2 u, \cdots, \partial^p u) \partial_{u^\nu_{ij \cdots p}}.$$  

In the case of the Einstein vacuum equations,

$$ H_A = 0 \iff R_{ij} = 0, $$

where $R_{ij}$ denotes the Ricci tensor. Now the unknown functions $u^\mu$ are the components $g_{ij}$ of the metric tensor, so that the generator of a Lie point symmetry of Einstein’s equations should be written as

$$ X = \xi^i(x, g) \partial_{x^i} + \eta_{ij}(x, g) \partial_{g_{ij}}. $$

It turns out [3] that the above symmetry generator is restricted to the following form, where $a$ is an arbitrary real number:

$$ X = \xi^i(x) \partial_{x^i} + (g_{ik} \xi^k_j + g_{jk} \xi^k_i - 2a g_{ij}) \partial_{g_{ij}}. $$

Once (some of) the symmetries of a given system have been determined, one can exploit them in one of the following ways:

(i) To generate new solutions from known, simpler ones.
(ii) To find solutions that remain invariant under the action of the symmetry group.

The latter are called similarity solutions and the method of their construction is referred to as similarity reduction of the system.

If one represents the solution of the system $H_A = 0$ as the $n$-dimensional submanifold

$$ u^\mu = \Phi^\mu(x), \ \mu = 1, 2, \ldots, m. $$

of the $\mathbb{R}^n \times \mathbb{R}^m$ space, then the condition of its invariance under the action of the vector field

$$ X = \xi^i(x, u) \partial_{x^i} + \eta^\mu(x, u) \partial_{u^\mu} $$

reads

$$ \eta^\mu(x, \Phi^\nu(x)) = \xi^i(x, \Phi^\nu(x)) \partial_i \Phi^\mu(x). $$

This set of first order quasilinear equations for the functions $\Phi^\mu$ can, in principle, be solved. The substitution of the result into the original system of PDEs leads to the latter’s similarity or group invariant solutions.

In the case of the Einstein vacuum equations the above invariant surface condition becomes

$$ \mathcal{L}_\xi g_{ij} := g_{ij, k} \xi^k + g_{ik} \xi^k_j + g_{jk} \xi^k_i = 2a g_{ij} $$

where $\mathcal{L}_\xi g_{ij}$ denotes the Lie derivative of the metric tensor. This relation shows that the similarity solutions of Einstein’s equations are those which admit a Killing vector ($a = 0$) or a homothetic vector ($a \neq 0$). “This connection with the symmetries explains the outstanding rôle Killing vectors and homothetic vectors play compared with other vector fields.” [2], p.132.

### 4. Symmetries Einstein’s vacuum equations for the Bondi metric

In order to find the symmetries of a given system of PDEs, one has to set up and solve the system’s determining equations. The latter is a (usually, very large) system of linear PDEs for the $n + m$ components $\xi^i(x, u), \eta^\mu(x, u)$ of the generator

$$ X = \xi^i(x, u) \partial_{x^i} + \eta^\mu(x, u) \partial_{u^\mu}. $$
In the case of Einstein’s vacuum equations, the determining equations concern the four functions $\xi^i(x)$. The system of determining equations is specified in an algorithmic way. However, this can be done by hand only in very simple cases, involving scalar PDEs of low order. In all other cases one has to turn to algebraic computing packages which have been specifically developed for that purpose. One such package, operating in the context of Mathematica, has been developed by our group recently. It is called SYM [4] and is freely available upon request. So far, it has been proven very effective in both obtaining and solving systems of the determining equations of very complicated systems of linear and nonlinear PDEs.

This includes the sixty (60) determining equations corresponding to the Einstein vacuum equations for the Bondi metric:

$$
\xi_u^2 = 0, \xi_r^1 = 0, \xi_r^2 = 0, \eta_{1, r} = 0, \eta_{2, r} = 0, \eta_3, r = 0, \eta_2, r = 0, \eta_1, r = 0, \eta_2, \rho = 0, \xi_3, \rho = 0, \\
\eta_{2, \rho} = 0, \eta_{1, u, u} = 0, \xi_{1, U}^3 = 0, \xi_{1, U}^1 = 0, \eta_{1, u} = 0, \eta_{3, U} = 0, \eta_{2, U} = 0, \xi_{1, U}^1 = 0, \\
\xi_{U}^1 = 0, \eta_{4, \nu} = 0, \eta_{5, \nu} = 0, \nu_{1, \nu} = 0, \xi_{3, \nu}^1 = 0, \xi_{3, \nu}^2 = 0, \xi_{3, \nu}^3 = 0, \xi_{3, \nu} = 0, \eta_{3, \nu} = 0, \\
\eta_{2, \beta} = 0, \eta_{1, \beta} = 0, \xi_{3, \beta} = 0, \xi_{3, \beta} = 0, \eta_{4, \gamma} = 0, \eta_{3, \gamma} = 0, \eta_{2, \gamma} = 0, \eta_{1, \gamma} = 0, \\
r \xi_u^2 - \xi^2 = 0, V \eta_{2, \nu} - \eta_2 = 0, r \eta_2 - 2 V \xi^2 + r V \xi_u^1 = 0, \\
- \csc \theta \sec \theta \xi^3 - \tan \theta \xi_{1, \theta} + \xi^3, \theta = 0, 2 \cot \theta \eta_{1, u} + \xi^3, u + \eta_{4, u} \theta = 0, \\
4 \csc(2 \theta) \xi^3 + (\cos(2 \theta) + 5) \csc(2 \theta) \eta_{4, \theta} + \eta_{4, \theta} \theta = 0, V(2 \eta_{3, \theta} + \xi^2, \theta) - r \eta_{2, \theta} = 0, \\
r \eta_2 - V(2 \eta_{3} - 2 \eta_{4} + \xi^2 - 2 r \xi^3, \theta) = 0, V(-2 r \eta_{3} + 2 \eta_{4} + 3 \xi^2 + 2 r \eta_{1, U}) - r \eta_2 = 0, \\
- \cot \theta \eta_1 + U \cot \theta \eta_{1, U} - \eta_{1, \theta} + U \eta_{1, \theta} U = 0, \\
2 \tan \theta \eta_{3, u} - 2 \tan \theta \eta_{4, u} + \xi^3, u - \tan \theta \eta_{1, a} U = 0, \\
r V \cot \theta \eta_1 - r U \cot \theta \eta_2 + V(-r U \xi^3 \cot^2 \theta + 2 U \cot \theta \xi^2 + r \eta_{1, \theta} = 0, \\
-2 r V \eta_1 + r U \eta_2 + V(2 r U \eta_{3} - 2 r U \eta_{4} - 3 U \xi^2 + 2 r \xi^3, u) = 0, \\
r \cot \theta \eta_2 + V(2 r \xi^3 \cot^2 \theta - 2 r \cot \theta \eta_{3} + 2 r \cot \theta \eta_4 - \cot \theta \xi^2 + 2 r \eta_{4, \theta} = 0, \\
-2 r V \cot \theta \eta_1 + r U \cot \eta_2 + \\
V(2 r U \cot \theta \eta_3 - 2 r U \cot \theta \eta_{4} - 3 U \cot \theta \xi^2 + 2 r \eta_{4, u} = 0, \\
cot \theta \eta_2 + V(3 \xi^3 \cot^2 \theta - 4 \cot \theta \eta_{3} + 4 \cot \theta \eta_4 + \cot \theta \eta_{1, U} + 2 r \eta_{4, \theta} - \eta_{1, \theta}) U = 0, \\
2 r V \cot \theta \eta_1 - r U \cot \eta_2 + V(-2 r U \cot \theta \eta_{3} + 2 r U \cot \theta \eta_{4} + 3 U \cot \theta \xi^2 - r \cot \theta \xi^3, u + \xi^3, u \theta) = 0, \\
r \eta_1 - U(2 r \eta_{3} + 2 r \eta_{4} + \xi^2 + r \csc \theta \sec \theta \xi^3 + r \tan \eta_{4, \theta} = 0 + 2 r \cot \theta \eta_{4, u} + r \eta_{4, u} \theta = 0, \\
2 e^{2 \gamma} r U \cot \theta \eta_2 + V(-4 e^{2 \gamma} r U \cot \theta \eta_{3} + 4 e^{2 \gamma} r U \cot \theta \eta_{4} - 2 e^{2 \gamma} r U \cot \theta \xi^2 + 3 e^{2 \beta} \cot \theta \eta_{4, \theta} + 2 e^{2 \gamma} \xi^3, \theta + 4 e^{2 \gamma} r U \cot \theta \xi^3, \theta + e^{2 \theta} \eta_{4, \theta} = 0, \\
2 r V \cot \theta \eta_1 - r U \cot \eta_2 - 2 r U V \cot \theta \eta_{3} + 2 r U V \cot \theta \eta_{4} + 3 V \cot \theta \xi^2 - 3 r U \eta_{2, \theta} - 2 r U V \eta_{3, \theta} + 7 U V \xi^2, \theta - 2 r \eta_{2, u} + 4 V \xi^2, u + 2 r V \eta_{1, a} U = 0.
5. Similarity reductions

The solution of the above system determines the Lie point symmetries of the Einstein vacuum equations for the Bondi metric. In terms of the generators of the corresponding symmetry algebra it reads

\[
\begin{align*}
X_1 &= \partial_\beta + \partial_\gamma, \\
X_2 &= \partial_\beta + 2r\partial_r + 4V\partial_V, \\
X_3 &= \partial_\beta - \cot \theta \partial_\theta - \cot^2 \theta \partial_\gamma + U \csc^2 \theta \partial_U, \\
X_4 &= F_1(u)\partial_u - F'_1(u)\left[\frac{1}{2}\partial_\beta + U\partial_U + V\partial_V\right], \\
X_5 &= \csc \theta F_2(u)\partial_\theta + \cot \theta \csc \theta F_2(u)\partial_\gamma + \csc \theta [F'_2(u) - U \cot \theta F_2(u)]\partial_U,
\end{align*}
\]

where \( F_i(u) \) are arbitrary functions of \( u \).

Here, \( \xi^i \) and \( \eta^i \) are functions of \((u, r, \theta, \beta, \gamma, U, V)\).

The solution of the above system respects the boundary conditions at infinity, 

\[
\beta \to 0, \quad \gamma \to 0, \quad r U \to 0, \quad \frac{V}{r} \to 1,
\]

is given by 

\[
X = \partial_u + \lambda (r \partial_r + u \partial_u - U \partial_U + V \partial_V)
\]

where \( \lambda \) is an arbitrary real parameter.

Using this symmetry, the field equations reduce to the following system

**The main equations**

\[
\begin{align*}
\zeta F_{1,\zeta}^2 - 2F_{2,\zeta} = 0 \\
e^{2F_1} F_{4,\zeta}^3 - 2e^{2F_1} (\zeta F_{2,\zeta} - 2) F_{4,\zeta}^2 + 2F_{1,\zeta} (e^{2F_1} F_{4,\zeta}^2 + 2e^{2F_2} (\cot \theta - F_{1,\theta})^3 \zeta + 2e^{2F_2} F_{1,\theta}^2 \zeta - 2e^{2F_2} F_{2,\theta} \zeta + 4e^{2F_2} F_{2,\theta} \zeta = 0 \\
e^{4F_1} \sin \theta F_{4,\zeta}^2 - 2e^{2F_1 + F_2} \cos \theta F_{4,\zeta}^2 - 2e^{2F_1 + F_2} \sin \theta F_{4,\zeta}^2 - 8e^{2F_1 + F_2} \cos \theta F_{4,\zeta}^2 - 8e^{2F_1 + F_2} \sin \theta F_{4,\zeta}^2 + 8e^{2F_2} \sin \theta F_{1,\theta} - 4e^{2F_2} \cos \theta F_{1,\theta} + 4e^{2F_2} \cos \theta F_{2,\theta} - 8e^{2F_2} \sin \theta F_{1,\theta} + 4e^{2F_2} \sin \theta F_{1,\theta} - 4e^{2F_2} \sin \theta F_{2,\theta} - 4e^{2F_2} \sin \theta F_{1,\theta} + 4e^{2F_2} \sin \theta F_{2,\theta} = 0 \\
e^{2F_1} \lambda F_{1,\zeta}^3 + e^{2F_1} \cot \theta F_{4,\zeta}^2 - 2F_{1,\zeta} F_{4,\zeta} F_{1,\theta}^2 + 2e^{2F_1} F_{1,\zeta} F_{1,\theta} + e^{2F_1} F_{3,\zeta} + e^{2F_1} F_{4,\zeta} - \end{align*}
\]
In the process of analyzing further the above system, we were led to the following results.

\[ e^{2F_1} F_4 (\zeta F_{1,\zeta} \cot \theta - 3 \cot \theta + 2 F_{1,\theta} + 2 \zeta (F_{1,\theta}) \zeta + e^{2F_2} - 2 e^{2F_2} F_{1,\theta}^2 - e^{2F_1} F_{3,\zeta} + 3 e^{2F_2} \cot \theta F_{1,\theta} - 2 e^{2F_2} \cot \theta F_{2,\theta} + e^{2F_1} F_{1,\zeta} (F_3 + \zeta (4 \zeta \lambda + F_{3,\zeta} - \zeta (F_{4,\theta}))) + e^{2F_2} F_{1,\theta} = 0 \]

The supplementary equations

\[
8 e^{2(F_1+F_2)} \cot \theta (F_{4})^2 \zeta - 3 - F_4 (e^{4F_1} F_{4,\zeta}^2 \zeta^4 - 4 e^{2(F_1+F_2)} F_{4,\zeta} (\cot \theta + F_{1,\theta} - F_{2,\theta}) \zeta^2 + 4 e^{2(F_1+F_2)} F_{3,\zeta} + 4 e^{2F_2} (2+F_2) (\cot \theta - F_{1,\theta}) F_{2,\theta} - e^{2F_1} (2 F_{4,\theta} + \zeta (F_{4,\theta}))) \zeta^2 + 2 e^{2F_1} (- e^{2F_1} F_{3,\zeta}^2 + 2 e^{2F_1} F_{4,\zeta} (\zeta F_{2,\zeta} - \lambda + F_{3,\theta})) \zeta^4 + 2 e^{2F_2} \lambda F_{1,\theta} \zeta^3 - 2 e^{2F_2} F_{1,\theta} \zeta^3 - 2 e^{2F_2} \zeta^2 F_{3,\theta} \zeta^2 + 2 e^{2F_2} F_{3,\theta} \zeta - 2 F_{1,\zeta} (e^{2F_1} F_{1,\zeta} - e^{2F_2} (2 \lambda \cot \theta^2 + 2 \lambda F_{1,\theta} F_{1,\zeta} + e^{2F_2} F_{3,\theta} = 0
\]

\[
-4 e^{2(F_1+F_2)} (\zeta F_{1,\zeta} - 1) (F_{1,\zeta} + \zeta F_{1,\zeta}) F_{3} - (3 e^{4F_1} F_{4,\zeta}^2 \zeta^4 - 8 e^{2(F_1+F_2)} \lambda F_{1,\zeta} \zeta^3 + 8 e^{2(F_1+F_2)} F_{1,\zeta} \zeta^2 - 2 e^{2(F_1+F_2)} F_{4,\zeta} F_{1,\theta} \zeta^2 - 4 e^{2(F_1+F_2)} F_{4,\zeta} F_{2,\theta} \zeta^2 + 2 e^{2(F_1+F_2)} F_{3,\zeta}^2 (4 \zeta \lambda + F_{3,\theta} - F_{4,\theta} \zeta^2 + 2 e^{2(F_1+F_2)} F_{4,\theta} \zeta^2 + 4 e^{2(F_1+F_2)} \lambda \zeta + 2 e^{2(F_1+F_2)} F_{1,\theta} \zeta^3 + 6 e^{2(F_1+F_2)} F_{1,\zeta} \zeta^2 F_{4,\theta} \zeta + 4 e^{2(F_1+F_2)} F_{1,\zeta} \zeta^2 (2 \lambda F_{1,\zeta} \zeta^2 - 4 \lambda \zeta + F_{4,\theta} \zeta - F_{3,\zeta}) \zeta - 2 e^{2(F_1+F_2)} F_{4,\zeta} \zeta^2 \cot \theta F_{1,\zeta} + 2 \zeta (2 F_{1,\theta} F_{1,\zeta} \cot \theta) F_{1,\zeta} + \cot \theta - 4 F_{1,\theta} + 4 F_{2,\theta} - 4 F_{1,\theta} \zeta + e^{4F_1} F_{2,\theta} \zeta + 8 e^{4F_2} (\cot \theta - 8 e^{4F_2} F_{1,\theta} F_{2,\theta}) F_{3} - e^{2F_2} \csc \theta
\]

\[
(4 \zeta F_{1,\theta} \zeta + 4 e^{4F_1} F_{2,\theta} \zeta + 8 e^{4F_2} \cot \theta F_{2,\theta} \zeta - 8 e^{4F_2} F_{1,\theta} F_{2,\theta}) F_{3} - e^{2F_2} \csc \theta
\]

Here, \( F_i = F_i (\zeta, \theta) \) and \( \zeta = \frac{r}{1 + \lambda u} \).

6. Conclusions

In the process of analyzing further the above system, we were led to the following results.

(i) The Schwarzschild solution is contained in the solution set of the above reduced system, and can be obtained by demanding that the functions involved are independent of the angular variable \( \theta \).

(ii) The solution based on the Bondi form of the metric, which was given by Hobill [5], can be obtained from our reduced system above via the ansatz

\[
F_3(\zeta, \theta) = e^{2(F_2-F_1)} f_3(\zeta, \theta) + f(\theta), \quad F_4(\zeta, \theta) = e^{2(F_2-F_1)} f_4(\zeta, \theta)
\]

together with the assumption that, in the resulting equations the coefficients of different powers of \( \exp(F_2 - F_1) \) are identically zero.
The above results strengthen our hope that an asymptotically flat and regular on the symmetry axis solution can be obtained from the Lie point symmetry reduction presented above. If this hope does not materialize, one would have to turn for help to other kind of symmetries (potential, conditional, generalized or Bäcklund, etc.) that are probably admitted by the Bondi metric.

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