LOW REGULARITY LOCAL WELL-POSEDNESS FOR THE
CHERN-SIMONS-HIGGS SYSTEM IN TEMPORAL GAUGE

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Abstract. The Cauchy problem for the Chern-Simons-Higgs system in the
(2+1)-dimensional Minkowski space in temporal gauge is locally well-posed
for low regularity initial data improving a result of Huh. The proof uses the
bilinear space-time estimates in wave-Sobolev spaces by d’Ancona, Foschi and
Selberg and takes advantage of a null condition.

1. Introduction and main results
Consider the Chern-Simons-Higgs system in the Minkowski space \( \mathbb{R}^{1+2} = \mathbb{R} \times \mathbb{R}^2 \) with metric \( g_{\mu\nu} = \text{diag}(1, -1, -1) \):
\[
F_{\mu\nu} = 2\epsilon^{\mu\nu\rho} \text{Im}(\bar{\phi} D^\rho \phi) \quad (1)
\]
\[
D_\mu \phi = -\phi V'(|\phi|^2), \quad (2)
\]
with initial data
\[
A_\nu(0) = a_\nu, \quad \phi(0) = \phi_0, \quad (\partial_0 \phi)(0) = \phi_1, \quad (3)
\]
where we use the convention that repeated upper and lower indices are summed,
Greek indices run over 0,1,2 and Latin indices over 1,2. Here
\[
D^\mu := \partial_\mu - iA_\mu
\]
\[
F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.
\]
Here \( F_{\mu\nu} : \mathbb{R}^{1+2} \to \mathbb{R} \) denotes the curvature, \( \phi : \mathbb{R}^{1+2} \to \mathbb{C} \) a scalar field and
\( A_\nu : \mathbb{R}^{1+2} \to \mathbb{R} \) the gauge potentials. We use the notation \( \partial_\mu = \frac{\partial}{\partial x^\mu} \), where
we write \( (x^0, x^1, ..., x^n) = (t, x_1, ..., x_n) \) and also \( \partial_0 = \partial_t \) and \( \nabla = (\partial_1, \partial_2) \). \( \epsilon^{\mu\nu\rho} \)
is the totally skew-symmetric tensor with \( \epsilon^{012} = 1 \), and the Higgs potential \( V \)
is assumed to fulfill \( V \in C^\infty(\mathbb{R}^+, \mathbb{R}) \), \( V(0) = 0 \) and all derivatives of \( V \) have
polynomial growth.

This model was proposed by Hong, Kim and Pac [HKP] and Jackiw and
Weinberg [JW] in the study of vortex solutions in the abelian Chern-Simons theory.

The equations are invariant under the gauge transformations
\[
A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi, \quad \phi \to e^{\chi} \phi, \quad D_\mu \to D'_\mu = D_\mu - iA'_\mu.
\]
The most common gauges are the Coulomb gauge \( \partial^j A_j = 0 \), the Lorenz gauge
\( \partial^\mu A_\mu = 0 \) and the temporal gauge \( A_0 = 0 \). In this paper we exclusively study the
temporal gauge for low regularity data.

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Global well-posedness in the Coulomb gauge was proven by Chae and Choe [CC] for data \( a_\mu \in H^a \), \( \phi_0 \in H^b \), \( \phi_1 \in H^{b-1} \) where \((a,b) = (l,l+1)\) with \( l \geq 1 \), satisfying a compatibility condition and a class of Higgs potentials. Huh [H] showed local well-posedness in the Coulomb gauge for \((a,b) = (e,1+\epsilon)\), in the Lorenz gauge for \((a,b) = (\frac{4}{3}+\epsilon, \frac{2}{3}+\epsilon)\) or \((a,b) = (\frac{1}{2},\frac{1}{2})\), and in the temporal gauge for \((a,b) = (l,l)\) with \( l \geq \frac{3}{2} \). He also showed global well-posedness in the temporal gauge for \( l \geq 2 \). The local well-posedness result in the Lorenz gauge was improved to \((a,b) = (l,l+1)\) and \( l > \frac{5}{4} \) by Bournaveas [B] and by Yuan [Y]. Also in Lorenz gauge the very important global well-posedness result in energy space, where \( a_\mu \in H^{\frac{5}{4}} \), \( \phi_0 \in H^1 \), \( \phi_1 \in L^2 \), was proven by Selberg and Tesfahun [ST] under suitable assumptions on the potential \( V \), and even unconditional well-posedness could be proven by Selberg and Oliveira da Silva [SO]. In [ST] the regularity assumptions on the data could also be lowered down in Lorenz gauge to \((a,b) = (l,l+1)\) and \( l > \frac{5}{2} \). This latter result was improved to \( l > \frac{5}{4} \) by Huh and Oh [HO]. Global well-posedness in energy space and local well-posedness for \( a_\mu \in H^{\frac{5}{4}} \), \( \phi_0 \in H^{\frac{11}{4}} \), \( \phi_1 \in H^1 \) for \( \frac{5}{4} \geq l \geq \frac{5}{2} \) in Coulomb gauge was very recently obtained by Oh [O]. For all these results up to the paper by Chae [118] the compatibility assumption for data \( a \) of a null condition which appears in the nonlinearity. Most of the crucial arguments follow from the bilinear estimates in wave-Sobolev spaces established by d’Ancona, Foschi and Selberg [AFS].

We denote the Fourier transform with respect to space and time by \( \hat{\cdot} \).

\[ a_+ := a + \epsilon \text{ for a sufficiently small } \epsilon > 0, \text{ so that } a < a+ < a+ +, \text{ and similarly } a_-- < a- < a+, \text{ and } \langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}. \]

We now formulate our main result and begin by defining the standard spaces \( X^{s,b}_\pm \) of Bourgain-Klainerman-Machedon type belonging to the half waves as the completion of the Schwartz space \( \mathcal{S}(\mathbb{R}^4) \) with respect to the norm

\[ ||u||_{X^{s,b}_\pm} := ||\langle \xi \rangle^s (\tau \pm |\xi|) \hat{u}(\tau,\xi)||_{L^2_{\tau \xi}}. \]

Similarly we define the wave-Sobolev spaces \( X^{s,b}_{|\tau|=|\xi|} \) with norm

\[ ||u||_{X^{s,b}_{|\tau|=|\xi|}} = ||\langle \xi \rangle^s |\tau| - |\xi|| \hat{u}(\tau,\xi)||_{L^2_{\tau \xi}} \]

and also \( X^{s,b}_{|\tau|=0} \) with norm

\[ ||u||_{X^{s,b}_{|\tau|=0}} = ||\langle \xi \rangle^s \langle \tau \rangle \hat{u}(\tau,\xi)||_{L^2_{\tau \xi}}. \]

We also define \( X^{s,b}_{[0,T]} \) as the space of the restrictions of functions in \( X^{s,b}_\pm \) to \([0,T] \times \mathbb{R}^2\) and similarly \( X^{s,b}_{|\tau|=|\xi|}[0,T] \) and \( X^{s,b}_{|\tau|=0}[0,T] \). We frequently use the estimates \( ||u||_{X^{s,b}_{|\tau|=|\xi|}} \leq ||u||_{X^{s,b}_{|\tau|=|\xi|}} \) for \( b \leq 0 \) and the reverse estimate for \( b \geq 0 \).

Our main theorem reads as follows:

**Theorem 1.1.** Assume \( s > \frac{1}{4} \). The Chern-Simons-Higgs system \([1], [2], [3]\) in temporal gauge \( A_0 = 0 \) with data \( \phi_0 \in H^{s+1}(\mathbb{R}^2) \), \( \phi_1 \in H^s(\mathbb{R}^2) \), \( a_\mu \in H^{s+\frac{1}{2}}(\mathbb{R}^2) \) satisfying the compatibility condition \( \partial_1 a_2 - \partial_2 a_1 = 2Im(\phi_0 \phi_1) \) has a unique local
solution \( \phi \in C^0([0, T], H^{s+1}(\mathbb{R}^2)) \cap C^1([0, T], H^s(\mathbb{R}^2)) \), \( A \in C^0([0, T], H^{s+\frac{1}{2}}(\mathbb{R}^2)) \). More precisely we have \( \phi \in X^{s+\frac{1}{2}}([0, T] + X^{s+\frac{1}{2}}([0, T] \mapsto A = A^f + A^d \), where \( \nabla A^f \in X^{s+\frac{1}{2}}([0, T]), A^f \in C^0([0, T], L^2(\mathbb{R}^2)) \), \( \nabla A^d \in X^{s+\frac{1}{2}}([0, T], A^d \in C^0([0, T], L^2(\mathbb{R}^2)) \) and in these spaces uniqueness holds, and moreover \( \nabla A^f \in X^{s+\frac{1}{2}}([0, T]) \).

2. Reformulation of the problem

In the temporal gauge \( A_0 = 0 \) the Chern-Simons-Higgs system (1), (2) is equivalent to the following system

\[
\partial_t A_j = 2 \epsilon_{ij} \text{Im}(\overline{\phi}D^i \phi) \quad (4)
\]

\[
\partial^2_t \phi - D^j D_j \phi = -\phi V'(\phi^2) \quad (5)
\]

\[
\implies \Box \phi = 2i A^f \partial_j \phi - i \partial_j A^j \phi + A^j A_j \phi - \phi V'(\phi^2)
\]

\[
\partial_t A_2 - \partial_2 A_1 = 2 \text{Im}(\overline{\phi} \partial_2 \phi),
\]

where \( i, j = 1, 2 \), \( \epsilon_{12} = 1 \), \( \epsilon_{21} = -1 \) and \( \Box = \partial_t^2 - \partial_1^2 - \partial_2^2 \). We remark that (5) is fulfilled for any solution of (1), (5), if it holds initially, i.e., if the following compatibility condition holds, which we assume from now on:

\[
\partial_t A_2(0) - \partial_2 A_1(0) = 2 \text{Im}(\overline{\phi}(0)(\partial_t \phi)(0)).
\]

Indeed, we have by (1) and (5):

\[
\partial_t (\partial_1 A_2 - \partial_2 A_1) = 2 \text{Im}(\overline{\phi}(D^2_1 \phi + D^2_2 \phi)) = 2 \text{Im}(\overline{\phi} \partial_i \phi) = 2 \partial_i \text{Im}(\overline{\phi} \partial_i \phi).
\]

Thus we only have to solve (1) and (5), and can assume that (5) is fulfilled. We make the standard decomposition of \( A = (A_1, A_2) \) into its divergence-free part \( A^d \) and its curl-free part \( A^f \), namely \( A = A^d + A^f \), where

\[
A^d = (-\Delta)^{-1}(\partial_1 \partial_2 A_2 - \partial_2^2 A_1, \partial_1 \partial_2 A_1 - \partial_1^2 A_2),
\]

\[
A^f = (\Delta)^{-1}(\partial_1 \partial_1 A_2 + \partial_2^2 A_1, \partial_1 \partial_2 A_1 + \partial_2^2 A_2).
\]

Let \( B \) be defined by \( A_1^d = \partial_2 B \), \( A_2^d = \partial_1 B \). Then by (1) and \( \partial_1 A_2^f - \partial_2 A_1^f = 0 \) we obtain

\[
\Delta B = \partial_1 A_2^d - \partial_2 A_1^d = 2 \text{Im}(\overline{\phi} \partial_1 \phi),
\]

so that

\[
A_1^d = -2 \Delta^{-1} \partial_2 \text{Im}(\overline{\phi} \partial_1 \phi), A_2^d = 2 \Delta^{-1} \partial_1 \text{Im}(\overline{\phi} \partial_1 \phi).
\]

Next we calculate \( \partial_t A^f \) for solutions \((A, \phi)\) of (1), (5):

\[
\partial_t A_1^f = \Delta^{-1} \partial_1 (\partial_2 \partial_1 A_1 + \partial_1 \partial_1 A_1)
\]

\[
= 2 \Delta^{-1} \partial_1 (\partial_2 \text{Im}(\overline{\phi} D_1 \phi) - \partial_1 \text{Im}(\overline{\phi} D_2 \phi))
\]

\[
= 2 \Delta^{-1} \partial_1 \text{Im}(\partial_2 \overline{\phi} D_1 \phi - \partial_1 \overline{\phi} D_2 \phi - \overline{\phi} D_1 \phi - \overline{\phi} D_2 \phi)
\]

\[
= 2 \Delta^{-1} \partial_1 \text{Im}(\overline{\phi} D_1 \phi - \overline{\phi} D_2 \phi - i A_1 \partial_1 \overline{\phi} + i A_2 \partial_2 \overline{\phi}
\]

\[
+ i \overline{\phi} (-i A_1 \partial_2 \phi + i A_2 \partial_1 \phi - i \partial_1 A_1 \phi + i \partial_1 A_2 \phi)
\]

\[
= 2 \Delta^{-1} \partial_1 \text{Im}(\overline{\phi} D_1 \phi - \overline{\phi} D_2 \phi + i \overline{\phi} (A_2 \partial_1 \phi - A_1 \partial_2 \phi)
\]

\[
+ i \overline{\phi} (A_2 \partial_2 \phi - A_1 \partial_1 \phi) + i \partial_1 A_2 \partial_2 \phi + i \partial_2 A_1 \partial_1 \phi))
\]

\[
= 2 \Delta^{-1} \partial_1 \text{Im}(\overline{\phi} D_1 \phi - \overline{\phi} D_2 \phi) + 2 \Delta^{-1} \partial_1 (A_2 \partial_1 |\phi|^2 - A_1 \partial_2 |\phi|^2)
\]

\[
+ 4 \Delta^{-1} \partial_1 \text{Im}(\overline{\phi} \partial_1 \phi)|\phi|^2.
\]
Similarly
\[ \partial_t A^{df}_2 = 2\Delta^{-1}\partial_2 Im(\partial_2 D_1 \phi) + 2\Delta^{-1}\partial_2 (A_1 \partial_2 |\phi|^2 - A_2 \partial_1 |\phi|^2) + 4\Delta^{-1}\partial_2 Im(\bar{\partial}\phi)|\phi|^2. \] (10)

Moreover from (5) we obtain using \( \partial^\dagger A^{df}_2 = 0 \):
\[ \square \phi = 2iA^{df}_1 \nabla \phi + 2iA^{df}_2 \nabla \phi - i\partial^\dagger A^{cf}_1 \phi + (A^{df,j}_1 + A^{cf,j}_1)(A^{df}_j + A^{cf}_j) \phi - \phi V'(|\phi|^2). \] (11)

We also obtain from (8) and (13), which was shown to be a consequence of (8) and (11).
\[ \partial_t A^{df}_2 = 2\Delta^{-1}\partial_1 Im(\bar{\partial}\phi) = 2\Delta^{-1}\partial_1 Im(\bar{\partial} D_1 \phi) = 2\Delta^{-1}\partial_1 Im(\bar{\partial} D_2 \phi). \]

Now
\[ Im(\bar{\partial} D_1 \phi) = Im(\bar{\partial} D_2 \phi) - 2\Delta^{-1}\partial_2 A^{df}_2 \phi, \]
so that
\[ \partial_t A^{df}_2 = 2\Delta^{-1}\partial_1 \partial^\dagger Im(\bar{\partial} D_1 \phi). \] (12)

Similarly we obtain
\[ \partial_t A^{df}_1 = -2\Delta^{-1}\partial_2 \partial^\dagger Im(\bar{\partial} D_2 \phi). \] (13)

Reversely defining \( A := A^{df} + A^{cf} \) we show that our new system (8), (9), (10), (11) implies (1), (5), and also (8), provided the compatibility condition (7) is fulfilled. (5) is obvious. (8) is fulfilled because by use of (9) and (10) one easily checks
\[ \partial_t A^{cf}_2 - \partial_2 A^{cf}_1 = 0, \]
so that by (8)
\[ \partial_t (\partial_t A_2 - \partial_2 A_1) = \partial_t (\partial_t A^{df}_2 - \partial_2 A^{df}_1) = \partial_t Im(\bar{\partial}\phi) \]

Thus (4) is fulfilled, if (7) holds. Finally we obtain
\[ \partial_t A_1 = \partial_t A^{df}_1 + \partial_t A^{df}_1 \]
\[ = 2\Delta^{-1}\partial_1(\partial_2 Im(\bar{\partial} D_1 \phi) - \partial_1 Im(\bar{\partial} D_2 \phi)) - 2\Delta^{-1}\partial_2(\partial_t Im(\bar{\partial} D_1 \phi) + \partial_2 Im(\bar{\partial} D_2 \phi)) \]
\[ = -2Im(\bar{\partial} D_2 \phi), \]
where we used (9) and also (13), which was shown to be a consequence of (5) and (11). Similarly we also get
\[ \partial_t A_2 = 2Im(\bar{\partial} D_1 \phi), \]
so that (11) is shown to be satisfied.

Summarizing we have shown that (1), (5), (9) are equivalent to (8), (9), (10), (11) (which also implies (12), (13)).

Concerning the initial conditions assume we are given initial data for our system (1), (5), (9):
\[ A_j(0) = a_j, \phi(0) = \phi_0, (\bar{\partial}\phi)(0) = \phi_1 \]
satisfying \( a_j \in H^{s+\frac{3}{2}} \), \( \phi_0 \in H^{s+1} \), \( \phi_1 \in H^s \) and (7). Then by (8) and (7) we obtain
\[ A^{df}_1(0) = -2\Delta^{-1}\partial_2 Im(\bar{\partial}\phi_0) = -\Delta^{-1}\partial_2(\partial_1 a_2 - \partial_2 a_1) \in H^{s+\frac{3}{2}} \]
\[ A^{df}_2(0) = 2\Delta^{-1}\partial_1 Im(\bar{\partial}\phi_0) = \Delta^{-1}\partial_1(\partial_1 a_2 - \partial_2 a_1) \in H^{s+\frac{3}{2}} \]
and thus
\[ A^{cf}_j(0) = a_j - A^{df}_j(0) \in H^{s+\frac{3}{2}}. \]

In the sequel we construct a solution of the Cauchy problem for (8), (9), (10), (11) with data \( \phi_0 \in H^{s+1} \), \( \phi_1 \in H^s \), \( A^{cf}_j(0) \in H^{s+\frac{3}{2}} \). We have shown that
whenever we have a local solution of this system with data \( \phi_0, \phi_1 \) and \( A_{ij}^{cf}(0) = a_j - A_{ij}^{df}(0) \), where \( A_{ij}^{df}(0) = -2\Delta^{-1}\partial_2 Im(\phi_0\phi_1) \), \( A_{ij}^{df}(0) = 2\Delta^{-1}\partial_1 Im(\phi_0\phi_1) \), we also have that \((\phi, A)\) with \( A := A_{ij}^{df} + A_{ij}^{cf}\) is a local solution of (4), (5) with data \((\phi_0, \phi_1, a_1, a_2)\). If (7) holds then (6) is also satisfied.

Defining

\[
\phi_\pm = \frac{1}{2}(\phi \pm i^{-1}(\nabla)^{-1}\partial_t \phi) \iff \phi = \phi_+ + \phi_- , \partial_t \phi = i(\nabla)(\phi_+ - \phi_-)
\]

the equation (11) transforms to

\[
(i\partial_t \pm (\nabla))\phi_\pm = \pm 2^{-1}(\nabla)^{-1}(2iA_{ij}^{cf}\nabla \phi + 2iA_{ij}^{df}\nabla \phi - i\partial_j A_{ij}^{cf} + (A_{ij}^{df} + A_{ij}^{cf})\phi - \phi V'(|\phi|^2) + \phi)
\]

Fundamental for the proof of our theorem are the following bilinear estimates in wave-Sobolev spaces which were proven by d’Ancona, Foschi and Selberg in the two-dimensional case \( n = 2 \) in [AFS] in a more general form which include many limit cases which we do not need.

**Theorem 2.1.** Let \( n = 2 \). The estimate

\[
\|uv\|_{X_t^{s_0, b_0}} \lesssim \|u\|_{X_t^{s_1, b_1}} \cdot \|v\|_{X_t^{s_2, b_2}}
\]

holds, provided the following conditions hold:

\[
\begin{align*}
b_0 + b_1 + b_2 &> \frac{1}{2} \\
b_0 + b_1 &> 0 \\
b_0 + b_2 &> 0 \\
b_1 + b_2 &> 0 \\
s_0 + s_1 + s_2 &> \frac{3}{2} - (b_0 + b_1 + b_2) \\
s_0 + s_1 + s_2 &> 1 - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2) \\
s_0 + s_1 + s_2 &> \frac{1}{2} - \min(b_0, b_1, b_2) \\
s_0 + s_1 + s_2 &> \frac{3}{4} \\
(s_0 + b_0) + 2s_1 + 2s_2 &> 1 \\
2s_0 + (s_1 + b_1) + 2s_2 &> 1 \\
2s_0 + 2s_1 + (s_2 + b_2) &> 1 \\
s_1 + s_2 &\geq \max(0, -b_0) \\
s_0 + s_2 &\geq \max(0, -b_1) \\
s_0 + s_1 &\geq \max(0, -b_2).
\end{align*}
\]

3. **Proof of Theorem 1.1**

Taking the considerations of the previous section into account Theorem 1.1 reduces to the following proposition and its corollary.
Proposition 3.1. Assume $s > \frac{1}{4}$. The system
\begin{equation}
(i\partial_t \pm (\nabla))\phi_\pm = \pm 2^{-1}(-\nabla)^{-1}(2A^f\partial_t \partial_1 \phi + 2iA^f\nabla \phi - i\partial_t A^f_{-})\phi + (A^f_{+} + A^f_{-})A^f_{-})\phi - \phi V'(\phi^2) + \phi \\tag{15}
\end{equation}

\begin{align}
A^f_{+} = -2\Delta^{-1}\partial_t \Im(\bar{\phi}\partial_t \phi), \quad A^f_{-} = 2\Delta^{-1}\partial_t \Im(\bar{\phi}\partial_t \phi) & \quad \tag{16} \\
\partial_t A^f_{+} = 2\Delta^{-1}\partial_t \Im(\bar{\phi}\partial_t \phi - \partial_t \partial_2 \phi) + 2\Delta^{-1}\partial_t (A_2 \partial_t |\phi|^2 - A_1 \partial_t |\phi|^2) & + 4\Delta^{-1}\partial_t \Im(\bar{\phi}\partial_t \phi)|\phi|^2 \quad \tag{17} \\
\partial_t A^f_{-} = 2\Delta^{-1}\partial_t \Im(\bar{\phi}\partial_t \phi - \partial_t \partial_2 \phi) + 2\Delta^{-1}\partial_t (A_2 \partial_t |\phi|^2 - A_1 \partial_t |\phi|^2) & + 4\Delta^{-1}\partial_t \Im(\bar{\phi}\partial_t \phi)|\phi|^2 \quad \tag{18}
\end{align}

with data $\phi_\pm(0) \in H^{s+1}$ and $A^f(0) \in H^{s+\frac{1}{2}}$ has a unique local solution

$$\phi_\pm \in X^{s+1}_{X_{\tau=0} = 0}, \nabla A^f \in X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}, A^f \in C^0([0,T], L^2).$$

Here $\phi = \phi_+ + \phi_-$, $\partial_t \phi = i(\nabla)(\phi_+ - \phi_-)$ . Moreover $A^f$ satisfies $\nabla A^f \in X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}$ and $A^f \in C^0([0,T], L^2)$ and also $\nabla A^f \in X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}$.

We obtain immediately

Corollary 3.1. The solution has the property $\phi \in C^0([0,T], H^{s+1}) \cap C^1([0,T], H^s)$, $A^f \in C^0([0,T], H^{s+\frac{1}{2}})$ and $A^f \in C^0([0,T], H^{s+1})$.

Proof. We want to apply the contraction mapping principle for

$$\phi_\pm \in X^{s+1}_{X_{\tau=0} = 0}, \nabla A^f \in X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}, A^f \in C^0([0,T], L^2).$$

By well-known arguments this is reduced to the estimates of the right hand sides of (15), (17) and (18) stated as claims 1-9 below. We start to control $\nabla A^f$ in $X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}$.

Claim 1:

$$\|\partial_t \bar{\phi}\partial_t \phi - \partial_t \bar{\phi}\phi_t \|_{X^{s+\frac{1}{2}, \frac{1}{2}+}_{X_{\tau=0} = 0}} \leq \|\nabla \phi\|^2_{X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}}.$$

Using $(\pm_1$ and $\pm_2$ denote independent signs

$$\partial_t \bar{\phi}\partial_t \phi - \partial_t \bar{\phi}\phi_t = \sum_{\pm_1, \pm_2} (\partial_t \bar{\phi}_{\pm_1} \partial_t \phi_{\pm_2} - \partial_t \bar{\phi}_{\pm_1} \partial_t \phi_{\pm_2})$$

it suffices to show

$$\|\partial_t \bar{\phi}\partial_t \phi - \partial_t \bar{\phi}\phi_t \|_{X^{s+\frac{1}{2}, \frac{1}{2}+}_{X_{\tau=0} = 0}} \leq \|\nabla \phi\|_{X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}} \|\nabla \psi\|_{X^{s, \frac{1}{2}+}_{X_{\tau=0} = 0}}.$$

We now use the null structure of this term in the form that for vectors $\xi = (\xi^1, \xi^2)$, $\eta = (\eta^1, \eta^2) \in \mathbb{R}^2$ the following estimate holds

$$|\xi^1 \eta^2 - \xi^2 \eta^1| \leq |\xi||\eta| \angle(\xi, \eta),$$

where $\angle(\xi, \eta)$ denotes the angle between $\xi$ and $\eta$, and the following lemma gives the decisive bound for the angle:

Lemma 3.1. ([S], Lemma 2.1 or [ST], Lemma 3.2)

$$\angle(\pm_1, \pm_2) \leq \left(\frac{\tau_1 \pm_1 |\xi_1| + \tau_2 \pm_2 |\xi_2| + |\tau_3| - |\xi_3|}{\min(|\xi_1|, |\xi_2|)}\right)^{\frac{1}{2}} \quad \tag{19}
$$

$\forall \xi, \xi_2, \xi_3 \in \mathbb{R}^2$, $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$ with $\xi_1 + \xi_2 + \xi_3 = 0$ and $\tau_1 + \tau_2 + \tau_3 = 0$. 
Thus the claimed estimate reduces to

\[
\int_{\mathcal{S}} \frac{\hat{u}_1(t_1, \xi_1)}{|\xi_1|^{\frac{3}{2}+\frac{1}{r}(\xi_1)+\frac{1}{r}(\xi_2)}|\xi_2|^{\frac{3}{2}+\frac{1}{r}(\xi_3)}} \frac{\hat{u}_2(t_2, \xi_2)}{|\xi_2|^{\frac{3}{2}+\frac{1}{r}(\xi_3)}} \hat{u}_3(t_3, \xi_3) \lesssim \sum_{\xi_1} |u_1|_{L^2_{\xi_1}} |u_2|_{L^2_{\xi_2}} |u_3|_{L^2_{\xi_3}},
\]

(20)

where * denotes integration over \(\xi_1 + \xi_2 + \xi_3 = 0\) and \(t_1 + t_2 + t_3 = 0\). We assume without loss of generality \(|\xi_1| \leq |\xi_2|\) and the Fourier transforms are nonnegative.

We distinguish three cases according to which of the modules on the right hand side of (19) is dominant.

Case 1: \(|\xi_3| - |\xi_1|\) dominant.

In this case (20) reduces to

\[
\int_{\mathcal{S}} \frac{\hat{u}_1(t_1, \xi_1)}{|\xi_1|^{\frac{1}{r}(\xi_1)+\frac{1}{r}(\xi_2)}} \frac{\hat{u}_2(t_2, \xi_2)}{|\xi_2|^{\frac{3}{2}+\frac{1}{r}(\xi_3)}} \hat{u}_3(t_3, \xi_3) \lesssim |u_1|_{L^2_{\xi_1}} |u_2|_{L^2_{\xi_2}} |u_3|_{L^2_{\xi_3}},
\]

This follows from Theorem 2.1 with \(s_0 = \frac{1}{4} - s\), \(b_0 = 0\), \(s_1 = s + \frac{1}{2}\), \(b_1 = 2 = \frac{1}{2} + s_2 = s\). Its assumptions are satisfied, because \(s_0 + s_1 + s_2 = \frac{3}{4} + s > 1 = \frac{1}{2} - b_0\) under our assumption \(s > \frac{1}{2}\).

Case 2: \(|\xi_1| - \xi_2|\) dominant.

This case is reduced to

\[
\int_{\mathcal{S}} \frac{\hat{u}_1(t_1, \xi_1)}{|\xi_1|^{\frac{1}{r}(\xi_1)+\frac{1}{r}(\xi_2)}} \frac{\hat{u}_2(t_2, \xi_2)}{|\xi_2|^{\frac{3}{2}+\frac{1}{r}(\xi_3)}} \hat{u}_3(t_3, \xi_3) \lesssim |u_1|_{L^2_{\xi_1}} |u_2|_{L^2_{\xi_2}} |u_3|_{L^2_{\xi_3}},
\]

which follows from Theorem 2.1 with \(s_0 = \frac{1}{4} - s\), \(b_0 = 0\), \(s_1 = s + \frac{1}{2}\), \(b_1 = 0 + s_2 = s\).

Case 3: \(|\xi_2| - |\xi_3|\) dominant.

This case is reduced to

\[
\int_{\mathcal{S}} \frac{\hat{u}_1(t_1, \xi_1)}{|\xi_1|^{\frac{1}{r}(\xi_1)+\frac{1}{r}(\xi_2)}} \frac{\hat{u}_2(t_2, \xi_2)}{|\xi_2|^{\frac{3}{2}+\frac{1}{r}(\xi_3)}} \hat{u}_3(t_3, \xi_3) \lesssim |u_1|_{L^2_{\xi_1}} |u_2|_{L^2_{\xi_2}} |u_3|_{L^2_{\xi_3}},
\]

which follows from Theorem 2.1 with \(s_0 = \frac{1}{4} - s\), \(b_0 = 0\), \(s_1 = s + \frac{1}{2}\), \(b_1 = \frac{1}{2} + s_2 = s\).

The cubic terms are easier to handle, because they contain one derivative less.

**Claim 2:**

\[
\|A_i \partial_j (|\phi|^2)\|_{X^{s-\frac{1}{2}+\frac{1}{r}, 0}_x} \lesssim (\|\nabla A_i\|_{X^{s, 0}_x} + \|A_i\|_{L^{\infty}_x(L^2_x)})|\phi|^2_{X^{s+1, \frac{1}{2}+}_x}.
\]

(21)

\[
(\|\phi\|_{X^{s+1, \frac{1}{2}_x} + \|\partial_t \phi\|_{X^{s+1, 0}_x} + \|\nabla A_j \partial^f \|_{X^{s-\frac{1}{2}+\frac{1}{r}, 0}_x} + \|A_j \partial^f \|_{L^{\infty}_x(L^2_x)})|\phi|^2_{X^{s+1, \frac{1}{2}_x} +}.
\]

(22)

(21) is proven by Sobolev and by splitting \(A_i = A_i^l + A_i^h\) into low and high frequency parts, i.e., \(supp \tilde{A}_i^l \subset \{\xi \mid |\xi| \leq 2\}\) and \(supp \tilde{A}_i^h \subset \{\xi \mid |\xi| \geq 1\}\). The low frequency part is easily taken care of as follows

\[
\|A_i \partial_j (|\phi|^2)\|_{L^2_t H^{s-\frac{1}{2}}_x} \lesssim \|A_i \partial_j \|_{L^{\infty}_x H^{s-\frac{1}{2}}_x} |\phi|^2_{L^{\infty}_x H^{s+1}_x} \lesssim \|A_i \|_{L^{\infty}_x L^2_x} |\phi|^2_{X^{s+1, \frac{1}{2}_x} +}.
\]
For the high frequency part we obtain
\[
\|A_t^f \partial_j (|\phi|^2)\|_{L_t^2 L_x^{\frac{4}{3}}} \leq \|\nabla (\nabla^2)\|_{L_t^\infty L_x^\infty} \|A_t^f\|_{L_t^2 L_x^{\frac{4}{3}}} + \|A_t^f\|_{L_t^2 L_x^{\frac{4}{3}}} \|\nabla (\nabla^2)\|_{L_t^\infty L_x^\infty}
\]
\[
\leq \|\nabla (\nabla^2)\|_{L_t^\infty L_x^\infty} \|\partial_j (|\phi|^2)\|_{L_t^2 L_x^{\frac{4}{3}}} + \|A_t^f\|_{L_t^2 H_x^{s+1}} \|\phi\|^2_{L_t^\infty H_x^{s+1}}
\]
\[
\leq \|\nabla A_t^f\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \|\phi\|^2_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}}.
\]

In order to obtain (22) from these estimates it remains to estimate \(A_t^f\). We obtain by (13) and Sobolev’s embedding \(H^{1,1} \subset L^2\):
\[
\|A_t^f\|_{L_t^2 (L_x^2)} \lesssim \|\phi \partial_t \phi\|_{L_t^2 L_x^1} \lesssim \|\partial_t \phi\|_{L_t^\infty L_x^1}
\]
\[
\lesssim \|\phi\|_{L_t^\infty L_x^2} \|\partial_t \phi\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \|\partial_t \phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}}.
\]

Moreover for sufficiently small \(\epsilon > 0\) we obtain
\[
\|\nabla A_t^f\|_{X_{|\tau|=|\xi|}^{s,\frac{4}{3}}} \lesssim \|\phi \partial_t \phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \lesssim \|\phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \|\partial_t \phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}}.
\]

by Theorem 2.4 with \(s_0 = -s\), \(b_0 = -\frac{1}{2} - \frac{s_1}{2}\), \(s_1 = s+1\), \(s_2 = s\), \(b_1 = b_2 = \frac{1}{2} + \epsilon\). This is more than we need here.

Claim 3:
\[
\|Im (\bar{\phi} \partial_t \phi)\|^2_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \lesssim \|\phi\|^3_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \|\partial_t \phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}}.
\]

This follows from
\[
\|Im (\bar{\phi} \partial_t \phi)\|^2_{L_t^2 L_x^\infty} \lesssim \|\phi\|^3_{L_t^\infty L_x^\infty} \|\partial_t \phi\|_{L_t^2 L_x^2} + \|\phi\|^3_{L_t^\infty L_x^\infty} \|\partial_t \phi\|_{L_t^2 L_x^2}
\]
\[
\lesssim \|\phi\|^3_{L_t^\infty H_x^{s+1}} \|\partial_t \phi\|_{L_t^2 H_x^{s+1}} + \|\phi\|^3_{L_t^\infty H_x^{s+1}} \|\partial_t \phi\|_{L_t^2 H_x^{s+1}}
\]
\[
\lesssim \|\phi\|^3_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \|\partial_t \phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}}.
\]

In order to control \(\|A_t^f\|_{L_t^\infty (0,T, L_x^2)}\) in the fixed point argument we have to estimate the \(L_t^1 (0,T, L_x^2)\) - norm of the right hand side in (17), (13).

Claim 4 (a)
\[
\int_0^T \|\nabla \dot{\phi} \nabla \phi\|_{H_x^{s+1}} dt \leq \int_0^T \|\nabla \dot{\phi} \nabla \phi\|_{L_t^2 L_x^2} dt \lesssim T \|\nabla \phi\|^2_{X_{|\tau|=|\xi|}^{s,\frac{4}{3}}}
\]

(b)
\[
\int_0^T \|A_t^f \nabla (|\phi|^2)\|_{H_x^{s+1}} dt \leq \int_0^T \|A_t^f \nabla (|\phi|^2)\|_{L_t^2 L_x^2} dt \lesssim \int_0^T \|A_t^f \Delta^{-1} \nabla (\phi \partial_t \phi) \nabla (|\phi|^2)\|_{L_t^2 L_x^2} dt
\]
\[
\lesssim \int_0^T \|\Delta^{-1} \nabla (\phi \partial_t \phi)\|_{L_t^2 L_x^2} \|\nabla (|\phi|^2)\|_{L_t^2 L_x^2} dt \lesssim \int_0^T \|\phi\|_{L_t^2 L_x^2} \|\partial_t \phi\|_{L_t^2 L_x^2} \|\phi\|_{L_t^\infty L_x^\infty} \|\nabla \phi\|_{L_t^2 L_x^2}
\]
\[
\lesssim T \|\phi\|^3_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}} \|\partial_t \phi\|_{X_{|\tau|=|\xi|}^{s+1,\frac{4}{3}}}.
\]

(c)
\[
\int_0^T \|A_t^f \nabla (|\phi|^2)\|_{H_x^{s+1}} dt \leq \int_0^T \|A_t^f \nabla (|\phi|^2)\|_{L_t^2 L_x^2} dt \lesssim T \|A_t^f\|_{L_t^\infty L_x^\infty} \|\nabla \phi\|_{L_t^\infty L_x^\infty} \|\nabla \phi\|_{L_t^\infty L_x^\infty} \|\nabla \phi\|_{L_t^\infty L_x^\infty}
\]
\[
\lesssim T \|A_t^f\|_{L_t^\infty L_x^\infty} \|\phi\|_{L_t^\infty L_x^\infty} \|\nabla \phi\|_{L_t^\infty L_x^\infty} \|\nabla \phi\|_{L_t^\infty L_x^\infty} \|A_t^f\|_{L_t^\infty L_x^\infty}.
\]
\[
\int_0^T \| \phi(\partial_t \phi) \|_{H_{x,t}^{-1}}^2 dt \lesssim \int_0^T \| \phi(\partial_t \phi) \|_{L^1}^2 dt
\]
\[
\lesssim T \| \phi \|_{L^\infty_t L^2_x}^2 \| \phi \|_{L^\infty_t L^2_x} \| \partial_t \phi \|_{L^\infty_t L^2_x} \lesssim T \| \phi \|_{X_{\tau=\xi}^{s+1,1}}^2 \| \partial_t \phi \|_{X_{\tau=\xi}^{s+1,1}}.
\]

Next in order to estimate \( \| \phi \|_{X_{\tau=\xi}^{s+1,1}} \) and \( \| \partial_t \phi \|_{X_{\tau=\xi}^{s+1,1}} \) we have to control the right hand side of (14).

**Claim 5:**
\[
\| A^{d g} \nabla \phi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| \phi \|_{X_{\tau=\xi}^{s+1,1}}^2 \| \partial_t \phi \|_{X_{\tau=\xi}^{s+1,1}}.
\]

We obtain by Sobolev in the case \( s \leq 1 \):
\[
\| A^{d g} \nabla \phi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| A^{d g} \|_{L^2_t H_{x,t}^{s+1}} \| \nabla \phi \|_{L^\infty_t L^2_x} + \| A^{d g} \|_{L^2_t L^\infty_x} \| \nabla \phi \|_{L^\infty_t H_{x,t}^{s}}.
\]
where \( \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = \frac{1}{2} \). Using (23) and (24) we obtain the claimed estimate. In the case \( s > 1 \) we simply use that \( H^s \) is a Banach algebra to obtain
\[
\| A^{d g} \nabla \phi \|_{L^2_t H_{x,t}^{s}} \lesssim \| A^{d g} \|_{L^2_t H_{x,t}^{s}} \| \nabla \phi \|_{L^\infty_t H_{x,t}^{s}},
\]
which also suffices in view of (23) and (24).

**Claim 6:**
\[
\| A^{c f} \nabla \phi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| A^{c f} \|_{L^2_t H_{x,t}^{s+1}} \| \nabla \phi \|_{L^\infty_t L^2_x} \lesssim \| A^{c f} \|_{L^2_t L^\infty_x} \| \nabla \phi \|_{X_{\tau=\xi}^{s+1,1}},
\]

where we split \( A^{c f} \) into its low and high frequency parts \( A^{c f}_l \) and \( A^{c f}_h \). The low frequency part is estimated as follows:
\[
\| A^{c f}_l \nabla \phi \|_{L^2_t H_{x,t}^{s}} \lesssim \| A^{c f}_l \|_{L^2_t H_{x,t}^{s}} \| \nabla \phi \|_{L^\infty_t H_{x,t}^{s}} \lesssim \| A^{c f}_l \|_{L^2_t L^\infty_x} \| \nabla \phi \|_{X_{\tau=\xi}^{s+1,1}},
\]
whereas for the high frequency part we use the trivial identity \( X_{\tau=\xi}^{s,0} = X_{\tau=\xi}^{s,0} \) and Theorem 2.1 with \( s_0 = -s, b_0 = \frac{1}{2}, s_1 = s + \frac{1}{2}, b_1 = 0, s_2 = s, b_2 = \frac{1}{2} \), which requires \( 2s_0 + s_1 + b_1 + 2s_2 = s + \frac{1}{2} > 1 \), thus our assumption \( s > \frac{1}{2} \).

**Claim 7:**
\[
\| \nabla A^{c f} \phi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| \nabla A^{c f} \|_{X_{\tau=\xi}^{s+1,1}} \| \phi \|_{X_{\tau=\xi}^{s+1,1}}.
\]

By duality this is equivalent to
\[
\| \omega \phi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| \omega \|_{X_{\tau=\xi}^{s+1,1}} \| \phi \|_{X_{\tau=\xi}^{s+1,1}}.
\]

We use the estimate \( \| \xi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| \xi \|_{X_{\tau=\xi}} \) and obtain
\[
\| \omega \phi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| \omega \|_{X_{\tau=\xi}^{s+1,1}} \| \phi \|_{X_{\tau=\xi}^{s+1,1}},
\]
where the last estimate follows from Theorem 2.1 with \( s_0 = s + \frac{1}{2}, b_0 = -\frac{1}{2} \), \( s_1 = s + 1, s_2 = -s, b_1 = \frac{1}{2}, b_2 = \frac{1}{2} \).

**Claim 8:**
\[
\| A^{c f} A^{c f} \phi \|_{X_{\tau=\xi}^{s,0}} \lesssim \| \nabla A^{c f} \|_{X_{\tau=\xi}^{s+1,1}}^2 + \| A^{c f} \|_{L^2_t L^\infty_x}^2 \| \phi \|_{X_{\tau=\xi}^{s+1,1}}.
\]
Splitting $A^{cf} = A^{cf}_h + A^{cf}_t$ we first consider
\[
\|A^{cf}_h A^{cf}_t \phi\|_{X_{1,\tau}^{0,0}} \lesssim \|\langle \nabla \rangle^s A^{cf}_h A^{cf}_t \phi\|_{L^2_t L^2_x} + \|A^{cf}_h A^{cf}_t \langle \nabla \rangle^s \phi\|_{L^2_t L^2_x}
\]
\[
\lesssim \|\langle \nabla \rangle^s A^{cf}_h \|_{L^2_t L^2_x} \|A^{cf}_t\|_{L^2_t L^2_x} + \|A^{cf}_h\|_{L^2_t L^2_x} \|\langle \nabla \rangle^s \phi\|_{L^2_t L^2_x}
\]
\[
\lesssim \|\langle \nabla \rangle^s A^{cf}_h \|_{L^2_t H^s_x} \|A^{cf}_t\|_{L^2_t L^2_x} + \|A^{cf}_h\|_{L^2_t H^s_x} \|\langle \nabla \rangle^s \phi\|_{L^2_t L^2_x}
\]
\[
\lesssim \|\langle \nabla \rangle A^{cf}_t \|_{L^2_t L^2_x} \|\langle \nabla \rangle^{s+\frac{1}{2}} \phi\|_{X_{1,\tau}^{0,0}} + \|A^{cf}_h\|_{L^2_t H^s_x} \|\langle \nabla \rangle^{s+\frac{1}{2}} \phi\|_{X_{1,\tau}^{0,0}}
\]

Next we consider
\[
\|A^{cf}_t A^{cf}_t \phi\|_{X_{1,\tau}^{0,0}} \lesssim \|A^{cf}_t\|_{L^2_t H^s_x} \|\phi\|_{L^2_t L^2_x} \lesssim \|A^{cf}_t\|_{L^2_t L^2_x} \|\phi\|_{X_{1,\tau}^{0,0}}
\]
and also
\[
\|A^{cf}_h A^{cf}_h \phi\|_{X_{1,\tau}^{0,0}} \lesssim \|A^{cf}_t\|_{L^2_t H^s_x} \|\phi\|_{L^2_t L^2_x} \lesssim \|A^{cf}_t\|_{L^2_t L^2_x} \|\phi\|_{X_{1,\tau}^{0,0}}
\]

which completes the proof of claim 8.

If one combines similar estimates with (23) and (24) we also obtain the required bounds for $\|A^{df} A^{cf} \|_{X_{1,\tau}^{0,0}}$ and $\|A^{cf} A^{cf} \|_{X_{1,\tau}^{0,0}}$.

**Claim 9:** For a suitable $N \in \mathbb{N}$ the following estimate holds:
\[
\|\phi V(\phi^2)\|_{X_{1,\tau}^{0,0}} \lesssim \|\phi\|_{X_{1,\tau}^{0,0}} + \|\phi\|_{X_{1,\tau}^{0,0}}^N (1 + \|\phi\|_{X_{1,\tau}^{0,0}}^N)^{N+rac{1}{2}+1}.
\]

Using the polynomial bounds of all derivatives of $V$ we crudely estimate using that $H^{s+1}$ is a Banach algebra:
\[
\|\phi V(\phi^2)\|_{L^2_t L^2_x} \lesssim \|\phi\|_{L^2_t H^{s+1}} \|\phi\|_{X_{1,\tau}^{0,0}} + \|\phi\|_{X_{1,\tau}^{0,0}}^N \|\phi\|_{X_{1,\tau}^{0,0}} (1 + \|\phi\|_{X_{1,\tau}^{0,0}}^N)^{N+rac{1}{2}+1}.
\]

Now the contraction mapping principle applies. The claimed properties of $A^{df}$ follow immediately from (23) and (24), and the property $\nabla A^{cf} \in X_{1,\tau}^{0,0}$ from claims 1-3. The proof of Theorem [11] is complete.

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