Cell division is fundamental to all living organisms. The last stage of cell division is called cytokinesis, where closure of a polymeric ring, made of actin filaments and myosin molecular motors [1, 2] completes the physical partitioning of the cell. In one mode of partitioning an intercellular membrane forms (see Fig-1a). This is common in mitotic cell divisions (eg., in C. elegans embryo, a widely studied model system for eukaryotes) and also in some compact tissues [3]. In the other mode (see Fig-1b), the contact area between the daughter cells gradually shrinks to zero, as the division furrow (the cusp in Fig-1b) caves in [4]. Here we focus on the development of the intercellular membrane (the first mode) which starts out as an annulus at the equatorial plane (see Fig-1a and inset-c) and gradually closes itself, as its inner boundary grows radially inward. The growth is assisted by the flow of actomyosin, beneath the cell surface (the cortical flow) [5]. Experiments suggest [1, 2, 6] that the ATP driven interaction between actin and myosin lead to the generation of active contractile stresses in the cytokinetic ring. How this stress changes with time during the course of the constriction however is not clear. Earlier models [1, 7] explain the observed contraction rate by assuming a constant contractile stress. Ref [8] had in addition assumed, an ad hoc intrinsic dynamic friction, to account for the eventual slowdown of the contraction process.

Such an approach, that considers the actin ring to be a separate entity attached with the growing active membrane, cannot explain the recent experimental observations [9] where the ring is found to reorganize and contract even after part of it is destroyed by localized laser ablation. This motivates us to consider the cortical ring to be part of the acto-myosin continuum spread over the growing membrane interface. In Ref [10], the authors developed an active gel model of the cytoskeletal flows to discuss wound healing in Xenopus oocyte [11]. Such a description involves solution of coupled equations for the actin alignment field $Q_{\alpha\beta}(r)$ (the order parameter OP) and the velocity field $v_\alpha(r)$. The ring was assumed to be a narrow annular zone with higher level of myosin activity $\zeta \Delta \mu$ than the rest of the growing interface.

In this Letter, we follow a similar continuum gel theory approach and first solve the coupled equations for the OP and the velocity fields numerically (Fig-1), retaining flow coupling. But instead of assuming an active contractility gradient, which is standard in the literature [10], we use the observation, that actin filaments are aligned tangentially to the inner boundary of the closing annulus [12, 13] as a boundary condition. This is motivated by recent experiments [14, 15] which indicate that local assembly kinetics, like guided polymerization, can drive rapid filament alignment at the ring, at a much faster rate compared to the relatively slow hydrodynamic modes of the OP and the flow fields. The mean (time averaged) effect of this molecular level, fast, alignment kinetics can be incorporated in the hydrodynamic equation for the OP field as a boundary condition. Such a boundary driven alignment was used in Ref [10, 13] to solve for the OP field. Ref [16] reported that acto-myosin filaments at open cell boundaries can respond to the curvature of the boundary, and align parallel or perpendicular to the concave or convex boundaries, respectively. Encouraged by these observations, on boundary driven alignment, we set out to compute, a) the constriction rate of the cytokinetic ring, and b) its stability with respect to non-axisymmetric deformations, which has wide applicability across eukaryotic cell division.

**Model**: The actomyosin gel on the growing interface is modeled as a nematic fluid. Orientational order in a nematic fluid, in 3-dimensions, is defined by the tensor order parameter $Q_{\alpha\beta} = \langle n_\alpha n_\beta - \delta_{\alpha\beta}/3 \rangle$, where $n_\alpha$ is the nematic director field, and $\alpha, \beta = (x, y, z)$. As the acto-myosin filaments (nematic directors) lie in the flat interface $(x - y)$, symmetry and tracelessness of $Q_{\alpha\beta}$...
dictate that the non-diagonal matrix elements involving z are zero, $Q_{xy} = Q_{yx} = q$, $Q_{zz} = -1/3$, and $Q_{xx} + Q_{yy} = 1/3$. Further, if the orientation distribution is isotropic in the $x-y$ plane then the resulting matrix $Q^0_{\alpha\beta}$ is diagonal, with $Q^0_{xx} = Q^0_{yy} = 1/6$, and $Q^0_{zz} = -1/3$. In the presence of cortical flows or due to specific boundary conditions the isotropic distribution is modified to $Q_{\alpha\beta} = Q^0_{\alpha\beta} + Q^1_{\alpha\beta}$. Again symmetric and tracelessness of $Q_{\alpha\beta}$ require (see Supplementary information -SI) that, $Q^0_{xx} = Q^0_{yy} = Q$, $Q^0_{xy} = Q^0_{yx} = q$, and rest of the elements are zero. This form remains invariant as we transform from cartesian to 2D polar coordinates later.

Active gel model for acto-myosin filaments: The free energy of the inhomogeneous nematic field can be described by the Landau-De Gennes form [17], using the $Q^r$ matrix, $F = \int d^3r \left( \frac{1}{2} Q^r_{ij} Q^r_{ij} + \frac{1}{2} \partial_t Q^r_{ij} \partial_t Q^r_{ij} \right)$. This enforces an isotropic arrangement of the director field in the bulk of the 2D growing cortical layer with a correlation length $L_c = \sqrt{2L/\chi}$. Later, we will see that this turns out to be the width of the actomyosin ring, which has been measured [18] to be $1 \mu m$.

Constitutive equations of the active gel can be described by a linear relationship between thermodynamic fluxes and forces [10, 11, 21]. We choose stress tensor $\sigma_{\alpha\beta}$, the rate of change of nematic order parameter $DQ_{\alpha\beta}/Dt$, and the rate of ATP consumption as the fluxes. The conjugate forces are the strain rate $v_{\alpha\beta} = \frac{1}{2} (\partial_\alpha v_\beta + \partial_\beta v_\alpha)$, the traceless nematic force field $H_{\alpha\beta} = \frac{Q_{\alpha\beta}}{\beta} = \frac{Q^r_{\alpha\beta}}{\beta}$, and the chemical potential difference generated due to ATP hydrolysis $\Delta \mu$. Following [11], the hydrodynamic equations in the liquid limit can be expressed as follows:

$$\sigma_{\alpha\beta} = 2\eta v_{\alpha\beta} - \beta_1 H_{\alpha\beta} + \zeta \Delta \mu Q_{\alpha\beta},$$

$$\frac{D}{Dt} Q_{\alpha\beta} = \beta_1 v_{\alpha\beta} + \frac{1}{\beta_2} H_{\alpha\beta}.$$  

$D/\partial t$ here implies material derivative [11], $\zeta \Delta \mu Q_{\alpha\beta}$ is the active stress and contractility of the cortical layer enforces $\zeta > 0$ [11, 21]. We ignored any explicit active term in the second equation because it just renormalizes the inverse susceptibility $\chi^{-1}$. Here $\eta$ is the fluid viscosity while $\beta_1$ and $\beta_2$ are Onsager coefficients [10], and give the flow coupling and nematic relaxation strengths, respectively [11].

Following [11], we define a 2D “tension tensor” $t_{ij}$ via the relation $t_{ij} = \int (\sigma_{ij} - \delta_{ij} P)dz$. Imposing the net normal stress on the interface $t_{zz}$ to be zero yields pressure $P = \sigma_{zz}$. Further, ignoring variation of stress across the thin interface, we get [11] $t_{ij} = e(\sigma_{ij} - \delta_{ij} \sigma_z)$, where $e$ is the effective thickness of the interface, assumed to be a constant here. This tension tensor allows us to write a two-dimensional hydrodynamic theory with the force balance equation as $\partial_t (\rho v_i) = \partial_j t_{ij} - \alpha v_i$. Here $\alpha$ is the cytoplasmic friction external to the growing membrane interface. The flat growing interface has an annular shape, see inset of Fig.1c. The shrinking cytokinetic ring of radius $R_0(t)$ lies at its inner perimeter, while its outer perimeter is fixed at radius $r_0$. After changing to 2D polar co-ordinates, and dropping the time derivative in highly viscous regime, the force balance equations are

$$\partial_r (t_{rr} + \frac{1}{r} (t_{r\theta} + t_{\theta r}) + \frac{1}{r^2} t_{\theta\theta}) = \alpha v_r,$$

$$\partial_r (t_{r\theta} + t_{\theta r}) + \frac{1}{r} (t_{\theta\theta} - t_{rr}) = \alpha v_\theta.$$

The $2 \times 2 (xy)$ block of $Q_{\alpha\beta}$ matrix (anisotropic part) remains traceless and symmetric, parameterised by two variables $\hat{q}$ and $q$, although their values change in the polar frame. The $2 \times 2$ block of the isotropic matrix however remains unchanged, $Q^0_{\alpha\beta} = 1/6$, where $I$ is the identity matrix (see SI).

Rotationally symmetric solutions for $Q^r_{\alpha\beta}(r)$ and $v_\alpha(r)$: We first consider the special case where the circular ring is at $r = R_0$, with our domain of interest $r \geq R_0$. We start with $\alpha = 0$, set stress free boundary condition at the open edge, i.e., normal stress $\sigma_{rr}(R_0) = 0$, and $v_r = 0$ at $r \to \infty$. The nematic directors are assumed to be parallel to the inner boundary, i.e., $\hat{n}(R_0) = \hat{\theta}$, and isotropic as $r \to \infty$. It implies, that at $r = R_0$, the anisotropic $Q_{\alpha\beta}$ matrix is diagonal with $Q^r_{rr} = -Q^r_{\theta\theta} = \hat{Q} = -1/2$ (see SI), and $Q^r_{rr}(r = \infty) = 0$.

We assume a quasi-steady state where the material derivative $DQ_{\alpha\beta}/Dt = 0$ in Eq.2. Note that $DQ_{\alpha\beta}/Dt \neq 0$ since the the inner edge $R_0$ keeps moving, but the convection term $\nabla Q_{\alpha\beta}$ counters this change to keep $Q_{\alpha\beta}$ unaltered in the material frame. This yields $H_{\alpha\beta} = -\beta_1 \beta_2 v_{\alpha\beta}$. When expressed in polar form the diagonal elements of this equation gives Eq.3 below. However the non-diagonal part yields, $q = 0$ (see SI). Here we used $\beta_2 \approx \eta [10]$ and $\zeta \Delta \mu/\chi \approx 1$.

$$\frac{1}{r} \partial_r (r \partial_r) \hat{Q} - \left( \frac{1}{L^2_c} + \frac{4}{r^2} \right) \hat{Q} = -\frac{\beta_1}{2L^2_c} \frac{\eta}{\zeta \Delta \mu} \left( \partial_r v_r - \frac{v_r}{r} \right)$$

Substitution of $H_{\alpha\beta} = -\beta_1 \beta_2 v_{\alpha\beta}$ into Eq.4 simply renormalizes the viscosity to $\tilde{\eta} = \eta (1 + \frac{1}{2} \beta_1^2)$. The resulting velocity equation (in polar form) using force balance yields

$$\partial_r \left( \partial_r + \frac{1}{r} \right) v_r = -\zeta \Delta \mu \left( \partial_r + \frac{2}{r} \right) \hat{Q},$$

Using zero influx $v_r(r_0) = 0$ at the outer boundary, and a stress free inner boundary $\sigma_{rr}(R_0) = 2\eta \partial_r v_r + \frac{\Delta \mu}{6} + \zeta \Delta \mu \hat{Q} = 0$, we solve these two coupled equations numerically (using Mathematica), for different values of the flow coupling strength $\beta_1$. The solutions are shown in Fig.11 using $L_c$ as unit of length and $\frac{\Delta \mu}{6}$ as unit of time. It shows damping of the velocity field $v_r$ with increase in flow coupling strength $\beta_1$. Therefore, stronger flow coupling delays the ring closure time, however the order parameter profile, shown in the inset of Fig.11, appears to be almost unaffected by flow coupling strength $\beta_1$. Note that, in this moving boundary problem, the major role
of the flow coupling on the OP is to move the boundary inward where the actin field gets realigned quickly. By setting \( \dot{Q}(R_0) = -1/2 \) we have already captured this effect indirectly. This important observation allows us to ignore flow coupling in the OP equation here (r.h.s. of Eq.3) which can now be solved exactly. The general solution is \( \dot{Q}(r) = c_1 K_2(r/L_c) + c_2 I_2(r/L_c) \), where \( K_2 \) and \( I_2 \) are modified Bessel functions (see SI). For outer boundary \( r_0 \to \infty \), we get

\[
\dot{Q}(r) = -K_2(r/L_c)/2K_2(R_0/L_c)
\]

The solution for finite \( r_0 \) is given in the SI. The sharp rise in the magnitude of \( \dot{Q} \) (irrespective of \( \beta_1 \)) at the inner edge can be interpreted as the acto-myosin ring, of width \( L_c \). Using this solution we can now solve for \( v_r \) (Eq.4) with arbitrary \( \beta_1 \). For \( r_0 \to \infty \), the solution reads,

\[
v_r(r) = -\frac{\zeta \Delta \mu / \eta}{3K_1(R_0/L_c) 4K_2(R_0/L_c)} \frac{R_0^2 + L_c K_1(r/L_c)}{6r} + \frac{L_c K_1(r/L_c)}{8 K_2(R_0/L_c)}
\]

Note that the velocity at \( r = R_0 \), is the ring closure rate \( v_r(R_0) = -\frac{\zeta \Delta \mu / \eta}{3K_1(R_0/L_c) 4K_2(R_0/L_c)} \frac{R_0^2 + L_c K_1(R_0/L_c)}{6r} + \frac{L_c K_1(R_0/L_c)}{8 K_2(R_0/L_c)} \), which is directly damped by the flow coupling strength \( \beta_1 \) via the effective viscosity \( \tilde{\eta} \). Inclusion of cytoplasmic friction ourselves to radial motion only \((v_\theta, \text{ nonzero}, v_y = 0)\) and assuming azimuthal symmetry, we get

\[
\frac{\partial_r \left( \frac{\partial_r + \frac{1}{r}}{r} \right) v_r}{\frac{\zeta \Delta \mu}{4 \eta}} = -\frac{\zeta \Delta \mu}{4 \eta} \left( \frac{\partial_r + \frac{2}{r}}{r} \right) \dot{Q}.
\]

With boundary conditions \( \dot{Q}(r_0) = v_r(r_0) = 0 \), and those at \( r = R_0 \) remaining same as before, we solve Eq.7 both using Green’s function (see SI) and numerically in Mathematica. As expected, see Fig.2 inset cytoplasmic friction damps the flow at the growing interface and slows down the ring closure speed (inset of Fig.2).

The above analysis is carried out quasi-statically for a fixed \( R_0 \). We can use these results to obtain the ring closure kinetics. We integrate the kinematic boundary condition \( \frac{\partial}{\partial t} R_0 = v_r(R_0) \) to derive the time dependence of the radius of the contracting ring i.e., \( R_0 \) versus \( t \). In Fig.2 inset we compare this closure rate with experimental data on \( C. \) elegans embryo \([6, 22]\). Note that this is a three parameter fit with \( \alpha, \beta_1 \) and the active time scale \( \frac{\eta}{\zeta \Delta \mu} \). Reasonable fits can be obtained for several combinations of these parameters in the range \( \alpha, \beta_1 \in [0.1, 0.5] \) and \( \frac{\eta}{\zeta \Delta \mu} \in [1.5, 2.5] \) sec. One such example is shown in Fig.2 inset. Here we used \( L_c = 1 \mu m \) \([22]\). Membrane tension \( \sigma_0 \) in the growing membrane can be linked to the activity as \( \sigma_0 = \zeta \Delta \mu / 2 \) \([10]\). Using \( \frac{\eta}{\zeta \Delta \mu} \approx 2 \) secs, measured value of cortical tension \( \sigma_0 = 3 \times 10^{-4} \) N/m \([22]\) and the thickness of the growing actomyosin cortex \( e \approx 0.3 \mu m \) \([22]\), we get \( \eta \approx 4 \times 10^3 \) Pa.sec, which is similar to.

![Figure 1](image1.png)

Figure 1: Solutions for the radially symmetric velocity field \( v_r(r) \) (main figure) and the OP field \( \dot{Q}(r) \) (inset) are shown as a function of \( r/R_0 \), for \( R_0 = 5 \mu m \). Schematic diagram ‘a’ shows sideview of the growing interface at the middle of the cell and ‘c’ shows its cross-sectional view (‘b’ shows partitioning without an interface). Alignment of filaments increases sharply near the inner boundary of the annulus at \( R_0 \). The outer boundary is fixed at \( r_0 = 15 \mu m \) for these plots.

friction \((\alpha \nu)\), the velocity influx \( v_r(r_0) \) at a finite outer boundary \( r = r_0 > R_0 \) (instead of \( r_0 \to \infty \)) can also influence the flow and the closure speed. Solutions for the boundary conditions \( \dot{Q}(r_0) = 0 \) and \( v_r(r_0) = 0 \) are given in the SI.

Cytoplasmic friction adds \( \alpha v_r \) to the right hand side of Eq.4 but does not alter the equation for \( \dot{Q} \).

![Figure 2](image2.png)

Figure 2: Cytoplasmic friction slows down the flow: \( v_r \) versus \( r \) in the main plot for different friction coefficients \( \alpha \). Inset: lines show scaled radius of the ring \( R_0(t)/r_0 \) versus time (sec), for \( \alpha, \beta_1 = 0 \) and nonzero values (see legends), with \( \frac{\eta}{\zeta \Delta \mu} = 2.06 \) secs for both. Furthermore, \( r_0 = 14 \mu m \), and \( v_r(r_0) = 0 \). Circles are the experimental data on \( C. \) elegans embryo \([6, 22]\).

The ring closure rate in euukaryotes shows an intriguing slow down at late times (Fig.2 inset), which has not been understood yet. In Ref.8 an adhoc intrinsic dynamic friction \( \zeta_L \) was added, to the ring tension to account for
Figure 3: The line tension, \((\text{Eq.} 5)\), at the ring \(\Sigma = \sigma_{\theta \theta}(R_0)\), in units of activity \(\zeta \Delta \mu\), shown as a function of ring radius \(R_0\) (in a), and as a function of \(r\), at a fixed \(R_0 = 5 \mu m\). The outer radius is fixed at \(r_0 = 14 \mu m\), appropriate for C. elegans embryo \([9, 22, 25]\). Friction (nonzero \(\alpha\)) does not affect ring tension significantly.

The outer radius is fixed at \(r_0 = 14 \mu m\), appropriate for C. elegans embryo \([9, 22, 25]\). Friction (nonzero \(\alpha\)) does not affect ring tension significantly.

Figure 4: \(\omega_n\), scaled by the inner radius \(R_0\), as a function of the mode number \(n\) for different values of \(R_0\) (\(\mu m\)), see legends. Low wave number modes switches from unstable to stable at smaller \(R_0\).

any given time, as \(r(\theta) = R_0 + \delta R(\theta)\), and using Fourier decomposition \(\delta R(\theta, t) = \sum_{n=0}^{\infty} Q_n e^{in\theta + \omega_n t}\) . We study stability of these deformation modes \([26]\) by computing \(\omega_n\), up to \(n = 10\). Note that the \(n = 1\) mode corresponds to an uniform translation of the inner circular boundary and therefore \(\omega_1 = 0\). The system has translational symmetry provided the outer boundary \(r_0 \rightarrow \infty\), which we exploit for this calculation. The results below are unlikely to change qualitatively when \(r_0\) is finite, except that \(\omega_1\) will be nonzero.

The change at the inner edge leads to change in all the dynamical variables \(Q(r, \theta, t) = Q_0(r) + \delta Q(r, \theta, t)\), and similarly, \(q(r, \theta, t) = \delta q(r, \theta, t)\), \(v_r(r, \theta, t) = v_0^0(r) + \delta v_r(r, \theta, t)\), and \(v_\theta(r, \theta, t) = \delta v_\theta(r, \theta, t)\).

Further, the perturbation fields \(\delta Q, \delta q, \delta v_r, \delta v_\theta\) can be decomposed into Fourier modes as \(\delta Q(r, \theta, t) = \sum_{n=0}^{\infty} Q_n(r)e^{in\theta + \omega_n t}\), \(\delta v_r(r, \theta, t) = \sum_{n=0}^{\infty} \delta v_{r,n}(r)e^{in\theta + \omega_n t}\), and similarly for the other two fields.

We substitute these perturbed fields in the dynamical equations and do a linear stability analysis to obtain \(\{\omega_n\}\), where \(\omega_n = \partial_r v_0^0(R_0) + \delta v_{r,n}(R_0) / \delta R_n\) following Ref \([26]\). Details of our calculations are given in the SI.

Fig \([4]\) reveals interesting behaviour for the growth rates of the Fourier modes \(\{\omega_n\}\) for different inner radius \(R_0\). At large \(R_0\) several modes are unstable \((\omega_n > 0)\), however they subsequently turn stable \((\omega_n < 0)\) as \(R_0\) becomes small, absolutely consistent with experimental observations. Note that \(\omega_0 < 0\), irrespective of \(R_0\), implies stability with respect to uniform contraction or expansion of the circular inner boundary. While in our theory \(\omega_0\) is exactly proportional to the activity, Fig \([4]\) shows that \(\omega_0\) is approximately proportional to \(R_0\). Also note that the higher modes decay relatively faster which would make any sharp distortion of the ring heal fast. This could be relevant for would healing in cells as well. But the fact that larger number of modes are unstable at larger ring...
size indicates that very large rings, if distorted, will fail to contract.

In summary, our phenomenological approximation on the boundary driven actomyosin alignment, was useful in obtaining exact solutions for the OP and the velocity field. The stability calculation, which produced several insights, exploited these solutions to perturb around them. Also we could identify three separate sources of slow down near the end of the contractions, namely, a) the curvature at the ring \( (1/R_0) \), b) the cytoplasmic friction \( (\alpha) \), and c) the flow coupling strength \( (\beta) \). Experiments along the lines of Ref\[14\] which probed poly/depolymerization processes near the ring and Ref\[9\] which studied healing of the perturbed ring after laser ablation, might be useful to assess the role of of boundary in maintaining actin alignment in the dynamic ring.

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Supplementary Information : Part-I
Dynamics and stability of the contractile actomyosin ring in the cell

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1. STRUCTURE OF THE ORDER PARAMETER MATRIX $Q_{\alpha\beta} = n_\alpha n_\beta - \delta_{\alpha\beta}/3$

1.1. $Q_{\alpha\beta}$ for filaments lying in the $x-y$ plane

When the filaments lie entirely in the $x-y$ plane, i.e., $\hat{n} = (n_x, n_y, 0)$ then the symmetric, traceless matrix,

$$Q = \begin{bmatrix} Q_1 & q & 0 \\ q & Q_2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}, \text{ with } Q_1 + Q_2 = 1/3$$

When the distribution of filaments is random (isotropic) in the X-Y plane then due to $\langle n_x^2 \rangle = \langle n_y^2 \rangle = 1/2$ and $n_z = 0$, the resulting $Q^0$ is diagonal, with $Q_{xx} = Q_{yy} = 1/6$ and $Q_{zz} = -1/3$. Due to flow or boundary condition $Q$ will deviate from its isotropic form and thus, $Q = Q^0 + Q'$. Explicitly,

$$\begin{bmatrix} Q_1 & q & 0 \\ q & Q_2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} + \begin{bmatrix} q_1 & q & 0 \\ q & q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

preserving symmetric structure of the L.H.S.

Further, $Q'$ also must be traceless, since $Q$ and $Q^0$ are already traceless, and thus $q_1 = -q_2 (= \tilde{Q}$ in the main text).

Figure 1: Nematic aligned along the azimuthal direction

1.2. Boundary condition for the order parameter $Q_{\alpha\beta}$

We impose that the acto-myosin filaments (red lines in Fig.1 below) be completely aligned with the circular inner boundary, at $r = R_0$, i.e., they are parallel to the azimuthal direction $\hat{\theta}$. At the outer boundary, either at a finite radius $r_0 > R_0$ or at $r \to \infty$, the filaments are randomly oriented. This is a phenomenological input to our theory, ensuring that an acto-myosin ring forms at the open edge of the inward growing membrane. Below, we derive the corresponding order parameter tensor in the cylindrical polar coordinates.
Let the nematic director on the ring, at an angle $\theta$, be oriented along $\hat{n}(\theta)$, (see Fig.[1]). In the Cartesian reference frame $(x, y)$, $\hat{n}(\theta) = (n_x, n_y, 0) = \hat{\theta} = (-\sin \theta, \cos \theta, 0)$. The resulting order parameter matrix at $r = R_0$ is then

$$Q(R_0, \theta) = \begin{bmatrix} n_x^2 - \frac{1}{3} & n_x n_y & 0 \\ n_x n_y & n_y^2 - \frac{1}{3} & 0 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} \sin^2 \theta - \frac{1}{3} & -\frac{\sin 2\theta}{2} & 0 \\ -\frac{\sin 2\theta}{2} & \cos^2 \theta - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

But $Q$ has a simple form in the polar coordinate frame $(\hat{r}, \hat{\theta})$, which is a obtained from the original cartesian frame $(x, y)$ via a counter-clockwise rotation (by angle $\theta$). In the polar frame, $\hat{n}(\theta) = (0, 1, 0)$, and thus,

$$Q^p(R_0, \theta) = \begin{bmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}.$$ 

Here the superscript $p$ denotes polar. We seek the same decomposition of $Q^p$ into isotropic and deviation parts: $Q^p = Q^0p + Q^p$. However $Q^0p$ remains same in the two coordinate frames. Thus,

$$Q^p(R_0, \theta) = -\frac{1/3}{0} \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (1)$$

This serves as the boundary condition at $r = R_0$ in our radially symetric formulation. Here we obtained the polar forms of the $Q$ matrices using the transformed form of director $\hat{n}$ in the polar frame. One could also transform the $Q$ matrix from the cartesian frame $(x, y)$ to the polar frame $(\hat{r}, \hat{\theta})$, using $Q^p = RQ^0R^\top$, where $R$ is the rotation matrix for counter-clockwise passive rotation about $\hat{z}$ axis, by angle $\theta$. More explicitly,

$$Q^p = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin^2 \theta - \frac{1}{3} & -\frac{\sin 2\theta}{2} & 0 \\ -\frac{\sin 2\theta}{2} & \cos^2 \theta - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}. \quad (2)$$

Using the same transformation on $Q^0$ one can show that the isotropic part remain unchanged in the polar frame.

### 2. Constitutive Relations and the Force-Balance Equation in 2D Polar Coordinates

Recall the constitutive equations (Eq. (2),(3) in the maintext) for the stress and the nematic order parameter

$$\sigma_{\alpha\beta} = 2\eta v_{\alpha\beta} - \beta_1 H_{\alpha\beta} + \zeta \Delta \mu Q_{\alpha\beta}, \quad (3)$$

$$\frac{D}{Dt} Q_{\alpha\beta} = \beta_1 v_{\alpha\beta} + \frac{1}{\beta_2} H_{\alpha\beta}. \quad (4)$$

In the steady state $H_{\alpha\beta} = -\beta_1 \beta_2 v_{\alpha\beta}$. Substitution of $H_{\alpha\beta}$ into Eq.3 renormalises the viscosity to $\tilde{\eta} = \eta(1 + \frac{\beta^2 \beta_2}{2\eta})$. Further, using the definition $Q_{\alpha\beta} = Q^0_{\alpha\beta} + Q'_{\alpha\beta}$ and retaining the same notation $\eta$ (henceforth) for the renormalised viscosity, the components of the stress tensor given in Eq. (3) can be written in 2D polar coordinates $(r, \theta)$ as:

$$\sigma_{rr} = 2\eta \partial_r v_r + \zeta \frac{\Delta \mu}{6} + \zeta \Delta \mu \tilde{Q}, \quad (5)$$

$$\sigma_{r\theta} = \eta \left( \frac{1}{r} \partial_\theta v_r + \partial_r v_\theta - \frac{v_\theta}{r} \right) + \zeta \Delta \mu q, \quad (6)$$

$$\sigma_{\theta r} = \sigma_{r\theta}, \quad (7)$$

$$\sigma_{\theta\theta} = \frac{2}{r} \left( \partial_\theta v_\theta + v_r \right) + \zeta \frac{\Delta \mu}{6} - \zeta \Delta \mu \tilde{Q}. \quad (8)$$

Here $\sigma_{rz} = \sigma_{zr} = \sigma_{z\theta} = 0$. The components of the strain rate tensor $v_{\alpha\beta}$, in polar coordinates, are

$$v_{\alpha\beta} = \begin{bmatrix} \frac{1}{r} \left( \frac{1}{r} \partial_\theta v_r + \partial_r \left( \frac{v_\theta}{r} \right) \right) \\ \frac{1}{r} \left( \frac{1}{r} \partial_\theta v_r + \partial_r \left( \frac{v_\theta}{r} \right) \right) \end{bmatrix} \quad (9)$$
Furthermore we obtain $\sigma_{zz}$ using the traceless condition.

$$\sigma_{zz} = -(\sigma_{rr} + \sigma_{\theta\theta}) = -2\eta \left( \frac{1}{r} v_r + \partial_r v_r + \frac{1}{r} \partial_\theta v_\theta \right) - \frac{\zeta \Delta \mu}{3}$$

(10)

Now we define the effective 2D tension tensor for the cortical layer $t_{ij} = \int dz (\sigma_{ij} - P \delta_{ij})$. As explained in main text (originally from Ref [1]), $P = \sigma_{zz}$.

The corresponding force balance equations, $Eqs. (13–14)$, further simplifies to:

$$\partial_r t_{rr} + \frac{1}{r} (t_{rr} - t_{\theta\theta}) + \frac{1}{r} \partial_\theta t_{r\theta} = \alpha v_r,$$

(11)

$$\partial_r t_{\theta r} + \frac{1}{r} (t_{\theta r} + t_{\theta\theta}) + \frac{1}{r} \partial_\theta t_{\theta\theta} = \alpha v_\theta.$$  

(12)

Substituting Eqs. (5–10) and Eq. 6 from maintext into Eqs. (11–12), we get the full expression of the force-balance equation

$$2\eta \left( \frac{2}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} v_r \right) + \frac{1}{r} \right) v_r + 3 \frac{\partial}{\partial r} \left( \frac{\partial}{\partial \theta} v_r \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} v_r \right) - \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} v_\theta \right) + \zeta \Delta \mu \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) q + \frac{\partial}{\partial \theta} q = \alpha v_r,$$

(13)

and

$$\eta \left( \frac{2}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} v_\theta \right) + \frac{1}{r} \right) v_\theta + 3 \frac{\partial}{\partial r} \left( \frac{\partial}{\partial \theta} v_\theta \right) + 5 \frac{\partial}{\partial r} \left( \frac{\partial}{\partial \theta} v_\theta \right) + 4 \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} v_r \right) - v_\theta + \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta + \zeta \Delta \mu \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) q - \frac{\partial}{\partial \theta} q = \alpha v_\theta.$$  

(14)

These equations are formidable to solve. We look for solutions with azimuthal symmetry and also assume $v_\theta = 0$. In the main text we had obtained solutions for $v_r(r)$ and $\tilde{Q}(r)$ by numerically solving Eq.3.4 (one can show $q = 0$, see below). We showed that, in steady-state ($\frac{D}{Dt}Q_{\alpha\beta} = 0$), with the boundary condition that the nematics are perfectly aligned with the inner perimeter ($r = R_0$), the solutions for $\tilde{Q}$ are very weakly dependent on $\beta_1$, the flow coupling parameter. However, $v_r(r)$ strongly depends on $\beta_1$, via viscosity renormalisation. Therefore we simplify Eq.4 by setting $\beta_1 = 0$ and get $H_{\alpha\beta} = 0$. Further using Landau-De Gennes energy functional (mentioned in the maintext), for $H_{\alpha\beta}$ we get,

$$\nabla^2 \tilde{Q} = \frac{\tilde{Q}}{L_c^2}, \quad \nabla^2 q = \frac{q}{L_c^2}.$$  

(15)

Here $L_c = \sqrt{L/\chi}$ is the nematic correlation length scale. Converting Eq. (15) into polar form we get (derivation given in section-4)

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{Q}}{\partial r} \right) - \left( \frac{1}{L_c^2} + \frac{4}{r^2} \right) \tilde{Q} = 0$$  

(16)

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial q}{\partial r} \right) - \left( \frac{1}{L_c^2} + \frac{4}{r^2} \right) q = 0.$$  

(17)

The force-balance equations, $Eqs. (13–14)$, further simplifies to:

$$4\eta \left( \frac{2}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} v_r \right) + \frac{1}{r} \right) v_r + \zeta \Delta \mu \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \tilde{Q} = \alpha v_r$$  

(18)

$$\frac{\partial q}{\partial r} + \frac{2q}{r} = 0.$$  

(19)

Eqs. (16,17) and Eqs. (18,19) are the main equations used in our study for $\beta_1 = 0$. We now look at several limiting cases using these set of equations.

### 2.1. Nematic order with $\beta_1 = 0$.

Substituting Eq. (19) in Eq. (17) we get $q = 0$. Furthermore we notice that Eq. (16) is a modified Bessel equation of order 2 [2]. This has solution of the form

$$\tilde{Q}(r) = c_1 K_2(r/L_c) + c_2 I_2(r/L_c),$$  

(20)
where $I_2$ and $K_2$ are the modified Bessel functions of the second kind $^{2}$, Since $I_2$ blows up as $r \to \infty$, we set $c_2 = 0$. Using the other boundary condition i.e., $\dot{Q}(r = R_0) = \frac{1}{2}$ (see Sec.), we get $c_1 = -\frac{1}{2K_2(R_0/L_c)}$. The zeroth-order solution for the nematic order parameter is then given by

$$\dot{Q}^0(r) = -\frac{K_2(r/L_c)}{2K_2(R_0/L_c)},$$

which is Eq. 5 in the main-text.

Instead of having the outer boundary at infinity, if it is located at $r = r_0 > R_0$ (which is the realistic case since at the division plane the cell has a finite cross-section), then using $\dot{Q} = 0$ at $r = r_0$ (keeping the other boundary condition at $R_0$ unchanged), we get

$$\dot{Q}^0(r) = -\frac{I_2(r)K_2(r_0) - K_2(r)I_2(r_0)}{2[I_2(R_0)K_2(r_0) - K_2(R_0)I_2(r_0)]}$$

(22)

### 2.2. Flow field in the absence of substrate friction

In the absence of substrate friction (i.e., $\alpha = 0$), Eq. 18 becomes

$$\frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) v_r + \frac{\zeta \Delta \mu}{4\eta} \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \dot{Q} = 0.$$  

(23)

Substituting Eq. 21 in Eq. 23 we get a second order differential equation in $v_r$ with a source term, namely,

$$\frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) v_r = -\frac{\zeta \Delta \mu}{4\eta} \frac{K_1(r/L_c)}{2K_2(R_0/L_c)}.$$  

(24)

The homogeneous solution of Eq. 24 is of the form $(Ar + B/r)$. To obtain the particular solution, we use an ansatz $v_p(r) = \Omega K_1(r/L_c)$. This is inspired by the form of the source term in Eq. 24. Substituting this particular solution in Eq. 24 gives us $\Omega = -\frac{\zeta \Delta \mu}{4\eta} \frac{L_c}{2K_2(R_0/L_c)}$. The general solution for $v_r$ is then given by

$$v_r(r) = Ar + \frac{B}{r} - \frac{\zeta \Delta \mu}{4\eta} \frac{L_c}{2K_2(R_0/L_c)} K_1(r/L_c).$$

(25)

We put $A = 0$, since $v_r \to 0$ as $r \to \infty$. The constant $B$ is calculated using the free boundary condition at $r = R_0$, i.e. $\sigma_{rr}(R_0) = 0$. Using this condition in Eq. 25 we get a boundary condition for $v_r$ at $R_0$, which is given by

$$2\eta v_r'(R_0) + \frac{\zeta \Delta \mu}{6} - \frac{\zeta \Delta \mu}{2} = 0.$$  

(26)

Using Eq. 22 and Eq. 26 we get $B = -\frac{\zeta \Delta \mu R_0^2}{6\eta} \left[ 1 + \frac{K_1'(R_0/L_c)}{K_2(R_0/L_c)} \right]$, which leads to the final expression for the zeroth-order velocity field $v_0^0(r)$ (Eq. 6 in the main text), namely,

$$v_0^0(r) = -\frac{\zeta \Delta \mu}{\eta} \left[ \left( 1 + \frac{3K_1'(R_0/L_c)}{4K_2(R_0/L_c)} \right) \frac{R_0^2}{6\eta} + \frac{L_c}{8} \frac{K_1(r/L_c)}{K_2(R_0/L_c)} \right]$$  

(27)

The corresponding velocity equation, for the case of outer boundary located at finite $r = r_0$, and $v_r(r_0) = 0$ (keeping the other stress boundary condition at $r = R_0$ unchanged) is solved numerically using Mathematica.

### 2.3. Flow-field in the presence of substrate friction

Here we consider the rotationally symmetric case but in the presence of substrate friction ($\alpha \neq 0$). The force-balance equation given by Eq. 18 can be re-written as

$$\left[ \partial_r \left( \partial_r + \frac{1}{r} \right) - \alpha \right] v_r = f(r).$$

(28)
Here \( \hat{\alpha} = \frac{\alpha}{4\eta} \) and \( f(r) = -\frac{\xi \Delta \mu}{4\eta} \left( \partial_r + \frac{2}{r} \right) \hat{Q}_0 = -\frac{\xi \Delta \mu}{4\eta} \frac{K_1(r/L_\alpha)}{2K_2(R_0/L_\alpha)} \). Here \( \hat{Q}_0(r) \) is the zeroth-order solution obtained for \( \beta_1 = 0 \) (see Eq. 21). Eq. 28 is a linear non-homogenous differential equation and can be solved using the method of Green's function with boundary conditions \( v_r(r \to \infty) = 0 \), and \( \sigma_{rr}(R_0) = 0 \).

The Green's function \( G(r, r') \) for the differential operator \( L \) appearing in Eq. 28 satisfies the differential equation:

\[
L G(r, r') = \left[ \frac{\partial^2}{\partial r'^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} (\hat{\alpha} r^2 + 1) \right] G(r, r') = \delta(r - r'). \tag{29}
\]

Eq. 29 is a modified Bessel equation of first order. This general solution is written in terms of the modified Bessel functions of first and second kind.

- For \( r < r' \) : \( G(r, r') = G_\prec(r, r') = A_1 I_1(\sqrt{\hat{\alpha}} r) + A_2 K_1(\sqrt{\hat{\alpha}} r) \), \( \text{Eq. (30)} \)
- For \( r > r' \) : \( G(r, r') = G_\succ(r, r') = B_1 I_1(\sqrt{\hat{\alpha}} r) + B_2 K_1(\sqrt{\hat{\alpha}} r) \), \( \text{Eq. (31)} \)

The full solution to Eq. 28 can be written as,

\[
v_r(r) = v_h(r) + \int_{R_0}^{\infty} G(r, r') f(r') \mathrm{d}r'. \tag{32}\]

Here \( v_h(r) \) represents the homogeneous solution obtained by solving \( L v_r = 0 \). It is easy to check that it is given by

\[
v_h(r) = -\frac{\xi \Delta \mu}{3\eta \sqrt{\hat{\alpha}}} \frac{K_1(\sqrt{\hat{\alpha}} r)}{K_0(R_0 \sqrt{\hat{\alpha}}) + K_2(R_0 \sqrt{\hat{\alpha}})} \tag{33}\]

The unknown constants in Eqs. 30, 31 are determined from the boundary conditions. Here \( G(r, r') \) is defined in the range \([R_0, \infty)\). By using the known boundary conditions for \( v_r \) and substituting the expression of Eq. 33 in Eq. 32 we notice that \( G(r, r') \) should satisfy the following boundary conditions at \( r = R_0 \) and \( r \to \infty \) respectively

\[
\left. \frac{\partial v_r}{\partial r} \right|_{r = R_0} = 0, \quad \left. G(r \to \infty, r') \right|_{r = R_0} = 0. \tag{34, 35}\]

Eqs. 34, 35 give \( B_1 = 0 \) and \( A_1 = A_2 \frac{K_0(\sqrt{\hat{\alpha}} R_0) + K_2(\sqrt{\hat{\alpha}} R_0)}{I_1(\sqrt{\hat{\alpha}} R_0) + I_2(\sqrt{\hat{\alpha}} R_0)} \), respectively. The remaining two boundary conditions are obtained by integrating Eq. 29 twice over an infinitesimally small interval around \( r' \). The gives the continuity conditions of \( G(r, r') \) and its derivative at \( r = r' \), namely,

\[
\left. \frac{\partial G_\succ}{\partial r} \right|_{r = r'} - \left. \frac{\partial G_\prec}{\partial r} \right|_{r = r'} = 1, \tag{36}\]

\[
\left. G_\prec(r, r') \right|_{r = r'} = \left. G_\succ(r, r') \right|_{r = r'}. \tag{37}\]

Eqs. (36, 37) give \( B_2 = \frac{A_2(K_0(\sqrt{\hat{\alpha}} R_0) + K_2(\sqrt{\hat{\alpha}} R_0))I_1(\sqrt{\hat{\alpha}} r')}{(I_0(\sqrt{\hat{\alpha}} R_0) + I_2(\sqrt{\hat{\alpha}} R_0))K_0(\sqrt{\hat{\alpha}} r')} + A_2 \), where

\[
A_2 = -\frac{2(I_0(\sqrt{\hat{\alpha}} R_0) + I_2(\sqrt{\hat{\alpha}} R_0))}{\sqrt{\hat{\alpha}}(K_0(\sqrt{\hat{\alpha}} R_0) + K_2(\sqrt{\hat{\alpha}} R_0))} \frac{1}{K_0(\sqrt{\hat{\alpha}} r') + K_2(\sqrt{\hat{\alpha}} r')}. \tag{38}\]

The full solution is now written as:

\[
v_r(r) = -\frac{\xi \Delta \mu}{3\eta \sqrt{\hat{\alpha}}} \frac{K_1(\sqrt{\hat{\alpha}} r)}{K_0(R_0 \sqrt{\hat{\alpha}}) + K_2(R_0 \sqrt{\hat{\alpha}})} + \int_{R_0}^{\infty} G(r, r') f(r') \mathrm{d}r' + \int_{r}^{\infty} G_\prec(r, r') f(r') \mathrm{d}r'. \tag{39}\]

The two integrals in Eq. 39 are evaluated numerically, for different values of \( r \), using ”NIntegrate” in Mathematica. The constants \( R_0, \alpha \) are kept fixed. In this manner we obtain a table for \( v_r \) (in units of \( \frac{\xi \Delta \mu}{\eta} \)) versus \( r \).
3. CORRECTION TO THE NEMATIC FIELD DUE TO NON-ZERO $\beta_1$

We have already shown numerically (main-text) that the correction to $\tilde{Q}(r)$ is small even for finite $\beta_1$. This correction can be estimated for small $\beta_1$ perturbatively, up to different orders in $\beta_1$, using Greens function approach. For that we use the zeroth order solution $v_0^\beta$ (Eq.27 with bare viscosity $\eta$, corresponding to $\beta_1 = 0$), on the right hand side of Eq.3, in the main-text, which now reads,

$$\partial_r (r \partial_r) \tilde{Q} - \left( \frac{r}{L_c^2} + \frac{4}{r} \right) \tilde{Q} = -\frac{1}{2L} \beta_1 \beta_2 r \left( \partial_r v_r^0 - \frac{v_0^0}{r} \right).$$

(39)

Substituting for $v_0^0$ we get,

$$\partial_r (r \partial_r) \tilde{Q} - \frac{r}{L_c^2} \tilde{Q} - \frac{4}{r} \tilde{Q} = -\frac{\beta_1 \beta_2}{2L} \left( \frac{R_0^2}{24r^2} + \frac{1}{2} \left( \frac{K_2(r/L_c)}{K_0(r/L_c)} + \frac{K_2(r/L_c)}{K_0(r/L_c)} \right) \right)$$

(40)

which can be re-written as follows

$$\left[ \partial_r (r \partial_r) - \frac{r}{L_c^2} - \frac{4}{r} \right] \tilde{Q} = u(r),$$

(41)

where $u(r)$ is the right hand side of Eq.39. The boundary conditions on $\tilde{Q}$ are: $\tilde{Q}(r \to \infty) = 0$, and $\tilde{Q}(r \to R_0) = -0.5$. The Green’s function $G(r, r')$ for operator $L$ in Eq.41 satisfies the modified Bessel equation of order 2.

$$LG(r,r') = \left[ \partial_r (r \partial_r) - \frac{r}{L_c^2} - \frac{4}{r} \right] G(r,r') = \delta(r - r'),$$

(42)

having a solution of the form

$$r < r' : \quad G(r,r') = G_<(r,r') = A_1 I_2(r) + A_2 K_2(r),$$

(43)

$$r > r' : \quad G(r,r') = G_>(r,r') = B_1 I_2(r) + B_2 K_2(r).$$

(44)

The four unknown constants can be determined from the boundary conditions on $G(r, r')$ which as before are given by,

$$G(r \to R_0, r') = 0,$$

(45)

$$G(r \to \infty, r') = 0,$$

(46)

$$\frac{\partial G_>}{\partial r} \bigg|_{r \to r'_+} - \frac{\partial G_<}{\partial r} \bigg|_{r \to r'_-} = 1,$$

(47)

$$G_<(r,r') \bigg|_{r \to r'_-} = G_>(r,r') \bigg|_{r \to r'_+}.$$  

(48)

Eq.45 and 48 gives $B_1 = 0$, $A_2 = -\frac{A_1 I_2(R_0)}{K_2(R_0)}$, $B_2 = A_1 \left( \frac{I_2(r)}{K_2(r)} - \frac{I_2(R_0)}{K_2(R_0)} \right)$, and $A_1 = \frac{K_2(r)}{r[I_2(r) - I_2(R_0)K_2(r)]}$. The full solution to Eq.41 is therefore

$$\tilde{Q}(r) = \tilde{Q}_h(r) + \int_{R_0}^{\infty} G(r,r')u(r')dr',$$

(49)

where $\tilde{Q}_h(r) = -\frac{K_2(r/L_c)}{2K_2(R_0/L_c)}$ is homogeneous solution obtained by solving $L\tilde{Q} = 0$. The full solution can then be rewritten as:

$$\tilde{Q} = \frac{K_2(r/L_c)}{2K_2(R_0/L_c)} + \int_{R_0}^{r} G>(r,r')u(r')dr' + \int_{r}^{\infty} G<(r,r')u(r')dr'.$$

(50)

The corresponding velocity equation (Eq.18) is easy to handle in the small $\beta_1$ limit. Substituting $\tilde{Q}_0^\beta$ in place of $\tilde{Q}$ gives us

$$4\eta \partial_r \left( \partial_r + \frac{1}{r} \right) v_r = -\frac{\zeta \Delta \mu}{1 + \frac{\beta_1 \beta_2}{2\eta}} \left( \partial_r + \frac{2}{r} \right) \tilde{Q}_0,$$

(51)
Note that this equation has the same structure as Eq[23] with the viscosity \( \eta \) enhanced by a factor of \( (1 + \frac{\beta^2 q_0^2}{2\eta}) \). Therefore the solution is same as Eq[27] where \( \eta \) has to be replaced by \( \eta(1 + \frac{\beta^2 q_0^2}{2\eta}) \).

4. DERIVATION OF DYNAMICAL EQUATIONS FOR \( \tilde{Q}, q \) (EQ.16,17 FROM EQ.15)

The polar form of \( Q'_{\alpha\beta} \) matrix, \( \begin{bmatrix} Q'_{rr} & Q'_{r\theta} \\ Q'_{\theta r} & Q'_{\theta\theta} \end{bmatrix} \), is given by

\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\end{bmatrix}
\begin{bmatrix}
\tilde{Q} & q \\
q & -\tilde{Q} \\
\end{bmatrix}
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{Q} \cos 2\theta + q \sin 2\theta & q \cos 2\theta - \tilde{Q} \sin 2\theta \\
q \cos 2\theta - \tilde{Q} \sin 2\theta & -\tilde{Q} \cos 2\theta + q \sin 2\theta \\
\end{bmatrix}
\begin{bmatrix}
Q_1 & q_1 \\
q_1 & -Q_1 \\
\end{bmatrix}
\]

(52)

where \( \begin{bmatrix} \tilde{Q} & q \\ q & -\tilde{Q} \end{bmatrix} \) is its corresponding cartesian form (in \( x-y \)). Noting that the basic structure of the matrix is retained, we express the new matrix elements in terms of the old ones as, \( Q_1 = \tilde{Q} \cos 2\theta + q \sin 2\theta \) and, \( q_1 = -\tilde{Q} \sin 2\theta + q \cos 2\theta \). Inverting these we get, \( Q = Q_1 \cos 2\theta - q_1 \sin 2\theta \) and, \( q = Q_1 \sin 2\theta + q_1 \cos 2\theta \). Note that \( Q_1, q_1 \) are functions of \( r \) via \( \tilde{Q}, q \). Now substituting the polar form for the laplacian operator and \( \tilde{Q} \) into Eq[15] \( \nabla^2 \tilde{Q} = \frac{\tilde{Q}}{L_c^2} \) (whose polar form we are interested in), we get

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q_1}{\partial r} \right) \cos 2\theta - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial q_1}{\partial r} \right) \sin 2\theta - \frac{Q_1}{r^2} 4 \cos 2\theta + \frac{q_1}{r^2} 4 \sin 2\theta = \frac{Q_1 \cos 2\theta - q_1 \sin 2\theta}{L_c^2}
\]

(53)

Further, separating the coefficients of \( \cos 2\theta \) and \( \sin 2\theta \) from the two sides, yields

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q_1}{\partial r} \right) - \frac{Q_1}{L_c^2} \cos 2\theta - \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial q_1}{\partial r} \right) \right] - 4 \frac{q_1}{r^2} - \frac{q_1}{L_c^2} \sin 2\theta = 0
\]

(54)

Since \( \cos 2\theta \) and \( \sin 2\theta \) are independent functions their coefficients, separately, must be zero. Thus we obtain the order parameter equations Eq[16] and Eq[17] where we retained the same notations as in the cartesian frame, by changing \( (Q_1, q_1) \to (\tilde{Q}, q) \).

5. COMPUTATION OF CLOSURE RATE \( R_0(t) \) VS \( t \).

This requires the velocity at the ring \( v_r(R_0) \) for different values of \( R_0 \). When the outer radius is located at infinity we have a closed form expression for \( v_r(R_0) \) which can be integrated to obtain the closure rate. But for the case of finite outer boundary (at \( r_0 \)) we could solve the velocity equation (Eq.18) only numerically (using "NDSolve" in Mathematica) after incorporating the analytic expression for \( Q^0(r) \) (Eq.22). Therefore we first prepared a table for \( v_r(R_0) \) versus \( R_0 \) by solving the velocity equation for different values of \( R_0 \). We then obtained an interpolation function for this table. This function is further integrated numerically to obtain \( R_0(t) \) versus \( t \).

All numerical calculations have been performed using Mathematica version 11.3.

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Supplementary Information Part-II: Dynamics and stability of the cytokinetic ring

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In the main text and Supplementary material part-1 (SI-I) we studied the contraction of the cytokinetic ring with radial symmetry as a simplifying assumption. Here, we extend the analysis to a deformed hole in order to investigate the stability of the symmetric solution derived previously. The motivation for this study arises from experimental observation about the typical shape of the ring during cytokinesis (see Fig.1 and Refs. [1–3]). As shown in Fig.1, the ring contraction does not occur in a radially symmetric fashion. At the outset, the ring appears to be far from a perfect circle, but as it contracts, it becomes increasingly circular. We would like to address this observation by studying the stability of the different angular modes of the contraction dynamics. In order to make this exercise analytically tractable (in the sense of linear stability analysis), we assume small deviation from the circular shape. This is a strategy analogous to that adopted in [4] and [5] (chapter-7). By using linear stability analysis for the quasi-static ring contraction dynamics, we are able to calculate the growth/decay rate of the angular modes using perturbation theory. We find that the lowest-order breathing mode and most of the higher-order modes are stable for experimentally relevant regime of parameter values. Furthermore, for large values of ring radius, there is a window of unstable modes in between; however this window shrinks as the ring contracts and eventually all modes become stable.

We study deformations of the circular ring by decomposing into Fourier modes as follows:

\[ \delta R(\theta, t) = \sum_{n=0}^{\infty} \delta R_n(t) e^{in\theta} \] (1)

This perturbative deformation to the ring implies that the circular ring defined by the polar equation \( r(\theta) = R_0 \) is now changed to \( r(\theta) = R_0 + \delta R(\theta, t) \) (the time dependence is suppressed for simplicity of notation).

Due to this deformation, velocity and order parameter fields change perturbatively:

\[ \tilde{Q}(r, \theta, t) = \tilde{Q}_0(r) + \delta \tilde{Q}(r, \theta, t) \] (2)
\[ q(r, \theta, t) = q_0(r, \theta, t) + \delta q(r, \theta, t) \] (3)
\[ v_r(r, \theta, t) = v^{0}_r(r) + \delta v_r(r, \theta, t) \] (4)
\[ v_\theta(r, \theta, t) = v^{0}_\theta(r, \theta, t) + \delta v_\theta(r, \theta, t) \] (5)

The perturbation fields \( \delta \tilde{Q}, \delta q, \delta v_r, \delta v_\theta \) can be expanded similar to \( \delta R(\theta, t) \) as

\[ \delta \tilde{Q}(r, \theta, t) = \sum_{n=0}^{\infty} \delta \tilde{Q}_n(r, t) e^{in\theta} \] (6)
\[ \delta q(r, \theta, t) = \sum_{n=0}^{\infty} \delta q_n(r, t) e^{in\theta} \] (7)
\[ \delta v_r(r, \theta, t) = \sum_{n=0}^{\infty} \delta v_{r,n}(r, t) e^{in\theta} \] (8)
\[ \delta v_\theta(r, \theta, t) = \sum_{n=0}^{\infty} \delta v_{\theta,n}(r, t) e^{in\theta} \] (9)

We substitute these expansions in the dynamical equations and use the exact forms of the zeroth order solutions to solve for the Fourier amplitudes of all the four fields.

**Dynamical equations for \( \delta \tilde{Q} \) and \( \delta q \) —** The Fourier amplitudes \( \delta \tilde{Q}_n \) and \( \delta q_n \) of the order parameter fields...
In terms of the variables $q$, $\tilde{q}$, and $\delta q$, we can diagonalise these equations by a change of variables,

$$
\begin{align*}
\delta Q_+ &= \delta \tilde{Q}_n + i \delta q_n \\
\delta Q_- &= i \delta \tilde{Q}_n + \delta q_n
\end{align*}
$$

In terms of the variables $\delta Q_+$ and $\delta Q_-$, (10) and (11) give

$$
\begin{align*}
\left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{L_c^2} - \frac{(n+2)^2}{r^2} \right) \delta Q_+ &= 0 \\
\left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{L_c^2} - \frac{(n-2)^2}{r^2} \right) \delta Q_- &= 0
\end{align*}
$$

These are modified Bessel equations with solutions of the form

$$
\begin{align*}
\delta Q_+ &= e_n^+ K_{n+2}(r/L_c) \\
\delta Q_- &= e_n^- K_{n-2}(r/L_c)
\end{align*}
$$

where we have left out the Bessel-I functions since $\delta Q_\pm$ must go to 0 as $r \to \infty$. Changing back to $\tilde{Q}_n$ and $\delta q_n$, we get

$$
\begin{align*}
\tilde{Q}_n &= e_n^+ K_{n+2}(r/L_c) - e_n^- i K_{n-2}(r/L_c) \\
\delta q_n &= e_n^- K_{n-2}(r/L_c) - i e_n^+ K_{n+2}(r/L_c)
\end{align*}
$$

To evaluate the constants $e_n^\pm$, we need to use the boundary conditions for the order parameter at the deformed ring $r = R(\theta)$,

$$
\begin{align*}
\dot{Q}(R(\theta)) &= -\frac{1}{2} \quad \text{and} \quad q(R(\theta)) = \frac{1}{2 R_0} \frac{d\delta R}{d\theta}
\end{align*}
$$

By substituting (2), (3), (6) and (7) in (20), and expanding both sides to lowest order, we can obtain the following boundary conditions for the perturbation fields

$$
\begin{align*}
\tilde{Q}(R_0, \theta) &= -\partial_r \tilde{Q}_0(R_0) \delta R(\theta) \\
\delta q(R_0, \theta) &= \frac{1}{2 R_0} \frac{d\delta R}{d\theta}
\end{align*}
$$

which imply, for the Fourier modes,

$$
\begin{align*}
\tilde{Q}_n(R_0) &= -\partial_r \tilde{Q}_0(R_0) \delta R_n \\
\delta q_n(R_0) &= \frac{1}{2 R_0} \frac{d\delta R}{d\theta}
\end{align*}
$$

Using these boundary conditions, we can express the exact analytical solutions for the perturbation fields $\tilde{Q}_n$ and $\delta q_n$,

$$
\begin{align*}
\tilde{Q}_n(r) &= \frac{\delta R_n}{2 R_0} \left[ \left( n - 1 - \frac{R_0}{2 L_c} K_1(R_0/L_c) \right) K_{n-2}(r/L_c) - \left( n + 1 + \frac{R_0}{2 L_c} K_1(R_0/L_c) \right) K_{n+2}(r/L_c) \right] \\
\delta q_n(r) &= \frac{i \delta R_n}{2 R_0} \left[ \left( n - 1 - \frac{R_0}{2 L_c} K_1(R_0/L_c) \right) K_{n-2}(r/L_c) + \left( n + 1 + \frac{R_0}{2 L_c} K_1(R_0/L_c) \right) K_{n+2}(r/L_c) \right]
\end{align*}
$$

**Dynamical equations for $\delta v_r$ and $\delta v_\theta$** — As discussed in the main text and SI-1, the velocity dynamics is expressed in terms of Navier-Stokes-like equations for the stress tensor. To perform our perturbative analysis, we expand those equations in terms of the perturbation fields $\delta v_r$ and $\delta v_\theta$ and then expand in terms of their Fourier modes. After some algebra, we can find the following dynamical equations for $\delta v_{r,n}$ and $\delta v_{\theta,n}$:
We apply \( \delta v \) \( \delta \) involves the exact solutions of differential equations. We set a change of variables is needed to solve these coupled equations, which involves the exact solutions of \( \delta Q_n \) and \( \delta q_n \), given in (25) and (26). For the expression to not blow up at \( r \to \infty \), we must enforce the following equalities:

\[
\delta v_{r,n}(x) = \frac{-4 + 9n^2}{72\eta} \left[ C_1 \frac{1}{r^{n+1}} + C_2 \frac{1}{r^{n-1}} + C_3 r^{n-1} + C_4 r^{n+1} \right] + \frac{1}{(n-1)r^{n-1}} \int_{R_0}^{r} x^{n-2} f_n(x) dx - \frac{r^{n-1}}{(n-1)} \int_{R_0}^{r} \frac{f_n(x)}{x^n} dx + \frac{r^{n+1}}{(n+1)} \int_{R_0}^{r} \frac{f_n(x)}{x^{n+2}} dx
\]

where \( \hat{n} \) is the normal vector at the deformed ring \( R(\theta) \).
We can evaluate \( \hat{n} \) to find:

\[
\hat{n}(\theta) = -\hat{r} + \frac{1}{R_0} \frac{d\delta R}{d\theta} \hat{\theta}
\]

where terms only up to first order in the perturbation have been kept. Hence, we can use (38) in the stress boundary conditions to obtain the following boundary conditions:

\[
\delta \sigma_{rr}(R_0) = -\partial_r \sigma_{rr}(R_0) \delta R(\theta)
\]

\[
\delta \sigma_{\theta r}(R_0) = \frac{1}{R_0} \frac{d\delta R}{d\theta} \sigma_{\theta r}
\]

which implies, for the Fourier components,

\[
\delta \sigma_{\theta r,n}(R_0) = -\partial_r \sigma_{\theta r,n}(R_0) \delta R_n
\]

\[
\delta \sigma_{\theta \theta,n}(R_0) = \frac{in}{R_0} \delta R_n \sigma_{\theta \theta,n}
\]

Using the above boundary conditions, \( C_1 \) and \( C_2 \) can be calculated, thereby producing the full exact solution expressed in (33).

For the modes \( n = 0, 1 \), the above four solutions are not independent hence these modes require a separate
where $\delta v_{r,0} = C_1 r + \frac{C_2}{r} + r \int_{R_0}^{r} \frac{g_0(x)}{x^2} dx - \frac{1}{r} \int_{R_0}^{r} g_0(x) dx \ (43)$

where $g_0(x) = -\frac{\Delta \mu}{4\eta} (\partial_r + \frac{2}{r}) \delta \tilde{Q}_0$. To prevent $\delta v_{r,0}$ from blowing up at $r \rightarrow \infty$, we must enforce

$$C_1 = - \int_{R_0}^{\infty} \frac{g_0(x)}{x^2} dx \ (44)$$

To evaluate $C_2$, we use the stress-free boundary conditions at $r = R(\theta)$ as discussed already. This gives us the full solution for $\delta v_{r,0}$. The rest of the analysis can be continued as for other modes.

**Kinematic boundary condition** — We define the ring velocity $V(\theta) = (v_r \hat{r} + v_\theta \hat{\theta}) \cdot \hat{n}_{\theta \! | R(\theta)}$, where $R(\theta) = R_0 + \delta R(\theta)$. Splitting $\vec{v}$ into zeroth order solution and perturbation, we find:

$$V(\theta) = (v_r^0(\theta) \hat{r} + \delta v_r(\theta) \hat{r} + \delta v_\theta(\theta) \hat{\theta}) \cdot \hat{n}_{\theta \! | R(\theta)}$$

$$= -v_r^0(R_0) - \delta v_r(R) - \frac{\delta v_\theta(R)}{R_0} \frac{d\delta R}{d\theta}$$

$$\approx -v_r^0(R_0) - \partial_r v_r^0(R_0) \delta R(\theta) - \delta v_r(R_0, \theta)$$

$$\therefore \delta V(\theta) \equiv V(\theta) - (-v_r^0(R_0))$$

$$= -\partial_r v_r^0(R_0) \delta R(\theta) - \delta v_r(R_0, \theta) \ (45)$$

Converting to Fourier modes again, with $\delta V_n = \frac{1}{2\pi} \int_0^{2\pi} \delta V(\theta) e^{-in\theta}$, we have:

$$\delta V_n = -\partial_r v_r^0(R_0) \delta R_n - \delta v_{r,n}(R_0) \ (46)$$

We now need to set up a kinematic boundary condition. This relates $\delta V$ to $\delta R$ as:

$$\delta V_n = -\frac{d\delta R_n}{dt} \ (47)$$

which gives us a way to evaluate $\omega_n$ - the growth/decay rate of the $n$th Fourier mode. The time dependence of the Fourier modes can be put in as follows:

$$\delta R(\theta, t) = \sum_{n=0}^{\infty} \delta R_n e^{in\theta + \omega_n t} \ (48)$$

Therefore, $\frac{d\delta R_n}{dt} = \omega_n R_n$. So we derive, from (47),

$$\omega_n = \partial_r v_r^0(R_0) + \frac{\delta v_{r,n}(R_0)}{\delta R_n} \ (49)$$

We performed the calculation outlined above on Mathematica, choosing a set of parameter values that are experimentally relevant (given in main text). We have plotted $\omega_n$ as a function of mode number $n$. We find that the unstable modes are present in a small window for large values of the dimensionless radius $R_0/L_c$. As the ring radius becomes smaller, the window of instability slowly diminishes in size and finally, at small enough radius, there are no unstable modes left. This parallels the experimental observation that asymmetrical modes are prominent when the contracting ring is large but they essentially disappear when the ring contracts to a small size, turning it into an increasingly perfect circle.

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