SEMl-WAVE AND SPREADING SPEED OF THE NONLOCAL FISHER-KPP EQUATION WITH FREE BOUNDARIES

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ABSTRACT. In Cao, Du, Li and Li [8], a nonlocal diffusion model with free boundaries extending the local diffusion model of Du and Lin [12] was introduced and studied. For Fisher-KPP type nonlinearities, its long-time dynamical behaviour is shown to follow a spreading-vanishing dichotomy. However, when spreading happens, the question of spreading speed was left open in [8]. In this paper we obtain a rather complete answer to this question. We find a threshold condition on the kernel function such that spreading grows linearly in time exactly when this condition holds, which is achieved by completely solving the associated semi-wave problem that determines this linear speed; when the kernel function violates this condition, we show that accelerating spreading happens.

Keywords: Nonlocal diffusion; Free boundary; Semi-wave; Spreading speed; Accelerating spreading.

AMS Subject Classification (2000): 35K57, 35R20

1. Introduction

In [8], a nonlocal diffusion model with free boundaries extending the local diffusion model of [12] was introduced, and its global existence and uniqueness was established for a rather general class of nonlinearities in the model. Moreover, for Fisher-KPP type nonlinearities, it is shown in [8] that a spreading-vanishing dichotomy holds for its long-time dynamical behaviour. However, when spreading happens, the question of spreading speed was not considered in [8]. The main purpose of this paper is to determine the spreading speed left open there. We will find the threshold condition on the kernel function such that the spreading grows linearly in time exactly when this condition holds; and when this condition is violated, we show that the spreading is accelerating.

The nonlocal diffusion model with free boundaries considered in [8] has the following form:

\[
\begin{align*}
\begin{cases}
  u_t &= d \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du(t,x) + f(u), & t > 0, \quad x \in (g(t), h(t)), \\
  u(t, g(t)) &= u(t, h(t)) = 0, & t > 0, \\
  h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{x}^{+\infty} J(x-y)u(t,x)dydx, & t > 0, \\
  g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{x} J(x-y)u(t,x)dydx, & t > 0, \\
  u(0, x) &= u_0(x), \quad h(0) = -g(0) = h_0, & x \in [-h_0, h_0],
\end{cases}
\]

where \( x = g(t) \) and \( x = h(t) \) are the moving boundaries to be determined together with \( u(t, x) \), which is always assumed to be identically 0 for \( x \in \mathbb{R} \setminus [g(t), h(t)] \); \( d \) and \( \mu \) are positive constants.

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The initial function $u_0(x)$ satisfies
\begin{equation}
(1.2) \quad u_0(x) \in C([-h_0, h_0]), \quad u_0(-h_0) = u_0(h_0) = 0 \quad \text{and} \quad u_0(x) > 0 \quad \text{in} \quad (-h_0, h_0),
\end{equation}
with $[-h_0, h_0]$ representing the initial population range of the species. The kernel function $J : \mathbb{R} \to \mathbb{R}$ has the properties

(J): $J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad J \geq 0$, $J(0) > 0$, $\int_{\mathbb{R}} J(x)dx = 1$, $J$ is even.

The growth term $f : \mathbb{R}^+ \to \mathbb{R}$ is assumed to be continuous and satisfies

(f1): $f(0) = 0$ and $f(u)$ is locally Lipschitz in $u \in \mathbb{R}^+$, i.e., for any $L > 0$, there exists a constant $K = K(L) > 0$ such that

$$|f(u_1) - f(u_2)| \leq K|u_1 - u_2| \quad \text{for} \quad u_1, u_2 \in [0, L];$$

(f2): There exists $K_0 > 0$ such that $f(u) < 0$ for $u \geq K_0$.

The nonlocal free boundary problem (1.1) may be viewed as describing the spreading of a new or invasive species with population density $u(t, x)$, whose population range $[g(t), h(t)]$ expands according to the free boundary conditions

$$
\begin{cases}
    h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx, \\
g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx,
\end{cases}
$$

that is, the expanding rate of the range $[g(t), h(t)]$ is proportional to the outward flux of the population across the boundary of the range (see [8] for further explanations and justification).

Under the assumptions (J), (f1) and (f2), the well-posedness and global existence of (1.1) has been established in [8]. If further, $f$ is a Fisher-KPP type function, namely it satisfies

(f3): $f \in C^1$, $f > 0 = f(0) = f(1)$ in $(0, 1)$, $f'(0) > 0 > f'(1)$, and $f(u)/u$ is nonincreasing in $u > 0$.

then the long-time dynamical behaviour of (1.1) is determined by a “spreading-vanishing dichotomy” (see Theorem 1.2 in [8]): As $t \to \infty$, either

(i) Spreading: $\lim_{t \to +\infty} (g(t), h(t)) = \mathbb{R}$ and $\lim_{t \to +\infty} u(t, x) = 1$ locally uniformly in $\mathbb{R}$, or
(ii) Vanishing: $\lim_{t \to +\infty} (g(t), h(t)) = (g_\infty, h_\infty)$ is a finite interval and $\lim_{t \to +\infty} u(t, x) = 0$ uniformly for $x \in [g(t), h(t)]$.

Criteria for spreading and vanishing are also obtained in [8]; see Theorem 1.3 there. In particular, if the supporting set of the initial function $u_0$ is large enough, then spreading always happens.

In order to describe the main results of this paper, we introduce a key condition on the kernel function $J$, namely

(J1): $\int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)dydx < +\infty$, i.e., $\int_{-\infty}^{0} \int_{-\infty}^{x} J(y)dydx < +\infty$.

**Theorem 1.1.** Suppose that (J) and (f3) are satisfied, and spreading happens to the unique solution $(u, g, h)$ of (1.1). Then the following conclusions hold:

(i) If (J1) is satisfied, then there exists a unique $c_0 > 0$ such that

$$\lim_{t \to \infty} \frac{h(t)}{t} = -\lim_{t \to \infty} \frac{g(t)}{t} = c_0.$$

(ii) If (J1) does not hold, then

$$\lim_{t \to \infty} \frac{h(t)}{t} = -\lim_{t \to \infty} \frac{g(t)}{t} = \infty.$$
As usual, when (J1) holds, we call $c_0$ the spreading speed of (1.1). The proof of Theorem 1.1 relies on the existence of semi-wave solutions to (1.1). These are pairs $(c, \phi)$ determined by the following two equations:

$$
(1.3) \quad \begin{cases}
    d \int_{-\infty}^{0} J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & -\infty < x < 0, \\
    \phi(-\infty) = 1, \quad \phi(0) = 0,
\end{cases}
$$

and

$$
(1.4) \quad c = \mu \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)\phi(y)dydx.
$$

If $(c, \phi)$ solves (1.3), then we call $\phi$ a semi-wave with speed $c$, since the function $v(t,x) := \phi(x-ct)$ satisfies

$$
(1.5) \quad \begin{cases}
    v_t = d \int_{-\infty}^{0} J(x-y)v(t,y)dy - dv(t,x) + f(v(t,x)), & t > 0, \quad x < ct, \\
    v(t,-\infty) = 1, \quad v(t,ct) = 0, & t > 0.
\end{cases}
$$

However, only the semi-wave satisfying (1.4) meets the free boundary condition along the moving front $x = ct$, and hence useful for determining the long-time dynamical behaviour of (1.1).

**Theorem 1.2.** Suppose that (J) and (f3) are satisfied. Then (1.3)-(1.4) has a solution pair $(c, \phi) = (c_0, \phi^{c_0})$ with $\phi^{c_0} \in C^1((-\infty, 0])$ and $\phi^{c_0}(x)$ nonincreasing in $x$ if and only if (J1) holds. Moreover, when (J1) holds, there exists a unique solution pair, and $c_0 > 0$, $\phi^{c_0}(x)$ is strictly decreasing in $x$.

The uniquely determined $c_0 > 0$ in Theorem 1.2 is the spreading speed for (1.1) given in part (i) of Theorem 1.1.

To put these results into perspective, we now recall some related results for the corresponding nonlocal diffusion problem without free boundaries, namely

$$
(1.5) \quad \begin{cases}
    u_t = d \int_{\mathbb{R}} J(x-y)u(t,y)dy - du(t,x) + f(u), & t > 0, \quad x \in \mathbb{R}, \\
    u(0,x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
$$

Problem (1.5) and its many variations have been extensively studied in the literature; see, for example, 1, 5, 9, 10, 16, 17, 19, 22, 25 and the references therein. In particular, if (J) and (f3) are satisfied, and if the nonnegative initial function $u_0$ has non-empty compact support, then the basic long-time dynamical behaviour of (1.5) is given by

$$
\lim_{t \to \infty} u(t,x) = 1 \quad \text{locally uniformly for } x \in \mathbb{R}.
$$

To understand the fine spreading behaviour of (1.5), one examines the level set

$$
E_\lambda(t) := \{x \in \mathbb{R} : u(t,x) = \lambda \} \text{ with fixed } \lambda \in (0,1),
$$

by considering the large time behaviour of

$$
x^+_\lambda(t) := \sup E_\lambda(t) \quad \text{and} \quad x^-_\lambda(t) = \inf E_\lambda(t).
$$

For this purpose, the following additional condition, apart from (J), on the kernel function is important:

**(J2):** There exists $\lambda > 0$ such that

$$
\int_{-\infty}^{+\infty} J(x)e^{\lambda x}dx < \infty.
$$
Yagisita [25] has proved the following result on traveling wave solutions to (1.5):

**Proposition 1.3.** Suppose that $f$ satisfies (f3) and $J$ satisfies (J). If additionally $J$ satisfies (J2), then there is a constant $c_*>0$ such that (1.5) has a traveling wave solution with speed $c$ if and only if $c \geq c_*$. To be more precise, the problem

\[
\begin{aligned}
&
\frac{d}{dt}J(x-t)\phi(y)dy - d\phi(x) + cf'(x) + f(\phi(x)) = 0, \quad x \in \mathbb{R}, \\
&\phi(-\infty) = 1, \quad \phi(+\infty) = 0
\end{aligned}
\]

has a solution $\phi \in L^\infty(\mathbb{R})$ which is nonincreasing if and only if $c \geq c_*$. Moreover, for each $c \geq c_*$, the solution has the following properties: $\phi \in C^1(\mathbb{R})$, $\phi'(x) < 0$ in $\mathbb{R}$, and it is unique up to translations. On the other hand, if $J$ does not satisfy (J2), then (1.5) does not have a traveling wave solution, that is, for any constant $c$, (1.6) has no solution $\phi \in L^\infty(\mathbb{R})$ which is nonincreasing.\]

Condition (J2) is often called a “thin tail” condition for $J$, and if it is not satisfied, then $J$ is said to have a “fat tail”. When $f$ satisfies (f3), and $J$ satisfies (J) and (J2), it is well known (see, for example, [24]) that

\[
\lim_{t \to \infty} \frac{|x^\pm(t)|}{t} = c_*,
\]

with $c_*$ given by Proposition 1.3. On the other hand, if (f3) and (J) hold but (J2) is not satisfied, then it follows from Theorem 6.4 of [24] that $|x^\pm(t)|$ grows faster than any linear function of $t$ as $t \to \infty$, namely,

\[
\lim_{t \to \infty} \frac{|x^\pm(t)|}{t} = \infty.
\]

Such a behaviour is usually called “accelerating spreading”. See also [13,7,15,18] and references therein for further progress on this and related questions.

We can easily show that (J2) implies (J1). Indeed, from

\[
a(x) := \int_0^\infty J(x-y)dy = \int_x^\infty J(z)dz = \int_0^\infty J(z)dz
\]

for $x \leq 0$, we obtain

\[
a(x) \leq e^{-\lambda|x|} \int_x^\infty J(z)e^{\lambda z}dz \leq e^{-\lambda|x|} \int_0^\infty J(z)e^{\lambda z}dz,
\]

which clearly implies

\[
\int_{-\infty}^0 a(x)dx < \infty,
\]

i.e., (J1) holds. On the other hand, it is easily checked that $J(x) = (1 + x^2)^{-\sigma}$ with $\sigma > 1$ satisfies (J1) but not (J2).

Therefore, for $f$ satisfying (f3), there exist kernel functions $J$ satisfying (J) and (J1) but not (J2) such that the free boundary problem (1.1) spreads linearly with speed $c_0$, but the corresponding problem (1.5) has accelerating spreading.

Theorem 1.1 indicates that for (1.1), under conditions (f3) and (J), accelerating spreading happens exactly when (J1) is not satisfied. In sharp contrast, let us recall that, for the corresponding local diffusion problem of (1.1), when spreading happens, the spreading speed is always finite; see [6],[12],[14].

More can be said about the relationship between (1.1) and (1.5). For the local diffusion versions of (1.1) (the free boundary problem) and (1.5) (the Cauchy problem), it is known that

\[\text{Theorem 2 in [25] actually provides a stronger nonexistence result.}\]
as $\mu \to +\infty$, the spreading speed of the free boundary problem converges to the spreading speed of the Cauchy problem (see [13]), and moreover, it follows from [11] that the Cauchy problem can be viewed as the limiting problem of the free boundary problem as $\mu \to +\infty$. Here we show that similar results hold for the nonlocal diffusion problems (1.1) and (1.5); see Theorems 5.1, 5.2 and 5.3 for details.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.2 on the semi-wave solution, which paves the ground for this research. In Section 3 we prove the first part of Theorem 1.1, by making use of the semi-wave solution established in Section 2. Section 4 is devoted to the proof of the second part of Theorem 1.1, on accelerating spreading, by making use of the first part of the theorem proved in Section 3 and an approximation argument. In Section 5, we consider the limiting profile of (1.1) and its semi-wave solution as $\mu \to +\infty$.

2. A unique semi-wave with the desired speed

The purpose of this section is to prove Theorem 1.2. We first prove the existence of a family of semi-waves to (1.1), namely for any speed $c$ in a certain range, there exists a unique positive solution $\phi = \phi_c \in C^1((\infty, 0])$ to the problem (1.3). We will further show that for any given $\mu > 0$, there exists a unique $c_0 = c(J)$ satisfying

$$c = \mu \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)\phi_c(x)dydx. \tag{2.1}$$

As we will see later, the unique solution $\phi_{c_0}(x)$ is strictly decreasing in $x$ for $x \in (-\infty, 0]$, and hence from the condition (J1), it is easily seen that

$$c_0 < \mu c(J) \quad \text{with} \quad c(J) := \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)dydx. \tag{2.2}$$

In the following, we will first prove the existence and uniqueness of $(c_0, \phi_{c_0})$ for the case that $J$ satisfies (J2), and then use an approximation argument to show that the conclusion also holds when $J$ satisfies (J1). It is easy to show that (J1) is a necessary condition for the existence of such a pair $(c_0, \phi_{c_0})$.

2.1. A perturbed problem. Suppose that (J2) holds. Fix $\sigma \in (0, 1)$, $c \in (0, c_*)$, and consider the auxiliary problem

$$\begin{cases}
    \frac{d}{dx} \int_{-\infty}^{+\infty} J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & -\infty < x < 0, \\
    \phi(-\infty) = 1, \quad \phi(x) = \sigma, & 0 \leq x < +\infty.
\end{cases} \tag{2.3}$$

We will show that (2.3) has a solution $\phi_\sigma$, which converges to the unique solution of (1.3) as $\sigma \to 0$.

If $\phi$ solves (2.3), then clearly, for $x < 0$,

$$-c\phi'(x) = d \int_{-\infty}^{0} J(x-y)\phi(y)dy + d \int_{0}^{+\infty} J(x-y)\phi(x)dy - d\phi(x) + c\phi'(x) + f(\phi(x)).$$

Choose $M > 0$ large so that

$$u \mapsto \tilde{f}(u) := (cM - d)u + f(u) \text{ is increasing for } u \in [0, 1],$$

and denote

$$a(x) = \int_{0}^{+\infty} J(x-y)dy = \int_{-\infty}^{x} J(y)dy.$$
Then for $x < 0$,

$$-c(e^{-Mx} \phi)' = e^{-Mx} \left[ d \int_{-\infty}^{0} J(x-y) \phi(y) dy + d\sigma a(x) + \tilde{f}(\phi(x)) \right],$$

and hence

$$\phi(x) = e^{Mx} \phi + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ d \int_{-\infty}^{0} J(\xi-y) \phi(y) dy + d\sigma a(\xi) + \tilde{f}(\phi(\xi)) \right] d\xi.$$  

We now define an operator $A$ over

$$\Omega := \{ \phi \in C(\mathbb{R}) : 0 \leq \phi \leq 1 \}$$

by

$$A[\phi](x) = \begin{cases} e^{Mx} \phi + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ d \int_{-\infty}^{0} J(\xi-y) \phi(y) dy + d\sigma a(x) + \tilde{f}(\phi(\xi)) \right] d\xi, & x < 0, \\ \sigma, & x \geq 0. \end{cases}$$

Then $\phi \in \Omega$ solves (2.3) if and only if $\phi$ is a fixed point of $A$ in $\Omega$.

Let $\phi_*$ denote the traveling wave solution with minimal speed $c_*$ given by Proposition 1.3. Clearly $\phi_* \in \Omega$, and by a suitable translation we may assume that

$$\phi_*(0) = \sigma.$$  

Then define

$$\tilde{\phi}_*(x) = \max \{ \phi_*(x), \sigma \} = \begin{cases} \phi_*(x), & x < 0, \\ \sigma, & x \geq 0. \end{cases}$$

We show next that

$$(2.4) \quad A[\tilde{\phi}_*](x) \geq \tilde{\phi}_*(x), \quad A[1](x) < 1 \text{ for } x \in \mathbb{R}.$$  

Evidently $A[\tilde{\phi}_*](x) = \tilde{\phi}_*(x) = \sigma$ for $x \geq 0$. For $x < 0$, we have

$$A[\tilde{\phi}_*](x) = e^{Mx} \sigma + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ d \int_{-\infty}^{0} J(\xi-y) \tilde{\phi}_*(y) dy + d\sigma a(x) + \tilde{f}(\tilde{\phi}_*(\xi)) \right] d\xi$$

$$\geq e^{Mx} \sigma + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ d \int_{-\infty}^{0} J(\xi-y) \tilde{\phi}_*(y) dy + \tilde{f}(\tilde{\phi}_*(\xi)) \right] d\xi$$

$$= e^{Mx} \sigma + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ cM \tilde{\phi}_*(\xi) - c^* \tilde{\phi}_*(\xi) \right] d\xi$$

$$> e^{Mx} \sigma + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ cM \tilde{\phi}_*(\xi) - c \tilde{\phi}_*(\xi) \right] d\xi$$

$$= e^{Mx} \sigma - e^{Mx} \int_{x}^{0} e^{-M\xi} \tilde{\phi}_*(\xi)' d\xi = \tilde{\phi}_*(x).$$

Therefore the first inequality in (2.4) holds. To prove the second inequality, again we only need to check it for $x < 0$, where we have

$$A[1](x) = e^{Mx} \sigma + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ d \int_{-\infty}^{0} J(\xi-y) dy + d\sigma a(x) + \tilde{f}(1) \right] d\xi$$

$$< e^{Mx} \sigma + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} \left[ d \int_{-\infty}^{+\infty} J(\xi-y) dy + \tilde{f}(1) \right] d\xi$$

$$= e^{Mx} \sigma + \frac{e^{Mx}}{c} \int_{x}^{0} e^{-M\xi} cM d\xi$$

$$= 1 + (\sigma - 1)e^{Mx} < 1.$$

This proves (2.4).
We now define inductively
\[ \phi_0(x) = \tilde{\phi}(x), \quad \phi_{n+1}(x) = A[\phi_n](x) = A^n[\tilde{\phi}](x), \quad n = 0, 1, 2, \ldots, x \in \mathbb{R}. \]
The monotonicity of \( \tilde{f} \) implies that the operator \( A \) is monotone increasing, namely
\[ \phi, \tilde{\phi} \in \Omega \text{ and } \phi \leq \tilde{\phi} \text{ imply } A[\phi](x) \leq A[\tilde{\phi}](x). \]
Using this property of \( A \) and (2.4) we obtain
\[ \phi_0(x) \leq \phi_n(x) \leq \phi_{n+1}(x) < 1 \text{ for } n = 1, 2, \ldots, x \in \mathbb{R}. \]
We now define
\[ \phi_\sigma(x) := \lim_{n \to \infty} \phi_n(x). \]
Clearly \( \phi_\sigma(x) = \sigma \) for \( x \geq 0 \), and for \( x < 0 \), by the Lebesque dominated convergence theorem, we deduce from \( \phi_{n+1}(x) = A[\phi_n](x) \) that
\[ \phi_\sigma(x) = A[\phi_\sigma](x). \]
Since \( \tilde{\phi}(x) = \phi_0(x) \leq \phi_\sigma(x) \leq 1 \) and \( \phi_0(-\infty) = 1 \), we necessarily have \( \phi_\sigma(-\infty) = 1 \). From the expression of \( A[\phi_\sigma](x) \) and \( \phi_\sigma(x) = A[\phi_\sigma](x) \), we see that \( \phi'_\sigma(x) \) exists and is continuous for \( x < 0 \), and hence \( \phi = \phi_\sigma \) satisfies (2.3).

We have thus proved the following conclusion.

**Lemma 2.1.** Suppose that (J2) holds. Then for any \( \sigma \in (0, 1) \) and \( c \in (0, c_*) \), (2.3) has a solution \( \phi_\sigma \), which can be obtained by an iteration process.

We show next that the \( \phi_\sigma(x) \) obtained in this way is nonincreasing in \( x \) and nondecreasing in \( \sigma \).

**Lemma 2.2.** \( \phi_\sigma(x) \geq \phi_\sigma(y) \) if \( x \leq y \); \( \phi_{\sigma_1}(x) \leq \phi_{\sigma_2}(x) \) if \( 0 < \sigma_1 \leq \sigma_2 < 1 \).

**Proof.** To show the monotonicity in \( x \), it suffices to show that for any \( \delta > 0 \),
\[ \phi_\sigma(x - \delta) \geq \phi_\sigma(x). \]
Clearly we have
\[ \tilde{\phi}_\sigma(x - \delta) := \max \{ \phi_\sigma(x - \delta), \sigma \} \geq \tilde{\phi}_\sigma(x). \]
The monotonicity of \( A \) then infers
\[ A^n[\tilde{\phi}_\sigma(x + \delta)](x) \geq A^n[\tilde{\phi}_\sigma](x). \]
Letting \( n \to \infty \) we obtain
\[ \phi_\sigma(x - \delta) \geq \phi_\sigma(x), \]
as we wanted.

To show the monotonicity in \( \sigma \), assume that \( 0 < \sigma_1 < \sigma_2 < 1 \). If \( \tilde{\phi}_\sigma(0) = \sigma_1 \), then there exists a unique \( x_0 < 0 \) such that \( \tilde{\phi}_\sigma(x_0) = \sigma_2 \). We thus have
\[ \phi_{\sigma_1}(x) = \lim_{n \to \infty} A^n[\tilde{\phi}_\sigma](x), \quad \phi_{\sigma_2}(x) = \lim_{n \to \infty} A^n[\tilde{\phi}_\sigma(x + x_0)](x). \]
Since \( \tilde{\phi}_\sigma(x) \leq \tilde{\phi}_\sigma(x + x_0) \), it follows from the monotonicity of \( A \) that \( A^n[\tilde{\phi}_\sigma](x) \leq A^n[\tilde{\phi}_\sigma(x + x_0)](x) \) for all \( n \geq 1 \). Hence
\[ \phi_{\sigma_1}(x) \leq \phi_{\sigma_2}(x), \]
as desired. \( \square \)

**Remark 2.3.** For later use we observe that if \( 0 < c_1 < c_2 < c_* \) and \( \phi_\sigma^i \) denotes the solution obtained from the iteration process with \( c = c_i, i = 1, 2 \), then \( \phi_\sigma^1 \geq \phi_\sigma^2 \), since the operator \( A \) over \( \Omega \) is nonincreasing in \( c \) by its definition.
2.2. Existence of semi-waves and spatial monotonicity. In this subsection, we make use of $\phi_\sigma$ obtained above to prove the following result.

**Theorem 2.4.** Suppose that (J2) holds. Then for every $c \in (0, c_*)$, there exists a semi-wave $\phi^c$ to the problem \((1.3)\) with speed $c$, and $\phi^c(x)$ is nonincreasing for $x \in (-\infty, 0]$.

*Proof.* Let $\sigma_n$ be a decreasing sequence in $(0, 1/2)$ satisfying $\sigma_n \to 0$ as $n \to \infty$. Then due to the monotonicity of $\phi_{\sigma_n}(x)$ in $x$, and the fact that $\phi_{\sigma_n}(-\infty) = 1$, $\phi_{\sigma_n}(0) = \sigma_n \in (0, 1/2)$, there exists a unique $\tilde{x}_n < 0$ such that

$$\phi_{\sigma_n}(\tilde{x}_n) = 1/2, \quad \phi_{\sigma_n}(x) < 1/2 \text{ for } x > \tilde{x}_n.$$  

By Lemma 2.2 we easily deduce $\tilde{x}_m \leq \tilde{x}_n$ for $m > n$. Set

$$\tilde{\phi}_n(x) = \phi_{\sigma_n}(x + \tilde{x}_n) \text{ for } x < -\tilde{x}_n.$$ 

Then $\tilde{\phi}_n$ satisfies, for $x < -\tilde{x}_n$

$$d \int_{-\infty}^{-\tilde{x}_n} J(x - y)\tilde{\phi}_n(y)dy + d \int_{-\tilde{x}_n}^{+\infty} J(x - y)\sigma_n dy - d\tilde{\phi}_n + c\tilde{\phi}_n' + f(\tilde{\phi}_n) = 0. \tag{2.5}$$

In view of $\tilde{x}_m \leq \tilde{x}_n < 0$ for $m > n$, there are two possible cases:

- Case 1. $-\tilde{x}_n \to +\infty$ as $n \to +\infty$.
- Case 2. $-\tilde{x}_n \to x_0$ as $n \to +\infty$ for some $x_0 \in (0, +\infty)$.

Since $\tilde{\phi}_n$ and by the equation subsequently $\tilde{\phi}_n$ are uniformly bounded, by the Arzela-Ascoli Theorem and a standard argument involving a diagonal process of choosing subsequences, there exist $\tilde{\phi}_\infty \in C(\mathbb{R})$ and a subsequence of $\{\tilde{\phi}_n\}_{n \geq 1}$, still denoted by $\{\tilde{\phi}_n\}_{n \geq 1}$, such that $\tilde{\phi}_n$ converges to $\tilde{\phi}_\infty$ locally uniformly in $\mathbb{R}$. (Here we extend $\tilde{\phi}_n(x)$ by 0 for $x \geq -\tilde{x}_n$.) Moreover, $\tilde{\phi}_\infty(x)$ is nonincreasing in $x$, and $\tilde{\phi}_\infty(0) = \frac{1}{2}$.

Moreover, if Case 1 happens, we can verify that $\tilde{\phi}_\infty$ satisfies

$$d \int_{-\infty}^{\infty} J(x - y)\tilde{\phi}_\infty(y)dy - d\tilde{\phi}_\infty + c\tilde{\phi}_\infty' + f(\tilde{\phi}_\infty) = 0. \tag{2.6}$$

Indeed, by (2.5), we have

$$c \left( \tilde{\phi}_n(x) - \frac{1}{2} \right) = -d \int_0^x \int_{-\infty}^{-\tilde{x}_n} J(z - y)\tilde{\phi}_n(y)dydz - d \int_x^{\infty} \int_{-\tilde{x}_n}^{+\infty} J(z - y)\sigma_n dydz + d \int_0^x \tilde{\phi}_n(z)dz - \int_0^x f(\tilde{\phi}_n(z))dz. \tag{2.7}$$

Fix $x \in \mathbb{R}$; by the dominated convergence theorem, one easily sees that by letting $n \to \infty$, the above equation yields

$$c \left( \tilde{\phi}_\infty(x) - \frac{1}{2} \right) = -d \int_0^x \int_{-\infty}^{+\infty} J(z - y)\tilde{\phi}_\infty(y)dydz \quad \quad +d \int_0^x \tilde{\phi}_\infty(z)dz - \int_0^x f(\tilde{\phi}_\infty(z))dz,$$

and thus (2.6) follows by differentiating this equation. However, (2.6) contradicts to the fact that $c_*$ is the minimal speed. Hence Case 1 cannot happen.

Therefore Case 2 must happen. Similarly fix $x \in \mathbb{R}$ and let $n \to \infty$ in (2.7); we obtain

$$c \left( \tilde{\phi}_\infty(x) - \frac{1}{2} \right) = -d \int_0^x \int_{-\infty}^{+x_0} J(z - y)\tilde{\phi}_\infty(y)dydz \quad \quad +d \int_0^x \tilde{\phi}_\infty(z)dz - \int_0^x f(\tilde{\phi}_\infty(z))dz,$$
which yields
\[
\begin{cases}
    d \int_{-\infty}^{x_0} J(x - y) \tilde{\phi}_\infty(y) dy - d\tilde{\phi}_\infty + c\tilde{\phi}'_\infty + f(\tilde{\phi}_\infty) = 0, & x < x_0, \\
    \tilde{\phi}_\infty(x_0) = 0.
\end{cases}
\]

Set \( \phi(x) = \tilde{\phi}_\infty(x + x_0) \); then \( \phi(x) \) satisfies
\[
\begin{cases}
    d \int_{-\infty}^{0} J(x - y) \phi(y) dy - d\phi + c\phi' + f(\phi) = 0, & x < 0, \\
    \phi(-\infty) = 1, \phi(0) = 0.
\end{cases}
\]

Since \( \tilde{\phi}_\infty(x) \in [0, 1] \) and is monotone in \( x \) with \( \tilde{\phi}_\infty(0) = 1/2 \), it follows that \( \phi(x) \) is nonincreasing in \( x < 0 \) and \( \lim_{x \to -\infty} \phi(x) \in [1/2, 1] \). The above equation and the property of \( f \) then imply that \( \phi(-\infty) = 1 \).

2.3. Uniqueness of semi-wave and its monotonicity. Fix \( c \in (0, c_*), \) and suppose that \( \phi_i, i = 1, 2, \) are nonnegative solutions of (1E3) with speed \( c \). Then
\[
\begin{cases}
    d \int_{-\infty}^{0} J(x - y) \phi_i(y) dy - d\phi_i + c\phi'_i + f(\phi_i) = 0, & x < 0, \\
    \phi_i(-\infty) = 1, \phi_i(0) = 0,
\end{cases}
\]
or equivalently
\[
\begin{cases}
    d(J * \phi_i)(x) - d\phi_i + c\phi'_i + f(\phi_i) = 0 & x < 0, \\
    \phi_i(-\infty) = 1, \phi_i(x) = 0, & x \geq 0.
\end{cases}
\]

To prove the uniqueness, it suffices to show that \( \phi_1 \equiv \phi_2 \). We first prove a strong maximum principle for later use. We remark that in the following lemma and the uniqueness proof, only condition (J) for the kernel function \( J \) is needed.

**Lemma 2.5.** Assume that \( w \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \) satisfies
\[
\begin{cases}
    d(J * w)(x) - dw + a(x)w' + b(x)w \leq 0, & x < 0, \\
    w(x) \geq 0, & x \geq 0,
\end{cases}
\]
where \( d \) is a positive constant, \( J \) satisfies (J), and \( a, b \in L^\infty_{loc}(\mathbb{R}) \). If \( w(x) \geq 0 \) and \( w(x) \not\equiv 0 \), then \( w(x) > 0 \) for \( x < 0 \).

**Proof.** Suppose that there exists \( x_0 < 0 \) such that \( w(x_0) = 0 \). Then \( w'(x_0) = 0 \) and it follows from the differential-integral inequality satisfied by \( w \) that at \( x = x_0 \),
\[
d(J * w)(x_0) \leq 0,
\]
which indicates that \( w(y) = 0 \) when \( y \) is close to \( x_0 \). This implies that \( w(x) \equiv 0 \) when \( x < 0 \), since \( \{x < 0 \mid w(x) = 0\} \) is now both open and closed.

We are now ready to show \( \phi_1 \equiv \phi_2 \). Similar to (23), for small \( \epsilon > 0 \), define
\[
K_\epsilon = \{k \geq 1 : k\phi_1(x) \geq \phi_2(x) - \epsilon \text{ for } x \leq 0\}.
\]
\( K_\epsilon \neq \emptyset \) since \( \phi_1(-\infty) = 1, \phi_1(0) = 0, i = 1, 2 \). Set
\[
k_\epsilon = \inf K_\epsilon \geq 1.
\]
It is clear that \( k_\epsilon \) is decreasing in \( \epsilon \) and thus we may define
\[
k^* = \lim_{\epsilon \to 0^+} k_\epsilon \in [1, +\infty].
\]
From the equation satisfied by \( \phi_i, i = 1, 2 \), we deduce
\[
\phi_i'(0^-) = \lim_{x \to 0^-} \frac{\phi_i(x) - \phi_i(0)}{x} = \lim_{x \to 0^-} \frac{1}{cx} \left[ -d \int_0^x \int_{-\infty}^0 J(z-y)\phi_i(y)dydz + d \int_0^x \phi_i(z)dz - \int_0^x f(\phi_i(z))dz \right] = -\frac{d}{c} \int_{-\infty}^0 J(0-y)\phi_i(y)dy = \lim_{x \to 0^-} \phi_i'(x) < 0.
\]
This implies that \( k^* < +\infty \). We also have
\[
k^* \phi_1(x) \geq \phi_2(x), \quad x \leq 0.
\]
We claim that \( k^* = 1 \). Otherwise, suppose that \( k^* > 1 \) and thus for \( \epsilon > 0 \) small, \( k_\epsilon > 1 \). Since
\[
k_\epsilon \phi_1(0) - \phi_2(0) + \epsilon = \epsilon > 0 \quad \text{and} \quad \lim_{x \to -\infty} k_\epsilon \phi_1(x) - \phi_2(x) = k_\epsilon - 1 + \epsilon > 0,
\]
by the definition of \( k_\epsilon \), there exists \( x_\epsilon \in (-\infty, 0) \) such that
\[
(2.8) \quad k_\epsilon \phi_1(x_\epsilon) - \phi_2(x_\epsilon) + \epsilon = 0.
\]
Now there are three possible cases:
- Case (i): \( x_\epsilon_n \to -\infty \) along some sequence \( \epsilon_n \to 0^+ \).
- Case (ii): \( x_\epsilon_n \to 0 \) along some sequence \( \epsilon_n \to 0^+ \).
- Case (iii): \( x_\epsilon_n \to x^* \in (-\infty, 0) \) along some sequence \( \epsilon_n \to 0^+ \).

In Case (i), from (2.8) we obtain
\[
0 = \lim_{\epsilon_n \to 0^+} (k_{\epsilon_n} \phi_1(x_{\epsilon_n}) - \phi_2(x_{\epsilon_n}) + \epsilon_n) = k^* - 1 > 0,
\]
which is impossible. Hence Case (i) leads to a contradiction.

Next, we consider Cases (ii) and (iii). Define
\[
w_\epsilon(x) = k_\epsilon \phi_1(x) - \phi_2(x) + \epsilon, \quad w^*(x) = k^* \phi_1(x) - \phi_2(x).
\]
Then
\[
w_\epsilon(x_\epsilon) = 0, \quad w_\epsilon(x) \geq 0, \quad w^*(x) \geq 0 \quad \text{for} \ x \in (-\infty, 0].
\]
Moreover, \( w^* \) satisfies
\[
d(J * w^*)(x) - dw^* + cw^*_x + k^* f(\phi_1) - f(\phi_2) = 0.
\]

In Case (ii), it follows from (2.8) that \( w^*(0) = 0 \). Moreover, at \( x = x_{\epsilon_n}, w_\epsilon'(x_{\epsilon_n}) = 0 \), i.e.,
\[
k_{\epsilon_n} \phi_1'(x_{\epsilon_n}) = \phi_2'(x_{\epsilon_n}).
\]
Then by letting \( \epsilon_n \to 0^+ \), one has \( k^* \phi_1'(0^-) = \phi_2'(0^-) \), and so \( w^*_x(0^-) = 0 \).

On the other hand, from the equation satisfied by \( w^* \) and the assumption (f3), one sees that for \( x < 0 \),
\[
0 = d(J * w^*)(x) - dw^* + cw^*_x + k^* f(\phi_1) - f(\phi_2) \\
\geq d(J * w^*)(x) - dw^* + cw^*_x + f(k^* \phi_1) - f(\phi_2) \\
= d(J * w^*)(x) - dw^* + cw^*_x + b(x)w^*,
\]
where the assumption \( k^* > 1 \) is used, and
\[
b(x) := \begin{cases} \frac{f(k^* \phi_1) - f(\phi_2)}{k^* \phi_1 - \phi_2}, & \text{if } k^* \phi_1 - \phi_2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}
\]
We thus obtain, by letting \( x \to 0^- \),
\[
(2.9) \quad d(J * w^*)(0) \leq 0.
\]
Since $k^* > 1$, obviously $w^* \neq 0$. Then by Lemma 2.5, $w > 0$ for $x < 0$. This is a contradiction to (2.9). Therefore, Case (ii) also leads to a contradiction.

In Case (iii), similar to the arguments in Case (ii), a contradiction can be derived at $x = x^*$. Since every possible case leads to a contradiction, we conclude that $k^* = 1$ must happen, which means $\phi_1 \geq \phi_2$. Similarly, we can show $\phi_2 \geq \phi_1$. Therefore $\phi_1 \equiv \phi_2$ and the uniqueness of the semi-wave (if exists) is verified.

We will now on assume additoinally (J2) is satisfied and denote the unique solution of (1.3) by $\phi^c(x)$. We are ready to consider the monotonicity of $\phi^c(x)$ in $x$ and in $c$, respectively.

If $\delta > 0$, then by Theorem 2.4 $w(x) = \phi^c(x - \delta) - \phi^c(x) \geq 0$. Applying Lemma 2.5 to $\phi^c$ we see that $\phi^c(x) > 0$ for $x < 0$. It follows that $w(x) \neq 0$. We may now apply Lemma 2.5 to $w$ to conclude that $w(x) > 0$ for $x < 0$. This proves the strict monotonicity of $\phi^c(x)$ in $x$ for $x \in (-\infty, 0]$.

We show next that $\phi^{c_1}(x) > \phi^{c_2}(x)$ for $x < 0$ if $0 < c_1 < c_2 < c_*$. By Remark 2.3 and the proof of Theorem 2.4, we see that for such $c_1$ and $c_2$, $w(x) := \phi^{c_1}(x) - \phi^{c_2}(x)$ is nonnegative. Moreover, $(\phi^{c_1})_x \leq 0$ for $x < 0$. Thus $\phi^{c_1}$ satisfies

$$\begin{cases} d(J * \phi^{c_1})(x) - d\phi^{c_1} + c_2(\phi^{c_1})_x + f(\phi^{c_1}) \leq 0, & x < 0, \\ \phi^{c_1}(-\infty) = 1, & \phi^{c_1}(x) = 0, \quad x \geq 0. \end{cases}$$

We may now apply Lemma 2.5 to $w(x) = \phi^{c_1}(x) - \phi^{c_2}(x)$ to conclude that either $w(x) > 0$ for $x < 0$ or $w(x) \equiv 0$. If the latter happens, then the above inequality for $\phi^{c_1}$ becomes an equality, which implies that $\phi^{c_1} \equiv 0$. But this is a contradiction to $\phi^{c_1}(-\infty) = 1 > \phi^{c_1}(0) = 0$. Therefore $w(x) > 0$ for $x < 0$ and the strict monotonicity of $\phi^c(x)$ in $c$ is proved.

Summarizing, we have proved the following result.

**Theorem 2.6.** Suppose that (J), (J2) and (f3) hold. Then for any $c \in (0, c_*)$, the problem (1.3) has a unique solution $\phi = \phi^c$, and $\phi^c(x)$ is strictly decreasing in $c \in (0, c_*)$ for fixed $x < 0$, and is strictly decreasing in $x \in (-\infty, 0]$ for fixed $c \in (0, c_*)$.

We conclude this section with the following theorem, which uniquely determines the spreading speed $c_0$.

**Theorem 2.7.** Suppose that (J), (J2) and (f3) hold. Then the unique semi-wave $\phi^c(x)$ satisfies

$$\lim_{c \to c_*} \phi^c(x) = 0 \text{ locally uniformly in } x \in (-\infty, 0].$$

Moreover, for any $\mu > 0$, there exists a unique $c = c_0 = c_0(\mu) \in (0, c_*)$ such that

$$c_0 = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x - y)\phi^{c_0}(x)dydx.$$  \hspace{1cm} (2.10)

**Proof.** Let $c_n$ be an arbitrary increasing sequence in $(0, c_*)$ converging to $c_*$ as $n \to \infty$. Denote $\phi_n(x) := \phi^{c_n}(x)$. Then $\phi_n(x)$ is uniformly bounded, and from the equation satisfied by $\phi_n$ we see that $\phi_n'(x)$ is also uniformly bounded. Therefore we can find a subsequence of $\phi_n$, still denoted by itself, such that $\phi_n(x) \to \phi(x)$ in $C_{loc}((-\infty, 0])$ as $n \to \infty$. As in the proof of Theorem 2.4, we can verify that $\phi$ satisfies

$$\begin{cases} d \int_{-\infty}^{0} J(x - y)\phi(y)dy - d\phi + c_\star \phi_x + f(\phi) = 0, & x < 0, \\ \phi(0) = 0. \end{cases}$$

Clearly we also have $0 \leq \phi(x) < \phi_n(x)$ for $x \leq 0$. We extend $\phi(x)$ by $0$ for $x > 0$. 

Fix $k \in (0, 1)$ and define $\tilde{\phi}(x) := k\phi(x)$. Then by (f3) we obtain $f(k\phi) \leq kf(\phi)$ and hence

\[
\begin{cases}
d(J * \tilde{\phi})(x) - d\tilde{\phi}(x) + c_s \tilde{\phi}'(x) + f(\tilde{\phi}(x)) \geq 0, & x < 0, \\
\tilde{\phi}(x) = 0, & x \geq 0.
\end{cases}
\]

For any $\eta > 0$ define $\phi^\eta_*(x) = \phi_*(x - \eta)$. Then

$$\phi^\eta_*(x) \geq \phi_*(-\eta) \text{ for } x \leq 0.$$ 

Since $\phi_*(-\infty) = 1$ and $\tilde{\phi}(x) \leq k < 1$, we find that for all large $\eta > 0$,

$$w^\eta(x) := \phi^\eta_*(x) - \tilde{\phi}(x) \geq 0 \text{ for } x \leq 0.$$ 

Hence we can define

$$\eta_* := \inf \{\xi \in \mathbb{R} : w^\eta(x) \geq 0 \text{ for } x \leq 0 \text{ and all } \eta \geq \xi\}.$$ 

If $\eta_* = -\infty$, then $\tilde{\phi}(x) \leq \phi_*(x - \eta)$ for all $\eta \in \mathbb{R}$. Letting $\eta \to -\infty$ and recalling $\phi_*(+\infty) = 0$ we immediately obtain $\tilde{\phi}(x) \leq 0$, which implies $\phi(x) \equiv 0$.

If $\eta_* > -\infty$, then

$$w^{\eta_*}(x) \geq 0 \text{ for } x \leq 0,$$

and since $w^{\eta_*}(-\infty) \geq 1 - k > 0$ and $w^{\eta_*}(0) = \phi_*(-\eta_*) > 0$, the definition of $\eta_*$ indicates that there exists $x_* \in (-\infty, 0)$ such that

$$w^{\eta_*}(x_*) = 0.$$ 

From

$$d\int_{-\infty}^{+\infty} J(x - y)\phi^{\eta_*}_*(y)dy - d\phi^{\eta_*}_* + c_s(\phi^{\eta_*}_*)' + f(\phi^{\eta_*}) = 0 \text{ for } x \in \mathbb{R},$$

we obtain

$$\begin{cases}
d(J * \phi^{\eta_*}_*)(x) - d\phi^{\eta_*}_*(x) + c_s(\phi^{\eta_*}_*)'(x) + f(\phi^{\eta_*}(x)) = 0, & x < 0, \\
\phi^{\eta_*}_*(x) > 0, & x \geq 0.
\end{cases}$$

Therefore we can apply Lemma 2.5 to $w^{\eta_*}$ to conclude that $w^{\eta_*}(x) > 0$ for $x < 0$, which is a contradiction to $w^{\eta_*}(x_*) = 0$. Therefore $\eta_* > -\infty$ cannot occur and we always have $\phi(x) \equiv 0$.

Since $c_0$ is an arbitrary increasing sequence converging to $c_*$, this implies that (2.10) holds.

It remains to prove (2.11). For $c \in (0, c_*)$ define

$$M(c) := \mu \int_{-\infty}^{0} \int_0^{\infty} J(x - y)\phi^c(x)dydx.$$ 

The monotonicity of $\phi^c(x)$ in $c$ indicates that $M(c)$ is strictly decreasing in $c$. Due to the uniqueness of $\phi^c$, one may use a similar argument to that used to show the convergence of $\phi_0(x)$ above to deduce that $\phi^c(x)$ is continuous in $c$ uniformly for $x$ in any bounded set of $(-\infty, 0]$. It follows that $M(c)$ is continuous in $c$. Now we consider the function $c \mapsto c - M(c)$ for $c \in (0, c_*)$. Clearly it is continuous and is strictly increasing. By (2.10) and the dominated convergence theorem, we see that as $c \to c_*$, $c - M(c) \to c_* > 0$. For all small $c > 0$, $c - M(c) \leq c - M(c_*)/2 < 0$. Therefore there exists a unique $c = c_0 \in (0, c_*)$ such that $c_0 - M(c_0) = 0$, i.e., (2.11) holds.$\square$
2.4. Semi-wave and condition (J1). In the previous subsection we have proved that when \( f \) satisfies (f3) and \( J \) satisfies (J) and (J2), then (1.3)-(2.1) has a unique solution pair \((c, \phi) = (c_0, \phi^{c_0})\), with \( \phi^{c_0}(x) \) decreasing in \( x \). Now we show that under condition (J), such a pair \((c_0, \phi^{c_0})\) exists if and only if (J1) holds.

**Theorem 2.8.** Suppose that \( J \) satisfies (J) and \( f \) satisfies (f3). Then (1.3)-(2.1) has a solution pair \((c, \phi) = (c_0, \phi^{c_0})\) with \( \phi^{c_0}(x) \) nonincreasing in \( x \) if and only if (J1) holds. Moreover, when (J1) holds, such a pair is unique and \( c_0 > 0 \), \( \phi^{c_0}(x) \) is strictly decreasing in \( x \).

We prove Theorem 2.8 (which is a restatement of Theorem 1.2) by two lemmas.

**Lemma 2.9.** Suppose that (J) and (J1) hold. Then (1.3)-(2.1) has a unique solution pair \((c, \phi) = (c_0, \phi^{c_0})\), and \( c_0 > 0 \), \( \phi^{c_0}(x) \) is strictly decreasing in \( x \).

**Proof.** We only need to consider the case that (J2) does not hold. Let \( J_n(x) \) be a sequence satisfying (J) and (J2) such that

\[
\lim_{n \to \infty} J_n(x) = J(x) \text{ locally uniformly in } \mathbb{R}
\]

and

\[
\lim_{n \to \infty} \int_{-\infty}^{0} \int_{0}^{\infty} |J_n(x - y) - J(x - y)|dydx = 0.
\]

Such a sequence can be easily obtained by letting \( J_n(x) = J(x)\xi_n(x) \) with \( \xi_n(x) \) a suitable sequence of smooth cut-off functions.

Since (J2) is satisfied by \( J_n \), from Proposition 2.1 we obtain a minimal wave speed \( c_* = c_*^n > 0 \). We must have \( \lim_{n \to \infty} c_*^n = +\infty \), for otherwise by passing to a subsequence we may assume \( \lim_{n \to \infty} c_*^n = c_*^\infty \in [0, +\infty) \), and then by a similar argument \(^2\) to the proof of Theorem 2.4, we can show that the Fisher-KPP equation in Proposition 1.3 with kernel function \( J \) satisfying (J) but not (J2) has a traveling wave with speed \( c_*^\infty \), which is a contradiction to the second part of the conclusion in that proposition. Therefore, for any fixed \( c > 0 \) and all large \( n \), we have \( 0 < c < c_*^n \) and so (1.3) with \( J \) replaced by \( J_n \) has a unique solution \( \phi = \phi^{c_n} \). Moreover, we may argue as in the proof of Theorem 2.4 to conclude that \( \phi^{c_n}_n(x) \to \hat{\phi}(x) \) and for some \( x_0 > 0 \), \( \phi^{c_n}(x) := \hat{\phi}(x - x_0) \) satisfies (1.3). The monotonicity of \( \phi^{c_n}(x) \) in \( x \) and in \( c \) then implies that \( \phi^{c_n}(x) \) is nonincreasing in \( x \in (-\infty, 0] \) for fixed \( c > 0 \), and nonincreasing in \( c \in (0, +\infty) \) for fixed \( x < 0 \).

We may now use Lemma 2.5 as in the proof of Theorem 2.6 where the property (J2) is not needed, to conclude that all the conclusions for \( \phi^{c_n} \) in Theorem 2.6 still hold for the current \( \phi^{c_n} \).

By Theorem 2.4, for each \( n \), (1.3)-(2.1) with \( J \) replaced by \( J_n \) has a unique solution pair \((c_{0,n}, \phi^{c_{0,n}})\). By (2.2), we have

\[
c_{0,n} \in (0, \mu c(J_n)) \quad \text{with } c(J_n) \to c(J) > 0 \text{ as } n \to \infty.
\]

Therefore, by passing to a subsequence we may assume that

\[
c_{0,n} \to c_0 \in [0, \mu c(J)] \text{ as } n \to \infty.
\]

If \( c_0 > 0 \), then we may argue as in the proof of Theorem 2.4 to conclude that \( \phi^{c_{0,n}}(x) \to \hat{\phi}(x) \) and for some \( x_0 > 0 \), \( (c, \phi(x)) = (c_0, \hat{\phi}(x - x_0)) \) satisfies (1.3)-(2.1). As the conclusions in Theorem 2.6 still hold we necessarily have \( \hat{\phi}(x - x_0) = \phi^{c_0}(x) \). Moreover, \( c_0 > 0 \) is the unique \( c \) such that (2.1) holds.

If \( c_0 = 0 \), we show that a contradiction occurs. For convenience, we denote

\[
\phi_n = \phi^{c_{0,n}}, \quad c_n = c_{0,n}.
\]

\(^2\)The case \( c_*^\infty = 0 \) has to be proved differently, as in the last part of the proof of this lemma.
Then \( \lim_{n \to \infty} c_n = 0 \) and
\[
\begin{cases}
\int_{-\infty}^{0} J_n(x-y)\phi_n(y)dy - d\phi_n + c_n\phi_n' + f(\phi_n) = 0, \quad \phi_n' < 0, \quad \text{for } x < 0, \\
\phi_n(0) = 0, \quad \phi_n(-\infty) = 1, \\
\mu \int_{-\infty}^{0} \int_{0}^{\infty} J_n(x-y)\phi_n(x)dydx = c_n.
\end{cases}
\]
Using \( c_n \to 0 \) we easily see that
\[
\lim_{n \to \infty} \int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)\phi_n(x)dydx = 0,
\]
and hence, due to the monotonicity of each \( \phi_n \), and the assumption that \( J \) does not satisfy \((J2)\) (and so it does not have compact support), we obtain
\[
\lim_{n \to \infty} \phi_n(x) = 0 \text{ uniformly on every bounded interval in } (-\infty, 0].
\]
Choose \( x_n < 0 \) such that \( \phi_n(x_n) = 1/2 \). Then \( x_n \to -\infty \) as \( n \to \infty \). We now define
\[
\tilde{\phi}_n(x) := \phi_n(x + x_n).
\]
Then
\[
\begin{cases}
\int_{-\infty}^{x} J_n(x-y)\tilde{\phi}_n(y)dy - d\tilde{\phi}_n + c_n\tilde{\phi}_n' + f(\tilde{\phi}_n) = 0, \quad \tilde{\phi}_n' < 0, \quad \text{for } x < -x_n, \\
\tilde{\phi}_n(0) = 1/2, \quad \tilde{\phi}_n(-\infty) = 1.
\end{cases}
\]
Since \( \tilde{\phi}_n' < 0 \) and \( 0 \leq \tilde{\phi}_n \leq 1 \), by Helly’s theorem, \( \{\tilde{\phi}_n\} \) has a subsequence, which for convenience we still denote by itself, such that, as \( n \to \infty \), \( \tilde{\phi}_n(x) \to \tilde{\phi}(x) \) for almost every \( x \in \mathbb{R} \). Clearly \( \tilde{\phi}(x) \) is nonincreasing and \( \tilde{\phi}(0) = 1/2 \).

From the above equations for \( \tilde{\phi}_n \), we obtain, for any \( z \in \mathbb{R} \) and all large \( n \),
\[
d \int_{0}^{z} \int_{-\infty}^{-x_n} J_n(x-y)\tilde{\phi}_n(y)dydx - d \int_{0}^{z} \tilde{\phi}_n(x)dx + c_n\tilde{\phi}_n(z) - c_n/2 + \int_{0}^{z} f(\tilde{\phi}_n(x))dx = 0.
\]
Letting \( n \to \infty \) and making use of the dominated convergence theorem, we deduce
\[
d \int_{0}^{z} \int_{-\infty}^{\infty} J(x-y)\tilde{\phi}(y)dydx - d \int_{0}^{z} \tilde{\phi}(x)dx + \int_{0}^{z} f(\tilde{\phi}(x))dx = 0.
\]
Since \( z \in \mathbb{R} \) is arbitrary, this implies that
\[
d \int_{-\infty}^{\infty} J(x-y)\tilde{\phi}(y)dy - d\tilde{\phi}(x) + f(\tilde{\phi}(x)) = 0 \text{ for a.e. } x \in \mathbb{R}.
\]
But this implies that \( \tilde{\phi} \) is a traveling wave with speed \( c = 0 \), a contradiction to Proposition 1.3, since we have assumed that \( J \) does not satisfy \((J2)\). Thus we have proved that \( c_0 = 0 \) cannot happen, and the proof is complete. \( \square \)

**Lemma 2.10.** Suppose that \((J)\) holds and \((1.1)\) has a solution pair \((c, \phi) = (c_0, \phi^{c_0})\) with \( \phi^{c_0}(x) \) nonincreasing in \( x \). Then \( J \) satisfies \((J1)\).

**Proof.** Since \( \phi^{c_0}(x) \) is nonincreasing in \( x \), we have
\[
c_0 = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)\phi^{c_0}(x)dydx \geq \mu \int_{-\infty}^{0} \int_{0}^{c_0}(x)dydx.
\]
Thus
\[
\int_{-\infty}^{c_0} \int_{0}^{\infty} J(x-y)dydx = \int_{-\infty}^{c_0} \int_{0}^{a(x)}dx < +\infty.
\]
Since \( a(x) \) is continuous, clearly
\[
\int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)dydx = \int_{-\infty}^{-1} a(x)dx + \int_{-1}^{0} a(x)dx < +\infty.
\]
Hence (J1) holds. \( \square \)

3. Spreading speed of (1.1)

Suppose that \( f \) satisfies (F3) and \( J \) satisfies (J) and (J1). Then there exists a unique pair \((c_0, \phi^{c_0})\) satisfying (1.3)-(2.1). Let \((u, g, h)\) be the unique solution of (1.1) and suppose that spreading happens, that is
\[
\lim_{t \to \infty} h(t) = -\lim_{t \to \infty} g(t) = \infty, \quad \text{and} \quad \lim_{t \to \infty} u(t, x) = 1 \text{ locally uniformly in } x \in \mathbb{R}.
\]
We are going to show that
\[
\lim_{t \to \infty} \frac{h(t)}{t} = - \lim_{t \to \infty} \frac{g(t)}{t} = c_0,
\]
which is part (i) of Theorem 1.1.

It suffices to show the conclusion for \( h(t) \), as \( \tilde{u}(t, x) := u(t, -x) \) satisfies (1.1) with free boundaries \( x = \tilde{h}(t) := -g(t), \ x = \tilde{g}(t) := -h(t) \) and initial function \( \tilde{u}_0(x) := u_0(-x) \).

**Lemma 3.1.** Under the above assumptions, we have
\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq c_0.
\]

**Proof.** For any given \( \epsilon > 0 \) we define
\[
\delta := 2\epsilon c_0, \quad \overline{h}(t) := (c_0 + \delta)t + L, \quad \overline{u}(t, x) := (1 + \epsilon)\phi^{c_0}(x - \overline{h}(t)),
\]
with \( L > 0 \) to be determined. A simple comparison argument with the ODE problem
\[
v' = f(v), \ v(0) = \|u_0\|_{\infty}
\]
shows that \( u(t, x) \leq v(t) \) and hence
\[
\limsup_{t \to \infty} u(t, x) \leq 1 \text{ uniformly for } x \in [g(t), h(t)].
\]
Thus there exists \( T > 0 \) large so that
\[
u(T + t, x) \leq 1 + \frac{\epsilon}{2} \text{ for } t \geq 0, \ x \in [g(T + t), h(T + t)].
\]
Since \( \phi^{c_0}(-\infty) = 1 \), we may choose \( L > 0 \) large such that \( \overline{h}(0) = L > h(T) \) and
(3.1) \[
\overline{u}(0, x) = (1 + \epsilon)\phi^{c_0}(x - L) > 1 + \frac{\epsilon}{2} \geq u(T, x) \text{ for } x \in [g(T), h(T)],
\]
We show next that
(3.2) \[
\overline{u}_t \geq d \int_{g(T)}^{\overline{h}(t)} J(x-y)\overline{u}(t, y)dy - d\overline{u}(t, x) + f(\overline{u}(t, x))
\]
for \( t > 0 \) and \( x \in [g(T), \overline{h}(t)] \), and
(3.3) \[
\overline{h}(t) > \mu \int_{g(T)}^{\overline{h}(t)} \int_{\overline{h}(t)}^{\infty} J(x-y)\overline{u}(t, x)dydx \text{ for } t > 0.
\]
Indeed,
\[
\overline{u}_t = -(1 + \epsilon)(c_0 + \delta)(\phi^{c_0})'(x - \overline{h}(t)) > -(1 + \epsilon)c_0(\phi^{c_0})'(x - \overline{h}(t))
\]
\[
= (1 + \epsilon) \left[ d \int_{-\infty}^{\overline{h}(t)} J(x-y)\phi^{c_0}(y - \overline{h}(t))dy - d\phi^{c_0}(x - \overline{h}(t)) + f(\phi^{c_0}(x - \overline{h}(t)) \right]
\]
= \int_{-\infty}^{\tilde{\eta}(t)} J(x-y)\overline{\eta}(t,y)dy - d\overline{\eta}(t,x) + (1 + \epsilon)f(\phi^{\epsilon_0}(x-\tilde{\eta}(t)))
\geq \int_{-\infty}^{\tilde{\eta}(t)} J(x-y)\overline{\eta}(t,y)dy - d\overline{\eta}(t,x) + f(\overline{\eta}(t,x))
\geq \int_{g(t+T)}^{\tilde{\eta}(t)} J(x-y)\overline{\eta}(t,y)dy - d\overline{\eta}(t,x) + f(\overline{\eta}(t,x))

for t > 0 and x < \tilde{\eta}(t), where we have used (F3). This proves (3.2).

To show (3.3) we calculate
\[
\int_{-\infty}^{\tilde{\eta}(t)} J(x-y)\overline{\eta}(t,y)dy
\leq \int_{-\infty}^{\tilde{\eta}(t)} J(x-y)\overline{\eta}(t,y)dy
\leq \mu(1 + \epsilon)\int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)\phi^{\epsilon_0}(x)dydx
= (1 + \epsilon)c_0 < c_0 + \delta = \tilde{\eta}(t).
\]

Thus (3.3) holds.

We are now ready to show that
\[ h(t + T) < \tilde{\eta}(t) \text{ and } u(t + T, x) < \overline{\eta}(t, x) \text{ for } t > 0, x \in [g(t + T), h(t + T)]. \]

By (3.1) and \( \tilde{\eta}(0) > h(T) \), we see that the above inequalities hold for \( t > 0 \) small. If the above inequalities do not hold for all \( t > 0 \), then there is a first time moment \( t^* > 0 \) such that at least one of them is violated at \( t = t^* \), i.e., the above inequalities hold for \( t \in (0, t^*) \), and

(i) \( h(t^* + T) = \tilde{\eta}(t^*) \), or

(ii) \( h(t^* + T) < \tilde{\eta}(t^*) \) and \( u(t^* + T, x^*) = \overline{\eta}(t^*, x^*) \) for some \( x^* \in [g(t^* + T), h(t^* + T)]. \)

If (i) happens, then necessarily \( h'(t^* + T) \geq \tilde{\eta}'(t^*) \). On the other hand,
\[
\tilde{\eta}'(t^*) > \mu\int_{g(t^* + T)}^{\tilde{\eta}(t^*)} \int_{\tilde{\eta}(t^*)}^{\infty} J(x-y)\overline{\eta}(t^*,x)dydx
\leq \mu\int_{g(t^* + T)}^{\tilde{\eta}(t^*)} \int_{\tilde{\eta}(t^*)}^{\infty} J(x-y)\overline{\eta}(t^*,x)dydx
\leq \mu\int_{g(t^* + T)}^{\tilde{\eta}(t^*)} \int_{h(t^* + T)}^{\infty} J(x-y)u(t^* + T, x)dydx
= h'(t^* + T),
\]
where we have used
\[ u(t^* + T, x) \leq \overline{\eta}(t^*, x) \text{ for } x \in [g(t^* + T), h(t^* + T)]. \]

Thus (i) leads to a contradiction.

If (ii) happens, then due to \( \overline{\eta}(t, x) > 0 \) for \( x \in [g(t + T), h(t + T)] \) for \( t \in (0, t^*) \), and \( \overline{\eta}(0, x) > u(T, x) \) for \( x \in [g(T), h(T)] \), we can use the comparison principle in [8] to conclude that
\[ \overline{\eta}(t^*, x) > u(t^* + T, x) \text{ for } x \in [g(t + T), h(t + T)], \]
and so we again reach a contradiction. Therefore (3.4) holds, and
\[ \limsup_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{h(t - T)}{t} = c_0 + \delta = c_0 + 2\epsilon c_0. \]
Letting \( \epsilon \to 0 \), we immediately obtain \( \limsup_{t \to \infty} h(t)/t \leq c_0 \).

**Lemma 3.2.** Under the assumptions of Lemma 3.1, we have
\[ \liminf_{t \to \infty} \frac{h(t)}{t} \geq c_0. \]

**Proof.** Since \( f'(1) < 0 \), there exists \( \delta_0 > 0 \) small such that \( f'(u) < 0 \) for \( u \in [1 - \delta_0, 1] \). For any given \( \epsilon \in (0, \sigma_0] \), we define
\[ \delta := 2\epsilon c_0, \quad h(t) := (c_0 - \delta)t + L \quad \text{and} \]
\[ u(t, x) := (1 - \epsilon) \left[ \phi^c_0(x - h(t)) + \phi^c_0(-x - h(t)) - 1 \right], \]
with \( L > 0 \) a large constant to be determined. Clearly, for \( t \geq 0 \),
\[ 0 > u(t, \pm h(t)) = (1 - \epsilon) \left[ \phi^c_0(-2h(t)) - 1 \right] \geq (1 - \epsilon) \left[ \phi^c_0(-2L) - 1 \right] \to 0 \text{ as } L \to \infty. \]
We show next that, if \( L \) is chosen large enough, then
\[ h'(t) \leq \mu \int_{-h(t)}^{h(t)} \int_{h(t)}^\infty J(x - y) u(t, x) dy dx \quad \text{for } t > 0, \]
and
\[ u_t \leq d \int_{-h(t)}^{h(t)} J(x - y) u(t, y) dy - du + f(u) \quad \text{for } x \in (-h(t), h(t)), \quad t > 0. \]

To show (3.5), we calculate
\[
\begin{align*}
\mu \int_{-h(t)}^{h(t)} \int_{h(t)}^\infty J(x - y) u(t, x) dy dx & \quad \\
& = \mu(1 - \epsilon) \int_{-2h(t)}^{0} \int_{0}^{\infty} J(x - y) \phi^c_0(x) dy dx \\
& \quad + \mu(1 - \epsilon) \int_{-2h(t)}^{0} \int_{0}^{\infty} J(x - y) \phi^c_0(-x - 2h(t)) - 1 \right) dy dx \\
& = (1 - \epsilon)c_0 - \mu(1 - \epsilon) \int_{-\infty}^{-2h(t)} \int_{0}^{\infty} J(x - y) \phi^c_0(x) dy dx \\
& \quad - \mu(1 - \epsilon) \int_{-2h(t)}^{0} \int_{0}^{\infty} J(x - y) \left[ 1 - \phi^c_0(-x - 2h(t)) \right] dy dx.
\end{align*}
\]
By (J1), we have, for all \( t \geq 0 \),
\[
0 \leq \mu(1 - \epsilon) \int_{-\infty}^{-2h(t)} \int_{0}^{\infty} J(x - y) \phi^c_0(x) dy dx \\
\quad \leq \mu(1 - \epsilon) \int_{-\infty}^{-2L} \int_{0}^{\infty} J(x - y) dy dx < \frac{1}{4} \epsilon c_0
\]
provided that \( L \) is large enough, say \( L \geq L_1 \). Moreover,
\[
0 \leq \mu(1 - \epsilon) \int_{-2h(t)}^{0} \int_{0}^{\infty} J(x - y) \left[ 1 - \phi^c_0(-x - 2h(t)) \right] dy dx \\
\quad \leq \mu(1 - \epsilon) \int_{-\infty}^{0} \int_{0}^{\infty} J(x - y) \left[ 1 - \phi^c_0(-x - 2h(t)) \right] dy dx
\]
Secondly we fix several constants for later use. Due to (3.8), which implies, in particular, this proves (3.5).

Next we prove (3.6). Firstly we need to extend \( f(u) \) by defining
\[
f(u) = f'(0)u \quad \text{for} \quad u < 0.
\]

Secondly we fix several constants for later use. Due to \( f(1) = 0 \) and \( f'(u) < 0 \) for \( u \in [1 - \epsilon, 1] \), we can choose \( \hat{\epsilon} > 0 \) small enough such that
\[
2(1 - \epsilon) f(1 - \epsilon) < f(1 - \epsilon) \quad \text{and} \quad f'(u) < 0 \quad \text{for} \quad u \in [(1 - \epsilon)(1 - \hat{\epsilon}), 1].
\]

Then using \( \phi(x) = 1 \) we can find \( M > 0 \) large enough such that
\[
\phi(x) > 1 - \frac{\hat{\epsilon}}{2},
\]
which implies, in particular,
\[
(3.8) \quad \phi(x - h(t)), \phi(x - h(t)) \in (1 - \frac{\hat{\epsilon}}{2}, 1) \quad \text{for} \quad x \in [-h(t) + M, h(t) - M].
\]

Define
\[
\epsilon_0 := \inf_{x \in [-M, 0]} |(\phi(x))'| > 0;
\]
then clearly
\[
(3.9) \quad \begin{cases}
(\phi(x))'|(x - h(t))| \leq -\epsilon_0 & \text{for} \quad x \in [h(t) - M, h(t)]; \\
(\phi(x))'|(-x - h(t))| \leq -\epsilon_0 & \text{for} \quad x \in [-h(t), -h(t) + M].
\end{cases}
\]

Finally we set
\[
M_0 := \max_{u \in [0,1]} |f'(u)|, \quad \hat{\epsilon} := \frac{1 - \epsilon}{2M_0} \delta \epsilon_0.
\]

To simplify notations, in the following we write \( \phi = \phi(x) \). We have
\[
u_0 = -(1 - \epsilon)(c_0 - \delta) \left[ \phi(x - h(t)) + \phi(-x - h(t)) \right]
\]
\[
\begin{align*}
&= (1 - \epsilon)\delta \left[ \phi'(x - h(t)) + \phi'(-x - h(t)) \right] \\
&\quad + (1 - \epsilon) \left[ \int_{-\infty}^{-h(t)} J(x - h(t) - y) \phi(y) dy - \phi(-x - h(t)) \right] \\
&\quad + (1 - \epsilon) \left[ \int_{-\infty}^{-h(t)} J(-x - h(t) - y) \phi(y) dy - \phi(-x - h(t)) \right] \\
&= (1 - \epsilon)\delta \left[ \phi'(x - h(t)) + \phi'(-x - h(t)) \right] \\
&\quad + (1 - \epsilon) \left[ \int_{-h(t)}^{0} J(x - h(t) - y) \phi(y) dy - \phi(x - h(t)) \right] \\
&\quad + (1 - \epsilon) \left[ \int_{-h(t)}^{0} J(-x - h(t) - y) \phi(y) dy - \phi(-x - h(t)) \right].
\end{align*}
\]
\[
+ d \int_{-\hat{h}(t)}^{\hat{h}(t)} J(-x + y)\phi(-y - \hat{h}(t))dy - d\phi(-x - \hat{h}(t))
\]

\[
+ (1 - \epsilon) \left[ f(\phi(x - \hat{h}(t))) + f(\phi(-x - \hat{h}(t))) \right]
\]
\[
= (1 - \epsilon) \delta \left[ \phi'(x - \hat{h}(t)) + \phi'(-x - \hat{h}(t)) \right]
\]
\[
+ d \int_{-\hat{h}(t)}^{\hat{h}(t)} J(x - y)u(t, y)dy - du(t, x)
\]
\[
+ (1 - \epsilon)d \left[ \int_{-\hat{h}(t)}^{\hat{h}(t)} J(x - y)[\phi(y - \hat{h}(t)) - 1]dy + \int_{\hat{h}(t)}^{\infty} J(x - y)[\phi(-y - \hat{h}(t)) - 1]dy \right]
\]
\[
+ (1 - \epsilon) \left[ f(\phi(x - \hat{h}(t))) + f(\phi(-x - \hat{h}(t))) \right]
\]
\[
\leq d \int_{-\hat{h}(t)}^{\hat{h}(t)} J(x - y)u(t, y)dy - du(t, x)
\]
\[
+ (1 - \epsilon)d \left[ \phi'(x - \hat{h}(t)) + \phi'(-x - \hat{h}(t)) \right] + (1 - \epsilon) \left[ f(\phi(x - \hat{h}(t))) + f(\phi(-x - \hat{h}(t))) \right]
\]
\[
= d \int_{-\hat{h}(t)}^{\hat{h}(t)} J(x - y)u(t, y)dy - du(t, x) + f(u(t, x)) + \delta(t, x)
\]

with
\[
\delta(t, x) : = (1 - \epsilon) \delta \left[ \phi'(x - \hat{h}(t)) + \phi'(-x - \hat{h}(t)) \right]
\]
\[
+ (1 - \epsilon) \left[ f(\phi(x - \hat{h}(t))) + f(\phi(-x - \hat{h}(t))) \right] - f(u(t, x)).
\]

To prove (3.6), it suffices to show that
\[
(3.10) \quad \delta(t, x) \leq 0 \text{ for } x \in [-\hat{h}(t), \hat{h}(t)], \ t \geq 0.
\]

We start by checking the case \( x \in [\hat{h}(t) - M, \hat{h}(t)] \) and \( t \geq 0 \). For such \( x \) and \( t \), we have
\[
0 > \phi(-x - \hat{h}(t)) - 1 \geq \phi(-2\hat{h}(t) + M) - 1 \geq \phi(-2L + M) - 1 \geq -\epsilon
\]
provided that \( L \) is large enough, say \( L \geq L_3 \geq L_2 \). It follows that
\[
f(u(t, x)) \geq f(1 - \epsilon)\phi(x - \hat{h}(t)) - M_0(1 - \epsilon)\epsilon,
\]
\[
f(\phi(-x - \hat{h}(t))) = f(\phi(-x - \hat{h}(t))) - f(1) \leq M_0\epsilon,
\]
and hence, by (3.9) and (f3),
\[
\delta(t, x) \leq -(1 - \epsilon)\delta\epsilon_0 + (1 - \epsilon)[f(\phi(x - \hat{h}(t))) + M_0\epsilon]
\]
\[
- f((1 - \epsilon)\phi(x - \hat{h}(t))) + M_0(1 - \epsilon)\epsilon
\]
\[
\leq -(1 - \epsilon)\delta\epsilon_0 + 2(1 - \epsilon)M_0\epsilon < 0.
\]

Since \( \delta(t, -x) = \delta(t, x) \), the above inequality also holds for \( x \in [-\hat{h}(t), -\hat{h}(t) + M] \) and \( t \geq 0 \).

It remains to check the case \( x \in [-\hat{h}(t) + M, \hat{h}(t) - M] \) and \( t \geq 0 \). Now (3.8) holds and so
\[
u(t, x) \in [(1 - \epsilon)(1 - \epsilon), 1 - \epsilon].
\]

Since \( f(u) \) is decreasing for \( u \in [(1 - \epsilon)(1 - \epsilon), 1 - \epsilon] \), it follows that, for such \( x \) and \( t \),
\[
\delta(t, x) < (1 - \epsilon)[f(1 - \frac{\epsilon}{2}) + f(1 - \frac{\epsilon}{2})] - f(1 - \epsilon) < 0
\]
due to (3.7). Thus (3.10) holds. This proves (3.6).
Since \( J(-x) = J(x) \) and \( u(t, -x) = u(t, x) \), from (3.5) we easily deduce
\[
(3.11) \quad -h'(t) \geq - \mu \int_{-h(t)}^{h(t)} \int_{-\infty}^{\infty} J(x-y)g(t,y)dydx \quad \text{for} \ t > 0.
\]

We are now ready to compare \((u, g, h)\) with \((u, \frac{-h}{\mu}, \frac{h}{\mu})\) by a comparison argument. Since spreading happens for \((u, g, h)\), there exists \( T > 0 \) large enough such that
\[
g(T) < -L = -h(0), \ h(T) > L = h(0), \ u(T, x) > 1 - \epsilon > \underline{u}(0, x) \quad \text{for} \ x \in [-L, L].
\]

In view of (3.5), (3.6) and (3.11), we may now use the lower solution version of Theorem 3.1 in [8] to deduce
\[
g(T + t) \leq -h(t), \ h(t + T) \geq h(t) \quad \text{and} \quad u(t + T, x) \geq \underline{u}(t, x) \quad \text{for} \ t > 0, \ x \in [-h(t), h(t)].
\]

In particular,
\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq \frac{\lim_{t \to \infty} \frac{h(t) - T}{t}}{c_0 - \delta} = (1 - 2\epsilon) c_0.
\]

Letting \( \epsilon \to 0 \) we obtain \( \liminf_{t \to \infty} \frac{h(t)}{t} \geq c_0 \). This completes the proof. \( \square \)

4. ACCELERATING SPREADING OF (1.1)

In this section, we prove part (ii) of Theorem [1.1] So throughout this section, we assume that (f3) and (J) hold, but (J1) is not satisfied. Moreover, we assume that \((u, g, h)\) is the unique solution of (1.1), and spreading happens, namely
\[
\lim_{t \to \infty} h(t) = -\lim_{t \to \infty} g(t) = \infty, \ \lim_{t \to \infty} u(t, x) = 1 \text{ locally uniformly for} \ x \in \mathbb{R}.
\]

We firstly choose a sequence \( \{J_n(x)\} \) such that each \( J_n(x) \) is nonnegative, continuous, even, has nonempty compact support, and
\[
J_n(x) \leq J_{n+1}(x) \leq J(x) \quad \text{for all} \ n \geq 1, \ x \in \mathbb{R}, \ J_n(x) \to J(x) \text{ in} \ L^1(\mathbb{R}).
\]

Such a sequence can be easily constructed by defining \( J_n(x) = J(x) \xi_n(x) \), with \( \xi_n(x) \) a suitable sequence of smooth cut-off functions. We then consider the auxiliary problem which is obtained by replacing \( J \) by \( J_n \) in (1.1), namely
\[
\begin{cases}
\begin{aligned}
\frac{u_t}{d} & = \int_{g(t)}^{h(t)} J_n(x-y)u(t,y)dy - du(t, x) + f(u), \quad t > 0, \ x \in (g(t), h(t)), \\
u(t, g(t)) & = u(t, h(t)) = 0, \quad t > 0, \\
h'(t) & = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J_n(x-y)u(t,x)dydx, \quad t > 0, \\
g'(t) & = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_n(x-y)u(t,x)dydx, \quad t > 0, \\
u(0, x) & = u_0(x), \ h(0) = -g(0) = h_0, \quad x \in [-h_0, h_0], \\
\end{aligned}
\end{cases}
\]

Lemma 4.1. For every large \( n \), problem (4.1) has a unique positive solution \((u_n, g_n, h_n)\) defined for all \( t > 0 \). Moreover,
\[
h(t) \geq h_{n+1}(t) \geq h_n(t), \ g(t) \leq g_{n+1}(t) \leq g_n(t) \quad \text{for all} \ t > 0, \ n \geq 1.
\]

Proof. Define \( \sigma_n := \int_{\mathbb{R}} J_n(x)dx \). Then \( \sigma_n \in (0, 1) \), is nondecreasing in \( n \), and \( \lim_{n \to \infty} \sigma_n = 1 \). Set
\[
\tilde{J}_n(x) := \frac{1}{\sigma_n} J_n(x).
\]
Clearly $\tilde{J}_n$ satisfies (J). Moreover, since $\tilde{J}_n$ has compact support, it also satisfies (J1) (and (J2) as well). Set

$$f_n(u) := f(u) - (1 - \sigma_n)u.$$  

Then at least for all large $n$, $f_n$ satisfies (F3) except that $f(1) = 0 > f'(1)$ should be replaced by $f_n(\eta_0) = 0 > f'(\eta_0)$ for some uniquely determined $\eta_0 \in (0, 1)$, and $\lim_{n \to \infty} \eta_0 = 1$.

We may now rewrite (4.1) in the equivalent form

$$\left\{ \begin{array}{l}
u_t = d\sigma_n \int_{g(t)}^{h(t)} \tilde{J}_n(x-y)u(t,y)dy - d\sigma_n u(t,x) + f_n(u), \quad t > 0, \ x \in (g(t), h(t)), \\
u(t, g(t)) = u(t, h(t)) = 0, \quad t > 0, \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \
as \( n \to \infty \). Therefore we can find \( n_0 > 0 \) large enough so that
\[
\lambda_p(\mathcal{L}_n f'_n(0)) = \lambda_p(\mathcal{L}_n f(0)) > f'(0)/4 \quad \text{for all } n \geq n_0.
\]
Using this fact we see from the proof of Theorem 1.3 in [3] that spreading happens to (4.2) with
\( n \geq n_0 \), provided that there exists \( T \geq 0 \) such that
\[
(4.3) \quad h_n(T) - g_n(T) > 2\ell_0.
\]
We show next that there exists \( T > 0 \) large so that (4.3) holds for all large \( n \), say \( n \geq n_1 \geq n_0 \). Indeed, since spreading happens to (1.1) by assumption, we can find \( T > 0 \) large such that \( h(T) - g(T) > 4\ell_0 \).

Since \((d\sigma_n, J_n, f_n) \to (d, J, f)\) as \( n \to \infty \) in the obvious sense, by the continuous dependence of the solution of (1.1) on \((d, J, f)\) (which follows easily from the uniqueness of the solution), we see that, as \( n \to \infty \), the solution \((u_n, g_n, h_n)\) of (4.2) converges to the solution \((u, g, h)\) of (1.1) over any bounded time interval; in particular,
\[
g_n(t) \to g(t), \quad h_n(t) \to h(t) \quad \text{uniformly for } t \in [0, T].
\]
It follows that
\[
h_n(T) - g_n(T) > \frac{1}{2}[h(T) - g(T)] > 2\ell_0 \quad \text{for all large } n.
\]
This proves (4.3) and hence spreading happens to \((u_n, g_n, h_n)\) for every large \( n \).

Since each \( \bar{J}_n \) satisfies (J) and (J1), we are in a position to apply part (i) of Theorem 1.1 to (4.2) to conclude that
\[
(4.4) \quad \lim_{t \to \infty} \frac{h_n(t)}{t} = \lim_{t \to \infty} \frac{-g_n(t)}{t} = c_n
\]
for every large \( n \), and \( c_n > 0 \) is determined by the following two equations
\[
(4.5) \quad \begin{cases}
  d \int_{-\infty}^{0} J_n(x-y) \phi(y)dy - d\phi(x) + c_n\phi'(x) + f(\phi(x)) = 0, & -\infty < x < 0, \\
  \phi(-\infty) = \eta_n, \quad \phi(0) = 0,
\end{cases}
\]
and
\[
(4.6) \quad c_n = \mu \int_{-\infty}^{0} \int_{0}^{+\infty} J_n(x-y)\phi(x)dydx,
\]
namely \( c_n > 0 \) is the unique value such that (4.5) and (4.6) have a solution \( \phi = \phi_n \in C^1((-\infty, 0]) \) which is strictly decreasing in \( x \). This last fact is a consequence of Theorem 1.2 applied to (4.2), but with the corresponding equations of (1.3) and (1.4), which now involve \((d\sigma_n, \bar{J}_n, f_n)\), rewritten in terms of \((d, J_n, f)\), much as in the equivalent form (4.1) of (4.2).

By Lemma 4.1 and (4.4), we have \( c_n \leq c_{n+1} \) and
\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq c_n, \quad \liminf_{t \to \infty} \frac{-g(t)}{t} \geq c_n \quad \text{for all large } n.
\]
Therefore part (ii) of Theorem 1.1 is a consequence of the following lemma.

**Lemma 4.3.** \( \lim_{n \to \infty} c_n = \infty. \)

**Proof.** Arguing indirectly we assume that the conclusion of the lemma does not hold. Then the nondecreasing positive sequence \( c_n \) must converge to some positive constant \( c_\infty \) as \( n \to \infty \). We show next that this leads to a contradiction.

We note that (4.5) and (4.6) have a solution \( \phi_n \) which is strictly decreasing. So we are in a position to argue as in the proof of Theorem 2.1 (with simple minor changes) to conclude that, either (4.6) has a nonincreasing solution \( \phi(x) \) for \( c = c_\infty \), or (4.3) and (4.4) have a solution pair
(c, \phi) with c = c_\infty and \phi(x) nonincreasing in x. Since J does not satisfy (J1) and hence also does not satisfy (J2), we have a contradiction to either Proposition 1.3 or Theorem 1.2. This completes the proof. \qed

5. LIMITING PROFILE AS \( \mu \to +\infty \)

In this section, we discuss the convergence of the semi-wave pair \((c, \phi)\) when \(\mu \to +\infty\) under the conditions \((J)\), \((J1)\) and \((F3)\). We will also show that \((1.5)\) can be viewed as the limiting problem of \((1.1)\) as \(\mu \to +\infty\). For this latter conclusion, we only require \((J)\) and \((f1)-(f2)\).

It is obvious that the semi-wave pair \((c, \phi)\) depends on \(\mu\). To stress this dependence, we will denote it by \((c_\mu, \phi_\mu)\) throughout this section. For each fixed \(\mu > 0\), we know that \(\phi_\mu(x)\) is strictly decreasing in \((-\infty, 0]\), and hence there exists a unique \(l_\mu > 0\) such that \(\phi_\mu(-l_\mu) = \frac{1}{2}\).

Define
\[
\hat{\phi}_\mu(x) := \phi_\mu(x - l_\mu) \text{ for } x \leq l_\mu, \text{ and so } \hat{\phi}_\mu(0) = \frac{1}{2}.
\]
Let us also define \(\phi_\mu(x) = 0\) for \(x > 0\).

Firstly, we consider the case that condition \((J2)\) holds.

**Theorem 5.1.** Suppose that \((J)\), \((J2)\) and \((F3)\) are satisfied. Then, as \(\mu \to +\infty\),
\[
c_\mu \to c_*, \quad l_\mu \to +\infty, \quad \phi_\mu(x) \to 0 \quad \text{and} \quad \hat{\phi}_\mu(x) \to \phi_*(x) \text{ locally uniformly in } \mathbb{R},
\]
where \((c_*, \phi_*)\) is the minimal speed solution pair of \((1.6)\) with \(\phi_*(0) = 1/2\).

**Proof.** Choose \(u_0\) such that spreading happens to \((1.1)\), and let \((u_\mu, g_\mu, h_\mu)\) denote the unique solution of \((1.1)\). Let \(u\) denote the unique solution of \((1.5)\) with the same initial function \(u_0\) (extended by 0 outside \([-h_0, h_0]\)). Then the comparison principle infers that \(u_\mu(t, x) \leq u(t, x)\) for \(t > 0, x \in [g(t), h(t)]\). From the proof of Lemma 3.2 we see that \(u_\mu(t, x) \geq \underline{u}(t - T, x)\) for \(x \in [-\overline{h}(t - T), \overline{h}(t - T)]\) for all \(t > T\). Therefore
\[
u(t, x) \geq \underline{u}(t - T, x) \text{ for } x \in [-\overline{h}(t - T), \overline{h}(t - T)] \text{ for all } t > T.
\]
From this and \((1.7)\) we easily deduce \(c_\mu \leq c_*\) for all \(\mu > 0\). By the comparison principle, \(h_\mu(t)\) is increasing in \(\mu\), which implies that \(c_\mu\) is nondecreasing in \(\mu\). Therefore
\[
c_\infty := \lim_{\mu \to +\infty} c_\mu \text{ exists, and } c_\infty \leq c_*.
\]

We are now ready to show that \(\lim_{\mu \to +\infty} l_\mu = +\infty\). Indeed, since
\[
0 \leq \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)\phi_\mu(x)dydx = \frac{c_\mu}{\mu} \leq \frac{c_*}{\mu},
\]
and \(\phi_\mu(x)\) is strictly decreasing in \(x\), in the case that \(J\) does not have compact support, we must have \(\lim_{\mu \to +\infty} \phi_\mu(x) \to 0\) locally uniformly in \((-\infty, 0]\), which immediately implies \(l_\mu \to +\infty\). If \(J\) has compact support, and \(L := \inf\{x > 0 : J(x) = 0\}\), then \((6.1)\) implies
\[
\lim_{\mu \to +\infty} \phi_\mu(x) = 0 \text{ uniformly for } x \in [-L, 0].
\]
We show that in this case we also have \(\lim_{\mu \to +\infty} \phi_\mu(x) \to 0\) locally uniformly in \((-\infty, 0]\). Indeed, from the equation \((1.3)\) and the monotonicity of \(c_\mu > 0\), it is easily seen that \(\phi'_\mu\) is uniformly bounded for \(\mu > 1\) and \(x \in (-\infty, 0]\). Therefore, for any sequence \(\mu_n \to +\infty\), \(\phi_{\mu_n}\) has a subsequence, still denoted by itself for convenience of notation, such that \(\phi_{\mu_n}\) converges to some \(\phi_\infty\) locally uniformly in \((-\infty, 0]\). Moreover, \(\phi_\infty(x)\) is nonincreasing in \(x\), is \(C^1\) for \(x \leq 0\), and satisfies
\[
d \int_{-\infty}^{0} J(x-y)\phi_\infty(y)dy - d\phi_\infty(x) + c_\infty \phi'_\infty(x) + f(\phi_\infty(x)) = 0 \text{ for } x \leq 0.
\]
It suffices to show that $\phi_\infty \equiv 0$. From (5.2) we have $\phi_\infty(x) = 0$ for $x \in [-L, 0]$. If $\phi_\infty \neq 0$, then by its monotonicity there exists $L_0 \leq -L$ such that

$$\phi_\infty(x) = 0 \text{ in } [L_0, 0], \quad \phi_\infty(x) > 0 \text{ in } (-\infty, L_0).$$

It follows that $\phi'_\infty(L_0) = 0$. On the other hand, from (5.3) and the definition of $L_0$ we also have

$$c_\infty \phi'_\infty(L_0) = -d \int_0^\infty J(L_0 - y) \phi_\infty(y) dy < 0.$$

This contradiction shows that we must have $\phi_\infty \equiv 0$ and hence we always have $\lim_{\mu \to +\infty} l_\mu = +\infty$.

We may now use the equation (1.3) and the monotonicity of $c_\mu > 0$ to see that $\hat{\phi}_\mu'$ is uniformly bounded for $\mu > 1$ and $x \in (-\infty, l_\mu]$. Repeating the argument of the last paragraph we can conclude that, for any sequence $\mu_n \to +\infty$, $\hat{\phi}_{\mu_n}$ has a subsequence, still denoted by itself, such that $\hat{\phi}_{\mu_n}$ converges to some $\hat{\phi}_\infty$ locally uniformly in $\mathbb{R}$, and $\hat{\phi}_\infty(x)$ is nonincreasing in $x$, $\hat{\phi}_\infty(0) = \frac{1}{2}$, and

$$d \int_\mathbb{R} J(x - y) \hat{\phi}_\infty(y) dy - d\phi_\infty(x) + c_\infty \phi'_\infty(x) + f(\hat{\phi}_\infty(x)) = 0 \text{ for } x \in \mathbb{R}.$$

Obviously, $\hat{\phi}_\infty \neq 1/2$, for otherwise $\hat{\phi}_\infty$ does not satisfy (5.4). Since 0 and 1 are the only nonnegative zeros of $f$ under (F3), we necessarily have $\hat{\phi}_\infty(-\infty) = 1$ and $\hat{\phi}_\infty(+\infty) = 0$. The strong maximum principle then infers $\hat{\phi}_\infty > 0$ in $\mathbb{R}$ (for example, we may apply Lemma 2.5 with $x < 0$ replaced by $x < l$ for any $l > 0$). Therefore, $(c_\infty, \hat{\phi}_\infty)$ is a solution of (1.6). Since $c_\infty$ is the minimal speed of (1.3) and $c_\infty \leq c_*$, we necessarily have $c_* = c_\infty$ and $\phi_* = \hat{\phi}_\infty$. The uniqueness of $\phi_*$ implies that $\hat{\phi}_\mu$ converges to $\phi_*$ locally uniformly in $\mathbb{R}$ as $\mu \to +\infty$. \hfill \qed

Next, let us consider the case (J1) holds but (J2) does not.

**Theorem 5.2.** Suppose that (J), (J1) and (F3) are satisfied but (J2) does not hold. Then $c_\mu \to +\infty$ as $\mu \to +\infty$.

**Proof.** As before, the comparison principle implies that $c_\mu$ increases in $\mu$, and so we can define $c_\infty := \lim_{\mu \to +\infty} c_\mu \in (0, +\infty]$. If $c_\infty < +\infty$, then we can repeat the argument in the proof of Theorem 5.1 to conclude that (5.4) has a solution pair $(c_\infty, \hat{\phi}_\infty)$ with $\hat{\phi}_\infty > 0$, $\hat{\phi}_\infty(-\infty) = 1$, $\hat{\phi}_\infty(+\infty) = 0$ and $\hat{\phi}_\infty(0) = 1/2$. Thus $(c_\infty, \hat{\phi}_\infty)$ is a solution of (1.6) with finite speed, which is a contradiction with the assumption that (J2) is not satisfied. Therefore we necessarily have $c_\infty = +\infty$. \hfill \qed

Finally, let us discuss the relationship between the solution of (1.1) and that of (1.5).

**Theorem 5.3.** Suppose that (J) and (F1)-(F2) are satisfied. Let $(u_\mu, g_\mu, h_\mu)$ be the solution of (1.1) for $\mu > 0$ with initial datum $u_0$, and $u_*$ be the solution of (1.5) with the same initial datum (extended by zero outside $[-h_0, h_0]$). Then $-g_\mu, h_\mu \to +\infty$ locally uniformly in $(0, +\infty)$ and $u_\mu \to u_*$ locally uniformly in $\mathbb{R}^+ \times \mathbb{R}$ as $\mu \to +\infty$.

**Proof.** By the comparison principle we know that $-g_\mu, h_\mu$ and $u_\mu$ are increasing in $\mu$. Therefore for each $t > 0$,

$$g_\infty(t) := \lim_{\mu \to +\infty} g_\mu(t) \in (-\infty, -h_0), \quad h_\infty(t) := \lim_{\mu \to +\infty} h_\mu(t) \in (h_0, +\infty],$$

$$\text{and } u_\infty(t, x) := \lim_{\mu \to +\infty} u_\mu(t, x), \quad g_\infty(t) < x < h_\infty(t), \quad t > 0,$$

are well-defined. Moreover,

$$0 < u_\infty(t, x) \leq M_0 := \max\{|u_0|_\infty, K_0\} \text{ for } t > 0, \quad g_\infty(t) < x < h_\infty(t),$$
By the uniqueness of solutions to (1.5), we must have
\[ u \in \text{to obtain} \]
Then letting \( t \to \infty \) we deduce
\[ \lim_{t \to \infty} u_\infty(t, x) = 0 = u_0(x) \] uniformly for \( x \in \mathbb{R} \setminus [-h_0, h_0] \).
By the uniqueness of solutions to (1.5), we must have \( u_\infty \equiv u_* \).
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