ON THE ABELIAN GROUPS WHICH OCCUR AS
GALOIS COHOMOLOGY GROUPS OF GLOBAL UNIT
GROUPS

MANABU OZAKI

1. Introduction

For any number field $K$ (number field means a finite extension fields of $\mathbb{Q}$ in what follows), we denote by $E_K$ the unit group of $K$. If $K/k$ is a Galois extension of number fields, then we define the Galois cohomology group $\hat{H}^i(K/k, E_K) := H^i(\text{Gal}(K/k), E_K)$ for $i \in \mathbb{Z}$. Here, $H^i(G, M)$ stands for the $i$-th Tate cohomology group for a finite group $G$ and a $G$-module $M$.

We are interested in Galois cohomology groups $\hat{H}^i(K/k, E_K)$ for many reasons. For example, when $K/k$ is an unramified extension, such cohomology groups are directly related to the capitulation of ideals, which is one of the major themes of algebraic number theory; If $K/k$ is unramified, then we have

\begin{align*}
H^1(K/k, E_K) &\simeq \ker(j_{K/k} : \text{Cl}_k \longrightarrow \text{Cl}_K^{\text{Gal}(K/k)}), \\
H^2(K/k, E_K) &\simeq \text{coker}(j_{K/k} : \text{Cl}_k \longrightarrow \text{Cl}_K^{\text{Gal}(K/k)}),
\end{align*}

where $j_{K/k}$ is the natural map from the class group $\text{Cl}_k$ of $k$ to that $\text{Cl}_K$ of $K$ induced by the inclusion $k \subseteq K$.

In the present paper, we shall investigate the following problem:

**Problem.** For a finite group $G$ and $i \in \mathbb{Z}$, we define

$\mathcal{H}^i(G) := \{[\hat{H}^i(K/k, E_K)] \mid K/k: \text{unramified } G\text{-ext.’n. of number fields}\},$

where $[A]$ denotes the isomorphism class of $A$ for any abelian group $A$. What is $\mathcal{H}^i(G)$?

We first note that a simple observation using Dirichlet’s unit theorem and properties of Tate cohomology groups shows

$\mathcal{H}^i(G) \subseteq \mathcal{A}(\#G),$

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where $\mathcal{A}(n)$ stands for the set of all the isomorphism classes of the finite abelian groups killed by $n$ for $n \in \mathbb{Z}$.

The case $i = -1$ and $G = \mathbb{Z}/p$, $p$ being a prime number, is the most classical situation, which was first considered by D. Hilbert [7]. He showed $[0] \not\in \mathcal{H}^{-1}(\mathbb{Z}/p)$ in [7, Satz 92], which is equivalent to $\ker j_{K/k} \neq 0$ for any unramified $\mathbb{Z}/p$-extension $K/k$ of number fields by (1) (Note that $\hat{H}^{-1}(G, M) \simeq \hat{H}^{1}(G, M)$ if $G$ is cyclic). This latter claim is stated in [7, Satz 94].

We will reduce our problem to a problem of group theory. At present, this reduction has been completely done only in the case where $G$ is a finite $p$-group, $p$ being a prime number. Then, in the case where $G$ is a finite $p$-group, we will determine $\mathcal{H}^i(G)$ for $i = 0, 1, 2, \text{and} 4$. We owe determination of $\mathcal{H}^1(G)$ to a series of extensive works by K. W. Gruenberg and A. Weiss [4], [5], [6].

2. Description of $\hat{H}^i(K/k, E_K)$

We recall the notion of a splitting module to describe cohomology groups of unit groups in terms of structures of certain Galois groups.

Let

$$
(\varepsilon) \quad 1 \longrightarrow A \longrightarrow G \longrightarrow G \longrightarrow 1
$$

be a group extension of finite groups with abelian kernel $A$. We denote by $\gamma_\varepsilon \in H^2(G, A)$ the cohomology class associated to $(\varepsilon)$. Define the $G$-module $M_\varepsilon$ so that

$$
M_\varepsilon = A \oplus \bigoplus_{1 \neq \tau \in G} \mathbb{Z}b_\tau
$$

as $\mathbb{Z}$-modules. Here $\{b_\tau\}$ is a free $\mathbb{Z}$-basis, and that $G$-action on $M_\varepsilon$ is given by the natural $G$-module structure of $A$ defined by $(\varepsilon)$ and

$$
\sigma b_\tau = b_{\sigma \tau} - b_\sigma + f(\sigma, \tau)
$$

for $\sigma, \tau \in G$, where $f$ is a 2-cocycle in the cohomology class $\gamma_\varepsilon$ and we set $b_1 := f(1, 1) \in A$. We call $M_\varepsilon$ the splitting module associated to $(\varepsilon)$, and note that the $G$-module isomorphism class of $M_\varepsilon$ is independent of the choice of a 2-cocycle $f$ of the cohomology class $\gamma_\varepsilon$.

Then we derive from group extension $(\varepsilon)$ the exact sequence of $G$-modules

$$
(2) \quad (\varepsilon^*) \quad 0 \longrightarrow A \longrightarrow M_\varepsilon \longrightarrow I_G \longrightarrow 0,
$$

where $I_G$ denotes the augmentation ideal of $\mathbb{Z}[G]$ and the map $M_\varepsilon \longrightarrow I_G$ is given by $b_\sigma \mapsto \sigma - 1$ ($\sigma \in G$).
Conversely, from exact sequence of $G$-modules
\[ 0 \to A \to M \to I_G \to 0, \]
we get the group extension (modulo isomorphisms as group extensions of $G$ by $A$)
\[ (e^1) \quad 1 \to A \to G \to G \to 1 \]
associated to the cohomology class $[f] \in H^2(G, A)$ of the cocycle $f$ defined by
\[ f(\sigma, \tau) := \sigma s(\tau - 1) - s(\sigma \tau - 1) + s(\sigma - 1) \in A \quad (\sigma, \tau \in G), \]
where $s$ is a fixed section of $\varphi$ as $\mathbb{Z}$-modules. Then we find that
\[ (e^*1) \simeq (e), \quad (e^1) \simeq (e). \]
as group extensions of $G$ by $A$ and $G$-module extensions of $I_G$ by $A$, respectively.

Now we will describe the Galois cohomology groups $\hat{H}^i(K/k, E_K)$ for unramified Galois extensions $K/k$ in terms of group extensions naturally arising from certain towers of unramified Galois extensions.

Let $K/k$ be an unramified Galois extension of number fields, and we denote by $H_K$ the maximal unramified abelian extension of $K$. Put $G := \text{Gal}(K/k)$, $\mathcal{G} := \text{Gal}(H_K/k)$, and $A := \text{Gal}(H_K/K)$. Then we have the natural group extension
\[ (\varepsilon_{K/k}) \quad 1 \to A \to G \to G \to 1. \]

For any finite group $G$, we define $\mathcal{M}(G)$ to be the class of all the $G$-modules $M$ fitting into an exact sequence $0 \to B \to M \to I_G \to 0$ with a finite $G$-module $B$, and put
\[ \mathcal{X}^i(G) := \{ [\hat{H}^i(G, M)] | M \in \mathcal{M}(G) \}. \]

**Proposition 1.** For any unramified Galois extension $K/k$ of number fields with $G = \text{Gal}(K/k)$ and $i \in \mathbb{Z}$, we have
\[ \hat{H}^i(K/k, E_K) \simeq \hat{H}^{i-2}(G, M_{(\varepsilon_{K/k})}). \]
Hence $\mathcal{H}^i(G) \subseteq \mathcal{X}^{i-2}(G)$ holds.

**Proof.** In fact, this proposition follows from a special case of Tate sequence given by Ritter-Weiss [11]. However we will give a direct proof, which is based on essentially same method as theirs specialized to the most simple situation, namely, the Galois extension considered is unramified.
Let $J_K$, $U_K$, $C_K$, and $\text{Cl}_K$ be the idele group, the unit idele group, the idele class group, and the ideal class group of $K$, respectively. We will observe the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & \downarrow & \downarrow & U_K/E_K & \sim & H & \downarrow & \downarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & C_K & \longrightarrow & M(\mathfrak{v}) & \longrightarrow & I_G & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \text{Cl}_K & \longrightarrow & M(\varepsilon) & \longrightarrow & I_G & \longrightarrow & 0 \\
\end{array}
\]

where $(\mathfrak{v})$ is the group extension

\[1 \rightarrow C_K \rightarrow \mathfrak{g} \rightarrow G \rightarrow 1\]

associated to the fundamental class $c_{K/k} \in H^2(G, C_K)$, and $(\varepsilon)$ is the group extension

\[1 \rightarrow \text{Cl}_K \rightarrow \mathfrak{g} \rightarrow G \rightarrow 1\]

associated to the image of $c_{K/k}$ under the natural map $H^2(G, C_K) \rightarrow H^2(G, \text{Cl}_K)$. It follows from Shafarevich's theorem ([1, Chapter 15, Theorem 6]) that the group extensions $(\varepsilon)$ and $(\varepsilon_{K/k})$ of $G$ are isomorphic via a morphism inducing the Artin map $\text{Cl}_K \simeq \text{Gal}(H_K/K)$, hence $M(\varepsilon) \simeq M(\varepsilon_{K/k})$ as $G$-modules. The $G$-module $U_K$ is cohomologically trivial since $K/k$ is unramified, and we know $M(\mathfrak{v})$ is also cohomologically trivial (see, for example, [9, Theorem (3.1.4)]). Therefore we have

\[
\hat{H}^i(G, E_K) \simeq \hat{H}^{i-1}(G, U_K/E_K) \simeq \hat{H}^{i-1}(G, H) \\
\simeq \hat{H}^{i-2}(G, M(\varepsilon)) \simeq \hat{H}^{i-2}(G, M(\varepsilon_{K/k})) \in \mathcal{X}^{i-2}(G),
\]

since $M(\varepsilon_{K/k}) \in \mathcal{M}(G)$. 

\[\square\]
We make a remark on the case where $G$ is a $p$-group. In this case, we naturally define the $p$-quotient of a group extension

$$(\varepsilon) \quad 1 \rightarrow A \rightarrow \mathcal{G} \rightarrow G \rightarrow 1$$

with finite abelian $A$ to be

$$(\varepsilon)_p \quad 1 \rightarrow A \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \mathcal{G}_p \rightarrow G \rightarrow 1,$$

where $\mathcal{G}_p$ is the maximal $p$-quotient of $\mathcal{G}$. Under this situation, we see that $\hat{H}^i(G, M(\varepsilon)) \simeq \hat{H}^i(G, M(\varepsilon)_p)$. Hence it is enough for our problem to take account of only the extensions of the forms

$$1 \rightarrow A \rightarrow \mathcal{G} \rightarrow G \rightarrow 1$$

and

$$0 \rightarrow A \rightarrow M \rightarrow I_G \rightarrow 0$$

with $\#A$ being a power of $p$ if $G$ is a $p$-group.

### 3. Reduction to group theory

In the case where $G$ is a $p$-group, we will reduce our problem in Introduction to a problem of group theory by using the following fact:

**Theorem 1** ([II]). For any finite $p$-group $\mathcal{G}$, there exists a number field $k$ such that

$$\text{Gal}(L_p(k)/k) \simeq \mathcal{G},$$

where $L_p(k)$ stands for the maximal unramified $p$-extension of $k$.

We can immediately derive the following from the above theorem:

**Corollary 1.** For any given group extension of finite $p$-groups

$$(\varepsilon) \quad 1 \rightarrow A \rightarrow \mathcal{G} \rightarrow G \rightarrow 1$$

with abelian $A$, there exists an unramified $G$-extension $K/k$ such that $(\varepsilon_{K/k})_p \simeq (\varepsilon)$, namely, there exists a group isomorphism $\alpha : \mathcal{G} \simeq \text{Gal}(L_p(K)/k)$ such that

$$1 \rightarrow \text{Gal}(L_p(K)/K) \rightarrow \text{Gal}(L_p(K)/k) \rightarrow \text{Gal}(K/k) \rightarrow 1$$

is an exact commutative diagram, where $\beta$ and $\gamma$ are the isomorphisms induced by $\alpha$.

**Theorem 2.** For any finite $p$-group $G$, we have $\mathcal{H}^i(G) = \mathfrak{X}^{i-2}(G)$ for $i \in \mathbb{Z}$.
Proof. It follows from Proposition 1 that $\mathcal{H}^i(G) \subseteq \mathfrak{X}^{i-2}(G)$.

Conversely, let $[\hat{H}^{i-2}(G, M)] \in \mathfrak{X}^{i-2}(G)$ ($M \in \mathcal{M}(G)$) be any element. Then there exists a group extension $\langle \varepsilon \rangle$ such that $M \simeq M_{\langle \varepsilon \rangle}$ by (3). We derive from Corollary 1 that there exists an unramified $G$-extension $K/k$ of number fields such that $(\varepsilon)_p \simeq (\varepsilon_{K/k})_p$. Hence, by using Proposition 1 and the remark after it, we have

$$[\hat{H}^{i-2}(G, M)] = [\hat{H}^{i-2}(G, M_{\langle \varepsilon \rangle_k})] = [\hat{H}^{i}(K/k, E_K)] \in \mathcal{H}^i(G).$$

Thanks to this theorem, our problem is completely reduced to a purely group theoretic problem in the case where $G$ is a finite $p$-group.

It is highly interesting and difficult to search whether $\mathcal{H}^i(G) = \mathfrak{X}^{i-2}(G)$ holds for general finite groups $G$.

4. Cases $i = 2$ and $i = 4$.

Now we will investigate our problem for finite $p$-groups $G$ by using Theorem 2. We start with rather easy cases:

Theorem 3. For any finite $p$-group $G$, we have

$$\mathcal{H}^2(G) = \mathcal{H}^4(G) = \mathcal{A}(\#G).$$

Proof. We first show that $\mathcal{H}^2(G) = \mathcal{A}(\#G)$. For any $[X] \in \mathcal{A}(\#G)$, let $M := X \oplus I_G \in \mathcal{M}(G)$, where we make $G$ act on $X$ trivially. Then we have $[X] = [\hat{H}^0(G, M)] \in \mathfrak{X}^0(G) = \mathcal{H}^2(G)$ by Theorem 2 because $\hat{H}^0(G, I_G) = 0$ and $\hat{H}^0(G, X) \simeq X$. This shows $\mathcal{A}(\#G) \subseteq \mathcal{H}^2(G)$. Converse inclusion clearly holds.

Next we prove $\mathcal{H}^4(G) = \mathcal{A}(\#G)$. For any $[X] \in \mathcal{A}(\#G)$, we view $X$ as a $G$-module with trivial $G$-action as above. We choose a surjection $(\mathbb{Z}/\#G)[G]^{\oplus r} \to X$ as $G$-modules and get an exact sequence

$$0 \to Y \to (\mathbb{Z}/\#G)[G]^{\oplus r} \to X \to 0$$

of $G$-modules. By the same manner, we get a exact sequence

$$0 \to Z \to (\mathbb{Z}/\#G)[G]^{\oplus s} \to Y \to 0$$

of $G$-modules for some $s \geq 0$ and finite $G$-module $Z$. Then we have $Z \oplus I_G \in \mathcal{M}(G)$ and

$$\hat{H}^2(G, Z \oplus I_G) \simeq \hat{H}^2(G, Z) \simeq \hat{H}^1(G, Y) \simeq \hat{H}^0(G, X) \simeq X$$

because $H^2(G, I_G) = 0$. Thus we have shown $[X] \in \mathfrak{X}^2(G) = \mathcal{H}^4(G)$ by Theorem 2. This implies our claim. \qed

5. Case \( i = 1 \).

The case \( i = 1 \) is the most classical situation and many group theoretic approaches have been available:

The following theorem is one of the most striking results given after the principal ideal theorem was proved by Frutwängler:

**Theorem 4** (H. Suzuki [12]). For any finite abelian group of order \( n \), we have

\[
\mathfrak{X}^{-1}(G) \subseteq \{ [X] \in \mathcal{A}(n) \mid n \mid \#X \}. 
\]

□

**Corollary 2.** If \( K/k \) is an unramified abelian extension, then we have

\[
[K : k] \mid \# \ker(\text{Cl}_k \to \text{Cl}_K).
\]

\[
\text{Proof. } [\ker(\text{Cl}_k \to \text{Cl}_K)] = [\hat{H}^1(K/k, E_K)] \in \mathcal{H}^1(G) \subseteq \mathfrak{X}^{-1}(G) \subseteq \{ [X] \in \mathcal{A}(n) \mid n \mid \#X \} \text{ by Proposition 1 and Theorem 4. } \]

We note that this corollary implies both of Hilbert’s Satz 94 and the principal ideal theorem. In fact, Suzuki’s theorem can be strengthened as follows:

**Theorem 5** ((K.W.Gruenberg–A.Weiss [4]). For any abelian group \( G \), we have

\[
\mathfrak{X}^{-1}(G) = \{ [X] \in \mathcal{A}(\#G) \mid \#G \mid \#X \}. 
\]

□

Thanks to this theorem and Theorem 2, we obtain;

**Theorem 6.** For any finite abelian \( p \)-group \( G \), we have

\[
\mathcal{H}^1(G) = \{ [X] \in \mathcal{A}(\#G) \mid \#G \mid \#X \}. 
\]

□

For any given finite group \( G \), Gruenberg-Weiss [5] showed that there exists an effectively computable finite subset \( \mathfrak{X}_{\text{min}}^{-1}(G) \) of \( \mathfrak{X}^{-1}(G) \) such that

\[
\mathfrak{X}^{-1}(G) = \{ [X] \in \mathcal{A}(\#G) \mid [X] \text{ has a quotient in } \mathfrak{X}_{\text{min}}^{-1}(G) \}.
\]

Therefore we can determine \( \mathcal{H}^1(G) \) effectively when \( G \) is a finite \( p \)-group by using Theorem 2 and the work of Gruenberg and Weiss.

6. Case \( i = 0 \).

In this section, we will show;

**Theorem 7.** For any finite \( p \)-group \( G \), we have

\[
\mathcal{H}^0(G) = \mathcal{A}(\#G).
\]
We reduce the above theorem as follows:

**Lemma 1.** If \([0] \in \mathfrak{X}^{-2}(G)\) for a finite \(p\)-group \(G\), then we have \(\mathcal{H}^0(G) = \mathcal{A}(\#G)\).

**Proof.** It is sufficient to show \(\mathcal{A}(\#G) \subseteq \mathcal{H}^0(G)\). For any \([X] \in \mathcal{A}(\#G)\), we view \(X\) as a \(G\)-module with trivial \(G\)-action and choose an embedding

\[
X \hookrightarrow (\mathbb{Z}/\#G)^{\oplus r} \cong ((\mathbb{Z}/\#G)[G]^{\oplus r})^G \subseteq (\mathbb{Z}/\#G)[G]^{\oplus r}
\]

as \(G\)-modules for some \(r \geq 0\). Then we get the exact sequence

\[
0 \rightarrow X \rightarrow (\mathbb{Z}/\#G)[G]^{\oplus r} \rightarrow Y \rightarrow 0
\]

of \(G\)-modules with finite \(Y\).

Our assumption implies that there is a exact sequence of \(G\)-modules

\[
0 \rightarrow Z \rightarrow M \rightarrow I_G \rightarrow 0
\]

such that \(\#Z < \infty\) and \(\hat{H}^{-2}(G, M) = 0\). Then we have the exact sequence of \(G\)-modules

\[
0 \rightarrow Z \oplus Y \rightarrow M \oplus Y \rightarrow I_G \rightarrow 0,
\]

which means \(M \oplus Y \in \mathcal{M}(G)\), and it follows from (4) that

\[
\hat{H}^{-2}(G, M \oplus Y) \cong \hat{H}^{-2}(G, Y) \cong \hat{H}^{-1}(G, X) \cong X.
\]

This implies \([X] \in \mathfrak{X}^{-2}(G) = \mathcal{H}^0(G)\) by Theorem 2. Thus we have shown \(\mathcal{A}(\#G) \subseteq \mathcal{H}^0(G)\). \(\square\)

To show \([0] \in \mathfrak{X}^{-2}(G)\), we give the following purely group theoretic proposition:

**Proposition 2.** (a) For any given finite \(p\)-group \(G\), there exists a surjective group homomorphism \(\pi : \overline{G} \rightarrow G\) such that \(\overline{G}\) is a pro-\(p\)-FAB group (namely, every open subgroup of \(\overline{G}\) has the finite abelianization) with \(H_2(\overline{G}, \mathbb{Z}_p) = 0\).

(b) Let \(\pi : \overline{G} \rightarrow G\) be any surjective homomorphism with properties stated in (a), and put \(N = \ker \pi\). Then, for group extension

\[
(\varepsilon) \quad 1 \rightarrow N/(N, N) \rightarrow \overline{G}/(N, N) \xrightarrow{\pi} G \rightarrow 1,
\]

\(\pi\) being the map induced by \(\pi\), we have \(\hat{H}^{-2}(G, M_(\varepsilon)) = 0\), especially, \([0] \in \mathfrak{X}^{-2}(G)\).

Though Proposition 2 seems purely group theoretic, our proof of it employs largely number theory.
We recall two facts from number theory: We define $G_{\mathbb{Q},S}(p)$ to be the Galois group of the maximal $p$-extension over $\mathbb{Q}$ unramified outside $S$ for any set $S$ of primes of $\mathbb{Q}$.

**Theorem 8.** For any finite $p$-group $G$, there are a finite set $S$ of primes away from $p$ and a surjection $G_{\mathbb{Q},S}(p) \twoheadrightarrow G$.

**Proof.** This theorem follows from Shafarevich’s theorem (see [9, Chapter 9, Section 6]) for inverse Galois problem on solvable extensions over number fields, noting that we can make the prime $p$ unramified in a constructed $G$-extension over $\mathbb{Q}$. $\square$

**Theorem 9.** ([3, Theorem 4.9]) Let $S$ be a finite set of primes of $\mathbb{Q}$. We assume that the archimedean prime is contained in $S$ if $p = 2$. Then we have $H_2(G_{\mathbb{Q},S}(p), \mathbb{Z}_p) = 0$. $\square$

Part (a) follows immediately from the above two theorems because $G_{\mathbb{Q},S}(p)$ is a pro-$p$-FAB group if $p \not\in S$.

To prove part (b), we further recall the following facts from number theory:

**Theorem 10** (Folk [2]). Let $K/k$ be a finite $p$-extension of number fields of finite degree, and we denote by $H_{K,p}/K$ the maximal unramified abelian $p$-extension. Then we have $E_k \cap N_{H_{K,p}/k} (J_{K,p}) \subseteq N_{K/k}(E_K)$. $\square$

**Lemma 2.** Let $L/M$ and $M/k$ be unramified $p$-extensions of number fields of finite degree such that $L/k$ is normal. Then we have the commutative diagram

\[
\begin{array}{ccc}
E_k/(E_k \cap N_{L/k}(L^\times)) & \xleftarrow{\text{nat.proj.}} & H_2(L/k, \mathbb{Z}_p) \\
\downarrow & & \downarrow \text{co-inf} \\
E_k/(E_k \cap N_{M/k}(M^\times)) & \subseteq & H_2(M/k, \mathbb{Z}_p),
\end{array}
\]

where the right vertical map is the co-inflation map, namely, the map induced by the natural projection $\text{Gal}(L/k) \twoheadrightarrow \text{Gal}(M/k)$ and the identity map on $\mathbb{Z}_p$.

**Proof.** Since $L/k$ is unramified, the theory of number knot gives a certain canonical isomorphism

$H_2(L/k, \mathbb{Z}_p) \simeq (k^\times \cap N_{L/k}(J_L)) / N_{L/k}(L^\times)$

(see [13, Section 11.3]) and we naturally embed $E_k/(E_k \cap N_{L/k}(L^\times))$ into the right hand term of the above isomorphism. Thus we get the
upper horizontal map. We also obtain the lower horizontal map similarly. The commutativity follows from Horie-Horie [8, p.618, diagram (3)]. □

Proof of Proposition 2 (b). Let \( \pi : G \to G \) and \( N = \ker \pi \) be as in the statement of Proposition 2. The fact \( H_2(\mathcal{G}, \mathbb{Z}_p) = 0 \) implies that there exists an open normal subgroup \( H \) of \( G \) such that \( H \subseteq (N, N) \) and \( H_2(\mathcal{G}/H, \mathbb{Z}_p) \xrightarrow{\text{co-inf}} H_2(\mathcal{G}/(N, N), \mathbb{Z}_p) \) is the zero map. Choose \( k \) such that there exists an isomorphism \( \delta : \text{Gal}(L_p(k)/k) \simeq G/H \) by using Theorem 1, and define the tower of number fields \( k \subseteq K \subseteq M \subseteq L := L_p(k) \) so that \( \delta \) induces \( \text{Gal}(L/M) \simeq (N, N)/H \) and \( \text{Gal}(L/K) \simeq N/H \). Then we see that \( \delta \) induces \( \text{Gal}(K/k) \simeq \mathcal{G}/N \simeq G \) and \( M = H_{K,p} \), and by using Lemma 2 we get the commutative diagram

\[
\begin{array}{c}
E_k/(E_k \cap N_{L/k}(L^\times)) \xrightarrow{\text{nat.proj.}} H_2(\mathcal{G}/H, \mathbb{Z}_p) \\
E_k/(E_k \cap N_{H_{K,p}/k}(H_{K,p}^\times)) \xrightarrow{\text{co-inf}} H_2(\mathcal{G}/(N, N), \mathbb{Z}_p).
\end{array}
\]

Because the right vertical map in the above diagram is the zero map from our assumption, we find that \( E_k/(E_k \cap N_{H_{K,p}/k}(H_{K,p}^\times)) = 0 \), which implies \( E_k \subseteq N_{K/K}(E_K) \) by Theorem 10. Therefore we have \( \delta^{-1}(G, M_{(\varepsilon)}) \simeq \hat{H}^0(K/k, E_K) = 0 \) by Proposition 1, where \( (\varepsilon) \) is the group extension \( 1 \to N/(N, N) \to \mathcal{G}/(N, N) \xrightarrow{\varepsilon} G \to 1 \). □

Thus Theorem 7 follows from Lemma 1 and Proposition 2. □

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Manabu Ozaki,
Department of Mathematics,
School of Fundamental Science and Engineering,
Waseda University,
Ohkubo 3-4-1, Shinjuku-ku, Tokyo, 169-8555, Japan
e-mail: ozaki@waseda.jp