A PARTICLE IN THE BIO-SAVART-LAPLACE MAGNETIC FIELD: EXPLICIT SOLUTIONS

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Abstract. We consider the Schrödinger operator $H = (i\nabla + A)^2$ in the space $L_2(\mathbb{R}^3)$ with a magnetic potential $A$ created by an infinite straight current. We perform a spectral analysis of the operator $H$ almost explicitly. In particular, we show that the operator $H$ is absolutely continuous, its spectrum has infinite multiplicity and coincides with the positive half-axis. Then we find the large-time behavior of solutions $\exp(-iHt)f$ of the time dependent Schrödinger equation. Equations of classical mechanics are also integrated. Our main observation is that both quantum and classical particles have always a preferable (depending on its charge) direction of propagation along the current and both of them are confined in the plane orthogonal to the current.

1. Introduction

They are very few examples of explicit solutions of the Schrödinger equation with a magnetic potential. Probably the only ones are a constant magnetic field, $B(x, y, z) = B_0$, (see, e.g., [3]) and, in the two dimensional case, a magnetic field localized at the origin, $B(x, y) = B_0\delta(x, y)$ where $\delta(x, y)$ is the Dirac function (see [1]). The solution is expressed in terms of Hermite functions in the first case and in terms of Bessel functions in the second case. Here we suggest a third example of an explicitly solvable Schrödinger equation. Actually, we consider the magnetic field $B(x, y, z)$ created by an infinite straight current. Physically, this case is opposite to the case of an (infinitely) thin straight solenoid considered in [1] by Aharonov and Bohm where the field is concentrated inside the solenoid.

Suppose that the current coincides with the axis $z$ and that the axes $x$, $y$ and $z$ are positively oriented. According to the Bio-Savart-Laplace law (see, e.g., [5])

$$B(x, y, z) = \alpha(-r^{-2}y, r^{-2}x, 0), \quad r = (x^2 + y^2)^{1/2},$$

where $|\alpha|$ is proportional to the current strength and $\alpha > 0$ ($\alpha < 0$) if the current streams in the positive (negative) direction. The magnetic potential is defined by the equation

$$B(x, y, z) = \text{curl} A(x, y, z)$$

and can be chosen as

$$A(x, y, z) = -\alpha(0, 0, \ln r).$$

Thus, the corresponding Schrödinger operator in the space $L_2(\mathbb{R}^3)$ has the form

$$H = H_\gamma = -\partial_x^2 - \partial_y^2 + (i\partial_z - \gamma \ln r)^2, \quad \gamma = e\alpha,$$
where $e$ is the charge of a quantum particle of the mass $m = 1/2$ and the speed of the light $c = 1$.

Since magnetic potential (1.1) grows as $r \to \infty$, the Hamiltonian $H$ does not fit to the well elaborated framework of spectral and scattering theory. Nevertheless, we perform in Section 2 its spectral analysis almost explicitly. To be more precise, we reduce the problem to an ordinary differential equation with the potential $\gamma^2 \ln^2 r$ (let us call it logarithmic oscillator). We show that the operator $H$ is absolutely continuous, its spectrum has infinite multiplicity and coincides with the positive half-axis. Then we find in Section 3 the large-time behavior of solutions $\exp(-iHt)f$ of the time dependent Schrödinger equation. In Section 4 we integrate equations of classical mechanics. Our main observation is that positively (negatively) charged quantum and classical particles always move in the direction of the current (in the opposite direction) and are localized in the orthogonal plane.

A detailed presentation of the results of this note can be found in [6].

2. Spectral analysis of the operator $H$

Let us first consider a more general magnetic potential

\begin{equation}
A(x, y, z) = (0, 0, A(x, y))
\end{equation}

with an arbitrary (we disregard here domain questions) real function $A(x, y)$ which tends to infinity (either $+\infty$ or $-\infty$) as $r = (x^2 + y^2)^{1/2} \to \infty$. The corresponding Schrödinger operator is

$$H = -\Delta + (i\partial_z + eA(x, y))^2,$$

where $\Delta$ is always the Laplacian in the variables $(x, y)$. Since $A$ does not depend on $z$, we make the Fourier transform $\Phi = \Phi_z$ in the variable $z$. Then the operator $H = \Phi H \Phi^*$ acts in the space $L_2(\mathbb{R}^2 \times \mathbb{R})$ as

$$(Hu)(x, y, p) = (h(p)u)(x, y, p),$$

where

\begin{equation}
\begin{aligned}
h(p) &= -\Delta + (p - eA(x, y))^2. \\
\end{aligned}
\end{equation}

Here $p \in \mathbb{R}$ (the momentum in the direction of the $z$-axis) is the variable dual to $z$ and the operator $h(p)$ acts in the space $L_2(\mathbb{R}^2)$.

Since $A(x, y) \to \infty$ or $A(x, y) \to -\infty$ as $r \to \infty$, the spectrum of each operator $h(p)$ is positive and discrete. Let $\lambda_n(p)$, $n = 1, 2, \ldots$, be its eigenvalues, numerated in such a way that $\lambda_n(p)$ are analytic functions of $p$. The spectrum of the operator $H$, and hence of $H$, consists of the sets (branches) covered by the functions $\lambda_n(p)$, $n = 1, 2, \ldots$, as $p$ runs from $-\infty$ to $\infty$. This is similar both to the cases of the constant field where $\lambda_n(p) = |eB_0|(2n + 1) + p^2$ and to the periodic problem where the role of the momentum $p$ in the direction of the $z$-axis is played by the quasimomentum (see, e.g. [4]). Thus, the general Floquet theory implies that the spectrum of $H$ is absolutely continuous, up eventually to some eigenvalues of infinite multiplicity. Such eigenvalues appear if at least one of the functions $\lambda_n(p)$
is a constant on some interval. Then this function is a constant for all \( p \in \mathbb{R} \). On the other hand, if, say, the function \(-e\mathcal{A}(x, y)\) is semibounded from below, then

\[
\lim_{p \to \infty} \inf_{(x,y) \in \mathbb{R}^2} (p - e\mathcal{A}(x, y))^2 = \infty
\]

as \( p \to \infty \), and hence \( \lim_{p \to \infty} \lambda_n(p) = \infty \) for all \( n \). Thus, we have the following simple result.

**Theorem 2.1.** Suppose that \( \mathcal{A}(x, y) \) is a semi-bounded function which tends either to \(+\infty\) or to \(-\infty\) as \( r \to \infty \). Then the operator \( H \) is absolutely continuous.

Note that the Thomas arguments (see, e.g., [4]) relying on the study of the operator-function \( h(p) \) for complex \( p \) are not necessary here.

The problem may be further simplified if \( \mathcal{A}(x, y) = \mathcal{A}(r) \). Then we can separate variables in the polar coordinates \((r, \theta)\). Denote by \( \mathcal{H}_m \) the space of functions \( u(r) e^{im\theta} \) where \( u \in L^2(\mathbb{R}; rdr) \) and \( m = 0, \pm 1, \pm 2, \ldots \) is the orbital quantum number. Then

\[
L^2(\mathbb{R}^2) = \bigoplus_{m=-\infty}^{\infty} \mathcal{H}_m.
\]

Every subspace \( \mathcal{H}_m \) is invariant with respect to the operator \( h(p) \). The spectra of their restrictions \( h_m(p) \) on \( \mathcal{H}_m \) consist of positive simple eigenvalues \( \lambda_{m,1}(p) < \lambda_{m,2}(p) < \ldots \), which are analytic functions of \( p \). We denote by \( \psi_{m,1}(r,p), \psi_{m,2}(r,p), \ldots \) the corresponding eigenfunctions which are supposed to be normalized and real.

Let us return to the operator \( H \) with the potential \( \mathcal{A}(r) = -\alpha \ln r \). In this case operator (2.2) equals

\[
h(p) = -\Delta + \ln^2(\mu^{p}\gamma).
\]

Since \( \mathcal{H}_\gamma \mathcal{A} = H_{-\gamma} \mathcal{H} \), it suffices to consider the case \( \gamma > 0 \). It is convenient to transfer the dependence on the momentum \( p \) into the kinetic energy and to introduce the parameter \( a = e^{p/\gamma} \in (0, \infty) \) instead of \( p \). Let us set

\[
K(a) = -a^2\Delta + \gamma^2 \ln^2 r,
\]

and let \( w(a), (w(a)f(x,y) = af(ax,ay) \) be the unitary operator of dilations in the space \( L^2(\mathbb{R}^2) \). Then

\[
w^*(a)h(p)w(a) = K(a), \quad a = e^{p/\gamma}.
\]

We denote by \( \mu_{m,n}(a) \) and \( \phi_{m,n}(r,a) \) eigenvalues and eigenfunctions of the restrictions of the operators \( K(a) \) on the subspaces \( \mathcal{H}_m \). It follows from (2.6) that \( \mu_{m,n}(a) = \lambda_{m,n}(p) \) and \( \phi_{m,n}(a) = w^*(a)\psi_{m,n}(p) \). Actually, decomposition (2.4) is needed only to avoid crossings between different eigenvalues of the operators \( h(p) \). It allows us to use formulas of perturbation theory (see, e.g., [4]) for simple eigenvalues. We fix \( m \) and omit it from the notation.

The next assertion is quite elementary but plays the crucial role in the following.

**Lemma 2.2.** For every \( n \), we have that \( \mu'_n(a) > 0 \) for all \( a > 0 \).
Indeed, analytic perturbation theory shows that

\begin{equation}
\mu_n'(a) = (K_n'(a)\phi_n(a),\phi_n(a)) = 2a \int_{\mathbb{R}^2} |\nabla \phi_n(x,y,a)|^2 \, dx \, dy.
\end{equation}

This expression is obviously positive since otherwise \( \phi_n(x,y,a) = \text{const.} \)

The next assertion realizes an obvious idea that the spectrum of \( K(a) \) converges as \( a \to 0 \) (in the quasiclassical limit) to that of the multiplication operator by \( \gamma^2 \ln^2 r \), which is continuous and starts from zero.

**Lemma 2.3.** For every \( n \), we have that \( \lim_{a \to 0} \mu_n(a) = 0 \).

Since the function \( \ln r \) is not semibounded, relation (2.3) is not true in our case. Nevertheless, taking into account the kinetic energy, we obtain the following result.

**Lemma 2.4.** For every \( n \), we have that \( \lim_{a \to \infty} \mu_n(a) = \infty \).

In terms of eigenvalues \( \lambda_n(p) \) of the operators \( h(p) \), Lemmas 2.2 – 2.4 mean that \( \lambda_n(p) > 0 \) for all \( p \in \mathbb{R} \) and \( \lim_{p \to -\infty} \lambda_n(p) = 0, \lim_{p \to \infty} \lambda_n(p) = \infty \) (for \( \gamma > 0 \)).

Let \( \Lambda_n \) be multiplication operator by the function \( \lambda_n(p) \) in the space \( L^2(\mathbb{R}) \). It follows from the results on the function \( \lambda_n(p) \) that the spectrum of \( \Lambda_n \) is absolutely continuous, simple and coincides with the positive half axis. Let us introduce a unitary mapping

\[ \Psi : L^2(\mathbb{R}_+ \times \mathbb{R}; r \, dr \, dp) \to \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}) \]

by the formula \( (\Psi f)_n(p) = \int_{0}^{\infty} f(r,p)\bar{\psi}_n(r,p) r \, dr \). Then

\begin{equation}
\Psi \Phi H \Phi^* \Psi^* = \bigoplus_{n=1}^{\infty} \Lambda_n
\end{equation}

(of course \( H = H_m \) and \( \Lambda_n = \Lambda_{n,m} \)), and we obtain the following

**Theorem 2.5.** The spectra of all operators \( H_m \) and \( H \) are absolutely continuous, have infinite multiplicity and coincide with the positive half axis.

As a by-product of our considerations, we have constructed a complete set of eigenfunctions of the operator \( H \). They are parametrized by the orbital quantum number \( m \), the momentum \( p \) in the direction of the \( z \)-axis and the number \( n \) of an eigenvalue \( \lambda_{m,n}(p) \) of the operator \( h_m(p) \) defined by formula (2.2) on the subspace \( H_m \). Thus, if we set

\[ u_{m,n,p}(r,z,\theta) = e^{ipz} e^{im\theta} \psi_{m,n}(r,p), \]

then \( Hu_{m,n,p} = \lambda_{m,n}(p)u_{m,n,p} \).
3. Time Evolution

Explicit formulas obtained in the previous section allow us to find the asymptotics for large \( t \) of solutions \( u(t) = \exp(-iHt)u_0 \) of the time dependent Schrödinger equation. On every subspace with a fixed orbital quantum number \( m \), the problem reduces to the asymptotics of the function \( u(t) = \exp(-iHmt)u_0 \). Below we fix \( m \) and suppose that \( \gamma > 0 \).

Assume that

\[
(\Phi u_0)(r,p) = \psi_n(r,p)f(p),
\]

where \( f \in C_0^\infty(\mathbb{R}) \). Then it follows from formula (2.8) that

\[
u(r,z,t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ipz-i\lambda_n(p)t}\psi_n(r,p)f(p)dp.
\]

The stationary points of this integral are determined by the equation

\[
z = \lambda'_n(p)t.
\]

Since \( \lambda'_n(p) > 0 \), the equation (3.3) has a solution only if \( zt > 0 \). We need the following information on the eigenvalues \( \mu_n(a) \) of the operator (2.5).

**Lemma 3.1.** For every \( n \), we have that \( \lim_{a \to 0} a\mu'_n(a) = 0 \).

Indeed, it follows from equation (2.7) that \( a\mu'_n(a) \leq 2\mu_n(a) \). Therefore it remains to use Lemma 2.3.

Lemma 3.1 means that \( \lim_{p \to -\infty} \lambda'_n(p) = 0 \). The following conjecture is physically quite plausible and is used mainly to formulate Theorem 3.3 below in a simpler form.

**Conjecture 3.2.** For every \( n \), we have that \( \lambda''_n(p) > 0 \) for all \( p \in \mathbb{R} \) and \( \lim_{p \to \infty} \lambda'_n(p) = \infty \).

Therefore equation \( \lambda'_n(p) = v \) has a unique solution \( p_n = \varphi_n(v) \) for every \( v > 0 \). Clearly,

\[
\lambda''_n(\varphi_n(v))\varphi'_n(v) = 1.
\]

Let \( \Phi_n(v) = \varphi_n(v)\alpha - \lambda_n(\varphi_n(v)), \theta(v) = 1 \) for \( v > 0 \), \( \theta(v) = 0 \) for \( v < 0 \) and \( \pm i = e^{\pm \pi i/2} \). Applying to the integral (3.2) the stationary phase method and taking into account identity (3.4), we find that

\[
u(r,z,t) = e^{i\Phi_n(z/t)}\varphi_n(r,\varphi_n(z/t))\varphi'_n(z/t)^{1/2} f(\varphi_n(z/t))(it)^{-1/2}\theta(z/t) + u_\infty(r,z,t),
\]

where

\[
\lim_{t \to \pm \infty} ||u_\infty(\cdot,t)|| = 0.
\]

Note that the norm in the space \( L_2(\mathbb{R}_+ \times \mathbb{R}) \) of the first term in the right-hand side of (3.5) equals \( ||u_0|| \). The asymptotics (3.5) extends of course to all \( f \in L_2(\mathbb{R}) \) and to linear combinations of functions (3.1) over different \( n \). Thus, we have proven
Theorem 3.3. Assume that Conjecture 3.2 is fulfilled. Suppose that $\gamma > 0$. Let $u(t) = \exp(-iH_m t)u_0$ where $u_0$ satisfies (3.1). Then the asymptotics as $t \to \pm \infty$ of this function is given by relations (3.5), (3.6). Moreover, if $f \in C^\infty_0(\mathbb{R})$ and $\mp z > 0$, then the function $u(r, z, t)$ tends to zero faster than any power of $|t|^{-1}$ as $t \to \pm \infty$.

Conversely, for any $g \in L_2(\mathbb{R}_+)$ define the function $u_0$ by the equation

$$(\Phi u_0)(r, p) = \psi_n(r, \lambda_n'(p))\lambda_n''(p)^{1/2}g(\lambda_n'(p)).$$

Then $u(t) = \exp(-iH_m t)u_0$ has the asymptotics as $t \to \pm \infty$

$$u(r, z, t) = e^{i\Phi_n(z/t)}\psi_n(r, z/t)g(z/t)(it)^{-1/2}\theta(z/t) + u_\infty(r, z, t),$$

where $u_\infty$ satisfies (3.6).

4. Classical mechanics

Let us consider the motion of a classical particle of mass $m = 1/2$ and charge $e$ in a magnetic field created by potential (2.1) where $A(x, y) = A(r)$, $r = (x^2 + y^2)^{1/2}$. We suppose that $A(r)$ is an arbitrary $C^2$-function such that $A(r) = o(r^{-1})$ as $r \to 0$ and $|A(r)| \to \infty$ as $r \to \infty$. The solution given below is, to a large extent, similar to the Kepler solution of equations of motion for a particle in a spherically symmetric electric field. However, in the electric case the motion is always restricted to a plane, whereas in the magnetic case it is confined in the plane $z = 0$ but the propagation of a particle in the $z$-direction has a non-trivial character. We proceed here from the Hamiltonian formulation. An approach based on the Newton equations can be found in [6].

Let $r$ be a position of a particle and $p$ be its momentum. Let us write down the Hamiltonian

$$H(r, p) = (p^2 - eA(r))^2$$

in the cylindrical coordinates $(r, \varphi, z)$. In the case (2.1) where $A(x, y) = A(r)$, we have that

$$H(r, p) = (p_r^2 + r^{-2}p_\varphi^2) + (pq - eA(r))^2,$$

where $p_r$, $p_\varphi$ and $p_z$ are momenta conjugate to the coordinates $r$, $\varphi$ and $z$. Since $H(r, p)$ does not depend on $\varphi$ and $z$, the momenta $p_\varphi(t)$ and $p_z(t)$ are conserved, i.e., $p_\varphi(t) = M$ ($M$ is the moment of momentum with respect to the $z$-axis) and $p_z(t) = P$ (the magnetic momentum in the $z$-direction). Therefore Hamiltonian equations read as

$$\begin{align*}
r'(t) &= 2p_r(t), \\
p_r'(t) &= -V'(r(t))
\end{align*}$$

where

$$V(r) = M^2r^{-2} + (P - eA(r))^2,$$

and

$$\varphi'(t) = 2Mr(t)^{-2}$$

$$z'(t) = 2(P - eA(r(t))).$$
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It suffices to solve the system (4.1) since, given \( r(t) \), the solutions of equations (4.3) and (4.4) are constructed by the formulas

\[
\varphi(t) = \varphi(0) + 2M \int_0^t r(s)^{-2} ds
\]

and

\[
z(t) - z(0) = 2 \int_0^t (P - eA(r(s))) ds.
\]

The solution of the system (4.1) is quite similar to the solution of the Kepler problem although in our case the effective potential energy (4.2) depends additionally on the momentum \( P \) in the \( z \)-direction. In the solutions of the quantum problems, it is reflected by the fact that, for electric spherically symmetric potentials, the variables can be separated (in the spherical coordinates), whereas in our case the operators \( h(p) \) depend on \( p \).

Thus, to solve (4.1), we remark that

\[
4^{-1}r'(t)^2 + V(r(t)) = K,
\]

where \( K = 4^{-1}r'(0)^2 + M^2r(0)^{-2} + 4^{-1}z'(0)^2 \) is a constant kinetic energy of a particle. Clearly, (4.7) is the equation of one-dimensional motion (see, e.g., [2]) with the effective potential energy \( V(r) \) and the total energy \( K \). It admits the separation of variables and can be integrated by the formula

\[
t = \pm 4 \int \left( K - V(r) \right)^{-1/2} dr.
\]

Note that \( V(r) \to \infty \) as \( r \to 0 \) and \( r \to \infty \). Let \( r_{\text{min}} \) and \( r_{\text{max}} \) be the roots of the equation \( V(r) = K \) (\( r_{\text{min}} \) and \( r_{\text{max}} \) are the nearest to \( r(0) \) roots such that \( r_{\text{min}} \leq r(0) \leq r_{\text{max}} \)). It follows from (4.8) that the function \( r(t) \) is periodic with period

\[
T = 8 \int_{r_{\text{min}}}^{r_{\text{max}}} \left( K - V(r) \right)^{-1/2} dr
\]

and \( r_{\text{min}} \leq r(t) \leq r_{\text{max}} \). One can imagine, for example, that on the period the function \( r(t) \) increases monotonically from \( r_{\text{min}} \) to \( r_{\text{max}} \) and then decreases from \( r_{\text{max}} \) to \( r_{\text{min}} \). Thus, we have integrated the system (4.1) and (4.3), (4.4).

**Theorem 4.1.** In the variable \( r \) a classical particle moves periodically according to equation (4.8) with period (4.9). The angular variable is determined by equation (4.5) so that \( \varphi(t) \) is a monotone function of \( t \) and \( \varphi(t) = \varphi_0 t + O(1) \), where \( \varphi_0 = 2MT^{-1} \int_0^T r(s)^{-2} ds \), as \( |t| \to \infty \). The variable \( z(t) \) is determined by equation (4.6).

According to equation (4.4) a particle can move in the direction of the current as well as in the opposite direction. Nevertheless one can give simple sufficient conditions for the inequality

\[
\pm (z(t + T) - z(t)) > 0
\]
(for all $t$). Indeed, it follows from the Newton equation $r''(t) = -2V'(r(t))$ (which is a consequence of (4.1)) and expression (4.2) that

$$r''(t) = 4M^2 r^{-3}(t) + 4e\mathcal{A}'(r(t))(P - e\mathcal{A}(r(t))).$$

Using equation (4.4), we see that

$$2e\varepsilon(t) = (r''(t) - 4M^2 r^{-3}(t))\mathcal{A}'(r(t))^{-1}. \quad (4.11)$$

Integrating this equation and taking into account periodicity of the function $r(t)$, we see that, for all $t$,

$$2e(z(t + T) - z(t)) = \int_0^T r''(s)\mathcal{A}'(r(s))^{-1}ds - 4M^2 \int_0^T r(s)^{-3}\mathcal{A}'(r(s))^{-1}ds$$

$$= \int_0^T r'(t)^2\mathcal{A}'(r(t))^{-2}\mathcal{A}''(r(t))dt - 4M^2 \int_0^T r(s)^{-3}\mathcal{A}'(r(s))^{-1}ds. \quad (4.11)$$

Let us formulate the results obtained.

**Theorem 4.2.** The increment of the variable $z$ on every period is determined by equation (4.11). In particular, if $\pm e\mathcal{A}'(r) < 0$ and $\pm e\mathcal{A}''(r) > 0$ for all $r$, then inequality (4.10) holds. In this case $z(t) = z_0 + O(1)$ with $z_0 = T^{-1}(z(T) - z(0))$, $\pm z_0 > 0$, as $|t| \to \infty$.

In particular, for potentials $\mathcal{A}(r) = -\alpha \ln r$ and $\mathcal{A}(r) = -\alpha r^a$ where $a \in (0, 1)$, inequality (4.10) holds if $\pm e\alpha > 0$. Note that in these cases the fields $B(x, y, z) = \mathcal{A}'(r)r^{-1}(y, -x, 0)$ tend to 0 as $r \to \infty$.

It follows from equation (4.4) that if, say, $e\mathcal{A}'(r) < 0$ and the point $r_{cr}$ is determined by the equation $p = e\mathcal{A}(r_{cr})$, then $z(t)$ increases for $r(t) \in (r_{min}, r_{cr})$ and decreases for $r(t) \in (r_{cr}, r_{max})$. Of course, it is possible that $r_{cr} < r_{min}$. In this case, $z(t)$ always increases. Let us discuss this phenomena in more details on our leading example $\mathcal{A}(r) = -\alpha \ln r$. Then $r_{cr} = e^{-P/\gamma}$ where $\gamma = e\alpha$. The points $r_{min}$ and $r_{max}$ are determined from the equation

$$V(r) = M^2 r^{-2} + \ln^2(e^P r^\gamma) = K.$$

The function $z(t)$ is increasing for all $t$ if $r_{cr} < r_{min}$ or, equivalently, $V(r_{cr}) \geq K$ and $r_{cr} \leq r(0)$. The first of these conditions is equivalent to $M^2 e^{2P/\gamma} \geq K$ or, since in view of (4.4) $e^{2P/\gamma} = r(0)^{-2} e^{z(0)/\gamma}$, to

$$M^2 r(0)^{-2} e^{z(0)/\gamma} \geq 4^{-1} r'(0)^2 + M^2 r(0)^{-2} + 4^{-1} z'(0)^2.$$

Thus, $z'(0)$ should be a sufficiently large positive number ($z'(0) \leq 0$ is definitely excluded). In this case the condition $r_{cr} \leq r(0)$ which is equivalent to $z'(0) \geq 0$ is automatically satisfied. Note finally that always $r_{cr} \leq r_{max}$, that is the function $z(t)$ cannot be everywhere decreasing (this is of course also a consequence of Theorem 4.2). Indeed, inequality $r_{cr} \geq r_{max}$ is equivalent to $V(r_{cr}) \geq K$ and $r_{cr} \geq r(0)$. The first of them require that $z'(0) > 0$ while the second require that $z'(0) \leq 0$.

Thus, positively (negatively) charged classical and quantum particles always move asymptotically in the direction of the current (in the opposite direction). In the plane orthogonal to the direction of the current classical and quantum particles are essentially localized.
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