DIAGRAM CATEGORIES FOR $U_q$-TILTING MODULES AT ROOTS OF UNITY

HENNING HAAHR ANDERSEN AND DANIEL TUBBENHAUER

ABSTRACT. In this paper we give a diagrammatic category (inspired by Elias’ dihedral cathedral) that is a diagrammatic presentation of the category of $U_q(sl_2)$-tilting modules $\mathcal{T}$ for $q$ being an $l$-th root of unity. Moreover, we, following an approach of Soergel and Stroppel, introduce a grading turning $\mathcal{T}$ into a graded category $\mathcal{T}^{gr}$. This grading is a purely “root of unity” phenomena and might lead to new insights about the link, tangle and 3-manifold invariants deduced from $\mathcal{T}$.

In addition, we also give a diagrammatic category for the (now graded) projective endofunctors $p\text{End}(\mathcal{T}^{gr})$, indicate how our results could generalize to $U_q(sl_n)$ and maybe other types and recall/collect some “well-known” facts to give a reasonably self-contained exposition.

Acknowledgements. We thank Ben Elias, Catharina Stroppel, Pedro Vaz and Geordie Williamson for many helpful discussion. Special thanks to Geordie Williamson for pointing out a connection of the tilting category to an anti-spherical quotient of Elias’ dihedral cathedral. The second author thanks the Danish summer for, positively formulated, not distracting him from typing this paper.

CONTENTS

1. Introduction 2
1.1. The framework 2
1.2. An outline of the paper 7
2. The tilting category 9
2.1. Quantum groups at roots of unity 9
2.2. Weyl modules, dual Weyl modules and simple modules 11
2.3. Tilting modules and the tilting category $\mathcal{T}$ 13
2.4. The linkage principle and blocks 16
2.5. Translation functors 19
3. The KS quiver algebra and gradings 23
3.1. The quiver algebras $A_m$ and $A_\infty$ 23
3.2. Combinatorics of the graded, right $A_m$- and $A_\infty$-modules 26
3.3. Endofunctors on $p\text{Mod}_{gr}A_m$ and $p\text{Mod}_{gr}A_\infty$ 27
3.4. $A_\infty$ and the tilting category $\mathcal{T}$ 29
3.5. The tilting category $\mathcal{T}$ and gradings 35
4. Diagrams for the tilting category 41
4.1. Some basic remarks: Diagrammatic categories and additive closures 42
4.2. The dihedral cathedral $\mathcal{D}(\infty)$ 43
4.3. The quotient $\mathcal{D}(\infty)$ 49
4.4. $\mathcal{D}(\infty)$ and the tilting category $\mathcal{T}^{gr}$: Its an equivalence! 54
4.5. Diagram categories for $p\text{End}(\mathcal{T}^{gr})$ 61
References 64

The authors were partially supported by the center of excellence grant “Centre for Quantum Geometry of Moduli Spaces (QGM)” from the “Danish National Research Foundation (DNRF)”.
1. Introduction

1.1. The framework. In this paper we study the quantum group $U_q(\mathfrak{sl}_2) = U_q$ where $q$ is an $l$-th root of unity, its category of tilting modules $\mathcal{T}$ and the category of projective endofunctors $\mathcal{pEnd}(\mathcal{T})$ combinatorially and diagrammatically.

Everything we do is completely explicit and “down to earth”, but motivated and deduced from a general machinery that comes, from the side of representation theory, from pioneering work of Soergel [62], Beilinson, Ginzburg and Soergel [10], Kazhdan-Lusztig [33] and Stroppel [65], and from the side of combinatorics and diagrammatics, from Soergel [64], Khovanov-Lauda [38] and Khovanov-Elias [20]. We expect that everything will generalize to $\mathfrak{sl}_n$ (see Remarks 3.17, 3.23 and 4.39) and maybe to other types. But, of course, it will be much more complicated.

Let us motivate and explain our approach.

1.1.1. Quantum groups at roots of unity: Non-semisimplicity, modular representation theory and affine Weyl groups. Fix a simple complex Lie algebra $\mathfrak{g}$. Then the finite dimensional representation theory of the quantum deformation $U_q(\mathfrak{g})$ of $U(\mathfrak{g})$ is, for generic parameter $v$, semisimple and very similar to the classical representation theory for $U(\mathfrak{g})$. This drastically changes when specializing $v$ to an $l$-th root of unity $q$: The representation theory of $U_q(\mathfrak{g})$ is non-semisimple. This is mostly due to the fact that the so-called Weyl modules at roots of unity are, in general, not simple and filtrations by Weyl modules do not necessarily split. A lot of questions remain open about the representation theory at roots of unity. In fact, some magic happens: The representation theory of $U_q(\mathfrak{g})$ over $\mathbb{C}$ has many similarities to the representation theory of a corresponding almost simple, simply connected algebraic group $G$ over an algebraically closed field $K$ of prime characteristic, see for example [3] or [47]. On the other hand, the “combinatorics” (the irreducible characters) of $G$ and $U_q(\mathfrak{g})$ was conjectured by Lusztig (see [49] for $G$ and [47] for $U_q(\mathfrak{g})$) to be related to values at 1 of the Kazhdan-Lusztig polynomial associated to the affine Weyl group for $\mathfrak{g}$. In addition, Kazhdan and Lusztig proved later that the category of finite dimensional $U_q(\mathfrak{g})$-modules (of type 1) is equivalent to a category of modules for the corresponding affine Kac-Moody algebra, see [33].

This combined with the solution of Kashiwara and Tanisaki [32] of the Kazhdan-Lusztig conjecture in the affine Kac-Moody case then solved the above mentioned conjecture for the irreducible characters of $U_q(\mathfrak{g})$. Even closer related to our work: Soergel first conjectured in [63] and later proved in [61] a corresponding statement about indecomposable tilting modules. In our little $\mathfrak{sl}_2$ case we do not need these deep results because we can work out both, the irreducible characters and the indecomposable tilting modules, “by hand”.

Although we do not use or exploit the first relation to the algebraic groups in prime characteristic in this paper much, we certainly use the second here by “shifting problems to infinity”: There is “no” root of unity $q$ anymore in Sections 3 and 4. For $\mathfrak{g} = \mathfrak{sl}_2$ the above means that the “combinatorics” in our case is governed by the infinite dihedral group $D_\infty$. This is the main reason to expect connections (as we work them out in Section 4) to Elias’ dihedral cathedral, see [19].

1.1.2. Tilting modules at roots of unity, modular categories and 2 + 1-TQFT’s. It turns out, when studying the representation theory of $U_q(\mathfrak{g})$, a certain category of tilting modules $\mathcal{T}$ comes up

\footnote{If not otherwise stated: Modules in this paper are assumed to be finite dimensional. There are only few exceptions in this paper, e.g. $T_q(\infty)$ from Definition 2.24. We hope its clear from the context.}
naturally (we recall the definition in Section 2). The category $\mathcal{X}$ is inspired by the corresponding category of tilting modules for reductive algebraic groups due to Donkin [26] (see also Ringel [58]) and shares most of its properties, see for example [1].

And, although $\mathcal{X}$ is not a semisimple category, it is a so-called ribbon (tensor) category (roughly: It is monoidal and has duality) that behaves in many aspects similar to the semisimple monoidal category of $\mathcal{U}_v(\mathfrak{g})$-modules. Such categories have an underlying topological behavior. In particular, as a ribbon (tensor) category it provides invariants of oriented, framed links and tangles without relying on a braid presentation, see for example [71].

On the other hand, ribbon (tensor) categories that are semisimple and have a finite number of simple objects are the basic ingredients of so-called modular categories. The latter, as demonstrated by Turaev [71], provide an abstract framework leading to $2+1$-TQFT’s and the Witten-Reshetikhin-Turaev invariants of 3-manifolds. Moreover, they are known to organize various algebraic structures arising in for example topological quantum field theory, conformal field theory, von Neumann algebras and vertex operator algebras (a good treatment can be found in e.g. [9] or [71]). Thus, a basic question is how to obtain modular categories.

This is, among the connections mentioned above, another fact that makes its reasonable to study $\mathcal{X}$: As explained in details in for example [1] or [60], one can “semisimplify” $\mathcal{X}$ using an explicit process by setting so-called “negligible morphisms” to zero. The quotient is semisimple and has only finitely many simple modules parametrized by the fundamental alcove $A_0$ (which we, in the $\mathfrak{sl}_2$ case, recall in Subsection 2.4). The quotient of $\mathcal{X}$ provides therefore a modular category giving rise to $2+1$-TQFT’s. These $2+1$-TQFT’s, by “evaluating” closed 3-manifolds, provide then invariants of 3-manifolds.

1.1.3. Categorification and graded categories. A ground-breaking development towards proving the so-called Kazhdan-Lusztig conjectures was initiated by Soergel in [64]. He defines a combinatorial category $\mathcal{S}$ consisting of objects that are bimodules over a polynomial ring $R$. These bimodules are nowadays commonly called Soergel bimodules and are indecomposable direct summands of tensor products of modules denoted by $B_i$.

His category is additive, monoidal and graded and he proves that the Grothendieck group $K_0$ of it is isomorphic to an integral form of the Hecke algebra $H_v(W)$ associated to the Weyl group $W$ of the simple Lie algebra $\mathfrak{g}$ in question. Here the grading and the corresponding shifting functors give on the level of Grothendieck groups rise to the indeterminate $v$ of the Hecke algebra $H_v(W)$.

Thus, we can say that Soergel’s construction categorifies $H_v(W)$. In fact, the categorification works for any Coxeter group $W$ and its associated Hecke algebra $H_v(W)$. For us $W$ will be the affine Weyl group for $\mathfrak{sl}_2$ (i.e. the infinite dihedral group $D_\infty$).

In fact, in the spirit of categorification outlined by Crane and Frenkel in the 90’s, graded categories $\mathcal{C}$ (or 2-categories) give rise to a structure of a $\mathbb{Z}[v, v^{-1}]$-module on $K_0(\mathcal{C})$. Many examples of this kind of categorification are known. For example, Khovanov’s categorification of the Jones polynomial (also called $\mathfrak{sl}_2$ polynomial) [34], Khovanov-Rozansky’s categorification of the $\mathfrak{sl}_n$ polynomial in [41] or Khovanov-Lauda and Rouquier’s categorification of $\mathcal{U}_v(\mathfrak{g})$ and its highest weight modules, see for example [38] and [59], are among the more popular ones and have opened new directions of research.

Thus, it is natural to ask if we can introduce a non-trivial grading on $\mathcal{X}$ as well. We do this in Section 3 by using an argument pioneered by Soergel (see [62]) in the ungraded and Stroppel
(see [65]) in the graded case for category $O$. Namely, the usage of Soergel’s combinatorial functor $V_m$ that gives rise to an equivalence of a block of $O$ (for $g$) and a certain full subcategory of $\text{Mod}-A$. The algebra $A$ is the endomorphism ring of the anti-dominant projective in the block and it can be explicitly (when the block is regular) identified with the algebra of coinvariants for the Weyl group associated to $g$. This algebra can be given a $\mathbb{Z}$-grading and, as Stroppel explains in [65], this set-up gives rise to graded versions of blocks of category $O$ and the categories of graded endofunctors on these blocks.

In fact, as Stroppel explains in [65], her approach is a combinatorial alternative to the approach of Beilinson, Ginzburg and Soergel given in [10].

As in the other cases above, the grading is the crucial point: In category $O$ the multiplicity $[\Delta_q(\lambda) : L_q(\mu)]$ of the simple module $L_q(\mu)$ inside of the Verma $\Delta_q(\lambda)$ is given by evaluating the corresponding Kazhdan-Lusztig polynomial at 1. The Kazhdan-Lusztig polynomial is a polynomial and not just a number and the grading of $O$ “explains” now the individual coefficients of these polynomials as well.

In our case the role of $A$ is, as we explain in Section 3, played by an “infinite version” $A_\infty$ of a quiver algebra $A_m$ that Khovanov and Seidel introduced in [42] in their study of Floer homology. Its “Koszul version” appears in various contexts related to symplectic topology, algebraic geometry and representation theory. In particular, it appears as a subquotient of Khovanov’s arc algebra that he introduced in [55] to give an algebraic structure underlying Khovanov homology and whose representation theory is known to be highly interesting as outlined in a series of papers by Brundan and Stroppel, see [12], [13], [14], [15] and [16]. Khovanov’s arc algebra is naturally graded by the Euler characteristic of cobordisms as Khovanov explains in [35]. This grading introduces a grading on the subquotient that agrees with the Koszul grading on $A_m$ and thus, since we use this grading to introduce a graded version $\mathcal{T}^{gr}$ of $\mathcal{T}$ (and its endofunctors), we tend to say that $\mathcal{T}$ can be given a natural grading from the viewpoint of topology and algebra.

We stress that this is a purely “root of unity” phenomena now: The category of finite dimensional $U_q$-modules is semisimple and has therefore no interesting grading. On the other hand, the grading on $\mathcal{T}^{gr}$ is non-trivial and gives for example rise (as mentioned above) to a “topological” grading for similar modules of reductive algebraic groups over algebraical closed fields $K$ of prime characteristic.

Moreover, in the spirit above, graded category $O$ can be used in various contexts. For example, Stroppel explains in [66] and [67] how graded (parabolic) category $O$ associated to $\mathfrak{sl}_{m+1}$ (or rather the category of graded endofunctors) can be used to categorify the $m + 1$-strand Temperley-Lieb algebra $TL_{m+1}$ (as conjectured by Bernstein, Frenkel and Khovanov in [11]) and gives rise to a method to obtain a generalization of Khovanov homology. Her method was later generalized by Mazorchuk and Stroppel in [53] (which is “Koszul dual” to related work of Sussan [68]). It is only indirectly known by work of Webster in [72] (or, alternatively, recent work of Cautis [18]) that (a restriction of) their generalization agrees with Khovanov’s (and Rozansky’s for the more general $\mathfrak{sl}_n$ case) approach to link homologies.

Thus, an intriguing question is if one can use the grading on $\mathcal{T}^{gr}$ to obtain new information about invariants of links and tangles coming from the ribbon structure of $\mathcal{T}$ or about the Witten-Reshetikhin-Turaev invariants. In fact, as we point out in Remark 3.24 the main point is that one needs to see the structure of a graded ribbon (tensor!) category. This is a non-trivial problem:
As we explain in Section 1.1.4, we introduce our grading on $\mathcal{S}^\text{gr}$ block-wise. But the tensor product does not respect the blocks. Note that, as we deduce in Remark 3.31 each block $\mathcal{S}^\text{gr}_\lambda$ decategorifies to the Burau representation of the braid group $B_\infty$ with $\infty$-many strands (the split Grothendieck group $K_0^\text{gr}$ carries an action of $B_\infty$).

Thus, each block $\mathcal{S}^\text{gr}_\lambda$ separately can be used to obtain invariants of links and tangles - there are very explicit relations to Khovanov homology (with the path length grading we use), sutured Khovanov homology (with the negative path length grading) and bordered Floer homology (with Khovanov-Seidel’s original grading), see for example the work of Auroux, Grigsby and Wehrli [7] and [8] or Stroppel [66] and [67]. Hence, each block $\mathcal{S}^\text{gr}_\lambda$ separately yields information about link and tangle invariants in the non-root-of-unity case, while the ribbon/modular structure of $\mathcal{S}$ yields the Witten-Reshetikhin-Turaev invariants. An interpretation of this is missing. Moreover, it is not clear if the grading on $\mathcal{S}^\text{gr}$ (which is purely a “root of unity” phenomena) and the graded (!) tensor structure yield new and interesting information.

Life is short, but this paper is not: Hopefully these questions will be addressed in a sequel of this paper.

Furthermore, another open question is how our work connects to the way more sophisticated categorification of tensor products from [72] is also very likely, see also Remark 3.17.

1.1.4. Diagram categories, biadjoint functors and diagrammatic categorification. Recall that the Schur-Weyl duality says that, if $V$ is a $k$ dimensional vector space, then there are commuting actions of $U_v(\mathfrak{sl}_k)$ and $S_n$ on $V^\otimes n$ that turn the quotients $U(\mathfrak{sl}_k)/I_1$ and $K[S_n]/I_2$ by the kernels of the actions into a dual pair. If $v$ is an indeterminate, then the same works for $U_v(\mathfrak{sl}_k)$ and $H_v(S_n)$.

In the flavour of what could be called geometric categorification: Grojnowski and Lusztig have categorified the dual pair $U_v(\mathfrak{sl}_k)/I_1$ and $H_v(W)/I_2$ using certain perverse sheaves on products of partial flag varieties, see [27].

Later, as mentioned above, Khovanov and Lauda have categorified $U_v(\mathfrak{sl}_k)$, see [38], and not just the finite dimensional quotients $U_v(\mathfrak{sl}_k)/I_1$. Their approach, that has turned out to be very fruitful, was to use diagrammatic categorification: They defined a certain 2-category $\mathcal{U}(\mathfrak{sl}_k)$ consisting of a certain type of so-called string diagrams whose (split) Grothendieck group $K_0^\text{gr}(\mathcal{U}(\mathfrak{sl}_k))$ gives the idempotented, integral form $\mathcal{U}_v(\mathfrak{sl}_k)$ of $U_v(\mathfrak{sl}_k)$. One of their main observations was that the $E$’s and $F$’s of $\mathcal{U}_v(\mathfrak{sl}_k)$ behave like biadjoint(!) induction and restriction functors on certain categories. As outlined in an even more general framework by Khovanov in [56] (although it was folklore knowledge for some years and appears in a more rigorous form in for example [31] or [54]), biadjoint functors have a “built-in topology” since, roughly, biadjointness means that we can straighten out diagrams. To end this, recall that two functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ are adjoint (with $F$ being the left adjoint of $G$) iff there exist natural transformations called unit $\eta: \text{id}_\mathcal{C} \Rightarrow GF$ and counit $\varepsilon: FG \Rightarrow \text{id}_\mathcal{D}$ such that

$$(1.1.1) \quad F \xrightarrow{id_\mathcal{C} \circ \eta} FGF \xrightarrow{\varepsilon \circ id_\mathcal{F}} F \quad \text{ and } \quad G \xrightarrow{\varepsilon \circ id_\mathcal{G}} GFG \xrightarrow{id_\mathcal{D} \circ \eta} G$$

commute.
In the string-2 framework (a good introduction is for example Section 2 in [43]) these equations reveal their topological nature: If we picture the categories $\mathcal{C}, \mathcal{D}$ as faces, the functors $F, G$ as oriented strings and the natural transformations as (often not pictured) 0-dimensional coupons, for example

$$
\begin{align*}
\text{id}_F &= \mathcal{D} \xrightarrow{F} \mathcal{C}, & \text{id}_G &= \mathcal{C} \xrightarrow{G} \mathcal{D}, & \iota &= \mathcal{C} \xrightarrow{F} \mathcal{C} & \text{and} & \varepsilon &= \mathcal{D} \xrightarrow{G} \mathcal{D}
\end{align*}
$$

(where we read from bottom to top and right to left), then the conditions in 1.1.1 are

$$
\begin{align*}
\mathcal{C} \xrightarrow{D} \mathcal{F} = \mathcal{D} \xrightarrow{F} \mathcal{C} & \text{ and } \mathcal{D} \xrightarrow{C} \mathcal{G} = \mathcal{C} \xrightarrow{G} \mathcal{D}
\end{align*}
$$

If, in addition, $F$ is also right adjoint to $G$ (thus, they are biadjoint), we get similar pictures as above (as we encourage the reader to verify). Thus, very roughly: “Biadjointness=planar isotopy”.

A main feature of the $B_i$’s from Soergel categorification of the Hecke algebra $H_v(W)$ is that tensoring with $B_i$ is an endofunctor that is self-adjoint and, even stronger, a Frobenius object, i.e. there are morphisms

$$
B_i \to R, \quad R \to B_i, \quad B_i \to B_i \otimes_R B_i & \quad \text{and} \quad B_i \otimes_R B_i \to B_i
$$

pictured as (we read from bottom to top and right to left again)

$$
\begin{align*}
\bullet & \xrightarrow{R} B_i, & B_i & \xrightarrow{B_i} B_i \otimes_R B_i, & B_i \otimes_R B_i & \xrightarrow{B_i} B_i
\end{align*}
$$

that satisfy the Frobenius relations (plus reflections of these)

$$
\begin{align*}
\text{Frob1:} & = \quad \text{Frob2:} & =
\end{align*}
$$

Thus, it is tempting to ask if one can give a diagrammatic categorification in the spirit of $U_q(\mathfrak{sl}_n)$-strings diagrams of Khovanov-Lauda (see [38]) for Soergel’s categorification as well. The observations from above, as Khovanov explains in Section 3 of [36], were the main reason why Khovanov and Elias started to look for such a description.

They were (very) successful in their search and their diagrammatic categorification given in [20] has inspired many successive work (most mentionable for this paper: Elias’ categorification of the Hecke algebra $H_v(D_\infty)$ from [19] called the dihedral cathedral).

Moreover, the “down-to-earth” approach using a diagrammatic description has already led to seminal results: As Elias and Williamson explain in Subsection 1.3 in [24], their algebraic proof that the Kazhdan-Lusztig polynomials have positive coefficients for arbitrary Coxeter systems
was discovered using the diagrammatic framework in [23] and [25]. Both papers are based on Khovanov-Elias work (the paper [24] itself does not contain any diagram).

In our context: The combinatorics of the blocks $\Xi_\lambda$ of $\Xi$, as we explain in Subsection 2.5, is mostly governed by two functors $\Theta_s$ and $\Theta_t$ called translation through the $s$ and $t$-wall respectively. Here, following Kazhdan-Lusztig approach from [33], $s$ and $t$ are the two reflections that generate the affine Weyl group $W_i = \langle s, t \rangle \cong D_\infty$ of $\mathfrak{sl}_2$.

These functors, motivated from the category $O$ analoga, are biadjoint and satisfy Frobenius relations. Moreover, we show in Lemma 3.8 and Theorem 3.12 that the same still holds in the graded setting.

Thus, it seems reasonable to expect that $\Xi_{\lambda}^{gr}$ and $p\text{End}(\Xi_{\lambda}^{gr})$ have a diagrammatic description as well. And, since $W_i \cong D_\infty$ (where the latter is the infinite dihedral group), it seems reasonable to expect that these diagrammatic descriptions are related to Elias’ dihedral cathedral $\mathcal{D}(\infty)$ from [19]. We prove this in Section 4. We point out that, even in our small $\mathfrak{sl}_2$ case, the diagrammatic description, due to its “built-in” isotopy invariance and Frobenius properties (as explained above), eases to work with $\Xi_{\lambda}^{gr}$ and $p\text{End}(\Xi_{\lambda}^{gr})$.

Moreover, as we explain in Subsection 2.4, the combinatorics in our case is completely determined by the “shifts” of the fundamental alcove $A_0$ in one direction: This gives rise to a slight asymmetry that we call the “dead-end condition”

1.2. An outline of the paper. The outline of the paper is as follows.

- Most of Section 2 is well-known but we have also included some “new” observations related to our approach (in particular, in Subsection 2.5). Section 2 is organized as:
  - We recall in Subsection 2.1 some basic facts about quantum groups at roots of unity.
  - In Subsection 2.2 we recall the building blocks of the category of finite dimensional
    $\mathcal{U}_q$-modules, called (dual) Weyl modules $\Delta_q(i)$ (or $\nabla_q(i)$), their simple heads (or socles) $L_q(i)$ and recall some basic, but important, properties of these.
  - In Subsection 2.3 we introduce the category $\Xi$ (and give some basic properties of it) that we study in this paper: The category of $\mathcal{U}_q$-tilting modules. In particular, we recall the construction of the indecomposable tilting modules $T_q(i)$. We introduce the tilting generator $T_q(\infty)$ and its $m$-th cut-off $T_q(\leq m)$ in Definition 2.24.
  - In Subsection 2.4 we recall the linkage principle in Theorem 2.23. This leads to a block decomposition of $\Xi$. Furthermore, we give a classification of all $\mathcal{U}_q$-intertwiners between the $T_q(i)$’s in Proposition 2.28 that is crucial for the rest of the paper.
  - In Subsection 2.5 we recall the for us most important functors called onto $T_{\lambda}^{\mu}$ and out of $T_{\mu}^{\lambda}$ and through the wall $\Theta_{\lambda,\mu}$ (see Definitions 2.30 and 2.31). Their combinatorics (see Proposition 2.32) is crucial for us. Moreover, we introduce the category $p\text{End}(\Xi)$ of projective endofunctors on $\Xi$, see Definition 2.35 and deduce some properties of it, see for example Proposition 2.38.

\footnote{Throughout the paper: “Well-known” means for us that strictly more than one person know it (not necessarily including the authors).}
In Section 3 we introduce a graded version of $\Sigma$ that we denote by $\Sigma^{gr}$. The section is organized as follows:

- We recall in Subsection 3.1 Khovanov-Seidel’s $m$-quiver algebra $A_m$ and introduce in Definition 3.4 the “infinite version” $A_\infty$ (sorry for the terrible notation: This is not an $A$-infinity-algebra) of it that gives rise to the grading on tilting modules. Both algebras are $\mathbb{Z}$-graded algebras where we, in contrast to Khovanov and Seidel, use the Koszul grading (aka path-length grading).

- In Subsection 3.2 we introduce some very important functors $U_{even}$ and $U_{odd}$ (certain sums of Khovanov-Seidel’s functors $U_i$ with the Koszul grading) and show in the key Lemma 3.8 that they are graded(!) biadjoint. We also introduce some refined versions $U_{even}'$ and $U_{odd}'$ and deduce some properties of these as well.

- In the important Subsection 3.4 we first show in Propositions 3.9 and 3.10 that

$$\text{End}_{U_q}(T_q(\leq m)) \cong A_m \quad \text{and} \quad \text{End}_{U_q}(T_q(\infty)) \cong A_\infty$$

as algebras which introduces a grading on the left two algebras. Then we use analogs of Soergel’s combinatorial functor that we denote by $\mathbb{V}_m$ and $\mathbb{V}_\infty$ to show in Theorems 3.11 and 3.12 that

$$\Sigma_\lambda(\leq m) \cong p\text{Mod}-A_m \quad \text{and} \quad \Sigma_\lambda(\infty) \cong p\text{Mod}-A_\infty,$$

where the $p$ indicates projective modules and the $\lambda$ a fixed block. Moreover, we show in the same theorems that $U_{even}$ and $U_{odd}$ correspond to $\Theta_s$ and $\Theta_t$ under this equivalence.

- Using the results from Subsection 3.4 we finally introduce in Subsection 3.5 $\Sigma^{gr}$ as a graded version of $\Sigma$ and show that all modules in $\Sigma$ can be equipped with a grading, see Proposition 3.19. In addition, we show in Proposition 3.28 that the same works for $p\text{End}(\Sigma)$ as well. Using both, we refine in Proposition 3.21 and Corollary 3.27 some of the properties from Section 2. Moreover, we show in Corollary 3.30 that there is an isomorphism of graded rings $\text{End}_{gr}(\text{id}) \cong Z(A_\infty)$, where $Z(\cdot)$ denotes the center, $\text{id}$ is the identity functor on $\Sigma^{gr}$ and the endomorphism ring is to be taken in $p\text{End}(\Sigma^{gr})$.

- Section 4 finally provides the diagrammatic interpretation of the results from before. Section 4 is organized as:

- In order to help the reader, we recall in Subsection 4.1 some basic definition and facts about diagrammatic categories, Karoubi envelopes and additive closures that we need.

- In Subsection 4.1 we recall in Definition 4.7 the graded(!) category $\mathcal{D}(\infty)$: Elias’ beautiful dihedral cathedral $\mathcal{D}(\infty)$ and deduce some basic properties of it.

- In Subsection 4.3 we introduce our graded(!) quotient $\Omega\mathcal{D}(\infty)$ of $\mathcal{D}(\infty)$, see Definition 4.18 and show in Proposition 4.22 that $\Omega\mathcal{D}(\infty)$ satisfies the relations of Khovanov and Seidel’s $\infty$-quiver algebra $A_\infty$.

- In Subsection 4.4 we give in Definition 4.25 an explicit functor $\mathcal{D}_\infty$ that, as we show in Theorem 4.28 of Subsection 4.4, that this functor is an equivalence of graded(!) categories $\mathcal{D}_\infty: \text{Mat}(\Omega\mathcal{D}(\infty)) \rightarrow \Sigma^{gr}$.

- In the last subsection, that is, Subsection 4.5, we give an extension of the results from before by giving a diagrammatic category for $p\text{End}(\Sigma^{gr})$ as well, see Theorem 4.37.

We point out that everything is completely “low-tech” and very explicit. We have, hoping to help the reader, illustrated this with many explicit examples along the way.
2. THE TILTING CATEGORY

In this section we shall describe the category $\mathcal{T}$ we are interested in, that is, the category of finite dimensional tilting modules for the quantum group of $\mathfrak{sl}_2$ at a fixed root of unity $q$ in $\mathbb{Q}(q)$. We sometimes also restrict to blocks $\mathcal{T}_\lambda$, $\mathcal{T}_\mu$ of $\mathcal{T}$. We point out right away our convention will be to let $\lambda \in A_0$ denote an element in the fundamental alcove $A_0$, see 2.4, while $\mu$ will denote an element on one of the two walls of $A_0$. Note that $\mathcal{T}_\mu$ is semisimple, while $\mathcal{T}_\lambda$ is far away from being so.

Most parts of this section are known and can be found in the literature cited throughout the text. In fact, we give a hopefully self-containing summary of the results in the $\mathfrak{sl}_2$ case since most results are either spread over the literature or only mentioned implicitly. But we note that we have also included some new observations related to our context as well.

We start in Subsection 2.1 by recalling the notion of $U_q$, see Definition 2.3 which is obtained by specializing Lusztig’s so-called $A$-form to a fixed root of unity $q \in \mathbb{Q}(q)$. Moreover, Subsection 2.1 contains some basic facts about $U_q$ and its modules.

In Subsection 2.2 we recall the Weyl $\Delta_q(i)$ and the dual Weyl $\nabla_q(i)$ modules and the simple modules $L_q(i)$. Then, in Proposition 2.7 and Corollary 2.8 we recall their relationship.

In Subsection 2.3 we recall the notion of tilting modules in Definition 2.13 and show some basic properties about the category of tilting modules $\mathcal{T}^{\text{all}}$ in Proposition 2.16. Moreover, we introduce the category $\mathcal{T}$ we are interested in, see Definition 2.17, and show some basic properties of it in Lemma 2.18.

Then, in Subsection 2.4 we recall in Theorem 2.23 the linkage principle in our case and use it to classify the hom-spaces between tilting modules in Proposition 2.28 and Corollary 2.29. These statements will be crucial in Section 3. We also introduce the indecomposable tilting modules $T_q(i)$ and the tilting generator $T_q(\infty)$ and discuss some properties of these, see Definition 2.24.

Last but not least, in Subsection 2.5 we introduce in Definition 2.35 the various categories of projective endofunctors on $\mathcal{T}$ or its blocks $\mathcal{T}_\lambda$, $\mathcal{T}_\mu$, denoted by $p\text{End}(\mathcal{T})$, $p\text{End}(\mathcal{T}_\lambda)$ and $p\text{End}(\mathcal{T}_\mu)$. We discuss the combinatorics of the so-called translation functors $\Theta^\lambda_s$ and $\Theta^\lambda_t$ (these are functors in $p\text{End}(\mathcal{T}_\lambda)$) in Proposition 2.32.

We note that we mostly follow Jantzen’s book [29] with our notation and conventions and $v$ denotes an indeterminate while $q \in \mathbb{Q}(q)$ denotes a fixed root of unity. Furthermore, we note that all module categories in this section are categories of left modules and, in contrast to Sections 3 and 4 where we read from right to left and use right modules.

2.1. Quantum groups at roots of unity.

**Definition 2.1.** The quantum special linear algebra, denoted by $U_q(\mathfrak{sl}_2)$, is the associative, unital $\mathbb{Q}(v)$-algebra generated by $K$ and $K^{-1}$ and $E$, $F$ subject to the relations

\[ KK^{-1} = K^{-1}K = 1, \]
\[ EF - FE = \frac{K - K^{-1}}{v - v^{-1}}, \]
\[ KE = v^2EK \quad \text{and} \quadKF = v^{-2}FK.\]

\[ ^3\text{We like to work in the cyclotomic field } \mathbb{Q}(q) \text{ instead of } \mathbb{C} \text{ because it has a more combinatorial flavour, but we need a field of characteristic zero containing } q \text{ in this section.} \]
We denote the algebra $U_v(sl_2)$ as a shorthand notation just by $U_v$.

It is worth noting that $U_v$ is a Hopf algebra with coproduct $\Delta$ given by

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F \quad \text{and} \quad \Delta(K) = K \otimes K.$$ 

The antipode $S$ and the counit $\varepsilon$ are given by

$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}, \quad \varepsilon(E) = \varepsilon(F) = 0 \quad \text{and} \quad \varepsilon(K) = 1.$$ 

Recall that the Hopf algebra structure allows to extend actions to tensor products of representations, to duals of representations and there is a trivial representation.

Note that there are different, but isomorphic, possible conventions for the Hopf algebra structure. As mentioned above, we follow Jantzen \cite{29} and Lusztig \cite{46}.

We are interested in the root of unity case. Thus, we want to “specialize” the $v$ of $U_v$ to be a root of unity $q \in \mathbb{Q}(q)$ (note our notation again: $v$ is a generic parameter and $q$ is a fixed root of unity). In order to do so, we consider Lusztig’s $A$-form $U_A$, see \cite{48}. Thus, we set $A = \mathbb{Z}[v, v^{-1}]$. Furthermore, we use for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ the convention that $[a]$ denotes the quantum integer (with $[0] = 1$), $[b]!$ denotes the quantum factorial, that is

$$[a] = \frac{v^a - v^{-a}}{v - v^{-1}} = v^{a-1} + v^{a-3} + \cdots + v^{-a+1} + v^{-a+1} \quad \text{and} \quad [b]! = [0][1] \cdots [b - 1][b],$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a][a-1] \cdots [a-b+2][a-b+1]}{[b]!} \in A$$

denotes the quantum binomial. Observe that $[-a] = -[a]$.

**Definition 2.2.** (Lusztig’s $A$-form $U_A$) Define for all $j \in \mathbb{N}$ the $j$-th divided powers

$$E^{(j)} = \frac{E^j}{[j]!} \quad \text{and} \quad F^{(j)} = \frac{F^j}{[j]!}.$$ 

Then $U_A$ is defined as the $A$-subalgebra of $U_v$ generated by $K, K^{-1}, E^{(j)}$ and $F^{(j)}$.

Now we can specialize.

**Definition 2.3.** Fix a root of unity $q \in \mathbb{Q}(q)$ of order $l' > 2$. Denote by $l$ the order of $q^2$, i.e. if $l'$ is even set $l = \frac{l'}{2}$ and if $l'$ is odd set $l = l'$ in the following\footnote{The reason why these cases are slightly different is due to the fact that $[a] = 0$ iff $q^{2a} = 1$.}. Consider $\mathbb{Q}(q)$ as an $A$-module by specializing $v$ to $q$. Define

$$U_q(sl_2) = U_q = U_A \otimes_A \mathbb{Q}(q).$$

We abuse notation and write $E^{(j)}$ instead of $E^{(j)} \otimes 1$. Analogously for the other generators.

**Remark 2.4.** It is easy to check that $U_A$ is a Hopf subalgebra of $U_v$. Thus, $U_q$ inherits a Hopf algebra structure from $U_v$. In particular, if one has a $U_q$-module $M$, then $M^*$ has an induced action given by

$$X f : m \mapsto f(S(X)m), \quad \text{for} \ X \in U_q, \ m \in M \quad \text{and} \quad f \in M^* = \text{Hom}_{U_q}(M, \mathbb{Q}(q)).$$
where $Q(q)$ is a $U_q$-module, called trivial, with an action induced by the antipode $\varepsilon$. It follows that, if $m \in M$ is an eigenvector of $K$ with eigenvalue $\alpha$ and $m^* \in M^*$ a dual vector with respect to some $K$ stable complement of $Q(q)m$ in $M$, then

\begin{equation}
K m = \alpha m \iff K m^* = \alpha^{-1} m^*.
\end{equation}

Moreover, note that $[j] = 0 \in U_q$ iff $l|j$. Still $E(l), F(l)$ stay well-defined, since $E(l) = E(l) \otimes 1$ with left part in $U_A$ (where $[j]$ is never zero). Likewise for $F(l)$. But this implies $E(l) = [l]!E(l) = 0$ and again similarly $F(l) = 0$. It also follows that we have $K^{2l} = 1$ (which can be deduced from the equations $E(l) = F(l) = 0$ together with the relations in the second and third line of Definition 2.1).

2.2. Weyl modules, dual Weyl modules and simple modules. As usual in representation theory one wants to determine for example the simple modules by giving an explicit definition of modules whose unique simple quotients determine the simple modules. For us these are the so-called Weyl modules $\Delta_q(i)$ which play the same role as the Verma modules do in category $O$ for $sl_2$. Note that we even use the same notation as some authors do for the Verma’s.

**Definition 2.5. (Weyl, dual Weyl and simple modules)** Let $i \in \mathbb{N}$ and denote by $\Delta_q(i)$ the $i$-th Weyl module. This is the $i + 1$-dimensional $U_q$-module with a basis given by $m_0, \ldots, m_i$ and an action defined by

\begin{equation}
K m_k = q^{i-2k} m_k, \quad E(j) m_k = \left[ \begin{array}{c} i - k + j \\ j \end{array} \right] m_{k-j} \quad \text{and} \quad F(j) m_k = \left[ \begin{array}{c} k + j \\ j \end{array} \right] m_{k+j},
\end{equation}

with the convention that $m_{<0} = m_{>i} = 0$. The $i$-th dual Weyl module, denoted by $\nabla_q(i)$, is obtained from $\Delta_q(i)$ by taking the dual, that is, $\text{Hom}_{U_q}(\Delta_q(i), Q(q)) = \Delta_q(i)^* = \nabla_q(i)$.

It is easy to check directly that $\Delta_q(i)$ has a unique simple quotient (the so-called head of $\Delta_q(i)$) that we denote by $L_q(i)$. See also Section 4 in [5].

**Example 2.6.** Let us illustrate the difference to the case where $q$ is not a root of unity. So let us fix $l = 3 = i$ and let $q = \frac{1+\sqrt{3}i}{2}$. We use this $q$ in a lot of examples in the following: Every time we fix $l = 3$ we work with this particular $q$.

The Weyl module $\Delta_q(3)$ is four dimensional with basis given by $m_0, m_1, m_2, m_3$. The action of $F$ on these is

\begin{align*}
F m_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} m_1 = m_1, \\
F m_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} m_2 = 2m_2 = -m_2, \\
F m_2 &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} m_3 = 3m_3 = 0.
\end{align*}

Similar, but mirrored for the action of $E$, that is

\begin{align*}
E m_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} m_2 = m_2, \\
E m_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} m_1 = 2m_1 = -m_1, \\
E m_2 &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} m_0 = 3m_0 = 0.
\end{align*}

Thus, we can visualize this as (the occurrence of 0’s here is the main difference to the non-root of unity case where these 0’s do not appear)

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\{ \begin{array}{c}
\ddots \\
\delta^{-3} \end{array} \\
\begin{array}{c}
m_3 \\
m_2 \\
m_1 \\
m_0
\end{array}
\end{array}
\end{array}
\end{equation}
where the action of $E$ points to the right and the action of $F$ to the left and $K$ acts as a loop.

We also have $E^{(2)}m_1 = 0$ and $E^{(2)}m_2 = 0$. Thus, the $A$-span of \{m_1, m_2\} is now, in contrast to the “classical” case, stable under the action of $U_q$.

Another example we encourage the reader to check is \[
\begin{pmatrix}
q^{-4} & +1 & q^{-2} & -1 & 0 \\
+1 & m_4 & +1 & m_3 & m_2 \\
-1 & 0 & m_1 & +1 & m_0,
\end{pmatrix}
\]
where $l$ and $q$ are as before and $i = 4$. Moreover, we have $E^{(2)}m_2 = F^{(2)}m_2 = 0$. Thus, $\Delta_q(4)$ has the trivial $U_q$-module spanned by $m_2$ as a submodule.

Note that $K$ acts on $\Delta_q(i)$ via the eigenvalues $q^{-i}, q^{-i+2}, \ldots, q^{-i-2}, q^i$. Thus, by Equation 2.1.1, the same is true for its dual $\nabla_q(i)$. Moreover, the $L_q(i)$ are self-dual, see e.g. Section 4 in [5].

**Proposition 2.7.** We have the following.
\begin{enumerate}[\(a\)]
\item $\Delta_q(i) = L_q(i)$ iff $i < l$ or $i \equiv -1 \mod l$.
\item Suppose $i = al + b$ for some $a, b \in \mathbb{N}$ with $b \leq l - 2$. Set $i' = (a + 2)l - b - 2$. Then there exists an exact sequence
\[
0 \longrightarrow L_q(i) \longrightarrow \Delta_q(i') \longrightarrow L_q(i') \longrightarrow 0.
\]
Moreover, $L_q(i')$ is the head and $L_q(i)$ is the socle of $\Delta_q(i')$.
\end{enumerate}

**Proof.** See Corollary 4.6 in [5]. \qed

**Corollary 2.8.** We have the following.
\begin{enumerate}[\(a\)]
\item $\nabla_q(i) \cong L_q(i)$ iff $i < l$ or $i \equiv -1 \mod l$.
\item Suppose $i = al + b$ for some $a, b \in \mathbb{N}$ with $b \leq l - 2$. Set $i' = (a + 2)l - b - 2$. Then there exists an exact sequence
\[
0 \longrightarrow L_q(i') \longrightarrow \nabla_q(i') \longrightarrow L_q(i) \longrightarrow 0.
\]
Moreover, $L_q(i')$ is the socle and $L_q(i)$ is the head of $\nabla_q(i')$.
\end{enumerate}

**Proof.** This follows from the self-duality of the $L_q(i)$’s and the fact that * is an exact and contravariant functor. \qed

**Example 2.9.** For $i = 0$ and $l, q$ as in Example 2.6, we have $i' = 4$. As predicted by Proposition 2.7, the trivial submodule $L_q(0)$ appears as a submodule of $\Delta_q(4)$, see 2.2.2.

We should note here that there are two different types of $U_q$-modules known as type 1 and $-1$, see for example Section 1 in [5]. For us the difference between the two types is not important in this paper and we only consider $U_q$-modules of type 1. We only note the (more general) treatment in Section 1 of [5] ensures that the consideration of only type 1 is still enough to get results for both types. The following corollary is well-known and can be found for example for the general case in Corollary 6.3 of [5].

**Corollary 2.10.** The set $\{L_q(i) \mid i \in \mathbb{N}\}$ is an up to isomorphisms a complete set of simple, pairwise non-isomorphic $U_q$-modules (of type 1). \qed
Recall from Definition 2.5 that $\Delta_q(3)$ is indecomposable, but not simple. In fact, Proposition 2.7 says that $\Delta_q(i)$ is never simple whenever we are in the case (b).

2.3. Tilting modules and the tilting category $\mathcal{X}$.

Definition 2.12. (Δ- and ∇-filtration) We say that a $U_q$-module $M$ has a $\Delta$-filtration if there exists a descending sequence of submodules

$$M = F_0 \supset F_1 \supset \cdots \supset F_i \supset \cdots,$$

and $\bigcap_{i=0}^{\infty} F_i = 0$

such that for all $i = 0, 1, \ldots$ we have $F_i/F_{i+1} \cong \Delta_q(i')$ for some $i' \in \mathbb{N}$. A $\nabla$-filtration is defined similarly, but using $\nabla_q(i')$ instead of $\Delta_q(i')$ and an ascending sequence of submodules, that is

$$0 = F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots,$$

and $\bigcup_{i=0}^{\infty} F_i = M$

such that for all $i = 0, 1, \ldots$ we have $F_{i+1}/F_i \cong \nabla_q(i')$ for some $i' \in \mathbb{N}$.

Note that such filtrations are unique up to reordering, see e.g. Proposition II.4.16 in [30]. Moreover, it is clear that a finite dimensional $U_q$-module $M$ has a $\Delta$-filtration iff $M^*$ has a $\nabla$-filtration.

Definition 2.13. (Tilting module) We call a $U_q$-module $M$ a tilting module if it has a $\Delta$- and a $\nabla$-filtration. We say short that $M$ is tilting.

Note that finite dimensional tilting modules $M$ have finite filtrations, i.e. $F_N = F_{N+k}$ for some $N \in \mathbb{N}$ and all $k \in \mathbb{N}$.

Example 2.14. Consider $l = 3$ again. Clearly $\mathbb{Q}(q) = \Delta_q(0) \cong \nabla_q(0)$ is tilting. Furthermore, recall from Definition 2.5 that $\Delta_q(1)$ has two basis vectors $m_0$ and $m_1$. Moreover, $Km_0 = qm_0$, $Em_0 = 0$, $Fm_0 = m_1$ and $Km_1 = q^{-1}m_1$, $Em_1 = m_0$, $Fm_1 = 0$. Thus, we have the tilting module

\begin{equation}
\begin{pmatrix}
q^{-1} \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\begin{pmatrix}
q+1 \\
\end{pmatrix}
\begin{pmatrix}
m_0 \\
m_1
\end{pmatrix}.
\end{equation}

More general: For $i < l$ or $i \equiv -1 \mod l$, by Proposition 2.7 and Corollary 2.8 we see that $\Delta_q(i) = L_q(i) \cong \nabla_q(i)$ is tilting.

Since the category of $U_q$-modules is semisimple it follows that all finite dimensional $U_q$-modules are tilting and this notion is somehow redundant for the non-root of unity case.

Recall that the comultiplication $\Delta : U_q \to U_q \otimes_{\mathbb{Q}(q)} U_q$ allows us to make the tensor product of two $U_q$-modules into a $U_q$-module. It is easy to see (in our $\mathfrak{sl}_2$ case - in general this is a non-trivial theorem, see e.g. Theorem 3.3 in [55]) that $\Delta_q(i) \otimes_{\mathbb{Q}(q)} \Delta_q(i')$ has a $\Delta$-filtration with factors

\begin{equation}
\Delta_q(i-i') \otimes \Delta_q(i+i'-2) \otimes \Delta_q(i+i'), \ i, i' > 0
\end{equation}

and likewise for $\nabla_q(i) \otimes_{\mathbb{Q}(q)} \nabla_q(i')$. In fact, what makes the tilting modules useful is the important point that the indecomposable tilting modules are exactly the summands of the tensor products of
(dual) Weyl modules $\Delta_q(\lambda), \nabla_q(\lambda)$ for $\lambda$ in the so-called fundamental alcove $A_0$ (for our $\mathfrak{sl}_2$-case see Subsection 2.4 and in general see e.g. Section 6 in [60]).

Definition 2.15. (Full tilting category) We denote by $\mathfrak{T}^{\text{all}}$ the full subcategory of all $U_q$-modules that are tilting. We denote its class of objects by $\text{Ob}(\mathfrak{T}^{\text{all}})$.

Proposition 2.16. The category $\mathfrak{T}^{\text{all}}$ has the following properties.

(a) If $M, M' \in \text{Ob}(\mathfrak{T}^{\text{all}})$, then $M \oplus M' \in \text{Ob}(\mathfrak{T}^{\text{all}})$.
(b) If $M \oplus M' \in \text{Ob}(\mathfrak{T}^{\text{all}})$, then $M, M' \in \text{Ob}(\mathfrak{T}^{\text{all}})$.
(c) If $M, M' \in \text{Ob}(\mathfrak{T}^{\text{all}})$, then $M \otimes_{\mathbb{Q}(q)} M' \in \text{Ob}(\mathfrak{T}^{\text{all}})$.

Proof. The parts (a) is immediate, part (b) is easy to verify and left to the reader (if he/she likes) and the non-trivial part (c) can be deduced from [2.3.2] and the corresponding statement for the dual Weyl modules $\nabla_q(i)$. \hfill $\Box$

In fact, the category $\mathfrak{T}^{\text{all}}$ is the category one would like to understand, but it is also very complicated. This motivates the definition of the category we want to consider.

Definition 2.17. (The tilting category $\mathfrak{T}$) Let $\mathfrak{T}$ be the full subcategory of $\mathfrak{T}^{\text{all}}$ of all finite dimensional tilting modules. That is, it consists of:

- The objects $\text{Ob}(\mathfrak{T})$ are all finite dimensional $U_q$-tilting modules $M \in \text{Ob}(\mathfrak{T}^{\text{all}})$.
- The morphisms $\text{Hom}_{\mathfrak{T}}(M, N)$ are all $U_q$-intertwiners $f \in \text{Hom}_{U_q}(M, N)$.

Lemma 2.18. We have the following.

(a) The category $\mathfrak{T}$ is Krull-Schmidt and every indecomposable tilting in $\mathfrak{T}$ is of the form $T_q(i)$ for some $i \in \mathbb{N}$.
(b) The category $\mathfrak{T}$ is closed under finite sums and finite tensor products. Moreover, the category $\mathfrak{T}$ is additive.
(c) Let $M \in \text{Ob}(\mathfrak{T})$. Then $M^* \in \text{Ob}(\mathfrak{T})$, i.e. $\mathfrak{T}$ is closed under duals.
(d) The Weyl modules $\Delta_q(i)$, the dual Weyl modules $\nabla_q(i)$, the simple modules $L_q(i)$ (for all $i < l$ or $i \equiv -1 \pmod{l}$) and the tilting modules $T_q(i)$ are (for all $i \in \mathbb{N}$) all in $\mathfrak{T}$.

Proof. Note that most of these statements can be verified by following for example similar statements in related categories, see for example Section 1 in [28].

Part (a): Being a full subcategory of finite dimensional $U_q$-modules $\mathfrak{T}$ is clearly Krull-Schmidt. Moreover, by Proposition 2.20, all finite dimensional indecomposable tiltings are of the form $T_q(i)$ for some $i \in \mathbb{N}$.

Part (b) follows from (a) and Proposition 2.16. We point out that $\mathfrak{T}$ is not closed under submodules or quotients since for example the simple $U_q$-modules $L_q(i)$ are not tilting in general.

Since $\text{Hom}_{U_q}(\cdot, \mathbb{Q}(q))$ commutes with finite sums, part (c) follows also from Proposition 2.16.

Part (d) is a direct consequence of Proposition 2.7 and Corollary 2.8. \hfill $\Box$

Remark 2.19. It follows from Lemma 2.18 part (b) and (c) that $\mathfrak{T}$ is a rigid category (a monoidal category with dual and certain compatibility properties). Moreover, $\mathfrak{T}$ is even a so-called ribbon (tensor) category. Furthermore, $\mathfrak{T}$ gives rise to a so-called modular category (roughly: One

\footnote{In fact, the subfactors $\Delta_q(i), \nabla_q(i)$ and $L_q(i)$ of $T_q(i)$ are \emph{not} tilting modules unless $i < l$ or $i \equiv -1 \pmod{l}$ which makes our category only additive and not abelian.}
mods out by negligible tiltings whose quantum trace is zero) and thus, gives a 2 + 1-dimensional TQFT and can be used to define the Witten-Reshetikhin-Turaev invariants of 3-manifolds. Good treatments about this is Section 4 in [1] or Section 7 in [60].

Define, using Propositions 2.7 and 2.16 and Equation 2.3.2 a family \((T_q(i))_{i \in \mathbb{N}}\) of indecomposable tilting modules as follows. We start by setting \(T_q(0) = L_q(0) = \Delta_q(0) \cong \nabla_q(0)\) and \(T_q(1) = L_q(1) = \Delta_q(1) \cong \nabla_q(1)\). Then we denote by \(m_0 \in T_q(1)\) any eigenvector for \(K\) with eigenvalue \(q\). For each \(i > 1\) we define \(T_q(i)\) to be the indecomposable summand of \((T_q(1))^\otimes i\) which contains the vector \(m_0 \otimes \cdots \otimes m_0 \in (T_q(1))^\otimes i\).

Denote by \((M : \Delta_q(i)) \in \mathbb{N}\) the filtration multiplicity for an \(U_q\)-tilting module \(M\). It is clear by using Equation 2.3.2 that

\[(T_q(1))^\otimes i : \Delta_q(i) = 1 \quad \text{and} \quad (T_q(1))^\otimes i : \Delta_q(i') = 0 \quad \text{for} \quad i' > i.
\]

Hence, \(T_q(i)\) may also be described as the unique indecomposable summand of \((T_q(1))^\otimes i\) that contains \(\Delta_q(i)\). We note the following more precise statement.

**Proposition 2.20.**

(a) \(\Delta_q(i) = L_q(i) = T_q(i) \cong \nabla_q(i)\) iff \(i < l\) or \(i \equiv -1 \mod l\).

(b) Suppose \(i = al + b\) for some \(a, b \in \mathbb{N}\) with \(b \leq l - 2\). Set \(i' = (a + 2)l - b - 2\). Then there exist exact sequences

\[
0 \rightarrow \Delta_q(i') \rightarrow T_q(i') \rightarrow \Delta_q(i) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \nabla_q(i) \rightarrow T_q(i') \rightarrow \nabla_q(i') \rightarrow 0.
\]

(c) We have \(T_q(i) \cong T_q(i')\) iff \(i = i'\). Moreover, if \(M \in \text{Ob}({\mathcal{F}})\) is indecomposable (of type 1), then there exists an \(i \in \mathbb{N}\) such that \(M \cong T_q(i)\).

**Proof.** Part (a) is clear by Proposition 2.7 and Corollary 2.8. For part (b) we use in addition Equation 2.3.2 repeatedly for the indecomposable tilting \(T_q(1) = \Delta_q(1)\) from Example 2.14.

To see the first statement of (c) assume, without loss of generality, that \(i' > i\). Then, by Equation 2.3.3 we see that \(\Delta_q(i')\) appears with multiplicity 1 in \(T_q(i')\), but not in \(T_q(i)\). Thus, they are not isomorphic.

That every finite dimensional indecomposable tilting is of this form needs a little bit more treatment. But it is a standard argument that appears in various contexts and can be adopted for example from Section 11.2 in [28] or Section II.E.6 in [30].

**Example 2.21.** Let us consider our favourite example \(l = 3\) again. The tilting module \(T_q(2)\) can be visualized by considering the tensor product \(T_q(1) \otimes_{Q(q)} T_q(1)\) from 2.3.1 which looks like

\[
\begin{array}{ccc}
q^{-1} & q^{+1} \\
\otimes & & \otimes \\
1 \quad m_1 & \quad 1 \quad m_0 \\
q^{-1} & q^{-2} & q^{-1} \\
1 \quad m_0 & \quad 1 \quad m_1 \\
q^{-1} & q^{-2} & q^{-1} \\
1 \quad m_0 & \quad 1 \quad m_1 \\
q^{-1} & q^{-2} & q^{-1} \\
1 \quad m_0 & \quad 1 \quad m_1 \\
q^{-1} & q^{-2} & q^{-1} \\
1 \quad m_0 & \quad 1 \quad m_1
\end{array}
\]

where we have simplified notation \(m_{ij} = m_i \otimes m_j\). By construction, \(T_q(2)\) contains \(m_{00}\) and it therefore has to be the span of \(\{m_{00}, q^{-1}m_{10} + m_{01}, m_{11}\}\) as indicated above (keep in mind that
Remark 2.22. One can in addition to part (f) show that the category of all finite dimensional projective $U_q$-modules, denoted by $U_q\text{-Mod}$, is a full subcategory of $\mathcal{I}$ (see e.g. Section 5 in [1]). The same is true for finite dimensional injective $U_q$-modules: $\mathcal{I}$ has enough injective and projective modules. In fact, something stronger is true (see e.g. Section 5 in [1]), namely that $T_q(i)$ is injective and projective for all $i \geq l - 1$. Hence, $\{T_q(i) \mid i \geq l - 1\}$ is a complete set of pairwise non-isomorphic, projective (injective) indecomposable, finite dimensional $U_q$-modules.

2.4. The linkage principle and blocks. Consider the alcove $A_0$ and its closure $\overline{A}_0$ given by

$$A_0 = \{k \in \mathbb{Z} \mid -1 < k < l - 1\} \quad \text{and} \quad \overline{A}_0 = \{k \in \mathbb{Z} \mid -1 \leq k \leq l - 1\}.$$  

We call $A_0$ the fundamental alcove. Any other alcove in $\mathbb{Z}$ is clearly of the form $A_0 + il$ for some $i \in \mathbb{N}$. Moreover, we call $-1, l - 1 \in \overline{A}_0 - A_0$ walls of $A_0$.

We denote by $s$ and $t$ the reflections in these walls, i.e.

$$s.k = -k - 2 \quad \text{and} \quad t.k = -k - 2 + 2l \quad \text{for} \quad k \in \mathbb{Z}.$$  

The affine Weyl group is $W_l = \langle s,t \rangle$ in this case. Both generators are of order 2 (in the group of permutations of $\mathbb{Z}$). Because of $st.k = s.(t.k) = s.(-k - 2 + 2l) = k - 2l$, we see that $st$ is not of finite order. Thus, $W_l \simeq D_{\infty}$ where the latter is the infinite dihedral group. This observation makes it reasonable to expect connections to Elias’ dihedral cathedral, see [19] or Section 4.

We denote by $W_l.x$ the orbits in $\mathbb{Z}_{\geq -1}$ under the action of the affine Weyl group on $x \in \mathbb{Z}_{\geq -1}$ where we, by convention, define the action $W_l.x$ for $x \in A_0 + il$ by letting the affine Weyl group act on the $A_0$-part and then shift by $il$. For our favourite case $l = 3$ this can be visualized as

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
\hline
& & & & & & & & & & \\
\hline
\text{Dead-end} & -1 & s\text{-wall} & t\text{-wall} & s\text{-wall} & & & & & & \\
\hline
\end{array}
\]

with red (or top) action by $s$ and green (or bottom) action by $t$. For this restriction we can read of the orbit of 0 as $0 \xrightarrow{t} 4 \xrightarrow{t} 6 \xrightarrow{t} 10 \xrightarrow{t} \ldots$.

The following is known as the $\mathfrak{sl}_2$-linkage principle. In the $\mathfrak{sl}_2$ case this can be easily derived from Propositions 2.7 and 2.20 and Corollary 2.8. The more general highly non-trivial statement can for example be found in Theorem 3.1 in [2].

Theorem 2.23. (Linkage principle) A simple module $L_q(i)$ can occur as a composition factor in $\Delta_q(i')$, $\nabla_q(i')$ or $T_q(i')$ only if $i' \in W_l.i$ and $i \leq i'$.

Proof. Use Propositions 2.7 and 2.20 and Corollary 2.8. \hfill \square

The linkage principle motivates the following notational convention and definition.

Notation. We always use $\lambda$ for elements in the fundamental alcove $A_0$ and $\mu$ for elements on the walls of the fundamental alcove $\overline{A}_0 - A_0$. Set $A_i = \{i' \mid il - 1 < i' < (i + 1)l - 1\} = A_0 + il$ for each $i \in \mathbb{N}$. Suppose $\lambda \in A_0$ and $\mu \in \overline{A}_0 - A_0$. We write $\lambda_i$ and $\mu_i$ for the unique elements

\[E, F \text{ act on tensor products via the comultiplication). This matches Proposition 2.20 part (a) since we see that this span isomorphic to } L_q(2).\]
in $\mathcal{A}_i \cap W_1.\lambda$ and $\mathcal{A}_i \cap W_1.\mu$. Note that, if $\mu = -1$, then $\mu_1 = \mu_2, \mu_3 = \mu_4$ and so forth, while for $\mu = l - 1$ we have $\mu_0 = \mu_1, \mu_2 = \mu_3$ etc. See also \textit{\textsuperscript{2.4.1}}

We define the tilting generator $T_q(\infty)$ and its cut-off’s $T_q(\leq m)$ that will play a crucial role in Section \textit{3}. By abuse of notation, we denote them in the same way for all $\lambda$ and $\mu$ and hope that it is clear from the context which ones we consider.

\textbf{Definition 2.24. (The tilting generator)} We call for each $m \in \mathbb{N} \cup \{\infty\}$ the $U_q$-modules given by

$$T_q(\infty) = \bigoplus_{i=0}^{\infty} T_q(\lambda_i) \quad \text{and} \quad T_q(\leq m) = \bigoplus_{i=0}^{m} T_q(\lambda_i)$$

the \textit{tilting generator} and its $m$-th cut-off, respectively. Likewise for the $\mu$’s instead of $\lambda$’s.

Denote by $\mathcal{S}_\lambda$ for $\lambda \in \mathcal{A}_0$ the $\lambda$’s block of $\mathcal{S}$: All indecomposable summands of modules of $\mathcal{S}_\lambda$ should be of the form $T_q(i)$ for $i \in W_1.\lambda$. We, using a similar notation on walls, have the following decomposition.

We note again that $T_q(\infty)$ is not finite dimensional and therefore not in $\mathcal{S}$, but only in $\mathcal{S}^{\text{all}}$. Still: Its combinatorics, as we show in Subsection \textit{3.4}, governs the category $\mathcal{S}$. On the other hand, the cut-offs $T_q(\leq m)$ are in $\mathcal{S}$.

\textbf{Lemma 2.25.} We have

$$\mathcal{S} = \bigoplus_{\lambda \in \mathcal{A}_0} \mathcal{S}_\lambda \oplus \mathcal{S}_{-1} \oplus \mathcal{S}_{l-1}$$

with semisimple categories $\mathcal{S}_{-1}$ and $\mathcal{S}_{l-1}$ equivalent to the corresponding $U_\nu$-modules categories (non-root of unity case). Moreover, for all $\lambda \in \mathcal{A}_0$, we have the following.

(a) \textit{The categories $\mathcal{S}_\lambda$ are Krull-Schmidt subcategories of $\mathcal{S}$. Moreover, every indecomposable tilting in $\mathcal{S}_\lambda$ is of the form $T_q(i)$ for some $i \in \mathcal{N}$ and $i \in W_1.\lambda$.}

(b) \textit{The categories $\mathcal{S}_\lambda$ are closed under finite sums and are additive categories.}

(c) \textit{Let $M \in \text{Ob}(\mathcal{S}_\lambda)$. Then $M^* \in \text{Ob}(\mathcal{S}_\lambda)$.}

Analoga of the statements (a)-(c) are also true for the categories $\mathcal{S}_{-1}$ and $\mathcal{S}_{l-1}$.

\textbf{Proof.} This is now only a combination of Theorem \textit{2.23} and Lemma \textit{2.18}. Note that $\mathcal{S}_{-1}$ and $\mathcal{S}_{l-1}$ are equivalent to categories of $U_\nu$-modules by Proposition \textit{2.20} part (a).

\textbf{Notation.} If it is clear from the context which $\lambda$ we consider, then we, by abuse of notation, denote the indecomposable tilting modules for $\lambda_i$ just by $T_q(\lambda_i) = T_q(i)$. Similarly for simple, Weyl and dual Weyl modules and on walls.

\textbf{Example 2.26.} Take our favourite example $l = 3$ again, see \textit{2.4.1}. Then we only have two $\lambda$’s, namely $0, 1$. Moreover, we have two $\mu$’s, namely $-1, 2$.

Then the indecomposable tilting modules in $\mathcal{S}_0$ are $T_q(i)$ for $i = 0, 4, 6, \ldots$ as a look at \textit{2.4.1} indicates. For $\mathcal{S}_1$ they are $T_q(i)$ for $i = 1, 3, 7, \ldots$. The two blocks $\mathcal{S}_{-1}$ and $\mathcal{S}_2$ are semisimple and consist of direct sums of $T_q(i) = L_q(i)$ for $i = 5, 11, \ldots$ and for $i = 2, 8, \ldots$ respectively.

\textbf{Remark 2.27.} As in the usual case for an indeterminate $v$, we have a \textit{triangular decomposition} $U_q = U_q^- U_q^0 U_q^+$, see for example Section 1 of \textit{[5]}.

If we set $B_q^+ = U_q^+ U_q^0$, then, for any $i \in \mathbb{N}$, the Weyl module $\Delta_q(i)$ has the following universal property: For any $U_q$-module $M$ there is an isomorphism of vector spaces

$$\text{Hom}_{U_q}(\Delta_q(i), M) \cong \{m \in M \mid \chi_i(b)m, b \in B_q^+\}.$$
How the character $\chi_i : B^- \rightarrow \mathbb{Q}(q)$ is determined by $i$ can be found in Lemma 1.1 in [5]. This together with Definition 2.5 and Proposition 2.7 imply that
\[
\text{Hom}_{U_q}(\Delta_q(i), L_q(j)) \cong \text{Hom}_{U_q}(\Delta_q(i), \nabla_q(j)) \cong \delta_{ij} \mathbb{Q}(q)
\]
for all $i, j \in \mathbb{N}$. In particular,
\[
\text{Hom}_{U_q}(L_q(i), L_q(j)) \cong \delta_{ij} \mathbb{Q}(q),
\]
i.e. Schur Lemma holds in our set-up. Note that this is in fact true for general quantum groups over arbitrary fields, see Corollary 7.4 in [5].

In addition, we get in our case, by using Proposition 2.7 and Corollary 2.8, that
\[
\text{Hom}_{U_q}(\nabla_q(i), \Delta_q(i)) \cong \mathbb{Q}(q),
\]
for all $i \in \mathbb{N}$.

The following maps are very important for us in Sections 3 and 4. We call them up $u_i^\lambda$, down $d_i^\lambda$ and loop $\varepsilon_i^\lambda$ (or simply $u_i, d_i$ and $\varepsilon_i$) respectively.

**Proposition 2.28.** There exist up to scalars unique $U_q$-intertwiners $u_i^\lambda, d_i^\lambda$ with
\[
\begin{align*}
\text{Hom}_{U_q}(\Delta_q(i), L_q(\lambda_i)) & \rightarrow T_q(\lambda_i), \quad i = 0, 1, \ldots, \\
\text{Hom}_{U_q}(L_q(\lambda_i), \nabla_q(i)) & \rightarrow T_q(\lambda_{i+1}), \quad i = 1, 2, \ldots,
\end{align*}
\]
They satisfy
\[
\begin{align*}
\text{Hom}_{U_q}(\nabla_q(i), L_q(\lambda_i)) & \rightarrow T_q(\lambda_i), \quad i = 0, 1, \ldots, \\
\text{Hom}_{U_q}(L_q(\lambda_i), \nabla_q(i)) & \rightarrow T_q(\lambda_{i-1}), \quad i = 1, 2, \ldots,
\end{align*}
\]
and the left dead-end relation
\[
d_i^\lambda \circ u_i^\lambda = 0
\]
for each $i \in \mathbb{N}$ and $\lambda \in \mathcal{A}_0$. Moreover, $\varepsilon_i^\lambda \circ \varepsilon_i^\lambda = 0$ for $i = 1, 2, \ldots$.

**Proof.** First let us assume that we are not in the dead-end relation (in fact, we leave it to the reader to verify this special case), that is, the indices $i$ are at least 1. We want to use Proposition 2.7 and Corollary 2.8. Moreover, we note that the $U_q$-morphisms below will only be unique up to scalars and their precise form does not matter. We only assume that they are non-zero. In fact, by abuse of notation, we always use the same symbols, but the maps are in general of course different.

We consider
\[
\begin{array}{ccc}
L_q(\lambda_i) & \xrightarrow{a} & L_q(\lambda_{i-1}) \\
\xrightarrow{b} & & \xrightarrow{c} \\
L_q(\lambda_{i-1}) & \xrightarrow{e} & L_q(\lambda_i)
\end{array}
\]
Hence, we see that $\text{Hom}_{U_q}(\nabla_q(\lambda_i), \Delta_q(\lambda_i))$ is one dimensional: As noted in Remark 2.27, the hom-spaces between Weyl and dual Weyl modules are at most 1-dimensional. The composite $eba$ ensures that the dimension is exactly 1 since it spans the hom-space.

Likewise for $\text{Hom}_{U_q}(\Delta_q(\lambda_i), \nabla_q(\lambda_i))$ by using $f ed$. Note that the composite $fedcba = 0$ due to the fact that the middle row is exact. Same of course for exchanged roles of $\Delta_q(\lambda_i)$ and $\nabla_q(\lambda_i)$.

---

6We point out that this fails for arbitrary quantum groups. An explicit counterexample can be found for example in Section 5 of [4].
Moreover, note that the morphism from $\nabla q(\lambda_i)$ to $\Delta q(\lambda_i)$ uses only $L_q(\lambda_{i-1})$ while the other way around uses only $L_q(\lambda_i)$. Thus, up to scalars, morphisms from $\nabla q(\lambda_i)$ to $\Delta q(\lambda_i)$ are the “same” as morphisms from $\Delta q(\lambda_{i-1})$ to $\nabla q(\lambda_{i-1})$.

We can now construct up $u^\lambda_i$ and down $d_i^\lambda$ by using Proposition [2,20] as a composition of the maps as follows.

$$
\begin{array}{c}
\Delta q(\lambda_i) \xrightarrow{a} T_q(\lambda_i) \xrightarrow{\alpha} \Delta q(\lambda_{i-1}) \\
\Delta q(\lambda_{i-1}) \xrightarrow{d} T_q(\lambda_{i-2}) \xrightarrow{\beta} \Delta q(\lambda_{i-3}).
\end{array}
$$

We define $d_i^\lambda$ to be the composite $cba$. Similar for $u^\lambda_i$ but using the left side of part (b) of Proposition [2,20]. By the same reasoning as above we see that they are unique up to scalars.

We have to check the relations between the various up $u^\lambda_i$ and down $d_i^\lambda$ maps now. To see that $u_{i+1}^\lambda \circ u_i^\lambda = 0 = d_i^\lambda \circ d_{i+1}^\lambda$ we can simply use the second diagram above and its dual counterpart and the fact that the rows are exact again. Now consider

$$
\begin{array}{c}
T_q(\lambda_{i-1}) \xrightarrow{\gamma} \Delta q(\lambda_{i-1}) \xrightarrow{\delta} T_q(\lambda_i) \xrightarrow{\epsilon} \nabla q(\lambda_i) \xrightarrow{\zeta} T_q(\lambda_{i+1}) \\
T_q(\lambda_{i-1}) \xrightarrow{\mu} \nabla q(\lambda_{i-1}) \xrightarrow{\nu} T_q(\lambda_i) \xrightarrow{\omega} \Delta q(\lambda_i) \xrightarrow{\eta} T_q(\lambda_{i+1})
\end{array}
$$

Combining everything, we see that (up to scalars) $d_{i+1}^\lambda \circ u_i^\lambda = \varepsilon_i^\lambda = u_{i-1}^\lambda \circ d_i^\lambda \neq 0$. Moreover, by the reasoning above, the $\varepsilon_i^\lambda$ is an up to scalars unique non-zero $U_q$-morphism $T_q(\lambda_i)$ to $T_q(\lambda_i)$ that squares to zero.

This finishes the proof since we leave the special dead-end case to the reader. □

**Corollary 2.29.** Let $i, i' \in \mathbb{N}$. Then we have the following.

(a) **Outside of walls:**

$$
\text{Hom}_{U_q}(T_q(\lambda_i), T_q(\lambda_{i'})) \cong \begin{cases}
\mathbb{Q}(q)[\varepsilon], & \text{if } |i - i'| = 0 \text{ and } i = i' \neq 0, \\
\mathbb{Q}(q), & \text{if } |i - i'| = 1 \text{ or } i = i' = 0, \\
0, & \text{if } |i - i'| > 1,
\end{cases}
$$

where $\mathbb{Q}(q)[\varepsilon] \cong \mathbb{Q}(q)[X]/X^2$ denotes the $\mathbb{Q}(q)$-algebra of dual numbers.

(b) **On walls:**

$$
\text{Hom}_{U_q}(T_q(\mu_i), T_q(\mu_{i'})) \cong \begin{cases}
\mathbb{Q}(q), & \text{if } |i - i'| = 0, \\
0, & \text{if } |i - i'| > 0.
\end{cases}
$$

**Proof.** Because of Proposition [2.28] we only need to verify (b). But since the $T_q(\mu_i)$ are simple on walls (by Proposition [2.20], part (b) follows from Remark [2.27]. □

2.5. **Translation functors.** Set $\nu = |\lambda - \mu|^\frac{1}{2}$. Recall that there is, by Corollary [2.10] and (a) of Proposition [2.20], a unique simple, tilting module $T_q(\nu) \cong L_q(\nu)$ corresponding to $\nu$. Moreover, for a fixed $\lambda \in \mathcal{A}_0$ denote by $p_{\lambda} : \Sigma \rightarrow \Sigma_\lambda$ the projection to the block $\Sigma_\lambda$ functor. Similarly for $\mu$. 

\footnote{In fact, $\nu = |\lambda_i - \mu_i|$ for all $i \in \mathbb{N}$ and $\nu$ will stay in $\mathcal{A}_0$.}
Definition 2.30. (Onto and out of the wall) Given $\lambda$ and $\mu$, we define two functors called onto the $\mu$-wall $T^\mu_\lambda$ and out of the $\mu$-wall $T^\lambda_\mu$ via

$$T^\mu_\lambda : \mathcal{X}_\lambda \to \mathcal{X}_\mu, \; M \mapsto p_\mu(M \otimes_{Q(q)} T_q(\nu))$$ and $$T^\lambda_\mu : \mathcal{X}_\mu \to \mathcal{X}_\lambda, \; M \mapsto p_\lambda(M \otimes_{Q(q)} T_q(\nu)).$$

Note that $T_q(\nu) \cong L_q(\nu)$ is a simple module for $\nu \in \bar{A}_0$.

We are now ready to define the endofunctors $\Theta^\lambda_\mu, \Theta^\lambda_\mu$, called translation through the $s$-wall and $t$-wall respectively, whose combinatorics play an important role in Sections 3 and 4. If it is clear which $\lambda$ we are using we, abusing notation, denote them by $\Theta_s, \Theta_t$.

Definition 2.31. (Translation through the walls) Define $\Theta^\lambda_s = T^\lambda_-^\lambda \circ T^\lambda_-^1$ and $\Theta^\lambda_t = T^\lambda_-^1 \circ T^\lambda_-^1$.

Proposition 2.32. For all $i \in \mathbb{N}$ we have the following (with $T_q(-1) = T_q(\lambda_-1) = T_q(\mu_-1) = 0$).

(a) The functors $T^\mu_\lambda$ and $T^\lambda_\mu$ are well-defined (their definition gives tilting modules in the right blocks), adjoints (left and right) and exact. Thus, $\Theta^\lambda_s$ and $\Theta^\lambda_t$ are exact and self-adjoint.

(b) We have

$$T^\mu_\lambda(T_q(\lambda_i)) \cong \begin{cases} T_q(\mu_{i-1}) \oplus T_q(\mu_{i+1}), & \text{if } \mu_i > \lambda_i, \\ T_q(\mu_i) \oplus T_q(\mu_i), & \text{if } \mu_i < \lambda_i, \end{cases}$$

(recall that $\Delta_q(\mu_i) = L_q(\mu_i) = T_q(\mu_i) \cong \nabla_q(\mu_i)$) and

$$T^\lambda_\mu(T_q(\mu_i)) \cong \begin{cases} T_q(\lambda_{i+1}), & \text{if } \mu_i > \lambda_i, \\ T_q(\lambda_i), & \text{if } \mu_i < \lambda_i. \end{cases}$$

(c) The dead-end relations $\Theta^\lambda_s(T_q(\lambda_0)) \cong 0$, $\Theta^\lambda_s(T_q(\lambda_1)) \cong T_q(\lambda_2)$, and $\Theta^\lambda_t(T_q(\lambda_0)) \cong T_q(\lambda_1)$. Moreover, we have

$$\Theta^\lambda_s \circ \Theta^\lambda_t(T_q(\lambda_i)) \cong \begin{cases} T_q(\lambda_{i-1}) \oplus T_q(\lambda_{i+1}), & \text{if } i > 1 \text{ is odd for } s \text{ and even for } t, \\ T_q(\lambda_i) \oplus T_q(\lambda_i), & \text{if } i > 0 \text{ is odd for } t \text{ and even for } s. \end{cases}$$

It is worthwhile to note that in the case $\mu = -1$ we have $\mu_i > \lambda_i$ iff $i$ is odd whereas in the case $\mu = l - 1$ we have $\mu_i > \lambda_i$ iff $i$ is even as a look at 2.4.1 should convince the reader.

Proof. That the functors are well-defined follows from (c) of Proposition 2.16 and Lemma 2.25 i.e. tensor products of tilting modules are tilting modules. Thus, projecting to the corresponding blocks gives a decomposition as in part (a) of Lemma 2.25. The other statements in (a) can be verified as in the usual case. For example the proof in Sections 7.1 and 7.2 of [28] can be adopted without difficulties. Note that biadjoint functors have in addition some other nice properties, see e.g. Section 2 in [36].

The lists in part (b) can be verified using Propositions 2.7 and 2.20, Corollary 2.8 and Theorem 2.23.

Part (c) is just a direct application of the finer list of statements in (b).

For example $\Theta^\lambda_s(T_q(\lambda_0)) \cong 0$ follows from $T^\lambda_-^1(T_q(\lambda_0)) \cong T_q(-1) \oplus T_q(-1) \cong 0$ while $\Theta^\lambda_t(T_q(\lambda_0)) \cong T_q(\lambda_1)$ follows from $T^\lambda_-^1(T_q(\lambda_0)) \cong T_q(-1) \oplus T_q(-1) \cong T_q(l-1) \oplus T_q(-1)$ combined with $T^\lambda_-^1(T_q(l-1)) \cong T_q(\lambda_1)$. This finishes the proof.

Example 2.33. To give an explicit example consider our favourite case $l = 3$ again. Moreover, remember that we have calculated $T_q(1) \otimes T_q(1) \cong T_q(0) \oplus T_q(2)$ in Example 2.21 and all of these $U_q$-modules are simple. Thus, for $\lambda = 1$ and $\mu = 2$, we see that $T^2_1(M) = p_2(M \otimes_{Q(q)} T_q(1))$ and
where we use the same shorthand notation as in Example 2.21. We now have to look for the
therefore (recall that in those cases essentially the same result as before, namely \( T_0^2(T_q(0)) \cong T_q(2) \).

If we apply \( T_2^2 \) now to \( T_q(2) \) (tensor with \( T_q(1) \)) we get a picture as (with \( T_q(2) \) in the top row)

\[
\begin{array}{c}
\circlearrowright \quad \bullet \\
1 & \quad 0 & \quad 1 \\
-1 & \quad 1 & \quad -1 \\
1 & \quad 1 & \quad 1 \\
\end{array}
\]

where we use the same shorthand notation as in Example 2.21. We now have to look for the
indecomposable summands in the Weyl orbit of 1, namely for \( T_q(1), T_q(3), T_q(7), \ldots \).

In the non-root of unity case (and also for \( l > 3 \)) we have that \( L_q(1) \otimes L_q(2) \cong L_q(1) \oplus L_q(3) \)
(recall that in those cases \( T_q(1) = L_q(1) \) and \( T_q(2) = L_q(2) \)).

But the result above is indecomposable: It has a unique simple submodule \( L_q(3) \) which is
spanned by \( \{m_{21}, q^{-2}m_{20} + m_{11}, q^{12}m_{10} + m_{01}, m_{00} \} \) and the remaining part is clearly not a submodule.
The reason for this is that \( E, F \) act differently in rows and columns. Thus, this is \( T_q(3) \) and we therefore get \( T_2^3(T_q(\lambda_0)) = T_2^3(T_q(2)) \cong T_q(3) = T_q(\lambda_1) \) as predicted.

We use the convention that \( \Theta^{k-\text{stst}} \) denotes an alternating composition of length \( k \) of translation functors starting with \( \Theta^{1}_k \). Likewise for \( \Theta^{k-\text{stst}} \).

**Corollary 2.34. (The combinatorics of the translation functors)** We have the following for all \( \lambda \).

(a) \( \Theta^{\lambda}_s \circ \Theta^{\lambda}_s \cong \Theta^{\lambda}_s \oplus \Theta^{\lambda}_s \) and \( \Theta^{\lambda}_t \circ \Theta^{\lambda}_t \cong \Theta^{\lambda}_t \oplus \Theta^{\lambda}_t \) as functors.

(b) We have for \( i \in \mathbb{N} \) even, that (with \( k \geq 0 \) terms \( \Theta^{\lambda}_{t,or,s} \)) there exist multiplicities \( m_j \in \mathbb{N} \) (that can be zero) such that

\[
\Theta^{\lambda}_{k-\text{stst}} T_q(\lambda_i) = (\Theta^{\lambda}_{s,or,t} \circ \ldots \Theta^{\lambda}_{t} \circ \Theta^{\lambda}_{s} \circ \Theta^{\lambda}_{t}) T_q(\lambda_i) \cong T_q(\lambda_{i+k}) \oplus \bigoplus_{j<i+k} T_q(\lambda_j)^{\oplus m_j}
\]

and similarly for \( i \in \mathbb{N} \) odd and \( \Theta^{\lambda}_{k-\text{stst}} \). On the other hand for \( i \in \mathbb{N} \) odd we have

\[
\Theta^{\lambda}_{k-\text{stst}} T_q(\lambda_i) = (\Theta^{\lambda}_{s,or,t} \circ \ldots \Theta^{\lambda}_{t} \circ \Theta^{\lambda}_{s} \circ \Theta^{\lambda}_{t}) T_q(\lambda_i) \cong T_q(\lambda_{i-1+k})^{\oplus 2} \oplus \bigoplus_{j<i-1+k} T_q(\lambda_j)^{\oplus m_j}
\]

and similarly for \( i \in \mathbb{N} \) even and \( \Theta^{\lambda}_{k-\text{stst}} \).

**Proof.** The parts (a)+(b) follow directly from (c) of Proposition 2.32. \( \square \)

The results above motivate the following definition. Note that the Lemmata 2.18 and 2.25 show
that the definition is well-defined: The functors we define below are endofunctors on \( \mathfrak{X} \), i.e. they send tilting modules to tilting modules (in the correct blocks).

**Definition 2.35.** A functor \( \mathcal{F} : \mathfrak{X} \to \mathfrak{X} \) is called **projective** if there exists a finite dimensional tilting module \( T \) such that \( \mathcal{F} \cong T \otimes \mathbb{Q}(\mathfrak{X}) \).

We denote by \( \text{pEnd}(\mathfrak{X}) \) the category (it follows from Lemma 2.36 that this is actually a category) of projective endofunctors \( \mathcal{F} : \mathfrak{X} \to \mathfrak{X} \) whose morphisms \( \text{Hom}_{\text{End}(\mathfrak{X})}(\mathcal{F}, \mathcal{F}') \) are natural
transformations $\eta: \mathcal{F} \to \mathcal{F}'$. Moreover, we denote by $\text{pEnd}(\Sigma)$ the category (it follows from Lemma 2.36 that this is actually a category) of endofunctors $\mathcal{F}: \Sigma \to \Sigma$ that come from a projective functor via projection to the block $\Sigma$. Similarly for walls $\mu$.

**Lemma 2.36.** The category $\text{pEnd}(\Sigma)$ is additive and closed under composition and direct sums. Furthermore, it contains the identity functor and each projective functor $\mathcal{F} \in \text{Ob}(\text{pEnd}(\Sigma))$ is exact. Likewise for the $\lambda$ and $\mu$ versions.

**Proof.** That $\text{pEnd}(\Sigma)$ is preserved under direct sums (showing that the category is additive) follows from Lemma 2.18 part (a). That functors in $\text{pEnd}(\Sigma)$ are exact follows because tensoring over a field always respects exact sequences.

The identity functor is projective because of $\text{id} \cong T_q(0) \otimes_{Q(q)}$. That $\text{pEnd}(\Sigma)$ is closed under composition follows from part (b) of Lemma 2.18 together with the fact that finite tensor products of finite dimensional $U_q$-modules are finite dimensional.

The statement for the $\lambda$ and $\mu$ versions follow similar by using Lemma 2.25.

The following lemma will be very useful later. It is true in more generality (see for example Chapter 4, Section 6 in [6]), but we restrict to our case here. Recall that a functor between additive categories $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is called indecomposable if any decomposition $\mathcal{F} \cong \mathcal{F}_1 \oplus \mathcal{F}_2$ implies that $\mathcal{F}_1 \cong 0$ or $\mathcal{F}_2 \cong 0$.

**Lemma 2.37.** A projective functor $\mathcal{F} \in \text{Ob}(\text{pEnd}(\Sigma))$ is indecomposable iff $T$ is an indecomposable $U_q$-module. Same for the $\lambda$ and $\mu$ versions. Thus, $\text{pEnd}(\Sigma)$ is Krull-Schmidt.

**Proof.** Assume that $T$ decomposes into $T_1 \oplus T_2$. Then $\mathcal{F}$ decomposes as

$$\mathcal{F} \cong (T_1 \otimes_{Q(q)} \cdot) \oplus (T_2 \otimes_{Q(q)} \cdot).$$

On the other hand, by Yoneda and the tensor-hom adjunction, the $T$ representing $\mathcal{F}$ is uniquely determined up to isomorphism. Thus, a decomposition of $\mathcal{F}$ induces a decomposition of $T$. This implies, using by part (a) of Lemma 2.18 that $\text{pEnd}(\Sigma)$ is Krull-Schmidt.

**Proposition 2.38.** The functors onto $T^{\mu}_\lambda$, out of $T^{\lambda}_\mu$ and translation functors $\Theta^\lambda s o r t$ are all indecomposable. Moreover, every functor $\mathcal{F} \in \text{Ob}(\text{pEnd}(\Sigma))$ appears as a direct summand of a direct sum of compositions of functors $T^{\mu}_\lambda$ or $T^{\lambda}_\mu$ for some $\lambda, \mu$. Every functor $\mathcal{F} \in \text{Ob}(\text{pEnd}(\Sigma))$ appears as a direct summand of a direct sum of compositions of functors $\Theta^\lambda s o r t$ or $\Theta^\lambda t$. Same for the $\mu$ version.

**Proof.** By Lemma 2.37 we see that it is enough to consider indecomposable projective functors who are parametrized by indecomposable tiltings.

Note that $T^{\mu}_\lambda$ and $T^{\lambda}_\mu$ are indecomposable by Lemma 2.37. To see that $\Theta^\lambda s o r t$ are indecomposable note that any non-trivial decomposition $\Theta^\lambda s o r t = F_1 \oplus F_2$ gives rise to a non-trivial idempotent $\eta \in \text{pEnd}(\Theta^\lambda s o r t)$. By using Proposition 2.32 together with Corollary 2.29 we see that such a natural transformation can not exist (most of the $U_q$-intertwiners are nilpotent).

All other statements follow now from the construction of the indecomposable tilting modules $T_q(i)$ as a summand of a tensor product of $T_q(1)$’s together with part (a) of Lemmata 2.18 and 2.25 and the fact, that the indecomposable tilting modules $T_q(i)$ form a complete set of finite dimensional, pairwise non-isomorphic tilting modules, see Proposition 2.20 part (c).
In this section we recall first in Subsection 3.1 the quiver algebras introduced by Khovanov and Seidel in [42]. We call them KS m-quinver algebras and denote them by \( A_m \).

Recall that the KS m-quinver algebra comes with two gradings: The one used by Khovanov and Seidel in [42] and one given by the path length. We use the path length grading. Using the path length grading, we recall in Subsection 3.2 the categories of graded (right) \( A_m \)-modules, which we denote by \( \text{Mod}_{gr-A_m} \). Moreover, we recall/define Khovanov-Seidel’s endofunctors \( U_i \) (Section 2b in [42]) in Subsection 3.3 with respect to the path length grading. Recall that the \( U_i \) give rise to a functorial action of the braid group \( B_{m+1} \) on \( m+1 \)-strands that they used to categorify the Burau representation of \( B_{m+1} \), see Subsections 2.d and 2.e in [42].

In order to relate the \( A_m \)’s to our tilting category \( \mathfrak{X} \), we introduce in Definition 3.4 what we call the KS \( \infty \)-quinver algebra and denote by \( A_\infty \). We show in Proposition 3.10 an analogon of Soergel’s Endomorphismensatz from [62].

We relate \( A_m, A_\infty \) and \( U_i \) further to our tilting category \( \mathfrak{X} \) from Subsection 2.3 in Subsection 3.4. In particular, we define functors \( \nabla_m \) and \( \nabla_\infty \) and prove analoga of Soergel’s Struktursatz from [62] in Theorems 3.11 and 3.12. In the same theorems we show how the \( U_i \) can be obtained from the translations through the wall functors \( \Theta_s, \Theta_t \).

We use this in Subsection 3.5 to transfer the path length grading to our category of tilting modules \( \mathfrak{X} \) which gives rise to a graded version that we denote by \( \mathfrak{X}^\text{gr} \). We show in Proposition 3.19 that all tilting modules in \( \mathfrak{X} \) are gradable. This grading will be essential in Section 4.

In addition, we show in Proposition 3.28 that all projective endofunctors \( F \in \text{Ob}(p\text{End}(\mathfrak{X})) \) are also gradable. Using these gradings, we refine for example Proposition 2.32 from Section 2. Last but not least: At the end of this section we derive also some consequences for the natural transformations between the graded(?) projective endofunctors.

Thus, by following Khovanov-Seidel [42], Bernstein-Frenkel-Khovanov [11] or Stroppel [66], as we note in Remark 3.32 the categories \( p\text{End}(\mathfrak{X}^\text{gr}_i(\leq m)) \) (see in Subsections 3.4 and 3.5 for the definition) categorify the \( m+1 \)-strand Temperley-Lieb algebra \( TL^v_{m+1} \) for all \( m \). Here the grading “categorifies” the indeterminate \( v \). Thus, taking the limit \( m \to \infty \), we have that \( p\text{End}(\mathfrak{X}^\text{gr}_\infty) \) categorifies the Temperley-Lieb algebra \( TL^v_{\infty} \) in \( \infty \)-many strands.

3.1. The quiver algebras \( A_m \) and \( A_\infty \). Let \( m \in \mathbb{N} \). We consider the following quiver

\[
\begin{align*}
\bullet &\;\overset{u_{-1}}{\longrightarrow}\; \bullet & \;\overset{u_{-2}}{\longrightarrow}\; \cdots \;\overset{u_{2}}{\longrightarrow}\; \bullet & \;\overset{u_{1}}{\longrightarrow}\; \bullet & \;\overset{u_{0}}{\longrightarrow}\; \bullet \\
& \;\overset{d_{m}}{\longrightarrow}\; \bullet & \;\overset{d_{m-1}}{\longrightarrow}\; \cdots & \;\overset{d_{3}}{\longrightarrow}\; \bullet & \;\overset{d_{2}}{\longrightarrow}\; \bullet & \;\overset{d_{1}}{\longrightarrow}\; \bullet & \;\overset{d_{0}}{\longrightarrow}\; \bullet 
\end{align*}
\]

having \( m+1 \) vertices denoted by \( 0, 1, \ldots, m \) and \( 2m \) arrows

\[
u_i: \; i \mapsto i+1, \; i = 0, \ldots, m-1 \quad \text{and} \quad d_i: \; i \mapsto i-1, \; i = 1, \ldots, m
\]

that we call up and down respectively.

The path algebra \( P_m \) of the quiver from (3.1.1) is defined to be the \( \mathbb{Q}(q) \)-algebra whose underlying \( \mathbb{Q}(q) \)-module is the \( \mathbb{Q}(q) \)-vector space spanned by all finite paths with multiplication given by composition of paths if possible and zero otherwise.

In fact, Soergel’s combinatorial functor \( \nabla_m \), see Section 3.4, turns left into right modules - that is why we, in contrast to Section 2, use right modules in this section and read from right to left: We think of the paths in the quiver as applying morphisms/functors to something on the right.
The path algebra $P_m$ is a $\mathbb{Z}$-graded $\mathbb{Q}(q)$-algebra\footnote{Note that Khovanov and Seidel use free $\mathbb{Z}$-modules instead of $\mathbb{Q}(q)$-vector spaces. In order to avoid too many different overlapping notations, we only use the $\mathbb{Q}(q)$-vector space version, but it is not a big problem to work over $\mathbb{Z}$ if the reader prefers to do so.} that is

$$P_m = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} P_m^k \text{ with } P_m^k = \{ \text{All paths in } P_m \text{ of length } k \},$$

since we clearly have $P_m^k \circ P_m^{k'} \subset P_m^{k+k'}$. We write $\ell(\cdot)$ for the grading on $P_m$ (the “length”).

Since we do not use other gradings: We call all $\mathbb{Z}$-graded algebras simply graded. Moreover, an algebra homomorphism $f : A \to B$ between graded algebras $A$ and $B$ is called a homomorphism of graded algebras, if $f(A^k) \subset B^k$ for all $k \in \mathbb{Z}$.

**Definition 3.1.** (Khovanov-Seidel’s $m$-quiver algebra) Let $A_m$ denote the quotient algebra obtained from the path algebra $P_m$ for the quiver from \cite{KaSe} by the defining relations

$$u_i \circ u_{i-1} = 0 = d_i \circ d_{i+1}, \ i = 1, \ldots, m - 1 \text{ and } d_{i+1} \circ u_i = u_{i-1} \circ d_i, \ i = 1, \ldots, m - 1$$

and the right dead-end relation

$$d_1 \circ u_0 = 0.$$  

Given two paths $p, p' \in A_m$ we write $p' \circ p$ instead of $p' \circ p$. The algebra $A_m$ inherits the grading $\ell(\cdot)$ from $P_m$ since all the relations are homogeneous.

We denote, by abuse of notation, the path of length 0 that starts and ends at $i$ also by $i$. Note that the $i$ are projectors or idempotents, because $i^{2} = i$, $p \circ i = p$ if $p$ starts in $i$ and 0 else and $i \circ p = p$ iff $p$ ends in $i$ and 0 else. Moreover, they form a complete set of pairwise orthogonal idempotents, that is $1 = 0 + 1 + \cdots + m$ and $ij = \delta(i,j)$, where $1 \in A_m$ is the unit. Note that $0 \neq 0, 1 \neq 1$ and the $i$’s are not central for $m > 0$, since e.g. $0 \neq u_i \neq u_i \circ i = u_i$ for $i = 0, \ldots, m - 1$.

Moreover, we denote for $i = 1, \ldots, m$ by $\varepsilon_i = u_{i-1}d_i$ the loop that starts at $i$ and goes via $d_i$ to $i-1$ and back via $u_{i-1}$. Note that the relations imply that $\varepsilon_i = d_{i+1}u_i$ (if possible, i.e. if $i+1 \leq m$). Thus, the $\mathbb{Q}(q)$-algebra $A_m$ has a basis given by $i$ (for $i = 0, \ldots, m$) and $u_i$ (for $i = 0, \ldots, m - 1$) and $d_i, \varepsilon_i$ (for $i = 1, \ldots, m$) with $\ell(i) = 0, \ell(u_i) = \ell(d_i) = 1$ and $\ell(\varepsilon_i) = 2$.

**Example 3.2.** The algebra $A_0$ consists just of $\mathbb{Q}(q)$-multiples of 0. The algebra $A_1$ is a $\mathbb{Q}(q)$-vector space with basis $\{ 0, 1, u_0, d_1, \varepsilon_1 \}$ and the only non-zero multiplications of these basis elements are $00 = 0, u_00 = u_0, 0d_1 = d_1$ and $11 = 1, d_11 = d_1, 1u_0 = u_0$ and finally $1 \varepsilon_1 = \varepsilon_1 = \varepsilon_1 1$.

The algebra $A_2$ is of dimension 9. Note that there are inclusions of algebras $\iota_0 : A_0 \hookrightarrow A_1$ and $\iota_1 : A_1 \hookrightarrow A_2$ that send the corresponding basis elements to the ones that we denote by the same symbols. This can be visualized as

$$\iota_0 : 0 \hookrightarrow 1 \xrightarrow{d_1} 0 \quad \text{and} \quad \iota_1 : 0 \hookrightarrow 1 \xrightarrow{d_1} 2 \xrightarrow{d_2} 1 \xrightarrow{d_1} 0.$$

In fact, for all $m > 0$, the algebra $A_m$ can be visualized as

$$\cdots \xrightarrow{d_{m-1}} \xrightarrow{d_m} i \xrightarrow{d_i} \cdots \xrightarrow{d_3} 2 \xrightarrow{d_2} 1 \xrightarrow{d_1} 0.$$  

We note the following easy lemma.
Lemma 3.3. There is a sequence of (non-unital!) inclusions of graded algebras

\[ A_0 \subseteq \cdots \subseteq A_i \subseteq \cdots \subseteq A_m \subseteq \cdots, \]

where \( \iota_m \colon A_m \to A_{m+1} \) is defined by \( i, u_i, d_i, \varepsilon_i \mapsto i, u_i, d_i, \varepsilon_i \) for all suitable indices \( i \).

Proof. This follows because \( P_m \) clearly includes into \( P_{m+1} \) and the set of relations for \( A_m \) is included in the ones for \( A_{m+1} \). That these morphisms respect the grading and are injective is immediate. \( \square \)

This motivates the following definition.

Definition 3.4. (Khovanov-Seidel’s \( \infty \)-quiver algebra) Define \( A_\infty \) to be the inductive limit of the sequence of inclusions of graded algebras from Lemma 3.3 that is

\[ A_\infty = \lim_{\to} A_m. \]

Note that \( A_\infty \) is a graded \( \mathbb{Q}(q) \)-vector space of countable dimension. Moreover, \( A_\infty \) is an associative algebra with a complete set of pairwise orthogonal idempotents \( \{ i \mid i \in \mathbb{N} \} \), but \( A_\infty \) is a non-unital algebra, since the unit would have to be an infinite sum of the \( i \)'s. Such an algebra is sometimes called idempotented. Note that one can see these algebras as categories whose objects are the idempotents \( i \) and whose morphism spaces \( \text{Hom}(i, j) \) are \( i A_\infty j \) - and this is essentially what the quiver approach is doing. Idempotented algebras turn up naturally within the setting of (categorified) quantum groups, see for example [38].

The KS \( \infty \)-quiver algebra \( A_\infty \) can be visualized as

\[
\begin{array}{c}
\cdots \bullet u_4 \bullet \vdots \bullet u_{i-1} \bullet \cdots \bullet u_2 \bullet \vdots \bullet u_1 \bullet \vdots \bullet u_0 \bullet \\
\longrightarrow \quad d_{i+1} \quad 1 \quad d_i \quad 2 \quad d_3 \quad 2 \quad d_2 \quad 1 \quad d_1 \quad 0
\end{array}
\]

We point out that \( A_\infty \) only has one asymmetry coming from the right dead-end relation. Moreover, \( A_\infty \) is graded by the path length \( l(\cdot) \) again due to the fact that infinite paths do not exists (the relations imply that each non-zero path has length at most two).

In order to be able to also consider the semisimple blocks \( \mathcal{F}_{-1} \) and \( \mathcal{F}_{l-1} \) of our category \( \mathcal{F} \), we also introduce another quotient of the path algebra \( P_m \), denoted by \( A_m^{\text{triv}} \), (and take a limit as above). Since the corresponding module categories should be semisimple, the quotient \( A_m^{\text{triv}} \) is rather trivial and we call it KS trivial \( m \)-quiver algebra.

Definition 3.5. (Khovanov-Seidel’s trivial \( m \)-quiver algebra) Let \( A_m^{\text{triv}} \) denote the quotient algebra obtained from the path algebra \( P_m \) for the quiver from 3.1.1 by the defining relations

\[ u_i = 0 = d_{i+1}, \quad 0 = 1, \ldots, m - 1. \]

The algebra \( A_m^{\text{triv}} \) inherits the grading \( l(\cdot) \) from \( P_m \), but this grading is trivial, that is, all of its elements are of degree zero. In fact, \( A_m^{\text{triv}} \) consist only of orthogonal idempotents that we denote by \( i \) for \( i = 0, \ldots, m \). This time the idempotents commute and form a set of pairwise non-isomorphic, central, orthogonal idempotents which shows that \( A_m^{\text{triv}} \) is a semisimple algebra. Moreover, the algebra \( A_m^{\text{triv}} \) is clearly isomorphic to \( \mathbb{Q}(q) \times \cdots \times \mathbb{Q}(q) \) (with \( m + 1 \) factors).

Thus, the category \( \text{Mod}_{A_m^{\text{triv}}} \) is semisimple and all of its simple modules \( \mathbb{Q}(q) i \) are concentrated in degree zero (up to a shift). We call \( A_m^{\text{triv}} \) Khovanov-Seidel’s trivial \( m \)-quiver algebra.
As before we define Khovanov-Seidel’s trivial $\infty$-quiver algebra via $A^\text{triv}_\infty = \varinjlim A^\text{triv}_m$. We note that this can be visualized as

$$\cdots \overset{1}{\bullet} \cdots \overset{2}{\bullet} \overset{1}{\bullet} \overset{0}{\bullet}$$

where we invite the reader to check that $A^\text{triv}_m$ embeds into $A^\text{triv}_{m+1}$ for all $m \geq 0$ and thus, taking the inductive limit makes sense.

For convenience we set $u_m = d_0 = \varepsilon_0 = 0$ for all $A_m$ and $d_0 = \varepsilon_0 = 0$ for $A_\infty$ in the following.

### 3.2. Combinatorics of the graded, right $A_m$- and $A_\infty$-modules.

Recall that, if $A$ denotes some $\mathbb{Z}$-graded $\mathbb{Q}(q)$-algebra, then a (right) $A$-module $M$ is called $\mathbb{Z}$-graded (or simply graded), if

$$M = \bigoplus_{k \in \mathbb{Z}} M^k \quad \text{and} \quad M^k \cdot A^k \subset M^{k+k'}$$

and an $A$-module homomorphism between graded modules $f: M \to N$ is called degree preserving, if $f(M^k) \subset N^k$ for all $k \in \mathbb{Z}$ and homogeneous (of degree $d \in \mathbb{Z}$) if $f(M^k) \subset N^{k+d}$ for all $k \in \mathbb{Z}$.

We denote by $\text{Mod}_{gr}^- A$ the category of graded, finitely generated $A$-modules whose morphisms from $M$ to $M'$ are given by

$$\text{Hom}_{\text{Mod}_{gr}^- A}(M, M') = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_A(M, M'\langle i \rangle)_0,$$

where the zero should mean degree preserving morphisms. Thus, all morphisms in $\text{Mod}_{gr}^- A$ are finite direct sums of homogeneous morphisms.

The endofunctor $\cdot \langle s \rangle: \text{Mod}_{gr}^- A \to \text{Mod}_{gr}^- A$, called shift by $s \in \mathbb{Z}$, sends the $k-s$-th degree part of a module to the $k$-th of the shift $M\langle s \rangle$, that is, $M\langle s \rangle^k = M^{k-s}$.

We use similar notions for left modules (we denote such categories by e.g. $A\text{-Mod}_{gr}^+$) or projective modules (we denote such categories by e.g. $p\text{Mod}_{gr}^- A$). We only work in categories of (graded) finitely generated (projective) modules.

Note that $A_m$ acts on itself from the right by pre-composition of paths and from the left by post-composition of paths.

This makes a difference: Let us denote by $P_i$ the left ideal of $A_m$ generated by $i$. Similar, $iP$ denotes the right ideal generated by $i$. We have as $\mathbb{Q}(q)$-vector spaces

$$P_i = \mathbb{Q}(q) i \oplus \mathbb{Q}(q) u_i \oplus \mathbb{Q}(q) d_i \oplus \mathbb{Q}(q) e_i \quad \text{and} \quad iP = \mathbb{Q}(q) i \oplus \mathbb{Q}(q) u_{i-1} \oplus \mathbb{Q}(q) d_{i+1} \oplus \mathbb{Q}(q) e_i,$$

with homogeneous components of degree 0, 1, 1, 2 (from left to right).

The $P_i$ and the $iP$ are graded $A_m$-modules (left and right respectively) that can be visualized as

$$P_i = \cdots \overset{\cdot u_i}{\bullet} \overset{\cdot d_i}{\bullet} \overset{\cdot e_i}{\bullet} \cdots \quad \text{and} \quad iP = \cdots \overset{\cdot u_{i-1}}{\bullet} \overset{\cdot d_{i+1}}{\bullet} \overset{\cdot e_{i-1}}{\bullet} \cdots.$$

Thus, the $P_i$’s and the $iP$’s are projective, since we have (as left and right $A_m$-modules)

$$A_m = \bigoplus_{i=0}^m P_i \quad \text{and} \quad A_m = \bigoplus_{i=0}^m iP.$$

\footnote{Because it is always confusing: In our convention a positive number shifts the degree up.}
Moreover, they are all indecomposable and it is easy to see that all indecomposable left, respectively right, $A_m$-modules are of the form $P_i(s)$, respectively $i^{-1}P(s)$, for some $s \in \mathbb{Z}$: This follows directly from the fact that $A_m$ is finite dimensional. Thus, all indecomposable $A_m$-modules are direct summands of $A_m$ considered as a (left or right) $A_m$-module.

**Example 3.6.** One easily checks that the same is true for the corresponding notions for $A_\infty$ and that one gets a decomposition into $iP$’s as follows (we have indicated the neighbouring $iP$’s using alternating colors and we for simplicity assume that the $i$ below is odd)

\[
\begin{array}{c}
\vdots \\
\bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \\
\hline
\circ \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \\
\hline
\circ \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet
\end{array}
\]

We encourage the reader to draw the decomposition into $P_i$’s. Again, all finitely generated, graded, projective, indecomposable right $A_\infty$-modules are up to a shift of the form $iP$. Same for the left modules and the $P_i$’s. This can in this case be deduced directly, but is true in more generality for idempotent algebras, see for example Proposition 5.3.1 in [39].

Moreover, we also encourage the reader to work out the decomposition of $A_{\text{triv}}^m$ and $A_{\text{triv}}^\infty$ into the corresponding (left and right) modules denoted by $P_{\text{triv}}^i$ and $i^{-1}P_{\text{triv}}$.

### 3.3. Endofunctors on $p\text{Mod}_{\text{gr}}-A_m$ and $p\text{Mod}_{\text{gr}}-A_\infty$.

Set $B_i = P_i \otimes_{Q(q)} i^{-1}P(-1)$ for all indices $i = 1, \ldots, m$. Note that the $B_i$’s are graded $A_m$-bimodules with the tensor product $p \otimes p'$ of degree $l(p \otimes p') = l(p) + l(p')$ for $p$ of degree $l(p)$ and $p'$ of degree $l(p')$. Following Khovanov and Seidel we define functors

\[
U_i: p\text{Mod}_{\text{gr}}-A_m \to p\text{Mod}_{\text{gr}}-A_m, \quad U_i = \cdot \otimes_{A_m} B_i \quad \text{for} \quad i = 1, \ldots, m.
\]

Note that we have

\[
\varepsilon_i P \otimes_{A_m} P_i \cong \begin{cases} 
Q(q) i \oplus Q(q) \varepsilon_i, & \text{if } |i - i'| = 0, \\
Q(q) u_i, & \text{if } i' - i = 1, \\
Q(q) d_i, & \text{if } i - i' = 1, \\
0, & \text{if } |i - i'| > 1.
\end{cases}
\]

This can be easily seen by considering (here $i = i'$ as an example)

\[
\begin{array}{c}
\vdots \\
\bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \\
\hline
\circ \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \\
\hline
\circ \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet
\end{array}
\]

where we have illustrated the overlapping pieces of $iP$ and $P_i$. This clearly implies

\[
U_i(iP) \cong \begin{cases} 
iP(-1) \oplus iP(+1), & \text{if } |i - i'| = 0, \\
iP, & \text{if } |i - i'| = 1, \\
0, & \text{if } |i - i'| > 1.
\end{cases}
\]

We note that we can see $iP \otimes_{A_m} P_i$ as a graded(!), right(!) $A_{\text{triv}}^m$-module where the action of $i$ kills everything that does not start in $i$. Note that, in this notation, $Q(q) i$, $Q(u_i)$, $Q(d_i)$, and $Q(q) \varepsilon_i$ are all one dimensional $A_{\text{triv}}^m$-modules, but concentrated in degrees $0, 1, 1$ and $2$ respectively.

Hence, we have as graded, right $A_{\text{triv}}^m$-modules $Q(q) u_i \cong Q(q) i(+1)$, $Q(q) d_i \cong Q(q) i(+1)$ and $Q(q) \varepsilon_i \cong Q(q) i(+2)$.

\[\text{Note that we use a different convention than Khovanov and Seidel for the gradings by shifting $iP$ down by one.}\]
Corollary 3.7. The functors $\mathcal{U}_i: \text{pMod}^{\text{gr}}_{-A_m} \to \text{pMod}^{\text{gr}}_{-A_m}$ satisfy the following.

(a) $\mathcal{U}_i \circ \mathcal{U}_j \cong \mathcal{U}_i \langle -1 \rangle \oplus \mathcal{U}_j \langle +1 \rangle$ for $i = 1, \ldots, m$.
(b) Let $i = i' \pm 1$ such that $i, i' \in \{1, \ldots, m\}$. Then $\mathcal{U}_i \circ \mathcal{U}_i' \circ \mathcal{U}_i \cong \mathcal{U}_i$.
(c) $\mathcal{U}_i \circ \mathcal{U}_{i'} \cong 0$ if $|i - i'| > 1$.

Proof. This follows directly from [3.3.2] \hfill \square

Finally we set $\mathcal{U}_{\text{even}} = \bigoplus_{1 \leq i, 2i \leq m} \mathcal{U}_{2i}$ and $\mathcal{U}_{\text{odd}} = \bigoplus_{0 \leq i, 2i+1 \leq m} \mathcal{U}_{2i+1}$.

By Corollary 3.7 we see that

\begin{equation}
(3.3.3) \quad \mathcal{U}_{\text{even}} \circ \mathcal{U}_{\text{even}} \cong \mathcal{U}_{\text{even}} \langle -1 \rangle \oplus \mathcal{U}_{\text{even}} \langle +1 \rangle \quad \text{and} \quad \mathcal{U}_{\text{odd}} \circ \mathcal{U}_{\text{odd}} \cong \mathcal{U}_{\text{odd}} \langle -1 \rangle \oplus \mathcal{U}_{\text{odd}} \langle +1 \rangle.
\end{equation}

It is clear that we can easily adopt the definition of the $\mathcal{U}_i$’s to the $A_{\infty}$ case. We denote these functors by $\mathcal{U}_i^\infty$ for all $i \in \mathbb{N}_{>0}$. The reader is invited to check that these functors satisfy completely similar relations as before.

We also define $\mathcal{U}_{\text{even}}^\infty$ and $\mathcal{U}_{\text{odd}}^\infty$ completely analogously by taking the direct sum over all $i \in \mathbb{N}_{>0}$. The reader is still invited to check that these two satisfy similar relations as in [3.3.3].

To make this observation useful, we refine the definitions of the functors $\mathcal{U}_i$ by factoring through the category of graded (right) $A_{m}^{\text{triv}}$-modules that we denote by $\text{Mod}^{\text{gr}}_{m}$. To be precise, we define $\mathcal{U}_i^t$ and $\mathcal{U}_i^b$ via

$\mathcal{U}_i^t: \text{Mod}^{\text{gr}}_{m} \to \text{Mod}^{\text{gr}}_{-A_m^{\text{triv}}}$, $\mathcal{U}_i^b = \cdot \otimes_{A_m} P_i$

and

$\mathcal{U}_i^t: \text{Mod}^{\text{gr}}_{m} \to \text{Mod}^{\text{gr}}_{-A_m^{\text{triv}}}$, $\mathcal{U}_i^b = \cdot \otimes_{\mathbb{Q}(q)} iP_{i-1}$.

It is immediate that $\mathcal{U}_i = \mathcal{U}_i^t \circ \mathcal{U}_i^b$. We define even and odd versions of these refinements as before where we use $\mathcal{U}_{\text{even}}^t$ and $\mathcal{U}_{\text{even}}^b$ as notations (similar for the odd versions).

The following lemma can be seen, by Remark 3.16 below, as an analog of Stroppel’s Theorem 8.4 in [65].

Lemma 3.8. As graded functors: $\mathcal{U}_i^t \langle +1 \rangle$ is the right adjoint of $\mathcal{U}_i^b$, $\mathcal{U}_i^b \langle -1 \rangle$ is the left adjoint of $\mathcal{U}_i^t \langle -1 \rangle$ and similar for the even and the odd versions. Moreover, the functors $\mathcal{U}_i$, $\mathcal{U}_{\text{even}}^t$, and $\mathcal{U}_{\text{even}}^b$ are all (graded) self adjoint.

Proof. This is a case-by-case check where we only do the first case and leave the rest to the reader. In all cases we use that, as graded $\mathbb{Q}(q)$-vector spaces, we have

\begin{equation}
(3.3.4) \quad \text{Hom}_{\text{pMod}^{\text{gr}}_{m}}(iP, \mathcal{U}_i^t \langle \mathbb{Q}(q) \rangle) \cong \text{Hom}_{\text{pMod}^{\text{gr}}_{m}}(\mathcal{U}_i^b \langle \mathbb{Q}(q) \rangle, iP).
\end{equation}

The isomorphism is given by mapping a path $p$ on the right hand side to the homomorphism $iP \to \mathcal{U}_i^t \langle \mathbb{Q}(q) \rangle$ given by left multiplication (post-composition) with $p$. Note that this is a homogeneous morphism of degree $l(p)$.

We have to show that there are isomorphisms of graded vector spaces

$$\text{Hom}_{\text{pMod}^{\text{gr}}_{m}}(\mathcal{U}_i^b \langle \mathbb{Q}(q) \rangle, \mathcal{U}_i^t \langle +1 \rangle \langle \mathbb{Q}(q) \rangle) \cong \text{Hom}_{\text{pMod}^{\text{gr}}_{m}}(\mathcal{U}_i^t, \mathcal{U}_i^b \langle \mathbb{Q}(q) \rangle \langle +1 \rangle),$$
where we can restrict to check only for objects as above by additivity. Note that we have three cases depending on $|i - i'|$. The case $|i - i'| > 1$ is clear, since both hom-spaces will be zero by the discussion above.

For $i = i'$ the left side is $\text{Hom}_{\text{Mod}_-} (iP(-1), iP) \cong \mathbb{Q}(q)i(1) \oplus \mathbb{Q}(q)\varepsilon_i(1)$ by [3.3.4]. Moreover, by [3.3.1] and the shift by +1, we get the same for the right side.

For $i' = i + 1$ the left side gives $\text{Hom}_{\text{Mod}_-} (iP(-1), i+1P) \cong \mathbb{Q}(q)u_i(1)$ again by [3.3.4]. The right side gives, again by [3.3.1] and the shift by +1, the same result. Similar for $i' = i - 1$. Thus, we see that they are an adjoint pair. The other cases are analogous and left to the reader. □

We again have no problems to define the $\infty$-versions. We denote them by $U_i^\infty$ and $U_i^{\text{even}, \infty}$. The even versions of these are denoted by $U_i^{\text{even}, \infty}$ and $U_i^{\text{even}, \infty}$ and similar for the odd versions.

3.4. $A_\infty$ and the tilting category $\mathcal{T}$. Recall that the category $\mathcal{T}_\lambda$ denotes the $\lambda$-block of $\mathcal{T}$ for $\lambda$ in the fundamental alcove $A_0$, see Subsection 2.4. We fix any such $\lambda$ and denote everything using the simplified notation without the $\lambda$’s. In particular, recall our notation from Definition 2.24 for the tilting generators $T_q(\infty)$ and $T_q(\leq m)$.

We denote by $\mathcal{T}_\lambda(\leq m)$ the full subcategory of $\mathcal{T}_\lambda$ consisting of objects whose indecomposable summands are all from the set $\{T_q(\lambda_0), \ldots, T_q(\lambda_m)\}$ for some fixed $m \in \mathbb{N}$.

In order to confuse the reader even more with bad notation, we also use completely similar notations for the walls $\mu$ by indicating this case with a superscript $t$. Note that

$$T_q(\leq m)^t \cong \begin{cases} \bigoplus_{i=0}^{m/2} T_q(\mu_{2i}), & \text{if } \mu = -1, \\ \bigoplus_{i=0}^{m/2} T_q(\mu_{2i+1}), & \text{if } \mu = l - 1, \end{cases}$$

and similarly for $T_q(\infty)^t$.

**Proposition 3.9.** We have an isomorphism of $\mathbb{Q}(q)$-algebras

$$\text{End}_{U_q}(T_q(\leq m)) \cong A_m \text{ and } \text{End}_{U_q}(T_q(\leq m)^t) \cong A^{\text{even}, \infty}[m/2].$$

**Proof.** We are going to construct these isomorphisms explicitly. Moreover, if we simply write $i$ for $T_q(i)$, then we can visualize $\text{End}_{U_q}(T_q(\leq m))$ using Proposition 2.28 for all $m > 0$ as

$$\begin{array}{cccccccc}
\bullet & m & \cdots & u_i & \cdots & u_{m-1} & \bullet \\
\bullet & d_m & \cdots & \bullet & \cdots & \bullet & \bullet \\
\end{array}$$

Thus, we consider the (by Proposition 2.28 well-defined) $\mathbb{Q}(q)$-algebra homomorphism

$$\phi: A_m \to \text{End}_{U_q}(T_q(\leq m)), u_i \mapsto u_i, d_i \mapsto d_i, \varepsilon_i \mapsto \varepsilon_i$$

for all suitable indices $i$. Using Corollary 2.29 we see that this is an isomorphism. The reader is (as usual) invite to check that the case $m = 0$ and the “trivial case” (where he/she can use part (b) of Corollary 2.29) work out as well. □

The following propositions together with the Struktursatz in Theorem 3.12 show that $T_q(\infty)$ governs the behaviour of $\mathcal{T}_\lambda$. It fact, its endomorphism ring $\text{End}_{U_q}(T_q(\infty))$ is even too huge. We consider instead $\text{End}_{U_q}^{\text{fs}}(T_q(\infty))$: Note that morphisms in $\text{End}_{U_q}(T_q(\infty))$ are given by matrices $f = \bigoplus_{i, i' \in \mathbb{N}} f_{i, i'}$ with $f_{i, i'}: T_q(i) \to T_q(i')$ and each row of $f$ consists only of finitely many non-zero entries (this follows from Corollary 2.29). Now $\text{End}_{U_q}^{\text{fs}}(T_q(\infty))$ should be the (non-unital!)
It is a non-unital subalgebra since the identity matrix is not in \( \text{End}_{U_q}(Q_\infty) \). Similar to the “trivial” case.

**Proposition 3.10.** We have an isomorphism of \( \mathbb{Q}(q) \)-algebras

\[
\text{End}_{U_q}^\text{fs}(Q_\infty) \cong \bigoplus_{i,i' \in \mathbb{N}} \text{Hom}_{U_q}(Q_i, Q_{i'}),
\]

and

\[
\text{End}_{U_q}^\text{fs}(Q_\infty)^t \cong \bigoplus_{i,i' \in \mathbb{N}} \text{Hom}_{U_q}(Q_i^t, Q_{i'}^t) \cong A^{\text{triv}}.
\]

**Proof.** We only show the first isomorphism and leave the other (easier) to the reader again.

By using Corollary 2.29 and the definition of \( \text{End}_{U_q}^\text{fs}(Q_\infty) \) from above we see that, for each \( f \in \text{End}_{U_q}^\text{fs}(Q_\infty)^t \), there exists some \( m \in \mathbb{N} \) such that \( f_{i'\ell} = 0 \) for \( i, i' > m \).

Those matrices describe elements in \( \text{End}_{U_q}(Q_{\leq m}) \) by “forgetting” the parts for \( i, i' \gg 0 \). Thus, we have

\[
\text{End}_{U_q}^\text{fs}(Q_\infty) \cong \lim_{\longrightarrow} \text{End}_{U_q}(Q_{\leq m}).
\]

By Proposition 3.9 we see that this implies

\[
\text{End}_{U_q}^\text{fs}(Q_\infty) \cong \lim_{\longrightarrow} A_m.
\]

But this is just the definition of \( A_\infty \), so

\[
\text{End}_{U_q}^\text{fs}(Q_\infty) \cong \lim_{\longrightarrow} A_\infty.
\]

This isomorphism is clearly homogeneous.

We leave the “trivial case” to the reader again. This finishes the proof. \( \square \)

We are now able to relate \( \mathfrak{T}(\leq m) \) to \( \text{pMod} \)-\( A_m \) and thus, \( \mathfrak{T}_\lambda \) to \( \text{pMod} \)-\( A_\infty \). To do so we use analogs of Soergel’s combinatorial functor (“Gold foil experiment” with \( M \) being the gold foil) that we denote by \( \mathbb{V}_m \) and \( \mathbb{V}_\infty \). We define

\[
\mathbb{V}_m : \mathfrak{T}(\leq m) \to \text{Mod} \cdot A_m, \mathbb{V}_m(M) = \text{Hom}_{U_q}(Q_{\leq m}, M)
\]

and

\[
\mathbb{V}_\infty : \mathfrak{T}_\lambda \to \text{Mod} \cdot A_\infty, \mathbb{V}_\infty(M) = \text{Hom}_{U_q}^\text{fs}(Q_\infty, M) \cong \bigoplus_{i=0}^\infty \text{Hom}_{U_q}(Q_i, M)
\]

for \( M \) in either \( \text{Ob}(\mathfrak{T}(\leq m)) \) or \( \text{Ob}(\mathfrak{T}_\lambda) \) respectively where the isomorphism follows from Proposition 3.10. Similarly for the walls \( \mu \) where we indicate the functors \( \mathbb{V}_m^t \) and \( \mathbb{V}_\infty^t \) (and everything else) with superscripts \( t \). We note that the hom-spaces above are right \( A_m \)-modules (or \( A_\infty \), \( A'_{m} \) or \( A'_{\infty} \)-modules) by pre-composition.

Note that \( \Theta_s, \Theta_t \in \text{Ob}(\text{End}(\mathfrak{T}_\lambda)) \) from Definition 2.31 clearly descent down to \( \mathfrak{T}_\lambda(\leq m) \) for all \( m \in \mathbb{N} \). We denote these restrictions, by abuse of notation, also by \( \Theta_s, \Theta_t \). Furthermore, define
\[ \Theta^i_{s, t} \text{ for } i \in \mathbb{N}_{>0} \text{ via } (p_t \text{ projects to the } T_q(i)\text{-part}) \]

\[ \Theta^i_{s, t} = \begin{cases} 
   p_t \circ \Theta_s, & \text{if } i \text{ is even}, \\
   p_t \circ \Theta_t, & \text{if } i \text{ is odd},
\end{cases} \]

where we use the convention that everything below 0 is defined to be zero. We use the same conventions and notations in the finite case by killing everything above \( m \).

Similarly: The onto \( \mathcal{T}^\mu_\lambda \) and out of \( \mathcal{T}^\lambda_\mu \) the wall functors from Definition 2.30 also descent down to the finite case where we again denote them by the same symbols.

We consider the right side for the Struktursatz (finite and infinite) as ungraded modules/functors. Moreover \( \text{pMod} \) always denotes categories of projective modules.

**Theorem 3.11. (Struktursatz - finite version)** We have the following.

(a) The functor \( V_m \) is fully faithful.

(b) The functor \( V_m \) sends objects \( M \in \text{Ob}(\Sigma(\leq m)) \) to projective \( A_m \)-modules and is an equivalence between \( \Sigma(\leq m) \) and \( \text{pMod}-A_m \).

(c) We have the following commuting diagram.

\[ \begin{array}{ccc}
   \Sigma(\leq m) & \xrightarrow{V_m} & \text{pMod}-A_m \\
   \Theta^i_{s, t} \downarrow & & \downarrow U_i \\
   \Sigma(\leq m) & \xrightarrow{V_m} & \text{pMod}-A_m .
\end{array} \]

(d) We have the following commuting diagrams.

\[ \begin{array}{ccc}
   \Sigma(\leq m) & \xrightarrow{V_m} & \text{pMod}-A_m \\
   \Theta_s \downarrow & & \downarrow U_{t, \text{even}} \text{ and } \Theta_t \downarrow \\
   \Sigma(\leq m) & \xrightarrow{V_m} & \text{pMod}-A_m .
\end{array} \]

(e) The same as in (a)+(b) works for the walls \( \mu \) as well by using \( V^t_m \).

(f) We have the following commuting diagrams.

\[ \begin{array}{ccc}
   \Sigma(\leq m) & \xrightarrow{V_m} & \text{pMod}-A_m \\
   \tau^{-1}_\lambda \downarrow & & \downarrow U_{t, \text{even}} \text{ and } \tau^{-1}_\lambda \downarrow \\
   \Sigma(-1)(\leq m) & \xrightarrow{V_m} & \text{pMod}^{\text{triv}}-A_m \\
\end{array} \]

(g) We have the following commuting diagrams.

\[ \begin{array}{ccc}
   \Sigma(-1)(\leq m) & \xrightarrow{V_m} & \text{pMod}^{\text{triv}}-A_m \\
   \tau^{-1}_\lambda \downarrow & & \downarrow U_{t, \text{even}} \text{ and } \tau^{-1}_\lambda \downarrow \\
   \Sigma(\leq m) & \xrightarrow{V_m} & \text{pMod}-A_m \\
\end{array} \]
Proof. We are very careless with the indices in the proof and hope that this does not cause confusion. All appearing indices should be such that everything below 0 or above m is defined to be zero.

(a): By Proposition \[\text{2.20}\] part (c) and Lemma \[\text{2.18}\] part (a) it is enough to verify the statement for \(M = T_q(i)\) for \(i = 0, \ldots, m\). Moreover, by using Corollary \[\text{2.29}\] together with the discussion in Subsection \[3.2\] we see that \(\mathbf{V}_m(T_q(i)) \cong \text{Hom}_{U_q}(T_q(\leq m), T_q(i)) \cong iP\). Thus, \(\mathbf{V}_m\) is full and faithful.

(b): We already discussed in Subsection \[3.2\] that the \(iP\)'s (for \(i = 0, \ldots, m\)) form a complete set of pairwise non-isomorphic, indecomposable projective, right \(A_m\)-modules. Thus, (b) follows from (a).

(c): This follows by combining \(\mathbf{V}_m(T_q(i)) = \text{Hom}_{U_q}(T_q(\leq m), T_q(i)) \cong iP\) together with \[\text{3.3.2}\] and part (c) of Proposition \[\text{2.32}\].

(d): This follows directly from (c) and the definition of \(U_{\text{odd}}\) and \(U_{\text{even}}\).

(e): We encourage the reader to work out the “trivial cases” as well. In those cases, by part (b) of Corollary \[\text{2.29}\] we see that \(\mathbf{V}_{m_i}(T_q(\mu_i)) \cong Q(q)1\).

(f): We do the left case and leave the right to the reader. Note that \(\mathbf{V}_m(T_q(i)) \cong iP\). Thus, using \[\text{3.3.1}\] we see that \(\mathbf{U}_{m_{\text{even}}}^{\text{even}}\mathbf{V}_{m_{\text{odd}}}^{\text{even}}\) sends \(T_q(i)\) to \(Q(q)i \oplus Q(q)e_i\) for even \(i\) and \(Q(q)u_{i-1} \oplus Q(q)d_{i+1}\) for odd \(i\).

This, by the discussion in Subsection \[3.3\], is isomorphic as right \(A_m^{\text{triv}}\)-modules to \(Q(q)i \oplus Q(q)i\) and \(Q(q)i-1 \oplus Q(q)i+1\) respectively. In addition, \(T_q^{-1}\) sends \(T_q(i)\) to \(T_q(\mu_i) \oplus T_q(\mu_i)\) (for \(i\) even) or to \(T_q(\mu_{i-1}) \oplus T_q(\mu_{i+1})\) (for \(i\) odd). These are then sent via \(\mathbf{V}_{m_{\text{odd}}}^{\text{even}}\mathbf{V}_{m_{\text{odd}}}^{\text{even}}\mathbf{V}_{m_{\text{odd}}}^{\text{odd}}\) to \(Q(q)i \oplus Q(q)i\) (for \(i\) even) or \(Q(q)i-1 \oplus Q(q)i+1\) (for \(i\) odd).

(g): Works as (f) and is left to the reader. \(\square\)

Theorem 3.12. (Struktursatz - infinite version) We have the following.

(a) The functor \(\mathbf{V}_\infty\) is fully faithful.

(b) The functor \(\mathbf{V}_\infty\) sends objects \(M \in \text{Ob}(\Sigma)\) to projective \(A_{\infty}\)-modules and is an equivalence between \(\Sigma_\lambda\) and \(\text{pMod}\cdash A_{\infty}\).

(c) We have the following commuting diagram.

\[
\begin{array}{ccc}
\Sigma_\lambda & \xrightarrow{V_\infty} & \text{pMod}\cdash A_{\infty} \\
\downarrow{\Theta}_{i_{\text{str}}} \quad & & \quad \downarrow{U}_i^{\infty} \\
\Sigma_\lambda & \xrightarrow{V_\infty} & \text{pMod}\cdash A_{\infty}.
\end{array}
\]

(d) We have the following commuting diagrams.

\[
\begin{array}{ccc}
\Sigma_\lambda & \xrightarrow{V_\infty} & \text{pMod}\cdash A_{\infty} \quad & \Sigma_\lambda & \xrightarrow{V_\infty} & \text{pMod}\cdash A_{\infty} \\
\downarrow{\Theta}_s \quad & & \quad \downarrow{U}_s^{\text{even}} \quad \downarrow{U}_t^{\text{odd}} \\
\Sigma_\lambda & \xrightarrow{V_\infty} & \text{pMod}\cdash A_{\infty} \quad & \Sigma_\lambda & \xrightarrow{V_\infty} & \text{pMod}\cdash A_{\infty}.
\end{array}
\]

(e) The same as in (a)+(b) works for the walls \(\mu\) as well by using \(V_\infty^t\).
We have the following commuting diagrams.

\[
\begin{align*}
\mathfrak{S}_\lambda(\infty) \xrightarrow{V_\infty} p\text{Mod-}A_\infty & \quad \mathfrak{S}_\lambda(\infty) \xrightarrow{V_\infty} p\text{Mod-}A_\infty \\
T_\lambda^{-1} & \quad U^\text{even}_t & \quad T_\lambda^{-1} & \quad U^\text{odd}_t \\
\mathfrak{S}_{l-1}(\infty) \xrightarrow{V_\infty} p\text{Mod-}A^\text{triv}_\infty & \quad \mathfrak{S}_{l-1}(\infty) \xrightarrow{V_\infty} p\text{Mod-}A^\text{triv}_\infty.
\end{align*}
\]

We have the following commuting diagrams.

\[
\begin{align*}
\mathfrak{S}_{l-1}(\infty) \xrightarrow{V_\infty^l} p\text{Mod-}A^\text{triv}_\infty & \quad \mathfrak{S}_{l-1}(\infty) \xrightarrow{V_\infty^l} p\text{Mod-}A^\text{triv}_\infty \\
T_\lambda & \quad U^\text{even}_t & \quad T_\lambda & \quad U^\text{odd}_t \\
\mathfrak{S}_\lambda(\infty) \xrightarrow{V_\infty} p\text{Mod-}A_\infty & \quad \mathfrak{S}_\lambda(\infty) \xrightarrow{V_\infty} p\text{Mod-}A_\infty.
\end{align*}
\]

Proof. The proof of (a)-(g) is similar to the proof of Theorem 3.11 by using arguments as in Proposition 3.10. We leave it to the reader.

Remark 3.13. Although we have not done it explicitly, we encourage the reader to work out the translation onto and out of the wall commuting diagrams similar to part (c) of the theorems above.

We note the following corollary for completeness.

Corollary 3.14. For all \( \lambda, \lambda' \in \mathcal{A}_0 \) we have \( \mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda'} \). Thus,

\[
\mathfrak{S}(\leq m) \cong \bigoplus_{i=0}^{l-2} p\text{Mod-}A_m \oplus p\text{Mod-}A^\text{triv}_{m/2} \quad \text{and} \quad \mathfrak{S}_r \cong \bigoplus_{i=0}^{l-2} p\text{Mod-}A_\infty \oplus p\text{Mod-}A^\text{triv}_\infty
\]

with semisimple categories \( p\text{Mod-}A^\text{triv}_m \) and \( p\text{Mod-}A^\text{triv}_\infty \).

Proof. This follows from Lemma 2.25 and Theorems 3.11 and 3.12 because the right hand sides are always the same independent of \( \lambda \in \mathcal{A}_0 \).

Example 3.15. (How to compute the images of the translation functors on \( U_q \)-morphisms) Combining the commuting diagrams of the Theorems 3.11 and 3.12 we obtain an explicit way to compute \( \Theta_s \) and \( \Theta_t \) just on the \( T_q(\hat{1}) \)'s (as we already know by part (c) of Proposition 2.32), but also on the \( U_q \)-intertwiners in \( \text{Hom}_{U_q}(T_q(\hat{1}), T_q(\hat{1}')) \).

This follows by using 3.3.4 together with \( \Theta_s \sim \sim U_\text{even} \) and \( \Theta_t \sim \sim U_\text{odd} \). For example, given \( u_0: T_q(\hat{0}) \to T_q(\hat{1}) \), we can compute \( \Theta_t(u_0) \) as follows. First note that \( V_\infty \Theta_t(T_q(\hat{0})) \cong 1 P \) and \( V_\infty \Theta_t(T_q(\hat{1})) \cong 1 P \oplus 2 P \) by 3.3.2 and Theorem 3.12. By using 3.3.4 we see

\[
\Theta_t(u_0) = \begin{pmatrix} 0 & 1 \\ \text{id} & \text{id} \end{pmatrix} : T_q(\hat{1}) \to T_q(\hat{1}) \oplus T_q(\hat{1}).
\]

Using the results we are going to explain in Subsection 3.5 we see that this is actually

\[
\Theta_t(u_0) = \begin{pmatrix} 0 & 1 \\ \text{id} & \text{id} \end{pmatrix} : T_q(\hat{1}) \to T_q(\hat{1}) \langle -1 \rangle \oplus T_q(\hat{1}) \langle +1 \rangle.
\]

Remark 3.16. Let \( g = \mathfrak{sl}_{m+1} \) and \( p = p_1 \) be a maximal parabolic subalgebra for \( S_1 \times S_{m-1} \) (for a definition see for example Chapter 9 in [28]). Denote by \( O^p \) the corresponding parabolic category \( O \) (for a definition and some properties see e.g. 9.3 in [28]). Moreover, denote by \( O_0^p \) the principle
block. As Khovanov and Seidel show in Proposition 2.9 of [42] (see also e.g. Example 1.1 in [67]) there is an equivalence of categories $O_p^0 \cong \text{Mod-}A_m$. Thus, by Theorems 3.11 and 3.12 we see that $\mathfrak{T}$ governs $O^0_p$ for all $m$.

The same is true for the projective endofunctors: As Khovanov and Seidel explain after Proposition 2.9 in [42], the functors $U_i$ correspond to the translation through the $i$-th wall (note that, under the equivalence above, the $\mathfrak{sl}_{m+1}$ has Weyl group generators $s_1, \ldots, s_m$ and therefore translation functors indexed by $i = 1, \ldots, m$). Moreover, $U^d_i$ corresponds to the onto and $U^o_i$ corresponds to the out of the wall functors.

Hence, by Theorems 3.11 and 3.12 we can say that these are governed by the combinatorics of the onto $T^\mu_\lambda$, the out of $T^\lambda_\mu$ and the through $\Theta^\lambda_{s\text{ or } t}$ the wall functors in our tilting case.

Another point is, as Khovanov and Seidel explain after Proposition 2.9 in [42], that the path length grading gives rise to a Koszul grading in the sense of [10] on the $A_m$'s, see also Example 1.1 in [67]. The same holds for $A_\infty$. This can either be seen “by hand” or by using a more general theory for (quotients of) not necessary finite quiver algebras that can be found for example in [52].

Remark 3.17. Khovanov and Seidel’s quiver is also related to Khovanov’s arc algebra $H^m$ (for $m$ even) that he introduced in [35] to give an algebraic interpretation of Khovanov homology. The arc algebra categorifies the invariant tensors of $V \otimes^m$ where $V$ is the irreducible two dimensional representation of $U_v$ (note that there are no invariant tensors for $m$ odd). Thus, a relation to Webster’s algebra [72] (coming from categorification of tensor products) is likely.

In addition, he together with Chen in [17] extended this categorification to the full tensor product $V \otimes^m$ by defining a certain subquotient $\prod_k A^{k,m-k}$ (this quotient can be seen as the quasi-hereditary cover of $H^m$). All of these have a topological interpretation as certain algebras consisting of cobordisms, see e.g. Section 2 in [17], and they are graded by the Euler characteristic of these cobordisms.

As Chen and Khovanov explain in Section 3 in [17], the $A^{1,m-1}$-part of this subquotient is graded isomorphic to $A_m$. Thus, by Theorems 3.11 and 3.12 we see that $\mathfrak{T}$ governs $A^{1,m-1}$ for all $m$.

Moreover, this gives a hint for a combinatorial and topological generalization of our work: We conjecture that analoga of the Theorems 3.11 and 3.12 can be proven for certain subquotients of the $\mathfrak{sl}_n$ generalizations of Khovanov’s arc algebra studied recently in e.g. [50], [51], [69] or [70].

For $l = 3$ we can summarize this subsection via

$$
\begin{array}{cccc}
-1 & 0 & 1 & 2 \\
U_q(\mathfrak{sl}_2) & \mathfrak{T}_\lambda(\leq m) & \mathfrak{T}_\lambda(\leq m) & \mathfrak{T}_\lambda(\leq m) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{semisimple} & \text{pMod-}A_m & \text{pMod-}A_m & \text{pMod-}A_m \\
\end{array}
$$

with respect to our picture in 2.4.1.
3.5. The tilting category $\mathcal{X}$ and gradings. In this subsection we discuss the consequences of Theorems 3.11 and 3.12 with respect to the question how the path length grading $l(\cdot)$ (see Definition 3.1) of Khovanov-Seidel’s $\infty$-quiver algebra relates to the tilting category $\mathcal{X}$. We point out that this is non-trivial since Theorems 3.11 and 3.12 only say that $\mathcal{X}_\lambda(\leq \mathfrak{m})$ and $\mathcal{X}_\tau$ are isomorphic to subcategories of $\text{pMod}_A$ and $\text{pMod}_A$.

The latter two have subcategories of graded modules $\text{pMod}_{gr}^A$ and $\text{pMod}_{gr}^A$ : Not all modules over a graded algebra $A$ can be given a grading, e.g. the reader is invited to check that a finite dimensional $\mathbb{Q}(q)[X]$-module $M$ (with $X$ of degree 1) can be given a grading iff $X$ acts nilpotently on $M$. Thus, the question can be visualized as

![Diagram](image)

and we want to describe the shaded region. To do so recall that there is a forgetful functor $\text{forget}: \text{Mod}_{gr}^A \to \text{Mod}^A$ for every graded algebra $A$ that forgets the grading of an $A$-module. Using the forgetful functor, we say an $A$-module $M \in \text{Ob} (\text{Mod}^A)$ is gradable if there exists an $A$-module $\tilde{M} \in \text{Ob} (\text{Mod}_{gr}^A)$ such that $\text{forget}(\tilde{M}) = M$. Note that, by abuse of notation, we usually do not distinguish between $\tilde{M}$ and $M$.

Recall that $\text{Hom}_{\text{Mod}^A}(M, M')$ consists of all (right) $A$-module homomorphisms $f: M \to M'$, while $\text{Hom}_{\text{Mod}_{gr}^A}(M, M')$ consists of (right) $A$-module homomorphisms $\tilde{f}: \tilde{M} \to \tilde{M}'$ as in 3.2.1. Hence, in the same vein as above, we can call an $A$-module homomorphism $f: M \to M'$ gradable, if there exists $\tilde{f}: \tilde{M} \to \tilde{M}'$ such that $\text{forget}(\tilde{f}) = f$.

Thus, using the Theorems 3.11 and 3.12, we make the following definition.

**Definition 3.18.** We call a (left) $U_q$-tilting module $M \in \text{Ob}(\mathcal{X})$ gradable, if $\mathcal{V}_\infty(M)$ is a gradable (right) $A_\infty$-module. We call a $U_q$-intertwiner $f: M \to M' \in \text{Hom}(\mathcal{X}, M')$ gradable, if $\mathcal{V}_\infty(f): \mathcal{V}_\infty(M) \to \mathcal{V}_\infty(M')$ is a gradable $A_\infty$-module homomorphism.

We denote by $\mathcal{X}^{gr}$ the subcategory of $\mathcal{X}$ consisting of gradable $U_q$-tilting modules and gradable $U_q$-intertwiners. We use similar notations and conventions for $\leq \mathfrak{m}$, for fixed $\lambda$, on walls for fixed $\mu$ or categories of projective modules.

Lucky, it turns out to be surprisingly simple in our case. We note that, with respect to the example above and the rather complicated situation for graded $\mathcal{O}^p_0$ discussed in e.g. Theorem 4.1 in [65], the following proposition is quite remarkable. It says that the shaded region above is everything in $\mathcal{V}_\infty(\mathcal{X}_\lambda)$.

**Proposition 3.19.** Every object $M \in \text{Ob}(\mathcal{X}_\lambda(\leq \mathfrak{m}))$, every object $M \in \text{Ob}(\mathcal{X}_\lambda)$ and every object $M \in \text{Ob}(\mathcal{X})$ is gradable. The grading of each irreducible module $T_q(i)$ is unique up to shifts.

**Proof.** We only prove the $\infty$ case and leave the other case to the reader. We start by considering $\lambda \in A_0$ and discuss the semisimple case on the walls afterwards.

It is easy to check that, for any graded algebra $A$ and any two gradable $A$-modules $M$ and $M'$, the direct sum $M \oplus M'$ is also gradable. Thus, by Lemma 2.18 part (a), it is enough to consider
only the \( T_q(i) \)'s. As in the proof of the part (b) of Theorem \ref{thm:main} (or Theorem \ref{thm:finite} in the finite case) we see that \( \mathbb{V}_\infty(T_q(i)) \cong iP \). As discussed in Subsection \ref{subsec:ungraded}, the \( iP \) can be given a grading coming from the path length \( l(\cdot) \). Thus, all the \( T_q(i) \)'s are gradable.

On the walls: Since this is the semisimple case by Lemma \ref{lem:semisimple}, we can just assign to any simple \( \mathbb{V}_\infty(T_q(\mu_i)) \) a degree by demanding that all elements of \( \mathbb{V}_\infty(T_q(\mu_i)) \) are concentrated in this particular degree. Thus, all \( T_q(\mu_i) \) are gradable and concentrated in (up to shifts) degree zero.

The uniqueness (up to shifts) can be proven as in Lemma 2.5.3 in \cite{10}.

**Corollary 3.20.** We have isomorphisms of ungraded(!) categories \( \mathbb{X}_\lambda(\leq m) \cong_{\text{iso}} \mathbb{X}^{gr}(\leq m) \) and \( \mathbb{X} \cong_{\text{iso}} \mathbb{X}^{gr} \). That is, also all morphisms are gradable and the gradings are unique up to a shift.

**Proof.** As before, it is straightforward to check that, for any graded algebra \( A \) and any two gradable \( A \)-homomorphisms \( f \) and \( f' \), the direct sum \( f \oplus f' \) is also gradable. Thus, by Lemma \ref{lem:direct-sum} part (a) again together with Proposition \ref{prop:graded-direct-sum} it is enough to consider only \( \text{Hom}_{U_q}(T_q(i), T_q(i')) \). By Corollary \ref{cor:graded-direct-sum} we see that this space has a basis consisting of a subset of \( i_1, d_i, \) and \( \varepsilon_i \) (in the non-semisimple case who are all gradable by Proposition \ref{prop:graded-direct-sum}). The semisimple case and the uniqueness of the grading follow as before. Note that this gives only rise to an isomorphism after collapsing the grading of \( \mathbb{X}^{gr} \) since the objects in \( \mathbb{X}^{gr} \), as a graded category, exists in multiple copies for all possible shifts via \( \langle s \rangle \) for \( s \in \mathbb{Z} \).

Using Corollary \ref{cor:graded-direct-sum} and abuse of language, we do not distinguish between \( U_q \)-tilting modules or their graded versions.

Moreover, as a consequence of Corollary \ref{cor:graded-direct-sum}, we can choose a *standard grading* (since it will be unique up to shifts) by demanding that simple tilting modules should be concentrated in degree zero and proceed inductively along the quiver. Note that, by Propositions \ref{prop:graded-direct-sum} and \ref{prop:graded-direct-sum} and Corollary \ref{cor:graded-direct-sum} this induces inductively a grading on \( L_q(i), \Delta_q(i) \) and \( \nabla_q(i) \) as well. This choice gives rise to the following graded refinements of Propositions \ref{prop:graded-direct-sum} and \ref{prop:graded-direct-sum} and Corollary \ref{cor:graded-direct-sum}.

**Proposition 3.21.** Suppose \( i = al + b \) for some \( a, b \in \mathbb{N} \) with \( b \leq l - 2 \). Set \( i' = (a + 2)b - b - 2 \). Then there exist exact sequences

\[
0 \longrightarrow L_q(i)\langle +1 \rangle \longrightarrow \Delta_q(i') \longrightarrow L_q(i') \longrightarrow 0; \quad 0 \longrightarrow L_q(i') \longrightarrow \nabla_q(i') \longrightarrow L_q(i)\langle -1 \rangle \longrightarrow 0
\]

and

\[
0 \longrightarrow \Delta_q(i') \longrightarrow T_q(i') \longrightarrow \Delta_q(i)\langle -1 \rangle \longrightarrow 0; \quad 0 \longrightarrow \nabla_q(i)\langle +1 \rangle \longrightarrow T_q(i') \longrightarrow \nabla_q(i') \longrightarrow 0
\]

with degree preserving morphisms.

**Proof.** This follows from our choice for the grading convention and the degree (under the equivalence in Theorems \ref{thm:finite} and \ref{thm:main}) of the morphisms in \( \text{Hom}_{U_q}(T_q(i), T_q(i')) \). To be more precise, we know that the \( T_q(i) \)'s will be, under Soergel’s combinatorial functor \( \mathbb{V}_\infty \), mapped to \( iP \). Then for example, as explained in Proposition \ref{prop:graded-direct-sum} the unique morphism

\[
T_q(i) \rightarrow \nabla_q(i) \rightarrow T_q(i+1) \quad u_i \quad i+1 \quad i
\]

is of degree 1. By our convention we see that moving to the left along the KS \( \infty \)-quiver always increases the degree (starting in degree zero for the first simple module). Thus,

\[
0 \longrightarrow \nabla_q(i)\langle +1 \rangle \longrightarrow T_q(i') \longrightarrow \nabla_q(i') \longrightarrow 0.
\]
All other cases follow similarly and are left to the reader. Note again that moving to the left along the KS $\infty$-quiver always increases the degree, but, by duality, moving right decreases the degree.

**Example 3.22.** Propositions 3.19 and 3.21 provide an inductive way to compute the gradings on the $T_q(i)$’s and thus, on all the modules in $\mathfrak{S}$. Let us do an explicit example. We use our favourite case $l = 3$ again and we consider the block for $1 \in A_0$. We need to proceed inductively.

Hence, we consider first $L_q(1) = \Delta_q(1) = T_q(1) \cong \nabla_q(1)$ from 2.3.1. Since this is a simple $U_q$-module, all of its elements (it is of dimension 2) have to be concentrated in one fixed degree. By our convention, this is degree zero.

We consider $\Delta_q(4)$ next which is 5 dimensional. By Proposition 3.21 we see that it has two basis elements in degree 1 (the ones for $L_q(1)$) and three in degree 0 (the ones for $L_q(4)$). Vice versa for $\nabla_q(4)$: It has two elements in degree $-1$ and three in degree 0.

Next we have to consider $T_q(4)$ (compare to 2.4.1). By Proposition 2.20 this is 7 dimensional. Using Proposition 3.21 and the results for $\Delta_q(4)$ from above, we see that $T_q(4)$ has three elements in degree zero and two elements in degree $-1$ and 1 respectively. The same of course works by using $\nabla_q(4)$ and the other short exact sequence as the reader is (as usual) invited to check. Thus, we have the following table

|       | $l(\cdot) = -1$ | $l(\cdot) = 0$ | $l(\cdot) = +1$ |
|-------|----------------|----------------|----------------|
| $T_q(1)$ | 0              | 2              | 0              |
| $\Delta_q(4)$ | 0              | 3              | 2              |
| $\nabla_q(4)$ | 2              | 3              | 0              |
| $T_q(4)$ | 2              | 3              | 2              |
| $T_q(7)$ | 3              | 5              | 3              |

We encourage the reader to verify the $T_q(7)$ case illustrated above.

**Remark 3.23.** As we already mentioned in Remark 3.17 the grading induced on the tilting category $\mathfrak{S}$ comes from an Euler characteristic on a certain cobordism category associated to Khovanov’s arc algebra $H^m$. Thus, this is a “natural” grading from the viewpoint of topology.

As we mentioned above, we think that this should generalize in type $A$. The degree will there be given by an Euler characteristic on a certain “foam” category associated to the $\mathfrak{sl}_n$-web algebras which generalize Khovanov’s arc algebra. See for example [44], [51], [57] or [70].

**Remark 3.24.** The category $\mathfrak{S}$ is a ribbon (tensor) category and can be truncated to a modular (tensor) category, as noted in Remark 2.19. A intriguing question is, if we can use the grading introduced above to obtain extra informations about the links and tangle invariants coming from the ribbon structure or the Witten-Reshetikhin-Turaev invariants of 3-manifolds. The main problem with this is that $\mathfrak{S}^{gr}$ is not a graded tensor category: One can show as in Example 3.22 that each $T_q(i)$ for $i > l - 1$ and $i \not\equiv -1 \mod l$ (thus, in the non-simple case) is not concentrated in one degree. But by construction, see Subsection 2.3, each $T_q(i)$ appears as a summand of $T_q(1)^{\otimes i}$. Thus, the usual notion of graded tensor product does not work, since $T_q(1)$ is concentrated in one fixed degree as explained above, but $T_q(1)^{\otimes i}$ can not be concentrated in one fixed degree since it contains $T_q(i)$ as a summand.

Hence, in order to make $\mathfrak{S}^{gr}$ into a graded tensor category another idea is missing (but this would be needed to define invariants of links, ribbons or 3-manifolds in the sense of Witten-Reshetikhin-Turaev, see [71]).
We conclude this section by lifting the onto $T^\mu_\lambda$, the out of $T^\lambda_\mu$ and the translation functors $\Theta_s$ and $\Theta_t$ to their graded versions (we use the same notation for these). To understand our notation, recall that a category $\tilde{C}$ is called a $\mathbb{Z}$-graded category, if there exists a functor $\text{deg} : \tilde{C} \to \mathbb{Z}$ where we consider $\mathbb{Z}$ as a category $\mathcal{Z}$ with one object $\text{Ob}(\mathcal{Z}) = \{\bullet\}$ and $\text{End}_\mathcal{Z}(\bullet) = \mathbb{Z}$. In addition, there is a $\mathbb{Z}$-action $\langle s \rangle$ on the objects $O \in \text{Ob}(\tilde{C})$, called shift by $s \in \mathbb{Z}$, such that a morphism $f \in \text{Hom}_\tilde{C}(O, O')$ of degree $d$ is of degree $d + s' - s$ in $\text{Hom}_\mathcal{Z}(\langle s \rangle, O' \langle s' \rangle)$ (this is the fancy way of saying that every morphism in $\tilde{C}$ has a degree which behaves additively under composition and objects can be shifted via $\langle s \rangle$). Moreover, denote by $C$ the ungraded version (after composing with the functor forget : $\tilde{C} \to C$ that forgets the $\mathbb{Z}$-grading). We say a functor $\mathcal{F} : C \to D$ is gradable if there exists a lift $\tilde{\mathcal{F}}$ such that

$$
\begin{array}{ccc}
\tilde{C} & \xrightarrow{\tilde{\mathcal{F}}} & \tilde{D} \\
\text{forget} & & \text{forget} \\
\downarrow & & \downarrow \\
C & \xrightarrow{\mathcal{F}} & D
\end{array}
$$

commutes and we say that $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are the same up to a shift $s \in \mathbb{Z}$ if there is a natural isomorphism\footnote{In the 2-category whose objects are graded categories, whose morphisms are graded functors between these categories and whose 2-morphisms are graded natural transformations.} between $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}' \cdot \langle s \rangle$.

As before, we are usually very careless with our distinction of $\tilde{\mathcal{F}}$ and $\mathcal{F}$. We are lucky again: All the (for us) important functors (onto, out of, through the wall and in general all projective functors) are gradable.

**Corollary 3.25.** The functors $T^\mu_\lambda$, $T^\lambda_\mu \in \text{Ob}(\mathbb{pEnd}(\Sigma))$ are gradable (same in the finite case for $T^\mu_\lambda$, $T^\lambda_\mu \in \text{Ob}(\mathbb{pEnd}(\Sigma_\lambda(\leq m)))$). Moreover, any two lifts of $T^\mu_\lambda$, $T^\lambda_\mu$ are the same up to a shift.

**Proof.** This follows by using parts (f) and (g) of the Theorems 3.11 and 3.12 and the fact that $\mathcal{U}_i^t$ and $\mathcal{U}_i^s$ are graded functors (similarly in the $\infty$ case). The uniqueness up to shifts can again be verified as in Lemma 2.5.3 in \cite{10} since onto and out of the wall functors are indecomposable as functors by Proposition 2.38. \qed

**Corollary 3.26.** The functors $\Theta^\lambda_s$, $\Theta^\lambda_t \in \text{Ob}(\mathbb{pEnd}(\Sigma_\lambda))$ are gradable (same in the finite case for $\Theta^\lambda_s$, $\Theta^\lambda_t \in \text{Ob}(\mathbb{pEnd}(\Sigma_\lambda(\leq m)))$). Moreover, any two lifts of $\Theta^\lambda_s$ or $t$ are the same up to a shift.

**Proof.** We can use Corollary 3.25 since the translation through the wall functors are compositions of the onto $T^\mu_\lambda$ and out of $T^\lambda_\mu$ the wall functors. \qed

We get the following refinement of Proposition 2.32.

**Corollary 3.27.** In the graded category $\Sigma^{gr}$ we have for all $i \in \mathbb{N}$ the following (here we use the conventions $T_q(-1) = T_q(\lambda-1) = T_q(\mu-1) = 0$).

(a) The functors $T^\mu_\lambda$ and $T^\lambda_\mu$ are well-defined (their definition gives tilting modules in the right blocks), (up to shifts) adjoints (left and right) and exact. Thus, $\Theta^\lambda_s$ and $\Theta^\lambda_t$ are exact and graded self-adjoint.

(b) We have

$$
T^\mu_\lambda(T_q(\lambda_i)) \cong \begin{cases} 
T_q(\mu_{i-1}) \langle +1 \rangle \oplus T_q(\mu_{i+1}) \langle +1 \rangle, & \text{if } \mu_i > \lambda_i, \\
T_q(\mu_i) \oplus T_q(\mu_i) \langle +2 \rangle, & \text{if } \mu_i < \lambda_i,
\end{cases}
$$

$$
\text{for } q \in \mathbb{C}, \mu_{i-1} \leq \lambda_i \leq \mu_{i+1}.
$$
Thus, in the spirit of 3.2.1, we set

\[ T^\lambda_{\mu}(T_q(\mu_i)) \cong \begin{cases} T_q(\lambda_{i+1})\langle -1 \rangle, & \text{if } \mu_i > \lambda_i, \\ T_q(\lambda_i)\langle -1 \rangle, & \text{if } \mu_i < \lambda_i. \end{cases} \]

(c) The (left) dead-end relations give \( \Theta^\lambda_{s or t}(T_q(\lambda_i)) \cong 0 \), \( \Theta^\lambda_s(T_q(\lambda_0)) \cong T_q(\lambda_2) \) for \( s \) and for \( t \) we get \( \Theta^\lambda_t(T_q(\lambda_0)) \cong T_q(\lambda_1) \). Moreover, we have

\[ \Theta^\lambda_{s or t}(T_q(\lambda_i)) \cong \begin{cases} T_q(\lambda_{i-1}) \oplus T_q(\lambda_{i+1}), & \text{if } i > 1 \text{ is odd for } s \text{ and even for } t, \\ T_q(\lambda_i)\langle -1 \rangle \oplus T_q(\lambda_i)\langle +1 \rangle, & \text{if } i > 0 \text{ is odd for } t \text{ and even for } s. \end{cases} \]

Proof. This follows directly from Theorem 3.12 parts (d), (f) and (g) together with 3.3.1 and the discussion in Subsection 3.3 for example Lemma 3.8.

**Proposition 3.28.** All functors in \( \mathcal{PEnd}(\Sigma) \) are gradable. Moreover, the grading is unique up to shifts on the indecomposable projective functors.

Proof. The uniqueness on the indecomposable factors can be proven as before. Moreover, by the Krull-Schmidt property of \( \mathcal{PEnd}(\Sigma) \) (see Lemma 2.36) and the easy to deduce fact that any direct sum of gradable, additive functors in an additive category is gradable, it suffices to show that indecomposable projective functors are gradable. By Lemma 2.37 these are given by tensoring with a \( T_q(i) \) for some \( i \in \mathbb{N} \). By Proposition 3.19 all objects of \( \Sigma \) are gradable and, by Lemma 2.18 part (b), the category \( \Sigma \) is preserved by finite tensor products. Thus, the indecomposable projective functors are gradable.

Proposition 3.28 motivates the definition of \( \mathcal{PEnd}(\Sigma^{gr}) \). Each \( F \in \text{Ob}(\mathcal{PEnd}(\Sigma^{gr})) \) is given by tensoring with some graded (!) tilting module \( T_F \) which, by Yoneda, is uniquely associated to \( F \). Thus, in the spirit of 3.2.1 we set

\[ \text{Hom}_{\mathcal{PEnd}(\Sigma^{gr})}(F, F') = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\Sigma^{gr}}(T_F, T_{F'}\langle i \rangle)_0, \]

as graded hom-spaces of natural transformations. This is the same as saying that the category \( \mathcal{PEnd}(\Sigma^{gr}) \) is graded in the sense from above.

In order to state the proposition, we write \( \bigvee_{\infty} F(T_q(\langle \infty \rangle)) \) short for \( \bigoplus_{\lambda} \bigvee_{\infty} F(T_q(\langle \lambda \rangle)) \). This is an \( A_{\infty} \)-bimodule via pre-composition (right) and post-composition (left) with \( F(\cdot) \) or \( F^{(\ell)}(\cdot) \).

Moreover, recall that the superscript \( fs \) indicates that we are only considering finitely supported homomorphisms. We have the following (similar of course for \( \mathcal{PEnd}(\Sigma^{gr}(\leq m)) \) and for a fixed \( \lambda \), that is, for \( \mathcal{PEnd}(\Sigma^{gr}_\lambda) \) and \( \mathcal{PEnd}(\Sigma^{gr}_\lambda(\leq m)) \).

**Proposition 3.29.** Let \( F, F' \) be two functors in \( \mathcal{PEnd}(\Sigma^{gr}_\lambda) \). Then there exists an isomorphism of graded \( \mathbb{Q}(q) \)-vector spaces

\[ \text{Hom}_{\mathcal{PEnd}(\Sigma^{gr}_\lambda)}(F, F') \cong \text{Hom}_{A_{\infty}-\text{Mod}_{\mathcal{A}_{\infty}}}(\bigvee_{\infty} F(T_q(\langle \infty \rangle)), \bigvee_{\infty} F'(T_q(\langle \infty \rangle))). \]

This is an isomorphism of graded rings, if \( F = F' \).
Proof. By Proposition 2.38 it suffices to verify the statement for the functors onto \( \mathcal{T}_\lambda^1 \), out of \( \mathcal{T}_\mu^1 \) and the translation functors \( \Theta_s \) for \( s \in t \). By Theorem 3.12 part (d), (f) and (g) it suffices to show the statement for the corresponding \( U \)'s.

Without loss of generality, we only discuss the “even” case, that is, \( U^e \), \( U^e_t \) and \( U^e_{\infty} \) which correspond to \( \mathcal{T}_\lambda^1 \), \( \mathcal{T}_\mu^1 \) and \( \Theta_s \) respectively. Thus, by additivity, it suffices to show the statement for \( U^e_t \), \( U^e_t \) and \( U^e_t \) for \( i > 0 \) even.

Fix \( i \) and assume that \( i \ll m \). Recall that Khovanov and Seidel have shown in Proposition 2.9 in [42] that \( \text{pMod}_{-}\delta \) and \( \mathcal{O}_0^p \) for \( \delta_{m+1} \) and parabolic \( p \) of type \( S_1 \times S_{m-1} \) are equivalent. Moreover, as they explain below Proposition 2.9 in [42], \( U^e_t \), \( U^e_t \) and \( U^e_t \) correspond to onto, out off and through the \( i \)-th wall functors (the corresponding Weyl group is isomorphic to \( S_m \)).

Note now that the same holds for the graded version of \( \mathcal{O}_0^p \) introduced by Stroppel in [65] as she explains in Section 1 of [67]. Thus, we can use her Theorem 1.12 in [67] to finish the proof. □

**Corollary 3.30.** There is an isomorphism of graded rings

\[
\text{End}_{\text{gr}}(\text{id}) \cong Z(A_{\infty}),
\]

where \( \text{id} \) is the identity functor on \( \mathcal{S}_x^x \), \( Z(A_{\infty}) \) is the center of \( A_{\infty} \) and the endomorphism ring is to be taken in \( \text{pEnd}(\mathcal{S}_x^x) \).

**Proof.** From Proposition 3.10 combined with Proposition 3.29. Note that we get the center \( Z(A_{\infty}) \) and not \( A_{\infty} \) itself because the endomorphism ring from Proposition 3.10 is as \( U_{\infty} \)-intertwiners while the one induced from Proposition 3.29 is as \( A_{\infty} \)-\( \text{pMod}_{\text{gr}} \)-\( A_{\infty} \)-bimodule intertwiners. □

**Remark 3.31.** It is well-known, as Khovanov and Seidel already proved in Proposition 2.8 in [42], that the split Grothendieck group \( \mathcal{K}_0^p(\cdot) = \mathcal{K}_0^p(\cdot) \otimes \mathbb{Z}[v,v^{-1}] \mathbb{Q}(q) \) (viewed as a module over \( \mathbb{Z}[v,v^{-1}] \) where the formal parameter \( v \) comes for the grading) of \( \text{pMod}_{\text{gr}} \) \( A_{\infty} \) categorifies the Burau representation \( B_{m+1} \) of the \( m+1 \)-strand braid group \( B_{m+1} \) (the action of \( B_{m+1} \) is induced via functors constructed from \( U_t \), see Subsection 2.d in [42]). Using Theorem 3.11 we see that the same is true for \( \mathcal{S}_x^x(\leq m) \), that is \( \mathcal{K}_0^p(\mathcal{S}_x^x(\leq m)) \cong B_{m+1} \) \( B_{m+1} \)-modules. Note now that the limit therefore categorifies the corresponding \( \infty \) version of the Burau representation of the braid group \( B_{\infty} \) with \( \infty \)-many strands.

**Remark 3.32.** Fix \( \delta_{m+1} \) and denote by \( p \), the parabolic for \( S_1 \times S_{m-1} \). Moreover, we denote by \( \mathcal{O}_{\text{max}}^p = \bigoplus_{i=0}^m \mathcal{O}_0^p \) the sum. We note that, as Bernstein-Frenkel-Khovanov have conjectured in Conjectures 1-4 in [11] and Stroppel proves in [66], parabolic category \( \mathcal{O}_{\text{max}}^p \) for \( \delta_{m+1} \) (compare to Remark 3.16) can be used to categorify the \( m+1 \)-strand Temperley-Lieb algebra \( TL_{m+1}^v \). The split Grothendieck group \( \mathcal{K}_0^p(\cdot) \) of the category of projective endofunctors on \( \mathcal{O}_{\text{max}}^p \) gives \( TL_{m+1}^v \). The Grothendieck group behaves additive, i.e.

\[
TL_{m+1}^v \cong \mathcal{K}_0^p(\text{pEnd}(\mathcal{O}_{\text{max}}^p)) \cong \bigoplus_{i=0}^m \mathcal{K}_0^p(\text{pEnd}(\mathcal{O}_i^p)),
\]

and thus, \( \text{pEnd}(\mathcal{O}_i^p) \) gives a summand \( \mathcal{T}_L^v_{m+1} \) that corresponds to the “next to highest weight” summand of \( V_{1}^\infty(\delta_{m+1}) \), where \( V_1 \) is the two dimensional vector representation of \( U_v(\delta_2) \), see Section 6 of [17].

\[\text{Remark 3.32.}\] The parameter \( q \) in the notation of Bernstein-Frenkel-Khovanov comes, as Stroppel explains, from the grading of the categories of projective endofunctors.
Thus, as explained in Remark 3.16 by Proposition 3.29, the same is true for $p\text{End}(\Sigma^{gr}_{\lambda}(\leq m))$. Moreover, it is easy to see that the embedding of categories $p\text{End}(\Sigma^{gr}_{\lambda}(\leq m)) \hookrightarrow p\text{End}(\Sigma^{gr}_{\lambda}(\leq m+1))$ gives rise to an embedding $\overline{T}_L^{m+1} \simeq K^{\oplus}_{\Theta}(p\text{End}(\Sigma^{gr}_{\lambda}(\leq m))) \hookrightarrow K^{\oplus}_{\Theta}(p\text{End}(\Sigma^{gr}_{\lambda}(\leq m+1))) \simeq \overline{T}_L^{m+2}$.

If we denote by $\Sigma^{\infty}$ the corresponding summand of the Temperley-Lieb algebra with $\infty$-many strands $i = 0, 1, \ldots$, then, by Proposition 3.10, we see that $K^{\oplus}_{\Theta}(p\text{End}(\Sigma^{gr}_{\lambda})) \simeq \overline{T}_L^{\infty}$ as graded $\mathbb{Q}(q)$-algebras.

Note now the following: As Stroppel explains in Section 7 of [66] or Section 2 of [67], the category of projective endofunctors on $O_{\text{max}}^p$ can be used to define link homologies. Thus, the same holds for $p\text{End}(\Sigma^{gr}_{\lambda})$, although we only see a certain restriction of the link homologies described by Stroppel.

Moreover, it follows from work of Auroux, Grigsby and Wehrli, see [7] and [8], that $\Sigma^{gr}_{\lambda}$ and $p\text{End}(\Sigma^{gr}_{\lambda})$ are related to sutured Khovanov and bordered Floer homology.

On the other hand, we are coming from the modular case. As explained in Remarks 2.19 and 3.24, the category $\Sigma^{gr}_{\lambda}$ can be used to define link and tangle invariants in the root of unity case and the Witten-Reshetikhin-Turaev invariance of 3-manifolds. A connection of these to the link homologies is a very interesting question in the opinion of the authors, but we do not have an interpretation yet.

4. Diagrams for the tilting category

We start this section by recalling in Subsection 4.1 some basic notions from abstract category theory that we use in the following, namely the *additive closure of a category* $\mathcal{C}$, denoted by $\text{Mat}(\mathcal{C})$, and the *Karoubi envelope of $\mathcal{C}$*, denoted by $\text{Kar}(\mathcal{C})$.

The diagrammatic part starts afterwards in Subsection 4.2. We recall Elias’ *dihedral cathedral* $\mathcal{D}(\infty)$ in Definition 4.7 (or a slightly modified version) and show some basic properties of it. This is the underlying diagrammatic for our (graded) tilting category $\Sigma^{gr}_{\lambda}$ and the category of its projective endofunctors $p\text{End}(\Sigma^{gr}_{\lambda})$. In fact, as we point out in Lemma 4.20, a main difference is that the tilting story has a *build in infinity* due to the fact that the center of the category is not a polynomials ring in two variables $\mathbb{Q}(q)[r, g]$ (as for the dihedral cathedral), but $Z(A_{\infty})$ that we can describe diagrammatically by an “infinite” diagram category, see Corollary 4.30.

We introduce the diagrammatic category we are working with, denoted by $\widehat{\Omega}\mathcal{D}(\infty)$, in Subsection 4.3. It is a certain quotient of $\mathcal{D}(\infty)$ that has a *global relation* coming from the (right) *dead-end relation*. We show how this diagram category relates to $\Sigma^{gr}_{\lambda}$ by defining in Definition 4.25 an explicit functor $\mathcal{D}_\infty : \text{Mat}(\widehat{\Omega}\mathcal{D}(\infty)) \rightarrow \Sigma^{gr}_{\lambda}$ (where $\widehat{\Omega}\mathcal{D}(\infty)$ can literally be thought of as $\Omega\mathcal{D}(\infty)$) that is a graded equivalence of categories, see Theorem 4.28. Thus, this gives a diagrammatic description of the tilting category $\Sigma^{gr}_{\lambda}$.

Last but not least: In Subsection 4.5 we give a diagrammatic category $\text{Mat}^{fs}_{\infty}(\widehat{\Omega}\mathcal{D}(\infty))$ for $p\text{End}(\Sigma^{gr}_{\lambda})_\Theta$ as before by giving an explicit functor $\mathcal{D}^{\infty}$ in Definition 4.35.
The category $p\text{End}(\Sigma^g_\lambda)_\Theta$ is the category of natural transformations between compositions of $\Theta_\lambda$ and $\Theta_\lambda'$s. By Proposition 2.38, the Karoubi envelope of $p\text{End}(\Sigma^g_\lambda)_\Theta$ is the category we are interested in, namely the category of all projective endofunctors $p\text{End}(\Sigma^g_\lambda)$. Thus, morally at least, we give a diagrammatic category for the projective endofunctors on $\Sigma^g_\lambda$ and natural transformations between them.

We note again our reading conventions: Since we think of a diagram as an application of certain morphisms or functors to a module or category “living” in a marked face, we read as in Section 3 again everything from right to left.

4.1. Some basic remarks: Diagrammatic categories and additive closures. We consider diagrammatic categories in the following. In fact, what we implicitly mostly consider are so-called string-2-categories. These categories consist of “strings” which are diagrammatic representations of 2-morphisms that are “Poincaré dual” to the “traditional” presentation from higher category theory (see e.g. [45]). String categories, although the idea was around before, were formalized by Joyal and Street in [31] and are our main combinatorial/topological tool in the following. We would like to give a good treatment of these, but in order to keep the length of the paper within (more or less) reasonable boundaries, we encourage the reader to take a look in the rather extensive literature (in the authors opinion, a good start is for example Section 2 in [43]).

The main advantage of these categories can be summarized as follows. They are usually given by generators and local relations making it “easy” to manipulate these 2-hom-spaces combinatorially. Moreover, they usually have a “built-in isotopy invariance” and often even further a Frobenius structure which makes it even more handy to manipulate these 2-hom-spaces. The categories we consider are almost of this nice type: We have only one additional global relation, see 4.3.1 coming from the (right) dead-end relation.

We note that, although string diagram categories have a very natural 2-categorical structure, we phrase everything in terms of 1-categories.

Furthermore, we note that we only use $\mathbb{Q}(q)$-linear categories. Recall that a category $C$ is called $\mathbb{Q}(q)$-linear if each hom-space $\text{Hom}_C(O, O')$ for $O, O' \in \text{Ob}(C)$ has the structure of a $\mathbb{Q}(q)$-vector space and the composition of morphisms is $\mathbb{Q}(q)$-bilinear. It is worthwhile to note that this includes the existence of a zero morphism denoted by $0$. We use $\mathbb{Q}(q)$-linear functors for such categories: Functors which induce $\mathbb{Q}(q)$-linear maps on each hom-space. And, as in Subsection 3.5 we only consider $\mathbb{Z}$-graded categories.

Note the reason why we need $\mathbb{Q}(q)$-linear categories in this section is that we need to invert 2 in order to prove the crucial Lemmata 4.12 and 4.13 and need to invert any prime for the Jones-Wenzl projectors, see 4.15. In fact, as we point out in Remark 4.32 this is “weird” from the viewpoint of the quiver, where we can work over $\mathbb{Z}$ instead of $\mathbb{Q}$.

Moreover, recall the following definition of the additive closure of a category $C$ which we think of as the “minimum” structure that allows a matrix calculus on $C$.

**Definition 4.1. (Additive closure of $C$)** Given a $\mathbb{Q}(q)$-linear category $C$, we define its additive closure, denoted by $\text{Mat}(C)$, to be the $\mathbb{Q}(q)$-linear category consisting of the following.

- The objects are finite (possibly empty), formal direct sums $\bigoplus_{i=1}^N O_i$ with $O_i \in \text{Ob}(C)$.
- Given two objects $O = \bigoplus_{k=1}^N O_k$, $O' = \bigoplus_{k=1}^{N'} O'_k$, then a morphism $F \in \text{Hom}_{\text{Mat}(C)}(O, O')$ is a $N \times N'$ matrix $F = (f_{i,i})$ consisting of morphisms $f_{i,i} \in \text{Hom}_C(O_i, O'_i)$.
• One can add the matrices component-wise and scalar multiplication with elements from \( \mathbb{Q}(q) \) is also component-wise.
• Composition of morphisms is multiplication of matrices. Here one needs to be able to add \textit{and} compose the \( f_{i'i} \)’s.

The reader may deduce for him/herself that this gives a category. He/she needs to check that that “matrix multiplication” in the sense above is well-defined, \( \mathbb{Q}(q) \)-bilinear and associative.

\textbf{Example 4.2.} One can think of additive closures of any category \( \mathcal{C} \) as a generalization of the algebra of matrices \( \bigoplus_{N,N'} \text{Mat}_{N \times N'}(R) \) over a ring \( R \). To be precise, given any ring \( R \), we can see \( R \) as a category \( \mathcal{R} \) consisting of one object \( * \) and \( \text{End}_\mathcal{R}(*) = R \). Then \( \text{Mat}(\mathcal{R}) \) has objects indexed by natural numbers \( N \in \mathbb{N} \) and morphisms \( F: N \to N' \) are \( N \times N' \) matrices \( F = (f_{i'i}) \) with entries in \( f_{i'i} \in \text{Hom}_{\text{Mat}(\mathcal{R})}(N, N') \cong R \). Thus, we have to \( \text{Mat}(\mathcal{R}) \cong \bigoplus_{N,N'} \text{Mat}_{N \times N'}(R) \).

Another category theoretical definition we need is the \textit{Karoubi envelope} \( \text{Kar}(\mathcal{C}) \) of a category \( \mathcal{C} \). The reason for this is that diagrammatically defined categories are \textit{rarely} idempotent complete while the algebraically defined module categories (which we want to describe using diagrams) usually are. Thus, one often has to take the Karoubi envelope of a diagrammatic category. This is a downside of diagrammatic categories: Only in a very few cases can one give the Karoubi envelope by a diagrammatic calculus. In \( \mathfrak{sl}_2 \) on the other hand, this is usually indeed possible (see for example [39] for a quite sophisticated description of the Karoubi envelope for categorified \( U_q(\mathfrak{sl}_2) \) from Khovanov-Lauda [38]). In our case we can use the so-called Jones-Wenzl projectors to give a diagrammatic description, see Definition 4.14.

\textbf{Definition 4.3.} (\textit{Karoubi envelope of} \( \mathcal{C} \)) Let \( \mathcal{C} \) be a category and \( A \in \text{Ob}(\mathcal{C}) \) be an object of \( \mathcal{C} \). Let \( e, e': A \to A \) denote idempotents in \( \text{End}_\mathcal{C}(A) \). The \textit{Karoubi envelope of} \( \mathcal{C} \), denoted by \( \text{Kar}(\mathcal{C}) \), is the following category.

• Objects are ordered pairs \( (A, e) \) consisting of an object \( A \) and an idempotent \( e \in \text{End}_\mathcal{C}(A) \).
• Morphisms \( f: (A, e) \to (A', e') \) in \( \text{Kar}(\mathcal{C}) \) are all morphisms \( f: A \to A' \) of \( \mathcal{C} \) such that the equations \( f = f \circ e = e' \circ f \) hold.
• Compositions are induced by compositions in \( \mathcal{C} \).

It is immediate that this is indeed a category and that the identity of an object \( (A, e) \) is \( e \) itself. Moreover, there is an embedding of categories \( \text{im}: \mathcal{C} \to \text{Kar}(\mathcal{C}) \), called \textit{the image}, that sends \( A \) to \( (A, \text{id}) \) and \( f: A \to A' \) to \( f: (A, \text{id}) \to (A', \text{id}) \).

\textbf{Example 4.4.} An example related to our context is the following. Define a category \( \mathcal{T}^\otimes \) whose objects are all (possible empty) tensor products \( T_q(1)^{\otimes i} \) and morphisms are \( U_q \)-intertwiners between these tensor products.

Thus, an object would be for instance \( T_q(1) \otimes T_q(1) \) from Example 2.21. This is not indecomposable as we already calculated in Example 2.21 because \( T_q(1) \otimes T_q(1) \cong T_q(0) \oplus T_q(2) \). Hence, \( \mathcal{T}^\otimes \) is \textit{not} idempotent complete: The identity on \( T_q(1) \otimes T_q(1) \) splits, but not in \( \mathcal{T}^\otimes \).

By construction, that we have recalled before Proposition 2.20 every \( T_q(i) \) appears as a direct summand of \( T_q(1)^{\otimes i} \) and, by part (c) of Proposition 2.20, the \( T_q(i) \)’s are a complete set of indecomposable \( U_q \)-modules. Thus, \( \text{Kar}(\mathcal{T}^\otimes) \cong \mathbb{T} \). Morally: Having an explicit description of \( \text{Kar}(\mathcal{C}) \) is the “same” as being able to decompose objects into indecomposable objects.

4.2. \textbf{The dihedral cathedral} \( \mathcal{D}(\infty) \). Recall from Subsection 2.4 that the affine Weyl group in our case is \( W_i = \langle s,t \rangle \cong D_\infty \). For the reader familiar with [19]: Be careful that Elias’ “root of unity
case” does not correspond to our root of unity case from Section 2. This comes from the duality we are describing here, since our root of unity corresponds to the affine case. Moreover, we slightly rescale his “barbell forcing relation”, see BF2 4.2.3.

Following [19], we encode the two different generators using two colors. Our convention, that is different from the one used by Elias, is that \(s\) is displayed in red and \(t\) in green. The “third color” is for \(1 \in W_l\) which is displayed as white colored or empty. Following Elias we start by defining Soergel graphs. These are certain isotopy classes of colored, planar graphs that we, by abuse of language, usually always consider as given by a certain representative.

**Definition 4.5.** (Soergel graph) A pre-Soergel graph \(\tilde{G}\) is a colored, planar graph embedded in a rectangle \([0, 1] \times [0, 1]\) whose bottom (or source) boundary is in \([0, 1] \times \{0\}\) and whose top (or target) boundary is in \([0, 1] \times \{1\}\). The only vertices are univalent (called dots) or trivalent. The edges of a Soergel graph are colored by red (or \(s\)) and green (or \(t\)) such that at each trivalent vertex all adjacent edges have the same color. We display dots and trivalent vertices locally as

- **Dots:** \(l(\cdot) = 1\)
- **Trivalent vertices:** \(l(\cdot) = -1\)

where we tend (as above) not to display the \(s\) and \(t\). Moreover, the illustration above is locally: Soergel graphs also include for example horizontal reflections of the ones above. In addition, given a pre-Soergel graph \(\tilde{G}\), we define its degree \(l(\tilde{G})\) to be the sum of the local degrees as above.

A Soergel graph \(G\) is an equivalence class of pre-Soergel graphs modulo boundary preserving isotopies of colored, planar graphs. Note that the degree is still well-defined for a Soergel graph and we denote it by \(l(G)\). We point out that the empty diagram, that we denote by \(0\), is a Soergel graph and of degree \(l(0) = 0\).

Each Soergel graph gives rise to a bottom sequence \(b(G) = b_i \ldots b_2 b_1\) and a top sequence \(t(G) = t_i' \ldots t_2 t_1\) of colors \(b_k, t_k \in \{s, t\}\) by reading the colors at the bottom or top boundary from right to left respectively. We also allow the empty sequence, if the boundary is empty (at bottom or top). We call a Soergel graph (or a local piece of it) floating, if both boundaries are empty. We call floating Soergel graphs consisting of only one edge barbells.

Our reading conventions are thus from right to left (we think of the pictures as applying functors to a module) and bottom to top. Moreover, we think of Soergel graphs as embedded into a rectangle (although we rarely illustrate the boundary), and thus, it make sense to speak about faces of Soergel graphs. Each Soergel graph has a unique rightmost face \(F_r\) and a unique leftmost face \(F_l\).

**Example 4.6.** An example of two representatives of a Soergel graph \(G\) are

- \(\tilde{G}\)
- \(\tilde{G}\)

Here \(l(G) = 4\), \(b(G) = ssstts\) and \(t(G) = ststss\). The graph above has one (green) barbell. We have marked the unique right and leftmost faces with \(F_r\) and \(F_l\) respectively.
Note that, since a Soergel graph $G$ completely determines $b(G)$ and $t(G)$, we do not display these in our pictures anymore.

We define now Elias’ dihedral cathedral $\mathcal{D}(\infty)$. The reader familiar with [19] should be careful since we use a slightly different convention.

**Definition 4.7. (Elias’ dihedral cathedral $\mathcal{D}(\infty)$)** We consider the $\mathbb{Q}(q)$-linear, graded, monoidal category called the *free* dihedral cathedral, denote by $\mathcal{D}(\infty)_f$, consisting of the following.

- **Objects** $\text{Ob}(\mathcal{D}(\infty)_f)$ are finite sequences $x = x_i \ldots x_2 x_1$ with $x_i \in \{s, t\}$. Moreover, the empty sequence $\emptyset$ is also an object.
- The space of morphisms $\text{Hom}_{\mathcal{D}(\infty)_f}(x, x')$ for $x, x' \in \text{Ob}(\mathcal{D}(\infty)_f)$ is the $\mathbb{Q}(q)$-linear span of all Soergel graphs $G$ with $b(G) = x$ and $t(G) = x'$.
- Composition of morphisms $G' \circ G$ is defined by gluing $G'$ on top of $G$ if the boundary matches and zero otherwise.
- The spaces $\text{Hom}_{\mathcal{D}(\infty)_f}(x, x')$ are graded $\mathbb{Q}(q)$-vector spaces where the degree of a Soergel graph $G$ is given by $l(G)$.
- The monoidal product $\otimes$ is, for $x = x_i \ldots x_2 x_1$ and $x' = x'_i \ldots x'_{2} x'_{1}$, given by concatenation $x' \otimes x = x' x$ and for Soergel graphs $G' \otimes G$ via placing $G'$ to the left of $G$.

The dihedral cathedral $\mathcal{D}(\infty)$ is the quotient category of $\mathcal{D}(\infty)_f$ obtained by taking the quotient of the hom-spaces by the following local (!) relations and their color-inverted (red $\leftrightarrow$ green) counterparts.

The **two Frobenius relations**:

(4.2.1) Frob1: $\begin{array}{c} \text{Frob2:} \end{array}$

The **needle relation**:

(4.2.2) Needle: $\begin{array}{c} \end{array}$

The **barbell forcing relations**:

(4.2.3) BF1: $\begin{array}{c} \end{array}$ BF2: $\begin{array}{c} \end{array}$

Note that the isotopy invariance can be locally displayed via the **isotopy relations**:

(4.2.4) Iso1: $\begin{array}{c} \end{array}$ Iso2: $\begin{array}{c} \end{array}$

together with a horizontal reflection of the two relations.

We also note that all relations are homogeneous with respect to $l(G)$ and thus, $\mathcal{D}(\infty)$ inherits the grading from $\mathcal{D}(\infty)_f$. The $\otimes$ also carries over to $\mathcal{D}(\infty)$ without difficulties.

Moreover, we use the following convention: We call elements of $\text{Hom}_{\mathcal{D}(\infty)_f}(x, x')$ Soergel graphs and, on the other hand, elements of $\text{Hom}_{\mathcal{D}(\infty)}(x, x')$ Soergel diagrams (for all possible
A Soergel diagram $G$ that has an internal cycle that bounds an empty face is zero.

**Proof.** Use repeatedly the first Frobenius relations $\text{Frob1}$ as illustrated below (note that each step reduces the number of adjacent edges of the face and we have indicated where to use the first Frobenius relation $\text{Frob1}$ in the last step)

\[
\cdots \xrightarrow{\text{Frob1}} \xrightarrow{\text{Frob1}} \xrightarrow{\text{Frob1}} \xrightarrow{\text{Frob1}} \xrightarrow{\text{Needle}} 0
\]

until we can use the needle relation $\text{4.2.2}$ to see that we get zero. \qed

A **tree** is a connected Soergel graph $G$ with no cycles. Moreover, we call a Soergel diagram **reduced** if all of its connected components are trees or barbells and all barbells are in the rightmost face.

The following is Proposition 5.19 in [19] converted to our conventions.

**Proposition 4.10.** Each Soergel diagram can be written as a $\mathbb{Q}(q)$-linear sum of reduced Soergel diagrams. Thus, reduced Soergel diagrams form spanning sets of the hom-spaces of $\mathcal{D}(\infty)$.

**Proof.** Note that it follows from a repeated application of the Frobenius relations $\text{4.2.1}$ that all floating trees are barbells and, by combining Lemma 4.9 with BF1 and BF2 $\text{4.2.3}$, all other floating Soergel diagrams are $\mathbb{Q}(q)$-linear combinations of barbells.

Thus, we can restrict to Soergel diagrams whose only floating components are barbells.

Now, given a Soergel diagram $G$, we can first use BF1 and BF2 $\text{4.2.3}$ to express $G$ as a direct sum of Soergel diagrams $\tilde{G}_i$ (with $i = 1, \ldots, k$) with only barbells in the rightmost face. By Lemma 4.9, all Soergel diagrams $G_i$ with an internal cycle are zero now.

This finishes the proof. \qed

We point out the following corollary. This can be compared to Corollary 5.20 in [19] and the corresponding statement, in our case for $\mathcal{F}^{gr}$, will later be completely different, see Lemma 4.20.
Corollary 4.11. We have $\mathbb{Q}(q)[r, g] \cong \text{End}_{\mathcal{D}(\infty)}(\emptyset)$ as graded rings where the degree of $r, g$ is 2.

Proof. By Proposition 4.10 the map that sends $g \mapsto \text{green barbell}$ and $r \mapsto \text{red barbell}$ is an isomorphism of graded rings. To see this note that a bunch of barbells do not satisfy any extra relations (the barbell forcing relations 4.2.3 do not give anything new).

The following lemma is similar to 5.18 in [19] and very useful.

Lemma 4.12. Soergel diagrams satisfy

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\hline
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 2} \\
\hline
\end{array}
\end{align*}
\]

Proof. Use Frob1 4.2.1 on the right side followed by BF1 4.2.3 for the middle edge.

This implies the useful fact that we can focus on alternating sequences (the case where the sequence is a reduced word in $\mathcal{D}(\infty)$ of red and green if we go to $\text{Mat}(\mathcal{D}(\infty))$). That is, we have the following (similar to 5.19 in [19]).

Lemma 4.13. There are isomorphisms in $\text{Mat}(\mathcal{D}(\infty))$ of the form\footnote{Recall that $\text{Mat}(\mathcal{D}(\infty))$ is a graded category as in Subsection 3.5 and has therefore a shift action on its objects.}

$rr \cong r(-1) \oplus r(+1)$ and $gg \cong g(-1) \oplus g(+1)$.

Proof. The isomorphism for the red case is induced by

\[
rr \xrightarrow{\begin{pmatrix} 1 \cdot \\ \frac{1}{2} \cdot \\ \end{pmatrix}} r(-1) \oplus r(+1) \xrightarrow{\begin{pmatrix} \frac{1}{2} \cdot \\ 1 \cdot \\ \end{pmatrix}} rr,
\]

where the reader is invited to check (with Lemma 4.12 for “right-left” and Lemma 4.9 together with BF1 4.2.3 for “left-right”) that the two matrices above are inverses.

Note that the scaling of the entries is crucial: The one with the barbell has to be multiplied by $\frac{1}{2}$, but the other one not. The green case is analogous and left to the reader.

We need the Jones-Wenzl projectors in the following. Note that we do not need non-alternating Jones-Wenzl projectors, since the projectors should correspond to indecomposable modules or functors and for example $\Theta_t \circ \Theta_t$ is not indecomposable (see Corollary 2.34).

But since we are going to work in the additive closure, Lemma 4.13 ensures that we are free to sometimes only consider alternating sequences in the following.

Definition 4.14. (Jones-Wenzl projectors) Denote by $x_{k-tst}$ a sequence $x_k \ldots x_2 x_1$ of length $k$ that alternates in the colors red and green and starts with $x_1 = g$ (thus, ends in $r$ iff $k$ is even).

We define for each $i \geq 0$ the $i$-th green Jones Wenzl projector $\text{JW}_{g_i}$ recursively as the element of $\text{End}_{\mathcal{D}(\infty)}(x_{i-tst})$ obtained via the convention that $\text{JW}_{g_0} = \emptyset : \emptyset \to \emptyset$, $\text{JW}_{g_1}$ is id : $g \to g$, $\text{JW}_{g_2}$

47
is id: \( rg \to rg \) and recursion rule for \( i > 2 \) given by

\[
JW^g_i = \cdots
\]

where we have only illustrated the case for odd \( i \).

We use a similar definition for the \( i \)-th red Jones Wenzl projector \( JW^r_i \).

**Example 4.15.** The first four (green) Jones-Wenzl projectors are

\[
\begin{align*}
JW^g_0 &= 0 \\
JW^g_1 &= \cdots \\
JW^g_2 &= \cdots \\
JW^g_3 &= \cdots 
\end{align*}
\]

For each \( i' > 2 \) we call a diagram of the form

a **green alternating** \( i' \)-pitchfork (here \( i' = 7 \)). We denote it by \( P^g_{i'} \) and similarly for \( P^r_{i'} \), which is the \( i' \)-th red alternating pitchfork. We note that \( b(P^g_{i'}) \) as well as \( b(P^r_{i'}) \) consist of \( i' \) elements. Moreover, we write \( D_i(P^g_{i'}) \) for a bigger diagram with \( i \geq i' > 2 \) that has at the bottom somewhere a green alternating pitchfork, i.e.

(4.2.5)

\[
D_i(P^g_{i'}) = D \circ (id_x P^g_{i'} id_{x'})
\]

for some sequences \( x, x' \) and some diagram \( D \). Similarly for red again.

**Lemma 4.16.** Let \( i \geq i' > 2 \). Then \( D_i(P^g_{i'}) \circ JW^g_i = 0 \). Similarly for red.

**Proof.** This follows recursively. To see the first step \( i = i' = 3 \), we compose \( P^g_3 \) with the second term of \( JW^g_3 \) from Example 4.15 and obtain

\[
-\frac{1}{2} \cdot \cdots = -\frac{1}{2} \cdot 2 \cdot \cdots = -
\]
by using BF2 4.2.3 and Lemma 4.9 for the first step and Frob2 4.2.1 for the second. This is exactly minus the composite of $P^g_3$ with the first term of $JW^{g_3}$ and thus, $D_3(P^g_3) \circ JW^{g_3} = P^g_3 \circ JW^{g_3} = 0$.

Now we use induction on $i$ and fix $i' = 3$. If $i > 3$ and $D_i(P^g_3)$ has $x \neq \emptyset$ (see 4.2.5), then the recursion rule from Definition 4.14 (the 3-pitchfork is on top of the smaller box that corresponds to the $i - 1$-th Jones-Wenzl projector), show by induction that we get zero. If, on the other hand, $x = \emptyset$, then we use a similar argument as above (we leave it to the reader who should be warned that in this case the fraction is important). Hence, we always have $D_i(P^g_3) \circ JW^{g_i} = 0$.

But the case $i' = 3$ suffices as the following illustrates.

Here we used Frob2 4.2.1 “backwards” on a bigger alternating pitchfork followed by BF1 4.2.3. We see now two smaller pitchforks and thus, the case $i' = 3$ suffices to verify the lemma.

It is easy to deduce that the Jones-Wenzl projectors are (degree zero) idempotents of $x_{i-\text{sts}}$ and $x_{i-\text{sts}}$, respectively. We associate these to the corresponding alternating sequences. Moreover, by Lemma 4.13 we can, by going to Mat($D(\infty)$), associate to each $x \in \text{Ob}(D(\infty))$ a (shifted) direct sum of Jones-Wenzl projectors. We denote this sum that consists of these projectors for $x$ by JWx. This motivates the following category.

Definition 4.17. Denote by Mat($D(\infty)$) the full subcategory of Kar(Mat($D(\infty)$)) generated by im(JWx) for all $x \in \text{Ob}(D(\infty))$.

Diagrammatically Mat($D(\infty)$) is given very similar to Mat($D(\infty)$), but has some extra relations. For example, Lemma 4.16 says that, whenever we see a pitchfork at the bottom (or its reflection at the top) boundary, then the corresponding Soergel diagram is zero. In fact, one should think that each sequence $x \in \text{Ob}(D(\infty))$ has a (sum of) Jones-Wenzl projector(s) attached to it. We do not picture them in order to save space and hope that the reader does not get confused.

4.3. The quotient $QD(\infty)$. We are now able to define the category whose combinatorics and diagrammatic govern the tilting category $D^{\text{gr}}$. Following Williamson, we call it the more than anti-spherical quotient (or short the quotient) and denote it by $QD(\infty)$.

Definition 4.18. (The quotient) We denote by $QD(\infty)_f$ the $\mathbb{Q}(q)$-linear full subcategory of $D(\infty)$, called “free quotient”, consisting of the following.

- Objects $\text{Ob}(QD(\infty)_f)$ are finite sequences $x = x_i \ldots x_2 x_1$ with $x_k \in \{s, t\}$. Moreover, the empty sequence $\emptyset$ is also an object.
- The hom-space $\text{Hom}_{QD(\infty)_f}(x, x')$ for $x, x' \in \text{Ob}(QD(\infty)_f)$ is the $\mathbb{Q}(q)$-linear span of all Soergel graphs $G$ with $b(G') = x$ and $t(G') = x'$ whose rightmost face is marked with the dead-end symbol.
- Composition $G' \circ G$ is defined by glueing $G'$ on top of $G$ if the boundary matches. The dead-end symbol is just a marker and behaves as an idempotent under composition and can not be moved from the rightmost face.
- The spaces $\text{Hom}_{QD(\infty)_f}(x, x')$ are graded $\mathbb{Q}(q)$-vector spaces with degree given by $l(G)$.
- The morphisms in $\text{Hom}_{QD(\infty)_f}(x, x')$ satisfy the relations of the corresponding morphisms (without the dead-end marker) is $\text{Hom}_D(x, x')$ given in Definition 4.7.
The quotient $\mathcal{Q}_D(\infty)$ is the quotient of $\mathcal{Q}_D(\infty)_f$ by the following extra relations which we call the dead-end relations:

\begin{align}
& (4.3.1) \\
& \text{DE1: } \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = 0 \\
& \text{DE2: } \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = 0
\end{align}

We stress that we do not allow the color inverted counterparts of $(4.3.1)$ (otherwise the whole story would be rather trivial!). All relations are homogeneous with respect to $l(G)$ and thus, $\mathcal{Q}_D(\infty)$ inherits the grading from $\mathcal{D}(\infty)$. But since the choice of a fixed “dead-end” face is global (!), the monoidal structure $\otimes$ from Definition 4.7 does not carry over.

In order to distinguish from the dihedral cathedral: We call the morphisms in $\mathcal{Q}_D(\infty)$ marked Soergel diagrams.

**Example 4.19.** An example of a Soergel diagram that is zero as a marked Soergel diagrams is

\[
\begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = 2 \\
\end{array}
- \begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} = 0
\end{array}
\]

To see this, note that, after applying BF1 4.2.3, the two remaining parts are both zero due to DE2 4.3.1 for the left and DE1 4.3.1 for the right (after isotopy).

To get started again, we note that following easy to deduce consequence of the dead-end relations. It says, by Corollary 4.11, that we do not have an interesting action of $\text{End}_{\mathcal{D}(\infty)}(\emptyset)$ - in contrast to what takes place in Elias’ dihedral cathedral.

**Lemma 4.20.** Each marked Soergel diagram can be written as a $\mathbb{Q}(q)$-linear sum of marked, reduced Soergel diagrams. Thus, marked, reduced Soergel diagrams without barbells form spanning sets of the hom-spaces of $\mathcal{Q}_D(\infty)$. The right action of $\text{End}_{\mathcal{D}(\infty)}(\emptyset)$ by placing barbells in the rightmost face is the trivial action.

**Proof.** Use repeatedly BF1 and BF2 4.2.3 to shift all existing barbells to the rightmost region. Then use the two dead-end relations DE1 and DE2 4.3.1 to see that only terms without barbells are not killed. The remaining diagrams are reduced by Proposition 4.10.

Note that, as a consequence of Lemma 4.20 a marked, reduced Soergel diagram does not have any floating components anymore.

As before, we are usually working in the following category. Note that, by definition of $\mathcal{Q}_D(\infty)$ as a certain quotient of $\mathcal{D}(\infty)$, we can easily adopt the notion (and notation) for the Jones-Wenzl projectors $\text{JW}_x$ from Subsection 4.2.

**Definition 4.21.** Denote by $\text{Mat}(\mathcal{Q}_D(\infty))$ the full subcategory of $\text{Kar}(\text{Mat}(\mathcal{Q}_D(\infty)))$ generated by $\text{im}(\text{JW}_x)$ for all $x \in \text{Ob}(\mathcal{Q}_D(\infty))$.

Thus, by using Lemma 4.13 and DE1 4.3.1 we usually can focus on alternating sequences $x$ that start with $g$ in the following.

Given a marked, reduced Soergel diagram $G$, we call two edges $E_1, E_2$ face sharing, if $E_1$ and $E_2$ are adjacent to some common face. Marked, reduced Soergel diagrams $G$ (with color
alternating boundary) with face sharing edges of the same color come in four types, where we, by Lemma 4.16, ignore diagrams with pitchforks. The first two are

(4.3.2) Type 1: \[ \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 1
\end{array}
\end{array} \quad \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k' \geq 1
\end{array}
\end{array} \quad \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 1
\end{array}
\end{array} \quad \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k' \geq 1
\end{array}
\end{array}
\]

where the dots \( \cdots \) mean an arbitrary (possible empty) diagram in between and we do not require the picture to be symmetric. Moreover, \( k, k' \) are odd in Type 1 and even in Type 2. We should note that only the first edge has to be green (otherwise the diagram is zero by DE1 4.3.1), but we allow possible color inversion of the rest of the picture (that is, after the rightmost dots).

The other two are

(4.3.3) Type 3: \[ \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 2
\end{array}
\end{array} \quad \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 0
\end{array}
\end{array} \quad \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 2
\end{array}
\end{array} \quad \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 0
\end{array}
\end{array}
\]

where the difference between the cases 1-2 and 3-4 is that 1-2 have face sharing edges for (some) non-extremal faces, while 3-4 have face sharing edges for the leftmost face. We should note that type 3 should also include the horizontal reflection of the first diagram in 4.3.3.

We note the following important lemma. It says that face sharing edges of the same color are sufficient for \( G \) to be zero in the cases 1, 2 and 3.

**Lemma 4.22.** If a marked, reduced Soergel diagram \( G \in \text{Hom}_{\text{Mat}(\widehat{QD}(\infty))}(x, x') \) has two face sharing edges of one of the three types 1, 2 or 3, then it is zero.

**Proof.** A case-by-case check. We assume that there is at least one such face. Recall that, while working in \( \text{Mat}(\widehat{QD}(\infty)) \), implicitly (we do not picture them) there are Jones-Wenzl projectors at the bottom and top.

Type 1: This follows via

\[ \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 1
\end{array}
\end{array} = \frac{1}{2} \cdot \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 1
\end{array}
\end{array} + \frac{1}{2} \cdot \begin{array}{c}
\cdots \\
\begin{array}{c}
\rightarrow \\
k \geq 1
\end{array}
\end{array} = 0 \]

which is an application of Frob2 4.2.1 followed by an isotopy (to get from the diagram in 4.3.2 to the leftmost diagram), then BF1 4.2.3, Frob1 4.2.1 and Lemma 4.16.

Type 2: This follows similarly to type 3 below and is left to the reader.

Type 3: Note that the number of bottom edges at the end does not matter (as the reader is invited to verify) and we thus, by simplicity, assume that there are none of these. In addition, we only
illustrate the case with two dotted edges at the top since all the others will be similar. And, without loss of generality, we assume that this is the only face with adjacent edges of the same color.

We proceed by induction on $j$, where $j > 0$ is the minimal number of cuts that a line drawn from the rightmost face to the face in question needs. The case $j = 1$ can be deduced via

$$= \frac{1}{2}.$$

where we used Frob2 4.2.1 followed by an isotopy and BF1 4.2.3. Note that the rightmost diagram is zero by Lemma 4.16. Thus, we continue by BF1 4.2.3

$$= - \frac{1}{2}.$$

where the last diagram is zero due to DE1 4.3.1. All others are zero because of Lemma 4.16, but we need the extra relation DE1 4.3.1 for the last one to be zero.

This shows that our starting diagram was already zero. The induction $j > 1$ works now in a similar fashion where we perform the same sequence of moves until we are in a situation as the last step above. Then we can use induction, since we have created a face with edges of the same color, but “closer” to the rightmost face. This finishes the proof.

We conclude the subsection by giving the diagrammatic analogue of Corollary 2.29 and of 3.3.4. We point out that the rescaling of the generators “$\tilde{\varepsilon}_i = 2^i \varepsilon_i$” and “$\tilde{d}_i = 2^i d_i$” is mysterious.

**Corollary 4.23.** Let $i, i' \in \mathbb{N}$. Then we have the following for $i, i' \neq 0$.

$$\text{Hom}_{\text{Mat}(\hat{QD}(\infty))}(x_{i - \text{tst}}, x_{i' - \text{tst}}) \cong \begin{cases} 
\mathbb{Q}(q) i \oplus \mathbb{Q}(q) \varepsilon_i, & \text{if } |i - i'| = 0, \\
\mathbb{Q}(q) u_i, & \text{if } i' - i = 1, \\
\mathbb{Q}(q) d_i, & \text{if } i - i' = 1, \\
0, & \text{if } |i - i'| > 1,
\end{cases}$$

with diagrams for $i = 1, 2, \ldots$ (and possible color inverted left sides)

$$i = \begin{array}{c}
\cdots \\
\varepsilon_i = \frac{1}{2^i} \\
\cdots \\
x_i, x_{i-1}, x_1
\end{array}, \quad \begin{array}{c}
\cdots \\
u_{i-1} = \begin{array}{c}
\cdots \\
x_i, x_{i-1}, x_1
\end{array} \\
\cdots \\
d_i = \frac{1}{2^i} \\
\cdots \\
x_i, x_{i-1}, x_1
\end{array}, \quad \begin{array}{c}
\cdots \\
x_i, x_{i-1}, x_1
\end{array}$$

of degree $l(i) = 0$, $l(u_i) = l(d_i) = 1$ and $l(\varepsilon_i) = 2$. In addition, $\text{Hom}_{\text{Mat}(\hat{QD}(\infty))}(\emptyset, \emptyset)$ is one dimensional and spanned by the empty diagram $0$ of degree $l(0) = 0$. 

52
Proof. A case-by-case check using Lemma 4.22. That $\text{Hom}_{\text{Mat}(\Sigma \mathcal{D}(\infty))}(\emptyset, \emptyset) \cong \mathbb{Q}(q) \emptyset$ follows from DE1 and DE2 4.3.1 since no floating diagrams can exist.

If $|i - i'| > 1$, then each Soergel diagram has a pitchfork or is of Type 3 4.3.3. Both are zero by Lemmata 4.16 and 4.22.

If $|i - i'| = 1$, then, by Lemma 4.22 again, only diagrams of the form $u_i$ or $d_i$ as above can be non-zero and these are clearly non-zero.

Last but not least: If $|i - i'| = 0$, then, again by Lemma 4.22, only diagrams of the form $i$ or $\varepsilon_i$ as above can be non-zero (which they clearly are).

All of these diagrams are linear independent due to degree reasons. □

The Soergel diagrams from above satisfy the quiver relations!

**Proposition 4.24.** Using the notation from Corollary 4.23, we have

$$u_i \circ u_{i-1} = 0 = d_i \circ d_{i+1}, \ i = 1, 2, \ldots \text{ and } d_{i+1} \circ u_i = \varepsilon_i = u_{i-1} \circ d_i, \ i = 1, 2, \ldots$$

and the right dead-end relation

$$d_1 \circ u_0 = 0.$$

Proof. Recall that we compose by stacking on the top. Then the dead-end-relation is exactly DE2 4.3.1 and we leave it to the reader to draw the pictures. For $u_i \circ u_{i-1} = 0$ the corresponding diagrams are

![Diagram](image-url)

which is zero, since it is of Type 3 4.3.3. The case $0 = d_i \circ d_{i+1}$ is similar and left to the reader. Moreover $u_{i-1} \circ d_i = \varepsilon_i$ can be verified directly as

![Diagram](image-url)

while $d_{i+1} \circ u_i = \varepsilon_i$ follows via

![Diagram](image-url)

where we use BF2 4.2.3 together with Lemma 4.22 to see that the only surviving term is the “broken” one from BF2 4.2.3. This finishes the proof. □
4.4. \(\hat{\mathcal{Q}}D(\infty)\) and the tilting category \(\mathfrak{T}^g\): Its an equivalence! We are now ready to define the functor \(\mathcal{D}_\infty\) that relates the diagrammatic category \(\hat{\mathcal{Q}}D(\infty)\) and the graded tilting category \(\mathfrak{T}^g\).

Recall that \(\hat{\mathcal{Q}}D(\infty)\) (and therefore \(\text{Mat}(\hat{\mathcal{Q}}D(\infty))\)) is given by generators and relations. Thus, it suffices to define \(\mathcal{D}_\infty\) on the generators and we show afterwards that our assignment is well-defined.

Given a sequence \(x \in \text{Ob}(\hat{\mathcal{Q}}D(\infty))\) of the form \(x_i \ldots x_1\) with \(x_k \in \{r, g\}\) we denote by \(\ell(x)\) the total number of color changes (including the first from empty to green), e.g. \(\ell(g) = \ell(gg) = 1\), \(\ell(ggrgrgrr) = 6\) and \(\ell(x) = \ell(x)\) iff \(x\) is alternating.

**Definition 4.25. (Diagrams for tilting modules)** Let (as always \(\lambda \in \mathcal{A}_0\)). Define a \(\mathbb{Q}(q)\)-linear functor

\[
\mathcal{D}_\infty: \text{Mat}(\hat{\mathcal{Q}}D(\infty)) \to \mathfrak{T}_\lambda^g
\]

of graded categories by the following convention.

On objects (we treat degree shifted objects in the evident way):

- Send the empty sequence \(\emptyset\) to \(T_q(0)\).
- If \(x \in \text{Ob}(\hat{\mathcal{Q}}D(\infty))\) is of the form \(x_i \ldots x_2x_1\) with \(x_k \in \{r, g\}\), then define

\[
\mathcal{D}_\infty(x) = p_{T_q(\lambda_{\ell(x)})} \circ \Theta_{x_i} \circ \cdots \circ \Theta_{x_2} \circ \Theta_{x_1} T_q(0),
\]

where we use the convention \(r = s, g = t\) and \(p_{T_q(\lambda_{\ell(x)})}\) projects to the \(T_q(\lambda_{\ell(x)})\) part.

- Send a general \(x \in \text{Ob}(\text{Mat}(\hat{\mathcal{Q}}D(\infty)))\) to the direct sum of the images of its components.

Recall that the five local generators are identities, dotted edges (up and down) and trivalent vertices (merges and splits) in green and red respectively. Thus, all marked Soergel diagrams will be compositions of diagrams of the form \(D'GD\) with identity diagrams \(D\) and \(D'\) and a generator \(G\) in between. On morphisms:

- We describe the image of marked Soergel diagrams inductively from right to left. That is, we describe what the functor does if one has already an identity diagram \(D: x \to x\) and \(\mathcal{D}_\infty(D): M_k \to M_k\) (with \(\mathcal{D}_\infty(x) \cong M_k\)) and one adds a generator to the left.
- By construction of the image on objects and Corollaries \(\ref{2.34}\) and \(\ref{3.27}\), \(M_k\) will be just one \(T_q(\lambda_k)\) with some multiplicity and grading shifts and some \(k \in \{0, \ldots, i\}\) giving two cases, namely \(k = \tilde{\ell}(x)\) even or odd. That is, we have

\[
M_k = T_q(\lambda_k)(s_1) + \cdots + T_q(\lambda_k)(s_k).
\]

Note that the translation functors \(\Theta_s\) and \(\Theta_t\) act on the summands as in Corollaries \(\ref{2.34}\) and \(\ref{3.27}\) separately. To simplify notation, we write \(\Theta_i(M_k) = M_{k+1}\) if \(k\) is even and \(\Theta_i(M_k) = M_k(-1) + M_k(+1) = \bar{M}\) if \(k\) is odd and vice versa for \(\Theta_s\). Moreover, in the case \(k = 0\) we send any red generator to zero.
- The inductive description has four cases. We call these green-even, green-odd, red-even and red-odd where we only give the list for the first two since the other two are similar with exchanged conventions for even and odd. Each case has five sub-cases (for the five generators) giving twenty cases in total. We write \(D_0\) (even) and \(D_1\) (odd) for the two different cases. An important note: Since the \(M_k\)’s could already consist of multiple, shifted copies of \(T_q(\tilde{i})\)’s, the entries in the matrices below are shorthand notations for matrices themselves.
• Basic pieces: Send the empty sequence $0$ to $\operatorname{id}: T_q(0) \to T_q(0)$. Moreover, for a green identity we assign 
\[ D_0 \mapsto \operatorname{id}: M_{k+1} \to M_{k+1} \quad \text{and} \quad D_1 \mapsto \begin{pmatrix} \operatorname{id} & 0 \\ 0 & \operatorname{id} \end{pmatrix}: \tilde{M}_k \to \tilde{M}_k \]

For the up dotted edge we assign (recall the rescaling $\tilde{\varepsilon}_k = 2^k \varepsilon_k$)
\[ D_0 \mapsto u_k: M_k \to M_{k+1} \quad \text{and} \quad D_1 \mapsto \begin{pmatrix} \tilde{\varepsilon}_k \\ \operatorname{id} \end{pmatrix}: M_k \to \tilde{M}_k \]

For the down dotted edge we assign (recall the rescaling $\tilde{d}_k = 2^k d_k$)
\[ D_0 \mapsto \tilde{d}_{k+1}: M_{k+1} \to M_k \quad \text{and} \quad D_1 \mapsto \begin{pmatrix} \operatorname{id} & \tilde{\varepsilon}_k \end{pmatrix}: \tilde{M}_k \to M_k \]

For the merges and splitters in the even case we assign
\[ D_0 \mapsto (0 \quad \operatorname{id}): \tilde{M}_{k+1} \to M_{k+1} \quad \text{and} \quad D_0 \mapsto \begin{pmatrix} \operatorname{id} \\ 0 \end{pmatrix}: M_{k+1} \to \tilde{M}_{k+1} \]

For the merges in the odd case we assign
\[ D_1 \mapsto \begin{pmatrix} 0 & 0 & \operatorname{id} & 0 \\ 0 & 0 & 0 & \operatorname{id} \end{pmatrix}: \tilde{M}_k \to \tilde{M}_k \]

(with $\tilde{M}_k = M_k(-2) \oplus M_k(0) \oplus M_k(0) \oplus M_k(+2)$) and last but not least
\[ D_1 \mapsto \begin{pmatrix} \operatorname{id} & 0 \\ 0 & \operatorname{id} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}: \tilde{M}_k \to \tilde{M}_k. \]

• “Fill up the generators to the left”: For each red identity strand to the left of the local generators from above apply a $\Theta_s$ to the elements of the matrices component-wise and likewise for green strands and $\Theta_t$. This can be made explicit as explained in Example 3.15.
• If $f$ is a homomorphism $f \in \operatorname{Hom}(\mathcal{QD}(\infty))$, then decompose it into generators and define $D_\infty(f)$ to be the composition of its local pieces.
• Extend linearly for a general $F \in \operatorname{Hom}(\mathcal{Mat}(\mathcal{QD}(\infty)))$.

**Example 4.26.** The functor is explicitly defined on any generator, but rather cumbersome to work out for bigger cases due to the inductive definition. This is in fact the power of the diagrammatic category, that is, the sophisticated matrix calculus in $\mathfrak{A}_\lambda^\mathbb{Z}$ can be done locally by manipulating diagrams combinatorial or topological.
But it is easy to see (with respect to Section 3) how the assignment is for alternating sequences. For example, we have
\[ \Rightarrow u_0 : T_q(0) \to T_q(1) \text{ and } \Rightarrow \tilde{d}_1 : T_q(1) \to T_q(0) \]
and similar for all other diagrams of this kind. Moreover, we have
\[ \Rightarrow (0 \ 	ext{id}) : T_q(1)(-1) \oplus T_q(1)(+1) \to T_q(1) \]
and
\[ \Rightarrow \begin{pmatrix} \text{id} & \tilde{e}_1 \\ 0 & 0 \end{pmatrix} : T_q(1) \to T_q(1)(-1) \oplus T_q(1)(+1) \]
where we see that even the degree is preserved (on both sides the morphisms are of degree $-1$). Moreover, the entries in the matrices of the list above are in general already matrices. For example, the matrix that corresponds to (here $M_1 = T_q(1)$)
\[ \Rightarrow \begin{pmatrix} \text{id} & \tilde{e}_1 \\ 0 & 0 \end{pmatrix} : \tilde{M}_1 \to \tilde{M}_1 \]
To see this note that, following the list above, the two strands (without the right dot) correspond to a $2 \times 2$-identity matrix, while the left part corresponds to $(\text{id} \ 	ilde{e}_1)$. Thus, the rules give the matrix as above, because we have to replace the entries inductively. Likewise for the horizontally flipped diagram.

The following lemma is one of the most important ingredients of this subsection.

**Lemma 4.27.** The functor $D_\infty : \text{Mat}(\widehat{\mathcal{O}}(\infty)) \to \mathcal{S}_\lambda^{gr}$ is a well-defined functor of $\mathbb{Q}(q)$-linear, graded categories.

**Proof.** Note that it is immediate, under the assumption that we have already proven that the assignment from Definition 4.25 does not depend on the representative we pick for a marked Soergel diagram, that $D_\infty$ is $\mathbb{Q}(q)$-linear. That it, under the same assumption, preservation of degrees can be read off from the list in Definition 4.25 as already indicated in Example 4.26.

Moreover, note that, by part (b) of Corollary 2.34 and part (c) of Corollary 3.27, the assignment is well-defined on objects $x$, since we send $x$ to a repeated application of $\Theta_s$ and $\Theta_t$ to the trivial $U_q$-module $T_q(0)$ together with a projection to the leading (that is, with the highest $i$) factor $T_q(i)$. Thus, without taking relations into account, we get a well-defined functor from the free dihedral cathedral $\mathcal{D}(\infty)_f$ (or rather its additive closure $\text{Mat}(\mathcal{D}(\infty)_f)$) to $\mathcal{S}_\lambda^{gr}$ since our assignment for the generating Soergel diagrams are $U_q$-intertwiners.

Thus, the main part is to show that the relations are satisfied. This is now a case-by-case check where we only do a few as examples and leave the rest to the reader.

Frobenius and isotopy relations: A good point is, although we can also (in principle) prove it directly from the assignment, that we do not have to check the Frobenius relations Frob1 and Frob2 4.2.1 and the isotopy relations Iso1 and Iso2 4.2.4. The reason for this is the following.
By part (a) of Corollary 3.27 we see that $\Theta_s$ and $\Theta_t$ are (graded) self-adjoint functors. By a more general principle such functors always give rise to a Frobenius structure, see e.g. Lemma 3.4 in [54] or the discussion in Section 1 in [36], and thus, satisfies Frobenius and isotopy relations. Our assignment is a decategorification of this: We evaluate compositions of these functors at a specific module. Thus, in order to show that the Frobenius relations Frob1 and Frob2 4.2.1 and the isotopy relations Iso1 and Iso2 4.2.4 are preserved, we can just adopt the proof of Lemma 3.4 in [54] to our set-up.

Thus, let us just check one case of it in our set-up, namely Iso1 4.2.4. This reads locally as

$$D_\infty \left( \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) = D_\infty \left( \begin{array}{c} \bullet \\ \bullet \end{array} \right) = D_\infty \left( \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right)$$

Let us for simplicity of notation assume that the right face is actually the marked face. Then the middle is $d_1: T_q(1) \to T_q(0)$. The three pieces for the right side are

$$T_q(1) \to T_q(1)^{(1)} \oplus T_q(1)^{(1)} \to (0 \quad \text{id}) \to T_q(1) \quad d_1 \to T_q(0)$$

which gives the correct result. The left side gives

$$T_q(1) \to T_q(1)^{(1)} \oplus T_q(1)^{(1)} \to (0 \quad \text{id}) \to T_q(1) \quad d_1 \to T_q(0)$$

where the first matrix follows as in Example 3.15. This gives the correct result again. We leave the other cases (for the Frobenius and isotopy relations) to the reader.

Needle: Again for simplicity of notation, we assume that we are in the even case. We show something stronger, namely

$$D_\infty \left( \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) = 0$$

This follows from

$$M_{k+1} \to (\text{id}) \quad \bar{M}_{k+1} \to (0 \quad \text{id}) \quad M_{k+1}$$

57
The odd case is similar. This clearly implies the needle relation.

Dead-end 1+2: DE1 follows by construction since a red strand adjacent to the rightmost face means applying $\Theta_s$ to $T_q(0)$ which is zero by part (c) of Proposition $2.32$. The DE2 relations is

$$D_\infty \begin{pmatrix} 1 & 1 \\ \epsilon_1 & \epsilon_1 \end{pmatrix} = 0$$

This follows from

$$T_q(0) \xrightarrow{\mu_0} T_q(1) \xrightarrow{\tilde{d}_1} T_q(0)$$

which is the dead-end relation in the tilting case (up to a scalar!).

Barbell forcing 1: We only illustrate the “harder” case (the one with equal colors) and leave the other to the reader. And again, for simplicity of notation, we focus on

$$D_\infty \begin{pmatrix} 1 & 1 & 1 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 \end{pmatrix} = 2 \cdot D_\infty \begin{pmatrix} 1 & 1 & 1 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 \end{pmatrix} - D_\infty \begin{pmatrix} 1 & 1 & 1 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 \\ \epsilon_1 & \epsilon_1 & \epsilon_1 \end{pmatrix}$$

The general case follows completely similar, but with bigger matrices. We have already explained in Example $4.26$ how to obtain the left two matrices. Thus, we obtain

$$\begin{pmatrix} 1 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \text{id} & \tilde{\epsilon}_1 \\ 0 & 0 & \text{id} & \tilde{\epsilon}_1 \end{pmatrix} \circ \begin{pmatrix} \epsilon_1 & 0 \\ \text{id} & 0 \\ 0 & \tilde{\epsilon}_1 \\ 0 & \text{id} \end{pmatrix} = \begin{pmatrix} 2\tilde{\epsilon}_1 & 0 \\ 0 & 2\tilde{\epsilon}_1 \end{pmatrix}$$

for the left side. The middle (without the factor 2) can be directly read off as

$$\begin{pmatrix} \tilde{\epsilon}_1 \\ \text{id} \end{pmatrix} \circ \begin{pmatrix} \text{id} & \tilde{\epsilon}_1 \end{pmatrix} = \begin{pmatrix} \tilde{\epsilon}_1 & 0 \\ \text{id} & \tilde{\epsilon}_1 \end{pmatrix}$$

The rightmost term is the result of applying $\Theta_t$ to $2\tilde{\epsilon}_1$ (which is the composite of the diagram for the first two strands). This can be computed as explained in Section $3$. Thus, we obtain

$$\begin{pmatrix} 2\tilde{\epsilon}_1 & 0 \\ 0 & 2\tilde{\epsilon}_1 \end{pmatrix} = 2 \cdot \begin{pmatrix} \tilde{\epsilon}_1 & 0 \\ \text{id} & \tilde{\epsilon}_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 \cdot \text{id} & 0 \end{pmatrix}$$

The other case is left to the reader.

Barbell forcing 2: This is very similar to BF1 and we only sketch it here. As before we only consider one particular case (the same as above for BF1) and leave the others for the reader to verify (where the reader has to be careful with the subscripts for the $\tilde{d}_k$’s!). What makes this case similar to the other is that, up to scalars, two of the four matrices for this case are the same as
before, namely the two of the rightmost part of BF2 4.2.3. The remaining two are now

\[
D_{\infty} \left( \begin{array}{c}
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\end{array} \right) \quad \text{and} \quad D_{\infty} \left( \begin{array}{c}
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\end{array} \right)
\]

These two cases give the matrices (be careful with the scaling!)

\[
\left( \begin{array}{cc}
2 \cdot \tilde{\varepsilon}_1 & 0 \\
0 & 2 \cdot \tilde{\varepsilon}_1 \\
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{cc}
0 & 0 \\
2 \cdot \text{id} & 0 \\
\end{array} \right)
\]

and the equation BF2 4.2.3 then reads as

\[
\left( \begin{array}{cc}
2 \cdot \tilde{\varepsilon}_1 & 0 \\
0 & 2 \cdot \tilde{\varepsilon}_1 \\
\end{array} \right) = \left( \begin{array}{cc}
0 & 0 \\
2 \cdot \text{id} & 0 \\
\end{array} \right) + 2 \left( \begin{array}{cc}
\tilde{\varepsilon}_1 & 0 \\
\text{id} & \tilde{\varepsilon}_1 \\
\end{array} \right) - 2 \left( \begin{array}{cc}
0 & 0 \\
2 \cdot \text{id} & 0 \\
\end{array} \right)
\]

This finishes the proof, since this is clearly a functor of graded categories (the reader may check that the assignment in Definition 4.25 is degree preserving). □

We are finally ready to state one of our main theorems.

**Theorem 4.28.** *(Diagram categories for \( T_{gr} \)) The functor \( D_{\infty} : \text{Mat(\( \widehat{QD}(\infty) \))} \rightarrow T_{gr} \) is an equivalence of graded categories.

**Proof.** We have to show that \( D_{\infty} \) is full, faithful and essentially surjective.

Essentially surjective: We have to show that, given some arbitrary object \( M \in \text{Ob}(T_{gr}) \), then there is an object \( x \in \text{Ob}(\text{Mat(\( \widehat{QD}(\infty) \)))} \) such that \( D_{\infty}(x) \cong M \). To see this, note that, by part (c) of Proposition 2.20, it is enough to verify this for indecomposable \( U_q \)-modules \( T_q(i) \) in the ungraded setting (using Proposition 3.19 the same is still true in the graded setting).

Now, because of our construction and part (b) of Corollary 2.34, we see that \( D_{\infty}(x_{i-tst}) \cong T_q(i) \) which shows that the functor is essentially surjective.

Fully faithful: By Lemmata 4.27 and 4.13 it is enough to show that

\[
\text{Hom}_{\text{Mat(\( \widehat{QD}(\infty) \)))}(x, x') \cong \text{Hom}_{T_{gr}}(D_{\infty}(x), D_{\infty}(x'))
\]

holds as graded \( Q(q) \)-vector spaces for alternating sequences \( x = x_{i-tst} \) and \( x'_{i'-tst} \). Thus, it is enough to show that

\[
\text{Hom}_{\text{Mat(\( \widehat{QD}(\infty) \)))}(x, x') \cong \text{Hom}_{T_{gr}}(T_q(i), T_q(i'))
\]

Lucky, we have already computed both sides: The right side was computed in Corollary 2.29 and the left in Corollary 4.23. By our construction, the (graded) isomorphism is induced by the assignment \( i \mapsto i \), \( u_i \mapsto u_i \), \( d_i \mapsto d_i \) and \( \varepsilon_i \mapsto \tilde{\varepsilon}_i \) whenever this makes sense. □

**Corollary 4.29.** The category \( \text{Mat(\( \widehat{QD}(\infty) \))) \) is its own Karoubi envelope.

**Proof.** This follows from Theorem 4.28 because in \( T_{\lambda} \cong T_{\lambda} \) every module decomposes into a direct sum of (shifted copies of) \( T_q(i) \)'s. □

As already mentioned above in and before Lemma 4.20, in contrast to Elias’ dihedral cathedral, the grading in our case comes from the (more sophisticated) graded ring \( A_{\infty} \) instead of \( Q(q)[r, g] \).
Corollary 4.30. (Diagram categories for the center $\text{End}_{gr}(\text{id})$) The graded ring $\text{End}_{gr}(\text{id})$ of natural transformations in $\text{End}(\Sigma^g_{\lambda})$ of the identity functor $\text{id}$ is given by the diagonal part of finitely supported (only a finite number of non-zero entries) matrices

\[(4.4.1)\quad F: \emptyset \oplus g \oplus rg \oplus grg \oplus rgrg \oplus \ldots \rightarrow \emptyset \oplus g \oplus rg \oplus grg \oplus rgrg \oplus \ldots\]

of marked Soergel diagrams with no identity diagram entry. In contrast, $\text{End}_{U_q(T_q(\infty))}^{fs}(T_q(\infty)) \cong A_{\infty}$ is given by all such matrices and not just the diagonal part.

Proof. An alternating sequence of red and green of length $i$ corresponds under the equivalence from Theorem 4.28 to $T_q(i)$, since, by Proposition 4.24, the quiver relations are satisfied. This module corresponds under the isomorphisms from Corollary 3.30 to $iP$. Thus, the claim follows by Theorem 4.28 and Proposition 3.29 since (graded) equivalent categories have (graded) isomorphic centers and the observation that an $F$ as in (4.4.1) is a natural transformation if and only if it commutes with all other such $F$’s (compare also to (4.5.2)). That the center of the matrix ring as stated above is then an easy to deduce fact: The $i$ are not central (as already noted after Definition 3.1) while the $\varepsilon_i$ compose with everything to zero (and are therefore in the center). □

Example 4.31. An immediate consequence of Corollary 4.30 is that we can also give a diagram category for the center in the cut-off case $\leq m$. In those cut-off cases everything will be finite which makes it easy to give a full description of the center in terms of diagrams. We note that $1 = 0 + 1 + \cdots + m$ will be central as well (recall that $A_m$, in contrast to $A_{\infty}$, is unital).

For example if $m = 2$, then the center is fully classified by

|       | $\emptyset$ | $g$ | $rg$ |
|-------|------------|-----|------|
| $\emptyset$ | $\emptyset$ | $\bullet$ | None |
| $g$ | $\bullet$ | $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$ | |
| $rg$ | None | $\begin{array}{c} \bullet \\ \bullet \end{array}$ | $\begin{array}{c} \bullet \\ \bullet \end{array}$ |

where we read the marked Soergel diagrams as starting at the sequences in the rows and ending at the sequences in the columns. We point out again that $A_m$ also includes the off-diagonal entries, but $Z(A_m)$ does not. But that the table above gathers around the main diagonal is in fact, by Corollaries 2.29 and 4.23, no coincidence.

Remark 4.32. As we already said in Footnote 6 in Section 2, if the reader prefers to relax the conditions to work over $\mathbb{Q}(q)$, then this is actually a problem in Section 2 since the indecomposable tilting modules and their hom-spaces are much more complicated in positive characteristic. For the KS quiver algebras from Section 3 working over $\mathbb{Z}$ is not a huge problem, while for the diagrammatic category $\mathcal{QD}(\infty)$ we, in addition, need primes to be invertible due to Lemma 4.13 and the Definition of the Jones-Wenzl projectors from Definition 4.14. Thus, we have to work over $\mathbb{Q}$, also the quiver approach suggests that this can be avoided.
4.5. Diagram categories for $p\text{End}(X^\mathbb{C}_r)$. Motivated by Proposition 3.29 and Corollary 4.11 we define in this subsection the category $\text{Mat}^{\mathbb{C}}_{\infty}(\hat{\Sigma}\mathbb{D}(\infty))$ that gives rise to a diagrammatic presentation of the projective, graded endofunctor category $p\text{End}(X^\mathbb{C}_r)$.

In fact, we have to extend the notion of the additive closure from Definition 4.1 slightly: We denote by $\text{Mat}(\mathcal{C})$ literally the same category as $\text{Mat}(\mathcal{C})$, but we allow countable direct sums as objects and finitely supported matrices as morphisms.

We carefully distinguish between red $r$ and green $g$ on one hand and $s, t \in W_i$ on the other. We think of an alternating sequence (of length $i \geq 0$) of the former $x_{i-grg}$ to correspond (under Theorem 4.28) to $T_q(i)$ and of any sequence of the latter $x = x_1 \ldots x_2 x_1$ to correspond to $\Theta_x$ with $\Theta_x = \Theta_{x_1} \cdots \Theta_{x_1}$. Using this interpretation, we are going to make Proposition 3.29 explicit.

We define $\infty$ to be the sequence of alternating $x_{i-grg}$ sequences

$$\infty = (\ldots, rgrg, grg, rg, g), \quad \infty_i = x_{i-grg} \quad \text{for} \quad i = 0, 1, \ldots,$$

where we, following our convention of this section, read the vector from right to left.

Moreover, given a finite sequence $x = x_1 \ldots x_2 x_1$ with $x_i \in \{s, t\}$ we define

$$x \cdot \infty = (\ldots, x rgrg, x grg, x rg, x g, x), \quad (x \cdot \infty)_i = xx_{i-grg} \quad \text{for} \quad i = 0, 1, \ldots,$$

where each entry is given by concatenation. In addition, we define

$$xx_{i-grg} = \bigoplus_{\ell} x_{i-grg}^\ell \cdot s_{\ell}^\prime$$

where the multiplicity $m_{\ell}$ is the multiplicity of $T_q(\ell)$ in $\Theta_x(T_q(i))$ given inductively by part c of Corollary 3.27 (from where we also get the shifts $s_{\ell}$). Thus, we see $xx_{i-grg}$ as an object $\text{Ob}(\text{Mat}(\hat{\Sigma}\mathbb{D}(\infty)))$. Note that $xx_{0-grg} = 0$ if $x$ starts with $r$ because of $\Theta_s(T_q(0)) = 0$.

**Definition 4.33.** ($\text{Mat}^{\mathbb{C}}_{\infty}(\hat{\Sigma}\mathbb{D}(\infty))$: An infinity of diagrams) Denote by $\text{Mat}^{\mathbb{C}}_{\infty}(\hat{\Sigma}\mathbb{D}(\infty))$ the $\mathbb{Q}(q)$-linear, graded category consisting of the following.

- Objects $\text{Ob}(\text{Mat}^{\mathbb{C}}_{\infty}(\hat{\Sigma}\mathbb{D}(\infty)))$ are finite sequences $x = x_1 \ldots x_2 x_1$ with $x_i \in \{s, t\}$ (plus shifts). Moreover, the empty sequence $\emptyset$ is also an object.
- The space of morphisms $\text{Hom}_{\text{Mat}^{\mathbb{C}}_{\infty}(\hat{\Sigma}\mathbb{D}(\infty))}(x, x')$ for $x, x' \in \text{Ob}(\text{Mat}^{\mathbb{C}}_{\infty}(\hat{\Sigma}\mathbb{D}(\infty)))$ is the $\mathbb{Q}(q)$-linear span of finitely supported (only a finite number of non-zero entries) matrices

  $$F = (F_{k,k'}, k, k' \in \mathbb{N}) \in \text{Hom}_{\text{Mat}(\hat{\Sigma}\mathbb{D}(\infty))}(x \cdot \infty, x' \cdot \infty)$$

  of marked Soergel graphs $F_{k,k'}: (x \cdot \infty)_k \rightarrow (x' \cdot \infty)_{k'}$.
- The diagonal part of $F$ consists of matrices $F_{k,k}: (x \cdot \infty)_k \rightarrow (x' \cdot \infty)_k$.
- Composition of morphisms is multiplication of matrices.
- The spaces $\text{Hom}_{\text{Mat}^{\mathbb{C}}_{\infty}(\hat{\Sigma}\mathbb{D}(\infty))}(x, x')$ are graded $\mathbb{Q}(q)$-vector spaces where the degree is given by $l(F_{k,k'})$ (the degree or length from Definition 4.7) for each entry of matrices $F$.

\[\text{Double } \infty \text{ symbol is justified: There are “a lot of” natural transformations due to the fact that the center of the category is already huge.}\]
Remark 4.34. We are interested in a diagrammatic description of $\mathfrak{pEnd}(\check{\Sigma}_A^g)$. As before in Corollary 4.30 the category $\text{Mat}^{fs}(\check{\Sigma}(\infty))$ is “richer”: It consist of matrices whose entries are matrices of marked Soergel diagrams. But $\mathfrak{pEnd}(\Sigma_A^g)$ only sees some diagonal matrices. The off-diagonal entries (which are gathered around the main diagonal by Corollaries 2.29 and 4.23) exists only in $\text{Mat}^{fs}(\check{\Sigma}(\infty))$

Denote by $\mathfrak{pEnd}(\Sigma_A^g)_\Theta$ the full subcategory of $\mathfrak{pEnd}(\Sigma_A^g)$ consisting of only compositions of $\Theta_s$ and $\Theta_t$. Note that, by Proposition 4.38, we have $\text{Kar}[\mathfrak{pEnd}(\Sigma_A^g)_\Theta] \cong \mathfrak{pEnd}(\Sigma_A^g)$.

The reader may compare this to Example 4.4. Moreover, in the spirit of Remark 4.34, we denote by $\text{Mat}^{fs}(\check{\Sigma}(\infty))_c$ the subcategory of $\text{Mat}^{fs}(\check{\Sigma}(\infty))$ whose hom-spaces are

(4.5.1) $\text{Hom}_{\text{Mat}^{fs}(\check{\Sigma}(\infty))_c}(x, x') = \{ F \in \text{Hom}_{\text{Mat}^{fs}(\check{\Sigma}(\infty))}(x, x') \mid FG = G'F \}$,

for $G \in \text{Hom}_{\text{Mat}^{fs}(\check{\Sigma}(\infty))}(x, x)$ and $G' \in \text{Hom}_{\text{Mat}^{fs}(\check{\Sigma}(\infty))}(x', x')$ such that entry-wise there exists a $U_q$-intertwiner $f : M \to M'$ with $D_\infty(G_{\epsilon_i}) = \Theta_x(f)$ and $D_\infty(G'_{\epsilon_i}) = \Theta_{x'}(f)$. The reader may check that $\text{Mat}^{fs}(\check{\Sigma}(\infty))_c$ is really a subcategory.

Definition 4.35. We define a functor

$$D_\infty : \mathfrak{pEnd}(\Sigma_A^g)_\Theta \to \text{Mat}^{fs}(\check{\Sigma}(\infty))_c$$

on objects (we treat shifts again in the evident way) and morphisms as

$$\Theta_x \mapsto x, \quad \text{and} \eta : \Theta_x \to \Theta_{x'}, \eta \tau_q(\iota) : \Theta_x T_q(i) \to \Theta_{x'} T_q(i) \mapsto \text{diag}(F) = (F^i)_{i \in \mathbb{N}} : x \to x',$$

where diag($F$) is a diagonal matrix consisting of the various $F^i$’s, and, for each $i$, the matrix of marked Soergel diagrams $F^i$ is component-wise given by $i \mapsto i, u_i \mapsto u_i, d_i \mapsto d_i$ and $\varepsilon_i \mapsto \varepsilon_i$ for all suitable indices $i$ and marked Soergel diagrams as in Corollary 4.23.

Lemma 4.36. The functor $D_\infty : \mathfrak{pEnd}(\Sigma_A^g)_\Theta \to \text{Mat}^{fs}(\check{\Sigma}(\infty))_c$ is a well-defined functor between $\mathbb{Q}(q)$-linear, graded categories.

Proof. As in Lemma 4.27 the linearity is clear, if we already know that the functor is well-defined. In addition, the assignment is clearly degree preserving and, since the quiver relations are satisfied, see Proposition 4.24 a well-defined component-wise.

To see that it is well-defined note that $\tau_q(\iota)$ is a matrix consisting of $U_q$-intertwiners between the direct summands of $\Theta_x T_q(i)$ and $\Theta_{x'} T_q(i)$ (for each $T_q(i)$). These are sums of $i, u_i, d_i$ and $\varepsilon_i$ by the isomorphism of Proposition 3.10. To see that the relations are satisfied, note that, given any $U_q$-intertwiner $f : M \to M'$, the naturality of a transformation $\eta : \Theta_x \to \Theta_{x'}$ says that

(4.5.2) $\begin{array}{ccc}
\Theta_x M & \xrightarrow{\Theta_x f} & \Theta_x M' \\
\eta_M & & \eta_M' \\
\Theta_{x'} M & \xrightarrow{\Theta_{x'} f} & \Theta_{x'} M'
\end{array}$

commutes. Thus, the matrices coming from the assignment in Definition 4.35 satisfy the condition 4.5.1. \hfill $\square$

\footnote{In fact, how to decompose a $\Theta_x$ into indecomposable functors is a question that could be (at least in principle) answered using the diagrammatic description from this subsection.}
We note that the morphisms in $\text{Mat}^{\infty}_c(\widehat{\mathcal{O}}(\infty))$ are matrices of matrices and the morphisms in $\text{Mat}^{\infty}_c(\widehat{\mathcal{O}}(\infty))_{\sigma}$ are certain diagonal (compare to 4.5.2) matrices in $\text{Mat}^{\infty}_c(\widehat{\mathcal{O}}(\infty))$.

**Theorem 4.37.** *(Diagram categories for $p\text{End}(\mathcal{T}_{\lambda}^g)$)* The functor $D^\infty$ is an equivalence of graded categories.

**Proof.** As in Theorem 4.28, we have to show that $D^\infty$ is full, faithful and essentially surjective.

With the work already done, this is not a big deal anymore. That $D^\infty$ is essentially surjective follows from the definition of the objects in $\text{Mat}^{\infty}_c(\widehat{\mathcal{O}}(\infty))_{\sigma}$. That it is faithful is a direct consequence of Proposition 3.29 combined with Theorem 4.28. That $D^\infty$ is full is just a direct comparison of 4.5.1 and 4.5.2. \(\square\)

**Example 4.38.** Let us consider $x = \emptyset$ and $x' = ts$. Then $\Theta_x = \text{id}$ and $\Theta_{x'} = \Theta_t \circ \Theta_s$. We denote by $T_q(i)_{\text{sh}}$ the module $T_q(i)(\text{sh})$ and likewise for $g$ and $r$. Using Corollary 3.27 we see that $\Theta_{x'}(T_q(0)) \cong 0$ and $\Theta_{x'}(T_q(1)) \cong T_q(1) \oplus T_q(3)$, $\Theta_{x'}(T_q(2)) \cong T_q(1)_{-1} \oplus T_q(1)_{+1} \oplus T_q(3)_{-1} \oplus T_q(3)_{+1} \ldots$.

Thus, $x \cdot \infty = \infty$ and $x' \cdot \infty = (\ldots, g \oplus g \oplus grg, g \oplus grg, r)$. Then Theorem 4.28 says that a natural transformation in $\text{Hom}_{p\text{End}(\mathcal{T}_{\lambda}^g)}(\text{id}, \Theta_x)$ is determined by what it does component-wise $\infty_i \rightarrow x' \cdot \infty_i$. For $i = 0$ there are no marked Soergel diagrams since starting in red is zero. For $i = 1$ we have (up to scalars) one possible diagram $g \rightarrow g$ and thus one possible non-zero matrix

$$
\begin{pmatrix}
g & \begin{pmatrix}1 \cr 0\end{pmatrix} \\
g \oplus grg
\end{pmatrix}
$$

since there is no marked Soergel diagram from green to green-red-green (see Lemma 4.22). For $i = 2$ we have two possible non-zero diagrams and thus two possible non-zero matrices

$$
\begin{pmatrix}
rg & \begin{pmatrix}0 \cr 0\end{pmatrix} \\
g_{-1} \oplus g_{+1} \oplus grg_{-1} \oplus grg_{+1}
\end{pmatrix}
$$

where we have illustrated the sum of these two above. To determine which configurations of matrices are legal, one picks any of $i, u_i, d_i$ and $\varepsilon_i$ and checks which configurations ensure that (here with $u_i$ as an illustration)

$$
\begin{pmatrix}
g & \begin{pmatrix}u_1 \cr 1\end{pmatrix} \\
g \oplus grg \Theta_{x' u_1}
\end{pmatrix}
\begin{pmatrix}r g \\
g_{-1} \oplus g_{+1} \oplus grg_{-1} \oplus grg_{+1}
\end{pmatrix}
$$

commutes for all $i$. We note that the $\Theta_{x' f}$ can be computed explicitly as explained in Subsection 4.4.
The reader is invited to do more cases, but beware: There are plenty of them (namely ∞-many). Note that the degree shifts work out as they should since the whole natural transformation will be of degree 0.

Remark 4.39. As already noted in Remarks 3.17 and 3.23, a possible extension of our work to type $A_n$ can, on the level of gradings and algebras, obtained via certain generalizations of Khovanov’s arc algebra. A possible generalization of the diagrammatic categories in this section could follow from recent work of Elias and Libedinsky on universal Coxeter groups, see [21]. Using the equivalence of Kazhdan and Lusztig (the one given in [33]), the underlying Coxeter group $W$ will for $A_n$ be the affine Weyl group, that is, the one given by the cyclic Dynkin diagram with $n + 1$-nodes

where the green nodes indicate the affine nodes. The main open question is what the appropriate dead-end relations (there have to be more than two) are.

REFERENCES

[1] H.H. Andersen, Tensor products of quantized tilting modules, Comm. Math. Phys. 149-1 (1992), 149-159.
[2] H.H. Andersen, The strong linkage principle for quantum groups at roots of 1, J. Algebra 260-1 (2003), 2-15.
[3] H.H. Andersen, J.C. Jantzen and W. Soergel, Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: Independence of p, Astérisque 220 (1994).
[4] H.H. Andersen and M. Kaneda, Rigidity of tilting modules, Mosc. Math. J. 11-1 (2011), 1-39, online available arXiv:0909.2935.
[5] H.H. Andersen, P. Polo and K. Wen, Representations of quantum algebras, Invent. Math. 104-1 (1991), 1-59.
[6] I. Assem, D. Simson and A. Skowronski, Elements of the Representation Theory of Associative Algebras, Students Texts 65, Volume 1, Lond. Math. Soc. (2006).
[7] D. Auroux, J.M. Grigsby and S.M. Wehrli, On Khovanov-Seidel quiver algebras and bordered Floer homology, Selecta Math. (N.S.) 20-1 (2014), 1-55, online available arXiv:1107.2841.
[8] D. Auroux, J.M. Grigsby and S.M. Wehrli, Sutured Khovanov homology, Hochschild homology, and the Ozsváth-Szabó spectral sequence, online available arXiv:1303.1986.
[9] B. Bakalov and A. Kirillov Jr., Lectures on tensor categories and modular functors, University Lecture Series Vol. 21, Amer. Math. Soc. (2001).
[10] A. Beilinson, V. Ginzburg and W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9-2 (1996), 473-527.
[11] J. Bernstein, I. Frenkel and M. Khovanov, A categorification of the Temperley-Lieb algebra and Schur quotients of $U(sl(2))$ via projective and Zuckerman functors, Selecta Math. (N.S.) 5-2 (1999) 199-241, online available arXiv:math/0002087.
[12] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra I: cellularity, Mosc. Math. J. 11-4 (2011), 685-722, online available arXiv:0806.1532.
[13] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra II: Koszulity, Transform. Groups 15-1 (2010), 1-45, online available arXiv:0806.3472.
[14] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra III: category $O$, Represent. Theory 15 (2011), 170-243, online available arXiv:0812.1090.
[15] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra IV: the general linear supergroup, J. Eur. Math. Soc. 14-2 (2012), 373-419, online available arXiv:0907.2543.
[16] J. Brundan and C. Stroppel, Gradings on walled Brauer algebras and Khovanov’s arc algebra, Adv. Math. 231-2 (2012), 709-773, online available arXiv:1107.0999.
[17] Y. Chen and M. Khovanov, An invariant of tangle cobordisms via subquotients of arc rings, Fund. Math. 225 (2014), 23-44, online available arXiv:math/0610054.
S. Cautis, Rigidity in higher representation theory, online available arXiv:1409.0827.

B. Elias, The two-color Soergel calculus, online available arXiv:1308.6611.

B. Elias and M. Khovanov, Diagrammatics for Soergel categories, Int. J. Math. Math. Sci. Vol. 2010 (2010), Article ID 978635, 58 pages, online available arXiv:1308.6611.

B. Elias and N. Libedinsky, Soergel bimodules for universal Coxeter groups, online available arXiv:1401.2467.

B. Elias and Y. Qi, An approach to categorification of some small quantum groups II, online available arXiv:1302.5478.

B. Elias and G. Williamson, Diagrammatics for Coxeter groups and their braid groups, online available arXiv:1405.4928.

B. Elias and G. Williamson, The Hodge theory of Soergel bimodules, to appear in Ann. of Math., online available arXiv:1212.0791.

B. Elias and G. Williamson, Soergel calculus, online available arXiv:1309.0865.

S. Donkin, On tilting modules for algebraic groups, Math. Z. 212-1 (1993), 39-60.

I. Grojnowski and G. Lusztig, On bases of irreducible representations of quantum $GL_n$, Contemp. Math. 139 (1992), 167-174.

J.E. Humphreys, Representations of semisimple Lie algebras in the BGG category $O$, Graduate Studies in Mathematics Vol. 94, Amer. Math. Soc. (2008).

J.C. Jantzen, Lectures on quantum groups, Graduate Studies in Mathematics Vol. 6, Amer. Math. Soc. (1996).

J.C. Jantzen, Representations of Algebraic Groups, Mathematical Surveys and Monographs 107, Second edition, Amer. Math. Soc. (2007).

A. Joyal and R. Street, The geometry of tensor calculus. I, Adv. Math. 88-1 (1991), 55-112.

M. Kashiwara and T. Tanisaki, Kazhdan-Lusztig conjecture for affine Lie algebras with negative level, Duke Math. J. 77-1 (1996), 21-62.

D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras I-IV, J. Amer. Math. Soc. 6-4 (1994), 905-947, 949-1011 and J. Amer. Math. Soc. 7-2 (1994), 335-381, 383-453.

M. Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101 (2000), 359-426, online available arXiv:math/9908171.

M. Khovanov, A functor-valued invariant of tangles, Algebr. Geom. Topol. 2 (2002), 665-741 (electronic), online available arXiv:math/0103190.

M. Khovanov, Categorifications from planar diagrammatics, Jpn. J. Math. 5-2 (2010), 153-181, online available arXiv:1008.5084.

M. Khovanov, Hopfological algebra and categorification at a root of unity: the first steps, to appear in Commun. Contemp. Math., online available arXiv:math/0509083.

M. Khovanov and A.D. Lauda, A categorification of quantum $sl_n$, Quantum Topol. 2-1 (2010), 1-92, online available arXiv:0807.3250.

M. Khovanov, A.D. Lauda, M. Mackaay and M. Stošić, Extended Graphical Calculus for Categorified Quantum $sl_n$, Mem. Amer. Math. Soc. Vol. 219-1029 (2012), online available arXiv:1006.2866.

M. Khovanov and Y. Qi, An approach to categorification of some small quantum groups, online available arXiv:1208.0616.

M. Khovanov and L. Rozansky, Matrix factorizations and link homology I, Fund. Math. 199-1 (2008), 1-91, online available arXiv:math/0401268.

M. Khovanov and P. Seidel, Quivers, Floer cohomology, and braid group actions, J. Amer. Math. Soc. 15-1 (2002), 203-271 (electronic), online available arXiv:math/0006056.

A.D. Lauda, An introduction to diagrammatic algebra and categorified quantum $sl(2)$, Bull. Inst. Math. Acad. Sin. (N.S.) 7-2 (2012) (2012), 165-270, online available arXiv:1106.2128.

A.D. Lauda, H. Queffelec and D.E.V. Rose, Khovanov homology is a skew Howe 2-representation of categorified quantum $sl(m)$, online available arXiv:1212.6076.

T. Leinster, Higher Operads, Higher Categories, Lond. Math. Soc. Lecture Note Series Vol. 298 (2004), online available arXiv:math/0305049.

G. Lusztig, Introduction to Quantum Groups, Progress in Mathematics, Birkhäuser (1993).

G. Lusztig, Modular representations and quantum groups, Contemp. Math. 82 (1989), 59-77.

G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata 35-1-3 (1990), 89-114.
[50] M. Mackaay, The \( \text{sl}(N) \)-web algebras and dual canonical bases, J. Algebra 409 (2014), 54-100, online available arXiv:1308.0566.

[51] M. Mackaay, W. Pan and D. Tubbenhauer, The \( \text{sl}_3 \)-web algebra, Math. Z. 277-1-2 (2014), 401-479, online available arXiv:1206.2118.

[52] V. Mazorchuk, S. Ovsienko and C. Stroppel, Quadratic duals, Koszul dual functors, and applications, Trans. Amer. Math. Soc. 361-3 (2009), 1129-1172, online available arXiv:math/0603475.

[53] V. Mazorchuk and C. Stroppel, A combinatorial approach to functorial quantum \( \text{sl}(k) \) knot invariants, Amer. J. Math. 131-6 (2009), 1679-1713, online available arXiv:0709.1971.

[54] M. Müger, From Subfactors to Categories and Topology I. Frobenius algebras in and Morita equivalence of tensor categories, J. Pure Appl. Alg. 180-1-2 (2003), 81-157, online available arXiv:math/0111204.

[55] J. Paradowski, Filtration of modules over the quantum algebra, Proc. Sympos. Pure Math. 56, part 2 (1994), 93-108.

[56] Y. Qi, Hopfological algebra, Compos. Math. 150-1 (2014), 1-45, online available arXiv:1205.1814.

[57] H. Queffelec and D.E.V. Rose, The \( \text{sl}(n) \)-foam 2-category: A combinatorial formulation of Khovanov-Rozansky homology via categorical skew-Howe duality, online available arXiv:1405.5920.

[58] C.M. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Z. 208-2 (1991), 209-223.

[59] R. Rouquier, 2-Kac-Moody algebras, online available arXiv:0812.5023.

[60] S.F. Sawin, Quantum Groups at Roots of Unity and Modularity, J. Knot Theor. Ramif. 15-10 (2006), 1245-1277, online available arXiv:math/0308281.

[61] W. Soergel, Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren, Represent. Theory 1 (1997), 115-132 (electronic).

[62] W. Soergel, Kategorie O, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, J. Amer. Math. Soc. 3-2 (1990), 421-445.

[63] W. Soergel, Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln, Represent. Theory 1 (1997), 37-68 (electronic).

[64] W. Soergel, The combinatorics of Harish-Chandra bimodules, J. Reine Angew. Math. 429, (1992) 49-74.

[65] C. Stroppel, Category O: Gradings and translation functors, J. Algebra 268-1 (2003), 301-326.

[66] C. Stroppel, Categorification of the Temperley-Lieb category, tangles and cobordisms via projective functors, Duke Math. J. 126-3 (2005), 547-596.

[67] C. Stroppel, TQFT with corners and tilting functors in the Kac-Moody case, online available arXiv:math/0605103.

[68] J. Sussan, Category O and \( \text{sl}(k) \) link invariants, online available arXiv:math/0701045.

[69] D. Tubbenhauer, \( \text{sl}_3 \)-web bases, intermediate crystal bases and categorification, to appear in J. Algebr. Comb., online available arXiv:1310.2779.

[70] D. Tubbenhauer, \( \text{sl}_n \)-webs, categorification and Khovanov-Rozansky homologies, online available arXiv:1404.5752.

[71] V. Turaev, Quantum invariants of knots and 3-manifolds, second revised edition, de Gruyter Studies in Mathematics (2010).

[72] B. Webster, Knot invariants and higher representation theory, online available arXiv:1309.3796.

H.H.A.: Centre for Quantum Geometry of Moduli Spaces (QGM), University Aarhus, Denmark email: mathha@qgm.au.dk

D.T.: Centre for Quantum Geometry of Moduli Spaces (QGM), University Aarhus, Denmark email: dtubben@qgm.au.dk