THE INTERSECTION OF 3-MAXIMAL SUBMONIDS

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ABSTRACT. Very little is known about the structure of the intersection of two k-generated monoids of words, even for k = 3. Here we investigate the case of k-maximal monoids, that is, monoids whose basis of cardinality k cannot be non-trivially decomposed into at most k words. We characterize the intersection in the case of two 3-maximal monoids.

1. Introduction

In this paper, we investigate the intersection of three-generated monoids of words in a special case when these monoids are 3-maximal. A monoid of words is k-maximal if its generating set cannot be non-trivially decomposed into at most k (shorter) words. Obviously, the intersection of two finitely generated monoids of words is regular. However, already in the case of free monoids generated by two words, the structure of the intersection can be quite complex as we recall in Theorem 7, see [9, 7]. While monoids of three words have been classified (see [4] for a survey), there is no classification of their intersection. It is useful to note, and we shall use this fact in the paper, that the general question about the structure of the intersection of two k-generated monoids is in fact a question about maximal solvable systems of equations over 2k unknowns, where the left hand sides and right hand sides are formed from disjoint sets of k unknowns respectively. This indicates why the question is so difficult for k = 3, where we have to deal with six unknowns.

It turns out, however, that when the condition of being k-maximal is added, the problem simplifies considerably. In [3], a kind of defect theorem is shown for 2-maximal monoids, see Theorem 9 below. In case of 3-maximal monoids, studied in this paper, we encounter a situation which rather resembles the general case of two two-generated monoids. In fact, there is a close similarity to the related problem of binary equality sets. In [4], it was shown that the binary equality set is either generated by at most two words, or it is of the form \((uwv)*\). While it was later shown in [6] that the latter possibility never takes place for binary equality words, we show in this paper that the set of possibilities given in the previous sentence is the exact description of intersection of two 3-maximal monoids. This setting therefore fits, from the point of its complexity, somewhere between binary equality words, and the intersection of free two-generated monoids.

2. Preliminaries

Let \(\Sigma^* (\Sigma^+ = \Sigma^* \setminus \{\epsilon\})\) resp.) be the free monoid (free semigroup resp.) freely generated by a countable set \(\Sigma\) which will be fixed throughout the paper. As usually, we shall call the set \(\Sigma\) an alphabet, and understand elements of \(\Sigma^*\) (resp. \(\Sigma^+\)) as finite words (finite nonempty words resp.) over \(\Sigma\) with the monoid operation of concatenation. Note however, that \(\Sigma\), understood as the set of generators satisfying \(\Sigma \subseteq \Sigma^*\), is the set of words of length one, rather than a set of letters.

We say that a word \(u\) is a prefix (res. proper prefix) of \(w\) and we write \(u \leq w\) (resp. \(u < w\)), if \(w = uz\) for some \(z \in \Sigma^*\) (resp. \(z \in \Sigma^+\)). We say that \(u\) is a

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suffix of \( w \) if \( w = zu \) for some \( z \in \Sigma^* \). Two words \( v \) and \( w \) are prefix comparable iff either \( v \leq w \) or \( w \leq v \). A word \( w \) is primitive if \( w = v^n \) implies \( n = 1 \) and \( w = v \), otherwise it is called a power. If we consider pairs of words, we say that \((u, v) \in \Sigma^* \times \Sigma^*\) is a prefix (resp. proper prefix) of \((r, s) \in \Sigma^* \times \Sigma^*\), and we write \((u, v) \leq (r, s)\) (resp. \((u, v) \prec (r, s)\)), if \( u \leq r \) and \( v \leq s \) (resp. \( u \prec r \) and \( v \prec s \)).

Given \( u, v \in \Sigma^* \), by \( u \wedge v \) we denote the longest common prefix of \( u \) and \( v \). Let \( u \in \Sigma^* \), by \( \text{first}(u) \) we denote the first letter of \( u \).

Given a subset \( X \) of \( \Sigma^* \), by \( X^* \) we denote the submonoid of \( \Sigma^* \) generated by \( X \). Conversely, given a submonoid \( M \) of \( \Sigma^* \), there exists a unique minimal (w.r.t. the set inclusion) generating set \( B(M) \) of \( M \), called the basis of \( M \), namely

\[
B(M) = (M \setminus \{ \varepsilon \}) \setminus (M \setminus \{ \varepsilon \})^2.
\]

That is, the basis of \( M \) is the set of all nonempty words of \( M \) that cannot be written as a concatenation of two nonempty words of \( M \). For an arbitrary set \( X \subseteq \Sigma^* \), we shall write \( B(X) \) instead of \( B(X^*) \). The cardinality of \( B(X) \) is the rank of \( X \), denoted \( r(X) \).

A submonoid \( M \) of \( \Sigma^* \) with the basis \( B \) is said to be free if any word of \( M \) can be uniquely expressed as a product of elements of \( B \). The basis of a free monoid is called a code.

It is well-known (see [13]) that for any set \( X \subseteq \Sigma^* \) there exists the smallest free submonoid \( (X)_f \) of \( \Sigma^* \) containing \( X \). It is called the free hull of \( X \). The basis of \( (X)_f \) is called the free basis of \( X \), denoted by \( B_f(X) \). The cardinality of \( B_f(X) \) is called the free rank of \( X \) and denoted by \( r_f(X) \).

For \( w \in (X)_f \), let \( \text{first}_X(w) = b_1 \) where \( w = b_1b_2 \cdots b_n \), \( b_i \in B_f(X) \), be the unique factorization of \( w \) into elements of \( B_f(X) \). The words \( b_1, b_1b_2, \ldots, b_1b_2 \cdots b_n \) are called \( X \)-prefixes of \( w \). We write \( u \prec_X w \) if \( u \) is a \( X \)-prefix of \( w \). Moreover, given \( u, w \in (X)_f \), by \( u \wedge_X w \) we denote the longest common \( X \)-prefix of \( u \) and \( w \).

**Example 1.** Let \( X = \{ abcac, bab, ab, cacabcacb, ca \} \). The free basis is \( B = \{ ab, b, ca, cac \} \), hence \( r_f(X) = 4 \). For \( u = ab \cdot ca \cdot b \cdot cac \cdot b \) and \( w = ab \cdot cac \cdot ca \cdot ca \cdot ab \in X^* \), we have \( \text{first}_X(u) = ab \), \( u \wedge_X w = abcac \) and \( u \wedge_X w = ab \).

We have the following well-known lemma.

**Lemma 2.** Let \( X \) a finite set of \( \Sigma^* \) and \( B \) its free basis. Then for each \( y \in B \) there exists \( u \in X \) such that \( \text{first}_X(u) = y \).

In order to see the importance of the above lemma, let us define the free graph of a finite set \( X \subseteq \Sigma^* \) as the undirected graph \( G(X) = (X, E_X) \) without loops where \( E_X = \{ [u, w] \in X \times X \mid u \neq w \text{ and } \text{first}_X(u) = \text{first}_X(w) \} \).

Let \( c(X) \) be the number of connected components of \( G(X) \). By Lemma 2 we now have that

\[
r_f(X) = c(X),
\]

which immediately implies the Defect Theorem claiming that \( r_f(X) < |X| \) if \( X \) is not a code (cf. [11] and [2]).

**Example 3.** Consider \( X \) of the Example [1] The free graph \( G(X) \) has a unique edge \([abcc, ab]\) connecting the only two words starting with \( ab \in B \). Note that there is no edge between \( ca \) and \( cacabcacb \), since \( \text{first}_X(ca) = ca \neq cac = \text{first}_X(cacabcacb) \).

The free graph is a frequently used tool in the paper because it allows us to easily establish the free rank of a set by considering the properties of the edges of the associated free graph.
3. \( k \)-maximal Monoids

In this section we study \( k \)-maximal submonoids introduced in [3]. With \( \mathcal{M}_k \) we denote the family of submonoids \( M \) of \( \Sigma^* \) of rank at most \( k \).

Definition 4. (cf. [3]) A submonoid \( M \in \mathcal{M}_k \) is \( k \)-maximal if for every \( M' \in \mathcal{M}_k \), \( M \subseteq M' \) implies \( M = M' \).

In other words, the elements of the basis of \( M \) cannot be nontrivially factored into at most \( k \) words.

Example 5. For every word \( v \in \Sigma^+ \), the submonoid \( \{v\}^* \) (denoted simply by \( v^* \)) is 1-maximal if and only if \( v \) is a primitive word.

The submonoid \( \{a, c, b, d\}^* \) is 3-maximal, whereas \( \{a, c, b, d, cd\}^* \) is not 3-maximal since it is contained in \( \{a, c, b, d\}^* \).

Let \( |X| = |\text{alph}(X)| = k \), where \( \text{alph}(X) \) is the subset of letters of \( \Sigma \) occurring in the words of \( X \). Then \( X^* \) is \( k \)-maximal if and only if \( X^* = \text{alph}(X)^* \). Also, a \( k \)-maximal submonoid is obviously generated by primitive words. On the other hand, a finite set of \( k \) primitive words does not necessarily generate a \( k \)-maximal submonoid of \( \Sigma^* \) as we can see in Example [5].

In [3], it is proved that the basis of a \( k \)-maximal submonoid of \( \Sigma^* \) is a bifix code, in particular its free rank is \( k \). We repeat the proof here.

Proposition 6. Let \( X \) be the basis of a \( k \)-maximal submonoid. Then, \( X \) is a bifix code.

Proof. If \( uv, u \in X \) then \( X^* \subseteq Y^* \) where \( Y \) is obtained from \( X \) by replacing \( uv \) with \( v \). Hence \( X^* \) is not \( k \)-maximal, since \( v \notin X \). Similarly for suffixes. \( \square \)

The inverse is not true, see again Example [5] where \( \{a, c, b, d, cd\} \) is a bifix code.

Submonoids generated by two words, i.e., elements of \( \mathcal{M}_2 \), have been extensively studied in the literature (cf. [10], [9], [12], [4]) and play an important role in many fundamental aspects of combinatorics on words. It is known (see [9] and [7]) that if \( X \) and \( U \) have free rank 2, then the intersection \( X^* \cap U^* \) is a free monoid generated either by at most two words or by an infinite set of words. More formally, we have the following theorem.

Theorem 7. Let \( X = \{x, y\} \) and \( U = \{u, v\} \) be two sets of \( \Sigma^* \) with free rank 2, then \( X^* \cap U^* \) is one of the forms

\[ X^* \cap U^* = \{\gamma, \beta\}^*, \quad \text{for some } \gamma, \beta \in \Sigma^*; \]

\[ X^* \cap U^* = (\beta_0 + \beta(\gamma(1 + \delta + \cdots + \delta^t))\tau)^*, \quad \text{for some } \beta_0, \beta, \gamma, \delta, \tau \in \Sigma^* \quad \text{and some } t \in \mathbb{N}. \]

Example 8. Let \( X_1 = \{abca, bc\} \) and \( U_1 = \{a, bcab\} \). One can verify that \( X_1^* \cap U_1^* = \{abcabc, bcabc\}^* \). Let \( X_2 = \{aab, aba\} \) and \( U_2 = \{a, baaba\} \), then \( X_2^* \cap U_2^* = \{a(ababa)^*baaba\}^* \). Note that the submonoids here considered are not \( 2 \)-maximal. Indeed, \( X_1^*, U_1^* \subseteq \{a, bc\}^* \) and \( X_2^*, U_2^* \subseteq \{a, b\}^* \).

To our knowledge nothing is proved in general for the intersection of two monoids of free rank 3. In [8], some properties of codes with three elements are studied.

Let us turn our attention to the intersection of \( k \)-maximal submonoids. For the intersection of 1-maximals, that is, for the submonoids in \( \mathcal{M}_1 \), we have the following important property: If \( x^* \) and \( u^* \) are 1-maximal submonoids (i.e., \( x \) and \( u \) are primitive words) then \( x^* \cap u^* = \{\varepsilon\} \). A generalization of this result, given in [3], to the case of 2-maximal submonoids is the following.
Theorem 9. Let $X = \{x, y\}$ and $U = \{u, v\}$, with $X \neq U$, be such that $X^*$ and $U^*$ are 2-maximal submonoids of $\Sigma^*$. If $X^* \cap U^* \neq \epsilon$, then there exists a unique primitive word $z \in \Sigma^+$ such that $X^* \cap U^* = z^*$.

Example 10. Let $X^* = \{abc, cb\}$ and $U^* = \{abc, bcb\}$ be two 2-maximal submonoids of $\Sigma^*$, then their intersection is $\{abc, bcb\}^*$.

The following example shows that Theorem 9 cannot be generalized to any $k > 2$.

Example 11. For $k > 2$, let $X = \{a, b, cd, ce, cf\}$ and $U = \{ac, bc, da, ea, fa\}$ be two 2-maximal submonoids and $X^* \cap U^* = \{acda, acea, acfa, bcda, bcea, bcfa\}^*$. Similar examples are easily found for $k > 5$.

In the following section, we characterize the intersection of two 3-maximal submonoids.

4. The Intersection of Two 3-maximal Submonoids

In what follows, $X = \{x, y, z\}$ and $U = \{u, v, w\}$ will be two distinct three-element subsets of $\Sigma^+$ such that $X^*$ and $U^*$ are 3-maximal. Let also $Z = X \cup U$.

We have the following lemma.

Lemma 12. The free rank of $Z$, that is, the number of connected components of $G(Z)$, is more than three. Formally, $3 < r_f(Z) = c(Z)$.

Proof. If $r_f(Z) \leq 3$ then the inclusions $U^* \subseteq Z^*$ and $X^* \subseteq Z^*$ imply that $X^*$ and $U^*$ are not 3-maximal unless $X^* = U^* = Z^*$ which is excluded by the hypothesis that $X$ and $U$ are distinct.

When we search for the elements of $X^* \cap U^*$ we are searching for those words that can be decomposed both into words $x$, $y$ and $z$, and into words $u$, $v$, $w$. Consider, as an example, the sets $X = \{abbc, da, db\}$ and $U = \{abca, b, cdad\}$. Then $abbcabcbdb$ is such a word as can be seen from its factorizations $abbc \cdot abbc \cdot da \cdot db = abca \cdot b \cdot cdad \cdot b$.

Double factorizations of this kind are best dealt with using two ternary morphisms as follows. We set $A = \{a, b, c\}$ and define morphisms $g, h : A^* \to \Sigma^*$ by

$$
g(a) = x \quad h(a) = u$$
$$g(b) = y \quad h(b) = v$$
$$g(c) = z \quad h(c) = w.
$$

For better readability, we use the boldface style for elements of $A^*$. The example above is then captured by the equality $g(\text{aabc}) = h(\text{abcb}) = \text{abcbabcbdb}$. That is, the word $\text{abcbabcbdb}$ has the structure $\text{aabc}$ if considered in $X^*$ and $\text{abcb}$ if considered in $U^*$.

We say that a morphism $g : A^* \to \Sigma^*$ is marked if for each pair of letters $a_1 \neq a_2 \in A$ we have $\text{first}(g(a_1)) \neq \text{first}(g(a_2))$. Furthermore, if $X$ is a finite set of $\Sigma^*$ and $B$ its free basis we say that the morphism $g$ is $X$-marked if for each pair of letters $a_1 \neq a_2 \in A$ we have $\text{first}_X(g(a_1)) \neq \text{first}_X(g(a_2))$.

Let $g, h : A^* \to \Sigma^*$ be two morphisms. The coincidence set of $g$ and $h$ is the set defined as follows

$$C(g, h) = \{(r, s) \in A^+ \times A^+ | g(r) = h(s)\}.$$
The pairs of the coincidence set are called solutions. A solution is minimal if it cannot be written as the concatenation of other solutions. That is, if \((u, v) < (r, s)\), then \((u, v)\) is not a solution. Clearly, \(C(g, h)\) is freely generated by the set of minimal solutions.

The property of \(k\)-maximality guarantees (by Proposition 4) the following lemmas that are responsible for a relatively simple structure of the intersection. In particular, complications related to the second case of Theorem 7 are avoided.

**Lemma 13.** \(h(u) \leq h(u')\) iff \(u \leq u'\).

**Proof.** If \(u < u'\) trivially \(h(u) \leq h(u')\). Viceversa, let \(h(u) \leq h(u')\), if there exist \(a \neq a' \in A\) such that \(u = pau_1\) and \(u' = pa'u'_1\). Then \(h(au_1) < h(a'u'_1)\) which implies that \(h(a)\) and \(h(a')\) are prefix comparable, a contradiction with Proposition 4.

**Lemma 14.** Let \((r, s)\) and \((r', s')\) be two distinct minimal solutions. Then \(r\) and \(r'\) are not prefix comparable, and \(s\) and \(s'\) are not prefix comparable.

**Proof.** Assume that \(r\) and \(r'\) are prefix comparable, and assume, without loss of generality, that \(r' = rq\), with \(q \in A^+\). Then \(g(r') = g(r)g(q) = h(s)g(q) = h(s')\) and \(h(s) < h(s')\). It follows by Lemma 13 that \(s < s'\), hence \((r', s')\) is not minimal. Similarly, we prove that \(s\) and \(s'\) are not prefix comparable.

| A | C | C | A |
|---|---|---|---|
| a | b | c | d |
| a | b | c | d |

| A | B |
|---|---|
| a | b |
| a | c |

**Figure 1.** A representation of the solution \((acca,acb)\) and the free graph of morphisms of Example 15.

**Example 15.** Let

\[
g(a) = ab \quad h(a) = abc \\
g(b) = cb \quad h(b) = dab \\
g(c) = cd \quad h(c) = dc.
\]

The pair \((acca,acb)\) is a solution. Indeed \(g(acca) = abcdab = h(ach)\). See Figure 1 for a representation of the solution and the free graph. The free basis of \(Z\) is \(B = \{ab, c, cb, d\}\) and we highlight the decomposition into the free basis of \(Z\) by different colors. The edges of \(G_Z\) are \(E_z = \{[g(a), h(a)], [h(b), h(c)]\}\). One can verify that the set of minimal solutions is \(\{ac^ib, ac^{i+1}a\mid i \geq 0\}\) and the intersection is therefore \((abc(dc) dab)^+\).

This way, the problem of finding the intersection \(X^+ \cap U^+\) is reduced to the problem of finding minimal elements of the coincidence set of morphisms \(g\) and \(h\). Indeed, when we find a minimal solution \((r, s)\), with \(r, s \in A^+\), then \(g(r)\) (which is equal to \(h(s)\)) is an element of the minimal generating set of the intersection \(X^+ \cap U^+\).

As we have seen in Example 15, the intersection of two \(3\)-maximal submonoids can be infinitely generated. We shall see that in the case of a finite number of generators, the cardinality is at most two. A trivial example of a two generated intersection is \(\{a, b, c\}^+ \cap \{a, b, d\}^+ = \{a, b\}^+\). A less trivial example is the following.
Example 16. Let

\[ g(a) = ab \quad h(a) = abbc \]
\[ g(b) = bcdd \quad h(b) = abcb \]
\[ g(c) = cbdd \quad h(c) = ddab. \]

There are only two minimal solutions, namely \((aba, ac)\) and \((aca, bc)\), hence the submonoid intersection is finitely generated by \(\{abbcddab, abcbddab\}\). The free basis of \(Z\) is \(B = \{ab, ac, bd, cd, da\}\), see Figure 2 for representations of the two solutions and the free graph.

\[
\begin{array}{cccc|ccc}
  & a & b & b & c & d & d & a & b \\
 a & b & b & c & d & d & a & b \\
 a & b & b & c & d & d & a & b \\
\end{array}
\]

Figure 2. A representation of the solutions \((aba, ac)\) and \((aca, bc)\) and the free graph of morphisms of Example 16.

In what follows, we equivalently refer to \(X\) (resp. \(U\)) and \(g(A)\) (resp. \(h(A)\)). Since \(g(a), g(b), g(c) \in (Z)_f\) and \(h(a), h(b), h(c) \in (Z)_f\) by definition, we have that \(w \in (Z)_f\) for any element \(w\) of the intersection.

As mentioned before, we often use the free graph of \(G_Z\) as the source of information about the free basis of \(Z\). The set \(V_Z\) of nodes is the union of the images \(g(A)\) and \(h(A)\). In figures, we graphically arrange nodes in \(V_Z\) in two rows containing elements from \(g(A)\) and \(h(A)\) respectively. We know that the number of connected components is the free rank of \(Z\), which is at least four. Moreover, we naturally distinguish two different kinds of edges. The edges that involve nodes in the same set, either \(g(A)\) or \(h(A)\), are horizontal edges, and the edges that involve one node of \(g(A)\) and one of \(h(A)\) are vertical edges.

The following two observations are immediate:

- A morphism \(g\) is \(Z\)-marked iff there are no horizontal edges in the corresponding row. Indeed, by definition, \([g(a_1), g(a_2)] \in E_Z\) iff \(\text{first}_Z(g(a_1)) = \text{first}_Z(g(a_2))\). Analogously for the morphism \(h\).
- A solution creates a vertical edge. Indeed, if \((r, s) \in C(g, h)\) we have \(\text{first}_Z(g(r_1)) = \text{first}_Z(h(s_1))\) and \([g(r_1), g(s_1)] \in E_Z\), where \(r_1 = \text{first}(r)\) and \(s_1 = \text{first}(s)\).

This implies the following property of our morphisms.

**Lemma 17.** If \(C(g, h) \neq \emptyset\) then either \(g\) or \(h\) is \(Z\)-marked. Moreover, if \(h\) is not marked then there exist exactly two letters \(a_1, a_2 \in A\) such that \(\text{first}_Z(h(a_1)) = \text{first}_Z(h(a_2))\).

Figure 3. Free graphs with a nonempty coincidence set and two horizontal edges.
Proof. Since the set of solutions is nonempty, there is at least one vertical edge in the free graph of $Z$. Since the free rank of $Z$ is at least four, the free graph cannot contain two horizontal edges (see Figure 3). The claim follows. □

Examples 15 and 16 shows two cases in which the morphism $g$ is $Z$-marked and $h$ is not. The following example shows two $Z$-marked morphisms $g$ and $h$ which have two minimal solutions.

![Figure 4](image-url)

**Figure 4.** A representation of the solutions $(ab, a)$, $(c, cb)$ and the free graph of two $Z$-marked morphisms from Example 18.

**Example 18.** Let

\[
\begin{align*}
g(a) &= aa & h(a) &= aabc \\
g(b) &= bc & h(b) &= ab \\
g(c) &= dab & h(c) &= d.
\end{align*}
\]

The free basis of $Z$ is $B = \{aa, ab, bc, d\}$, $g$ and $h$ are both $Z$-marked, and the only two minimal solutions are $(ab, a)$, $(c, cb)$. In such a case each minimal solution introduces a vertical edge, which yields four connected components (cf. Figure 4).

By symmetry, we shall suppose in what follows that $g$ is $Z$-marked, first$_Z(h(a)) \neq$ first$_Z(h(c))$ and first$_Z(h(b)) \neq$ first$_Z(h(c))$.

Now we introduce the key ingredient of the proof of our theorem, namely the definition of the critical overflow which was first introduced in [4] (see also [13, pp. 347–351]). We say that the word $o \in \Sigma^*$ is a critical overflow if $g(u) = h(v)o$, for some $u, v \in A^*$, and there are pairs $(u_1, u_2), (v_1, v_2)$ in $A^* \times A^*$ such that first$(u_1) \neq$ first$(u_2), first(v_1) \neq first(v_2)$ and both $g(uu_1) = h(vv_1)$ and $g(uu_2) = h(vv_2)$. Moreover, we say that $o$ is a critical overflow on $(u, v)$.

Informally, if $o$ is a critical overflow on a pair $(u, v)$, then $(u, v)$ is a prefix of at least two distinct minimal solutions $(uu_1, vv_1)$ and $(uu_2, vv_2)$. It represents the situation when the continuation of $(u, v)$ is not given uniquely during the construction of the minimal solution neither for $u$ nor for $v$.

![Figure 5](image-url)

**Figure 5.** A critical overflow of morphisms of Example 19.

**Example 19.** Let $g(a) = abc, g(b) = bab$ and $g(c) = dcb, h(a) = ab, h(b) = cb$ and $h(c) = cd$. Then $c$ is a critical overflow on $(u, v) = (a, a)$ and two minimal solutions are $(ab, aba)$ and $(ac, acb)$. See Figure 5 for a representation.
Remark 20. Since $g$ and $h$ are morphism and $(Z)_f$ is free, it follows that the critical overflows belongs to $(Z)_f$.

The previous remark is a basic trivial property of free monoids and its free basis but it is fundamental for the proof of the following results that characterize the critical overflows in our setting and the corresponding properties of the free graph.

Remark 21. For sake of completeness, we should also consider the case when $o$ is nonempty and $g(u)o = h(v)$ in the definition of the critical overflow. Note however, that such a situation is excluded by the hypothesis that $g$ is marked.

Proposition 22. If $(r, s)$ and $(r', s')$ are two distinct minimal solutions then there is a critical overflow $o$ on $(u, v)$, with $u = r \land r'$ and $v = s \land s'$. Therefore, $h$ is $Z$-marked iff $o$ is an empty overflow.

Proof. By lemma [14] the components of the two minimal solutions are not prefix comparable respectively. Therefore $r = uu_1, r' = uu_2, s = vv_1$ and $s' = vv_2$ where $u = r \land r', v = s \land s'$, and all $u_1, u_2, v_1, v_2$ are nonempty. Let $a = first(u_1), a' = first(u_2), b = first(v_1)$ and $b' = first(v_2)$ where $a \neq a'$ and $b \neq b'$. The case $u = \varepsilon$ and $v \neq \varepsilon$ is excluded by the assumption that $g$ is marked. We have the following cases:

- If $u = v = \varepsilon$, then the empty word is a critical overflow on $(\varepsilon, \varepsilon)$. Since $G_Z$ has two vertical edges $[g(a), h(b)]$ and $[g(a'), h(b')]$, it cannot have an horizontal edge, hence $h$ is $Z$-marked.
- If $u \neq \varepsilon$ and $v = \varepsilon$, then $g(u)$ is a nonempty critical overflow on $(u, \varepsilon)$. We have $h(b) \neq h(b')$, but $first_Z(h(b)) = first_Z(h(b'))$ i.e., $h$ is not $Z$-marked.
- Finally, if both $u \neq \varepsilon$ and $v \neq \varepsilon$, then we have $h(v) < g(u)$ because $g$ is marked and moreover there is a nonempty critical overflow $o$ with $g(u) = h(v)o$. By Remark [20] we have that $first_Z(o) = first_Z(h(b)) = first_Z(h(b'))$, hence $h$ is not $Z$-marked.

\[\]

Figure 6. Free graphs in the three cases of critical overflows.

Remark 23. We can reformulate the three cases of critical overflows of the previous proof in terms of properties of $G_Z$ as follows. Let $(r, s)$ and $(r', s')$ be two distinct minimal solutions and $u = r \land r', v = s \land s'$. Let $r = uu_1, r' = uu_2, s = vv_1$ and $s' = vv_2, a = first(r_1), a' = first(r_1'), b = first(s_1)$ and $b' = first(s_1')$ where $a \neq a'$ and $b \neq b'$. Then,

1. If $u = v = \varepsilon$, $G_Z$ has two vertical edges $[g(a), h(b)]$ and $[g(a'), h(b')]$ (see the first case in Figure [6]. Note that Example [15] with Figure [4] show such a situation.\[\]
2. If $u \neq \varepsilon$ and $v = \varepsilon$, $G_Z$ has two vertical edges $[g(first(u)), h(b)]$ and $[g(first(u)), h(b')]$ and an horizontal edge $[h(b), h(b')]$ creating a connected component of three nodes (see the second case in Figure [6]. Example [16] with Figure [2] verify such a case.\[\]
3. Finally, if both $u \neq \varepsilon$ and $v \neq \varepsilon$, $G_Z$ has a vertical edge $[g(first(u)), h(first(v))]$ and a horizontal edge $[h(b), h(b')]$ (see the first case in Figure [6]. Note that Example [15] and Figure [4] show such a situation.\[\]

Note that in any of these cases $G_Z$ cannot have further edges.
Lemma 24. Let \( o \) be a critical nonempty overflow such that \( g(u) = h(v)o \) with \( u, v \in A^* \). Let \( u_1, u_2, v_1, v_2 \in A^+ \) be such that \( g(uu_1) = h(vv_1) \) and \( g(uu_2) = h(vv_2) \)

\[
\begin{align*}
\alpha &= first(u_1) \neq first(u_2) = a', \\
\beta &= first(v_1) \neq first(v_2) = b'.
\end{align*}
\]

Then

\[
o = h(b) \land_Z h(b').
\]

Proof. Note that \( o \) is prefix comparable with both \( h(b) \) and \( h(b') \). If \( h(b) \leq o \) or \( h(b') \leq o \), then also \( h(b) \) and \( h(b') \) are prefix comparable, contradicting Proposition 20. Moreover \( o \leq h(b) \land_Z h(b') \), by Remark 20.

Let \( o < h(b) \land_Z h(b') \) and let \( o = (h(b) \land_Z h(b'))o' \). Then first_Z(g(a')) = first_Z(g(a')'), a contradiction with \( g \) being \( Z \)-marked. Then \( o = h(b) \land_Z h(b') \).

By Lemma 17 and Lemma 24, we have the uniqueness of the critical overflow.

We can now prove the main result of the paper.

Theorem 25. Let \( X = \{x, y, z\}^* \) and \( U = \{u, v, w\}^* \) be different 3-maximal submonoids of \( \Sigma^* \). Then

\[
X^* \cap U^* = \{\alpha, \beta\}^*, \text{ for some } \alpha, \beta \in \Sigma^*
\]
or

\[
X^* \cap U^* = \{\alpha\gamma\beta\}^*, \text{ for some } \alpha, \beta, \gamma \in \Sigma^*.
\]

Proof. Let \((r_1, s_1)\), \((r_2, s_2)\) and \((r_3, s_3)\) be three minimal solutions. If two of them, say \((r_1, s_1)\) and \((r_2, s_2)\) are such that \( r_1 \land r_2 = \varepsilon \) then, by Remark 23 case 1 \( G_Z \) has two vertical edges and cannot have others edges. Hence, if \( r_1 \land r_3 = \varepsilon \) and \( r_2 \land r_3 = \varepsilon \) then, by Remark 23 case 1 \( G_Z \) must have another vertical edge hence we have a contradiction. If \( r_1 \land r_3 = \varepsilon \) and \( r_2 \land r_3 \neq \varepsilon \) (resp. \( r_1 \land r_3 \neq \varepsilon \) and \( r_2 \land r_3 = \varepsilon \)), then, by Remark 23 case 3 there is a nonempty critical overflow i.e. \( h \) is not \( Z \)-marked and a horizontal edge exists, again a contradiction.

It follows that \( r_1 \land r_2 \neq \varepsilon \). Let

\[
u = r_1 \land r_2 \text{ and } v = s_1 \land s_2
\]

and

\[
u' = r_2 \land r_3 \text{ and } v' = s_2 \land s_3.
\]

By Proposition 22, we have

\[
g(u) = h(v)o \text{ and } g(u') = h(v')o.
\]

where \( o \) is the (unique) critical overflow.

Moreover, by Lemma 14 there exist \( u_1 \neq u_2 \neq \varepsilon, v_1 \neq v_2 \neq \varepsilon \) with

\[
a_1 = first(u_1) \neq first(u_2) = a_2, \quad b_1 = first(v_1) \neq first(v_2) = b_2
\]

**Figure 7.** The representation of three minimal solutions.
such that
\[(r_1, s_1) = (uu_1, vv_1), \quad (r_2, s_2) = (uu_2, vv_2)\]
(cf. Figure 7(a)), and there exist \(u'_1 \neq u'_2 \neq \varepsilon, v'_1 \neq v'_2 \neq \varepsilon\) with
\[a'_1 = \text{first}(u'_1) \neq \text{first}(u'_2) = a'_2, \quad b'_1 = \text{first}(v'_1) \neq \text{first}(v'_2) = b'_2\]
such that
\[(r_2, s_2) = (u'u'_1, v'v'_1), \quad (r_3, s_3) = (u'u'_2, v'v'_2)\]
(cf. Figure 7(b)).

First, we prove that \(u \neq u'\). Indeed, if \(u = u'\) then, by (2), we have \(v = v'\), with \(b_1 \neq b_2 \neq b'_2\). By Lemma (24) we have \(o = h(b_1) \land_Z h(b_2) = h(b_2) \land_Z h(b'_2)\), i.e., we have three different elements of \(h(A)\) having a nonempty common prefix. Then \(G_Z\) has two distinct horizontal edges, a contradiction.

If \((u, v)\) is such that for any \((u, v) < (u, v), g(\overline{u}) \neq h(\overline{v})o\).
1. \((u, v) < (u, v)\) and \((u, v) < (u, v) < (uu_1, vv_1), g(\overline{uu_1}) \neq h(\overline{vv_1})\).
2. \((u, v) < (upu_1, vq\overline{v_1}) \mid i \geq 0\) is the set of all the minimal solutions, i.e., we prove that a pair \((r, s)\) is a minimal solution if \((r, s) = (upu_1, vq\overline{v_1})\) for a certain \(i \geq 0\).
3. If \((r, s) \neq (uu_1, vv_1)\) and \((r, s) \neq (upu_1, vq\overline{v_1})\), we have that \(r \land uu_1 \neq \varepsilon\) and \(r \land upu_1 \neq \varepsilon\). Indeed we saw that for any three minimal solutions the first components must have a nonempty common prefix.

\[
\begin{array}{c|c|c}
\hline
\text{g(u)} & \text{g(u)} & \text{h(v)} \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\hline
\text{g(u)} & \text{g(p)} & \text{g(u)} & \text{h(v)} \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
\hline
\text{g(u)} & \text{g(p)} & \text{g(p)} & \text{g(u)} & \text{h(v)} \\
\hline
\end{array}
\]

Figure 8. Three minimal solutions \((upu_1, vq\overline{v_1})\), with \(i = 0, 1, 2\).

If \(r \land uu_1 = \overline{u} < u\), then, by Proposition (22) and Lemma (24) we have \(g(\overline{u}) = h(\overline{v})o\), where \(\overline{v} = s \land vv_1\). By (2), we have \(\overline{v} < v\) which is a contradiction with the assumption (1).

By Proposition (22) and Lemma (24) we have \(g(\overline{u}) = h(\overline{v})o\), with \(q_1 < q\), against assumption (2).

If there exists \(k \in \{0, 1\}\) such that \(\overline{u} = r \land up^ku_1 = up^kt_1\), with \(t_1 < u_1\), then, by Proposition (22) and Lemma (24) we have \(g(\overline{u}) = h(\overline{v})o\), where \(\overline{v} = vq^kw_1\), with \(w_1 < v_1\) from (2). Since \(g(up^kt_1) = h(vq^kw_1,o)\) it follows that \(g(ut_1) = h(vw_1,o)\) against assumption (3).

We can conclude that \((r, s) = (upt, vqw)\), where \(t, w \neq \varepsilon\), and \((ut, vw)\) is a minimal solution. By induction, it follows that \((t, w) = (p'u_1, q'v_1)\), for some \(i > 0\).
Finally, \((up'u_1, vq'v_1)\), is minimal for each \(i \geq 0\). Indeed, if for some \(i\) there exists a minimal solution \((r, s) < (up'u_1, vq'v_1)\), as we have seen, \((r, s) = (up'u_1, vq'v_1)\) for some \(j \geq 0\). It follows that either \(u_1 < p\) or \(p < u_1\). In the first case we have the contradiction \(uu_1 < upu_1\), i.e. \(r_1\) and \(r_2\) are prefix comparable, in the second case we contradict \(u = r_1 \land r_2\). The thesis follows with \(\alpha = g(u), \gamma = g(p)\) and \(\beta = g(u_1)\). □

5. Conclusions

The hypothesis of \(k\)-maximality considerably simplifies the structure of the intersection of monoids and gives an interesting connection with the generation of binary equality set, in the case \(k = 3\). Our proof also shows the importance of the free graph in this context. Together with combinatorial properties of \(k\)-maximal monoids investigated in [3], this is promising for further investigation of cases with arbitrary \(k\).

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