Coefficient Functional and Bohr-Rogosinski Phenomenon for Analytic functions involving Semigroup Generators

Surya Giri and S. Sivaprasad Kumar

Abstract

This paper examines the coefficient problems for the class of semigroup generators, a topic in complex dynamics that has recently been studied in context of geometric function theory. Further, sharp bounds of coefficient functional such as second order Hankel determinant, third order Toeplitz and Hermitian-Toeplitz determinants are derived. Additionally, the sharp growth estimates and the bounds of difference of successive coefficients are determined, which are used to prove the Bohr and the Bohr-Rogosinski phenomenon for the class of semigroup generators.

Keywords: Holomorphic generators; Hankel determinant; Toeplitz determinant; Zalcman functional; Successive coefficient difference; Bohr and Bohr-Rogosinski radius.

AMS Subject Classification: 30C45, 30C50, 30C55, 47H20, 37L05.

1 Introduction

Let $H$ be the class of holomorphic functions in the unit disk $D$ and $A \subset H$ containing functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hfill (1.1)

By $B$, we denote the class of holomorphic self mappings from $D$ to $D$. A family $\{u_t(z)\}_{t \geq 0} \subset B$ is called a one parameter continuous semigroup if (i) $\lim_{t \to 0} u_t(z) = z$, (ii) $u_{t+s}(z) = u_t(z) \circ u_s(z)$, and (iii) $\lim_{t \to s} u_t(z) = u_s(z)$ for each $z \in D$ hold.

Berkson and Porta \cite{3} showed that each one parameter semigroup is locally differentiable in parameter $t \geq 0$ and moreover, if

$$\lim_{t \to 0} \frac{z - u_t(z)}{t} = f(z),$$

which is a holomorphic function, then $u_t(z)$ is the solution of the the Cauchy problem

$$\frac{\partial u_t(z)}{\partial t} + f(u_t(z)) = 0, \quad u_0(z) = z.$$

The function $f$ is called the infinitesimal (holomorphic) generator of semigroup $\{u_t(z)\} \subset B$. The class of all holomorphic generators is denoted by $\mathcal{G}$. Also, note that each element of $\{u_t(z)\}$ generated by $f \in H(D, \mathbb{C})$ is univalent function while $f$ is not necessarily univalent \cite{13}. Different analytic criteria are available to show that a function is semigroup generator \cite{3, 37, 6}. Berkson and Porta \cite{3} proved:

**Theorem 1.1.** The following assertions are equivalent:

(a) $f \in \mathcal{G}$;

(b) $f(z) = (z - \sigma)(1 - z\bar{\sigma})p(z)$ with some $\sigma \in \mathbb{D}$ and $p \in H$, $\text{Re}(p(z)) \geq 0$.  

1
The point $\sigma \in \mathbb{D} := \{ z \in \mathbb{C} : |z| \leq 1 \}$ is unique and called the Denjoy–Wolff point of the semigroup generated by $f(z)$. By Denjoy-Wolff theorem \cite{13, 37, 30} for continuous semigroup, $\lim_{t \to \infty} u(t, z) = \sigma$ if the semigroup generated by $f$ is neither an elliptic automorphism of $\mathbb{D}$ nor an identity map for at least one $t \in [0, \infty)$. We denote the class of infinitesimal generators with Denjoy-Wolff point $\sigma$ by $\mathcal{G}[\sigma]$. For $\sigma = 0$, we obtain the following subclass

$$\mathcal{G}[0] = \{ f \in \mathcal{G} : f(z) = zp(z), \ Re\ p(z) \geq 0 \}.$$ 

Bracci et al. \cite{7} considered the class $\mathcal{G}_0 = \mathcal{G}[0] \cap \mathcal{A}$. In the study of non-autonomous problem such as Loewner theory, the class $\mathcal{G}_0$ plays a significant role \cite{8, 12}. Various subclasses of $\mathcal{G}_0$ are considered with parameter, which is also called filtration. For instance, the class

$$\mathcal{A}_\beta = \left\{ f \in \mathcal{A} : \Re\left( \frac{\beta f(z)}{z} + (1 - \beta)f'(z) \right) > 0, \ \beta \in [0, 1] \right\}$$

is a subclass of $\mathcal{G}_0$. In \cite{7}, the authors proved that $\mathcal{A}_{\beta_1} \subseteq \mathcal{A}_{\beta_2} \subseteq \mathcal{G}_0$ for $0 \leq \beta_1 < \beta_2 < 1$ and whenever $f \in \mathcal{A}_\beta$, \n
$$\Re\left( \frac{f(z)}{z} \right) \geq \int_0^1 \frac{1 - t^{1-\beta}}{1 + t^{1-\beta}} dt.$$ 

Clearly, when $\beta = 0$, the class $\mathcal{A}_\beta$ reduces to the class $\mathcal{R} := \{ f \in \mathcal{A} : \Re f'(z) > 0 \}$, which is called the class of bounded turning functions. It can be easily seen that $\mathcal{R} \subset \mathcal{G}_0$ and each $f \in \mathcal{R}$ satisfies the Noshiro-Warshawski condition \cite{12}, thus every member of $\mathcal{R}$ generates a semigroup that is univalent. Elin et al. \cite{14} solved the radii problems for the class $\mathcal{A}_\beta$. They found the radii $r \in (0, 1)$ for $f \in \mathcal{A}_\beta$ such that $f(rz)/r$ belong to the class of starlike functions, denoted by $S^*$, and some other subclasses of starlike functions. This problem arises from the fact that neither $S^* \subset \mathcal{A}_\beta$ nor $\mathcal{A}_\beta \subset S^*$. Generalizing this work, Giri and Kumar \cite{16} obtained $r$ such that $f(rz)/r$ belong to the class of Ma-Minda starlike functions.

Coefficient problems, growth estimates and others were still open for the class $\mathcal{A}_\beta$. In this paper, we focus on these problems. We find the bound of $n^{th}$ coefficient of $f \in \mathcal{A}_\beta$ and coefficient functional such as second Hankel determinant, third order Toeplitz and Hermitian Toeplitz determinant and Zalcman functional. Later, Bohr and Bohr-Rogosinski phenomenon with growth estimates are also discussed for this class.

In 1914, Bohr \cite{5} proved that, if $\omega(z) = \sum_{n=0}^\infty c_n z^n \in \mathcal{B}$, then $\sum_{n=0}^\infty |c_n|^n \leq 1$ for all $z \in \mathbb{D}$ with $|z| = r \leq 1/3$. The constant $1/3$ is known as Bohr radius. This inequality was first derived by Bohr for $r \leq 1/6$ and sharpened independently to $r \leq 1/3$ by Wiener, Riesz, and Schur. In recent years, a lot of study have been carried out on the Bohr inequality for functions, which map $\mathbb{D}$ onto other domains, say $\Omega$ \cite{4} \cite{15}. Different generalizations of the Bohr inequality are taken into consideration \cite{29} \cite{37}. We say that

**Definition 1.2.** The class $\mathcal{A}_\beta$ satisfies the Bohr phenomenon if there exists $r_\beta$ such that

$$|z| + \sum_{n=2}^\infty |a_n||z|^n \leq d(f(0), \partial f(\Omega))$$

holds in $|z| = r \leq r_\beta$, where $\partial f(\Omega)$ is the boundary of image domain of $\mathbb{D}$ under $f$ and $d$ denotes the Euclidean distance between $f(0)$ and $\partial f(\Omega)$.

For $m = 1$, Muhanna \cite{32} showed that the Bohr phenomenon holds for the class of univalent functions and the class of convex functions, when $|z| = r \leq 3 - 2\sqrt{2}$ and $|z| = r \leq 1/3$ respectively. We refer to the survey article \cite{31} for further details on this topic. There is also the concept of Rogosinski radius along with the Bohr radius, although a little is known about Rogosinski radius in comparison to Bohr radius \cite{17} \cite{22} \cite{35}. It says that, if $\omega(z) = \sum_{n=0}^\infty c_n z^n \in \mathcal{B}$, then

$$\sum_{n=0}^{N-1} |c_n||z|^n \leq 1 \quad (N \in \mathbb{N})$$

in the disk $|z| = r \leq 1/2$. The radius $1/2$ is called the Rogosinski radius. Kayumov et al. \cite{18} considered the following expression

$$R_\beta^N(z) := |f(z)| + \sum_{n=N}^\infty |a_n||z|^n$$
and found the radius \( r_N \) such that \( R_N^f(z) \leq 1 \) in \(|z| = r \leq r_N\) for the Cesáro operators on the space of bounded analytic functions. Here, we say that:

**Definition 1.3.** The class \( \mathcal{A}_\beta \) satisfies the Bohr-Rogosinski phenomenon if there exist \( r_N \) such that

\[
|f(z^n)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\Omega))
\]

holds in \(|z| = r \leq r_N\).

Section 5 is devoted to find the \( r_\beta \) and \( r_N \) for the class \( \mathcal{A}_\beta \).

For \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A} \), the \( m^{th} \) Hankel, Toeplitz and Hermitian Toeplitz determinant for \( m \geq 1 \) and \( n \geq 0 \) are respectively given by

\[
H_m(n)(f) = \begin{vmatrix}
 a_n & a_{n+1} & \cdots & a_{n+m-1} \\
 a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2}
\end{vmatrix},
\]

\[
T_m(n)(f) = \begin{vmatrix}
 a_n & a_{n+1} & \cdots & a_{n+m-1} \\
 a_{n+1} & a_n & \cdots & a_{n+m-2} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n+m-1} & a_{n+m-2} & \cdots & a_n
\end{vmatrix},
\]

\[
T_m,n(f) = \begin{vmatrix}
 a_n & a_{n+1} & \cdots & a_{n+m-1} \\
 a_{n+1} & a_n & \cdots & a_{n+m-2} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n+m-1} & a_{n+m-2} & \cdots & a_n
\end{vmatrix},
\]

where \( \overline{a_n} = \overline{a_{n+m}} \). Toeplitz matrices have constant entries along their diagonals, while Hankel matrices have constant entries along their reverse diagonals. In particular,

\[
H_2(n)(f) = a_n a_{n+2} - a_{n+1}^2, \quad T_3(1)(f) = 1 - 2a_2^2 + 2\text{Re}(a_2^2 \overline{a_3}) - |a_3|^2.
\]

Finding the sharp bound of \(|H_2(2)(f)|\) for the class \( \mathcal{S} \) and its subclasses has always been the focus of many researchers. Although, investigations concerning Toeplitz and Hermitian Toeplitz are recently introduced in \([2, 11]\), a summary of some of the more significant results is given in \([3, 8]\). For more work in this direction (see \([21, 20, 24, 33]\)).

In 1999, Ma \([28]\) proposed a conjecture for \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S} \) that

\[
|J_{m,n}| := |a_n a_m - a_{n+m-1}| \leq (m-1)(n-1).
\]

They proved this conjecture for the class of starlike functions and univalent functions with real coefficients. It is also called generalized Zalcman conjecture as it generalizes the Zalcman conjecture \(|a_n^2 - a_{2n-1}| \leq (2n-1)^2\) for \( f \in \mathcal{S} \). Recently, bounds of \(|J_{2,3}|\) are obtained for various subclasses of \( \mathcal{A} \) \([11, 10]\). In section 2 and 3 we obtain the sharp bound of \(|H_2(2)(f)|, |T_3(1)(f)|\) and \(|J_{2,3}|\) for \( f \in \mathcal{A}_\beta \).

## 2 Hankel Determinant and Zalcman Functional

**Theorem 2.1.** If \( f \in \mathcal{A}_\beta \) is of the form \([1,1]\), then

\[
|a_n| \leq \frac{2}{n - \beta(n-1)}.
\]

Further, this inequality is sharp for each \( n \).
Proof. Let $f \in A_\beta$ is given by (1.1), then we have

$$\beta \frac{f(z)}{z} + (1 - \beta)zf'(z) = p(z) \quad (z \in \mathbb{D}),$$

where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ such that $\text{Re} \, p(z) > 0$ is a member of the Carathéodory class $\mathcal{P}$. Upon comparing the coefficients of same powers on either side with the series expansion of $f$ and $p$ yields

$$(n - (n - 1)\beta)a_n = p_{n-1}$$  \quad (2.2)

for $n = 2, 3, 4, \cdots$, which gives the needed bound of $|a_n|$ using the Carathéodory coefficient bounds $|p_n| \leq 2$ (see [12]). The function $\tilde{f} : \mathbb{D} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = z \left( -1 + 2 \left( 2F_1 \left[ 1, \frac{1}{1 - \beta}, \frac{2 - \beta}{1 - \beta}, iz \right] \right) \right) = z + \sum_{n=2}^{\infty} \frac{2}{n - (n - 1)\beta} z^n$$  \quad (2.3)

satisfies the condition $\text{Re} \left( \beta \tilde{f}(z)/z + (1 - \beta)\tilde{f}'(z) \right) > 0$, hence $\tilde{f}$ is a member of $A_\beta$, where $2F_1$ denotes the Gauss hypergeometric function. Equality in (2.1) occurs for $\tilde{f}$, which proves the sharpness of the bound.

Corollary 2.2. If $f \in A_\beta$, then for any real $\mu \geq 0$

$$|a_n a_{n+2} - \mu a_{n+1}^2| \leq \frac{4}{(n - (n - 1)\beta)(n + 2 - (n + 1)\beta) + \frac{4\mu}{(n + 1 - n\beta)^2}}.$$

The bound is sharp.

Proof. Since $|a_n a_{n+2} - \mu a_{n+1}^2| \leq |a_n| a_{n+2} + \mu |a_{n+1}|^2$. The bound simply follows from (2.1). To see the sharpness, consider

$$\tilde{f}_1(z) = z \left( -1 + 2 \left( 2F_1 \left[ 1, \frac{1}{1 - \beta}, \frac{2 - \beta}{1 - \beta}, iz \right] \right) \right) = z + \sum_{n=2}^{\infty} \frac{2^{n-1}}{(n - (n - 1)\beta) z^n}.$$  \quad (2.4)

It can be easily seen that $\tilde{f}_1(z)$ satisfies (1.2), thus $\tilde{f}_1 \in A_\beta$.

For $\mu = 1$, Corollary 2.2 gives the following sharp bound:

Corollary 2.3. If $f \in A_\beta$ is of the form (2.2), then

$$|H_2(n)(f)| \leq \frac{4((2n^2 - 1)\beta^2 - (4n^2 + 4n - 2)\beta + 2n^2 + 4n + 1)}{(n - (n - 1)\beta)(n + 2 - (n + 1)\beta)(n + 1 - n\beta)^2}.$$

For $n = 2$ and $3$, the following sharp bound of second order Hankel determinant follows:

Corollary 2.4. If $f \in A_\beta$ is of the form (2.2), then

$$|H_2(2)(f)| \leq \frac{4(7\beta^2 - 22\beta + 17)}{(4 - 3\beta)(3 - 2\beta)^2(2 - \beta)}, \quad |H_2(3)(f)| \leq \frac{4(17\beta^2 - 46\beta + 31)}{(5 - 4\beta)(4 - 3\beta)^2(3 - 2\beta)}.$$

Theorem 2.5. If $f \in A_\beta$ is of the form (1.1), then

$$|J_{2,3}(f)| \leq \frac{2}{4 - 3\beta}.$$

The bound is sharp.

Proof. Let $f \in A_\beta$ is given by (1.1), then from (2.2), we have

$$|J_{2,3}(f)| = |a_2a_3 - a_4| = \left| \frac{p_1p_2}{(3 - 2\beta)(2 - \beta)} - \frac{p_3}{4 - 3\beta} \right|.$$  \quad (2.5)
Thus where \( p \) rotationally invariant, therefore, without losing generality, we can take \( p \).

For \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P} \), Libera et al. [26] proved that

\[
2p_2 = p_1^2 + x(4 - p_1^2), \quad 4p_3 = p_1^3 + 2xp_1(4 - p_1^2) - x^2p_1(4 - p_1^2) + 2z(1 - |x|^2)(4 - p_1^2),
\]

where \(|x| \leq 1\) and \(|z| \leq 1\). Substituting these values of \( p_2 \) and \( p_3 \) in (2.5), we obtain

\[
|J_{2,3}(f)| = \left| \frac{p_1^3}{4} \left( \frac{2}{2\beta^2 - 7\beta + 6} + \frac{1}{3\beta - 4} \right) - \frac{p_1(4 - p_1^2)(1 - \beta)^2}{(2 - \beta)(3 - 2\beta)(4 - 3\beta)} x \right. \right.
\]

\[
\left. + \frac{p_1(4 - p_1^2)x^2}{4(4 - 3\beta)} - \frac{(4 - p_1^2)(1 - |x|^2)z}{2(4 - 3\beta)} \right|.
\]

Since the class \( \mathcal{P} \) is rotationally invariant and it is an easy exercise to check that the class \( \mathcal{A}_\beta \) is also rotationally invariant, therefore, without losing generality, we can take \( p_1 = p \in [0, 2] \). Now, applying the triangle inequality with \( |x| = \rho \), we obtain

\[
|J_{2,3}(f)| \leq \frac{p^3}{4} \left( \frac{2}{2\beta^2 - 7\beta + 6} + \frac{1}{3\beta - 4} \right) + \frac{p(4 - p^2)(1 - \beta)^2}{(2 - \beta)(3 - 2\beta)(4 - 3\beta)} \rho + \frac{4 - p^2}{2(4 - 3\beta)} \rho^2.
\]

To determine the maximum value of \( F(p, \rho) \), first we find out the stationary points, given by the roots of \( \partial F/\partial \rho = 0 \) and \( \partial F/\partial p = 0 \), where

\[
\frac{\partial F(p, \rho)}{\partial p} = \frac{3p^2(r^2(2\beta^2 - 7\beta + 6) + 4r(1 - \beta)^2 + 2\beta^2 - \beta - 2)}{4(-2 + \beta)(-3 + 2\beta)(-4 + 3\beta)} + p \left( \frac{r^2}{4 - 3\beta} - \frac{1}{4 - 3\beta} \right)
\]

\[
+ \frac{r^2}{4 - 3\beta} \left( \frac{4r(1 - \beta)}{(4 - 3\beta)(3 - 2\beta)(2 - \beta)} \right).
\]

\[
\frac{\partial F(p, \rho)}{\partial \rho} = 2r \left( \frac{p(4 - p^2)}{4(4 - 3\beta)} + \frac{-4 + p^2}{2(4 - 3\beta)} \right) + \frac{p(4 - p^2)(1 - \beta)^2}{(4 - 3\beta)(3 - 2\beta)(2 - \beta)}.
\]

A simple calculation shows that for \( p \in [0, 2] \) and \( r \in [0, 1] \), the stationary point is \((0, 0)\) and

\[
\left( \frac{\partial^2 F \partial^2 F}{\partial p^2 \partial \rho^2} \right)_{(p, \rho) = (0, 0)} = \frac{4(8 - 11\beta + 4\beta^2)}{(3 - 2\beta)^2(2 - \beta)^2(4 - 3\beta)} > 0 \quad \text{for all} \quad \beta \in [0, 1].
\]

Thus \( F(p, \rho) \) attains either maximum or minimum at \((p, \rho) = (0, 0)\). Since, we have

\[
\left( \frac{\partial^2 F}{\partial p^2} \right)_{(0, 0)} = -\frac{1}{4 - 3\beta} < 0, \quad \left( \frac{\partial^2 F}{\partial \rho^2} \right)_{(0, 0)} = -\frac{4}{4 - 3\beta} < 0 \quad \text{for all} \quad \beta \in [0, 1].
\]

Therefore, \( F(p, \rho) \) attain its maximum value at \((p, \rho) = (0, 0)\), which is \( 2/(4 - 3\beta) \).

Now, to prove the sharpness of the bound, consider the function \( \hat{f}_2 : \mathbb{D} \to \mathbb{C} \) given by

\[
\alpha \frac{\hat{f}_2(z)}{z} + (1 - \alpha) \hat{f}_2(z) = \frac{1 + z^3}{1 - z^3}. \quad (2.7)
\]

If \( \hat{f}_2(z) = z + \sum_{n=2}^{\infty} a_n z^n \), then \( a_2 = a_3 = 0 \) and \( a_4 = 2/(4 - 3\beta) \), thus \( |J_{2,3}(f)| = 2/(4 - 3\beta) \).

3 Toeplitz and Hermitian-Toeplitz Determinant

Theorem 3.1. If \( f \in \mathcal{A}_\beta \) is of the form (1.1), then
(i) \(|T_{2,n}(f)| \leq 4 \left( \frac{1}{(n - \beta(n - 1))^2} + \frac{1}{(n + 1 - n\beta)^2} \right)\),

(ii) \(|T_{3,1}(f)| \leq \frac{4\beta^4 - 28\beta^3 + 101\beta^2 - 196\beta + 140}{(3 - 2\beta)^2(\beta - 2)^2} .

The bounds are sharp.

**Proof.** From (1.3), it follows that

\[ |T_{2,n}(f)| = |a_n^2 - a_{n+1}^2| \leq |a_n|^2 + |a_{n+1}|^2 .\]

Using the bound of \(|a_n|\) from (2.1), required bound of \(|T_{2,n}(f)|\) follows directly and equality case holds for the function \(f_1\) given by (2.4).

Now we proceed for \(|T_{3,1}(f)|\). Again from (1.3), we have

\[ |T_{3,1}(f)| = |1 - 2a_2^2 + 2a_2^2a_3 - a_3^2| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2| . \quad (3.1) \]

By (2.2),

\[ |a_3 - 2a_2^2| = \frac{1}{3 - 2\beta} \left| p_2 - \frac{2(3 - 2\beta)}{(2 - \beta)^2} p_1 \right| . \]

Applying the well known result \(|p_2 - \mu p_1|^2 \leq 4\mu - 2\) for \(\mu > 1\) (see [29]), we obtain

\[ |a_3 - 2a_2^2| \leq \frac{8}{(2 - \beta)^2} - \frac{2}{3 - 2\beta} . \]

Using this bound of \(|a_3 - 2a_2^2|\) and the bounds of \(|a_2|, |a_3|\) from (2.1) in (3.1), required bound of \(|T_{3,1}(f)|\) follows. Sharpness of the bound of \(|T_{3,1}(f)|\) follows from the function \(f_1\).

**Remark 3.1.** The bounds of \(|T_{2,n}(f)|\) and \(|T_{3,1}(f)|\) for the class \(\mathcal{R}\) follow from Theorem 3.1 when \(\beta = 0\) [21, Theorem 2.12].

**Theorem 3.2.** If \(f \in A_\beta\) is of the form (1.1), then

\[
T_{3,1}(f) \leq \begin{cases} 
\frac{4\beta^4 - 28\beta^3 + 37\beta^2 - 4\beta - 4}{(3 - 2\beta)^2(2 - \beta)^2}; & 0 \leq \beta \leq \frac{10 - \sqrt{10}}{9} , \\
1; & \frac{10 - \sqrt{10}}{9} \leq \beta \leq 1 ,
\end{cases} \quad (3.2)
\]

The bounds are sharp.

**Proof.** For \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A_\beta\), Theorem 2.1 yields

\[ |a_2| \leq \frac{2}{2 - \beta} \text{ and } |a_3| \leq \frac{2}{3 - 2\beta} . \]

Hence \(|a_2| \in [0, 2]\) and \(|a_3| \in [0, 2]\) for \(\beta \in [0, 1]\). From (1.4), we have

\[
T_{3,1}(f) = 1 + 2 Re(a_2^2a_3) - 2|a_2|^2 - |a_3|^2 \\
\leq 1 + 2|a_2|^2|a_3| - 2|a_2|^2 - |a_3|^2 =: g(|a_3|),
\]

where \(g(x) = 1 + 2|a_2|^2x - 2|a_2|^2 - x^2\) with \(x = |a_3| \in [0, 2]\). Since \(g'(x) = 0\) at \(x_0 := |a_2|^2\) and \(g''(x_0) < 0\), therefore \(g(x)\) attains its maximum value at \(x = x_0\), whenever \(|a_2|^2\) belongs to the range of \(x\), that means \(|a_2|^2 \leq 2\). Thus

\[
T_{3,1}(f) \leq g(|a_2|^2) = (|a_2|^2 - 1)^2 \\
\leq 1 \text{ when } |a_2|^2 \leq 2,
\]

\(\text{6} \)
Now, the other case, when \(|a_2|^2\) does not lie in the range of \(x\), that is \(|a_2|^2 > 2 \text{ or } 2 - \sqrt{2} \leq \beta \leq 1\), then
\[
T_{3,1}(f) \leq \max g(x) = \frac{2}{3 - 2\beta}
\]
\[
= 1 - 2a_2^2 - \frac{4}{(3 - 2\beta)^2} + \frac{4a_2^2}{3 - 2\beta}
\]
\[
\leq \frac{4\beta^4 - 28\beta^3 + 37\beta^2 - 4\beta - 4}{(3 - 2\beta)(2 - \beta)^2}.
\]
Using all these above arguments, we obtain
\[
T_{3,1}(f) \leq \left\{ \begin{array}{ll}
1, & \beta \leq \beta_0; \\
\frac{1}{4\beta^4 - 28\beta^3 + 37\beta^2 - 4\beta - 4}, & \beta_0 \leq \beta \leq 1,
\end{array} \right.
\]
where \(\beta_0 = (10 - \sqrt{10})/9\) is the root of the equation \(9\beta^2 - 20\beta + 10 = 0\).

The sharpness of the bound follows from \(f(z) = z\) when \(0 \leq \beta \leq (10 - \sqrt{10})/9\). However, for \((10 - \sqrt{10})/9 \leq \beta \leq 1\), equality in (3.2) holds for the function \(\tilde{f}\) given in (2.3).

Remark 3.2. For \(\beta = 0\) in Theorem 3.2 we obtain \(T_{3,1}(f) \leq 1\) for \(f \in R\) [20 Example 2.4].

**Theorem 3.3.** If \(f \in A_\beta\) is of the form (1.1), then
\[
T_{3,1}(f) \geq 1 - \frac{4\beta - 9}{\beta^4 - 4\beta^3 + 2\beta^2 + 8\beta - 8}.
\]
The bound is sharp.

**Proof.** Let \(f \in A_\beta\), then from (2.2), we have
\[
a_2 = \frac{p_1}{2 - \beta}, \quad a_3 = \frac{p_2}{3 - 2\beta}.
\]
Now, by replacing \(p_2\) in terms of \(p_1\) using (2.6), we get
\[
2 \text{Re}(a_2^2a_3) = \frac{p_1^4 + (4 - p_1^2)\text{Re}\zeta}{(3 - 2\beta)(2 - \beta)^2}, \quad -|a_2|^2 = \frac{|p_1|^2}{(2 - \beta)^2},
\]
\[
-|a_3|^2 = -\frac{p_1^4 + (4 - p_1^2)^2|\zeta|^2 + 2p_1^2(4 - p_1^2)\text{Re}\zeta}{4(3 - 2\beta)^2}.
\]
A simple computation yields that
\[
T_{3,1}(f) = 1 + \frac{1}{4(3 - 2\beta)^2(2 - \beta)^2} \left( p_1^4(8 - 4\beta - \beta^2) - 8p_1^2(3 - 2\beta)^2 \
- (4 - p_1^2)^2(2 - \beta)^2|\zeta|^2 + 2p_1^2(4 - p_1^2)(2 - \beta)\text{Re}\zeta \right) \
=: g(p_1, \zeta, \text{Re}\zeta).
\]
Since the classes \(A_\beta\) and \(P\) are rotationally invariant, we can take \(p = p_1 \in [0, 2]\). Using \(\text{Re}\zeta \geq -|\zeta|\) with notation \(|\zeta| = y\), we have \(g(p_1, |\zeta|, \text{Re}\zeta) \geq g_1(p, y)\), where
\[
g_1(p, y) = 1 + \frac{1}{4(3 - 2\beta)^2(2 - \beta)^2} \left( p_1^4(8 - 4\beta - \beta^2) - 8p_1^2(3 - 2\beta)^2 \
- (4 - p_1^2)^2(2 - \beta)^2y^2 + 2p_1^2(4 - p_1^2)(2 - \beta)y \right).
\]
Also, note that
\[
\frac{\partial g_1(p,y)}{\partial y} = -\frac{2(4-p^2)y(2-\beta)^2 + 2p^2(4-p^2)(2-\beta^2)}{4(3-2\beta)^2(2-\beta^2)} < 0
\]
for all \( p \in [0,2] \) and \( \beta \in [0,1] \). Hence \( g_1(p,y) \) is a decreasing function of \( y \) with \( g_1(p,y) \leq g_1(p,1) =: g_2(p) \). Minimum of \( g_2(p) \) is the lower bound of \( \det T_{3,1}(f) \). The equation \( g_2''(p) = 0 \) gives the following critical points
\[
p^{(1)} = 0, \quad p^{(2)} = \pm \sqrt{\frac{(2\beta^2 - 8\beta + 7)}{(2-\beta^2)}}.
\]
Using the basic calculus rule, it can be easily observed that the function \( g_2(p) \) attains its minimum value at \( p^{(2)} \) as \( g''(p^{(2)}) > 0 \) for all \( \beta \in [0,1] \). Thus
\[
\det T_{3,1}(f) \geq g_2(p^{(2)}) = 1 - (4\beta - 9)/(2\beta^4 - 4\beta^3 + 2\beta^2 + 8\beta - 8).
\]
To show the sharpness consider the function \( \tilde{f}_3 \in \mathcal{A} \) given by
\[
\frac{\beta \tilde{f}_3(z)}{z} + (1 - \beta)\tilde{f}_3(z) = \frac{1 - z^2}{1 - z\sqrt{(2\beta^2 - 8\beta + 7)/(2-\beta^2)} + z^2}.
\]
For \( \tilde{f}_3(z) = z + \sum_{n=2}^{\infty} a_n z^n \), we have
\[
a_2 = \frac{1}{2-\beta} \sqrt{\frac{2\beta^2 - 8\beta + 7}{2-\beta^2}}, \quad a_3 = \frac{1 - 2\beta}{2-\beta^2}
\]
and \( T_{3,1}(\tilde{f}_3) = 1 - (4\beta - 9)/(2\beta^4 - 4\beta^3 + 2\beta^2 + 8\beta - 8) \).

**Remark 3.3.** For \( \beta = 0 \) in Theorem 3.2 we obtain \( \det T_{3,1}(f) \geq -1/8 \) for \( f \in \mathcal{R} \) [20, Example 2.4].

## 4 Coefficient Difference

Robertson [34] proved that \( 3|a_{n+1} - a_n| \leq (2n+1)|a_2 - 1| \) for the class of convex functions. Recently, Li and Sugawa [25] obtained the bound of \( |a_{n+1} - a_n| \) for particular choices of \( n \) for the class of convex function with fixed second coefficient. In this section, we find the the bound of \( |a_{n+1} - a_n| \) \((N \in \mathbb{N})\) depending on the second coefficient for \( f \in \mathcal{A}_\beta \). In fact, it is more convenient to express our result in terms of \( p_1 = p \), applying the correspondence
\[
(2-\beta)a_2 = p_1 = p.
\]
To make the results more legible, we define the class \( \mathcal{A}_\beta(p), p \in [-2,2] \) as follows
\[
\mathcal{A}_\beta(p) = \{ f \in \mathcal{A}_\beta : f''(0) = p \}.
\]
Clearly,
\[
\bigcup_{-2 \leq p \leq 2} \mathcal{A}_\beta(p) \subset \mathcal{A}_\beta \quad \text{and} \quad \bigcup_{-2 \leq p \leq 2} \mathcal{A}_\beta(p) \neq \mathcal{A}_\beta.
\]
The following lemmas are used to establish our main results.

**Lemma 4.1.** [23] If \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P} \), then the following estimate holds:
\[
|p_{n+1}^N - p_n^N| \leq 2^N \sqrt{2 - 2^{1-N} \text{Re}(p_1^N)} \quad (N \in \mathbb{N}).
\]
*Equality holds for the function \((1 + e^{i\alpha}z)/(1 - e^{i\alpha}z)\), where \( \alpha = \cos^{-1}(b/2) \) and \( \text{Re} p_1 = 2b \).*

**Lemma 4.2.** [23] Fix \( \zeta \in \overline{D} \). If \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P} \), then
\[
|\zeta p_{n+1} - p_n| \leq \frac{2(1 - |\zeta|^n)(1 + |\zeta|^2 - \text{Re}(\zeta p_1))}{1 - |\zeta|} + |2 - \zeta p_1||\zeta|^n \quad \text{for} \ |\zeta| < 1.
\]
*The bounds are sharp for \( p(z) = (1 + z)/(1 - z) \).*
According to Komatu [19], if \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \) and \( q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \) both are the members of \( \mathcal{P} \), then the weighted Hadamard product, \( f \ast g \), also belongs to \( \mathcal{P} \), where

\[
f \ast g = 1 + \sum_{n=1}^{\infty} \frac{p_n q_n}{2} z^n.
\]

Let us define \( F_j(z) = F_{j-1} \ast p(z) \) for \( j \in \mathbb{N} \) with \( F_0(z) = p(z) \), then using the above result, we have \( F_j \in \mathcal{P} \). Particulary, for \( N \in \mathbb{N} \), the function

\[
F_{N-1}(z) = 1 + \sum_{n=1}^{\infty} \frac{p_n N}{2^{N-1}} z^n \in \mathcal{P}.
\]

Replacing \( p(z) \) in Lemma 4.2 by \( F_{N-1} \), the result is as follows:

**Lemma 4.3.** Fix \( \xi \in \mathbb{D} \) and \( N \in \mathbb{N} \). If \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \), then

\[
|\xi p_{n+1}^N - p_n^N| \leq \frac{2(1 - |\xi|^n) (2N-1 + 2^{N-1}|\xi|^2 - \text{Re}(\xi p_1^N)) + |2N - \xi p_1^N| \cdot |\xi|^n}{1 - |\xi|} \quad \text{for } |\xi| < 1,
\]

Equality holds for the function \((1 + z)/(1 - z)\).

**Theorem 4.1.** If \( f \in A_\beta(p) \), then the following inequalities hold:

\[
|a_{n+1}^N - a_n^N| \leq \begin{cases} 
\frac{2(n - (n - 1)\beta)^N}{(n + 1 - n\beta)^N} \left( \frac{n - (n - 1)\beta}{n + 1 - n\beta} \right) p_n^N - p_{n-1}^N, & \beta \neq 1, \\
\frac{2N}{\sigma} \sqrt{2^{2-2N} p_1^N}, & \beta = 1,
\end{cases}
\]

where \( \sigma = (n - (n - 1)\beta)^N \) and \( \mu = (n + 1 - n\beta)^N \). Bounds for \( \beta \in [0,1) \) is sharp for \( p = 2 \) whereas for \( \beta = 1 \), bound is sharp for odd \( N \) and \( p = -2 \).

**Proof.** For \( f \in A_\beta(p) \), from (2.2), we have

\[
(n - (n - 1)\beta)^N |a_{n+1}^N - a_n^N| = \left| \left( \frac{n - (n - 1)\beta}{n + 1 - n\beta} \right)^N p_n^N - p_{n-1}^N \right|.
\]

From Lemma 4.3 with \((n - (n - 1)\beta)/((n + 1 - n\beta))^N =: \xi \), bound in (4.1) for \( \beta \in [0,1) \) follows since \( \xi \in (0,1) \) whenever \( \beta \in (0,1) \). For \( \beta = 1 \) we have \( \xi = 1 \). Bounds for \( \beta = 1 \) are obtained using Lemma 4.1.

To show the sharpness for \( \beta \in [0,1) \), consider the function \( \hat{f}(z) \) given in (2.3). As for \( f \), we have

\[
|a_{n+1} - a_n| = \frac{2N}{(n - (n - 1)\beta)^N} \left| \left( \frac{n - (n - 1)\beta}{n + 1 - n\beta} \right)^N - 1 \right|,
\]

which is same as in (4.1) for \( p = 2 \). In case of \( \beta = 1 \), for the function \( \hat{f}(-z) \), we have

\[
|a_{n+1} - a_n| = \frac{2^{N+1}}{(n - (n - 1)\beta)^N},
\]

which coincides with the bounds in (4.1) for odd \( N \) and \( p = -2 \).

For \( N = 1 \), Theorem 4.1 yields the following bounds:
Corollary 4.2. If \( f \in \mathcal{A}_\beta(p) \) is of the form \((1.1)\), then

\[
|a_{n+1} - a_n| \leq \begin{cases} 
  \frac{2(\sigma^n - \mu^n)(\sigma^2 + \mu^2 - \sigma \mu p)}{\sigma \mu^{n+1}} + \frac{\sigma^n |2 \mu - \sigma p|}{\sigma \mu^{n+1}}; & \beta \in [0, 1), \\
  \frac{2\sqrt{2 - p}}{p}; & \beta = 1.
\end{cases}
\]

The class \( \mathcal{A}_\beta \) reduces to the class \( \mathcal{R} \) for \( \beta = 0 \). Let us take corresponding class \( \mathcal{R}(p) = \{ f \in \mathcal{R} : f''(0) = p \} \). Theorem 4.1 gives the following result for the class \( \mathcal{R}(p) \) when \( \beta = 0 \).

Corollary 4.3. If \( f \in \mathcal{R}(p) \) is of the form \((1.1)\), then the following sharp bounds hold:

\[
|a_{n+1}^N - a_n^N| \leq \frac{2(\sigma^n - \mu^n)(2^{N-1}\sigma^2 + 2^{N-1}\mu^2 - \sigma \mu p^N)}{(\sigma - \mu)\mu^{n+1}} + \frac{\sigma^n |2 \mu - \sigma p^N|}{\sigma \mu^{n+1}}.
\]

5 Growth Theorem and Bohr Phenomenon

Theorem 5.1. If \( f \in \mathcal{A}_\beta \) is of the form \((1.1)\), then for \( |z| \leq r \), the following hold:

(i) \( -\frac{\tilde{f}(-r)}{r} \leq \text{Re} \left( \frac{f(z)}{z} \right) \leq \frac{\tilde{f}(r)}{r} \),

(ii) \( -\tilde{f}(-r) \leq |f(z)| \leq \tilde{f}(r) \),

where \( \tilde{f}(z) \) is given by \((2.3)\). All these estimations are sharp.

Proof. (i) Let \( f \in \mathcal{A}_\beta \). Consider \( p(z) = f(z)/z \), then we have

\[
\text{Re}(p(z) + (1 - \beta)zp'(z)) > 0.
\]

It can be viewed as \( p(z) + (1 - \beta)zp'(z) \prec (1 + z)/(1 - z) \). Further, by Hallenbeck and Rusheweyeh [30] Theorem 3.1b, it follows that

\[
p(z) \prec q(z) \prec \frac{1 + z}{1 - z},
\]

where \( q(z) \) is convex and best dominant, given by

\[
q(z) = \frac{1}{(1 - \beta)z^{(1 - \beta)}} \int_0^z (1 + t) t^{(1 - \beta) - 1} dt = \frac{\tilde{f}(z)}{z},
\]

where \( \tilde{f}(z) \) is defined in \((2.3)\). Since \( q(z) \) is convex and all coefficients are real for \( \beta \in [0, 1] \), therefore image domain of \( D \) under the function \( q(z) \) is symmetric with respect to real axis and

\[
q(-r) \leq \text{Re}(q(z)) \leq q(r), \quad |z| = r < 1.
\]

As \( p(z) = f(z)/z \prec q(z) \), so required bound of \( \text{Re}(f(z)/z) \) follows. This completes the first part. Sharpness of the bounds follow as \( q(z) \) is the best dominant. (ii) From [7] Lemma 4.10, \( f \in \mathcal{A}_\beta \) if and only if

\[
f(z) = z \int_0^1 p(t^{1-\beta}) dt, \quad (5.1)
\]

where \( p \in \mathcal{P} \). Using the well known bound \( |p(z)| \leq (1 + r)/(1 - r) \) of Carathéodory functions, we have

\[
|f(z)| \leq r \int_0^1 \frac{1 + rt^{1-\beta}}{1 - rt^{1-\beta}} dt = \tilde{f}(r),
\]

10
Now, we proceed for the lower bound of $|f(z)|$. After solving the integration in (5.1) for $p(z) = (1+z)/(1-z)$, we get

$$f(z) = z(-1 + 2H(z)),$$

where

$$H(z) = zF\left[1, \frac{1}{1-\beta}; \frac{2-\beta}{1-\beta}; z\right].$$

Thus for $z = re^{i\theta}$,

$$|f(z)| = |z(-1 + 2H(z))| \geq \min_{\theta \in [0, 2\pi]} g(\theta)$$

where

$$g(\theta) = \sqrt{\text{Re}(re^{i\theta}(-1 + 2H(re^{i\theta})))^2 + \text{Im}(re^{i\theta}(-1 + 2H(re^{i\theta})))^2},$$

Since for different choices of $\beta$ in $[0, 1)$, $H(z)$ reduces to different functions. For instance, when $\beta = 0$, it becomes $-2\log(1-z)/z$ and for $\beta = 1/2$, it reduces to $-4(z + \log(1-z))/z^2$. By a simple calculation, we find that the function $g(\theta)$ is decreasing from $[0, \pi]$ and increasing from $[\pi, 2\pi]$ for $r \in (0, 1)$ and $\beta \in (0, 1)$.

Hence $g(\theta)$ attains its minimum value at $\theta = \pi$. Thus from (5.2), we get

$$|f(z)| \geq |r(-1 + 2H(-r))| = r(-1 + 2H(-r)) = -r\tilde{f}(-r),$$

which completes the proof. Bounds are sharp for the function $\tilde{f}(z)$.

**Theorem 5.2.** If $f \in A_\beta$ is of the form (4.1), then for $m \in \mathbb{N}$

$$|\omega(z^m)| + \sum_{n=2}^{\infty} |a_nz^n| \leq d(0, \partial f(D))$$

in $|z| \leq r^*$, where $r^*$ is the smallest positive root of

$$r^m + \tilde{f}(r) - r + \tilde{f}(-1) = 0.$$  \hspace{1cm} (5.3)

**The radius $r^*$ is sharp.**

**Proof.** Let $f \in A_\beta$, then by Theorem 5.1 the Euclidean distance between $f(0) = 0$ and the boundary of $f(D)$ satisfies

$$d(0, \partial f(D)) \geq \lim_{r \to 1^-} |f(z)| = -\tilde{f}(-1).$$

Let $|z| \leq r$. Now using (4.1) with the above inequality, we have

$$|\omega(z^m)| + \sum_{n=2}^{\infty} |a_nz^n| \leq r^m + \sum_{n=2}^{\infty} \left(\frac{2}{n-\beta(n-1)}\right) r^n,

= r^m + \tilde{f}(r) - r

\leq -\tilde{f}(-1) \leq d(0, \partial f(D)).$$

which is true in $|z| = r \leq r^*$, where $r^*$ is the root of $H(r) = r(r^{m-1} - 1) + \tilde{f}(r) + \tilde{f}(-1)$. Note that, $H(0) = \tilde{f}(-1) < 0$ and $H(1) = \tilde{f}(1) + \tilde{f}(-1) > 0$ for all $\beta \in [0, 1]$, therefore by the Intermediate value property for continuous functions there must exist a $r^* \in (0, 1)$ such that $H(r^*) = 0$.

Sharpness holds for the functions $\tilde{f}(z)$ and $\omega(z) = z$. Since at $z = r^*$,

$$|\omega(z^m)| + \sum_{n=2}^{\infty} |a_nz^n| = (r^*)^m + \sum_{n=2}^{\infty} \left(\frac{2}{n-(n-1)\beta}\right) (r^*)^n

= (r^*)^m + \tilde{f}(r^*) - r^* = -\tilde{f}(-1).$$

Hence the radius is sharp.
For \( w(z) = z \) and \( m = 1 \), Theorem 5.2 gives the following Bohr-radius for the class \( A_\beta \).

**Corollary 5.3.** If \( f \in A_\beta \), then \( |z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(D)) \) in \( |z| \leq r_b \), where \( r_b \) is root of \( \hat{f}(r) + \hat{f}(-1) = 0 \). The radius \( r_b \) is sharp.

For various values of \( \beta \in [0, 1] \), the root \( r_b \) is shown in Figure 1 and Table 1.

![Figure 1](image)

**Table 1: Radius \( r^* \) for various choices of \( \beta \)**

| \( \beta \) | 0.1 | 0.2 | 0.3 | 0.5 | 0.7 | 0.8 | 0.9 |
|-------------|-----|-----|-----|-----|-----|-----|-----|
| \( r_b \)   | 0.267139 | 0.24766 | 0.22655 | 0.178366 | 0.119726 | 0.085113 | 0.0457777 |

**Theorem 5.4.** If \( f \in A_\beta \), then

\[
|f(z^m)| + \sum_{k=N}^{\infty} |a_k z^k| \leq d(0, \partial f(D)) \quad (5.4)
\]

hold for \( |z| = r \leq r_N \), where \( r_N \) is the root of the equation

\[
\hat{f}(r^m) + \hat{f}(r) - \hat{f}(r) + \hat{f}(-1) = 0,
\]

with

\[
\hat{f}(r) = \begin{cases} 
0 & N = 1, \\
r & N = 2, \\
r + \sum_{n=2}^{N-1} \frac{2}{n - (n-1)\beta} r^n & N \geq 3.
\end{cases}
\]

The radius is sharp.

**Proof.** Suppose \( f \in A_\beta \), then from (1.2) and Theorem 5.1 we have

\[
|f(z^m)| + \sum_{k=N}^{\infty} |a_k z^k| \leq \hat{f}(r^m) + \sum_{n=N}^{\infty} \frac{2}{n - (n-1)\beta} r^n
\]

\[
= \hat{f}(r^m) - \hat{f}(r) + \hat{f}(r)
\]

\[
\leq -\hat{f}(-1)
\]

\[
\leq d(0, \partial f(D))
\]

holds in \( |z| = r_N \), where \( r_N \) is the root of

\[
G(r) := \hat{f}(r^m) - \hat{f}(r) + \hat{f}(r) + \hat{f}(-1) = 0.
\]
Since $G(0) = \tilde{f}(-1) < 0$ and $G(1) = (\tilde{f}(1) - \tilde{f}(1)) + (\tilde{f}(1) + \tilde{f}(-1)) > 0$, therefore there exist a $r_N \in (0, 1)$ such that (5.4) holds. Note that, for the function $\tilde{f}(z)$ at $|z| = r_N$,

$$|f(z^m)| + \sum_{k=N}^{\infty} |a_k z^k| = \tilde{f}(r_N^m) + \sum_{n=N}^{\infty} \frac{2}{n - (n - 1)\beta} r_N^n$$

which proves the sharpness of radius.

6 Declarations

Funding

The work of the Surya Giri is supported by University Grant Commission, New-Delhi, India under UGC-Ref. No. 1112/(CSIR-UGC NET JUNE 2019).

Conflict of interest

The authors declare that they have no conflict of interest.

Author Contribution

Each author contributed equally to the research and preparation of manuscript.

Data Availability

Not Applicable.

References

[1] V. Allu, A. Lecko and D. K. Thomas, Hankel, Toeplitz, and Hermitian-Toeplitz determinants for certain close-to-convex functions, Mediterr. J. Math. 19 (2022), no. 1, Paper No. 22, 17 pp.

[2] M. F. Ali, D. K. Thomas and A. Vasudevarao, Toeplitz determinants whose elements are the coefficients of analytic and univalent functions, Bull. Aust. Math. Soc. 97 (2018), no. 2, 253–264.

[3] E. Berkson and H. Porta, Semigroups of analytic functions and composition operators, Michigan Math. J. 25 (1978), no. 1, 101–115.

[4] C. Bénétieau, A. Dahlner and D. Khavinson, Remarks on the Bohr phenomenon, Comput. Methods Funct. Theory 4 (2004), no. 1, 1–19.

[5] H. Bohr, A Theorem Concerning Power Series, Proc. London Math. Soc. (2) 13 (1914), 1–5.

[6] F. Bracci, M. D. Contreras and S. Díaz-Madrigal, Pluripotential theory, semigroups and boundary behavior of infinitesimal generators in strongly convex domains, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 1, 23–53.

[7] F. Bracci, M. D. Contreras, S. Díaz-Madrigal, M. Elin and D. Shoikhet, Filtrations of infinitesimal generators, Funct. Approx. Comment. Math. 59 (2018), no. 1, 99–115.

[8] F. Bracci, M. D. Contreras and S. Díaz-Madrigal, Evolution families and the Loewner equation I: the unit disc, J. Reine Angew. Math. 672 (2012), 1–37.
[9] J. E. Brown, Successive coefficients of functions with positive real part, Int. J. Math. Anal. (Ruse) 4 (2010), no. 49-52, 2491–2499.

[10] N. E. Cho, O. S. Kwon, A. Lecko and Y. J. Sim, Sharp estimates of generalized Zalcman functional of early coefficients for Ma-Minda type functions, Filomat 32 (2018), no. 18, 6267–6280.

[11] K. Cudna, O. S. Kwon, A. Lecko, Y. J. Sim and B. Śmiarowska, The second and third-order Hermitian Toeplitz determinants for starlike and convex functions of order $\alpha$, Bol. Soc. Mat. Mex. (3) 26 (2020), no. 2, 361–375.

[12] P. L. Duren, Univalent functions, Grundlehren der mathematischen Wissenschaften, 259, Springer-Verlag, New York, 1983.

[13] M. Elin and D. Shoikhet, Linearization models for complex dynamical systems, Operator Theory: Advances and Applications, 208, Birkhäuser Verlag, Basel, 2010.

[14] M. Elin, D. Shoikhet and N. Tuneski, Radii problems for starlike functions and semigroup generators, Comput. Methods Funct. Theory 20 (2020), no. 2, 297–318.

[15] K. Gangania and S. S. Kumar, Bohr Radius for Some Classes of Harmonic Mappings, Iranian Journal of Science and Technology, Transactions A: Science (2022): 1-8.

[16] S. Giri and S. S. Kumar, Radius and Convolution problems of analytic functions involving Semigroup Generators, arXiv:2205.10777 (2022).

[17] K. Gangania and S. S. Kumar, Bohr-Rogosinski Phenomenon for $S^*(\psi)$ and $C(\psi)$, Mediterr. J. Math. 19 (2022), no. 4, Paper No. 161.

[18] I. R. Kayumov, D. M. Khammatova and S. Ponnusamy, Bohr-Rogosinski phenomenon for analytic functions and Cesáro operators, J. Math. Anal. Appl. 496 (2021), no. 2, Paper No. 124824, 17 pp.

[19] Y. Komatu, On convolution of power series, Kodai Math. Sem. Rep. 10 (1958), 141–144.

[20] V. Kumar and N. E. Cho, Hermitian-Toeplitz determinants for functions with bounded turning, Turkish J. Math. 45 (2021), no. 6, 2678–2687.

[21] V. Kumar and S. Kumar, Bounds on Hermitian-Toeplitz and Hankel determinants for strongly starlike functions, Bol. Soc. Mat. Mex. (3) 27 (2021), no. 2, Paper No. 55, 16 pp.

[22] E. Landau and D. Gaier, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, third edition, Springer-Verlag, Berlin, 1986.

[23] A. Lecko, On coefficient inequalities in the Carathéodory class of functions, Annales Polonici Mathematici. Vol. 75. Instytut Matematyczny Polskiej Akademii Nauk, 2000.

[24] A. Lecko, Y. J. Sim and B. Śmiarowska, The fourth-order Hermitian Toeplitz determinant for convex functions, Anal. Math. Phys. 10 (2020), no. 3, Paper No. 39, 11 pp.

[25] M. Li and T. Sugawa, A note on successive coefficients of convex functions, Comput. Methods Funct. Theory 17 (2017), no. 2, 179–193.

[26] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc. 85 (1982), no. 2, 225–230.

[27] M.-S. Liu, Y.-M. Shang and J.-F. Xu, Bohr-type inequalities of analytic functions, J. Inequal. Appl. 2018, Paper No. 345, 13 pp.

[28] W. Ma, Generalized Zalcman conjecture for starlike and typically real functions, J. Math. Anal. Appl. 234 (1999), no. 1, 328–339.
[29] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA.

[30] S. S. Miller and P. T. Mocanu, Differential subordinations, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York, 2000.

[31] Y. A. Muhanna, R. M. Ali and S. Ponnusamy, On the Bohr inequality, in Progress in approximation theory and applicable complex analysis, 269–300, Springer Optim. Appl., 117, Springer, Cham.

[32] Y. A. Muhanna, Bohr’s phenomenon in subordination and bounded harmonic classes, Complex Var. Elliptic Equ. 55 (2010), no. 11, 1071–1078.

[33] M. Obradović and N. Tuneski, Hermitian Toeplitz determinants for the class $S$ of univalent functions, Armen. J. Math. 13 (2021), Paper No. 4, 10 pp.

[34] M. S. Robertson, Univalent functions starlike with respect to a boundary point, J. Math. Anal. Appl. 81 (1981), no. 2, 327–345.

[35] W. Rogosinski, Über image barriers in power series and their sections, Math. Z. 17 (1923), no. 1, 260–276.

[36] S. Reich and D. Shoikhet, The Denjoy–Wolff theorem, Math. Encyclopaedia, Supplement 3, Kluwer Academic Publishers, (2002), 121–123.

[37] D. Shoikhet, Semigroups in geometrical function theory, Kluwer Academic Publishers, Dordrecht, 2001.

[38] D. K. Thomas, N. Tuneski and A. Vasudevarao, Univalent functions, De Gruyter Studies in Mathematics, 69, De Gruyter, Berlin, 2018.

[39] L. Wu, Q. Wang and B. Long, Some Bohr-type inequalities with one parameter for bounded analytic functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 116 (2022), no. 2, Paper No. 61, 13 pp.

Department of Applied Mathematics, Delhi Technological University, Delhi–110042, India

E-mail address: spkumar@dtu.ac.in

E-mail address: suryagiri456@gmail.com