The problem of the equilibrium state of the charged many-particle system above dielectric surface is formulated. We consider the case of the presence of the external attractive pressing field and the case of its absence. The equilibrium distributions of charges and the electric field, which is generated by these charges in the system in the case of ideally plane dielectric surface, are obtained. The solution of electrostatic equations of the system under consideration in case of small spatial heterogeneities caused by the dielectric surface, is also obtained. These spatial inhomogeneities can be caused both by the inhomogeneities of the surface and by the inhomogeneous charge distribution upon it. In particular, the case of the wavy, spatially periodic surface is considered taking into account the possible presence of the surface charges.

Keywords: charged fermions, surface, solid and liquid dielectrics, equilibrium distribution of charges and electric field.

PACS: 05.30.Fk, 05.70.Np, 41.20.Cv, 71.10.Ca, 73.20.At.

1 Introduction

The problems concerned with the research of the charges above dielectric surface belong to classical electrodynamics and electrostatics problems. A special interest to such problems appeared due to the phenomenon of the Wigner crystallization. These researches were initiated in 1934 by Wigner in his theoretical work [1], per se. In this work, the possibility of the existence of periodic structures in the systems with repulsive forces between particles was demonstrated by the example of the crystallization of the three-dimensional low-density gas of electrons in the field generated by the spatial-homogeneous positive charge. This field played exactly a role of the compensative factor for the repulsive forces. The Landau-Silen Fermi-liquid theory also enables to predict the existence of spatially-periodic state of electrons in metals and to describe its structure (see in this case Ref. [2]). The experimental improvement of Wigners' prediction of the three-dimensional crystal-like structures still does not exist (see, e.g., Refs. [3 4 5]). This is caused by difficulties in the achievement of the experimental conditions for the mentioned phenomenon, which is also called as "Wigner crystallization".

However, as it is well known, different two-dimension periodic electron structures above the surface of a fluid helium are experimentally realized (the so-called "Wigner crystals"). The works [3 4 5 6] may be referred as the first publications containing theoretical and real experimental results of different properties of the surface electrons. The large number of works related to the theoretical and experimental research in this area has appeared by now.

The theoretical papers that are devoted to the microscopic description of the charge state above dielectric surface are usually based on the conception of an isolated charge above dielectric surface interacting with its electrostatic reflection in dielectric ("levitate electron", see, e.g., Refs. [3 4 5]). In this case, the quantum-mechanical state "charge - electrostatic reflection" is described as the hydrogen like one-dimensional state with the corresponding energy structure. Very often the localization of such quantum-mechanical object in the ground state is considered (see Refs. [3 4 5 6 7]) occurring at some distance from the surface (first "Bohr radius"). This, particularly in most cases, allows not to take into account the influence of the surface inhomogeneity on the single charge state. However, at the description of many-particle charge system above dielectric surface the mentioned approach inevitably faces some difficulties. For example, such difficulty appears when the electron density above dielectric surface does not allow to consider the charged particles as isolated, i.e., it is necessary to take into account the interparticle interaction.

The references, which are devoted to the two-dimensional Wigner crystallization in the phenomenological approach, predominantly consider the system that consists of a large number of charged particles near the surface of the fluid dielectric as a two-dimensional structure (see, e.g., Refs. [3 4 5 6 7 8]).

Basing on the premises, it becomes clear that the complete description of charges above dielectric surface needs to take into account their spatial distribution in vacuum. The possibility of charges adsorption by the surface must also be taken into consideration (in this case the surface inhomogeneities must play a
crucl role). The possibility of charge spatial distribution above dielectric surface comes from the fact that a charged particle is always attracted by a dielectric surface. Moreover, in the experiments [3, 4] concerned with the registration of two-dimensional Wigner crystallization an external electric field attracts charges to the surface and affects on their spatial distribution.

The present paper is devoted to the problem of equilibrium charge distribution above dielectric surface as in the external pressing electric field as in its absence. This problem is considered in the case of ideally plane vacuum-dielectric boundary and in the case of "wavy" spatial-periodic surface with account of the possibility of existence of the "sticked" to the surface charges. In our opinion, the formulated problem is interesting as from purely academic side as from the research side of the influence of volume charges located closely to the fluid helium surface on the spatial-inhomogeneous state of the charges adsorbed on the helium surface.

2 Equations of electrostatics for many-charge system above dielectric surface

Let us consider the equilibrium system of charged particles (Fermi-particles) with the charge \( Q \) per particle that is situated in vacuum above dielectric surface with the permittivity \( \varepsilon \). We describe below the surface profile by function \( \xi(\rho) \equiv \xi(x, y) \), where \( \rho \equiv \{x, y\} \) is the radius-vector in the plane \( z = 0 \) of Cartesian coordinates \( \{x, y, z\} \). The vacuum - dielectric boundary lies in the plane \( z = 0 \) and we consider it unbounded below. All physical quantities considered in the area above dielectric, i.e., at \( z > \xi(\rho) \) we mark by the index "1" and all quantities concerned with the dielectric \( (z < \xi(\rho)) \) we mark by the index "2". Let us assume that the external pressing electric field \( E \) acts on particles and is directed along the \( z \)-axis. We also assume the existence of some potential barrier that forbids the charges to penetrate inside the dielectric.

As it is mentioned above, the charged particles are always attracted by the dielectric. Therefore, even in the absence of the external pressing electric field there is a reason to believe that there are conditions under what the stable equilibrium distribution along \( z \)-axis is developed. To avoid the questions on the repulsion of likely charged particles along the plane \( \rho \) we shall consider the system located in a vessel with the walls at \( \rho \rightarrow \infty \). These walls forbid the charges to leave the system. Let us describe the equilibrium charge distribution above the dielectric surface by the distribution function \( f(p; z, \rho) \).

The electric field potential \( \varphi_1 \) in vacuum above the dielectric surface must satisfy the Poisson’s equation

\[
\Delta \varphi_1(z, \rho) = -4\pi Q n(z, \rho) \theta(z - \xi(\rho)), \tag{1}
\]

where \( \Delta \) is the Laplace operator,

\[
\Delta = \frac{\partial^2}{\partial z^2} + \Delta_p, \quad \Delta_p = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2}
\]

\( \theta(z - \xi(\rho)) \) is the Heaviside function. In eq. (1) the quantity \( n(z, \rho) \) is the charge density above the dielectric surface, which can be expressed in terms of the distribution function \( f(p; z, \rho) \) as

\[
n(z, \rho) = \int d^3p f(p; z, \rho). \tag{3}
\]

As charges are considered as Fermi-particles, the distribution function \( f(p; z, \rho) \) has the following form:

\[
f(p; z, \rho) = \frac{g}{(2\pi\hbar)^3} \times
\]

\[
\times \left\{ \exp \beta \left[ \frac{p^2}{2m} + Q\varphi_1(z, \rho) - \mu \right] + 1 \right\}^{-1}, \tag{4}
\]

where \( g = (2S_Q + 1) \), \( S_Q \) is the spin of the charged particle, \( \beta = 1/T \), \( T \) is the temperature in the energy units, \( m \) is the charge mass and \( \mu \) is the chemical potential of charges. We emphasize that taking into account relations [3] and [4] the eq. (1) is often called the Thomas-Fermi equation.

The electric field potential \( \varphi_2 \) is the result of charge absence in the dielectric. In the assumption of the dielectric homogeneity and isotropy it must satisfy the Laplace’s equation

\[
\varepsilon \Delta \varphi_2(z, \rho) = 0. \tag{5}
\]

If the system is placed in the external static homogeneous electric field, the potentials \( \varphi_1 \) and \( \varphi_2 \) can be written in the form

\[
\varphi_1 = \varphi_1^{(i)} + \varphi_1^{(e)}, \quad \varphi_2 = \varphi_2^{(i)} + \varphi_2^{(e)}, \tag{6}
\]

where \( \varphi_1^{(i)}, \varphi_2^{(i)} \) are the potentials induced by the system of charges in vacuum and in the dielectric, respectively, \( \varphi_1^{(e)} \) and \( \varphi_2^{(e)} \) are the potentials of the external field in vacuum and dielectric. According to eqs. (1), (2), these fields satisfy the following equations:

\[
\Delta \varphi_1^{(i)}(z, \rho) = -4\pi Q n(z, \rho), \quad \Delta \varphi_1^{(e)} = 0,
\]

\[
\Delta \varphi_2^{(i)} = 0, \quad \Delta \varphi_2^{(e)} = 0. \tag{7}
\]

Eqs. (1), (5) for the potentials must be expanded with the boundary conditions on the vacuum-dielectric border (as usual, these conditions can be obtained directly from eqs. (1), (5), see e.g.[10]):

\[
\varphi_1(z, \rho) \big|_{z=\xi(\rho)} = \varphi_2(z, \rho) \big|_{z=\xi(\rho)},
\]

\[
n_1(\rho) \left( \varepsilon \nabla_1 \varphi_2(z, \rho) - \varepsilon \nabla_1 \varphi_1(z, \rho) \right) = 4\pi \sigma(\rho, \xi(\rho)), \tag{8}
\]

where \( n(\rho) \) is the unit vector of the surface normal in the point \( \rho \), \( \sigma(\rho, \xi(\rho')) \) is the surface charge density.
in the point $\rho$ (here we emphasize the functional dependence of this value on the surface profile $\xi(\rho')$).

The surface charge density $\sigma(\rho, \xi(\rho'))$ must satisfy the following relation:

$$\int dS_\xi \sigma(\rho, \xi(\rho')) = Q N_\xi,$$  

(9)

where $N_\xi$ is the complete charge number on the dielectric surface and $dS_\xi$ is the surface element with the profile $\xi(\rho')$:

$$dS_\xi = d^2 \rho \sqrt{1 + (\partial \xi(\rho)/\partial \rho)^2}.$$

The surface charge density can appear due to several reasons. For example, these charges may be specially placed on the dielectric surface and can stay there for arbitrary long time. At that time, the surface charges and charges above the dielectric surface can differ in sign. But in this case we need to consider the possibility of the formation of bound states of the opposite charged particles. Taking into account the presence of such bound states represents a separate rather complicated problem. In the present paper such case of surface charges is not considered. The case, in which some part of charges condenses on the surface from the volume distribution, stays for some period of time and then back to the volume, is possible too. In this case, the equilibrium distribution of charges above the surface that coexist with the “sticked” surface charges for some period of time (the lifetime of the charge staying on the surface) is possible. Next, we shall take into account only the possibility of the presence of the surface charges that have the same sign as the volume ones.

It is easy to see that in eq. (5) the directional cosines of the surface normal vector $n(\rho)$ in the point $\rho$ play the main role: cos $\nu$ is cosine of the angle between the normal and z-axis, cos $\lambda$ is cosine of the angle between the normal and x-axis and cos $\mu$ is cosine of the angle between the normal and y-axis. In the case when the surface profile is given explicitly (in our case $z = \xi(\rho)$), these cosines are determined by the following relations:

$$\cos \nu = \frac{1}{\sqrt{1 + (\partial \xi(\rho)/\partial \rho)^2}},$$

$$\cos \lambda = -\frac{\partial \xi(\rho)/\partial x}{\sqrt{1 + (\partial \xi(\rho)/\partial \rho)^2}},$$

$$\cos \mu = -\frac{\partial \xi(\rho)/\partial y}{\sqrt{1 + (\partial \xi(\rho)/\partial \rho)^2}}.$$

(11)

The derived electrostatics equations (1), (3) with the boundary conditions (8), (11) can be solved analytically in a very low case count. Some of these cases are considered below. Before solving eqs. (1), (3), let us consider the simplification of the boundary conditions (8) in the case, when the surface profile differs a little from the plane one. In this case we essentially have the effective boundary conditions. From eqs. (10), (11) it is obvious that the surface slightly differs from the plane one, when the surface profile slowly varies on coordinate, i.e., when the following inequalities take place:

$$|\partial \xi(\rho)/\partial x| \ll 1, \quad |\partial \xi(\rho)/\partial y| \ll 1.$$  

(12)

Let us also consider that the surface profile $\xi(\rho)$ can be given as:

$$\xi(\rho) = \xi + \tilde{\xi}(\rho), \quad |\xi| \gg |\tilde{\xi}(\rho)|.$$  

(13)

It is easy to see that the inequality (12) is provided in this case by the conditions

$$|\partial \xi(\rho)/\partial x| \ll 1, \quad |\partial \tilde{\xi}(\rho)/\partial y| \ll 1.$$  

(14)

The directional cosines (11) with accuracy up to the second order over $\partial \xi(\rho)/\partial \rho$ have the following form:

$$\cos \nu \approx 1, \quad \cos \lambda = -\partial \xi(\rho)/\partial x,$$

$$\cos \mu = -\partial \tilde{\xi}(\rho)/\partial y.$$  

(15)

If the relations (13) - (15) take place, we can expect that the charge and the field distributions in the system slightly differ from the distributions that take place in the case of the plane dielectric surface. Then, the potentials $\varphi_1(z, \rho)$ and $\varphi_2(z, \rho)$ (see eqs. (1), (3)) can be written as

$$\varphi_1(z, \rho) = \varphi_1(z) + \tilde{\varphi}_1(z, \rho),$$

$$\varphi_2(z, \rho) = \varphi_2(z) + \tilde{\varphi}_2(z, \rho),$$

(16)

where $\varphi_1(z)$ and $\varphi_2(z)$ are the potentials of some electric field above the dielectric and inside of it (but not on the surface!) in the case of the plane surface. The small distortions of the field above the dielectric and inside of it are described by the potentials $\tilde{\varphi}_1(z, \rho)$ and $\tilde{\varphi}_2(z, \rho)$ due to the surface inhomogeneity in the mentioned above sense. The meaning of the introduced potentials $\varphi_1(z)$ and $\varphi_2(z)$, and also $\tilde{\varphi}_1(z, \rho)$ and $\tilde{\varphi}_2(z, \rho)$ becomes more clear after the obtaining of the Poisson’s equations and effective boundary conditions for them.

According to the assumption of small field perturbations provided by the wave surface, the following inequalities take place:

$$|\varphi_1(z)| \gg |\varphi_1(z, \rho)|, \quad |\varphi_2(z)| \gg |\varphi_2(z, \rho)|.$$  

(17)

Let us also consider that the distribution of charges that can be condensed on surface slightly differs from the homogeneous one:

$$\sigma(\rho, \xi) = \sigma(\xi) + \tilde{\sigma}(\rho, \xi) + \frac{\partial \sigma(\xi)}{\partial \xi} \tilde{\xi}(\rho),$$

$$|\sigma(\xi)| \gg |\tilde{\sigma}(\rho; \xi)|, \quad |\sigma(\xi)| \gg \left|\frac{\partial \sigma(\xi)}{\partial \xi} \tilde{\xi}(\rho)\right|.$$  

(18)

In the expressions (18) the quantity $\tilde{\sigma}(\rho, \xi)$ corresponds to the impact of the weakly inhomogeneous charge distribution on the plane dielectric surface with $z = \xi$ profile. The quantity $\xi(\rho)|\partial \sigma(\xi)/\partial \xi|$ in eq. (18) describes
the surface charge inhomogeneity related to the weak irregularity of the surface itself.

In the expressions (12)-(18), we consider that in the case of uniformly charged surface, which is ideally plane and infinitely extended with charge density $\sigma(\xi)$ both the field and the charge distributions are homogeneous along $\rho$ plane. In other words, the spatial charge and field distribution depends only on $z$ coordinates.

From the relations (13)-(18) it is easy to obtain the effective boundary conditions for field potentials on vacuum-dielectric boundary in the case, when the dielectric surface weakly differs from the ideally plane. To this end, we must develop the perturbation theory up to the first or-

ing of limits

$$\{
\begin{aligned}
\varphi_1(z) + \varphi_1(z, \rho) &= \varphi_2(z) + \varphi_2(z, \rho), \\
\varphi_2(z) + \varphi_2(z, \rho) &= \varphi_1(z) + \varphi_1(z, \rho),
\end{aligned}
\} \xi (\rho) \\
\{ \varepsilon \partial_2 \varphi_2(z) + \varphi_2(z) \partial_2 \varphi_2(z, \rho) - \partial_2 \varphi_1(z) + \varphi_1(z, \rho) \} \xi (\rho)
\}
$$

Making the necessary calculations up to the first order of the perturbation theory from the first relation of eq. (19) we obtain

$$\varphi_1(z)|_{z=\xi} = \varphi_2(z)|_{z=\xi},$$

$$\{ \varphi_2(z) - \varphi_2(z, \rho) \} \xi (\rho) = \{ \partial_2 \varphi_2(z) - \partial_2 \varphi_1(z) \} \xi (\rho).$$

The use of the perturbation theory up to the first order for the second relation of eq. (19) results in the following equalities:

$$\{ \varepsilon \partial_2 \varphi_2(z) - \partial_2 \varphi_2(z, \rho) \} z=\xi = 4\pi \sigma(\xi),$$

$$\{ \varepsilon \partial_2 \varphi_2(z) - \partial_2 \varphi_2(z, \rho) \} z=\xi = -4\pi \bar{\sigma}(\rho; \xi).$$

Let us remind that in electrostatics the denotations like $\{ \partial^2 \varphi_1(z) / \partial z^2 \} z=\xi$, $\{ \partial^2 \varphi_2(z) / \partial z^2 \} z=\xi$ have the meaning of limits

$$\{ \partial^2 \varphi_1(z) / \partial z^2 \} z=\xi = \lim_{h \to 0} \{ \partial^2 \varphi_1(z) / \partial z^2 \} z=\xi + h,$$

$$\{ \partial^2 \varphi_2(z) / \partial z^2 \} z=\xi = \lim_{h \to 0} \{ \partial^2 \varphi_2(z) / \partial z^2 \} z=\xi - h.$$

Then, for the further simplification of the obtained effective boundary conditions (20), (21), according to eqs. (11), (15), (16) we can use the following equations, which are satisfied by the potentials $\varphi_1(z), \varphi_2(z)$:

$$\frac{\partial^2 \varphi_1(z)}{\partial z^2} = -4\pi Q n(z) \theta(z - \xi), \quad \frac{\partial^2 \varphi_2(z)}{\partial z^2} = 0, \quad (22)$$

where

$$n(z) = \int d^3 p f(p, z),$$

$$f(p, z) = \frac{g}{(2\pi\hbar)^3} \left\{ \exp \beta \left[ \frac{p^2}{2m} + Q \varphi_1(z) - \mu \right] + 1 \right\}^{-1}.$$  

Taking into account eq. (21), the conditions (20) can be expressed in the following form (the first one of them remains the same):

$$\{ \varepsilon \partial_2 \varphi_2(z) - \partial_2 \varphi_1(z) \} \xi (\rho) = 4\pi \sigma(\xi),$$

$$4\pi \{ Q n(z) - \partial_2 \sigma(\xi) \} \xi (\rho) - 4\pi \bar{\sigma}(\rho; \xi) = \{ \partial_2 \varphi_2(z, \rho) - \partial_2 \varphi_2(z) \} \xi (\rho).$$

Thus, we obtain the effective boundary conditions (20), (24) for the fields in the system of charges above the dielectric surface with the surface profile that slightly differs from plane

$$\frac{\partial^2 \varphi_1(z, \rho)}{\partial z^2} + \Delta_\rho \varphi_1(z, \rho) = 4\pi Q \frac{\partial n(z)}{\partial \mu} \varphi_1(z, \rho),$$

$$\frac{\partial^2 \varphi_2(z, \rho)}{\partial z^2} + \Delta_\rho \varphi_2(z, \rho) = 0.$$  

(25)

3 System of charges above the ideally plane dielectric surface

It is easy to see that the obtained equations (23), (24) of electrostatics and the effective boundary conditions (20), (21) for these equations are much simpler than the initial electrostatic equations (11), (15) and the boundary conditions (8). Firstly, to solve the equations that determine the charge and the field distribution above the vacuum-dielectric boundary one needs consider the case of ideally plane surface of this border that lies in $z = \xi$. Let us start withe the considering the case of the surface in the absence of the charge $\sigma = 0$. Then, the solution of (22) must satisfy the following boundary conditions:

$$\varphi_1(z)|_{z=\xi} = \varphi_2(z)|_{z=\xi},$$

$$\{ \varepsilon \partial_2 \varphi_2(z) - \partial_2 \varphi_1(z) \} \xi (\rho) = 0.$$  

(26)

To simplify the further calculations, let us write the first formula from eq. (22) in the following form:

$$\frac{\partial^2 \varphi_1(z)}{\partial z^2} = -4\pi Q \nu \int_0^{\infty} d\varepsilon \varepsilon^{1/2} \{ \exp \beta (\varepsilon - \psi) + 1 \},$$  

(27)

where we denote

$$\psi(z) \equiv \mu - Q \varphi_1(z), \quad \nu \equiv (2m)^{3/2}/2\pi^2 \hbar^3.$$  

(28)
Here we also consider the spin of a charged particle equal to $1/2$, $\psi$ is so-called electrochemical potential.

Multiplying eq. (27) by the derivative $(\partial \varphi_1(z) / \partial z)$ and using the following equality

$$
\left(\frac{\partial \varphi_1}{\partial z}\right) e^{\beta(z - \psi)} + 1 = -\frac{1}{\beta q} \partial \ln \left[ e^{-\beta(z - \psi)} + 1 \right],
$$

after simple calculations we obtain the first-order differential equation:

$$
\left(\frac{\partial \varphi_1}{\partial z}\right)^2 = \frac{16\pi}{3} \nu \int_0^\infty d\varepsilon \varepsilon^{3/2} \left[ e^{\beta(z - \psi)} + 1 \right]^{-1} + C,
$$

where $C$ is an arbitrary integration constant. Thus, the need of the following equation solving arises:

$$
\frac{\partial \varphi_1}{\partial z} = \pm \left\{ \frac{16\pi}{3} \nu \int_0^\infty d\varepsilon \varepsilon^{3/2} \left[ e^{\beta(z - \psi)} + 1 \right]^{-1} + C \right\}^{1/2}.
$$

(29)

The sign before the square root in eq. (29) must be chosen from the following consideration. The force acting on the charges at $z > \xi$ presses these charges to the dielectric surface. Thus, in the case of positive charges above the dielectric we choose the positive sign, and in the case of negative charges we choose the negative one. Let us consider below the distribution of negative charges above the dielectric surface, $Q = -e, e > 0$. Hence, the potential $\varphi_1$ satisfies the relation:

$$
\frac{\partial \varphi_1}{\partial z} = -\left\{ \frac{16\pi}{3} \nu \int_0^\infty d\varepsilon \varepsilon^{3/2} \left[ e^{\beta(z - \psi)} + 1 \right]^{-1} + C \right\}^{1/2}.
$$

(30)

Now we make the following denotations:

$$
\varphi_1(z = 0) \equiv \varphi_0, \quad \psi(z = 0) \equiv \mu + e \varphi_1(z = 0),
$$

$$
E_0 \equiv -\left(\frac{\partial \varphi_1(z)}{\partial z}\right)_{z=0} \quad (31)
$$

Let us remind that we consider the case of the electric forces that attract charges to the dielectric surface. Thus, at $z \to \infty$ there is no charges, $f(p; z) \to 0$, or

$$
\{ \exp \beta (z - \psi) + 1 \}^{-1} \to 0. \quad (32)
$$

The action of the electrostatic image force along $z$-axis must vanish at $z \to \infty$:

$$
\frac{\partial \varphi_1^{(i)}(z)}{\partial z} \to 0. \quad (33)
$$

As the result, it is essential to say that at $z \to \infty$ the following relation takes place:

$$
\frac{\partial \varphi_1(z)}{\partial z} \to -\frac{\partial \varphi_1^{(e)}(z)}{\partial z} \equiv E, \quad (33)
$$

where $E$ is the external field intensity that attracts charges to the dielectric surface.

At $z = 0$ from eq. (30) one can get

$$
E_0^2 = \frac{16\pi}{3} \nu \int_0^\infty d\varepsilon \varepsilon^{3/2} \{ \exp \beta (z - \psi_0) + 1 \}^{-1} + C.
$$

On the other hand, from the same equation and taking into account eqs. (32), (33) at $z \to \infty$ we obtain:

$$
C = E^2. \quad (34)
$$

Comparing the last two expressions, we come to the relation between the constants $\psi_0, E_0$ (see eq. (31)) and the external electromagnetic field $E$:

$$
E_0^2 - E^2 = \frac{16\pi}{3} \nu \int_0^\infty d\varepsilon \varepsilon^{3/2} \{ \exp \beta (z - \psi_0) + 1 \}^{-1}.
$$

(35)

Then, after integration of the first expression from eq. (23) over $z$ within the limits from $\xi$ to $\varepsilon$ and using eqs. (32), (34), we get:

$$
E_0 - E = 4\pi e n_s, \quad e > 0, \quad (36)
$$

where $n_s$ is the number of the volume charges per unit of the plane dielectric surface:

$$
n_s = \int_\xi^\infty dz n(z), \quad (37)
$$

Let us emphasize that for the equilibrium charge system above dielectric the value of the number $n_s$ depends neither on the coordinates, neither on the fields’ distribution. It is determined only by the entire number $N$ of the charges above dielectric. We also point out that this value characterizes the additional field intensity that presses the charges to the dielectric surface. Besides that, this field is generated by the charges themselves.

Thus, eqs. (35), (36) allow to express the unknown quantities $\psi_0$ and $E_0$ (integration constants of eq. (27)) in terms of the external pressing electric field $E$ and the number of charges above the unit item of the dielectric surface $n_s$ (see eq. (37)).

The second equation in (38) can be solved trivially in general case, because the electric field intensity in dielectric does not depend on $z$. Using the boundary conditions (26), we can express the potential of electric field in dielectric in the following form:

$$
\varphi_2 = -\frac{E_0}{\varepsilon z} + \varphi_0, \quad E_2 = \frac{E_0}{\varepsilon}, \quad (38)
$$

where $E_2$ is the electric field intensity in the dielectric, $E_0$ can be expressed from eqs. (35), (36) and $\varphi_0$ is the potential on the surface. As in the electrostatic case,
the field potential is determined accurate within a constant. Therefore, we set the potential \( \varphi_0 \) equal to zero below. In this case, the value of the electrochemical potential on the dielectric surface coincides with the chemical one:

\[
\psi(z = 0) = \psi_0 = \mu, \quad \varphi_0 = 0. \tag{39}
\]

Taking into account eq. (31), the spatial distribution of the potential (see eq. (30)) can be written as follows:

\[
\frac{\partial \varphi_1}{\partial z} = - 16\pi \int_0^\infty \delta \varepsilon^{3/2} \times \left\{ \exp \beta (\varepsilon - \psi) + 1 \right\}^{-1} + E^2 \right\}^{1/2}, \tag{40}
\]

\[
\psi(z) \equiv \mu + e\varphi_1(z).
\]

It is easy to see that in general case the solution of this equation can be found only in quadratures (see below). However, the gas of charged Fermi-particles above the dielectric surface is nondegenerate. Therefore, the solution of eq. (40) can be obtained analytically. Indeed, in the case of nondegenerate gas its distribution function has the form that weakly differs from Boltzmann's one

\[
\{ \exp \beta (\varepsilon - \psi) + 1 \}^{-1} \sim \exp \beta (\psi - \varepsilon) .
\]

Accordingly, the expression for the density distribution of a gas along the \( z \) coordinate (see eq. (37)) becomes:

\[
n(z) = \sqrt{\frac{\pi}{2}} \nu \beta^{-3/2} \exp(\beta \psi) . \tag{41}
\]

As the Fermi-particle gas is degenerate at low temperature and high density ranges (see e.g. [11]) from eq. (41) one can get the gas nondegeneracy condition:

\[
\exp(\beta \psi) \ll 1,
\]

As the electrochemical potential depends on \( z \), this condition is obviously realized in the case when the following inequality takes place:

\[
\exp(\beta \psi_0) \ll 1, \tag{42}
\]

where \( \psi_0 \) is the electrochemical potential on the dielectric surface (see eqs. (28), (39) in this case). The last statement takes place because of the assumption of the particle absence at \( z \rightarrow \infty \), see above. So, according to eq. (11) one can express \( \beta \psi \) as:

\[
\frac{\partial \psi}{\partial z} = - \left\{ 4\pi^{3/2} \varepsilon^2 \beta^{-5/2} \nu \exp(\beta \psi) + e^2 E^2 \right\}^{1/2} . \tag{43}
\]

This equation has the analytical solution:

\[
\frac{\sqrt{\pi}}{2} \nu \beta^{-3/2} \exp(\beta \psi(z)) = \frac{\beta E^2}{8\pi} \frac{4\chi(z)}{1 - \chi(z)} , \tag{44}
\]

\[
\psi(z) = \mu + e\varphi_1(z),
\]

where the function \( \chi(z) \) is defined by the relation:

\[
\chi(z) = \frac{E_0 - E}{E_0 + E} \exp \left\{ - \frac{(z - \xi)}{z_0} \right\} , \tag{45}
\]

\[
z_0 \equiv (\beta e E)^{-1} , \quad \beta^{-1} = T.
\]

Let us emphasize that the multiplier before the exponent in eq. (45) according to eq. (36) can be expressed in terms of the intensity of the external electric field and the number of charges \( n_s \) in the "column" above the surface unit element:

\[
\frac{E_0 - E}{E_0 + E} = \frac{2\pi e n_s}{E + 2\pi e n_s}.
\]

From the eqs. (41), (43) and (44) follows that the charge density above the dielectric surface has the distribution:

\[
n(z) = \beta E^2 \frac{4\chi(z)}{8\pi (1 - \chi(z))^2} , \tag{46}
\]

and the electric field intensity above the dielectric \( E_1(z) \) is expressed as

\[
E_1(z) = E \frac{1 + \chi(z)}{1 - \chi(z)} . \tag{47}
\]

It is easy to see that at high values of \( z, z \gg z_0 \) (see eq. (46)), the charge distribution above the dielectric surface is close to the Boltzmann distribution and the electric field density exponentially tends to the external pressing electric field density. This fact confirms the above assumptions(see eqs. (32), (33)).

The inequality (42) that determines the nondegeneracy condition of the charge gas can be written in terms of the obtained solutions:

\[
en_s \nu^{-1} \beta^{5/2} (E + 2\pi e n_s) \ll 1. \tag{48}
\]

It is obvious that this inequality is not accomplished in the case of low temperature range or high values of the external pressing field.

Expressions (44), (47) allow to make the limit process at \( E \rightarrow 0 \). In the case of the absence of the external pressing field, these solutions have the following form:

\[
E_1(z) \rightarrow E_0 \left\{ 1 + \frac{z - \xi}{2z_0} \right\}^{-1} , \tag{49}
\]

\[
n(z) \rightarrow \frac{E_0^2}{8\pi} \left\{ 1 + \frac{z - \xi}{2z_0} \right\}^{-2} , \quad E_2 = E_0 / \varepsilon,
\]

where

\[
z_0 \equiv (\beta e E_0)^{-1} , \quad E_0 = 4\pi e n_s . \tag{50}
\]

If to compare the expressions (49), (47) and (49), it is easy to see that in the case of the absence of the external pressing field the exponential law of the electric field and charge density above the dielectric surface changes to the weaker power dependence. In this case, the inequality (48) can be written as:

\[
(en_s)^2 \nu^{-1} \beta^{5/2} \ll 1. \tag{51}
\]
Note, that it takes place in the region of the relatively high temperatures and low charge number in the volume above a surface area unit, see eq. (37).

The obtained formulae (38), (41)-(51) are the solution of the problem of the field and nondegenerate charged gas distribution in charged particle system above the plane dielectric surface as in the external pressing field as in its absence. Let us emphasize that the dielectric permittivity does not appear in these expressions. The reason is that the problem is homogeneous along the surface coordinate $\rho$. In the case of inhomogeneity along $\rho$, the solution of equations essentially depends on the sort of the dielectric, i.e., on its permittivity $\varepsilon$. These inhomogeneities may be caused by the inhomogeneities of the surface itself or by inhomogeneity charge distribution on it (or the both reasons simultaneously, see eqs. (8), (11), (20), (21)).

In the case of the degenerate gas, i.e., when the condition (48) or (51) fails, the solution that is obtained earlier is inapplicable. Let us make the following remark relating to this fact. As it is mentioned earlier, the charge density distribution decreases with the distance from the surface. For this reason, in general case described by eq. (10) the gas can be degenerate in the area near the dielectric surface and nondegenerate far from it. The typical distance from the surface that separates these cases can be obtained using the following considerations. As it is well known (see e. g. Ref. [11]), in low temperatures region the temperature expansions are widely used for the calculus of the thermodynamical quantities characterizing the gas. Applying such expansion to the integral over the energy in eq. (10), we obtain:

$$\int_0^{\infty} d\varepsilon \varepsilon^{3/2} \left\{ \exp \beta (\varepsilon - \psi) + 1 \right\}^{-1} \approx \int_0^{\infty} d\varepsilon \left\{ \exp \beta \varepsilon + 1 \right\}^{-1} \approx \frac{2}{5} \psi^{5/2} + \frac{\pi^2}{4} \beta^{-2} \psi^{1/2} - \frac{7\pi^4}{960} \beta^{-4} \psi^{-3/2} + ...$$

(52)

From this expression it is easy to see that such expansion is absolutely useless near the point $z_1$ obtained from the condition

$$\psi(z_1) = \mu + e\varphi_1(z_1) = 0.$$  

(53)

The solution of eq. (10) obtained in quadratures is given by

$$z - \xi = -z_0 \int_{\psi_0}^{\psi} \frac{16\pi}{3} \nu \beta^{-5/2} E^{-2} \int_0^{\infty} dy y^{3/2} \times$$

$$\times \left\{ \exp (y - \zeta) + 1 \right\}^{-1/2} \times \left\{ \exp (y - \zeta) + 1 \right\}^{-1} + 1 \right\}^{-1/2},$$

(54)

where the distance $z_0$ is determined by eq. (19). Taking into account eqs. (33), (51), the expression of the border distance $z_1$ can be written as

$$z_1 = \xi + z_0 \int_{\psi_0}^{\psi} \frac{16\pi}{3} \nu \beta^{-5/2} E^{-2} \int_0^{\infty} dy y^{3/2} \times$$

$$\times \left\{ \exp (y - \zeta) + 1 \right\}^{-1/2} \times \left\{ \exp (y - \zeta) + 1 \right\}^{-1} + 1 \right\}^{-1/2},$$

(55)

where the electrochemical potential $\psi_0$ as the function of temperature $T = \beta^{-1}$ and the external electric field is obtained from the equation (see eqs. (55), (36), (39))

$$4\pi e n_o E^2 - E^2 = \frac{16\pi}{3} \nu \int_0^{\infty} d\varepsilon \varepsilon^{3/2} \left\{ \exp \beta (\varepsilon - \psi_0) + 1 \right\}^{-1}.$$  

(56)

As it is mentioned above, the potential $\varphi_1(z)$ is defined accurate within an arbitrary constant, which can be set equal to zero. Hence, eq. (56) is the expression defining the chemical potential $\mu$, see eq. (49).

As expected, the typical distance $z_1$ (see eq. (55)) is defined by the temperature, the external pressing field and the number of charges above the dielectric surface area unit. Thus, the charge gas is nondegenerate in the region $z \gg z_1$ and degenerate at $z \ll z_1$. Let us point out that the solutions (45)-(50) are obtained assuming the charge gas nondegeneracy in the entire area above the dielectric surface. Therefore, in general case the mentioned expressions describe the charge system only in the region $z \gg z_1$. The charge gas above the dielectric surface can be degenerate even in the case of the absence of the external pressing field. It is easy to see if to analyze the expression (55) at $E \to 0$ with the account of eq. (45).

In the case of the generate charge gas above dielectric surface, eq. (10) according to eq. (52) can be written in more simple form:

$$\frac{\partial \varphi_1}{\partial z} = -\left\{ \frac{32\pi \nu \psi^{5/2} + E^2}{15} \right\}^{1/2},$$  

(57)

$$\psi(z) = \mu + e\varphi_1(z).$$

However, in this case eq. (54) can not be solved analytically, and the numerical integration methods are needed.

Let us show now the influence of the plane dielectric surface charges on the obtained results in the present section of this paper. It is well known that infinitely thin homogeneously charged plate with charge density $\sigma(\xi)$ induces the homogeneous field intensity $E_0 = 2\pi \sigma(\xi)$ in vacuum (in this case the expression $\sigma(\xi)$ shows that the surface plane is described by the equation $z = \xi$). This field has the opposite direction in the opposite sides of the plane. In the case of the charged plane dielectric surface the situation is absolutely similar. E.g., the negatively charged dielectric surface induces the field intensity $E_{2\sigma} = 2\pi |\sigma(\xi)| / \varepsilon$ in
the dielectric and $E_{1\sigma} = -2\pi |\sigma(\xi)|$ above the dielectric surface (see eq. (20)). As mentioned earlier in the present paper, we consider only the cases of the same signs of charges as on the dielectric surface, as in the volume above it (in our case we consider the negative charges). In this case, the field induced by the surface charges repulses the volume charges from the surface. So, the results obtained in the present partition remain useful if we substitute the external electric field in vacuum $E$ for $E - 2\pi |\sigma(\xi)|$ in the expressions (31)-(56),

$$E \to E - 2\pi |\sigma(\xi)|.$$  

(58)

It is easy to see that it is necessary to satisfy the condition

$$E_0 - 2\pi |\sigma(\xi)| > 0$$  

(59)

that provides the possibility of existence of the equilibrium volume charge distribution above the dielectric surface in the repulsive field of the surface charges.

## 4 The charge system above the spatially inhomogeneous dielectric surface

As already mentioned, the spatial inhomogeneities can be caused by the surface heterogeneities or by the inhomogeneous charge distribution on it (or the both reasons simultaneously, see eqs. (11), (15), (20), (24)). Let us consider the mentioned surface inhomogeneities that slightly distort the electric field induced by the charge system above the plane dielectric:

$$\varphi_1(z, \rho) = \varphi_1(z) + \tilde{\varphi}_1(z, \rho), \quad |\varphi_1(z)| \gg |\tilde{\varphi}_1(z, \rho)|,$$

$$\varphi_2(z, \rho) = \varphi_2(z) + \tilde{\varphi}_2(z, \rho), \quad |\varphi_2(z)| \gg |\tilde{\varphi}_2(z, \rho)|,$$

where $\varphi_1(z), \varphi_2(z)$ are the potentials above the dielectric and inside of it, respectively, in the case of the ideally plane dielectric surface with the equation of the profile $z = \xi$ (see eqs. (10), (15)). The obtaining procedure for the potentials $\varphi_1(z), \varphi_2(z)$ and charge density $n(z)$ is described in details in the previous section of the present paper, see eqs. (20) - (59).

The next problem is concerned with the potentials $\tilde{\varphi}_1(z, \rho)$ and $\tilde{\varphi}_2(z, \rho)$ obtaining. For these potentials one can use the eq. (24) and the boundary conditions (20), (24). In terms of the Fourier-transforms $\tilde{\varphi}_1(z, \mathbf{q}), \tilde{\varphi}_2(z, \mathbf{q})$ over coordinate $\rho$ of the potentials $\varphi_1(z, \rho)$ and $\varphi_2(z, \rho)$

$$\tilde{\varphi}_1(z, \rho) = \int d^2q \exp (i\mathbf{q}\rho) \varphi_1(z, q),$$

$$\tilde{\varphi}_2(z, \rho) = \int d^2q \exp (i\mathbf{q}\rho) \varphi_2(z, q)$$  

(60)

the equations (24) have the following form:

$$\frac{\partial^2 \tilde{\varphi}_1(z, q)}{\partial z^2} - q^2 \tilde{\varphi}_1(z, q) = 4\pi e^2 \frac{\partial n(z)}{\partial \mu} \tilde{\varphi}_1(z, q),$$

$$\frac{\partial^2 \tilde{\varphi}_2(z, q)}{\partial z^2} - q^2 \tilde{\varphi}_2(z, q) = 0.$$  

(61)

According to eqs. (20), (24), the boundary conditions concerned with these equations can be written as:

$$\{\tilde{\varphi}_1(z, q) - \tilde{\varphi}_2(z, q)\} = \xi =$$

$$= \left\{\frac{\partial \tilde{\varphi}_2(z, q)}{\partial z} - \xi \frac{\partial \tilde{\varphi}_1(z, q)}{\partial z}\right\} \varepsilon(z),$$

$$-4\pi \left\{\frac{\partial \tilde{\varphi}_1(z, q)}{\partial z} - \xi \frac{\partial \tilde{\varphi}_2(z, q)}{\partial z}\right\} = \xi =$$

(62)

where $\xi(q), \tilde{\sigma}(q, \xi)$ are the Fourier-transforms of the quantities $\xi(\rho)$ and $\tilde{\sigma}(\rho; \xi)$, respectively (see eqs. (14), (18)):

$$\tilde{\sigma}(\rho; \xi) = \int d^2q \exp (i\mathbf{q}\rho) \tilde{\sigma}(q),$$

$$\tilde{\sigma}(\rho; \xi) = \int d^2q \exp (i\mathbf{q}\rho) \tilde{\sigma}(q; \xi).$$  

(63)

Let us consider the electric fields intensity perturbations caused by inhomogeneities of the dielectric surface rapidly decreasing at $z \to \pm \infty$. It is easy to see that the first equation in eq. (61) in general case cannot be solved analytically. But in two particular cases the analytical solution exists. In the first case, we solve eq. (61) at $z \sim \xi$ setting $\partial n(z)/\partial \mu$ equal to its value on the plane surface, $z = \xi$:

$$\frac{\partial n(z)}{\partial \mu} \approx \frac{\partial n(\xi)}{\partial \mu}.$$  

(64)

Such consideration is possible in the case when the typical size of spatial inhomogeneities of the unperturbed charge density $n(\xi)$ considerably larger than the typical size of the spatial inhomogeneities of the potential $\tilde{\varphi}_1(z, q)$ along $z$-axis:

$$\left| \left(\frac{\partial n(z)}{\partial \mu}\right)^{-1} \frac{\partial \tilde{\varphi}_1(z, q)}{\partial z} \right|_{z = \xi} \ll$$

$$\ll \left| \left(\frac{\partial \tilde{\varphi}_1(z, q)}{\partial z}\right) \right|_{z = \xi}.$$  

(65)

Let us return to the discussion of the condition (61) below.

Then, taking into account the assumption of rapidly fading field densities at $z \to \pm \infty$, the solution of eq. (61) can be given in the following form:

$$\tilde{\varphi}_1(z, q) = A_1(q) \exp (-zb(q)),$$

$$\tilde{\varphi}_2(z, q) = A_2(q) \exp (zb(q)),$$  

(66)

where (see eq. (61))

$$b(q) \equiv \sqrt{q^2 + 4\pi e^2 \frac{\partial n(\xi)}{\partial \mu}},$$  

(67)

and $A_1(q), A_2(q)$ are obtained from the boundary conditions (62). To this end, we put the expressions (66)
into the boundary conditions (62) and obtain the following relations for the potentials \( \tilde{\varphi}_1(z, q) \), \( \tilde{\varphi}_2(z, q) \):

\[
\tilde{\varphi}_1(z, q) = \exp\left(-\frac{(z - \xi) b(q)}{\varepsilon q + b(q)}\right) \left\{ [\varepsilon q (E_1(\xi) - E_2(\xi)) + 4\pi (\varepsilon_0 \xi (\xi) + (\partial \sigma(\xi)/\partial z))] \tilde{\xi}(\xi; \xi) + 4\pi \sigma(q; \xi) \right\},
\]

\[
\tilde{\varphi}_2(z, q) = -\exp\left(\frac{(z - \xi) q}{\varepsilon q + b(q)}\right) \left\{ [b(q) (E_1(\xi) - E_2(\xi)) - 4\pi \varepsilon_0 \sigma(q; \xi)] \right\},
\]

where (see eqs. (22), (23))

\[
E_1(\xi) = -\left(\frac{\partial \varphi_1(z)}{\partial z}\right)_{z=\xi}, \quad E_2(\xi) = -\left(\frac{\partial \varphi_2(z)}{\partial z}\right)_{z=\xi}.
\]

Let us emphasize that according to the boundary conditions (20), (21) (see also eqs. (58), (65)) the values of the quantities \( E_1(\xi) \), \( E_2(\xi) \) can be expressed as

\[
E_1(\xi) = E_0 - 2\pi |\sigma(\xi)| > 0, \quad E_2(\xi) = (E_0 + 2\pi |\sigma(\xi)|) / \varepsilon,
\]

where the field intensity \( E_0 \) is defined by the relation (60):

\[
E_0 = E + 4\pi \varepsilon n_s.
\]

Let us remind that the values of the potentials \( \tilde{\varphi}_1(z, q) \), \( \tilde{\varphi}_2(z, q) \) at \( z = \xi \) do not coincide due to the fact that the potential continuity on the surface in the case of its inhomogeneous wavy structure is provided by the inequalities (see eqs. (10)-(20)): \( \varphi_1(z)|_{z=\xi} = \varphi_2(z)|_{z=\xi}, \) \( \delta \varphi_1(\xi, q) = \delta \varphi_2(\xi, q), \)

where \( \delta \varphi_1(\xi, q) \equiv \tilde{\xi}(q) (\partial \varphi_1(z)/\partial z)_{z=\xi} + \tilde{\varphi}_1(\xi, q). \) According to eq. (68), one can get:

\[
\delta \varphi_1(\xi, q) = -\frac{1}{\varepsilon q + b(q)} \left\{ [\varepsilon q E_1(\xi) + b(q) E_1(\xi)] - 4\pi (\varepsilon_0 \xi (\xi) + (\partial \sigma(\xi)/\partial z))] \tilde{\xi}(\xi; \xi) + 4\pi \sigma(q; \xi) \right\},
\]

where \( E_1(\xi), E_2(\xi) \) are still defined by the relations (69). It is easy to see from the obtained formulae (68), (70) that the gas of the volume charges can sufficiently affect on the potential of the electric field near the dielectric surface.

Now let us show that the solution of eq. (61) in the form of (60), (68) is correct. As it is mentioned above, the condition of the existence of such solution is defined by the relation (60). According to eq. (67) it can be expressed as follows:

\[
\left| \frac{\partial n(\xi)}{\partial \mu} \right|^{-1} \frac{\partial}{\partial z} \frac{\partial n(\xi)}{\partial \mu} \ll \sqrt{q^2 + 4\pi e^2 (\varepsilon_0 n(\xi)/\partial \mu)}.
\]

The explicit expression for the derivative \( \partial n(\xi)/\partial \mu \) can be obtained from eqs. (51), (55), (56), (61), (66)

\[52, \ 56\] as in the case of degenerate charge gas above the dielectric surface, as in the case of nondegenerate one. In the second case the condition (71) has a rather simple form:

\[
q^2 \gg \beta e^2 \left\{ (E + 2\pi \varepsilon n_s)^2 + 4\pi^2 e^2 n_s^2 \right\}. \quad (72)
\]

In the case, when the gas of charged Fermi-particles is degenerate at \( z \approx z_1 \) (see eq. (55)) and low temperature expansions (62) take place, we can obtain the following expressions for the volume charge density \( n(z) \) at \( z \approx \xi \) and the electrochemical potential \( \psi_0 \) at \( z = \xi \) (see eqs. (55), (52), (56)):

\[
n(z) \approx \frac{2}{3} \nu \psi^{3/2}, \quad \psi_0 \approx \left\{ \frac{15 E_0^2 - E^2}{32 \pi \nu} \right\}^{2/5}, \quad (73)
\]

where \( \nu \) and \( \psi \) are still defined by the relations (28) with \( Q = -e \), and \( E_0, E \) are expressed by (30). By the use of the expression (73), one can write the condition (71):

\[
E^2 - 52\pi \varepsilon n_s E - 8\pi^2 e n_s^2 \ll \quad (74)
\]

Accouting the solutions (68) are the Fourier-transforms of the potentials \( \tilde{\varphi}_1(z, \rho), \tilde{\varphi}_2(z, \rho) \) (see eq. (51)), the relations (71), (73) in general case are correct for any value of \( q \), including also the value \( q = 0 \).

It is easy to see that the relation (72) does not satisfy such a requirement. The relation (71) can take place at all values of \( q \) in the case of the external pressing field \( E \) that satisfies the following inequality:

\[
0 \leq E \leq E', \quad E' \approx 52\pi \varepsilon n_s. \quad (75)
\]

At \( E > E' \) the expressions (68) do not take place. In this case, as in the case of the condition (72) realization, the equations (61) must be solved by the use of the numerical methods.

The case of the particular interest is the spatially periodic inhomogeneities caused by the dielectric surface. As it is already mentioned in the present paper, such inhomogeneities are concerned with two-dimensional Wigner crystallization. In the most simple case of spatial periodic inhomogeneities the Fourier-transforms of the quantities \( \tilde{\xi}(\rho), \tilde{\sigma}(\rho; \xi) \) (see eq. (69)) can be expressed in the form:

\[
\tilde{\sigma}(q; \xi) = \frac{1}{2} \sum_{\alpha=1}^2 \tilde{\sigma}(q_\alpha; \xi) \left\{ \delta (q + q_\alpha) + \delta (q - q_\alpha) \right\},
\]

\[
\tilde{\xi}(q) = \frac{1}{2} \sum_{\alpha=1}^2 \tilde{\xi}(q_\alpha) \left\{ \delta (q + q_\alpha) + \delta (q - q_\alpha) \right\},
\]

\[
(76)
\]

where \( q_\alpha (\alpha = 1, 2) \) are the vectors of the reciprocal two-dimensional lattice concerned with the spatial
periodic charge distribution on the dielectric surface, \(q_{\alpha\xi}(\alpha = 1, 2)\) are the vectors of the reciprocal two-dimensional lattice concerned with the spatial periodic wavy surface type, and \(\tilde{\sigma}(q_{\alpha\sigma}; \xi), \xi(q_{\alpha\xi})\) are the amplitudes of the corresponding surface heterogeneities. Of course, it is necessary to consider that the conditions \(13\) take place, which in this case can be written as:

\[
q_{\alpha\xi}\xi(q_{\alpha\xi}) \ll 1, \quad q_{\alpha\sigma}\xi(q_{\alpha\sigma}) \ll 1. \tag{77}
\]

Then, putting the expressions \(10\) into eq. \(68\) and making inverse Fourier transformation according to eqs. \(72\), \(74\) for two-dimensional reciprocal lattice with the weak (see eq. \(77\)) spatially periodic inhomogeneities problem (so, the charge density of spatial-periodic homogeneities it is necessary to take into account the possibility of the surface charge presence on it. The influence of the external pressing electric field acting on the system is also taken into consideration. It is shown that the presence of the gas of volume charges essentially influences on the value of the electric field potential in the area near the dielectric surface. Mostly, this fact plays an important role in the description of the deformation of the liquid dielectric surface caused by near-surface charges pressure on it. Authors of the present paper are working on this problem now.

However, in our opinion, the solved problem is useful not only in two-dimensional Wigner crystallization aspect. The problem is worth concerning purely with the academic purposes as it can be related to the number of classical problems of electrodynamics and statistical physics. Due to this fact, in this paper we do not use the results of the real experiments on two-dimensional Wigner crystallization research. The formulations and obtained results in this paper can be used for the research of the influence of the volume charges near the liquid helium surface on the spatial inhomogeneous states of charges, which are adsorbed on the helium surface at the system parameters close to the experimental ones.

**Acknowledgements**

Authors acknowledge financial support from the Consolidated Foundation of Fundamental Research of Ukraine under grant No. 25.2/102 and thank S.V. Peletminsky for valuable discussions.

**References**

[1] E. Wigner. On the interaction of electrons in metals. - Phys. Rev., 46, no.11, p. 1002, (1934).

[2] A.S. Peletminsky, S.V. Peletminsky, Yu.V. Slusarenko. On phase transitions in a Fermi-liquid. II Transition associated with translational symmetry breaking. - Low Temp. Phys., 25, no.5, p. 303, (1999).

[3] Yu.P. Monarkha and V.B. Shikin. Low-dimensional electronic systems on a liquid helium surface (Review). - Sov. J. Low Temp. Phys. 8, no. 6, p. 279, (1982).

[4] Y. Monarkha and K. Kono. Two-dimensional Coulomb liquids and solids. - Berlin: Springer - Verlag, p. 346, (2003).
[5] M.W. Cole, M.H. Cohen. Image-potential-induced surface bands in insulators. - Phys. Rev. Lett., 23, no. 21, p. 1238, (1969).

[6] V.B. Shikin. On helium ions motion near vapour-liquid boundary. - JETP, 58, no. 5, p. 1748, (1970) [in Russian].

[7] T.R. Brown, C.C. Grimes. Observation of cyclotron resonance in surface-bound electrons in liquid helium. - Phys. Rev. Lett., 29, no. 18, p. 1233, (1972).

[8] C.C. Grimes, T.R. Brown. Direct spectroscopy observation of electrons in image-potential state outside liquid helium. - Phys. Rev. Lett., 32, no. 6, p. 280, (1974).

[9] V.S. Edelman. Levitated Electrons. - Sov. Phys. Usp. 23 no.4, p. 227, (1980).

[10] I.E. Tamm. Fundamentals of the theory of electricity, Central Books Ltd, 684 pages, (1980).

[11] L.D. Landau and E.M. Lifshitz. Statistical Physics, 3rd Ed., Pergamon Pres, Oxford (1981), Nauka, Moscow (1980).