VARIATIONAL CALCULATION OF LAPLACE TRANSFORMS VIA ENTROPY
ON WIENER SPACE AND SOME APPLICATIONS

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Abstract. Let \((W, H, \mu)\) be the classical Wiener space where \(H\) is the Cameron-Martin space which consists of the primitives of the elements of \(L^2([0,1], dt) \otimes \mathbb{R}^d\), we denote by \(L^2_0(\mu, H)\) the equivalence classes w.r.t. \(dt \times d\mu\) whose Lebesgue densities \(s \mapsto \dot{u}(s, w)\) are almost surely adapted to the canonical Brownian filtration. If \(f\) is a Wiener functional s.t. \(\frac{1}{E[e^{-f}]e^{-f}d\mu}\) is of finite relative entropy w.r.t. \(\mu\), we prove that

\[
J_* = \inf \left( E[\mu]\left[f \circ U + \frac{1}{2} |u|^2_H\right] : u \in L^2_0(\mu, H) \right)
\]

\[
\geq - \log E[\mu][e^{-f}] = \inf \left( \int_W f d\gamma + H(\gamma|\mu) : \nu \in P(W) \right)
\]

where \(P(W)\) is the set of probability measures on \((W, \mathcal{B}(W))\) and \(H(\gamma|\mu)\) is the relative entropy of \(\gamma\) w.r.t. \(\mu\). We call \(f\) a tamed functional if the inequality above can be replaced with equality, we characterize the class of tamed functionals, which is much larger than the set of essentially bounded Wiener functionals. We show that for a tamed functional the minimization problem of l.h.s. has a solution \(u_0\) if and only if \(U_0 = I_W + u_0\) is almost surely invertible and

\[
\frac{dU_0\mu}{d\mu} = \frac{e^{-f}}{E[\mu][e^{-f}]}
\]

and then \(u_0\) is unique. To do this we prove the theorem which says that the relative entropy of \(U_0\mu\) is equal to the energy of \(u_0\) if and only if it has a \(\mu\)-a.s. left inverse. We use these results to prove the strong existence of the solutions of stochastic differential equations with singular (functional) drifts and also to prove the non-existence of strong solutions of some stochastic differential equations.

Keywords: Invertibility, entropy, Girsanov theorem, variational calculus, Malliavin calculus, large deviations

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1. Introduction

Let \((W, H, \mu)\) be the classical Wiener space, i.e., \(W = C_0([0,1], \mathbb{R}^d)\), \(H\) is the corresponding Cameron-Martin space consisting of \(\mathbb{R}^d\)-valued absolutely continuous functions on \([0,1]\) with square integrable derivatives w.r.t. the Lebesgue measure. Denote by \((\mathcal{F}_t, t \in [0,1])\) the filtration of the canonical Wiener process, completed w.r.t. \(\mu\)-negligeable sets. Let \(V : W \rightarrow W\) be a mapping of
the form \( V = I_W + v, \ v : W \to H, \) i.e.,

\[ V_t(u) = w(t) + v(t, w) = W_t(w) + \int_0^t \dot{v}_s(w)\,ds, \]

where \( w \to \dot{v}_s(w) \) is \( \mathcal{F}_s \)-measurable \( ds \)-a.s., and \( (s, w) \to \dot{v}_s(w) \) is measurable w.r.t. the product sigma algebra \( \mathcal{B}([0,1]) \otimes \mathcal{F} \). Heuristically, the existence of the strong solution of the following stochastic differential equation:

\[ dU_t = -\dot{v}_t \circ U \, dt + dW_t \]

can be interpreted as the existence of an optimal element of the following minimization problem:

\[ K_* = \inf \left( \frac{1}{2} E \left[ |v \circ (I_W + \xi) + \xi^2_H| : \xi \in L^2_\mu(H) \right] \right) \]

where \( L^2_\mu(H) \) is the set of functionals as \( v \) described above with square integrable \( H \)-norm. The difficulty in this method lies in the fact that, due to the quadratic character of the cost function, the classical variational approach requires very strong regularity hypothesis about the vector field \( v \), which make the things unrealistic. Let us write this problem in a different form: assume that the Girsanov exponential of the vector field \( v \), denoted as \( \rho(-\delta \nu) \) is a probability density, i.e., \( E[\rho(-\delta \nu)] = 1 \), let \( f = -\log \rho(-\delta \nu) \). If \( U = I_W + u, \) with \( u \in L^2_\mu(H) \), we have

\[ f \circ U = \int_0^1 \dot{v}_s \circ U \, dU_s + \frac{1}{2} |v \circ U|_H^2. \]

If \( v \circ U \in L^2_\mu(H) \), taking the expectation of both sides, we get

\[ E \left[ f \circ U + \frac{1}{2} |u|_H^2 \right] = \frac{1}{2} E \left[ |v \circ U + u|_H^2 \right]. \]

Hence, again heuristically, the minimization problem \( K_* \) should be equivalent to the minimization problem

\[ J_* = \inf \left( E \left[ f \circ (I_W + u) + \frac{1}{2} |u|_H^2 \right] : u \in L^2_\mu(H) \right) \]

\[ = \inf \left( J(u) : u \in L^2_\mu(H) \right). \]

In this latter formulation there is no more the quadratic term provided that the function \( f \) is given directly as it happens quite often in physics, in optimization, in the calculation of Laplace transforms, in large deviations theory, etc. Of course, if one studies this problem, he has to verify that the optimal solution, if there is any, corresponds to the solution of the corresponding stochastic differential equation. This is precisely what we do in this paper by establishing the entropic characterization of the \( \mu \)-almost sure left invertibility of the adapted perturbations of identity. Let us explain in more general terms the premises of the problem: assume that \( f : W \to \mathbb{R} \) is a measurable function such that the relative entropy of the measure \( d\nu = e^{-f}(E_\mu[e^{-f}])^{-1} \, d\mu \) w.r.t. \( \mu \) is finite. Then it is easy to show the validity of the following expression:

\[ -\log E[e^{-f}] = \inf \left( \int_W f \, d\gamma + H(\gamma|\mu) : \gamma \in P(W) \right) \]

and the measure \( \nu \) is the unique minimiser of the right hand side of this equality. In case \( f \) is bounded, it has been shown in \([1]\) that

\[ \inf \left( E \left[ f \circ (I_W + \xi) + \frac{1}{2} |\xi|_H^2 \right] : \xi \in L^2_\mu(H) \right) = -\log E[e^{-f}]. \]

In case this relation holds for those \( f \) which one may encounter in the problems of invertibility mentioned above, any minimizer \( u \) for \( J_* \) will very likely have the property that \( U\mu = (I_W + u)\mu = \nu \), where \( (I_W + u)\mu \) means the push forward of the measure \( \mu \) by the map \( U = I_W + u \). At this point an
important concept comes up, namely we shall call $f$ a **tamed functional** if one has the following identity:

$$
(1.1) \quad \inf \left( \int_W f d\gamma + H(\gamma|\mu) : \gamma \in P(W) \right) = \inf \left( E \left[ f \circ (I_W + \xi + \frac{1}{2} |\xi|^2_H) \right] : \xi \in L^2_\alpha(\mu, H) \right).
$$

We prove in Theorem 7 that if $f \in L^{1+\varepsilon}(\mu)$ for some $\varepsilon > 0$, is such that the corresponding measure is of finite relative entropy w.r.t. $\mu$, then it is a tamed functional. To prove the equality between the minimizing measure of the left side of (1.1) and the $(I_W + u)\mu$, where $u$ is the minimizing vector of the right hand side of (1.1), if there is any such element, we shall need the extension of the results of [17] as well as some of Ph. D. Thesis of R. Lassalle (cf.[9]). Namely, the following result will be essential in the sequel: Assume that $U = I_W + u$ is an API, then it is $\mu$-a.s. left invertible if and only if the following equality holds true:

$$
\frac{1}{2} E[|u|^2] = H(U\mu|\mu),
$$

where $H(U\mu|\mu)$ is the relative entropy of $U\mu = (I_W + u)\mu$ (push-forward of $\mu$ under $U$) w.r.t $\mu$.

Using this result we prove on the one hand the equivalence between the existence of a minimizer $u$ and the $\mu$-almost sure invertibility of the corresponding API, namely, of $U = I_W + u$ (which is necessarily unique) and on the other hand that $u \in L^2_\alpha(\mu, H)$ is a minimizing element for $J_\star$ if and only if the measure $U\mu = (I_W + u)\mu$ is the unique minimizer of $K_\star$. It is a remarkable fact that using this method we can solve stochastic differential equations with very singular (functional) drifts, e.g., we can show that the following equation is well-defined and has a unique strong solution

$$
X_t = W_t - \int_0^t \left( \frac{E[D_{\tau}e^{-f} | F_t]}{E[e^{-f} | F_t]} \right) \circ X d\tau,
$$

where $f$ is any 1-convex tamed functional of $W$ and where $D_{\tau}F$ denotes the density of the Sobolev derivative of $F : W \to \mathbb{R}$ w.r.t. the Lebesgue measure of $[0, 1]$. Note that it is not even evident to justify $D_{\tau}e^{-f}$ using the classical Malliavin calculus since this derivative may exist only in the sense of distributions. Of course the next step is to characterize the class of functions $f$ for which the minimization problem is well-defined. This requires a version of calculus of variations on the space of adapted, $dt \times d\mu$-square integrable processes combined with the Malliavin calculus and with the notion of $H$-convex functions; a concept which is specific to the Wiener space(s).

As a final application of these results, we prove that, for a given $H$-convex subset $A \subset W$ with $\mu(A) \in (0, 1)$, there is no API of the form $U = I_W + u$ which is $\mu$-a.s. left invertible and that $dU\mu = d\nu = \frac{1}{\mu(A)} 1_A d\mu$. To understand the meaning of this result, let us write, via Itô representation theorem,

$$
\frac{1_A}{\mu(A)} = \rho(-\delta v)
$$

where $v \in L^2(\nu, H)$. Then, under $\nu$, $V = I_W + v$ is a Brownian motion and the SDE

$$
dU_t = -\dot{v}_t \circ U dt + dV_t
$$

has a weak solution but has no strong solution. As the reader can realize, this result is a consequence of the variational calculus developed here and hence its nature and its philosophy are quite different from the example of B. Tsirelson, cf. [11].

Let us finally add that the results of this paper have immediate extensions to the infinite dimensional case (i.e., the cylindrical Brownian motion) and also to the abstract Wiener spaces via the theory developed in [10].

2. Preliminaries and notation

Let $W$ be the classical Wiener space with the Wiener measure $\mu$. The corresponding Cameron-Martin space is denoted by $H$. Recall that the injection $H \hookrightarrow W$ is compact and its adjoint is the natural injection $W^* \hookrightarrow H^* \subset L^2(\mu)$. A subspace $F$ of $H$ is called regular if the corresponding
orthogonal projection has a continuous extension to $W$, denoted again by the same letter. It is well-known that there exists an increasing sequence of regular subspaces $(F_n, n \geq 1)$, called total, such that $\cup_n F_n$ is dense in $H$ and in $W$. Let $\sigma(\pi_{F_n})$ be the $\sigma$-algebra generated by $\pi_{F_n}$, then for any $f \in L^p(\mu)$, the martingale sequence $(E[f|\sigma(\pi_{F_n})], n \geq 1)$ converges to $f$ (strongly if $p < \infty$) in $L^p(\mu)$. Observe that the function $f_n = E[f|\sigma(\pi_{F_n})]$ can be identified with a function on the finite dimensional abstract Wiener space $(F_n, \mu_n, F_n)$, where $\mu_n = \pi_n \mu$.

Since the translations of $\mu$ with the elements of $H$ induce measures equivalent to $\mu$, the Gâteaux derivative in $H$ direction of the random variables is a closable operator on $L^p(\mu)$-spaces and this closure will be denoted by $\nabla$ cf., for example [13][14]. The corresponding Sobolev spaces (the equivalence classes) of the real random variables will be denoted as $I^p_D$, and they are denoted as $I^p_D$ (Φ)

For any $t \geq 0$ and measurable $f : W \to \mathbb{R}_+$, we note by

$$P_tf(x) = \int_W f \left( e^{-t}x + \sqrt{1 - e^{-2ty}} \right) \mu(dy),$$

it is well-known that $(P_t, t \in \mathbb{R}_+)$ is a hypercontractive semigroup on $L^p(\mu), p > 1$, which is called the Ornstein-Uhlenbeck semigroup (cf.[13][14]). Its infinitesimal generator is denoted by $-\mathcal{L}$ and we call $\mathcal{L}$ the Ornstein-Uhlenbeck operator (sometimes called the number operator by the physicists).

The norms defined by

$$\|\phi\|_{p,k} = \|(I + \mathcal{L})^{k/2}\phi\|_{L^p(\mu)}$$

are equivalent to the norms defined by the iterates of the Sobolev derivative $\nabla$. This observation permits us to identify the duals of the space $D_{p,k}(\Phi): p > 1, k \in \mathbb{N}$ by $D_{q,-k}(\Phi^q)$, with $q^{-1} = 1 - p^{-1}$, where the latter space is defined by replacing $k$ in (2.2) by $-k$, this gives us the distribution spaces on the Wiener space $W$ (in fact we can take as $k$ any real number). An easy calculation shows that, formally, $\delta \circ \nabla = \mathcal{L}$, and this permits us to extend the divergence and the derivative operators to the distributions as linear, continuous operators. In fact $\delta : D_{q,k}(\mathcal{H} \otimes \Phi) \to D_{q,k-1}(\mathcal{H} \otimes \Phi)$ and $\nabla : D_{q,k}(\Phi) \to D_{q,k-1}(\mathcal{H} \otimes \Phi)$ continuously, for any $q > 1$ and $k \in \mathbb{R}$, where $\mathcal{H} \otimes \Phi$ denotes the completed Hilbert-Schmidt tensor product (cf., for instance [13][14]). Finally, in the case of classical Wiener space, we denote by $D_{p,k}^a(H)$ the subspace defined by

$$D_{p,k}^a(H) = \{ \xi \in D_{p,k}(H) : \xi \text{ is adapted} \}$$

for $p \geq 1, k \in \mathbb{R}$, for $p = 2, k = 0$, we shall write $L^2_2(\mu, H)$.

A measurable function $f : W \to \mathbb{R} \cup \{\infty\}$ is called $\alpha$-convex, $\alpha \in \mathbb{R}$, if the map

$$h \to f(x + h) + \alpha \frac{1}{2} |h|^2_H = F(x, h)$$

is convex on the Cameron-Martin space $H$ with values in $L^0(\mu)$. Note that this notion is compatible with the $\mu$-equivalence classes of random variables thanks to the Cameron-Martin theorem. It is proven in [3] that this definition is equivalent the following condition: Let $(\pi_n, n \geq 1)$ be a sequence of regular, finite dimensional, orthogonal projections of $H$, increasing to the identity map $I_H$. Denote also by $\pi_n$ its continuous extension to $W$ and define $\pi_n^\perp = I_W - \pi_n$. For $x \in W$, let $x_n = \pi_n x$ and $x_n^\perp = \pi_n^\perp x$. Then $f$ is 1-convex if and only if

$$x_n \to \frac{1}{2} |x_n|^2_H + f(x_n + x_n^\perp)$$

is $\pi_n^\perp \mu$-almost surely convex.

\footnote{For the notational simplicity, in the sequel we shall denote it by $\pi_n$.}
We shall use also the following result, which makes part of the folklore of the Wiener measure:

**Lemma 1.** Denote by $\rho(\delta h), h \in H$, the Wick exponential

$$\rho(\delta h) = \exp \left( \delta h - \frac{1}{2} |h|^2_H \right)$$

where $\delta h = \int_0^1 h(s) dW_S$ (i.e., Wiener integral). The map

$$h \to \rho(\delta h)$$

is weakly continuous on $H$ with values in $L^p(\mu)$, for any $p > 1$.

**Proof:** Assume that $F$ is of finite Wiener chaos, then, for any $h \in H$, it follows from Cameron-Martin theorem that

$$E[F\rho(\delta h)] = E[F(\cdot + h)] = \sum_{n=0}^{\infty} \frac{1}{n!} E[\nabla^n F, h^{\otimes n}]$$

(cf. [13, 14]). If $h_k \to h$ weakly in $H$, then $h_k^{\otimes m} \to h^{\otimes m}$ weakly in $H^{\otimes m}$ for any $m \geq 1$, hence $E[F\rho(\delta h_k)] \to E[F\rho(\delta h)]$ as $k \to \infty$ if $F$ is chosen as above. If $F \in L^p(\mu)$, then there exists a sequence $(F_n, n \geq 1)$ converging to $F$ in $L^p(\mu)$ with each $F_n$ being of finite Wiener chaos. Let $c_n(h) = E[F_n\rho(\delta h)]$, $c(h) = E[F\rho(\delta h)]$, then

$$|c(h) - c(k)| \leq |c(h) - c_n(h)| + |c_n(h) - c_n(k)| + |c_n(k) - c(k)|$$

hence $\lim c(h) = c(k)$ as $h \to k$ weakly in $H$. \qed

**Definition 1.** A map $u : W \to H$ is called an $H - C^1$-map if the map $h \to u(w + h)$ is Fréchet differentiable on $H$ for almost all $w$.

**Remark 1.** This is a very strong property, in particular it implies that the set of the elements $w$ of $W$ should be $H$-invariant. Let us note that, if $u$ is in some space $L^p(\mu, H)$, then $P_\tau u$ is an $H - C^1$-map for any $\tau > 0$, where $P_\tau$ denotes the Ornstein-Uhlenbeck semigroup on $W$ (in fact it is even $H$-analytic, cf. [18], Chapter 2).

A more relaxed notion is given as

**Definition 2.** The map $u$ is called an $(H - C^1)_{loc}$ map if there exists an almost surely strictly positive map $q$ such that $h \to u(w + h)$ is continuously differentiable on the set $\{h \in H : |h|_H < q(w)\}$.

We have the following result about the change of variables formula for $(H - C^1)_{loc}$-maps proved in Theorem 4.4.1 of Chapter 4 of [18]

**Theorem 1.** Assume that $u \in L^2(\mu, H)$ is an $(H - C^1)_{loc}$-mapping, define $U : W \to W$ as $U = I_W + u$, let $Q$ be the set $\{w \in W : q(w) > 0\}$, where $q$ is the mapping given in Definition 2, then for any $f, g \in C_B(W)$, the following identity holds true:

$$E[f \circ U g(-\delta u)] = E \left[ f \sum_{y \in U^{-1}(\{w\} \cap Q)} g(y) \right].$$
3. Characterization of the invertible shifts

We begin with the definition of the notion of almost sure invertibility with respect to a measure. This notion is extremely important since it makes the things work.

**Definition 3.**

- A measurable map $T : W \to W$ is called $(\mu)$-almost surely left invertible if there exists a measurable map $S : W \to W$ such that and $S \circ T = I_W \mu$-a.s.
- Moreover, in this case it is trivial to see that $T \circ S = I_W T \mu$-a.s., where $T \mu$ denotes the image of the measure $\mu$ under the map $T$.
- If $T \mu$ is equivalent to $\mu$, then we say in short that $T$ is $\mu$-a.s. invertible.
- Otherwise, we may say that $T$ is $(\mu, T \mu)$-invertible in case precision is required or just $\mu$-a.s. left invertible and $S$ is called the $\mu$-left inverse of $T$.

**Theorem 2.** For any $u \in L^2(\mu, H)$, we have the following inequality

$$H(U \mu|\mu) \leq \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 ds,$$

where $H(U \mu|\mu)$ is the relative entropy of the measure $U \mu$ w.r.t. $\mu$.

**Proof:** Let $L$ be the Radon-Nikodym density of $U \mu$ w.r.t. $\mu$. For any $0 \leq g \in C_b(W)$, using the Girsanov theorem, we have

$$E[g \circ U] = E[g L] \geq E[g \circ U L \circ U \rho(-\delta u)],$$

hence

$$L \circ U E[\rho(-\delta u)|U] \leq 1 \mu$$-a.s. Consequently, using the Jensen inequality

$$H(U \mu|\mu) = E[L \log L] = E[\log L \circ U]$$

$$\leq -E[\log E[\rho(-\delta u)|U]]$$

$$\leq -E[\log \rho(-\delta u)]$$

$$= \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 ds.$$

**Theorem 3.** Assume that $U = I_W + u$ is an API, i.e., $u \in L^2(\mu, H)$ such that $s \to \dot{u}(s, w)$ is $\mathcal{F}_s$-measurable for almost all $s$. Then $U$ is almost surely left invertible with a left inverse $V$ if and only if

$$H(U \mu|\mu) = \frac{1}{2} E[|u|^2_H] = \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 ds,$$

i.e., if and only if the entropy of $U \mu$ is equal to the energy of the drift $u$.

**Proof:** Due to Theorem 2, the relative entropy is finite as soon as $u \in L^2(\mu, H)$. Let us suppose now that the equality holds and let us denote by $L$ the Radon-Nikodym derivative of $U \mu$ w.r.t. $\mu$. Using the Itô representation theorem, we can write

$$L = \exp \left( -\int_0^1 \dot{\nu}_s dW_s - \frac{1}{2} \int_0^1 |\dot{\nu}_s|^2 ds \right)$$

$U \mu$-almost surely. Let $V = I_W + v$, as described in [7], from the Itô formula and Paul Lévy’s theorem, it is immediate that $V$ is an $U \mu$-Wiener process, hence

$$(3.3) \quad E[L \log L] = \frac{1}{2} E[L |v|^2_H].$$

Now, for any $f \in C_b(W)$, we have from the Girsanov theorem

$$E[f \circ U] = E[f L] \geq E[f \circ U L \circ U \rho(-\delta u)]$$
consequently
\[ L \circ U E[\rho(-\delta u)|U] \leq 1 \]
\( \mu \)-a.s. Let us denote \( E[\rho(-\delta u)|U] \) by \( \hat{\rho} \). We have then \( \log L \circ U + \log \hat{\rho} \leq 0 \) \( \mu \)-a.s. Taking the expectation w.r.t. \( \mu \) and the Jensen inequality give
\[
H(U\mu|\mu) = E[L \log L] \leq -E[\log \hat{\rho}]
\leq -E[\log \rho(-\delta u)] = \frac{1}{2} E[|u|_H^2].
\]
Since \( \log \) is a strictly concave function, the equality \( E[\log \hat{\rho}] = E[\log \rho(-\delta u)] \) implies that \( \rho(-\delta u) = \hat{\rho} \) \( \mu \)-a.s. Hence we obtain
\[
E[\log L + \log \rho(-\delta u)] = E[\log L \circ U \rho(-\delta u)] = 0,
\]
so \( L \circ U \rho(-\delta u) \leq 1 \) \( \mu \)-a.s., the equation (3.3) implies
\[
L \circ U \rho(-\delta u) = 1
\]
\( \mu \)-a.s. Combining the exponential representation of \( L \) with the relation (3.5) implies
\[
0 = \left( \int_0^1 \epsilon_s dW_s \right) \circ U + \frac{1}{2} |v \circ U|_H^2 + \delta u + \frac{1}{2} |u|_H^2
\leq \delta(v \circ U) + \delta u + (v \circ U, u)_H + \frac{1}{2} (|u|_H^2 + |v \circ U|_H^2)
= \delta(v \circ U + u) + \frac{1}{2} |v \circ U + u|_H^2
\]
\( \mu \)-a.s. From the relation (3.3) it follows that \( v \circ U \in L_2^{0}(\mu, H) \), hence taking the expectations of both sides of (3.6) w.r.t. \( \mu \) is licit and this implies \( v \circ U + u = 0 \) \( \mu \)-a.s., which means that \( V = I_W + v \) is the \( \mu \)-left inverse of \( U \).

To show the necessity, let us denote by \( (L_t, t \in [0,1]) \) the martingale
\[
L_t = E[L|\mathcal{F}_t] = E \left[ \frac{dU\mu}{d\mu}|\mathcal{F}_t \right]
\]
and let
\[
T_n = \inf \left( t : L_t < \frac{1}{n} \right).
\]
Since \( U \circ V = I_W (U\mu)\)-a.s., \( V \) can be written as \( V = I_W + v (U\mu)\)-a.s. and that \( v \in L_1^0(U\mu, H) \), i.e.,
\[
v(t, w) = \int_0^1 \epsilon_s(w)ds, \quad \epsilon \text{ is adapted to the filtration } (\mathcal{F}_t) \text{ completed w.r.t. to } U\mu \text{ and } \int_0^1 |\epsilon_s|^2 ds < \infty \text{ (U\mu)-a.s.}
\]
Since \( \{t \leq T_n\} \subset \{L > 0\} \) and since on this latter set \( \mu \) and \( U\mu \) are equivalent, we have
\[
\int_0^{T_n} |\epsilon_s|^2 ds < \infty
\]
\( \mu \)-almost surely. Consequently the inequality
\[
E_\mu[\rho(-\delta v^n)] \leq 1
\]
holds true for any \( n \geq 1 \), where \( v^n(t, w) = \int_0^1 1_{[0,T_n]}(s,w)\epsilon_s(w)ds \). By positivity we also have
\[
E_\mu[\rho(-\delta v^n)1_{\{L > 0\}}] \leq 1.
\]
Since \( \lim_n T_n = \infty (U\mu)\)-a.s., we also have \( \lim_n T_n = \infty \) \( \mu \)-a.s. on the set \( \{L > 0\} \) and the Fatou lemma implies
\[
E_\mu[\rho(-\delta v)1_{\{L > 0\}}] = E_\mu[\lim_n \rho(-\delta v^n)1_{\{L > 0\}}] \leq \liminf_n E_\mu[\rho(-\delta v^n)1_{\{L > 0\}}] \leq 1.
\]
for any \( n \geq 1 \). From the identity \( U \circ V = I_W (U\mu)\)-a.s., we have \( v + u \circ V = 0 (U\mu)\)-a.s., hence \( v \circ U + u = 0 \) \( \mu \)-a.s. An algebraic calculation gives immediately
\[
\rho(-\delta u) \circ U \rho(-\delta u) = 1
\]
\(\mu\)-a.s. Now applying the Girsanov theorem to \(API U\) and using the relation \((3.8)\), we obtain
\[
E[g \circ U] = E[g L] = E \left[ g \circ U(\rho(-\delta v)1_{\{L > 0\}}) \circ U(\rho(-\delta u)) \right]
\leq E \left[ g \rho(-\delta v)1_{\{L > 0\}} \right],
\]
for any positive \(g \in C_b(W)\) (note that on the set \(\{L > 0\}\) \(\rho(-\delta v)\) is perfectly well-defined w.r. to \(\mu\)). Therefore
\[
L \leq \rho(-\delta v)1_{\{L > 0\}}\mu\text{-a.s.}
\]
Now, this last inequality, combined with the inequality \((3.7)\) entails that
\[
L = \rho(-\delta v)1_{\{L > 0\}}\mu\text{-a.s.}
\]
\(\mu\)-a.s. To complete the proof it suffices to remark then that
\[
H(U\mu|\mu) = E[L \log L] = E[\log L \circ U]
= E[- \log \rho(-\delta u)]
= \frac{1}{2} E[|u|^2_H].
\]

The following theorem, although it has strong hypothesis, is at the heart of the further developments:

**Theorem 4.** Assume that \(u \in L^2_0(\mu, H)\) is an \((H - C^1)\) such that \(E[\rho(-\delta u)] = 1\), then the mapping \(U = I_W + u\) is \(\mu\)-a.s. invertible.

**Proof:** We have, from Theorem \([1]\) taking \(g = 1\) and defining the multiplicity of \(U\) on the set \(Q\) (cf. the notations of Theorem \([1]\) as
\[
N(w, Q) = \sum_{y \in U^{-1}(w) \cap Q} 1(y),
\]
the relation
\[
E[f \circ U \rho(-\delta u)] = E[f N(\cdot, Q)].
\]
On the other hand the Girsanov theorem implies that
\[
E[f \circ U \rho(-\delta u)] = E[f],
\]
for any \(f \in C_b(W)\), hence \(N(w, Q) = 1\) \(\mu\)-a.s., this implies that the map \(U\) is almost surely injective. The hypothesis \(E[\rho(-\delta u)] = 1\) implies also that \(U(W) = W\) \(\mu\)-a.s., hence \(U\) is almost surely surjective.

**Remark 2.** Again using the Girsanov theorem, it is immediate to show that the inverse of \(U\) is of the form \(V = I_W + v\), with \(v \in L^2_0(\mu, H)\).
4. Some variational problems related to entropy and large deviations

The following is an extension of a well-known result in large deviations theory:

**Theorem 5.** Let \((A, \mathcal{A})\) be a measurable space and let \(f : A \to \mathbb{R}\) be a measurable function, denote by \(P(A)\) the set of probability measures on \((A, \mathcal{A})\). Suppose that for some \(\gamma \in P(A)\), \(f\) satisfies

\[
\int_A |f|(1 + e^f) d\gamma < \infty.
\]

Then the following identity holds:

\[
\log \gamma(e^f) = \sup \left( \int f d\nu - H(\nu|\gamma) : \nu \in P(A) \right)
\]

and the unique supremum is attained at the measure

\[
d\nu = \frac{e^f}{\gamma(e^f)} d\gamma,
\]

where \(H(\nu|\gamma)\) denotes the relative entropy of \(\nu\) w.r.t. \(\gamma\).

**Proof:** When \(f\) is bounded, this theorem is well-known (cf. [1] and the references there). First, suppose that \(f\) is lower-bounded, let \(f_n = f \wedge n\), it follows from the bounded case that

\[
\log \int e^f d\gamma \geq \sup \left( \int f d\nu - H(\nu|\gamma) \right).
\]

Since \(\nu(f_n) \leq \nu(f)\) for any \(\nu\) with \(H(\nu|\gamma) < \infty\), we also have

\[
\log \gamma(e^{f_n}) = \sup_{\nu} \left( \int f_n d\nu - H(\nu|\gamma) \right) \leq \sup_{\nu} \left( \int f d\nu - H(\nu|\gamma) \right)
\]

and passing to the limit we get

\[
\log \gamma(e^f) \leq \sup_{\nu} \left( \int f d\nu - H(\nu|\gamma) \right)
\]

and this proves the claim when \(f\) is lower bounded. For the general case, define \(g_\varepsilon = \log(e^f + \varepsilon)\), then \(g_\varepsilon\) is lower bounded, hence

\[
\log \gamma(e^{g_\varepsilon}) \geq \int g_\varepsilon d\nu - H(\nu|\gamma)
\]

for any \(\varepsilon\) and for any \(\nu\) with finite \(H(\nu|\gamma)\). Passing to the limit as \(\varepsilon \to 0\) and taking the supremum w.r.t. \(\nu\), we get

\[
\log \gamma(e^f) \geq \sup_{\nu} \left( \int f d\nu - H(\nu|\gamma) \right).
\]

To see the equality, it suffices to remark that for the measure

\[
d\nu_0 = \frac{e^f}{\gamma(e^f)} d\gamma
\]

the supremum is attained.

\(\square\)

**Remark 3.** In the sequel, we shall use a variation of this theorem where \(f\) will be replaced by \(-f\) and the corresponding equality is

\[- \log \int e^{-f} d\gamma = \inf \left( \int f d\nu + H(\nu|\gamma) : \nu \in P(A) \right)\]
Theorem 6. Assume that
\[ \int_W (|f| + 1)e^{-f} d\mu < \infty. \]
Then the following inequality holds true:
\[ -\log E[e^{-f}] \leq \inf \left( E \left[ f \circ (I_W + u) + \frac{1}{2} |u|^2_H \right] : u \in L^2(\mu, H) \right). \]

Proof: Combining Theorem 5 with Remark 3, we see already that \( f \) is quasi-integrable for any measure \( \nu \) which is of finite relative entropy w.r.t. the Wiener measure \( \mu \). In particular, if \( \nu = (I_W + u)(\mu) \), with \( u \in L^2(\mu, H) \), then, from Theorem 2 we have \( H((I_W + u)\mu) \leq \frac{1}{2} \|u\|_{L^2(\mu, H)}^2 \), hence the inequality follows.

Theorem 7. Assume that \( f \in L^p(\mu), e^{-f} \in L^q(\mu) \) with \( p^{-1} + q^{-1} = 1 \), hence
\[ \int (|f| + 1)e^{-f} d\mu < \infty. \]
Then the following equalities hold true
\[ J_\ast = -\log \mu(e^{-f}) \]
\[ = \inf \left( \int_W f d\nu + H(\nu|\mu) : \nu \in P(W) \right) \]
\[ = \inf \left[ \int [f \circ (I_W + u) + \frac{1}{2} |u|^2_H] d\mu : u \in L^2(\mu, H) \right]. \]

Proof: We just need to prove the last equality; we shall proceed the proof by showing that each side of this last equality is less than the other one. First, it is immediate from the definition of infimum and from Theorem 2 that
\[ -\log \mu(e^{-f}) \leq \inf \left[ \int [f \circ (I_W + u) + \frac{1}{2} |u|^2_H] d\mu : u \in L^2(\mu, H) \right]. \]
To show the reverse inequality is more delicate: let \( (\epsilon_n, n \geq 1) \) be a complete, orthonormal basis of the Cameron-Martin space \( H \), denote by \( V_n, n \geq 1 \), the sigma-algebra generated by the Gaussian random variables \( \delta e_1, \ldots, \delta e_n \). Define now \( f_n \) as
\[ f_n = E[P_{1/n} f|V_n] \]
where \( P_{1/n} \) is the Ornstein-Uhlenbeck semi-group on \( W \). Denote by \( l_n \) the density \( e^{-f_n}/E[e^{-f}] \) and define
\[ \hat{v}_n^m = \frac{E[D_l e^{-f_n}|F_t]}{E[e^{-f_n}|F_t]} \]
where \( D_l e^{-f_n} \) denotes the Lebesgue density of the \( H \)-derivative \( \nabla e^{-f_n} \) which is perfectly well-defined thanks to the hypothesis. Let \( v^n \) be the primitive of \( \hat{v}^n \), i.e.,
\[ v^n(t, w) = \int_0^t \hat{v}_n^m(w) ds. \]
Let us indicate that the mapping
\[ w \to \int_0^t E[D_l e^{-f_n}|F_t](w) dt \]
is an \( H - C^1 \)-map due to the regularization with the Ornstein-Uhlenbeck semi-group (cf. Remark 1). Let \( H_\varphi \) denote the Cameron-Martin space equipped with its weak topology, then, for any \( h \in H, \)
\[ E[f_n|F_t](w + h) = \int_W E[E[f|V_n]|F_t](e^{-1/n}(w + h) + \sqrt{1 - e^{-2/n} y}) \mu(dy) \]
\[ = \int_W \rho(\alpha_n \delta h(y)) E[E[f|V_n]|F_t](e^{-1/n} w + \sqrt{1 - e^{-2/n} y}) \mu(dy), \]
where $\alpha_n = e^{-1/n}/\sqrt{1 - e^{-2/n}}$. It follows from Lemma 1 and from the hypothesis about $f$ that the mapping 

$$(t, h) \to E[E[P_{1/n}f|V_n]|F_t](w + h)$$

is $\mu$-a.s. continuous on the space $[0, 1] \times H_\sigma$. Consequently

$$\sup_{t \in [0, 1], h \in B} E[E[P_{1/n}f|V_n]|F_t](w + h) > 0$$

$\mu$-a.s. for any bounded, weakly closed set $B \in H$. and the set of such $w$'s are again $H$-invariant

$$\inf_{t \in [0, 1], h \in B} E[E[P_{1/n}e^{-f}|V_n]|F_t](w + h) > 0$$

$\mu$-a.s. and the set of such $w$'s are again $H$-invariant. This observation, combined with the $H - C^1$-property of $w \to \int_0^1 E[D_t e^{-f}|F_t](w)dt$ implies that $v^n$ is an $H-C^1$-map and it follows from Theorem 3 that the mapping $w \to w + v^n(w) = V_n(w)$ is $\mu$-a.s. invertible. Let $U_n = I_W + u^n$ be its inverse, then clearly

$$\frac{dU_n}{d\mu} = \rho(-\delta y^n) = l_n = \frac{e^{-f_n}}{e^{-f_n}}.$$

It is now trivial to see from Jensen’s inequality that $u^n \in L^2_a(\mu, H)$. Moreover

$$- \lim_n \log E[e^{-f_n}] = \lim_n \left( \frac{1}{E[e^{-f_n}]} \int_W f_ne^{-f_n}d\mu + \frac{1}{2} E[|u^n|_H^2] \right)$$

$$= \lim_n \left( \frac{1}{E[e^{-f_n}]} \int_W e^{-f_n}d\mu + \frac{1}{2} E[|u^n|_H^2] \right)$$

$$= \lim_n \left( \int_W f \circ U_n d\mu + \frac{1}{2} E[|u^n|_H^2] \right)$$

$$\geq \inf \left( \int_W (f \circ U + |u|_H^2) d\mu : u \in L^2_a(\mu, H) \right)$$

and this completes the proof.

\[ \square \]

**Definition 4.** We call a measurable map $f : W \to \mathbb{R} \cup \{\infty\}$ with the property $E[(1 + |f|)e^{-f}] < \infty$, a **tamed functional** if the conclusion of Theorem 3 is valid for $f$, mainly if

$$- \log E[e^{-f}] = \inf \left( \int (f \circ (I_W + u) + \frac{1}{2} |u|_H^2) d\mu : u \in L^2_a(\mu, H) \right)$$

An immediate consequence of the logarithmic Sobolev inequality (cf. [2]) for the Wiener measure gives

**Proposition 1.** Assume that $f \in L^{1+\varepsilon}(\mu)$ such that $E[e^{-f}] < \infty$. Let $f^- = \max(-f, 0)$, if $E[f^-e^{-f^-}] < \infty$, then $f$ is a tamed functional, in particular the latter condition is satisfied if

$$E[e^{-f} | \nabla f|^2_H] < \infty.$$

5. **Characterization of the minimizers**

We come to the minimization problem for:

$$J_* = - \log \mu(e^{-f}) = \inf \left( \int (f \circ (I_W + u) + \frac{1}{2} |u|_H^2) d\mu : u \in L^2_a(\mu, H) \right)$$

The following result gives a complete characterization of the attainability of $J_*$ in the situation of finite entropy:
Theorem 8. Assume that \( f \) is a tamed functional, then the infimum \( J_* \) is attained at some \( u \in L^2_0(\mu, H) \) if and only if the API defined as \( U = I_W + u \) has a left inverse \( V = I_W + v \) with \( v \in L^2_0(U\mu, H) \) and
\[
\frac{dU\mu}{d\mu} = \frac{e^{-f}}{E[e^{-f}]} = L = \rho(-\delta v).
\]
Moreover \( U \) is the unique strong solution of the following SDE
\[
dU_i = -\dot{v}_i \circ U dt + dW_i
\]
and if \( E[e^{-(1+\varepsilon)f}] < \infty \) for some \( \varepsilon > 0 \), then \( \dot{v} \) can be expressed as
\[
\dot{v}_\tau = \frac{E[D_\tau L|F_\tau]}{E[L|F_\tau]}
\]
d\( \tau \times dU\mu \)-almost surely.

Proof: Sufficiency: since \( U \) is a.s. left invertible, we have from Theorem 3
\[
H(U\mu|\mu) = H(L\cdot \mu|\mu) = \frac{1}{2} E[|u^2_H|],
\]
and it is a trivial calculation of the entropy to see that
\[
E[f \circ U + \frac{1}{2} |u^2_H|] = -\log E[e^{-f}],
\]
hence \( J_* = J(u) \).

To prove the necessity, suppose that there exists some \( u \in L^2_0(\mu, H) \) with \( J_* = J(u) \). Assume that \( U = I_W + u \) is not a.s. left invertible, then from Theorem 3 we have
\[
H(U\mu|\mu) \leq \frac{1}{2} E[|u^2_H|].
\]
Hence
\[
J_* = E[f \circ U + \frac{1}{2} |u^2_H|] > E[f \circ U] + H(U\mu|\mu),
\]
but \( f \) is a tamed functional, hence \( J_* = K_* \) and the last inequality is a contradiction to the fact that \( K_* \) is the infimum of such expressions. Therefore \( H(U\mu|\mu) \) should be equal to the energy of \( u \), i.e., to \( \frac{1}{2} E[|u^2_H|] \), which is equivalent to the left a.s. invertibility. Since the minimizing measure of \( K_* \) is unique, we should have evidently
\[
\frac{dU\mu}{d\mu} = \frac{e^{-f}}{E[e^{-f}]} = L.
\]
The expression for \( L \) is obvious from the stochastic integral representation of Wiener functionals which are not necessarily Sobolev differentiable, cf. [12].

Remark 4. Notice that if \( f < \infty \) \( \mu \)-a.s. then \( U \) is \( \mu \)-a.s. invertible.

Theorem 9. Assume that \( f \) is a tamed functional, if \( J_* \) is attained at some \( u \in L^2_0(\mu, H) \), then \( u \) is unique.

Proof: Suppose that there are two such elements of \( L^2_0(\mu, H) \), say \( u_1, u_2 \) such that \( J(u_1) = J(u_2) = J_* \). Since, from Theorem 3
\[
\frac{e^{-f}}{E[e^{-f}]} = \frac{dU_1\mu}{d\mu} = \frac{dU_2\mu}{d\mu},
\]
where \( U_i = I_W + u_i, i = 1, 2 \). Moreover, if we denote \( L \) as \( \rho(-\delta v) \) \( \nu \)-a.s., where \( \nu = U_1\mu \), we see that \( V \circ U_1 = V \circ U_2 \) \( \mu \)-a.s., where \( V = I_W + v \). Consequently \( U_1 \circ V = U_2 \circ V \) \( \nu \)-a.s., and it follows then that \( U_1 \circ V \circ U_1 = U_2 \circ V \circ U_1 \) \( \mu \)-a.s., consequently \( U_1 = U_2 \) \( \mu \)-a.s.
Theorem 10. Assume that $\nu$ be a probability on $(W, \mathcal{F})$ absolutely continuous w.r.t. $\mu$, denote by $K$ the corresponding Radon-Nikodym derivative. Let $f : W \to \mathbb{R}$ be a Borel function such that $\nu(\{|f| \exp f| < \infty$. Assume that $-\log K$ is a tamed functional. Then we have

\[-\log \nu(e^{-f}) = \inf \left( \int_W f dB + H(\beta \nu); \beta \in P(W) \right) = \inf \left( E_{\mu} \left[ (f - \log K) \circ (I_W + u) + \frac{1}{2} |u|_{H}^2 \right]; u \in L^2_\nu(\mu, H) \right)\]

and the second infimum is attained if and only if $U = I_W + u$ is $\mu$-a.s. left invertible.

**Proof:** The first equality follows from Theorem 9, the second follows from the hypothesis by noting that $\nu(e^{-f}) = E_{\mu}[e^{-f + \log K}]$, hence the proof follows from Theorem 8. \hfill $\square$

6. Existence for $H$-convex functionals

Theorem 11. Assume that $f \in L^0(\mu)$ is 1-convex and that $f^- = \max(-f, 0)$ is exponentially integrable, i.e., $E[\exp cf^-] < \infty$ for some $c > 1$. Then $J_f$ is attained at some $u \in L^2_\nu(\mu, H)$ provided that $E[f \circ (I_W + \xi)] < \infty$ for at least one $\xi \in L^2_\nu(\mu, H)$. Moreover, if $f \in L^{1+}(\mu)$ for some $\varepsilon > 0$, then $f$ is a tamed functional, consequently the conclusions of Theorem 8 hold true for $f$, in particular

\[\frac{dU}{d\mu} = \frac{e^{-f}}{E[e^{-f}]} = \rho(-\delta v)\]

where

\[\dot{v}_t = \frac{E[D_t e^{-f} | F_t]}{E[e^{-f} | F_t]}\]

and $V = I_W + v$ is the $\mu$-left inverse to $U = I_W + u$ which is the unique strong solution of the following stochastic differential equation:

\[dU_t = -\dot{v}_t \circ U dt + dW_t.\]

Finally $U$ is also the solution of the following Monge-Ampère equation:

\[\frac{dU}{d\mu} = \exp \left( - \int_0^1 E_{U^t}[D_t f | F_t] dW_t - \frac{1}{2} \int_0^1 |E_{U^t}[D_t f | F_t]|^2 dt \right),\]

where $U \mu$ denotes the image of $\mu$ under $U$.

**Proof:** Let $A_\lambda$, for $\lambda > 0$ be defined as

\[A_\lambda = \{ \alpha \in L^2_\nu(\mu, H) : J(\alpha) \leq \lambda \},\]

by the 1-convexity of $f$, $A_\lambda$ is a, non-empty, convex subset of $L^2_\nu(\mu, H)$. Assume that $(\alpha_n, n \in \mathbb{N}) \subset A_\lambda$ converges to some $\alpha$ in $L^2_\nu(\mu, H)$, let $T_n = I_W + \alpha_n$ and $T = I_W + \alpha$. From Theorem 2

\[H(T_n \mu | \mu) \leq \frac{1}{2} E[|\alpha_n|_{H}^2],\]

hence the sequence $(dT_n \mu / d\mu : n \in \mathbb{N})$ is uniformly integrable. From Lusin’s lemma ($f \circ T_n, n \geq 1$) converges in $L^0(\mu)$ to $f \circ T$. The Fatou Lemma gives

\[\alpha \geq \liminf_n E[f \circ T_n + \frac{1}{2} |\alpha_n|_{H}^2] \geq E[f \circ T + \frac{1}{2} |\alpha|^2_H],\]
i.e., \( \alpha \in A_\lambda \), consequently \( A_\lambda \) is closed in \( L^2(\mu, H) \), by convexity it is weakly closed, which implies the weak lower semi continuity of \( J \). We claim that \( A_\lambda \) is also bounded; in fact we have

\[
\frac{1}{2} \| \alpha \|^2_{L^2(\mu, H)} = J(\alpha) - E[f \circ T] \\
\leq J(\alpha) + E[f^* \circ T] \\
\leq J(\alpha) + E[e^{c f^*}] + \frac{1}{c} H(T \mu | \mu) \\
\leq J(\alpha) + E[e^{c f^*}] + \frac{1}{c} \| \alpha \|^2_{L^2(\mu, H)},
\]

where the last inequality follows again from Theorem 12 and this estimate proves the boundedness of \( A_\lambda \). Consequently \( A_\lambda \) is a weakly compact subset of \( L^2(\mu, H) \), hence \( J \) attains its infimum on it, convexity implies that this minimum is global. To see that \( f \) is a tamed functional under the last hypothesis, it suffices to remark that this hypothesis and the assumption \( E[e^{f^*}] < \infty \) imply that \( E[(|f| + 1)e^{-f}] < \infty \). \( \square \)

The hypothesis of integrability of \( f \) seems to be indispensable for the existence of a minimizing element \( u \) as the following theorem shows:

**Theorem 12.** Let \( A \subset W \) be a measurable, \( H \)-convex set with positive Wiener measure, let \( f \) be defined as

\[
e^{-f} = \mu(A) 1_A,
\]

usually \( f \) is denoted by \( \chi_A \) in analysis. Then there is no \( \mu \)-a.s. left invertible API, say \( U = I_W + u \), which maps \( W \) to \( A \) unless \( \mu(A) = 1 \). In other words \( \chi_A \) is not a tamed Wiener functional for \( \mu(A) \in (0,1) \).

**Proof:** Suppose the contrary, then there exists an \( a \in L^2(\mu, H) \) such that \( U = I_W + u \) is \( \mu \)-a.s. left invertible and that \( L = d U / d \mu = e^{-f} / \mu(A) \), besides, from Theorem 8 we should have \( J_* = J(u) \). Moreover, we can write \( L = \rho(-\delta u) \), where \( v \in L^2(\mu, H) \). with \( L \circ U \rho(-\delta u) = 1 \) \( \mu \)-a.s. These equalities imply immediately that

\[
\rho(-\delta u) = \exp \left( -\delta u - \frac{1}{2} |u|_H^2 \right) = \mu(A)
\]

\( \mu \)-a.s. Consequently \( \delta u = -\log \mu(A) - \frac{1}{2} |u|_H^2 \) \( \mu \)-a.s. Then the Kazamaki condition for the Girsanov exponential (cf. [13]) of \( u \) implies that

\[
1 = E \left[ \exp \left( \lambda \delta u - \frac{\lambda^2}{2} |u|_H^2 \right) \right] \\
= E \left[ \exp \left( -\lambda \log \mu(A) - \frac{\lambda}{2} (1 + \lambda) |u|_H^2 \right) \right]
\]

for any \( \lambda \geq 0 \) and this relation is possible only when \( u = 0 \) \( \mu \)-a.s. and \( \mu(A) = 1 \). \( \square \)

**Remark 5.** Theorem 12 gives an explicit negative result about the representability of positive random variables via API (cf. [6]), although the same question has a positive answer if we forget about the adaptedness, then there exists always an invertible perturbation of identity \( T = I_W + \nabla \phi \) which maps \( W \) onto \( A \), where \( \phi \in D_{2,1} \) such that the operator norm of \( \nabla^2 \phi \) is essentially bounded, cf. [6].

**Corollary 1.** For any \( H \)-convex subset \( A_\chi_A \) is tamed if and only if \( \mu(A) = 1 \)

The corollary below gives an explicit example of a stochastic differential equation having a weak but no strong solution in a frame which is totally different than the example of Tsirelson (cf. [8]):
Corollary 2. Let \( A \subset W \) be an \( H \)-convex set with \( \mu(A) \in (0,1) \), define
\[
d\nu = \frac{1}{\mu(A)} I_A d\mu,
\]
and define \( v \) as
\[
\frac{1_A}{\mu(A)} = \rho(-\delta v)
\]
defined \( \nu \)-a.s., where \( v \in L^2_0(\nu, H) \). Let \( V_t = W_t + \int_0^t \dot{v}_s ds \). Then under the probability \( \nu \), \( V \) is a Brownian motion and the canonical path \( (W_t, t \in [0,1]) \) is the weak solution of the following stochastic differential equation
\[
W_t = -\int_0^t \dot{v}_s \circ W + V_t
\]
and this equation has no strong solution.

Proof: The fact that \( (W_t, t \in [0,1]) \) is a weak solution follows from the fact that \( V \) is a Brownian motion under \( \nu \), the fact that it is not a strong solution follows from Theorem 12.

7. Variational techniques to calculate the minimizers

In this section we shall derive a necessary and sufficient condition for a large class of adapted perturbation of identity. We begin with some technical results:

Lemma 2. Assume that \( g \in D_{p,1}, p > 1 \) with \( E[\exp \varepsilon |\nabla g|_H] < \infty \) for some \( \varepsilon > 0 \). Suppose that \( \lambda \to \xi_{\lambda} \) is an absolutely continuous curve from \([0,1]\) to \( L^2_0(\mu, H) \) such that
\[
\xi_{\lambda}' = \frac{d\xi_{\lambda}}{d\lambda} \in L^\infty(\mu, H).
\]
Then we have
\[
(7.9) \quad g(w + \xi_{\lambda}(w)) = g(w + \xi_0(w)) + \int_0^\lambda (\nabla g(w + \xi_t(w)), \xi_t'(w))_H dt
\]
\( \mu \)-almost surely.

Proof: Let \( (P_t, t \geq 0) \) be the Ornstein-Uhlenbeck semigroup on \( W \), let \( g_n = P_{1/n}g \), denote \( w \to w + \xi_{\lambda}(w) \) by \( T_{\lambda}(w) \). Since \( g_n \) is an \( H - \mathcal{C}^\infty \)-map (cf. [15]), (7.9) holds for \( g_n \). To pass to the limit, it suffices to show that \( (\nabla g_n \circ T_{\lambda}, n \geq 1) \) is uniformly integrable w.r.t. \( d\lambda \times d\mu = d\eta \). To see this, let \( L_{\lambda} \) be the Radon-Nikodym density of \( T_{\lambda} \) w.r.t. \( \mu \) and write
\[
E_\eta[|\nabla g_n \circ T_{\lambda}|_H 1_{\{|\nabla g_n|_H > c\}}] = E_\eta[|\nabla g_n|_H 1_{\{|\nabla g_n|_H > c\}} L_{\lambda}] \leq E_\eta \left[ e^{c|\nabla g_n|_H} 1_{\{|\nabla g_n|_H > c\}} \right] + E_\eta \left[ \frac{1}{c} L \log L 1_{\{|\nabla g_n|_H > c\}} \right].
\]
The first term of the last line tends to zero uniformly in \( n \) as \( c \to \infty \) due to the uniform integrability of \( (\exp |\nabla g_n|_H, n \geq 1) \), the second term also has the same behaviour by the dominated convergence theorem, i.e., for any \( \gamma > 0 \), there exists some \( c_\gamma > 0 \) such that \( \sup_n \eta\{|\nabla g_n|_H > c\} < \beta \) as soon as \( c > c_\gamma \).

This lemma says that under its regularity assumptions, to find the candidate elements of \( L^2(\mu, H) \) for the solution of the minimization problems, we have to verify if they satisfy the functional equation \( u + \Phi(u) = 0 \), where \( \Phi \) is defined by
\[
\Phi(\xi) = -\pi(\nabla f \circ (I_W + \xi)),
\]
and where \( \pi \) denotes the dual predictable projection. Suppose that \( \|\nabla^2 f\|_{op} \leq c < 1 \) almost surely, where \( c > 0 \) is a fixed constant and the norm is the operator norm on \( H \). Then the map
\[ \Phi : L^2_2(\mu, H) \to L^2_2(\mu, H) \text{ is a strict contraction, hence there exists a unique } u \in L^2_{2,0}(H) \text{ which satisfies the equation} \]
\[ \dot{u}_t + E[D_t f \circ U | \mathcal{F}_t] = 0 \]
\[ dt \times d\mu \text{-almost surely. However, this condition is too restrictive to be applicable and we reduce these hypothesis in the next theorem.} \]

Although the conditions are strong, the following theorem justifies rigorously the ideas explained in the introduction and by itself it is an interesting property. Briefly it tells that if \( f \) is sufficiently regular, then every local minimum of \( J \) is a global one, hence it is unique and the corresponding API is almost surely invertible:

**Theorem 13.** Assume that \( f \in L^p_1 \) for some \( p > 1 \) and
\[ (7.10) \quad E[e^{\varepsilon f}] < \infty . \]
Let \( L \) denote the density \( e^{-f} / E[e^{-f}] \). Define \( v \) as to be
\[ v(t, w) = \int_0^t \dot{v}_s(w) ds \]
where
\[ \dot{v}_t = E_L[D_t f | \mathcal{F}_t] \]
and where \( E_L \) denotes the expectation operator w.r.t. the measure \( L d \mu \). Assume that
\[ (7.11) \quad E[\exp \varepsilon \| \nabla v \|_{op}^2 + \exp \varepsilon |v|_{H}^2] < \infty , \]
for some arbitrary constant \( \varepsilon > 0 \), where \( \| \cdot \|_{op} \) denotes the operator norm on the Cameron-Martin space \( H \). Let \( u \in L^2_2(\mu, H) \) be a solution of the functional equation
\[ u + \Phi(u) = 0 . \]
Then \( u \) is a global minimizer of \( J \), hence, in particular it is unique and the conclusions of Theorem 8 hold true for \( U = I_W + u \).

**Proof:** Let us note immediately that, using the Young inequality and the hypothesis \( (7.10) \), \( (7.11) \) imply that \( v \circ U \circ T \) is again in \( L^2_2(\mu, H) \) for any \( T = I_W + \xi \) with \( \xi \in L^\infty(\mu, H) \) and the hypothesis of Lemma 2 are satisfied so that we have the relation
\[ |v \circ (U + \lambda \xi) + \lambda \xi|_{H}^2 = |v \circ U + u|_{H}^2 + 2 \int_0^\lambda \left( \nabla v \circ (U + t\xi) \xi + \xi, v \circ (U + t\xi + t\xi) \right)_H dt , \]
which justifies the calculation of the directional derivative of \( J \) which is done below. From the Itô representation theorem, we have
\[ L = \exp \left( -\delta v - \frac{1}{2} |v|_{H}^2 \right) , \]

hence
\[ f = -\log E[e^{-f}] + \delta v + \frac{1}{2} |v|_{H}^2 . \]

Using the last expression we get
\[ J(u) = E[f \circ U + \frac{1}{2} |u|_{H}^2] \]
\[ = -\log E[e^{-f}] + E[(\delta v + \frac{1}{2} |v|_{H}^2) \circ U + \frac{1}{2} |u|_{H}^2] \]
\[ = -\log E[e^{-f}] + \frac{1}{2} E[|v \circ U + u|_{H}^2] . \]

Note that the last equality above follows from the fact
\[ E[|v \circ U|_{H}^2] = E[|v|_{H}^2 \frac{dU}{d\mu}] \leq E[e^{\varepsilon |v|_{H}^2}] + \frac{1}{\varepsilon} H(U \mu | \mu) < \infty , \]
hence $E[\delta(v \circ U)] = 0$. Using Lemma 2, we obtain

\begin{equation}
E[(\nabla v \circ U[\eta] + \eta, v \circ U + u)_H] = 0
\end{equation}

for any $\eta \in L^2_0(\mu, H)$ bounded. Note that, for any $\xi \in L^2_0(\mu, H)$,

\[
E[\nabla v \circ U[\eta], \xi]_H = E\int_0^1 ds \int_s^1 D_s \dot{v} \tau \circ U \dot{\eta} \tau \dot{\xi} d\tau.
\]

Since $\eta$ is adapted, $\tau \rightarrow \nabla \eta \dot{v} \tau$ is adapted, consequently $\nabla v \circ U$ is a quasi-nilpotent operator, therefore $I_H + \nabla v \circ U$ is almost surely invertible. Relation (7.12) implies then that $(I_H + \nabla v \circ U)(v \circ U + u) = 0$ a.s., by the invertibility of $I_H + \nabla v \circ U$, we obtain $v \circ U + u = 0$ a.s., and this proves the a.s. left invertibility of $U$ and the rest of the proof follows from Theorem 8.

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