A NOTE ON COLLAPSIBILITY OF ACYCLIC 2-COMPLEXES

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Abstract. We present a Morse-theoretic characterization of collapsibility for 2-dimensional acyclic simplicial complexes by means of the values of normalized optimal combinatorial Morse functions.

Let $K$ be a finite connected simplicial complex and let $f : K \to \mathbb{R}$ be a discrete Morse function over $K$. The normalization of $f$ is the function $h_f : K \to \mathbb{Z}_{\geq 0}$ equivalent to $f$ satisfying $h_f(\sigma) \leq g(\sigma)$ for every $\sigma \in K$ whenever $g : K \to \mathbb{Z}_{\geq 0}$ is a combinatorial Morse function equivalent to $f$ (see [1]). The purpose of this note is to give a characterization of collapsibility for 2-dimensional acyclic simplicial complexes by means of the values of $h_f$.

**Lemma I** ([1, Lemma 7]). Let $f$ be a combinatorial Morse function over $K$. The normalization $h_f$ of $f$ satisfies:

1. $h_f(\sigma) \geq \dim(\sigma)$ for all $\sigma \in K$.
2. $h_f(\sigma) = 0$ if and only if $\sigma$ is a critical vertex for $f$.
3. If $\sigma \prec \tau$ and $f(\sigma) \geq f(\tau)$ then $h_f(\sigma) = h_f(\tau)$.
4. $h_f$ takes all the integer values between 0 and $\max_{\sigma \in K} h_f(\sigma)$.

For a given discrete Morse function $f : K \to \mathbb{R}$ consider the number

$$\mathcal{N}(K,f) := \sum_{\sigma \in K} (-1)^{\dim(\sigma)} h_f(\sigma).$$

This definition is motivated by property (3) of Lemma I, which in turn implies that the sum may be taken over the critical simplices alone. We have the following result.

**Proposition II.** If $K$ is collapsible then there exists a combinatorial Morse function $f : K \to \mathbb{R}$ such that $\mathcal{N}(K,f) = 0$.

**Proof.** If $K$ is collapsible then there exist a discrete Morse function $f$ over $K$ with only one critical simplex, which must be vertex $v$ (see e.g. [2, Lemma 4.3]). Therefore $\mathcal{N}(K,f) = h_f(v) = 0$, the last equality holding by property (2) of Lemma I. \[\square\]

In the case of graphs, the other implication also holds.

**Proposition III.** A connected graph $G$ is collapsible if and only if there exists a combinatorial Morse function $f : G \to \mathbb{R}$ such that $\mathcal{N}(G,f) = 0$.

**Proof.** Let $f$ be a Morse function with $\mathcal{N}(G,f) = 0$. Write

$$0 = \sum_{\text{critical vertices}} h_f(v) - \sum_{\text{critical edges}} h_f(e).$$

By Lemma I the first sum is zero and the second sum is positive if there is a critical edge. We conclude that $f$ has no critical edges. Since $G$ is connected there must be only one
critical vertex. Hence $G$ is homotopy equivalent to CW with only a 0-cell and thus it is a tree.

![Figure 1](image)

It is easy to see that Proposition [III] does not hold in this generality for complexes of dimension greater than 1. Note however that the alleged functions appearing in these last two propositions can be taken to be optimal; i.e. they have the least possible number of critical simplices (among all discrete Morse functions over that complex). It is therefore natural to associate to a complex $K$ the number

$$N(K) := \min\{|N(K, f)| : f : K \to \mathbb{R} \text{ optimal Morse function}\}.$$  

With this definition, Proposition [II] may be restated as follows: “If $K$ is collapsible then $N(K) = 0$.” The converse of this statement does not hold in dimension greater than 1 either (see Figure [I]). However, the number $N$ can be used to characterize collapsibility for acyclic 2-complexes. The main result of this note is the following.

**Theorem IV.** Let $K$ be an acyclic 2-complex. Then, $K$ is collapsible if and only if $N(K) = 0$.

Before we prove Theorem [IV] recall that, given a discrete Morse function $f : K \to \mathbb{R}$, the **Morse complex associated to $f$** is the chain complex of $\mathbb{R}$-vector spaces

$$0 \to \mathcal{M}_k \xrightarrow{\partial_k} \mathcal{M}_{k-1} \xrightarrow{\partial_{k-1}} \mathcal{M}_{k-2} \xrightarrow{\partial_{k-2}} \cdots,$$

where $\mathcal{M}_k$ is the span of the critical $k$-simplices of $f$. By [2] Theorem 8.2, this complex has the same homology with real coefficients as $K$. Also, [2] Theorem 8.10] shows that the boundary map $\partial_k : \mathcal{M}_k \to \mathcal{M}_{k-1}$ can be written

$$\partial_k(\tau) = \sum_{\sigma \in \mathcal{M}_{k-1}} \lambda_{\tau}^\sigma \sigma,$$

where the coefficients $\lambda_{\tau}^\sigma$ depend on the set $\Gamma(\tilde{\sigma}, \sigma)$ of gradient paths between $\sigma$ and the immediate faces $\tilde{\sigma}$ of $\tau$ (see [2] §8]). In particular, if $\Gamma(\tilde{\sigma}, \sigma) = \emptyset$ for every $\tilde{\sigma} < \tau$ then $\lambda_{\tau}^\sigma = 0$.

**Proof of Theorem [IV]** Let $L$ be a non-collapsible 2-complex satisfying the hypotheses of the theorem. We shall show that $N(L) > 0$. Let $f$ be an optimal discrete Morse function over $L$ and let $m_i(f)$ stand for the number of critical $i$-simplices of $f$. On one hand, $m_0(f) = 1$ by [2] Corollary 11.2. On the other hand, $m_1(f) = m_2(f) \geq 1$ by the weak
Morse inequalities (see [2 Corollary 3.7]) and the non-collapsibility of L. Let A be the set of critical edges of f, B the set of critical 2-simplices of f and form the (balanced) bipartite graph $G = (A \cup B, E)$, where we put an edge between $e \in A$ and $\sigma \in B$ if there exists a gradient path from $e$ to an immediate face of $\sigma$ (see [2, §8]). We claim that $G$ admits a complete matching (i.e. a matching involving every vertex of $G$). If this was not true, there exists by Hall’s Theorem a subset $S \subset B$ such that $|S| \geq |N(S)|$, where $N(S) = \{e \in A| \{e, \sigma\} \in E \text{ for some } \sigma \in S\}$. Write $S = \{\sigma_1, \ldots, \sigma_r\}$. By the above remarks, $\{\partial_2(\sigma_1), \ldots, \partial_2(\sigma_r)\} \subset \text{span}(N(S))$. Since $r > \dim(\text{span}(N(S)))$ we can write

$$0 = \sum_{j=1}^{r} b_j \partial_2(\sigma_j),$$

for some $b_i \in \mathbb{R}$, not all zero. But in this case, $\sum_{j=1}^{r} b_j \sigma_j$ is a generating cycle of $H_2(\mathbb{N}_*, \partial_*) \simeq H_2(L)$ and we reach a contradiction to our hypotheses. This proves that there exists a complete matching $M$ in $G$. Order $\mathcal{A} = \{e_1, \ldots, e_k\}$ and $B = \{\sigma_1, \ldots, \sigma_k\}$ so that $(e_i, \sigma_i) \in \mathcal{M}$ for every $i = 1, \ldots, k$. By construction, there is a gradient path between $e_i$ and a boundary edge of $\sigma_i$ for every $i = 1, \ldots, k$. In particular, $h_f(e_i) < h_f(\sigma_i)$ for every $i = 1, \ldots, k$. We conclude that

$$\mathcal{M}(L, f) = -\sum_{j=1}^{k} h_f(e_j) + \sum_{j=1}^{k} h_f(\sigma_j) = \sum_{j=1}^{k} (h_f(\sigma_j) - h_f(e_j)) > 0.$$

□

Remark V. The hypotheses in the statement of the previous theorem can be slightly relaxed. The same proof can be carried out for connected 2-complexes fulfilling $\chi(K) = 1$ and $H_2(K) = 0$. In particular, $\mathcal{M}(\mathbb{R}P^2) > 0$.

It is straightforward to produce similar results for PL-collapsibility. A complex is PL-collapsible if it has a collapsible subdivision. For a complex $K$ one can define the number

$$\tilde{\mathcal{M}}(K) := \min\{\mathcal{M}(L) : L \text{ is a subdivision of } K\}.$$

As a direct corollary to Theorem IV we have the following result.

Corollary VI. An acyclic 2-complex $K$ is PL-collapsible if and only if $\tilde{\mathcal{M}}(K) = 0$.

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References

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