Prefab posets’ Whitney numbers

A. Krzysztof Kwaśniewski
High School of Mathematics and Applied Informatics
Kamienna 17, PL-15-021 Bialystok, Poland
e-mail: kwandr@wp.pl

Summary
We introduce a natural partial order $\leq$ in structurally natural finite subsets the cobweb prefabs sets recently constructed by the present author. Whitney numbers of the second kind of the corresponding subposet which constitute Stirling-like numbers’ triangular array - are then calculated and the explicit formula for them is provided. Next - in the second construction - we endow the set sums of prefabiants with such an another partial order that their their Bell-like numbers include Fibonacci triad sequences introduced recently by the present author in order to extend famous relation between binomial Newton coefficients and Fibonacci numbers onto the infinity of their relatives among which there are also the Fibonacci triad sequences and binomial-like coefficients (incidence coefficients included).

AMS Classification Numbers: 05C20, 11C08, 17B56.

Key Words: prefab, cobweb poset, Whitney numbers, Bell like numbers, Fibonacci like sequences

presentation (November 2005) at the Gian-Carlo Rota Polish Seminar
http://ii.uwb.edu.pl/akk/index.html
to appear in Bull. Soc. Sci. Lett. Lodz. October 2005.

1 Introduction
The clue algebraic concept of combinatorics - prefab (with associative and commutative composition) was introduced in [1], see also [2,3]. In [4] the present author hadconstructed a new broader class of prefab’s notion extending combinatorial structure based on the so called cobweb posets (see Section 1. [4] for the definition of a cobweb poset as well as a combinatorial interpretation of its characteristic binomial-type coefficients - for example- fibonomial ones [5,6]).

Here we introduce two natural partial orders: one $\leq$ in grading-natural subsets of cobweb’s prefabs sets [4] and in the second proposal we endow the set sums of prefabiants with such another partial order that one arrives at Bell-like numbers including Fibonacci triad sequences introduced by the present author in [7].

2 Prefab based posets and their Whitney numbers.

Let the family $S$ of combinatorial objects (prefabiants) consists of all layers $\langle \Phi_k \rightarrow \Phi_n \rangle$, $k < n$, $k, n \in N \cup \{0\} \equiv Z_\geq$ and an empty prefabiant $i$.

The set $\varphi$ of prime objects consists of all sub-posets $\langle \Phi_0 \rightarrow \Phi_m \rangle$ i.e. all $P_m$’s $m \in N \cup \{0\} \equiv Z_\geq$ constitute from now on a family of prime prefabiants [4].

Layer is considered here to be the set of all max-disjoint isomorphic copies (iso-copies) of $P_{n-k} = P_n$ [4].
As a matter of illustration we quote after [4] examples of cobweb posets’ Hasse Diagrams so that the layers become visualized.

Fig.1. Display of Natural numbers’ cobweb poset.

Fig.2. Display of Even Natural numbers $\cup \{1\}$ - cobweb poset.
Fig 3. Display of Odd natural numbers' cobweb poset.

and so on up

\[ n_4 = 7 \]
\[ n_3 = 5 \]
\[ n_2 = 3 \]
\[ n_1 = 1 \]

Fig 4. Display of divisible by 3 natural numbers \( \cup \{1\} \) - cobweb poset.

and so on up

\[ n_4 = 9 \]
\[ n_3 = 6 \]
\[ n_2 = 3 \]
\[ n_1 = 1 \]
Consider then now the partially ordered family $S$ of these layers considered to be sets of all max-disjoint isomorphic copies (iso-copies) of prime prefabints $P_m = P_{n-k}$ displayed by Fig 1. - Fig. 5. above [4]. Let us define in $S$ the partial order relation as follows.

**Definition 1**

$$\langle \Phi_k \rightarrow \Phi_n \rangle \leq \langle \Phi_k \blacktriangleleft \rightarrow \Phi_n \blacktriangleleft \rangle \equiv k \leq k \blacktriangleleft \land n \leq n \blacktriangleleft .$$

For convenience reasons we shall also adopt and use the following notation:

$$\langle \Phi_k \rightarrow \Phi_n \rangle = p_{k,n}.$$ 

The interval $[p_{k,n}, p_{k,n} \blacktriangleleft]$ is of course a subposet of $(S, \leq)$. We shall consider in what follows the subposet $(P_{k,n}, \leq)$ where

$$P_{k,n} = [p_{a,o}, p_{k,n}].$$

**Observation 1.** The size $|P_{k,n}|$ of $P_{k,n} = |\{(l,m), \ 0 \leq l \leq k \land 0 \leq m \leq n \land k \leq n\}| = (n-k)(k+1) + \frac{k(k+1)}{2}$.

Proof: Obvious. Just draw the picture $\{(l,m), \ 0 \leq l \leq k \land 0 \leq m \leq n \land k \leq n\}$ of $P_{k,n}$ grid.

**Observation 2.** The number of maximal chains in $(P_{k,n}, \leq)$ is equal to the number $d(k,n)$ of 0-dominated strings of binary i.e. 0’s and 1’s sequences

$$d(k,n) = \frac{n+1-k}{n} \binom{k+n}{n}.$$ 

Proof. The number we are looking for equals to the number of minimal walk-paths in Manhattan grid $[8] [k \times n]$ restricted by the condition $k \leq n$ i.e. it equals to the number of 0-dominated strings of 0’s and 1’s sequences.
Recall that \((d(k, n))\) infinite matrix’s diagonal elements are equal to the **Catalan** numbers \(C(n)\)

\[
C(n) = \frac{1}{n} \binom{2n}{n}.
\]

as the Catalan numbers count the number of 0-dominated strings of 0’s and 1’s with equal number of 0’s and 1’s. Recall that a 0-dominated string of length \(n\) is such a string that the first \(k\) digits of the string contain at least as many 0’s as 1’s for \(k = 1, ..., n\) i.e. 0’s prevail in appearance, dominate 1’s from the left to the right end of the string.

0-dominated strings correspond bijectively to minimal bottom-left to the right upper corner paths in an integer grid \(\mathbb{Z}_+ \times \mathbb{Z}_+\) rectangle part called Manhattan [8] with the restriction imposed on those minimal paths to obey the “safety” condition \(k \leq n\).

**Comment 1.** Observation 2. equips the poset \(< P_{k,n}, \leq >\) with clear cut combinatorial meaning.

The poset \(< P_{k,n}, \leq >\) is naturally graded. \(< P_{k,n}, \leq >\) poset’s maximal chains are of all of equal size (Dedekind property) therefore the rang function is defined.

**Observation 3.** The rang \(r(P_{k,n})\) of \(P_{k,n}\) = number of elements in maximal chains \(P_{k,n}\) minus one \(= k + n - 1\). The rang \(r(P_{l,m})\) is defined accordingly: \(r(P_{l,m}) = l + m - 1\).

Proof: obvious. Just draw the picture \(\{(l, m), \ 0 \leq l \leq k \ \land \ 0 \leq m \leq \ l \ \land \ k \leq n\}\) of \(P_{k,n}\)’ grid and note that maximal means paths without at a slant edges.

Accordingly Whitney numbers \(W_k(P_{l,m})\) of the second kind are defined as follows (association: \(n \leftrightarrow \langle l, m \rangle\))

**Definition 2**

\[
W_k(P_{l,m}) = \sum_{\pi \in P_{l,m}, r(\pi) = k} 1 \equiv S(k, \langle l, m \rangle).
\]

Here now and afterwards we identify \(W_k(P_{l,m})\) with \(S(k, \langle l, m \rangle)\) called and viewed at as Stirling-like numbers of the second kind of the naturally graded poset \(< P_{k,n}, \leq >\) - note the association: \(n \leftrightarrow \langle l, m \rangle\).

**Right now challenge problems. 1.**

I. Let us define now Whitney numbers \(w_k(P_{l,m})\) of the first kind as follows (association: \(n \leftrightarrow \langle l, m \rangle\)). Note the text-book notation for Möbius function \(\mu\)

**Definition 3**

\[
w_k(P_{l,m}) = \sum_{\pi \in P_{l,m}, r(\pi) = k} \mu(0, \pi) \equiv s(k, \langle l, m \rangle).
\]

Here now and afterwards we identify \(w_k(P_{l,m})\) with \(s(k, \langle l, m \rangle)\) called and viewed at as Stirling-like numbers of the first kind of the poset \(< P_{k,n}, \leq >\) - note the association: \(n \leftrightarrow \langle l, m \rangle\).

**Problem 1** Find an explicit expression for \(w_k(P_{l,m}) \equiv s(k, \langle l, m \rangle) =?\)

and \(W_k(P_{l,m}) \equiv S(k, \langle l, m \rangle) =?\)
Occasionally note that $S(k, \langle l, m \rangle)$ equals to the number of the grid points counted at a slant (from the up-left to the right-down) accordingly to the $l + m = k$ requirement.

**Problem 2** Find the recurrence relations for

$$w_k(P_l, m) \equiv s(k, \langle l, m \rangle) \quad \text{and} \quad W_k(P_l, m) \equiv S(k, \langle l, m \rangle).$$

We define now (note the association: $n \leftrightarrow \langle l, m \rangle$) the corresponding Bell-like numbers $B(\langle l, m \rangle)$ of the naturally graded poset $\langle P_{k,n}, \leq \rangle$ as follows.

**Definition 4**

$$B(\langle l, m \rangle) = \sum_k S(k, \langle l, m \rangle).$$

**Observation 4.**

$$B(\langle l, m \rangle) = |P_{l,m}| = \frac{k(k + 1)}{2} + (n - k)(k + 1).$$

Proof: Just draw the picture $\{(l, m), \ 0 \leq l \leq k \land 0 \leq m \leq n \land k \leq n\}$ of $P_{k,n}$ grid and note that $S(k, \langle l, m \rangle)$ equals to the number of the grid points counted at a slant (from the up-left to the right-down) accordingly to the $l + m = k$ requirement. Summing them up over all gives the size of $P_{k,n}$.

**Comment 2.** Observation 4. equips the poset’s $\langle P_{k,n}, \leq \rangle$ Bell-like numbers $B(\langle l, m \rangle)$ with clear cut combinatorial meaning.

### 3 Set Sums of prefabiants’ posets and their Whitney numbers.

In this part we consider prefabiants’ set sums with an appropriate another partial order so as to arrive at Bell-like numbers including Fibonacci triad sequences introduced recently by the present author in [7] - see also [9].

Let $F$ be any "GCD-morphic" sequence [4]. This means that $\text{GCD}[F_n, F_m] = F_{\text{GCD}[n,m]}$ where GCD stays for Greatest Common Divisor operator. We define the finite partial ordered set $P(n, F)$ as the set of prime prefabiants $P_l$ given by the sum below.

**Definition 5**

$$P(n, F) = \bigcup_{0 \leq p} \langle \Phi_p \rightarrow \Phi_{n-p} \rangle = \bigcup_{0 \leq l} P_{n-l}$$

with the partial order relation defined for $n - 2l \leq 0$ according to

**Definition 6**

$$P_l \leq P_l \quad \equiv \quad l \leq \hat{l}, \ P_l, P_{\hat{l}} \in \langle \Phi_l \rightarrow \Phi_{n-l} \rangle.$$  

Recall that rang of $P_l$ is $l$. Note that $\langle \Phi_l \rightarrow \Phi_{n-l} \rangle = \emptyset$ for $n - 2l \leq 0$. The Whitney numbers of the second kind are introduce accordingly.

**Definition 7**

$$W_k(P_n, F) = \sum_{\pi \in P_n, F, r(\pi) = k} S(n, k, F).$$
Right from the definitions above we infer that: (recall that \textbf{rang of } P_i \textbf{ is } l.)

\textbf{Observation 5.}

\[ W_k(P_n,F) = \sum_{\pi \in P_n, \pi(F) = k} S(k, n, k, F) = \binom{n-k}{k}. \]

Here now and afterwards we identify \( W_k(P_n,F) = S(n, k, F) \) viewed at and called as Stirling - like numbers of the second kind of the \( P \) defined in [4]. \( P \) by construction (see Figures above) displays self-similarity property with respect to its prime prefabrians sub- posets \( P_n = P(n, F) \).

\textbf{Right now challenge problems. II.}

We repeat with obvious replacements of corresponding symbols, names and definitions the same problems as in "Right now challenge problems. I".

Here now consequently - for any \( GCD \)-morphic sequence \( F \) (see: [4]) we define the corresponding Bell-like numbers \( B_n(F) \) of the poset \( P(n, F) \) as follows.

\textbf{Definition 8}

\[ B_n(F) = \sum_{k \geq 0} S(n, k, F). \]

Due to the investigation in [9,7] we have right now at our disposal all corresponding results of [7,9] as the following identification with special case of \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequence \( F^{\alpha, \beta, \gamma}_n \) defined in [7] holds.

\textbf{Observation 6.}

\[ B_n(F) \equiv F^{\alpha=0, \beta=0, \gamma=0}_{n+1}. \]

\textbf{Proof:} See the Definition 2.2. from [7]. Compare also with the special case of formula (6) in [9].

\textbf{Recurrence relations.} Recurrence relations for \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequences \( F^{\alpha, \beta, \gamma}_n \) are to be found in [7] - formula (9). Compare also with the special case formula (7) in [9].

\textbf{Closing-Opening Remark.} The study of further properties of these Bell-like numbers as well as the study of consequences of these identifications for the domain of the widespread data types [7] and perhaps for eventual new dynamical data types we leave for the possibly coming future. Examples of special cases - a bunch of them - one finds in [7] containing [9] as a special case. As seen from the identification Observation 6. the special cases of \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequences \( F^{\alpha, \beta, \gamma}_n \) gain \textbf{additional} with respect to [9,7] combinatorial interpretation in terms Bell-like numbers as sums over rang = \( k \) parts of the poset i.e. just sums of Whitney numbers of the poset \( P(n, F) \). This adjective "additional" shines brightly over Newton binomial connection constants between bases \( \langle (x-1)^k \rangle_{k \geq 0} \) and \( \langle x^n \rangle_{n \geq 0} \) as these are Whitney numbers of the numbers from \( [n] \) chain i.e. Whitney numbers of the poset \( ([n], \leq) \). For other elementary "shining brightly" examples see Joni \, Rota and Sagan excellent presentation in [10].

\textbf{Acknowledgements}

Discussions with Participants of Gian-Carlo Rota Polish Seminar on all related topics \url{http://ii.uwb.edu.pl/akk/index.html} - are appreciated with pleasure.

\textbf{References}

[1] E. Bender, J. Goldman \textit{Enumerative uses of generating functions} , Indiana Univ. Math.J. \textbf{20} 1971), 753-765.
[2] D. Foata and M. Schützenberger, Théorie géométrique des polynomes euleriens, (Lecture Notes in Math., No. 138). Springer-Verlag, Berlin and New York, 1970.

[3] A. Nijenhuis and H. S. Wilf, Combinatorial Algorithms, 2nd ed., Academic Press, New York, 1978.

[4] A. K. Kwaśniewski, Cobweb posets as noncommutative prefabs submitted for publication ArXiv: [math.CO/0503286] (2005)

[5] A. K. Kwaśniewski, Information on combinatorial interpretation of Fibonomial coefficients Bull. Soc. Sci. Lett. Lodz Ser. Rech. Deform. 53, Ser. Rech.Deform. 42 (2003), 39-41. ArXiv: [math.CO/0402291] v1 18 Feb 2004

[6] A. K. Kwaśniewski, The logarithmic Fib-binomials Advanced Stud. Contemp. Math. 9 No 1 (2004), 19-26. ArXiv: [math.CO/0406258] 13 June 2004.

[7] A. K. Kwaśniewski, Fibonacci-triad sequences Advan. Stud. Contemp. Math. 9 (2) (2004), 109-118.

[8] Z. Palka, A. Ruciski Lectures on Combinatorics I. WNT Warsaw 1998 (in polish)

[9] A. K. Kwaśniewski, Fibonacci q-Gauss sequences Advanced Studies in Contemporary Mathematics 8 No 2 (2004), 121-124. ArXiv: [math.CO/0405591] 31 May 2004.

[10] S.A. Joni, G.C. Rota, B. Sagan From sets to functions: three elementary examples Discrete Mathematics 37 (1981), 193-2002.