THE BRANCHED DEFORMATIONS OF THE SPECIAL LAGRANGIAN SUBMANIFOLDS

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Abstract. In this paper, we investigate the branched deformations of immersed compact special Lagrangian submanifolds. If there exists a nondegenerate $\mathbb{Z}_2$ harmonic 1-form over a special Lagrangian submanifold $L$, we construct a family of immersed special Lagrangian submanifolds $\tilde{L}_t$, that are diffeomorphic to a branched covering of $L$ and $\tilde{L}_t$ converge to $2L$ as current. This answers a question suggested by Donaldson (Deformations of multivalued harmonic functions, 2019. arXiv:1912.08274). As a corollary, we discover examples of special Lagrangian submanifolds that are rigid in the classical sense but exhibit branched deformations. In conjunction with the work of Abouzaid and Imagi in Nearby special lagrangians, 2021. arXiv:2112.10385, we derive constraints on the existence of nondegenerate $\mathbb{Z}_2$ harmonic 1-forms.

1 Introduction

Calabi-Yau n-folds $(X, J, \omega, \Omega)$ are compact complex manifold $(X, J)$ of complex dimension $n$, with a Kähler metric $g$, a compatible symplectic form $\omega$, and a parallel holomorphic volume form $\Omega$ with specific constant norm. Harvey and Lawson [HL82] introduced the concepts of calibrated submanifolds, with special Lagrangian submanifolds being an important class.

Let $L$ be a closed special Lagrangian submanifold, then the normal bundle $N_L$ can be identified with the cotangent bundle $T^*L$ using the complex structure. The exponential map could identify small 1-forms $\alpha$ with submanifolds $L_\alpha$ close to $L$ and we could regard this as a deformation. According to McLean’s deformation theorem [MCL98], the $C^1$ deformations of the special Lagrangian submanifold $L$ can be parameterized by harmonic 1-forms on $L$ with the induced metric. For generalizations in different circumstances, see [JOY05, MAR02, SAL06].

In this work, we’ll look at a generalization of McLean’s theorem to branched covering deformations using the $\mathbb{Z}_2$ harmonic 1-forms, which can be thought of as a multivalued 1-forms satisfy extra constraints. The study of $\mathbb{Z}_2$ harmonic 1-forms started from the work of Taubes [TAU13A] on characterized the non-compactness behavior of flat $\text{PSL}(2, \mathbb{C})$ connections, see also [HW15, HE20, WZ19] for different generalizations.

Let $(L, g)$ be a compact Riemannian manifold and $\Sigma$ a codimension 2 oriented embedded submanifold of $L$. Let $\mathcal{I}$ be a flat $\mathbb{R}$ bundle over $L \setminus \Sigma$, such that every
small loop locally linking \( \Sigma \) has monodromy \(-1\). The monodromy of \( \mathcal{I} \) will define a branched covering \( p : \bar{L} \to L \), that branches along \( \Sigma \). A section \( \alpha \in T^*L \otimes \mathcal{I} \) is called a multivalued harmonic 1-form if \( \alpha \in L^2 \) satisfies the harmonic equations \( d\alpha = d\star \alpha = 0 \), where \( \star \) the metric Hodge star operator on \( L \) and the derivative is taken using the flat structure on \( \mathcal{I} \).

According to the work of Donaldson [DON], the multivalued harmonic 1-forms have an asymptotic description near \( \Sigma \). Fixed \( p \in \Sigma \), we could find suitable complex coordinates \( z \) on a slice through \( p \) transverse to \( \Sigma \) with the leading asymptotic
\[
\alpha = \Re d(Az^2) + \Re d(Bz^2) + o(|z|^{3-\gamma}),
\]
where \( A \) and \( B \) are sections of suitable bundles, \( d \) is taking the derivative using the flat structure and \( 0 < \gamma < \frac{1}{2} \). \( \alpha \) is called a \( \mathbb{Z}_2 \) harmonic 1-form, named in [TAU13B], if \( |\alpha| \) is bounded, and nondegenerate if \( A = 0 \) and \( B \) is nowhere vanishing along \( \Sigma \).

For a more detailed explanation, see Sect. 4.

The deformation theory for nondegenerate \( \mathbb{Z}_2 \) harmonic 1-forms have been studied by Donaldson [DON], see also [MAZ, TAK15] for different settings and generalizations. Donaldson predicts that the nondegenerate multivalued harmonic 1-forms can be suitable candidates for the branched deformations of the special Lagrangian submanifolds, with the following question:

**Question 1** [DON, Page 3]. Let \( \iota_0 : L \to X \) be the inclusion map of an immersed special Lagrangian submanifold, suppose over \( \iota_0(L) \) there exists a nondegenerate \( \mathbb{Z}_2 \) harmonic 1-form with branched covering \( p : \bar{L} \to L \). Could we construct a family of immersed special Lagrangian submanifolds \( \bar{i}_t : \bar{L} \to X \) such that \( \bar{i}_t \) converges to the double branched covering \( \iota_0 \circ p \)?

We could give a positive answer to Donaldson’s question, in particular, we proved:

**Theorem 1.** Let \( (X, J, \omega, \Omega) \) be a Calabi–Yau manifold and \( \iota_0 : L \to X \) be an immersed special Lagrangian submanifold with induced metric \( g_L \). Suppose there exists a nondegenerate \( \mathbb{Z}_2 \) harmonic 1-form \( \alpha \) on \( L \) with induced branched covering \( p : \bar{L} \to L \), then there exists a positive constant \( T \) and a family of special Lagrangian submanifold \( \bar{i}_t : \bar{L} \to X \) for \( |t| \leq T \) such that

(i) \( \bar{i}_t(\bar{L}) \) converge to \( 2\iota_0(L) \) as current,

(ii) \( \lim_{t \to 0} \|\bar{i}_t - \iota_0 \circ p\|_{C^\gamma} = 0 \),

(iii) In a Weinstein neighborhood \( U_L \), there exist diffeomorphisms \( \phi_t : L \to L \) such that, if we write \( \bar{i}_{t\phi_t^*(\alpha)} : \bar{L} \to U_L \) be the graph of \( t\phi^*_t(\alpha) \), then \( \|\bar{i}_{t\phi_t^*(\alpha)} - \bar{i}_t\|_{C^\gamma} \leq Ct^2 \),

where \( C_{\gamma} \) is the Hölder norm taken with respect to the metric \( p^*g_L \) over \( \bar{L} \) with \( 0 < \gamma < \frac{1}{2} \).

When \( L \) is Riemann surface, nondegenerate harmonic 1-forms can be identified with quadratic differentials with simple zeros. Given a quadratic differential, the
family of spectral curves will be an example of branched deformations for a suitable Calabi–Yau structure in a neighborhood of the zero section in $T^*L$, which we refer Sect. 5 for more details of this construction. However, when $\dim(L) \geq 3$, as the graph of a $\mathbb{Z}_2$ harmonic 1-form might not be an embedded submanifold in a Weinstein neighborhood of $L$, we don’t expect the branched deformation family we constructed above are embedded submanifolds.

By [HE], there exist examples of real analytic rational homology 3-spheres that admits nondegenerate $\mathbb{Z}_2$ harmonic 1-form, which can be realized as a special Lagrangian submanifolds in a neighborhood of the zero section in the cotangent bundle by the Calabi–Yau neighborhood theorem [BRY98]. Then for these special Lagrangian submanifolds, as the first Betti number vanishes, the McLean’s deformations will not exist. We obtain the following corollary.

**Corollary 1.** There exist special Lagrangian submanifolds which are rigid in the classical sense but have branched deformations.

There are two major challenges in answering this question. The unbounded geometry of the branched deformation family presented the first challenge. Let $t$ be the parameter for the family, we may expect the injective radius, as well as the curvatures of the branched submanifolds, to grow to infinity when $t \to 0$. This phenomenon also occurred in previous desingularization problems of calibrated submanifolds with singularities. For example, the isolated conical singularity desingularization [JOY04A] and the Lawlor neck problem, see [BUT04, IJS16, LEE04, NOR].

The key observation for the branched deformation problem is that, despite the family’s unbounded geometry, the first eigenvalues of the linearization operators over the family are bounded by a uniform positive number. In addition, we have a good understanding of the blow-up order of the geometry of the family, including the second fundamental form, the injective radius and the size of the Weinstein tubular neighborhood.

The second challenge comes from the singular set moving. The multivalued forms branched along submanifold can be considered as a free boundary problem, while the branch locus itself needs to be determined. For additional information on the linearized problem, we refer [DON, TAK15]. The linearization of the special Lagrangian equation is the harmonic equation for multivalued 1-forms. Even we start with a multivalued harmonic 1-form which solves the linearization equations with branch locus $\Sigma$, it doesn’t guarantee that we will be able to solve the non-linear equation using the same branch locus. In actuality, for each $t$, the branched deformation family may branch along a different codimension 2 submanifold $\Sigma_t$ which will be close to the initial branch locus of the $\mathbb{Z}_2$ harmonic 1-form.

The above observation contributes to the construction of approximate special Lagrangian submanifolds, where the method we used is very similar to Donaldson in [DON, DON21]. Given an initial nondegenerate $\mathbb{Z}_2$ harmonic 1-form $\alpha$, we proved that after moving the singular set in a specified direction, we could make good enough approximate special Lagrangian submanifolds $\widetilde{L}^{app}_t$, which is close to $t \phi^*_t \alpha$,
for some diffeomorphisms $\phi_t$ of $L$. Then we could apply a version of Joyce’s nearby special Lagrangian theorem [JOY04C] to obtain the branched deformation family.

According to the celebrated work of Taubes [TAU13B, TAU13A], the $\mathbb{Z}_2$ harmonic 1-form plays an important role in $\text{PSL}(2, \mathbb{C})$ gauge theory, which characterized the non-compactness behavior of the moduli space of $\text{PSL}(2, \mathbb{C})$ flat connections. It would be fascinating to learn if there are any topological constraints on the existence of nondegenerate $\mathbb{Z}_2$ harmonic 1-forms. Combing our main theorem with the work of nearby special Lagrangian theorem by Abouzaid and Imani [AI21], we obtain a criterion for non-existence of nondegenerate $\mathbb{Z}_2$ harmonic 1-form.

**Theorem 2.** Let $(L, g_L)$ be a real analytic Riemannian manifold and $(\Sigma, I, \alpha)$ be a multivalued harmonic 1-form with $\Sigma$ a smooth codimension 2 submanifold and the associate branched covering $p : \tilde{L} \to L$. Suppose the following holds:

(i) $\pi_1(L)$ is finite or contains no nonabelian free subgroup,

(ii) Let $R$ be the canonical involution on $T^*L$ by sending a vector $v$ to $-v$, there exists a Calabi–Yau neighborhood $(U_L, J, \omega, \Omega)$ of the zero section in $T^*L$ with $R$ an anti-holomorphic involution,

(iii) All $R$-invariant immersed special Lagrangian submanifolds that are diffeomorphic to $\tilde{L}$ is unobstructed,

then $(\Sigma, I, \alpha)$ is not a nondegenerate $\mathbb{Z}_2$ harmonic 1-form.

We should emphasize that our assumption of the multivalued harmonic 1-form having singularity on a codimension 2 compact submanifold $\Sigma$ is a strong assumption. In general, the singular set of the general multivalued harmonic 1-form, as described in the work of Taubes [TAU13B, TAU13A], may only be a Hausdorff codimension 2 subset, which means that we may not be able to define the double branched covering. As a result, the above theorem may not provide any information about this situation.

According to the work of Bryant [BRY98] and Doicu [DOI15], the condition (ii) holds when $\chi(L) = 0$. For the condition (iii), in general, it is very hard to check whether a Lagrangian is unobstructed or not. We do, however, have an additional $R$-symmetry given by the anti-holomorphic involution, which allows us to check the assumption (iii) for specific situations. When $\dim(L) = 2$, according to [SOL20], the condition (i) and (iii) holds for $L = S^2$ and $T^2$. For a single special Lagrangian submanifold, we could introduce topological constraint to avoid obstruction, which give us a Betti number constraint for the branched covering.

**Theorem 3.** Let $(\Sigma, I, \alpha)$ be a nondegenerate $\mathbb{Z}_2$ harmonic 1-form on $L$ with branched covering $p : \tilde{L} \to L$, suppose $\pi_1(L)$ is either finite or contains no non-abelian free subgroups and the Euler characteristic $\chi(L) = 0$, then for the second Betti number, we have the strict inequality $b_2(\tilde{L}) > b_2(L)$.

The first Betti number inequality $b_1(\tilde{L}) > b_1(L)$ is already implied by the presence of nondegenerate $\mathbb{Z}_2$ harmonic 1-forms, hence the preceding theorem is trivial when
This paper is organized as follows. In Sect. 2, we introduce the background of the special Lagrangian geometry. In Sect. 3, we study the structure of the special Lagrangian equation in a Weinstein neighborhood. In Sect. 4, we introduce the analytic theory of $Z_2$ harmonic 1-form. In Sects. 5, 6 and 7, we construct approximate solutions to the special Lagrangian equation and study their geometry of them. In Sect. 8, we prove a nearby special Lagrangian theorem to prove the main theorem. In Sect. 9, we discuss the possible applications of our branched deformation theorem.

Conventions. We denote by $C > 0$ a constant, which depends only on the background Calabi–Yau structure $(X, J, \omega, \Omega)$, the initial special Lagrangian submanifold or the initial $Z_2$ harmonic 1-form. All the $C^{\gamma}$ we discuss will assume that $0 < \gamma < \frac{1}{2}$. The values of $C$ could change from one line to the next, and we always specify when a constant depends on further data. In addition, we write $x \lesssim y$ if $x \leq C y$ and $O(y)$ denotes a quantity $x$ such that $|x| \lesssim C y$. Moreover, without specific statement, all manifolds we considered in this paper will be compact.

2 Special Lagrangian geometry

In this section, we will define the special Lagrangian submanifolds of a Calabi–Yau manifold and introduce the McLean’s deformation theorem. Some references for this section are Harvey and Lawson [HL82], Joyce [JOY05] and McLean [MCL98].

2.1 Calabi–Yau manifold and the special Lagrangian submanifolds. In this subsection, we will define the immersed special Lagrangian submanifolds in a Calabi–Yau manifold, follows from Harvey and Lawson [HL82].

Definition 1. A Calabi–Yau manifold is a quadruple $(X, J, \omega, \Omega)$ such that $(X, J, \omega)$ is a Kähler manifold with a Kähler metric $g_X$ and Kähler class $\omega$. Let $n$ be the complex dimension of $X$, $\Omega$ is a nowhere vanishing holomorphic $(n, 0)$-form on $X$ satisfies

$$\omega^n = (-1)^{\frac{n(n-1)}{2}} (\sqrt{-1})^n \Omega \wedge \bar{\Omega}. \quad (1)$$

An immersed submanifold is a pair $[L] = (L, \iota)$, where $L$ is a closed oriented manifold and $\iota : L \to X$ is an immersion. $[L]$ is called a Lagrangian submanifold if $\iota^* \omega = 0$. For any immersed Lagrangian submanifold $[L]$, for the holomorphic volume form, we have $|\iota^* \Omega| = 1$, where the norm is taken using the pull-back metric $\iota^* g_X$. Therefore, we could write

$$\iota^* \Omega = e^{i\theta} \text{Vol} \iota^* g_X, \quad (2)$$

where $e^{i\theta} : L \to S^1$ is called a phase function.

Harvey and Lawson [HL82] introduced the concepts of special Lagrangian submanifolds.
**Definition 2.** Let \((X, J, \omega, \Omega)\) be a Calabi–Yau manifold, an immersed oriented submanifold \([L] = (L, \iota)\) is called a special Lagrangian submanifold if \(e^{i\theta} = 1\).

By (2), \(e^{i\theta} = 1\) is equivalent to \(\iota^*3\Omega = 0\) or \(\iota^*\Re \Omega = \text{Vol}_{\iota^*g_X}\). For a general Lagrangian submanifold, the \(S^1\) valued function \(e^{i\theta}\) doesn’t necessarily have a lift to a real valued function \(\theta\), while \(d\theta\) is well-defined, which will satisfy \(d\theta = \iota_H \omega\), where \(H\) is the mean curvature vector field. Suppose the lifting \(\theta: L \to \mathbb{R}\) exists, then we call \(\theta\) the Lagrangian angle. In particular, any Lagrangian that sufficiently close to a special Lagrangian will have a global well-defined Lagrangian angle, which are the cases that considered in this paper.

Special Lagrangian submanifolds are an important class of calibrated submanifold. A Lagrangian submanifold \([L] = (L, \iota: L \to X)\) is called calibrated by \(\Re \Omega\) if \(\iota^*\Re \Omega = d\text{Vol}_{\iota^*g_X}\), where \(d\text{Vol}_{\iota^*g_X}\) is the volume form for the pullback metric \(\iota^*g_X\).

The special Lagrangian submanifold is calibrated by \(\Re \Omega\) based on the definition.

There are many interesting examples of special Lagrangian submanifolds have been constructed, which we refer to [HL82, HAS04, HK07, JOY01, JOY02].

### 2.2 McLean’s deformation for the special Lagrangian submanifolds.

The deformation theory of compact special Lagrangian submanifold is first studied by McLean [MCL98], see also [JOY05, MAR02, SAL06] in different settings.

#### 2.2.1 Weinstein tubular neighborhood.

Now, we will briefly discuss the Weinstein tubular neighborhood for immersed Lagrangian submanifolds, for more details, we refer [SIL01].

**Definition 3.** Let \([L] = (L, \iota)\) be an immersed submanifold, we define the normal bundle \(N_L := \iota^*T_L \cong TL^\perp \subset \iota^*TX\), where \(TL^\perp\) is the complement bundle defined by the pullback Riemannian metric \(\iota^*g_X\) on \(\iota^*TX\).

The complex structure induces an isomorphism between \(TL\) and \(N_L\) and as \(\iota\) is an immersion, the pull-back Riemannian metric \(\iota^*g_X\) induces an isomorphism between \(TL\) and \(T^*L\) and we write \(\varphi : T^*L \to N_L\) be this isomorphism. We could define

\[
\Phi : U_L \to X \quad \text{as} \quad \Phi(\alpha) = \exp_{\iota\circ\pi(\alpha)} \circ \varphi(\alpha),
\]

where exp is the exponential map on \(X\), \(\alpha \in T^*L\) and \(\pi : T^*L \to L\) is the projection map of the bundle. Therefore, a neighborhood of the zero section of \(T^*L\) can be regarded a tubular neighborhood of the immersion. Composing \(\Phi\) with suitable self diffeomorphism of \(U_L\), we have the Weinstein tubular neighborhood theorem.

**Theorem 4 [SIL01].** Let \((X, \omega)\) be a symplectic manifold and \([L] = (L, \iota)\) is a compact Lagrangian submanifold, then there exists an open tubular neighborhood \(U_L\) with \(\Phi : U_L \to X\) such that \(\Phi^*\omega = \omega_0\), where \(\omega_0\) is the canonical symplectic structure on \(T^*L\).
2.2.2 McLean’s deformation theorem. Let \([L_0]\) be an immersed special Lagrangian submanifold, \(U_L \subset T^*L_0\) be a Weinstein neighborhood of \([L_0]\) and \(\iota_0 : L \to U_L\) be the inclusion map of the zero section, then any Lagrangian immersion that is sufficiently \(C^1\)-close to \(\iota_0\) will be given by the graph of a 1-form on \(L\). Suppose we have a family of special Lagrangian submanifolds \([L_t] = (L, \iota_t)\) on \(U_L\) such that \(\iota_t\) are given by the graph of 1-forms on \(L\) and \(\lim_{t \to 0} \|\iota_t - \iota_0\|_{C^1} = 0\), then we call \([L_t]\) a \(C^1\) deformations of \([L_0]\).

**Theorem 5 [MCL98].** Let \((X, J, \omega, \Omega)\) be a Calabi–Yau manifold with metric \(g_X\), \([L_0] = (L, \iota_0)\) be an immersed special Lagrangian submanifold on \(X\), then \(C^1\) deformations of \([L_0]\) are parameterized by harmonic 1-forms on \(L\) for the metric \(\iota_0^*g_X\).

**Proof.** We will give a sketch of proof of McLean’s theorem by an implicit functional argument. Let \(\Phi : U_L \to X\) be a Weinstein neighborhood of \([L_0]\), let \(\alpha\) be a closed 1-form on \(L\), for \(t \in \mathbb{R}\) a real parameter, we define \([L_t] := (L, \iota_t)\), with \(\iota_t(x) = \Phi(\iota_0|x)\), which is the graph of \(t\alpha\).

We construct a map \(F_t : \Omega^1(L) \to \Omega^2(L) \oplus \Omega^n(L)\), \(F_t(v) := (\iota_t^*\omega, \iota_t^*\Omega)\), and by [MCL98, Page 722], we obtain \(\frac{d}{dt}|_{t=0}\iota_t^*\omega = d\alpha\), \(\frac{d}{dt}|_{t=0}\iota_t^*\Omega = d\ast \alpha\), where \(\ast\) is the Hodge star operator on \([L_0]\) w.r.t. the pull-back metric \(\iota^*g_X\). In addition, by the special Lagrangian condition \(\iota_0^*\omega = 0\) and \(\iota_0^*\Omega = 0\), \(\iota_t^*\omega\) and \(\iota_t^*\Omega\) are trivial in the cohomology group \(H^2(L; \mathbb{R})\) and \(H^n(L; \mathbb{R})\). Therefore, \(\text{Im} F_t \subset d\Omega^1(L) \oplus d\ast \Omega^1(L)\). In a suitable norm \(dF_t\) will be surjective and \(F_t^{-1}(0, 0)\) will be a smooth manifold model by the harmonic 1-forms. \(\square\)

The McLean’s deformation theorem works straight forward if we consider unbranched smooth deformation. Let \(p : \tilde{L} \to L\) be any unbranched k-fold covering of \(L\), then \([\tilde{L}_0] = (\tilde{L}, \iota \circ p)\) is also an immersed submanifold. Applying Theorem 5 for \([\tilde{L}_0]\), we obtain the following corollary.

**Corollary 2.** Under the previous assumptions, given an harmonic 1-form \(v\) on \(\tilde{L}\) w.r.t. the pull-back smooth metric \((\iota \circ p)^*g_X\), then there exists a family of special Lagrangian submanifolds \([\tilde{L}_t] = (\tilde{L}, \tilde{\iota}_t)\) such that \(\lim_{t \to 0} \tilde{\iota}_t = \iota \circ p\).

The unbranched smooth covering condition is an essential condition for Mclean’s argument. Suppose \(p : \tilde{L}' \to L\) is a branched covering of \(L\), then \((\iota \circ p)^*g\) is no longer a smooth metric along the branch locus and a harmonic 1-form for the singular metric doesn’t guarantee a deformation. Suppose the Ricci curvature of \(L\) is positive, then by [TW18], see also [AI21], that the branched deformation will not exist.

3 The special Lagrangian equation over the cotangent bundle

In a Weinstein neighborhood, the special Lagrangian equation characterizes whether the graph of a 1-form \(\alpha\) on \(T^*L\) is a special Lagrangian submanifold. The structure of this equation up to quadratic terms has been studied in previous work.
In this section, we will examine the higher order expansion of the special Lagrangian equation using a pseudo-holomorphic volume form on $T^*L$, which approximates the real holomorphic volume form. While this section contains some technical details, the main result is Proposition 7. Readers can skip most of the details without affecting their understanding of the rest of the paper.

### 3.1 Special Lagrangian equations in $\mathbb{C}^n$ as a graph.

We first consider the special Lagrangian equation in $\mathbb{C}^n$ for a graph manifold, introduced by Harvey and Lawson [HL82]. Let $\mathbb{C}^n$ be the $n$-dimensional complex plane with complex coordinates $(z_1, \ldots, z_n)$. Let $z_i = x_i + \sqrt{-1}y_i$, then the canonical Calabi–Yau structure on $\mathbb{C}^n$ is:

$$g_0 = \sum_{i=1}^{n} |dz_i|^2, \quad \omega_0 = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i, \quad \text{and} \quad \Omega_0 = dz_1 \wedge \cdots \wedge dz_n.$$  

Let $\mathbb{R}^n$ be the $n$-plane span by $(x_1, x_2, \ldots, x_n)$ and we identified $T^*\mathbb{R}^n \cong \mathbb{C}^n$. Let $f$ be a smooth function on $\mathbb{R}^n$, then the graph manifold of $df$ is a Lagrangian. Let $\iota_f$ be the inclusion of the graph, then $\iota_f^* \Omega = \Im \det(I + \text{Hess}f)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

The special Lagrangian equation $\iota_f^* \Omega = 0$ for $df$ can be written as

$$\Im \det(I + \text{Hess}f) = 0. \quad (4)$$

### 3.2 The geometry of the cotangent bundle.

In this subsection, we will introduce the Sasaki metric on the cotangent bundle of a Riemannian manifold and the readers are referred to [YL73] for further details. Moreover, using the Sasaki metric, we will define an approximation to the holomorphic volume form.

Let $(L, g)$ be a Riemannian manifold with Riemannian metric $g$ and cotangent bundle $T^*L$. Let $p \in T^*L$ with $\pi(p) = x$, using the Levi-Civita connection, we have the following splitting of the tangent bundle of $T^*L$, which is $T_p(T^*L) = H \oplus V$, where the horizontal part $H|_p \cong T_xL$ and the vertical part $V|_p \cong T^*_xL$, and we write the isomorphism induced by the Levi-Civita connection as

$$\Gamma : T_p(T^*L) \to T_xL \oplus T^*_xL. \quad (5)$$

Let $x = (x_1, \ldots, x_n)$ be a local coordinate system of $L$, and $y = (y_1, \ldots, y_n)$ by the fiber coordinate of $T^*_xL$, such that $\sum_{i=1}^{n} y_idx_i$ is a section of $T^*_xL$. Let $\nabla$ be the Levi-Civita connection with $\nabla_{\partial_{x_j}} \partial_{x_i} = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_{x_k}$, then for $\partial_x$ and $dx_i$, under the identification in (5), we write the horizontal and vertical lifting in local coordinates $(x, y)$ as

$$\partial^H_{x_i} = \partial_{x_i} + \sum_{j,k=1}^{n} y_j \Gamma_{ik}^j \partial_{y_k}, \quad dx^V_i = \partial_{y_i}. \quad (6)$$

**Proposition 1.** Let $\alpha$ be a 1-form on $L$, we write $\iota_\alpha : L \to T^*L$ be the inclusion induced by the graph of $\alpha$, then for any vector field $v$ on $L$, we have $\Gamma \circ (\iota_\alpha)_* v = (v, \nabla_v \alpha) \in TL \oplus T^*L$. 
Proof. Without losing generality, we take \( v = \partial_{x_i} \), and write \( \alpha = \alpha_i dx_i \). We compute 
\[
(\iota_\alpha) \ast \partial_{x_i} = \partial_{x_i} + \sum_{j=1}^n \partial_{x_j} \alpha_j \partial_{x_j} = \partial_{x_i} + (\nabla_{\partial_{x_i}} \alpha)^V,
\]
thus \( \Gamma((\iota_\alpha) \ast \partial_{x_i}) = (\partial_{x_i}, \nabla_{\partial_{x_i}} \alpha) \). \( \square \)

Let \( X, Y \) be vectors on \( L \) with horizontal lifting \( X^H, Y^H \) and \( \alpha, \beta \) be 1-forms on \( L \) with vertical lifting \( \alpha^V, \beta^V \), then the Sasaki metric \( g_0 \) on \( T^*L \) is defined as 
\[
g_0(X^H, Y^H) = g(X, Y) \circ \pi, \quad g_0(\alpha^V, \beta^V) = g^{-1}(\alpha, \beta) \circ \pi, \quad g_0(X^H, \alpha^V) = 0.
\]
In addition, there exists a natural almost complex structure \( J_0 \) on \( T^*L \). Let \( \varphi_g : TL \to T^*L \) be the induced isomorphism of the vector bundle, then we define the almost complex structure as \( J_0(X^H) := (\varphi_g_0(X))^V \), \( J_0(\varphi_g_0(X)) = -X^H \), where \( X \) is a vector field on \( L \). Let \( \omega_0 \) is the canonical symplectic structure on \( T^*L \), then by a straightforward computation, these three structures \( g_0, \omega_0, J_0 \) are compatible: 
\[
\omega_0(X, Y) = g_0(X, J_0 Y).
\]

Using the Sasaki metric, we could give an isomorphism of the tangent and cotangent bundle of \( T^*L \), which we write as \( \Phi_{g_0} : T(T^*L) \to T^*(T^*L) \). As the decomposition \( T(T^*L) \cong H \oplus V \) is orthogonal under the Sasaki metric and \( T^*(T^*L) \cong H^* \oplus V^* \), \( \Phi_{g_0} \) preserves the decomposition and gives the identification \( \Phi_{g_0}|_H : H \to H^* \) and \( \Phi_{g_0}|_V : V \to V^* \).

We define the dual basis to \( \partial^H_{x_i}, dy_i \) and compute 
\[
dx_i^H := g_0(\partial^H_{x_i}, \cdot) = \sum_j g_{ij} dx_j, \quad dy_i^V := g_0(dy_i, \cdot) = \sum_j g^{ij} dy_j - \sum_{j,k} y_k \Gamma^k_{ij} g^{is} dx_j.
\]

3.3 The pseudo holomorphic volume form. Now we start to define a canonical \( n \)-form.

3.3.1 The pseudo holomorphic volume form. We write \( \xi : V \to T^*L \) be the identification and \( \xi_k := \Lambda^k \xi : \Lambda^k V \to \Lambda^k T^*L \). We could also identify \( T^*L \) and \( H^* \) by the following map 
\[
\eta : T^*L \to TL \to H \to H^*,
\]
where the first map above is given by the Riemannian metric on \( L \) and the third map above is given by the Sasaki metric. A straightforward computation gives \( \eta(dx_i) = \pi^* dx_i \), with projection \( \pi : T^*L \to L \). To avoid notation, we also write \( \pi^* dx_i \) as \( dx_i \). In addition, we define \( \eta_k := \Lambda^k \eta : \Lambda^k T^*L \to \Lambda^k H^* \).

Using \( \xi_k \) and \( \eta_{n-k} \), we define an endomorphism \( \kappa_k \) by 
\[
\kappa_k : \Lambda^k V \xrightarrow{\xi_k} \Lambda^k T^*L \star \Lambda^{n-k} T^*L \xrightarrow{\eta_{n-k}} \Lambda^{n-k} H^*,
\]
where \( \star \) is Hodge star operator induced by the Riemannian metric on \( L \) and we write \( K_k \) be the corresponding section of \( \Lambda^k V^* \otimes \Lambda^{n-k} H^* \).

In local coordinates, \( K_k \) can be written as 
\[
K_k := \sum_{j_1, \ldots, j_n} \frac{\sqrt{\det(g)}}{n!} \epsilon_{j_1 \ldots j_n} dy^{V}_{j_1} \wedge \cdots \wedge dy^{V}_{j_p} \wedge dx_{j_{p+1}} \wedge \cdots \wedge dx_{j_n},
\]
where $\epsilon_{j_1,\ldots,j_n}$ is the Levi-Civita symbol. Moreover, as $\Lambda^n(T^*(T^*L)) = \bigoplus_{k=1}^n \Lambda^k V^* \otimes \Lambda^{n-k} H^*$, $K_k$ is a n-form on $T^*L$.

**Definition 4.** The pseudo holomorphic volume form $\Omega_0 \in \Lambda^n(T^*(TL) \otimes \mathbb{C})$ is defined by

$$\Im \Omega_0 := \sum_{k \text{ odd}} (-1)^{\frac{k+1}{2}} K_k, \ \Re \Omega_0 := \sum_{k \text{ even}} (-1)^{\frac{k}{2}} K_k, \ \Omega_0 := \Re \Omega_0 + i \Im \Omega_0. \quad (11)$$

**Proposition 2.** The pseudo holomorphic volume form $\Omega_0$ has the following properties:

(i) Let $\iota_0$ the zero section inclusion, then $\iota_0^* \Im \Omega_0 = 0$, $\iota_0^* \Re \Omega_0 = \text{Vol}_L$.

(ii) Suppose at $m \in L$, $(x_1,\ldots,x_n)$ are normal coordinates w.r.t. the Riemannian metric on $L$, then at $q = (m,0) \in T^*L$, $\Omega_0|_q = \Lambda^n_{i=1}(dx_i + \sqrt{-1} dy_i)$.

(iii) $\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \frac{\omega^n}{2^n} \Omega_0 \wedge \check{\Omega}_0$ on the zero section.

(iv) $d\Omega_0|_{\iota_0(L)} = 0$.

Proof. For (i), as $\iota_0^* dy_i^V = 0$ and every term of $\Im \Omega$ consists of $dy_i^V$, we obtain $\iota_0^* \Im \Omega = 0$. Moreover, $\iota_0^* \Re \Omega = \iota_0^* K_0 = \frac{\sqrt{\text{det}(g)}}{n!} \epsilon_{k_1\ldots k_n} dx_{k_1} \wedge \cdots \wedge dx_{k_n}$, which is the volume form on $L$. As in the normal coordinates $(x_1,\ldots,x_n)$, we have $dy_i^V|_q = dy_i$, which implies (ii). (iii) follows directly from (ii). For (iv), in the normal coordinates above, we obtain $d(dy_i^V)|_q = dy^{ij} \wedge dy_j|_q - d(y_k \Gamma^k_{ij} g^{js}) \wedge dx_j|_q = 0$, which implies $d\Omega_0|_{\iota_0(L)} = d(\text{Vol}_X)|_{\iota_0(L)} = 0$. \qed

3.3.2 The pseudo special Lagrangian equation. Given a closed 1-form $\alpha$, we write $[L_\alpha] = (L, \iota_\alpha)$ be the graph of $\alpha$ and we will define the pseudo special Lagrangian equation.

**Definition 5.** $[L_\alpha]$ is called a pseudo special Lagrangian submanifold if $[L_\alpha]$ is symplectic and $\iota_\alpha^*(\Im \Omega) = 0$. Moreover, we define the pseudo special Lagrangian equation for $1$-form $\alpha$ as $\text{pSL}(\alpha) := \star_{0}^*(\Im \Omega)$, where $\star$ is the Hodge star operator w.r.t. the induced Riemannian metric on the zero section.

We could explicitly compute pseudo special Lagrangian equation. Let $\alpha$ be a closed 1-form on $L$, then $\nabla \alpha$ is a symmetric $(0,2)$ tensor, we could use the Riemannian metric to define a $(1,1)$ tensor $A_\alpha : TL \to TL$. We define the $k$-th symmetric power of $A_\alpha$ as $P_k(\nabla \alpha) := \text{Tr}(A^k \alpha)$. If we choose an orthonormal base of $T^*L$, we could regard $\nabla \alpha$ as a $n \times n$ matrix, then we have $\text{det}(\text{Id} + t \nabla \alpha) = \sum_{k=1}^n P_k(\nabla \alpha) t^k$. In addition, we define

$$P(\nabla \alpha) := \sum_{k=1}^{\left\lceil \frac{n-1}{2} \right\rceil} (-1)^k P_{2k+1}(\nabla \alpha). \quad (12)$$

**Proposition 3.** The pseudo special Lagrangian equation of $[L_\alpha]$ on $T^*L$ can be written as $\text{pSL}(\alpha) = -d^* \alpha + P(\nabla \alpha)$, with $P$ defined in (12).
Proof. We will compute in local coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) under previous conventions. The linearization of \(pSL(\alpha)\) would be \(*_\alpha K_1\), where

\[
K_1 = \sum_{k_1, \ldots, k_n=1}^n \frac{\sqrt{\det g}}{n!} \epsilon_{k_1 \cdots k_n} dy_{k_1}^V \wedge dx_{k_2} \wedge \cdots \wedge dx_{k_n}.
\]

We write \(\alpha = \sum_{k=1}^n \alpha_k dx_k\), then by a straight computation, we found

\[
*_\alpha dy_i^V = \sum_{j,l=1}^n g^{il}(\partial_x \alpha_l - \sum_{k=1}^n \alpha_k \Gamma^k_{ij}) dx_j = \sum_{j,l=1}^n g^{il} \nabla \alpha(\partial_{x_j}, \partial_{x_l}) dx_j.
\]

Therefore, \(*_\alpha K_1 = -d^* \alpha\).

For the non-linear terms, we define \(H^i_j := \sum_{k=1}^n g^{ik} \nabla \alpha(\partial_{x_j}, \partial_{x_k})\), then we compute

\[
*_\alpha (i_j K_{2l+1}) = \sum \epsilon_{k_1 \cdots k_{2l+1}} \epsilon_{j_1 \cdots j_{2l+1}} H^k_{j_1} H^{k_2}_{j_2} \cdots H^{k_{2l+1}}_{j_{2l+1}} = P_{2l+1}(\nabla \alpha),
\]

which is the desire term. \(\square\)

When \(L = \mathbb{R}^n, T^*L \cong \mathbb{C}^n\). When we associating \(T^*L\) with the canonical Calabi–Yau structure on \(\mathbb{C}^n\), the pseudo special Lagrangian equation will coincident with (4).

3.3.3 The relationship between the holomorphic volume forms. Over \((U_L, J, \omega, \Omega)\) with Calabi–Yau metric \(g\), we write \(g_0\) be the induced Sasaki metric on \(U_L\) together with almost complex structure \(J_0\) and pseudo holomorphic volume form \(\Omega_0\), then we have:

**Proposition 4.**

(i) Over the zero section \(L_0 \subset U_L\), we have \(g_0|_{L_0} = g|_{L_0}\).

(ii) Let \(m \in L_0\), we write \(T^\perp_m L\) be the orthogonal complement of \(T_m L\) in \(T_m U\) w.r.t \(g\), then \(T^\perp_m L\) is also a Lagrangian subspace of \(\omega\).

(iii) Let \(\theta\) be the Lagrangian angle of \([L]\), then \(\Omega|_{L_0} = e^{i\theta} \Omega_0|_{L_0}\).

Proof. Let \(m \in L_0\), we choose normal coordinate \((x_1, \ldots, x_n)\) of an open set of \(L\) centered at \(m\) with fiber coordinates \((y_1, \ldots, y_n)\), we will check the statement in this coordinate system. As \(i_0^* g = i_0^* g_0, (x_1, \ldots, x_n)\) is a normal coordinate for both metrics. For (i), based on the definition of the Sasaki metric, \(\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}\}\) is an orthonormal frame on \(T_m^* L\) for \(g_0\) and \(\{\partial_{x_1}, \ldots, \partial_{x_n}, J\partial_{x_1}, \ldots, J\partial_{x_n}\}\) is an orthonormal frame for \(g\). As \(L_0\) is a Lagrangian submanifold, we have \(g(J\partial_{x_1}, \partial_{x_2}) = 0\). Thus, \(J\partial_{x_1}\) lies in the plane span by \(\{\partial_{y_1}, \ldots, \partial_{y_n}\}\). As \(\omega\) is the canonical symplectic structure, we have \(g(\partial_{x_1}, J\partial_{y_2}) = -\omega(\partial_{x_1}, \partial_{y_2}) = -\delta_{12}\), which implies \(J\partial_{y_2} = -\partial_{x_2}\). Therefore, \(\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}\}\) is also an orthonormal frame for \(g\), which implies (i). As \(\omega(\partial_{y_1}, \partial_{y_2}) = -g(J\partial_{y_1}, \partial_{y_2}) = 0\), we obtain (ii). As \(\{dx_1, \ldots, dx_n, dy_1, \ldots, dy_n\}\) is an orthonormal frame at \(m\), by (1), we have

\[
\Omega|m = e^{\sqrt{-1} \theta} \wedge_{i=1}^n (dx_i + \sqrt{-1} dy_i)|_m = e^{\sqrt{-1} \theta} \Omega_0|m,
\]

which implies (iii). \(\square\)
3.4 The special Lagrangian equation in a neighborhood of a Lagrangian. Let $\alpha$ be a 1-form on $U_L$ with $\iota_\alpha : L \to U_L$ be the graph manifold defined by $\alpha$ and $\iota_0$ be the inclusion of the zero section. The graph of $\alpha$ is a Lagrangian submanifold if and only if $\alpha$ is closed. In addition, the special Lagrangian equation for $\alpha$ is defined as

$$SL(\alpha) := *\iota_\alpha^* \Omega,$$

where $*$ is the Hodge star operator using the induced metric on the zero section.

**Proposition 5** [JOY04B, Prop 2.21]. The special Lagrangian equation can be written as

$$SL(\alpha) = \sin \theta - d^*(\cos \theta \alpha) + Q(\alpha, \nabla \alpha),$$

where $Q(\alpha, \nabla \alpha)$ is a smooth function with $|Q(\alpha, \nabla \alpha)| = O(|\alpha|^2 + |\nabla \alpha|^2)$ for small $\alpha$.

**Proof.** Let $\theta$ be the Lagrangian degree defined in (2), then we have $SL(0) = *\iota_0^* \Omega = *\Im(\exp i\theta \text{Vol}_{\iota_0^*g}) = \sin \theta$. By [JOY04B, Prop 2.21], $\frac{d}{dt}|_{t=0} SL(t\alpha) = -d^*(\cos \theta \alpha)$, the proposition follows straightforward. \qed 

Now, we will focus on the case when $[L]$ is a special Lagrangian submanifold, while the following observations are due to Joyce [JOY04C, Page 15]. The value of $SL(\alpha)$ at $x \in L$ depends on the $T^*_{\iota_\alpha(p)}(\iota_\alpha(L))$, where $p = \iota_\alpha(x)$, and the cotangent bundle depends on $\alpha|_x$ and $\nabla \alpha|_x$. Therefore, $SL(\alpha)|_x$ depends pointwisely on $\alpha|_x$ and $\nabla \alpha|_x$.

Let $\Omega_0$ be the pseudo holomorphic volume form, we define a n-form over $U_L$ as

$$T := \Im - \Im \Omega_0,$$

then by Proposition 4, $T|_{t=0} = 0$. The error term $S$ of the quadratic terms of the special Lagrangian equation can be expressed as

$$S(\alpha, \nabla \alpha) := *\iota_\alpha^* T.$$

As $S$ depends on the tangent space of the graph manifold of $\alpha$, $S$ is a smooth function of both $\alpha$ and $\nabla \alpha$.

After fixing $x \in L$, define variable $y \in T^*_x L$ and $z \in \otimes^2 T^*_x L$. Under the identification $\Gamma : T(x,y)U_L \to T_x L \oplus T^*_x L$, for any $z \in \otimes^2 T^*_x L$, we define the map

$$I_z : T_x L \to T_x L \oplus T^*_x L, \ I_z(y) := (y, \iota_y z).$$

We define

$$\bar{SL}, \ \bar{S} : \{(x,y,z) : x \in L, \ y \in T^*_x L, \ z \in \otimes^2 T^*_x L\} \to \mathbb{R},$$

$$\bar{SL}(x,y,z) := *I_z^*(\Gamma^{-1})^* \Im \Omega(x,y), \ \bar{S}(x,y,z) := *I_z^*(\Gamma^{-1})^* T|_{(x,y)}.$$
After fixing $x \in L$, the variable $y, z$ lies in the vector spaces $T_x^* L$ and $\otimes^2 T_x^* L$. So we could take partial derivative in $y$ and $z$ direction without using a connection. Then we have

$$\partial_y^{k_1} \partial_z^{k_2} \tilde{S}|_x, \partial_y^{k_1} \partial_z^{k_2} \tilde{S}|_x \in S^{k_1} T_x L \otimes S^{k_2} (\otimes^2 T_x L).$$

We have the following direct descriptions for $\tilde{S}L$ and $\tilde{S}$.

**Proposition 6.** $\tilde{S}L(x, \alpha, \nabla \alpha) = S L(\alpha)|_x$, $\tilde{S}(x, \alpha, \nabla \alpha) = S(\alpha, \nabla \alpha)|_x$. Suppose $[L]$ is a special Lagrangian submanifold, then $\tilde{S}L, \tilde{S}$ are real analytic functions.

**Proof.** By Proposition 1, we compute $I_{\nabla \alpha}(y) = (y, \nabla y \alpha) = \Gamma \circ (\iota_{\alpha})_* y$. Therefore, we compute

$$\tilde{S}L(x, \alpha, \nabla \alpha) = \star I_{\nabla \alpha} (\Gamma^{-1})^* \Omega = \star (\Gamma^{-1} \circ I_{\nabla \alpha})^* \Omega = \star \iota_{\alpha}^* \Omega,$$

while a similar computation holds for $\tilde{S}$.

As $\Omega$ is holomorphic, over $U_L$, $\tilde{S}L$ is real analytic. When $[L] = (L, \iota)$ is a special Lagrangian submanifold, by [MOR66], $\iota^* g_X$ is a real analytic metric on $L$. Thus, the pseudo holomorphic volume form $\Omega_0$ and $T = \Omega - \Omega_0$ are real analytic forms. Therefore, $\tilde{S}$ is also a real analytic function.  

The following proposition explains the structure of the special Lagrangian equation.

**Proposition 7.** Let $[L]$ be a special Lagrangian submanifold, over the Weinstein neighborhood $(U_L, J, \omega, \Omega)$ with Calabi–Yau metric $g$, the following holds:

(i) The special Lagrangian equation for a closed 1-form $\alpha$ can be written as

$$SL(\alpha) := \star \iota_{\alpha}^* \Omega = -d^* (\alpha) + P(\nabla \alpha) + S(\alpha, \nabla \alpha),$$

where $\star$ is the Hodge star operator on the zero section, $P$ is a real analytic function of $\nabla \alpha$ defined in (12) and $S$ is a real analytic function of $\alpha, \nabla \alpha$.

(ii) Let $x = (x_1, \ldots, x_n)$ be a coordinate on a neighborhood center at $m \in L_0$ with fiber coordinates $y = (y_1, \ldots, y_n)$. We define a function $A_{ij} = (\nabla \partial_{x_i} \alpha, \partial_{x_j})$ with $\nabla$ the Levi-Civita connection of $g$, then we could write

$$S(\alpha, \nabla \alpha) = \sum_{k \geq 0} \sum_{1 \leq i_1 \neq i_2 \ldots \neq i_k \leq n} F_{i_1 \cdots i_k j_1 \cdots j_k} (\alpha) A_{i_1 j_1} \cdots A_{i_k j_k},$$

where $F_{i_1 \cdots i_k j_1 \cdots j_k} (\alpha)$ are real analytic functions of $\alpha$. In addition, $F_0(0) = F_0'(0) = 0$ and $F_{1i j_1}(0) = 0$ for any $1 \leq i, j_1 \leq n$. 
Proof. For (i), as \( \iota^{*}_{\alpha} \Omega = \iota^{*}_{\alpha} \Omega_{0} + \iota^{*}_{\alpha} T \), and \( \frac{d}{dt}|_{t=0} \iota^{*}_{t\alpha} \Omega = \frac{d}{dt}|_{t=0} \iota^{*}_{t\alpha} \Omega_{0} \), we could write
\[
Q(\alpha, \nabla \alpha) = P(\nabla \alpha) + S(\alpha, \nabla \alpha).
\]
By Proposition 6, \( P, S \) are real analytic functions of \( \alpha \) and \( \nabla \alpha \).

For (ii), as \( \alpha \) is closed, we have \( A_{ij} = A_{ji} \). Over a neighborhood with coordinate \((x, y)\), we could write
\[
T = \sum T_{i_{1} \ldots i_{k} j_{1} \ldots j_{n-k}}(x, y)dy_{i_{1}} \wedge \cdots \wedge dy_{i_{k}} \wedge dx_{j_{1}} \wedge \cdots \wedge dx_{j_{n-k}},
\]
where \( T_{i_{1} \ldots i_{k} j_{1} \ldots j_{n-k}} \) is real analytic function with variable \((x, y)\). Then for the pullback, we compute
\[
\iota^{*}_{\alpha} dy_{i} = \sum_{j=1}^{n} \partial_{x_{i}} \alpha_{i} dx_{j} = \sum_{j=1}^{n} (\partial_{x_{i}} \alpha_{i}) dx_{j} = \sum_{j=1}^{n} (A_{ij} + \sum_{k=1}^{n} \alpha_{k} \Gamma^{k}_{ij}) dx_{j},
\]
\[
\iota^{*}_{\alpha} dx_{i} = dx_{i}, \quad (\iota^{*}_{\alpha} T_{i_{1} \ldots i_{k} j_{1} \ldots j_{n-k}})|_{x} = T_{i_{1} \ldots i_{k} j_{1} \ldots j_{n-k}}(x, \alpha|_{x}).
\]
In addition, as \( S(0, 0) = 0 \) and \( S(\alpha, \nabla \alpha) = \ast \iota^{*}_{\alpha} T \), \( S \) can be written as the form
\[
S(\alpha, \nabla \alpha) = \sum_{k \geq 0} \sum_{1 \leq i_{1} \neq i_{2} \ldots \neq i_{k} \leq n} \sum_{1 \leq j_{1} \neq j_{2} \ldots \neq j_{k} \leq n} F_{i_{1} \ldots i_{k} j_{1} \ldots j_{k}}(\alpha) A_{i_{1} j_{1}} \cdots A_{i_{k} j_{k}}.
\]
In addition, as \( \frac{d}{dt}|_{t=0} S(t\alpha, t\nabla \alpha) = 0 \). For the \( k = 0 \) term \( F_{0} \) in the above expression, we have \( F_{0}(0) = F_{0}'(0) = 0 \) and for any \( k = 1 \) terms, we have \( F_{i_{1} j_{1}}(0) = 0 \) for any \( 1 \leq i_{1}, j_{1} \leq n \).

\[ \square \]

4 Multivalued harmonic function and the analytic theory

In this section, we will introduce the analytic theory for the multivalued harmonic function developed in [DON] and we also refer [TAU14] for different settings.

4.1 Multivalued harmonic function and multivalued harmonic 1-form.

Let \((L, g)\) be a compact oriented Riemannian manifold, \( \Sigma \subset L \) be a smooth embedded oriented codimensional two submanifold. Let \( \Xi \) be the group of isomorphism of the real line, which satisfies the exact sequence \( 0 \to (\mathbb{R}, +) \to \Xi \to \{\pm 1\} \to 0 \).

Let \( \chi : \pi_{1}(L \setminus \Sigma) \to \Xi \) be a representation such that for any small loop linking \( \Sigma \), \( \chi \) maps to a reflection. We define \( \chi' \) be the composition of \( \chi \) with the projection \( \Xi \to \{\pm\} \). We write \( V^{-} \) be the associate affine \( \mathbb{R} \) bundle defined on \( L \setminus \Sigma \) and \( \mathcal{I} \) be the vertical line bundle of \( V^{-} \), where \( \mathcal{I} \) is the flat bundle associate with the representation \( \chi' \). Using the flat structure and the Riemannian metric on \( L \), the Laplacian operator make sense as
\[
\Delta_{g} : \Gamma(V^{-}) \to \Gamma(\mathcal{I}).
\]
Definition 6. A multivalued harmonic function consists of \((\Sigma, \chi, f)\), where

(i) \(\Sigma\) is an oriented smooth embedded codimension 2 submanifold of \(L\),
(ii) \(\chi\) is a representation \(\chi : \pi_1(L \setminus \Sigma) \to \Xi\) such that for any small loop linking \(\Sigma\), \(\chi\) maps to a reflection,
(iii) If we write \(V^-\) be the associative affine bundle of \(\chi\) with induced flat structure, then \(f\) is a section of \(V^-\) such that \(\Delta_g f = 0\), where \(\Delta_g\) is defined w.r.t. the flat structure on \(V^-\) and \(df \in L^2\).

Similarly, we could define the following.

Definition 7. A multivalued harmonic 1-form \(\alpha \in L^2\) is a triple \((\Sigma, I, \alpha)\) consisting of

(i) A flat \(\mathbb{R}\) bundle \(I\) with holonomy \(-1\) on small loop linking \(\Sigma\),
(ii) A section \(\alpha \in \Gamma(L \setminus \Sigma, T^*L \otimes I)\) such that \(d\alpha = d\star \alpha = 0\), where \(d\) and \(d\star\) are taken w.r.t. the flat structure of \(I\).

Moreover, \(\alpha\) is called a \(\mathbb{Z}_2\) harmonic 1-form if \(|\alpha|\) is bounded near \(\Sigma\).

A model example of multivalued function would be \(L = \mathbb{C}, \Sigma = \{0\}\), \(f = \Re(z^{k+\frac{1}{2}})\) with \(k \geq 1\) and \(v = df = \frac{2k+1}{2}\Re(z^{k-\frac{1}{2}}dz)\). We could either think of \(v\) as a two valued 1-from over \(\mathbb{C} \setminus \{0\}\) or a section of a \(\mathbb{R}\) vector bundle which has monodromy \(-1\) along loops linking \(\{0\}\).

In general, the multivalued harmonic 1-form are purely topological and we will briefly review the construction in [DON].

First, we will construct the double branched covering of \(L\) along \(\Sigma\). The kernel of \(\chi' : \pi_1(L \setminus \Sigma) \to \{\pm 1\}\) is a index two normal subgroup of \(\pi_1(L \setminus \Sigma)\), which gives us a covering \(p : L' \to L \setminus \Sigma\). In addition, the deck transformation defines an involution on \(L'\). We choose a Riemannian metric on \(L\) and write \(N_{\Sigma}\) be the normal bundle. As \(\Sigma\) is oriented, using the co-orientation, \(N_{\Sigma}\) is a complex vector bundle and the flat \(I\) bundle defines a square root \(N_{\Sigma}^{\frac{1}{2}}\). Using the normal exponential map, a neighborhood of \(\Sigma\) in \(L\) can be identified with a neighborhood of the zero section in \(N_{\Sigma}\).

We write \(\tilde{L} := L' \cup \Sigma\) and we could associate a smooth structure on \(\tilde{L}\). Let \(U\) be an open set of \(m \in \Sigma\) with trivialization \(N_{\Sigma}|_U \cong U \times \mathbb{C}\). Let \(\tilde{x} = (\tilde{x}_3, \ldots, \tilde{x}_n)\) be the coordinate on \(U\), \(\tilde{z}\) be the coordinate on \(\mathbb{C}\), then for \(\tilde{z} \neq 0\), the covering map is locally given by

\[
p : \mathbb{C} \setminus \{0\} \times U \to \mathbb{C} \setminus \{0\} \times U, \quad (\tilde{z}, \tilde{x}) \to (\tilde{z}^2, \tilde{x}),
\]

which naturally extends to the zero section and gives \(\tilde{L}\) a smooth structure. In addition, the involution on \(L'\) extends naturally to an involution \(\sigma\) on \(\tilde{L}\) with fixed point the branch locus. To save notation, we also write \(\Sigma\) be the branch locus on \(\tilde{L}\).

A neighborhood of \(\Sigma \subset \tilde{L}\) can be identified with a neighborhood of the zero section of \(N_{\Sigma}^{\frac{1}{2}}\). Without loss of generality, we could assume that \(\sigma\) acts on each fibre of \(N_{\Sigma}^{\frac{1}{2}}\) by changing the sign and \(\Sigma\) is the fixed point of the involution.
Let \((\Sigma, \mathcal{I}, \alpha)\) be a multivalued harmonic 1-form and with \(p : \tilde{L} \rightarrow L\) the double branched covering constructed above. Based on our assumptions in Definition 7, \(p^*\mathcal{I}\) is trivial on \(\tilde{L} \setminus \Sigma\) and \(p^*\alpha\) is a 1-form over \(\tilde{L}\). As \(p\) is only Lipschitz along \(\Sigma\), the pull-back Riemannian metric \(p^*g\) will be a Lipschitz metric in a neighborhood of \(\Sigma \subset \tilde{L}\). As \(\alpha\) is a harmonic section, \(p^*\alpha\) will satisfy the harmonic equations for the pull-back metric \(p^*g\).

By [WAN93, Lemma 1.5], the space of \(L^2\) cohomology
\[
\{\alpha \in \Omega^1(\tilde{L})|\alpha \in L^2, \ d\alpha = d \ast p^*g \alpha = 0 \text{ over } \tilde{L} \setminus Z\}
\]
is naturally isomorphism to the singular cohomology \(H^1(\tilde{L}; \mathbb{R})\). From another side, the involution \(\sigma : \tilde{L} \rightarrow \tilde{L}\) induces a decomposition \(H^i(\tilde{L}; \mathbb{R}) = H^i(\tilde{L}; \mathbb{R})^+ \oplus H^i(\tilde{L}; \mathbb{R})^-\), while \(H^1(\tilde{L}; \mathbb{R})^\pm\) is the \(\pm 1\) eigenvalue of \(\sigma^*\) and \(p^*\alpha\) represents an element in \(H^1(\tilde{L}; \mathbb{R})^-\).

The following observation is due to Donaldson [DON]. By the Hodge theorem for \(p^*g\), given an element \([\delta] \in H^1(\tilde{L}; \mathbb{R})^-\), there exists a harmonic representative \(\alpha_\delta\) and a harmonic section \(f\) of suitable affine bundle with \(\alpha_\delta = df\). Moreover, the representation \(\chi : \pi_1(L \setminus \Sigma) \rightarrow \Xi\) is up to conjugate determined by \(\chi' : \pi_1(L \setminus \Sigma) \rightarrow \{\pm 1\}\) and \([\delta] \in H^1(\tilde{L}; \mathbb{R})^-\).

**Proposition 8 [DON, Section 4].** Given \(\chi : \pi_1(L \setminus \Sigma) \rightarrow \Xi\) satisfies (ii) of Definition 6, then there exists an unique harmonic section \(f\) of \(V^\perp\). In addition, \(\sigma^*\pi^*df = -\pi^*df\).

We synthesize all of the above discussions to arrive at the following proposition.

**Proposition 9 [DON, Page 2].** Let \(\Sigma\) be an oriented embedded codimensional 2 smooth submanifold, the following four concepts are equivalent:

(i) a representation \(\chi : \pi_1(L \setminus \Sigma) \rightarrow \Xi\) such that for any small loop linking \(\Sigma\), \(\chi\) maps to a reflection,

(ii) a multivalued harmonic function \((\Sigma, V^\perp, f)\),

(iii) a multivalued harmonic 1-form \((\Sigma, \mathcal{I}, \alpha)\),

(iv) a double branched covering \(\tilde{L} \rightarrow L\) branched along a codimension 2 submanifold \(\Sigma\) together with \([\delta] \in H^1(\tilde{L}; \mathbb{R})^-\).

### 4.2 Topological constraints for multivalued harmonic 1-form.

In this subsection, we will discuss the topology for the branched covering \(p : \tilde{L} \rightarrow L\). Let \(\sigma\) be the involution on \(\tilde{L}\), \(C_*(L), C_*(\tilde{L})\) be the singular chains on \(L\) and \(\tilde{L}\), then the transformation morphism for the branched covering \(p : \tilde{L} \rightarrow L\) is a map of singular chain
\[
T : C_*(L) \rightarrow C_*(\tilde{L}), \ T(\gamma) = \tilde{\gamma} + \sigma^*\tilde{\gamma},
\]
where \(\gamma\) is a small enough chain and \(\tilde{\gamma}\) is any lifting of \(\gamma\). In other word, the transformation morphism map a singular chain to the sum of two distinct lifts. If we write \(H_*(\tilde{L}; \mathbb{R})^+\) be the homology invariant under the involution, then the transformation map induces morphism \(T_* : H_*(L; \mathbb{R}) \rightarrow H_*(\tilde{L}; \mathbb{R})^+\).
**Lemma 1.** Im $p^* \subset H^i(\tilde{L}; \mathbb{R})^+$ and $p^* : H^i(L; \mathbb{R}) \to H^i(\tilde{L}; \mathbb{R})^+$ induces an isomorphism.

*Proof.* As $p \circ \sigma = p$, for $x \in H^i(L; \mathbb{R})$, we have $p^* x \in H^i(L; \mathbb{R})^+$. Let $T_* : H_i(L; \mathbb{R}) \to H_*(\tilde{L}; \mathbb{R})^+$ be the induced map of the transformation, then we have $p_* \circ T_* = 2$ and $T_* \circ p_*|_{H_*(\tilde{L}; \mathbb{R})^+} = 2$. Therefore, $p_* : H_1(\tilde{L}; \mathbb{R})^+ \to H_1(L; \mathbb{R})$ is both injective and surjective thus induces an isomorphism, which implies that $p^*$ is also an isomorphism. □

Let $b_1(\tilde{L})^\pm := \dim H^1(\tilde{L}; \mathbb{R})^\pm$, then $b_1(\tilde{L}) = b_1(\tilde{L})^+ + b_1(\tilde{L})^-$ with $b_1(\tilde{L})^+ = b_1(L)$. Combing with the Hodge theorem for the pull-back metric, we obtain:

**Corollary 3.** Suppose there exists a multivalued harmonic function $(\Sigma, \chi, f)$ over $L$, then for the double branched covering $p : \tilde{L} \to L$, we have $b_1(\tilde{L}) > b_1(L)$.

We could construct multivalued harmonic 1-form in the following way.

**Example 1.** Let $(L, g)$ be a Riemannian 3-dimensional rational homology sphere and $\Sigma$ be an oriented link of $L$, there exists a canonical morphism $\mu : \pi_1(L \setminus \Sigma) \to \{\pm 1\}$, which defines a double branched covering $p : \tilde{L} \to L$ branched along $\Sigma$. By [Lic97, Corollary 9.2], suppose the determinant of the link is zero, then $H^1(L; \mathbb{R}) \neq 0$. By Lemma 1, we have $H^1(\tilde{L}; \mathbb{R})^-= H^1(\tilde{L}; \mathbb{R})$. By the Hodge theorem for $p^* g$, for any $[\delta] \in H^1(\tilde{L}; \mathbb{R})^-$, we could find a harmonic representative $\alpha_{[\delta]}$ that gives a multivalued harmonic 1-form on $L$.

The determinant is a topological invariant of knots and links over rational homology sphere. The determinant of a link is zero implies the link has more than one component, Haydys [Hay20] showed that the branch locus $\Sigma$ in this case is disconnected, which can be generalized to higher dimension.

**Lemma 2 [LW95].** There exists a long exact sequence

$$\cdots \to H_i(L, \Sigma; \mathbb{Z}_2) \xrightarrow{T_*} H_i(\tilde{L}; \mathbb{Z}_2) \xrightarrow{p_*} H_i(L; \mathbb{Z}_2) \xrightarrow{\partial_i} H_{i-1}(L, \Sigma; \mathbb{Z}_2) \to \cdots , \quad (17)$$

where $T_*$ is the transformation map.

*Proof.* We consider the chain with $\mathbb{Z}_2$ coefficients. As $T_*|_{C_*(\Sigma)} = 0$, we have short exact sequence $0 \to C_*(L, \Sigma; \mathbb{Z}_2) \xrightarrow{T_*} C_*(\tilde{L}; \mathbb{Z}_2) \xrightarrow{p_*} C_*(L; \mathbb{Z}_2) \to 0$, which implies the claim. □

**Proposition 10.** Suppose there exists a multivalued harmonic 1-form on $L$ with $H_1(L; \mathbb{Z}) = 0$, then $\Sigma$ has to be disconnected.

*Proof.* Let $k$ be the number of connected components, by the long exact sequence of relative homology of $(L, \Sigma)$, we obtain

$$\to H_1(L; \mathbb{Z}) \to H_1(L, \Sigma; \mathbb{Z}) \to H_0(\Sigma; \mathbb{Z}) \to H_0(L; \mathbb{Z}) \to H_0(L, \Sigma; \mathbb{Z}) \to 0.$$

As $H_1(L; \mathbb{Z}) = 0$, we obtain $H_1(L, \Sigma; \mathbb{Z}) \cong \mathbb{Z}^{k-1}$ and (17) implies $H_1(\tilde{L}; \mathbb{Z}_2) \xrightarrow{T_*} H_1(L; \mathbb{Z}_2) \to 0$. Suppose $\Sigma$ is connected, $k = 1$, then $H_1(\tilde{L}; \mathbb{Z}_2) = 0$ which implies $b_1(\tilde{L}) = 0$. However, this violates the existence of multivalued harmonic 1-form. □
4.3 The differential forms on the branched covering. Let \( p : \tilde{L} \to L \) be the branched covering map and \( \sigma \) be the involution map defined on \( \tilde{L} \). The involution induces a decomposition of \( k \)-forms \( \Omega^k(\tilde{L}) = \Omega^k(\tilde{L})^- \oplus \Omega^k(\tilde{L})^+ \), where \( \Omega^k(\tilde{L})^\pm \cong \Omega^k(L) \), \( \Omega^k(\tilde{L})^- \cong \Omega^k(L, \mathcal{I}) \), and \( \Omega^k(L, \mathcal{I}) \) is \( \mathcal{I} \) valued \( k \)-form on \( L \setminus \Sigma \). Therefore, given any \( \tilde{\alpha} \in \Omega^k(\tilde{L}) \), we could write \( \tilde{\alpha} = p^*\alpha^+ + p^*\alpha^- \) where \( \alpha^+ \in \Omega^1(\tilde{L}) \) and \( \alpha^- \in \Omega^k(L, \mathcal{I}) \). We define \( \alpha := \alpha^+ + \alpha^- \) and \( p^*\alpha = \tilde{\alpha} \). Moreover, as \( \alpha^- \) can be understood as a two valued 1-form over \( L \), we could also regard \( \alpha \) be a two valued 1-form on \( \tilde{L} \). As \( \alpha \) consists of two parts, we usually call \( \alpha \) ”a pair”.

Near the branch locus, let \( m \in \Sigma \subset L \) with \( U \subset L \) a neighborhood of \( m \), we choose \( x = (z = x_1 + \sqrt{-1}x_2, x_3, \ldots, x_n) \) such that \( \Sigma \cap U = \{z = 0\} \). Let \( \tilde{m} = p^{-1}(m) \) and we could choose a coordinate system \( (\tilde{z} = \tilde{x}_1 + \sqrt{-1}\tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n) \) on \( \tilde{U} = p^{-1}(U) \) such that \( p \) is given by \( p(\tilde{z}, \tilde{x}_3, \ldots, \tilde{x}_n) = (\tilde{z}^2, \tilde{x}_3, \ldots, \tilde{x}_n) \).

We compute \( p^*dz = 2\tilde{z}d\tilde{z}, p^*(z^{k-\frac{1}{2}}dz) = 2\tilde{z}^{2k}d\tilde{z}, p^*dx_i = d\tilde{x}_i \). and an unfavorable situation is when \( k = 0 \), where the pullback would be \( p^*(z^{k-\frac{1}{2}}dz) = 2d\tilde{z} \in \Omega^1(\tilde{L})^- \).

For \( \alpha = p^*\alpha = p^*\alpha^+ + p^*\alpha^- \in \Omega^1(\tilde{L}) \), suppose \( \alpha^- \) is bounded, then by the assumption on the holonomy of the bundle, we have \( |\alpha^-|_{\Sigma} = 0 \) and \( |\alpha|_m = p^*\alpha^+|_m \). Recall that \( p^* \) is an isomorphism over \( L \setminus \Sigma \), for \( \alpha \) will corresponding \( \alpha^- \) bounded, we could define the following inclusion

\[
\iota_{\alpha} = \iota_{\alpha} : \tilde{L} \to T^*L, \quad \iota_{\alpha}(\tilde{x}) = \iota_{\alpha} := (p(\tilde{x}), (p^*)^{-1}\alpha|_{\tilde{x}}).
\]

4.4 Norms of multivalued forms. In this subsection, we will discuss functional spaces introduced by Donaldson [DON]. Let \( \Sigma \) be an embedded codimension 2 submanifold of \((L, g)\), \( \mathcal{I} \) be a flat \( \mathbb{R} \) bundle satisfies Definition 7 with branched covering \( p : \tilde{L} \to L \).

4.4.1 Norms on \( \mathcal{I} \) valued forms. Let \( U \) be a open neighborhood of a point \( m \in \Sigma \) in \( L \) and \( U' \) be an open neighborhood of the zero section of \( N_\Sigma \), then for a suitable choice of \( U \) and \( U' \), there exists a diffeomorphism \( z : U \to U' \) which is the inverse of the normal exponential map. The co-orientation of \( \Sigma \) and the real line bundle \( \mathcal{I} \) makes \( N_\Sigma \) a complex vector bundle with square root. If \( p \) is a half integer, for \( \sigma_p \in \Gamma(N_\Sigma^{-p}) \), \( \sigma_pz^p \) is a section of the complexified bundle \( N_\Sigma \) over \( U \setminus \Sigma \). Let \( t = (t_3, \ldots, t_n) \) be coordinate on \( \Sigma \), then together with \( z = re^{i\theta} \), we obtain a coordinate system on \( U \).

Fix \( \gamma \in (0, \frac{1}{2}) \), we define the Hölder norm on sections of \( V \) to be

\[
\|s\|_{C^{\gamma}} = \sup_{p \neq p'} \frac{|s(p) - s(p')|}{|p - p'|^{\gamma}},
\]

where the supremum is taken for \( p = (z, t), p' = (z', t') \) satisfy \( |p - p'| \leq \frac{1}{2} \min(|z|, |z'|) \), which means that there is no ambiguity in defining \( |s(p) - s(p')| \). By parallel transport around a polygon, one sees that \( |s(z, t)| \leq C|s|_{C^{\gamma}}|z|^\gamma \), which controls the \( C^0 \) norm.
We define $T_k$ be the set of differential operators given by elements of degree $k$ in vectors fields of the form $\{r \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}\}$. The $D^{k,\gamma}$ norm of sections of $V^-$ can be written as

$$\|s\|_{D^{k,\gamma}} = \max_{0 \leq j \leq k, D \in T_k} \|Ds\|_{C^\gamma}.$$  

Similarly, the $D^{k,\gamma}$ norm can be defined for $\Omega^k(L,I)$ using the Riemannian metric on $L$.

4.4.2 Norms on pairs. In most cases considered in our paper, we only have a natural metric $g$ on $L$ instead of $\tilde{L}$. When we try to measure forms on $\tilde{L}$, we usually treat them as a form on $L$ and a section on $I$. One might also be able to define similar norms using $p^*g$ over $\tilde{L}$.

Let $f$ be a continuous function on $(L, g)$ with $\nabla$ the Levi-Civita connection, we define the $C^{k,\gamma}$ norm $\|\cdot\|_{C^{k,\gamma}}$ on $L$ in the usual way as

$$\|f\|_{C^{k,\gamma}} = \sum_{i=0}^k \sup_{i} |\nabla^i f| + \sup_{p \neq p'} \frac{|\nabla^i f(p) - \nabla^i f(p')|}{|p - p'|^\gamma},$$  

which extends to differential forms on $L$.

Let $\tilde{\alpha}$ be a p-form on $\tilde{L}$, then as Sect. 4.3, we could write $\tilde{\alpha} = p^*\alpha^+ + p^*\alpha^-$ with $\alpha^+ \in \Omega^p(L)$ and $\alpha^- \in \Omega^p(L,I)$. We also write $\alpha = \alpha^+ + \alpha^-$, then we define the $C^{k,\gamma}$ norm $\|\cdot\|_{C^{k,\gamma}}$ of $\tilde{\alpha}$ or $\alpha$ as $\|\tilde{\alpha}\|_{C^{k,\gamma}} = \|\alpha\|_{C^{k,\gamma}} := \|\alpha^+\|_{C^{k,\gamma}} + \|\alpha^-\|_{C^{k,\gamma}}$.

4.4.3 Polyhomogeneous expansions. Now we will introduce the definition polyhomogeneous, we refer [MAZ91, Section 2A] for more details. Let $(\Sigma, \chi, f)$ be a multivalued function with associate affine bundle $V^-$ and $f \in \Gamma(V^-)$ be a section. We call $f$ is conormal with order $\lambda_0$ if

$$r^{-\lambda_0} |(r \partial_r)^i \partial^j_\theta \partial^k_t f| \leq C,$$

where $i, j, k$ are non-negative integers. We say $f$ is polyhomogeneous at $\Sigma$ if $f$ is conormal and there exists a discrete index set $E = \{\gamma_j\}$ with $\lim_{j \to \infty} \gamma_j = \infty$ such that $f$ has an asymptotic expansion

$$f \sim \sum r^{\gamma_j} (\log r)^p f_{jp}(\theta, t),$$  

where the exponents $\gamma_j$ lies in some discrete index set $E \subset \mathbb{R}$ with $\lim_{j \to \infty} \gamma_j = \infty$. In addition, the powers $p$ of $\log r$ are all positive integers which are only finitely many for each $\gamma_j$. Moreover, the $\sim$ means

$$|f - \sum_{j \leq N} r^{\gamma_j} (\log r)^p f_{jp}(\theta, t)| \leq C r^{\gamma_N + 1} (\log r)^q,$$

where the term on the right is the next most singular term and we requires that the corresponding statement holds by differentiating any finite number of times.
4.5 Elliptic theory for the multivalued harmonic equations. The elliptic theory for multivalued functions, established in [DON], will be introduced in the following.

Definition 8. Given a representation \( \chi \) satisfies Definition 6 (ii), we write \( V^- \) be the associative affine bundle with \( \mathcal{I} \) the vertical line bundle. Given a multivalued function \( f \in \Gamma(V^-) \), \( f \) is called nondegenerate if \( f \) has an expansion \( f = \Re(Bz^\frac{3}{2}) + o(r^2) \), such that \( B \) is nowhere vanishing section along \( \Sigma \).

In addition, \( \alpha^- \in \Omega^1(L, \mathcal{I}) \) is called nondegenerate if there exists \( V^- \) and \( f \in \Gamma(V^-) \) with \( df = \alpha^- \) such that \( f \) is nondegenerate. Moreover, a pair \( \alpha = \alpha^+ + \alpha^- \) is called nondegenerate and harmonic if \( \alpha^\pm \) are both harmonic.

The following example explains the motivation to introduce the concept of non-degenerate.

Example 2. Let \((L, g)\) be a Riemannian surface and let \((\Sigma, g, I, v)\) be a \( \mathbb{Z}_2 \) harmonic 1-form, with \( \Sigma \) a collection of points on \( L \). Then there exists a two valued holomorphic 1-form \( \alpha \) such that \( \Re(\alpha) = v \) and \( \alpha \otimes \alpha \) is a meromorphic quadratic differential. Let \( p \in \Sigma \), we use \( z \) be a local coordinates around \( p \) with \( p = \{ z = 0 \} \). Then near \( p \), \( \alpha \otimes \alpha \) will have the asymptotic \( \alpha \otimes \alpha = Az^{-1}dz \otimes dz + Bzdz \otimes dz + O(z^2) \).

The condition \( A = 0 \) means \( \alpha \otimes \alpha \) is a holomorphic quadratic differential and \( B \neq 0 \) means \( \alpha \otimes \alpha \) has simple zeroes. Therefore, we could regard the definition of nondegenerate as a generalization of quadratic differential with simple zeroes.

There are also examples of nondegenerate \( \mathbb{Z}_2 \) harmonic 1-forms over 3-manifolds.

Example 3 [HE, Theorem 1.3, Corollary 3.3]. Let \( L \) be a closed 3-manifold and \( \Sigma \) be a link on \( L \) with determinant \( \det(\Sigma) = 0 \), then for generic metric, there exists nondegenerate \( \mathbb{Z}_2 \) harmonic 1-form over the three cyclic branched covering of \( L \) along \( \Sigma \). Moreover, [HE, Corollary 1.4] construct infinite numbers of rational homology spheres that admit nondegenerate \( \mathbb{Z}_2 \) harmonic 1-forms.

Now, we will introduce the elliptic theory. The Laplacian operator is understood as

\[ \Delta_g : D^{k+2,\gamma}(V^-) \to D^{k,\gamma}(\mathcal{I}) \]

and we have

Theorem 6 [DON, MAZ91]. Let \( \rho \in \Gamma(\mathcal{I}) \) be a \( D^{k,\gamma} \) section, then there exists a \( f \in D^{k+2,\gamma}(V^-) \) satisfies \( \Delta_g f = \rho \). In addition, \( f \) has an asymptotic expansion

\[ f = \Re(Az^\frac{3}{2} + Bz^\frac{5}{2}) + E, \tag{20} \]

with \( |E| \leq C\gamma^2+\gamma \| \rho \|_{D^{2,\gamma}} \) near \( \Sigma \). Moreover, suppose \( \rho \in C^{\gamma}(\mathcal{I}) \) is polyhomogeneous with order \( \frac{1}{2} \) and index set \( \{ k + \frac{1}{2} | k \in \mathbb{N} \} \), then \( f \) is polyhomogeneous with \( E \) an expansions

\[ E \sim \sum_{k \geq 2} \sum_{0 \leq p \leq p_k} E_k(p, \theta, t)p^{k+\frac{1}{2}}(\log r)^p, \]
where $p_k$ are finite integers.

The above result is a combination of the results found in [DON, MAZ91]. More precisely, the leading expansion estimates can be obtained from [DON, Section 2], as shown in (20). In [DON, Section 3], the elliptic estimates of the operator $\Delta_g$ are presented using norms $D^{k,\gamma}$ and using these estimates, one can find suitable solutions that lie within the space $D^{\infty,\gamma}$. Furthermore, [MAZ91, Section 2A, Section 7] demonstrates that for every functions in $D^{\infty,\gamma}$ satisfying an elliptic edge equation like $\Delta_g f = \rho$, the existence of the polyhomogeneous expansion can be proven using the Mellin transformation [MAZ91, Theorem 7.3].

5 The special Lagrangian equation for multivalued 1-forms

Motivated by Donaldson [DON] as in Question 1, we will explicit discuss the special Lagrangian equation for multivalued 1-forms.

5.1 Examples of branched deformations. To warm up, we introduce some toy examples of the branched deformation of special Lagrangian submanifolds. Let $\mathbb{C}^2$ be the two dimensional complex plane with complex coordinates $z = x_1 + \sqrt{-1}y_1, w = x_2 + \sqrt{-1}y_2$. We consider the following Calabi–Yau structure $(\mathbb{C}^2, J, \omega, \Omega)$ with metric $g$, where

$$g = dx_1^2 + dx_2^2 + dy_1^2 + dy_2^2, \quad J(\partial_{x_1}) = \partial_{x_2}, \quad J(\partial_{y_1}) = -\partial_{y_2}, \quad \omega = dx_1 \wedge dx_2 - dy_1 \wedge dy_2,$$

$$\Im \Omega = dx_1 \wedge dy_1 + dy_1 \wedge dx_2, \quad \Re \Omega = -dx_1 \wedge dy_1 - dx_2 \wedge dy_2,$$

which is a hyperKähler rotation of the standard Calabi–Yau structure and it is straightforward to see that $dz \wedge dw = \omega - i\Im \Omega$.

As a real 2-dimensional submanifold $L \subset \mathbb{C}^2$ is a special Lagrangian submanifold if $\omega|_L = \Im \Omega|_L = 0$. Therefore, $L$ is a special Lagrangian if and only if $dz \wedge dw|_L = 0$, which holds when $L$ is a holomorphic submanifold with respect to the complex coordinates $(z, w)$.

Let $L_0$ be the complex $z$ plane, which is also a special Lagrangian submanifold. We consider a multivalued harmonic function $f^k_t = t^{\frac{2}{2k+1}} \Re(z^{2k+1})$ over $L_0 \setminus \{z = 0\}$ and write $z = re^{i\theta}$, then the defining equations of the graph of $df^k_t$ are $L^k_t := \{(z, w) \in \mathbb{C}^2| w^2 = t^2 z^{2k-1}\}$.

When $k = 1$, $L^1_t$ will be a family of smooth special Lagrangian submanifolds such that $L^1_t \to 2L_0$ as current when $t \to 0$. When $k > 1$, we found $L^k_t$ will not be a smooth submanifold and when $k = 0$, as $df^k_t$ behave as $z^{-1}$ along $\{z = 0\}$, $L^0_t$ will be a large deformation of $L_0$.

5.2 The splitting of the special Lagrangian equation. Let $U_L \subset T^*L$ be Weinstein neighborhood with $\theta$ denoting the Lagrangian angle of the zero section. Given a pair $a = \alpha^+ + \alpha^-$ and we write $\tilde{\alpha} = p^*a \in \Omega^1(\tilde{L})$ with the inclusion $[\tilde{L}_{\tilde{a}}] := (\tilde{L}, \iota_{\tilde{a}})$ defined in (18), which is the graph of $\tilde{\alpha}$ on $U_L$. Then $[\tilde{L}_{\tilde{a}}]$ is a symplectic manifold if $\iota_{\tilde{a}}^*\omega = 0$, which is equivalent to $d\alpha = 0$. The special Lagrangian condition for $\tilde{L}$ would be $\iota_{\tilde{a}}^*\Im \Omega = 0$. 

**Definition 9.** The special Lagrangian equation for \([L_\alpha]\) is defined as \(\text{SL}(\tilde{\alpha}) := \star_p g \iota^*_\alpha \Omega\).

The above equation can be understood as an equation on \(L\). As \(\iota^*_\alpha \Omega\) is an n-form over \(\tilde{L}\), under the splitting induced by the involution, we could write

\[
i^*_\alpha \Omega = p^* \beta^+ + p^* \beta^-,
\]

where \(\beta^+ \in \Omega^n(L)\) and \(\beta^- \in \Omega^n(L, \mathcal{I})\). We define \(\text{SL}(\alpha)^\pm = \star_g \beta^\pm\), where \(\text{SL}(\alpha)^+\) is a function on \(L\) and \(\text{SL}(\alpha)^-\) is an \(\mathcal{I}\) valued function on \(L\). As \(\star_p g \beta^\pm = p^*(\star_g \beta^\pm)\), we obtain

\[
\text{SL}(\tilde{\alpha}) = p^*(\text{SL}(\alpha)^+) + p^*(\text{SL}(\alpha)^-). \tag{21}
\]

If we write \(\sigma\) be the involution over \(\tilde{L}\), then \(\sigma^* p^*(\text{SL}(\alpha)^\pm) = \pm p^*(\text{SL}(\alpha)^\pm)\)

In particular, we could define \(\text{SL}(\alpha) := \text{SL}(\alpha)^+ + \text{SL}(\alpha)^-\), which can be considered as a two-valued function on \(L\). Moreover, as \(\alpha\) can be regarded as a two-valued form, analogy to Proposition 5, we could write

\[
\text{SL}(\alpha) = \sin \theta - d^*(\cos \theta \alpha) + Q(\alpha),
\]

where each terms could also be understood as a two-valued function.

**Proposition 11.** \([\tilde{L}_\alpha]\) is a special Lagrangian if and only if \(\text{d} \alpha = 0\) and \(\text{SL}(\alpha) = 0\).

**Proof.** The non-trivial part is the "if" part. By (21), \(\text{SL}(\alpha) = 0\) implies \(\text{SL}(\alpha)^\pm = 0\), thus \(\star_g \beta^\pm = 0\). In addition, as \(p^*(\star_g \beta^\pm) = \star_p g p^* \beta^\pm\), \(\text{SL}(\alpha) = 0\) implies \(\iota^*_\alpha \Omega = 0\). \(\square\)

Similarly, for \(\alpha = \alpha^+ + \alpha^-\), for each terms in the decomposition of special Lagrangian equation in Propositions 5, 7, we could define \(Q(\alpha), P(\alpha), S(\alpha)\) with the decomposition

\[
P(\alpha) = P^+(\alpha) + P^-(\alpha), \quad S(\alpha) = S^+(\alpha) + S^-(\alpha).
\]

As \((d^*\alpha)^- = d^* \alpha^-, (d^*\alpha)^+ = d^* \alpha^+\), then the decomposition of special Lagrangian equation can be written as

\[
\text{SL}(\alpha)^+ = \sin \theta - d^*(\cos \theta \alpha^+) + Q(\alpha)^+, \quad \text{SL}(\alpha)^- = -d^*(\cos \theta \alpha^-) + Q(\alpha)^-. \tag{22}
\]

Let \(\iota_0 : L \to U_L \subset T^* L\) be the inclusion of the zero section, we write \([\iota_0^* \Omega]\) be the homology class in \(H^n(L; \mathbb{R})\), then we have the following.

**Proposition 12.** Suppose \([\iota_0^* \Omega] = 0 \in H^n(L; \mathbb{R})\), then \(\int_L \text{SL}(\alpha)^+ d\text{Vol}_g = 0\).
Proof. As \([\iota_0^*\overline{\Omega}] = 0\), then \([(\iota_0 \circ p)^*\overline{\Omega}] = 0\). We write \(\iota_t: \tilde{L} \to U_L(s)\) be the family of submanifolds defined by the graph of \(t\tilde{\alpha}\) with \(t \in \mathbb{R}\), then

\[
\int_{\tilde{L}} SL(\tilde{\alpha})dVol_{p^*g} = \int_{\tilde{L}} (\iota_t\tilde{\alpha})^*\overline{\Omega} = \int_{\tilde{L}} (\iota_0 \circ p)^*\overline{\Omega} = 0.
\]

In addition, by (21), we have

\[
\int_{\tilde{L}} SL(\tilde{\alpha})dVol_{p^*g} = \int_{\tilde{L}} p^*(SL(\alpha)^+)dVol_{p^*g} + \int_{\tilde{L}} p^*(SL(\alpha)^-)dVol_{p^*g}.
\]

As \(\sigma^*dVol_{p^*g} = dVol_{p^*g}\) and \(\sigma^*p^*SL(\alpha)^- = -p^*SL(\alpha)^-\), we obtain \(\int_{\tilde{L}} p^*SL(\alpha)^+dVol_{p^*g} = 0\). By definition, we obtain

\[
\int_{\tilde{L}} SL(\alpha)^+dVol_g = \frac{1}{2}\int_{\tilde{L}} p^*SL(\alpha)^+dVol_{p^*g} = 0. \tag*{\qed}
\]

5.3 Special Lagrangian equations over a fixed real locus. An antiholomorphic involution \(R\) over a Calabi–Yau manifold \((X, J, \omega, \Omega)\) with Calabi–Yau metric \(g\) will satisfy \(R^*\omega = -\omega\), \(R_\ast \circ J = -J \circ R_\ast\), \(R^*g = g\), \(R^*\Omega = \overline{\Omega}\). By [HIT97], the fixed point of \(R\) is a special Lagrangian. Moreover, over a cotangent bundle \(T^*L\), there exists a canonical involution map \(R_0: T^*L \to T^*L\) which maps \((x, v) \to (x, -v)\), where \(v \in T^*_xL\).

Definition 10. A special Lagrangian manifold \([L]\) is called locally a fixed real locus if there exists a \(R_0\)-invariant Weinstein neighborhood \(U_L \subset T^*L\) such that \(R_0\) will be an anti-holomorphic involution for the pull-back Calabi–Yau structure over \(U_L\).

We should note that being the fixed point of an anti-holomorphic involution will imply that the special Lagrangian submanifold is totally geodesic. There are a lot of special Lagrangians that are not totally geodesic. For example, every holomorphic curve over \(\mathbb{C}^2\) which is not a complex line is not totally geodesic but a special Lagrangian. Moreover, for a fixed real locus, the Weinstein neighborhood also have extra symmetry.

Proposition 13. Suppose \([L]\) is the fixed real locus of a global anti-holomorphic involution \(R\), then we could find a \(R_0\)-invariant neighborhood \(U_L\) on \(T^*L\), a \(R\)-invariant neighborhood \(U\) of \([L]\) in \(X\) and a diffeomorphism \(\Phi: U_L \to U\) such that \(\Phi^*\omega = \omega_0\), \(\Phi^*R = R_0\).

Proof. The claim is followed by a straight forward step-by-step check of the original proof of the Weinstein neighborhood theorem, we will leave the verification to the reader. \(\tag*{\qed}\)

Proposition 14. Let \(L\) be a locally fixed real locus and let \(\alpha^-\) be a multivalued 1-form over \(L\) with \(\iota_{\alpha^-}: \tilde{L} \to T^*L\) be the inclusion, then the special Lagrangian equation for \(\alpha^-\) will satisfy \(SL(\alpha^-)^+ = 0\) and \(Q(\alpha^-)^+ = 0\).
Proof. As \( R \circ i_\alpha^- = i_\alpha^- \circ \sigma \), where \( \sigma \) is the involution on \( \tilde{L} \), we compute

\[-i_\alpha^- R^* \Omega = i_\alpha^- R^* \Omega = \sigma^* i_\alpha^- \Omega,\]

which implies \( \text{SL}(\alpha^-)^+ = 0 \). Therefore, \( Q(\alpha^-)^+ = \text{SL}(\alpha^-)^+ - (\alpha^-)^+ = 0 \). \( \square \)

6 Construction of the approximation solutions

In this section, given a nondegenerate harmonic pair \( a = \alpha^+ + \alpha^- \), for each small \( t \in \mathbb{R} \), we will construct a family of approximate special Lagrangian submanifolds submanifolds close to the graph of \( ta \).

6.1 Main results of the approximation constructions. We first summarize the main results that will be proved in this section. As usual, let \( U_L \) be a neighborhood of the zero section on \( T^* L \), \( (U_L, J, \omega, \Omega) \) be a Calabi–Yau structure and \( t_0 : L \to U_L \) be the inclusion map of the zero section. We write \( g_{U_L} \) be the Calabi–Yau metric and \( g \) be the induced metric on the zero section.

As in this section, we will use diffeomorphisms on \( L \) to change the based metric \( g \), for different terms in the special Lagrangian equation, we will write \( \text{SL}_g, \gamma, P_g, S_g \) will label ”\( g \)” to emphasize which Calabi–Yau structure we are considering to define these terms.

Given a nondegenerate pair \( a = \alpha^+ + \alpha^- \), \( \alpha^- \) defines a branched covering \( p : \tilde{L} \to L \). As previously, we first choose a suitable coordinates near \( \Sigma \), then using normal exponential map, we identify a neighborhood of \( \Sigma \) in \( L \) with a neighborhood \( U \subset N \Sigma \) contains the zero section. We choose a local coordinate system \( (x_1, x_2, \ldots, x_n) \) such that \( z = x_1 + \sqrt{-1} x_2 \) is a complex coordinate on the normal bundle of \( \Sigma \) and \((x_3, \ldots, x_n) \) are coordinates on \( \Sigma \) such that locally \( \Sigma \cap U = \{ z = 0 \} \). We also write \( z = re^{i\theta} \) and \( r \) measures the distance to the branch locus \( \Sigma \). We extends \( r \) to \( L \), which we still denote as \( r \), such that the only zeros of \( r \) is the branch locus. We state our main result in this section as follows:

**Theorem 7.** Given an nondegenerate harmonic pair \( a = \alpha^+ + \alpha^- \) over \( L \) and a positive integer \( N \), there exists a family of nondegenerate pairs \( \mathcal{A}_t = \mathcal{A}_t^+ + \mathcal{A}_t^- \), smooth diffeomorphisms \( \phi_t : L \to L \) with \( r_t := \phi_t^* r \) and a positive number \( T_N \), such that for all \( t < T_N \), there exists a t-independent constant \( C \), the following holds:

(i) \( \|r_t \text{SL}_g(\phi_t^* \mathcal{A}_t)\|_{C^\gamma(L)} \leq C t^{N+1} \).

(ii) \( \|\mathcal{A}_t - ta\|_{C^\gamma(L)} \leq Ct^2 \).

(iii) \( \mathcal{A}_t \) is closed, \( d \mathcal{A}_t = 0 \), and smooth on any open set of \( L \setminus \Sigma \). Near \( \Sigma \), \( \mathcal{A}_t \in C^{1,\gamma} \) is nondegenerate and polyhomogeneous near \( \Sigma \) with expansions

\[
\mathcal{A}_t^- \sim a_1 r^{\frac{1}{2}} + \sum_{k=1}^p a_{2, k} r^{\frac{3}{2}} (\log r)^k + \cdots , \tag{23}
\]

\[
\mathcal{A}_t^+ \sim b_0 + b_1 r + b_2 r^2 + \sum_{k=1}^p b_{2, k} r^2 (\log r)^k + \cdots .
\]
(iv) \( \phi_t(\Sigma) \) is an embedded submanifold of \( L \) and \( \phi_t \) convergence smoothly to \( \Id \).

(v) Let \( \tilde{i}_t : \tilde{L} \to U_L \) be the graph of nondegenerate pair \( \phi_t^*\mathfrak{A}_t \) and \( \theta_t \) be the Lagrangian angle of \( \tilde{i}_t \), then we have \( \|\theta_t\|_{C^\gamma(\tilde{L})} \leq C t^{N-1} \).

(vi) \( \tilde{i}_t(\tilde{L}) \) converge to \( 2_{t_0}(L) \) as current and \( \|\tilde{i}_t - \tilde{i}_0 \circ p\|_{C^\gamma(\tilde{L})} \leq C t \).

Here the \( C^\gamma(\tilde{L}) \) norm on (v) and (vi) will the the induced metric \( \tilde{i}_t^* g_{U_L} \), later by Theorem 9, the \( C^\gamma(\tilde{L}) \) norm will be independent of \( t \).

### 6.2 Primary estimates along the branch locus \( \Sigma \)

In this subsection, we will give estimate for quadratic terms of a pair \( a = \alpha^+ + \alpha^- \). We call \( a \) a polyhomogeneous nondegenerate pair if both \( \alpha^-, \alpha^+ \) are polyhomogeneous and \( \alpha^- \) is nondegenerate along \( \Sigma \).

**Lemma 3.** Let \( a = \alpha^+ + \alpha^- \) be a polyhomogeneous nondegenerate pair, then \( \|r Q_g(ta)\|_{C^\gamma} \leq C t^2 \), where \( C \) depends on \( a \) but independent of \( t \). In addition, near \( \Sigma \), \( Q_g(ta)^- \) and \( Q_g(ta)^+ \) has leading expansions with order \( Q_g(ta)^- \sim t^2 r^{-\frac{1}{2}}, Q_g(ta)^+ \sim t^2 r^{-1} \).

**Proof.** By Proposition 5, we only need to consider the behavior near the branch locus.

As \( Q_g(ta) = P_g(ta) + S_g(ta) \), we will give similar estimates on both \( P_g(ta) \) and \( S_g(ta) \). Based on the choice of coordinates, we have \( \partial_{x_i} z^s = O(z^s)^{-1} \) for \( i = 1, 2 \).

Recall that we have \( P_g(ta) = \sum_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^k P_{2k+1}(ta) \), with

\[
P_{2l+1}(ta) = \ast(t_{ta}^* K_{2l+1}) = \epsilon_{k_1 \ldots k_{2l+1}} \epsilon_{j_1 \ldots j_{2l+1}} H_{j_1}^{k_1}(ta) H_{j_2}^{k_2}(ta) \ldots H_{j_{2l+1}}^{k_{2l+1}}(ta),
\]

where \( H_j^{j}(ta) = H_j^i(t\alpha^+) + H_j^i(t\alpha^-) \), with \( H_j^i(t\alpha^+) = t \sum_k g^{ik} \nabla \alpha^+(\partial_{x_j}, \partial_{x_k}) \).

Therefore, \( H_j^i(t\alpha^-) \) have the following asymptotic behaviors near the branch locus: when \( i, j \in \{1, 2\} \), \( H_j^i(t\alpha^-) \sim tr^{-\frac{1}{2}} \), when \( i \in \{1, 2\}, j \in \{3, \ldots, n\} \), \( H_j^i(t\alpha^-) \sim tr^\frac{1}{2} \) and when \( i, j \in \{3, \ldots, n\} \), \( H_j^i(t\alpha^-) \sim tr^\frac{1}{2} \).

The most singular terms of \( P_g(ta)^- \) will be \( H_1^1(t\alpha^-) H_2^2(t\alpha^+) H_3^3(t\alpha^+) \) and other similar combinations. \( P_g(ta)^- \) have the leading asymptotic \( P_g(ta)^- \sim tr^{-\frac{1}{2}} \) near \( \Sigma \). Similarly, the most singular terms of \( P_g(ta)^+ \) would be \( H_1^1(t\alpha^-) H_2^2(t\alpha^-) H_3^3(t\alpha^+) \) and other similar combinations and we have the worst leading asymptotic \( P_g(ta)^+ \sim tr^{-1} \).

Now, we will estimate \( S(ta) \). By Proposition 7 and a similar argument as the "P" terms, we have \( S_g(ta)^- \sim t^2 r^{-\frac{1}{2}}, S_g(ta)^+ \sim t^2 r^{-1} \), which implies the claim. □

**Lemma 4.** For \( a, b \) polyhomogeneous nondegenerate closed pairs, we have

\[
\|r(Q_g(ta + t^k b) - Q_g(ta))\|_{C^\gamma} \leq C t^{k+1},
\]

where \( C \) depends on \( a, b \) but independent of \( t \).
Proof. By the previous Lemma 3, we have \( \| rQ_g(ta + tkb) \|_{C^1} \leq C \) and \( \| rQ_g(ta) \|_{C^1} \leq C \), which means we don’t need to concern the singularity coming from the branch locus. By (12) and Proposition 7, we have \( P_g(a) = O(|\nabla a|^3) \), \( S_g(a) = O(|a|^2 + |\nabla a|^2) \), which imply the desire bound with order \( t^{k+1} \).

In summary, we obtain the following estimates.

**Proposition 15.** Let \( a = \alpha^+ + \alpha^- \) be a polyhomogeneous nondegenerate pair, then \( \| rSL_g(a) \|_{C^1} \leq C \), where \( C \) depends on \( a \). In addition, near \( \Sigma \), we have the leading expansions,

\[
SL_g(a)^- \sim r^{-\frac{1}{2}}, \quad SL_g(a)^+ \sim r^{-1}.
\]

**Proof.** As \( d^* a = d^* \alpha^- + d^* \alpha^+ \) and \( \alpha^- \sim r^{\frac{1}{2}} \), thus \( \| rd^* a \|_{C^1} \leq C \). In addition, as \( SL_g(a) = d^* a + Q_g(a) \), by Lemma 3, we obtain the desire estimate. \( \square \)

### 6.3 Variation of the singular set.

Let \( (L, \iota_0) \) be the zero section of \( (U_L, J, \omega, \Omega) \) with induced Riemannian metric \( g \). Let \( \Sigma \) be the codimension 2 submanifold of \( L \) with normal bundle \( N_\Sigma \) and let \( v \in \Gamma(N_\Sigma) \) be a section. We choose an extension of \( v \) to a vector field over \( L \), which we still denote by \( v \) to save notation.

The vector field \( v \) generates 1-parameter family of diffeomorphisms

\[
\lambda_s : L \to L, \quad \text{with} \quad \frac{d}{ds} \lambda_s = v \circ \lambda_s, \quad \lambda_0 = \text{Id},
\]

and we write \( \lambda := \lambda_s|_{s=1} \). \( \lambda_s \) could also induced a family of diffeomorphisms on \( T^*L \): let \( (x, y) \in T^*L \) with \( x \in L \) and \( y \in T^*_x L \), we define

\[
\hat{\mu}_s : T^*L \to T^*L, \quad \hat{\mu}_s(x, y) := (\lambda_s(x), \lambda^*_s(y)),
\]

then under the identification \( \Gamma : T^*_x L \oplus T_x L \to T_{(x,y)}(T^*L) \), we have

\[
\hat{V} = \left. \frac{d}{ds} \right|_{s=0} \hat{\mu}_s(x, y) = \Gamma(v, -Lv y) \in T_x L \oplus T^*_x L,
\]

where \( L_y \) is taking the Lie derivative for 1-form \( y \).

Let \( U'_L \) be a proper subset of \( U_L \), \( \chi \) be a cut-off function such that \( \chi|_{U'_L} = 1 \) and \( \chi|_{U_L} = 0 \). We define \( V = \chi \hat{V} \) and write \( \mu_s : U_L \to U_L \) be the 1-parameter family of diffeomorphisms such that \( \frac{d}{ds} \mu_s = V \circ \mu_s \) with \( \mu := \mu_s|_{s=1} \).

We would like to consider the special Lagrangian equation over the Calabi–Yau structure \( (U_L, \mu^* J, \mu^* \omega, \mu^* \Omega) \) for a nondegenerate pair \( a := \alpha^+ + \alpha^- \subset U'_L \), which is defined to be

\[
SL_{\lambda^* g}(a) := \ast \lambda^* g \ell^*_a(\mu^* \Omega).
\]

Similarly, we could define

\[
S_{\lambda^* g}(a) := \ast \lambda^* g \ell^*_a(\mu^* T), \quad P_{\lambda^* g}(a) = P_g(\nabla_{\lambda^* g} a), \quad Q_{\lambda^* g}(a) = S_{\lambda^* g}(a) + P_{\lambda^* g}(a),
\]

where \( \nabla^\lambda u \) is the Levi-Civita connection for the metric \( \lambda^* g \).
Proposition 16. For \( a = \alpha^+ + \alpha^- \subset U' \), we have

\[
\lambda^*SL_g(a) = SL_{\lambda^*g}(\lambda^*a), \quad \lambda^*P_g(a) = P_{\lambda^*g}(\lambda^*a), \quad \lambda^*S_g(a) = S_{\lambda^*g}(\lambda^*a). \tag{27}
\]

Moreover, we could write \( SL_{\lambda^*g}(a) = -d^{\lambda^*}a + Q_{\lambda^*g}(a) \).

Proof. By the definition of \( \lambda \) and \( \mu \), we obtain \( \iota_\lambda \circ \lambda = \mu \circ \iota_{\lambda^*a} \). We compute

\[
\lambda^*SL_g(a) = \lambda^*(\ast_{\lambda^*g} \Omega) = *_{\lambda^*g}(\iota_\lambda \circ \lambda)^* \Omega = *_{\lambda^*g}(\mu \circ \iota_{\lambda^*a})^* \Omega = SL_{\lambda^*g}(\lambda^*a).
\]

In addition, we compute

\[
SL_{\lambda^*g}(a) = \lambda^*SL_g((\lambda^{-1})^*a) = -\lambda^*d^*((\lambda^{-1})^*a) + \lambda^*Q_g((\lambda^{-1})^*a) = -d^{\lambda^*}a + Q_{\lambda^*g}(a).
\]

\( \square \)

The terms in the special Lagrangian equations under the variation will be estimate now.

Lemma 5. (i) We write \( g_s = \lambda^*_s g \), then the following estimates hold,

\[
||\frac{d}{ds}g_s||_{C^\gamma} \leq C||v||_{C^{1,\gamma}}, \quad ||\frac{d^2}{ds^2}g_s||_{C^\gamma} \leq C||v||_{C^{2,\gamma}}, \quad ||\frac{d}{ds}\Gamma_{ij}^k||_{C^\gamma} \leq C||v||_{C^{2,\gamma}},
\]

where \( \Gamma_{ij}^k \) is the Christoffel symbols w.r.t metric \( g_s \) in any smooth coordinates.

(ii) Let \( \alpha \) be an \( n \)-form on \( L \), then \( \frac{d}{ds}\ast_s \alpha = -\frac{1}{2}\text{Tr}(g_s^{-1}\partial_s g_s)\ast_s \alpha \).

Proof. (i) follows directly from the definition of Lie derivative on tensors. For (ii), let \( (x_1, \ldots, x_n) \) be a coordinates on \( L \), \( \ast_s \) be the Hodge star operator for the metric \( g_s \), then we write \( \alpha = fdx_1 \wedge \cdots \wedge dx_n \) and \( \ast_s \alpha = |g_s|^{-\frac{1}{2}}f \), where \( |g_s| \) is the determinant of the metric.

We compute

\[
\frac{d}{ds}\ast_s \alpha = \frac{d}{ds}f|g_s|^{-\frac{1}{2}} = -\frac{1}{2}|g_s|^{-\frac{3}{2}}\partial_s |g_s|f = -\frac{1}{2}|g_s|^{-1}\partial_s |g_s||g_s|^{-\frac{1}{2}}f = -\frac{1}{2}\text{Tr}(g_s^{-1}\partial_s g_s)\ast_s \alpha,
\]

where we use \( \partial_s |g_s| = |g_s|\text{Tr}(g_s^{-1}\partial_s g_s) \) for the last identity. \( \square \)

Lemma 6. Over the Riemannian manifold \( (L, g) \), for \( f \) is either a function or a section of \( I \), we have the following expressions:

\[
\Delta_{\lambda^*g}f = \Delta_g f + \nabla_v(\Delta_g f) - \Delta_g(\nabla_v f) + e_\Delta(f, \lambda, g), \tag{28}
\]

with \( ||r^\frac{1}{2}e_\Delta(f, \lambda, g)||_{C^\gamma} \leq C||r^\frac{1}{2}\nabla \nabla f||_{C^\gamma} ||v||_{C^{2,\gamma}} \).
Proof. For $x \in L$, we define $F(s, x) := \Delta_{\lambda^*_s g} f$, then the Taylor expansion in terms of $s$ would be

$$\Delta_{\lambda^*_s g} f = \Delta_g f + \frac{d}{ds}|_{s=0} \Delta_{\lambda^*_s g} f s + e_\Delta,$$

where $e_\Delta$ is the integral reminder term which can be written as

$$e_\Delta = \int_0^1 \partial_s^2 F(s, x)(1 - s)ds = \int_0^1 \partial_s^2 \Delta_{\lambda^*_s g} f ds.$$

As $\lambda^*_s(\Delta_g f) = \Delta_{\lambda^*_s g} \lambda^*_s f$, we take derivative of $s$ in both side at $s = 0$, we obtain

$$(\frac{d}{ds}|_{s=0} \Delta_{\lambda^*_s g}) f = \nabla_v(\Delta f) - \Delta(\nabla_v f).$$

Let $(x_1, \ldots, x_n)$ be local coordinate on $L$ and now we compute $\partial_s^2 F(s, x)$ in this coordinate. We write $g = \sum_{i,j=1}^n g_{ij} dx_i \otimes dx_j$, then the Laplacian operator in this coordinate can be written as

$$\Delta_g f = \sum_{i,j=1}^n -|g|^{-\frac{1}{2}} \partial_i(|g|^{\frac{1}{2}} g^{ij} \partial_j f) = \sum_{i,j=1}^n -\partial_i(g^{ij} \partial_j f) - \frac{1}{2} \partial_i \log |g| g^{ij} \partial_j f,$$

where $|g| = \det(g_{ij})$. We compute

$$\partial_s^2 F(s, x) = \frac{d^2}{ds^2} \Delta_{\lambda^*_s g} f = \sum_{i,j=1}^n -\frac{d^2}{ds^2}(\lambda^*_s g)^{ij} \partial_i \partial_j f - \frac{d^2}{ds^2} \partial_i(\lambda^*_s g^{ij}) \partial_j f - \frac{1}{2} \partial_i \frac{d^2}{ds^2}(\log |\lambda^*_s g| \lambda^*_s g^{ij}) \partial_j f. \tag{29}$$

A straight forward computation shows that

$$\| \frac{d}{ds} \lambda^*_s g \|_{C^\gamma} = \| \lambda^*_s (L_v g) \|_{C^\gamma} \leq C \| v \|_{C^{1,\gamma}}, \quad \| \frac{d^2}{ds^2}(\lambda^*_s g) \|_{C^\gamma} = \| \lambda^*_s L^2_v g \|_{C^\gamma} \leq C \| v \|_{C^{2,\gamma}}.$$

Therefore for each $s$, $\| r^{\frac{1}{2}} \partial_s^2 F(s, x) \|_{C^\gamma} \leq C \| r^{\frac{1}{2}} \nabla \nabla f \|_{C^\gamma} \| v \|_{C^{2,\gamma}}$, which implies the claim. \hfill \square

**Lemma 7.** Let $a$ be a polyhomogeneous nondegenerate pair, then

(i) $\| r(P_{\lambda^*_s g}(ta) - P_{g}(ta)) \|_{C^\gamma} \leq Ct^3 \| v \|_{C^{2,\gamma}},$

(ii) $\| r(S_{\lambda^*_s g}(ta) - S_{g}(ta)) \|_{C^\gamma} \leq Ct^2 \| v \|_{C^{2,\gamma}},$

(iii) $\| r(Q_{\lambda^*_s g}(ta) - Q_{g}(ta)) \|_{C^\gamma} \leq Ct^2 \| v \|_{C^{2,\gamma}},$

where $C$ is a constant which depends on $a$ but is independent of $v$ and $t$. 

Proof. For (i), we write \( g_s = \lambda^*_s g \), then \( P_{\lambda^*_s g}(ta) - P_g(ta) = \int_0^1 \partial_s P_g(\kappa t a) ds \). By Lemma 5, we have
\[
\|\partial_s g^i_k\|_{C^\gamma} \leq C\|v\|_{C^{1,\gamma}}, \quad \|\partial_s T_{ij}^s\|_{C^\gamma} \leq C\|v\|_{C^{2,\gamma}}.
\]
By the expression of \( \partial_s P_g(ta) \) and the same argument of Lemma 4, near \( \Sigma \), we obtain \( r\partial_s P_g(ta) \sim t^2 r^{-\frac{3}{2}} \), which implies \( ||r\partial_s P_g(ta)||_{C^\gamma} \leq C\|v\|_{C^{2,\gamma}} t^3 \).

For (ii), as \( \tau t a \circ \lambda_s = \mu_s \circ \tau t \lambda_s a \), we compute
\[
S_{\lambda^*_s g}(t\alpha) - S_g(t\alpha) = \int_0^1 \frac{d}{ds} [\star g, \tau t a (\mu^*_s t) ds = \int_0^1 \frac{d}{ds} \lambda^*_s (\star g) (\tau t \lambda^*_s a T) ds. (30)
\]
As \( \star g (\tau t \lambda^*_s a T) = S_g(t\lambda^*_s a) \), by Lemma 3, the leading expansions of \( S_g(t\lambda^*_s a) \) will be
\[
S_g(t\lambda^*_s a)^- \sim \lambda^*_s r^{-\frac{3}{2}}, \quad S_g(t\lambda^*_s a)^+ \sim \lambda^*_s r^{-1}.
\]
As \( \lambda^*_s \lambda^*_s r = r \), which is independent of the order of \( r \) direction. Therefore, the singular terms of \( \frac{d}{ds} \lambda^*_s (\star g) (\tau t \lambda^*_s a T) \) will have order \( r^{-1} \) or \( r^{-\frac{3}{2}} \) near \( \Sigma \), which implies \( r \frac{d}{ds} \lambda^*_s (\star g) (\tau t \lambda^*_s a T) \) is bounded near \( \Sigma \).

On the other hand, we compute
\[
\frac{d}{ds} \lambda^*_s (\star g) (\tau t \lambda^*_s a T) = \lambda^*_s (\nabla_v \star g (\tau t \lambda^*_s a T)) + \lambda^*_s (\star g (\tau t \lambda^*_s (-L_v a) T)) = \lambda^*_s \nabla_v S_g(t\lambda^*_s a) + \lambda^*_s S_g(t\lambda^*_s (-L_v a)). (31)
\]

Near \( \Sigma \), the most singular terms of \( \lambda^*_s \nabla_v S_g(t\lambda^*_s a) \) and \( \lambda^*_s S_g(t\lambda^*_s (-L_v a)) \) will be order \( r^{-2} \) or \( r^{-\frac{3}{2}} \). However, as \( r \frac{d}{ds} \lambda^*_s (\star g) (\tau t \lambda^*_s a T) \) is bounded, the singular terms at order \( r^{-2} \) or \( r^{-\frac{3}{2}} \) will cancel with each other.

By Proposition 7, we could schematically write
\[
\lambda^*_s \nabla_v S_g(t\lambda^*_s a) + \lambda^*_s S_g(t\lambda^*_s (-L_v a)) = t^2 v \otimes A + t^2 \nabla v \otimes B,
\]
where \( A, B \) are \( C^{\gamma} \) terms depends on \( v \) and \( a \). We obtain \( ||r \frac{d}{ds} \lambda^*_s (\star g) (\tau t \lambda^*_s a T)||_{C^\gamma} \leq Ct^2 ||v||_{C^\gamma} \), which implies \( ||S_{\lambda^*_s g}(t\alpha) - S_g(t\alpha)||_{C^\gamma} \leq Ct^2 ||v||_{C^{2,\gamma}} \). (iii) follows by (i) and (ii).

We consider \( a = df - +df - , b = dh = dh^+ + dh^- \) be nondegenerate pairs, where \( f^-, h^- \) are sections of \( V^- \) and \( f^+, h^+ \) are functions on \( L \). We have the following estimates.

**Lemma 8.** Let \( v \) be vector on \( L \) which generates the 1-parameter family of diffeomorphisms \( \lambda_s \) with \( \lambda : = \lambda_s |_{s = 1} \), then we could write
\[
\text{SL}_{\lambda^*_s g}(a + b) = \text{SL}_g(a) - (\Delta g h + \nabla_v \Delta g f - \Delta g \nabla_v f) + E_{\delta Q} + E_{Q_{s g}} + E_{\Delta}. (32)
\]
where
\[
E_{\delta Q} := Q_g(a + b) - Q_g(a), \quad E_{Q_s} := Q_{\lambda^* g}(a + b) - Q_g(a + b),
\]
\[
E_\Delta := e_\Delta(f + \delta, \lambda, \gamma) + \nabla_\gamma \Delta_g \delta - \Delta_g \nabla_\gamma \delta,
\]
where \(e_\Delta(a + b, \lambda, \gamma)\) is the error term defined in (28).

**Proof.** We write \(F := f + \delta\) and we write \(SL_{\lambda^* g}(a + b) = \Delta_{\lambda^* g} F + Q_{\lambda^* g}(a + b)\).

In addition, we compute
\[
\Delta_{\lambda^* g} F = \Delta_g F + \nabla_\gamma \Delta_g F - \Delta \nabla_\gamma F + E_\Delta(F, \lambda, g)
\]
\[
= \Delta_g f + (\Delta_g \delta + \nabla_\gamma \Delta f - \Delta \nabla_\gamma f) + E_\Delta,
\]
where \(E_\Delta = \nabla_\gamma \Delta_g \delta - \Delta_g \nabla_\gamma \delta + e_\Delta(F, \lambda, g)\). We compute
\[
SL_{\lambda^* g}(a + b) = SL_g(a) - (\Delta_g \delta + \nabla_\gamma \Delta f - \Delta \nabla_\gamma f) + E_\Delta + Q_{\lambda^* g}(a + b) - Q_g(a).
\]
As \(Q_{\lambda^* g}(a + b) - Q_g(a) = E_{\delta Q} + E_{Q_s}\), we obtain (32).

Moreover, we have the following proposition for solving the Laplace equation.

**Proposition 17.** (i) Let \(\rho_0^+ \in \mathcal{C}^{2,\gamma}, \rho_1^+ \in \mathcal{C}^{\gamma}\) be polyhomogeneous along \(\Sigma\) with index set \(\{0\}\), suppose \(\int_L \rho_0^+ d\text{Vol} = 0\), then there exists a polyhomogeneous solution \(f^+ \in \mathcal{C}^{2,\gamma}\) to \(\Delta f^+ = r^{\gamma} \rho_0^+ + \rho_1^+\) with expansion
\[
f^+ \sim b_0 + b_1 r + b_2 r^2 + \sum_{k=0}^p b_{3,k} r^3 (\log r)^k + \cdots.
\]

(ii) Let \(\rho_0^-, \rho_1^- \in \mathcal{C}^{\gamma}\) be sections of \(\Gamma(L)\) which is polyhomogeneous along \(\Sigma\) with index set \(\{0, \frac{1}{2}\}\), then there exists a polyhomogeneous solution \(f^- \in \Gamma(L)\) to \(\Delta f^- = r^{\gamma} \rho_0^- + \rho_1^-\) with expansion
\[
f^- \sim a_1 r^{\frac{1}{2}} + a_2 r^{\frac{3}{2}} + \sum_{k=0}^p a_{3,k} r^3 (\log r)^k + \cdots.
\]

**Proof.** For (i), note that \(r^{-1} \rho_0^+\) is not a \(L^2\) function, we could not directly apply the classical theory. However, we could choose \(\mu = r \rho_0^+\), then we compute \(\Delta \mu = r^{-1} \rho_0^+ + \mu\), with \(\mu \in \mathcal{C}^{\gamma}\). The equation we would like to solve becomes \(\Delta(f^+ - \mu) = \rho_1^+ - \mu\) with \(\int (\rho_1^+ - \mu) d\text{Vol} = 0\). In addition, as \(\rho_1^+ - \mu \in \mathcal{C}^{\gamma}\) and by Theorem 6, we could find a solution with \(f^+ \in \mathcal{C}^{2,\gamma}\). \(f^+\) is polyhomogeneous follows by classical result of elliptic edge operator, which we refer to [MAZ91, Theorem 7.14]. For (ii), we define \(\mu^- = r^{\frac{3}{2}} \rho_0^-\), then we could write \(\Delta \mu^- = r^{-\frac{1}{2}} \rho_0^- + \mu^-\) with \(\mu^- \in \mathcal{C}^{\gamma}\). For the equation \(\Delta(f^- - \mu^-) = \rho_1^- - \mu^-\), by Theorem 6, we could find a polyhomogeneous solution with the desire leading expansion. \(\square\)
6.4 Constructing approximate solution. We initiate the construction of Theorem 7 using an inductive approach. Given a nondegenerate harmonic pair \( a_1 = \alpha_1^+ + \alpha_1^- \), we aim to construct the following data for each integer \( k \) and \( t \geq 0 \):

(i) An nondegenerate closed pair \( \mathfrak{A}_k \) with expression \( \mathfrak{A}_k = \sum_{i=1}^k a_i t^i \), with

\[
\alpha_i = \alpha_i^+ + \alpha_i^- = df_i^+ + df_i^-,
\]

where \( f_i^+ \) is a function on \( L \) for \( i \geq 2 \) and \( f_i^- \) is a section of \( V^- \). We also write \( f_i = f_i^+ + f_i^- \) and \( \mathfrak{A}_k = \sum_{i=1}^k f_i \).

(ii) A diffeomorphism \( \varphi_{k,t} \) and a positive number \( T_k \) such that for \( t < T_k \), \( \varphi_{k,t}^{-1}(\Sigma) \) is an embedded codimension 2 submanifold and \( \| rSL_{g_k,t}(\mathfrak{A}_k) \|_{C^\gamma} \lesssim t^{k+1} \), where \( SL_{g_k,t}(\mathfrak{A}_k) \) is the special Lagrangian equation for the variation of Calabi–Yau structure \( \varphi_{k,t}(U_L, J, \omega, \Omega) \) defined in (26) with Calabi–Yau metric \( \varphi_{k,t}^* g_{UL} \) and \( g_{k,t} := \varphi_{k,t}^* g \) is the induced Riemannian metric on the zero section.

Since \( \alpha_1^+ \) is a harmonic 1-form, we can express it as \( \alpha_1^+ = df_1^+ \), where \( f_1^+ \) is a section of an affine line bundle. It is important to note that \( f_1^+ \) and \( f_2^+ \) are sections of distinct bundles; however, to simplify the notation, we continue to write \( f_1^+ + f_2^+ \). This is acceptable because our results only depend on \( df_1^+ + df_2^+ \), which is a meaningful 1-form on \( L \).

The construction of the approximate solution process is divided into several steps, which we briefly outline here before delving into the details. In Step 1, we estimate the quadratic term of the special Lagrangian equation for a harmonic pair. The crux of the construction lies in Step 2, where we perturb the singular set to obtain a nondegenerate solution. In Step 3, we complete the inductive steps and construct the approximate solution, ensuring that the error term is properly addressed. Although the entire construction may seem redundant, as the quadratic term of the special Lagrangian equation might not be symmetric under the involution, it is beneficial for the reader to grasp the main idea of the construction by assuming \( \alpha_1^+ = 0 \) and that all “+” parts of these equations vanish.

We now proceed with the detailed construction:

- **Step 1**: We set \( \phi_1 = \text{Id} \) and demonstrate that \( |rSL_g(\mathfrak{A}_1)|_{C^\gamma} \leq t^2 \).
  
  Let \( \mathfrak{A}_1 := t a_1 \), then we compute \( SL_g(\mathfrak{A}_1) = td^* a_1 + Q_g(t a_1) = Q_g(t a_1) \). By Lemma 3, \( Q_g(t a_1) \) is polyhomogeneous with leading expansions

\[
Q_g(t a_1)^- \sim t^2 r^{-\frac{3}{2}}, \quad Q_g(t a_1)^+ \sim t^2 r^{-1}.
\]

Consequently, we conclude that \( \| rSL_g(\mathfrak{A}_1) \|_{C^\gamma} \lesssim t^2 \).

- **Step 2**: Constructing the nondegenerate multivalued form \( \mathfrak{A}_2 = t a_1 + t^2 a_2 \), the diffeomorphism \( \varphi_{2,t} \) with \( \| rSL_{g_2,t}(\mathfrak{A}_2) \|_{C^\gamma} \lesssim t^3 \).

- **Step 2.1**: We aim to construct a vector field \( v_1 \), a pair \( \mathfrak{A}_2 = \mathfrak{A}_1 + t^2 a_2 \) and a positive number \( T_2 \) such that for \( t < T_2 \), \( \mathfrak{A}_2 \) is nondegenerate and \( \varphi_{2,t}^{-1}(\Sigma) \) is an embedded codimension 2 submanifold.
By Step 1, we have constructed $\mathfrak{A}_1$ such that $\|r\SL_g(\mathfrak{A}_1)\|_{C^\gamma} \lesssim t^2$. We define
\[
\rho_2 := \lim_{t \to 0} t^{-2} \SL_g(\mathfrak{A}_1),
\] (36)
which is the $t^2$ order term in the $t$ expansion. By Step 1, $\rho_2 = \rho_2^+ + \rho_2^-$ has leading expansions $\rho_2^- \sim r^{-1/2}$, $\rho_2^+ \sim r^{-1}$. In addition, we could write
\[
\SL_g(\mathfrak{A}_1) = \rho_2 t^2 + E_{\SL}.
\] (37)

We would like to solve the following equations for $v_1$, $f_2^\pm$ and we define $a_2 := (d\rho_2^+, d\rho_2^-)$:
\[
\rho_2^- - \Delta g(f_2^- - \nabla v_1 f_1^-) = 0, \quad \rho_2^+ - \Delta g(f_2^+ - \nabla v_1 f_1^+) = 0.
\] (38)

To begin with, we will find $v_1$ and $f_2$ to solve the first equation. By Theorem 6, there exists $P_2$ such that $-\Delta g P_2 + \rho_2^- = 0$, with $P_2 = \Re(az^{3/2} + bz^{3/2}) + E P_2$.

Recall that $f_1^-$ is a nondegenerate 2-valued harmonic function with expansion
\[
f_1^- = \Re(B_1 z^{3/2}) + E f_1^-,
\] (39)
and $B_1$ is nowhere vanishing along $\Sigma$. Therefore $\frac{2a}{3B_1}$ is a well-defined section of the normal bundle $N_\Sigma$. We take $v_1$ be an extension of $-\frac{2a}{3B_1}$ over $L$ and compute
\[
\nabla v_1 f_1^- = -\Re(az^{3/2}) + \nabla v_1 E f_1^-.
\]
We define $f_2^- := P_2 + \nabla v_1 f_1^-$ and compute
\[
f_2^- = \Re(az^{12} + bz^{12}) + E P_2 - \Re(az^{12}) + \nabla v_1 E f_1^- = \Re(bz^{3/2}) + E P_2 + \nabla v_1 E f_1^-.
\] (40)

Thus, we could write $f_2^- = \Re(B_2 z^{3/2}) + E f_2^-$ with $|E f_2^-| \leq C r^{3/2 + \epsilon}$. Moreover, the $r^{3/2}$ term of $f_2^- := tf_1^- + t^2 f_2^-$ can be written as $\Re((tB_1 + t^2 B_2)z^{3/2})$. We take $T_2' = \frac{\min_{B}}{2 \max_{B} |B|^2}$, then for any $t \leq T_2'$, $f_2^-$ will be a nondegenerate $\mathbb{Z}_2$ function.

Next we will solve the following equation for a function $f_2^+$:
\[
\rho_2^+ - \Delta g(f_2^+ - \nabla v_1 f_1^+) = 0.
\] (41)

By Proposition 12 and the definition of $\rho_2^+$, we have $\int_L \rho_2^+ d\Vol_g = 0$. In addition, as $\nabla v_1 f_1^+$ is a smooth function and $\int_L \Delta g(\nabla v_1 f_1^+) d\Vol_g = 0$. By Proposition 17, we could find a polyhomogeneous solution $f_2^+ \in C^{2, \gamma}$ to (41) with expansions
\[
f_2^+ \sim f_{2,0}^+ + f_{2,1}^+ r + f_{2,2}^+ r^2 + \sum_{k \geq 3} \sum_{0 \leq \rho \leq p_k} f_{2,k,\rho}^+ r^k (\log r)^p,
\]
for $p_k$ finite integers. We define $f_2 = f_2^+ + f_2^-$ and $a_2 = d f_2$.

We define the $t$-depending 1-parameter family of diffeomorphisms as $\frac{d}{dt} \lambda_{1,s} = t v_1 \circ \lambda_{1,s}$, and $\lambda_1 := \lambda_{1,s} |_{s=1}$ and define $\varphi_{2,t} := \lambda_1$. As $\lim_{t \to 0} \varphi_{2,t} = \Id$, there exists a positive number $T_2''$ such that for any $t < T_2''$, $\varphi_{2,t}^{-1}(\Sigma)$ is an embedded submanifold.

**Step 2.2**: We will prove for $\|r\SL_{\lambda_1,g}(\mathfrak{A}_2)\|_{C^\gamma} \lesssim t^3$. 


We define $\mathfrak{A}_2 := \mathfrak{A}_1 + t^2 a_2$ and we will compute the formal power series expansions of the special Lagrangian equation in terms of $t$. By Proposition 8, we could write

$$\text{SL}_{\lambda^*_t g}(\mathfrak{A}_2) = \text{SL}_g(\mathfrak{A}_1) - t^2(\Delta_g f_2 - \Delta_g \nabla_{v_1} f_1) - t \nabla_{v_1} \Delta_g f_1 + E\delta Q + E Q_{ss} + E \Delta$$

$$= t^2(\rho_2 - \Delta_g (f_2 - \nabla_{v_1} f_1)) - t \nabla_{v_1} \Delta_g f_1 + E\text{SL} + E\delta Q + E Q_{ss} + E \Delta,$$

where

$$E_{\Delta} = e_\Delta(d\mathfrak{A}, \lambda_1, g) + t^3(\nabla_{v_1} \Delta f_2 - \Delta \nabla_{v_1} f_2), \quad E_{Q_{ss}} := Q_{\lambda^*_t g}(\mathfrak{A}_1) - Q_g(\mathfrak{A}_1),$$

$$E_{\delta Q} := Q_g(\mathfrak{A}_1) - Q_g(\mathfrak{A}_0), \quad E\text{SL} = \text{SL}_g(\mathfrak{A}_1) - \rho_2 t^2.$$

We define $E := E_{\Delta} + E_{\delta Q} + E Q_{ss} + E \text{SL} + t \nabla_{v_1} \Delta_g \mathfrak{A}_1$, then (42) can be rewritten as

$$\text{SL}_{\lambda^*_t g}(\mathfrak{A}_2) = t^2(\rho_2 - \Delta_g \nabla_{v_1} f_1 + \Delta_f f_2) + E.$$  (44)

By Lemma 4, 6, 7 and Proposition 15, we could write $E = t^3 E'$ with $\|r E'\|_{C, \gamma} \lesssim 1$, which implies $\|r \text{SL}_{\lambda^*_t g}(\mathfrak{A}_2)\|_{C, \gamma} \lesssim t^3$.

**Step 3**: We will finish the inductive step. Suppose we have construct the nondegenerate pair $\mathfrak{A}_k = d\mathfrak{A}_k$, metric $g_k$ and $T_k$ such that for any $t < T_k$, $\mathfrak{A}_k$ is nondegenerate and $\|r \text{SL}_{g_k}(\mathfrak{A}_k)\|_{C, \gamma} \lesssim t^{k+1}$. We will construct $\mathfrak{A}_{k+1}$, $g_{k+1}$ and $T_{k+1}$ with the desire property.

We could write $\text{SL}_{g_k}(\mathfrak{A}_k) = \rho_{k+1} t^{k+1} + E\text{SL}$ and we will find $a_{k+1} = (df_{k+1}^+, df_{k+1}^-)$, a vector field $v_{k+1}$ such that $f_{k+1}^-$ has leading asymptotic $f_{k+1}^- \sim \Re(B_{f_{k+1}^-} z^\frac{3}{2})$ and $a_{k+1}$ solves the following equations:

$$\rho_{k+1}^+ - \Delta g_k (f_{k+1}^- - \nabla_{v_k} f^-_1) = 0, \quad \rho_{k+1}^+ - \Delta g_k (f_{k+1}^+ - \nabla_{v_k} f^+_1) = 0.\quad (45)$$

Similarly to Step 2.1, we first solve $-\Delta g_k P_{k+1} + \rho_{k+1}^- = 0$ with

$$P_{k+1} = \Re(a_{k+1} z^\frac{1}{2} + b_{k+1} z^\frac{3}{2}) + E P_{k+1}.$$  

We choose $v_k$ be an extension of $-2a_{k+1} \overline{z} B_{l_k}$ over $L$, then $\nabla_{v_k} f_1^-$ has leading expansion $-\Re(a_{k+1} z^{\frac{3}{2}})$. We define $f_{k+1}^- = P_{k+1} + \nabla_{v_k} f_1^-$. By Proposition 12, 17, we could find $f_{k+1}^+$ solves the second equation.

Let $\frac{d}{ds} \lambda_{k,s} = t^k v_k \circ \lambda_{k,s}$ be the 1-parameter family and define $\varphi_{k+1,t} = \varphi_{k,t} \circ \lambda_{k,1}$, then $g_{k+1,t} := \varphi_{k+1,*} g_{k,t} = \lambda_{k,1} g_{k,t}$. We define $\mathfrak{A}_{k+1} := \mathfrak{A}_k + t^{k+1} a_{k+1}$, then by Lemma 8, we have

$$\text{SL}_{g_{k+1}}(\mathfrak{A}_{k+1}) = t^{k+1}(\rho_{k+1} - \Delta_g (f_{k+1}^- - \nabla_{v_k} f_1^-)) + E.$$  (46)

By Lemma 4, 6, 7, we could write $E = t^{k+2} E'$ and by Proposition 15, $\|r E'\|_{C, \gamma} \lesssim 1$. Therefore, we obtain $\|r \text{SL}_{g_{k+1}}(\mathfrak{A}_{k+1})\|_{C, \gamma} \lesssim t^{k+2}$.

As $\lim_{t \to 0} \lambda_{k,1} = \text{Id}$, there exists $T_{k+1}'$ such that $t < T_{k+1}'$, $\varphi_{k+1,t}^{-1}(\Sigma)$ is an embedded submanifold. We write the leading expansion as $\mathfrak{A}_{k+1} \sim \Re((tB_{\mathfrak{A}_k} + t^{k+1} B_{f_{k+1}^-} z^\frac{3}{2}),$
where $B_{f_{k+1}}$ is the coefficient of order $r^{\frac{3}{2}}$ for $f_{k+1}$. As $\mathfrak{A}_k$ is nondegenerate, $\min |B_{\mathfrak{A}_k}| > 0$. Thus we could define a positive number

$$T_{k+1} := \min\{T_k, \frac{1}{4} \max |B_{f_{j+1}}|, T'_{k+1}\},$$

such that for any $t < T_{k+1}$, $\mathfrak{A}_{k+1}$ will be nondegenerate. This completes Step 3.

Now, we will give a proof of Theorem 7.

**Proof of Theorem 7.** Given positive integer $N$, we define $\phi_t := \varphi_{N,t}^{-1}$ and $\mathfrak{A}_t := \mathfrak{A}_{t,N}$. Recall that $g_{N,t} := \varphi_{N,t}^* g$, then by Proposition 16, we have $\phi_t^* SL_{g_{N,t}}(\mathfrak{A}_t) = SL_g(\phi_t^* \mathfrak{A}_t)$. In addition, as $\phi_t$ is a smooth diffeomorphism with bounded derivative, we have

$$\|r_t SL_g(\phi_t^* \mathfrak{A}_t)\|_{C^r} = \|\phi_t^* (r SL_{g_{N,t}}(\mathfrak{A}_t))\|_{C^r} \lesssim \|r SL_{\varphi_{N,t}^* g}(\mathfrak{A}_t)\|_{C^r},$$

where $r_t := \phi_t^* r$. Therefore, we obtain (i): $\|r_t SL_g(\phi_t^* \mathfrak{A}_t)\|_{C^r} \lesssim t^{N+1}$.

(ii), (iii) and (iv) follows straight forward by the construction. For (v), as $t \to 0$, $\lambda_i$ converge smoothly to the identity map, when $t \to 0$, we have $\phi_t$ converge smoothly to the identity map.

We still need to check (v). Near $p \in \Sigma$, let $(x_1, x_2, \ldots, x_n)$ be the coordinate system on a neighborhood $U$ of $p \in \Sigma$, which we used in 6.2 with $z = x_1 + \sqrt{-1}x_2$ and $\Sigma \cap U = \{z = 0\}$. Over $T^*L_{\Sigma}$, we write the corresponding fiber coordinate as $(y_1, \ldots, y_n)$, then the Riemannian metric can be written as

$$g_{U, \Sigma} = \sum_{i,j=1}^n A_{ij} dx_i \otimes dx_j + B_{ij} dx_i \otimes dy_j + C_{ij} dy_i \otimes dy_j.$$

As $[\tilde{L}_t]$ is the graph manifold of $\phi_t^* \mathfrak{A}$ and WLOG, we could assume $\phi_t = Id$ and write $\phi_t^* \mathfrak{A} = d\mathfrak{F}$, $\mathfrak{F}_{ij} = \partial_{x_i} \partial_{x_j} \mathfrak{F}$. Suppose we write $\tilde{g}^t$ be the induced metric on $\tilde{L}_t$ and $\tilde{g}_{ij}^t = \tilde{g}^t(\partial_{x_i}, \partial_{x_j})$, then

$$\tilde{g}_{ij}^t = A_{ij} + \sum_{p=1}^n B_{ip} \mathfrak{F}_{pj} + \sum_{p,q=1}^n \mathfrak{F}_{ip} \mathfrak{F}_{jq} C_{pq}.$$

As $\mathfrak{F}_{ij} \sim r^{-\frac{1}{2}}$ when $i, j \in \{1, 2\}$, $\mathfrak{F}_{ij} \sim O(1)$ in other situation, the worst singularity comes from the third terms, which will be order $r^{-1}$. We are in particular interested in the behavior of $\lim_{r \to 0} r \tilde{g}_{ij}^t$ and it is straight forward to check that suppose one of $i$ or $j \notin \{1, 2\}$, we have $\lim_{r \to 0} r \tilde{g}_{ij}^t = 0$.

Let $\mathfrak{f}$ be the leading term of $\mathfrak{F}$, then we could write $\mathfrak{f} = R(Bz^{-\frac{1}{2}})$ with $B$ is no where vanishing along $\Sigma$. We write $f_{ij} = \partial_{x_i} \partial_{x_j} \mathfrak{f}$, then we compute

$$f_{11} = R(Bz^{-\frac{1}{2}}), \quad f_{22} = -R(Bz^{-\frac{1}{2}}), \quad f_{12} = -\mathfrak{F}(Bz^{-\frac{1}{2}}).$$

Moreover, for $i, j \in \{0, 1\}$, we compute $\lim_{r \to 0} r \tilde{g}_{ij}^t = \lim_{r \to 0} t^2 \sum_{p,q=1}^2 C_{pq} R_p f_{pi} f_{qj}$. 
We define the vector fields \( v_i = r^\frac{1}{2}(f_{1i}\partial_{y_1} + f_{2i}\partial_{y_2}) \), then by the previous computations, for \( i \in \{1, 2\} \), \( v_i \) are nowhere vanishing in this neighborhood. In addition, for \( i, j \in \{1, 2\} \), we have \( \lim_{r \to 0} r \tilde{g}_{ij}^t = t^2 g(v_i, v_j) \), which is also nowhere vanishing near \( \Sigma \).

Therefore, we could find a positive function \( \mu \) such that

\[
r \det(g_{ij}^t)dx_1 \wedge \cdots \wedge dx_n = \mu dx_1 \wedge \cdots \wedge dx_n,
\]

with a lower bound \( \mu \geq t^2 c_0 \). Let \( g_0 \) be the Riemannian metric on the zero section, then

\[
t^2 d\text{Vol}_{g_0} = t^2 \det(g_0) dx_1 \wedge \cdots \wedge dx_n \lesssim r \det(g_{ij}^t) dx_1 \wedge \cdots \wedge dx_n = r d\text{Vol}_{\tilde{g}^t}.
\]

In addition, we have \( \tilde{t}^* \tilde{\Omega} = \sin \theta_t d\text{Vol}_{\tilde{g}^t} \) and \( t^* \tilde{\Omega} = \text{SL}(\phi^*_t \mathfrak{A}_t) d\text{Vol}_{g_0} \). Therefore, we obtain \( \|\theta_t\|_{C^{2\gamma}} \lesssim t^{-2} \|r_t \text{SL}(\phi^*_t \mathfrak{A}_t)\|_{C^{2\gamma}} \lesssim t^{N-1} \).

For (vi), over a neighborhood \( U \) of the zero section \( T^* \tilde{L} \), as currents, \( [\tilde{L}_t] \) will be the graph of \( p^* \mathfrak{A}_t \) and \( 2[L] \) is the zero section in \( T^* \tilde{L} \). By (iii), \( p^* \mathfrak{A}_t \) have the following asymptotic expansions

\[
p^* \mathfrak{A}_t \sim \tilde{a}_0 + \tilde{a}_1 \tilde{r} + \tilde{a}_2 \tilde{r}^2 + \sum_{k=1}^{p} \tilde{a}_{3,k} \tilde{r}^3 (\log \tilde{r})^k + \cdots,
\]

where \( \tilde{r}^2 = p^* r \). Therefore, for a smooth Riemannian metric \( g' \) over \( \tilde{L} \), \( p^* \mathfrak{A}_t \) is a \( C^{2,\gamma} \) 1-form with \( \lim_{t \to 0} \|p^* \mathfrak{A}_t\|_{C^{2,\gamma}} = 0 \). Therefore, for any differential form \( \omega \) on \( U \), we have \( \lim_{t \to 0} \int_U t^*_L \omega = \int_U (\iota_0 \circ p)^* \omega \), which implies the current convergence \( \iota_t(\tilde{L}) \to 2\iota_0(L) \). \( \Box \)

6.5 The construction over fixed real locus. Suppose our initial special Lagrangian submanifold is locally a fixed real locus as in Definition 10, the approximate solutions constructed in Theorem 7 will have extra symmetry. Let \( R_0 : T^* L \to T^* L \) be the canonical involution and \((U_L, J, \omega, \Omega)\) be a \( R_0 \)-invariant Calabi–Yau structure, we obtain:

**Theorem 8.** Suppose \([L] \) is locally a fixed real locus and \( \mathfrak{a} = \alpha^- \in T^* L \otimes \mathcal{I} \) be a nondegenerate \( \mathbb{Z}_2 \) harmonic 1-form, then in Theorem 7, we could choose the construction in Theorem 7 such that \( \mathfrak{A}_t = \mathfrak{A}_t^- \in T^* L \otimes \mathcal{I} \). In addition, let \([\tilde{L}_t] = (\tilde{L}, \tilde{\iota}_t)\) be the graph of \( \mathfrak{A}_t \), then \( R_0 \circ \tilde{\iota}_t = \tilde{\iota}_t \circ \sigma \).

**Proof.** By Proposition 14, we have \( Q(\alpha^-)^+ = 0 \). Suppose we start with \( \mathfrak{a} = \alpha^- \in T^* L \otimes \mathcal{I} \), then in each step of the construction in Sect. 6.4, we could always choose the correction to be \( \mathfrak{A}_i = \alpha_i^- \in T^* L \otimes \mathcal{I} \). The rest follows straightforward. \( \Box \)

If we write \( \theta_t \) be the Lagrangian angle of \([\tilde{L}_t]\) and \( \tilde{g}_t := \tilde{\iota}_t^* \tilde{g} \) be the induced metric, then as \( R_0 \circ \tilde{\iota}_t = \tilde{\iota}_t \circ \sigma \), we obtain \( \sigma^* \tilde{g}_t = \tilde{g}_t \), \( \sigma^* e^{i\theta_t} = e^{-i\theta_t} \). In particular, over the branch locus \( \Sigma \), we have \( \sin \theta_t |_{\Sigma} = 0 \).

By Proposition 19, \([\tilde{L}_t]\) is an immersed submanifold and we could choose a Weinstein neighborhood of \([\tilde{L}_t]\) and consider the special Lagrangian equation \( \text{SL}_{\tilde{g}_t}(\alpha) \) for the graph of 1-form \( \alpha \) on \([\tilde{L}_t]\).
Corollary 4. \( \sigma^* \text{SL}\tilde{g}_t(\alpha) = -\text{SL}\tilde{g}_t(-\sigma^*\alpha) \).

Proof. We define \( \tilde{\iota}_\alpha : \tilde{L} \to U_L \) as \( \tilde{\iota}_\alpha(x) := \exp_{\tilde{i}_t(x)}(J(\tilde{i}_t)_*V_\alpha) \), where \( V_\alpha \) is the vector field on \( \tilde{L} \) dual to \( \alpha \) given by the Riemannian metric \( \tilde{g}_t \). We will first prove that \( R_0 \circ \tilde{\iota}_\alpha(\tilde{x}) = \tilde{\iota}_{-\sigma^{-1}*\alpha} \circ \alpha \). As \( R_0 \) is an isomorphism and \( R_0 \circ \tilde{i}_t = \tilde{i}_t \circ \alpha \), we compute
\[
R_0 \circ \tilde{\iota}_\alpha(x) = R_0 \exp_{\tilde{i}_t(x)}(J(\tilde{i}_t)_*V_\alpha) = \exp_{R_0\tilde{i}_t(x)}((R_0)_*J(\tilde{i}_t)_*V_\alpha) = \exp_{\tilde{i}_t\sigma(x)}(-J(\tilde{i}_t)_*\sigma_*V_\alpha).
\]
As \( \sigma^*g_t = g_t \), we obtain \( -\sigma_*V_\alpha = V_{(-\sigma^{-1})*\alpha} \). Therefore, we obtain
\[
R_0 \circ \tilde{\iota}_\alpha(x) = \exp_{\tilde{i}_t\sigma(x)}(J(\tilde{i}_t)_*V_{(-\sigma^{-1})*\alpha}) = \tilde{\iota}_{-\sigma^-1*\alpha} \circ \sigma.
\]
As \( \tilde{\iota}_\alpha^*\Omega = \text{SL}\tilde{g}_t(\alpha)d\text{Vol}\tilde{g}_t \), we compute
\[
-\text{SL}\tilde{g}_t(\alpha)d\text{Vol}\tilde{g}_t = -\tilde{\iota}_\alpha^*\Omega = \tilde{\iota}_\alpha^*R_0^*\Omega = \sigma^*\tilde{\iota}_{-\sigma^{-1}*\alpha}^*\Omega = \sigma^*\text{SL}\tilde{g}_t(-\sigma^{-1}*\alpha)d\text{Vol}\tilde{g}_t,
\]
which implies the claim. \( \square \)

7 The geometry of the approximate special Lagrangian submanifolds

In this section, we will study the geometry for the family of graphic special Lagrangian submanifolds constructed in Theorem 7. Let \( (U_L, J, \omega, \Omega) \) be the pull-back Calabi–Yau structure on the Weinstein neighborhood with Calabi–Yau metric \( g \), and \( [\tilde{L}_t] = (\tilde{L}, \tilde{i}_t) \) be the family of submanifolds constructed in Theorem 7, such that \( \tilde{i}_t \) is given by the graph of a nondegenerate pair \( \mathfrak{A}_t = \mathcal{A}_t^+ + \mathcal{A}_t^- \). We will study the induced Riemannian metric, the second fundamental form and the Schauder estimates on \( [\tilde{L}_t] \).

7.1 The induced metric on the approximate family. Now, we will prove the induced metric on \( [\tilde{L}_t] \) converge to a cone metric with cone angle \( 4\pi \).

7.1.1 Cone Metric. Let \( \mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-2} \) with coordinates \( z = re^{i\theta}, x_3, \ldots, x_n \), consider the model metric
\[
g_{4\pi} = 4r^2(dr^2 + r^2d\theta^2) + \sum_{i=3}^{n} dx_i^2,
\]
which has cone singularity of angle \( 4\pi \) along the submanifold \( \{z = 0\} \). Let \( g_1, g_2 \) be two metric, we write \( g_1 \leq g_2 \) if for any \( v \), we have \( g_1(v,v) \leq g_2(v,v) \).

Definition 11. Let \( \Sigma \subset L \) be a codimension two embedded submanifold, we call \( g \) is a metric with cone angle \( 4\pi \) along \( \Sigma \) if for every \( p \in \Sigma \), we could find coordinates \( (z, x_3, \ldots, x_n) \) centered at \( p \) with \( \Sigma = \{z = 0\} \) and constant \( C \) such that \( C^{-1}g_{4\pi} \leq g \leq Cg_{4\pi} \).
For (49), passing to the branched covering, we write \( r = \sqrt{r}, \) then the model metric become \( d\tilde{r}^2 + 4\tilde{r}^2 d\theta^2 + \sum_{i=3}^n dx_i^2, \) which is quasi-isometric to the Euclidean metric. Therefore, when we define the Hölder norm or \( W^{1,2} \) norms using the cone metric, it will be the same as defining using the smooth Riemannian metric.

**Proposition 18.** \( p^*g_L \) is a cone metric over \( \tilde{L} \) with cone angle \( 4\pi. \)

**Proof.** Let \( \mathcal{N}_\Sigma \) be the normal bundle of \( \Sigma \subset L \), we identified a neighborhood \( U \) of \( \Sigma \) in \( \tilde{L} \) with a neighborhood of zero section of \( \mathcal{N}_\Sigma \) and write \( z \) be the fibre coordinates of \( \mathcal{N}_\Sigma \). We could define another Riemannian metric \( g_0 \) on \( U \) as \( g_0 = |dz|^2 + g_\Sigma \), then \( g_0 \) is also smooth Riemannian metric which satisfies \( C^{-1}g_0 \leq g_L \leq Cg_0. \)

We could write the branched covering map \( p : \tilde{L} \to L \) in this local coordinate as \( p(\tilde{z}, x_3, \ldots, x_n) = (\tilde{z}^2, x_3, \ldots, x_n), \) then \( \quad p^*g_0 = 4\tilde{r}^2(d\tilde{r}^2 + \tilde{r}^2 d\theta^2) + p^*g_\Sigma, \) which is a cone metric along \( \tilde{\Sigma} \) with cone angle \( 4\pi \). As \( C^{-1}p^*g_0 \leq p^*g_L \leq Cp^*g_0, p^*g_L \) is also a cone metric with cone angle \( 4\pi. \) \( \square \)

**7.1.2 Induced metric and converge.** Recall that \( \tilde{\iota}_t : \tilde{L} \to U_L \) is given by the graph of the nondegenerate pair \( \mathcal{A}_t = \mathcal{A}_t^+ + \mathcal{A}_t^- \) and we would like to understand the map. By the toy model in Sect. 5.1, we need to prove \( \tilde{\iota}_t \) is well-behaved near \( \Sigma \) using the nondegenerate condition.

**Proposition 19.** \( \tilde{\iota}_t : \tilde{L} \to U_L \) is a \( C^{2,\gamma} \) immersion and \( \tilde{\iota}_t^*g \) is a \( C^{1,\gamma} \) Riemannian metric.

**Proof.** Outside of the branch locus, \( \tilde{\iota}_t \) will locally given by graph of different 1-forms which is an immersion. Therefore, we only need to consider the map near the branch locus. Near the branch locus, we choose coordinates \( (\tilde{z}, \tilde{x}_3, \ldots, \tilde{x}_n) \) on \( \tilde{L} \) and \( (z, x_3, \ldots, x_n) \) on \( L \) such that the branched covering map can be written as \( p(\tilde{z}, \tilde{x}_3, \ldots, \tilde{x}_n) = (\tilde{z}^2, x_3, \ldots, x_n). \)

Then under the identification of \( \Gamma : T(T^*L) \cong TL \oplus T^*L, \) we have
\[
\Gamma \circ \tilde{\iota}_t(\partial_{\tilde{z}_i}) = (p_*\partial_{\tilde{z}_i}, \nabla_{p_*\partial_{\tilde{z}_i}}\mathcal{A}_t^- + \nabla_{p_*\partial_{\tilde{z}_i}}\mathcal{A}_t^+). \tag{50}
\]

As \( p_*\partial_{\tilde{z}_i} = 2\tilde{z}\partial_{\tilde{z}_i} \) and \( \mathcal{A}_t^- \) is nondegenerate with expansion \( \mathcal{A}_t^- \sim \Re(B\tilde{z}\partial_{\tilde{z}}dz) + \cdots, \) where \( B \) is nowhere vanishing along \( \Sigma. \) We compute
\[
\nabla_{p_*\partial_{\tilde{z}_i}}\mathcal{A}_t^-|_{\Sigma} = 2B\tilde{z}\nabla_{\partial_{\tilde{z}_i}}(z\frac{1}{2}dz)|_{\Sigma} = Bdz \neq 0. \]

In addition, we compute \( \nabla_{p_*\partial_{\tilde{z}_i}}(\mathcal{A}_t^+)|_{\Sigma} = 2\tilde{z}\nabla_{\partial_{\tilde{z}_i}}\mathcal{A}_t^+|_{\Sigma} = 0. \) Therefore, \( (\tilde{\iota}_t)_* \) is injective thus an immersion. \( \square \)

Recall that \( U_L \) is a tubular neighborhood of \( L \) as a submanifold of \( X \) and we write \( g_{U_L} \) be induced metric on \( U_L, \) we have the following theorem.

**Theorem 9.** Let \( g' \) be any smooth Riemannian metric on \( \tilde{L}, g_L \) be the Riemannian metric on the zero section of \( U_L \subset T^*L \) and \( \tilde{g}_t := \tilde{\iota}_t^*(g_{U_L}) \) be the pullback metric, then \( \lim_{t \to 0} \| \tilde{g}_t - p^*g_L \|_{C^{1,\gamma}}^2 = 0, \) where \( C^{1,\gamma}_g \) is defined using \( g'. \)
Proof. We prove the above theorem using local charts. For $U_L \subset T^*L$, let $(x_1, \ldots, x_n)$ be a local coordinate on $L$ and $(y_1, \ldots, y_n)$ be coordinates on fiber. The Riemannian metric on $U_L$ in these coordinates can be written as

$$g_{U_L} = g_L + B_{ij}dx_i \otimes dy_j + C_{ij}dy^i \otimes dy^j + O(|y|),$$

then the pullback metric $\tilde{g}_t := t^*g$ can be written as

$$\tilde{g}_t = p^*g_L + t(B_{ij}x_i \otimes d\Theta_j) + t^2C_{ij}dx_i \otimes d\Theta_j + O(|t|),$$

where $\Theta_i := p^*\langle \partial_x, \frac{1}{t}\Theta_t \rangle$. Therefore, we could write $\tilde{g}_t = p^*g_L + t\tilde{g}_t$ with

$$\|\tilde{g}_t\|_{C^{1,\gamma}} \lesssim \sum_{i,j=1}^{n} (\|B_{ij}\|_{C^{1,\gamma}} + \|C_{ij}\|_{C^{1,\gamma}}) + \frac{1}{t}\|\mathcal{A}_t\|_{C^{2,\gamma}} \lesssim 1.$$
metric is quasi-isomorphism to the Euclidean metric, the coefficients of the Laplace operator will be \( L^\infty \), then by the elliptic theory for \( L^\infty \) coefficients elliptic operator, \( f_\infty \) is an eigenfunction with eigenvalue \( \lambda_0 \).

We write \( \lambda_t \) be the first eigenvalue of \( D_{\theta_t} \), then \( \lambda_t \) can be realized by the following variation

\[
\lambda_t = \inf_f \left\{ \frac{\int \cos \theta_t |df|_g^2 \, dVol_t}{\int |f|^2_{g_t} \, dVol_t} \right\} ,
\]

where \( dVol_t \) is the volume form for the metric \( g_t \).

**Theorem 11.** Let \( \lambda_t \) be the first eigenvalue of \( D_{\theta_t} \), then \( \lim_{t \to 0} \lambda_t = \lambda_0 > 0 \). In particular, there exists a constant \( c_0 \) independent of \( t \) such that \( \lambda_t \geq c_0 > 0 \).

**Proof.** Let \( Vol_t \) be the volume form of \( g_t \) over \( \tilde{L} \) and \( Vol_0 \) be the volume form of \( g_0 \) over \( L \). In local coordinates \((x_1, \ldots, x_n)\), we could write

\[
\cos \theta_t |df|_{\tilde{g}_t}^2 \, dVol_t = \sum_{i,j=1}^n \cos \theta_t (\tilde{g}_t)^{ij} \partial_{x_i} f \partial_{x_j} f \, \det(\tilde{g}_t) \, dx_1 \wedge \ldots \wedge dx_n .
\]

As \( \lim_{t \to 0} \tilde{g}_t = g_0 \) and \( \lim_{t \to 0} \cos \theta_t = 1 \), we obtain

\[
(1 - \epsilon) \int |df|_{g_0}^2 \, dVol_0 \leq \int \cos \theta_t |df|_{\tilde{g}_t}^2 \, dVol_t \leq (1 + \epsilon) \int |df|_{g_0}^2 \, dVol_0 .
\]  

Let \( f_t \) be the eigenfunction \( D_{\theta_t} f_t = \lambda_t f_t \), then we compute

\[
\int \cos \theta_t |df_t|_{\tilde{g}_t}^2 \, dVol_t = \lambda_t \int |f_t|_{\tilde{g}_t}^2 \, dVol_t \leq \lambda_t (1 + \epsilon) \int |f_t|_{g_0}^2 \, dVol_0 ,
\]

\[
\int \cos \theta_t |df_t|_{\tilde{g}_t}^2 \, dVol_t \geq (1 - \epsilon) \int |df_t|_{g_0}^2 \, dVol_0 \geq (1 - \epsilon) \lambda_0 \int |f_t|_{g_0}^2 \, dVol_0 .
\]

Therefore, \( \lambda_t \geq \frac{1 - \epsilon}{1 + \epsilon} \lambda_0 \).

On the other hand, by the definition of \( \lambda_0 \), for every \( \epsilon \), we can find a function \( f \) such that

\[
\int |df|_{g_0}^2 \, dVol_0 \leq (\lambda_0 + \epsilon) \int |f|_{g_0}^2 \, dVol_0 .
\]

In addition, we have

\[
\int |df|_{g_0}^2 \, dVol_0 \geq (1 - \epsilon) \int |df|_{\tilde{g}_t}^2 \, dVol_t \geq (1 - \epsilon) \lambda_t \int |f|^2 \, dVol_t \geq (1 - \epsilon)^2 \lambda_t \int |f|^2 \, dVol_0 .
\]

Therefore, we conclude that \( \lambda_t \leq \frac{\lambda_0 + \epsilon}{(1 - \epsilon)^2} \), which implies the claim. \( \square \)
7.3 Curvature and injective radius estimate for the graph family. In this subsection, we will compute the second fundamental form and the injective radius for $[\tilde{L}_t]$.

**Definition 12.** Let $(X,g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$, let $\iota : L \to X$ be an immersion, the second fundamental form $K_m$ of $(L,\iota)$ at $m \in L$ is defined as

$$K_m : T_ml \times T_ml \times N_m \to \mathbb{R},$$

$$K_m(X,Y,n) := \langle \nabla_{\iota X} (\iota Y), n \rangle,$$

where $N_m$ is the normal bundle of the immersion at $\iota(m)$ with section $n$ of $N_m$, $X, Y$ are vector fields on $L$.

Let $[\tilde{L}_t]$ be the family of immersed Lagrangians constructed in Theorem 7 on $(U_L, J, \omega, \Omega)$ and $t_0 : L \to U_L$ be the inclusion of the zero section.

**Proposition 20.** For $[\tilde{L}_t]$, let $\tilde{m} \in \tilde{L}_t$, let $K_{\tilde{m}}^t$ be the second fundamental form at $\tilde{m}$, then:

(i) For $\tilde{m} \notin \Sigma$, we write $m = p(\tilde{m}) \in L$ and $K_m^0$ be the second fundamental form of $(L, \iota_0)$ at $m$ and suppose $n_t$ is a sequence of smooth section of $N_{\iota_t(\tilde{m})}$ which converge continuously to $n_0 \in N_{\iota_0(m)}$ on $T^*U_L$, then for any $X, Y \in T_m L$, we have

$$\lim_{t \to 0} K_{\tilde{m}}^t(X,Y,n_t) := K_m^0(p_*X, p_*Y, n_0).$$

(ii) In a suitable neighborhood of $\Sigma$, we write $K^t := \max|X|=|Y|=|n|=1, \tilde{m} \in \Sigma |K_{\tilde{m}}^t (X,Y,n)|$, then we have $|K^t| \leq \frac{C}{r^2 + t^2} \leq Ct^{-2}.$

**Proof.** Let $(\tilde{x}_1, \ldots, \tilde{x}_n)$ be a coordinate on a neighborhood of $\tilde{m}$ and $(x_1, \ldots, x_n)$ be a coordinate on a neighborhood of $m$ and let $(y_1, \ldots, y_n)$ be the fiber coordinates of $T^*L$. We define $E_i^t := (\tilde{\iota}_t), \partial_{\tilde{x}_1},$ then under the identification $\Gamma : T(T^*L) \to TL \oplus T^*L$, we could write

$$\Gamma(E_i^t) = (p_*\partial_{\tilde{x}_i}, \nabla_{p_*\partial_{\tilde{x}_i}} A_t^+ + \nabla_{p_*\partial_{\tilde{x}_i}} A_t^-)$$

For (i), as $\tilde{m} \notin \Sigma$, $\lim_{t \to 0} \|\nabla_{p_*\partial_{\tilde{x}_i}} A_t^+\|_{C^{0,\gamma}} + \|\nabla_{p_*\partial_{\tilde{x}_i}} A_t^-\|_{C^{0,\gamma}} = 0$ and we could write $\lim_{t \to 0} E_i^t = p_*\partial_{\tilde{x}_i}$. If we write $\tilde{\nabla}$ be the Levi-Civita connection of $(U_L, g_{U_L}),$ then $\lim_{t \to 0} \tilde{\nabla}_j E_i^t = \nabla_{p_*\partial_{\tilde{x}_i}} p_*(\partial_{\tilde{x}_i}).$ As $\lim_{t \to 0} n_t = n_0,$ we compute

$$\lim_{t \to 0} K_{\tilde{m}}^t(\partial_{\tilde{x}_i}, \partial_{\tilde{x}_j}, n_t) = \lim_{t \to 0} \langle \tilde{\nabla}_j E_i^t, n_t \rangle = \langle \nabla_{p_*\partial_{\tilde{x}_i}} p_*(\partial_{\tilde{x}_j}), n_0 \rangle = K_m^0(p_*\partial_{\tilde{x}_i}, p_*\partial_{\tilde{x}_j}, n_0),$$

which implies (i).

For (ii), in a suitable neighborhood of $m \in \Sigma$, we write $z = x_1 + i x_2, \tilde{z} = \tilde{x}_1 + i \tilde{x}_2$ and $\Sigma = \{z = 0\}$, then we could write the expansion of $A_t^- \sim t \Re(B_t \tilde{z}^2 dz)$ with $|B_t| \geq C_0 > 0$ on $\Sigma$. We compute

$$p_*\partial_{\tilde{z}} = 2\tilde{z}\partial_z, \nabla_{p_*\partial_{\tilde{z}}} A^-|_{\Sigma} = t B_t dz, \nabla_{p_*\partial_{\tilde{z}}} (A^+) = 0,$$
which implies \(|K|^t_m(X, Y, n)| \leq C\) for any \(|n| = 1\). Moreover, we have
\[|E^i_t|^2 \leq r + t^2,\] for \(i = 1\) or \(2\) and \(|E^i_t|^2 \sim O(1)\) for \(i \neq 1, 2\), which implies the claim. \(\Box\)

We have the following estimate of the injective radius using the second fundamental form.

**Proposition 21.** Let \((X, g)\) be a Riemannian manifold, \(\iota: Y \to X\) be an immersion and let \(K_Y\) be the second fundamental form of \(Y\) and \(\tau_Y := \max|K_Y|\), then the injective radius \(\text{inj}_p\) of \(Y\) at any point \(p \in Y\) satisfies \(\text{inj}_p \geq \frac{\pi}{C + \tau_Y}\), where \(C\) is a constant independent of \(\tau_Y\).

**Proof.** By the Gauss equation, let \(\text{Riem}\) be the Riemannian curvature tensor of \((Y, \iota^*g)\) and \(\tau_Y = \max|K_Y|\), then we have \(\max|\text{Riem}| \leq C\tau_Y^2\). As \((Y, \iota^*g)\) is a complete Riemannian manifold, for any point \(p \in Y\), we have
\[\text{inj}_p \geq \min\{\frac{\pi}{\tau_Y}, \frac{1}{2}\}l(Y),\]
where \(l(Y)\) is the infimum of lengths of closed geodesics.

Now, we will control \(l(Y)\). We take a isometric embedding \((X, g) \to \mathbb{R}^N\), if we write \(\nabla^{\mathbb{R}^N}\) be the Levi-Civita connection on \(\mathbb{R}^N\), then for \(\alpha, \beta \in TY\), we could write \(\nabla^{\mathbb{R}^N}_{\alpha, \beta} = \nabla^X_{\alpha, \beta} + K_X\), where \(\nabla^X\) is the Levi-Civita connection on \(X\) and \(K_X\) is a section of the normal bundle of \(X\) in \(\mathbb{R}^N\). Similarly, we have \(\nabla^{\mathbb{R}^N}_\alpha \beta = \nabla^Y_{\alpha, \beta} + K_Y\), where \(\nabla^Y\) is the Levi-Civita connection on \(Y\) and \(K_Y\) is a section of the normal bundle of \(Y\) in \(X\). Therefore, we obtain \(\nabla^{\mathbb{R}^N}_{\alpha, \beta} = \nabla^Y_{\alpha, \beta} + K_Y + K_X\), where \(\nabla^{\mathbb{R}^N}\) is the covariant derivative on \(\mathbb{R}^N\) and \(\nabla^X_{\alpha, \beta}\) is the covariant derivative on \(Y\).

Let \(\gamma\) be a closed geodesic on \(Y\) and we could consider \(\gamma\) is a curve on \(\mathbb{R}^N\), then the extrinsic curvature \(k_\gamma\) of \(\gamma\) is defined as \(k_\gamma = |\nabla_{\gamma'}^Y|\). In addition, we compute
\[|\nabla_{\gamma'}^Y| = |\nabla_{\gamma'}^Y + K_Y(\gamma', \gamma') + K_X(\gamma', \gamma')| \leq \tau_X + \tau_Y,\]
where \(\tau_X\) is the maximal of the second fundamental form of \(X\).

Let \(k_\gamma\) be the extrinsic curvature of \(\gamma\) in \(\mathbb{R}^N\), which satisfies \(k_\gamma \leq |\nabla_{\gamma'}^Y| \leq \tau_X + \tau_Y\). Moreover, by Fenchel’s theorem on \(\mathbb{R}^N\) [ALF65], we have \(\int_{\gamma} k_\gamma ds \geq 2\pi\). Let \(l(\gamma)\) be the arc length of \(\gamma\), then we have \(2\pi \leq \int_{\gamma} k_\gamma ds \leq l(\gamma)(\tau_X + \tau_Y)\), which implies \(l(\gamma) \geq \frac{2\pi}{\tau_X + \tau_Y}\) for any closed geodesic on \(Y\). Therefore, \(\text{inj}_p \geq \frac{\pi}{\tau_X + \tau_Y}\). \(\Box\)

Over \((\tilde{L}, g_t)\), let \(\text{inj}_t\) be minimal of injective radius along different points of \(\tilde{L}\), then we obtain the following estimate.

**Corollary 5.** There exist constants \(t_0, C\) independent of \(t\) such that for any \(t \leq t_0\), we have \(\text{inj}_t \geq Ct^2\).

**Proof.** By Proposition 20, for the second fundamental form \(K_t\), we have \(\max_{p \in \tilde{L}}|K_t| \leq Ct^{-2}\). By Proposition 21, we have \(\text{inj}_t \geq \frac{\pi}{C + \tau_X t^2}\), where \(\tau_X\) is the maximal of the second fundamental form for an isomorphic embedding into \(\mathbb{R}^n\). As the ambient manifold \((X, g)\) is fixed, we take \(t_0 < \frac{1}{\sqrt{\tau_X}}\), then \(\text{inj}_t \geq Ct^2\). \(\Box\)
7.4 Size estimate for the Weinstein neighborhood. Let \([L] = (L, \iota)\) be an immersed Lagrangian submanifold in a Calabi–Yau manifold \((X, J, \omega, \Omega)\) with Calabi–Yau metric \(g_X\). By Theorem 4, we could find a Weinstein neighborhood of \([L]\) and in this subsection, we will prove a Weinstein neighborhood theorem with size controlled by \(\tau\), where \(\tau\) is maximum of the second fundamental form of \([L]\). Moreover, we assume \(\tau \geq 1\).

We define \(g_L := \iota^*g_X\) and \(U(s) := \{x \in T^*L| \text{dist}(x, L) < s\}\) be the size \(s\) neighborhood of the zero section in \(T^*L\). We start with introducing several lemmas.

**Lemma 9** [SIL01, Proposition 8.3]. Let \(V\) be a 2\(n\)-dimensional vector space, let \(\Omega_0\) and \(\Omega_1\) be symplectic forms on \(V\). Let \(S\) be a subspace of \(V\) and is Lagrangian for \(\Omega_0\) and \(\Omega_1\), and let \(W\) be any complement to \(S\) in \(V\), then from \(W\), we can canonically construct a linear isomorphism \(A : V \to V\) such that \(A|_S = \text{Id}_S\) and \(A^*\Omega_1 = \Omega_0\).

**Lemma 10** [SIL01, Proposition 6.8]. Let \(U\) be a tubular neighborhood of \(L\) in \(X\) and \(\beta\) is a closed \(k\)-form. Suppose the homology class of \(\beta\) restricted to \(L\) vanishes, then we could find a \((k - 1)\)-form \(\mu\) on \(U\) such that \(\beta = d\mu\) and \(\mu|_x = 0\), where \(x \in L\).

**Theorem 12.** Under the previous assumptions, for the immersed Lagrangian submanifold \([L] = (L, \iota) : L \to X\), there exists a neighborhood \(V\) of \([L]\), a neighborhood \(U\) of the zero section of \(T^*L\), a diffeomorphism \(\varphi : U \to V\) and a constant \(C\) independent of \([L]\) such that

(i) \(\varphi^*(\omega) = \omega_{\text{can}}\),
(ii) we could take \(U = U(C\tau^{-1})\), to be a size \(C\tau^{-1}\) neighborhood of \(T^*L\).

**Proof.** Let \(U(s)\) be an size \(s\) neighborhood on \(T^*L\), let \(\Phi : U(s) \to X\) be decomposition of the complex structure and the normal exponential map defined in (3). Let \(\text{inj}_X\) be the injective radius of \(X\), then for any \(s < \text{inj}_X\), \(\Phi\) will be an immersion. Therefore, we could obtain a pullback Calabi–Yau structure \((U(s), \Phi^*J, \Phi^*\omega, \Phi^*\Omega)\) with pull-back metric \(g_U = \Phi^*g_X\).

Let \(x \in L\) and we write \((x, 0)\) be a point on the zero section of \(U(s)\), we write \(S = T_xL\) and \(W = (T_pL)^\perp\), where the \(\perp\) is taken using the metric \(g\) over \(T_{(x, 0)}U(s)\). By Lemma 9, there exists a linear isomorphism \(A_x : T_{(x, 0)}U(s) \to T_{(x, 0)}U(s)\) such that \(A_x|_S = \text{Id}\) and \(A_x^*\omega|_{(x, 0)} = \omega_{\text{can}}\). In addition, as the choice of \(A_x\) is canonical, \(A_x\) varies smoothly with respect to \(x\). Using \(A_x\), we define a diffeomorphism \(h : U(s) \to U(s)\) as

\[h(p) := \exp_x(A_xv), \text{ where } p = (x, v) \in U(s) \subset T^*L.\]

Then we have \(h|_{(x, 0)} = (x, 0)\) and \(dh|_{(x, 0)} = A_x\) for \(p \in U\). Therefore, for \(x \in L\), we have \(h^*\omega|_{(x, 0)} = \omega_{\text{can}}|_{(x, 0)}\).

Over the neighborhood of \(L\), we write \(\omega_s = sh^*\omega + (1 - s)\omega_{\text{can}}\) and we wish to find the size of the tubular neighborhood when \(\omega_s\) is symplectic. As \(h^*\omega|_{(x, 0)} = \omega_{\text{can}}|_{(x, 0)}\) for \(x \in L\), \(\omega_s\) is also a symplectic form in a small neighborhood of \(L\).
Lemma 10, there exists a 1-form $\omega$ could choose $U$ such that for each $p \in L$, we also have $\|\omega_{\text{can}}\|_{C^0} \lesssim 1$, thus $\|\omega_s\|_{C^0} \lesssim 1$.

Let $R = \tau^{-1}$, we cover $L$ by radius $R$ ball and over each ball $B_R$, we rescale the size of ball by $R^{-1}$, then the Riemannian metric is in Euclidean size. After rescaling back, we obtain the following estimate $\|\det(\omega_s)\|_{C,\gamma} \lesssim \tau^\gamma$.

Let $C_0 = \min_{p \in L} \|\det(\omega_s)\|_{C^0}$, over a neighborhood $U_L(C\tau^{-1}) \subset T^*L$, for $p = (x, v) \in U_L(C\tau^{-1})$, we compute

$$
|\det(\omega_s)(p)| \geq |\det(\omega_s)(x)| - \|\det(\omega_s)\|_{C,\gamma} \|v||_{C^0} \\
\geq C_0 - C\tau^{-\gamma} \|\det(\omega_s)\|_{C,\gamma} \\
\geq C_0 - CC',
$$

where the last inequality, we use $\|\det(\omega_s)\|_{C,\gamma} \leq C'\tau^\gamma$. Therefore, for $C \leq \frac{C_0}{2C'}$, over $U_L(C\tau^{-1})$, we have $|\det(\omega_s)| > 0$ which implies $\det(\omega_s)$ is symplectic.

As the 2-form $h^*\omega - \omega_{\text{can}}$ is closed and $(h^*\omega - \omega_{\text{can}})_p = 0$ for all $p \in L$, then by Lemma 10, there exists a 1-form $\beta$ on $U_L(C\tau^{-1})$ such that $h^*\omega - \omega_{\text{can}} = d\beta$ and $\beta|_{(x,0)} = 0$ for $x \in L$. We solve the Moser equation for vector field $v_s$: $v_s, \omega_s = -\beta$ with $v_s = 0$ on $L$. Let $\rho_s : U_L(C\tau^{-1}) \to U_L(C\tau^{-1})$ be family of diffeomorphisms generates by $v_s$ then $\rho_s^*\omega_s = \omega_{\text{can}}$ over $U_L(C\tau^{-1})$. Let $\varphi := \Phi \circ h \circ \rho_1$, then $\varphi$ satisfies the claim.

Let $\tau_t$ be the maximal of the second fundamental form for the family of approximate Lagrangian submanifolds $[\hat{L}_t]$ constructed in Theorem 7 and we assume $\tau_t \geq 1$. By Proposition 20, we have $\tau_t \leq Ct^{-2}$, then we obtain the following corollary.

**Corollary 6.** There exists constant $C$ independent of $t$ such that for each $t$, we could choose $U_L(t) := \{\alpha \in T^*\hat{L}||\alpha||_{C^0} \leq Ct^{-2}\}$ to be a Weinstein neighborhood of $[\hat{L}_t]$ and $C^0$ is taken by the induced metric $\hat{g}_t$ on $[\hat{L}_t]$.

Note that by Theorem 9, the induced metric $\hat{g}_t$ on $[\hat{L}_t]$ will converge to a cone metric, thus the $C^0$ norms define by different metric will be equivalent.

### 7.5 Schauder estimate and quadratic term estimate on submanifolds.

In this subsection, we will study the Schauder estimate and quadratic term estimate for the immersed Lagrangian submanifolds. We will first state the Schauder estimate for a general immersed Lagrangian in a Calabi–Yau and for the family that constructed in Theorem 7.

#### 7.5.1 Schauder estimate for submanifold.

Let $(X, J, \omega, \Omega)$ be a Calabi–Yau manifold with Calabi–Yau metric $g$, and let $[L] = (L, \iota : L \to X)$ be a Lagrangian submanifold, where $\iota$ is a $C^2$ immersion with $\iota^*g$ a $C^1$ metric. Let $\tau$ be the maximal of the second fundamental form of $[L]$ and we assume that $\tau \geq 1$. Let $\theta$ be the Lagrangian angle for $[L]$ and $D_\theta f := -d^*(\cos \theta df)$, where $*$ is taken using $\iota^*g$. In
addition, as $\iota^*g$ is only a $C^{1,\gamma}$ metric, $D_\theta$ is an elliptic operator with $C^\gamma$ coefficients. We choose a smooth metric $\hat{g}$ such that if we write $\hat{\tau}$ be the maximal of the second fundamental form of $\hat{g}$, then $\tau \leq C\hat{\tau}$. The different choices of $\hat{g}$ will not influence the following estimates as we only need to obtain a bound in terms of the second fundamental form.

**Proposition 22.** For any function $f$ over the Riemannian manifold $(L, \iota^*g)$, we have

$$
\tau \|df\|_{C^\gamma} + \|\nabla df\|_{C^\gamma} \lesssim \|D_\theta f\|_{C^\gamma} + \tau^{(2+\gamma+\frac{n}{2})}\|f\|_{L^2},
$$

(53)

where $\nabla, D_\theta$ are taking for the metric $\iota^*g$.

Let $U \subset T^*L$ be a Weinstein neighborhood of $[L]$ and we still write $\Omega$ to denote the holomorphic volume form on $U$. Let $\nabla$ be the Levi-Civita connection induced by the Sasaki metric of $\hat{g}$ to $U$, we have $\|\nabla^k\Omega\|_{C^\gamma} \lesssim \tau^{(k+\gamma)}$.

**Proof.** Let $B_R$ be a radius $R$ ball centered at point $p$, then set $R = \tau^{-1}$ and scale by $\tau$, we obtain a radius 1 ball with Euclidean geometry. Applying the classical Schauder estimate for elliptic operator with $C^\gamma$ coefficients and rescale back, we obtain

$$
R^{1+\gamma}\|\nabla f\|_{C^\gamma(B_R^2)} + R^{2+\gamma}\|\nabla^2 f\|_{C^\gamma(B_R^2)} \lesssim R^{2+\gamma}\|D_\theta f\|_{C^\gamma(B_R)} + R^{\frac{n}{2}}\|f\|_{L^2(B_R)},
$$

which implies the desire estimate.

Moreover, let $\pi : T^*L \to L$ be the projection map, we define $D_R := \pi^{-1}(B_R) \cap U$. Then over $D_R$, by a rescale argument, we have $R^{k+\gamma}\|\nabla^k\Omega\|_{C^\gamma(B_R)} \lesssim 1$ for any integer $k \geq 0$. We cover $L$ by size $\tau^{-1}$ balls, which implies the claim.

**Corollary 7.** Under the previous assumption, let $f$ be a function on $L$ with $\int f d\text{Vol} = 0$, $\lambda$ be the first eigenvalue of $D_\theta$ and $\text{Vol}(L)$ be the volume of $L$, then we have

$$
\tau \|\nabla f\|_{C^\gamma} + \|\nabla^2 f\|_{C^\gamma} \leq C(1 + \lambda^{-1}\text{Vol}(L)\tau^{2+\gamma+\frac{n}{2}})\|D_\theta f\|_{C^\gamma}.
$$

(54)

**Proof.** We compute $\|f\|_{L^2(L)} \leq \lambda^{-1}\|D_\theta f\|_{L^2(L)} \leq \lambda^{-1}\text{Vol}(L)\|D_\theta f\|_{C^\gamma}$. The corollary follows by the previous computation and Proposition 22. □

### 7.5.2 Quadratic term estimate.

Let $\alpha$ be a 1-form on $L$, for the immersed submanifold $[L]$ discussed above, we will estimate the quadratic terms $Q(\alpha)$ of the special Lagrangian equation in a Weinstein tubular neighborhood of $[L]$, which follows from Joyce [JOY04C].

**Proposition 23 [JOY04C, Proposition 5.8].** Let $\alpha_1, \alpha_2$ be 1-forms on $L$ with $\|\alpha_1\|_{C^0} + \|\alpha_1\|_{C^0} \leq C\tau^{-1}$ and $\|\nabla \alpha_1\|_{C^0} + \|\nabla \alpha_2\|_{C^0} \leq C$, we have

$$
\|Q(\alpha_1) - Q(\alpha_2)\|_{C^\gamma} \lesssim \tau^\gamma (\|\alpha_1 - \alpha_2\|_{C^\gamma} + \|\nabla \alpha_1 - \nabla \alpha_2\|_{C^\gamma}).
$$

(55)
Proof. Let $\alpha_1, \alpha_2$ be two 1-forms, for each $x \in L$, we write $P(r, s) = Q(r(\alpha_1 - \alpha_2) + s\alpha_2)$.

Then we compute $Q(\alpha_1) - Q(\alpha_2) = P(1, 1) - P(0, 1) = \int_0^1 \frac{\partial P}{\partial r}(r, 1) dr$. As $dQ(0) = 0$, we obtain $\frac{\partial P}{\partial r}(0, 0) = 0$. Therefore, we obtain

$$
\frac{\partial P}{\partial r}(u, 1) = \int_0^1 \left( \frac{d}{ds}\left( \frac{\partial P}{\partial r}(us, s)) \right) \right) ds = \int_0^1 u \frac{\partial^2 P}{\partial r^2}(us, s) + \frac{\partial^2 P}{\partial r \partial s}(us, s) ds.
$$

By a change of variable, we obtain

$$
Q(\alpha_1) - Q(\alpha_2) = \int_0^1 \int_0^s \left[ r \frac{\partial^2 P}{s^2 \partial r^2}(r, s) + \frac{1}{s} \frac{\partial^2 P}{\partial r \partial s}(r, s) \right] dr ds.
$$

By (14), we have $\tilde{SL} : \{(x, y, z) : x \in L, y \in T^*_x L, z \in \otimes^2 T^*_x L \} \to \mathbb{R}$ such that $\text{SL}(\alpha)_x = \tilde{SL}(x, \alpha, \nabla \alpha)$. Let $\partial_1 \text{SL}$ be taking the derivative for $y$ variable and $\partial_2 \text{SL}$ be taking the derivative for $z$ variable, then

$$
\frac{\partial^2 P}{\partial r^2}(r, s) = \otimes^2 (\alpha_1 - \alpha_2) \cdot \partial_1^2 \text{SL} + 2(\alpha_1 - \alpha_2) \otimes (\nabla \alpha_1 - \nabla \alpha_2) \cdot \partial_1 \partial_2 \text{SL} + \otimes^2 (\nabla \alpha_1 - \nabla \alpha_2) \cdot \partial_2^2 \text{SL},
$$

and

$$
\frac{\partial^2 P}{\partial r \partial s}(r, s) = (\alpha_1 - \alpha_2) \otimes \alpha_2 \cdot \partial_1^2 \text{SL} + ((\alpha_1 - \alpha_2) \otimes \nabla \alpha_2 + \alpha_1 \otimes (\nabla \alpha_1 - \nabla \alpha_2)) \cdot \partial_1 \partial_2 \text{SL} + (\nabla \alpha_1 - \nabla \alpha_2) \otimes \nabla \alpha_2 \cdot \partial_2^2 \text{SL},
$$

where we are evaluating $\partial_1 \partial_2 \tilde{SL}$ at $(x, r(\alpha_1 - \alpha_2) + s\alpha_2, r(\nabla \alpha_1 - \nabla \alpha_2) + s\nabla \alpha_2)$.

We define

$$
E_1 := \frac{1}{2} \frac{1}{C} r s^{-2} \|
\alpha_1 - \alpha_2\|^2_{C^2, \gamma} + s^{-1} \|
\alpha_1 - \alpha_2\|_{C^1, \gamma} \|
\alpha_2\|_{C^\gamma},
$$

$$
E_2 := \frac{1}{2} \frac{1}{C} r s^{-2} ||\nabla \alpha_1 - \nabla \alpha_2||^2_{C^2, \gamma} + s^{-1} ||\nabla (\alpha_1 - \alpha_2)||_{C^1, \gamma} \|
\nabla \alpha_2\|_{C^\gamma},
$$

$$
E_3 := \frac{1}{2} \frac{1}{C} r s^{-2} ||\alpha_1 - \alpha_2||_{C^1, \gamma} ||\nabla (\alpha_1 - \alpha_2)||_{C^1, \gamma} + s^{-1} ||\alpha_1 - \alpha_2||_{C^1, \gamma} ||\nabla \alpha_2||_{C^\gamma}
$$

$$
+ s^{-1} ||\alpha_2||_{C^\gamma} ||\nabla (\alpha_1 - \alpha_2)||_{C^\gamma}.
$$

Then (56) implies

$$
\|Q(\alpha_1) - Q(\alpha_2)\|_{C^\gamma} \leq \int_0^1 \int_0^s \left( E_1 \|
\partial_1^2 \tilde{SL}\|_{C^2, \gamma} + E_2 \|
\partial_2^2 \tilde{SL}\|_{C^1, \gamma} + E_3 ||\partial_1 \partial_2 \tilde{SL}||_{C^\gamma} \right) dr ds.
$$

As $\|
\nabla \Omega\|_{C^\gamma} \lesssim \tau^{k+\gamma}$, we have $\|
\partial_1^2 \tilde{SL}\|_{C^\gamma} \lesssim \tau^{2+\gamma}$, $\|
\partial_1 \partial_2 \tilde{SL}\|_{C^1, \gamma} \lesssim \tau^{1+\gamma}$, $\|
\partial_2^2 \tilde{SL}\|_{C^\gamma} \lesssim \tau^{1+\gamma}$, which implies (55).

$\square$
7.5.3 **Corresponding estimates for the family.** Let \([\tilde{L}_t] = (\tilde{L}, \tilde{\iota}_t)\) be the family of special Lagrangian submanifolds we constructed in Theorem 7 with \(\tilde{g}_t := \tilde{\iota}_t g\) be the induced \(C^{1,\gamma}\) metric and \(\tau_t \geq 1\) be the maximal of the second fundamental form.

By Theorem 9, \(\tilde{g}_t\) converge to a cone metric with bounded coefficients. If we take \(g'\) be a smooth background metric on \(\tilde{L}\), then there exists a \(t\)-independent constant \(C\) such that \(\|\tilde{g}_t\|_{C^0} \leq C\), where the \(C^0\) norm is taken using \(g'\). For each \(t\), we choose \(\hat{g}_t\) be a smooth Riemannian metric close to \(\tilde{g}_t\) on \(\tilde{L}\) such that \(\|\tilde{g}_t - \hat{g}_t\|_{C^{1,\gamma}} \leq \tau_t^{-1}\). If we write \(\hat{\tau}\) be the maximal of the second fundamental form of \(\hat{g}_t\), then we have

\[
|\hat{\tau}_t - \tau_t| \leq \|\tilde{g}_t\|_{C^0} (\|\tilde{g}_t - \hat{g}_t\|_{C^{1,\gamma}} + \|\hat{g}_t\|_{C^0} \|\hat{g}_t - \tilde{g}_t\|_{C^{1,\gamma}}) \leq C \tau_t^{-1},
\]

where \(C\) is a \(t\)-independent constant. Therefore, we obtain \(\hat{\tau}_t \leq \tau_t + C \tau_t^{-1}\). Our following estimates will be independent of the choice \(\hat{g}_t\) and \(t\) as we will estimate using \(\hat{\tau}_t\), which is bounded by \(C \tau_t\) for some constant \(C\).

We write \(\tilde{\nabla}_t, \nabla_t\) be the Levi-Civita connections of the Sasaki metric on \(T^*\tilde{L}\) defined by \(\tilde{g}_t, \hat{g}_t\). Let \(D_{\theta_t}(f) := d^*(\cos \theta_t df)\) and \(\text{Vol}_t\) be the volume of \((\tilde{L}, \tilde{g}_t)\).

**Proposition 24.** Let \(f\) be a function on \(\tilde{L}\) with \(\int_{\tilde{L}} f d\text{Vol}_t = 0\), we have

\[
t^{-2}\|df\|_{C^{1,\gamma}} + \|\nabla_t df\|_{C^{1,\gamma}} \lesssim (1 + t^{-(k+2\gamma+n)}) \|D_{\theta_t} f\|_{C^{1,\gamma}}.
\]

Moreover, for the holomorphic volume form, we have \(\|\tilde{\nabla}_t^k \Omega\|_{C^{1,\gamma}} \lesssim t^{-2(k+\gamma)}\).

**Proof.** Let \(\lambda_t\) be the 1st eigenvalue on \(\tilde{L}_t\), then by Theorem 11, there exists \(c_0 > 0\) such that \(\lambda_t \geq c_0 > 0\). Moreover, by Theorem 9, \(\text{Vol}_t \leq 2\text{Vol}(L) + 1\) which implies the claim. \(\square\)

By Proposition 5, the special Lagrangian equation on \([\tilde{L}_t]\) can be written as

\[
\text{SL}_t(\alpha) = \sin \theta_t - d^*(\cos \theta_t \alpha) + Q_t(\alpha).
\]

We have the following estimate for the quadratic terms \(Q_t(\alpha)\).

**Corollary 8.** For 1-forms \(\alpha_1, \alpha_2\) on \([\tilde{L}_t]\) with \(\|\alpha_1\|_{C^0} + \|\alpha_2\|_{C^0} \leq Ct^2\) and \(\|\nabla_t \alpha_1\|_{C^0} + \|\nabla_t \alpha_2\|_{C^0} \leq C\), we have

\[
\|Q_t(\alpha_1) - Q_t(\alpha_2)\|_{C^{1,\gamma}} \lesssim t^{-2\gamma} (t^{-2}(\|\alpha_1 - \alpha_2\|_{C^{1,\gamma}} + \|\nabla_t \alpha_1 - \nabla_t \alpha_2\|_{C^{1,\gamma}}) + (t^{-2}\|\alpha_1\|_{C^{1,\gamma}} + t^{-2}\|\alpha_2\|_{C^{1,\gamma}} + \|\nabla_t \alpha_1\|_{C^{1,\gamma}} + \|\nabla_t \alpha_2\|_{C^{1,\gamma}}).
\]

8 Branched deformation family constructions and nearby special Lagrangian theorem

In this section, we will prove our main theorem and finish our construction of branched deformation family. We will first prove a nearby special Lagrangian theorem in our setting, which is first obtained by Joyce [JOY04B], then we will apply the nearby special Lagrangian theorem to the family we constructed in Theorem 7 to obtain the branched deformations.
8.1 Nearby special Lagrangian theorem. Let \((X, J, \omega, \Omega)\) be a Calabi–Yau manifold, \([L] = (L, \iota : L \to X)\) be an immersed Lagrangian submanifold with \(C^{2,\gamma}\) structure and \(\theta\) be the Lagrangian angle. By Theorem 12, we could choose the Weinstein neighborhood \(U_L\) be \(U_L = \{\alpha \in T^*L \| \|\alpha\|_{C^0} \leq C_\tau^{-1}\}\), where \(C\) is a \(\tau\) independent constant.

Let \(f\) be a function on \(L\), such that \(df \in U_L\), then by Proposition 5, we could write the special Lagrangian equation for \(df\) as
\[
\sin \theta - D_\theta f + Q(df) = 0, \tag{61}
\]
where \(D_\theta f := d^*(\cos \theta df)\). We start with a lemma follows from Aubin [AUB82, Thorem 4.7].

**Lemma 11.** For each \(h \in C^{\gamma}(L)\) with \(\int_L h d\text{Vol} = 0\), there exists \(f \in C^{2,\gamma}\) such that \(\int_L f d\text{Vol} = 0\) and \(D_\theta f = h\).

Therefore, we define \(f := D_\theta^{-1} h\), then we are looking for suitable \(h\) with \(\int_L h d\text{Vol} = 0\) to solve \(-h + Q(dD_\theta^{-1}h) = -\sin \theta\). Let \(C^{\gamma}(L)\) be the space of \(C^{\gamma}\) functions on \(L\) and \(C_0^{\gamma}(L)\) be the \(C^{\gamma}\) functions with mean value zero, then we could define the operator
\[
T : C_0^{\gamma}(L) \to C^{\gamma}(L), \; Th := Q(dD_\theta^{-1}h) + \sin \theta.
\]

**Proposition 25.** Let \(\lambda\) be the first eigenvalue of \(D_\theta\), then there exists a constant \(C\) such that for \(N := C\tau^{\gamma}(1 + \lambda^{-1}\text{Vol}(L)^{2^{2^{\gamma}+\frac{3}{2}}} + 2)\) and \(B_{N^{-1}} := \{f \in C^{\gamma} \| \|f\|_{C^{\gamma}} \leq N^{-1}\}\), suppose \(\|\sin \theta\|_{C^{\gamma}} \leq \frac{1}{4N}\), then \(T : B_{N^{-1}} \to B_{N^{-1}}\) is a contraction mapping.

**Proof.** For \(h_1, h_2 \in B_N\), we compute
\[
\|Th_1 - Th_2\|_{C^{\gamma}} = \|Q_g(dD_\theta^{-1}h_1) - Q_g(dD_\theta^{-1}h_2)\|_{C^{\gamma}} \\
\lesssim \tau^{\gamma}(\tau\|dD_\theta^{-1}(h_1 - h_2)\|_{C^{\gamma}} + \|\nabla dD_\theta^{-1}(h_1 - h_2)\|_{C^{\gamma}}).
\]
where for the second inequality we use Proposition 23. By Corollary 7, we have
\[
\tau\|dD_\theta^{-1}h_1\|_{C^{\gamma}} + \|\nabla dD_\theta^{-1}h_1\|_{C^{\gamma}} \lesssim (1 + \lambda^{-1}\text{Vol}(L)^{2^{2^{\gamma}+\frac{3}{2}}})\|h_1\|_{C^{\gamma}},
\]
and
\[
\tau\|dD_\theta^{-1}(h_1 - h_2)\|_{C^{\gamma}} + \|\nabla dD_\theta^{-1}(h_1 - h_2)\|_{C^{\gamma}} \lesssim (1 + \lambda^{-1}\text{Vol}(L)^{2^{2^{\gamma}+\frac{3}{2}}})\|h_1 - h_2\|_{C^{\gamma}}.
\]

Therefore, there exists a constant \(C_0\) such that
\[
\|Th_1 - Th_2\|_{C^{\gamma}} \leq C_0\tau^{\gamma}(1 + \lambda^{-1}\text{Vol}(L)^{2^{2^{\gamma}+\frac{3}{2}}})^2\|h_1 - h_2\|_{C^{\gamma}}(\|h_1\|_{C^{\gamma}} + \|h_2\|_{C^{\gamma}})
\]
\[
\leq \frac{C_0}{C}\|h_1 - h_2\|_{C^{\gamma}}.
\]
Thus, for any $C \geq 4C_0$, we have $\|Th_1 - Th_2\|_{C, \gamma} \leq \frac{1}{4} \|h_1 - h_2\|_{C, \gamma}$. Moreover, we still need to check $T : B_N \to B_N$ is a well-defined map. By a similar computation, we obtain

$$\|Th\|_{C, \gamma} \leq \|Q(dD_0^{-1}h)\|_{C, \gamma} + \|\sin \theta\|_{C, \gamma} \leq C_0'\tau\lambda^{-1}\Vol(L)\tau^{2+\gamma+\frac{n}{2}} \|h\|_{C, \gamma} + \|\sin \theta\|_{C, \gamma} \leq (C_0'C^{-1} + \frac{1}{4})N^{-1}.$$  

We take $C = \max\{4C_0', 4C_0\}$, then $T : B_{N^{-1}} \to B_{N^{-1}}$ is a well-defined contraction map. 

**Corollary 9.** There exists an unique solution to (61) which satisfies

$$\tau\|df\|_{C, \gamma} + \|\nabla df\|_{C, \gamma} \lesssim \tau^{-\gamma}(1 + \lambda^{-1}\Vol(L)\tau^{2+\gamma+\frac{n}{2}})^{-1}.$$  

In particular, $df$ lies in the Weinstein neighborhood $U_L$.

**Proof.** Let $h$ be the fixed point of the contraction map $T : B_{N^{-1}} \to B_{N^{-1}}$, then $\|h\|_{C, \gamma} \leq N^{-1}$. By Corollary 7, we obtain

$$\tau\|df\|_{C, \gamma} + \|\nabla df\|_{C, \gamma} \lesssim (1 + \lambda^{-1}\Vol(L)\tau^{2+\gamma+\frac{n}{2}})\|h\|_{C, \gamma} \lesssim N^{-1}(1 + \lambda^{-1}\Vol(L)\tau^{2+\gamma+\frac{n}{2}}).$$

Let $\iota_f$ be the inclusion given by the graph of $df$, then $\iota_f$ will be a $C^{1,\gamma}$ immersion with $[L_f] = (L, \iota_f)$ a $C^{1,\gamma}$ special Lagrangian. Moreover, by Morrey [MOR66, Chapter 5.3, Chapter 5.4 and Chapter 6.6], every $C^1$ special Lagrangian is real analytic. Therefore, $[L_f]$ is real analytic. In summary, we obtain the following nearby special Lagrangian theorem, which will be a special case of Joyce [JOY04B].

**Theorem 13.** Let $[L] = (L, \iota)$ be an immersed Lagrangian submanifold, let $\lambda$ be the first eigenvalue of the linearization operator $D$ and $\tau$ be the maximal of the second fundamental form. There exists a constant $C$ such that for $\Upsilon := C(1 + \lambda^{-1}\Vol(L)\tau^{2+\gamma+\frac{n}{2}})^2\tau^{-\gamma}$, suppose $\|\iota^*\Theta\|_{C, \gamma} \leq \Upsilon^{-1}$, then there exists a function $f$ on $L$ with $\int_L df\Vol = 0$ and

$$\|df\|_{C^{1,\gamma}} \lesssim \tau^{-\gamma}(1 + \lambda^{-1}\Vol(L)\tau^{2+\gamma+\frac{n}{2}})^{-1},$$

such that the graph submanifold $[L_f]$ is a real analytic special Lagrangian.

### 8.2 Existence of special Lagrangian submanifolds near the approximate family.

Let $(X, J, \omega, \Omega)$ be a Calabi–Yau manifold with a special Lagrangian submanifold $[L] = (L, \iota_0)$. Suppose over $[L]$ there exists a nondegenerate harmonic pair, $[\tilde{L}_t^{\text{app}}] = (\tilde{L}, \tilde{\iota}_t^{\text{app}})$ be the family of approximate special Lagrangian submanifolds constructed in Theorem 7 and $\theta_t$ be the Lagrangian angle. In this subsection, we will apply the nearby special Lagrangian theorem for the family of Lagrangian submanifold $[\tilde{L}_t^{\text{app}}]$, to construct special Lagrangian submanifolds.
Let $\tau_t$ be the maximal of second fundamental form of $[\tilde{L}^\text{app}]$, then by Proposition 20, we have $\tau_t \lesssim t^{-2}$. We define $\tilde{g}_t = (\tilde{L}^\text{app})^*g$ be the pull-back Riemannian metric. By Theorem 9, $\tilde{g}_t$ converge to a cone metric which is quasi-isomorphic to any smooth Riemannian metric on $\tilde{L}$. The $C^\gamma$ norm defined using different $\tilde{g}_t$ or the cone metric are equivalent. Therefore, for the rest of this subsection, we don’t label $t$ for different $C^\gamma$ norms. Moreover, by Theorem 12, for each $t$, we could choose the Weinstein neighborhood of $[\tilde{L}^\text{app}]$ to be

$$ U_t := \{ \alpha \in T^*\tilde{L} \| \alpha \|_{C, \gamma} \leq Ct^2 \}. $$

Let $f_t$ be a function on $\tilde{L}$ such that $df_t \in U_t$, the special Lagrangian equation for $df_t$ in this Weinstein neighborhood will be

$$ \sin \theta_t - D_{\theta_t}f_t + Q_t(df_t) = 0. $$

Let $f_t := -D_{\theta_t}^{-1}h_t$ with $\int_{\tilde{L}} h_t d\text{Vol}_t = 0$, where $d\text{Vol}_t$ is the volume form w.r.t. the Riemannian metric $g_t$, then the special Lagrangian equation for $h_t$ will be

$$ \sin \theta_t + h_t + Q_t(dD_{\theta_t}^{-1}h_t) = 0. $$

We define $T_t h_t := \sin \theta_t + Q_t(dD_{\theta_t}^{-1}h_t)$, then by Proposition 25 and Corollary 9, we obtain

**Proposition 26.** For $N_t = t^{8+4\gamma+n}$, there exists a constant $C$ independent of $t$ such that suppose $\| \sin \theta_t \|_{C, \gamma} \leq \frac{N_t}{t}$, then $T_t : B_t \to B_t$ is a contracting map for $B_t := \{ \| \alpha \|_{C, \gamma} \leq C^{-1}N_t \}$. Moreover, let $h_t$ be the fixed point of $T_t$ with $f_t = D_{\theta_t}h_t$, then $f_t$ satisfies

$$ t^{-2}\| df_t \|_{C, \gamma} + \| \nabla_t df_t \|_{C, \gamma} \leq C^{-1}t^{4+n+4\gamma}. $$

By Theorem 7, for any $k$, we can choose $[\tilde{L}^\text{app}]$ such that $\| \sin \theta_t \|_{C, \gamma} \leq C_0 t^k$. When $t$ small enough and $t^k \geq t^{8+4\gamma+n+1}$, we could obtain the desire approximate solutions.

From the above estimate, we know that $df_t \subset U_t$. We could write the inclusion induces by $df_t$ as $\tilde{t}_t : \tilde{L} \to U_t \subset T^*\tilde{L}$, which is a special Lagrangian submanifold. Using $\Phi_t : U_t \to U'_t \subset X$, we define $\tilde{\alpha}_t := \Phi_t \circ \tilde{t}_t$, then $[\tilde{L}_t] = (\tilde{L}, \tilde{\alpha}_t)$ will be a family of special Lagrangian submanifolds on $X$.

Now, let’s discuss the rate of the converge. By Proposition 26, we have

$$ \| \tilde{t}_t - \tilde{t}_t^\text{app} \|_{C, \gamma} \lesssim \| df_t \|_{C, \gamma} \leq Ct^{4+n+4\gamma}. $$

In addition, by Theorem 7 (vi), we have $\| \tilde{t}_t^\text{app} - \iota_0 \circ p \|_{C, \gamma} \leq Ct$. Therefore, we obtain

$$ \| \tilde{t}_t - \iota_0 \circ p \|_{C, \gamma} \leq C\| \tilde{t}_t - \tilde{t}_t^\text{app} \|_{C, \gamma} + \| \tilde{t}_t^\text{app} - \iota_0 \circ p \|_{C, \gamma} \leq Ct. $$

On the other hand, let $a$ be the nondegenerate harmonic pair and $\phi_t$ be the diffeomorphism constructed in Theorem 7. We write $\tilde{\phi}_t^\circ (ta) : \tilde{L} \to U_L$ be the
inclusion map induces by the graph of \( t\phi_t^*(a) \), then by Theorem 7 (ii), we have
\[ \|\tilde{\iota}_t^{app} - \tilde{\iota}_{t\phi_t^*(a)}\|_{C,\gamma} \leq Ct^2, \]
which implies
\[ \|\tilde{\iota}_t - \tilde{\iota}_{t\phi_t^*(a)}\|_{C,\gamma} \leq Ct^2. \]

In summary, we finish the proof the proof of Theorem 1.

Suppose the special Lagrangian is a fixed real locus, then the branched deformations we constructed in Theorem 1 could also have extra symmetry.

**Theorem 14.** Let \((X,J,\omega,\Omega)\) be a Calabi–Yau manifold and \([L] = (L,\iota_0 : L \to X)\) be a special Lagrangian submanifold which is locally a fixed real locus of anti-holomorphic involution \(R\). Suppose there exists a \(\mathbb{Z}_2\) harmonic 1-form \(\alpha^-\) over \([L]\), then the family of special Lagrangian submanifolds \([\tilde{L}_t] = (\tilde{L},\tilde{\iota}_t)\) constructed in Theorem 1 will satisfy \(R \circ \tilde{\iota}_t = \tilde{\iota}_t \circ \sigma\), where \(\sigma\) is the involution on \(\tilde{L}\).

**Proof.** By Theorem 8 and Corollary 4, the special Lagrangian equation over \([\tilde{L}_t^{app}]\) for \(df_t\) over \(\tilde{L}\) will satisfy \(\sigma^*\text{SL}(df_t) = -\text{SL}(-\sigma^*df_t)\). Therefore, suppose \(f_t\) is a solution, then \(-\sigma^*df_t\) is also a solution. By Corollary 9, the solution is unique for the contracting map, therefore \(df_t = -\sigma^*df_t\). Recall that \(\tilde{\iota}_t : \tilde{L} \to X\) is given by the graph of \(df_t\), which defined as \(\tilde{\iota}_t(x) = \exp_{\tilde{\iota}_0}^{\text{app}}(J(\tilde{\iota}_t^{app})_*V_{df_t})\), where \(V_{df_t}\) is the dual vector of \(df_t\) given by the Riemannian metric. By the same computation as (48), we obtain \(R \circ \tilde{\iota}_t = \tilde{\iota}_t \circ \sigma\). \(\square\)

### 9 Applications of the branched deformation theorem

In this section, we will discuss several directions related to our main theorem.

#### 9.1 The moduli space of the branched deformation family.

The moduli space of special Lagrangian submanifolds have been studied in [JOY05, MCL98, HIT97]. In this subsection, we will briefly discuss the meaning of the branched deformation theorem for the moduli space.

Let \([L] = (L,\iota_0)\) be a special Lagrangian submanifold in a Calabi–Yau manifold \((X,J,\omega,\Omega)\). Let \(\mathcal{M}_[L]\) be the connected component of the set of special Lagrangian submanifolds containing \([L]\). Then McLean’s deformation theorem 5 implies that \(\mathcal{M}_[L]\) is a smooth manifold with dimension \(b^1(L)\), which is the first Betti number of \(L\).

Suppose there exists a nondegenerate harmonic pair \((\alpha^+,\alpha^-)\) on \([L]\), and we write \([\tilde{L}_t] = (\tilde{L},\tilde{\iota}_t)\) be the family of branched deformation constructed in Theorem 1. We could consider the moduli space \(\mathcal{M}_{[\tilde{L}_t]}\), which is the connected components containing all the \([\tilde{L}_t]\), then \(\mathcal{M}_{[\tilde{L}_t]}\) is also a smooth moduli space with dimension \(b^1(\tilde{L})\). By Proposition 1, \(b^1(\tilde{L}) = b^1_+ + b^1(L)\). Therefore, we might enlarge \(\mathcal{M}_{[\tilde{L}_t]}\) to a compactification \(\overline{\mathcal{M}_{[\tilde{L}_t]}}\) by adding \(\mathcal{M}_{[L]}\) as a codimension \(b^1_-\) boundary.

The inverse of the above discussions is incorrect that not every element of \(b^1(\tilde{L})\) could generates a deformation. Let \([L] = (L,\iota_0)\) be an embedded special Lagrangian...
submanifold such that the induced metric on $[L]$ is Ricci positive, then by [AI21, TW18], $[L]$ doesn’t admit any deformation. However, by Example 1, in the case when $L = S^3$, we could construct branched covering $p : \tilde{L} \to S^3$ such that $b_1(\tilde{L}) \geq 1$. We could also regard $(\tilde{L}, \iota_0 \circ p)$ be an branched immersed special Lagrangian submanifold. However, there will be no deformation of $(\tilde{L}, \iota_0 \circ p)$ that is generating by $L^2$ harmonic 1-forms on $b_1(\tilde{L})$.

9.2 Branched deformations in a Calabi–Yau neighborhood. In this subsection, we will introduce examples that our deformation theorem can be applied using the Calabi–Yau neighborhood theorem.

**Definition 13.** Let $(L, g_L)$ be a real analytic Riemannian manifold, let $U_L$ be a neighborhood of the zero section in $T^*L$. A Calabi–Yau neighborhood of $L$ is a Calabi–Yau structure $(U_L, J, \omega, \Omega)$ with Calabi–Yau metric $g$ such that

(i) $\omega$ is the canonical symplectic form,
(ii) The restriction of $g$ to the zero section is $g_L$,
(iii) The zero section is a special Lagrangian submanifold in this Calabi–Yau structure.

**Theorem 15** [BRY98, DOI15]. Let $(L, g_L)$ be a real analytic Riemannian manifold, suppose $\chi(L) = 0$, then $L$ admits a Calabi–Yau neighborhood $(U_L, J, \omega, \Omega)$. In addition, let $R$ be the canonical involution on $T^*L$, then $R$ is an anti-holomorphic involution on $U_L$.

Using Theorem 14, we obtain:

**Theorem 16.** Let $(L, g_L)$ be a real analytic Riemannian manifold with $\chi(L) = 0$, let $(U_L, J, \omega, \Omega)$ be a Calabi–Yau structure constructed in Theorem 15, suppose there exists a nondegenerate multivalued harmonic form on $(L, g_L)$ with associated branched covering $p : L \to L$, then there exists a family of immersed special Lagrangian submanifolds $\tilde{\iota}_t : \tilde{L} \to U_L$ such that $\lim_{t \to 0} \|\tilde{\iota}_t - \iota_0 \circ p\|_{C^\gamma} = 0$, where $\iota_0$ is the inclusion of the zero section. In addition, $\iota_t$ will satisfy $R \circ \tilde{\iota}_t = \tilde{\iota}_t \circ \sigma$.

By [HE], the author construct nondegenerate $Z_2$ harmonic 1-forms over some rational homology 3-spheres. As a rational homology 3-sphere has vanishing first Betti number, it is rigid in McLean sense as in Theorem 5. Therefore, we obtain the following.

**Corollary 10.** There exist special Lagrangian submanifolds that are rigid in classical sense but could have branched deformations.

9.3 Topological constraint of the existence of nondegenerate $Z_2$ harmonic 1-forms. Let $(L, g)$ be a Riemannian manifold and $(\Sigma, J, \alpha)$ be a nondegenerate $Z_2$ harmonic 1-form. By the work of Taubes [TAU13B, TAU13A], nondegenerate $Z_2$ harmonic 1-forms are the ideal boundary of the gauge theory compactification of PSL$(2, \mathbb{C})$ flat connection and it is still very mysterious. We would like to understand the following question.
**Question 2.** Is there any topological constraint to the existence of nondegenerate \(\mathbb{Z}_2\) harmonic 1-forms?

Using the branched deformation theorem we proved, we could actually give a topological constraint for the existence of nondegenerate \(\mathbb{Z}_2\) harmonic 1-forms. Our starting point come from the work of Abouzaid and Imagi [AI21].

**Theorem 17** [AI21] Let \(U_L\) be a neighborhood of the zero section of \(T^*L\), and \((U_L, J, \omega, \Omega)\) be a Calabi–Yau structure such that the zero section \([L_0]\) a special Lagrangian submanifold, then for any unobstructed nearby immersed special Lagrangian \([L']\) which is \(C^0\) close to \([L_0]\):

(i) Suppose \(\pi_1(L)\) is finite, then \([L'] = [L_0]\).

(ii) Suppose \(\pi_1(L)\) has no nonabelian free subgroup, then \([L']\) is an unbranched deformation of \([L_0]\) that constructed in Corollary 2.

The definition of a Lagrangian submanifold is unobstructed is in [FOO09], which is not relevant to most of the parts in the paper, so we omit the definition. We will list a few cases where the unobstructed condition is satisfied. Let \((L, g)\) be a Riemannian manifold, suppose there exists a nondegenerate \(\mathbb{Z}_2\) harmonic 1-form on \(L\) and suppose there exists a Calabi–Yau neighborhood \((U_L, J, \omega, \Omega)\). We write \([L_0] = (L, \iota_0)\) be the inclusion of the zero section. Then by Theorem 16, we obtain a family of special Lagrangian submanifolds \([\tilde{L}_t]\) which will be \(C^0\) close to \([L_0]\). If any \([\tilde{L}_t]\) is unobstructed and suppose \(\pi_1(L)\) is either finite or has no nonabelian free subgroup, this will lead to a contradiction based on Theorem 17.

**Proposition 27.** Let \((L, g_L)\) be a real analytic Riemannian manifold and \((\Sigma, J, \alpha)\) be a multivalued harmonic 1-form on \(L\) with associate branched covering \(p : \tilde{L} \to L\). Suppose

(i) \(\pi_1(L)\) is finite or contains no nonabelian free subgroup,

(ii) \((L, g_L)\) has a Calabi–Yau neighborhood \((U_L, J, \omega, \Omega)\) in \(T^*L\),

(iii) every special Lagrangian submanifold over \(U_L\) which is diffeomorphic to \(\tilde{L}\) is unobstructed,

then \((\Sigma, J, \alpha)\) is not nondegenerate. In addition, suppose the canonical involution \(R\) on \(T^*L\) induces an anti-holomorphic involution, then (iii) can be changed to

(iii) every \(R\)-invariant special Lagrangian submanifold over \(U_L\) which is diffeomorphic to \(\tilde{L}\) is unobstructed.

In general, it is very hard to check whether a Lagrangian submanifold is unobstructed or not and by the work of FOOO [FOO09], there is an easy topological condition to be verified.

**Proposition 28** [FOO09]. Let \((\tilde{L}, \tilde{\iota} : \tilde{L} \to U_L)\) be a Lagrangian submanifold, suppose \(\tilde{\iota}^* : H^2(U_L; \mathbb{R}) \to H^2(\tilde{L}; \mathbb{R})\) is surjective, then \((\tilde{L}, \tilde{\iota})\) is unobstructed.
As a corollary, we obtain extra constraints of nondegenerate $\mathbb{Z}_2$ harmonic 1-form.

**Theorem 18.** Let $(L, g_L)$ be a real analytic Riemannian manifold and let $(\Sigma, \mathcal{I}, \alpha)$ be a nondegenerate $\mathbb{Z}_2$ harmonic 1-form on $L$ with branched covering $p : \tilde{L} \to L$. Suppose

(i) $\pi_1(L)$ is either finite or contains no nonabelian free subgroup,

(ii) $(L, g_L)$ has a Calabi–Yau neighborhood on $T^*L$,

then $b_2(\tilde{L}) > b_2(L)$, where $b_2$ is the second Betti number.

Unfortunately, the above theorem is trivial when $\dim(L) = 3$. By Corollary 3, the existence of $\mathbb{Z}_2$ harmonic 1-form will require that $b_1(\tilde{L}) > b_1(L)$ which implies $b_2(\tilde{L}) > b_2(L)$ by Poincare duality. However, this topological constraint is non-trivial in for four dimensional or higher and we will introduce an example. Let $\sigma_n : S^n \to S^n$ be the involution map fixing the equator and let $\tilde{L} := S^1 \times S^{n-1}$. We take $\sigma := (\sigma_1, \sigma_{n-1})$, then $\sigma$ is an involution action on $\tilde{L}$ with fixed point $\text{Fix}(\sigma)$ is diffeomorphic to $S^0 \times S^{n-2}$ which is two copies of $S^{n-2}$. The quotient $L := \tilde{L}/\langle \sigma \rangle$ under the involution, would be $L := S^n$. Moreover, the quotient map $\tilde{p} : \tilde{L} \to L$ will be a branched covering map. When $n \geq 4$ and $n$ is odd, as $H^1(\tilde{L}; \mathbb{R})^\perp \neq 0$, over $L$, there exists a $\mathbb{Z}_2$ harmonic 1-form $\alpha$ over $L$. However, as $H^2(\tilde{L}; \mathbb{R}) = 0$ and $\chi(\tilde{L}) = 0$, by Theorem 18, this $\mathbb{Z}_2$ harmonic 1-form will not be nondegenerate.

When $n$ is even, over $L = S^n$, as $\chi(L) \neq 0$, we don’t know the existence of Calabi–Yau neighborhood for generic metric on $L$. However, over $T^*S^n$ we have the Stenzel metric [STE93], which extend the round metric on $S^n$ to a complete Riemannian metric over the whole $T^*S^n$. Thus over $T^*L$, we could associate a Calabi–Yau structure. By Theorem 18, the $\mathbb{Z}_2$ harmonic 1-form will not be nondegenerate. In summary, we obtain

**Corollary 11.** Let $p : S^1 \times S^{n-1} \to S^n$ be the branched covering map considered above, $\Sigma$ be the branch locus, $\chi \in H^1(S^1 \times S^{n-1}; \mathbb{R})$, then:

(i) If $n$ is odd, then for any Riemannian metric over $S^n$, the $\mathbb{Z}_2$ harmonic 1-form corresponding to $\chi$ will not be nondegenerate.

(ii) If $n$ is even, then for the round Riemannian metric over $S^n$, the $\mathbb{Z}_2$ harmonic 1-form corresponding to $\chi$ will not be nondegenerate.

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