GROWTH OF GROUPS AND WREATH PRODUCTS

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To Wolfgang Woess, in fond remembrance of many a visit to Graz

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Introduction

These notes are an expanded version of a mini-course given at “Le Louverain”, June 24-27, 2014. Its main objective was to gather together useful facts about wreath products, and especially their geometry, in its application to problems and questions about growth of groups. The wreath product is a fundamental construction in group theory, and I hope to help make the reader more familiar with it.

It has proven very useful, in the recent years, in better understanding asymptotics of the word growth function on groups, namely the function assigning to $R \in \mathbb{N}$ the number of group elements that may be obtained by multiplying at most $R$ generators. The papers [4–8] may be hard to read, and contain many repetitions as well as references to outer literature; so that, by providing a unified treatment of these articles, I may provide the reader with easier access to the results and methods.

I have also attempted to define all notions in their most natural generality, while restricting the statements to the most important or fundamental cases. In this manner, I would like the underlying ideas to appear more clearly, with fewer details that obscure the line of sight. I avoided as much as possible reference to literature, taking the occasion of reproving some important results along the way.

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I have also allowed myself, exceptionally, to cheat. I do so only under three conditions: (1) I clearly mark where there is a cheat; (2) the complete result appears elsewhere for the curious reader; (3) the correct version would be long and uninformative.

I have attempted to make the text suitable for a short course. In doing so, I have included a few exercises, some of which are hopefully stimulating, and a section on open problems. What follows is a brief tour of the highlights of the text.

**Wreath products.** The wreath product construction, described in §3, is an essential operation, building a new group $W$ out of a group $H$ and a group $G$ acting on a set $X$. Assuming that $G$ is a group of permutations of $X$, the wreath product is the group $W \rtimes_{X} G$ of $H$-decorated permutations in $G$: if elements of $G$ are written in the arrow notation, with elements of $X$ lined in two identical rows above each other and an arrow from each $x \in X$ to its image, then an element of $H \rtimes_{X} G$ is an arrow diagram with an element of $H$ attached to each arrow, e.g.

\[
\begin{array}{ccc}
  h_1 & h_2 \\
  \downarrow & \downarrow
\end{array}
\quad
\begin{array}{ccc}
  h_3 & h_5 & h_4 \\
  \downarrow & \downarrow & \downarrow
\end{array}
\]

One of the early uses of wreath products is as a classifier for extensions, as discovered by Kaloujnine, see Theorem A.4: there is a bijective correspondence between group extensions with kernel $H$ and quotient $G$ on the one hand, and appropriate subgroups of the wreath product $H \rtimes G$, with $G$ seen as a permutation group acting on itself by multiplication. We extend this result to permutational wreath products:

**Theorem** (Theorem A.6). Let $G, H$ be groups, and let $G$ act on the right on a set $X$. Denote by $\pi : H \rtimes G \to G$ the natural projection. Then the map $E \mapsto E$ defines a bijection between

\[
\{ E : E \leq H \text{ and the } G \text{-sets } X \text{ and } H \setminus E \text{ are isomorphic } \} \\
\text{via a homomorphism } E \to G
\]

isomorphism $E \to E'$ of groups intertwining the actions on $H \setminus E$ and $H \setminus E'$

and

\[
\{ E \leq H \rtimes G : \pi(E) \text{ transitive on } X \text{ and } \ker(\pi) \cap E \cong H \text{ via } f \mapsto f(x) \text{ for all } x \in X \}.
\]

The wreath product $H \rtimes_{X} G$ is uncountable, if $H \neq 1$ and $X$ is infinite. It contains some important subgroups: $H \wr_{X} G$, defined as those decorated permutations in which all but finitely many labels are trivial; and $H \wr^{-}_{X} G$, defined as those decorated permutations in which the labels take finitely many different values. Clearly $H \wr_{X} G \leq H \wr^{-}_{X} G \leq H \rtimes_{X} G$, and $H \wr_{X} G$ is countable as soon as $H, G, X$ are countable.

\footnote{There is no need to require the action of $G$ on $X$ to be faithful; this is merely a visual aid. See §3 for the complete definition.}
Growth of groups. Let us summarise here the main notions; for more details, see [14]. A choice of generating set $S$ for a group $G$ gives rise to a graph, the Cayley graph: its vertex set is $G$, and there is an edge from $g$ to $gs$ for each $g \in G$, $s \in S$. The path metric on this graph defines a metric $d$ on $G$ called the word metric. The Cayley graph is invariant under left translation, and so is the word metric.

One of the most naive invariants of this graph is its growth, namely the function $v_{G,S}(R)$ measuring the cardinality of a ball of radius $R$ in the Cayley graph. If the graph exhibits some kind of regularity, then it should translate into some regularity of the function $v_{G,S}$.

For example, Klarner [47,48] studied the growth of crystals (that expand according to a precise and simple rule) via what turns out to be the growth of an abelian group. A convenient tool to study various forms of regularity of a function $v_{G,S}$ is the associated generating function $\Gamma_{G,S}(z) = \sum_{R \in \mathbb{N}} (v(R) - v(R - 1))z^R$. The regularity of $v_{G,S}$ translates then into a property of $\Gamma_{G,S}$ such as being a rational, algebraic, $D$-finite, ... function of $z$.

We may rewrite $\Gamma_{G,S}(z) = \sum_{g \in G} z^{d(1,g)}$; then a richer power series keeps track of more regularity of $G$:

\[
\hat{\Gamma}_{G,S}(z) = \sum_{g \in G} g z^{d(1,g)}.
\]

This is a power series with coefficients in the group ring $\mathbb{Z}G$, and again we may ask whether $\hat{\Gamma}_{G,S}$ is rational or algebraic\(^2\).

If $G$ has an abelian subgroup of finite index [52], or if $G$ is word-hyperbolic [33], then $\hat{\Gamma}_{G,S}$ is a rational function of $z$ for all choices of $S$. We give a sufficient condition for $\hat{\Gamma}_{G,S}$ to be algebraic:

**Theorem** (Theorem [C.2].) Let $H = \langle T \rangle$ be a group such that $\hat{\Gamma}_{H,T}$ is algebraic, and let $F$ be a free group. Consider $G = H \wr F$, generated by $S = T \cup \{a \text{ a basis of } F\}$. Then $\hat{\Gamma}_{G,S}$ is algebraic.

We then turn to studying the asymptotics of the growth function $v_{G,S}$. Let us write $v \preceq w$ to mean that $v(R) \leq w(CR)$ for some constant $C \in \mathbb{R}^+$ and all $R \geq 0$, and $v \sim w$ to mean $v \preceq w \preceq v$. Then the $\sim$-equivalence class of $v_{G,S}$ is independent of the choice of $S$, so we may simply talk about $v_G$.

For “most” examples of groups, either $v_G(R)$ is bounded by a polynomial in $R$ or $v_G(R)$ is exponential in $R$. This is, in particular, the case for soluble, linear and word-hyperbolic groups. There exist, however, examples of groups for which $v_G(R)$ admits an intermediate behaviour between polynomial and exponential; they are called groups of intermediate growth. The question of their existence was raised by Milnor [57], was answered positively by Grigorchuk [29], and has motivated much group theory in the second half of the 20th century.

\(^2\)The rational subring of $\mathbb{Z}[t]$ is the smallest subring of $\mathbb{Z}[t]$ containing $\mathbb{Z}[t]$ and closed under Kleene’s star operation $A^* = 1 + A + A^2 + \cdots$, for all $A(z)$ with $A(0) = 0$.

The algebraic subring of $\mathbb{Z}[t]$ is the set of power series that may be expressed as the solution $A_1$ of a non-trivial system of non-commutative polynomial equations \{ $P_i(A_1, \ldots, A_n) = 0, \ldots, P_n(A_1, \ldots, A_n) = 0$ \} with coefficients in $\mathbb{Z}[t]$. The solution is actually rational if furthermore the $P_i$ are of the form $c_{i,0} + \sum_{j=1}^m c_{i,j} A_j$ with $c_{i,j} \in \mathbb{Z}[t]$. 
Let \( \eta_+ \approx 2.46 \) be the positive root of \( T^3 - T^2 - 2T - 4 \), and set \( \alpha = \log 2 / \log \eta_+ \approx 0.76 \). We shall show that, for every sufficiently regular function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \exp(R^\alpha) < f < \exp(R) \), there exists a group with growth function equivalent to \( f \):

**Theorem** (Theorem [F.2]). Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be a function satisfying

\[
f(2R) \leq f(R)^2 \leq f(\eta_+ R) \text{ for all } R \text{ large enough.}
\]

Then there exists a group \( G \) such that \( \nu_G \sim f \).

Thus, groups of intermediate growth abound, and the space of asymptotic growth functions of groups is as rich as the space of functions. Furthermore, we shall show that there is essentially no restriction on the subgroup structure of groups of intermediate growth. Let us call a group \( H \) locally of subexponential growth if every finitely generated subgroup of \( H \) has growth function \( \leq \exp(R) \). Clearly, if \( H \) is a subgroup of a group of intermediate growth then it has locally subexponential growth. We show, conversely:

**Theorem** (Theorem [G.1]). Let \( B \) be a countable group locally of subexponential growth. Then there exists a finitely generated group of subexponential growth in which \( B \) imbeds as a subgroup.

Finally, it may happen that a group \( G \) has exponential growth, namely that the growth rate \( \lim_{R \to \infty} \nu_{G,S}(R)^{1/R} \) is \( > 1 \) for all \( S \), but that the infimum of these growth rates, over all \( S \), is \( 1 \). Such a group is called of non-uniform exponential growth. The question of their existence was raised by Gromov [35, Remarque 5.12]. Again, soluble, linear and word-hyperbolic groups cannot have non-uniform exponential growth; but, again, it turns out that such groups abound. We shall show:

**Theorem** (Theorem [H.2]). Every countable group may be imbedded in a group of non-uniform exponential growth.

Furthermore, the group \( W \) in which the countable group imbeds may be required to have the following property: there is a constant \( K \) such that, for all \( R > 0 \), there exists a generating set \( S \) of \( W \) with

\[
\nu_{W,S}(r) \leq \exp(Kr^\alpha) \text{ for all } r \in [0, R].
\]

(Self-)similar groups and branched groups. All the constructions mentioned in the previous subsection take place in the universe of (self-)similar groups. Here is a brief description of these groups; see [1] for details.

Just as a self-similar set, in geometry, is a set describable in terms of smaller copies of itself, a self-similar group is a group describable in terms of “smaller” copies of itself. A self-similar structure on a group \( G \) is a homomorphism \( \phi : G \to G \wr_X P \) for a permutation group \( P \) of \( X \). Thus elements of \( G \) may be recursively written in terms of \( G \)-decorated permutations of \( X \). For this description to be useful, of course, the homomorphism \( \phi \) must satisfy some non-degeneracy condition (in particular be injective), and \( P \) should be manageable, say finite.

The fact that the copies of \( G \) in \( G \wr_X P \) are “smaller” than the original is expressed as follows: there is a norm on \( G \) such that, for \( g \in G \) and \( \phi(g) \) a permutation with labels \( (g_x : x \in X) \), the elements \( g_x \) are shorter than \( g \), at least as soon as \( g \) is long enough. For example, consider \( G \) finitely generated, and denote by \( \| \cdot \| \) the word norm on \( G \). One requires \( \|g_x\| < \|g\| \) for all \( x \in X \) and all \( \|g\| > 1 \); this is equivalent to the existence of \( \lambda \in (0, 1) \) and \( K > 0 \) such that \( \|g_x\| \leq \lambda\|g\| + K \) for all \( x \in X, g \in G \).
Furthermore, in cases that interest us, the map $\phi$ is almost an isomorphism, in that its image $\phi(p)$ has finite index in $G \wr X P$. Thus $\phi$ may be thought of as a virtual isomorphism between $G$ and $G \wr X P$, namely an isomorphism between finite-index subgroups. When one endows $G \wr X P$ with the $\ell^\infty$ metric $\|g_x\| = \max_{x \in X} \|g_x\|$, the condition above requires that this virtual isomorphism be a contraction. On the other hand, endowing $G \wr X P$ with the $\ell^1$ metric $\|g_x\| = \sum_{x \in X} \|g_x\|$, the optimal Lipschitz constant of the virtual isomorphism plays a fundamental role in estimating the growth of $G$.

Similar groups are a natural generalization: one is given a set $\Omega$ and a self-map $\sigma: \Omega \to \Omega$; for each $\omega \in \Omega$, a group $G_\omega$ and a permutation group $P_\omega$ of a set $X_\omega$; and homomorphisms $\phi_\omega: G_\omega \to G_{\sigma\omega} \wr X_\omega P_\omega$. Taking for $\Omega$ a singleton recovers the notion of self-similar group. Taking $\Omega = \mathbb{N}$ and $\sigma(n) = n+1$ defines in full generality a similar group $G_0$; but it is often more convenient to consider a larger family of groups in which $(G_n)_{n \in \mathbb{N}}$ imbeds. In particular, one obtains a topological space of groups, in such a manner that close groups have close properties (for example, their Cayley graphs coincide on a large ball).

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Open problems. This text presents a snapshot of what is known on growth of groups in 2014; there remain a large number of open problems. Here are some promising directions for further research.

(1) Which groups $G$ are such that, for all generating sets $S$, the complete growth series $\Gamma_{G,S}$ is a rational function of $z$?

This is known to hold for virtually abelian groups, and for word-hyperbolic groups. Conjecturally, this holds for no other group.

The related question of which groups have a rational (classical) growth function is probably more complicated, see §B.1.

(2) Is the analytic continuation $1/\Gamma_{G,S}(1)$ related to the complete Euler characteristic of $G$, just as $1/\Gamma_{G,S}(1)$ is (under some additional conditions) the Euler characteristic of $G$? See [74, §1.8] for complete Euler characteristic.

(3) Does there exist a group $G$ with two generating sets $S_1, S_2$ such that $\hat{\Gamma}_{G,S_1}$ is rational but $\hat{\Gamma}_{G,S_2}$ is transcendental?

Such an example could be $G = F_2 \times F_2$. Set $S = \{x,y\}^+\{1\}$ a free generating set of $F_2$, and $S_1 = S \times \{1\} \cup \{1\} \times S$ and $S_2 = S_1 \cup \{(s,s) : s \in S\}$.

The same properties probably hold for the usual generating series $\Gamma_{G,S_1}$ and $\Gamma_{G,S_2}$. The radius of convergence of $\Gamma_{G,S_1}$ is $1/3$, but that of $\Gamma_{G,S_2}$ is unknown.

This problem is strongly related to the “Matching subsequence problem”, which asks for the longest length of a common subsequence among two independently and uniformly chosen words of length $n$ over a $k$-letter alphabet;
see [16]. It is easy to see that, for two uniformly random reduced words of length \( n \) in \( F_2 \), the longest common subword has length \( \approx \gamma n \) for some constant \( \gamma \), as \( n \to \infty \). Thus a pair \((g, h) \in G\) with \( \|g\| = \|h\| = n \) has length \( 2n \) with respect to \( S_1 \), but approximately \( (2 - \alpha)n \) with respect to \( S_2 \). We might call \( \gamma \) the Chv´atal-Sankoff constant of \( F_2 \).

(4) Do there exist infinite simple groups of subexponential growth?

There is no reason for such groups not to exist; but the construction methods described in this text yield groups acting on rooted trees, which therefore are as far as possible from being simple.

There is also no reason for finitely presented groups of subexponential growth not to exist; again, the obstacle is probably more our mathematical limitations than fundamental mathematical reasons.

The following question, by de la Harpe [38], is still open at the time of writing: “Do there exist groups with Kazhdan’s property (T) and non-uniform exponential growth?”

Similarly, it is not known whether there exist simple finitely generated groups of non-uniform exponential growth, and whether there exist finitely presented groups of non-uniform exponential growth.

(5) Do there exist groups whose growth function lies strictly between polynomials and \( \exp(R^{1/2}) \)?

See the discussion in §B.3. There exists a superpolynomial function \( f(R) \geq R^{(\log R)^{1/100}} \) such that no group has growth strictly between polynomials and \( f(R) \). There exists no residually nilpotent group whose growth is strictly between polynomials and \( \exp(R^{1/2}) \), see Theorem E.2.

(6) What is the asymptotic growth of the first Grigorchuk group? What is its exact growth, for the generating set \( \{a, b, c, d\} \)? Does the growth series of the Grigorchuk group exhibit some kind of regularity?

Some experiments indicate that this must be the case. For example, consider the quotient \( G_n \) of the first Grigorchuk group that acts on \( \{0, 1\}^n \). It is a finite group of cardinality \( 2^{3 \cdot 2^{n-3} + 2} \). For \( n \leq 7 \), the diameter \( D_n \) of its Cayley graph (for the natural generating set \( \{a, b, c, d\} \)) is the sequence 1, 4, 8, 24, 56, 136, 344 and satisfies the recurrence \( D_n = D_{n-1} + 2D_{n-2} + 4D_{n-3} \). If this pattern went on, the growth of the first Grigorchuk group would be asymptotically \( \exp(R^{(\log R)^{1/56}}) \approx \exp(R^{0.76}) \).

If a group has subexponential growth, then its growth series is either rational or transcendental, and if the group has intermediate growth, then the growth series must be transcendental; see [B.1] in [B.1]. Thus Grigorchuk group’s growth series is transcendental. Does the series satisfy a functional equation? That would make it akin to the classical partition function \( \sum_{n \geq 0} p(n)z^n = \prod_{n \geq 1}(1 - z^n)^{-1} \), which (up to scaling and multiplying by \( z^{1/24} \)) is a modular function.

Ghys asked me once: “Is the growth series \( \Gamma(z) \) of Grigorchuk’s group modular?”

(7) For every \( k \in \mathbb{N} \), the space of marked \( k \)-generated groups \( \mathcal{X}_k \) may be defined as the space of normal subgroups of the free group \( F_k \), by identifying \( G = \langle s_1, \ldots, s_k \rangle \) with the kernel of the natural map \( F_k \to G \) sending generator to

\[ A(z) = A(\exp(2\pi i\tau)) \]

on the upper half plane is invariant under a finite-index subgroup of \( \text{SL}_2(\mathbb{Z}) \).
generator. It is a compact space. What properties does the set \( \mathcal{I} \) of groups of intermediate growth, and the set \( \mathcal{N} \) of groups of non-uniform exponential growth, enjoy in this space? For example,

"Is there an uncountable open subset of \( \mathcal{S}_k \) in which \( \mathcal{N} \), or \( \mathcal{I} \), is dense? Is \( \mathcal{N} \) dense in the complement of groups of polynomial growth?"

Recall that similar groups are families of groups \( \mathcal{P}_G \circledcirc \mathcal{G} \circledcirc \mathcal{W} \) indexed by a space \( \Omega \). If all \( \mathcal{G}_w \) are \( k \)-generated, we obtain a map \( \Omega \rightarrow \mathcal{S}_k \), which under favourable circumstances is continuous. This has been exploited e.g. in [60] to produce groups of non-uniform exponential growth.

It had actually been doubted, before Grigorchuk’s discovery [29], whether there exist groups of intermediate growth. This text tries to convince the reader that they are abundant. Giving a precise meaning to the above question would quantify, in some manner, the extent to which they are abundant.

**Notational conventions.** I try to adhere to standard group-theoretical notation. In particular, the right action of a group element \( g \) on a point \( x \) is written \( xg \), and a left action would be written \( gx \). The stabilizer of \( x \) is written \( \mathcal{G}_x \). The conjugation action of a group on itself is written \( gh \) - \( h^{-1}gh \), and the commutator of two elements is \( [g, h] = g^{-1}h^{-1}gh = g^{-1}g^{-1}h = h^{-1}gh \).

I also introduce a minimal amount of new, “fancy” notation to represent elements of wreath products or of self-similar groups, and hope that it helps in achieving clarity and conciseness.

**A. Wreath products**

We start by the basic construction. Let \( H \) be a group, and let \( G \) be a group acting on the right on a set \( X \). We construct two groups

\[
H \circledcirc_X G = \left( \prod_X H \right) \times G \quad \text{the restricted wreath product,}
\]

\[
H \circledast_X G = \left( \prod_X H \right) \times G \quad \text{the unrestricted wreath product.}
\]

Here the unrestricted product \( \prod_X H \) may be viewed as the group of functions \( X \rightarrow H \), with pointwise composition and with left \( G \)-action given by precomposition: \( g \) is the function given by \( (g)(x) = f(xg) \). The restricted product \( \prod_X H \) is then identified with finitely supported functions \( X \rightarrow H \). In both cases, this product is a subgroup of the wreath product, and is called its **base group**.

In the particular case of \( X = G \) with natural right action by multiplication, one calls \( H \circledast_G G \) the **regular unrestricted wreath product**, and writes it simply \( H \circledast G \); and similarly for the **regular restricted wreath product** \( H \circledcirc_G G = H \circledcirc G \).

Assume that the action of \( G \) on \( X \) is faithful; so that elements of \( G \) may be identified with permutations of \( X \). The best way to describe elements of \( H \circledast_X G \) or its subgroup \( H \circledcirc_X G \) is by **decorated permutations**: one writes a permutation of \( X \), decorated by elements of \( H \), such as

\[4\text{Note that the side of the action changes! It is best to always use the appropriate side, so as to avoid inverses. Recall however that every left action can be converted into a right action by setting } f^g := g^{-1}f \text{ and vice versa.} \]
Permutations are multiplied as usual: by stacking them, and pulling the arrows tight. Likewise, decorated permutations are multiplied by stacking them and multiplying the labels along the composed arrows. We do not write the labels when they are the identity. Here is a graphical computation of a product:

When writing formulæ, we must sometimes depart from the graphical notation, in which permutations are written top-to-bottom or left-to-right and thanks to their arrows there is no ambiguity in knowing in which order to compose the labels. We invariably let permutations act on the right on sets, and thus \( \sigma \tau \) means ‘first \( \sigma \), then \( \tau \).

Exercise A.1. In the wreath product \( W = \{ \pm 1 \} \wr \text{Sym}(2) \), consider the element

\[
\begin{array}{c}
\text{h} \\
\end{array}
\]

Show by concatenating the diagram with itself that \( h \) has order exactly 4. The group \( W \) has order 8; which of the order-8 groups is it?

Let us consider a wreath product \( W = H \wr \mathcal{X} G \). In writing elements \( w, w' \in W \) algebraically, we may express them in the form \( w = fg \) with \( f : X \to H \) and \( g \in G \), or in the form \( w' = gf \). In both cases, the element \( g \) and the function \( f \) are unique, by definition of the semidirect product. The compositions \( (fg)(f'g') \) and \( (gf)(g'f') \) are, in all cases, computed using the relation

\[
g \cdot f = g' \cdot g,
\]

namely

\[
(fg)(f'g') = (f \cdot g')(gg') \quad \text{and} \quad (gf)(g'f') = (gg')(g'^{-1}f \cdot f').
\]

Exercise A.2. Let \( R \) be a ring, viewed as a group under addition, and let \( RG \) denote the group ring of \( G \), on which \( G \) acts by right multiplication. Show that \( R \wr G \) is isomorphic to \( RG \rtimes G \). More generally, let \( X \) be a \( G \)-set; then \( RX \) is a \( G \)-module. Show that \( R \wr X \) and \( RX \rtimes X \) are isomorphic.

A.1. Actions. Assume now moreover that \( H \) acts from the right on a set \( Y \). Then there are two natural sets on which \( W = H \wr X \) acts:

- There is an action on \( Y \times X \), given by \((y, x) \cdot fg = (gf(x), xg)\) for \((y, x) \in Y \times X\); it is called the \textit{imprimitive} action;
- There is an action on \( Y^X \), the set of functions \( X \to Y \), given by \((\phi \cdot fg)(xg) = \phi(x) f(x)\) for \(\phi : X \to Y\); it is called the \textit{primitive} action.
Exercise A.3. There are natural bijections between the sets \((Z \times Y) \times X = Z \times (Y \times X)\) and \((Z^Y)^X = Y \times X\). Assume now that a group \(G\) acts on \(X\), a group \(H\) acts on \(Y\) and a group \(I\) acts on \(Z\). Show that the bijections above give isomorphisms between the groups \((I \wr (Y \wr H)) \wr X \mathcal{G} (I \wr (Y \wr (H \wr X))\) as permutation groups, both of \(Z \times Y \times X\) and of \((Z^Y)^X\).

A.2. History. Leo Kaloujnine understood the importance of wreath products in the early 1940’s. It is said that he worked, during the second World War, in a uniform factory and observed a rivet machine; it was made of a rotating ring containing many rotating disks in it. Identifying the movement of the ring with an action of \(G\) and each subdisk with an action of \(H\), one sees that motions of the machine are described by wreath product elements. In his dissertation (under Élie Cartan, [44]), he studied the Sylow subgroups of symmetric groups, and showed that they were iterated wreath products.

However, this description of maximal \(p\)-subgroups of symmetric groups already appears in the classical 1870 treatise by Camille Jordan [43, II.I.41], who implicitly defined wreath products there. This is all the more remarkable since Sylow’s theorems were only published two years later [78]!

Kaloujnine returned to Soviet Union after the war, and contributed greatly to the development of mathematics in Ukraine, founding in 1959 the department of algebra and mathematical logic. He is remembered for the following important result classifying group extensions; namely, that the wreath product is a universal object containing all extensions:

**Theorem A.4** (Kaloujnine-Krasner, [45]). Let \(G, H\) be groups. Denote by \(\pi: H \wr G \rightarrow G\) the natural projection. Then the map \(\{E \colon 1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1\}\) isomorphism of extensions and \(\{E \leq H \wr G : \pi(E) = G \text{ and } \ker(\pi) \cap E \rightarrow H \text{ via } c \in H^G \mapsto c(1)\}\) conjugacy of subgroups of \(H \wr G\).

**Exercise A.5.** Prove Theorem A.4.

Hint: given an extension \(1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1\), choose a set-theoretic section \(g \mapsto \tilde{g}\) of \(\tau\), and define an imbedding \(E \rightarrow H \wr G\) by \(e \mapsto \tilde{g}(e \tau(e))^{-1}\) \(\tau(e)\).

Theorem A.4 may be interpreted more abstractly as saying that, if \(1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1\) is an exact sequence and \(f_0: H \rightarrow K\) is a group homomorphism, then \(f_0\) extends naturally to a homomorphism \(f: E \rightarrow K \wr G\). The case \(f = \text{id}\) and \(H = K\) is exactly the statement of the theorem, and the generalised version is proven in exactly the same manner. The advantage of this formulation is that one need not require that \(H\) be normal in \(E\); and the more general statement is

**Theorem A.6.** Let \(H \leq E\) be groups, and let \(f_0: H \rightarrow K\) be a homomorphism. Then \(f_0\) extends naturally to a homomorphism \(f: E \rightarrow K \wr E/\text{core}(H)\).

\(^5\)namely, a map satisfying \(\tau(\tilde{g}) = g\) for all \(g \in G\)

\(^6\)recall that the core of the subgroup \(H\) is the intersection of its conjugates. \(E/\text{core}(H)\) is the natural permutation group acting faithfully on the left coset space \(H/E\).
More precisely, let $G, H$ be groups, and let $G$ act on the right on a set $X$. Denote by $\pi : H \wr G \to G$ the natural projection. Then the map $E \mapsto E$ defines a bijection between

$$\{ E : H \leq E\text{ and the } G\text{-sets } X \text{ and } H\backslash E \text{ are isomorphic } \}$$

via a homomorphism $E \to G$

isomorphism $E \to E'$ of groups intertwining the actions on $H\backslash E$ and $H\backslash E'$

and

$$\{ E \leq H \wr G : \pi(E) \text{ transitive on } X \text{ and } \ker(\pi) \cap E \to H \text{ via } c \in H^X \mapsto c(x) \text{ for all } x \in X \}$$

conjugacy of subgroups of $H \wr G$

Note that we could have written $E$ instead of $E/\text{core}(H)$ everywhere above. We included it to obtain a smaller group acting on the set $X$, thus reinforcing the parallel with Theorem A.4 and expressing $E$ as a subgroup of a presumably easier-to-construct group. In particular, if $[E : H] = d < \infty$, then $E/\text{core}(H)$ is a group of cardinality between $d$ and $d!$, since it acts faithfully and transitively on the set $H\backslash E$ of cardinality $d$.

**Proof.** We follow the sketched proof of Theorem A.4 rather than a section of Theorem A.4, rather than a section involving the O'Nan-Scott theorem classifying maximal subgroups of the symmetric group. We skip subcases (iii.3) . . . (iii.viii) which are too technical:

**Theorem A.7** (O’Nan-Scott, [70]). Let $G$ be a maximal subgroup of $\text{Sym}(X)$ for a finite set $X$. Then either

1. the action of $G$ is not transitive\(^7\) and $X = Y \uplus Z$ and $G = \text{Sym}(Y) \times \text{Sym}(Z)$; or
2. the action of $G$ is transitive, but not primitive\(^8\) and $G = \text{Sym}(Y) \wr \text{Sym}(Z)$ in its imprimitive action; or
3. the action of $G$ is primitive; and then either
   \begin{itemize}
   \item (3.i) $G$ is an affine group over a finite field; or
   \item (3.ii) $X = Y^2$ and $G = \text{Sym}(Y) \wr \text{Sym}(Z)$ in its primitive action; or
   \item (3.iii) . . . (3.viii) . . .
   \end{itemize}

\[^7\text{i.e. there are at least two orbits on } X\]
\[^8\text{i.e. there is a non-trivial } G\text{-invariant equivalence relation on } X\]
A.3. Generators for wreath products. We now consider restricted wreath products $W = H \wr_X G$ in more detail. We introduce the following notation: if $X$ be a set and $H$ be a group, then for all $x \in X$ and $h \in H$,

we define $h@x : X \to H$ by

\[
(h@x)(y) = \begin{cases} 
  h, & \text{if } y = x, \\
  1 & \text{for all } y \neq x.
\end{cases}
\]

One has the important formula relating conjugation in $W$ with translation:

\[(h@x)^g = h@gx\]

(check it by drawing the decorated permutations!)

Let $S$ be a generating set for $G$, and let $T$ be a generating set for $H$. Let $Y \subseteq X$ be a choice of one representative from each $G$-orbit on $X$. Then $W$ is generated by

\[\{t@y : t \in T, y \in Y\} \cup S.\]

A good, but also slightly misleading example, is the “lamplighter group”: this is the group $W = \{\pm 1\} \wr \mathbb{Z}$. It is best understood by its primitive action (on the space of functions $\{\pm 1\}^\mathbb{Z}$): imagine an infinite street with at each integer position a lamppost. The lamp there can be “on” or “off”. Imagine also that the street is viewed from the perspective of the “lamplighter”, namely the person in charge of turning various lamps on and off. The generator $s$ of $\mathbb{Z}$ means “move up the street”, or equivalently “shift all lamps down” relatively to the lamplighter; and the generator $t@1$ means “change the state of the lamp currently in front of the lamplighter”.

This is what is expressed by the cartoon above: relativistically speaking, it makes no difference to think that a train moves on its tracks, or that the tracks move under the train. The lamplighter is on the train, and the lamp configurations are on the track.

The misleading aspect of this example is that one must remember that, when the lamplighter moves in one direction, the lamps actually move in the opposite direction. In a regular wreath product, this makes no difference; but in the general case it does.

Definition A.8. Let $G = \langle S \rangle$ be a finitely generated group acting on the right on a set $X$. The associated Schreier graph has vertex set $X$, and for each $x \in X, s \in S$ an edge from $x$ to $xs$.

In the case $X = G$ of a group $G$ acting on itself by (right or left) multiplication, one obtains the (right or left) Cayley graph of $G$. The group $G$ acts then by graph isometries on its Cayley graph by (left or right) multiplication.

Exercise A.9. Consider $G = \text{Sym}(4)$ generated by $\{(1, 2), (2, 3), (3, 4)\}$. Draw the Schreier graphs of (in order of difficulty)

1. the action of $G$ on $\{1, 2, 3, 4\}$;
2. the action on the collection of 2-element subsets of $\{1, 2, 3, 4\}$;
3. the action of $G$ on itself by conjugation.
It would be a great mistake to think that a path in the permutational wreath product \( W = H \wr X G \) is a path in the Schreier graph of \( G \) with decorations in \( H \) along the path. We shall see where exactly the decorations appear, but their positions are rather at the inverses of path points. To complicate matters, in case \( G \) is abelian the map \( x \mapsto x^{-1} \) is an automorphism of \( G \), whence the above “mistaken” description luckily gives the correct answer.

Let us fix generating sets \( G = \langle S \rangle \) and \( H = \langle T \rangle \), and assume for simplicity that \( G \) acts transitively on \( X \). We also choose a base point \( x_0 \in X \). Then \( W \) is naturally generated by \( T \circ x \sqcup S \).

We consider first the description of elements of \( W \) as \( fg \) with \( f : X \to H \) and \( g \in G \). This is the “lamplighter moving” version, because the “lamp vector” \( f \) is not shifted, but the position at which it is changed varies. By (2), the right action on \( X \) acts transitively on inverses along the path. We shall see where exactly the decorations appear, but their positions are rather at the inverses of path points. To complicate matters, in case \( G \) is abelian the map \( x \mapsto x^{-1} \) is an automorphism of \( G \), whence the above “mistaken” description luckily gives the correct answer.

Thus if \( w = (t_0 \circ x)s_1(t_1 \circ x)\cdots s_t(t_t \circ x) \) gives \( ws = f(gs) \); a generator \( s \in S \) gives \( ws = f(g) \); a generator \( t \circ x \in T \circ x \) gives \( w(t \circ x) = (f \cdot t \circ xg^{-1})g \).

Thus if \( w = (t_0 \circ x)s_1(t_1 \circ x)\cdots s_t(t_t \circ x) \) gives \( y \), then the support of \( f \) is included in \( \{x, x^{-1}, \ldots, x(s_1 \ldots s_t)^{-1}\} \).

We may also describe elements of \( W \) as \( gf \); this is the “earth moving” version, because the “lamp vector” \( f \) is shifted, and the lamplighter always changes the lamp at a fixed position. By (2), the right action on \( w = gf \) of

- a generator \( s \in S \) gives \( ws = (gs)^{-1}f \);
- a generator \( t \circ x \in T \circ x \) gives \( w(t \circ x) = g(f \cdot t \circ x) \).

Thus if \( w = (t_0 \circ x)s_1(t_1 \circ x)\cdots s_t(t_t \circ x) \) gives \( y \), then the support of \( f \) is included in \( \{x, x^{-1}, \ldots, x(s_1 \ldots s_t)^{-1}\} \).

The semidirect product description of \( W = H \wr X G \) gives a presentation of \( W \) by generators and relations; see [17]. Generating as above \( W \) by \( T \circ Y \sqcup S \), we get

\[
W = \langle T \circ Y \sqcup S \rangle \mid \text{relations of } G, \text{ relations of } H, \text{ and} \]

\[
\forall t, t' \in T, \forall y \neq y' \in Y : [t \circ y, t' \circ y'], \text{ and} \]

\[
\forall t \in T, \forall y \in Y, \forall g \in G_y : [t \circ y, g], \text{ and} \]

\[
\forall t, t' \in T, \forall y \in Y, \forall g \in (G_y \setminus \{G_y\}) : [t \circ y, (t' \circ y)^9].
\]

The exact criterion, assuming \( X \neq \emptyset \) and \( H \neq 1 \), is:

**Theorem A.10** (Cornulier [17 Theorem 1.1]). *The wreath product \( W \) is finitely presented if and only if both \( G, H \) are finitely presented, \( G \) acts on \( X \) with finitely generated stabilizers, and \( G \) acts diagonally on \( X \times X \) with finitely many orbits.*

**Diestel-Leader graphs.** We present a particularly intuitive description of the Cayley graph of “lamplighter groups” \( W = F \wr \mathbb{Z} \), for a finite group \( F \) of cardinality \( q \), see [3].

Let \( \mathcal{T} \) denote the \((q + 1)\)-regular tree, choose a basepoint \( o \), and a geodesic ray \( \omega : \mathbb{N} \to \mathcal{T} \) starting at \( o \). Imagine \( \mathcal{T} \) as “hanging from \( \omega \)”: orient the edges on \( \omega \) from \( \omega(n) \) to \( \omega(n + 1) \), and orient every other edge from its furthest point to \( \omega \) to its closest. Define then \( h : \mathcal{T} \to \mathbb{Z} \) as follows: for each \( x \in \mathcal{T} \), there is a unique path in \( \mathcal{T} \) from \( o \) to \( x \), and set \( h(x) = \) (number of edges oriented forward) – (number of edges oriented backward) on this path.

\[\text{This is usually called a Busemann function}\]
Consider now the following graph $B(p,q)$. Its vertex set is $\{(x,y) \in \mathcal{T} \times \mathcal{T} : h(x) + h(y) = 0\}$. There is an edge from $(x,y)$ to $(x',y')$ precisely if $\{x,x'\}$ and $\{y,y'\}$ are connected in $\mathcal{T}$. Here is a portion of $B(2,2)$, with one tree pointing up and one tree pointing down:

![Graph](image)

**Proposition A.11.** Consider a wreath product $G = F \wr \mathbb{Z}$ with a finite group $F$ of cardinality $q$. Denote by $s$ a generator of $\mathbb{Z}$, and consider for $G$ the generating set $S = (F \circ 1)s \sqcup s^{-1}(F \circ 1)$. Then the Cayley graph of $(G,S)$ is $B(q,q)$.

**Proof.** Only in this proof, for a subset $A$ of the integers, let us denote by $F^A$ the set of finitely-supported functions $A \to F$. The vertex set of $\mathcal{T}$ may be identified with $\bigcup_{n \in \mathbb{Z}} F^{(-\infty,n]} \times \{n\}$, in such a manner that there is an edge from $(\sigma,n)$ to $(\sigma|_{(-\infty,n-1)},n-1)$ for all $n \in \mathbb{Z}$ and all $\sigma \in F^{(\infty,n]}$. Vertices on the ray $\omega$ are those of the form $(1,n)$ with $n \leq 0$.

Therefore, the vertex set of $B(q,q)$ may be identified with $\bigcup_{n \in \mathbb{Z}} F^{(-\infty,n]} \times F^{(-\infty,-n]} \times \{n\}$. This is easily put in correspondence with $F^\omega \times \mathbb{Z}$ via the map $(\sigma,\sigma',n) \mapsto (\tau,n)$ with $\tau : \mathbb{Z} \to F$ the finitely supported function given by $\tau(k) = \sigma(k)$ if $k \leq n$ and $\tau(k) = \sigma'(1-k)$ if $k > n$.

Thus we put the vertex set of $B(q,q)$ is bijection with $G$. It is now routine to check that the generators in $S$ produce the edges of $B(q,q)$. 

This description of lamplighter groups makes some of their geometric features quite transparent. For example, let us consider a finitely generated group $G = \langle S \rangle$, and its Cayley graph. A vertex $v \in G$ is called a **dead end** if all neighbours of $v$ are at least as close to 1 as $v$. More generally, let us say $v$ is on a **$k$-hill** if all paths from $v$ to an element of norm $\|v\| + 1$ has to go through a vertex of norm $\|v\| - k$. In other words, from the top of the hill the only way of going to infinity is to first go down at least $k$ steps.

Consider the two dots on the picture above. Say one of them is the origin 1. Then the other one is on a 1-hill. More generally, any element in the lamplighter group that is reached from the origin by going down $k$ steps, up $2k$ steps and down again $k$ steps along a reduced path reaches the top of a $k$-hill.

This is an fact a familiar phenomenon: consider a long street and two remote addresses we want to visit on that street, in whichever order, from our starting point on the street. The shortest way of doing this is to first go to the closest, and then the other one. The **worst** possible place to start is at equal distance from both addresses.

### B. Growth of groups

This section is not a treatise on growth of groups; for that, see rather [53]. We do recall some elementary, basic notions, and provide some motivation for the material.
to appear later. Let $G$ be a group generated by a finite symmetric set $(S = S^-1)$. One defines the word norm on $G$ by
$$\|g\| = \min\{n : g = s_1 \cdots s_n, s_i \in S\},$$
and a distance $d$ on $G$ that is invariant under left translation
$$d(g, h) = \| g^{-1} h \|.$$

Thus $G$ is viewed as a normed space, and as a metric space on which $G$ acts by isometries via left translation.

**B.1. Formal growth.** One may be interested in regularity properties of the metric space $G$; these are best studied via the growth series, the formal power series
$$\Gamma_G(z) = \sum_{g \in G} z^{\|g\|} \in \mathbb{Z}[[z]].$$

The natural questions that arise are: what is the domain of convergence of $\Gamma_G(z)$? What can be said of analytic continuations of $\Gamma_G(z)$? What are its singularities? Is the function $\Gamma_G(z)$ rational (i.e. in $\mathbb{Q}[z]$) or at least algebraic (i.e. there exists a two-variable polynomial $F(y, z) \in \mathbb{Z}[y, z]$ with $F(\Gamma_G(z), z) = 0$)?

The consideration of the power series $\Gamma_G(z)$, and of the above questions, is justified by the answers that have been given:

- $\Gamma_G(z)$ converges in a disk of radius at least $1/\#S$. If $S$ is symmetric, then the convergence radius is in fact at least $1/(\#S - 1)$, with equality if and only if $G$ is a free product of $\mathbb{Z}$’s and $C_2$’s with its natural generating set, as in Definition B.9 below.
- If groups $G, H$ are respectively generated by $S, T$, then the direct product $G \times H$ is naturally generated by $S \sqcup T$. One then has
$$\Gamma_{G \times H}(z) = \Gamma_G(z) \Gamma_H(z),$$
see Proposition B.4.
- If groups $G, H$ are respectively generated by $S, T$, then the free product $G \ast H$ is naturally generated by $S \sqcup T$. One then has
$$\frac{1}{\Gamma_{G \ast H}(z)} = \frac{1}{\Gamma_G(z)} + \frac{1}{\Gamma_H(z)} - 1,$$
see Proposition B.5.
- If the group $G$ is virtually abelian \[9\], or word-hyperbolic \[36\], or the discrete Heisenberg group \[18\] $H_3 = \left\langle \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$, then $\Gamma_G$ is a rational function of $z$ for all choices of the finite generating set $S$.
- Wreath products give some examples of power series $\Gamma_G$ that are algebraic functions, as we shall see in Corollary C.3 below.
- If $G$ is a 2-step nilpotent group with cyclic derived subgroup, then there exist generating sets for $G$ such that $\Gamma_G$ is a rational function of $z$. However, if $G$ is the 5-dimensional Heisenberg group $H_5 = \left\langle \begin{pmatrix} 1 & z & z^2 & z^3 & z^4 \\ 0 & 1 & z & z^2 & z^3 \\ 0 & 0 & 1 & z & z^2 \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$, then there exist generating sets for which $\Gamma_G(z)$ is transcendental; see \[75\].

\[\text{It is only here that we use the fact that } S \text{ is symmetric, to obtain } d(g, h) = d(h, g); \text{ in fact, it suffices in all that follows to assume that } S \text{ generates } G \text{ as a monoid.} \]

\[\text{i.e. } d(tg, th) = d(g, h) \]
In respect to this last point, note that, for $G$ nilpotent, the growth of $G$ is polynomial so $\Gamma_G(z)$ converges in the unit disk. It is either rational or transcendental, by the Fatou theorem \[21\]. More is known:

**Theorem B.1** (Pólya-Carlson \[14\]). Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a power series with integer coefficients. If $A$ is not rational, then $A$ does not extend analytically beyond the unit circle.

Most importantly, the growth series is a convenient object that encodes information on $G$. A quite satisfactory theory of “Euler characteristic” has been developed for groups, see \[15\]. Here is a special case: if $G$ is the fundamental group of a cellular complex $\mathcal{X}$ with contractible universal cover, one declares $\chi(G)$ to be $\chi(\mathcal{X})$.

More generally, if $G$ has a finite-index subgroup $H$ which is the fundamental group of a space $\mathcal{Y}$, one sets $\chi(G) = \chi(\mathcal{Y})/[G:H]$; this makes sense because if $G$ is the fundamental group of $\mathcal{X}$ then $H$ is the fundamental group of a $\{G:H\}$-sheeted covering of $\mathcal{X}$, whose Euler characteristic is $[G:H]\chi(\mathcal{X})$. In particular, if $G$ is finite then $\chi(G) = 1/\#G$. This led to the idea that $1/\Gamma_G(z)$ could behave like an Euler characteristic, and that its limit $1/\Gamma_G(1)$ could express $\chi(G)$. This is not always true, but it does hold in some illustrative cases.

Let us compute for instance the growth series of a free group $F_k$ generated by a basis.\[12\] It follows from the formula for free products, or by direct counting if one notes, for all $\ell \geq 1$, that there are $2k(2k-1)^{\ell-1}$ elements of norm $\ell$ in $F_k$, that

$$\Gamma_{F_k}(z) = \frac{1+z}{1-(2k-1)z}.$$

The value $\Gamma_{F_k}(1)$ is uniquely defined by analytic continuation, and one has $1/\Gamma_{F_k}(1) = 1-k$, in agreement with $F_k$ being the fundamental group of a graph with 1 vertex and $k$ edges. See \[23\], \[32\], \[51\], \[73\] for more such examples of the ‘$1/\Gamma_G(1) = \chi(G)$’ phenomenon.

**B.2. Complete growth series.** There exists a stronger property than having a rational growth series: a group $G = \langle S \rangle$ has a rational geodesic combing if there exists a finite directed graph with edge labels in $S$ and a fixed “initial” vertex, such that the set $L \subseteq S^*$ of words read from the initial vertex along paths in the graph has the following property: $L$ maps bijectively to $G$ by the natural evaluation map of words as elements of $G$, and the words in $L$ have minimal length among all words in $S^*$ having the same evaluation in $G$. Kervaire suggested to consider the complete growth series

$$\widehat{\Gamma}_G(z) = \sum_{g \in G} g z^{|g|} \in \mathbb{Z}[z].$$

Note that $\widehat{\Gamma}_G$ depends on the choice of generating set $S$, even though we do not mention it explicitly. The series $\widehat{\Gamma}_G(z)$ is a power series with coefficients in the group ring, and one may again ask whether it is rational or algebraic. Since $\mathbb{Z}G$ need not be commutative, let us define more precisely these notions; we refer to \[68\] for details.

Let $\Lambda \subseteq \overline{\Lambda}$ be rings. An algebraic system over $\Lambda$ in variables $X_1, \ldots, X_n$ is a non-degenerate\[13\] $n$-tuple of polynomials $P_1, \ldots, P_n$ in non-commuting indeterminates $X_1, \ldots, X_n$ and coefficients in $\Lambda$. In a linear system over $\Lambda$, the polynomials are

\[12\] i.e. $S = \{x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}\}$ and $F_k$ may be identified with reduced words over $S$.

\[13\] Let us not detail this too much; suffice it to say that the system must have a unique solution once its initial terms $f_1(0), \ldots, f_n(0)$ have been fixed.
restricted to have degree 1 and contain the indeterminate on the right; i.e. the $P_i$ are sums of monomials all belonging to $\Lambda \cup \bigcup_{1 \leq i \leq n} \Lambda X_i$. A solution is an $n$-tuple $(f_1, \ldots, f_n) \in \mathbb{X}_n^*$ such that all $P_i(f_1, \ldots, f_n) = f_i$ for all $i = 1, \ldots, n$.

We then say that a power series $F(z) \in \mathbb{Z}[[z]]$ is rational, respectively algebraic, if it is the first coordinate of the solution of a linear, respectively algebraic system over the polynomial ring $\mathbb{Z}[z]$. A more direct definition of the ring of rational functions is that it is the smallest subring of $\mathbb{Z}[[z]]$ containing $\mathbb{Z}[z]$ and closed under Kleene’s quasi-inversion, the operation $F(z)^* = (1 - F(z))^{-1} = 1 + F(z) + F(z)^2 + \cdots$ defined for all $F(z) \in \mathbb{Z}[[z]]$ with $F(0) = 0$.

**Exercise B.2.** If $G$ admits a rational geodesic combing, then its growth series is rational.

Hint: define one variable $X_i$ for each vertex $i$ of the graph defining the combing, and encode the edges of the graph into polynomials.

**Exercise B.3.** If the growth series of $G$ is rational, then $G$ admits a quasi-geodesic combing: a language $L \subset S^*$, recognised by a finite graph as above, with the property that, for some constant $C \in \mathbb{N}$, all words $s_1 \ldots s_\ell \in L$ have the property $\ell \leq C\|s_1 \cdots s_\ell\|$. Let $C$ be the maximal length of all these words over $S$ that appear in the polynomial system. A solution to the polynomial system will be a sum of monomials of the form $s_1 \cdots s_\ell z^n$, where $n = \|s_1 \cdots s_\ell\| \geq \ell/C$.

These notions strengthen the ones for the classical growth series: if $\hat{\Gamma}_G(z)$ is rational or algebraic, then its image under the augmentation map $\mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ is rational or algebraic. On the other hand, statements concerning the complete growth series are usually not much harder to prove than the analogous ones concerning the classical growth series:

**Proposition B.4.** Let the groups $G, H$ and $G \times H$ be respectively generated by $S, T$ and $S \sqcup T$. One then has

$$\hat{\Gamma}_{G \times H}(z) = \hat{\Gamma}_G(z)\hat{\Gamma}_H(z).$$

**Proof.** Every element $(g, h) \in G \times H$ satisfies $\|(g, h)\| = \|g\| + \|h\|$; so

$$\hat{\Gamma}_{G \times H}(z) = \sum_{(g, h) \in G \times H} gh z^{\|g\| + \|h\|} = \sum_{g \in G} g z^{\|g\|} \sum_{h \in H} h z^{\|h\|} = \hat{\Gamma}_G(z)\hat{\Gamma}_H(z). \quad \Box$$

**Proposition B.5.** Let the groups $G, H$ and $G \ast H$ be respectively generated by $S, T$ and $S \sqcup T$. One then has

$$\frac{1}{\hat{\Gamma}_{G \ast H}(z)} = \frac{1}{\hat{\Gamma}_G(z)} + \frac{1}{\hat{\Gamma}_H(z)} - 1.$$

**Proof.** Every element of $w \in G \ast H$ may be uniquely written in the form $w = h_0g_1h_1 \cdots g_\ell$ with $h_0 \in H$, $h_1, \ldots, h_{\ell-1} \in H \setminus \{1\}$, $g_1, \ldots, g_{\ell-1} \in G \setminus \{1\}$, $g_\ell \in G$. Thus

$$\hat{\Gamma}_{G \ast H}(z) = \sum_{\ell \geq 0} \hat{\Gamma}_H(z)(\hat{\Gamma}_G(z) - 1)(\hat{\Gamma}_H(z) - 1)^\ell \hat{\Gamma}_G(z)$$

$$= \hat{\Gamma}_H(z) \frac{1}{1 - (\hat{\Gamma}_G(z) - 1)(\hat{\Gamma}_H(z) - 1)} \hat{\Gamma}_G(z);$$
so
\[
\frac{1}{\Gamma_{G*H}(z)} = \frac{1}{\Gamma_{G}(z)}(\hat{\Gamma}_{G}(z) + \hat{\Gamma}_{H}(z) - \hat{\Gamma}_{G}(z)\hat{\Gamma}_{H}(z)) \frac{1}{\Gamma_{H}(z)} = \frac{1}{\Gamma_{H}(z)} + \frac{1}{\Gamma_{G}(z)} - 1.
\]

These results are generalized to graph products in [3]: given a graph \( \Gamma \) with vertex set \( V \) and a group \( G_v \) for each \( v \in V \), the graph product of the \( G_v \) is

\[
G_{\Gamma} := \bigast_{v \in V} G_v / \langle [G_v, G_w] \rangle \quad \text{for each edge} \quad (v, w).
\]

Recall that a clique in a graph is a subset of the vertices any two of which are connected by an edge. They show:

**Proposition B.6** ([3, Theorem 3.8]). Let each group \( G_v \) have generating set \( S_v \), and consider the generating set \( \bigcup_{v \in V} S_v \) of \( G_{\Gamma} \). Then

\[
\frac{1}{\Gamma_{G_{v}}(z)} = \sum_{\text{clique } W \subseteq V} \prod_{v \in W} \left( \frac{1}{\Gamma_{G_v}} - 1 \right).
\]

There are few classes of groups in which \( \hat{\Gamma}_{G}(z) \) is rational for all choices of generating set:

**Proposition B.7** (Liardet [52]). If \( G \) is virtually abelian then \( \hat{\Gamma}_{G}(z) \) is rational for all choices of generating set. \[ \square \]

**Proposition B.8** (Grigorchuk-Nagnibeda [33]). If \( G \) is word-hyperbolic then \( \hat{\Gamma}_{G}(z) \) is rational for all choices of generating set. \[ \square \]

Note that there is no need to consider rings such as \( ZG \); the definition is more naturally phrased in terms of a semiring such as \( NG \). The polynomials \( P_i \) are restricted to be sums of products of monomials, and no subtraction is allowed. The notions of \( Z \)-rationality and \( N \)-rationality differ subtly, see e.g. [10].

Let us now compute explicitly the complete growth series of a free group. It simplifies a little the notation to answer a slightly more general question. We denote throughout the text the cyclic group of order \( p \) by \( C_p \):

**Definition B.9.** A free-like group is a finite free product of \( Z \)'s and \( C_2 \)'s.

Say that a free-like group \( G \) has \( m_1 \) factors isomorphic to \( C_2 \) and \( m_2 \) factors isomorphic to \( Z \); then it has a symmetric generating set of size \( m_1 + 2m_2 \), consisting of one generator for each \( C_2 \) and a generator and its inverse for each \( Z \). The group \( G \) is characterised by the property that its Cayley graph (see after Definition A.8) is an \( (m_1 + 2m_2) \)-regular undirected tree. We call such an \( S \) a natural generating set for \( G \). \[ \triangle \]

For instance, the free group \( F_{m/2} \) is free-like for \( m \) even, and \( *^mC_2 \) is free-like.

Let \( G \) be free-like, and let \( S \) denote a natural generating set of \( G \) with cardinality \( m \). We shall see that \( G \) has rational complete growth series. For ease of notation, we write \( \bar{s} \) for \( s^{-1} \). We identify elements of \( G \) with reduced words over \( S \), i.e. words not containing consecutive \( s\bar{s} \).

For all \( s \in S \), define \( F_s \in ZG[[z]] \) by

\[
F_s = \sum_{w \in G, \text{not starting with } s} w \bar{w}^{|w|};
\]
so \( \hat{\Gamma}_G(z) = 1 + \sum_{s \in S} szF_s \). We have the linear system in non-commutative unknowns \( F_s \)

\[
F_s = 1 + \sum_{t \in S, t \neq s} tzF_t,
\]

with solution \( F_s = (1 - sz)(1 - \sum_{s \in S} tz + (m - 1)z^2)^{-1} \), so finally

\[
(4) \quad \hat{\Gamma}_G(z) = \frac{1 - z^2}{1 - \sum_{s \in S} sz + (m - 1)z^2},
\]

compare with (3).

B.3. **Asymptotic growth.** We return to a group \( G \) with generating set \( S \), and view it as a metric space for the word metric. We consider the volume growth of balls in the metric space \( v_G : \mathbb{R}_+ \rightarrow \mathbb{N} \) given by

\[
v_G(R) = \#\{ g \in G : \| g \| \leq R \}.
\]

It is naturally related to the formal power series \( \Gamma(z) \): indeed \( v_G(R) \) is the sum of all coefficients of \( \Gamma(z) \) of degree \( \leq R \); equivalently, for \( R \in \mathbb{N} \) it is the degree-\( R \) coefficient of \( \Gamma(z)/(1 - z) \). Thus, by Tauberian and Abelian theorems (see e.g. [59]), asymptotics of \( v_G(R) \) as \( R \rightarrow \infty \) may be related to asymptotics of \( \hat{\Gamma}_G(z) \) as \( z \rightarrow 1 \) the convergence radius. In particular, the function \( v_G(R) \) grows as \( R^d \) if and only if \( \hat{\Gamma}_G(z) \) converges in the unit disk and has an order-\( d \) pole singularity at 1.

The norm \( \| \cdot \| \) depends on the choice of generating set \( S \), but only mildly: different choices of generating sets give equivalent norms, and equivalent metrics. If for \( v, w : \mathbb{R}_+ \rightarrow \mathbb{N} \) we write \( v \asymp w \) to mean that \( v(R) \leq w(CR) \) for a constant \( C \in \mathbb{R}_+ \) and all \( R \gg 0 \), and we write \( v \sim w \) to mean \( v \asymp w \asymp v \), then

**Lemma B.10.** The \( \sim \)-equivalence class of \( v_G \) is independent of the choice of generating set.

**Proof.** Let \( S, S' \) be two finite generating sets for \( G \), and let us temporarily write \( \| g \|_S, \| g \|_{S'} \) and \( v_G, v_{G, S}, v_{G, S'} \) for the norms and growth functions with respect to \( S, S' \). There exists then a constant \( C \in \mathbb{N} \) such that \( \| g \|_{S'} \leq C \| g \|_S \) for all \( s' \in S' \), and thus \( \| g \|_{S'} \leq C \| g \|_S \). This gives \( v_G, S(R) \leq v_{G, S'}, (CR) \). The reverse inequality holds by symmetry. \( \square \)

Note, as a consequence, that all exponentially-growing functions are equivalent, and that \( R^d \) and \( C \cdot R^d \) are equivalent as soon as \( d > 0 \). The exponential growth rate

\[
(5) \quad \lambda_{G, S} = \lim v_{G, S}(R)^{1/R}
\]

is nevertheless worthy of consideration, and will be discussed in §H.

B.4. **History.** Interest in asymptotic growth of groups dates back at least to the early 1950’s, in the works of Krause [50], Efremovich [19] and ˇSvarc [76]; they were seeking coarse invariants of manifolds based on their fundamental group. Milnor noted in [56] that, if \( G \) is the fundamental group of a compact riemannian manifold \( M \), then \( v_G \) is equivalent to the volume growth of balls in the universal cover of \( M \).
Here is a schematic of the known equivalence classes of growth functions of groups. Note the two dots for the two groups of order 4, respectively the two groups \( \mathbb{Z}^4 \) and the Heisenberg group \( H_3 \) with quartic growth:

\[
\begin{array}{cccc}
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 \\
1 & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
1 & 2 & 3 & 4 & 5 & \cdots & R \cdot \frac{R^3}{R^2} \cdot \frac{R^4}{R^3} \cdots \\
\end{array}
\]

The left of the graph is occupied by finite groups; the growth of a finite group is equivalent to the constant function taking value the order of the group. Abelian groups, and more generally virtually nilpotent groups have polynomial growth of type \( R^d \) for an integer \( d \). The converse is a deep result by Gromov:

**Theorem B.11** (Gromov [34]). A finitely generated group has growth function bounded by a polynomial if and only if it is virtually nilpotent.

It follows also from Gromov’s argument that there exists a superpolynomial function \( v_{\min}(R) \) such that all groups with growth \( \leq v_{\min} \) are virtually nilpotent; so there are no functions with growth strictly between polynomial and \( v_{\min} \). Explicit estimates in [72] imply that one may take \( v_{\min}(R) = R^{(\log R)^{1/100}} \), although the gap is probably larger. Note also that there need not exist a \( \zeta \)-largest function \( v_{\min} \). If one restricts oneself to virtually residually nilpotent groups, then the gap extends at least to \( \exp(R^{1/2}) \), see [31]; if one restricts oneself to virtually residually solvable groups, then the gap extends at least to \( \exp(R^{1/6}) \), see [82].

Milnor asked in 1968, in a famous problem in the “American Math Monthly” [57], whether there exist groups whose growth function is neither polynomial nor exponential. He also conjectured in that note that groups of polynomial growth are precisely the virtually nilpotent groups. Milnor and Wolf showed in [55, 84] that virtually solvable groups have either polynomial or exponential growth, and the inexistence of groups with growth between polynomial and exponential became known as “Wolf’s conjecture”. Recall the celebrated “Tits’ alternative” [79]: a finitely generated subgroup of a linear group in characteristic 0 either is virtually solvable or contains a non-abelian free subgroup; from this it follows that linear groups always have polynomial or exponential growth (see furthermore [71]).

However, groups of intermediate growth exist, and Grigorchuk [29] gave such an example, known as the first Grigorchuk group \( G_{012} \); see [13].

The growth of \( G_{012} \) is not known, even up to \( \sim \)-equivalence; conjecturally, it is the same as the growth of the group \( W_{012}(C_2) \), which will be introduced in (4.3). The hatched region above indicates that, in fact, there are many groups of intermediate growth, and that any “reasonable” function between \( \exp(R^{0.76\ldots}) \) and \( \exp(R) \) is equivalent to the growth function of a group, see Theorem (F.2).

There are at least two arguments for considering asymptotic growth rather than exact growth of groups. Firstly, the asymptotics of the growth function does not depend on the generating set, by Lemma (3.10); so is an invariant of the group itself.

---

14 A property is said to virtually hold if it holds for a finite-index subgroup.

15 A property is said to hold residually if for every non-trivial element there exists a quotient in which this element remains non-trivial and the property holds.
Secondly, we expect “most” growth series to be transcendental power series, so that they are probably difficult to describe, manipulate or expand; this happens e.g. for groups of subexponential growth, whose growth series converges in the unit disk so is either rational or transcendental, by Fatou’s theorem (see Theorem B.1).

C. Growth of regular wreath products

We consider in this section a wreath product $W = H \wr X G$, and compute its growth series. We assume that generating sets $S, T$ for $G, H$ respectively have been chosen, and that the growth series of $G$ is finite, say $X = \{x_1, \ldots, x_d\}$. Then, as generating set for $W$, we may take $\{t@x : t \in T, x \in X\} \cup S$. For this generating set, we have

$$\Gamma_W(z) = \Gamma_H(z)^* \Gamma_G(z),$$

and by a small abuse of notation the same relation on complete growth series:

$$\hat{\Gamma}_W(z) = \left( \prod_{i=1}^d \hat{\Gamma}_H(z) @ x_i \right) \hat{\Gamma}_G(z).$$

Indeed, every element $w \in W$ may uniquely be written in the form $w = (h_1@x_1) \cdots (h_d@x_d)g$ for some $h_1, \ldots, h_d \in H, g \in G$, and the growth series of $(h_i@x_i : h_i \in H)$ naturally coincides with that of $H$. In particular, if $\Gamma_G(z)$ and $\Gamma_H(z)$ are rational, then so is $\Gamma_W(z)$.

Johnson obtained in [42] the same conclusion for more complicated generating sets of $W$.

C.2. Lamplighter groups. The next case we consider is $G = X = \mathbb{Z}$, and in particular “lamplighter groups”. Since the computations will be generalised in the next section, we content ourselves with a brief description of the growth series, and for simplicity assume that $H$ is a finite group. We consider $W = H \wr \mathbb{Z}$, denote a generator of $\mathbb{Z}$ by $s$, and let $W$ be generated by the set $\{s, s^{-1}\} \cup H @ 1$.

Consider an element $w \in W$. If its image under the natural map $W \to \mathbb{Z}$ is non-negative, then it may be written minimally in the form $s^{-m}(h_0@1)s(h_1@1) \cdots s(h_p@1)s^{-n}$ with $h_i \in H$, $m, n \geq 0$ and $p \geq m + n$, while if its image in $\mathbb{Z}$ is negative, then it may be written minimally in the form $s^m(h_0@1)s^{-1}(h_1@1) \cdots s^{-1}(h_p@1)s^n$ with $p > m + n$. Furthermore, $h_0$ must be non-trivial unless $m = 0$, and $h_p$ must be non-trivial unless $n = 0$.

All these constraints are local and therefore rational, except the long-range relation between $m, n, p$. However, in terms of computing growth series, the letters in the expression $s^{-m}(h_0@1)s(h_1@1) \cdots s(h_p@1)s^{-n}$ can be permuted at no cost; and the set of expressions of the form

$$(h_0@1)s^{-1}s(h_1@1) \cdots s^{-1}s(h_m@1)s(h_{m+1}@1) \cdots s(h_{p-n}@1)ss^{-1}(h_{p-n+1}@1) \cdots ss^{-1}(h_p@1)$$

is indeed a rational language, so that its growth function is rational.

Exercise C.1. Compute the growth series of $C_2 \wr \mathbb{Z}$ with the standard generators $C_2 @ 1 \cup \{s, s^{-1}\}$. Note that the growth function grows exponentially, at the same rate as Fibonacci numbers. Could you have guessed the appearance of Fibonacci numbers without going through the calculations?
C.3. Regular wreath products with free groups. We compute in this subsection the complete growth series of a wreath product of the form $W = H \wr G$ for $G$ a free group. In fact, we suppose more generally that $G$ is free-like, see Definition B.9, so that its Cayley graph for the generating set $S$ is an $m$-regular tree $\mathcal{T}$. We keep the convention of writing $\pi$ for $s^{-1} \in S$. We suppose as usual that $W$ is generated by $T \cup S$. 

Consider $w \in W$, written as $w = (h_0 \circ 1)g_1(h_1 \circ 1) \cdots g_t(h_t \circ 1)$ with $g_i \in G$ and $h_i \in H$. Following the arguments in [4.3] one may write it as

$$w = \prod_{i=0}^{t} (h_i \circ e_i) \cdot g_1 \cdots g_t, \quad \text{with } e_i = (g_1 \cdots g_i)^{-1}.$$ 

The support of $w$ is the subgraph of $\mathcal{T}$ traced by inverses of prefixes of the word $g_1 \cdots g_t$; it is the convex hull of $\{e_0, \ldots, e_t\}$ in $\mathcal{T}$. We shall count elements of $W$ by examining their possible supports and summing over them.

For each $s \in S$, let $\Theta_s$ denote the set of finite subtrees of $\mathcal{T}$, containing 1 and no element of $S$ except possibly $s$. Each $\theta \in \Theta_s$ has outer vertices, with at most one neighbour in $\theta$, and inner vertices, with at least two neighbours in $\theta$. We introduce non-commutative power series $E_s(x,y,z)$ with coefficients in $\mathbb{Z}[G[z]]$, which count the number of Eulerian cycles in trees $\Theta_s$, weighted by length in $z$; the variables $x, y$ belong to $G$ and in particular are not assumed central. The series $E_s(x,y,z)$ are defined by the algebraic system

$$E_s(x,y,z) = 1 + sy\pi z^2 + sx \left( \prod_{t \in S, t \neq \pi} E_t(x,y,z) - 1 \right) \pi z^2.$$ 

The monomials in $E_s$ are in bijection with (Eulerian cycles tracing) trees in $\Theta_s$; if a tree $\theta$ with $p$ edges has inner vertices at $f_1, \ldots, f_n$ and non-trivial outer vertices at $f'_1, \ldots, f'_{n'}$, then the monomial corresponding to it is the product, in some order, of $z^{2p}, x^{f_1}, \ldots, x^{f_n}, y^{f'_1}, \ldots, y^{f'_{n'}}$. Indeed the equation defining $E_s$ says that a monomial counted by $E_s$ is either the empty tree (counted as 1), or a single edge from 1 to $s$ (counted as $sz\pi z^2$), or an edge from 1 to $s$, followed by $m-1$ subtrees counted recursively by $E_t$ for all $t \neq \pi$, which are not all empty. Note that, if $\theta$ has $p$ edges, then a minimal closed path that explores all vertices of $\theta$ has length $2p$.

Let $D_s(z)$ denote the sum of $wz^{|w|}$ over all elements $w \in W$ which belong to the base group $H^G$ and whose support is an element of $\Theta_s$. Such elements may be counted as follows: starting from a support $\theta \in \Theta_s$, choose a word $g_1 \ldots g_t$ in $G$ of minimal length that visits all vertices of $\theta$; and, each time a vertex is first visited, insert an element of $H$, which furthermore must be non-trivial if the vertex is outer. Therefore,

$$D_s(z) = E_s(\Gamma_H(z), \Gamma_H(z) - 1, z).$$

\[16\text{i.e. cycles that traverse each edge once} \]
Next, let $F_s(z)$ denote the sum of $wz^{|w|}$ over all elements $w = fg \in W$ with $f : G \to H$, $g \in G$ not beginning in $\pi$, and whose support does not contain $\pi$. We have a linear system

$$ F_s(z) = \prod_{t \neq \pi} D_t(z) + \sum_{t \neq \pi} \left( \prod_{u \neq \pi} D_u(z) \right) t z F_t, \tag{8} $$

since in every such element either $g = 1$ and the support explores all the neighbours $t$ of 1 except $\pi$, or $g$ begins by a generator, say $t$, and then its support explores all neighbours of 1 except $\pi, t$, then moves to $t$, and continues by an element not starting by $\pi$. Finally,

$$ \hat{\Gamma}_W(z) = \prod_{s \in S} D_s(z) + \sum_{s \in S} \left( \prod_{t \neq s} D_t(z) \right) sz F_s(z), \tag{9} $$

for the same reasoning as above. Combining Equations (6–9), we deduce:

**Theorem C.2.** If $H$ is a finitely generated group whose complete growth series $\tilde{\Gamma}_H(z)$ is algebraic, and $G$ is a free-like group, then the complete growth series of $W$ is also algebraic.

**Corollary C.3** (Parry, [65]). If $H$ is a finitely generated group whose growth series $\Gamma_H(z)$ is algebraic, and $G$ is a free-like group, then the growth series of $W$ is also algebraic.

If furthermore $\Gamma_H(z)$ is rational and $m \leq 2$, then $\hat{\Gamma}_W(z)$ is also rational.

On the other hand, if $m \geq 3$ then $\hat{\Gamma}_W(z)$ does not belong to the field generated by $z$ and $\Gamma_H(z)$.

**Proof.** Apply the augmentation map $\varpi : g \mapsto 1$ to Equations (6–9); this gives an algebraic system of degree $\max(m - 1, 1)$ expressing $\hat{\Gamma}_W(z)$ in terms of $z$ and $\Gamma_H(z)$. In particular, for $m = 2$ it is a linear system.

Conversely, assume $m > 3$ and let $\rho$ denote the convergence radius of the series of the image of $D_s$ under $\varpi$. Note that $\lim_{z \to \rho^{-}} \varpi(D_s)(z)$ is finite: if the limit were infinite, convergence to infinity would be order $(m - 1) \times$ itself, a contradiction. Therefore $\varpi(D_s)$ has a non-pole singularity, so is not in $\mathbb{Q}(z, \Gamma_H(z))$. \hfill $\square$

**Exercise C.4.** Show that, for the lamplighter group $C_2 \wr \mathbb{Z}$, the complete growth series is not rational.

**Hint:** use Exercise B.3

C.4. **Traveling salesmen.** To glimpse at the limit of what can be computed, consider now the case $G = \mathbb{Z}^2$. No property of $\Gamma_W(z)$ is known, and this is due to the fact that there is no good description of words of minimal norm describing group elements.

In fact, the problem can be quite precisely stated as follows. One is given a point $p_{\infty}$ and a set $\{p_1, \ldots, p_t\}$ in $\mathbb{Z}^2$, and is required to find a walk of minimal length on the grid that starts at $(0, 0)$, visits all the points $p_1, \ldots, p_t$ in some order, and ends at $p_{\infty}$. This is a classical travelling salesman problem, and is known to be NP-complete, see [24, 25]. It is a small step to venture that finding a good description of minimal paths is at least as hard as finding those paths length.

C.5. **Asymptotic growth.** Regular wreath products, in non-degenerate cases, all have exponential growth. This is in stark contrast to the case of permutational wreath products, as we shall see in §E.
Proposition C.5. If \( H \neq 1 \) and \( G \) is infinite, then \( W = H \wr G \) has exponential growth.

Proof. Choose \( h \neq 1 \in H \), and without loss of generality assume that \( h \) is a generator of \( H \). Since \( G \) is infinite, there exists an infinite word \( g_1 g_2 \ldots \) that traces a geodesic in the Cayley graph (see after [A.8]) of \( G \), with \( g_1, g_2, \ldots \) generators of \( G \) and also of \( W \). In particular, all \( (g_1 \cdots g_i)^{-1} \) are distinct. Consider then, for any \( \ell \in \mathbb{N} \), the set of elements
\[
\{(h@1)^{\epsilon_0} g_1 (h@1)^{\epsilon_1} \cdots g_\ell (h@1)^{\epsilon_\ell} : \epsilon_0, \ldots, \epsilon_\ell \in \{0, 1\}\}.
\]
All these elements have norm at most \( 2\ell + 1 \), and there are \( 2^{\ell+1} \) such elements. They are all distinct, since when they are rewritten in the form \( f g_1 \cdots g_\ell \) one has \( f((g_1 \cdots g_i)^{-1}) = h^{\epsilon_i} \) so that the \( \epsilon_i \) can be recovered from the element. Therefore, \( v_W(2\ell + 1) \geq 2^{\ell+1} \).

\( \square \)

D. (Self-)similar groups

We begin by introducing self-similar groups. They are groups with an additional structure:

Definition D.1. A group \( G \) is self-similar if it is endowed with a homomorphism \( \phi: G \to G \wr_X \text{Sym}(X) \) for some set \( X \). The map \( \phi \) is called the wreath recursion of \( G \).

In this text, we shall always assume that the set \( X \) is finite, and shall (unless stated otherwise) also assume that the homomorphism \( \phi \) is injective. A self-similar group is a group \( G \) in which elements may be recursively described by a permutation of \( X \), decorated by elements of \( G \) itself. In case \( X = \{0, 1, \ldots, d - 1\} \), we also write elements of \( G \wr_X \text{Sym}(X) \) in the form \( \langle g_0, \ldots, g_{d-1} \rangle \pi \) for group elements \( g_0, \ldots, g_{d-1} \) and a permutation \( \pi \in \text{Sym}(X) \).

It is essential to understand that being self-similar is an attribute of a group, and not a property. Thus, for example, a topological group is a group endowed with a topology; and every group is a topological group, for the discrete and the coarse topology. In the same vein, every group is self-similar, merely for the reason that it is similar to itself. Taking \( X = \{0\} \) and \( \phi(g) = \langle g \rangle \) is uninteresting, but is not illegal.

D.1. Finite-state self-similar groups. We describe two fundamental constructions of self-similar groups.

For the first, start by a well-understood group \( F \), such as a free group; and choose a (not necessarily injective!) homomorphism \( \tilde{\phi}: F \to F \wr_X \text{Sym}(X) \). There exists then a maximal quotient of \( F \) on which the map \( \tilde{\phi} \) induces an injective wreath recursion. To wit, one defines an increasing sequence \( N_i \) of normal subgroups of \( F \) by
\[
N_0 = 1, \quad N_{i+1} = \tilde{\phi}^{-1}(N_i^X),
\]
and sets \( G = F / \bigcup N_i \). By construction, the map \( \tilde{\phi} \) induces an injective map \( \phi: G \to G \wr_X \text{Sym}(X) \).

An important example of group defined by this method — and which, essentially, cannot be defined differently — is the first Grigorchuk group, introduced in [28] and based on [2]. Consider
\[
F = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, b c d \rangle \cong C_2 * (C_2 \times C_2).
\]
and define $\tilde{\phi}: F \to F \wr \text{Sym}(2)$ by

$$
\tilde{\phi}(a) = \begin{array}{c}
\begin{array}{c}
\text{a}
\end{array}
\end{array},
\tilde{\phi}(b) = \begin{array}{c}
\begin{array}{c}
\downarrow\text{c}
\end{array}
\end{array},
\tilde{\phi}(c) = \begin{array}{c}
\begin{array}{c}
\downarrow\text{d}
\end{array}
\end{array},
\tilde{\phi}(d) = \begin{array}{c}
\begin{array}{c}
\downarrow\text{b}
\end{array}
\end{array}.
$$

It is straightforward to see that $\tilde{\phi}$ is a homomorphism — just compute the images of the relators. It is, however, remarkable that one may compute efficiently in $G$ just using this description. We use the same letters $a, b, c, d$ for the corresponding generators of $G$. As an illustration, let us check that the relation $(ad)^4$ holds in $G$.

Writing the permutation diagrams horizontally, one has

$$
\phi((ad)^4) = \begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array}^4 = \begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array}^2 = \begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array}^2 = 1,
$$

so $(ad)^4 = 1$ in $G$ because $\phi$ is injective.

**Exercise D.2.** Using similar calculations, compute the exponent of $ab$ and $ac$ in $G$.

Note that, in that example, $G = \langle S \rangle$ for the set $S = \{1, a, b, c, d\}$ which has the property that $\phi(S)$ is contained in $S \times S \times \text{Sym}(2)$. More generally,

**Definition D.3.** Let $G$ be a self-similar group. A subset $S \subseteq G$ is state-closed if $\phi(S)$ is contained in $S \times S \times \text{Sym}(X)$. An element $g \in G$ is finite-state if there exists a state-closed subset of $G$ containing $g$. A subset of $G$ is finite-state if all its elements are finite-state. \(\triangle\)

**Exercise D.4.** Let $Z = \langle t \rangle$ be endowed with the self-similar structure $\phi(t) = \text{t}^4$. Show that only $t^0$ is finite-state.

**Lemma D.5.** The product and inverse of finite-state elements is again finite-state.

*Proof.* If $g, h$ are finite-state contained respectively in state-closed sets $S, T$, then $gh^{-1}$ is finite-state, since it belongs to the finite state-closed set $ST^{-1}$. \(\square\)

Therefore, a finitely generated self-similar group $G$ is finite-state if and only if its generators are finite-state, and one may assume that $G$ is generated by a state-closed set.

In that case, the wreath recursion of $G$ may conveniently be represented by an automaton, more precisely a Mealy automaton. This is a directed graph with vertex set $S$ called its states, and with an edge from $s \in S$ to $t \in S$, with label $'x|y'$, whenever the decorated permutation $\phi(s)$ maps $x \in X$ to $y \in X$ and has label $t$ on the edge $x \to y$. Thus, in a sense, the graph is the dual of the permutation diagram, with the roles of $X$ and $S$ exchanged. The automaton generating the first
Grigorchuk is, with the convention $X = \{0, 1\}$,

Assume that the self-similar group was obtained as above as a quotient of a self-similar group $F$ with a map $\tilde{\phi}: F \to F \wr \text{Sym}(X)$. Recall that the word problem asks, given a word in the generators of a finitely generated group, to determine whether the group element that it defines is trivial. There are groups, even finitely presented, in which the word problem is unsolvable \[62\]; however,

**Lemma D.6.** Let $F$ be a finite-state finitely generated self-similar group with wreath recursion $\tilde{\phi}$ and solvable word problem, and $G$ be the maximal quotient of $F$ on which the induced wreath recursion $G \to G \wr \text{Sym}(X)$ is injective. Then $G$ also has solvable word problem.

**Proof.** Assume without loss of generality that $F$ is generated by the finite state-closed set $S$, and denote also by $S$ the corresponding generating set of $G$. Given a word $w \in S^*$ of length $\ell$, it defines a state-closed element of $F$, belonging to the state-closed set $S^\ell$. Consider the corresponding automaton with vertex set $S^\ell$.

Let $U \subseteq S^\ell$ denote the set of states that are reachable from $w \in S^\ell$ by arbitrarily long paths. This set is computable: set $U_0 := \{w\}$, and for $i \geq 0$ set $U_{i+1} := U'_i \cup \{\text{endpoints of edges starting in } U'_i\}$; then the $U_i$ form an increasing sequence of subsets of $S^\ell$, hence stabilize, say to $U_0$. Note that $U_0$ is the set of states reachable from $w$. For all $i \geq 0$, let $U''_i$ denote those endpoints of edges starting in $U'_i$; then the $U''_i$ form a decreasing sequence of subsets of $S^\ell$, hence stabilize, to $U$.

The element of $G$ defined by $w$ is trivial in $G$ if and only if both all the edges starting in $U''_0$ have labels of the form $'x|x'$ for some $x \in X$, and all elements of $U$ define trivial elements of $F$ under the evaluation map $S^* \to F$.

More precisely, let $m \in \mathbb{N}$ be minimal such that every element of $U$ may be reached from $w$ by a path of length at most $m$. Then the conditions above imply that $w$ belongs to the normal subgroup $N_m$ of $F$, see \[10\]. \[\Box\]

The above proof amounts to constructing a Mealy automaton for the action of $S^\ell$, and examining it to determine which of its states are trivial in $G$.

**D.2. Linear groups.** Here is another construction of self-similar groups. Consider a group $G$, a subgroup $H$, and a homomorphism $\phi_0: H \to G$. By the “permutational Kaloujnine-Krasner theorem” \[\underline{A.6}\] there exists a natural extension $\tilde{\phi}: G \to G \wr \text{Sym}(X)$, with $X = H \setminus G$, in such a manner that $\phi(h) = \langle \ldots , \phi_0(h) , \ldots \rangle \ldots$ for all $h \in H$, with the ‘$\phi_0(h)$’ in position $H \in H \setminus G$. \[\underline{A.6}\]
Alternatively, this map \( \phi \) may be directly constructed as follows: choose a transversal \( T \) of \( H \) in \( G \), namely a subset \( T \subseteq G \) such that every \( g \in G \) may uniquely be written in the form \( ht \) with \( h \in H, t \in T \). Identify \( X \) with \( T \). Let then \( \phi(g) \) be the decorated permutation that sends \( t \in T \) to \( u \in T \) with label \( \phi_0(tgu^{-1}) \) whenever \( tgu^{-1} \) belongs to \( H \).

Here is a fundamental example: choose a prime number \( p \), and consider \( \Gamma = \Gamma_0(p) = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{array} \right) \cap \text{SL}_2(\mathbb{Z}) \).

Consider also the matrix \( \Phi = \left( \begin{array}{cc} p & 0 \\ 0 & p^{-1} \end{array} \right) \in \text{SL}_2(\mathbb{Q}) \), and \( H = G \cap G^\Phi \). Set \( \phi_0(h) = h^\Phi \). This example generalizes naturally to any matrix group, such as for instance a congruence subgroup of \( \text{SL}_n(\mathbb{Z}) \) for arbitrary \( n \), or even \( \text{SL}_n(\mathbb{Z}) \) itself. This shows that the class of linear groups over \( \mathbb{Z} \) is contained in the class of self-similar groups.

In fact, the only essential ingredient of the above construction is the element \( \Phi \) in the commensurator of \( G \). Recall that, for a subgroup \( G \) of a group \( L \), the commensurator of \( G \) is the subgroup of those \( x \in L \) such that \( G \cap Gx \) has finite index in \( G \) and in \( Gx \). If \( G \) is an irreducible lattice in a Lie group \( G \), then \( G \) may be called arithmetic \([54]\) if its commensurator is dense in \( L \); e.g. the commensurator of \( \text{SL}_n(\mathbb{Z}) \) in \( \text{SL}_n(\mathbb{R}) \) is \( \text{SL}_n(\mathbb{Q}) \). Then all arithmetic lattices admit self-similar actions on rooted trees \([46]\).

D.3. Rooted trees. Let \( X \) be a set, and consider the associated rooted regular tree \( \mathcal{T} \): its vertex set is \( X^* = \{ x_1 \ldots x_i : x_i \in X \} = \bigsqcup_{i \geq 0} X^i \), and it has an edge between \( x_{i+1}x_1 \ldots x_i \) and \( x_i \ldots x_1 \) for all \( x_i \in X \). The tree is rooted at the empty word, the unique element of \( X^0 \); the set of vertices at distance \( i \) from the root is identified with \( X^i \), and the Cartesian product \( X^\infty \) is naturally interpreted as the boundary \( \partial \mathcal{T} \) of the tree, namely the set of infinite rays emanating from the root. Here for illustration is the top of the binary tree:

Let \( W \) denote the isometry group of \( \mathcal{T} \), namely the set of bijections of \( X^* \) that fix the root \( \emptyset \) and preserve the edge structure of \( \mathcal{T} \). Given \( g \in W \), let \( \sigma \in \text{Sym}(X) \) denote the action of \( g \) on \( X = X^1 \), and for all \( x \in X \) define an element \( g_x \in W \) by \( (x_n \cdots x_1)g = (x_n \cdots x_1)g_x(x^\sigma) \); namely, \( g_x \) describes the action of \( g \) on the subtree \( X^*x \) as is it carried to \( X^*x^\sigma \) by \( g \).

**Lemma D.7.** The map

\[
\phi: \begin{cases}
W & \to W \wr \text{Sym}(X) \\
g & \to \langle g_x : x \in X \rangle \sigma
\end{cases}
\]

is a group isomorphism.
Proof. Given \( g_\sharp \in W \) and \( \sigma \in \text{Sym}(X) \), an element \( g \in W \) may uniquely be defined by \( (x_1, \ldots, x_N) g = (x_n, \ldots, x_1) g_\sharp (x^\prime) \). This proves that \( \phi \) is bijective.

To see that \( \phi \) is a homomorphism, consider the \( \#X \) subtrees below the root. They are permuted according to the permutation part \( \sigma \) of \( \phi \), and simultaneously acted upon by the decorations \( g_\sharp \). Composition of decorated permutations therefore coincides with composition of tree isometries. \( \square \)

Let us now start with a self-similar group \( G \). Its wreath recursion \( \phi : G \to G \wr X \text{Sym}(X) \) then defines an action of \( G \) on \( X \). Furthermore, the wreath recursion can be \textquotedblleft iterated\textquotedblright;: one has maps

\[
G \to G \wr X \text{Sym}(X) \xrightarrow{\phi^X} (G \wr X \text{Sym}(X)) \wr X \text{Sym}(X)
\]

so that \( G \) acts on \( X^i \) for all \( i \in \mathbb{N} \). Furthermore, these actions are compatible with each other, in the sense that the map \( (x_{i+1}, x_i, \ldots, x_1) \to (x_i, \ldots, x_1) \) interlaces the actions on \( X^{i+1} \) and \( X^i \). Taking the inverse limit of \( X^i \) under these projection maps gives an action on the Cartesian product \( X^\infty \). Note that sequences in \( X^\infty \) are infinite on the left, namely are of the form \((\ldots, x_{i+1}, x_i, \ldots, x_1)\).

The compatibility between the actions on \( X^i \) and \( X^{i+1} \) precisely means that \( G \) acts by tree isometries on the rooted regular tree \( \mathcal{T} \) with vertex set \( X^* \).

Note that, even if the wreath recursion \( \phi \) is injective, the action of the \( G \) on the tree \( \mathcal{T} \) need not be faithful. This is, however, the case for the examples of self-similar action of the first Grigorchuk group (see Proposition \[E.3\]) and of the congruence subgroup \( \Gamma_0(p) \).

We introduced the self-similar structure on \( \Gamma_0(p) \) and not on \( \text{SL}_2(\mathbb{Z}) \) because the latter does not act on the \textit{rooted \( p \)-regular tree}, in which each vertex has degree \( p + 1 \) except the root which has degree \( p \). If we add an edge upwards from the root, and a rooted \( p \)-regular tree above it, to the rooted \( p \)-regular tree, we obtain a \((p + 1)\)-regular tree on which the action of \( \Gamma_0(p) \) extends to an action of \( \text{SL}_2(\mathbb{Z}) \). In fact, this action is already well-known, see [7] II.1: the \((p + 1)\)-regular tree is the Bruhat-Tits tree of \( \text{SL}_2(\mathbb{Z}_p) \). Its vertices are homothety classes of lattices \( \cong \mathbb{Z}_p^2 \) in \( \mathbb{Q}_p \), and there is an edge between classes \( \mathbb{Q}_p^\Lambda \cap \mathbb{Q}_p^\Lambda \) if they admit representatives \( \alpha \Lambda, \alpha' \Lambda' \) with \( \alpha \Lambda \subset \alpha' \Lambda' \) and \( [\alpha' \Lambda' : \alpha \Lambda] = p \). The group \( \text{SL}_2(\mathbb{Q}_p) \) naturally acts on lattices, and \( \text{SL}_2(\mathbb{Z}_p) \) acts as the stabilizer of the root \( \mathbb{Q}_p^\Lambda \). The congruence subgroup \( \Gamma_0(p) \) fixes an edge adjacent to the root, and the rooted \( p \)-regular tree \( \mathcal{T} \) is spanned by those lattices of the form \( \langle (p^n, 0), (x_0 + x_1 p + \cdots + x_{n-1} p^{n-1}, 1) \rangle \) for \( n \in \mathbb{N} \) and \( x_0, \ldots, x_{n-1} \in \{0, \ldots, p - 1\} \).

Let us remark in passing that obtaining an action on a rooted tree is not spectacular in itself: every countable residually-\( p \) group acts on a rooted \( p \)-regular tree. Indeed, choose a descending sequence \( G = G_0 > G_1 > \cdots \) of subgroups with \( |G : G_i| = p^i \) and \( \bigcap G_i = 1 \). Let the vertices of \( \mathcal{T} \) be the set of right cosets of all \( G_i \), with an edge between \( G_i g \) and \( G_{i+1} g \) for all \( i \in \mathbb{N}, g \in G \); and let \( G \) act by right multiplication on \( \mathcal{T} \).

This action is in general not self-similar, nor is it \textit{“economical”}, in the sense that the permutation group acting on \( X^i \) may have order comparable to \((\#X)^i \) rather than \((\#X)! \#X^i \).

Finally, let us return to the construction of a self-similar group \( G \) as a quotient of a self-similar group \( F \) so that the wreath recursion becomes injective. Knowing
that the action of $G$ is faithful helps in solving the word problem, in case $G$ is not finite-state:

**Lemma D.8.** Let $F$ be a self-similar group with wreath recursion $\hat{\phi}$ and solvable word problem, and let $G$ be the maximal quotient of $F$ on which the induced wreath recursion is injective. Assume that the action of $G$ on the tree $T$ is faithful. Then $G$ also has solvable word problem.

**Proof.** Let $S$ be a generating set for $F$, and consider $w \in S^*$. We start two semi-algorithms in parallel; the first one will stop if $w$ is non-trivial in $G$, and the second one will stop if $w$ is trivial in $G$.

If $w$ is non-trivial in $G$, then it will act non-trivially on some vertex of $T$, and this vertex may be found by enumerating all vertices of $T$ and computing the action of $w$ on it by applying $\hat{\phi}$.

If $w$ is trivial, then it belongs to one of the normal subgroups $N_i$. Going through all $i = 0, 1, \ldots$ in sequence, and iterating $i$ times $\hat{\phi}$ on $w$ yields $\#X^i$ elements of $F$. If all of them are trivial in $F$, then $w$ is trivial; otherwise continue with the next $i$. □

Here are two fundamental examples of self-similar groups. Let $T = X^*$ be a rooted regular tree, and let $Q$ be a group acting transitively on $X$. Consider the iterated wreath products $Q = Q \wr X \wr Q \wr \cdots \wr Q$, with $i$ factors; these groups act naturally on $X^i$ by the imprimitive action.

On the one hand, there is a natural map $Q \twoheadrightarrow Q_{i+1}$, given by deleting the leftmost factor, i.e. naturally mapping $Q_{i+1} = Q \wr X Q_i$ to $Q_i \cong 1 \wr X Q_i$. Set then $\overline{Q} = \lim\limits_{\longrightarrow} Q_i$, the projective limit being taken along these epimorphisms. The self-similarity structure $\phi: \overline{Q} \to \overline{Q} \wr X Q$ is induced by the identity map $Q_i \twoheadrightarrow Q_i \wr X Q$. Since it ‘peels off’ the rightmost factor, it is compatible with the inverse limit. It defines a profinite self-similar group $\overline{G}$.

On the other hand, there is a natural map $Q_i \hookrightarrow Q_{i+1}$, given by inserting a trivial leftmost factor, i.e. naturally mapping $Q_{i+1} = Q \wr X Q_i$ to $Q_i \cong 1 \wr X Q_i$. Set then $L = \lim\limits_{\longleftarrow} Q_i$, the union (= injective limit) being taken along these monomorphisms. The self-similarity structure $\phi: L \to L \wr X Q$ is induced by the identity map $Q_i \hookrightarrow Q_i \wr X Q$. Since it ‘peels off’ the rightmost factor, it is compatible with the union. It defines a locally finite group $L$.

**Exercise D.9.** Show that $L$ is a dense subgroup of $\overline{Q}$.

A law for a group $G$ is a word $w(x_1, x_2, \ldots)$ in variables $x_1, x_2, \ldots$ such that, whenever the elements $x_1, x_2, \ldots$ are replaced by group elements from $G$, the word evaluates to 1 in $G$. For example, abelian groups are characterised as those groups satisfying the law $w = [x_1, x_2]$.

**Exercise D.10.** Show that $L$ satisfies no non-trivial law.

Hint: it suffices to look at the case $X = \{1, 2\}$. By Theorem [A.4] every finite 2-group imbeds in $Q_i$ for some $i \in \mathbb{N}$, and therefore in $G$. Finally, the free group is residually 2.

See [1] for a general result about inexistence of group laws, which covers the group $L$. 

D.4. **Similar families of groups.** The notion of self-similar group may be generalised to a family of similar groups.

**Definition D.11.** Let $\Omega$ be a set, and let $\sigma: \Omega \to \Omega$ be a map. A **similar family of groups** over $\Omega$ is a family $(G_\omega)_{\omega \in \Omega}$ of groups and a family of homomorphisms $\phi_\omega: G_\omega \to G_\sigma \wr \text{Sym}(X_\omega)$, for a family of sets $(X_\omega)_{\omega \in \Omega}$.

Just as before, each group $G_\omega$ acts on a tree $T_\omega$ with vertex set $\bigsqcup_{i \geq 0} X_{\sigma^{-i}\omega} \times \cdots \times X_\omega$. This rooted tree is now not anymore regular, but it is still **spherically homogeneous**, in that its isometry group is transitive on the set of vertices at given distance from the root.

As before, there are two fundamental examples of similar families of groups. Let $(X_\omega)_{\omega \in \Omega}$ be a family of sets, and let $(Q_\omega)_{\omega \in \Omega}$ be a family of groups, with $Q_\omega$ acting on $X_\omega$, say transitively for simplicity.

Consider the iterated wreath products $Q_{\omega,i} = Q_{\sigma^{-i} \omega} \wr Q_{\sigma^{-i-1} \omega} \cdots \wr Q_{\omega}$, with $i$ factors; these groups act naturally on $X_{\sigma^{-i} \omega} \times \cdots \times X_\omega$ by the imprimitive action.

On the other hand, there is a natural map $Q_{\omega,i+1} \to Q_{\omega,i}$, given by deleting the leftmost factor, i.e. naturally mapping $Q_{\omega,i+1} = Q_{\sigma^i \omega} \wr Q_{\omega,i}$ to $Q_{\omega,i} = 1 \wr Q_{\omega,i}$.

Set then $L_\omega = \varprojlim Q_{\omega,i}$, the inverse limit being taken along these monomorphisms. The self-similarity structure $\phi_\omega: L_\omega \to G_\sigma \wr \text{Sym}(X_\omega)$ is induced by the identity map $\sigma^i: Q_{\sigma^i \omega} \to Q_{\sigma^i \omega}$. Since it ‘peels off’ the rightmost factor, it is compatible with the inverse limit. It defines a profinite self-similar group $\overline{G}_\omega$.

On the other hand, there is a natural map $Q_{\omega,i} \hookrightarrow Q_{\omega,i+1}$, given by inserting a trivial leftmost factor, i.e. naturally mapping $Q_{\omega,i} = 1 \wr Q_{\omega,i}$ to $Q_{\omega,i+1} = Q_{\sigma^i \omega} \wr Q_{\omega,i}$. Set then $T_\omega = \varinjlim Q_{\omega,i}$, the projective limit being taken along these epimorphisms. The self-similarity structure $\phi_\omega: T_\omega \to G_\sigma \wr \text{Sym}(X_\omega)$ is induced by the identity map $\sigma^i: Q_{\sigma^i \omega} \to Q_{\sigma^i \omega}$.

D.5. **The Grigorchuk family $G_\omega$.** We shall concentrate particularly on one specific example. Write $\{0, 1, 2\}$ for the three non-trivial homomorphisms $C_2 \times C_2 \to C_2$, identified for definiteness as follows. We view the source $C_2 \times C_2 = \{1, b, c, d\}$ as a subgroup of the group $F$ given in (11), and the range $C_2 = \{1, a\}$ in that same group $F$. The three homomorphisms are then uniquely defined by $\ker(0) = \langle b \rangle$ and $\ker(1) = \langle c \rangle$ and $\ker(2) = \langle d \rangle$.

Set then $\Omega = \{0, 1, 2\}^\omega$, $\sigma(\omega_0 \omega_1 \omega_2 \ldots) = \omega_1 \omega_2 \ldots$.

We start by the similar family $(F_\omega)_{\omega \in \Omega}$ with maps $\phi_\omega: F \to F \wr \text{Sym}(2)$ given by $\phi_\omega(a) = \bigtimes_a$ and for all $x \in \{b, c, d\}$: $\phi_\omega(x) = \bigtimes x$.

We define normal subgroups $(N_{\omega,i})_{i \in \mathbb{N}, \omega \in \Omega}$ of $F$ by $N_{\omega,0} = 1$ and $N_{\omega,i+1} = \phi^{-1}_{\omega}(N_{\omega,i})$, and set $G_\omega = F/ \bigcup_{i \in \mathbb{N}} N_{\omega,i}$. This is the same construction as above, and computes $G_\omega$ as the maximal quotient of $F$ such that the maps $\bar{\phi}_\omega: F \to F \wr \text{Sym}(2)$ descend to injective maps $\phi_\omega: G_\omega \to G_\sigma \wr \text{Sym}(2)$.

In particular, letting $G$ denote the Grigorchuk group defined in [D.1], we have isomorphisms

\[
G \cong G_{\{012\}} \cong G_{\{120\}} \cong G_{\{201\}}.
\]
identifying the generating sets as follows:

\{a, b, c, d\} \quad \{a, d, c, b\} \quad \{a, b, d, c\} \quad \{a, c, b, d\}.

The groups \(G_\omega\) all act on the binary rooted tree, and it is easy to see that the orbit of the ray \(1^\infty\) is dense. Therefore, the groups \(G_\omega\) could just as well have been defined by their actions on their respective orbit \(1^\omega G_\omega\). These are naturally graphs, called Schreier graphs, with vertex set \(1^\omega G_\omega\), and with an edge from \(1^\omega g\) to \(1^\omega gs\) for each generator \(s \in \{a, b, c, d\}\), see Definition A.8.

**Proposition D.12.** The graph \(1^\omega G_\omega\) is isometric to the half-infinite line \(\mathbb{N}\) with multiple edges and loops. Under this identification with \(\mathbb{N}\), the action of \(G_\omega\) is given by

\[ a(2j) = 2j + 1, \quad a(2j + 1) = 2j, \]

and for all \(x \in \{b, c, d\}\), \(x(0) = 0\), \(x(2^j(2j + 1)) = 2^j(2\omega_i(j) + 1)\).

**Proof.** Consider the infinite dihedral group \(D = \langle a, x \mid a^2, x^2 \rangle\), and the wreath recursion \(\phi: D \to D \rtimes \text{Sym}(2)\) defined by

\[ \phi(a) = a, \quad \tilde{\phi}(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x = 0 \end{cases} \cdot x. \]

It defines a faithful action of \(D\) on the binary rooted tree \(T\) with vertex set \(\{0, 1\}^*\), and the action on the ray \(1^\omega\) is isomorphic to the action on the set of cosets \(\langle x \rangle / D\), since the stabilizer in \(D\) of \(1^\omega\) is \(\langle x \rangle\). We abbreviate \(1^\omega = 1\). The Schreier graph of the latter is a half-infinite line

\[
\begin{array}{c}
\vdots \\
T \quad 10 \quad 100 \quad 1001 \quad 10010 \quad 100100 \quad 1001001 \\
\vdots
\end{array}
\]

Since the action of generators of \(D\) change only a single symbol on sequences in \(\{0, 1\}^\infty\), the identification of the Schreier graph’s vertices with \(\mathbb{N}\) is explicit: it is the “Gray code” enumeration starting from the left-infinite word \(\overline{1}\). Thus the sequence \(\ldots 1110\ldots 1\) is identified with the integer \(\sum_{j=1}^\infty (1 - x_j)2^{j-1}\), reading the number in base 2 with 0’s and 1’s switched.

Now, to obtain the Schreier graph of \(G_\omega\), one replaces each ‘\(x\)’ edge by a pair of edges labeled by two letters out of \(\{b, c, d\}\), and puts loops at the extremities of the edge labeled by the remaining letter. The choice of which letter becomes a loop is determined by the position of the edge on the graph and the sequence \(\omega\).

For example, here is the Schreier graph of the action of the first Grigorchuk group \(G_{012} = \langle a, b, c, d \rangle\) on \(1^\omega G_\omega\):

\[
\begin{array}{c}
\vdots \\
T \quad 10 \quad 100 \quad 1001 \quad 10010 \quad 100100 \quad 1001001 \\
\vdots
\end{array}
\]

**E. Growth estimates for self-similar groups**

One of the purposes of this section is to reprove the following result. Let \(\Omega'\) denote the subset of \(\Omega\) consisting of sequences containing infinitely many of each of the symbols 0, 1, 2.
Theorem E.1 (Grigorchuk, [30]). If \( \omega \in \Omega' \), namely if \( \omega \) contains infinitely many of each of the symbols 0, 1, 2, then \( G_\omega \) has intermediate word growth.

We shall in fact prove much more, in preparation for the construction, in §6 of groups with prescribed growth. We mainly follow [6].

E.1. A lower bound via algebras. We begin by a general lower bound on growth, coming from the theory of Hopf algebras. Recall that the lower central series of a group \( G \) is defined by \( \gamma_1(G) = G \) and \( \gamma_{n+1}(G) = [\gamma_n(G), G] \) for all \( n \geq 1 \).

Theorem E.2 (Grigorchuk, [31]). Let \( G \) be a finitely generated group, and assume that there is a subgroup \( H \triangleleft G \) such that \( \gamma_n(H) \neq \gamma_{n+1}(H) \) for all \( n \in \mathbb{N} \). Then \( G \)'s growth function satisfies

\[ \gamma_G \geq \exp(\sqrt{R}). \]

In particular, if \( G \) is residually virtually nilpotent, then either \( G \) is virtually nilpotent (in which case \( \gamma_G \) is polynomial) or \( \gamma_G \geq \exp(\sqrt{R}) \).

Before embarking on the proof, let us set up some algebraic notions. Let \( K \) be a field, and let \( G \) be a group. The group ring \( \mathcal{A} = KG \) is the \( K \)-vector space with basis \( G \), and multiplication extended linearly. It is a Hopf algebra: it admits a coproduct, which is an algebra homomorphism \( \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \) defined on the basis by \( \Delta(g) = g \otimes g \), a counit, which is an algebra homomorphism \( \varepsilon : \mathcal{A} \to K \) defined on the basis by \( \varepsilon(g) = 1 \); and an antipode, which is an antihomomorphism \( \sigma : \mathcal{A} \to \mathcal{A} \) defined on the basis by \( \sigma(g) = g^{-1} \). Various axioms are satisfied, in particular the coproduct is coassociative: \((1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta : \mathcal{A} \to (\mathcal{A} \otimes 1)^{\otimes 3} \), and cocommutative: \( \Delta = \Delta \circ \tau \), for \( \tau : \mathcal{A}^{\otimes 2} \to \mathcal{A}^{\otimes 2} \) the map \( x \otimes y \mapsto y \otimes x \) flipping both factors. See [77] for details.

Denote by \( \varpi \) the kernel of \( \varepsilon \), called the augmentation ideal. The associated graded of \( \mathcal{A} \) is the vector space

\[ \overline{\mathcal{A}} = \bigoplus_{n \geq 0} \varpi^n/\varpi^{n+1}. \]

Lemma E.3. The associated graded \( \overline{\mathcal{A}} \) is a graded, cocommutative Hopf algebra.

Proof. We have \( \Delta(\varpi) \leq \varpi \otimes \varpi + \varpi \otimes \mathcal{A} \); so \( \Delta(\varpi^n) \leq \sum_{i=0}^{n} \varpi^i \otimes \varpi^{n-i} \). Now given \( \pi \in \varpi^n/\varpi^{n+1} \), choose \( x \in \varpi^n \) representing it; write \( \Delta(x) = \sum_{i=0}^{n} y_i \otimes z_{n-i} \) with \( y_i, z_i \in \varpi^i \); and set \( \Delta(\pi) = \sum_{i=0}^{n} \varpi^i \otimes \varpi^{n-i} \) where \( \varpi^i, \varpi_{i} \in \varpi^i/\varpi^{i+1} \) are the images of their respective representatives. It is easy to show that this definition does not depend on the choices of \( x, y_i, z_i \); and it defines a coassociative and cocommutative coproduct since \( \mathcal{A} \)'s coproduct was already coassociative and cocommutative.

Let \( H \) be a Hopf algebra. An element \( x \in H \) is called primitive if \( \Delta(x) = x \otimes 1 + 1 \otimes x \). The set of primitive elements in a Hopf algebra forms a Lie subalgebra of \( H \) for the usual bracket \([x, y] = xy - yx\). Conversely, if \( L \) is a Lie algebra, then its universal enveloping algebra is a Hopf algebra whose primitive elements are \( L \). Note that, in characteristic \( p \), one should consider restricted Lie algebras.

Proposition E.4 ([58, Theorem 6.11]). Let \( H \) be a cocommutative, primitively generated, graded Hopf algebra. Then it is the universal enveloping algebra of its primitive elements.
Proof. Let \( P \) denote the Lie algebra of primitive elements in \( H \). By the universal property of \( U(P) \), there is a map \( f : U(P) \to H \) which is graded, and surjective because \( P \) generates \( H \). We show that \( f \) is also injective. Consider a homogeneous element \( x \in U(P) \), say of degree \( n \). If \( n = 0 \), then \( x \in \ker(f) \) if and only if \( x = 0 \). Assume then that \( f \) is injective on elements of degree \( < n \).

We have \( \Delta(x) = 1 \otimes x + x \otimes 1 + y \) for some \( y \in U(P)_{<n} \otimes U(P)_{<n} \). If \( f(x) = 0 \), then \( f(y) = 0 \); but we had assumed \( f \) to be injective on elements of degree \( < n \), so \( y = 0 \) and \( x \in P \). By assumption, \( f \) is injective on \( P \), so \( x = 0 \) and therefore \( f \) is injective on elements of degree \( n \) as well. \( \square \)

We apply these considerations to \( \mathcal{A} = KG \). First, we identify the primitive elements in \( KG \). Let us define the series \( \gamma_n^K(G) = \{ g \in G \mid g - 1 \in \varpi^n \} \) of normal subgroups of \( G \).

**Proposition E.5.** The space of primitive elements in \( KG \) is

\[
\mathcal{L}^K(G) = \bigoplus_{n \geq 1} (\gamma_n^K(G)/\gamma_{n+1}^K(G)) \otimes \mathbb{K}.
\]

Proof. The natural map \( g \mapsto g - 1 \) from \( \gamma_n^K(G) \) to \( \varpi^n/\varpi^{n+1} \) extends to a Lie algebra isomorphism from \( \gamma_n^K(G) \) onto primitive elements of \( KG \). \( \square \)

In fact, \( \gamma_n^K(G) \) only depends on the characteristic \( p \) of \( \mathbb{K} \) and, up to extension of scalars, \( \mathcal{L}^K(G) \) depends only on \( p \). Furthermore, it may be identified directly within \( G \), and is a variant of the lower central series [40][41]. Indeed one has \( \gamma_1^p(G) = G \), and if \( p = 0 \) then

\[
\gamma_{n+1}^0(G) = \langle g \in G \mid g^t \in [G, \gamma_n^0(G)] \text{ for some } t > 0 \rangle,
\]

while if \( p > 0 \) then

\[
\gamma_{n+1}^p(G) = \langle [G, \gamma_n^p(G)] \{ x^p \mid x \in \gamma_{[n/p]}(G) \} \rangle.
\]

**Proposition E.6.** Let \( S \) be such that \( S \cup S^{-1} \) generates \( G \), and let \( v_{G,S} \) denote the corresponding growth function. Let \( w(n) = \dim_{\mathbb{K}} \varpi^n/\varpi^{n+1} \) denote the growth function of \( KG \). Then

\[
v_{G,S}(n) \geq w(0) + w(1) + \cdots + w(n) \text{ for all } n \in \mathbb{N}.
\]

Proof. Consider first an element \( x = 1 - g \in \varpi \), and write \( g = s_1^{i_1} \cdots s_\ell^{i_\ell} \) as a product of generators and inverses. Using the identities

\[
1 - gh = (1 - g) + (1 - h) - (1 - g)(1 - h),
\]

\[
1 - g^{-1} = -(1 - g) + (1 - g)(1 - g^{-1}),
\]

we get \( x = \sum_{i=1}^{\ell} \epsilon_i (1 - s_i) \) modulo \( \varpi^2 \).

The ideal \( \varpi^n \) is generated, qua ideal, by all \( x = (1 - g_1) \cdots (1 - g_n) \) with \( g_i \in G \). By the above, \( \varpi^n/\varpi^{n+1} \) is generated, again qua ideal, by all \( x = (1 - s_1) \cdots (1 - s_n) \) with \( s_i \in S \). Now \( \varpi^n/\varpi^{n+1} \) has trivial multiplication, so the \( (1 - s_1) \cdots (1 - s_n) \) also generate \( \varpi^n/\varpi^{n+1} \) as a vector space.

This generating set is contained in the linear span of \( B_{G,S}(n) \subseteq KG \); so for \( 0 \leq i \leq n \) we have \( w(i) \leq \dim(\varpi^i \cap KB_{G,S}(n)) - \dim(\varpi^{i+1} \cap KB_{G,S}(n)) \) and the claim follows. \( \square \)
Proof of Theorem [E.2]. Consider a subgroup $H$ such that $\gamma_n(H) \neq \gamma_{n+1}(H)$ for all $n \in \mathbb{N}$. Without loss of generality, suppose $H$ is finitely generated. For $n \in \mathbb{N}$, let $\mathcal{P}(n)$ be the set of prime numbers $p$ such that $(\gamma_n(H)/\gamma_{n+1}(H)) \otimes \mathbb{F}_p \neq 0$. Each $\mathcal{P}(n)$ is non-empty because $\gamma_n(H)/\gamma_{n+1}(H)$ is a finitely generated abelian group, and $\mathcal{P}(n+1) \subseteq \mathcal{P}(n)$ because the commutator map $(\gamma_n(H)/\gamma_{n+1}(H)) \times H \rightarrow \gamma_{n+1}(H)/\gamma_{n+2}(H)$ is onto. There exists therefore a prime number $p$ such that $\gamma_n(H)/\gamma_{n+1}(H) \otimes \mathbb{F}_p \neq 0$ for all $n \in \mathbb{N}$. In particular, $\gamma_n^n(H) \neq \gamma_{n+1}^n(H)$ for all $n \in \mathbb{N}$.

Let $\mathbb{K}$ be a field of characteristic $p$. It follows that $\mathbb{K}H$ contains a primitive element $x_n$ of degree $n$ for all $n \in \mathbb{N}$, namely $x_n = g_n - 1$ for some $g_n \in \gamma_n^p(H) \setminus \gamma_{n+1}^p(H)$; so $\mathbb{K}H$ contains $\pi(n)$ linearly independent elements of degree $n$, where $\pi(n)$ denotes the partition function. Indeed to every partition $n = i_1 + \cdots + i_k$ with $i_1 \leq \cdots \leq i_k$ associate the element $x_{i_1} \cdots x_{i_k}$; these elements are linearly independent by Proposition E.4. Therefore, the growth function of $\mathbb{K}H$ is at least $\pi(n)$.

It now follows from Proposition E.6 that $v_{H,S}(R) \geq \pi(R)$ holds for any generating set $S$ of $H$. Classical results on partitions [37] tell us that $\pi(n) \propto \exp(\sqrt{n})$; so a fortiori $v_{G}(R) \geq \exp(\sqrt{n})$.

Note that, for Grigorchuk’s group $G = G_{012}$, the quotients $\gamma_n(G)/\gamma_{n+1}(G)$ have bounded rank, in fact 1 or 2, see [67]; so that no improvement on the lower bound can be obtained using Proposition E.6.

E.2. Metrics on $G_\omega$. We already saw that the groups $G_\omega$ are contracting, namely if $\phi(g) = \langle g_0, g_1 \rangle \pi$ then $g_0$ and $g_1$ are shorter than $g$. We shall need a strengthening of this property: we assign norms $\| \cdot \|_\omega$ to the groups $G_\omega$ to obtain relations of the form

$$\|g_0\|_\sigma + \|g_1\|_\sigma \leq \frac{2}{\eta_\omega} (\|g\|_\omega + \|a\|_\omega)$$

(14)

with $\eta_\omega > 2$ as large as possible.

We do this by assigning norms $e \in \mathbb{R}_+$ to the generators $a, b, c, d$ of $G_\omega$, and extend $\| \cdot \|_\omega$ to $G_\omega$ by the triangular inequality:

$$\|g\|_\omega = \min \{\|s_1 \| + \cdots + \|s_n\| : g = s_1 \cdots s_n, \ s_i \in S\}.$$

For this purpose, consider the open 2-simplex

$$\Delta = \{ (\beta, \gamma, \delta) \in \mathbb{R}^3 : \max \{\beta, \gamma, \delta\} \leq \frac{1}{2}, \beta + \gamma + \delta = 1 \}.$$

Its extremal points are $(\frac{1}{2}, \frac{1}{2}, 0)$ and its permutations. A choice of $p_\omega = (\beta, \gamma, \delta) \in \Delta$ defines the following norm on the generators of $G_\omega$:

$$\|a\|_\omega = 1 - 2\max \{\beta, \gamma, \delta\} , \quad \|b\|_\omega = \beta - \|a\|_\omega, \quad \|c\|_\omega = \gamma - \|a\|_\omega, \quad \|d\|_\omega = \delta - \|a\|_\omega.$$
In particular, note that the triangular inequality $\|x\|_\omega + \|y\|_\omega \leq \|xy\|_\omega$ is sharp for $x, y \in \{b, c, d\}$.

We extend the family of groups $(G_\omega)_{\omega \in \{0,1,2\}^3}$ to a family $(G_\omega)_{\omega \in \{0,1,2\}^\mathbb{Z}}$; namely, the parameter space is now $\Omega = \{0,1,2\}^\mathbb{Z}$ with the two-sided shift map $\sigma : \Omega \to \Omega$. The group $G_\omega$ itself only depends on the restriction of $\omega$ to $\mathbb{N}$, but the norm on $G_\omega$ depends on the restriction of $\omega$ to $\mathbb{N}$.

Analogously to before, we denote by $\Omega'$ the subset of $\Omega$ consisting of sequences that contain infinitely many $0, 1, 2$ in both directions. For $\omega \in \Omega'$, we construct the point $\eta_\omega \in \Delta$ as follows. Consider the matrices

$$M_0 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}. $$

Define then

$$\eta : \Delta \times \{0,1,2\} \to (2,3), \quad \overline{M}_\lambda : \Delta \to \mathbb{R}, \quad \mu : \Delta \to (0, \frac{1}{2})$$

by setting, for all $\lambda \in \{0,1,2\}$ and all $p \in \Delta$,

$$\eta(p, \lambda) = (1 \, 1 \, 1) \cdot M_\lambda(p), \quad \text{the } \ell^1\text{-norm of } M_\lambda(p),$$

$$\overline{M}_\lambda(p) = \frac{M_\lambda(p)}{\eta(p, \lambda)}, \quad \text{the projection of } M_\lambda(p) \text{ to } \Delta,$$

$$\mu(p) = \min\{\beta, \gamma, \delta\} \quad \text{the minimal distance of } p \text{ to a vertex of } \Delta.$$

Endow $\Delta$ with the Hilbert metric $d_{\Delta}(V_1, V_2) = \log(V_1, V_2; V_-, V_+)$, computed using the cross-ratio of the points $V_1, V_2$ and the intersections $V_-, V_+$ of the line containing $V_1, V_2$ with the boundary of $\Delta$. The transformations $\overline{M}_\lambda$ are projective transformations of $\Delta$, and are therefore contracting:

**Lemma E.7** (Essentially [11]). Let $K$ be a convex subset of affine space, and let $A : K \to K$ be a projective map. Then $A$ contracts the Hilbert metric.

If furthermore $A(K)$ contains no lines from $K$ (that is, $A(K) \cap \ell \neq K \cap \ell$ for every line $\ell$ intersecting $K$), then $A$ is strictly contracting.

**Proof.** Since $A$ is projective, it preserves the cross-ratio on lines, so we have $d_{A(K)}(A(V_1), A(V_2)) = d_K(V_1, V_2)$ for all $V_1, V_2 \in K$. Furthermore, on the line $\ell$ through $V_1, V_2$, the intersection points $\ell \cap \partial A(K)$ are not further from $V_1, V_2$ than $\{V_+, V_-\} = \ell \cap \partial K$; the
Hilbert metric decreases as \( V_\perp \) are moved further apart from \( V_1, V_2 \), and this gives strict contraction under the condition \( A(K) \cap \ell \neq K \cap \ell \).

We are ready to define the points \( p_\omega \in \Delta \), and therefore the metrics \( \| \cdot \|_\omega \). Choose an arbitrary point \( p \in \Delta \), and set

\[
p_\omega = \lim_{n \to \infty} M_{\omega_{n-1}} \circ M_{\omega_{n-2}} \circ \cdots \circ M_{\omega_1}(p).
\]

Note that, since the transformations \( \overline{M}_\lambda \) are contracting, the limit \( p_\omega \) is independent of the choice of \( p \). Note also that, by our assumption that the negative part of \( \omega \) contains infinitely many 0, 1 and 2, the limit \( p_\omega \) does not belong to the boundary of \( \Delta \). From now on, we write

\[
\eta_\omega := \eta(p_\omega, \omega_0).
\]

**Lemma E.8.** For all \( g \in G_\omega \) with \( \phi(g) = \langle \langle g_0, g_1 \rangle \rangle_\pi \), we have the inequality

\[
\|g_0\|_\sigma \omega + \|g_1\|_\sigma \omega \leq \frac{2}{\eta_\omega} (\|g\|_\omega + \|a\|_\omega);
\]

and furthermore, if \( g \notin \{b, c, d\} \) then up to replacing \( g \) with a conjugate we have

\[
\|g_0\|_\sigma \omega + \|g_1\|_\sigma \omega \leq \frac{2}{\eta_\omega} \|g\|_\omega.
\]

**Proof.** Without loss of generality, we suppose \( \omega_0 = 0 \), and write \( p_\omega = (\beta, \gamma, \delta) \) and

\[
p_{\sigma \omega} = \overline{M}_{\omega_0} p = (\beta', \gamma', \delta').
\]

Then

\[
\eta_\omega = 3 - 2\beta, \quad (\beta', \gamma', \delta') = \frac{(\beta + \gamma + \delta, 2\gamma, 2\delta)}{\eta_\omega} = \left( \frac{1}{\eta_\omega}, \frac{2\gamma}{\eta_\omega}, \frac{2\delta}{\eta_\omega} \right).
\]

Thus

\[
\|b\|_{\sigma \omega} + \|\omega_0(a)\|_{\sigma \omega} = \|b\|_{\sigma \omega} = \beta' - \|a\|_{\sigma \omega} = \beta' - (1 - 2\beta') = 3\beta' - 1
\]

\[
= \frac{3}{\eta_\omega} - 1 = 2\beta \frac{\beta}{\eta_\omega} = \frac{2}{\eta_\omega} \|ab\|_\omega,
\]

\[
\|c\|_{\sigma \omega} + \|\omega_0(a)\|_{\sigma \omega} = \frac{2\gamma}{\eta_\omega} = \frac{2}{\eta_\omega} \|ac\|_\omega,
\]

\[
\|d\|_{\sigma \omega} + \|\omega_0(a)\|_{\sigma \omega} = \frac{2\delta}{\eta_\omega} = \frac{2}{\eta_\omega} \|ad\|_\omega.
\]

Now given \( g \in G_\omega \), write it as a word of minimal norm as \( g = \alpha x_1 \cdots x_d \alpha' \), with \( x_i \in \{b, c, d\} \), and the \( \alpha' \) mean that the initial and final 'a' may be present or absent. Thus \( \|g\|_\omega \geq \|\alpha x_1\|_\omega + \cdots + \|\alpha x_d\|_\omega - \|\alpha\|_\omega \). On the other hand, each 'a' in the expression of \( g \) contributes nothing to \( g_0 \) and \( g_1 \), while each 'x_i' contributes an 'x_0(x_i)' to \( g_0 \) and \( g_1 \), in some order. Summing together the inequalities above gives the claimed inequality above.

The second claim follows, because the extra '\|a\|_\omega' term occurs only if \( g \) both starts and ends with a letter in \( \{b, c, d\} \), and this case can be prevented by conjugating \( g \) by its last letter.

**Lemma E.8** can be used to prove statements on \( G_\omega \) by induction. For example,

**Proposition E.9.** If \( \omega \in \Omega \), then the action of \( G_\omega \) on the tree \( \mathcal{T} = X^* \) is faithful, and it is transitive on each orbit \( X^* \). In particular, \( G_\omega \) is infinite.
Proof. We first use induction on $\| \cdot \|_\omega$, simultaneously on all $\omega \in \Omega'$, to show that if $g \in G_\omega$ acts trivially on $T$ then $g = 1$.

The induction starts by noting that the generators $b, c, d$ act non-trivially by our assumption that $\omega$ contains infinitely many $0, 1, 2$. Indeed, without loss of generality consider $b$; let $k \in \mathbb{N}$ be minimal such that $\omega_k(b) \neq 1$; then $b$ acts non-trivially on $X^{k+1}$.

Consider then $g \in G_\omega$ acting trivially on $T$. In particular, $g$ fixes $x$, so $\phi_\omega(g) = \langle g_0, g_1 \rangle$. By Lemma E.8 both $g_0$ and $g_1$ are shorter, so by induction they are trivial; thus $g = 1$ because $\phi_\omega$ is injective.

To check that $G_\omega$ acts transitively on $X^i$, it suffices to show that the stabilizer $H$ of $1$ acts transitively on $X^{i-1}1$; because then $X^{i-1}1G_\omega = X^{i-1}1(a) = X^i$. Now $H$ contains $b, c, d, y^a$ for a letter $y \in \{b, c, d\}$ such that $\omega(y) = a$; and the action of $b, c, d, y^a$ on $w1$ is $wb1, wc1, wd1, wa1$ respectively, so that the $H$-orbit of $w1$ is $W1$ for a $G_{a_\omega}$-orbit $W \subseteq X^{i-1}$. Again we are done by induction. □

Note that the action of $G_\omega$ is still faithful if $\omega$ only contains infinitely many of two symbols; it is not faithful if $\omega$ contains finitely many of two symbols.

E.3. The $G_\omega$ are infinite torsion groups. One of Burnside’s questions [13] asks whether there exist infinite, finitely generated groups in which every element has finite order. The first such examples were constructed by Golod [26]; here, we show that the groups $G_\omega$ are other examples:

Theorem E.10. If $\omega \in \Omega'$, then $G_\omega$ is an infinite torsion 2-group.

Proof. The group $G_\omega$ is infinite by Proposition E.9. We prove the claim by induction on $\|g\|$. It is easy to check that the generators $a, b, c, d$ all have order 2. Consider $g \in G_\omega$, with $\phi_\omega(g) = \langle g_0, g_1 \rangle \pi$. If $\pi = ()$, then $g_0, g_1$ are shorter than $g$ by Lemma E.8 so have finite order, say $2^{n_0}, 2^{n_1}$ respectively. Then $g$ has order $2^\max\{n_0, n_1\}$. If $\pi = (0, 1)$, then $g^2 = \langle g_0g_1, g_1g_0 \rangle$, and $g_0g_1$ is shorter than $g$ again by Lemma E.8 so has finite order, say $2^p$. Then $g$ has order $2^{n+1}$. □

Note that, if $\omega$ contains finitely many copies of a symbol, then $G_\omega$ is not a torsion group anymore. In fact, suppose that $\omega$ contains no 0; then $\omega_1(b) = a$ for all $i$, and the element $ab$ has infinite order, since $\phi_\omega((ab)^2n) = \langle (ba)^n, (ab)^n \rangle$ for all $n \in \mathbb{Z}$.

We also note that the groups $G_\omega$ resemble very much infinitely iterated wreath products; namely the map $\phi_\omega : G_\omega \to G_{a_\omega} \wr \text{Sym}(2)$ is almost an isomorphism:

Definition E.11. Let $(G_\omega)_{\omega \in \Omega'}$ be a similar sequence of groups. It is called branched if every $G_\omega$ has a finite-index subgroup $K_\omega$ such that

$$K_{a_\omega} \leq \phi_\omega(K_\omega).$$

If the subgroups $K_\omega$ are merely required to be non-trivial, then $(G_\omega)$ is called weakly branched.

Every group in a (weakly) branched family of groups is also called (weakly) branched. △

Proposition E.12. The groups $G_\omega$ are branched for all $\omega \in \Omega'$.

Proof. For each $\omega \in \Omega'$, let $x_\omega \in \{b, c, d\}$ be such that $\omega(x_\omega) = 1$, and set $K_\omega = \langle [x_\omega, a] \rangle_{G_\omega}$. Choose also $y_\omega \in \{b, c, d\}\setminus\{x_\omega\}$. Then $1 \times K_{a_\omega}$ is normally generated by $\langle 1, [x_{a_\omega}, a] \rangle$, and

$$\langle 1, [x_{a_\omega}, a] \rangle = \phi_\omega([x_\omega, y_\omega^a]) = \phi_\omega([x_\omega, a][x_\omega, a]^{a_\omega}) \in \phi_\omega(K_\omega);$$
the same computation holds for $K_{p\omega} \times 1$.

We now show that the groups $K_{p\omega}$ have finite index. Consider first the quotient $G_\omega/\langle x_\omega \rangle^{G_\omega}$. This group is generated by two involutions $a$ and $y_\omega$, so is a finite dihedral group, because $G_\omega$ is torsion. It follows that $\langle x_\omega \rangle^{G_\omega}$ has finite index in $G_\omega$. Then $\langle x_\omega \rangle^{G_\omega}/K_\omega = \langle x_\omega \rangle K_\omega$ has order 2, so $K_\omega$ also has finite index. 

**Proposition E.13.** If $G$ is a $p$-torsion weakly branched group, then it contains $L = \bigcup \hat{\iota}C_p$ as a subgroup.

*Sketch of proof.* Let $(G_\omega)_{\omega \in \Omega}$ be a similar family of groups, with $G = G_\omega$, and let $K_\omega \leq G_\omega$ be the subgroups given by the condition that $(G_\omega)_{\omega \in \Omega}$ is branched. For each $\omega \in \Omega$, let $g_\omega \in K_\omega$ be an element of order $p$, and let $n(\omega) \in \mathbb{N}$ be such that $g_\omega$ acts non-trivially on $X_{n(\omega)-1,\omega} \times \cdots \times X_\omega$; let $v_\omega \in X_{n(\omega)-1,\omega} \times \cdots \times X_\omega$ be a point on a non-trivial orbit.

Define then simultaneously and recursively $L_\omega = \langle L_{n(\omega),\omega} @ v_\omega, g_\omega \rangle$. It contains an element $g_\omega$ permuting $p$ copies of $L_{n(\omega),\omega}$, so is isomorphic to $L$. 

If $G$ contains no torsion, or torsion of different primes, analogous (but harder-to-state) results hold. In particular, by Exercise D.10 branched groups satisfy no law. This recovers a result by Abért [1].

**E.4. Lower growth estimates for $G_\omega$.** The proof of Theorem E.3 requires upper and lower bounds on the growth function of $G_\omega$. A lower bound is easily provided by Theorem E.2 the group $G_\omega$ is infinite (Proposition E.9) and torsion (Theorem E.10), so cannot be virtually nilpotent, since nilpotent groups with finite-order generators are finite. However, in some cases a direct argument also gives the bound $v_{G_\omega}(R) \gtrsim \exp(\sqrt{R})$.

Define indeed maps $\hat{\theta}_\omega : F \to F$ by

\[
\begin{aligned}
\hat{\theta}_\omega(a) &= a y a \text{ if } \omega_0 = \cdots = \omega_{k-1} \neq \omega_k, \text{ and } \omega_0(y) = a, \omega_k(y) = 1, \\
\hat{\theta}_\omega(x) &= x \text{ for all } x \in \{b, c, d\}.
\end{aligned}
\]

A direct calculation shows that $\hat{\theta}_\omega$ induces a map $\theta_\omega : G_{p\omega} \to G_\omega$, with

\[
\phi_\omega(\theta_\omega(g)) = \left\lfloor \frac{1}{2} \rho \right\rfloor
\]

for some $\ast \in \langle a, y \rangle$ using the notation introduced in (16). Furthermore, $\langle a, y \rangle$ is a dihedral group of order $2^{k+2}$.

**Exercise E.14.** Prove that $\theta_\omega$ is a group homomorphism.

Let us now cheat, and assume that the element $\ast$ is always trivial, rather than an element of a finite group of order $2^{k+2}$. If $k$ is bounded, this is unimportant; however, if $k$ is unbounded then an additional argument is really required.

Denote by $B_\omega(R)$ the ball of radius $R$ in $G_\omega$, and abbreviate $v_\omega(R) = v_{G_\omega}(R)$. Consider the map $G_{2\omega}^2 \to G_\omega$ given by $(g_0, g_1) \mapsto \theta_\omega(g_0) \cdot \theta_\omega(g_1)$. For $R_0, R_1$ even, it defines an injective map $B_{2\omega}(R_0) \times B_{\omega}(R_1) \to B_\omega(2(R_0 + R_1))$, hence $v_{2\omega}(R)^2 \leq v_{\omega}(4R)$ for all $R$ even. We conclude $v_{\omega}(2 \cdot 4^k) \geq v_{2\omega+A}(2^{2k}) \geq 5^{2k}$, so $v_\omega(R) \geq \exp(\frac{1}{2} \log 5\sqrt{R})$. 
Lemma E.15. For every $A > 0$ there exists $B \in \mathbb{N}$ such that, for all $\omega \in \Omega'$, we have $v_\omega(A_{\mu_\omega}) \leq B$.

Proof. It suffices to bound the number of elements of $G_\omega$ whose minimal expression has length $\leq A_{\mu_\omega}$ and has the form $g = ax_1 \cdots ax_t$ for some $x_i \in \{b, c, d\}$, since there are at most 8 times more elements of norm $\leq A_{\mu_\omega}$, namely all $\{1, b, c, d\}g(1, a)$. Now by definition $g$ has norm at least $t_{\mu_\omega}$, and there are at most $3^t$ such $g$'s, so one may take $B = 8 \cdot 3^t$.

Lemma E.16. For all $\omega \in \Omega'$ we have $\eta_\omega \mu_{\sigma \omega} \geq \mu_\omega + \|a\|_\omega$.

Proof. Write $p_\omega = (\beta, \gamma, \delta)$, and assume without loss of generality $\omega_0 = 0$ so $\mu_{\sigma \omega} = 2/\eta_\omega \min\{\gamma, \delta\}$. Consider the six orderings $\beta < \gamma < \delta$ etc. in turn to check the inequalities

$$
\eta_\omega \mu_{\sigma \omega} = 2 \min\{\gamma, \delta\} \geq \min\{\beta, \gamma, \delta\} + (1 - \max\{\beta, \gamma, \delta\}) = \mu_\omega + \|a\|_\omega.
$$

Lemma E.17. Let $f$ be a positive sublinear function, namely $f(n)/n \to 0$ as $n \to \infty$. Then $f$ is bounded from above by a concave sublinear function.

Proof. For every $\theta \in (0, 1)$, let $n_\theta$ be such that $f(n) - \theta n$ is maximal. Given $n \in \mathbb{R}$, let $\zeta < \theta$ be such that $n \in [n_\theta, n_\zeta]$ with maximal $\zeta$ and minimal $\theta$, and define $\tilde{f}(n)$ on $[n_\theta, n_\zeta]$ by linear interpolation between $(n_\theta, f(n_\theta))$ and $(n_\zeta, f(n_\zeta))$. Clearly $\tilde{f} > f$, and $\tilde{f}(n)/n$ is decreasing and coincides infinitely often with $f(n)/n$, so it converges to 0.

Proposition E.18. There is an absolute constant $B$ such that, for all $\omega \in \Omega'$ and all $k \in \mathbb{N}$,

$$
v_\omega(\eta_\omega \cdots \eta_{\sigma^{k-1}_\omega} \mu_{\sigma^k_\omega}) \leq B^{2^k}.
$$

Proof. For all $i \in \mathbb{N}$, set $\alpha_i(R) = 18(R + 2)v_{\sigma^i_\omega}(R + \mu_{\sigma^i_\omega})$. We cheat, in assuming that the functions $\alpha_i$ are log-concave, i.e. satisfy $\alpha_i(R_0)\alpha_i(R_1) \leq \alpha_i((R_0 + R_1)/2)^2$. This assumption is in fact harmless, since each function $\alpha_i$ can be replaced by its log-concave majorand: the smallest log-concave function that is pointwise larger than $\alpha_i$, given by Lemma [E.17].

The proposition will follow from the inequalities $\alpha_i(R) \leq \alpha_{i+1}((R/\eta_{\sigma^i_\omega})^2$ for all $i, R$; because then

$$
v_\omega(\eta_\omega \cdots \eta_{\sigma^{k-1}_\omega} \mu_{\sigma^k_\omega}) \leq \alpha_0(\eta_\omega \cdots \eta_{\sigma^{k-1}_\omega} \mu_{\sigma^k_\omega})
= \alpha_1(\eta_{\sigma^i_\omega} \cdots \eta_{\sigma^{k-1}_\omega} \mu_{\sigma^k_\omega})^2
\leq \cdots \leq \alpha_k(\mu_k)^{2^k} \leq B^{2^k}
$$

by Lemma [E.15].
To simplify notation, we consider only the case $i = 0$, since all cases are the same. We have
\[
\alpha_0(R) = 18(R + 2)v(R + \mu) \leq 18(R + 2) \sum_{R_0 + R_1 \leq \frac{2}{\eta_\omega}(R + \mu + \|a\|_\omega)} 2v_{\sigma\omega}(R_0)v_{\sigma\omega}(R_1)
\]
\[
\leq 36(R + 2)^2 \max_{R_0 + R_1 \leq \frac{2}{\eta_\omega}(R + \mu + \|a\|_\omega)} v_{\sigma\omega}(R_0)v_{\sigma\omega}(R_1)
\]
\[
\leq 6^2(3/\eta_\omega)^2(R + 2)^2v_{\sigma\omega}((R + \mu + \|a\|_\omega)/\eta_\omega)^2
\]
\[
\leq (18(R + 2)/\eta_\omega v_{\sigma\omega}(R/\eta_\omega + \mu_{\sigma\omega}))^2
\]
\[
\leq \alpha_1(R/\eta_\omega)^2.
\]

We are now ready to conclude the proof of Theorem 1 by showing that the groups $G_\omega$ have subexponential growth. There are in fact different methods for this. Let $\lambda_\omega$ be the exponential growth rate of $G_\omega$, as in $[5]$; we are to show $\lambda_\omega = 1$.

Since by assumption $\omega$ contains infinitely many $0,1,2$, there are infinitely many positions $k \in \mathbb{N}$ with $\omega_k = 0$ and that are separated by 1 and 2 in $\omega$. For these $k$, the point $p_{\sigma_k\omega} \in \Delta$ belongs to the subsimplex $\beta < \gamma \wedge \beta < \delta$. Thus $\beta < \frac{1}{6}$ on that subsimplex, $\eta_{\sigma_k\omega} = 3 - 2\beta > 7/3$ is uniformly bounded away from 2, and $\gamma, \delta > \frac{1}{6}$ so $\mu_{\sigma_k\omega} > \frac{1}{18}$. Thus by Proposition E.18
\[
\log \lambda_\omega = \lim \log \frac{v_\omega(R)}{R} \leq \liminf \frac{2k \log B}{\eta_\omega \cdots \eta_{\sigma^{k-1}\omega} \mu_{\sigma_k\omega}} = 0
\]

since on a subsequence the $\mu_{\sigma_k\omega}$ are bounded away from 0, all terms $2/\eta_{\sigma_k\omega}$ are bounded by 1, and infinitely many of them are bounded by $6/7$.

A more “abstract” proof may be obtained by noting that the map $\omega \to \lambda_\omega$ is continuous and bounded by 3, and that the proof of Proposition E.18 gives $\log \lambda_\omega \leq 2/\eta_\omega \log \lambda_{\sigma\omega}$. Since the action of $\sigma$ on $\Omega'$ is ergodic, we must have $\log \lambda_\omega = 0$ for all $\omega \in \Omega'$.

Let us compute more precisely an upper bound for the growth of the first Grigorchuk group $G_{0(12)}$. Since the sequence $\omega = (012)^\mathbb{N}$ is 3-periodic, we can find $p_\omega \in \Delta$ explicitly. The calculation is made even simpler by noting that $p_{\sigma\omega}$ and $p_{\sigma^2\omega}$ are cyclic permutations of $p_\omega$; thus $p_\omega$ is the normalised eigenvector of $M_0 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and its spectral radius is $\eta_\omega \approx 2.46$, the positive root of the characteristic polynomial $T^3 - T^2 - 2T - 4$. Thus for the first Grigorchuk group we get $v_\omega(\eta_\omega^k \mu_\omega) \leq B^{2k}$ for all $k$, and therefore
\[
v_\omega(R) \leq \exp(R^{\log 2/\log \eta_\omega}) \approx \exp(R^{0.76}).
\]

F. Growth of permutational wreath products

The upper and lower bounds on the growth of $G_\omega$ are both of intermediate type $\exp(R^a)$, but do not match. We consider, in this section, permutational wreath products based on the groups $G_\omega$.

Choose a sequence $\omega \in \Omega'$ and a ray $\xi \in \{0, 1\}^\mathbb{N}$, and consider the ray’s orbit $X = \xi G_\omega$. Choose a group $H$. Set then
\[
W_\omega(H) := H \wr_X G_\omega.
\]

(Even though the notation does not make it clear, the group $W_\omega(H)$ depends on $\xi$.) We shall show, in this section:
Theorem F.1. If $H$ has subexponential growth, then so does $W_\omega(H)$.

Theorem F.2. Let $\eta_+ \approx 2.46$ be the positive root of $T^3 - T^2 - 2T - 4$. Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying
\[ f(2R) \leq f(R)^2 \leq f(\eta_+, R) \text{ for all } R \text{ large enough.} \]
Then there exists $\omega \in \Omega'$ such that $v_{W_\omega(C_2)} \sim f$.

We shall give more illustrations of the growth functions that may occur in \[F.3\]
We content ourselves with the following:

Theorem F.3. For any finite group $H$, the group $W_{012}(H)$ has growth
\[ v_{W_{012}(H)} \sim \exp\left(R^{\log 2/\log \eta_+}\right). \]

The proofs of Theorems \[F.1\] and \[F.2\] rely on estimates of the support $\subset X$ of an element of $W_\omega(H)$ of norm $\leq R$. Recall that every element of $W_\omega(H)$ may be written in the form $cg$ with $c: X \to H$ and $g \in G_\omega$; its support is $\{x \in X : c(x) \neq 1\}$. To better understand the support of elements of $W_\omega(H)$, let us introduce the following

Definition F.4. Let $G$ be a group acting on the right on a set $X$ with basepoint $\xi$. For a word $w = w_1 \ldots w_\ell \in G^*$, its inverted orbit is the set
\[ O(w) = \{\xi w_{\ell+1} \ldots w_\ell : 0 \leq i \leq \ell\}. \]
If furthermore $G$ is given with a metric $\| \cdot \|$, then its inverted orbit growth is the function $\Delta: \mathbb{R}_+ \to \mathbb{N}$ given by
\[ \Delta(R) = \max\{\#O(w) : \|w\| \leq R\}. \]

We write $O_\omega(w)$ and $\Delta_\omega(R)$ in the case of $G = G_\omega$ with its metric $\| \cdot \|_\omega$. Thus, for example, taking $\xi = 1^\infty$, the inverted orbit of $\text{acadab}$ is
\[ O(\text{acadab}) = \{\text{acadab}, \text{cadab}, \text{dab}, \text{ab}, \xi\} = \{\text{T010, T, T00}\}, \]
see the Schreier graph at the end of \[F.3\].

Exercise F.5. Assume that $G$ acts transitively on $X$. Show that $\Delta(R)$ depends only mildly of the choice of $\xi$, in the following sense: if $\xi, \xi' \in X$ are two choices of a basepoint and $\Delta(R), \Delta'(R)$ are the corresponding inverted orbit growth functions, then there exists a constant $C \in \mathbb{R}$ such that $\Delta(R) \leq \Delta'(R + C)$ and $\Delta'(R) \leq \Delta(R + C)$ for all $R$.

Proposition F.6. There exists a universal constant $C$ such that, for all $\omega \in \Omega'$ and all $k \in \mathbb{N}$,
\[ 2^k \leq \Delta_\omega(\eta_\omega \cdots \eta_{\sigma^{-1}\omega} \mu_{\sigma^{k}\omega}) \leq C 2^k. \]

Proof. For the upper bound, we note that Lemma \[E.8\] applies just as well to the group $G_\omega$ as to the monoid
\[ R := S^*/\{b^2 = c^2 = d^2 = 1, bc = d, cd = h, db = c\}, \]
see Equation \[11\]. Indeed, in a minimal-length representative of an element of $R$, the number of "a" is at least the number of $b, c, d$-letters minus one, and this is the only property required for Lemma \[E.8\]. Now given a word $w \in S^*$, its inverted orbit may be read from the image of $w$ in $R$. Every element of $R$ has a unique reduced form: the reduced form of a word $w \in S^*$ is the word $\overline{w} \in S^*$ obtained by
replacing every subword equal to a left-hand side of a relation by the corresponding right-hand side.

Without loss of generality and merely at the cost of increasing the constant $C$, we may suppose $\xi = 1^x$. We claim that the inverted orbit of a word $w \in S^*$ coincides with the inverted orbit of its reduction $\overline{w}$. To see this, consider $w = w_1 \ldots w_\ell \in S^*$, a subword $w_i w_{i+1}$ equal to a left-hand side of a relation, and the word $w'$ obtained by replacing $w_i w_{i+1}$ by the right-hand side of the relation. All terms $\xi w_{i+1} \ldots w_\ell$ with $i \neq j$ clearly appear both in $O(w)$ and $O(w')$. For the remaining term in $O(w)$, we have $\xi w_{i+1} \ldots w_\ell = \xi w_{i+2} \ldots w_\ell$ because $w_{i+1}$ fixes $\xi$, so this term also belongs to $O(w')$.

If $w \in F$ satisfies $\hat{\phi}_\omega(w) = \langle w_0, w_1 \rangle \pi$ and $\xi = \xi'0$, then

$$O_\omega(w) = O_{\xi'0}(w_0) \sqcup O_{\xi'1}(w_1),$$

where the inverted orbits $O_{\xi'0}$ are computed with respect to the basepoint $\xi'$, and similarly if $\xi = \xi'1$. We therefore get

$$|\Delta_\omega(R)| \leq \max_{R_0 + R_1 \leq 2/|\omega|} (|\Delta_{\xi'0}(R_0)| + |\Delta_{\xi'1}(R_1)|).$$

The same argument as in Proposition E.18 finishes the proof of the upper bound.

For the lower bound, it suffices to exhibit for all $k \in \mathbb{N}$ a word of length at most $\eta_k \cdots \eta_k+1 \mu_{\sigma^k}\omega$ and inverted orbit of size at least $2^k$. For that purpose, define self-substitutions $\zeta_x$ of $\{ab, ac, ad\}^*$, for $x \in \{0, 1, 2\}$, by

- $\zeta_0 : ab \mapsto adabac, \ ac \mapsto acac, \ ad \mapsto adad$,
- $\zeta_1 : ab \mapsto abab, \ ac \mapsto abacad, \ ad \mapsto adad$,
- $\zeta_2 : ab \mapsto abab, \ ac \mapsto acac, \ ad \mapsto acadab$,

and note that for any word $w \in \{ab, ac, ad\}^*$ representing an element of $F$ we have

$$\hat{\phi}_\omega(\zeta_0w) = \begin{cases} a \mapsto w, \ w \mapsto w & \text{if } \zeta_0w \text{ contains an even number of } 'a', \\ a \mapsto a, \ w \mapsto a & \text{if } \zeta_0w \text{ contains an odd number of } 'a'. \end{cases}$$

In particular, $\zeta_0w$ induces a homomorphism $G_{\sigma^k}\omega \rightarrow G_{\omega}$.

By induction, we see that for any non-trivial $w \in \{ab, ac, ad\}^*$ (representing an element of $G_{\sigma^k}\omega$) we have

$$|\Delta_\omega(\zeta_0 \cdots \zeta_{k-1}(w))| \geq 2^k.$$

Note then that, if $Z \in \mathbb{N}^3$ count the numbers of $ab, ac, ad$ respectively in $w$, then $M_z^sZ$ counts the numbers of $ab, ac, ad$ respectively in $\zeta_x(w)$. Indeed without loss of generality consider $x = 0$; then every $ab$ in $w$ contributes one each of $ab, ac, ad$ to $\zeta_x(w)$, while every $ac$ and $ad$ in $w$ contributes two copies of itself to $\zeta_x(w)$.

Let $as \in \{ab, ac, ad\}$ be such that $\|as\|_{\sigma^k}\omega$ is minimal — recall the notation $p_{\omega}$ from (13); if $p_{\sigma^k}\omega = (\beta, \gamma, \delta)$ and $\beta \leq \gamma, \delta$ then $s = b$, etc. Let $W$ be the basis vector in $\mathbb{R}^3$ with a ‘1’ at the position which as has in $\{ab, ac, ad\}$; if $\beta \leq \gamma, \delta$ then $W = (1, 0, 0)^t$, etc. Set $w = \zeta_0 \cdots \zeta_{k-1}(as)$. We have $|\Delta_\omega(w)| \geq 2^k$, and

$$\|w\|_{\omega} = Z^t P_{\omega} = W^t M_{\omega_{k-1}} \cdots M_{\omega_0} P_{\omega} = \eta_{\omega} \cdots \eta_{\sigma^k-1}\omega W^t \eta_{\sigma^k}\omega = \eta_{\omega} \cdots \eta_{\sigma^k-1}\omega \eta_{\sigma^k}\omega.$$
Finally, let us introduce the “choice of inverted orbits growth” function, first generally for a group $G$, with given metric $\| \cdot \|$, acting on a set $X$ with basepoint $\xi$:

$$\Sigma(R) := \# \{ \mathcal{O}(w) : \| w \| \leq R \}.$$ 

This function counts the number of subsets that may occur as inverted orbit of a word of length at most $R$. Since $\mathcal{O}(w)$ is a subset of cardinality at most $R + 1$ of the Schreier graph $X$, and furthermore lies in the ball of radius $R$ about $\xi$ in $X$, we get the crude estimate $\Sigma(R) \leq \binom{(v_{X,\xi}(R) + R)}{2}$ based on the growth function $v_{X,\xi}$ of balls centered at $\xi$ in the graph $X$. However, in the particular case of the groups $G_\omega$, we can do better:

**Proposition F.7.** There is an absolute constant $D$ such that for all $\omega \in \Omega$ and all $k \in \mathbb{N}$ we have

$$\Sigma_\omega(\eta_\omega \cdots \eta_{\sigma - 1} \omega \mu_{\sigma^*} \omega) \leq D^2.$$ 

**Proof.** Consider $w \in G^*$ with $\hat{\phi}_\omega(w) = \langle w_0, w_1 \rangle$. The inverted orbit of $w$ is determined by the inverted orbits of $w_0$ and $w_1$, two words of total $\sigma\omega$-length at most $2/\eta_\omega(\| w_0 \| + \| a \|)$ by Lemma E.8. Therefore,

$$\Sigma_\omega(\eta_\omega R) \leq \sum_{R_0 + R_1 \leq \frac{2}{\eta_\omega}(\| w_0 \| + \| a \|)} \Sigma_{\sigma\omega}(R_0) \Sigma_{\sigma\omega}(R_1),$$

and the same argument as in Proposition E.18 applies. \hfill \square

### F.1. The growth of $W_\omega(H)$

We start by general estimates on the growth of a permutational wreath product:

**Proposition F.8.** Let $H$ be a group with growth function $v_H$, and suppose that $v_H$ is log-concave.

Let $G$ be a group acting transitively on a set $X$ with basepoint $\xi$, and let $v_G$ denote the growth function of $G$. Denote the inverted orbit growth of $G$ on $(X, \xi)$ by $\Delta$, and denote its inverted orbit choice growth by $\Sigma$.

Consider the wreath product $W = H \wr_X G$, generated by $S \cup T \circ \xi$ for the generating sets $S, T$ of $G, H$ respectively. Then

$$v_G(R) v_H(R/\Delta(R))^{\Delta(R)} \leq v_W(3R),$$

$$v_W(R) \leq v_G(R) v_H(R/\Delta(R))^{\Delta(R)} (2R)^{\Delta(R)} \Sigma(R),$$

$$v_W(R) \leq v_G(R) (\# H)^{\Delta(R)} \Sigma(R) \text{ if } H \text{ is finite.}$$

**Proof.** We begin by the lower bound. For every $R \in \mathbb{N}$, consider a word $w \in G^*$ of norm $\leq R$ realizing the maximum $\Delta(R)$; write $\mathcal{O}(w) = \{ x_1, \ldots, x_k \}$ for $k = \Delta(R)$. Choose then $k$ elements $a_1, \ldots, a_k$ of norm $\leq R/k$ in $A$. Define $f \in \sum_X H$ by $f(x_i) = a_i$, all unspecified values being 1. Then $w f \in W$ may be expressed as a word of norm $R + |a_1| + \cdots + |a_k| \leq 2R$ in the standard generators of $W$, by inserting $a_1 \circ \xi, \ldots, a_k \circ \xi$ appropriately into the word $w$.

Furthermore, different choices of $a_i$ yield different elements of $W$. Finally multiplying $w f$ with an arbitrary $g \in G$ of length at most $R$, we obtain $v_G(R) v_H(R/k)^k$ elements in the ball of radius $3R$ in $W$. 

For the upper bound, consider a word \( w \) of norm \( R \) in \( W \), and let \( f = \sum_X H \) denote its value in the base of the wreath product. The support of \( f \) has cardinality at most \( \Delta(R) \), and may take at most \( \Sigma(R) \) values.

Write then \( \sup_p f \) for some \( k \leq \Delta(R) \), and let \( a_1, \ldots, a_k \in H \) be the values of \( w \) at its support; write \( \ell_i = \|a_i\| \).

Since \( \ell_i \leq R \), the norms of the different elements on the support of \( f \) define a composition of a number not greater than \( R \) into at most \( k \) summands; such a composition is determined by \( k \) “marked positions” among \( R + k \), so there are at most \( \binom{R+k-1}{k} \) possibilities, which we bound crudely by \( (2R)^k \). Each of the \( a_i \) is then chosen among \( v_H(\ell_i) \) elements, and (by the assumption that \( v_H \) is log-concave) there are \( \prod v_H(\ell_i) \leq v_H(R/k)^k \) total choices for the elements in \( H \).

We have now decomposed \( w \) into data that specify it uniquely, and we multiply the different possibilities for each of the pieces of data. Counting the possibilities for the value of \( w \) in \( G \), the possibilities for its support in \( X \), and the possibilities for the elements in \( H \) on its support, we get

\[
v_W(R) \leq v_G(R) v_A(R/k)^k (2R)^k \Sigma(R),
\]

which is maximised by \( k = \Delta(R) \).

Finally, if \( H \) is finite then we may more simply bound the possible values of \( f \) by \( (#H)^k \).

**Corollary F.9.** Let \( G, H \) be groups of subexponential growth. Let \( G \) act transitively on a set \( X \) with basepoint \( \xi \), with sublinear inverted orbit growth and subexponential inverted orbit choice growth. Then the wreath product \( W = H \wr_X G \) has subexponential growth.

**Proof of Theorem F.1.** Assume that \( H \) has subexponential growth; then by Lemma E.17 there exists a log-concave subexponentially growing function \( v_H \) bounding the growth of \( H \) from above.

By Propositions F.6 and F.6 the function \( \Delta \) is sublinear and the function \( \Sigma \) is subexponential. By Proposition E.18 the growth of \( G_\omega \) is subexponential. Corollary F.9 then shows that \( W_\omega(H) \) has subexponential growth.

In the special case of \( H \) finite and \( G = G_\omega \), Proposition F.8 gives the

**Corollary F.10.** Let \( H \) be a non-trivial finite group. There are then two absolute constants \( F, E > 1 \) such that the growth function \( v \) of \( W_\omega(H) = H \wr_X G_\omega \) satisfies

\[
E^{2^k} \leq v(\eta_\sigma \cdots \eta_{\sigma^{k-1}} \mu_{\sigma^k \omega}) \leq F^{2^k}.
\]

**Proof.** Take together the upper bound on the growth of \( G_\omega \) from Proposition E.18 the bounds on the inverted orbit growth from Proposition F.6 and the choices for the inverted orbits from Proposition F.7. The conclusion follows from Proposition F.8.

**Proof of Theorem F.2.** This follows directly from Corollary F.10 using the fact that \( \eta_{\sigma^i \omega} = \eta_+ \) for all \( i \in \mathbb{N} \).

**F.2. Proof of Theorem F.2.** Our approach will be construct, out of the function \( f \) satisfying (17), a sequence \( \omega \in \{0, 012\}^\infty \) with long stretches of 0 when \( f \) grows fast, and long stretches of 012 when \( f \) grows slowly.
We start by introducing some shorthand notation. For a finite sequence \( \omega = \omega_0 \ldots \omega_{n-1} \in \{0, 1, 2\}^n \) and \( p \in \Delta \), we write by extension
\[
\overline{M}_\omega = \overline{M}_{\omega_0} \cdots \overline{M}_{\omega_{n-1}} : \Delta \supset
\]
and
\[
\eta(p, \omega_0 \ldots \omega_{n-1}) = \eta(p, \omega_0) \eta(\overline{M}_{\omega_0} p, \omega_1) \cdots \eta(\overline{M}_{\omega_0 \ldots \omega_{n-2}} p, \omega_{n-1}).
\]

For \( \omega \in \{0, 1, 2\}^\mathbb{Z} \), recall the construction of \( p_\omega \in \Delta \) from (15), and \( \eta_\omega = \eta(p_\omega, \omega_0) \) and \( \mu_\omega = \mu(p_\omega) \). Assume that a sequence \( \omega \in \{0, 1, 2\}^\mathbb{Z} \) is under construction, and that there exists \( k \in \mathbb{N} \) such that \( \omega_i \) has been determined for all \( i \leq k \). Then \( p_\omega \), \( \eta_\omega \) and \( \mu_\omega \) are determined, and so are \( p_{\sigma^i \omega}, \eta_{\sigma^i \omega}, \mu_{\sigma^i \omega} \) for all \( i \leq k \). We abbreviate
\[
p_i = p_{\sigma^i \omega}, \quad \eta_i = \eta_{\sigma^i \omega}, \quad \mu_i = \mu_{\sigma^i \omega}.
\]

**Lemma F.11.** If the restriction to \( \mathbb{N} \) of the sequence \( \omega \) has the form
\[
\omega = (012)^{i_1} 2^{j_1} (012)^{i_2} 2^{j_2} (012)^{i_3} 2^{j_3} \ldots,
\]
with \( i_1, j_1, i_2, j_2, \ldots \geq 1 \), then the \( \mu_k \) are all bounded away from 0.

**Proof.** In fact, the image of \( \overline{M}_{012} \) is the open triangle spanned by \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\), \((\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\) and \((\frac{4}{9}, \frac{4}{9}, \frac{4}{9})\), so after each \( 012 \) the \( \mu_k \) belongs to \((\frac{4}{9}, \frac{1}{3})\).

The image of that triangle under \( \overline{M}_2 \) is contained in the convex quadrilateral spanned by \((\frac{2}{3}, \frac{2}{3}, \frac{1}{3})\), \((\frac{4}{9}, \frac{4}{9}, \frac{1}{3})\), \((\frac{4}{9}, \frac{1}{9}, \frac{4}{9})\) and \((\frac{4}{9}, \frac{1}{9}, \frac{1}{9})\), so \( \mu_k \in (\frac{4}{9}, \frac{1}{3}) \) for all \( k \).

The heart of the argument is the following lemma, which shows that \( \eta \) approaches very quickly its limiting values 2 and \( \eta \) as the sequence \( \omega \) contains long segments of 2 or of 012:

**Lemma F.12.** There exist constants \( A' \leq 1, B' \geq 1 \) such that,

1. For all \( p \in \Delta \) and all \( n \in \mathbb{N} \),
\[
\eta(p, (012)^n) \geq \eta_+^{2n} A';
\]
2. For all \( p \in \Delta \) and all \( n \in \mathbb{N} \),
\[
\eta(p, 2^n) \leq 2^n B'.
\]

**Proof.** Let \( \mathcal{U} \) denote the image of \( \overline{M}_{012} \), and let \( p_+ \in \mathcal{U} \) denote the fixed point of \( \overline{M}_{012} \). Note first that \( \eta(\cdot, 012) \) is differentiable at \( p_+ \), and that \( \overline{M}_{012} \) is uniformly contracting on \( \mathcal{U} \); let \( \rho < 1 \) be such that \( \overline{M}_{012} \) is \( \rho \)-Lipschitz on \( \mathcal{U} \), and let \( D \) be an upper bound for the derivative of \( \log \eta(\cdot, 012) \) on \( \mathcal{U} \). Recall that \( \eta_+^3 = \eta(p_+, 012) \).

For all \( k \in \mathbb{N} \), write \( p_k = \overline{M}_{012}^k p \). For \( k \geq 1 \) we have \( \|p_k - p_+\| \leq \rho^{k-1} \), so \( \log \eta(p_k, 012) - 3 \log \eta_+ < D \rho^{k-1} \), while for \( k = 0 \) we have \( \log \eta(p_0, 012) - 3 \log \eta_+ \leq 3 \log 3 \). Therefore,
\[
\| \log \eta(p, (012)^n) - 3n \log \eta_+ \| \leq 3 \log 3 + \sum_{k=1}^{n-1} \| \log \eta(p_k, 012) - 3 \log \eta_+ \| \leq 3 \log 3 + D/(1 - \rho)
\]
is bounded over all \( n \) and \( p \). The estimate (1) follows, with \( A' = 3^3 \exp(D/(1 - \rho)) \).

For the second part, consider \( p_k = \overline{M}_{2^k} p \), and note that \( p_k \) converges at exponential (Euclidean) speed to a point \( p_\infty \) on the side \( \{\delta = \frac{1}{2}\} \) of \( \partial \Delta \), since \( \partial \overline{M}_{2}(\beta, \gamma, \delta)/\partial \delta(\ast, \ast, \frac{1}{2}) = \frac{1}{2} \); so we have \( \|p_k - p_\infty\| < \rho^{k-1} \) for some \( \rho < 1 \). As above, \( \eta(\cdot, -2) \) is differentiable in a neighbourhood \( \mathcal{U} \) of \( \{\delta = \frac{1}{2}\} \), and the derivative
of \( \log \eta(-2) \) is bounded on \( \mathcal{U} \), say by \( D \). Recall that \( \eta(p, 2) = 2 \) for all \( p \in \partial \Delta \).

The same computation as above yields \( 2 \), with \( B' = 3 \exp(D/(1 - \rho)) \).

We now reformulate the statement of Theorem F.2 as follows. Set \( g(R) = \log f(R) \), so that we have

\[ g(2R) \leq g(R) \leq g(\eta R) \tag{19} \]

for all \( R \) large enough. For simplicity (since growth is only an asymptotic property) we assume that (19) holds for all \( R \). Without loss of generality (since we are allowed to replace \( g \) by an equivalent function), we also assume that \( g \) is increasing and satisfies \( g(1) = 1 \).

We are ready to construct \( \omega \). Fix arbitrarily the value of \( \omega \) on its negative part, say \( \omega_{-N} = (012)^{-\infty} \). This determines an initial metric \( \eta_0 \in \Delta \). Out of the function \( g \), we will construct a sequence \( \omega \) such that, for constants \( A, B \), we have

\[ A \leq \frac{g(\eta(p_0, \omega_0 \ldots \omega_{k-1}))}{2^k} \leq B \text{ for all } k; \tag{20} \]

in fact, it will suffice to obtain this inequality for a set of values \( k_0, k_1, \ldots \) of \( k \) such that \( \sup_i (k_{i+1} - k_i) < \infty \). Indeed, the orbit \( p_i \) of \( p_0 \) in \( \Delta' \) will remain bounded, so we will have \( \mu(p_i) \in [C, 1] \) for some \( C > 0 \). By Corollary F.10,

\[ E^{2^k} \leq v(\eta(p_0, \omega_0 \ldots \omega_{k-1}) \mu_k) \leq v(Bg^{-1}(2^k)), \]

and therefore \( v(R) \sim \exp(g(R)) = f(R) \).

**Proof of (20).** We will extend the sequence \( \omega \) to be, on the positive integers, of the form \( (012)^{i_1} 2^{i_2} (012)^{i_3} 2^{i_4} (012)^{i_5} 2^{i_6} \ldots \),

with \( i_1, i_2, i_3, \ldots \geq 1 \). The \( \mu_k \) are bounded away from 0 by Lemma F.11.

Assuming by induction that \( \omega' = \omega_0 \ldots \omega_{k-1} \) has been constructed, we repeat the following:

- while \( g(\eta(p_0, \omega')) < 2^k \), we append 012 to \( \omega' \);
- while \( g(\eta(p_0, \omega')) > 2^k \), we append 2 to \( \omega' \).

For our induction hypothesis, we assume that the stronger condition

\[ \frac{1}{2} 2^k \leq g(\eta(p_0, \omega')) \leq 2^k \]

holds for each \( k \) of the form \( i_1 + j_1 + \cdots + i_m + j_m \), and that

\[ 2^k \leq g(\eta(p_0, \omega')) \leq 3^2 2^k \]

holds for each \( k \) of the form \( i_1 + j_1 + \cdots + i_m \); these conditions apply whenever \( \omega \)

is a product of ‘syllables’ \( (012)^{i_1} \) and \( 2^{i_2} \).

Consider first the case \( \frac{1}{2} 2^k \leq g(\eta(p_0, \omega')) \leq 2^k \); and let \( n \) be minimal such that \( g(\eta(p_0, \omega'(012)^n)) > 2^{k+3n} \). Then, for all \( i \in \{1, \ldots, n\} \), Lemma F.12 gives \( \eta(p_0, \omega'(012)^i) \geq g(\eta(p_0, \omega')\eta_{\omega'}^{i} A') \). Let \( u \in \mathbb{N} \) be minimal such that \( A' \geq \eta_{\omega'}^{i+u} \); this, combined with \( g(\eta_{\omega'}R) \geq 2g(R) \), gives

\[ g(\eta(p_0, \omega'(012)^i)) \geq g(\eta(p_0, \omega')\eta_{\omega'}^{i+u}) \geq 2^{1-u} 2^{k+3i}. \]
By minimality of $n$, we have $g(\eta(p_0, \omega'(012)^{n-1})) \leq 2^{k+3(n-1)}$; since $g$ is sublinear and $\eta \leq 3$, we get

$$g(\eta(p_0, \omega'(012)^n)) \leq 3^3 2^{k+3n}.$$ 

Consider then the case $2^k \leq g(\eta(p_0, \omega')) \leq 3^3 2^k$, which is similar, and let $n$ be minimal such that $g(\eta(p_0, \omega'2^n)) < 2^{k+n}$. Then, for all $i \in \{1, \ldots, n\}$, Lemma [F.12] gives $g(\eta(p_0, \omega'2^i)) \leq \eta(p_0, \omega')2^i B'$; this, combined with $g(2R) \leq 2g(R)$, gives

$$g(\eta(p_0, \omega'2^n)) \leq 3^3 2^{k+i} B'.$$

By minimality of $n$, we have $g(\eta(p_0, \omega'2^{n-1})) \geq \frac{1}{2} 2^{k+n}$. We have proved the claim [20], with $A = 2^{-1-n}$ and $B = 3^3 B'$.

**Remark F.13.** The construction of $\omega$ from $f$ is algorithmic, in the following sense. The initial point $p_0$ may be computed to arbitrary precision by an algorithm. If there exists an algorithm that computes values of $f$, then there exists an algorithm that, with $k \in \mathbb{N}$ as input, computes $g(\eta(p_0, \omega_0 \ldots \omega_{k-1}))$ to arbitrary precision; so there exists an algorithm that computes the digits of $\omega$.

It then follows via Theorem A.10 that the groups $G_\omega$ and $W_\omega(H) = H \wr G_\omega$ are recursively presented, for recursively presented $H$.

**F.3. Illustrations.** We now consider illustrations of Theorem F.2 and examples of growth functions that may occur for groups $W_\omega(C_2)$. In fact, the examples can be constructed in both directions: either choose a “nice” function $f$ that satisfies [17], or choose a “nice” sequence $\omega$ and estimate the corresponding growth using Corollary F.10 and Lemma F.11 and Lemma F.12. We follow both approaches.

- For every $\alpha \in [\log 2 / \log \eta_+, 1]$, there exists a group of growth $\sim \exp(R^\alpha)$. Furthermore, for a dense set of $\alpha$ in that interval, there exists a periodic sequence $\omega$ such that $W_\omega(C_2)$ has growth $\sim \exp(R^\alpha)$.

  For every $\alpha \leq \beta \in [\log 2 / \log \eta_+, 1]$, one may construct a function $f$ satisfying [17] that coincides, on arbitrarily large intervals, sometimes with the function $\exp(R^\alpha)$ and sometimes with the function $\exp(R^\beta)$. Therefore, there exists a group whose growth function accumulates both at $\exp(R^\alpha)$ and at $\exp(R^\beta)$. This recovers a result by Brieussel, see [12].

- there exists groups of growth $\sim \exp(R/\log R)$, of growth $\sim \exp(R/\log \log R)$, of growth $\exp(R/\log \log \log R)$.

- Consider conversely the sequence $\omega = (012)^2 (012)^2 (012)^2 (012)^2 (012)^4 \ldots$. Among the first $k$ entries, approximately $\sqrt{k}$ instances of 012 will have been seen; therefore $\eta(p_0, \omega_0 \ldots \omega_{k-1}) \approx 2^k + O(1) \sqrt{k}$. This gives a growth function of the order of

$$\exp \left( R/\exp(O(1) \sqrt{\log R}) \right).$$

Consider next the sequence $\omega = (012)^2 (012)^2 (012)^2 (012)^4 (012)^8 \ldots$. Among the first $k$ entries, approximately $\log k$ instances of 012 will have been seen; therefore $\eta(p_0, \omega_0 \ldots \omega_{k-1}) \approx 2^k + O(1) \log k$. This gives a growth function of the order of

$$\exp \left( R/(\log R)^{O(1)} \right).$$
Consider further the sequence $\omega = (012)2^{2^1}(012)2^{2^2}(012)2^{2^4}(012)2^{2^8}\ldots$. Among the first $k$ entries, approximately $\log \log k$ instances of 012 will have been seen; therefore $\eta(\omega_0, \omega_0, \ldots, \omega_k)$ $\approx 2^{k+O(1)\log \log k}$. This gives a growth function of the order of

$$\exp\left(\frac{R}{\log \log R}O(1)\right).$$

These constructions generalise easily to give “nice” sequences $\omega$ such that $W_\omega(C_2)$ has growth of the order of $\exp\left(\frac{R}{\log \log R}O(1)\right)$.

**G. Imbeddings and subgroups**

A classical result by Higman, Neumann and Neumann [39] states that every countable group imbeds in a finitely generated group. It was then shown that many properties of the group can be inherited by the imbedding: in particular, solvability (Neumann-Neumann [61]), torsion (Phillips [66]), residual finiteness (Wilson [80]), and amenability (Olshansky-Osin [63]).

Seen the other way round, these results show that there is little restriction, apart from being countable, on the subgroups of a finitely generated group; even if that group is furthermore assumed to be residually finite, amenable, or solvable.

On the other hand, very strong restrictions exist on the subgroups of a virtually nilpotent group: they are all finitely generated, for example.

Since by Gromov’s Theorem [B.11] the finitely generated virtually nilpotent groups are precisely the groups of polynomial growth, we naturally ask what conditions are imposed on the subgroups of a group of subexponential word growth. As we shall see, there are essentially none.

In particular, there are torsion branched groups of subexponential growth such as the group $G_\omega$. They contain iterated wreath products, by Proposition [E.13] so contain infinitely generated subgroups.

Let us say that a group has **locally subexponential growth** if all of its finitely generated subgroups have subexponential growth. Clearly, if $G$ has subexponential growth then all its subgroups have locally subexponential growth. This is the only restriction, and the objective of this section is to prove the following result:

**Theorem G.1** ([5]). Let $B$ be a group. Then there exists a finitely generated group of subexponential growth in which $B$ imbeds as a subgroup if and only if $B$ is countable and locally of subexponential growth.

For example, this implies that there exists a group of subexponential growth containing $\mathbb{Q}$ as a subgroup. I do not know any explicit such example of group.
G.1. **Neumann’s proof.** By way of motivation, we start with the classical result by Higman and the Neumanns:

**Theorem G.2** (Higman, B.H. Neumann and H. Neumann [39]). *Every countable group imbeds in a finitely generated group.*

We shall not follow the original proof (which proceeds by a sequence of “HNN extensions”), but rather that by the two Neumanns [61], which uses wreath products. It follows immediately from combining the following two propositions:

**Proposition G.3.** *Every countable group \( B \) imbeds in the commutator subgroup \([G, G]\) of a countable group \( G \).*

**Proposition G.4.** *For every countable group \( G \), there exists a 2-generated group \( W \) such that \([G, G]\) imbeds in \([W, W]\).*

**Proof of Proposition G.3.** Consider the following subgroup \( G \) of the unrestricted wreath product \( B \wr \mathbb{Z} \). The group \( G \) is generated by \( \mathbb{Z} \) and, for all \( b \in B \), the function \( f_b: \mathbb{Z} \to B \) defined by \( f_b(m) = b^m \). Denoting by \( t \) the generator of \( \mathbb{Z} \), we see that \([t, f_b]\) is the constant function \( b \); so \( B \) is in fact imbedded in \([t, G]\). □

**Exercise G.5.** Could we also have defined \( f_b(p) = b \) for \( m \geq 0 \) and \( f_b(m) = 1 \) for \( m < 0 \)? What would be the advantages and disadvantages of this alternative construction?

**Proof of Proposition G.4.** Consider the following subgroup \( W \) of the unrestricted wreath product \( G \wr \mathbb{Z} \). Denote by \( u \) a generator of \( \mathbb{Z} \), and by \( \{b_1, b_2, \ldots\} \) a generating set of \( G \). Choose a sparse-enough sequence of elements \( x_1, x_2, \ldots \) of \( \mathbb{Z} \), and define \( f: \mathbb{Z} \to G \) by \( f(x_i) = b_i, \) all other values being trivial. The group \( W \) is then \( W = \langle f, u \rangle \).

Let us spell out below what it means to be “sparse enough”. We write \( \mathbb{Z} \) additively. Since in fact by Proposition G.3 we only need to imbed \([t, G]\) in \( W \), we may set \( f(0) = t \) and a sufficient condition on the \( x_i \in \mathbb{Z} \) is that \( x_i \neq 0 \) for all \( i \); all \( x_i \) are distinct; and \( x_i + x_j \notin \{0, x_k\} \) for all \( i, j, k \in \mathbb{N} \). One then sees that \([f, f^{u^{-x_i}}]\) is a function supported only at 0, with value \([t, b_i]\) there. This defines the imbedding of \([t, G]\) in \( W \). □

Note that the group \( W \) contains the standard wreath product \( B \wr \mathbb{Z} \), so always has exponential growth.

The above construction shows that every countable solvable group imbeds in a 2-generated solvable group.

**Exercise G.6.** Show first that not every countable nilpotent group imbeds in a 2-generated nilpotent group.

Show then that every finitely generated nilpotent group imbeds in a 2-generated nilpotent group.

Here is a useful, small improvement on Theorem G.2:

**Theorem G.7.** *Let \( p \geq 5 \) be an integer. Then every countable group imbeds in a 2-generated group both of whose generators have order \( p \).*

**Proof.** Piggybacking on Propositions G.3 and G.4, it suffices to consider a 2-generated group \( G = \langle x, y \rangle \), and to imbed \([G, G]\) into a 2-generated group \( W \).
with generators of order \( p \). Write \( C_p = \langle t | t^p \rangle \), and define \( f : C_p \to G \) by

\[
f(1) = x, \quad f(t^{-1}) = y, \quad f(t^2) = y^{-1}x^{-1}, \quad f(g) = 1 \quad \text{for all other } g \in C_p.
\]

Consider then the group \( W = \langle t, ft \rangle \). It is clearly generated by two elements of order \( p \), since in \( (ft)^p \) all the coordinates contain some cyclic permutation of the product \( x \cdot y \cdot y^{-1}x^{-1} \). Now \( W \) contains \( f \), so it also contains \([f, ft]\), which is the function \( C_p \to G \) taking value \([x, y]\) at 1 and 1 elsewhere. Furthermore conjugating \([f, ft]\) by an arbitrary word in \( f \) and \( ft \), one obtains the function taking value an arbitrary conjugate of \([x, y]\) at 1; so \( W \) contains \([G, G]@1\).

**Exercise G.8.** Where have we used the assumption that \( p \geq 5 \)? Can you improve the above result to arbitrary \( p \geq 3 \)? Can you imbed any countable group into a group generated by an involution and an element of order \( p \)?

**G.2. Finite-valued permutational wreath products.** Our goal is, starting from a countable group \( B \) locally of subexponential growth, to construct a finitely generated group \( W \) of subexponential growth containing \( B \). We take inspiration from Neumann’s proof given above, with two modifications: first, we consider permutational wreath products rather than standard ones; secondly, we consider finite-valued permutational wreath products:

**Definition G.9.** Let \( H \) be a group acting on a set \( X \), and let \( H \) be a group. Their finite-valued permutational wreath product is the group \( H \wr_X G \), defined as the extension of functions \( X \to H \) with finite image by \( G \):

\[
H \wr_X G = \{ (\phi, g) \in H^X \times G : \#\phi(X) < \infty \}.
\]

\( \triangleleft \)

Note that it is a subgroup, because if \((\phi, g)^{-1}(\phi', g') = (\phi'', g^{-1}g')\) then \(\phi''(X) \subseteq \phi(X)^{-1}\phi'(X)\) is finite. Clearly, we have

\[
H \wr_X G \leq H \wr_X G \leq H \wr G.
\]

We also introduce a condition on imbeddings that guarantees control on growth:

**Definition G.10.** Let \( B \) be a group. A group \( G \) is called hyper-\( B \) if it is a directed union of finite extensions of finite powers of \( B \).

\( \triangleleft \)

Clearly, if a group \( B \) is locally of subexponential growth and a group \( G \) is hyper-\( B \), then \( G \) is also locally of subexponential growth. Indeed, for every finite subset \( S \) of \( G \), there exists a finite extension of a finite power of \( B \) that contains \( S \).

**Lemma G.11.** Let \( G \) be a hyper-\( B \) group, and let \( H \) be a hyper-\( G \) group. Then \( H \) is hyper-\( B \).

**Proof.** Consider \( h \in H \); then \( h \) belongs to a finite extension of a finite power of \( G \), which may be assumed of the form \( G \wr F \) for a finite group \( F \). Let us write \( h = \phi f \) with \( \phi : F \to G \) and \( f \in F \); then \( \phi(f) \) belongs for all \( f \in F \) to a finite extension of a finite power of \( B \), which can be assumed to be the same for all \( f \). This extension may be assumed to be of the form \( B \wr E \) for a finite group \( E \). It follows that \( h \) belongs to \( B \wr_{E \times F} (E \wr F) \), a finite extension of a finite power of \( B \); so \( H \) is hyper-\( B \).

\( \square \)

**Lemma G.12.** If \( H \) is a hyper-\( B \) group and \( U \) is locally finite, then \( H \wr \hat{X} \cdot U \) is a hyper-\( B \) group.
Proof. We first show that $H \wr U$ is hyper-$H$. By hypothesis, $U$ is a directed union of finite subgroups $E$. The partitions $\mathcal{P}_0$ of $U$ into finitely many parts also form a directed poset; and for every such partition $\mathcal{P}_0$ and every finite subgroup $E \leq U$ there exists a finite partition $\mathcal{P}$ of $U$ that is invariant under $E$ and refines $\mathcal{P}_0$, namely the wedge (= least upper bound) of all $E$-images of $\mathcal{P}_0$.

Consider now the directed poset of pairs $(E, \mathcal{P})$ consisting of finite subgroups $E \leq U$ and $E$-invariant partitions of $U$. Consider the corresponding subgroups $H \wr E \leq H \wr U$. If $(E, \mathcal{P}) \leq (E', \mathcal{P}')$ then $H \wr E$ is naturally contained in $H \wr E' \times E'$, so these subgroups of $H \wr U$ form a directed poset, which exhausts $H \wr U$.

It follows that $H \wr U$ is a hyper-$H$ group, and we are done by Lemma G.11.

G.3. Imbedding in the derived subgroup. Our main goal, in this section, is to prove the following proposition, which replaces Proposition G.3.

Proposition G.13. Let $B$ be a group. Then there exists a hyper-$B$ group $G$ such that $[G, G]$ contains $B$ as a subgroup.

If $B$ is infinite, then $G$ may furthermore be supposed to have the same cardinality as $B$.

Lemma G.14. Let $B$ be a group. Then there exists a subgroup $C$ of $B$, containing $[B, B]$, such that $B/C$ is torsion and $C/[B, B]$ is free abelian.

Proof. $B/[B, B] \otimes_\mathbb{Q} \mathbb{Q}$ is a $\mathbb{Q}$-vector space, hence has a basis, call it $X$. It generates a free abelian group $\mathbb{Z}X$ within $B/[B, B]$, whose full preimage in $B$ we call $C$. Then $B/C \otimes_\mathbb{Z} \mathbb{Q} = 0$ so $B/C$ is torsion.

We set up the following notation for the proof of Proposition G.13. We choose a subgroup $C \leq B$ as in Lemma G.14 and write $T := B/C$. We choose a basis $X$ of $C/[B, B]$, for every $x \in X$ we choose an element $b_x \in C$ representing it, and we define a homomorphism $\theta_x: C \to \langle b_x \rangle \leq B$, trivial on $[B, B]$, by $\theta_x(b_x) = b_x^{-1}$ and $\theta_x(b_y) = 1$ for all $y \neq x \in X$. In particular, we have for all $b \in C$

$$b \cdot \prod_{x \in X} \theta_x(b) \in [B, B]$$

and the product is finite.

We write $\pi: B \to T$ the natural projection, and define a section $\sigma$ of $\pi$ with the following property, which we single out as a lemma:

Lemma G.15. There exists a set-theoretic section $\sigma: T \to B$ such that, for every $t \in T$, the subset $\{ \sigma(tu)\sigma(u)^{-1} : u \in T \}[B, B]$ of $B/[B, B]$ is finite.

Proof. Since every abelian torsion group is the direct sum of its $p$-subgroups, we may first define the section on each of $T$’s $p$-components, and then extend it to $T$ multiplicatively (in any order).

We therefore suppose that $T$ is a $p$-group. Recall the notation $\Omega_n(T) = \{ t \in T : t^p = 1 \}$. Each quotient $V_n := \Omega_n(T)/\Omega_{n-1}(T)$ is a vector space over $\mathbb{F}_p$, and the homomorphism $t \mapsto t^p$ induces an injective linear map $V_n \to V_{n-1}$. Choose inductively subsets $X_1, X_2, \ldots$ of $T$ such that $X_n$ maps to a basis of $V_n$ and such that $t \mapsto t^p$ induces an injective map $X_n \to X_{n-1}$. Set $X = \bigcup_n X_n$, and give an arbitrary total order to $X$. Choose for each $x \in X$ a $\pi$-preimage $\sigma(x) \in B$. 

Since $T$ is torsion, every element $t \in T$ belongs to $\Omega_n(T)$ for some $n \in \mathbb{N}$, so can be uniquely written as a product $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with ordered $x_i \in X$ and $0 < \alpha_i < p$ for all $i$. Extend then $\sigma$ by $\sigma(t) = \sigma(x_1)^{\alpha_1} \cdots \sigma(x_n)^{\alpha_n}$.

Consider now $t \in T$, and write it in the form $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ as above. Extend \{x_1, \ldots, x_n\} to a finite subset $Y = \{x_1, \ldots, x_n, \ldots, x_s\}$ of $X$, by adding all $p^i$-th powers of all $x_i$ to it. The set
\[ T' := \{x_1^{\gamma_1} \cdots x_s^{\gamma_s} : 0 \leq \gamma_i < p\} \]
is then a finite subgroup of $T$. Consider next $u \in T$, and note that it may be written uniquely in the form $u = y_1^{\beta_1} \cdots y_m^{\beta_m}z$ with $y_i \in X \setminus Y$ and $z \in T'$. Then $tu = y_1^{\beta_1} \cdots y_m^{\beta_m}(zt)$ and this representation is unique; so $\sigma(tu)\sigma(u)^{-1}[B, B]$ belongs to the finite set \{\sigma(t')[B, B] : t' \in T'\}.

**Exercise G.16.** Rephrase Lemma G.15 in terms of the cohomology of $T$ with coefficients in $\mathbb{Z} \times X$.

Let $F$ be a locally finite group of cardinality $\#X$, and fix an imbedding of $X$ in $F \setminus \{1\}$. As a first step, we consider the group $G_0 = B^{\hat{\cdot}x} \cdot (T \times F)$, and define a map $\Phi_0 : B \to G_0$ as follows:

\[
\Phi_0(b) = (\phi, \pi(b), 1) \quad \text{with} \quad \phi(t, f) = \begin{cases} b & \text{if } f = 1, \\ \theta_f(\sigma(t)\sigma(t\pi(b))^{-1}) & \text{if } f \in X, \\ 1 & \text{otherwise.} \end{cases}
\]

**Lemma G.17.** The map $\Phi_0$ is well-defined and is an injective homomorphism.

*Proof.* To see that $\Phi_0$ is well-defined, note that the argument $\sigma(t)\sigma(t\pi(b))^{-1}$ belongs to $\ker(\pi) = C$, so that $\theta_f$ may be applied to it. Note then that, by Lemma G.15, the expression $\sigma(t)\sigma(t\pi(b))^{-1}$ takes finitely many values in $C/[B, B]$, so that $\phi(t, f)$ takes finitely many values for varying $t$ and fixed $f$. Finally, $\theta_f(\sigma(t)\sigma(t\pi(b))^{-1}) = 1$ except for finitely many values of $f \in X$. In summary, the function $\phi \in B^{T \times F}$ is such that $\phi(t, f)$ takes only finitely many values.

It is clear that $\Phi_0$ is injective: if $b \neq 1$ and $\Phi_0(b) = (\phi, \pi(b), 1)$ then $\phi(1, 1) = b \neq 1$. It is a homomorphism because all $\theta_f$ are homomorphisms. \hfill $\square$

**Lemma G.18.** We have $\Phi_0(C) \leq [G_0, G_0]$.

*Proof.* If $b \in [B, B]$ then clearly $\Phi_0(b) \in [G_0, G_0]$. Since $C$ is generated by $[B, B] \cup \{b_x\}_{x \in X}$, it suffices to consider $b = b_x$.

We define $g \in G_0$ by
\[
g = (\psi, 1, 1) \quad \text{with} \quad \psi(t, f) = \begin{cases} b_x & \text{if } f = 1, \\ 1 & \text{otherwise.} \end{cases}
\]

Then $\Phi_0(b_x) = (\phi, 1, 1)$ with $\phi(t, 1) = b_x$ and $\phi(t, x) = b_x^{-1}$, all other values being trivial, according to (21): so, as was to be shown,
\[
\Phi_0(b_x) = (\phi, 1, 1) = (x^{-1}\psi^{-1} \cdot \psi, 1, 1) = [(1, 1, x^{-1}), g] \in [G_0, G_0].
\]

We finally define
\[
G = G_0^{\hat{\cdot}x} \cdot (\mathbb{Q}/\mathbb{Z})
\]
and a map $\Phi : B \to G$ by
\[
\Phi(b) = (\phi, 0) \quad \text{with} \quad \phi(r) = \Phi_0(b) \quad \text{for all } r \in \mathbb{Q}/\mathbb{Z}.
\]
Lemma G.19. The map $\Phi$ is an injective homomorphism, and $\Phi(B) \leq [G,G]$.

Proof. Clearly $\Phi$ is an injective homomorphism, since $\Phi_0$ is an injective homomorphism by Lemma G.17.

We identify $\mathbb{Q}/\mathbb{Z}$ with $\mathbb{Q} \cap [0,1)$. For every $n \in \mathbb{N}$, consider the map $\Psi_n : B \to G$ defined by

$$\Psi_n(b) = (\phi,0) \text{ with } \phi(r) = \begin{cases} \Phi_0(b) & \text{if } r \in [0,1/n), \\ 1 & \text{otherwise}; \end{cases}$$

so $\Phi = \Psi_1$. We know from Lemma G.18 that $\Psi_n(C)$ is contained in $[G,G]$.

Consider now $b \in B$. Since $B/C$ is torsion, there exists $n \in \mathbb{N}$ such that $b^n \in C$. We define $g \in G$ by

$$g = (\psi,0) \text{ with } \psi(r) = \Phi_0(b)^{[\ell_n]} \text{ for } r \in [0,1) \cap \mathbb{Q}.$$

Let us write $h = \Phi_0(b)$, and consider the element $[(1,1/n),g] \cdot \Psi_n(b^n) = (\phi,0)$.

If $r \in [0,1/n)$ then $\phi(r) = \psi(r) - 1/n)^{-1} \psi(h^n) = h$, while if $r \in [1/n,1)$ then $\phi(r) = \psi(r) - 1/n)^{-1} \psi(r) = h$; therefore

$$\Phi(b) = [(1,1/n),g] \cdot \Psi_n(b^n) \in [G,G]. \quad \square$$

Proof of Proposition G.13. The first assertion is simply Lemma G.19. For the last one: if $B$ is infinite, we wish to find a subgroup $H$ of $G$ with the same cardinality as $B$, such that $\Phi$ maps into $[H,H]$. For each $b \in B$, choose a finite subset $S_b$ of $G$ such that $\Phi(b) \in \langle S_b \rangle$, and a subgroup $G_b$, containing $S_b$, that is virtually a finite power of $B$. Consider the group $H$ generated by the union of all the $G_b$. As soon as $B$ is infinite, all $G_b$ have the same cardinality as $B$, and so does $H$. \quad \square

G.4. Spreading, stabilizing, rectifiable sequences. We also extend Proposition G.4 by replacing the wreath product $G \wr \mathbb{Z}$ by permutational wreath products of the form $G \wr \langle P \rangle$ for a group $P$ acting on a set $X$, and considering subgroups of the form $W = \langle f,P \rangle$ for a function $f : X \to G$.

We already encountered in Proposition F.8 sufficient conditions for such a group $W$ to be of subexponential growth, in case $f$ is finitely generated. As we shall see, the group $W$ may also have subexponential growth if $f$ is infinitely supported, but its support is sufficiently sparse, in a sense that we describe now.

In this section, we assume a finitely generated group $P = \langle S \rangle$ acting on the right on a set $X$ has been fixed. We use the same notation for $X$ as a set and as a Schreier graph, namely as the graph with vertex set $X$, and with for all $x \in X, s \in S$ an edge labelled $s$ from $x$ to $xs$, see Definition A.8. We denote by $d$ the path metric on this graph.

Definition G.20. A sequence $(x_0, x_1, \ldots)$ in $X$ is spreading if for all $R$ there exists $N$ such that if $i, j \geq N$ and $i \neq j$ then $d(x_i, x_j) \geq R$. \quad \triangle

Example G.21. If all $x_i$ lie in order on a geodesic ray starting from $x_0$ (for example if $X$ itself is a ray starting from $x_0$) and for all $i$ we have $d(x_0, x_{i+1}) \geq 2d(x_0, x_i)$, then $(x_i)$ is spreading.

Exercise G.22. Show that a sequence $(x_0, x_1, \ldots)$ in $X$ locally stabilises if for all $R$ there exists $N$ such that if $i \neq j$ and $i \geq N$ then $d(x_i, x_j) \geq R$.

Definition G.23. A sequence $(x_i)$ in $X$ locally stabilises if for all $R$ there exists $N$ such that if $i, j \geq N$ then the $S$-labelled radius-$R$ balls centered at $x_i$ and $x_j$ in $X$ are isomorphic as labelled graphs. \quad \triangle
**Definition G.24.** A sequence of points \((x_i)\) in \(X\) is rectifiable if for all \(i, j\) there exists \(g \in P\) with \(x_i g = x_j\) and \(x_k g \neq x_\ell\) for all \(k \neq \{i, \ell\}\). \(\triangle\)

For example, if \(X = \mathbb{Z}\) and \(P = \mathbb{Z}\) acting by translations, then \(\Sigma = \{2^i : i \in \mathbb{N}\}\) is rectifiable, since \(2^i - 2^i = 2^\ell - 2^k\) only has trivial solutions \(i = k, j = \ell\) and \(i = j, k = \ell\). It is also spreading and locally stabilizing.

**Exercise G.25.** Show that the sequence \(\Sigma = (x_i) \subseteq X\) is rectifiable if and only if for all \(i, j\) there exists \(g \in P\) with \(x_i g = x_j\) and \(\Sigma \cap \Sigma g \subseteq \{x_j\} \cup \text{fixed points}(g)\).

**Definition G.26.** Fix a point \(z \in X\). A sequence \((g_i)\) in \(P\) is parallelogram-free at \(z\) if, for all \(i, j, k, \ell\) with \(i \neq j\) and \(j \neq k\) and \(k \neq \ell\) and \(\ell \neq i\) one has \(z g_i^{-1} g_j g_k^{-1} g_\ell \neq z\).

**Lemma G.27.** If \(z \in X\) and \((g_i)\) is parallelogram-free at \(z\), then \((z g_i^{-1})\) is a rectifiable sequence in \(X\).

**Proof.** Set \(x_i = z g_i^{-1}\) for all \(i \in \mathbb{N}\). Given \(i, j \in \mathbb{N}\), consider \(g = g_i g_j^{-1}\), so \(x_i g = x_j\). If furthermore we have \(x_k g = x_\ell\), then we have \(z g_k^{-1} g_j g_i^{-1} g_\ell = z\), so either \(k = i\), or \(i = j\) which implies \(k = \ell\), or \(j = \ell\) which implies \(k = i\), or \(\ell = k\). In all cases \(k \in \{i, \ell\}\) as was to be shown. \(\square\)

It is clear that, if \(P\) is finitely generated and \(X\) is infinite, then it admits spreading and locally stabilizing sequences. Indeed every sequence contains a spreading subsequence, and every sequence contains a stabilizing subsequence, and a subsequence of a spreading or locally stabilizing sequence is again spreading, respectively locally stabilizing.

We will content ourselves with the following rectifiable sequences, based on the Grigorchuk groups \(G_\omega = \langle a, b, c, d \rangle\), with \(\omega \in \Omega\), for example the first Grigorchuk group \(G_{012}\). Recall the description of \(G_\omega\) from \([13]\) and in particular the Schreier graph of its action on \(1^\mathbb{Z}\) in \([12]\). We construct explicitly a spreading, locally stabilizing, rectifiable sequence for the action of \(G_\omega\) on \(X\): for all \(i \in \mathbb{N}\), let us define \(x_i = 1^\mathbb{Z}0^i\),

the point at distance \(2^i\) from the origin on the Schreier graph.

**Lemma G.28.** For all \(\omega \in \Omega\) containing infinitely many \(0, 1, 2\)'s, and for all \(i, j \in \mathbb{N}\),

1. the marked balls of radius \(2^{\min(i,j)}\) in \(X\) around \(x_i\) and \(x_j\) coincide;
2. the distance \(d(x_i, x_j)\) is \(|2^i - 2^j|\);
3. there exists \(g_{i,j} \in G_\omega\) of length \(|2^i - 2^j|\) with \(x_i g_{i,j} = x_j\) and \(x_k g_{i,j} \neq x_\ell\) for all \((k, \ell) \neq (i, j)\).
Proof. (1), (2) Consider the map $\theta_\omega$ from Equation (16). It maps the stabilizer of $1^\omega$ in $G_\omega$ to the stabilizer of $1^\omega$ in $G_\omega$, and therefore defines a self-map of $X$ by sending $1^\omega g$ to $1^\omega \theta_\omega(g)$. A direct calculation shows that it sends $x \in X$ to $x0$.

Since $\theta_\omega$ is 2-Lipschitz on words of even length in $\{a, b, c, d\}$, it maps the ball of radius $n$ around $x$ to the ball of radius $2n$ around $x0$. Its image is in fact a net in the ball of radius $2n$: two points at distance 1 in the ball of radius $n$ around $x$ will be mapped to points at distance 1 or 3 in the image, connected either by a segment over $a$ or by a path over $a$ for some $\{x,y\} \subseteq \{b,c,d\}$. In particular, the $2^n$-neighbourhoods of the balls about the $x_m$ coincide for all $m \geq n$.

(3) Note, first, that there exists $g_{i,j}$ with $x_i g_{i,j} = x_j$, because the rays ending in $1^\omega$ form a single orbit. Note, also, that we have $x_k g_{i,j} = x_l$ for either finitely many $(k, \ell) \neq (i, j)$ or for all but finitely many $(k, \ell)$, because there is a level $N$ at which the decomposition of $g_{i,j}$ consists entirely of generators; if the entry at $0^N$ of $g_{i,j}$ is trivial or ‘d’ then all but finitely many of the $x_k$ are fixed; while otherwise (up to increasing $N$ by at most one) we may assume it is an ‘a’; then $0^N g_{i,j} = 0^N 1$, so $x_k \neq x_t$ for all $k > N + 1$.

We use the following property of the Grigorchuk groups $G_\omega$: for every finite sequence $u \in \{0, 1\}^*$ there exists an element $h_u \in G_\omega$ whose fixed points are precisely those sequences in $\{0, 1\}^\omega$ that do not start with $u$. One may take for $h_u$ the element $[a, b], [a, c]$ or $[a, d]$ inside the copy of the branching subgroup of $G_\omega$ that acts on $\{0, 1\}^\omega u$, see Proposition [E.12]

If the entry at $0^N$ of $g_{i,j}$ is trivial, then we multiply $g_{i,j}$ with $h_{0^M} = 0^M 1$ for some $M > \max(N, i)$, so as to fall back to the second case.

Then, for each pair $(k, \ell) \neq (i, j)$ with $x_k g_{i,j} = x_\ell$, we multiply $g_{i,j}$ with $h_{0^M}$, so as to destroy the relation $x_k g_{i,j} = x_\ell$. The resulting element $g_{i,j}$ satisfies the required conditions. □

G.5. Subexponential growth of wreath products. The next step in the proof is an argument controlling the growth of a subgroup of the form $W = \langle P, f \rangle \leq G \wr_X P$, for a function $f : X \to G$ with sparse-enough (but infinite!) support.

We select a finitely generated group $P$ of subexponential growth acting on a set $X$ with sublinear inverted orbit growth (see Definition [F.4] recall, from Exercise [F.5] that the property of having sublinear orbit growth is independent of a choice of basepoint) and subexponential inverted orbit choice growth. These are the hypotheses for Corollary [F.9] which guarantee that $G \wr_X P$ has subexponential growth as soon as $G$ has subexponential growth. The main example we have in mind is a Grigorchuk group $P = G_\omega$ acting on $X = 1^\omega P$, for $\omega \in \{0, 1, 2\}^\omega$ containing infinitely many times each symbol, see Lemma [G.28]

We also assume that there are rectifiable sequences in $X$, and (using the results of the previous section) we fix a rectifiable, spreading, locally stabilizing sequence $(x_0, x_1, \ldots)$ of elements of $X$.

Finally, we fix a countable group $G$, and a finite or infinite sequence of elements $(b_1, b_2, \ldots)$ generating $G$.

Definition G.29. For an increasing finite or infinite sequence $0 \leq n(1) < n(2) < \ldots$ of integers, define $f : X \to G$ by

$$f(x_{n(1)}) = b_1, \quad f(x_{n(2)}) = b_2, \quad \ldots, \quad f(x) = 1 \quad \text{for other } x.$$
The group $W_{(n)} = W_{n(1),n(2),\ldots}$ is then defined as the subgroup $\langle P, f \rangle$ of the unrestricted wreath product $G \ltimes X P$.

If all but finitely many $b_i$ are trivial, then $W_{(n)}$ has subexponential growth since then $G$ is finitely generated and $W_{(n)}$ is a subgroup of $G \ltimes X P$. However, if $(n(i))$ is sparse enough then $W_{(n)}$ may have subexponential growth even if $f$ has infinite support:

**Proposition G.30.** If $G$ has locally subexponential growth, then there exists an infinite sequence $(n(i))$ such that the group $W_{(n)}$ has subexponential growth.

The proof of Proposition G.30 follows from a stronger, and independently interesting, statement: arbitrarily large balls in $W_{(n)}$ are approximable by groups of the form $W_{(n(1),\ldots,n(i))}$, which have subexponential growth by the remark above:

**Proposition G.31.** Assume that the sequence $(x_i)$ in $X$ is spreading and locally stabilizing, and that all elements $b_i$ have the same order.

Then for every increasing sequence $(m(i))$ there exists an increasing sequence $(n(i))$ with the following property: the ball of radius $m(i)$ in $W_{(n)}$ coincides with the ball of radius $m(i)$ in $W_{n(1),n(2),\ldots,n(i)}$, via the natural identification $P \rightarrow P, f \rightarrow f$ between $W_{(n)}$ and $W_{n(1),\ldots,n(i)}$.

Furthermore, the term $n(i)$ depends only on the previous terms $n(1),\ldots,n(i-1)$, on the initial terms $m(1),\ldots,m(i)$, and on the ball of radius $m(i)$ in the subgroup $\langle b_1,\ldots,b_{i-1} \rangle$ of $G$.

**Proof.** Choose $n(i)$ such that $d(x_j,x_k) \geq m(i)$ for all $j \neq k$ with $k \geq n(i)$, and such that the balls of radius $m(i)$ around $x_{n(i)}$ and $x_j$ coincide for all $j > n(i)$.

Consider then an element $h \in W_{(n)}$ in the ball of radius $m(i)$, and write it in the form $h = (c,g)$ with $c: X \rightarrow G$ and $g \in P$. The function $c$ is a product of conjugates of $f$ by words of length $< m(i)$. Its support is therefore contained in the union of balls of radius $m(i) - 1$ around the $x_j$, with $j$ either $\geq n(i)$ or of the form $n(k)$ for $k < i$. In particular, the entries of $c$ are in $\langle b_1,\ldots,b_{i-1} \rangle \cup \bigcup_{j \geq 1} \langle b_j \rangle$. For $j > n(i)$, the restriction of $c$ to the ball around $x_j$ is determined by the restriction of $c$ to the ball around $x_{n(i)}$, via the identification $b_i \mapsto b_j$, because the neighbourhoods in $X$ coincide and all cyclic groups $\langle b_j \rangle$ are isomorphic.

It follows that the element $h \in W$ is uniquely determined by the corresponding element in $W_{n(1),\ldots,n(i)}$. □

**Proof of Proposition G.30.** Let $Z = \langle z \rangle$ be a cyclic group whose order (possibly $\infty$) is divisible by the order of all the $b_i$’s. We replace $G$ by $G \times Z$ and each $b_i$ by $b_i z$, so as to guarantee that all generators in $G$ have the same order.

Let $\epsilon_i$ be a decreasing sequence tending to 1. We now construct a sequence $(m(i))$ inductively, and obtain the sequence $(n(i))$ by Proposition G.31 making always sure that $m(i)$ depends only on $m(j), n(j)$ with $j < i$.

Denote by $v_i$ the growth function of the group $W_{n(1),\ldots,n(i)}$. Since the group $W_{n(1),\ldots,n(i)}$ is contained in $G \ltimes X P$, it has subexponential growth. Therefore, there exists $m(i)$ be such that

$$v_i(m(i)) \leq \epsilon_i^{m(i)}.$$

By Proposition G.31, the terms $n(i+1), n(i+2), \ldots$ can be chosen in such a manner that the balls of radius $m(i)$ coincide in $W_{(n)}$ and $W_{n(1),\ldots,n(i)}$. 

\[ \text{Proof.} \]
Denote now by \( w \) the growth function of \( W_{(n)} \). We then have \( w(m(i)) \leq \epsilon_i^{m(i)} \) for all \( i \in \mathbb{N} \). Therefore,

\[
w(R) \leq \epsilon_i^{R+m(i)} \text{ for all } R > m(i),
\]

so \( \lim \sup (Fekete \ [22]) \) Lemma H.1 for all \( i \in \mathbb{N} \). Thus the growth of \( W_{(n)} \) is subexponential. \( \square \)

Finally, the rectifiability of the sequence \( (x_i) \) guarantees that functions with singleton support and arbitrary values in \( [G, G] \) belong to \( W_{(n)} \) for all sequences \( n \):

**Lemma G.32.** If the sequence \( (x_i) \) is rectifiable, then \( [W_{(n)}, W_{(n)}] \) contains \( [G, G] \) as a subgroup for all choices of \( n = n(1) < n(2) < \cdots \).

**Proof.** We denote by \( \iota : G \rightarrow G^X \times P \) the imbedding of \( G \) mapping the element \( b \in G \) to the function \( X \rightarrow G \) with value \( b \) at \( x_0 \) and 1 elsewhere. We abbreviate \( W = W_{(n)} \). We shall show that \( [W, W] \) contains \( \iota([G, G]) \). For this, denote by \( H \) the subgroup \( \iota([G, G]) \cap [W, W] \).

We first consider an elementary commutator \( g = [b_i, b_j] \). Let \( g_i, g_j \in P \) respectively map \( x_i, x_j \) to \( x_0 \), and be such that \( g_i g_j^{-1} \) maps no \( x_k \) to \( x_\ell \) with \( k \neq \ell \), except for \( x_i g_j^{-1} = x_j \). Consider \( [f^{\gamma_i}, f^{\gamma_j}] \in [W, W] \); it belongs to \( G^X \), and has value \( [b_i, b_j] \) at \( x_0 \) and is trivial elsewhere, so equals \( \iota(g) \) and therefore \( \iota(g) \in H \).

We next show that \( H \) is normal in \( G^X \). For this, consider \( h \in H \). It suffices to show that \( h^{\iota(b_i)} \) belongs to \( H \) for all \( i \). Now \( h^{\iota(b_i)} = h^{f^{\gamma_i}} \) belongs to \( H \), and we are done. \( \square \)

We are now ready to complete the proof of Theorem [G.1]. By Proposition [G.13] the countable, locally subexponentially growing group \( B \) imbeds in \( [G, G] \) for a countable, locally subexponentially growing group \( G \). Let \( (b_1, b_2, \ldots) \) be a generating set for \( G \). By Proposition [G.30] there exists an increasing sequence \( (n(i)) \) such that the group \( W = W_{(n)} \) has subexponential growth. By Lemma [G.32] \( [G, G] \) imbeds in \( [W, W] \), so \( B \) imbeds in \( [W, W] \) and we are done.

**Exercise G.33.** Give examples of sequences \( (x_i) \) that are only spreading, or only stabilizing, and such that \( W_{(n)} \) has exponential growth, even when the sequence \( (n(i)) \) grows arbitrarily fast.

**H. Groups of non-uniform exponential growth**

Recall that \( v_{G,S}(R) \) denotes the growth function of a group \( G \) generated by a finite set \( S \). The *volume entropy* of \( (G, S) \) is

\[
\lambda_{G,S} := \lim_{R \to \infty} \frac{\log v_{G,S}(R)}{R}.
\]

The limit exists because the function \( v_{G,S} \) is submultiplicative (namely, \( v_{G,S}(R_1 + R_2) \leq v_{G,S}(R_1)v_{G,S}(R_2) \)). Indeed, apply the following lemma to the sequence \( (\log v_{G,S}(n))_{n \geq 1} \):

**Lemma H.1 (Fekete [22]).** Let \( (a_n) \) be a subadditive sequence: \( a_{n+m} \leq a_n + a_m \) for all \( m, n \geq 1 \). Then \( \lim a_n/n \) exists and equals \( \inf a_n/n \).

**Proof.** Set \( A = \inf a_n/n \) and consider any \( B > A \). Choose \( k \geq 1 \) such that \( a_k/k < B \). Every \( n \in \mathbb{N} \) may be written in the form \( n = rk + s \) with \( r \in \mathbb{N} \) and \( s \in \{0, \ldots, k-1\} \). Thus

\[
\frac{a_n}{n} = \frac{a_{rk+s}}{n} \leq \frac{ra_k + a_s}{n} = \frac{a_k n - s}{k n} + \frac{a_s}{n},
\]

for all \( n \geq k \). Therefore, \( \lim a_n/n \) exists and equals \( \inf a_n/n \).
so \( \limsup_{n \to \infty} a_n / n \leq a_k / k \leq B. \) Since \( B > A \) was arbitrary, \( \limsup a_n / n \leq L \) and we are done. \( \square \)

Furthermore, the following are equivalent: \( G \) has exponential word growth; \( \lambda_{G,S} > 0 \) for some generating set \( S; \) and \( \lambda_{G,S} > 0 \) for all generating sets \( S. \)

Let \( G \) be a group of exponential growth. Note that, even though \( \lambda_{G,S} > 0 \) for all \( S, \) one might have \( \lambda_{G,S_i} \to 0 \) along a sequence of generating sets \( S_i. \) It is easy to see that this cannot happen for \( G \) a free group of rank \( k \geq 2; \) indeed then each \( S_i \) contains a subset of cardinality \( k \) generating a free subgroup, so \( \lambda_{G,S_i} \geq \log(2k - 1) > 0 \) for all \( i. \) Let us say that a finitely generated group \( G \) has uniform exponential growth if \( \inf_S \lambda_{G,S} > 0, \) and non-uniform exponential growth if \( \lambda_{G,S} > 0 \) for all \( S \) yet \( \inf_S \lambda_{G,S} = 0. \)

The existence of groups of non-uniform exponential growth is asked by Gromov in \([35, \text{Remarque 5.12}]; \) see \([38] \) for a survey. There have been quite a few positive results: Osin showed in \([64] \) that virtually solvable groups have uniform exponential growth unless they are virtually nilpotent; Eskin, Mozes and Oh obtained the same result in \([20] \) for finitely generated linear groups in characteristic 0; Koubi showed in \([49] \) that word-hyperbolic groups have exponential growth unless they are virtually cyclic.

A Golod-Shafarevich group is a residually-p group such that the associated Hopf algebra \( \bigoplus_{n \geq 0} \mathbb{z}^n / \mathbb{z}^{n+1} \) from \([5,t] \) has exponential growth; equivalently, the Lie algebra \( \bigoplus_{n \geq 1} \gamma_n(G)/\gamma_{n+1}(G) \) from \([13] \) has exponential growth. Since by Proposition \([1.6] \) the growth of \( kG \) is always a lower bound for the growth of \( G, \) such groups have uniformly exponential growth.

Among groups \( G \) of uniform exponential growth, one may ask whether the infimal entropy \( \inf_{(S) = G} \lambda_{G,S} \) is realised. Recall that a group \( G \) is Hopfian if it is not isomorphic to a proper quotient of itself. Sambusetti proves in \([69] \) that if \( G \) is a free product \( G = G_1 \ast G_2 \) with \( G_1 \) non-Hopfian and \( G_2 \) non-trivial, then \( \inf_{(S) = G} \lambda_{G,S} \) is not attained.

It was widely suspected, since the appearance of groups of intermediate growth, that examples of groups of non-uniform exponential growth should exist. The first examples of groups of non-uniform exponential growth were exhibited by Wilson, see \([81] \). We now give a simple construction showing that such groups abound:

**Theorem H.2** ([2, Theorem E]). Every countable group may be imbedded in a group of non-uniform exponential growth.

Furthermore, let \( \eta_+ \approx 2.46 \) denote the positive root of the polynomial \( T^3 - T^2 - 2T - 4. \) Then the group \( W \) in which the countable group imbeds may be required to have the following property: there is a constant \( K \) such that, for all \( R > 0, \) there exists a generating set \( S \) of \( W \) with

\[
v_{W,S}(r) \leq \exp(K r^{\log 2 / \log \eta_+}) \text{ for all } r \in [0, R].
\]

**Proof.** Let \( B \) be a countable group. By Theorem \([G.7] \) one may imbed \( B \) into a group \( G \) generated by two elements \( s, t \) of order 5. Without loss of generality, assume that \( G \) has exponential growth (if needed, replace first \( B \) by \( B \times F_2 \)).

The group \( W \) in which \( G \) imbeds is the wreath product \( G \wr_{\mathbb{z}} G_{012} \) of \( G \) with the first Grigorchuk group. We also consider \( A = C_5 \times C_5 = \langle s', t' \rangle, \) and the wreath product \( W' = A \wr_{\mathbb{z}} G_{012}. \)
We consider the points $x_0 = 0^x$ and $x_i = 0^x 10^i$ for all $i \geq 1$ in the Schreier graph $X$, and the generating sets $S_i = \{a, b, c, d, s@x_0, t@x_i\}$ of $W$ and $S' = \{a, b, c, d, s'@x_0, t'@x_0\}$ of $W'$.

Note that the sequence $(x_i)$ is spreading and locally stabilizing (see Definitions G.20 and G.23); better, for all $R$ the radius-$R$ balls around $x_0$ and $x_i$ in $X$ are isomorphic as labelled graphs for all $i$ large enough, because the action on a sequence $\in \{0, 1\}^\mathbb{Z}$ of an element of $G_{012}$ of length $R$ depends only on the last $\lfloor \log_2(R) \rfloor$ symbols of the sequence.

We now claim that, for all $R \in \mathbb{N}$, there exists $i$ such that the balls of radius $R$ in the Cayley graphs of $(W, S_i)$ and $(W', S'_i)$ coincide. By Theorem F.3 there exists a constant $K$ such that, for all $R \in \mathbb{N}$, the ball of radius $R$ in the Cayley graph of $(W', S'_i)$ has cardinality $v_{W',S'_i}(R) \leq \exp(K R^{\log 2/\log \eta_i})$. Assuming the claim, we get $v_{W,S_i}(R) \leq \exp(K R^{\log 2/\log \eta_i})$, from which the second claim of the theorem follows.

For every $\varepsilon > 0$ there exists $R$ such that $K R^{\log 2/\log \eta_i} < \varepsilon R$, so $v_{W,S_i}(R) \leq \exp(\varepsilon R)$ and $\Delta_{W,S_i} \leq \varepsilon$ for some $i$. It follows then that $W$ has non-uniform exponential growth.

It remains to prove the claim. Given $R \in \mathbb{N}$, let $i$ be large enough so that the distance between $x_0$ and $x_i$ in $X$ is at least $2R$. Consider a word $w$ in $S_i$ of length $\leq R$, and let $w'$ be the corresponding word in $S'$ obtained by replacing $s@x_0, t@x_0$ respectively by $s'@x_0, t'@x_0$. We show that $w$ represents the identity in $W$ if and only if $w'$ represents the identity in $W'$.

Write $w = (c, g)$ in $W$, with $c: X \to G$ and $g \in G_{012}$. Similarly, write $w' = (c', g)$ in $W'$, with $c': X \to A$ and the same $g \in G_{012}$. Note that the support of $c$ is contained in the union of the balls of radius $R$ around $x_0$ and $x_i$, and these balls are isomorphic and disjoint. Therefore, $c$ can be written in the form $c = c_1 c_2$, with $c_1: X \to \langle s \rangle$ and $c_2: X \to \langle t \rangle$, and $c_1, c_2$ have disjoint support so they commute. The function $c'$ may correspondingly be written as $c' = c'_1 c'_2$ with $c'_1: X \to \langle s' \rangle$ obtained by composing $c_1$ with the isomorphism $\langle s \rangle \to \langle s' \rangle$, and $c'_2: X \to \langle t' \rangle$ obtained by composing the isomorphism from the radius-$R$ ball around $x_0$ to the radius-$R$ ball around $x_i$, the map $c_2$, and the isomorphism $\langle t \rangle \to \langle t' \rangle$. Therefore, $c' = 1$ if and only if $c = 1$, so the balls in $W$ and $W'$ are isomorphic.

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