The Energy Density in the Maxwell-Chern-Simons Theory

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A two-dimensional nonrelativistic fermion system coupled to both electromagnetic gauge fields and Chern-Simons gauge fields is analysed. Polarization tensors relevant in the quantum Hall effect and anyon superconductivity are obtained as simple closed integrals and are evaluated numerically for all momenta and frequencies. The correction to the energy density is evaluated in the random phase approximation (RPA), by summing an infinite series of ring diagrams. It is found that the correction has significant dependence on the particle number density.

In the context of anyon superconductivity, the energy density relative to the mean field value is minimized at a hole concentration per lattice plaquette $(0.05 \sim 0.06) \cdot \frac{p_c a}{\hbar}$ where $p_c$ and $a$ are the momentum cutoff and lattice constant, respectively. At the minimum the correction is about $-5 \% \sim -25 \%$, depending on the ratio $2m \omega_c / p_c^2$ where $\omega_c$ is the frequency cutoff.

In the Jain-Fradkin-Lopez picture of the fractional quantum Hall effect the RPA correction to the energy density is very large. It diverges logarithmically as the cutoff is removed, implying that corrections beyond RPA become important at large momentum and frequency.

1. Introduction

There is a growing interest in Chern-Simons theory. Gauge fields with a pure Chern-Simons term, which we call Chern-Simons gauge fields, effectively alter the statistics of matter fields.[1-5] In Jain’s picture of the fractional quantum Hall effect,[6] an electron and magnetic flux form a bound state, or a composite fermion. The fractional quantum Hall effect for electrons is understood as integral quantum Hall effect for these composite fermions. Further, Fradkin and Lopez have shown that these composite fermions are regarded as fermions interacting with Chern-Simons gauge fields.[7] Halperin, Lee, and Read have advocated this picture further to investigate physics near half filling in the quantum Hall effect.[8,9]

In anyon superconductivity, physics of essentially the same system of fermions coupled to Chern-Simons and Maxwell gauge fields is explored, but with a different Chern-Simons coefficient.[10-26] It is known that this system shows a rather unique behavior at finite
temperature, particularly around $T \sim 100\text{K}$.[25,26] So far there has been no evidence for its possible connection to high $T_c$ superconductors. In fact, the theory of anyon superconductivity is still in its infancy to the extent that it provides only few predictions which can be tested experimentally.

In the relativistic case the Chern-Simons gauge fields induce boson-fermion transmutation.[27] If gauge fields have both Maxwell and Chern-Simons terms, the Chern-Simons term gives the gauge fields a topological mass.[28] Further it has been recently shown that the presence of Dirac fermions leads, under certain conditions, to the spontaneous breakdown of the Lorentz invariance by dynamical generation of a magnetic field ($B \neq 0$).[29] There quantum fluctuations play a crucial role in lowering the energy density of the true ground state with $B \neq 0$. Lopez, Rojo, and Fradkin have shown that an effective relativistic Dirac theory with Maxwell-Chern-Simons gauge interactions naturally arises in the quantum Heisenberg antiferromagnet on a square lattice.[30]

In this paper we examine a nonrelativistic fermion system interacting with both Maxwell and Chern-Simons gauge fields. Employing the technique developed in ref. [29], we first cast one-loop polarization tensors for gauge fields in simple integral forms suited for both analytical and numerical evaluation. Then we derive the exact formula for the energy density in terms of gauge field propagators. Applying our one-loop formula for the polarization tensors, which corresponds to the random phase approximation (RPA), or to summing up an infinite series of ring diagrams in perturbation theory, we examine contributions of quantum fluctuations to the energy density.

The detailed numerical evaluation is given in the absence of dynamical electromagnetic interactions. We shall find that RPA corrections generate non-trivial dependence of the energy density on the particle (or hole) number density in the context of anyon superconductivity. The energy is minimized at a number density precisely where high $T_c$ material is superconducting. In the case of the fractional quantum Hall effect, RPA gives a large correction to the energy density, in accord with the equivalence between the electron picture and the composite fermion picture.

2. The model

The model we consider consists of a nonrelativistic fermion field $\psi$, electromagnetic field $A_\mu$, and Chern-Simons gauge field $a_\mu$. Its Lagrangian is given by

\begin{align}
\mathcal{L} &= \frac{1}{2} (\epsilon E_j^2 - \chi B^2) - \frac{\kappa}{2} \epsilon^{\mu\nu\rho\sigma} a_\mu \partial_\nu a_\rho + i\psi^\dagger D_0 \psi - \frac{1}{2m} (D_k \psi)^\dagger (D_k \psi) + e\bar{\rho}A_0 \\
E_j &= F_{0j}, \quad B = -F_{12} = \partial_1 A^2 - \partial_2 A^1 \\
D_\mu &= \partial_\mu + i(a_\mu + eA_\mu).
\end{align}

The last term represents a contribution from a background neutralizing charge. Note that $a^k = -a_k$ and $A^k = -A_k$. The Euler equations are

\begin{align}
\epsilon \partial_k E_k &= e (j^0 - \bar{\rho}) \\
\epsilon^{kl} \chi \partial_l B &= e j^k \\
-\frac{\kappa}{2} \epsilon^{\mu\nu\rho} f_{\nu\rho} &= j^\mu \\
i\partial_0 \psi &= \left\{ -\frac{1}{2m} D_k^2 + a_0 + eA_0 \right\} \psi.
\end{align}
where the currents are given by
\begin{align}
  j^0 &= \psi^\dagger \psi \\
  j^k &= -\frac{i}{2m} \left\{ \psi^\dagger D_k \psi - (D_k \psi)^\dagger \psi \right\} .
\end{align}

(2.3)

Suppose a uniform external magnetic field $B_{\text{ext}}$ is applied and the system remains translation invariant. Eq. (2.2) implies that $\psi$ feels a total magnetic field
\begin{equation}
  b_{\text{tot}} \equiv b^{(0)} + eB^{(0)} = \frac{\bar{\rho}}{\kappa} + eB_{\text{ext}} .
\end{equation}

(2.4)

The Landau level densities associated with the external magnetic field and the total magnetic field are given by
\begin{align}
  \rho^\text{ext}_L &= \frac{|eB_{\text{ext}}|}{2\pi} \\
  \rho^\text{tot}_L &= \frac{|b_{\text{tot}}|}{2\pi} .
\end{align}

(2.5)

Something special happens when the total filling factor is an integer:
\begin{equation}
  \nu = \frac{\bar{\rho}}{\rho^\text{tot}_L} = n .
\end{equation}

(2.6)

The change in the statistics phase induced by Chern-Simons gauge fields is $\Delta \theta_s = 1/2\kappa$. Combining (2.4), (2.5), and (2.6), one finds
\begin{equation}
  \frac{\Delta \theta_s}{\pi} = \frac{1}{2\pi \kappa} = \epsilon(b_{\text{tot}}) \frac{1}{n} - \epsilon(eB_{\text{ext}}) \frac{1}{\nu_{\text{ext}}} , \quad \nu_{\text{ext}} = \frac{\bar{\rho}}{\rho^\text{ext}_L} .
\end{equation}

(2.7)

In the Jain-Fradkin-Lopez picture, the system exhibits the fractional quantum Hall effect when $\Delta \theta_s$ is a multiple of $2\pi$. Suppose that $eB_{\text{ext}} > 0$. Then the condition is satisfied if
\begin{equation}
  \frac{1}{2\pi \kappa} = -2p \\
  \nu_{\text{ext}} = \frac{n}{2pn \pm 1}
\end{equation}

(2.8)

where $\pm$ corresponds to $\epsilon(b_{\text{tot}})$. In the case $eB_{\text{ext}} < 0$ the signs of $\kappa$ and $b_{\text{tot}}$ are reversed. The main sequence in fractional quantum Hall effect is given by $p = 1$, or $\nu_{\text{ext}} = n/(2n \pm 1)$. In this paper we focus on the special case $\nu = 1$. Generalization to the case $\nu = n$ is straightforward, but is left for a future investigation. Note that for $\nu = 1$
\begin{equation}
  \left\{ \frac{1}{2\pi \kappa} - \epsilon(b_{\text{tot}}) \right\} \bar{\rho} = -\frac{eB_{\text{ext}}}{2\pi} .
\end{equation}

(2.9)

3. Fermion propagator

The perturbation theory is defined around $b_{\text{tot}}$. We rewrite $a_\mu \rightarrow a_\mu^{(0)} + a_\mu$ and $A_\mu \rightarrow A_\mu^{\text{ext}} + A_\mu$, i.e. $a_\mu$ and $A_\mu$ now represent fluctuation parts. Then the Lagrangian becomes
\begin{equation}
  \mathcal{L} = -\frac{\chi}{2} B_{\text{ext}}^2 + \mathcal{L}_0^{\text{gauge}}[a, A] + \mathcal{L}_0^{\text{matter}}[\psi, \psi^\dagger] + \mathcal{L}_0^{\text{int}}[a, A, \psi, \psi^\dagger] .
\end{equation}

(3.1)
$L^\text{gauge}_0 + L^\text{matter}_0$ defines the zeroth order “free” part. The gauge field part is given by

$$L^\text{gauge}_0 = \frac{1}{2} (\epsilon B^2) - \frac{\kappa}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + L_\text{g.f.}[a, A]$$  \hspace{1cm} (3.2)

where $L_\text{g.f.}[a, A]$ is a gauge fixing term to be specified in the next section. The matter field part is

$$L^\text{matter}_0 = i\psi^\dagger \partial_0 \psi - \frac{1}{2m} (\mathcal{D} - k) \psi^\dagger (\mathcal{D} - k) \psi$$

$$\mathcal{D} = \partial_k - i(a^{(0)k} + eA_k)$$ \hspace{1cm} (3.3)

The interaction part is given by

$$L^\text{int} = -\chi B^\text{ext} - (\psi^\dagger \psi - \bar{\rho}) (a_0 + eA_0)$$

$$- (a^k + eA_k) \left\{ \psi^\dagger (\mathcal{D}_k \psi) - (\mathcal{D}_k \psi)^\dagger \psi \right\} - \frac{1}{2m} (a^k + eA_k)^2 \psi^\dagger \psi$$ \hspace{1cm} (3.4)

The $B^\text{ext}$ term in $L^\text{int}$, being a total derivative, does not contribute in perturbation theory and may be dropped.

The zeroth order fermion propagator has been well described in the literature. Usually it is given in the form of an infinite sum over the Landau level index. Our task here is to recast it in an integral form for later convenience.

Let us quote the result from Ref. [26]. In the Landau gauge

$$a^{(0)k} + eA_k^\text{ext} = -\epsilon(b^\text{tot}) \delta^{k_1} \frac{x_2}{l^2}$$

$$\frac{1}{l^2} = |b^\text{tot}|$$ \hspace{1cm} (3.5)

$l$ is the magnetic length. When $\nu$ lowest Landau levels are completely filled, the fermion propagator is given by

$$G(x, y) = -i\left( T[\psi(x) \psi^\dagger(y)] \right)$$

$$= e^{-i\epsilon(b^\text{tot})(x_1 - y_1)(x_2 + y_2)/2l^2} \cdot G_0(x - y)$$

$$G_0(x) = -i \left\{ \theta(x_0) \sum_{n=\nu}^\infty -\theta(-x_0) \sum_{n=0}^{\nu-1} \right\} e^{-ix_n x_0}$$

$$\times \frac{1}{2\pi l^2} \int_{-\infty}^\infty dz \ e^{-izx_1/l} \ v_n[z - \bar{z}(x_2)] \ v_n[z + \bar{z}(x_2)]$$ \hspace{1cm} (3.6)

where

$$\epsilon_n = \left( n + \frac{1}{2} \right) \frac{1}{ml^2} \quad (n = 0, 1, 2, \cdots)$$ \hspace{1cm} (3.7)

is the $n$-th Landau energy level, and

$$v_n(x) = \frac{(-1)^n}{2^{n/2} \pi^{1/4} (n!)^{1/2}} \frac{d^n}{dx^n} e^{-x^2}$$

$$\bar{z}(x_2) = \epsilon(b^\text{tot}) \ x_2^2 \frac{1}{2l}$$ \hspace{1cm} (3.8)

The phase factor in the propagator $G(x, y)$ is not translation invariant, but does not contribute to any physical quantities as we see below.
We now utilize the Feynman-Hibbs summation formula\[31\]
\[
\sum_{n=0}^{\infty} e^{-int} v_n(x)v_n(y) = \frac{e^{it/2}}{\sqrt{2\pi i \sin t}} \exp \left\{ \frac{i}{2 \sin t} \left[ (x^2 + y^2) \cos t - 2xy \right] \right\}
\]
(3.9)
in order to perform the infinite sum and obtain an explicit integral expression for the propagator. We also quote another formula
\[
\sum_{n=0}^{\infty} v_n(x)v_n(y) = i \int_0^{\infty} dt \frac{e^{it} \lambda - i\epsilon}{\sqrt{2\pi i \sin t}} \exp \left\{ \frac{i}{2 \sin t} \left[ (x^2 + y^2) \cos t - 2xy \right] \right\}
\]
(3.10)
which follows from (3.9).

First we evaluate the propagator in the $\nu = 0$ case. Employing (3.9) and taking the Fourier transform
\[
G_0(p) = G_0(\omega, \vec{p}) = \int dx G_0(x) e^{i(\omega x_0 - \vec{p} \vec{x})},
\]
one finds that
\[
G_0(p)|_{\nu=0} = -iml^2 \int_0^{\infty} \frac{dt}{\cos(t/2)} e^{-i(p_1^2 + p_2^2)l^2 \tan(t/2) + i\omega ml^2 t}. \tag{3.11}
\]
For $\nu \geq 1$ we observe that
\[
G_0(x)|_{\nu} - G_0(x)|_{\nu=0}
= \sum_{n=0}^{\nu-1} e^{-i\epsilon_0 x_0} \int_0^{\infty} dz e^{-i\epsilon_0 z} v_n(z - \bar{z}(x_2))v_n(z + \bar{z}(x_2)). \tag{3.12}
\]
Using the explicit expression for $v_n$, one finds that for $\nu = 1$, for instance,
\[
G_0(p)|_{\nu=1} = 4\pi i e^{-\vec{p}^2 l^2} \delta(\omega - \epsilon_0)
- iml^2 \int_0^{\infty} \frac{dt}{\cos(t/2)} e^{-i\vec{p}^2 l^2 \tan(t/2) + i\omega ml^2 t}. \tag{3.13}
\]

4. Gauge field propagators

Free gauge field propagators are obtained from $\mathcal{L}_0^{\text{gauge}}$ in (3.2). For the sake of unified treatment of both electromagnetic and Chern-Simons gauge fields, let us consider gauge fields described by
\[
N_0 = \frac{1}{2} (\epsilon E^2 - \chi B^2) - \frac{\kappa}{2} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{1}{2\chi} (\epsilon \partial_0 A^0 + \chi \partial_k A^k)^2. \tag{4.1}
\]

The last term represents gauge-fixing. Electromagnetic fields are described with $\kappa = 0$, whereas Chern-Simons fields in the $\epsilon, \chi \to 0$ limit:
\[
\mathcal{L}_0^{\text{gauge}}[a, A] = N_0[A; \kappa = 0] + N_0[a; \epsilon, \chi \to 0]. \tag{4.2}
\]
As is obvious, the propagator is singular in the $\epsilon, \chi \to 0$ limit, which is merely a gauge artifact. Physical quantities such as the energy density are well defined in the $\epsilon, \chi \to 0$ limit. We take this limit when convenient and safe to do so.
In passing, we mention that for pure Chern-Simons gauge fields the alternative gauge fixing term
\[ L_{g.f.} = \frac{1}{2\alpha} (\partial_k a^k)^2 \] (4.3)
is also very useful and convenient.\[26\] The radiation gauge is obtained in the \( \alpha \to 0 \) limit. We adopt the gauge fixing in (4.1) in this paper.

The Lagrangian (4.1) has the form
\[ N_0 = \frac{1}{2} A_\mu K^{\mu\nu} A_\nu , \]
\[ K^{\mu\nu} = \begin{pmatrix} (\epsilon/\chi) D & \kappa \partial_2 & -\kappa \partial_1 \\ -\kappa \partial_2 & -D & \kappa \partial_0 \\ \kappa \partial_1 & -\kappa \partial_0 & -D \end{pmatrix} , \]
\[ D = \epsilon \partial_0^2 - \chi \nabla^2 . \] (4.4)

For this Lagrangian, the propagator, the inverse of \( K^{\mu\nu} \), is straightforwardly found to be
\[ (D_0)_{\mu\nu} = -\frac{1}{\epsilon D + \kappa^2} \begin{pmatrix} \chi & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix} + \chi \kappa^2 \frac{\partial_2 \partial_0}{D^2} + \frac{\kappa}{D} \begin{pmatrix} 0 & \chi \partial_2 & -\chi \partial_1 \\ -\chi \partial_2 & 0 & \epsilon \partial_0 \\ \chi \partial_1 & \epsilon \partial_0 & 0 \end{pmatrix} \] . (4.5)

Equal time commutation relations are given by
\[ [A_\mu(t, x), \Pi^\nu(t, y)] = i \delta^\nu_\mu \delta(x - y) \]
\[ [A_\mu(t, x), A_\nu(t, y)] = [\Pi^\mu(t, x), \Pi^\nu(t, y)] = 0 \]
\[ \Pi^0 = -\frac{\epsilon^2}{\chi} \partial_0 A_0 \] (4.6)
\[ \Pi^j = \epsilon \partial_0 A_j - \frac{\kappa}{2} \epsilon^{jk} A_k \]

In evaluating the energy density we need the full gauge field propagators. The interaction, given by (3.4), mixes the electromagnetic and Chern-Simons gauge fields. The propagators have a matrix structure:
\[ \hat{D}_{\mu\nu} = -\frac{i}{\epsilon D + \kappa^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix} + \chi \kappa^2 \frac{\partial_2 \partial_0}{D^2} + \frac{\kappa}{D} \begin{pmatrix} 0 & \chi \partial_2 & -\chi \partial_1 \\ -\chi \partial_2 & 0 & \epsilon \partial_0 \\ \chi \partial_1 & \epsilon \partial_0 & 0 \end{pmatrix} \] . (4.7)

When the lowest Landau level is completely filled so that the perturbative ground state is of Hartree-Fock type, one can develop a Feynman diagram method. Since both gauge fields couple to fermions in the combination of \( a_\mu + e A_\mu \), one can write
\[ \hat{D}_{\mu\nu}^{-1} = \hat{D}_0^{-1} - \Gamma_{\mu\nu} \\ = \begin{pmatrix} D_{0EM}^{-1} & 0 \\ 0 & D_{0CS}^{-1} \end{pmatrix} - \Gamma_{\mu\nu} \begin{pmatrix} e^2 & \epsilon \\ \epsilon & 1 \end{pmatrix} \] . (4.8)

Here \( \Gamma_{\mu\nu} \) represents the sum of one-particle irreducible diagrams common to both \( a_\mu \) and \( e A_\mu \).
Gauge invariance, translational invariance, and rotational invariance imply that $\Gamma^{\mu\nu}$ is expressed in terms of three independent invariant functions $\Pi_k$’s:
\[
\Gamma^{\mu\nu}(p) = (p^\mu p^\nu - p^2 g^{\mu\nu}) \Pi_0 + i e^{\mu\nu\rho\sigma} p_\rho \Pi_1 \\
+ (1 - \delta^{\mu0})(1 - \delta^{\nu0})(p^\mu p^\nu - \vec{p}^2 \delta^{\mu\nu})(\Pi_2 - \Pi_0).
\] (4.9)

All $\Pi_k$’s are functions of $p_0^2$ and $\vec{p}^2$.

5. Evaluation of $\Pi_k$’s

We evaluate the kernel $\Gamma^{\mu\nu}(\omega, \vec{q})$, or equivalently the invariant functions $\Pi_k$’s, to the leading order in the case $\nu = 1$. The interaction is given by (3.4). One-loop contributions are depicted in Fig. 1.

To calculate $\Pi_k$’s we take a frame $\vec{q} = (q, 0)$. Then
\[
q^2 \Pi_0 = \Gamma^{(b)00} = i q \Pi_1 = \Gamma^{(c)02}
\] (5.1)
\[
\omega^2 \Pi_0 - q^2 \Pi_2 = \Gamma^{22} = \Gamma^{(a)22} + \Gamma^{(d)22} = \Pi_2.
\]

Diagram (a) is easily evaluated to give
\[
\Gamma^{(a)jk}(\omega, \vec{q}) = -\frac{\vec{p}}{m} \delta^{jk}.
\]

As is well known, other diagrams yield
\[
\Gamma^{(b)00}(q) = i \int \frac{d^3p}{(2\pi)^3} G_0(p)G_0(p - q)
\]
\[
\Gamma^{(c)0j}(q) = -\frac{1}{2m} \int \frac{d^3p}{(2\pi)^3} \left\{ G_0(p) \cdot D^-_j G_0(p - q) + D^+_j G_0(p) \cdot G_0(p - q) \right\}
\]
\[
\Gamma^{(d)22}(q) = -\frac{i}{4m^2} \int \frac{d^3p}{(2\pi)^3} \left\{ D^+_2 G_0(p) \cdot D^-_2 G_0(p - q) + D^+_2 G_0(p) \cdot D^-_2 G_0(p - q) + D^-_2 D^+_2 G_0(p - q) \cdot G_0(p) \cdot D^-_2 D^+_2 G_0(p - q) \right\}
\] (5.2)
\[
+ D^+_2 D^-_2 G_0(p) \cdot G_0(p - q) + G_0(p) \cdot D^-_2 D^+_2 G_0(p - q)
\]
\[
D^+_2 G_0(p) = \left( ip_2 \pm \epsilon(b_{+0}) \frac{\partial}{2l^2 \partial p_1} \right) G_0(p) .
\]

The phase factor in $G(x, y)$ (3.6), does not contribute to $\Gamma^{\mu\nu}$.

The next step is to insert (3.13) into (5.2) and perform the momentum integrals. All integrals involve heavy, but similar manipulations. We shall describe the computation of $\Pi_0$ in detail.

There are four terms in the integral
\[
q^2 \Pi_0(q) = i \int \frac{d^3p}{(2\pi)^3} \\
\times \left\{ 4 \pi i e^{-\vec{p}^2 l^2} \delta(p_0 - \epsilon_0) - iml^2 \int_0^{\infty} \frac{dt}{\cos \frac{t}{2}} e^{-ip^2 t^2 \tan(t'/2) + ip_0 m^2 t^2} \right\}
\]
\[
\times \left\{ 4 \pi i e^{-2 \vec{r}^2 l^2} \delta(q_0 - \omega - \epsilon_0) - iml^2 \int_0^{\infty} \frac{dt'}{\cos \frac{t'}{2}} e^{-i(\vec{p} - \vec{q})^2 t^2 \tan(t'/2) + i(p_0 - \omega) ml^2 t'} \right\}
\] (5.3)
The product of the two \( \delta \)-function terms is easily evaluated to be

\[
- \frac{i}{l^2} e^{-q^2l^2/2} \delta(\omega) .
\]  

(5.4)

Two cross terms of the \( \delta \)-function and \( t \)-integral pieces are, after \( p_0 \) integration,

\[
4\pi im^2 \int \frac{dp}{(2\pi)^3} e^{-\vec{p}^2l^2} \int_0^\infty \frac{dt}{\cos \frac{\omega l}{2}} e^{-i(\mathbf{p} - \mathbf{q})^2l^2 \tan(t/2) + i(\epsilon_0 - \omega)ml^2t} \\
+ \{ (\omega, \mathbf{q}) \to (\omega, -\mathbf{q}) \}
\]

\[
= i \frac{m}{2\pi} \int_0^\infty \frac{dt}{\cos \frac{\omega l}{2}} \frac{1}{1 + i \tan \frac{\omega l}{2}} \exp \left( i(\epsilon_0 - \omega)ml^2t - i\frac{q^2l^2}{2} \tan \frac{\omega l}{2} \right) \\
+ \{ (\omega, \mathbf{q}) \to (\omega, -\mathbf{q}) \}
\]

\[
= i \frac{m}{\pi} \int_0^\infty dt e^{-q^2l^2(1-e^{-it})/2} \cos(\omega ml^2t) .
\]

(5.5)

The product of the two \( t \)-integrals in (5.3) vanishes:

\[
-(ml^2)^2 \int_0^\infty \frac{dt}{\cos \frac{\omega l}{2}} \int_0^\infty \frac{dt'}{\cos \frac{\omega l}{2}} \int \frac{d^3p}{(2\pi)^3} \\
\times e^{-i\vec{p}^2l^2 \tan \frac{\omega l}{2} + i\mathbf{p} \cdot \mathbf{m}} e^{-i(\mathbf{p} - \mathbf{q})^2l^2 \tan \frac{\omega l}{2} + i\mathbf{p} \cdot \mathbf{m} - i\omega ml^2t} \\
= \int_0^\infty dt dt' \int \frac{d^3p}{(2\pi)^3} \cdots \delta(t + t') \cdots \\
= 0 .
\]

(5.6)

The manipulation in the last equality is justified since the integrand is regular at \( t = t' = 0 \).

\( \Pi_0(q) \) is a regular function of \( \omega \) so that the \( \delta(\omega) \) term in (5.4) must be cancelled by a part of (5.5). To see this we rewrite (5.5) as

\[
i \frac{m}{2\pi} \int_0^\infty dt e^{-q^2l^2(1-e^{-it})/2} \left( e^{i(\omega + i\epsilon)ml^2t} + e^{-i(\omega - i\epsilon)ml^2t} \right)
\]

\[
= \frac{m}{2\pi} \int_0^\infty dt e^{-q^2l^2(1-e^{-it})/2} \partial \frac{\partial}{\partial t} \left\{ \frac{e^{i(\omega + i\epsilon)ml^2t}}{(\omega + i\epsilon)ml^2} - \frac{e^{-i(\omega - i\epsilon)ml^2t}}{(\omega - i\epsilon)ml^2} \right\} .
\]

Integration by parts yields

\[
- \frac{1}{2\pi l^2} \left( \frac{1}{\omega + i\epsilon} - \frac{1}{\omega - i\epsilon} \right)
\]

\[
- \frac{m}{2\pi} \int_0^\infty dt \left( -i \frac{q^2l^2}{2} e^{-it} \right) e^{-q^2l^2(1-e^{-it})/2} \left\{ \frac{e^{i(\omega + i\epsilon)ml^2t}}{(\omega + i\epsilon)ml^2} - \frac{e^{-i(\omega - i\epsilon)ml^2t}}{(\omega - i\epsilon)ml^2} \right\}
\]

\[
= i l^2 \delta(\omega) + \frac{iq^2}{4\pi} \int_0^\infty dt e^{-it - q^2l^2(1-e^{-it})/2} \left\{ -2i\omega \delta(\omega) + \frac{2i}{\omega} \sin(\omega ml^2t) \right\} .
\]

(5.7)

Now the second \( \delta(\omega) \) piece is

\[
\frac{q^2}{2} \delta(\omega) \int_0^\infty dt e^{-it - q^2l^2(1-e^{-it})/2} = \frac{q^2}{2} \delta(\omega) \int_0^\infty (-ids) e^{-s - q^2l^2(1-e^{-s})/2}
\]

\[
= i l^2 (e^{-q^2l^2/2} - 1) \delta(\omega) .
\]

(5.8)
Hence (5.5) becomes
\[
\frac{i}{l^2} e^{-q^2l^2/2} \delta(\omega) - \frac{q^2}{2\pi} \int_0^\infty dt e^{-it-q^2(1-e^{-it})/2} \sin(\omega ml^2t) \ .
\]

(5.9)

The first term exactly cancels (5.4). Combining (5.4), (5.6), and (5.9), one finds that
\[
\Pi_0 = \frac{ml^2}{2\pi} F_0(\frac{1}{2}q^2l^2, \omega ml^2)
\]

(5.10)

\[
F_0(x, y) = -\frac{1}{y} \int_0^\infty dt \sin yt e^{-it-x(1-e^{-it})} \ .
\]

The evaluation of \(\Pi_1\) and \(\Pi_2\) proceeds similarly. The cancellation of \(\delta\)-function pieces takes place as in the case of \(\Pi_0\). The result is
\[
\Pi_1 = \frac{e(b_{tot})}{2\pi} F_1(\frac{1}{2}q^2l^2, \omega ml^2)
\]

\[
\Pi_2 = \frac{1}{2\pi ml^2} F_2(\frac{1}{2}q^2l^2, \omega ml^2)
\]

(5.11)

\[
F_1(x, y) = \frac{\partial}{\partial x} x \cdot F_0(x, y)
\]

\[
F_2(x, y) = -1 + \left\{ x + \left( \frac{\partial}{\partial x} x \right)^2 \right\} F_0(x, y) \ .
\]

In refs. [11,13,21,26,32] these \(\Pi_k\)'s are obtained in the form of infinite sums over the Landau level index \(n\). Applying the second Feynman-Hibbs formula (3.10) to the result in ref. [26], the results (5.10) and (5.11) are reproduced with some labor.

Notice that all \(\Pi_k\)'s are related to one function \(F_0\). For small \(x\) [33]
\[
F_0(x, y) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{-1}{y^2 - (k + 1)^2 + i\epsilon} \ .
\]

(5.12)

In particular
\[
F_0(0, y) = \frac{1}{1 - y^2 - i\epsilon}
\]

\[
F_0(x, 0) = e^{-x} x \int_0^x dw \frac{e^w - 1}{w} \ .
\]

(5.13)

One can derive an alternative integral representation for \(F_j\) which is more suited for both analytical and numerical evaluation. First we Wick-rotate the \(y\) variable: \(\hat{F}_j(x, iz) = F_j(x, iz)\). Then
\[
\hat{F}_0(x, z) = \frac{e^{-x}}{x} \cdot h(x, z)
\]

\[
h(x, z) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} \frac{1}{z^2 + (k + 1)^2} \ .
\]

(5.14)

\(h(x, z)\) satisfies
\[
\left\{ \left( x \frac{\partial}{\partial x} \right)^2 + z^2 \right\} h(x, z) = xe^x
\]

\[
h(0, z) = 0
\]

\[
\partial_z h(0, z) = \frac{1}{z^2 + 1} \ .
\]

(5.15)
In terms of $s = \ln x$, $h(x, z) = z^{-1} \int_{-\infty}^{x} du \rho(u) \sin(z(s - u))$ where $\rho(s) = xe^s$. Or

\[
h(x, z) = \frac{1}{z} \int_{0}^{x} dw \sin \left( z \ln \frac{x}{w} \right) \cdot e^w = -\frac{x}{z} \int_{0}^{1} dt \sin(z \ln t) \cdot e^{xt}.
\] (5.16)

This $h(x, z)$ satisfies the boundary condition in (5.15).

Hence one finds that

\[
\hat{F}_0(x, z) = -\frac{1}{z} \int_{0}^{1} dt \sin(z \ln t) \cdot e^{-(1-t)x}
\]

\[
\hat{F}_1(x, z) = -\frac{1}{z} \int_{0}^{1} dt \{1 + x(t - 1)\} \sin(z \ln t) \cdot e^{-(1-t)x}
\]

\[1 + \hat{F}_2(x, z) = -\frac{1}{z} \int_{0}^{1} dt \{1 - 2x + 3xt + x^2(t - 1)^2\} \sin(z \ln t) \cdot e^{-(1-t)x}.
\] (5.17)

The behavior of these functions is depicted in Fig. 2. The asymptotic behavior is given by

\[
\hat{F}_0 \sim \frac{1}{x^2 + z^2}
\]

\[
\hat{F}_1 \sim \frac{z^2 - x^2}{(x^2 + z^2)^2}
\]

\[1 + \hat{F}_2 \sim \frac{x}{x^2 + z^2} + \frac{1}{x^2 + z^2} - \frac{8x^2z^2}{(x^2 + z^2)^3}.
\] (5.18)

These approximate expressions are valid to the accuracy of 2% for $x$ or $z > 10$.

6. Energy density

Quantum fluctuations shift the energy density of the ground state from the mean field value. This shift can be related to the full gauge field propagators. We first derive an exact formula for the energy density\,[34], and estimate it by utilizing the result in the previous section. The mean field energy density for $\nu = 1$ is given by

\[\mathcal{E}_{\text{mean}} = \frac{1}{2} B_{\text{ext}}^2 + \frac{\pi \bar{\rho}^2}{m}.
\] (6.1)

Here $B_{\text{ext}}$ is related to $\bar{\rho}$ and $\kappa$ by (2.9). The interactions (3.4) give corrections to $\mathcal{E}_{\text{mean}}$. The corresponding interaction Hamiltonian is

\[
H_{\text{int}} = H^{(1)} + H^{(2)}\]

\[
H^{(1)} = \int dx \left\{ (a^k + eA^k) \frac{i}{2m} \left( \psi^+ \bar{D}_k \psi - (\bar{D}_k \psi^+) \psi \right) + (a_0 + eA_0) (\psi^+ \psi - \bar{\rho}) \right\}
\] (6.2)

\[
H^{(2)} = \int dx \frac{1}{2m} (a^k + eA^k)^2 \psi^+ \psi.
\]

Now we consider a Hamiltonian given by

\[H(g) = H_0 + gH^{(1)} + g^2 H^{(2)}\] (6.3)
where $H_0$ is the free part of the Hamiltonian. With an auxiliary parameter $g$, $H(g)$ connects the mean field Hamiltonian $H(0)$ and the full Hamiltonian $H(1)$. Its ground state satisfies

$$H(g)|\Psi(g)\rangle = E(g)|\Psi(g)\rangle$$
$$E(g) = \langle \Psi(g)|H(g)|\Psi(g)\rangle.$$  (6.4)

The desired change in energy density with inclusion of the interaction is $E(1) - E(0)$. Since $\langle \Psi(g)|\Psi(g)\rangle = 1$,

$$E(1) - E(0) = \int_0^1 dg \frac{d}{dg} E(g)$$
$$= \int_0^1 dg \langle \Psi(g)|\frac{\partial H(g)}{\partial g}|\Psi(g)\rangle$$
$$= \int_0^1 dg \langle \Psi(g)|H^{(1)} + 2gH^{(2)}|\Psi(g)\rangle.$$  (6.5)

The currents associated with the fluctuation parts of the gauge fields are defined by

$$K_{\rho \mu}^{CS,a} a_\nu = K_{\rho \mu}^{CS,a_{\nu}}$$
$$K_{\rho \mu}^{EM,a} A_\nu = K_{\rho \mu}^{EM,a_{\nu}}.$$  (6.6)

$K^{\mu \nu}$ has been defined in (4.4). In the theory described by the Hamiltonian $H(g)$,

$$\tilde{j}^0 = g (\bar{\psi} \gamma^0 \psi - \bar{\rho})$$
$$\tilde{j}^k = -\frac{ig}{2m} \{ \bar{\psi} \gamma^k \bar{D} \psi - (\bar{D} \psi) \gamma^k \psi \} - \frac{g^2}{m} (a^k + eA^k) \bar{\psi} \gamma^k \psi.$$  (6.7)

Straightforward substitution of these currents yields

$$\int d\mathbf{x} \tilde{j}^\mu(x) \{ a_\mu(x) + eA_\mu(x) \} = gH^{(1)} + 2g^2H^{(2)}.$$  (6.8)

Thus

$$E(1) - E(0) = \int_0^1 \frac{dg}{g} \int d\mathbf{x} \langle \Psi(g)| \tilde{j}^\mu(a_\mu + eA_\mu)|\Psi(g)\rangle.$$  (6.9)

If the gauge fields $A_\mu$ have (4.1) as a free Lagrangian, their equal-time commutation relations are given by (4.6). Elementary algebra shows that

$$K_2^{\rho \mu} \{ T[A_\mu(x)A_\nu(y)] \} = (T[K^{\rho \mu}A_\mu(x)A_\nu(y)])$$
$$+ g^{\rho \mu} \left \{ 1 + \delta^\rho \iota^0 \left ( \frac{\epsilon}{\chi} - 1 \right ) \right \} \epsilon \delta(x_0 - y_0) [A_\mu(x), A_\nu(y)]$$
$$= (T[K^{\rho \mu}A_\mu(x)A_\nu(y)]) + ig^{\rho \nu} \delta^\mu(x-y).$$  (6.10)

The difference between the full and bare propagators, therefore, satisfies

$$K_2^{\rho \mu} \left \{ \langle T[A_\mu(x)A_\nu(y)] \rangle - \langle T[A_\mu(x)A_\nu(y)] \rangle_0 \right \} = (T[K^{\rho \mu}A_\mu(x)A_\nu(y)]).$$  (6.11)
Applying (6.6) and (6.11) to (6.9), one obtains that

\[ E(1) - E(0) = \int_0^1 \frac{dg}{g} \int d\mathbf{x} \lim_{y \to x} \left( K_{xEM,\nu}^{EM} \left\{ \langle T [A_\mu(x)A_\nu(y)] \rangle - \langle T [A_\mu(y)A_\nu(x)] \rangle \rangle_0 \right\} + K_{xCS,\nu}^{CS} \left\{ \langle T [a_\mu(x)a_\nu(y)] \rangle - \langle T [a_\mu(y)a_\nu(x)] \rangle \rangle_0 \right\} \right) \]  

(6.12)

Fourier-transforming this expression yields, for the density,

\[ \mathcal{E}(1) - \mathcal{E}(0) = \int_0^1 \frac{dg}{g} \int \frac{d^3p}{(2\pi)^3} \left\{ \text{tr} D^{-1}_{0,EM} D[AA] - D_{0,EM}^{EM} \right\} + \text{tr} D^{-1}_{0,CS} D[aa] - D_{0,CS}^{CS} \right\} \]  

(6.13)

Here the two-by-two matrices \( \hat{D}, \hat{D}_0, \) and \( \hat{\Gamma} \) are given by (4.7) and (4.8). The trace \( \text{tr} \) is taken over only Lorentz indices, whereas \( \text{Tr} \) is taken over both Lorentz and gauge field species. We stress that the formula (6.13) is exact, no approximation being involved.

We have evaluated \( \hat{\Gamma}_{\mu\nu} \) to the leading order in Section 5. With the Hamiltonian (6.3), all \( \Pi_k \)'s there must be multiplied by \( g^2 \). In other words, \( \hat{\Gamma} = O(g^3) \). The \( g \)-integral in (6.13), then, can be easily performed:

\[ \mathcal{E}(1) - \mathcal{E}(0) = -\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \ln \left\{ 1 + \hat{\Gamma}_{\mu=1} \hat{D}_0 \right\} \]  

(6.14)

\( \Gamma^{(2)} \) is precisely the kernel evaluated in Section 5, which represents the sum of one-loop diagrams. The final expression in the above equation has simple diagramatic interpretation. It represents the sum of a series of ring diagrams connected by either Chern-Simons or electromagnetic fields. (See Fig. 3.) The ‘ln’ takes care of combinatoric factors.

In the frame \( p^\mu = (\omega, q, 0) \)

\[ \Gamma^{\mu\nu}(p) = \begin{pmatrix}
q^2\Pi_0 & \omega q \Pi_0 & iq \Pi_1 \\
\omega q \Pi_0 & \omega^2 \Pi_0 & i\omega \Pi_1 \\
-iq \Pi_1 & -i\omega \Pi_1 & \Pi_2
\end{pmatrix} \]  

(6.15)

With a general propagator \( D_0 \) given by (4.5),

\[ -\Gamma D_0 = \frac{1}{c_8 - \kappa^2} \begin{pmatrix}
-\chi q^2 (\Pi_0 + \frac{\kappa}{s} \Pi_1) & \epsilon \omega q (\Pi_0 + \frac{\kappa}{s} \Pi_1) & iq (\epsilon \Pi_1 + \kappa \Pi_0) \\
-\chi \omega q (\Pi_0 + \frac{\kappa}{s} \Pi_1) & \epsilon \omega^2 (\Pi_0 + \frac{\kappa}{s} \Pi_1) & i\omega (\epsilon \Pi_1 + \kappa \Pi_0) \\
i \chi q (\Pi_1 + \frac{\kappa}{s} \Pi_2) & -i \epsilon \omega (\Pi_1 + \frac{\kappa}{s} \Pi_2) & c \Pi_2 + \kappa \Pi_1
\end{pmatrix} \]  

(6.16)
where \( s \equiv \omega^2 - \chi q^2 \). The electromagnetic part is obtained by taking the \( \kappa \to 0 \) limit in the above formula. For the Chern-Simons part we take the limit \( \epsilon, \chi \to 0 \) with the ratio \( \beta = \epsilon / \chi \) fixed. We find

\[
-\Gamma(\epsilon^2 D_0^{\text{EM}} + D_0^{\text{CS}}) = \begin{pmatrix}
-q^2 a_1 & -\omega q a_2 & iqc \\
-\omega q a_1 & \omega^2 a_2 & i\omega c \\
-iq b_1 & -i\omega b_2 & d
\end{pmatrix}
\]

\[a_1 = \frac{\chi e^2}{\epsilon} \Pi_0 - \frac{\beta}{\kappa u} \Pi_1, \quad a_2 = \frac{\epsilon^2}{\pi} \Pi_0 - \frac{1}{\kappa u} \Pi_1
\]

\[b_1 = \frac{\chi e^2}{\epsilon} \Pi_1 - \frac{\beta}{\kappa u} \Pi_2, \quad b_2 = \frac{\epsilon^2}{\pi} \Pi_1 - \frac{1}{\kappa u} \Pi_2
\]

\[c = \frac{\epsilon^2}{\pi} \Pi_1 - \frac{1}{\kappa} \Pi_0, \quad d = \frac{\epsilon^2}{\pi} \Pi_2 - \frac{1}{\kappa} \Pi_1
\]

(6.17)

where \( u = \omega^2 - \beta q^2 \). The \( \beta \)-dependence in the expression for the Chern-Simons part should disappear in physical quantities.

Indeed, for the energy density (6.14), we have

\[
\Delta E = E(1) - E(0)
\]

\[= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \ln \det \left\{ 1 - \Gamma(\epsilon^2 D_0^{\text{EM}} + \epsilon^2 D_0^{\text{CS}}) \right\}
\]

\[= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \ln \left\{ 1 + \frac{\epsilon^2}{s} \Pi_2 + \frac{\epsilon^2}{\epsilon} \Pi_0 - \frac{2}{\kappa} \Pi_1 + \left( \frac{\epsilon^4}{\epsilon s} - \frac{1}{\kappa^2} \right) (\Pi_0 \Pi_2 - \Pi_1^2) \right\}
\]

(6.18)

The shift in the energy density in pure Chern-Simons or pure Maxwell theory is found as a special case:

\[
\Delta E_{\text{pure CS}}^{\text{EM}} = \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \ln \left\{ \left( \Pi_1 - \frac{1}{\kappa} \Pi_0 \Pi_2 \right) \right\}
\]

\[\Delta E_{\text{pure EM}}^{\text{CS}} = \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \ln \left\{ \left( 1 + \frac{\epsilon^2}{s} \Pi_0 \right) \left( 1 + \frac{\epsilon^2}{s} \Pi_2 \right) - \frac{\epsilon^4}{\epsilon s} \Pi_1^2 \right\}
\]

(6.19)

In terms of \( \hat{F}_j(x, z) = F_j(x, iz) \) introduced in (5.10) and (5.11),

\[
\Delta E = \frac{1}{4\pi^2 ml^4} \int_0^{\Lambda_x} dx \int_0^{\Lambda_z} dz \ln \left\{ 1 + \frac{\epsilon^2}{2\pi ml^2 s} \hat{F}_2 + \frac{\epsilon^2 ml^2}{2\pi \epsilon} \hat{F}_0 - \frac{\epsilon (b_{\text{tot}})}{\pi \kappa} \hat{F}_1 + \frac{1}{(2\pi)^2} \left( \frac{\epsilon^4}{\epsilon s} - \frac{1}{\kappa^2} \right) (\hat{F}_0 \hat{F}_2 - \hat{F}_1^2) \right\}
\]

(6.20)

where \( s = -(2\chi / l^2) x - (\epsilon / m^2 l^4) z^2 \). Here we have introduced ultraviolet cutoffs, \( \Lambda_x \) and \( \Lambda_z \), supposing that the model (2.1) is an effective theory valid at low energies. They are related to the momentum and frequency cutoffs by \( \Lambda_x = p_0^2 l^2 / 2 \) and \( \Lambda_z = ml^2 \omega_c \).

Expression (6.20) contains many parameters. Let us take anyon superconductivity as a typical example. \( \rho \sim 10^{14} \text{ cm}^{-2} \) so that \( b_{\text{tot}} \sim 100 \text{ T} \) and \( l \sim 10 \text{ A} \). The lattice spacing \( a \) is about 5\( \text{A} \). For high \( T_c \) superconductors one expects \( p_c \sim a^{-1} \) and \( \omega_c \sim 1 \text{ eV} \). The bare electron mass is \( m = 2.6 \times 10^{-26} \text{ cm}^{-1} \). With these values \( \Lambda_x \sim 2 \), and \( \Lambda_z \sim 2.6 \). Note that \( \Lambda_z / \Lambda_x = 2m \omega_c / p_0^2 \) and that the effective mass \( m \) may be different from the bare electron mass. The coupling constant \( e^2 / 4\pi \) is \( \sim a / d \) where \( a = 1 / 137 \) is the fine structure constant and \( d \sim 5 \text{A} \) is the interplanar spacing. This gives \( e^2 / 4\pi \sim 1.5 \times 10^3 \text{ cm}^{-1} \), \( e^2 l / 4\pi \sim 0.015 \).
and $e^2ml^2/4\pi \sim 40$. However, in the Fradkin-Lopez picture of the fractional quantum Hall effect, there are no cutoffs. We shall come back to this point in Section 8.

In pure Chern-Simons theory

$$\Delta \mathcal{E} = \frac{\pi \bar{\rho}^2}{m} R(\Lambda_x, \Lambda_z; c)$$

$$R(\Lambda_x, \Lambda_z; c) = \frac{1}{\pi} \int_0^{\Lambda_x} dx \int_0^{\Lambda_z} dz \ln \Re(x, z; c) \quad (6.21)$$

$$\Re(x, z; c) = (1 - c\tilde{F}_1)^2 - c^2\tilde{F}_0\tilde{F}_2, \quad c = \frac{\epsilon(b_{tot})}{2\pi \kappa}.$$  

Here we have made use of the relation $\nu = 1$ and $\bar{\rho} = (2\pi l^2)^{-1}$. $R(\Lambda_x, \Lambda_z; c)$ represents the RPA correction relative to the mean field value. It has an important dependence on the number density $\bar{\rho}$ through $\Lambda_x$ and $\Lambda_z$. A detailed numerical study is presented in the following sections.

The behavior of $R(\Lambda_x, \Lambda_z; c)$ at large $\Lambda_x$ and $\Lambda_z$ is analytically estimated with the aid of (5.18). One finds that for $\Lambda_x, \Lambda_z > \Lambda_0 \geq 10$

$$R(\Lambda_x, \Lambda_z; c) = R(\Lambda_0, \Lambda_0; c) + c\left(\frac{1}{2} - \frac{2}{\pi} \tan^{-1} \frac{\Lambda_z}{\Lambda_x}\right)$$

$$+ c^2 \left\{\frac{1}{2} \ln \frac{\Lambda_x}{\Lambda_0} - \frac{1}{\pi} \int_1^{\Lambda_z/\Lambda_x} du \frac{1}{u} \tan^{-1} u\right\} \quad (6.22)$$

Notice that it grows logarithmically as $\Lambda$, or as the density $\bar{\rho}$ gets smaller.

### 7. Anyon superconductivity

A neutral anyon gas with pure Chern-Simons interactions has an application to anyon superconductivity.[10-26] Previously, the energy density in the neutral anyon gas has been evaluated in the Hartree-Fock approximation by Hanna, Laughlin and Fetter.[12,35] The correction to the energy density is $\sim 1/2 (\frac{1}{2})$ of the mean field value in the $\nu = 1 (\nu = 2)$ case, independent of the particle number density $\bar{\rho}$.

It is known that high $T_c$ cuprate superconductors become superconducting only when a hole concentration $x_c$, namely the number of holes per lattice plaquette, is in the range 0.05 to 0.25. Within the context of anyon superconductivity it has been an unanswered question why superconductivity is achieved only in a limited range of $x_c$.

We shall show that in RPA the energy density is minimized exactly at a number density in the range mentioned above. At a lower or higher number density the energy is increased, and the system is expected to lose superconductivity.

Since $c = 1$ for $\nu = 1$ and $B_{ext} = 0$, the total energy density of a neutral anyon gas is, from (6.21),

$$\mathcal{E}(\bar{\rho})^\text{tot} = \frac{\pi \bar{\rho}^2}{m} \left\{1 + R(\Lambda_x, \Lambda_z; 1)\right\}$$

$$R(\Lambda_x, \Lambda_z; 1) = \frac{1}{\pi} \int_0^{\Lambda_x} dx \int_0^{\Lambda_z} dz \ln \Re(x, z; 1) \quad (7.1)$$

The density dependence comes in through $\Lambda_x = p_c^2/4\pi \bar{\rho}$ and $\Lambda_z = m\omega_c/2\pi \bar{\rho}$. $R$ is evaluated numerically.
First we depict, in Fig. 4, the behaviour of the argument of the logarithm, $\Re(x, z; 1)$, in the integrand. It vanishes at the origin, behaving as $2x + \frac{1}{12}x^2 + z^2$. It is a smooth function, approaching 1 as $x$ or $z$ gets large.

As explained in the previous section, the momentum and frequency cutoffs in anyon superconductivity are approximately $p_c \sim a^{-1}$ and $\omega_c \sim 1$ eV, where $a$ is the lattice spacing. This gives $\Lambda_z/\Lambda_x \sim 1$, but there remains an ambiguity. We have evaluated $R$ as a function of $\Lambda_x$, or equivalently of $(p_c a)^{-2}x_c = \bar{\rho}/p_c^2 = 1/(4\pi \Lambda_x)$, with $\Lambda_z/\Lambda_x \equiv r$ fixed.

Fig. 5 (a) shows the behavior of $R$ for a wide range of $\Lambda_x$. When $\Lambda_x, \Lambda_z > 10$ (small density) it grows logarithmically ($\sim \frac{1}{2} \ln \Lambda_x$) as expected from (6.22).

In Fig. 5 (b), $R$ is plotted in the range $0 < x_c/(p_c a) < 0.5$ where the minimum takes place. We recognize that the minimum is located around $x_c/(p_c a/h)^2 = 0.05 \sim 0.06$ for a wide range of $r$ ($0.2 < r < 5$). If $p_c a/h \sim 2$, $x_c^\text{min} \sim 0.2$. Although the value of $p_c$ is ambiguous, it is safe to conclude that at a lower number density the energy density sharply increases and therefore that the superconductivity is lost at low densities. The correction at the minimum ranges from $-5\%$ to $-25\%$, depending on $r$. It is rather surprising, but also encouraging, that the minimum occurs approximately at a number density where cuprate material is superconducting.

Certainly, more elaboration and detailed study of the anyon superconductivity model is necessary in order to see if it can describe high $T_c$ superconductors or yet-to-be-discovered new material. We leave it for future investigation.

### 8. Fractional quantum Hall effect

In the Jain-Fradkin-Lopez picture of the fractional quantum Hall effect, electrons in an external magnetic field with a filling factor $\nu_{\text{ext}}$ are described by composite fermions interacting with, in addition to the external magnetic field, Chern-Simons gauge fields, as was explained in Section 2.[6-9] Our previous result is valid for the total filling factor $\nu = n = 1$, which applies, from (2.8), to a sequence $\nu_{\text{ext}} = 1/(2p \pm 1)$.

In the electron picture the lowest Landau level has an energy $|eB_{\text{ext}}|/2m$ so that the mean field energy density is

$$\varepsilon_{\text{electron}}^\text{MF} = \frac{1}{\nu_{\text{ext}}} \cdot \frac{\pi \bar{\rho}^2}{m} .$$

In the composite fermion picture the lowest Landau level has an energy $|\bar{b}_{\text{tot}}|/2m$ so that the mean field energy for $\nu = 1$ is

$$\varepsilon_{\text{composite}}^\text{MF} = \frac{\pi \bar{\rho}^2}{m} .$$

Clearly $\varepsilon_{\text{MF}}^\text{composite} < \varepsilon_{\text{MF}}^\text{electron}$. There appears a large discrepancy, by a factor of $\nu_{\text{ext}}^{-1}$.

Since the transformation from the original electron system to the Chern-Simons-fermion system is exact, the discrepancy observed above is merely an artifact of the mean field approximation. We are going to show that quantum fluctuations give a big correction. It is expected that if one includes all corrections, the energy densities in the two systems should be exactly the same.

Without loss of generality we take $eB_{\text{ext}} > 0$ as before. From (2.8) one finds that for $\nu = 1$

$$\nu_{\text{ext}} = \frac{1}{2p \pm 1} \quad \text{where } \pm = \epsilon(b_{\text{tot}})$$

$$c = \mp 2p .$$

(8.3)
As shown by Halperin, Lee and Read [8] and by Simon and Halperin [9], RPA gives a substantial correction to the excitation energy spectrum. In the absence of the Coulomb interaction, the spectrum is determined by zeros of \( \det D^{-1}_{\text{CS}} \propto (\Pi_1 - \kappa)^2 - \Pi_0 \Pi_2 \). In terms of \( F_j \)
\[
(1 - cF_1) - \kappa^2 F_0 F_2 = 0 \quad \text{(8.4)}
\]
At a zero momentum \( x = \frac{1}{2} q^2 l^2 = 0 \),
\[
F_0(0, y) = F_1(0, y) = 1 + F_2(0, y) = \frac{1}{1 - y^2 - i\epsilon} \quad \text{(8.5)}
\]
Substitution of (8.5) into (8.4) yields
\[
y = ml^2 \omega = |c - 1| = 2p \pm 1 = \frac{1}{\nu_{\text{ext}}} \quad \text{(8.6)}
\]
or
\[
\omega(q = 0)_{\text{RPA}} = \frac{1}{\nu_{\text{ext}}} \frac{2\pi \bar{\rho}}{m} \quad \text{(8.7)}
\]
This is exactly the Landau gap in the original electron picture. The RPA correction yields a factor \( \nu_{\text{ext}}^{-1} \) compared with the mean field value in the Chern-Simons picture.

The energy density in RPA is given by
\[
\mathcal{E}(\bar{\rho})_{\text{RPA}} = \frac{\pi \bar{\rho}^2}{m} (1 + R)
\]
\[
R(\Lambda_x, \Lambda_z; c) = \frac{1}{\pi} \int_0^{\Lambda_x} dx \int_0^{\Lambda_z} dz \ln \Re(x, z; c) \quad \text{(8.8)}
\]
We have evaluated \( R \) as a function of \( \Lambda_x \) for various values of \( r = \Lambda_x/\Lambda_z \) and \( c \). There correspond two values of \( c \) to each \( \nu_{\text{ext}} \). For instance, \( \nu_{\text{ext}} = \frac{1}{3} (\pm) \) corresponds to \( c = -2 \) and \( +4 \) (\(-4 \) and \(+6 \)). The function \( \Re(x, z; c) \) in (8.8) is depicted in Fig. 6.

For a large cutoff \( \Lambda_x \), \( R \sim \frac{1}{4} c^2 \ln \Lambda_x \) as follows from (6.22). Its global behaviour is depicted in Fig. 7. For a given \( \nu_{\text{ext}} \), the RPA correction exceeds the saturation value \( \nu_{\text{ext}}^{-1} - 1 \) at around \( \Lambda_x = 10 \sim 100 \). This implies that RPA is not accurate for large momenta and frequencies.

In the light of the equivalence between the original electron picture and the composite fermion picture the cutoffs must be removed. In one respect our result shows that higher order corrections beyond RPA become crucial in the computation of the energy density. It asserts that quantum fluctuations give a large correction of order \( \nu_{\text{ext}}^{-1} \).

9. Summary
We have examined the nonrelativistic Maxwell-Chern-Simons gauge theory relevant in the quantum Hall effect and anyon superconductivity. We have derived compact integral representations for the polarization tensors \( \Gamma^{\mu\nu}(p) \) or \( \Pi_k(p) \). Quantum fluctuations give important contributions to the energy density of the system, which is most conveniently expressed in terms of the gauge field propagators. The energy density was evaluated in RPA.
One of the important consequences in RPA is that there results a non-trivial dependence of the energy density on the particle number density. In particular, in anyon superconductivity, the ratio of the energy density in RPA to that in the mean field approximation is minimized at a hole concentration $x_c = (0.05 \sim 0.06)(p_c a/h)^2$. In light of the application to high $T_c$ superconductors this is extremely intriguing and encouraging. Typical high $T_c$ cuprates become superconducting for $0.05 < x_c < 0.25$. Neither the mean field approximation nor the Hartree-Fock approximation have been successful in explaining why superconductivity is achieved only in a limited range of $x_c$.

In the application to the fractional quantum Hall effect, the RPA correction to the energy density in the Fradkin-Lopez picture diverges as the cutoffs are removed. The divergence comes from large momenta and frequencies where RPA is expected to break down. We may conclude that higher order quantum fluctuations give substantial corrections to the energy density, presumably restoring the equivalence between the original electron picture and the composite fermion picture.

In this paper we have concentrated on the case of a unit filling $\nu = 1$ with respect to the total magnetic field $b_{\text{tot}} = \rho/\kappa + eB_{\text{ext}}$. The generalization to arbitrary integer filling $\nu = n$ is important both for the anyon superconductivity and for the fractional quantum Hall effect. It is left for future investigation.

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Figure captions

Fig. 1 One-loop corrections $\Gamma^{\mu\nu}$ defined in (4.8) and (5.2). Three kinds of vertices are given by the last three terms in (3.4).

Fig. 2 The behaviour of $\tilde{F}_j(x, z)$ in (5.17). (a) $\tilde{F}_1(x, z)$, (b) $\tilde{F}_2(x, z)$ and (c) $1 + \tilde{F}_2(x, z)$. All are smooth functions of $x$ and $z = iy$. They start out with unity at the origin and approach zero asymptotically for large $x$ or $z$ (5.18).

Fig. 3 The energy density in RPA. See Eq. (6.14). A doubly dashed line represents the sum of Chern-Simons and electromagnetic field propagators.

Fig. 4 Plot of the argument of the natural logarithm in (7.1) as a function of $x$ and $z = iy$. It vanishes at the origin and approaches unity asymptotically for large $x$ or $z$. (See text for more details on the asymptotic behaviour.) This is related to the RPA energy
density of a neutral anyon gas. Note that conventional order of \( x \) and \( z \) has been switched for visual clarity.

Fig. 5 Magnitude of the RPA correction to the mean field energy density \( R(\Lambda_x, \Lambda_z; 1) \) for a neutral anyon gas normalized to the mean field value for various choices of the lattice cutoff \( \Lambda_x \) and \( \Lambda_z \) (7.1). The solid line denotes cutoff ratio \( r = \Lambda_z / \Lambda_x = \frac{1}{5} \), the dashed line denotes \( r = 1 \) and the dotted line denotes \( r = 5 \). (a) Plotted as a function of \( \Lambda_x \) for fixed values of \( r \) on a logarithmic scale. Linearity for large values of \( \Lambda_x \) is due to the logarithmic growth of \( R \) with cutoff (6.22). Departure from linearity at very large values of \( \Lambda_x \) is understood to be a numerical artifact. Note that the minima for \( r = 1 \) and \( r = 5 \) are not resolved. They are resolved in the following figure. Numerical estimates of \( R \) are accurate to about 5%. (b) Alternative presentation of Fig. 5(a) around the minima. \( R \) has been plotted as a function of \( x_c(p_c a / \hbar)^{-2} = 1/(4\pi \Lambda_x) \) (scaled hole concentration), where \( p_c \) and \( a \) are the momentum cutoff and lattice spacing, respectively. The minima are adequately resolved and their locations are more or less independent of the cutoff ratio \( r \). For large values of concentration, \( R \) approaches zero. We note intriguing similarity with the cuprate superconductors. Numerical estimates of \( R \) are accurate to about 10%.

Fig. 6 Plot of the argument of the natural logarithm in (8.8) for \( \nu_{\text{ext}} = \frac{1}{3} \) as functions of \( x \) and \( z = iy \). Corresponding values of the constant \( c \) are \(-2\) and \(+4\). Both start with \( \Re = 9 \) at the origin. See text for more details on the asymptotic behaviour. Note that conventional order of \( x \) and \( z \) has been switched for visual clarity. (a) Plot for \( c = +4 \). (b) Plot for \( c = -2 \).

Fig. 7 Magnitude of the RPA correction to the mean field energy density \( R(\Lambda_x, \Lambda_z; c) \) for a quantum Hall system normalized to the mean field value for various choices of the lattice cutoff \( \Lambda_x \) and \( \Lambda_z \) (8.8). \( R \) is plotted as a function of \( \Lambda_x \) for fixed values of \( r = \Lambda_z / \Lambda_x \) on a logarithmic scale. Linearity for large values of \( \Lambda_x \) is due to the logarithmic growth of \( R \) with cutoff (6.22). \( R \) is always positive in all cases. Departure from linearity at very large values of \( \Lambda_x \) is understood to be a numerical artifact. Numerical estimates of \( R \) are accurate to about 5%. (a) Behaviour for a \( c = 4 \) (\( \nu_{\text{ext}} = \frac{1}{3} \)) system. The solid line denotes cutoff ratio \( r = \frac{1}{5} \), the dashed line denotes \( r = 1 \) and the dotted line denotes \( r = 5 \). (b) Behaviour for various choices of external filling fraction with \( r = 1 \). The solid line denotes \( c = -2 \) (\( \nu_{\text{ext}} = \frac{1}{3} \)), the dashed line denotes \( c = +6 \) (\( \nu_{\text{ext}} = \frac{1}{3} \)), the dotted line denotes \( c = -4 \) (\( \nu_{\text{ext}} = \frac{1}{3} \)) and the dash-dotted line denotes \( c = +4 \) (\( \nu_{\text{ext}} = \frac{1}{3} \)).
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