Attractiveness of Invariant Manifolds✩

Lijun Pei ,✩✩a

aDepartment of Mathematics, Zhengzhou University, 450001 Zhengzhou, Henan, China

Abstract

In this paper an operable, universal and simple theory on the attractiveness of the invariant manifolds is first obtained. It is motivated by the Lyapunov direct method. It means that for any point $\vec{x}$ in the invariant manifold $M$, $n(\vec{x})$ is the normal passing by $\vec{x}$, and $\forall x \in n(\vec{x})$, if the tangent $f(x)$ of the orbits of the dynamical system intersects at obtuse (sharp) angle with the normal $n(\vec{x})$, or the inner product of the normal vector $n(\vec{x})$ and tangent vector $f(x)$ is negative (positive), i.e., $\vec{n}(\vec{x}).\vec{f}(\vec{x}) < (>)0$, then the invariant manifold $M$ is attractive (repulsive). Some illustrative examples of the invariant manifolds, such as equilibria, periodic solution, stable and unstable manifolds, other invariant manifold are presented to support our result.

Keywords:

✩This work was supported by the NNSF of China under the Grant No. 10702065.
✩✩Corresponding Author. Tel: +86 371 67783167.
Email Address: peilijun@zzu.edu.cn, lijunpei@yahoo.com.cn.

Preprint submitted to Elsevier April 29, 2013
1. Introduction

The theory of invariant manifolds (shorten for IMs) is very important to the reduction of the higher-dimensional and complex systems, synchronization of the coupled and complex chaotic systems. The existence, uniqueness and multivaluedness of the IMs can be solved by the theory of Partial Differential Equations (shorten for PDEs) since the differential equations governing the IMs are in fact the first-ordered quasi-linear PDEs. So the existence and number of IMs are equivalent to those of the analytic solutions of the first ordered quasi-linear PDEs. The existence and uniqueness of the IM can be determined by Cauchy-Kawalewskaja Theorem \(^1\). And several IMs appear if the above theorem doesn’t hold. Since the general analytic solutions of the first ordered quasi-linear PDEs can’t be solved explicitly, the expression of IM can be approximated only by the numerical method. The another important question of the IMs is their attractiveness.

The attractiveness of the IMs is a difficult and unsolved question. Some authors present the different methods to solve it. Fenichel \(^2\) obtained the
sufficient conditions for persistence of a diffeomorphic IM under perturbation of the flow in terms of generalized Lyapunov type numbers, and the smoothness of the perturbed manifold. The concept of normally hyperbolicity has been proposed in \([2] - [5]\), i.e., the contraction of the flow is \(r \geq 1\) times exponentially stronger in the direction normal to the manifold than within the manifold. Josić introduced a modification of Fenichel theory which applied to chaotic synchronization, proposed a necessary and sufficient condition for such persistence was \(r\) normally hyperbolicity, and discussed the Lyapunov-exponent-like quantities used to determine the transverse stability of synchronization manifolds \([6]\). But these methods are too abstract to be employed. The other gave the operable but not general method to attack it, for example, Gorban put forward the approach that the stability of the equilibria of the invariance equation, which correspond to the slow IMs, is equivalent to that or attractiveness of the corresponding slow IMs, thus the slow IMs’ stability can be obtained by the corresponding stability \([7]\). There is a generalization that the IMs’ stability or attractiveness is also equivalent to stability of the corresponding equilibrium of the invariance equation. But if the IM is not also the equilibrium of the invariance equation, such as the global periodic solutions, the attractiveness can’t be solved in this way. Thus
an operable, simple and general method is looked forward to appearing to consider the attractiveness of the IMs.

In this paper a new method of the attractiveness of the IMs is first proposed. The attractiveness of the IMs is equivalent to that the nearby orbits will intersect continuously inward with the normals of IM, or the tangent of the orbits passing the normals intersect at obtuse angle. Obviously the latter suggests the inner product of the tangent vectors of the orbits and the normal vectors of IM is negative. This idea is motivated from the attractiveness of the equilibria by the theory of Lyapunov direct method \cite{8}. Then some examples are presented to verify this idea.

The structure of this paper is the following: the theory of the attractiveness of the IMs of the two-dimensional dynamical systems is derived in Section 2; then some illustrative examples are presented in Section 3; at last the conclusion and discussion are given.
2. Attractiveness of the IMs of the two-dimensional dynamical systems

There are several equivalent arguments on the attractiveness of the IMs of the two-dimensional dynamical systems. And an operable, simple and general method can be achieved by them. If the IM is attractive, then the orbits in some neighbor field will be attracted to it and close to it, i.e., the distance of the nearby orbits to the IM will tend to zero as time tends to the positive infinity. The latter is equivalent to that the nearby orbits will intersect continuously inward with the normals of IM, or the tangent of the orbits passing the normals intersect at obtuse angle. Obviously the latter suggests the inner product of the normal vectors of IM and the tangent vectors of the orbits passing the normal is negative. This idea is motivated from the attractiveness of the equilibria by the theory of Lyapunov direct method [8]. It means that if the gradient $\text{grad}V$ of the normal along the equipotential surfaces $V \equiv c$ of the $V$ function cross the tangent of the orbit, then the orbit won’t traverse outward the equipotential surfaces $V \equiv c$, thus the equilibria are asymptotically stable.

Remark 1. For the attractiveness of the IMs, it is required that the
separation angle of the tangent of the orbit passing every point on the normal of every point on the IM and the normal is only obtuse angle but not other angles including the right angle, i.e., their inner product is negative but not positive or 0.

Remark 2. Here the separation angle of the tangent and the gradient $\nabla V$ of the IM is not considered, since it judges if the orbit cross the IM. By the uniqueness of the solution of the ODEs, the orbit is impossible to cross the IM. Thus we must consider if the the orbit cross inward the normal and close to the IM.

Remark 3. If the IM is the closed curve, such as the periodic solution or limit cycle, then not only the inward attractiveness but also the outward attractiveness must be both considered.

Then the theory for the attractiveness of the IMs is obtained.

Theorem 1. Assuming the manifold $M$ is the IM of the two-dimensional dynamical system

$$\dot{x} = f(x)$$

for $\forall \vec{x} \in M$, $n(\vec{x})$ is the normal passing by $\vec{x}$, and $\forall \vec{x} \in n(\vec{x})$, $f(\vec{x})$ is the tangent of the orbits of the dynamical system passing $\vec{x}$, then the attractiveness and repulsiveness are obtained,
1. if the tangent \( f(\vec{x}) \) intersects at obtuse angle with the normal \( n(\vec{x'}) \), or the inner product of the normal vector \( n(\vec{x'}) \) and tangent vector \( \vec{f}(\vec{x}) \) is negative, i.e., \( \vec{f}(\vec{x}) . n(\vec{x'}) < 0 \), then the IM \( M \) is attractive; 

2. otherwise, if the tangent \( f(\vec{x}) \) intersects at sharp angle with the normal \( n(\vec{x'}) \), or the inner product of the normal vector \( n(\vec{x'}) \) and tangent vector \( \vec{f}(\vec{x}) \) is positive, i.e., \( \vec{f}(\vec{x}) . n(\vec{x'}) > 0 \), then the IM \( M \) is repulsive; 

3. if the tangent \( f(\vec{x}) \) intersects at right angle with the normal \( n(\vec{x'}) \), or the inner product of the normal vector \( n(\vec{x'}) \) and tangent vector \( \vec{f}(\vec{x}) \) is zero, i.e., \( \vec{f}(\vec{x}) . n(\vec{x'}) = 0 \), is neither attractive nor repulsive. 

Then the attractiveness of the IMs in the two-dimensional dynamical systems, such as that of the equilibria, their stable and unstable manifolds, the periodic solutions (i.e., global solutions) and other IMs, is considered by Theorem 1. It displays the correctness, universality and simpleness of Theorem 1.
3. Examples of attractiveness of different IMs

3.1. Attractiveness of equilibria

The equilibria can be assumed to be zero without the loss of generality. Their normal is $y = kx, \forall k \in \mathbb{R}$ since their tangent is the point $(0, 0)$ and they are orthogonal. The attractiveness of all kinds of simple equilibria, i.e., the focus, node, saddle and center, will be considered by Theorem 1 in this section.

3.1.1. Focus

The considered system is

$$
\begin{align*}
\dot{x} &= -x + y, \\
\dot{y} &= -x - y,
\end{align*}
$$

(0, 0) is the stable focus of system (2) and attractive. (0, 0) is also it’s IM. The normal of the focus (0, 0) is $y = kx, \forall k \in \mathbb{R}$. Let $(x', y')$ is any point in the normal $y = kx$, i.e., $y' = kx'$. The tangent vector of the orbit passing $(x', y')$ is $\vec{f}(\vec{x'}) = ((k - 1)x', (-k - 1)x')^T$. In the first quadrant, the normal vector is $\vec{n}(\vec{x'}) = (1, k)^T$ and their inner product is $\vec{n}(\vec{x'}) \cdot \vec{f}(\vec{x'}) = -(1 + k^2)x' < 0$ since $x' > 0$. By Theorem 1, the focus is attractive in the first quadrant. In the second quadrant, the normal vector is $\vec{n}(\vec{x'}) = (-1, -k)^T$ and the
inner product is $\overrightarrow{n}(\overrightarrow{x}) = (1 + k^2)x' < 0$ since $x' < 0$. In the third quadrant, the normal vector is $\overrightarrow{n}(\overrightarrow{x}) = (-1, -k)^T$ and the inner product is $\overrightarrow{n}(\overrightarrow{x}).\overrightarrow{f}(\overrightarrow{x}) = (1 + k^2)x' < 0$ since $x' < 0$. In the fourth quadrant, the normal vector is $\overrightarrow{n}(\overrightarrow{x}) = (1, k)^T$ and the inner product is $\overrightarrow{n}(\overrightarrow{x}).\overrightarrow{f}(\overrightarrow{x}) = -(1 + k^2)x' < 0$ since $x' > 0$. So the focus is attractive in the other quadrants. Thus, the focus is attractive in the all quadrants of the plane.

The repulsiveness of the unstable focus is also considered here. For the system

$$\begin{cases} \dot{x} = x + y, \\ \dot{y} = -x + y, \end{cases}$$

(0,0) is the unstable focus of system (4) and repulsive. The normal of the focus (0,0) is still $y = kx, \forall k \in \mathbb{R}$. The tangent vector of the orbit passing $(x', y')$ is $\overrightarrow{f}(\overrightarrow{x}) = ((k + 1)x', (k - 1)x')^T$. In the first quadrant, the normal vector is $\overrightarrow{n}(\overrightarrow{x}) = (1, k)^T$ and their inner product is $\overrightarrow{n}(\overrightarrow{x}).\overrightarrow{f}(\overrightarrow{x}) = (1 + k^2)x' > 0$ since $x' > 0$. By Theorem 1, the focus is repulsive in the first quadrant. Thus the focus is repulsive in the full plane.
3.1.2. Node

The considered system is

\[
\begin{align*}
\dot{x} &= -x, \\
\dot{y} &= -2y,
\end{align*}
\]  

(4)

(0, 0) is the stable node of system (4) and attractive. (0, 0) is also it's IM. The normal of the node (0, 0) is still \( y = kx, \forall k \in \mathbb{R} \). The tangent vector of the orbit passing \((x', y')\) is \( f'(x') = (-x', -2kx')^T \). In the first quadrant, the normal vector is \( n(\vec{x}) = (1, k)^T \) and their inner product is \( n(\vec{x}).f'(x') = -(1 + 2k^2)x' < 0 \) since \( x' > 0 \). In the second and third quadrants, the normal vector is \( n(\vec{x}) = (-1, -k)^T \) and the inner product is \( n(\vec{x}).f'(x') = (1 + 2k^2)x' < 0 \) since \( x' < 0 \). In the fourth quadrant, the normal vector is \( n(\vec{x}) = (1, k)^T \) and the inner product is \( n(\vec{x}).f'(x') = -(1 + 2k^2)x' < 0 \) since \( x' > 0 \). Thus, the node is attractive in the full plane.

The repulsiveness of the unstable node is also considered here. For the system

\[
\begin{align*}
\dot{x} &= x, \\
\dot{y} &= 2y,
\end{align*}
\]  

(5)

(0, 0) is the unstable node of system (5) and repulsive. The normal of the node (0, 0) is still \( y = kx, \forall k \in \mathbb{R} \). The tangent vector of the orbit passing
\((x', y')\) is \(\overrightarrow{f}(\overrightarrow{x}) = (x', 2kx')^T\). In the first quadrant, the normal vector is \(\overrightarrow{n}(\overrightarrow{x}) = (1, k)^T\) and their inner product is \(\overrightarrow{n}(\overrightarrow{x}).\overrightarrow{f}(\overrightarrow{x}) = (1 + 2k^2)x' > 0\) since \(x' > 0\). By Theorem 1, the focus is repulsive in the first quadrant. Thus similarly the unstable node is repulsive in the full plane.

3.1.3. Saddle

The considered system is

\[
\begin{aligned}
\dot{x} &= x, \\
\dot{y} &= -2y,
\end{aligned}
\]  

\((0, 0)\) is the saddle of system (6) and unattractive. The tangent vector of the orbit passing \((x', y')\) is \(\overrightarrow{f}(\overrightarrow{x}) = (x', -2kx')^T\). In the first quadrant, the normal vector is \(\overrightarrow{n}(\overrightarrow{x}) = (1, k)^T\) and their inner product is \(\overrightarrow{n}(\overrightarrow{x}).\overrightarrow{f}(\overrightarrow{x}) = (1 - 2k^2)x'\) and is not sign definite, since \(k\) is varying. Thus, the saddle is unattractive.

3.1.4. Center

The considered system is

\[
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -x,
\end{aligned}
\]

\((0, 0)\) is the center of system (7), stable and but unattractive. The tangent vector of the orbit passing \((x', y')\) is \(\overrightarrow{f}(\overrightarrow{x}) = (kx', -x')^T\). In the first
quadrant, the normal vector is $\overrightarrow{n}(\overrightarrow{x}) = (1, k)T$ and their inner product is $\overrightarrow{n}(\overrightarrow{x}).\overrightarrow{f}(\overrightarrow{x}) = 0$. In the other quadrants, their inner product is always $\overrightarrow{n}(\overrightarrow{x}).\overrightarrow{f}(\overrightarrow{x}) = 0$. Thus, the center is neither attractive nor repulsive. It is just stable but not asymptotically stable.

3.2. Attractiveness of periodic solution

For the canonical system

\[
\begin{cases}
\dot{x} = -y - x(x^2 + y^2 - 1), \\
\dot{y} = x - x(x^2 + y^2 - 1),
\end{cases}
\]

obviously there is a stable periodic solution, or limit cycle, a global solution $x^2 + y^2 = 1$. It’s attractiveness can’t be considered by the stability of the corresponding equilibrium of the invariance equation since it isn’t the equilibrium of the invariance equation. Let $y = f(x)$ is the IM of system (8) and the invariance equation is

\[
\frac{df(x(t))}{dt} = x - f(x)[x^2 + f(x)^2 - 1].
\]

In fact the stable periodic solution $x^2 + y^2 = 1$ isn’t the equilibrium of the invariance equation (9) and it’s attractiveness can’t be considered in terms of the method in [7]. But it’s attractiveness can be deduced by Theorem 1 in Section 2. Since it is the closed curve, not only the inward attractiveness but also the outward attractiveness must be considered.
Let any point \((x_0, y_0)\) in the periodic solution is in the third quadrat, where \(x_0, y_0 < 0\). It’s normal is \(y = \frac{y_0}{x_0}x\). Let any point \((x', y')\) in the normal, where \(y' = \frac{y_0}{x_0}x'\), the inward normal vector is \(\mathbf{n}_i(x') = (1, \frac{y_0}{x_0})^T\). The tangent vector of the point \((x', y')\) is

\[
\mathbf{f}(x') = \left(-\frac{y_0}{x_0}x' - x'\left(\frac{x'^2}{x_0^2} - 1\right), x' - \frac{y_0}{x_0}x'\left(\frac{x'^2}{x_0^2} - 1\right)\right)^T.
\]  

(10)

The inner product of the the inward normal vector \(\mathbf{n}_i(x')\) and the tangent vector \(\mathbf{f}(x')\) is

\[
\mathbf{n}_i(x'). \mathbf{f}(x') = \frac{x'}{x_0^2} \left(\frac{x'^2}{x_0^2} - 1\right) < 0,
\]

(11)

since \(x_0 < x' < 0\). The outward normal vector is \(\mathbf{n}_o(x') = (-1, -\frac{y_0}{x_0})^T\). The inner product of the the outward normal vector \(\mathbf{n}_o(x')\) and the tangent vector \(\mathbf{f}(x')\) is

\[
\mathbf{n}_o(x'). \mathbf{f}(x') = \frac{x'}{x_0^2} \left(\frac{x'^2}{x_0^2} - 1\right) < 0,
\]

(12)

since \(x' < x_0 < 0\). By Theorem 1, the periodic solution is attractive inward and outward in the third quadrant. Similarly the attractiveness in the other quadrants can be derived. In the second quadrant, the inward normal vector is \(\mathbf{n}_i(x') = (1, \frac{y_0}{x_0})^T\). The inner product of the the inward normal vector \(\mathbf{n}_i(x')\) and the tangent vector \(\mathbf{f}(x')\) is

\[
\mathbf{n}_i(x'). \mathbf{f}(x') = -\frac{x'}{x_0^2} \left(\frac{x'^2}{x_0^2} - 1\right) < 0,
\]

(13)
since $x_0 < x' < 0$. The inner product of the the outward normal vector $\vec{n}_o(\vec{x}')$ and the tangent vector $\vec{f}(\vec{x})$ is

$$\vec{n}_o(\vec{x}'). \vec{f}(\vec{x}) = \frac{x'}{x_0^2}(\frac{x'^2}{x_0^2} - 1) < 0,$$

(14)
since $x' < x_0 < 0$. In the first quadrant, the inward normal vector is $\vec{n}_i(\vec{x}') = (-1, \frac{y_0}{x_0})^T$. The inner product of the the inward normal vector $\vec{n}_i(\vec{x}')$ and the tangent vector $\vec{f}(\vec{x})$ is

$$\vec{n}_i(\vec{x}'). \vec{f}(\vec{x}) = \frac{x'}{x_0^2}(\frac{x'^2}{x_0^2} - 1) < 0,$$

(15)
since $x_0 > x' > 0$. The inner product of the the outward normal vector $\vec{n}_o(\vec{x}')$ and the tangent vector $\vec{f}(\vec{x})$ is

$$\vec{n}_o(\vec{x}'). \vec{f}(\vec{x}) = -\frac{x'}{x_0^2}(\frac{x'^2}{x_0^2} - 1) < 0,$$

(16)
since $x' > x_0 > 0$. In the fourth quadrant, the inward normal vector is $\vec{n}_i(\vec{x}') = (-1, -\frac{y_0}{x_0})^T$. The inner product of the the inward normal vector $\vec{n}_i(\vec{x}')$ and the tangent vector $\vec{f}(\vec{x})$ is

$$\vec{n}_i(\vec{x}'). \vec{f}(\vec{x}) = \frac{x'}{x_0^2}(\frac{x'^2}{x_0^2} - 1) < 0,$$

(17)
since $x_0 > x' > 0$. The inner product of the the outward normal vector $\vec{n}_o(\vec{x}')$ and the tangent vector $\vec{f}(\vec{x})$ is

$$\vec{n}_o(\vec{x}'). \vec{f}(\vec{x}) = -\frac{x'}{x_0^2}(\frac{x'^2}{x_0^2} - 1) < 0,$$

(18)
since $x' > x_0 > 0$. By Theorem 1, the periodic solution is attractive inward and outward in the full plane.

3.3. Attractiveness of stable and unstable manifolds of the equilibria

For the canonical system

$$ \begin{cases} \dot{x} = x, \\ \dot{y} = -y, \end{cases} \tag{19} $$

obviously the zero is it’s saddle equilibrium, $x = 0$ and $y = 0$ are respectively the zero’s stable and unstable manifolds. These manifolds are also the IMs, $x = 0$ is repulsive and $y = 0$ is attractive. Now let’s verify their attractiveness and repulsiveness by Theorem 1.

Firstly let’s talk about the attractiveness of the right half of the IM $y = 0$, where $x > 0$. Let any point $(x_0, 0)$ in the IM $y = 0$, where $x_0 > 0$, the tangent passing $(x_0, 0)$ is IM $y = 0$ itself, thus the normal is $x \equiv x_0$. The tangent vector passing any point $(x_0, y')$ in the normal, where $y' > 0$, is $\vec{f}'(x) = (x_0, -y')^T$. The upward normal vector is $\vec{n}_u(\vec{x'}) = (0, 1)^T$. The inner product of the the upward normal vector $\vec{n}_u(\vec{x'})$ and the tangent vector $\vec{f}'(x')$ is

$$ \vec{n}_u(\vec{x'}). \vec{f}'(x') = -y' < 0, \tag{20} $$

15
since $y' > 0$. The downward normal vector is $\vec{n_d}(\vec{x'}) = (0, -1)^T$. The inner product of the downward normal vector $\vec{n_d}(\vec{x'})$ and the tangent vector $\vec{f}(x)$ is

$$\vec{n_d}(\vec{x'}).\vec{f}(\vec{x'}) = y' < 0,$$

(21)

since $y' < 0$. Thus the right half of the unstable manifold $y = 0$ is attractive by Theorem 1. The attractiveness of the left half of the unstable manifold $y = 0$ is can be deduced similarly by Theorem 1. So the unstable manifold $y = 0$ is attractive.

Then let’s consider the repulsiveness of the upper half of the IM $x = 0$, where $y > 0$. Let any point $(0, y_0)$ in the IM $x = 0$, where $y_0 > 0$, the tangent passing $(0, y_0)$ is IM $x = 0$ itself, thus the normal is $y \equiv y_0$. The tangent vector passing any point $(x', y_0)$ in the normal, where $x' > 0$, is $\vec{f}(\vec{x'}) = (x', -y_0)^T$. The rightward normal vector is $\vec{n_r}(\vec{x'}) = (1, 0)^T$. The inner product of the rightward normal vector $\vec{n_r}(\vec{x'})$ and the tangent vector $\vec{f}(\vec{x'})$ is

$$\vec{n_r}(\vec{x'}).\vec{f}(\vec{x'}) = x' > 0,$$

(22)

since $x' > 0$. The leftward normal vector is $\vec{n_l}(\vec{x'}) = (-1, 0)^T$. The inner product of the leftward normal vector $\vec{n_l}(\vec{x'})$ and the tangent vector
\( f(x) \) is

\[
\vec{m}(\vec{x}) \cdot \vec{f}(\vec{x}) = -x' > 0,
\]

(23)
since \( x' < 0 \). Thus the upper half of the stable manifold \( x = 0 \) is repulsive by

Theorem 1. The repulsiveness of the lower half of the stable manifold \( x = 0 \) is can be deduced similarly by Theorem 1. So the stable manifold \( x = 0 \) is repulsive.

3.4. Attractiveness of the "real" IM

There are the IMs satisfy the definition of IM other than the equilibria, limit cycle, stable and unstable manifold. They are called as the real IM.

For the system

\[
\begin{align*}
\dot{x} &= xy^3, \\
\dot{y} &= -y - x - xy^3,
\end{align*}
\]

(24)
obviously \( y = -x \) is the IM of system \( (24) \). Let any point \( (x_0, y_0) \) in the IM, where \( y_0 = -x_0 \), the upward normal passing point \( (x_0, y_0) \) is \( y = x - x_0 + y_0 = x - 2x_0 \). Let any point \( (x', y') \) in the upper normal, where \( y' = x' - 2x_0 \), the tangent vector passing point \( (x', y') \) is \( \vec{f}(\vec{x}) = (x'y'^3, -y' - x' - x'y'^3)^T \), where \( x' > x_0 \). The upward normal vector is \( \vec{n}_u(\vec{x}) = (1, 1)^T \). The inner product of the the upward normal vector \( \vec{n}_u(\vec{x}) \) and the tangent vector
\( \mathbf{f}(x) \) is

\[ \mathbf{n}_d(x).\mathbf{f}(x) = -x' - y' = 2(x_0 - x') < 0, \quad (25) \]

since \( x' > x_0 \). The downward normal vector is \( \mathbf{n}_d(x) = (-1, -1)^T \). The inner product of the downward normal vector \( \mathbf{n}_d(x) \) and the tangent vector \( \mathbf{f}(x) \) is

\[ \mathbf{n}_d(x).\mathbf{f}(x) = x' + y' = -2(x_0 - x') < 0, \quad (26) \]

since \( x_0 > x' \). Thus the IM \( y = -x \) is attractive upward and downward by Theorem 1.

4. Conclusion and Discussion

In this paper, we first present an operable but not abstract, universal and simple theory, Theorem 1, on the attractiveness of the IMs, i.e., for any point \( x \) in the IM \( M \), \( \mathbf{n}'(x) \) is the normal passing by \( x \), and \( \forall x \in \mathbf{n}'(x) \), if the tangent \( \mathbf{f}(x) \) of the orbits of the dynamical system intersects at obtuse angle with the normal \( x \), i.e., \( \mathbf{f}(x).\mathbf{n}'(x) < 0 \), then the IM \( M \) is attractive. The conclusion of the repulsiveness is also obtained similarly. Some illustrative examples of the IMs, such as equilibria, periodic solution,
stable and unstable manifolds, other IM are presented to support our result. This method is simple and universal for different kinds of IMs.

This method provides an operable theory to consider the attractiveness of the IMs. But there are two problems to be solved. If the considered systems or the IMs are higher-dimensional, including the delayed differential dynamical systems, the normal vector $\vec{n}(\vec{x})$ and the tangent vector $\vec{f}(\vec{x})$ will be too difficult to be obtained. Their inner product is hard to be derived. On the other hand, this method can be applied only in the case that the expression of IM is analytically known. But the closed form of IM, such as the synchronization manifolds in the generalized synchronization, is usually hard to be solved since it is in fact the solution of a first-ordered quasi-linear PDEs. But we can obtain their approximated expression and consider their attractiveness or repulsiveness by Theorem 1. These will my next work.

Acknowledgment

The author would like to acknowledge the financial support for this research via the Natural National Science Foundation of China (No. 10702065). He also thanks the reviewers for their valuable reviews and suggestions.
References

[1] J. R. Ockendon, S. D. Howison, A. A. Lacery, et al, Applied Partial Equations, Oxford: Oxford University Press (2003).

[2] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, Indiana University Mathematics Journal, 21(3) (1971) 193–225.

[3] I. U. Bronstein, A. Y. Kopanskii, Smooth invariant manifolds and normal forms, Singapore: World Scientific (1994).

[4] S. Wiggins, Normally Hyperbolic Invariant Manifolds in Dynamical Systems, Applied Mathematical Sciences (1994).

[5] S. N. Chow, W. Liu, Synchronization, stability and normal hyperbolicity, Resenhas IME-USP 3(1997) 139–158.

[6] K. Josić, Synchronization of chaotic systems and invariant manifolds, Nonlinearity, 13(4)(2000) 1321-1336.

[7] A. N. Gorban, I. V. Karlin, A. Y. Zinovyev, Constructive methods of invariant manifolds for kinetic problems, Physics Reports 396(4-6)(2004) 197-403.
[8] A. M. Lyapunov, General Problem of the Stability of Motion, Taylor and Francis Books Ltd (1992).