Subset-lex: did we miss an order?

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Abstract

We generalize a well-known algorithm for the generation of all subsets of a set in lexicographic order with respect to the sets as lists of elements (subset-lex order). We obtain algorithms for various combinatorial objects such as the subsets of a multiset, compositions and partitions represented as lists of parts, and for certain restricted growth strings. The algorithms are often loopless and require at most one extra variable for the computation of the next object. The performance of the algorithms is very competitive even when not loopless. A Gray code corresponding to the subset-lex order and a Gray code for compositions that was found during this work are described.

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Figure 1: Two lexicographic orders for the subsets of a 5-element set: ordering using the sets as lists of elements (left) and ordering using the characteristic words (right). Dots are used to denote zeros in the characteristic words.

1 Two lexicographic orders for binary words

Two simple ways to represent subsets of a set \( \{0, 1, 2, \ldots, n-1\} \) are as lists of elements and as the characteristic word, the binary word with ones at the positions corresponding to the elements in the subset. Sorting all subsets by list of characteristic words gives the right columns of figure 1, sorting by the list of elements gives the left columns. We will concern ourselves with the generalization of the ordering by the lists which we call \textit{subset-lex} order.

The algorithms for computing the successor or predecessor of a (nonempty) subset \( S \) in \textit{subset-lex} order are well-known\(^1\). In the following let \( z \) be the last, and \( y \) the next to last element in \( S \). We call an element minimal or maximal if it is respectively the smallest or greatest element in the superset

\begin{algorithm}[Next-SL2] Compute the successor of the subset \( S \).
\begin{enumerate}
  \item If there is just one element, and it is maximal, stop.
  \item If \( z \) is not maximal, append \( z + 1 \).
  \item Otherwise remove \( z \) and \( y \), then append \( y + 1 \).
\end{enumerate}
\end{algorithm}

\begin{algorithm}[Prev-SL2] Compute the predecessor of the subset \( S \).
\begin{enumerate}
  \item If there is just one element, and it is minimal, stop.
  \item If \( z \) is not minimal, append \( z - 1 \).
  \item Otherwise remove \( z \) and \( y \), then append \( y - 1 \).
\end{enumerate}
\end{algorithm}

\(^1\)For example, an algorithm for the \( k \)-subsets of an \( n \)-set is given in \textbar{} Algorithm LEXSUB, p.18\textbar{}, see section \([\text{?}]\) for an implementation.
2. If \( z - 1 \in S \), remove \( z \).

3. Otherwise remove \( z \), append \( z - 1 \) and the maximal element.

C++ implementations of all algorithms discussed here are given in the FXT library \[1\]. Here and elsewhere we give slightly simplified versions of the actual code, like omitting modifiers such as public, protected, and private. The type ulong is short for unsigned long.

```cpp
// FILE: src/comb/subset-lex.h
class subset_lex
// Nonempty subsets of the set \{0,1,2,...,n-1\} in subset-lex order.
// Representation as list of parts.
// Loopless generation.
{
ulong n_; // number of elements in set, should have n>=1
ulong k_; // index of last element in subset
ulong n1_; // == n - 1 for n >1, and == 0 for n==0
ulong *x_; // x[0...k-1]: subset of \{0,1,2,...,n-1\}
subset_lex(ulong n)
// Should have n>=1,
// for n==0 one set with the element zero is generated.
{
    n_ = n;
    n1_ = (n_ ? n_ - 1 : 0);
    x_ = new ulong[n_ + (n_==0)];
    first();
}
ulong first()
{
k_ = 0;
x_[0] = 0;
return k_ + 1;
}
ulong last()
{
k_ = 0;
x_[0] = n1_;
return k_ + 1;
}
ulong next()
// Return number of elements in subset.
// Return zero if current is last.
{
    if ( x_[k_] == n1_ ) // last element is max?
        if ( k_==0 ) return 0;
    --k_; // remove last element
    x_[k_] += 1; // increment last element
}
else // add next element from set:
{
    ++k_; 
    x_[k_] = x_[k_-1] + 1;
}
return k_ + 1;
}
ulong prev()
// Return number of elements in subset.
// Return zero if current is first.
{
```
26 if ( k_ == 0 ) // only one element ?
27 {
28   if ( x_[0]==0 ) return 0;
29   x_[0] -= 1; // decrement last (and only) element
30   ++k_;
31   x_[k_] = n1_; // append maximal element
32 }
33 else
34 {
35   // remove last element
36   if ( x_[k_] == x_[k_-1] + 1 ) --k_;
37   else
38   {
39     x_[k_] -= 1; // decrement last element
40     ++k_;
41     x_[k_] = n1_; // append maximal element
42   }
43   }
44 return k_ + 1;
45 }

A program demonstrating usage of the class is
1 // FILE: demo/comb/subset-lex-demo.cc
2 long n = 6;
3 subset_lex S(n);
4 do
5 { // visit subset
6   }
7 while ( S.next() );

2 Subset-lex order for multisets

For the internal representation of the subsets of a multiset, we could use lists of pairs \((e, m)\) where \(e\) is the element and \(m\) its multiplicity. This choice would lead to loopless algorithms as will become clear in a moment. The disadvantage of this representation, however, is that the generalizations we will find would have a slightly more complicated form. We will instead use generalized characteristic words, where nonzero entries can be at most the multiplicity of the element in question. For a multiset \([0^{m(0)}, 1^{m(1)}, \ldots, k^{m(k)}]\) these are the mixed radix numbers with radix vector \([m(0)+1, m(1)+1, \ldots, m(k)+1]\). The subsets of the set \(\{0^1, 1^2, 2^2, 3^3\}\) in subset-lex order are shown in figure 2.

The algorithms for computing the successor or predecessor of a (nonempty) subset \(S\) of a multiset in subset-lex are now given. In the following let \(z\) be the last element in \(S\).

Algorithm 3 (Next-SL). Compute the successor of the subset \(S\).
1. If the multiplicity of \(z\) is not maximal, increase it; return.
2. If \(z\) is not maximal, append \(z + 1\) with multiplicity 1; return.
3. If \(z\) is the only element, stop.
4. Remove \(z\) and decrease the multiplicity of next to last nonzero element \(y\), then append \(y + 1\) with multiplicity 1.

The description omits the case of the empty set, the implementation takes care of this case by initially pointing to the leftmost digit of the characteristic word, which is zero. With the first call to the routine it is changed to 1.

Algorithm 4 (Prev-SL). Compute the predecessor of the subset \(S\).
Figure 2: Subsets of the multiset \{0^1, 1^2, 2^2, 3^3\} in subset-lex order. Dots are used to denote zeros in the (generalized) characteristic words.
1. If the set is empty, stop.
2. Decrease the multiplicity of $z$.
3. If the new multiplicity is nonzero, return.
4. Increase the multiplicity of $z - 1$ and append the maximal element.

Clearly both algorithms are loopless for the representation using a list of pairs. With the representation as characteristic words algorithm 3 involves a loop in the last step, a scan for the next to last nonzero element. This could be prevented by maintaining an additional list of positions where the characteristic word is nonzero. Algorithm 4 does stay loopless, the position of the maximal element does not need to be sought. Our implementations will always use a variable holding the position of the last nonzero position in the characteristic word. This is called the “current track” in the implementations.

We will not make the equivalents of algorithm 3 loopless, as the maintenance of the additional list would render algorithm 3 slower (at least as long as mixed calls to both shall be allowed), but see section 3 for one such example.

A step in either direction has either one or three positions in the characteristic word changed. The number of transitions with one change are more frequent with higher multiplicities. The worst case occurs if all multiplicities are one (corresponding to binary words), when (about) half of the steps involve only one transition.

The C++ implementation uses the sentinel technique to reduce the number of conditional branches.

```cpp
// FILE: src/comb/mixedradix-subset-lex.h
class mixedradix_subset_lex
{
ulong n_; // number of digits (n kinds of elements in multiset)
ulong tr_; // aux: current track
ulong *a_; // digits of mixed radix number
ulong *m1_; // nines (radix minus one) for each digit

mixedradix_subset_lex(ulong n, ulong mm, const ulong *m=0)
{
    n_ = n;
a_ = new ulong[n+2]; // two sentinels, one left, one right
    a_[0] = 1; a_[n+1] = 1;
    ++a_; // nota bene
    m1_ = new ulong[n+2];
    m1_[0] = 0; m1_[n+1] = 0; // sentinel with n==0
    ++m1_; // nota bene
    mixedradix_init(n_, mm, m, m1_); // set up m1[], omitted
    first();
}

void first()
{
    for (ulong k=0; k<n_; ++k) a_[k] = 0;
    tr_ = 0; // start by looking at leftmost (zero) digit
}

void last()
{
    for (ulong k=0; k<n_; ++k) a_[k] = 0;
}
```

The omitted routine mixedradix_init() sets all elements of the array m1[] to mm (fixed radix case), unless the pointer m is nonzero, then the elements behind m are copied into m1[] (mixed radix case).
ulong n1 = ( n_ ? n_ - 1 : 0 );
a_[n1] = m1_[n1];
tr_ = n1;
}

Note the while-loop in the next() method.

bool next()
{
ulong j = tr_;
if ( a_[j] < m1_[j] ) // easy case 1: increment
{
    a_[j] += 1;
    return true;
}
// here a_[j] == m1_[j]
if ( j+1 < n_ ) // easy case 2: append (move track to the right)
{
    ++j;
    a_[j] = 1;
    tr_ = j;
    return true;
}
a_[j] = 0;
// find first nonzero digit to the left:
--j;
while ( a_[j] == 0 ) { --j; } // may read sentinel a_[-1]
if ( (long)j < 0 ) return false; // current is last
a_[j] = 1; // decrement digit to the left
++j;
a_[j] = 1;
tr_ = j;
return true;
}

The method prev() is indeed loopless:

bool prev()
{
ulong j = tr_;
if ( a_[j] > 1 ) // easy case 1: decrement
{
    a_[j] -= 1;
    return true;
}else
{
    if ( tr_ == 0 )
    {
        if ( a_[0] == 0 ) return false; // current is first
        a_[0] = 0; // now word is first (all zero)
        return true;
    }
    a_[j] = 0;
    --j; // now looking at next track to the left
    if ( a_[j] == m1_[j] ) // easy case 2: move track to left
    {
        tr_ = j; // move track one left
    }
    else
    {
        a_[j] += 1; // increment digit to the left
        j = n_ - 1;
        a_[j] = m1_[j]; // set rightmost digit = nine
        tr_ = j; // move to rightmost track
    }
    return true;
}
| colex | Gray code | subset-lex | iset |
|-------|-----------|------------|------|
| 0: $[\ldots \ldots]$ | $[4.2.]$ | $[\ldots \ldots]$ | $\{\}$ |
| 1: $[1.\ldots]$ | $[3.2.]$ | $[1.\ldots]$ | $\{0\}$ |
| 2: $[\ldots 2.]$ | $[2.2.]$ | $[2.\ldots]$ | $\{0\}$ |
| 3: $[3.2.]$ | $[1.\ldots]$ | $[3.\ldots]$ | $\{0\}$ |
| 4: $[\ldots 2.]$ | $[2.\ldots]$ | $[4.\ldots]$ | $\{0\}$ |
| 5: $[\ldots 1.]$ | $[\ldots 1.]$ | $[4.1.]$ | $\{0,2\}$ |
| 6: $[\ldots 2.]$ | $[1.\ldots]$ | $[4.2.]$ | $\{0,2\}$ |
| 7: $[3.\ldots]$ | $[2.1.]$ | $[4.\ldots]$ | $\{0,3\}$ |
| 8: $[\ldots 1.]$ | $[3.1.]$ | $[3.\ldots]$ | $\{0,2\}$ |
| 9: $[1.1.]$ | $[4.1.]$ | $[3.2.]$ | $\{0,2\}$ |
| 10: $[2.1.]$ | $[4.\ldots]$ | $[3.1.]$ | $\{0,3\}$ |
| 11: $[3.1.]$ | $[3.\ldots]$ | $[2.1.]$ | $\{0,2\}$ |
| 12: $[\ldots 2.]$ | $[2.\ldots]$ | $[2.2.]$ | $\{0,2\}$ |
| 13: $[\ldots 2.]$ | $[2.\ldots]$ | $[2.1.]$ | $\{0,3\}$ |
| 14: $[1.2.]$ | $[\ldots \ldots]$ | $[1.1.]$ | $\{0,2\}$ |
| 15: $[2.2.]$ | $[\ldots 1.]$ | $[1.2.]$ | $\{0,2\}$ |
| 16: $[3.2.]$ | $[2.\ldots]$ | $[1.\ldots]$ | $\{0,3\}$ |
| 17: $[\ldots 1.]$ | $[3.1.]$ | $[3.\ldots]$ | $\{1\}$ |
| 18: $[\ldots 1.]$ | $[3.1.]$ | $[3.\ldots]$ | $\{1\}$ |
| 19: $[1.1.]$ | $[2.1.]$ | $[3.\ldots]$ | $\{1\}$ |
| 20: $[2.1.]$ | $[1.\ldots]$ | $[3.1.]$ | $\{1,3\}$ |
| 21: $[3.1.]$ | $[1.\ldots]$ | $[3.1.]$ | $\{1,3\}$ |
| 22: $[\ldots 1.]$ | $[1.1.]$ | $[1.\ldots]$ | $\{1,3\}$ |
| 23: $[\ldots 1.]$ | $[1.1.]$ | $[1.\ldots]$ | $\{2\}$ |
| 24: $[\ldots 1.]$ | $[3.1.]$ | $[2.\ldots]$ | $\{2\}$ |
| 25: $[\ldots 1.]$ | $[4.1.]$ | $[\ldots 1.]$ | $\{3\}$ |

Figure 3: Mixed radix numbers of length 4 in falling factorial base that are non-adjacent forms (NAF), in co-lexicographic order, minimal-change order (Gray code), and subset-lex order (with the sets of positions of nonzero digits).

The performance of the generator is quite satisfactory. A system based on an AMD Phenom(™) II X4 945 processor clocked at 3.0 GHz and the GCC C++ compiler version 4.9.0 [12] were used for measuring. The updates using `next()` cost about 12 cycles for base 2 (the worst case) and 9 cycles for base 16. Using `prev()` takes about 8.5 cycles with base 2 and 4.2 cycles with base 16, the latter figure corresponding to a rate of 780 million subsets per second.

These and all following such figures are average values, obtained by measuring the total time for the generation of all words of a certain size and dividing by the number of words.

### 3 Loopless computation of the successor for non-adjacent forms

The method of using a list of nonzero positions in the words to obtain loopless methods for both `next()` and `prev()` has been implemented for words where no two adjacent digits are nonzero (non-adjacent forms, NAF).

In the following algorithm for the computation of the successor let $a_i$ be the digits of the length-$n$ NAF where $0 \leq i < n$.

**Algorithm 5** (Next-NAF-SL). *Compute the successor of a non-adjacent form in subset-lex order.*

1. Let $j$ be the position of the last nonzero digit.
2. If $a_j$ is not maximal, increment it and return.
3. If \( j + 2 < n \), set \( a_{j+2} = 1 \) (append new digit) and return.

4. Set \( a_j = 0 \) (set last nonzero digit to zero).

5. If \( j + 1 < n \), set \( a_{j+1} = 1 \) (move digit right) and return.

6. If there is just one nonzero digit, stop.

7. Let \( k \) be the position of the nearest nonzero digit left of \( j \).

8. Set \( a_k = a_{k-1} - 1 \) (decrement digit to the left).

9. If \( a_k = 0 \), set \( a_{k+1} = 1 \) (move digit right) and return.

10. Otherwise, \( a_{k+2} = 1 \) (append digit two positions to the right).

In the following implementation the array iset[] holds the positions of the nonzero digits. We only read the last or second last element, the adjustments of the length (variable ni) have been left out in the description above. The implementation handles all \( n \geq 0 \) correctly, for the all-zero word iset[] is of length one and its only element points to the leftmost digit.

```c
// FILE: src/comb/mixedradix-naf-subset-lex.h
class mixedradix_naf_subset_lex
{
    ulong *iset_; // Set of positions of nonzero digits
    ulong *a_; // digits
    ulong *m1_; // nines (radix minus one) for each digit
    ulong ni_; // number of elements in iset[]
    ulong n_; // number of digits

    void first()
    {
        for (ulong k=0; k<n_; ++k) a_[k] = 0;
        // iset[] initially with one element zero:
        iset_[0] = 0;
        ni_ = 1;
    }
}
```
The computation of the successor is

```cpp
bool next()
{
    ulong j = iset_[ni_-1];
    const ulong aj = a_[j] + 1;
    if ( aj <= m1_[j] ) // can increment last digit
    {
        a_[j] = aj;
        return true;
    }

    if ( j + 2 < n_ ) // can append new digit
    {
        iset_[ni_] = j + 2;
        a_[j+2] = 1; // assume all m1[] are nonzero
        ++ni_;
        return true;
    }

    if ( j + 1 < n_ ) // can move last digit to the right
    {
        a_[j+1] = 1;
        iset_[ni_-1] = j + 1;
        return true;
    }

    if ( ni_ == 1 ) return false; // current is last

    // Now we look to the left:
    const ulong k = iset_[ni_-2]; // nearest nonzero digit to the left
    const ulong ak = a_[k] - 1; // decrement digit to the left
    a_[k] = ak;
    if ( ak == 0 ) // move digit one to the right
    {
        a_[k+1] = 1;
        iset_[ni_-2] = k + 1;
        --ni_;
        return true;
    }
    else // append digit two positions to the right
    {
        a_[k+2] = 1;
        iset_[ni_-1] = k + 2;
        return true;
    }
}
```

The update via `next()` takes about 7 cycles. The updates for co-lexicographic order and the Gray code respectively take about 15 and 21 cycles.

We note that the falling factorial NAFs of length \( n \) are one-to-one with involutions (self-inverse permutations) of \( n + 1 \) elements. Process the NAF from left to right; for \( a_i = 0 \) let the next unused element of the permutation be a fixed point (and mark it as used); for \( a_i \neq 0 \) put the next unused element in a cycle with the \( a_i \)th unused element (and mark both as used). A (simplistic) implementation of this method is given in the program `demo/comb/perm-involution-naf-demo.cc`.

The binary NAFs of length 8 are shown in figure 5. Algorithms for the generation of these NAFs as binary words for both lexicographic order and the Gray code are
|       | colex  | Gray code | subset-lex |
|-------|--------|-----------|------------|
| 0:    | .......| .1....1   | { 0 }      |
| 1:    | .......| .1....1   | { 0, 2 }   |
| 2:    | .......| .1....1   | { 0, 2, 4 }|
| 3:    | .......| .1....1   | { 0, 2, 4, 6}|
| 4:    | .......| .1....1   | { 0, 2, 4, 7}|
| 5:    | .......| .1....1   | { 0, 2, 5 }|
| 6:    | .......| .1....1   | { 0, 2, 5, 7}|
| 7:    | .......| .1....1   | { 0, 2, 6 }|
| 8:    | .......| .1....1   | { 0, 2, 7 }|
| 9:    | .......| .1....1   | { 0, 3 }   |
| 10:   | .......| .1....1   | { 0, 3, 5 }|
| 11:   | .......| .1....1   | { 0, 3, 5, 7}|
| 12:   | .......| .1....1   | { 0, 3, 6 }|
| 13:   | .......| .1....1   | { 0, 3, 7 }|
| 14:   | .......| .1....1   | { 0, 4 }   |
| 15:   | .......| .1....1   | { 0, 4, 6 }|
| 16:   | .......| .1....1   | { 0, 4, 7 }|
| 17:   | .......| .1....1   | { 0, 5 }   |
| 18:   | .......| .1....1   | { 0, 5, 7 }|
| 19:   | .......| .1....1   | { 0, 6 }   |
| 20:   | .......| .1....1   | { 0, 7 }   |
| 21:   | .......| .1....1   | { 1 }      |
| 22:   | .......| .1....1   | { 1, 3 }   |
| 23:   | .......| .1....1   | { 1, 3, 5 }|
| 24:   | .......| .1....1   | { 1, 3, 5, 7}|
| 25:   | .......| .1....1   | { 1, 3, 6 }|
| 26:   | .......| .1....1   | { 1, 3, 7 }|
| 27:   | .......| .1....1   | { 1, 4 }   |
| 28:   | .......| .1....1   | { 1, 4, 6 }|
| 29:   | .......| .1....1   | { 1, 4, 7 }|
| 30:   | .......| .1....1   | { 1, 5 }   |
| 31:   | .......| .1....1   | { 1, 5, 7 }|
| 32:   | .......| .1....1   | { 1, 6 }   |
| 33:   | .......| .1....1   | { 1, 7 }   |
| 34:   | .......| .1....1   | { 2 }      |
| 35:   | .......| .1....1   | { 2, 4 }   |
| 36:   | .......| .1....1   | { 2, 4, 6 }|
| 37:   | .......| .1....1   | { 2, 4, 7 }|
| 38:   | .......| .1....1   | { 2, 5 }   |
| 39:   | .......| .1....1   | { 2, 5, 7 }|
| 40:   | .......| .1....1   | { 2, 6 }   |
| 41:   | .......| .1....1   | { 2, 7 }   |
| 42:   | .......| .1....1   | { 3 }      |
| 43:   | .......| .1....1   | { 3, 5 }   |
| 44:   | .......| .1....1   | { 3, 5, 7 }|
| 45:   | .......| .1....1   | { 3, 6 }   |
| 46:   | .......| .1....1   | { 3, 7 }   |
| 47:   | .......| .1....1   | { 4 }      |
| 48:   | .......| .1....1   | { 4, 6 }   |
| 49:   | .......| .1....1   | { 4, 7 }   |
| 50:   | .......| .1....1   | { 5 }      |
| 51:   | .......| .1....1   | { 5, 7 }   |
| 52:   | .......| .1....1   | { 6 }      |
| 53:   | .......| .1....1   | { 7 }      |
| 54:   | .......| .1....1   |           |

Figure 5: Binary non-adjacent forms of length 8 in co-lexicographic order, minimal-change order (Gray code), and subset-lex order (with the corresponding sets without consecutive elements).
given in [2, p.75-77]. Here we give the implementations for computing successor and predecessor for the nonempty NAFs in subset-lex order.

The routine for the successor needs to start with the word that has a single set bit at the highest position of the desired word length.

```c
// FILE: src/bits/fibrep-subset-lexrev.h
ulong next_subset_lexrev_fib(ulong x)
{
    ulong x0 = x & -x; // lowest bit
    ulong xs = x0 >> 2;
    if ( xs != 0 ) // easy case: set bit right of lowest bit
    {
        x |= xs;
        return x;
    }
    else // lowest bit at index 0 or 1
    {
        if ( x0 == 2 ) // at index 1
        {
            x -= 1;
            return x;
        }
        x ^= x0; // clear lowest bit
        x0 = x & -x; // new lowest bit ...
        x0 >>= 1; x -= x0; // ... is moved one to the right
        return x;
    }
}
```

The all-zero word is returned as successor of the word whose value is 1. The routine for the predecessor can be started with the all-zero word.

```c
ulong prev_subset_lexrev_fib(ulong x)
{
    ulong x0 = x & -x; // lowest bit
    if ( x & (x0<2) ) // easy case: next higher bit is set
    {
        x ^= x0; // clear lowest bit
        return x;
    }
    else
    {
        x += x0; // move lowest bit to the left and
        x |= (x0!=1); // set rightmost bit unless blocked by next bit
        return x;
    }
}
```
Figure 6: The compositions of 7 together with their run-length encodings as binary words (where dots denote zeros), lexicographic order. A succession of $k$ ones, followed by a zero, in the run-length encoding stands for a part $k+1$, one trailing zero is implied.
4 Compositions

One of the most simple algorithms in combinatorial generation may be the computation of the successor of a composition represented as a list of parts for the lexicographic ordering. In the following let \( z \) be the last element of the composition.

**Algorithm 6 (Next-Comp).** Compute the successor of a composition in lexicographic order.

1. If there is just one part, stop (this is the last composition).
2. Add 1 to the second last part (and remove \( z \)) and append \( z - 1 \) ones at the end.

**Algorithm 7 (Prev-Comp).** Compute the predecessor of a composition in lexicographic order.

1. If the number of parts is maximal (composition into all ones), stop.
2. If \( z > 1 \), replace \( z \) by \( z - 1 \), 1 (move one unit right); return
3. Otherwise, replace the tail \( y, 1^q \) (\( y \) followed by \( q \) ones) by \( y - 1, q + 1 \).

Figure 6 shows the compositions of 7 in lexicographic order. The corresponding run-length encodings appear in lexicographic order as well.

The algorithm for the successor can be made loopless when care is taken that only ones are left beyond the end of the current composition.

```cpp
// FILE: src/comb/composition-nz.h
class composition_nz // Compositions of n into positive parts, lexicographic order.
{
    ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
    ulong n_; // composition of n
    ulong m_; // current composition is into m parts

    ulong next() // Return number of parts of generated composition.
        // Return zero if the current is the last composition.
    {
        if ( m_ <= 1 ) return 0; // current is last
        // [*., Y, Z] --> [*., Y+1, 1, 1, 1, ..., 1] (Z-1 trailing ones)
        a_[m_-1] += 1;
        const ulong z = a_[m_-];
        a_[m_] = 1;
        // all parts a[m+1], a[m+2], ..., a[n] are already ==1
        m_ += z - 2;
        return m_;
    }
}
```

Figure 7 shows two orderings for the compositions. The ordering corresponding to the (complemented) binary Gray code is shown in the left columns. We will call this order **RL-order**, as the compositions correspond to the run-lengths of the binary words in lexicographic order.
Figure 7: The compositions of 7 together with their run-length encodings as binary words (where dots denote zeros), in an order corresponding to the complemented binary Gray code (left) and in subset-lex order (right). A succession of \( k \) ones, followed by a zero, in the run-length encoding stands for a part \( k+1 \), one trailing zero is implied (left). The roles of ones and zeros are reversed for the subset-lex order (right).
For comparison with the new algorithm we give the (loopless) algorithms. In the following let \( m \) be the number of parts in the composition and \( x, y, z \) the last three parts.

**Algorithm 8** (Next-Comp-RL). *Compute the successor of a composition in RL-order.*

1. If \( m \) is odd: if \( z \geq 2 \), replace \( z \) by \( z - 1,1 \) and return; otherwise \( (z = 1) \) replace \( y, 1 \) by \( y + 1 \) and return.
2. If \( m \) is even: if \( y \geq 2 \), replace \( y, z \) by \( y - 1,1, z \) and return; otherwise \( (y = 1) \) replace \( x, 1, z \) by \( x + 1, z \) and return.

The next algorithm is obtained from the previous simply by swapping “even” and “odd” in the description.

**Algorithm 9** (Prev-Comp-RL). *Compute the predecessor of a composition in RL-order.*

1. If \( m \) is even: if \( z \geq 2 \), replace \( z \) by \( z - 1,1 \) and return; otherwise \( (z = 1) \) replace \( y, 1 \) by \( y + 1 \) and return.
2. If \( m \) is odd: if \( y \geq 2 \), replace \( y, z \) by \( y - 1,1, z \) and return; otherwise \( (y = 1) \) replace \( x, 1, z \) by \( x + 1, z \) and return.

With each transition, at most three parts (at the end of the composition) are changed. The number of parts changes by 1 with each step.

An alternative algorithm for this ordering is given in [19, ex.12, sect.7.2.1.1, p.308], see also [24].

Now we give the algorithms for subset-lex order (the first is essentially given in [23]).

**Algorithm 10** (Next-Comp-SL). *Compute the successor of a composition in subset-lex order.*

1. If \( z = 1 \) and there are at most two parts, stop.
2. If \( z \geq 2 \), replace \( z \) by \( 1, z - 1 \) (move all but one unit to the right); return.
3. Otherwise \( (z = 1) \) add 1 to the third last part and remove \( z \) (move one unit two places left: \( x, y, 1 \) is replaced by \( x + 1, y \)).

For the next algorithm let \( y, z \) be the two last elements.

**Algorithm 11** (Prev-Comp-SL). *Compute the predecessor of a composition in subset-lex order.*

1. If there is just one part, stop.
2. If \( y = 1 \), replace \( y, z \) by \( z + 1 \) (add \( z \) to the left); return.
3. Otherwise \( (y \geq 2) \) replace \( y, z \) by \( y - 1, z, 1 \) (move one unit two places right).

At most two parts are changed by either method and these span at most those three parts at the end of the composition. Again, the number of parts changes by 1 with each step.

Note that also the initializations are loopless for the preceding two orderings.

We give the crucial parts of the implementation.

```cpp
// FILE: src/comb/composition-nz-subset-lex.h
class composition_nz_subset_lex
{
    // See src/comb/composition-nz-rl.h for an implementation.
```
ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
ulong n_; // composition of n
ulong m_; // current composition is into m parts

void first()
{
    a_[0] = 0;
    a_[1] = n_;
    m_ = ( n_ ? 1 : 0 );
}

void last()
{
    if ( n_ >= 2 )
    {
        a_[1] = n_ - 1;
        a_[2] = 1;
        for (ulong j=2; j<n_; ++j) a_[j] = 1;
        m_ = 2;
    }
    else
    {
        a_[1] = n_;
        m_ = n_;
    }
}

For the methods next() and prev() we give one of the two implementations found in the file. Both methods return the number of parts in the generated composition and return zero if there are no more compositions.

ulong next()
{
    const ulong z = a_[m_-];
    if ( z<=1 ) // move one unit two places left
    {
        if ( m_ <= 2 ) return 0; // current is last
        a_[m_-1] += 1;
        return m_;
    }
    else // move all but one unit right
    {
        a_[m_] = z - 1;
        return m_;
    }
}

ulong prev()
{
    if ( m_ <= 1 ) return 0; // current is first
    const ulong y = a_[m_-1];
    if ( y==1 ) // add Z to left place
    {
        const ulong z = a_[m_-];
        a_[m_] = z + 1;
        a_[m_-1] = 1;
        m_ = z + 1;
        return m_;
    }
    else // move one unit two places right
    {
        a_[m_-1] = y - 1;
        m_ += 1;
        return m_;
    }
}

The method next() takes about 5 cycles for lexicographic order, 9 cycles for RL-order, and 6 cycles for subset-lex order. The method prev() takes about 10.5 cycles
for lexicographic order and is identical in performance to `next()` for the other orders.

Note that the last parts in the successive compositions in lexicographic order give the
(one-based) ruler function (sequence [A001511](https://oeis.org/A001511) in [28]), see `src/comb/ruler-func1.h`
for the trivial implementation, compare to `src/comb/ruler-func.h` (giving sequence
[A007814](https://oeis.org/A007814) in [28]) which uses the techniques from [10] and [3] as described in [19],
Algorithm L, p.290).

An loopless implementation for the generation of the compositions into odd parts in
subset-lex order is given in `src/comb/composition-nz-odd-subset-lex.h`.

### Ranking and unranking

An unranking algorithm for the subset-lex order is obtained by observing (see figure
[7]) that the composition into one part has rank 0 and otherwise the first part and the
remaining parts are easily determined.

**Algorithm 12** (Unrank-Comp-SL). Recursive routine $U(r, n, C[], m)$ for the com-
putation of the composition $C[]$ of $n$ with rank $r$ in subset-lex order. The auxiliary
variable $m \geq 0$ is the index of the part about to be written in $C[]$. The initial call is
$F(r, n, C[], 0)$.

1. If $r = 0$, set $C[m] = n$ and return $m + 1$ (the number of parts).
2. Set $f = 0$ (the tentative first part).
3. Set $f = f + 1$ and $t = 2^{n-f-1}$.
4. If $r < t$ (can use part $f$), set $C[m] = f$ and return $F(r, n - f, C[], m + 1)$.
5. Set $r = r - t$ and go to step 3.

We give an iterative implementation.

```c
ulong composition_nz_subset_lex_unrank(ulong r, ulong *x, ulong n)
{
    if ( n==0 ) return 0;
    ulong m = 0;
    while ( true )
    {
        if ( r==0 ) // composition into one part
        {
            x[m++] = n;
            return m;
        }
        r -= 1;
        ulong t = 1UL << (n-1); // == number of compositions of n with first part f
        for (ulong f=1; f<n; ++f) // find first part f >= 1
        {
            t >>= 1; // == 2**(n-f-1)
            if ( r < t ) // first part is f
            {
                x[m++] = f;
                n -= f;
                break;
            }
            r -= t;
        }
    }
}
```

The following algorithm for computing the rank is modeled as inverse of the method
above.
Algorithm 13 (Rank-Comp-SL). Computation of the rank $r$ of a composition $C[]$ of $n$ in subset-lex order.

1. Set $r = 0$ and $e = 0$ (position of part under consideration).
2. Set $f = C[e]$ (part under consideration).
3. If $f = n$, return $r$.
4. Set $r = r + 1$, $t = 2^{n-1}$, and $n = n - f$.
5. While $f > 1$, set $t = t/2$, $r = r + t$, and $f = f - 1$.
6. Set $e = e + 1$ and go to step 2.

An implementation is

```c
ulong composition_nz_subset_lex_rank(const ulong *x, ulong m, ulong n)
{
    ulong r = 0; // rank
    ulong e = 0; // position of first part
    while ( e < m )
    {
        ulong f = x[e];
        if ( f==n ) return r;
        r += 1;
        ulong t = 1UL << (n-1);
        n -= f;
        while ( f > 1 )
        {
            t >>= 1;
            r += t;
            f -= 1;
        }
        e += 1;
    }
    return r; // return r==0 for the empty composition
}
```

Conversion functions between binary words in lexicographic order and subset-lex order are shown in [2, pp.71-72].

```c
ulong negidx2lexrev(ulong k)
{
    ulong z = 0;
    ulong h = highest_one(k);
    while ( k )
    {
        while ( 0==(h&k) ) h >>= 1;
        z ^= h;
        ++k;
        k &= h - 1;
    }
    return z;
}

ulong lexrev2negidx(ulong x)
{
    if ( 0==x ) return 0;
    ulong h = highest_one(x);
    while ( x^=h )
    {
        h = x & -x; // next higher one
    }
    r += h; // highest bit
    return r;
}
```
Based on these, alternative ranking and unranking functions can be given.

Ranking and unranking methods for the compositions into odd parts can be obtained by replacing $2^{n-f-1}$ (number of compositions of $n$ with first part $f$) in the algorithms \[2\] and \[3\] by the Fibonacci numbers $F_{n-f-1}$ (number of compositions of $n$ into odd parts with first part $f$), where $n \geq 1$, $f < n$, and $f$ odd (see sequence A242086 in [28]).

5 Partitions as weakly increasing lists of parts

We now give algorithms for the computation of the successor for partitions represented as weakly increasing lists of parts. The generation of all partitions in this representation is the subject of [16].

5.1 All partitions

Figure 8 shows the partitions of 11 as weakly increasing lists of parts, in lexicographic order (left) and in subset-lex order (right). We first give a description of the computation of the successor in lexicographic order. The three last parts of the partition are denoted by $x$, $y$, and $z$.

Algorithm 14 (Next-Part-Asc). Compute the successor of a partition in lexicographic order.

1. If there is just one part, stop.
2. If $z - 1 < y + 1$, replace $y, z$ by $y + z$; return.
3. Otherwise, change $y$ to $y + 1$ and append parts $y + 1$ as long as there are at least $y + 1$ units left.
4. Add the remaining units to the last part.

The second step of the algorithm can be merged into the other steps, as it is a special case of them.

\[
\begin{array}{l}
\text{// FILE: src/comb/partition-asc.h} \\
\text{class partition_asc} \\
\text{\{ \\
ulong *a_; // partition: a[1] + a[2] + ... + a[m] = n \\
ulong n_; // integer partitions of n \\
ulong m_; // current partition has m parts \\
\}}
\end{array}
\]

The implementation is correct for all $n \geq 0$, for $n = 0$ the empty list of parts is generated.

\[
\begin{array}{l}
\text{ulong next()} \\
\text{// Return number of parts of generated partition.} \\
\text{// Return zero if the current partition is the last.} \\
\text{\{ \\
\text{if ( m_ <= 1 ) return 0; // current is last} \\
\text{ulong z1 = a_[m_] - 1; // take one unit from last part} \\
\text{m_- = 1;} \\
\text{const ulong y1 = a_[m_] + 1; // add one unit to previous part} \\
\text{while ( y1 <= z1 ) // can put part Y+1} \\
\text{\{ \\
\text{a_[m_] = y1;} \\
\text{}} \\
\text{}} \\
\end{array}
\]

\[\text{3See src/comb/composition-nz-rank.cc where the corresponding routines for all three orders shown here are implemented.}\]
Figure 8: The partitions of 11 as weakly increasing lists of parts, lexicographic order (left) and subset-lex order (right).
The computation of the successor in subset-lex order is loopless.

Algorithm 15 (Next-Part-Asc-SL). Compute the successor of a partition in subset-lex order. A sentinel zero shall precede list of elements.

1. If \( z \geq 2 \cdot y \), replace \( y, z \) by \( y, y, z-y \) (extend to the right); return.
2. If \( z-1 \geq y+1 \), replace \( y, z \) by \( y+1, z-1 \) (add one unit to the left); return.
3. If \( z = 1 \) (the all-ones partition) do the following. Stop if the number of parts is \( \leq 3 \), otherwise replace the tail \([1, 1, 1, 1]\) by \([2, 2]\) and return.
4. If the number of parts is 2, stop.
5. Replace \( x, y, z \) by \( x+1, y+z-1 \) (add one unit to second left, add rest to end) and return.

Note that with each update all but the last two parts are the same as in the predecessor. This can be an advantage over the lexicographic order for computations where partial results for prefixes can be reused.

The implementation is again correct for all \( n \geq 0 \).
The updates via `next()` take about 15 cycles for lexicographic order and about 13 cycles for subset-lex order.

### 5.2 Partitions into odd and into distinct parts

It is not very difficult to adapt the algorithms to specialized partitions. We give the algorithms for partitions into distinct and odd parts by their implementations.

**Computation of the successor for partitions into distinct parts in lexicographic order:**

```
ulong next()
{
    if ( m_ <= 1 ) return 0; // current is last
    ulong s = a_[m_] + a_[m_-1];
    ulong k = a_[m_-1] + 1;
    m_- = 1;
    // split s into k, k+1, k+2, ..., y, z where z >= y + 1:
    while ( s >= ( k + (k+1) ) )
    {
        a_[m_] = k;
        s -= k;
        k += 1;
        m_ += 1;
    }
    a_[m_] = s;
    return m_;  
}
```

**Computation of the successor for partitions into odd parts in lexicographic order:**

```
ulong next()
{
    const ulong z = a_[m_]; // can read sentinel a[0] if n==0
    const ulong y = a_[m_-1]; // can read sentinel a[0] (a[-1] for n==0)
    ulong s = y + z; // sum of parts we scan over
    ulong k; // min value of next term
    if ( z >= y+4 ) // add last 2 terms
    {
        if ( m_ == 1 ) return 0; // current is last
        k = y + 2;
        a_[m_-1] = k;
        s -= k;
    }  
}
```
Figure 9: The partitions of 19 into distinct parts (left) and odd parts (right) in subset-lex order.
else // add last 3 terms
{
    if ( m_ <= 2 ) return 0; // current is last
    const ulong x = a_[m_-2];
    s += x;
    k = x + 2;
    m_ -= 2;
}
const ulong k2 = k + k;
while ( s >= k2 + k )
{
    a_[m_] = k; s -= k; m_ += 1;
    a_[m_] = k; s -= k; m_ += 1;
}
    a_[m_] = s;
    return m_;
}

Computation of the successor for partitions into distinct parts in subset-lex order:

All routines in this section return the number of parts in the generated partition and return zero if there are no more partitions.

Computation of the successor for partitions into odd parts in subset-lex order:
if ( z >= 3*y ) // can extend to the right
{ // [*, Y, Z] --> [*, Y, Y, Y, Z-2*Y]
a_[m_] = y;
a_[m_+1] = y;
a_[m_+2] = z - 2 * y;
m_ += 2;
return m_; }

if ( m_ >= n_ ) // all-ones partition
{ if ( n_ <= 5 ) return 0; // current is last
  // [1, ..., 1, 1, 1, 1, 1] --> [1, ..., 3, 3]
m_ = 4;
a_[m_] = 3;
a_[m_-1] = 3;
return m_; }

ulong z2 = z - 2;
ulong y2 = y + 2;

if ( z2 >= y2 ) // add 2 units to the left
{ // [*, Y, Z] --> [*, Y+2, Z-2]
a_[m_-1] = y2;
a_[m_] = z2;
return m_; }

if ( m_==2 ) // current is last (happens only for n even)
return 0;

// here m >= 3; add to second or third left:
ulong x2 = a_[m_-2] + 2;
ulong s = z + y - 2;

if ( x2 <= z2 ) // add 2 units to third left, repeat part, put rest at end
{ // [*, X, Y, Z] --> [*, X+2, X+2, Y+Z-2 -X-2]
a_[m_-2] = x2;
a_[m_-1] = x2;
a_[m_] = s - x2;
return m_; }

if ( m_==3 ) // current is last (happens only for n odd)
return 0;

// add 2 units to third left, combine rest into second left
// [*, W, X, Y, Z] --> [*, W+2, X+Y+Z-2]
a_[m_-3] += 2;
a_[m_-2] += s;
m_ -= 2;
return m_; }

The updates via next() of partitions into distinct parts take about 11 cycles for lexicographic order and about 9 cycles for subset-lex order. The respective figures for the partitions into odd parts are 13 and 13.5 cycles.

Algorithms for ranking and unranking are obtained by replacing $2^n - f - 1$ (the number of compositions of $n$ with first part $f$) in algorithms [12] and [13] by the expression for the number of partitions of $n$ (of the desired kind) with first part $f$. 
6 Subset-lex order for restricted growth strings (RGS)

By modifying the algorithms for the successor and predecessor (3 and 4) for subset-lex order to adhere to certain conditions, one obtains the equivalents for RGS. These can often be given without any difficulty. We give two examples, RGS for set partitions and RGS for \( k \)-ary Dyck words.

6.1 RGS for set partitions

We consider the restricted growth strings (RGS) \( a_0, a_1, \ldots, a_{n-1} \) such that \( a_0 = 0 \) and \( a_k \leq 1 + \max(a_0, a_1, \ldots, a_{k-1}) \). These RGS are counted by the Bell numbers, see sequence \( \text{A000110} \) in \[28\]. Figure 10 shows the all such RGS of length 5 in lexicographic and subset-lex order. The set partitions are obtained by putting all \( i \) such that \( a_i = k \) into the same set.

In the following algorithm we assume that \( m_k = 1 + \max(a_0, a_1, \ldots, a_{k-1}) \) and that there is a sentinel \( a_{-1} = 1 \).

**Algorithm 16** (Next-Setpart-RGS). *Compute the successor in subset-lex order.*

1. Let \( j \) be the index of the last nonzero digit (\( j = 1 \) for the all-zero RGS).
2. If \( a_j < m_j \), set \( a_j = a_j + 1 \) and return.
3. If \( j + 1 < n \), set \( a_{j+1} = 1 \) and return.
4. Set \( a_j = 0 \).
5. Set \( j \) to the position of the next nonzero digit to the left. If \( j = -1 \), stop.
6. Set \( a_j = a_j - 1 \) and \( a_{j+1} = 1 \).

The implementation has to take care of updating the array \( m[] \) which has an additional (write-only) element at its end.

```cpp
1 // FILE: src/comb/setpart-rgs-subset-lex.h
2 class setpart_rgs_subset_lex
3 {
4  ulong *a_; // digits of the RGS
5  ulong *m_; // maximum + 1 in prefix
6  ulong tr_; // current track
7  ulong n_; // number of digits in RGS

8 bool next()
9 {
10    ulong j = tr_;  // easy case 1: can increment track
11    if ( a_[j] < m_[j] ) {  // handle n <= 1 correctly
12      if ( n_ <= 1 ) return false;
13      a_[j] += 1;
14      return true;
15    }
16    const ulong j1 = j + 1;
17    if ( j1 < n_ ) {  // easy case 2: can attach
18      m_[j1] = m_[j] + 1;
19      a_[j1] = +1;
20      tr_ = j1;
21      return true;
22    }
23  }
```
Figure 10: Restricted growth strings and corresponding set partitions in lexicographic and subset-lex order.
20 \quad a_\[j\] = 0;
21 \quad m_\[j\] = m_\[j-1\];
22
23 // Find nonzero track to the left:
24 do { --j; } while ( a_\[j\] == 0 ); // can read sentinel
25 if ( (long)j < 0 ) return false; // current is last
26 if ( a_\[j\] == m_\[j\] ) m_\[j+1\] = m_\[j\];
27 a_\[j\] -= 1;
28 ++j;
29 a_\[j\] = 1;
30 tr_ = j;
31 return true;
32
An update takes about 9.5 cycles for subset-lex order and 12.5 cycles for lexicographic order.

6.2 RGS for $k$-ary Dyck words

Figure [1] shows the 55 RGS for the 3-ary Dyck words of length 12, in both lexicographic and subset-lex order. The $j$th value in each RGS contains the distance of the position $j$th one in the Dyck word from its maximal value $k \cdot j$ The sequences of numbers of $k$-ary Dyck words are entries $A000108$ ($k = 2$, Catalan numbers), $A001764$ ($k = 3$), $A002293$ ($k = 4$), and $A002294$ ($k = 5$) in [28].

It shall suffice to give the implementation for the (loopless) computation of the predecessor in subset-lex order. A sentinel $a[-1]=+1$ is used.

```c
1 // FILE: src/comb/dyck-rgs-subset-lex.h
2 class dyck_rgs_subset_lex
3 {
4     ulong *a_; // digits of the RGS: a_\[k\] <= as\[k-1\] + 1
5     ulong tr_; // current track
6     ulong n_; // number of digits in RGS
7     ulong i_; // k-ary Dyck words: i = k - 1
8
9     void last()
10     {
11         for (ulong k=0; k<n_; ++k) a_\[k\] = 0;
12         tr_ = n_ - 1;
13         // make things work for n <= 1:
14         if ( n_=0 )
15             {
16                 tr_ = 0;
17                 a_\[0\] = 1;
18             }
19         if ( n_>=2 ) a_\[tr_\] = i_;
20     }
21
22 bool prev()
23     // Loopless algorithm.
24     {
25         if ( n_<=1 ) return false; // just one RGS
26         ulong j = tr_;
27         if ( a_\[j\] > 1 ) // can decrement track
28             {
29                 a_\[j\] -= 1;
30                 return true;
31             }
32         const ulong aj = a_\[j\]; // zero or one
33         a_\[j\] = 0;
```
1: [ . . . . ] 1..1..1..1.. [ . . . . ] 1..1..1..1..  
2: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
3: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
4: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
5: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
6: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
7: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
8: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
9: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
10: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
11: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
12: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
13: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
14: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
15: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
16: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
17: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
18: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
19: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
20: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
21: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
22: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
23: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
24: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
25: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
26: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
27: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
28: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
29: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
30: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
31: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
32: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
33: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
34: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
35: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
36: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
37: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
38: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
39: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
40: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
41: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
42: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
43: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
44: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
45: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
46: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
47: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
48: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
49: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
50: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
51: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
52: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
53: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
54: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  
55: [ . . . . 1 ] 1..1..1..1.. [ . . . . ] 1..2..1..1..  

Figure 11: The 55 RGS for the 3-ary Dyck words of length 12, in lexicographic order (left) and subset-lex order (right).
---j;
if ( a_[j] == a_[j-1] + i_ ) // move track to the left
{
  --tr_;
  return true;
}
if ( j==0 ) // current or next is last
{
  if ( aj == 0 ) return false;
  return true;
}
a_[j] += 1; // increment left digit
tr_ = n_ - 1; // move to right end
a_[tr_] = a_[tr_-1] + i_; // set to max value
return true;
}

One update takes about 10 cycles for both lexicographic and subset-lex order.

For the generation of other restricted growth strings in subset-lex order, see the following files.

// Ascent sequences, see OEIS sequence A022493:
src/comb/ascent-rgs-subset-lex.h
src/comb/ascent-rgs.h // lexicographic order

// RGS for Catalan objects, see OEIS sequence A000108:
src/comb/catalan-rgs-subset-lex.h (loopless prev())
src/comb/catalan-rgs.h // lexicographic order

// Standard Young tableaux, represented as ballot sequences,
// see OEIS sequence A000085:
src/comb/young-tab-rgs-subset-lex.h
src/comb/young-tab-rgs.h // lexicographic order

7 A variant of the subset-lex order

An order obtained by processing the positions in the characteristic words as for subset-lex order but with reversed priorities is shown in figure 12. We will call this ordering subset-lexrev. The algorithms for computing successor and predecessor are easily obtained from those for the subset-lex order, we just give the implementation for prev(), which is (again) loopless.

```c
// FILE: src/comb/mixedradix-subset-lexrev.h
bool prev()
{
ulong j = tr_;
  if ( a_[j] > 1 ) // easy case: just decrement
    {
      a_[j] -= 1;
      return true;
    }
a_[j] = 0;
++j; // now looking at next track to the right
if ( j >= n_ ) // was on rightmost track (last two steps)
  {
    bool q = ( a_[j] != 0 );
a_[j] = 0;
    return q;
  }
if ( a_[j] == m1_[j] ) // semi-easy case: move track to left
  {
    tr_ = j; // move track one right
    return true;
  }
```
Figure 12: Subsets of the set \( \{0^3, 1^2, 2^2, 3^3\} \) in subset-lexrev order. Dots denote zeros in the (generalized) characteristic words. Note the sets are printed starting with the largest element.
One use of the subset-lexrev order is the development of algorithms for the generation of orderings for objects where the subset-lex order exhibits no simple structure.

As an example we consider the restricted growth strings corresponding to lattice paths from $(0,0)$ to $(n,n)$ with steps $(+1,0)$ and $(+1,+1)$ that do not go below the diagonal. These RGS are words $a_0, a_1, \ldots, a_{n-1}$ such that $a_0 = 0$, $a_j \leq j$, and $a_{j+1} \geq a_j$ (the final $a_n = n$ is omitted). These RGS are counted by the Catalan numbers, see sequence A000108 in [28].

Figure 13 shows the all such RGS of length 5 in lexicographic, co-lexicographic, subset-lexrev, and subset-lex order.
We now assume a sentinel $a_n = +\infty$ at the end of the word.

**Algorithm 17** (Next-Catalan-Step-RGS). *Compute the successor in subset-lexrev order.*

1. Let $a_t$ be the first nonzero digit.
2. If $a_t < a_{t+1}$ and $a_t < t$, increment $a_t$ and return.
3. If $t \geq 2$, set $a_{t-1} = 1$ and return.
4. Remove all leading ones (setting them to zero), let $j$ be the position of the first digit $a_j \neq 1$.
5. If the word is all-zero (that is, $j = n$), stop.
6. Set $a_j = a_j - 1$ and set $a_{j-1} = 1$.

The implementation correctly handles all cases $n \geq 0$.

```c
ulong next()
{
    const ulong a0 = a_[tr_];
    if ( a0 < a_[tr_+1] ) // may read sentinel
    {
        if ( a0 < tr_ ) // can increment
        {
            a_[tr_] = a0 + 1;
            return tr_;
        }
    }
    if ( tr_ != 1 ) // can move left and increment (from 0 to 1)
    {
        --tr_;
        a_[tr_] = 1;
        return tr_;
    }
    // remove ones:
    ulong j = tr_;
    do { a_[j] = 0; } while ( a_[j+j] == 1 );
    if ( j==n2_ ) return 0; // current was last
    // decrement first value != 1:
    ulong aj = a_[j] - 1;
    a_[j] = aj;
    // move left and restore the 1:
    tr_ = j - 1;
    a_[tr_] = 1;
    return j; // rightmost change at j==tr+1
}
```

An update takes about 10.5 cycles, whereas the updates for lexicographic and co-lexicographic order respectively take 9 and 8 cycles.

We mention that the subset-lexrev order coincides for certain objects with well-known
Figure 14: The partitions of 10 in subset-lexrev order, as weakly increasing lists of parts (left) and as weakly decreasing lists of parts (right).
orderings. For example, for partitions as lists of parts as shown in figure 14, both orderings are by falling length of partitions as major order, the minor order is lexicographic in case of weakly increasing parts and reverse co-lexicographic for weakly decreasing parts.

Similar observations can be made (for example) for compositions into a fixed number of parts, for both subset-lex and subset-lexrev order.

8 SL-Gray order for binary words

A well-known construction for the binary reflected Gray code proceeds by successively reversing ranges having identical prefixes that end in a one, using increasingly longer prefixes, see top of figure 13.

The same construction, now using prefixes ending in a zero, gives a Gray code for subset-lex order, see bottom of figure 13. We will call this ordering the SL-Gray order.

For the computation of the successor we keep a variable \( t \) for the current track and a variable \( d \) indicating the direction in which the track will be moved if necessary. Initially \( t = 0 \) and \( d = +1 \).

Algorithm 18 (Next-SL-Gray). Compute the successor in SL-Gray order.

1. If \( d = +1 \), do the following (try to append trailing ones):
   (a) If \( a_t = 0 \), set \( a_t = 1 \), \( t = t + 1 \), and return.
   (b) Otherwise, set \( d = -1 \) (change direction), \( j = n - 2, a_j = 1 - a_j \), and return.

2. Otherwise \( (d = -1) \), do the following (try to remove trailing ones):
   (a) If \( a_{t-1} = 1 \), set \( a_t = 0 \), \( t = t - 1 \), and return.
   (b) Otherwise, set \( d = +1 \) (change direction), \( j = t - 2, a_j = 1 - a_j, t = t + 1 \) (move right), and return.

The algorithm is loopless. In the implementation, the variables \( tr \) and \( dt \) respectively correspond to \( t \) and \( d \) in the algorithm. Two sentinels \( a_{-1} = a_n = +1 \) are used.

```c
// FILE: src/comb/binary-sl-gray.h
class binary_sl_gray
{
  ulong n_; // number of digits
  ulong tr_; // aux: current track (0 \( \leq \) tr \( < \) n)
  ulong dt_; // aux: direction in which track tries to move
  ulong *a_; // digits

  void first()
  {
    for (ulong k=0; k<n_; ++k) a_[k] = 0;
    tr_ = 0;
    dt_ = +1;
  }

  explicit binary_sl_gray(ulong n)
  {
    n_ = n;
    a_ = new ulong[n_*2]; // sentinels at both ends
    a_[n_*1] = +1; // != 0
    a_[0] = +1;
    ++a_; // nota bene
  }
};
```
Figure 15: Construction of Gray codes by reversing sublists: for the binary reflected Gray code (top) and for the binary SL-Gray order (bottom). The symbols $v$ and $\wedge$ respectively mark begin and end of the reversed sublists.
first();
}

void first()
{
    for (ulong k=0; k<n_; ++k) a_[k] = 0;
    tr_ = 0;
    dt_ = +1;
    j_ = ( n_>=2 ? 1 : 0); // wrt. last word
    dm_ = -1; // wrt. last word
}

The routines for successor and predecessor handle all cases \( n \geq 0 \) correctly.

bool next()
{
    if ( dt_ == +1 ) // try to append trailing ones
    {
        if ( a_[tr_] == 0 ) // can append // may read sentinel a[n]
        {
            a_[tr_] = 1;
            ++tr_;
        }
    }
    else // dt_ == -1 // try to remove trailing ones
    {
        if ( a_[tr_-1] != 0 ) // can remove // tr - 1 >= 0
        {
            a_[tr_] = 0;
            --tr_;
        }
    }
    else // dt_ == -1 // try to remove trailing ones
    {
        dt_ = -1UL; // change direction
        tr_ = n_ - 1;
        ulong j_ = tr_ - 1;
        // Is current last (only for n_ <= 1)?
        if ( ( j_ > n_ ) return false;
            a_[j_] = 1 - a_[j_];
            ++tr_;
        }
    }
    return true;
}

The routine to compute the predecessor is obtained by (essentially) negating the direction the track tries to move (variable \( dt \)):

bool prev()
{
    if ( dt_ != +1 ) // dt==1 // try to append trailing ones
    {
        if ( a_[tr_+1] == 0 ) // can append // may read sentinel a[n]
        {
            a_[tr_+1] = 1;
            ++tr_;
        }
    }
    else // dt_ == -1 // try to remove trailing ones
    {
        dt_ = +1; // change direction
        tr_ = n_ - 1;
        ulong j_ = tr_ - 2;
        // Is current last (only for n_ <= 1)?
        if ( (long)j_ < 0 ) return false;
            a_[j_] = 1 - a_[j_];
            ++tr_;
else // dt_ == +1  // try to remove trailing ones
{
  if ( tr_ == 0 )
  {
    if ( a_[0]==0 ) return false;  // (only for n_ <= 1)
    a_[0] = 0;
    return true;
  }

  // tr - 1 >= -1 (can read low sentinel)
  if ( a_[tr_-2] != 0 ) // can remove
  {
    a_[tr_-1] = 0;
    --tr_;
  }
  else
  {
    dt_ = -1UL;  // change direction
    ulong j_ = tr_ - 3;
    a_[j_] = 1 - a_[j_];
    --tr_;
  }   
}

return true;

One update in either direction takes about 7.5 cycles. An implementation for the generation of the SL-Gray order in a binary word is given in src/bits/bit-sl-gray.h.

The ranking algorithm is easily obtained by observing that if the highest bit is not set (and the word is nonzero), the order for the remaining word is reflected, as shown in figure 15.

Algorithm 19 (Binary-SL-Gray-Rank). Recursive routine $F(k, n)$ for the computation of the rank of an $n$-bit word in binary SL-Gray order.

1. If $k = 0$, return 0.

2. Set $w = k$ and unset the highest bit in $w$.

3. If the highest bit of $k$ (at position $n - 1$) is set, return $1 + F(w, n - 1)$.

4. Otherwise return $2^n - F(w, n - 1)$.

In the following implementation the variable ldn must give the number of bits in the SL-Gray code.

A routine for the conversion of a binary word into the corresponding word in SL-Gray order (unranking algorithm) is

A routine for the conversion of a binary word into the corresponding word in SL-Gray order (unranking algorithm) is
while ( b != 0 )
{
    const ulong h = k & b; // bit under consideration
    z ^= h; // with one, switch bit in Gray code
    if ( !h ) k ^= m; // reverse direction with zero
    k += 1; // SL-Gray
    b >>= 1; // next lower bit
    m >>= 1; // next smaller mask
}
return z;

The delta sequence (sequence of positions of changes), starting after the initial slope and indexing positions from the end, is shown in figure 16, this is sequence A217262 in [28]. It can be obtained from the ruler function (sequence A007814) by replacing 0 by \( w = 01210 \), 1 by 3, 2 by 141, 3 by 12521, 4 by 1236321, \( \ldots \), \( n \) by \( 123 \ldots (n - 1)(n + 2)(n - 1) \ldots 321 \):

\[
\begin{align*}
0 & 1 0 2 0 1 0 3 0 1 0 2 0 1 0 4 0 \ldots \\
0 & 3 \ 0 3 \ 0 12521 \ 0 & 3 \ 0 141 \ 0 & 3 \ 0 1236321 \ 0 \ldots \end{align*}
\]

(ruler function)

01210 3 01210 141 01210 3 01210 12521 01210 3 01210 141 01210 3 01210 \ldots

It is easy to see that (for \( n \geq 4 \)) the SL-Gray order for the \( 2^n \) words of length \( n \) has \( 2^{n-2} - 2 \) successive transitions that are 3-close (have a distance of 3), the rest are 1-close (adjacent changes).

A Gray code where the maximal distance between successive transitions is \( k \) is called \( k \)-close. In [2] p.400 it has been observed that 1-close Gray codes exists for \( n \leq 6 \) but not for \( n = 7 \) (and it appears unlikely that for any \( n \geq 8 \) such a Gray code exists). Does a 2-close Gray code exist? We remark that the SL-Gray order is 2-close if the radices for all digits are even and \( \geq 4 \).

### 9 SL-Gray order for mixed radix words

The generalization of the SL-Gray order to mixed radix words is obtained by successive reversion of the sublists with identical prefix for all prefixes ending in an even digit.

For the computation of the successor we keep a variable \( t \) for the current track and variables \( d_j = \pm 1 \) (an array) for the direction the digit \( a_j \) is currently moving in (whether its is incremented or decremented). We further use \( m_j \) for the maximal value the digit \( a_j \) can have (the “nines”). Initially all digits \( a_k \) are zero, all directions \( d_k \) are +1, and \( t = 0 \).
Figure 17: Mixed radix words (left) and corresponding subsets (right) of the multiset \{0, 1, 2, 3\} in SL-Gray order, and the array of directions used in the computation (middle).
Algorithm 20 (Next-SL-Gray). Compute the successor SL-Gray order.

1. Set \( j = t \) and \( b = a_j + d_j \).
2. If \( b \neq 0 \) and \( b \leq m_j \), set \( a_j = b \) and return (easy case).
3. Set \( d_j = -d_j \) (change direction for digit \( j \)).
4. If \( d_j = +1 \) and \( a_{j+1} = 0 \), set \( a_{j+1} = +1, \ t = j + 1 \) (move track right), and return.
5. If \( d_j = -1 \) and \( a_{j-1} = m_{j-1} \), set \( a_j = 0, \ d_{j-1} = -1, \ t = j - 1 \) (move track left), and return.
6. Find the position \( p \) of the nearest digit \( a_k \) to the left such that \( a_k + d_k \) is in the range \( 0, 1, \ldots, m_k \) (a valid digit). In the process, change \( d_k \) for all \( k \) where \( p < k < j \).
7. Set \( a_p = a_p + d_p \) (change digit, keep track).

The details for termination have been omitted, this is handled with the help of sentinels in the implementation.

```cpp
// FILE: src/comb/mixedradix-sl-gray.h
class mixedradix_sl_gray
{
ulong n_; // number of digits
ulong tr_; // aux: current track
ulong *a_; // digits of mixed radix number
ulong *d_; // directions (either +1 or -1)
ulong *m1_; // nines (radix minus one) for each digit

mixedradix_sl_gray(ulong n, ulong mm, const ulong *m=0) { mixedradix_init(n_, mm, m, m1_); first(); }

void first()
{
for (ulong k=0; k<n_; ++k) a_[k] = 0;
for (ulong k=0; k<n_; ++k) d_[k] = +1;
tr_ = 0;
}

// Must have n>=1
{
ulong n; // number of digits
a_ = new ulong[n+2]; // all with sentinels at both ends
d_ = new ulong[n+2];
m1_ = new ulong[n+2];

// sentinels on the right:
a_[n+1] = +1; // != 0
m1_[n+1] = +1; // same as a_[n+1]
d_[n+1] = +1; // positive

// sentinels on the left:
a_[0] = +2; // >= +2
m1_[0] = +2; // same as a_[0]
d_[0] = 0; // zero
++a_; ++d_; ++m1_; // nota bene
mixedradix_init(a_, mm, m, m1_);
first();
}
```

Sentinels are used in all arrays at index \(-1\) and at index \(n\) to handle termination.

The method for the successor handles all cases \( n \geq 0 \) correctly. The variable \( a_1 \) in the implementation corresponds to \( b \) in the algorithm.

42
10  A Gray code for compositions

We now describe a Gray code for compositions where at most one unit is moved in each step, all moves are 1-close or 2-close (these always cross a part 1), and all moves are at the end of the current composition.

The compositions of 7 in this ordering are shown in figure 18 (left), it can be obtained from the lexicographic order by successively reversing the sublists with identical prefixes that end in an odd part. To keep matters simple for the iterative algorithm, we arrange the compositions for \( n \) odd such that the first part is only decreasing (starting with the composition \([n]\)) and otherwise such that the first part is increasing (starting
Figure 18: The 64 compositions of 7 together with their binary encodings as in figure 6 (left) and the 56 compositions of 9 into exactly 6 parts (right).
with the composition $[1, n-1]$). The compositions of 6 in this ordering are obtained by dropping the first part 1 in all compositions of 7 in the second half of the list, see figure 18.

A recursive routine can be obtained by switching between the recursive functions for lexicographic and reversed lexicographic order whenever an odd part has been written. In the following a global array $a[]$ is used.

```c
// FILE: src/comb/composition-nz-gray-rec-demo.cc
void F(ulong n, ulong m)
{
    if (n==0) { visit(m); return; }
    for (ulong f=n; f!=0; --f) // first part decreasing
        a[m] = f;
    if ( 0 == (f & 1) ) F(n-f, m+1);
    else B(n-f, m+1);
}

void B(ulong n, ulong m)
{
    if (n==0) { visit(m); return; }
    for (ulong f=1; f<=n; ++f) // first part increasing
        a[m] = f;
    if ( 0 == (f & 1) ) B(n-f, m+1);
    else F(n-f, m+1);
}
```

The initial call is $F(n, 0)$ for $n$ odd and $G(n, 0)$ for $n$ even.

For the iterative algorithm we use a function $D(x)$ that shall return $+1$ if $x$ is odd and otherwise $-1$. We will refer to the last three parts as respectively $x$, $y$, and $z$.

**Algorithm 21** (Next-Comp-Gray). Compute the successor of a composition.

1. If $z = n - 1$ and $n$ is odd, or $z = n$ and $n$ is even, stop.
2. If $z = 1$, do the following:
   (a) If $D(y) = +1$, set $y = y + 1$ and drop $z$; return.
   (b) Otherwise ($D(y) = -1$), set $y = y - 1$ and append a part 1; return.
3. Otherwise ($z > 1$) do the following:
   (a) If $z$ is odd, set $z = z - 1$ and append a part 1; return.
   (b) If $y > 1$, set $y = y - D(y)$ and $z = z + D(y)$; return.
   (c) If $x > 1$, set $x = x + D(x)$ and $z = z - D(x)$; return.
   (d) Otherwise ($x = 1$) set $x = x + 1$ and $z = z - 1$.

The algorithm is loopless. Note that the initialization is loopless as well.

The implementation handles all $n \geq 0$ correctly.
n_ = n;
a_ = new ulong[n_+1+(n_==0)];
a_[0] = 0; // returned by last_part() when n==0
a_[1] = 0; // returned by first_part() when n==0
if ( n_ <= 1 ) e_ = n_;
else e_ = ( oddq(n_) ? n_ - 1 : n_);
first();

void first()
{
  if ( n_ <= 1 )
  {
    a_[1] = n_;  
    m_ = n_; 
  }
  else
  {
    if ( oddq(n_) )
    { 
      a_[1] = n_; 
      m_ = 1; 
    }
    else
    { 
      a_[1] = 1; 
      a_[2] = n_ - 1; 
      m_ = 2; 
    }
  }
}

A few auxiliary routines are used.

ulong next_zeq1() // for Z == 1
{
  const ulong y = a_[m_-1];
  const ulong dy = par_to_dir_odd(y);
  if ( dy == +1 )
  {  // [*, Y, 1 ] --> [*, Y+1 ]
    a_[m_-1] = y + 1;
    m_ -= 1;
    return m_; 
  }
  else // dy == -1
  {  // [*, Y, 1 ] --> [*, Y-1, 1, 1 ]
    a_[m_-1] = y - 1;
    m_ += 1;
    a_[m_] = 1;
    return m_; 
  }
}

ulong next_zgt1() // for Z > 1
{
}
23   const ulong z = a_[m_];
24   const ulong y = a_[m_-1];
25   if ( oddq(z) )
26   { // [*, Z ] --> [*, Z-1, 1 ]
27     a_[m_] = z - 1;
28     m_ += 1;
29     a_[m_] = 1;
30     return m_;
31   }
32   else // Y >= 1
33   {
34     if ( y != 1 ) // Y > 1
35       { // [*, Y, Z ] --> [*, Y+-1, Z+-1 ]
36         const ulong dy = par_to_dir_even(y);
37         a_[m_-1] = y + dy;
38         a_[m_] = z - dy;
39         return m_;
40       }
41     else // Y == 1
42       {
43         const ulong x = a_[m_-2];
44         if ( x != 1 )
45           { // [*, X, 1, Z ] --> [*, X+-1, 1, Z-+1 ]
46             const ulong dx = par_to_dir_odd(x);
47             a_[m_-2] = x + dx;
48             a_[m_] = z - dx;
49             return m_;
50           }
51         else // X == 1
52           { // [*, X, 1, Z ] --> [*, X+1, 1, Z-1 ]
53             a_[m_-2] = x + 1;
54             a_[m_] = z - 1;
55             return m_;
56           }
57         }
58       }
59   }

The routine next() returns number of parts in the new composition and zero if there are no more compositions.

ulong next()
{
  ulong z = a_[m_];
  if ( z == e_ ) return 0; // current is last
  if ( z != 1 ) return next_zgt1();
  else return next_zeq1();
}

One update takes about 11 cycles.

We remark that the sublists of compositions into a fixed number of parts correspond to the combinations in enup order. The list of binary words on the right of figure 18 are those shown in [2, fig.6.6-B (left), p.189].

Ranking and unranking

Recursive routines for ranking and unranking are obtained easily. The following implementations are for the ordering with the first part decreasing.

// FILE: src/comb/composition-nz-rank.cc
ulong composition_nz_gray_rank(const ulong *x, ulong m, ulong n)
// Return rank r of composition x[], 0 <= r < 2**(n-1)
// where n is the sum of all parts.
{
  if ( m <= 1 ) return 0;
  ulong f = x[0]; // first part
  return composition_nz_gray_rank(x, m-1, n-f) + f * (1 << m-1);
1: 1..... { 0 }  
2: 11.... { 0, 1 }  
3: 111... { 0, 1, 2 }  
4: 111.. { 0, 1, 3 }  
5: 111.1 { 0, 1, 4 }  
6: 111.1 { 0, 1, 5 }  
7: 11...1 { 0, 2 }  
8: 11...1 { 0, 2, 3 }  
9: 11...1 { 0, 2, 4 }  
10: 11...1 { 0, 2, 5 }  
11: 11...1 { 0, 2, 6 }  
12: 11...1 { 0, 2, 7 }  
13: 11...1 { 0, 2, 8 }  
14: 11...1 { 0, 2, 9 }  
15: 11...1 { 0, 2, 10}  
16: 11...1 { 0, 2, 11}  
17: 11...1 { 0, 2, 12}  
18: 11...1 { 0, 2, 13}  
19: 11...1 { 0, 2, 14}  
20: 11...1 { 0, 2, 15}  
21: 11...1 { 0, 2, 16}  
22: 11...1 { 0, 2, 17}  
23: .1.1. { 1, 3, 4 }  
24: .1.1.1 { 1, 3, 5 }  
25: .1.1.1 { 1, 4 }  
26: .1.1.1 { 1, 4, 5 }  
27: .1.1...1 { 1, 5 }  
28: .1.1...1 { 2 }  
29: .1.1...1 { 2, 3 }  
30: .1.1...1 { 2, 3, 4 }  
31: .1.1...1 { 2, 3, 5 }  
32: .1.1...1 { 2, 4 }  
33: .1.1...1 { 2, 4, 5 }  
34: .1.1...1 { 2, 5 }  
35: .1.1...1 { 3 }  
36: .1.1...1 { 3, 4 }  
37: .1.1...1 { 3, 4, 5 }  
38: .1.1...1 { 3, 5 }  
39: .1.1...1 { 4 }  
40: .1.1...1 { 4, 5 }  
41: .1.1...1 { 5 }  

Figure 19: Nonempty subsets of the set \{0,1,\ldots,5\} with at most 3 elements.

10 ulong s = n - f; // remaining sum
11 ulong y = composition_nz_gray_rank(x+1, m-1, s);
12 if ( 0 == ( f & 1 ) ) // first part even
13 return ( 1UL << (s-1) ) + y;
14 else
15 return ( 1UL << s ) - 1 - y;
16 }
17
18 ulong
19 composition_nz_gray_unrank(ulong r, ulong *x, ulong n)
20 // Generate composition x[] of n with rank r.
21 // Return number of parts m of generated composition, 0 <= m <= n.
22 {
23 if ( r == 0 )
24 {
25 if ( n==0 ) return 0;
26 x[0] = n;
27 return 1;
28 }
29 }
30 ulong h = highest_one_idx(r);
31 ulong f = n - 1 - h; // first part
32 x[0] = f;
33 ulong b = 1UL << h; // highest one
34 r -= b; // delete highest one
35 bool p = f & 1; // first part f odd ?
36 if ( p ) r = b - 1 - r; // change direction with odd f
37 return 1 + composition_nz_gray_unrank( r, x+1, n-f );
38 }

11 Appendix: nonempty subsets with at most $k$ elements

The modifications needed in algorithm 1 for restricting the subsets to those with a prescribed maximal number of elements are quite small. Without ado, we give the implementation.

1 // FILE: src/comb/xsubset-lex.h

48
class ksubset_lex
{
    ulong n_; // number of elements in set, should have n>=1
    ulong j_; // number of elements in subset
    ulong m_; // max number of elements in subsets
    ulong *x_; // x[0...j-1]: subset of {0,1,2,...,n-1}
}

The computation of the successor is
ulong next()
{
    ulong j1 = j_ - 1;
    ulong z1 = x_[j1] + 1;
    if ( z1 < n_ ) // last element is not max
    {
        if ( j_ < m_ ) // append element
        {
            x_[j_] = x_[j1] + 1;
            ++j_;
            return j_;
        }
        x_[j1] = z1; // increment last element
        return j_;
    }
    else // last element is max
    {
        if ( j1 == 0 ) return 0; // current is last
        --j_;
        x_[j_-1] += 1;
        return j_;
    }
}

We omit the routine for the predecessor. Updates take no more time than the updates for the unrestricted subsets.

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References

[1] Jörg Arndt: FXT, a library of algorithms, (1996-2014). URL: \texttt{http://www.jjj.de/fxt/}
[2] Jörg Arndt: Matters computational – Ideas, Algorithms, Source Code. Springer-Verlag, (2011). URL: \texttt{http://www.jjj.de/fxt/#fxtbook}
[3] James R. Bitner, Gideon Ehrlich, Edward M. Reingold: Efficient generation of the binary reflected Gray code and its applications, Communications of the ACM, vol.19, no.9, pp.517-521, (September-1976).
[4] Bette Bultena, Frank Ruskey: An Eades-McKay Algorithm for Well-Formed Parentheses Strings, Information Processing Letters, vol.68, no.5, pp.255-259, (1998).
[5] Phillip J. Chase: Algorithm 383: permutations of a set with repetitions, Communications of the ACM, vol.13, no.6, pp.368-369, (June-1970).
[6] Phillip J. Chase: Combination generation and graylex ordering, Congressus Numerantium, vol.69, pp.215-242, (1989).
[7] Stephane Durocher, Pak Ching Li, Debajyoti Mondal, Aaron Williams: **Ranking and Loopless Generation of k-ary Dyck Words in Cool-lex Order**, The 22nd International Workshop on Combinatorial Algorithms, Victoria, Canada, IWOCA, (2011). URL: [http://webhome.csc.uvic.ca/~haron/](http://webhome.csc.uvic.ca/~haron/).

[8] Stanislav Dvořák: **Combinations in lexicographic order**, The Computer Journal, vol.33, no.1, p.188, (1990). URL: [http://comjnl.oxfordjournals.org/content/33/2.toc](http://comjnl.oxfordjournals.org/content/33/2.toc).

[9] Peter Eades, Brendan McKay: **An algorithm for generating subsets of fixed size with a strong minimal change property**, Information Processing Letters, vol.19, pp.31-133, (1986). URL: [http://cs.anu.edu.au/~bdm/publications.html](http://cs.anu.edu.au/~bdm/publications.html).

[10] Gideon Ehrlich: **Loopless Algorithms for Generating Permutations, Combinations, and Other Combinatorial Configurations**, Journal of the ACM, vol.20, no.3, pp.500-513, (July-1973).

[11] Gideon Ehrlich: **Four Combinatorial Algorithms**, Communications of the ACM, vol.16, no.11, pp.690-691, (November-1973).

[12] The Free Software Foundation (FSF): GCC, the GNU Compiler Collection, version 4.9.0. URL: [http://gcc.gnu.org/](http://gcc.gnu.org/).

[13] T. A. Jenkyns: **Loopless Gray Code Algorithms**, Technical Report CS-95-03, Brock University, Canada, (July-1995). URL: [http://www.cosc.brocku.ca/files/downloads/research/cs9503.ps](http://www.cosc.brocku.ca/files/downloads/research/cs9503.ps).

[14] Selmer M. Johnson: **Generation of permutations by adjacent transposition**, Mathematics of Computation, vol.17, no.83, pp.282-285, (July-1963). URL: [http://www.ams.org/journals/mcom/1963-17-083/S0025-5718-1963-0159764-2/](http://www.ams.org/journals/mcom/1963-17-083/S0025-5718-1963-0159764-2/).

[15] Richard Kaye: **A Gray Code For Set Partitions**, Information Processing Letters, vol.5, no.6, pp.171-173, (December-1976). URL: [http://www.kaye.to/rick/jkellehe/downloads/k06.pdf](http://www.kaye.to/rick/jkellehe/downloads/k06.pdf).

[16] Jerome Kelleher, Barry O’Sullivan: **Generating All Partitions: A Comparison Of Two Encodings**, arXiv:0909.2331v1 [cs.DS], (12-September-2009). URL: [http://arxiv.org/abs/0909.2331](http://arxiv.org/abs/0909.2331).

[17] Paul Klingsberg: **A Gray Code for Compositions**, Journal of Algorithms, vol.4, pp.41-44, (1982).

[18] Donald E. Knuth: **The Art of Computer Programming**, Volume 4A: Combinatorial Algorithms, Part 1, Addison-Wesley, (2011).

[19] Donald L. Kreher, Douglas R. Stinson.: **Combinatorial algorithms: generation, enumeration, and search**, CRC Press, (1998).

[20] Clement W. H. Lam, Leonard H. Soicher: **Three new combination algorithms with the minimal change property**, Communications of the ACM, vol.25, no.8, pp.555-559, (August-1982).

[21] Toufik Mansour, Ghalib Nassar: **Gray Codes, Loopless Algorithm and Partitions**, Journal of Mathematical Modelling and Algorithms, vol.7, pp.291-301, (2008).

[22] Toufik Mansour, Ghalib Nassar: **Loop-Free Gray Code Algorithms for the Set of Compositions**, Journal of Mathematical Modelling and Algorithms, vol.9, no.4, (December-2010).

[23] Jayadev Misra: **Remark on algorithm 246: Graycode [Z]**, ACM Transactions on Mathematical Software (TOMS), vol.1, no.3, p.285, (September-1975).

[24] Albert Nijenhuis, Herbert S. Wilf: **Combinatorical Algorithms for Computers and Calculators**, Academic Press, second edition, (1978). URL: [http://www.math.upenn.edu/~wilf/website/CombAlgDownld.html](http://www.math.upenn.edu/~wilf/website/CombAlgDownld.html).
[26] W. H. Payne, F. M. Ives: **Combination Generators**, ACM Transactions on Mathematical Software (TOMS), vol.5, no.2, pp.163-172, (June-1979).

[27] Edward M. Reingold, Jurg Nievergelt, Narsingh Deo: **Combinatorial algorithms – theory and practice**, Prentice-Hall, Englewood Cliffs, (1977).

[28] N. J. A. Sloane: **The On-Line Encyclopedia of Integer Sequences**, (1964-2014).
URL: [http://oeis.org/?blank=1](http://oeis.org/?blank=1).

[29] Ivan Stojmenović, M. Miyakawa: Applications of a subset generating algorithm to base enumeration, knapsack and minimal covering problems, The Computer Journal, vol.31, no.1, pp.65-70, (1988). URL: [http://www.site.uottawa.ca/~ivan/comb.html](http://www.site.uottawa.ca/~ivan/comb.html).

[30] H. F. Trotter: Algorithm 115: Perm, Communications of the ACM, vol.5, no.8, pp.434-435, (August-1962).

[31] Vincent Vajnovszki, Timothy R. Walsh: A loop-free two-close Gray-code algorithm for listing k-ary Dyck words, Journal of Discrete Algorithms, vol.4, no.4, pp.633-648, (December-2006). URL: [http://dx.doi.org/10.1016/j.jda.2005.07.003](http://dx.doi.org/10.1016/j.jda.2005.07.003).

[32] Timothy R. Walsh: A simple sequencing and ranking method that works on almost all Gray codes, Research Report 243, Department of Mathematics and Computer Science, University of Quebec at Montreal, (1995). URL: [http://www.info2.uqam.ca/~walsh_t/pages/papers.html](http://www.info2.uqam.ca/~walsh_t/pages/papers.html).

[33] Timothy R. Walsh: Generation of well-formed parenthesis strings in constant worst-case time, Journal of Algorithms, vol.29, no.1, pp.165-173, (1998). URL: [http://www.info2.uqam.ca/~walsh_t/pages/papers.html](http://www.info2.uqam.ca/~walsh_t/pages/papers.html).

[34] Timothy R. Walsh: Loop-free sequencing of bounded integer compositions, Journal of Combinatorial Mathematics and Combinatorial Computing, vol.33, pp.323-345, (2000). URL: [http://www.info2.uqam.ca/~walsh_t/pages/papers.html](http://www.info2.uqam.ca/~walsh_t/pages/papers.html).

[35] Timothy R. Walsh: Gray codes for involutions, Journal of Combinatorial Mathematics and Combinatorial Computing, vol.36, pp.95-18, (2001). URL: [http://www.info2.uqam.ca/~walsh_t/pages/papers.html](http://www.info2.uqam.ca/~walsh_t/pages/papers.html).

[36] Timothy R. Walsh: Generating Gray codes in $O(1)$ worst-case time per word, In: DMTCS 2003, C. S. Calude et al. (eds.), Lecture Notes in Computer Science, vol.2731, pp.73-88, (2003). URL: [http://www.info2.uqam.ca/~walsh_t/pages/papers.html](http://www.info2.uqam.ca/~walsh_t/pages/papers.html).

[37] Mark B. Wells: Generation of Permutations by Transposition, Mathematics of Computation, vol.15, no.74, pp.192-195, (April-1961). URL: [http://www.ams.org/journals/scom/1961-15-074/S0025-5718-1961-0127507-2/](http://www.ams.org/journals/scom/1961-15-074/S0025-5718-1961-0127507-2/).

[38] Aaron Williams: **Shift Gray Codes**, PhD thesis, University of Victoria, (2009). URL: [http://webhome.csc.uvic.ca/~haron/](http://webhome.csc.uvic.ca/~haron/).

[39] Aaron Williams: The coolest order of binary strings, 6th International Conference on Fun with Algorithms (FUN 2012), San Servolo, Italy. LNCS 7288, pp.322-333., (2012). URL: [http://www.math.mcgill.ca/haron/](http://www.math.mcgill.ca/haron/).

[40] Limin Xiang, Kazuo Ushijima: On O(1) Time Algorithms for Combinatorial Generation, The Computer Journal, vol.44, pp.292-302, (2001).

[41] Katsuhisa Yamanaka, Shin-ichiro Kawano, Yosuke Kikuchi, Shin-ichi Nakano: Constant Time Generation of Integer Partitions, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, vol.E90-A, no.5, pp.888-895, (May-2007).

[42] Antoine Zoghbi, Ivan Stojmenović: Fast algorithms for generating integer partitions, International Journal of Computer Mathematics, vol.70, no.2, pp.319-332, (1998). URL: [http://www.site.uottawa.ca/~ivan/comb.html](http://www.site.uottawa.ca/~ivan/comb.html).