An existence result for singular nonlocal fractional Kirchhoff–Schrödinger–Poisson system

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ABSTRACT
In this paper, we study the existence of infinitely many weak solutions to a fractional Kirchhoff–Schrödinger–Poisson system involving a weak singularity, i.e. when $0 < \gamma < 1$. Further, we obtain the existence of a solution with a strong singularity, i.e. when $\gamma > 1$. We employ variational techniques to prove the existence and multiplicity results. Moreover, an $L^\infty$ estimate is obtained by using the Moser iteration method.

1. Introduction

In this paper, we study the fractional Kirchhoff–Schrödinger–Poisson system involving a singular term

$$
(a + b \int_Q \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}) (-\Delta)^s u + \phi u = \lambda h(x) u^{-\gamma} + f(x, u) \text{ in } \Omega,
$$

$$
(-\Delta)^s \phi = u^2 \text{ in } \Omega,
$$

$$
u > 0 \text{ in } \Omega,
$$

$$
u = \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $a, b \geq 0, a + b > 0, \gamma > 0, \lambda > 0, h \in L^1(\Omega), h(x) > 0 \text{ a.e. in } \Omega$ and $f$ satisfies some growth conditions.

In the recent time elliptic PDEs involving singularity has drawn interest to many researchers for both as well local as nonlocal operators. Noteworthy applications involving nonlocal operators can be found in [1–5] and the references therein. Further applications can be seen in the field of fluid dynamics, in particular in the study of thin boundary layer properties for viscous fluids [6], in probability theory to study the Levy process [7], in

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finance [2], in free boundary obstacle problems [8], in the fractional quantum mechanics [3,4]. Another application of PDEs involving these type of nonlocal operator is due to Buades et al. [9] and Kindermann et al. [10] in the field of image processing to find a clear image $u$ from a given noisy image $f$. Readers who are interested to know further details on applications of PDEs involving nonlocal operators can also refer to [1,8,11–13] and the references therein.

The main driving equation to study problem of the type (1) is due to the parabolic Schrödinger–Poisson equation

$$-i\frac{\partial \psi}{\partial t} = -\Delta \psi + V(x)\psi + \phi\psi - g(x, \psi) \text{ in } \Omega,$$

$$-\Delta \phi = |\psi|^2 \text{ in } \Omega,$$

$$\phi = \psi = 0 \text{ on } \partial \Omega.$$  \hspace{1cm} (2)

On substituting $\psi(x, t) = e^{-it}u(x)$, the equation (2) reduces to the classical stationary Schrödinger–Poisson system

$$-\Delta u + V(x)u + \phi(x)u = f(x, u) \text{ in } \Omega,$$

$$-\Delta \phi = u^2 \text{ in } \Omega,$$

$$u = \phi = 0 \text{ in } \partial \Omega.$$  \hspace{1cm} (3)

From the application point of view, the equation of type (2) arises in quantum mechanics which depicts a system of identical charged particles, where each particle interacts in such a way that one may abdicate the magnetic field effects and the corresponding solution becomes a standing wave. In particular, the coupled equation in (3) illustrates that the charge of the wave function determines the potential. For further details on the derivation of such system, one may refer to Benci and Fortunato [14,15].

On the other hand, the fractional Schrödinger–Poisson system emanates from the fractional quantum mechanics. Similar to the classical case, the following fractional Schrödinger–Poisson system can be obtained as standing wave from the fractional parabolic Schrödinger–Poisson system

$$(-\Delta)^s u + V(x)u + \phi(x)u = f(x, u) \text{ in } \Omega,$$

$$(-\Delta)^s \phi = u^2 \text{ in } \Omega,$$

$$u = \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$  \hspace{1cm} (4)

That is, one can obtain (4) as a solution of the form $\psi(x, t) = e^{-it}u(x)$ from the fractional parabolic Schrödinger–Poisson system

$$-i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi + \phi\psi - g(x, \psi) \text{ in } \Omega,$$

$$(-\Delta)^s \phi = |\psi|^2 \text{ in } \Omega,$$

$$\phi = \psi = 0 \text{ on } \mathbb{R}^N \setminus \Omega.$$  \hspace{1cm} (5)

For $\phi = 0$ and $\Omega = \mathbb{R}^N$, the fractional Schrödinger equation (5) was introduced by Laskin [3,4] as an extension of the Feynman path integral in the quantum field from...
the Brownian-like paths to Lévy-like paths. A detailed study elaborating the fractional Schrödinger equation over a bounded domain can be found in [16, Appendix], where the authors explain a motivation to study such equations. For further insight in the study of Schrödinger systems, the readers are referred to [3, 4, 12–19] and the references therein.

Another motivation to study the Kirchhoff type problem (1) is due to the model problem

\[ M(\|u\|)(-\Delta)u + V(x)u + \phi(x)u = f(x, u) \text{ in } \Omega, \]

\[ (-\Delta)\phi = u^2 \text{ in } \Omega, \]

\[ u = \phi = 0 \text{ on } \partial \Omega, \]

where \( M \) is known as the Kirchhoff function. A typical example of \( M \) is \( M(t) = a + bt^p \) for \( t \geq 0 \), where \( a, b \), \( \geq 0 \). The problem (6) is known as degenerate if \( M(0) = 0 \) and is nondegenerate if \( M(0) > 0 \). The origin of the problem of type (6) emerged first in the study due to Kirchhoff [20] as a generalization of the D’Alembert wave equation.

\[ \rho \frac{\partial^2 u}{\partial t^2} - \left( a + b \int_0^1 \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = g(x, u) \]

where \( a, b, \rho \) are positive constants and \( l \) is the changes of the length of the strings due to the vibrations.

The fractional Kirchhoff type problem has drawn interest of many researchers following the seminal work due to Fiscella and Valdinoci [21, Appendix A] describing the vibration of the fractional length of a string in the nonlocal sense. Here, the Kirchhoff function \( M \) refers to the change of tension of the string due to the change of the length of the string during the vibration and \( M(0) \) defines the initial tension on the string at zero. In [21], the authors considered the nonlocal stationary Kirchhoff model involving the fractional Laplacian

\[ M([u]_s^2)(-\Delta)^s u = f(x, u) + |u|^{2^*_s - 2} u \text{ in } \Omega, \]

\[ u = 0 \text{ on } \bar{\Omega} \setminus \Omega, \]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain, \( 2^*_s = \frac{2N}{N-2s} \) is the critical Sobolev exponent. The authors in [21] established the existence of a nonnegative solution to the problem (7) by employing mountain pass theorem together with the concentration compactness lemma.

For \( b = f(x, u) = \phi(x) = 0 \), problem (1), reduces to a purely singular problem. PDEs involving singularities have gained attention post the pioneering study due to Lazer and McKenna [22], where the authors have studied a purely singular problem involving the Laplacian operator, i.e. for \( s = 1 \) and \( b = f(x, u) = \phi(x) = 0 \). The problem considered in [22] is

\[ -\Delta u = \frac{p(x)}{u^{\gamma}} \& u > 0 \text{ in } \Omega, \]

\[ u = 0 \text{ on } \partial \Omega. \]

The authors in [22], have proved that the problem has a unique \( C^1(\bar{\Omega}) \) solution iff \( 0 < \gamma < 1 \) and it has a \( H^1_0(\Omega) \) solution iff \( \gamma < 3 \). Later in [23], the author proved that if \( \gamma \geq 3 \), then
the singular problem can not have an \( H^1_0(\Omega) \) solution. The problem of type (8) first arose in the study of pseudo-plastic fluids due to [24].

Similar to the study with the Laplacian operator, Canino et al. [25], have studied the nonlocal PDE involving singularity. In [25], the authors considered the problem

\[
(-\Delta_p)^s u = \frac{a(x)}{u^{\gamma}} &\text{in } \Omega, \\
u = 0 &\text{in } \mathbb{R}^N \setminus \Omega.
\]  

The authors in [25], has guaranteed the existence of unique solution in \( W^{s,p}_0(\Omega) \) for \( 0 < \gamma \leq 1 \) and in \( W^{s,p}_{loc}(\Omega) \) for \( \gamma > 1 \). For \( a(x) \equiv 1, p = 2 \), Fang [26], has proved the existence of a unique \( C^{2,\alpha}(\Omega) \) solution for \( 0 < \alpha < 1 \).

For \( b = \phi = 0 \), a vast amount of study to prove the existence, multiplicity and regularity of solutions to the problem of type (1) has been done involving as well the local operator \( (s = 1) \) as the nonlocal operator \( (0 < s < 1) \) with a singularity and a power nonlinearity or an \( L^1 \) data or both. The literature is so vast that it is almost impossible to enlist all of them here. A few of such studies can be found in [22,23,25,27–40] and the references therein.

One of the earliest study to show the existence of multiple solutions was made by Crandall et al. [28] involving the Laplacian operator. Further references on multiplicity involving local operator can be found in [31] and the references therein. Recently, Saoudi et al. in [37] have guaranteed the existence of at least two solutions by using min-max method with the help of modified Mountain Pass theorem involving fractional \( p \)-Laplacian operator. Saoudi in [36], obtained two solutions involving fractional Laplacian operator. For further references on the study of multiple solutions, consult [31,37] and the references therein.

In the last decades, a huge amount of study of the Kirchhoff type problem coupled with the Schrödinger–Poisson system have been studied to show the existence and multiplicity of the solutions via critical point theory assuming different potential functions \( V \) and power nonlinearities. It is almost impossible to specify all of them here. Therefore, we refer the readers to [19,32–34,38,39,41–49] and the references therein for a detailed study of existence, uniqueness and multiplicity of Kirchhoff type problem. In most of these studies, the authors have used the variational techniques, in particular min-max method, sub-super solution method, Nehari manifold method and Mountain Pass theorem to guarantee the existence and multiplicity of solutions.

A vast amount of problems of type (1) for \( 0 < \gamma < 1 \) has been considered in the literature pertaining to the study of existence and multiplicity (finitely many) of solutions. In the recent past, for \( a = 0 \) and \( 0 < \gamma < 1 \), Fiscella [41] has obtained two distinct solutions involving fractional Laplacian operator by variational technique. Later, for \( a > 0, 0 < \gamma < 1 \), Fiscella and Mishra [42] proved the multiplicity by Nehari manifold method. In Ref. [33,34], the authors studied the multiplicity of solutions involving singular nonlinearity. In [44], Li and Zhang have studied the existence, uniqueness and multiplicity of solution(s) for Schrödinger–Poisson system without compactness conditions. On the other hand, Zhang [47] has studied a Schrödinger–Poisson system. For a detailed study on Schrödinger–Poisson system, one can see [17,18,50] and there references therein. Liao et al. [46] have guaranteed the existence and uniqueness of solutions for the Kirchhoff type problem involving singularity.
The earliest introduction to obtain infinitely many solutions is due to Clark [51] by considering the symmetric mountain pass theorem. In the pioneering work, Ambrosetti and Rabinowitz [52] have guaranteed the existence of infinitely many solutions to the problem of type (1) for \(a = s = 1, b = \lambda = \phi = 0\) by introducing the well known (AR) condition on \(f\) for a \(C^1\) energy functional. In fact the (AR) condition has proved to be an important tool to obtain infinitely many solutions. Finally, Kajikiya [53] has well exposed the symmetric mountain pass theorem for sublinear data. With these introductions the existence of infinitely many solutions emerged in the study of elliptic PDEs.

Recently the existence of infinitely many solutions to Kirchhoff type problems has drawn interest from many researchers. Some of the well known expository works can be found in [49,54–58] and the references therein. For \(\lambda = 0\), Zhang et al. [54], have proved the existence of infinitely many solutions for a superlinear data \(f\). The study of Kirchhoff type problems with regard to infinitely many solutions can be found in [49,56,56] and the references therein. For \(\lambda = 0\), Li et al. [57] guaranteed the existence of infinitely many solutions to the problem (1) for a sublinear data \(f\).

In general, solutions obtained by the symmetric mountain pass theorem either converge to zero or diverge to infinity depending upon the sublinear or superlinear data, respectively. Another crucial assumption in the literature to study infinitely many solutions is that the data should be odd and the functional should be \(C^1\). In the present article, we have considered singular term \(u^{-\gamma}, 0 < \gamma < 1\) and an odd, sublinear power nonlinearity \(f\). The main difficulty arising due to the singular term is that the corresponding energy functional fails to be \(C^1\). Further, the singular term is also not odd. Hence we cannot apply the symmetric mountain pass theorem immediately as in Kajikiya [53]. We use the advantage of continuity of the functional and the Gateaux differentiability with delicacy to extract a \((PS)\) sequence and hence apply the symmetric mountain pass theorem for a modified problem. The following assumptions are made on the data \(f\) for \(0 < \gamma < 1\).

(A1) \(f \in C(\Omega \times \mathbb{R}, \mathbb{R})\) and \(\exists \delta > 0\) such that \(\forall x \in \Omega\) and \(|t| \leq \delta, f(x, -t) = -f(x, t)\).
(A2) \(\lim_{t \to 0} \frac{f(x,t)}{t} = +\infty\) uniformly on \(\Omega\).
(A3) There exists \(r > 0\) and \(p \in (1 - \gamma, 2)\) such that \(\forall, x \in \Omega\) and \(|t| \leq r, tf(x, t) \leq pF(x, t), \) where \(F(x, t) = \int_0^t f(x, \tau)d\tau\).

Considering the present interest toward the study of fractional PDEs involving singularity and motivated from the work due to [49,53,57], we have obtained infinitely many small solutions to the fractional Kirchhoff–Schrödinger–Poisson system (1) for \(0 < \gamma < 1\). To the best of my knowledge, there in no study of infinitely many solutions for a fractional Kirchhoff–Schrödinger–Poisson system involving a power nonlinearity and a singularity \((0 < \gamma < 1)\) in the literature. Below stated is the first theorem proved in this article which guarantees the existence of infinitely many small solutions.

**Theorem 1.1:** Assume \(a, b \geq 0, a + b > 0, h \in L^1(\Omega), h > 0\) a.e. in \(\Omega\) and (A1)–(A3) hold. Then for \(0 < \gamma < 1\) and for any \(\lambda \in (0, \Lambda),\) problem (1) has a sequence of positive weak solutions \(u_n \subset X_0 \cap L^\infty(\Omega)\) such that \(I(u_n) < 0, I(u_n) \to 0^-\) and \(u_n \to 0\) in \(X_0\). (See Section 2 for notations).
Remark 1.2: Note that in Theorem 1.1 there is no any restriction condition for $f$ in $t$ at infinity.

The problem (1) is more subtle due to the presence of strong singularity ($\gamma > 1$). The major obstacle to overcome is to tackle the energy functional as in this case it is not even continuous. Therefore, we cannot proceed to use the critical point theory. Hence, we approach to show the existence of a solution in a Nehari type manifold and some a-priori inequality with the help of Dini’s notion of derivatives. To the best of author’s knowledge, the literature involving strong singularity, i.e. $\gamma > 1$ is meagre even involving local operators. The earliest result on the existence of solution with a strong singularity involving the Laplacian operator is due to Lazer and McKenna [22]. Later in [38], the author provided a compatibility criterion to obtain the existence of solution involving the Laplacian operator for the purely singular problem of type (8) with strong singularity. The main result proved in [38], states that the problem (1) has an $H_0^1(\Omega)$ solution if and only if the compatibility condition for the pair $(h, \gamma)$ hold:

$$\int_{\Omega} h(x)|u_0|^{1-\gamma} < \infty \text{ for some } u_0 \in H_0^1(\Omega). \quad (10)$$

Some noteworthy studies involving strong singularity can be found in [22,23,38,39,47,48]. Recently, Zhang [48], proved a necessary and sufficient condition for the existence of solution for a Kirchhoff–Schrödinger–Poisson system involving the Laplacian operator with strong singularity. Hence, motivated from the study due to Sun [38], Zhang [48] and the growing interest toward fractional PDEs, we have considered the fractional Kirchhoff–Schrödinger–Poisson system (1) involving strong singularity. The next theorem proved in this article states necessary and sufficient condition for the existence of solutions.

**Theorem 1.3:** Assume $a, b \geq 0, a + b > 0, \, h \in L^1(\Omega), \, h > 0 \text{ a.e. in } \Omega$ and $f(x,u) = k(x)u^p$ such that $k \in L^\infty(\Omega)$ with $k > 0, \, 0 < p < 1$. Then, for $\gamma > 1$ and for any $\lambda \in (0, \Lambda)$, problem (1) has a weak solution in $X_0$ iff the compatibility condition holds true:

$$\int_{\Omega} h(x)|u_0|^{1-\gamma} \, dx < \infty \text{ for some } u_0 \in X_0. \quad (11)$$

**Remark 1.4:** If we assume $f(x, \cdot) \equiv 0$, then problem (1) possesses a unique solution.

The paper is organized as follows. In Section 2, we will first give some mathematical formulations and define the space $X_0$. Moreover, we will discuss some preliminary properties of $\phi$ and prove that $\Lambda$ has a finite range. In the subsequent sections, Sections 3 and 4, we will obtain the results as stated in Theorems 1.1 and 1.3, respectively.

### 2. Mathematical formulations

This section is devoted to give a few important results of fractional Sobolev spaces, embeddings, variational formulations and space setup. Let $\Omega$ be open bounded domain of $\mathbb{R}^N$ and $Q = \mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$. For $0 < s < 1$, the space $(X, \| \cdot \|)$, which is an
intermediary Banach space between $H^1(\Omega)$ and $L^2(\Omega)$, is defined as

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, } u|_{\Omega} \in L^2(\Omega) \text{ and } \frac{|u(x) - u(y)|}{|x - y|} \in L^2(Q) \right\}$$

equipped with the norm

$$\|u\|_X = \|u\|_2 + [u]_2,$$

where $[u]_2 = \left( \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}$ refers to the Gagliardo semi norm. Due to the zero Dirichlet boundary condition, it is natural to consider the space

$$X_0 = \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \},$$

equipped with the following Gagliardo norm on it.

$$\|u\| = \left( \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}.$$

The space $(X_0, \|\cdot\|)$ is a Hilbert space [59]. The best Sobolev constant is defined as

$$S = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy}{\left( \int_\Omega |u|^2 \, dx \right)^{2/2s}}$$

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Then for every $q \in [1, 2^*_s)$, the space $X_0$ is continuously embedded in $L^q(\Omega)$ and for every $q \in [1, 2^*_s)$, the space $X_0$ is compactly embedded in $L^q(\Omega)$, where $2^*_s = \frac{2N}{N-2s}$. Prior to define the weak solution to our problem, let us first consider the following problem

$$(-\Delta)^s \phi = u^2 \text{ in } \Omega,$$
$$\phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$  \hspace{1cm} (13)

In light of the Lax-Milgram theorem, for every $u \in X_0$, the problem (13) has a unique solution $\phi_u \in X_0$ and we have the following Lemma consisting some properties of the solution $\phi_u$.

**Lemma 2.1:** For each solution $\phi_u \in X_0$ of (13), we have

(i) $\|\phi_u\|^2 = \int_\Omega \phi_u u^2 \, dx = \int_\Omega \frac{|(-\Delta)^{1/2} \phi_u|^2}{|x-y|^{N+2s}} \, dx \leq C\|u\|^4$, $\forall u \in X_0$;

(ii) $\phi_u \geq 0$. Moreover, $\phi_u > 0$ if $u \neq 0$;

(iii) for all $t \neq 0$, $\phi_{tu} = t^2 \phi_u$;

(iv) $\|u_n - u\| \to 0$ implies that $\|\phi_{u_n} - \phi_u\| \to 0$ and $\int_\Omega \phi_{u_n} u_n v \, dx \to \int_\Omega \phi_u v \, dx$, for any $v \in X_0$;

(v) for any $u, v \in X_0$, we have $\int_\Omega (\phi_u u - \phi_v v)(u - v) \, dx \geq \frac{1}{2}\|\phi_u - \phi_v\|^2$. 

Now by replacing $\phi_u$ in place of $\phi$ in (1), the problem (1) reduces to the following Dirichlet boundary value problem

$$
(a + b \int_Q \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}) (-\Delta)^{s/2} u + \phi_u u = \lambda h(x) u^{-\gamma} + f(x, u) \text{ in } \Omega,
$$

$$
\begin{align*}
&u > 0 \text{ in } \Omega, \\
&u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\end{align*}
$$

We now define a weak solution to the problem (14).

**Definition 2.2:** A function $u \in X_0$ is a weak solution to the problem (14), if $u > 0$ and

$$
(a + b[u]^2) \int_Q (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \psi + \int \phi_u u \psi - \lambda \int h(x) u^{-\gamma} \psi - \int f(x, u) \psi = 0,
$$

for every $\psi \in X_0$.

The associated energy functional to the problem (14) is defined as

$$
I(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \frac{1}{4} \int \phi_u u^2 - \frac{\lambda}{1 - \gamma} \int h(x)|u|^{1-\gamma} - \int F(x, u), \quad u \in X_0,
$$

where $F(x, u) = \int_0^u f(x, t) dt$. Observe that for $0 < \gamma < 1$, the term $\int h(x)|u|^{1-\gamma} < \infty$ but the functional $I$ fails to be $C^1$. Therefore, by modifying the problem (1), we will use the Kajikiya’s Symmetric mountain pass theorem [53] and a cut-off technique developed in [51] to obtain a $C^1$ functional to guarantee the existence of infinitely many solutions. On the other hand, for $\gamma > 1$ the integral $\int h(x)|u|^{1-\gamma} \, dx$ is not finite for $u \in X_0$. Therefore, the energy functional $I$ fails to be continuous and we cannot use the usual variational technique to guarantee the existence of solution. We will use arguments from [38] to obtain a weak solution to the problem (14). Similar type of results can also be found in [47]. We now state and prove the following Lemma to guarantee a finite range for $\Lambda$, which is defined as

$$
\Lambda = \inf \{\lambda > 0 : \text{The problem (1) has no solution}\}.
$$

**Lemma 2.3:** Assume $a, b, \gamma > 0$, (A1)–(A3) and (11) holds. Then $0 \leq \Lambda < \infty$.

**Proof:** By definition, $\Lambda \geq 0$. Let $\phi_1 > 0$ be the first eigenfunction [60] corresponding to the first eigenvalue $\lambda_1$ for the fractional Laplacian operator. Then we have

$$
(-\Delta)^{s/2} \phi_1 = \lambda_1 \phi_1 \text{ in } \Omega
$$

$$
\begin{align*}
&\phi_1 > 0 \text{ in } \Omega, \\
&\phi_1 > 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\end{align*}
$$

Therefore, by putting $\phi_1$ as the test function in Definition 2.2, we obtain

$$
\begin{align*}
\lambda_1 \int \Omega (a + b \|u\|^2) u \phi_1 \, dx &= \int \Omega (a + b \|u\|^2) (-\Delta)^{s/2} \phi_1 u \, dx \\
&= \int \Omega (\lambda h(x) u^{-\gamma} + f(x, u) - \phi_u u) \phi_1 \, dx
\end{align*}
$$
At this stage, we choose $\tilde{\Lambda} > 0$ such that
\[\tilde{\Lambda} h(x_0) t^{-\gamma} + f(x_0, t) > 2\lambda_1 t(a + bt^2) + \phi_t t\]
for all $t > 0$ and for some $x_0 \in \Omega$, which gives a contradiction to (18). Hence $\Lambda < \infty$. ■

In the subsequent two sections, we establish the existence of solution(s).

### 3. Existence of infinitely many solutions for $0 < \gamma < 1$

We begin this section with the definition of genus of a set.

**Definition 3.1 (Genus):** Let $X$ be a Banach space and $A \subset X$. A set $A$ is said to be symmetric if $u \in A$ implies $(-u) \in A$. Let $A$ be a closed, symmetric subset of $X$ such that $0 \not\in A$. We define a genus $\gamma(A)$ of $A$ by the smallest integer $k$ such that there exists an odd continuous mapping from $A$ to $\mathbb{R}^k \setminus \{0\}$. We define $\gamma(A) = \infty$, if no such $k$ exists.

We now define the following family of sets,

\[\Gamma_n = \{A_n \subset X : A_n \text{ is closed, symmetric and } 0 \not\in A_n \text{ such that } \gamma(A_n) \geq n\}.

Further, we will use the following version of the symmetric mountain pass theorem from Kajikiya [53].

**Theorem 3.2:** Let $X$ be an infinite dimensional Banach space and $\tilde{I} \in C^1(X, \mathbb{R})$ satisfies the following

1. $\tilde{I}$ is even, bounded below, $\tilde{I}(0) = 0$ and $\tilde{I}$ satisfies the $(PS)_c$ condition.
2. For each $n \in \mathbb{N}$, there exists an $A_n \in \Gamma_n$ such that $\sup_{u \in A_n} \tilde{I}(u) < 0$.

Then for each $n \in \mathbb{N}$, $c_n = \inf_{A_n \in \Gamma_n} \sup_{u \in A_n} \tilde{I}(u) < 0$ is a critical value of $\tilde{I}$.

We will modify the problem (14) to apply the symmetric mountain pass theorem as follow

\[\left( a + b \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \right) (-\Delta)^s u + \phi_u u = \lambda h(x) \text{sign}(u)|u|^{-\gamma} + f(x, u) \text{ in } \Omega,
\]
\[u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\]

(19)

The associated energy functional to the problem (19) is defined as

\[J(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \frac{1}{4} \int_{\Omega} \phi_u u^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} h(x)|u|^{1-\gamma} - \int_{\Omega} F(x, u), \quad u \in X_0,
\]

(20)

where $F(x, u) = \int_0^{|u|} f(x, t)dt$. Observe that the functional $J$ is even by using the assumption $(A1)$ and Lemma 2.1(iii). We now define a weak solution to the modified problem (19).
Definition 3.3: A function $u \in X_0$ is a weak solution of (19), if $\phi |u|^{-\gamma} \in L^1(\Omega)$ and
\[
(a + b[u]^2) \int_Q (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \psi + \int_\Omega \phi_u u \psi
- \int_\Omega (\lambda h(x) \text{sign}(u)|u|^{-\gamma} + f(x, u)) \psi = 0,
\]
for every $\psi \in X_0$. Observe that if $u > 0$ a.e. in $\Omega$, then weak solutions to the problem (19) and to the problem (14) coincide. Therefore, it is sufficient to obtain a sequence of nonnegative weak solutions to the problem (14). We now extend and modify $f(x, u)$ for $u$ outside a neighborhood of 0 by $\tilde{f}(x, u)$ as follow. We will follow [51] by considering a cut-off problem. Choose $l > 0$ sufficiently small such that $0 < l \leq \frac{1}{2} \min\{\delta, r\}$, where $\delta$ and $r$ are same as in the assumptions on $f$. We now define a $C^1$ function $\xi : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $0 \leq \xi(t) \leq 1$ and
\[
\xi(t) = \begin{cases} 
1, & \text{if } |t| \leq l \\
1 - \frac{t}{l}, & \text{if } l \leq t \leq 2l \\
0, & \text{if } |t| \geq 2l.
\end{cases}
\]
We now consider the following cut-off problem by defining $\tilde{f}(x, u) = f(x, u)\xi(u)$.
\[
\left( a + b \int_Q \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \right) (-\Delta)^s u + \phi_u u = \lambda h(x) \text{sign}(u)|u|^{-\gamma} + \tilde{f}(x, u) \text{ in } \Omega,
\]
\[
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\]
The associated energy functional to the problem (22) is defined as
\[
\tilde{I}(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \frac{1}{4} \int_\Omega \phi_u u^2 - \frac{\lambda}{1-\gamma} \int_\Omega h(x)|u|^{1-\gamma} - \int_\Omega \tilde{F}(x, u) \, dx, \quad u \in X_0.
\]
We define a weak solution to the problem (22) as follows.

Definition 3.4: A function $u \in X_0$ is a weak solution of (22), if $\phi |u|^{-\gamma} \in L^1(\Omega)$ and
\[
(a + b[u]^2) \int_Q (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \psi + \int_\Omega \phi_u u \psi
- \int_\Omega (\lambda h(x) \text{sign}(u)|u|^{-\gamma} + \tilde{f}(x, u)) \psi = 0
\]
for every $\psi \in X_0$. Again, if $\|u\|_{\infty} \leq l$ holds, then the weak solutions of (22) and the weak solutions of (19) coincide. We establish the existence result for the problem (22). Finally, we prove our main theorem by showing that the solutions to (22) are positive and $\|u\|_{\infty} \leq l$.

We first prove the following Lemmas which are the hypotheses to the symmetric mountain pass theorem.

Lemma 3.5: The functional $\tilde{I}$ is bounded from below and satisfies (PS)$_c$ condition.
**Proof:** By the definition of $\xi$ and using the Hölder’s inequality, we get

$$
\tilde{I}(u) \geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \frac{1}{4} \int_{\Omega} \phi u^2 - C\|u\|^{1-\gamma} - C_1
$$

where, $C, C_1$ are nonnegative constants. Since $a, b > 0$, this implies that $\tilde{I}$ is coercive and bounded from below in $X_0$. Let $\{u_n\} \subset X_0$ be a Palais-Smale sequence for the functional $\tilde{I}$. Therefore, by using the coerciveness property of $\tilde{I}$ we have $\{u_n\}$ is bounded in $X_0$. Thus, we may assume that $\{u_n\}$ has a subsequence (still denoted by $\{u_n\}$) such that $u_n \rightharpoonup u$ in $X_0$. Therefore, we have

$$
\int_{Q} (-\Delta)^{5/2} u_n \cdot (-\Delta)^{5/2} \psi \, dx \, dy \longrightarrow \int_{Q} (-\Delta)^{5/2} u \cdot (-\Delta)^{5/2} \psi \, dx \, dy
$$

for all $\phi \in X_0$. By the embedding result [59], we can assume for every $q \in [1, 2^*_s)$$
\tag{25}
\begin{align}
    u_n &\longrightarrow u \text{ in } L^q(\Omega), \\
    u_n(x) &\longrightarrow u(x) \text{ a.e. } L^q(\Omega).
\end{align}

\tag{26}
\tag{27}
\tag{28}
\tag{29}
Therefore, from Lemma A.1 [61], we get that there exists $g \in L^q(\Omega)$ such that

$$
\tag{28}
|u_n(x)| \leq g(x) \text{ a.e. in } \Omega, \quad \forall n \in \mathbb{N}.
$$

Now on using (26), (27), (28) and applying the Lebesgue dominated convergence theorem, we obtain

$$
\int_{\Omega} \tilde{f}(x, u_n) u \, dx \rightarrow \int_{\Omega} \tilde{f}(x, u) u \, dx \text{ and } \int_{\Omega} \tilde{f}(x, u_n) u_n \, dx \rightarrow \int_{\Omega} \tilde{f}(x, u) u_n \, dx.
$$

Moreover,

$$
\int_{\Omega} \phi u_n u_n u \, dx \rightarrow \int_{\Omega} \phi u_n u \, dx \text{ and } \int_{\Omega} \phi u_n u_n^2 \, dx \rightarrow \int_{\Omega} \phi u_n^2 \, dx.
$$

Again, on using the Hölder’s inequality and passing the limit $n \to \infty$, we get

$$
\int_{\Omega} u_n^{1-\gamma} \, dx \leq \int_{\Omega} u^{1-\gamma} \, dx + \int_{\Omega} |u_n - u|^{1-\gamma} \, dx
$$

$$
\leq \int_{\Omega} u^{1-\gamma} \, dx + C\|u_n - u\|^{1-\gamma}_{L^2(\Omega)}
$$

$$
= \int_{\Omega} u^{1-\gamma} \, dx + o(1).
$$

Similarly, we have

$$
\int_{\Omega} u^{1-\gamma} \, dx \leq \int_{\Omega} u_n^{1-\gamma} \, dx + \int_{\Omega} |u_n - u|^{1-\gamma} \, dx
$$

$$
\leq \int_{\Omega} u_n^{1-\gamma} \, dx + C\|u_n - u\|^{1-\gamma}_{L^2(\Omega)}
$$

$$
= \int_{\Omega} u_n^{1-\gamma} \, dx + o(1).
$$
Therefore,
\[ \int_{\Omega} u_n^{1-\gamma} \, dx = \int_{\Omega} u^{1-\gamma} \, dx + o(1). \] (33)
Since, \( \{u_n\} \) is a Palais-Smale sequence of \( \tilde{I} \) therefore, by weak convergence, we get
\[ \langle \tilde{I}'(u_n) - \tilde{I}'(u), u_n - u \rangle = o(1) \text{ as } n \to \infty. \] (34)
On the other hand,
\[
\langle \tilde{I}'(u_n) - \tilde{I}'(u), (u_n - u) \rangle \\
= (a + b[u_n]^2) \langle u_n, (u_n - u) \rangle - (a + b[u]^2) \langle u, (u_n - u) \rangle \\
+ \int_{\Omega} \left[ (\phi u_n - \phi u) - \lambda h(x)(\text{sign}(u_n)|u_n|^{\gamma} - \text{sign}(u)|u|^{\gamma}) \right] (u_n - u) \\
- \int_{\Omega} (\tilde{f}(x, u_n) - \tilde{f}(x, u))(u_n - u) 
\] (35)
Now, on using (29), (30) and (33) we get
\[
\langle \tilde{I}'(u_n) - \tilde{I}'(u), (u_n - u) \rangle \\
= (a + b[u_n]^2) \langle u_n, (u_n - u) \rangle - (a + b[u]^2) \langle u, (u_n - u) \rangle + o(1) \] (36)
as \( n \to \infty \). Observe that
\[
(a + b[u_n]^2) \langle u_n, (u_n - u) \rangle - (a + b[u]^2) \langle u, (u_n - u) \rangle \\
= (a + b[u_n]^2)[u_n - u]^2 + b([u_n]^2 - [u]^2) \langle u, (u_n - u) \rangle. \] (37)
Since, the sequence \( (a + b[u_n]^2) \) is bounded in \( X_0 \). Thus by using the definition of weak convergence, we get
\[ b([u_n]^2 - [u]^2) \langle u, (u_n - u) \rangle = o(1) \text{ as } n \to \infty. \] (38)
Therefore, from (37) and (38), we obtain
\[
(a + b[u_n]^2) \langle u_n, (u_n - u) \rangle - (a + b[u]^2) \langle u, (u_n - u) \rangle \geq a[u_n - u]^2 \text{ as } n \to \infty. \] (39)
Finally, on using (34), (36) and (39), we conclude that
\[ o(1) \geq \min\{a, 1\} \|u_n - u\|^2 + o(1) \text{ as } n \to \infty. \] (40)
Hence, \( u_n \to u \) strongly in \( X_0 \) and this completes the proof. \( \blacksquare \)

**Lemma 3.6:** For any \( n \in \mathbb{N} \), there exists a closed, symmetric subset \( A_n \subset X_0 \) with \( 0 \not\in A_n \) such that the genus \( \gamma(A_n) \geq n \) and \( \sup_{u \in A_n} \tilde{I}(u) < 0 \).

**Proof:** We will first obtain the existence of a closed, symmetric subset \( A_n \) of \( X_0 \) over every finite dimensional subspace such that \( \gamma(A_n) \geq n \). Let \( X_k \) be a subspace of \( X_0 \) such that...
dim($X_k$) = $k$. Since, every norm over a finite dimensional Banach space are equivalent then there exists a positive constant $M = M(k)$ such that $\|u\| \leq M\|u\|_{L^2(\Omega)}$ for all $u \in X_k$.

**Claim:** There exists a positive constant $R$ such that

$$\frac{1}{2} \int_{\Omega} |u|^2 \, dx \geq \int_{\{u > l\}} |u|^2 \, dx, \quad \forall u \in X_k \text{ such that } \|u\| \leq R. \quad (41)$$

We proof it by contradiction. Let $\{u_n\}$ be a sequence in $X_k \setminus \{0\}$ such that $u_n \to 0$ in $X_0$ and

$$\frac{1}{2} \int_{\Omega} |u_n|^2 \, dx < \int_{\{u_n > l\}} |u_n|^2 \, dx. \quad (42)$$

Choose, $\nu_n = \frac{u_n}{\|u_n\|_{L^2(\Omega)}}$. Then (42) reduces to

$$\frac{1}{2} < \int_{\{u_n > l\}} |\nu_n|^2 \, dx. \quad (43)$$

Since, $X_k$ is finite dimensional and $\{\nu_n\}$ is bounded, we can assume $\nu_n \to \nu$ in $X_0$ upto a subsequence. Therefore, $\nu_n \to \nu$ also in $L^2(\Omega)$. Further observe that,

$$m\{x \in \Omega : |u_n| > l\} \to 0 \text{ as } n \to \infty,$$

since $u_n \to 0$ in $X_0$, where $m$ refers to the Lebesgue measure. This is a contradiction to Equation (43). Hence, the claim is established. Again, from the assumption (A2), one can choose $0 < l \leq 1$ sufficiently small such that,

$$\tilde{F}(x,t) = F(x,t) \geq 4 \left( \frac{a}{2} + \frac{b}{4} + C_\phi \right) M^2 t^2, \quad \forall (x,t) \in \Omega \times [0,l].$$

Hence, for all $u \in X_k \setminus \{0\}$ such that $\|u\| \leq R$ and by using (41), we get

$$\tilde{I}(u) \leq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \int_{\Omega} \phi u^2 \, dx - \frac{\lambda}{1 - \gamma} \int_{\Omega} |h(x)||u|^{1-\gamma} \, dx - \int_{\{u \leq l\}} \tilde{F}(x,u) \, dx$$

$$\leq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + C_\phi \|u\|^4 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |h(x)||u|^{1-\gamma} \, dx$$

$$- 4 \left( \frac{a}{2} + \frac{b}{4} + C_\phi \right) M^2 \int_{\{u \leq l\}} |u|^2 \, dx$$

$$\leq \left( \frac{a}{2} + \frac{b}{4} + C_\phi \right) \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |h(x)||u|^{1-\gamma} \, dx$$

$$- 4 \left( \frac{a}{2} + \frac{b}{4} + C_\phi \right) M^2 \left( \int_{\Omega} |u|^2 \, dx - \int_{\{u > l\}} |u|^2 \, dx \right)$$

$$\leq \left( \frac{a}{2} + \frac{b}{4} + C_\phi \right) \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |h(x)||u|^{1-\gamma} \, dx$$

$$- 2 \left( \frac{a}{2} + \frac{b}{4} + C_\phi \right) M^2 \int_{\Omega} |u|^2 \, dx$$
\[ \leq - \left( \frac{a}{2} + \frac{b}{4} + C\phi \right) \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |h(x)||u|^{1 - \gamma} \, dx \]

< 0, for all \( u \in X_0 \) such that \( \|u\| \leq \min\{1, R\} \).

We now choose, \( 0 < \rho \leq \min\{1, R\} \) and \( A_n = \{u \in X_n : \|u\| = \rho\} \). Thus \( \Gamma_n \neq \phi \). This concludes that \( A_n \) is symmetric, closed with \( \gamma(A_n) \geq n \) such that \( \sup_{u \in A_n} I(u) < 0 \). \( \blacksquare \)

We now state the following Lemmas which are essential to prove the boundedness of the solutions to the problem (22). The Lemma 3.7 and Lemma 3.8 are taken from [60] and a simple proof can be found in [29].

**Lemma 3.7:** Let \( g : \mathbb{R} \to \mathbb{R} \) be a convex \( C^1 \) function. Then for every \( c, d, C, D \in \mathbb{R} \) with \( C, D > 0 \) the following inequality holds.

\[ (g(c) - g(d))(C - D) \leq (c - d)(Cg'(c) - Dg'(d)) \] (44)

**Lemma 3.8:** Let \( \tilde{h} : \mathbb{R} \to \mathbb{R} \) be an increasing function, then for \( c, d, \tau \in \mathbb{R} \) with \( \tau > 0 \) we have

\[ [\tilde{H}(c) - \tilde{H}(d)]^2 \leq (c - d)(\tilde{h}(c) - \tilde{h}(d)) \] (45)

where, \( \tilde{H}(t) = \int_0^t \sqrt{\tilde{h}^\prime(\tau)} \, d\tau \), for \( t \in \mathbb{R} \).

The following Lemma is based on the Moser iteration technique, which gives an uniform \( L^\infty \) bound to the weak solutions of the problem (22).

**Lemma 3.9:** Let \( u \in X_0 \) be a positive weak solution to the problem in (22), then \( u \in L^\infty(\Omega) \).

**Proof:** The proof is based on arguments as in [29]. We will make use of the fact that

\[ \int_{Q} \frac{(u(x) - u(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} \, dx \, dy = C \int_{Q} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \psi \, dx \, dy, \]

for \( \psi \in X_0 \). For every small \( \epsilon > 0 \), consider the smooth function

\[ g_\epsilon(t) = (\epsilon^2 + t^2)^{1/2}. \]

Note that the function \( g_\epsilon \) is convex as well as Lipschitz. We choose \( \psi = \tilde{\psi} g_\epsilon'(u) \) as the test function in (22) for all positive \( \tilde{\psi} \in C_0^\infty(\Omega) \). Now by taking \( c = u(x), d = u(y), C = \psi(x) \) and \( D = \psi(y) \) in Lemma 3.7, we get

\[ (a + b\|u\|^2) \int_{Q} \frac{(g_\epsilon(u(x)) - g_\epsilon(u(y)))(\tilde{\psi}(x) - \tilde{\psi}(y))}{|x - y|^{N+2s}} \, dx \, dy \]

\[ \leq \int_{\Omega} \left( |\lambda h(x)u^{-\gamma} + \tilde{f}(x, u)| - \phi u \right) |g_\epsilon'(u)| \tilde{\psi} \, dx \]

\[ \leq \int_{\Omega} \left( |\lambda h(x)u^{-\gamma} + \tilde{f}(x, u)| \right) |g_\epsilon'(u)| \tilde{\psi} \, dx \] (46)
Since, \( g_\epsilon(t) \to |t| \) as \( t \to 0 \), hence \( |g_\epsilon'(t)| \leq 1 \) for all \( t \geq 0 \). Therefore, on using the Fatou's Lemma and passing the limit \( \epsilon \to 0 \) in (46), we obtain

\[
(a + b\|u\|^2) \int_Q \frac{|(u(x) - u(y))| (\tilde{\psi}(x) - \tilde{\psi}(y))}{|x - y|^{N+2s}} \, dx \, dy \leq \int_\Omega \left( |\lambda h(x)u^{-\gamma} + \tilde{f}(x, u)| \right) \tilde{\psi} \, dx
\]

for all \( \tilde{\psi} \in C_c^\infty(\Omega) \) with \( \tilde{\psi} > 0 \). The inequality (47) remains true for all \( \tilde{\psi} \in X_0 \) with \( \tilde{\psi} \geq 0 \). We define the cut-off function \( u_k = \min\{(u - 1)^+, k\} \in X_0 \) for \( k > 0 \). Now for any given \( \beta > 0 \) and \( \delta > 0 \), we choose \( \tilde{\psi} = (u_k + \delta)^\beta - \delta^\beta \) as the test function in (47) and get

\[
(a + b\|u\|^2) \int_Q \frac{|(u(x) - u(y))| ((u_k(x) + \delta)^\beta - (u_k(y) + \delta)^\beta)}{|x - y|^{N+2s}} \, dx \, dy \\
\leq \int_\Omega \left( |\lambda h(x)u^{-\gamma} + \tilde{f}(x, u)| \right) ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]

Now applying the Lemma 3.8 to the function \( \tilde{h}(u) = (u_k + \delta)^\beta \), we get

\[
\leq \frac{(\beta + 1)^2}{4\beta} \int_Q \frac{|(u(x) - u(y))| ((u_k(x) + \delta)^\beta - (u_k(y) + \delta)^\beta)}{|x - y|^{N+2s}} \, dx \, dy
\]

\[
\leq \frac{(\beta + 1)^2}{4\beta} \int_\Omega \left( |\lambda h(x)u^{-\gamma} + \tilde{f}(x, u)| \right) ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]

\[
\leq \frac{(\beta + 1)^2}{4\beta} \int_{\{u \geq 1\}} \left( |\lambda h(x)u^{-\gamma}| + |\tilde{f}(x, u)| \right) ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]

\[
\leq \frac{(\beta + 1)^2}{4\beta} \int_{\{u \geq 1\}} \left( |\|h\|_\infty + (|c_1| + |c_2||u|^p) \right) ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]

\[
\leq C_1 \frac{(\beta + 1)^2}{4\beta} \int_{\{u \geq 1\}} \left( 1 + |u|^p \right) ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]

\[
\leq 2C_1 \frac{(\beta + 1)^2}{4\beta} \int_{\{u \geq 1\}} |u|^p ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]

\[
\leq C \frac{(\beta + 1)^2}{4\beta} |u|_p^q (u_k + \delta)^\beta |q
\]

where, \( q = \frac{2^*_q}{2^*_q - p} \) and \( C = \max\{1, |\lambda|\} \). The rest of the proof is similar to the Lemma 2.7 in [29] to obtain

\[
|u_k|_\infty \leq C_2 \eta^{\frac{n}{(n-1)q}} \left( |\Omega|^{1 - \frac{1}{q}} \left( \frac{2^*_q}{q^*} \right)^{\frac{n}{q-1}} \left( |(u - 1)^+|_q + \delta |\Omega|^{\frac{1}{q}} \right) \right)
\]

(50)
Now letting $k \to \infty$ in (50), we have

$$
\| (u - 1)^+ \|_\infty \leq C n^{\frac{q}{(q-1)r}} \left( |\Omega|^{1-\frac{1}{q}-\frac{2}{N}} \right) \frac{n}{r-1} \left( |(u - 1)^+|_q + \delta |\Omega|^{\frac{1}{q}} \right) \quad (51)
$$

Hence, we conclude that $u \in L^\infty(\Omega)$. 

**Proof of Theorem 1.1:** By using the assumption (A1) and the definition of $\xi$, we get the functional $\tilde{I}$ is even and $\tilde{I}(0) = 0$. Thus, on using Theorem 3.2, Lemmas 3.5 and 3.6, we conclude that $\tilde{I}$ has sequence of critical points $\{u_n\}$ such that $\tilde{I}(u_n) < 0$ and $\tilde{I}(u_n) \to 0^-$. We now prove that the critical points of $\tilde{I}$ are nonnegative.

**Claim:** Let $u_n$ be a critical point of $\tilde{I}$, then $u_n \geq 0$ a.e. in $X_0$ for every $n \in \mathbb{N}$.

**Proof:** We first divide the domain as $\Omega = \Omega^+ \cup \Omega^-$, where $\Omega^+ = \{x \in X_0 : u_n(x) \geq 0\}$ and $\Omega^- = \{x \in X_0 : u_n(x) < 0\}$. We define $u_n = u_n^+ - u_n^-$, where $u_n^+(x) = \max\{u_n(x), 0\}$ and $u_n^-(x) = \max\{-u_n(x), 0\}$. We proceed through a contradiction by taking $u_n < 0$ a.e. in $\Omega$. Then on choosing, $\phi = u_n^-$ as the test function in Equation (22) in association with the inequality $(a - b)(a^- - b^-) \leq -(a^- - b^-)^2$, we obtain

$$
\int_{\Omega} \left( \lambda h(x) \frac{\text{sign}(u_n)u_n^-}{|u_n|^\gamma} + \tilde{f}(x, u_n)u_n^- \right) \, dx
$$

$$
= (a + b[u_n]^2) \int_{\Omega} \frac{(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+2s}} \, dy + \int_{\Omega} \phi_{u_n} u_n u_n^- \, dx
$$

$$
= -(a + b||u_n||^2)||u_n^-||^2 - \int_{\Omega} \phi_{u_n} ||u_n^-||^2 \, dx
$$

$$
\Rightarrow \lambda \int_{\Omega^-} h(x)|u_n^-|^{1-\gamma} \, dx < 0.
$$

Therefore, $|\Omega^-| = 0$, which is a contradiction to the assumption $u_n < 0$ a.e. in $\Omega$. 

We now prove $u_n \to 0$ in $X_0$. Indeed by the definition of $\tilde{I}$, we obtain

$$
\frac{1}{p} (\tilde{I}'(u_n), u_n) - \tilde{I}(u_n)
$$

$$
= \frac{1}{p} \left[ (a + b||u_n||^2)||u_n||^2 + \int_{\Omega} \phi_{u_n} u_n^2 - \int_{\Omega} \left( \lambda \frac{h(x)\text{sign}(u_n)u_n}{|u_n|^\gamma} + \tilde{f}(x, u_n)u_n \right) \, dx \right]
$$

$$
- \left[ \frac{a}{2}||u_n||^2 + \frac{b}{4}||u_n||^4 + \frac{1}{4} \int_{\Omega} \phi_{u_n} u_n^2 - \int_{\Omega} \left( \frac{\lambda h(x)}{1-\gamma} |u_n|^{1-\gamma} + \tilde{F}(x, u_n) \right) \, dx \right]
$$

$$
= a \left( \frac{1}{p} - \frac{1}{2} \right) ||u_n||^2 + b \left( \frac{1}{p} - \frac{1}{4} \right) ||u_n||^4 + \left( \frac{1}{p} - \frac{1}{4} \right) \int_{\Omega} \phi_{u_n} u_n^2
$$

$$
- \lambda \left( \frac{1}{p} - \frac{1}{1-\gamma} \right) \int_{\Omega} h(x)|u_n|^{1-\gamma} \, dx + \frac{1}{p} \int_{\Omega} (p\tilde{F}(x, u_n) - \tilde{f}(x, u_n)) \, dx
$$

$$
\geq a \left( \frac{1}{p} - \frac{1}{2} \right) ||u_n||^2 + b \left( \frac{1}{p} - \frac{1}{4} \right) ||u_n||^4 + \frac{1}{p} (\frac{1}{1-\gamma} - \frac{1}{4}) \int_{\Omega} h(x)|u_n|^{1-\gamma} \, dx
$$
Lemma 4.1: Let \begin{align*}
    \frac{1}{p} (\tilde{I}(u_n), u_n) - \tilde{I}(u_n) = o(1)
    \Rightarrow \left( \frac{1}{p} - \frac{1}{2} \right) \|u_n\|^2 \leq o(1),
\end{align*}
as \( n \to \infty \). Since, \( 1 - \gamma < p < 2 \), we conclude that \( u_n \to 0 \) in \( X_0 \). Thus from the Lemma 3.9, we can obtain \( \|u_n\|_{L^\infty(\Omega)} \leq l \) as \( n \to \infty \), thanks to the Moser iteration method. Hence, the problem (19) has infinitely many solutions. Moreover, by using \( u_n \geq 0 \) and \( \tilde{I}(u_n) < 0 \), we conclude that the problem (1) has infinitely many weak solutions in \( X_0 \). Thus Theorem 1.1 is proved.

4. Existence of solution for \( \gamma > 1 \)

This section is fully devoted to establish the existence of a weak solution to the problem (1) in \( X_0 \). The results obtained in this section is the fractional version of the main result of [48]. For the sake of the readers, the proof involving the nonlocal operator is given. Further, we will prove that for \( k \equiv 0 \), the problem (1) possesses a unique solution. Let us define the following two subsets of \( X_0 \) similar to the Nehari manifold:

\begin{align*}
    N_1 &= \{ u \in X_0 : (a + b[u]^2)\|u\|^2 + \int_{\Omega} \phi_u u^2 - \lambda \int_{\Omega} h(x) |u|^{1-\gamma} - \int_{\Omega} k(x) |u|^{1+p} \geq 0 \}, \\
    N_2 &= \{ u \in X_0 : (a + b[u]^2)\|u\|^2 + \int_{\Omega} \phi_u u^2 - \lambda \int_{\Omega} h(x) |u|^{1-\gamma} - \int_{\Omega} k(x) |u|^{1+p} = 0 \}.
\end{align*}

We will show that the fractional Kirchhoff–Schrödinger–Poisson system with a strong singularity has a weak solution in \( N_2 \). One can see that \( N_2 \) is not closed. We will prove that \( N_1 \) is closed in \( X_0 \) and the functional \( \tilde{I} \) is coercive and bounded below on \( N_1 \). Further we will obtain a minimizing sequence \( \{ u_n \} \) of \( c = \inf_{N_1} \tilde{I} \) such that \( \{ u_n \} \) converges to \( u \in X_0 \). Finally, we will show that \( u \in N_2 \) and hence \( u \) is a weak solution to the problem (14). We begin with the following Lemmas.

**Lemma 4.1:** Let \( \int_{\Omega} \lambda h(x) |u|^{1-\gamma} < \infty \) for some \( u \in X_0 \). Then there exists a unique \( t_0 > 0 \) such that \( t_0 u \in N_2 \) and \( tu \in N_1 \) for \( t \geq t_0 \), i.e. \( N_1 \cap N_2 \neq \emptyset \). Moreover, for \( t > 0 \), \( \psi \in X_0 \) the function \( f \) defined as \( \theta(t) = t(u + t\psi) \) is continuous on \([0, \infty)\).

**Proof:** Let \( \int_{\Omega} \lambda h(x) |u|^{1-\gamma} < \infty \) for some \( u \in X_0 \). Now for \( t > 0 \), we get

\begin{align*}
    I(tu) &= \frac{at^2}{2} \|u\|^2 + \frac{bt^4}{4} \|u\|^4 + \frac{t^4}{4} \int_{\Omega} \phi_u u^2 - \frac{t^{1-\gamma}}{1-\gamma} \int_{\Omega} \lambda h(x) |u|^{1-\gamma} \\
    &\quad - \frac{t^{p+1}}{p} \int_{\Omega} k(x) |u|^{1+p}.
\end{align*}
It is easy to see that \( tu \in N_1 \Leftrightarrow I'(tu) \geq 0 \) and \( tu \in N_2 \Leftrightarrow I'(tu) = 0 \). Also, since \( 0 < p < 1 < \gamma, I(tu) \to +\infty \), if \( t \to 0^+ \) as well as \( t \to +\infty \) and there exists a unique \( t_0 > 0 \) such that \( I'(t_0 u) = 0, I(tu) \geq 0, t \geq t_0 \) and \( I(t_0 u) = \min_{t \geq 0} I(tu) \). Therefore, \( tu \in N_1, t_0 u \in N_2 \) for \( t \geq t_0 \), and \( I(tu) \geq I(t_0 u) \).

Again, observe that for \( t, \psi \geq 0, \int_{\Omega} \lambda h(x)|u + t\psi|^{1-\gamma} < \infty \). Now, consider a nonnegative sequence \( \{t_n\} \) such that \( t_n \to t \) as \( n \to \infty \). On using the arguments as above, there exists \( \theta(t_n), \theta(t) \geq 0 \) such that \( \theta(t_n)(u + t_n \psi), \theta(t)(u + t \psi) \in N_2 \). Thus, we get

\[
a\theta^{1+\gamma}(t_n)\|u + t_n \psi\|^2 + \theta^{3+\gamma}(t_n) \left( b\|u + t_n \psi\|^4 + \int_{\Omega} \phi_{u + t_n \psi}(u + t_n \psi)^2 \right) - \theta^{p+\gamma}(t_n) \int_{\Omega} k(x)|u + t_n \psi|^{1+p} = \int_{\Omega} \lambda h(x)|u + t_n \psi|^{1-\gamma} \tag{52}
\]

and

\[
a\theta^{1+\gamma}(t)\|u + t \psi\|^2 + \theta^{3+\gamma}(t) \left( b\|u + t \psi\|^4 + \int_{\Omega} \phi_{u + t \psi}(u + t \psi)^2 \right) - \theta^{p+\gamma}(t) \int_{\Omega} k(x)|u + t \psi|^{1+p} = \int_{\Omega} \lambda h(x)|u + t \psi|^{1-\gamma}. \tag{53}
\]

Now for all \( n \in \mathbb{N} \), we have \( \lambda h(x)|u + t_n \psi|^{1-\gamma} \leq \lambda h(x)|u|^{1-\gamma} \). and for each \( x \in \Omega \), we have the pointwise convergence \( \lambda h(x)|u + t_n \psi|^{1-\gamma} \to \lambda h(x)|u + t \psi|^{1-\gamma} \) as \( n \to \infty \). Therefore, by using Lebesgue’s dominated convergence theorem, we get \( \int_{\Omega} \lambda h(x)|u + t_n \psi|^{1-\gamma} \to \int_{\Omega} \lambda h(x)|u + t \psi|^{1-\gamma} \) as \( n \to \infty \). Further, from (52), one can see that the sequence \( \{\theta(t_n)\} \) is bounded. Therefore, it has a convergent subsequence. Let \( \{\theta(t_{n_k})\} \) converge to \( s \). Then, on using (52) and (53) and the above arguments, we can conclude that \( s = \theta(t) \). Hence \( \theta \) is continuous.

In the following Lemma, we establish that \( N_1 \) is closed in \( X_0 \) and the the functional \( I \) is coercive and bounded below on \( N_1 \).

**Lemma 4.2:** \( N_1 \) is closed in \( X_0 \) and for all for \( u \in N_1 \), there exists \( C > 0 \) such that \( \|u\| \geq C \). Moreover, the functional \( I \) is coercive and bounded below on \( N_1 \).

**Proof:** We first show that \( N_1 \) is closed. Let \( \{u_n\} \subset N_1 \) be such that \( u_n \to u \) in \( X_0 \). Since, \( \{u_n\} \subset N_1 \) and \( \int_{\Omega} \lambda h(x)|u_n|^{1-\gamma} < \infty \), then \( u_n(x) > 0 \) a.e. in \( \Omega \) and then up to a subsequence, \( u_n(x) \to u(x) \) a.e. in \( \Omega \). Therefore, on applying the Fatou’s lemma and then using Sobolev embedding, we get

\[
\int_{\Omega} \lambda h(x)|u|^{1-\gamma} \leq \lim_{n \to \infty} \inf_{u} \int_{\Omega} \lambda h(x)|u_n|^{1-\gamma} \\
\leq \lim_{n \to \infty} \inf \left( (a + b\|u_n\|^2)\|u_n\|^2 + \int_{\Omega} \phi_{u_n} u_n^2 - \int_{\Omega} k(x)|u_n|^{1+p} \right) \\
\leq (a + b\|u\|^2)\|u\|^2 + \int_{\Omega} \phi_u u^2 - \int_{\Omega} k(x)|u|^{1+p}.
\]
Thus we have $u \in N_1$ and hence $N_1$ is closed in $X_0$. We prove the functional $I$ is bounded below on $N_1$ by using the method of contradiction. Suppose, there exists $\{u_n\} \subset N_1$ such that $u_n \to 0$ in $X_0$ as $n \to \infty$. Then, on using the Reverse Hölder inequality, we get

$$
(a + b\|u_n\|^2)\|u_n\|^2 + \int \Omega \phi_{u_n}u_n^2 \geq \int \Omega \lambda h(x)|u_n|^{1-\gamma} + \int \Omega k(x)|u_n|^{1+p}
\geq \left(\int \Omega |\lambda h(x)|^{\frac{1}{\gamma}}\right) \left(\int \Omega |u_n|\right)^{1-\gamma}
\geq c \left(\int \Omega |\lambda h(x)|^{\frac{1}{\gamma}}\right)^{\gamma} \|u_n\|^{1-\gamma}
$$

where, $c$ is a positive constant. This gives a contradiction, since $\gamma > 1$. Therefore, there exists $C > 0$ such that $\|u\| \geq C$ for all $u \in N_1$.

Since, $u \in N_1$ implies that $\int \Omega \lambda h(x)|u|^{1-\gamma} \leq (a + b\|u\|^2)\|u\|^2 + \int \Omega \phi_{u}u^2 - \int \Omega k(x) u^{1+p} < \infty$. Therefore, from the definition (16) of $I$, we have

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{4} \int \Omega \phi_{u}u^2 - \frac{1}{1-\gamma} \int \Omega \lambda h(x)|u|^{1-\gamma} - \frac{1}{p+1} \int \Omega k(x)|u|^{1+p}
\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c|k|_{\infty}\|u\|^{1+p}.
$$

(54)

Now, since $0 < p < 1$ and $a + b \geq 0$, therefore by using the Lemma 4.1, we get that $I$ is coercive and bounded below on $N_1$.

**Lemma 4.3:** Assume the compatibility condition (11) holds. Then there exists a minimizing sequence $\{u_n\} \subset N_1$ of $c = \inf_{N_1} I$, i.e. there exists $u \in N_1$ such that $I(u) = c$. Moreover, $u \in N_2$.

**Proof:** It is easy to see that the functional $I$, defined as in (16) is lower semicontinuous. Since, $N_1 (\neq \emptyset)$ is closed, then by using Ekeland’s variational principle for $c = \inf_{N_1} I$, we can extract a minimizing sequence $\{u_n\} \subset N_1$ such that

(i) $I(u_n) \leq c + \frac{1}{n}$;
(ii) $I(u_n) \leq I(v) + \frac{1}{n}\|u_n - v\|$, $\forall v \in N_1$.

Now, from the fact $I(|u|) = I(u)$, one can assume that $u_n > 0$ a.e. in $\Omega$. By Lemma 4.2, on using the coerciveness of $I$, we have $\{u_n\}$ is bounded. Therefore, up to a subsequence, we have

(i) $u_n \to u$ weakly in $X_0$
(ii) $u_n \to u$ strongly in $L^q(\Omega)$ for $q \in [1, 2_+^*[$, $2_+ = \frac{2N}{N-2s}$, and
(iii) $u_n(x) \to u(x)$ pointwise a.e. in $\Omega$.

Since, $N_1$ is closed and $\gamma > 1$, then from Fatou’s lemma and $\{u_n\} \subset N_1$, we get $u > 0$ a.e. in $\Omega$, $\int \Omega \lambda h(x)|u|^{1-\gamma} < \infty$ and $u \in N_1$. On using Fatou’s lemma and Lemma 4.1, we have

$$\inf_{N_1} I = \liminf_{n \to \infty} I(u_n)$$
\[
= \liminf_{n \to \infty} \left( \frac{a}{2} \|u_n\|^2 + \frac{b}{4} \|u_n\|^4 + \frac{1}{4} \int_\Omega \phi_{u_n} u_n^2 - \frac{1}{1 - \gamma} \int_\Omega \lambda h(x) u_n^{1-\gamma} - \frac{1}{1 + p} \int_\Omega k(x) u_n^{1+p} \right)
\]
\[
\geq \liminf_{n \to \infty} \frac{a}{2} \|u_n\|^2 + \frac{b}{4} \liminf_{n \to \infty} \|u_n\|^4 + \frac{1}{4} \phi_\alpha u_\alpha^2
\]
\[
\geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \frac{1}{4} \int \phi_\alpha u_\alpha^2 - \frac{1}{1 - \gamma} \int \lambda h(x) u^{1-\gamma} - \frac{1}{p + 1} \int k(x) u^{1+p}
\]
\[
= I(u) \geq I(t_0 u)
\]
\[
\geq \inf_{N_2} I \geq \inf_{N_1} I. \quad (55)
\]

Hence, \( t_0 = 1 \), i.e. \( u \in N_2 \) and hence \( I(u) = c \). This completes the proof. \( \blacksquare \)

**Proof of Theorem 1.3:** Suppose \( u \) is a solution to the problem (1), then the compatibility condition (11) must be true. We will prove the other part. Let (11) be true. We first prove the following inequality which is essential to guarantee the existence of solution.

\[
(a + b \|u\|^2) \int (-\Delta)^{1/2} u \cdot (-\Delta)^{1/2} \psi + \int \phi_\alpha u_\alpha \psi - \int k(x) u_\alpha^p \psi \geq \int \lambda h(x) u^{-\gamma} \psi.
\]

for every nonnegative \( \psi \in X_0 \). We will divide the proof of (56) in two cases, i.e. either \( \{u_n\} \subset N_1 \setminus N_2 \) or \( \{u_n\} \subset N_2 \).

\( \blacksquare \)

**Case 1. \( \{u_n\} \subset N_1 \setminus N_2 \) for \( n \) large enough.**

For a given nonnegative function \( \psi \in X_0 \), by \( \{u_n\} \subset N_1 \setminus N_2 \), we derive that

\[
(a + b \|u_n\|^2) \|u_n\|^2 + \int \phi_{u_n} u_n^2 - \int k(x) |u_n|^{1+p}
\]
\[
> \int \lambda h(x) u_n^{1-\gamma} \geq \int \lambda h(x)(u_n + t\psi)^{1-\gamma}, \quad t \geq 0,
\]
then by the continuity, there exists \( t > 0 \) small enough such that

\[
(a + b \|u_n + t\psi\|^2) \|u_n + t\psi\|^2 + \int \phi_{u_n + t\psi} (u_n + t\psi)^2 - \int k(x)(u_n + t\psi)^{1+p}
\]
\[
\geq \int \lambda h(x)(u_n + t\psi)^{1-\gamma},
\]
that is \( (u_n + t\psi) \in N_1 \). Then, by (ii) of Ekeland’s variational principle, we have

\[
\frac{1}{n} ||t\psi|| + I(u_n + t\psi) - I(u_n) \geq 0.
\]

That is,

\[
\frac{||t\psi||}{n} + \frac{a}{2} (||u_n + t\psi||^2 - ||u_n||^2) + \frac{b}{4} (||u_n + t\psi||^4 - ||u_n||^4)
\]
\[
\frac{1}{4} \int_{\Omega} (\phi_{u_n+t\psi} (u_n + t\psi)^2 - \phi_{u_n} u_n^2) - \frac{1}{1 + p} \int_{\Omega} k(x) \left( (u_n + t\psi)^{1+p} - u_n^{1+p} \right) \\
\geq \frac{1}{1 - \gamma} \int_{\Omega} \lambda h(x) \left( (u_n + t\psi)^{1-\gamma} - u_n^{1-\gamma} \right).
\]

Dividing by \(t > 0\) and by Fatou’s lemma, we conclude that

\[
\frac{1}{n} \|\psi\| + (a + b\|u_n\|^2) \int_{\Omega} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} \psi + \int_{\Omega} \phi_{u_n} u_n \psi - \int_{\Omega} k(x) u_n^p \psi \\
\geq \liminf_{t \to 0} \int_{\Omega} \lambda h(x) \left( (u_n + t\psi)^{1-\gamma} - u_n^{1-\gamma} \right) \\
\geq \int_{\Omega} \lambda h(x) u_n^{1-\gamma} \psi.
\]

Now by Lemma 4.3, we have \(I(u) = c\) for some \(u \in N_2\). Therefore, by using (55), we obtain \(\|u_n\|^2 \to \|u\|^2\) for every \(a > 0, b \geq 0\) and similarly, \(\|u_n\|^4 \to \|u\|^4\) for every \(b > 0\) with \(a = 0\). In both of the cases, \(\|u_n\| \to \|u\|\) as \(n \to \infty\). Thus, by applying Fatou’s lemma once again, we get

\[
(a + b\|u\|^2) \int_{\Omega} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \psi + \int_{\Omega} \phi_{u} u \psi - \int_{\Omega} k(x) u^p \psi \geq \int_{\Omega} \lambda h(x) u^{1-\gamma} \psi.
\]

**Case 2.** There exists a subsequence of \(\{u_n\}\) (still denoted by \(u_n\)) belonging to \(N_2\).

In this case, we can also show that (56) holds. For given nonnegative \(\psi \in X_0\), for each \(u_n \in N_2\) and \(t \geq 0\),

\[
\int_{\Omega} \lambda h(x) (u_n + t\psi)^{1-\gamma} \leq \int_{\Omega} \lambda h(x) u_n^{1-\gamma} < \infty.
\]

By Lemma 4.1, there exists \(t(u_n + t\psi) > 0\) satisfying \(t(u_n + t\psi)(u_n + t\psi) \in N_2\). For clarity, we denote \(\theta_n(t) = t(u_n + t\psi)\), it is obvious that \(\theta_n(0) = 1\). By \(\theta_n(t)(u_n + t\psi) \in N_2\), we have

\[
a \theta_n^2(t) \|u_n + t\psi\|^2 + b \theta_n^4(t) \|u_n + t\psi\|^4 + \theta_n^4(t) \int_{\Omega} (\phi_{u_n+t\psi} (u_n + t\psi)^2 \\
- \theta_n^{1-\gamma}(t) \int_{\Omega} \lambda h(x)(u_n + t\psi)^{1-\gamma} - \theta_n^{1+p}(t) \int_{\Omega} k(x)(u_n + t\psi)^{1+p} = 0.
\]

By Lemma 4.1, for given \(n, \theta_n\) is continuous on \([0, \infty)\). We denote \(D_+\theta_n(0)\) the right lower Dini derivative of \(\theta_n\) at zero. Next, we shall show that \(\theta_n\) has uniform behavior at zero with respect to \(n\), i.e. \(|D_+\theta_n(0)| \leq C\) for suitable \(C > 0\) independent of \(n\). By the definition of \(D_+\theta_n(0) = \lim_{t \to 0^+} \inf \frac{\theta_n(t) - \theta_n(0)}{t}\), there exists a sequence \(\{t_k\}\) with \(t_k > 0\) and \(t_k \to 0\) as \(k \to \infty\) such that

\[
D_+\theta_n(0) = \lim_{k \to \infty} \frac{\theta_n(t_k) - \theta_n(0)}{t_k}.
\]

By \(u_n \in N_2\) and (57), for \(t > 0\), we get that

\[
0 = \frac{1}{t} \left[ a (\theta_n^2(t) - 1) \|u_n + t\psi\|^2 + (\theta_n^4(t) - 1) \left( b \|u_n + t\psi\|^4 + \int_{\Omega} \phi_{u_n+t\psi} (u_n + t\psi)^2 \right) \right]
\]
By Lemma 4.1, we get that
\[
D_+\theta_n(0)(a(1-p)\alpha^2 + (3-p)b\alpha^4) + (2a + 4b\|u_n\|^2)\int_{\Omega} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} \psi + 4\int_{\Omega} \phi_{u_n}u_n\psi - (p + 1)\int_{\Omega} k(x)u_n^p \psi \leq 0.
\]
Since, \( p \in (0, 1) \), this implies that \( D_+ \theta_n(0) \neq +\infty \) and \( D_+ \theta_n(0) \) is bounded from above uniformly in \( n \). That is \( D_+ f_n(0) \in [-\infty, +\infty) \) and

\[
D_+ \theta_n(0) \leq C_1 \text{ uniformly for } n
\]  

for some \( C_1 > 0 \).

On the other hand, we can obtain the lower bound for \( D_+ \theta_n(0) \). If \( D_+ \theta_n(0) \geq 0 \) for \( n \) large, this gives the results. Otherwise, up to a subsequence, still denoted by \( D_+ \theta_n(0) \) such that \( D_+ \theta_n(0) \) are negative (possibly \( -\infty \)). Then by (ii) of Ekeland’s variational principle, for \( t > 0 \), we have

\[
(1 - \theta_n(t)) \| u_n \| + t\theta_n(t) \| \psi \|
\]

\[
\geq \frac{\| u_n - \theta_n(t)(u_n + t\psi) \|}{n}
\]

\[
\geq I(u_n) - I(f_n(t)(u_n + t\psi))
\]

\[
= \frac{a(\gamma + 1)}{2(\gamma - 1)} (\| u_n \|^2 - \| u_n + t\psi \|^2)
\]

\[
+ \frac{\gamma + 3}{4(\gamma - 1)} \left[ b(\| u_n \|^4 - \| u_n + t\psi \|^4) + \int_{\Omega} (\phi_{u_n} u_n^2 - \phi_{u_n + t\psi} (u_n + t\psi)^2) \right]
\]

\[- \frac{p + \gamma}{(\gamma - 1)(p + 1)} \int_{\Omega} k(x) (u_n^{p+1} - (u_n + t\psi)^{p+1})
\]

\[- \frac{a(\gamma + 1)}{2(\gamma - 1)} (f_n^2(t) - 1) \| u_n + t\psi \|^2
\]

\[- \frac{\gamma + 3}{4(\gamma - 1)} (f_n^4(t) - 1) \left( b \| u_n + t\psi \|^4 + \int_{\Omega} (\phi_{u_n + t\psi} (u_n + t\psi)^2) \right)
\]

\[+ \frac{p + \gamma}{(\gamma - 1)(p + 1)} (f_n^{p+1}(t) - 1) \int_{\Omega} k(x) (u_n + t\psi)^{p+1}.
\]

Then,

\[
\frac{t\theta_n(t)}{n} \| \psi \| \geq I(u_n) - I(\theta_n(t)(u_n + t\psi)) + \frac{\theta_n(t) - 1}{n} \| u_n \|
\]

\[= (\theta_n(t) - 1) \left[ \| u_n \| - \frac{a(\gamma + 1)}{2(\gamma - 1)} (\theta_n(t) + 1) \| u_n + t\psi \|^2
\]

\[- \frac{\gamma + 3}{4(\gamma - 1)} \frac{\theta_n^4(t) - 1}{\theta_n(t) - 1} \left( b \| u_n + t\psi \|^4 + \int_{\Omega} (\phi_{u_n + t\psi} (u_n + t\psi)^2) \right)
\]

\[+ \frac{p + \gamma}{(\gamma - 1)(p + 1)} \frac{\theta_n^{p+1}(t) - 1}{\theta_n(t) - 1} \int_{\Omega} k(x) (u_n + t\psi)^{p+1}
\]

\[+ \frac{a(\gamma + 1)}{2(\gamma - 1)} (\| u_n \|^2 - \| u_n + t\psi \|^2)
\]

\[+ \frac{\gamma + 3}{4(\gamma - 1)} \left[ b(\| u_n \|^4 - \| u_n + t\psi \|^4) + \int_{\Omega} (\phi_{u_n} u_n^2 - \phi_{u_n + t\psi} (u_n + t\psi)^2) \right]
\]
Then, replacing \( t \) in (60), dividing \( t_k \) and letting \( k \to \infty \), we deduce that
\[
\frac{\|\psi\|}{n} \geq D_+ \theta_n(0) \left[ \frac{\|u_n\|^2}{n} - \frac{a(y + 1)}{y - 1} \|u_n\|^2 - \frac{\gamma + 3}{\gamma - 1} \left( b\|u_n\|^4 + \int_{\Omega} \phi_{u_n} u_n^2 \right) \right. \\
+ \left. \frac{p + \gamma}{\gamma - 1} \int_{\Omega} k(x) u_n^{p+1} \right]
\]
\[
\geq \frac{\gamma}{\gamma - 1} \left[ a(1 - p)\|u_n\|^2 + (3 - p) \left( b\|u_n\|^2 + \int_{\Omega} \phi_{u_n} u_n^2 \right) \right. \\
+ \left. (\gamma + p) \int_{\Omega} \lambda h(x) u_n^{1-\gamma} \right] \\
\leq \frac{-a(1-p)}{\gamma - 1} \|u_n\|^2 \\
\leq \frac{-a(1-p)}{\gamma - 1} C^2.
\]
So, from (61), we have
\[
\frac{\|\psi\|}{n} \geq D_+ \theta_n(0) \left( \frac{\|u_n\|^2}{n} - \frac{(1 - p)ac^2}{\gamma - 1} \right) - \frac{a(y + 1)}{y - 1} \int_{\Omega} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} \psi \\
- \frac{\gamma + 3}{\gamma - 1} \left( b\|u_n\|^2 \int_{\Omega} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} \psi + \int_{\Omega} \phi_{u_n} u_n \psi \right) \\
+ \frac{p + \gamma}{\gamma - 1} \int_{\Omega} k(x) u_n^p \psi.
\]
We choose \( n \) large enough such that \( \frac{\|u_n\|^2}{n} - \frac{(1 - p)ac^2}{\gamma - 1} < 0 \), we know from (62) that \( D_+ \theta_n(0) \neq -\infty \) as \( n \to \infty \). That is \( D_+ \theta_n(0) \) is bounded from below uniformly for \( n \) large enough. Hence from (59), we have
\[
|D_+ \theta_n(0)| \leq C \text{ for } n \text{ large enough,}
\]
for some \( C > 0 \). Again, by (ii) of Ekeland’s variation principle, we also have
\[
\frac{|\theta_n(t) - 1|\|u_n\| + |t\theta_n(t)|\|\psi\|}{n}
\]
Thus from $\psi \geq t$, the above inequality also holds for $t = 0$, that is, the inequality (56) holds.

We now prove that $a + b\|u\| + \|\psi\| \geq 1$. Let $n \to \infty$, then by using $\|u_n\| \to \|u\|$ as $n \to \infty$. Hence, by using $|D_{++}f_n(0)| \leq C$, Fatou’s lemma and the strong convergence, we obtain $\int_\Omega \lambda h(x) u^{-\gamma} \psi < \infty$ and

$$(a + b\|u\|^2) \int_\Omega (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \psi + \int_\Omega \phi u_n \psi + \int_\Omega k(x) u_n^p \psi \geq 0.$$
Thus, \( t > 0 \) and letting \( t \to 0 \), we obtain that

\[
\int \Omega \phi_u u \psi_t - \int \Omega \lambda h(x) u^{-\gamma} \psi_t - \int \Omega k(x) u^p \psi_t
\]

\[
= (a + b\|u\|)^2 \int \Omega_1 (\Delta)^{s/2} u \cdot (\Delta)^{s/2} (u + t \psi) + \int \Omega_1 \phi_u u (u + t \psi)
\]

\[
- \int \Omega_1 \lambda h(x) u^{-\gamma} (u + t \psi) - \int \Omega_1 k(x) u^p (u + t \psi)
\]

\[
= (a + b\|u\|)^2 \int \Omega_2 (\Delta)^{s/2} u \cdot (\Delta)^{s/2} (u + t \psi) + \int \Omega_2 \phi_u u (u + t \psi)
\]

\[
- \int \Omega_2 \lambda h(x) u^{-\gamma} (u + t \psi) - \int \Omega_2 k(x) u^p (u + t \psi)
\]

\[
\leq t \left[ (a + b\|u\|)^2 \int \Omega_2 (\Delta)^{s/2} u \cdot (\Delta)^{s/2} \psi
\right.

\[
+ \int \Omega \phi_u u \psi - \int \Omega \lambda h(x) u^{-\gamma} \psi - \int \Omega k(x) u^p \psi
\]

\[
- (a + b\|u\|)^2 \int \Omega_2 (\Delta)^{s/2} u \cdot (\Delta)^{s/2} \psi - \int \Omega_2 \phi_u u \psi \right]
\]

Since, \( u > 0 \) almost everywhere in \( \Omega \) and the measure of \( \Omega_2 \) tends to zero as \( t \to 0 \), then dividing by \( t > 0 \) and letting \( t \to 0 \), we obtain that

\[
(a + b\|u\|)^2 \int \Omega_2 (\Delta)^{s/2} u \cdot (\Delta)^{s/2} \psi + \int \Omega \phi_u u \psi
\]

\[
- \int \Omega \lambda h(x) u^{-\gamma} \psi - \int \Omega k(x) u^p \psi \geq 0, \quad \psi \in X_0.
\]

This inequality also holds for \(-\psi\), so we have

\[
(a + b\|u\|)^2 \int \Omega_2 (\Delta)^{s/2} u \cdot (\Delta)^{s/2} \psi + \int \Omega \phi_u u \psi
\]

\[
- \int \Omega \lambda h(x) u^{-\gamma} \psi - \int \Omega k(x) u^p \psi = 0, \quad \psi \in X_0.
\]

Thus, \( u \in X_0 \) is a solution of system (14).

**Uniqueness of solution.** Assume the compatibility condition (11) holds. Let \( u, v \in X_0 \) be two weak solutions to system (14). Then from Definition 2.2, we have

\[
(a + b\|u\|)^2 \int \Omega_2 (\Delta)^{s/2} u \cdot (\Delta)^{s/2} (u - v) + \int \Omega \phi_u u (u - v)
\]

\[
= \int \Omega \lambda h(x) u^{-\gamma} (u - v) + \int \Omega f(x, u)(u - v). \tag{63}
\]
and
\[
(a + b\|v\|^2) \int_{\Omega} (-\Delta)^{3/2} v \cdot (-\Delta)^{3/2} (u - v) + \int_{\Omega} \phi_v v(u - v) \\
= \int_{\Omega} \lambda h(x) v^{-\gamma} (u - v) + \int_{\Omega} f(x, u)(u - v).
\]
(64)

On subtracting (64) from (63), we get
\[
a\|u - v\|^2 + b(\|u\|^4 + \|v\|^4 - \|u\|^2(u, v) - \|v\|^2(u, v)) + \int_{\Omega} (\phi_u u - \phi_v v)(u - v) \\
= \int_{\Omega} \lambda h(x) (u^{-\gamma} - v^{-\gamma})(u - v) + \int_{\Omega} (f(x, u) - f(x, v))(u - v) \\
= \int_{\Omega} \lambda h(x) (u^{-\gamma} - v^{-\gamma})(u - v), \text{ (by using } f(x; \cdot) \equiv 0). \tag{65}
\]

Now, on applying Hölder’s inequality, we have
\[
\|u\|^4 + \|v\|^4 - \|u\|^2(u, v) - \|v\|^2(u, v) \geq (\|u\| - \|v\|)^2(\|u\|^2 + \|u\|\|v\| + \|v\|^2) \geq 0.
\]
and \(\int_{\Omega} \lambda h(x) (u^{-\gamma} - v^{-\gamma})(u - v) \leq 0\), since \(\gamma > 0\). Therefore, on using \(a, b \geq 0\) with \(a + b > 0\) and from Lemma 2.1, we deduce that \(\|u - v\|^2 \leq 0\). Hence, the solution to the system (14) is unique.

**Remark 4.4:** Observe that, if we replace \(f\) by \(-f\) in the problem (14), then from (65) one can conclude that the problem (14) possesses a unique solution.

**Remark 4.5:** The existence of solution to the problem (14), for \(\gamma = 1\) can be obtained as a limit of the following sequence of problems
\[
(a + b[u]^2) (-\Delta)^{3/2} u + \phi_u u = \lambda \frac{h(x)}{(u + \frac{1}{n})} + f(x, u) \text{ in } \Omega, \\
\]
\[
u > 0 \text{ in } \Omega, \\
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\]

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