Relative crystalline representations and \( p \)-divisible groups

Yong Suk Moon

Abstract

Let \( k \) be a perfect field of characteristic \( p > 2 \), and let \( R \) be a relative base ring over \( W(k) \) which is unramified. Examples of \( R \) include \( R = W(k)[X_1, \ldots, X_d] \) and \( R = W(k)(X_1^{\pm 1}, \ldots, X_d^{\pm 1}) \). We define relative \( B \)-pairs and study their relations to weakly admissible \( R[\frac{1}{p}] \)-modules and \( \mathbb{Q}_p \)-representations. As an application, we show that when \( R = W(k)[X] \) with \( k = \overline{k} \), every horizontal crystalline representation of rank 2 with Hodge-Tate weights in \([0, 1]\) arises from a \( p \)-divisible group over \( \text{Spec} R \). Furthermore, we characterize all admissible \( R[\frac{1}{p}] \)-modules of rank 2 which are generated by parallel elements, and give an example of a \( B \)-pair which arises from a weakly admissible \( R[\frac{1}{p}] \)-module but does not arise from a \( \mathbb{Q}_p \)-representation.

Contents

1 Introduction 2

2 \( p \)-adic Hodge Theory in the Relative Case 3 1.1 Crystalline and de Rham Period Rings 3
2.2 Relative \( B \)-pairs 8

3 \( p \)-divisible Groups over \( R \) and Strongly Divisible Lattices 10
3.1 Relative Fontaine-Laffaille Modules 10
3.2 \( p \)-divisible Groups 12
3.3 Lattices of Punctually Weakly Admissible Modules 15

4 Dimension 2 Case 16
4.1 Horizontal Crystalline Representations 16
4.2 Horizontal Crystalline Representations of Rank 2 when \( R = W(k)[X] \) and \( k = \overline{k} \) 20
1 Introduction

Let $k$ be a perfect field of characteristic $p > 2$, and let $W(k)$ be its ring of Witt vectors. Let $R$ be a relative base ring over $W(k)$ which is unramified and satisfies some properties (cf. Section 2.1). Important cases of $R$ include the formal power series ring $R = W(k)[X_1, \ldots, X_d]$, and $R = W(k)[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ which is the $p$-adic completion of $R = W(k)[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$.

Brinon developed $p$-adic Hodge theory in this relative case in [Bri08], which is studied further by Scholze in [Sch13] and Kedlaya-Liu in [KL15]. Let $R$ denote the union of finite $R$-subalgebras $R'$ of a fixed separable closure of Frac($R$) such that $R'[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Then Spec$R[\frac{1}{p}]$ is a pro-universal covering of Spec$R[\frac{1}{p}]$, and $R$ is the integral closure of $R$ in Spec$R[\frac{1}{p}]$. Let $G_R := \text{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi^\text{et}(\text{Spec}R[\frac{1}{p}])$. In [Bri08], the notions of horizontal crystalline and crystalline representations of $G_R$ and filtered $(\varphi, \nabla)$-modules over $R[\frac{1}{p}]$ are defined, generalizing those in the case when the base is a $p$-adic field. In loc. cit., punctually weakly admissible modules and weakly admissible modules are also defined to generalize weakly admissible modules over a $p$-adic field. An open question concerning these objects is the following:

**Question 1.1.** Which filtered $(\varphi, \nabla)$-modules over $R[\frac{1}{p}]$ arise from crystalline representations of $G_R$?

When the base is a $p$-adic field $K$, it is proved in [CF00] that a filtered $\varphi$-module over $K$ with zero monodromy arises from a crystalline representation of the absolute Galois group $\text{Gal}(\overline{K}/K)$ if and only if it is weakly admissible.

Another interesting question in relative $p$-adic Hodge theory concerns representations arising from a $p$-divisible group. For a $p$-divisible group $G$ over Spec$R$, let $T_p(G) := \text{Hom}_R(Q_p/\mathbb{Z}_p, G \times_R \overline{R})$ be the associated Tate module. Then by [Kim15, Corollary 5.4.2], $T_p(G) \otimes_{\mathbb{Z}_p} Q_p$ is a crystalline $G_R$-representation whose Hodge-Tate weights lie in $[0, 1]$. This raises the following natural question.

**Question 1.2.** Which crystalline representations of $G_R$ whose Hodge-Tate weights lie in $[0, 1]$ arise from $p$-divisible groups over Spec$R$?

When the base is a $p$-adic field $K$, it is proved in [Kis06, Corollary 2.2.6] that every crystalline $\text{Gal}(\overline{K}/K)$-representation whose Hodge-Tate weights lie in $[0, 1]$ arises from a $p$-divisible group over the ring of integers $\mathcal{O}_K$ of $K$.

In this paper, our objects of study center around the questions 1.1 and 1.2. By studying Fontaine-Laffaille modules in the relative case, we first establish the following linear algebraic criterion for a punctually weakly admissible module to arise from a $p$-divisible group.

**Theorem 1.3.** Let $D$ be a punctually weakly admissible filtered $(\varphi, \nabla)$-module over $R[\frac{1}{p}]$ whose Hodge-Tate weights lie in $[0, 1]$. Then $D$ arises from a $p$-divisible group over Spec$R$.
if and only if $D$ admits a strongly divisible lattice compatible with filtration, Frobenius, and connection structures.

Here, a strongly divisible lattice means the data $(M, \text{Fil}^1 M)$ such that $M \subset D$ is a projective $R$-submodule stable under Frobenius and connection, $\text{Fil}^1 M \subset M$ is an $R$-direct summand such that $\varphi(\text{Fil}^1 M) \subset pM$, and $M[\frac{1}{p}] \cong D$ as filtered $(\varphi, \nabla)$-modules.

On the other hand, we define the category of $B$-pairs in the relative case and study the relations with $\mathbb{Q}_p$-representations and weakly admissible modules. When $R = W(k)[[X]]$ with $k = \overline{k}$, we compute $B$-pairs corresponding to certain weakly admissible modules and show the following theorem.

**Theorem 1.4.** Let $R = W(k)[[X]]$ and suppose $k$ is algebraically closed. Then every horizontal crystalline $G_R$-representation of rank 2 with Hodge-Tate weights in $[0,1]$ arises from a $p$-divisible group over Spec$R$. If $D$ is an admissible $R[\frac{1}{p}]$-module of rank 2 with Hodge-Tate weights in $[0,1]$ which is generated by its parallel elements, then $D$ is either étale, multiplicative, type I, or type II (cf. Section 4.2). Furthermore, there exists a $B$-pair which arises from a weakly admissible $R[\frac{1}{p}]$-module but does not arise from a $\mathbb{Q}_p$-representation.

In particular, the last statement of Theorem 1.4 shows that the relative case is different from the case when the base is a $p$-adic field where every semi-stable $B$-pair of slope 0 arises from a $\mathbb{Q}_p$-representation. It also answers negatively the question raised in [Bri08] whether weakly admissible implies admissible in the relative case.

**Acknowledgement**

I would like to express my sincere gratitude to Tong Liu for many helpful discussions and suggestions on this topic.

## 2 $p$-adic Hodge Theory in the Relative Case

### 2.1 Crystalline and de Rham Period Rings

We follow the notation as in the Introduction. We first recall the constructions and results of relative $p$-adic Hodge theory developed in [Bri08], using the same terminologies such as punctually weakly admissible modules and weakly admissible modules. Denote by $W(k)[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ the $p$-adic completion of the polynomial ring $W(k)[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. Throughout this paper, We work over a base ring $R$ which is unramified, obtained from $W(k)[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ by iterations of the following operations:

- $p$-adic completion of an étale extension;
• $p$-adic completion of a localization;
• completion with respect to an ideal containing $p$.

We further assume that either $W(k)(X^{±1}_1, \ldots, X^{±1}_d) \to R$ has geometrically regular fibers or $R$ has Krull dimension less than 2, and that $k \to R/pR$ is geometrically integral and $R$ is an integral domain. Let $\hat{\Omega}_R = \lim_{\varphi} R_{\varphi}(R/p^{n})/W(k)$ be the module of $p$-adically continuous Kähler differentials. Then $\hat{\Omega}_R \cong \bigoplus_{d}^{d} R \cdot d(\log X_i)$. If $\nabla : R[1/\rho] \to R[1/\rho] \otimes_R \hat{\Omega}_R$ is the universal continuous derivation, then $(R[1/\rho])^{\nabla=0} = W(k)[1/\rho]$. The Witt vector Frobenius on $W(k)$ extends (not necessarily uniquely) to $R$. We fix such a Frobenius endomorphism $\varphi : R \to R$.

The relative de Rham period ring and the crystalline period ring are constructed as follows. Let $\hat{R} = \lim_{\varphi} R/p\hat{R}$. There exists a natural $W(k)$-linear surjective map $\theta : W(\hat{R}) \to \hat{R}$ which lifts the projection onto the first factor. Here, $\hat{R}$ denotes the $p$-adic completion of $R$. For integers $n \geq 0$, we choose compatibly $p_n \in \hat{R}$ such that $p_0 = p$ and $p_n = p_{n+1}$. Write $\rho = (p_n)_{n \geq 0} \in \hat{R}$, and let $[\rho] \in W(\hat{R})$ be its Teichmüller lift. Then $\theta$ is generated by $\xi := p - [\rho]$. Define $B_{\text{dr}}^{\nabla+}((R) \cong \lim_{\varphi} W((R))[1/\rho]/(\ker(\theta))^n$. Choose compatibly $\epsilon_n \in \hat{R}$ such that $\epsilon_0 = 1, \epsilon_n = \epsilon_{n+1}^p$ with $\epsilon_1 \neq 1$, and let $\bar{\epsilon} = (\epsilon_n)_{n \geq 0} \in \hat{R}$. Then $t := \log [\bar{\epsilon}] \in B_{\text{dr}}^{\nabla+}(R)$ and $B_{\text{dr}}^{\nabla+}(R)$ is $t$-torsion free. The horizontal de Rham period ring is defined to be $B_{\text{dr}}^{\nabla+}(R) = B_{\text{dr}}^{\nabla+}(R)[1/t]$, equipped with the filtration $\text{Fil}^{-j} B_{\text{dr}}^{\nabla+}(R) = t^j B_{\text{dr}}^{\nabla+}(R)$ for $j \in \mathbb{Z}$. The $G_{R}$-action on $W(\hat{R})$ extends uniquely to $B_{\text{dr}}^{\nabla+}(R)$. Let $\theta_R : R \otimes W(k) \to \hat{R}$ be the $R$-linear extension of $\theta$, and denote by $A_{\text{inf}}(\hat{R}/R)$ the completion of $R \otimes W(k) W(\hat{R})$ for the topology given by the ideal $\theta^{-1}_R(p)$. Let $B_{\text{dr}}^{\nabla+}(R) = \lim_{\varphi} A_{\text{inf}}(\hat{R}/R)[1/\rho]/(\ker(\theta_R))^n$. Define the de Rham period ring to be $B_{\text{dr}}(R) = B_{\text{dr}}^{\nabla+}(R)[1/\epsilon]$. For $j \geq 0$, we let $\text{Fil}^{-j} B_{\text{dr}}^{\nabla+}(R) = (\ker(\theta_R))^{j}$ and $\text{Fil}^{0} B_{\text{dr}}^{\nabla+}(R) = \sum_{n=0}^{\infty} \frac{1}{\epsilon^n} \text{Fil}^{n} B_{\text{dr}}^{\nabla+}(R)$. For $j \in \mathbb{Z}$, let $\text{Fil}^{j} B_{\text{dr}}^{\nabla+}(R) = t^{j} \text{Fil}^{0} B_{\text{dr}}^{\nabla+}(R)$. $B_{\text{dr}}^{\nabla}(R)$ is equipped with the connection $\nabla : B_{\text{dr}}^{\nabla}(R) \to B_{\text{dr}}^{\nabla}(R) \otimes_R \hat{\Omega}_R$ which is $W(\hat{R})$-linear and extends the universal continuous derivation of $R$. $\nabla$ satisfies the Griffiths transversality. The $G_{R}$-action on $R \otimes W(k) W(\hat{R})$ extends uniquely to $B_{\text{dr}}^{\nabla+}(R)$, and commutes with $\nabla$. We have a natural embedding $B_{\text{dr}}^{\nabla+}(R) \hookrightarrow B_{\text{dr}}^{\nabla}(R)$ compatible with the filtrations and $G_{R}$-actions, and $B_{\text{dr}}^{\nabla}(R) = (B_{\text{dr}}^{\nabla+}(R))^\nabla=0$. Furthermore, $B_{\text{dr}}^{\nabla}(R)^{G_{R}} = R[1/\rho]$ and $(B_{\text{dr}}^{\nabla}(R))^{G_{R}} = W(k)[1/\rho]$.

For $i = 1, \ldots, d$, choose compatibly $X_{i,n} \in \hat{R}$ such that $X_{i,0} = X_i$ and $X_{i,n} = X_{i,n}^{p}$, and let $[\hat{X}_i] \in \hat{R}$. Let $u_i = X_i \otimes 1 - 1 \otimes [\hat{X}_i] \in R \otimes W(k) W(\hat{R})$. The following proposition is proved in [Bri08].
Proposition 2.1. (cf. [Bri08, Proposition 5.1.4, 5.2.2]) The natural embedding
\[ B_{\text{dr}}^+(R)[u_1, \ldots, u_d] \to B_{\text{dr}}^+(R) \]
is an isomorphism. Furthermore, \( \text{Fil}^0 B_{\text{dr}}(R) = B_{\text{dr}}^+(R)[\frac{u_1}{t}, \ldots, \frac{u_d}{t}] \).

For the horizontal crystalline and crystalline period rings, we first construct the integral ones. Let \( A_{\text{cris}}(R) \) be the \( p \)-adic completion of the divided power envelope of \( W(\overline{R}) \) with respect to \( \ker(\theta) \). The Witt vector Frobenius and \( G_R \)-action on \( W(\overline{R}) \) extend uniquely to \( A_{\text{cris}}(R) \). Define \( A_{\text{cris}}(R) \) to be the \( p \)-adic completion of the divided power envelope of \( R \otimes_{W(k)} W(\overline{R}) \) with respect to \( \ker(\theta_R) \). The \( G_R \)-action on \( R \otimes_{W(k)} W(\overline{R}) \) extends uniquely to \( A_{\text{cris}}(R) \). The Frobenius endomorphism on \( R \otimes_{W(k)} W(\overline{R}) \) given by \( \varphi \) on \( R \) and the Witt vector Frobenius on \( W(\overline{R}) \) extends uniquely to \( A_{\text{cris}}(R) \). We have the connection
\[ \nabla : A_{\text{cris}}(R) \to A_{\text{cris}}(R) \otimes_{R} \Omega_{R} \] which is \( W(\overline{R}) \)-linear and extends the universal continuous derivation of \( R \). The Frobenius on \( A_{\text{cris}}(R) \) is horizontal. We have a natural \( G_R \)-equivariant embedding \( A_{\text{cris}}(R) \hookrightarrow A_{\text{cris}}(R) \), and \( A_{\text{cris}}(R)^{\varphi = 0} = A_{\text{cris}}(R) \). Moreover, \( (A_{\text{cris}}(R))^{G_R} = W(k) \) and \( A_{\text{cris}}(R)^{G_R} = R \). Note that \( t \in A_{\text{cris}}(\mathbb{Z}_p) \) and \( p \) divides \( t^{p-1} \) in \( A_{\text{cris}}(\mathbb{Z}_p) \). \( A_{\text{cris}}(R) \) is \( t \)-torsion free, and we define \( B_{\text{cris}}^\nabla(R) = A_{\text{cris}}(R)[\frac{1}{t}] \) and \( B_{\text{cris}}(R) = A_{\text{cris}}(R)[\frac{1}{t}] \), equipped with the Frobenius and \( G_R \)-action extending those on \( A_{\text{cris}}(R) \). We extend the connection on \( A_{\text{cris}}(R) \) to \( B_{\text{cris}}(R) \) \( t \)-linearly. \( B_{\text{cris}}(R) \) naturally embeds into \( B_{\text{dr}}(R) \) compatibly with the connections and \( G_R \)-actions. We equip the crystalline period rings with the filtration induced by this embedding. Let \( U_1 = \{ x \in B_{\text{cris}}(R) \cap B_{\text{dr}}^+(R), \varphi(x) = px \} \). The following propositions are shown in [Bri08].

Proposition 2.2. (cf. [Bri08, Lemma 6.2.22, Proposition 6.2.23]) The following sequences are exact:
\[ 0 \to \mathbb{Q}_p \cdot t \to U_1 \overset{\theta}{\to} B_{\text{dr}}^+(R) / \text{Fil}^1 B_{\text{dr}}^+(R) \to 0, \]
\[ 0 \to \mathbb{Q}_p \to (B_{\text{cris}}^\nabla(R))^{\varphi = 1} \to B_{\text{dr}}^+(R) / B_{\text{dr}}^+(R) \to 0. \]

Proposition 2.3. (cf. [Bri08, Lemma 6.3.3 Proof, Lemma 6.3.5]) \( A_{\text{cris}}(R)/pA_{\text{cris}}(R) \) is free over \( \overline{R}/p\overline{R} \). Furthermore, for any finite \( R \)-module \( M \), the natural map
\[ A_{\text{cris}}(R) \otimes_{R} M \to \varprojlim_n (A_{\text{cris}}(R)/p^n A_{\text{cris}}(R) \otimes_{R} M) \]
is surjective.

For a continuous \( G_R \)-representation \( V \) over \( \mathbb{Q}_p \), we denote \( D_{\text{cris}}^\nabla(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\nabla(R))^{G_R} \), \( D_{\text{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}}(R))^{G_R} \) and \( D_{\text{dr}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{dr}}(R))^{G_R} \). The natural morphisms
\[ \alpha_{\text{cris}} : D_{\text{cris}}^\nabla(V) \otimes_{W(k)}[\frac{1}{t}] B_{\text{cris}}^\nabla(R) \to V \otimes_{\mathbb{Q}_p} B_{\text{cris}}^\nabla(R), \]
\[ \alpha_{\text{cris}} : D_{\text{cris}}(V) \otimes_{R}[\frac{1}{t}] B_{\text{cris}}(R) \to V \otimes_{\mathbb{Q}_p} B_{\text{cris}}(R), \]
\[ \alpha_{\text{dr}} : D_{\text{dr}}(V) \otimes_{R}[\frac{1}{t}] B_{\text{dr}}(R) \to V \otimes_{\mathbb{Q}_p} B_{\text{dr}}(R) \]
are injective. We say $V$ is horizontal crystalline (resp. crystalline, de Rham) if $\alpha_{\text{cris}}$ (resp. $\alpha_{\text{cris}}, \alpha_{\text{dr}}$) is an isomorphism. For any $\mathbb{Q}_p$-representation $V$, we have natural embeddings $R[1/p]^\dag \otimes_{W(k)[1/p]} D_{\text{cris}}(V) \hookrightarrow D_{\text{cris}}(V)$ and $D_{\text{cris}}(V) \hookrightarrow D_{\text{dr}}(V)$. If $V$ is horizontal crystalline, then $V$ is crystalline and the map $R[1/p]^\dag \otimes_{W(k)[1/p]} D_{\text{cris}}^\nabla(V) \rightarrow D_{\text{cris}}(V)$ is an isomorphism of $R[1/p]$-modules. If $V$ is crystalline, then $V$ is de Rham and the map $D_{\text{cris}}(V) \rightarrow D_{\text{dr}}(V)$ is an isomorphism.

We study the linear algebraic structure of $D_{\text{cris}}(V)$ in the following way. A filtered $(\varphi, \nabla)$-module over $R[1/p]$ is defined to be a tuple $(D, \varphi_D, \nabla_D, \Fil^j D)$ such that

- $D$ is a finite projective $R[1/p]$-module;
- $\varphi_D : D \rightarrow D$ is $\varphi$-semilinear endomorphism such that $1 \otimes \varphi_D$ is an isomorphism;
- $\nabla_D : D \rightarrow D \otimes_R \widehat{\Omega}_R$ is an integrable connection which is topologically quasi-nilpotent, i.e., there exists a finitely generated $R$-submodule $M \subset D$ stable under $\nabla$ such that $M[\frac{1}{p}] = D$ and the induced connection on $M/pM$ is nilpotent. The Frobenius $\varphi_D$ is horizontal with respect to $\nabla_D$;
- $\Fil^j D$ is a decreasing separated and exhaustive filtration by $R[1/p]$-submodules of $D$ such that the graded module $\gr^j D$ is projective over $R[1/p]$. Furthermore, Griffiths transversality holds: $\nabla_D(\Fil^j D) \subset \Fil^{j-1} D \otimes_R \widehat{\Omega}_R$.

Denote by $\text{MF}(R)$ be the category of filtered $(\varphi, \nabla)$-modules over $R[1/p]$, whose morphisms are $R[1/p]$-module morphisms compatible with all structures. It is equipped with tensor product and duality structures as given in [Bri08, Section 7]. For $D \in \text{MF}(R)$, its Hodge-Tate weights are defined to be integers $w \in \mathbb{Z}$ such that $\gr^w D \neq 0$. We define its Hodge number

$$t_H(D) := \sum_{j \in \mathbb{Z}} j \cdot \text{rank}_{R[1/p]}(\gr^j D).$$

Let $\mathfrak{p} \in \text{Spec} R/pR$, and let $\kappa_\mathfrak{p}$ be the direct perfection $\lim_{\rightarrow R/pR}/\mathfrak{p}$. By the universal property of $p$-adic Witt vectors, there exists a unique map $b_\mathfrak{p} : R \rightarrow W(\kappa_\mathfrak{p})$ lifting $R \rightarrow \kappa_\mathfrak{p}$ which is compatible with Frobenius (with Witt vector Frobenius on $W(\kappa_\mathfrak{p})$). Then $D_{\mathfrak{p}} := D \otimes_{R, b_\mathfrak{p}} W(\kappa_\mathfrak{p})$ is a filtered $\varphi$-module over $W(\kappa_\mathfrak{p})[\frac{1}{p}]$ with the induced filtration and Frobenius. We define the Newton number of $D$ at $\mathfrak{p}$ to be the Newton number of $D_{\mathfrak{p}}$, and denote it by $t_N(D, \mathfrak{p})$. We say $D$ is punctually weakly admissible if for all $\mathfrak{p} \in \text{Spec}(R/pR)$, the following conditions hold:

- $t_H(D) = t_N(D, \mathfrak{p})$;
- For each sub-object $D'$ of $D$ in $\text{MF}(R)$, $t_H(D') \leq t_N(D', \mathfrak{p})$.
Denote by $\text{MF}^{\text{pwa}}(R)$ the full subcategory of $\text{MF}(R)$ consisting of punctually weakly admissible modules.

For a $\mathcal{G}_R$-representation $V$ over $\mathbf{Q}_p$, we equip $D^{\text{cris}}(V)$ with the Frobenius induced from $B_{\text{cris}}(R)$, $D_{\text{cris}}(V)$ with the Frobenius and connection induced from $B_{\text{cris}}(R)$, and $D_{\text{dr}}(V)$ with the filtration and connection induced from $B_{\text{dr}}(R)$. Then $D_{\text{cris}}(V)^{\nabla=0} = D^{\text{cris}}(V)$ and the map $D_{\text{cris}}(V) \to D_{\text{dr}}(V)$ is compatible with connections. If $V$ is horizontal crystalline, then the isomorphism $R[\frac{1}{p}] \otimes_{W(k)[\frac{1}{p}]} D^{\text{cris}}(V) \to D_{\text{cris}}(V)$ is compatible with Frobenius. Note that if $V$ is crystalline, then it is horizontal crystalline if and only if the map $R[\frac{1}{p}] \otimes_{W(k)[\frac{1}{p}]} D^{\text{cris}}(V) \to D_{\text{cris}}(V)$ is an isomorphism, i.e., if and only if $D_{\text{cris}}(V)$ is generated by its parallel elements.

If $V$ is crystalline, we further equip $D_{\text{cris}}(V)$ with the filtration induced by $D_{\text{cris}}(V) \cong D_{\text{dr}}(V)$. Then we have $D_{\text{cris}}(V) \in \text{MF}^{\text{pwa}}(R)$.

For $D \in \text{MF}(R)$, define

$$V_{\text{cris}}(D) := (D \otimes_{R[\frac{1}{p}]} B_{\text{cris}}(R))^{\nabla=0,\varphi=1} \cap \text{Fil}^0((D \otimes_{R[\frac{1}{p}]} B_{\text{dr}}(R))^{\nabla=0})$$

where $D \otimes_{R[\frac{1}{p}]} B_{\text{cris}}(R)$ is equipped with the Frobenius and connection given by the tensor product, and $D \otimes_{R[\frac{1}{p}]} B_{\text{dr}}(R)$ is equipped with the filtration and connection given by the tensor product. Then $V_{\text{cris}}(D)$ is a continuous $\mathbf{Q}_p$-representation of $\mathcal{G}_R$. We say $D$ is admissible if there exists a crystalline representation $V$ such that $D \cong D_{\text{cris}}(V)$ in $\text{MF}(R)$, and denote by $\text{MF}^a(R)$ the full subcategory of $\text{MF}^{\text{pwa}}(R)$ consisting of admissible modules. Then $D_{\text{cris}}$ and $V_{\text{cris}}$ are quasi-inverse equivalences of Tannakian categories between the category of crystalline representations and $\text{MF}^a(R)$. It is not known precisely which punctually weakly admissible modules are admissible.

We say $D \in \text{MF}(R)$ is weakly admissible if $D$ is punctually weakly admissible and there exists a finite étale extension $R'$ over $R$ such that $R'[\frac{1}{p}] \otimes_{R[\frac{1}{p}]} D_{\text{cris}}(V)$ is free over $R'[\frac{1}{p}]$. For characters, $D_{\text{cris}}$ induces an equivalence between the category of crystalline characters of $\mathcal{G}_R$ and the category of weakly admissible $R[\frac{1}{p}]$-modules of rank 1. However, it is not known whether $D_{\text{cris}}(V)$ is weakly admissible for any crystalline representation $V$.

For our study, it is useful to consider the following natural morphism between unramified base rings over $W(k)$. Let $R_{\varphi}$ be the $p$-adic completion of $\lim_{\phi} R_{(p)}$. By the universal property of $p$-adic Witt vectors, we have a $\varphi$-equivariant isomorphism $R_{\varphi} \cong W(k_{\varphi})$ where $k_{\varphi}$ is the perfect closure $\lim_{\phi} \text{Frac}(R/pR)$ of $\text{Frac}(R/pR)$. Let $b_{\varphi} : R \to R_{\varphi}$ be the natural morphism compatible with Frobenius. Note that $b_{\varphi}$ is injective since $R$ is an integral domain and $\varphi : R \to R$ is injective, and it is flat. Choose a lifting $b_{\natural} : \overline{R} \to \overline{R}_{\varphi}$. Such a choice induces the commutative diagram
Thus, $b_g$ extends uniquely to embeddings from the period rings over $R$ to the corresponding period rings over $R_g$ compatible with all structures, and induces $b_g : G_{R_g} \to G_R$. If $V$ is a crystalline $G_R$-representation, then it is a crystalline $G_{R_g}$-representation via $b_g$, and $D_{\text{cris},R}(V) \otimes_{R,b_g} R_g \cong D_{\text{cris},R_g}(V)$ compatibly with the filtrations and Frobenius.

### 2.2 Relative $B$-pairs

In [Ber08], $B$-pairs are studied when the base is a $p$-adic field. There is a natural fully faithful functor from the category of $\mathbb{Q}_p$-representations to the category of $B$-pairs, and the category of $B$-pairs is equivalent to that of $(\phi, \Gamma)$-modules over the Robba ring. We define the category of $B$-pairs in the relative case and study its relations to $\mathbb{Q}_p$-representations and admissible modules.

Let $B_e(R) := (B_{\text{cris}}(R))^\varphi = 1$. A $B$-pair $W = (W_e, W_{\text{dR}})$ is given by a finite free $B_e(R)$-module $W_e$ equipped with a semi-linear $G_R$-action and a finite free $B_{\text{dR}}(R)$-module $W_{\text{dR}}$ equipped with a semi-linear $G_R$-action such that

$$W_e \otimes_{B_e(R)} B_{\text{dR}}(R) \cong W_{\text{dR}} \otimes_{B_{\text{dR}}(R)} B_{\text{dR}}(R)$$

as $B_{\text{dR}}(R)$-modules compatible with $G_R$-actions. Denote by $B\text{-Pair}(R)$ the category of $B$-pairs whose morphisms are pairs of $B_e(R)$-module and $B_{\text{dR}}(R)$-module morphisms compatible with $G_R$-actions and with isomorphisms over $B_{\text{dR}}(R)$.

We have a natural functor $W_R$ from the category of $\mathbb{Q}_p$-representations of $G_R$ to $B\text{-Pair}(R)$ given by $W_R(V) = (V \otimes_{\mathbb{Q}_p} B_e(R), V \otimes_{\mathbb{Q}_p} B_{\text{dR}}(R))$.

**Proposition 2.4.** The functor $W_R$ is fully faithful. Furthermore, if $V$ is a crystalline $G_R$-representation, then

$$W_R(V) \cong ((D_{\text{cris}}(V) \otimes_{R[\frac{1}{p}]} B_{\text{cris}}(R))^\varphi = 1, \text{Fil}^0(D_{\text{dR}}(V) \otimes_{R[\frac{1}{p}]} B_{\text{dR}}(R))^\varphi = 0)$$

as $B$-pairs.

**Proof.** We have $B_e(R) \cap B_{\text{dR}}(R) = \mathbb{Q}_p$ by Proposition 2.2, so $W_R$ is fully faithful.

If $V$ is crystalline, then the maps

$$\alpha_{\text{cris}} : D_{\text{cris}}(V) \otimes_{R[\frac{1}{p}]} B_{\text{cris}}(R) \to V \otimes_{\mathbb{Q}_p} B_{\text{cris}}(R)$$

are respectively $W_e \circ \psi_{\text{cris}}(V)$ and $W_{\text{dR}} \circ \psi_{\text{dR}}(V)$.
embedding
morphisms of period rings
image of
Furthermore, the diagram connecting \( \alpha \)
are isomorphisms. Thus,

\[(V \otimes_{\mathbb{Q}_p} B_{\text{cris}}(R))^{\nabla = 0, \varphi = 1} = V \otimes_{\mathbb{Q}_p} B_{\epsilon}(R) \cong (D_{\text{cris}}(V) \otimes_{R[\frac{1}{p}]} B_{\text{cris}}(R))^{\nabla = 0, \varphi = 1}\]

and

\[\text{Fil}^0(V \otimes_{\mathbb{Q}_p} B_{\text{dR}}(R))^{\nabla = 0} = V \otimes_{\mathbb{Q}_p} B_{\text{dR}}^{\nabla +}(R) \cong \text{Fil}^0(D_{\text{dR}}(V) \otimes_{R[\frac{1}{p}]} B_{\text{dR}}(R))^{\nabla = 0}.\]

Furthermore, the diagram connecting \( \alpha_{\text{cris}} \) and \( \alpha_{\text{dR}} \) induced by \( D_{\text{cris}}(V) \cong D_{\text{dR}}(V) \) and the embedding \( B_{\text{cris}}(R) \to B_{\text{dR}}(V) \) is commutative. This proves the second statement.

If we denote by \( B-\text{Pair}^{\text{rep}}(R) \) the full subcategory of \( B-\text{Pair}(R) \) given by the essential image of \( W_R \), then the functor \( (W_e, W_{\text{dR}}^{\nabla +}) \mapsto \text{Fil}^0(W_e \otimes_{R[\frac{1}{p}]} B_{\text{cris}}(R))^{\nabla = 0} \) to the category of \( \mathbb{Q}_p \)-representations is the quasi-inverse to \( W_R \). We say a \( B \)-pair \( (W_e, W_{\text{dR}}^{\nabla +}) \) is crystalline if there exists a crystalline representation \( V \) such that \( (W_e, W_{\text{dR}}^{\nabla +}) \cong W_R(V) \).

For a weakly admissible \( R[\frac{1}{p}] \)-module \( D \), we denote \( W_e(D) := (D \otimes_{R[\frac{1}{p}]} B_{\text{cris}}(R))^{\nabla = 0, \varphi = 1} \) and \( W_{\text{dR}}^{\nabla +}(D) := \text{Fil}^0(D \otimes_{R[\frac{1}{p}]} B_{\text{dR}}(R))^{\nabla = 0} \). In the case when \( (W_e(D), W_{\text{dR}}^{\nabla +}(D)) \in B-\text{Pair}(R) \), we denote by \( W(D) = (W_e(D), W_{\text{dR}}^{\nabla +}(D)) \) the corresponding \( B \)-pair.

Let \( R_1 \) and \( R_2 \) be unramified base rings over \( \mathbb{W}(k) \) satisfying the conditions in Section 2.1. Suppose we have a ring map \( b : R_1 \to R_2 \) compatible with Frobenius and connections which lifts to \( b : \mathcal{O}_1 \to \mathcal{O}_2 \). This uniquely induces a map of Galois groups \( \mathcal{G}_{R_2} \to \mathcal{G}_{R_1} \), and morphisms of period rings \( B_{\text{cris}}(R_1) \to B_{\text{cris}}(R_2) \) and \( B_{\text{dR}}(R_1) \to B_{\text{dR}}(R_2) \) compatible with all structures. Thus, the base change map gives a functor from \( B-\text{pair}(R_1) \) to \( B-\text{pair}(R_2) \) by

\[(W_e, W_{\text{dR}}^{\nabla +}) \mapsto (W_e \otimes_{B_{\text{cris}}(R_1), b} B_{\epsilon}(R_2), W_{\text{dR}}^{\nabla +} \otimes_{B_{\text{dR}}^{\nabla +}(R_1), b} B_{\text{dR}}^{\nabla +}(R_2)),\]

which is compatible with the functors \( W_{R_1} \) and \( W_{R_2} \). This sends crystalline \( B \)-pairs over \( R_1 \) to crystalline \( B \)-pairs over \( R_2 \).

We end this section by studying the properties of certain subrings of \( B_{\text{cris}}^{\nabla +}(R) \). Let \( w \in \mathbb{Q}_p^{\times} \) be a generator of a normal basis over \( \mathbb{Q}_p \). Then \( \varphi^2(w) = w \) and \( w^2 - (\varphi(w))^2 \neq 0 \).

**Lemma 2.5.** \( (B_{\text{cris}}^{\nabla +}(R))^{\varphi^2 = 1} \) is a free rank 2 \( B_{\epsilon}(R) \)-module with a basis given by \( \{w, \varphi(w)\} \).

**Proof.** It suffices to show that the natural map \( \mathbb{Q}_p^{\times} \otimes_{\mathbb{Q}_p} B_{\epsilon}(R) \to (B_{\text{cris}}^{\nabla +}(R))^{\varphi^2 = 1} \) of \( B_{\epsilon}(R) \)-modules is surjective. This follows from that for any \( x \in (B_{\text{cris}}^{\nabla +}(R))^{\varphi^2 = 1} \) and both \( wx + \varphi(w)\varphi(x) \) and \( w\varphi(x) + \varphi(w)x \) lie in \( B_{\epsilon}(R) \).

By [Col08, Section 2.4], there exists an element \( t_2 \in B_{\text{cris}}(\mathbb{Z}_p) \) such that \( \varphi^2(t_2) = pt_2, \ t_2 \in \text{Fil}^1 B_{\text{dR}}(\mathbb{Z}_p) \setminus \text{Fil}^2 B_{\text{dR}}(\mathbb{Z}_p) \), and \( t_2\varphi(t_2) \in t_2 \cdot \mathbb{Q}_p^{\times} \).

**Corollary 2.6.** \( (B_{\text{cris}}^{\nabla +}(R))^{\varphi^2 = p} \) is free of rank 2 over \( B_{\epsilon}(R) \) with a basis given by \( \{wt_2, \varphi(w)t_2\} \).

**Proof.** The map \( (B_{\text{cris}}^{\nabla +}(R))^{\varphi^2 = 1} \to (B_{\text{cris}}^{\nabla +}(R))^{\varphi^2 = p} \) given by \( x \mapsto t_2 \cdot x \) is an isomorphism of \( B_{\epsilon}(R) \)-modules. Thus, it follows from Lemma 2.5.
3 p-divisible Groups over $R$ and Strongly Divisible Lattices

3.1 Relative Fontaine-Laffaille Modules

We define relative Fontaine-Laffaille modules which are projective over $R$. We first recall torsion Fontaine-Laffille modules which are defined in [Fal88] to study torsion crystalline cohomologies of smooth schemes over $R$. Let $h \leq p - 2$ be a positive integer. Denote by $\text{FL}^{[0,h]}_{\text{tor}}(R)$ the category whose objects are tuples $(M, \text{Fil}^i M, \varphi^i, \nabla)$ satisfying the following conditions:

- $M$ is a finite $p$-power torsion $R$-module;
- $\text{Fil}^i M$ is a decreasing filtration on $M$ such that $\text{Fil}^0 M = M$ and $\text{Fil}^{h+1} M = 0$;
- For each $i$, $\varphi^i : \text{Fil}^i M \to M$ is a $\varphi$-semilinear map. The composition of $\varphi^{i-1}$ with the injection $\text{Fil}^i M \to \text{Fil}^{i-1} M$ is $p\varphi^i$;
- Denote by $\tilde{M}$ the $R$-module given by the colimit of the following diagram
  $$
  \cdots \to \text{Fil}^{i+1} M \leftarrow \text{Fil}^i M \rightarrow \text{Fil}^i M \leftarrow \text{Fil}^i M \rightarrow \text{Fil}^{i-1} M \leftarrow \cdots
  $$
  where the right arrows are the injections and the left arrows are multiplication by $p$. Then the $R$-linear map $\varphi_M \otimes 1 : \tilde{M} \otimes_R \varphi R \to M$ induced by $\varphi^i$’s is an isomorphism;
- $\nabla : M \to M \otimes_R \hat{\Omega}_R$ is an integrable connection satisfying the Griffiths transversality $\nabla(\text{Fil}^i M) \subset \text{Fil}^{i-1} M \otimes_R \hat{\Omega}_R$ such that $\varphi^i$’s are parallel: $\nabla \circ \varphi^i = (\varphi^{i-1} \otimes_R d\varphi/p) \circ \nabla$ as a map from $\text{Fil}^i M$ to $M \otimes_R \hat{\Omega}_R$.

Morphisms in $\text{FL}^{[0,h]}_{\text{tor}}(R)$ are $R$-module morphisms compatible with all structures. For any $(M, \text{Fil}^i M, \varphi^i, \nabla) \in \text{FL}^{[0,h]}_{\text{tor}}(R)$, we have the induced connection on $\tilde{M} \otimes_{R,\varphi} R$ such that $\varphi_M \otimes 1 : \tilde{M} \otimes_{R,\varphi} R \to M$ is parallel. It follows that $\nabla$ on $M$ is nilpotent.

The following proposition about $R$-module structures and filtrations of torsion Fontaine-Laffaille modules are proved in [Fal88].

**Proposition 3.1.** (cf. [Fal88] Theorem 2.1) If $M \in \text{FL}^{[0,h]}_{\text{tor}}(R)$, then $M$ is locally a direct sum of $R$-modules of the form $R/p^e R$ where $e$ is a positive integer. Furthermore, $\text{Fil}^i M$’s are direct summands of $M$ as $R$-modules. Any morphism $M \to N$ in $\text{FL}^{[0,h]}_{\text{tor}}(R)$ is strict for filtrations.

We now define projective Fontaine-Laffaille modules over $R$. Denote by $\text{FL}^{[0,h]}_{\text{proj}}(R)$ the category whose objects are tuples $(M, \text{Fil}^i M, \varphi, \nabla)$ satisfying the following conditions:
• $M$ is a finite projective $R$-module;

• $\Fil^i M$ is a decreasing filtration on $M$ such that $\Fil^0 M = M$ and $\Fil^{i+1} M = 0$. For each $i$, $\Fil^{i+1} M$ is an $R$-direct summand of $\Fil^i M$;

• $\varphi : M \to M$ is a $\varphi$-semilinear endomorphism such that $\varphi(\Fil^i M) \subset p^i M$. The $R$-submodule of $M$ given by $\sum_{i=0}^{h} \varphi_p^i (\Fil^i M)$ generates $M$ as $R$-modules;

• $\nabla : M \to M \otimes_R \tilde{\Omega}_R$ is a topologically quasi-nilpotent integrable connection such that $\varphi$ is horizontal and Griffiths transversality holds.

Morphisms in $\FL^{[0,h]}_{\text{proj}}(R)$ are $R$-module morphisms compatible with all structures.

**Proposition 3.2.** Let $M$ be a finite projective $R$-module. Then $M \in \FL^{[0,h]}_{\text{proj}}(R)$ if and only if for each $n \geq 1$, $M/p^n M \in \FL^{[0,h]}_{\text{tor}}(R)$ such that the natural $R$-module map $M/p^{n+1} M \to M/p^n M$ induces a morphism in $\FL^{[0,h]}_{\text{tor}}(R)$, and $M \cong \varinjlim_n M/p^n M$ compatibly with filtration, Frobenius, and connection.

**Proof.** By Proposition 3.1, it suffices to show that if $M \in \FL^{[0,h]}_{\text{proj}}(R)$, then for $M/p^n M$ with the induced filtration, Frobenius, and connection structures,

$$\varphi_{M/p^n M} \otimes 1 : \tilde{M/p^n M} \otimes_{R,\varphi} R \to M/p^n M$$

is an isomorphism. We induct on $n$.

For the case $n = 1$, note that $\varphi_{M/pM} \otimes 1$ is surjective since $\sum_{i=0}^{h} \varphi_p^i (\Fil^i M)$ generates $M$. Since $\Fil^{i+1}(M/pM)$ is an $R/pR$-direct summand of $\Fil^i(M/pM)$ for each $i$, we have $M/pM \cong \text{gr}_{\Fil^i}(M/pM) \cong M/pM$ as $R$-modules. So $\varphi_{M/pM} \otimes 1$ is a surjective map of projective $R/pR$-modules of the same rank, hence an isomorphism.

For $n \geq 1$, suppose $\varphi_{M/p^n M} \otimes 1$ is an isomorphism. Since $M$ is flat over $R$ and $\varphi : R \to R$ is flat by [Bri08, Lemma 7.1.9], we have a commutative diagram of $R$-modules

$$
\begin{array}{c}
\tilde{M/p^n M} \otimes_{R,\varphi} R \\
\downarrow \\
M/p^n M & \xrightarrow{\times p} & M/p^{n+1} M
\end{array}
\begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow & \longrightarrow & \\
0 & \longrightarrow & M/pM
\end{array}
$$

whose rows are exact. The left and right vertical maps are isomorphisms, so the middle vertical map is an isomorphism by the snake lemma. This proves the assertion. \(\square\)

In the following sections, we will consider the case when $h = 1$, i.e., the category $\FL^{[0,1]}_{\text{proj}}(R)$. Note that in this case, Griffiths transversality does not impose any additional condition.
3.2 $p$-divisible Groups

We show that the category of $p$-divisible groups over $R$ is anti-equivalent to $\text{FL}_{\text{proj}}^{[0,1]}(R)$. We remark that for the case when $R$ is further assumed to be smooth over $W(k)$, such a result is a direct consequence of Proposition 3.2 and [Fal88, Theorem 7.1]. Instead of following the argument in [Fal88], we take a slightly different approach.

We first recall the result in [Kim15] about $p$-divisible groups over $\text{Spec} R$ and relative Breuil modules. Let $\mathcal{S} = R[u]$, and let $S$ be the $p$-adic completion of the divided power envelope of $\mathcal{S}$ with respect to the ideal $(u - p) \subset \mathcal{S}$. Note that $\mathcal{S}/(u - p) = R$. Explicitly, the elements of $S$ can be described as

$$S = \left\{ \sum_{n \geq 0} a_n \frac{u^n}{n!} \mid a_n \in R, a_n \to 0 \text{ p-adically} \right\}$$

inside the ring $(R^{[1/p]}[u])$. The Frobenius on $\mathcal{S}$ extending that on $R$ given by $u \mapsto u^p$ extends uniquely to $S$. Let $\text{Fil}^1 S \subset S$ be the $p$-adically completed ideal generated by the divided powers $\frac{(u - p)^n}{n!}$, $n \geq 1$. Equip $S$ with the connection $\nabla : S \to S \otimes_R \hat{\Omega}_R$ by $u$-linearly extending the universal continuous derivation of $R$. Let $I \subset S$ be the ideal topologically generated by $\frac{u^n}{n!}$ for $n \geq 1$. Then $S/I \cong R$ and the natural induced projection $f_0 : S \to R$ is $\varphi$-equivariant. On the other hand, let $f_p : S \to S/\text{Fil}^1 S \cong R$.

A relative Breuil module is a tuple $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi, \nabla)$ such that

- $\mathcal{M}$ is a finite projective $S$-module;
- $\text{Fil}^1 \mathcal{M} \subset \mathcal{M}$ is an $S$-submodule containing $\text{Fil}^1 S \cdot \mathcal{M}$ such that $\mathcal{M}/\text{Fil}^1 \mathcal{M}$ is projective over $R$;
- $\varphi : \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear endomorphism such that $(1 \otimes \varphi)(\varphi^* \text{Fil}^1 \mathcal{M}) = p\mathcal{M}$ as $S$-modules.
- If we denote $M := \mathcal{M} \otimes_{S,f_0} R$, then $\nabla : M \to M \otimes_R \hat{\Omega}_R$ is a topologically quasi-nilpotent integrable connection commuting with $\varphi_M := \varphi_M \otimes \varphi_R$.

Denote by $\text{Mod}_{\text{Br}}(R)$ the category of Breuil modules whose morphisms are $S$-module morphisms compatible with Frobenius, filtration, and connection. There is a natural contravariant functor $\mathcal{M}^*$ from the category of $p$-divisible groups over $\text{Spec} R$ to $\text{Mod}_{\text{Br}}(R)$, obtained by evaluating the associated Dieudonné crystals at $S$. The following result is proved in [Kim15] generalizing the classical case when the base ring is the ring of integers of a $p$-adic field.
Proposition 3.4. given in Section 2.1. To make use of that, we consider the base change map $b$ for each integer $n$.

Proof. Thus, it defines a functor from $\text{FL}^{[0,1]}_\text{proj}(R) \rightarrow \text{Mod}_{\text{Br}}(R)$ as follows. For $M \in \text{FL}^{[0,1]}_\text{proj}(R)$, let $\mathcal{M}_{\text{Br}}(M) = M \otimes_R S$ equipped with Frobenius endomorphism $\varphi : \mathcal{M}_{\text{Br}}(M) \rightarrow \mathcal{M}_{\text{Br}}(M)$ given by $\varphi_M \otimes \varphi_S$. Equip $\mathcal{M}_{\text{Br}}(M) \otimes_{S,f_0} R \cong M$ with the connection $\nabla_M$. The filtration is defined by the $S$-submodule

$$\text{Fil}^1 \mathcal{M}_{\text{Br}}(M) = M \otimes_R \text{Fil}^1 S + \text{Fil}^1 M \otimes_R S.$$ Then $\text{Fil}^1 \mathcal{M}_{\text{Br}}(M)$ contains $\text{Fil}^1 S \cdot \mathcal{M}_{\text{Br}}(M)$, and $\mathcal{M}_{\text{Br}}(M)/\text{Fil}^1 \mathcal{M}_{\text{Br}}(M) \cong M/\text{Fil}^1 M$ as $R$-modules. We have $(1 \otimes \varphi)(\varphi^* \text{Fil}^1 M) = pM$ as $S$-modules since $\varphi(M) + \frac{2}{p}(\text{Fil}^1 M) = M$. Thus, it defines a functor from $\text{FL}^{[0,1]}_\text{proj}(R)$ to $\text{Mod}_{\text{Br}}(R)$.

Proposition 3.4. The functor $\mathcal{M}_{\text{Br}} : \text{FL}^{[0,1]}_\text{proj}(R) \rightarrow \text{Mod}_{\text{Br}}(R)$ is fully faithful.

Proof. Faithfulness follows from the definition of $\mathcal{M}_{\text{Br}}$. Let $M_1, M_2 \in \text{FL}^{[0,1]}_\text{proj}(R)$, and suppose $h : \mathcal{M}_{\text{Br}}(M_1) = M_1 \otimes_R S \rightarrow \mathcal{M}_{\text{Br}}(M_2) = M_2 \otimes_R S$ is a morphism in $\text{Mod}_{\text{Br}}(R)$. Then

$$h_0 := h \otimes 1 : \mathcal{M}_{\text{Br}}(M_1) \otimes_{S,f_0} R = M_1 \rightarrow \mathcal{M}_{\text{Br}}(M_2) \otimes_{S,f_0} R = M_2$$

is an $R$-module morphism compatible with Frobenius and connections. Let $h' = h_0 \otimes 1 : M_1 \otimes_R S \rightarrow M_2 \otimes_R S$ be the induced $S$-module morphism which is compatible with Frobenius.

Note that $(h - h')(\mathcal{M}_{\text{Br}}(M_1)) \subset I\mathcal{M}_{\text{Br}}(M_2)$. Since $(1 \otimes \varphi)[\frac{1}{p}] : \varphi^*(\mathcal{M}_{\text{Br}}(M_i)) [\frac{1}{p}] \rightarrow \mathcal{M}_{\text{Br}}(M_i)[\frac{1}{p}]$ is an isomorphism for $i = 1, 2$ and both $h$ and $h'$ are $\varphi$-compatible, we have

$$(h - h')(\mathcal{M}_{\text{Br}}(M_1)) \subset (h - h')(1 \otimes \varphi)^n(\varphi^* \mathcal{M}_{\text{Br}}(M_1)[\frac{1}{p}])$$

$$= (1 \otimes \varphi)^n(\varphi^n((h - h')(\mathcal{M}_{\text{Br}}(M_1)[\frac{1}{p}])) \subset \varphi^n(I)\mathcal{M}_{\text{Br}}(M_2)[\frac{1}{p}]$$

for each integer $n \geq 1$. Since $\bigcap_{n \geq 1} \varphi^n(I) = 0$ and therefore $\bigcap_{n \geq 1} \varphi^n(I)\mathcal{M}_{\text{Br}}(M_2)[\frac{1}{p}] = 0$, we have $h = h'$. It follows that $h_0 : M_1 \rightarrow M_2$ is compatible with filtration, and the functor $\mathcal{M}_{\text{Br}}$ is full. \hfill \Box

When the base ring is $W(k)$, the category of Breuil modules and the category of Fontaine-Laffaille modules of Hodge-Tate weights in $[0, 1]$ is shown to be equivalent in \cite{Gao17} via an explicit method. To make use of that, we consider the base change map $b_g : R \rightarrow R_g$ given in Section [2.1].

13
Lemma 3.5. $R[p^{1/2}] \cap R_{g} = R$ as subrings of $R_{g}[1/p]$.

Proof. It suffices to show that if $r \in R$ with $p \nmid r$ in $R$, then $p \nmid r$ in $R_{g}$. This follows from $R_{g} \cong W(k_{g})$ and the map $R/pR \to k_{g}$ is injective since $R/pR$ is an integral domain.

Let $\mathfrak{S}_{g} = R_{g}[u]$ equipped with Frobenius given by $u \mapsto u^{p}$, and let $S_{g}$ be the $p$-adic completion of the divided power envelope of $\mathfrak{S}_{g}$ with respect to the ideal $(u - p)$, equipped with Frobenius extending that of $\mathfrak{S}_{g}$. Let $\text{Fil}^{1} S_{g} \subset S_{g}$ be the filtration defined similarly as above. Then $b_{g} : R \to R_{g}$ induces the map $b_{g} : S \to S_{g}$ compatible with Frobenius and filtration. Denote by $\text{Mod}_{\mathfrak{B}_{g}}(R_{g})$ the corresponding category of Breuil modules over $S_{g}$.

Note that we have a natural functor $\text{Mod}_{\mathfrak{B}_{g}}(R) \to \text{Mod}_{\mathfrak{B}_{g}}(R_{g})$ given by $\mathcal{M} \mapsto \mathcal{M} \otimes_{S_{b_{g}}} S_{g}$.

For a Breuil module $\mathcal{M} \in \text{Mod}_{\mathfrak{B}_{g}}(R)$, let $M = \mathcal{M} \otimes_{S_{b_{g}}} R$ be the associated $R$-module equipped with the induced Frobenius $\varphi_{M} = \varphi_{\mathcal{M}} \otimes \varphi_{R}$ and connection.

**Proposition 3.6.** There exists a unique $\varphi$-compatible section $s : M \to \mathcal{M}$. Furthermore, $s$ is horizontal, and the map $s \otimes 1 : M \otimes_{R} S \to \mathcal{M}$ is an isomorphism.

Proof. By [Kim15, Lemma 3.3.5], there exists a unique $\varphi$-compatible section $s_{1} : M \to \mathcal{M}[1/p]$. On the other hand, since $\mathcal{M} \otimes_{S_{b_{g}}} S_{g} \in \text{Mod}_{\mathfrak{B}_{g}}(R_{g})$, there exists a unique $\varphi$-compatible section $s_{2} : M \otimes_{R,b_{g}} R_{g} \to \mathcal{M} \otimes_{S_{b_{g}}} S_{g}$ by [Gao17, Proposition 3.2.3]. Let $s_{2} : M \to \mathcal{M} \otimes_{S_{b_{g}}} S_{g}$ be the composite of $s_{2}$ with the embedding $M \to M \otimes_{R,b_{g}} R_{g}$. By the unicity of $s_{2}$, the maps $s_{1}$ and $s_{2}$ agree as maps from $M$ to $\mathcal{M} \otimes_{S_{b_{g}}} S_{g}[1/p]$. Thus, we get a $\varphi$-equivariant $R$-module morphism $s : M \to \mathcal{M}[1/p] \cap (\mathcal{M} \otimes_{S_{b_{g}}} S_{g})$.

By Lemma 3.5, $S_{g} \cap S[1/p] = S$ as subrings of $S[1/p]$. Since $\mathcal{M}$ is projective over $S$, we have $\mathcal{M}[1/p] \cap (\mathcal{M} \otimes_{S_{b_{g}}} S_{g}) = \mathcal{M}$. Thus, $s$ is a $\varphi$-compatible section $s : M \to \mathcal{M}$. The unicity of $s$ follows from that of $s_{1}$, and $s$ is horizontal since $s_{1}$ is horizontal by [Kim15, Lemma 3.3.5]. Furthermore, by Nakayama’s lemma, the map $s \otimes 1 : M \otimes_{R} S \to \mathcal{M}$ is a surjective morphism of finite projective $S$-modules of the same rank, and thus it is an isomorphism.

Using Proposition 3.6, we can construct a functor $M_{\text{FL}} : \text{Mod}_{\mathfrak{B}_{g}}(R) \to \text{FL}[0,1]_{\text{proj}}$. For $\mathcal{M} \in \text{Mod}_{\mathfrak{B}_{g}}(R)$, let $M_{\text{FL}}(\mathcal{M}) = \mathcal{M} \otimes_{S_{b_{g}}} R$ equipped with the induced connection and Frobenius $\varphi : M_{\text{FL}}(\mathcal{M}) \to M_{\text{FL}}(\mathcal{M})$ given by $\varphi_{\mathcal{M}} \otimes \varphi_{R}$. Consider the projection $f_{p} : S \to S/\text{Fil}^{1} S \cong R$. Let $\text{Fil}^{1} M_{\text{FL}}(\mathcal{M}) \subset M_{\text{FL}}(\mathcal{M})$ be the $R$-submodule given by the image of the composite

$$\text{Fil}^{1} \mathcal{M} \to \mathcal{M} \xrightarrow{(s \otimes 1)^{-1}} M_{\text{FL}}(\mathcal{M}) \otimes_{R} S \xrightarrow{f_{p}} M_{\text{FL}}(\mathcal{M}).$$

Here, $(s \otimes 1)^{-1} : \mathcal{M} \to M_{\text{FL}}(\mathcal{M}) \otimes_{R} S$ is the inverse of the isomorphism $s \otimes 1$ given by Proposition 3.6. Note that $M_{\text{FL}}(\mathcal{M})/\text{Fil}^{1} M_{\text{FL}}(\mathcal{M}) \cong \mathcal{M}/\text{Fil}^{1} \mathcal{M} \otimes_{S_{b_{g}}} R \cong \mathcal{M}/\text{Fil}^{1} \mathcal{M}$ since $\text{Fil}^{1} S : \mathcal{M} \subset \text{Fil}^{1} \mathcal{M}$, and $\varphi(\text{Fil}^{1} M_{\text{FL}}(\mathcal{M})) \subset \mathcal{M}$. 

\[ \text{End of page 14} \]
**Proposition 3.7.** Under the isomorphism \( s \otimes 1 : M_{\text{FL}}(\mathcal{M}) \otimes_R S \to \mathcal{M} \) given by Proposition 3.6, we have
\[
\text{Fil}^1 M_{\text{FL}}(\mathcal{M}) \otimes_R S + M_{\text{FL}}(\mathcal{M}) \otimes_R \text{Fil}^1 S = \text{Fil}^1 \mathcal{M}
\]
as \( S \)-modules. Furthermore,
\[
\varphi(M_{\text{FL}}(\mathcal{M})) + \frac{\varphi}{p}(\text{Fil}^1 M_{\text{FL}}(\mathcal{M})) = M_{\text{FL}}(\mathcal{M})
\]
as \( R \)-modules.

**Proof.** Consider the short exact sequence
\[
0 \to \text{Fil}^1 \mathcal{M} \to \mathcal{M} \to \mathcal{M}/\text{Fil}^1 \mathcal{M} \to 0.
\]
Via taking the tensor product \( \cdot \otimes_{S,f} R \), we obtain a short exact sequence
\[
0 \to \text{Fil}^1 M_{\text{FL}}(\mathcal{M}) \to \mathcal{M} \otimes_{S,f} R \to \mathcal{M}/\text{Fil}^1 \mathcal{M} \to 0.
\]
Hence, \( \text{Fil}^1 M = \text{Fil}^1 S \cdot \mathcal{M} + S \cdot \text{Fil}^1 M_{\text{FL}}(\mathcal{M}) \). Note that \( \mathcal{M} \cong M_{\text{FL}}(\mathcal{M}) \otimes_R S \) and \( M_{\text{FL}}(\mathcal{M})/\text{Fil}^1 M_{\text{FL}}(\mathcal{M}) \) is flat over \( R \). Thus, \( \text{Fil}^1 S \cdot \mathcal{M} \cong M_{\text{FL}}(\mathcal{M}) \otimes_R \text{Fil}^1 S \) and \( S \cdot \text{Fil}^1 M_{\text{FL}}(\mathcal{M}) \cong \text{Fil}^1 M_{\text{FL}}(\mathcal{M}) \otimes_R S \), and the first part of the proposition follows.

Since \( \varphi(\text{Fil}^1 M) = p M \), it follows that \( \varphi(M_{\text{FL}}(\mathcal{M})) + \frac{\varphi}{p}(\text{Fil}^1 M_{\text{FL}}(\mathcal{M})) = M \) as \( S \)-modules. By taking the tensor product \( \cdot \otimes_{S,f} R \), we get \( \varphi(M_{\text{FL}}(\mathcal{M})) + \frac{\varphi}{p}(\text{Fil}^1 M_{\text{FL}}(\mathcal{M})) = M_{\text{FL}}(\mathcal{M}) \).

**Proposition 3.8.** We have an equivalence of categories \( \mathcal{M}_{\text{Br}} : \text{FL}^{[0,1]}_{\text{proj}}(R) \cong \text{Mod}_{\text{Br}}(R) \) with a quasi-inverse \( M_{\text{FL}} \). Hence,
\[
M^* := M_{\text{FL}} \circ M^* : \{p\text{-divisible groups over Spec}R\} \to \text{FL}^{[0,1]}_{\text{proj}}(R)
\]
is an anti-equivalence of categories.

**Proof.** By Theorem 3.3 and Proposition 3.4, it suffices to show that \( \mathcal{M}_{\text{Br}}(M_{\text{FL}}(\mathcal{M})) = \mathcal{M} \) for any \( \mathcal{M} \in \text{Mod}_{\text{Br}}(R) \). This follows from Proposition 3.6 and 3.7.

### 3.3 Lattices of Punctually Weakly Admissible Modules

We now study which filtered \((\varphi, \nabla)\)-modules over \( R^{[1]}_{\mathbb{P}} \) arise from a \( p \)-divisible group over Spec\( R \). Let \( D \) be a filtered \((\varphi, \nabla)\)-module over \( R^{[1]}_{\mathbb{P}} \) with Hodge-Tate weights in \([0, 1]\). We define a **strongly divisible lattice** of \( D \) to be a tuple \((M, \text{Fil}^1 M)\) such that

- \( M \) is a finite projective \( R \)-submodule of \( D \) stable under Frobenius and connection;
- \( \text{Fil}^1 M \subset M \) is an \( R \)-direct summand such that \( \varphi(\text{Fil}^1 M) \subset p M \);
• $M[\frac{1}{p}] \cong D$ as filtered $(\varphi, \nabla)$-modules over $R[\frac{1}{p}]$.

Let $G$ be a $p$-divisible group over $\text{Spec} R$, and denote by $\mathbb{D}(G)$ the associated Dieudonné crystal. Let $D(G) := (\mathbb{D}(G)(R)[\frac{1}{p}], \text{Fil}^1 \mathbb{D}(G)(R)[\frac{1}{p}])$. Then, $D(G)$ with the induced Frobenius and connection is a filtered $(\varphi, \nabla)$-module over $R[\frac{1}{p}]$ with Hodge-Tate weights in $[0, 1]$, and by [Kim15, Corollary 5.4.2], $D(G)$ is admissible. Furthermore, from the constructions of the functors in Section 3.2, we have $D(G) \cong M^*(G)[\frac{1}{p}]$ as filtered $(\varphi, \nabla)$-modules. Thus, $M^*(G)$ is a strongly divisible lattice of $D(G)$.

**Theorem 3.9.** Let $D$ be a punctually weakly admissible filtered $(\varphi, \nabla)$-module over $R[\frac{1}{p}]$ whose Hodge-Tate weights lie in $[0, 1]$. Then $D$ arises from a $p$-divisible group over $\text{Spec} R$ if and only if $D$ admits a strongly divisible lattice.

**Proof.** For each maximal ideal $m$ of $R$, denote by $\kappa_m$ the residue field at $m$. Note that there exists a unique $\varphi$-compatible map $b_m : R \to W(\kappa_m)$, and it factors through $b_m : R \to R_m \to W(\kappa_m)$ $\varphi$-compatibly.

Let $D$ be a punctually weakly admissible filtered $(\varphi, \nabla)$-module over $R[\frac{1}{p}]$ whose Hodge-Tate weights lie in $[0, 1]$, and let $h = t_H(D)$. Suppose $D$ admits a strongly divisible lattice $(M, \text{Fil}^1 M)$. For each $m \in m\text{Spec} R$, we have $h = t_N(D, m)$. Thus, since $R_m$ is a regular local ring and therefore factorial, the determinant of the matrix associated with the map of finite free $R_m$-modules

$$1 \otimes \varphi : \varphi^*(M \otimes_R R_m) \to M \otimes_R R_m$$

is $p^h u$ for some unit $u$ of $R_m$. This implies that the submodule $\varphi(M) + \frac{\varphi}{p}(\text{Fil}^1 M)$ generates $M \otimes_R R_m$ as $R_m$-modules. Hence,

$$\varphi(M) + \frac{\varphi}{p}(\text{Fil}^1 M) = M$$

as $R$-modules. $(M, \text{Fil}^1 M)$ with the induced Frobenius and connection is therefore an object of $\text{Fl}^{[0,1]}_{\text{proj}}(R)$. By Proposition 3.8, $D$ arises from a $p$-divisible group over $\text{Spec} R$. \hfill $\square$

**4 Dimension 2 Case**

**4.1 Horizontal Crystalline Representations**

We suppose that the Krull dimension of $R$ is 2 and study some properties of admissible modules associated to horizontal crystalline representations of $G_R$.

**Lemma 4.1.** $\mathcal{R}$ is flat over $R$. 

16
Proof. Let $R' \subset \overline{R}$ be a finite normal $R$-algebra such that $R'[\frac{1}{p}]/R[\frac{1}{p}]$ is étale. Let $\mathfrak{m}$ be a maximal ideal of $R$. $R_{\mathfrak{m}}$ is a regular local ring, so it is factorial. Note that $R'_{\mathfrak{m}} := R' \otimes_R R_{\mathfrak{m}}$ is a normal domain which is a finite integral extension of $R_{\mathfrak{m}}$.

Let $X \in R_{\mathfrak{m}}$ be a prime element such that $(p, X) = \mathfrak{m}R_{\mathfrak{m}}$. Let $y \in R'_{\mathfrak{m}}[\frac{1}{p}] \cap R'_{\mathfrak{m}}[\frac{1}{X}]$. If $f(Y) \in (R_{\mathfrak{m}}[\frac{1}{p}])[Y]$ is a monic polynomial such that $f(y) = 0$ which has the minimal degree, then $f(Y)$ is also the minimal monic polynomial of $y$ over Frac($R_{\mathfrak{m}}$), since $R_{\mathfrak{m}}$ is factorial. Similarly, if $g(Y)$ is a monic polynomial of $y$ over $R_{\mathfrak{m}}[\frac{1}{X}]$ with the minimal degree, then it is the minimal monic polynomial of $y$ over Frac($R_{\mathfrak{m}}$). Thus, $f(Y) = g(Y)$, and $f(Y) \in R_{\mathfrak{m}}[Y]$. This implies $y \in R'_{\mathfrak{m}}$ and $R'_{\mathfrak{m}}[\frac{1}{p}] \cap R'_{\mathfrak{m}}[\frac{1}{X}] = R'_{\mathfrak{m}}$.

Since $R_{\mathfrak{m}}$ has depth 2, $R'_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$ by the Auslander-Buchbaum formula, and hence $R'$ is flat over $R$. $\overline{R}$ is a filtered colimit of such $R'$'s, so $\overline{R}$ is flat over $R$. \qed

**Corollary 4.2.** $A_{\text{cris}}(R)$ and $\text{Fil}^1 A_{\text{cris}}(R)$ are flat over $R$.

**Proof.** By Proposition 2.3 and Lemma 4.1, $A_{\text{cris}}(R)/pA_{\text{cris}}(R)$ is flat over $R/pR$. Since $A_{\text{cris}}(R)$ is $p$-torsion free, $A_{\text{cris}}(R)/p^n A_{\text{cris}}(R)$ is flat over $R/p^n R$ for any positive integer $n$. Let $\mathfrak{a} \subset R$ be an ideal. To show $A_{\text{cris}}(R)$ is flat over $R$, it suffices to check that $A_{\text{cris}} \otimes_R \mathfrak{a}$ is separated for the $p$-adic topology.

There exists a finite free $R$-module $L$ and a short exact sequence of $R$-modules

$$0 \to N \to L \to \mathfrak{a} \to 0$$

where $N$ is finite over $R$. Since $\mathfrak{a}$ is $p$-torsion free, we have a short exact sequence

$$0 \to N/p^n \to L/p^n \to \mathfrak{a}/p^n \to 0$$

for any positive integer $n$, which induces the short exact sequence

$$0 \to A_{\text{cris}}(R)/p^n \otimes_R N \to A_{\text{cris}}(R)/p^n \otimes_R L \to A_{\text{cris}}(R)/p^n \otimes_R \mathfrak{a} \to 0.$$

Thus, we have the commutative diagram

$$
\begin{array}{cccccc}
A_{\text{cris}}(R) \otimes_R N & \to & A_{\text{cris}}(R) \otimes_R L & \to & A_{\text{cris}}(R) \otimes_R \mathfrak{a} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\lim_{\leftarrow n} (A_{\text{cris}}(R)/p^n \otimes_R N) & \to & \lim_{\leftarrow n} (A_{\text{cris}}(R)/p^n \otimes_R L) & \to & \lim_{\leftarrow n} (A_{\text{cris}}(R)/p^n \otimes_R \mathfrak{a}) & \to & 0
\end{array}
$$

whose rows are exact, and the vertical maps are surjective by Proposition 2.3. The middle vertical map is an isomorphism since $L$ is finite free over $R$. By the snake lemma, the right vertical map is also an isomorphism, and $A_{\text{cris}} \otimes_R \mathfrak{a}$ is separated for the $p$-adic topology.

The map $\theta_R : R \otimes_{W(k)} W(\overline{R}) \to \overline{R}$ induces a short exact sequence of $R$-modules

$$0 \to \text{Fil}^1 A_{\text{cris}}(R) \to A_{\text{cris}}(R) \to \overline{R} \to 0.$$
Since $R$ is Noetherian, $p$-adically complete, and $\hat{R}$ is $p$-torsion free, $\hat{R}$ is flat over $R$. Thus, $\text{Fil}^1 A_{\text{cris}}(R)$ is flat over $R$. \hfill \Box

Consider the base change map $b_g : R \to R_g \cong W(k_g)$ as above, and choose a lifting $b_g : \hat{R} \to \hat{R_g}$. We will also denote by $b_g$ for maps of rings induced by the choice of lifting. In particular, note that we have an embedding $b_g : A^\n_{\text{cris}}(R) \hookrightarrow A^\n_{\text{cris}}(R_g)$. Let $\pi_e = [\epsilon] - 1 \in W(\hat{R})$, and let $\pi_e^* \subset \hat{R}$ be the image of $\pi_e$ under $W(\hat{R}) \to W(\hat{R})/pW(\hat{R}) \cong \hat{R}/p\hat{R}$.

**Lemma 4.3.** The map $b_g : W(\hat{R})/\pi_e W(\hat{R}) \to W(\hat{R}_g)/\pi_e W(\hat{R}_g)$ is injective. Furthermore, the induced map $b_g : \hat{R}/\pi_e \to \hat{R}_g/\pi_e$ by reducing mod $p$ is injective.

**Proof.** Under the multiplicative isomorphism $\lim_{\xrightarrow{x \to x_p^p}} \hat{R} \cong \hat{R}_p$, $\pi_e$ corresponds to $((\epsilon - 1)_0, (\epsilon - 1)_1, \ldots)$ where $(\epsilon - 1)_0 = \lim_{\xrightarrow{n \to \infty}} (\epsilon_n + (-1)^n)\pi^n \in \hat{R}$. Note that $(\epsilon - 1)_0 = p^n u$ where $\frac{p^n}{p - 1} \in \hat{R}$ is a $(p - 1)$-th root of $p^n$ and $u \in W(k) \subset \hat{R}$ is a unit.

Since $\hat{R}_g$ is $\pi_e$-torsion free and $W(\hat{R})$ is $p$-adically complete, it suffices to show that the map $b_g : \hat{R}/\pi_e \to \hat{R}_g/\pi_e$ is injective, or equivalently, the map $b_g : \hat{R}/(\epsilon - 1)_0 \to \hat{R}_g/(\epsilon - 1)_0$ is injective. Equip $\hat{R}_g \cong W(k_g)$ with the $p$-adic valuation $v_p(\cdot)$ normalized by $v_p(p) = 1$. Let $x \in \hat{R}$ such that $\frac{x}{p^n} \in W(k_g)$, and let $R' \subset \hat{R}$ be a finite normal $R$-algebra containing $x, \frac{p^n}{p - 1}$, and $u$ such that $R'[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Let $m \subset R$ be a maximal ideal. Note that $b_g : R' \hookrightarrow W(k_g)$ factors through $R'_m$. Since $v_p(x) \geq \frac{p^n}{p - 1}$ and $R_m$ is factorial, we have $\frac{x}{p^n} \in R'_m$. Thus,

$$\frac{x}{p^n} \in \bigcap_{m \subset R \text{ maximal}} (R' \otimes_R R_m) = R' \otimes_R \bigcap_{m \subset R} R_m = R'$$

as $R'$ is flat over $R$ by the proof of Lemma 4.1 and $R$ is an integral domain. This proves the statement. \hfill \Box

For non-negative integers $n$, let $q(n) = \lfloor \frac{n}{p - 1} \rfloor$ and $\pi_e^{(n)} = \frac{\pi_e^q}{q(n)! \cdot p^{n(n)}}$. Let $A_0$ be the $\mathbb{Z}_p$-algebra given by

$$A_0 = \left\{ \sum_{n=0}^\infty a_n \pi_e^{(n)} \mid a_n \in \mathbb{Z}_p, \ a_n \xrightarrow{n \to \infty} 0 \text{ $p$-adically} \right\}.$$ 

By [Bri08, Lemma 6.2.12 Proof], we have $t \in \pi_e A_0$ and $\pi_e \in tA_0$. Furthermore, we have inclusions $\mathbb{Z}_p[\pi_e] \subset W(\hat{R})$ and $A_0 \subset A^\n_{\text{cris}}(R)$, and the natural morphism of $W(k)$-algebras $W(\hat{R}) \otimes_{\mathbb{Z}_p[\pi_e]} A_0 \to A^\n_{\text{cris}}(R)$.

18
is an isomorphism by \cite{Bri08} Proposition 6.2.13. Here, the completed tensor product is taken with respect to the $p$-adic topology.

**Lemma 4.4.** We have

\[
A^\nabla_{\text{cris}}(R)[\frac{1}{t}] \cap A_{\text{cris}}(R_g) = A^\nabla_{\text{cris}}(R)
\]

as subrings of $A_{\text{cris}}(R_g)[\frac{1}{t}]$.

**Proof.** Let $x \in A^\nabla_{\text{cris}}(R)$ such that $x \in t \cdot A_{\text{cris}}(R_g)$. We need to show that $x \in t \cdot A^\nabla_{\text{cris}}(R)$, or equivalently, that $x \in \pi_{t} \cdot A^\nabla_{\text{cris}}(R)$.

We can write $x = \sum_{m=0}^{\infty} x_m \pi_t^{(m)}$ for some $x_m \in W(\overline{R}^g)$ such that $x_m \rightarrow 0$ as $m \rightarrow \infty$ in the $p$-adic topology. Note that for $n \geq 0$, $\varphi^n(\pi_t) = [\bar{\epsilon}]^p - 1 \in \pi_t \cdot W(\overline{R}^g)$. Since $x \in \pi_t \cdot A_{\text{cris}}(R_g)$ and $\theta(\pi_t) = 0$, we have $\theta(\varphi^n(x_0)) = 0 \forall n \geq 0$. Thus, $x_0 \in \pi_t \cdot W(\overline{R}^g)$ by \cite{Bri08} Lemma 6.2.14. We can therefore write $x = \sum_{m=1}^{\infty} x_m \pi_t^{(m-1)}$ with $x_m \in W(\overline{R}^g)$ such that either $x_m \notin \pi_t \cdot W(\overline{R}^g)$ or $x_m = 0$ for each $m \geq 1$, and $x_m \xrightarrow{m \rightarrow \infty} 0$.

Then $x = \pi_t y$ with $y = \sum_{m=1}^{\infty} \frac{p^q(m-1)!}{m^q(m)!} x_m \pi_t^{(m-1)}$. We have $y \in A_{\text{cris}}(R_g)$. By Lemma 4.3, either $x_m \notin \pi_t \cdot W(\overline{R}^g)$ or $x_m = 0$, and if $x_m = pa_m + \pi_t b_m$ for some $a_m, b_m \in W(\overline{R}_g)$, then $x_m = pa'_m + \pi_t b'_m$ for some $a'_m, b'_m \in W(\overline{R}^g)$. Hence, since $y \in A_{\text{cris}}(R_g) \cong W(\overline{R}_g) \otimes_{\mathbb{Z}_p} \Lambda_0$, we can inductively choose $x'_m \in W(\overline{R}^g)$ such that $y = \sum_{m=1}^{\infty} x'_m \pi_t^{(m-1)}$.

Note that for any $a \in W(\overline{R}^g)$, we have $a \in pW(\overline{R}^g)$ if and only if $a \in pW(\overline{R}_g)$. Since $y \in A_{\text{cris}}(R_g)$, we therefore have $x'_m \rightarrow 0$ as $m \rightarrow \infty$ in the $p$-adic topology. This implies $y \in A^\nabla_{\text{cris}}(R)$, and $x \in \pi_t \cdot A^\nabla_{\text{cris}}(R)$. \hfill \Box

Now, let $V$ be a horizontal crystalline $G_R$-representation over $\mathbb{Q}_p$, whose Hodge-Tate weights lie in $[0, 1]$. Let $T \subset V$ be a $G_R$-stable $\mathbb{Z}_p$-lattice. Via $b_g : G_{R_g} \rightarrow G_R$, $V$ is also a crystalline $G_{R_g}$-representation with Hodge-Tate weights in $[0, 1]$, and $D_{\text{cris}}(V) \otimes_R R_g \cong D_{\text{cris},R_g}(V)$. By \cite{Liu15} Proposition 4.1.2, $D_{\text{cris},R_g}(V) = (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R_g))^{1/p}\mathbb{G}_R[1/p]$. Since $D_{\text{cris}}(V) = (T \otimes_{\mathbb{Z}_p} B^\nabla_{\text{cris}}(R))^{1/p}\mathbb{G}_R \subset D_{\text{cris},R_g}(V)$, we have $D_{\text{cris}}(V) = (T \otimes_{\mathbb{Z}_p} A^\nabla_{\text{cris}}(R))^{1/p}\mathbb{G}_R[1/p]$ by Lemma 4.4. Hence, $D_{\text{cris}}(V) = (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^{1/p}\mathbb{G}_R[1/p]$ since $D_{\text{cris}}(V) = D_{\text{cris}}(V) \otimes_{W(k)} R$ and $R \otimes_{W(k)} A^\nabla_{\text{cris}}(R) \subset A_{\text{cris}}(R)$. This implies moreover that $(T \otimes_{\mathbb{Z}_p} \text{Fil}^1 A_{\text{cris}}(R))^{1/p}\mathbb{G}_R[1/p] = \text{Fil}^1 D_{\text{cris}}(V)$. For any finite $R[1/p]$-module $D$, we say that a finite $R$-submodule $M \subset D$ is an $R$-lattice of $D$ if $M$ is flat over $R$ and $M \otimes_R R[1/p] = D$. 

19
Proposition 4.5. \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \) and \((T \otimes_{\mathbb{Z}_p} \text{Fil}^1 A_{\text{cris}}(R))^\otimes_{R} \) are finite flat over \(R\). Thus, \(D_{\text{cris}}(V)\) admits an \(R\)-lattice stable under the Frobenius and connection, and \(\text{Fil}^1 D_{\text{cris}}(V)\) admits an \(R\)-lattice. Furthermore, \(D_{\text{cris}}(V)/\text{Fil}^1 D_{\text{cris}}(V)\) also admits an \(R\)-lattice.

Proof. We have an injective map \(b_{\varphi} : (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \hookrightarrow (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \). By Proposition 4.1.2, \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \) is in particular a finite free \(W(k_g)\)-module. Let \(M \subset D_{\text{cris}}(V)\) be a finite \(R\)-submodule such that \(M \otimes_{R} R[\frac{1}{p}] = D_{\text{cris}}(V)\). Then \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \subset \frac{1}{p^n} M \otimes_{R} R_g\) for some integer \(n\). Thus, \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R}\) is contained inside \((\frac{1}{p^n} M \otimes_{R} R_g) \cap (\frac{1}{p^n} M \otimes_{R} R[\frac{1}{p}])\), and hence contained inside the image of \(\frac{1}{p^n} M \otimes_{R} R_g \cap R[\frac{1}{p}] \rightarrow \frac{1}{p^n} M \otimes_{R} R_g[\frac{1}{p}]\). By Lemma 3.5, we get \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \subset \frac{1}{p^n} M\). Thus, \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \) is a finite \(R\)-module since \(R\) is Noetherian.

Let \(m\) be a maximal ideal of \(R\). Extend the \(G_R\)-action to \(A_{\text{cris}}(R) \otimes_{R} R_m\) in the \(R_m\)-linear way. Then \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R) \otimes_{R} R_m))^\otimes_{R} = (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m\). Let \(X \in R_m\) be a prime element such that \((p,X) = m R_m\). \(A_{\text{cris}}(R)\) is flat over \(R\) by Corollary 4.2, so \((A_{\text{cris}}(R) \otimes_{R} R_m[\frac{1}{p}]) \cap (A_{\text{cris}}(R) \otimes_{R} R_m[\frac{1}{X}]) = A_{\text{cris}}(R) \otimes_{R} R_m\). Thus,

\[
(T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m[\frac{1}{p}] \bigcap (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m[\frac{1}{X}] =
\]

\[
(T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m[\frac{1}{p}] \bigcap (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m[\frac{1}{X}] =
\]

\[
(T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m = (T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m.
\]

By the Auslander-Buchbaum formula, \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R} \otimes_{R} R_m\) is flat over \(R_m\). Hence, \((T \otimes_{\mathbb{Z}_p} A_{\text{cris}}(R))^\otimes_{R}\) is a finite flat \(R\)-module, and is an \(R\)-lattice of \(D_{\text{cris}}(V)\) stable under the Frobenius and connection.

Since \(\text{Fil}^1 A_{\text{cris}}(R)\) is flat over \(R\) by Corollary 4.2, a similar argument shows that \((T \otimes_{\mathbb{Z}_p} \text{Fil}^1 A_{\text{cris}}(R))^\otimes_{R}\) is finite flat over \(R\), and is therefore an \(R\)-lattice of \(\text{Fil}^1 D_{\text{cris}}(V)\). The \(G_R\)-representation \(V^\vee \otimes_{Q_p} \chi^{-1}\) is horizontal crystalline with Hodge-Tate weights in \([0,1]\), and \(D_{\text{cris}}(V)/\text{Fil}^1 D_{\text{cris}}(V) \cong \text{Fil}^1 D_{\text{cris}}(V^\vee \otimes_{Q_p} \chi^{-1})\) as \(R[\frac{1}{p}]\)-modules. Here, \(V^\vee\) is the dual representation of \(V\) and \(\chi\) is the cyclotomic character. Thus, by applying the above result to \(V^\vee \otimes_{Q_p} \chi^{-1}\), we see that \(D_{\text{cris}}(V)/\text{Fil}^1 D_{\text{cris}}(V)\) admits an \(R\)-lattice.

4.2 Horizontal Crystalline Representations of Rank 2 when \(R = W(k)[X]\) and \(k = \overline{k}\)

We consider the special case when \(R = W(k)[X]\) with the residue field \(k\) being algebraically closed, and study admissible modules associated to horizontal crystalline representations of rank 2 with Hodge-Tate weights in \([0,1]\). Equip \(R\) with the Frobenius given by \(X \mapsto X^p\). Note that \(R\) is isomorphic to the completion of \(W(k)[Y^{\pm 1}]\) with respect to the ideal \((p,Y-1)\) via \(Y \mapsto X + 1\). Thus, by Proposition 2.1, if we let \(u = (X + 1) \otimes 1 - 1 \otimes [\overline{X} + 1] \in\)
$R \otimes_{W(k)} W(R)\), then
\[ B_{\text{dr}}^+ (R)[u] = B_{\text{dr}}^+ (R), \quad \text{Fil}^0 B_{\text{dr}}(R) = B_{\text{dr}}^+ (R)[\frac{u}{t}]. \]

Let $V$ be a horizontal crystalline $G_R$-representation of rank 2 whose Hodge-Tate weights lie in $[0, 1]$. $D_{\text{crys}}^\nabla (V)$ is an isocrystal over $W(k)[\frac{1}{p}]$, and we have a $\varphi$-equivariant isomorphism $D_{\text{crys}}^\nabla (V) \otimes_{W(k)} R \cong D_{\text{crys}}(V)$. We say $D_{\text{crys}}(V)$ is étale (resp. multiplicative) if $\text{Fil}^1 D_{\text{crys}}(V) = D_{\text{crys}}(V)$ (resp. $	ext{Fil}^1 D_{\text{crys}}(V) = 0$). If $D_{\text{crys}}(V)$ is étale (resp. multiplicative), then it is induced as a filtered $(\varphi, \nabla)$-module over $R[\frac{1}{p}]$ from an étale (resp. a multiplicative) filtered $\varphi$-module $D_{\text{crys}}^\nabla (V)$. In particular, $V$ arises from a $p$-divisible group over $R$ in both étale and multiplicative cases.

Now, assume $\text{Fil}^1 D_{\text{crys}}(V)$ has rank 1 over $R[\frac{1}{p}]$, and denote $D = D_{\text{crys}}(V), \ D^1 = \text{Fil}^1 D_{\text{crys}}(V)$. Then $t_H(D) = 1$, and both $D^1$ and $D/D^1$ are free over $R[\frac{1}{p}]$ by Proposition 4.5. First consider the case when $D_{\text{crys}}^\nabla (V)$ is reducible as an isocrystal over $W(k)[\frac{1}{p}]$. Since $k = \overline{k}$, we can apply the Dieudonné-Manin classification. Note that $D$ is punctually weakly admissible and thus the slopes of all isoclinic subobjects of $D_{\text{crys}}^\nabla (V)$ are non-negative, since each isoclinic subobject of $D_{\text{crys}}^\nabla (V)$ induces a subobject of $D$. Hence, we can choose a $W(k)[\frac{1}{p}]$-basis $(e_1, e_2)$ of $D_{\text{crys}}^\nabla (V)$ such that
\[
\varphi(e_1) = pe_1, \\
\varphi(e_2) = e_2.
\]

Then
\[
W_e(D) = (D \otimes_R B_{\text{crys}}(R))^\nabla = (D \otimes_R B_{\text{crys}}(R))^\varphi = 1 = \frac{1}{t} e_1 \cdot B_e \oplus e_2 \cdot B_e.
\]

On the other hand, since $D^1$ and $D/D^1$ are free of rank 1 over $R[\frac{1}{p}]$, we have
\[
D^1 = (f(X)e_1 + g(X)e_2) \cdot R[\frac{1}{p}]
\]
for some $f(X), g(X) \in W(k)[X]$ such that either $p \nmid f(X)$ or $p \nmid g(X)$ in $W(k)[X]$ and that there exist $h(X), r(X) \in W(k)[X]$ with $(f(X)r(X) - g(X)h(X))$ being a unit in $R[\frac{1}{p}]$. Then,
\[
D \cong (f(X)e_1 + g(X)e_2) \cdot R[\frac{1}{p}] \oplus (h(X)e_1 + r(X)e_2) \cdot R[\frac{1}{p}]
\]
as $R[\frac{1}{p}]$-modules, and
\[
\text{Fil}^0 (D \otimes_R B_{\text{dr}}(R)) = \frac{f(X)e_1 + g(X)e_2}{t} \cdot \text{Fil}^0 B_{\text{dr}}(R) \oplus (h(X)e_1 + r(X)e_2) \cdot \text{Fil}^0 B_{\text{dr}}(R).
\]
Denote \( c = \lfloor X + 1 \rfloor - 1 \), so that \( X = u + c \). Write \( f(X) = f(u + c) = f(c) + uf_1(u) \) with \( f(c) \in B_{\text{dr}}^{\nabla+}(R) \) and \( f_1(u) \in B_{\text{dr}}^+(R) = B_{\text{dr}}^{\nabla+}[u] \), and similarly for \( g(X), h(X), r(X) \). For any \( a(u), b(u) \in B_{\text{dr}}^+(R) \), we have

\[
\frac{f(X)e_1 + g(X)e_2}{t} (1 + ua(u)) + (h(X)e_1 + r(X)e_2)\frac{u}{t}b(u) = \frac{f(c)e_1 + g(c)e_2}{t} + \frac{u}{t}((f(X)a(u) + h(X)b(u) + f_1(u))e_1 + (g(X)a(u) + r(X)b(u) + g_1(u))e_2).
\]

The system of equations

\[
\begin{align*}
  f(X)a(u) + h(X)b(u) &= -f_1(u), \\
  g(X)a(u) + r(X)b(u) &= -g_1(u)
\end{align*}
\]

has a unique solution \( a(u), b(u) \in B_{\text{dr}}^+(R) \), since \( f(X)r(X) - g(X)h(X) \) is a unit in \( R_{[1]} \). Thus,

\[
\frac{f(c)e_1 + g(c)e_2}{t} \in W_{\text{dr}}^{\nabla+}(D) = \text{Fil}^0(D \otimes_R B_{\text{dr}}(R))^{\nabla = 0}
\]

Similarly, for any \( a(u), b(u) \in B_{\text{dr}}^+(R) \),

\[
\frac{f(X)e_1 + g(X)e_2}{t} \cdot tua(u) + (h(X)e_1 + r(X)e_2)(1 + ub(u)) = h(c)e_1 + r(c)e_2 + u((f(X)a(u) + h(X)b(u) + h_1(u))e_1 + (g(X)a(u) + r(X)b(u) + r_1(u))e_2),
\]

and the system

\[
\begin{align*}
  f(X)a(u) + h(X)b(u) &= -h_1(u), \\
  g(X)a(u) + r(X)b(u) &= -r_1(u)
\end{align*}
\]

has a unique solution \( a(u), b(u) \in B_{\text{dr}}^+(R) \). Thus, \( h(c)e_1 + r(c)e_2 \in W_{\text{dr}}^{\nabla+}(D) \). We then have

\[
W_{\text{dr}}^{\nabla+}(D) = \frac{f(c)e_1 + g(c)e_2}{t} \cdot B_{\text{dr}}^{\nabla+}(R) \oplus (h(c)e_1 + r(c)e_2) \cdot B_{\text{dr}}^{\nabla+}(R).
\]

Note that \( W_e(D) \otimes_{B_e(R)} B_{\text{dr}}^{\nabla+}(R) = W_{\text{dr}}^{\nabla+}(D) \otimes_{B_{\text{dr}}^{\nabla+}(R)} B_{\text{dr}}^{\nabla+}(R) = e_1 \cdot B_{\text{dr}}^{\nabla+}(R) \oplus e_2 \cdot B_{\text{dr}}^{\nabla+}(R) \), since \( f(c)r(c) - g(c)h(c) \) is a unit in \( B_{\text{dr}}^{\nabla+}(R) \). In particular, \( (W_e(D), W_{\text{dr}}^{\nabla+}(D)) \) is a B-pair.

The intersection \( W_e(D) \cap W_{\text{dr}}^{\nabla+}(D) \) is given by the set of solutions \( (x, y, s, z) \) with \( x, y \in B_e(R), s, z \in B_{\text{dr}}^{\nabla+}(R) \) satisfying

\[
\frac{x}{t} = \frac{f(c)}{t}s + h(c)z, \\
\frac{y}{t} = \frac{g(c)}{t}s + r(c)z.
\]
Then \( x = f(c)s + th(c)z \in B_e(R) \cap B_{\text{dr}}(R) = Q_p \) by Proposition 2.2, and \( y = \frac{y_1}{t} \) with \( y_1 \in U_1 \).

Suppose \( f(X) \) is not a unit in \( R[\frac{1}{p^k}] \). Since \( \theta(c) = X \), by applying \( \theta \) to the equations

\[
\begin{align*}
x &= f(c)s + th(c)z, \\
y_1 &= g(c)s + tr(c)z,
\end{align*}
\]

we obtain

\[
\begin{align*}
x &= f(X)\theta(s), \\
\theta(y_1) &= g(X)\theta(s).
\end{align*}
\]

Since \( x \in Q_p \) and \( \theta(s) \in \hat{R}[\frac{1}{p}] \), we have \( x = 0 = \theta(s) \). Then \( \theta(y_1) = 0 \), and by Proposition 2.2, the \( Q_p \)-module \( W_e(D) \cap W_{\text{dr}}(D) \) has rank 1. This contradicts to \( D \) being admissible.

Hence, \( f(X) \) is a unit in \( R[\frac{1}{p}] \), and we can write \( D^1 = (e_1 + g(X)e_2) \cdot R[\frac{1}{p}] \) for some \( g(X) \in R[\frac{1}{p}] \). Write \( g(X) = \frac{1}{p^n}g_0(X) \) such that \( n \) is a non-negative integer and \( g_0(X) \in W(k)[X] \). Then \( (M, \text{Fil}^1 M) \) given by \( M = e_1 \cdot R \oplus \frac{e_2}{p^{n+1}} \cdot R \) with \( \text{Fil}^1 M = (e_1 + g(X)e_2) \cdot R \)

is a strongly divisible lattice of \( D \). Furthermore, since \( t_N(D, (p, X)) = 1 = t_H(D) \), we see from the proof of Theorem 3.9 that \( (M, \text{Fil}^1 M) \in \text{FL}^{[0,1]}(R) \). Thus, such \( D \) arises from a \( p \)-divisible group over \( R \) by Proposition 3.8. In this case, we call the admissible modules \( D \) to be of Type I.

We can use the above computations to construct \( B \)-pairs which are induced from weakly admissible \( R[\frac{1}{p}] \)-modules but do not arise from \( Q_p \)-representations. For example, let \( D = e_1 \cdot R[\frac{1}{p}] \oplus e_2 \cdot R[\frac{1}{p}] \) equipped with the filtration \( \text{Fil}^0 D = D, \text{Fil}^1 D = ((X + p)e_1 + e_2) \cdot R[\frac{1}{p}] \)

and \( \text{Fil}^2 D = 0 \). Equip \( D \) with Frobenius endomorphism given by

\[
\begin{align*}
\varphi(e_1) &= pe_1, \\
\varphi(e_2) &= e_2.
\end{align*}
\]

Equip \( D \) with the connection given by \( \nabla(e_1) = \nabla(e_2) = 0 \). We have \( t_H(D) = 1 \) and \( t_N(D, p) = 1 \) for any \( p \in \text{Spec} R/pR \). To check \( D \) is weakly admissible, it suffices to show that the induced filtered-\( \varphi \)-modules \( D_0 := D \otimes_{R,b_0} W(k) \) and \( D_g := D \otimes_{R,b_0} W(k_g) \) are weakly admissible, where \( b_0 : R = W(k)[X] \to W(k) \) is the projection given by \( X \mapsto 0 \). We have \( D_0 = \varphi_1 \cdot W(k)[\frac{1}{p}] \oplus \varphi_2 \cdot W(k)[\frac{1}{p}] \) with \( \text{Fil}^1 D_0 = (p\varphi_1 + \varphi_2) \cdot W(k)[\frac{1}{p}] \). It admits a strongly divisible \( W(k) \)-lattice \( M_0 = pe_1 \cdot W(k) \oplus \varphi_x \cdot W(k) \) with \( \text{Fil}^1 M_0 = (p\varphi_1 + \varphi_2) \cdot W(k) \), so \( D_0 \) is weakly admissible. On the other hand, \( D_g = e_1 \cdot W(k_g)[\frac{1}{p}] \oplus e_2 \cdot W(k_g)[\frac{1}{p}] \) with \( \text{Fil}^1 D_g = ((X + p)e_1 + e_2) \cdot W(k_g)[\frac{1}{p}] \). It admits a strongly divisible \( W(k_g) \)-lattice \( M_g = e_1 \cdot W(k_g) \oplus \varphi \cdot W(k_g) \) with \( \text{Fil}^1 M_g = ((X + p)e_1 + e_2) \cdot W(k_g) \), so \( D_g \) is weakly admissible. Hence, \( D \) is a weakly admissible \( R[\frac{1}{p}] \)-module. However, above computations show that
$(W_c(D), W_{\text{dR}}^\nabla(D))$ is a $B$-pair which does not arise from a $\mathbb{Q}_p$-representation, since $(X + p)$ is not a unit in $R[\frac{1}{p}]$. Thus, the relative case is different from the case when the base ring is a $p$-adic field where every $B$-pair semi-stable of slope 0 arises from a $\mathbb{Q}_p$-representation. In particular, this answers negatively the question raised in [Bri08, Section 8] whether weakly admissible implies admissible in the relative case.

We now consider the case when $D_{\text{cris}}^\nabla(V)$ is irreducible as an isocrystal over $W(k)[\frac{1}{p}]$. By the Dieudonné-Manin classification, we can choose a $W(k)[\frac{1}{p}]$-basis $(e_1, e_2)$ of $D_{\text{cris}}^\nabla(V)$ such that

$$\varphi(e_1) = pe_2, \quad \varphi(e_2) = e_1.$$ 

By Corollary 2.6 we have

$$W_c(D) = \left( \frac{\varphi(w)\varphi(t_2)}{pt} e_1 + \frac{wt_2}{t} e_2 \right) \cdot B_c(R) \oplus \left( \frac{w\varphi(t_2)}{pt} e_1 + \frac{\varphi(w)t_2}{t} e_2 \right) \cdot B_c(R).$$

As above, $D^1 = \langle f(X)e_1 + g(X)e_2 \rangle \cdot R[\frac{1}{p}]$ for some $f(X), g(X) \in W(k)[X]$ such that either $p \nmid f(X)$ or $p \nmid g(X)$ in $W(k)[X]$ and that there exist $h(X), r(X) \in W(k)[X]$ with $(f(X)r(X) - g(X)h(X))$ being a unit in $R[\frac{1}{p}]$. By changing the basis $(e_1, e_2)$ to $(e_2, \frac{1}{p}e_1)$ if necessary, we can further assume $g(X) = pg_0(X)$ for some $g_0(X) \in W(k)[X]$ and $p \nmid f(X)$.

Suppose that $f(X)$ is not a unit in $W(k)[X]$. We claim that $D$ is not admissible in this case. Suppose otherwise. Consider the base change $b_g : R \rightarrow W(k_g)$, and choose a lifting $b_g : \overline{R} \rightarrow \overline{W(k_g)}$. Then the natural map $V := W_c(D) \cap W_{\text{dR}}^\nabla(D) \rightarrow V_{\text{cris}, R_g}(D_g)$ is an isomorphism of $\mathbb{Q}_p$-modules, where $D_g = D \otimes_{R, b_g} W(k_g)$ is the induced weakly admissible $W(k_g)$-module.

The $\mathbb{Q}_p$-representation $V = W_c(D) \cap W_{\text{dR}}^\nabla(D)$ is given by the set of solutions $(x, y, s, z)$ with $x, y \in B_c(R)$, $s, z \in B_{\text{dR}}^\nabla$ satisfying

$$\frac{\varphi(w)\varphi(t_2)}{pt} x + \frac{w\varphi(t_2)}{pt} y = \frac{f(c)}{t} s + h(c)z, \quad \frac{wt_2}{t} x + \frac{\varphi(w)t_2}{t} y = \frac{g(c)}{t} s + r(c)z. \tag{4.1}$$

Note that we have an embedding $B_c(R) \hookrightarrow B_c(W(k_g)) = (B_{\text{cris}}(W(k_g)))^{\varphi=1}$ induced by $b_g$, and

$$V_{\text{cris}, R_g}(D_g) = \left\{ \left( \frac{\varphi(w)\varphi(t_2)}{pt} e_1 + \frac{wt_2}{t} e_2 \right) x + \left( \frac{w\varphi(t_2)}{pt} e_1 + \frac{\varphi(w)t_2}{t} e_2 \right) y \mid x, y \in B_c(R) \text{ solutions of (4.1)} \right\}. \tag{4.2}$$
On the other hand, consider a strongly divisible \( W(k_g) \)-lattice of \( D_g \) given by

\[
M_g = e'_1 \cdot W(k_g) \oplus e'_2 \cdot W(k_g),
\]

\[
\text{Fil}^1 M_g = e'_1 \cdot W(k_g)
\]

where \( e'_1 = f(X)e_1 + pg_0(X)e_2, e'_2 = e_2 \). We have

\[
\frac{\varphi(e'_1)}{p} = \frac{\varphi(g_0(X))}{f(X)}e'_1 + \left(-p\frac{\varphi(g_0(X))g_0(X)}{f(X)} + \varphi(f(X))\right)e'_2,
\]

\[
\frac{\varphi(e'_2)}{p} = \frac{1}{f(X)}e'_1 - p\frac{g_0(X)}{f(X)}e'_2.
\]

We follow \cite{Fal88} to compute \( \text{Hom}_{W(k_g), \text{Fil}, \varphi}(M_g, A_{\text{cris}}(R_g)/pA_{\text{cris}}(R_g)) \). Let

\[
\text{pr}_2 : W(R_g)/pW(R_g) \cong \overline{R} \to \overline{R}/p\overline{R}
\]

be the projection onto the second component of \( \overline{R} = \lim_{\leftarrow \varphi} R/pR \). This extends uniquely to \( \text{pr}_2 : A_{\text{cris}}(R)/pA_{\text{cris}}(R) \to \overline{R}/p\overline{R} \), and \( b_g \) induces the commutative diagram

\[
\begin{array}{ccc}
A_{\text{cris}}(R)/pA_{\text{cris}}(R) & \xrightarrow{b_g} & A_{\text{cris}}(W(k_g))/pA_{\text{cris}}(W(k_g)) \\
\text{pr}_2 & & \text{pr}_2 \\
\overline{R}/p\overline{R} & \xrightarrow{b_g} & \overline{W(k_g)}/p\overline{W(k_g)}.
\end{array}
\]

Let \( E(R_g) = \overline{W(k_g)}/p\overline{W(k_g)} \) equipped with the \( W(k_g) \)-module structure given by the inverse of Frobenius on \( W(k_g) \) and decreasing filtration \( \text{Fil}^i E(R_g) = p^{\frac{i}{p}} E(R_g) \) for \( i = 0, \ldots, p-1 \) and \( \text{Fil}^p E(R_g) = 0 \). The Frobenius on \( E(R_g) \) is given by \( \varphi^i : \text{Fil}^i E(R_g) \to E(R_g) \), \( \varphi^i(x) = \frac{x^p}{(p)} \) for \( i = 0, \ldots, p-1 \). Then by \cite{Fal88} Theorem 2.4 Proof], the map \( \text{Hom}_{W(k_g), \text{Fil}, \varphi}(M_g, A_{\text{cris}}(R_g)/pA_{\text{cris}}(R_g)) \to \text{Hom}_{W(k_g), \text{Fil}, \varphi}(M_g, E(R_g)) \) induced by \( \text{pr}_2 \) is an isomorphism.

Note that \( f(X) \equiv uX^m \mod p \) in \( W(k)[X] \) for some unit \( u \) and integer \( m \geq 1 \). By \cite{Fal88} Theorem 2.4 Proof], \( \text{Hom}_{W(k_g), \text{Fil}, \varphi}(M_g, E(R_g)) \) is given by the set of solutions \( (y_1, y_2) \) with \( y_1, y_2 \in \overline{W(k_g)} \) such that

\[
y_1^p = -p\frac{g_0(X)}{u^p X^m} y_1 - puX^m y_2,
\]

\[
y_2^p = \frac{1}{u^p X^m} y_1.
\]

Thus, if \( y_2 \neq 0 \), then

\[
y_2^{p-1} + p\frac{g_0(X)}{uX^m} y_2^{p-1} + p = 0.
\]
This is an Eisenstein polynomial over $W(k_g)$. Since $\frac{1}{X} \notin \widehat{R}[\frac{1}{p}]$, there exists a solution $(y_1, y_2)$ such that the image of $\frac{y_1}{uX^m} = \frac{1}{(uX^m)^{p-1}p}y_2^p$ in $W(k_g)/pW(k_g)$ does not lie in $\overline{R}/p\overline{R}$.

Since $e_1 = \frac{1}{f(X)}(e_1' - pg_0(X)e_2')$, this implies that the $\mathbb{Z}_p$-lattice $\text{Hom}_{W(k_g),\text{Fil},\varphi}(M, A_{\text{cris}}(R_g))$ of $V_{\text{cris},R_g}(D_g)^\vee$ is not contained in $V^\vee$. This contradicts to equation (4.2), and proves the claim that such $D$ is not admissible. Note in particular that such $D$ gives a $B$-pair which does not arise from a $\mathbb{Q}_p$-representation.

Hence, $f(X)$ is a unit in $R$. Then $M = e_1 \cdot R \oplus e_2 \cdot R$ with $\text{Fil}^1M = (f(X)e_1 + pg_0(X)e_2) \cdot R$ is a strongly divisible lattice of $D$, so $D$ arises from a $p$-divisible group over $R$. In this case, we say $D$ is of type II.

We summarize above results in the following theorem.

**Theorem 4.6.** Let $R = W(k)[[X]]$ and suppose $k$ is algebraically closed. Then every horizontal crystalline $G_R$-representation of rank 2 with Hodge-Tate weights in $[0, 1]$ arises from a $p$-divisible group over $\text{Spec}R$. If $D$ is an admissible $R[\frac{1}{p}]$-module of rank 2 with Hodge-Tate weights in $[0, 1]$ which is generated by its parallel elements, then $D$ is either étale, multiplicative, type I, or type II. Furthermore, there exists a $B$-pair which arises from a weakly admissible $R[\frac{1}{p}]$-module but does not arise from a $\mathbb{Q}_p$-representation.

**References**

[Ber08] Laurent Berger, *Construction de $(\varphi, \gamma)$-modules: représentations $p$-adiques et b-paires*, Algebra and Number Theory 2 (2008), no. 1, 91–120.

[Bri08] Olivier Brinon, *Représentations $p$-adiques cristallines et de de rham dans le cas relatif*, Mém. Soc. Math. Fr. 112 (2008).

[CF00] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations $p$-adiques semi-stables*, Inventiones Mathematicae 140 (2000), no. 1, 1–43.

[Col08] Pierre Colmez, *Espaces vectoriels de dimension finie et représentations de de rham*, Astérisque 319 (2008), 117–186.

[Fal88] Gerd Faltings, *Crystalline cohomology and $p$-adic galois representations*, Algebraic Analysis, Geometry, and Number Theory (Baltimore), vol. , The Johns Hopkins University Press, 1988, pp. 25–80.

[Gao17] Hui Gao, *Fontaine-laffaille modules and strongly divisible modules*, Annales Mathématiques du Québec (2017), 1–15.
[Kim15] Wansu Kim, *The relative breuil-kisin classification of p-divisible groups and finite flat group schemes*, International Mathematics Research Notices 2015 (2015), no. 17, 8152–8232.

[Kis06] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic Geometry and Number Theory (Boston), Progress in Mathematics, vol. 253, Birkhäuser, 2006, pp. 459–496.

[KL15] Kiran Kedlaya and Ruochuan Liu, *Relative p-adic hodge theory: Foundations*, Astérisque 371 (2015).

[Liu15] Tong Liu, *Compatibility of Kisin modules for different uniformizers*, Journal für die reine und angewandte Mathematik (2015).

[Sch13] Peter Scholze, *p-adic hodge theory for rigid-analytic varieties*, Forum of Mathematics, Pi 1 (2013), no. e1.