Multi-scale Modeling for Piezoelectric Composite Materials

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Abstract

In this paper, we focus on multi-scale modeling and simulation of piezoelectric composite materials. A multi-scale model for piezoelectric composite materials under the framework of Heterogeneous Multi-scale Method (HMM) is proposed. For materials with periodic microstructure, macroscopic model is derived from microscopic model of piezoelectric composite material by asymptotic expansion. Convergence analysis under the framework of homogenization theory is carried out. Moreover, error estimate between HMM solutions and homogenization solutions is derived. A 3-D numerical example of 1-3 type piezoelectric composite materials is employed to verify the error estimate.

\textbf{Keywords:} Piezoelectric composite materials, Multi-scale modeling, Homogenization theory, Asymptotic expansion, Heterogeneous multi-scale method

1. Introduction

Piezoelectricity is the ability of some materials to generate an electric field or electric potential in response to mechanical strain applied. The piezoelectric effect is reversible in that materials exhibiting the direct piezoelectric effect (the production of an electric potential when stress is applied) also exhibit the reverse piezoelectric effect (the production of stress and/or strain when an electric field is applied). The effect is found useful in applications such as the production and detection of sound, generation of high voltages,

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electronic frequency generation, microbalances, and ultra fine focusing of optical assem-
blys. It is also the basis of a number of scientific instrumental techniques with atomic
resolution, the scanning probe microscopies such as STM, AFM, MTA, SNOM, etc., and
everyday uses such as acting as the ignition source for cigarette lighters and push-start
propane barbecues. Moreover, the piezoelectric effect and the reverse piezoelectric effect
can be reflected by the following two constitutive relationships respectively

\[
D_m = \varepsilon_{mn} E_n + d_{mkl} \sigma_{kl} \\
S_{ij} = s_{ijkl} \sigma_{kl} + d_{mij} E_m
\]

(1)

where \( D \) is electric displacement, \( \varepsilon_{mn} \) is permittivity, \( E \) is electric field strength, \( S_{ij} \) is
strain, \( s_{ijkl} \) is compliance and \( \sigma_{kl} \) is stress, \( d_{mkl} \) is piezoelectric constant.

It is apparent that some composite materials could be designed so as to attain prop-
erties desired in industry. In this way, the best properties from each constituent phase
within the composite may be utilized to create an improved material. In general, by
replacing a portion of a piezoceramic with a lightweight, flexible, polymer, the resulting
density, acoustic impedance, mechanical quality factor and dielectric constant can be
decreased. If the phases can be arranged so the piezoelectric charge coefficients of the
composite are maintained at reasonable levels, its voltage coefficients can be substantially
improved. As a result, the merit of the piezocomposites can actually surpass those of
single phase materials.

Therefore, according to aspects mentioned above, multi-scale modeling is necessary
to describe the behavior of piezoelectric composites for the following two reasons. On
one hand, in engineering, we are interested in relationship between micro-structure of
the composite materials and macro-properties of the composite materials because in this
way we can make the composite materials with desired macro-properties by controlling
its micro-structure. On the other hand, in numerical computation, the computation
of the parameters describing a micro-nonhomogeneous medium is an extremely difficult
task, since the coefficients of the corresponding differential equations are given by rapidly
oscillating functions. Many methods has been raised to get the multi-scale model, such
as MsFEM, RVE, HMM and etc.

A number of work have been done in this field by former researchers. Most of them
focus on the piezoelectric composite materials with some special microstructures. The
modeling of the piezoelectric composites with shell, perforated, periodic micro-structure
with theoretical analysis has been held by Marius et al. [5] and Bernadette et al. [1]. The modeling of 1–3 type piezoelectric composite without theoretical analysis has been held by Harald et al. [4] using RVE.

In this paper, a multi-scale model under the framework of HMM is designed. Corresponding theoretical analysis is derived. Differing from the elliptic equations whose solution is a minimum point of its energy functional, the solution of piezoelectric equations is a saddle point of its energy functional, which is caused by the coupling of mechanical field and electrical field. This difference brought main obstacle for theoretical analysis. We also deduced the macroscopic model from the microscopic model of piezoelectric composite material for materials with periodic microstructure and carried out the corresponding convergence analysis under the framework of homogenization. For the analysis of convergence, noticing that the regularity of the solutions of cell problems in the model cannot reach $W^{1,\infty}$, which makes the traditional treatment to this kind of problem not work on our problem. To solve this problem, we employ two useful lemmas, whose original idea is from Tatyana et al. [8] under the assumption that the regularity of solutions of cell problems in the model can reach $L^\infty$. Moreover, error estimate between HMM solutions and homogenization solutions is derived. A 3-D numerical example was given out to verify the error estimate.

The rest of the paper is organized as follows. In Section 2, we introduction the multi-scale problem of piezoelectric composite material briefly. In Section 3, a multi-scale model under the framework of HMM is designed. The macroscopic model from the microscopic model of piezoelectric composite material for materials with periodic microstructure is derived by asymptotic expansion in Section 4. We also give out the corresponding convergence analysis for the model we derived under the framework of homogenization in Section 4. In Section 5, both error estimate between HMM solutions and homogenization solutions and error estimate of the effective coefficients is derived. Numerical examples are employed to verify the error estimate in Section 6. The paper concludes in Section 7.
2. Piezoelectric Equations

Under the action of applied volume loading \( f \in L^2(\Omega) \) and without electric charges, the electroelastic state of piezoelectric medium is governed by the following system of equations.

- **Equations of motion:**
  \[
  -\frac{\partial \sigma_{ij}}{\partial x_j} = f_i \quad \text{in} \quad \Omega \quad (2)
  \]

- **Maxwell’s equations (in the quasistatic approximation):**
  \[
  \frac{\partial D_i}{\partial x_i} = 0 \quad E_j = -\frac{\partial \phi}{\partial x_i} \quad \text{in} \quad \Omega \quad (3)
  \]

- **Constitutive relations:**
  \[
  \sigma_{ij} = c_{ijkn} \frac{\partial u_k}{\partial x_n} - e_{kij} E_k \quad (4)
  \]
  \[
  D_i = e_{ikn} \frac{\partial u_k}{\partial x_n} + \epsilon_{ij} E_j \quad (5)
  \]

Without lose of generalization, we just take the Dirichlet Boundary condition as follows.

\[
  u_i = 0, \quad \varphi = 0 \quad \text{on} \quad \partial \Omega \quad (6)
\]

where \( \sigma_{ij} \) is stress tensor, \( u \) is elastic displacement field, \( D \) is electrical displacement, \( E \) is electric field, \( \varphi \) is potential field.

Hence, characteristics of the piezoelectric material are given by elastic tensor \((c_{ijkn})\), dielectric tensor \((\epsilon_{ij})\) and piezoelectric tensor \((e_{kij})\). These three tensors have the following properties:

- The elastic tensor \((c_{ijkn})\) is symmetric and positive defined, that is,
  \[
  c_{ijkl} = c_{jikl} = c_{kitj}
  \]
  and there exists \( \alpha > 0 \), such that \( c_{ijkl}X_{ij}X_{kl} \geq \alpha X_{ij}X_{kl}, \forall X_{ij} = X_{ji} \in \mathbb{R} \)

- The dielectric tensor \((\epsilon_{ij})\) is symmetric and positive defined, that is,
  \[
  \epsilon_{ij} = \epsilon_{ji}
  \]
  and there exists \( \beta > 0 \), such that \( \epsilon_{ij}X_{ij}X_{ij} \geq \beta X_{ij}^2, \forall X_{ij} \in \mathbb{R} \)
• The piezoelectric tensor \((e_{kl})\) is symmetric in the sense that \(e_{kl} = e_{lk}\).

• \(c_{ijkl} \in L^\infty(\Omega), d_{ij} \in L^\infty(\Omega), e_{ijk} \in L^\infty(\Omega)\)

Thus, in the rest parts of this paper, we consider the following equation system,

\[
\begin{aligned}
-\frac{\partial \sigma^{(c)}_{ij}}{\partial x_j} &= f_i & x \in \Omega \subset \mathbb{R}^d \\
\frac{\partial D^{(c)}_{ij}}{\partial x_i} &= 0 & x \in \Omega \subset \mathbb{R}^d \\
\sigma^{(c)}_{ij} &= c^{(c)}_{ijkl} \frac{\partial u^{(c)}_k}{\partial x_k} + e^{(c)}_{kl} \frac{\partial \varphi^{(c)}}{\partial x_k} & x \in \Omega \\
D^{(c)}_{ij} &= e^{(c)}_{ikn} \frac{\partial u^{(c)}_n}{\partial x_n} - \epsilon^{(c)}_{ij} \frac{\partial \varphi^{(c)}}{\partial x_j} & x \in \Omega \\
u^{(c)}_i &= 0 & \varphi^{(c)} = 0 & x \in \partial \Omega
\end{aligned}
\tag{7}
\]

where \(\varepsilon \ll 1\) is the characteristic non-homogeneity dimension.

Moreover, through this paper, Latin indices and exponents take their values in the set \(\{1,2,3\}\), if there is no special illustration. The average symbol \(\langle \cdot \rangle\) is defined as

\[
\langle \cdot \rangle = \frac{1}{|\square|} \int_{\square} \cdot \, dy,
\]

where \(|\square|\) is the volume of \(\square\). Let \(\Omega\) be a polyhedral domain in \(\mathbb{R}^d\) with a Lipschitz boundary \(\partial \Omega\) whose unit outer normal is denoted by \(n\). In the derivations below, we use \(L^2(\Omega)\) based Sobolev spaces \(H^k(\Omega)\) equipped with norms and seminorms

\[
\| u \|_{k, \Omega} = \left( \int_{\Omega} \sum_{|\alpha| \leq k} |D^\alpha u|^2 \right)^{1/2}, \quad | u |_{k, \Omega} = \left( \int_{\Omega} \sum_{|\alpha| = k} |D^\alpha u|^2 \right)^{1/2}.
\]

3. HMM modeling of piezoelectric composite materials

HMM (Heterogeneous multiscale method) by W. E et al. \cite{11} is a general methodology for designing sublinear algorithms by exploiting scale separation and other special features of the problem. It consists of two components: selection of a macroscopic solver and estimating the missing macroscale data by solving locally the fine scale problem.

For \(\Omega\), the macroscopic solver can be chosen as standard piecewise linear finite element method on a triangulation \(\mathcal{T}_H\) of element size \(H\) which should resolve the macroscale features of \(c^{(c)}_{ijkl}, e^{(c)}_{klj}\) and \(\epsilon^{(c)}_{ij}\). The missing data are the effective stiffness matrices
at this scale. Assuming that the effective coefficients at this scale are $c_{ijkn}^H$, $e_{ij}^H$, and $\epsilon_{ij}^H$, if we know $c_{ijkn}^H$, $e_{ij}^H$, and $\epsilon_{ij}^H$ explicitly, macroscopic piezoelectric equation system could be solved by FEM in the following variational form.

Find $u \in X_H \times X_H \times X_H$, $\varphi \in X_H$ s.t.

\[
\begin{align*}
&c_H(u, v) + e_H(v, \varphi) = (f, v), \quad \forall \ v \in X_H \times X_H \times X_H \\
-e_H(u, \psi) + d_H(\varphi, \psi) = 0, \quad \forall \ \psi \in X_H
\end{align*}
\]

(8)

where

\[
c_H(u, v) = \int_{\Omega} c_{ijkl}^H s_{ij}(u) s_{kl}(v) dx \simeq \sum_{K \in T_H} |K| \sum_{x_{\alpha} \in K} w_{\alpha} (c_{ijkl}^H s_{ij}(u) s_{kl}(v))(x_{\alpha})
\]

\[
e_H(v, \varphi) = \int_{\Omega} e_{ij}^H s_{ij}(v) \partial_k \varphi dx \simeq \sum_{K \in T_H} |K| \sum_{x_{\alpha} \in K} w_{\alpha} (e_{ij}^H s_{ij}(v) \partial_k \varphi)(x_{\alpha})
\]

\[
d_H(\varphi, \psi) = \int_{\Omega} d_H(\varphi, \psi) dx \simeq \sum_{K \in T_H} |K| \sum_{x_{\alpha} \in K} w_{\alpha} (d_H(\varphi, \psi))(x_{\alpha})
\]

(9)

where $x_{\alpha}$ and $w_{\alpha}$ are the quadrature points and weights in $K$, $|K|$ is the volume of $K$.

However, we cannot get $c_{ijkl}^H$, $e_{ij}^H$ and $\epsilon_{ij}^H$ explicitly in most cases. In the absence of explicit knowledge of $c_{ijkl}^H$, $e_{ij}^H$ and $\epsilon_{ij}^H$, we approximation $c_H(u, v)$, $e_H(v, \varphi)$ and $d_H(\varphi, \psi)$ by solving two microscopic problems as follows on the samples we have chosen.

\[
\begin{align*}
-\frac{\partial \sigma_{ij}^{(c)}}{\partial x_j} &= 0 \quad x \in I_\delta(x_{\alpha}) \\
\frac{\partial D_i^{(c)}}{\partial x_i} &= 0 \quad x \in I_\delta(x_{\alpha}) \\
\sigma_{ij}^{(c)} &= c_{ijkl}^{(c)} \frac{\partial (v_{\alpha})_k^{(c)}}{\partial x_n} + \epsilon_{ij}^{(c)} \frac{\partial (\varphi_{\alpha})^{(c)}}{\partial x_k} \\
D_i^{(c)} &= \epsilon_{ijkn}^{(c)} \frac{\partial (v_{\alpha})_k^{(c)}}{\partial x_n} - \epsilon_{ij}^{(c)} \frac{\partial (\varphi_{\alpha})^{(c)}}{\partial x_j} \\
v_{\alpha}^{(c)} &= V_\alpha(x) \quad \varphi_{\alpha}^{(c)} = 0 \quad x \in \partial I_\delta(x_{\alpha})
\end{align*}
\]

(10)
Estimation of the macroscopic effective coefficients could be given directly by solving the

\[
\begin{align*}
-\frac{\partial \sigma_{ij}(c)}{\partial x_j} &= 0 & x &\in I_\delta(x_\alpha) \\
\frac{\partial D_i(c)}{\partial x_i} &= 0 & x &\in I_\delta(x_\alpha) \\
\sigma_{ij}(c) &= e_{ijkn}(c) \frac{\partial (v_{\alpha x_k})}{\partial x_n} + \epsilon_{ijk} \frac{\partial u_{\alpha i}}{\partial x_k} \\
D_i(c) &= e_{ikn}(c) \frac{\partial (v_{\alpha x_n})}{\partial x_k} - \epsilon_{ij} \frac{\partial \varphi_{\alpha c}}{\partial x_j} \\
v_{\alpha c} &= 0 & \varphi_{\alpha c} &= \varphi_\alpha(x) & x &\in \partial I_\delta(x_\alpha)
\end{align*}
\]

where \( I_\delta(x_\alpha) = x_\alpha + [-\frac{\delta}{2}, \frac{\delta}{2}]^d \) is a cubic of size \( \delta \) centered at \( x_\alpha \) and \( V_\alpha \) is the linear approximation of \( v \) at \( I_\delta(x_\alpha) \). For the macroscopic finite element space \( X_H \) we have chosen, \( V_\alpha \) is \( v \). Similarly, \( \varphi_\alpha(x) \) is the linear approximation of \( \varphi \) at \( I_\delta(x_\alpha) \).

![Illustration of HMM](image)

Figure 1. Illustration of HMM for solving (7). The dots are the quadrature points in \( K \).

we define the corresponding bilinear forms as follows.

For any \( u \in X_H \times X_H \times X_H, v \in X_H \times X_H \times X_H, \varphi \in X_H, \psi \in X_H \)

\[
\begin{align*}
c_H(u, v) &= \sum_{K \in \mathcal{T}_H} |K| \sum_{x_\alpha \in K} w_\alpha \frac{1}{|I_\delta(x_\alpha)|} \int_{I_\delta(x_\alpha)} (e_{ikl} s_{ij}(v_{\alpha k}) s_{kl}(v_{\alpha l}) + e_{ij} s_{ij}(u_{\alpha i}) \partial_{k} \varphi_{\alpha c}) dx \\
e_H(v, \varphi) &= \sum_{K \in \mathcal{T}_H} |K| \sum_{x_\alpha \in K} w_\alpha \frac{1}{|I_\delta(x_\alpha)|} \int_{I_\delta(x_\alpha)} (e_{ikl} s_{ij}(v_{\alpha k}) \partial_{k} \varphi_{\alpha c} - e_{ij} \partial_{k} \varphi_{\alpha c} \partial_{l} \varphi_{\alpha c}) dx \\
e_H(v, \varphi) &= \sum_{K \in \mathcal{T}_H} |K| \sum_{x_\alpha \in K} w_\alpha \frac{1}{|I_\delta(x_\alpha)|} \int_{I_\delta(x_\alpha)} (e_{ikl} s_{ij}(v_{\alpha k}) \partial_{k} \varphi_{\alpha c} + e_{ij} \partial_{k} \varphi_{\alpha c} \partial_{l} \varphi_{\alpha c}) dx \\
d_H(\varphi, \psi) &= -\sum_{K \in \mathcal{T}_H} |K| \sum_{x_\alpha \in K} w_\alpha \frac{1}{|I_\delta(x_\alpha)|} \int_{I_\delta(x_\alpha)} (e_{ikl} s_{ij}(v_{\alpha k}) \partial_{k} \psi_{\alpha c} - e_{ij} \partial_{k} \varphi_{\alpha c} \partial_{l} \psi_{\alpha c}) dx
\end{align*}
\]

Moreover, cell problems of HMM held above are equal to the following cell problems.

Estimation of the macroscopic effective coefficients could be given directly by solving the
microscopic equation systems as follows.

\[
\begin{cases}
- \frac{\partial}{\partial x_j} \left( c^{(e)}_{ij} \frac{\partial P^{kl}}{\partial x_h} + c^{(e)}_{hi} \frac{\partial \Phi^{kl}}{\partial x_h} \right) = 0 & x \in I_\delta (x_a) \\
\frac{\partial}{\partial x_j} \left( e^{(c)}_{inh} \frac{\partial P^{kl}}{\partial x_h} - e^{(c)}_{ih} \frac{\partial \Phi^{kl}}{\partial x_h} \right) = 0 & x \in I_\delta (x_a) \\
P_k = x_l \quad P_n = 0 \quad (n \neq k) & x \in \partial I_\delta (x_a) \\
\Phi_{kl} = 0 & x \in \partial I_\delta (x_a)
\end{cases}
\]  

(12)

Estimation of \( e_{ijkl}^H \) and \( e_{ijkl}^H \) are given by

\[
\begin{align*}
e_{ijkl}^H (x_a) &= \int_{I_\delta (x_a)} \left( c^{(c)}_{ijnh} \frac{\partial P^{kl}}{\partial x_h} \right) dx_a \\
e_{ijkl}^H (x_a) &= \int_{I_\delta (x_a)} \left( c^{(c)}_{inlh} \frac{\partial P^{kl}}{\partial x_h} \right) dx_a 
\end{align*}
\]

(13)

and

\[
\begin{cases}
- \frac{\partial}{\partial x_j} \left( c^{(c)}_{i} \frac{\partial Q^{l}}{\partial x_h} + e^{(c)}_{h} \frac{\partial \Psi^{l}}{\partial x_h} \right) = 0 & x \in I_\delta (x_a) \\
\frac{\partial}{\partial x_i} \left( e^{(c)}_{i} \frac{\partial Q^{l}}{\partial x_h} - e^{(c)}_{h} \frac{\partial \Psi^{l}}{\partial x_h} \right) = 0 & x \in I_\delta (x_a) \\
Q_{n}^{l} = 0 & x \in \partial I_\delta (x_a) \\
\Psi_{l} = x_l & x \in \partial I_\delta (x_a)
\end{cases}
\]

(15)

Estimation of \( e_{ij} \) and \( e_{ikl} \) are given by

\[
\begin{align*}
e_{ij}^H (x_a) &= \int_{I_\delta (x_a)} \left( c^{(c)}_{ij} \frac{\partial \Psi^{l}}{\partial x_j} \right) dx_a \\
e_{ikl}^H (x_a) &= \int_{I_\delta (x_a)} \left( c^{(c)}_{ikl} \frac{\partial \Psi^{l}}{\partial x_k} \right) dx_a 
\end{align*}
\]

(16)

The following results assert that estimate of the effective coefficients \( e_{ijkl}^H (x), e_{ii}^H (x) \) and \( e_{ijkl}^H (x) \) give above share the same symmetric property and elliptic property with \( e_{i}^{(c)}_{ijkn} (x), e_{ij}^{(c)} (x) \) and \( e_{ij}^{(c)} (x) \) as follows.
Lemma 3.1. If P, Q, Φ, Ψ are solutions of (12) and (15), $e^H_{ikl}$, $c^H_{ijkl}$ and $\epsilon^H_{ij}$ defined by (13) and (16) have the following properties,

(I) 
$$
e^H_{ikl} = \langle e^{(c)}_{inh} \frac{\partial P_n^k}{\partial x_h} - e^{(c)}_{ih} \frac{\partial \Phi_{kl}}{\partial x_h} \rangle_{I_s} = \langle e^{(c)}_{klk} \frac{\partial Q_n^i}{\partial x_h} + e^{(c)}_{kkl} \frac{\partial \Psi_{i}}{\partial x_h} \rangle_{I_s}$$

(II) 
$$c^H_{ijkl} = c^H_{jikl} = c^H_{klij} = e^H_{klij} = e^H_{kjli} = \epsilon^H_{ij}$$

(III) 
$$\exists \alpha_H > 0, s.t. c^H_{ijkl}X_{ij}X_{kl} \geq \alpha_H X_{ij}X_{kl} \quad \text{and} \quad \exists \beta_H > 0, s.t. \epsilon^H_{ij}X_iX_j \geq \beta_H X_iX_j$$

Proof. The main idea for proving (I) and (II) is to take proper test functions in the variation forms of (12) and (15). The main idea for proving (III) is to make full use of property of energy functional of piezoelectric equation system, which is caused by the coupling of mechanical field and electric field.

In (12), taking $N_{nk}^l = P_n^l$ for $n \neq k$ and $N_{kn}^l = P_k^l - x_l$ we have,

$$
\begin{align*}
\frac{\partial}{\partial x_j} (e^{(c)}_{ijnh} \frac{\partial N_{nk}^l}{\partial x_h} + e^{(c)}_{inh} \frac{\partial \Phi_{kl}}{\partial x_h}) &= - \frac{\partial e^{(c)}_{ijkl}}{\partial x_j} \quad &x \in I_s(x_a) \\
\frac{\partial}{\partial x_i} (e^{(c)}_{inh} \frac{\partial N_{nk}^l}{\partial x_h} - e^{(c)}_{ih} \frac{\partial \Phi_{kl}}{\partial x_h}) &= - \frac{\partial e^{(c)}_{ijkl}}{\partial x_i} \quad &x \in I_s(x_a) \\
N_{nk}^l &= 0 \quad \Phi_{kl} = 0 \quad &x \in \partial I_s(x_a)
\end{align*}
$$

The variational form of (18) is,

$$
\begin{align*}
\int_{I_s(x_a)} (e^{(c)}_{ijnh} \frac{\partial N_{nk}^l}{\partial x_h} \frac{\partial G_i}{\partial x_j} + e^{(c)}_{ih} \frac{\partial \Phi_{kl}}{\partial x_h} \frac{\partial G_i}{\partial x_j}) dx &= \int_{I_s(x_a)} e^{(c)}_{ijkl} \frac{\partial G_i}{\partial x_j} dx \\
- \int_{I_s(x_a)} (e^{(c)}_{ijnh} \frac{\partial N_{nk}^l}{\partial x_h} \frac{\partial \zeta}{\partial x_i} - e^{(c)}_{ih} \frac{\partial \Phi_{kl}}{\partial x_h} \frac{\partial \zeta}{\partial x_i}) dx &= \int_{I_s(x_a)} e^{(c)}_{ijkl} \frac{\partial \zeta}{\partial x_j} dx
\end{align*}
$$

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for any $G \in H^1_0(I_\delta) \times H^1_0(I_\delta) \times H^1_0(I_\delta)$ and $\zeta \in H^1_0(I_\delta)$.

In (15), taking $\psi_t = \Psi - x_t$, we have,

$$
\begin{aligned}
& -\frac{\partial}{\partial x_j}(e_{ijh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} + e_{ih}^{(c)} \frac{\partial \psi_t}{\partial x_h}) = -\frac{\partial e_{ijh}^{(c)}}{\partial x_j} x \in I_\delta(x_a) \\
& \frac{\partial}{\partial x_i}(e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} - e_{ih}^{(c)} \frac{\partial \psi_t}{\partial x_h}) = \frac{\partial e_{inh}^{(c)}}{\partial x_i} x \in I_\delta(x_a) \\
& Q_n^i = 0 \quad \psi_t = 0 \quad x \in \partial I_\delta(x_a)
\end{aligned}
$$

(20)

The variational form of (20) is,

$$
\begin{aligned}
& \int_{I_\delta(x_a)} (e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} - e_{ih}^{(c)} \frac{\partial \psi_t}{\partial x_h}) dx = \int_{I_\delta(x_a)} e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} dx \\
& \int_{I_\delta(x_a)} (e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} - e_{ih}^{(c)} \frac{\partial \psi_t}{\partial x_h}) dx = \int_{I_\delta(x_a)} e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} dx
\end{aligned}
$$

(21)

for any $F \in H^1_0(I_\delta(x_a)) \times H^1_0(I_\delta(x_a)) \times H^1_0(I_\delta(x_a))$ and $\xi \in H^1_0(I_\delta(x_a))$

(1) Taking $(G, \zeta) = (Q^t, \psi_t)$ in (19), we have

$$
\begin{aligned}
& \int_{I_\delta(x_a)} (e_{inh}^{(c)} \frac{\partial N^k_l}{\partial x_m} \frac{\partial Q^i_n}{\partial x_h} + e_{mnh}^{(c)} \frac{\partial \Phi^k_l}{\partial x_m} \frac{\partial Q^i_n}{\partial x_h}) dx = \int_{I_\delta(x_a)} e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} dx \\
& - \int_{I_\delta(x_a)} (e_{inh}^{(c)} \frac{\partial N^k_l}{\partial x_m} \frac{\partial \psi_t}{\partial x_h} + e_{mnh}^{(c)} \frac{\partial \Phi^k_l}{\partial x_m} \frac{\partial \psi_t}{\partial x_h}) dx = \int_{I_\delta(x_a)} e_{inh}^{(c)} \frac{\partial \psi_t}{\partial x_h} dx
\end{aligned}
$$

(22)

Taking $(F, \zeta) = (N^k_l, \Phi^k_l)$ in (21), we obtain

$$
\begin{aligned}
& \int_{I_\delta(x_a)} (e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} \frac{\partial N^k_l}{\partial x_m} + e_{mnh}^{(c)} \frac{\partial \psi_t}{\partial x_h} \frac{\partial N^k_l}{\partial x_m}) dx = \int_{I_\delta(x_a)} e_{inh}^{(c)} \frac{\partial N^k_l}{\partial x_m} dx \\
& \int_{I_\delta(x_a)} (e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} \frac{\partial \Phi^k_l}{\partial x_m} - e_{mnh}^{(c)} \frac{\partial \psi_t}{\partial x_h} \frac{\partial \Phi^k_l}{\partial x_m}) dx = \int_{I_\delta(x_a)} e_{inh}^{(c)} \frac{\partial \Phi^k_l}{\partial x_m} dx
\end{aligned}
$$

(23)

Since

$$
\begin{aligned}
& \langle e_{inh}^{(c)} \frac{\partial N^k_l}{\partial x_h} - e_{ih}^{(c)} \frac{\partial \Phi^k_l}{\partial x_h}, I_\delta(x_a) \rangle = \langle e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} - e_{ih}^{(c)} \frac{\partial \psi_t}{\partial x_h}, I_\delta(x_a) \rangle \\
& \langle e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} + e_{ih}^{(c)} \frac{\partial \psi_t}{\partial x_h}, I_\delta(x_a) \rangle = \langle e_{inh}^{(c)} \frac{\partial Q^i_n}{\partial x_h} + e_{ih}^{(c)} \frac{\partial \psi_t}{\partial x_h}, I_\delta(x_a) \rangle
\end{aligned}
$$

(24)
Therefore, we get
\[
e^{(c)}_{ijkl} = (c^{(c)}_{inh} \frac{\partial P_{ij}}{\partial x_h} - c^{(c)}_{ih} \frac{\partial \Phi_{kl}}{\partial x_h})_{I_s} = (c^{(c)}_{kl} \frac{\partial Q_{ij}}{\partial x_h} + c^{(c)}_{hl} \frac{\partial \Psi_{ijkl}}{\partial x_h})_{I_s}
\]

by the symmetric property of \(c^{(c)}_{ijkl}(x), c^{(c)}_{ij}(x), c^{(c)}_{ijk}(x)\) and subtracting of (22) and (23) respectively.

(II) Taking \((\mathbf{G}, \zeta) = (\mathbf{N}^{ij}, \Phi_{ij})\) in (19) we obtain
\[
c^{(H)}_{ijkl} = \langle c^{(c)}_{smnk} \frac{\partial P_{ij}}{\partial x_h} + c^{(c)}_{shm} \frac{\partial \Phi_{kl}}{\partial x_h} \rangle_{I_s(x_a)} = \langle c^{(c)}_{ijkl} + c^{(c)}_{ih} \frac{\partial N_{ij}}{\partial x_h} + c^{(c)}_{ij} \frac{\partial \Phi_{kl}}{\partial x_h} \rangle_{I_s(x_a)}
\]

Therefore, \(c^{(H)}_{ijkl} = c^{(H)}_{klij}\)

Since
\[
c^{(H)}_{ijkl} = \langle c^{(c)}_{ijkl} \frac{\partial N_{ij}}{\partial x_h} + c^{(c)}_{ij} \frac{\partial \Phi_{kl}}{\partial x_h} \rangle_{I_s(x_a)} = \langle c^{(c)}_{ijkl} + c^{(c)}_{kh} \frac{\partial N_{ij}}{\partial x_h} + c^{(c)}_{ik} \frac{\partial \Phi_{kl}}{\partial x_h} \rangle_{I_s(x_a)}
\]

we have \(c^{(H)}_{ijkl} = c^{(H)}_{klij}\) by the symmetric property of \(c^{(c)}_{ijkl}(x), c^{(c)}_{ij}(x), c^{(c)}_{ijk}(x)\) in (18).

Similarly, it is easy to prove that \(e^{(H)}_{klj} = e^{(H)}_{kjy}, e^{(H)}_{ij} = e^{(H)}_{ji}\)

(III) By (20), for any \(X = (X_{ij}) \neq 0\), we have
\[
e^{(H)}_{ijkl}X_{ij}X_{kl} = \langle c^{(c)}_{ijkl}X_{ij}X_{kl} + c^{(c)}_{ij} \frac{\partial N_{ij}}{\partial x_h}X_{ij}X_{kl} + c^{(c)}_{ik} \frac{\partial \Phi_{kl}}{\partial x_h}X_{ij}X_{kl} \rangle_{I_s(x_a)}
\]

Setting \((\mathbf{N}, \Phi) = (N^{kl}_{ij}, \Phi_{ij}X_{kl})\) in (19), we obtain
\[
\begin{cases}
\int_{I_s(x_a)} (c^{(c)}_{ijkl} \frac{\partial N_{ij}}{\partial x_h} + c^{(c)}_{ij} \frac{\partial \Phi_{kl}}{\partial x_h})_{I_s(x_a)} = \int_{I_s(x_a)} c^{(c)}_{ijkl} \frac{\partial G_{ij}}{\partial x}X_{kl}
\end{cases}
\]

Since \((\mathbf{N}, \Phi)\) is the saddle point associated to the energy functional defined by
\[
I[\mathbf{V}, \Phi] = \frac{1}{2} \int_{I_s(x_a)} (c^{(c)}_{ijkl} \frac{\partial V_{ij}}{\partial x_h} + X_{nh})(\frac{\partial V_{ij}}{\partial x} + X_{ij})dx - \frac{1}{2} \int_{I_s(x_a)} e^{(c)}_{ih} \frac{\partial \Psi_{ijkl}}{\partial x_h} + X_{nh}
\]

Thus, we obtain
\[
I[\mathbf{N}, \Phi] \leq I[\mathbf{N}, \Phi] \leq I[\mathbf{V}, \Phi], \forall \mathbf{V} \in (H^1_0(I_\delta))^3, \Phi \in H^1_0(I_\delta)
\]
which implies \( I[N, \Phi] \geq I[N, 0] > 0 \)

Taking \((G, \zeta) = (N, \Phi)\) in (30), we get

\[
e^H_{ijkl} X_{ij} X_{kl} = 2I[N, \Phi] > 0 \tag{33}
\]

\( \mathcal{B} = \{X = (X_{ij}) : X\) is symmetric and \(X_{ij} X_{ij} = 1\}\) and consider \(\Psi : \mathcal{B} \mapsto \mathbb{R}\) defined by \(\Psi(X_{ij}) = e^H_{ijkl} X_{ij} X_{kl}\), it is easy to see \(\Psi\) is continuous on \(\mathcal{B}\) and \(\Psi > 0\), which implies there exists \(\alpha_H > 0\) such that

\[
\Psi(\frac{X_{ij}}{\|X\|}) \geq \alpha_H, \text{ for any } X = (X_{ij}) \neq 0 \tag{34}
\]

Therefore, there exists \(\alpha_H > 0\) such that \(e^H_{ijkl} X_{ij} X_{kl} \geq \alpha_H X_{ij} X_{kl}\)

Similarly, it is easy to prove that there exists \(\beta_H > 0\) such that \(e^H_{ij} X_{ij} \geq \beta_H X_{ij} X_{ij}\).

This completes the proof of Lemma 3.1.

In Section 5, we will give out both error estimate between the HMM solutions and the homogenization solutions and error estimate of the effective coefficients for piezoelectric composite materials with periodic microstructure.

4. Homogenization theory

For those piezoelectric composite materials with periodic microstructure we consider the following equation system,

\[
\begin{aligned}
\frac{\partial \sigma^{(c)}_{ij}}{\partial x_j} &= f_i, & x \in \Omega \subset \mathbb{R}^d \\
\frac{\partial D^{(c)}_{i}}{\partial x_i} &= 0, & x \in \Omega \subset \mathbb{R}^d \\
\sigma^{(c)}_{ij} &= c^{(c)}_{ijkn} \frac{\partial u^{(c)}_k}{\partial x_n} + e^{(c)}_{kij} \frac{\partial \varphi^{(c)}}{\partial x_k}, & \sigma^{(c)}_{ij} = c^{(c)}_{ij} \frac{\partial u^{(c)}_k}{\partial x_n} - e_{ij} \frac{\partial \varphi^{(c)}}{\partial x_j}, & \sigma^{(c)}_{ij} = \sigma^{(c)}_{ij} + \frac{\partial \varphi^{(c)}}{\partial x_j}, & D^{(c)}_{i} = e^{(c)}_{ijkn} \frac{\partial u^{(c)}_k}{\partial x_n} - \epsilon_{ij} \frac{\partial \varphi^{(c)}}{\partial x_j}, & u^{(c)}_{i} = 0, & \varphi^{(c)} = 0, & x \in \partial \Omega
\end{aligned}
\]

where \(c^{(c)}_{ijkn}(x) = c_{ijkn}(x, y), e^{(c)}_{ij}(x) = c_{ij}(x, y), e^{(c)}_{ij}(x) = \epsilon_{ij}(x, y)\) are periodic in \(y = \frac{x}{\varepsilon}\) with reference cell \(Y = [0, 1]^d\) as their periodic. For understanding easily, we assume that the coefficients are periodic globally, i.e.

\[
\begin{aligned}
c^{(c)}_{ijkn}(x) &= c_{ijkn}(\frac{x}{\varepsilon}), & e^{(c)}_{ij}(x) &= e_{ij}(\frac{x}{\varepsilon}), & \epsilon^{(c)}_{ij}(x) &= \epsilon_{ij}(\frac{x}{\varepsilon})
\end{aligned}
\]

(36)
Following the step in homogenization theory by Jikov et al. [10], we get the homogenized equation system of (35) as follows,

\[
\begin{align*}
-\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} &= f_i & x \in \Omega \\
\frac{\partial D_i^{(0)}}{\partial x_i} &= 0 & x \in \Omega \\
\sigma_{ij}^{(0)} &= c_{ij}^{(0)} \frac{\partial u_k^{(0)}}{\partial x_n} + e_{ij}^{(0)} \frac{\partial \varphi^{(0)}}{\partial x_k} \\
D_i^{(0)} &= e_{ik}^{(0)} \frac{\partial u_k^{(0)}}{\partial x_n} - e_{ij}^{(0)} \frac{\partial \varphi^{(0)}}{\partial x_n} \\
u_i^{(0)} &= 0 & \varphi^{(0)} = 0 & x \in \partial \Omega
\end{align*}
\]

where \(c_{ij}^{(0)}, e_{ij}^{(0)}\) and \(\epsilon_{ij}^{(0)}\) are the homogenized effective coefficients given by,

\[
\begin{align*}
c_{ij}^{(0)} &= \langle c_{ijkl}(y) + \tau_{ij}^{kl}(y) \rangle_Y \\
e_{ij}^{(0)} &= \langle e_{ij}(y) + \xi_{ij}^l(y) \rangle_Y \\
\epsilon_{ij}^{(0)} &= \langle \epsilon_{ij}(y) - \gamma_{ij}(y) \rangle_Y
\end{align*}
\]

where \(\tau_{ij}^{kl}(y), \xi_{ij}^l(y), \gamma_{ij}^l(y), \gamma_{ij}(y)\) can be got by solving the following two cell problems,

\[
\begin{align*}
-\frac{\partial \tau_{ij}^{kl}}{\partial y_j} &= -\frac{\partial c_{ijkl}(y)}{\partial y_j} & y \in Y \\
-\frac{\partial d_i^{kl}}{\partial y_i} &= -\frac{\partial e_{ijkl}(y)}{\partial y_i} & y \in Y \\
\tau_{ij}^{kl} &= c_{ijnh}(y) \frac{\partial N_n^{kl}(y)}{\partial y_n} + e_{hij}(y) \frac{\partial \phi_{kl}(y)}{\partial y_n} \\
d_i^{kl} &= e_{inh}(y) \frac{\partial N_n^{kl}(y)}{\partial y_n} - \epsilon_{ih}(y) \frac{\partial \phi_{kl}(y)}{\partial y_n}
\end{align*}
\]

\(N_n^{kl}(y)\) is periodic on \(R^d\) and \(\int_Y N_n^{kl}(y) dy = 0\)

\(\phi_{kl}(y)\) is periodic on \(R^d\) and \(\int_Y \phi_{kl}(y) dy = 0\)
\[
\frac{\partial e_{ij}(y)}{\partial y_j} = -\frac{\partial e_{ij}(y)}{\partial y_j} \\
\frac{\partial g}{\partial y_i} = \frac{\partial e_{ij}(y)}{\partial y_i} \\
\xi_{ij} = \epsilon_{ijkh}(y) \frac{\partial W^i_k(y)}{\partial y_h} + \epsilon_{ijh}(y) \frac{\partial \psi_i(y)}{\partial y_h} \\
g_{kl} = \epsilon_{ikh}(y) \frac{\partial W^i_k(y)}{\partial y_h} - \epsilon_{ikl}(y) \frac{\partial \psi_i(y)}{\partial y_h} \\
W^i_k(y) \text{ is periodic on } R^d \text{ and } \int_Y W^i_k(y) dy = 0 \\
\psi_i(y) \text{ is periodic on } R^d \text{ and } \int_Y \psi_i(y) dy = 0
\]

Remark 4.1. we may deduce that \(\langle e_{ij}(y) + \xi_{ij}(y) \rangle_Y = \langle e_{ij}(y) + d_{ij}(y) \rangle_Y \) and \(c^{(0)}_{ijkl} = c^{(0)}_{jikl}, c^{(0)}_{kjil}, c^{(0)}_{ijl}, c^{(0)}_{ij} \) by the techniques we use in the proof of Lemma 3.1.

So far we have derived the homogenization equations of piezoelectric composite materials with periodic microstructure in formal by asymptotic expansion. The following Lemmas and Theorems asserts the convergence of asymptotic expansion. The difficulty for the theoretical analysis of this part lays on the point that the regularity of the solutions of cell problems (41) and (42) cannot reach \(W^{1,\infty}(R^d)\), which implies traditional treatment in second order elliptic equations in Jikov et al. [10] does not work.

The following two lemmas play an important role in getting the convergence conclusion of asymptotic expansion. The original idea of them is from Tatyana et al. [8]. Our result is the promotion of Corollary 8.3 in Tatyana et al. [8] in piezoelectric equation system.

**Lemma 4.1.** If \(N^{kl}_{ij} \in H^1_{per}(Y) \cap L^\infty(R^d)\) and \(\phi_{kl} \in H^1_{per}(Y) \cap L^\infty(R^d)\), the operator \(\nabla N^{kl}_{i} \) of multiplication by the matrix \(\nabla N^{kl}_{i} \) and the operator \(\nabla \phi_{kl} \) of multiplication by the column \(\nabla N^{kl}_{i} \) are continuous from \(H^1(R^d)\) to \(R = L_2(R^d; C^d)\) and

\[
\| \nabla N^{kl}_{i} \|_{H^1(R^d) \to R} + \| \nabla \phi_{kl} \|_{H^1(R^d) \to R} \leq C_1
\]

where \(C_1\) depends on the coefficient of the material only.

**Lemma 4.2.** If \(W^i_k \in H^1_{per}(Y) \cap L^\infty(R^d)\) and \(\psi_i \in H^1_{per}(Y) \cap L^\infty(R^d)\), the operator \(\nabla W^i_k \) of multiplication by the matrix \(\nabla W^i_k \) and the operator \(\nabla \psi_i \) of multiplication
by the column \(|\nabla \psi_l|\) are continuous from \(H^1(\mathbb{R}^d)\) to \(\mathbb{R} = L_2(\mathbb{R}^d; C^d)\) and
\[
\| |\nabla W| |_{H^1(\mathbb{R}^d)\to\mathbb{R}} + \| |\nabla \psi| |_{H^1(\mathbb{R}^d)\to\mathbb{R}} \leq C_2
\]
where \(C_2\) depends on the coefficient of the material only.

The following Lemma is the main result that leads the convergence conclusion of asymptotic expansion. For expression convenience, we introduce
\[
u_{k(1)}^{(e)} = u_{k(1)}^{(0)}(x) + \varepsilon(N_n^{(e)}(y) \frac{\partial u_{k(1)}^{(0)}}{\partial x_l} + W_l^{(e)}(y) \frac{\partial \varphi_{k(1)}^{(0)}}{\partial x_l})
\]
\[
\varphi_{k(1)}^{(e)} = \varphi_{k(1)}^{(0)}(x) + \varepsilon(\frac{\partial u_{k(1)}^{(0)}}{\partial x_l} + \frac{\partial \varphi_{k(1)}^{(0)}}{\partial x_l})
\]

**Lemma 4.3.** If \(u_{k(1)}^{(0)} \in C^2(\Omega), \varphi_{k(1)}^{(0)} \in C^2(\Omega), N_{k(1)}^{(e)} \in H^1_{per}(Y) \cap L^\infty(\mathbb{R}^d), W_l^{(e)} \in H^1_{per}(Y) \cap L^\infty(\mathbb{R}^d), \varphi_{k(1)}^{(0)} \in H^1_{per}(Y) \cap L^\infty(\mathbb{R}^d), \psi_l \in H^1_{per}(Y) \cap L^\infty(\mathbb{R}^d), \) we have
\[
\| u_{k(1)}^{(e)} - u_{k(1)}^{(0)} \|_{H^1(\Omega)} + \| \varphi_{k(1)}^{(e)} - \varphi_{k(1)}^{(0)} \|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}
\]

**Proof.** Take a truncation function \(\tau^\varepsilon(x)\) satisfying,
1. \(\tau^\varepsilon(x) \in C^\infty_0(\Omega), \ 0 \leq \tau^\varepsilon(x) \leq 1 \ and \ \tau^\varepsilon(x) = 1, \ if \ \rho(x, \partial \Omega) > \varepsilon\)
2. \(\varepsilon|\nabla \tau^\varepsilon(x)| \leq C, \ in \ \Omega\)

Set
\[
W_{k(1)}^{(e)} = u_{k(1)}^{(e)} - \varepsilon(1 - \tau^\varepsilon(x))(N_{k(1)}^{(e)} \frac{\partial u_{k(1)}^{(0)}}{\partial x_l} + W_l^{(e)} \frac{\partial \varphi_{k(1)}^{(0)}}{\partial x_l})
\]
\[
W_{k(1)}^{(e)} = \varphi_{k(1)}^{(e)} - \varepsilon(1 - \tau^\varepsilon(x))(\frac{\partial u_{k(1)}^{(0)}}{\partial x_l} + \frac{\partial \varphi_{k(1)}^{(0)}}{\partial x_l})
\]

To get the conclusion, we separate the proof into two steps,

**Step1.** Prove \(\| u_{k(1)}^{(e)} - W_{k(1)}^{(e)} \|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}, \ \| \varphi_{k(1)}^{(e)} - W_{k(1)}^{(e)} \|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}\)

**Step2.** Prove \(\| u_{k(1)}^{(e)} - W_{k(1)}^{(e)} \|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}, \ \| \varphi_{k(1)}^{(e)} - W_{k(1)}^{(e)} \|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}\)
For step 1,
\[
\| u_k^{(1)} - W_k^{(1)} \|_{H^1(\Omega)}^2 = \int_{\Omega} \left[ \varepsilon (1 - \tau^\varepsilon(x)) (N^{k,l}_n \frac{\partial u_n^{(0)}}{\partial x_l} + W_k \frac{\partial \varphi^{(0)}}{\partial x_l}) \right]^2 dx
\]
\[
+ \sum_j \int_{\Omega} \left[ \varepsilon \frac{\partial \tau^\varepsilon(x)}{\partial x_j} (N^{k,l}_n \frac{\partial u_n^{(0)}}{\partial x_l} + W_k \frac{\partial \varphi^{(0)}}{\partial x_l}) \right]^2 dx
\]
\[
+ \sum_j \int_{\Omega} \left[ (1 - \tau^\varepsilon(x)) \left( \frac{\partial N^{m,l}_n}{\partial y_j} \frac{\partial u_m^{(0)}}{\partial x_l} + \frac{\partial W_k}{\partial y_j} \frac{\partial \varphi^{(0)}}{\partial x_l} \right) \right]^2 dx
\]
\[
+ \sum_j \int_{\Omega} \left[ \varepsilon (1 - \tau^\varepsilon(x)) (N^{m,l}_k \frac{\partial^2 u_m^{(0)}}{\partial x_j \partial x_l} + W_k \frac{\partial^2 \varphi^{(0)}}{\partial x_j \partial x_l}) \right]^2 dx
\]
\[
= (I) + (II) + (III) + (IV)
\]

For (I), (II), (IV), there is no essential difference from the analysis of corresponding parts in second order elliptic equations in Jikov et al. [10]. So it is easy to get
\[
(I) + (II) + (IV) \leq C_1 \varepsilon^2 + \varepsilon \rho^{d-1} (\| u_1^{(0)} \|_{H^2(\Omega)} + \| \varphi^{(0)} \|_{H^2(\Omega)})
\]

For (III), we notice that the regularity of the solutions of cell problems (41) and (42) cannot reach $W^{1,\infty}(R^d)$, which makes the traditional treatment in second order elliptic equations in Jikov et al. [10] not work. In this case, we lent help from Lemma 4.1 and Lemma 4.2. We take the first term of (III) as example to give out proof in details. The proof of other terms are similar.

By lemma 4.1, we have
\[
\int_{R^d} \left[ \frac{\partial N^{k,l}_n}{\partial y_j} \frac{\partial N^{m,l}_n}{\partial y_j} \right] v(y) \ dy \leq C (\int_{R^d} | v |^2 dy + \int_{R^d} | \nabla v |^2 dy) \quad \forall v \in H^1(\Omega)
\]

which implies
\[
\int_{R^d} \left[ \frac{\partial N^{k,l}_n}{\partial y_j} \frac{\partial N^{k,l}_n}{\partial y_j} \right] w(x) \ dx \leq \varepsilon^d \int_{R^d} \left[ \frac{\partial N^{k,l}_n}{\partial x_j} \frac{\partial N^{k,l}_n}{\partial x_j} \right] w(\varepsilon x) \ dx
\]
\[
\leq \int_{R^d} | w(\varepsilon x) |^2 dx + \int_{R^d} | \nabla w(\varepsilon x) |^2 dx
\]
\[
= \int_{R^d} | w(x) |^2 dx + \varepsilon^2 \int_{R^d} | \nabla w(x) |^2 dx
\]
Taking \( w(x) = (1 - \tau^\varepsilon(x)) \frac{\partial u_i^{(0)}}{\partial x_i}, \) we obtain
\[
\begin{align*}
\int_{\Omega} \left[ (1 - \tau^\varepsilon(x)) \left( \frac{\partial N_{n}^{k l}}{\partial y_j} \frac{\partial u_i^{(0)}}{\partial x_i} \right) \right]^2 dx & \\
\leq \int_{R^3} \left[ (1 - \tau^\varepsilon(x)) \left( \frac{\partial N_{n}^{k l}}{\partial y_j} \frac{\partial u_i^{(0)}}{\partial x_i} \right) \right]^2 dx & \\
\leq \int_{R^3} | (1 - \tau^\varepsilon(x)) \frac{\partial u_i^{(0)}}{\partial x_i} |^2 dx + \varepsilon^2 \int_{R^3} | \nabla((1 - \tau^\varepsilon(x)) \frac{\partial u_i^{(0)}}{\partial x_i} ) |^2 dx & \\
= \int_{Q^*} | (1 - \tau^\varepsilon(x)) \frac{\partial u_i^{(0)}}{\partial x_i} |^2 dx + \varepsilon^2 \int_{Q^*} | \nabla((1 - \tau^\varepsilon(x)) \frac{\partial u_i^{(0)}}{\partial x_i} ) |^2 dx & \\
\leq C \varepsilon \rho^{-1}
\end{align*}
\]

Therefore,
\[
(I) + (II) + (III) + (IV) \leq C_1 (\varepsilon^2 + \varepsilon \rho^{-1}) (\| u_i^{(0)} \|_{H^2(\Omega)}^2 + \| \varphi^{(0)} \|_{H^2(\Omega)}^2)
\]

Similarly, we can get \( \| \varphi^{(e)}(1) - W_i^{(e)} \|_{H^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}} \)

This completes the proof of Step 1.

For Step 2, there is no essential difference from the analysis of corresponding parts in second order elliptic equations in Jikov et al. [10]. Therefore, we omit the proof of this part. Thus, it is a direct corollary from Step 1 and Step 2 that
\[
\| u_i^{(e)} - u_i^{(0)} \|_{H^1(\Omega)} + \| \varphi^{(e)} - \varphi^{(0)} \|_{H^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}}
\]

This completes the proof of Lemma 4.3.

Thus, the convergence conclusion of asymptotic expansion is got in the sense of H convergence as follows.

**Theorem 4.1.** If \( u_i^{(0)} \in C^2(\Omega), \varphi^{(0)} \in C^2(\Omega), N_{n}^{k l} \in H^1_{\text{per}}(Y) \cap L^\infty(R^d), W_i^{(e)} \in H^1_{\text{per}}(Y) \cap L^\infty(R^d), \varphi_{k l}^{(e)} \in H^1_{\text{per}}(Y) \cap L^\infty(R^d), \psi_{l}^{(e)} \in H^1_{\text{per}}(Y) \cap L^\infty(R^d), \) we have
\[
\begin{align*}
&u_i^{(e)} \rightharpoonup u_i^{(0)}, \quad \varphi^{(e)} \rightharpoonup \varphi^{(0)} \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad \varepsilon \to 0 \\
&\sigma_{ij}^{(e)} \rightharpoonup \sigma_{ij}^{(0)}, \quad D_i^{(e)} \rightharpoonup D_i^{(0)} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \to 0
\end{align*}
\]

**Proof.** The property of the mean value yields
\[
\begin{align*}
&u_i^{(e)} \rightharpoonup u_i^{(0)}, \quad \varphi^{(e)} \rightharpoonup \varphi^{(0)} \quad \text{in} \quad H^1(\Omega) \quad \text{as} \quad \varepsilon \to 0 \quad (46) \\
&\sigma_{ij}^{(e)} \rightharpoonup \sigma_{ij}^{(0)}, \quad D_i^{(e)} \rightharpoonup D_i^{(0)} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \to 0 \quad (47)
\end{align*}
\]
following the steps in Jikov et al. \cite{10}.

Then, by the conclusions in Lemma 4.3, we have

\[ u_i^{(c)} \to u_i^{(0)}, \quad \phi_i^{(c)} \to \phi_i^{(0)} \quad \text{in} \quad H^1(\Omega) \quad \text{as} \quad \varepsilon \to 0 \tag{48} \]

Thus, it is obviously that

\[ u_i^{(c)} \to u_i^{(0)}, \quad \phi_i^{(c)} \to \phi_i^{(0)} \quad \text{in} \quad H_0^1(\Omega) \quad \text{as} \quad \varepsilon \to 0 \]

\[ \sigma_{ij}^{(c)} \to \sigma_{ij}^{(0)}, \quad D_i^{(c)} \to D_i^{(0)} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \to 0 \]

by (46) and (48). This completes the proof of Theorem 4.1. \hfill \Box

**Remark 4.2.** It is not necessary to assume the solutions of the cell problems (41) and (42) belong to \( L^\infty(\mathbb{R}^d) \) to certify the convergence of asymptotic expansion. However, to get the order of \( \varepsilon \) in Lemma 4.3, the conditions \( N_{n}^{k,l} \in H_{\text{per}}^1(Y) \cap L^\infty(\mathbb{R}^d), \quad W_{n}^{l} \in H_{\text{per}}^1(Y) \cap L^\infty(\mathbb{R}^d), \quad \varphi_{kl} \in H_{\text{per}}^1(Y) \cap L^\infty(\mathbb{R}^d), \quad \psi_{l} \in H_{\text{per}}^1(Y) \cap L^\infty(\mathbb{R}^d) \) should be involved.

So far, for those materials with periodic microstructure, the homogenized equation have been deduced from the microscopic model of piezoelectric composite material under the frame of homogenization theory and the corresponding convergence analysis is complected.

5. Error estimate of multi-scale modeling for piezoelectric composite materials

In Section 3, we have designed multi-scale model for piezoelectric composite materials under the framework of HMM. However, HMM is an algorithm estimating the missing macro-scale data by solving the fine scale problem locally. Therefore, both error estimate between HMM solutions and homogenization solutions and error estimate of the effective coefficients will be held in the following part. We restrict our discussion for these analysis to the periodic case, i.e., we assume that \( \text{[56]} \) holds.

**Theorem 5.1.** Denote the solution of (37) and the HMM solution by \((u^{(0)}, \phi^{(0)})\) and
\((\mathbf{u}^H, \varphi^H)\), respectively. Let

\[
e^c_{\text{HMM}} = \max_{x_i \in K, K \in \mathcal{T}_H} \| c_{ijkl}^{(0)} - c_{ijkl}^H \|
\]

\[
e^e_{\text{HMM}} = \max_{x_i \in K, K \in \mathcal{T}_H} \| e_{ij}^{(0)} - e_{ij}^H \|
\]

\[
e^d_{\text{HMM}} = \max_{x_i \in K, K \in \mathcal{T}_H} \| e_{ij}^{(0)} - e_{ij}^H \|
\]

where \(\| \cdot \|\) is the Euclidean norm. If \((\mathbf{u}^{(0)}, \varphi^{(0)})\) are sufficiently smooth, then there exists a constant \(C\) independent of \(\varepsilon, \delta, H\), such that

\[
\| u_i^{(0)} - u_i^H \|_1 + \| \varphi^{(0)} - \varphi^H \|_1 \leq C(H + e^c_{\text{HMM}} + e^e_{\text{HMM}} + e^d_{\text{HMM}})
\]

**Proof.** It is obviously that

\[
\| u_i^{(0)} - u_i^H \|_{H^1(\Omega)} \leq \| u_i^{(0)} - \Pi u_i^{(0)} \|_{H^1(\Omega)} + \| \Pi u_i^{(0)} - u_i^H \|_{H^1(\Omega)} \tag{49}
\]

\[
\| \varphi^{(0)} - \varphi^H \|_{H^1(\Omega)} \leq \| \varphi - \Pi \varphi^{(0)} \|_{H^1(\Omega)} + \| \Pi \varphi^{(0)} - \varphi^H \|_{H^1(\Omega)} \tag{50}
\]

where \(\Pi\) is 1st-order Lagrange interpolate operator.

Using interpolation error estimate in Susanne et al. \cite{7}, we have

\[
\| u_i^{(0)} - \Pi u_i^{(0)} \|_{H^1(\Omega)} \leq CH | u_i^{(0)} |_2 \tag{51}
\]

\[
\| \varphi^{(0)} - \Pi \varphi^{(0)} \|_{H^1(\Omega)} \leq CH | \varphi^{(0)} |_2 \tag{52}
\]

Then, it remains analysis of \(\| \Pi u_i^{(0)} - u_i^H \|_{H^1(\Omega)}\) and \(\| \varphi^H - \Pi \varphi^{(0)} \|_{H^1(\Omega)}\)

By adding, \(51, \tag{53}\) yields

\[
c_H (\mathbf{u}^H, \mathbf{v}^H) + e_H (\mathbf{v}^H, \varphi^H) - e_H (\mathbf{u}^H, \psi^H) + d_H (\varphi^H, \psi^H) = (f, \mathbf{v}^H) \tag{53}
\]

For homogenized equations \cite{37}, the variational form is

\[
\begin{cases}
  c(\mathbf{u}^{(0)}, \mathbf{v}^{(0)}) + e(\mathbf{v}^{(0)}, \varphi^{(0)}) = (f, \mathbf{v}^{(0)}) & \forall \mathbf{v}^{(0)} \in H_0^1 \times H_0^1 \times H_0^1 \\
  -e(\mathbf{u}^{(0)}, \psi^{(0)}) + d(\varphi^{(0)}, \psi^{(0)}) = 0 & \forall \psi^{(0)} \in H_0^1
\end{cases} \tag{54}
\]

where

\[
c(\mathbf{u}^{(0)}, \mathbf{v}^{(0)}) = \int_{\Omega} c_{ijkl}^{(0)} s_{ijkl}(\mathbf{u}^{(0)}) s_{kl}(\mathbf{v}^{(0)}) dx
\]

\[
e(\mathbf{v}^{(0)}, \varphi^{(0)}) = \int_{\Omega} e_{ijkl}^{(0)} s_{ijkl}(\mathbf{v}^{(0)}) \partial_k \varphi^{(0)} dx
\]

\[
d(\varphi^{(0)}, \psi^{(0)}) = \int_{\Omega} e_{ij}^{(0)} \partial_i \varphi^{(0)} \partial_j \psi^{(0)} dx
\]
By adding and $X_H \subset H^1_0$, [54] yields

$$c(u(0), v^H) + e(v^H, \varphi(0)) - e(u(0), \psi^H) + d(\varphi(0), \psi^H) = (f, v^H) \quad (55)$$

for any $v^H \in X_H \times X_H \times X_H$ and $\psi^H \in X_H$ [55] - [53], we have,

$$(c(u(0), v^H) - c_H(u^H, v^H)) + (d(\varphi(0), \psi^H) - d_H(\varphi^H, \psi^H))$$

$$+ (e(v^H, \varphi(0)) - e_H(v^H, \varphi^H)) - (e(u(0), \psi^H) - e_H(u^H, \psi^H)) = 0 \quad (56)$$

By (56), it is obviously that

$$(c(u(0), v^H) - c_H(u(0), v^H)) + c_H(u(0) - \Pi u(0), v^H)$$

$$+ (d(\varphi(0), \psi^H) - d_H(\varphi(0), \psi^H)) + d_H(\varphi^H - \Pi \varphi(0), \psi^H)$$

$$+ (e(v^H, \varphi(0)) - e_H(v^H, \varphi^H)) + e_H(v^H, \varphi^H - \Pi \varphi(0))$$

$$- (e(u(0), \psi^H) - e_H(u(0), \psi^H)) = 0 \quad (57)$$

Setting $v^H = u^H - \Pi u(0), \psi^H = \varphi^H - \Pi \varphi(0)$ in (57), we have

The right hand side of (57) $\geq \alpha \| u^H - \Pi u(0) \|_1^2 + \beta \| \varphi^H - \Pi \varphi(0) \|_1^2$

by Fridriches’s inequality and Lemma 3.1

Using Hölder inequality, we obtain

The left hand side of (57)

$$\leq c_c(HMM) | u(0) |_1 | u^H - \Pi u(0) |_1 + C_1 | u^H - \Pi u(0) |_1 | u(0) - \Pi u(0) |_1$$

$$+ c_d(HMM) | \varphi(0) |_1 | \varphi^H - \Pi \varphi(0) |_1 + C_2 | \varphi^H - \Pi \varphi(0) |_1 | \varphi^H - \Pi \varphi(0) |_1$$

$$+ c_e(HMM) | u(0) |_1 | u^H - \Pi u(0) |_1 + C_3 | \varphi(0) - \Pi \varphi(0) |_1 | u^H - \Pi u(0) |_1$$

$$+ c_e(HMM) | u(0) |_1 | \varphi^H - \Pi \varphi(0) |_1 + C_4 | \varphi^H - \Pi \varphi(0) |_1 | u(0) - \Pi u(0) |_1$$

Then, by Cauchy inequality and interpolation error estimate in Susanne et al. [7], we have

$$\| \Pi u(0) \|_{H^1(\Omega)} + \| \varphi(0) - \Pi \varphi(0) \|_{H^1(\Omega)} \leq C_H + c_c(HMM) + c_d(HMM) + c_e(HMM))$$

This completes the proof of Theorem 5.1
Remark 5.1. At this stage, no assumption on the form of \( c^{(\varepsilon)}_{ijkl}, e^{(\varepsilon)}_{ijk} \) and \( \varepsilon^{(\varepsilon)}_{ij} \) is needed. \((u^{(0)}, \varphi^{(0)})\) can be the solution of an arbitrary macroscopic equation system with the same right-hand side as in [7].

Theorem 5.2. For those piezoelectric composite materials with periodic microstructure, if \( N^{kl}_n \in H^1_{per}(Y) \cap L^\infty(R^d) \) and \( \phi_{kl} \in H^1_{per}(Y) \cap L^\infty(R^d) \), we have the following estimate

\[
e^{(\varepsilon)} c(HMM) \leq C_c \frac{\varepsilon \delta}{\delta} e^{(\varepsilon)} e(HMM) \leq C_e \frac{\varepsilon}{\delta} e^{(\varepsilon)} d(HMM) \leq C_d \frac{\varepsilon}{\delta}
\]

where \( C_c, C_e, C_d \) are constants independent of \( \varepsilon \) and \( \delta \).

Proof. By (12) and (13), we obtain

\[
c^{(\varepsilon)}_{ijkl} = \langle c^{(\varepsilon)}_{smnh} \frac{\partial P^{kl}_{m}}{\partial x_{h}} - c^{(\varepsilon)}_{hsm} \frac{\partial P^{ij}_{n}}{\partial x_{m}} \rangle_{I_8}
\]

(58)

\[
= \langle c^{(\varepsilon)}_{smnh} \frac{\partial P^{kl}_{n}}{\partial x_{h}} - c^{(\varepsilon)}_{hm} \frac{\partial P^{ij}_{m}}{\partial x_{m}} \rangle_{I_8}
\]

(59)

Similarly, by (41) and (38), we have

\[
c^{(0)}_{ijkl} = \langle c_{smnb}(y) \frac{\partial N^{ij}_{m}}{\partial y_{n}} + \delta_{ik} \delta_{jm} \rangle_{Y}
\]

(60)

\[
= \langle c_{smnb}(y) \frac{\partial N^{ij}_{m}}{\partial y_{n}} + \delta_{ik} \delta_{jm} \rangle_{Y}
\]

(61)

The solution of (12), \((P^{kl}, \Phi_{kl})\) has the following expansion,

\[
P^{kl}_n = P^{kl}_{n} + \varepsilon (\chi^{kl}(y) \frac{\partial P^{kl}_{n}(x)}{\partial x_{j}} + W^{ij}_{n}(y) \frac{\partial \Phi^{kl}_{n}(x)}{\partial x_{j}}) + \varepsilon \theta^{kl}
\]

(62)

\[
\Phi_{kl} = \phi^{kl}(y) + \varepsilon (\phi^{ij}(y) \frac{\partial P^{kl}_{n}(x)}{\partial x_{j}} + \psi^{ij}(y) \frac{\partial \Phi^{kl}_{n}(x)}{\partial x_{j}}) + \varepsilon \theta^{kl}
\]

(63)
where \((P_{kl}^{0}, \Phi_{kl}^{0})\) is the solution of the following equation system

\[
\begin{aligned}
\frac{\partial}{\partial x_j} (c_{ijnh}^{0} \frac{\partial P_{n}^{kl}(0)}{\partial x_h} + c_{hij}^{0} \frac{\partial \Phi_{kl}^{0}(0)}{\partial x_h}) &= 0 \quad x \in I_{\delta}(x_{\alpha}) \\
\frac{\partial}{\partial x_i} (c_{inh}^{0} \frac{\partial P_{n}^{kl}(0)}{\partial x_h} - c_{ih}^{0} \frac{\partial \Phi_{kl}^{0}(0)}{\partial x_h}) &= 0 \quad x \in I_{\delta}(x_{\alpha}) \\
\end{aligned}
\]  

(64)

\[
\begin{align*}
P_{k}^{0}(0) &= x_{l} \quad P_{n}^{0}(0) = 0(n \neq k) \quad x \in \partial I_{\delta}(x_{\alpha}) \\
\Phi_{kl}^{0}(0) &= 0 \quad x \in \partial I_{\delta}(x_{\alpha})
\end{align*}
\]

Then,

\[
\begin{align*}
\nabla P_{k}^{0} &= e_{l} + \nabla y N_{k}^{0}(y) + \nabla (\varepsilon \theta_{k}^{0}) = W_{k}^{0} + \nabla (\varepsilon \theta_{k}^{0}) \\
\nabla P_{n}^{0} &= 0 + \nabla y N_{n}^{0}(y) + \nabla (\varepsilon \theta_{n}^{0}) = W_{n}^{0} + \nabla (\varepsilon \theta_{n}^{0}) \\
\nabla \Phi_{kl} &= 0 + \nabla y \phi_{kl}(y) + \nabla (\varepsilon \Theta_{kl}) = M_{kl} + \nabla (\varepsilon \Theta_{kl})
\end{align*}
\]

(65)

(66)

(67)

(68)

where \((\theta_{kl}, \Theta_{kl})\) satisfies the following equation system

\[
\begin{aligned}
\frac{\partial}{\partial x_j} (c_{ijnh}^{c} \frac{\partial \theta_{n}^{kl}}{\partial x_h} + c_{hij}^{c} \frac{\partial \Theta_{kl}}{\partial x_h}) &= 0 \quad x \in I_{\delta}(x_{\alpha}) \\
\frac{\partial}{\partial x_i} (c_{inh}^{c} \frac{\partial \theta_{n}^{kl}}{\partial x_h} - c_{ih}^{c} \frac{\partial \Theta_{kl}}{\partial x_h}) &= 0 \quad x \in I_{\delta}(x_{\alpha}) \\
\theta_{n}^{kl} &= -N_{k}^{kl} \quad x \in \partial I_{\delta}(x_{\alpha}) \\
\Theta_{kl} &= -\phi_{kl} \quad x \in \partial I_{\delta}(x_{\alpha})
\end{aligned}
\]  

(69)

We know that \(c_{ijnh}^{0}, c_{inh}^{0}, c_{ih}^{0}\) are constants under the assumption (36). Thus, we can write out the solution of (64) explicitly as follows by the existent and unique property of the solution of (65).

\[
\begin{align*}
P_{k}^{0}(0) &= x_{l} \quad P_{n}^{0}(0) = 0(n \neq k) \quad \Phi_{kl}^{0}(0) = 0 \quad \text{on } I_{\delta}(x_{\alpha})
\end{align*}
\]

(65)
We have

\[ e_{ijkl}^H - e_{ijkl}^{(0)} = \langle e_{s_{mnh}}^c(W_n^{kl})_h + \varepsilon \frac{\partial \theta_n^{kl}}{\partial x_h}((W_s^{ij})_m + \varepsilon \frac{\partial \theta_s^{ij}}{\partial x_m}) \rangle + \varepsilon \langle e_{s_{mnh}}^c((M_{kl})_h + \varepsilon \frac{\partial \Theta_{kl}}{\partial x_h})((W_s^{ij})_m + \varepsilon \frac{\partial \Theta_{kl}}{\partial x_m}) \rangle \]

\[ - (c_{s_{mnh}}(y)(W_n^{kl})_h(W_s^{ij})_m + e_{s_{mnh}}(y)(M_{kl})_h(W_s^{ij})_m)\gamma \]

\[ = \langle e_{s_{mnh}}^c(W_n^{kl})_h(W_s^{ij})_m + e_{s_{mnh}}^c(y)(M_{kl})_h(W_s^{ij})_m \rangle \]

\[ + \varepsilon \langle e_{s_{mnh}}^c((W_n^{kl})_h + e_{s_{mnh}}^c(M_{kl})_h)\frac{\partial \theta_s^{ij}}{\partial x_m} \rangle + \varepsilon^2 \langle e_{s_{mnh}}^c(W_n^{kl})_h \frac{\partial \theta_s^{ij}}{\partial x_m} + e_{s_{mnh}}^c(M_{kl})_h \frac{\partial \theta_s^{ij}}{\partial x_m} \rangle \]

\[ \tag{70} \]

We obtain

\[ e_{ijkl}^H - e_{ijkl}^{(0)} = \langle e_{s_{mnh}}^c(W_n^{kl})_h + \varepsilon \frac{\partial \theta_n^{kl}}{\partial x_h}((W_s^{ij})_m + \varepsilon \frac{\partial \theta_s^{ij}}{\partial x_m}) \rangle + \varepsilon \langle e_{s_{mnh}}^c((M_{ij})_m + \varepsilon \frac{\partial \Theta_{ij}^1}{\partial x_m})((M_{kl})_h + \varepsilon \frac{\partial \Theta_{kl}}{\partial x_h}) \rangle \]

\[ - (c_{s_{mnh}}(y)(W_n^{kl})_h(W_s^{ij})_m + e_{s_{mnh}}(y)(M_{ij})_m(M_{kl})_h)\gamma \]

\[ = \langle e_{s_{mnh}}^c(W_n^{kl})_h(W_s^{ij})_m + e_{s_{mnh}}^c(y)(M_{ij})_m(M_{kl})_h \rangle \]

\[ + \varepsilon \langle e_{s_{mnh}}^c((W_n^{kl})_h + e_{s_{mnh}}^c(M_{ij})_m)\frac{\partial \theta_s^{ij}}{\partial x_m} \rangle + \varepsilon^2 \langle e_{s_{mnh}}^c(W_n^{kl})_h \frac{\partial \theta_s^{ij}}{\partial x_m} + e_{s_{mnh}}^c(M_{ij})_m \frac{\partial \Theta_{ij}^1}{\partial x_m} \rangle \]

\[ \tag{73} \]
Due to (41) and (69), we obtain,

\[\begin{align*}
&= \langle e_{\text{smh}}^{(c)}(y)(W_{n}^{l})_{h}(W_{s}^{ij})_{m} + e_{\text{smh}}^{(c)}(M_{kl})_{h}(W_{s}^{ij})_{m} \rangle_{I_{s}} \\
&= -\langle e_{\text{smh}}^{(c)}(y)(W_{n}^{l})_{h}(W_{s}^{ij})_{m} + e_{\text{smh}}^{(c)}(M_{kl})_{h}(W_{s}^{ij})_{m} \rangle_{Y} \\
&+ \varepsilon^{2} (\varepsilon_{n}^{(c)} \frac{\partial \Theta_{kl}^{(c)}}{\partial x_{n}} \frac{\partial \Theta_{ij}^{(c)}}{\partial x_{m}})_{I_{s}} - \varepsilon^{2} (\varepsilon_{n}^{(c)} \frac{\partial \Theta_{kl}^{(c)}}{\partial x_{n}} \frac{\partial \Theta_{ij}^{(c)}}{\partial x_{m}})_{I_{s}} \\
&+ 2\varepsilon (\langle e_{\text{smh}}^{(c)}(W_{s}^{ij})_{m} - e_{\text{mh}}^{(c)}(M_{ij})_{m} \rangle_{I_{s}} + \langle e_{\text{smh}}^{(c)}(W_{n}^{l})_{h} + e_{\text{smh}}^{(c)}(M_{kl})_{h} \rangle_{I_{s}}) \\
&\quad (74)
\end{align*}\]

Setting

\[\begin{align*}
I_{0}(\varepsilon^{0}) &= \langle e_{\text{smh}}^{(c)}(y)(W_{n}^{l})_{h}(W_{s}^{ij})_{m} + e_{\text{smh}}^{(c)}(M_{kl})_{h}(W_{s}^{ij})_{m} \rangle_{I_{s}} \\
&= -\langle e_{\text{smh}}^{(c)}(y)(W_{n}^{l})_{h}(W_{s}^{ij})_{m} + e_{\text{smh}}^{(c)}(M_{kl})_{h}(W_{s}^{ij})_{m} \rangle_{Y} \\
I_{1}(\varepsilon^{1}) &= 2\varepsilon \langle e_{\text{smh}}^{(c)}(W_{s}^{ij})_{m} - e_{\text{mh}}^{(c)}(M_{ij})_{m} \rangle_{I_{s}} + 2\varepsilon (\langle e_{\text{smh}}^{(c)}(W_{n}^{l})_{h} + e_{\text{smh}}^{(c)}(M_{kl})_{h} \rangle_{I_{s}}) \\
I_{2}(\varepsilon^{2}) &= -2\varepsilon^{2} (\varepsilon_{n}^{(c)} \frac{\partial \Theta_{kl}^{(c)}}{\partial x_{n}} \frac{\partial \Theta_{ij}^{(c)}}{\partial x_{m}})_{I_{s}} - \varepsilon^{2} (\varepsilon_{n}^{(c)} \frac{\partial \Theta_{kl}^{(c)}}{\partial x_{n}} \frac{\partial \Theta_{ij}^{(c)}}{\partial x_{m}})_{I_{s}} \\
&\quad (75)
\end{align*}\]

For \(I_{0}(\varepsilon^{0})\), referring to the ideas of Lemma 4.1, Lemma 4.2 and X.H et al. [13], we get

\[| I_{0}(\varepsilon^{0}) | \leq C_{0} \frac{\varepsilon}{\delta} \] (76)

For \(I_{1}(\varepsilon^{1})\), we denote the first term and the second term of \(I_{1}(\varepsilon^{1})\) as \(I_{1}(1)\) and \(I_{1}(2)\) respectively.

By Green Formula, we have

\[I_{1}(1) = \frac{2\varepsilon}{|I_{s}|} \int_{\partial I_{s}} \Theta_{kl}^{(c)}(e_{\text{smh}}^{(c)}(W_{s}^{ij})_{m} - e_{\text{mh}}^{(c)}(M_{ij})_{m}) \cdot \vec{n} \, ds \]

\[- \int_{I_{s}} \Theta_{kl} \frac{\partial}{\partial x_{n}} (e_{\text{smh}}^{(c)}(W_{s}^{ij})_{m} - e_{\text{mh}}^{(c)}(M_{ij})_{m}) \, dx \]

Due to (41) and (69), we obtain,

\[| I_{1}(1) | = \frac{2\varepsilon}{|I_{s}|} \int_{\partial I_{s}} \phi_{kl}^{(c)}(e_{\text{smh}}^{(c)}(W_{s}^{ij})_{m} - e_{\text{mh}}^{(c)}(M_{ij})_{m}) \cdot \vec{n} \, ds | \]

\[24\]
Using the singularity of the solutions of (41) and Hölder inequality, we get

\[ |I_1(1)| \leq \frac{2\varepsilon}{|I_\delta|} C_1 |\partial I_\delta|^{\frac{4}{3}} \leq C_1 \varepsilon \]

Similarly, we can get

\[ |I_1(2)| \leq C_2 \varepsilon \]

Thus,

\[ |I_1(\varepsilon^1)| \leq \max(C_1, C_2) \varepsilon \]

(77)

For \(I_2(\varepsilon^2)\), we may just follow the steps in the proof of Lemma 4.3 to get the following estimate,

\[ |I_2(\varepsilon^2)| \leq C_3 \varepsilon \]

(78)

Therefore, we obtain

\[ e_c(HMM) \leq \max\{C_0, C_1, C_2, C_3\} \varepsilon \]

\[ = C_c \varepsilon \]

Similarly, we can also get

\[ e_d(HMM) \leq C_d \varepsilon \]

For the estimation of \(e_c(HMM)\), we do not have essential difference with the estimation of \(e_c(HMM)\) and \(e_d(HMM)\), except some techniques on constructing suitable expressions for analytical convenience, which is not necessary to be written out in details here.

This completes the proof of Theorem 5.2

\(\square\)

6. Numerical simulation

6.1. Numerical example

In order to illustrate the multi-scale methods mentioned above in this paper, we introduce an numerical example of 1-3 type piezoelectric composite materials made of piezoceramic(PZT) fibers embedded in a soft non-piezoelectric matrix(polymer). Since, for a transversely isotropic piezoelectric solid, the stiffness matrix, the piezoelectric matrix and the dielectric matrix simplify so that there remain 11 independent coefficients, we
take the piezoelectric composite materials aligned fibers made of a transversely isotropic piezoelectric solid (PZT), embedded in an isotropic polymer matrix. Moreover, it is easy to verify that the resulting composite is a transversely isotropic piezoelectric material too. Therefore, if we take

$$\begin{align*}
11 & \rightarrow 1 \\
22 & \rightarrow 2 \\
33 & \rightarrow 3 \\
23 & \rightarrow 4 \\
13 & \rightarrow 5 \\
12 & \rightarrow 6 \\
\end{align*}$$

then the constitutive relations of both each component and the resulting composite have the following form.

$$
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12} \\
D_1 \\
D_2 \\
D_3 \\
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{13} \\
c_{12} & c_{22} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{13} \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} \\
0 & 0 & 0 & c_{44} & 0 & 0 & 0 & -e_{15} & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 & 0 & -e_{15} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66} & 0 & 0 & 0 \\
e_{15} & 0 & 0 & 0 & e_{15} & 0 & e_{11} & 0 & 0 \\
e_{13} & 0 & 0 & 0 & 0 & e_{13} & 0 & e_{11} & 0 \\
e_{13} & 0 & 0 & 0 & 0 & 0 & e_{33} & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
s_{11} \\
s_{22} \\
s_{33} \\
s_{12} \\
s_{11} \\
s_{12} \\
E_1 \\
E_2 \\
E_3 \\
\end{pmatrix}
$$

In our numerical example, we take the material properties of the composite constituents fiber (PZT-5) and matrix (polymer) as Table 1.

|                | \(c_{11}(10^{10})\) | \(c_{12}(10^{10})\) | \(c_{23}(10^{10})\) | \(c_{44}(10^{10})\) | \(c_{66}(10^{10})\) | \(e_{15}\)  | \(e_{13}\)  | \(e_{11}(10^{-9})\) | \(e_{33}(10^{-9})\) |
|----------------|---------------------|---------------------|---------------------|---------------------|---------------------|-------------|-------------|---------------------|---------------------|
| PZT-5          | 12.1                | 7.54                | 7.52                | 11.1                | 2.11                | 2.28        | 12.3        | -5.4                | 15.8                |
| Polymer        | 0.386               | 0.257               | 0.257               | 0.386               | 0.064               | 0.064       | -           | -                   | -                   |

Table 1: Material properties of the composite constituents fiber(PZT-5) and matrix(polymer)

And the multi-scale structure of the composite is shown in .

We take \(\Omega = [0, 5] \times [0, 5] \times [0, 5]\) as the computation domain, and \(R\) denotes the radius of the fiber in one microscopic periodic. \(\varepsilon = 0.015625\) is the microscopic periodic. Ignoring volume forces, we apply boundary condition as follows,

\[
u_i = 0 \quad \varphi(5, y, z) = 1000 \quad \varphi(x, y, z) = 0 \quad for \quad x \neq 5, \quad on \ \partial \Omega
\]
The computation domain $\Omega$ is divided into $40 \times 40 \times 40$ elements, the coarse mesh. Each element contains $8 \times 8 \times 8$ microscopic periodicals, see Picture 2. The computation process for given $R$ and $\delta$ is as follows:

1. Solving the two cell problems (41) and (42) and calculating the effective coefficients as (38).
2. Solving (37) on $\Omega$ with the effective coefficients got in Step 1.
3. Given $\alpha$, solving the cell problems of HMM (12) and (15) on $I_\delta(x_\alpha)$. Then we give out the estimation of the effective coefficients at scale $H$ by (13) and (16). In this numerical example, we take one sample in each Macroscopic element $K$.
4. Solving (8) on $\Omega$ by the estimation of the effective coefficients got in Step 3.

Picture 2. The multi-scale structure of 1-3 type piezoelectric composite materials
6.2. Numerical results and analysis

6.2.1. Order of the effective coefficients

For given \( R = 0.3125 \varepsilon \) and \( R = 0.4375 \varepsilon \), we take \( \delta = 4 \varepsilon \), \( \delta = 3 \varepsilon \), \( \delta = 2 \varepsilon \) respectively. We take the scale of elements on \( I_3 \) as \( h = 0.00078125 \). Then, following the steps in Section 3.1, we give out the order of error estimate between the effective coefficients got in Step 1 and the estimation of the effective coefficients got in Step 3.

| \( \frac{\varepsilon}{\varepsilon} \) | \( c_{11}^H \) | order | \( c_{12}^H \) | order | \( c_{13}^H \) | order |
|---|---|---|---|---|---|---|
| 1/4 | 1.11e+10 | 0.93 | 6.18e+09 | 0.99 | 6.70e+09 | 0.95 |
| 1/3 | 1.23e+10 | 0.97 | 6.99e+09 | 1.03 | 7.53e+09 | 0.99 |
| 1/2 | 1.51e+10 | – | 8.70e+09 | – | 9.23e+09 | – |
| \( c_{11}^{(0)} \) | 0.67e+10 | – | 3.71e+09 | – | 4.10e+09 | – |

| \( \frac{\varepsilon}{\varepsilon} \) | \( c_{33}^H \) | order | \( c_{44}^H \) | order | \( c_{66}^H \) | order |
|---|---|---|---|---|---|---|
| 1/4 | 2.28e+10 | 1.06 | 2.25e+09 | 1.15 | 2.08e+09 | 0.92 |
| 1/3 | 2.35e+10 | 1.07 | 2.61e+09 | 1.26 | 2.38e+09 | 0.94 |
| 1/2 | 2.48e+10 | – | 3.47e+09 | – | 2.98e+09 | – |
| \( c_{11}^{(0)} \) | 2.11e+10 | – | 1.33e+09 | – | 1.10e+09 | – |

| \( \frac{\varepsilon}{\varepsilon} \) | \( c_{13}^H \) | order | \( c_{13}^H \) | order | \( c_{15}^H \) | order |
|---|---|---|---|---|---|---|
| 1/4 | - 0.31 | 0.95 | 5.88 | 1.06 | 0.46 | 1.18 |
| 1/3 | - 0.37 | 0.99 | 5.83 | 1.07 | 0.63 | 1.28 |
| 1/2 | - 0.49 | – | 5.74 | – | 1.04 | – |
| \( c_{11}^{(0)} \) | - 0.11 | – | 6.01 | – | 0.04 | – |

| \( \frac{\varepsilon}{\varepsilon} \) | \( c_{11}^H \) | order | \( c_{13}^H \) | order | \( c_{15}^H \) | order |
|---|---|---|---|---|---|---|
| 1/4 | 6.40e-10 | 0.83 | 2.38e-09 | 1.03 |
| 1/3 | 7.67e-10 | 0.78 | 2.38e-09 | 1.05 |
| 1/2 | 9.90e-10 | – | 2.38e-09 | – |
| \( c_{11}^{(0)} \) | 1.72e-10 | – | 2.40e-09 | – |

Table 2: Accuracy of HMM on coefficients for 1-3 type composites with \( R = 0.3125 \varepsilon \)
6.2.2. Analysis of the results

In the numerical result, we can see that the order of error of the effective coefficients we have got in the numerical examples support the corresponding conclusions in our theoretical analysis in section 5. Therefore, we can predict that the estimation of the effective coefficients in the multi-scale model we designed under the framework of HMM can be better and better as the number of the cells taken in one sample increases larger and larger.

7. Conclusions

In this paper, a multi-scale model for piezoelectric composite materials under the framework of Heterogeneous Multi-scale Method (HMM) is proposed. In order to verify the capability of the multi-scale model we developed, macroscopic model is derived from microscopic model of piezoelectric composite material by asymptotic expansion for materials with periodic microstructure. Convergence rate of asymptotic expansion is proved to be $\sqrt{\varepsilon}$ under the framework of homogenization theory. We then give out both error
estimate between HMM solutions and homogenization solutions and error estimate of the
effective coefficients for piezoelectric composite materials with periodic microstructure.
Therefore, HMM solutions is shown to convergence to homogenization solutions in the
order of ($H + \varepsilon \delta$) and the effective coefficients got by HMM modeling is shown to conver-
gence to effective coefficients got by homogenization theory in the order of $\varepsilon$. Moreover,
our numerical simulation result support the corresponding theoretical conclusions we got
above very well.

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