Extended Kepler–Coulomb quantum superintegrable systems in three dimensions

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Abstract

The quantum Kepler–Coulomb system in three dimensions is well known to be second order superintegrable, with a symmetry algebra that closes polynomially under commutators. This polynomial closure is also typical for second order superintegrable systems in 2D and for second order systems in 3D with nondegenerate (four-parameter) potentials. However, the degenerate three-parameter potential for the 3D Kepler–Coulomb system (also second order superintegrable) is an exception, as its symmetry algebra does not close polynomially. The 3D four-parameter potential for the extended Kepler–Coulomb system is not even second order superintegrable, but Verrier and Evans (2008 J. Math. Phys. 49 022902) showed it was fourth order superintegrable, and Tanoudis and Daskaloyannis (2011 arXiv:11020397v1) showed that, if a second fourth order symmetry is added to the generators, the symmetry algebra closes polynomials. Here, based on the Tremblay, Turbiner and Winternitz construction, we consider an infinite class of quantum extended Kepler–Coulomb three- and four-parameter systems indexed by a pair of rational numbers \((k_1, k_2)\) and reducing to the usual systems when \(k_1 = k_2 = 1\). We show these systems to be superintegrable of arbitrarily high order and determine the structure of their symmetry algebras. We demonstrate that the symmetry algebras close algebraically; only for systems admitting extra discrete symmetries is polynomial closure achieved. Underlying the structure theory is the existence of raising and lowering operators, not themselves symmetry operators or even defined independent of basis, that can be employed to construct the symmetry operators and their structure relations.

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1. Introduction

A quantum system is defined by a Schrödinger operator $H = \Delta + V(x)$ where $\Delta = \sum_{ij} \frac{\partial}{\partial x_i} (\sqrt{g^{ij}}) \frac{\partial}{\partial x_j}$ is the Laplace–Beltrami operator on an $n$-dimensional Riemannian manifold, in local coordinates $x_j$. The system is maximal superintegrable of order $\ell$ if it admits $2n - 1$ algebraically independent, globally defined differential symmetry operators (the maximal number possible) $S_j$, $1 \leq j \leq 2n - 1$, $n \geq 2$, with $S_1 = H$ and $[H, S_j] = HS_j - S_jH = 0$, such that $\ell$ is the maximum order of the generating symmetries (other than $H$) as a differential operator. Systems associated with Lie algebras ($\ell = 1$) and separation of variables ($\ell = 2$) are the simplest and best studied. An integrable system has $n$ algebraically independent commuting symmetry operators whereas a superintegrable system has $2n - 1$ independent symmetry operators which cannot all commute and this non-Abelian structure is critical for finding the spectral resolution of $H$ by algebraic methods alone. The importance of these systems is that they can be solved exactly. Progress in classifying and elucidating the structure of these systems has been impressive in the last two decades, see, e.g., [1–4] for some representative works related to the present paper.

The three-parameter extended Kepler–Coulomb system is defined by the Hamiltonian operator

$$H_3 = \partial_r^2 + \partial_\theta^2 + \partial_\phi^2 + \frac{\alpha}{r} + \frac{\beta}{r^2}, \quad r = \sqrt{x^2 + y^2 + z^2}. \quad (1)$$

It is second order superintegrable, with generators $H = L_1$,

$$L_2 = (x \partial_x - y \partial_y)^2 + (y \partial_y - z \partial_z)^2 + (z \partial_z - x \partial_x)^2 + \frac{\beta r^2}{x^2} + \frac{\gamma r^2}{y^2},$$

$$L_3 = (x \partial_y - y \partial_x)^2 + \frac{\beta (x^2 + y^2)}{x^2} + \frac{\gamma (x^2 + y^2)}{y^2}, \quad L_4 = (x \partial_z - z \partial_x)^2 + \frac{\beta z^2}{x^2 + z^2},$$

$$L_5 = -\frac{1}{2}(\partial_r x \partial_x - z \partial_z) - \frac{1}{2}(\partial_r y \partial_y - z \partial_z) + \frac{\alpha}{2r} + \frac{1}{2} \left( \frac{\beta}{x^2} + \frac{\gamma}{y^2} \right).$$

Here $\{A, B\} = AB + BA$. In spherical coordinates $x = r \cos(\theta_2) \sin(\theta_1)$, $y = r \sin(\theta_2) \sin(\theta_1)$, $z = r \cos(\theta_1)$ the operators $L_2$, $L_3$ become

$$L_2 = \partial_{\theta_1}^2 + \cot(\theta_1) \partial_{\theta_1} + \frac{L_3}{\sin^2(\theta_1)}, \quad L_3 = \partial_{\theta_1}^2 + \frac{\beta \cos^2(\theta_2)}{\sin^2(\theta_1)} + \frac{\gamma}{\sin^2(\theta_1)},$$

and we see that these are the symmetries responsible for the separation of the eigenvalue equation $H \Psi = E \Psi$ in spherical coordinates. Now we use the same idea as in the papers [5, 6] by expressing $H$ in spherical coordinates and replacing $\theta_1$ by $k_1 \theta_1$ and $\theta_2$ by $k_2 \theta_2$ where $k_1$, $k_2$ are arbitrary positive rational numbers. We obtain a family of extended three-parameter Kepler–Coulomb operators

$$H = \partial_r^2 + \frac{\alpha}{r} + \frac{1 - k_1^2}{4r^2} + L_2^2, \quad (2)$$

where

$$L_2 = \partial_{\theta_1}^2 + k_1 \cot(k_1 \theta_1) \partial_{\theta_1} + \frac{L_3}{\sin^2(k_1 \theta_1)}, \quad L_3 = \partial_{\theta_1}^2 + \frac{\beta \cos^2(k_2 \theta_2)}{\sin^2(k_1 \theta_1)} + \frac{\gamma}{\sin^2(k_2 \theta_2)} \quad (3)$$

Again, $L_2$, $L_3$ are symmetry operators that determine multiplicative separation of the Schrödinger eigenvalue equation $H \Psi = E \Psi$, and $k_j = p_j/q_j$ where $p_j$, $q_j$ are nonzero relatively prime positive integers for $j = 1, 2$, respectively. Note that $[L_2, L_3] = 0$ so $L_2$ and $L_3$ are in involution. Also note that the potential here is

$$V = \frac{\alpha}{r} + \frac{1 - k_1^2}{4r^2} + \frac{1}{r^2} \left( \frac{\beta}{\sin^2(k_1 \theta_1)} \cos^2(k_2 \theta_2) + \frac{\gamma}{\sin^2(k_2 \theta_2)} \right).$$
differing from the classical potential $V$ by the additive term $\frac{1-k_1^2}{2r^2}$. This term corresponds to $R/8$ where $R$ is the scalar curvature of the manifold, just the correction term needed for the conformally invariant Laplacian, e.g., [7], and needed here for superintegrability. We will verify the superintegrability of this system for all rational $k_1, k_2$ by explicit construction of two additional independent symmetry operators.

The four-parameter extended Kepler–Coulomb system is defined by the Hamiltonian

$$H_4 = \partial_r^2 + \partial_{\theta}^2 + \frac{\alpha}{r^2} + \frac{\beta}{r^2} + \frac{\gamma}{r^2} + \frac{\delta}{r^2}. \tag{4}$$

The eigenvalue equation is separable in spherical coordinates so the system admits three commuting symmetry operators $L_1 = H, L_2, L_3$, responsible for the separation of variables:

$$L_2 = (\alpha \partial_r - \beta \partial_{\theta})^2 + (\gamma \partial_r - \delta \partial_{\theta})^2 + \frac{\beta r^2}{x^2} + \frac{\gamma r^2}{y^2} + \frac{\delta r^2}{z^2}, \tag{5}$$

$$L_3 = (\alpha \partial_r - \beta \partial_{\theta})^2 + \frac{\beta (x^2 + y^2)}{x^2} + \frac{\gamma (x^2 + y^2)}{y^2}. \tag{6}$$

This system is not second order superintegrable, but as shown in [8], it is fourth order superintegrable. We will again verify this in section 2.4. We extend the system by passing to spherical coordinates and replacing each angular coordinate $\theta_i$ by $k_i \theta_i$, where $k_i$ is a fixed rational number. The extended Kepler–Coulomb operator is now $H \Psi = E \Psi$, where $[L_2, L_3] = [L_1, H] = 0$ and

$$H_4 = \partial_r^2 + \frac{2}{r} \partial_r + \frac{\alpha}{r^2} + \frac{1-k_1^2}{4r^2} + \frac{L_2}{r^2}, \tag{7}$$

$$L_2 = \partial_r^2 + k_1 \cot(k_1 \theta_1) \partial_{\theta_1} + \frac{L_3}{\sin^2(k_1 \theta_1)} + \frac{\delta}{\cos^2(k_1 \theta_1)}, \quad L_3 = \partial_r^2 + \frac{\beta}{\cos^2(k_2 \theta_2)} + \frac{\gamma}{\sin^2(k_2 \theta_2)}.$$

Again the second order operators $L_2, L_3$ are just those that determine multiplicative separation of the Schrödinger equation. The scalar potential is

$$\tilde{V} = \frac{\alpha}{r} + \frac{1-k_1^2}{4r^2} + \frac{1}{r^2} \left( \frac{\beta}{\sin^2(k_1 \theta_1) \cos^2(k_2 \theta_2)} + \frac{\gamma}{\sin^2(k_1 \theta_1) \sin^2(k_2 \theta_2)} + \frac{\delta}{\cos^2(k_1 \theta_1)} \right).$$

It differs from the classical potential $V$ by the term $\frac{1-k_1^2}{4r^2}$, which corresponds to $-R/8$ where $R$ is the scalar curvature of the manifold. Note that for $k_1 \neq 1$ the space is not flat. We will show that this system is superintegrable for all rational $k_1, k_2$.

We construct the missing symmetry operators by exploiting the following observation [9, 10]: the separated eigenfunctions of these Hamiltonians in spherical coordinates are products of hypergeometric-type functions. The formal eigenspaces of the Hamiltonian are invariant under the action of any symmetry operator, so the operator must induce recurrence relations for the basis of separated eigenfunctions. Our strategy is to use the known recurrence relations for hypergeometric functions to reverse this process and determine a symmetry operator from the recurrence relations. We look for recurrence operators that change the eigenvalues of $L_2, L_3$ but preserve $E$; hence they map the eigenspace into itself. All of the special functions used in this paper are studied in [11], for example.

Before taking up our examples in section 1.1, we describe how to compute with higher order symmetry operators on an $n$-dimensional Riemannian or pseudo-Riemannian manifold with the Schrodinger eigenvalue equation that separates multiplicatively in an orthogonal subgroup coordinate system. We show that such operators can be put in a canonical form, which will be critical in verifying that differential operators that commute with $H$ on all formal eigenspaces must commute with $H$ in general.
In the following sections we show that the extended three- and four-parameter extended Kepler–Coulomb systems are superintegrable of arbitrarily high order, the order depending on \( k_1, k_2 \), and determine the structure of their symmetry algebras. We demonstrate that in general the symmetry algebras close algebraically; only for systems admitting extra discrete symmetries is polynomial closure achieved. Underlying the structure theory is the existence of raising and lowering operators, not themselves symmetry operators or defined independent of basis, that can be employed to construct the symmetry operators and their structure relations. We demonstrate that our structure equations lead to two-variable models of representations of the symmetry algebras in terms of difference operators. In general the eigenfunctions in these models are rational functions of the variables, not polynomials as in symmetry algebras with polynomial closure, e.g., [12]. We note that the proof of superintegrability for the extended four-parameter system gives a proof of superintegrability of the three-parameter system, simply by setting \( \delta = 0 \). However, it does not give the minimum order generators for the three-parameter system or the full structure. That is why we have to study each system separately.

In the special case \( k_1 = k_2 = 1 \), for the four-parameter potential we complete the results of [13] and verify polynomial closure. For this system there are six linearly independent generators but we show that these generators must satisfy a symmetrized polynomial equation of order 12.

Many of the issues in this paper are quite technical. We think that the exercise is very worthwhile for several reasons. (1) The root three- and four-parameter systems have attracted a lot of attention during this last decade and families of their extensions are of interest. (2) To our knowledge this is the first explicit computation of the structure of the symmetry algebras for families of higher order superintegrable systems in three variables. That it is practical to carry out the computation is not obvious. The results show how some properties of second order superintegrable systems and for superintegrable systems in two variables break down in higher dimensional cases. (3) The explicit two-variable models suggest important connections with rational special functions for higher order superintegrable systems, rather than with polynomial special functions as for second order systems.

In the related paper [14] we have determined the structure equations for the extended classical Kepler superintegrable systems, both three and four parameter. The methods used to find the structure equations are very different from the quantum case but there is some similarity in the results. Classically we can prove generic rational closure rather than polynomial closure, whereas in the quantum case the closure is algebraic (due to noncommutivity of quantum operators). Of course, the classical potentials must be modified to achieve quantization. In the short proceedings paper [15] some of the results for the four-parameter quantum system are announced. However, all of the three-parameter work and the details of the four-parameter computation are contained here.

### 1.1. The canonical form for a 3D symmetry operator

In the special case of a three-dimensional Riemannian or pseudo-Riemannian manifold the defining equations for the Hamiltonian operator \( H \), expressed in separable orthogonal ‘subgroup coordinates’ \( q_1, q_2, q_3 \), take the form [16]

\[
H = L_1 = \partial_1^2 - \frac{f'_1}{f_1} \partial_1 + V_1 + f_1(q_1)L_2, \quad L_2 = \partial_2^2 - \frac{1}{2} \frac{f'_2}{f_2} \partial_2 + V_2 + f_2L_3, \quad L_3 = \partial_3^2 + V_3,
\]  

(8)

where the functions \( f_j, V_j \) depend only on coordinate \( q_j \). Thus \( H = \Delta_3 + \tilde{V} \) where \( \Delta_3 \) is the Laplace–Beltrami operator on the manifold with metric \( dx^2 = dq_1^2 + \frac{1}{f_1} dq_2^2 + \frac{1}{f_1 f_2} dq_3^2 \),
The operators $L_1, L_2, L_3$ are formally self-adjoint on the space with measure $dv = dq_1 dq_2 dq_3 / \sqrt{f_1^2 f_2}$. We write $\bar{V} = V_1 + f_1 V_2 + f_1 f_2 V_3$ for the scalar potential because the quantum analogue of the classical Hamiltonian system with potential $V$ on the manifold may have a different scalar potential. Thus the Schrödinger eigenvalue equation is $H \Psi \equiv (\Delta + \bar{V}) \Psi = E \Psi$. Any finite order differential operator $\bar{L}$ on the manifold can be written uniquely in the standard form [17]

$\bar{L} = \sum_{j, j_1, j_2} (A^{j-j_1, j_2} (q) \partial_{123} + B_1^{j-j_1, j_2} (q) \partial_{23} + B_2^{j-j_1, j_2} (q) \partial_{13} + B_3^{j-j_1, j_2} (q) \partial_{12} + C_1^{j-j_1, j_2} (q) \partial_{q_1} + C_2^{j-j_1, j_2} (q) \partial_{q_2} + C_3^{j-j_1, j_2} (q) \partial_{q_3} + D^{j-j_1, j_2} (q)) \bar{L}_1^{j} \bar{L}_2^{j_1} \bar{L}_3^{j_2}$. \hspace{1cm} (9)

Note that if the formal operators (9) contained partial derivatives in any of $q_1, q_2, q_3$ of orders $\geq 2$ we could use the identities (8), recursively, and rearrange terms to achieve the unique standard form (9). In this view we can write

$\bar{L}(q, L_1, L_2, L_3) = A(q, L_1, L_2, L_3) \partial_{123} + B_1(q, L_1, L_2, L_3) \partial_{23} + B_2(q, L_1, L_2, L_3) \partial_{13} + B_3(q, L_1, L_2, L_3) \partial_{12} + \sum_{i=1}^3 C_i(q, L_1, L_2, L_3) \partial_{q_i} + D(q, L_1, L_2, L_3)$, \hspace{1cm} (10)

and consider $\bar{L}$ as an at most third-order order differential operator in $q$ that is polynomial in the parameters $L_1, L_2, L_3$. Note that $A$ is the only term of third order. Then the terms in $L_1, L_2$ and $L_3$ must be interpreted as (9) to be considered as partial differential operators.

1.2. Recurrence relations for the quantum Kepler–Coulomb system $H_3$

For system (2), which separates in spherical coordinates, $q_1 = r, q_2 = \theta_1, q_3 = \theta_2$ we have the metric definitions

$f_1(r) = \frac{1}{r^2}, \hspace{1cm} V_1(r) = \frac{\alpha}{r} + \frac{1 - k_1^2}{4r^2}, \hspace{1cm} f_2(\theta_1) = \frac{1}{\sin^2(k_1 \theta_1)}, \hspace{1cm} \hspace{1cm} f_3(\theta_1) = 0, \hspace{1cm} V_3(\theta_2) = \frac{\beta}{\cos^2(k_2 \theta_2)} + \frac{\gamma}{\sin^2(k_2 \theta_2)}$.

The difference between the quantum and classical potentials is the term $\frac{1-k_1^2}{4r^2} = -\frac{\mathcal{R}}{r^2}$ where $\mathcal{R}$ is the scalar curvature of the manifold. This is the quantization condition; it is 0 for flat space $k_1 = 1$. The separation equations for the equation $H \Psi = E \Psi$ are:

$$\Phi(\theta_2) = -\mu^2 \Phi(\theta_2), \hspace{1cm} L_3 \Psi = -\mu^2 \Psi,$$

where $\beta = k_2^2 (\frac{1}{4} - b^2), \gamma = k_2^2 (\frac{1}{4} - c^2)$, and $\mu = k_2 (2m + b + c + 1)$;

$$\Theta(\theta_1) = -k_1^2 \nu (\nu + 1) \Theta(\theta_1), \hspace{1cm} L_2 \Psi = -k_1^2 \nu (\nu + 1) \Psi,$$

where $\nu = (\rho - 1)/2$;

$$R(r) = 0, \hspace{1cm} H \Psi = E \Psi.$$
In (3) it is convenient to introduce a new variable $R = 2\sqrt{E}r$ so that the separation equation becomes

$$
(3)'
\left( \frac{\alpha}{2\sqrt{E}} - 1 \right) S(R) = 0, \quad R(R) = S(R)/R.
$$

The separated solutions are $\Psi_{n,\rho,\mu} = R_{n}\rho(r) \Phi_{m}^{(c,b)}(\cos(2k_{2}\theta_{2})) \Theta_{\nu}^{\mu}(\cos(k_{1}\theta_{1}))$ with

$$
\Phi_{m}^{(c,b)}(\cos(2k_{2}\theta_{2})) = \sin^{c+1/2}(k_{2}\theta_{2}) \cos^{b+1/2}(k_{2}\theta_{2}) P_{m}^{(c,b)}(\cos(2k_{2}\theta_{2})),
$$

where the $P_{m}^{(c,b)}(\cos(2k_{2}\theta_{2}))$ are Jacobi functions;

$$
\Theta_{\nu}^{\mu}(\cos(k_{1}\theta_{1})) = P_{\nu}^{\mu/k_{1}}(\cos(k_{1}\theta_{1})),
$$

where the $P_{\nu}^{\mu/k_{1}}(\cos(k_{1}\theta_{1}))$ are associated Legendre functions; and

$$
R_{n}\rho(r) = \frac{S(r)}{2\sqrt{E}r}, \quad S(r) = (2\sqrt{E})^{k_{1}\rho/2}e^{-\sqrt{E}r}(k_{1}\rho+1/2) L_{n}^{k_{1}\rho}(2\sqrt{E}r),
$$

where the $L_{n}^{k_{1}\rho}(2\sqrt{E}r)$ are associated Laguerre functions, and the relation between $E$ and $n$ is the quantization condition

$$
E = \frac{\alpha^{2}}{(2n + k_{1}\rho + 1)^{2}}.
$$

**Note.** We are only interested in the space of generalized eigenfunctions, not the normalization of any individual eigenfunction. Thus the relations to follow are valid on generalized eigenspaces and do not necessarily agree with the normalization of, say, common polynomial eigenfunctions.

Now we look for recurrence operators acting on the separated eigenfunctions that will change the eigenvalues of $L_{2}$ and $L_{3}$ but preserve $E$. From (11) we see that $E$ is preserved under either of the transformations:

$$
n \to n + p_{1}, \quad \rho \to \rho - 2q_{1} \quad \text{or} \quad n \to n - p_{1}, \quad \rho \to \rho + 2q_{1}.
$$

Writing $S(r) = S_{n}^{k_{1}\rho}(R)$, we can use recurrence relations for the associated Laguerre functions [11] to obtain

$$
\left[ 2(k_{1}\rho + 1) \partial_{r} + (2n + k_{1}\rho + 1) - \frac{(k_{1}\rho + 1)^{2}}{R} \right] S_{n}^{k_{1}\rho}(R) = -2S_{n-1}^{k_{1}\rho}(R).
$$

$$
\left[ 2(-k_{1}\rho + 1) \partial_{r} + (2n + k_{1}\rho + 1) - \frac{(k_{1}\rho - 1)^{2}}{R} \right] S_{n}^{k_{1}\rho}(R) = -2(n + 1)(n + k_{1}\rho)S_{n+1}^{k_{1}\rho}(R).
$$

Recalling that $R = 2\alpha/(2n + k_{1}\rho + 1)$, we see that these relations can be written as

$$
\tilde{Y}(1)^{n}_{r}T_{n}^{k_{1}\rho}(r) = \left[ 2(k_{1}\rho + 1) \partial_{r} + \left( 2\alpha - \frac{(k_{1}\rho + 1)^{2}}{r} \right) \right] T_{n}^{k_{1}\rho}(r) = -\frac{4\alpha}{2n + k_{1}\rho + 1} T_{n+1}^{k_{1}\rho+2}(r),
$$

$$
\tilde{Y}(1)^{n}_{r}T_{n}^{k_{1}\rho}(r) = \left[ 2(-k_{1}\rho + 1) \partial_{r} + \left( 2\alpha - \frac{(k_{1}\rho - 1)^{2}}{r} \right) \right] T_{n}^{k_{1}\rho}(r) = -\frac{4\alpha}{2n + k_{1}\rho + 1} (n + 1)(n + k_{1}\rho) T_{n+1}^{k_{1}\rho-2}(r),
$$

where $T_{n}^{k_{1}\rho}(r) = S_{n}^{k_{1}\rho}(R)$. To get the action of the recurrences on the separated solutions $R(r)_{n}^{k_{1}\rho}(r) = S(r)/(2\sqrt{E}r)$ we have to perform the gauge transformation of multiplying by $R^{-1}$ so that the recurrences transform as $r^{-1}\tilde{Y}(1)^{n}_{r}r = Y(1)^{n}_{r}$, $r^{-1}\tilde{Y}(1)^{n-1}_{r}r = Y(1)^{n}_{r}$. Note
that the factor $\sqrt{E}$ cancels out of the operator transformation formulas since the recurrences preserve energy. We obtain

\[ Y(1)^n R_n^{k,\rho}(r) = \left[ 2(k_1 \rho + 1) \partial_\rho + \left( 2\alpha + \frac{1 - k^2 \rho^2}{r} \right) \right] R_n^{k,\rho}(r) = -\frac{4\alpha}{2n + k_1 \rho + 1} R_{n-1}^{k,\rho+2}(r), \]

\[ Y(1)^n R_n^{k,\rho}(r) = \left[ 2(-k_1 \rho + 1) \partial_\rho + \left( 2\alpha + \frac{1 - k^2 \rho^2}{r} \right) \right] R_n^{k,\rho}(r) = -\frac{4\alpha}{2n + k_1 \rho + 1} (n + 1)(n + k_1 \rho) R_{n+1}^{k,\rho-2}(r). \]  

(13)

We can also find raising and lowering operators for the $\theta_1$-dependent part of the separable solutions, using differential relations for the associated Legendre functions

\[ X(1)^\mu p^\mu_\mu \equiv (1 - x^2) \partial_x + \frac{1 - \rho}{2} x \]  

\[ X(1)^\mu p^\mu_{-\mu} \equiv (1 - x^2) \partial_x + \frac{1 + \rho}{2} x \]  

(14)

where $x = \cos(k \theta_1)$. We now construct the two operators

\[ J^+ = (Y(1)^{n-p} Y(1)^{n-p-1}) \cdots (Y(1)^{n-p_{\mu}} Y(1)^{n-p_{\mu+1}}) \times \Theta_{\mu}^{\mu+2q_1}, \]

\[ J^- = (Y(1)^{n-p} Y(1)^{n-p-1}) \cdots (Y(1)^{n-p_{\mu}} Y(1)^{n-p_{\mu+1}}) \times \Theta_{\mu}^{\mu-2q_1}, \]

(15)

(16)

each with $p_1 + q_1$ factors. When applied to a basis function $\Psi^{n,p,m} \equiv \Psi_n$ for fixed $2n + k_1 \rho$ and $m$, these operators raise and lower indices according to

\[ J^+ \Psi_n = \left( \frac{4\alpha}{u} \right)^{p_1} \left( \frac{u - k_1 \rho + 1}{2} \right)_{p_1} \left( \frac{u - k_1 \rho + 1}{2} \right)_{p_1} \left( \frac{-\rho - k_1 + 1}{2} \right)_{q_1} \Psi_{n+p_1}, \]

\[ J^- \Psi_n = \left( \frac{4\alpha}{u} \right)^{p_1} \left( \frac{\rho - k_1 + 1}{2} \right)_{q_1} \Psi_{n-p_1}, \quad u = 2n + k_1 \rho + 1, \]

(17)

(18)

where $(\alpha)_q \equiv (\alpha)(\alpha + 1) \cdots (\alpha + q - 1)$ for nonnegative integer $q$.

Similarly we can raise and lower $m$ while fixing $\rho$ and $n$, thus preserving the energy eigenspace. The recurrences for $x = \cos(k \theta_1)$,

\[ Y(2)^m \Theta^m_\rho (x) = \left[ \frac{\sqrt{1 - x^2}}{k^2} \frac{d}{dx} + \frac{x}{k^2 \sqrt{1 - x^2}} \right] \Theta^m_\rho (x) = -\Theta^{m+k/2k_2}_\rho (x), \]

\[ Y(2)^m \Theta^m_\rho (x) = \left[ \frac{\sqrt{1 - x^2}}{k^2} \frac{d}{dx} + \frac{x}{k^2 \sqrt{1 - x^2}} \right] \Theta^m_\rho (x) = \left( \frac{\rho - 1}{2} + \frac{\mu}{k_1} \right) \left( \frac{\rho + 1}{2} - \frac{\mu}{k_1} \right) \Theta^{m-k/2k_2}_\rho (x), \]

(19)

and the recurrences for $z = \cos(2k_2 \theta_2)$,

\[ X(2)^m \Phi^m_\alpha (z) = \left[ \left( 1 + \frac{\mu}{k_2} \right) (1 - z^2) \frac{d}{dz} - \frac{\mu}{2k_2} \left( 1 + \frac{\mu}{k_2} \right) z - \frac{1}{2} (c^2 - b^2) \right] \Phi^m_\alpha (z) = \frac{1}{2} \left( \frac{\mu}{k_2} - b - c + 1 \right) \left( \frac{\mu}{k_2} + b + c + 1 \right) \Phi^m_{\alpha+1} (z), \]

\[ X(2)^m \Phi^m_\alpha (z) = \left[ \left( 1 - \frac{\mu}{k_2} \right) (1 - z^2) \frac{d}{dz} + \frac{\mu}{2k_2} \left( 1 - \frac{\mu}{k_2} \right) z - \frac{1}{2} (c^2 - b^2) \right] \Phi^m_\alpha (z) = \frac{1}{2} (\mu - b + c - 1) (\mu + b - c - 1) \Phi^m_{\alpha-1} (z). \]
We construct the two operators

\[ K^+ = (Y(2)^{m_+}(2q_1p_1-1)k_1/2k_2)Y(2)^{m_+(2q_2p_2-2)k_1/2k_2} \ldots Y(2)^{m+k_1/2k_2}Y(2)^{m+1}) \times (X(2)^{m+p_1q_1-1}X(2)^{m+p_1q_2-2} \ldots X(2)^{m+p_1}X(2)^{m+1}), \]  

(20)

\[ K^- = (Y(2)^{m-(2q_1p_1-1)k_1/2k_2)Y(2)^{m-(2q_2p_2-2)k_1/2k_2}) \ldots Y(2)^{m-k_1/2k_2}Y(2)^{m}) \times (X(2)^{-m-p_1q_1+1}X(2)^{-m-p_1q_2+2} \ldots X(2)^{-m+1}X(2)^{m}), \]  

(21)
each with \( p_1q_2 + 2p_2q_1 \) factors. When applied to a basis function \( \Psi^{n,m} \) for fixed \( n \) and \( m \), these operators raise and lower lower indices according to

\[ K^+ \Psi^{m} = (-2)^{p_1q_2} \left( \frac{\mu}{2k_2} - \frac{b}{2} - \frac{c}{2} + \frac{1}{2} \right)_{p_1q_2} \left( \frac{\mu}{2k_2} + \frac{b}{2} + \frac{c}{2} + \frac{1}{2} \right)_{p_1q_2} \Psi^{m+p_1}, \]

\[ K^- \Psi^{m} = 2^{p_1q_2} \left( \frac{1 - \rho}{2k_1} \right)_{2p_1q_2} \left( \frac{1 + \rho}{2k_1} - \frac{\mu}{2k_1} \right)_{2p_1q_2} \left( -\frac{\mu}{2k_2} + \frac{b}{2} + \frac{c}{2} + \frac{1}{2} \right)_{p_1q_2} \Psi^{m-p_1q_2}, \]

(22)

Note that each of the separated solutions \( R_\rho^{(i)}(x), \Theta_{\rho}^{(i)}(x), \Phi_{\rho}^{(i,c)}(z) \) is characterized as an eigenfunction of a second order ordinary differential operator. Hence, except for isolated values of the parameters, it is possible to find bases \( \Psi^{n,m} \equiv \Psi^{m} \) for each of these eigenspaces such that each element of the basis set satisfies exactly the same recurrence formulas. In particular, the solution space of \( H\Psi = E\Psi \) is eight dimensional.

From the explicit expressions (15), (16) for the \( J^\pm \) operators it is easy to verify that under the transformation \( \rho \rightarrow -\rho \) we have \( J^+ \rightarrow J^- \) and \( J^- \rightarrow J^+ \). Then we see that

\[ J_1 = \frac{J^- - J^+}{\rho} \quad \text{and} \quad J_2 = J^- + J^+ \]  

(23)

are each unchanged under this transformation. Note: When applied to an eigenbasis we interpret \( J_1 \) ordered as \( (J^- - J^+) \frac{1}{\rho} \). Therefore each is a polynomial in \( \rho^2 \). As a consequence of the eigenvalue equation \( L_2 \Psi = -k_2^2(\rho^2 - 1)\Psi \), in the expansions of \( J_1, J_2 \) in terms of powers of \( \rho^2 \) we can replace \( \rho^2 \) by \( 1 - 4L_2/k_2^2 \) everywhere it occurs, and express \( J_1, J_2 \) as pure differential operators, independent of parameters. Clearly each of these operators commutes with \( H \) on its eigenspaces.

Under the reflection \( \mu \rightarrow -\mu \) we have \( K^+ \rightarrow K^- \) and \( K^- \rightarrow K^+ \). Thus

\[ K_1 = \frac{K^- - K^+}{\mu} \quad \text{and} \quad K_2 = K^- + K^+ \]  

(24)

are each unchanged under this transformation. Therefore each is a polynomial in \( \mu^2 \). As a consequence of the eigenvalue equation \( L_2 \Psi = -\mu^2 \Psi \), in the expansions of \( K_1, K_2 \) in terms of even powers of \( \mu \) we can replace \( \mu^2 \) by \( -L_3 \) everywhere it occurs, and express \( K_1, K_2 \) as pure differential operators, independent of parameters. Again each of these operators commutes with \( H \) on its eigenspaces.

We have now constructed partial differential operators \( J_1, J_2, K_1, K_2 \), each of which commutes with the Hamiltonian \( H \) on all its eight-dimensional formal eigenspaces. Thus they act like symmetry operators. However, to prove that they are true symmetry operators we must show that they commute with \( H \) when acting on any analytic functions, not just separated eigenfunctions. To establish this fact we use the canonical form of section 1.1. To be specific,
we consider the commutator \([H, J_1]\). When acting on the eight-dimensional space of formal eigenfunctions \(\Psi_0\) of \(H\) the commutator gives 0. We want to show that it vanishes identically. To do this we write \([H, J_1]\) in the canonical form (10):

\[
[H, J_1]\Psi_0 = \left(A \partial_{123} + B_1 \partial_{23} + B_2 \partial_{13} + B_3 \partial_{12} + \sum_{\ell=1}^{3} C_\ell \partial_\ell + D\right) \Psi_0 = 0. 
\]

(25)

Noting that we have eight linearly independent choices for \(\Psi_0 = R_n^{(k, \rho)} \Phi_n^{(m, h)} \Phi^{(c, b)}\), we can express (25) as a system of eight homogeneous equations for the eight unknowns \(A, B_1, B_2, B_3, C_1, C_2, C_3, D\). We write this system in the matrix form \(Vv = 0\) where \(v^T = (A, B_1, B_2, B_3, C_1, C_2, C_3, D)\) and \(V\) is an 8 \times 8 matrix built out of the possible products of the three basis eigenfunctions and their first derivatives. By a straightforward computation we can show that

\[
\det V = \pm W(R_n^{(k, \rho)}, \tilde{R}_n^{(k, \rho)})^4 W(\Phi_n^{(m, h)}, \tilde{\Phi}_n^{(m, h)})^4 W(\Phi^{(c, b)}, \tilde{\Phi}^{(c, b)})^4 \neq 0
\]

(26)

where \(W(R_n^{(k, \rho)}, \tilde{R}_n^{(k, \rho)}) \neq 0\) is the Wronskian of the two basis solutions. Thus we conclude that \(A = B_1 = C_1 = D = 0\). Consequently, \([H, J_1] = 0\) identically. A similar argument shows that \(J_2, K_1, K_2\) are also true symmetry operators. We will work out the structure of the symmetry algebra generated by \(H, L_2, L_3, J_0, J_1, J_3\) from this it will be clear that the system is superintegrable.

To determine the structure relations it is sufficient to establish them on the generalized eigenbases. Then an argument analogous to the treatment of (25) shows that the relations hold when operating on general analytic functions. We start by using the definitions (23) and computing on an eigenbasis. It is straightforward to verify the relations:

\[
[J_1, L_2] = k_1^2 q_1 J_2 + k_1^2 q_1^2 J_1, \quad [J_2, L_2] = -2 q_1 J_1, L_2 = \rho^2 J_2 - k_1^2 q_1 (1 + 2 q_1^2) J_1, \\
[J_1, L_3] = [J_2, L_3] = 0.
\]

Here \([A, B] = AB + BA\) is the anticommutator. We set \(u = 2n + k_1 \rho + 1\) and note that \(E = \alpha^2/u^2\). Making use of the relation \((-c)_q = (-1)^q (c - q + 1)_q\) we can obtain

\[
J^- J^+ \Psi_n = \left(\frac{4 \alpha}{u}\right)^{2 \nu_1} \left(\frac{\rho + \mu + 1}{q_1}\right)_{q_1} \left(\frac{\rho -\mu + 1}{q_1}\right)_{q_1} \left(\frac{u + k_1 \rho + 1}{p_1}\right)_{p_1} \left(\frac{u - k_1 \rho + 1}{p_1}\right)_{p_1} \Psi_n
\]

\[
= \xi_n \Psi_n,
\]

\[
J^+ J^- \Psi_n = \left(\frac{4 \alpha}{u}\right)^{2 \nu_1} \left(\frac{\rho + \mu + 1}{q_1}\right)_{q_1} \left(\frac{\rho -\mu + 1}{q_1}\right)_{q_1} \left(\frac{u + k_1 \rho + 1}{p_1}\right)_{p_1} \left(\frac{u - k_1 \rho + 1}{p_1}\right)_{p_1} \Psi_n
\]

\[
= \eta_n \Psi_n.
\]

Note that each of \(\xi_n, \eta_n\) is invariant under the transformation \(u \to -u\), hence each is a polynomial in \(1/u^2 \sim E\). Similarly, each of \(\xi_n, \eta_n\) is invariant under the transformation \(\mu \to -\mu\), so each is a polynomial in \(\mu^2 \sim L_2\). Further, under the transformation \(\rho \to -\rho\) we see that \(\xi_n, \eta_n\) switch places. Thus \(\xi_n + \eta_n\) is a polynomial in each of \(1/u^2, \mu^2, \rho^2\), hence a polynomial in \(H, L_2\) and \(L_3\):

\[
J^+ J^- + J^- J^+ = P^+(H, L_2, L_3).
\]

Here the coefficients of the polynomial \(P^+\) depend on the parameters \(\alpha, b, c, p_1, q_1, p_2, q_2\). Thus the basis dependent operator \(J^+ J^- + J^- J^+\) extends to a true globally defined symmetry operator \(P^+\).

By a similar argument, \((\xi_n - \eta_n)/\rho\) is invariant under the transformation \(\rho \to -\rho\), hence a polynomial in \(\rho^2\). Thus

\[
\frac{J^+ J^- - J^- J^+}{\rho} = P^-(H, L_2, L_3)
\]

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and the basis dependent operator \((J^+ J^- - J^- J^+) / \rho\) extends to a true globally defined symmetry operator \(P^-\), a polynomial in \(H, L_2, L_3\). Then direct computation gives the structure equations

\[ [J_1, J_2] = -2q_1 J_1^2 - 2P^- . \]

Interim results are \(J_2^2 = J_1^2 (1 - 4q_1^2 J_2^2) + 2P^+ + 2q_1 J_1 J_2\), and the symmetrizations \([J_1, J_1, J_2] = 6J_1^2 J_2^2 + 4p_1^2 J_1^2 - 3k_1^2 q_1 J_1 J_2 + 2k_1^2 q_1 P^-\), \(J_1 J_2 = \frac{1}{2} [J_1, J_2] - q_1 J_1^2 - P^-\). Thus we have the symmetrized form

\[ J_2^2 = \left( 1 - \frac{14q_1^2}{3} \right) J_1^2 - \frac{2}{3k_1} (J_1 J_1, J_2) - q_1 (J_1, J_2) = -\frac{2q_1}{3} P^- + 2P^+ . \]

Here \([A, B, C] = ABC + ACB + BAC + BCA\).

Similarly, for the \(K\) operators it is straightforward to verify:

\[ [K_1, L_3] = 4p_1^2 p_2^2 K_1 - 4p_1 p_2 K_2 , \]

\[ [K_2, L_3] = 2p_1 p_2 K_2 [K_1, L_3] + 8p_1^3 p_2^2 K_1 - 4k_1^2 p_1^2 p_2 q_2 K_2 , \]

\[ [K_1, L_2] = [K_2, L_2] = 0 . \]

We find

\[ K^+ K^- \Psi^m = (-4)^{p_1 q_2} \left( -\frac{\mu}{2k_2} - \frac{b}{2} - \frac{c}{2} + \frac{1}{2} \right)_{p_1 q_2} \left( \frac{\mu}{2k_2} - \frac{b}{2} + \frac{c}{2} + \frac{1}{2} \right)_{p_1 q_2} \left( \frac{1 - \rho}{2} - \frac{\mu}{k_1} \right)_{2p_2 q_2} \left( \frac{1 + \rho}{2} - \frac{\mu}{k_1} \right)_{2p_2 q_2} \Psi^m = \xi_m \Psi_m , \]

\[ K^- K^+ \Psi^m = (-4)^{p_1 q_2} \left( \frac{\mu}{2k_2} + \frac{b}{2} - \frac{c}{2} + \frac{1}{2} \right)_{p_1 q_2} \left( \frac{\mu}{2k_2} - \frac{b}{2} + \frac{c}{2} + \frac{1}{2} \right)_{p_1 q_2} \left( \frac{1 + \rho}{2} + \frac{\mu}{k_1} \right)_{2p_2 q_2} \left( \frac{1 - \rho}{2} + \frac{\mu}{k_1} \right)_{2p_2 q_2} \Psi^m = \omega_m \Psi_m . \]

Note that each of \(\xi_m, \omega_m\) is invariant under the transformation \(\rho \to -\rho\), hence each is a polynomial in \(\rho^2 \sim L_2\). Further, under the transformation \(\mu \to -\mu\) we see that \(\xi_m, \omega_m\) switch places. Thus \(\xi_m + \omega_m\) is a polynomial in each of \(\mu^2\) and \(\rho^2\), hence a polynomial in \(L_3\) and \(L_2\):

\[ K^+ K^- + K^- K^+ = S^+ (L_2, L_3) . \]

Again the coefficients of the polynomial \(S^+\) depend on the parameters \(a, b, c, p_1, q_1, p_2, q_2\). Thus the basis dependent operator \(K^+ K^- + K^- K^+\) extends to a true globally defined symmetry operator \(S^+\).

Similarly, \((\xi_m - \omega_m) / \mu\) is a polynomial in each of \(\mu^2\) and \(\rho^2\), hence also a polynomial in \(L_3\) and \(L_2\):

\[ \frac{K^+ K^- - K^- K^+}{\mu} = S^- (L_2, L_3) . \]

Thus the basis dependent operator \((K^+ K^- - K^- K^+) / \mu\) extends to the true globally defined symmetry operator \(S^-\). Additional computations yield

\[ [K_1, K_2] = 2p_1 p_2 K_2^2 - 2S^- , \quad K_1 K_2 = \frac{1}{2} [K_1, K_2] + p_1 p_2 K_1^2 - S^- , \]

\[ -K_2^2 L_3 = K_2^2 - 2S^+ + 2p_1 p_2 K_1 K_2 , \]

\[ [K_1, K_1, L_3] = 6K_1^2 L_3 - 24p_1^2 p_2^2 K_1^2 + 12p_1 p_2 [K_1, K_2] + 8k_1^2 p_1^2 p_2 q_2 K_1^2 - 8p_1 p_2 S^- . \]
The last three identities are needed for the symmetrized result
\[ K_2^\pm = \frac{1}{6} (K_1, K_1, L_3) + 4K_2^2 p_1^2 q_2 \left( q_2 - \frac{P_2}{3} \right) K_1^+ - 3 p_1 p_2 (K_1, K_2) + \frac{10}{3} p_1 p_2 S^- + 2 S^+ . \]

We write
\[ J^+ \psi_{n, p, m} = J^+ (n, \rho, m) \psi_{n, p, \rho - 2q_2, m}, \quad J^- \psi_{n, p, m} = J^- (n, \rho, m) \psi_{n, p, \rho + 2q_2, m}, \]
\[ K^+ \psi_{n, p, m} = K^+ (m) \psi_{n, p, m + p_1 q_2}, \quad K^- \psi_{n, p, m} = K^- (m, \rho) \psi_{n, p, m - p_1 q_2}, \]
where \( J^\pm, K^\pm \) are defined by the right-hand sides of equations (17), (18), (22), (23). Then
\[ [J^+, K^+] \psi_{n, p, m} = 1 - A(\rho, \mu), \quad [J^+, K^+] \psi_{n, p, m} = 1 + A(\rho, \mu), \]
\[ [J^-, K^-] \psi_{n, p, m} = 1 - A(-\rho, -\mu), \quad [J^-, K^-] \psi_{n, p, m} = 1 + A(-\rho, -\mu), \]
\[ [J^+, K^-] \psi_{n, p, m} = 1 - A(-\mu), \quad [J^-, K^+] \psi_{n, p, m} = 1 + A(-\mu), \]
where
\[ A(\rho, \mu) = \left( -\frac{\rho}{2} - \frac{\mu}{k_1} + \frac{1}{2} \right) Q_i. \]

From this we can compute the relations
\[ [K_2, J_2] = Q_{12}^{12} [J_1, K_1] + Q_{22}^{12} [J_1, K_2] + Q_{22}^{22} [J_2, K_1] + Q_{22}^{22} [J_2, K_2], \]
where the \( Q_{ij}^{kl} \) are rational functions of \( \rho^2 \) and \( \mu^2 \). In particular,
\[ Q_{22}^{12} = \frac{1}{4} (C(\rho,\mu) + C(-\rho,\mu) + C(\rho,-\mu) + C(-\rho,-\mu)), \]
\[ Q_{12}^{12} = \frac{\mu}{4} (C(\rho,\mu) - C(-\rho,\mu) - C(\rho,-\mu) + C(-\rho,-\mu)), \]
\[ Q_{12}^{22} = \frac{\mu}{4} (-C(\rho,\mu) + C(-\rho,\mu) - C(\rho,-\mu) + C(-\rho,-\mu)), \]
\[ Q_{22}^{22} = \frac{\mu}{4} (C(\rho,\mu) - C(-\rho,\mu) + C(\rho,-\mu) + C(-\rho,-\mu)). \]

Similarly we find
\[ [K_1, J_1] = Q_{11}^{11} [J_1, K_1] + Q_{12}^{11} [J_1, K_2] + Q_{21}^{11} [J_2, K_1] + Q_{22}^{11} [J_2, K_2], \]
\[ [K_2, J_2] = Q_{12}^{12} [J_1, K_1] + Q_{22}^{12} [J_1, K_2] + Q_{22}^{22} [J_2, K_1] + Q_{22}^{22} [J_2, K_2], \]
\[ [K_1, J_2] = Q_{11}^{11} [J_1, K_1] + Q_{12}^{11} [J_1, K_2] + Q_{21}^{11} [J_2, K_1] + Q_{22}^{11} [J_2, K_2], \]
where the \( Q_{ij}^{kl} \) are rational functions of \( \rho^2 \) and \( \mu^2 \). These \( Q \) functions are related to one another. Most particularly, \( Q_{11}^{11} = Q_{11}^{22} = Q_{22}^{12} = Q_{22}^{22} \). These relations make sense only on a generalized eigenbasis, but can be cast into the pure operator form
\[ [K, J]Q = [J_1, K_1] P_{11}^{12} + [J_1, K_2] P_{12}^{11} + [J_2, K_1] P_{21}^{22} + [J_2, K_2] P_{22}^{22}, \quad h, \ell = 1, 2, \]
where \( Q \) and \( P_{ij}^{kl} \) are polynomial operators in \( L_2, L_3 \). In particular, \( Q \) is the symmetry operator defined by the product \( B(\rho, \mu) B(-\rho, -\mu) \), where
\[ B(\rho, \mu) = \left( -\frac{\rho}{2} - \frac{\mu}{k_1} - 2q_1 p_2 + \frac{1}{2} \right) q_i + \left( -\frac{\rho}{2} - \frac{\mu}{k_1} + \frac{1}{2} \right) q_i, \]
on a generalized eigenbasis.
1.3. The symmetry $K_0$

Now we investigate the fact that our method does not always give generators of minimal order. In particular, for the case $k_1 = k_2 = 1$ there we have found a system of generators of orders $(2, 2, 2, 3)$ whereas it is known that a generating set of orders $(2,2,2,2)$ exists. In the standard case $k_1 = k_2 = 1$ it is easy to see that $J_1$ is of order 2, $K_1$ is of order 3 and $K_2$ is of order 4. We know that there is a second order symmetry operator $K_0$ for this case, independent of $L_3$, $H$, such that $[L_3, K_0] = K_1$, $[L_2, K_0] = 0$. We will show how to obtain this symmetry from the raising and lowering operators $K^\pm$, without making use of multiseparability. Thus, for general rational $k_1, k_2$ we look for a symmetry operator $K_0$ such that $[L_3, K_0] = K_1$, $[L_2, K_0] = 0$. Applying this condition to a formal eigenbasis of functions $\Psi^{n, p, m}$ we obtain the general solution

$$
K_0 = -\frac{1}{4p_1q_2} \left( \frac{K^+}{\mu(\mu + p_1p_2)} + \frac{K^-}{\mu(\mu - p_1p_2)} \right) + \xi_m
$$

(27)

where $\xi_m$ is a rational scalar function. It is easy to check that the quantity in parentheses is a rational scalar function of $\mu^2$. Thus we will have a true constant of the motion, polynomial in the momenta, provided we can choose $\xi_m$ such that the full quantity (28) is polynomial in $\mu^2$.

To determine the possibilities we need to investigate the singularities of this solution at $\mu = 0$ and $\mu = \pm p_1, p_2$, i.e. $m = -(b + c + 1)/2$, $m = (\pm p_1q_2 - b - c - 1)/2$. Noting that

$$
Y(2)^{(b+c+1)/2+lp_2q_2/(2p_2q_2)} = Y(2)^{(b+c+1)/2+lp_2q_2/(2p_2q_2)}
$$

$$
X(2)^{(b+c+1)/2+j} = X(2)^{(b+c+1)/2-j}
$$

for arbitrary $l, j$ we see that $K^+ \to K^-$ as $\mu \to 0$ so that the pole at $\mu = 0$ is removable. For general $p_1, p_2, q_1, q_2$, odd we set

$$
\xi_m = -\frac{D_3(L_2)}{2p_1q_2(\mu + p_1p_2)(\mu - p_1p_2)}
$$

and determine a polynomial $D_3$ such that the operator $K_0$ has a removable singularity at $\mu = p_1p_2$, i.e. such that the residue is 0. (Since the solution is a polynomial in $\mu^2$ we do not have to worry about the singularity at $\mu = -p_1p_2$.) Thus to compute the residue we consider the product of operators that forms $K^-$ as applied to basis vectors for which $\mu = p_1p_2$. First consider the product of the $p_1q_2$ factors of $X(2)^m$:

$$(X(2)_{-}^{p_1q_2-b-c/2} \cdots X(2)_{-}^{(b-c)/2} \cdots X(2)_{-}^{p_1q_2-b-c-1/2}).$$

We make use of the relations

$$
X(2)_{+}^{(b+c+1)/2+j} = X(2)_{-}^{-(b+c+1)/2-j}, \quad X(2)_{+}^{m-1}X(2)_{-}^m = 4m(m + b)(m + c)(m + b + c).
$$

There are an odd number of $X(2)^m$ factors in $K^-$ and the central factor is $X(2)_{-}^{(b+c)/2} = (b^2 - c^2)/2$, a constant. The operators on either side of this central term pair up:

$$
X(2)_{-}^{(b+c)/2-1}X(2)_{-}^{-(b+c)/2+1} = X(2)_{-}^{(b+c)/2}X(2)_{-}^{-(b+c)/2+1} = 4 \begin{pmatrix} b - c + 2 \cr 2 \end{pmatrix} \begin{pmatrix} -b + c + 2 \cr 2 \end{pmatrix} \begin{pmatrix} b + c + 2 \cr 2 \end{pmatrix} \begin{pmatrix} -b - c \cr 2 \end{pmatrix}
$$

which acts as multiplication by a constant. Then we consider the next pair $X(2)_{-}^{(b+c)/2-2}X(2)_{-}^{-(b+c)/2+2}$, and so on to evaluate the product

$$
\frac{b^2 - c^2}{2} \prod_{m = -(b+c)/2+1}^{m = -(b+c)/2-1} \left( 4m(m + b)(m + c)(m + b + c) \right).
$$
Now we consider the $2p_1q_2 Y(2)_m$ factors in $K^-$:

$$Y(2)_m = (-2p_1q_2 - b - c + 1/2) Y(2)_{m_0} \cdot \cdots \cdot Y(2)_{m_{2k_3}/2k_1} Y(2)_{m_0}$$

where $m_0 = (p_1q_2 - b - c - 1)/2$. We make use of the relations

$$Y(2)_+^{-(b+c+1)/2+\rho/(2p_2q_1)} = Y(2)_-^{-(b+c+1)/2-\rho/(2p_2q_1)},$$

and pair up the first and last factors, the second and next to last factors, and so on, to obtain the product $\Pi_{\ell=1}^{\rho/2} (\ell - \frac{1}{2})^{2 - c_\ell^2}$. Thus we obtain the residue of the pole and determine that

$$D_2(L_2) = 2(c^2 - b^2) \Pi_{\ell=1}^{\rho/2} (\ell - \frac{1}{2})^{2 - c_\ell^2} \sum_{j=1}^{\rho/2} (2j - b - c)(2j + b - c)$$

$$\times (2j - b + c)(2j + b + c).$$

The full operator

$$K_0 = -\frac{1}{4p_1q_2} \left( \frac{K^+}{\mu + p_1p_2} + \frac{K^-}{\mu - p_1p_2} \right) = \frac{D_2(L_2)}{2p_1q_2(\mu + p_1p_2)(\mu - p_1p_2)}$$

is thus a polynomial in $\rho^2$ and $\mu^2$, as well as a differential operator in $x, z$, hence it can be defined as a pure differential symmetry operator, independent of basis. From the explicit expression for $K_0$ we find easily that

$$4p_1q_2K_0(\rho^2_1 + L_3) = K_2 + p_1p_2K_1 + D_2(L_2),$$

which can be written in the symmetrized form

$$2p_1q_2(\rho^2_1 + L_3, K_0) = K_2 + 3p_1p_2K_1 + D_2(L_2).$$

By explicit computation we have checked that this gives the correct second order symmetry operator in the case $k_1 = k_2 = 1$. Note. A similar construction using the operators $J^\pm$ fails to produce a true symmetry operator $J_0$.

Now we consider the symmetry algebra generated by $H, L_2, L_3, K_0, K_1, J_1$. Using the results of the last two sections we can find algebraic relations between $[J_1, K_0]$ and the generators, so that the symmetry algebra closes algebraically. However, it does not close polynomially.

### 1.4. Two-variable models of the three-parameter symmetry algebra action

The recurrence operators introduced via special function theory lead directly to two-variable function space models representing the symmetry algebra action in terms of difference operators on spaces of rational functions $f(\rho, \mu)$. Indeed, from relations (17), (18), (22), (23) it is easy to write down a function space model for irreducible representations of the symmetry algebra. Note that since $H$ commutes with all elements of the algebra, it corresponds to multiplication by a constant $E$ where $E = \alpha^2/\mu^2$ in the model. We let complex variables $\rho, \mu$ correspond to the multiplication realization

$$H = L_1 = E, \quad L_2 = k_1^4 \frac{1 - \rho^2}{4}, \quad L_3 = -\mu^2.$$

We take a generalized basis function corresponding to eigenvalues $\rho_0, \mu_0$ in the form $\delta(\rho - \rho_0)\delta(\mu - \mu_0)$. Then the action of the symmetry algebra on the space of functions $f(\rho, \mu)$ is given by equations

$$J^\pm f(\rho, \mu) = \left( \frac{4\rho_0}{u} \right)^{p_1} \left( -\frac{\rho}{2} - \frac{\mu}{k_1} + \frac{1}{2} \right)_{q_1} \left( \frac{u}{2} - k_1 \rho \right)^{p_1} f(\rho - 2q_1, \mu).$$

$$f(\rho, \mu) = \left( \frac{4\rho_0}{u} \right)^{p_1} \left( -\frac{\rho}{2} - \frac{\mu}{k_1} + \frac{1}{2} \right)_{q_1} \left( \frac{u}{2} - k_1 \rho \right)^{p_1} f(\rho - 2q_1, \mu).$$

(31)
Now we study the four-parameter extended Kepler–Coulomb system (4). The separation
of rational functions \( f(\rho, \mu) \) determines a function space realization of the symmetry algebra. The space of 
\( J^\pm f(\rho, \mu) = (\frac{4\lambda}{u})^{p_1} (\frac{\rho - \mu + 1}{2})^{q_1} (\frac{\mu + k_1 \rho + 1}{2})^{p_1} f(\rho + 2q_1, \mu), \) 
where we have taken \( p_1 = (24), (28) \) determine a function space realization of the symmetry algebra. The space of 
where we have performed a gauge transformation to make the expressions for 
therefore \( J^\pm \) more symmetric. The operators \( J_1, J_2, K_0, K_1, K_2 \) defined in terms of \( J^\pm, K^\pm, L_1, L_2, L_3 \) by (23), 
(24), (28) determine a function space realization of the symmetry algebra. The space of 
rational functions \( f(\rho, \mu) \) is invariant under the symmetry algebra action, but the space of the 
polynomial functions is not.

2. The Kepler–Coulomb system with four-parameter potential

Now we study the four-parameter extended Kepler–Coulomb system (4). The separation
of the equations, \( H\Psi = E\Psi, L_3\Psi = -\mu^2\Psi, L_2\Psi = -\mu^2(1 - \rho^2)\Psi \) with 
\( \Psi = R(\rho)\Theta(\Omega)\Phi(z) \), are:

\[
\begin{align*}
(1) & \quad \left( \partial_{\rho}^2 + k_2^2 \frac{(1 - b^2)}{\cos^2(k_2z)} + k_2^2 \frac{(1 - c^2)}{\sin^2(k_2z)} \right) \Phi(z_2) = -\mu^2 \Phi(z_2), \\
(2) & \quad \left( \partial_{\rho}^2 + k_1 \cot(k_1\rho) \partial_{\rho} - \frac{\mu^2}{\sin^2(k_1\rho)} + k_1^2 \frac{(1 - d^2)}{\cos^2(k_1\rho)} \right) \Theta(\Omega) = \frac{k_1^2}{4} (1 - \rho^2) \Theta(\Omega),
\end{align*}
\]

where we have taken \( \beta = k_2^2 \frac{(1 - b^2)}{\cos^2(k_2\rho)}, \gamma = k_2^2 \frac{(1 - c^2)}{\sin^2(k_2\rho)}, \) and we write \( \mu = k_2(2\alpha + b + c + 1) \);

\[
\Theta(\Omega) = \frac{\Psi(\Omega)}{\sqrt{\sin(k_1\rho)}}
\]
we see that the differential equation satisfied by $\Psi$ is

\[(2^\prime) \left( \partial^2_r + \frac{k^2}{r^2} - \frac{\mu^2}{\sin^2(k_1 \theta_1)} + \frac{k_1^2}{\cos^2(k_1 \theta_1)} + \frac{k_1^2 \rho^2}{4} \right) \Psi(\theta_1) = 0.\]

Further

\[(3) \left( \partial^2_r + \frac{2}{r} \partial_r + \frac{1 - k_1^2 \rho^2}{4r^2} + \frac{\alpha}{r} - E \right) R(r) = 0.\]

In (3) we introduce a new variable $R = 2\sqrt{E}r$ so that the separation equation becomes

\[(3^\prime) \left( \partial^2_{R^2} + \frac{1 - k_1^2 \rho^2}{4R^2} + \frac{\alpha}{2\sqrt{ER}} - \frac{1}{4} \right) S(R) = 0,

with $R(r) = S(R)/R$. The separated solutions are

\[S_{n,m,p} = R_n^{k_1,\rho}(r) \Phi_m^{(c,b)}(\cos(2k_2 \theta_2)) \frac{\Psi_p^{(\mu/k_1,d)}(\cos(k_1 \theta_1))}{\sqrt{\sin(k_1 \theta_1)}}, \]

\[\Phi_m^{(c,b)}(\cos(2k_2 \theta_2)) = \sin^{\rho+1/2}(k_2 \theta_2) \cos^{\rho+1/2}(k_2 \theta_2) P_m^{(c,b)}(\cos(2k_2 \theta_2)) \]

\[\Psi_p^{(\mu/k_1,d)}(\cos(k_1 \theta_1)) = \sin^{\rho+1/2}(k_1 \theta_1) \cos^{\rho+1/2}(k_1 \theta_1) P_p^{(\mu/k_1,d)}(\cos(2k_1 \theta_1)),\]

where the $P_m^{(c,b)}(\cos(2k_2 \theta_2))$, $P_p^{(\mu/k_1,d)}(\cos(2k_1 \theta_1))$ are Jacobi functions;

\[R_n^{k_1,\rho}(r) = \frac{S(r)}{2\sqrt{ER}}, \quad S(r) = (2\sqrt{E})^{k_1,\rho/2} e^{-\sqrt{E}r(k_1,\rho+1/2)} \frac{L_n^{k_1,\rho}(2\sqrt{ER})}{2},\]

and $\rho = 2(2p + \frac{k_1}{k_1} + d + 1)$, where the $L_n^{k_1,\rho}(2\sqrt{ER})$ are associated Laguerre functions, and the relation between $E$, $\rho$ and $n$ is the quantization condition (where we set $u = 2n + k_1, \rho + 1$)

\[E = \frac{\alpha^2}{u^2} = \left( \frac{2n + 2k_1, p + 2k_2, m + 2k_1, d + 1 + 2k_2, b + c + 1 + 1}{(2k_1)^2} \right). \quad (39)\]

As in the computations with the three-parameter potential, we are interested in the space of generalized eigenfunctions, not the normalization of any individual eigenfunction. Thus our relations are valid on generalized eigenspaces and do not agree with the normalization of the usual polynomial eigenfunctions.

There are transformations that preserve $E$ and imply quantum superintegrability. Indeed for $k_1 = p_1/q_1$, $k_2 = p_2/q_2$ the following transformations will accomplish this:

1. \[n \rightarrow n + 2p_1, \quad m \rightarrow m, \quad p \rightarrow p - q_1,\]
2. \[n \rightarrow n - 2p_1, \quad m \rightarrow m, \quad p \rightarrow p + q_1,\]
3. \[n \rightarrow n, \quad m \rightarrow m - p_2q_2, \quad p \rightarrow p + q_1p_2,\]
4. \[n \rightarrow n, \quad m \rightarrow m + p_1q_2, \quad p \rightarrow p - q_1p_2.\]

To effect the $r$-dependent transformations (1), (2) we use $Y_1(z)^{\pm \pm}$ as in (13). To incorporate the $\theta_1$-dependent parts of (1), (2) we use recurrence formulas for the functions $\Psi_p^{(\mu/k_1,d)}(z)$ where $z = \cos(2k_1 \theta_1)$:

\[Z(1)^p \frac{\Psi_p^{(\mu/k_1,d)}(z)}{\sqrt{\sin(k_1 \theta_1)}} \equiv -\left(1 - z^2 \right) \left(1 - \frac{\rho}{2} \right) \partial_\rho + \frac{1}{2} \left( \left(1 - \frac{\rho}{2} \right) \left( -\frac{\rho}{2} \right) z + \frac{\mu^2}{k_1^2} - a^2 \right) \frac{\Psi_p^{(\mu/k_1,d)}(z)}{\sqrt{\sin(k_1 \theta_1)}} \]

\[= -2 \left( \frac{\rho}{4} + \frac{\mu}{2k_1} - d - \frac{1}{2} \right) \left( \frac{\rho}{4} - \frac{\mu}{2k_1} + d - \frac{1}{2} \right) \frac{\Psi_p^{(\mu/k_1,d)}(z)}{\sqrt{\sin(k_1 \theta_1)}}.\]
\[ Z(1)_\rho^p \frac{\psi_{\mu/k_1}^{i,k_1}(z)}{\sqrt{\sin(k_1 \theta_1)}} = \left( - (1 - z^2) \left( 1 + \frac{\mu}{2} \right) - \frac{1}{2} \left( \left( 1 + \frac{\mu}{2} \right) \left( \frac{\rho}{2} z + \frac{\mu^2}{k_1^2} - d^2 \right) \right) \right) \frac{\psi_{\mu/k_1}^{i,k_1}(z)}{\sqrt{\sin(k_1 \theta_1)}} + 2 \left( \frac{\rho}{4} + \frac{\mu}{2 k_1} + \frac{d}{2} + \frac{1}{2} \right) \left( \frac{\rho}{4} - \frac{\mu}{2 k_1} - \frac{d}{2} + \frac{1}{2} \right) \frac{\psi_{\rho+1,k_1}^{i,k_1}(z)}{\sqrt{\sin(k_1 \theta_1)}}. \]

We see that the operators \( Z(1)_\rho^p \) depend on \( \mu^2 \) (which can be interpreted as a differential operator) and are polynomial in \( \rho \). We now form the operators

\[ J^+ = (Y(1)^{\rho+2,p+1}_+ Y(1)^{\rho+2,p+2}_+ \ldots Y(1)^{\rho+1,p}_+ Y(1)^{\rho,p}_+)(Z(1)^{\rho-(p-1)}_+ \ldots Z(1)^{\rho}_+), \]

\[ J^- = (Y(1)^{\rho-2,p+1}_- Y(1)^{\rho-2,p+2}_- \ldots Y(1)^{\rho-1,p}_- Y(1)^{\rho,p}_-)(Z(1)^{\rho+(p+1)}_+ \ldots Z(1)^{\rho}_+). \]

Since \( J^+ \) and \( J^- \) switch places under the reflection \( \rho \rightarrow -\rho \) we see that \( J_2 = J^+ + J^- \), \( J_1 = (J^- - J^+)/\rho \) are even functions in both \( \rho \) and \( \mu \) and can be interpreted as differential operators.

To implement the \( \theta_1 \)-dependent parts of (3), (4) we set \( w = \sin^2(k_1 \theta_1) \) and use

\[ W_\rho^p(1) \frac{\psi_{\mu/k_1}^{i,k_1}(w)}{w^{1/4}} = \left( \left( 1 + \frac{\mu}{k_1} \right) (w - 1) \frac{d}{dw} + \frac{\mu}{4 k_1} \left( 1 - \frac{2}{w} \right) \left( 1 + \frac{\mu}{k_1} \right) + \frac{d^2 - \rho^2/4}{4} \right) \frac{\psi_{\mu/k_1}^{i,k_1}(w)}{w^{1/4}} \]

\[ = \left( \frac{\mu}{k_1} + \frac{\mu}{4 k_1} + d + 1 \right) \frac{\psi_{\mu/k_1+2}^{i,k_1}(w)}{w^{1/4}}. \]

\[ W_\rho^p(1) \frac{\psi_{\mu/k_1}^{i,k_1}(w)}{w^{1/4}} = \left( \left( 1 - \frac{\mu}{k_1} \right) (w - 1) \frac{d}{dw} + \frac{\mu}{4 k_1} \left( 1 - \frac{2}{w} \right) \left( 1 - \frac{\mu}{k_1} \right) + \frac{d^2 - \rho^2/4}{4} \right) \frac{\psi_{\mu/k_1}^{i,k_1}(w)}{w^{1/4}} \]

\[ = \frac{1}{4} \left( \frac{\mu}{k_1} + \frac{\mu}{4 k_1} + d + 1 \right) \frac{\psi_{\mu/k_1-2}^{i,k_1}(w)}{w^{1/4}}. \]

To implement the \( \theta_2 \)-dependent parts of (3) and (4) we use the recurrences \( X(2)_m^p \) already employed for the three-parameter potential, (19).

We define

\[ K^+ = (W(1)^{\rho+(p+1)}_+ \ldots W(1)^{\rho+1}_+)(X(2)^{\rho-(p+1)}_m \ldots X(2)^{\rho}_m), \]

\[ K^- = (W(1)^{\rho-(p+1)}_- \ldots W(1)^{\rho-1}_-)(X(2)^{\rho+(p+1)}_m \ldots X(2)^{\rho}_m). \]

From the form of these operators we see that they are even functions of \( \rho^2 \) and they switch places under the reflection \( \mu \rightarrow -\mu \). Thus \( K^+_2 = K^+_1 + K^-_1 = (K^- - K^+)/\mu \) are even functions in both \( \rho \) and \( \mu \) and can be interpreted as differential operators.

We have the action of \( J^+, K^+ \) on a generalized eigenbasis:

\[ J^+ \Xi_{m,p} = \frac{(-\frac{\mu}{k_1} - \frac{\mu}{4 k_1} + \frac{d}{2} + \frac{1}{2}) q_1 \left( \frac{\rho}{4} - \frac{\mu}{2 k_1} - \frac{d}{2} + \frac{1}{2} \right) q_{1,p_1} \left( \frac{\rho}{4} - \frac{\mu}{2 k_1} + \frac{d}{2} + \frac{1}{2} \right) q_{1,p_2}}{(2)^{-4p+q_1} (-1)^n \alpha^{-2p+4p_1}} \]

\[ \times \Xi_{-2p+1,m,p+q_1}, \]

\[ J^- \Xi_{m,p} = \frac{(-\frac{\mu}{k_1} - \frac{\mu}{4 k_1} + \frac{d}{2} + \frac{1}{2}) q_1 \left( \frac{\rho}{4} + \frac{\mu}{2 k_1} + \frac{d}{2} + \frac{1}{2} \right) q_{1,p_1} \left( \frac{\rho}{4} + \frac{\mu}{2 k_1} - \frac{d}{2} + \frac{1}{2} \right) q_{1,p_2}}{(2)^{-4p+q_1} (-1)^n \alpha^{-2p+4p_1}} \]

\[ \times \Xi_{2p+1,m-p+q_1}, \]

\[ K^+ \Xi_{m,p} = (-1)^{q_1} (2)^{4p+q_1} \left( \frac{\mu}{k_1} \right) \left( \frac{\rho}{4} - \frac{\mu}{2 k_1} - \frac{d}{2} + \frac{1}{2} \right) q_{1,p_1} \left( \frac{\rho}{4} - \frac{\mu}{2 k_1} + \frac{d}{2} + \frac{1}{2} \right) q_{1,p_2} \]

\[ \times \left( -\frac{\mu}{2 k_2} + \frac{b}{2} + \frac{c}{2} + \frac{1}{2} \right)_{p_1,p_2} \left( -\frac{\mu}{2 k_2} - \frac{b}{2} + \frac{c}{2} + \frac{1}{2} \right)_{p_1,p_2} \Xi_{m-p+q_1,p+q_1}, \]

\[ J^- \Xi_{m,p} = (-1)^{q_1} (2)^{-4p+q_1} (-1)^n \alpha^{-2p+4p_1} \]

\[ \times \left( \frac{\rho}{4} + \frac{\mu}{2 k_1} + \frac{d}{2} + \frac{1}{2} \right) q_{1,p_1} \left( \frac{\rho}{4} + \frac{\mu}{2 k_1} - \frac{d}{2} + \frac{1}{2} \right) q_{1,p_2} \]

\[ \times \left( \frac{\mu}{k_1} \right) \left( \frac{\rho}{4} - \frac{\mu}{2 k_1} - \frac{d}{2} + \frac{1}{2} \right) q_{1,p_1} \left( \frac{\rho}{4} - \frac{\mu}{2 k_1} + \frac{d}{2} + \frac{1}{2} \right) q_{1,p_2} \Xi_{m,p-q_1,p+q_1}. \]
Further we find

\[ H \Xi_{n,m,p} = \frac{\alpha^2}{(2n + k_1 \rho + 1)^2} \Xi_{n,m,p}, \quad L_2 \Xi_{n,m,p} = \frac{k_1^2}{4} (1 - \rho^2) \Xi_{n,m,p}, \]

\[ L_3 \Xi_{n,m,p} = -\mu^2 \Xi_{n,m,p}, \]

\[ u = 2n + k_1 \rho + 1 = \frac{\alpha}{\sqrt{E}}, \quad \rho = 2(2p + \mu/k_1 + d + 1), \quad \mu = k_2(2m + b + c + 1). \]

Using these definitions and computing on an eigenbasis, it is straightforward to verify the relations:

\[ [J_1, L_2] = 2k_1^2 q_1 J_2 + 4p_1^2 J_1, \quad [J_2, L_2] = -4q_1 [J_1, L_2] - 4q_1^2 k_1^2 J_2 + 2k_1^2 q_1 (1 - 8q_1^2) J_1, \]

\[ [J_1, L_3] = [J_2, L_3] = 0, \quad [K_1, L_3] = 4p_1 p_2 K_2 + 4p_1^2 p_2^2 K_1, \]

\[ [K_2, L_3] = -2 p_1 p_2 [L_3, K_1] - 4p_1^2 p_2^2 K_2 - 8p_3 p_2^3 K_1, \quad [K_1, L_2] = [K_2, L_2] = 0. \]

Further we find

\[ J^+ J^- \Xi_{n,m,p} = 4^{n_1+q_1} E^{2n_1} \left( \frac{\rho/2 - \frac{\mu}{k_1} - d + 1}{2} \right)^{q_1} \left( \frac{\rho/2 + \frac{\mu}{k_1} + d + 1}{2} \right)^{q_1} \Xi_{n,m,p}, \]

\[ J^- J^+ \Xi_{n,m,p} = 4^{n_1+q_1} E^{2n_1} \left( \frac{-\rho/2 - \frac{\mu}{k_1} - d + 1}{2} \right)^{q_1} \left( \frac{-\rho/2 + \frac{\mu}{k_1} + d + 1}{2} \right)^{q_1} \Xi_{n,m,p}, \]

\[ K^+ K^- \Xi_{n,m,p} = 2^{n_1+q_1} \left( \frac{\rho/2 + \frac{\mu}{k_1} + d + 1}{2} \right)^{q_1} \left( \frac{\rho/2 + \frac{\mu}{k_1} - d + 1}{2} \right)^{q_1} \Xi_{n,m,p}. \]
\[ K^-K^+ \Xi_{n,m,p} = 2^{2p_1q_1} \left( \frac{\rho - \frac{\mu}{\nu} + d + 1}{2} \right)_{q_1p_2} \left( \frac{\rho - \frac{\mu}{\nu} - d + 1}{2} \right)_{q_1p_2} \times \left( \frac{-\rho - \frac{\mu}{\nu} + d + 1}{2} \right)_{q_1p_2} \left( \frac{-\rho - \frac{\mu}{\nu} - d + 1}{2} \right)_{q_1p_2} \times \left( \frac{-\mu - b + c + 1}{2} \right)_{p_1q_2} \left( \frac{-\mu - b + c + 1}{2} \right)_{p_1q_2} \times \left( \frac{-\mu - b + c + 1}{2} \right)_{p_1q_2} \left( \frac{-\mu - b + c + 1}{2} \right)_{p_1q_2} \Xi_{n,m,p}. \]

From these expressions it is easy to see that each of \( J^+J^-, J^-J^+ \) is a polynomial in \( \mu^2 \) and \( E \) and that these operators switch places under the reflection \( \rho \to -\rho \). Thus \( P_1(H, L_2, L_3) = J^+J^- + J^-J^+ \) and \( P_2(H, L_2, L_3) = (J^+J^- - J^-J^+)/\rho \) are each polynomials in \( H, L_2, L_3 \).

Similarly, each of \( K^-K^-, K^-K^- \) is a polynomial in \( \rho^2 \) and in \( \mu \) and they switch places under the reflection \( \mu \to -\mu \). Thus \( P_3(L_2, L_3) = K^-K^- + K^-K^+ \), \( P_4(L_2, L_3) = (K^-K^- - K^-K^+)/\mu \) are polynomials in \( L_2, L_3 \).

Straightforward consequences of these formulas are the structure relations

\[ [J_1, J_2] = -2q_1J_1^2 - 2P_2(H, L_2, L_3), \quad [K_1, K_2] = -2p_1p_2K_1^2 - 2P_4(L_2, L_3), \]

and the unsymmetrized structure relations

\[ J_1^2 \left( J_1^2 - k_1^{-2}L_2 \right) = J_2^2 - 2P_4(H, L_2, L_3) - 2q_1J_1J_2, \quad K_1^2 = \frac{K^2_1}{2} + 2P_3(L_2, L_3) + 2p_1p_2K_1K_2. \]

### 2.1. Lowering the orders of the generators

Just as for the classical analogues, we can find generators that are of order one less than \( J_1 \) and \( K_1 \). First we look for a symmetry operator \( J_0 \) such that \([L_2, J_0] = J_1 \) and \([L_3, J_0] = 0\):

\[ J_0 = \frac{1}{2k_1^2q_1} \left( J^- \frac{J^-}{\mu(\rho + 2q_1)} + \frac{J^+}{\rho(\rho - 2q_1)} \right) + \frac{S_1(H, L_3)}{\rho^2 - 4q_1^2}, \]

where the symmetry operator \( S_1 \) is to be determined. From this it is easy to show that

\[ 2k_1^2q_1(\rho^2 - 4q_1^2)J_0 = -J_2 + 2q_1J_1 + 2k_1^2q_1S_1(H, L_3). \]

To determine \( S_1 \) we evaluate both sides of this equation for \( \rho = -2q_1 \); \( S_1(H, L_3) = \frac{1}{k_1^2q_1}J^+ \mid \rho = -2q_1 \), and apply \( J^- \) to a specific eigenfunction. Consider the product of the 2\( p_1 \) factors of \( Y(1)^n \):

\[ Y(1)^{m-p_1}Y(1)^{m-p_1}Y(1)^{m-p_1} \ldots Y(1)^{m-p_1}Y(1)^{m-p_1} \]

acting on an eigenbasis function corresponding to \( m_0 = p_1 + (\alpha/\sqrt{E} - 1)/2 \). There are an even number of \( Y(1)^m \) factors in \( J^- \) and the central pair is \( Y(1)^{m-p_1}Y(1)^{m-p_1} \). The left-hand operator corresponds to \( k_1\mu = 0 \) and the right-hand operator corresponds to \( k_1\mu = -2 \). Thus we have

\[ Y(1)^{m-p_1}Y(1)^{m-p_1} = Y(1)^{m-p_1}Y(1)^{m-p_1} = 4(\alpha^2 - E). \]
Operators on either side of the central term can be grouped in nested pairs
$Y(1)_-^{n_0-p_1-1}Y(1)_-^{n_0-(p_1-1)+s}$, where $s = 0, 1, \ldots, p_1 - 1$, corresponding to $k_1 \rho = 2s$, $k_1 \rho = -2(s + 1)$, respectively. Each pair gives

$$Y(1)_-^{n_0-p_1-1}Y(1)_-^{n_0-(p_1-1)+s} = \left(2(2s + 1)\partial_r + \left( 2\alpha + \frac{1 - 4 \mu^2}{r} \right) \right) \left(2(-2s - 1)\partial_r + \left( 2\alpha + \frac{1 - 4(s + 1)^2}{r} \right) \right)$$

$$= Y(1)_-^{n_0-p_1-1}Y(1)_-^{n_0-(p_1-1)+s} = 4(2(\alpha^2 - (1 + 2s)^2)E).$$

Thus,

$$Y(1)_-^{n_0-(2p_1-1)}Y(1)_-^{n_0-(2p_1-2)} \cdots Y(1)_-^{n_0} = 4^{p_1} \Pi_{s=0}^{p_1-1} (\alpha^2 - (1 + 2s)^2)E.$$

Now we consider the $q_1$ factors $Z(1)_+^{I_q}$ in $J^*$:

$$Z(1)_+^{p_0+(q_1-1)}Z(1)_+^{p_0+(q_1-2)} \cdots Z(1)_+^{p_0}$$

acting on an eigenbasis function corresponding to $p_0 = -(q_1 + \mu/k_1 + d + 1)/2$. There are an odd number of factors and the central factor is $(\mu^2/k_1^2 - d^2)/2$. Operators on either side of this central term can be grouped in nested pairs to give

$$Z(1)_+^{p_0+(q_1-1)}Z(1)_+^{p_0+(q_1-2)} \cdots Z(1)_+^{p_0}$$

$$= -2(\mu^2/k_1^2 - d^2) (-4)^{(q_1-1)/2} \Pi_{s=1}^{p_1-1} \left( \left( s - \frac{d}{2} \right)^2 - \frac{\mu^2}{4k_1^2} \right) \left( \left( s + \frac{d}{2} \right)^2 - \frac{\mu^2}{4k_1^2} \right),$$

$$S_1(H, L_3) = \frac{1}{2k_1^2 q_1} \left( \frac{L_3}{k_1^2} + d^2 \right) (-4)^{(q_1-1)/2} 4^{p_1} \Pi_{s=0}^{p_1-1} \left( (\alpha^2 - (1 + 2s)^2)H \right)$$

$$\times \Pi_{s=1}^{p_1-1} \left( \left( s - \frac{d}{2} \right)^2 + \frac{L_3}{4k_1^2} \right) \left( \left( s + \frac{d}{2} \right)^2 + \frac{L_3}{4k_1^2} \right),$$

$$[L_2, J_0] = J_1, \quad [L_3, J_0] = 0, \quad 2k_1^2 q_1 J_0 \left( 1 - \frac{4L_2}{k_1^2} - 4q_1^2 \right) = -J_2 + 2q_1 J_1 + 2k_1^2 q_1 S_1(H, L_3).$$

Next we look for a symmetry operator $K_0$ such that $[L_3, K_0] = K_1$ and $[L_2, K_0] = 0$:

$$K_0 = -\frac{1}{4p_1 p_2} \left( \frac{K^-}{\mu + p_1 p_2} + \frac{K^+}{\mu - p_1 p_2} \right) + \frac{S_2(L_2)}{\mu^2 - p_1^2 p_2^2},$$

where the symmetry operator $S_2$ is to be determined. From this it is easy to show that

$$4p_1 p_2 K_0 (\mu^2 - p_1^2 p_2^2) = -K_2 + p_1 p_2 K_1 + 4p_1 p_2 S_2(L_2).$$

To determine $S_2$ we evaluate both sides of this equation for $\mu = -p_1 p_2$: $S_2(L_2) = \frac{1}{2p_1 p_2} K^-_{\mu=-p_1 p_2}$ and apply $K^-$ to a specific eigenfunction. Consider the product of the $q_1 p_2$ factors of $W(1)_-^{p_0}$:

$$(W(1)_-^{p_0-(q_1 p_2-1)} \cdots W(1)_-^{p_0-(q_1 p_2-2)} \cdots W(1)_-^{p_0})$$

where $p_0 = (\rho - q_1 p_2 - d - 1)/2$. There are an odd number of $W(1)_-^{q_1 p_2}$ factors in $K^-$ and the central factor is $(d^2 - \rho^2/4)/4$. Operators on either side of this central term can be grouped in nested pairs:

$$(W(1)_+^{q_1 p_2-1}) \cdots W(1)_+^{q_1 p_2-2} \cdots W(1)_+^{p_0})$$

$$= \left( \frac{d^2}{4} - \frac{\rho^2}{4} \right) \Pi_{s=1}^{p_1-1} \left( \left( \frac{d}{2} + \frac{\rho}{4} \right)^2 - s^2 \right) \left( \left( \frac{d}{2} + \frac{\rho}{4} \right)^2 - s^2 \right).$$
Now we consider the \( p_1 q_2 \) factors \( X(2)_i \) in \( K^- \):

\[
(X(2)_i)^{m_0+1} X(2)_i^{m_0+q_1-1} \ldots X(2)_i^{m_0+q_2-2} X(2)_i^{m_0} = 2^{m_0+q_2-2} \Pi_{s=1}^{(p_1 q_2-1)/2} (s + \frac{c+b}{2})
\]

where \( m_0 = -(p_1 q_2 + b + c + 1)/2 \). There are an odd number of factors and the central factor is \((b^2 - c^2)/2\). The operators on either side of this central term can be grouped in nested pairs to give:

\[
(X(2)_i)^{m_0+1} X(2)_i^{m_0+q_1-1} \ldots X(2)_i^{m_0+q_2-2} X(2)_i^{m_0} = 2^{m_0+q_2-2} \Pi_{s=1}^{(p_1 q_2-1)/2} (s + \frac{c+b}{2})
\]

\[
\times \left( s - \frac{c+b}{2} \right) \left( s - 1 - \frac{b-c}{2} \right) \left( s - 1 + \frac{b-c}{2} \right).
\]

\[
S_2(L_2) = \left( d^2 - \frac{\rho^2}{4} \right) 2^{p_1 q_2-5} \frac{(b^2 - c^2)}{p_1 p_2} \Pi_{s=1}^{(p_1 q_2-1)/2} \left( \left( \frac{d}{2} + \frac{\rho}{4} \right)^2 - s^2 \right) \left( \left( \frac{d}{2} - \frac{\rho}{4} \right)^2 - s^2 \right)
\]

\[
\times \Pi_{s=1}^{(p_1 q_2-1)/2} \left( s + \frac{c+b}{2} \right) \left( s - \frac{c+b}{2} \right) \left( s - 1 - \frac{b-c}{2} \right) \left( s - 1 + \frac{b-c}{2} \right).
\]

We conclude that

\[
[L_3, K_0] = K_1, \quad [L_2, K_0] = 0, \quad 4p_1 p_2 K_0 (L_3 + p_1^2 p_2^2) = K_2 - p_1 p_2 K_1 - 4p_1 p_2 S_2(L_2).
\]

### 2.2. The structure equations

Now we determine the commutators of the \( J \)-operators with the \( K \)-operators. We write

\[
J^+ \Xi_{n,m,p} = J^+ (n, m, p) \Xi_{n+2p_1 m, p-q_1}, \quad J^- \Xi_{n,m,p} = J^- (n, m, p) \Xi_{n-2p_1 m, p+q_1},
\]

\[
K^+ \Xi_{n,m,p} = K^+ (m, p) \Xi_{n+m-p_2 q_2, p+q_2}, \quad K^- \Xi_{n,m,p} = K^- (m, p) \Xi_{n+m+p_2 q_2, p-q_2},
\]

where \( J^\pm, K^\pm \) are defined by the right-hand sides of equations (40), (41). Then, we find

\[
[J^+, K^+] \Xi_{n,m,p} = (1 - A(\rho, -\mu)) J^+ K^+ \Xi_{n,m,p},
\]

\[
[J^-, K^-] \Xi_{n,m,p} = (1 - A(\rho, -\mu)) J^- K^- \Xi_{n,m,p},
\]

or

\[
[J^+, K^+] \Xi_{n,m,p} = \frac{1 - A(-\rho, \mu)}{1 + A(-\rho, \mu)} \Xi_{n,m,p} \equiv C(-\rho, \mu) \{J^+, K^+\} \Xi_{n,m,p},
\]

\[
[J^-, K^+] \Xi_{n,m,p} = \frac{1 - A(\rho, \mu)}{1 + A(\rho, \mu)} \Xi_{n,m,p} \equiv C(\rho, \mu) \{J^- K^+\} \Xi_{n,m,p},
\]

\[
[J^+, K^-] \Xi_{n,m,p} = \frac{1 - A(-\rho, -\mu)}{1 + A(-\rho, -\mu)} \Xi_{n,m,p} \equiv C(-\rho, -\mu) \{J^+, K^-\} \Xi_{n,m,p},
\]

\[
[J^-, K^-] \Xi_{n,m,p} = \frac{1 - A(\rho, -\mu)}{1 + A(\rho, -\mu)} \Xi_{n,m,p} \equiv C(\rho, -\mu) \{J^- K^-\} \Xi_{n,m,p},
\]

\[
A(\rho, \mu) = \frac{(\frac{d}{2} + \frac{\mu}{q_2} + \frac{1}{2} - \frac{d}{2} q_1) \frac{\mu}{q_2} + \frac{1}{2} + \frac{d}{2} q_1}{(\frac{d}{2} + \frac{\mu}{q_2} + \frac{1}{2} - \frac{d}{2} q_1) \frac{\mu}{q_2} + \frac{1}{2} + \frac{d}{2} q_1}.
\]

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From this we can compute the relations
\[ [J_1, K_1] = Q_{11}^{11} [J_1, K_1] + Q_{12}^{11} [J_1, K_2] + Q_{21}^{11} [J_2, K_1] + Q_{22}^{11} [J_2, K_2], \]
\[ [J_2, K_2] = Q_{11}^{22} [J_1, K_1] + Q_{12}^{22} [J_1, K_2] + Q_{21}^{22} [J_2, K_1] + Q_{22}^{22} [J_2, K_2], \]
where the \( Q_{ij}^{kl} \) are rational functions of \( \rho^2 \) and \( \mu^2 \). In particular,
\[ Q_{11}^{11} = Q_{22}^{22} = \frac{1}{4} (C(-\rho, \mu) + C(\rho, \mu) + C(-\rho, -\mu) + C(\rho, -\mu)), \]
\[ Q_{12}^{11} = \frac{1}{\mu^2} Q_{21}^{21} = \frac{1}{4\mu} (-C(-\rho, \mu) - C(\rho, \mu) + C(-\rho, -\mu) + C(\rho, -\mu)), \]
\[ Q_{22}^{11} = \frac{1}{\mu^2 \rho^2} Q_{21}^{21} = \frac{1}{4\mu \rho} (C(-\rho, \mu) - C(\rho, \mu) - C(-\rho, -\mu) + C(\rho, -\mu)). \]
Similarly we find
\[ [J_1, K_2] = Q_{11}^{12} [J_1, K_1] + Q_{12}^{12} [J_1, K_2] + Q_{21}^{12} [J_2, K_1] + Q_{22}^{12} [J_2, K_2], \]
\[ [J_2, K_1] = Q_{11}^{21} [J_1, K_1] + Q_{12}^{21} [J_1, K_2] + Q_{21}^{21} [J_2, K_1] + Q_{22}^{21} [J_2, K_2], \]
\[ Q_{11}^{12} = Q_{22}^{22}, \quad Q_{12}^{12} = Q_{21}^{21} = Q_{11}^{11}, \quad Q_{21}^{21} = Q_{12}^{12}, \quad Q_{22}^{22} = \mu^2 Q_{21}^{21}, \quad Q_{22}^{21} = \mu^2 Q_{21}^{22}. \]
These relations make sense only on a generalized eigenbasis, but can be cast into operator form
\[ [K_\ell, J_\ell] Q = [J_1, K_1] P_{1\ell}^{1\ell} + [J_1, K_2] P_{1\ell}^{2\ell} + [J_2, K_1] P_{2\ell}^{1\ell} + [J_2, K_2] P_{2\ell}^{2\ell}, \quad h, \ell = 1, 2, \]
where \( Q \) and \( P_{ij}^{kl} \) are polynomial operators in \( L_2, L_3 \). In particular, \( Q \) is the symmetry operator defined by the product
\[ B(\rho, \mu) B(-\rho, -\mu) B(\rho, -\mu) B(-\rho, \mu), \]
\[ B(\rho, \mu) = \left( \frac{\rho}{4} + \frac{\mu}{2k_1} + \frac{1}{2} - \frac{d}{2} \right) q, \quad \left( \frac{\rho}{4} + \frac{\mu}{2k_1} + \frac{1}{2} + \frac{d}{2} \right) q, \]
\[ + \left( \frac{\rho}{4} + \frac{\mu}{2k_1} + \frac{1}{2} - q_1 p_2 - \frac{d}{2} \right) q, \quad \left( \frac{\rho}{4} + \frac{\mu}{2k_1} + \frac{1}{2} - q_1 p_2 + \frac{d}{2} \right) q, \]
on a generalized eigenbasis. Thus for general \( k_1, k_2 \) the symmetry algebra closes only algebraically. The basis generators are \( H, L_2, L_3, J_0, K_0 \) and the commutators \( J_1, K_1 \) are appended to the algebra.

2.3. Two-variable models of the four-parameter symmetry algebra action

Our recurrence operators lead directly to two-variable function space models that represent the symmetry algebra action in terms of the difference operators on spaces of rational functions \( f(\rho, \mu) \). Using relations (40), (41) we can determine a function space model for irreducible representations of the symmetry algebra. Since \( H \) commutes with all elements of the algebra, it corresponds to multiplication by \( E = \alpha^2/\beta^2 \) in the model. We let complex variables \( \rho, \mu \) correspond to the realization
\[ H = L_1 = E, \quad L_2 = k_1^2 \frac{1 - \rho^2}{4}, \quad L_3 = -\mu^2. \]
We take a generalized basis function corresponding to eigenvalues $\rho_0, \mu_0$ in the form
\[ \delta(\rho - \rho_0)\delta(\mu - \mu_0). \]
Then the action of the symmetry algebra on the space of functions $f(\rho, \mu)$ is given by
\[
J^+ f(\rho, \mu) = \frac{\alpha^2 f(\rho, \mu)}{2} \left( \frac{\rho - \mu}{2k_1} + \frac{d}{2} + \frac{1}{2} \right) q_1 \left( \frac{\rho + \mu}{2k_1} - \frac{d}{2} + \frac{1}{2} \right) q_2 \left( \frac{\mu}{2k_2} - \frac{1}{2} \right) f(\rho, \mu),
\]
(45)
\[
J^- f(\rho, \mu) = \frac{\alpha^2 f(\rho, \mu)}{2} \left( \frac{\rho - \mu}{2k_1} + \frac{d}{2} + \frac{1}{2} \right) q_1 \left( \frac{\mu}{2k_2} - \frac{1}{2} \right) f(\rho, \mu).
\]
(46)
\[
K^+ f(\rho, \mu) = \left( \frac{\rho - \mu}{2k_1} - \frac{d}{2} + \frac{1}{2} \right) q_1 p_2 \left( \frac{\rho - \mu}{2k_1} + \frac{d}{2} + \frac{1}{2} \right) q_2 p_2 \left( \frac{\mu}{2k_2} - \frac{1}{2} \right) f(\rho, \mu, -2p_1 q_2).
\]
(47)
\[
K^- f(\rho, \mu) = \left( \frac{\rho - \mu}{2k_1} + \frac{d}{2} + \frac{1}{2} \right) q_1 p_2 \left( \frac{\rho - \mu}{2k_1} - \frac{d}{2} + \frac{1}{2} \right) q_2 p_2 \left( \frac{\mu}{2k_2} - \frac{1}{2} \right) f(\rho, \mu, 2p_1 q_2).
\]
(48)

The space of rational functions is invariant under the algebra action, but polynomials are not.

2.4. The special case $k_1 = k_2 = 1$

In the case $k_1 = k_2 = 1$ we are in Euclidean space and our system has additional symmetry. In terms of Cartesian coordinates $x = r \sin \theta_1 \cos \theta_2$, $y = r \sin \theta_1 \sin \theta_2$, $z = r \cos \theta_1$, the Hamiltonian is
\[
H = \frac{\alpha^2}{r} + \frac{\beta^2}{y^2} + \frac{\gamma^2}{z^2} + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}.
\]
(49)

Note that any permutation of the ordered pairs $(x, \beta), (y, \gamma), (z, \delta)$ leaves the Hamiltonian unchanged. This leads to an additional structure in the symmetry algebra. The basic symmetries are (5), (6). The permutation symmetry of the Hamiltonian shows that $I_{xy}, I_{xz}$ are also symmetry operators, and $L_2 = I_{xy} + I_{xz} + I_{yz} = (\beta + \gamma + \delta)$. The constant of the motion $K_0$ is second order:
\[
K_0 = -\frac{1}{8} I_{yz} - \frac{1}{16} I_{zz} + \frac{1}{16} (L_2 + \beta + \gamma + \delta) = -\frac{1}{32} (I_{yz} - I_{zz}),
\]
and $J_0$ is fourth order:
\[
J_0 = -4M_3^2 - \frac{1}{2} \left( \{(x, \beta) + (y, \gamma) + (z, \delta)\}^2, \frac{\delta}{z^2} \right) + 2H \left( I_{zz} + I_{yz} - \left( \beta + \gamma + \frac{\delta}{4} \right) \right)
\]
+ $5 \frac{\delta}{z^2} + \frac{\alpha^2}{2}.$
(50)
\[
M_3 = \frac{1}{2} \{(y \beta_z - z \beta_y), \beta_x\} - \frac{1}{2} \{(z \beta_x - x \beta_z), \beta_y\} - z \left( \frac{\alpha}{2r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} \right).
\]
(51)

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but it is straightforward to check that the symmetries given by these Cartesian expressions agree exactly with the special cases \( k_1 = k_2 = 1 \) of the spherical coordinate expressions for these operators derived in the preceding sections.) If \( \beta = \gamma = \delta = 0 \) then \( M_3 \) is itself a symmetry operator.

The symmetries \( H, L_2, L_3, J_0, K_0 \) form a generating (rational) basis for the symmetry operators. Under the transposition \((x, \beta) \leftrightarrow (z, \delta)\) this basis is mapped to an alternate basis \( H, L_2', L_3', J_0', K_0' \):

\[
\begin{align*}
L_2' &= L_2, \\
L_3' &= -16K_0 + \frac{1}{2}L_2 - \frac{1}{2}L_3 + \frac{\beta + \gamma + \delta}{2}, \\
K_0' &= \frac{1}{2}K_0 + \frac{1}{64}L_2 - \frac{3}{64}L_3 + \frac{1}{64}(\beta + \gamma + \delta), \\
R_1' &= [L_2, J_0'], \ R_2' = [L_3', K_0'] = -[L_3, K_0] = -R_2, \ R_3' = [J_0', K_0'] = -\frac{1}{32}R_1' + \frac{1}{32}[L_3, J_0'].
\end{align*}
\]

(52)

All of the identities derived earlier hold for the primed symmetries. It is easy to see that the \( K' \) symmetries are simple polynomials in the \( L, K \) symmetries already constructed, e.g., \( K_1' = [L_3', K_0'] = -K_1 \). However, the \( J' \) symmetries are new. In particular,

\[
\begin{align*}
J_0' &= -4M_2^2 - \frac{1}{2} \left\{ \left( x \partial_x + y \partial_y + z \partial_z \right) + \left( \frac{\beta}{x^2} \right)^2 \right\} + 2H \left( I_y + I_z - \left( \gamma + \delta + \frac{3}{4} \right) \right) \\
&\quad + 5\frac{\beta}{x^2} + \frac{\alpha^2}{2}, \\
M_1 &= \frac{1}{2} \left\{ \left( y \partial_y - x \partial_x \right), \partial_z \right\} - \frac{x}{2} \left( \frac{\alpha}{2r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} \right).
\end{align*}
\]

(53)

Note that the transposition \((y, \gamma) \leftrightarrow (z, \delta)\) does not lead to anything new. Indeed, under the symmetry we would obtain a constant of the motion

\[
\begin{align*}
J_0' &= -4M_2^2 - \frac{1}{2} \left\{ \left( x \partial_x + y \partial_y + z \partial_z \right), \partial_y \right\} + 2H \left( I_x + I_z - \left( \beta + \delta + \frac{3}{4} \right) \right) \\
&\quad + 5\frac{\gamma}{y^2} + \frac{\alpha^2}{2}, \\
M_2 &= \frac{1}{2} \left\{ \left( yz \partial_y - y \partial_x \right), \partial_z \right\} - \frac{1}{2} \left( \left( y \partial_y - x \partial_x \right), \partial_z \right) - \gamma \left( \frac{\alpha}{2r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} \right) \\
&\quad + \left( \frac{\alpha}{2r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} \right).
\end{align*}
\]

(55)

but it is straightforward to check that

\[
J_0 + J_0' + J_0'' = -\frac{1}{2}H + \frac{\alpha^2}{2},
\]

(56)

so that the new constant depends linearly on the previous constants. For further use, we remark that under the symmetry \((x, \beta) \leftrightarrow (y, \gamma)\) the constants of the motion \( L_2, L_3, J_0, J_1 \) are invariant, whereas \( K_0, K_1 \) change sign. The action on \( J_0' \) is more complicated: \( J_0' \rightarrow J_0'' = \frac{\alpha^2}{2} - \frac{1}{2}H - J_0 - J_0' \).

Note here that, with respect to the standard inner product on Euclidean space, the \( L \) symmetries and \( J_0, J_0', K_0 \) are all formally self-adjoint whereas the commutators \( K_1, J_1 \) are skew-adjoint:

\[
H^* = H, \ L_2^* = L_2, \ L_3^* = L_3, \ K_0^* = K_0, \ J_0^* = J_0, \ (J_0')^* = J_0', \ J_1^* = -J_1, \ K_1^* = -K_1.
\]

(52)
Further the commutators $J_3 + 2J_1, K_2 + K_1$ are formally self-adjoint. Hence $J_2^* = J_2 + 4J_1, K_2^* = K_2 + 2K_1$. Finally, when acting on a generalized eigenbasis the raising and lowering operators satisfy
\[
(J^+)^* = J^- \left( 1 + \frac{4}{\rho} \right), \quad (J^-)^* = J^+ \left( 1 - \frac{4}{\rho} \right),
\]
\[
(K^+)^* = K^- \left( 1 + \frac{2}{\mu} \right), \quad (K^-)^* = K^+ \left( 1 - \frac{2}{\mu} \right).
\]

Note also the operator identities on a generalized eigenbasis:
\[
K^+(\mu - 2) = \mu K^+, \quad K^- (\mu + 2) = \mu K^-, \quad J^+(\rho - 4) = \rho J^+, \quad J^- (\rho + 4) = \rho J^-.
\]

These adjoint properties and symmetry with respect to permutations can be used to greatly simplify the determination of structure equations. For example, one can see that it is not possible for a linear term in the generator $J_0'$ to be invariant under the transposition $(x, \beta) \leftrightarrow (y, \gamma)$ and if such a term changes sign it must be proportional to $J_0 + 2J_0' + \frac{1}{2}H - \frac{\omega}{2}$.

For
\[
Q = (L_3 - L_2 - \delta)^2 - 2(2\delta + 1)L_2 - 3L_3 + 2\delta + 1,
\]
and $k_1 = k_2 = 1$ the relevant rational structure equations simplify to
\[
[J_1, K_1]Q = \{J_1, K_1\}(2L_2 - 2L_3 + 2\delta + 1) + \{J_2, K_1\}(L_2 - L_3 - \delta - 1),
\]
\[
[J_1, K_2]Q = -\{J_1, K_1\}L_3(2L_2 - 2L_3 + 2\delta + 1) + \{J_2, K_2\}(L_2 - L_3 - \delta - 1),
\]
\[
4K_0(L_3 + 1) = K_2 - K_1 - \frac{1}{2}(L_2 - \delta)(\gamma - \beta), \quad [K_1, L_3] = 4K_2 + 4K_1, \quad [L_2, J_0] = J_1,
\]
\[
2J_0(4L_2 + 3) = J_2 - 2J_1 - 4(L_3 - \delta + \frac{1}{2})(H - \alpha^2), \quad [J_1, L_3] = 2J_2 + 4J_1, \quad [L_3, K_0] = K_1.
\]

In the paper [13], Tanoudis and Daskaloyannis show that the quantum symmetry algebra generated by the six functionally dependent symmetries $H, L_2, L_3, J_0, K_0$ and $J_0'$ closes polynomially, in the sense that all double commutators of the generators are again expressible as noncommuting polynomials in the generators. However more is true. All formally self-adjoint symmetry operators, such as $\{K_1, J_1\}, K_1^2, J_1^2, [K_1, J_1]$, are expressible as symmetrized polynomials in the generators. Further there is a polynomial relation among the six generators. To see this let us first consider the symmetries $K_1^2, J_1^2$. We expect these order six and order ten symmetries to be expressed as polynomials in the generators. Using the corresponding expression in the classical case to get the highest order terms, as well as the adjoint and permutation symmetry constraints, we find the expansions to take the form
\[
K_0^2 + 8[L_3 + 5, K_0^2] - 4(\beta - \gamma)K_0(L_2 - \delta) - C(L_2, L_3) = 0,
\]
\[
C(L_2, L_3) = \frac{1}{4} \gamma + \frac{1}{8} \beta + \frac{1}{8} \delta - \frac{1}{8} L_3 L_2 \delta + \frac{1}{2} L_3 L_2 \beta + \frac{1}{2} L_3 \delta \beta + \frac{1}{2} L_3 L_2 \gamma + \frac{1}{4} L_3 \delta \gamma
+ \frac{1}{2} L_2 \delta \gamma - \frac{1}{4} L_3 \beta \gamma + \frac{1}{4} L_2 \beta \gamma - \frac{1}{4} L_3 - \frac{1}{2} L_2 + \frac{1}{2} L_3 + \frac{1}{8} L_3 \beta \delta - \frac{1}{8} L_2 \beta \delta
+ \frac{1}{2} L_3 \gamma - \frac{1}{4} L_3 \beta \gamma + \frac{1}{8} \delta ^2 + \frac{1}{8} \beta ^2 - \frac{1}{8} \gamma ^2 - \frac{1}{8} \delta ^2 \beta - \frac{1}{8} \delta ^2 \gamma - \frac{1}{8} \delta ^2 \gamma - \frac{1}{8} \delta ^2 \gamma
+ \frac{1}{4} L_3 \gamma + \frac{1}{4} L_3 \delta - \frac{1}{8} L_3 \beta \delta - \frac{1}{8} L_3 \beta \delta + \frac{1}{8} L_3 \beta \delta + \frac{1}{8} L_2 \delta + \frac{1}{8} L_2 \gamma - \frac{1}{8} L_3 \delta + \frac{1}{8} L_3 \delta ^2 - \frac{1}{8} L_3 \beta ^2 - \frac{1}{8} L_3 \beta ^2 - \frac{1}{8} L_3 \gamma - \frac{1}{8} L_3 \gamma
+ \frac{1}{8} L_3 \gamma - \frac{1}{8} L_3 \gamma
\]
\[
J_1^2 + [8L_2 + 38, J_0^2] + 4J_0(4HL_3 + (1 - 4\delta)H - 4\alpha^2 L_3 + \alpha^2 (4\delta - 1)) - D(H, L_2, L_3) = 0.
\]
\[
D(H, L_2, L_3) = 5a^4 - 8L_3a^4 + 296E^2L_3 + 80H^2L_3^2 + 4a^4L_2 - 156H^2L_2 - 560H^2L_3^2 - 8a^3\delta \\
+ 216H^2\delta + 80H^2\delta^2 - 16HL_3a^2 + 32HL_3^2a^2 + 64H^2L_4L_2 - 72H^2L_2 \\
+ 288H^2L_3L_2 + 32H^2L_2^2 - 128H^2L_3L_2^2 - 160H^2L_3\delta + 224H^2L_2\delta \\
- 128H^2L_2^2\delta - 48HL_2^2\delta + 64H^2L_2^3\delta^2 + 32H^2\delta^2 + 64H^2L_3^2 - 64HL_3a^2L_2 \\
- 64HL_3a^2\delta - 64H^2L_2^3\delta - 74H^2\delta^2 + 229H^2.
\]

Here \(C\) is a symmetry of order at most 6, and \(D\) is a symmetry of order at most 8.

Using the fact that \(J_3\) must be formally self-adjoint and computing \([K_0, J_1] + 4[K_0, J_0]\) on a generalized basis of eigenvectors \(X[\mu, \rho] = \Sigma_{n,m,p}\) we define \(J\) by

\[
JX[\mu, \rho] = J_{pp}X[\mu - 2, \rho - 4] + J_{pm}X[\mu - 2, \rho + 4] \\
+ J_{mp}X[\mu + 2, \rho - 4] + J_{mm}X[\mu + 2, \rho + 4] + J_{p}X[\mu, \rho - 4] \\
+ J_{m}X[\mu, \rho + 4] + J_{p}X[\mu - 2, \rho] + J_{m}X[\mu + 2, \rho] + J_{p}X[\mu, \rho],
\]

(60)

\[
J_{pp}X[\mu - 2, \rho - 4] = \{K^+, J^+\}P_{++}(\mu, \rho)X[\mu, \rho], \\
J_{pm}X[\mu - 2, \rho + 4] = \{K^+, J^-\}P_{+-}(\mu, \rho)X[\mu, \rho], \\
J_{mp}X[\mu + 2, \rho - 4] = \{K^-, J^+\}P_{-+}(\mu, \rho)X[\mu, \rho], \\
J_{mm}X[\mu + 2, \rho + 4] = \{K^-, J^-\}P_{--}(\mu, \rho)X[\mu, \rho], \\
J_{p}X[\mu, \rho - 4] = J^+P_{0+}(\mu, \rho)X[\mu, \rho], \\
J_{m}X[\mu, \rho + 4] = J^-P_{0-}(\mu, \rho)X[\mu, \rho], \\
J_{p}X[\mu - 2, \rho] = K^+P_{+0}(\mu, \rho)X[\mu, \rho], \\
J_{m}X[\mu + 2, \rho] = K^-P_{-0}(\mu, \rho)X[\mu, \rho], \\
J_{p}X[\mu, \rho] = P_{00}(\mu, \rho)X[\mu, \rho].
\]

In particular,

\[
P_{-+}(\mu, \rho) = \frac{1}{2((\rho - 2\mu)^2 + 48 + 3)\rho(\rho + 2\mu)(\mu + 1)}, \\
P_{++}(\mu, \rho) = \frac{1}{2((\rho - 2\mu)^2 + 48 + 3)\rho(\rho - 2\mu)(\mu - 1)}, \\
P_{+-}(\mu, \rho) = \frac{1}{2((\rho + 2\mu)^2 + 48 + 3)\rho(\rho + 2\mu)(\mu - 1)}, \\
P_{--}(\mu, \rho) = \frac{1}{2((\rho + 2\mu)^2 + 48 + 3)\rho(\rho - 2\mu)(\mu + 1)}.
\]

\[
P_{0+}(\mu, \rho) = \frac{\beta - \gamma}{16\rho(\rho - 2)(\mu + 1)(\mu - 1)}, \\
P_{0-}(\mu, \rho) = \frac{\beta - \gamma}{16\rho(\rho + 2)(\mu + 1)(\mu - 1)}, \\
P_{+0}(\mu, \rho) = \frac{-H - a^2}{(\rho + 2)(\rho - 2)(\mu + 1)(\mu - 1)}, \\
P_{-0}(\mu, \rho) = -\frac{H - a^2}{(\rho + 2)(\rho - 2)(\mu + 1)(\mu - 1)}.
\]

Then we can derive the polynomial identities

\[
[K_0, J_1] + 4[K_0, J_0] = \frac{\gamma - \beta}{2}J_0 + 4(H - a^2)K_0 - \frac{\gamma - \beta}{4}(H - a^2) \\
+ \left\{2\left(L_2 - L_3 + \frac{1}{2}\right)J\right\} = 0,
\]

(61)
expressing the double commutator $[K_0, J_1]$ and the symmetrized product of commutators $[K_1, J_1]$ in terms of the generators. We guess that $J = \zeta (J_0 - 2J_0' - 2\alpha H - 4J_0)$ for some scalar $\zeta$. A direct computation gives $\xi = -8$ and verifies the identities. Similar methods can be used to show polynomial closure of all terms of the form $[[S_1, S_2], [S_3, S_4]]$ where $S_1, \ldots, S_4$ are generators, but the details are complicated and we do not provide them here.

We conclude by sketching the determination of the algebraic relation between the six generators. The key here is the expression of all operators in terms of their action on a generalized basis of eigenvectors $X[\mu, \rho]$. Note that each of the operators $J^2, [K_0, J_0], [K_0, J_0]^2, [J, [K_0, J_0]]$ can be expanded in the eigenbasis with only terms

$$
X[\mu \pm 4, \rho \pm 8], \quad X[\mu \pm 4, \rho \mp 8], \quad X[\mu \pm 4, \rho], \quad X[\mu, \rho \pm 8],
$$

$$
X[\mu \pm 4, \rho], \quad X[\mu \pm 2, \rho], \quad X[\mu, \rho \pm 4], \quad X[\mu, \rho].
$$

Furthermore, the first differential operator is of order 8, the second and third are of order 12 and the fourth is of order 10. From these facts and the adjoint properties of the operators it is fairly straightforward to conclude that the operator

$$
M_1 = [Q, J^2] + 32[K_0, J_0]^2 - 16[L_2 + L_3 - \delta - 1, [J, [K_0, J_0]]
$$

is the unique minimal order linear combination of these symmetries such that all terms in $X[\mu \pm 4, \rho \pm 8]$ are zero. Next we compute the coefficients of $X[\mu, \rho \pm 8], X[\mu \pm 4, \rho]$ in the action of $M_1$ and show that these can be canceled by an operator of the form $M_2 = [Q_1, K_0^2] + [Q_2, J^2_0]$, where $Q_1$ is a polynomial in $H, L_2, L_3$ of order at most 2 and $Q_2$ is of order at most 4. Then $M_1 + M_2$ has only terms in $X[\mu \pm 2, \rho], X[\mu, \rho \pm 4], X[\mu, \rho]$. Continuing in this way, we can cancel these remaining terms by an operator of the form $M_3 = [Q_3, K_0] + [Q_4, J_0] + Q_5$ where $Q_3$ is a polynomial in $H, L_2, L_3$ of order at most 5, $Q_4$ is of order at most 4, and $Q_5$ is of order at most 6. Thus we have the nontrivial differential operator identity of order 12:

$$
M_1 + M_2 + M_3 = 0
$$

which is quadratic in $J_0'$. This is the algebraic relation that we seek. The final expression is too lengthy to list in a paper.

### 3. Conclusions and Outlook

One conclusion that we can reach from our analysis of the these extended Kepler–Coulomb systems is that for any rational pair $k_1, k_2$ the three-parameter potential is never just the restriction of the four-parameter potential obtained by setting $\delta = 0$. The three-parameter system always has additional symmetries not inherited from the four-parameter system. Note that the $K$ raising and lowering operators fix $\rho$ and raise and lower $\mu$ by $\pm 2p_1p_2$ for each system, so the $K$ operators in the three-parameter case are simply the restricted $K$ operators from the four-parameter case. However, the three-parameter $J$ operators fix $\mu$ and change $\rho$ by $\pm 2q_1$, whereas the four-parameter $J$ operators fix $\mu$ and change $\rho$ by $\pm 4q_1$. In essence, the
four-parameter $J$ operators become perfect squares upon restriction to $\delta = 0$. This changes the structure of the symmetry algebra.

More generally, we have demonstrated that the recurrence relation method developed in [10] for proving superintegrability and determining the structure equations for families of 2D quantum superintegrable systems can be extended to the 3D case. The construction appears to be quite general and not restricted to Kepler–Coulomb analogues. For these higher order superintegrable systems it appears that algebraic closure is the norm. For polynomial closure, extra symmetry is needed. We have not proved in all cases that there do not exist other generators of lower order but, if they exist, they must also be obtainable in terms of recurrence relations of hypergeometric functions. A crucial role is played by the raising and lowering operators. They are not defined independent of eigenbasis and are not even symmetries, but all symmetries are built from them. The two variable models introduced here show promise in uncovering properties of rational orthogonal special functions analogous to properties of orthogonal polynomials related to second order superintegrable systems [12, 18].

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