ON MEASURE CONTRACTION PROPERTY
WITHOUT RICCI CURVATURE LOWER BOUND

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ABSTRACT. Measure contraction properties $MCP(K, N)$ are synthetic Ricci curvature lower bounds for metric measure spaces which do not necessarily have smooth structures. It is known that if a Riemannian manifold has dimension $N$, then $MCP(K, N)$ is equivalent to Ricci curvature bounded below by $K$. On the other hand, it was observed in [20] that there is a family of left invariant metrics on the three dimensional Heisenberg group for which the Ricci curvature is not bounded below. Though this family of metric spaces equipped with the Harr measure satisfy $MCP(0, 5)$.

In this paper, we give sufficient conditions for a $2n + 1$ dimensional weakly Sasakian manifold to satisfy $MCP(0, 2n + 3)$. This extends the above mentioned result on the Heisenberg group in [20].

1. INTRODUCTION

In the past decade, there is a surge of interest in studying synthetic Ricci curvature lower bounds. These are reformulations of Ricci curvature lower bounds on Riemannian manifolds without using the underlying smooth structure. As a consequence, they can be used as the definitions of Ricci curvature lower bounds on more general metric measure spaces.

There are quite a few synthetic Ricci curvature lower bounds defined via different approaches. This includes the one in [2] via the formalism of Dirichlet forms, the one in [14, 23, 24] via the theory of optimal transportation, and the one in [19] via coupling of Markov chains.

In this paper, we consider another synthetic Ricci curvature lower bound, called measure contraction property $MCP(K, N)$, discussed in [24, 18]. Here, we recall that a length space $(M, d)$ equipped with a measure $\mu$ satisfies $MCP(0, N)$ if, for each Borel set $U_0$ and each point $x_0$ in $M$, the contraction $U_t$ of $U_0$ along geodesics ending at $x_0$ satisfies

$$\mu(U_t) \geq (1 - t)^N \mu(U_0).$$

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The condition $MCP(K, N)$ is defined in a similar way. For Riemannian manifolds with dimension $N$, the condition $MCP(K, N)$ is equivalent to the Ricci curvature bounded below by $K$. On the other hand, it was observed in [20] that there is a family of left invariant metrics on the three dimensional Heisenberg group for which the Ricci curvature is not bounded below. Though this family of metric spaces equipped with the Harr measure satisfy $MCP(0, 5)$.

In this paper, we give sufficient conditions on a family of Riemannian manifolds, called weakly Sasakian manifolds, of dimension $2n+1$ which guarantee that the condition $MCP(0, 2n+3)$ holds. More precisely, let $M$ be a contact manifold of dimension $2n+1$ equipped a contact form $\eta$ and a Reeb field $V$. Let $J$ be a $(1, 1)$-tensor which is almost complex on the distribution $\ker \eta$ and $JV = 0$. The Riemannian metric $\langle \cdot, \cdot \rangle$ is defined by the 2-form $d\eta$ and the tensor $J$ on $\ker \eta$. Outside $\ker \eta$, the Riemannian metric is defined by $|V| = 1$. On such a manifold, one can define a convenient connection, called the Tanaka-Webster connection. The corresponding curvature tensor, denoted by $\overline{Rm}$, is called the Tanaka-Webster curvature (see Section 2 for the detail).

The geometric structure $(M, J, V, \eta, \langle \cdot, \cdot \rangle)$ is a Sasakian manifold if additional compatibility and integrability conditions are satisfied (see Section 2 for the precise definition). We call $(M, J, V, \eta, \langle \cdot, \cdot \rangle)$ a weakly Sasakian manifold if all the above mentioned conditions except $|V| = 1$ are satisfied. We show that the Ricci curvature $Rc$ blows up in some directions as $|V| = \epsilon \to \infty$. On the other hand, we show that

**Theorem 1.1.** Let $(M, J, V, \eta, \langle \cdot, \cdot \rangle)$ be a weakly Sasakian manifold of dimension $2n+1$ such that $|V|$ is constant. Assume that the Tanaka-Webster curvature $\overline{Rm}$ satisfies

1. $\langle \overline{Rm}(Jv, v)v, Jv \rangle \geq 0$,
2. $\sum_{i=1}^{2n-2} \langle \overline{Rm}(w_i, v)v, w_i \rangle \geq 0$,

for any orthonormal basis $\{v, Jv, w_1, \ldots, w_{2n-2}\}$ of $\ker \eta$. Then the metric measure space $(M, d, \text{vol})$ satisfies $MCP(0, 2n+3)$, where $d$ and $\text{vol}$ are, respectively, the Riemannian distance and the Riemannian volume of $\langle \cdot, \cdot \rangle$.

Note that the curvature conditions in Theorem 1.1 are satisfied by the Heisenberg group. In fact, all inequalities become equalities in this case.

Note also that, under the same assumptions as in Theorem 1.1, it was shown in [10] that $(M, d_{CC}, \text{vol}_P)$ satisfies $MCP(0, 2n+3)$, where $d_{CC}$ is the Carnot-Caratheordory distance and $\text{vol}_P$ is the Popp measure (see also [8, 1] for the earlier results).
Metric measure spaces satisfying measure contraction property $\text{MCP}(0,N)$, in particular the ones defined in Theorem 1.1, satisfy doubling property and Poincaré inequality.

**Corollary 1.2.** (Doubling) Assume that the conditions in Theorem 1.1 hold. Then there is a constant $C > 0$ such that
\[
\text{vol}(B_x(2R)) \leq C \text{vol}(B_x(R))
\]
for all $x$ in $M$ and all $R > 0$, where $B_x(R)$ is the ball of radius $R$ centered at $x$.

**Corollary 1.3.** (Poincaré inequality) Assume that the conditions in Theorem 1.1 hold. Then, for each $p > 1$, there is a constant $C > 0$ such that
\[
\int_{B_x(R)} \left| f(x) - \frac{1}{\text{vol}(B_x(R))} \int_{B_x(R)} f(x) \text{dvol}(x) \right|^p \text{dvol}(x) \leq CR^p \int_{B_x(R)} |\nabla f|^p \text{dvol}(x).
\]

Here Corollary 1.2 easily follows from the measure contraction property. For a proof of Corollary 1.3 which relies on a result in [7], see [10].

With the doubling property and the Poincaré inequality, numerous results follow. For instance, it follows from the above corollaries and the results in [15, 4] that

**Corollary 1.4.** (Harnack inequality) Assume that the conditions in Theorem 1.1 hold. Then, for each $p > 1$, there is a constant $C > 0$ such that any positive solution to the $p$-Laplace equation $\text{div}_\text{vol}(\|\nabla f\|^{p-2}\nabla f) = 0$ on $B_x(R)$ satisfies
\[
\sup_{B_x(R/2)} f \leq C \inf_{B_x(R/2)} f.
\]

**Corollary 1.5.** (Liouville theorem) Assume that the conditions in Theorem 1.1 hold. Then any non-negative solution to the $p$-Laplace equation is a constant.

The following parabolic Harnack inequality also holds (see [16, 17, 0, 22]).

**Corollary 1.6.** (Parabolic Harnack inequality) Assume that the conditions in Theorem 1.1 hold. Then, for each $R > 0$, there is a constant $C > 0$ such that any positive solution to the heat equation $\dot{f} = \Delta f$ on $(s-r^2, s) \times B_x(R)$ with $0 < r < R$ satisfies
\[
\sup_{(s-\frac{3}{4}r^2, s-\frac{1}{4}r^2) \times B_x(R/2)} f \leq C \inf_{(s-\frac{1}{4}r^2, s) \times B_x(R/2)} f.
\]
for all points $x$ in $M$.

For a converse result of the above corollary, see also [9, 6, 22]. The above parabolic Harnack inequality is also equivalent to a two sided Gaussian bound for the heat kernel (see [5]).

**Corollary 1.7.** *(Two-sided Gaussian bound)* Assume that the conditions in Theorem 1.1 hold. Then there are positive constants $C_1, C_2, C_3, C_4$ such that the heat kernel $h$ satisfies

$$\frac{C_1}{\text{vol}(B_{\sqrt{t}}(x))} e^{-\frac{C_2 d(x, y)^2}{t}} \leq h_t(x, y) \leq \frac{C_3}{\text{vol}(B_{\sqrt{t}}(x))} e^{-\frac{C_4 d(x, y)^2}{t}}$$

for all points $x$ and $y$ in $M$.

Finally, we remark that there are also consequences following from Corollary 1.2 and 1.3 about quasi-regular mappings. For this, see [1] and references therein.

The paper is organized as follows. In Section 2, we introduce the weakly Sasakian manifolds and summarize some facts that are needed for this paper. In Section 3, we begin the proof of Theorem 1.1 by introducing one of the key ingredients of the proof, a moving frame adapted to the given geometry defined along a geodesic. We also rewrite the measure contraction property as estimates on solutions of a matrix Riccati equation using this moving frame. This approach was also used by the author in various other situations (see [11, 12]). In Section 4, the case of the Heisenberg group is discussed. The proof of Theorem 1.1 is summarized in Section 5. Finally, the proofs of the results mentioned in Section 2 are discussed in the appendix.

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2. **Weakly Sasakian manifolds**

In this section, we introduce what we call weakly Sasakian manifolds and discuss some of the properties that are needed in this paper.
Let \( \eta \) be a contact form on a manifold \( M \) of dimension \( 2n + 1 \). This means that the restriction of the two-form \( d\eta \) to the distribution \( \ker \eta \) is symplectic. Let \( V \) be the Reeb field defined by \( \eta(V) = 1 \) and \( d\eta(V, \cdot) = 0 \). Let \( J \) be a \((1, 1)\)-tensor satisfying \( J\mathcal{J} = 0 \) and \( J\mathcal{J}^2 X = -X \) for all vector field \( X \) contained in the distribution \( \ker \eta \). Let \( \langle \cdot, \cdot \rangle \) be a Riemannian metric such that

\[
d\eta(X_1, X_2) = \langle X_1, \mathcal{J}X_2 \rangle.
\]

Note that this implies, in particular, that the Reeb field \( V \) is orthogonal to the distribution \( \ker \eta \). We call the structure \((\mathcal{J}, V, \eta, \langle \cdot, \cdot \rangle)\) weakly contact metric structure. We also say the structure \((\mathcal{J}, V, \eta, \langle \cdot, \cdot \rangle)\) is weakly Sasakian if the following holds for all vector fields \( Y_1 \) and \( Y_2 \):

\[
[Y_1, Y_2] + \mathcal{J}[\mathcal{J}Y_1, Y_2] + \mathcal{J}[Y_1, \mathcal{J}Y_2] - [\mathcal{J}Y_1, \mathcal{J}Y_2] = (Y_1 \cdot \eta(Y_2) - Y_2 \cdot \eta(Y_1))V.
\]

In other words,

\[
d\eta(Y_1, Y_2)V = -\mathcal{J}^2[Y_1, Y_2] + \mathcal{J}[\mathcal{J}Y_1, Y_2] + \mathcal{J}[Y_1, \mathcal{J}Y_2] - [\mathcal{J}Y_1, \mathcal{J}Y_2].
\]

Note that the structure \((\mathcal{J}, V, \eta, \langle \cdot, \cdot \rangle)\) is Sasakian if \(|V| = 1\). In this paper, we consider weakly Sasakian manifolds such that the length \(|V|\) of the Reeb field \( V \) is constant. The proof of the following result is contained in the appendix.

**Proposition 2.1.** Assume that the structure \((\mathcal{J}, V, \eta, \langle \cdot, \cdot \rangle)\) is weakly Sasakian and \(|V| = \epsilon\). Then

1. \( \eta(Y) = \frac{1}{\epsilon} \langle V, Y \rangle \)
2. \( \mathcal{L}_V \mathcal{J} = 0 \)
3. \( \mathcal{L}_V g = 0 \)
4. \( \nabla_Y V = -\frac{\epsilon}{2} \mathcal{J}Y \)
5. \( \nabla_{X_1} \mathcal{J}(X_2) = \frac{\langle X_1, X_2 \rangle}{2} V \)
6. \( \nabla_X \mathcal{J}(V) = -\frac{\epsilon}{2} \mathcal{J}X \)
7. \( \nabla_V \mathcal{J} = 0 \)
8. \( (\nabla_{X_1} X_2)_{\text{hor}} \) is independent of \( \epsilon \)
9. \( \nabla_{X_1} X_2 = (\nabla_{X_1} X_2)_{\text{hor}} + \frac{1}{2} \langle \mathcal{J}X_1, X_2 \rangle V \)
10. \( \nabla_Y V = -\frac{\epsilon^2}{2} \mathcal{J}Y \)
11. \( \nabla_V X_1 = ([V, X_1])_{\text{hor}} - \frac{\epsilon^2}{2} \mathcal{J}X_1 \)

for all vector fields \( X_1, X_2, \) and \( Y \) such that \( X_1 \) and \( X_2 \) are contained in the distribution \( \ker \eta \). Here \( Y_{\text{hor}} \) denotes the orthogonal projection of the vector field \( Y \) onto the distribution \( \ker \eta \).
Let \( \nabla \) be the connection defined by

\[
\nabla_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + \frac{1}{2} \langle J Y_1, Y_2 \rangle V + \frac{1}{2} \langle J Y_2, V \rangle Y_1 - \frac{1}{4} \langle J Y_1, Y_2 \rangle J Y_1 - \frac{1}{4} \langle J Y_2, V \rangle J Y_2.
\]

Note that \( \nabla \) is the Tanaka-Webster connection when \( \epsilon = 1 \).

**Proposition 2.2.** The connection \( \nabla \) is independent of \( \epsilon \).

**Proof.** Note that the following formula holds for all vector fields \( Y_1 \) and \( Y_2 \)

\[
\nabla_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + \frac{1}{2} \langle J Y_1, Y_2 \rangle V + \frac{1}{2} \langle J Y_2, V \rangle Y_1 - \frac{1}{4} \langle J Y_1, Y_2 \rangle J Y_1 - \frac{1}{4} \langle J Y_2, V \rangle J Y_2
\]

Therefore, the result follows from Proposition 2.1.

Let \( Rm \) and \( \overline{Rm} \) be the curvature tensors defined by the connections \( \nabla \) and \( \nabla \), respectively. The two curvatures are related as follows (see Appendix for the proof).

**Proposition 2.3.** Assume that the structure \((J, V, \eta, \langle \cdot, \cdot \rangle)\) is weakly Sasakian and \( |V| = \epsilon \). Then

1. \( \overline{Rm}(Y_1, Y_2)V = \frac{\epsilon^2 \langle Y_2, V \rangle(Y_1)_{\text{hor}}}{4} - \frac{\epsilon^2 \langle Y_1, V \rangle(Y_2)_{\text{hor}}}{4}, \)
2. \( \overline{Rm}(X_2, X_3)X_1 = \overline{Rm}(X_2, X_1)X_3 + \frac{\epsilon^2 \langle J X_2, X_1 \rangle}{4} J X_3 - \frac{\epsilon^2 \langle J X_3, X_1 \rangle}{4} J X_2, \)
3. \( \overline{Rm}(Y_1, Y_2)V = 0, \)
4. \( \overline{Rm}(X_1, V)X_2 = \overline{Rm}(X_1, V)X_2 + \frac{\epsilon^2}{4} \langle X_1, X_2 \rangle V, \)
5. \( \overline{Rm}(X_1, V)X_2)_{\text{hor}} = 0, \)

for all vector fields \( X_1, X_2, Y_1, \) and \( Y_2 \) such that \( X_1 \) and \( X_2 \) are contained in the distribution \( \ker \eta \).

Let \( v_0, v_1, \ldots, v_{2n} \) be an orthonormal frame such that \( v_0 = \frac{1}{\epsilon} V, v_1 = \frac{1}{|Y_1|_{\text{hor}}} Y_1, \) \( v_2 = \frac{1}{|Y_2|_{\text{hor}}} J Y_1, \) and \( J v_{2k-1} = v_{2k} \) for each \( k = 1, \ldots, n \). The following is a consequence of Proposition 2.3

**Proposition 2.4.** Assume that the structure \((J, V, \eta, \langle \cdot, \cdot \rangle)\) is weakly Sasakian and \( |V| = \epsilon \). Then, for each \( i, j \neq 0, \)

1. \( \langle \overline{Rm}(v_i, Y)Y, v_0 \rangle = -\frac{\epsilon \langle V, Y \rangle |Y_{\text{hor}}|^2}{4} \delta_{i1}, \)
2. \( \langle \overline{Rm}(v_0, Y)Y, v_0 \rangle = \frac{\epsilon^2 |Y_{\text{hor}}|^2}{4} \)
3. \( \langle \overline{Rm}(v_i, Y)Y, v_j \rangle, \)
\[
= \frac{\epsilon^2 |Y_{\text{hor}}|^2}{4} \delta_{ij} - \frac{3 \epsilon^2 \delta_{i2} \delta_{j2} |Y_{\text{hor}}|^2}{4} + \langle \overline{Rm}(v_i, Y)Y, v_j \rangle
\]
(4) $Rc(Y,Y) = \frac{\langle Y, Y \rangle^2}{2} - \frac{3\epsilon^2 |Y|_{\text{hor}}^2}{4} + \overline{Rc}(Y,Y)$

where $Rc(Y,Y)$ and $\overline{Rc}(Y,Y)$ are the traces of $v \mapsto \langle Rm(v,Y)Y, v \rangle$ and $v \mapsto \langle Rm(v,Y)Y, v \rangle$, respectively.

Note that, for each tangent vector $Y$ with $Y_{\text{hor}} \neq 0$, $Rc(Y,Y) \to -\infty$ as $\epsilon \to \infty$.

3. On conjugate points and measure contraction

From now on, we assume that the structure $(\mathcal{J}, V, \eta, \langle \cdot, \cdot \rangle)$ on the manifold $M$ is weakly Sasakian with $|V| = \epsilon$. In this section, we prove some preliminary results on conjugate points and measure contraction properties of these manifolds.

Let $t \mapsto \gamma_\epsilon(t)$ be a family of geodesics parameterized by the variable $\epsilon$ such that $\gamma_\epsilon(0) = x$. It follows that $\frac{d^2}{dt^2} \gamma_\epsilon(t) = 0$ and so

$$0 = \frac{D}{d\epsilon} \frac{d^2}{dt^2} \gamma_\epsilon(t) \bigg|_{\epsilon=0} = \frac{D}{dt} \frac{d\gamma}{dt} + Rm(\gamma_\epsilon(t), v_0(t)) \gamma_\epsilon(t)$$

Let $v_0(t) = \frac{1}{\epsilon} V(\gamma_\epsilon(t))$ and let $v_{2i}(t) = \mathcal{J}v_{2i-1}(t)$. We also assume that $v_1(t) = \frac{1}{|\gamma_\epsilon(t)|_H} \gamma_\epsilon(t)$ and $v_2(t) = \frac{1}{|\gamma_\epsilon(t)|_H} \mathcal{J} \gamma_\epsilon(t)$. It follows that

$$\dot{v}_0(t) = \frac{1}{\epsilon} \nabla_{\dot{\gamma}_0} V(\gamma_0(t)) = -\frac{\epsilon |(\gamma_0(0))_H|}{2} v_2(t),$$

$$v_1(t) = \frac{1}{|(\gamma_0(0))_H|} (\dot{\gamma}(t) - \langle \dot{\gamma}(t), v_0(t) \rangle v_0(t)),$$

$$\dot{v}_1(t) = -\frac{\langle \dot{\gamma}(0), v_0(0) \rangle}{\langle \gamma_0(0) \rangle_H} \dot{v}_0(t) = \frac{\epsilon \langle \dot{\gamma}(0), v_0(0) \rangle}{2} v_2(t),$$

and

$$\dot{v}_2(t) = \nabla_{\dot{\gamma}_0} \mathcal{J}(v_1(t)) + \mathcal{J} \dot{v}_1(t)$$

$$= \frac{\epsilon |(\dot{\gamma}(0))_H|}{2} v_0(t) - \frac{\epsilon \langle \dot{\gamma}(0), v_0(0) \rangle}{2} v_1(t).$$

Finally, we can choose $v_3(t), ..., v_{2n}(t)$ such that $\dot{v}_i(t)$ is contained in the span of $v_0(t), v_1(t), v_2(t)$ for each $i = 3, ..., 2n$. It follows that $\langle \dot{v}_i(t), v_j(t) \rangle = -\langle v_i(t), \dot{v}_j(t) \rangle = 0$ for each $j = 0, 1, 2$ and $i = 3, ..., 2n$. Therefore, $\dot{v}_i(t) = 0$. 
Let $W(t)$ be the matrix defined by $\dot{v}(t) = W(t)v(t)$ and let $\nabla_H f = (\nabla f)_{\text{hor}}$. Since $|\dot{\gamma}(t)|$ and $\langle V(\gamma(t)), \dot{\gamma}(t) \rangle$ are independent of $t$,

$$
W = \begin{pmatrix}
0 & 0 & -\langle \nabla_H f_0 \rangle |\nabla_H f_0| & 0 \\
0 & 0 & \frac{(\nabla f_0, v)}{|\nabla f_0|^2} & 0 \\
\frac{|\nabla_H f_0|}{2} & \frac{(\nabla f_0, v)}{|\nabla f_0|^2} & 0 & 0 \\
0 & 0 & 0 & O_{2^{n-2}}
\end{pmatrix}.
$$

Let $a(t)$ be the matrix defined by $\gamma_0'(t) = a(t)v(t)$. It follows that

$$
\frac{D}{dt} \gamma_0'(t) = \dot{a}(t)v(t) + a(t)Wv(t)
$$

and

$$
-a(t)R(t)v(t) = -Rm(\gamma_0'(t), \gamma_0(t))\gamma_0(t) = \frac{D^2}{dt^2} \gamma_0(t) = \ddot{a}(t)v(t) + 2\dot{a}(t)Wv(t) + a(t)W^2v(t),
$$

where $R_{ij}(t) = \langle Rm(v_i(t), \gamma_0(t))\gamma_0(t), v_j(t) \rangle$.

Let $\mathcal{A}(t)$ be solution of the following equation

$$
\ddot{\mathcal{A}}(t) + 2\dot{\mathcal{A}}(t)W + \mathcal{A}(t)W^2 + \mathcal{A}(t)R(t) = 0
$$

with initial conditions $\mathcal{A}(0) = 0$ and $\dot{\mathcal{A}}(0) = I$.

Let $\mathcal{F}(t) = \mathcal{A}(t)^{-1}\dot{\mathcal{A}}(t) + W$. Then

$$
\ddot{\mathcal{F}}(t) = -\mathcal{A}(t)^{-1}\dot{\mathcal{A}}(t)\mathcal{A}(t)^{-1}\dot{\mathcal{A}}(t) + \mathcal{A}(t)^{-1}\ddot{\mathcal{A}}(t)
= -\mathcal{A}(t)^{-1}\dot{\mathcal{A}}(t)\mathcal{A}(t)^{-1}\dot{\mathcal{A}}(t) + \mathcal{A}(t)^{-1}\dot{\mathcal{A}}(t)
= -(\mathcal{F}(t) - W)^2 - 2(\mathcal{F}(t) - W)W - W^2 - R(t)
= -\mathcal{F}(t)^2 - \mathcal{F}(t)W - W^T\mathcal{F}(t) - R(t).
$$

(3.1)

Since $\gamma_0(0)$ and $\gamma_0(\tau)$ are conjugate along $\gamma$ if and only if $\langle \mathcal{F}(t) v, v \rangle \to -\infty$ as $t \to \tau$ for some vector $v$, we have the following

**Proposition 3.1.** Assume that $\gamma_0$ is a minimizing geodesic between its endpoints $\gamma_0(0)$ and $\gamma_0(1)$. Then $\text{tr} \mathcal{F}(t)$ stays bounded for all $t$ in $(0, 1]$.

Next, we consider the contraction of the measure $\text{vol}$ along geodesics ending at the same point $x_0$. Let $d(x, x_0)$ be the Riemannian distance between the points $x$ and $x_0$. It is locally semi-concave and so twice differentiable Lebesgue almost everywhere. Let $\exp$ be the Riemannian exponential map and let $\varphi_t = \exp(t\nabla f_0)$, where $f_0(x) = -\frac{d^2(x, x_0)}{2}$. For the rest of this section, we discuss the volume contraction $\text{vol}(\varphi_t(U))$, where $U$ is a fixed Borel set.
For this, let $x$ be a point where $f_0$ is twice differentiable. Let $v_0(t), ..., v_{2n}(t)$ be a orthonormal frame along $\varphi_t(x)$ defined in the same way as the beginning of this section. Let $v(t) = (v_0(t), ..., v_{2n}(t))^T$ and let $A(t)$ be the matrix defined by

$$d\varphi_t(v(0)) = A(t)v(t).$$

It follows that

$$\frac{D}{dt}d\varphi_t(v(0)) = \dot{A}(t)v(t) + A(t)\dot{v}(t)$$

$$= \left(\dot{A}(t) + A(t)W\right)v(t)$$

and

$$\frac{D^2}{dt^2}d\varphi_t(v(0)) = \left(\ddot{A}(t) + 2\dot{A}(t)W + A(t)W^2\right)v(t).$$

Let $f_t(y) = \frac{-d^2(y,x_0)}{2(1-t)}$. Then $f_t$ satisfies

$$\dot{f}_t + \frac{1}{2}|\nabla f_t|^2 = 0$$

at $x$ and so $\varphi_t(x) = \nabla f_t(\varphi_t(x))$. Therefore, we also have

$$\frac{D}{dt}d\varphi_t(v_i(0)) = \frac{d}{ds}\nabla f_t(\varphi_t(\gamma_i(s))) \bigg|_{s=0}$$

$$= \nabla^2 f_t(d\varphi_t(v_i(0))) = \sum_{j,k} A_{ij}(t) F_{jk}(t) v_k(t)$$

and

$$\frac{D^2}{dt^2}d\varphi_t(v_i(0)) = \frac{D}{dt}\frac{D}{ds}\nabla f_t(\varphi_t(\gamma_i(s))) \bigg|_{s=0}$$

$$= Rm(\nabla f_t(\varphi_t), d\varphi_t(v_i(0))) \nabla f_t(\varphi_t) = -\sum_{j,k} A_{ij}(t) R_{jk}(t) v_k(t),$$

where $F_{ij}(t) = \langle \nabla^2 f_t(v_i(t)), v_j(t) \rangle$ and

$$R_{ij}(t) = \langle Rm(v_i(t), \nabla f_t(\varphi_t)) \nabla f_t(\varphi_t), v_j(t) \rangle.$$ 

It also follows that

$$-R(t) = A(t)^{-1}\dot{A}(t) + 2A(t)^{-1}\dot{A}(t)W + W^2.$$ 

Hence,

$$F(t) = A(t)^{-1}\dot{A}(t) + W$$

and

$$\dot{F}(t) = -R(t) - F(t)^2 - F(t)W - W^T F(t).$$

(3.2)
It also follows that \( \det A(t) = e^{\int_0^t \text{tr} F(s) \, ds} \). Hence, by applying [25, Theorem 11.3], we obtain

**Proposition 3.2.**

\[
\int_{\varphi_t(U)} d\text{vol} = \int_U e^{\int_0^t \text{tr} F(s) \, ds} d\text{vol}.
\]

Finally, we record the following formula.

**Proposition 3.3.** Let \( \bar{R}(t) \) be the matrix defined by

\[
\bar{R}_{ij}(t) = \langle \bar{R}m(v_i(t), \nabla f_t(\varphi_t))\nabla f_t(\varphi_t), v_j(t) \rangle.
\]

Then \( R(t) \) satisfies

\[
R(t) = \bar{R}(t) + \begin{pmatrix}
  b^2 & bc & 0 & 0 \\
  bc & c^2 & 0 & 0 \\
  0 & 0 & c^2 - 3b^2 & 0 \\
  0 & 0 & 0 & c^2I
\end{pmatrix}
\]

where \( b = -\frac{|\nabla H f_0|}{2} \) and \( c = \frac{\langle \nabla f_0, V \rangle}{2} \).

4. **The Heisenberg group**

In this section, we discuss the Heisenberg group which is the model case of our results.

First, recall that the underlying manifold of the Heisenberg group is \( M = \mathbb{R}^{2n+1} \) with coordinates \( \{ x_1, \ldots, x_n, y_1, \ldots, y_n, z \} \). The contact form \( \eta \) and the Reeb field \( V \) are given by \( \eta = dz - \frac{1}{2} \sum_{i=1}^{2n} x_i dy_i + \frac{1}{2} \sum_{i=1}^{2n} y_i dx_i \) and \( V = \partial_z \), respectively. Let \( X_i = \partial_{x_i} - \frac{1}{2} y_i \partial_z \) and \( Y_i = \partial_{y_i} + \frac{1}{2} x_i \partial_z \). The tensor \( J \) is defined by \( J(X_i) = Y_i, J(Y_i) = -X_i, \) and \( J(V) = 0 \). The Riemannian metric \( \langle \cdot, \cdot \rangle \) is defined by the condition that \( \{ \frac{1}{2} V, X_1, \ldots, X_n, Y_1, \ldots, Y_n \} \) is orthonormal. In this case, we have \( \bar{R}m \equiv 0 \). Below, we let \( b = -\frac{|\nabla H f_0|}{2} \) and \( c = \frac{\langle \nabla f_0, V \rangle}{2} \).

**Theorem 4.1.** Assume that \( \gamma : [0,1] \to M \) is a minimizing geodesic between its endpoints. Then \( |\langle \gamma(0), V(\gamma(0)) \rangle| \leq 2\pi \).

**Proof.** Let \( F(t) = \begin{pmatrix} F_1(t) & F_2(t) \\ F_2(t)^T & F_3(t) \end{pmatrix} \), where \( F_1(t) \) is a 3 \times 3 block. A computation shows that

\[
\text{tr} F_1(t) = -\frac{(b^2 + c^2)(\cos(2ct) - 1) + 2t^2 b^2 c^2 \cos(2ct) - 2tc^3 \sin(2ct)}{t(b^2 + c^2)(\cos(2ct) - 1) + t^2 b^2 c \sin(2ct)}.
\]
Theorem 4.2. For each Borel set $U$ from Proposition 3.1.

The method of proof is the same as that of $\text{tr}F_1(t)$ which can be found in the proof of Theorem 4.2.

Since
\[
(b^2 + c^2)(\cos(2ct) - 1) + tb^2c\sin(2ct)
= -2\sin(ct)((b^2 + c^2)\sin(ct) - tb^2c\cos(ct)),
\]
$\text{tr}F_1(t)$ blows up for some $t < 1$ if $c > \pi$. Therefore, the result follows from Proposition 3.1.

**Theorem 4.2.** For each Borel set $U$ in the Heisenberg group,

\[
\int_{\tilde{\varphi}(U)} d\text{vol} = \int_U \frac{\sin^{2n-2}(c(1-t))}{\sin^{2n-2}(c)} d\text{vol}
+ \int_U \frac{(1-t)\sin(c(1-t))[((1-t)b^2c\cos(c(t-t)) - (b^2 + c^2)]}{\sin(c)b^2c\cos(c) - (b^2 + c^2)} d\text{vol}
\geq (1-t)^{2n+3} \int_U d\text{vol}.
\]

**Proof.** Let $F(t) = \left( \begin{array}{ccc}
F_1(t) & F_2(t) & F_3(t) \\
F_2(t)^T & F_1(t) & F_3(t) \\
F_3(t)^T & F_2(t) & F_1(t)
\end{array} \right)$, $G(t) = F(1-t)^{-1}$, and $G(t) = \left( \begin{array}{ccc}
G_1(t) & G_2(t) & G_3(t) \\
G_2(t)^T & G_1(t) & G_3(t) \\
G_3(t)^T & G_2(t) & G_1(t)
\end{array} \right)$, where $F_1(t)$ and $G_1(t)$ are $3 \times 3$ blocks.

It follows that $G(0) = 0$ and
\[
\dot{G}_1(t) = -G_1(t)R_1(1-t)G_1(t) - I - G_2(t)R_3(1-t)G_2(t)^T - W_1G_1(t) - G_1(t)W_1^T
\dot{G}_2(t) = -G_1(t)R_1(1-t)G_2(t) - G_2(t)R_3(1-t)G_3(t) - W_1G_2(t)
\dot{G}_3(t) = -G_2(t)^TR_1(1-t)G_2(t) - G_3(t)R_3(1-t)G_3(t) - I.
\]

Therefore, $G_2 \equiv 0$ and
\[
\dot{G}_1(t) = -G_1(t)R_1(1-t)G_1(t) - I - W_1G_1(t) - G_1(t)W_1^T
\dot{G}_3(t) = -G_3(t)R_3(1-t)G_3(t) - I.
\]

It follows that $F_2 \equiv 0$. A computation using the method in [13] shows that
\[
(4.1)
F_1(1-t) = G_1(t)^{-1}
\]
\[
= \frac{1}{K_1(t)} \left( \begin{array}{ccc}
\frac{c^2}{b} & \frac{bK_2(t)}{t} & \frac{b^3K_2(t)}{t} \\
\frac{bK_2(t)}{t} & \frac{b^2K_3(t)+c^2t\cot(tc)}{t} & \frac{c^2K_2(t)}{t} \\
\frac{b^3K_2(t)}{t} & \frac{c^2K_2(t)}{t} & tbc^2 - c\cot(tc)K_1(t)
\end{array} \right).
\]
and

\[(4.2) \quad F_3(1-t) = G_3(t)^{-1} = -c \cot(ct) I,\]

where

\[K_1(t) = tc b^2 \cot(tc) - b^2 - c^2 = b^2 K_2(t) - c^2,\]
\[K_2(t) = tc \cot(tc) - 1.\]

The following two inequalities and Proposition 3.2 give the result:

\[(4.3) \quad \text{tr} F_1(1-t) = -\frac{(b^2 + c^2)(\cos(2ct) - 1) + 2t^2 b^2 c^2 \cos(2ct) - 2t c^3 \sin(2ct)}{t(b^2 + c^2)(\cos(2ct) - 1) + t^2 b^2 c \sin(2ct)}
\]
\[= -\frac{d}{dt} \ln \left( t(b^2 + c^2)(\cos(2ct) - 1) + t^2 b^2 c \sin(2ct) \right) \geq -\frac{5}{t},\]

and

\[(4.4) \quad \text{tr} F_3(1-t) = -(2n - 2)c \cot(ct)
\]
\[= -\frac{d}{dt} \ln \sin^{2n-2}(ct) \geq -\frac{2n - 2}{t}.\]

The inequality (4.4) follows from \(x \cot(x) \leq 1\) for all \(x\) in the open interval \((-\pi, \pi)\) and Theorem 4.1.

For (4.3), we first minimize over \(b\) and then over \(c\) (using again Theorem 4.1) to obtain

\[\text{tr} F_1(1-t) \geq -\lim_{b \to 0} \frac{(b^2 + c^2)(\cos(2ct) - 1) + 2t^2 b^2 c^2 \cos(2ct) - 2t c^3 \sin(2ct)}{t(b^2 + c^2)(\cos(2ct) - 1) + t^2 b^2 c \sin(2ct)}
\]
\[= -\frac{(\cos(2ct) - 1) + 2t^2 c^2 \cos(2ct)}{t(\cos(2ct) - 1) + t^2 c \sin(2ct)}
\]
\[\geq -\lim_{c \to 0} \frac{(\cos(2ct) - 1) + 2t^2 c^2 \cos(2ct)}{t(\cos(2ct) - 1) + t^2 c \sin(2ct)} = -\frac{5}{t}.\]

\[\square\]

5. Proof of Theorem 1.1

Let \(\tilde{F}_1(t)\) be the matrix defined by (1.1) and let

\[\tilde{f}_3(t) = -(2n - 2)c \cot(ct).\]
Then
\[ \dot{F}_1(t) = -\bar{R}_1(t) - F_1(t)^2 - \tilde{F}_1(t)W_1 - W_1^T\tilde{F}_1(t) \]
\[ \dot{f}_3(t) = -(2n-2)c^2 - \frac{1}{2n-2}\tilde{f}_3(t)^2, \]
respectively. Moreover, we also have \( \tilde{F}_1^{-1} \to 0 \) and \( \frac{1}{f_3} \to 0 \) as \( t \to 1 \).

Since \( \bar{R}_1(t) \geq 0 \), it follows that
\[ \frac{d}{dt} (F_1(t-\epsilon) - \tilde{F}_1(t)) = \tilde{F}_1(t)^2 - F_1(t-\epsilon)^2 + (\tilde{F}_1(t) - F_1(t-\epsilon))W_1 \]
\[ + W_1^T(\tilde{F}_1(t) - F_1(t-\epsilon)) - \bar{R}_1(t) - F_2(t-\epsilon)F_2(t-\epsilon)^T \]
\[ \leq (\tilde{F}_1(t) - F_1(t-\epsilon))(W_1 + \tilde{F}_1(t)) + (W_1^T + F_1(t-\epsilon))(\tilde{F}_1(t) - F_1(t-\epsilon)) \].

Note also that \( F_1(t-\epsilon) \geq \tilde{F}_1(t) \) for all \( t \) close enough to 1. It follows from this and [21, Proposition 1] that \( F_1(t-\epsilon) \geq \tilde{F}_1(t) \) for all \( t \geq \epsilon \).

By letting \( \epsilon \to 0 \), we obtain \( F_1(t) \geq \tilde{F}_1(t) \) for all \( t \) in \([0, 1]\). Therefore, by (4.3),
\[ \text{tr}F_1(t) \geq \text{tr}\tilde{F}_1(t) \geq -\frac{5}{1-t}. \]

Similarly, by using \( \text{tr}\bar{R}_3(t) \geq 0 \), we also have
\[ \frac{d}{dt}\text{tr}F_3(t) = -\text{tr}\bar{R}_3(t) - |F_3(t)|^2 - \text{tr}(F_2(t)^TF_2(t)) \]
\[ \leq -(2n-2)c^2 - \frac{1}{2n-2}(\text{tr}F_3(t))^2. \]

An argument as above shows that
\[ \text{tr}F_3(t) \geq \tilde{f}_3(t) \geq -\frac{2n-2}{1-t}. \]

Finally, the result follows from Proposition 3.2 and the above estimates on \( \text{tr}F_1(t) \) and \( \text{tr}F_3(t) \).

6. Appendix

In this appendix, we give the proofs of Proposition 2.1 and 2.3. They are very mild modification of the corresponding ones in the Sasakian case (see [3]). First, we prove the following result for more general weakly Sasakian manifolds.

**Proposition 6.1.** Assume that the structure \((J, V, \eta, \langle \cdot, \cdot \rangle)\) is weakly Sasakian. Let \( X_1 \) and \( X_2 \) be vector fields contained in the distribution \( \ker \eta \). Then

1. \( \mathcal{L}_VJ = 0 \),
2. \( \mathcal{L}_Vg(X_1, \cdot) = 0 \),
\[ (3) \langle \nabla_{X_1} V, X_2 \rangle = -\langle \nabla_{X_2} V, X_1 \rangle, \]
\[ (4) \nabla V = \frac{\mathcal{L}_V |V|^2}{2|V|^2} V + \frac{1}{2} J^2 |V|^2, \]
\[ (5) \langle X_1, J X_2 \rangle = \nabla_{X_1} \eta(X_2) - \nabla_{X_2} \eta(X_1), \]
\[ (6) \nabla_{X_1} J(X_2) = \frac{\langle X_2, \mathcal{J} \nabla_{X_1} \eta \rangle}{|V|^2} V, \]
\[ (7) \nabla_{X} \mathcal{J}(V) = -\mathcal{J} \nabla X V, \]
\[ (8) \nabla V \mathcal{J}(X) = \nabla_{J X} V - \mathcal{J} \nabla X V, \]
\[ (9) \nabla_{X} \mathcal{J}(V) = \frac{1}{2} \mathcal{J} \nabla |V|^2. \]

**Proof.** By (2.2), we have
\[ \mathcal{J}^2[V, X] = \mathcal{J}[V, \mathcal{J} X]. \]
It follows that \( \mathcal{J}(\mathcal{L}_V \mathcal{J}) X = 0 \) and the horizontal part of \( (\mathcal{L}_V \mathcal{J}) X \) vanishes for any \( X \). Since \( \eta \circ \mathcal{J} = 0 \), we have, by Cartan’s formula, the following for the vertical part
\[ \eta((\mathcal{L}_V \mathcal{J}) X) = -\mathcal{L}_V \eta(\mathcal{J} X) = 0. \]
The first assertion follows.
By the first assertion and (2.1), we have
\[ \mathcal{L}_V g(X_1, \mathcal{J} X_2) = \mathcal{L}_V (d\eta)(X_1, X_2) = 0. \]
The second assertion follows.
The third and the fourth assertions follow from Koszul’s formula. Assertion five follows from (2.1).
By (2.2), we have
\[
\begin{align*}
\mathcal{J} \nabla_{X_1} \mathcal{J}(X_2) & = -\mathcal{J}^2[X_1, X_2] + \mathcal{J} [\mathcal{J} X_1, X_2] + \mathcal{J} [X_1, \mathcal{J} X_2] - [\mathcal{J} X_1, \mathcal{J} X_2] \\
& = -\mathcal{J}^2 \nabla X_1 X_2 + \mathcal{J}^2 \nabla X_2 X_1 + \mathcal{J} \nabla_{J X_1} X_2 \\
& - \mathcal{J} \nabla_{X_2} (\mathcal{J} X_1) + \mathcal{J} \nabla_{X_1} (J X_2) - \mathcal{J} \nabla_{J X_2} X_1 - \nabla_{J X_1} (J X_2) + \nabla_{J X_2} (J X_1).
\end{align*}
\]
Since \( X_1 \), \( X_2 \), and \( X_3 \) are in ker \( \eta \), it follows that
\[
\begin{align*}
& \langle \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - \nabla_{J X_1} (J X_2) + \nabla_{J X_2} (J X_1), X_3 \rangle \\
& = \langle \nabla_{J X_1} X_2 - \nabla_{X_2} (J X_1) + \nabla_{X_1} (J X_2) - \nabla_{J X_2} X_1, J X_3 \rangle.
\end{align*}
\]
In other words,
\[ \langle -\nabla_{J X_1} \mathcal{J} X_2 + (\nabla_{J X_2} \mathcal{J}) X_1, X_3 \rangle = \langle -\nabla_{X_2} \mathcal{J} X_1 + (\nabla_{X_1} \mathcal{J}) X_2, \mathcal{J} X_3 \rangle \]

It follows that
\[ \langle -\nabla_{J X_1} \mathcal{J} X_2 + (\nabla_{J X_2} \mathcal{J}) X_1, \mathcal{J} X_3 \rangle = \langle (\nabla_{X_2} \mathcal{J}) X_1 - (\nabla_{X_1} \mathcal{J}) X_2, X_3 \rangle. \]

(6.1)
On the other hand, we have, by taking exterior derivative of \( d\eta \), the following
\[
\langle X_3, (\nabla_{X_2} J) X_1 \rangle + \langle X_1, (\nabla_{X_3} J) X_2 \rangle - \langle X_3, (\nabla_{X_1} J) X_2 \rangle = 0.
\]
By combining this with (6.1), we obtain
\[
\langle X_1, (\nabla_{X_3} J) X_2 \rangle = \langle X_1, (\nabla_{X_3} J) X_2 \rangle - \langle X_3, (\nabla_{X_2} J) X_1 \rangle
\]
\[
= \langle (\nabla_{JX_1} J) X_2 - (\nabla_{JX_2} J) X_1, J X_3 \rangle.
\]
By (6.2), we also have
\[
\langle X_1, (\nabla_{X_3} J) X_2 \rangle = \langle J X_3, (\nabla_{JX_2} J) X_1 - (\nabla_{JX_1} J) X_2 \rangle.
\]
It follows that \( \langle X_1, (\nabla_{X_3} J) X_2 \rangle = 0 \).

A calculation shows that \( \nabla_{X_1} J (X_2) = \frac{\langle X_2, \nabla_{X_1} V \rangle}{|V|^2} V \) for all tangent vectors \( X_1 \) and \( X_2 \) in \( \ker \eta \). The sixth assertion follows. A similar calculation gives the seventh assertion. Using the formula at the beginning of this proof, we obtain
\[
\nabla_V J (X) = \nabla_{JX} V - J \nabla_X V
\]
which is the eighth assertion. The last assertion follows from \( J V = 0 \). \( \square \)

**Proof of Proposition 2.1.** The first nine assertions follow from the above Proposition. By Koszul’s formula, \( \langle \nabla_{X_1} X_2, X_3 \rangle \) is independent of \( \epsilon \) if \( X_i \) are contained in \( \ker \eta \). It follows that
\[
\nabla_{X_1} X_2 = (\nabla_{X_1} X_2)_{\text{hor}} - \frac{1}{\epsilon^2} \langle X_2, \nabla_{X_1} V \rangle V
\]
\[
= (\nabla_{X_1} X_2)_{\text{hor}} + \frac{1}{2} \langle X_2, J X_1 \rangle V.
\]
This also gives \( \nabla_Y V = -\frac{\epsilon^2}{2} J Y \) for all \( Y \). Finally,
\[
\langle \nabla_V X_1, X_2 \rangle = \langle [V, X], X_2 \rangle - \frac{1}{2} \langle [X, X_2], V \rangle
\]
\[
= \langle [V, X], X_2 \rangle - \frac{\epsilon^2}{2} \eta ([X_1, X_2])
\]
\[
= \langle [V, X], X_2 \rangle + \frac{\epsilon^2}{2} d\eta (X_1, X_2)
\]
\[
= \langle [V, X], X_2 \rangle - \frac{\epsilon^2}{2} \langle J X_1, X_2 \rangle
\]
for all sections \( X_1 \) and \( X_2 \) of \( \ker \eta \). Therefore, the last assertion follows. \( \square \)
**Proof of Proposition 2.3.** By Proposition 2.1, we have

\[
\nabla Y_1 \nabla Y_2 V = \frac{\epsilon^2}{2} \nabla Y_1 (JY_2) = \frac{\epsilon^2}{2} \langle \nabla Y_1, J \rangle Y_2 - \frac{\epsilon^2}{2} J \nabla Y_1 Y_2
\]

\[
= \frac{\epsilon^2}{2} (\nabla (Y_1) \nabla J)(Y_2) \mathbf{hor} + \frac{\epsilon^2}{2} Y_2 \mathbf{V} - \frac{\epsilon^2}{2} J \nabla Y_1 Y_2
\]

\[
= \frac{\epsilon^2}{2} (\nabla (Y_1) \nabla J)(Y_2) \mathbf{hor} - \frac{\epsilon^2}{2} (\nabla (Y_1) \nabla J) V - \frac{\epsilon^2}{2} J \nabla Y_1 Y_2
\]

\[
= \frac{\epsilon^2}{4} (Y_1, Y_2) V + \frac{\epsilon^2}{4} \langle Y_2, \mathbf{V} \rangle (Y_1) \mathbf{hor} - \frac{\epsilon^2}{2} J \nabla Y_1 Y_2.
\]

Therefore,

\[
\text{Rm}(Y_1, Y_2) V = \nabla Y_1 \nabla Y_2 V - \nabla Y_2 \nabla Y_1 V - \nabla [Y_1, Y_2] V
\]

\[
= \frac{\epsilon^2}{4} \langle Y_2, \mathbf{V} \rangle (Y_1) \mathbf{hor} - \frac{\epsilon^2}{4} \langle Y_1, \mathbf{V} \rangle (Y_2) \mathbf{hor}.
\]

This is the first assertion.

If \(X_1\) and \(X_2\) are in ker \(\eta\), then

\[
\nabla X_3 X_1 = (\nabla X_3 X_1) \mathbf{hor} = \nabla X_3 X_1 - \frac{\langle JX_2, X_1 \rangle}{2} \mathbf{V}.
\]

It follows that

\[
\nabla X_3 \nabla X_2 X_1 = \nabla X_3 \nabla X_2 X_1 - \frac{\langle \nabla X_2 X_1, JX_3 \rangle}{2} V
\]

\[
= \nabla X_3 \left( \nabla X_2 X_1 - \frac{\langle JX_2, X_1 \rangle}{2} V \right) - \frac{\langle \nabla X_2 X_1, JX_3 \rangle}{2} V
\]

\[
= \nabla X_3 \nabla X_2 X_1 + \frac{\epsilon^2}{4} \langle JX_2, X_1 \rangle JX_3 - \frac{\langle J \nabla X_2 X_1, X_3 \rangle}{2} V
\]

\[
- \frac{\langle JX_2, \nabla X_3 X_1 \rangle}{2} V - \frac{\langle \nabla X_2 X_1, JX_3 \rangle}{2} V.
\]

and

\[
\nabla [X_2, X_3] X_1 = \nabla [X_2, X_3] X_1 - \frac{1}{2} \langle J[X_2, X_3], X_1 \rangle V + \frac{\langle V, [X_2, X_3] \rangle}{2} JX_1.
\]

Therefore,

\[
\text{Rm}(X_2, X_3) X_1 = \text{Rm}(X_2, X_3) X_1 + \frac{\epsilon^2}{4} \langle JX_3, X_1 \rangle JX_2
\]

\[
- \frac{\epsilon^2}{4} \langle JX_2, X_1 \rangle JX_3 - \frac{\epsilon^2}{2} \langle JX_2, X_3 \rangle JX_1.
\]

This gives the second assertion. The proofs of the remaining claims follow in a similar manner and are omitted. \(\square\)
ON MCP WITHOUT RICCI CURVATURE LOWER BOUND

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