STATE-SPACE CONTROLLER DESIGN FOR THE FRACTIONAL-ORDER REGULATED SYSTEM

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Abstract: In this paper we will present a mathematical description and analysis of a fractional-order regulated system in the state space and the state-space controller design based on placing the closed-loop poles on the complex plane. Presented are the results of simulations and stability investigation of this system.

Key words: fractional-order regulated system, fractional calculus, model, state space

1. Introduction

Research of the fractional-order derivatives continued in the last decades not only as a mathematical branch [1, 2, 3, 4], etc., but in many applied domains, e.g. [5, 6] and in control theory too [7, 8, 9], etc. In works [7, 8, 9] the first generalizations of analysis methods for fractional-order control systems were made (s-plane, frequency response, etc.). The following of the above-mentioned works were oriented to the methods of fractional-order system parameters identification, methods of fractional-order controllers synthesis, methods of stability analysis, methods of control of chaotic fractional-order systems, and so on.

In works [14, 15] was presented a state space model described in vector and matrix relations expressing the fractional-order derivatives

\[
x^{(\alpha)}(t) = A \, x(t) + B \, u(t),
\]
\[
y(t) = C \, x(t), \quad t \geq 0.
\]

This description is convenient only for simple models of systems with only one fractional-order derivation, or for spatial type of systems. In work [14] we proposed a state space model of the linear time-invariant one dimensional system which expresses...
the first derivatives in the state space equations and has the **classical state space interpretation** for the fractional-order system too. On the right side of these equations we can then transfer more than one fractional-order derivatives of the state space variables. A disadvantage of this expression is that in time domain we cannot express the state space equations in vector and matrix relations as in previous description. But we can do this in s-plane.

This contribution deals with a mathematical description and analysis of a fractional-order regulated system in the state space and the state-space controller design based on placing the closed-loop poles on the complex plane.

2. **Definition of the system**

For the definition of the control system we consider a simple unity feedback control system illustrated in Fig.1, where \( G_s(s) \) denotes the transfer function of the controlled system and \( G_r(s) \) is the controller transfer function, both integer- or fractional-order. The differential equation of the above closed regulation system for the transfer function

\[
W(s) + G_r(s) G_s(s) = W(s) + G_r(s) = K w(t) + T_d w^{(\delta)}(t) \tag{2}
\]

where \( \alpha, \beta, \delta \) are generally real numbers and \( a_0, a_1, a_2, K, T_d \) are arbitrary constants.

3. **Fractional-order control system with \( PD^\delta \) controller in the state space**

Consider the system described by differential equation (2). After its modification and substitution of state space variables \( x(t) = x_1(t), \dot{x}(t) = x_2(t), \ddot{x}(t) = \ddot{x}_2(t) \) we can derive the following state space model equivalent to model (2)

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) , \\
\dot{x}_2(t) &= -\frac{a_0 + K}{a_2} x_1^{(2-\alpha)}(t) - \frac{T_d}{a_2} x_2^{(1+\delta-\alpha)}(t) - \frac{a_1}{a_2} x_2^{(1+\beta-\alpha)}(t) + \frac{1}{a_2} w^{(2-\alpha)}(t) , \\
y(t) &= K x_1(t) + T_d x_2^{(\delta-1)}(t) ,
\end{align*}
\tag{3}
\]

Of course, we can make other alternative state space models for the same system. For the approximation of the fractional-order derivatives on the right-hand side of equations (3) we can take the relation (4) from e.g. [2, 3, 10]

\[
y^{(\alpha)}(t) \approx h^{-\alpha} \sum_{j=0}^{N(t)} b_j y(t - jh) ,
\tag{4}
\]
where $L$ is "memory length", $h$ is time step of calculation, $N(t) = \min \left\{ \left\lfloor \frac{t}{h} \right\rfloor, \left\lfloor \frac{L}{h} \right\rfloor \right\}$, 
$[z]$ is the integer part of $z$, $b_j = (-1)^j \binom{\alpha}{j}$, and $\binom{\alpha}{j}$ is binomial coefficient. To calculate $b_j$ it is convenient to use the following recurrent relation

$$b_0 = 1, \quad b_j = (1 - \frac{1 + \alpha}{j}) b_{j-1}$$

(5)

After the above-mentioned fractional-order derivatives discretisation (4) and discretisation of the first derivatives on the left-hand side of equations (3) we obtained the simple Euler methods for solving the state space model

$$x_{1,k+1} = x_{1,k} + h x_{2,k},$$

$$x_{2,k+1} = x_{2,k} + h \left( -\frac{a_0 + K}{a_2} h^{\alpha - 2} \sum_{j=0}^{k} b_j x_{1,k-j} - \frac{T_d}{a_2} h^{\alpha - \delta - 1} \sum_{j=0}^{k} c_j x_{2,k-j} \right. \left. + \frac{1}{a_2} h^{\alpha - 2} \sum_{j=0}^{k} b_j w_{k-j} \right)$$

(6)

$$y_k = K x_{1,k} + T_d h^{(1-\delta)} \sum_{j=0}^{k} e_j x_{2,k-j} \quad k \geq 0.$$  

From these equations we can compute state trajectories of the fractional-order control system described above.

4. Design of the $PD^\delta$ controller in the state space

After Laplace transformation we can write state space equations (3) in vector and matrix relations in $s$-plane

$$pX(s) = A(s) X(s) + B(s) W(s),$$

$$Y(s) = C(s) X(s), \quad t \geq 0.$$  

(7)

where matrix $A(s)$ and vectors $B(s), \ C(s)$ are

$$A(s) = \begin{bmatrix} 0 & 1 \\ -\frac{a_0 + K}{a_2} s^{2-\alpha} & -\frac{a_1 s^{1+\beta-\alpha} + T_d s^{1+\delta-\alpha}}{a_2} \end{bmatrix}, \quad B(s) = \begin{bmatrix} 0 \\ \frac{1}{a_2} s^{2-\alpha} \end{bmatrix}, \quad C(s) = \begin{bmatrix} K \\ T_d s^{\delta-1} \end{bmatrix}^T$$

From equations (8) we can derive the overall transfer function

$$\frac{Y(s)}{W(s)} = \frac{C \text{adj}(sI - A) B}{\text{det}(sI - A)}$$

(8)

The characteristic equation of the closed-loop system is determined by solving the determinant in the denominator of the transfer function, which is a fractional-order polynomial in $s$

$$a_2 s^\alpha + a_1 s^\beta + T_d s^\delta + (a_0 + K) = 0$$

(9)
Solving this equation we can find the poles of the closed-loop system with known parameters, or design the PD\(\delta\) controller parameters for desired control system poles.

Consider the system described by differential equation (2) with system coefficients 
\[a_2 = 0.8, a_1 = 0.5, a_0 = 1, \alpha = 2.2, \beta = 0.9\]. The task is to determine the controller parameters \(K, T_d\) and \(\delta\) for desired control system poles \(s_{1,2} = -1\pm 6i\) and the steady state error less than 4\%. Parameter \(K = 24\) can be computed from the equation based on the steady state error \(e_{ss}(\infty)\) of the closed-loop control system 
\[K = (100/e_{ss} - 1)a_0\].

If we introduce the desired system poles \(s_{1,2}\) to the characteristic equation (9) we obtain system of two nonlinear equations from which we can easy derive two equations 
\[\delta = \arctan(2.9839/1.6098)\] and \(T_d^2 = 645,2174\). For the calculation of the controller parameters \(T_d = 6,9407\) and \(\delta = 0,71859\). In Fig.2 and Fig.3 are depicted the unit step responses of the classical numerical solution \([10]\) and the numerical solution of the state space model \([10]\). The obtained state trajectory represents stable focal point for the above-mentioned coefficients of the system.

Assuming the integer-order PD controller we obtain a system of two linear equations for direct calculation of the controller parameters \(K = 36,0854\) and \(T_d = 4,0141\).

If we increased in case of fractional-order PD\(\delta\) controller the requirement on steady state error less than 2\% we would obtain the following controller parameters \(K = 49, T_d = -79,74427\) and \(\delta = -0,55194\). That means, we obtained the controller with weak integrator. But the characteristic equation of such a system has one additional pole \(s_3 = 1.98\) and it follows from the stability analysis that such closed-loop system is unstable. We can verify this fact also with frequency methods for stability investigation of the fractional-order system \([12, 13]\). In this case we can change the desired control system poles \(s_{1,2}\), or we have to proceed in a different way, e.g. as follows.

5. Fractional-order control system with \(PI^\lambda\) controller in the state space

The differential equation of the closed regulation system for the transfer function of the controlled system \(G_s(s) = 1/(a_2s^\alpha + a_1s^\beta + a_0)\) and the fractional-order \(PI^\lambda\) controller \(G_r(s) = K + T_i s^{\lambda - \alpha}\) has the form 
\[a_2 y^{(\alpha+\lambda)}(t) + a_1 y^{(\beta+\lambda)}(t) + (a_0 + K) y^{(\lambda)}(t) + T_i y(t) = K w^{(\lambda)}(t) + T_i w(t)\] (10)
where \(\alpha, \beta, \lambda\) are generally real numbers and \(a_0, a_1, a_2, K, T_i\) are arbitrary constants.
The state space model equivalent to model (10) has the following form:

\[
\begin{align*}
\dot{x}_1(t) &= -x_2(t) + w, \\
\dot{x}_2(t) &= +x_3(t), \\
\dot{x}_3(t) &= \frac{T_i}{a_2} x_1^{(3-\alpha-\lambda)}(t) - \frac{a_0 + K}{a_2} x_2^{(2-\alpha)}(t) - \frac{a_1}{a_2} x_1^{(1+\beta-\alpha)}(t) + \frac{K}{a_2} u_1^{(2-\alpha)}(t), \\
y(t) &= x_2(t), \quad t \geq 0.
\end{align*}
\]

where \( \dot{x}_1(t) = \dot{z}(t) \) is the actuating error signal and \( \dot{x}_2(t) = \dot{v}_1(t), \dot{x}_3(t) = \dot{v}_2(t) \) are the state variables of the controlled system (Fig.4). As above, we can derive the following characteristic equation of the closed-loop system:

\[
a_2 s^{\alpha+\lambda} + a_1 s^{\beta+\lambda} + (a_0 + K)s^\lambda + T_i = 0
\]  

Assume the same controlled system as in the previous section. If we introduce the desired system poles \( s_1, s_2, s_3 \) to the characteristic equation (12) we obtain a system of three nonlinear equations from which we can numerically calculate the controller parameters \( K, T_i \) and \( \lambda \).

6. Conclusion

We have presented a mathematical description of a fractional-order control system in the state space and the state-space controller design based on placing the closed-loop poles on the complex plane. In the design of a fractional-order \( PD^\delta \) regulator it is necessary to pay attention to the question of stability of the control system. In the case of negative values of \( \delta \) the order of the system increases. In addition to the required poles, the system can develop new poles that can render the system unstable.

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