ON THE COHOMOLOGY OF STRATA OF ABELIAN DIFFERENTIALS

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Abstract. For any $g \geq 3$ we show that the pull-backs of the Mumford Morita Miller classes of the moduli space $\mathcal{M}_g$ of curves of genus $g$ to a component of a stratum of projective abelian differentials over $\mathcal{M}_g$ vanish. We deduce that strata are affine.

1. Introduction

For $g \geq 3$ the moduli space $\mathcal{M}_g$ of complex curves of genus $g$ is a complex orbifold. It is the quotient of Teichmüller space $\mathcal{T}_g$ of genus $g$ under the action of the mapping class group $\text{Mod}(S_g)$. The following question can be found in [FL08], see also [HL98] for a motivation.

Question. Does $\mathcal{M}_g$ admit a stratification with all strata affine subvarieties of codimension $\leq g - 1$?

Here a variety is affine if it does not contain any complete proper subvariety. The complement of an irreducible effective ample divisor in the Deligne Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ is affine. This was used by Fontanari and Looijenga to show that the complement of the Theta null divisor in $\mathcal{M}_g$ parameterizing curves with an effective even theta characteristic is affine for every $g \geq 4$ (Proposition 2.1 of [FL08]). They also show that the answer to the question is yes for all $g \leq 5$. Another approach towards an answer to this question which is closer to our viewpoint is due to Chen [Ch19].

The main goal of this article is to give some additional evidence that the answer to the above question is affirmative. To this end consider the Hodge bundle over $\mathcal{M}_g$ whose fiber over a complex curve $X$ is just the $g$-dimensional vector space of holomorphic one-forms on $X$. The projectivization $P : \mathcal{P} \to \mathcal{M}_g$ of the Hodge bundle admits a natural stratification whose strata consist of projective differentials with the same number and multiplicities of zeros. These strata need not be connected, but the number of connected components is at most 3 [KtZ03].

The tautological ring of $\mathcal{M}_g$ is the subring of the rational cohomology ring of $\mathcal{M}_g$ generated by the Mumford Morita Miller classes $\kappa_k \in H^{2k}(\mathcal{M}_g, \mathbb{Q})$ (see [M87] and [Lo95] for a comprehensive discussion of these classes). Denote by $\psi$ the Chern class of the universal line bundle over the fibers of $\mathcal{P}$. The following is the main result of this article.

Theorem 1. Let $\mathcal{Q} \subset \mathcal{P}$ be a component of a stratum of projective abelian differentials; then $P^*\kappa_k|\mathcal{Q} = \psi|\mathcal{Q} = 0$ for all $k \geq 1$. 

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The projective Hodge bundle extends to the Deligne Mumford compactification $\overline{M}_g$ of $M_g$. Teichmüller curves in $Q$ extend to complete curves in the closure of $Q$ in this extension which violate the statement of Theorem 1 for $k = 1$.

Since all but the first Mumford Morita Miller classes vanish on $M_3$ [Lo95], Theorem 1 for two strata in $g = 3$ is due to Looijenga and Mondello [LM14].

As an application of Theorem 1, we give a positive answer to a question of Chen [Ch19].

**Corollary 2.** Components of strata of projective abelian differentials are affine.

The article [Ch19] contains some partial results in the direction of Corollary 2, using a completely different approach. It also contains a complete analysis of the case $g = 3$. Additional related results can be found in [FL08] and [Mo17].

Corollary 2 can be used to construct a new stratification of $M_g$ with all strata affine subvarieties of codimension $\leq g - 2$ in the case $g = 3$ and $g = 4$. We also obtain some information for arbitrary $g$. Namely, the closure in $P$ of the component $\mathcal{P}H(2, \ldots, 2)$ of the stratum of projective abelian differentials with all zeros of order two and odd spin structure projects onto $M_g$. The closure in $P$ of the component $\mathcal{P}H(2, \ldots, 2, 4)$ of projective differentials with a single zero of order 4, all remaining zeros of order 2 and odd spin structure projects to a divisor $D$ in $M_g$. Similarly, the union of the closures in $P$ of the components of strata of projective differentials with all zeros of even order, odd spin structure and either at least one zero of order at least 6 or at least two zeros of order at least 4 projects to a divisor $D_2$ in $D$ (see Section 4 for details). Corollary 2 is used to show

**Corollary 3.** The locus $M_g - D$ is affine, and $D - D_2$ does not contain a complete curve.

As another consequence of Theorem 1, we obtain an alternative proof of the following well known result (a proof is for example contained in [FL08]). For its formulation, the *hyperelliptic locus* in $M_g$ is the subset of all curves which are hyperelliptic, that is, they admit a degree two branched cover over $\mathbb{C}P^1$.

**Corollary 4.** The hyperelliptic locus in $M_g$ is affine.

There exists a natural fiber bundle $C \to M_g$, the so-called *universal curve*, whose fiber over a complex curve $X$ is just $X$. A *surface bundle* $E$ with base a simplicial complex $B$ and fiber a closed surface $S_g$ of genus $g$ is the pull-back of the universal curve by a continuous map $f : B \to M_g$, a so-called *classifying map*. Homotopic maps give rise to homeomorphic surface bundles.

Denote by $\nu^*$ the *vertical cotangent bundle* of a surface bundle $\Pi : E \to B$, that is, the cotangent bundle of the fibers. The bundle $\nu^*$ admits a natural structure of a complex line bundle over $E$ whose Chern class $c_1(\nu^*) \in H^2(E, \mathbb{Z})$ (or, alternatively, its Euler class) is defined. The main tool for the proof of Theorem 1 is an analysis of the Poincaré dual of the cohomology class $c_1(\nu^*)$ in such a surface bundle $E$. Along the way we give a purely topological proof of the following result of Korotkin and Zograf [KZ11]. For its formulation, let $\mathcal{P}_1 \subset \mathcal{P} \to M_g$ be the divisor in the projectivized Hodge bundle consisting of projective abelian differentials with at least one zero which is not simple. The projectivized Hodge bundle is a Poincaré duality space, so we can ask for the dual of $\mathcal{P}_1$, viewed as a homology class relative to the boundary of the Deligne Mumford compactification of $\mathcal{P}$. 
Theorem 5 (Korotkin and Zograf [KZ11]). The class in $H^2(\mathcal{P}, \mathbb{Q})$ which is dual to the divisor $\mathcal{P}_1$ equals $2P^*\kappa_1 - (6g - 6)\psi$.

The work [KZ11] also contains a computation of the dual of the extension of $\mathcal{P}_1$ to the projective Hodge bundle over the Deligne Mumford compactification of $\mathcal{M}_g$ which we do not duplicate here. The methods of proof in [KZ11] stem from mathematical physics. An algebraic geometric proof of Theorem 5 is due to Chen [Ch13].

The organization of this article is as follows. In Section 2, we introduce branched multisections of a surface bundle over a surface. We use the Hodge bundle over $[\text{Ch13}]$. The moduli space of curves of genus $g \geq 2$. This a complex orbifold. The moduli space of curves of genus $g$ with a single marked point (puncture) is the universal curve $\mathcal{C} \to \mathcal{M}_g$, a fiber bundle (in the orbifold sense) over $\mathcal{M}_g$ whose fiber over the point $X \in \mathcal{M}_g$ is just the complex curve $X$.

The moduli spaces $\mathcal{M}_g$ and $\mathcal{C}$ are quotients of the Teichmüller spaces $\mathcal{T}_g$ and $\mathcal{T}_{g,1}$ of marked complex curves of genus $g$ and of marked complex curves of genus $g$ with one marked point, respectively, under the corresponding mapping class group $\text{Mod}(S_g)$ and $\text{Mod}(S_{g,1})$. The marked point forgetful map induces a surjective [FM12] homomorphism $\Theta : \text{Mod}(S_{g,1}) \to \text{Mod}(S_g)$. This homomorphism fits into the Birman exact sequence

$$1 \to \pi_1(S_g) \to \text{Mod}(S_{g,1}) \to \text{Mod}(S_g) \to 1.$$  

Any surface bundle $E$ over a surface $B$ with fiber genus $g$ is a topological manifold which can be represented as the pull-back of $\mathcal{C}$ by a continuous map $\varphi : B \to \mathcal{M}_g$, called a classifying map for $E$. Up to homeomorphism, the bundle only depends on the homotopy class of $\varphi$. Equivalently, it only depends on the conjugacy class of the induced monodromy homomorphism $\varphi_* = \rho : \pi_1(B) \to \text{Mod}(S_g)$. As a consequence, we may choose the map $\varphi$ to be smooth (as maps between orbifolds). Then $\Pi : E \to B$ is a smooth fiber bundle. In the remainder of this section we always assume that this is the case.

A section of a surface bundle $\Pi : E \to B$ is a smooth map $\sigma : B \to E$ such that $\Pi \circ \sigma = \text{Id}$. The following is well known and reported here for completeness. For its formulation, define a lift of the monodromy homomorphism $\rho : \pi_1(B) \to \text{Mod}(S_g)$ to be a homomorphism $\tilde{\rho} : \pi_1(B) \to \text{Mod}(S_{g,1})$ with the property that $\Theta \circ \tilde{\rho} = \rho$. Note that there may be elements of the kernel of $\rho$ which are not contained in the kernel of $\tilde{\rho}$.

Lemma 2.1. The surface bundle $E \to B$ has a section if and only if there is a lift $\tilde{\rho} : \pi_1(B) \to \text{Mod}(S_{g,1})$ of the monodromy homomorphism $\rho : \pi_1(B) \to \text{Mod}(S_g)$.

Proof. If $\sigma : B \to E$ is a section and if $x \in B$ is an arbitrarily chosen point, then the image under $\sigma$ of any based loop $\alpha$ at $x$ is a based loop at $\sigma(x)$. Via the classifying
map $\varphi : B \to M_g$, this loop defines a lift of the element $\varphi_* \alpha \in \text{Mod}(S_g)$ to $\text{Mod}(S_g, 1)$. As this construction is compatible with group multiplication, it defines a lift of $\rho$ to $\text{Mod}(S_g, 1)$.

Vice versa, let us assume that the monodromy homomorphism $\rho : \pi_1(B) \to \text{Mod}(S_g)$ admits a lift $\tilde{\rho} : \pi_1(B) \to \text{Mod}(S_g, 1)$. Since both $B$ and $C$ are classifying spaces for their orbifold fundamental groups, there exists a smooth map $F : B \to C$ with monodromy $F_* = \tilde{\rho}$. Then the projection of $F$ to a map $f : B \to M_g$ induces the homomorphism $\rho : \pi_1(B) \to \text{Mod}(S_g)$. As $\rho$ is the monodromy of a classifying map for $E$, the surface bundle defined by $f$ is diffeomorphic to $E$. Since the map $F$ is a lift of $f$ to $C$ by construction, it defines a section of $E$. 

Recall that a branched covering of a surface $\Sigma$ over a surface $B$ is a finite to one surjective map $f : \Sigma \to B$ such that there exists a finite set $A \subset B$, perhaps empty, with the property that $f| f^{-1}(B - A)$ is an ordinary covering projection.

We will use the following generalization of the notion of a section.

**Definition 2.2.** A branched multi-section of degree $d$ of a surface bundle $\Pi : E \to B$ is defined to be a smooth injective immersion $f : \Sigma \to E$ where $\Sigma$ is a (not necessarily connected) closed surface and such that $\Pi \circ f : \Sigma \to B$ is a branched covering of degree $d$.

Any section of a surface bundle is a branched multisection as in Definition 2.2. A multisection of $E$ is a smooth injective immersion $f : \Sigma \to E$ so that $\Pi \circ f$ is an unbranched covering.

Surface bundles may or may not admit multisections, although it is difficult to construct examples of surface bundles which do not admit multisections. Some evidence for the existence of examples is a result of Chen and Salter [CS18], inspired by earlier work of Mess: For $g \geq 5$ and $m \geq 1$, there does not exist a finite index subgroup of $\text{Mod}(S_g)$ which admits a lift to $\text{Mod}(S_{g,m})$. In particular, the Birman exact sequence (1) does not virtually split.

Complete complex curves in $M_g$ constructed from complete intersections provided examples of surface bundles over surfaces so that the image of the monodromy homomorphism is a finite index subgroup of $\text{Mod}(S_g)$. As this monodromy homomorphism necessarily has a large kernel, such surface bundles may admit sections in spite of [CS18].

From now on we assume that all surfaces are oriented. Then the image $f(\Sigma)$ of a branched multisection $f : \Sigma \to E$ is a cycle in $E$ which defines a homology class $[f(\Sigma)] \in H_2(E, \mathbb{Z})$. Recall from the introduction that the vertical cotangent bundle $\nu^*$ of $E$ is the cotangent bundle of the fibers of the surface bundle $\Pi : E \to B$. This is a smooth complex line bundle on $E$. The main goal of this section is to show.

**Theorem 2.3.** A surface bundle over a surface admits a branched multisection whose homology class is Poincaré dual to the Chern class $c_1(\nu^*) \in H^2(E, \mathbb{Z})$ of the vertical cotangent bundle $\nu^*$.

Our strategy is to construct explicitly a cycle in $E$ representing the Poincaré dual of $c_1(\nu^*)$ using the moduli space of abelian differentials. We begin with introducing the objects we need.

The moduli space of abelian differentials for a surface $S_g$ of genus $g$ is the complement of the zero section in the Hodge bundle $\mathcal{H} \to M_g$ over the moduli space of curves.
A holomorphic one-form on a Riemann surface $X$ of genus $g$ has precisely $2g - 2$ zeros counted with multiplicity. Denote as in the introduction by $P : \mathcal{P} \to \mathcal{M}_g$ the projectivized Hodge bundle over $\mathcal{M}_g$ and let $\mathcal{P}_1 < \mathcal{P}$ be the closure of the subspace of all projective holomorphic one-forms with at least one zero which is not simple. Then $\mathcal{P}_1$ is an complex subvariety of $\mathcal{P}$ of complex codimension one. More explicitly, the set of all projective abelian differentials with precisely one zero of order two and $2g - 4$ zeros of order one is a smooth complex suborbifold of $\mathcal{P}$ of complex codimension one which is contained in $\mathcal{P}_1$ by definition. Its complement in $\mathcal{P}_1$ is a complex subvariety $\mathcal{P}_2$ of codimension one which is a union of strata of smaller dimension. Here by a smooth complex orbifold we mean an orbifold with a finite orbifold cover which is a complex manifold.

The Hodge bundle $\mathcal{H}$ is a complex vector bundle of rank $g \geq 3$ (in the orbifold sense) and hence the fiber of its sphere subbundle $\mathcal{S}$ is a sphere of real dimension $2g - 1 \geq 3$. Let $Q : \mathcal{S} \to \mathcal{P}$ be the natural projection.

Let $\Pi : E \to B$ be a surface bundle over a surface $B$ with fibre $S_g$. We may assume that $E$ is defined by a smooth map $\varphi : B \to \mathcal{M}_g$. In particular, each fibre $\pi^{-1}(x)$ has a complex structure which depends smoothly on $x$. We have (see Lemma 2.7 of [H19] for a proof of this standard fact).

**Lemma 2.4.** There exists a smooth map $\theta : B \to \mathcal{S}$ such that $P \circ Q \circ \theta = \varphi$.

As $Q^{-1}\mathcal{P}_2 \subset \mathcal{S}$ is of real codimension 4, by transversality we may assume that $Q(\theta(B))$ is disjoint from $\mathcal{P}_2$. We may furthermore assume that $Q(\theta(B))$ intersects the divisor $\mathcal{P}_1$ transversely in isolated smooth points.

For each $x \in B$ let $\delta(x) \subset \Pi^{-1}(x)$ be the set of zeros of the holomorphic one-form $\theta(x)$ on the Riemann surface $x$, counted with multiplicities. Then $\delta(x)$ is a divisor of degree $2g - 2$ on $\Pi^{-1}(x)$ which defines the canonical bundle of $x$. Write $\Delta(x)$ to denote the unweighted set $\delta(x)$, that is, the support of the divisor $\delta(x)$. This set consists of either $2g - 2$ or $2g - 3$ points, and points $x \in B$ so that the cardinality of $\Delta(x)$ equals $2g - 3$ are isolated. Moreover,

$$\Delta = \bigcup_{x \in B} \Delta(x)$$

is a closed subset of $E$.

Call a point $y \in \Delta$ singular if it is a double zero of the abelian differential $\theta(\Pi(y))$. A point in $\Delta$ which is not singular is called regular. By the choice of the map $\theta$, the set $A \subset \Delta$ of singular points of $\Delta$ is finite. Moreover, if $y \in \Delta$ is singular, then $y$ is the only singular point in $\Pi^{-1}(\Pi(y))$. We have

**Lemma 2.5.** There exists a closed oriented surface $\Sigma$ and an injective continuous map $f : \Sigma \to E$ with image $\Delta$ which is smooth on $\Sigma - f^{-1}(A)$. The map $\Pi \circ f : \Sigma \to B$ is a branched covering of degree $2g - 2$, branched at $f^{-1}(A)$, and each branch point has branch index 2.

**Proof.** The zeros of an abelian differential depend smoothly on the differential. Thus if $x \in B$ and if $y \in \Delta(x)$ is a simple zero of the differential $\theta(x)$, then there is a neighborhood $U$ of $y$ in $E$ such that the intersection $U \cap \Delta$ is diffeomorphic to a disk and that the restriction of the projection $\Pi$ to $U \cap \Delta$ is a diffeomorphism onto a neighborhood of $x$ in $B$. In particular, if $A \subset \Delta$ is the finite set of double zeros of the differentials $\theta(x)$ ($x \in B$), then the restriction of the projection $\Pi : E \to B$ to $\Delta - \Pi^{-1}(\Pi(A))$ is a $2g - 2$-sheeted covering of $B - \Pi(A)$.
Choose a triangulation $T$ of the base surface $B$ into $k > 0$ triangles such that each of the finitely many points of $\Pi(A)$ is a vertex of the triangulation and that no triangle contains more than one of these points. Then each of the triangles of $T$ has precisely $2g - 2$ preimages under the map $\Pi|_A$. In particular, the preimage of $T$ in $\Delta$ defines a decomposition of $\Delta$ into $(2g - 2)k$ triangles with disjoint interiors. Each edge of such a triangle is adjacent to an edge of precisely one other triangle. But this just means that $\Delta$ is a topological surface (see [Hat01] for a nice exposition of this fact). The orientation of $B$ pulls back to an orientation of $\Delta$. As a consequence, there exists a closed oriented surface $\Sigma$ and an injective continuous map $f : \Sigma \to \Delta$ so that $\Pi \circ f$ is a branched covering of degree $2g - 2$. Using again the fact that zeros of abelian differentials depend smoothly on the differential, the map $f$ can be chosen to be smooth away from the preimage of points in $A$.

The lemma now follows from the fact that the restriction of the map $\Pi \circ f$ to $\Sigma - f^{-1}(\Pi^{-1}(A))$ is a covering of degree $2g - 2$, and a point in $\Pi(A)$ has precisely $2g - 3$ preimages under $\Pi \circ f$. □

Lemma 2.5 does not explicitly state that the map $f$ is a branched multisection as this requires that $f$ is a smooth immersion. To show that we may assume that this is indeed the case we have to analyze the map $f$ near the points in $f^{-1}(A)$. This is carried out in the next lemma. In its statement and later on, we allow to modify the map $\theta : B \to \mathcal{S}$ by a smooth homotopy which also changes the classifying map $P \circ \theta$.

**Lemma 2.6.** The map $f : \Sigma \to \Delta$ is a branched multisection. At each point $y \in A$ which corresponds to a positive (or negative) intersection point of $Q\theta(B)$ with $P_1$, the oriented tangent plane of $f(\Sigma)$ at $y$ equals the oriented tangent plane of the fibre (or the tangent plane of the fibre with the reversed orientation).

**Proof.** Choose a complex structure for $B$. Let $x \in B$ be such that $Q\theta(x)$ is a positive transverse intersection point with $P_1$. Then up to changing $\theta$ with a homotopy supported in a small neighborhood of $x$, we may assume that there is a neighborhood $W$ of $x$ in $B$ such that the restriction of $\theta$ to $W$ is holomorphic as a map into the complex orbifold $\mathcal{H} \supset \mathcal{S}$. Note that since $\theta$ is holomorphic in $W$ and the projection $\mathcal{H} \to \mathcal{M}_g$ is holomorphic, the local surface bundle $\Pi^{-1}(W) \subset E$ is a complex manifold.

Let $y \in A \subset \Delta$ be the double zero of the abelian differential $\theta(x)$. Choose holomorphic coordinates $(u, v) \in \mathbb{C}^2$ on a neighborhood of $y$ in the complex surface $\Pi^{-1}(W)$ so that in these coordinates, the projection $\Pi : E \to B$ is the projection $(u, v) \to v$. We may assume that the range of the coordinates $(u, v)$ contains a set of the form $U \times V$ for open disks $U, V \subset \mathbb{C}$ centered at 0, that the singular point $y$ corresponds to the origin 0 and that $y$ is the only singular point in $U \times V$. Moreover, we may assume that for every $0 \neq v \in V$ the intersection $(U \times \{v\}) \cap \Delta$ consists of precisely two points.

Any holomorphic one-form $\omega$ on a Riemann surface $X$ can be represented in a holomorphic local coordinate $z$ on $X$ in the form $\omega(z) = h(z)dz$ with a holomorphic function $h$. Zeros of $\omega$ of order one or two correspond to zeros of the function $h$ of the same order. Thus up to a biholomorphic coordinate change and perhaps decreasing the sizes of the sets $U$ and $V$, we may expand the holomorphic differentials $\theta(v)$ $(v \in V)$ on the coordinate disk $U$ as a power series about the point 0. This
expansion is of the form
\[ \theta(v)(u) = \left( \sum_{n=0}^{\infty} a_n(v)u^n \right) du \]

with holomorphic functions \( a_n : V \to \mathbb{C} \) which satisfy \( a_0(0) = a_1(0) = 0 \) and \( a_2(0) = 1 \). By transversality, we also may assume that \( a'_0(0) \neq 0 \).

Apply the holomorphic implicit function theorem to the equation \( q(u, v) = \sum_{n=0}^{\infty} a_n(v)u^n = 0 \) at the point \( v = u = 0 \). This is possible since \( a'_0(0) \neq 0 \) by assumption. We find that locally near \( v = u = 0 \), the solution of this equation is a complex curve which is tangent at \( v = u = 0 \) to the fiber \( v \equiv 0 \). (A model case is the family \( \theta(v)(u) = (v + u^2)du \). As this complex curve is contained in the topological surface \( \Delta \), we conclude that \( \Delta \) is smooth near the point \( y \). Moreover, the map \( f \) maps the tangent plane of \( \Sigma \) at \( f^{-1}(y) \) orientation preserving onto the tangent plane at \( y \) of the fiber.

The above reasoning extends to the case that \( \theta(x) \) is a negative transverse intersection point with \( \mathcal{P}_1 \). Namely, in this case we may assume that the map \( \theta \) is antiholomorphic near \( x \) in suitable complex coordinates. We then can write \( \theta \) as a composition of a holomorphic map with complex conjugation.

To summarize, up to perhaps modifying \( \theta \) and hence \( \Pi \circ \theta = \varphi \) with a smooth homotopy, we may assume that the map \( f \) is a smooth embedding which is tangent to the fibers exactly at the double zeros of the differentials \( \theta(x) (x \in B) \). Moreover, the differential of \( f \) at the preimage of such a double zero preserves the orientation if and only if \( \theta(x) \) is a positive transverse intersection point with \( \mathcal{P}_1 \). Then \( f : \Sigma \to \Delta \) has all the properties stated in the lemma. \( \square \)

The next observation is the last remaining step for the proof of Theorem 2.3. As \( \Sigma \) is a compact oriented surface, the map \( f : \Sigma \to E \) defines a second homology class \( \delta \in H_2(E, \mathbb{Z}) \). We have

**Proposition 2.7.** The homology class \( \delta \in H_2(E, \mathbb{Z}) \) is Poincaré dual to \( c_1(\nu^*) \).

**Proof.** Recall that the set \( A \subset \Delta \) of singular points of \( \Delta \) is finite, and that \( \Delta \subset E \) is a smoothly embedded surface. Thus to show that the homology class \( \delta \) is Poincaré dual to \( c_1(\nu^*) \), in view of the fact that every second integral homology class in \( E \) can be represented by a smooth map from a closed oriented surface (for smoothness, note that \( E \) is a classifying space for its fundamental group and recall the proof of Lemma 2.1), it suffices to show the following.

Let \( M \) be a smooth closed oriented surface and let \( \alpha : M \to E \) be a smooth map such that \( \alpha(M) \) intersects \( \Delta \) transversely in finitely many regular points. Then the number of such intersection points, counted with sign (and multiplicity, however the multiplicity is one by assumption on transversality) equals the degree of the pull-back line bundle \( \alpha^*(\nu^*) \) on \( M \).

That this holds true can be seen as follows. For each \( z \in E - \Delta \), the restriction of the holomorphic one-form \( \theta(\Pi(z)) \) to the tangent space of the fiber of \( E \to B \) at \( z \) does not vanish and hence it defines a nonzero element \( \beta(z) \) of the fiber of \( \nu^* \) at \( z \). Associating to \( z \) the linear functional \( \beta(z) \) defines a trivialization of \( \nu^* \) on \( E - \Delta \).

We claim that at each regular point \( y \in \Delta \), the restriction of this trivialization to the oriented fiber \( \Pi^{-1}(\Pi(y)) \) has rotation number 1 about \( y \) with respect to a trivialization which extends across \( y \). However, this is equivalent to the statement
that the divisor on the Riemann surface $\Pi^{-1}(\Pi(y))$ defined by $\theta(\Pi(y))$ defines the holomorphic cotangent bundle of $\Pi^{-1}(\Pi(y))$.

Namely, the holomorphic one-form $\theta(\Pi(y))$ on the surface $\Pi^{-1}(\Pi(y))$ defines an euclidean metric on $\Pi^{-1}(\Pi(y)) - \Delta$ which extends to a singular euclidean metric on all of $\Pi^{-1}(\Pi(y))$. As $y$ is a simple zero of the abelian differential $\theta(\Pi(y))$ by assumption, it is a four-pronged singular point for this singular euclidean metric. Now let $D \subset \Pi^{-1}(\Pi(y))$ be a small disk about $y$ not containing any other point of $\Delta$, with boundary $\partial D$. Choose a nowhere vanishing vector field $\xi$ on $\partial D$ with $\theta(\Pi(y))(\xi) > 0$. As $y$ is a four-pronged singular point, the rotation number of $\xi$ with respect to a vector field which extends to a trivialization of the tangent bundle of $D$ equals $-1$. By duality, the rotation number of the trivialization of the cotangent bundle $\nu^*$ of $\Pi^{-1}(\Pi(y)) - \Delta$ defined by $\theta(\Pi(y))$ on $\partial D$ with respect to a trivialization of the cotangent bundle which extends to all of $D$ equals $1$.

Now if $M$ is a closed oriented surface and if $\alpha : M \to E$ is a smooth map which intersects $\Delta$ transversely in finitely many regular points, then via modifying $\alpha$ with a smooth homotopy we may assume that the following holds true. Let $u \in M$ be such that $\alpha(u) \in \Delta$. Then $\alpha$ maps a neighborhood of $u$ in $M$ diffeomorphically to a neighborhood of $\alpha(u)$ in the fibre $\Pi^{-1}(\Pi(\alpha(u)))$.

As the restriction of $\nu^*$ to $E - \Delta$ admits a natural trivialization, the same holds true for the pull-back of $\nu^*$ to $M - \alpha^{-1}(\Delta)$. Furthermore, for each $u \in M$ with $\alpha(u) \in \Delta$, the induced trivialization of the pull-back bundle $\alpha^*(\nu^*)$ on $M - \alpha^{-1}(\Delta)$ has rotation number 1 or $-1$ at $u$ with respect to a trivialization of $\alpha^*\nu^*$ on a neighborhood $u$. Here the sign depends on whether $\alpha$, viewed as a diffeomorphism from a neighborhood of $u$ in $M$ onto a neighborhood of $\alpha(u)$ in $\Pi^{-1}(\Pi(\alpha(u)))$, is orientation preserving or orientation reversing. This just means that the degree of the line bundle $\alpha^*\nu^*$ on $M$ equals the number of intersection points of $\alpha(M)$ with $\Delta$ counted with signs (and multiplicities- by the assumption on $\alpha$, these multiplicities are all one). In other words, the degree of the line bundle $\alpha^*(\nu^*)$ on $M$ equals the intersection number $\alpha(M) \cdot \Delta$. □

3. The Poincaré dual of $\mathcal{P}_1$

In this section we apply the results of Section 2 to represent the signature of a surface bundle as an intersection number. This yields a purely topological proof of Theorem 5 [KZ11].

The projectivized Hodge bundle $P : \mathcal{P} \to \mathcal{M}_g$ extends to a bundle over the Deligne Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ which we denote by $P : \overline{\mathcal{P}} \to \overline{\mathcal{M}}_g$. A standard spectral sequence argument shows that the second rational cohomology group of $\overline{\mathcal{P}}$ is generated by the pull-back of the second rational cohomology group of $\overline{\mathcal{M}}_g$ together with the cohomology class $\psi$ of the universal line bundle of the fibre (see Lemma 1 of [KZ11] for details). Since $\overline{\mathcal{P}}$ is a Poincaré duality space, there is a cohomology class $\eta \in H^2(\overline{\mathcal{P}}, \mathbb{Q})$ which is Poincaré dual to the closure $\overline{\mathcal{P}}_1$ of $\mathcal{P}_1$. The class $\eta$ can be expressed as a rational linear combination of the class $\psi$ and the pull-back of a set of generators of $H^2(\overline{\mathcal{M}}_g, \mathbb{Q})$. Such a set of generators consists of the first Chern class $\lambda$ of the Hodge bundle as well as the Poincaré duals $\delta_j$ ($0 \leq j \leq \lfloor g/2 \rfloor$) of the boundary divisors. Here $\delta_0$ is dual to the divisor of stable curves with a single non-separating node, and for $1 \leq j \leq g/2$, the class $\delta_j$ is dual to the divisor of stable curves with a node separating the stable curve into a curve of genus $j$ and a curve of genus $g - j$. 
Korotkin and Zograf calculated this linear combination (the formula before Remark 2 on p.456 of [KZ11]).

**Theorem 3.1** (Korotkin and Zograf [KZ11]).

\[ \eta = 24P^*\lambda - (6g - 6)\psi - P^*(2\delta_0 - 3 \sum_{j=1}^{\lfloor g/2 \rfloor} \delta_j) \]

Another proof of Theorem 3.1 using tools from algebraic geometry is due to Chen [Ch13]. In view of the identity \( \kappa_1 = 12\lambda \) on \( \mathcal{M}_g \) [HaM98], the formula in Theorem 5 is a special case of Theorem 3.1 for which we present below a topological argument.

Let as before \( \Pi : E \to B \) be a surface bundle over a surface, defined by a smooth map \( \varphi : B \to \mathcal{M}_g \). Let \( \theta : B \to S \) be a lift of \( \varphi \) to the sphere subbundle of the Hodge bundle and let \( \delta \in H_2(E, \mathbb{Z}) \) be the Poincaré dual for the Chern class \( c_1(\nu^*) \) of the vertical cotangent bundle defined by \( \theta \) in Proposition 2.7. We have

**Corollary 3.2.** \( c_1(\nu^*) \cup c_1(\nu^*)(E) = \delta \cdot \delta = c_1(\nu^*)(\delta) \).

**Proof.** Both equations follow from Poincaré duality for the bundle \( E \). \( \Box \)

The first Mumford Morita Miller class \( \kappa_1 \in H^2(\mathcal{M}_g, \mathbb{Q}) \) is defined as follows [M87]. Let as before \( \mathcal{C} \to \mathcal{M}_g \) be the universal curve and let \( c_1(\nu^*) \) be the first Chern class of the relative dualizing sheaf of \( \mathcal{C} \). On fibres over smooth points \( x \in \mathcal{M}_g \), the relative dualizing sheaf is just the sheaf of sections of the vertical cotangent bundle \( \nu^* \). Then

\[ \kappa_1 = \Pi_* (c_1(\nu^*) \cup c_1(\nu^*)) \]

where \( \Pi_* \) is the Gysin push-forward map obtained by integration over the fiber. In particular, for any smooth map \( \varphi : B \to \mathcal{M}_g \) which defines the surface bundle \( E \to B \) we have

\[ \kappa_1(\varphi(B)) = c_1(\nu^*) \cup c_1(\nu^*)(E) \]

and hence

**Corollary 3.3.** \( \varphi^*\kappa_1(B) = c_1(\nu^*)(\delta) \).

**Lemma 3.4.** \( Q\theta(B) \cdot \mathcal{P}_1 = 2\delta \cdot \delta = 2c_1(\nu^*)(\delta) \); in particular, the restriction of the class \( \eta \) to the \( \mathcal{P}|\mathcal{M}_g \) satisfies

\[ \eta = 2P^*\kappa_1 + a\psi = 24P^*\lambda + a\psi \]

for some \( a \in \mathbb{Q} \).

**Proof.** Using the notations from Section 2, let \( \Delta \subset E \) be the set of zeros of the differentials in \( \theta(B) \) and let \( A \subset \Delta \) be the set of singular points. By Lemma 2.6, there is a closed oriented surface \( \Sigma \) and a smooth embedding \( f : \Sigma \to E \) so that \( f(\Sigma) = \Delta \). The map \( \Pi \circ f \) is a covering branched at the points in \( A \). Each such branch point has branch index two, and the index is positive (or negative) if the corresponding intersection point of \( Q\theta(B) \) with \( \mathcal{P}_1 \) is a positive (or negative) intersection point.

Since \( A \) is precisely the set of branch points of the map \( \Pi \circ f \), the Hurwitz formula shows that the tangent bundle of \( \Sigma \) can be represented in the form \( f^*(\Pi|\Delta)^*(TB \otimes \)
$(-H)$ where $H$ is the line bundle on $\Sigma$ with divisor $f^{-1}(A)$. Then the normal bundle $N$ of $f(\Sigma)$ can be written as $N = \nu \otimes H^+(\otimes H^-)^{-1}$ where $H^+$ corresponds to the intersection points of $Q\theta(B)$ with $P_1$ of positive intersection index, and $H^-$ corresponds to the points with negative intersection index. This implies that the self-intersection number in $E$ of the surface $f(\Sigma) \subset E$ equals $c_1(f^*\nu)(\Sigma) + b$ where $b = Q\theta(B) \cdot P_1$

is the number of branch points of $\Pi \circ f$, counted with sign.

By Poincaré duality (see Corollary 3.2), we have

$$c_1(\nu^*)(\delta) = \delta \cdot \delta = c_1(\nu)(\delta) + b = -c_1(\nu^*)(\delta) + b$$

and hence $b = 2c_1(\nu^*)(\delta)$. Together with Corollary 3.3 and the fact that $\kappa_1 = 12\lambda$ as classes in $H^2(M_g, \mathbb{Q})$ [HaM98], this completes the proof of the lemma.

**Proof of Theorem 5.** By Lemma 3.4, for the proof of Theorem 3.1 we are left with calculating the coefficient $a \in \mathbb{Q}$ in the expression in Lemma 3.4.

First note that in the case $g = 2$, we have $\lambda = 0$ [HaM98] and

$$\eta = 2P^*\kappa_1 - 6\psi = -6\psi.$$

Namely, the complex dimension of the Hodge bundle equals 2 and hence the fibre of the bundle $\mathcal{P} \to \mathcal{M}_2$ over the moduli space of genus 2 complex curves is just $\mathbb{CP}^1$. A **Weierstrass point** on a genus 2 complex curve $X$ is a double zero of a holomorphic one-form on $X$, and there are no other double zeros of any holomorphic one-forms on $X$. Now $X$ has precisely $6 = -3\chi(S_2)$ Weierstrass points and hence the intersection number of the fibre of the bundle $\mathcal{P} \to \mathcal{M}_2$ with the divisor $P_1$ equals 6. As the evaluation on $\mathbb{CP}^1$ of the Chern class of the universal bundle on $\mathbb{CP}^1$ equals $-1$, the formula in the theorem follows from Poincaré duality.

For arbitrary $g \geq 3$ choose a complex curve $X \in \mathcal{M}_g$ which admits an unbranched cover of degree $d$ onto a curve $Y \in \mathcal{M}_2$. Note that $d = g - 1$. The projective line $\mathbb{CP}^1$ of projective holomorphic one-forms on $Y$ pulls back to a projective line of projective holomorphic one-forms on $X$. The preimage of a projective differential with simple zeros is a differential with only simple zeros, but the preimage of a differential with a double zero is a holomorphic differential with $d$ double zeros. As an abelian differential $q$ with a double zero at a point $p$ can be deformed to a differential with two simple zeros with a deformation which keeps $q$ fixed outside of an arbitrarily small neighborhood of $p$, this complex line of projective abelian differentials can be deformed to a complex line of differentials with at most one double zero each without changing the total intersection index. As a consequence, the intersection number with $P_1$ of this lifted sphere equals $6d = 6g - 6$. This completes the proof of Theorem 5.

4. The second cohomology group of strata

In this final section we apply the results from Section 2 to prove Theorem 1, Corollary 2, Corollary 3 and Corollary 4.

We begin with the proof of Theorem 1. Thus let $Q \subset \mathcal{P}$ be a component of a stratum of projective abelian differentials on a surface of genus $g \geq 3$ where as before, $P : \mathcal{P} \to \mathcal{M}_g$ is the projectivized Hodge bundle. For $k \geq 0$ the cohomology group $H^{2k}(Q, \mathbb{Q})$ can be thought of as the space of linear functionals on the $2k$-th rational homology group of $Q$. As $H_{2k}(Q, \mathbb{Q}) = H_{2k}(Q, \mathbb{Z}) \otimes \mathbb{Q}$, it suffices to evaluate for $k \geq 1$ the class $P^*\kappa_k \in H^{2k}(Q, \mathbb{Q})$ on a class in $H_{2k}(Q, \mathbb{Z})$. Such a
homology class can be represented as the image of a finite simplicial complex $B$ of homogeneous dimension $2k$ with $H_{2k}(B, \mathbb{Z}) = \mathbb{Z}$ by a continuous map $\varphi : B \to \mathcal{Q}$. We may furthermore assume that $B$ is a Poincaré duality space. In the case $k = 1$, we may assume that that $B$ is a closed oriented surface and that $\varphi$ is smooth.

The pull-back of the universal curve by the map $P \circ \varphi : B \to \mathcal{M}_g$ is a surface bundle $\Pi : E \to B$. Since the image of $B$ under $\varphi$ is contained in the component $\mathcal{Q}$, the zeros of the projective differentials in $\varphi(B)$ define a multisection $\Delta$ of $E$. A component of this multisection only contains zeros of the same order. Thus the multisection $\Delta$ has at least as many components as the number of different orders of zeros of the differentials in $\mathcal{Q}$.

The universal line bundle over the fibers of $\mathcal{P}$ pulls back via the map $(\varphi \circ \Pi)^*$ to a line bundle $\tau$ on $E$. In the proof of the following lemma and later on, we always denote by $\zeta^*$ the dual of a line bundle $\zeta$, or, equivalently, the inverse of $\zeta$ in the group of all complex line bundles on $E$. In particular, $\nu^*$ denotes the vertical cotangent bundle of $E$. We have

**Lemma 4.1.** Let $\Delta_j$ be a component of $\Delta$ with the property that the order of the zeros of the points in $\Delta_j$ equals $k_j$. Then $\nu^*|\Delta_j = \tau^{\otimes(k_j+1)}|\Delta_j$.

**Proof.** The lemma is well known, but we were not able to locate it in the literature, so we provide a proof.

A point $y \in \Delta_j$ is a zero of order $k_j$ of a projective holomorphic one-form on the fiber of $E$ through $y$. The fiber $\tau_y$ of $\tau$ at $y$ can be identified with the complex line of holomorphic one-forms in this projective class.

As $y$ is a zero of this holomorphic one-form of order $k_j$, there are local coordinates $z$ on $\Pi^{-1}(\Pi(y))$ near $y$, with $y$ corresponding to $z = 0$, such that a nonzero differential in $\tau_y$ can locally near $y$ be written in the form $az^{k_j}dz$ for some $a \in \mathbb{C}^*$. This differential then defines a singular euclidean metric near $y$, which has a cone point of cone angle $2\pi(k_j + 1)$ at $y$.

A tangent vector $Y \in T_y\Pi^{-1}(\Pi(y))$ of the fiber of $E$ through $y$ defines a $\mathbb{C}$-valued functional $\beta_Y$ on $\tau_y$ by associating to a differential $\omega \in \tau_y$ the complex length of $Y$ with respect to the singular euclidean metric defined by $\omega$, that is, we distinguish real and imaginary part of this length, and we distinguish the orientation.

If $0 \neq Y$ then we have $\beta_{aY} = \beta_Y$ for some $a \in \mathbb{C}$ if and only if $a = e^{2\pi i\ell/(k_j+1)}$ for some $\ell \in \mathbb{Z}$. As a consequence, $Y \to \beta_Y$ defines an nonzero element in the fiber of the bundle $\nu^* \otimes (\tau^{k_j+1})^*$ at $y$. Since this element depends continuously on $y \in \Delta_j$ by construction, we conclude that the line bundle $\nu^* \otimes (\tau^{k_j+1})^*|\Delta_j$ is trivial. In other words, $\nu^*|\Delta_j$ is isomorphic to $\tau^{\otimes(k_j+1)}|\Delta_j$ which shows the lemma. \hfill $\square$

In the proof of Proposition 2.7, we used triviality of the vertical cotangent bundle on the complement of the branched multisection to identify the Poincaré dual of the class $c_1(\nu^*)$. The following observation serves as a substitute.

**Lemma 4.2.** The line bundle $\nu \otimes \tau$ is trivial on $E - \Delta$.

**Proof.** Let $\alpha \in \nu^*$ be any vector in the vertical cotangent bundle of $E$ at a point $y \in E - \Delta$. We may view $\alpha$ as a $\mathbb{C}$-linear functional on the holomorphic tangent space of the fiber at $y$. As the dimension of the complex vector space of $\mathbb{C}$-linear functionals $T_y\Pi^{-1}(\Pi(y)) \to \mathbb{C}$ equals one, there is precisely one holomorphic one-form $\Lambda(\alpha)$ in the fiber $\tau_y$ of the bundle $\tau$ at $y$ whose restriction to $T_y\Pi^{-1}(\Pi(y))$ coincides with $\alpha$. Then $\alpha \to \Lambda(\alpha)$ defines an isomorphism between $\nu^*$ and $\tau$ on
Let $\Delta_1, \ldots, \Delta_m$ be the components of $\Delta$. The restriction of the projection $\Pi$ to any of these components is a finite unbranched cover $\Delta_j \to B$. Hence each of these components defines a homology class $[\Delta_j] \in H_{2k}(E, \mathbb{Z})$. For each $j$ let $k_j$ be the order of the zero of the differentials in $\varphi(B)$ at the points in $\Delta_j$. Denote as before by $\psi$ the Chern class of the universal bundle over the fibers of $\mathcal{P}$. The following lemma is an analog of Proposition 2.7 and is the key step towards the proof of Theorem 1.

Lemma 4.3. Assume that either

1. $B$ is a closed surface or
2. $\varphi^* \psi = 0$;

then the Chern class $c_1(\nu^* \otimes \tau^*)$ of the complex line bundle $\nu^* \otimes \tau^* \to E$ is Poincaré dual to $\sum k_j [\Delta_j]$.

Proof. Since $B$ is a Poincaré duality space by assumption, the same holds true for $E$. Thus there exists a cohomology class $\alpha \in H^2(E, \mathbb{Z})$ which is Poincaré dual to $\delta = \sum k_j [\Delta_j]$.

The class $\alpha \in H^2(E, \mathbb{Z})$ is the first Chern class of a complex line bundle $L$ on $E$. Namely, complex line bundles on $E$ are classified up to topological equivalence by classes in the cohomology group $H^1(E, \mathbb{R}^*)$ where $\mathbb{R}^*$ is the sheaf of continuous nowhere vanishing $\mathbb{C}$-valued functions on $E$. Since the sheaf $\mathbb{R}$ of all continuous $\mathbb{C}$-valued functions on $E$ is fine, associating to a line bundle its Chern class defines an isomorphism between the group of line bundles on $E$ and the cohomology group $H^2(E, \mathbb{Z})$ via the long exact sequence for sheaf cohomology defined by the short exact sequence $0 \to \mathbb{R} \to \mathbb{R}^* \to \mathbb{Z} \to 0$.

As we will need some more precise information about the line bundle $L$, we also sketch the explicit construction of $L$ as for example described on p. 141 of [GH78].

Let $\mathcal{F}$ be the sheaf of continuous complex valued functions on $E$ whose restrictions to a fibre of $E$ are holomorphic and not identically zero. Let $\mathcal{F}^*$ be the sheaf of functions in $\mathcal{F}$ which vanish nowhere. A global section of $\mathcal{F}/\mathcal{F}^*$ is given by an open cover $\{U_\alpha\}$ of $E$ and a function $f_\alpha \in \mathcal{F}$ on $U_\alpha$ for each $\alpha$ so that

$$\frac{f_\alpha}{f_\beta} \in \mathcal{F}^*(U_\alpha \cap U_\beta).$$

We first claim that a component $\Delta_j$ of $\Delta$ defines a section of $\mathcal{F}/\mathcal{F}^*$. Namely, choose a cover $U = \{U_\alpha\}$ of $E$ which consists of open contractible sets with the following additional properties. The intersection of each set $U_\alpha$ with $\Delta_j$ is connected. There is a function $f_\alpha \in \mathcal{F}$ on $U_\alpha$ such that for each point $y \in U_\alpha \cap \Delta_j$, the restriction of $f_\alpha$ to $\Pi^{-1}(\Pi(y)) \cap U_\alpha$ has a simple zero at $y$. Moreover, these are the only zeros of $f_\alpha$.

By the construction of $\Delta_j$, such functions $f_\alpha$ exists provided that the sets $U_\alpha$ are sufficiently small. Namely, let $z$ be a local holomorphic coordinate on the intersection of $U_\alpha$ with the fibers of $E \to B$ depending continuously on the base. The functions $f_\alpha$ can be chosen as the fiberwise coordinate, normalized in such a way that they vanish precisely at $\Delta_j \cap U_\alpha$. Any two such sections of $\mathcal{F}/\mathcal{F}^*$ differ by a section of $\mathcal{F}^*$ and hence these sections define a class in $H^1(U, \mathcal{F}^*)$. A
standard refinement argument (see [GH78] for details) then yields a cohomology class $\xi(\Delta_j) \in H^1(E, F^*)$ and hence a line bundle $L_j$ on $E$.

The line bundle $L_j$ has the following properties.

1. $L_j|E - \Delta_j$ is trivial.
2. The restriction of $L_j$ to each fiber of $E$ is holomorphic.
3. $L_j$ has a continuous fiberwise holomorphic section which vanishes to first order precisely at the points of $\Delta_j$. In particular, the degree of the restriction of $L_j$ to a fiber of $E$ equals the degree of the covering $\Pi|\Delta_j : \Delta_j \to B$.

Furthermore, these properties characterize the line bundle $L_j$ uniquely.

The first property is immediate from the construction. The second property also follows from the construction. Namely, as the functions $f_\alpha$ are fiberwise holomorphic by assumption, the same holds true for $f_\alpha/f_\beta$ and hence for the restriction of the transition functions of $L_j$ to a fiber of the surface bundle.

A section of $L_j$ with the properties stated in the third property is given by the defining functions $f_\alpha$ on the sets $U_\alpha$. As they only vanish on $\Delta_j$ and are compatible with the transition functions which define the line bundle $L_j$, they define indeed a global section of $L_j$. This section is moreover fiberwise holomorphic, and it vanishes to first order at the intersection of a fiber with $\Delta_j$.

We next show that properties (1)-(3) above imply that $c_1(L_j)$ is Poincaré dual to $[\Delta_j]$ from which uniqueness follows. Namely, let $B$ be a closed surface and let $\varphi : B \to E$ be a continuous map which is transverse to $\Delta_j$ (viewed as a map between simplicial complexes). Then $\varphi(B)$ intersects $\Delta_j$ in finitely many points, counted with multiplicity. The pull-back by $\varphi$ of the global section of $L_j$ is a section of $\varphi^*L_j$ which vanishes precisely at the points in $\varphi^{-1}(\Delta_j)$, with multiplicity equal to the multiplicity of the intersection points. This shows that $c_1(L_j)$ is Poincaré dual to $[\Delta_j]$. We refer to the proof of Proposition 2.7 for a more detailed discussion in a similar situation.

Now assume that $B$ is a closed surface. Assume furthermore without loss of generality that the map $\varphi$ is smooth. Then $\Delta$ is a smoothly embedded surface in $E$ transverse to the fibers. The pull-back of the universal bundle over the fibers of $P \to M_g$ is a line bundle $\xi$ on $B$ which pulls back to the line bundle $\tau$.

Line bundles on a surface are uniquely determined by their degree up to topological equivalence. Let $d \in \mathbb{Z}$ be the degree of $\xi$. Choose a point $p \in B$ and a trivialization of $\xi$ on $B - \{p\}$ as well as a trivialization of $\xi$ on a disk neighborhood $D$ of $B$. These trivializations determine a global fiberwise holomorphic section of $\nu^*$ on $\Pi^{-1}(B - \{p\})$ and on $\Pi^{-1}(D)$ which vanishes precisely on the components $\Delta_j$ of $\Delta$, to the order $k_j$. Namely, they define a lift of the map $\varphi|B - \{p\}$ and $\varphi|D$ to the Hodge bundle over $M_g$.

The line bundle $\xi$ is obtained from its restrictions to $B - \{p\}$ and $D$ by gluing the fibers over a circle $S^1$ in $D - \{p\} \subset B - \{p\}$ surrounding the point $p$. This fiberwise gluing map is given by a homomorphism $S^1 \to S^1 \subset \mathbb{C}^*$ whose degree is the degree of the line bundle. Furthermore, gluing the fibers of $\xi$ over the circle $S^1$ extends to a gluing of the holomorphic sections of $\nu^*$ over the circle $S^1$ which are uniquely determined by the points in these fibers (and, of course, the map $\varphi$).

By naturality of the tensor product of line bundles, a global section of the tensor product $\nu^* \otimes \tau^*$ is given by the same construction, but with the gluing map corresponding to the trivial line bundle on $B$. As a consequence, the bundle $\nu^* \otimes \tau^*$
admits a global fiberwise holomorphic section which vanishes precisely on the components of $\Delta$, and the vanishing degree on the component $\Delta_j$ equals $k_j$. By the beginning of this proof and the fact that the Poincaré dual of $k_j[\Delta_j]$ is the first Chern class of the line bundle $\otimes_j L_j^{k_j}$, this means that $\nu^* \otimes \tau^*$ is Poincaré dual to $\Delta$ which is what we wanted to show.

Similarly, if $B$ is arbitrary and if $\varphi^* \psi = 0$, then the pull-back by $\varphi$ of the universal bundle on $\mathcal{P}$ is trivial and therefore its pull-back $\tau$ is trivial as well. Therefore the map $\varphi$ admits a lift to the Hodge bundle, and as in the proof of Proposition 2.7, we conclude that $\nu^*$ admits a global fiberwise holomorphic section which only vanishes on the components of $\Delta_j$ to the correct order. This then implies as before that $\nu^*$ is Poincaré dual to $\sum_j k_j[\Delta_j]$. The lemma is proven. \hfill \box

We use this to show

**Proposition 4.4.** Let $Q$ be a component of a stratum, let $B$ be a closed surface and let $\varphi : B \to Q$ be a smooth map. Then $\kappa_1(P\varphi(B)) = \psi(\varphi(B)) = 0$.

**Proof.** We continue to use the assumptions and notations from Lemma 4.3. Consider in particular the components $\Delta_j$ of the cycle $\Delta$. These are smoothly embedded surfaces in the surface bundle $\Pi : E \to B$ defined by $P \circ \varphi$.

By Lemma 4.3, the first Chern class $c_1(\nu^*) - c_1(\tau)$ of the line bundle $\nu^* \otimes \tau^*$ is Poincaré dual to the homology class $\delta = \sum_j k_j[\Delta_j]$ where as before, $k_j \geq 1$ is the order of the zero of the points in $\Delta_j$. We compute the self-intersection number of $\delta$ as follows.

As $\Delta_j \subset E$ is a smoothly embedded surface transverse to the fibers of $E \to B$, the vertical tangent bundle $\nu$ is the normal bundle of $\Delta_j$ and hence the self-intersection number of $[\Delta_j]$ equals $c_1(\nu)[\Delta_j]$. Since the components of $\Delta_j$ are pairwise disjoint we then have

\begin{equation}
\delta \cdot \delta = \left( \sum_j (\sum_j k_j[\Delta_j]) \right) \cdot \left( \sum_j k_j[\Delta_j] \right) = \sum_j k_j^2 c_1(\nu)(\Delta_j).
\end{equation}

On the other hand, by Lemma 4.3, the class $\delta$ is Poincaré dual to $c_1(\nu^*) - c_1(\tau)$. Moreover, by Lemma 4.1, the restriction of $\nu^* \otimes \tau^*$ to $\Delta_j$ is equivalent to the line bundle $\tau^{k_j}$. Thus we also have

\begin{equation}
\delta \cdot \delta = \sum_j k_j(c_1(\nu^*) - c_1(\tau))[\Delta_j] = \sum_j k_j^2 c_1(\tau)(\Delta_j) = \sum_j k_j^2 d_j b
\end{equation}

where $b = \psi(\varphi(B))$ and where $d_j$ is the degree of the map $\Pi|\Delta_j : \Delta_j \to B$. The last equality holds true since $\tau$ is the pull-back of the line bundle on $B$ with Chern class $\varphi^* \psi$.

Substituting the identity $c_1(\nu)[\Delta_j] = -(k_j + 1)c_1(\tau)[\Delta_j] = -(k_j + 1)d_j b$ from Lemma 4.1 in equation (3) and comparison with equation (4) then yields

\begin{equation}
b \left( \sum_j d_j (k_j^2 (k_j + 1) + k_j^2) \right) = 0.
\end{equation}

But the numbers $d_j, k_j$ are all positive and hence this is only possible if $b = 0$.

As a consequence, the line bundle $\tau$ on $E$ is trivial, and $\delta$ is Poincaré dual to $c_1(\nu^*)$. Moreover, equation (4) yields that $\delta \cdot \delta = c_1(\nu^*) \cup c_1(\nu^*)(E) = \kappa_1(P\varphi(B)) = 0$ whence the proposition. \hfill \box
Proof of Corollary 2. Let $Q$ be a component of a stratum of projective abelian varieties. Let $V$ be a complex variety of dimension $k$. We have to show that any holomorphic map $\eta : V \to Q$ is constant.

To this end we evoke the following result of Wolpert [W86]: There exists a holomorphic line bundle $L$ on $M_g$ with Chern class $\kappa_1$, and there is a Hermitian metric on $L$ with curvature form $\omega = \frac{1}{2\pi\iota} \omega_{WP}$ where $\omega_{WP}$ is the Weil Peterssen Kähler form on $M_g$. In particular, $\omega$ is positive. As a consequence, if $V$ is a complex variety of dimension $k \geq 1$ and if $\zeta : V \to M_g$ is a holomorphic map which does not factor through a map from a variety of smaller dimension, then

$$
\kappa_1^k(\zeta(V)) = \int_V (\zeta^* \omega)^k > 0.
$$

Now if $\eta : V \to Q$ is holomorphic, then the same holds true for $P \circ \eta$. Proposition 4.4 shows that the pull-back by $P$ of the cohomology class $\kappa_1$ vanishes on $Q$ and hence $P^* \kappa_1^k(\eta(V)) = \kappa_1^k(P \circ \eta)(V) = 0$.

By possibly modifying the variety $V$ we may assume that the holomorphic map $\eta$ does not factor through a map from a variety of smaller dimension. However, by positivity of the curvature form $\omega$, if the dimension of $V$ is positive, then the map $P \circ \eta$ factors through a map of a variety of smaller dimension. As a consequence, the dimension of the generic fiber of $P \circ \eta$ is positive.

Let $C \subset V$ be such a generic fiber of positive dimension. Since by assumption the map $\eta$ does not factor through a map from a variety of smaller dimension, the restriction of $\eta$ to $C$ is a nonconstant holomorphic map $C \to Q$ whose image is entirely contained in a fiber of the bundle $P \to M_g$. Moreover, we may assume that $\eta|C$ does not factor through a map from a variety of smaller dimension.

On the other hand, by Proposition 4.4, if we denote as before by $\psi$ the Chern class of the universal bundle of the fiber of $P \to M_g$, then $(\eta|C)^* \psi = 0$. But the dual of the universal bundle over a complex projective space admits a Hermitian metric with fiberwise positive curvature form (which up to a positive constant is just the Fubini Study metric) and hence using the same argument as in the previous paragraph, we deduce that $\eta|C$ factors through a map from a variety of smaller dimension. This is a contradiction which completes the proof of the corollary. □

Proof of Corollary 3. Let $M_{g,\text{odd}}$ be the finite orbifold cover of $M_g$ which is the moduli space of curves with odd theta characteristic. By definition, this is the quotient of Teichmüller space by the finite index subgroup of $\text{Mod}(S_g)$ which preserves an odd spin structure on $S_g$. Such an odd spin structure is defined as a quadratic form on $H_2(S_g, \mathbb{Z}/2\mathbb{Z})$ with odd Arf invariant (see [KtZ03] for more information). Each of the curves $X \in M_{g,\text{odd}}$ admits an odd theta characteristic which by definition is a holomorphic line bundle $L$ whose square equals the canonical bundle and such that $h^0(X,L)$ is odd. The square of a holomorphic section of $L$ is a holomorphic one-form on $X$ with all zeros of even multiplicity.

All bundles over $M_g$ will be pulled back to $M_{g,\text{odd}}$ and will be denoted by the same symbols. Let $Q$ be the closure in $P$ of the stratum $PH(2,\ldots,2)_{\text{odd}}$ of projective abelian differentials with all zeros of order two and odd spin structure. Then the restriction of the projection $P : P \to M_{g,\text{odd}}$ to $Q$ is surjective.

By a result of Teixidor i Bigas [TiB87], the locus of pairs $(X,L) \in M_{g,\text{odd}}$ with $h^0(X,L) \geq 3$ has codimension three in $M_{g,\text{odd}}$. This implies that there exists a
subset $A \subset \mathcal{M}_{g, \text{odd}}$ of codimension 3 such that the restriction of $P$ to $Q - P^{-1}(A)$ is a bijective holomorphic morphism.

Let $\mathcal{D} \subset \mathcal{M}_{g, \text{odd}}$ be the image of the set of all points in $Q$ which are contained in the boundary of $\mathbb{P}H(2, \ldots, 2)^{\text{odd}}$. For reasons of dimension, $\mathcal{D}$ is a divisor in $\mathcal{M}_{g, \text{odd}}$. Moreover, if $h^0(X, L) \geq 3$ then there is at least one holomorphic section of $L$ whose square is a differential with at least one zero of order at least four. This yields that $A \subset \mathcal{D}$.

As a consequence, if $V$ is any complex variety and if $\eta : V \to \mathcal{M}_{g, \text{odd}} - \mathcal{D}$ is any holomorphic map, then $\eta$ lifts to a holomorphic map into $\mathbb{P}H(2, \ldots, 2)^{\text{odd}}$. By Corollary 2, $\eta$ is constant. This shows that indeed, $\mathcal{M}_{g, \text{odd}} - \mathcal{D}$ is affine.

Similarly, define $\mathcal{D}_2 \subset \mathcal{D}$ to be the image under the map $P$ of the union of all strata of projective abelian differentials with all zeros of even order and either at least one zero of order at least 6 or at least two zeros or order at least 4. Then $\mathcal{D}_2 \subset \mathcal{D}$ is of codimension one. Furthermore, as $A \subset \mathcal{D}$ is of complex codimension 2, if $B$ is a closed surface and if $\varphi : B \to \mathcal{D} - \mathcal{D}_2$ is any smooth map, then with a small homotopy, we can modify $\varphi$ to a map $\tilde{\varphi}$ whose image is entirely contained in $\mathcal{D} - (\mathcal{D}_2 \cup A)$. Then $\tilde{\varphi}$ lifts to a smooth map $B \to \mathbb{P}H(2, \ldots, 2, 4)^{\text{odd}}$ and hence by Proposition 4.4, we have $\kappa_1(\tilde{\varphi}(B)) = \kappa_1(\varphi(B)) = 0$. By the discussion in the proof of Corollary 2, this implies that $\varphi$ can not be non-constant and holomorphic. This shows that $\mathcal{D} - \mathcal{D}_2$ does not contain a complete curve. □

**Example 4.5.** Using the notations from the proof of Corollary 3, a result of Harris (see Theorem 0.1 of [TiB87]) shows that for $g = 3$ and $g = 4$, the locus of all curves $X \in \mathcal{M}_{g, \text{odd}}$ with odd theta characteristic $L$ and such that $h^0(X, L) \geq 3$ is empty. As a consequence, the restriction of the projection $P : \mathcal{P} \to \mathcal{M}_{g, \text{odd}}$ to the closure of $\mathbb{P}H(2, 2)^{\text{odd}}$ (for $g = 3$) and of $\mathbb{P}H(2, 2, 2)^{\text{odd}}$ (for $g = 4$) is a biholomorphism.

By Corollary 3, the projections of the components of strata $\mathbb{P}H(2, 2)^{\text{odd}}$ and $\mathbb{P}H(4)^{\text{odd}}$ define a stratification of depth $2 = g - 1$ of $\mathcal{M}_{3, \text{odd}}$ with affine strata. This is however well known and is discussed in [FL08] and [Ch19].

Similarly, the components of the strata $\mathbb{P}H(2, 2, 2)^{\text{odd}}$, $\mathbb{P}H(2, 4)^{\text{odd}}$ and $\mathbb{P}H(4)^{\text{odd}}$ project to a stratification of $\mathcal{M}_{4, \text{odd}}$ of depth $3 = g - 1$ with affine strata.

Corollary 4 from the introduction also is a consequence of Theorem 1. Namely, the hyperelliptic locus in $\mathcal{M}_g$ is the moduli space of all hyperelliptic complex curves. In the case $g = 2$, this is just the entire moduli space.

**Proof of Corollary 4.** By Corollary 3 and its proof, it suffices to show that the restriction of the first Mumford Morita Miller class to the hyperelliptic locus vanishes.

Thus let $B$ be a closed oriented surface and let $\varphi : B \to \mathcal{M}_g$ be a smooth map whose image is contained in the hyperelliptic locus. The pull-back under $\varphi$ of the universal curve is a surface bundle $\Pi : E \to B$.

Choose a basepoint $x_0 \in B$ and a Weierstrass point $z_0 \in \Pi^{-1}(x_0)$ in the fibre. Since Weierstrass points are distinct, every loop $\gamma$ in $B$ based at $x_0$ admits a unique lift to $E$ beginning at $z_0$ whose image consists of Weierstrass points. The endpoint is another Weierstrass point in $\Pi^{-1}(x_0)$ which only depends on the homotopy class of the loop. Thus this construction defines a homomorphism of $\pi_1(B)$ into the permutation group of the $2g + 2$ Weierstrass points of $\Pi^{-1}(x_0)$.

Let $\Gamma < \pi_1(B)$ be the kernel of this homomorphism and let $\theta : B_0 \to B$ be the finite cover of $B$ with fundamental group $\Gamma$. Then $\kappa_1(\varphi\theta(B_0)) = p\kappa_1(\theta(B))$
where \( p \geq 1 \) is the degree of the covering \( B_0 \to B \). Hence it suffices to show that 
\[
\kappa_1(\varphi \theta(B_0)) = 0.
\]
The homomorphism of \( \pi_1(B_0) \) into the permutation group of \( 2g+2 \) points defined above is trivial by construction and hence the pullback \( \Pi : E_0 \to B_0 \) of the universal curve by \( \varphi \circ \theta \) admits \( 2g+2 \) pairwise disjoint sections whose images consist of Weierstrass points.

Let \( \alpha : B_0 \to E_0 \) be one of these sections. Then for each \( x \in B_0 \), there is an abelian differential \( q(x) \) on the Riemann surface \( \varphi \circ \theta(x) \), unique up to a multiple by an element in \( \mathbb{C}^* \), which has a zero of order \( 2g-2 \) at \( x \). This differential is the pull-back under the hyperelliptic involution of a meromorphic quadratic differential on \( \mathbb{C}P^1 \) which has a single zero of order \( 2g-3 \) at the image of the distinguished Weierstrass point, and a simple pole at each of the other \( 2g+1 \) Weierstrass points (see [KtZ03] for a detailed account on this construction). The projective class of this differential depends smoothly on \( x \) and hence the differentials \( q(x) \) \( (x \in B_0) \) define a section of the bundle \( \mathcal{P} \) whose image is contained in the stratum of projective abelian differentials with a single zero of order \( 2g-2 \).

Theorem 4 is now an immediate consequence of Proposition 4.4. \( \square \)

**Proof of Theorem 1.** Let \( Q \subset \mathcal{P} \) be a component of a stratum of projective abelian differentials. It suffices to show that for every \( k \geq 1 \), for every finite \( 2k \)-dimensional simplicial Poincaré duality complex \( B \) of homogeneous dimension \( 2k \) and for every continuous map \( \varphi : B \to Q \) we have \((P \circ \varphi)^* \kappa_k(B) = 0\).

To this end consider the surface bundle \( \Pi : E \to B \) defined by \( P \circ \varphi \). The zeros of the differentials in \( \varphi(B) \) define a multisection \( \Delta \) of \( E \). Let as before \( \Delta_1, \ldots, \Delta_m \) be the components of \( \Delta \) and for \( j \leq m \) let \( k_j \) be the multiplicity of the zero of a point in \( \Delta_j \). By Lemma 4.3 and Proposition 4.4, the homology class \( \delta = \sum_j k_j[\Delta_j] \) is Poincaré dual to \( c_1(\nu^*) \). In particular, by the definition of the \( k \)-th Mumford–Morita–Miller class [M87], we have
\[
(7) \quad \kappa_k(P\varphi(B)) = c_1(\nu^*)^{k+1}(E) = c_1(\nu^*)^k(\delta).
\]

The restriction of \( \Pi \) to each of the sets \( \Delta_j \subset \Delta \) is a covering. Moreover, by Lemma 4.1, we have \( \nu^*[\Delta_j] = \tau^{k_j+1} \) where \( \tau \) is the pull-back by \( \varphi \circ \Pi \) of the universal bundle over the fibers of \( \mathcal{P} \to \mathcal{M}_g \). But by Proposition 4.4, the Chern class of the line bundle \( \tau \) on \( E \) vanishes and hence the bundle \( \tau \) is trivial. Then the same holds true for \( \nu^*[\Delta_j] \) and therefore \( c_1(\nu^*)^k(\Delta_j) = 0 \). Since this holds true for all \( j \), we have \( \kappa_k(P\varphi(B)) = 0 \) by equation (7) which completes the proof of the theorem. \( \square \)

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