Group analysis of a semi-linear general bond-pricing equation

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Abstract
The complete group classification of a semi-linear generalization of a general bond-pricing equation is carried out by employing the underlying equivalence and additional equivalence transformations. Examples of invariant solutions are given under the terminal and the barrier option condition.

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1. Introduction
Recently, Sinkala et al. have introduced a general bond-pricing equation
\[
u_t + \frac{1}{2} \rho^2 x^{-\gamma} u_{xx} + (\alpha + \beta x - \lambda \rho x^\delta) u_x - xu = 0, \quad \rho \neq 0, \quad \delta \neq 0, 1 \quad (1)
\]
where $\alpha, \beta, \gamma, \delta, \rho$ and $\lambda$ are real constants and $u = u(x,t), x > 0$ [19]. The equation have an interesting property: it contains some of the classical models of financial mathematics, namely the Longstaff model ($\gamma = \delta = 1/2, \alpha =$...
\(\rho^2/4\), the Vacicek model \((\gamma = \delta = 0, \beta \neq 0)\) and the Cox–Ingersoll–Ross model \((\gamma = \delta = 1/2, \lambda = 0)\) [13, 22, 5].

In this paper we study a semi-linear generalization of Eq. (1)

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \rho^2 x^{2\gamma} u_{xx} + (\alpha + \beta x - \lambda \rho x^\delta) u_x - f(x, u) = 0, \quad \rho \neq 0, \ \delta \neq 0, 1
\]

(2)

using the analytic machinery provided by symmetry analysis. We use elements from the enhanced group analysis, namely equivalence transformations, in order to simplify the main task at hand, the complete group classification of Eq. (2). Recall that to perform a complete group classification of a differential equation (or a system of differential equations) involving arbitrary functions or/and parameters, means to find the Lie point symmetry group \(G\) for the most general case, and then to find specific forms of the differential equation for which \(G\) can be enlarged, [17, p. 178]. Having the group classification specific cases of Eq. (2) can be pinpointed admitting a larger set of symmetries, and consequently, a better likelihood of constructing invariant solutions. A fact that plays a prominent role when one considers also a set of initial or boundary conditions along the PDE.

In the present work two distinct conditions of financial interest will be considered along with (2). The terminal condition

\[ u(x, T) = 1, \tag{3} \]

where \(T\) is the terminal time, and the barrier option condition

\[
\begin{align*}
  u(H(t), t) &= R(t), \\
  u(x, T) &= \max(x - K, 0),
\end{align*}
\]

(4a, 4b)

where the barrier option \(u(x, t)\) satisfies Eq. (2) for \(x > H(t), 0 < t < T\).\(^2\) The constant \(T\) again is the terminal time where the barrier option is exercised and \(K\) is the strike price.

The first condition (4a) describes the evolution of standard or “vanilla” products; the price of a zero-coupon bond (or of a financial option), \(u(x, t)\), which expires when \(t = T\) [1, 2, 14, 21].

\(^1\)Immediately one can observe that eq. (2) contains also the celebrated Black–Scholes–Merton model [2] when \(\gamma = \delta = 1, \alpha = 0\).

\(^2\)It is evident that the terminal condition (3) is included as a special situation when the asset price exceeds a value defined by the barrier function \(H\) on time \(T\).
The second condition (4b) takes into account the possibility of an unacceptable change in the price of the underlying option. It is considered an exotic type of option possessing features that makes it more complex than the “vanilla” option, [15, 12, 16]: The underlying idea is that now a barrier $H(t)$ exists and when the asset price $x$ crosses it, the barrier option $u(x,t)$ becomes extinguished or comes into existence. Those two types are also known as down-and-out and down-and-in respectively. Often a rebate, $R(t)$, is paid if the option is extinguished. In what follows we shall consider the down-and-out type. A common assumption for the barrier function $H$ is to have the exponential form

$$H(\tau) = bK e^{-a\tau},$$

(5)

where $a \geq 0$, $0 \leq b \leq 1$ and $\tau = t - T$ [12, p. 187].

As previously mentioned the key analytical tool used in this work is the symmetry analysis of Eq. (2). One of the advantages of this approach is that it provides a well defined algorithmic procedure which essentially enables one to find the involved linearizing transformations, conservation laws, invariant solutions, etc. In fact, various works on the classical financial models mentioned above have a connection with the heat equation [8, 7]. As probably expected Eq. (2) shares a connection with the heat equation, and more precisely with the heat equation with nonlinear source,

$$u_t = u_{xx} + f(x,u).$$

(6)

For all the calculations involved the symbolic package SYM for Mathematica was extensively used, both for the interactive manipulation of the found symmetries as well as for determining the equivalence transformation and the classification of Eq. (2) [6].

This paper is organized as follows. In section 2 the basic concepts of the Lie point symmetry approach to differential equations used in the paper are presented. In section 3 the continuous equivalence transformations are found and with their help the complete group classification of Eq. (2) is obtained. In section 4 specific examples of invariant solutions under the specific boundary problems studied, the “vanilla” option and the barrier option, are given. Finally, in section 5 the results of this work are discussed.

2. Preliminaries

In this section we expose some notions of the modern group analysis that will be encountered in the main sections of the article suitably adapted to
the article’s needs. For a full treatise of the subject we direct the interested reader to the classical texts \([17, 3, 11, 18, 10, 20]\).

A Lie point symmetry\(^3\) is a differential operator, named \textit{infinitesimal generator},

\[
\mathfrak{X} = \xi^1(x, t, u) \frac{\partial}{\partial x} + \xi^2(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u},
\]

that satisfies the condition

\[
\mathfrak{X}^{(2)} \left[ u_t + \frac{1}{2} \rho^2 x^{2\gamma} u_{xx} + (\alpha + \beta x - \lambda \rho x \gamma) u_x - f(x, u) \right] \bigg|_{(2)} \equiv 0,
\]

where \(\mathfrak{X}^{(2)}\) is the suitable prolongation of the differential operator up to order two. The set of all the symmetries admitted by a differential equation constitute a Lie algebra.

Having the symmetries there is a wealth of things that can be done with. In the present paper, we use them to obtain \textit{invariant} or \textit{similarity solutions} of the Eq. (2). By invariant solutions we mean solutions of (2) that are invariant under one of the found symmetries \(\mathfrak{X}\), that is,

\[
\mathfrak{X}[u - \varphi(x, t)]|_{u=\varphi(x,t)} \equiv 0.
\]

The Eq. (8) is a linear PDE called \textit{invariant surface condition} and by solving it we obtain a way to reduce the order of Eq. (2). Similarly, when we look for a similarity solution of Eq. (2) along with a initial/boundary condition we have to choose the subalgebra that leaves also invariant that condition and its boundary:

\[
\mathfrak{X}(t - T)|_{t=T} \equiv 0
\]

and

\[
\mathfrak{X}(u - 1)|_{t=T, u=1} \equiv 0.
\]

for the boundary condition (3). And

\[
\mathfrak{X}(x - H(t))|_{x=H(t)} \equiv 0
\]

and

\[
\mathfrak{X}(u - R(t))|_{x=H(t), u=R(t)} \equiv 0.
\]

\(^3\)Henceforth referred simply as symmetry.
for the boundary condition (4).

For obtaining the equivalence transformations there are two possible ways. The first one is by a direct substitution of the equivalence transformations, an approach that gives in theory the most general equivalence set of transformations, the equivalence group. But it has the same pitfall as when trying to obtain the symmetry group for the equation using the direct method: computational difficulties and a set of conditions equally difficult to solve as the initial equation. The second approach is through the calculation of the *equivalence algebra* from which the continuous equivalence group can be obtained. In the present work the second road will be realized by complementing the usual prolongation of the infinitesimal generator with a *secondary prolongation* [11].

To calculate the equivalence algebra, an extension of Eq. (2) must be considered with the arbitrary elements \( \alpha, \beta, \gamma, \delta, \lambda, \rho, f \), now considered as functions of \( x, t, u \), and by including the following constraints on them,

\[
\begin{align*}
\alpha_x &= \alpha_t = \alpha_u = \beta_x = \beta_t = \beta_u = \gamma_x = \gamma_t = \gamma_u = \delta_x = \delta_t = \delta_u = \\
\lambda_x &= \lambda_t = \lambda_u = \rho_x = \rho_t = \rho_u = f_t = 0.
\end{align*}
\] (13)

For this extended system the infinitesimal generator is

\[
\mathfrak{X} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial \alpha} + \eta_3 \frac{\partial}{\partial \beta} + \eta_4 \frac{\partial}{\partial \gamma} + \eta_5 \frac{\partial}{\partial \delta} + \eta_6 \frac{\partial}{\partial \lambda} + \eta_7 \frac{\partial}{\partial \rho} + \eta_8 \frac{\partial}{\partial f},
\] (14)

where now the coefficients depend on the extended space of variables: \( \xi^i = \xi^i(x, t, u, \alpha, \beta, \gamma, \delta, \lambda, \rho, f) \) and \( \eta_j = \eta_j(x, t, u, \alpha, \beta, \gamma, \delta, \lambda, \rho, f) \).

Having the equivalence algebra by exponentiation one can obtain the continuous part of the equivalence group. In addition, by using the method proposed in [10, pp. 187 c.f.], [9] one can also obtain the discrete part and hence retrieve the whole set of equivalence transformations permissible by this class of equations.

Another useful notion is that of the *additional equivalence transformation*. An additional equivalence transformation is a point transformation that connects inequivalent classes of differential equations that possess the same Lie algebra of symmetries. The knowledge of such transformations greatly facilitates the classification.
3. Group classification

In this section we proceed with the group classification of Eq. (2). First, the best representative for the class of equations (2) is obtained utilizing its equivalence algebra. To do that, the continuous part of the equivalence group is constructed and with its help as many as possible arbitrary elements are zeroed.

Firstly, observe the trivial transformation
\[
\tilde{x} = x, \quad \tilde{t} = \frac{\rho^2}{2} t, \quad \tilde{u} = u, \quad \tilde{\alpha} = \frac{2}{\rho^2} \alpha, \quad \tilde{\beta} = \frac{2}{\rho^2} \beta, \quad \tilde{\gamma} = \gamma, \quad \tilde{\delta} = \delta, \quad \tilde{\lambda} = \frac{\sqrt{2}}{\rho} \lambda,
\]
\[\tilde{f} = \frac{2}{\rho^2} f.\]  
(15)
transforms Eq. (2) into the equation
\[u_t + x^{2\gamma} u_{xx} + (\tilde{\alpha} + \tilde{\beta} x - \sqrt{2} \lambda x^\delta) u_x - \tilde{f}(x, u) = 0.\]  
(16)
We proceed in obtaining the equivalence algebra for Eq. (16) without making any assumption on the coefficients of the extended operator (14), now adapted for the extended space \((x, \tilde{t}, u, \tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda}, \tilde{f})\)^4.

**Theorem 3.1.** The equivalence algebra \(\hat{L}_E\) of the class of equations (16) is generated by the following vector fields
\[
F_1(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda}) \partial_{\tilde{t}}
\]
(17)
\[
F_2(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda})(\tilde{f} \partial_{\tilde{f}} + u \partial_u)
\]
(18)
\[
F_3(x, \tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda}) \partial_u + \frac{((\tilde{\alpha} + \tilde{\beta} x - \sqrt{2} \lambda x^\delta) F_{3x}(x, \tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda}) + x^{2\gamma} F_{3xx}(x, \tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda}))}{\gamma - 1} \partial_{\tilde{f}}
\]
(19)
\[
F_4(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda}) \left( \frac{x^{2(1-\gamma)} u \partial_u + 4 \partial_{\tilde{\beta}} + \frac{x^{-2\gamma}\left(\tilde{f} x^2 + 2(\gamma - 1) \left((2\gamma - 1)x^{2\gamma} + \sqrt{2} \lambda x^{1+\delta} - \tilde{\alpha} x - \tilde{\beta} x^2\right) u\right)}{\gamma - 1} \partial_{\tilde{f}} \right)
\]
(20)
\[
F_5(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \tilde{\lambda}) \left( \frac{x^{1-2\gamma}}{2\gamma - 1} u \partial_u + 2 \partial_{\tilde{\alpha}} + \frac{x^{-1-2\gamma}\left(\tilde{f} x^2 + (2\gamma - 1) \left(2\gamma x^{2\gamma} + \sqrt{2} \lambda x^{1+\delta} - \tilde{\alpha} x - \tilde{\beta} x^2\right) u\right)}{2\gamma - 1} \partial_{\tilde{f}} \right)
\]
(21)

\[^4\text{In other words we look for generalized equivalence transformations.}\]
\[ F_6(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \lambda) \left( \frac{\sqrt{2} x^{1-2\gamma+\delta}}{1-2\gamma+\delta} u \partial_u + 2\partial_x + \frac{x^{\delta-1-2\gamma}}{\delta + 1-2\gamma} \left( \sqrt{2} \tilde{f} x^2 + (\delta + 1-2\gamma) \left( \sqrt{2} \tilde{\alpha} x + \sqrt{2} \tilde{\beta} x^2 + \sqrt{2} x^{2\gamma} (\delta - 2\gamma) - 2 \tilde{\lambda} x^{1+\delta} \right) u \right) \partial_f \right) \]

\[ F_7(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \lambda) \left( \frac{\sqrt{2} \lambda x^{1-2\gamma+\delta} \left( (1-2\gamma + \delta) \log x - 1 \right)}{(1-2\gamma + \delta)^2} u \partial_u + 2\partial_x + \tilde{\lambda} x^{-1+\delta} \left( u - \frac{\tilde{f} x^{1-\gamma}}{(1-2\gamma + \delta)^2} + \frac{x^{-2\gamma} \log x}{1-2\gamma + \delta} \left( \tilde{f} x^2 + (1-2\gamma + \delta) \left( \tilde{\alpha} x + \tilde{\beta} x^2 + (\delta - 2\gamma) x^{2\gamma} - \sqrt{2} \lambda x^{1+\delta} \right) u \right) \right) \partial_f \right) \]

\[ F_8(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \lambda) \left( 2 x^\gamma \partial_x + u x^{-1-\gamma} \left( \gamma x^{2\gamma} + \sqrt{2} \lambda x^{1+\delta} - \tilde{\alpha} x - \tilde{\beta} x^2 \right) \partial_u + x^{-3-\gamma} \left( \tilde{f} \left( \gamma x^{2(1+\gamma)} + \sqrt{2} \lambda x^{3+\delta} - \tilde{\alpha} x^3 - \tilde{\beta} x^4 \right) + u \left( \tilde{\beta}^2 (\gamma - 1) x^4 - 2 \tilde{\alpha} \gamma x^{1+2\gamma} + \tilde{\alpha}^2 \gamma x^2 + \tilde{\alpha} \tilde{\beta} (2\gamma - 1) x^3 + \gamma (2 - 3\gamma + \gamma^2) x^{4\gamma} - \sqrt{2} (2\gamma - \delta) (\delta - 1) \lambda x^{1+2\gamma+\delta} + \sqrt{2} \tilde{\alpha} (\delta - 2\gamma) \tilde{\lambda} x^{2+\delta} + \sqrt{2} \tilde{\beta} (1 - 2\gamma + \delta) \tilde{\lambda} x^{3+\delta} - 2(\delta - \gamma) \tilde{\lambda}^2 x^{2(1+\delta)} \right) \right) \partial_f \right) \]

\[ F_9(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta, \lambda) \left( \frac{2 x}{1-\gamma} \partial_x + 4 \tilde{\alpha} \partial_y + \frac{u x^{-2\gamma} \left( \tilde{\alpha} x + \tilde{\beta} x^2 + 4(\gamma - 1) x^{2\gamma} - \sqrt{2} \lambda x^{1+\delta} \right)}{\gamma - 1} \partial_u + \frac{x^{1-2\gamma}}{\gamma - 1} \left( \tilde{f} x^2 \left( \tilde{\alpha} + \tilde{\beta} x - \sqrt{2} \lambda x^3 \right) - u \left( 2 \tilde{\beta}^2 (\gamma - 1) x^3 + \tilde{\alpha}^2 (2\gamma - 1) x - 2 \tilde{\alpha} \gamma (2\gamma - 1) x^{2\gamma} + \tilde{\alpha} \tilde{\beta} (4\gamma - 3) x^2 - 2 \tilde{\beta} (1 - 3\gamma + 2\gamma^2) x^{1+2\gamma} + \sqrt{2} \tilde{\alpha} (2 - 4\gamma + \delta) \tilde{\lambda} x^{1+\delta} + \sqrt{2} \tilde{\beta} (3 - 4\gamma + \delta) \tilde{\lambda} x^{2+\delta} + \sqrt{2} (2\gamma - \delta) (\delta - 2\gamma + 1) \tilde{\lambda} x^{2\gamma+\delta} - 2(1 - 2\gamma + \delta) \tilde{\lambda}^2 x^{1+2\delta} \right) \right) \partial_f \right) \]
modulo the extended system itself, we get the system of determining equa-

\[ F_{10}(\tilde{\alpha}, \bar{\beta}, \gamma, \delta, \lambda) \left( 4\partial_x - \frac{4x(1 + (\gamma - 1)\log x)}{(\gamma - 1)^2} \partial_x + \frac{x^{-2\gamma}u}{(2\gamma - 1)^2(1 - 2\gamma + \delta)^2} \left( x (2\tilde{\alpha} (2\gamma^2 - 1) (1 - 2\gamma + \delta)^2 + (1 - 2\gamma)^2 \left( \tilde{\beta}(1 - 2\gamma + \delta)^2x - 2\sqrt{2} (2\gamma^2 + \delta(2 - 4\gamma + \delta) - 1) \tilde{x}^2 \right) \right) + 2 (1 - 3\gamma + 2\gamma^2) (1 - 2\gamma + \delta) ((1 - 2\gamma)(1 - 2\gamma + \delta)x^{2\gamma} + \tilde{\alpha}(1 - 2\gamma + \delta)x - \sqrt{2}(2\gamma - 1)(\delta - 1)\tilde{x}^{1+\delta}) \right) \right) \partial_u + \]

\[ \frac{(2\gamma - 1)^2(1 - 2\gamma + \delta)^2}{2u(1 - 2\gamma)^2(1 - 2\gamma + \delta)^2} \left( \tilde{\alpha}(1 - 2\gamma)(1 - 2\gamma + \delta) - \tilde{\alpha}^2(1 - 2\gamma)^2x^2 + 2\tilde{\alpha}(1 - 2\gamma + \delta)x^{2\gamma} - \sqrt{2}\tilde{\alpha}(1 - 2\gamma + \delta)\tilde{x}^{1+\delta} - \frac{\sqrt{2} \left( 2 + 2\gamma - 2\gamma^2 - 2\gamma(1 + \delta) \right) \tilde{x}^{1+2\gamma+\delta} + 2(\delta - \gamma)^2 \tilde{x}^{2(1+\delta)} \right) + f x^3 \left( 2\tilde{\alpha}(2\gamma^2 - 1)(1 - 2\gamma + \delta)^2 + (1 - 2\gamma)^2 \left( \tilde{\beta}(1 - 2\gamma + \delta)^2x - 2\sqrt{2}(1 - 2\gamma + \delta)\tilde{x}^{1+\delta} \right) \right) \right) - 2x (1 - 3\gamma + 2\gamma^2) (2\gamma - \delta - 1) \left( \tilde{f} x \left( (2\gamma - 1)(2\gamma - \delta - 1)x^{2\gamma} + \tilde{\alpha}(1 - 2\gamma + \delta)x - \sqrt{2}(2\gamma - 1)(\delta - 1)\tilde{x}^{1+\delta} \right) \right) + u(2\gamma - 1)(2\gamma - \delta - 1) \left( \tilde{\alpha}^2 x + \tilde{\alpha}\tilde{\beta} x^2 = 2\tilde{\alpha}\gamma x^{2\gamma} + \sqrt{2}\tilde{\alpha}(\delta - 2)\tilde{x}^{1+\delta} + \sqrt{2}\tilde{\alpha}(\delta - 1)\tilde{x}^{2+\delta} - \sqrt{2}(2\gamma - \delta)(\delta - 1)\tilde{x}^{2\gamma+\delta} - 2(\delta - 1)\tilde{x}^{1+2\delta} \right) \left( \log x \right) \partial_{\tilde{f}} \right) \]

where \( F_i, \ i = 1, \ldots, 10 \) are arbitrary real functions.

**Proof.** By applying the second order prolongation to the extended system

\[ u_{\tilde{t}} + x^{2\gamma}(x, \tilde{t}, u) u_{xx} - \tilde{f}(x, \tilde{t}, u) + \]

\[ (\tilde{\alpha}(x, \tilde{t}, u) + \tilde{\beta}(x, \tilde{t}, u)x - \sqrt{2}\tilde{\lambda}(x, \tilde{t}, u)x^{\delta(x, \tilde{t}, u)}u_x) = 0, \]

\[ \tilde{\alpha}_x = \tilde{\alpha}_{\tilde{t}} = \tilde{\alpha}_u = \tilde{\beta}_x = \tilde{\beta}_{\tilde{t}} = \tilde{\beta}_u = \gamma_x = \gamma_{\tilde{t}} = \gamma_u = \delta_x = \delta_{\tilde{t}} = \delta_u = \]

\[ \tilde{\lambda}_x = \tilde{\lambda}_{\tilde{t}} = \tilde{\lambda}_u = \tilde{f}_{\tilde{t}} = 0, \]

modulo the extended system itself, we get the system of determining equa-
\( \eta_3 = 0, \eta_4 = 0, \eta_5 = 0, \eta_6 = 0, \eta_7 = 0, \xi_2 = 0, \xi_{2f} = 0, \eta_{3u} = 0, \eta_{4u} = 0, \eta_{5u} = 0, \eta_{6u} = 0, \xi_2 = 0, \xi_{2u} = 0, \eta_{6i} = 0, \eta_{7i} = 0, \xi_2 = 0, \eta_{3x} = 0, \eta_{4x} = 0, \eta_{5x} = 0, \eta_{7x} = 0, \xi_2 = 0, \eta_{1f} - \tilde{f} \xi_f = 0, \)

\( \eta_{1f} - \tilde{f} \xi_f = 0, \eta_1 - (\tilde{\alpha} + \tilde{\beta}x - \sqrt{2}x^\delta) \xi_f = 0, 2\xi_1^2 - \eta_1f + \tilde{f} \xi_f = 0, \)

\( \xi_1^1 - (\tilde{\alpha} + \tilde{\beta}x - \sqrt{2}x^\delta) \xi_1^2 = 0, \eta_{1f} - 2\xi_2^2 - \tilde{f} \xi_f = 0, \)

\( \xi_1^1 - (\tilde{\alpha} + \tilde{\beta}x - \sqrt{2}x^\delta) \xi_1^2 = 0, \eta_{1f} - 2\xi_2^2 - \tilde{f} \xi_f = 0, \)

\( \xi_{uu}^1 - (\tilde{\alpha} + \tilde{\beta}x - \sqrt{2}x^\delta) \xi_{uu}^2 = 0, \eta_{1f} - 2\xi_2^2 - \tilde{f} \xi_f = 0, \)

\( \tilde{f} \xi_f - \tilde{f} \left( \tilde{\alpha} + \tilde{\beta}x - \sqrt{2}x^\delta \right) \xi_f^2 + 2x^{2\gamma} \left( \xi_2 - \eta_{1f} + \tilde{f} \xi_f \right) = 0, \)

\( \left( 3\tilde{\alpha}x + 3\tilde{\beta}x^2 + 4\gamma x^{2\gamma} - 3\sqrt{2}x^\delta \right) \xi_f^2 + 2x^{1+2\gamma} \xi_{xf}^2 - 2x\xi_f = 0, \)

\( \left( \tilde{\alpha} + \tilde{\beta}x - \sqrt{2}x^\delta \right) \eta_{1j} - 2\left( \tilde{\beta}x - \sqrt{2}x^\delta \right) \xi_f^2 + 2x^{2\gamma} \left( \xi_2 - \eta_{1f} + \tilde{f} \xi_f \right) = 0, \)

\( 2\left( \tilde{\beta}x - \sqrt{2}x^\delta \right) \xi_f^2 + x \left( \eta_{uu} - \tilde{f} \xi_u^2 - 2\xi_{xf}^1 + 2\tilde{\alpha} \xi_{xf}^2 + 2\tilde{\beta}x \xi_{xf}^2 - 2\sqrt{2}x^\delta \xi_{xf}^2 \right) = 0, \)

\( \left( \tilde{\beta}x - \sqrt{2}x^\delta \right) \xi_f^2 + x \left( \eta_{uf} - \tilde{f} \xi_u^2 - \xi_{xf}^1 + \tilde{\alpha} \xi_{xf}^2 + \tilde{\beta}x \xi_{xf}^2 - \sqrt{2}x^\delta \xi_{xf}^2 - \xi_{xf}^2 \right) = 0, \)

\( 2x \log x \eta_5 + 2\gamma \xi_1^1 + \tilde{f} x \xi_u^2 + x \xi_f^1 - 2x \xi_x^1 + x^{1+2\gamma} \xi_{xx}^2 + (3\tilde{\alpha}x + 3\tilde{\beta}x^2 + 4\gamma x^{2\gamma} - 3\sqrt{2}x^\delta) \xi_f^2 = 0, \)

\( \tilde{f} \eta_{1u} - \eta_2 - f^2 \xi_u^2 + \eta_i - \tilde{f} \xi_i^1 + \tilde{\alpha} \eta_x + \tilde{\beta}x \eta_x - \sqrt{2}x^\delta \eta_1 - \tilde{f} \tilde{\alpha} \xi_x^2 - \tilde{f} \tilde{\beta}x \xi_x^2 + \sqrt{2}f \xi^2 \xi_x^2 + x^{2\gamma} \eta_{1xx} - \tilde{f} x^{2\gamma} \xi_x^2 = 0, \)

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\[
\left(\hat{\alpha}^2 x + 2\tilde{\beta} x^{1+2\gamma} + 2\tilde{\alpha} \tilde{\beta} x^2 + \tilde{\beta}^2 x^3 - 2\sqrt{2}\hat{\alpha} \hat{\lambda} x^{1+\delta} - 2\sqrt{2}\tilde{\beta} \hat{\lambda} x^{2+\delta} - 2\sqrt{2}\tilde{\delta} \lambda x^{2+\delta}\right) \\
2\sqrt{2}\Delta \hat{\lambda} x^{2+\delta} + 2\hat{\lambda}^2 x^{1+2\delta}\right) \xi^2 - x \left(\hat{\alpha} + \hat{\beta} x - \sqrt{2}\hat{\lambda} x^\delta\right) \xi^1 + 2x^{1+2\gamma} \left(\eta_{u\gamma} - \xi^2 - \tilde{f} \xi^2 - \xi_{x\gamma}^2 + \hat{\alpha} \xi_{x\gamma}^2 + \hat{\beta} x^2 \xi_{x\gamma}^2 - \sqrt{2}\hat{\lambda} x^\delta \xi_{x\gamma}^2\right) = 0,
\]

where \(2\tilde{\alpha} \tilde{\beta} x^2 - 2\sqrt{2}\tilde{\lambda} \hat{\lambda} x^\delta\). By setting \(\delta = 2\gamma - 1\), and afterwards we treat each special case separately.

**Lemma 3.2.** For \((\gamma - 1)(\gamma - 1/2)(\delta - 2\gamma + 1) \neq 0\) there is an equivalence transformation that zeroes the parameters \(\hat{\alpha}, \hat{\beta}, \lambda, \lambda\):

\[
\hat{\alpha} = A_1 \alpha, \quad \hat{\beta} = A_1 \beta, \quad \hat{\lambda} = \frac{\alpha + \beta x}{\gamma}, \quad \hat{\lambda} = \frac{2\sqrt{2}\tilde{\lambda} \hat{\lambda} x^\delta}{x},
\]

where

\[
A_1 = \exp \left(\frac{1}{4} x^{-2\gamma} \left(\frac{2\tilde{\alpha} x^2}{1 - 2\gamma} - 2\sqrt{2}\tilde{\lambda} \hat{\lambda} x^\delta - \frac{\hat{\beta}}{\gamma - 1} x\right) \right).
\]

**Proof.** By exponentiating the vectors \((20), (21) and (22)\) (after setting the arbitrary functions to unity) we have that \(\hat{\beta} = \beta + 4\delta_0, \ \hat{\alpha} = \alpha + 2\delta_1, \ \hat{\lambda} = \lambda + 2\delta_2\), respectively. By setting \(\delta_0 = -\beta/4, \ \delta_1 = -\tilde{\alpha}/2\) and \(\delta_2 = -\tilde{\lambda}/2\) and substituting to the rest of the transformations we get the said result.

Applying the above transformation to Eq. (16) we get

\[
\hat{u}_t + x^{2\gamma} \hat{u}_{xx} - \hat{f}(x, \hat{u}) = 0.
\]
Once again we repeat the same process, now for Eq. (27), also assuming that \( \xi^i = \xi^i(x, \tilde{t}, u), \eta_1 = \eta_1(x, \tilde{t}, u) \). For brevity, we skip the intermediate steps and give directly the equivalence transformations in the following lemma.

**Lemma 3.3.** The continuous part of the equivalence group, \( \hat{E}_C \), of Eq. (27) consists of the transformations

\[
\begin{align*}
\bar{x} &= x^{\frac{1}{2}}, \quad \bar{\xi} = \frac{1}{\delta_2} \xi, \quad \bar{\eta} = x^\frac{4}{\delta_2 - 1} (\delta_1 \hat{u} + F(x)), \quad \bar{\gamma} = 1 + \delta_2^2 (\gamma - 1), \\
\bar{\tilde{t}} &= \tilde{t} - 1 + \frac{\delta_2}{2} \tilde{\xi}, \\
\bar{\tilde{t}} &= \frac{1}{4} x^{\frac{1}{2} (\frac{4}{\delta_2 - 5} - 3)} \left( (\delta_2^2 - 1)x^{2\gamma}(\delta_1 \hat{u} + F(x)) + 4 \delta_2^2 x^2 (\delta_1 \hat{f} + x^{2\gamma} F''(x)) \right),
\end{align*}
\]

where \( \delta_1, \delta_2 \) are arbitrary constants with \( \delta_1 \delta_2 \neq 0 \) and \( F \) is an arbitrary real function.

Immediately, one can choose a specific equivalence transformation\(^5\) that zeroes \( \bar{\gamma} \):

\[
\begin{align*}
\bar{x} &= x^{1-\gamma}, \quad \bar{\tau} = -(\gamma - 1)^2 \bar{\xi}, \quad \phi = x^{-\frac{3}{2}} \hat{u}, \\
\bar{f} &= -\frac{1}{4(\gamma - 1)^2 x^{-\frac{1}{2} (\gamma + 4)}} \left( (\gamma - 2 - \gamma)x^{2\gamma} \hat{u} + 4x^2 \hat{f} \right).
\end{align*}
\]

Via this transformation Eq. (27) transforms into the equation

\[
\phi_{\tau} - \phi_{\bar{x}x} - \bar{f}(\bar{x}, \phi) = 0,
\]

the heat equation with nonlinear source. Hence the problem of the complete group classification of Eq. (2) — for \( (\gamma - 1)(\gamma - 1/2)(\delta - 2\gamma + 1) \neq 0 \) — is reduced to the group classification of the heat equation with nonlinear source, (6).

Before continuing with the group classification of Eq. (6) we give, omitting the detailed calculations, the equivalence transformations for each one of the special cases.

\(^5\)At this point we have also included the discrete equivalence transformation \( t \to -t, f \to -f \).
3.1. The case $\gamma = 1$

**Lemma 3.4.** There is a equivalence transformation that zeroes the parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}$:

\[ \hat{u} = A_2 u, \]
\[ \hat{f} = \frac{1}{4x^2} A_2 \left( 4x^2 \hat{f} + \left( \tilde{\alpha}^2 + 2\tilde{\lambda}^2 x^{2\delta} - 2\sqrt{2\tilde{\lambda}} x^{\delta} (\tilde{\alpha} + (\tilde{\beta} + \delta - 2)x + \tilde{\lambda}^2 x^2 - 2\tilde{\beta} x^2 + 2\tilde{\alpha}\tilde{\beta}x - 4\tilde{\alpha}x) u \right) \right), \]

where

\[ A_2 = x^{3/2} \exp \left( -\frac{\tilde{\alpha} + \sqrt{2}\tilde{\lambda} x^{\delta}}{2x} \right). \]

Applying this transformation we arrive again to Eq. (27), now for $\gamma = 1$. For this case an additional equivalence transformation exists,

\[ \bar{x} = \log x, \quad \tau = -\tilde{t}, \quad \phi = x^{-1/2} \hat{u}, \quad \bar{f} = -\frac{1}{\sqrt{x}} (\frac{1}{4} \hat{u} + \hat{f}), \]

that turns it into Eq. (29).

3.2. The case $\gamma = 1/2$

**Lemma 3.5.** There is a equivalence transformation that zeroes the parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}$:

\[ \hat{u} = A_3 u, \]
\[ \hat{f} = \frac{1}{4x} A_3 \left( 4x \hat{f} + (2\tilde{\alpha} - 2)\tilde{\alpha} u + u \left( 2\tilde{\lambda}^2 x^{2\delta} - 2\sqrt{2\tilde{\lambda}} x^{\delta} (\tilde{\alpha} + \delta - 1 + \tilde{\beta} x) + \tilde{\beta} x (2\tilde{\alpha} + \tilde{\beta} x) \right) \right), \]

where

\[ A_3 = x^{5/2} \exp \left( \frac{\tilde{\beta} x - \tilde{\lambda}}{\sqrt{2}\delta} x^{\delta} \right). \]

Similarly, utilizing this transformation we reach Eq. (27), now for $\gamma = 1/2$. By using now the equivalence transformation (28) for $\gamma = 1/2$,

\[ \bar{x} = \sqrt{x}, \quad \tau = -\tilde{t}/4, \quad \phi = x^{-1/4} \hat{u}, \quad \bar{f} = -x^{-\frac{1}{4}} \left( \frac{3}{4} \hat{u} + 4x \hat{f} \right), \]

we again arrive at Eq. (29).
3.3. The case $\delta = 2\gamma - 1 \neq 0, 1$

**Lemma 3.6.** There is an equivalence transformation that zeroes the parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}$:

\[
\hat{u} = A_4 u, \\
\hat{f} = \frac{1}{4} x^{-2\gamma-2} A_4 \left( 4 x^{2\gamma+2} \tilde{f} + \left( 2 \tilde{\lambda} \left( \tilde{\lambda} + \sqrt{2} \right) x^{4\gamma} + x^2 (\tilde{\alpha} + \tilde{\beta} x)^2 - 2 x^{2\gamma+1} \left( \tilde{\alpha} \left( 2\gamma + \sqrt{2} \tilde{\lambda} \right) + \tilde{\beta} \left( 2\gamma + \sqrt{2} \tilde{\lambda} - 1 \right) x \right) \right) u \right),
\]

where

\[
A_4 = x^{-\frac{\lambda}{\sqrt{2}}} \exp \left( -\frac{1}{4} x^{1-2\gamma} \left( \frac{2\tilde{\alpha}}{2\gamma - 1} + \frac{\tilde{\beta} x}{\gamma - 1} \right) \right).
\]

Similarly, using this transformation we arrive to Eq. (27) and by using equivalence transformation (28) we end up once again to Eq. (29).

3.4. Group classification of the heat equation with nonlinear source

As shown previously Eq. (2) is linked via a series of point transformations to the heat equation with nonlinear source (29). Hence, the problem of the group classification of (2) is reduced to obtaining the group classification of (29). This classification was done in [4] and is included, as a subset, in the work of Zhdanov et al. [23]. In their work, the group classification of the heat conductivity equation with a nonlinear source

\[
u_t = u_{xx} + F(t, x, u, u_x)
\]

is performed taking advantage of the fact that the abstract Lie algebras of dimensions up to five are already classified.

4. Examples of invariant solutions

Having accessible the complete group classification for Eq. (29) — and consequently for Eq. (2) — we can look for invariant solutions under the terminal condition (3) and the barrier option condition (4): Given a specific algebra from the classification the appropriate subalgebra (and the functions $H(t), R(t)$ for the barrier option problem) admitted by each problem are determined using the two required conditions (9), (10) and (11), (12) adapted now to Eq. (29).
Namely,

\[ \mathcal{X}(\tau - T') \big|_{\tau = T'} \equiv 0, \]  
\[ \mathcal{X} \left( \phi - \Phi(X^{-1}(\bar{x}), 1) \right) \big|_{\tau = T', \phi = \Phi(X^{-1}(\bar{x}), 1)} \equiv 0, \]  
where \( T' = \Psi(T) \), and

\[ \mathcal{X} \left( \bar{x} - X(H^*(\tau)) \right) \big|_{\bar{x} = X(H^*(\tau))} \equiv 0, \]  
\[ \mathcal{X} \left( \phi - \Phi(H^*(\tau), R^*(\tau)) \right) \big|_{\phi = \Phi(H^*(\tau), R^*(\tau))} \equiv 0, \]

where \( H^*(\tau) = H(\Psi^{-1}(\tau)), \) \( R^*(\tau) = R(\Psi^{-1}(\tau)) \) and \( \bar{x} = X(x), \) \( \tau = \Psi(t) \), \( \phi = \Phi(x, u) \) denote the point transformation in each case. Finally, by using the found subalgebra similarity solutions are constructed as per usual.

In [4] we have illustrated in detail the process, here we avoid exposing the cumbersome calculations and present a few illustrative examples that by no means exhaust all the possible solutions that can be found using the classification of eq. (29).

4.1. The terminal condition

Due to the restrictions imposed to the admitted symmetries by this condition the chances of obtaining a nontrivial analytic solution, a solution depending explicitly on both variables \( \bar{x}, \tau \), using less than a four-dimensional algebra are slim. Hence and the fact that the following examples are restricted to the higher dimensional algebras only.

4.1.1. \((\gamma - 1)(\gamma - 1/2)(\delta - 2\gamma + 1) \neq 0\)

By using the Lie algebra \( A_4 \),

\[ \text{span} \left( \partial_\tau, e^{-\frac{2\beta}{\rho^2}} \phi \partial_\phi, \frac{2\beta}{\rho^2} \partial_\bar{x} + B \phi \partial_\phi, 2e^{-\frac{2\beta}{\rho^2}} \tau \partial_\phi + e^{-\frac{2\beta}{\rho^2}} \left( \frac{2\beta}{\rho^2} \bar{x} + 2B \tau \right) \phi \partial_\phi \right), \]

with \( A = -2\beta/\rho^2 \), we get the similarity solution

\[ u(x, t) = \exp \left( \frac{1}{8\beta^3 \rho^2} \left( \rho^2 \left( \beta^2 B^2 \rho^6 \omega^2 - 2B^2 \rho^6 \left( e^{\beta \omega} - 1 \right) + 2\beta B^2 \rho^6 \omega \right) \right) \right), \]

where \( \omega = t - T \) for \( f(x, u) = -\frac{1}{2\rho^2} u \left( \alpha^2 + \beta \rho^2 + B \rho^4 x - 2\beta \rho^2 \log|u| \right) \) and \( \gamma = \lambda = 0. \)
4.1.2. $\gamma = 1$

By using the Lie algebra $A_{3,8}^3$, span $(\partial_\tau, 2\bar{x}\partial_x + 4\tau\partial_\tau, 4\bar{x}\tau\partial_x + 4\tau^2\partial_\tau - \bar{x}^2\phi\partial_\phi)$, with $A = 1/2$, $\Gamma = 0$, we get the similarity solution

$$u(x, t) = e^{\log^2 x},$$

for $f(x, u) = \frac{\rho^2 u \log|u|}{\log x}$ and $\alpha = \lambda\rho$, $\beta = \frac{\rho^2}{2}, \delta = 0$.

4.1.3. $\gamma = 1/2$

By using the Lie algebra $A_{3,8}^3$, span $(\partial_\tau, 2\bar{x}\partial_x + 4\tau\partial_\tau + \frac{(4\alpha - \rho^2)\phi}{\rho^2} \partial_\phi, 4\bar{x}\tau\partial_x + 4\tau^2\partial_\tau - \left(\bar{x}^2 + \frac{2(\rho^2 - 4\alpha)t}{\rho^2}\right) \phi\partial_\phi)$, with $A = 2\alpha/\rho^2$, $B = \Gamma = 0$, we get the similarity solution

$$u(x, t) = e^{-\frac{2x(t-T)}{\rho^2 T^2}},$$

for $f(x, u) = \frac{\rho^2 u \log|u|}{4x} - \frac{2xu}{\rho^2 T^2}$ and $\alpha = \frac{\rho^2}{4}, \beta = \frac{2}{T}, \lambda = 0$.

4.1.4. $\delta = 2\gamma - 1$

By using the Lie algebra $A_4^1$, span $(\partial_\tau, e^{A\tau}\phi\partial_\phi, A\partial_x - B\phi\partial_\phi, 2e^{A\tau}\partial_x + e^{A\tau}(2B\tau - A\bar{x})\phi\partial_\phi)$, we get the similarity solution

$$u(x, t) = \exp\left(\frac{B}{A^3} \left( B e^{A(\gamma-1)^2\rho^2(T-t)} - A^2 x^{1-\gamma} + Ax^{-\gamma} e^{\frac{1}{2}A(\gamma-1)^2\rho^2(T-t)} (Ax + B(\gamma - 1)^2 \rho^2(t - T)x^{\gamma}) - B \right) \right),$$

for $f(x, u) = -\frac{1}{2}(\gamma - 1)^2 \rho^2 u (A \log|u| + Bx^{1-\gamma})$ and $\alpha, \beta = 0$, $\lambda = -\frac{2\rho}{2}$.  

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4.2. The condition for the barrier option

Contrary to the terminal condition, for the barrier option — due to the two arbitrary functions \( H(t), R(t) \) — any choice of Lie Algebras from the classification of (29) can be utilized in order to obtain nontrivial similarity solutions. Furthermore, the majority of the symmetries admit the form of the barrier function \( H(t) \), (5), used in the literature. A strong indication that this choice for the barrier function has a significant physical — in our case financial — meaning.

4.2.1. \((\gamma - 1)(\gamma - 1/2)(\delta - 2\gamma + 1) \neq 0\)

By using the Lie algebra \( A_{2,2}^1 \),

\[
\text{span} \left( \partial_x, e^\tau x \partial_x + 2e^\tau \partial_x - \frac{e^\tau}{4} (x^2 - 2) \phi \partial_\phi \right),
\]

with \( A = -1/2 \), we get the similarity solution

\[
u(x, t) = \frac{\sqrt{x}}{A} \log \left| \frac{x^{8(\gamma-1)} e^{-2(\gamma-1)^2 \rho^2 t}}{256c} + c x^{8(1-\gamma)} e^{2(\gamma-1)^2 \rho^2 t} \right|
\]

for

\[
f(x, u) = \frac{x^{-2\gamma - \frac{3}{2}}}{32 \rho^2} \times \left( \frac{16(\gamma - 1)^2 \rho^4 x^{4\gamma + 1}}{A} \exp \left( -\frac{2Au}{\sqrt{x}} \right) \right) +
\]

\[
u \left( 32\alpha \gamma \rho^2 x^{2\gamma + \frac{5}{2}} + 32\alpha \lambda \rho x^{4\gamma + \frac{7}{2}} - 16\alpha^2 x^{5/2} - 32\alpha \beta x^{7/2} + +32\beta \lambda \rho x^{6 + \frac{7}{2}} + 
\]

\[
x^{9/2} \left( (\gamma - 1)^2 \rho^4 - 16\beta^2 \right) - 8\rho^2 x^{2\gamma + \frac{5}{2}} \left( \beta(2 - 4\gamma) + (\gamma - 1)^2 \rho^2 \right) - 
\]

\[
16\lambda \rho^3 (2\gamma - \delta) x^{2\gamma + \delta} + \frac{4}{4} (\gamma - 1)^2 \rho^2 x^{2\gamma + \frac{7}{2}} - 16\lambda^2 \rho^2 x^{2\gamma + 2\delta + \frac{7}{2}} \right),
\]

where

\[
A = \exp \left( \frac{1}{8} x^{1-2\gamma} \left( 4 \left( \frac{2\alpha}{1-2\gamma} - \frac{2\lambda px^3}{-2\gamma+\delta+1} - \frac{\beta x}{\gamma-1} \right) + x \right) \right),
\]

\( c \neq 0 \) is a constant and \( H(t) = bKe^\frac{1}{2}(\gamma-1)\rho^2(t-T) \).
4.2.2. γ = 1

By using the Lie algebra $A_{3,5}^9$,

$$
\text{span}\left( \partial_\tau, \partial_x + \frac{a}{\rho^2} \phi \partial_\phi, \right)
\left( x - \frac{2a}{\rho^2} \tau \right) \partial_x + 2\tau \partial_\tau - \left( 2 - \frac{a}{\rho^2} x + \frac{2a^2}{\rho^4} \tau \right) \phi - \frac{a^2}{\rho^4} e^{-Bx} \partial_\phi, \right),
$$

with $B = -a/\rho^2$, we get the similarity solution

$$
u(x, t) = \frac{x^{a-\beta+1/2} \left( a^4 t^2 + a^2 (\log x - 2at) \log x + 12 \rho^4 \right) e^{\frac{a + \lambda \rho x^\delta}{\rho^2}}}{2 \rho^4 (\log x - at)^2},$$

for

$$f(x, u) = \frac{1}{8} \left( a^4 x^\beta \phi \frac{\partial}{\partial x} + \frac{a^2}{\rho^2} + \frac{1}{2} e^{\frac{a (\delta - 1) + \lambda \rho x^\delta}{(\delta - 1) \rho^2 x^\delta}} + 4 \rho^2 u^2 x \frac{\partial}{\partial x} - \frac{a}{\rho^2} + \frac{1}{2} \frac{1}{\rho^2} \frac{e^{\frac{a \delta + \alpha - \lambda \rho x^\delta}{\rho^2}}}{\rho^2} - \frac{u}{\rho^2 x^2} \left( 4 a^2 - 4 \lambda \rho x^\delta + 1 \left( 2 \beta + (\delta - 2) \rho^2 \right) + 4 \lambda^2 \rho^2 x^{2\delta} - 8 \alpha \left( \lambda \rho x^\delta + x \left( \rho^2 - \beta \right) \right) + x^2 \left( \rho^2 - 2 \beta \right)^2 \right) \right)$$

and $H(t) = b Ke^{a(t-T)}, a < 0$.

4.2.3. γ = 1/2

By using the Lie algebra $A_{3,8}^2$,

$$\text{span}\left( \partial_\tau, \frac{4a}{\rho^2} e^{-\frac{8a}{\rho^2} \tau} x \partial_x - e^{-\frac{8a}{\rho^2} \tau} \partial_\tau + \frac{2a}{\rho^2} e^{-\frac{8a}{\rho^2} \tau} \left( 2 \left( \frac{2a}{\rho^2} + \Gamma \right) \Delta e^{-\frac{1}{4} \Gamma x^2} x^2 \phi + \left( \frac{4a}{\rho^2} x^2 + 1 \right) \phi \right) \right) \partial_\phi + \frac{4a}{\rho^2} e^{-\frac{8a}{\rho^2} \tau} x \partial_x + \frac{2a}{\rho^2} e^{-\frac{8a}{\rho^2} \tau} \left( 2 \left( \frac{2a}{\rho^2} - \Gamma \right) \Delta e^{-\frac{1}{4} \Gamma x^2} x^2 \phi + \left( \frac{4a}{\rho^2} x^2 - 1 \right) \phi \right) \partial_\phi, \right),$$
with \( A = 1, \ B = -\frac{16\alpha^2}{\rho^2} \), we get the similarity solution

\[
\begin{align*}
  u(x, t) &= x^{\frac{1}{3}} \frac{4}{\rho^2} e^{\frac{16\alpha^2a}{\rho^2}} \exp \left( \frac{1}{32} \left( \frac{4at - \frac{\alpha^2}{6} + 2}{\rho^2} \right) \right) \\
  &\times \exp \left( \frac{1}{64} \left( \frac{c^2\rho^4}{a^2} + 16\alpha^2t^2 - \frac{64ax}{\rho^2} - 16\Gamma - 8c\rho^2t + 16 \log^2 x + 28 \right) \right) - \sqrt{x} \Delta e^{\frac{1}{2} \Delta x},
\end{align*}
\]

where \( c \) is a constant, for

\[
f(x, u) = \frac{1}{32\rho^2} \left( 16\alpha^2 \left( ux + \Delta x^{\frac{3}{2}} \frac{4}{\rho^2} e^{\frac{16\alpha^2a}{\rho^2}} \right) \right) - 32\alpha\beta u - 8\Gamma^4 u + 32\alpha\lambda\rho u x^{\delta - 1} - 16\lambda^2\rho^2 u x^{2\delta - 1} - 16\lambda\rho^3 u x^{\delta - 1} + 16\delta\lambda\rho^4 u x^{\delta - 1} + 32\beta\lambda\rho u x^{\delta} + 16\rho^4 \left( \frac{u}{x} + \Delta x^{\frac{3}{2}} \frac{4}{\rho^2} e^{\frac{16\alpha^2a}{\rho^2}} \right) \log \left( \frac{x}{\rho^2} - \frac{1}{2} e^{\frac{16\alpha^2a}{\rho^2}} + 16 \Delta x \right) - \frac{16\alpha^2u}{x} + \frac{16\alpha\rho^2u}{x} - 16\beta^2 u x + \frac{4\Gamma\rho^4u}{x}
\]

By using the Lie algebra \( A_4 \),

\[
\text{span} \left( \partial_\tau, e^{\frac{2a}{(1 - \gamma)}\rho^2\tau} \phi \partial_\phi, \frac{2a}{(1 - \gamma)}\rho^2 \partial_\phi - B\phi \partial_\phi, 2e^{\frac{2a}{(1 - \gamma)}\rho^2\tau} \partial_\tau - e^{\frac{2a}{(1 - \gamma)}\rho^2\tau} \left( \frac{2a}{(1 - \gamma)}\rho^2 x + 2B\tau \right) \phi \partial_\phi \right),
\]

and \( H(t) = bKe^{-a(t-T)}, a > 0 \).

4.2.4. \( \delta = 2\gamma - 1 \)

By using the Lie algebra \( A_4 \),
with $A = \frac{2a}{(1-\gamma)\rho}$, we get the similarity solution

$$u(x, t) = x^{\frac{2}{\gamma} + \frac{1}{\rho}} \exp \left( \frac{1}{8\rho^2} \left( \frac{B^2(\gamma - 1)^3\rho^8}{a^3} + \frac{4B(\gamma - 1)\rho^4x^{-\gamma}}{a} \right) - 4B(\gamma - 1)^2\rho^4b^{1-\gamma}K^{1-\gamma}te^{a(\gamma - 1)(t-T)} + 4x^{1-2\gamma} \left( \frac{2\alpha}{2\gamma - 1} + \frac{\beta x}{\gamma - 1} \right) + \frac{4ab^{1-2\gamma}K^{1-2\gamma}x^{-\gamma}\rho^4x^{-\gamma}}{\gamma - 1} e^{a(\gamma - 1)(t-T)} \left( 2xb^\gamma K^{\gamma} - bKxe^{a(\gamma - 1)(t-T)} \right) \right),$$

for

$$f(x, u) = \frac{x^{-2(\gamma + 1)u}}{8(2\gamma - 1)\rho^2} \left( 4ax^3(2\alpha - 2\gamma(\alpha + \beta x) + \beta x) - 4a(\gamma - 1)(2\gamma - 1)\rho x^{2\gamma + 2}((\gamma \rho + 2\lambda) \log x - 2\rho \log|u|) + (2\gamma - 1)(4\rho x^{2\gamma + 1}(2\alpha(\gamma \rho + \lambda) + \beta x((2\gamma - 1)\rho + 2\lambda)) - 4B(\gamma - 1)^2\rho^4x^{\gamma + 3} + \rho^2x^{4\gamma} ((\gamma - 2)\gamma \rho^2 - 4\lambda^2 - 4\lambda \rho) - 4x^2(\alpha + \beta x)^2) \right)$$

and $H(t) = bK e^{-a(t-T)}, a > 0$.

5. Conclusion

In the present paper a generalization of a general bond–pricing equation (1) was proposed and studied under the view of the modern group analysis. To that end, we harnessed the advantage that the equivalence transformations offer when studying classes of differential equations: the knowledge of the best representative(s) for this class of equations. This fact substantially simplifies the task of classifying it and obtaining its point symmetries.

Through this classification interesting cases, from the point of view of symmetries arise. Actually, the classical models of financial mathematics mentioned briefly in the introduction are resurfaced, this further solidifies the fact that their significance and place in financial mathematics is well justified. Nonetheless the significance of the case $\delta = 2\gamma - 1$,

$$u_t + \frac{1}{2}\rho^2x^{2\gamma}u_{xx} + (\alpha + \beta x - \lambda \rho x^{2\gamma - 1})u_x - f(x, u) = 0,$$

is yet to be determined in the literature.
Nonlinear equations in general have few or no symmetries so cases that augment the set of symmetries at disposal are like an oasis in the desert. After all, it is evident in the related literature that a dynamical system possessing an ample number of symmetries is more probable to relate with a physical system or model a more realistic process.

This fact is even more decisive when we wish to study a boundary problem: since not all of the symmetries admit the boundary and its condition some — or all — of the symmetries will be excluded. Hence the bigger the set of symmetries the bigger the probability that some will survive the scrutiny of the boundary conditions and give an invariant solution for the problem in its entirety.

This is evident for the terminal condition. As it can be seen by the examples three– and four–dimensional algebras were used in order to arrive to a nontrivial solution. A lower dimensional algebra seems to be unable to yield a nontrivial solution.

Things are different when the barrier option is considered, because of the two arbitrary functions $H(t), R(t)$ a broader range of cases can yield interested solutions. It is worth mentioning at this point that, as can be seen by the examples, the barrier function $H$ usually used in the related literature is admitted by the symmetries. A fact that further strengthens the belief that symmetries can be a valuable tool in investigating this kind of financial problems and the importance of this particular choice for the barrier function in financial mathematics.

Indeed, the insight provided through the above symmetry analysis might prove practical to anyone looking for a more realistic economic model without departing from the reasoning behind the proposed general bond–pricing model. Even more when one studies more exotic kinds of options. Options that have gained ground in the Asian markets which in turn play an ever increasing role in the world market. The provided nonlinear variant of this model might be deemed useful in that respect. We leave to the interested reader the possible economical interpretation and use of the obtained results.

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