Parametrix for wave equations on a rough background IV: control of the error term

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Abstract. This is the last of a sequence of four papers [15], [16], [17], [18] dedicated to the construction and the control of a parametrix to the homogeneous wave equation \( \Box_g \phi = 0 \), where \( g \) is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes \( L^2 \) bounds on the curvature tensor \( R \) of \( g \) is a major step of the proof of the bounded \( L^2 \) curvature conjecture proposed in [8], and solved by S. Klainerman, I. Rodnianski and the author in [12]. On a more general level, this sequence of papers deals with the control of the eikonal equation on a rough background, and with the derivation of \( L^2 \) bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

1 Introduction

We consider the Einstein vacuum equations,

\[ R_{\alpha\beta} = 0 \]  

(1.1)

where \( R_{\alpha\beta} \) denotes the Ricci curvature tensor of a four dimensional Lorentzian space time \( (\mathcal{M}, g) \). The Cauchy problem consists in finding a metric \( g \) satisfying (1.1) such that the metric induced by \( g \) on a given space-like hypersurface \( \Sigma_0 \) and the second fundamental form of \( \Sigma_0 \) are prescribed. The initial data then consists of a Riemannian three dimensional metric \( g_{ij} \) and a symmetric tensor \( k_{ij} \) on the space-like hypersurface \( \Sigma_0 = \{t = 0\} \). Now, (1.1) is an overdetermined system and the initial data set \( (\Sigma_0, g, k) \) must satisfy the constraint equations

\[
\begin{align*}
\nabla^j k_{ij} - \nabla_i \text{Tr} k &= 0, \\
R - |k|^2 + (\text{Tr} k)^2 &= 0,
\end{align*}
\]

(1.2)

where the covariant derivative \( \nabla \) is defined with respect to the metric \( g \), \( R \) is the scalar curvature of \( g \), and \( \text{Tr} k \) is the trace of \( k \) with respect to the metric \( g \).

The fundamental problem in general relativity is to study the long term regularity and asymptotic properties of the Cauchy developments of general, asymptotically flat, initial data sets \( (\Sigma_0, g, k) \). As far as local regularity is concerned it is natural to ask what are the minimal regularity properties of the initial data which guarantee the existence and uniqueness of local developments. In [12], we obtain the following result which solves bounded \( L^2 \) curvature conjecture proposed in [8]:
Theorem 1.1 (Theorem 1.10 in [12]) Let \((M, g)\) an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation by space-like hypersurfaces \(\Sigma_t\) defined as level hypersurfaces of a time function \(t\). Let \(r_{\text{vol}}(\Sigma_0, 1)\) the volume radius on scales \(\leq 1\) of \(\Sigma_1\). Assume that the initial slice \((\Sigma_0, g, k)\) is such that:

\[
\|R\|_{L^2(\Sigma_0)} \leq \varepsilon, \quad \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \varepsilon \quad \text{and} \quad r_{\text{vol}}(\Sigma_0, 1) \geq \frac{1}{2}.
\]

Then, there exists a small universal constant \(\varepsilon_0 > 0\) such that if \(0 < \varepsilon < \varepsilon_0\), then the following control holds on \(0 \leq t \leq 1\):

\[
\begin{align*}
\|R\|_{L^\infty[0,1]L^2(\Sigma_t)} &\lesssim \varepsilon, \\
\|k\|_{L^\infty[0,1]L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty[0,1]L^2(\Sigma_t)} &\lesssim \varepsilon \\
\inf_{0 \leq t \leq 1} r_{\text{vol}}(\Sigma_t, 1) &\geq \frac{1}{4}.
\end{align*}
\]

Remark 1.2 While the first nontrivial improvements for well posedness for quasilinear hyperbolic systems (in spacetime dimensions greater than \(1 + 1\)), based on Strichartz estimates, were obtained in [2], [1], [19], [20], [6], [10], [13], Theorem 1.1, is the first result in which the full nonlinear structure of the quasilinear system, not just its principal part, plays a crucial role. We note that though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to its causal geometry, i.e. \(L^2\) bounds on the curvature is the minimum requirement necessary to obtain lower bounds on the radius of injectivity of null hypersurfaces. We refer the reader to section 1 in [12] for more motivations and historical perspectives concerning Theorem 1.1.

Remark 1.3 The regularity assumptions on \(\Sigma_0\) in Theorem 1.1 - i.e. \(R\) and \(\nabla k\) bounded in \(L^2(\Sigma_0)\) - correspond to an initial data set \((g, k)\) \(H^2_{\text{loc}}(\Sigma_0) \times H^1_{\text{loc}}(\Sigma_0)\).

Remark 1.4 In [12], our main result is stated for corresponding large data. We then reduce the proof to the small data statement of Theorem 1.1 relying on a truncation and rescaling procedure, the control of the harmonic radius of \(\Sigma_0\) based on Cheeger-Gromov convergence of Riemannian manifolds together with the assumption on the lower bound of the volume radius of \(\Sigma_0\), and the gluing procedure in [5], [4]. We refer the reader to section 2.3 in [12] for the details.

Remark 1.5 We recall for the convenience of the reader the definition of the volume radius of the Riemannian manifold \(\Sigma_t\). Let \(B_r(p)\) denote the geodesic ball of center \(p\) and radius \(r\). The volume radius \(r_{\text{vol}}(p, r)\) at a point \(p \in \Sigma_t\) and scales \(\leq r\) is defined by

\[
r_{\text{vol}}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r^3},
\]

with \(|B_r|\) the volume of \(B_r\) relative to the metric \(g_t\) on \(\Sigma_t\). The volume radius \(r_{\text{vol}}(\Sigma_t, r)\) of \(\Sigma_t\) on scales \(\leq r\) is the infimum of \(r_{\text{vol}}(p, r)\) over all points \(p \in \Sigma_t\).

The proof of Theorem 1.1, obtained in the sequence of papers [12], [15], [16], [17], [18], [11], relies on the following ingredients:

1. See Remark 1.5 below for a definition
2. We also need trilinear estimates and an \(L^4(M)\) Strichartz estimate (see the introduction in [12])
A. Provide a system of coordinates relative to which (1.1) exhibits a null structure.

B. Prove appropriate bilinear estimates for solutions to $\Box_g \phi = 0$, on a fixed Einstein vacuum background.\(^3\)

C. Construct a parametrix for solutions to the homogeneous wave equations $\Box_g \phi = 0$ on a fixed Einstein vacuum background, and obtain control of the parametrix and of its error term only using the fact that the curvature tensor is bounded in $L^2$.\(^4\)

Steps A and B are carried out in [12]. In particular, the proof of the bilinear estimates rests on a representation formula for the solutions of the wave equation using the following plane wave parametrix:\(^5\)

$$Sf(t, x) = \int_{\mathbb{R}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} f(\lambda \omega) \lambda^2 d\lambda d\omega, \ (t, x) \in \mathcal{M} \quad (1.3)$$

where $u(\cdot, \cdot, \omega)$ is a solution to the eikonal equation $g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ such that $u(0, x, \omega) \sim x \omega$ when $|x| \to +\infty$ on $\Sigma_0$.\(^5\) Therefore, in order to complete the proof of the bounded $L^2$ curvature conjecture, we need to carry out step C with the parametrix defined in (1.3).

Remark 1.6 Note that the parametrix (1.3) is invariantly defined, i.e. without reference to any coordinate system. This is crucial since coordinate systems consistent with $L^2$ bounds on the curvature would not be regular enough to control a parametrix.

Remark 1.7 In addition to their relevance to the resolution of the bounded $L^2$ curvature conjecture, the methods and results of step C are also of independent interest. Indeed, they deal on the one hand with the control of the eikonal equation $g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ at a critical level,\(^6\) and on the other hand with the derivation of $L^2$ bounds for Fourier integral operators with significantly lower differentiability assumptions both for the corresponding phase and symbol compared to classical methods (see in particular the discussion below (1.6)).

In view of the energy estimates for the wave equation, it suffices to control the parametrix at $t = 0$ (i.e. restricted to $\Sigma_0$)

$$Sf(0, x) = \int_{\mathbb{R}^2} \int_0^{+\infty} e^{i\lambda u(0, x, \omega)} f(\lambda \omega) \lambda^2 d\lambda d\omega, \ x \in \Sigma_0 \quad (1.4)$$

---

\(^3\)Note that the first bilinear estimate of this type was obtained in [9].

\(^4\)(1.3) actually corresponds to a half-wave parametrix. The full parametrix corresponds to the sum of two half-parametrix. See [16] for the construction of the full parametrix.

\(^5\)The asymptotic behavior for $u(0, x, \omega)$ when $|x| \to +\infty$ is used in [16] to generate with the parametrix any initial data set for the wave equation.

\(^6\)Our choice is reminiscent of the one used in [13] in the context of $H^{2+\epsilon}$ solutions of quasilinear wave equations. Note however that the construction in that paper is coordinate dependent.

\(^7\)We need at least $L^2$ bounds on the curvature to obtain a lower bound on the radius of injectivity of the null level hypersurfaces of the solution $u$ of the eikonal equation, which in turn is necessary to control the local regularity of $u$ (see [17]).
and the error term
\[ E f(t, x) = \square_g S f(t, x) = \int_{S^2} \int_0^{+\infty} e^{i\lambda u(t,x,\omega)} \square_g u(t, x, \omega) f(\lambda \omega) \lambda^3 d\lambda d\omega, \quad (t, x) \in \mathcal{M}. \quad (1.5) \]

This requires the following ingredients, the two first being related to the control of the parametrix restricted to \( \Sigma_0 \) (1.4), and the two others being related to the control of the error term (1.5):

**C1** Make an appropriate choice for the equation satisfied by \( u(0, x, \omega) \) on \( \Sigma_0 \), and control the geometry of the foliation generated by the level surfaces of \( u(0, x, \omega) \) obtained in C1.

**C2** Prove that the parametrix at \( t = 0 \) given by (1.4) is bounded in \( \mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma_0)) \) using the estimates for \( u(0, x, \omega) \) obtained in C1.

**C3** Control the geometry of the foliation generated by the level hypersurfaces of \( u \) on \( \mathcal{M} \).

**C4** Prove that the error term (1.5) satisfies the estimate \( \| E f \|_{L^2(\mathcal{M})} \leq C \| \lambda f \|_{L^2(\mathbb{R}^3)} \) using the estimates for \( u \) and \( \square_g u \) proved in C3.

Step C1 has been carried out in [15], step C2 has been carried out in [16], and step C3 has been carried out in [17]. In the present paper, we focus on step C4. Note that the error term (1.5) is a Fourier integral operator (FIO) with phase \( u(t, x, \omega) \) and symbol \( \square_g u(t, x, \omega) \). Now, we only assume \( L^2 \) bounds on the curvature tensor \( R \) in order to be consistent with the statement of Theorem 1.1. This severely limits the regularity \((t, x)\) we are able to obtain in step C3 for the solution \( u(t, x, \omega) \) of the Eikonal equation \( g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0 \) on \( \mathcal{M} \) (see [17] and section 2.4). Although \( R \) does not depend on the parameter \( \omega \), the regularity in \( \omega \) we are able to obtain in step C3 for \( u(t, x, \omega) \) is very limited as well\(^8\). In particular, we obtain for the symbol of \( E \) in (1.5):

\[
\sup_{\omega, u} \left( \| \square_g u \|_{L^\infty(\mathcal{H}_u)} + \| \mathbf{D} \square_g u \|_{L^2(\mathcal{H}_u)} + \| \partial_\omega \square_g u \|_{L^2(\mathcal{H}_u)} \right) \lesssim \varepsilon, \quad (1.6)
\]

where \( \mathcal{H}_u \) denotes the level hypersurfaces of the function \( u(t, x, \omega) \). Let us note that the classical arguments for proving \( L^2 \) bounds for FIO are based either on a \( TT^* \) argument, or a \( T^*T \) argument, which requires in our setting taking at least 3 derivatives of the symbol in \( L^\infty(\mathcal{M} \times S^2) \) either with respect to \((t, x)\) for \( T^*T \), or with respect to \((\lambda, \omega)\) for \( TT^* \) (see for example [14]). Both methods would fail by a large margin, in particular in view of the regularity (1.6) obtained for the symbol of the error term \( E \). In order to obtain the control required in step C4 with the regularity of the symbol of the FIO \( E \) given by (1.6), we rely in particular on the following ingredients:

- geometric integrations by parts taking full advantage of the better regularity properties in certain directions tied to the level hypersurfaces \( \mathcal{H}_u \) of \( u \),

\(^8\)This is due to the fact that our estimates are sensitive to certain directions tied to the \( u \)-foliation of \( \mathcal{M} \). Now, after differentiation with respect to \( \omega \), derivatives in ”good” directions pick up a nonzero component along ”bad” directions (see [17] for details)
• the standard first and second dyadic decomposition in frequency and angle (see [14]),
• an additional decomposition in physical space relying on the geometric Littlewood-Paley projections of [7].

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2 Main results

The error term $E$ in (1.5) is a Fourier integral operator on $\mathcal{M}$ with phase $u(t, x, \omega)$ and symbol $\Box_g u(t, x, \omega)$. The regularity assumptions on $u(t, x, \omega)$ will be crucial to complete step C4, that is prove the following estimate for the error term:

$$\|Ef\|_{L^2(\mathcal{M})} \lesssim \|\lambda f\|_{L^2(\mathbb{R}^3)}.$$  

In this section, we state our assumptions on $u(t, x, \omega)$ before stating our main result.

2.1 Maximal foliation on $\mathcal{M}$

We foliate the space-time $\mathcal{M}$ by space-like hypersurfaces $\Sigma_t$ defined as level hypersurfaces of a time function $t$. Denoting by $T$ the unit, future oriented, normal to $\Sigma_t$ and $k$ the second fundamental form

$$k_{ij} = - <\nabla_i T, \partial_j>$$  

we find,

$$k_{ij} = - \frac{1}{2} \mathcal{L}_T g_{ij}$$

with $\mathcal{L}_X$ denoting the Lie derivative with respect to the vectorfield $X$. Let $\text{Tr}(k) = g^{ij}k_{ij}$ where $g$ is the induced metric on $\Sigma_t$ and Tr is the trace. In order to be consistent with the statement of Theorem 1.1, we impose a maximal foliation

$$\text{Tr}(k) = 0.$$  

We also define the lapse $n$ as

$$n^{-1} = T(t).$$  

We have (see for example [17]):

$$\mathbf{D}_T T = n^{-1} \nabla n,$$
where $\nabla$ denotes the gradient with respect to the induced metric on $\Sigma_t$.

Finally, the lapse $n$ satisfies the following elliptic equation on $\Sigma_t$ (see [3] p. 13):

$$\Delta n = |k|^2 n,$$

(2.5)

where one uses (2.1), (2.4), Einstein vacuum equations (1.1) and the fact that the foliation generated by $t$ on $\mathcal{M}$ is maximal (2.2).

### 2.2 Geometry of the foliation generated by $u$ on $\mathcal{M}$

Remember that $u$ is a solution to the eikonal equation $g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ depending on an extra parameter $\omega \in \mathbb{S}^2$. The level hypersurfaces $u(t,x,\omega) = u$ of the optical function $u$ are denoted by $\mathcal{H}_u$. Let $L'$ denote the space-time gradient of $u$, i.e.:

$$L' = -g^{\alpha\beta} \partial_\beta u \partial_\alpha.$$

(2.6)

Using the fact that $u$ satisfies the eikonal equation, we obtain:

$$D_{L'} L' = 0,$$

(2.7)

which implies that $L'$ is the geodesic null generator of $\mathcal{H}_u$.

We have:

$$T(u) = \pm |\nabla u|$$

where $|\nabla u|^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame $e_i$ on $\Sigma_t$. Since the sign of $T(u)$ is irrelevant, we choose by convention:

$$T(u) = |\nabla u|.$$  

(2.8)

We denote by $P_{t,u}$ the surfaces of intersection between $\Sigma_t$ and $\mathcal{H}_u$. They play a fundamental role in our discussion.

**Definition 2.1 (Canonical null pair)**

$$L = b L' = T + N, \quad \bar{L} = 2T - L = T - N$$

(2.9)

where $L'$ is the space-time gradient of $u$ (2.6), $b$ is the lapse of the null foliation (or shortly null lapse)

$$b^{-1} = -<L', T> = T(u),$$

(2.10)

and $N$ is a unit normal, along $\Sigma_t$, to the surfaces $P_{t,u}$. Since $u$ satisfies the eikonal equation $g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$, this yields $L'(u) = 0$ and thus $L(u) = 0$. In view of the definition of $L$ and (2.8), we obtain:

$$N = -\frac{\nabla u}{|\nabla u|}.$$  

(2.11)

**Definition 2.2** A null frame $e_1, e_2, e_3, e_4$ at a point $p \in P_{t,u}$ consists, in addition to the null pair $e_3 = \bar{L}, e_4 = L$, of arbitrary orthonormal vectors $e_1, e_2$ tangent to $P_{t,u}$. 

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Definition 2.3 (Ricci coefficients) Let \( e_1, e_2, e_3, e_4 \) be a null frame on \( P_{t,u} \) as above. The following tensors on \( P_{t,u} \)

\[
\chi_{AB} = \langle D_A e_4, e_B \rangle, \quad \chi_{AB} = \langle D_A e_3, e_B \rangle, \tag{2.12}
\]

\[
\zeta_A = \frac{1}{2} \langle D_3 e_4, e_A \rangle, \quad \zeta_A = \frac{1}{2} \langle D_4 e_3, e_A \rangle,
\]

are called the Ricci coefficients associated to our canonical null pair.

We decompose \( \chi \) and \( \chi \) into their trace and traceless components.

\[
\begin{align*}
tr \chi &= g^{AB} \chi_{AB}, \\
\hat{\chi}_{AB} &= \chi_{AB} - \frac{1}{2} tr \chi g_{AB}, \tag{2.13}
\end{align*}
\]

\[
\begin{align*}
\hat{\chi}_{AB} &= \chi_{AB} - \frac{1}{2} tr \chi g_{AB}, \quad \hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} tr \chi g_{AB},
\end{align*}
\]

Observe that all tensors defined above are \( P_{t,u} \)-tangent.

Definition 2.4 We decompose the symmetric traceless 2 tensor \( k \) into the scalar \( \delta \), the \( P_{t,u} \)-tangent 1-form \( \epsilon \), and the \( P_{t,u} \)-tangent symmetric 2-tensor \( \eta \) as follows:

\[
\begin{align*}
k_{NN} &= \delta, \\
k_{AN} &= \epsilon_A, \\
k_{AB} &= \eta_{AB}. \tag{2.15}
\end{align*}
\]

Note that \( Tr(k) = tr(\eta) + \delta \) which together with the maximal foliation assumption (2.2) yields:

\[
tr(\eta) = -\delta. \tag{2.16}
\]

The following Ricci equations can be easily derived from the properties of \( T \) (2.1) (2.4), the fact that \( L' \) is geodesic (2.7), and the definition (2.12) of the Ricci coefficients (see [3] p. 171):

\[
\begin{align*}
D_A e_4 &= \chi_{AB} e_B - \epsilon_A e_4, \\
D_A e_3 &= \chi_{AB} e_B + \epsilon_A e_3, \\
D_4 e_4 &= -\delta e_4, \\
D_3 e_4 &= 2\zeta_A e_A + (\delta + n^{-1} \nabla N n) e_4, \\
D_4 e_3 &= \nabla_4 e_A + \zeta_A e_4, \\
D_4 e_A &= \nabla_4 e_A + \zeta_A e_4, \\
D_B e_A &= \nabla_B e_A + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \chi_{AB} e_4
\end{align*}
\]

where, \( \nabla_3, \nabla_4 \) denote the projection on \( P_{t,u} \) of \( D_3 \) and \( D_4 \), \( \nabla \) denotes the induced covariant derivative on \( P_{t,u} \) and \( \delta, \zeta \) are defined by:

\[
\delta = \delta - n^{-1} N(n), \quad \zeta_A = \epsilon_A - n^{-1} \nabla_A n. \tag{2.18}
\]
Also,

\[
\chi_{AB} = -\chi_{AB} - 2k_{AB}, \\
\zeta_A = -\epsilon_A, \\
\xi_A = \epsilon_A + n^{-1}\nabla_A n - \zeta_A.
\]  

(2.19)

Let \( \theta \) is the second fundamental form of \( P_{t,u} \) in \( \Sigma_t \). Since \( L = T + N \), \( \theta \) is connected to the second fundamental form \( k \) of \( \Sigma_t \) and the null second fundamental form \( \chi \) of \( P_{t,u} \) through the formula:

\[
\theta_{AB} = \chi_{AB} + \eta_{AB}.
\]  

(2.20)

In view of the Ricci equations (2.17), we have:

\[
\begin{cases}
\nabla_A N = \theta_{AB} e_B, \\
\nabla_N N = -b^{-1}\nabla b.
\end{cases}
\]  

(2.21)

Recall that \( \text{tr}\chi \) satisfies a transport equation called the Raychaudhuri equation:

\[
L(\text{tr}\chi) + \frac{1}{2}(\text{tr}\chi)^2 = -|\hat{\chi}|^2 - \delta\text{tr}\chi.
\]  

(2.22)

We also recall the transport equation satisfied by the null lapse \( b \):

\[
L(b) = -\delta b.
\]  

(2.23)

The following lemma will allow us to identify the symbol \( \Box_g u \) of the error term (1.5):

**Lemma 2.5** For any scalar function \( \phi \) on \( \mathcal{M} \), we have:

\[
\Box_g \phi = -L(L(\phi)) + \Delta \phi + 2\zeta \cdot \nabla \phi + (\delta + n^{-1}\nabla_N n)L(\phi) + \frac{1}{2}\text{tr}\chi L(\phi) + \frac{1}{2}\text{tr}\chi L(\phi).
\]  

(2.24)

**Proof** We have:

\[
\Box_g \phi = D^a D_a \phi = -D^2 \phi(L, L) + D^A D_A \phi.
\]  

(2.25)

Now, using the Ricci equations (2.17), we obtain:

\[
D_A D_B \phi = e_A (e_B (\phi)) - D_{D_A e_B} \phi
= e_A (e_B (\phi)) - \nabla_A \nabla_B \phi - \frac{1}{2} \chi_{AB} L(\phi) - \frac{1}{2} \chi_{AB} L(\phi)
= \nabla_A \nabla_B \phi - \frac{1}{2} \chi_{AB} L(\phi) - \frac{1}{2} \chi_{AB} L(\phi).
\]

Taking the trace, this yields:

\[
D^A D_A \phi = \Delta \phi - \frac{1}{2}\text{tr}\chi L(\phi) - \frac{1}{2}\text{tr}\chi L(\phi).
\]  

(2.26)

Using again the Ricci equations (2.17), we also have:

\[
D^2 u(L, L) \phi = L(L(\phi)) - D_{D_L L} \phi = L(L(\phi)) - 2\zeta \cdot \nabla \phi - (\delta + n^{-1}\nabla_N n)L(\phi).
\]  

(2.27)
Finally, (2.25), (2.26) and (2.27) yield the conclusion of the lemma.

We conclude this section with the identification of the symbol \( \Box g u \) of the error term (1.5). In view of (2.9), (2.10), the fact that \( e_A, A = 1, 2 \) are tangent to \( P_{t,u} \), and the fact that \( \Delta \) is the Laplace-Beltrami on \( P_{t,u} \), we have:

\[
L(u) = 0, e_A(u) = 0, A = 1, 2, \Delta(u) = 0, \text{ and } \overline{L}(u) = 2b^{-1}.
\]

Together with (2.24), this yields:

\[
\Box g u = b^{-1} \text{tr} \chi. \tag{2.28}
\]

Thus, we may rewrite the error term \( E \) as:

\[
Ef(t, x) = \int_{S^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} b^{-1}(t, x, \omega) \text{tr} \chi(t, x, \omega) f(\lambda \omega) \lambda^3 d\lambda d\omega. \tag{2.29}
\]

### 2.3 Commutation formulas

From the Ricci equations (2.17), we immediately deduce the following four useful commutation formulas:

**Lemma 2.6** Let \( f \) a scalar function on \( \mathcal{M} \). Then,

\[
\nabla_B \nabla_4 f - \nabla_4 \nabla_B f = \chi_{BC} \nabla_C f - n^{-1} \nabla_B n \nabla_4 f, \tag{2.30}
\]

\[
\nabla_B \nabla_3 f - \nabla_3 \nabla_B f = \xi_{BC} \nabla_C f - \xi_B \nabla_4 f - b^{-1} \nabla_B b \nabla_3 f \tag{2.31}
\]

\[
[L, L] f = -\delta \nabla_2 f + (\delta + n^{-1} \nabla N n) \nabla_4 f + 2(\zeta_B - \zeta_B) \nabla_B f. \tag{2.32}
\]

Finally, (2.30), (2.31) together with the fact that \( N = \frac{1}{2}(L - \overline{L}) \) yield:

\[
\nabla_B \nabla_N f - \nabla_N \nabla_B f = (\chi_{BC} + k_{BC}) \nabla_C f - b^{-1} \nabla_B b \nabla_N f. \tag{2.33}
\]

For some applications we have in mind, we would like to get rid of the term containing a \( \nabla_4 \) derivative in the Right-hand side of (2.30). This is achieved by considering the commutator \([\nabla, \nabla_{nL}]\) instead of \([\nabla, \nabla_4]\):

\[
\nabla_B \nabla_{nL} f - \nabla_{nL} \nabla_B f = n \chi_{BC} \nabla_C \Pi_A. \tag{2.34}
\]

Also, we would like to get rid of the term containing a \( \nabla_N \) derivative in the right-hand side of (2.33). This is achieved by considering the commutator \([\nabla, \nabla_{bN}]\) instead of \([\nabla, \nabla_N]\):

\[
\nabla_B \nabla_{bN} f - \nabla_{bN} \nabla_B f = b(\chi_{BC} + k_{BC}) \nabla_C f. \tag{2.35}
\]
2.4 Regularity assumptions on the phase $u(t, x, \omega)$

We define some norms on $\mathcal{H}$. For any $1 \leq p \leq +\infty$ and for any tensor $F$ on $\mathcal{H}_u$, we have:

$$\|F\|_{L^p(\mathcal{H}_u)} = \left( \int_0^1 dt \int_{P_{t,u}} |F|^p d\mu_{t,u} \right)^{\frac{1}{p}},$$

where $d\mu_{t,u}$ denotes the area element of $P_{t,u}$. We also introduce the following norms:

$$\mathcal{N}_1(F) = \|F\|_{L^2(\mathcal{H}_u)} + \|\nabla F\|_{L^2(\mathcal{H}_u)} + \|\nabla L F\|_{L^2(\mathcal{H}_u)},$$

$$\mathcal{N}_2(F) = \mathcal{N}_1(F) + \|\nabla^2 F\|_{L^2(\mathcal{H}_u)} + \|\nabla \nabla L F\|_{L^2(\mathcal{H}_u)}.$$

Let $x'$ a coordinate system on $P_{0,u}$. By transporting this coordinate system along the null geodesics generated by $L$, we obtain a coordinate system $(t, x')$ of $\mathcal{H}$. We define the following norms:

$$\|F\|_{L^2_{x'} L^1_t} = \sup_{x' \in P_{0,u}} \left( \int_0^1 |F(t, x')|^2 dt \right)^{\frac{1}{2}},$$

$$\|F\|_{L^1_{x'} L^\infty_t} = \sup_{0 \leq t \leq 1} \|F(t, x')\|_{L^\infty(P_{0,u})}.$$

We now state our assumptions for the phase $u(t, x, \omega)$ and the symbol $b^{-1}(t, x, \omega)\text{tr}\chi(t, x, \omega)$ of the error term $E$ which is given by (2.29). These assumptions are compatible with the regularity obtained for the functions $u(t, x, \omega)$ constructed in [17] (this construction corresponds to step C3). The constant $\varepsilon > 0$ below is the one appearing in the statement of Theorem 1.1. In particular, it satisfies $0 < \varepsilon < 1$ and is small.

**Assumption 1 (regularity with respect to $(t, x)$):**

$$\|n - 1\|_{L^\infty(\mathcal{H}_u)} + \|\nabla n\|_{L^\infty(\mathcal{H}_u)} + \|\nabla^2 n\|_{L^2_{x'} L^2_t} + \|\nabla T(n)\|_{L^\infty_{x'} L^2_t} \lesssim \varepsilon, \quad (2.36)$$

$$\mathcal{N}_1(k) + \|\nabla L\|_{L^2(\mathcal{H}_u)} + \|L(\delta)\|_{L^2(\mathcal{H}_u)} + \|\epsilon\|_{L^\infty_{x'} L^2_t} + \|\delta\|_{L^\infty_{x'} L^2_t} \lesssim \varepsilon, \quad (2.37)$$

$$\|b - 1\|_{L^\infty(\mathcal{H}_u)} + \mathcal{N}_2(b) + \|L(b)\|_{L^2_{x'} L^\infty_t} + \|\nabla(b)\|_{L^2_{x'} L^\infty_t} + \|L(b)\|_{L^4_{x'} L^\infty_t} \lesssim \varepsilon, \quad (2.38)$$

$$\|\text{tr} \chi\|_{L^\infty(\mathcal{H}_u)} + \|\nabla \text{tr} \chi\|_{L^2_{x'} L^\infty_t} + \|L \text{tr} \chi\|_{L^2_{x'} L^\infty_t} \lesssim \varepsilon, \quad (2.39)$$

$$\|\hat{\chi}\|_{L^2_{x'} L^\infty_t} + \mathcal{N}_1(\hat{\chi}) + \|\nabla L \hat{\chi}\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon, \quad (2.40)$$

$$\|\zeta\|_{L^2_{x'} L^\infty_t} + \mathcal{N}_1(\zeta) \lesssim \varepsilon. \quad (2.41)$$

**Assumption 2 (regularity with respect to $\omega$):**

$$\|\partial_\omega N\|_{L^\infty(\mathcal{H}_u)} \lesssim 1, \quad (2.42)$$

$$\|N(x, \omega) - N(x, \omega')\| - |\omega - \omega'| \lesssim \varepsilon |\omega - \omega'|, \quad \forall x \in \sigma, \omega, \omega' \in S^2, \quad (2.43)$$

$$\|\partial_\omega b\|_{L^\infty(\mathcal{H}_u)} + \|\partial_\omega \chi\|_{L^2_{x'} L^\infty_t} \lesssim \varepsilon. \quad (2.44)$$
Furthermore, we have the following decomposition for $\hat{\chi}$:

$$\hat{\chi} = \chi_1 + \chi_2, \tag{2.45}$$

where $\chi_1$ and $\chi_2$ are two symmetric traceless $P_{t,u}$-tangent 2-tensors satisfying:

$$N_1(\chi_1) + \|\omega \chi_1\|_{L^p_t L^2_x} + N_1(\chi_2) + \|\chi_2\|_{L^\infty_t L^2_x} + \|\partial_x \omega \chi_2\|_{L^\infty_t L^2_x} \lesssim \varepsilon \tag{2.46}$$

and for any $2 \leq p < +\infty$, we have:

$$\|\chi_1\|_{L^p_t L^\infty_x} + \|\partial_x \omega \chi_2\|_{L^6_{t}(H_u)} \lesssim \varepsilon. \tag{2.47}$$

**Assumption 3** (additional regularity with respect to $x$):

We introduce the family of intrinsic Littlewood-Paley projections $P_j$ which have been constructed in [7] using the heat flow on the 2-surfaces $P_{t,u}$ (see section 2.6). There exists a function $\mu$ in $L^2(\mathbb{R})$ satisfying:

$$\|\mu\|_{L^2(\mathbb{R})} \leq 1$$

such that for all $j \geq 0$, we have:

$$\|\nabla N P_j \nabla N tr \chi\|_{L^2(H_u)} \lesssim 2^j \varepsilon + 2^j \varepsilon \mu(u). \tag{2.48}$$

**Remark 2.7** In Assumptions 1-3, all inequalities hold for any $\omega \in S^2$ with the constant in the right-hand side being independent of $\omega$. Thus, one may take the supremum in $\omega$ everywhere. To ease the notations, we do not explicitly write down this supremum.

**Remark 2.8** The fact that we may take a small constant $\varepsilon > 0$ in Assumptions 1-3 is directly related to the conclusions of Theorem 1.1.

**Remark 2.9** In the flat case, we have $\mathcal{M} = (\mathbb{R}^{1+3}, \mathbf{m})$, where $\mathbf{m}$ is the Minkowski metric, $u(t,x,\omega) = t + x \cdot \omega$, $b = 1$, $N = -\omega$, $L = \partial_t - \omega \cdot \partial_x$, $L = \partial_t + \omega \cdot \partial_x$, and $\chi = \hat{\chi} = \zeta = \zeta = \xi = k = 0$. Thus, Assumptions 1-3 are clearly satisfied with $\varepsilon = 0$.

### 2.5 Estimates on $P_{t,u}$ and $\mathcal{M}$

In this section, we state the embeddings on $P_{t,u}$ and $\mathcal{M}$ that will be needed for the proof of the main theorem. We refer to section 3 in [17], as well as [7] for (2.49), for their proof within the regularity assumptions of section 2.4.

We have the following Gagliardo-Nirenberg inequality on $P_{t,u}$ (see [7]). For an arbitrary tensorfield $F$ on $P_{t,u}$ and any $2 \leq p < \infty$, we have:

$$\|F\|_{L^p(P_{t,u})} \lesssim \|\nabla F\|_{L^2(P_{t,u})}^{1-\frac{2}{p}} \|F\|_{L^2(P_{t,u})}^{\frac{2}{p}} + \|F\|_{L^2(P_{t,u})}. \tag{2.49}$$

We have the classical Sobolev inequality on $\mathcal{H}$ (see [17]):
Lemma 2.10 For any tensor $F$ on $\mathcal{H}_u$, we have:
\[
\|F\|_{L^p(\mathcal{H}_u)} \lesssim \mathcal{N}_1(F),
\]
and
\[
\|F\|_{L^\infty_tL^2_x} \lesssim \mathcal{N}_1(F).
\]

On $\mathcal{H}_u$, we also have the following estimate of the $L^\infty_tL^2_x$ norm for any tensor $F$ on $\mathcal{H}_u$:
\[
\|F\|_{L^\infty_tL^2_x} \lesssim \int_{\mathcal{H}_u} |F||D_tF|dtd\mu_{t,u} + \|F\|_{L^2(\mathcal{H}_u)}^2.
\]

The following lemma will be useful to estimate transport equations (see [17] for a proof).

Lemma 2.11 Let $W$ and $F$ two $P_{t,u}$-tangent tensors such that $\nabla_L W = F$. Then, for any $p \geq 1$, we have:
\[
\|W\|_{L^p_{x'}L^\infty_t} \lesssim \|W(0)\|_{L^p(P_{0,u})} + \|F\|_{L^p_{x'}L^1_t}.
\]

Finally, we have the Sobolev embedding on $\mathcal{M}$. Given an arbitrary tensorfield $F$ on $\mathcal{M}$, we have (see [17])
\[
\|F\|_{L^4(\mathcal{M})} \lesssim \|DF\|_{L^2(\mathcal{M})}.
\]

2.6 Geometric Littlewood-Paley projections on $P_{t,u}$

In [7], Littlewood-Paley projections have been constructed relying on the heat flow on 2-surfaces, within our low regularity assumptions. They recover the basic properties of the standard Littlewood-Paley projections. We denote by $P_j$ such a Littlewood-Paley projection on the 2-surface $P_{t,u}$. In particular, we have from [7]
\[
\sum_j P_j = I.
\]

Also, the following properties of the LP-projections $P_j$ have been proved in [7]:

Theorem 2.12 The LP-projections $P_j$ verify the following properties:

i) $L^p$-boundedness For any $1 \leq p \leq \infty$, and any interval $I \subset \mathbb{Z}$,
\[
\|P_jF\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})}.
\]

ii) Bessel inequality
\[
\sum_j \|P_jF\|_{L^2(P_{t,u})}^2 \lesssim \|F\|_{L^2(P_{t,u})}^2.
\]

iii) Finite band property For any $1 \leq p \leq \infty$,
\[
\|\Delta P_jF\|_{L^p(P_{t,u})} \lesssim 2^{2j} \|F\|_{L^p(P_{t,u})},
\]
\[
\|P_jF\|_{L^p(P_{t,u})} \lesssim 2^{-2j} \|\Delta F\|_{L^p(P_{t,u})},
\]
In addition, the $L^2$ estimates
\[
\|\nabla P_j F\|_{L^2(P_{t,u})} \lesssim 2^j \|F\|_{L^2(P_{t,u})}, \\
\|P_j F\|_{L^2(P_{t,u})} \lesssim 2^{-j} \|\nabla F\|_{L^2(P_{t,u})}
\] (2.58)
hold together with the dual estimate
\[
\|P_j \nabla F\|_{L^2(P_{t,u})} \lesssim 2^j \|F\|_{L^2(P_{t,u})}
\]

iv) Weak Bernstein inequality  For any $2 \leq p < \infty$
\[
\|P_j F\|_{L^p(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j} + 1) \|F\|_{L^2(P_{t,u})}, \\
\|P_{<0} F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})}
\]
together with the dual estimates
\[
\|P_j F\|_{L^2(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j} + 1) \|F\|_{L^p(P_{t,u})}, \\
\|P_{<0} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^p(P_{t,u})}
\]

We also have the following sharp Bernstein inequality for a scalar function $f$ on $P_{t,u}$:
\[
\|P_j f\|_{L^\infty(P_{t,u})} \lesssim 2^j \|f\|_{L^2(P_{t,u})}, \\
\|P_{<0} f\|_{L^\infty(P_{t,u})} \lesssim \|f\|_{L^2(P_{t,u})}
\] (2.59) (2.60)
and the following Bochner inequality:
\[
\int_{P_{t,u}} |\nabla^2 f|^2 \lesssim \int_{P_{t,u}} |\Delta f|^2 + \varepsilon \int_{P_{t,u}} |\nabla f|^2.
\] (2.61)

There is an equivalent of (2.61) for tensors, which yields the following consequence. There exists a function $\mu$ in $L^2(\mathbb{R})$ satisfying:
\[
\|\mu\|_{L^2(\mathbb{R})} \lesssim 1
\]
such that for any scalar function $f$ on $P_{t,u}$, we have:
\[
\|\nabla^3 f\|_{L^2(P_{t,u})} \lesssim \|\nabla \Delta f\|_{L^2(P_{t,u})} + \mu(t)\varepsilon \|\Delta f\|_{L^2(P_{t,u})} + \mu^2(t)\varepsilon \|\nabla f\|_{L^2(P_{t,u})}.
\] (2.62)

Finally, we have the following lemma (see Lemma 5.10 in [17]):

**Lemma 2.13** For any 1-form $F$ on $P_{t,u}$, for any $1 < p \leq 2$ and for all $j \geq 0$, we have:
\[
\|P_j \delta \mu(F)\|_{L^2(P_{t,u})} \lesssim 2^{\frac{2}{p}j} \|F\|_{L^p(P_{t,u})}.
\] (2.63)
2.7 Commutator estimates

In this section, we state the commutator estimates, as well as two additional estimates for trχ, that will be needed for the proof of the main theorem. We refer to section 9 in [17] for their proof within the regularity assumptions of section 2.4.

Let f a scalar function on M. Then, we have the following commutator estimates:
\[ \|[bN, P_j]f\|_{L^2(H_u)} + 2^{-j}\|\nabla[bN, P_j]f\|_{L^2(H_u)} \lesssim \varepsilon N^1_1(f). \] (2.64)

and
\[ \|[nL, P_j]f\|_{L^2(H_u)} + 2^{-j}\|\nabla[nL, P_j]f\|_{L^2(H_u)} \lesssim \varepsilon N^1_1(f). \] (2.65)

We also have the following commutator estimates acting on trχ.
\[ 2^j\|[nL, P_j]tr\chi\|_{L^1_tL^\infty_x} + \|\nabla[nL, P_j]tr\chi\|_{L^1_tL^2_x} \lesssim \varepsilon, \] (2.66)
\[ 2^j\|[nL, P_j]tr\chi\|_{L^\infty_tL^2_x} \lesssim \varepsilon, \] (2.67)
\[ 2^j\|[bN, P_j]tr\chi\|_{L^\infty_tL^2_x} \lesssim 2^j\varepsilon. \] (2.68)

Finally, we have the following estimate for \( P_m tr\chi \):
\[ \|P_m tr\chi\|_{L^2_tL^\infty_x} + \|P_m(nLtr\chi)\|_{L^2_tL^1_t} \lesssim 2^{-m}\varepsilon, \] (2.69)

and the following estimate for \( P_{\leq m} tr\chi \):
\[ \|\nabla P_{\leq m} tr\chi\|_{L^2_tL^\infty_x} + \|\nabla(P_{\leq m}(nLtr\chi))\|_{L^2_tL^1_t} \lesssim \varepsilon. \] (2.70)

2.8 Dependence of the norm \( L^\infty_tL^2(\mathcal{H}_u) \) on \( \omega \in S^2 \)

Let \( \omega \) and \( \nu \) in \( S^2 \) such that
\[ |\omega - \nu| \lesssim 2^j. \]

Let \( u = u(., \omega) \) and \( u_{\nu} = u(., \nu) \). In this section, we morally evaluate the norm in \( L^\infty_tL^2(\mathcal{H}_u) \) of the difference between various scalars and tensors evaluated at \( \omega \) and their corresponding evaluation at \( \nu \). Consider a FIO where the integration in \( \omega \) is localized in a patch of size \( 2^{-j/2} \) and of center \( \nu \). This will be used to morally replace the symbol of this FIO depending on \( \omega \) by its value at the middle of the patch \( \nu \), so that one may take the symbol outside of the integral in \( \omega \). The following decompositions are proved in section 8 of [17] within the regularity assumptions of section 2.4.

- We have the following decomposition for \( N(., \omega) - N(., \nu) \):
\[ 2^{j/2}(N(., \omega) - N(., \nu)) = F^j_1 + F^j_2 \] (2.71)
where the tensor $F_1^j$ only depends on $\nu$ and satisfies:

$$\|F_1^j\|_{L^\infty} \lesssim 1,$$

and where the tensor $F_2^j$ satisfies:

$$\|F_2^j\|_{L_\infty^2 L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}.$$

• We have following decomposition for $\text{tr}\chi$:

$$\text{tr}\chi(., \omega) = f_1^j + f_2^j$$

(2.72)

where the scalar $f_1^j$ only depends on $\nu$ and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim \varepsilon,$$

and where the scalar $f_2^j$ satisfies:

$$\|f_2^j\|_{L_\infty^2 L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

• Let $p \in \mathbb{Z}$. We have following estimate for $b^p$:

$$\|b^p(., \omega) - b^p(., \nu)\|_{L_\infty^2 L^2(\mathcal{H}_u)} \lesssim \varepsilon |\omega - \nu|.$$  

(2.73)

• We have following decomposition for $\hat{\chi}$:

$$\hat{\chi}(., \omega) = F_1^j + F_2^j$$

(2.74)

where the tensor $F_1^j$ only depends on $\nu$ and satisfies:

$$\|F_1^j\|_{L_\infty^2 L^2(\mathcal{H}_u)} \lesssim \varepsilon,$$

and where the tensor $F_2^j$ satisfies:

$$\|F_2^j\|_{L_\infty^2 L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

• We have following decomposition for $\chi_2$:

$$\|\chi_2(., \omega) - \chi_2(., \nu)\|_{L_\infty^2 L^4 - (\mathcal{H}_u)} \lesssim \varepsilon |\omega - \nu|.$$  

(2.75)

• We have following decomposition for $\chi$:

$$\chi(., \omega) = \chi_2(., \nu) + F_1^j + F_2^j$$

(2.76)

where the tensor $F_1^j$ only depends on $\nu$ and satisfies for any $2 \leq p < +\infty$:

$$\|F_1^j\|_{L_\infty^p L^p \times L_\infty^p} \lesssim \varepsilon,$$

and where the tensor $F_2^j$ satisfies:

$$\|F_2^j\|_{L_\infty^2 L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$
• We have the following decomposition for $|\hat{\chi}|^2$:

$$
|\hat{\chi}|^2(\cdot, \omega) = |\chi_2(\cdot, \nu)|^2 + \chi_2(\cdot, \nu) \cdot F^j_1 + \chi_2(\cdot, \nu) \cdot F^j_2 + f^j_3 + f^j_4 + f^j_5, \tag{2.77}
$$

where the tensor $F^j_1$ and the scalar $f^j_3$ only depend on $\nu$ and satisfy:

$$
\|F^j_1\|_{L^\infty_tL^2_x(P_{t,\nu})} + \|f^j_3\|_{L^\infty_tL^2_x(P_{t,\nu})} \lesssim \varepsilon,
$$

where the tensor $F^j_2$, and the scalar $f^j_4$ satisfy:

$$
\|F^j_2\|_{L^\infty_tL^2_x(H_{\omega})} + \|f^j_4\|_{L^\infty_tL^2_x(H_{\omega})} \lesssim \varepsilon 2^{-\frac{j}{2}},
$$

and where the scalar $f^j_5$ satisfies:

$$
\|f^j_5\|_{L^2(M)} \lesssim \varepsilon 2^{-j}.
$$

• We have the following decomposition for $\hat{\chi}(\cdot, \omega)^3$:

$$
\hat{\chi}(\cdot, \omega)^3 = \chi_2(\cdot, \nu)^3 + \chi_2(\cdot, \nu)^2 F^j_1 + \chi_2(\cdot, \nu)^2 F^j_2 + \chi_2(\cdot, \nu) F^j_3 + \chi_2(\cdot, \nu) F^j_6 + F^j_7 + F^j_8 + F^j_9 \tag{2.78}
$$

where $F^j_1$, $F^j_3$ and $F^j_6$ do not depend on $\omega$ and satisfy:

$$
\|F^j_1\|_{L^\infty_tL^2_x(L^\infty_t)} + \|F^j_3\|_{L^\infty_tL^2_x(L^\infty_t)} + \|F^j_6\|_{L^\infty_tL^2_x(L^\infty_t)} \lesssim \varepsilon,
$$

where $F^j_2$, $F^j_4$ and $F^j_7$ satisfy:

$$
\|F^j_2\|_{L^\infty_tL^2_x(H_{\omega})} + \|F^j_4\|_{L^\infty_tL^2_x(H_{\omega})} + \|F^j_7\|_{L^\infty_tL^2_x(H_{\omega})} \lesssim 2^{-\frac{j}{2}} \varepsilon,
$$

where $F^j_5$ and $F^j_8$ satisfy

$$
\|F^j_5\|_{L^2(M)} + \|F^j_8\|_{L^2(M)} \lesssim \varepsilon 2^{-j},
$$

and where $F^j_9$ satisfies

$$
\|F^j_9\|_{L^2_-(M)} \lesssim \varepsilon 2^{-\frac{3j}{2}}.
$$

• We have the following decomposition for $b(\cdot, \omega) - b(\cdot, \nu)$:

$$
2^{\frac{j}{2}}(b(\cdot, \omega) - b(\cdot, \nu)) = f^j_1 + f^j_2 \tag{2.79}
$$

where the scalar $f^j_1$ only depends on $\nu$ and satisfies:

$$
\|f^j_1\|_{L^\infty} \lesssim \varepsilon,
$$

and where the scalar $f^j_2$ satisfies:

$$
\|f^j_2\|_{L^\infty_tL^2_x(H_{\omega})} \lesssim 2^{-\frac{4j}{3}} \varepsilon.
$$
• We have following decomposition for \( \zeta \) and \( \nabla(b) \):

\[
\zeta(., \omega), \nabla(b)(., \omega) = F^j_1 + F^j_2
\]

(2.80)

where the tensor \( F^j_1 \) only depends on \( \nu \) and satisfies for any \( 2 \leq p < +\infty \):

\[
\| F^j_1 \|_{L^\infty_u L^2_t L^p_x} \lesssim \varepsilon,
\]

and where the tensor \( F^j_2 \) satisfies:

\[
\| F^j_2 \|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-4j}. \]

Remark 2.14 Let us give some insight on these decompositions by considering the particular example of the decomposition for \( \text{tr} \chi \) (2.72). A naive approach consists in writing the following decomposition

\[
\text{tr} \chi(t, x, \omega) = \text{tr} \chi(t, x, \nu) + (\text{tr} \chi(t, x, \omega) - \text{tr} \chi(t, x, \nu)) = f^j_1 + f^j_2.
\]

\( f^j_1 \) does not depend on \( \omega \) and satisfies, in view of the estimate (2.39)

\[
\| f^j_1 \|_{L^\infty} \lesssim \| \text{tr} \chi(., \nu) \|_{L^\infty} \lesssim \varepsilon.
\]

Also, we have

\[
f^j_2 = (\omega - \nu) \int_0^1 \partial_\omega \text{tr} \chi(t, x, \omega) d\sigma,
\]

which together with the fact that \( |\omega - \nu| \lesssim 2^{-\frac{j}{2}} \) yields

\[
\| f^j_2 \|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-j} \left\| \int_0^1 \partial_\omega \text{tr} \chi(t, x, \omega) d\sigma \right\|_{L^\infty_u L^2(\mathcal{H}_u)}.
\]

Unfortunately, we cannot obtain the desired estimate for \( f^j_2 \) since we have \( \partial_\omega \text{tr} \chi(., \omega) \in L^\infty_u L^2(\mathcal{H}_u) \), and \( L^\infty_u L^2(\mathcal{H}_u) \) and \( L^\infty_u L^2(\mathcal{H}_u) \) are not directly comparable. Nevertheless, in section 8 of [17], we are able to improve on this naive approach, in order to obtain the above decompositions within the regularity assumptions of section 2.4.

### 2.9 The boundedness of the error term

The main result of this paper is the following \( L^2 \) bound on the error term (2.29). It achieves step C4 and therefore, together with the results in [15], [16], [17] completes step C.

**Theorem 2.15** Let \( u \) be a function on \( \mathcal{M} \times \mathbb{S}^2 \) satisfying Assumption 1, Assumption 2 and Assumption 3, as well as the assumptions of sections 2.5-2.8. Let \( E \) the Fourier integral operator with phase \( u(t, x, \omega) \) and symbol \( b^{-1}(t, x, \omega) \text{tr} \chi(t, x, \omega) \):

\[
Ef(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} b^{-1}(t, x, \omega) \text{tr} \chi(t, x, \omega) f(\lambda \omega) \lambda^3 d\lambda d\omega.
\]

(2.81)

Then, \( E \) satisfies the estimate:

\[
\| Ef \|_{L^2(\mathcal{M})} \lesssim \varepsilon \| \lambda f \|_{L^2(\mathbb{R}^3)}.
\]

(2.82)
3 Proof of Theorem 2.15 (control of the error term)

3.1 The basic computation

We start the proof of Theorem 2.15 with the following instructive computation:

\[ \|Ef\|_{L^2(M)} \leq \int_{\mathbb{S}^2} \left\| b(t,x,\omega)^{-1}\text{tr}\chi(t,x,\omega) \int_0^{+\infty} e^{i\lambda u} f(\lambda \omega) \lambda^2 d\lambda \right\|_{L^2(M)} d\omega \]

\[ \leq \int_{\mathbb{S}^2} \|b(t,x,\omega)^{-1}\text{tr}\chi(t,x,\omega)\|_{L^\infty_x(L^2(H_u))} \left\|\int_0^{+\infty} e^{i\lambda u} f(\lambda \omega) \lambda^2 d\lambda \right\|_{L^2_\omega} d\omega \]

\[ \leq \varepsilon \|\lambda^2 f\|_{L^2(\mathbb{R}^3)}, \]

where we have used Plancherel with respect to \( \lambda \), Cauchy-Schwarz with respect to \( \omega \), and the estimates (2.38) for \( b \) and (2.39) for \( \text{tr}\chi \). (3.1) misses the conclusion (2.82) of Theorem 2.15 by a power of \( \lambda \). Now, assume for a moment that we may replace a power of \( \lambda \) by a derivative on \( b(t,x,\omega)^{-1}\text{tr}\chi(t,x,\omega) \). Then, the same computation yields:

\[ \left\|\int_{\mathbb{S}^2} \int_0^{+\infty} D(b(t,x,\omega)^{-1}\text{tr}\chi(t,x,\omega)) e^{i\lambda u} f(\lambda \omega) \lambda d\lambda d\omega \right\|_{L^2(M)} \]

\[ \leq \int_{\mathbb{S}^2} \|D(b(t,x,\omega)^{-1}\text{tr}\chi(t,x,\omega))\|_{L^\infty_x(L^2(H_u))} \left\|\int_0^{+\infty} e^{i\lambda u} f(\lambda \omega) \lambda^2 d\lambda \right\|_{L^2_\omega} d\omega \]

\[ \leq \varepsilon \|\lambda f\|_{L^2(\mathbb{R}^3)}, \]

which is (2.82). This suggests a strategy which consists in making integrations by parts to trade powers of \( \lambda \) against derivatives of the symbol \( b(t,x,\omega)^{-1}\text{tr}\chi(t,x,\omega) \).

3.2 Structure of the proof of Theorem 2.15

The proof of Theorem 2.15 proceeds in three steps. We first localize in frequencies of size \( \lambda \sim 2^j \). We then localize the angle \( \omega \) in patches on the sphere \( \mathbb{S}^2 \) of diameter \( 2^{-j/2} \). Finally, we estimate the diagonal terms.

3.2.1 Step 1: decomposition in frequency

For the first step, we introduce \( \varphi \) and \( \psi \) two smooth compactly supported functions on \( \mathbb{R} \) such that:

\[ \varphi(\lambda) + \sum_{j \geq 0} \psi(2^{-j}\lambda) = 1 \text{ for all } \lambda \in \mathbb{R}. \]  

(3.3)

We use (3.3) to decompose \( Ef \) as follows:

\[ Ef(t,x) = \sum_{j \geq -1} E_j f(t,x), \]

(3.4)

where for \( j \geq 0 \):

\[ E_j f(t,x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} b(t,x,\omega)^{-1}\text{tr}\chi(t,x,\omega) \psi(2^{-j}\lambda) f(\lambda \omega) \lambda^2 d\lambda d\omega, \]

(3.5)
and
\[
E_{-1}f(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} b(t, x, \omega)^{-1} \text{tr}\chi(t, x, \omega) \varphi(\lambda) f(\lambda \omega) \lambda^2 d\lambda d\omega. \tag{3.6}
\]

This decomposition is classical and is known as the first dyadic decomposition (see [14]). The goal of this first step is to prove the following proposition:

**Proposition 3.1** The decomposition (3.4) satisfies an almost orthogonality property:
\[
\|Ef\|_{L^2(M)}^2 \lesssim \sum_{j \geq -1} \|E_j f\|_{L^2(M)}^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2. \tag{3.7}
\]

The proof of Proposition 3.1 is postponed to section 4.

### 3.2.2 Step 2: decomposition in angle

Proposition 3.1 allows us to estimate \(\|E_j f\|_{L^2(M)}\) instead of \(\|Ef\|_{L^2(M)}\). The analog of computation (3.1) for \(\|E_j f\|_{L^2(M)}\) yields:
\[
\|E_j f\|_{L^2(M)} \leq \varepsilon \|\psi(2^{-j} \lambda) f\|_{L^2(\mathbb{R}^3)} \lesssim 2^j \|\psi(2^{-j} \lambda) f\|_{L^2(\mathbb{R}^3)}, \tag{3.8}
\]
which misses the wanted estimate by a power of \(2^j\). We thus need to perform a second dyadic decomposition (see [14]). We introduce a smooth partition of unity on the sphere \(\mathbb{S}^2\):
\[
\sum_{\nu \in \Gamma} \eta_j^{\nu}(\omega) = 1 \text{ for all } \omega \in \mathbb{S}^2, \tag{3.9}
\]
where the support of \(\eta_j^{\nu}\) is a patch on \(\mathbb{S}^2\) of diameter \(\sim 2^{-j/2}\). We use (3.9) to decompose \(E_j f\) as follows:
\[
E_j f(t, x) = \sum_{\nu \in \Gamma} E_j^{\nu} f(t, x), \tag{3.10}
\]
where:
\[
E_j^{\nu} f(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u} b(t, x, \omega)^{-1} \text{tr}\chi(t, x, \omega) \psi(2^{-j} \lambda) \eta_j^{\nu}(\omega) \varphi(\lambda \omega) \lambda^2 d\lambda d\omega. \tag{3.11}
\]

We also define:
\[
\gamma_{-1} = \|\varphi(\lambda) f\|_{L^2(\mathbb{R}^3)}, \quad \gamma_j = \|\psi(2^{-j} \lambda) f\|_{L^2(\mathbb{R}^3)}, \quad j \geq 0, \\
\gamma_j^{\nu} = \|\psi(2^{-j} \lambda) \eta_j^{\nu}(\omega) f\|_{L^2(\mathbb{R}^3)}, \quad j \geq 0, \ \nu \in \Gamma, \tag{3.12}
\]
which satisfy:
\[
\|f\|_{L^2(\mathbb{R}^3)}^2 = \sum_{j \geq -1} \gamma_j^2 = \sum_{j \geq -1} \sum_{\nu \in \Gamma} (\gamma_j^{\nu})^2. \tag{3.13}
\]

The goal of this second step is to prove the following proposition:

**Proposition 3.2** The decomposition (3.10) satisfies an almost orthogonality property:
\[
\|E_j f\|_{L^2(M)}^2 \lesssim \sum_{\nu \in \Gamma} \|E_j^{\nu} f\|_{L^2(M)}^2 + \varepsilon^2 \gamma_j^2. \tag{3.14}
\]

The proof of Proposition 3.2 is postponed to section 6.
3.2.3 Step 3: control of the diagonal term

Proposition 3.2 allows us to estimate \( \|E^\nu_j f\|_{L^2(M)} \) instead of \( \|E_j f\|_{L^2(M)} \). The analog of computation (3.1) for \( \|E^\nu_j f\|_{L^2(M)} \) yields:

\[
\begin{align*}
\|E^\nu_j f\|_{L^2(M)} & \leq \int_{\mathbb{R}^2} \|b(t, x, \omega)^{-1} \text{tr} \chi(t, x, \omega)\|_{L^\infty L^2(H_u)} \left\| \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j} \lambda) \eta^\nu_j(\omega)f(\lambda \omega) \lambda^2 d\lambda \right\|_{L^2}\ d\omega \\
& \leq \varepsilon \sqrt{\text{vol}(\text{supp}(\eta^\nu_j))} \|\lambda \psi(2^{-j} \lambda) \eta^\nu_j(\omega)f\|_{L^2(\mathbb{R}^3)} \\
& \lesssim \varepsilon 2^{i/2} \gamma^\nu_j,
\end{align*}
\]

where the term \( \sqrt{\text{vol}(\text{supp}(\eta^\nu_j))} \) comes from the fact that we apply Cauchy-Schwarz in \( \omega \). Note that we have used in (3.15) the fact that the support of \( \eta^\nu_j \) is 2 dimensional and has diameter \( 2^{-j/2} \) so that:

\[
\sqrt{\text{vol}(\text{supp}(\eta^\nu_j))} \lesssim 2^{-j/2}.
\]

Now, (3.15) still misses the wanted estimate by a power of \( 2^{i/2} \). Nevertheless, we are able to estimate the diagonal term:

**Proposition 3.3** The diagonal term \( E^\nu_j f \) satisfies the following estimate:

\[
\|E^\nu_j f\|_{L^2(M)} \lesssim \varepsilon \gamma^\nu_j.
\]

The proof of Proposition 3.3 is postponed to section 5.

3.2.4 Proof of Theorem 2.15

Proposition 3.1, 3.2 and 3.3 immediately yield the proof of Theorem 2.15. Indeed, (3.7), (3.13), (3.14) and (3.17) imply:

\[
\|Ef\|_{L^2(M)}^2 \lesssim \sum_{j \geq -1} \|E_j f\|_{L^2(M)}^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\
\lesssim \sum_{j \geq -1} \sum_{\nu \in \Gamma} \|E^\nu_j f\|_{L^2(M)}^2 + \varepsilon^2 \sum_{j \geq -1} \gamma^j_j + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\
\lesssim \varepsilon^2 \sum_{j \geq -1} \sum_{\nu \in \Gamma} (\gamma^\nu_j)^2 + \varepsilon^2 \sum_{j \geq -1} \gamma^j_j + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2 \\
\lesssim \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2,
\]

which is the conclusion of Theorem 2.15.

The rest of the paper is dedicated to the proof of Proposition 3.1, 3.2 and 3.3. In section 4, we prove Proposition 3.1. In section 5, we prove Proposition 3.3. Finally, we turn to the proof of Proposition 3.2 which constitutes the most technical part of this paper and occupies sections 6 to 10.
4 Proof of Proposition 3.1 (almost orthogonality in frequency)

We have to prove (3.7):

\[ \|E f\|_{L^2(M)}^2 \lesssim \sum_{j \geq -1} \|E_j f\|_{L^2(M)}^2 + \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}^2. \]  \hfill (4.1)

This will result from the following inequality using Shur’s Lemma:

\[ \left| \int_M E_j f(t, x) E_k f(t, x) dM \right| \lesssim \varepsilon^2 2^{-\frac{|j-k|}{4}} \gamma_j \gamma_k \text{ for } |j-k| > 2. \]  \hfill (4.2)

Before we proceed with the proof of (4.2), let us recall that the volume element on \( M \) expressed in the coordinate system \((u, t, x')\) is given by:

\[ dM = nbdu dt d\mu_{t,u}, \]

where \( d\mu_{t,u} \) denotes the volume element on \( P_{t,u} \). Since we have:

\[ \|n - 1\|_{L^\infty} \lesssim \varepsilon, \]

from the estimates for \( n \) (2.36), the \( L^2(M) \) norm of a tensor \( F \) on \( M \) is equivalent to:

\[ \int_M |F(t, x)|^2 b du dt d\mu_{t,u}, \]  \hfill (4.3)

where we have removed the lapse \( n \) in the definition of the volume element. To avoid unnecessary terms containing derivatives of \( n \) in the numerous integrations by parts of this paper, we will estimate the equivalent (4.3) of the \( L^2(M) \) norm for \( Ef, E_j f \) and \( E^\nu f \).

By a slight abuse of notation which we shall do throughout the paper, this is equivalent to modifying the volume element by removing the time lapse \( n \):

\[ dM = b du dt d\mu_{t,u}. \]  \hfill (4.4)

Remark 4.1 One may want to further simplify the expression of the volume element (4.4) by removing the null lapse \( b \). However, this is not possible since the decomposition (4.4) depends of the angle \( \omega \in S^2 \) under consideration. Indeed, the coordinate system is \((u, t, x')\) where \( u = u(t, x, \omega) \) and thus \( b = b(t, x, \omega) \) in (4.4), while the time lapse \( n \) is independent of \( \omega \).

Also, recall that we have for all integrable scalar functions \( f \):

\[ \frac{d}{du} \left( \int_{P_{t,u}} f d\mu_{t,u} \right) = \int_{P_{t,u}} b(\nabla_N f + \text{tr} \theta) d\mu_{t,u} \]  \hfill (4.5)

where \( \theta \) is the second fundamental form of \( P_{t,u} \) in \( \Sigma_t \), i.e. \( \theta_{ij} = \nabla_i N_j \). Note that from the definition of \( k, \chi \) and \( \theta \), we have:

\[ \chi_{AB} = <D_A L, e_B> = <\nabla_A T, e_B> + <\nabla_A N, e_B> = -k_{AB} + \theta_{AB}. \]  \hfill (4.6)
Together with the estimate (2.39) (2.40) for $\chi$ and (2.37) for $k$, we obtain:
\[ \mathcal{N}_1(\theta) \lesssim \varepsilon. \] (4.7)

Finally, we recall that we have:
\[ \begin{align*}
\nabla_A N &= \theta_{AB} e_B, \\
\nabla_N N &= -b^{-1} \nabla b.
\end{align*} \] (4.8)

### 4.1 A first integration by parts

From now on, we focus on proving (4.2). We may assume $j \geq k + 3$. We have:
\[ \int_{\mathcal{M}} E_j f(t, x) E_k f(t, x) d\mathcal{M} = \int_{\mathcal{M}} \int_{0}^{+\infty} \int_{0}^{+\infty} \frac{1}{2} \text{tr} \chi(t, x, \omega) (-1)^{j-1} \text{tr} \chi(t, x, \omega') d\mathcal{M} \] (4.9)

\[ \times \psi^2 t \psi (2^{-j} \lambda) f(\lambda \omega) \lambda^2 \psi(2^{-k} \lambda') f(\lambda' \omega') \lambda^2 d\lambda d\lambda' d\omega. \]

In view of the expression of the volume element (4.4) on $\mathcal{M}$, we integrate by parts with respect to $\partial_u$ in
\[ \int_{\mathcal{M}} \frac{1}{2} \text{tr} \chi(t, x, \omega) (-1)^{j-1} \text{tr} \chi(t, x, \omega') \chi \partial_u(e^{i\lambda u-i\lambda' u'}) d\mathcal{M} \]
\[ = \int_{\mathcal{M}} \frac{1}{2} \text{tr} \chi(t, x, \omega) (-1)^{j-1} \text{tr} \chi(t, x, \omega') \chi \partial_u(e^{i\lambda u-i\lambda' u'}) d\mathcal{M} \]
\[ = \int_{\mathcal{M}} \frac{1}{2} \text{tr} \chi(t, x, \omega) (-1)^{j-1} \text{tr} \chi(t, x, \omega') \chi \partial_u(e^{i\lambda u-i\lambda' u'}) d\mathcal{M} \]
\[ = \int_{\mathcal{M}} \frac{1}{2} \text{tr} \chi(t, x, \omega) (-1)^{j-1} \text{tr} \chi(t, x, \omega') \chi \partial_u(e^{i\lambda u-i\lambda' u'}) d\mathcal{M} \]
\[ = \int_{\mathcal{M}} \frac{1}{2} \text{tr} \chi(t, x, \omega) (-1)^{j-1} \text{tr} \chi(t, x, \omega') \chi \partial_u(e^{i\lambda u-i\lambda' u'}) d\mathcal{M} \]

using the fact that:
\[ e^{i\lambda u-i\lambda' u'} = -\frac{i}{\lambda - \lambda' g(N, N')} \partial_u(e^{i\lambda u-i\lambda' u'}), \] (4.10)

where we use the notation $u$ for $u(t, x, \omega)$, $b$ for $b(t, x, \omega)$, $N$ for $N(t, x, \omega)$, $u'$ for $u(t, x, \omega')$, $b'$ for $b(t, x, \omega')$ and $N'$ for $N(t, x, \omega')$. We will also use the notation $\text{tr} \chi$ for $\text{tr} \chi(t, x, \omega)$, $\text{tr} \chi'$ for $\text{tr} \chi(t, x, \omega')$, $\text{tr} \theta$ for $\text{tr} \theta(t, x, \omega)$, and $\text{tr} \theta'$ for $\text{tr} \theta(t, x, \omega')$. Using (4.10) and the expression for the volume element (4.4), we obtain:
\[ \int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'} b \bar{\omega} d\mathcal{M} \] (4.11)

\[ = \int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'} \frac{b^{-1} \partial_u \chi b^{-1} \partial_u \chi'}{\lambda - \lambda' g(N, N')} d\mathcal{M} + \int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'} \frac{b^{-1} \partial_u \chi b^{-1} \partial_u \chi'}{\lambda - \lambda' g(N, N')} d\mathcal{M} \]
\[ + i \int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'} \frac{b^{-1} \partial_u \chi b^{-1} \partial_u \chi'}{\lambda - \lambda' g(N, N')} d\mathcal{M} \]
\[ + i \lambda' \int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'} \frac{b^{-1} \partial_u \chi b^{-1} \partial_u \chi'}{\lambda - \lambda' g(N, N')} d\mathcal{M} \]
\[ + i \lambda' \int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'} \frac{b^{-1} \partial_u \chi b^{-1} \partial_u \chi'}{\lambda - \lambda' g(N, N')} d\mathcal{M} \]
\[ + i \lambda' \int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'} \frac{b^{-1} \partial_u \chi b^{-1} \partial_u \chi'}{\lambda - \lambda' g(N, N')} d\mathcal{M}, \]
where we have used (4.5) to obtain the third term in the right-hand side of (4.11). Since $|\lambda^b g(N, N')| < \lambda$, we may expand the fractions in (4.11):

\[
\frac{1}{\lambda - \lambda^b g(N, N')} = \sum_{p \geq 0} \frac{1}{\lambda} \left( \frac{\lambda^b g(N, N')}{\lambda} \right)^p, \tag{4.12}
\]

and

\[
\frac{1}{(\lambda - \lambda^b g(N, N'))^2} = \sum_{p \geq 0} \frac{p + 1}{\lambda^2} \left( \frac{\lambda^b g(N, N')}{\lambda} \right)^p. \tag{4.13}
\]

For $p \in \mathbb{Z}$, we introduce the notation $F_{j,p}(u)$:

\[
F_{j,p}(u) = \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j}\lambda) f(\lambda \omega) (2^{-j}\lambda)^p \lambda^2 d\lambda. \tag{4.14}
\]

Together with (4.9), (4.11) and (4.12), this implies:

\[
\int_{\mathcal{M}} E_j f(t, x) \overline{E_k f(t, x)} d\mathcal{M} = \sum_{p \geq 0} A^1_p + \sum_{p \geq 0} A^2_p + \sum_{p \geq 0} A^3_p + \sum_{p \geq 0} A^4_p, \tag{4.15}
\]

where $A^1_p$, $A^2_p$, $A^3_p$ and $A^4_p$ are given by:

\[
A^1_p = 2^{-j-p(j-k)} \int_{\mathcal{M}} \left( \int_{\mathbb{R}^2} (b^{-1} \nabla N \text{tr} \chi + b^{-1} \text{tr} \chi \text{tr} \theta) b^{p+1} N^p F_{j,-p-1}(u) d\omega \right) \\
\cdot \left( \int_{\mathbb{R}^2} b^{-1} \text{tr} \chi b^{-p} N^p F_{k,p}(u') d\omega' \right) d\mathcal{M}, \tag{4.16}
\]

\[
A^2_p = 2^{-j-p(j-k)} \int_{\mathcal{M}} \left( \int_{\mathbb{R}^2} b^{-1} \text{tr} \chi b^{p+1} N^p F_{j,-p-1}(u) d\omega \right) \\
\cdot \left( \int_{\mathbb{R}^2} \nabla (b^{-1} \text{tr} \chi) b^{-p} N^p F_{k,p}(u') d\omega' \right) d\mathcal{M}. \tag{4.17}
\]

\[
A^3_p = (p+1)2^{-j-(p+1)(j-k)} \int_{\mathcal{M}} \left( \int_{\mathbb{R}^2} b^{-1} \text{tr} \chi (\nabla_b N b N + b \nabla N b) b^p N^p F_{j,-p-2}(u) d\omega \right) \\
\cdot \left( \int_{\mathbb{R}^2} b^{-1} \text{tr} \chi b^{-p-1} N^{p+1} F_{k,p+1}(u') d\omega' \right) d\mathcal{M}, \tag{4.18}
\]

and

\[
A^4_p = (p+1)2^{-j-(p+1)(j-k)} \int_{\mathcal{M}} \left( \int_{\mathbb{R}^2} b^{-1} \text{tr} \chi b^{p+1} N^p F_{j,-p-2}(u) d\omega \right) \\
\cdot \left( \int_{\mathbb{R}^2} b^{-1} \text{tr} \chi (\nabla \log(b') N + \nabla N') b^{-p-1} N^p F_{k,p+1}(u') d\omega' \right) d\mathcal{M}. \tag{4.19}
\]

**Remark 4.2** The expansion (4.12) allows us to rewrite $\int_{\mathcal{M}} E_j f(t, x) \overline{E_k f(t, x)} d\mathcal{M}$ in the form (4.15), i.e. as a sum of terms $A^1_p$, $A^2_p$, $A^3_p$, $A^4_p$. The key point is that in each of these terms - according to (4.16)-(4.19) - one may separate the terms depending of $(\lambda, \omega)$ from the terms depending on $(\lambda', \omega')$. 

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4.2 Estimates for $A^1_p$ and $A^2_p$

Let $H(t,x,\omega)$ a tensor on $\mathcal{M}$ such that $\|H\|_{L^\infty_0 L^2(\mathcal{H}_u)} \lesssim \varepsilon$. Then proceeding as in the basic computation (3.1), we have for any $p \in \mathbb{Z}$:

\[
\left\| \int_{S^2} H(x,\omega) F_{j,p}(u) d\omega \right\|_{L^2(\mathcal{M})} \leq \int_{S^2} \|H\|_{L^\infty_0 L^2(\mathcal{H}_u)} \|F_{j,p}(u)\|_{L^2_u} d\omega \\
\leq \|H\|_{L^\infty_0 L^2(\mathcal{H}_u)} \|\psi(2^{-j} \lambda) f(\lambda \omega) (2^{-j} \lambda)^p \lambda\|_{L^2(\mathbb{R}^3)}
\]

where we have used the fact that $1/2 \leq 2^{-j} \lambda \leq 2$ on the support of $\psi(2^{-j} \lambda)$. Now, the estimates (2.38) on $b$, (4.7) on $\theta$, and the equation for $\nabla N$ (4.8) yield:

\[
\| (b^{-1} \nabla_N \text{tr} \chi + b^{-1} \text{tr} \chi \text{tr} \theta)b^{p+1} N^p \|_{L^\infty_0 L^2(\mathcal{H}_u)} + \| (b^{-1} \text{tr} \chi') b^{-p} N^p \|_{L^\infty_0 L^2(\mathcal{H}_u)} + \| b^{-1} \text{tr} \chi (\nabla_N b N + b \nabla_N N)b^{p} N^p \|_{L^\infty_0 L^2(\mathcal{H}_u)} + \| b^{-1} \text{tr} \chi' (\nabla \log(b') N' + \nabla N')b^{-p-1} N^p \|_{L^\infty_0 L^2(\mathcal{H}_u)} \lesssim \varepsilon,
\]

which together with (4.20) implies:

\[
\left\| \int_{S^2} (b^{-1} \nabla_N \text{tr} \chi + b^{-1} \text{tr} \chi \text{tr} \theta)b^{p+1} N^p F_{j,-p-1}(u) d\omega \right\|_{L^2(\mathcal{M})} + \left\| \int_{S^2} \nabla (b^{-1} \text{tr} \chi') b^{-p} N^p F_{k,p}(u') d\omega \right\|_{L^2(\mathcal{M})} \\
+ \left\| \int_{S^2} b^{-1} \text{tr} \chi (\nabla_N b N + b \nabla_N N)b^{p} N^p F_{j,-p-2}(u) d\omega \right\|_{L^2(\mathcal{M})} + \left\| \int_{S^2} b^{-1} \text{tr} \chi' (\nabla \log(b') N' + \nabla N')b^{-p-1} N^p F_{k,p+1}(u') d\omega \right\|_{L^2(\mathcal{M})} \\
\lesssim \varepsilon 2^{p+j} \gamma_j.
\]

Note that Proposition 3.2 together with Proposition 3.3 yields the estimate:

\[
\|E_j f\|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j,
\]

for any symbol satisfying the same regularity assumptions than $b^{-1} \text{tr} \chi$ where $b$ satisfies (2.38), and $\text{tr} \chi$ satisfies (2.39). Now, the terms containing no derivative in (4.16)-(4.19) have a symbol given respectively by $b^{-1} b^{-p} N^p$, $b^{-1} \text{tr} \chi b^{p+1} N^{p+1}$, $b^{-1} \text{tr} \chi' b^{-p-1} N^{p+1}$ and $b^{-1} \text{tr} \chi b^{p+1} N^{p+2}$. Since $N$ satisfies regularity assumptions which are at least as good as $\text{tr} \chi$, these symbols satisfies the same regularity assumptions than $b^{-1} \text{tr} \chi$. Applying (4.23), we obtain:

\[
\left\| \int_{S^2} b^{-1} \text{tr} \chi' b^{-p} N^p F_{k,p}(u') d\omega \right\|_{L^2(\mathcal{M})} + \left\| \int_{S^2} b^{-1} \text{tr} \chi' b^{-p-1} N^{p+1} F_{k,p+1}(u') d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^p \gamma_k,
\]

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and

\[
\| \int_{S^2} b^{-1} \text{tr} \chi b^{p+1} N^{p+1} F_{j,-p-1}(u) d\omega \|_{L^2(M)} + \| \int_{S^2} b^{-1} \text{tr} \chi b^{p+1} N^{p+2} F_{j,-p-2}(u) d\omega \|_{L^2(M)} \lesssim \varepsilon 2^{p \gamma_j}, \quad (4.25)
\]

where we have used the fact that \(1/2 \leq 2^{-j} \lambda \leq 2\) on the support of \(\psi(2^{-j} \lambda)\).

Finally, the definition of \(A^1_p - A^4_p\) given by (4.16)-(4.19) and the estimates (4.22), (4.24) and (4.25) yield:

\[
|A^1_p| \lesssim \varepsilon 2^{2p-(j-k)} \gamma_j \gamma_k, \quad \forall p \geq 0, \quad (4.26)
\]

and

\[
|A^2_p| + |A^3_p| + |A^4_p| \lesssim \varepsilon 2^{2p-(p+1)(j-k)} \gamma_j \gamma_k, \quad \forall p \geq 0. \quad (4.27)
\]

(4.26) and (4.27) imply:

\[
\sum_{p \geq 1} |A^1_p| + \sum_{p \geq 0} (|A^2_p| + |A^3_p| + |A^4_p|) \lesssim \varepsilon 2^{-(j-k)} \left( \sum_{p \geq 0} 2^{-p(j-k-2)} \right) \gamma_j \gamma_k \lesssim \varepsilon 2^{-(j-k)} \gamma_j \gamma_k, \quad (4.28)
\]

where we have used the assumption \(j - k - 2 > 0\). (4.15) and (4.28) will yield (4.2) provided we obtain a similar estimate for \(A^1_0\). Now, the estimate of \(A^1_0\) provided by (4.26) is not sufficient since it does not contain any decay in \(j - k\). We will need to perform a second integration by parts for this term.

### 4.3 A more precise estimate for \(A^1_0\)

From (4.16) with \(p = 0\), we have:

\[
A^1_0 = 2^{-j} \int_M \left( \int_{S^2} (\nabla_N \text{tr} \chi + \text{tr} \chi \text{tr} \theta) F_{j,-1}(u) d\omega \right) E_k f(t,x) dM. \quad (4.29)
\]

Using the geometric Littlewood-Paley projections on the 2-surfaces \(P_{t,u}\), we decompose \(\nabla_N \text{tr} \chi\) as:

\[
\nabla_N \text{tr} \chi = P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) + P_{> \frac{j+k}{2}} (\nabla_N \text{tr} \chi).
\]

In turn, this yields a decomposition for \(A^1_0\):

\[
A^1_0 = A^1_{0,1} + A^1_{0,2} \quad (4.30)
\]

where:

\[
A^1_{0,1} = 2^{-j} \int_M \left( \int_{S^2} P_{> \frac{j+k}{2}} (\nabla_N \text{tr} \chi) F_{j,0}(u) d\omega \right) E_k f(t,x) dM,
\]

\[
A^1_{0,2} = 2^{-j} \int_M \left( \int_{S^2} (P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) + \text{tr} \chi \text{tr} \theta) F_{j,0}(u) d\omega \right) E_k f(t,x) dM. \quad (4.31)
\]
We first estimate $A_{0,1}^1$. The finite band property yields:

\[
P_{> \frac{j+k}{2}}(\nabla_N \text{tr} \chi) = \sum_{l > \frac{j+k}{2}} P_l(\nabla_N \text{tr} \chi)
= \sum_{l > \frac{j+k}{2}} 2^{-2l} \Delta P_l(\nabla_N \text{tr} \chi),
\]

which yields the following decomposition for $A_{0,1}^1$:

\[
A_{0,1}^1 = \sum_{l > \frac{j+k}{2}} A_{0,1,l}^1
\]

where $A_{0,1,l}^1$ is given by:

\[
A_{0,1,l}^1 = 2^{-j-2l} \int_{\mathcal{M}} \left( \int_{S^2} \Delta P_l(\nabla_N \text{tr} \chi) F_{j,0}(u) d\omega \right) E_k f(t, x) d\mathcal{M}.
\]

Now, the decomposition of the volume element (4.4) yields:

\[
A_{0,1,l}^1 = 2^{-j-2l} \int_{S^2} \int_{t,u} \left( \int_{P_{t,u}} \Delta P_l(\nabla_N \text{tr} \chi) E_k f(t, x) b d\mu_{t,u} \right) F_{j,0}(u) du dt d\omega.
\]

Integrating by parts $\Delta$ on $P_{t,u}$, we obtain:

\[
A_{0,1,l}^1 = -2^{-j-2l} \int_{S^2} \int_{t,u} \left( \int_{P_{t,u}} \nabla P_l(\nabla_N \text{tr} \chi) \nabla E_k f(t, x) b d\mu_{t,u} \right) F_{j,0}(u) du dt d\omega
= -2^{-j-2l} \int_{S^2} \left( \int_{\mathcal{M}} \nabla P_l(\nabla_N \text{tr} \chi) F_{j,0}(u) \nabla E_k f(t, x) b d\mathcal{M} \right) d\omega,
\]

where we used again the decomposition of the volume element (4.4) in the last equality. We apply Cauchy-Schwartz to the integral on $\mathcal{M}$ and obtain:

\[
|A_{0,1,l}^1| \leq 2^{-j-2l} \int_{S^2} \| \nabla P_l(\nabla_N \text{tr} \chi) F_{j,0}(u) \|_{L^2(\mathcal{M})} \| \nabla (E_k b) b^{-1} \|_{L^2(\mathcal{M})} d\omega
\leq 2^{-j-2l} \int_{S^2} \| \nabla P_l(\nabla_N \text{tr} \chi) \|_{L^\infty} F_{j,0}(u) \|_{L^2(\mathcal{M})} \| \nabla (E_k b) b^{-1} \|_{L^\infty} d\omega
\leq 2^{-j-1} \int_{S^2} \| \nabla N \text{tr} \chi \|_{L^\infty} F_{j,0}(u) \|_{L^2(\mathcal{M})} \| \nabla (E_k b) \|_{L^2(\mathcal{M})} d\omega
\leq \varepsilon 2^{-j-1} \int_{S^2} \| F_{j,0}(u) \|_{L^2(\mathcal{M})} \| \nabla (E_k b) \|_{L^2(\mathcal{M})} d\omega,
\]

where we used the finite band property for $P_l$, the estimates (2.38) for $b$, and the estimates (2.39) for $\text{tr} \chi$. Plancherel yields:

\[
\| F_{j,0} \|_{L^2_{t,u}} \leq \| \psi(2^{-j} \lambda) f(\lambda) \lambda \|_{L^2(\mathbb{R}^3)} \approx 2^j \gamma_j.
\]
In view of (4.33), we also need to estimate \( \| \nabla (E_k b) \|_{L^2(\mathcal{M})} \). We have:

\[
\| \nabla (E_k b) \|_{L^2(\mathcal{M})} \lesssim \| E_k \nabla b \|_{L^2(\mathcal{M})} + \| b \nabla E_k \|_{L^2(\mathcal{M})} \tag{4.35}
\]

where we used in the last inequality the estimates (2.38) for \( b \), the Sobolev embedding on \( \mathcal{H}_u \) (2.50), and the Sobolev inequality on the 4-dimensional manifold \( \mathcal{M} \) (2.54). We still need to estimate \( \| D E_k \|_{L^2(\mathcal{M})} \). We have:

\[
D E_k f(t, x) = \int_{S^2} \int_0^{+\infty} e^{i\lambda u} D(b^{-1} \text{tr} \chi) \psi((2^{-k}) \lambda) f(\lambda \omega) \lambda^2 d\lambda d\omega
\]

\[
+ i 2^k \int_{S^2} \int_0^{+\infty} e^{i\lambda u} \text{tr} \chi L \psi((2^{-k}) \lambda) (2^{-k}) \lambda f(\lambda \omega) \lambda^2 d\lambda d\omega. \tag{4.36}
\]

Using the basic computation (3.1) for the first term together with the fact that \( D(b^{-1} \text{tr} \chi) \in L_u^\infty \tilde{L}_t^2(\mathcal{H}_u) \) from the estimates (2.38) for \( b \) and (2.39) for \( \text{tr} \chi \), and (4.23) for the second term together with the fact that \( \text{tr} \chi N \) satisfies the same regularity assumptions than \( b^{-1} \text{tr} \chi \), we obtain:

\[
\| D E_k \|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^k \gamma_k. \tag{4.37}
\]

(4.33), (4.34), (4.35) and (4.37) yield:

\[
| A_{0,1,l}^1 | \lesssim \varepsilon^2 2^{-l+k} \varepsilon^2 \gamma_j \gamma_k.
\]

Together with (4.32), this yields:

\[
| A_{0,1,l}^1 | \lesssim \left( \sum_{l > \frac{l+k}{2}} 2^{-l} \right) \varepsilon 2^k \varepsilon^2 \gamma_j \gamma_k \lesssim \varepsilon^2 2^{-l+k} \varepsilon^2 \gamma_j \gamma_k. \tag{4.38}
\]

### 4.4 A second integration by parts

We now estimate the term \( A_{0,2}^1 \) defined in (4.31).

We perform a second integration by parts relying again on (4.10). We obtain:

\[
A_{0,2}^1 = 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} \left( b(\nabla_N P_{\leq \frac{i+k}{2}} (\nabla_N \text{tr} \chi)) + \nabla_N (\text{tr} \chi \text{tr} \theta) + \nabla_N (\text{tr} \chi \text{tr} \theta) \right) F_{j,0}(u) d\omega \right) E_k f(t, x) d\mathcal{M}
\]

\[
+ 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} (P_{\leq \frac{i+k}{2}} (\nabla_N \text{tr} \chi) + \text{tr} \chi \text{tr} \theta) b NF_{j,0}(u) d\omega \right) \nabla E_k f(t, x) d\mathcal{M} + \cdots,
\]

where we only mention the first term generated by the expansion (4.12). In fact, the other terms generated by (4.12) and the ones generated by (4.13) are estimated in the same
way and generate more decay in $j - k$ similarly to the estimates (4.26) (4.27). In view of (4.39), we decompose the main part of $A^1_{0,2}$ as the sum of three terms:

$$
A^1_{0,2} = A^1_{0,2,1} + A^1_{0,2,2} + A^1_{0,2,3} + \cdots , \tag{4.40}
$$

where $A^1_{0,2,1}$ is given by:

$$
A^1_{0,2,1} = 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} \left( b \nabla_N P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) + \nabla_N (b) P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) + b \text{tr} \theta P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) \right) F_{j,0}(u) d\omega \right) E_k f(t,x) d\mathcal{M},
$$

where $A^1_{0,2,2}$ is given by:

$$
A^1_{0,2,2} = 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} (b \nabla_N (\text{tr} \chi \text{tr} \theta) + \nabla_N (b) \text{tr} \chi \text{tr} \theta + b \text{tr} \theta^2 (\text{tr} \chi) \right) F_{j,0}(u) d\omega \right) E_k f(t,x) d\mathcal{M},
$$

and where $A^1_{0,2,3}$ is given by:

$$
A^1_{0,2,3} = 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} (P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) + \text{tr} \chi \text{tr} \theta) bNF_{j,0}(u) d\omega \right) \cdot \nabla E_k f(t,x) d\mathcal{M}. \tag{4.43}
$$

We first estimate $A^1_{0,2,1}$. We have:

$$
\left\| b \nabla_N P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) + \nabla_N (b) P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) + b \text{tr} \theta P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) \right\|_{L^2(\mathcal{H}_u)} 
\lesssim \| b \|_{L^\infty} \left( \sum_{l \leq \frac{j+k}{k}} \| \nabla_N P_l (\nabla_N \text{tr} \chi) \|_{L^2(\mathcal{H}_u)} \right) 
\quad + \left( \| \nabla_N b \|_{L^\infty L^2(\mathcal{H}_u)} + \| b \|_{L^\infty} \| \text{tr} \theta \|_{L^\infty L^2(\mathcal{H}_u)} \right) \| P_{\leq \frac{j+k}{2}} (\nabla_N \text{tr} \chi) \|_{L^\infty} 
\lesssim \sum_{l \leq \frac{j+k}{k}} (2^{j+\frac{k}{2}} + 2^{j+\frac{k}{2}} \varepsilon \mu(u)) + \varepsilon 2^{j+\frac{k}{2}} \| \nabla_N \text{tr} \chi \|_{L^2_{t,x}} 
\lesssim \varepsilon 2^{j+\frac{k}{2}} + 2^{j+\frac{k}{2}} \varepsilon \mu(u),
$$

where we used the estimates (2.38) for $b$, the estimates (4.7) for $\theta$, the estimate (2.48) for $\nabla_N P_l (\nabla_N \text{tr} \chi)$, the strong Bernstein inequality (2.59), and the estimates (2.39) for $\text{tr} \chi$, and where $\mu$ in a function satisfying:

$$
\| \mu \|_{L^2(\mathbb{R})} \lesssim 1,
$$

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according to (2.48). In view of (4.41), this yields:

\[ |A_{0,2,1}^1| \lesssim 2^{-2j} \|b\|_{L^2(M)} \int_{S^2} \left\| b \nabla_N P_{\leq \frac{j+k}{2}} (\nabla_N \chi) \right\| \, d\omega \]

\[ + \| \nabla_N (b) P_{\leq \frac{j+k}{2}} (\nabla_N \chi) + b \tau \nabla P_{\leq \frac{j+k}{2}} (\nabla_N \chi) \|_{L^2(H_u)} \| F_{j,0}(u) \|_{L^2} \, d\omega \]

\[ \lesssim 2^{-2j} \varepsilon \gamma_k \int_{S^2} \left\| (\varepsilon^{2j} + \varepsilon^{2j+k}) \| b \|_{L^2} \, d\omega \right\| + 2^{j+k} \| F_{j,0}(u) \|_{L^2} \, d\omega \]

\[ \lesssim 2^{-2j} \varepsilon \gamma_k \left( \varepsilon^{2j+k} \int_{S^2} \| F_{j,0}(u) \|_{L^2} \, d\omega + 2^{j+k} \| \mu \|_{L^2(\mathbb{R})} \| F_{j,0}(u) \|_{L^2} \, d\omega \right) \]

\[ \lesssim 2^{-2j} \varepsilon \gamma_k \varepsilon^2 \gamma_k, \]

where we used (4.23) for \( E_k f \), Plancherel with respect to \( \lambda \) for \( \| F_{j,0}(u) \|_{L^2} \), Cauchy-Schwarz in \( \lambda \) for \( \| F_{j,0}(u) \|_{L^\infty} \), and Cauchy-Schwarz in \( \omega \).

Next, we estimate \( A_{0,2,2}^1 \). We have:

\[ \left\| b \nabla_N (\nabla_N \chi \nabla \chi) \right\|_{L^\infty L^\frac{3}{2}} \lesssim \left\| b \right\|_{L^\infty} \left( \left\| \nabla_N \chi \right\|_{L^\infty L^2(H_u)} \right) \left\| \nabla \chi \right\|_{L^\infty L^2(H_u)} \]

\[ + \left\| \nabla_N b \right\|_{L^\infty L^2(H_u)} \left\| \nabla \chi \right\|_{L^\infty L^2(H_u)} \left\| \nabla \chi \right\|_{L^\infty} \left\| \nabla \theta \right\|_{L^\infty L^3(H_u)} \left\| \nabla \chi \right\|_{L^\infty} \]

\[ \lesssim \varepsilon, \]

where we used the Sobolev embedding (2.50) on \( H_u \), the estimates (2.38) for \( b \), the estimates (2.39) for \( \nabla \chi \) and the estimates (4.7) for \( \theta \). In view of (4.42), this yields:

\[ |A_{0,2,2}^1| \lesssim 2^{-2j} \int_{S^2} \left\| b \nabla_N (\nabla_N \chi \nabla \chi) \right\|_{L^\infty L^\frac{3}{2}(H_u)} \left\| F_{j,0}(u) \right\|_{L^2} \| E_k f \|_{L^2 L^3(H_u)} \, d\omega \]

\[ \lesssim 2^{-2j} \varepsilon \left( \int_{S^2} \| F_{j,0}(u) \|_{L^2} \, d\omega \right) \| E_k f \|_{L^4(M)} \]

\[ \lesssim 2^{-2j} \varepsilon \gamma_j \| D E_k f \|_{L^2(M)} \]

\[ \lesssim 2^{-2j} \varepsilon \gamma_j \varepsilon^2 \gamma_k, \]

where we used Plancherel with respect to \( \lambda \) for \( \| F_{j,0}(u) \|_{L^2} \), Cauchy-Schwarz in \( \omega \), the Sobolev embedding on \( M \) (2.54), and (4.37) for \( D E_k f \).

Finally, we estimate \( A_{0,2,3}^1 \). We have:

\[ \left\| (P_{\leq \frac{j+k}{2}} (\nabla_N \chi) + \nabla \chi \nabla \chi) \right\|_{L^\infty L^2(H_u)} \lesssim \left\| b \right\|_{L^\infty} \left( \left\| P_{\leq \frac{j+k}{2}} (\nabla_N \chi) \right\|_{L^\infty L^2(H_u)} + \left\| \nabla \chi \right\|_{L^\infty} \left\| \nabla \theta \right\|_{L^\infty L^2(H_u)} \right) \]

\[ \lesssim \left\| \nabla_N \chi \right\|_{L^\infty L^2(H_u)} + \varepsilon \]

\[ \lesssim \varepsilon, \]

where we used the estimates (2.38) for \( b \), (2.39) for \( \nabla \chi \) and (4.7) for \( \theta \). In view of (4.43),
this yields:

\[ |A_{1,0,2,3}| \lesssim 2^{-2j} \left( \int_{S^2} \| (P_{j-k} (\nabla_N \text{tr} \chi) + \text{tr} \chi \text{tr} \theta) b\nu \|_{L^\infty}\|F_{j,0}(u)\|_{L^2_u} d\omega \right) \| \nabla E_k f \|_{L^2(M)} \]

\[ \lesssim \varepsilon 2^{-2j}\gamma_k \left( \int_{S^2} \| F_{j,0}(u) \|_{L^2_u} d\omega \right) \]

\[ \lesssim \varepsilon 2^{-j-k}\gamma_k\gamma_j, \]

where we used (4.23) for \( E_k f \), Plancherel with respect to \( \lambda \) for \( \| F_{j,0}(u) \|_{L^2_u} \), and Cauchy-Schwarz in \( \omega \). Finally, (4.40), (4.44), (4.45) and (4.46) imply:

\[ |A_{1,0}| \lesssim \varepsilon 2^{-\frac{j-k}{4}}\gamma_j\gamma_k. \] (4.47)

### 4.5 End of the proof of Proposition 3.1

Since \( A_1 = A_{1,1} + A_{1,2} \), the estimate (4.38) of \( A_{1,1} \) and the estimate (4.47) of \( A_{1,2} \) yield:

\[ |A_1| \lesssim \varepsilon 2^{-\frac{j-k}{4}}\gamma_j\gamma_k. \] (4.48)

Together with (4.15) and (4.28), this implies:

\[ \left| \int_M E_j f(t,x) E_k f(t,x) dM \right| \lesssim \varepsilon 2^{-\frac{|j-k|}{4}}\gamma_j\gamma_k \text{ for } |j-k| > 2. \] (4.49)

Finally, (4.49) together with Shur’s Lemma yields:

\[ \| E f \|_{L^2(M)}^2 \lesssim \sum_{j \geq -1} \| E_j f \|_{L^2(M)}^2 + \varepsilon^2 \| f \|_{L^2(\mathbb{R}^3)}^2. \] (4.50)

This concludes the proof of Proposition 3.1. \( \blacksquare \)

### 5 Proof of Proposition 3.3 (control of the diagonal term)

Since the orthogonality argument in angle in the core of the paper, we choose to deal first with the control of the diagonal term in this section. We will then proceed with the orthogonality argument in angle in the rest of the paper.

In order to control the diagonal term, we have to prove (3.17):

\[ \| E_j^\nu f \|_{L^2(M)} \lesssim \varepsilon \gamma_j^\nu. \] (5.1)

Recall that \( E_j^\nu \) is given by:

\[ E_j^\nu f(t,x) = \int_{S^2} b^{-1}(t,x,\omega) \text{tr} \chi(t,x,\omega) F_j(u) \eta_j^\nu(\omega) d\omega, \] (5.2)
where \( F_j(u) \) is defined by:

\[
F_j(u) = \int_0^{\infty} e^{i \lambda u} \psi(2^{-j} \lambda) f(\lambda \omega) \lambda^2 d\lambda.
\]  
(5.3)

In view of the decompositions (2.72), and the decomposition (2.73) with \( p = -1 \), we have the following decomposition for \( b^{-1} \text{tr} \chi \):

\[
b^{-1} \text{tr} \chi = f^j_1 + f^j_2,
\]
(5.4)

where \( f^j_1 \) only depends on \( (t, x, \nu) \) and satisfies:

\[
\| f^j_1 \|_{L^\infty} \lesssim \varepsilon,
\]
(5.5)

and where \( f^j_2 \) satisfies

\[
\| f^j_2 \|_{L^\infty L^2(\mathcal{H}_\nu)} \lesssim 2^{-\frac{j}{2}} \varepsilon.
\]
(5.6)

In view of (5.2), (5.4) yields the following decomposition for \( \mathcal{E}_\nu^j f \):

\[
\mathcal{E}_\nu^j f(t, x) = f^j_1(t, x, \nu) \int_{\mathbb{S}^2} F_j(u) \eta^\nu_j(\omega) d\omega + \int_{\mathbb{S}^2} F_j(u) f^j_2(t, x, \omega, \nu) \eta^\nu_j(\omega) d\omega.
\]

which together with the estimates (5.5) and (5.6) implies:

\[
\| \mathcal{E}_\nu^j f \|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j^\nu,
\]
(5.7)

where we used in the last inequality Plancherel in \( \lambda \), Cauchy-Schwarz in \( \omega \), and the size of the patch.

The following proposition allows us to estimate the right-hand side of (5.7).

**Proposition 5.1** We have the following bound:

\[
\| \int_{\mathbb{S}^2} F_j(u) \eta^\nu_j(\omega) d\omega \|_{L^2(\mathcal{M})} \lesssim \gamma_j^\nu.
\]
(5.8)

The proof of Proposition 5.1 is postponed to the end of this section. (5.8) and (5.7) yield:

\[
\| \mathcal{E}_\nu^j f \|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j^\nu,
\]

which is the wanted estimate (5.1). This concludes the proof of Proposition 3.3.

**Remark 5.2** In order to control the diagonal term, it suffices to have a bound of the \( L^2(\mathcal{M}) \) norm for the left-hand side of (5.8). The improvement to a bound for the \( L^2_{u, x, \nu} L^\infty_{t, \nu} \) norm will be crucial when proving the almost orthogonality in angle.
We still need to prove Proposition 5.1. Note that it suffices to show:

$$\left\| L_\nu \left( \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(M)} \lesssim \gamma_j^\nu.$$  \hspace{1cm} (5.9)

Now, since the space-time gradient of $u$ is given by $b^{-1}L$, we have:

$$L_\nu \left( \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) = \int_{S^2} b^{-1} g(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega,$$  \hspace{1cm} (5.10)

where $F_j^1$ is given by:

$$F_j^1(u) = \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j} \lambda) f(\lambda \omega) \lambda^3 d\lambda.$$

We have:

$$g(L(t, x, \omega), L(t, x, \nu)) = g(N(t, x, \omega) - N(t, x, \nu), N(t, x, \omega) - N(t, x, \nu)).$$  \hspace{1cm} (5.11)

Thus, the estimate (2.42) for $\partial_\omega N$ and the size of the patch yields:

$$\| g(L(t, x, \omega), L(t, x, \nu)) \|_{L^\infty(H_u)} \lesssim 2^{-j},$$  \hspace{1cm} (5.12)

which implies:

$$\left\| \int_{S^2} (b^{-1}(t, x, \omega) - b^{-1}(t, x, \nu)) g(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \int_{S^2} \| b^{-1}(t, x, \omega) - b^{-1}(t, x, \nu) \|_{L^\infty L^2(H_u)} \| g(L(t, x, \omega), L(t, x, \nu)) \|_{L^\infty} \| F_j^1(u) \|_{L^2} \eta_j^\nu(\omega) d\omega \lesssim \varepsilon \gamma_j^\nu,$$  \hspace{1cm} (5.13)

where we used in the last inequality (5.12), the estimate (2.44) for $\partial_\omega b$, Plancherel in $\lambda$, Cauchy-Schwarz in $\omega$, and the size of the patch.

Now, in view of (5.10), we have:

$$L_\nu \left( \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) = b^{-1}(t, x, \nu) \int_{S^2} b^{-1} g(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega$$

$$\quad + \int_{S^2} (b^{-1}(t, x, \omega) - b^{-1}(t, x, \nu)) g(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega,$$

which together with (5.13) and the estimate (2.44) for $\partial_\omega b$ yields:

$$\left\| L_\nu \left( \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(M)} \lesssim \left\| \int_{S^2} g(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} + \varepsilon \gamma_j^\nu,$$  \hspace{1cm} (5.14)
Next, we estimate the right-hand side of (5.14). Using the decomposition (2.71), we have, taking into account (5.11):

$$g(L(t, x, \omega), L(t, x, \nu)) = (f_j^1 + f_j^2)(\omega - \nu)^2,$$  \hspace{1cm} (5.15)

where $f_j^1$ only depends on $\nu$ and satisfies:

$$\|f_j^1\|_{L^\infty} \lesssim 1,$$

and where $f_j^2$ satisfies:

$$\|f_j^2\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}},$$

where we took into account the size of the patch in the last inequality. Thus, we may rewrite the oscillatory integral in the right-hand side of (5.14) as:

$$\int_{S^2} g(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega = f_j^1(t, x, \nu) \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega$$

$$+ \int_{S^2} f_j^2(t, x, \omega, \nu)(\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega,$$

which yields:

$$\left\| \int_{S^2} g(L(t, x, \omega), L(t, x, \nu)) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \|f_j^1\|_{L^\infty} \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \|f_j^2\|_{L^\infty L^2(\mathcal{H}_u)} |\omega - \nu|^2 \|F_j^1(u)\|_{L^2} \eta_j^\nu(\omega) d\omega$$

$$\lesssim \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \gamma_j^\nu,$$

where we used in the last inequality the estimates for $f_j^1$ and $f_j^2$, Plancherel in $\lambda$, Cauchy-Schwarz in $\omega$, and the size of the patch. Together with (5.14), this implies:

$$\left\| L_\nu \left( \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \lesssim \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \gamma_j^\nu \hspace{1cm} (5.16)$$

Finally, we need to estimate the first term in the right-hand side of (5.14). We will rely on the energy estimate for the wave equation\(^9\). Recall from (2.28) that:

$$\Box_g u = b^{-1} \text{tr} \chi.$$  \hspace{1cm} (5.17)

Thus, we have:

$$\Box_g \left( \int_{S^2} (\omega - \nu)^2 F_j(u) \eta_j^\nu(\omega) d\omega \right) = \int_{S^2} b^{-1}(t, x, \omega) \text{tr} \chi(t, x, \omega)(\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega.$$
Arguing as in (5.4)-(5.7), we have:

\[
\left\| \int_{S^2} b^{-1}(t, x, \omega) \text{tr} \chi(t, x, \omega)(\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} 
\lesssim \varepsilon \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} + \varepsilon \gamma_j^\nu,
\]

which together with (5.17) implies:

\[
\left\| \Box_g \left( \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(M)} \lesssim \varepsilon \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} + \varepsilon \gamma_j^\nu.
\]

Let us now define the scalar function \( \phi \) on \( M \) as:

\[
\phi(t, x) = \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega.
\]

Then, \( \phi \) satisfies the following wave equation on \( M \):

\[
\begin{aligned}
\Box_g \phi &= F, \\
\phi|_{\Sigma_0} &= \phi_0, \quad \partial_0(\phi)|_{\Sigma_0} = \phi_1,
\end{aligned}
\]

where in view of (5.18), \( F \) satisfies:

\[
\| F \|_{L^2(M)} \lesssim \varepsilon \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} + \varepsilon \gamma_j^\nu.
\]

Note also that \( \phi_0 \) and \( \phi_1 \) correspond to the initial data of the half wave parametrix \( \phi \). The corresponding control is the subject of step C2 and has been obtained in [16]:

\[
\| \nabla \phi_0 \|_{L^2(\Sigma_0)} + \| \phi_1 \|_{L^2(\Sigma_0)} \lesssim \gamma_j^\nu.
\]

Next, we recall how to derive the energy estimate for the wave equation (5.20). Recall that \( T \), the future unit normal to the \( \Sigma_t \) foliation. Let \( \pi \) be the deformation tensor of \( T \), that is the symmetric 2-tensor on \( M \) defined as:

\[
\pi_{ij} = -2k_{ij}, \quad \pi_{iT} = \pi_{Ti} = n^{-1} \nabla_i n, \quad \pi_{TT} = 0.
\]

We also introduce the energy momentum tensor \( Q_{\alpha\beta} \) on \( M \) given by:

\[
Q_{\alpha\beta} = Q_{\alpha\beta}[\phi] = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi).
\]

We have the following energy estimate for the scalar wave equation:
Lemma 5.3 Let $F$ a scalar function on $\mathcal{M}$, and let $\phi_0$ and $\phi_1$ two scalar functions on $\Sigma_0$. Let $\phi$ the solution of the wave equation (5.20). Then, $\phi$ satisfies the following energy estimate:

$$
\|D\phi\|_{L^\infty_t L^2_x(\Sigma_t)} \lesssim \|\nabla \phi_0\|_{L^2_x(\Sigma_0)} + \|\phi_1\|_{L^2_x(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})} + \left| \int_M Q_{\alpha\beta} \pi^{\alpha\beta} dM \right|^{\frac{1}{2}},
$$

where $Q_{\alpha\beta}$ is the energy momentum tensor of $\phi$, and where $\pi$ is the deformation tensor of $T$.

**Proof** In view of the equation (5.20) satisfied by $\phi$, we have:

$$
D^\alpha Q_{\alpha\beta} = F \partial_\beta \phi.
$$

Now, we form the 1-tensor $P$:

$$
P_\alpha = Q_{\alpha 0},
$$

and we obtain:

$$
D^\alpha P_\alpha = D^\alpha Q_{\alpha 0} + Q_{\alpha\beta} D^\beta = F \partial_\beta \phi + \frac{1}{2} Q_{\alpha\beta} \pi^{\alpha\beta},
$$

where $\pi$ is the deformation tensor of $e_0$. Integrating over the region $0 \leq t \leq 1$, we obtain:

$$
\|D\phi\|_{L^\infty_t L^2_x(\Sigma_t)}^2 \lesssim \|\nabla \phi_0\|_{L^2_x(\Sigma_0)}^2 + \|\phi_1\|_{L^2_x(\Sigma_0)}^2 + \left| \int_M F T(\phi) dM \right| + \left| \int_M Q_{\alpha\beta} \pi^{\alpha\beta} dM \right|,
$$

which concludes the proof of the lemma. 

We are now in position to estimate $\phi$ given by (5.19). In view of the estimate (5.21) for $F$ and (5.22) for $\phi_0, \phi_1$, Lemma 5.3 implies:

$$
\|D\phi\|_{L^\infty_t L^2_x(\Sigma_t)} \lesssim \varepsilon \left( \int_{\mathbb{S}^2} (\omega - \nu)^2 F_j^1 (u) n_j (\omega) d\omega \right)_{L^2(\mathcal{M})} + \left| \int_M Q_{\alpha\beta} \pi^{\alpha\beta} dM \right|^{\frac{1}{2}} + \gamma^\nu. \quad (5.25)
$$

Then, note from the decomposition of $\pi$ (5.23) and the maximal foliation assumption (2.2) that:

$$
g_{\alpha\beta} \pi^{\alpha\beta} = 0
$$

which together with the definition of the energy momentum tensor $Q$ yields:

$$
Q_{\alpha\beta} \pi^{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi \pi^{\alpha\beta}.
$$
Together with the definition of $\phi$ (5.19), we obtain:

\[
\int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} dM = \pi^{\alpha\beta} \left( \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) L_\alpha(t, x, \omega) F_j^1(u) \eta_j' (\omega) d\omega \right) (5.26)
\]

\[
\times \left( \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) L_\beta(t, x, \omega) F_j^1(u) \eta_j' (\omega) d\omega \right)
\]

\[
= 2n^{-1} \nabla_i n \left( \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) F_j^1(u) \eta_j' (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_i(t, x, \omega) F_j^1(u) \eta_j' (\omega) d\omega \right)
\]

\[
- 2k_{ij} \left( \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_j(t, x, \omega) F_j^1(u) \eta_j' (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_j(t, x, \omega) F_j^1(u) \eta_j' (\omega) d\omega \right),
\]

where we used in the last equality the decomposition of $\pi$ (5.23) and the fact that $g(T, L) = -1$ and $L_i = N_i$. Now, we have:

\[
\left\| k_{ij} \left( \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_i(t, x, \omega) - b^{-1}(t, x, \nu) N_i(t, x, \nu) \right) F_j^1(u) \eta_j' (\omega) d\omega \right\|_{L^2(\mathcal{M})}
\]

\[
\lesssim \int_{S^2} (\omega - \nu)^2 \| b^{-1}(t, x, \omega) N_i(t, x, \omega) - b^{-1}(t, x, \nu) N_i(t, x, \nu) \|_{L^\infty} \| k \|_{L^\infty_x L^2(\mathcal{H}_\omega)} \| F_j^1(u) \|_{L^2_x} \eta_j' (\omega) d\omega
\]

\[
\lesssim \varepsilon \gamma_j^\nu,
\]

where we used in the last inequality the estimates (2.37) for $k$, (2.38) for $b$, (2.44) for $\partial_\nu b$ and (2.42) for $\partial_\nu N$, Plancherel in $\lambda$, Cauchy-Schwarz in $\omega$, and the size of the patch. Treating the other terms in the right-hand side of (5.26) similarly, we obtain:

\[
\int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} dM = 2n^{-1}(t, x) \nabla N_i n(t, x) b^{-2}(t, x, \nu) \left( \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j' (\omega) d\omega \right)^2
\]

\[
- 2\delta(t, x, \nu) b^{-2}(t, x, \nu) \left( \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j' (\omega) d\omega \right)^2 + O(\varepsilon \gamma_j^\nu).
\]

which yields:

\[
\left| \int_{\mathcal{M}} Q_{\alpha\beta} \pi^{\alpha\beta} dM \right| \lesssim \left| n^{-1} \nabla N_i n b^{-2} \right|_{L^\infty} \left| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j' (\omega) d\omega \right|_{L^2(\mathcal{M})}^2
\]

\[
+ \left| \delta \right|_{L^\infty_x L^2_\nu} b^{-2} \left| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j' (\omega) d\omega \right|_{L^2_x L^\infty_\nu}
\]

\[
\times \left| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j' (\omega) d\omega \right|_{L^2(\mathcal{M})}
\]

\[
\lesssim \varepsilon \left| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j' (\omega) d\omega \right|_{L^2(\mathcal{M})}^2 + \varepsilon \gamma_j^\nu
\]

\[
+ \varepsilon \left| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j' (\omega) d\omega \right|_{L^2_x L^\infty_\nu}^2 + \varepsilon \gamma_j^\nu,
\]

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where we used in the last inequality the estimates (2.37) for \( \delta \), (2.36) for \( n \), and (2.38) for \( b \). Together with (5.25), this yields:

\[
\|D\phi\|_{L^2(\Sigma_t)} \lesssim \varepsilon \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v}L^\infty_t} + \gamma_j^\nu. \quad (5.27)
\]

In view of the definition of \( \phi \) (5.19), we have:

\[
D\phi(t, x) = \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega)L(t, x, \omega) F_j^1(u) \eta_j^\nu(\omega) d\omega.
\]

Also:

\[
\left\| \int_{S^2} (\omega - \nu)^2 b^{-1}(t, x, \omega) N_i(t, x, \omega) - b^{-1}(t, x, \nu) N_i(t, x, \nu) F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \gamma_j^\nu,
\]

where we used in the last inequality the estimates (2.38) for \( b \), (2.44) for \( \partial, b \) and (2.42) for \( \partial_L = \partial_\omega N \), Plancherel in \( \lambda \), Cauchy-Schwarz in \( \omega \), and the size of the patch. Thus, we obtain:

\[
\left\| D\phi(t, x) - b^{-1}(t, x, \nu)L(t, x, \nu) \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \gamma_j^\nu,
\]

which together with the estimate (2.38) for \( b \) implies:

\[
\left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \|D\phi\|_{L^2(\mathcal{M})} + \gamma_j^\nu.
\]

Together with (5.27), we obtain:

\[
\left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v}L^\infty_t} + \gamma_j^\nu.
\]

Together with (5.16), this implies:

\[
\left\| L_\nu \left( \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v}L^\infty_t} + \gamma_j^\nu,
\]

and thus:

\[
\left\| \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v}L^\infty_t} \leq C\varepsilon \left\| \int_{S^2} (\omega - \nu)^2 F_j^1(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v}L^\infty_t} + C\gamma_j^\nu,
\]

for some universal constant \( C > 0 \). Iterating, we obtain for any \( q \geq 0 \):

\[
\left\| \int_{S^2} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v}L^\infty_t} \leq C^q \varepsilon^q \left\| \int_{S^2} (\omega - \nu)^2 F_j^q(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v}L^\infty_t} + C \left( \sum_{l=0}^{q-1} C^l \varepsilon^l \right)^{\gamma_j^\nu},
\]

(5.28)
where $F^q_j(u)$ is defined as:

$$F^q_j(u) = \int_0^{+\infty} e^{i\lambda u} \psi(2^{-j}\lambda) f(\lambda \omega) \lambda^{2q} d\lambda.$$ 

We have:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F^q_j(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2_{\nu, x\nu} L^\infty_{\nu, \lambda}} \lesssim L_\nu \left( \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F^q_j(u) \eta^\nu_j(\omega) d\omega \right)_{L^2(M)},$$

which together with the analog of (5.16) yields:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F^q_j(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2_{\nu, x\nu} L^\infty_{\nu, \lambda}} \lesssim \left( \int_{\mathbb{S}^2} (\omega - \nu)^{2(q+1)} F^{q+1}_j(u) \eta^\nu_j(\omega) d\omega \right)_{L^2(M)} + \gamma^\nu_j,$$

This implies the non sharp estimate:

$$\left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F^q_j(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2_{\nu, x\nu} L^\infty_{\nu, \lambda}} \lesssim 2^{\frac{j}{2}} \gamma^\nu_j,$$

where we used Plancherel in $\lambda$, Cauchy-Schwarz in $\omega$, and the size of the patch. Thus, we have:

$$C^q \varepsilon^q \left\| \int_{\mathbb{S}^2} (\omega - \nu)^{2q} F^q_j(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2_{\nu, x\nu} L^\infty_{\nu, \lambda}} \lesssim C^q \varepsilon^q 2^{\frac{j}{2}} \gamma^\nu_j \to 0 \text{ as } q \to +\infty,$$

(5.29)

since $\varepsilon > 0$ is small and may be chosen to ensure $0 < C\varepsilon < 1$. Finally, letting $q \to +\infty$ in (5.28) and taking (5.29) into account yields:

$$\left\| \int_{\mathbb{S}^2} F_j(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2_{\nu, x\nu} L^\infty_{\nu, \lambda}} \lesssim \gamma^\nu_j.$$

This concludes the proof of Proposition 5.1.

6 Proof of Proposition 3.2 (almost orthogonality in angle)

We have to prove (3.14):

$$\| E_j f \|_{L^2(M)}^2 \lesssim \sum_{\nu \in \Gamma} \| E_j^\nu f \|_{L^2(M)}^2 + \varepsilon^2 \gamma^2_j.$$

(6.1)

This will result from an estimate for:

$$\left| \int_{\mathcal{M}} E_j^\nu f(t, x) \overline{E_j^\nu f(t, x)} d\mathcal{M} \right|.$$

(6.2)

Remark 6.1 In [13], the authors rely on a partial Fourier transform with respect to a coordinate system on $P_{t, u}$ to prove almost orthogonality in angle for their parametrix. In our case, coordinate systems on $P_{t, u}$ are not regular enough, which forces us to work invariantly. More precisely, we will use geometric integrations by parts tied to the $u$-foliation on $\mathcal{M}$ in order to estimate (6.2).

Let us first explain why proceeding directly by integration by parts in (6.2) results in a log-loss.
6.1 Presence of a log-loss

Let us first introduce integrations by parts with respect to tangential derivatives. By definition of $\nabla$, we have $\nabla h = \nabla h - (\nabla_N h)N$ for any function $h$ on $\sigma$. In particular, we have $\nabla(u) = 0$ and $\nabla(u') = \nu^{\prime -1}N^{\prime} - \nu^{\prime -1}g(N',N)N$. Now, since $|N' - g(N',N)N|^2 = 1 - g(N',N)^2$, this yields:
\[
e^{i\lambda u - i\lambda u'} = \frac{ib'}{\lambda(1 - g(N',N)^2)} \nabla_{N' - g(N,N')N}(e^{i\lambda u - i\lambda u'}),
\]
where we have used the fact that $N' - g(N,N')N$ is a tangent vector with respect of the level surfaces of $u$. Similarly, we have:
\[
e^{i\lambda u - i\lambda u'} = -\frac{ib}{\lambda(1 - g(N,N')^2)} \nabla_{N - g(N,N')N'}(e^{i\lambda u - i\lambda u'}),
\]
where we have used the fact that $N - g(N,N')N'$ is a tangent vector with respect of the level surfaces of $u'$.

Next, we also introduce integrations by parts with respect to $L$. Since $L(u) = 0$ and $L(u') = \nu^{\prime -1}g(L,L')$, we have:
\[
e^{i\lambda u - i\lambda u'} = \frac{ib'}{\lambda g(L,L')} L(e^{i\lambda u - i\lambda u'}). \tag{6.5}
\]
Similarly, we have:
\[
e^{i\lambda u - i\lambda u'} = \frac{ib}{\lambda g(L,L')} L'(e^{i\lambda u - i\lambda u'}). \tag{6.6}
\]

We have:
\[
\int_{M} E_j^{\nu} f(t,x) \overline{E_j^{\nu}} f(t,x) d\mathcal{M} = \int_{S^2 \times S^2} \int_0^{+\infty} \int_0^{+\infty} \left( \int_{M} e^{i\lambda u - i\lambda u'} b^{-1} \text{tr} \chi b^{-1} \text{tr} \chi' d\mathcal{M} \right)
\times \eta_j^{\nu}(\omega) \eta_j^{\nu'}(\omega') \psi(2^{-j}\lambda) \psi(2^{-j}\lambda') f(\lambda) f(\lambda') \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega'.
\]
We integrate by parts tangentially using (6.3). Since $\lambda' \sim 2^j$, and
\[
1 - g(N,N') = \frac{g(N - N',N - N')}{2} \sim |\omega - \omega'|^2 \sim |\nu - \nu'|^2 \tag{6.7}
\]
in view of (2.43), we see that integrating by parts using (6.3) gains roughly $2^j |\nu - \nu'|$ at the expense of a tangential derivative. Consider the term where the tangential derivative falls on $\text{tr} \chi$, which is roughly of the form:
\[
\frac{1}{2^j|\nu - \nu'|} \int_{S^2 \times S^2} \int_0^{+\infty} \int_0^{+\infty} \left( \int_{M} e^{i\lambda u - i\lambda u'} b^{-1} \nabla \text{tr} \chi b^{-1} \text{tr} \chi' d\mathcal{M} \right)
\times \eta_j^{\nu}(\omega) \eta_j^{\nu'}(\omega') \psi(2^{-j}\lambda) \psi(2^{-j}\lambda') f(\lambda) f(\lambda') \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega'.
\]
Since $L\nabla \text{tr} \chi$ is the only derivative of $\nabla \text{tr} \chi$ for which we have an estimate, our next integration by parts must be with respect to $L$, that is we use (6.5). Since $\lambda' \sim 2^j$, and since
\[
g(L,L') = -1 + g(N,N') \sim |\nu - \nu'|^2, \tag{6.8}
\]
in view of (6.7), we see that integrating by parts using (6.5) gains roughly $2^j |\nu - \nu'|^2$ at the expense of an $L$ derivative. Consider the term where the $L$ derivative falls on $\operatorname{tr} \chi'$, which is roughly of the form:

$$\frac{1}{2^{2j} |\nu - \nu'|^2} \int_{S^2} \int_{S^2} \int_{0}^{+\infty} \int_{0}^{+\infty} \left( \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b^{-1} \nabla \operatorname{tr} \chi b^{-1} L(\operatorname{tr} \chi') d\mathcal{M} \right) \times \eta_j'(\omega) \eta_j''(\omega') d\omega d\omega'.$$

Now, note in view of (6.8) and the estimate (2.42) for $\partial_{u} N$, that:

$$g(L, L') \sim |\nu - \nu'|^2, \quad g(L, e'_A) = g(L - L', e'_A) \sim |\nu - \nu'|$$

and

$$g(L, L') = -2 + g(L, L') \sim 1.$$

Thus, decomposing $L$ on the frame $L', L', e'_A$, we obtain:

$$L \sim L' + |\nu - \nu'| |\nabla'\chi' + |\nu - \nu'|^2 L'.$$

We finally consider the term $|\nu - \nu'| |\nabla\operatorname{tr} \chi'$ in the expansion of $L(\operatorname{tr} \chi')$, and we obtain a term which is roughly of the form:

$$\frac{1}{2^{2j} |\nu - \nu'|^2} \int_{\mathcal{M}} \left( \int_{S^2} b^{-1} \nabla \operatorname{tr} \chi F_j(u) \eta_j'(\omega) d\omega \right) \cdot \left( \int_{S^2} b^{-1} \nabla \operatorname{tr} \chi F_j(u') \eta_j'(\omega') d\omega' \right) d\mathcal{M}.$$

We claim that such a term leads to a log-loss. Indeed, we have:

$$\lesssim \frac{1}{2^{2j} |\nu - \nu'|^2} \int_{\mathcal{M}} \left( \int_{S^2} b^{-1} \nabla \operatorname{tr} \chi F_j(u) \eta_j'(\omega) d\omega \right) \cdot \left( \int_{S^2} b^{-1} \nabla \operatorname{tr} \chi F_j(u') \eta_j'(\omega') d\omega' \right) d\mathcal{M} \lesssim$$

$$\lesssim \frac{1}{2^{2j} |\nu - \nu'|^2} \left( \int_{S^2} \|b^{-1} \nabla \operatorname{tr} \chi F_j(u)\|_{L^2(\mathcal{M})} \eta_j'(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} \|b^{-1} \nabla \operatorname{tr} \chi F_j(u')\|_{L^2(\mathcal{M})} \eta_j'(\omega') d\omega' \right)$$

$$\lesssim \frac{1}{2^{2j} |\nu - \nu'|^2} \left( \int_{S^2} \|b^{-1} \nabla \operatorname{tr} \chi\|_{L^\infty L^2(\mathcal{M})} \|F_j(u)\|_{L^2} \eta_j'(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} \|b^{-1} \nabla \operatorname{tr} \chi\|_{L^\infty L^2(\mathcal{M})} \|F_j(u')\|_{L^2} \eta_j'(\omega') d\omega' \right)$$

$$\lesssim \frac{\varepsilon^2 \gamma_j \gamma_j'}{(2^{j/2} |\nu - \nu'|^2)^2}, \quad (6.10)$$

where we used in the last inequality Plancherel in $\lambda$ and $\lambda'$, Cauchy-Schwartz in $\omega$ and $\omega'$ which gains the square root of the volume of the patch, the estimates (2.38) for $b$, and the estimates (2.39) for $\operatorname{tr} \chi$. This corresponds to a log-loss since we have$^{10}$:

$$\sup_{\nu, \nu'/1 \leq 2^{j/2} |\nu - \nu'| \leq 2^{2j/2}} \frac{1}{(2^{j/2} |\nu - \nu'|)^2} \sim j. \quad (6.11)$$

$^{10}$The log divergence in (6.11) is due to the fact that we are working at the level of $H^2$ solutions for Einstein equations. Indeed, summations similar to (6.11) appear in particular in [13] in the context of $H^{2+\epsilon}$ solutions of quasilinear wave equations, albeit with a power strictly larger than 2, and hence without log divergence.
Indeed, note that $\nu'$ runs on a lattice on $S^2$ of basic size $2^{-j/2}$ so that (6.11) corresponds to the sum

$$\sum_{l \in \mathbb{Z}^2, 1 \leq \|l\| \leq 2^{j/2}} \frac{1}{\|l\|^2} \sim j.$$ 

### 6.2 A physical space decomposition for $E^\nu_j f$

To remove the log-loss exhibited in (6.10) (6.11), we need a further decomposition. The same problem was present when dealing with the parametrix at initial time in [16]. In that case, we introduced a second decomposition in $\lambda$ and exploited the corresponding gain of the size of the patch in $\lambda$ when estimating $F_j(u)$ in $L^\infty$ and taking Cauchy-Schwartz in $\lambda$. In turn, we always estimate $F_j(u)$ in $L^2$ using Plancherel and can therefore not exploit the size of the patch in $\lambda$.

Instead, we rely here on a decomposition of $\text{tr} \chi$ using the geometric Littlewood-Paley projections $P_j$. We have:

$$\text{tr} \chi = P_{\leq j/2} (\text{tr} \chi) + \sum_{l > j/2} P_l \text{tr} \chi$$

which in turn yields the following decomposition for $E^\nu_j f$:

$$E^\nu_j f(t, x) = \sum_{l \geq j/2} E^\nu_{j,l} f(t, x),$$

where:

$$E^\nu_{j,l} f(t, x) = \int_{S^2} b(t, x, \omega)^{-1} P_l \text{tr} \chi(t, x, \omega) F_j(u) \eta^\nu_j(\omega) d\omega \quad \forall l > \frac{j}{2}$$

and:

$$E^\nu_{j,j/2} f(t, x) = \int_{S^2} b(t, x, \omega)^{-1} P_{\leq j/2} \text{tr} \chi(t, x, \omega) F_j(u) \eta^\nu_j(\omega) d\omega.$$ 

### 6.3 The mechanism to remove the log-loss

In order to prove almost orthogonality in angle, i.e. (6.1), we will estimate:

$$\left| \sum_{l,m} \int_{\mathcal{M}} E^\nu_{j,l} f(t, x) \overline{E^\nu_{j,m} f(t, x)} d\mathcal{M} \right|. $$

Let us assume for convenience that $m \leq l$ in (6.15). In order to remove the log-loss, our goal will be to always put more tangential derivatives on the lowest frequency, i.e. $P_m \text{tr} \chi'$ (as opposed to the higher frequency $P_l \text{tr} \chi$). This will be achieved as follows:

1. Integrate by parts with respect to $L$ using (6.5).

2. One term corresponds to the case where the $L$ derivative falls on the largest frequency $P_l \text{tr} \chi$, while the other term corresponds to the case where $L$ falls on the lowest frequency $P_m \text{tr} \chi'$. For the second term, decompose the $L$ derivative on the frame $L', N', e'_A$ as in (6.9).
3. We claim the terms involving $L$ and $L'$ do not contain any log-loss. Indeed, instead of the sum (6.11) containing the log-loss, they will ultimately yield

$$\sup_{\nu} \sum_{\nu'/1 \leq 2^{j/2} \lvert \nu - \nu' \rvert \leq 2^{j/2}} \frac{1}{(2^{j/2} \lvert \nu - \nu' \rvert)^3} \leq 1.$$ 

4. We claim the term involving $N'$ does not contain any log-loss. Indeed, instead of the sum (6.11) containing the log-loss, it will ultimately yield

$$\sup_{\nu} \sum_{\nu'/1 \leq 2^{j/2} \lvert \nu - \nu' \rvert \leq 2^{j/2}} \frac{1}{2^{j/2} (2^{j/2} \lvert \nu - \nu' \rvert)} \leq 1.$$ 

5. Finally, the last term is the one containing the $\nabla'$ derivative. This term is the only one which contains the log-loss exhibited in (6.11). Now, we have achieved our goal since after integration by parts, the tangential derivative fell on $P_m \text{tr} \chi'$ which is the lowest frequency.

**Remark 6.2** Due to the decomposition (6.12), we now not only need to obtain summability in $(\nu, \nu')$, but also in $(l, m)$.

### 6.4 The main estimates

Recall that in order to prove almost orthogonality in angle, i.e. (6.1), we will estimate:

$$\left| \sum_{l,m} \int_{\mathcal{M}} E_{j}^{\nu,l} f(t,x) \overline{E_{j}^{\nu',m} f(t,x)} dM \right|.$$ 

We will distinguish the following two regions:

$$2^{\min(l,m)} > 2^j \lvert \nu - \nu' \rvert \quad \text{and} \quad 2^{\min(l,m)} \leq 2^j \lvert \nu - \nu' \rvert.$$ 

We start with the estimate in the first region.

**Proposition 6.3** If $\nu \neq \nu'$ and $2^{\min(l,m)} > 2^j \lvert \nu - \nu' \rvert$, we have the following estimate:

$$\left| \int_{\mathcal{M}} E_{j}^{\nu,l} f(t,x) \overline{E_{j}^{\nu',m} f(t,x)} dM \right| \lesssim 2^{-j} \lVert \mu_{j,\nu,l} \rVert_{L^2(\mathbb{R} \times S^2)} \lVert \mu_{j,\nu',m} \rVert_{L^2(\mathbb{R} \times S^2)}, \quad (6.16)$$

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times S^2$ satisfies:

$$\sum_{\nu} \sum_{l>j/2} 2^{2j} \lVert \mu_{j,\nu,l} \rVert_{L^2(\mathbb{R} \times S^2)}^2 \lesssim \varepsilon 2^{2j} \lVert f \rVert_{L^1(\mathbb{R}^3)}^2.$$
\textbf{Proof} We have:

\[
\left| \int_{\mathcal{M}} E_{j}^{\nu,l} f(t,x) \overline{E_{j}^{\nu,m} f(t,x)} d\mathcal{M} \right| \leq \| E_{j}^{\nu,l} f \|_{L^2(\mathcal{M})} \| E_{j}^{\nu,m} f \|_{L^2(\mathcal{M})}
\]

\[
\lesssim \left( \int_{\mathcal{M}} \| b(t,x,\omega) \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{\mathcal{M}} \| b(t,x,\omega') \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega') d\omega' \right)
\]

\[
\lesssim b^{-1} \| P \|_{L^\infty} \left( \int_{\mathcal{M}} \| P \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{\mathcal{M}} \| P \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega') d\omega' \right)
\]

\[
\lesssim 2^{-j} \left( \int_{\mathcal{M}} \| P \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{\mathcal{M}} \| P \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega') d\omega' \right)
\]

where we used in the last inequality the estimates (2.38) for \( b \), Cauchy Schwarz in \( \omega \) and \( \omega' \), and the size of the patch.

Now, we have:

\[
\sum_{\nu} \sum_{l>j/2} 2^{2l} \left( \| P \|_{L^2(\mathcal{M})} \eta_j^\nu(\omega) \right)^2 \lesssim \sum_{\nu} \int_{\mathcal{M}} \left( \int_{\mathcal{M}} 2^{2l} \| P \|_{L^2(\mathcal{M})}^{2} \right) |F_j(u)|^2 d\omega \eta_j^\nu(\omega) d\omega
\]

\[
\lesssim \sum_{\nu} \int_{\mathcal{M}} \left( \int_{\mathcal{M}} \| \nabla \chi \|^2_{L^2(\mathcal{M})} |F_j(u)|^2 d\omega \right) \eta_j^\nu(\omega) d\omega
\]

\[
\lesssim \sum_{\nu} \int_{\mathcal{M}} \| \nabla \chi \|^2_{L^2(\mathcal{M})} |F_j(u)|^2 \eta_j^\nu(\omega) d\omega
\]

\[
\lesssim \varepsilon^2 2^{2j} \sum_{\nu} (\gamma_j^\nu)^2 \lesssim 2^{2j} \varepsilon^2 \| f \|_{L^2(\mathbb{R}^3)}^2,
\]

where we used the finite band property for \( P \), the estimates (2.39) for \( \text{tr} \chi \) and Plancherel in \( \lambda \). (6.17) and (6.18) yield the proof of the proposition. \( \square \)

\textbf{Remark 6.4} In (6.18), we used the estimate:

\[
\sup_{\omega,u} \left( \sum_{l>j/2} 2^{2l} \| P \|_{L^2(\mathcal{M})} \right) \lesssim \sup_{\omega,u} \| \nabla \chi \|^2_{L^2(\mathcal{M})} \lesssim \varepsilon^2,
\]

which is true in view of the finite band property for \( P \) and the estimates (2.39) for \( \text{tr} \chi \). Note that the sum in \( l \) has to be taken before the sup in \( \omega \) and \( u \) for the estimate to hold.
This explains why the \( L^2 \) norm on \( \mathbb{R} \times S^2 \) is present in (6.16) and estimated only after letting the sum in \( l \) enter the integral as in (6.18).

Next, we consider the second region. We have the following decomposition:

**Proposition 6.5** If \( \nu \neq \nu' \) and \( 2^{\min(l,m)} \leq 2^j|\nu - \nu'| \), we have the following decomposition:

\[
\int_{\mathcal{M}} E_j^{\nu,l} f(t,x) \overline{E_j^{\nu',l}} f(t,x) d\mathcal{M} = A_{j,\nu,\nu',l,m} + B_{j,\nu,\nu',l,m},
\]

where \( B_{j,\nu,\nu',l,m} \) satisfies:

\[
\left| \sum_{(l,m)\neq (\nu',l,m)} \left( B_{j,\nu,\nu',l,m} + B_{j,\nu,\nu',l,m} \right) \right| \leq \left[ \frac{1}{(2^\frac{1}{2}|\nu - \nu'|)^3} + \frac{1}{(2^\frac{1}{2}|\nu - \nu'|)^2} + \frac{1}{2^\frac{1}{2}|\nu - \nu'|} \right] \varepsilon^{2\gamma_j}.
\]

Next, we estimate \( A_{j,\nu,\nu',l,m} \) for \((l,m)\) such that \( 2^{\min(l,m)} \leq 2^j|\nu - \nu'| \). We consider the following two subregions:

\[
2^{\min(l,m)} \leq 2^j|\nu - \nu'| < 2^{\max(l,m)} \text{ and } 2^{\max(l,m)} \leq 2^j|\nu - \nu'|
\]

starting with the first one:

**Proposition 6.6** If \( \nu \neq \nu' \) and \( 2^{\min(l,m)} \leq 2^j|\nu - \nu'| < 2^{\max(l,m)} \), we have the following estimate:

\[
\left| \sum_{(l,m)\leq 2^j|\nu - \nu'|} A_{j,\nu,\nu',l,m} \right| \leq \left[ \frac{1}{(2^\frac{1}{2}|\nu - \nu'|)^3} + \frac{2^{-\frac{1}{4}}}{(2^\frac{1}{2}|\nu - \nu'|)^2} + \frac{1}{(2^\frac{1}{2}|\nu - \nu'|)^2} \right] \varepsilon^{2\gamma_j}.
\]

where the sequence of functions \( (\mu_{j,\nu,l})_{l>j/2} \) on \( \mathbb{R} \times S^2 \) satisfies:

\[
\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times S^2)}^2 \lesssim \varepsilon^{22^j} \|f\|_{L^2(\mathbb{R} \times S^2)}^2.
\]

Finally, we estimate \( A_{j,\nu,\nu',l,m} \) for \((l,m)\) such that \( 2^{\max(l,m)} \leq 2^j|\nu - \nu'| \).
Proposition 6.7 If \( \nu \neq \nu' \) and \( 2^{\max(l,m)} \leq 2^j|\nu - \nu'| \), we have the following estimate:

\[
\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^j|\nu - \nu'|} A_{j,\nu,\nu',l,m} \right| \leq 2^{\frac{5j}{2}} 2^{l+m+\min(l,m)} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times S^2)} \| \mu_{j,\nu',m} \|_{L^2(\mathbb{R} \times S^2)}
\]

\[
+ \left[ \frac{1}{(2^j|\nu - \nu'|)^3} + \frac{1}{(2^j|\nu - \nu'|)^\frac{5}{2}} + \frac{2 - (\frac{1}{2})^{-j}}{(2^j|\nu - \nu'|)^2} + \frac{1}{2^j(2^j|\nu - \nu'|)} + 2^{-j} \right] \varepsilon^{2j\gamma_{\nu,\nu'}}
\]

where the sequence of functions \((\mu_{j,\nu,l})_{l>j/2}\) on \( \mathbb{R} \times S^2 \) satisfies:

\[
\sum_{\nu} \sum_{l>j/2} 2^j \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times S^2)}^2 \lesssim \varepsilon^{2j} \| f \|_{L^2(\mathbb{R}^3)}^2.
\]

The proof of Proposition 6.5 is postponed to section 8, the proof of Proposition 6.6 is postponed to section 9, and the proof of Proposition 6.7 is postponed to section 10.

6.5 End of the proof of Proposition 3.2

We conclude the proof of Proposition 3.2 by using Proposition 6.3, Proposition 6.5, Proposition 6.6 and Proposition 6.7. In view of (6.12), we have:

\[
\sum_{\nu \neq \nu'} \left| \int_{M} E_{j}^{\nu} f(t, x) \overline{E_{j}^{\nu'}} f(t, x) dM \right|
\]

\[
\lesssim \sum_{\nu \neq \nu'} \left| \sum_{l,m} \int_{M} E_{j}^{\nu,l} f(t, x) \overline{E_{j}^{\nu',m}} f(t, x) dM \right|
\]

\[
\lesssim \sum_{\nu \neq \nu'} \sum_{2^{\min(l,m)} > 2^j|\nu - \nu'|} \left| \int_{M} E_{j}^{\nu,l} f(t, x) \overline{E_{j}^{\nu',m}} f(t, x) dM \right|
\]

\[
+ \sum_{\nu \neq \nu'} \left| \int_{M} E_{j}^{\nu,l} \lesssim j/2 f(t, x) \overline{E_{j}^{\nu',l} \lesssim j/2} f(t, x) dM \right|
\]

\[
+ \sum_{\nu \neq \nu'} \sum_{2^{\min(l,m)} \leq 2^j|\nu - \nu'|} \left| \int_{M} E_{j}^{\nu,l} f(t, x) \overline{E_{j}^{\nu',m}} f(t, x) dM \right|.
\]
In view of Proposition 6.3 and Proposition 6.5, we obtain:

\[
\sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_\nu^\nu f(t, x) \overline{E_{\nu'}^{\nu'} f(t, x)} d\mathcal{M} \right| \leq 2^{-j} \sum_{\nu \neq \nu'} 2^{-j \frac{1}{2}} \|\mu_{\nu, \nu'}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{\nu', m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} + \sum_{\nu \neq \nu'} \sum_{l, m \leq 2j |\nu - \nu'|} \int_{\mathcal{M}} A_{j, \nu, \nu', l, m} d\mathcal{M}
\]

\[
+ \sum_{\nu \neq \nu'} \sum_{l, m \leq 2j |\nu - \nu'|} \int_{\mathcal{M}} A_{j, \nu, \nu', l, m} d\mathcal{M}
\]

Now, we have:

\[
\left( \sum_{2^{4} |\nu - \nu'| < 2^{\text{min}(l, m)} - \frac{j}{2}} 1 \right)^{\frac{1}{2}} \lesssim 2^{\text{min}(l, m) - j},
\]

and:

\[
\sum_{\nu \neq \nu'} \left[ \frac{1}{(2^4 |\nu - \nu'|)^{3}} + \frac{1}{(2^4 |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^{4} (2^{4} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^{4} (2^{4} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^{4} (2^{4} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{j}{4}) - j}}{(2^4 |\nu - \nu'|)^{2}} \right] \lesssim 2^{2} \|f\|_{L^2(\mathbb{R}^3)} \leq 2^{j} \|f\|_{L^2(\mathbb{R}^3)}.
\]

Together with (6.23), we obtain:

\[
\sum_{\nu \neq \nu'} \left| \int_{\mathcal{M}} E_\nu^\nu f(t, x) \overline{E_{\nu'}^{\nu'} f(t, x)} d\mathcal{M} \right| \leq 2^{-2j} \sum_{l, m} 2^{-j l - m} \left( \sum_{\nu} 2^{2j} \|\mu_{\nu, l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \right)^{\frac{1}{2}} \left( \sum_{\nu'} 2^{2j} \|\mu_{\nu', m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \right)^{\frac{1}{2}}
\]

\[
+ \sum_{\nu \neq \nu'} \sum_{l, m \leq 2j |\nu - \nu'|} \int_{\mathcal{M}} A_{j, \nu, \nu', l, m} d\mathcal{M}
\]

\[
+ \sum_{\nu \neq \nu'} \sum_{l, m \leq 2j |\nu - \nu'|} \int_{\mathcal{M}} A_{j, \nu, \nu', l, m} d\mathcal{M}
\]

\[
+ \varepsilon^2 \|f\|_{L^2(\mathbb{R}^3)}.
\]
Together with Proposition 6.6 and Proposition 6.7, this yields:

\[
\sum_{\nu \neq \nu'} \left| \int_M E_\nu^* f(t, x) \overline{E_{\nu'}^* f(t, x)} d\mathcal{M} \right| \leq 2^{-2j} \left( \sum_{l, \nu} 2^{2j} ||\mu_{j, \nu, l}||_{L^2(\mathbb{R} \times S^2)}^2 \right)^{\frac{1}{2}} \left( \sum_{m, \nu'} 2^{2m} ||\mu_{j, \nu', m}||_{L^2(\mathbb{R} \times S^2)}^2 \right)^{\frac{1}{2}} \\
+ \sum_{\nu \neq \nu'} \sum_{2^{\min(l, m)} \leq 2^{j} |\nu - \nu'| < 2^{\max(l, m)}} \frac{2^{-2j} 2^{2 \min(l, m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} ||\mu_{j, \nu, l}||_{L^2(\mathbb{R} \times S^2)} ||\mu_{j, \nu', m}||_{L^2(\mathbb{R} \times S^2)} \\
+ \sum_{\nu \neq \nu'} \sum_{2^{\max(l, m)} \leq 2^{j} |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l + m + \min(l, m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} ||\mu_{j, \nu, l}||_{L^2(\mathbb{R} \times S^2)} ||\mu_{j, \nu', m}||_{L^2(\mathbb{R} \times S^2)} \\
+ \sum_{\nu \neq \nu'} \left[ \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-\left(\frac{5}{6}\right) - j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon \gamma_j \gamma_{\nu'} \\
+ \varepsilon^2 ||f||_{L^2(\mathbb{R}^3)}^2.
\]

Now, we have:

\[
\left( \sum_{2^{\min(l, m)} - \frac{j}{2} \leq 2^{j} |\nu - \nu'| < 2^{\max(l, m)} - \frac{j}{2}} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} \right)^{\frac{1}{2}} \leq \log(2^{\max(l, m) - \frac{j}{2}}) - \log(2^{\min(l, m) - \frac{j}{2}}) \\
\leq \max(l, m) - \min(m, l),
\]

and:

\[
\left( \sum_{2^{\frac{j}{2}} |\nu - \nu'| \geq 2^{\max(l, m)} - \frac{j}{2}} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} \right)^{\frac{1}{2}} \leq 2^{-\max(l, m) + \frac{j}{2}},
\]

and:

\[
\sum_{\nu \neq \nu'} \left[ \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{5}{2}}} + \frac{2^{-\left(\frac{5}{6}\right) - j}}{(2^{\frac{j}{2}} |\nu - \nu'|)^2} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} + 2^{-j} \right] \varepsilon \gamma_j \gamma_{\nu'} \\
\leq \varepsilon^2 ||f||_{L^2(\mathbb{R}^3)}^2.
\]
Together with (6.24), we obtain:

\[
\sum_{\nu \neq \nu'} \left| \int_\mathcal{M} E_j^\nu f(t, x) \overline{E_j^{\nu'}} f(t, x) d\mathcal{M} \right| \\
\lesssim 2^{-2j} \sum_{l,m} (1 + |l - m|) 2^{-l-m} \left( \sum_{\nu} 2^{2l} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left( \sum_{\nu'} 2^{2m} \| \mu_{j,\nu',m} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \\
+ 2^{-2j} \left( \sum_{l,\nu} 2^{2l} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left( \sum_{m,\nu'} 2^{2m} \| \mu_{j,\nu',m} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} + \varepsilon^2 \| f \|_{L^2(\mathbb{R}^3)}^2 \\
\lesssim 2^{-2j} \left( \sum_{l,\nu} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} \left( \sum_{m,\nu'} \| \mu_{j,\nu',m} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \right)^{\frac{1}{2}} + \varepsilon^2 \| f \|_{L^2(\mathbb{R}^3)}^2.
\]

Since the sequence of functions \((\mu_{j,\nu,l})_{t \geq j/2}\) on \(\mathbb{R} \times \mathbb{S}^2\) satisfies:

\[
\sum_{\nu} \sum_{l \geq j/2} 2^{2l} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \| f \|_{L^2(\mathbb{R}^3)}^2,
\]

we finally obtain:

\[
\sum_{\nu \neq \nu'} \left| \int_\mathcal{M} E_j^\nu f(t, x) \overline{E_j^{\nu'}} f(t, x) d\mathcal{M} \right| \lesssim \varepsilon^2 \| f \|_{L^2(\mathbb{R}^3)}^2.
\]

This concludes the proof of Proposition 3.2.

The rest of the paper is as follows. In section 7, we derive estimates for oscillatory integrals in various norms, as well as integrations by parts formulas tied to the \(u\)-foliation on \(\mathcal{M}\). In section 8, we prove Proposition 6.5. In section 9, we prove Proposition 6.6. Finally, we prove Proposition 6.7 in section 10.

7 The key estimates

7.1 Estimate of the \(L^p(\mathcal{M})\) norm of oscillatory integrals

Lemma 7.1 Let \(H\) a tensor on \(\mathcal{M}\). Then, we have the following estimate:

\[
\left\| \int_{\mathbb{S}^2} HF_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \| H \|_{L^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu. \tag{7.1}
\]

More generally, for \(2 \leq p \leq +\infty\), we have:

\[
\left\| \int_{\mathbb{S}^2} HF_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left( \sup_{\omega} \| H \|_{L^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu. \tag{7.2}
\]
**Proof** We have:

\[
\left\| \int_{\mathbb{S}^2} HF_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^p(\mathcal{M})} \lesssim \int_{\mathbb{S}^2} \|HF_j(u)\|_{L^p(\mathcal{M})} \eta_j^\nu(\omega) d\omega
\]

\[
\lesssim \int_{\mathbb{S}^2} \|H\|_{L^\infty L^p(\mathcal{H}_u)} \|F_j(u)\|_{L^\infty L^p(\mathcal{H}_u)} \eta_j^\nu(\omega) d\omega
\]

\[
\lesssim \left( \sup_{\omega} \|H\|_{L^\infty L^p(\mathcal{H}_u)} \right) \int_{\mathbb{S}^2} \|F_j(u)\|_{L^2} \|F_j(u)\|_{L^\infty}^{1-\frac{2}{p}} \eta_j^\nu(\omega) d\omega.
\]

Using Plancherel to estimate \(\|F_j(u)\|_{L^2}\), Cauchy-Schwarz in \(\lambda\) to estimate \(\|F_j(u)\|_{L^\infty}\), Cauchy-Schwarz in \(\omega\) and the size of the patch, we obtain:

\[
\left\| \int_{\mathbb{S}^2} HF_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left( \sup_{\omega} \|H\|_{L^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p}) \gamma_j^\nu}
\]

which concludes the proof of the lemma. \(\blacksquare\)

**Corollary 7.2** Let \(H\) a tensor on \(\mathcal{M}\). Then, we have the following estimate:

\[
\left\| \int_{\mathbb{S}^2} HP_m(\text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \|H\|_{L^\infty} \right) \varepsilon 2^{-m} 2^{\frac{1}{2}} \gamma_j^\nu.
\] \hspace{1cm} (7.3)

More generally, for \(2 \leq p \leq +\infty\), we have:

\[
\left\| \int_{\mathbb{S}^2} HP_m(\text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left( \sup_{\omega} \|H\|_{L^\infty} \right) \varepsilon 2^{-m} 2^{\frac{1}{p}} 2^{j(1-\frac{1}{p})} \gamma_j^\nu.
\] \hspace{1cm} (7.4)

**Proof** In view of (7.2), we have:

\[
\left\| \int_{\mathbb{S}^2} HP_m(\text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left( \sup_{\omega} \|HP_m(\text{tr} \chi)\|_{L^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu
\]

\[
\lesssim \left( \sup_{\omega} \|H\|_{L^\infty} \right) \left( \sup_{\omega} \|P_m(\text{tr} \chi)\|_{L^\infty L^p(\mathcal{H}_u)} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu.
\]

Using Bernstein on \(P_{t,u}\) and the finite band property for \(P_m\), we obtain:

\[
\left\| \int_{\mathbb{S}^2} HP_m(\text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left( \sup_{\omega} \|H\|_{L^\infty} \right) 2^{-m} \left( \sup_{\omega} \|\text{tr} \chi\|_{L^\infty L^2_t} \right) 2^{j(1-\frac{1}{p})} \gamma_j^\nu.
\]

Together with the estimates (2.39) for \(\text{tr} \chi\), we obtain:

\[
\left\| \int_{\mathbb{S}^2} HP_m(\text{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^p(\mathcal{M})} \lesssim \left( \sup_{\omega} \|H\|_{L^\infty} \right) \varepsilon 2^{-m} 2^{(1-\frac{1}{p})} \gamma_j^\nu
\]

which concludes the proof of the corollary. \(\blacksquare\)
7.2 Estimates of the $L^1(M)$ norm of oscillatory integrals

Lemma 7.3 Let $\nu, \nu'$ in $S^2$ such that $\nu \neq \nu'$. Recall the decomposition $\tilde{\chi} = \chi_1 + \chi_2$ in (2.45). Let $H$ a tensor on $M$. Then, we have the following estimate:

$$
\int_M \left| \int_{S^2} HD(L(\text{tr} \chi)) F_j(u) \eta^\nu_j(\omega) d\omega \right| dM \lesssim \left( \sup_{\omega \in \text{supp}(\eta^\nu_j)} \left( \|H\|_{L^2 L^4 L^2_s} + |\nu - \nu'| \|H\|_{L^3(M)} + \|\chi_2 \nu H\|_{L^2(M)} \right) \right)^{\frac{3}{2}} \varepsilon \gamma_j^\nu.
$$

Proof We have:

$$
\int_M \left| \int_{S^2} HD(L(\text{tr} \chi)) F_j(u) \eta^\nu_j(\omega) d\omega \right| dM \lesssim \int_{S^2} \|HD(L(\text{tr} \chi)) F_j(u)\|_{L^1(M)} \eta^\nu_j(\omega) d\omega \lesssim \int_{S^2} \|HD(L(\text{tr} \chi))\|_{L^4 L^2(M) H} \|F_j(u)\|_{L^2(M)} \eta^\nu_j(\omega) d\omega.
$$

Differentiating the Raychaudhuri equation (2.22), we obtain:

$$
D(L(\text{tr} \chi)) = -(\text{tr} \chi + \bar{x}) D\text{tr} \chi - 2\tilde{\chi} D\tilde{\chi} - D(\bar{x}) \text{tr} \chi.
$$

Using the decomposition $\tilde{\chi} = \chi_1 + \chi_2$ in (2.45), we obtain:

$$
D(L(\text{tr} \chi)) = G_1 + \chi_2 G_2
$$

where

$$
G_1 = -(\text{tr} \chi + \bar{x}) D\text{tr} \chi - 2\chi_1 D\tilde{\chi} - D(\bar{x}) \text{tr} \chi
$$

and

$$
G_2 = -2D\tilde{\chi}.
$$

In particular, we have:

$$
\|G_1\|_{L^\infty L^4 L^2_s} + \|G_2\|_{L^\infty L^2(M)} \lesssim \varepsilon
$$

where we used the estimate (2.39) for $\text{tr} \chi$, the estimate $\tilde{\chi}$ for $\tilde{\chi}$, the estimate (2.37) and (2.36) for $\bar{x}$, and the estimate (2.47) for $\chi_1$. In view of (7.6) and (7.7), we have:

$$
\int_M \left| \int_{S^2} HD(L(\text{tr} \chi)) F_j(u) \eta^\nu_j(\omega) d\omega \right| dM \lesssim \int_{S^2} \|HG_1\|_{L^2 L^1(H\omega)} \|F_j(u)\|_{L^2(M)} \eta^\nu_j(\omega) d\omega + \int_{S^2} \|\chi_2 HG_2\|_{L^2 L^1(H\omega)} \|F_j(u)\|_{L^2(M)} \eta^\nu_j(\omega) d\omega \lesssim \int_{S^2} \|H\|_{L^2 L^4 L^2_s} \|G_1\|_{L^\infty L^4 L^2_s} \|F_j(u)\|_{L^2(M)} \eta^\nu_j(\omega) d\omega + \int_{S^2} \|\chi_2 H\|_{L^2(M)} \|G_2\|_{L^\infty L^2(H\omega)} \|F_j(u)\|_{L^2(M)} \eta^\nu_j(\omega) d\omega \lesssim \sup_{\omega \in \text{supp}(\eta^\nu_j)} \left( \|H\|_{L^2 L^4 L^2_s} \|F_j(u)\|_{L^2(M)} \eta^\nu_j(\omega) d\omega + \varepsilon \right) \int_{S^2} \|\chi_2 H\|_{L^2(M)} \|F_j(u)\|_{L^2(M)} \eta^\nu_j(\omega) d\omega,
$$

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where we used in the last inequality the estimate (7.8). In view of the estimate (2.47) for $\chi_2$, we have:

$$\|\chi_2 - \chi_2\|_{L^6(M)} \lesssim |\nu - \nu'| \|\partial_\omega \chi_2\|_{L^6(M)} \lesssim |\nu - \nu'| \varepsilon,$$

which yields:

$$\|\chi_2 H\|_{L^2(M)} \lesssim \|\chi_2 H\|_{L^2(M)} + \|\chi_2 - \chi_2\|_{L^6(M)} \|H\|_{L^3(M)}$$

$$\lesssim \|\chi_2 H\|_{L^2(M)} + \varepsilon |\nu - \nu'| \|H\|_{L^3(M)}.$$

Together with (7.9), we obtain:

$$\left\| \int_M \int_{S^2} H \nabla L (\operatorname{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \left( \sup_{\omega \in \operatorname{supp}(\eta_j^\nu)} \left( \|H\|_{L^2_u L^2_t L^6_s} + |\nu - \nu'| \|H\|_{L^3(M)} + \|\chi_2 H\|_{L^2(M)} \right) \right)^2 \varepsilon \gamma_j^\nu,$$

where we used in the last inequality Plancherel in $\lambda$ for $\|F_j(u)\|_{L^2_x}$, Cauchy-Schwarz in $\omega$, and the size of the patch. This concludes the proof of the lemma.

**Lemma 7.4** Let $\nu, \nu'$ in $S^2$ such that $\nu \neq \nu'$. Let $l$ an integer. Recall the decomposition $\tilde{\chi} = \chi_1 + \chi_2$ in (2.45). Let $H$ a tensor on $M$. Then, we have the following estimate:

$$\int_M \int_{S^2} H L (P_t \operatorname{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \lesssim \left( \sup_{\omega \in \operatorname{supp}(\eta_j^\nu)} \|H\|_{L^2_u L^2_t L^6_s} \right)^2 \varepsilon \gamma_j^\nu. \quad (7.10)$$

**Proof** We have:

$$\int_M \left| \int_{S^2} H L (P_t \operatorname{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right| dM \lesssim \int_{S^2} \|H L (P_t \operatorname{tr} \chi) F_j(u)\|_{L^2(M)} \eta_j^\nu(\omega) d\omega$$

$$\lesssim \int_{S^2} \|H L (P_t \operatorname{tr} \chi)\|_{L^2_u L^1_t(H_u)} \|F_j(u)\|_{L^2_e \eta_j^\nu(\omega)} d\omega$$

$$\lesssim \int_{S^2} \|H n L (P_t \operatorname{tr} \chi)\|_{L^2_u L^1_t(H_u)} \|F_j(u)\|_{L^2 \eta_j^\nu(\omega)} d\omega,$$

where we used the estimate (2.36) on $n$ in the last inequality. This yields:

$$\int_M \left| \int_{S^2} H L (P_t \operatorname{tr} \chi) F_j(u) \eta_j^\nu(\omega) d\omega \right| dM \lesssim \left( \sup_{\omega \in \operatorname{supp}(\eta_j^\nu)} \|H\|_{L^2_u L^2_t L^6_s} \right)^2 \int_{S^2} \|n L (P_t \operatorname{tr} \chi)\|_{L^2_u L^1_t} \|F_j(u)\|_{L^2_\omega \eta_j^\nu(\omega)} d\omega. \quad (7.11)$$
Next, we estimate $nL(P_{\text{tr}}\chi)$. We have:

$$nL(P_{\text{tr}}\chi) = [nL, P_t](\text{tr}\chi) + P_t(nL(\text{tr}\chi)),$$

which yields:

$$\|nL(P_{\text{tr}}\chi)\|_{L^2_tL^1_x} \lesssim \|[nL, P_t](\text{tr}\chi)\|_{L^1_tL^2_x} + \|P_t(nL(\text{tr}\chi))\|_{L^2_tL^1_x} \quad (7.12)$$

where we used in the last inequality the commutator estimate (2.66) and the estimate (2.69). Now, (7.11) and (7.12) imply:

$$\int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} \overline{H}L(P_{\text{tr}}\chi)F_j(u)\eta_j^\nu(\omega)d\omega \right| d\mathcal{M} \lesssim \left( \sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L^2_{u,x}, L^\infty} \right)^{2^{-l}} \left( \int_{\mathbb{S}^2} \|F_j(u)\|_{L^2_{\delta^j}}\eta_j^\nu(\omega)d\omega \right) \lesssim \left( \sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L^2_{u,x}, L^\infty} \right)^{2^{-l}} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality Plancherel in $u$, Cauchy-Schwarz in $\omega$ and the size of the patch. This concludes the proof of the lemma.

Lemma 7.5 Let $\nu, \nu'$ in $\mathbb{S}^2$ such that $\nu \neq \nu'$. Let $l$ an integer. Recall the decomposition $\tilde{\chi} = \chi_1 + \chi_2$ in (2.45). Let $H$ a tensor on $\mathcal{M}$. Then, we have the following estimate:

$$\int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H\nabla(L(P_{\leq l}\text{tr}\chi))F_j(u)\eta_j^\nu(\omega)d\omega \right| d\mathcal{M} \lesssim \left( \sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L^2_{u,x}, L^\infty} \right) 2^{\frac{1}{2}} \varepsilon \gamma_j^\nu. \quad (7.13)$$

**Proof** We have:

$$\int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H\nabla(L(P_{\leq l}\text{tr}\chi))F_j(u)\eta_j^\nu(\omega)d\omega \right| d\mathcal{M} \lesssim \int_{\mathbb{S}^2} \|H\nabla(L(P_{\leq l}\text{tr}\chi))F_j(u)\|_{L^1(\mathcal{M})}\eta_j^\nu(\omega)d\omega$$

$$\lesssim \int_{\mathbb{S}^2} \|H\nabla(L(P_{\leq l}\text{tr}\chi))\|_{L^2_{u,x}L^1} \|F_j(u)\|_{L^2_{\delta^j}}\eta_j^\nu(\omega)d\omega$$

$$\lesssim \int_{\mathbb{S}^2} \|Hn\nabla(L(P_{\leq l}\text{tr}\chi))\|_{L^2_{u,x}L^1(H_u)} \|F_j(u)\|_{L^2_{\delta^j}}\eta_j^\nu(\omega)d\omega,$$

where we used the estimate (2.36) for $n$ in the last inequality. This yields:

$$\int_{\mathcal{M}} \left| \int_{\mathbb{S}^2} H\nabla(L(P_{\leq l}\text{tr}\chi))F_j(u)\eta_j^\nu(\omega)d\omega \right| d\mathcal{M} \lesssim \left( \sup_{\omega \in \text{supp}(\eta_j^\nu)} \|H\|_{L^2_{u,x}, L^\infty} \right) \int_{\mathbb{S}^2} \|H\nabla(L(P_{\leq l}\text{tr}\chi))\|_{L^2_{u,x}L^1} \|F_j(u)\|_{L^2_{\delta^j}}\eta_j^\nu(\omega)d\omega. \quad (7.14)$$
Next, we estimate $\nabla(nL(P_{\leq t} \text{tr} \chi))$. We have:

$$n\nabla(L(P_{\leq t} \text{tr} \chi)) = -\nabla nL(P_{\leq t} \text{tr} \chi) + \nabla[nL, P_{\leq t}](\text{tr} \chi) + \nabla(P_{\leq t}(nL(\text{tr} \chi))),$$

which yields:

$$\| n \nabla(L(P_{\leq t} \text{tr} \chi)) \|_{L^2_{x,t} L^1} \lesssim \| n^{-1} \nabla n \|_{L^\infty} \| nL(P_{\leq t} \text{tr} \chi) \|_{L^1_t L^2_x}$$

$$\quad + \| \nabla[nL, P_{\leq t}](\text{tr} \chi) \|_{L^1_t L^2_x} + \| \nabla(P_{\leq t}(nL(\text{tr} \chi))) \|_{L^2_t L^1_x}$$

$$\lesssim \varepsilon,$$

where we used in the last inequality the estimate (2.36) for $n$, the commutator estimate (2.66) and the estimate (2.70). Now, (7.14) and (7.15) imply:

$$\int_M \left| \int_{S^2} H \nabla(L(P_{\leq t} \text{tr} \chi))(u) \eta_j^\nu(\omega) d\omega \right| dM \lesssim \left( \sup_{\omega \in \text{supp}(\eta_j^\nu)} \| H \|_{L^2_{x,t} L^\infty_t} \right) \varepsilon \left( \int_{S^2} \| F_j(u) \|_{L^2_{x} \eta_j^\nu(\omega)} d\omega \right)$$

$$\lesssim \left( \sup_{\omega \in \text{supp}(\eta_j^\nu)} \| H \|_{L^2_{x,t} L^\infty_t} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality Plancherel in $u$, Cauchy-Schwarz in $\omega$ and the size of the patch. This concludes the proof of the lemma. 

\[\blacksquare\]

### 7.3 Estimate of the $L^2_{u,x'} L^\infty_t$ norm of oscillatory integrals

**Lemma 7.6** Let $p \in \mathbb{N}$. We have:

$$\left\| \int_{S^2} b^{-1} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,x',1} L^\infty_t} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu. \quad (7.16)$$

**Proof** Note that it suffices to show:

$$\left\| \mathcal{L}_\nu \left( \int_{S^2} b^{-1} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2_{u,x',1} L^1_t} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu. \quad (7.17)$$

We have:

\[\mathcal{L}_\nu \left( \int_{S^2} b^{-1} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right) = i \int_{S^2} b^{-2} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p \omega \mathbf{g}(L, \mathcal{L}_\nu) F_{j,1}(u) \eta_j^\nu(\omega) d\omega + \int_{S^2} \mathcal{L}_\nu(b^{-1} \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega + p \int_{S^2} b^{-1} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p-1} \left( 2^{\frac{j}{2}} (\mathbf{D}_{\mathcal{L}_\nu} \mathbf{L} - \mathbf{D}_{\mathcal{L}_\nu}) \right) F_j(u) \eta_j^\nu(\omega) d\omega,\]
where we used the fact that \( N - N_\nu = L - L_\nu \) since \( T \) does not depend on \( \omega \), and the fact that \( b^{-1}L \) is the space-time gradient of \( u \) so that:

\[
L_\nu(u) = b^{-1}g(L_\nu, L).
\]

Next, we evaluate the various terms in the right-hand side of (7.18). First, recall the identity (5.11):

\[
g(L, L_\nu) = g(N - N_\nu, N - N_\nu). \tag{7.19}
\]

Next, decompose \( L_\nu \) on the frame \( L, L_\nu, e_A, A = 1, 2 \) which yields:

\[
L_\nu = \frac{1}{2} (1 + g(N, N_\nu))L + (N_\nu - g(N, N_\nu)N) + \frac{1}{2}(1 - g(N, N_\nu)L, \tag{7.20}
\]

which yields:

\[
L_\nu(b^{-1}\text{tr}\chi) = \frac{1}{2} (1 + g(N, N_\nu))L(b^{-1}\text{tr}\chi) + (N_\nu - g(N, N_\nu)N)(b^{-1}\text{tr}\chi) \tag{7.21}
\]

\[
+ \frac{1}{2}(1 - g(N, N_\nu)L)(b^{-1}\text{tr}\chi).
\]

Also, in view of the decomposition (7.19) and the Ricci equations (2.17), we have:

\[
D_{L_\nu}L - D_{L_\nu}L_\nu \tag{7.22}
\]

\[
= \frac{1}{2}(1 + g(N, N_\nu))D_L L + D_{N_\nu - g(N, N_\nu)N}L + \frac{1}{2}(1 - g(N, N_\nu)D_L L - D_{L_\nu}L_\nu
\]

\[
= -\frac{1}{2}(1 + g(N, N_\nu))\delta L + \chi(N_\nu - g(N, N_\nu)N, e_A)e_A - e_{N_\nu - g(N, N_\nu)N}L
\]

\[
+ \frac{1}{2}(1 - g(N, N_\nu)(\zeta_A e_A + (\delta + n^{-1}\nabla_N n)L) - \delta L_\nu.
\]

Now, (7.18), (7.19), (7.21) and (7.22) yield:

\[
L_\nu \left( \int_{S^2} b^{-1}\text{tr}\chi \left( 2^{\frac{p}{2}}(N - N_\nu) \right)^p F_j(u)\eta^\nu_j(\omega)d\omega \right)
\]

\[
= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7,
\]

where \( A_1, A_2, A_3, A_4, A_5, A_6 \) and \( A_7 \) are respectively given by:

\[
A_1 = \int_{S^2} b^{-2}\text{tr}\chi \left( 2^{\frac{p}{2}}(N - N_\nu) \right)^{p+2} F_j(u)\eta^\nu_j(\omega)d\omega, \tag{7.24}
\]

\[
A_2 = \int_{S^2} \frac{1}{2}(1 + g(N, N_\nu))L(b^{-1}\text{tr}\chi) \left( 2^{\frac{p}{2}}(N - N_\nu) \right)^p F_j(u)\eta^\nu_j(\omega)d\omega, \tag{7.25}
\]

\[
A_3 = \int_{S^2} (N_\nu - g(N, N_\nu)N)(b^{-1}\text{tr}\chi) \left( 2^{\frac{p}{2}}(N - N_\nu) \right)^p F_j(u)\eta^\nu_j(\omega)d\omega, \tag{7.26}
\]

\[
A_4 = \int_{S^2} \frac{1}{2}(1 - g(N, N_\nu)L)(b^{-1}\text{tr}\chi) \left( 2^{\frac{p}{2}}(N - N_\nu) \right)^p F_j(u)\eta^\nu_j(\omega)d\omega. \tag{7.27}
\]
\[ A_5 = \frac{1}{2} p \int_{S^2} (1 + g(N, N_\nu)) b^{-1} \text{tr} \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p-1} (2^{\frac{1}{2}} (-\delta L + \delta_\nu L_\nu)) \times F_j(u) \eta_j^\nu(\omega) d\omega, \]  
\[ A_6 = p \int_{S^2} b^{-1} \text{tr} \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p-1} \times (2^{\frac{1}{2}} (\chi(N_\nu - g(N, N_\nu)N, e_A) e_A - \epsilon_{N_\nu - g(N, N_\nu)N L}) F_j(u) \eta_j^\nu(\omega) d\omega, \]

and:

\[ A_7 = p \int_{S^2} \frac{1}{2} (1 - g(N, N_\nu)) b^{-1} \text{tr} \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p-1} \times (2^{\frac{1}{2}} (\zeta_A e_A + (\delta + n^{-1} \nabla N n)L) F_j(u) \eta_j^\nu(\omega) d\omega. \]

We estimate \( A_1, A_2, A_3, A_4, A_5, A_6 \) and \( A_7 \) starting with \( A_1 \). Recall the decomposition (2.71) for \( 2^{\frac{1}{2}} (N - N_\nu) \):

\[ 2^{\frac{1}{2}} (N - N_\nu) = F_1^j + F_2^j \]

where the tensor \( F_1^j \) only depends on \( \nu \) and satisfies:

\[ \| F_1^j \|_{L^\infty} \lesssim 1, \]

and where the tensor \( F_2^j \) satisfies:

\[ \| F_2^j \|_{L^\infty L^2(H_\omega)} \lesssim 2^{-\frac{1}{2}}. \]

This yields:

\[ \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p+2} = \sum_{m=0}^{p+1} F_{1,j}^m \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p-m+1} F_{2,j}^m + F_{1,j}^{p+2} \]

and thus:

\[ A_1 = \sum_{m=0}^{p+1} F_{1,j}^m \left( \int_{S^2} b^{-2} \text{tr} \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p-m+1} F_{2,j} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right) \]

\[ + F_{1,j}^{p+2} \left( \int_{S^2} b^{-2} \text{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right), \]

where we used the fact that \( F_1^j \) does not depend on \( \omega \). We obtain:

\[ \| A_1 \|_{L^2(\mathcal{M})} \lesssim \sum_{m=0}^{p+1} \| F_{1,j} \|_{L^\infty(\mathcal{M})} \left\| \int_{S^2} b^{-2} \text{tr} \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p-m+1} F_{2,j} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \]

\[ + \| F_{1,j} \|_{L^\infty(\mathcal{M})} \left\| \int_{S^2} b^{-2} \text{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \]

\[ \lesssim \sum_{m=0}^{p+1} \left\| \int_{S^2} b^{-2} \text{tr} \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^{p-m+1} F_{2,j} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \]

\[ + \left\| \int_{S^2} b^{-2} \text{tr} \chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \]

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where we used (7.32) in the last inequality. The estimate in $L^2(\mathcal{M})$ (7.1) for oscillatory integrals yields:

$$\|A_1\|_{L^2(\mathcal{M})} \lesssim \sum_{m=0}^{p+1} \left( \sup_\omega \left\| b^{-2} \text{tr}_\chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p-m+1} F_{2,j} \right\|_{L^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu$$

$$\lesssim \sum_{m=0}^{p+1} \left( \sup_\omega \left\| b^{-2} \text{tr}_\chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p-m+1} \right\|_{L^\infty \| L^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu$$

$$+ \left\| \int_{S^2} b^{-2} \text{tr}_\chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\lesssim (1+p) \varepsilon \gamma_j^\nu + \left\| \int_{S^2} b^{-2} \text{tr}_\chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

(7.34)

where we used in the last inequality the estimate (7.33) and the estimates (2.38) for $b$ and (2.39) for tr$\chi$.

Next, we estimate the second term in the right-hand side of (7.34). In view of the decomposition (2.72) for tr$\chi$, and the decomposition (2.73) for $b$, we have:

$$b^{-2} \text{tr}_\chi = f_1^j + f_2^j$$

(7.35)

where the scalar $f_1^j$ only depends on $\nu$ and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim \varepsilon.$$ 

(7.36)

and where the scalar $f_2^j$ satisfies:

$$\|f_2^j\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$ 

(7.37)

This yields:

$$\int_{S^2} b^{-2} \text{tr}_\chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega = f_1^j \int_{S^2} F_{j,1}(u) \eta_j^\nu(\omega) d\omega + \int_{S^2} f_2^j F_{j,1}(u) \eta_j^\nu(\omega) d\omega$$

which together with (7.36) implies:

$$\left\| \int_{S^2} b^{-2} \text{tr}_\chi F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \|f_1^j\|_{L^\infty} \left\| \int_{S^2} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \left\| \int_{S^2} f_2^j F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \varepsilon \left\| \int_{S^2} F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \left\| \int_{S^2} f_2^j F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}.$$
Using the estimate (5.8) and the estimate in $L^2(\mathcal{M})$ (7.1) for oscillatory integrals, we finally obtain:

\[
\left\| \int_{\mathbb{S}^2} b^{-2} \text{tr}\chi F_{j,1}(u)\eta^\nu_j(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma^\nu_j + \varepsilon \gamma^\nu_j 2^{\frac{3}{2}} \left( \sup_{\omega} \| F_j^2 \|_{L^2(\mathcal{H}_u)} \right) \lesssim \varepsilon \gamma^\nu_j
\]

where we used (7.37) in the last estimate. Together with (7.34), we obtain:

\[
\| A_1 \|_{L^2(\mathcal{M})} \lesssim (1 + p)\varepsilon \gamma_j^\nu. \tag{7.38}
\]

Next, we estimate $A_2$ defined by (7.25). In view of the Raychaudhuri equation (2.22) satisfied by $\text{tr}\chi$ and the transport equation (2.23) satisfied by $b$, we have:

\[
L(b^{-1}\text{tr}\chi) = -b^{-1}\frac{1}{2}(\text{tr}\chi)^2 - b^{-1}|\tilde{\chi}|^2.
\]

Together with the decomposition (2.72) for $\text{tr}\chi$, (2.77) for $|\tilde{\chi}|^2$ and (2.73) for $b^{-1}$, and with the $L^\infty$ estimates for $b$ and $\text{tr}\chi$ provided respectively by (2.38) and (2.39), we obtain the following decomposition for $L(b^{-1}\text{tr}\chi)$:

\[
L(b^{-1}\text{tr}\chi) = |\chi_2^\nu|^2 + \chi_2^\nu \cdot F_1^j + \chi_2^\nu \cdot F_2^j + f_3^j + f_4^j + f_5^j, \tag{7.39}
\]

where the tensor $F_1^j$ and the scalar $f_3^j$ only depends on $\nu$ and satisfy:

\[
\| F_1^j \|_{L^\infty L^2(\mathcal{H}_u)} + \| f_3^j \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon, \tag{7.40}
\]

where the tensor $F_2^j$, and the scalar $f_4^j$ satisfy:

\[
\| F_2^j \|_{L^\infty L^2(\mathcal{H}_u)} + \| f_4^j \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}, \tag{7.41}
\]

and where the scalar $f_5^j$ satisfies:

\[
\| f_5^j \|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-j}. \tag{7.42}
\]

Together with the definition (7.25) for $A_2$, this yields the following decomposition for $A_2$:

\[
A_2 = (|\chi_2^\nu|^2 + \chi_2^\nu \cdot F_1^j + f_3^j) \left( \int_{\mathbb{S}^2} \frac{1}{2}(1 + g(N, N^\nu)) \left( 2^{\frac{3}{2}}(N - N^\nu) \right)^p F_j^2(u)\eta_j^\nu(\omega) d\omega \right)
+ \chi_2^\nu \cdot \left( \int_{\mathbb{S}^2} \frac{1}{2}(1 + g(N, N^\nu)) \left( 2^{\frac{3}{2}}(N - N^\nu) \right)^p F_2^j F_j^2(u)\eta_j^\nu(\omega) d\omega \right)
+ \left( \int_{\mathbb{S}^2} \frac{1}{2}(1 + g(N, N^\nu)) \left( 2^{\frac{3}{2}}(N - N^\nu) \right)^p f_4^j F_j^2(u)\eta_j^\nu(\omega) d\omega \right)
+ \left( \int_{\mathbb{S}^2} \frac{1}{2}(1 + g(N, N^\nu)) \left( 2^{\frac{3}{2}}(N - N^\nu) \right)^p f_5^j F_j^2(u)\eta_j^\nu(\omega) d\omega \right).
\]
We may now estimate $A_2$. We have:

\[
||A_2||_{L^{2}_{u,v',t'}L^{1}_t} \lesssim \left( \|\chi_{2r}\|_{L^{\infty}_{u,v',t'}L^{2}_t} + \|F^j_1\|_{L^{\infty}_{u,v'}L^{\infty}(P_{t,u,v})} + \|f^j_3\|_{L^{\infty}_{u,v'}L^{\infty}(P_{t,u,v})} \right) \times \left\| \int_{\mathbb{R}^2} \frac{1}{2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{2}_{u,v',t'}L^{\infty}_t} \\
+ \|\chi_{2r}\|_{L^{\infty}_{u,v',t'}L^{2}_t} \left\| \int_{\mathbb{R}^2} \frac{1}{2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F^3_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
+ \|\int_{\mathbb{R}^2} \frac{1}{2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p f^j_3 F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
\lesssim \varepsilon \left\| \int_{\mathbb{R}^2} \frac{1}{2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{2}_{u,v',t'}L^{\infty}_t} \\
+ \varepsilon \left\| \int_{\mathbb{R}^2} \frac{1}{2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F^3_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
+ \varepsilon \left\| \int_{\mathbb{R}^2} \frac{1}{2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p f^j_3 F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} ,
\]

where we used in the last inequality the estimate (7.40) for $F^j_1$ and $F^3_j$ and the estimate (2.46) for $\chi_{2r}$. Using the estimate in $L^2(M)$ (7.1) for oscillatory integrals we obtain:

\[
||A_2||_{L^{2}_{u,v',t'}L^{1}_t} \lesssim \varepsilon \left\| \int_{\mathbb{R}^2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{2}_{u,v',t'}L^{\infty}_t} \\
+ \sup \omega \left( (\|F^j_1\|_{L^{\infty}_{u,v'}L^{2}(H_{\omega})} + \|f^j_3\|_{L^{\infty}_{u,v'}L^{2}(H_{\omega})}) \left\| (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
+ \int_{\mathbb{R}^2} \left\| (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \right\|_{L^\infty} \|f^j_3\|_{L^2(M)} \|F_j(u)\|_{L^{\infty}_{u,v'}} \eta_j^\nu(\omega) d\omega \\
\lesssim \varepsilon \left\| \int_{\mathbb{R}^2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{2}_{u,v',t'}L^{\infty}_t} \\
+ \varepsilon \gamma_j^\nu + \varepsilon 2^{-j} \int_{\mathbb{R}^2} \|F_j(u)\|_{L^{\infty}_{u,v'}} \eta_j^\nu(\omega) d\omega \\
\lesssim \varepsilon \left\| \int_{\mathbb{R}^2} (1 + g(N, N_{\nu})) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^{2}_{u,v',t'}L^{\infty}_t} + \varepsilon \gamma_j^\nu ,
\]

where we used the estimates (7.41) and (7.42), Cauchy-Schwarz in $\lambda$ to estimate $\|F_j(u)\|_{L^{\infty}_{u,v'}}$. 

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Cauchy-Schwartz in $\omega$ and the size of the patch, and the fact that:

\[ \|2^\frac{j}{2}(N - N_\nu)\|_{L^\infty} \lesssim 1 \]

in view of the estimate (2.42) for $\partial_\omega N$ and the size of the patch.

Next, we estimate $A_3$ and $A_4$ defined respectively by (7.26) and (7.27). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we obtain:

\[
\|A_3\|_{L^2(\mathcal{M})} + \|A_4\|_{L^2(\mathcal{M})} \lesssim \sup_\omega \left( \|\text{D} \text{tr}_\chi\|_{L^\infty L^2(\mathcal{H}_\omega)} \left( \left\| (N_\nu - g(N, N_\nu)N) \left( 2^\frac{j}{2}(N - N_\nu) \right)^p \right\|_{L^\infty} \right. \right. \\
+ \left. \left. \left\| \frac{1}{\nu} (1 - g(N, N_\nu)) \left( 2^\frac{j}{2}(N - N_\nu) \right)^p \right\|_{L^\infty} \right) \right)^{\frac{1}{2}} \epsilon \gamma_j' \]

where we used in the last inequality the estimate (2.39) for $\text{tr}_\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch.

Next, we estimate $A_5$ defined in (7.28). We first decompose $-\delta L + \delta_\nu L_\nu$. We have:

\[
2^\frac{j}{2} (-\delta L + \delta_\nu L_\nu) = 2^\frac{j}{2} \left( -\delta(L - L_\nu) + (-\delta + \delta_\nu) L_\nu \right) = 2^\frac{j}{2} \left( -\delta_\nu(N - N_\nu) + (-\delta + \delta_\nu)(N - N_\nu + L_\nu) \right).
\]

Furthermore:

\[
-\delta + \delta_\nu = -k_{NN} + n^{-1} n \nabla N n + k_{N_\nu N_\nu} - n^{-1} \nabla N_\nu n \\
= n^{-1} \nabla n \cdot (N - N_\nu) - k_{N_\nu}(N - N_\nu) + k_{N_\nu N_\nu} - k_{NN} \\
= \left( n^{-1} \nabla n - 2k_{N_\nu} \right) \cdot (N - N_\nu) - k(N - N_\nu, N - N_\nu) \\
= \left( n^{-1} \nabla n - 2\delta_\nu N_\nu - 2\epsilon_\nu \right) \cdot (N - N_\nu) - k(N - N_\nu, N - N_\nu).
\]

(7.45), (7.46) and the definition (7.28) of $A_5$ yield:

\[
A_5 = -p\delta_\nu \int_{S^2} \frac{1}{2} (1 + g(N, N_\nu)) b^{-1} \text{tr}_\chi \left( 2^\frac{j}{2}(N - N_\nu) \right)^p F_j(u) \nu_j' (\omega) d\omega \\
+ p \left( n^{-1} \nabla n - 2\delta_\nu N_\nu - 2\epsilon_\nu \right) \int_{S^2} (N - N_\nu + L_\nu) \frac{1}{2} (1 + g(N, N_\nu)) b^{-1} \text{tr}_\chi \left( 2^\frac{j}{2}(N - N_\nu) \right)^p \\
\times F_j(u) \nu_j' (\omega) d\omega \\
+ p 2^{\frac{j}{2}} \int_{S^2} (N - N_\nu + L_\nu) k(N - N_\nu, N - N_\nu) \frac{1}{2} (1 + g(N, N_\nu)) b^{-1} \text{tr}_\chi \left( 2^\frac{j}{2}(N - N_\nu) \right)^{p-1} \\
\times F_j(u) \nu_j' (\omega) d\omega
\]

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which implies:
\[
\|A_5\|_{L^2_{\omega'\nu,L_1}} \lesssim \int_{S^2} \left(1 + g(N,N_{\nu})\right) b^{-1} \text{tr}_\chi \left(2^\frac{1}{2} (N - N_{\nu})\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \leq 0.
\]

\[
\|A_5\|_{L^2_{\omega'\nu,L_1}} \lesssim \|\delta_{\nu}\|_{L^\infty_{\omega'\nu,L^1}} \left\|\int_{S^2} \left(1 + g(N,N_{\nu})\right) b^{-1} \text{tr}_\chi \left(2^\frac{1}{2} (N - N_{\nu})\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\]

\[
+ p\left\|n^{-1}\nabla n\right\|_{L^\infty(M)} + \|\delta_{\nu}\|_{L^\infty_{\omega'\nu}} + \|\epsilon_{\nu}\|_{L^\infty_{\omega'\nu}}
\]

\[
\times \left\|\int_{S^2} (N - N_{\nu} + L_{\nu})(1 + g(N,N_{\nu})) b^{-1} \text{tr}_\chi \left(2^\frac{1}{2} (N - N_{\nu})\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\]

\[
+ p2^\frac{1}{2} \left\|\int_{S^2} (N - N_{\nu} + L_{\nu}) k(N - N_{\nu}, N - N_{\nu})(1 + g(N,N_{\nu})) b^{-1} \text{tr}_\chi \right.
\]

\[
\times \left(2^\frac{1}{2} (N - N_{\nu})\right)^{p-1} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\]

\[
\lesssim p\varepsilon \left\|\int_{S^2} \left(1 + g(N,N_{\nu})\right) b^{-1} \text{tr}_\chi \left(2^\frac{1}{2} (N - N_{\nu})\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\]

\[
+ p\varepsilon \left\|\int_{S^2} (N - N_{\nu} + L_{\nu})(1 + g(N,N_{\nu})) b^{-1} \text{tr}_\chi \left(2^\frac{1}{2} (N - N_{\nu})\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\]

\[
+ p2^\frac{1}{2} \left\|\int_{S^2} (N - N_{\nu} + L_{\nu}) k(N - N_{\nu}, N - N_{\nu})(1 + g(N,N_{\nu})) b^{-1} \text{tr}_\chi \right.
\]

\[
\times \left(2^\frac{1}{2} (N - N_{\nu})\right)^{p-1} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\]

where we used in the last inequality the estimates (2.36) for \(n\) and the estimates (2.37) for \(\delta\) and \(\epsilon\). The first two terms in the right-hand side of (7.47) are similar to \(A_1\) and can be estimated in the same way. In view of (7.38), we obtain:

\[
\|A_5\|_{L^2_{\omega'\nu,L_1}} \lesssim (1 + p^2) \gamma_j^\nu + p2^\frac{1}{2} \left\|\int_{S^2} (N - N_{\nu} + L_{\nu}) k(N - N_{\nu}, N - N_{\nu}) \right.
\]

\[
\times (1 + g(N,N_{\nu})) b^{-1} \text{tr}_\chi \left(2^\frac{1}{2} (N - N_{\nu})\right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\]

Using the basic estimate in \(L^2(M)\) (7.1), this yields:

\[
\|A_5\|_{L^2_{\omega'\nu,L_1}} \lesssim (1 + p^2) \gamma_j^\nu + p2^\frac{1}{2} \sup_\omega \left\|k\right\|_{L^\infty L^2(H^\omega)}
\]

\[
\times \left\|(N - N_{\nu} + L_{\nu})(N - N_{\nu})^2 (1 + g(N,N_{\nu})) b^{-1} \text{tr}_\chi \left(2^\frac{1}{2} (N - N_{\nu})\right)^{p-1} F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^\infty}
\]

\[
\lesssim (1 + p^2) \gamma_j^\nu,
\]

where we used in the last inequality the estimates (2.37) for \(k\), (2.38) for \(b\), (2.39) for \(\text{tr}_\chi\) and (2.42) for \(\partial_\omega N\), and the size of the patch.
Next, we estimate $A_6$ defined in (7.29). We first decompose $\epsilon$. We have, schematically:

$$
\epsilon = k_{N_\nu} + k(N - N_\nu, \cdot) = \delta_\nu N_\nu + \varepsilon_\nu + k(N - N_\nu, \cdot). 
$$

(7.49)

Together with the decompositions (2.72) for $\text{tr} \chi$ and (2.74) $\tilde{\chi}$, this yields:

$$
2^{\frac{j}{2}}(\chi(N_\nu - g(N_\nu)N, e_A)e_A - \epsilon_{N_\nu - g(N_\nu)N}L)
= F_1^{j}2^{\frac{j}{2}}(N - N_\nu) + F_1^{j}2^{\frac{j}{2}}(N - N_\nu)
$$

where the tensor $F_1^{j}$ only depends on $\nu$ and satisfies:

$$
\| F_1^{j} \|_{L^2_{\nu,x,L_1^1}} \lesssim \varepsilon, 
$$

(7.50)

and where the tensor $F_2^{j}$ satisfies:

$$
\| F_2^{j} \|_{L^\infty L^2(H_\omega)} \lesssim \varepsilon 2^{-\frac{j}{2}}.
$$

(7.51)

In view of the definition (7.29) of $A_6$, we obtain:

$$
A_6 = pF_1^{j} \int_{S^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u)\eta^\nu_j(\omega) d\omega
$$

$$
+ p \int_{S^2} b^{-1} \text{tr} \chi F_2^{j} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u)\eta^\nu_j(\omega) d\omega.
$$

This yields:

$$
\| A_6 \|_{L^2_{\nu,x,L_1^1}} \lesssim p \| F_1^{j} \|_{L^\infty L^2_{\nu,x,L_1^1}} \left\| \int_{S^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u)\eta^\nu_j(\omega) d\omega \right\|_{L^2(M)}
$$

$$
+ p \left\| \int_{S^2} b^{-1} \text{tr} \chi F_2^{j} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u)\eta^\nu_j(\omega) d\omega \right\|_{L^2(M)},
$$

(7.52)

where we used the estimate (7.50) in the last inequality. The first term in the right-hand side of (7.52) are similar to $A_1$ and can be estimated in the same way. In view of (7.38), we obtain:

$$
\| A_6 \|_{L^2_{\nu,x,L_1^1}} \lesssim (1 + p^2)\varepsilon \gamma^\nu_j + p \left\| \int_{S^2} b^{-1} \text{tr} \chi F_2^{j} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u)\eta^\nu_j(\omega) d\omega \right\|_{L^2(M)}.
$$

Using the basic estimate in $L^2(M)$ (7.1), this yields:

$$
\| A_6 \|_{L^2_{\nu,x,L_1^1}} \lesssim (1 + p^2)\varepsilon \gamma^\nu_j + p \sup_{\omega} \left( \| F_2^{j} \|_{L^\infty L^2(H_\omega)} \right) \left\| b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right\|_{L^\infty} 2^{\frac{j}{2}} \gamma^\nu_j
$$

$$
\lesssim (1 + p^2)\varepsilon \gamma^\nu_j.
$$

(7.53)
where we used in the last inequality the estimate (7.51) for $F_2^j$, the estimate (2.34) for $b$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch.

Finally, we estimate $A_7$. In view of the definition (7.30) for $A_7$ and the basic estimate in $L^2(M)$ (7.1), we have:

$$\|A_7\|_{L^2(M)} \lesssim p2^{\frac{3}{2}} \sup_{\omega} \left( (\|\zeta\|_{L^\infty} + \|\delta\|_{L^\infty} + \|n^{-1}\nabla n\|_{L^\infty}) \right) \tag{7.54}$$

where we used in the last inequality the estimate (7.51) for $\| \|$, the size of the patch, and the fact that:

$$1 - g(N, N) = \frac{g(N - N, N - N)}{2}. \tag{7.55}$$

We have:

$$\left\| \int_{S^2} b^{-1} \text{tr} \chi \left( 2^{\frac{1}{2}} \left( N - N_\nu \right) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u_\nu, x_\nu'} L^\infty_{t_l}} \tag{7.56}$$

where we used (7.23) in the last inequality. Together with (7.38), (7.43), (7.44), (7.48), (7.53) and (7.54), we obtain:

$$\left\| \int_{S^2} b^{-1} \text{tr} \chi \left( 2^{\frac{1}{2}} \left( N - N_\nu \right) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u_\nu, x_\nu'} L^\infty_{t_l}} \tag{7.56}$$

Now, we have:

$$\left\| \int_{S^2} (1 - g(N, N)) \left( 2^{\frac{1}{2}} \left( N - N_\nu \right) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u_\nu, x_\nu'} L^\infty_{t_l}}$$

where we used (2.41) for $\zeta$, the estimate (2.37) for $\delta$, the estimate (2.36) for $n$, the estimate (2.38) for $b$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, the size of the patch, and the fact that:

$$1 - g(N, N) = \frac{g(N - N, N - N)}{2}. \tag{7.55}$$
where we used (7.55), the estimate (2.42) for $\partial_\nu N$, the size of the patch, Cauchy-Schwarz in $\lambda$ to estimate $\|F_j(u)\|_{L^2_{\nu}}$, and Cauchy-Schwarz in $\omega$. Together with (7.56), this yields:

$$\left\| \int_{S^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^p(\omega) d\omega \right\|_{L^2_{u,N_\nu} L^\infty_t} \lesssim \varepsilon \left\| \int_{S^2} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^p(\omega) d\omega \right\|_{L^2_{u,N_\nu} L^\infty_t} + (1 + p^2)\varepsilon \gamma_j^\nu.$$  

Note that the first term in the right-hand side corresponds to the left-hand side where $b^{-1}\text{tr} \chi$ has been replaced by 1. In particular, we have the analog of (7.23):

$$L^\nu \left( \int_{S^2} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^p(\omega) d\omega \right) = A'_1 + A'_5 + A'_6 + A'_7,$$

where $A'_1, A'_5, A'_6$ and $A'_7$ are respectively given by:

$$A'_1 = \int_{S^2} b^{-1} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p+2} F_{j,1}(u) \eta_j^p(\omega) d\omega,$$

$$A'_5 = p \int_{S^2} \frac{1}{2} (1 + g(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-1} (2^{\frac{j}{2}}(-\delta L + \delta_\nu L_\nu)) F_j(u) \eta_j^p(\omega) d\omega,$$

$$A'_6 = p \int_{S^2} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-1} \times (2^{\frac{j}{2}}(\chi(N_\nu - g(N, N_\nu) N, e_A) e_A - \varepsilon_{N_\nu - g(N, N_\nu) N}) F_j(u) \eta_j^p(\omega) d\omega,$$

and:

$$A'_7 = p \int_{S^2} \frac{1}{2} (1 - g(N, N_\nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{p-1} \times (2^{\frac{j}{2}}(\zeta_A e_A + (\delta + n^{-1} \nabla_N n) L) F_j(u) \eta_j^p(\omega) d\omega.$$

The analog of the estimates (7.38), (7.48), (7.53) and (7.54) for $A'_1, A'_5, A'_6$ and $A'_7$ yield:

$$\left\| \int_{S^2} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^p(\omega) d\omega \right\|_{L^2_{u,N_\nu} L^\infty_t} \lesssim \|A'_1\|_{L^2_{u,N_\nu} L^1_t} + \|A'_5\|_{L^2_{u,N_\nu} L^1_t} + \|A'_6\|_{L^2_{u,N_\nu} L^1_t} + \|A'_7\|_{L^2_{u,N_\nu} L^1_t}$

$$\lesssim (1 + p^2)\gamma_j^\nu.$$

Together with (7.57), we obtain:

$$\left\| \int_{S^2} b^{-1} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_j(u) \eta_j^p(\omega) d\omega \right\|_{L^2_{u,N_\nu} L^\infty_t} \lesssim (1 + p^2)\varepsilon \gamma_j^\nu.$$

This concludes the proof of the lemma. ■
Lemma 7.7 Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. We have:

$$
\left\| \int_{S^2} b^q \left( 2^q (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v} L_t^\infty} \lesssim (1 + p^2) \gamma_j^\nu. \tag{7.63}
$$

and:

$$
\left\| \int_{S^2} b^q \xi_n^\nu \left( 2^q (N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u,v} L_t^\infty} \lesssim (1 + p^2) \xi \gamma_j^\nu. \tag{7.64}
$$

The proof of Lemma 7.7 is completely analogous to the proof of Lemma 7.6 and is left to the reader.

Next, we obtain estimates evaluating the $L^2_{u,x} L^\infty_t$ of $H$ where $u = u(t, x, \omega)$, and where $H$ is naturally defined with respect to the foliation of $u' = u(t, x, \omega')$. We start with a basic lemma.

Lemma 7.8 Let $H$ a tensor on $M$, and $\omega, \omega'$ two angles in $S^2$. Let $u = u(t, x, \omega)$, and $L$ corresponding to $u$. Let $u' = u(t, x, \omega')$, and $L', L', \nabla'$ corresponding to $u'$. Then, we have the following estimate for the $L^2_{u,x} L^\infty_t$ of $H$:

$$
\|H\|_{L^2_{u,x} L^\infty_t} \lesssim \|H\|_{L^2(M)} \|D_L H\|_{L^2(M)} |\omega - \omega'|^2 + \|H\|_{L^2(M)} \|\nabla' H\|_{L^2(M)} |\omega - \omega'|
+ \int_M |H| \|D_L H\| d\mathcal{M} + \|H\|^2_{L^2(M)}. \tag{7.65}
$$

Proof Recall the estimate (2.52) on $H_u$:

$$
\|H\|^2_{L^2_{u,x} L^\infty_t} \lesssim \int_{H_u} |H| \|D_L H\| dtd\mu_{t,u} + \|H\|^2_{L^2(H_u)}.
$$

Integrating in $u$, and using the expression of the volume element $d\mathcal{M}$ in the coordinate system $(u, t, x')$ (4.4) and the control of $b$ in $L^\infty$ given by (2.38), we obtain:

$$
\|H\|^2_{L^2_{u,x} L^\infty_t} \lesssim \int_M |H| \|D_L H\| d\mathcal{M} + \|H\|^2_{L^2(M)}. \tag{7.66}
$$

Next, we decompose $L$ on the frame $L', L', e_A', A = 1, 2$. We have:

$$
L = -\frac{1}{2} g(L, L') L' - \frac{1}{2} g(L, L') L' + g(L, e_A') e_A'. \tag{7.67}
$$

Now, we have:

$$
1 - g(N, N') = \frac{g(N - N', N - N')}{2} \sim |\omega - \omega'|^2, \tag{7.68}
$$

where we used (2.43), (7.68) and the estimate (2.42) for $\partial_\omega N$ yield:

$$
\begin{align*}
g(L, L') & \sim |\omega - \omega'|^2, 
g(L, e_A') = g(L - L', e_A') \sim |\omega - \omega'| \text{ and } g(L, L') = -2 + g(L, L').
\end{align*}
\tag{7.69}
$$

Together with (7.67), this yields:

$$
L = (-2 + O(|\omega - \omega'|^2)) L' + O(|\omega - \omega'|) \nabla' + O(|\omega - \omega'|^2) L'. \tag{7.70}
$$
Finally, plugging (7.70) in (7.66) and using Cauchy-Schwartz yields (7.65). This concludes the proof of the lemma.

We have the following corollary of Lemma 7.8

**Corollary 7.9** Let $\omega, \nu, \omega', \nu'$ four angles in $\mathbb{S}^2$ such that $\omega$ belongs to the patch of center $\nu$ and $\omega'$ belongs to the patch of center $\nu'$. Let $u = u(t, x, \omega)$. Let $u' = u(t, x, \omega')$, and $L', L', \nabla'$ corresponding to $u'$. Let a tensor $G$ on $\mathcal{M}$. Then, we have the following estimate:

$$
\left\| \int_{\mathbb{S}^2} GP'_t \text{tr} \chi' F_j(u') \, d\omega' \right\|_{L^2_{u,x,L^\infty_t}} \lesssim \left( \sup_{\omega} \|G\|_{L^\infty} \right) 2^{-\frac{1}{2}} \left\| \sqrt{\eta''(\omega')} \|P'_t \text{tr} \chi' F_j(u')\|_{L^2_\omega L^2_{t,x,\omega}} \right\|_{L^2_{\omega,j}},
$$

(7.72)

where we used in the last inequality the estimate (2.38) for $b$, Cauchy-Schwartz in $\omega'$ and the size of the patch.

Next, we apply (7.65) with the choice $H = P'_t \text{tr} \chi' F_j(u')$:

$$
\|P'_t \text{tr} \chi' F_j(u')\|_{L^2_{u,x,L^\infty_t}}^2 \lesssim \|P'_t \text{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})} \|P'_t \text{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})} |\nu - \nu'|^2 \\
+ \|P'_t \text{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})} \|\nabla' (P'_t \text{tr} \chi' F_j(u'))\|_{L^2(\mathcal{M})} |\nu - \nu'| \\
+ \int_{\mathcal{M}} |P'_t \text{tr} \chi' F_j(u')| \|L' (P'_t \text{tr} \chi' F_j(u'))| \, d\mathcal{M} + \|P'_t \text{tr} \chi' F_j(u')\|_{L^2(\mathcal{M})}^2,
$$

where we used the fact that $|\omega - \omega'| \sim |\nu - \nu'|$ and $L'(u') = \nabla'(u') = 0$. Also, since $L'(u') = -2b'_{-1}$ and $\lambda' \sim 2^i$, we have:

$$
L'(F_j(u')) \sim 2^i b'_{-1} F_j(u')
$$

and we obtain:

$$
\|P'_t \text{tr} \chi' F_j(u')\|_{L^2_{u,x,L^\infty_t}}^2 \lesssim \left( \|P'_t \text{tr} \chi\|_{L^\infty_{\omega,L^2_t(H_u)}} \|P'_t \text{tr} \chi\|_{L^\infty_{\omega,L^2_t(H_u)}} |\nu - \nu'|^2 \\
+ \|L'(P'_t \text{tr} \chi')\|_{L^\infty_{\omega,L^2_t(H_u)}} |\nu - \nu'|^2 + \|\nabla' P'_t \text{tr} \chi'\|_{L^\infty_{\omega,L^2_t(H_u)}} |\nu - \nu'| \\
+ \int_{H_u} |P'_t \text{tr} \chi'| \|L' (P'_t \text{tr} \chi')| \, dH_u + \|P'_t \text{tr} \chi'\|_{L^2_{\omega,L^2_t(H_u)}}^2 \right) \|F_j(u')\|_{L^2_{\omega}}^2 \\
\lesssim \left( 2^{-i} \varepsilon |\nu - \nu'| + \|L'(P'_t \text{tr} \chi')\|_{L^\infty_{\omega,L^2_t(H_u)}} |\nu - \nu'|^2 + \varepsilon |\nu - \nu'| \right) \\
+ \int_{H_u} |P'_t \text{tr} \chi'| \|L' (P'_t \text{tr} \chi')| \, dH_u + 2^{-2i} \varepsilon^2 \right) \|F_j(u')\|_{L^2_{\omega}}^2.
$$

(7.73)
where we used in the last inequality the finite band property for $P'_t$ and the estimate (2.39) for $\text{tr}\chi$.

Next, we evaluate $\|L'(P'_t\text{tr}\chi')\|_{L^2_uL^2(\mathcal{H}_u)}$. We have:

$$
\|L'(P'_t\text{tr}\chi')\|_{L^2_uL^2(\mathcal{H}_u)} \lesssim \|L'(P'_t\text{tr}\chi')\|_{L^\infty_uL^2(\mathcal{H}_u)} + \|N'(P'_t\text{tr}\chi')\|_{L^\infty_uL^2(\mathcal{H}_u)} \\
\lesssim ||nL'(P'_t\text{tr}\chi')\|_{L^\infty_uL^2(\mathcal{H}_u)} + ||nL'_t\text{tr}\chi'\|_{L^\infty_uL^2(\mathcal{H}_u)} + ||nL'_t\text{tr}\chi'\|_{L^\infty_uL^2(\mathcal{H}_u)} + ||nL'_t\text{tr}\chi'\|_{L^\infty_uL^2(\mathcal{H}_u)} + \mathcal{N}_1(\text{tr}\chi')
$$

where we used the fact that $L' = L' - 2N'$, the estimate (2.38) for $b$ and the estimate (2.36) for $n$. Together with the estimates (2.39) for $\text{tr}\chi$, (2.38) for $b$ and (2.36) for $n$, and the commutator estimates (2.64) and (2.65), we obtain:

$$
\|L'(P'_t\text{tr}\chi')\|_{L^\infty_uL^2(\mathcal{H}_u)} \lesssim ||nL'(\text{tr}\chi')\|_{L^\infty_uL^2(\mathcal{H}_u)} + ||b'N'(\text{tr}\chi')\|_{L^\infty_uL^2(\mathcal{H}_u)} + \mathcal{N}_1(\text{tr}\chi') \lesssim \varepsilon.
$$

(7.74)

Next, we estimate the integral over $\mathcal{H}_u$ in the right-hand side of (7.73). We have:

$$
\int_{\mathcal{H}_u} |P'_t\text{tr}\chi'|L'(P'_t\text{tr}\chi')| d\mathcal{H}_u \lesssim \int_{\mathcal{H}_u} |P'_t\text{tr}\chi'|nL'(P'_t\text{tr}\chi')| d\mathcal{H}_u
$$

$$
\lesssim \|P'_t\text{tr}\chi'\|_{L^2_uL^\infty_t} ||P'_t(n\text{tr}\chi')\|_{L^2_uL^1_t} \\
+ \|P'_t\text{tr}\chi'\|_{L^2_uL^\infty_t} ||[nL'_t,P'_t](\text{tr}\chi')\|_{L^2_uL^1_t} \\
\lesssim 2^{-2l}\varepsilon^2,
$$

where we used in the last inequality the finite band property for $P'_t$, the commutator estimate (2.66), the estimate (2.39) for $\text{tr}\chi$, and the estimate (2.69) for $P_t\text{tr}\chi$ and $P_t(nL\text{tr}\chi)$.

Finally, (7.73), (7.74) and (7.75) imply:

$$
\|P'_t\text{tr}\chi'F_j(u')\|_{L^2_uL^\infty_t}^2 \lesssim (2^j|\nu - \nu'|^2 + 2^j|\nu - \nu'| + 1) \varepsilon^22^{-2l}\|F_j(u')\|_{L^2_u}^2,
$$

which together with (7.72) implies:

$$
\left\| \int_{\mathbb{R}^2} GP'_t\text{tr}\chi'F_j(u')\eta_j''(\omega')d\omega' \right\|_{L^2_uL^\infty_t} \lesssim \left\| \sup_{\omega}\|G\|_{L^\infty}(2^{\frac{1}{2}}|
u - \nu'| + 2^{\frac{1}{2}}|\nu - \nu'| + 1) \varepsilon^{2-l-\frac{1}{2}} \|\eta_j''(\omega')\|_{L^2_t}\|F_j(u')\|_{L^2_t} \right\|_{L^2_uL^\infty_t} \\
\lesssim \left\| \sup_{\omega}\|G\|_{L^\infty}(2^\frac{1}{2}|
u - \nu'|2^{-l-\frac{1}{2}} + 2^{-l-\frac{1}{2}}(2^{\frac{1}{2}}|\nu - \nu'|)|\frac{1}{2} \right\| \varepsilon \gamma_j',
$$

where we used in the last inequality Plancherel in $u'$ and the fact that

$$
2^{\frac{1}{2}}|\nu - \nu'| \gtrsim 1
$$

since $\nu \neq \nu'$. This conclude the proof of the corollary.

\[\blacksquare\]

Lemma 7.8 yields also a second corollary.
Corollary 7.10 Let $\omega, \nu, \omega', \nu'$ four angles in $S^2$ such that $\omega$ belongs to the patch of center $\nu$ and $\omega'$ belongs to the patch of center $\nu'$. Let $u = u(t, x, \omega)$. Let $u' = u(t, x, \omega')$, and $L', L', \nabla'$ corresponding to $u'$. Let a tensor $G$ on $\mathcal{M}$. Then, we have the following estimate:

\[
\left\| \int_{S^2} G \nabla' P'_{\leq l} \text{tr}' \chi' F_j (u') \eta_j' (\omega') d\omega' \right\|_{L^2_{a, x}, L^\infty_t}
\lesssim \left( \sup_{\omega'} \| G \|_{L^\infty} \right) \epsilon \left( 2^{\frac{3}{2}} |\nu - \nu'| 2^\frac{1}{2} + (2^\frac{3}{2} |\nu - \nu'|)^\frac{3}{2} 2^\frac{1}{2} \right) \gamma_j' .
\]

Proof We have:

\[
\left\| \int_{S^2} G \nabla' P'_{\leq l} \text{tr}' \chi' F_j (u') \eta_j' (\omega') d\omega' \right\|_{L^2_{a, x}, L^\infty_t}
\lesssim \int_{S^2} \| G \|_{L^\infty} \| \nabla' P'_{\leq l} \text{tr}' \chi' F_j (u') \|_{L^2_t L^\infty_x L^2_a} \| \eta_j' (\omega') d\omega'
\lesssim \left( \sup_{\omega'} \| G \|_{L^\infty} \right) 2^{-\frac{1}{2}} \left\| \nabla' P'_{\leq l} \text{tr}' \chi' F_j (u') \|_{L^2_t L^\infty_x L^2_a} \right\|_{L^2_x},
\]

where we used in the last inequality the estimate (2.38) for $b$, Cauchy-Schwartz in $\omega'$ and the size of the patch.

Next, we apply (7.65) with the choice $H = \nabla' P'_{\leq l} \text{tr}' \chi' F_j (u')$, and we obtain the analogous estimate to (7.73):

\[
\| \nabla' P'_{\leq l} \text{tr}' \chi' F_j (u') \|_{L^2_{a, x}, L^\infty_t}^2
\lesssim \left( \| \nabla' P'_{\leq l} \text{tr}' \|_{L^\infty_a L^2_x (H_a)} (2^j \| \nabla' P'_{\leq l} \text{tr}' \|_{L^\infty_a L^2_x (H_a)}) |\nu - \nu'|^2
\right.
\]
\[+ \| L' \nabla' (P'_{\leq l} \text{tr}') \|_{L^\infty_a L^2_x (H_a)} |\nu - \nu'|^2 + \| \nabla'^2 P'_{\leq l} \text{tr}' \|_{L^\infty_a L^2_x (H_a)} |\nu - \nu'|^2
\]
\[+ \int_{H_a} \| \nabla' P'_{\leq l} \text{tr}' \|_{L^\infty_a L^2_x (H_a)} |\nu - \nu'|^2 + \epsilon 2^j \| F_j (u') \|_{L^2_x}^2
\]
\[\lesssim \left( \epsilon \left( 2^{\frac{3}{2}} |\nu - \nu'|^2 + \| L' \nabla' (P'_{\leq l} \text{tr}') \|_{L^\infty_a L^2_x (H_a)} |\nu - \nu'|^2 + \epsilon 2^j |\nu - \nu'| \right)
\right)
\[+ \int_{H_a} \| \nabla' P'_{\leq l} \text{tr}' \|_{L^\infty_a L^2_x (H_a)} |\nu - \nu'|^2 + \epsilon 2^j |\nu - \nu'| \right)
\]
where we used the fact that $L' = L' - 2N'$, the estimate (2.38) for $b$ and the estimate (2.36) for $n$. Together with the finite band property for $P'_{\leq t}$, the estimates (2.39) for $\text{tr} \chi'$, (2.38) for $b$ and (2.36) for $n$, and the commutator estimates (2.64) and (2.65), we obtain:

\[
\| L' \nabla' (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} \\
\lesssim 2 l' \| nL' (\text{tr} \chi') \|_{L^\infty_t L^2(H_u)} + 2 l' \| b' N' (\text{tr} \chi') \|_{L^\infty_t L^2(H_u)} + N_1 (\text{tr} \chi') \\
+ \| [nL', \nabla'] P'_{\leq t} (\text{tr} \chi') \|_{L^\infty_t L^2(H_u)} + \| [b' N', \nabla'] (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} \\
\lesssim 2 l' \varepsilon + \| [nL', \nabla'] P'_{\leq t} (\text{tr} \chi') \|_{L^\infty_t L^2(H_u)} + \| [b' N', \nabla'] (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)}. \quad (7.79)
\]

To estimate the commutator terms in the right-hand side of (7.79), we use the commutator formulas (2.34) and (2.35):

\[
\| [nL', \nabla'] P'_{\leq t} (\text{tr} \chi') \|_{L^\infty_t L^2(H_u)} + \| [b' N', \nabla'] (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} \\
\lesssim \| \chi' \nabla' (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} + \| k \nabla' (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} \\
\lesssim (\| \chi' \|_{L^\infty_t L^4} + \| k \|_{L^\infty_t L^4}) \| \nabla' (P'_{\leq t} \text{tr} \chi') \|_{L^2_t L^4}.
\]

Together with the estimate (2.51) and the Gagliardo-Nirenberg inequality (2.49), we obtain:

\[
\| [nL', \nabla'] P'_{\leq t} (\text{tr} \chi') \|_{L^\infty_t L^2(H_u)} + \| [b' N', \nabla'] (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} \\
\lesssim (N_1 (\chi') + N_1 (k)) \| \nabla^2 (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} \\
\lesssim 2 l' \varepsilon,
\]

where we used in the last inequality the Bochner inequality for scalars (2.61), the finite band property for $P'_{\leq t}$, the estimates (2.39) and (2.40) for $\chi'$ and the estimates (2.37) for $k$. Together with (7.79), this yields:

\[
\| L' \nabla' (P'_{\leq t} \text{tr} \chi') \|_{L^\infty_t L^2(H_u)} \lesssim 2 l' \varepsilon. \quad (7.80)
\]

Next, we estimate the integral over $H_u$ in the right-hand side of (7.78). We have:

\[
\int_{H_u} | \nabla' P'_{\leq t} \text{tr} \chi' | \| L' \nabla' (P'_{\leq t} \text{tr} \chi') \| d H_u \\
\lesssim \int_{H_u} | \nabla' P'_{\leq t} \text{tr} \chi' | \| nL' \nabla' (P'_{\leq t} \text{tr} \chi') \| d H_u \\
\lesssim \| \nabla' P'_{\leq t} \text{tr} \chi' \|_{L^2_t L^\infty_{\nu}} \| \nabla' P'_{\leq t} (nL' (\text{tr} \chi')) \|_{L^2_t L^1_{\nu}} + \| \nabla' P'_{\leq t} \text{tr} \chi' \|_{L^\infty_t L^2(H_w)} \\
\times \| [nL', \nabla'] P'_{\leq t} (\text{tr} \chi') \|_{L^\infty_t L^2(H_w)} + \| \nabla' P'_{\leq t} \text{tr} \chi' \|_{L^2_t L^\infty_{\nu}} \| \nabla' [nL', P'_{\leq t}] (\text{tr} \chi') \|_{L^2_{\nu} L^1}
\lesssim \varepsilon \| [nL', \nabla'] P'_{\leq t} (\text{tr} \chi') \|_{L^\infty_t L^2(H_w)} + \varepsilon^2,
\]

where we used in the last inequality the finite band property for $P'_{\leq t}$, the commutator estimate (2.66), the estimate (2.39) for $\text{tr} \chi'$, and the estimate (2.70) for $P_{\leq t} \text{tr} \chi$ and $P_{\leq t} (nL \text{tr} \chi)$.
Next, we estimate the right-hand side of (7.81):
\[
\| [nL', \nabla'] P_{\leq l}^t (\text{tr} \chi') \|_{L^\infty_x L^2_t (H_u)} \lesssim \| \chi' \nabla' (P_{\leq l}^t \text{tr} \chi') \|_{L^\infty_x L^2_t (H_u)} \\
\lesssim \| \chi' \|_{L^\infty_x L^2_t} \| \nabla' (P_{\leq l}^t \text{tr} \chi') \|_{L^2_x L^\infty_t} \\
\lesssim \varepsilon,
\]
where we used in the last inequality the estimates (2.39) (2.40) for $\chi'$ and the estimate (2.70) for $\nabla' (P_{\leq l}^t \text{tr} \chi')$. Together with (7.81), we obtain:
\[
\int_{H_u} |\nabla' P_{\leq l}^t \text{tr} \chi'||L' \nabla' (P_{\leq l}^t \text{tr} \chi')| dH_u \lesssim \varepsilon^2.
\] (7.82)

Finally, (7.78), (7.80) and (7.82) imply:
\[
\| P_{\leq l}^t \text{tr} \chi' F_j (u') \|_{L^2_t L^\infty_x L^\infty_t} \lesssim (2^j |\nu - \nu'| + 2^j |\nu - \nu'| + 1) \varepsilon^2 \| F_j (u') \|_{L^2_t}^2,
\]
which together with (7.77) implies:
\[
\left\| \int_{S^2} G P_{\leq l}^t \text{tr} \chi' F_j (u') \eta_j (\omega') d\omega' \right\|_{L^2_t L^\infty_x L^\infty_t} \\
\lesssim \left( \sup_{\omega'} \| G \|_{L^\infty} \right) \left( 2^{\frac{j}{2}} |\nu - \nu'| + 2^{\frac{j}{2}} |\nu - \nu'| + 1 \right) \varepsilon 2^{-\frac{j}{2}} \left\| \eta_j (\omega') \right\|_{L^2_t} \left\| F_j (u') \right\|_{L^2_t} \\
\lesssim \left( \sup_{\omega'} \| G \|_{L^\infty} \right) \left( 2^{\frac{j}{2}} |\nu - \nu'| + 2^{\frac{j}{2}} + 2^{\frac{j}{2}} \left( 2^{\frac{j}{2}} |\nu - \nu'| \right)^{\frac{1}{2}} \right) \varepsilon \gamma_j^{\nu'},
\]
where we used in the last inequality Plancherel in $u'$ and the fact that
\[
2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1
\]
since $\nu \neq \nu'$. This conclude the proof of the corollary.  

Lemma 7.8 also yields a third corollary.

**Corollary 7.11** Let $\omega, \nu, \omega', \nu'$ four angles in $S^2$ such that $\omega$ belongs to the patch of center $\nu$ and $\omega'$ belongs to the patch of center $\nu'$. Let $u = u(t, x, \omega)$. Let $u' = u(t, x, \omega')$, and $L', L', \nabla'$ corresponding to $u'$, and let $L, L', \nabla'$ corresponding to $u(t, x, \nu')$. Let $q \in \mathbb{N}$. Then, we have the following estimate:
\[
\left\| \int_{S^2} b^{-1} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j (u') \eta_j (\omega') \right\|_{L^2_t L^\infty_x L^\infty_t} \\
\lesssim (1 + q^{\frac{3}{2}}) \varepsilon \left( 2^{\frac{j}{2}} |\nu - \nu'| + 1 \right) \gamma_j^{\nu'}.
\] (7.83)

**Proof** We apply (7.65) with the choice
\[
H = \int_{S^2} b^{-1} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j (u') \eta_j (\omega') d\omega'.
\] (7.84)
We have:

\[ \|H\|_{L^2(u,v,L_t^\infty)}^2 \lesssim \|H\|_{L^2(M)}\|L_{\nu}(H)\|_{L^2(M)}|\nu - \nu'|^2 + \|H\|_{L^2(M)}\|\nabla_{\nu}(H)\|_{L^2(M)}|\nu - \nu'| + \int_M |H|L_{\nu}(H)|dM + \|H\|_{L^2(M)}^2 \]

\[ \lesssim \|H\|_{L^2(M)}\|L_{\nu}(H)\|_{L^2(M)}|\nu - \nu'|^2 + \|H\|_{L^2(M)}\|\nabla_{\nu}(H)\|_{L^2(M)}|\nu - \nu'| + \|H\|_{L^2_{u,v,L_t^\infty}}\|L_{\nu}(H)\|_{L^2_{u,v,L_t^\infty}} L_1 + \|H\|_{L^2(M)}^2, \quad (7.85) \]

where we used the fact that $|\omega - \nu'| \sim |\nu - \nu'|$. Now, the estimate of the $L^2_{u,v,L_t^\infty}$ norm of oscillatory integrals (7.16) yields:

\[ \|H\|_{L^2_{u,v,L_t^\infty}} \lesssim (1 + q^2)\varepsilon \gamma_j'. \quad (7.86) \]

Furthermore in order to prove the estimate (7.16), we actually obtain the following estimate (see (7.17)):

\[ \|L_{\nu}(H)\|_{L^2_{u,v,L_t^\infty}} L_1 \lesssim (1 + q^2)\varepsilon \gamma_j'. \quad (7.87) \]

Together with (7.85) and (7.86), we obtain:

\[ \|H\|_{L^2_{u,v,L_t^\infty}}^2 \lesssim (1 + q^2)\varepsilon \gamma_j' \left( \|L_{\nu}(H)\|_{L^2(M)}|\nu - \nu'|^2 + \|\nabla_{\nu}(H)\|_{L^2(M)}|\nu - \nu'| \right) + (1 + q^2)^2\varepsilon^2(\gamma_j')^2. \quad (7.88) \]

Next, we estimates the various term in the right-hand side of (7.88) starting with the one involving $L_{\nu}(H)$. In view of (7.84), we have:

\[ L_{\nu}(H) = 2i \int_{S^2} g(L_{\nu},L')b^{-1}\text{tr}\chi' \left( 2^{1/2}(N' - N_{\nu}) \right)^q F_{j,1}(u')\eta_j'(\omega') d\omega' \]

\[ + \int_{S^2} L_{\nu}(b^{-1}\text{tr}\chi') \left( 2^{1/2}(N' - N_{\nu}) \right)^q F_j(u')\eta_j'(\omega') d\omega' \]

\[ + q2^{1/2} \int_{S^2} b^{-1}\text{tr}\chi' (D_{L_{\nu}} L' - D_{L_{\nu}} L_{\nu}) \left( 2^{1/2}(N' - N_{\nu}) \right)^{q-1} F_j(u')\eta_j'(\omega') d\omega' \]

\[ = 2i \int_{S^2} g(L_{\nu},L')b^{-1}\text{tr}\chi' \left( 2^{1/2}(N' - N_{\nu}) \right)^q F_{j,1}(u')\eta_j'(\omega') d\omega' \]

\[ + \int_{S^2} L_{\nu}(b^{-1}\text{tr}\chi') \left( 2^{1/2}(N' - N_{\nu}) \right)^q F_j(u')\eta_j'(\omega') d\omega' \]

\[ + q2^{1/2} \int_{S^2} b^{-1}\text{tr}\chi' D_{L_{\nu}} L' \left( 2^{1/2}(N' - N_{\nu}) \right)^{q-1} F_j(u')\eta_j'(\omega') d\omega' \]

\[ - q2^{1/2} D_{L_{\nu}} L_{\nu} \int_{S^2} b^{-1}\text{tr}\chi' \left( 2^{1/2}(N' - N_{\nu}) \right)^{q-1} F_j(u')\eta_j'(\omega') d\omega' \]
where we used the fact that \( L_{\nu'}(u) = g(L_{\nu'}, L') \) and \( N' - N_{\nu'} = L' - L_{\nu'} \). This yields:

\[
\left\| L_{\nu'}(u) \right\|_{L^2(M)} \lesssim \left\| \int_{S^2} g(L_{\nu'}, L') b^{-1} \text{tr} \chi' \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} + \left\| \int_{S^2} L_{\nu'}(b^{-1} \text{tr} \chi') \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \\
+ q^2 \left\| \int_{S^2} b^{-1} \text{tr} \chi' \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^{-1} F_{j}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} + q^2 \left\| D L_{\nu'} \left( L_{\nu'} \right) \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^{-1} F_{j}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)}
\]

Using the estimate of the \( L^2_{u,v} L^\infty \) norm of oscillatory integrals (7.16) for the first and the last term in the right-hand side, and the basic estimate in \( L^2(M) \) (7.1) for the second and the third term in the right-hand side, we obtain:

\[
\left\| L_{\nu'}(H) \right\|_{L^2(M)} \lesssim 2^j (1 + q^2) \varepsilon \gamma_j^{\nu'} + \sup_{\omega} \left\| \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^{-1} \gamma_j^{\nu'} \right\|_{L^2(M)}
\]

where we used in the last inequality the Ricci equations (2.17) for \( D L_{\nu'} \) and \( D L' \), the estimate (2.38) for \( b \), the estimate (2.36) for \( n \), the estimate (2.37) for \( k \), the estimates (2.39) (2.40) for \( \chi \), the estimate (2.41) for \( \zeta \), and the estimate (2.42) for \( \partial_\omega N \).

Next, we estimate the term in the right-hand side of (7.88) involving \( \nabla_{\nu'}(H) \). In view of (7.84), we have for \( A = 1, 2 \):

\[
\nabla_{(e_{\nu'})_A}(H) = i2^j \int_{S^2} g((e_{\nu'})_A, L') b^{-1} \text{tr} \chi' \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,1}(u')\eta_j^{\nu'}(\omega') d\omega' + \left\| \int_{S^2} (e_{\nu'})_A (b^{-1} \text{tr} \chi') \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \\
+ q^2 \left\| \int_{S^2} b^{-1} \text{tr} \chi' \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^{-1} F_{j}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} + q^2 \left\| D (e_{\nu'})_A \left( L' \right) \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^{-1} F_{j}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)}
\]
where we used the fact that:
\[(e_{\nu})(u) = g((e_{\nu}), L') = g((e_{\nu}), L' - L_{\nu}) = g((e_{\nu}), N' - N_{\nu})\]
and \(N' - N_{\nu} = L' - L_{\nu}\). This yields:
\[
\| \nabla (e_{\nu})(H) \|_{L^2(M)} \\
\lesssim 2^\frac{1}{2} \int_{S^2} b^{-1} \text{tr} \chi' \left( 2^\frac{1}{2} (N' - N_{\nu}) \right) q^{1+} F_{j,1}(u') \eta_j' (\omega') d\omega' \\
+ \int_{S^2} (e_{\nu})(b^{-1} \text{tr} \chi) \left( 2^\frac{1}{2} (N' - N_{\nu}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \\
+ q 2^\frac{1}{2} \int_{S^2} b^{-1} \text{tr} \chi' D(e_{\nu}) L' \left( 2^\frac{1}{2} (N' - N_{\nu}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \\
+ q 2^\frac{1}{2} \| D(e_{\nu}) L' \|_{L^\infty_{u,\nu,\nu'}} \left( \int_{S^2} b^{-1} \text{tr} \chi' \left( 2^\frac{1}{2} (N' - N_{\nu}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right)_{L^2(M)}.
\]
Using the estimate of the \(L^2_{u,\nu'} L^\infty_t\) norm of oscillatory integrals (7.16) for the first and the last term in the right-hand side, and the basic estimate in \(L^2(M)\) (7.1) for the second term in the right-hand side, we obtain:
\[
\| \nabla (e_{\nu})(H) \|_{L^2(M)} \lesssim 2^\frac{1}{2} \left( 1 + q^2 \right) \epsilon \gamma_j' + \sup_{\omega} \left( \| D(b^{-1} \text{tr} \chi) \|_{L^\infty_{u,\nu}} \int \left( 2^\frac{1}{2} (N' - N_{\nu}) \right)^q \right)_{L^\infty_t} 2^\frac{1}{2} \gamma_j' \\
+ q 2^\frac{1}{2} \int_{S^2} b^{-1} \text{tr} \chi' D(e_{\nu}) L' \left( 2^\frac{1}{2} (N' - N_{\nu}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \\
+ q 2^\frac{1}{2} \| D(e_{\nu}) L' \|_{L^\infty_{u,\nu,\nu'}} \left( \int_{S^2} b^{-1} \text{tr} \chi' \left( 2^\frac{1}{2} (N' - N_{\nu}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right)_{L^2(M)} \\
\lesssim 2^\frac{1}{2} \left( 1 + q^3 \right) \epsilon \gamma_j' \\
+ q 2^\frac{1}{2} \epsilon \int_{S^2} b^{-1} \text{tr} \chi' D(e_{\nu}) L' \left( 2^\frac{1}{2} (N' - N_{\nu}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \|_{L^2(M)}
\]
where we used in the last inequality the Ricci equations (2.17) for \(D(e_{\nu}) L'\), the estimate (2.38) for \(b\), the estimate (2.37) for \(k\), the estimates (2.39) (2.40) for \(\chi\), and the estimate (2.42) for \(\partial_\nu N\). Next, we decompose the term \(D(e_{\nu}) L'\) in the right-hand side of (7.90). First, we decompose \((e_{\nu})\) on the frame \(L', L', \epsilon_{B}, B = 1, 2\). We have:
\[
(e_{\nu}) = -\frac{1}{2} g((e_{\nu}), L') L' - \frac{1}{2} g((e_{\nu}), L') L' + ((e_{\nu}) - g((e_{\nu}), N')(N').
\]
Together with the Ricci equations (2.17), this yields, schematically:
\[
D(e_{\nu}) L' = (\chi + \epsilon)(e_{\nu}) A + (N' - N_{\nu})(\chi + \epsilon + \xi + \delta + n^{-1} \nabla N n).
\]
In view of the decompositions (2.74) (2.72) for \(\chi\), the fact that \(\epsilon_A = k_{N,A}\) with \(k\) independent of \(\omega\), the estimate (2.36) for \(n\), the estimate (2.37) for \(\delta, \epsilon A\) and \(k\), the estimate (2.39) (2.40) for \(\chi\), and the estimate (2.41) for \(\xi\), we obtain the following decomposition:
\[
D(e_{\nu}) L' = F_1^i + F_2^j,
\]
where the tensor $F_{j}^{1}$ only depends on $\nu'$ and satisfies:

$$\|F_{j}^{1}\|_{L_{u,v',\omega',t}^{\infty}} \lesssim \varepsilon \tag{7.92}$$

and the tensor $F_{j}^{2}$ satisfies:

$$\|F_{j}^{2}\|_{L_{u,v,t}^{2}(H_{u})} \lesssim 2^{-\frac{1}{2}} \varepsilon. \tag{7.93}$$

In view of (7.91), we obtain:

$$\int_{S_{2}} b'^{-1} \text{tr} \chi'(D_{(e',\nu')}_{\omega} L' \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega'$$

$$= F_{j}^{1} \int_{S_{2}} b'^{-1} \text{tr} \chi' \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega'$$

$$+ \int_{S_{2}} b'^{-1} \text{tr} \chi' F_{j}^{2} \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega'$$

which yields:

$$\left\| \int_{S_{2}} b'^{-1} \text{tr} \chi'(D_{(e',\nu')}_{\omega} L' \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega' \right\|_{L^{2}(\mathcal{M})}$$

$$\lesssim \|F_{j}^{1}\|_{L_{u,v',\omega',t}^{\infty}} \left\| \int_{S_{2}} b'^{-1} \text{tr} \chi' \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega' \right\|_{L^{2}(\mathcal{M})}$$

$$+ \left\| \int_{S_{2}} b'^{-1} \text{tr} \chi' F_{j}^{2} \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega' \right\|_{L^{2}(\mathcal{M})}$$

$$\lesssim \varepsilon \left\| \int_{S_{2}} b'^{-1} \text{tr} \chi' \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega' \right\|_{L^{2}(\mathcal{M})}$$

$$+ \left\| \int_{S_{2}} b'^{-1} \text{tr} \chi' F_{j}^{2} \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega' \right\|_{L^{2}(\mathcal{M})}$$

where we used in the last inequality the estimate (7.92). Using the estimate of the $L_{u,v',\omega',t}^{\infty}$ norm of oscillatory integrals (7.16) for the first term in the right-hand side, and the basic estimate in $L^{2}(\mathcal{M})$ (7.1) for the second term in the right-hand side, we obtain:

$$\left\| \int_{S_{2}} b'^{-1} \text{tr} \chi'(D_{(e',\nu')}_{\omega} L' \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} F_{j}(u') \eta_{j}'(\omega') d\omega' \right\|_{L^{2}(\mathcal{M})}$$

$$\lesssim \varepsilon (1 + q^{2}) \gamma_{j}' + \sup_{\omega'} \left( \|F_{j}^{2}\|_{L_{u,v',\omega',t}^{\infty}(H_{u})} \left(2^{\frac{1}{2}} (N' - N_{\nu'})\right)^{q-1} \right\|_{L^{2}(\mathcal{M})} 2^{\frac{1}{2}} \gamma_{j}'$$

$$\lesssim \varepsilon (1 + q^{2}) \gamma_{j}' ,$$

where we used in the last inequality the estimate (7.93), the estimate (2.42) for $\partial_{\omega}N$, and the size of the patch. Together with (7.90), we finally obtain:

$$\|\mathbf{\nabla}_{(e',\nu')}_{\omega} (H)\|_{L^{2}(\mathcal{M})} \lesssim 2^{\frac{1}{2}} (1 + q^{2}) \varepsilon \gamma_{j}'. \tag{7.94}$$
Together with (7.88) and (7.89), this yields:
\[
\|H\|_{L^2_{\nu}, L^\infty_{\nu}}^2 \lesssim (1 + q^2)\varepsilon^2 (\gamma_j')^2 \left(2^j |\nu - \nu'| + 2^j |\nu - \nu'| + 1\right).
\]
This concludes the proof of the lemma.

We have finally a last corollary of Lemma 7.8.

**Corollary 7.12** Let \(\omega, \nu, \omega', \nu'\) four angles in \(S^2\) such that \(\omega\) belongs to the patch of center \(\nu\) and \(\omega'\) belongs to the patch of center \(\nu'\). Let \(u = u(t, x, \omega)\). Let \(u' = u(t, x, \omega')\), and \(L', L', \nabla'\) corresponding to \(u'\), and let \(L_{\nu'}, L_{\nu'}, \nabla_{\nu'}\) corresponding to \(u(t, x, \nu')\). Let \(q \in \mathbb{N}\). Then, we have the following estimate:

\[
\boxed{
\left\| \int_{S^2} (b' - b_{\nu'}) \text{tr} \chi' \left(2^j (N' - N_{\nu'})\right)^q F_j(u')\eta^{\nu'}_j (\omega') d\omega' \right\|_{L^2_{\nu}, L^\infty_{\nu}} \lesssim 2^{-\frac{j}{4}} (1 + 2^j |\nu - \nu'| + 1) \gamma_j'.
\]

**Proof** We apply (7.65) with the choice
\[
H = \int_{S^2} (b' - b_{\nu'}) \text{tr} \chi' \left(2^j (N' - N_{\nu'})\right)^q F_j(u')\eta^{\nu'}_j (\omega') d\omega'.
\]
As in (7.85), we have:
\[
\|H\|_{L^2_{\nu}, L^\infty_{\nu}}^2 \lesssim \|H\|_{L^2(M)} \|L_{\nu'}(H)\|_{L^2(M)} |\nu - \nu'|^2 + \|H\|_{L^2(M)} \|\nabla_{\nu'}(H)\|_{L^2(M)} |\nu - \nu'|
+ \|H\|_{L^2_{\nu}, L^\infty_{\nu}} \|L_{\nu'}(H)\|_{L^2_{\nu}, L^\infty_{\nu}} L^2_{\nu} + \|H\|_{L^2(M)}^2.
\]
We first estimate \(\|H\|_{L^2(M)}\). Recall the decomposition (2.79) for \(b - b_{\nu}\). We have:
\[
b - b_{\nu} = 2^{-\frac{j}{2}} (f^j_1 + f^j_2)
\]
where the tensor \(f^j_1\) only depends on \(\nu\) and satisfies:
\[
\|f^j_1\|_{L^\infty} \lesssim \varepsilon,
\]
and where the tensor \(f^j_2\) satisfies:
\[
\|f^j_2\|_{L^\infty, L^2(H_u)} \lesssim \varepsilon 2^{-\frac{j}{4}}.
\]
This yields:
\[
\left\| \int_{S^2} (b' - b_{\nu'}) \text{tr} \chi' \left(2^j (N' - N_{\nu'})\right)^q F_j(u')\eta^{\nu'}_j (\omega') d\omega' \right\|_{L^2(M)} \leq 2^{-\frac{j}{4}} \|f^j_1\|_{L^\infty(M)} \left\| \int_{S^2} \text{tr} \chi' \left(2^j (N' - N_{\nu'})\right)^q F_j(u')\eta^{\nu'}_j (\omega') d\omega' \right\|_{L^2(M)}
+ 2^{-\frac{j}{4}} \left\| \int_{S^2} f^j_2 \text{tr} \chi' \left(2^j (N' - N_{\nu'})\right)^q F_j(u')\eta^{\nu'}_j (\omega') d\omega' \right\|_{L^2(M)}
\leq 2^{-\frac{j}{4}} \varepsilon \left\| \int_{S^2} \text{tr} \chi' \left(2^j (N' - N_{\nu'})\right)^q F_j(u')\eta^{\nu'}_j (\omega') d\omega' \right\|_{L^2(M)}
+ 2^{-\frac{j}{4}} \left\| \int_{S^2} f^j_2 \text{tr} \chi' \left(2^j (N' - N_{\nu'})\right)^q F_j(u')\eta^{\nu'}_j (\omega') d\omega' \right\|_{L^2(M)}.
\]
where we used in the last inequality the estimate (7.99) for \( f_1^j \). We control the first term in the right-hand side of (7.101) using the estimate (7.64), and the second term in the right-hand side of (7.101) using the basic estimate in \( L^2(\mathcal{M}) \) (7.1). We obtain:

\[
\left\| \int_{S^2} (b' - b_{\nu'}) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta'_j (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
\lesssim \left( \sup_{\omega'} \left\| \int_{S^2} (b' - b_{\nu'}) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^2(\mathcal{H}_{\nu'})} \right) \gamma_j' \\
\lesssim \left( \sup_{\omega'} \left\| \int_{S^2} (b' - b_{\nu'}) \text{tr} \chi' \right\|_{L^\infty} \left\| \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) \gamma_j' \\
\lesssim 2^{-\frac{j}{4}} (1 + q^2) \gamma_j', \quad (7.102)
\]

where we used in the last inequality the estimate (7.100) for \( f^j_2 \), the estimate (2.39) for \( \text{tr} \chi' \), the estimate (2.42) for \( \partial_{\nu} N \) and the size of the patch. In view of the definition (7.96) of \( H \), this yields:

\[
\| H \|_{L^2(\mathcal{M})} \lesssim 2^{-\frac{j}{4}} (1 + q^2) \gamma_j'. \quad (7.103)
\]

Together with (7.97), we obtain:

\[
\| H \|_{L^2_{u',\nu',x'\mid L\infty}}^2 \lesssim \left( \| L_{\nu'}(H) \|_{L^2(\mathcal{M})} |\nu - \nu'|^2 + \| \nabla_{\nu'}(H) \|_{L^2(\mathcal{M})} |\nu - \nu'| \right) 2^{-\frac{j}{4}} (1 + q^2) \gamma_j' \\
+ \| H \|_{L^2_{u',\nu,x',\nu',L\infty}} L^2_{u',x',\nu,L^2} + 2^{-\frac{j}{4}} (1 + q^4) \gamma_j' \| H \|_{L^2_{u',x',\nu'}} \quad (7.104)
\]

Next, we define:

\[
H_1 = \int_{S^2} (L_{\nu'}(b') - L_{\nu'}(b_{\nu'})) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta'_j (\omega') d\omega', \quad (7.105)
\]

\[
H_2 = \int_{S^2} (\nabla_{\nu'}(b') - \nabla_{\nu'}(b_{\nu'})) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta'_j (\omega') d\omega', \quad (7.106)
\]

\[
H_3 = \int_{S^2} (L_{\nu'}(b') - L_{\nu'}(b_{\nu'})) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta'_j (\omega') d\omega', \quad (7.107)
\]

and:

\[
H'_1 = L_{\nu'}(H) - H_1, \quad H'_2 = \nabla_{\nu'}(H) - H_2 \quad \text{and} \quad H'_3 = L_{\nu'}(H) - H_3. \quad (7.108)
\]

The terms \( H_1, H_2, H_3 \) denote the contributions in the right-hand side of (7.110) where the derivatives \( L_{\nu'}, \nabla_{\nu'}, L_{\nu'} \) fall on \( b \). The terms \( H'_1, H'_2, H'_3 \) are the ones already treated in the proof of Corollary 7.11 up to the presence of the extra term \( b' - b_{\nu'} \) which is evaluated in \( L^\infty \) norm. In view of the estimate (2.44) for \( \partial_{\nu} b \) and the size of the patch, we have:

\[
\| b' - b_{\nu'} \|_{L^\infty} \lesssim 2^{-\frac{j}{4}}. \quad (7.109)
\]

Thus, in view of the estimates (7.87), (7.89) and (7.94) of the proof of Corollary 7.11, and taking into account the extra \( 2^{-\frac{j}{4}} \) factor coming from (7.109), we obtain the analog of (7.87) (7.89) (7.94):

\[
\| H'_3 \|_{L^2_{u',x',\nu',L^4}} \lesssim 2^{-\frac{j}{4}} (1 + q^2) \epsilon \gamma_j'.
\]
\[ \|H'_1\|_{L^2(M)} \lesssim 2^{\frac{j}{2}}(1 + q^3)\varepsilon \gamma_j', \]

and:
\[ \|H'_2\|_{L^2(M)} \lesssim (1 + q^3)\varepsilon \gamma_j'. \]

Together with (7.104) and in view of the decompositions (7.108), we get:

\[ \|H\|^2_{L^2_{u,\nu},t_t^\infty} \lesssim ((\|H_1\|_{L^2(M)}|\nu - \nu'|^2 + \|H_2\|_{L^2(M)}|\nu - \nu'|)^{2-\frac{j}{2}}\varepsilon (1 + q^2)\gamma_j')^{2-\frac{j}{2}}(\gamma_j')^2. \tag{7.110} \]

Next, we evaluate both terms in the right-hand side of (7.111) starting with the first one. Using the basic estimate in \( L^2(M) \), we have:

\[ \left\| \int_{S^2} L_{\nu'}(b')\operatorname{tr}\chi' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u')\eta_j'(\omega')d\omega' \right\|_{L^2(M)} \]  
\[ \lesssim \left( \sup_{\omega'} \left\| L_{\nu'}(b')\operatorname{tr}\chi' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty L^2(H_u)} \right)^{2^{\frac{j}{2}}(\gamma_j')^2}. \tag{7.112} \]

Now, we have:

\[ \left\| L_{\nu'}(b')\operatorname{tr}\chi' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty_{u} L^2(H_u)} \lesssim \|D\|_{L^\infty_{u} L^2(H_u)}\|\operatorname{tr}\chi'\|_{L^\infty} \left\| \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \lesssim \varepsilon \]

where we used in the last inequality the estimate (2.38) for \( b' \), the estimate (2.39) for \( \operatorname{tr}\chi' \), the estimate (2.42) for \( \partial_u N \) and the size of the patch. Together with (7.112), we obtain:

\[ \left\| \int_{S^2} L_{\nu'}(b')\operatorname{tr}\chi' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u')\eta_j'(\omega')d\omega' \right\|_{L^2(M)} \lesssim \varepsilon 2^{\frac{j}{2}}\gamma_j'. \tag{7.113} \]

Next, we estimate the second term in the right-hand side of (7.111). We have:

\[ \left\| L_{\nu'}(b') \left( \int_{S^2} \operatorname{tr}\chi' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u')\eta_j'(\omega')d\omega' \right) \right\|_{L^2(M)} \]  
\[ \lesssim \|L_{\nu'}(b')\|_{L^2(M)} \left\| \int_{S^2} \operatorname{tr}\chi' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u')\eta_j'(\omega')d\omega' \right\|_{L^4(M)} \]  
\[ \lesssim \varepsilon \left\| \int_{S^2} \operatorname{tr}\chi' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u')\eta_j'(\omega')d\omega' \right\|_{L^4(M)}, \]
where we used in the last inequality the estimate (2.38) for \( b \). Now, we have:

\[
\left\| \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^2(M)} \\
\lesssim \left\| \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j''(\omega') d\omega' \right\|_{L^2_{u', x', \nu', L_\infty}} \\
\lesssim \varepsilon \gamma_j''
\]

(7.115)

where we used in the last inequality the estimate (2.64) of the \( L^2_{u, x, \nu, L_\infty} \) of oscillatory integrals. Also, we have:

\[
\left\| \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^\infty(M)} \\
\lesssim \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \| F_{j,-1}(u') \|_{L^\infty N_j''(\omega') d\omega'} \\
\lesssim \varepsilon \left( \int_{S^2} \| F_{j,-1}(u') \|_{L^\infty N_j''(\omega') d\omega'} \right) \\
\lesssim \varepsilon 2^j \gamma_j''
\]

(7.116)

where we used, the estimate (2.39) for \( \text{tr} \chi \), the estimate (2.42) for \( \partial_\omega N \), Cauchy-Schwarz in \( \lambda' \) to estimate \( \| F_{j,-1}(u') \|_{L^\infty} \), Cauchy-Schwarz in \( \omega' \) and the size of the patch. Interpolating (7.115) and (7.116), we obtain:

\[
\left\| \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^4(M)} \lesssim 2^{\frac{j}{4}} \varepsilon \gamma_j''
\]

(7.117)

Finally, (7.111), (7.113) and (7.117) imply:

\[
\| H_1 \|_{L^2(M)} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j''
\]

(7.118)

Next, we estimates the term \( H_2 \) in the right-hand side of (7.110). In view of the definition (7.106) for \( H_2 \), we have:

\[
\| H_2 \|_{L^2(M)} \lesssim \left\| \int_{S^2} \nabla_{\nu'}(b') \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^2(M)} \\
+ \left\| \nabla_{\nu'}(b_{\nu'}) \left( \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right) \right\|_{L^2(M)}
\]

(7.119)

Next, we evaluate both terms in the right-hand side of (7.119) starting with the last one. We have:

\[
\left\| \nabla_{\nu'}(b_{\nu'}) \left( \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right) \right\|_{L^2(M)} \\
\lesssim \left\| \nabla_{\nu'}(b_{\nu'}) \right\|_{L^\infty_{u', x', \nu', L_\infty}} \left\| \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^2_{u', x', \nu', L_\infty}} \\
\lesssim \varepsilon (1 + q^2) \gamma_j''
\]

(7.120)
where we used in the last inequality the estimate (2.38) for $b$ and the estimate in $L^2_{u,v',x',\nu',t'} L^\infty_t$ (7.64). Next, we evaluate the first term in the right-hand side of (7.111). Decomposing $\nabla_{\nu'}$ on the frame $L', L', e_A$ and using the fact that:

$$|\omega' - \nu'| \lesssim 2^\frac{\nu}{2},$$

we have schematically:

$$\nabla_{\nu'}(b') = \nabla'(b') + 2^{-\frac{\nu}{2}} D(b')$$

and thus:

$$\left\| \int_{S^2} \nabla_{\nu'}(b') \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)}$$

\begin{align*}
&\lesssim \left\| \int_{S^2} \nabla'(b') \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} \\
&\quad + 2^{-\frac{\nu}{2}} \left\| \int_{S^2} D(b') \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} \\
&\lesssim \left\| \int_{S^2} \nabla'(b') \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} + \varepsilon \gamma_j',
\end{align*}

where we used in the last inequality an estimate analogous to (7.113). In order to estimate the right-hand side of (7.121), we use the decomposition (2.80) of $\nabla'(b')$. We have:

$$\nabla(b) = F_1^j + F_2^j$$

where the tensor $F_1^j$ only depends on $\nu$ and satisfies:

$$\| F_1^j \|_{L^\infty_{\nu',L^2_{u,v'},L^8_{x',\nu'}}} \lesssim \varepsilon,$$

(7.123)

and where the tensor $F_2^j$ satisfies:

$$\| F_2^j \|_{L^\infty_{\nu',L^2(H_{\nu'})}} \lesssim \varepsilon 2^{-\frac{\nu}{2}}.$$

(7.124)

In view of (7.122), we have:

\begin{align*}
&\left\| \int_{S^2} \nabla'(b') \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} \\
&\lesssim \| F_1^j \|_{L^\infty_{\nu',L^2_{u,v'},L^8_{x',\nu'}}} \left\| \int_{S^2} \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^\infty_{\nu',L^\infty_t,L^8_{x',\nu'}}} \\
&\quad + \left\| \int_{S^2} F_2^j \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} \\
&\lesssim \varepsilon \left\| \int_{S^2} \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^\infty_{\nu',L^\infty_t,L^8_{x',\nu'}}} \\
&\quad + \left\| \int_{S^2} F_2^j \text{tr} \chi' \left( 2^\frac{\nu}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)},
\end{align*}

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where we used the estimate (7.123) for $F^j_1$ in the last inequality. Now, interpolating between the the estimate in $L^2_{u',x',t'}$, $L^\infty_t$ (7.64) and the $L^\infty$ estimate (7.116), we obtain:

$$\left\| \int_{S_2^2} \text{tr} \chi' \left( 2^{\frac{3}{2}} (N' - N_u) \right)^q F_j(u') \eta_j^\nu' (\nu') d\omega' \right\|_{L^2_{u',x',t'}} \lesssim 2^{\frac{3}{2}} \varepsilon \gamma_j^\nu'. \quad (7.126)$$

For the second term in the right-hand side of (7.125), we have:

$$\left\| \int_{S_2^2} F_j^2 \text{tr} \chi' \left( 2^{\frac{3}{2}} (N' - N_u) \right)^q F_j(u') \eta_j^\nu' (\nu') d\omega' \right\|_{L^2(M)}/L^2(M) \lesssim \int_{S_2^2} \| F_j^2 \text{tr} \chi' \left( 2^{\frac{3}{2}} (N' - N_u) \right)^q F_j(u') \|_{L^2(M)} \| \text{tr} \chi' \left( 2^{\frac{3}{2}} (N' - N_u) \right)^q \|_{L^\infty(M)} \eta_j^\nu' (\nu') d\omega'. \quad (7.127)$$

Together with the estimate (7.124) for $F_j^2$, the estimate (2.39) for tr $\chi$, the estimate (2.42) for $\partial_\nu N$ and the size of the patch, we obtain:

$$\left\| \int_{S_2^2} F_j^2 \text{tr} \chi' \left( 2^{\frac{3}{2}} (N' - N_u) \right)^q F_j(u') \eta_j^\nu' (\nu') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon \int_{S_2^2} 2^{-\frac{1}{2}} \varepsilon \| F_j(u') \|_{L^2(\nu), \eta_j^\nu' (\nu')} d\omega' \lesssim \varepsilon 2^{\frac{1}{2}} \gamma_j^\nu', \quad (7.128)$$

where we used in the last inequality Plancherel in $\nu'$ for $\| F_j(u') \|_{L^2(\nu)}$, Cauchy Schwartz in $\omega'$ and the size of the patch. Finally, (7.121), (7.125), (7.126) and (7.127) imply:

$$\left\| \int_{S_2^2} \nabla_{\nu'} (u') \text{tr} \chi' \left( 2^{\frac{3}{2}} (N' - N_u) \right)^q F_j(u') \eta_j^\nu' (\nu') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon 2^{\frac{1}{4}} \gamma_j^\nu'. \quad (7.129)$$

Together with (7.119) and (7.120), this yields:

$$\| H_2 \|_{L^2(M)} \lesssim \varepsilon 2^{\frac{1}{4}} \gamma_j^\nu'. \quad (7.130)$$

Next, we estimates the term $H_3$ in the right-hand side of (7.110). Decomposing $L_{\nu'}$ on the frame $L', L', e'_A$ and using the fact that:

$$|\omega' - \nu'| \lesssim 2^{\frac{1}{2}},$$

we have schematically:

$$L_{\nu'} (b') = L'(b') + 2^{-\frac{1}{2}} \nabla'(b') + 2^{-j} L'(b').$$

Together with the transport equation (2.23) satisfied by $b$, we obtain:

$$L_{\nu'} (b') - L_{\nu'} (b_{\nu'}) = - \delta b' + \delta_{\nu'} b_{\nu'} + 2^{-\frac{1}{2}} \nabla'(b') + 2^{-j} L'(b').$$

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In view of the definition (7.107) for $H_3$, this yields:

$$
\|H_3\|_{L^2(\mathcal{M})} \lesssim \left\| \int_{S^2} (\bar{\delta}' b' + \bar{\delta}_\nu b_\nu) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
+ 2^{-\frac{j}{2}} \left\| \int_{S^2} \nabla'(b') \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
+ 2^{-j} \left\| \int_{S^2} L'(b') \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}. 
$$

Together with an estimate analog to (7.128) and an estimate analog to (7.113), we get:

$$
\|H_3\|_{L^2(\mathcal{M})} \lesssim \left\| \int_{S^2} (\bar{\delta}' b' + \bar{\delta}_\nu b_\nu) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
+ \varepsilon 2^{-\frac{j}{2}} \gamma_j' + \varepsilon 2^{-\frac{j}{2}} \gamma_j'.
$$

Next, we estimate the right-hand side of (7.130). We have:

$$
\left\| \int_{S^2} (\bar{\delta}' b' + \bar{\delta}_\nu b_\nu) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\lesssim \left\| \int_{S^2} (\bar{\delta}' + \bar{\delta}_\nu) b' \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
+ \|\bar{\delta}_\nu\|_{L^\infty_{u_\nu', q_\nu'}} \left\| \int_{S^2} (b' + b_\nu) \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2_{u_\nu', q_\nu'} L^\infty}
\lesssim \left\| \int_{S^2} (\bar{\delta}' + \bar{\delta}_\nu) b' \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_j(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
+ \varepsilon \|H\|_{L^2_{u_\nu', q_\nu'} L^\infty}.
$$

where we used in the last inequality the estimates (2.36) (2.37) for $\bar{\delta}_\nu$ and the definition (7.96) for $H$. Now, recall the decomposition (7.46):

$$
\bar{\delta}' + \bar{\delta}_\nu = \left( n^{-1} \nabla n - 2\delta_\nu N_\nu - 2\varepsilon_\nu \right) \cdot (N' - N_\nu) - k(N' - N_\nu, N' - N_\nu).
$$

(7.132)
This yields:
\[
\left\| \int_{\mathbb{S}^2} (-\vec{\delta} + \vec{\delta}_\nu) b' \partial_x' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \leq 2^{-\frac{j}{2}} \left\| n^{-1} \nabla n - 2 \delta_\nu N_{\nu'} - 2 \epsilon_\nu \right\|_{L^\infty_{u_\nu', \nu'} L^2_u} \times \left\| \int_{\mathbb{S}^2} b' \partial_x' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^{q+1} F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^2_{u_\nu', \nu'} L^\infty_u} + 2^{-j} \| k \|_{L^6(\mathcal{M})} \left\| \int_{\mathbb{S}^2} b' \partial_x' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^{q+2} F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^3(\mathcal{M})}
\]
for \( n \), the estimates (2.36) (2.37) for \( \vec{\delta} \) and \( \epsilon \) and the estimate (2.37) for \( k \). (7.133) together with the estimate (7.115) and the interpolation of (7.115) with (7.116) implies:
\[
\left\| \int_{\mathbb{S}^2} (-\vec{\delta} + \vec{\delta}_\nu) b' \partial_x' \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_j(u') \eta_j''(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{-\frac{j}{2}} \epsilon \gamma_j''.
\]
Finally, (7.130), (7.131) and (7.134) imply:
\[
\| H_3 \|_{L^2(\mathcal{M})} \lesssim \| H \|_{L^2_{u_\nu', \nu'} L^\infty_u} + \epsilon 2^{-\frac{j}{2}} \gamma_j''.
\]
Finally, (7.110), (7.118), (7.129) and (7.135) yield:
\[
\| H \|_{L^2_{u_\nu', \nu'} L^\infty_u} \lesssim (2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|^2)^2 + 2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|) 2^{-\frac{j}{2}} \epsilon^2 (1 + q^2) (\gamma_j'')^2 + \epsilon \| H \|_{L^2_{u_\nu', \nu'} L^\infty_u} + \epsilon 2^{-\frac{j}{2}} \gamma_j'' \| H \|_{L^2_{u_\nu', \nu'} L^\infty_u} + (1 + q^5) 2^{-\frac{j}{2}} (1 + (2^{\frac{j}{2}} |\nu - \nu'|)^2) \epsilon^2 (\gamma_j'')^2.
\]
This implies:
\[
\| H \|_{L^2_{u_\nu', \nu'} L^\infty_u} \lesssim (2^{-\frac{j}{2}} + 2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|^2) + 2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|^2)) (1 + q^5) \epsilon^2 (\gamma_j'')^2.
\]
This concludes the proof of the lemma.

\section*{7.4 Integration by parts}

\subsection*{7.4.1 Integration by parts in tangential directions}

\textbf{Lemma 7.13} We consider an oscillatory integral of the following form:
\[
\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b'^{-1} h(t, x) F_j(u) F_j(u') \eta_j''(\omega) \eta_j''(\omega') d\omega' d\mathcal{M},
\]
where \( h \) is a scalar function on \( M \). Integrating by parts once using (6.3) yields:

\[
\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b^{-1} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M} \quad (7.136)
\]

\[
= -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - g(N, N')^2} \left( (N' - g(N, N')N)(h) + \left( \text{tr} \theta' - g(N, N') \text{tr} \theta \right) 
- \theta'(N' - g(N, N')N, N - g(N, N')N) - g(N, N') b^{-1} (N - g(N, N')N)(b) \right) h 
+ \frac{2g(N, N')}{1 - g(N, N')^2} \left( \theta'(N' - g(N, N')N, N - g(N, N')N) 
- g(N, N') \theta'(N - g(N, N')N, N - g(N, N')N) \right) h \right) 
\times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

Also, integrating by parts once using (6.4) yields:

\[
\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b^{-1} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M} \quad (7.137)
\]

\[
= i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - g(N, N')^2} \left( (N - g(N, N')N)(h) + \left( \text{tr} \theta - g(N, N') \text{tr} \theta' \right) 
- \theta(N' - g(N, N')N, N' - g(N, N')N) - g(N, N') b^{-1} (N' - g(N, N')N)(b) \right) h 
+ \frac{2g(N, N')}{1 - g(N, N')^2} \left( \theta(N' - g(N, N')N, N - g(N, N')N) 
- g(N, N') \theta(N - g(N, N')N, N - g(N, N')N) \right) h \right) 
\times F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

**Proof** We have:

\[
\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b^{-1} h(t, x) F_j(u) F_j(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}
\]

\[
= \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left( \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b^{-1} b^{-1} h d\mathcal{M} \right)
\times \eta_j^\nu(\omega) \eta_j'^\nu(\omega') \psi(2^{-j} \lambda)(2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega'.
\]
We integrate by parts in tangential directions using (6.3). We obtain:

\[
\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} h(t, x) F_j(u) F_j(u') \eta^j_{\omega'} \omega') d\omega d\omega' d\mathcal{M}
\]

\[
= -i 2^j \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{\infty} \int_0^{\infty} \left( \int_{\mathcal{M}} e^{i \lambda u - i \lambda u'} b^{-1} \frac{1}{1 - g(N, N')^2} (N' - g(N, N') N)(h) \right. \\
+ (b(N' - g(N, N') N)(b^{-1}) + \text{div}_g(N' - g(N, N') N)) h \\
+ 2g(N, N')(N' - g(N, N') N)(g(N, N') h) \left. \right) \left( \frac{1}{1 - g(N, N')^2} \right) d\omega d\omega' d\mathcal{M}
\]

\[
\times \eta^j_{\omega'} (\omega')(2^{-j} \lambda')^{-1} \psi(2^{-j} \lambda')(2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda') f(\lambda') \lambda^2 \lambda^2 d\lambda d\lambda' d\omega d\omega',
\]

where \(\text{div}_g(N' - g(N, N') N)\) denotes the space-time divergence of \(N' - g(N, N') N\).

Next, we consider the various terms in the right-hand side of (7.138). Using (2.21), we have:

\[
b(N' - g(N, N') N)(b^{-1}) + \text{div}_g(N' - g(N, N') N) = \text{tr} \theta - g(N, N') \theta - \theta (N - g(N, N') N', N - g(N, N') N') \\
+ g(N, N')(N - g(N, N') N)(b')
\]

where we used the decomposition of \(N\) in the frame \(N', e_A\):

\[
N = g(N, N') N' + (N - g(N, N') N'),
\]

and the decomposition of \(N'\) in the frame \(N, e_A\):

\[
N' = g(N, N') N + (N' - g(N, N') N),
\]

and where \(\theta\) is the second fundamental form of \(P_t, u\) in \(\Sigma_t\). We also have in view of (2.21), (7.140) and (7.141):

\[
(N' - g(N, N') N)(g(N, N')) \]

\[
= (g(N, N')^2 - 1)b^{-1} (N - g(N, N') N')(b') + \theta (N' - g(N, N') N, N' - g(N, N') N) \\
- g(N, N') \theta (N - g(N, N') N', N - g(N, N') N').
\]

Using (7.138), (7.139) and (7.142), we obtain:

\[
\int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} h(t, x) F_j(u) F_j(u') \eta^j_{\omega'} (\omega') d\omega d\omega' d\mathcal{M}
\]

\[
= -i 2^j \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - g(N, N')^2} \left( \frac{1}{1 - g(N, N')^2} \right) (N' - g(N, N') N)(h) + \left( \text{tr} \theta - g(N, N') \text{tr} \theta \\
- \theta (N - g(N, N') N', N - g(N, N') N') - g(N, N') b^{-1} (N - g(N, N') N')(b') \right) h \\
+ 2g(N, N') \left( \frac{1}{1 - g(N, N')^2} \right) \left( \theta (N' - g(N, N') N, N' - g(N, N') N) \\
- g(N, N') \theta (N - g(N, N') N', N - g(N, N') N') \right) h \\
\times F_j(u) F_j(u') \eta^j_{\omega'} (\omega') d\omega d\omega' d\mathcal{M},
\]
which concludes the proof of (7.136).

In order to obtain (7.137), we integrate by parts in tangential directions using (6.4) instead of (6.3). The proof is completely analogous by exchanging the role played by \( N \) and \( N' \), so we omit it. This concludes the proof of the lemma. ■

7.4.2 Integration by parts in \( L \)

**Lemma 7.14** We consider an oscillatory integral of the following form:

\[
\int_\mathcal{M} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b^{-1} h(t, x) F_j(u) F_j(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M},
\]

where \( h \) is a scalar function on \( \mathcal{M} \). Integrating by parts once using (6.5) yields:

\[
\int_\mathcal{M} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b^{-1} h(t, x) F_j(u) F_j(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}
= -i2^{-j} \int_\mathcal{M} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{g(L, L')} \left( L(h) + tr\chi h - \delta h - \tilde{\delta} h - (1 - g(N, N')) \delta' h \right)
-2 \zeta_{N - g(N, N') N} h - \frac{\chi(N - g(N, N') N', N - g(N, N') N') h}{g(L, L')}
F_j(u) F_{j, -1}(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}.
\]

Also, integrating by parts once using (6.6) yields:

\[
\int_\mathcal{M} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b^{-1} h(t, x) F_j(u) F_j(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}
= -i2^{-j} \int_\mathcal{M} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{g(L, L')} \left( L'(h) + tr\chi' h - \delta h - \tilde{\delta} h - (1 - g(N, N')) \delta' h \right)
-2 \zeta_{N' - g(N, N') N} h - \frac{\chi(N' - g(N, N') N, N' - g(N, N') N) h}{g(L, L')}
F_{j, -1}(u) F_j(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}.
\]

**Proof** We have:

\[
\int_\mathcal{M} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} b^{-1} h(t, x) F_j(u) F_j(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}
= \int_{\mathbb{S}^2 \times \mathbb{S}^2} \int_0^{+\infty} \int_0^{+\infty} \left( \int_\mathcal{M} e^{i\lambda u - i\lambda' u'} b^{-1} b^{-1} h d\mathcal{M} \right)
\times \eta_j'(\omega) \eta_j'(\omega') \psi(2^{-j} \lambda)(2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda) f(\lambda') f(\lambda'' \lambda') \lambda^2 \lambda'^2 d\lambda d\lambda d\omega d\omega'.
\]

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We integrate by parts in $L$ using (6.5). We obtain:

$$\int_M \int_{S^2 \times S^2} h(t, x) F_j(u) F_j(u') \eta^\nu_j(\omega) \eta^\nu_j'(\omega') d\omega d\omega' \, d\mathcal{M}$$

(7.145)

$$= -i 2^{-j} \int_{S^2 \times S^2} \int_0^{+\infty} \int_0^{+\infty} \left( \int_M \epsilon^{j \lambda u - i \lambda u'} \frac{b^{-1}}{g(L, L')} \left( L(h) + (bL(b^{-1}) + \text{div}_g(L))\right) \right.$$ 

$$\left. - \frac{L(g(L, L'))}{g(L, L') h} \right) d\mathcal{M}$$

$$\times \eta^\nu_j(\omega) \eta^\nu_j'(\omega') \psi(2^{-j} \lambda)(2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda\omega) f(\lambda'\omega') \lambda^2 \lambda' d\lambda d\lambda' d\omega d\omega' ,$$

where $\text{div}_g(L)$ denotes the space-time divergence of $L$.

Next, we consider the various terms in the right-hand side of (7.145). Using the Ricci equations (2.17), we have:

$$L(b^{-1}) + \text{div}_g(L) = b^{-1} \text{tr}_L$$

(7.146)

and:

$$L(g(L, L')) = -\bar{\delta}_g(L, L') + g(L, D_L L').$$

(7.147)

We decompose $L$ on the frame $(L', L', e_A')$:

$$L = \frac{1}{2}(1 + g(N, N')) L' + \frac{1}{2}(1 - g(N, N')) L' + N - g(N, N') N' ,$$

(7.148)

where the vector $N - g(N, N') N'$ is tangent to $P_{t, u'}$. (7.147), (7.148) and the Ricci equations (2.17) yields:

$$L(g(L, L'))$$

(7.149)

$$= -\bar{\delta}_g(L, L') - \frac{1}{2}(1 + g(N, N')) \bar{\delta}_g(L, L') + \frac{1}{2}(1 - g(N, N')) \zeta_{N - g(N, N') N'}$$

$$+ \frac{1}{2}(1 - g(N, N'))((\delta' + n^{-1} \nabla_N n) g(L, L') + \chi(N - g(N, N') N', N - g(N, N') N')$$

$$- \zeta_{N - g(N, N') N'} g(L, L')$$

Using (7.145), (7.146) and (7.149), we obtain:

$$\int_M \int_{S^2 \times S^2} h(t, x) F_j(u) F_j(u') \eta^\nu_j(\omega) \eta^\nu_j'(\omega') d\omega d\omega' \, d\mathcal{M}$$

$$= -i 2^{-j} \int_{S^2 \times S^2} \int_0^{+\infty} \int_0^{+\infty} \left( \int_M \epsilon^{j \lambda u - i \lambda u'} \frac{b^{-1}}{g(L, L')} \left( L(h) + \text{tr}_L h - \bar{\delta}_h - \bar{\delta}_h' - (1 - g(N, N')) \delta' h$$

$$- 2\zeta_{N - g(N, N') N'} h - \frac{\chi(N - g(N, N') N', N - g(N, N') N')}{g(L, L')} h \right)$$

$$F_j(u) F_j(u') \eta^\nu_j(\omega) \eta^\nu_j'(\omega') d\omega d\omega' \, d\mathcal{M} ,$$

where we also used the identity:

$$g(L, L') = -1 + g(N, N') .$$
This concludes the proof of (7.143).

In order to obtain (7.144), we integrate by parts in $L'$ using (6.6) instead of (6.5). The proof is completely analogous by exchanging the role played by $L$ and $L'$, so we omit it. This concludes the proof of the lemma.

\section{Proof of Proposition 6.5}

Since $2^{\min(l,m)} \leq 2^j|\nu - \nu'|$, we may assume that $l > m$ and thus:

\begin{equation}
m < l \text{ and } 2^m \leq 2^j|\nu - \nu'|.
\end{equation}

In order to prove Proposition 6.5, recall that we need to exhibit a decomposition:

\begin{equation}
\int_{\mathcal{M}} E^{\nu l}_j f(t,x) E^{\nu' m}_j f(t,x) d\mathcal{M} = A_{j,\nu,\nu',l,m} + B_{j,\nu,\nu',l,m},
\end{equation}

where $B_{j,\nu,\nu',l,m}$ satisfies:

\begin{equation}
\mathcal{L}
\left[
\sum_{(l,m)/2^{\min(l,m)} \leq 2^j|\nu - \nu'|} (B_{j,\nu,\nu',l,m} + B_{j,\nu,\nu',l,m})
\right]
\end{equation}

\begin{align*}
&\leq \left[ \frac{1}{(2^j|\nu - \nu'|)^3} + \frac{1}{(2^{j}|\nu - \nu'|)^{3/2}} + \frac{1}{2^{j}(2^j|\nu - \nu'|)^{3/2}} + \frac{2^{-(1/2) - j}}{(2^j|\nu - \nu'|)^2} + \frac{1}{2^{j}|\nu - \nu'|^{1/2}} \right] \varepsilon^{2^j|\nu - \nu'|}.
\end{align*}

We have:

\begin{equation}
\int_{\mathcal{M}} E^{\nu l}_j f(t,x) E^{\nu' m}_j f(t,x) d\mathcal{M} = \int_{\mathcal{M}} \int_{S^2 \times S^2} b^{-1} p_{\nu l} \nu' - 1 p_{m \nu'} \nu' \nabla_f(u) \nabla_f(u') \eta_{\nu l} \nabla_{\nu} \eta_{\nu'} \nabla_{\nu'} \omega \omega' d\omega d\omega' d\mathcal{M}.
\end{equation}

We first integrate by parts in $L$ using (7.143) with the choice $h = p_{\nu l} p_{m \nu'}$. We obtain:

\begin{equation}
\int_{\mathcal{M}} E^{\nu l}_j f(t,x) E^{\nu' m}_j f(t,x) d\mathcal{M} = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} b^{-1} p_{\nu l} \nu' - 1 p_{m \nu'} \nu' \nabla_f(u) \nabla_f(u') \eta_{\nu l} \nabla_{\nu} \eta_{\nu'} \nabla_{\nu'} \omega \omega' d\omega d\omega' d\mathcal{M}.
\end{equation}
Next, we decompose $L$ on the frame $(L', N', e_A')$:

$$L = L' + (g(N, N') - 1)N' + N - g(N, N')N', \quad (8.5)$$

which yields:

$$L(P_m' \text{tr} \chi') = L'(P_m' \text{tr} \chi') + (g(N, N') - 1)N'(P_m' \text{tr} \chi') + (N - g(N, N')N')(P_m' \text{tr} \chi'). \quad (8.6)$$

Now, (8.4) and (8.6) yield:

$$\int_{\mathcal{M}} E_{\nu, l, m}^{\nu, l} f(t, x)E_{\nu, l, m}^{\nu, l} f(t, x) d\mathcal{M} = A_{j, \nu, \nu', l, m} + B_{j, \nu, \nu', l, m} \quad (8.7)$$

where $A_{j, \nu, \nu', l, m}$ is given by:

$$A_{j, \nu, \nu', l, m} = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{P_1 \text{tr} \chi(N - g(N, N')N')(P_m \text{tr} \chi')}{g(L, L')} \times F_j(u)F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M} \quad (8.8)$$

and $B_{j, \nu, \nu', l, m}$ may be decomposed as:

$$B_{j, \nu, \nu', l, m} = B_{j, \nu, \nu', l, m}^1 + B_{j, \nu, \nu', l, m}^2 \quad (8.9)$$

where $B_{j, \nu, \nu', l, m}^1$ and $B_{j, \nu, \nu', l, m}^2$ are given by:

$$B_{j, \nu, \nu', l, m}^1 = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( L(P_1 \text{tr} \chi)P_m \text{tr} \chi' + P_1 \text{tr} \chi L'(P_m \text{tr} \chi') \right) \times F_j(u)F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}, \quad (8.10)$$

and:

$$B_{j, \nu, \nu', l, m}^2 = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( (g(N, N') - 1)P_1 \text{tr} \chi N'(P_m \text{tr} \chi') + \left( \text{tr} \chi - \delta - \delta' \right) - (1 - g(N, N'))(\omega' - 2\zeta_{N-N(N, N')N'}) - \frac{\chi'(N - g(N, N')N', N - g(N, N')N')}{g(L, L')} \right) \times P_1 \text{tr} \chi P_m \text{tr} \chi' \right) F_j(u)F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. \quad (8.11)$$

The estimates satisfied by $B_{j, \nu, \nu', l, m}^1$ and $B_{j, \nu, \nu', l, m}^2$ are provided by the following propositions.

**Proposition 8.1** Let $B_{j, \nu, \nu', l, m}^1$ be given by (8.10). Then, we have the following estimate:

$$\sum_{(l, m)/2^{\min(l, m)} \leq 2|\nu - \nu'|} \left| B_{j, \nu, \nu', l, m}^1 + B_{j, \nu, \nu', l, m}^2 \right| \leq \frac{1}{(2^{\frac{3}{2}}|\nu - \nu'|)^3} + \frac{1}{(2^{\frac{3}{2}}|\nu - \nu'|)^{\frac{3}{2}}} + \frac{2^{-(\frac{11}{2})-j}}{(2^{\frac{3}{2}}|\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^j(2^{\frac{3}{2}}|\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^j(2^{\frac{3}{2}}|\nu - \nu'|)^{\frac{1}{2}}} \varepsilon^{2}\gamma_j^\nu \gamma_j^{\nu'} \quad (8.12)$$
Proposition 8.2 Let $B^2_{j,\nu,\nu',l,m}$ be given by (8.11). Then, we have the following estimate:

$$\left| \sum_{(l,m)/2\min(l,m) \leq 2|\nu-\nu'|} (B^2_{j,\nu,\nu',l,m} + B^2_{j,\nu,\nu',l,m}) \right| \lesssim \sum_{(l,m)/2\min(l,m) \leq 2|\nu-\nu'|} (B^1_{j,\nu,\nu',l,m} + B^2_{j,\nu,\nu',l,m})$$

Together with (8.12) and (8.13), this yields the estimate (8.3) and thus concludes the proof of Proposition 6.5. The rest of this section is devoted to the proof of Proposition 8.1 and Proposition 8.2.

We start with the proof of Proposition 8.1. We rewrite $B^1_{j,\nu,\nu',l,m}$ as:

$$B^1_{j,\nu,\nu',l,m} = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{g(L,L')} \left( L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right)$$

$$\times \eta_j^\nu(\omega) \eta_j^\nu(\omega')(2^{-j} \lambda)^{-1} \psi(2^{-j} \lambda)(2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \lambda^2 \lambda' \lambda^2 d\lambda d\lambda'$$

$$= B^{1,1}_{j,\nu,\nu',l,m} + B^{1,2}_{j,\nu,\nu',l,m}$$

where $B^{1,1}_{j,\nu,\nu',l,m}$ and $B^{1,2}_{j,\nu,\nu',l,m}$ are given by:

$$B^{1,1}_{j,\nu,\nu',l,m} = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{g(L,L')} \left( L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right)$$

$$\times \eta_j^\nu(\omega) \eta_j^\nu(\omega')(2^{-j} \lambda)^{-1} \psi(2^{-j} \lambda)(2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \lambda^2 \lambda'^2 d\lambda d\lambda'$$

$$\times \lambda^2 \lambda^2 d\lambda' \lambda' d\lambda d\lambda'$$

and:

$$B^{1,2}_{j,\nu,\nu',l,m} = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{g(L,L')} \left( L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi') \right)$$

$$\times \eta_j^\nu(\omega) \eta_j^\nu(\omega')(2^{-j} \lambda)^{-1} \psi(2^{-j} \lambda)(2^{-j} \lambda') \psi(2^{-j} \lambda') f(\lambda \omega) f(\lambda' \omega') \lambda^2 \lambda'^2 d\lambda d\lambda'$$

$$\times \lambda^2 \lambda'^2 d\lambda' \lambda' d\lambda d\lambda'$$

$B^{1,1}_{j,\nu,\nu',l,m}$ and $B^{1,2}_{j,\nu,\nu',l,m}$ satisfy the following estimates:
Proposition 8.3 Let $B_{j,\nu,\nu',l,m}^{1,1}$ be given by (8.15). Then, we have the following estimate:

$$
\left| \sum_{(m,l) \ 2m \leq 2|\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} + \sum_{(m,l) \ 2m \leq 2|\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} \right| \leq |B_{j,\nu,\nu',l,m}^{1,1} + B_{j,\nu,\nu',l,m}^{1,1} + B_{j,\nu,\nu',l,m}^{1,1} + \frac{\varepsilon^2 \nu_j \gamma_j'}{(2^{\frac{1}{2}}|\nu - \nu'|)^3} | \tag{8.17}
$$

Proposition 8.4 Let $B_{j,\nu,\nu',l,m}^{1,2}$ be given by (8.16). Then, we have the following estimate:

$$
\left| \sum_{(l,m) / 2\min(l,m) \leq 2|\nu - \nu'|} \left( B_{j,\nu,\nu',l,m}^{1,2} + B_{j,\nu,\nu',l,m}^{1,2} \right) \right| \leq \left[ \frac{1}{2^{\frac{1}{2}}|\nu - \nu'|^3} + \frac{j 2^{-\frac{1}{3}}}{(2^{\frac{1}{2}}|\nu - \nu'|)^2} + \frac{1}{2^j (2^{\frac{1}{2}}|\nu - \nu'|)^2} + \frac{1}{2^j (2^{\frac{1}{2}}|\nu - \nu'|)^2} \right] \varepsilon^2 \gamma_j \gamma_j'. \tag{8.18}
$$

Now, the decomposition (8.14) of $B_{j,\nu,\nu',l,m}^{1,2}$ yields:

$$
\left| \sum_{(l,m) / 2\min(l,m) \leq 2|\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} \right| \leq \left| \sum_{(l,m) / 2\min(l,m) \leq 2|\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,1} \right| + \left| \sum_{(l,m) / 2\min(l,m) \leq 2|\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2} \right|.
$$

Together with the estimates (8.17) and (8.18), we obtain:

$$
\left| \sum_{(l,m) / 2\min(l,m) \leq 2|\nu - \nu'|} \left( B_{j,\nu,\nu',l,m}^{1,1} + B_{j,\nu,\nu',l,m}^{1,2} \right) \right| \leq \left[ \frac{1}{2^{\frac{1}{2}}|\nu - \nu'|^3} + \frac{1}{(2^{\frac{1}{2}}|\nu - \nu'|)^2} + \frac{2^{\frac{1}{2}}|\nu - \nu'|^2}{2^j (2^{\frac{1}{2}}|\nu - \nu'|)^2} + \frac{1}{2^j (2^{\frac{1}{2}}|\nu - \nu'|)^2} \right] \varepsilon^2 \gamma_j \gamma_j'.
$$

This concludes the proof of Proposition 8.1.

The rest of this section is organized as follows. Proposition 8.3 is proved in section 8.1, Proposition 8.4 is proved in section 8.2, and Proposition 8.2 is proved in section 8.3.

8.1 Proof of Proposition 8.3 (Control of $B_{j,\nu,\nu',l,m}^{1,1}$)

Recall the definition (8.15) of $B_{j,\nu,\nu',l,m}^{1,1}$:

$$
B_{j,\nu,\nu',l,m}^{1,1} = -i 2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{b^{-1}}{g(L, L')} \left( L(P_{tr} \chi) P_{m} tr \chi' + P_{tr} \chi' L(P_{m} tr \chi') \right) \times \eta_j^\nu(\omega) \eta_j^\nu(\omega') \left( (2^{-j} \lambda')^{-1} + (2^{-j} \lambda)^{-1} \right) \times \frac{1}{2} \psi(2^{-j} \lambda)(2^{-j} \lambda')(2^{-j} \chi') f(\lambda) f(\chi') \times \lambda^2 \lambda' d\lambda d\lambda' d\omega d\omega' d\mathcal{M}.
$$
We have:
\[
\sum_{(l,m)/2^i\min(l,m)\leq 2^j|\nu-\nu'|} \left( L(P_l\text{tr}_X)P_m\text{tr}_Y' + P_l\text{tr}_X L'(P_m\text{tr}_Y') \right) \tag{8.19}
\]
\[
= L(\text{tr}_X)\text{tr}_Y' + \text{tr}_X L(\text{tr}_Y') - \sum_{(l,m)/2^i\min(l,m) > 2^j|\nu-\nu'|} \left( L(P_l\text{tr}_X)P_m\text{tr}_Y' + P_l\text{tr}_X L'(P_m\text{tr}_Y') \right).
\]
Now, the difference between $B^{1,1}_{j,\nu,\nu',l,m}$ and $B^{1,1}_{j,\nu,\nu',l,m}$ is the fact that the term $(2^{-j}\lambda')^{-1}$ has been replaced by:
\[
\frac{(2^{-j}\lambda')^{-1} + (2^{-j}\lambda)^{-1}}{2}
\]
such as to obtain an expression which is totally symmetric in $(\lambda, \lambda')$ and $(\omega, \omega')$. In turn, we may sum over $l, m$ belonging to the region $2^m \leq 2^j|\nu - \nu'|$. Together with (8.19) we obtain:
\[
\sum_{(m,l)/2^m \leq 2^j|\nu-\nu'|} B^{1,1}_{j,\nu,\nu',l,m} \tag{8.20}
\]
\[
= B^{1,1,1}_{j,\nu,\nu',l,m} + B^{1,1,2}_{j,\nu,\nu',l,m} + \sum_{(m,l)/2^m > 2^j|\nu-\nu'|} B^{1,1,3}_{j,\nu,\nu',l,m} + \sum_{(m,l)/2^m > 2^j|\nu-\nu'|} B^{1,1,4}_{j,\nu,\nu',l,m}
\]
where $B^{1,1,1}_{j,\nu,\nu',l,m}$, $B^{1,1,2}_{j,\nu,\nu',l,m}$, $B^{1,1,3}_{j,\nu,\nu',l,m}$, and $B^{1,1,4}_{j,\nu,\nu',l,m}$ are given by:
\[
B^{1,1,1}_{j,\nu,\nu'} = -i2^{-j-1} \int_M \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')}(L(\text{tr}_X)\text{tr}_Y' + \text{tr}_X L(\text{tr}_Y')) \tag{8.21}
\]
\[
\times F_j(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'dM,
\]
\[
B^{1,1,2}_{j,\nu,\nu'} = -i2^{-j-1} \int_M \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')}(L(\text{tr}_X)\text{tr}_Y' + \text{tr}_X L(\text{tr}_Y')) \tag{8.22}
\]
\[
\times F_{j,-1}(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'dM,
\]
\[
B^{1,1,3}_{j,\nu,\nu',l,m} = i2^{-j-1} \int_M \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')}(L(P_l\text{tr}_X)P_m\text{tr}_Y' + P_l\text{tr}_X L'(P_m\text{tr}_Y')) \tag{8.23}
\]
\[
\times F_j(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'dM,
\]
and:
\[
B^{1,1,4}_{j,\nu,\nu',l,m} = i2^{-j-1} \int_M \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')}(L(P_l\text{tr}_X)P_m\text{tr}_Y' + P_l\text{tr}_X L'(P_m\text{tr}_Y')) \tag{8.24}
\]
\[
\times F_{j,-1}(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'dM.
\]
We have the following propositions:
Proposition 8.5 Let $B_{j,v,\nu',l,m}^{1,1,3}$ be given by (8.23), and let $B_{j,v,\nu',l,m}^{1,1,4}$ be given by (8.24). Then, we have the following estimate:

$$
\sum_{(m,l)/2^m>2|\nu-\nu'|} \left( |B_{j,v,\nu',l,m}^{1,1,3}| + |B_{j,v,\nu',l,m}^{1,1,4}| \right) \lesssim \frac{\varepsilon^2 \gamma_{j} \gamma_{\nu'}}{(2^2 |\nu-\nu'|)^3}.
$$

(8.25)

Proposition 8.6 Let $B_{j,v,\nu'}^{1,1,1}$ be given by (8.21), and let $B_{j,v,\nu'}^{1,1,2}$ be given by (8.22). Then, we have the following estimate:

$$
|B_{j,v,\nu'}^{1,1,1} + B_{j,v,\nu'}^{1,1,1}| + |B_{j,v,\nu'}^{1,1,2} + B_{j,v,\nu'}^{1,1,2}| \lesssim \left[ \frac{2^{-\left(\frac{1}{2}\right)j}}{(2^2 |\nu-\nu'|)^2} + \frac{1}{(2^2 |\nu-\nu'|)^2} \right] \varepsilon^2 \gamma_{j} \gamma_{\nu'}.
$$

(8.26)

In view of the decomposition (8.20), we have:

$$
\lesssim \sum_{(m,l)/2^m>2|\nu-\nu'|} \left| B_{j,v,\nu',l,m}^{1,1} - (B_{j,v,\nu'}^{1,1,1} + B_{j,v,\nu'}^{1,1,2}) \right|
$$

$$
\lesssim \sum_{(m,l)/2^m>2|\nu-\nu'|} \left| B_{j,v,\nu',l,m}^{1,1,3} \right| + \sum_{(m,l)/2^m>2|\nu-\nu'|} \left| B_{j,v,\nu',l,m}^{1,1,4} \right|
$$

$$
\lesssim \varepsilon^2 \gamma_{j} \gamma_{\nu'},
$$

(8.27)

where we used the estimate (8.25) in the last inequality. Together with (8.26), this yields:

$$
\lesssim \left| B_{j,v,\nu',l,m}^{1,1,1} + B_{j,v,\nu'}^{1,1,1} \right| + \left| B_{j,v,\nu'}^{1,1,2} + B_{j,v,\nu'}^{1,1,2} \right| + \frac{\varepsilon^2 \gamma_{j} \gamma_{\nu'}}{(2^2 |\nu-\nu'|)^3}.
$$

This concludes the proof of Proposition 8.3.

The rest of this section is as follows. In section 8.1.1, we give a proof of Proposition 8.5, and In section 8.1.2, we give a proof of Proposition 8.6.

8.1.1 Proof of Proposition 8.5 (Control of $B_{j,v,\nu',l,m}^{1,1,3}$ and $B_{j,v,\nu',l,m}^{1,1,4}$)

We further decompose. We have:

$$
B_{j,v,\nu',l,m}^{1,1,3} = B_{j,v,\nu',l,m}^{1,1,3,1} + B_{j,v,\nu',l,m}^{1,1,3,2},
$$

(8.27)

where $B_{j,v,\nu',l,m}^{1,1,3,1}$ and $B_{j,v,\nu',l,m}^{1,1,3,2}$ are given by:

$$
B_{j,v,\nu',l,m}^{1,1,3,1} = i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{g(L, L')} L(P_l \text{tr} \chi) P_m \text{tr} \chi' F_j(u) F_{j,-1}(u')
$$

$$
\times \eta_{j}^{\nu}(\omega) \eta_{j}^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},
$$

(8.28)
and:

\[ B_{j,\nu',l,m}^{1,1,3,2} = i2^{-j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} P_{\nu} \chi L'(P_{\nu} \chi') F_j(u) F_{j-1}(u') \times \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' dM, \]  

(8.29)

The terms \( B_{j,\nu',l,m}^{1,1,3,1} \) and \( B_{j,\nu',l,m}^{1,1,3,2} \) are estimated in the same way, so we focus on \( B_{j,\nu',l,m}^{1,1,3,1} \).

We first deal with \( g(L, L') \). We have the identities:

\[ g(L, L') = -1 + g(N, N') \]  

(8.30)

and

\[ 1 - g(N, N') = \frac{g(N - N', N - N')}{2}. \]  

(8.31)

Furthermore, the estimates on \( N \) (2.42) and (2.43) yield:

\[ |N - N| \lesssim |\omega - \nu|, \quad |N' - N'| \lesssim |\omega' - \nu'| \quad \text{and} \quad |N - N'| \gtrsim |\nu - \nu'|, \]  

(8.32)

where we have used the following notation for any vectorfield tangent to \( \Sigma_l \):

\[ |X| = g(X, X)^{1/2}. \]

Since \( \omega \) belongs to the patch of center \( \nu \), \( \omega' \) belongs to the patch of center \( \nu' \), and \( \nu \neq \nu' \),

we obtain in view of (8.30), (8.31) and (8.32):

\[ \frac{1}{g(L, L')} = \frac{1}{|N - N'|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N}{|N - N'|} \right)^p \left( \frac{N' - N'}{|N - N'|} \right)^q \right), \]  

(8.33)

for some explicit real coefficients \( c_{pq} \) such that the series

\[ \sum_{p,q \geq 0} c_{pq} x^p y^q \]

has radius of convergence 1.

In view of (8.28) and (8.33), we may rewrite \( B_{j,\nu',l,m}^{1,1,3,1} \) as:

\[ B_{j,\nu',l,m}^{1,1,3,1} = i2^{-j-1} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{|N - N'|^2} \left( \int_{S^2} b^{-1} L(P_{\nu}) \chi \left( \frac{N}{|N - N'|} \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} P_{\nu'} \chi' \left( \frac{N'}{|N - N'|} \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) dM. \]

Using the estimate (7.10) with the choice:

\[ H_{pq} = \frac{b^{-1}}{|N - N'|^2} \left( \frac{N}{|N - N'|} \right)^p \left( \int_{S^2} P_{\nu'} \chi' \left( \frac{N'}{|N - N'|} \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right), \]

(8.34)
we obtain:

\[
|B_{j,\nu,\nu',l,m}^{1,1,3,1}| \lesssim \left( \sum_{p,q \geq 0} c_{pq} \left( \sup_{\omega \in \text{supp} (\eta_j^\nu)} (\|H_{pq}^{1,1,3,1}\|_{L^2_{u,v},L^\infty}) \right) \right)^2 \frac{1}{2^{j-l} \varepsilon \gamma_j^\nu}. \tag{8.35}
\]

Next, we evaluate the right-hand side of (8.35). In view of (8.32), we have:

\[
\left\| \frac{b^{-1}}{|N_\nu - N_{\nu'}|^2} \left( \frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \right\|_{L^\infty(M)} \lesssim \frac{\|b^{-1}\|_{L^\infty(M)}}{|\nu - \nu'|^2} \left( \frac{|\omega - \nu|}{|\nu - \nu'|} \right)^p \tag{8.36}
\]

\[
\lesssim \frac{1}{|\nu - \nu'|^2} \left( \frac{1}{2^{j} |\nu - \nu'|} \right)^p,
\]

where we used in the last inequality the estimate (2.38) for $b$, and the fact that $\omega$ is in the patch centered around $\nu$ of diameter $\sim 2^j$. Let:

\[
H_{pq}^1 = \int_{S^2} P_m \tgamma'(\frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|})^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega'.
\]

Then, (8.34), (8.35) and (8.36) yield:

\[
|B_{j,\nu,\nu',l,m}^{1,1,3,1}| \lesssim \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{1}{2^{j} |\nu - \nu'|} \right)^p \left( \sup_{\omega \in \text{supp} (\eta_j^\nu)} (\|H_{pq}^{1}\|_{L^2_{u,v},L^\infty}) \right) \right)^2 \frac{2^{2j-l} \varepsilon \gamma_j^\nu}{(2^{j} |\nu - \nu'|)^2}. \tag{8.37}
\]

Next, we evaluate $H_{pq}^1$. In view of (8.32), we have:

\[
\left\| \left( \frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right\|_{L^\infty(M)} \lesssim \left( \frac{|\omega - \nu'|}{|\nu - \nu'|} \right)^q \lesssim \left( \frac{1}{2^{j} |\nu - \nu'|} \right)^q, \tag{8.38}
\]

where we used in the last inequality the fact that $\omega'$ is in the patch centered around $\nu'$ of diameter $\sim 2^j$. Now, (8.38) together with Corollary 7.9 yields:

\[
\|H_{pq}^1\|_{L^2_{u,v},L^\infty} \lesssim \left( \frac{1}{2^{j} |\nu - \nu'|} \right)^q \varepsilon (2^{j} |\nu - \nu'|2^{-m+\frac{j}{2}} + (2^{j} |\nu - \nu'|)^{\frac{j}{2}2^{-\frac{m}{2}+\frac{j}{4}}})^{\gamma_j^{\nu'}}, \tag{8.39}
\]

Finally, (8.37) and (8.39) imply:

\[
|B_{j,\nu,\nu',l,m}^{1,1,3,1}| \lesssim \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{1}{2^{j} |\nu - \nu'|} \right)^{p+q} \left( \frac{1}{2^{j} |\nu - \nu'|} \right)^{2^{j-l} \varepsilon \gamma_j^{\nu'}} \left( \frac{1}{2^{j} |\nu - \nu'|} \right)^{\gamma_j^{\nu'}} \right) \frac{2^{2j-l} \varepsilon \gamma_j^{\nu'} \gamma_j^{\nu'}}{(2^{j} |\nu - \nu'|)^2} \tag{8.40}
\]

\[
\lesssim \left( 2^{j} |\nu - \nu'|2^{-m+\frac{j}{2}} + (2^{j} |\nu - \nu'|)^{\frac{j}{2}2^{-\frac{m}{2}+\frac{j}{4}}} \right) \frac{2^{2j-l} \varepsilon \gamma_j^{\nu'} \gamma_j^{\nu'}}{(2^{j} |\nu - \nu'|)^2}.
\]

(8.40) implies:

\[
\sum_{(m,l)/2^m(m,l) > 2j |\nu - \nu'|} |B_{j,\nu,\nu',l,m}^{1,1,3,1}| \lesssim \frac{\varepsilon \gamma_j^{\nu'} \gamma_j^{\nu'}}{(2^{j} |\nu - \nu'|)^3}. \]
The term $B_{j,\nu,\nu',l,m}^{1,1,3,2}$ is completely analogous, so we obtain in view of (8.27):

$$
\sum_{(m,l)/2^{\min(m,l)} > 2^{j}|\nu - \nu'|} |B_{j,\nu,\nu',l,m}^{1,1,3}| \lesssim \frac{e^{2\gamma j_{j}^{\nu} j_{j}^{\nu'}}}{(2^{j}|\nu - \nu'|)^3}.
$$

The term $B_{j,\nu,\nu',l,m}^{1,1,4}$ is completely analogous to $B_{j,\nu,\nu',l,m}^{1,1,3}$. This concludes the proof of Proposition 8.6.

8.1.2 Proof of Proposition 8.6 (Control of $B_{j,\nu,\nu'}^{1,1,1}$ and $B_{j,\nu,\nu'}^{1,1,2}$)

We need to estimate $B_{j,\nu,\nu'}^{1,1,1}$ and $B_{j,\nu,\nu'}^{1,1,2}$. These terms are estimated in the same way, so we focus on $B_{j,\nu,\nu'}^{1,1,1}$. We further decompose:

$$
B_{j,\nu,\nu'}^{1,1,1} = B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2}
$$

(8.41)

where $B_{j,\nu,\nu'}^{1,1,1,1}$ and $B_{j,\nu,\nu'}^{1,1,1,2}$ are given by:

$$
B_{j,\nu,\nu'}^{1,1,1,1} = -i2^{-j-1}\int_{\mathcal{M}}\int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} L(\text{tr} \chi) \text{tr} \chi' \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},
$$

(8.42)

and:

$$
B_{j,\nu,\nu'}^{1,1,1,2} = -i2^{-j-1}\int_{\mathcal{M}}\int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \text{tr} \chi L'(\text{tr} \chi') \times F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.
$$

(8.43)

The terms $B_{j,\nu,\nu',l,m}^{1,1,1}$ and $B_{j,\nu,\nu',l,m}^{1,1,2}$ are estimated in the same way, so we focus on $B_{j,\nu,\nu',l,m}^{1,1,1}$. We integrate by parts in $B_{j,\nu,\nu',l,m}^{1,1,1}$ using (7.137).

**Lemma 8.7** Let $B_{j,\nu,\nu'}^{1,1,1}$ be defined by (8.42). Integrating by parts using (7.137) yields:

$$
B_{j,\nu,\nu'}^{1,1,1} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^{j} |N_\nu - N_{\nu'}|)^{p+q}}
$$

$$
\times \left[ \frac{1}{|N_\nu - N_{\nu'}|^2} h_{1,p,q} + \frac{1}{|N_\nu - N_{\nu'}|^3} (h_{2,p,q} + h_{3,p,q} + h_{4,p,q}) \right] d\mathcal{M}
$$

$$
+ 2^{-2j} \int_{\mathcal{M}}\int_{S^2 \times S^2} \frac{(\chi - \chi')L(\text{tr} \chi) \text{tr} \chi'}{g(L, L')^2} F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},
$$

where $c_{pq}$ are explicit real coefficients such that the series

$$
\sum_{p,q \geq 0} c_{pq} x^p y^q
$$

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has radius of convergence 1, where the scalar functions \(h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}\) on \(\mathcal{M}\) are given by:

\[
h_{1,p,q} = \left(\int_{S^2} N(L(\text{tr}\chi)) \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) (8.45)
\]

\[\times \left(\int_{S^2} \text{tr}' \left(2^j (N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega'\right),\]

\[
h_{2,p,q} = \left(\int_{S^2} \nabla L(\text{tr}\chi) \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) (8.46)
\]

\[\times \left(\int_{S^2} \text{tr}' \left(2^j (N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega'\right),\]

\[
h_{3,p,q} = \left(\int_{S^2} L(\text{tr}\chi) \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) (8.47)
\]

\[\times \left(\int_{S^2} H_1 \left(2^j (N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega'\right),\]

\[
h_{4,p,q} = \left(\int_{S^2} H_2 \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right) (8.48)
\]

\[\times \left(\int_{S^2} \text{tr}' \left(2^j (N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega'\right),\]

where the tensor \(H_1\) on \(\mathcal{M}\) involved in the definition of \(h_{3,p,q}\) is a linear combination of terms in the following list:

\[b^{-1}\nabla'(b'tr\chi'), \theta'tr\chi', (8.49)\]

and where the tensor \(H_2\) on \(\mathcal{M}\) involved in the definition of \(h_{4,p,q}\) is a linear combination of terms in the following list:

\[\theta L(tr\chi), b^{-1}\nabla(b)L(tr\chi). (8.50)\]

The proof of lemma 8.7 is postponed to Appendix A. In the rest of this section, we use Lemma 8.7 to obtain the control of \(B_{1,1,1,1}^{1,1,1,1}\).

We first estimate \(h_{1,p,q}\). We have:

\[
h_{1,p,q} = \int_{S^2} H N(L(\text{tr}\chi)) F_{j,-1}(u)\eta_j^\nu(\omega)d\omega, (8.51)\]

where the tensor \(H\) is given by:

\[
H = \left(2^j (N - N_\nu)\right)^p \left(\int_{S^2} \text{tr}' \left(2^j (N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega'\right), (8.52)\]
In view of (8.51), the estimate (7.5) in $L^1(M)$ yields:

$$\|h_{1,p,q}\|_{L^1(M)} \lesssim \left( \sup_{\omega \in \text{supp}(\eta_{\nu}^j)} (\|H\|_{L^2_t L^4_u L^\infty_x} + |\nu - \nu'| \|H\|_{L^3+(M)} + \|\chi_{2\nu} H\|_{L^2(M)}) \right)^{2^j \varepsilon \gamma_{\nu}^j}.$$  \hfill (8.53)

Let the tensor $H_1$ be defined by:

$$H_1 = \int_{S^2} \text{tr} \chi'(2^j (N' - N_{\nu})) q F_{\nu,-1}(u') \eta_{\nu}^j(\omega') d\omega'.$$  \hfill (8.54)

Then, we have:

$$H = \left(2^j (N - N_{\nu})\right)^p H_1$$

which together with (8.53) yields:

$$\|h_{1,p,q}\|_{L^1(M)} \lesssim \left( \sup_{\omega \in \text{supp}(\eta_{\nu}^j)} (\|H_1\|_{L^2_t L^4_u L^\infty_x} + |\nu - \nu'| \|H_1\|_{L^3+(M)} + \|\chi_{2\nu} H_1\|_{L^2(M)}) \right)^{2^j \varepsilon \gamma_{\nu}^j},$$  \hfill (8.55)

where we used the estimate (2.42) for $\partial_\omega N$, and the size of the patch.

Next, we estimate the various terms in the right-hand side of (8.55) starting with the last one. In view of (8.54), we have:

$$\|\chi_{2\nu} H_1\|_{L^2(M)} \lesssim \|\chi_{2\nu} H_{u_{\nu},x_{\nu},x'} L_t^2\left|S^2\right| \text{tr} \chi'(2^j (N' - N_{\nu})) q F_{\nu,-1}(u') \eta_{\nu}^j(\omega') d\omega' \|_{L_{u_{\nu},x_{\nu},x'} L_t^\infty}$$

$$\lesssim \varepsilon(1 + q^2)^{\gamma_{\nu}^j},$$  \hfill (8.56)

where we used in the last inequality the estimate (7.64) of the $L^2_{u_{\nu},x_{\nu},x'} L_t^\infty$ of oscillatory integrals together with the estimate (2.46) for $\chi_2$.

Next, we estimate the second term in the right-hand side of (8.55). In view of the definition (8.54) of $H_1$, and in view of the estimates (7.115) and (7.116), we have:

$$\|\chi_{2\nu} H_1\|_{L^2(M)} \lesssim \varepsilon \gamma_{\nu}^j,$$

and

$$\|H_1\|_{L^\infty(M)} \lesssim 2^j \varepsilon \gamma_{\nu}^j.$$  \hfill (8.57)

Interpolating between these two estimates, we obtain:

$$\|H_1\|_{L^{3+(M)}} \lesssim 2^{(\frac{3}{2})j + j} \varepsilon \gamma_{\nu}^j.$$  \hfill (8.57)

Next, we estimate the first term in the right-hand side of (8.55). The estimate (7.83) applied to $H$ yields:

$$\sup_{\omega \in \text{supp}(\eta_{\nu}^j)} (\|H_1\|_{L^2_t L^\infty_u L^2_x}) \lesssim (1 + q^2)^{\varepsilon} 2^j |\nu - \nu'| \gamma_{\nu}^j.$$
Interpolating with (7.115), we obtain:

\[ \sup_{\omega \in \text{supp}(\eta_j^\nu)} (\| H_1 \|_{L^2_x H^4_t}) \lesssim (1 + q^2) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'|^{\frac{1}{2}} \gamma_j^\nu). \]  

(8.58)

Finally, (8.55), (8.56), (8.57) and (8.58) imply:

\[ \| h_{1,p,q} \|_{L^1(M)} \lesssim (1 + q^2) \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-\left(\frac{j}{2}\right) - j} + (2^{\frac{j}{2}} |\nu - \nu'|)^\frac{1}{2}\right) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^\nu. \]  

(8.59)

Next, we estimate \( h_{2,p,q} \) defined in (8.46). We have:

\[ h_{2,p,q} = \int_{\mathbb{S}^2} H \nabla L(\text{tr} \chi) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega, \]  

(8.60)

where the tensor \( H \) is given by:

\[ H = \left(2^{\frac{j}{2}} (N - N_{\nu})\right)^p \left(\int_{\mathbb{S}^2} \text{tr} \chi' \left(2^{\frac{j}{2}} (N' - N_{\nu})\right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega'\right). \]  

(8.61)

In view of (8.60), the estimate (7.5) in \( L^1(M) \) yields:

\[ \| h_{2,p,q} \|_{L^1(M)} \lesssim \left( \sup_{\omega \in \text{supp}(\eta_j^\nu)} (\| H \|_{L^2_x L^4_t}) + |\nu - \nu'| \| H \|_{L^2(M)} + \| \chi_{2\nu} H \|_{L^2(M)} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu. \]  

(8.62)

In view of (8.62), the estimate (7.5) in \( L^1(M) \) yields:

\[ \| h_{2,p,q} \|_{L^1(M)} \lesssim (1 + q^2) \left(2^{\frac{j}{2}} |\nu - \nu'| 2^{-\left(\frac{j}{2}\right) - j} + (2^{\frac{j}{2}} |\nu - \nu'|)^\frac{1}{2}\right) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j^\nu. \]  

(8.63)

Next, we estimate \( h_{3,p,q} \). In view of the Raychaudhuri equation (2.22) satisfied by \( \text{tr} \chi \), the decomposition (2.72) for \( \text{tr} \chi \), the decomposition (2.77) for \( |\hat{\chi}|^2 \), and with the \( L^\infty \) estimates for \( b \) and \( \text{tr} \chi \) provided respectively by (2.38) and (2.39), we obtain the following decomposition for \( L(\text{tr} \chi) \):

\[ L(\text{tr} \chi) = \chi_{2\nu} \cdot (2\chi_1 + \hat{\chi}) + f_1^j + f_2^j, \]  

(8.64)

where the scalar \( f_1^j \) only depends on \( \nu \) and satisfies:

\[ \| f_1^j \|_{L^\infty \text{supp}(\eta_j^\nu)} \lesssim \varepsilon, \]  

(8.65)

where the scalar \( f_2^j \) satisfies:

\[ \| f_2^j \|_{L^2(M)} \lesssim \varepsilon 2^{-\frac{j}{2}}. \]  

(8.66)
This implies the following decomposition:

\[
\int_{S^2} L(\text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega
\]

\[
= -\chi_{2\nu} \cdot \left( \int_{S^2} (2 \chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right)
\]

\[
+ f_1^j \left( \int_{S^2} \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right)
\]

\[
+ \int_{S^2} f_2^j \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega
\]

\[
= -\chi_{2\nu} \cdot \left( \int_{S^2} (2 \chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right)
\]

\[-(\chi_{2\nu} - \chi_{2\nu}) \cdot \left( \int_{S^2} (2 \chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right)
\]

\[
+ f_1^j \left( \int_{S^2} \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right)
\]

\[
+ \int_{S^2} f_2^j \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega
\]

We obtain the following estimate for \( h_{3,p,q} \):

\[
\| h_{3,p,q} \|_{L^1(M)} \lesssim \left\| \int_{S^2} (2 \chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right\|_{L^2(M)}
\]

\[
\times \left\| \int_{S^2} \chi_{2\nu} H_1 \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^q(\omega') d\omega' \right\|_{L^2(M)}
\]

\[
+ \int_{S^2} \|\chi_{2\nu} - \chi_{2\nu}\|_{L^{a,-}(M)} \left\| \int_{S^2} (2 \chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right\|_{L^2(M)}
\]

\[
\times \left\| H_1 \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^q(\omega') d\omega' \right\|_{L^2(M)}
\]

\[
+ \left( \| f_1^j \|_{L^\infty_{u,v,x',y',L^2_t}}^2 \right) \int_{S^2} \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right\|_{L^2(M)}
\]

\[
+ \left( \| f_2^j \|_{L^\infty_{u,v,x',y',L^2_t}}^2 \right) \int_{S^2} \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^p(\omega) d\omega \right\|_{L^2(M)}
\]

\[
\times \left\| H_1 \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^q(\omega') d\omega' \right\|_{L^2(M)}
\]

which together with the estimate (8.65) for \( f_1^j \), the estimate in \( L^2(M) \) (8.56), the estimates (2.46) and (2.47) for \( \chi_2 \), and the estimate (7.63) of the \( L^2_{u,v,x',y',L^\infty_t} \) of oscillatory integrals

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yields:

$$\| h_{3,p,q} \|_{L^1(M)}$$

(8.68)

$$\lesssim \left\| \int_{S^2} (2\chi_1 + \bar{\chi}) \left( 2^{j}(N - N_\nu) \right)^p F_{j-1}(u) \eta'_j(\omega) d\omega \right\|_{L^2(M)}$$

$$\times \left\| \int_{S^2} \chi_2^2 H_1 \left( 2^{j}(N' - N_\nu) \right)^q F_{j-1}(u') \eta'_j(\omega') d\omega' \right\|_{L^2(M)}$$

$$+ \varepsilon |\nu - \nu'| \left( \int_{S^2} \left| \int_{S^2} (2\chi_1 + \bar{\chi}) \left( 2^{j}(N - N_\nu) \right)^p F_{j-1}(u) \eta'_j(\omega) d\omega \right| d\omega' \right)$$

$$\times \| H_1 \|_{L^\infty L^2(\mathcal{H}_u)} \left\| \left( 2^{j}(N' - N_\nu) \right)^q \right\|_{L^\infty} \| F_{j-1}(u') \|_{L^q_{\omega} \eta'_j(\omega') d\omega'}$$

$$+ \varepsilon \gamma_j^\nu \left( \int_{S^2} H_1 \left( 2^{j}(N' - N_\nu) \right)^q F_{j-1}(u') \eta'_j(\omega') d\omega' \right) \right\|_{L^2(M)}$$

Next, we estimate the $L^{3+}(M)$ norm in the right-hand side of (8.68). Using the estimate for the $L^p(M)$ norm (7.2) with $p = 6$, we have:

$$\left\| \int_{S^2} (2\chi_1 + \bar{\chi}) \left( 2^{j}N - N_{\nu} \right)^p F_j(u) \eta_j'(\omega) d\omega \right\|_{L^6(M)}$$

(8.69)

$$\lesssim \sup_{\omega} \left( \| \chi_1 \|_{L^\infty L^6(\mathcal{H}_u)} + \| \bar{\chi} \|_{L^\infty L^6(\mathcal{H}_u)} \right) \left\| \left( 2^{j}(N - N_\nu) \right)^p \right\|_{L^\infty} 2^{3j} \gamma_j^\nu$$

$$\lesssim 2^{3j} \varepsilon \gamma_j^\nu,$$

where we used in the last inequality the estimate (2.40) for $\bar{\chi}$, the estimate (2.42) for $\partial_\omega N$, and the estimate (2.46) for $\chi_1$. Next, recall the decomposition (2.74) for $\bar{\chi}$ and the decomposition (2.75) for $\chi_2$ which yield:

$$2\chi_1 + \bar{\chi} = F_1^j + F_2^j$$

where the tensor $F_1^j$ only depends on $\nu$ and satisfies:

$$\| F_1^j \|_{L^\infty_{\omega_1,\omega_2} L^2_{\nu}} \lesssim \varepsilon,$$

(8.70)

where the scalar $F_2^j$ satisfies:

$$\| F_2^j \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

(8.71)

This yields:

$$\int_{S^2} (2\chi_1 + \bar{\chi}) \left( 2^{j}(N - N_\nu) \right)^p F_j(u) \eta_j'(\omega) d\omega$$

$$= F_1^j \left( \int_{S^2} \left( 2^{j}(N - N_\nu) \right)^p F_j(u) \eta_j'(\omega) d\omega \right) + \int_{S^2} F_2^j \left( 2^{j}(N - N_\nu) \right)^p F_j(u) \eta_j'(\omega) d\omega$$

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and thus:

\[
\left\| \int_{S^2} (2\chi + \tilde{\chi}) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
\lesssim \| F_j^p \|_{L^\infty_{\nu,\nu'}} \left\| \int_{S^2} \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
+ \left\| \int_{S^2} F_j^p \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
\lesssim \varepsilon \gamma_j^\nu + \left\| \int_{S^2} F_j^p \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)},
\]

where we used in the last inequality the estimate (8.70) and the estimate (7.63) of the $L^2_{\nu,\nu'} L^\infty_t$ of oscillatory integrals. Then, using the basic estimate in $L^2(M)$ (7.1), we obtain:

\[
\left\| \int_{S^2} (2\chi + \tilde{\chi}) b^{-1} \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \varepsilon \gamma_j^\nu + \sup_{\omega} \left\| \int_{S^2} F_j^p \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p b^{-1} \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p \right\|_{L^\infty} 2^{\frac{j}{2}} \gamma_j^\nu, 
\]

where we used in the last inequality the estimate (8.71). Next, interpolating (8.69) and (8.72), we obtain:

\[
\left\| \int_{S^2} (2\chi + \tilde{\chi}) b^{-1} \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^3+(M)} \lesssim 2^{\left(\frac{3}{2}\right) + j} \varepsilon \gamma_j^\nu.
\]

Together with (8.68), the estimate (2.42) for $\partial_\omega N$, the size of the patch, and the estimate in $L^2(M)$ (8.72), we obtain:

\[
\| h_{3,p,q} \|_{L^1(M)} \lesssim \varepsilon \gamma_j^\nu \left\| \int_{S^2} \chi_2^\nu H_1 \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon \gamma_j^\nu + \varepsilon |\nu - \nu'| 2^{\left(\frac{j}{2}\right) + j} \gamma_j^\nu \left( \int_{S^2} \| H_1 \|_{L^\infty_{\omega'} L^2(H_{\omega'})} \| F_{j,-1}(u') \|_{L^2_{\omega'} \eta_j^\nu(\omega') d\omega'} \right) \lesssim \varepsilon \gamma_j^\nu \left\| \int_{S^2} H_1 \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)}.
\]

Next, using the definition (8.49) of $H_1$, the estimate (2.38) for $b$, the estimate (2.39) and (2.40) for $\chi$, and the estimate (2.46) for $\chi_2$, we have:

\[
\| H_1 \|_{L^\infty_{\omega'} L^2(H_{\omega'})} + \| \chi_2^\nu H_1 \|_{L^\infty_{\omega'} L^2(H_{\omega'})} \lesssim \varepsilon.
\]
Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

\[
\left\| \int_{\mathbb{S}^2} \chi_{\omega}^2 H_1 \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
+ \left\| \int_{\mathbb{S}^2} H_1 \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
\lesssim \sup_{\omega} \left( \left\| \chi_{\omega}^2 H_1 \right\|_{L^\infty(\mathcal{H}_\omega)} + \left\| H_1 \right\|_{L^\infty(\mathcal{H}_\omega)} \left\| \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
\lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu
\]

where we used in the last inequality the estimate (8.74), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (8.73) and (8.74), this yields:

\[
\left\| h_{3,p,q} \right\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu + \varepsilon^2 |\nu - \nu'| 2^j \left| \mathbf{H}_j \right| \left( \int_{\mathbb{S}^2} \left\| F_{j-1}(u') \right\|_{L^\infty(\mathcal{H}_\omega)} \eta_j^{\nu'}(\omega') d\omega' \right) (8.75)
\]

\[
\lesssim 2^{\frac{j}{2}} \left( 1 + 2^{\frac{j}{2}} |\nu - \nu'| 2^{-\frac{j}{2}} \right) \varepsilon \gamma_j^\nu,
\]

where we used in the last inequality Cauchy-Schwarz in $\lambda'$ to evaluate $\left\| F_{j-1}(u') \right\|_{L^\infty(\mathcal{H}_\omega)}$, Cauchy-Schwarz in $\omega'$, and the size of the patch.

Next, we estimate $h_{4,p,q}$. We have:

\[
\left\| h_{4,p,q} \right\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} H_2 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} (8.76)
\]

\[
\times \left\| \int_{\mathbb{S}^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})},
\]

The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

\[
\left\| \int_{\mathbb{S}^2} H_2 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} (8.77)
\]

\[
\lesssim \sup_{\omega} \left( \left\| H_2 \right\|_{L^\infty(\mathcal{H}_\omega)} \left\| \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty} \right) 2^{\frac{j}{2}} \gamma_j^\nu \\
\lesssim \sup_{\omega} \left( \left\| H_2 \right\|_{L^\infty(\mathcal{H}_\omega)} \right) 2^{\frac{j}{2}} \gamma_j^\nu,
\]

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of (8.50), the estimate (2.38) for $b$, the estimates (2.36) (2.37) for $\overline{a}$, the estimates (2.39) (2.40) for $\chi$, and the Raychaudhuri equation (2.22) satisfied by $\text{tr} \chi$, we have:

\[
\left\| H_2 \right\|_{L^\infty(\mathcal{H}_\omega)} \lesssim \varepsilon,
\]

which together with (8.77) yields:

\[
\left\| \int_{\mathbb{S}^2} H_2 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu.
\]
Together with (8.76) and the estimate (7.63) of the $L^2_{u',x'}L^\infty_t$ of oscillatory integrals, we obtain:

$$
\| h_{4,p,q} \|_{L^1(M)} \lesssim (1 + q^2)2^{\frac{j}{2}} \varepsilon^{2} \gamma_j^{\nu} \gamma_{\nu}'.
$$

(8.78)

Finally, in view of (8.44), (8.59), (8.63), (8.75) and (8.78), we obtain for $B^{1,1,1,1}_{j,\nu,\nu'}$ the following decomposition:

$$
B^{1,1,1,1}_{j,\nu,\nu'} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_M \frac{1}{(2^{j}|N_{\nu} - N_{\nu'}|)^{p+q}} \times \left[ \frac{1}{|N_{\nu} - N_{\nu'}|^2} h_{1,p,q} + \frac{1}{|N_{\nu} - N_{\nu'}|^3} (h_{2,p,q} + h_{3,p,q} + h_{4,p,q}) \right] dM
$$

$$
+ 2^{-2j} \int_M \int_{S^2 \times S^2} \frac{(\chi - \chi')L(\text{tr} \gamma') \text{tr} \gamma'}{g(L, L')^2} F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu (\omega) \eta_j^\nu' (\omega') d\omega d\omega' dM,
$$

where $c_{pq}$ are explicit real coefficients such that the series

$$
\sum_{p,q \geq 0} c_{pq} x^p y^q
$$

has radius of convergence 1, and where $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}$ and $h_{4,p,q}$ satisfy the following estimate:

$$
\| h_{1,p,q} \|_{L^1(M)} + \| h_{2,p,q} \|_{L^1(M)} + \| h_{3,p,q} \|_{L^1(M)} + \| h_{4,p,q} \|_{L^1(M)} \lesssim (1 + q^2) \left( 1 + 2^{\frac{j}{2}} |\nu - \nu'| \right) (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{\frac{j}{2}} \varepsilon^{2} \gamma_j^{\nu} \gamma_{\nu}'.
$$

(8.80)

The term $B^{1,1,1,2}_{j,\nu,\nu'}$ defined by (8.43) is estimated in the same way. Indeed, proceeding as for $B^{1,1,1,1}_{j,\nu,\nu'}$, we integrate by parts in $B^{1,1,1,2}_{j,\nu,\nu',l,m}$ using (7.137).

**Lemma 8.8** Let $B^{1,1,1,2}_{j,\nu,\nu'}$ be defined by (8.43). Integrating by parts using (7.137) yields:

$$
B^{1,1,1,2}_{j,\nu,\nu'} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_M \frac{1}{(2^{j}|N_{\nu} - N_{\nu'}|)^{p+q}} \times \left[ \frac{1}{|N_{\nu} - N_{\nu'}|^2} h'_{1,p,q} + \frac{1}{|N_{\nu} - N_{\nu'}|^3} (h'_{2,p,q} + h'_{3,p,q} + h'_{4,p,q}) \right] dM
$$

$$
+ 2^{-2j} \int_M \int_{S^2 \times S^2} \frac{(\chi - \chi')L' \text{tr} \gamma' \text{tr} \gamma'}{g(L, L')^2} F_{j,-1}(u) F_{j,-1}(u') \eta_j^{\nu'} (\omega) \eta_j^{\nu'} (\omega') d\omega d\omega' dM,
$$

where $c_{pq}$ are explicit real coefficients such that the series

$$
\sum_{p,q \geq 0} c_{pq} x^p y^q
$$

has radius of convergence 1.
has radius of convergence 1, where the scalar functions \(h'_{1,p,q}, h'_{2,p,q}, h'_{3,p,q}, h'_{4,p,q}\) on \(\mathcal{M}\) are given by:

\[
h'_{1,p,q} = \left( \int_{S^2} N(tr\chi) \left( 2^\frac{1}{2} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \tag{8.82}
\]

\[
\times \left( \int_{S^2} L'(tr\chi') \left( 2^\frac{1}{2} (N' - N_\nu') \right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega' \right),
\]

\[
h'_{2,p,q} = \left( \int_{S^2} tr\chi \left( 2^\frac{1}{2} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \tag{8.83}
\]

\[
\times \left( \int_{S^2} \nabla' L'(tr\chi') \left( 2^\frac{1}{2} (N' - N_\nu') \right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega' \right),
\]

\[
h'_{3,p,q} = \left( \int_{S^2} H_1'(2^\frac{1}{2} (N - N_\nu)) F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \tag{8.84}
\]

\[
\times \left( \int_{S^2} L'(tr\chi') \left( 2^\frac{1}{2} (N' - N_\nu') \right)^q F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega' \right),
\]

\[
h'_{4,p,q} = \left( \int_{S^2} tr\chi \left( 2^\frac{1}{2} (N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \tag{8.85}
\]

\[
\times \left( \int_{S^2} H_2'(2^\frac{1}{2} (N' - N_\nu')) F_{j,-1}(u')\eta_j^\nu'(\omega')d\omega' \right),
\]

where the tensor \(H'_1\) on \(\mathcal{M}\) involved in the definition of \(h'_{3,p,q}\) is a linear combination of terms in the following list:

\[
b^{-1}\nabla(btr\chi), \theta tr\chi,
\]

and where the tensor \(H'_2\) on \(\mathcal{M}\) involved in the definition of \(h'_{4,p,q}\) is a linear combination of terms in the following list:

\[
\theta'L'(tr\chi'), b^{-1}\nabla'(b')L'(tr\chi').
\]

The proof of Lemma 8.8 is postponed to Appendix B. Next, we use Lemma 8.8 to obtain the control of \(B_{j,\omega,\omega'}^{1,1,1,2}\). Now, note that exchanging the role of \(\omega\) and \(\omega'\), we obtain that \(h'_{2,p,q}\) corresponds to \(h_{2,p,q}\), \(h'_{3,p,q}\) corresponds to \(h_{3,p,q}\), and \(h'_{4,p,q}\) corresponds to \(h_{4,p,q}\). Also, exchanging the role of \(\omega\) and \(\omega'\), we obtain that \(h'_{1,p,q}\) corresponds to \(h_{3,p,q}\) where \(H_1\) has been replaced with \(N(tr\chi)\) which satisfies (8.74) in view of the estimate (2.39) for \(tr\chi\), and the estimate (2.46) for \(\chi_2\). Thus, since \(h_{1,p,q}, h_{2,p,q}, h_{3,p,q}\) and \(h_{4,p,q}\) satisfy the estimate (8.80), we obtain that \(h'_{1,p,q}, h'_{2,p,q}, h'_{3,p,q}\) and \(h'_{4,p,q}\) satisfy the following estimate:

\[
\|h'_{1,p,q}\|_{L^1(\mathcal{M})} + \|h'_{2,p,q}\|_{L^1(\mathcal{M})} + \|h'_{3,p,q}\|_{L^1(\mathcal{M})} + \|h'_{4,p,q}\|_{L^1(\mathcal{M})} \tag{8.88}
\]

\[
\lesssim (1 + q^2 \left( 1 + 2^{\frac{2}{4}} |\nu - \nu'| 2^{-\left(\frac{1}{12}\right)} + (2^{\frac{2}{4}} |\nu - \nu'|)\right) 2^{rac{2}{2}} \varepsilon \gamma_j^\nu \gamma_j^\nu'.
\]
Summing (8.79) and (8.81), we obtain:

$$B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \left( \frac{1}{(2^\frac{1}{2} |\nu - \nu'|)^{p+q}} \left[ \frac{1}{|\nu - \nu'|} \right]^2 (h_{1,p,q} + h'_{1,p,q}) + \frac{1}{|\nu - \nu'|} (h_{2,p,q} + h'_{2,p,q} + h_{3,p,q} + h'_{3,p,q} + h_{4,p,q} + h'_{4,p,q}) \right) d\mathcal{M} + 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi - \chi')(L(\text{tr}\chi)(\text{tr}\chi') + \text{tr}\chi L'(\text{tr}\chi'))}{g(L, L')^2} \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}.$$ 

Note that the last term in the right-hand side is antisymmetric in ($\nu, \nu'$) and thus vanishes when considering the sum:

$$B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} + B_{j,\nu',\nu}^{1,1,1,1} + B_{j,\nu',\nu}^{1,1,1,2}. \qquad (8.90)$$

This cancellation together with (8.89) yields:

$$|B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} + B_{j,\nu',\nu}^{1,1,1,1} + B_{j,\nu',\nu}^{1,1,1,2}| \lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^\frac{1}{2} |\nu - \nu'|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \left\| \frac{1}{|\nu - \nu'|} \right\|_{L^\infty(\mathcal{M})} \times (\| h_{1,p,q} \|_{L^1(\mathcal{M})} + \| h'_{1,p,q} \|_{L^1(\mathcal{M})}) + \left\| \frac{1}{|\nu - \nu'|} \right\|_{L^\infty(\mathcal{M})} (\| h_{2,p,q} \|_{L^1(\mathcal{M})} + \| h'_{2,p,q} \|_{L^1(\mathcal{M})} + \| h_{3,p,q} \|_{L^1(\mathcal{M})} + \| h'_{3,p,q} \|_{L^1(\mathcal{M})} + \| h_{4,p,q} \|_{L^1(\mathcal{M})} + \| h'_{4,p,q} \|_{L^1(\mathcal{M})}).$$

Together with the estimate (8.32) for $|\nu - \nu'|$, and the estimates (8.80) and (8.88), we obtain:

$$|B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2} + B_{j,\nu',\nu}^{1,1,1,1} + B_{j,\nu',\nu}^{1,1,1,2}| \lesssim \left( \sum_{p,q \geq 0} c_{pq} (2^\frac{1}{2} |\nu - \nu'|)^{p+q} \right) \left[ \frac{2^{-j}}{(2^\frac{1}{2} |\nu - \nu'|)^2} + \frac{2^{-\frac{1}{2}}}{(2^\frac{1}{2} |\nu - \nu'|)^2} \right] \times \left( 1 + 2^\frac{1}{2} |\nu - \nu'| 2^{-\frac{1}{2}j} 2^{-\frac{1}{2}j} + (2^\frac{1}{2} |\nu - \nu'|)^{\frac{1}{2}} \right) 2^\frac{1}{2} \varepsilon^{2} \gamma_j^\nu \gamma_j'^\nu \gamma_j^\nu \gamma_j'^\nu \varepsilon^{2} \gamma_j^\nu \gamma_j'^\nu.$$ 

Since we have:

$$B_{j,\nu,\nu'}^{1,1,1} = B_{j,\nu,\nu'}^{1,1,1,1} + B_{j,\nu,\nu'}^{1,1,1,2}$$

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in view of the decomposition (8.41), this yields:

\[
|B_{j,v,v'} + B_{j,v',v}| \leq \left( \sum_{p,q \geq 0} C_{pq} \left( \frac{1 + q^2}{(2^q |v - v'|)^{\nu + q}} \right) \left[ \frac{2^{-j}}{(2^{\frac{j}{2}} |v - v'|)^2} + \frac{2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |v - v'|)^3} \right] \right) \times \left( 1 + 2^{\frac{j}{2}} |v - v'| 2^{\frac{j}{2}} - \frac{j}{2} - \frac{j}{2} - \frac{j}{2} \right) \times 2^{\frac{j}{2}} \varepsilon \gamma'_j \gamma_j.
\]

(8.91)

Remark 8.9 The cancellation of the last term of (8.89) when considering the sum (8.90) in view of the antisymmetry in \((\nu, \nu')\) is crucial. Indeed, we would not be able to estimate this term directly.

Note that exchanging the role of \(\omega\) and \(\omega'\), we obtain that the term \(B_{j,v,v',l,m}^{1,1,2}\) corresponds to \(B_{j,v,v',l,m}^{1,1,1}\). Thus, we obtain in view of (8.91):

\[
|B_{j,v,v'} + B_{j,v',v}| \leq \left( \sum_{p,q \geq 0} C_{pq} \left( \frac{1 + q^2}{(2^q |v - v'|)^{\nu + q}} \right) \left[ \frac{2^{-j}}{(2^{\frac{j}{2}} |v - v'|)^2} + \frac{2^{-\frac{j}{2}}}{(2^{\frac{j}{2}} |v - v'|)^3} \right] \right) \times \left( 1 + 2^{\frac{j}{2}} |v - v'| 2^{\frac{j}{2}} - \frac{j}{2} - \frac{j}{2} - \frac{j}{2} \right) \times 2^{\frac{j}{2}} \varepsilon \gamma'_j \gamma_j.
\]

Together with (8.91), this concludes the proof of Proposition 8.6.

8.2 Proof of Proposition 8.4 (Control of \(B_{j,v,v',l,m}^{1,2}\))

Recall from (8.16) that \(B_{j,v,v',l,m}^{1,2}\) is given by:

\[
B_{j,v,v',l,m}^{1,2} = -i 2^{-j} \int_M \int_{\mathbb{R}^2} \int_0^\infty \int_0^\infty \frac{b^{-1}}{g(L, L')} \left( L(P_{l} \text{tr}_\chi) P_m \text{tr}_\chi' + P_l \text{tr}_\chi L'(P_m \text{tr}_\chi') \right)
\]

\[
\times \eta_j'(\omega) \eta_j''(\omega) \left( \frac{2^{-j} \lambda'}{2} \right)^{-1} \frac{2^{-j} \lambda}{2} \psi(2^{-j} \lambda)(2^{-j} \lambda') \psi(2^{-j} \lambda')(\psi(2^{-j} \lambda) \psi(2^{-j} \lambda')), \lambda \lambda', \omega \omega'.
\]

\[
\times \lambda^2 \lambda'^2 d\lambda d\lambda' d\omega d\omega' dM.
\]

(8.92)

Since \(\nabla u = b^{-1} N\) and \(\nabla u' = b'^{-1} u'\), we have:

\[
-i N(e^{i\lambda u - i\lambda' u'}) = e^{i\lambda u - i\lambda' u'} \left( b^{-1} \lambda - b'^{-1} \lambda \right) g(N, N') \lambda'
\]

\[
= e^{i\lambda u - i\lambda' u'} b^{-1}(\lambda - \lambda') + e^{i\lambda u - i\lambda' u'} (b^{-1} - b'^{-1} g(N, N')) \lambda'.
\]

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This yields:

\[
(2^{-j} \lambda')^{-1} - (2^{-j} \lambda) = e^{i \lambda u - i \lambda' u} b^{-1} = -i \frac{2j}{\lambda'} N(e^{i \lambda u - i \lambda' u}) + \frac{2j}{\lambda} e^{i \lambda u - i \lambda' u} (b^{-1} - b^{-1}) g(N, N') \]

\[
= -i \frac{2j}{\lambda' \lambda} N(e^{i \lambda u - i \lambda' u}) + \frac{2j}{\lambda} e^{i \lambda u - i \lambda' u} (b^{-1} - b^{-1}) + \frac{2j}{\lambda} e^{i \lambda u - i \lambda' u} y^{-1} (1 - g(N, N')).
\]

In view of (8.92), this implies the following decomposition for \( B_{j,u,v,l,m}^{1,2} \):

\[
B_{j,u,v,l,m}^{1,2} = B_{j,u,v,l,m}^{1,2,1} + B_{j,u,v,l,m}^{1,2,2} + B_{j,u,v,l,m}^{1,2,3}
\]

(8.93)

where \( B_{j,u,v,l,m}^{1,2,1} \), \( B_{j,u,v,l,m}^{1,2,2} \) and \( B_{j,u,v,l,m}^{1,2,3} \) are respectively given by:

\[
B_{j,u,v,l,m}^{1,2,1} = -2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{R}^2} \int_0^\infty \frac{1}{g(L, L')} \left( L(P_{tr\chi} P_{m \chi} + P_{tr\chi} L'(P_{m \chi})) \right) \times \eta_j^\nu (\omega) \eta_j^\nu (\omega') (2^{-j} \lambda)^{-1} (2^{-j} \lambda)^{-1} \psi(2^{-j} \lambda) \psi(2^{-j} \lambda') f(\lambda) f(\lambda') \times \lambda^2 \delta_\lambda d\lambda d\omega d\omega' \ d\mathcal{M},
\]

\[
B_{j,u,v,l,m}^{1,2,2} = -2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{R}^2} \int_0^\infty \frac{1}{g(L, L')} \left( L(P_{tr\chi} P_{m \chi} + P_{tr\chi} L'(P_{m \chi})) \right) \times (b^{-1} - b^{-1}) F_{j,-1}(u) F_j(u') \eta_j^\nu (\omega) \eta_j^\nu (\omega') d\omega d\omega' d\mathcal{M}. \quad (8.95)
\]

and:

\[
B_{j,u,v,l,m}^{1,2,3} = -2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{R}^2} \int_0^\infty \frac{1}{g(L, L')} \left( L(P_{tr\chi} P_{m \chi} + P_{tr\chi} L'(P_{m \chi})) \right) \times b^{-1} (1 - g(N, N')) F_{j,-1}(u) F_j(u') \eta_j^\nu (\omega) \eta_j^\nu (\omega') d\omega d\omega' d\mathcal{M}. \quad (8.96)
\]

We have the following propositions:

**Proposition 8.10** Let \( B_{j,u,v,l,m}^{1,2,1} \) be given by (8.94). Then, we have the following estimate:

\[
\left| \sum_{(l,m)/m < 1 \text{ and } 2^m \leq 2^j |v - v'|} B_{j,u,v,l,m}^{1,2,1} + \sum_{(l,m)/m < 1 \text{ and } 2^m \leq 2^j |v - v'|} B_{j,u,v,l,m}^{1,2,1} \right| \leq \frac{e^{2 \gamma_j^\nu \gamma_j^\nu'}}{2^j |v - v'|} + \frac{j 2^{-j} e^{2 \gamma_j^\nu \gamma_j^\nu'}}{(2^j |v - v'|)^2} + \frac{e^{2 \gamma_j^\nu \gamma_j^\nu'}}{(2^j |v - v'|)^3}.
\]

**Proposition 8.11** Let \( B_{j,u,v,l,m}^{1,2,2} \) be given by (8.95). Then, we have the following estimate:

\[
\left| \sum_{(l,m)/2^{min(l,m)} \leq 2^j |v - v'|} (B_{j,u,v,l,m}^{1,2,2} + B_{j,u,v,l,m}^{1,2,2}) \right| \leq 2^{-j} + \frac{1}{2^j (2^j |v - v'|)} + \frac{1}{2^j (2^j |v - v'|)^2} + \frac{2^{-\frac{1}{2}j} - j}{(2^j |v - v'|)^3}.
\]

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Proposition 8.12 Let $B_{j,\nu,\nu',l,m}^{1,2,3}$ be given by (8.96). Then, we have the following estimate:

$$\left\| \sum_{(l,m) \leq 2^j [\nu - \nu']} B_{j,\nu,\nu',l,m}^{1,2,3} \right\| \lesssim \left\| \frac{j^{2-\frac{1}{2}}}{(2^j |\nu - \nu'|)^3} + \frac{1}{2^j (2^j |\nu - \nu'|)^2} + \frac{1}{2^j (2^j |\nu - \nu'|)^{\frac{3}{2}}} + 2^{-j} \right\| 2^{\gamma_j' \gamma_{j'}}.$$

Together with the estimates (8.97), (8.98) and (8.99), we obtain:

$$\leq \left\| \sum_{(l,m) \leq 2^j [\nu - \nu']} (B_{j,\nu,\nu',l,m}^{1,2} + B_{j,\nu,\nu',l,m}^{1,3}) \right\| \lesssim \left\| \frac{1}{(2^j |\nu - \nu'|)^3} + \frac{j^{2-\frac{1}{2}}}{(2^j |\nu - \nu'|)^2} + \frac{1}{2^j (2^j |\nu - \nu'|)^2} + \frac{1}{2^j (2^j |\nu - \nu'|)^{\frac{3}{2}}} + 2^{-j} \right\| 2^{\gamma_j' \gamma_{j'}}.$$

This concludes the proof of Proposition 8.4.

The rest of this section is as follows. In section 8.2.1, we give a proof of Proposition 8.10, in section 8.2.2, we give a proof of Proposition 8.11, and in section 8.2.3, we give a proof of Proposition 8.12.

8.2.1 Proof of Proposition 8.10 (Control of $B_{j,\nu,\nu',l,m}^{1,2,1}$)

Integrating by parts the $N$ derivative in (8.94), we obtain:

$$B_{j,\nu,\nu',l,m}^{1,2,1} = 2^{-2j-1} \int_{M} \int_{S^2 \times S^2} \frac{1}{g(L, L')} \left( N(L(P_{\nu} \chi)) P_{m} P_{\nu}' + P_{\nu} N(L'(P_{\nu} \chi')) + L(P_{\nu} \chi) N(P_{m} P_{\nu}') + N(P_{\nu} \chi) L'(P_{m} P_{\nu}') \right. \times F_{j-1}(u) F_{j-1}(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' dM. \quad (8.100)$$
Recall the decomposition of $N$ in the frame $N', e_A'$:

$$N = g(N, N')N' + (N - g(N, N')N').$$  \hfill (8.101)

and the decomposition of $N'$ in the frame $N, e_A$:

$$N' = g(N, N')N + (N' - g(N, N')N).$$  \hfill (8.102)

(8.101) yields:

$$P_l g(N, N')N'(P_m g(N, N')) + L(P_l g(N, N')N'(P_m g(N, N')))

= g(N, N')P_l g(N, N')N'(P_m g(N, N')) + P_l g(N, N')N' + L(P_l g(N, N')N'(P_m g(N, N')))

= g(N, N')N + (N' - g(N, N')N).$$  \hfill (8.103)

Also, recall that:

$$g(L, L') = -1 + g(N, N')$$

which together with (2.21), (8.101) and (8.102) yields:

$$N(g(L, L')) = -g(N, N') + g(N, N') - \theta(N - g(N, N')N', N - g(N, N')N').$$  \hfill (8.104)

In view of (8.100), (8.103) and (8.104), we obtain:

$$B_{j,\nu,\nu',l,m}^{1,2,1} = B_{j,\nu,\nu',l,m}^{1,2,1,1} + B_{j,\nu,\nu',l,m}^{1,2,1,2} + B_{j,\nu,\nu',l,m}^{1,2,1,3}.$$  \hfill (8.105)

where $B_{j,\nu,\nu',l,m}^{1,2,1,1}$ is given by:

$$B_{j,\nu,\nu',l,m}^{1,2,1,1} = 2^{-2j-1} \int_M \int_{S^2 \times S^2} \frac{N(L(P_l g(N, N'))N'(P_m g(N, N')))}{g(L, L')}\times F_{j-1}(u)F_{j-1}(u')\eta_j^\nu(u)\eta_j^\nu(u')d\omega d\omega'd\mathcal{M},$$  \hfill (8.106)

where $B_{j,\nu,\nu',l,m}^{1,2,1,2}$ is given by:

$$B_{j,\nu,\nu',l,m}^{1,2,1,2} = 2^{-2j} \int_M \int_{S^2 \times S^2} HF_{j-1}(u)F_{j-1}(u')\eta_j^\nu(u)\eta_j^\nu(u')d\omega d\omega'd\mathcal{M},$$  \hfill (8.107)

with the tensor $H$ on $\mathcal{M}$ given, schematically, by:

$$H = \frac{1}{g(L, L')} \left( P_l g(N, N')N'(P_m g(N, N')) + L(P_l g(N, N')N'(P_m g(N, N'))

+ L(P_l g(N, N')N'(P_m g(N, N')) + (\frac{\theta(N - N')^2}{g(L, L')} + tr\theta) (L(P_l g(N, N')N'(P_m g(N, N'))) \right).$$  \hfill (8.108)
and where \( B^{1,2,1,3}_{j,\nu,\nu',l,m} \) is given by:

\[
B^{1,2,1,3}_{j,\nu,\nu',l,m} = -2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} b^{-1} \nabla'_{N' - g(N,N')} N(b) + b'^{-1} \nabla'_{N - g(N,N')} N'(b') \frac{g(L, L')^2}{g(L, L')^2} \times (L(P_{l} \text{tr} \chi) P_{m} \text{tr} \chi' + P_{l} \text{tr} \chi L'(P_{m} \text{tr} \chi')) F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

Next, we estimate the three terms in the right-hand side of (8.105) starting with \( B^{1,2,1,1}_{j,\nu,\nu',l,m} \). Recall from (8.1) that \((l, m)\) satisfy:

\[
m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|.
\]

Summing in \((l, m)\), we obtain:

\[
\sum_{(l, m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} \frac{N(L(P_{l} \text{tr} \chi)) P_{m} \text{tr} \chi' + P_{l} \text{tr} \chi N'(L'(P_{m} \text{tr} \chi'))}{g(L, L')} \quad + \quad \sum_{(l, m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} \frac{N(L(P_{m} \text{tr} \chi)) P_{l} \text{tr} \chi' + P_{m} \text{tr} \chi N'(L'(P_{l} \text{tr} \chi'))}{g(L, L')} = \frac{N(L(\text{tr} \chi)) \text{tr} \chi' + \text{tr} \chi N'(L'(\text{tr} \chi'))}{g(L, L')} - \frac{N(L(P_{2^j |\nu - \nu'|} \text{tr} \chi)) P_{2^j |\nu - \nu'|} \text{tr} \chi' + P_{2^j |\nu - \nu'|} \text{tr} \chi N'(L'(P_{2^j |\nu - \nu'|} \text{tr} \chi'))}{g(L, L')}.
\]

Thus, using the symmetry in \((\omega, \omega')\) of the integrant in \( B^{1,2,1,1}_{j,\nu,\nu',l,m} \), we obtain:

\[
\sum_{(l, m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B^{1,2,1,1}_{j,\nu,\nu',l,m} + \sum_{(l, m)/m < l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B^{1,2,1,1}_{j,\nu',\nu,l,m} = 2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{N(L(\text{tr} \chi)) \text{tr} \chi' + \text{tr} \chi N'(L'(\text{tr} \chi'))}{g(L, L')} \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M} - 2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{N(L(P_{2^j |\nu - \nu'|} \text{tr} \chi)) P_{2^j |\nu - \nu'|} \text{tr} \chi' + P_{2^j |\nu - \nu'|} \text{tr} \chi N'(L'(P_{2^j |\nu - \nu'|} \text{tr} \chi'))}{g(L, L')} \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

Estimating the terms \( N(L(P_{2^j |\nu - \nu'|} \text{tr} \chi)) \) and \( N'(L'(P_{2^j |\nu - \nu'|} \text{tr} \chi')) \) would involve commutator terms which are difficult to handle. To avoid this issue, we commute \( L \) with \( N \) and \( L' \) with \( N' \), and then integrate the \( L \) and the \( L' \) derivative by parts. We obtain
schematically in view of (8.110),

\[
\sum_{(l,m)/m<l \text{ and } 2^m \leq 2^{2j|\nu-\nu'|}} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m<l \text{ and } 2^m \leq 2^{2j|\nu-\nu'|}} B_{j,\nu,\nu',l,m}^{1,2,1,1}
\]

\[
(8.111)
\]

\[
= 2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{N(L(\text{tr}\chi)) \text{tr}\chi'}{g(L, L')} \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M}
\]

\[
- 2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} [N, L](P_{>2^j|\nu-\nu'|\text{tr}\chi}) P_{>2^j|\nu-\nu'|\text{tr}\chi'} g(L, L') \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M}
\]

\[
- 2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \left( \frac{\text{div}_g(L)}{g(L, L')} - \frac{L(g(L, L'))}{g(L, L')^2} \right) N(P_{>2^j|\nu-\nu'|\text{tr}\chi}) P_{>2^j|\nu-\nu'|\text{tr}\chi'} \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) d\omega
\]

\[
- 2^{-j-1} \int_{\mathcal{M}} \left( \int_{S^2} N(P_{>2^j|\nu-\nu'|\text{tr}\chi}) F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} \frac{b^{-1} P_{>2^j|\nu-\nu'|\text{tr}\chi'} F_{j}(u') \eta_j^\nu'(\omega') d\omega'}{b^{-1} P_{>2^j|\nu-\nu'|\text{tr}\chi'} F_{j}(u') \eta_j^\nu'(\omega') d\omega'} d\mathcal{M} + \text{ terms interverting } (\nu, \nu'),
\]

where the last term in the right-hand side of (8.111) appears when the $L$ derivative falls on the phase in view of (6.5), and where we chose to ignore the terms which are obtained by interverting $\nu$ and $\nu'$ since they are treated in the exact same way.

We decompose $L$ in the frame $L', N', e'_A$:

\[
L = L' + (N - g(N, N')N') + (g(N, N') - 1)N',
\]

which yields the following decomposition:

\[
L(P_{>2^j|\nu-\nu'|\text{tr}\chi'}) = L'(P_{>2^j|\nu-\nu'|\text{tr}\chi'}) + (N - g(N, N')N')(P_{>2^j|\nu-\nu'|\text{tr}\chi'}) + (g(N, N') - 1)N'(P_{>2^j|\nu-\nu'|\text{tr}\chi'}).
\]

Recall the identities (8.30) and (8.31):

\[
g(L, L') = -1 + g(N, N') \quad \text{and} \quad 1 - g(N, N') = \frac{g(N - N', N - N')}{2}.
\]

We may thus expand

\[
\frac{1}{g(L, L')} \quad \text{and} \quad \frac{1}{g(L, L')^2}
\]

in the same fashion than (8.33), and in view of (8.111), (8.113), the formula (7.146) for
\[
\text{div}_g(L) \text{ and the formula (7.149) for } L(g(L, L')) \text{, we obtain, schematically:}
\]

\[
\sum_{(l, m)/m<1 \text{ and } 2^m \leq 2^{|\nu-\nu'|}} B^{1,2,1,1}_{j,\nu,\nu',l,m} + \sum_{(l, m)/m<1 \text{ and } 2^m \leq 2^{|\nu-\nu'|}} B^{1,2,1,1}_{j,\nu,\nu',l,m}
\]

\[
= 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_M \frac{1}{(2^{|\nu-N|})^{p+q}} \left[ \frac{1}{(2^{|\nu-N|})^2} (h_{1,p,q} + h_{2,p,q} + h_{3,p,q}) \right]
\]

\[
+ \frac{1}{2^{|\nu-N|}} h_{4,p,q} + 2^{-j} h_{5,p,q} \right] dM
\]

\[
-2^{-j-1} \int_M \left( \int_{S^2} N(P_{>2^{|\nu-\nu'|}}) F_j(u) \eta_j^\nu (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} h_{l-1}^{-1} P_{>2^{|\nu-\nu'|}} \text{tr} \chi' F_j(u') \eta_j^\nu (\omega') d\omega' \right) dM + \text{ terms interverting } (\nu, \nu'),
\]

where the scalar functions \( h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}, h_{5,p,q} \) on \( M \) are given by:

\[
h_{1,p,q} = \left( \int_{S^2} N(L(\text{tr} \chi)) \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} \text{tr} \chi' \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega') d\omega' \right),
\]

\[
h_{2,p,q} = \left( \int_{S^2} G_1 \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} \text{tr} \chi' \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega') d\omega' \right),
\]

\[
h_{3,p,q} = \left( \int_{S^2} N(P_{>2^{|\nu-\nu'|}}) \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} G_2 \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega') d\omega' \right),
\]

\[
h_{4,p,q} = \left( \int_{S^2} N(P_{>2^{|\nu-\nu'|}}) \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} \nabla' (P_{>2^{|\nu-\nu'|}}) \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega') d\omega' \right),
\]

and:

\[
h_{5,p,q} = \left( \int_{S^2} N(P_{>2^{|\nu-\nu'|}}) \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega) d\omega \right)
\]

\[
\times \left( \int_{S^2} N(P_{>2^{|\nu-\nu'|}}) \left( 2^{|\nu-\nu'|} \right) \eta_j^\nu (\omega') d\omega' \right),
\]

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where the tensors $G_1$ and $G_2$ are schematically given by:
\begin{equation}
G_1 = [N, L](P_{>2|\nu'\nu|\partial N} + \hat{\delta}(P_{>2|\nu'\nu|\partial N}))
\end{equation}
and:
\begin{equation}
G_2 = L'(P_{>2|\nu'\nu|\partial N} + (\hat{\delta} + \chi + \zeta')P_{>2|\nu'\nu|\partial N})
\end{equation}
and where $c_{pq}$ are explicit real coefficients such that the series
\begin{equation}
\sum_{p,q \geq 0} c_{pq} x^p y^q
\end{equation}
has radius of convergence 1.

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}, h_{5,p,q}$ starting with $h_{1,p,q}$. We have:
\begin{equation}
\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} N(L(\partial N)) \left(2^q(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^q(\omega) d\omega \right\|_{L^\frac{3}{2}(\mathcal{M})}
\end{equation}
\begin{equation}
\lesssim \left\| \int_{\mathbb{S}^2} \partial N(\partial N') \left(2^q(N' - N_\nu)\right)^q F_{j,-1}(u') \eta_j^q(\omega') d\omega' \right\|_{L^3(\mathcal{M})}.
\end{equation}
We estimate the $L^\infty(\mathcal{M})$ norm of the last term:
\begin{equation}
\left\| \int_{\mathbb{S}^2} \partial N(\partial N') \left(2^q(N' - N_\nu)\right)^q F_{j,-1}(u') \eta_j^q(\omega') d\omega' \right\|_{L^\infty(\mathcal{M})}
\lesssim \varepsilon \left\| F_{j,-1}(u') \right\|_{L^\infty(u_\nu)} \eta_j^q(\omega') d\omega'
\lesssim 2^\varepsilon \gamma_j^p',
\end{equation}
where we used the estimate (2.39) for $\partial N'$, the estimate (2.42) for $\partial N$, the size of the patch, Cauchy Schwartz in $\lambda'$ for $\|F_{j,-1}(u')\|_{L^\infty(u_\nu)}$, and Cauchy Schwartz in $\omega'$. On the other hand, the estimate (7.64) yields:
\begin{equation}
\left\| \int_{\mathbb{S}^2} \partial N(\partial N') \left(2^q(N' - N_\nu)\right)^q F_{j,-1}(u') \eta_j^q(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim (1 + q^2)\varepsilon \gamma_j^p'.
\end{equation}
Interpolating these two estimates, we obtain:
\begin{equation}
\left\| \int_{\mathbb{S}^2} \partial N(\partial N') \left(2^q(N' - N_\nu)\right)^q F_{j,-1}(u') \eta_j^q(\omega') d\omega' \right\|_{L^3(\mathcal{M})} \lesssim 2^\varepsilon (1 + q^2)\varepsilon \gamma_j^p'.
\end{equation}
Next, we estimate the first term in the right-hand side of (8.122). We have:
\begin{equation}
\left\| \int_{\mathbb{S}^2} N(L(\partial N)) \left(2^q(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^q(\omega) d\omega \right\|_{L^\frac{3}{2}(\mathcal{M})}
\lesssim \int_{\mathbb{S}^2} \|N(L(\partial N))\|_{L^\infty L^\frac{3}{2}(H_\omega)} \left\| \left(2^q(N - N_\nu)\right)^p \right\|_{L^\infty} \|F_{j,-1}(u)\|_{L^\frac{3}{2}(\mathcal{M})} \eta_j^q(\omega) d\omega
\lesssim \int_{\mathbb{S}^2} \|N(L(\partial N))\|_{L^\infty L^\frac{3}{2}(H_\omega)} \|F_{j,-1}(u)\|_{L^\frac{3}{2}(\mathcal{M})} \eta_j^q(\omega) d\omega,
\end{equation}
where we used in the last inequality the estimate (2.42) for $\partial_N N$ and the size of the patch. Next, we estimate $N(L(\text{tr}\chi))$. In view of the Raychaudhuri equation (2.22), we have:

$$N(L(\text{tr}\chi)) = -\text{tr}\chi N(\text{tr}\chi) - 2\hat{\chi} \cdot D_N \hat{\chi} - N(\delta)\text{tr}\chi - \bar{\delta}N(\text{tr}\chi),$$

which together with the Sobolev embedding (2.50), and the estimates (2.39) for $\text{tr}\chi$, (2.40) for $\hat{\chi}$, and (2.36) (2.37) for $\bar{\delta}$ yields:

$$\|N(L(\text{tr}\chi))\|_{L^\infty L^2(H_u)} \lesssim (\|D\chi\|_{L^\infty L^2(H_u)} + \|D\bar{\delta}\|_{L^\infty L^2(H_u)})(\|\chi\|_{L^\infty L^p(H_u)} + \|\bar{\delta}\|_{L^\infty L^p(H_u)}) \lesssim \varepsilon.$$

Together with (8.125), we obtain:

$$\left\| \int_{S^2} N(L(\text{tr}\chi)) \left(2^{\hat{\varepsilon}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega)d\omega \right\|_{L^2(M)} \lesssim \varepsilon \int_{S^2} \|F_{j,-1}(u)\|_{L^2} \eta_j^{\nu'}(\omega)d\omega \lesssim 2^{\hat{\varepsilon}}\varepsilon \gamma_j^{\nu'},$$

where we used in the last inequality Plancheer in $u$, Cauchy Schwarz in $\omega$ and the size of the patch. Finally, (8.122), (8.124) and (8.126) imply:

$$\|h_{1,p,q}\|_{L^1(M)} \lesssim (1 + q)2^{\hat{\varepsilon}}\varepsilon^{2} \gamma_j^{\nu'} \gamma_j^\nu.$$  

Next, we estimate $h_{2,p,q}$. We have:

$$\|h_{2,p,q}\|_{L^1(M)} \lesssim \left\| \int_{S^2} G_1 \left(2^{\hat{\varepsilon}}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^{\nu'}(\omega)d\omega \right\|_{L^2(M)} \lesssim \int_{S^2} P_{>2|\nu - \nu'|} \text{tr}\chi' \left(2^{\hat{\varepsilon}}(N' - N_\nu')\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^3(M)}.$$

Arguing as in (8.123), we have:

$$\left\| \int_{S^2} P_{>2|\nu - \nu'|} \text{tr}\chi' \left(2^{\hat{\varepsilon}}(N' - N_\nu')\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^\infty(M)} \lesssim 2^{\hat{\varepsilon}}\varepsilon \gamma_j^{\nu'}.$$

On the other hand, we have in view of the $L^2$ estimate (7.3):

$$\left\| \int_{S^2} P_{>2|\nu - \nu'|} \text{tr}\chi' \left(2^{\hat{\varepsilon}}(N' - N_\nu')\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(M)} \lesssim \left(\sup_{\omega} \left\| \left(2^{\hat{\varepsilon}}(N' - N_\nu')\right)^q \right\|_{L^\infty} \right) \frac{2^{\hat{\varepsilon}}}{2^{\hat{\varepsilon}}|\nu - \nu'|} \varepsilon \gamma_j^{\nu'} \lesssim \frac{\varepsilon \gamma_j^{\nu'}}{2^{\hat{\varepsilon}}|\nu - \nu'|},$$

where we used in the last inequality the estimate (2.42) for $\partial_N N$ and the size of the patch. Interpolating these two estimates, and using the fact that:

$$2^{\hat{\varepsilon}}|\nu - \nu'| \gtrsim 1,$$  

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we obtain:

\[
\left\| \int_{S^2} P_{>2|\nu-\nu'|} \text{tr} \chi' \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \delta_i^{j'}(\omega') d\omega' \right\|_{L^2(M)} \lesssim 2^{\frac{1}{2}} \varepsilon \gamma_j^{\nu'}.
\]  (8.130)

Next, we estimate the first term in the right-hand side of (8.128). Arguing as in (8.125), we have:

\[
\left\| \int_{S^2} G_1 \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \delta_i^j(\omega) d\omega \right\|_{L^2(M)} \leq \varepsilon \nabla n L P_{>2|\nu-\nu'|} \text{tr} \chi + (\zeta - \zeta') \cdot \nabla \left( P_{>2|\nu-\nu'|} \text{tr} \chi \right).
\]  (8.131)

Next, we estimate \( G_1 \). In view of the definition of \( G_1 \) (8.120), the commutator formulas (2.32) for \([L, L]\) and the fact that \( 2N = L - L \), we have schematically:

\[
G_1 = \bar{\delta} N (P_{>2|\nu-\nu'|} \text{tr} \chi) + n^{-1} \nabla N n L P_{>2|\nu-\nu'|} \text{tr} \chi + (\zeta - \zeta') \cdot \nabla \left( P_{>2|\nu-\nu'|} \text{tr} \chi \right).
\]

This yields:

\[
\|G_1\|_{L^\infty L^2(H_u)} \lesssim \left( \|\bar{\delta}\|_{L^\infty L^6(H_u)} + \|n^{-1} \nabla N n\|_{L^\infty L^6(H_u)} + \|\zeta\|_{L^\infty L^6(H_u)} + \|\zeta\|_{L^\infty L^6(H_u)} \right) \times \|DP_{>2|\nu-\nu'|} \text{tr} \chi\|_{L^\infty L^2(H_u)} \lesssim \varepsilon \|DP_{>2|\nu-\nu'|} \text{tr} \chi\|_{L^\infty L^2(H_u)},
\]

where we used in the last inequality the Sobolev embedding (2.50), and the estimates (2.37) (2.36) for \( n, \bar{\delta} \) and \( \zeta \), and the estimate (2.41) for \( \zeta \). Together with the basic properties of \( P_{>2|\nu-\nu'|} \), the commutator estimates (2.64) and (2.65), and the estimate (2.39) for \( \text{tr} \chi \), this implies:

\[
\|G_1\|_{L^\infty L^2(H_u)} \lesssim \varepsilon.
\]  (8.132)

Together with (8.131), and arguing as in (8.126), we obtain:

\[
\left\| \int_{S^2} G_1 \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \delta_i^j(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{\frac{1}{2}} \varepsilon \gamma_j^{\nu'}.
\]  (8.133)

Finally, (8.128), (8.130) and (8.133) imply:

\[
\|h_{2,p,q}\|_{L^1(M)} \lesssim 2^{\frac{1}{2}} \varepsilon \gamma_j^{\nu'} \gamma_j^{\nu'}.
\]  (8.134)

Next, we estimate \( h_{3,p,q} \). We have:

\[
\|h_{3,p,q}\|_{L^1(M)} \lesssim \left\| \int_{S^2} N (P_{>2|\nu-\nu'|} \text{tr} \chi) \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \delta_i^j(\omega) d\omega \right\|_{L^2(M)} + \left\| \int_{S^2} G_2 \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \delta_i^j(\omega') d\omega' \right\|_{L^2(M)}.
\]  (8.135)
We estimate the first term in the right-hand side of (8.135). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$
\left\| \int_{\mathbb{S}^2} N(P_{>2|v-\nu'|} \text{tr}\chi) \left(2^{\frac{v}{2}}(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \|N(P_{>2|v-\nu'|} \text{tr}\chi) \left(2^{\frac{v}{2}}(N - N_\nu)\right)^p \|_{L^\infty L^2(\mathcal{H}_u)} \right)^{\frac{1}{2}} 2^{\frac{v}{2}} \gamma_j^\nu.
$$

Together with the estimate (2.39) for $\text{tr}\chi$, the commutator estimate (2.64), the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we obtain:

$$
\left\| \int_{\mathbb{S}^2} N(P_{>2|v-\nu'|} \text{tr}\chi) \left(2^{\frac{v}{2}}(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{v}{2}} \gamma_j^\nu.
$$ (8.136)

Next, we estimate the second term in the right-hand side of (8.135). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$
\left\| \int_{\mathbb{S}^2} G_2 \left(2^{\frac{v}{2}}(N' - N_\nu')\right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega'} \|G_2 \left(2^{\frac{v}{2}}(N' - N_\nu')\right)^q \|_{L^\infty L^2(\mathcal{H}_u)} \right)^{\frac{1}{2}} 2^{\frac{v}{2}} \gamma_j^{\nu'}
$$

$$
\lesssim \left( \sup_{\omega'} \|G_2\|_{L^\infty L^2(\mathcal{H}_u)} \left\| \left(2^{\frac{v}{2}}(N' - N_\nu')\right)^q \right\|_{L^\infty} \right)^{\frac{1}{2}} 2^{\frac{v}{2}} \gamma_j^{\nu'}.
$$

In view of the definition of $G_2$ (8.121), we have:

$$
\|G_2\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim (\|\widehat{3}\|_{L^\infty L^4_x} + \|\chi\|_{L^\infty L^4_x} + \|\zeta\|_{L^\infty L^4_x}) \|P_{>2|v-\nu'|} \text{tr}\chi\|_{L^2_x L^4_x}.
$$ (8.138)

where we used in the last inequality the embedding (2.51), and the estimates (2.36) (2.37) for $\delta$, the estimates (2.39) (2.40) for $\chi$, and the estimate (2.41) for $\zeta$. Using the Bernstein inequality and the finite band property for $P_t$, we have:

$$
\|P_{>2|v-\nu'|} \text{tr}\chi\|_{L^2_x L^4_x} \lesssim \sum_{l > 2^l|v-\nu'|} \|P_t \text{tr}\chi\|_{L^2_x L^4_x},
$$ (8.139)

$$
\lesssim \sum_{l > 2^l|v-\nu'|} 2^{\frac{v}{2}} \|P_t \text{tr}\chi\|_{L^\infty L^2(\mathcal{H}_u)}
$$

$$
\lesssim \sum_{l > 2^l|v-\nu'|} 2^{-\frac{v}{2}} \|\nabla \text{tr}\chi\|_{L^\infty L^2(\mathcal{H}_u)}
$$

$$
\lesssim \frac{\varepsilon}{(2^{l}|v-\nu'|)^{\frac{1}{2}}},
$$
where we used the estimate (2.39) for \( \text{tr} \chi \) in the last inequality. In view of (8.138), we also need to estimate \( L(P_{>2|\nu-\nu'|}\text{tr} \chi) \). Using the estimate (2.36) for \( n \), we have:

\[
\|L(P_{\text{tr} \chi})\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \|nL(P_{\text{tr} \chi})\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \|P_l(n\text{tr} \chi)\|_{L^\infty L^2(\mathcal{H}_u)} + \|[nL,P_l]\text{tr} \chi\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \|P_l(n\text{tr} \chi)\|_{L^\infty L^2(\mathcal{H}_u)} + 2^{-\frac{1}{2}}\varepsilon, \tag{8.140}
\]

where we used in the last inequality the commutator estimate (2.67). Now, in view of the Raychaudhuri equation (2.22), the worst term in \( P_l(n\text{tr} \chi) \) is \( P_l(n|\hat{\chi}|^2) \). In view of the finite band property, we have:

\[
\|P_l(n|\hat{\chi}|^2)\|_{L^\infty L^2(\mathcal{H}_u)} = 2^{-2l}\|P_l((\text{div} \hat{\chi})(n|\hat{\chi}|^2))\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{1}{2}}2^{3l}\|\nabla(n|\hat{\chi}|^2)\|_{L^2 L^2},
\]

where we used (2.63) with \( p = \frac{4}{3} \) in the last inequality. This yields:

\[
\|P_l(n|\hat{\chi}|^2)\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \frac{1}{2}2^{\frac{1}{2}}\|\hat{\chi}\|_{L^2 L^2}(\|n\|_{L^\infty(\mathcal{M})}\|\hat{\chi}\|_{L^\infty L^2(\mathcal{H}_u)} + \|\nabla n\|_{L^\infty L^2(\mathcal{H}_u)}\|\hat{\chi}\|_{L^\infty L^6(\mathcal{H}_u)}) \lesssim \frac{1}{2}2^{-\frac{1}{2}}\varepsilon,
\]

where we used in the last inequality the Sobolev embedding (2.50) and the embedding (2.51), and the estimates (2.36) for \( n \) and (2.40) for \( \hat{\chi} \). Since \( P_l(n|\hat{\chi}|^2) \) is the worst term in \( P_l(n\text{tr} \chi) \) in view of the Raychaudhuri equation (2.22), we obtain:

\[
\|P_l(n\text{tr} \chi)\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{5}{4}}\varepsilon, \tag{8.141}
\]

which together with (8.140) yields:

\[
\|L(P_{\text{tr} \chi})\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{1}{2}}\varepsilon. \tag{8.142}
\]

Now, in view of (8.138), (8.139) and (8.142), and since \( 2^{\frac{5}{4}}|\nu-\nu'| \gtrsim 1 \), we obtain:

\[
\|G_2\|_{L^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{5}{4}}\varepsilon.
\]

Together with (8.137), the estimate (2.42) for \( \partial_\omega N \) and the size of the patch, we deduce:

\[
\left\| \int_{S^2} G_2 \left( 2^{\frac{5}{4}}(N' - N'') \right)^q F_{j,-1}(u')\eta_j''(\omega')d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{5}{4}}\gamma_j'' \tag{8.143}
\]

where we also used the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Finally, (8.135), (8.136) and (8.143) imply:

\[
\|h_{3,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{\frac{3}{4}}\gamma_j' \gamma_j''. \tag{8.144}
\]

Next, we estimate the \( L^1(\mathcal{M}) \) norm of \( h_{4,p,q} \). In view of the definition of \( h_{4,p,q} \) (8.118), we have:

\[
\|h_{4,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} N(P_{>2|\nu-\nu'|}\text{tr} \chi) \left( 2^{\frac{5}{4}}(N - N') \right)^p F_{j,-1}(u')\eta_j''(\omega')d\omega \right\|_{L^2(\mathcal{M})} \times \left\| \int_{S^2} \nabla'(P_{>2|\nu-\nu'|}\text{tr} \chi') \left( 2^{\frac{5}{4}}(N' - N'') \right)^q F_{j,-1}(u')\eta_j''(\omega')d\omega' \right\|_{L^2(\mathcal{M})}.
\]
Using the finite band property and the estimate (2.39) for $tr\nu$, where we used in the last inequality the estimate (2.42) for the second term, we obtain:

$$\|h_{4,p,q}\|_{L^1(M)} \lesssim \left( \sup_{\nu} \left\| \nabla' (P_{>2^j|\nu-\nu'|} tr\chi) \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty_t L^2(\mathcal{H}_{\nu'})} \right) \varepsilon 2^j \gamma_j^\nu \gamma_j^{\nu'}$$

(8.145)

where we used in the last inequality the estimate (2.42) for $\partial_\nu N$ and the size of the patch. Using the finite band property and the estimate (2.39) for $tr\chi$, we obtain:

$$\| \nabla' (P_{>2^j|\nu-\nu'|} tr\chi') \|_{L^\infty_t L^2(\mathcal{H}_{\nu'})} \lesssim \varepsilon.$$

Together with (8.145), we finally obtain:

$$\|h_{4,p,q}\|_{L^1(M)} \lesssim 2^j \varepsilon 2^j \gamma_j^\nu \gamma_j^{\nu'}$$

(8.146)

Next, we estimate the $L^1(M)$ norm of $h_{5,p,q}$. In view of the definition of $h_{5,p,q}$ (8.119), we have:

$$\|h_{5,p,q}\|_{L^1(M)} \lesssim \left\| \int_{S^2} N(P_{>2^j|\nu-\nu'|} tr\chi) \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^p F_{j-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(M)}$$

$$\times \left\| \int_{S^2} N'(P_{>2^j|\nu-\nu'|} tr\chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)}$$

Then, using the estimate (8.136) for both terms, we obtain:

$$\|h_{5,p,q}\|_{L^1(M)} \lesssim 2^j \varepsilon 2^j \gamma_j^\nu \gamma_j^{\nu'}$$

(8.147)

Now, we have in view of (8.114), we have:

$$\begin{align*}
2^{-2j} \sum_{p,q \geq 0} c_{pq} \left| \frac{1}{(2^{\frac{j}{2}} |N_{\nu} - N_{\nu'}|)^{p+q}} \right|_{L^\infty(M)} \left( \|h_{1,p,q}\|_{L^1(M)} + \|h_{2,p,q}\|_{L^1(M)} + \|h_{3,p,q}\|_{L^1(M)} \right) \\
\times \left( \|h_{4,p,q}\|_{L^1(M)} + 2^{-j} \|h_{5,p,q}\|_{L^1(M)} \right) \\
+ \left\| \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |N_{\nu} - N_{\nu'}|)} \right\|_{L^\infty(M)} \left( \|h_{1,p,q}\|_{L^1(M)} + \|h_{2,p,q}\|_{L^1(M)} + \|h_{3,p,q}\|_{L^1(M)} \right) \\
+ 2^{-j} \left\| \int_{S^2} N(P_{>2^j|\nu-\nu'|} tr\chi) F_{j-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(M)} \\
\times \left\| \int_{S^2} h^{-1} P_{>2^j|\nu-\nu'|} tr\chi' F_j(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)} ,
\end{align*}$$

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which together with (8.32), (8.127), (8.134), (8.144), (8.146) and (8.147) yields:

\[
\begin{aligned}
\sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} \\
\lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left( \frac{1}{(2^j |\nu - \nu'|)^{p+q}} \left[ \frac{(1+q)2^j}{2^j |\nu - \nu'|^2} + \frac{1}{2^j (2^j |\nu - \nu'|)} \right] \phi_j^{\nu,\nu'} \right) \\
+ 2^{-j} \left\| \int_{S^2} N(P_{>2^j |\nu - \nu'|} \text{tr} \chi) F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(M)} \\
\times \left\| \int_{S^2} b^{-1} P_{>2^j |\nu - \nu'|} \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \\
\lesssim 2^{-j} \varepsilon^{2,\gamma_j^{\nu,\nu'}} + 2^{-j} \left\| \int_{S^2} N(P_{>2^j |\nu - \nu'|} \text{tr} \chi) F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(M)} \\
\times \left\| \int_{S^2} b^{-1} P_{>2^j |\nu - \nu'|} \text{tr} \chi' F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)}.
\end{aligned}
\]

Using the corresponding analog of (8.136) and the corresponding analog of (8.129) to estimate the last term in the right-hand side, we deduce:

\[
\begin{aligned}
\sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} + \sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,1} \\
\lesssim 2^{-j} \varepsilon^{2,\gamma_j^{\nu,\nu'}} + 2^{-j} \varepsilon^{2,\gamma_j^{\nu,\nu'}}.
\end{aligned}
\]

Next, we estimate \( B_{j,\nu,\nu',l,m}^{1,2,1,2} \). Recall from (8.107) and (8.108) that \( B_{j,\nu,\nu',l,m}^{1,2,1,2} \) is given by:

\[
B_{j,\nu,\nu',l,m}^{1,2,1,2} = 2^{-2j} \int_M \int_{S^2 \times S^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^{\nu'}(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' dM,
\]

with the tensor \( H \) on \( M \) given, schematically, by:

\[
H = \frac{1}{g(L, L')} \left( P_{l \text{tr} \chi} \nabla'(L'(P_m \text{tr} \chi'))(N - N') + L(P_{l \text{tr} \chi}) N'(P_m \text{tr} \chi') \\
+ L(P_{l \text{tr} \chi}) \nabla'(P_m \text{tr} \chi')(N - N') + N(P_{l \text{tr} \chi}) L'(P_m \text{tr} \chi') \\
+ \left( \frac{b^{-1}(b') + \theta'(N - N')^2}{g(L, L')} + \text{tr} \theta \right) (L(P_{l \text{tr} \chi}) P_m \text{tr} \chi' + P_{l \text{tr} \chi} L'(P_m \text{tr} \chi')) \right).
\]

Expanding

\[
\frac{1}{g(L, L')} \text{ and } \frac{1}{g(L, L')^2}
\]

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in the same fashion than (8.33), and in view of (8.108), we obtain, schematically:

\[ H = \frac{1}{|N - N'|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N'}{|N - N'|} \right)^p \left( \frac{N' - N''}{|N - N'|} \right)^q \right) \times \left( L(P_{1tr}(H_1 + H_2L'(P_{mtr}\nu') + P_{1tr}(N'(P_{mtr}\nu'))) + (b^{-1}R(b') + \theta')L'(P_{mtr}\nu')) + \theta L(P_{1tr}(P_{mtr}\nu')) \right), \]

where the tensors \( H_1, H_2 \) on \( \mathcal{M} \) are schematically given by:

\[ H_1 = N'(P_{mtr}\nu') + N'(P_{mtr}\nu') + \theta' P_{mtr}\nu', \]

and:

\[ H_2 = N(P_{1tr}) + \theta P_{1tr}, \]

and where \( c_{pq} \) are explicit real coefficients such that the series

\[ \sum_{p,q \geq 0} c_{pq} x^p y^q \]

has radius of convergence 1. In turn, this yields in view of (8.107) and (8.108) the following decomposition for \( B_{j,j'\nu',l,m}^{1,2,1,2} \):

\[ B_{j,j'\nu',l,m}^{1,2,1,2} = 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^j|N - N'|)^{p+q+2}} \times [h_{1,p,q,l,m} + h_{2,p,q,l,m} + h_{3,p,q,l,m} + h_{4,p,q,l,m}] d\mathcal{M}, \]

where the scalar functions \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m} \) on \( \mathcal{M} \) are schematically given by:

\[ h_{1,p,q,l,m} = \left( \int_{S^2} L(P_{1tr}(H_1 + H_2L'(P_{mtr}\nu'))) \right)^{p} F_{j,-1}(u) \eta_j'(\omega) d\omega \]

\[ \times \left( \int_{S^2} H_1 \left( 2^j(N - N') \right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right), \]

\[ h_{2,p,q,l,m} = \left( \int_{S^2} H_2 \left( 2^j(N - N') \right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right) \]

\[ \times \left( \int_{S^2} L'(P_{mtr}\nu') \left( 2^j(N' - N'') \right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right), \]

\[ h_{3,p,q,l,m} = \left( \int_{S^2} P_{1tr}(H_1 + H_2L'(P_{mtr}\nu')) \right)^{p} F_{j,-1}(u) \eta_j'(\omega) d\omega \]

\[ \times \left( \int_{S^2} (b^{-1}R(b') + \theta')L'(P_{mtr}\nu') \left( 2^j(N' - N'') \right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right) \]

\[ + \left( \int_{S^2} \theta L(P_{1tr} \left( 2^j(N - N') \right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right) \]

\[ \times \left( \int_{S^2} P_{mtr}(2^j(N' - N'')) \left( 2^j(N' - N'') \right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right). \]
and:
\[
h_{4,p,q,l,m} = \left( \int_{S^2} P_{t \chi} \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) (8.156)
\]
\[
\times \left( \int_{S^2} \nabla'(L'(P_m t \chi')) \left( 2^{\frac{j}{2}}(N' - N_\nu') \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right).
\]

Next, we evaluate the \( L^1(\mathcal{M}) \) norm of \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m} \), starting with \( h_{1,p,q,l,m} \). We have:
\[
\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} L(P_{t \chi}) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} (8.157)
\]
\[
\times \left\| \int_{S^2} H_1 \left( 2^{\frac{j}{2}}(N' - N_\nu') \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.
\]

Next, we evaluate both terms in the right-hand side of (8.157) starting with the first one. Assume first that \( l > j/2 \). Then, the basic estimate in \( L^2(\mathcal{M}) \) (7.1) yields:
\[
\left\| \int_{S^2} L(P_{t \chi}) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_\omega \| L(P_{t \chi}) \|_{L^\infty L^2(H_\omega)} \right) \int_{S^2} \left( 2^{\frac{j}{2}} \right)^2 \gamma_j^\nu, (8.158)
\]
where we used in the last inequality the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. In view of (8.158) and the estimate (8.142) for \( L(P_{t \chi}) \), we obtain, in the case \( l > j/2 \):
\[
\left\| \int_{S^2} L(P_{t \chi}) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} - \frac{j}{2}} \gamma_j^\nu. (8.159)
\]

Next, we evaluate the first term in the right-hand side of (8.157) in the case \( l = j/2 \), which is given by:
\[
\left\| \int_{S^2} L(P_{\leq j/2 t \chi}) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}.
\]

We first decompose \( L(P_{\leq j/2 t \chi}) \) as:
\[
L(P_{\leq j/2 t \chi}) = L(t \chi) - \sum_{l > \frac{j}{2}} L(P_{l t \chi}),
\]
which together with (8.159) yields:
\[
\left\| \int_{S^2} L(P_{\leq j/2 t \chi}) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left\| \int_{S^2} L(t \chi) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \sum_{l \geq \frac{j}{2}} \varepsilon 2^{\frac{j}{2} - \frac{j}{2}} \gamma_j^\nu
\]
\[
\lesssim \left\| \int_{S^2} L(t \chi) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} + \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu.
\]
Now, recall the decomposition (8.64) (8.65) (8.66) for $L(\text{tr}\chi)$. We have:

$$L(\text{tr}\chi) = \chi_{2\nu} \cdot (2\chi_1 + \tilde{\chi}) + f^j_1 + f^j_2,$$

where the scalar $f^j_1$ only depends on $\nu$ and satisfies:

$$\|f^j_1\|_{L^\infty_s L^2_t L^\infty_{(P_{i\nu})}} \lesssim \varepsilon,$$

where the scalar $f^j_2$ satisfies:

$$\|f^j_2\|_{L^\infty_s L^2_t (H_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

Together with (8.165), this yields:

$$\left\| \int_{S^2} L(P_{\leq \frac{1}{2}} \text{tr}\chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2(M)} \lesssim \|\chi_{2\nu}\|_{L^6(M)} \left\| \int_{S^2} (2\chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta^\nu_j(\omega) d\omega \right\|_{L^3(M)} \lesssim \varepsilon \left\| \int_{S^2} (2\chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta^\nu_j(\omega) d\omega \right\|_{L^3(M)},$$

where we used in the last inequality the Sobolev embedding (2.50) and the estimate (2.46) for $\chi_2$. Interpolating (8.69) and (8.72), we obtain:

$$\left\| \int_{S^2} (2\chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j}(u) \eta^\nu_j(\omega) d\omega \right\|_{L^3(M)} \lesssim 2^{\frac{5j}{2}} \varepsilon \gamma^\nu_j.$$

Together with (8.165), this yields:

$$\left\| \chi_{2\nu} \cdot \int_{S^2} (2\chi_1 + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{\frac{5j}{2}} \varepsilon \gamma^\nu_j.$$

Next, we evaluate the second term in the right-hand side of (8.164). We have:

$$\left\| f^j_1 \int_{S^2} \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2(M)} \lesssim \varepsilon \left\| \int_{S^2} \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta^\nu_j(\omega) d\omega \right\|_{L^2_{u,v\cdot x_t} L^\infty_t} \lesssim (1 + p^2) \varepsilon \gamma^\nu_j.$$
where we used the estimate (8.162) for \( f^1 \), and the estimate (7.63) to bound the \( L^2_{u,v,x'} L^\infty_t \) norm.

Next, we evaluate the third term in the right-hand side of (8.164). In view of the basic estimate in \( L^2(\mathcal{M}) \) (7.1), we have:

\[
\left\| \int_{S^2} f^3_2 \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j' (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \left\| f^3_2 \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right) \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} \right) 2^{\frac{j}{2}} \gamma_j',
\]

where we used in the last inequality the estimate (8.162) for \( f^3_2 \), the estimate (2.42) for \( \partial_\omega N \), and the size of the patch. Finally, (8.164), (8.166), (8.167) and (8.168) imply:

\[
\left\| \int_{S^2} L(\mathcal{P}_{\frac{1}{2}} \text{tr}_\chi) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j' (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) 2^{\frac{j}{2}} \gamma_j'.
\]  

(8.169)

Now, in view of (8.159) in the case \( l > j/2 \) and (8.169) in the case \( l = j/2 \), we finally obtain for any \( l \geq j/2 \):

\[
\left\| \int_{S^2} L(\mathcal{P}_l \text{tr}_\chi) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j' (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) 2^{\frac{j}{2}} \gamma_j'.
\]  

(8.170)

Next, we evaluate the second term in the right-hand side of (8.157). In view of the basic estimate in \( L^2(\mathcal{M}) \) (7.1), we have:

\[
\left\| \int_{S^2} H_1 \left( 2^{\frac{j}{2}} (N' - N'_{\nu}) \right)^q F_{j,-1}(u') \eta_j'' (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega'} \left\| H_1 \left( 2^{\frac{j}{2}} (N' - N'_{\nu}) \right)^q \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} \right) 2^{\frac{j}{2}} \varepsilon \gamma_j''
\]

where we used in the last inequality the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. In view of the definition (8.150) of \( H_1 \), we have:

\[
\left\| H_1 \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} \lesssim \left\| N'(P_m \text{tr}_\chi') \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} + \left\| \nabla'(P_m \text{tr}_\chi') \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} + \left\| \theta' P_m \text{tr}_\chi' \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} \\
\lesssim \left\| P_m (b'N'(\text{tr}_\chi')) \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} + \left\| [b'N', P_m]\text{tr}_\chi' \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} + \left\| \theta' \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} \left\| P_m \text{tr}_\chi' \right\|_{L^\infty_{u,v} L^2(\mathcal{M})} \\
\lesssim \varepsilon
\]

where we used the finite band property and the boundedness on \( L^p(\mathcal{P}_t u) \) for \( P_m \), the commutator estimate (2.64) for \([b'N', P_m]\), and the estimates (2.38) for \( b \), (2.37) (2.39) (2.40) for \( \theta \), and (2.39) for \( \text{tr}_\chi \). In view of (8.171), this yields:

\[
\left\| \int_{S^2} H_1 \left( 2^{\frac{j}{2}} (N' - N'_{\nu}) \right)^q F_{j,-1}(u') \eta_j'' (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j''
\]  

(8.173)
Finally, (8.157), (8.170) and (8.173) imply:
\[
\| h_{1,p,q,l,m} \|_{L^1(M)} \lesssim (1 + p^2)^{\frac{11j}{12}} \varepsilon^2 \gamma_j \gamma_j'.
\] (8.174)

Next, we evaluate the \( L^1(M) \) norm of \( h_{2,p,q,l,m} \). In view of (8.154), we have:
\[
\| h_{2,p,q,l,m} \|_{L^1(M)} \lesssim \int_{S^2} H_2 \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j'(\omega) d\omega \left\|_{L^2(M)} \right.
\]
\[
\times \left\| \int_{S^2} L'(P_m \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)}.
\] (8.175)

Next, we evaluate both terms in the right-hand side of (8.175) starting with the first one. In view of the definition (8.151) of \( H_2 \), and proceeding as in (8.172), we have:
\[
\| H_2 \|_{L^\infty L^2(H_u)} \lesssim \| N(P \text{tr} \chi) \|_{L^\infty L^2(H_u)} + \| \text{tr} \theta P \text{tr} \chi \|_{L^\infty L^2(H_u)} \lesssim \varepsilon.
\]
Thus, proceeding as in (8.173), we obtain:
\[
\left\| \int_{S^2} H_2 \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j'.
\] (8.176)

Also, the analog of (8.170) yields:
\[
\left\| \int_{S^2} L'(P_m \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)} \lesssim (1 + q^2) 2^{\frac{j}{12}} \varepsilon^2 \gamma_j \gamma_j'.
\]
Together with (8.175) and (8.167), we deduce:
\[
\| h_{2,p,q,l,m} \|_{L^1(M)} \lesssim (1 + q^2) 2^{\frac{j}{12}} \varepsilon^2 \gamma_j \gamma_j'.
\] (8.177)

Next, we evaluate the \( L^1(M) \) norm of \( h_{3,p,q,l,m} \). In view of (8.155), we have:
\[
\| h_{3,p,q,l,m} \|_{L^1(M)} \lesssim \left\| \int_{S^2} P \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)}
\]
\[
\times \left\| \int_{S^2} (b' \cdot \nabla (b') + \theta') L'(P_m \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)}
\]
\[
+ \left\| \int_{S^2} \text{tr} \theta L(P \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j(\omega) d\omega \right\|_{L^2(M)}
\]
\[
\times \left\| \int_{S^2} P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)}.
\] (8.178)

Next, we estimate the various terms in the right-hand side of (8.178) starting with the first one. Assume first that \( l > j/2 \). Then, the basic estimate in \( L^2(M) \) (7.1) yields:
\[
\left\| \int_{S^2} P \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)} \lesssim \left( \sup_{\omega} \| P \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \|_{L^\infty L^2(H_u)} \right) 2^{\frac{j}{2}} \gamma_j'.
\]
\[
\lesssim \left( \sup_{\omega} \| P \text{tr} \chi \|_{L^\infty L^2(H_u)} \right) 2^{\frac{j}{2}} \gamma_j'.
\]

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where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the finite band property for $P_l$, this yields:

$$\left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left( 2^{\frac{l}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{-l} \left( \sup_{\omega} \| \nabla \text{tr} \chi \|_{L^\infty(H_u)} \right) 2^{\frac{l}{2}} \gamma_j^\nu$$

where we used the estimate (2.39) for $\text{tr} \chi$ in the last inequality.

Next, we evaluate the first term in the right-hand side of (8.178) in the case $l = j/2$, which is given by:

$$\left\| \int_{\mathbb{S}^2} P_{\leq j/2} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} .$$

We first decompose $P_{\leq j/2} \text{tr} \chi$ as:

$$P_{\leq j/2} \text{tr} \chi = \text{tr} \chi - \sum_{l > j/2} P_l \text{tr} \chi,$$

which together with (8.179) yields:

$$\left\| \int_{\mathbb{S}^2} P_{\leq j/2} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \int_{\mathbb{S}^2} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \left( \sup_{\omega} \| \nabla \text{tr} \chi \|_{L^\infty(H_u)} \right) 2^{\frac{j}{2}} \gamma_j^\nu + \sum_{l > j/2} \varepsilon 2^{\frac{j}{2} - l} \gamma_j^\nu$$

Together with the estimate (7.64), this yields:

$$\left\| \int_{\mathbb{S}^2} P_{\leq j/2} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu . \quad (8.180)$$

Now, in view of (8.179) in the case $l > j/2$ and (8.180) in the case $l = j/2$, we finally obtain for any $l \geq l/2$:

$$\left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu . \quad (8.181)$$

Arguing similarly for the third term in the right-hand side of (8.178), we obtain:

$$\left\| \int_{\mathbb{S}^2} P_m \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu') \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim (1 + q^2) \varepsilon \gamma_j^\nu , \quad (8.182)$$

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which together with (8.178) and (8.181) yields:

\[
\|h_{3,p,q,l,m}\|_{L^1(M)} \quad \text{(8.182)}
\]

\[
\lesssim (1 + p^2)\varepsilon |\gamma_j| \left\| \int_{S^2} (\theta' + b^{-1}\nabla(b')) L'(P_m\text{tr}\chi) \left( 2^j(N' - N_{\nu'}) \right)^\rho F_{j-1}(u')\eta_j' (\omega')d\omega' \right\|_{L^2(M)}
\]

\[
+(1 + q^2)\varepsilon |\gamma_j| \left\| \int_{S^2} \text{tr}\theta L(P\text{tr}\chi) \left( 2^j(N - N_{\nu}) \right)^\rho F_{j-1}(u)\eta_j' (\omega)d\omega \right\|_{L^2(M)} .
\]

We estimate the two terms in the right-hand side of (8.182) starting with the first one. Using the basic estimate in \( L^2(M) \) (7.1), we have:

\[
\left\| \int_{S^2} (b^{-1}\nabla(b') + \theta')L'(P_m\text{tr}\chi) \left( 2^j(N' - N_{\nu'}) \right)^\rho F_{j-1}(u')\eta_j' (\omega')d\omega' \right\|_{L^2(M)} \quad \text{(8.183)}
\]

\[
\lesssim \left( \sup_{\omega} \left\| (\nabla'(L'(P_m\text{tr}\chi))) + (b^{-1}\nabla(b') + \theta')L'(P_m\text{tr}\chi)) \left( 2^j(N' - N_{\nu'}) \right)^\rho \right\|_{L^\infty L^2(H_\omega)} \right)^{2^j|\gamma_j|'}
\]

where we used in the last inequality the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Now, using the estimate (2.36) for \( n \), we have for any tensor \( G \) and any integer \( r \):

\[
\|GLP_r(\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} \lesssim \|GnLP_r(\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} \quad \text{(8.184)}
\]

\[
\lesssim \|GP_r(nL\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} + \|G[nL, P_r](\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} \quad \text{(8.185)}
\]

\[
\lesssim \|G\|_{L^\infty L^2(H_\omega)}\|P_r(nL\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} + \|G\|_{L^\infty L^2(H_\omega)}\|[nL, P_r](\text{tr}\chi)\|_{L^1 L^2(H_\omega)} .
\]

Together with the embeddings (2.50) and (2.51), the \( L^p \) boundedness of \( P_r \), the Gagliardo-Nirenberg inequality (2.49) and the estimate (2.36) for \( n \), we obtain:

\[
\|GLP_r(\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} \lesssim N_1(G)(\|L\text{tr}\chi\|_{L^\infty L^3(H_\omega)} + \|[nL, P_r](\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} \left\| \nabla[nL, P_r](\text{tr}\chi) \right\|_{L^\infty L^2(H_\omega)}) .
\]

Together with the commutator estimate (2.67), we deduce:

\[
\|GLP_r(\text{tr}\chi)\|_{L^\infty L^2(H_\omega)} \lesssim (\|L\text{tr}\chi\|_{L^\infty L^3(H_\omega)} + \varepsilon)N_1(G) \lesssim \varepsilon N_1(G),
\]

where we used the fact that:

\[
\|L\text{tr}\chi\|_{L^\infty L^3(H_\omega)} \lesssim \varepsilon , \quad \text{(8.185)}
\]

in view of the Raychaudhuri equation (2.22), and the estimates (2.36) (2.37) for \( \tilde{\delta} \) and (2.39) (2.40) for \( \chi \). Choosing \( G = \theta' + b^{-1}\nabla(b') \), we obtain:

\[
\left\| (\theta' + b^{-1}\nabla(b'))L'(P_m\text{tr}\chi) \right\|_{L^\infty L^2(H_\omega)} \lesssim \varepsilon N_1(\theta') \lesssim \varepsilon ,
\]

(8.186)
where we used in the last inequality the estimates (2.36) (2.37) (2.39) (2.40) for \( \theta' \). Together with (8.183), we obtain:
\[
\left\| \int_{\mathbb{S}^2} (\theta' + b^{-1} \nabla(b')) L'(P_m \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j,-1}(u) \eta_j' (\omega') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j'.
\] (8.187)

Arguing similarly, we obtain:
\[
\left\| \int_{\mathbb{S}^2} \text{tr} \theta L(P_l \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j (\omega) d\omega \right\|_{L^2(M)} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j'.
\] (8.188)

Finally, (8.178), (8.187), and (8.188) imply:
\[
\|h_{3,p,q,l,m}\|_{L^1(M)} \lesssim (1 + p^2 + q^2) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j' \gamma_j'.
\] (8.189)

Next, we evaluate the \( L^1(M) \) norm of \( h_{4,p,q,l,m} \). In view of (8.155), we have:
\[
\|h_{4,p,q,l,m}\|_{L^1(M)} \lesssim \left\| \int_{\mathbb{S}^2} P_l \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j (\omega) d\omega \right\|_{L^2(M)}
\times \left\| \int_{\mathbb{S}^2} \nabla' L'(P_m \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j,-1}(u) \eta_j' (\omega') d\omega' \right\|_{L^2(M)}.
\] (8.190)

Let us first estimate the last term in the right-hand side of (8.190). Using the basic estimate in \( L^2(M) \) (7.1), we have:
\[
\left\| \int_{\mathbb{S}^2} \nabla' L'(P_m \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q F_{j,-1}(u) \eta_j' (\omega') d\omega' \right\|_{L^2(M)} \lesssim \left( \sup_{\omega} \left\| (\nabla' (L'(P_m \text{tr} \chi')) + \theta' L'(P_m \text{tr} \chi')) \left( 2^{\frac{j}{2}} (N' - N_\nu) \right)^q \right\|_{L^\infty L^2(H_u)} \right) 2^{\frac{j}{2}} \gamma_j'.
\] (8.191)

where we used in the last inequality the estimate (2.42) for \( \partial \omega N \) and the size of the patch. Using the estimate (2.36) for \( n \), we estimate the right-hand side of (8.191):
\[
\left\| \nabla' (L'(P_m \text{tr} \chi')) \right\|_{L^\infty L^2(H_u)} \lesssim \left\| \nabla' P_m (n L' \text{tr} \chi') \right\|_{L^\infty L^2(H_u)} + \left\| \nabla' [n L', P_m](\text{tr} \chi') \right\|_{L^\infty L^2(H_u)} + \left\| n^{-1} \nabla' n L' P_m(\text{tr} \chi') \right\|_{L^\infty L^2(H_u)}.
\]

Applying (8.184) with the choice \( G = n^{-1} \nabla' n \), we obtain:
\[
\left\| \nabla' (L'(P_m \text{tr} \chi')) \right\|_{L^\infty L^2(H_u)} \lesssim \left\| \nabla' P_m (n L' \text{tr} \chi') \right\|_{L^\infty L^2(H_u)} + \left\| \nabla' [n L', P_m](\text{tr} \chi') \right\|_{L^\infty L^2(H_u)} + \varepsilon N_1 (n^{-1} \nabla' n)
\lesssim \left\| \nabla' P_m (n L' \text{tr} \chi') \right\|_{L^\infty L^2(H_u)} + 2^\alpha \varepsilon,
\] (8.192)

where we used in the last inequality the estimate (2.36) for \( n \) and the commutator estimate (2.67). Next, using the finite band property for \( P_m \), we have:
\[
\left\| \nabla' (P_m (n L' \text{tr} \chi')) \right\|_{L^\infty L^2(H_u)} \lesssim 2^n \left\| P_m (n L' \text{tr} \chi') \right\|_{L^\infty L^2(H_u)} \lesssim 2^{\frac{n}{2}} \varepsilon,
\]
where we used (8.141) in the last inequality. Together with (8.192), we obtain:
\[ \| \nabla' (L'(P_m \text{tr} \chi')) \|_{L^\infty_t L^2(\mathcal{H}_\omega)} \lesssim 2^{\frac{m}{2}} \varepsilon. \] (8.193)
which in view of (8.191) yields:
\[ \left\| \int_{S^2} \nabla' L'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N^* - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{m}{2} + \frac{j}{2}} \gamma_j'. \]
Together with (8.190), we obtain:
\[ \| h_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{\frac{m}{2} + \frac{j}{2}} \gamma_j'. \] (8.194)
Assume first that \( l > j/2 \). Then, (8.179) and (8.194) yield:
\[ \| h_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{2j + \frac{m}{2} + j} \gamma_j' \gamma_{j}. \]
which together with the fact that \( l > m \) from (8.1) and the assumption \( l > j/2 \) yields:
\[ \| h_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{\frac{3j}{2}} \gamma_j' \gamma_{j}. \] (8.195)
Next, assume that \( l = j/2 \). Then, (8.180) and (8.194) yield:
\[ \| h_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim (1 + p^2) \varepsilon 2^{j + \frac{m}{2} + j} \gamma_j' \gamma_{j}. \]
which together with the fact that \( j = m \) from (8.1) and the assumption \( l = j/2 \) yields:
\[ \| h_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim (1 + p^2) \varepsilon 2^{\frac{3j}{2}} \gamma_j' \gamma_{j}. \] (8.196)
In view of (8.195) and (8.196), we finally obtain in all cases:
\[ \| h_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim (1 + p^2) \varepsilon 2^{\frac{3j}{2}} \gamma_j' \gamma_{j}. \] (8.197)
We are now ready to estimate \( B_{j,p,q,l,m}^{1,2,1,2} \). In view of (8.152), we have:
\[ |B_{j,p,q,l,m}^{1,2,1,2}| \lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^{\frac{j}{2}} |N_{\nu'} - N_{\nu''}|)^{p+q+2}} \right\|_{L^\infty(\mathcal{M})} \times \left[ \| h_{1,p,q,l,m} \|_{L^1(\mathcal{M})} + \| h_{2,p,q,l,m} \|_{L^1(\mathcal{M})} + \| h_{3,p,q,l,m} \|_{L^1(\mathcal{M})} + \| h_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \right], \]
which together with (8.32), (8.174), (8.177), (8.189) and (8.197) yields:
\[ |B_{j,p,q,l,m}^{1,2,1,2}| \lesssim \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{p+q+2}} (1 + p^2 + q^2) 2^{-\frac{j}{2}} \varepsilon 2^{2j} \gamma_j' \gamma_{j}. \] (8.198)
Note that summing the estimate (8.198) in \( m \) is not a problem. Indeed, we have from (8.1):
\[
2^m \leq 2^j|\nu - \nu'|.
\]

Now, we have:
\[
\#\{m \mid 2^m \leq 2^j|\nu - \nu'|\} \lesssim j \tag{8.199}
\]
so that the sum in \( m \) generates a \( O(j) \) term which is absorbed by the extra gain \( 2^{-\frac{j}{12}} \) in (8.198). On the other hand, there is no a priori bound on \( l \) so that summing the estimate (8.198) in \( l \) is problematic. To fix this issue, it suffices, since \( l \geq m \) in view of (8.1), to obtain an upper bound for
\[
\left| \sum_{l/l \geq m} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right|.
\]
To this end, it suffices to replace \( P_l \) with \( P_{l > 2^j|\nu - \nu'|} \) in the definition and the estimate of \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m} \). The estimates are completely analogous, and we obtain as in (8.198):
\[
\left| \sum_{l/l \geq m} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right| \lesssim \frac{2^{-\frac{j}{12}} \varepsilon^{\gamma_2 \gamma_j \gamma_j'}}{(2^\frac{j}{2}|\nu - \nu'|)^2}. \tag{8.200}
\]

Now, we have:
\[
\left| \sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j|\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right| \lesssim \sum_{2^m \leq 2^j|\nu - \nu'|} \sum_{l/l \geq m} B_{j,\nu,\nu',l,m}^{1,2,1,2}
\]
which together with (8.199) and (8.200) implies:
\[
\left| \sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j|\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,1,2} \right| \lesssim \frac{j2^{-\frac{j}{12}} \varepsilon^{\gamma_2 \gamma_j \gamma_j'}}{(2^\frac{j}{2}|\nu - \nu'|)^2}. \tag{8.201}
\]

Next, we estimate \( B_{j,\nu,\nu',l,m}^{1,2,1,3} \) defined in (8.109). Recall from (8.1) that \((l, m)\) satisfy:
\[
m < l \text{ and } 2^m \leq 2^j|\nu - \nu'|.
\]

Summing in \((l, m)\), we obtain:
\[
\sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j|\nu - \nu'|} L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi')
\]
\[
+ \sum_{(l,m)/m<l \text{ and } 2^m \leq 2^j|\nu - \nu'|} L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi L'(P_m \text{tr} \chi')
\]
\[
= L(\text{tr} \chi) \text{tr} \chi' + \text{tr} \chi L'(\text{tr} \chi')
\]
\[
- L(P_{> 2^j|\nu - \nu'|} \text{tr} \chi) P_{> 2^j|\nu - \nu'|} \text{tr} \chi' + P_{> 2^j|\nu - \nu'|} \text{tr} \chi L'(P_{> 2^j|\nu - \nu'|} \text{tr} \chi').
\]
Thus, using the symmetry in \((\omega, \omega')\) of the integrant in \(B_{j,\nu,\nu',l,m}^{1,2,1,3}\) defined in \((8.109)\), we obtain:

\[
\sum_{(l,m)/m<l \text{ and } 2^m \leq 2^{(\nu+\nu')}} B_{j,\nu,\nu',l,m}^{1,2,1,3} + \sum_{(l,m)/m<l \text{ and } 2^m \leq 2^{(\nu+\nu')}} B_{j,\nu',\nu,l,m}^{1,2,1,3} \quad (8.202)
\]

\[
= 2^{-2^j-1} \int \int_{S^2 \times S^2} \frac{b^{-1} \nabla_{N-N-g(N,N')N}(b)}{g(L, L')^2} \left( (L P_{2^j|\nu-\nu'|} \text{tr} \chi) P_{2^j|\nu-\nu'|} \text{tr} \chi' \right) \\
\times F_{j,-1}(u) F_{j,-1}(u') \eta_j' (\omega) \eta_j'' (\omega') d\omega d\omega' dM
\]

\[
- 2^{-2^j-1} \int \int_{S^2 \times S^2} \frac{b^{-1} \nabla_{N-N-g(N,N')N}(b)}{g(L, L')^2} \left( (L P_{2^j|\nu-\nu'|} \text{tr} \chi) P_{2^j|\nu-\nu'|} \text{tr} \chi' \right) \\
\times (L P_{2^j|\nu-\nu'|} \text{tr} \chi) P_{2^j|\nu-\nu'|} \text{tr} \chi' \right) F_{j,-1}(u) F_{j,-1}(u') \eta_j' (\omega) \eta_j'' (\omega') d\omega d\omega' dM + \text{ terms interverting (}\nu, \nu').
\]

We estimate the two terms in the right-hand side of \((8.202)\) starting with the last one. We have:

\[
\left| 2^{-2^j-1} \int \int_{S^2 \times S^2} \frac{b^{-1} \nabla_{N-N-g(N,N')N}(b)}{g(L, L')^2} \left( (L P_{2^j|\nu-\nu'|} \text{tr} \chi) P_{2^j|\nu-\nu'|} \text{tr} \chi' \right) \\
\times F_{j,-1}(u) F_{j,-1}(u') \eta_j' (\omega) \eta_j'' (\omega') d\omega d\omega' dM \right|
\]

\[
\lesssim 2^{-2^j} \int \int_{S^2 \times S^2} \left| \begin{vmatrix} N \times g(N,N')N \end{vmatrix} \frac{g(L, L')^2}{1_{L^2(M)}} \left( \left\| b^{-1} \nabla b \right\|_{L^2(M)} \left( \left\| b^{-1} \nabla b \right\|_{L^2(M)} \right) \eta_j' (\omega) \eta_j'' (\omega') d\omega d\omega' \right|.
\]

In view of the identities \((8.30)\) \((8.31)\) for \(g(L, L')\) and \(g(N, N')\), and in view of the estimate \((8.32)\), we obtain:

\[
\left| 2^{-2^j-1} \int \int_{S^2 \times S^2} \frac{b^{-1} \nabla_{N-N-g(N,N')N}(b)}{g(L, L')^2} \left( (L P_{2^j|\nu-\nu'|} \text{tr} \chi) P_{2^j|\nu-\nu'|} \text{tr} \chi' \right) \\
\times F_{j,-1}(u) F_{j,-1}(u') \eta_j' (\omega) \eta_j'' (\omega') d\omega d\omega' dM \right|
\]

\[
\lesssim \frac{1}{2^j \left(2^{2^j} \left| \nu - \nu' \right| \right)^3} \int \int_{S^2 \times S^2} \left( \left\| b^{-1} \nabla (b \left( P_{2^j|\nu-\nu'|} \text{tr} \chi) \right) \right\|_{L^\infty L^2(H_u)} \left\| F_{j,-1}(u) \right\|_{L^2} \\
\times \left\| F_{j,-1}(u') \right\|_{L^2} \left\| b^{-1} \nabla (b \left( P_{2^j|\nu-\nu'|} \text{tr} \chi \right) \right\|_{L^\infty L^2(H_u)} \left\| b^{-1} \nabla (b \left( P_{2^j|\nu-\nu'|} \text{tr} \chi \right) \right\|_{L^\infty L^2(H_u)} \left\| F_{j,-1}(u) \right\|_{L^2} \left\| F_{j,-1}(u') \right\|_{L^2} \right| \eta_j' (\omega) \eta_j'' (\omega') d\omega d\omega'.
\]

Next, we evaluate the various terms in the right-hand side of \((8.203)\). Choosing \(G = b^{-1} \nabla (b)\) in \((8.184)\), we have:

\[
\left\| b^{-1} \nabla (b \left( P_{2^j|\nu-\nu'|} \text{tr} \chi \right) \right\|_{L^\infty L^2(H_u)} \lesssim \varepsilon N_1 \left( b^{-1} \nabla (b) \right) \lesssim \varepsilon,
\]
where we used the estimate (2.38) for $\mathbf{b}$ in the last inequality. Also, (8.142) together with the estimate (2.36) for $n$ yields:

$$
\| L'(P_{2j|\nu-\nu'|}\text{tr}\chi') \|_{L^\infty_t L^2(\mathcal{H}_\nu')} \lesssim \sum_{2^m > 2^j|\nu-\nu'|} 2^{-m} \varepsilon \lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{3}{2}}}. \quad (8.205)
$$

Using the finite band property for $P_m$, we have:

$$
\| P_{2j|\nu-\nu'|}\text{tr}\chi' \|_{L^\infty_t L^2(\mathcal{H}_\nu')} \lesssim \sum_{2^m > 2^j|\nu-\nu'|} \| P_m \text{tr}\chi' \|_{L^\infty_t L^2(\mathcal{H}_\nu')} \quad (8.206)
$$

\lesssim \sum_{2^m > 2^j|\nu-\nu'|} 2^{-m} \| \nabla' \text{tr}\chi' \|_{L^\infty_t L^2(\mathcal{H}_\nu')}

\lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{1}{2}}},
$$

where we used the estimate (2.39) for $\text{tr}\chi$ in the last inequality. Also, we have:

$$
\| b^{-1} \nabla(b) P_{2j|\nu-\nu'|}\text{tr}\chi \|_{L^\infty_t L^2(\mathcal{H}_\nu')} \lesssim \| b^{-1} \nabla(b) \|_{L^\infty_t L^2_{\nu'}} \| P_{2j|\nu-\nu'|}\text{tr}\chi \|_{L^2_t L^2_{\nu'}}

\lesssim \mathcal{N}_1(b^{-1} \nabla(b)) \left( \sum_{2^m > 2^j|\nu-\nu'|} \| P_m \text{tr}\chi \|_{L^2_t L^2_{\nu'}} \right)

\lesssim \varepsilon \left( \sum_{2^m > 2^j|\nu-\nu'|} 2^{-m} \| \nabla \text{tr}\chi \|_{L^\infty_t L^2(\mathcal{H}_\nu')} \right),
$$

where we used the embedding (2.51), the estimate (2.38) for $\mathbf{b}$ and the Bernstein inequality for $P_m$. Together with the finite band property for $P_m$, this yields:

$$
\| b^{-1} \nabla(b) P_{2j|\nu-\nu'|}\text{tr}\chi \|_{L^\infty_t L^2(\mathcal{H}_\nu')} \lesssim \varepsilon \left( \sum_{2^m > 2^j|\nu-\nu'|} 2^{-m} \| \nabla \text{tr}\chi \|_{L^\infty_t L^2(\mathcal{H}_\nu')} \right) \quad (8.207)
$$

\lesssim \frac{\varepsilon}{(2^j|\nu-\nu'|)^{\frac{3}{2}}},
$$

where we used the estimate (2.39) for $\text{tr}\chi$ in the last inequality. Finally, (8.203) (8.204) (8.205) (8.206) (8.207) yield:

$$
\left| 2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1} \nabla_{\nu'-\nu} \mathbf{g}(X,X') \mathbf{g}(b)}{\mathbf{g}(L,L')}^2 (L(P_{2j|\nu-\nu'|}\text{tr}\chi') P_{2j|\nu-\nu'|}\text{tr}\chi') d\omega d\omega' d\mathcal{M} + P_{2|\nu-\nu'|}\text{tr}\chi L'(P_{2|\nu-\nu'|}\text{tr}\chi') F_{j-1}(u) F_{j-1}(u') \eta_j'^{\nu'}(\omega) \eta_j'^{\nu'}(\omega') d\omega d\omega' d\mathcal{M} \right|

\lesssim \varepsilon^2 \left( \int_{S^2} \| F_{j-1}(u) \|_{L^2_{\nu'}} \eta_j'^{\nu'}(\omega) d\omega \right) \left( \int_{S^2} \| F_{j-1}(u') \|_{L^2_{\nu'}} \eta_j'^{\nu'}(\omega') d\omega' \right)

\lesssim \varepsilon^2 \frac{\eta_j'^{\nu'} \eta_j'^{\nu'}}{(2^j|\nu-\nu'|)^{4}}.
$$
where we used in the last inequality Plancherel in $\lambda$ for $\|F_{j,-1}(u)\|_{L^2_\lambda}$, Plancherel in $\lambda'$ for $\|F_{j,-1}(u')\|_{L^2_{\lambda'}}$, Cauchy Schwarz in $\omega$ and $\omega'$, and the size of the patches.

Next, we estimate the first term in the right-hand side of (8.202), which is given by:

$$2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1} \nabla N' - g(N,N') N(b)}{g(L, L')^2} (L(\text{tr}\chi)\text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi')) (8.209)$$

$$\times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}.$$  

Recall the identities (8.30) and (8.31):

$$g(L, L') = -1 + g(N, N')$$ and 

$$1 - g(N, N') = \frac{g(N - N', N - N')}{2}.$$

We may thus expand

$$\frac{1}{g(L, L')^2}$$

in the same fashion than (8.33), and in view of (8.209), we obtain, schematically:

$$2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1} \nabla N' - g(N,N') N(b)}{g(L, L')^2} (L(\text{tr}\chi)\text{tr}\chi' + \text{tr}\chi L'(\text{tr}\chi')) (8.210)$$

$$\times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}$$

$$= 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^\frac{j}{2}|N_\nu - N_\nu'|)^{p+q+3}} [h_{1,p,q} + h_{2,p,q}] d\mathcal{M},$$

where the scalar functions $h_{1,p,q}, h_{2,p,q}$ on $\mathcal{M}$ are given by:

$$h_{1,p,q} = \left( \int_{S^2} b^{-1} \nabla(b) L(\text{tr}\chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} \text{tr}\chi' \left( 2^\frac{j}{2} (N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),$$

and:

$$h_{2,p,q} = \left( \int_{S^2} b^{-1} \nabla(b) \text{tr}\chi \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} L'(\text{tr}\chi') \left( 2^\frac{j}{2} (N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),$$

and where $c_{pq}$ are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1.

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q}, h_{2,p,q}$ starting with $h_{1,p,q}$. We have:

$$\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) \varepsilon \gamma_j^\nu \left[ \int_{S^2} b^{-1} \nabla(b) L(\text{tr}\chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right]_{L^2(\mathcal{M})}.$$
where we used the estimate (7.64) in the last inequality. Together with the basic estimate in $L^2(\mathcal{M})$ (7.1), we obtain:

$$
\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2)\varepsilon \gamma_j^\nu \left( \sup_{\omega} \|b^{-1}\nabla(b) L(\text{tr}\chi)\|_{L^\infty(\mathcal{H}_u)} \right)^{2^j \gamma_j^\nu}. \tag{8.213}
$$

Next, we estimate $b^{-1}\nabla(b)L(\text{tr}\chi)$. We have:

$$
\|b^{-1}\nabla(b)L(\text{tr}\chi)\|_{L^\infty(\mathcal{H}_u)} \lesssim \|b^{-1}\nabla(b)\|_{L^\infty(\mathcal{H}_u)} \|L(\text{tr}\chi)\|_{L^\infty(\mathcal{H}_u)} \lesssim \mathcal{N}_1(b^{-1}\nabla(b)) \|L(\text{tr}\chi)\|_{L^\infty(\mathcal{H}_u)} \lesssim \varepsilon,
$$

where we used the Sobolev embedding (2.50), the estimate (2.38) for $b$ and the estimate (8.185) for $L(\text{tr}\chi)$. Together with (8.213), we deduce:

$$
\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2)\varepsilon 2^j \gamma_j^\nu \gamma_j^\nu. \tag{8.214}
$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q}$. Recall the decomposition (8.67):

$$
\int_{S^2} L'((\text{tr}\chi')) \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u'')\eta_j^\nu' (\omega')d\omega' 
= -\chi_2 \cdot \left( \int_{S^2} (2\chi_{1'} + \tilde{\chi}') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u'')\eta_j^\nu' (\omega')d\omega' \right) 
- (\chi_{2\nu'} - \chi_2) \cdot \left( \int_{S^2} (2\chi_{1'} + \tilde{\chi}') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u'')\eta_j^\nu' (\omega')d\omega' \right) 
+ f_{1j}^1 \left( \int_{S^2} \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u'')\eta_j^\nu' (\omega')d\omega' \right) 
+ \int_{S^2} f_{2j}^j \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u'')\eta_j^\nu' (\omega')d\omega',
$$

where the scalar $f_{1j}^1$ only depends on $\nu'$ and satisfies:

$$
\|f_{1j}^1\|_{L^\infty(\mathcal{M}) L^2 L^\infty(H_{\omega'})} \lesssim \varepsilon, \tag{8.215}
$$

where the scalar $f_{2j}^j$ satisfies:

$$
\|f_{2j}^j\|_{L^\infty(\mathcal{M}) L^2(H_{\omega'})} \lesssim \varepsilon 2^{-\frac{j}{2}}. \tag{8.216}
$$
Together with the definition (8.212) of $h_{2,p,q}$, this yields:

$$
\|h_{2,p,q}\|_{L^1(M)} \lesssim \left( \left\| \int_{S^2} \chi_2 b^{-1} \nabla(b) \text{tr}(2^{\frac{1}{4}}(N - N_\nu)) F_{j,-1}(u) \eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} + \left\| \int_{S^2} (\chi_{2'} - \chi_2) b^{-1} \nabla(b) \text{tr}(2^{\frac{1}{4}}(N - N_\nu)) F_{j,-1}(u) \eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} \right) \varepsilon \gamma_j^\nu
$$

$$
\times \left( \left\| f_1^j \int_{S^2} \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j,-1}(u) \eta_j^\nu(\omega')d\omega' \right\|_{L^2(M)} + \left\| \int_{S^2} f_2^j \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j,-1}(u) \eta_j^\nu(\omega')d\omega' \right\|_{L^2(M)} \right).
$$

Together with the estimates (7.63) and (8.72), and the estimate (8.215) for $f_1^j$, we obtain:

$$
\|h_{2,p,q}\|_{L^1(M)} \lesssim \left( \left\| \int_{S^2} \chi_2 b^{-1} \nabla(b) \text{tr}(2^{\frac{1}{4}}(N - N_\nu)) F_{j,-1}(u) \eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} + \left\| \int_{S^2} (\chi_{2'} - \chi_2) b^{-1} \nabla(b) \text{tr}(2^{\frac{1}{4}}(N - N_\nu)) F_{j,-1}(u) \eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} \varepsilon \gamma_j^\nu
$$

$$
\times \left( (1 + q^2) \varepsilon \gamma_j^\nu + \left\| f_1^j \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j,-1}(u) \eta_j^\nu(\omega')d\omega' \right\|_{L^2(M)} \right).
$$

Next, we apply the basic estimate in $L^2(M)$ to the first, the third and the last term in the right-hand side of (8.217). We obtain:

$$
\|h_{2,p,q}\|_{L^1(M)} \lesssim \left( \sup_\omega \|\chi_2 b^{-1} \nabla(b) \text{tr}\|_{L^\infty L^2(H_\omega)} \right) 2^{\frac{1}{4}} \gamma_j^\nu
$$

$$
\times \left( \sup_\omega \|b^{-1} \nabla(b) \text{tr}\|_{L^\infty L^2(H_\omega)} \right) 2^{\frac{1}{4}} \varepsilon \gamma_j^\nu
$$

$$
\times \left( (1 + q^2) \varepsilon \gamma_j^\nu + \left( \sup_\omega \|f_1^j\|_{L^\infty L^2(H_\omega)} \right) 2^{\frac{1}{4}} \gamma_j^\nu \right).
$$
Now, the Sobolev embedding (2.50):

\[
\|\chi_2 b^{-1} \nabla(b) \text{tr} \chi\|_{L^\infty L^2(H_\omega)} + \|b^{-1} \nabla(b) \text{tr} \chi\|_{L^\infty L^2(H_\omega)} \\
\lesssim \left( \|\chi_2\|_{L^\infty L^1(H_\omega)} \|b^{-1} \nabla(b)\|_{L^\infty L^1(H_\omega)} + \|b^{-1} \nabla(b)\|_{L^\infty L^2(H_\omega)} \|\text{tr} \chi\|_{L^\infty(M)} \right) \\
\lesssim (N_1(\chi_2) + 1) N_1(\|b^{-1} \nabla(b)\|_{L^\infty L^2(M)} \|\text{tr} \chi\|_{L^\infty(M)}) \\
\lesssim \varepsilon,
\]

where we used in the last inequality the estimate (2.38) for \(b\), the estimate (2.39) for \(\text{tr} \chi\) and the estimate (2.46) for \(\chi_2\). Together with (8.218) and the estimate (8.216) for \(f_j^2\), this yields:

\[
\|h_{2,p,q}\|_{L^1(M)} \approx (8.219) \\
\lesssim \left( 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu + \left\| \int_{S^2} (\chi_{2\nu} - \chi_2) b^{-1} \nabla(b) \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \right) \varepsilon \gamma_j^\nu \\
+ 2^{\frac{j}{2}} (1 + q^2) \varepsilon \gamma_j^\nu \gamma_j^\nu \cdot
\]

Next, we estimate the right-hand side of (8.219). We have:

\[
\left\| \int_{S^2} (\chi_{2\nu} - \chi_2) b^{-1} \nabla(b) \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
\lesssim \int_{S^2} \left\| (\chi_{2\nu} - \chi_2) b^{-1} \nabla(b) \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
\lesssim \int_{S^2} \left\| \chi_{2\nu} - \chi_2 \right\|_{L^3(M)} \|b^{-1} \nabla(b)\|_{L^\infty L^6(H_\omega)} \|\text{tr} \chi\|_{L^\infty(M)} \left\| \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^P \right\|_{L^\infty(M)} \\
\|F_{j,1}(u)\|_{L^6} \eta_j^\nu(\omega) d\omega,
\]

which together with the Sobolev embedding (2.50) yields:

\[
\left\| \int_{S^2} (\chi_{2\nu} - \chi_2) b^{-1} \nabla(b) \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
\lesssim \int_{S^2} \left\| \omega - \nu' \right\|_{L^\infty(M)} \|\partial_\omega \chi_2\|_{L^3(M)} \|N_1(b^{-1} \nabla(b))\|_{L^\infty M} \left\| \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^P \right\|_{L^\infty(M)} \\
\|F_{j,1}(u)\|_{L^6} \|F_{j,1}(u)\|_{L^6} \eta_j^\nu(\omega) d\omega \\
\lesssim \left\| \nu - \nu' \right\| \varepsilon \int_{S^2} \left\| F_{j,1}(u) \right\|_{L^6} \|F_{j,1}(u)\|_{L^6} \eta_j^\nu(\omega) d\omega,
\]

where we used in the last inequality the estimate (2.47) for \(\chi_2\), the estimate (2.38) for \(b\), the estimate (2.39) for \(\text{tr} \chi\), the estimate (2.42) for \(\partial_\omega N\), and the size of the patch.

Using Cauchy Schwartz in \(\lambda\) for \(\|F_{j,1}(u)\|_{L^\infty}^\lambda\), Plancherel in \(u\) for \(\|F_{j,1}(u)\|_{L^2}^\lambda\), Cauchy Schwarz in \(\omega\) and the volume of the patch, we finally obtain:

\[
\left\| \int_{S^2} (\chi_{2\nu} - \chi_2) b^{-1} \nabla(b) \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^P F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \left\| \nu - \nu' \right\| \varepsilon 2^{\frac{j}{2}} \gamma_j^\nu.
\]
In view of (8.219), we deduce:

\[ \|h_{2,p,q}\|_{L^1(M)} \lesssim (1 + q^2)(1 + |\nu - \nu'|2^\gamma)2^\gamma e^{2^\gamma \gamma_j \gamma_j} \tag{8.220} \]

In view of (8.210), we have:

\[
2^{-2j-1} \int_M \int_{S^2 \times S^2} \frac{b^{-1} \nabla N' \cdot \mathbf{g}(N,N')N(b)}{g(L,L')} (L(\text{tr} \chi') + \text{tr} \chi' L'(\text{tr} \chi'))
\times F_j^{-1}(u) F_j^{-1}(u') \eta_j' (\omega) \eta_j' (\omega') d\omega d\omega' dM
\lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^\gamma |N_p - N_{p'}|)^{p+q+3}} \right\|_{L^\infty(M)} [\|h_{1,p,q}\|_{L^1(M)} + \|h_{2,p,q}\|_{L^1(M)}],
\]

which together with (8.32), (8.214) and (8.215) yields:

\[
2^{-2j-1} \int_M \int_{S^2 \times S^2} \frac{b^{-1} \nabla N' \cdot \mathbf{g}(N,N')N(b)}{g(L,L')} (L(\text{tr} \chi') + \text{tr} \chi' L'(\text{tr} \chi'))
\times F_j^{-1}(u) F_j^{-1}(u') \eta_j' (\omega) \eta_j' (\omega') d\omega d\omega' dM
\lesssim \sum_{p,q \geq 0} c_{pq} \left(2^\gamma |\nu - \nu'|^{p+q+3}(1 + q^2)(1 + |\nu - \nu'|2^\gamma)e^{2^\gamma \gamma_j \gamma_j} \right)
\lesssim \frac{\varepsilon^{2^\gamma \gamma_j \gamma_j}}{(2^\gamma |\nu - \nu'|)^3} + \frac{2^\gamma (2^\gamma |\nu - \nu'|)^2}.
\]

Finally, (8.202), (8.208) and (8.221) imply:

\[
\left\| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j|\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,3} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j|\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,3} \right\|
\lesssim \frac{\varepsilon^{2^\gamma \gamma_j \gamma_j}}{(2^\gamma |\nu - \nu'|)^3} + \frac{2^\gamma (2^\gamma |\nu - \nu'|)^2}.
\]

Now, recall (8.105):

\[
B_{j,\nu',\nu,l,m}^{1,2,1,1} = B_{j,\nu',\nu,l,m}^{1,2,1,1} + B_{j,\nu',\nu,l,m}^{1,2,1,2} + B_{j,\nu',\nu,l,m}^{1,2,1,3}
\]

Together with the estimates (8.148), (8.201) and (8.222), we finally obtain:

\[
\left\| \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j|\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,1} + \sum_{(l,m)/m < l \text{ and } 2^m \leq 2^j|\nu - \nu'|} B_{j,\nu',\nu,l,m}^{1,2,1,3} \right\|
\lesssim \frac{\varepsilon^{2^\gamma \gamma_j \gamma_j}}{2^\gamma |\nu - \nu'|} + \frac{j^2 \varepsilon^{2^\gamma \gamma_j \gamma_j}}{(2^\gamma |\nu - \nu'|)^2} + \frac{\varepsilon^{2^\gamma \gamma_j \gamma_j}}{(2^\gamma |\nu - \nu'|)^3}.
\]

This concludes the proof of Proposition 8.11.
8.2.2 Proof of Proposition 8.11 (Control of $B^{1,2}_{j,\nu,\nu',l,m}$)

Recall that $B^{1,2}_{j,\nu,\nu',l,m}$ is defined by (8.95):

$$B^{1,2}_{j,\nu,\nu',l,m} = -i2^{-j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{1}{g(L, L')} \left( L(P_l \text{tr} \chi) P_m \text{tr} \chi' + P_l \text{tr} \chi' P_m \right)$$

$$\times (b^{-1} - b'^{-1}) F_{j, -1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

Recall (8.1):

$$m < l$$

and $2^m \leq 2^j |\nu - \nu'|$. We first consider the range of $(l, m)$ such that:

$$2^m \leq 2^j |\nu - \nu'| < 2^l.$$

This yields:

$$\sum_{m/2^m \leq 2^j |\nu - \nu'| < 2^l} B^{1,2}_{j,\nu,\nu',l,m}$$

$$= -i2^{-j-1} \sum_{2^l > 2^j |\nu - \nu'|} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{1}{g(L, L')} \left( L(P_l \text{tr} \chi) P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' + P_l \text{tr} \chi' P_{\leq 2^j |\nu - \nu'|} \right)$$

$$\times (b^{-1} - b'^{-1}) F_{j, -1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

Next, recall the identities (8.30) and (8.31):

$$g(L, L') = -1 + g(N, N')$$

and $1 - g(N, N') = g(N - N', N - N')$. We may thus expand

$$\frac{1}{g(L, L')}$$

in the same fashion than (8.33), and in view of (8.223), we obtain, schematically:

$$\sum_{m/2^m \leq 2^j |\nu - \nu'| < 2^l} B^{1,2}_{j,\nu,\nu',l,m}$$

$$= \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} (2^j |N_\nu - N_{\nu'}|)^{p+q+2} [h_{1,p,q,l} + h_{2,p,q,l} + h_{3,p,q,l} + h_{4,p,q,l}] d\mathcal{M},$$

where the scalar functions $h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l}, h_{4,p,q,l}$ on $\mathcal{M}$ are given by:

$$h_{1,p,q,l} = \left( \int_{S^2} (b^{-1} - b'^{-1}) L(P_l \text{tr} \chi) \left( 2^j (N - N_\nu) \right)^p F_{j, -1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left( 2^j (N' - N_{\nu'}) \right)^q F_{j, -1}(u') \eta_j^{\nu'}(\omega') d\omega' \right),$$

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\[
    h_{2,p,q,l} = \left( \int_{S^2} (b^{-1} - b_{\nu}^{-1}) P_l \mathrm{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) (8.226)
    \times \left( \int_{S^2} L'(P_{\leq 2^{j-1}} |\nu - \nu'|) \mathrm{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right),
\]

\[
    h_{3,p,q,l} = \left( \int_{S^2} L(P_l \mathrm{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) (8.227)
    \times \left( \int_{S^2} (b_{\nu}^{-1} - b^{-1}) P_{\leq 2^{j-1}} |\nu - \nu'| \mathrm{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right),
\]

and:

\[
    h_{4,p,q,l} = - \left( \int_{S^2} P_l \mathrm{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) (8.228)
    \times \left( \int_{S^2} (b_{\nu}^{-1} - b^{-1}) L'(P_{\leq 2^{j-1}} |\nu - \nu'|) \mathrm{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right),
\]

and where \( c_{pq} \) are explicit real coefficients such that the series

\[
    \sum_{p,q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1.

Next, we evaluate the \( L^1(\mathcal{M}) \) norm of \( h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l}, h_{4,p,q,l} \) starting with \( h_{1,p,q,l}. \) We first estimate \( L(P_l \mathrm{tr} \chi) \). We have:

\[
    nL(P_l \mathrm{tr} \chi) = P_l(nL \mathrm{tr} \chi) + [nL, P_l] \mathrm{tr} \chi
\]

which together with the estimate (2.36) for \( n \) yields:

\[
    \| L(P_l \mathrm{tr} \chi) \|_{L^1_{\nu}, L^1_\nu} \lesssim \| [nL, P_l] \mathrm{tr} \chi \|_{L^1_{\nu}, L^2_\nu} + \| P_l(nL \mathrm{tr} \chi) \|_{L^1_{\nu}, L^1_\nu}.
\]

Together with the commutator estimate (2.66) for \( [nL, P_l] \mathrm{tr} \chi \) and the estimate (2.69) for \( P_l(nL \mathrm{tr} \chi) \), we obtain:

\[
    \| L(P_l \mathrm{tr} \chi) \|_{L^1_{\nu}, L^1_\nu} \lesssim 2^{-l}\varepsilon. \quad (8.229)
\]

Now, in view of the definition of \( h_{1,p,q,l} \) (8.225), we have:

\[
    \| h_{1,p,q,l} \|_{L^1(\mathcal{M})} \lesssim \int_{S^2} \left\| (b^{-1} - b_{\nu}^{-1}) L(P_l \mathrm{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \right\|_{L^1(\mathcal{M})} \eta_j^\nu(\omega) d\omega
    \times \left( \int_{S^2} P_{\leq 2^{j-1}} |\nu - \nu'| \mathrm{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right) \|_{L^1(\mathcal{M})} \eta_j^\nu(\omega') d\omega'
    \lesssim \int_{S^2} \left\| b^{-1} - b_{\nu}^{-1} \right\|_{L^\infty(\mathcal{M})} \| L(P_l \mathrm{tr} \chi) \|_{L^2_{\nu}, L^1_\nu} \right\| \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \|_{L^\infty(\mathcal{M})} \| F_{j-1}(u) \|_{L^2_{\nu}, L^\infty_\nu}
    \times \left( \int_{S^2} P_{\leq 2^{j-1}} |\nu - \nu'| \mathrm{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right) \right\|_{L^2_{\nu}, L^\infty_\nu} \eta_j^\nu(\omega') d\omega'.
\]

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Together with the estimate (2.44) for $\partial_w b$, the estimate (8.229) for $L(P_{l}\text{tri} \chi)$, the estimate (2.42) for $\partial_w N$, and the size of the patch, we obtain:

$$
\|h_{1,p,q,l}\|_{L^1(M)} \lesssim 2^{-l}|\nu - \nu'|\varepsilon \int_{S^2} \|F_{j,-1}(u)\|_{L^2_{\omega}} \cdot 2^{\frac{j}{2}}(N' - N_{\nu'}) \eta_j'(\omega')d\omega' \tag{8.230}
$$

and

$$
\times \left\| \int_{S^2} P_{\leq 2^l|\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega')d\omega' \right\|_{L^2_{u',x'} L^\infty_{\omega}} \eta_j'(\omega)d\omega.
$$

Next, we estimate the last term in the right-hand side of (8.230). We have:

$$
P_{\leq 2^l|\nu - \nu'|} \text{tr} \chi' = \text{tr} \chi' - \sum_{m > 2^l|\nu - \nu'|} P_m \text{tr} \chi'
$$

which yields the decomposition:

$$
\int_{S^2} P_{\leq 2^l|\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega')d\omega' \nonumber
$$

$$
= \int_{S^2} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega')d\omega' 
$$

$$
- \sum_{m > 2^l|\nu - \nu'|} \int_{S^2} P_m \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega')d\omega'.
$$

Together with (7.71) and (7.83), we obtain:

$$
\left\| \int_{S^2} P_{\leq 2^l|\nu - \nu'|} \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega')d\omega' \right\|_{L^2_{u',x'} L^\infty_{\omega}} \tag{8.231}
$$

$$
\lesssim (1 + q^{\frac{3}{2}}) \varepsilon (2^j|\nu - \nu'|)^{\gamma_j'} + \left( \sup_{\omega'} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty(M)} \right) \nonumber
$$

$$
\times \sum_{m > 2^l|\nu - \nu'|} \varepsilon (2^j|\nu - \nu'|2^{-m+\frac{1}{2}} + (2^j|\nu - \nu'|)^{\frac{1}{2}}2^{-\frac{m}{2} + \frac{1}{2}})^{\gamma_j'}
$$

$$
\lesssim (1 + q^{\frac{3}{2}}) \varepsilon (2^j|\nu - \nu'|)^{\gamma_j'},
$$

where we used in the last inequality the estimate (2.42) for $\partial_w N$ and the size of the patch. Finally, (8.230) and (8.231) imply:

$$
\|h_{1,p,q,l}\|_{L^1(M)} \lesssim (1 + q^{\frac{3}{2}}) \varepsilon 2^{-l}|\nu - \nu'|2^j|\nu - \nu'| \int_{S^2} \|F_{j,-1}(u)\|_{L^2_\omega} \eta_j'(\omega)d\omega \tag{8.232}
$$

$$
\lesssim (1 + q^{\frac{3}{2}}) \varepsilon 2^{-l}(2^j|\nu - \nu'|)^2 \gamma_j'\gamma_j',
$$

where we used in the last inequality Plancherel in $\lambda$ for $\|F_{j,-1}(u)\|_{L^2_\omega}$, Cauchy-Schwarz in $\omega$ and the size of the patch.

Next, we evaluate the $L^1(M)$ norm of $h_{2,p,q,l}$. In view of the definition (8.226) of $h_{2,p,q,l}$, we have:

$$
\|h_{2,p,q,l}\|_{L^1(M)} \lesssim \left\| \int_{S^2} (b^{-1} - b_{\nu'}^{-1}) P_{l} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_{\nu'})\right)^p F_{j,-1}(u) \eta_j'(\omega)d\omega' \right\|_{L^2_{u',x'} L^\infty_{\omega}} \tag{8.233}
$$

$$
\times \left\| \int_{S^2} L'(P_{\leq 2^l|\nu - \nu'|} \text{tr} \chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega')d\omega' \right\|_{L^2_{u',x'} L^\infty_{\omega}}.
$$
Next, we estimate the two terms in the right-hand side of (8.233) starting with the first one. Using the estimate (7.71) with $G = b^{-1} - b_{\nu}'$, we have:

$$\left\| \int_{S^2} (b^{-1} - b_{\nu}') P_{t, \chi} \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^{q} F_{j,-1}(u') \eta_j^\nu(\omega) d\omega \right\|_{L^2_{u', \nu, \nu'} L^\infty_t}$$

where we used in the last inequality the estimate (8.235) for $b$. Next, we estimate the second term in the right-hand side of (8.233). We have:

$$\left\| \int_{S^2} L^\prime \left( \text{tr} \chi^\nu \right) \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q} F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2_{u', \nu, \nu'} L^\infty_t}$$

In view of (7.17), we have:

$$(1 + q^2) \varepsilon \gamma_j^\nu.$$
Finally, (8.233), (8.234) and (8.238) imply:

\[
\left\| \int_{S^2} L'(P_{\leq 2|\nu - \nu'|}) \left( 2^\frac{i}{2} (N' - N_{\nu'}) \right) F_{j,-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2_{u', \nu', L^1}} \tag{8.239}
\]

\[
\lesssim (1 + q^2) \varepsilon \gamma_j'' + \frac{2^\frac{i}{2}}{(2^\frac{i}{2}|\nu - \nu'|)^\frac{1}{2}} \gamma_j''.
\]

Finally, (8.233), (8.234) and (8.239) yield:

\[
\| h_{2,p,q,l} \|_{L^1(M)} \tag{8.240}
\]

\[
\lesssim \left( 2^\frac{i}{2}|\nu - \nu'| \right)^2 2^{-l} + \left( 2^\frac{i}{2}|\nu - \nu'| \right)^2 2^{-\frac{1}{2} - \frac{i}{2}} \left( 1 + q^2 \right) + \frac{2^\frac{i}{2}}{(2^\frac{i}{2}|\nu - \nu'|)^\frac{1}{2}} \varepsilon^2 \gamma_j'' \gamma_j''
\]

Next, we evaluate the \(L^1(M)\) norm of \(h_{3,p,q,l}\). We decompose

\[
\int_{S^2} (b_{\nu, l}^{-1} - b^{-1}) P_{\leq 2|\nu - \nu'|} \text{tr} \chi' \left( 2^\frac{i}{2} (N' - N_{\nu'}) \right) F_{j,-1}(u') \eta_j''(\omega') d\omega',
\]

where \(H\) is given by

\[
H = \int_{S^2} (b_{\nu, l}^{-1} - b^{-1}) \text{tr} \chi' \left( 2^\frac{i}{2} (N' - N_{\nu'}) \right) F_{j,-1}(u') \eta_j''(\omega') d\omega'.
\]

Together with the definition (8.227) of \(h_{3,p,q,l}\), we obtain:

\[
\| h_{3,p,q,l} \|_{L^1(M)} \tag{8.241}
\]

\[
\lesssim \left| \int_{S^2} HL(P_t \text{tr} \chi) \left( 2^\frac{i}{2} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j''(\omega) d\omega \right|_{L^1(M)}
\]

\[
+ \left| \int_{S^2} L(P_t \text{tr} \chi) \left( 2^\frac{i}{2} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j''(\omega) d\omega \right|_{L^2(M)}
\]

\[
\times \left| \int_{S^2} (b_{\nu, l}^{-1} - b^{-1}) P_{> 2|\nu - \nu'|} \text{tr} \chi' \left( 2^\frac{i}{2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j''(\omega') d\omega' \right|_{L^2(M)}.
\]

Next, we evaluate the three terms in the right-hand side of (8.241) starting with the first one. In view of the estimate (7.10), we have

\[
\left| \int_{S^2} HL(P_t \text{tr} \chi) \left( 2^\frac{i}{2} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j''(\omega) d\omega \right|_{L^1(M)} \tag{8.242}
\]

\[
\lesssim \sup_{\omega \in \text{supp} \eta_j''} \left| H \left( 2^\frac{i}{2} (N - N_{\nu}) \right)^p \right|_{L^2_{u', \nu', L^\infty}} \varepsilon 2^{\frac{i}{2} - l} \gamma_j''
\]

\[
\lesssim \sup_{\omega \in \text{supp} \eta_j''} \left| H \right|_{L^2_{u', \nu', L^\infty}} \left| \left( 2^\frac{i}{2} (N - N_{\nu}) \right)^p \right|_{L^\infty} \varepsilon 2^{\frac{i}{2} - l} \gamma_j''
\]

\[
\lesssim \sup_{\omega \in \text{supp} \eta_j''} \left| H \right|_{L^2_{u', \nu', L^\infty}} \varepsilon 2^{\frac{i}{2} - l} \gamma_j''
\]
where we used in the last inequality the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. In view of the estimate (7.95) and the definition of \( H \), we have:

\[
\| H \|_{L^2_{u,v}, L^\infty} \lesssim 2^{-\frac{1}{4}} (1 + q^2) \varepsilon \left( 2^{\frac{1}{2}} |\nu - \nu'| + 1 \right) \gamma_j^{\nu'},
\]

which together with (8.242) implies

\[
\left\| \int_{S^2} HL(P_t \chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^1(M)} \lesssim (1 + q^2) (2^{\frac{1}{2}} |\nu - \nu'| + 1) \varepsilon 2^{\frac{1}{4} - l} \gamma_j^{\nu', \gamma_j^{\nu'}}.
\]

Next, we evaluate the second term in the right-hand side of (8.241). Using the basic estimate in \( L^2(M) \) (7.1), we have:

\[
\left\| \int_{S^2} L(P_t \chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(M)} \lesssim \left( \sup_\omega \left\| L(P_t \chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{1}{2}} \gamma_j^{\nu'}
\]

\[
\lesssim \left( \sup_\omega \| L(P_t \chi) \|_{L^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{1}{2}} \gamma_j^{\nu'},
\]

where we used in the last inequality the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Also, (8.142) together with the estimate (2.36) for \( n \) yields:

\[
\| L(P_t \chi) \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{1}{2}} \varepsilon,
\]

which together with (8.244) yields:

\[
\left\| \int_{S^2} L(P_t \chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^{\nu'}(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{\frac{1}{2} - \frac{1}{4}} \varepsilon \gamma_j^{\nu'}.
\]

Next, we evaluate the third term in the right-hand side of (8.241). Using the basic estimate in \( L^2(M) \) (7.3), we have:

\[
\left\| \int_{S^2} (b_\nu^{-1} - b'^{-1}) P_{2^j |\nu - \nu'|} \chi' \left( 2^{\frac{1}{2}} (N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \lesssim \left( \sup_\omega \left\| (b_\nu^{-1} - b'^{-1}) \left( 2^{\frac{1}{2}} (N' - N_\nu') \right)^q \right\|_{L^\infty} \right) \left( \sum_{2^m > 2^j |\nu - \nu'|} 2^{-m} \right) \varepsilon 2^{\frac{1}{4} - l} \gamma_j^{\nu'}
\]

\[
\lesssim \varepsilon 2^{\frac{1}{4} - l} \gamma_j^{\nu'} \frac{2^{\frac{1}{2}}}{2^{l} |\nu - \nu'|},
\]

where we used in the last inequality the estimate (2.44) for \( \partial_\omega b \), the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Finally, (8.241), (8.243), (8.246) and (8.247) imply:

\[
\| h_{3,p,q,l} \|_{L^1(M)} \lesssim (1 + q^2) (2^{\frac{1}{2}} |\nu - \nu'| + 1) \varepsilon 2^{\frac{1}{4} - l} \gamma_j^{\nu', \gamma_j^{\nu'}} + \frac{\varepsilon 2^{\frac{1}{2}} \gamma_j^{\nu', \gamma_j^{\nu'}}}{2^{l} |\nu - \nu'|}.
\]
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l}$. In view of the definition (8.228) of $h_{4,p,q,l}$, we have:

$$
\left\| h_{4,p,q,l} \right\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} P \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}
\times \left\| \int_{S^2} (b_{\nu}^{-1} - b^{-1}) L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.
$$

Next, we estimate the two terms in the right-hand side of (8.249) starting with the first one. Using the basic estimate in $L^2(\mathcal{M})$ (7.3), we have:

$$
\left\| \int_{S^2} P \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_\omega \left\| \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p \right\|_{L^\infty} \right) \varepsilon 2^{\frac{j}{2} - l} \gamma_j^\nu
\lesssim \varepsilon 2^{\frac{j}{2} - l} \gamma_j^\nu,
$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Next, we estimate the second term in the right-hand side of (8.249). Using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$
\left\| \int_{S^2} (b_{\nu}^{-1} - b^{-1}) L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega'} \left\| (b_{\nu}^{-1} - b^{-1}) L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty, L^2(\mathcal{H}_{u'})} \right) 2^{\frac{j}{2}} \gamma_j^\nu.
$$

Now, we have:

$$
\left\| (b_{\nu}^{-1} - b^{-1}) L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty, L^2(\mathcal{H}_{u'})} \lesssim \left\| b_{\nu}^{-1} - b^{-1} \right\|_{L^\infty} \left\| L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \right\|_{L^\infty, L^2(\mathcal{H}_{u'})} \left\| \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty}
\lesssim 2^{-\frac{j}{2}} \varepsilon \left\| L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \right\|_{L^\infty, L^2(\mathcal{H}_{u'})},
$$

where we used in the last inequality the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Furthermore, we have:

$$
\left\| L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \right\|_{L^\infty, L^2(\mathcal{H}_{u'})} \lesssim \varepsilon + \frac{\varepsilon}{(2|\nu - \nu'|)^{\frac{1}{2}}}
\lesssim \varepsilon + \frac{\varepsilon}{(2|\nu - \nu'|)^{\frac{1}{2}}}
$$

where we used in the last inequality the estimate (2.39) for $\text{tr} \chi'$ and the estimate (8.205) for $L'(P_{\geq 2|\nu - \nu'|}\text{tr} \chi')$. Together with (8.252) and the fact that $2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$, we obtain:

$$
\left\| (b_{\nu}^{-1} - b^{-1}) L'(P_{\leq 2|\nu - \nu'|}\text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty, L^2(\mathcal{H}_{u'})} \lesssim 2^{-\frac{j}{2}} \varepsilon.
$$
Together with (8.251), this yields:
\[
\left\| \int_{\mathbb{R}^2} (b_{y'}^{-1} - b^{-1}) L'(P_{\leq 2|\nu - \nu'|} \text{tr} \chi')(2^{\frac{1}{2}}(N' - N_\nu))^{q} F_{\nu - 1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon \gamma_j^\nu.
\]
(8.253)

Finally, (8.249), (8.250) and (8.253) yield:
\[
\|h_{4,p,q,l}\|_{L^1(M)} \lesssim \varepsilon^{2\frac{1}{2} - l} \gamma_j^\nu \gamma_j'.
\]
(8.254)

Finally, in view of (8.224), we have:
\[
\left\| \sum_{m/2^m \leq 2|\nu - \nu'| < 2^l} B_{j,\nu,\nu',l,m}^{1,2,2} \right\| \lesssim \sum_{p,q \geq 0} C_{pq} \left\| \frac{1}{(2^{\frac{1}{2}}|N_\nu - N_{\nu'}|)^{p+q+2}} \left[ \|h_{1,p,q,l}\|_{L^1(M)} + \|h_{2,p,q,l}\|_{L^1(M)} + \|h_{3,p,q,l}\|_{L^1(M)} + \|h_{4,p,q,l}\|_{L^1(M)} \right] \right\|
\]
which together with (8.32), (8.232), (8.240), (8.248) and (8.254) yields:
\[
\left\| \sum_{m/2^m \leq 2|\nu - \nu'| < 2^l} B_{j,\nu,\nu',l,m}^{1,2,2} \right\| \lesssim \varepsilon^{2\frac{1}{2} - l} \gamma_j^\nu \gamma_j'.
\]
(8.255)

Summing in $l$, we obtain:
In view of (8.1), we still need to estimate $B_{j,\nu,\nu',l,m}^{1,2,2}$ in the range of $(l, m)$ such that:

$2^m \leq 2^l \leq 2^j |\nu - \nu'|$.

Recall the definition (8.95) of $B_{j,\nu,\nu',l,m}^{1,2,2}$:

$$B_{j,\nu,\nu',l,m}^{1,2,2} = -i2^{-j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{1}{g(L, L')} \left( L(P_l \text{tr}\chi)P_m \text{tr}\chi' + P_l \text{tr}\chi'(P_m \text{tr}\chi') \right)$$

$$\times (b^{-1} - b'^{-1}) F_{j,-1}(u)F_{j}(u') \eta_j^\nu(\omega)\eta_j^\nu'(\omega')d\omega d\omega' d\mathcal{M}.$$ 

We integrate by parts tangentially using (7.136).

**Lemma 8.13** Let $B_{j,\nu,\nu',l,m}^{1,2,2}$ defined by (8.95). Integrating by parts using (7.136) yields:

$$B_{j,\nu,\nu',l,m}^{1,2,2} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^j |N_\nu - N_\nu'|)^{p+q}} \left[ \frac{1}{|N_\nu - N_\nu'|^2} h_{1,p,q,l,m} \right]$$

$$+ \frac{1}{|N_\nu - N_\nu'|^3} h_{2,p,q,l,m} \right] d\mathcal{M} + B_{j,\nu,\nu',l,m}^{1,2,2.1} + B_{j,\nu,\nu',l,m}^{1,2,2.2} + B_{j,\nu,\nu',l,m}^{1,2,2.3},$$

where $c_{pq}$ are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h_{1,p,q,l,m}, h_{2,p,q,l,m}$ on $\mathcal{M}$ are given by:

$$h_{1,p,q,l,m} = \left( \int_{S^2} L(P_l \text{tr}\chi) \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} N'(P_m \text{tr}\chi')(b' - b_\nu) \left( 2^j (N' - N_\nu) \right)^q F_{j,-1}(u') \eta_j^\nu'(\omega') d\omega' \right)$$

$$+ \left( \int_{S^2} L(P_l \text{tr}\chi)(b_\nu - b) \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} N'(P_m \text{tr}\chi') \left( 2^j (N' - N_\nu) \right)^q F_{j,-1}(u') \eta_j^\nu'(\omega') d\omega' \right),$$

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and:

\[ h_{2,p,q,l,m} = \left( \int_{S^2} (\theta + b^{-1}\nabla(b))L(P_t \chi) \left( 2^{\frac{j}{2}}(N - N_\nu) \right) F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) ^p \]

\[ + \left( \int_{S^2} P_m \chi' (b' - b_\nu) \left( 2^{\frac{j}{2}}(N' - N_\nu) \right) ^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right) \]

\[ + \left( \int_{S^2} L'(P_m \chi')(b' - b_\nu) \left( 2^{\frac{j}{2}}(N' - N_\nu) \right) ^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right) \]

\[ + \left( \int_{S^2} (\theta + b^{-1}\nabla(b))L(P_t \chi) (b_\nu - b) \left( 2^{\frac{j}{2}}(N - N_\nu) \right) ^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \]

\[ + \left( \int_{S^2} L'(P_m \chi') (b_\nu - b) \left( 2^{\frac{j}{2}}(N - N_\nu) \right) ^q F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \]

\[ \times \left( \int_{S^2} L'(P_m \chi') \left( 2^{\frac{j}{2}}(N' - N_\nu) \right) ^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \]

and where \( B_{j,v,v',l,m}^{1,2,2,1}, B_{j,v,v',l,m}^{1,2,2,2}, B_{j,v,v',l,m}^{1,2,2,3} \) are given, schematically, by:

\[ B_{j,v,v',l,m}^{1,2,2,1} = 2^{-2j} \int_{S^2 \times S^2} \frac{(N' - N)(b' - b)}{g(L, L')} \left( \nabla(L(P_t \chi))P_m \chi' + \nabla(P_t \chi)L'(P_m \chi') \right. \]

\[ + \left. L(P_t \chi) \nabla'(P_m \chi) + P_t \chi \nabla'(L'(P_m \chi')) \right) F_{j-1}(u) \eta_j^\nu(\omega) F_{j-1}(u') \eta_j^\nu(\omega') d\omega d\omega', \]

\[ B_{j,v,v',l,m}^{1,2,2,2} = 2^{-2j} \int_{S^2 \times S^2} \frac{P_t \chi \nabla N(L'(P_m \chi'))(b' - b)}{g(L, L')} F_{j-1}(u) \eta_j^\nu(\omega) F_{j-1}(u') \eta_j^\nu(\omega') d\omega d\omega', \]

\[ B_{j,v,v',l,m}^{1,2,2,3} = 2^{-2j} \int_{S^2 \times S^2} \frac{(\chi' - \chi)(b' - b)}{g(L, L')} \left( L(P_t \chi)P_m \chi' + P_t \chi L'(P_m \chi') \right) \]

\[ \times F_{j-1}(u) \eta_j^\nu(\omega) F_{j-1}(u') \eta_j^\nu(\omega') d\omega d\omega'. \]

The proof of lemma 8.13 is postponed to Appendix C. In the rest of this section, we use Lemma 8.13 to obtain the control of \( B_{j,v,v',l,m}^{1,2,2} \).

We first estimate the \( L^1(\mathcal{M}) \) norm of \( h_{1,p,q,l,m}, h_{2,p,q,l,m} \) starting with \( h_{1,p,q,l,m} \). In view
of the definition (8.257) of $h_{1,p,q,l,m}$, we have:

$$\| h_{1,p,q,l,m} \|_{L^1(M)} \leq \left\| \int_{S^2} L(P_t \chi)(2^{\frac{j}{2}}(N - N_\nu))^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}$$

$$\times \left\| \int_{S^2} N'(P_m \chi')(b' - b_\nu) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} + \left\| \int_{S^2} L(P_t \chi)(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}$$

Using the basic estimate in $L^2(M)$ (7.1) and the estimate (8.170), we obtain:

$$\| h_{1,p,q,l,m} \|_{L^1(M)} \leq \left( \sup_{\omega'} \left\| N'(P_m \chi')(b' - b_\nu) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty_{\omega'} L^2(H_{\nu'})} \right) \left(1 + p^2\right) 2^{146} \times 2^{11} \varepsilon \gamma_j \gamma_j' \gamma_j''.$$

Using the estimate (2.38) for $b'$, we have:

$$\| N'(P_m \chi') \|_{L^\infty_{\omega'} L^2(H_{\nu'})} \leq \| P_m(b' N' \chi') \|_{L^\infty_{\omega'} L^2(H_{\nu'})} + \| b' N' P_m \chi' \|_{L^\infty_{\omega'} L^2(H_{\nu'})}$$

which together with the estimate (2.39) for $\chi'$, the commutator estimate (2.64), and the boundedness of $P_m$ on $L^2(P_{t,u'})$ yields:

$$\| N'(P_m \chi') \|_{L^\infty_{\omega'} L^2(H_{\nu'})} \leq \varepsilon. \quad (8.863)$$

Now, we have:

$$\left\| N'(P_m \chi')(b' - b_\nu) \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty_{\omega'} L^2(H_{\nu'})} \leq \| N'(P_m \chi') \|_{L^\infty_{\omega'} L^2(H_{\nu'})} \| b' - b_\nu \|_{L^\infty} \left\| \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q \right\|_{L^\infty} \leq \varepsilon |\nu - \nu'|,$$

where we used in the last inequality the estimate (8.263), the estimate (2.44) for $\partial_{\omega} b$, the estimate (2.42) for $\partial_{\omega} N$, and the size of the patch. Using the same estimates, we obtain similar estimates for the last two terms in the right-hand side of (8.262):}

$$\| L(P_t \chi)(b_\nu - b) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \|_{L^\infty_{\omega'} L^2(H_{\nu'})} \leq 2^{-\frac{j}{2}} \varepsilon,$$
and:
\[
\|N'(P_m \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_\nu') \right)^q\|_{L^q_{H_\nu} L^2(H)} \lesssim \varepsilon.
\]
In the end, we obtain:
\[
\|h_{1,p,q,l,m}\|_{L^1(M)} \lesssim (1 + p^2) 2^{\frac{j}{2}} |\nu - \nu'| + 2^{\frac{j}{2}} \varepsilon^2 \gamma_j \gamma_j',
\]
(8.264)
\[
\lesssim (1 + p^2) 2^{\frac{j}{2}} \varepsilon^2 \gamma_j \gamma_j',
\]
where we used in the last inequality the fact that $|\nu - \nu'| \lesssim 1$.

Next, we estimate the $L^1(M)$ norm of $h_{2,p,q,l,m}$. In view of the definition (8.258) of $h_{2,p,q,l,m}$, we have:
\[
\|h_{2,p,q,l,m}\|_{L^1(M)} \lesssim \int_{S^2} \left(\theta + b^{-1} \nabla(b) \right) \|P \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)\|_{L^2(M)}
\times \int_{S^2} \|P_m \text{tr} \chi'(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu'(\omega')\|_{L^2(M)}
+ \int_{S^2} \left(\theta + b^{-1} \nabla(b) \right) \|P \text{tr} \chi \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)\|_{L^2(M)}
\times \int_{S^2} L'(P_m \text{tr} \chi'(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu'(\omega')\|_{L^2(M)}
+ \int_{S^2} \left(\theta + b^{-1} \nabla(b) \right) \|P \text{tr} \chi(b_\nu - b) \left(2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)\|_{L^2(M)}
\times \int_{S^2} L'(P_m \text{tr} \chi'(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu'(\omega')\|_{L^2(M)}
\times \int_{S^2} L'(P_m \text{tr} \chi'(b' - b_\nu) \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu'(\omega')\|_{L^2(M)}.
\]
Using the basic estimate in $L^2(M)$ (7.1) for the first term, the fourth term, the fifth term and the eighth term in the right-hand side of (8.265), the estimate (7.102) for the second term in the right-hand side of (8.265), and the estimate (8.181) for the sixth term in the
right-hand side of (8.265), we obtain:

\[
\begin{align*}
\|h_{2,p,q,t,m}\|_{L^1(M)} & \lesssim \left(\sup_{\omega} \left\| (\theta + b^{-1}\nabla(b))L(P \text{tr} \chi) \left(2^{\frac{1}{2}}(N - N_{\nu})\right) P_{\omega} \right\|_{L^\infty\times L^2(H_{\omega})}\right) 2^{\frac{1}{2}} \gamma_j' \varepsilon (1 + q^2) \gamma_j'' \\
& \quad + \left\| \int_{S^2} (\theta + b^{-1}\nabla(b))P \text{tr} \chi \left(2^{\frac{1}{2}}(N - N_{\nu})\right)^{P} F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)} \\
& \quad \times \left(\sup_{\omega} \left\| L'(P \text{tr} \chi')(b' - b) \left(2^{\frac{1}{2}}(N' - N_{\nu})\right) \right\|_{L^\infty\times L^2(H_{\omega'})}\right) 2^{\frac{1}{2}} \gamma_j'.
\end{align*}
\]

Together with the estimate (2.38) for \(b\), the estimates (2.37) (2.39) (2.40) for \(\theta\), the estimate (2.39) for \(\text{tr} \chi\), the estimate (8.186) for \(L(P \text{tr} \chi)\) and \(L'(P \text{tr} \chi')\), the estimate (2.44) for \(\partial_{\omega} b\), the estimate (2.42) for \(\partial_{\omega} N\) and the size of the patch, we obtain:

\[
\begin{align*}
\|h_{2,p,q,t,m}\|_{L^1(M)} & \lesssim (1 + q^2) 2^{\frac{1}{2}} \varepsilon^{2} \gamma_j' \gamma_j'' \\
& \quad + \left\| \int_{S^2} (\theta + b^{-1}\nabla(b))P \text{tr} \chi \left(2^{\frac{1}{2}}(N - N_{\nu})\right)^{P} F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)} 2^{\frac{1}{2}} |\nu - \nu'| \gamma_j'' \\
& \quad + 2^{\frac{1}{2}} |\nu - \nu'| (1 + q^2) \varepsilon^{2} \gamma_j' \gamma_j'' \\
& \quad + \left\| \int_{S^2} (\theta + b^{-1}\nabla(b))P \text{tr} \chi (b - b') \left(2^{\frac{1}{2}}(N - N_{\nu})\right)^{P} F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)} \varepsilon^{2} \gamma_j'.
\end{align*}
\]

Next, we estimate the two \(L^2(M)\) norms in the right-hand side of (8.266). Recall the definition of \(\theta\) (2.20):

\[
\theta = \chi + k.
\]

Now, since \(k\) does not depend on \(\omega\), and in view of the decomposition (2.72) (2.74) for \(\chi\), and the decomposition (2.80) for \(b^{-1}\nabla(b)\), we have the following decomposition for \(\theta + b^{-1}\nabla(b)\):

\[
\theta + b^{-1}\nabla(b) = F_1^j + F_2^j
\]

where the tensor \(F_1^j\) only depends on \(\nu\) and satisfies:

\[
\|F_1^j\|_{L^\infty \times L^2(H_{\omega})} \lesssim \varepsilon,
\]

and where the tensor \(F_2^j\) satisfies:

\[
\|F_2^j\|_{L^\infty \times L^2(H_{\omega})} \lesssim \varepsilon 2^{- \frac{1}{4}}.
\]

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We estimate the first term in the right-hand side of (8.266). In view of (8.267), we have:

\[ \left\| \int_{S^2} (\theta + b^{-1} \nabla (b)) P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_{j-1}(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(M)} \]

\[ \lesssim \| F_1 \|_{L_{t, x}^\nu L_{s}^2 L_{x}^\nu} \left\| \int_{S^2} P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_j(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(M)} \]

\[ + \left\| \int_{S^2} F_2^2 P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_j(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(M)} \]

where we used the estimate (8.268) for \( F_1 \) in the last inequality. Now, using (7.71) in the case \( l > j/2 \), and:

\[ P_{\leq j/2} \text{tr} \chi = \text{tr} \chi - \sum_{l > j/2} P_l \text{tr} \chi \]

together with (7.71) and (7.16) in the case \( l = j/2 \), we obtain:

\[ \left\| \int_{S^2} P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_j(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2_{t, x} L_{x}^\nu} \lesssim (1 + p^2) \varepsilon \gamma_j^\nu. \]

Also, we have:

\[ \left\| \int_{S^2} P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_j(u) \eta_j^\nu (\omega) d\omega \right\|_{L^\infty(M)} \]

\[ \lesssim \int_{S^2} \| P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_j(u) \|_{\infty} \eta_j^\nu (\omega) d\omega \]

\[ \lesssim \varepsilon \left( \int_{S^2} \| F_{j-1}(u) \|_{L^\infty} \eta_j^\nu (\omega) d\omega \right) \]

\[ \lesssim \varepsilon 2^j \gamma_j^\nu, \]

where we used the estimate (2.39) for \( \text{tr} \chi \), the estimate (2.42) for \( \partial_\omega N \), Cauchy-Schwarz in \( \lambda \) to estimate \( \| F_{j-1}(u) \|_{L^\infty} \), Cauchy-Schwarz in \( \omega \) and the size of the patch. Interpolating between (8.271) and (8.272), we obtain:

\[ \left\| \int_{S^2} P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_j(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2_{t, x} L_{x}^\nu} \lesssim 2^j \varepsilon \gamma_j^\nu. \]

For the second term in the right-hand side of (8.270), we have:

\[ \left\| \int_{S^2} F_2^2 P_t \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^{p} F_j(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(M)} \]

\[ \lesssim \varepsilon \int_{S^2} 2^{-\frac{j}{2}} \| F_j(u) \|_{L^2} \eta_j^\nu (\omega) d\omega \]

\[ \lesssim \varepsilon 2^j \gamma_j^\nu. \]
where we used in the last inequality Plancherel in $\lambda$, Cauchy Schwartz in $\omega$ and the size of the patch. Finally, (8.270), (8.273) and (8.274) imply:

$$
\left\| \int_{S^2} (\theta + b^{-1}\nabla(b)) P_1 \text{tr} \chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{\frac{j}{4}}\varepsilon \gamma_j^\nu.
$$

Next, we estimate the second term in the right-hand side of (8.266). In view of (8.267), we have:

$$
\left\| \int_{S^2} (\theta + b^{-1}\nabla(b)) P_1 \text{tr} \chi(b - b_\nu) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \| F_j^1 \|_{L^8_{\omega^1}L^4_{\nu^1}L^\infty_{\omega^1}} \left\| \int_{S^2} P_1 \text{tr} \chi(b - b_\nu) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
$$

$$
+ \left\| \int_{S^2} F_j^1 P_1 \text{tr} \chi(b - b_\nu) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
$$

$$
\lesssim \varepsilon \left\| \int_{S^2} P_1 \text{tr} \chi(b - b_\nu) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
$$

$$
+ \left\| \int_{S^2} F_j^1 P_1 \text{tr} \chi(b - b_\nu) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)},
$$

where we used the estimate (8.268) for $F_j^1$ in the last inequality. Now, using (7.71) in the case $l > j/2$, and:

$$
P_{\leq j/2} \text{tr} \chi = \text{tr} \chi - \sum_{l > j/2} P_l \text{tr} \chi
$$

together with (7.71) and (7.95) in the case $l = j/2$, we obtain:

$$
\left\| \int_{S^2} P_l \text{tr} \chi(b_\nu - b) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{\omega^1}L^4_{\nu}L^\infty_{\omega^1}} \lesssim 2^{-\frac{j}{4}}(1 + p^{\frac{3}{2}}) \varepsilon \gamma_j^\nu.
$$

Also, we have:

$$
\left\| \int_{S^2} P_l \text{tr} \chi(b_\nu - b) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \eta_j^\nu(\omega) d\omega \right\|_{L^\infty(M)} \lesssim \int_{S^2} \left\| P_l \text{tr} \chi(b_\nu - b) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_j(u) \right\|_{L^\infty} \eta_j^\nu(\omega) d\omega
$$

$$
\lesssim \varepsilon 2^{-\frac{j}{2}} \left( \int_{S^2} \left\| F_{j,-1}(u) \right\|_{L^\infty} \eta_j^\nu(\omega) d\omega \right)
$$

$$
\lesssim \varepsilon 2^{\frac{j}{4}} \gamma_j^\nu,
$$

where we used, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, Cauchy-Schwarz in $\lambda$ to estimate $\| F_{j,-1}(u) \|_{L^\infty}$, Cauchy-Schwarz in $\omega$ and the
size of the patch. Interpolating between (8.277) and (8.278), we obtain:

\[
\left\| \int_{S^2} P_t \text{tr} \chi (b_\nu - b) \left( 2^\frac{i}{4} (N - N_\nu) \right)^p \nabla_j (u) \omega_j (\omega) d\omega \right\|_{L^2_{\nu} L^8_{u} L^\infty} \lesssim 2^{-\frac{1}{4} \pi \varepsilon \gamma_j ^\nu}, \tag{8.279}
\]

For the second term in the right-hand side of (8.276), we have:

\[
\left\| \int_{S^2} F^j_2 P_t \text{tr} \chi (b_\nu - b) \left( 2^\frac{i}{4} (N - N_\nu) \right)^p \nabla_j (u) \omega_j (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \int_{S^2} \| F^j_2 \|_{L^2_{\nu} L^2(H_u)} \| P_t \text{tr} \chi (b_\nu - b) \left( 2^\frac{i}{4} (N - N_\nu) \right)^p \nabla_j (u) \omega_j (\omega) d\omega.
\]

Together with the estimate (8.269) for \( F^j_2 \), the estimate (2.39) for \( \text{tr} \chi \), the estimate (2.44) for \( b \), the estimate (2.42) for \( \partial_\omega N \) and the size of the patch, we obtain:

\[
\left\| \int_{S^2} F^j_2 P_t \text{tr} \chi (b_\nu - b) \left( 2^\frac{i}{4} (N - N_\nu) \right)^p \nabla_j (u) \omega_j (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{1}{4} \frac{i}{4}} \int_{S^2} \| F_j (u) \|_{L^2} \eta_j ^\nu (\omega) d\omega
\]

where we used in the last inequality Plancherel in \( \lambda \), Cauchy Schwartz in \( \omega \) and the size of the patch. Finally, (8.276), (8.279) and (8.280) imply:

\[
\left\| \int_{S^2} (\theta + b^{-1} \nabla (b)) P_t \text{tr} \chi (b_\nu - b) \left( 2^\frac{i}{4} (N - N_\nu) \right)^p \nabla_j (u) \omega_j (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{1}{4} \frac{i}{4}} \varepsilon \gamma_j ^\nu. \tag{8.281}
\]

Finally, (8.266), (8.281) and (8.275) yield:

\[
\| h_{2,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim (1 + q^2) (2^{\frac{i}{2}} (2^{\frac{i}{4}} |\nu - \nu'|) + 2^{\frac{7}{16}}) \varepsilon^2 \gamma_j ^\nu \gamma_j ^\nu. \tag{8.282}
\]

Next, we estimate \( B^{1,2,1}_{j,\nu,\nu',l,m} \). Recall that we are considering the range of \((l,m)\):

\[
2^m \leq 2^l \leq 2^j |\nu - \nu'|.
\]

Summing in \((l,m)\), we have:

\[
\sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} \left( \nabla (L (P_t \text{tr} \chi)) P_m \text{tr} \chi' + \nabla (P_t \text{tr} \chi) L' (P_m \text{tr} \chi') \right) + L (P_t \text{tr} \chi) \nabla' (P_m \text{tr} \chi') + P_t \text{tr} \chi \nabla' (L' (P_m \text{tr} \chi')) = \nabla (L (P_{\leq 2^j |\nu - \nu'| \text{tr} \chi})) P_{\leq 2^l |\nu - \nu'| \text{tr} \chi'} + \nabla (P_{\leq 2^j |\nu - \nu'| \text{tr} \chi}) L' (P_{\leq 2^l |\nu - \nu'| \text{tr} \chi'}) + L (P_{\leq 2^j |\nu - \nu'| \text{tr} \chi}) \nabla' (P_{\leq 2^l |\nu - \nu'| \text{tr} \chi'}) + P_{\leq 2^j |\nu - \nu'| \text{tr} \chi} \nabla' (L' (P_{\leq 2^l |\nu - \nu'| \text{tr} \chi})).
\]

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Thus, using the symmetry in \((\omega, \omega')\) of the integrant in \(B_{j,\nu,\nu',l,m}^{1,2,2,1}\), we obtain in view of the definition (8.259) of \(B_{j,\nu,\nu',l,m}^{1,2,2,1}\):

\[
\sum_{(l,m)/2^m \leq |\nu| \leq 2^{|\nu'|}} \left( B_{j,\nu,\nu',l,m}^{1,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,1} \right) \tag{8.283}
\]

\[
= 2^{-2j} \int_{S^2 \times S^2} \left( N' - N\right) \left( b' - b \right) \mathcal{Y}(L(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R})) \mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}
\]

\[
\mathcal{Y}(L(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}')) + L(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}) \mathcal{Y}'(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}')
\]

\[
+ P \leq 2|\nu-\nu'|\mathcal{R} \mathcal{Y}'(L'(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}')) F_{j-1}(u) \eta'_{j}(\omega) F_{j-1}(u') \eta'_{j}(\omega') d\omega d\omega'.
\]

Now, note that the right-hand side of (8.283) is the analog of the right-hand side of (8.223) provided one replaces

\[
P \mathcal{R}, 2^j > 2^j|\nu - \nu'|
\]

with:

\[
\frac{2^{-j'}(N' - N)}{g(L, L')} \mathcal{Y}(P \leq 2|\nu-\nu'|\mathcal{R}).
\]

We obtain the analog of (8.255):

\[
\bigg\lfloor \frac{2^{-\frac{3}{2}}}{(2^\frac{3}{2}|\nu - \nu'|)^\frac{3}{2}} + \frac{2^{-\frac{4}{3}}}{(2^\frac{4}{3}|\nu - \nu'|)^\frac{4}{3}} + \frac{1}{(2^\frac{5}{3}|\nu - \nu'|)^3} \bigg\rfloor \xi^{2j_{\nu'}j_{\nu'}}.
\]

The proof of (8.284) is essentially the same as the proof of the estimate (8.255) and is left to the reader. The similarity in these proofs originates from the fact that

\[
\sum_{2^j > 2^j|\nu - \nu'|} P \mathcal{R}
\]

and

\[
\frac{2^{-j'}(N' - N)}{g(L, L')} \mathcal{Y}(P \leq 2|\nu-\nu'|\mathcal{R})
\]

satisfy the same estimates. For instance, in view of the finite band property, the identities (8.30) (8.31), and the estimate (8.32), we have:

\[
\sum_{2^j > 2^j|\nu - \nu'|} P \mathcal{R} \sim \frac{\mathcal{Y}(L(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}))}{2^j|\nu - \nu'|} \text{ and } 2^{-j'}(N' - N) \mathcal{Y}(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}) \sim \frac{\mathcal{Y}(L(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}))}{2^j|\nu - \nu'|}.
\]

Next, we estimate \(B_{j,\nu,\nu',l,m}^{1,2,2,2}\). Recall the definition (8.260) of \(B_{j,\nu,\nu',l,m}^{1,2,2,2}\):

\[
B_{j,\nu,\nu',l,m}^{1,2,2,2} = 2^{-2j} \int_{S^2 \times S^2} \frac{P \mathcal{R}(L(\mathcal{P} \leq 2|\nu-\nu'|\mathcal{R}))(b' - b)}{g(L, L')} F_{j,-1}(u) \eta'_{j}(\omega) F_{j,-1}(u') \eta'_{j}(\omega') d\omega d\omega'.
\]
Estimating the term $N'(L'(P_m \text{tr} \chi'))$ would involve commutator terms which are difficult to handle. To avoid this issue, we commute $L'$ with $N'$, and then integrate the $L'$ derivative by parts. We obtain schematically in view of the definition (8.260) of $B_{j,\nu',\nu',l,m}^{1,2,2,2}$:

\[
B_{j,\nu',\nu',l,m}^{1,2,2,2} = \frac{P \text{tr} \chi[N', L'(P_m \text{tr} \chi'))(b' - b)}{g(L, L')} F_{j, -1}(u) \eta_j^\nu(\omega) F_{j, -1}(u') \eta_j^\nu(\omega') \, d\omega \, d\omega' - 2 - 2j \int_{S^2 \times S^2} \left( \frac{\text{div}_g(L') - L'(g(L, L'))}{g(L, L')} \right) P \text{tr} \chi[N'(P_m \text{tr} \chi')(b' - b)} F_{j, -1}(u) \eta_j^\nu(\omega) F_{j, -1}(u') \eta_j^\nu(\omega') \, d\omega \, d\omega' - 2 - 2j \int_{S^2 \times S^2} \frac{P \text{tr} \chi N'(P_m \text{tr} \chi')(L'(b') - L'(b))}{g(L, L')} F_{j, -1}(u) \eta_j^\nu(\omega) F_{j, -1}(u') \eta_j^\nu(\omega') \, d\omega \, d\omega' - 2 - 2j \int_{S^2 \times S^2} b^{-1} P \text{tr} \chi N'(P_m \text{tr} \chi')(b' - b) F_j(u) \eta_j^\nu(\omega) F_{j, -1}(u') \eta_j^\nu(\omega') \, d\omega \, d\omega',
\]

where the last term in the right-hand side of (8.285) appears when the $L'$ derivative falls on the phase in view of (6.6).

We decompose $L'$ in the frame $L, N, e_A$:

\[
L' = L + (N' - g(N, N')N) + (g(N, N') - 1)N,
\]

which yields the following decompositions:

\[
L'(P \text{tr} \chi) = L(P \text{tr} \chi) + (N' - g(N, N')N)(P \text{tr} \chi) + (g(N, N') - 1)N(P \text{tr} \chi),
\]

and:

\[
L'(b) = L(b) + (N' - g(N, N')N)(b) + (g(N, N') - 1)N(b).
\]

Recall the identities (8.30) and (8.31):

\[
g(L, L') = -1 + g(N, N') \text{ and } 1 - g(N, N') = \frac{g(N - N', N - N')}{2}.
\]

We may thus expand

\[
\frac{1}{g(L, L')} \text{ and } \frac{1}{g(L, L')^2}
\]
in the same fashion than (8.33), and in view of (8.285), (8.287), (8.288), the formula (7.146) for $\text{div}_g(L')$ and the formula (7.149) for $L'(g(L, L'))$, we obtain, schematically:

\[
\sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2} = 2 - j \sum_{p, q \geq 0} c_{pq} \int_{M} \frac{1}{(2 \frac{1}{2}|N_\nu - N_{\nu'}|)^{p+q}} \left[ \frac{1}{(2 \frac{1}{2}|N_\nu - N_{\nu'}|)^2} (h_{1, p, q} + h_{2, p, q}) + \frac{1}{2 \frac{1}{2}(2 \frac{1}{2}|N_\nu - N_{\nu'}|)} h_{3, p, q} + 2 - j h_{4, p, q} \right] dM + \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,3}),
\]

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where the scalar functions $h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q}$ on $\mathcal{M}$ are given by:

\[
h_{1,p,q} = \left( \int_{S^2} G_1 (b - b_\nu)^r \left( 2^\frac{2}{3} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu (\omega) d\omega \right)
\times \left( \int_{S^2} N' (P_{\leq l} \text{tr} \chi)' (b_\nu - b')^s \left( 2^\frac{2}{3} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'} (\omega') d\omega' \right),
\]  

(8.290)

\[
h_{2,p,q} = \left( \int_{S^2} \nabla (b) P_l \text{tr} \chi \left( 2^\frac{2}{3} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu (\omega) d\omega \right)
\times \left( \int_{S^2} G_2 \left( 2^\frac{2}{3} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'} (\omega') d\omega' \right),
\]  

(8.291)

\[
h_{3,p,q} = \left( \int_{S^2} \nabla (b) P_l \text{tr} \chi \left( 2^\frac{2}{3} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu (\omega) d\omega \right)
\times \left( \int_{S^2} N' (P_{\leq l} \text{tr} \chi)' \left( 2^\frac{2}{3} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'} (\omega') d\omega' \right),
\]  

(8.292)

\[
h_{4,p,q} = \left( \int_{S^2} \nabla (b) P_l \text{tr} \chi \left( 2^\frac{2}{3} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu (\omega) d\omega \right)
\times \left( \int_{S^2} N' (P_{\leq l} \text{tr} \chi)' \left( 2^\frac{2}{3} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'} (\omega') d\omega' \right),
\]  

(8.293)

where the integer $r, s$ satisfy:

\[ r + s = 1, \]

and:

\[
G_1 = L (P_l \text{tr} \chi) + (\bar{\delta} + \chi + \zeta + L (b)) P_l \text{tr} \chi,
\]  

(8.294)

\[
G_2 = [N', L'] (P_{\leq l} \text{tr} \chi') + (\bar{\delta}' + L'(b')) N' (P_{\leq l} \text{tr} \chi'),
\]  

(8.295)

where $B_{j_l,\nu',l,m}^{1,2,2,1}, B_{j_l,\nu',l,m}^{1,2,2,2}, B_{j_l,\nu',l,m}^{1,2,2,3}$ are given by:

\[
B_{j_l,\nu',l,m}^{1,2,2,1} = -2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{(N' - \mathbf{g}(N, N')) (P_l \text{tr} \chi) N' (P_m \text{tr} \chi')(b' - b)}{\mathbf{g}(L, L')}
\times F_{j,-1}(u) \eta_j^{\nu'} (\omega) F_{j,-1}(u') \eta_j^{\nu'} (\omega') d\omega d\omega' d\mathcal{M},
\]  

(8.296)

\[
B_{j_l,\nu',l,m}^{1,2,2,2} = -2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} N (P_l \text{tr} \chi) N' (P_m \text{tr} \chi')(b' - b) F_{j,-1}(u) \eta_j^{\nu'} (\omega) F_{j,-1}(u') \eta_j^{\nu'} (\omega') d\omega d\omega' d\mathcal{M},
\]  

(8.297)
\[ B_{j,p,q}^{1,2,2,2,3} \]

\[ = -i2^{-j} \int_{S^2} \int_{S^2} b^{-1} P_{tr\chi}N'(P_{m\tr\chi}'(b' - b)F_j(u)\eta_j^\nu(\omega)F_{j,-1}(u')\eta_{j'}^\nu(\omega')d\omega d\omega'd\mathcal{M}, \]

and where \( c_{pq} \) are explicit real coefficients such that the series

\[ \sum_{p,q \geq 0} c_{pq} p^q y^q \]

has radius of convergence 1. Note that the term \( b - b' \) present in all terms of (8.285) is present in the definition of \( h_{1,p,q} \), but absent from the definition of \( h_{2,p,q}, h_{3,p,q}, h_{4,p,q} \). Indeed, we do not need to exploit the gain \( b - b' \) in \( h_{2,p,q}, h_{3,p,q}, h_{4,p,q} \), and we just separate \( b \) and \( b' \) and estimate them in \( L^\infty \) using the estimate (2.38) for \( b \). To simplify the notations, we chose not to specify these factors of \( b \) and \( b' \).

Next, we estimate the \( L^1(M) \) norm of \( h_{1,p,q}, h_{2,p,q}, h_{3,p,q}, h_{4,p,q} \) starting with \( h_{1,p,q} \). We have:

\[ \|h_{1,p,q}\|_{L^1(M)} \lesssim \left\| \int_{S^2} G_1(b - b_v)^\nu \left( 2^{\frac{j}{2}}(N - N_v) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) \right\|_{L^2(M)} \]

and

\[ \left\| \int_{S^2} N'(P_{\leq \tr\chi}'(b_v - b')^\nu \left( 2^{\frac{j}{2}}(N' - N_v') \right)^q F_{j,-1}(u')\eta_j^\nu(\omega') \right\|_{L^2(M)}. \]

We estimate the second term in the right-hand side of (8.299). Using the basic estimate in \( L^2(M) \) (7.1), we have:

\[ \left\| \int_{S^2} N'(P_{\leq \tr\chi}'(b_v - b')^\nu \left( 2^{\frac{j}{2}}(N' - N_v') \right)^q F_{j,-1}(u')\eta_j^\nu(\omega') \right\|_{L^2(M)} \]

\[ \lesssim \left( \sup_{\omega} \left\| N'(P_{\leq \tr\chi}'(b_v - b')^\nu \left( 2^{\frac{j}{2}}(N' - N_v') \right)^q \right\|_{L^\infty_{\omega}L^2(R_{\nu'})} \right) 2^{\frac{j}{2}}\gamma_j^\nu \]

where we used in the last inequality the estimate (2.44) for \( \partial_v b \), the estimate (2.42) for \( \partial_v N \) and the size of the patch. Together with the estimate (2.39) for \( \tr\chi \) and the commutator estimate (2.64), we obtain:

\[ \left\| \int_{S^2} N'(P_{\leq \tr\chi}'(b_v - b')^\nu \left( 2^{\frac{j}{2}}(N' - N_v') \right)^q F_{j,-1}(u')\eta_j^\nu(\omega') \right\|_{L^2(M)} \lesssim \epsilon |\nu - \nu'|^{s2^{\frac{j}{2}}\gamma_j^\nu}. \]

Next, we estimate the first term in the right-hand side of (8.299). Using the basic estimate in \( L^2(M) \) (7.1), we have:

\[ \left\| \int_{S^2} G_1(b - b_v)^\nu \left( 2^{\frac{j}{2}}(N - N_v) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega) \right\|_{L^2(M)} \]

\[ \lesssim \left( \sup_{\omega} \left\| G_1(b - b_v)^\nu \left( 2^{\frac{j}{2}}(N - N_v) \right)^p \right\|_{L^\infty_{\omega}L^2(R_{\nu})} \right) 2^{\frac{j}{2}}\gamma_j^\nu \]

\[ \lesssim \left( \sup_{\omega} \|G_1\|_{L^\infty_{\omega}L^2(R_{\nu})} \right) 2^{-\frac{j}{2}}2^{\frac{j}{2}}\gamma_j^\nu, \]

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where we used in the last inequality the estimate (2.44) for \( \partial \omega b \), the estimate (2.42) for \( \partial \omega N \) and the size of the patch. In view of the definition of \( G_1 (8.294) \), we have:
\[
\| G_1 \|_{L^\infty_t L^2_x (H_u)} \lesssim \left( \| \delta \|_{L^\infty_t L^4_x} + \| \chi \|_{L^\infty_t L^4_x} + \| \xi \|_{L^\infty_t L^4_x} + \| L(b) \|_{L^\infty_t L^4_x} \right) \| P \text{tr} \chi \|_{L^2_t L^4_x} \\
\quad + \| L(P \text{tr} \chi) \|_{L^\infty_t L^2_x (H_u)} \lesssim \varepsilon \| P \text{tr} \chi \|_{L^2_t L^4_x} + \| L(P \text{tr} \chi) \|_{L^\infty_t L^2_x (H_u)},
\]
(8.302)
where we used in the last inequality the embedding (2.51), and the estimates (2.36) (2.37) for \( \delta \), the estimates (2.39) (2.40) for \( \chi \), the estimate (2.38) for \( b \), and the estimate (2.41) for \( \zeta \). If \( l > j/2 \), we have the analog of (8.139):
\[
\| P_l \text{tr} \chi \|_{L^2_t L^4_x} \lesssim 2^{-\frac{j}{2}} \varepsilon,
\]
(8.303)
while in the case \( l = j/2 \), the boundedness of \( P_{\leq j/2} \) on \( L^4(P_{t,u}) \) and the estimate (2.39) for \( \text{tr} \chi \) yields:
\[
\| P_{\leq j/2} \text{tr} \chi \|_{L^2_t L^4_x} \lesssim \varepsilon.
\]
(8.304)
In view of (8.302), we also need to estimate \( L(P \text{tr} \chi) \). In the case \( l > j/2 \), the estimate (8.142) yields:
\[
\| L(P \text{tr} \chi) \|_{L^\infty_t L^2_x (H_u)} \lesssim 2^{-\frac{j}{2}} \varepsilon,
\]
(8.305)
which together with the estimate (2.39) for \( \text{tr} \chi \) and the decomposition:
\[
P_{\leq j/2} \text{tr} \chi = \text{tr} \chi - \sum_{l > \frac{j}{2}} P_l \text{tr} \chi
\]
implies in the case \( l = j/2 \):
\[
\| L(P_{\leq j/2} \text{tr} \chi) \|_{L^\infty_t L^2_x (H_u)} \lesssim \varepsilon.
\]
(8.306)
Now, in view of (8.302), (8.303), (8.304), (8.305) and (8.306), we obtain:
\[
\| G_1 \|_{L^\infty_t L^2_x (H_u)} \lesssim \varepsilon 2^{-l(1-\delta_{l,j/2})},
\]
where we defined:
\[
\delta_{l,j/2} = 1 \text{ if } l = j/2 \text{ and } \delta_{l,j/2} = 0 \text{ otherwise.}
\]
Together with (8.301), we deduce:
\[
\left\| \int_{S^2} G_1(b - b_0)^r \left( 2^j(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^p(\omega) d\omega \right\|_{L^2(M)} \lesssim \varepsilon 2^{-\frac{j}{2}} 2^{\frac{j}{2} 2^{-l(1-\delta_{l,j/2})}} \gamma_j \gamma_j',
\]
(8.307)
Finally, (8.299), (8.300) and (8.307) imply:
\[
\| h_{1,p,q} \|_{L^1(M)} \lesssim \varepsilon^2 |\nu - \nu'| s 2^{-\frac{j}{2}} 2^{j-l(1-\delta_{l,j/2})} \gamma_j \gamma_j'.
\]
Since we have:
\[
2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1 \text{ and } r + s = 1,
\]
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this yields:
\[
\|h_{1,p,q}\|_{L^1(M)} \lesssim \varepsilon^2 |\nu - \nu'| 2^{j - l(1 - \delta_i \beta / 2)} \gamma_j \gamma_j'.
\] (8.308)

Next, we estimate the \(L^1(M)\) norm of \(h_{2,p,q}\). In view of the definition (8.291) of \(h_{2,p,q}\), we have:
\[
\|h_{2,p,q}\|_{L^2(M)} \lesssim \left\| \int_{S^2} P \text{tr}_\chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \quad (8.309)
\]
\[
\times \left\| \int_{S^2} G_2 \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)}.
\]
Recall (8.272):
\[
\left\| \int_{S^2} P \text{tr}_\chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^\infty(M)} \lesssim \varepsilon 2^j \gamma_j'.
\] (8.310)

On the other hand, (8.179) and (8.180) imply:
\[
\left\| \int_{S^2} P \text{tr}_\chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim (1 + p^2) 2^{\frac{j}{2} - \frac{1}{2}} \varepsilon \gamma_j'.
\] (8.311)
Interpolating between (8.310) and (8.311), we obtain:
\[
\left\| \int_{S^2} P \text{tr}_\chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^3(M)} \lesssim \varepsilon 2^{\frac{j}{2} + 2 - \frac{2}{3}} \gamma_j'.
\] (8.312)

Next, we estimate the second term in the right-hand side of (8.309). We have:
\[
\left\| \int_{S^2} G_2 \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^\frac{3}{2}(M)} \quad (8.313)
\]
\[
\lesssim \int_{S^2} \| G_2 \|_{L^\frac{3}{2}(H_u')} \left\| \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \| F_{j-1}(u') \|_{L^\frac{3}{2}(H_u')} \eta_j^{\nu'}(\omega') d\omega'
\lesssim \int_{S^2} \| G_2 \|_{L^\infty L^\frac{3}{2}(H_u')} \| F_{j-1}(u') \|_{L^\frac{3}{2}(H_u')} \eta_j^{\nu'}(\omega') d\omega',
\]
where we used in the last inequality the estimate (2.42) for \(\partial_u N\) and the size of the patch. Next, we estimate \(G_2\). In view of the definition of \(G_2\) (8.295), the commutator formulas (2.32) for \([L, \bar{L}]\) and the fact that \(2N = L - \bar{L}\), we have schematically:
\[
G_2 = n^{-1} \nabla_{N' n} L' (P_{\leq} \text{tr}_\chi') + (\zeta' - \zeta') \cdot \nabla'(P_{\leq} \text{tr}_\chi') + (\bar{\delta}' + L'(b')) N'(P_{\leq} \text{tr}_\chi').
\]
This yields:
\[
\| G_2 \|_{L^\infty L^\frac{3}{2}(H_u')} \lesssim \left( \| \nabla_{N' n} L' (P_{\leq} \text{tr}_\chi') \|_{L^\infty L^\frac{3}{2}(H_u')} + \| \zeta' \|_{L^\infty L^\frac{3}{2}(H_u')} + \| \bar{\delta}' + L'(b') \|_{L^\infty L^\frac{3}{2}(H_u')} \right) \| \text{tr}_\chi' \|_{L^\infty L^2(H_u')}.
\]
\[
\lesssim \varepsilon \| \text{tr}_\chi' \|_{L^\infty L^2(H_u')}.
\]
where we used in the last inequality the Sobolev embedding (2.50), and the estimates (2.37) (2.36) for $n$, $\delta$ and $\zeta$, the estimate (2.38) for $b$, and the estimate (2.41) for $\zeta$. Together with the basic properties of $P_{\leq t}$, the commutator estimates (2.64) and (2.65), and the estimate (2.39) for $\text{tr} \chi$, this implies:

$$\|G_2\|_{L^2_{\omega}L^{2/3}_{\omega}(H_u)} \lesssim \varepsilon.$$ 

Together with (8.313), we obtain:

$$\left\| \int_{S^2} G_2 \left(2^{j} (N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon \left( \int_{S^2} \|F_{j,-1}(u)\|_{L^2} \eta_j'(\omega) d\omega \right) \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j'',$n

where we used in the last inequality Plancherel in $\lambda$ for $\|F_{j,-1}(u')\|_{L^2_{\omega'}}$, Cauchy Schwarz in $\omega'$ and the size of the patch. Finally, (8.309), (8.312) and (8.314) imply:

$$\|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim 2^{\frac{j}{2}} 2^{-\frac{j}{2}} \varepsilon^2 \gamma_j \gamma_{j'}.$$ 

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{3,p,q}$. In view of the definition of $h_{3,p,q}$ (8.292), we have:

$$\|h_{3,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} \nabla(b) P_{t} \text{tr} \chi \left(2^{j} (N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon \left( \int_{S^2} \|N'(P_{\leq t} \text{tr} \chi') \left(2^{j} (N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.$$ 

Arguing as for the proof of (8.316), we have:

$$\left\| \int_{S^2} N'(P_{\leq t} \text{tr} \chi') \left(2^{j} (N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon^2 \gamma_j'.$$ 

Also, using the basic estimate in $L^2(\mathcal{M})$ (7.1), we have:

$$\left\| \int_{S^2} \nabla(b) P_{t} \text{tr} \chi \left(2^{j} (N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \left\| \nabla(b) P_{t} \text{tr} \chi \left(2^{j} (N - N_{\nu})\right)^p \right\|_{L^\infty_{\omega}L^2_{\omega}(H_u)} \right) 2^{\frac{j}{2}} \gamma_j' \lesssim \left( \sup_{\omega} \|\nabla(b)\|_{L^\infty_{\omega}L^2_{\omega}} \|P_{t} \text{tr} \chi\|_{L^2_{\omega}L^2_{\omega}} \right) 2^{\frac{j}{2}} \gamma_j',$n

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the estimate (2.38) for $b$, the embedding (2.51), and the estimates (8.303) and (8.304) for $P_{t} \text{tr} \chi$, we obtain:

$$\left\| \int_{S^2} \nabla(b) P_{t} \text{tr} \chi \left(2^{j} (N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{j}{2}(1-\delta_j/2)} 2^{\frac{j}{2}} \gamma_j'.$$
Finally, (8.316), (8.317) and (8.318) imply:

$$\|h_{3,p,q}\|_{L^1(\mathcal{M})} \lesssim 2^{-\frac{1}{2}(1-\delta/2,l)}2^j\gamma_j\gamma_{j'}.$$  (8.319)

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{4,p,q}$. In view of the definition of $h_{4,p,q}$ (8.293), we have:

$$\|h_{4,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} N(b)P_{tr\chi} \left(2^\frac{j}{2}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})}$$

$$\times \left\| \int_{\mathbb{S}^2} N'(P_{\leq l}tr\chi') \left(2^\frac{j}{2}(N' - N_\nu)\right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega')d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \varepsilon 2^\frac{j}{2}\gamma_j\gamma_{j'} \left\| \int_{\mathbb{S}^2} N(b)P_{tr\chi} \left(2^\frac{j}{2}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})},$$

where we used in the last inequality the estimate (8.317). In view of the estimate (2.38) for $b$, we have:

$$\|N(b)\|_{L^\infty_t L^4_{x,t}} \lesssim \varepsilon.$$

Thus, arguing as for the proof of (8.318), we obtain:

$$\left\| \int_{\mathbb{S}^2} N(b)P_{tr\chi} \left(2^\frac{j}{2}(N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{1}{2}(1-\delta/2,l)}2^j\gamma_j\gamma_{j'}.$$  (8.321)

Together with (8.320), this yields:

$$\|h_{4,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{-\frac{1}{2}(1-\delta/2,l)}2^j\gamma_j\gamma_{j'}.$$  (8.320)

Now, in view of the decomposition (8.289) of $B_{j,\nu,\nu',l,m}^{1,2,2,2}$, we have:

$$\left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,2,3}) \right|$$

$$\lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^\frac{j}{2}|N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(\mathcal{M})} \left\| \frac{1}{(2^\frac{j}{2}|N_\nu - N_{\nu'}|)^{2}} \right\|_{L^\infty(\mathcal{M})} (\|h_{1,p,q}\|_{L^1(\mathcal{M})})$$

$$+ \|h_{2,p,q}\|_{L^1(\mathcal{M})}) + \left\| \frac{1}{2^\frac{j}{2}(2^\frac{j}{2}|N_\nu - N_{\nu'}|)} \right\|_{L^\infty(\mathcal{M})} \|h_{3,p,q}\|_{L^1(\mathcal{M})} + 2^{-j} \|h_{4,p,q}\|_{L^1(\mathcal{M})}.$$
which together with (8.32), (8.308), (8.315), (8.319) and (8.321) yields:

\[ \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,2,3}) \]

\[ \lesssim 2^{-j} \sum_{p,q \geq 0} c_{pq} \left( \frac{2^j |\nu - \nu'|}{2^j |\nu - \nu'|} \right)^{p+q} \left( \frac{1}{(2^j |\nu - \nu'|)^2} \right) \left( |\nu - \nu'| 2^j - l(1-\delta_j/\gamma) + 2^j \gamma_j^{\nu'} \right) \\
+ \frac{1}{2^j (2^j |\nu - \nu'|)} 2^{-\frac{j}{2}(1-\delta_j/\gamma) 2^j + 2^{-j} 2^{-\frac{j}{2}(1-\delta_j/\gamma) 2^j} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu''}} \]

Summing in \( l \), we obtain:

\[ \sum_{(l,m)/2^m \leq 2^l \leq 2^{l+1} |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,2,2} - \sum_{(l,m)/2^m \leq 2^l \leq 2^{l+1} |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2,2,3}) \]

\[ \lesssim \left[ \frac{1}{2^j (2^j |\nu - \nu'|)} + \frac{1}{2^j (2^j |\nu - \nu'|)} 2^{-\frac{j}{2}(1-\delta_j/\gamma) 2^j + 2^{-j} 2^{-\frac{j}{2}(1-\delta_j/\gamma) 2^j} \varepsilon^2 \gamma_j^{\nu'} \gamma_j^{\nu''}}. \right] \]

In view of (8.322), we need to estimate \( B_{j,\nu,\nu',l,m}^{1,2,2,2,1}, B_{j,\nu,\nu',l,m}^{1,2,2,2,2}, B_{j,\nu,\nu',l,m}^{1,2,2,2,3} \). We start with \( B_{j,\nu,\nu',l,m}^{1,2,2,2,3} \) which is defined in (8.298) as:

\[ B_{j,\nu,\nu',l,m}^{1,2,2,2,3} = -i2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} b^{-1} P_{tr \chi N'}(P_m tr \chi')(b' - b) F_j(u) \eta_j' (\omega) F_{j-1}(u') \eta_j^{\nu'} (\omega') d\omega d\omega' d\mathcal{M}. \]

We integrate by parts tangentially using (7.137).

**Lemma 8.14** Let \( B_{j,\nu,\nu',l,m}^{1,2,2,2,3} \) defined in (8.298). Integrating by parts using (7.137) yields:

\[ \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,2,2,3} \]

\[ = 2^{-j} \sum_{p,q \geq 0} c_{pq} \left( \frac{1}{2^j |N_\nu - N_{\nu'}|} \right)^{p+q} \left( \frac{1}{(2^j |N_\nu - N_{\nu'}|)^2} \right) \left( h_{1,\nu,p,q}^{l} + h_{2,\nu,p,q}^{l} \right) \\
+ \frac{1}{2^j (2^j |N_\nu - N_{\nu'}|)} \left( h_{3,\nu,p,q}^{l} + h_{5,\nu,p,q}^{l} \right) + 2^{-j} h_{4,\nu,p,q}^{l} \right] d\mathcal{M} + \sum_{m/m \leq l} (B_{j,\nu,\nu',l,m}^{1,2,2,2,1} + B_{j,\nu,\nu',l,m}^{1,2,2,2,2}), \]

where the scalar functions \( h_{3,\nu,p,q}, h_{4,\nu,p,q} \) are given respectively by (8.292) and (8.293), where
the scalar functions \( h'_{1,p,q}, h'_{2,p,q}, h'_{5,p,q} \) on \( M \) are given by:

\[
\begin{align*}
    h'_{1,p,q} &= \left( \int_{S^2} G'_1(b - b_0)^r \left( 2^z (N - N_0) \right)^p F_{j-1}(u) \eta_j^r(\omega) d\omega \right) \\
    & \times \left( \int_{S^2} N'(P_{\leq 1} \text{tr} \chi') (b_0 - b')^s \left( 2^z (N' - N_0) \right)^q F_{j-1}(u') \eta_j^q(\omega') d\omega' \right), \\
    h'_{2,p,q} &= \left( \int_{S^2} P_l \text{tr} \chi \left( 2^z (N - N_0) \right)^p F_{j-1}(u) \eta_j^r(\omega) d\omega \right) \\
    & \times \left( \int_{S^2} G'_2 \left( 2^z (N' - N_0) \right)^q F_{j-1}(u') \eta_j^q(\omega') d\omega' \right), \\
    h'_{5,p,q} &= \left( \int_{S^2} P_l \text{tr} \chi \left( 2^z (N - N_0) \right)^p F_{j-1}(u) \eta_j^r(\omega) d\omega \right) \\
    & \times \left( \int_{S^2} G'_3 \left( 2^z (N' - N_0) \right)^q F_{j-1}(u') \eta_j^q(\omega') d\omega' \right),
\end{align*}
\]

where the integer \( r, s \) satisfy:

\[ r + s = 1, \]

where the tensors \( G'_1, G'_2 \) and \( G'_3 \) are schematically given by:

\[
\begin{align*}
    G'_1 &= (\chi + \theta) P_l \text{tr} \chi, \\
    G'_2 &= (\chi' + L'(b') + \theta' + \nabla'(b)) N'(P_{\leq 1} \text{tr} \chi'), \\
    G'_3 &= \nabla' N'(P_{\leq 1} \text{tr} \chi'),
\end{align*}
\]

and:

\[
\sum_{p,q\geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1.

The proof of Lemma 8.14 is postponed to Appendix D. We now use this lemma to estimate \( B_{j,\nu,\nu',\ell,\delta_m}^{1,2,2,2,1}, B_{j,\nu,\nu',\ell,\delta_m}^{1,2,2,2,2} \).

We estimate the \( L^1(M) \) norm of \( h'_{1,p,q}, h'_{2,p,q}, h'_{5,p,q} \). The estimate of \( h'_{1,p,q} \) is completely analogous to the one of \( h_{1,p,q} \) defined in (8.290). Thus, we obtain in view of (8.308):

\[
\| h'_{1,p,q} \|_{L^1(M)} \lesssim \varepsilon^2 |\nu - \nu'| 2^{j-l} (1-\delta_{j,l/2}) \gamma_j^{\nu_0} \gamma_j^{\nu'}. \]

(8.330)

Also, the estimate of \( h'_{2,p,q} \) is completely analogous to the one of \( h_{2,p,q} \) defined in (8.291). Thus, we obtain in view of (8.315):

\[
\| h'_{2,p,q} \|_{L^1(M)} \lesssim 2^{\frac{j_2}{2}} 2^{-\frac{2l}{3}} \varepsilon^2 \gamma_j^{\nu_0} \gamma_j^{\nu'}. \]

(8.331)
Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{5,p,q}'$. In view of (8.326), we have:

$$
\left\| h_{5,p,q}' \right\|_{L^1(\mathcal{M})} \lesssim \int_{S^2} P_t \text{tr} \chi \left( 2^s (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \times \int_{S^2} G_3' \left( 2^s (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega'
\lesssim \int_{S^2} \left\| P_t \text{tr} \chi \left( 2^s (N - N_\nu) \right)^p F_{j,-1}(u) \right\|_{L^2(\mathcal{M})} \eta_j'(\omega) d\omega \times \int_{S^2} \left\| G_3' \left( 2^s (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^2(\mathcal{M})} \eta_j'(\omega') d\omega'
\lesssim \left( \int_{S^2} \left\| P_t \text{tr} \chi F_{j,-1}(u) \right\|_{L^2(\mathcal{M})} \eta_j'(\omega) d\omega \right) \left( \int_{S^2} \left\| G_3' F_{j,-1}(u') \right\|_{L^2(\mathcal{M})} \eta_j'(\omega') d\omega' \right),
$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with the definition of $G_3'$ (8.329), this yields:

$$
\left\| h_{5,p,q}' \right\|_{L^1(\mathcal{M})} \lesssim \sum_{m/m \leq l} \left( \int_{S^2} \left\| P_t \text{tr} \chi \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u) \right\|_{L^2_u} \eta_j'(\omega) d\omega \right) \times \left( \int_{S^2} \left\| \nabla' N'(P_m \text{tr} \chi') \right\|_{L^2(\mathcal{H}_u)} F_{j,-1}(u') \right\|_{L^2_{u'}} \eta_j'(\omega') d\omega' \right).
$$

Next, we estimate $\nabla' N'(P_m \text{tr} \chi')$. Using the estimate (2.38) for $b'$, we have:

$$
\left\| \nabla' N'(P_m \text{tr} \chi') \right\|_{L^2(\mathcal{H}_u)} \lesssim \left\| b'^{-1} \nabla'(b') N'(P_m \text{tr} \chi') \right\|_{L^2(\mathcal{H}_u)} + \left\| b'^{-1} \nabla'(b' N'(P_m \text{tr} \chi')) \right\|_{L^2(\mathcal{H}_u)}
\lesssim \left\| b'^{-1} \nabla'(b') \right\|_{L^2_u L^2_{u'}} \left\| b N'(P_m \text{tr} \chi') \right\|_{L^2_u L^2_{u'}} + \left\| \nabla' P_m (b' N' \text{tr} \chi') \right\|_{L^2(\mathcal{H}_u)}
+ \left\| \nabla [b' N', P_m] \text{tr} \chi' \right\|_{L^\infty_u L^2(\mathcal{H}_u)}
\lesssim \left\| P_m (b' N' \text{tr} \chi') \right\|_{L^2_u L^2_{u'}} + \left\| [b N', P_m] \text{tr} \chi' \right\|_{L^2_u L^2_{u'}} + \left\| \nabla' P_m (b' N' \text{tr} \chi') \right\|_{L^2(\mathcal{H}_u)}
+ \left\| \nabla [b' N', P_m] \text{tr} \chi' \right\|_{L^\infty_u L^2(\mathcal{H}_u)},
$$

where we used in the last inequality the last property of $P_m$ and the commutator estimate (2.68), we obtain:

$$
\left\| \nabla' N'(P_m \text{tr} \chi') \right\|_{L^2(\mathcal{H}_u)} \lesssim 2^m \left\| P_m (b' N' \text{tr} \chi') \right\|_{L^2(\mathcal{H}_u)} + 2^m \varepsilon. \quad (8.333)
$$
Together with (8.332), this yields:

$$
\|h'_{5,p,q}\|_{L^1(\mathcal{M})} \lesssim \sum_{m/m \leq l} 2^{-|l-m|} \left( \int_{S^2} 2^l \| P_l \text{tr} \chi \|_{L^2(\mathcal{H}_u)} F_{j-1}(u) \| L^2_u \eta_j'(\omega) d\omega \right)
$$

(8.334)

\[ \times \left( \int_{S^2} \| (P_m(b'N'\text{tr} \chi'))\|_{L^2(\mathcal{H}_{u'})} + 2^{-\frac{m}{2}} \varepsilon F_{j-1}(u') \| L^2_u \eta_j'(\omega') d\omega' \right)
\]

$$
\lesssim 2^{-j} \sum_{m/m \leq l} 2^{-|l-m|} \left( \int_{S^2} 2^l \| P_l \text{tr} \chi \|_{L^2(\mathcal{H}_u)} F_{j-1}(u) \sqrt{\eta_j'(\omega)} \| L^2_u \eta_j'(\omega') d\omega' \right)
$$

where we used the finite band property for \( \omega \) and \( \omega' \), and the size of the patch. Now, we have:

$$
\sum_{l} \left( \int_{S^2} 2^l \| P_l \text{tr} \chi \|_{L^2(\mathcal{H}_u)} F_{j-1}(u) \sqrt{\eta_j'(\omega)} \| L^2_u \eta_j'(\omega') d\omega \right)
$$

= \int_{S^2} \left( \int_{u'} \left( \sum_{l} 2^l \| P_l \text{tr} \chi \|_{L^2(\mathcal{H}_u)} \right) |F_{j-1}(u) |^2 du \right) \eta_j'(\omega) d\omega
$$

\[ \lesssim \int_{S^2} \| \nabla \text{tr} \chi \|_{L^2(\mathcal{H}_u)}^{2} |F_j(u)|^{2} \| L^2_u \eta_j'(\omega) d\omega
\]

\[ \lesssim \varepsilon^{2} 2^{2j} (\gamma_j')^2,
\]

where we used the finite band property for \( P_{1} \), the estimates (2.39) for \( \text{tr} \chi \) and Plancherel in \( \lambda \). Also, we have:

$$
\sum_{m} \left( \| (P_m(b'N'\text{tr} \chi'))\|_{L^2(\mathcal{H}_{u'})} + 2^{-\frac{m}{2}} \varepsilon F_{j-1}(u') \| L^2_u \eta_j'(\omega') d\omega' \right)
$$

\[ \lesssim \int_{S^2} \left( \sum_{m} \left( \| P_m(b'N'\text{tr} \chi')\|_{L^2(\mathcal{H}_{u'})} + 2^{-\frac{m}{2}} \varepsilon \right)^2 |F_{j-1}(u') |^2 du \right) \eta_j'(\omega') d\omega'
\]

\[ \lesssim \varepsilon^{2} 2^{2j} (\gamma_j')^2,
\]

where we used the finite band property for \( P_{m} \), the estimates (2.39) for \( \text{tr} \chi' \) and Plancherel in \( \lambda' \). Finally, (8.334), (8.335) and (8.336) yield:

$$
\sum_{(l,m)/m \leq l} \| h'_{5,p,q} \|_{L^1(\mathcal{M})}
$$

(8.337)

\[ \lesssim 2^{-j} \left( \sum_{l} \left( \int_{S^2} 2^l \| P_l \text{tr} \chi \|_{L^2(\mathcal{H}_u)} F_{j-1}(u) \sqrt{\eta_j'(\omega)} \| L^2_u \eta_j'(\omega') d\omega' \right) \right)^{\frac{1}{2}}
\]

\[ \times \left( \sum_{m} \left( \| (P_m(b'N'\text{tr} \chi'))\|_{L^2(\mathcal{H}_{u'})} + 2^{-\frac{m}{2}} \varepsilon F_{j-1}(u') \| L^2_u \eta_j'(\omega') d\omega' \right) \right)^{\frac{1}{2}}
\]

\[ \lesssim \varepsilon^{2} 2^{2j} \gamma_j' \gamma_{j'}.
\]
Now, in view of the decomposition (8.323) of $B_{j,v,v',l,m}^{1,2,2,2,3}$ we have:

\[
\sum_{m/m \leq l} B_{j,v,v',l,m}^{1,2,2,2,3} - \sum_{m/m \leq l} (B_{j,v,v',l,m}^{1,2,2,2,1} + B_{j,v,v',l,m}^{1,2,2,2,2}) \leq 2^{-j} \sum_{p,q \geq 0} \left[ \left( \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^{p+q}} \left( \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^2} (|\nu - \nu'|2^{j-l(1-\delta_{l,j/2})} + 2^{\frac{7j}{2}} 2^{-\frac{j}{4}}) \right) 
\right. \\
+ \left. \frac{1}{2^\frac{j}{2} (|\nu - \nu'|^2) 2^{j} + 2^{-j} 2^{\frac{j}{2}} (1-\delta_{l,j/2}) 2^{j}} \right] \varepsilon^{2j} \gamma_{j}^{\nu} \gamma_{j}^{\nu'} + 2^{-j} \frac{1}{2^\frac{j}{2} (|\nu - \nu'|)^{p+q+1}} \left( \sum_{m/m \leq l} \left( \sum_{l,m} (h_{5,p,q}^{l}) \right) \right)
\]

which together with (8.32), (8.319), (8.321), (8.330) and (8.331) yields:

\[
\sum_{m/m \leq l} B_{j,v,v',l,m}^{1,2,2,2,3} - \sum_{m/m \leq l} (B_{j,v,v',l,m}^{1,2,2,2,1} + B_{j,v,v',l,m}^{1,2,2,2,2}) \leq 2^{-j} \sum_{p,q \geq 0} \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^{p+q+1}} \left( \sum_{l,m} \left( \sum_{m/m \leq l} \left( \sum_{l,m} (h_{5,p,q}^{l}) \right) \right) \right)
\]

Summing in $l$, we obtain:

\[
\sum_{l,m} B_{j,v,v',l,m}^{1,2,2,2,3} - \sum_{l,m} (B_{j,v,v',l,m}^{1,2,2,2,1} + B_{j,v,v',l,m}^{1,2,2,2,2}) \leq 2^{-j} \sum_{p,q \geq 0} \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^{p+q+1}} \left( \sum_{l,m} \left( \sum_{m/m \leq l} \left( \sum_{l,m} (h_{5,p,q}^{l}) \right) \right) \right)
\]
Together with (8.337), we get:

\[
\begin{align*}
\lambda \left[ \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu,l,m}^{1,2,2,3} - \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu,l,m}^{1,2,2,1} + B_{j,\nu,\nu,l,m}^{1,2,2,2}) \right] \\
\lambda \left[ \frac{1}{2^j (2^j |\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^j |\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'} \\
+ 2^{-\frac{3}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^j |\nu - \nu'|)^{p+q+1}} \varepsilon^2 2^j \gamma_j^\nu \gamma_j^{\nu'} \\
\lambda \left[ \frac{1}{2^j (2^j |\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^j |\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. 
\end{align*}
\]

(8.338)

Finally, (8.322) and (8.338) imply:

\[
\begin{align*}
\lambda \left[ \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu,l,m}^{1,2,2} - \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu,l,m}^{1,2,2,1} + B_{j,\nu,\nu,l,m}^{1,2,2,2}) \right] \\
\lambda \left[ \frac{1}{2^j (2^j |\nu - \nu'|)} + \frac{2^{-\frac{j}{6}}}{(2^j |\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j^\nu \gamma_j^{\nu'}. 
\end{align*}
\]

(8.339)

In view of (8.339), we still need to estimate \(B_{j,\nu,\nu,l,m}^{1,2,2,1}\) and \(B_{j,\nu,\nu,l,m}^{1,2,2,2}\). We start with \(B_{j,\nu,\nu,l,m}^{1,2,2,1}\) which is defined by (8.296) as:

\[
B_{j,\nu,\nu,l,m}^{1,2,2,1} = -2^{-2j} \int_M \int_{S^2 \times S^2} \frac{(N' - g(N,N')(P_{tr\chi}N')(P_{mtr\chi})(b' - b)}{g(L,L')} \\
\times F_{j-1}(u)|\eta_j^\nu(\omega)F_{j-1}(u')|\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M},
\]

We integrate by parts tangentially using (7.137).

**Lemma 8.15** Let \(B_{j,\nu,\nu,l,m}^{1,2,2,1}\) defined in (8.296). Integrating by parts using (7.137) yields:

\[
\sum_{m/m \leq \ell} B_{j,\nu,\nu,l,m}^{1,2,2,1} = \frac{3}{2}\sum_{p,q \geq 0} c_{pq} \int_M \frac{1}{(2^j |N_\nu - N_{\nu'}|)^{p+q+1}} \left[ \frac{1}{(2^j |N_\nu - N_{\nu'}|)^2} (h_{1,p,q}'' + h_{2,p,q}'') + 2^{-j} h_{3,p,q}'' + 2^{-j} h_{4,p,q}'' + 2^{-j} h_{5,p,q}'' \right] d\mathcal{M},
\]

where the scalar functions \(h_{1,p,q}'', h_{2,p,q}'', h_{3,p,q}'', h_{4,p,q}'', h_{5,p,q}''\) on \(\mathcal{M}\) are given by:

\[
h_{1,p,q}'' = \left( \int_{S^2} C_1'' \left( 2^j (N - N_\nu) \right)^p F_{j-1}(u)|\eta_j^\nu(\omega)d\omega \right)
\times \left( \int_{S^2} N'(P_{tr\chi})(2^j (N - N_{\nu'}))^{q} F_{j-1}(u')|\eta_j^{\nu'}(\omega')d\omega' \right),
\]

(8.341)
\[ h_{2,p,q}^{''} = \left( \int_{S^2} \nabla(P_{l}\chi) \left( 2^\frac{j}{2}(N - N_\nu) \right)^p F_{j-1,u}(\omega)d\omega \right) \times \left( \int_{S^2} G_2'' \left( 2^\frac{j}{2}(N' - N_{\nu'}) \right)^q F_{j-1,u}(\omega)d\omega' \right), \tag{8.342} \]

\[ h_{3,p,q}^{''} = \left( \int_{S^2} \nabla^2(P_{l}\chi) \left( 2^\frac{j}{2}(N - N_\nu) \right)^p F_{j-1,u}(\omega)d\omega \right) \times \left( \int_{S^2} N'(P_{\leq l}\chi') \left( 2^\frac{j}{2}(N' - N_{\nu'}) \right)^q F_{j-1,u}(\omega)d\omega' \right), \tag{8.343} \]

\[ h_{4,p,q}^{''} = \left( \int_{S^2} \nabla(P_{l}\chi) \left( 2^\frac{j}{2}(N - N_\nu) \right)^p F_{j-1,u}(\omega)d\omega \right) \times \left( \int_{S^2} N'(P_{\leq l}\chi') \left( 2^\frac{j}{2}(N' - N_{\nu'}) \right)^q F_{j-1,u}(\omega)d\omega' \right), \tag{8.344} \]

and:

\[ h_{5,p,q}^{''} = \left( \int_{S^2} \nabla(N(P_{l}\chi)) \left( 2^\frac{j}{2}(N - N_\nu) \right)^p F_{j-1,u}(\omega)d\omega \right) \times \left( \int_{S^2} N'(P_{\leq l}\chi') \left( 2^\frac{j}{2}(N' - N_{\nu'}) \right)^q F_{j-1,u}(\omega)d\omega' \right), \tag{8.345} \]

where the tensors \( G_1'' \) and \( G_2'' \) are schematically given by:

\[ G_1'' = (\chi + \theta + \nabla(b))\nabla(P_{l}\chi), \tag{8.346} \]

and:

\[ G_2'' = (\chi' + \theta' + \nabla'(b'))N'(P_{\leq l}\chi'), \tag{8.347} \]

and where \( c_{pq} \) are explicit real coefficients such that the series

\[ \sum_{p,q \geq 0} c_{pq}x^py^q \]

has radius of convergence 1.

The proof of Lemma 8.15 is postponed to Appendix E. We now use this lemma to estimate \( B_{j,l,u_{\nu},l,m}^{1,2,2,2,1} \).

We estimate the \( L^1(\mathcal{M}) \) norm of \( h_{1,p,q}^{''}, h_{2,p,q}^{''}, h_{3,p,q}^{''}, h_{4,p,q}^{''}, h_{5,p,q}^{''} \) starting with \( h_{1,p,q}^{''} \). In view of (8.341), we have:

\[ \|h_{1,p,q}^{''}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} G_1'' \left( 2^\frac{j}{2}(N - N_\nu) \right)^p F_{j-1,u}(\omega)d\omega \right\|_{L^2(\mathcal{M})} \times \left\| \int_{S^2} N'(P_{\leq l}\chi') \left( 2^\frac{j}{2}(N' - N_{\nu'}) \right)^q F_{j-1,u}(\omega)d\omega' \right\|_{L^2(\mathcal{M})}. \]

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Together with the basic estimate in $L^2(\mathcal{M})$ (7.1), this yields:

$$
\|h''_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \left( \sup_\omega \left\| \frac{G''}{L^\infty L^2(\mathcal{H}_u)} \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty L^2(\mathcal{H}_u)} \right) \times \left( \sup_\omega \left\| N'(P_{\leq} \text{tr} \chi') \left( 2^{\frac{1}{2}} (N' - N_\nu') \right)^q \right\|_{L^\infty L^2(\mathcal{H}_u')} \right) 2^{j_\gamma} \gamma',
$$

(8.348)

where we used in the last inequality the estimate (2.42) for $\partial_x N$ and the size of the patch. Now, the estimate (2.39) for $\text{tr} \chi$, the boundedness of $P_m$ on $L^2(P_{t,u})$ and the commutator estimate (2.68) yields:

$$
\| N'(P_{\leq} \text{tr} \chi') \|_{L^\infty L^2(\mathcal{H}_u')} \lesssim \varepsilon.
$$

(8.349)

Also, we have in view of the definition (8.346) of $G''$:

$$
\| G'' \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim (\| x \|_{L^\infty L^2_x} + \| \theta \|_{L^\infty L^2_y} + \| \nabla(b) \|_{L^\infty L^2_y} \| \nabla(P_t) \|_{L^1 L^2_y}) \lesssim (N_1(\chi) + N_1(\theta) + N_1(\nabla(b))) \| \nabla(P_t) \|_{L^1 L^2_y} \| \nabla^2(P_t) \|_{L^\infty L^2(\mathcal{H}_u)},
$$

where we used in the last inequality the embedding (2.51) and the Gagliardo-Nirenberg inequality (2.49). Together with the Bochner inequality (2.61) and the finite band property for $P_t$, we obtain:

$$
\| G'' \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim (N_1(\chi) + N_1(\theta) + N_1(\nabla(b))) 2^{\frac{3}{2}} \| \nabla \text{tr} \chi \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim 2^{\frac{3}{2}} \varepsilon
$$

(8.350)

where we used in the last inequality the estimates (2.39) (2.40) for $\chi$, the estimate (2.38) for $b$ and the estimates (2.37) (2.39) (2.40) for $\theta$. Finally, (8.348), (8.349) and (8.350) imply:

$$
\| h''_{1,p,q} \|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{l_\gamma} \gamma',
$$

(8.351)

Next, we estimate $h''_{2,p,q}$. In view of its definition (8.342), we have the analog of the estimate (8.348):

$$
\| h''_{2,p,q} \|_{L^1(\mathcal{M})} \lesssim \left( \sup_\omega \left\| \nabla(P_t) \right\|_{L^\infty L^2(\mathcal{H}_u)} \left( \sup_\omega \| G'' \|_{L^\infty L^2(\mathcal{H}_u')} \right) 2^{j_\gamma} \gamma' \right).
$$

(8.352)

The estimate (2.39) for $\text{tr} \chi$ together with the finite band property for $P_t$ yields:

$$
\| \nabla(P_t) \|_{L^\infty L^2(\mathcal{H}_u)} \lesssim \varepsilon.
$$

(8.353)

Also, in view of the definition (8.347) of $G''_2$ and the estimate (8.349), the analog of the estimate (8.350) yields:

$$
\| G''_2 \|_{L^\infty L^2(\mathcal{H}_u')} \lesssim 2^{\frac{3}{2}} \varepsilon
$$

(8.354)
Next, we estimate \( h''_{3,p,q} \). In view of its definition (8.343), we have the analog of the estimate (8.348):
\[
\| h''_{3,p,q} \|_{L^1(\mathcal{M})} \lesssim \left( \sup_{\omega} \| \nabla^2 (P_t \text{tr} \chi) \|_{L^\infty_{\omega} L^2(\mathcal{H}_w)} \right) \left( \sup_{\omega'} \| N'(P_{\leq t} \text{tr} \chi') \|_{L^\infty_{\omega'} L^2(\mathcal{H}_w)} \right) 2^j \gamma_j' \gamma_j'.
\]
(8.355)

Now, the Bochner inequality (2.61), the finite band property for \( P_t \), and the estimate (2.39) for \( \text{tr} \chi \) yield:
\[
\| \nabla^2 (P_t \text{tr} \chi) \|_{L^\infty_{\omega} L^2(\mathcal{H}_w)} \lesssim 2^j \varepsilon.
\]
Finally, (8.355), (8.356) and (8.349) yield:
\[
\| h''_{3,p,q} \|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^j \gamma_j' \gamma_j'.
\]
(8.357)

Next, we estimate \( h''_{4,p,q} \). In view of its definition (8.344), we have the analog of the estimate (8.348):
\[
\| h''_{4,p,q} \|_{L^1(\mathcal{M})} \lesssim \left( \sup_{\omega} \| \nabla (P_t \text{tr} \chi) \|_{L^\infty_{\omega} L^2(\mathcal{H}_w)} \right) \left( \sup_{\omega'} \| \nabla' (P_{\leq t} \text{tr} \chi') \|_{L^\infty_{\omega'} L^2(\mathcal{H}_w)} \right) 2^j \gamma_j' \gamma_j'.
\]
(8.358)

Now, we have in view of the estimate (8.333):
\[
\| \nabla' N'(P_{\leq t} \text{tr} \chi') \|_{L^2(\mathcal{H}_w)} \lesssim 2^j \| P_{\leq t} (b' N' \text{tr} \chi') \|_{L^2(\mathcal{H}_w)} + 2^j \varepsilon
\]
\[
\lesssim 2^j \varepsilon,
\]
where we used in the last inequality the boundedness of \( P_m \) on \( L^2(P_{tw}) \), the estimate (2.38) for \( b \) and the estimate (2.39) for \( \text{tr} \chi \). Finally, (8.358), (8.359) and (8.353) yield:
\[
\| h''_{4,p,q} \|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^j \gamma_j' \gamma_j'.
\]
(8.360)

Next, we estimate \( h''_{5,p,q} \). In view of its definition (8.345), we have the analog of the estimate (8.348):
\[
\| h''_{5,p,q} \|_{L^1(\mathcal{M})} \lesssim \left( \sup_{\omega} \| \nabla (P_t \text{tr} \chi) \|_{L^\infty_{\omega} L^2(\mathcal{H}_w)} \right) \left( \sup_{\omega'} \| N'(P_{\leq t} \text{tr} \chi') \|_{L^\infty_{\omega'} L^2(\mathcal{H}_w)} \right) 2^j \gamma_j' \gamma_j'.
\]
(8.354)

Together with the estimate (8.349) and the estimate (8.359), we obtain:
\[
\| h''_{5,p,q} \|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^j \gamma_j' \gamma_j'.
\]
(8.361)

Now, in view of the decomposition (8.340) of \( B^{1,2,2,2,1}_{j,\nu,\nu',l,m} \), we have:
\[
\sum_{m/m \leq l} B^{1,2,2,2,1}_{j,\nu,\nu',l,m} \lesssim 2^{-j/2} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^j |N_\nu - N_{\nu'}|)^{p+q+1}} \left( \frac{1}{|2^j |N_\nu - N_{\nu'}|^2} \right) \| h''_{1,p,q} \|_{L^1(\mathcal{M})} + \| h''_{2,p,q} \|_{L^1(\mathcal{M})} + \| h''_{3,p,q} \|_{L^1(\mathcal{M})} + \| h''_{4,p,q} \|_{L^1(\mathcal{M})} + \| h''_{5,p,q} \|_{L^1(\mathcal{M})} + 2^{-j} \| h''_{5,p,q} \|_{L^1(\mathcal{M})},
\]
(8.362)
which together with (8.32), (8.351), (8.354), (8.357), (8.360) and (8.361) implies:

\[
\left| \sum_{m/m \leq l} \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1.2.2.2.1} \right| \lesssim 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^{p+q+1}} \left[ \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^2} 2^j + \frac{1}{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)} 2^j + 2^{-j+l} \right] \varepsilon^2 \gamma_j \gamma_j'.
\]

In view of (8.363), we still need to estimate \( B_{j,\nu,\nu',l,m}^{1.2.2.2.2} \). Recall the definition (8.297) of \( B_{j,\nu,\nu',l,m}^{1.2.2.2.2} \).

\[
B_{j,\nu,\nu',l,m}^{1.2.2.2.2} = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} N(P_{1}\text{tr} \chi) N'(P_{1}\text{tr} \chi')(b' - b) F_{j,-1}(u) \eta_j(\omega) F_{j,-1}(u') \eta_j'(\omega') d\omega d\omega' d\mathcal{M}.
\]

Note that the integrand in the definition of \( B_{j,\nu,\nu',l,m}^{1.2.2.2.2} \) is antisymmetric in \(((l, \omega, \nu), (m, \omega', \nu'))\), and thus we have the following cancellation:

\[
B_{j,\nu,\nu',l,m}^{1.2.2.2.2} + B_{j,\nu,\nu',m,l}^{1.2.2.2.2} = 0.
\]

This yields:

\[
\sum_{(l,m)/2^{\max(m,l)} \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^{1.2.2.2.2} + B_{j,\nu,\nu',m,l}^{1.2.2.2.2}) = 0,
\]

which together with (8.363) implies:

\[
\left| \sum_{(l,m)/2^{\max(m,l)} \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1.2.2.2.2} \right| \lesssim 2^{-\frac{j}{2}} \left[ \frac{1}{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)} + \frac{2^{-\frac{j}{2}}}{(2^\frac{j}{2} |\nu - \nu'|)^2} + 2^{-j} \right] \varepsilon^2 \gamma_j \gamma_j'. \tag{8.364}
\]
Next, we estimate $B^{1,2,2,3}_{j,\nu,\nu',l,m}$. Recall the definition (8.261) of $B^{1,2,2,3}_{j,\nu,\nu',l,m}$:

$$B^{1,2,2,3}_{j,\nu,\nu',l,m} = 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi' - \chi)(b' - b)}{g(L, L')^2} \left( L(P_{l\text{tr}}\chi)P_{m\text{tr}}\chi' + P_{l\text{tr}}\chi L'(P_{m\text{tr}}\chi') \right) \times F_{j,-1}(u)\eta_j\nu(\omega)F_{j,-1}(u')\eta_j\nu'(\omega')d\omega d\omega' \, .$$

Recall also that we are considering the range of $(l, m)$:

$$2^m \leq 2^l \leq 2^j|\nu - \nu'| \, .$$

Summing in $(l, m)$, we have:

$$\sum_{(l,m)/2^{\max(l,m)} \leq 2^j|\nu - \nu'|} \left( L(P_{l\text{tr}}\chi)P_{m\text{tr}}\chi' + P_{l\text{tr}}\chi L'(P_{m\text{tr}}\chi') \right) = L(P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi)P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi' + P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi L'(P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi').$$

Thus, using the symmetry in $(\omega, \omega')$ of the integrant in $B^{1,2,2,3}_{j,\nu,\nu',l,m}$, we obtain in view of the definition (8.261) of $B^{1,2,2,3}_{j,\nu,\nu',l,m}$:

$$\sum_{(l,m)/2^{\max(l,m)} \leq 2^j|\nu - \nu'|} (B^{1,2,2,3}_{j,\nu,\nu',l,m} + B^{1,2,2,3}_{j,\nu,\nu',l,m}) \quad (8.365)$$

$$= 2^{-2j} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(\chi' - \chi)(b' - b)}{g(L, L')^2} L(P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi)P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi' \times F_{j,-1}(u)\eta_j\nu(\omega)F_{j,-1}(u')\eta_j\nu'(\omega')d\omega d\omega' + \text{ terms interverting } (\nu, \nu'),$$

where we chose to ignore the terms which are obtained by interverting $\nu$ and $\nu'$ since they are treated in the exact same way.

Recall the identities (8.30) and (8.31):

$$g(L, L') = -1 + g(N, N') \text{ and } 1 - g(N, N') = \frac{g(N - N', N - N')}{2} \, .$$

We may thus expand

$$\frac{1}{g(L, L')^2}$$

in the same fashion than (8.33), and in view of (8.365), we obtain, schematically:

$$\sum_{(l,m)/2^{\max(l,m)} \leq 2^j|\nu - \nu'|} (B^{1,2,2,3}_{j,\nu,\nu',l,m} + B^{1,2,2,3}_{j,\nu,\nu',l,m}) \quad (8.366)$$

$$= \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^l|\nu - \nu'|)^{p+q+4}} [h_{1,p,q} + h_{2,p,q}]d\mathcal{M} + \text{ terms interverting } (\nu, \nu'),$$

where the scalar functions $h_{1,p,q}, h_{2,p,q}$ on $\mathcal{M}$ are given by:

$$h_{1,p,q} = \left( \int_{\mathbb{S}^2} \chi L(P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi)(b - b')^r \left( 2^l(N - N') \right)^p F_{j,-1}(u)\eta_j\nu(\omega)d\omega \right) \times \left( \int_{\mathbb{S}^2} P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi(b - b')^s \left( 2^l(N - N') \right)^q F_{j,-1}(u')\eta_j\nu'(\omega')d\omega' \right),$$

$$h_{2,p,q} = \left( \int_{\mathbb{S}^2} \chi L(P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi)(b - b')^r \left( 2^l(N - N') \right)^p F_{j,-1}(u)\eta_j\nu(\omega)d\omega \right) \times \left( \int_{\mathbb{S}^2} P_{\leq 2^j|\nu - \nu'|\text{tr}}\chi(b - b')^s \left( 2^l(N - N') \right)^q F_{j,-1}(u')\eta_j\nu'(\omega')d\omega' \right).$$
where we used in the last inequality the estimate (2.44) for the case 
\( s = 1 \), and \( s = 0 \) which is easier. In view of the definition (8.367) of \( h_{1,p,q} \) in the case \( r = 1 \) and \( s = 0 \), we have:

\[
\|h_{1,p,q}\|_{L^1(M)} \lesssim \left\| \int_{S^2} \chi L(P_{\leq 2|\nu - \nu'|}|tr \chi)(b - b') \left( 2^\frac{j}{2} (N - N_b) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \\
\times \left\| \int_{S^2} P_{\leq 2|\nu - \nu'|}|tr \chi'(b' - b') \left( 2^\frac{j}{2} (N' - N_{b'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)}.
\]

We estimate the two terms in the right-hand side of (8.369) starting with the first one. The basic estimate in \( L^2(M) \) (7.1) yields:

\[
\left\| \int_{S^2} \chi L(P_{\leq 2|\nu - \nu'|}|tr \chi)(b - b') \left( 2^\frac{j}{2} (N - N_b) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \left( \sup_\omega \left\| \chi L(P_{\leq 2|\nu - \nu'|}|tr \chi)(b - b') \left( 2^\frac{j}{2} (N - N_b) \right)^p \right\|_{L^\infty L^2(H_u)} \right)^{2^\frac{j}{2} \gamma_j^\nu}.
\]

where we used in the last inequality the estimate (2.44) for \( \partial_\omega b \), the estimate (2.42) for \( \partial_\omega N \), and the size of the patch. Now, the estimate (8.184) yields:

\[
\| \chi L(P_{\leq 2|\nu - \nu'|}|tr \chi) \|_{L^\infty L^2(H_u)} \lesssim \varepsilon N_1(\chi) \lesssim \varepsilon,
\]

where we used in the last inequality the estimates (2.39) (2.40) for \( \chi \). Together with (8.370), we obtain:

\[
\left\| \int_{S^2} \chi L(P_{\leq 2|\nu - \nu'|}|tr \chi)(b - b') \left( 2^\frac{j}{2} (N - N_b) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim |\nu - \nu'|^{2^\frac{j}{2} \gamma_j^\nu}.
\]

(8.371)
Next, we estimate the second term in the right-hand side of (8.369). We have:

\[
\left\| \int_{S^2} P_{\leq 2|\nu - \nu'|} |tr\chi|^{2}(N' - N_{\nu}) \right)^q F_{j-1}(u') \eta_j''(\omega') d\omega \right\|_{L^2(M)} \\
\lesssim \left\| \int_{S^2} tr\chi'' \left(2^{j}(N' - N_{\nu})\right)^q F_{j-1}(u') \eta_j''(\omega') d\omega \right\|_{L^2(M)} \\
+ \left\| \int_{S^2} P_{>2|\nu - \nu'|} |tr\chi'|(2^{j}(N' - N_{\nu}))^q F_{j-1}(u') \eta_j''(\omega') d\omega \right\|_{L^2(M)},
\]

which together with the estimate (8.129), the estimate (7.64) and the fact that $2^{j}|\nu - \nu'| \gtrsim 1$ yields:

\[
\left\| \int_{S^2} P_{\leq 2|\nu - \nu'|} |tr\chi|^{2}(N' - N_{\nu}) \right)^q F_{j-1}(u') \eta_j''(\omega') d\omega \right\|_{L^2(M)} \lesssim \varepsilon(1 + q^2)\varepsilon \gamma_j''.
\]  
(8.372)

Finally, (8.369), (8.371) and (8.372) imply in the case $r = 1$ and $s = 0$:

\[
\|h_{1,p,q}\|_{L^1(M)} \lesssim (1 + q^2)|\nu - \nu'|^2 \varepsilon^2 \gamma_j'' \gamma_j'.
\]  
(8.373)

Next, we consider the case $r = 0$ and $s = 1$. We decompose $h_{1,p,q}$ as:

\[
h_{1,p,q} = h_{1,p,q,1} + h_{1,p,q,2} + h_{1,p,q,3},
\]  
(8.374)

where $h_{1,p,q,1}$, $h_{1,p,q,2}$ and $h_{1,p,q,3}$ are given respectively by

\[
h_{1,p,q,1} = \left( \int_{S^2} \chi L(P_{\leq 2|\nu - \nu'|} |tr\chi|) \left(2^{j}(N - N_{\nu})\right)^p F_{j-1}(u') \eta_j''(\omega) d\omega \right)
\times \left( \int_{S^2} P_{>2|\nu - \nu'|} |tr\chi| \left(2^{j}(N' - N_{\nu})\right)^q F_{j-1}(u') \eta_j''(\omega') d\omega' \right),
\]  
(8.375)

\[
h_{1,p,q,2} = \left( \int_{S^2} \chi L(P_{>2|\nu - \nu'|} |tr\chi|) \left(2^{j}(N - N_{\nu})\right)^p F_{j-1}(u') \eta_j''(\omega) d\omega \right)
\times \left( \int_{S^2} tr\chi''(b_{\nu'} - b') \left(2^{j}(N' - N_{\nu'})\right)^q F_{j-1}(u') \eta_j''(\omega') d\omega' \right),
\]  
(8.376)

and

\[
h_{1,p,q,3} = \left( \int_{S^2} \chi L(tr\chi) \left(2^{j}(N - N_{\nu})\right)^p F_{j-1}(u') \eta_j''(\omega) d\omega \right)
\times \left( \int_{S^2} tr\chi'(b_{\nu'} - b') \left(2^{j}(N' - N_{\nu'})\right)^q F_{j-1}(u') \eta_j''(\omega') d\omega' \right).
\]  
(8.377)

Next, we estimate the $L^1(M)$ norm of $h_{1,p,q,1}$, $h_{1,p,q,2}$ and $h_{1,p,q,3}$ starting with $h_{1,p,q,1}$. In view of the definition (8.375) of $h_{1,p,q,1}$, we have

\[
\|h_{1,p,q,1}\|_{L^1(M)} \lesssim \left\| \int_{S^2} \chi L(P_{\leq 2|\nu - \nu'|} |tr\chi|) \left(2^{j}(N - N_{\nu})\right)^p F_{j-1}(u') \eta_j''(\omega) d\omega \right\|_{L^2(M)}
\times \left\| \int_{S^2} P_{>2|\nu - \nu'|} |tr\chi|'(b_{\nu'} - b') \left(2^{j}(N' - N_{\nu'})\right)^q F_{j-1}(u') \eta_j''(\omega') d\omega' \right\|_{L^2(M)}.
\]  
(8.378)
We estimate the two terms in the right-hand side of (8.369) starting with the first one. Proceeding as for the proof of (8.371), and noticing that the only difference is the missing factor of $b-b_{\nu'}$, we obtain:

$$\left\| \int_{\mathcal{S}} \chi L(P_{\leq 2|\nu-\nu'|\text{tr}\chi}) \left( 2^\frac{j}{2}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^\frac{j}{2} \gamma_j^\nu. \quad (8.379)$$

Next, we estimate the second term in the right-hand side of (8.369). The analog of (8.247) yields:

$$\left\| \int_{\mathcal{S}} P_{>2|\nu-\nu'|\text{tr}\chi'}(b_{\nu'} - b') \left( 2^\frac{j}{2}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \frac{\varepsilon^2 2^{\frac{j}{2}} \gamma_j^{\nu'}}{2^\frac{j}{2}|\nu - \nu'|}. \quad (8.380)$$

(8.378), (8.379) and (8.380) imply:

$$\|h_{1,p,q,1}\|_{L^1(\mathcal{M})} \lesssim \frac{\varepsilon^2 2^{\frac{j}{2}} \gamma_j^{\nu} \gamma_j^{\nu'}}{2^\frac{j}{2}|\nu - \nu'|}. \quad (8.381)$$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,2}$. In view of the definition (8.376) of $h_{1,p,q,2}$, we have

$$h_{1,p,q,2} = \int_{\mathcal{S}} \chi HL(P_{>2|\nu-\nu'|\text{tr}\chi}) \left( 2^\frac{j}{2}(N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^{\nu}(\omega) d\omega,$$

where $H$ is given by

$$H = \int_{\mathcal{S}} \text{tr}\chi'(b_{\nu'} - b') \left( 2^\frac{j}{2}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega'.$$

This yields

$$\|h_{1,p,q,2}\|_{L^1(\mathcal{M})} \lesssim \int_{\mathcal{S}} \left\| \chi HL(P_{>2|\nu-\nu'|\text{tr}\chi}) \left( 2^\frac{j}{2}(N - N_{\nu}) \right)^p F_{j,-1}(u) \right\|_{L^1(\mathcal{M})} \eta_j^{\nu}(\omega) d\omega \lesssim \int_{\mathcal{S}} \left\| \chi \right\|_{L^\infty_{\nu,\nu'}} \left\| H \right\|_{L^2_{\nu,\nu'}} \left\| L(P_{>2|\nu-\nu'|\text{tr}\chi}) \right\|_{L^\infty_{\nu,\nu'}} \left\| \left( 2^\frac{j}{2}(N - N_{\nu}) \right)^p \right\|_{L^\infty} \times \left\| F_{j,-1}(u) \right\|_{L^2_{\nu,\nu'}} \eta_j^{\nu}(\omega) d\omega \lesssim \frac{\varepsilon}{2^\frac{j}{2}|\nu - \nu'|} \int_{\mathcal{S}} \left\| H \right\|_{L^2_{\nu,\nu'}} \left\| F_{j,-1}(u) \right\|_{L^2_{\nu,\nu'}} \eta_j^{\nu}(\omega) d\omega,$$

where we used in the last inequality the estimates (2.39) (2.40) for $\chi$, the estimate (2.42) for $\partial_\nu N$, the estimate (8.142) for $L(P_{>2|\nu-\nu'|\text{tr}\chi})$, and the size of the patch. In view of the definition of $H$ and the estimate (7.95), we have

$$\|H\|_{L^2_{\nu,\nu'}} \lesssim 2^{-\frac{j}{2}} (1 + q^\frac{j}{2}) \varepsilon (2^\frac{j}{2}|\nu - \nu'| + 1) \gamma_j^{\nu'},$$

which together with (8.382) implies

$$\|h_{1,p,q,2}\|_{L^1(\mathcal{M})} \lesssim \frac{\varepsilon}{2^\frac{j}{2}|\nu - \nu'|} \int_{\mathcal{S}} \left\| F_{j,-1}(u) \right\|_{L^2_{\nu,\nu'}} \eta_j^{\nu}(\omega) d\omega.$$
Taking Plancherel in $\lambda$, Cauchy-Schwarz in $\omega$ and using the size of the patch, we finally obtain:

$$\|h_{1,p,q,2}\|_{L^1(M)} \lesssim (1 + q^2)(2 \nu |\nu - \nu'|)^{1/2} \epsilon^2 \gamma_j' \gamma_j'. \quad (8.383)$$

Next, we estimate the $L^1(M)$ norm of $h_{1,p,q,3}$. In view of the Raychaudhuri equation (2.22) satisfied by $\chi_L$, the worst term in $\chi L(\chi_L)$ is of the form $\chi^3$. In view of the decomposition (2.78) for $\hat{\chi}^3$, we obtain the following decomposition for $\chi L(\chi_L)$:

$$\chi L(\chi_L) = \chi^2 + \chi^2 F_1^j + \chi^2 F_2^j + \chi^2 F_3^j + \chi^2 F_4^j + \chi F_5^j + F_6^j + F_7^j + F_8^j \quad (8.384)$$

where $F_1^j, F_3^j$ and $F_6^j$ only depend on $(t, x)$ and $\nu$ and satisfy:

$$\|F_1^j\|_{L^2(\nu)} = \|F_3^j\|_{L^2(\nu)} + \|F_6^j\|_{L^2(\nu)} \lesssim \epsilon, \quad (8.385)$$

where $F_2^j, F_4^j$ and $F_7^j$ satisfy:

$$\|F_2^j\|_{L^2(H_u)} + \|F_4^j\|_{L^2(H_u)} + \|F_7^j\|_{L^2(H_u)} \lesssim 2^{-1} \epsilon, \quad (8.386)$$

where $F_5^j$ and $F_8^j$ satisfy:

$$\|F_5^j\|_{L^2(M)} + \|F_8^j\|_{L^2(M)} \lesssim \epsilon 2^{-j}. \quad (8.387)$$

and where $F_9^j$ satisfies

$$\|F_9^j\|_{L^2(M)} \lesssim \epsilon 2^{-3j}. \quad (8.388)$$

In view of the definition (8.377) of $h_{1,p,q,3}$, this yields the following decomposition

$$h_{1,p,q,3} = h_{1,p,q,3,1} + h_{1,p,q,3,2} + h_{1,p,q,3,3} + h_{1,p,q,3,4} + h_{1,p,q,3,5} + h_{1,p,q,3,6} + h_{1,p,q,3,7}, \quad (8.389)$$

where $h_{1,p,q,3,1}, h_{1,p,q,3,2}, h_{1,p,q,3,3}, h_{1,p,q,3,4}, h_{1,p,q,3,5}, h_{1,p,q,3,6}$ and $h_{1,p,q,3,7}$ are given by:

$$h_{1,p,q,3,1} = \left(\chi_2^2 F_1^j + \chi_2^2 F_3^j + F_6^j\right) \left(\int_{S^3} \left(2^j(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega\right) H, \quad (8.390)$$

$$h_{1,p,q,3,2} = \chi_2^2 \left(\int_{S^2} F_2^j \left(2^j(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega\right) H, \quad (8.391)$$

$$h_{1,p,q,3,3} = \chi_2 \left(\int_{S^2} F_3^j \left(2^j(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega\right) H, \quad (8.392)$$

$$h_{1,p,q,3,4} = \chi_2 \left(\int_{S^2} F_4^j \left(2^j(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega\right) H, \quad (8.393)$$

$$h_{1,p,q,3,5} = \left(\int_{S^2} F_5^j \left(2^j(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega\right) H, \quad (8.394)$$

$$h_{1,p,q,3,6} = \left(\int_{S^2} F_6^j \left(2^j(N - N_\nu)\right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega\right) H, \quad (8.395)$$
and
\[ h_{1,p,q,3,7} = \left( \int_{S^2} F^j_9 \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) H, \quad (8.396) \]
with \( H \) given by:
\[ H = \int_{S^2} \text{tr} \left( b_{\nu} - b' \right) \left( 2^\frac{j}{2} (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega'. \quad (8.397) \]
Using the basic estimate (7.1), we have for \( n = 2, 4, 7 \)
\[ \left\| \int_{S^2} F^j_n \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \leq \varepsilon \gamma_j^\nu, \quad (8.398) \]
where we used in the last inequality the estimate (8.386), the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Also, we have for \( n = 5, 8 \)
\[ \left\| \int_{S^2} F^j_n \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \leq \varepsilon \gamma_j^\nu, \quad (8.399) \]
Also, we have
\[ \left\| \int_{S^2} F^j_9 \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \leq \varepsilon \gamma_j^\nu, \quad (8.400) \]
where we used in the last inequality the estimate (8.388), the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Taking Cauchy-Schwarz both in $\lambda$ and $\omega$ and using the size of the patch, we obtain:

$$\left\| \int_{S^2} F_{\delta}^j \left( 2^{\lambda} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim 2^{-\frac{1}{2}} \varepsilon \gamma_j^\nu. \quad \text{(8.400)}$$

Finally, let $G$ be given by

$$G = \int_{S^2} \left( 2^{\lambda} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega.$$ 

Then, we have in view of Lemma 7.6, we have

$$\left\| G \right\|_{L^2_{u^\nu,x^\nu} L^\infty_{t^\nu}} \lesssim (1 + p^2) \gamma_j^\nu. \quad \text{(8.401)}$$

(8.389)-(8.400) imply

$$\left\| h_{1,p,q,3} \right\|_{L^1(\mathcal{M})} \quad \text{(8.402)}$$

$$\lesssim \left\| (\chi_{2,\nu}, \chi_{2,\nu}, 1) H \right\|_{L^2(\mathcal{M})} \left( \left\| (\chi_{2,\nu}, F_{1}^{\delta}, F_{3}^{\delta}, F_{6}^{\delta}) G \right\|_{L^2(\mathcal{M})} + \varepsilon \gamma_j^\nu \right) + \left\| H \right\|_{L^2(\mathcal{M})} \left( 1 + p^2 \right) \varepsilon \gamma_j^\nu,$$

where we used in the last inequality the estimate (2.46) for $\chi_{2,\nu}$, the estimate (8.385) for $F_{1}^{\delta}$, $F_{3}^{\delta}$ and $F_{6}^{\delta}$, and the estimate (8.401) for $G$. Next, we estimate $H$. In view of its definition (8.397), we have from (7.103)

$$\left\| H \right\|_{L^2(\mathcal{M})} \lesssim 2^{-\frac{1}{4}} \varepsilon (1 + q^2) \gamma_j^\nu. \quad \text{(8.403)}$$

Also, we have

$$\left\| H \right\|_{L^\infty(\mathcal{M})} \lesssim \int_{S^2} \left\| \text{tr} \chi'(b_\nu - b') \left( 2^{\lambda} (N' - N_\nu) \right)^q F_{j,-1}(u') \right\|_{L^\infty(\mathcal{M})} \eta_j^\nu(\omega') d\omega' \lesssim 2^{-\frac{1}{2}} \varepsilon \int_{S^2} \left\| F_{j,-1}(u') \right\|_{L^\infty(\mathcal{M})} \eta_j^\nu(\omega') d\omega',$$

where we used in the last inequality the estimate (2.39) for $\text{tr} \chi'$, the estimate (2.44) for $\partial_\omega b'$, the estimate (2.42) for $\partial_\omega N'$, and the size of the patch. Taking Cauchy-Schwarz in $\lambda'$ and $\omega'$, and using the size of the patch, we obtain

$$\left\| H \right\|_{L^\infty(\mathcal{M})} \lesssim 2^{\frac{1}{4}} \varepsilon \gamma_j^\nu. \quad \text{(8.404)}$$

In particular, interpolating (8.403) and (8.404), we obtain

$$\left\| H \right\|_{L^3(\mathcal{M})} \lesssim (1 + q^2)^{\frac{1}{3}} \varepsilon \gamma_j^\nu,$$
which together with (8.402) and (8.403) implies
\[
\| h_{1,p,q,3} \|_{L^1(M)} \lesssim \left( \| \chi_2^2 H \|_{L^2(M)} + (1 + q^2)^{\frac{3}{2}} \varepsilon \right) (1 + p^2) \varepsilon \gamma_j^{\nu'}.
\] (8.405)

Thus, in view of (8.405), it remains to estimate \( \| \chi_2^2 H \|_{L^2(M)} \). We decompose \( \chi_2^2 H \):
\[
\chi_2^2 H = \chi_2 H_1 + H_2 + H_3,
\] (8.406)
where \( H_1, H_2 \) and \( H_3 \) are given by
\[
H_1 = \int_{\mathbb{S}^2} (\chi_2 - \chi_2') \text{tr}'(b_{\nu'} - b') \left( 2^\frac{j}{2} (N' - N_{\nu'}) \right)^{\frac{q}{2}} F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega',
\] (8.407)
\[
H_2 = \int_{\mathbb{S}^2} (\chi_2 - \chi_2') \chi_2' \text{tr}'(b_{\nu'} - b') \left( 2^\frac{j}{2} (N' - N_{\nu'}) \right)^{\frac{q}{2}} F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega',
\] (8.408)
and
\[
H_3 = \int_{\mathbb{S}^2} \chi_2^2 \text{tr}'(b_{\nu'} - b') \left( 2^\frac{j}{2} (N' - N_{\nu'}) \right)^{\frac{q}{2}} F_{j-1}(u') \eta_j^{\nu'}(\omega') d\omega'.
\] (8.409)

In view of (8.406), we have
\[
\| \chi_2^2 H \|_{L^2(M)} \lesssim \| \chi_2 H_1 \|_{L^2(M)} + \| H_2 \|_{L^2(M)} + \| H_3 \|_{L^2(M)}
\] (8.410)
\[
\lesssim \| \chi_2 \|_{L^p(M)} \| H_1 \|_{L^1(M)} + \| H_2 \|_{L^2(M)} + \| H_3 \|_{L^2(M)}
\lesssim \varepsilon \| H_1 \|_{L^1(M)} + \| H_2 \|_{L^2(M)} + \| H_3 \|_{L^2(M)}
\]
where we used in the last inequality the estimate (2.46) for \( \chi_2 \). Next, we estimate each term in the right-hand side of (8.410) starting with the first one. In view of the definition (8.407) of \( H_1 \) and the estimate (7.2), we have
\[
\| H_1 \|_{L^1(M)} \lesssim \left( \sup_{\omega} \left\| (\chi_2 - \chi_2') \text{tr}'(b_{\nu'} - b') \left( 2^\frac{j}{2} (N' - N_{\nu'}) \right)^{\frac{q}{2}} \right\|_{L^\infty_{\omega} L^1(H_{\nu'})} \right)^{\frac{2^j}{\gamma_j^{\nu'}}}
\lesssim \left( \sup_{\omega} \| \chi_2 - \chi_2' \|_{L^\infty_{\omega} L^1(H_{\nu'})} \| \text{tr}'(b_{\nu'} - b') \left( 2^\frac{j}{2} (N' - N_{\nu'}) \right)^{\frac{q}{2}} \|_{L^\infty} \right)^{\frac{2^j}{\gamma_j^{\nu'}}}
\lesssim | \nu - \nu'|^{\frac{1}{2}} \varepsilon \gamma_j^{\nu'},
\] (8.411)
where we used in the last inequality the estimate (2.75) for \( \chi_2 - \chi_2' \), the estimate (2.39) for \( \text{tr}' \), the estimate (2.44) for \( \partial_\omega b \), the estimate (2.42) for \( \partial_\omega N \), and the size of the patch. Next, we estimate \( H_2 \). In view of the definition (8.408) of \( H_2 \) and the estimate (7.1), we have
\[
\| H_2 \|_{L^2(M)} \lesssim \left( \sup_{\omega'} \left\| (\chi_2 - \chi_2') \chi_2' \text{tr}'(b_{\nu'} - b') \left( 2^\frac{j}{2} (N' - N_{\nu'}) \right)^{\frac{q}{2}} \right\|_{L^\infty_{\omega'} L^2(H_{\nu'})} \right)^{\frac{2^j}{\gamma_j^{\nu'}}}
\lesssim \left( \sup_{\omega'} \| \chi_2 - \chi_2' \|_{L^\infty_{\omega'} L^1(H_{\nu'})} \chi_2' \| \text{tr}'(b_{\nu'} - b') \left( 2^\frac{j}{2} (N' - N_{\nu'}) \right)^{\frac{q}{2}} \|_{L^\infty} \right)^{\frac{2^j}{\gamma_j^{\nu'}}}
\lesssim | \nu - \nu'| \varepsilon \gamma_j^{\nu'},
\] (8.412)
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where we used in the last inequality the estimate (2.75) for $\chi_{2p} - \chi\nu'$, the estimate (2.46) for $\chi\nu'$, the estimate (2.39) for $\tr\chi'$, the estimate (2.44) for $\partial_{\nu} b$, the estimate (2.42) for $\partial_{\nu} N$, and the size of the patch. Next, we estimate $H_3$. In view of the definition (8.409) of $H_3$ and the estimate (7.1), we have

$$\|H_3\|_{L^2(M)} \lesssim \left( \sup_{\omega'} \left\| \chi_{2p}^{\nu} \tr\chi'(b_{\nu'} - b') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{\nu} \right\|_{L^\infty L^2(H_\omega)} \right)^{2^{\frac{j}{2}} \gamma_{\nu}'} (8.413)$$

$$\lesssim \left( \sup_{\omega'} \left\| \chi_{2p}^{\nu} \right\|_{L^\infty L^4(H_\omega)}^2 \right) \left\| \tr\chi'(b_{\nu'} - b') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{\nu} \right\|_{L^\infty} \left( \frac{2^{\frac{j}{2}} \gamma_{\nu}'}{2^{\frac{j}{2}} \gamma_{\nu}'} \right)$$

$$\lesssim \varepsilon \gamma_{\nu}'.$$

where we used in the last inequality the estimate (2.46) for $\chi\nu'$, the estimate (2.39) for $\tr\chi'$, the estimate (2.44) for $\partial_{\nu} b$, the estimate (2.42) for $\partial_{\nu} N$, and the size of the patch. Finally, (8.410)-(8.413) yield

$$\|\chi_{2p}^{\nu} H\|_{L^2(M)} \lesssim (1 + |\nu - \nu'|2^{\frac{j}{2}}) \varepsilon \gamma_{\nu}'.$$

In view (8.405), we obtain:

$$\|h_{1,p,q,3}\|_{L^1(M)} \lesssim \left( |\nu - \nu'|2^{\frac{j}{2}} + (1 + q^2)\frac{3}{2} \right) (1 + p^2) \varepsilon \gamma_{\nu}^\nu \gamma_{\nu}^\nu,$$

which together with (8.374), (8.381), (8.383), and the fact that $2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1$, implies in the case $r = 0$ and $s = 1$

$$\|h_{1,p,q}\|_{L^1(M)} \lesssim \|h_{1,p,q,1}\|_{L^1(M)} + \|h_{1,p,q,2}\|_{L^1(M)} + \|h_{1,p,q,3}\|_{L^1(M)} \quad \quad (8.414)$$

$$\lesssim (1 + p^2) (1 + q^2) (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} \varepsilon \gamma_{\nu}^\nu \gamma_{\nu}^\nu.$$

Using (8.373) in the case $r = 1$ and $s = 0$, and (8.414) in the case $r = 0$ and $s = 1$, together with the fact that $2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1$, we finally obtain

$$\|h_{1,p,q}\|_{L^1(M)} \lesssim (1 + p^2) (1 + q^2) (2^{\frac{j}{2}} |\nu - \nu'| \varepsilon \gamma_{\nu}^\nu \gamma_{\nu}^\nu \quad \quad (8.415)$$

Next, we estimate the $L^1(M)$ norm of $h_{2,p,q}$. We start with the case $r = 1$ and $s = 0$ which is easier. In view of definition (8.368), we have:

$$\|h_{2,p,q}\|_{L^2(M)} \lesssim \left\| \int_{S^2} L(P_{\leq 2} |\nu - \nu'| \tr\chi) (b - b_{\nu'}) \left( 2^{\frac{j}{2}} (N - N_{\nu'}) \right)^{\nu} F_{j,-1}(u) \eta_{\nu}^\nu(\omega) d\omega \right\|_{L^2(M)} \quad \quad \quad (8.416)$$

$$\times \left\| \int_{S^2} \chi' P_{\leq 2} |\nu - \nu'| \tr\chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{\nu} F_{j,-1}(u) \eta_{\nu}'(\omega') d\omega' \right\|_{L^2(M)}.$$

We estimate the two terms in the right-hand side of (8.369) starting with the first one. The basic estimate in $L^2(M)$ (7.1) yields:

$$\left\| \int_{S^2} L(P_{\leq 2} |\nu - \nu'| \tr\chi)(b - b_{\nu'}) \left( 2^{\frac{j}{2}} (N - N_{\nu'}) \right)^{\nu} F_{j,-1}(u) \eta_{\nu}^\nu(\omega) d\omega \right\|_{L^2(M)} \quad \quad \quad (8.417)$$

$$\lesssim \left( \sup_{\omega} \left\| L(P_{\leq 2} |\nu - \nu'| \tr\chi)(b - b_{\nu'}) \left( 2^{\frac{j}{2}} (N - N_{\nu'}) \right)^{\nu} \right\|_{L^\infty L^2(H_\omega)} \right)^{2^{\frac{j}{2}} \gamma_{\nu}^\nu}$$

$$\lesssim |\nu - \nu'| \left( \sup_{\omega} \left\| L(P_{\leq 2} |\nu - \nu'| \tr\chi) \right\|_{L^\infty L^2(H_\omega)} \right)^{2^{\frac{j}{2}} \gamma_{\nu}^\nu}.$$
where we used in the last inequality the estimate \((2.44)\) for \(\partial_\omega b\), the estimate \((2.42)\) for \(\partial_\omega N\), and the size of the patch. Now, the estimate \((8.184)\) yields:
\[
\|L(P_{\leq 2_j}[\nu - \nu'][\text{tr}\chi])\|_{L^\infty_xL^2(\mathcal{H}_\omega)} \lesssim \varepsilon. \tag{8.418}
\]
Together with \((8.417)\), we obtain:
\[
\left\| \int_{\mathbb{S}^2} L(P_{\leq 2_j}[\nu - \nu'][\text{tr}\chi])(b - b_\nu) \left(2^j (N - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega')d\omega \right\|_{L^2(M)} \lesssim \varepsilon |\nu - \nu'|2^j \gamma_j^\nu. \tag{8.419}
\]
Next, we estimate the second term in the right-hand side of \((8.416)\). Recall the decomposition \((2.76)\) for \(\chi'\):
\[
\chi' = \chi_2 + F_1^j + F_2^j \tag{8.420}
\]
where the tensor \(F_1^j\) only depends on \(\nu'\) and satisfies for any \(2 \leq p < +\infty\):
\[
\|F_1^j\|_{L^\infty_{\nu'}L^p_{\nu'}L^\infty_{\nu'}} \lesssim \varepsilon, \tag{8.421}
\]
and where the tensor \(F_2^j\) satisfies:
\[
\|F_2^j\|_{L^\infty_{\nu'}L^2(\mathcal{H}_\omega)} \lesssim \varepsilon^{2^{j/2}}. \tag{8.422}
\]
The decomposition \((8.420)\) implies:
\[
\left\| \int_{\mathbb{S}^2} \chi' P_{\leq 2_j}[\nu - \nu'][\text{tr}\chi'] \left(2^j (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega')d\omega' \right\|_{L^2(M)} \lesssim \varepsilon \tag{8.423}
\]
where we used in the last inequality the estimate \((8.421)\) for \(F_1^j\) and the estimate \((2.46)\) for \(\chi_2\). Now, in view of the estimate \((7.71)\), we have for \(m > j/2\):
\[
\left\| \int_{\mathbb{S}^2} P_m[\text{tr}\chi] \left(2^j (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega')d\omega' \right\|_{L^\infty_{\nu'}x^j_{\nu'}L^\infty_t} \lesssim \varepsilon \left(2^{-l+\frac{j}{2}} + 2^{-\frac{j}{2} + \frac{j}{4}}\right) \gamma_j^\nu, \tag{8.424}
\]
where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Also, in view of the estimate (7.83), we have:

$$\left\| \int_{S^2} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2_{\nu', x', L^\infty}} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j'. \quad (8.425)$$

(8.424), (8.425), the fact that $2^{\frac{j}{2}} |\nu - \nu'| \gtrsim 1$, and the fact that:

$$P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' = \text{tr} \chi' - \sum_{m/2^m > 2^j |\nu - \nu'|} P_m \text{tr} \chi'$$

implies:

$$\left\| \int_{S^2} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2_{\nu', x', L^\infty}} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j'. \quad (8.426)$$

On the other hand, the basic estimate in $L^2(M)$ (7.1) yields:

$$\left\| \int_{S^2} F_2^j P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)} \lesssim \left( \sup_{\omega} \left\| F_2^j P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^\infty_{(H_2, \nu')}} \right)^{2^{\frac{j}{2}} \gamma_j'} \lesssim \varepsilon \gamma_j', \quad (8.427)$$

where we used in the last inequality the estimate (8.422) for $F_2^j$, the boundedness of $P_{\leq 2^j |\nu - \nu'|}$ on $L^\infty(P_{t,u})$, the estimate (2.39) for $\text{tr} \chi'$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Now, (8.423), (8.426) and (8.427) imply:

$$\left\| \int_{S^2} \chi' P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon (1 + q^{\frac{5}{2}}) \varepsilon \gamma_j'. \quad (8.428)$$

Finally, (8.416), (8.419) and (8.428) imply in the case $r = 1$ and $s = 0$:

$$\|h_{2,p,q}\|_{L^2(M)} \lesssim 2^\frac{j}{2} |\nu - \nu'| (1 + q^{\frac{5}{2}}) \varepsilon^{2} \gamma_j' \gamma_j'. \quad (8.429)$$

Next, we estimate $h_{2,p,q}$ in the case $r = 0$ and $s = 1$. In view of the definition (8.368), we may decompose $h_{2,p,q}$ as:

$$h_{2,p,q} = h_{2,p,q,1} + h_{2,p,q,2} + h_{2,p,q,3}, \quad (8.430)$$

where $h_{2,p,q,1}$, $h_{2,p,q,2}$ and $h_{2,p,q,3}$ are given respectively by

$$h_{2,p,q,1} = \left( \int_{S^2} L(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u') \eta_j'(\omega') d\omega' \right) \quad (8.431)$$

and

$$h_{2,p,q,2} = \left( \int_{S^2} \chi' P_{\geq 2^j |\nu - \nu'|} \text{tr} \chi' (b_{\nu'} - b') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right),$$

$$h_{2,p,q,3} = \left( \int_{S^2} \chi' P_{\geq 2^j |\nu - \nu'|} \text{tr} \chi' (b_{\nu'} - b') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right).$$

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\[ h_{2,p,q,2} = \left( \int_{S^2} L(P_{>2|\nu-\nu'|}\text{tr}\chi) \left( 2^{\frac{j}{2}}(N-N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \] (8.432)
\[ \times \left( \int_{S^2} \chi'\text{tr}\chi'(b_{\nu'} - b') \left( 2^{\frac{j}{2}}(N'-N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^\nu(\omega')d\omega' \right). \]

and
\[ h_{2,p,q,3} = \left( \int_{S^2} L(\text{tr}\chi) \left( 2^{\frac{j}{2}}(N-N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \] (8.433)
\[ \times \left( \int_{S^2} \chi'\text{tr}\chi'(b_{\nu'} - b') \left( 2^{\frac{j}{2}}(N'-N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^\nu(\omega')d\omega' \right). \]

Next, we estimate the \( L^1(M) \) norm of \( h_{2,p,q,1}, h_{2,p,q,2} \) and \( h_{2,p,q,3} \) starting with \( h_{2,p,q,1} \). In view of the definition (8.431) of \( h_{2,p,q,1} \), we have
\[ \|h_{2,p,q,1}\|_{L^1(M)} \leq \left\| \int_{S^2} L(P_{\leq 2|\nu-\nu'|}\text{tr}\chi) \left( 2^{\frac{j}{2}}(N-N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} \]
\[ \times \left\| \int_{S^2} \chi'P_{\geq 2|\nu-\nu'|}\text{tr}\chi'(b_{\nu'} - b') \left( 2^{\frac{j}{2}}(N'-N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j^\nu(\omega')d\omega' \right\|_{L^2(M)}. \]

We estimate the two terms in the right-hand side of (8.434) starting with the first one. The basic estimate (7.1) yields
\[ \left\| \int_{S^2} L(P_{\leq 2|\nu-\nu'|}\text{tr}\chi) \left( 2^{\frac{j}{2}}(N-N_\nu) \right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} \] (8.435)
\[ \lesssim \left( \sup_{\omega} \left\| L(P_{\leq 2|\nu-\nu'|}\text{tr}\chi) \left( 2^{\frac{j}{2}}(N-N_\nu) \right)^p \right\|_{L^\infty L^2(H_u)} \right)^{2^{\frac{j}{2}}\gamma_j^\nu} \]
\[ \lesssim \left( \sup_{\omega} \|L(P_{\leq 2|\nu-\nu'|}\text{tr}\chi)\|_{L^\infty L^2(H_u)} \left\| \left( 2^{\frac{j}{2}}(N-N_\nu) \right)^p \right\|_{L^\infty} \right)^{2^{\frac{j}{2}}\gamma_j^\nu} \]
\[ \lesssim 2^{\frac{j}{2}}\varepsilon_j^\nu, \]
where we used in the last inequality the estimate (2.42) for \( \partial_\omega N \), the size of the patch, and the following estimate
\[ \|L(P_{\leq 2|\nu-\nu'|}\text{tr}\chi)\|_{L^\infty L^2(H_u)} \lesssim \varepsilon, \]
which follows from the estimates (8.305) and (8.306). Next, we estimate the second term.
in the right-hand side of (8.434). Using the basic estimate (7.1), we have

\[
\left\| \int_{S^2} \chi' P_{>2l|\nu-\nu'|} \trch'(b, - b') \left( 2^j (N' - N\nu) \right)^q F_{j,-1}(u') \eta_{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \tag{8.436}
\]

\[
\lesssim \left( \sup_{\omega'} \left\| \chi' P_{>2l|\nu-\nu'|} \trch'(b, - b') \left( 2^j (N' - N\nu) \right)^q \right\|_{L^2(H, \nu)} \right) 2^j \gamma'_j
\]

\[
\lesssim \sum_{2^l > 2l|\nu-\nu'|} \left( \sup_{\omega'} \| \chi' \|_{L^\infty_{\nu, x, r, l}} \| P_l \trch' \|_{L^\infty_{\nu, x, r, l}} \right) \left( \int_{S^2} \trch'(b, - b') \left( 2^j (N' - N\nu) \right)^q \right) \lesssim \sum_{2^l > 2l|\nu-\nu'|} 2^{-l} \varepsilon \gamma'_j
\]

\[
\lesssim \frac{2^{-\frac{1}{2}}}{\left( 2^j |\nu-\nu'| \right)^{\frac{1}{2}}} \varepsilon \gamma'_j.
\]

where we used in the last inequality the estimates (2.39) (2.40) for \( \chi' \), the estimate (2.69) for \( P_l \trch' \), the estimate (2.44) for \( \partial_\omega b \), the estimate (2.42) for \( p_0 N \), and the size of the patch. (8.434), (8.435) and (8.436) imply:

\[
\| h_{2,p,q,1} \|_{L^1(M)} \lesssim \frac{\varepsilon 2^\gamma'_j \gamma'_j}{\left( 2^j |\nu-\nu'| \right)^{\frac{1}{2}}}. \tag{8.437}
\]

Next, we estimate the \( L^1(M) \) norm of \( h_{2,p,q,2} \). In view of the definition (8.432) of \( h_{2,p,q,2} \), we have

\[
\| h_{2,p,q,2} \|_{L^1(M)} \lesssim \left\| \int_{S^2} L(P_{>2l|\nu-\nu'|} \trch) \left( 2^j (N - N\nu) \right)^p F_{j,-1}(u) \eta_{\nu}(\omega) d\omega \right\|_{L^2(M)} \tag{8.438}
\]

\[
\times \left\| \int_{S^2} \trch'(b, - b') \left( 2^j (N' - N\nu) \right)^q F_{j,-1}(u') \eta_{\nu'}(\omega') d\omega' \right\|_{L^2(M)}.
\]

We estimate both terms in the right-hand side of (8.438) starting with the first one. In view of (8.159), we have

\[
\left\| \int_{S^2} L(P_{>2l|\nu-\nu'|} \trch) \left( 2^j (N - N\nu) \right)^p F_{j,-1}(u) \eta_{\nu}(\omega) d\omega \right\|_{L^2(M)} \tag{8.439}
\]

\[
\lesssim \left( \sum_{2^l > 2l|\nu-\nu'|} 2^{j-\frac{1}{2}} \right) \varepsilon \gamma'_j
\]

\[
\lesssim \frac{2^j}{\left( 2^j |\nu-\nu'| \right)^{\frac{1}{2}}} \varepsilon \gamma'_j.
\]

For the second term in the right-hand side of (8.438), we use the decomposition (2.76) for \( \chi' \) which yields

\[
\chi' = F_{1j} + F_{2j}, \tag{8.440}
\]

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where the tensor $F^i_1$ depends only on $(t, x)$ and $\nu'$ and satisfies
\[
\|F^i_1\|_{L^\infty_{\nu',x',\nu'U_*} L^2_t} \lesssim \varepsilon, \tag{8.441}
\]
and where the tensor $F^j_2$ satisfies
\[
\|F^j_2\|_{L^\infty_t L^2(H, \omega)} \lesssim 2^{-j/2} \varepsilon. \tag{8.442}
\]
Using the decomposition (8.440), we have
\[
\int_{S^2} \chi' \text{tr} \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' = F^j_1 \left( \int_{S^2} \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' \right)
+ \int_{S^2} F^j_2 \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega'.
\]
This yields
\[
\left\| \int_{S^2} \chi' \text{tr} \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \|F^j_1\|_{L^\infty_{\nu',x',\nu'U_*} L^2_t} \left\| \int_{S^2} \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
+ \left\| \int_{S^2} F^j_2 \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\lesssim 2^{-j/2} (1 + q^{1/2}) \varepsilon \gamma_j^{\nu'} + \left\| \int_{S^2} F^j_2 \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})},
\]
where we used in the last inequality the estimate (8.441) for $F^j_1$ and the estimate (7.95). The basic estimate (7.1) yields
\[
\left\| \int_{S^2} F^j_2 \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \left\| F^j_2 \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty_t L^2(H, \omega)} \right) 2^{j/2} \gamma_j^{\nu'}
\lesssim \left( \sup_{\omega} \left\| F^j_2 \right\|_{L^\infty_t L^2(H, \omega)} \left\| \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) 2^{j/2} \gamma_j^{\nu'}
\lesssim 2^{-j/2} \varepsilon \gamma_j^{\nu'},
\]
where we used in the last inequality the estimate (8.442) for $F^j_2$, the estimate (2.39) for $\text{tr} \chi'$, the estimate (2.44) for $\partial_\omega b$, the estimate (2.42) for $\partial_\omega N$, and the size of the patch. Finally, (8.443) and (8.444) yield
\[
\left\| \int_{S^2} \chi' \text{tr} \chi' (b_{\nu'} - b') \left(2^{j/2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{-j/2} (1 + q^{1/2}) \varepsilon \gamma_j^{\nu'}. \tag{8.445}
\]
(8.438), (8.439) and (8.445), together with the fact that $2^{\frac{j}{2}}|\nu - \nu'| \gtrsim 1$, imply

$$
\| h_{2,p,q,2} \|_{L^1(M)} \lesssim (1 + q^{\frac{5}{2}}) \varepsilon^2 \gamma_j \gamma_{j'}'.
$$

(8.446)

Next, we estimate the $L^1(M)$ norm of $h_{2,p,q,3}$. Recall the decomposition (8.64) (8.65) (8.66) for $L(\text{tr} \chi)$. We have:

$$
L(\text{tr} \chi) = \chi_{2\nu} \cdot (2\chi + \tilde{\chi}) + f_1^j + f_2^j,
$$

(8.447)

where the scalar $f_1^j$ only depends on $\nu$ and satisfies:

$$
\| f_1^j \|_{L^\infty_t L^2_x (P_{l,u})} \lesssim \varepsilon,
$$

(8.448)

where the scalar $f_2^j$ satisfies:

$$
\| f_2^j \|_{L^\infty_t L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.
$$

(8.449)

In view of the definition (8.433) of $h_{2,p,q,3}$, this yields the following decomposition

$$
h_{2,p,q,3} = h_{2,p,q,3,1} + h_{2,p,q,3,2} + h_{2,p,q,3,3},
$$

(8.450)

where $h_{2,p,q,3,1}$, $h_{2,p,q,3,2}$ and $h_{2,p,q,3,3}$ are given by:

$$
h_{2,p,q,3,1} = \chi_{2\nu} \left( \int_{\mathbb{R}^2} (2\chi + \tilde{\chi}) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_{j'}(\omega) d\omega \right) H,
$$

(8.451)

$$
h_{2,p,q,3,2} = f_1^j \left( \int_{\mathbb{R}^2} \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_{j}(\omega) d\omega \right) H,
$$

(8.452)

and

$$
h_{2,p,q,3,3} = \left( \int_{\mathbb{R}^2} f_2^j \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_{j}(\omega) d\omega \right) H,
$$

(8.453)

with $H$ given by:

$$
H = \int_{\mathbb{R}^2} \chi' \text{tr} \chi' (b_{\nu'} - b') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_{j'}(\omega') d\omega'.
$$

(8.454)

Using the basic estimate (7.1), we have

$$
\left\| \int_{\mathbb{R}^2} f_2^j \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_{j}(\omega) d\omega \right\|_{L^2(M)} \lesssim \varepsilon \gamma_j \gamma_{j'},
$$

(8.455)
where we used in the last inequality the estimate (8.449), the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Also, we have

\[
\left\| f_1^j \left( \int_{S^2} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \right\|_{L^2(M)} 
\leq \left\| f_1^j \right\|_{L^\infty_{\omega},L^2_{L^\infty(P_{\nu,u_\nu})}} \left\| \int_{S^2} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2_{\omega,u_\nu,x_\nu}L^2_t}
\leq (1 + p^2)\varepsilon\gamma_j^\nu,
\]

where we used in the last inequality the estimate (8.448) for \( f_1^j \), and Lemma 7.6. Also, recall (8.72):

\[
\left\| \int_{S^2} \left( 2\chi_1 + \tilde{\chi} \right) \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \leq (1 + p^2)\varepsilon\gamma_j^\nu.
\]

(8.450)-(8.457) imply

\[
\|h_{2,p,q,3}\|_{L^1(M)} \lesssim (\|\chi_{2\nu}H\|_{L^2(M)} + \|H\|_{L^2(M)}) (1 + p^2)\gamma_j^\nu.
\]

Next, we estimate \( H \). In view of the definition (8.454) of \( H \), the analog of (8.72) yields

\[
\|H\|_{L^2(M)} \lesssim (1 + q^2)\varepsilon\gamma_j^\nu.
\]

Also, in view of the definition (8.454) of \( H \), we have

\[
\chi_{2\nu}H = H_2 + H_3,
\]

where \( H_2 \) and \( H_3 \) have been defined respectively in (8.408) and (8.409). We deduce

\[
\|\chi_{2\nu}H\|_{L^2(M)} \lesssim \|H_2\|_{L^2(M)} + \|H_3\|_{L^2(M)} \lesssim \varepsilon\gamma_j^\nu,
\]

where we used in the last inequality the estimate (8.412) for \( H_2 \) and the estimate (8.413) for \( H_3 \). (8.458), (8.459) and (8.460) imply

\[
\|h_{2,p,q,3}\|_{L^1(M)} \lesssim (1 + p^2)(1 + q^2)\varepsilon^2\gamma_j^\nu\gamma_j^\nu,
\]

which together with (8.430), (8.437), (8.446), and the fact that \( 2^{\frac{1}{2}}|\nu - \nu'| \gtrsim 1 \), implies in the case \( r = 0 \) and \( s = 1 \)

\[
\|h_{2,p,q}\|_{L^1(M)} \lesssim \|h_{2,p,q,1}\|_{L^1(M)} + \|h_{2,p,q,2}\|_{L^1(M)} + \|h_{2,p,q,3}\|_{L^1(M)} \lesssim (1 + p^2)(1 + q^2)\varepsilon^2\gamma_j^\nu\gamma_j^\nu.
\]

Using (8.429) in the case \( r = 1 \) and \( s = 0 \), and (8.461) in the case \( r = 0 \) and \( s = 1 \), together with the fact that \( 2^{\frac{1}{2}}|\nu - \nu'| \gtrsim 1 \), we finally obtain

\[
\|h_{2,p,q}\|_{L^1(M)} \lesssim (1 + p^2)(1 + q^2)2^{\frac{1}{2}}|\nu - \nu'|\varepsilon^2\gamma_j^\nu\gamma_j^\nu.
\]
Now, in view of the decomposition (8.366), we have:

\[
\begin{align*}
&\left| \sum_{(l,m)/2\max(l,m) \leq 2^j|\nu - \nu'|} (B_{j,\nu',\nu',l,m}^{1,2,2,3} + B_{j,\nu',\nu',l,m}^{1,2,2,3}) \right| \\
&\lesssim \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^4|N_\nu - N_{\nu'}|)^{p+q+4}} \right\|_{L^\infty(M)} \left[ \|h_{1,p,q}\|_{L^1(M)} + \|h_{2,p,q}\|_{L^1(M)} \right],
\end{align*}
\]

which together with (8.32), (8.415) and (8.462) implies:

\[
\begin{align*}
&\left| \sum_{(l,m)/2\max(l,m) \leq 2^j|\nu - \nu'|} (B_{j,\nu',\nu',l,m}^{1,2,2,3} + B_{j,\nu',\nu',l,m}^{1,2,2,3}) \right| \\
&\lesssim \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^4|N_\nu - N_{\nu'}|)^{p+q+4}} \right\|_{L^\infty(M)} (1 + p^2)(1 + q^2)2^4|\nu - \nu'| \varepsilon^{2\gamma_{\nu,\nu'}} \\
&\lesssim \frac{\varepsilon^{2\gamma_{\nu,\nu'}}}{(2^4|\nu - \nu'|)^{5/2}}.
\end{align*}
\]

In view of (8.256), we have in the range \(2\max(l,m) \leq 2^j|\nu - \nu'|:\)

\[
\begin{align*}
&\left| B_{j,\nu',\nu',l,m}^{1,2,2,2} - (B_{j,\nu',\nu',l,m}^{1,2,2,1} + B_{j,\nu',\nu',l,m}^{1,2,2,2} + B_{j,\nu',\nu',l,m}^{1,2,2,3}) \right| \\
&\lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^4|N_\nu - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(M)} \left\| h_{1,p,q,l,m} \right\|_{L^1(M)} \left\| h_{2,p,q,l,m} \right\|_{L^1(M)}.
\end{align*}
\]

Together with (8.32), (8.264) and (8.282), we obtain:

\[
\begin{align*}
&\left| B_{j,\nu',\nu',l,m}^{1,2,2,2} - (B_{j,\nu',\nu',l,m}^{1,2,2,1} + B_{j,\nu',\nu',l,m}^{1,2,2,2} + B_{j,\nu',\nu',l,m}^{1,2,2,3}) \right| \\
&\lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^4|\nu - \nu'|)^{p+q}} \right\|_{L^\infty(M)} \left[ \frac{1}{|\nu - \nu'|^2} ((1 + p^2)2^\frac{7}{11}|\nu - \nu'| + 2^{\frac{7}{11}}) \varepsilon^{2\gamma_{\nu,\nu'}} + \varepsilon^{2\gamma_{\nu,\nu'}} \right] \\
&\quad + \frac{2^{-\frac{10j}{11}}}{(2^4|\nu - \nu'|)^3} + 2^{-\frac{4j}{11}} + 2^{-\frac{4j}{11}} \varepsilon^{2\gamma_{\nu,\nu'}}.
\end{align*}
\]
Together with (8.284), (8.364) and (8.463), we finally obtain:

\[
\sum_{\{l,m\}/2^{\max(l,m)} \leq 2^{|\mu-\nu'|}} (B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2}) \leq \sum_{\{l,m\}/2^{\max(l,m)} \leq 2^{|\mu-\nu'|}} (B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2}) + \sum_{\{l,m\}/2^{\max(l,m)} \leq 2^{|\mu-\nu'|}} (B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2}) \leq \left[ 2^{-j} \frac{1}{2^j (2^j |\nu-\nu'|)} + \frac{1}{2^j (2^j |\nu-\nu'|)^2} + \frac{2^{-(4^{-j})}}{2^j (2^j |\nu-\nu'|)^2} + \frac{1}{2^j (2^j |\nu-\nu'|)^3} \right] \varepsilon^{2} \gamma_j \gamma_j'.
\]

Together with (8.255), we obtain the estimate on the whole range $2^{\min(l,m)} \leq 2^j |\nu-\nu'|$:

\[
\sum_{\{l,m\}/2^{\min(l,m)} \leq 2^{j|\nu-\nu'|}} (B_{j,\nu,\nu',l,m}^{1,2,2} + B_{j,\nu,\nu',l,m}^{1,2,2}) \leq \left[ 2^{-j} \frac{1}{2^j (2^j |\nu-\nu'|)} + \frac{1}{2^j (2^j |\nu-\nu'|)^2} + \frac{2^{-(4^{-j})}}{2^j (2^j |\nu-\nu'|)^2} + \frac{1}{2^j (2^j |\nu-\nu'|)^3} \right] \varepsilon^{2} \gamma_j \gamma_j'.
\]

This concludes the proof of Proposition 8.11.

### 8.2.3 Proof of Proposition 8.12 (Control of $B_{j,\nu,\nu',l,m}^{1,2,3}$)

Recall that $B_{j,\nu,\nu',l,m}^{1,2,3}$ is defined by (8.96):

\[
B_{j,\nu,\nu',l,m}^{1,2,3} = -i2^{-j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{1}{g(L, L')} \left( L(P_l tr\chi) P_m tr\chi' + P_l tr\chi' L'(P_m tr\chi') \right) x' \left( 1 - g(N, N') \right) F_{j-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

Together with the identity (8.30):

\[
g(L, L') = -1 + g(N, N'),
\]

this yields:

\[
B_{j,\nu,\nu',l,m}^{1,2,3} = i2^{-j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} b'^{-1} \left( L(P_l tr\chi) P_m tr\chi' + P_l tr\chi' L'(P_m tr\chi') \right) F_{j-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_j^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

Recall (8.1):

\[
m < l \text{ and } 2^m \leq 2^l |\nu-\nu'|.
\]

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We first consider the range of \((l, m)\) such that:

\[2^m \leq 2^l |\nu - \nu'| < 2^l.\]

This yields:

\[
\sum_{(l, m)/2^m \leq 2^l |\nu - \nu'| < 2^l} B_{j, \nu, \nu', l, m}^{1, 2, 3} = -i 2^{-j-1} \int_M (h_1 + h_2) dM, \tag{8.465}
\]

where the scalar functions \(h_1, h_2\) on \(M\) are given by:

\[h_1 = \left(\int_{\mathbb{S}^2} L(P_{>2^j |\nu - \nu'|} \text{tr} \chi F_{j, -1}(u) \eta_j^\nu (\omega) d\omega\right) \left(\int_{\mathbb{S}^2} b^{-1} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' F_j (u') \eta_j^\nu' (\omega') d\omega'\right), \tag{8.466}\]

and:

\[h_2 = \left(\int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \text{tr} \chi F_{j, -1}(u) \eta_j^\nu (\omega) d\omega\right) \left(\int_{\mathbb{S}^2} b^{-1} L'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') F_j (u') \eta_j^\nu' (\omega') d\omega'\right). \tag{8.467}\]

Next, we estimate the \(L^1(M)\) norm of \(h_1\) and \(h_2\) starting with \(h_1\). In view of the definition of \(h_1\) (8.466), we have:

\[
\|h_1\|_{L^1(M)} \lesssim \left\| \int_{\mathbb{S}^2} L(P_{>2^j |\nu - \nu'|} \text{tr} \chi) F_{j, -1}(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(M)} \tag{8.468}
\times \left\| \int_{\mathbb{S}^2} b^{-1} P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' F_j (u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(M)}
\lesssim \frac{2^j}{(2^j |\nu - \nu'|)\frac{1}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j'^{\nu'},
\]

where we used in the last inequality the estimates (8.238) and (8.372).

Next, we estimate the \(L^1(M)\) norm of \(h_2\). In view of the definition of \(h_2\) (8.467), we have:

\[
\|h_2\|_{L^1(M)} \lesssim \left\| \int_{\mathbb{S}^2} P_{>2^j |\nu - \nu'|} \text{tr} \chi F_{j, -1}(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(M)} \tag{8.469}
\times \left\| \int_{\mathbb{S}^2} b^{-1} L'(P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi') F_j (u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(M)}
\lesssim \frac{2^j}{(2^j |\nu - \nu'|)\frac{1}{2}} \varepsilon^2 \gamma_j^\nu \gamma_j'^{\nu'},
\]

where we used in the last inequality the estimates (8.129) and (8.435).

Finally, (8.465), (8.468) and (8.469) imply the following estimate in the range of \((l, m)\) such that \(2^m \leq 2^l |\nu - \nu'| < 2^l\):

\[
\sum_{(l, m)/2^m \leq 2^l |\nu - \nu'| < 2^l} B_{j, \nu, \nu', l, m}^{1, 2, 3} \lesssim 2^{-j} (\|h_1\|_{L^1(M)} + \|h_2\|_{L^1(M)}) \tag{8.470}
\lesssim \left[\frac{1}{2^j (2^j |\nu - \nu'|)\frac{1}{2}} + \frac{1}{2^j (2^j |\nu - \nu'|)\frac{1}{2}}\right] \varepsilon^2 \gamma_j^\nu \gamma_j'^{\nu'}.
\]
Next, we estimate \( B_{j,v,v',l,m}^{1,2,3} \) in the range of \((l,m)\) such that:

\[
2^m \leq 2^j |\nu - \nu'|.
\]

We have the following decomposition for \( B_{j,v,v',l,m}^{1,2,3} \):

\[
B_{j,v,v',l,m}^{1,2,3} = B_{j,v,v',l,m}^{1,2,3,1} + B_{j,v,v',l,m}^{1,2,3,2},
\]

(8.471)

where \( B_{j,v,v',l,m}^{1,2,3,1} \) and \( B_{j,v,v',l,m}^{1,2,3,2} \) are given by:

\[
B_{j,v,v',l,m}^{1,2,3,1} = i2^{-j-1} \int_M \int_{S^2 \times S^2} b^{-1} L(P_l \text{tr} \chi P_m \text{tr} \chi' F_{j,-1}(u)F_j(u') \eta_j^\nu(\omega)\eta_j'^\nu(\omega')d\omega d\omega' dM,
\]

and

\[
B_{j,v,v',l,m}^{1,2,3,2} = i2^{-j-1} \int_M \int_{S^2 \times S^2} b^{-1} P_l \text{tr} \chi' P_m \text{tr} \chi' F_{j,-1}(u)F_j(u') \eta_j^\nu(\omega)\eta_j'^\nu(\omega')d\omega d\omega' dM.
\]

(8.472)

(8.473)

We first estimate \( B_{j,v,v',l,m}^{1,2,3,1} \). We integrate by parts tangentially using (7.136).

**Lemma 8.16** Let \( B_{j,v,v',l,m}^{1,2,3,1} \) defined by (8.472). Integrating by parts using (7.136) yields:

\[
B_{j,v,v',l,m}^{1,2,3,1} = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_M \frac{1}{(2^\frac{j}{2}|N_\nu - N_{\nu'}|)^{p+q}} \left[ \frac{1}{|N_\nu - N_{\nu'}|^2} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) + \frac{1}{|N_\nu - N_{\nu'}|} (h_{3,p,q,l,m} + h_{4,p,q,l,m}) + h_{5,p,q,l,m} \right] dM,
\]

(8.474)

where \( c_{pq} \) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} p^p q^q
\]

has radius of convergence 1, where the scalar functions \( h_{1,p,q,l,m} \), \( h_{2,p,q,l,m} \), \( h_{3,p,q,l,m} \), \( h_{4,p,q,l,m} \), \( h_{5,p,q,l,m} \) on \( M \) are given by:

\[
h_{1,p,q,l,m} = \left( \int_{S^2} \chi L(P_l \text{tr} \chi) \left(2^\frac{j}{2}(N - N_\nu)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \right)
\times \left( \int_{S^2} P_m \text{tr} \chi' \left(2^\frac{j}{2}(N' - N_{\nu'})^q F_{j,-1}(u')\eta_j'^\nu(\omega')d\omega' \right) \right),
\]

(8.475)

\[
h_{2,p,q,l,m} = \left( \int_{S^2} L(P_l \text{tr} \chi) \left(2^\frac{j}{2}(N - N_\nu)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega \right) \right)
\times \left( \int_{S^2} \chi' P_m \text{tr} \chi' \left(2^\frac{j}{2}(N' - N_{\nu'})^q F_{j,-1}(u')\eta_j'^\nu(\omega')d\omega' \right) \right),
\]

(8.476)
\[ h_{3,p,q,l,m} = \left( \int_{S^2} G_1 \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} P_m tr\chi' \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \quad (8.477) \]

\[ h_{4,p,q,l,m} = \left( \int_{S^2} L(P_t tr\chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} G_2 \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \quad (8.478) \]

and:

\[ h_{5,p,q,l,m} = \left( \int_{S^2} L(P_t tr\chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} N'(P_m tr\chi') \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \quad (8.479) \]

and where the tensors \( G_1 \) and \( G_2 \) on \( M \) are given by:

\[ G_1 = \nabla(L(P_t tr\chi)) + (\theta + b^{-1}\nabla(b))L(P_t tr\chi), \quad (8.480) \]

and:

\[ G_2 = \nabla'(P_m tr\chi') + (\theta' + b'^{-1}\nabla(b'))P_m tr\chi'. \quad (8.481) \]

The proof of lemma 8.16 is postponed to Appendix F. In the rest of this section, we use Lemma 8.16 to obtain the control of \( B_{j,\nu',\nu,l,m}^{1,2,3,1} \).

We estimate the \( L^1(M) \) norm of \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m} \) starting with \( h_{1,p,q,l,m} \). In view of the definition (8.475) of \( h_{1,p,q,l,m} \), we have:

\[ \|h_{1,p,q,l,m}\|_{L^1(M)} \lesssim \left\| \int_{S^2} \chi L(P_t tr\chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \times \left\| \int_{S^2} P_m tr\chi' \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim (1 + q^2)\varepsilon^\gamma_j^\nu \left\| \int_{S^2} \chi L(P_t tr\chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}, \quad (8.482) \]

where we used in the last inequality the estimate (8.181). Now, the analog of the estimate (8.371) with \( r = 0 \) yields:

\[ \left\| \int_{S^2} \chi L(P_t tr\chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \varepsilon^{2^{\frac{1}{2}} \gamma_j^\nu}, \quad (8.483) \]

which together with (8.482) implies:

\[ \|h_{1,p,q,l,m}\|_{L^1(M)} \lesssim (1 + q^2)2^{\frac{1}{2}}\varepsilon^{2^{\frac{1}{2}} \gamma_j^\nu} \gamma_j'^\nu. \quad (8.484) \]
Finally, (8.485), (8.486) and (8.487) imply:

In view of the estimates (8.179) and (8.180), we have:

\[ \left\| h_{2,p,q,l,m} \right\|_{L^1(M)} \lesssim \left\| \int_{S^2} L(P_{\text{tr}} \chi) \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \]
\[ \times \left\| \int_{S^2} \chi^\prime P_{m} \text{tr} \chi^\prime \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)}. \]

The basic estimate in \( L^2(M) \) (7.1) yields:

\[ \left\| \int_{S^2} L(P_{\text{tr}} \chi) \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \]
\[ \lesssim \left( \sup_{\omega} \left\| L(P_{\text{tr}} \chi) \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p \right\|_{L^\infty L^2(\mathcal{H}_u)} \right)^{2^{\frac{1}{2}} \gamma_j^\nu} \]
\[ \lesssim \varepsilon 2^{\frac{1}{2}} \gamma_j^\nu, \]

where we used in the last inequality the analog of the estimate (8.418), the estimate (2.42) for \( \partial_u N \), and the size of the patch. Also, the analog of the estimate (8.428) yields:

\[ \left\| \int_{S^2} \chi^\prime P_{m} \text{tr} \chi^\prime \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon (1 + q^2) \gamma_j^\nu. \]

Finally, (8.485), (8.486) and (8.487) imply:

\[ \left\| h_{2,p,q,l,m} \right\|_{L^1(M)} \lesssim (1 + q^2) 2^{\frac{1}{2}} \varepsilon 2^{\frac{1}{2}} \gamma_j^\nu \gamma_j^\nu. \]

Next, we estimate the \( L^1(M) \) norm of \( h_{3,p,q,l,m} \). In view of the definition (8.477) of \( h_{3,p,q,l,m} \), we have:

\[ \sum_{m \leq l} h_{3,p,q,l,m} = \left( \int_{S^2} G_1 \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \]
\[- \left( \int_{S^2} P_{\leq l} \text{tr} \chi^\prime \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right), \]

which yields:

\[ \left\| \sum_{m \leq l} h_{3,p,q,l,m} \right\|_{L^1(M)} \lesssim \left\| \int_{S^2} G_1 \left( 2^{\frac{1}{2}} (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \]
\[ \times \left\| \int_{S^2} P_{\leq l} \text{tr} \chi^\prime \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)}. \]

In view of the estimates (8.179) and (8.180), we have:

\[ \left\| \int_{S^2} P_{\leq l} \text{tr} \chi^\prime \left( 2^{\frac{1}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim (1 + q^2) \varepsilon 2^{\frac{1}{2}} \gamma_j^\nu. \]
Also, using the basic estimate in $L^2(M)$ (7.1), we have:

$$\left\| \int_{S^2} G_1 \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \approx (\sup_{\omega} \left\| G_1 \left( 2^\frac{j}{2} (N - N_\nu) \right)^p \right\|_{L^\infty_{\omega} L^2(H_u)}) 2^\frac{j}{2} \gamma^\nu_j$$

which together with (8.489) and (8.490) yields:

$$\left\| \sup_{\omega} \left\| G_1 \left( 2^\frac{j}{2} (N - N_\nu) \right)^p \right\|_{L^\infty_{\omega} L^2(H_u)} \right\|_{L^\infty_{\omega} L^2(H_u)} \approx 2^\frac{j}{2} \varepsilon,$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of the definition (8.478) of $G_1$, we have:

$$\| G_1 \|_{L^\infty_{\omega} L^2(H_u)} \lesssim \| \nabla (L(P_1 \text{tr} \chi)) \|_{L^\infty_{\omega} L^2(H_u)} + \| (\theta + b^{-1} \nabla (b)) L(P_1 \text{tr} \chi) \|_{L^\infty_{\omega} L^2(H_u)} \lesssim 2^\frac{j}{2} \varepsilon,$$

where we used the estimate (8.193) for the first term and the estimate (8.186) for the second term. Now, (8.491) and (8.492) imply:

$$\left\| \int_{S^2} G_1 \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim \varepsilon 2^\frac{j}{2} + \frac{1}{2} \gamma_j^\nu,$$

which together with (8.489) and (8.490) yields:

$$\left\| \sum_{m \leq l} h_{3,p,q,l,m} \right\|_{L^1(M)} \lesssim (1 + q^2) 2^\frac{j}{2} + \frac{1}{2} \varepsilon 2^\gamma_j^\nu \gamma_j^\nu.$$ (8.493)

Next, we estimate the $L^1(M)$ norm of $h_{4,p,q,l,m}$. In view of the definition (8.478) of $h_{4,p,q,l,m}$, we have:

$$\| h_{4,p,q,l,m} \|_{L^1(M)} \lesssim \left\| \int_{S^2} L(P_1 \text{tr} \chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}$$

$$\times \left\| \int_{S^2} G_2 \left( 2^\frac{j}{2} (N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j^\nu'(\omega') d\omega' \right\|_{L^2(M)}$$

$$\lesssim (1 + p^2) 2^\gamma_j \varepsilon 2^\gamma_j \left\| \int_{S^2} G_2 \left( 2^\frac{j}{2} (N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j^\nu'(\omega') d\omega' \right\|_{L^2(M)},$$

where we used in the last inequality the estimate (8.170). Now, the basic estimate in $L^2(M)$ (7.1) yields:

$$\left\| \int_{S^2} G_2 \left( 2^\frac{j}{2} (N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j^\nu'(\omega') d\omega' \right\|_{L^2(M)} \lesssim \left( \sup_{\omega'} \left\| G_2 \left( 2^\frac{j}{2} (N' - N_\nu') \right)^q \right\|_{L^\infty_{\omega'} L^2(H_{\omega'})} \right) 2^\gamma_j^\nu$$

$$\lesssim \left( \sup_{\omega'} \left\| G_2 \right\|_{L^\infty_{\omega'} L^2(H_{\omega'})} \right) 2^\gamma_j^\nu.$$ (8.495)
where we used in the last inequality the estimate (2.42) for $\partial \omega N$ and the size of the patch. Now, in view of the definition (8.481) for $G_2$, we have:

$$\|G_2\|_{L^\infty_\omega L^2(H_\omega)} \lesssim \|\nabla(P_{\leq t} \text{tr} \chi)\|_{L^\infty_\omega L^2(H_\omega)} + \|\theta' + b'-1 \nabla(b')\|_{L^\infty_\omega L^2(H_\omega)} \|P_{\leq t} \text{tr} \chi'\|_{L^\infty(M)} \lesssim \varepsilon,$$

where we used in the last inequality the finite band property and the boundedness on $L^\infty(P_{t,u})$ for $P_{\leq t}$, the estimate (2.39) for $\text{tr} \chi'$, the estimate (2.38) for $b'$, and the estimates (2.37) (2.39) (2.40) for $\theta'$. Together with (8.495), this yields:

$$\left\| \int_{S^2} G_2 \left(2 \frac{j}{2} (N' - N_{\nu'})^q F_{j,-1}(u') \eta_j' (\omega') d\omega' \right) \right\|_{L^2(M)} \lesssim \varepsilon 2^{\frac{j}{2}} \gamma_j'',$$

which in view of (8.494) implies:

$$\|h_{4,p,q,l,m}\|_{L^1(M)} \lesssim (1 + p^2) 2^{\frac{11j}{12}} \varepsilon 2^{\gamma_j' \gamma_{j'}}. \quad (8.496)$$

Next, we estimate the $L^1(M)$ norm of $h_{5,p,q,l,m}$. In view of the definition (8.479) of $h_{5,p,q,l,m}$, we have:

$$\|h_{5,p,q,l,m}\|_{L^1(M)} \lesssim \left\| \int_{S^2} L(P_{t} \text{tr} \chi) \left(2 \frac{j}{2} (N' - N_{\nu'})^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right) \right\|_{L^2(M)} \quad (8.497)$$

$$\times \left\| \int_{S^2} N'(P_{m} \text{tr} \chi') \left(2 \frac{j}{2} (N' - N_{\nu'})^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right) \right\|_{L^2(M)}$$

$$\lesssim (1 + p^2) 2^{\frac{11j}{12}} \varepsilon 2^{\gamma_j' \gamma_{j'}},$$

where we used in the last inequality the size of the patch.

Finally, we have in view of (8.474):

$$\left| \sum_{m/m' \leq l} B_{j,\nu',p,q,l,m}^{1,2,3,1} \right| \lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2 \frac{j}{2} |N_{\nu} - N_{\nu'}|)^{p+q}} \right\|_{L^\infty(M)}$$

$$\times \left[ \left\| \frac{1}{|N_{\nu} - N_{\nu'}|^2} \right\|_{L^\infty(M)} \sum_{m/m' \leq l} (\|h_{1,p,q,l,m}\|_{L^1(M)} + \|h_{2,p,q,l,m}\|_{L^1(M)}) \right.$$  

$$+ \left\| \frac{1}{|N_{\nu} - N_{\nu'}|} \right\|_{L^1(M)} \left( \sum_{m/m' \leq l} \|h_{3,p,q,l,m}\|_{L^1(M)} + \sum_{m/m' \leq l} \|h_{4,p,q,l,m}\|_{L^1(M)} \right)$$

$$\left. + \sum_{m/m' \leq l} \|h_{5,p,q,l,m}\|_{L^1(M)} \right] dM.$$  

Together with the estimates (8.32), (8.484), (8.488), (8.493), (8.496) and (8.497), and the
fact that we are in the range $2^m \leq 2^l \leq 2^j |\nu - \nu'|$, we obtain:

$$
\left| \sum_{m/m \leq l} B_{j,\nu,\nu',l,m}^{1,2,3,1} \right| \lesssim 2^{-2j} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^l |\nu - \nu'|)^{p+q}} \times \left( \frac{1}{|\nu - \nu'|^2} (1 + q^2) j 2^{\frac{4}{l} + \frac{j}{2}} \right. \\
+ (1 + p^2) j 2^{\frac{11}{l} + \frac{j}{2}} \left. \right) \varepsilon^{2j,j,j'} \gamma_j^{\nu} \gamma_j^{\nu'}
$$

Summing in $l$, we finally obtain:

$$
\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^l |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3,1} \right| \lesssim \left[ \frac{j 2^{\frac{4}{l} + \frac{j}{2}}}{(2^l |\nu - \nu'|)^2} + \frac{1}{2^{\frac{11}{l} + \frac{j}{2}} (2^l |\nu - \nu'|)^{\frac{j}{2}}} \right] + 2^{-j} \varepsilon^{2j,j,j'} \gamma_j^{\nu} \gamma_j^{\nu'} \tag{8.498}
$$

Next, we estimate $B_{j,\nu,\nu',l,m}^{1,2,3,2}$ in the range $2^m \leq 2^l \leq 2^j |\nu - \nu'|$. We obtain the analog of the estimate (8.498):

$$
\left| \sum_{(l,m)/2^{\max(l,m)} \leq 2^l |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3,2} \right| \lesssim \left[ \frac{j 2^{\frac{4}{l} + \frac{j}{2}}}{(2^l |\nu - \nu'|)^2} + \frac{1}{2^{\frac{11}{l} + \frac{j}{2}} (2^l |\nu - \nu'|)^{\frac{j}{2}}} \right] + 2^{-j} \varepsilon^{2j,j,j'} \gamma_j^{\nu} \gamma_j^{\nu'} \tag{8.499}
$$

To this end, we proceed exactly as for $B_{j,\nu,\nu',l,m}^{1,2,3,1}$, the only difference being that we integrate by parts tangentially using (7.137) instead of (7.136) to obtain the analog of Lemma 8.16. This is left to the reader.

Finally, the decomposition (8.471) of $B_{j,\nu,\nu',l,m}^{1,2,3}$ together with the estimates (8.498) and (8.499) imply:

$$
\sum_{(l,m)/2^{\max(l,m)} \leq 2^l |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3} \lesssim \left[ \frac{j 2^{\frac{4}{l} + \frac{j}{2}}}{(2^l |\nu - \nu'|)^2} + \frac{1}{2^{\frac{11}{l} + \frac{j}{2}} (2^l |\nu - \nu'|)^{\frac{j}{2}}} \right] + 2^{-j} \varepsilon^{2j,j,j'} \gamma_j^{\nu} \gamma_j^{\nu'}
$$

Together with the estimate (8.470) in the range of $(l,m)$ such that $2^m \leq 2^l |\nu - \nu'| < 2^l$, we obtain:

$$
\sum_{(l,m)/2^{\min(l,m)} \leq 2^l |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{1,2,3} \lesssim \left[ \frac{j 2^{\frac{4}{l} + \frac{j}{2}}}{(2^l |\nu - \nu'|)^2} + \frac{1}{2^{\frac{11}{l} + \frac{j}{2}} (2^l |\nu - \nu'|)^{\frac{j}{2}}} + \frac{1}{2^{\frac{11}{l} + \frac{j}{2}} (2^l |\nu - \nu'|)^{\frac{j}{2}}} \right] + 2^{-j} \varepsilon^{2j,j,j'} \gamma_j^{\nu} \gamma_j^{\nu'}.
$$
This concludes the proof of Proposition 8.12.

### 8.3 Proof of Proposition 8.2 (Control of $B^2_{j,\nu,\nu', l, m}$)

Recall the definition of $B^2_{j,\nu,\nu', l, m}$ (8.11):

$$B^2_{j,\nu,\nu', l, m} = -i2^{-j} \int_{M} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( (g(N, N') - 1)P_{tr\chi}N'(P_{m\chi}) + \left( \text{tr\chi} - \bar{\delta} \right. \right.$$

$$\left. - \bar{\delta}' - (1 - g(N, N'))\delta' - 2\zeta_N - g(N, N')N' \right) \frac{\chi'(N - g(N, N')N, N - g(N, N')N')}{g(L, L')} \left. \right)$$

$$\times P_{tr\chi}P_{m\chi} \left. \right) F_j(u) F_{j,-1}(u') \eta'_j(\omega) \eta''_j(\omega') d\omega d\omega' d\mathcal{M}.$$

We first consider the range of $(l, m)$ such that:

$$2^m \leq 2^{\nu - \nu'} < 2^j.$$

This yields:

$$\sum_{m/2^m \leq 2^{\nu - \nu'}} B^2_{j,\nu,\nu', l, m}$$

$$= -i2^{-j} \int_{M} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( (g(N, N') - 1)P_{tr\chi}N'(P_{\leq 2^{\nu - \nu'}}\chi') + \left( \text{tr\chi} - \bar{\delta} \right.$$

$$\left. - \bar{\delta}' - (1 - g(N, N'))\delta' - 2\zeta_N - g(N, N')N' \right) \frac{\chi'(N - g(N, N')N, N - g(N, N')N')}{g(L, L')} \left. \right)$$

$$\times P_{tr\chi}P_{\leq 2^{\nu - \nu'}}\chi' \left. \right) F_j(u) F_{j,-1}(u') \eta'_j(\omega) \eta''_j(\omega') d\omega d\omega' d\mathcal{M}.$$

Together with the identity (8.30):

$$g(L, L') = -1 + g(N, N'),$$

we obtain:

$$\sum_{m/2^m \leq 2^{\nu - \nu'}} B^2_{j,\nu,\nu', l, m}$$

$$= -i2^{-j} \int_{M} \left( \int_{S^2} b^{-1} P_{tr\chi}F_j(u) \eta'_j(\omega) d\omega \right) \left( N'(P_{\leq 2^{\nu - \nu'}}\chi') F_{j,-1}(u') \eta''_j(\omega') d\omega' \right) d\mathcal{M}$$

$$- i2^{-j} \int_{M} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( \text{tr\chi} - \bar{\delta} - \bar{\delta}' - (1 - g(N, N'))\delta' - 2\zeta_N - g(N, N')N' \right) \frac{\chi'(N - g(N, N')N, N - g(N, N')N')}{g(L, L')} \left. \right)$$

$$\times F_j(u) F_{j,-1}(u') \eta'_j(\omega) \eta''_j(\omega') d\omega d\omega' d\mathcal{M}.$$
Next, recall the identities (8.30) and (8.31):

\[ g(L, L') = -1 + g(N, N') \quad \text{and} \quad 1 - g(N, N') = \frac{g(N - N', N - N')}{2} \]

We may thus expand \( \frac{1}{g(L, L')} \) in the same fashion than (8.33), and in view of (8.500), we obtain, schematically:

\[
\sum_{m/2^m \leq 2^{j}|\nu - \nu'|} B^2_{j,\nu,\nu',l,m} = 2^{-\frac{j}{2}} \sum_{p, q \geq 0} c_{pq} \int_{\mathcal{M}} \left( \frac{1}{2^j |N_\nu - N_{\nu'}|} \frac{1}{|N_\nu - N_{\nu'}|} (h_{1,p,q,l} + h_{2,p,q,l} + h_{3,p,q,l}) \right) d\mathcal{M}
\]

\[
- i2^{-j} \int_{\mathcal{M}} \left( \int_{\mathbb{S}^2} b^{-1} P_l \text{tr} \chi F_j(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{\mathbb{S}^2} N'(P_{\leq 2^{j}|\nu - \nu'|} \text{tr} \chi') F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right) d\mathcal{M},
\]

where the scalar functions \( h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l} \) on \( \mathcal{M} \) are given by:

\[
h_{1,p,q,l} = \left( \int_{\mathbb{S}^2} (\text{tr} - \delta) P_l \text{tr} \chi \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{\mathbb{S}^2} P_{\leq 2^{j}|\nu - \nu'|} \text{tr} \chi' \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right),
\]

\[
h_{2,p,q,l} = \left( \int_{\mathbb{S}^2} P_l \text{tr} \chi \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{\mathbb{S}^2} (-\delta' - \chi') P_{\leq 2^{j}|\nu - \nu'|} \text{tr} \chi' \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right),
\]

and:

\[
h_{3,p,q,l} = \left( \int_{\mathbb{S}^2} P_l \text{tr} \chi \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{\mathbb{S}^2} \zeta' P_{\leq 2^{j}|\nu - \nu'|} \text{tr} \chi' \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right),
\]

where \( c_{pq} \) are explicit real coefficients such that the series

\[
\sum_{p, q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1.
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{1,p,q,l}, h_{2,p,q,l}, h_{3,p,q,l}$ starting with $h_{1,p,q,l}$. We have:

\[
\|h_{1,p,q,l}\|_{L^1(\mathcal{M})} \lesssim \left\| \frac{1}{\sqrt{2}} \right\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^p F_{j-1}(u) \eta_{j'}^p(\omega) \right\|_{L^2(\mathcal{M})} (8.505) \]

\[
\times \left\| \frac{1}{\sqrt{2}} \right\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^q F_{j-1}(u) \eta_{j'}^q(\omega) \right\|_{L^2(\mathcal{M})} ,
\]

where we used (8.372) in the last inequality. The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

\[
\left\| \frac{1}{\sqrt{2}} \right\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^p F_{j-1}(u) \eta_{j'}^p(\omega) \right\|_{L^2(\mathcal{M})} (8.506) \]

\[
\lesssim (1 + q^2) \varepsilon \gamma \left\| \frac{1}{\sqrt{2}} \right\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^q F_{j-1}(u) \eta_{j'}^q(\omega) \right\|_{L^2(\mathcal{M})} ,
\]

Now, for any tensor $G$, we have:

\[
\|GP_m \mathcal{F}_{\delta}\|_{L^p_{\omega}L^q(\mathcal{H}_\omega)} \lesssim \|G\|_{L^p_{\omega}L^q} \|P_m \mathcal{F}_{\delta}\|_{L^p_{\omega}L^q},
\]

which together with the estimate (2.69) for $P_m \mathcal{F}_{\delta}$ yields:

\[
\|GP_m \mathcal{F}_{\delta}\|_{L^p_{\omega}L^q(\mathcal{H}_\omega)} \lesssim 2^{-m} \varepsilon \|G\|_{L^p_{\omega}L^q} .
\]

Using the estimate (8.507) with $G = \mathcal{F}_{\delta}$, the estimate (2.39) for $\mathcal{F}_{\delta}$, the estimates (2.36) (2.37) for $\delta$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch, we have:

\[
\left\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^p F_{j-1}(u) \eta_{j'}^p(\omega) \right\|_{L^p_{\omega}L^q(\mathcal{H}_\omega)} \lesssim \varepsilon^{-1}.
\]

Together with (8.505) and (8.506), we obtain:

\[
\|h_{1,p,q,l}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2)2^{-l+\frac{1}{2}} \varepsilon^{2} \gamma \gamma_j'.
\]

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l}$. In view of the definition (8.503), we have:

\[
\|h_{2,p,q,l}\|_{L^1(\mathcal{M})} (8.509) \]

\[
\lesssim \left\| \frac{1}{\sqrt{2}} \right\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^p F_{j-1}(u) \eta_{j'}^p(\omega) \right\|_{L^2(\mathcal{M})} \times \left\| \frac{1}{\sqrt{2}} \right\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^q F_{j-1}(u) \eta_{j'}^q(\omega) \right\|_{L^2(\mathcal{M})}.
\]

The basic estimate in $L^2(\mathcal{M})$ yields:

\[
\left\| \frac{1}{\sqrt{2}} \right\| \left( \mathcal{F}_{\delta} \mathcal{F}_{\delta} \right)^p F_{j-1}(u) \eta_{j'}^p(\omega) \right\|_{L^2(\mathcal{M})} (8.510) \]

\[
\lesssim (1 + q^2)2^{-l+\frac{1}{2}} \varepsilon \gamma \gamma_j',
\]

\[
\lesssim 2^{-l+\frac{1}{2}} \varepsilon \gamma_j'.
\]
where we used in the last inequality the finite band property for $P_i$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. On the other hand, the decomposition (2.76) for $\chi'$, the decomposition (7.132) for $\delta'$, and the estimate (2.42) for $\partial_\omega N$ yield the following decomposition for $\delta' + \chi'$:

$$
\delta' + \chi' = F_1^j + F_2^j
$$

(8.511)

where the tensor $F_1^j$ only depends on $\nu'$ and satisfies:

$$
\|F_1^j\|_{L^\infty_{\nu'}, x'_\nu, L^2_t} \lesssim \varepsilon,
$$

(8.512)

and where the tensor $F_2^j$ satisfies:

$$
\|F_2^j\|_{L^\infty_{\nu}, L^2(\mathcal{H}_{\nu'})} \lesssim \varepsilon 2^{-\frac{j}{2}}.
$$

(8.513)

In view of (8.511), we obtain:

$$
\left\| \int_{S^2} (-\delta' - \chi') P_{\leq 2|\nu'\nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} 
\lesssim \|F_1^j\|_{L^\infty_{\nu'}, x'_\nu, L^2_t} \left\| \int_{S^2} P_{\leq 2|\nu'\nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} 
+ \left\| \int_{S^2} F_2^j P_{\leq 2|\nu'\nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} 
\lesssim (1 + q^2) \varepsilon \gamma_j' + \left\| \int_{S^2} F_2^j P_{\leq 2|\nu'\nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)},
$$

where we used in the last inequality the estimate (8.512) for $F_1^j$ and the estimate (8.426). Now, the basic estimate in $L^2(M)$ (7.1) yields:

$$
\left\| \int_{S^2} F_2^j P_{\leq 2|\nu'\nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} 
\lesssim \left( \sup_{\omega'} \left\| F_2^j P_{\leq 2|\nu'\nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty_{\nu}' L^2(\mathcal{H}_{\nu'})} \right) 2^{\frac{j}{2}} \gamma_j' 
\lesssim \varepsilon \gamma_j',
$$

where we used in the last inequality the estimate (8.513) for $F_2^j$, the boundedness of $P_{\leq 2|\nu'\nu'|}$ on $L^\infty(P_t, \omega')$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Together with (8.514), this yields:

$$
\left\| \int_{S^2} (-\delta' - \chi') P_{\leq 2|\nu'\nu'|} \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j' (\omega') d\omega' \right\|_{L^2(M)} \lesssim (1 + q^2) \varepsilon \gamma_j'.
$$

Together with (8.509) and (8.510), we finally obtain:

$$
\|h_{2, \rho, q,l}\|_{L^1(M)} \lesssim (1 + q^2) 2^{-l+\frac{j}{2}} \varepsilon^2 \gamma_j' \gamma_j'.
$$

(8.515)
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{3,p,q,t}$. In view of the definition (8.504), we have:

$$\|h_{3,p,q,t}\|_{L^2(\mathcal{M})} \lesssim \left\| \int_{S^2} P_{t \chi} \left( 2^{\frac{j}{2}} (N - N_{ts}) \right)^p F_{t,j-1} (u) \eta^{\nu}_j (\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\times \left\| \int_{S^2} \zeta' \chi' \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim 2^{-l+\frac{s}{2} \varepsilon \gamma^{\nu}_j} \left\| \int_{S^2} \zeta' \chi' \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^2(\mathcal{M})},$$

where we used (8.510) in the last inequality. In order to estimate the right-hand side of (8.516), we use the decomposition (2.80) of $\zeta'$. We have:

$$\zeta' = F_1^j + F_2^j$$

(8.517)

where the tensor $F_1^j$ only depends on $\nu'$ and satisfies:

$$\|F_1^j\|_{L^2_{\nu'} L^2_{t'}, L^8_{L^2_{\nu'}}} \lesssim \varepsilon,$$

(8.518)

and where the tensor $F_2^j$ satisfies:

$$\|F_2^j\|_{L^8_{\nu', t', \varepsilon_{\gamma^{\nu}_j}}} \lesssim \varepsilon^2 \gamma^{\nu}_j.$$

(8.519)

In view of (8.517), we have:

$$\left\| \int_{S^2} \zeta' \chi' \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \|F_1^j\|_{L^2_{\nu'} L^2_{t'}, L^8_{L^2_{\nu'}}} \left\| \int_{S^2} P_{t \chi} \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$+ \left\| \int_{S^2} F_2^j \chi' \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \varepsilon \left\| \int_{S^2} P_{t \chi} \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$+ \left\| \int_{S^2} F_2^j \chi' \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^2(\mathcal{M})},$$

where we used the estimate (8.518) for $F_1^j$ in the last inequality. We have the analog of (8.272):

$$\left\| \int_{S^2} P_{t \chi} \left( 2^{\frac{j}{2}} (N - N_{t}) \right)^p F_{t,j-1} (u) \eta^{\nu}_j (\omega) d\omega \right\|_{L^\infty(\mathcal{M})} \lesssim \varepsilon 2^j \gamma^{\nu}_j.$$

(8.521)

Now, interpolating between the the estimate in $L^2_{u_{\nu'}, x_{\nu'}, t'} L^\infty_{L^2_{t'}}$ (8.426) and the $L^\infty$ estimate (8.521), we obtain:

$$\left\| \int_{S^2} P_{t \chi} \left( 2^{\frac{j}{2}} (N' - N_{t'}) \right)^q F_{t,j-1} (u') \eta^{\nu'}_{j'} (\omega') d\omega' \right\|_{L^8_{u_{\nu'}, x_{\nu'}, t'} L^\infty_{L^2_{t'}}} \lesssim 2^j \varepsilon \gamma^{\nu}_j.$$
For the second term in the right-hand side of (8.520), we have:
\[
\left\| \int_{S^2} F_j^2 P_{\leq 2^l|\nu-\nu'|} \chi'_j \left( 2^\frac{l}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \\
\lesssim \int_{S^2} \left\| F_j^2 P_{\leq 2^l|\nu-\nu'|} \chi'_j \left( 2^\frac{l}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \\
\lesssim \left\| F_j^2 \right\|_{L^\infty L^2(\mathcal{H}_\nu)} \left\| F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \\ 
\lesssim 2^{-\frac{l}{2}} \varepsilon \int_{S^2} \left\| F_j(u') \right\|_{L^2(\mathcal{H}_\nu)} \eta_j^{\nu'}(\omega') d\omega' \\
\lesssim \varepsilon 2^\frac{l}{2} \gamma_j^\nu,
\]
where we used in the last inequality Plancherel in \( \lambda' \), Cauchy Schwartz in \( \omega' \) and the size of the patch. Finally, (8.520), (8.522) and (8.523) imply:
\[
\left\| \int_{S^2} \zeta^l P_{\leq 2^l|\nu-\nu'|} \chi'_j \left( 2^\frac{l}{2} (N' - N_{\nu'}) \right)^q F_j(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \lesssim \varepsilon 2^\frac{l}{2} \gamma_j^\nu.
\] (8.524)
Together with (8.516), this yields:
\[
\left\| h_{3,p,q,l} \right\|_{L^2(M)} \lesssim 2^{-l+\frac{1}{2}} 2^\frac{l}{2} \varepsilon 2^\frac{l}{2} \gamma_j^\nu.
\] (8.525)
Next, we estimate the last term in the right-hand side of (8.501):
\[
\int_M \left( \int_{S^2} b^{-1} P_l \chi_j F_j(u) \eta_j^{\nu}(\omega) d\omega \right) \left( N' \left( P_{\leq 2^l|\nu-\nu'|} \chi'_j \right) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) dM.
\]
We have:
\[
\left\| \int_M \left( \int_{S^2} b^{-1} P_l \chi_j F_j(u) \eta_j^{\nu}(\omega) d\omega \right) \left( N' \left( P_{\leq 2^l|\nu-\nu'|} \chi'_j \right) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right) dM \right\| \\
\lesssim \left\| \int_{S^2} b^{-1} P_l \chi_j F_j(u) \eta_j^{\nu}(\omega) d\omega \right\|_{L^2(M)} \left\| \int_{S^2} N' \left( P_{\leq 2^l|\nu-\nu'|} \chi'_j \right) F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(M)} \\
\lesssim 2^{-l+\frac{1}{2}} \varepsilon 2^\frac{l}{2} \gamma_j^\nu \gamma_j^\nu,\] (8.526)
where we used in the last inequality the estimate (8.510) and the estimate (8.317).
Together with (8.32), (8.508), (8.515), (8.525) and (8.526), we obtain:

Let Lemma 8.17
We integrate by parts in tangential directions using (7.137).

Next, we consider the range of \( (\nu - \nu') < 2^l \):

Summing in \( l \), we finally obtain in the range \( 2^m \leq 2^l |\nu - \nu'| < 2^l \):

Next, we consider the range of \( (l, m) \) such that:

We integrate by parts in tangential directions using (7.137).

**Lemma 8.17** Let \( B_{j,\nu,\nu',l,m}^2 \) be defined by (8.11). Integrating by parts using (7.137) yields:

\[
B_{j,\nu,\nu',l,m}^2 = 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \int_\mathcal{M} \frac{1}{(2^\frac{j}{2}|N_\nu - N_{\nu'}|)^{p+q+1}} \left[ \frac{1}{|N_\nu - N_{\nu'}|^2} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) + \frac{1}{|N_\nu - N_{\nu'}|^3} \right] d\mathcal{M}
\]

\[
+ \sum_{p,q \geq 0} c_{pq} \int_\mathcal{M} \frac{1}{(2^\frac{j}{2}|N_\nu - N_{\nu'}|)^{p+q+1}} \left[ \frac{1}{|N_\nu - N_{\nu'}|^2} (h_{3,p,q,l,m} + h_{4,p,q,l,m} + h_{5,p,q,l,m}) + \frac{1}{|N_\nu - N_{\nu'}|^2} \right] d\mathcal{M}
\]

\[
+ \sum_{p,q \geq 0} c_{pq} \int_\mathcal{M} \frac{1}{(2^\frac{j}{2}|N_\nu - N_{\nu'}|)^{p+q+1}} \left[ \frac{1}{|N_\nu - N_{\nu'}|^3} (h_{6,p,q,l,m} + h_{7,p,q,l,m} + h_{8,p,q,l,m} + h_{9,p,q,l,m} + h_{10,p,q,l,m}) \right] d\mathcal{M}
\]

\[
+ \text{terms interverting \((\nu, \nu')\)} + B_{j,\nu,\nu',l,m}^{2,1} + B_{j,\nu,\nu',l,m}^{2,2}.
\]
where \( c_{pq} \) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1, where the scalar functions \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m}, h_{6,p,q,l,m}, h_{7,p,q,l,m}, h_{8,p,q,l,m}, h_{9,p,q,l,m}, h_{10,p,q,l,m} \) on \( \mathcal{M} \) are given by:

\[
h_{1,p,q,l,m} = \left( \int_{S^2} \chi(\chi + \bar{\delta}) P_l tr \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} P_m tr \chi' \left( 2^{\frac{1}{2}} (N' - N'_\nu) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
\]

\[
h_{2,p,q,l,m} = \left( \int_{S^2} (\chi + \delta) P_l tr \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} (\chi' + \bar{\delta}) P_m tr \chi' \left( 2^{\frac{1}{2}} (N' - N'_\nu) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
\]

\[
h_{3,p,q,l,m} = \left( \int_{S^2} G_1 \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} P_m tr \chi' \left( 2^{\frac{1}{2}} (N' - N'_\nu) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
\]

\[
h_{4,p,q,l,m} = \left( \int_{S^2} \nabla(P_l tr \chi) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} (\chi' + \bar{\delta}) P_m tr \chi' \left( 2^{\frac{1}{2}} (N' - N'_\nu) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
\]

\[
h_{5,p,q,l,m} = \left( \int_{S^2} \chi(\chi + \bar{\delta}) P_l tr \chi \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla(P_m tr \chi') \left( 2^{\frac{1}{2}} (N' - N'_\nu) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
\]

\[
h_{6,p,q,l,m} = \left( \int_{S^2} G_2 \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla(P_m tr \chi') \left( 2^{\frac{1}{2}} (N' - N'_\nu) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
\]

\[
h_{7,p,q,l,m} = \left( \int_{S^2} (N(P_l tr \chi) + \nabla(P_l tr \chi)) \left( 2^{\frac{1}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \chi(\chi + \bar{\delta}) P_m tr \chi' \left( 2^{\frac{1}{2}} (N' - N'_\nu) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
\]

\[
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\]
\[
\begin{align*}
\delta_{h, p, q, l, m} &= \left( \int_{S^2} (\theta + b^{-1}(b)P_l \text{tr}\chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right) F_{j-1}(u)\eta_j^\nu(\omega)d\omega \right) \\
&\quad \times \left( \int_{S^2} \zeta' P_m \text{tr}\chi' \left( 2^{\frac{1}{2}}(N' - N_{\nu'}) \right) F_{j-1}(u')\eta_j'^\nu(\omega')d\omega' \right), \\
\delta_{h_{9, p, q, l, m}} &= \left( \int_{S^2} P_l \text{tr}\chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right) F_{j-1}(u)\eta_j^\nu(\omega)d\omega \right) \\
&\quad \times \left( \int_{S^2} \nabla'(N'(P_m \text{tr}\chi')) \left( 2^{\frac{1}{2}}(N' - N_{\nu'}) \right) F_{j-1}(u')\eta_j'^\nu(\omega')d\omega' \right), \\
\text{and:} \\
\delta_{h_{10, p, q, l, m}} &= \left( \int_{S^2} P_l \text{tr}\chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right) F_{j-1}(u)\eta_j^\nu(\omega)d\omega \right) \\
&\quad \times \left( \int_{S^2} b^{-1}(b')N'(P_m \text{tr}\chi') \left( 2^{\frac{1}{2}}(N' - N_{\nu'}) \right) F_{j-1}(u')\eta_j'^\nu(\omega')d\omega' \right),
\end{align*}
\]

where the tensors \( G_1 \) and \( G_2 \) on \( \mathcal{M} \) are given by:

\[
G_1 = (\chi + \bar{\delta})\nabla P_l \text{tr}\chi + (\nabla(\chi) + \nabla(\bar{\delta}) + (\chi + \bar{\delta})(\theta + b^{-1}(b)\nabla(\bar{b})))P_l \text{tr}\chi, \\
\]

and:

\[
G_2 = (\chi + \bar{\delta})N(P_l \text{tr}\chi) + \zeta\nabla P_l \text{tr}\chi + (\nabla(\chi) + N(\bar{\delta}) + \nabla(\zeta)\theta)P_l \text{tr}\chi,
\]

and where \( B_{j, \nu, \nu', l, m}^{2, 1} \) and \( B_{j, \nu, \nu', l, m}^{2, 2} \) are defined by:

\[
B_{j, \nu, \nu', l, m}^{2, 1} = 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} N(P_l \text{tr}\chi) F_{j-1}(u)\eta_j^\nu(\omega)d\omega \right) \\
\quad \times \left( \int_{S^2} \nabla'(P_m \text{tr}\chi') F_{j-1}(u')\eta_j'^\nu(\omega')d\omega' \right) d\mathcal{M},
\]

and:

\[
B_{j, \nu, \nu', l, m}^{2, 2} = 2^{-2j} \int_{\mathcal{M}} \int_{S^2} \left( \frac{N - g(N, N')N(P_l \text{tr}\chi)N'(P_m \text{tr}\chi')}{1 - g(N, N')^2} \right) \\
\quad \times F_{j-1}(u)F_{j-1}(u')\eta_j^\nu(\omega)\eta_j'^\nu(\omega')d\omega d\omega' d\mathcal{M}.
\]

The proof of Lemma 8.17 is postponed to Appendix G. In the rest of this section, we use Lemma 8.17 to control \( B_{j, \nu, \nu', l, m}^{2, 2} \) over the range of \( (l, m) \) such that \( 2^m \leq 2^l \leq 2^j|\nu - \nu'| \).

### 8.3.1 Control of the \( L^1(\mathcal{M}) \) norm of \( h_{1, p, q, l, m} \)

We estimate the \( L^1(\mathcal{M}) \) norm of \( h_{1, p, q, l, m}, h_{2, p, q, l, m}, h_{3, p, q, l, m}, h_{4, p, q, l, m}, h_{5, p, q, l, m}, h_{6, p, q, l, m}, h_{7, p, q, l, m}, h_{8, p, q, l, m}, h_{9, p, q, l, m}, h_{10, p, q, l, m} \) starting with \( h_{1, p, q, l, m} \). Consider first the case \( l > j/2 \).

Let \( H \) be defined by:

\[
H = \int_{S^2} P_m \text{tr}\chi \left( 2^{\frac{1}{2}}(N' - N_{\nu'}) \right) F_{j-1}(u')\eta_j'^\nu(\omega')d\omega'.
\]
Then, we have in view of the definition (8.529) of \( h_{1,p,q,l,m} \):
\[
\| h_{1,p,q,l,m} \|_{L^1(M)} \lesssim \left\| \int_{S^2} \chi H(\chi + \delta) P_{l} \text{tr} \left( 2^\frac{l}{2} (N - N_\nu) \right)^p F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^1(M)}
\]
(8.544)
\[
\lesssim \int_{S^2} \| \chi H \|_{L^2(M)} \left\| \left( \chi + \delta \right) P_{l} \text{tr} \left( 2^\frac{l}{2} (N - N_\nu) \right)^p F_{j,1}(u) \right\|_{L^2(M)} \eta_j^\nu(\omega) d\omega
\]
\[
\lesssim \int_{S^2} \| \chi \|_{L^\infty(M)} \| H \|_{L^1(M)} \| \chi + \delta \|_{L^\infty(M)} \| P_{l} \text{tr} \left( 2^\frac{l}{2} (N - N_\nu) \right)^p F_{j,1}(u) \|_{L^2(M)} \eta_j^\nu(\omega) d\omega
\]
\[
\times \left\| \left( 2^\frac{l}{2} (N - N_\nu) \right)^p \right\|_{L^\infty(M)} \| F_{j,1}(u) \|_{L^2(M)} \eta_j^\nu(\omega) d\omega
\]
\[
\lesssim \varepsilon^{-l} \int_{S^2} \| H \|_{L^2(M)} \| F_{j,1}(u) \|_{L^2(M)} \eta_j^\nu(\omega) d\omega,
\]
where we used in the last inequality the estimate (8.507) with the choice \( G = \chi + \delta \), the estimates (2.39) (2.40) for \( \chi \), the estimates (2.36) (2.37) for \( \delta \), the estimate (2.42) for \( \partial_{\omega} N \) and the size of the patch. Next we estimate the term in \( H \) in the right-hand side of (8.544). Using the estimate (7.71) in the case \( m > j/2 \), we have:
\[
\left\| \int_{S^2} P_{m} \text{tr} \left( 2^\frac{l}{2} (N' - N_\nu) \right)^q F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim (1 + q^\frac{2}{3}) \varepsilon (2^\frac{l}{2} |\nu - \nu'| + 1) \gamma_j^\nu.
\]
(8.545)
Also, using the decomposition:
\[
P_{\leq j/2} \text{tr} \chi' = \text{tr} \chi' - \sum_{m/m > j/2} P_m \text{tr} \chi',
\]
we obtain for all \( m \geq j/2 \):
\[
\left\| \int_{S^2} P_{\leq j/2} \text{tr} \left( 2^\frac{l}{2} (N' - N_\nu) \right)^q F_{j,1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim (1 + q^\frac{2}{3}) \varepsilon (2^\frac{l}{2} |\nu - \nu'| + 1) \gamma_j^\nu.
\]
(8.546)
In view of the definition (8.533) of \( H \), this yields:
\[
\| H \|_{L^1(M)} \lesssim \varepsilon (2^\frac{l}{2} |\nu - \nu'| + 2^{-m+\frac{l}{2}}) 2^{-l} \gamma_j^\nu.
\]
Together with (8.544), we obtain in the case \( l > j/2 \):
\[
\| h_{1,p,q,l,m} \|_{L^1(M)} \lesssim \varepsilon^2 (2^\frac{l}{2} |\nu - \nu'| + 2^{-m+\frac{l}{2}}) 2^{-l} \gamma_j^\nu.
\]
(8.547)
where we used in the last inequality Plancherel in $\lambda$ for $\|F_{j,-1}(u)\|_{L^2}$, Cauchy Schwartz in $\omega$ and the size of the patch.

Next, we consider the case $l = j/2$. Recall that in view of the decomposition for $\chi$ (2.76), we have:

$$\chi = F_1^j + F_2^j$$  \hspace{1cm} (8.548)

where the tensor $F_1^j$ only depends on $\nu$ and satisfies:

$$\|F_1^j\|_{L^\infty_{u,v,x,t}L^2_{\mathcal{E}}} \lesssim \varepsilon,$$  \hspace{1cm} (8.549)

and where the tensor $F_2^j$ satisfies:

$$\|F_2^j\|_{L^\infty_{u,v}L^2(H_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$  \hspace{1cm} (8.550)

In view of the definition (8.529) of $h_{1,p,q,l,m}$, the decomposition (8.548) and the definition of $H$ (8.543), we have in the case $l = j/2$:

\[
\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \|F_1^j\|_{L^\infty_{u,v,x,t}L^2_{\mathcal{E}}} \left|\int_{\mathbb{S}^2} (\chi + \delta)P_{\leq j/2} \text{tr} \chi \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} \|H\|_{L^\infty_{u,v,x,t}L^\infty} \\
+ \left|\int_{\mathbb{S}^2} F_2^j(\chi + \delta)HP_{\leq j/2} \text{tr} \chi \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} \\
\lesssim \varepsilon (2^j |\nu - \nu'|2^{-m+\frac{j}{2}} + (2^j |\nu - \nu'|)\frac{j}{2}2^{-\frac{m+j}{2}}) \gamma_j^\nu \\
\times \left|\int_{\mathbb{S}^2} (\chi + \delta)P_{\leq j/2} \text{tr} \chi \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} \\
+ \int_{\mathbb{S}^2} \left|F_2^j(\chi + \delta)HP_{\leq j/2} \text{tr} \chi \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right|_{L^1(\mathcal{M})},
\]

where we used in the last inequality the estimate (8.544) for $F_1^j$ and the estimate (8.546) for $H$. Next, we estimate the two terms in the right-hand side of (8.551) starting with the first one. In view of the decomposition (7.132) for $\delta$, $\delta$ also has a decomposition of the form (8.543) (8.544) (8.545), and thus so has $\chi + \delta$. Proceeding as in (8.423), (8.426) and (8.427), we obtain the analog of (8.428):

\[
\left|\int_{\mathbb{S}^2} (\chi + \delta)P_{\leq j/2} \text{tr} \chi \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} \lesssim \varepsilon (1 + q^2) \gamma_j^\nu. \hspace{1cm} (8.552)
\]

On the other hand, we have:

\[
\int_{\mathbb{S}^2} \left|F_2^j(\chi + \delta)HP_{\leq j/2} \text{tr} \chi \left(2^j (N - N_\nu)\right)^p F_{j,-1}(u)\eta_j^\nu(\omega)d\omega\right|_{L^1(\mathcal{M})} \]

\[
\lesssim \int_{\mathbb{S}^2} \left|F_2^j\right|_{L^\infty_{u,v}L^2(\mathcal{H}_u)} \|\chi + \delta\|_{L^\infty_{u,v,x,t}} \|H\|_{L^\infty_{u,v,x,t}} \left|P_{\leq j/2} \text{tr} \chi \left(2^j (N - N_\nu)\right)^p\right|_{L^\infty(\mathcal{M})} \\
\times \left|F_{j,-1}(u)\right|_{L^2_{u,v,x,t}} \|\eta_j^\nu(\omega)d\omega\right|_{L^\infty(\mathcal{M})} \\
\lesssim \varepsilon 2^{-\frac{j}{2}} \int_{\mathbb{S}^2} \|H\|_{L^2_{u,v,x,t}} \|F_{j,-1}(u)\|_{L^2_{u,v,x,t}} \|\eta_j^\nu(\omega)d\omega\right|_{L^\infty(\mathcal{M})},
\]

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where we used in the last inequality the estimate (8.550) for $F^2_j$, the estimates (2.39) (2.40) for $\chi$, the estimates (2.36) (2.37) for $\overline{\delta}$, the boundedness of $P_{\leq j/2}$ on $L^\infty(P_{t,u})$, the estimate (2.42) for $\partial_u N$ and the size of the patch. Together with the estimate (8.546) for $H$, Plancherel in $\lambda$ for $\|F_{j-1}(u)\|_{L^2_\omega}$, Cauchy Schwartz in $\omega$ and the size of the patch, we obtain:

$$\int_{S^2} \left| \int_{S^2} \left( \frac{1}{2} (N - N_\nu) \right) F_{j-1}(u) \right|_{L^1(\mathcal{M})} \eta_j^\nu(\omega) d\omega (8.553) \lesssim \varepsilon^2 \left( 2 \frac{j}{2} |\nu - \nu'| 2^{-m + \frac{j}{2}} + (2 \frac{j}{2} |\nu - \nu'| ) \frac{1}{4} 2^{-\frac{m}{2} + \frac{j}{4}} \right) \gamma_j^\nu \gamma_j'^\nu.$$

Now, (8.551), (8.552) and (8.553) yield in the case $l = j/2$:

$$\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) \varepsilon^2 \left( 2 \frac{j}{2} |\nu - \nu'| 2^{-m + \frac{j}{2}} + (2 \frac{j}{2} |\nu - \nu'| ) \frac{1}{4} 2^{-\frac{m}{2} + \frac{j}{4}} \right) \gamma_j^\nu \gamma_j'^\nu.$$

Together with (8.547), we finally obtain for all $l \geq j/2$:

$$\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) \varepsilon^2 \left( 2 \frac{j}{2} |\nu - \nu'| 2^{-m + \frac{j}{2}} + (2 \frac{j}{2} |\nu - \nu'| ) \frac{1}{4} 2^{-\frac{m}{2} + \frac{j}{4}} \right) 2^{\frac{j}{2} - l} \gamma_j^\nu \gamma_j'^\nu (8.554).$$

### 8.3.2 Control of the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l,m}$

Next, we estimate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l,m}$. In view of the definition (8.530) of $h_{2,p,q,l,m}$, we have:

$$\|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} \left( \frac{1}{2} (N - N_\nu) \right) F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} (8.555) \times \left\| \int_{S^2} \left( \frac{1}{2} (N' - N_\nu) \right) F_{j-1}(u') \eta_j'^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})},$$

In the case $l > j/2$, we use the basic estimate in $L^2(\mathcal{M})$:

$$\left\| \int_{S^2} \left( \frac{1}{2} (N - N_\nu) \right) F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \left\| \left( \frac{1}{2} (N - N_\nu) \right) \eta_j^\nu(\omega) \right\|_{L^\infty L^2(\mathcal{H}_u)} \right) 2^{\frac{j}{2} - l} \gamma_j^\nu \lesssim \varepsilon 2^{\frac{j}{2} - l} \gamma_j^\nu,$$

where we used in the last inequality the estimate (8.507) with the choice $G = \chi + \overline{\delta}$, the estimates (2.39) (2.40) for $\chi$, the estimates (2.36) (2.37) for $\overline{\delta}$, the estimate (2.42) for $\partial_u N$ and the size of the patch. Together with (8.549), we obtain for all $l \geq j/2$:

$$\left\| \int_{S^2} \left( \frac{1}{2} (N - N_\nu) \right) F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} - l} \gamma_j^\nu. (8.556)$$

Finally, (8.555), (8.556) and the analog of (8.556) for the second term in the right-hand side of (8.555) implies:

$$\|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{j - m} \gamma_j^\nu \gamma_j'^\nu. (8.557)$$
8.3.3 Control of the $L^1(\mathcal{M})$ norm of $h_{3,p,q,l,m}$

In view of the definition (8.531) of $h_{3,p,q,l,m}$ and in view of the definition (8.539) of $G_1$, we have:

$$
\sum_{l/l \geq m} h_{3,p,q,l,m} = \left( \int_{S^2} \tilde{G}_1 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right),
$$

where $\tilde{G}_1$ is given by:

$$
\tilde{G}_1 = (\chi + \delta) \nabla P_{\geq m} \text{tr} \chi + (\nabla(\chi) + \nabla(\delta) + (\chi + \delta)(\theta + b^{-1} \nabla(b))) P_{\geq m} \text{tr} \chi. \tag{8.558}
$$

This yields the following estimate:

$$
\left\| \sum_{l/l \geq m} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} \tilde{G}_1 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} 
\times \left\| \int_{S^2} P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \tag{8.559}
$$

Now, using (7.71) in the case $m > j/2$, and (8.271) in the case $m = j/2$, we obtain for all $m \geq j/2$:

$$
\left\| \int_{S^2} P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right\|_{L^2_{u,v',x', L^\infty}} \lesssim (1 + q^2) \varepsilon 2^{\frac{j}{2}-m} \gamma_j'. \tag{8.560}
$$

Together with (8.559), this yields:

$$
\left\| \sum_{l/l \geq m} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) 2^{\frac{j}{2}-m} \gamma_j' \left\| \int_{S^2} \tilde{G}_1 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}. \tag{8.561}
$$

Next, we estimate the right-hand side of (8.561). The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$
\left\| \int_{S^2} \tilde{G}_1 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega} \left\| \tilde{G}_1 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^\infty L^2(H_u)} \right) 2^{\frac{j}{2}} \gamma_j' \tag{8.562}
$$

$$
\lesssim \left( \sup_{\omega} \left\| \tilde{G}_1 \right\|_{L^\infty L^2(H_u)} \right) 2^{\frac{j}{2}} \gamma_j',
$$

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where we used in the last inequality the estimate (2.42) for $\partial_N N$ and the size of the patch.

In view of (8.558), we have:

$$\left\| \tilde{G}_1 \right\|_{L^\infty L^2(H_u)} \lesssim (\|\chi\|_{L^\infty L^2} + \|\tilde{\delta}\|_{L^\infty L^2})(\|\nabla \text{tr} \chi\|_{L^2 L^\infty} + \|\nabla P_{\leq m} \text{tr} \chi\|_{L^2 L^\infty}) \quad (8.563)$$

$$+ (\|\nabla (\chi)\|_{L^\infty L^2(H_u)} + \|\nabla (\tilde{\delta})\|_{L^\infty L^2(H_u)}) \quad (8.564)$$

where we used in the last inequality the estimates (2.39) (2.40) for $\chi$, the estimates (2.37) (2.36) for $\tilde{\delta}$, the estimate (2.38) for $b$, the estimates (2.37) (2.39) (2.40) for $\theta$, the decomposition:

$$\text{tr} \chi = P_{\leq m} \text{tr} \chi + P_{> m} \text{tr} \chi,$$

the estimate (2.70) for $\nabla P_{\leq m} \text{tr} \chi$, and the fact that $P_{\geq m}$ is bounded on $L^\infty(P_{t,u})$. Finally, (8.61), (8.62) and (8.63) imply:

$$\left\| \sum_{l/\ell \geq m} h_{3,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim (1 + q^2) \varepsilon 2j^{-m} \gamma_j \gamma_j'. \quad (8.64)$$

8.3.4 Control of the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l,m}$

In view of the definition (8.532) of $h_{4,p,q,l,m}$, we have:

$$\sum_{l/\ell \geq m} h_{4,p,q,l,m} = \left( \int_{S^2} \nabla (P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} (\chi' + \tilde{\delta}') P_m \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu})\right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right)$$

which yields:

$$\left\| \sum_{l/\ell \geq m} h_{4,p,q,l,m} \right\|_{L^1(\mathcal{M})} \leq \left( \int_{S^2} \nabla (P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right)_{L^2(\mathcal{M})}$$

$$\times \left( \int_{S^2} (\chi' + \tilde{\delta}') P_m \text{tr} \chi' \left(2^{\frac{j}{2}}(N' - N_{\nu})\right)^q F_{j,-1}(u') \eta_j'(\omega') d\omega' \right)_{L^2(\mathcal{M})}$$

$$\lesssim \varepsilon 2^{\frac{j}{2} - m} \gamma_j \left( \int_{S^2} \nabla (P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right)_{L^2(\mathcal{M})},$$

where we used in the last inequality the estimates (8.565).

Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$\left\| \int_{S^2} \nabla (P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(\mathcal{M})} \quad (8.66)$$

$$\lesssim \left( \sup_{\omega} \left\| \nabla (P_{\geq m} \text{tr} \chi) \left(2^{\frac{j}{2}}(N - N_{\nu})\right)^p \|_{L^\infty L^2(H_u)} \right) 2^{\frac{j}{2}} \gamma_j' \right.$$
where we used in the last inequality the finite band property for $P_{\geq m}$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (8.565) and (8.566) yield:

$$
\left\| \sum_{l / l \geq m} h_{4,p,q,l,m} \right\|_{L^1(M)} \lesssim \varepsilon^{2j - m} \gamma_j^\nu \gamma_j^\nu'.
$$

(8.567)

8.3.5 Control of the $L^1(M)$ norm of $h_{5,p,q,l,m}$

In view of the definition (8.533) of $h_{5,p,q,l,m}$, we have:

$$
\left\| h_{5,p,q,l,m} \right\|_{L^1(M)} \lesssim \left\| \int_{S^2} (\chi + \delta) P_l \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
\times \left\| \int_{S^2} (\theta' + \zeta') P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)}
\lesssim \varepsilon^{2j - l} \lambda_j^\nu \left\| \int_{S^2} (\theta' + \zeta') P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)},
$$

where we used in the last inequality the estimate (8.556).

Now, the basic estimate in $L^2(M)$ (7.1) yields:

$$
\left\| \int_{S^2} (\theta' + \zeta') P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim \left( \sup_{\omega'} \left\| (\theta' + \zeta') P_m \text{tr} \chi' \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L_{H_\nu}^\infty L^2(H_\nu)} \right)^{2^{\frac{j}{2}} \gamma_j^\nu},
$$

(8.569)

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. We have:

$$
\left\| (\theta' + \zeta') P_m \text{tr} \chi' \right\|_{L_{H_\nu}^\infty L^2(H_\nu)} \lesssim \left( \left\| \theta' \right\|_{L_{H_\nu}^\infty L^4(H_\nu)} + \left\| \zeta' \right\|_{L_{H_\nu}^\infty L^4(H_\nu)} \right) \left\| P_m \text{tr} \chi' \right\|_{L_{H_\nu}^\infty L^4(H_\nu)} \lesssim 2^{-\frac{m}{2}(1-\delta_{m,j/2})} \varepsilon,
$$

(8.570)

where we used in the last inequality the embedding (2.51), the estimates (2.37) (2.39) (2.40) for $\theta'$, the estimates (2.41) for $\zeta'$, the Bernstein inequality and the finite band property for $P_m$, and the estimate (2.39) for $\text{tr} \chi'$. Note that the factor $1 - \delta_{j/2,m}$ comes from the fact that we use the finite band property for $P_m$ in the case $m > j/2$, but only the boundedness of $P_{\leq j/2}$ in the case $m = j/2$. Finally, (8.568), (8.569) and (8.570) yield:

$$
\left\| h_{5,p,q,l,m} \right\|_{L^1(M)} \lesssim \varepsilon^{2j - l - \frac{m}{2}(1-\delta_{m,j/2})} \gamma_j^\nu \gamma_j^\nu'.
$$

(8.571)
8.3.6 Control of the $L^1(\mathcal{M})$ norm of $h_{6,p,q,l,m}$

In view of the definition (8.534) of $h_{6,p,q,l,m}$, we have:

$$
\| h_{6,p,q,l,m} \|_{L^1(\mathcal{M})} \leq \left\| \int_{\mathcal{S}^2} G_2 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \tag{8.572}
$$

\[
\times \left\| \int_{\mathcal{S}^2} P_m \text{tr}_x \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.
\]

\[
\lesssim \varepsilon^{2^{j} - m} \gamma_j^\nu \left\| \int_{\mathcal{S}^2} G_2 \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})},
\]

where we used in the last inequality the estimate (8.556).

Next, we estimate $G_2$. In view of the definition (8.540) of $G_2$, we have:

$$
\| G_2 \|_{L^\infty_t L^2(\mathcal{M}_x)} \lesssim \left( \| \chi \|_{L^\infty_t L^4_x} + \| \bar{\delta} \|_{L^\infty_t L^4_x} \right) \| N(P_t \text{tr}_x) \|_{L^2_t L^4_x} \tag{8.574}
$$

\[
\quad + \| \zeta \|_{L^\infty_t L^4_x} \| \nabla P_t \text{tr}_x \|_{L^2_t L^4_x} + \| D_N(\chi) \|_{L^\infty_t L^2(\mathcal{M}_x)}
\]

\[
\quad + \| N(\bar{\delta}) \|_{L^\infty_t L^2(\mathcal{M}_x)} + \| \nabla(\zeta) \theta \|_{L^\infty_t L^2(\mathcal{M}_x)} \| P_t \text{tr}_x \|_{L^\infty_t L^4_x}
\]

\[
\lesssim \varepsilon (\| N(P_t \text{tr}_x) \|_{L^2_t L^4_x} + \| \nabla P_t \text{tr}_x \|_{L^2_t L^4_x}) + \varepsilon,
\]

where we used in the last inequality the embedding (2.51), the estimates (2.39) (2.40) for $\chi$, the estimates (2.36) (2.37) for $\bar{\delta}$, the estimate (2.41) for $\zeta$, the estimates (2.37) (2.39) (2.40) for $\theta$, the estimate (2.39) for $\text{tr}_x$ and the boundedness of $P_t$ on $L^\infty_t (P_t \omega)$. Now, the estimate (2.38) for $b$ and the Gagliardo-Nirenberg inequality (2.49), yields:

$$
\| N(P_t \text{tr}_x) \|_{L^2_t L^4_x} + \| \nabla P_t \text{tr}_x \|_{L^2_t L^4_x}
\]

\[
\lesssim \| P_t(b \text{tr}_x) \|_{L^2_t L^4_x} + \| [b, P_t] \text{tr}_x \|_{L^2_t L^4_x} + \| \nabla^2 P_t \text{tr}_x \|_{L^\infty_t L^2(\mathcal{M}_x)} \| \nabla P_t \text{tr}_x \|_{L^\infty_t L^2(\mathcal{M}_x)}
\]

\[
\lesssim 2^{\frac{j}{2}} \| b \text{tr}_x \|_{L^\infty_t L^2(\mathcal{M}_x)} + \| \nabla [b, P_t] \text{tr}_x \|_{L^\infty_t L^2(\mathcal{M}_x)} \| [b, P_t] \text{tr}_x \|_{L^\infty_t L^2(\mathcal{M}_x)} + 2^{\frac{j}{2}} \| \nabla \text{tr}_x \|_{L^\infty_t L^2(\mathcal{M}_x)},
\]

where we used in the last inequality the Gagliardo-Nirenberg inequality (2.49), the Bernstein inequality for $P_t$, the Bohrcher inequality (2.61), and the finite band property for $P_t$. Together with the estimate (2.39) for $\text{tr}_x$ and the commutator estimate (2.68) for $[b, P_t] \text{tr}_x$, we obtain:

$$
\| N(P_t \text{tr}_x) \|_{L^2_t L^4_x} + \| \nabla P_t \text{tr}_x \|_{L^2_t L^4_x} \lesssim 2^{\frac{j}{2}} \varepsilon. \tag{8.575}
$$

Finally, (8.572), (8.573), (8.574) and (8.575) imply:

$$
\| h_{6,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim \varepsilon^{2^{j} + \frac{j}{2} - m} \gamma_j^\nu \gamma_j^{\nu'} \tag{8.576}
$$
8.3.7 Control of the $L^1(\mathcal{M})$ norm of $h_{\tau,p,q,l,m}$

In view of the definition (8.535) of $h_{\tau,p,q,l,m}$, we have:

\[
\|h_{\tau,p,q,l,m}\|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} \left( N(P_{\text{tr}}\chi) + \nabla(P_{\text{tr}}\chi) \right) \left( 2^{2\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
\times \left\| \int_{S^2} \left( \chi' + \delta' + \zeta' \right) P_m \text{tr} \chi' \left( 2^{2\frac{1}{2}}(N' - N_\nu) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.
\]  

(8.577)

The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

\[
\left\| \int_{S^2} \left( N(P_{\text{tr}}\chi) + \nabla(P_{\text{tr}}\chi) \right) \left( 2^{2\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u)\eta_j^{\nu}(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
\lesssim \left( \sup_\omega \left\| (N(P_{\text{tr}}\chi) + \nabla(P_{\text{tr}}\chi)) \left( 2^{2\frac{1}{2}}(N - N_\nu) \right)^p \right\|_{L^\infty L^2(\mathcal{H}_\omega)} \right)^{\frac{2}{2} \gamma_j^{\nu}} \\
\lesssim \left( \sup_\omega \left\| (N(P_{\text{tr}}\chi)) \left( 2^{2\frac{1}{2}}(N - N_\nu) \right)^p \right\|_{L^\infty L^2(\mathcal{H}_\omega)} + \left\| \nabla(P_{\text{tr}}\chi)) \right\|_{L^\infty L^2(\mathcal{H}_\omega)} \right)^{\frac{2}{2} \gamma_j^{\nu}} \\
\lesssim \varepsilon 2^{\frac{1}{2}} \gamma_j^{\nu},
\]

where we used in the last inequality the estimate (8.263) for $N(P_{\text{tr}}\chi)$, the finite band property for $P$, the estimate (2.39) for $\text{tr}\chi$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch.

Now, since $k$ does not depend on $\omega$, and in view of the decomposition (2.72) (2.74) for $\chi'$, and the decomposition (2.80) for $z'$, we have the following decomposition for $\chi' + \delta' + \zeta'$:

\[
\chi' + \delta' + \zeta' = F_1^j + F_2^j
\]

where the tensor $F_1^j$ only depends on $\nu'$ and satisfies:

\[
\|F_1^j\|_{L^\infty L^2 L^8_{\nu'} L^8_{\nu'}} \lesssim \varepsilon,
\]

and where the tensor $F_2^j$ satisfies:

\[
\|F_2^j\|_{L^\infty L^2(\mathcal{H}_\nu)} \lesssim \varepsilon 2^{-\frac{3}{4}}.
\]

Note that this decomposition has the same properties as the decomposition (8.267) (8.268) (8.269) for $\theta + b^{-1}\nabla(b)$. Thus, arguing as in (8.270)-(8.275), we obtain:

\[
\left\| \int_{S^2} \left( \chi' + \delta' + \zeta' \right) P_m \text{tr} \chi' \left( 2^{2\frac{1}{2}}(N' - N_\nu) \right)^q F_{j,-1}(u')\eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{1}{2}} \varepsilon \gamma_j^{\nu'}.
\]  

(8.579)

Finally, (8.577), (8.578) and (8.579) imply:

\[
\|h_{\tau,p,q,l,m}\|_{L^1(\mathcal{M})} \leq \varepsilon 2^{\frac{3}{2}} \gamma_j^{\nu'} \gamma_j^{\nu'}.
\]  

(8.580)
8.3.8 Control of the $L^1(M)$ norm of $h_{8,p,q,l,m}$

In view of the definition (8.536) of $h_{8,p,q,l,m}$, we have:

$$
\|h_{8,p,q,l,m}\|_{L^1(M)} \leq \left\| \int_{S^2} (\theta + b^{-1}\nabla(b)) P_t \chi \left( 2^j (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \tag{8.581}
$$

$$
\times \left\| \int_{S^2} \zeta' P_m \chi' \left( 2^j (N' - N_{\nu}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)}
$$

$$
\lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu \left\| \int_{S^2} \zeta' P_m \chi' \left( 2^j (N' - N_{\nu}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)},
$$

where we used the estimate (8.275) in the last inequality. Also, we have the analog of (8.579):

$$
\left\| \int_{S^2} \zeta' P_m \chi' \left( 2^j (N' - N_{\nu}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j^\nu. \tag{8.582}
$$

Finally, (8.581) and (8.582) imply:

$$
\|h_{8,p,q,l,m}\|_{L^1(M)} \lesssim 2^{\frac{j}{2}} \varepsilon^2 \gamma_j^\nu. \tag{8.583}
$$

8.3.9 Control of the $L^1(M)$ norm of $h_{9,p,q,l,m}$

In view of the definition (8.537) of $h_{9,p,q,l,m}$, we have:

$$
\|h_{9,p,q,l,m}\|_{L^1(M)} \leq \left\| \int_{S^2} P_t \chi \left( 2^j (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}
$$

$$
\times \left\| \int_{S^2} \nabla' (P_m \chi') \left( 2^j (N' - N_{\nu}) \right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(M)}
$$

$$
\lesssim \left( \int_{S^2} \left\| P_t \chi \left( 2^j (N - N_{\nu}) \right)^p F_{j,-1}(u) \right\|_{L^2(M)} \eta_j^\nu(\omega) d\omega \right)
$$

$$
\times \left( \int_{S^2} \left\| \nabla' (P_m \chi') \left( 2^j (N' - N_{\nu}) \right)^q F_{j,-1}(u') \right\|_{L^2(M)} \eta_j^\nu(\omega') d\omega' \right).
$$

Together with the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch, we obtain:

$$
\|h_{9,p,q,l,m}\|_{L^1(M)} \lesssim \left( \int_{S^2} \left\| P_t \chi \right\|_{L^2(H_\omega)} F_{j,-1}(u) \right\|_{L^2_\omega} \eta_j^\nu(\omega) d\omega \right)
$$

$$
\times \left( \int_{S^2} \left\| \nabla' (P_m \chi') \right\|_{L^2(H_\omega)} F_{j,-1}(u') \right\|_{L^2_{\omega'}} \eta_j^\nu(\omega') d\omega' \right)
$$

$$
\lesssim 2^{-j} \left\| P_t \chi \right\|_{L^2(H_\omega)} F_{j,-1}(u) \sqrt{\eta_j^\nu(\omega)} \right\|_{L^2_\omega}
$$

$$
\times \left\| \nabla' (P_m \chi') \right\|_{L^2(H_\omega)} F_{j,-1}(u') \sqrt{\eta_j^\nu(\omega')} \right\|_{L^2_{\omega'}}^2,
$$

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where we used in the last inequality Cauchy Schwarz in $\omega$ and $\omega'$, and the size of the patch. This yields:

$$\sum_{(l,m)/m\leq l} \| h_{g_{3},q,l,m} \|_{L^1(\mathcal{M})} \leq 2^{-j} \sum_{(l,m)} 2^{-|l-m|} \left( 2^l \| |\nu| \|_{L^2(\mathcal{H}_u)} |F_{j-1}(u)\sqrt{\eta_j^\prime(\omega)}\|_{L^2_{\omega,\nu}} \right) \times \left( 2^{-m} \| |\nu| (N'(P_m|\nu|'))\|_{L^2(\mathcal{H}_{u'})} F_{j-1}(u') \sqrt{\eta_j^\prime(\omega')}\right) \right) \right) \right)^{\frac{1}{2}}. $$

Now, we have:

$$\sum_{l} \| |\nu| \|_{L^2(\mathcal{H}_u)} |F_{j-1}(u)\sqrt{\eta_j^\prime(\omega)}\|_{L^2_{\omega,\nu}} \leq \int_{S^2} \left( \int_{u} \left( \sum_{l} 2^{l} \| |\nu| \|_{L^2(\mathcal{H}_u)} |F_{j}(u)|^2 \right) \| |\nu| \|_{L^2_{\omega,\nu}} \right)^{\frac{1}{2}} \sqrt{|\eta_j^\prime(\omega)}\right) d\omega$$

$$\leq \int_{S^2} \| |\nu| \|_{L^2_{\omega,\nu}} \| |\nu| \|_{L^2_{\omega,\nu}} \eta_j^\prime(\omega) d\omega$$

$$\leq \varepsilon^2 2^{2j} (\gamma_j^\prime)^2,$$

where we used the finite band property for $P_i$, the estimates (2.39) for $|\nu|$ and Plancherel in $\lambda$. Also, in view of the estimate (2.38) for $b'$, we have:

$$\| |\nu| (N'(P_m|\nu|'))\|_{L^2(\mathcal{H}_{u'})} \leq \| b' \|_{L^\infty \mathcal{E}_s} \| \eta_j^\prime \|_{L^2 \mathcal{E}_s} + \| |\nu| \|_{L^2_{\omega,\nu}} \| \eta_j^\prime \|_{L^2_{\omega,\nu}},$$

Together with the estimate (2.38) for $b$, the estimate (8.575) for $N'(P_m|\nu|')$, the finite band property for $P_m$, and the commutator estimate (2.68), we obtain:

$$\| |\nu| (N'(P_m|\nu|'))\|_{L^2(\mathcal{H}_{u'})} \leq 2^m \varepsilon + 2^m \| P_m(b' N'|\nu|')\|_{L^2(\mathcal{H}_{u'})}. $$

(8.586)
This yields:

\[
\sum_m 2^{-2m} \left\| \nabla' (N'(P_m \text{tr} \lambda')) \right\|_{L^2(H_{\omega'})}^2 \left\| \eta_j''(\omega') \right\|_{L^2_{\omega', \omega}}^2 \tag{8.587}
\]

\[
= \int_{S^2} \left( \int_{u'} \left( \sum_m \left( \| P_m (b'N' \text{tr} \lambda') \|_{L^2(H_{\omega'})} + 2^{-\frac{m}{2}} \varepsilon \right)^2 \right) |F_j(u')|^2 du' \right) \eta_j''(\omega') d\omega'
\]

\[
\lesssim \int_{S^2} \left( \| b'N' \text{tr} \chi \|_{L^\infty_{\omega'} L^2(H_{\omega'})} + \varepsilon^2 \right) \left[ F_j(u')^2 \right] \eta_j''(\omega') d\omega'
\]

\[
\lesssim \varepsilon^2 2^{2j}(\gamma_j')^2,
\]

where we used the finite band property for \( P_m \), the estimates (2.39) for \( \text{tr} \chi' \) and Plancherel in \( \lambda' \).

Finally, (8.584), (8.585) and (8.587) yield:

\[
\sum_{(l,m)/m \leq l} \| h_{l_0, p, q, l, m} \|_{L^1(\mathcal{M})} \lesssim 2^j \varepsilon^2 \gamma_j' \gamma_j'^{\prime}.
\tag{8.588}
\]

### 8.3.10 Control of the \( L^1(\mathcal{M}) \) norm of \( h_{l_0, p, q, l, m} \)

In view of the definition (8.538) of \( h_{l_0, p, q, l, m} \), we have:

\[
\| h_{l_0, p, q, l, m} \|_{L^1(\mathcal{M})}
\tag{8.589}
\]

\[
\leq \left\| \int_{S^2} P_l \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j''(\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\]

\[
\times \left\| \int_{S^2} b'^{-1} \nabla' (b') N'(P_m \text{tr} \lambda') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j''(\omega') d\omega' \right\|_{L^2(\mathcal{M})}.
\]

We estimate the first term in the right-hand side of (8.589). Using (8.510) for \( l > j/2 \) and (8.180) for \( l = j/2 \), we obtain for all \( l \geq j/2 \):

\[
\left\| \int_{S^2} P_l \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j''(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim (1 + p^2) 2^{-l+\frac{j}{2}} \varepsilon \gamma_j' \tag{8.590}
\]

Next, we estimate the second term in the right-hand side of (8.589). The basic estimate in \( L^2(\mathcal{M}) \) (7.1) yields:

\[
\left\| \int_{S^2} b'^{-1} \nabla' (b') N'(P_m \text{tr} \lambda') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j''(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \tag{8.591}
\]

\[
\lesssim \left( \sup_{\omega'} \left\| b'^{-1} \nabla' (b') N'(P_m \text{tr} \lambda') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^\infty_{\omega'} L^2(H_{\omega'})} \right) 2^{\frac{j}{2}} \gamma_j''
\]

\[
\lesssim \left( \sup_{\omega'} \left\| b'^{-1} \nabla' (b') N'(P_m \text{tr} \lambda') \right\|_{L^\infty_{\omega'} L^2(H_{\omega'})} \right) 2^{\frac{j}{2}} \gamma_j''
\]

Now, we have:

\[
\| b'^{-1} \nabla' (b') N'(P_m \text{tr} \lambda') \|_{L^\infty_{\omega'} L^2(H_{\omega'})} \lesssim \| b'^{-1} \nabla' (b') \|_{L^\infty_{\omega'} L^2(H_{\omega'})} \| N'(P_m \text{tr} \lambda') \|_{L^2 L^2_{\omega'}}
\]

\[
\lesssim \varepsilon^{2m}.
\]

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where we used in the last inequality the estimate (2.38) for \( b \) and the estimate (8.575) for \( N'(P_m \text{tr} \chi') \). Together with (8.591), we obtain:

\[
\left\| \int_{S^2} b^{-1} \nabla(b) N'(P_m \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu}) \right)^q F_{j,-1}(u') \eta'_{j} (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{\frac{j}{2} + \frac{q}{2}} \gamma_j'.
\] (8.592)

Finally, (8.589), (8.590) and (8.592) imply:

\[
\| h_{10, p, q, l, m} \|_{L^1(\mathcal{M})} \lesssim (1 + p^2) 2^{j - l + \frac{q}{2}} \varepsilon 2^{\frac{j}{2} + \frac{q}{2}} \gamma_j'.
\] (8.593)

### 8.3.11 Control of \( B^{2,1}_{j, \nu, \nu', l, m} \)

Recall the definition (8.541) of \( B^{2,1}_{j, \nu, \nu', l, m} \):

\[
B^{2,1}_{j, \nu, \nu', l, m} = 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} N(P \text{tr} \chi) F_{j,-1}(u) \eta_{j} (\omega) d\omega \right) \times \left( \int_{S^2} N'(P_m \text{tr} \chi') F_{j,-1}(u') \eta'_{j} (\omega') d\omega' \right) d\mathcal{M},
\]

Recall that we are considering the range of \((l, m)\):

\[
2^m \leq 2^l \leq 2^{|\nu - \nu'|}.
\]

Summing in \((l, m)\), we have:

\[
\sum_{(l, m)/2^m \leq 2^l \leq 2^{|\nu - \nu'|}} N(P \text{tr} \chi) N'(P_m \text{tr} \chi') = N(P_{\leq 2^{|\nu - \nu'|}} \text{tr} \chi) N'(P_{\leq 2^{|\nu - \nu'|}} \text{tr} \chi').
\]

Thus, using the symmetry in \((\omega, \omega')\) of the integrant in \( B^{2,1}_{j, \nu, \nu', l, m} \), we obtain in view of the definition \( B^{2,1}_{j, \nu, \nu', l, m} \):

\[
\sum_{(l, m)/2^m \leq 2^l \leq 2^{|\nu - \nu'|}} \left( B^{2,1}_{j, \nu, \nu', l, m} + B^{2,1}_{j, \nu, \nu', l, m} \right)
\]

\[
= 2^{-2j} \int_{\mathcal{M}} \left( \int_{S^2} N(P_{\leq 2^{|\nu - \nu'|}} \text{tr} \chi) F_{j,-1}(u) \eta_{j} (\omega) d\omega \right) \times \left( \int_{S^2} N'(P_{\leq 2^{|\nu - \nu'|}} \text{tr} \chi') F_{j,-1}(u') \eta'_{j} (\omega') d\omega' \right) d\mathcal{M},
\]

which yields:

\[
\left\| \sum_{(l, m)/2^m \leq 2^l \leq 2^{|\nu - \nu'|}} \left( B^{2,1}_{j, \nu, \nu', l, m} + B^{2,1}_{j, \nu, \nu', l, m} \right) \right\|_{L^2(\mathcal{M})} \leq 2^{-2j} \left\| \int_{S^2} N(P_{\leq 2^{|\nu - \nu'|}} \text{tr} \chi) F_{j,-1}(u) \eta_{j} (\omega) d\omega \right\|_{L^2(\mathcal{M})} \times \left\| \int_{S^2} N'(P_{\leq 2^{|\nu - \nu'|}} \text{tr} \chi') F_{j,-1}(u') \eta'_{j} (\omega') d\omega' \right\|_{L^2(\mathcal{M})}.
\] (8.594)
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we integrate by parts in tangential directions using (7.137). We finally obtain:

Together with the estimate (8.317) applied to both terms in the right-hand side of (8.594), we finally obtain:

\[
\left| \sum_{|l,m|/2^{\max(|l,m)|}\leq 2^{|\nu-\nu'|}} (B_{j,\nu',\nu,l,m}^2 + B_{j,\nu,\nu',l,m}^2) \right| \lesssim 2^{-j} 2^{2\nu_j' \nu_j'}. \tag{8.595}
\]

8.3.12 Control of \( B_{j,\nu',\nu,l,m}^2 \)

Recall the definition (8.542) of \( B_{j,\nu',\nu,l,m}^2 \):

\[
B_{j,\nu',\nu,l,m}^2 = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{(N' - g(N,N')(P_{\nu}\chi')N'(P_m\chi'))}{1 - g(N,N')^2} \times F_{j,-1}(u) F_{j,-1}(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}.
\]

We sum for \( m \leq l \), and we obtain:

\[
\sum_{m \leq l} B_{j,\nu',\nu,l,m}^2 = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{(N' - g(N,N')(P_{\nu}\chi')N'(P_{\nu'}\chi'))}{1 - g(N,N')^2} \times F_{j,-1}(u) F_{j,-1}(u') \eta_j'(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}. \tag{8.596}
\]

We cannot estimate \( \sum_{m \leq l} B_{j,\nu',\nu,l,m}^2 \) directly due to a lack of summability in \( l \). Instead, we integrate by parts in tangential directions using (7.137).

Lemma 8.18 Let \( \sum_{m \leq l} B_{j,\nu',\nu,l,m}^2 \) be defined by (8.596). Integrating by parts using (7.137) yields:

\[
\sum_{m \leq l} B_{j,\nu',\nu,l,m}^2 = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^2 |N_{\nu} - N_{\nu'}|)^{p+q+1}} \times \left[ \frac{1}{|N_{\nu} - N_{\nu'}|^2} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) + \frac{1}{|N_{\nu} - N_{\nu'}|} (h_{3,p,q,l,m} + h_{4,p,q,l,m}) + h_{5,p,q,l,m} + h_{6,p,q,l,m} + h_{7,p,q,l,m} + h_{8,p,q,l,m} \right] d\mathcal{M}, \tag{8.597}
\]

where \( c_{pq} \) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1, where the scalar functions \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m}, h_{6,p,q,l,m}, h_{7,p,q,l,m}, h_{8,p,q,l,m} \) on \( \mathcal{M} \) are given by:

\[
h_{1,p,q,l,m} = \left( \int_{S^2} \chi \nabla P_{\nu}' \left( 2^2 (N - N_{\nu}) \right) F_{j,-1}(u) \eta_j'(\omega) d\omega \right) \times \left( \int_{S^2} N'(P_{\nu'} \chi') \left( 2^2 (N' - N_{\nu'}) \right) F_{j,-1}(u') \eta_j'(\omega') d\omega' \right), \tag{8.598}
\]

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\[ h'_{2,p,q,l,m} = \left( \int_{S^2} \nabla (P_t \nabla \chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j' (\omega) d\omega \right) \]
\[ \times \left( \int_{S^2} \chi' N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N'_\nu) \right)^q F_{j-1}(u') \eta_j'' (\omega') d\omega' \right), \]
\[ h'_{3,p,q,l,m} = \left( \int_{S^2} \nabla^2 (P_t \nabla \chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j'' (\omega) d\omega \right) \]
\[ \times \left( \int_{S^2} N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N'_\nu) \right)^q F_{j-1}(u') \eta_j'' (\omega') d\omega' \right), \]
\[ h'_{4,p,q,l,m} = \left( \int_{S^2} (\theta + b^{-1} \nabla (b)) \nabla (P_t \nabla \chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j'' (\omega) d\omega \right) \]
\[ \times \left( \int_{S^2} N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N'_\nu) \right)^q F_{j-1}(u') \eta_j'' (\omega') d\omega' \right), \]
\[ h'_{5,p,q,l,m} = \left( \int_{S^2} (\theta + b^{-1} \nabla (b) + \theta') N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N_\nu) \right)^p F_{j-1}(u) \eta_j'' (\omega) d\omega \right) \]
\[ \times \left( \int_{S^2} N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N'_\nu) \right)^q F_{j-1}(u') \eta_j'' (\omega') d\omega' \right), \]
\[ h'_{6,p,q,l,m} = \left( \int_{S^2} \nabla (P_t \nabla \chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j'' (\omega) d\omega \right) \]
\[ \times \left( \int_{S^2} (b^{-1} \nabla (b') + \theta') N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N'_\nu) \right)^q F_{j-1}(u') \eta_j'' (\omega') d\omega' \right), \]
\[ h'_{7,p,q,l,m} = \left( \int_{S^2} \nabla (N(P_t \nabla \chi)) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j'' (\omega) d\omega \right) \]
\[ \times \left( \int_{S^2} N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N'_\nu) \right)^q F_{j-1}(u') \eta_j'' (\omega') d\omega' \right), \]
and:
\[ h'_{8,p,q,l,m} = \left( \int_{S^2} (\theta + N(b)) \nabla (P_t \nabla \chi) \left( 2^\frac{j}{2} (N - N_\nu) \right)^p F_{j-1}(u) \eta_j'' (\omega) d\omega \right) \]
\[ \times \left( \int_{S^2} N'(P_{\leq 1} \nabla \chi') \left( 2^\frac{j}{2} (N' - N'_\nu) \right)^q F_{j-1}(u') \eta_j'' (\omega') d\omega' \right). \]

The proof of Lemma 8.18 is postponed to Appendix H. In the rest of this section, we use Lemma 8.18 to control \( \sum_{m \leq t} B_{j,n,\nu,l,m}^{2,2} \) over the range of \((l,m)\) such that \(2^m \leq 2^l \leq 2^j |\nu - \nu'|\).
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{1,p,q,l,m}$, $h'_{2,p,q,l,m}$, $h'_{3,p,q,l,m}$, $h'_{4,p,q,l,m}$, $h'_{5,p,q,l,m}$, $h'_{6,p,q,l,m}$, $h'_{7,p,q,l,m}$, $h'_{8,p,q,l,m}$ starting with $h'_{1,p,q,l,m}$. In view of the definition (8.598), we have:

\[
\|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} \chi \nabla (P_{l}\chi) \left(2\frac{j}{j^*}(N - N_{\nu})\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}
\]

\[
\times \left\| \int_{S^2} N'(P_{\leq l}\chi'') \left(2\frac{j}{j^*}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j'^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\]

\[
\lesssim 2^{\frac{j}{j^*}} \varepsilon \gamma_j^\nu \left\| \int_{S^2} \chi \nabla (P_{l}\chi) \left(2\frac{j}{j^*}(N - N_{\nu})\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})},
\]

where we used in the last inequality the estimate (8.317). Now, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

\[
\left\| \int_{S^2} \chi \nabla (P_{l}\chi) \left(2\frac{j}{j^*}(N - N_{\nu})\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}
\]

\[
\lesssim \left( \sup_{\omega} \left\| \chi \nabla (P_{l}\chi) \left(2\frac{j}{j^*}(N - N_{\nu})\right)^p \right\|_{L^p(\mathcal{M})} \right) 2^{\frac{j}{j^*}} \gamma_j^\nu
\]

\[
\lesssim \left( \sup_{\omega} \|\chi\|_{L^p_{\nu} L^q_{\nu'}} \left\| \nabla (P_{l}\chi) \right\|_{L^q_{\nu'} L^p_{\nu}} \right) \left(2\frac{j}{j^*}(N - N_{\nu})\right)^p L^2(\mathcal{M}) 2^{\frac{j}{j^*}} \gamma_j^\nu
\]

\[
\lesssim \varepsilon 2^{1 + \frac{j}{j^*}} \gamma_j^\nu,
\]

where we used in the last inequality the estimates (2.39) (2.40) for $\chi$, the estimate (8.575) for $\nabla (P_{l}\chi)$, the estimate (2.42) for $\partial_\omega N$ and the size of the patch. In view of (8.606) and (8.607), we obtain:

\[
\|h'_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{j + j^*} \gamma_j^\nu \gamma_j^\nu.
\]

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{2,p,q,l,m}$. In view of the definition (8.599) of $h'_{2,p,q,l,m}$, we have:

\[
\|h'_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} \nabla (P_{l}\chi) \left(2\frac{j}{j^*}(N - N_{\nu})\right)^p F_{j,-1}(u)\eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}
\]

\[
\times \left\| \int_{S^2} \chi' N'(P_{\leq l}\chi') \left(2\frac{j}{j^*}(N' - N_{\nu'})\right)^q F_{j,-1}(u')\eta_j'^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\]

\[
\lesssim \varepsilon 2^{j + j^*} \gamma_j^\nu \gamma_j^\nu,
\]

where we used in the last inequality the analog of the estimate (8.317) for the first term and the analog of the estimate (8.607) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h'_{3,p,q,l,m}$. In view of the definition (8.600) of
\[ h'_{3,p,q,l,m} \text{, we have:} \]

\[
\| h'_{3,p,q,l,m} \|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} \nabla^2 (P \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1} (u) \eta^\nu_j (\omega) d\omega \right\|_{L^2(\mathcal{M})} (8.610)
\]

\[
\times \left\| \int_{S^2} N' (P_{\leq l} \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1} (u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\]

\[
\lesssim \varepsilon^{2^{\frac{j}{2}} \gamma_j^\nu} \left\| \int_{S^2} \nabla^2 (P \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1} (u) \eta^\nu_j (\omega) d\omega \right\|_{L^2(\mathcal{M})},
\]

where we used in the last inequality the estimate (8.317). Now, the basic estimate in \( L^2(\mathcal{M}) \) yields:

\[
\left\| \int_{S^2} \nabla^2 (P \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1} (u) \eta^\nu_j (\omega) d\omega \right\|_{L^2(\mathcal{M})} (8.611)
\]

\[
\lesssim \left( \sup_{\omega} \left\| \nabla^2 (P \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p \right\|_{L^2(\mathcal{M})} \right) 2^{\frac{j}{2}} \gamma_j^\nu
\]

\[
\lesssim \varepsilon^{2^{j+\frac{1}{2}} \gamma_j^\nu},
\]

where we used in the last inequality the estimate (8.356) for \( \nabla^2 (P \text{tr} \chi) \), the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Finally, (8.610) and (8.611) yield:

\[
\| h'_{3,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim \varepsilon^{2^{j+\frac{1}{2}} \gamma_j^\nu} \gamma_j^\nu. \quad (8.612)
\]

Next, we evaluate the \( L^1(\mathcal{M}) \) norm of \( h'_{4,p,q,l,m} \). In view of the definition (8.601) of \( h'_{4,p,q,l,m} \), we have:

\[
\| h'_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} \nabla^2 (P \text{tr} \chi) \left( 2^{\frac{j}{2}} (N - N_\nu) \right)^p F_{j-1} (u) \eta^\nu_j (\omega) d\omega \right\|_{L^2(\mathcal{M})} (8.613)
\]

\[
\times \left\| \int_{S^2} N' (P_{\leq l} \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1} (u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\]

\[
\lesssim \varepsilon^{2^{\frac{j}{2}} \gamma_j^\nu} \left\| \int_{S^2} \nabla^2 (N' (P_{\leq l} \text{tr} \chi')) \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1} (u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})},
\]

where we used in the last inequality the analog of estimate (8.317). Now, the basic estimate in \( L^2(\mathcal{M}) \) yields:

\[
\left\| \int_{S^2} \nabla^2 (N' (P_{\leq l} \text{tr} \chi')) \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q F_{j-1} (u') \eta_j^\nu' (\omega') d\omega' \right\|_{L^2(\mathcal{M})} (8.614)
\]

\[
\lesssim \left( \sup_{\omega} \left\| \nabla^2 (N' (P_{\leq l} \text{tr} \chi')) \left( 2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^q \right\|_{L^2(\mathcal{M})} \right) 2^{\frac{j}{2}} \gamma_j^\nu
\]

\[
\lesssim \varepsilon^{2^{j+\frac{1}{2}} \gamma_j^\nu},
\]

where we used in the last inequality the estimate (8.359) for \( \nabla^2 (N' (P_{\leq l} \text{tr} \chi')) \), the estimate (2.42) for \( \partial_\omega N \) and the size of the patch. Finally, (8.613) and (8.614) yield:

\[
\| h'_{4,p,q,l,m} \|_{L^1(\mathcal{M})} \lesssim \varepsilon^{2^{j+\frac{1}{2}} \gamma_j^\nu} \gamma_j^\nu. \quad (8.615)
\]
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{5,p,q,l,m}'$. In view of the definition (8.602) of $h_{5,p,q,l,m}'$, we have:

$$
\| h_{5,p,q,l,m}' \|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} (\theta + b^{-1} \nabla(b)) \nabla(P_l \chi) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}
\times \left\| \int_{S^2} N'(P_{\leq l} \chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\lesssim \varepsilon^{2^{j+\frac{1}{2}}} \gamma_j^\nu \left\| \int_{S^2} (\theta + b^{-1} \nabla(b)) \nabla(P_l \chi) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})},
$$

where we used in the last inequality the estimate (8.317). Proceeding as for the estimate of (8.607) and using the estimates (2.38) for $b$ and (2.37) (2.39) (2.40) for $\theta$, we obtain the following estimate:

$$
\int_{S^2} (\theta + b^{-1} \nabla(b)) \nabla(P_l \chi) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \lesssim \varepsilon^{2^{j+\frac{1}{2}} \gamma_j^\nu}. \quad (8.617)
$$

Finally, (8.616) and (8.617) imply:

$$
\| h_{5,p,q,l,m}' \|_{L^1(\mathcal{M})} \lesssim \varepsilon^{2^{j+\frac{1}{2}}} \gamma_j^\nu \gamma_j^{\nu'} \quad (8.618)
$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{6,p,q,l,m}'$. In view of the definition (8.603) of $h_{6,p,q,l,m}'$, we have:

$$
\| h_{6,p,q,l,m}' \|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} \nabla(P_l \chi) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}
\times \left\| \int_{S^2} (b^{-1} \nabla(b') + \theta') N'(P_{\leq l} \chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\lesssim \varepsilon^{2^{j+\frac{1}{2}}} \gamma_j^\nu \gamma_j^{\nu'},
$$

where we used in the last inequality the analog of the estimate (8.317) for the first term, and the analog of the estimate (8.617) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{7,p,q,l,m}'$. In view of the definition (8.604) of $h_{7,p,q,l,m}'$, we have:

$$
\| h_{7,p,q,l,m}' \|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} \nabla(N(P_l \chi)) \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}
\times \left\| \int_{S^2} N'(P_{\leq l} \chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\lesssim \varepsilon^{2^{j+\frac{1}{2}}} \gamma_j^\nu \gamma_j^{\nu'},
$$

where we used in the last inequality the analog of the estimate (8.614) for the first term, and the estimate (8.317) for the second term.
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{8,p,q,l,m}'$. In view of the definition (8.605) of $h_{8,p,q,l,m}'$, we have:

$$
\|h_{8,p,q,l,m}'\|_{L^1(\mathcal{M})} \leq \frac{1}{|2^j (N - N_{\nu})|^{p+q+1}} \left( \frac{1}{|N_{\nu} - N_{\nu'}|^2} \right) L_\infty(\mathcal{M}) \left( \frac{1}{|\gamma_{\nu} - \gamma_{\nu'}|^2} \right) (8.622)
$$

where we used in the last inequality the analog of the estimate (8.317). Proceeding as for the estimate of (8.607) and using the estimates (2.38) for $b$ and (2.37) (2.39) (2.40) for $\theta$, we obtain the following estimate:

$$
\|\int_{S^2} (\theta + N(b)) \nabla(P_{l,\gamma}(\theta)) \left( 2^j (N - N_{\nu}) \right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \|_{L^2(\mathcal{M})} \lesssim \varepsilon \gamma_j \gamma_{j'} (8.623)
$$

Finally, (8.621) and (8.622) imply:

$$
\|h_{8,p,q,l,m}'\|_{L^1(\mathcal{M})} \lesssim \varepsilon^{2j+\frac{1}{2}} \gamma_j \gamma_{j'} (8.623)
$$

Now, we have in view of the decomposition (8.597) of $\sum_{m \leq 1} B_{j,l,v',l,m}'^2$:

$$
\left| \sum_{m \leq 1} B_{j,l,v',l,m}'^2 \right| \lesssim 2^{-\frac{3}{4}} \sum_{p,q \geq 0} C_{pq} \left( \frac{1}{(2^j |\nu - \nu'|)^{p+q+1}} \right) \left( \frac{1}{|\nu - \nu'|^2} \right) L_\infty(\mathcal{M}) \left( \frac{1}{|\gamma_{\nu} - \gamma_{\nu'}|^2} \right) L_\infty(\mathcal{M}) \left( \frac{1}{|\gamma_{\nu} - \gamma_{\nu'}|^2} \right) L_\infty(\mathcal{M}) + \|h_{1,p,q,l,m}'\|_{L^1(\mathcal{M})} + \|h_{2,p,q,l,m}'\|_{L^1(\mathcal{M})} + \|h_{3,p,q,l,m}'\|_{L^1(\mathcal{M})} + \|h_{4,p,q,l,m}'\|_{L^1(\mathcal{M})} + \|h_{5,p,q,l,m}'\|_{L^1(\mathcal{M})} + \|h_{6,p,q,l,m}'\|_{L^1(\mathcal{M})} + \|h_{7,p,q,l,m}'\|_{L^1(\mathcal{M})} + \|h_{8,p,q,l,m}'\|_{L^1(\mathcal{M})} \right). \tag{8.623}
$$

Together with (8.32), (8.608), (8.609), (8.612), (8.615), (8.618), (8.619), (8.620) and (8.623), we obtain:

$$
\left| \sum_{m \leq 1} B_{j,l,v',l,m}'^2 \right| \lesssim \left[ \begin{array}{c} \frac{2^{-j+\frac{1}{2}}}{(2^j |\nu - \nu'|)^3} + \frac{2^{-j+l}}{2^j |\nu - \nu'|^2} + \frac{2^{-\frac{3}{2}+l}}{2^j |\nu - \nu'|} \end{array} \right] \varepsilon^{2j} \gamma_j \gamma_{j'} .
$$
Summing in $l$, we finally obtain in the range $2^m \leq 2^l \leq 2^j |\nu - \nu'|$.

\[
\left| \sum_{(l,m) : 2^m \leq 2^l \leq 2^j |\nu - \nu'|} B_{j,\nu,\nu',l,m}^{2,2} \right| \lesssim \left\lfloor \frac{2^{-\frac{j}{2}}}{(2^\frac{j}{2} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)} \right\rfloor + 2^{-j} \varepsilon^2 \gamma_j \gamma_j'. \tag{8.624}
\]

### 8.4 End of the proof of Proposition 8.2

In view of the decomposition (8.528) of $B_{j,\nu,\nu',l,m}^2$, the estimate (8.32), the estimates (8.547) (8.557) (8.564) (8.567) (8.571) (8.576) (8.580) (8.583) (8.593) for $h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m}, h_{6,p,q,l,m}, h_{7,p,q,l,m}, h_{8,p,q,l,m}, h_{9,p,q,l,m}, h_{10,p,q,l,m}$, the estimate (8.595) for $B_{j,\nu,\nu',l,m}^{2,1}$ and the estimate (8.624) for $B_{j,\nu,\nu',l,m}^{2,2}$, we obtain:

\[
\begin{align*}
\sum_{(l,m) : 2^m \leq 2^l \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^2 + B_{j,\nu,\nu',l,m}^{2,2}) & \lesssim 2^{-\frac{j}{2}} \sum_{p,q \geq 0} c_{pq} \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^{p+q+1}} \\
& \times \left[ \frac{(1 + q^2)2^\frac{j}{2}}{|\nu - \nu'|^2} + \frac{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)^{\frac{1}{2}}}{|\nu - \nu'|} + 2^j + 2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)^{\frac{1}{2}} \right] \varepsilon^2 \gamma_j \gamma_j' + 2^{-j} \varepsilon^2 \gamma_j \gamma_j' \\
& \lesssim \frac{1}{2^\frac{j}{2} |\nu - \nu'|^3} + \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)^{\frac{1}{2}}} + \frac{2^{-\frac{j}{2}}}{(2^\frac{j}{2} |\nu - \nu'|)^2} + \frac{1}{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)} + 2^{-j} \varepsilon^2 \gamma_j \gamma_j'.
\end{align*}
\]

Together with the estimate (8.527) for $B_{j,\nu,\nu',l,m}^2$ in $2^m \leq 2^j |\nu - \nu'| < 2^l$, we finally obtain the following control for $B_{j,\nu,\nu',l,m}^2$:

\[
\begin{align*}
\sum_{(l,m) : 2^\min(l,m) \leq 2^j |\nu - \nu'|} (B_{j,\nu,\nu',l,m}^2 + B_{j,\nu,\nu',l,m}^{2,2}) & \lesssim \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^3} + \frac{1}{(2^\frac{j}{2} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)^{\frac{1}{2}}} + \frac{2^{-\frac{j}{2}}}{(2^\frac{j}{2} |\nu - \nu'|)^2} + \frac{1}{2^\frac{j}{2} (2^\frac{j}{2} |\nu - \nu'|)} + 2^{-j} \varepsilon^2 \gamma_j \gamma_j'.
\end{align*}
\]

This concludes the proof of Proposition 8.2.

### 9 Proof of Proposition 6.6

Since $2^\min(l,m) \leq 2^j |\nu - \nu'| < 2^\max(l,m)$, we may assume that $l > m$ and thus:

\[
2^m \leq 2^j |\nu - \nu'| < 2^l. \tag{9.1}
\]
In order to prove Proposition 6.6, recall that we need to show:

\[
\sum_{\substack{(l,m)/2^\text{min}(l,m) \leq 2|\nu - \nu'| \leq 2^{\text{max}(l,m)}}} A_{j,\nu,\nu',l,m}
\]

where the sequence of functions \((\mu_{j,\nu,l})_{l \geq 0}\) on \(\mathbb{R} \times S^2\) satisfies:

\[
\sum_{\nu} \sum_{l \geq 0} 2^{2l} \|\mu_{j,\nu,l}\|^2_{L^2(\mathbb{R} \times S^2)} \lesssim \varepsilon 2^{j} \|f\|^2_{L^2(\mathbb{R}^3)}
\]

and where \(A_{j,\nu,\nu',l,m}\) is given by (8.8):

\[
A_{j,\nu,\nu',l,m} = -i 2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{P_{l} \text{tr}_\chi(N - g(N, N')N')(P_{m} \text{tr}_\chi')}{g(L, L')}
\]

\[
\times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_{j'}^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

We may sum over the region (9.1), and we obtain:

\[
\sum_{\substack{(l,m)/2^\text{min}(l,m) \leq 2|\nu - \nu'| \leq 2^{\text{max}(l,m)}}} A_{j,\nu,\nu',l,m}
\]

\[
= -i 2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{P_{>2^j|\nu - \nu'|} \text{tr}_\chi(N - g(N, N')N')(P_{\leq 2^j|\nu - \nu'|} \text{tr}_\chi')}{\gamma(L, L')}
\]

\[
\times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_{j'}^\nu(\omega') d\omega d\omega' d\mathcal{M}.
\]

We integrate by parts using (7.143).

**Lemma 9.1** Let \(\sum_{\substack{(l,m)/2^\text{min}(l,m) \leq 2|\nu - \nu'| \leq 2^{\text{max}(l,m)}}} A_{j,\nu,\nu',l,m}\) be defined by (9.2). Integrating by parts using (7.143) yields:

\[
\sum_{\substack{(l,m)/2^\text{min}(l,m) \leq 2|\nu - \nu'| \leq 2^{\text{max}(l,m)}}} A_{j,\nu,\nu',l,m}
\]

\[
= 2^{-\frac{3}{2}} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^j |N_{\nu} - N_{\nu'}|)^{p+q+1}} \left[ \frac{1}{|N_{\nu} - N_{\nu'}|^2} (h_{1,p,q} + h_{2,p,q} + h_{3,p,q} + h_{4,p,q})
\right.
\]

\[
+ \frac{1}{|N_{\nu} - N_{\nu'}|} (h_{5,p,q} + h_{6,p,q} + h_{7,p,q} + h_{8,p,q})
\]

\[
d\mathcal{M},
\]

where \(c_{pq}\) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

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has radius of convergence 1, where the scalar functions $h_{1,p,q}$, $h_{2,p,q}$, $h_{3,p,q}$, $h_{4,p,q}$, $h_{5,p,q}$, $h_{6,p,q}$, $h_{7,p,q}$, $h_{8,p,q}$ on $\mathcal{M}$ are given by:

$$h_{1,p,q} = \left( \int_{S^2} L(P_{>2|\nu-\nu'|tr\chi}) \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla'(P_{\leq 2|\nu-\nu'|tr\chi'}) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.4)$$

$$h_{2,p,q} = \left( \int_{S^2} P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla'(L'(P_{\leq 2|\nu-\nu'|tr\chi'}) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.5)$$

$$h_{3,p,q} = \left( \int_{S^2} H_1 P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla'(P_{\leq 2|\nu-\nu'|tr\chi'}) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.6)$$

$$h_{4,p,q} = \left( \int_{S^2} P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} H_2 \nabla'(P_{\leq 2|\nu-\nu'|tr\chi'}) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.7)$$

$$h_{5,p,q} = \left( \int_{S^2} P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla^2(P_{\leq 2|\nu-\nu'|tr\chi'}) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.8)$$

$$h_{6,p,q} = \left( \int_{S^2} H_3 P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla(P_{\leq 2|\nu-\nu'|tr\chi'}) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.9)$$

$$h_{7,p,q} = \left( \int_{S^2} P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} H_4 \nabla(P_{\leq 2|\nu-\nu'|tr\chi'}) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.10)$$

$$h_{8,p,q} = \left( \int_{S^2} P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla'(N'(P_{\leq 2|\nu-\nu'|tr\chi'})) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.11)$$

and:

$$h_{8,p,q} = \left( \int_{S^2} P_{>2|\nu-\nu'|tr\chi} \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega) d\omega \right) \times \left( \int_{S^2} \nabla'(N'(P_{\leq 2|\nu-\nu'|tr\chi'})) \left( 2^{\frac{1}{2}}(N' - N_\nu) \right)^q F_{j-1}(u') \eta_j^\nu(\omega') d\omega' \right), \quad (9.11)$$
where the tensor $H_1$ on $\mathcal{M}$ involved in the definition of $h_{3,p,q}$ is given by:

$$H_1 = \chi + \epsilon + \delta + n^{-1}\nabla n + L(b),$$

(9.12)

where the tensor $H_2$ on $\mathcal{M}$ involved in the definition of $h_{4,p,q}$ is given by:

$$H_2 = \chi' + \epsilon' + \delta' + n^{-1}\nabla n + L'(b').$$

(9.13)

where the tensor $H_3$ on $\mathcal{M}$ involved in the definition of $h_{6,p,q}$ is given by:

$$H_3 = k + n^{-1}\nabla n + \theta + b^{-1}\nabla(b) + \chi + \zeta,$$

(9.14)

and where the tensor $H_4$ on $\mathcal{M}$ involved in the definition of $h_{7,p,q}$ is given by:

$$H_4 = k + n^{-1}\nabla n + \theta' + b^{-1}\nabla'(b') + \zeta' + \nabla_{N'}(b').$$

(9.15)

The proof of lemma 9.1 is postponed to Appendix I. In the rest of this section, we use Lemma 9.1 to obtain the control of $\sum\frac{1}{m}(l,m) \leq 2\nu - \nu' \leq 2\max(l,m) A_{j,l,m},$

We evaluate the $L^1(\mathcal{M})$ norm of $h_{1,p,q}$, $h_{2,p,q}$, $h_{3,p,q}$, $h_{4,p,q}$, $h_{5,p,q}$, $h_{6,p,q}$, $h_{7,p,q}$, $h_{8,p,q}$ starting with $h_{1,p,q}$. In view of the definition (9.4) of $h_{1,p,q}$, we have:

$$h_{1,p,q} = \sum_{l>2|\nu-\nu|}^{} \int_{\mathbb{R}^2} G_1 L(P_{>2|\nu-\nu|} \text{tr}\chi) \left(2^{\frac{j}{2}}(N - N_{\nu})\right)^p F_{j-1}(\eta_j^\nu(\omega))d\omega,$$

where the tensor $G_1$ on $\mathcal{M}$ is given by:

$$G_1 = \int_{\mathbb{R}^2} \nabla'(P_{<2|\nu-\nu|} \text{tr}\chi') \left(2^{\frac{j}{2}}(N' - N_{\nu'})\right)^q F_{j-1}(\eta_j^{\nu'}(\omega'))d\omega'.$$

(9.16)

In view of the estimate (7.10), this yields:

$$\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|G_1\|_{L^2_{u',L^\infty}}\right) \left(\sum_{l>2|\nu-\nu|} 2^{\frac{j}{2}-l} \varepsilon \gamma_j^{\nu'}\right)$$

(9.17)

$$\lesssim \left(\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|G_1\|_{L^2_{u',L^\infty}}\right) \frac{\varepsilon \gamma_j^{\nu'}}{2^{j}|\nu - \nu'|}.$$

Now, in view of the definition (9.16) of $G_1$ and the estimate (7.76), we have:

$$\sup_{\omega \in \text{supp}(\eta_j^\nu)} \|G_1\|_{L^2_{u',L^\infty}} \lesssim \left(\sup_{\omega'} \left\|2^{\frac{j}{2}}(N' - N_{\nu'})\right\|_{L^\infty}\varepsilon 2^{j}|\nu - \nu'| \gamma_j^{\nu'}\right)$$

(9.18)

$$\lesssim \varepsilon 2^{j}|\nu - \nu'| \gamma_j^{\nu'},$$

where we used in the last inequality the estimate (2.42) for $\partial_{\nu} N$ and the size of the patch. Finally, (9.17) and (9.18) imply:

$$\|h_{1,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{2j} \gamma_j^{\nu'} \gamma_j^{\nu'}.$$

(9.19)
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{2,p,q}$. In view of the definition (9.5) of $h_{2,p,q}$, we have:

$$h_{2,p,q} = \sum_{l>2|\nu-\nu'|} \int_{\mathbb{S}^2} G_2 \nabla'(P_{\leq 2|\nu-\nu'|} \text{tr} \chi') \left(2^{\frac{l}{4}} (N' - N_{\nu'})\right)^q F_{j,-1}(u) \eta_j'(\omega) d\omega,$$

where the tensor $G_2$ on $\mathcal{M}$ is given by:

$$G_2 = \int_{\mathbb{S}^2} P_{>2|\nu-\nu'|} \text{tr} \chi \left(2^{\frac{l}{4}} (N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega. \quad (9.20)$$

In view of the estimate (7.13), this yields:

$$\|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim \left(\sup_{\omega' \in \text{supp}(\eta_j')} \|G_2\|_{L^2_{u,\omega}, L^\infty_t}\right) 2^\frac{l}{2} \varepsilon \gamma_j^{\nu'} . \quad (9.21)$$

Now, in view of the definition (9.20) of $G_2$ and the estimate (7.71), we have:

$$\sup_{\omega' \in \text{supp}(\eta_j')} \|G_2\|_{L^2_{u,\omega}, L^\infty_t} \lesssim \left(\sup_{\omega} \left\| \left(2^{\frac{l}{4}} (N - N_{\nu})\right)^p \right\|_{L^\infty_t}\right) \varepsilon \gamma_j^{\nu'}$$

$$\times \sum_{2^l > 2|\nu-\nu'|} \left(2^{\frac{l}{4}} |\nu - \nu'| 2^{-l+\frac{1}{2}} + \left(2^{\frac{l}{4}} |\nu - \nu'|\right)^{\frac{1}{2}} 2^{-\frac{l}{2} + \frac{1}{4}}\right) \lesssim \varepsilon \gamma_j^{\nu'},$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Finally, (9.21) and (9.22) imply:

$$\|h_{2,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 2^{\frac{l}{2}} \gamma_j^{\nu'} \gamma_j^{\nu'} . \quad (9.23)$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{3,p,q}$. In view of the definition (9.6) of $h_{3,p,q}$, we have:

$$\|h_{3,p,q}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} H_1 P_{>2|\nu-\nu'|} \text{tr} \chi \left(2^{\frac{l}{4}} (N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^1(\mathcal{M})} \quad (9.24)$$

$$\times \left\| \int_{\mathbb{S}^2} \nabla'(P_{\leq 2|\nu-\nu'|} \text{tr} \chi') \left(2^{\frac{l}{4}} (N' - N_{\nu'})\right)^q F_{j,-1}(u) \eta_j'(\omega) d\omega' \right\|_{L^1(\mathcal{M})}$$

$$\lesssim \left\| \int_{\mathbb{S}^2} H_1 P_{>2|\nu-\nu'|} \text{tr} \chi \left(2^{\frac{l}{4}} (N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^1(\mathcal{M})} 2^{\frac{l}{2}} \varepsilon \gamma_j^{\nu'},$$

where we used in the last inequality the analog of the estimate (8.317). The basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$\left\| \int_{\mathbb{S}^2} H_1 P_{\text{tr} \chi} \left(2^{\frac{l}{4}} (N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(\mathcal{M})} \quad (9.25)$$

$$\lesssim \left(\sup_{\omega} \left\| H_1 \right\|_{L^\infty_t L^2_{u,\omega}} \left\| P_{\text{tr} \chi} \right\|_{L^2_{u,\omega}, L^\infty_t} \right) \left(\left\| 2^{\frac{l}{4}} (N - N_{\nu})^p \right\|_{L^\infty_t} \right) 2^\frac{l}{2} \gamma_j^{\nu'}$$

$$\lesssim \left(\sup_{\omega} \left\| H_1 \right\|_{L^\infty_t L^2_{u,\omega}} \right) 2^{\frac{l}{2} - t} \varepsilon \gamma_j^{\nu'},$$
where we used in the last inequality the estimate (2.69) for $P_t \text{tr} \chi$, the estimate (2.42) for $\partial_\nu N$ and the size of the patch. Now, the definition of $H_1$ (9.12), the estimates (2.39) (2.40) for $\chi$, the estimate (2.37) for $\epsilon$ and $\delta$, the estimate (2.36) for $n$ and the estimate (2.38) for $b$ imply:

$$\|H_1\|_{L^\infty_t L^2_x} \lesssim \|\chi\|_{L^\infty_t L^2_x} + \|\epsilon\|_{L^\infty_t L^2_x} + \|\delta\|_{L^\infty_t L^2_x} + \|n^{-1}\nabla n\|_{L^\infty_t L^2_x} + \|L(b)\|_{L^\infty_t L^2_x}$$

(9.26)

which together with (9.25) yields:

$$\left\| \int_{S^2} H_1 P_t \text{tr} \chi \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{\frac{3}{2}j - l} \epsilon \gamma_j^\nu.$$  

(9.27)

Finally, (9.24) and (9.27) imply:

$$\|h_{3,p,q}\|_{L^1(M)} \lesssim 2^j \left( \sum_{2^j \geq 2^{|\nu - \nu'|}} 2^{-l} \right) \epsilon^2 \gamma_j^\nu \gamma_j'^\nu.$$

(9.28)

Next, we evaluate the $L^1(M)$ norm of $h_{4,p,q}$. In view of the definition (9.7) of $h_{4,p,q}$, we have:

$$\|h_{4,p,q}\|_{L^2(M)}$$

(9.29)

$$\lesssim \left\| \int_{S^2} P_{>2^j |\nu - \nu'|} \text{tr} \chi \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(M)}$$

$$\times \left\| \int_{S^2} H_2 \nabla' \left( P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \right) \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right\|_{L^2(M)}$$

$$\lesssim \frac{\epsilon \gamma_j'^\nu}{2^{\frac{3}{2}j - l}} \left\| \int_{S^2} H_2 \nabla' \left( P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \right) \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right\|_{L^2(M)}$$

where we used in the last inequality the estimate (8.129). The basic estimate in $L^2(M)$ (7.1) yields:

$$\left\| \int_{S^2} H_2 \nabla' \left( P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \right) \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right\|_{L^2(M)}$$

(9.30)

$$\lesssim \left( \sup_{\omega'} \left\| H_2 \nabla' \left( P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \right) \left( 2^j (N' - N_{\nu'}) \right)^q \right\|_{L^\infty_t L^2_x(H_\omega)} \right) \left( 2^j \gamma_j'^\nu \right)$$

$$\lesssim \left( \sup_{\omega'} \left\| H_2 \left\| L^\infty_t L^2_x \right\| \nabla' \left( P_{\leq 2^j |\nu - \nu'|} \text{tr} \chi' \right) \right\|_{L^2_t L^\infty_x} \left( \left( 2^j (N' - N_{\nu'}) \right)^q \right) \right) \left( 2^j \gamma_j'^\nu \right)$$

$$\lesssim \left( \sup_{\omega'} \left\| H_2 \right\|_{L^\infty_t L^2_x} \right) \left( 2^j \epsilon \gamma_j'^\nu \right)$$

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where we used in the last inequality the estimate (2.70) for $\mathbf{V}'(P_{\leq 2^j |u - \nu|} \text{tr} \chi')$, the estimate (2.42) for $\partial_u N$ and the size of the patch. Now, in view of the definition of $H_2$ (9.13), and proceeding as for the proof of (9.26), we have:

$$\|H_2\|_{L^2_\omega L^2_\tau} \lesssim \varepsilon,$$

which together with (9.30) yields:

$$\left\| \int_{S^2} H_2 \mathbf{V}'(P_{\leq 2^j |u - \nu|} \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N') \right)^q F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)} \lesssim 2^{\frac{j}{2}} \varepsilon \gamma_j' \gamma_j'. \quad (9.31)$$

Finally, (9.29) and (9.31) imply:

$$\|h_{4,p,q}\|_{L^1(M)} \lesssim \frac{2^{\frac{j}{2}}}{2^{\frac{j}{2}} |\nu - \nu'|} \varepsilon^2 \gamma_j' \gamma_j' \quad (9.32)$$

Next, we evaluate the $L^1(M)$ norm of $h_{5,p,q}$. In view of the definition (9.8) of $h_{5,p,q}$, we have:

$$\|h_{5,p,q}\|_{L^1(M)} \lesssim \left\| \int_{S^2} P_{\leq 2^j |u - \nu|} \text{tr} \chi \left( 2^{\frac{j}{2}} (N - N') \right)^p F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)}$$

$$\times \left\| \int_{S^2} \mathbf{V}^2(P_{\leq 2^j |u - \nu|} \text{tr} \chi') \left( 2^{\frac{j}{2}} (N' - N') \right)^q F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)}$$

$$\lesssim \sum_{2^m \leq 2^j |u - \nu'| < 2^l} \left( \int_{S^2} \left\| P_{\text{tr} \chi} \left( 2^{\frac{j}{2}} (N - N') \right)^p F_{j,-1}(u) \right\|_{L^2(M)} \eta_j'(\omega) d\omega \right)$$

$$\times \left( \int_{S^2} \left\| \mathbf{V}^2(P_{\text{m} \text{tr} \chi'}) \left( 2^{\frac{j}{2}} (N' - N') \right)^q F_{j,-1}(u) \right\|_{L^2(M)} \eta_j'(\omega) d\omega \right)$$

where we used in the last inequality the estimate (2.42) for $\partial_u N$ and the size of the patch. Taking Cauchy Schwartz in $\omega$ and $\omega'$, using the size of the patches, and using the Bochner inequality (2.61), we obtain:

$$\|h_{5,p,q}\|_{L^1(M)} \lesssim \sum_{2^m \leq 2^j |u - \nu'| < 2^l} 2^{2m-j} \left\| P_{\text{tr} \chi} \left\|_{L^2(H_\omega) F_{j,-1}(u)^{\sqrt{\eta_j'}(\omega)}} \right\|_{L^2_{\omega,u}} \right\|_{L^2_{\omega,u}} \right\|_{L^2_{\omega,u}} \right\|_{L^2_{\omega,u}} \right\|_{L^2_{\omega,u}} \right.$$ 

In view of (9.33) and the estimate (6.18), we finally obtain:

$$\|h_{5,p,q}\|_{L^1(M)} \lesssim \sum_{2^m \leq 2^j |u - \nu'| < 2^l} 2^{2m-j} \|\mu_{j,u,\tau,\ell}\|_{L^2(R \times S^2)} \|\mu_{j,u',m}\|_{L^2(R \times S^2)} \quad (9.34)$$

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where the sequence of functions \((\mu_{j,\nu,l})_{l>j/2}\) on \(\mathbb{R} \times S^2\) satisfies:
\[
\sum_\nu \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times S^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.
\]

Next, we evaluate the \(L^1(\mathcal{M})\) norm of \(h_{6,p,q}\). In view of the definition (9.9) of \(h_{6,p,q}\), we have:
\[
\|h_{6,p,q}\|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} H_3 P_{>2|\nu-\nu'|} \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j-1}(u)\eta_j(\omega)d\omega \right\|_{L^2(\mathcal{M})}
\leq \left(\sup \|H_3 P_l \text{tr} \chi \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p F_{j-1}(u)\eta_j(\omega;\nu)\right) |\omega|^{2\frac{j}{2}} \gamma_j^\nu
\leq \left(\sup \|H_3\|_{L^\infty(L^4;H_\nu)} \|P_l \text{tr} \chi\|_{L^2(L^4;\nu)} \left(2^{\frac{j}{2}}(N - N_\nu)\right)^p \right) |\omega|^{2\frac{j}{2}} \gamma_j^\nu.
\]

Finally, (9.35) and (9.38) imply:
\[
\|h_{6,p,q}\|_{L^1(\mathcal{M})} \lesssim \varepsilon^2 \left(\sum_{2^l>2|\nu-\nu'|} 2^{2\frac{j}{2} - \frac{1}{2}} \gamma_j^\nu \gamma_j^\nu\right)
\lesssim \frac{\varepsilon^2 2^{\frac{j}{2}} \gamma_j^\nu \gamma_j^\nu}{(2^j |\nu - \nu'|)^{\frac{1}{2}}}.
\]
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{7,p,q}$. In view of the definition (9.10) of $h_{7,p,q}$, we have:

$$\|h_{7,p,q}\|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} P_{>2|\nu-\nu'|} tr \chi \left(2^{\frac{3}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\times \left\| \int_{S^2} H_4 \nabla (P_{\leq 2|\nu-\nu'|} tr \chi') \left(2^{\frac{3}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \frac{\varepsilon \gamma_j^{\nu'}}{2^{\frac{3}{2}}|\nu - \nu'|} \left\| \int_{S^2} H_4 \nabla (P_{\leq 2|\nu-\nu'|} tr \chi') \left(2^{\frac{3}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})},$$

where we used in the last inequality the estimate (8.129). Proceeding as for the estimate of (8.607), we have:

$$\left\| \int_{S^2} H_4 \nabla (P_{\nu} tr \chi') \left(2^{\frac{3}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega'} \|H_4\|_{L^\infty L^4_{\nu'}} \right) 2^{\frac{3}{2} + \frac{3}{2}} \varepsilon \gamma_j^{\nu'}.$$ 

Now, in view of the definition of $H_4$, we have:

$$\|H_4\|_{L^\infty L^4_{\nu'}} \lesssim \|k\|_{L^\infty L^4_{\nu'}} + \|n^{-1} \nabla n\|_{L^\infty L^4_{\nu'}} + \|\theta'\|_{L^\infty L^4_{\nu'}}$$

$$+ \|b^{-1} \nabla'(b)\|_{L^\infty L^4_{\nu'}} + \|\zeta'\|_{L^\infty L^4_{\nu'}} + \|\nabla N'(b)\|_{L^\infty L^4_{\nu'}}$$

$$\lesssim \varepsilon,$$

where we used in the last inequality the embedding (2.51), the estimate (2.36) for $n$, the estimates (2.37) (2.39) (2.40) for $\theta'$, the estimate (2.38) for $b'$, and the estimate (2.41) for $\zeta'$. (9.41) and (9.42) yield:

$$\left\| \int_{S^2} H_4 \nabla (P_{\nu} tr \chi') \left(2^{\frac{3}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim 2^{\frac{3}{2} + \frac{3}{2}} \varepsilon \gamma_j^{\nu'}.$$ 

Finally, (9.40) and (9.43) imply:

$$\|h_{7,p,q}\|_{L^1(\mathcal{M})} \lesssim \frac{1}{2^\frac{3}{2} |\nu - \nu'|} \left( \sum_{2'|>2|\nu-\nu'|} 2^{\frac{3}{2}} \varepsilon \gamma_j^{\nu'} \gamma_j^{\nu'} \right)$$

$$\lesssim \frac{2^\frac{3}{2} \varepsilon \gamma_j^{\nu'} \gamma_j^{\nu'}}{(2^\frac{3}{2} |\nu - \nu'|)^{\frac{3}{2}}}.$$ 

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{8,p,q}$. In view of the definition (9.11) of $h_{8,p,q}$, we have:

$$\|h_{8,p,q}\|_{L^1(\mathcal{M})} \leq \left\| \int_{S^2} P_{>2|\nu-\nu'|} tr \chi \left(2^{\frac{3}{2}}(N - N_\nu)\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})}$$

$$\times \left\| \int_{S^2} \nabla' (P_{\leq 2|\nu-\nu'|} tr \chi') \left(2^{\frac{3}{2}}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^\nu(\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim 2^\frac{3}{2} \varepsilon \gamma_j^{\nu'} \gamma_j^{\nu'},$$

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where we used in the last inequality the estimate (8.129) for the first term, and the analog of the estimate (8.614) for the second term.

Now, we have in view of the decomposition (9.3) of \( \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'| < 2^{m+1}} A_{j,\nu,\nu',l,m} \):

\[
\left| \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'| < 2^{m+1}} A_{j,\nu,\nu',l,m} \right| \\
\lesssim 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^j |\nu - \nu'|)^{p+1}} \right\|_{L^\infty(\mathcal{M})} \\
\times \left[ \left\| \frac{1}{|\nu - \nu'|^2} \right\|_{L^\infty(\mathcal{M})} \left( \| h_{1,p.q} \|_{L^1(\mathcal{M})} + \| h_{2,p.q} \|_{L^1(\mathcal{M})} + \| h_{3,p.q} \|_{L^1(\mathcal{M})} + \| h_{4,p.q} \|_{L^1(\mathcal{M})} \right) \\
+ \left\| \frac{1}{|\nu - \nu'|^2} \right\|_{L^\infty(\mathcal{M})} \left( \| h_{5,p.q} \|_{L^1(\mathcal{M})} + \| h_{6,p.q} \|_{L^1(\mathcal{M})} + \| h_{7,p.q} \|_{L^1(\mathcal{M})} + \| h_{8,p.q} \|_{L^1(\mathcal{M})} \right) \right],
\]

Together with (8.32), (9.19), (9.23), (9.28), (9.32), (9.34), (9.39), (9.44), and (9.45), we obtain:

\[
\left| \sum_{(l,m)/2^m \leq 2^j |\nu - \nu'| < 2^{m+1}} A_{j,\nu,\nu',l,m} \right| \\
\lesssim 2^{-\frac{3j}{2}} \sum_{p,q \geq 0} c_{pq} \left\| \frac{1}{(2^j |\nu - \nu'|)^{p+1}} \right\|_{L^\infty(\mathcal{M})} \\
+ \left[ \left\| \frac{1}{|\nu - \nu'|^2} \right\|_{L^\infty(\mathcal{M})} \sum_{2^m \leq 2^j |\nu - \nu'| < 2^l} 2^{2m-j} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times S^2)} \| \mu_{j,\nu',m} \|_{L^2(\mathbb{R} \times S^2)} + \frac{2^j}{2^j} + 2^j \right] \epsilon^2 \gamma_j \gamma_j',
\]

where the sequence of functions \( (\mu_{j,\nu,l})_{l \geq j/2} \) on \( \mathbb{R} \times S^2 \) satisfies:

\[
\sum_{\nu \geq j/2} 2^{2l} \| \mu_{j,\nu,l} \|^2_{L^2(\mathbb{R} \times S^2)} \lesssim \epsilon^2 \sum_{\nu} 2^{2j} \| f \|^2_{L^2(\mathbb{R} \times S^2)}.
\]

This concludes the proof of Proposition 6.6.

10 Proof of Proposition 6.7

Since \( 2^{\max(l,m)} \leq 2^j |\nu - \nu'| \), we may assume that \( l \geq m \) and thus:

\[
2^m \leq 2^j \leq 2^j |\nu - \nu'|.
\]
In order to prove Proposition 6.7, recall that we need to show:

\[
\sum_{(l,m)/2^{\max(l,m)} \leq 2|\nu - \nu'|} A_{j,\nu',l,m} \\
\lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2|\nu - \nu'|} 2^{-\frac{1}{2}j}2^{l+m+\min(l,m)} \|A_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|A_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \\
+ \left[ \frac{1}{(2^\frac{j}{2}|\nu - \nu'|)^3} + \frac{1}{(2^\frac{j}{2}|\nu - \nu'|)^2} + \frac{1}{2^\frac{j}{2}|\nu - \nu'|} + 2^{-j} \right] \varepsilon^2 \gamma^\nu \gamma' \varepsilon,
\]

where the sequence of functions \((\mu_{j,\nu,l})_{l>\nu/2}\) on \(\mathbb{R} \times \mathbb{S}^2\) satisfies:

\[
\sum_{\nu} \sum_{l>\nu/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2,
\]

and where \(A_{j,\nu',l,m}\) is given by (8.8):

\[
A_{j,\nu',l,m} = -i 2^{-j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr} \chi (N - g(N, N') N') (P_m \text{tr} \chi')}{g(L, L')} \\
\times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M}.
\]

We integrate by parts using (7.137).

**Lemma 10.1** Let \(A_{j,\nu',l,m}\) be defined by (10.2). Integrating by parts using (7.137) yields:

\[
A_{j,\nu',l,m} = A^1_{j,\nu',l,m} + A^2_{j,\nu',l,m} + A^3_{j,\nu',l,m} + 2^{-j} \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^\frac{j}{2}|N_\nu - N_{\nu'}|)^{p+q+2}} \\
\times \left[ \frac{1}{|N_\nu - N_{\nu'}|} (h_{1,p,q,l,m} + h_{2,p,q,l,m} + h_{3,p,q,l,m} + h_{4,p,q,l,m}) \right] d\mathcal{M},
\]

where \(c_{pq}\) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1, where the scalar functions \(A^1_{j,\nu',l,m}, A^2_{j,\nu',l,m}, A^3_{j,\nu',l,m}\) on \(\mathcal{M}\) are given by:

\[
A^1_{j,\nu',l,m} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{P_l \text{tr} \chi (N - g(N, N') N') (N - g(N, N') N')}{g(L, L')(1 - g(N, N')^2)} \\
\times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M},
\]

\[
A^2_{j,\nu',l,m} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - g(N, N') N')(P_l \text{tr} \chi')(N - g(N, N') N')(P_m \text{tr} \chi')}{g(L, L')(1 - g(N, N')^2)} \\
\times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M},
\]

\[
A^3_{j,\nu',l,m} = 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - g(N, N') N')(P_l \text{tr} \chi')(N - g(N, N') N')}{g(L, L')(1 - g(N, N')^2)} \\
\times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M},
\]

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and:

\[ A^3_{j,\nu,\nu',l,m} = 2^{-2j} \int_M \int_{S^2 \times S^2} \frac{N(P_l \text{tr}_\chi)(N - g(N, N')(N)(P_m \text{tr}_\gamma')}{g(L, L')} \times F_j(u)F_{j-1}(u')\eta_j^\nu(\omega)\eta_j'^\nu(\omega')d\omega d\omega' dM, \]

and where the scalar functions \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m} \) on \( M \) are given by:

\[ h_{1,p,q,l,m} = \left( \int_{S^2} \chi P_l \text{tr}_\chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)d\omega \right) \times \left( \int_{S^2} \nabla'(P_m \text{tr}_\chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'^\nu(\omega')d\omega' \right), \]

\[ h_{2,p,q,l,m} = \left( \int_{S^2} P_l \text{tr}_\chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)d\omega \right) \times \left( \int_{S^2} \chi \nabla'(P_m \text{tr}_\chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'^\nu(\omega')d\omega' \right), \]

\[ h_{3,p,q,l,m} = \left( \int_{S^2} (\theta + b^{-1}\nabla(b)) P_l \text{tr}_\chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)d\omega \right) \times \left( \int_{S^2} \nabla'(P_m \text{tr}_\chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'^\nu(\omega')d\omega' \right), \]

and:

\[ h_{4,p,q,l,m} = \left( \int_{S^2} P_l \text{tr}_\chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)d\omega \right) \times \left( \int_{S^2} (\theta' + b'^{-1}\nabla(b')) \nabla'(P_m \text{tr}_\chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'^\nu(\omega')d\omega' \right). \]

The proof of lemma 10.1 is postponed to Appendix J. In the rest of this section, we use Lemma 10.1 to obtain the control of \( A_{j,\nu,\nu',l,m} \).

We evaluate the \( L^1(M) \) norm of \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m} \) starting with \( h_{1,p,q,l,m} \). In view of the definition (10.7) of \( h_{1,p,q,l,m} \), we have:

\[ \sum_{m \leq l} h_{1,p,q,l,m} = \left( \int_{S^2} \chi P_l \text{tr}_\chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)d\omega \right) \times \left( \int_{S^2} \nabla'(P_{\leq l} \text{tr}_\chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'^\nu(\omega')d\omega' \right). \]

This yields:

\[ \left\| \sum_{m \leq l} h_{1,p,q,l,m} \right\|_{L^1(M)} \leq \left\| \int_{S^2} \chi P_l \text{tr}_\chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} \]

\[ \times \left\| \int_{S^2} \nabla'(P_{\leq l} \text{tr}_\chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'^\nu(\omega')d\omega' \right\|_{L^2(M)} \]

\[ \leq \left\| \int_{S^2} \chi P_l \text{tr}_\chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu(\omega)d\omega \right\|_{L^2(M)} \varepsilon 2^{\frac{j}{2}} \gamma_j'^\nu, \]
where we used in the last inequality the analog of the estimate (8.317). Now, the analog of (8.556) implies:

$$\left\| \int_{\mathbb{S}^2} \chi_P \text{tr} \left( 2^j (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim 2^{j-l} \gamma_j^\nu. \quad (10.12)$$

Finally, (10.11) and (10.12) imply:

$$\left\| \sum_{m \leq l} h_{1,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim 2^{j-l} \gamma_j^\nu \gamma_j^\nu. \quad (10.13)$$

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{2,p,q,l,m}$. In view of the definition (10.8) of $h_{2,p,q,l,m}$, we have:

$$\sum_{m \leq l} h_{2,p,q,l,m} = \left( \int_{\mathbb{S}^2} P \text{tr}_\chi \left( 2^j (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu (\omega) d\omega \right) \times \left( \int_{\mathbb{S}^2} \chi' \text{tr}_\chi \left( 2^j (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j'^\nu (\omega') d\omega' \right).$$

This yields:

$$\left\| \sum_{m \leq l} h_{2,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{\mathbb{S}^2} P \text{tr}_\chi \left( 2^j (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(\mathcal{M})} \times \left\| \int_{\mathbb{S}^2} \chi' \text{tr}_\chi \left( 2^j (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j'^\nu (\omega') d\omega' \right\|_{L^2(\mathcal{M})}. \quad (10.14)$$

Using (8.179) for $l > j/2$ and (8.180) for $l = j/2$, we obtain for all $l \geq j/2$:

$$\left\| \int_{\mathbb{S}^2} P \text{tr}_\chi \left( 2^j (N - N_\nu) \right)^p F_{j-1}(u) \eta_j^\nu (\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim 2^{j-l} \epsilon \gamma_j^\nu. \quad (10.15)$$

Also, the basic estimate in $L^2(\mathcal{M})$ (7.1) yields:

$$\left\| \int_{\mathbb{S}^2} \chi' \text{tr}_\chi \left( 2^j (N' - N_\nu) \right)^q F_{j-1}(u') \eta_j'^\nu (\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega'} \left\| \chi' \text{tr}_\chi \left( 2^j (N' - N_\nu) \right)^q \right\|_{L^\infty L^2(\mathcal{M})} \right) 2^{j/2} \gamma_j'^\nu \quad (10.16)$$

$$\lesssim \left( \sup_{\omega'} \left\| \chi' \text{tr}_\chi \left( 2^j (N' - N_\nu) \right)^q \right\|_{L^\infty L^2(\mathcal{M})} \right) 2^{j/2} \gamma_j'^\nu \lesssim \epsilon 2^{j/2} \gamma_j'^\nu,$$

where we used in the last inequality the estimates (2.39) (2.40) for $\chi$, the estimate (2.70) for $\text{tr}_\chi (P_{\leq j} \text{tr}_\chi')$, the estimate (2.42) for $\partial_\nu N$ and the size of the patch. Finally, (10.14), (10.15) and (10.16) imply:

$$\left\| \sum_{m \leq l} h_{2,p,q,l,m} \right\|_{L^1(\mathcal{M})} \lesssim \epsilon 2^{j-l} \gamma_j^\nu \gamma_j'^\nu. \quad (10.17)$$
Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{3,p,q,l,m}$. In view of the definition (10.9) of $h_{3,p,q,l,m}$, we have:

$$\|h_{3,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} (\theta + \beta^{-1} \nabla(b)) P_l tr\chi L_j (N - N_\nu) F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(\mathcal{M})} \tag{10.18}$$

$$\times \left\| \int_{S^2} \nabla' (P_m tr\chi L_j (N' - N_{\nu'})) F_{j,-1}(u) \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \varepsilon^{2} \eta_j^{\nu} \gamma_j^{\nu'},$$

where we used in the last inequality the estimate (8.275) for the first term, and the analog of the estimate (8.317) for the second term.

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l,m}$. In view of the definition (10.10) of $h_{4,p,q,l,m}$, we have:

$$\|h_{4,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} P_l tr\chi L_j (N - N_\nu) F_{j,-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(\mathcal{M})} \tag{10.19}$$

$$\times \left\| \int_{S^2} (\theta' + \beta^{-1} \nabla'(b')) \nabla' (P_m tr\chi L_j (N' - N_{\nu'})) F_{j,-1}(u) \eta_j' (\omega') d\omega' \right\|_{L^2(\mathcal{M})}$$

$$\lesssim \varepsilon^{2} \gamma_j^{\nu} \eta_j^{\nu'},$$

where we used in the last inequality the estimate (10.15) for the first term, and the analog of the estimate (8.622) for the second term.

Next, we estimate $A_{3,j,\nu,\nu',l,m}$. In view of the definition (10.6) of $A_{3,j,\nu,\nu',l,m}$ and the definition (8.542) of $B_{2,2,j,\nu,\nu',l,m}$, and in view of the fact that $g(L, L') = -1 + g(N, N')$, we see that $A_{3,j,\nu,\nu',l,m}$ is essentially obtained from $B_{2,2,j,\nu,\nu',l,m}$ by exchanging the role of $\nu$ and $\nu'$. Proceeding for $A_{3,j,\nu,\nu',l,m}$ as we did for $B_{2,2,j,\nu,\nu',l,m}$, using the integration by parts (7.136) instead of (7.137), we obtain the analog of the estimate (8.624):

$$\left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} A_{3,j,\nu,\nu',l,m} \right| \lesssim \left[ \frac{2^{-\frac{3}{2}}}{(2^j |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^j (2^j |\nu - \nu'|)} + 2^{-j} \right] \varepsilon^{2} \gamma_j^{\nu} \gamma_j^{\nu'} \tag{10.20}$$

Next, we consider $A_{1,j,\nu,\nu',l,m}$. We have the following proposition.

**Proposition 10.2** Let $A_{1,j,\nu,\nu',l,m}$ be given by (10.4). Then, $A_{1,j,\nu,\nu',l,m}$ satisfies the following estimate:

$$\left| \sum_{(l,m)/2^m \leq 2^l \leq 2^j |\nu - \nu'|} A_{1,j,\nu,\nu',l,m} \right| \lesssim \left[ \frac{2^{-\frac{3}{2}}}{(2^j |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{(2^j |\nu - \nu'|)^3} + \frac{2^{-\frac{3}{2}j}}{(2^j |\nu - \nu'|)^2} + \frac{1}{2^j (2^j |\nu - \nu'|)^2} \right] \varepsilon^{2} \gamma_j^{\nu} \gamma_j^{\nu'} \tag{10.21}$$

$$+ \left[ \frac{2^{-\frac{3}{2}j}}{(2^j |\nu - \nu'|)^3} + \frac{1}{2^j (2^j |\nu - \nu'|)^2} + \frac{1}{2^j (2^j |\nu - \nu'|)^2} \right] \varepsilon^{2} \gamma_j^{\nu} \gamma_j^{\nu'},$$

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where the sequence of functions \((\mu_{j,\nu,l})_{l>j/2}\) on \(\mathbb{R} \times \mathbb{S}^2\) satisfies:

\[
\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2l} \|f\|_{L^2(\mathbb{R}^3)}^2.
\]

The proof of Proposition 10.2 is postponed to section 10.1.

Next, we consider \(A_{j,\nu',l,m}^2\). We have the following proposition.

**Proposition 10.3** Let \(A_{j,\nu',l,m}^2\) be given by (10.5). Then, \(A_{j,\nu',l,m}^2\) satisfies the following estimate:

\[
\sum_{(l,m)/2^{\max(l,m)} \leq 2^{j/2} |\nu - \nu'|} A_{j,\nu',l,m}^2
\]

\[
\lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^{j/2} |\nu - \nu'|} 2^{-\frac{3j}{2} l + m + \min(l,m)} \frac{\|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} 
\]

\[
+ \left[ \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^{2j \gamma_{j,l} \gamma_{j,l}}^y ,
\]

where the sequence of functions \((\mu_{j,\nu,l})_{l>j/2}\) on \(\mathbb{R} \times \mathbb{S}^2\) satisfies:

\[
\sum_{\nu} \sum_{l>j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 \lesssim \varepsilon^2 2^{2l} \|f\|_{L^2(\mathbb{R}^3)}^2.
\]

The proof of Proposition 10.3 is postponed to section 10.2.

Finally, we estimate \(A_{j,\nu',l,m}^2\). In view of the decomposition (10.3) of \(A_{j,\nu',l,m}^2\), the estimate (8.32), the estimates (10.13) (10.17) (10.18) (10.19) for \(h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}\), and the estimates (10.20) (10.21) (10.22) for \(A_{j,\nu',l,m}^1\), \(A_{j,\nu',l,m}^2\), and \(A_{j,\nu',l,m}^3\), we obtain:

\[
\sum_{(l,m)/2^{\max(l,m)} \leq 2^{j/2} |\nu - \nu'|} A_{j,\nu',l,m}^2
\]

\[
\lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^{j/2} |\nu - \nu'|} 2^{-\frac{3j}{2} l + m + \min(l,m)} \frac{\|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} 
\]

\[
+ \left[ \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^{2j \gamma_{j,l} \gamma_{j,l}}^y ,
\]

\[
\lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^{j/2} |\nu - \nu'|} 2^{-\frac{3j}{2} l + m + \min(l,m)} \frac{\|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} 
\]

\[
+ \left[ \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{3}{2}}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)} \right] \varepsilon^{2j \gamma_{j,l} \gamma_{j,l}}^y ,
\]

The proof of Proposition 10.3 is postponed to section 10.2.
where the sequence of functions \((\mu_{j,\nu,l})_{l>j/2}\) on \(\mathbb{R} \times S^2\) satisfies:

\[
\sum_{\nu} \sum_{l > j/2} 2^{2l} \|\mu_{j,\nu,l}\|^2_{L^2(\mathbb{R} \times S^2)} \lesssim \varepsilon 2^{2j} \|f\|^2_{L^2(\mathbb{R}^3)}.
\]

This concludes the proof of Proposition 6.7.

### 10.1 Proof of Proposition 10.2 (Control of \(A_{j,\nu,\nu',l,m}^1\))

In order to prove Proposition 10.2, recall that we need to show:

\[
\left| \sum_{(l,m)/2\max(l,m) \leq 2|\nu - \nu'|} A_{j,\nu,\nu',l,m} \right| \lesssim \sum_{(l,m)/2\max(l,m) \leq 2|\nu - \nu'|} 2^{-\frac{j}{2}l + m + \min(l,m)} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times S^2)} \|\mu_{j,\nu',m}\|_{L^2(\mathbb{R} \times S^2)}
\]

\[
\lesssim \sum_{(l,m)/2\max(l,m) \leq 2|\nu - \nu'|} \frac{2^{-\frac{j}{2}l + m + \min(l,m)}}{(2^\frac{j}{2}|\nu - \nu'|)^3} \times \left[ \frac{1}{(2^\frac{j}{2}|\nu - \nu'|)^3} + \frac{2^{-\left(\frac{j}{2}\right) - j}}{(2^\frac{j}{2}|\nu - \nu'|)^2} + \frac{1}{2^\frac{j}{2}(2^\frac{j}{2}|\nu - \nu'|)} \right] \varepsilon^{2|\nu - \nu'|},
\]

where the sequence of functions \((\mu_{j,\nu,l})_{l>j/2}\) on \(\mathbb{R} \times S^2\) satisfies:

\[
\sum_{\nu} \sum_{l > j/2} 2^{2l} \|\mu_{j,\nu,l}\|^2_{L^2(\mathbb{R} \times S^2)} \lesssim \varepsilon 2^{2j} \|f\|^2_{L^2(\mathbb{R}^3)}.
\]

and where \(A_{j,\nu,\nu',l,m}^1\) is given by (10.4):

\[
A_{j,\nu,\nu',l,m}^1 = 2^{-2j} \int_M \int_{S^2 \times S^2} P l \tr \chi \nabla^2 \Psi_m(\tr \chi')(N - g(N, N')N', N - g(N, N')N') \frac{g(L, L')(1 - g(N, N'))}{g(L', L')(1 - g(N, N'))} \times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' dM,
\]

(10.23)

We integrate by parts using (7.137).

**Lemma 10.4** Let \(A_{j,\nu,\nu',l,m}^1\) be defined by (10.23). Integrating by parts using (7.137) yields:

\[
A_{j,\nu,\nu',l,m}^1 = 2^{-2j} \sum_{p,q \geq 0} c_{pq} \int_M \frac{1}{|N_{\nu} - N_{\nu}'|^{p+q+2}} \left[ \frac{1}{|N_{\nu} - N_{\nu}'|^2} (h_{1,p,q,l,m} + h_{2,p,q,l,m}) 
\right.
\]

\[
+ \frac{1}{|N_{\nu} - N_{\nu}'|} (h_{3,p,q,l,m} + h_{4,p,q,l,m} + h_{5,p,q,l,m} + h_{6,p,q,l,m} + h_{7,p,q,l,m}) \right] dM,
\]

(10.24)

where \(c_{pq}\) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

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has radius of convergence 1, and where the scalar functions \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m}, h_{6,p,q,l,m}, h_{7,p,q,l,m} \) on \( \mathcal{M} \) are given by:

\[
\begin{align*}
    h_{1,p,q,l,m} &= \left( \int_{S^2} \chi P \text{tr} \chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
    &\quad \times \left( \int_{S^2} \nabla^2(P_m \text{tr} \chi') \left( 2^{\frac{1}{2}}(N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \\
    h_{2,p,q,l,m} &= \left( \int_{S^2} P \text{tr} \chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
    &\quad \times \left( \int_{S^2} \chi' \nabla^2(P_m \text{tr} \chi') \left( 2^{\frac{1}{2}}(N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \\
    h_{3,p,q,l,m} &= \left( \int_{S^2} \nabla(P \text{tr} \chi) \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
    &\quad \times \left( \int_{S^2} \nabla^2(P_m \text{tr} \chi') \left( 2^{\frac{1}{2}}(N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \\
    h_{4,p,q,l,m} &= \left( \int_{S^2} P \text{tr} \chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
    &\quad \times \left( \int_{S^2} \nabla^3(P_m \text{tr} \chi') \left( 2^{\frac{1}{2}}(N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \\
    h_{5,p,q,l,m} &= \left( \int_{S^2} (\theta + b^{-1} \nabla(b)) P \text{tr} \chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
    &\quad \times \left( \int_{S^2} \nabla^2(P_m \text{tr} \chi') \left( 2^{\frac{1}{2}}(N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \\
    h_{6,p,q,l,m} &= \left( \int_{S^2} P \text{tr} \chi \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
    &\quad \times \left( \int_{S^2} (\theta' + b'^{-1} \nabla(b')) \nabla^2(P_m \text{tr} \chi') \left( 2^{\frac{1}{2}}(N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right), \\
    
    \text{and:}
\end{align*}
\]

\[
\begin{align*}
    h_{7,p,q,l,m} &= \left( \int_{S^2} N(P \text{tr} \chi) \left( 2^{\frac{1}{2}}(N - N_\nu) \right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right) \\
    &\quad \times \left( \int_{S^2} \nabla^2(P_m \text{tr} \chi') \left( 2^{\frac{1}{2}}(N' - N_\nu') \right)^q F_{j,-1}(u') \eta_j'^\nu(\omega') d\omega' \right).
\end{align*}
\]
The proof of Lemma 10.4 is postponed to Appendix K. In the rest of this section, we use Lemma 10.4 to obtain the control of \( A_{1,\nu,\nu',l,m} \).

We evaluate the \( L^1(\mathcal{M}) \) norm of \( h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m}, h_{6,p,q,l,m}, \) and \( h_{7,p,q,l,m} \) starting with \( h_{1,p,q,l,m} \). In view of the definition (10.25) of \( h_{1,p,q,l,m} \), we have:

\[
\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} \chi P_t \chi \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta''_j(\omega) d\omega \right\|_{L^2(\mathcal{M})} \\
\times \left\| \int_{S^2} \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta''_j(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \\
\lesssim \varepsilon 2^{j-1} \gamma_j \left\| \int_{S^2} \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta''_j(\omega') d\omega' \right\|_{L^2(\mathcal{M})},
\]

where we used in the last inequality the estimate (10.12). The basic estimate in \( L^2(\mathcal{M}) \) (7.1) yields:

\[
\left\| \int_{S^2} \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta''_j(\omega') d\omega' \right\|_{L^2(\mathcal{M})} \lesssim \left( \sup_{\omega'} \left\| \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right) \right\|_{L\infty L^2(\mathcal{H}_{\omega'})} \right) 2^j \gamma_j
\]

where we used in the last inequality the Bochner inequality (2.61), the finite band property for \( P_m \), the estimate (2.39) for \( \chi P_t \), the estimate (2.42) for \( \partial_u N \), and the size of the patch. Finally, (10.32) and (10.33) imply:

\[
\|h_{1,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \varepsilon 2^{j+m-1} \gamma_j \gamma_j'.
\]

Next, we evaluate the \( L^1(\mathcal{M}) \) of \( h_{2,p,q,l,m} \). In view of the definition (10.26) of \( h_{2,p,q,l,m} \), we have:

\[
h_{2,p,q,l,m} = \int_{S^2} H \chi' \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \eta''_j(\omega') d\omega',
\]

where \( H \) is given by:

\[
H = \int_{S^2} P_t \chi \left( 2^j (N - N_\nu) \right)^p F_{j,-1}(u) \eta''_j(\omega) d\omega.
\]

This yields:

\[
\|h_{2,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \int_{S^2} \left\| H \chi' \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta''_j(\omega') d\omega' \\
\lesssim \int_{S^2} \left\| H \chi' \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta''_j(\omega') d\omega' \\
+ \int_{S^2} \left\| H \chi' \nabla^2 (P_m \chi') \left( 2^j (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta''_j(\omega') d\omega',
\]

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where we used in the last inequality the decomposition (2.45) of $\chi'$, and where we neglected the $\text{tr} \chi'$ contribution to $\chi'$ since it satisfies better estimates than $\chi'_1$. We start with the first term in the right-hand side of (10.36). We have:

$$\int_{S^2} \left\| H \chi'_1 \nabla^2 (P_m \text{tr} \chi') \left( 2^\frac{1}{2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(M)} \eta_j'(\omega') d\omega'$$  \hspace{1cm} (10.37)

$$\lesssim \|H\|_{L^2(M)} \left( \int_{S^2} \|\chi'_1\|_{L^\infty S_{\nu', \omega'}^2 L^\infty_{\nu',\omega'}} \|\nabla^2 (P_m \text{tr} \chi')\|_{L^\infty_{\nu',\omega'}} \times \left\| \left( 2^\frac{1}{2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^\infty \eta_j'(\omega') d\omega'} \right)$$

$$\lesssim \|H\|_{L^2(M)} e^{2^\frac{1}{2} + m \gamma_j'},$$

where we used in the last inequality the estimate (2.47) for $\chi'_1$, the Bochner inequality (2.61), the finite band property for $P_m$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, the size of the patch, Plancherel in $\lambda'$ for $F_{j,-1}(u')$, and Cauchy Schwarz in $\omega'$. Next, we estimate the second term in the right-hand side of (10.36). We have:

$$\int_{S^2} \left\| H \chi'_2 \nabla^2 (P_m \text{tr} \chi') \left( 2^\frac{1}{2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(M)} \eta_j'(\omega') d\omega'$$  \hspace{1cm} (10.38)

$$\lesssim \int_{S^2} \left\| H (\chi'_2 - \chi_{2\nu}) \nabla^2 (P_m \text{tr} \chi') \left( 2^\frac{1}{2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(M)} \eta_j'(\omega') d\omega'$$

$$+ \int_{S^2} \left\| H \chi_{2\nu} \nabla^2 (P_m \text{tr} \chi') \left( 2^\frac{1}{2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^1(M)} \eta_j'(\omega') d\omega'$$

$$\lesssim \left( \sup_{\omega'} \|\chi'_2 - \chi_{2\nu}\|_{L^\infty(M)} + \|\chi_{2\nu} H\|_{L^2(M)} \right)$$

$$\times \left( \int_{S^2} \left\| \nabla^2 (P_m \text{tr} \chi') \left( 2^\frac{1}{2} (N' - N_{\nu'}) \right)^q F_{j,-1}(u') \right\|_{L^2(M)} \eta_j'(\omega') d\omega' \right)$$

$$\lesssim (|\nu - \nu'| \|H\|_{L^2(M)} + \|\chi_{2\nu} H\|_{L^2(M)}) e^{2^\frac{1}{2} + m \gamma_j'},$$

where we used in the last inequality the estimate (2.47) for $\partial_\nu \chi'_2$, the Bochner inequality (2.61), the finite band property for $P_m$, the estimate (2.39) for $\text{tr} \chi$, the estimate (2.42) for $\partial_\omega N$, the size of the patch, Plancherel in $\lambda'$ for $F_{j,-1}(u')$, and Cauchy Schwarz in $\omega'$. Next, we estimate $H$. In view of the definition (10.35) of $H$ and the estimate (10.15), we have:

$$\|H\|_{L^2(M)} \lesssim e^{2^\frac{1}{2} - l \gamma_j'}. \hspace{1cm} (10.39)$$

Also, in view of the estimate (8.272), we have:

$$\|H\|_{L^\infty(M)} \lesssim e^{2^\frac{1}{2} \gamma_j'},$$

which by interpolation with (10.39) implies:

$$\|H\|_{L^3(M)} \lesssim e^{2^\left(\frac{1}{2}\right) + l \gamma_j'}. \hspace{1cm} (10.40)$$

Finally, we have:

$$\|\chi_{2\nu} H\|_{L^2(M)} \lesssim \|\chi_{2\nu}\|_{L^\infty_{\nu', \omega'} L^2_{\nu', \omega'}} \|H\|_{L^2_{\nu', \omega'} L^2_{\nu', \omega'}}$$

$$\lesssim (1 + p^2) e^{2^\frac{1}{2} - \frac{l}{2} \gamma_j'},$$

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where we used in the last inequality the estimate (2.46) for $\chi_{2\nu}$, the estimate (7.71) for $H$ in the case $l > j/2$, and the estimate (8.271) for $H$ in the case $l = j/2$. Finally, (10.36), (10.37), (10.38), (10.39), (10.40) and (10.41) imply:

$$
\|h_{2,p,q,l,m}\|_{L^1(M)} \lesssim (1 + p^2)\varepsilon^2 (2^{\frac{j}{2}} - \frac{j}{2} + |\nu - \nu'|2^{\frac{j}{2}} + j)2^{\frac{j}{2} + m} \gamma_j \gamma_j'.
$$

(10.42)

Next, we evaluate the $L^1(M)$ of $h_{3,p,q,l,m}$. We first consider the case where $m = j/2$. In view of the definition (10.27) of $h_{3,p,q,l,m}$ with $m = j/2$, we have:

$$
\|h_{3,p,q,l,j/2}\|_{L^1(M)} \lesssim \left\| \int_{S^2} \nabla(P_{l+1}\chi) \left( 2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)}
$$

$$
\times \left\| \int_{S^2} \nabla^2(P_{\leq j/2}\chi) \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)}
$$

$$
\lesssim 2^{\frac{j}{2}} \varepsilon^2 \gamma_j \gamma_j'.
$$

where we used in the last inequality the analog of the estimate (8.317) for the first term, and the analog of (10.33) for the second term. Since $l \geq j/2$, this yields in the case $m = j/2$:

$$
\|h_{3,p,q,l,j/2}\|_{L^1(M)} \lesssim 2^{\frac{j}{2}} \varepsilon^2 \gamma_j \gamma_j'.
$$

(10.43)

Next, we consider the case where $m > j/2$. Since $l \geq m$, we also have $l > j/2$. In view of the definition (10.27) of $h_{3,p,q,l,m}$, we have:

$$
\|h_{3,p,q,l,m}\|_{L^1(M)} \lesssim \left\| \int_{S^2} \nabla(P_{l+1}\chi) \left( 2^{\frac{j}{2}}(N - N_{\nu}) \right)^p F_{j-1}(u) \eta_j'(\omega) d\omega \right\|_{L^2(M)}
$$

$$
\times \left\| \int_{S^2} \nabla^2(P_{m}\chi) \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta_j'(\omega') d\omega' \right\|_{L^2(M)}
$$

$$
\lesssim \left\| \int_{S^2} \nabla(P_{l}\chi) \right\|_{L^2(H_u)} \left\| F_{j-1}(u) \right\|_{L^2(H_u)} \eta_j'(\omega) d\omega
$$

$$
\times \left\| \nabla^2(P_{m}\chi) \right\|_{L^2(H_u)} \left\| F_{j-1}(u') \right\|_{L^2(H_u)} \eta_j'(\omega') d\omega'
$$

, where we used in the last inequality the estimate (2.42) for $\partial_{\omega}N$ and the size of the patch. Taking Cauchy Schwartz in $\omega$ and $\omega'$, using the size of the patches, and using the Bochner inequality (2.61) and the finite band property for $P_l$ and $P_m$, we obtain:

$$
\|h_{3,p,q,l,m}\|_{L^1(M)} \lesssim 2^{2m+l-j} \| P_l \chi \|_{L^2(H_u)} \left\| F_j(u) \right\|_{L^2(H_u)} \eta_j'(\omega)\|_{L^2,H_u}
$$

$$
\times \| P_m \chi \|_{L^2(H_u)} \left\| F_j(u') \right\|_{L^2(H_u)} \eta_j'(\omega')\|_{L^2,H_u}.
$$

(10.44)

In view of (10.44) and the estimate (6.18), we finally obtain in the case $m > j/2$:

$$
\|h_{3,p,q,l,m}\|_{L^1(M)} \lesssim 2^{m+l+\min(l,m)} \| P_{l+1} \chi \|_{L^2(H_u)} \left\| F_j(u) \right\|_{L^2(H_u)} \eta_j'(\omega)\|_{L^2,H_u}
$$

$$
\times \| P_{l+1} \chi \|_{L^2(H_u)} \left\| F_j(u') \right\|_{L^2(H_u)} \eta_j'(\omega')\|_{L^2,H_u}.
$$

(10.45)

where the sequence of functions $(\mu_{j,\nu,l})_{l>j/2}$ on $\mathbb{R} \times S^2$ satisfies:

$$
\sum_{\nu} \sum_{l>j/2} 2^{2l} \| \mu_{j,\nu,l} \|_{L^2(\mathbb{R} \times S^2)}^2 \lesssim \varepsilon^2 2^{2j} \| f \|_{L^2(\mathbb{R}^3)}^2.
$$
Next, we evaluate the $L^1(\mathcal{M})$ of $h_{4,p,q,l,m}$. We first consider the case $m = j/2$. In view of the definition (10.28) of $h_{4,p,q,l,m}$ with $m = j/2$, we have:

$$h_{4,p,q,l,j/2} = \int_{\mathbb{S}^2} H \nabla^3 (P_{\leq j/2} \text{tr} \chi') \left(2^{j/2} (N' - N_\nu)\right)^q F_{j,-1}(u') \eta_j'(\omega')d\omega'.$$

where $H$ is given by:

$$H = \int_{\mathbb{S}^2} P_l \text{tr} \chi \left(2^{j/2} (N - N_\nu)\right)^p F_{j,-1}(u) \eta_j'(\omega)d\omega. \quad (10.46)$$

This yields:

$$\|h_{4,p,q,l,j/2}\|_{L^1(\mathcal{M})} \lesssim \int_{\mathbb{S}^2} \left\|H \nabla^3 (P_{\leq j/2} \text{tr} \chi') \left(2^{j/2} (N' - N_\nu)\right)^q F_{j,-1}(u')\right\|_{L^1(\mathcal{M})} \eta_j'(\omega')d\omega'$$

$$\lesssim \int_{\mathbb{S}^2} \left\|H\right\|_{L^2(P_{l,u'})} \left\|\nabla^3 (P_{\leq j/2} \text{tr} \chi')\right\|_{L^2(P_{l,u'})} F_{j,-1}(u') \eta_j'(\omega')d\omega',$$

where we used in the last inequality the estimate (2.42) for $\partial_\omega N$ and the size of the patch. Now, in view of the estimate (2.62), we have:

$$\|\nabla^3 (P_{\leq j/2} \text{tr} \chi')\|_{L^2(P_{l,u'})} \lesssim (2^j + \mu(t)2^{j/2} + \mu(t)^2)\varepsilon_\omega,$$

where we used in the last inequality the finite band property for $P_{\leq j/2}$ and the estimate (2.39) for $\text{tr} \chi$, and where $\mu$ in a function in $L^2(\mathbb{R})$ satisfying:

$$\|\mu\|_{L^2(\mathbb{R})} \lesssim 1. \quad (10.49)$$

(10.47), (10.48) and (10.49) yield:

$$\|h_{4,p,q,l,j/2}\|_{L^1(\mathcal{M})} \lesssim 2^j \|H\|_{L^2(\mathcal{M})} \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L^2_u \eta_j'(\omega')d\omega'}\right)$$

$$+ \left(\sup_{\omega'} \|H\|_{L^2_u L^\infty L^2_{\omega'}}\right) \left(\int_{\mathbb{S}^2} \|F_{j,-1}(u')\|_{L^2_u \eta_j'(\omega')d\omega'}\right)$$

$$\lesssim 2^{2j-l} \varepsilon^2 \gamma_j' \gamma_j'' + \left(\sup_{\omega'} \|H\|_{L^2_u L^\infty L^2_{\omega'}}\right) 2^{j} \varepsilon \gamma_j'',$$

where we used in the last inequality the estimate (10.15) for $H$ which holds in view of the definition (10.46) of $H$, Plancherel in $\lambda'$ for $\|F_{j,-1}(u')\|_{L^2_{\omega'}}$, Cauchy Schwarz in $\omega'$ and the size of the patch. Using the estimate (7.71) in the case $l > j/2$, and the estimates (7.71) (7.83) in the case $l = j/2$ together with the decomposition

$$P_{\leq j/2} \text{tr} \chi = \text{tr} \chi - \sum_{l > \frac{j}{2}} P_l \text{tr} \chi,$$
we obtain:

\[
\sup_{\omega'} \| H \|_{L^2(\omega',\eta L^\infty)} \lesssim (1 + p^2) \varepsilon (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{j}{2} - \frac{3}{4}}) \gamma_{j'} \gamma_{j'}. \tag{10.51}
\]

Together with (10.50), this yields in the case \( m = j/2 \):

\[
\| h_{4p,q,l,j/2} \|_{L^1(\mathcal{M})} \lesssim 2^{2j-l} \varepsilon^2 \gamma_{j} \gamma_{j'} + (1 + p^2) (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{j}{2} - \frac{3}{4}}) \varepsilon^2 \gamma_{j} \gamma_{j'}. \tag{10.52}
\]

Next, we consider the case where \( m > j/2 \). Since \( l \geq m \), we also have \( l > j/2 \). In view of the definition (10.28) of \( h_{4p,q,l,m} \), we have:

\[
h_{4p,q,l,m} = \int_{S^2} H \nabla^2 (P_{m} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q} F_{j,-1}(u') \eta_{j'}(\omega') d\omega'.
\]

where \( H \) is given by (10.46). This yields: This yields:

\[
\| h_{4p,q,l,m} \|_{L^1(\mathcal{M})} \tag{10.53}
\]

\[
\lesssim \int_{S^2} \left\| H \nabla^2 (P_{m} \text{tr} \chi') \left(2^{\frac{j}{2}} (N' - N_{\nu'}) \right)^{q} F_{j,-1}(u') \right\|_{L^1(\mathcal{M})} \eta_{j'}(\omega') d\omega'
\]

\[
\lesssim \int_{S^2} \| H \|_{L^2(P_{t,u})} \| \nabla^2 (P_{m} \text{tr} \chi') \|_{L^2(P_{t,u})} \| F_{j,-1}(u') \|_{L_{\omega,t}^2} \eta_{j'}(\omega') d\omega',
\]

where we used in the last inequality the estimate (2.42) for \( \partial_{\omega} N \) and the size of the patch.

Now, in view of the estimate (2.62), we have:

\[
\| \nabla^2 (P_{m} \text{tr} \chi') \|_{L^2(P_{t,u})} \tag{10.54}
\]

\[
\lesssim \| \nabla \Delta (P_{m} \text{tr} \chi') \|_{L^2(P_{t,u})} + \mu(t) \| \Delta (P_{m} \text{tr} \chi') \|_{L^2(P_{t,u})} + \mu^2(t) \| \nabla (P_{m} \text{tr} \chi') \|_{L^2(P_{t,u})}
\]

\[
\lesssim 2^{3m} \| P_{m} \text{tr} \chi' \|_{L^2(P_{t,u})} + 2^{m} \mu(t)^2 \varepsilon,
\]

where we used in the last inequality the finite band property for \( P_{m} \) and the estimate (2.39) for \( \text{tr} \chi' \), and where \( \mu \) in a function in \( L^2(\mathbb{R}) \) satisfying (10.49). (10.53), (10.54) and (10.49) yield:

\[
\| h_{4p,q,l,m} \|_{L^1(\mathcal{M})} \tag{10.55}
\]

\[
\lesssim 2^{3m} \| H \|_{L^2(\mathcal{M})} \left( \int_{S^2} \| P_{m} \text{tr} \chi' \|_{L^2(\omega,\nu')} \| F_{j,-1}(u') \|_{L_{\omega}} \eta_{j'}(\omega') d\omega' \right)
\]

\[
+ 2^{m} \varepsilon \left( \sup_{\omega'} \| H \|_{L^2(\omega',\eta L^\infty)} \right) \left( \int_{S^2} \| F_{j,-1}(u') \|_{L_{\omega}} \eta_{j'}(\omega') d\omega' \right)
\]

\[
\lesssim 2^{3m} \| H \|_{L^2(\mathcal{M})} \left( \int_{S^2} \| P_{m} \text{tr} \chi' \|_{L^2(\omega,\nu')} \| F_{j,-1}(u') \|_{L_{\omega}} \eta_{j'}(\omega') d\omega' \right)
\]

\[
+ (1 + p^2) (2^{\frac{j}{2}} |\nu - \nu'| 2^{-l+j+m} + (2^{\frac{j}{2}} |\nu - \nu'|)^{\frac{1}{2}} 2^{-\frac{j}{2} - \frac{3}{4} + m}) \varepsilon^2 \gamma_{j} \gamma_{j'} \gamma_{j'},
\]

where we used in the last inequality the estimate (10.51) for \( H \), Plancherel in \( \lambda' \) for \( \| F_{j,-1}(u') \|_{L_{\omega}^2} \), Cauchy Schwarz in \( \omega' \) and the size of the patch. In view of the definition (10.46) of \( H \), and using the estimate (2.42) for \( \partial_{\omega} N \) and the size of the patch, we have:

\[
\| H \|_{L^2(\mathcal{M})} \lesssim \int_{S^2} \| P_{t} \text{tr} \chi' \|_{L^2(\omega,\nu')} \| F_{j,-1}(u) \|_{L_{\omega}^2} \eta_{j'}(\omega) d\omega.
\]
Together with (10.55), this yields:
\[ ||h_{4,p,q,l,m}||_{L^1(M)} \lesssim 2^{3m} \left( \int_{S^2} ||P_t \text{tr} \chi||_{L^2(H_u)} F_{j,-1}(u) ||\eta_j''(\omega)||_{L^2_u} d\omega \right) \]
\[ \times \left( \int_{S^2} ||P_m \text{tr} \chi' ||_{L^2(H_{\omega})} F_{j,-1}(u') ||\eta_j''(\omega')||_{L^2_{\omega'}} d\omega' \right) \]
\[ + (1 + p^2)(2^{\frac{j}{2}}|\nu - \nu'|2^{-l+j+m} + (2^{\frac{j}{2}}|\nu - \nu'|)\frac{j}{2}2^{-\frac{j}{2} + \frac{j}{2} + m}) \varepsilon^2 \gamma_j'' \gamma_j'. \]

Taking Cauchy Schwartz in \( \omega \) and \( \omega' \), using the size of the patches, and using the Bochner inequality (2.61) and the finite band property for \( P_t \) and \( P_m \), we obtain:
\[ ||h_{4,p,q,l,m}||_{L^1(M)} \lesssim 2^{3m-j} \left( ||P_t \text{tr} \chi ||_{L^2(H_u)} F_{j}(u) \sqrt{\eta_j''(\omega)} \right) \]
\[ \times \left( ||P_m \text{tr} \chi' ||_{L^2(H_{\omega})} F_{j}(u') \sqrt{\eta_j''(\omega')} \right) \]
\[ + (1 + p^2)(2^{\frac{j}{2}}|\nu - \nu'|2^{-l+j+m} + (2^{\frac{j}{2}}|\nu - \nu'|)\frac{j}{2}2^{-\frac{j}{2} + \frac{j}{2} + m}) \varepsilon^2 \gamma_j'' \gamma_j' \] (10.56)

In view of (10.56) and the estimate (6.18), we finally obtain in the case \( m > j/2 \):
\[ ||h_{4,p,q,l,m}||_{L^1(M)} \lesssim 2^{m+l+\min(m,j)} ||\mu_{j,\nu,l}||_{L^2(\mathbb{R} \times S^2)} ||\mu_{j,\nu',\nu,m}||_{L^2(\mathbb{R} \times S^2)} \]
\[ + (1 + p^2)(2^{\frac{j}{2}}|\nu - \nu'|2^{-l+j+m} + (2^{\frac{j}{2}}|\nu - \nu'|)\frac{j}{2}2^{-\frac{j}{2} + \frac{j}{2} + m}) \varepsilon^2 \gamma_j'' \gamma_j' \] (10.57)

where the sequence of functions \( (\mu_{j,\nu,l})_{l>j/2} \) on \( \mathbb{R} \times S^2 \) satisfies:
\[ \sum_{\nu} \sum_{l>j/2} 2^l ||\mu_{j,\nu,l}||_{L^2(\mathbb{R} \times S^2)} \lesssim \varepsilon^2 2^j \|f\|_{L^2(\mathbb{R}^3)}. \]

Next, we evaluate the \( L^1(M) \) of \( h_{5,p,q,l,m} \). In view of the definition (10.29) of \( h_{5,p,q,l,m} \), we have:
\[ ||h_{5,p,q,l,m}||_{L^1(M)} \lesssim \left( \int_{S^2} (\theta + b^{-1}\nabla(b))P_t \text{tr} \chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u)\eta_j''(\omega)d\omega \right) \]
\[ \times \left( \int_{S^2} \nabla^2 (P_m \text{tr} \chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j''(\omega')d\omega' \right) \]
\[ \lesssim 2^{\frac{j}{2} + l_2 + m} \varepsilon^2 \gamma_j'' \gamma_j', \] (10.58)

where we used in the last inequality the estimate (8.275) for the first term and the estimate (10.33) for the second term.

Next, we evaluate the \( L^1(M) \) of \( h_{6,p,q,l,m} \). In view of the definition (10.30) of \( h_{6,p,q,l,m} \), we have:
\[ ||h_{6,p,q,l,m}||_{L^1(M)} \]
\[ \lesssim \left( \int_{S^2} P_t \text{tr} \chi \left( 2^{\frac{j}{2}}(N - N_\nu) \right)^p F_{j,-1}(u)\eta_j''(\omega)d\omega \right) \]
\[ \times \left( \int_{S^2} (\theta' + b'^{-1}\nabla'(b'))\nabla^2 (P_m \text{tr} \chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j''(\omega')d\omega' \right) \]
\[ \lesssim 2^{\frac{j}{2} - l_2} \varepsilon \gamma_j'' \left( \int_{S^2} (\theta' + b'^{-1}\nabla'(b'))\nabla^2 (P_m \text{tr} \chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j,-1}(u')\eta_j''(\omega')d\omega' \right), \] (10.59)
where we used in the last inequality the estimate (10.15). The basic estimate in $L^2(M)$ (7.1) implies:

$$\left\| \int_{S^2} (\theta' + b^{-1} \nabla'(b')) \nabla^2(P_m \text{tr} \chi') \left(2^{s}(N' - N_{\nu'}) \right)^q F_{j',-1}(u') \eta_{j'}^{\nu'}(\omega')d\omega' \right\|_{L^2(M)} \leq \left( \sup_{\omega'} \left\| (\theta' + b^{-1} \nabla'(b')) \nabla^2(P_m \text{tr} \chi') \left(2^{s}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty(M)} \right)^{2^{s}} \gamma_{j'}^{\nu'}$$

$$\leq \left( \sup_{\omega'} \left\| \theta' + b^{-1} \nabla'(b') \right\|_{L^\infty} \left\| \nabla^2(P_m \text{tr} \chi') \right\|_{L^2} \left\| \left(2^{s}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right)^{2^{s}} \gamma_{j'}^{\nu'}$$

$$\leq \left( \sup_{\omega'} \left\| \nabla^2(P_m \text{tr} \chi') \right\|_{L^2} \right)^{\varepsilon} 2^{s} \gamma_{j'}^{\nu'},$$

where we used in the last inequality the estimates (2.37) (2.39) (2.40) for $\chi'$, the estimate (2.38) for $b'$, the estimate (2.42) for $\partial_{\omega}N$ and the size of the patch. Now, the Gagliardo-Nirenberg inequality (2.49) yields:

$$\left\| \nabla^2(P_m \text{tr} \chi') \right\|_{L^4(P_{t,u'})} \leq \left\| \nabla^2(P_m \text{tr} \chi') \right\|_{L^2(P_{t,u'})} \left\| \nabla^3(P_m \text{tr} \chi') \right\|_{L^2(P_{t,u'})} \leq 2^{3m} \varepsilon(1 + \mu(t)),$$

where we used in the last inequality the Bochner inequality (2.61), the finite band property for $P_m$, the estimates (10.48) and (10.54) for $\nabla^3(P_m \text{tr} \chi')$, and the estimate (2.39) for $\text{tr} \chi$. Together with the estimate (10.49) for $\mu$, we obtain:

$$\left\| \nabla^2(P_m \text{tr} \chi') \right\|_{L^2} \leq 2^{3m} \varepsilon,$$

which together with (10.60) yields:

$$\left\| \int_{S^2} (\theta' + b^{-1} \nabla'(b')) \nabla^2(P_m \text{tr} \chi') \left(2^{s}(N' - N_{\nu'}) \right)^q F_{j',-1}(u') \eta_{j'}^{\nu'}(\omega')d\omega' \right\|_{L^2(M)} \leq 2^{3m} \varepsilon^{2} \gamma_{j'}^{\nu'}.$$

Finally, (10.59) and (10.61) imply:

$$\left\| h_{6,p,q,l,m} \right\|_{L^1(M)} \leq 2^{j-l+3m} \varepsilon^{2} \gamma_{j'}^{\nu'} \gamma_{j}^{\nu'}.$$

Next, we evaluate the $L^1(M)$ of $h_{7,p,q,l,m}$. In view of the definition (10.31) of $h_{7,p,q,l,m}$, we have:

$$\left\| h_{7,p,q,l,m} \right\|_{L^1(M)} \leq \left\| \int_{S^2} N(P_{t,x}) \left(2^{s}(N - N_{\nu}) \right)^p F_{j',-1}(u) \eta_{j'}^{\nu}(\omega)d\omega \right\|_{L^2(M)} \times \left\| \nabla^2(P_m \text{tr} \chi') \left(2^{s}(N' - N_{\nu'}) \right)^q F_{j',-1}(u') \eta_{j'}^{\nu'}(\omega')d\omega' \right\|_{L^2(M)} \leq 2^{j+m} \varepsilon^{2} \gamma_{j'}^{\nu'} \gamma_{j}^{\nu'},$$
where we used in the last inequality the analog of the estimate (8.317) for the first term, and the estimate (10.33) for the second term.

Finally, we estimate $A_{j,v,v',l,m}^1$. In view of the decomposition (10.24) of $A_{j,v,v',l,m}^1$, the estimate (8.32), and the estimates (10.34) (10.42) (10.43) (10.45) (10.52) (10.57) (10.58) (10.62) (10.63) for $h_{1,p,q,l,m}, h_{2,p,q,l,m}, h_{3,p,q,l,m}, h_{4,p,q,l,m}, h_{5,p,q,l,m}, h_{6,p,q,l,m}, h_{7,p,q,l,m}$, we obtain:

$$
\sum_{(l,m)/2^{\max(l,m)} \leq 2^j|v-v'|} A_{j,v,v',l,m}^1 \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j|v-v'|} \sum_{p,q \geq 0} C_{pq} \frac{2^{-\frac{5}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{1}{2}}|v-v'|)^{p+q+3}} \left| \mu_{j,v,l} \right| L^2(\mathbb{R} \times \mathbb{S}^2) \left| \mu_{j,v',m} \right| L^2(\mathbb{R} \times \mathbb{S}^2)
$$

$$
+ \sum_{p,q \geq 0} C_{pq} \frac{1}{(2^{\frac{1}{2}}|v-v'|)^{p+q+2}} \left[ 1 + p^2 \right] \left( \frac{1}{2^{\frac{1}{2}}|v-v'|} \right)^3 + \left( 1 + p^2 \right) 2^{-\left( \frac{1}{2} \right) - j} + 2^{-\left( \frac{1}{2} \right) \left( |v-v'| \right)} \right] \varepsilon^2 \gamma_j \gamma_j^v \gamma_j^v',
$$

where the sequence of functions $(\mu_{j,v,l})_{l \geq j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$
\sum_{v} \sum_{l \geq j/2} 2^{2l} \left| \mu_{j,v,l} \right| L^2(\mathbb{R} \times \mathbb{S}^2) \lesssim \varepsilon^2 2^{2j} \left| f \right| L^2(\mathbb{R}^3).
$$

This concludes the proof of Proposition 10.2.

### 10.2 Proof of Proposition 10.3 (Control of $A_{j,v,v',l,m}^2$)

In order to prove Proposition 10.3, recall that we need to show:

$$
\sum_{(l,m)/2^{\max(l,m)} \leq 2^j|v-v'|} A_{j,v,v',l,m}^2 \lesssim \sum_{(l,m)/2^{\max(l,m)} \leq 2^j|v-v'|} \frac{2^{-\frac{5}{2}} 2^{l+m+\min(l,m)}}{(2^{\frac{1}{2}}|v-v'|)^3} \left| \mu_{j,v,l} \right| L^2(\mathbb{R} \times \mathbb{S}^2) \left| \mu_{j,v',m} \right| L^2(\mathbb{R} \times \mathbb{S}^2)
$$

$$
+ \left[ \frac{1}{(2^{\frac{1}{2}}|v-v'|)^3} + \frac{1}{(2^{\frac{1}{2}}|v-v'|)^{\frac{3}{2}}} + \frac{1}{2^{\frac{1}{2}} (2^{\frac{1}{2}}|v-v'|)} \right] \varepsilon^2 \gamma_j \gamma_j^v \gamma_j^v',
$$

where the sequence of functions $(\mu_{j,v,l})_{l \geq j/2}$ on $\mathbb{R} \times \mathbb{S}^2$ satisfies:

$$
\sum_{v} \sum_{l \geq j/2} 2^{2l} \left| \mu_{j,v,l} \right| L^2(\mathbb{R} \times \mathbb{S}^2) \lesssim \varepsilon^2 2^{2j} \left| f \right| L^2(\mathbb{R}^3),
$$

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and where $A^2_{j,v,v',l,m}$ is given by (10.5):

$$A^2_{j,v,v',l,m} = 2^{-2j} \int \int_{\mathcal{M} \times S^2} \frac{(N' - g(N',N')(P_{tr\chi})(N - g(N',N')(P_m(tr\chi))}{g(L',1 - g(N,N')^2)}$$

$$\times F_j(u)F_{j-1}(u')\eta_j'(\omega)\eta_j''(\omega')d\omega d\omega' d\mathcal{M}, \quad (10.64)$$

We integrate by parts using (7.143).

**Lemma 10.5** Let $A^2_{j,v,v',l,m}$ be defined by (10.64). Integrating by parts using (7.143) yields:

$$A^2_{j,v,v',l,m} = \sum_{p,q \geq 0} c_{pq} \int_{\mathcal{M}} \frac{1}{(2^j |N_{\nu} - N_{\nu'}|)^{p+q+2}} \left[ \frac{1}{|N_{\nu} - N_{\nu'}|^2} (h^j_{1,p,q,l,m} + h^j_{2,p,q,l,m}) \right. \left. + h^j_{3,p,q,l,m} + h^j_{4,p,q,l,m} + h^j_{5,p,q,l,m} + h^j_{6,p,q,l,m} + h^j_{7,p,q,l,m} + h^j_{8,p,q,l,m} \right] d\mathcal{M},$$

where $c_{pq}$ are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1, where the scalar functions $h^j_{1,p,q,l,m}, h^j_{2,p,q,l,m}, h^j_{3,p,q,l,m}, h^j_{4,p,q,l,m}, h^j_{5,p,q,l,m}, h^j_{6,p,q,l,m}, h^j_{7,p,q,l,m}, h^j_{8,p,q,l,m}$ on $\mathcal{M}$ are given by:

$$h^j_{1,p,q,l,m} = \left( \int_{S^2} \nabla(L(P_{tr\chi})) (2^j (N - N_{\nu}))^p F_{j-1}(u)\eta_j'(\omega)d\omega \right) \times \left( \int_{S^2} \nabla' (P_{mtr\chi}) (2^j (N' - N_{\nu'})^q F_{j-1}(u')\eta_j''(\omega')d\omega' \right),$$

$$h^j_{2,p,q,l,m} = \left( \int_{S^2} \nabla'(L(P_{tr\chi})) (2^j (N - N_{\nu}))^p F_{j-1}(u)\eta_j'(\omega)d\omega \right) \times \left( \int_{S^2} \nabla' (P_{mtr\chi}) (2^j (N' - N_{\nu'})^q F_{j-1}(u')\eta_j''(\omega')d\omega' \right),$$

$$h^j_{3,p,q,l,m} = \left( \int_{S^2} H_1 \nabla(P_{tr\chi}) (2^j (N - N_{\nu}))^p F_{j-1}(u)\eta_j'(\omega)d\omega \right) \times \left( \int_{S^2} \nabla'(P_{mtr\chi}) (2^j (N' - N_{\nu'})^q F_{j-1}(u')\eta_j''(\omega')d\omega' \right),$$

$$h^j_{4,p,q,l,m} = \left( \int_{S^2} \nabla(P_{tr\chi}) (2^j (N - N_{\nu}))^p F_{j-1}(u)\eta_j'(\omega)d\omega \right) \times \left( \int_{S^2} H_2 \nabla'(P_{mtr\chi}) (2^j (N' - N_{\nu'})^q F_{j-1}(u')\eta_j''(\omega')d\omega' \right).$$
Lemma 10.5 to obtain the control of $A$. The proof of Lemma 10.5 is postponed to Appendix L. In the rest of this section, we use $G$ and $H$ starting with $p, q, l, m$.

We evaluate the $L^1(\mathcal{M})$ norm of $h'_{1,p,q,l,m}$, $h'_{2,p,q,l,m}$, $h'_{3,p,q,l,m}$, $h'_{4,p,q,l,m}$, $h'_{5,p,q,l,m}$, $h'_{6,p,q,l,m}$, $h'_{7,p,q,l,m}$ starting with $h'_{1,p,q,l,m}$. In view of the definition (10.66) of $h'_{1,p,q,l,m}$, we have:

$$h'_{1,p,q,l,m} = \left( \int_{S^2} \nabla(P_t \text{tr}\chi) \left( 2^{\frac{d}{2}}(N - N_v) \right)^p F_{j,-1}(u)\eta_j^p(\omega)d\omega \right)$$

(10.70)

$$\times \left( \int_{S^2} \nabla^2(P_m \text{tr}\chi') \left( 2^{\frac{d}{2}}(N' - N_{v'}) \right)^q F_{j,-1}(u')\eta_j^p(\omega')d\omega' \right),$$

$$h'_{6,p,q,l,m} = \left( \int_{S^2} H_3 \nabla(P_t \text{tr}\chi) \left( 2^{\frac{d}{2}}(N - N_v) \right)^p F_{j,-1}(u)\eta_j^p(\omega)d\omega \right)$$

(10.71)

$$\times \left( \int_{S^2} \nabla(P_m \text{tr}\chi') \left( 2^{\frac{d}{2}}(N' - N_{v'}) \right)^q F_{j,-1}(u')\eta_j^p(\omega')d\omega' \right),$$

and:

$$h'_{7,p,q,l,m} = \left( \int_{S^2} \nabla(P_t \text{tr}\chi) \left( 2^{\frac{d}{2}}(N - N_v) \right)^p F_{j,-1}(u)\eta_j^p(\omega)d\omega \right)$$

(10.72)

$$\times \left( \int_{S^2} H_4 \nabla(P_m \text{tr}\chi') \left( 2^{\frac{d}{2}}(N' - N_{v'}) \right)^q F_{j,-1}(u')\eta_j^p(\omega')d\omega' \right),$$

and where the tensors $H_1$, $H_2$, $H_3$ and $H_4$ are given by:

$$H_1 = \chi + \epsilon + \delta + n^{-1}\nabla n + L(b),$$

(10.74)

$$H_2 = \chi' + \epsilon' + \delta' + n^{-1}\nabla n + L'(b'),$$

(10.75)

$$H_3 = k + n^{-1}\nabla n + \theta + b^{-1}\nabla(b) + \chi + \zeta,$$

(10.76)

and:

$$H_4 = k + n^{-1}\nabla n + \theta' + b'^{-1}\nabla(b') + \zeta'.$$

(10.77)

The proof of Lemma 10.5 is postponed to Appendix L. In the rest of this section, we use Lemma 10.5 to obtain the control of $A_{j,v,v',l,m}^2$. We evaluate the $L^1(\mathcal{M})$ norm of $h'_{1,p,q,l,m}$, $h'_{2,p,q,l,m}$, $h'_{3,p,q,l,m}$, $h'_{4,p,q,l,m}$, $h'_{5,p,q,l,m}$, $h'_{6,p,q,l,m}$, $h'_{7,p,q,l,m}$ starting with $h'_{1,p,q,l,m}$. In view of the definition (10.66) of $h'_{1,p,q,l,m}$, we have:

$$h'_{1,p,q,l,m} = \int_{S^2} G \nabla(L(P_t \text{tr}\chi)) \left( 2^{\frac{d}{2}}(N - N_v) \right)^p F_{j,-1}(u)\eta_j^p(\omega)d\omega,$$

where $G$ is given by:

$$G = \int_{S^2} \nabla'(P_m \text{tr}\chi') \left( 2^{\frac{d}{2}}(N' - N_{v'}) \right)^q F_{j,-1}(u')\eta_j^p(\omega')d\omega'.$$

(10.78)
Using the analog of (7.13), this yields:

$$
\|h'_{1,p,q,l,m}\|_{L^1(M)} \lesssim \left( \sup_{\omega' \in \text{supp}(\eta'_j)} \|G \left( 2^{\frac{j}{2}}(N - N_{\nu'}) \right)^p \|_{L^2_{u,x} L^\infty_t} \right) 2^{\frac{j}{2}} \varepsilon \gamma'_j \ (10.79)
$$

$$
\lesssim \left( \sup_{\omega \in \text{supp}(\eta''_j)} \|G\|_{L^2_{u,x} L^\infty_t} \right) 2^{\frac{j}{2}} \varepsilon \gamma'_j,
$$

where we used in the last inequality the estimate (8.317). Now, in view of the definition (10.78) of $G$, the analog of the estimate (7.76) yields:

$$
\|G\|_{L^2_{u,x} L^\infty_t} \lesssim \left( \sup_{\omega} \left\| \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q \right\|_{L^\infty} \right) \varepsilon \left( 2^{\frac{j}{2}}|\nu - \nu'| 2^{\frac{j}{2}} + (2^{\frac{j}{2}}|\nu - \nu'|)^{\frac{1}{2}} 2^{\frac{m}{2} + \frac{j}{2}} \right) \gamma'_j \varepsilon 
$$

$$
\lesssim \varepsilon 2^{\frac{j}{2}} (2^{\frac{j}{2}}|\nu - \nu'|) \gamma'_j,
$$

where we used in the last inequality the fact that $2^m \leq 2^j |\nu - \nu'|$, the estimate (2.42) for $\partial_{\omega} N$ and the size of the patch. Together with (10.79), we obtain:

$$
\|h'_{1,p,q,l,m}\|_{L^1(M)} \lesssim 2^j (2^{\frac{j}{2}}|\nu - \nu'|) \varepsilon \gamma'_j \gamma'_j.
$$

Now, since $m \geq j/2$, we finally obtain:

$$
\|h'_{1,p,q,l,m}\|_{L^1(M)} \lesssim 2^{\frac{3j}{2}} 2^{\frac{m}{2}} (2^{\frac{j}{2}}|\nu - \nu'|) \varepsilon \gamma'_j \gamma'_j. \ (10.80)
$$

Next, we evaluate the $L^1(M)$ norm of $h'_{2,p,q,l,m}$. Comparing the definition (10.67) of $h'_{2,p,q,l,m}$ and (10.66) of $h'_{1,p,q,l,m}$, we notice that these terms are similar. We proceed as for $h'_{1,p,q,l,m}$, and we obtain the analog of (10.80):

$$
\|h'_{2,p,q,l,m}\|_{L^1(M)} \lesssim 2^{\frac{3j}{2}} 2^{\frac{m}{2}} (2^{\frac{j}{2}}|\nu - \nu'|) \varepsilon \gamma'_j \gamma'_j. \ (10.81)
$$

Next, we evaluate the $L^1(M)$ norm of $h'_{3,p,q,l,m}$. In view of the definition (10.68) of $h'_{3,p,q,l,m}$, we have:

$$
\|h'_{3,p,q,l,m}\|_{L^1(M)} \lesssim \left\| \int_{S^2} H_1 \nabla(P_t \chi) \left( 2^{\frac{j}{2}}(N - N_{\nu'}) \right)^p F_{j-1}(u) \eta'_j(\omega) d\omega \right\|_{L^2(M)}
$$

$$
\times \left\| \int_{S^2} \nabla'(P_m \chi') \left( 2^{\frac{j}{2}}(N' - N_{\nu'}) \right)^q F_{j-1}(u') \eta''_j(\omega') d\omega' \right\|_{L^2(M)}
$$

$$
\lesssim \left\| \int_{S^2} H_1 \nabla(P_t \chi) \left( 2^{\frac{j}{2}}(N - N_{\nu'}) \right)^p F_{j-1}(u) \eta'_j(\omega) d\omega \right\|_{L^2(M)} 2^{\frac{j}{2}} \varepsilon \gamma'_j,
$$

where we used in the last inequality the estimate the analog of the estimate (8.317). Now,
using the basic estimate (7.1) in $L^2(\mathcal{M})$, we have:

$$
\left\| \int_{\mathbb{S}^2} H_1 \nabla(P_{\ell}(\gamma', \chi, \nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{P} F_{j,-1}(u) \eta_j(\omega) d\omega \right\|_{L^2(\mathcal{M})} \leq C \epsilon_j \gamma_j
$$

(10.83)

where we used in the last inequality the estimate (2.70) for $\nabla(P_{\ell}(\gamma', \chi, \nu))$, the estimate (2.42) for $\partial_\nu N$ and the size of the patch. In view of the definition (10.74) of $H_1$, we have:

$$
\|H_1\|_{L^2(\mathcal{M})} \lesssim \|\chi\|_{L^2(\mathcal{M})} + \|\epsilon\|_{L^2} + \|\delta\|_{L^2} + \|n^{-1}\nabla n\|_{L^2} + \|L(b)\|_{L^2} \lesssim \epsilon,
$$

where we used in the last inequality the estimates (2.39) (2.40) for $\chi$, the estimates (2.36) (2.37) for $\epsilon$, the estimate (2.37) for $\delta$, the estimate (2.36) for $n$ and the estimate (2.38) for $b$. Together with (10.83), this yields:

$$
\left\| \int_{\mathbb{S}^2} H_1 \nabla(P_{\ell}(\gamma', \chi, \nu)) \left(2^{\frac{j}{2}}(N - N_\nu)\right)^{P} F_{j,-1}(u) \eta_j(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \epsilon_j \gamma_j.
$$

(10.84)

(10.82) and (10.84) imply:

$$
\|h_{3,p,q,l,m}'\|_{L^1(\mathcal{M})} \lesssim 2^{j} \epsilon_j \gamma_j\gamma_j'.
$$

Now, since $m \geq j/2$, we finally obtain:

$$
\|h_{3,p,q,l,m}'\|_{L^1(\mathcal{M})} \lesssim 2^{j} 2^{m} \epsilon_j \gamma_j\gamma_j'.
$$

(10.85)

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{4,p,q,l,m}'$. Comparing the definition (10.69) of $h_{4,p,q,l,m}'$ and (10.68) of $h_{3,p,q,l,m}'$, we notice that these terms are similar. We proceed as for $h_{3,p,q,l,m}'$, and we obtain the analog of (10.85):

$$
\|h_{4,p,q,l,m}'\|_{L^1(\mathcal{M})} \lesssim 2^{j} 2^{m} \epsilon_j \gamma_j\gamma_j'.
$$

(10.86)

Next, we evaluate the $L^1(\mathcal{M})$ norm of $h_{5,p,q,l,m}'$. Comparing the definition (10.70) of $h_{5,p,q,l,m}'$ and (10.27) of $h_{3,p,q,l,m}$, we notice that these terms are the same. Thus, in view of the estimates (10.43) and (10.45), we have in the case $m = j/2$:

$$
\|h_{5,p,q,l,m}'\|_{L^1(\mathcal{M})} \lesssim 2^{j} \epsilon_j \gamma_j\gamma_j'.
$$

(10.87)

and in the case $m > j/2$:

$$
\|h_{5,p,q,l,m}'\|_{L^1(\mathcal{M})} \lesssim 2^{m} \epsilon_j \gamma_j\gamma_j'.
$$

(10.88)
where the sequence of functions \((\mu_{j,\nu,l})_{l>|j|/2}\) on \(\mathbb{R} \times S^2\) satisfies:
\[
\sum_j \sum_{\nu} \sum_{l>|j|/2} 2^{2l} \|\mu_{j,\nu,l}\|^2_{L^2(\mathbb{R} \times S^2)} \lesssim \varepsilon 2^{2j} \|f\|^2_{L^2(\mathbb{R}^3)}.
\]

Next, we evaluate the \(L^1(\mathcal{M})\) norm of \(h'_{6,p,q,l,m}\). In view of the definition (10.71) of \(h'_{6,p,q,l,m}\), we have:
\[
\|h'_{6,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim \left\| \int_{S^2} H_3 \nabla (P_t \chi) \left(2^{j}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \tag{10.89}
\times \left\| \int_{S^2} \chi (P_n \chi'') \left(2^{j}(N' - N_{\nu'})\right)^q F_{j,-1}(u') \eta_j^{\nu'}(\omega') d\omega' \right\|_{L^2(\mathcal{M})}
\lesssim \left\| \int_{S^2} H_3 \nabla (P_t \chi) \left(2^{j}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} 2^{j} \varepsilon \gamma_j^{\nu'},
\]
where we used in the last inequality the analog of (8.317). Now, the basic estimate in \(L^2(\mathcal{M})\) (7.1) yields:
\[
\left\| \int_{S^2} H_3 \nabla (P_t \chi) \left(2^{j}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \tag{10.90}
\lesssim \left( \sup_{\omega} \left\| H_3 \nabla (P_t \chi) \left(2^{j}(N - N_{\nu})\right)^p \right\|_{L^\infty L^2(\mathcal{M})} \right) 2^{j} \gamma_j^{\nu}
\lesssim \left( \sup_{\omega} \left\| H_3 \left\| \nabla (P_t \chi) \right\|_{L^\infty L^2} \left\| \left(2^{j}(N - N_{\nu})\right)^p \right\|_{L^\infty} \right) 2^{j} \gamma_j^{\nu}
\lesssim \left( \sup_{\omega} \left\| H_3 \right\|_{L^\infty L^4} \right) \varepsilon 2^{j+\frac{1}{2}} \gamma_j^{\nu},
\]
where we used in the last inequality the estimate (8.575) for \(\nabla (P_t \chi)\), the estimate (2.42) for \(\partial_\omega N\) and the size of the patch. In view of the definition (10.76) of \(H_3\), we have:
\[
\left\| H_3 \right\|_{L^\infty L^4} \lesssim \|k\|_{L^\infty L^4} + \|n^{-1}\nabla n\|_{L^\infty L^4} + \|\theta\|_{L^\infty L^4} + \|n^{-1} \nabla (b)\|_{L^\infty L^4} + \|\chi\|_{L^\infty L^4} + \|\zeta\|_{L^\infty L^4},
\]
where we used in the last inequality the embedding (2.51), the estimate (2.37) for \(k\), the estimate (2.36) for \(n\), the estimates (2.37) (2.39) (2.40) for \(\theta\), the estimate (2.38) for \(b\), the estimates (2.39) (2.40) for \(\chi\), and the estimate (2.41) for \(\zeta\). Together with (10.90), this yields:
\[
\left\| \int_{S^2} H_3 \nabla (P_t \chi) \left(2^{j}(N - N_{\nu})\right)^p F_{j,-1}(u) \eta_j^\nu(\omega) d\omega \right\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^{j+\frac{1}{2}} \gamma_j^{\nu}. \tag{10.91}
\]
Finally, (10.89) and (10.91) imply:
\[
\|h'_{6,p,q,l,m}\|_{L^1(\mathcal{M})} \lesssim 2^{j+\frac{1}{2}} \varepsilon 2^{\frac{1}{2}} \gamma_j^{\nu'} \gamma_j^{\nu'} \tag{10.92}
\]

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Next, we evaluate the $L^1(M)$ norm of $h'_{\tau,p,q,l,m}$. Comparing the definition (10.72) of $h'_{\tau,p,q,l,m}$ and (10.71) of $h'_{6,p,q,l,m}$, we notice that these terms are similar. We proceed as for $h'_{6,p,q,l,m}$, and we obtain the analog of (10.92):

$$\|h'_{\tau,p,q,l,m}\|_{L^1(M)} \lesssim 2^{j + \frac{m}{2}} \varepsilon^2 \gamma_j \gamma'_j.$$  

(10.93)

Next, we evaluate the $L^1(M)$ norm of $h'_{8,p,q,l,m}$. In view of the definition (10.73) of $h'_{8,p,q,l,m}$, we have:

$$\|h'_{8,p,q,l,m}\|_{L^1(M)} \lesssim \left\| \int_{S^2} \nabla(P_1 tr \chi) \left(2^{\frac{j}{2}} (N - N_\nu)\right)^p F_{j-1}(u) \eta'_{\nu}(\omega) d\omega \right\|_{L^2(M)}$$

$$\times \left\| \int_{S^2} \nabla' (N'(P_m tr \chi')) \left(2^{\frac{j}{2}} (N' - N_{\nu'})\right)^q F_{j-1}(u') \eta''_{\nu'}(\omega') d\omega' \right\|_{L^2(M)}$$

$$\lesssim 2^{j + m} \varepsilon^2 \gamma_j \gamma'_j,$$

where we used in the last inequality the analog of the estimate (8.317) for the first term, and the analog of the estimate (8.614) for the second term.

Finally, we estimate $A_{j,\nu,\nu',l,m}^2$. In view of the decomposition (10.65) of $A_{j,\nu,\nu',l,m}^2$, the estimate (8.32), and the estimates (10.80) (10.81) (10.85) (10.86) (10.87) (10.88) (10.92) (10.93) (10.94) for $h'_{1,p,q,l,m}$, $h'_{2,p,q,l,m}$, $h'_{3,p,q,l,m}$, $h'_{4,p,q,l,m}$, $h'_{5,p,q,l,m}$, $h'_{6,p,q,l,m}$, $h'_{7,p,q,l,m}$, $h'_{8,p,q,l,m}$, we obtain:

$$\|A_{j,\nu,\nu',l,m}^2\| \lesssim \sum_{(l,m)/2^{2max(l,m)} \leq |\nu - \nu'|} \sum_{p,q,l,m \geq 0} \epsilon_{p,q,l,m} 2^{-\frac{5j}{2}} 2^{l+m+min(l,m)}$$

$$\left(\frac{2^{\frac{j}{2}} |\nu - \nu'|}{2^{\frac{j}{2}} |\nu - \nu'|^{p+q+3}} \|M_{j,\nu,l}\|_{L^2(\mathbb{R} \times S^2)} \|M_{j,\nu',m}\|_{L^2(\mathbb{R} \times S^2)}$$

$$+ \sum_{p,q,l,m \geq 0} \frac{1}{2^{\frac{j}{2}} |\nu - \nu'|^{p+q+2}} \left(\frac{1}{2^{\frac{j}{2}} |\nu - \nu'|^{p+q+1}} + \frac{1}{2^{\frac{j}{2}} |\nu - \nu'|^{p+q+2}}\right) 2^{-\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|) \right) \epsilon^2 \gamma_j \gamma'_j$$

$$\lesssim \sum_{(l,m)/2^{2max(l,m)} \leq |\nu - \nu'|} \frac{2^{-\frac{5j}{2}} 2^{l+m+min(l,m)}}{(2^{\frac{j}{2}} |\nu - \nu'|)} \|M_{j,\nu,l}\|_{L^2(\mathbb{R} \times S^2)} \|M_{j,\nu',m}\|_{L^2(\mathbb{R} \times S^2)}$$

$$+ \left(\frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)^3} + \frac{1}{(2^{\frac{j}{2}} |\nu - \nu'|)\frac{5}{2}} + \frac{1}{2^{\frac{j}{2}} (2^{\frac{j}{2}} |\nu - \nu'|)}\right) \epsilon^2 \gamma_j \gamma'_j,$$

where the sequence of functions $(\mu_{j,\nu,l})_{l \geq j/2}$ on $\mathbb{R} \times S^2$ satisfies:

$$\sum_{\nu} \sum_{l \geq j/2} 2^{2l} \|\mu_{j,\nu,l}\|_{L^2(\mathbb{R} \times S^2)}^2 \lesssim \varepsilon^2 2^{2j} \|f\|_{L^2(\mathbb{R}^3)}^2.$$

This concludes the proof of Proposition 10.3.
A Proof of Lemma 8.7

Recall from (8.42) that $B_{j,k'}^{1,1,1}$ is given by:

$$B_{j,k'}^{1,1,1} = -i2^{-j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{\mathbf{g}(L, L')} L(\tr x) \tr x' \times F_j(u) F_{j-1}(u') \eta_{\nu}^\nu (\omega) \eta_{\nu'}^\nu (\omega') d\omega d\omega' d\mathcal{M},$$

We integrate by parts in $B_{j,k',l,m}^{1,1,1}$ using (7.137) with

$$h = \frac{L(\tr x) b \tr x'}{\mathbf{g}(L, L')}.$$  \hspace{1cm} (A.1)

We obtain:

$$B_{j,k'}^{1,1,1}$$  \hspace{1cm} (A.2)

$$= -2^{-2j-1} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left[ (N - \mathbf{g}(N, N') N')(h) + \left( \tr b - \mathbf{g}(N, N') \tr b' \right. \\
\left. - \theta(N' - \mathbf{g}(N, N') N, N' - \mathbf{g}(N, N') N) - \mathbf{g}(N, N') b^{-1}(N' - \mathbf{g}(N, N') N)(b) \\
+ \frac{2 \mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left( \theta'(N - \mathbf{g}(N, N') N', N - \mathbf{g}(N, N') N') \\
- \mathbf{g}(N, N') \theta(N' - \mathbf{g}(N, N') N, N' - \mathbf{g}(N, N') N) \right) h \right) \\
\times F_{j-1}(u) F_{j-1}(u') \eta_{\nu}^\nu (\omega) \eta_{\nu'}^\nu (\omega') d\omega d\omega' d\mathcal{M}.$$  

Next, we compute the term $(N - \mathbf{g}(N, N') N')(h)$. We have:

$$\nabla_{N - \mathbf{g}(N, N') N'} (L(\tr x)) b \tr x' = \frac{(N - \mathbf{g}(N, N') N')(h)}{\mathbf{g}(L, L')} + \frac{L(\tr x)(N - \mathbf{g}(N, N') N')(b \tr x')}{\mathbf{g}(L, L')} - \frac{(N - \mathbf{g}(N, N') N')(g(L, L'))(L(\tr x)) b \tr x'}{\mathbf{g}(L, L')^2}.  \hspace{1cm} (A.3)$$

Decomposing $N - \mathbf{g}(N, N') N'$ on $N$ and $N' - \mathbf{g}(N, N') N$, we have:

$$N - \mathbf{g}(N, N') N' = (1 - \mathbf{g}(N, N')^2) N - \mathbf{g}(N, N')(N' - \mathbf{g}(N, N') N)$$  \hspace{1cm} (A.4)

which yields schematically for the first term in the right-hand side of (A.3):

$$\frac{\nabla_{N - \mathbf{g}(N, N') N'} (L(\tr x)) b \tr x'}{\mathbf{g}(L, L')} = \frac{N(L(\tr x)) b \tr x'(N - N')^2}{\mathbf{g}(L, L')} + \frac{\nabla (L(\tr x)) b \tr x'(N' - N)}{\mathbf{g}(L, L')},$$  \hspace{1cm} (A.5)

where we used the fact that:

$$1 - \mathbf{g}(N, N')^2 = (1 + \mathbf{g}(N, N'))(1 - \mathbf{g}(N, N')) = (1 + \mathbf{g}(N, N')) \frac{g(N - N', N - N')}{2} \sim (N - N')^2.$$
Finally, in order to estimate the third term in the right-hand side of (A.3), we need to compute \((N - g(N,N')N')(g(L,L'))\). Since \(g(L,L') = -1 + g(N,N')\), we need to compute \((N - g(N,N')N')(g(N,N'))\). Using the structure equation (2.21) for \(N\) and the decomposition (A.4), we obtain:

\[
\nabla_{N - g(N,N')N'}(g(N,N')) = g(\nabla_{N - g(N,N')N'}N,N') + g(N, \nabla_{N - g(N,N')N'}N') = -(1 - g(N,N')^2)g(b^{-1}\nabla b, N' - g(N,N')N) - g(N,N')\theta(N' - g(N,N')N, N' - g(N,N')N) + \theta'(N - g(N,N')N', N - g(N,N')N').
\]

Now, we have:

\[
(N - g(N,N')N') + (N' - g(N,N')N) = (N + N')(1 - g(N,N')) \sim (N - N')^2. \tag{A.7}
\]

Also, since \(\theta = \chi + k\) by definition, and since \(k\) does not depend on \(\omega\), we have:

\[
\theta - \theta' = \chi - \chi'. \tag{A.8}
\]

In view of (A.6), (A.7) and (A.8), we obtain schematically for the third term in the right-hand side of (A.3):

\[
\frac{-(N - g(N,N')N')(g(L,L'))L(\text{tr} \chi)b'b\text{tr} \chi'}{g(L,L')^2} = \frac{L(\text{tr} \chi)(\theta + b^{-1}\nabla(b))b'b\text{tr} \chi'(N - N')^3}{g(L,L')^2} + \frac{L(\text{tr} \chi)\theta'b'b\text{tr} \chi'(N - N')^3}{g(L,L')^2} + \frac{(\chi - \chi')L(\text{tr} \chi)b'b\text{tr} \chi'(N - N')^2}{g(L,L')^2}. \tag{A.9}
\]

Finally, (A.3), (A.5) and (A.9) imply, schematically:

\[
\frac{(N - g(N,N')N')(h)}{g(L,L')} + \frac{N(L(\text{tr} \chi)b'b\text{tr} \chi'(N - N')^2}{g(L,L')} + \frac{\nabla(L(\text{tr} \chi))b'b\text{tr} \chi'(N - N')}{g(L,L')} = \frac{L(\text{tr} \chi)\nabla(b'b\text{tr} \chi'(N - N')}{g(L,L')} + \frac{L(\text{tr} \chi)(\theta + b^{-1}\nabla(b))b'b\text{tr} \chi'(N - N')^3}{g(L,L')^2} + \frac{(\chi - \chi')L(\text{tr} \chi)b'b\text{tr} \chi'(N - N')^2}{g(L,L')^2} + \frac{(\chi - \chi')L(\text{tr} \chi)b'b\text{tr} \chi'(N - N')^2}{g(L,L')^2}. \tag{A.10}
\]

We consider the term multiplied by \(h\) in the right-hand side of (A.2). Using (A.8), we have schematically:

\[
\text{tr} \theta - g(N,N')\text{tr} \theta' - \theta(N' - g(N,N')N, N' - g(N,N')N) = \frac{2g(N,N')}{1 - g(N,N')^2} \left( \theta'(N - g(N,N')N', N - g(N,N')N) - g(N,N')\theta(N' - g(N,N')N, N' - g(N,N')N) \right) = \chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1}\nabla(b)(N - N'). \tag{A.11}
\]
Thus, in view of (A.1), (A.2), (A.10) and (A.11) we obtain:

\[
B_{j,\nu,\nu'}^{1,1,1,1} = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \mathcal{H}(u) \mathcal{F}_{j,1}(u') \eta_{j}^{\nu}(\omega) \eta_{j'}^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},
\]  

with the tensor \(H\) on \(\mathcal{M}\) given, schematically, by:

\[
H = \frac{1}{1 - g(N, N')^2} \left( \frac{N(L(tr\chi))tr\chi'(N - N')^2}{g(L, L')} + \frac{\nabla(L(tr\chi))tr\chi'(N - N')}{g(L, L')} \right)
\]

\[
+ \frac{(N - N')^2}{g(L, L')} \left( \frac{L(tr\chi)b'\nabla(b')(N - N')}{g(L, L')} + \frac{L(tr\chi)'(\theta + b^{-1} \nabla(b))tr\chi'(N - N')^3}{g(L, L')^2} \right)
\]

\[
+ \frac{(N - N')^2}{g(L, L')} \left( \frac{L(tr\chi)'(N - N')^3}{g(L, L')^2} + \frac{(\chi - \chi')L(tr\chi)tr\chi'(N - N')^2}{g(L, L')^2} \right)
\]

\[
+ \left( \chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1} \nabla(b)(N - N') \right) \frac{L(tr\chi)tr\chi'}{g(L, L')}. \tag{A.13}
\]

Recall the identities (8.30) and (8.31):

\[
g(L, L') = -1 + g(N, N') \text{ and } 1 - g(N, N') = \frac{g(N - N', N - N')}{2}.
\]

We may thus expand:

\[
\frac{1}{(1 - g(N, N')^2)g(L, L')} \left( \frac{1}{g(L, L')} \right) \frac{1}{(1 - g(N, N')^2)g(L, L')} \frac{1}{g(L, L')^2}
\]

in the same fashion than (8.33), and we obtain, schematically:

\[
H = \frac{1}{N_{\nu} - N_{\nu'}}^2 \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N_{\nu}}{|N_{\nu} - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_{\nu} - N_{\nu'}|} \right)^q \right) \tag{A.14}
\]

\[
\times \left( H_1 + \frac{1}{|N_{\nu} - N_{\nu'}|} H_2 + \frac{1}{|N_{\nu} - N_{\nu'}|^2} H_3 \right),
\]

where the tensors \(H_1, H_2, H_3\) on \(\mathcal{M}\) are given by:

\[
H_1 = N(L(tr\chi))tr\chi', \tag{A.15}
\]

\[
H_2 = \nabla(L(tr\chi))tr\chi' + L(tr\chi)b'\nabla(b')(tr\chi') + L(tr\chi)(\theta)tr\chi' \tag{A.16}
\]

\[
+ \left( \theta + \theta' + b^{-1} \nabla(b) \right) L(tr\chi)tr\chi',
\]

and:

\[
H_3 = (\chi - \chi')L(tr\chi)tr\chi', \tag{A.17}
\]

and where \(c_{pq}\) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1. In view of (A.12), (A.14), (A.15), (A.16) and (A.17), we obtain the decomposition (8.44) (8.45) (8.46) (8.47) (8.48) (8.49) (8.50) of \(B_{j,\nu,\nu'}^{1,1,1,1}\). This concludes the proof of Lemma 8.7.
B  Proof of Lemma 8.8

Recall from (8.43) that $B_{j,v,v'}^{1,1,1,2}$ is given by:
\[
B_{j,v,v'}^{1,1,1,2} = -i 2^{-j-1} \int_{\mathcal{M}} \frac{b^{-1}}{g(L, L')} \text{tr} \chi L'(\text{tr} \chi') \times F_j(u) F_{j,-1}(u') \eta_j^v(\omega) \eta_{j'}^{v'}(\omega') d\omega d\omega' d\mathcal{M},
\]
We integrate by parts in $B_{j,v,v',l,m}^{1,1,1,2}$ using (7.137) with
\[
h = \frac{\text{tr} \chi b' L'(\text{tr} \chi')}{g(L, L')}. \tag{B.1}
\]
We obtain:
\[
B_{j,v,v'}^{1,1,1,2} = -2^{-2j-1} \int_{\mathcal{M}} \frac{b^{-1}}{1 - g(N, N')^2} \left( (N - g(N, N') N')(h) + \left( \text{tr} \theta - g(N, N') \text{tr} \theta' - \text{tr} \theta' \right) (N - g(N, N') N)(b) \right)
\]
\[
+ \frac{2g(N, N')}{1 - g(N, N')^2} \left( \text{tr} \theta' (N - g(N, N') N', N - g(N, N') N) \right)
\]
\[
-g(N, N') \theta (N' - g(N, N') N, N' - g(N, N') N) \right) h \right)
\]
\[
\times F_{j,-1}(u) F_{j,-1}(u') \eta_j^v(\omega) \eta_{j'}^{v'}(\omega') d\omega d\omega' d\mathcal{M}.
\]
Next, we compute the term $(N - g(N, N') N')(h)$. Proceeding as in (A.3), (A.5), (A.9) and (A.10), we obtain schematically:
\[
(N - g(N, N') N')(h) = \frac{N (\text{tr} \chi) b' L'(\text{tr} \chi')(N - N')^2}{g(L, L')} + \frac{\text{tr} \chi \nabla' (b' L'(\text{tr} \chi')) (N - N')^2}{g(L, L')} + \frac{\text{tr} \chi \nabla (\text{tr} \chi' b' L'(\text{tr} \chi')(N - N')^3}{g(L, L')^2} + \frac{\text{tr} \chi \nabla' (\text{tr} \chi' b' L'(\text{tr} \chi') (N - N')^2}{g(L, L')^2}.
\]
Thus, in view of (B.1), (B.2), (B.3) and (A.11) we obtain:
\[
B_{j,v,v'}^{1,1,1,1} = 2^{-2j} \int_{\mathcal{M}} \frac{HF_{j,-1}(u) F_{j,-1}(u') \eta_j^v(\omega) \eta_{j'}^{v'}(\omega') d\omega d\omega' d\mathcal{M}}{g(L, L')} \tag{B.4}
\]
with the tensor $H$ on $\mathcal{M}$ given, schematically, by:

$$
H = \frac{1}{1 - g(N, N')^2} \left( \frac{N(\text{tr} \chi)L'(\text{tr} \chi')(N - N')^2}{g(L, L')} + \frac{\nabla(\text{tr} \chi)L'(\text{tr} \chi')(N - N')}{g(L, L')} 
\right.
\left. + \frac{\text{tr} \chi b^{-1} \nabla'(b' L'(\text{tr} \chi'))(N - N')}{g(L, L')} + \frac{\text{tr} \chi(\theta + b^{-1} \nabla(b))L'(\text{tr} \chi')(N - N')^3}{g(L, L')^2} 
\right.
\left. + \frac{\text{tr} \chi \theta' L'(\text{tr} \chi')(N - N')^3}{g(L, L')^2} + \frac{(\chi - \chi')\text{tr} \chi L'(\text{tr} \chi')(N - N')^2}{g(L, L')^2} 
\right.
\left. + \left( \chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1} \nabla(b)(N - N') \right) \text{tr} \chi L'(\text{tr} \chi') \frac{\text{tr} \chi L'(\text{tr} \chi')}{g(L, L')} \right).
$$

Proceeding in the same fashion than (A.14), we obtain, schematically:

$$
H = \frac{1}{|N_\nu - N_{\nu'}|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right)
\left( H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_3 \right),
$$

(B.5)

where the tensors $H_1$, $H_2$ and $H_3$ on $\mathcal{M}$ are given by:

$$
H_1 = N(\text{tr} \chi)L'(\text{tr} \chi'),
$$

(B.6)

$$
H_2 = \nabla(\text{tr} \chi)L'(\text{tr} \chi') + \text{tr} \chi b^{-1} \nabla'(b' L'(\text{tr} \chi')) + \text{tr} \chi(\theta)L'(\text{tr} \chi')
\left( \theta + \theta' + b^{-1} \nabla(b) \right) \text{tr} \chi L'(\text{tr} \chi'),
$$

(B.7)

and:

$$
H_3 = (\chi - \chi')\text{tr} \chi L'(\text{tr} \chi'),
$$

(B.8)

and where $c_{pq}$ are explicit real coefficients such that the series

$$
\sum_{p,q \geq 0} c_{pq} x^p y^q
$$

has radius of convergence 1. In view of (B.4), (B.5), (B.6), (B.7) and (B.8), we obtain the decomposition (8.81) (8.82) (8.83) (8.84) (8.85) (8.86) (8.87) of $B_{j,\nu,\nu'}^{1,1,1,2}$. This concludes the proof of Lemma 8.8.

C Proof of Lemma 8.13

Recall from (8.95) that $B_{j,\nu,\nu',l,m}^{1,2,2}$ is given by:

$$
B_{j,\nu,\nu',l,m}^{1,2,2} = -i2^{-j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{1}{g(L, L')} \left( L(\text{tr} \chi)P_m \text{tr} \chi' + P \text{tr} \chi L'(P_m \text{tr} \chi') \right)
\times (b^{-1} - b^{-1} g(N, N')) F_{j,-1}(u) F_j(u') \eta_j^\nu(\omega) \eta_{j'}^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.
$$

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We integrate by parts in $B^{1,2,2}_{j,v,v',l,m}$ using (7.136) with

$$h = \frac{\left( L(P_{l}tr\chi) P_{m}tr\chi' + P_{l}tr\chi L'(P_{m}tr\chi') \right)(b' - b)}{g(L, L')}.$$  \hspace{1cm} (C.1)

We obtain:

$$B^{1,2,2}_{j,v,v',l,m} \hspace{1cm} \text{(C.2)}$$

$$= -2^{-2j-1} \int_{M} \int_{S^{2} \times S^{2}} \frac{b^{-1}}{1 - g(N, N')^{2}} \left( (N' - g(N, N')N)(h) + \left( \text{tr} \theta' - g(N, N')\text{tr}\theta 

- \theta'(N - g(N, N')N', N - g(N, N')N') - g(N, N')b'^{-1}(N - g(N, N')N')(b') \right) 

+ \frac{2g(N, N')}{1 - g(N, N')^{2}} \left( \theta(N' - g(N, N')N, N' - g(N, N')N) 

- g(N, N')\theta'(N - g(N, N')N', N - g(N, N')N') \right) \right)(b') \right) 

\times F_{j,-1}(u)F_{j,-1}(u') \eta_{j}^{\nu}(\omega) \eta_{j}'^{\nu}(\omega') d\omega d\omega' dM. \right)$$

Next, we compute the term $(N' - g(N, N')N)(h)$. We have:

$$= \frac{(N' - g(N, N')N)(h)}{g(L, L')} + \frac{L(P_{l}tr\chi')(P_{m}tr\chi')(b' - b)}{g(L, L')} + \frac{L(P_{l}tr\chi)(N' - g(N, N')N)(P_{m}tr\chi')(b' - b)}{g(L, L')} + \frac{L(P_{l}tr\chi)(N' - g(N, N')N)(P_{m}tr\chi')(b' - b)}{g(L, L')} \hspace{1cm} (C.3)$$

$$= \frac{(N' - g(N, N')N)(h)}{g(L, L')} + \frac{L(P_{l}tr\chi')(P_{m}tr\chi')(b' - b)}{g(L, L')} + \frac{L(P_{l}tr\chi)(N' - g(N, N')N)(P_{m}tr\chi')(b' - b)}{g(L, L')} \hspace{1cm} (C.5)$$

Decomposing $N' - g(N, N')N$ on $N'$ and $N - g(N, N')N'$, we have:

$$N' - g(N, N')N = (1 - g(N, N')^{2})N' - g(N, N')(N - g(N, N')N') \hspace{1cm} (C.4)$$

which yields schematically for the second and the fourth term in the right-hand side of (C.3):

$$= \frac{L(P_{l}tr\chi)(N' - g(N, N')N)(P_{m}tr\chi')(b' - b)}{g(L, L')} + \frac{L(P_{l}tr\chi)(N' - g(N, N')N)(P_{m}tr\chi')(b' - b)}{g(L, L')} \hspace{1cm} (C.5)$$

where we used the fact that:

$$1 - g(N, N')^{2} = (1 + g(N, N'))(1 - g(N, N')) = (1 + g(N, N')) \frac{g(N - N', N - N')}{2} \sim (N - N')^{2}. \hspace{1cm} 258$$
Finally, in order to estimate the last term in the right-hand side of (A.3), we need to compute \((N' - g(N, N')N)(g(L, L'))\). We have the analog of (A.6):

\[
\nabla_{N' - g(N, N')N} \left( g(N, N') \right) = -(1 - g(N, N')^2)g(b'^{-1}\nabla b', N - g(N, N')N') - g(N, N')\theta'(N - g(N, N')N', N - g(N, N')N') + \theta(N' - g(N, N')N, N' - g(N, N')N)
\]

In view of (C.6), (A.7) and (A.8), we obtain schematically for the last term in the right-hand side of (C.3):

\[
\frac{(N' - g(N, N')N)(g(L, L'))(L(Ptr\chi)P_mtr\chi' + Pptr\chi'P_mtr\chi')(b' - b)}{g(L, L')^2} = \frac{(\chi' - \chi)(L(Ptr\chi)P_mtr\chi' + Pptr\chi'P_mtr\chi')(b' - b)(\theta' + b'^{-1}\nabla'(b') + \theta)(N - N')^3}{g(L, L')^2}.
\]

Finally, (C.3), (C.5) and (C.7) imply, schematically:

\[
\begin{align*}
\nabla_{N' - g(N, N')N} & \left( g(L, L') \right) = \frac{(N' - g(N, N')N)(h)}{g(L, L')} + \frac{(N' - g(N, N')N)(Pptr\chi)L'(P_mtr\chi')(b' - b)}{g(L, L')} \\
& - \frac{L(Ptr\chi)(N - g(N, N')N')(P_mtr\chi')(b' - b)}{g(L, L')} + \frac{L(Pptr\chi)(P_mtr\chi')(b' - b)(N - N')^2}{g(L, L')} \\
& - \frac{Pptr\chi\nabla_{N' - g(N, N')N'}(L'(P_mtr\chi'))(b' - b)}{g(L, L')} + \frac{Pptr\chi\nabla_{N'}(L'(P_mtr\chi'))(b' - b)(N - N')^2}{g(L, L')} \\
& + \frac{(L(Ptr\chi)P_mtr\chi' + Pptr\chi'L'(P_mtr\chi'))(b' - b)(\theta' + b'^{-1}\nabla'(b') + \theta)(N - N')^3}{g(L, L')^2} \\
& + \frac{(\chi' - \chi)(L(Ptr\chi)P_mtr\chi' + Pptr\chi'L'(P_mtr\chi'))(b' - b)(N - N')^2}{g(L, L')^2}.
\end{align*}
\]

We consider the term multiplied by \(h\) in the right-hand side of (C.2). Using (A.8), we have schematically:

\[
\begin{align*}
\text{tr} \theta - g(N, N')\text{tr}\theta' - \theta(N' - g(N, N')N, N' - g(N, N')N) & = C.9 \\
-\frac{g(N, N')b^{-1}(N' - g(N, N')N)(b)}{1 - g(N, N')^2} & + \frac{2g(N, N')}{1 - g(N, N')^2} \left( \theta'(N - g(N, N')N', N - g(N, N')N') \\
& - g(N, N')\theta(N' - g(N, N')N, N' - g(N, N')N) \right) \\
& = \chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1}\nabla(b)(N - N').
\end{align*}
\]
Thus, in view of (C.1), (C.2), (C.8) and (C.9) we obtain:

$$B_{j,v,v',l,m}^{1,2,2} = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} HF_{j,-1}(u)F_{j,-1}(u')\eta_{j}^{\nu}(u)\eta_{j}'^{\nu'}(u')d\omega d\omega' d\mathcal{M}, \quad (C.10)$$

with the tensor $H$ on $\mathcal{M}$ given, schematically, by:

$$H = \frac{1}{1 - g(N, N')^2} \times \left[ \nabla_{N'} - g(N, N')^N (L(P_l \chi))P_m \text{tr}_x'(b' - b) \frac{g(L, L')}{g(L, L')} - \frac{L(P_l \chi)(N - g(N, N')^N)(P_m \text{tr}_x')(b' - b)}{g(L, L')} + \frac{P_l \chi N' (L'(P_m \text{tr}_x'))(b' - b)(N - N')}{g(L, L')} \right] \left. \right\} \frac{\chi' - \chi}{g(L, L')^2} \left( L(P_l \chi)P_m \text{tr}_x' + P_l \chi L'(P_m \text{tr}_x') \right)(b' - b)(N - N')^2 + \left( \chi - \chi' + \theta(N - N') + \theta'(N - N') + b^{-1} \nabla(b')(N - N') \right) \times \frac{L(P_l \chi)P_m \text{tr}_x' + P_l \chi L'(P_m \text{tr}_x')}{g(L, L')} (b' - b) \right\].$$

Recall the identities (8.30) and (8.31):

$$g(L, L') = -1 + g(N, N') \quad \text{and} \quad 1 - g(N, N') = \frac{g(N - N', N - N')}{2}.$$ 

We may thus expand:

$$\frac{1}{(1 - g(N, N')^2)}g(L, L') = \frac{1}{g(L, L')} \quad \text{and} \quad \frac{1}{(1 - g(N, N')^2)}g(L, L')^2.$$
in the same fashion than (8.33), and we obtain, schematically:

\[ H = \frac{1}{|N_\nu - N_{\nu'}|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) \]  

\[ \times \left( H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 \right) \]

\[ + \frac{(N' - N)(b' - b)}{g(L, L')^2} \left( \nabla(L(P_{\text{tr}}\chi))P_{\text{tr}}\chi' + \nabla'(P_{\text{tr}}\chi)L'(P_{\text{tr}}\chi') \right) \]

\[ + L(P_{\text{tr}}\chi)\nabla'(P_{\text{tr}}\chi') + P_{\text{tr}}\chi\nabla'(L'(P_{\text{tr}}\chi')) \]

\[ + P_{\text{tr}}\chi L'(P_{\text{tr}}\chi') \]

\[ + \frac{(\chi - \chi')(b' - b)}{g(L, L')^2} \left( L(P_{\text{tr}}\chi)P_{\text{tr}}\chi' + P_{\text{tr}}\chi L'(P_{\text{tr}}\chi') \right), \]  

where the tensors \( H_1, H_2 \) on \( \mathcal{M} \) are given by:

\[ H_1 = L(P_{\text{tr}}\chi)N'(P_{\text{tr}}\chi')(b' - b), \]  

\[ H_2 = \left( \theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b') \right) \left( L(P_{\text{tr}}\chi)P_{\text{tr}}\chi' + P_{\text{tr}}\chi L'(P_{\text{tr}}\chi') \right) (b' - b), \]  

and where \( c_{pq} \) are explicit real coefficients such that the series

\[ \sum_{p, q \geq 0} c_{pq} x^p y^q \]

has radius of convergence 1. In view of (C.10), (C.11), (C.12), and (C.13), we obtain the decomposition (8.256) (8.257) (8.258) (8.259) (8.260) (8.261) of \( B_{1,2,2,2}^{j,\nu,\nu',l,m} \). This concludes the proof of Lemma 8.13.

D Proof of Lemma 8.14

Recall from (8.298) that \( B_{1,2,2,2,2}^{j,\nu,\nu',l,m} \) is given by:

\[ B_{1,2,2,2,2}^{j,\nu,\nu',l,m} = -i 2^{-j} \int_{\mathcal{M}} \int_{S^1 \times S^2} b^{-1} P_{\text{tr}}\chi N'(P_{\text{tr}}\chi')(b' - b) F_j(u) \eta_j'(\omega) F_{j-1}(u') \eta_j'(\omega') d\omega d\omega' d\mathcal{M}. \]

We integrate by parts in \( B_{1,2,2,2,2}^{j,\nu,\nu',l,m} \) using (7.137) with

\[ h = P_{\text{tr}}\chi b' N'(P_{\text{tr}}\chi')(b' - b). \]  

\[ (D.1) \]
We obtain:

\[ B^{1,2,2,2,3}_{j,m} = 2^{-2j} \int_{M} \int_{S^2 \times S^2} \frac{b'^{-1}}{1 - \mathbf{g}(N, N')^2} \left( (N - \mathbf{g}(N, N')N')(h) + \left( \mathbf{tr} \theta - \mathbf{g}(N, N')\mathbf{tr} \theta' \right. \right. \]

\[ \left. \left. - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b'^{-1}(N' - \mathbf{g}(N, N')N) \right) + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left( \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N') \right. \right. \]

\[ \left. \left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right) h \right) \times F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j'^\nu(\omega')d\omega'd\mathbf{d}M. \]

Next, we compute the term \((N - \mathbf{g}(N, N')N')(h)\). Proceeding as in (A.3), (A.5) and (A.10), we obtain schematically:

\[ (N - \mathbf{g}(N, N')N')(h) \]

\[ = b'(1 - \mathbf{g}(N, N')^2)N(P\mathbf{tr} \chi)'(P_m \mathbf{tr} \chi')(b' - b) \]

\[ + b'(N' - \mathbf{g}(N, N')N)(P\mathbf{tr} \chi)'(P_m \mathbf{tr} \chi')(b' - b) \]

\[ + b'P\mathbf{tr} \chi'\nabla'(N'(P_m \mathbf{tr} \chi')(b' - b)(N - N') + b'b^{-1}\nabla(b)P\mathbf{tr} \chi'N'(P_m \mathbf{tr} \chi')(N - N') \]

\[ + \nabla'(b')P\mathbf{tr} \chi'N'(P_m \mathbf{tr} \chi')(N - N') + b'N(b)P\mathbf{tr} \chi'N'(P_m \mathbf{tr} \chi')(N - N')^2. \]

Thus, in view of (D.1), (D.2), (D.3) and (A.11) we obtain:

\[ \int_{M} \int_{S^2 \times S^2} H F_{j,-1}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_j'^\nu(\omega')d\omega'd\mathbf{d}M, \]

with the tensor \(H\) on \(M\) given, schematically, by:

\[ H = \frac{1}{1 - \mathbf{g}(N, N')^2} \left( 1 - \mathbf{g}(N, N')^2 \right) \left( [(N - \mathbf{g}(N, N')N)(P\mathbf{tr} \chi)'(P_m \mathbf{tr} \chi')(b' - b) \right. \]

\[ \left. + (N' - \mathbf{g}(N, N')N)(P\mathbf{tr} \chi)'(P_m \mathbf{tr} \chi')(b' - b) + P\mathbf{tr} \chi'\nabla'(N'(P_m \mathbf{tr} \chi'))(b' - b)(N - N') \right. \]

\[ \left. + b^{-1}\nabla(b)P\mathbf{tr} \chi'N'(P_m \mathbf{tr} \chi')(N - N') + b^{-1}\nabla(b')P\mathbf{tr} \chi'N'(P_m \mathbf{tr} \chi')(N - N') \right. \]

\[ \left. + N(b)P\mathbf{tr} \chi'N'(P_m \mathbf{tr} \chi')(N - N')^2 + \left( \chi - \chi' + \theta(N - N') + \theta'(N - N') \right. \right. \]

\[ \left. \left. + b^{-1}\nabla(b)(N - N')P\mathbf{tr} \chi'N'(P_m \mathbf{tr} \chi')(b' - b) \right. \right). \]

Proceeding in the same fashion than (A.14), we obtain, schematically:

\[ H = \frac{1}{|N_\nu - N_\nu'|^2} \left( \sum_{|\mu| \leq 0} c_{pq} \left( \frac{N - N_\nu}{|N_\nu - N_\nu'|} \right)^p \left( \frac{N' - N_\nu}{|N_\nu - N_\nu'|} \right)^q \right) \]

\[ \times \left( \frac{1}{|N_\nu - N_\nu'|^p}H_1 + \frac{1}{|N_\nu - N_\nu'|}H_2 + H_3 \right) \]

\[ + N(P_{\mathbf{tr} \chi})N'(P_m \mathbf{tr} \chi')(b' - b) + \left( N' - \mathbf{g}(N, N')N(P_{\mathbf{tr} \chi})N'(P_m \mathbf{tr} \chi')(b' - b) \right) \]

\[ \mathbf{g}(L, L'). \]

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where the tensors $H_1, H_2$ and $H_3$ on $\mathcal{M}$ are given by:

$$H_1 = (\chi + \theta + \chi' + L'(b') + \theta')P_{l}tr\chi N'(P_{m}tr\chi')(b' - b), \quad (D.6)$$

$$H_2 = P_{l}tr\chi \nabla' N'(P_{m}tr\chi') + (\nabla'(b') + \nabla(b))P_{l}tr\chi N'(P_{m}tr\chi'), \quad (D.7)$$

and:

$$H_3 = N(b)P_{l}tr\chi N'(P_{m}tr\chi'), \quad (D.8)$$

and where $c_{pq}$ are explicit real coefficients such that the series

$$\sum_{p,q\geq 0} c_{pq}x^p y^q$$

has radius of convergence 1. In view of (D.4), (D.5), (D.6), (D.7) and (D.8), we obtain the decomposition (8.323) (8.324) (8.325) (8.326) (8.327) (8.328) (8.329) of $B^{1,2,2,2,3}_{j,j'},l,m$. This concludes the proof of Lemma 8.14.

### E Proof of Lemma 8.15

Recall from (8.296) that $B^{1,2,2,2,1}_{j,j',l,m}$ is given by:

$$B^{1,2,2,2,1}_{j,j',l,m} = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{(N' - \mathbf{g}(N, N')N)(P_{l}tr\chi)N'(P_{m}tr\chi')(b' - b)}{\mathbf{g}(L, L')}$$

$$\times F_{j,-1}(u)\eta_j^\nu(\omega)F_{j,-1}(u')\eta_{j'}^\nu'(\omega')d\omega d\omega' d\mathcal{M},$$

We integrate by parts in $B^{1,1,1,2}_{j,j',l,m}$ using (7.137) with

$$h = \frac{b(N' - \mathbf{g}(N, N')N)(P_{l}tr\chi)b'N'(P_{m}tr\chi')(b' - b)}{\mathbf{g}(L, L')} . \quad (E.1)$$

We obtain:

$$B^{1,1,1,2}_{j,j'}$$

$$= i2^{-3j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b^{-1}}{1 - \mathbf{g}(N, N')^2} \left( \left( N - \mathbf{g}(N, N')N \right)(h) + \left( \text{tr}\theta - \mathbf{g}(N, N')\text{tr}\theta' \right) \right.$$

$$\left. - \theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) - \mathbf{g}(N, N')b^{-1}(N' - \mathbf{g}(N, N')N)(b) \right.$$

$$\left. + \frac{2\mathbf{g}(N, N')}{1 - \mathbf{g}(N, N')^2} \left( \theta'(N - \mathbf{g}(N, N')N', N - \mathbf{g}(N, N')N) \right. \right.$$

$$\left. - \mathbf{g}(N, N')\theta(N' - \mathbf{g}(N, N')N, N' - \mathbf{g}(N, N')N) \right) \right)$$

$$\times F_{j,-2}(u)F_{j,-1}(u')\eta_j^\nu(\omega)\eta_{j'}^\nu'(\omega')d\omega d\omega' d\mathcal{M}. \quad (E.2)$$
Next, we compute the term \((N - g(N, N')N')(h)\). Proceeding as in (A.3), (A.5), (A.9) and (A.10), we obtain schematically:

\[
(N - g(N, N')N')(h) = \frac{N((N' - g(N, N')N(P_{l \triangleright})N'(P_m \triangleright')(N - N'))}{g(L, L')} + \frac{(N' - g(N, N')N((N' - g(N, N')N(P_{l \triangleright})N'(P_m \triangleright'))}{g(L, L')} + \frac{\nabla(P_{l \triangleright})\nabla(N'(P_m \triangleright'))(N - N')^2}{g(L, L')} + \left(\chi + \chi' + \theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla(b')\right)\frac{\nabla(P_{l \triangleright})N'(P_m \triangleright')(N - N')}{g(L, L')}.
\]

Next, we evaluate the first two terms of (E.3) starting with the first one. We have, schematically:

\[
N((N' - g(N, N')N)(P_{l \triangleright})) = \nabla(N'(P_{l \triangleright}))(N - N') + \nabla \nabla_{N - g(N, N')N}(P_{l \triangleright}). \quad (E.4)
\]

Using the structure equations for \(N\) (2.21) together with the decomposition of \(N\) given by (7.140), we obtain, schematically:

\[
\nabla_N(N' - g(N, N')N) = \theta'(N - N') + b^{-1}\nabla(b) + b'^{-1}\nabla(b'). \quad (E.5)
\]

Together with (E.4), this yields, schematically:

\[
N((N' - g(N, N')N)(P_{l \triangleright})) = \nabla(N'(P_{l \triangleright}))(N - N') + \theta'(N - N') + b^{-1}\nabla(b) + b'^{-1}\nabla(b'))D(P_{l \triangleright}). \quad (E.6)
\]

Next, we evaluate the second term in the right-hand side of (E.3). We have, schematically:

\[
(N' - g(N, N')N)((N' - g(N, N')N)(P_{l \triangleright})) = \nabla^2 P_{l \triangleright}(N - N')^2 + \nabla \nabla_{N - g(N, N')N}(N - g(N, N')N)(P_{l \triangleright}). \quad (E.7)
\]

Using the structure equations for \(N\) (2.21) together with (A.6) and (A.7), we obtain, schematically:

\[
\nabla_{N' - g(N, N')N}(N' - g(N, N')N) = \nabla_{N' - g(N, N')N}(N' - g(N, N')N) + \nabla \nabla_{N - g(N, N')N}(g(N, N'))N
\]

\[
= (1 - g(N, N')^2)\nabla_{N' - g(N, N')N}N' - g(N, N')\nabla_{N - g(N, N')N}N' - g(N, N')\theta(N' - g(N, N')N, \theta_{N' - g(N, N')N})e_{\theta'} - ((1 - g(N, N')^2)g(-\nabla' \log(a'), N) - g(N, N')\theta(N' - g(N, N')N, \theta(N - g(N, N')N, \theta_{N - g(N, N')N}))N
\]

\[
= b'^{-1}\nabla(b')(N - N')^2 + (\theta + \theta')(N - N').
\]

Together with (E.7), this yields, schematically:

\[
\nabla P_{l \triangleright}(N - N')^2 + (b'^{-1}\nabla(b')(N - N')^2 + (\theta + \theta')(N - N'))\nabla P_{l \triangleright}.
\]
Recall from (8.472) that

\[ B_{j,\nu,\nu',l,m}^{1,2,2,1} = 2^{-3j} \int_{\mathcal{M}} \int_{S^2 \times S^2} H F_{j,-2}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_{j'}^\nu'(\omega') d\omega d\omega' d\mathcal{M}, \quad (E.10) \]

with the tensor \( H \) on \( \mathcal{M} \) given, schematically, by:

\[
H = \frac{1}{1 + g(N,N')^2} \left( \nabla^2(P mtr\chi) N'(P mtr\chi')(N - N')^2 + \nabla(N(P mtr\chi)) N'(P mtr\chi')(N - N')^3 \right) \frac{g(L,L')}{g(L,L')}
\]

\[
+ \frac{\nabla(P mtr\chi') \nabla'(N'(P mtr\chi'))(N - N')^2}{g(L,L')} \left( \chi + \chi' + \theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla(b') \right)
\]

\[
\times \frac{\nabla(P mtr\chi) N'(P mtr\chi')}{g(L,L')} \left( (N - N') + (N - N')^3 \right) \right).
\]

Proceeding in the same fashion than (A.14), we obtain, schematically:

\[
H = \frac{1}{|N - N'|} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N'}{|N - N'|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_{\nu'} - N_{\nu'}} \right)^q \right)
\]

\[
\times \left( H_1 + \frac{1}{|N_{\nu} - N_{\nu'}|} H_2 + \frac{1}{|N_{\nu} - N_{\nu'}|^2} H_3 \right), \quad (E.11)
\]

where the tensors \( H_1, H_2 \) and \( H_3 \) on \( \mathcal{M} \) are given by:

\[
H_1 = \nabla(N(P mtr\chi)) N'(P mtr\chi'), \quad (E.12)
\]

\[
H_2 = \nabla^2(P mtr\chi) N'(P mtr\chi') + \nabla'(P mtr\chi) \nabla'(N'(P mtr\chi')), \quad (E.13)
\]

and:

\[
H_3 = \left( \chi + \chi' + \theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla(b') \right) \nabla(P mtr\chi) N'(P mtr\chi'), \quad (E.14)
\]

and where \( c_{pq} \) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1. In view of (E.10), (E.11), (E.12), (E.13) and (E.14), we obtain the decomposition (8.340), (8.341), (8.342), (8.343), (8.344), (8.345), (8.346), (8.347) of \( B_{j,\nu,\nu',l,m}^{1,2,2,1} \). This concludes the proof of Lemma 8.15.

**F Proof of Lemma 8.16**

Recall from (8.472) that \( B_{j,\nu,\nu',l,m}^{1,2,3,1} \) is given by:

\[
B_{j,\nu,\nu',l,m}^{1,2,3,1} = i 2^{-j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} b^{-1} L(P mtr\chi) P mtr\chi' F_{j,-2}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_{j'}^\nu'(\omega') d\omega d\omega' d\mathcal{M}.
\]

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We integrate by parts in $B_{j,l,v',l,m}^{1,2,3,1}$ using (7.136) with

$$h = bL(P_{l}tr\chi)P_{m}tr\chi'.$$  \hspace{1cm} (F.1)

We obtain:

$$B_{j,l,v',l,m}^{1,2,3,1} \hspace{1cm} (F.2)$$

$$= -2^{-2j-1} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1}}{1 - g(N, N')} \left( (N' - g(N, N')N)(h) + (tr\theta' - g(N, N')tr\theta$$

$$- \theta'(N - g(N, N')N', N - g(N, N')N') - g(N, N')b^{-1}(N - g(N, N')N')(b') \right)$$

$$+ \frac{2g(N, N')}{1 - g(N, N')} \left( \frac{(N' - g(N, N')N, N' - g(N, N')N)}{N,N} \right) h$$

$$\times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$ 

Next, we compute the term $(N' - g(N, N')N)(h)$. Proceeding as in (C.3), (F.3) and (C.9), we obtain:

$$(N' - g(N, N')N)(h)$$

$$= b\nabla(L(P_{l}tr\chi))P_{m}tr\chi(N - N') + bL(P_{l}tr\chi)\nabla'(P_{m}tr\chi)(N - N')$$

$$+ bL(P_{l}tr\chi)N'(P_{m}tr\chi)(N - N')^2 + (\chi - \chi')L(P_{l}tr\chi)P_{m}tr\chi$$

$$+ (\theta + \theta' + b^{-1}\nabla(b) + b^{-1}\nabla(b'))L(P_{l}tr\chi)P_{m}tr\chi(N - N').$$

Thus, in view of (F.1), (F.2), (F.3) and (C.9) we obtain:

$$B_{j,l,v',l,m}^{1,2,3,1} = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} HF_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},$$  \hspace{1cm} (F.4)

with the tensor $H$ on $\mathcal{M}$ given, schematically, by:

$$H = \frac{1}{1 - g(N, N')}^2$$

$$\times \left[ \nabla(L(P_{l}tr\chi))P_{m}tr\chi(N - N') + L(P_{l}tr\chi)\nabla'(P_{m}tr\chi)(N - N')$$

$$+ L(P_{l}tr\chi)N'(P_{m}tr\chi)(N - N')^2 + (\chi - \chi')L(P_{l}tr\chi)P_{m}tr\chi$$

$$+ (\theta + \theta' + b^{-1}\nabla(b) + b^{-1}\nabla(b'))L(P_{l}tr\chi)P_{m}tr\chi(N - N') \right].$$

Proceeding in the same fashion than (C.11), we obtain, schematically:

$$H = \frac{1}{|N_{\nu} - N_{\nu'}|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N_{\nu}}{|N_{\nu} - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_{\nu} - N_{\nu'}|} \right)^q \right)^2$$

$$\times \left( \frac{1}{|N_{\nu} - N_{\nu'}|^2} H_1 + \frac{1}{|N_{\nu} - N_{\nu'}|} H_2 + H_3 \right),$$  \hspace{1cm} (F.5)
where the tensors $H_1, H_2, H_3$ on $\mathcal{M}$ are given by:

$$H_1 = (\chi - \chi')L(P_l\text{tr}\chi)P_m\text{tr}\chi,$$

$$H_2 = \mathcal{N}(L(P_l\text{tr}\chi))P_m\text{tr}\chi + L(P_l\text{tr}\chi)\mathcal{N}'(P_m\text{tr}\chi) + (\theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla(b'))L(P_l\text{tr}\chi)P_m\text{tr}\chi,$$

and:

$$H_3 = L(P_l\text{tr}\chi)N'(P_m\text{tr}\chi),$$

and where $c_{pq}$ are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq}x^py^q$$

has radius of convergence 1. In view of (F.4), (F.5), (F.6), (F.7) and (F.8), we obtain the decomposition (8.474) (8.475) (8.476) (8.477) (8.478) (8.479) (8.480) (8.481) of $B_{j,\nu,\nu',l,m}^{1,2,3,1}$. This concludes the proof of Lemma 8.16.

G Proof of Lemma 8.17

Recall from (8.11) that $B_{j,\nu,\nu',l,m}^2$ is given by:

$$B_{j,\nu,\nu',l,m}^2 = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( (g(N, N') - 1)P_l\text{tr}\chi N'(P_m\text{tr}\chi') + \left( tr\chi - \bar{\delta} - \bar{\delta}' 
\right. 
\right. 
\left. \left. 
-(1 - g(N, N'))\delta' - 2\zeta'_{N-N'}N' - \frac{\chi'(N - g(N, N')N', N - g(N, N')N')}{g(L, L')} 
\right) 
\times P_l\text{tr}\chi P_m\text{tr}\chi' \right) F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$

Together with the identity (8.30):

$$g(L, L') = -1 + g(N, N'),$$

we obtain:

$$B_{j,\nu,\nu',l,m}^2 = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \left[ \frac{b^{-1}}{g(L, L')} \left( tr\chi - \bar{\delta} - \bar{\delta}' - (1 - g(N, N'))\delta' - 2\zeta'_{N-N'}N' - \frac{\chi'(N - g(N, N')N', N - g(N, N')N')}{g(L, L')} \right) P_l\text{tr}\chi P_m\text{tr}\chi' 
\right. 
\right. 
\left. 
+ b^{-1} P_l\text{tr}\chi N'(P_m\text{tr}\chi') \right] F_j(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}.$$
We integrate by parts in $B_{j,v,v',l,m}^{1,1,1,2}$ using (7.137) with

$$h = \frac{b'}{g(L, L')} \left( \text{tr} \chi - \tilde{\delta} - \tilde{\gamma}' - (1 - g(N, N')) \delta' - 2 \zeta'_{N-g(N,N')N'} \right) + \frac{\chi'(N - g(N, N')N', N - g(N, N')N')}{g(L, L')} P_l \text{tr} \chi P_m \text{tr} \chi' + b' \text{tr} \chi N' (P_m \text{tr} \chi').$$

We obtain:

$$B_{j,v,v',l,m}^2 = -2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{b'}{g(N, N')^2} \left( (N - g(N, N')N')(h) + \left( \text{tr} \theta - g(N, N') \text{tr} \theta' \right) - \theta(N' - g(N, N')N, N' - g(N, N')N) - g(N, N')^{-1} (N' - g(N, N')N)(b) + \frac{2g(N, N')}{1 - g(N, N')^2} \left( \theta'(N - g(N, N')N', N - g(N, N')N') - g(N, N') \theta(N' - g(N, N')N, N' - g(N, N')N) \right) \right) \right) h \right) \times F_{j,-1}(u) F_{j,-1}(u') \eta_j(\omega) \eta_j'(\omega') d\omega d\omega' d\mathcal{M}.$$

Next, we compute the term $(N - g(N, N')N')(h)$. Using the structure equation for $N$ (2.21), we have, schematically:

$$\nabla_{N-g(N,N')N'}(N - g(N, N')N')$$

$$= \nabla_{N-g(N,N')N'} N - g(N, N') \nabla_{N-g(N,N')N'} N' - \nabla_{N-g(N,N')N'} (g(N, N')) N'$$

$$= (1 - g(N, N')^2) \nabla N - g(N, N') \nabla_{N-g(N,N')N'} N - g(N, N') \theta(N - g(N, N')N, e_A) e_A'$$

$$- ((1 - g(N, N')^2) g(-\nabla \log(b), N') - g(N, N') \theta(N' - g(N, N')N, N' - g(N, N')N)$$

$$+ \theta(N - g(N, N')N', N - g(N, N')N')) \right) N'$$

$$= -(1 - g(N, N')^2) \nabla \log(b) - g(N, N') \theta(N' - g(N, N')N, e_A) e_A$$

$$- g(N, N') \theta(N' - g(N, N')N, e_A') e_A' + (1 - g(N, N')^2) \nabla_{N-g(N,N')N} \log(b) N'$$

$$+ g(N, N') \theta(N' - g(N, N')N, N' - g(N, N')N) N'$$

$$- \theta'(N - g(N, N')N', N - g(N, N')N')) \right) N',$$

which we rewrite schematically as:

$$\nabla_{N-g(N,N')N'}(N - g(N, N')N') = (\theta - \theta')(N - N') + (\theta + \theta' + b^{-1} \nabla(b))(N - N')^2.$$
Proceeding as in (A.3), (A.5), (A.9) and (A.10), and using (G.3), we obtain schematically:

\[
(N - g(N, N')N')(h) \quad \text{(G.4)}
\]

\[
= \left( \chi + \bar{\delta} + \chi' + \bar{\delta}' + \zeta'(N - N') \right) \frac{b'}{g(L, L')} \left( \nabla P_{tr}\chi P_{m}tr\chi'(N - N') \right) + P_{tr}\chi\nabla' P_{m}tr\chi'(N - N') + N(P_{tr}\chi)P_{m}tr\chi'(N - N')^2 + \left( \nabla(\chi)(N - N') \right. \\
+ \nabla'(\bar{\delta})(N - N') + D_N(\chi)(N - N')^2 + N(\bar{\delta})(N - N')^2 + \nabla'(\chi')(N - N') \\
\left. + \nabla'(\bar{\delta}')N - N') + \nabla'(\zeta)(N - N')^2 + b^{-1}\nabla(b')(N - N') + (\chi + \bar{\delta} + \chi' + \bar{\delta}') \\
+ \zeta'(N - N')(\theta - \theta' + (\theta + \theta' + b^{-1}\nabla(b))(N - N')) \right) \frac{b'}{g(L, L')} P_{tr}\chi P_{m}tr\chi' \\
+ (1 - g(N, N')) b'N(P_{tr}\chi)N'(P_{m}tr\chi') + b'(N' - g(N, N')N(P_{tr}\chi)N(P_{m}tr\chi') \\
+ (N - N') b'P_{tr}\chi\nabla'(N'(P_{m}tr\chi')) + (N - N') b^{-1}\nabla'(b')P_{tr}\chi N'(P_{m}tr\chi').
\]

Thus, in view of (G.1), (G.2), (G.4), (A.8) and (A.11) we obtain:

\[
B_{j,m}^{2j} = \int_{\mathcal{M}} \frac{1}{2^{2j} \mathcal{G}^2 \times \mathcal{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \epsilon_j'(\omega) \epsilon_j''(\omega') d\omega d\omega' d\mathcal{M}, \quad \text{(G.5)}
\]

with the tensor $H$ on $\mathcal{M}$ given, schematically, by:

\[
H = \frac{1}{1 - g(N, N')^2} \left( \chi + \bar{\delta} + \chi' + \bar{\delta}' + \zeta'(N - N') \right) \frac{1}{g(L, L')} \left( \nabla P_{tr}\chi P_{m}tr\chi'(N - N') \right) \\
+ P_{tr}\chi\nabla' P_{m}tr\chi'(N - N') + N(P_{tr}\chi)P_{m}tr\chi'(N - N')^2 + \left( \nabla(\chi)(N - N') \right. \\
\left. + \nabla'(\bar{\delta})(N - N') + D_N(\chi)(N - N')^2 + N(\bar{\delta})(N - N')^2 + \nabla'(\chi')(N - N') \\
+ \nabla'(\bar{\delta}')N - N') + \nabla'(\zeta)(N - N')^2 + b^{-1}\nabla(b')(N - N') + (\chi + \bar{\delta} + \chi' + \bar{\delta}') \\
+ \zeta'(N - N')(\theta - \theta' + (\theta + \theta' + b^{-1}\nabla(b))(N - N')) \right) \frac{1}{g(L, L')} P_{tr}\chi P_{m}tr\chi' \\
+ (1 - g(N, N') b'N(P_{tr}\chi)N'(P_{m}tr\chi') + (N' - g(N, N')N(P_{tr}\chi)N(P_{m}tr\chi') \\
+ (N - N') b'P_{tr}\chi\nabla'(N'(P_{m}tr\chi')) + (N - N') b^{-1}\nabla'(b')P_{tr}\chi N'(P_{m}tr\chi').
\]

Proceeding in the same fashion than (A.14), we obtain, schematically:

\[
H = \frac{1}{|N_{\nu} - N_{\nu'}|} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N_{\nu}}{|N_{\nu} - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_{\nu} - N_{\nu'}|} \right)^q \right) \quad \text{(G.6)}
\]

\[
\times \left( \frac{1}{|N_{\nu} - N_{\nu'}|^3} H_1 + \frac{1}{|N_{\nu} - N_{\nu'}|^2} H_2 + \frac{1}{|N_{\nu} - N_{\nu'}|} H_3 + H_4 \right) \\
+ N(P_{tr}\chi)N'(P_{m}tr\chi') + \frac{(N' - g(N, N')N(P_{tr}\chi)N(P_{m}tr\chi')}{1 - g(N, N')^2},
\]

where the tensors $H_1, H_2, H_3$ and $H_4$ on $\mathcal{M}$ are given by:

\[
H_1 = (\chi - \chi')(\chi + \bar{\delta} + \chi' + \bar{\delta}') P_{tr}\chi P_{m}tr\chi', \quad \text{(G.7)}
\]
\[ H_2 = (\chi + \delta + \chi') P_{tr} \chi P_{tr} \chi' + P_{tr} \chi \nabla' P_{tr} \chi', \quad \text{(G.8)} \]

\[ H_3 = (\chi + \delta + \chi') N(P_{tr} \chi) P_{tr} \chi' + \nabla' (\nabla P_{tr} \chi) P_{tr} \chi' + P_{tr} \chi \nabla' P_{tr} \chi', \quad \text{(G.9)} \]

and:

\[ H_4 = P_{tr} \chi (N'(P_{tr} \chi')) + b'^{-1} \nabla'(b') P_{tr} \chi N'(P_{tr} \chi'), \quad \text{(G.10)} \]

and where \( c_{pq} \) are explicit real coefficients such that the series

\[ \sum_{p,q \geq 0} c_{pq} x^p y^q \]

has radius of convergence 1. In view of (G.5), (G.6), (G.7), (G.8), (G.9) and (G.10), we obtain the decomposition (8.528) (8.529) (8.530) (8.531) (8.532) (8.533) (8.534) (8.535) (8.536) (8.537) (8.538) (8.539) (8.540) (8.541) (8.542) of \( B_{j,\nu',l,m}^2 \). This concludes the proof of Lemma 8.17.

**H Proof of Lemma 8.18**

Recall from (8.596) that \( \sum_{m \leq t} B_{j,\nu',l,m}^{2,2} \) is given by:

\[ \sum_{m \leq t} B_{j,\nu',l,m}^{2,2} = 2^{-2j} \int_\mathcal{M} \int_{S^2 \times S^2} \frac{(N' - g(N, N') N)(P_{tr} \chi) N'(P_{tr} \chi')}{1 - g(N, N')^2} \]

\[ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}. \]

We integrate by parts using (7.137) with

\[ h = \frac{b' N' - g(N, N') N (P_{tr} \chi) N(P_{tr} \chi')}{1 - g(N, N')^2}. \quad \text{(H.1)} \]

We obtain:

\[ \sum_{m \leq t} B_{j,\nu',l,m}^{2,2} = -i2^{-3j} \int_\mathcal{M} \int_{S^2 \times S^2} \frac{b'^{-1}}{1 - g(N, N')^2} \left( (N - g(N, N') N')(h) + \left( tr \theta - g(N, N') tr \theta' \right. \right. \]

\[ -\theta(N' - g(N, N') N, N' - g(N, N') N) - g(N, N') b^{-1}(N' - g(N, N') N)(b) \]

\[ + \frac{2g(N, N')}{1 - g(N, N')^2} \left( \theta'(N - g(N, N') N', N - g(N, N') N') \right. \]

\[ \left. - g(N, N') \theta(N' - g(N, N') N, N' - g(N, N') N) \right)

\[ \times F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^\nu(\omega') d\omega d\omega' d\mathcal{M}. \]
Next, we compute the term \((N - g(N, N')N')(h)\). Using the structure equation for \(N\) (2.21), we have, schematically:

\[
\nabla_N (N' - g(N, N')N) = b^{-1} \nabla(b) + b'^{-1} \nabla'(b') + (N - N')\theta'.
\]

(H.3)

Proceeding as in (A.3), (A.5), (A.9) and (A.10), and using (G.3) and (H.3), we obtain schematically:

\[
(N - g(N, N')N')(h) = \frac{bb'}{(1 - g(N, N')^2)^2}
\left((N - N')^2 \nabla^2(P\nabla\chi)N'(P_{\leq} \nabla\chi') + (N - N')^3 \nabla(N(P\nabla\chi))N'(P_{\leq} \nabla\chi') + (N - N')(\theta - \theta')\right)
\left((N - N')N'(P_{\leq} \nabla\chi')\right)
\]

Thus, in view of (H.1), (H.2), (H.4), (A.8) and (A.11) we obtain:

\[
\sum_{m \leq l} B_{j, l, m, t, l, m}^{2, 2} = 2^{-3j} \int \int_{S^2 \times S^2} H F_{j, t-1}(u) F_{j, t-1}(u')(\omega) \eta_j^{\nu}\eta_j^{\nu'}(\omega') d\omega d\omega' dM,
\]

(H.5)

with the tensor \(H\) on \(M\) given, schematically, by:

\[
H = \frac{1}{(1 - g(N, N')^2)^2}
\left((N - N')^2 \nabla^2(P\nabla\chi)N'(P_{\leq} \nabla\chi') + (N - N')^3 \nabla(N(P\nabla\chi))N'(P_{\leq} \nabla\chi') + (N - N')(\chi - \chi')\right)
\left((N - N')N'(P_{\leq} \nabla\chi')\right)
\]

Proceeding in the same fashion than (A.14), we obtain, schematically:

\[
H = \frac{1}{|N_\nu - N_{\nu'}|} \left(\sum_{p, q \geq 0} c_{pq} \left(\frac{N - N_\nu}{|N_\nu - N_{\nu'}|}\right)^p \left(\frac{N' - N_{\nu'}}{|N_{\nu} - N_{\nu'}|}\right)^q\right)
\times \left(\frac{1}{|N_\nu - N_{\nu'}|^2} H_1 + \frac{1}{|N_{\nu} - N_{\nu'}|} H_2 + H_3\right)
+ N(P\nabla\chi)N'(P_{\leq} \nabla\chi') + \frac{(N' - g(N, N')N(P\nabla\chi)N(P_{\leq} \nabla\chi'))}{1 - g(N, N')^2},
\]

(H.6)
where the tensors \( H_1, H_2 \) and \( H_3 \) on \( \mathcal{M} \) are given by:

\[
H_1 = (\chi - \chi')\nabla (P_1 \text{tr} \chi) N'(P_{\leq 1} \text{tr} \chi'), \tag{H.7}
\]

\[
H_2 = \nabla^2 (P_1 \text{tr} \chi) N'(P_{\leq 1} \text{tr} \chi') + \nabla (P_1 \text{tr} \chi) \nabla' N'(P_{\leq 1} \text{tr} \chi') + (\theta + \theta' + b^{-1}\nabla(b) + b^{-1}\nabla'(b')) \nabla (P_1 \text{tr} \chi) N'(P_{\leq 1} \text{tr} \chi'), \tag{H.8}
\]

and:

\[
H_3 = \nabla (N(P_1 \text{tr} \chi)) N'(P_{\leq 1} \text{tr} \chi') + (\theta + \theta' + N(b)) \nabla (P_1 \text{tr} \chi) N'(P_{\leq 1} \text{tr} \chi'), \tag{H.9}
\]

and where \( c_{pq} \) are explicit real coefficients such that the series

\[
\sum_{p,q \geq 0} c_{pq} x^p y^q
\]

has radius of convergence 1. In view of (H.5), (H.6), (H.7), (H.8) and (H.9), we obtain the decomposition (8.597) (8.598) (8.599) (8.600) (8.601) (8.602) (8.603) (8.604) (8.605) of \( \sum_{m \geq 1} B_{j,\nu,\nu',l,m}^{2,2} \). This concludes the proof of Lemma 8.18.

## I Proof of Lemma 9.1

Recall from (9.2) that \( \sum_{(l,m)/2 \min(l,m) \leq 2|\nu - \nu'| \leq 2 \max(l,m)} A_{j,\nu,\nu',l,m} \) is given by:

\[
\sum_{(l,m)/2 \min(l,m) \leq 2|\nu - \nu'| \leq 2 \max(l,m)} A_{j,\nu,\nu',l,m} = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} P_{> 2|\nu - \nu'|} \text{tr} \chi (N - g(N, N') N')(P_{\leq 2|\nu - \nu'|} \text{tr} \chi') \frac{g(L, L')}{\gamma(L, L')} \times F_j(u) F_{j-1}(u') \eta_j''(\omega') \eta_j''(\omega') d\omega d\omega' d\mathcal{M}.
\]

We integrate by parts in \( \sum_{(l,m)/2 \min(l,m) \leq 2|\nu - \nu'| \leq 2 \max(l,m)} A_{j,\nu,\nu',l,m} \) using (7.143) with

\[
h = \frac{bb'}{g(L, L')} \frac{P_{> 2|\nu - \nu'|} \text{tr} \chi (N - g(N, N') N')(P_{\leq 2|\nu - \nu'|} \text{tr} \chi')}{g(L, L')}.
\]

We obtain:

\[
\sum_{(l,m)/2 \min(l,m) \leq 2|\nu - \nu'| \leq 2 \max(l,m)} A_{j,\nu,\nu',l,m} = -2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( L(h) + \text{tr} \chi h - \tilde{\theta} h - \tilde{\theta}' h - (1 - g(N, N')) \delta h - 2 \xi(N - g(N, N') N', N - g(N, N') N') h \right) F_j(u) F_{j-1}(u') \eta_j''(\omega') \eta_j''(\omega') d\omega d\omega' d\mathcal{M}. \tag{I.2}
\]
Next, we compute the term $L(h)$. We have:

$$
L(h) = \frac{bb'L(P_{>2|\nu-\nu'|tr\chi})(N - g(N, N')(P_{\leq 2|\nu-\nu'|tr\chi') (g(L, L'))}{g(L, L')}
+ \frac{bb'P_{>2|\nu-\nu'|tr\chi L((N - g(N, N')N')(P_{\leq 2|\nu-\nu'|tr\chi')}}{g(L, L')}
+ \frac{(L(b) + L(b'))P_{>2|\nu-\nu'|tr\chi(N - g(N, N')N'(P_{\leq 2|\nu-\nu'|tr\chi'})}{g(L, L')}
- \frac{L(g(L, L'))bb'P_{>2|\nu-\nu'|tr\chi(N - g(N, N')N')(P_{\leq 2|\nu-\nu'|tr\chi')}}{g(L, L')^2},
$$

Decomposing $L$ on $L'$, $N'$ and $N - g(N, N')N'$, we have:

$$
L = L' + (N - g(N, N')N') + (g(N, N') - 1)N',
$$

which yields schematically for the derivative in the second term in the right-hand side of (I.3):

$$
L((N - g(N, N')N')(P_{\leq 2|\nu-\nu'|tr\chi'}))
= L'((N - g(N, N')N')(P_{\leq 2|\nu-\nu'|tr\chi'})) + (N - g(N, N')N'((N - g(N, N')N')(P_{\leq 2|\nu-\nu'|tr\chi'}))
+ (g(N, N') - 1)N'((N - g(N, N')N')(P_{\leq 2|\nu-\nu'|tr\chi'}))
= (N - N')\nabla(L'(P_{\leq 2|\nu-\nu'|tr\chi'})) + [L', N - g(N, N')N'](P_{\leq 2|\nu-\nu'|tr\chi'})
+ (N - N')^2\nabla^2(P_{\leq 2|\nu-\nu'|tr\chi'}) + \nabla\nabla_{N - g(N, N')N'}(N - N')^2(P_{\leq 2|\nu-\nu'|tr\chi'})
+ (N - N')^3\nabla'(N'(P_{\leq 2|\nu-\nu'|tr\chi'})) + (N - N')^2[N', N - g(N, N')N'](P_{\leq 2|\nu-\nu'|tr\chi'}),
$$

where we used in the last inequality the fact that, schematically, $1 - g(N, N') = (N - N')^2.$

Next, we compute the two commutators in the right-hand side of (I.5). Using the structure equation for $N$ (2.21), we have, schematically:

$$
[N', N - g(N, N')N'] = b^{-1}\nabla(b) + b'\nabla'(b') + (N - N')(\theta + \theta').
$$

Also, using the fact that $L = T + N$, $L' = T + N'$ and $g(L, L') = -1 + g(N, N')$, we have:

$$
[L', N - g(N, N')N'] = [L', L - g(N, N')L' + (g(N, N') - 1)T]
= [L', L] - [L'(g(L, L')))N' + (g(N, N') - 1)[L', T]
$$

which together with the Ricci equations (2.17) implies, schematically:

$$
[L', N - g(N, N')N'] = -\delta\nabla + \delta'\nabla' + \nabla(N - N')(\chi + \chi' + \epsilon + \epsilon')
+ (N - N')^2(\zeta + \zeta + n^{-1}\nabla n + \delta' + \chi).
$$

Using the analog of (7.45) (7.46) for $-\delta\nabla + \delta'\nabla'$, we finally obtain, schematically:

$$
[L', N - g(N, N')N'] = (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n)
+ (N - N')^2(k + n^{-1}\nabla n + \chi + \zeta).$$
Now, in view of (G.3), (I.5), (I.6) and (I.7), we obtain, schematically:

\[
L((N - g(N, N')N')(P_{2|\nu-\nu'}|tr\chi')) \\
= (N - N')\nabla(L'(P_{2|\nu-\nu'}|tr\chi')) + (N - N')^2\nabla^2(P_{2|\nu-\nu'}|tr\chi')) \\
+ (N - N')^2\nabla'(N'(P_{2|\nu-\nu'}|tr\chi')) \\
+ (N - N')'(N + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n)\nabla(P_{2|\nu-\nu'}|tr\chi') \\
+ (N - N')^2(k + n^{-1}\nabla n + \theta + \theta' + b^{-1}\nabla(b) + b^{-1}\nabla'(b) + \chi + \zeta)\nabla(P_{2|\nu-\nu'}|tr\chi').
\] (I.8)

Also, in view of (7.149), we have, schematically:

\[
L(g(L, L')) = (N - N')^2(\delta + \delta' + n^{-1}\nabla n + \chi') + (N - N')^3\zeta'.
\] (I.9)

Finally, (I.3), (I.8) and (I.9) yield, schematically:

\[
L(h) = \frac{1}{g(L, L')} \left[ bb'(N - N')L(P_{2|\nu-\nu'}|tr\chi')\nabla'(P_{2|\nu-\nu'}|tr\chi') \\
+ bb'P_{2|\nu-\nu'}|tr\chi' \left( (N - N')\nabla(L'(P_{2|\nu-\nu'}|tr\chi')) + (N - N')^2\nabla^2(P_{2|\nu-\nu'}|tr\chi')) \\
+ (N - N')^3\nabla'(N'(P_{2|\nu-\nu'}|tr\chi')) \\
+ (N - N')'(N + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1}\nabla n + L(b) + L'(b'))\nabla(P_{2|\nu-\nu'}|tr\chi') \\
+ (N - N')^2(k + n^{-1}\nabla n + \theta + \theta' + b^{-1}\nabla(b) + b^{-1}\nabla'(b) + \chi + \zeta \\
+ \nabla N'(b'))\nabla(P_{2|\nu-\nu'}|tr\chi') \right) \\
- \frac{((N - N')^3(\delta + \delta' + n^{-1}\nabla n + \chi') + (N - N')^4\zeta')} {g(L, L')} \right] \\
- \frac{bb'P_{2|\nu-\nu'}|tr\chi'} {g(L, L')}.
\] (I.10)

We consider the term multiplied by \( h \) in the right-hand side of (I.2). We have schematically:

\[
tr\chi - \delta - \delta' - (1 - g(N, N'))\delta' - 2\zeta_{N-g(N,N')}N' \\
= \frac{\chi(N - g(N, N')N, N - g(N, N')N')} {g(L, L')} \\
= \frac{\chi + \delta + \delta' + (N - N')\zeta' + \frac{(N - N')^2\chi'} {g(L, L')}}.
\] (I.11)

Thus, in view of (I.1), (I.2), (I.10) and (I.11) we obtain:

\[
\sum_{(l,m)/2\min(l,m) < 2|\nu-\nu'| < 2\max(l,m)} A_{j,\nu,\nu',l,m} \\
= 2^{-2j} \int_{\mathbb{S}^2 x \mathbb{S}^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j^\nu(\omega') d\omega d\omega' d\mathcal{M},
\] (I.12)
with the tensor $H$ on $\mathcal{M}$ given, schematically, by:

$$H = \frac{1}{\mathbf{g}(L, L')^2} \left[ (N - N') L(P_{>2|\nu-\nu'|\text{tr} \chi}) \nabla'(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) \right. $$

$$+ P_{>2|\nu-\nu'|\text{tr} \chi} \left( (N - N') \nabla'(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) \right) + (N - N')^2 \nabla'^2 (P_{\leq 2|\nu-\nu'|\text{tr} \chi}) $$

$$+ (N - N')^3 \nabla''(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) $$

$$+ (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n + L(b) + L'(b')) \nabla(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) $$

$$+ (N - N')^2(k + n^{-1} \nabla n + \theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b') + \chi + \zeta + \zeta' $$

$$+ \nabla_{\nu'}(b')) \nabla(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) \left. \right] $$

$$- ((N - N')^3(\delta + \delta' + n^{-1} \nabla n + \chi') + (N - N')^4 \zeta') P_{>2|\nu-\nu'|\text{tr} \chi} \nabla'(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) $$

$$\mathbf{g}(L, L')^3$$

Recall the identities (8.30) and (8.31):

$$\mathbf{g}(L, L') = -1 + \mathbf{g}(N, N')$$

and

$$1 - \mathbf{g}(N, N') = \frac{\mathbf{g}(N - N', N - N')}{2}.$$ 

We may thus expand:

$$\frac{1}{\mathbf{g}(L, L')^2}$$

and

$$\frac{1}{\mathbf{g}(L, L')^3}$$

in the same fashion than (8.33), and we obtain, schematically:

$$H = \frac{1}{|N_\nu - N_{\nu'}|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right) $$

$$\times \left( H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|} H_3 \right),$$

where the tensors $H_1, H_2$ and $H_3$ on $\mathcal{M}$ are given by:

$$H_1 = P_{>2|\nu-\nu'|\text{tr} \chi} \nabla'(P_{\leq 2|\nu-\nu'|\text{tr} \chi}),$$

$$H_2 = P_{>2|\nu-\nu'|\text{tr} \chi} \left( \nabla'^2(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) + (k + n^{-1} \nabla n + \theta + \theta' $$

$$+ b^{-1} \nabla(b) + b'^{-1} \nabla'(b') + \chi + \zeta + \zeta' + \nabla_{\nu'}(b')) \nabla(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) \right),$$

and:

$$H_3 = L(P_{>2|\nu-\nu'|\text{tr} \chi}) \nabla'(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) + P_{>2|\nu-\nu'|\text{tr} \chi} \left( \nabla(L'(P_{\leq 2|\nu-\nu'|\text{tr} \chi}) \right) $$

$$+ (\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n + L(b) + L'(b')) \nabla(P_{\leq 2|\nu-\nu'|\text{tr} \chi}),$$

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and where $c_{pq}$ are explicit real coefficients such that the series
\[ \sum_{p,q \geq 0} c_{pq} x^p y^q \]
has radius of convergence 1. In view of (I.12), (I.13), (I.14), (I.15) and (I.16), we obtain the decomposition (9.3) (9.4) (9.5) (9.6) (9.7) (9.8) (9.9) (9.10) (9.11) (9.12) (9.13) (9.14) (9.15) of $\sum_{(l,m)/2 \min(l,m) \leq 2 |\nu - \nu'| < 2 \max(l,m)} A_{j,\nu,\nu',l,m}$. This concludes the proof of Lemma 9.1.

J Proof of Lemma 10.1

Recall from (10.2) that $A_{j,\nu,\nu',l,m}$ is given by:

\[
A_{j,\nu,\nu',l,m} = -i2^{-j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{P_l \text{tr}_\chi(N - g(N, N')N')(P_m \text{tr}_\chi')}{g(L, L')} \times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M}.
\]

We integrate by parts using (7.137) with
\[
h = \frac{b b' P_l \text{tr}_\chi(N - g(N, N')N')(P_m \text{tr}_\chi')}{g(L, L')}.
\]

We obtain:

\[
A_{j,\nu,\nu',l,m} = -2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{j-1}}{1 - g(N, N')^2} \left( (N - g(N, N')N')(h) + \left( \text{tr} \theta - g(N, N')\text{tr} \theta' \right) - \theta(N' - g(N, N')N, N' - g(N, N')N) - g(N, N')b^{-1}(N' - g(N, N')N)(b) + 2g(N, N') \left( \theta'(N - g(N, N')N', N - g(N, N')N') \right) \right) h \times F_{j-1}(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j^\nu'(\omega') d\omega d\omega' d\mathcal{M}.
\]

Next, we compute the term $(N - g(N, N')N')(h)$. Proceeding as in (A.3), (A.5), (A.9)
and \((A.10)\), and using \((G.3)\), we obtain schematically:

\[
(N - g(N, N')N')(h) = \frac{bb'}{g(L, L')} \left( (N' - g(N, N')N)(P_{\text{tr}x})(N - g(N, N')N')(P_{\text{tr}x}') + P_{\text{tr}x} \nabla^2 P_m(\text{tr}x')(N - g(N, N')N', N - g(N, N')N') + (1 - g(N, N')^2)N(P_{\text{tr}x})(N - g(N, N')N')(P_{\text{tr}x}') + (N - N')(\theta - \theta'P_{\text{tr}x} \nabla'(P_{\text{tr}x}')') + (N - N')^2(\theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b'))P_{\text{tr}x} \nabla'(P_{\text{tr}x}') \right)
\]

Thus, in view of \((J.1)\), \((J.2)\), \((J.3)\), \((A.8)\) and \((A.11)\) we obtain:

\[
A_{j,\nu,\nu',l,m} = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_j^\nu(\omega) \eta_j'^{\nu'}(\omega') d\omega d\omega' d\mathcal{M},
\]

with the tensor \(H\) on \(\mathcal{M}\) given, schematically, by:

\[
H = \frac{1}{g(L, L')(1 - g(N, N')^2)} \left( (N' - g(N, N')N)(P_{\text{tr}x})(N - g(N, N')N')(P_{\text{tr}x}') + P_{\text{tr}x} \nabla^2 P_m(\text{tr}x')(N - g(N, N')N', N - g(N, N')N') + (1 - g(N, N')^2)N(P_{\text{tr}x})(N - g(N, N')N')(P_{\text{tr}x}') + (N - N')(\chi - \chi')P_{\text{tr}x} \nabla'(P_{\text{tr}x}') \right)
\]

\[
+ \frac{1}{g(L, L')(1 - g(N, N')^2)} \left( (N - N')^2(\chi - \chi') + (N - N')^4(\theta + \theta' + b^{-1}\nabla(b)) \right) P_{\text{tr}x} \nabla'(P_{\text{tr}x}') \times P_{\text{tr}x} \nabla'(P_{\text{tr}x}').
\]

Proceeding in the same fashion than \((A.14)\), we obtain, schematically:

\[
H = \frac{1}{|N' - N'|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N'}{|N' - N'|} \right)^p \left( \frac{N' - N}{|N - N'|} \right)^q \right) \left( \frac{1}{|N' - N'|} H_1 + H_2 \right)
\]

\[
+ \frac{(N' - g(N, N')N)(P_{\text{tr}x})(N - g(N, N')N')(P_{\text{tr}x}')}{g(L, L')(1 - g(N, N')^2)}
\]

\[
+ \frac{P_{\text{tr}x} \nabla^2 P_m(\text{tr}x')(N - g(N, N')N', N - g(N, N')N')}{g(L, L')(1 - g(N, N')^2)}
\]

\[
+ \frac{N(P_{\text{tr}x})(N - g(N, N')N')(P_{\text{tr}x}')}}{g(L, L')},
\]

\[\text{J.5}\]
where the tensors $H_1$ and $H_2$ on $\mathcal{M}$ are given by:

$$H_1 = (\chi - \chi')P_{\text{tr}}\nabla(P_{\text{tr}}\chi'), \quad (J.6)$$

and:

$$H_2 = (\theta + \theta' + b^{-1}\nabla(b) + b'^{-1}\nabla'(b'))P_{\text{tr}}\nabla'(P_{\text{tr}}\chi'), \quad (J.7)$$

and where $c_{pq}$ are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq} x^p y^q$$

has radius of convergence 1. In view of (J.4), (J.5), (J.6) and (J.7), we obtain the decomposition (10.3) (10.4) (10.5) (10.6) (10.7) (10.8) (10.9) (10.10) of $A_{j,\nu,\nu',l,m}$. This concludes the proof of Lemma 10.1.

K Proof of Lemma 10.4

Recall from (10.2) that $A_{j,\nu,\nu',l,m}^1$ is given by:

$$A_{j,\nu,\nu',l,m}^1 = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{P_{\text{tr}}\nabla^2 P_{m}(\nabla' \chi')(N - g(N,N')N', N - g(N,N')N')}{g(L,L')(1 - g(N,N')^2)}$$

$$\times F_j(u) F_{j-1}(u') \eta^{\nu'}(\omega) \eta^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. $$

We integrate by parts using (7.137) with

$$h = \frac{bb' P_{\text{tr}}\nabla^2 P_{m}(\nabla' \chi')(N - g(N,N')N', N - g(N,N')N')}{g(L,L')(1 - g(N,N')^2)}. \quad (K.1)$$

We obtain:

$$A_{j,\nu,\nu',l,m}^1 = -i2^{-3j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b'^{-1}}{1 - g(N,N')^2} \left( (N - g(N,N')N')(h) + \left( \text{tr} \theta - g(N,N')\text{tr}\theta' 
\right.ight.$$

$$\left. - \theta(N' - g(N,N')N, N' - g(N,N')N) - g(N,N')b^{-1}(N' - g(N,N')N)(b)
+ \frac{2g(N,N')}{1 - g(N,N')^2} \left( \theta'(N - g(N,N')N', N - g(N,N')N')
- g(N,N')\theta(N' - g(N,N')N, N' - g(N,N')N) \right) \right) h \right)$$

$$\times F_{j-1}(u) F_{j-1}(u') \eta^{\nu'}(\omega) \eta^{\nu'}(\omega') d\omega d\omega' d\mathcal{M}. $$
Next, we compute the term \((N - g(N,N')N')(h)\). Proceeding as in (A.3), (A.5), (A.9) and (A.10), and using (G.3), we obtain schematically:

\[
(N - g(N,N')N')(h) = \frac{bb'}{g(L,L')(1 - g(N,N')^2)} (N - N')^3 \nabla(P_{tix}(\nabla^2(P_m x'))
\]

\[
+ (N - N')^4 N(P_{tix})(\nabla^2(P_m x')) + (N - N')^3 P_{tix} \nabla^2(P_m x')
\]

\[
+ \left( (N - N')^2(\theta - \theta') + (N - N')^3(\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) \right) P_{tix} \nabla^2(P_m x')
\]

\[
+ \frac{bb'}{g(L,L')(1 - g(N,N')^2)} \left( \frac{1}{g(L,L')} + \frac{1}{1 - g(N,N')^2} \right) (N - N')^4(\theta - \theta')
\]

\[
+ (N - N')^5(\theta + \theta' + b^{-1} \nabla(b)) P_{tix} \nabla^2(P_m x')
\]

Thus, in view of (K.1), (K.2), (K.3), (A.8) and (A.11) we obtain:

\[
A_{1,j,\nu',\ell,m}^{} = 2^{-3j} \int_M \int_{S^2 \times S^2} H F_{j,-1}(u) F_{j,-1}(u') \eta_{j}^{\nu'}(\omega) \eta_{j}^{\nu'}(\omega') d\omega d\omega' dM,
\]

with the tensor \(H\) on \(M\) given, schematically, by:

\[
H = \frac{1}{g(L,L')(1 - g(N,N')^2)} (N - N')^3 \nabla(P_{tix})(\nabla^2(P_m x'))
\]

\[
+ (N - N')^4 N(P_{tix})(\nabla^2(P_m x')) + (N - N')^3 P_{tix} \nabla^2(P_m x')
\]

\[
+ \left( (N - N')^2(\chi - \chi') + (N - N')^3(\theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b')) \right) P_{tix} \nabla^2(P_m x')
\]

\[
+ \frac{1}{g(L,L')(1 - g(N,N')^2)} \left( \frac{1}{g(L,L')} + \frac{1}{1 - g(N,N')^2} \right) (N - N')^4(\chi - \chi')
\]

\[
+ (N - N')^5(\theta + \theta' + b^{-1} \nabla(b)) P_{tix} \nabla^2(P_m x')
\]

Proceeding in the same fashion than (A.14), we obtain, schematically:

\[
H = \frac{1}{|N_\nu - N_{\nu'}|^2} \left( \sum_{p,q \geq 0} c_{pq} \left( \frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q \right)
\]

\[
\times \left( \frac{1}{|N_\nu - N_{\nu'}|^2} H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + H_3 \right),
\]

where the tensors \(H_1, H_2\) and \(H_3\) on \(M\) are given by:

\[
H_1 = (\chi - \chi') P_{tix} \nabla^2(P_m x'),
\]

\[
(\text{K.5})
\]

\[
(\text{K.6})
\]
We obtain:

\[ H_2 = \nabla(P_{l \text{tr} \chi}) \nabla^2 (P_{m \text{tr} \chi'}) + P_{l \text{tr} \chi} \nabla^3 (P_{m \text{tr} \chi'}) \]

\[(\theta + \theta' + b^{-1} \nabla(b) + b^{-1} \nabla'(b')) P_{l \text{tr} \chi} \nabla^2 (P_{m \text{tr} \chi'}),\]

and:

\[ H_3 = N(P_{l \text{tr} \chi}) \nabla^2 (P_{m \text{tr} \chi'}), \]

and where \(c_{pq}\) are explicit real coefficients such that the series

\[ \sum_{p,q \geq 0} c_{pq} t^p y^q \]

has radius of convergence 1. In view of (K.4), (K.5), (K.6), (K.7) and (K.8), we obtain the decomposition (10.24) (10.25) (10.26) (10.27) (10.28) (10.29) (10.30) (10.31) of \(A_{j,\nu,\nu',t,m}^1\). This concludes the proof of Lemma 10.4.

### L Proof of Lemma 10.5

Recall from (10.64) that \(A_{j,\nu,\nu',t,m}^2\) is given by:

\[ A_{j,\nu,\nu',t,m}^2 = 2^{-2j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{(N' - g(N, N')) N(P_{l \text{tr} \chi})(N - g(N, N')) N'(P_{m \text{tr} \chi'})}{g(L, L')(1 - g(N, N'))^2} \]

\times F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j'^{\nu'}(\omega') d\omega d \omega' d \mathcal{M}, \]

We integrate by parts in \(A_{j,\nu,\nu',t,m}^2\) using (7.143) with

\[ h = \frac{bb' (N' - g(N, N') N)(P_{l \text{tr} \chi})(N - g(N, N') N')(P_{m \text{tr} \chi'})}{g(L, L')(1 - g(N, N'))^2}. \]  

(10.1)

We obtain:

\[ A_{j,\nu,\nu',t,m}^2 = -i2^{-3j} \int_{\mathcal{M}} \int_{S^2 \times S^2} \frac{b^{-1}}{g(L, L')} \left( L(h) + \text{tr} \chi h - \delta h - \bar{\delta} h - (1 - g(N, N')) \delta' h \right) \]

\[ -2 \zeta_{N' - g(N, N')} N' h - \frac{\chi(N - g(N, N') N', N - g(N, N') N')}{g(L, L')} \]

\[ F_j(u) F_{j-1}(u') \eta_j^\nu(\omega) \eta_j'^{\nu'}(\omega') d\omega d \omega' d \mathcal{M}. \]  

(10.2)

Next, we compute the term \(L(h)\). We have:

\[ L(h) = \frac{bb' L((N' - g(N, N') N)(P_{l \text{tr} \chi}))(N - g(N, N') N')(P_{m \text{tr} \chi'})}{g(L, L')(1 - g(N, N'))^2} \]

\[ + \frac{bb' (N' - g(N, N') N)(P_{l \text{tr} \chi}) L((N - g(N, N') N')(P_{m \text{tr} \chi'}))}{g(L, L')(1 - g(N, N'))^2} \]

\[ + \frac{L(b) + L(b') (N' - g(N, N') N)(P_{l \text{tr} \chi})(N - g(N, N') N')(P_{m \text{tr} \chi'})}{g(L, L')(1 - g(N, N'))^2} \]

\[ - \left( \frac{1}{g(L, L')} + \frac{1}{1 - g(N, N')^2} \right) \]

\[ \times \frac{L(g(L, L')) bb' (N' - g(N, N') N)(P_{l \text{tr} \chi}))(N - g(N, N') N')(P_{m \text{tr} \chi'})}{g(L, L')(1 - g(N, N'))^2}. \]  

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In view of (I.7), we have, schematically:

\[
L((N' - g(N, N')N)(P_\text{tr} \chi')) = (N - N') \nabla (L(P_\text{tr} \chi')) + (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n) \nabla (P_\text{tr} \chi') + (N - N')^2 (k + n^{-1} \nabla n + \chi + \zeta) \nabla (P_\text{tr} \chi').
\] (L.4)

Next, recall from (I.8) that we have, schematically:

\[
L((N - g(N, N')N')(P_\text{tr} \chi')) = (N - N') \nabla (L'(P_\text{tr} \chi')) + (N - N')^2 \nabla^2 (P_\text{tr} \chi') + (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n) \nabla (P_\text{tr} \chi') + (N - N')^2 (k + n^{-1} \nabla n + \theta + \theta' + b^{-1} \nabla (b) + b'^{-1} \nabla (b') + \chi + \zeta) \nabla (P_\text{tr} \chi').
\] (L.5)

Also, in view of (7.149), we have, schematically:

\[
L(g(L, L')) = (N - N')^2 (\delta + \delta' + n^{-1} \nabla n + \chi') + (N - N')^3 \zeta'.
\] (L.6)

Finally, (L.3), (L.4), (L.5) and (L.6) yield, schematically:

\[
L(h) = \frac{1}{g(L, L')(1 - g(N, N')^2)} \left[ bb' (N - N')^2 \nabla (L(P_\text{tr} \chi')) \nabla' (P_\text{tr} \chi') + (N - N')^2 \nabla^2 (P_\text{tr} \chi') + (N - N')(\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n + L(b) + L'(b')) \nabla (P_\text{tr} \chi') + (N - N')^2 (k + n^{-1} \nabla n + \theta + \theta' + b^{-1} \nabla (b) + b'^{-1} \nabla (b') + \chi + \zeta) + \nabla_{N'}(b') \nabla (P_\text{tr} \chi') \right] + \frac{1}{g(L, L')} \left[ \frac{1}{1 - g(N, N')^2} \right] \left[ (N - N')^4 (\delta + \delta' + n^{-1} \nabla n + \chi') + (N - N')^5 \zeta' \right] \frac{bb' \nabla (P_\text{tr} \chi') \nabla' (P_\text{tr} \chi')}{g(L, L')(1 - g(N, N')^2)}.
\] (L.7)

Thus, in view of (L.1), (L.2), (L.7) and (I.11) we obtain:

\[
A_{j,\nu,\nu',l,m}^2 = 2^{-3j} \int_{\mathcal{M}} \int_{S^2 \times S^2} HF_{j,-1}(u)F_{j,-1}(u')n_j'(\omega)n_j''(\omega')d\omega d\omega' d\mathcal{M},
\] (L.8)

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with the tensor $H$ on $\mathcal{M}$ given, schematically, by:

$$
H = \frac{1}{g(L, L')^2(1 - g(N, N')^2)} 
\left[ (N - N')^2 \nabla(L(P_t \text{tr} \chi)) \nabla'(P_m \text{tr} \chi') 
+ \nabla'(P_t \text{tr} \chi) \left( (N - N')^2 \nabla'(L'(P_m \text{tr} \chi')) + (N - N')^3 \nabla^2(P_m \text{tr} \chi') \right) 
+ (N - N')^4 \nabla'(N'(P_m \text{tr} \chi')) 
+ (N - N')^2 (\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n + L(b) + L'(b')) \nabla(P_m \text{tr} \chi') 
+ (N - N')^3 (k + n^{-1} \nabla n + \theta + \theta' + b^{-1} \nabla(b) + b'^{-1} \nabla'(b') + \chi + \zeta + \zeta') 
+ \nabla_{N'}(b') \nabla(P_m \text{tr} \chi') \right] 
\n+ \nabla(P_t \text{tr} \chi) \nabla'(P_m \text{tr} \chi')(N - N')^3 (k + n^{-1} \nabla n + \chi + \zeta) 
\n+ \left( \frac{1}{g(L, L')} + \frac{1}{1 - g(N, N')^2} \right) 
\times \left( \frac{(N - N')^4 (\delta + \delta' + n^{-1} \nabla n + \chi') + (N - N')^5 \zeta'}{g(L, L')^2(1 - g(N, N')^2)} \right) 
\n\n\frac{1}{g(L, L')^2(1 - g(N, N')^2)} \frac{1}{g(L, L')^3(1 - g(N, N')^2)} \frac{1}{g(L, L')^2(1 - g(N, N')^2)^2}
$$

Recall the identities (8.30) and (8.31):

$$
g(L, L') = -1 + g(N, N') \quad \text{and} \quad 1 - g(N, N') = \frac{g(N - N', N - N')}{2}.
$$

We may thus expand:

$$
H = \frac{1}{|N_\nu - N_{\nu'}|^2} \sum_{p, q \geq 0} c_{p q} \left( \frac{N - N_\nu}{|N_\nu - N_{\nu'}|} \right)^p \left( \frac{N' - N_{\nu'}}{|N_\nu - N_{\nu'}|} \right)^q 
\times \left( H_1 + \frac{1}{|N_\nu - N_{\nu'}|} H_2 + \frac{1}{|N_\nu - N_{\nu'}|^2} H_3 \right),
$$

where the tensors $H_1, H_2$ and $H_3$ on $\mathcal{M}$ are given by:

$$
H_1 = \nabla(P_t \text{tr} \chi) \nabla'(N'(P_m \text{tr} \chi')),
$$

$$
H_2 = \nabla'(P_t \text{tr} \chi) \nabla'(P_m \text{tr} \chi') + \left( k + n^{-1} \nabla n + \theta + \theta' \right) 
\n+ b^{-1} \nabla(b) + b'^{-1} \nabla'(b') + \chi + \zeta + \zeta' + \nabla_{N'}(b') \nabla(P_t \text{tr} \chi) \nabla(P_m \text{tr} \chi'),
$$

and:

$$
H_3 = \nabla(L(P_t \text{tr} \chi)) \nabla'(P_m \text{tr} \chi') + \nabla(P_t \text{tr} \chi) \left( \nabla'(L'(P_m \text{tr} \chi')) \nabla'(L'(P_m \text{tr} \chi')) \right) 
\n+ (\chi + \chi' + \epsilon + \epsilon' + \delta + \delta' + n^{-1} \nabla n + L(b) + L'(b')) \nabla'(P_m \text{tr} \chi'),
$$

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and where $c_{pq}$ are explicit real coefficients such that the series

$$\sum_{p,q \geq 0} c_{pq}x^p y^q$$

has radius of convergence 1. In view of (L.8), (L.9), (L.10), (L.11) and (L.12), we obtain the decomposition (10.65) (10.66) (10.67) (10.68) (10.69) (10.70) (10.71) (10.72) (10.73) (10.74) (10.75) (10.76) (10.77) of $A_{j,\nu,\nu',l,m}^2$. This concludes the proof of Lemma 10.5.

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