THE FUNDAMENTAL SOLUTION FOR THE HEAT EQUATION ON THE HALF-LINE WITH DRIFT AND DIRICHLET BOUNDARY CONDITION

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ABSTRACT. By a probabilistic method we provide an explicit fundamental solution of the Cauchy problem associated to the heat equation on the half-line with constant drift and Dirichlet boundary condition at zero.

1. INTRODUCTION

Explicit fundamental solutions for partial differential equations are quite rare in the literature. The most famous one is undoubtedly the fundamental solution of the heat equation on \( \mathbb{R} \)

\[
\begin{align*}
&\left\{ \begin{array}{l}
\partial_t \phi(t,u) = \frac{1}{2} \partial_{uu}^2 \phi(t,u), \\
\phi(0,u) = g(u),
\end{array} \right. & t \geq 0, & u \in \mathbb{R}, \\
& & u \in \mathbb{R},
\end{align*}
\]

(1.1)

which is given by

\[
p_t(u,v) = \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(v-u)^2}{2t} \right\}.
\]

(1.2)

By a fundamental solution \( p_t(u,v) \) of a partial differential equation, also called Green function or Green kernel, we mean that the solution of the corresponding Cauchy problem (also called initial value problem) is given by

\[
\phi(t,u) = \int_0^\infty g(v)p_t(v,u)vdv.
\]

The fundamental solution (1.2) of the heat equation (1.1) can be obtained by some ansatz on the structure of fundamental solutions (where such a structure is deduced from scale invariance, see e.g. [2, page 45]) or via Fourier transforms (see e.g. [3, Chapter 7]). However, aside the Fourier transform method in \( \mathbb{R}^n \), no general method to obtain fundamental solutions is known.

The kernel (1.2) can be used to deduce fundamental solutions of some related Cauchy problems, as for the heat equation in \( \mathbb{R}^d \) and the heat equation with drift:

\[
\begin{align*}
&\left\{ \begin{array}{l}
\partial_t \phi(t,u) = \frac{1}{2} \partial_{uu}^2 \phi(t,u) + \alpha \partial_u \phi(t,u), \\
\phi(0,u) = g(u),
\end{array} \right. & t \geq 0, & u \in \mathbb{R}, \\
& & u \in \mathbb{R}.
\end{align*}
\]

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A quite famous technique to deduce fundamental solutions from (1.2) to related Cauchy problems with boundary conditions is the image method (see [4, Chapter 2] for instance) largely used in Physics literature. This method consists in obtaining the fundamental solution for the partial differential equation with boundary conditions in some region \( S \subset \mathbb{R} \) (or \( \mathbb{R}^n \)) by extending the initial condition to \( \mathbb{R} \) in a suitable way where symmetries guarantee that the boundary conditions are satisfied.

The most standard applications of the image method are the fundamental solution for the heat equation on the half-line with Dirichlet boundary conditions

\[
\begin{align*}
\frac{\partial \phi(t,u)}{\partial t} &= \frac{1}{2} \frac{\partial^2 \phi(t,u)}{\partial u^2}, \quad t \geq 0, \quad u \in [0, \infty), \\
\phi(t,0) &= 0, \quad t > 0, \\
\phi(0,u) &= g(u), \quad u \in [0, \infty),
\end{align*}
\]

and the heat equation on the half-line with Neumann boundary conditions

\[
\begin{align*}
\frac{\partial \phi(t,u)}{\partial t} &= \frac{1}{2} \frac{\partial^2 \phi(t,u)}{\partial u^2}, \quad t \geq 0, \quad u \in \mathbb{R}^d, \\
\frac{\partial \phi(t,0)}{\partial u} &= 0, \quad t > 0, \\
\phi(0,u) &= g(u), \quad u \in \mathbb{R},
\end{align*}
\]

whose fundamental solutions are given, respectively, by

\[
p_t(u,v) = \frac{1}{\sqrt{2\pi t}} \left[ \exp \left\{ -\frac{(v-u)^2}{2t} \right\} - \exp \left\{ -\frac{(v+u)^2}{2t} \right\} \right].
\]

and

\[
p_t(u,v) = \frac{1}{\sqrt{2\pi t}} \left[ \exp \left\{ -\frac{(v-u)^2}{2t} \right\} + \exp \left\{ -\frac{(v+u)^2}{2t} \right\} \right].
\]

However, even apparently simple variations of the classical heat equation may not offer simple ways to deduce a fundamental solution. We consider in this paper the heat equation in the half-line with drift and Dirichlet boundary condition, see (1.5) below. On the one hand the domain being a subset of \( \mathbb{R} \) prevents to apply Fourier transform techniques, and on the other hand the lack of symmetry coming from the drift forbids an application of the image method. Although this Cauchy problem looks like a very classical problem, surprisingly no fundamental solution was known for it, to the best of our acquaintance.

We thus provide in this paper a curious probabilistic way to obtain the following:

**Theorem 1.1.** Assume that \( g : \mathbb{R}_+ \to \mathbb{R} \) is a bounded continuous function and \( \alpha \in \mathbb{R} \). Then, a solution of the Cauchy problem

\[
\begin{align*}
\frac{\partial \phi(t,u)}{\partial t} &= \frac{1}{2} \frac{\partial^2 \phi(t,u)}{\partial u^2} + \alpha \frac{\partial \phi(t,u)}{\partial u}, \quad t \geq 0, \quad u > 0, \\
\phi(t,0) &= 0, \quad t \geq 0, \\
\phi(0,u) &= g(u), \quad u \geq 0,
\end{align*}
\]
is given by
\[ \phi(t, u) = \int_0^\infty g(v) p_t(u, v) dv, \] (1.6)
where
\[ p_t(u, v) = \exp \left\{ \frac{-u^2 t + \alpha(u - v)}{\sqrt{2\pi t}} \right\} \left[ \exp \left\{ -\frac{(v-u)^2}{2t} \right\} - \exp \left\{ -\frac{(v+u)^2}{2t} \right\} \right]. \] (1.7)

Observe that, for \( \alpha = 0 \), the function \( p_t \) in (1.7) coincides with the fundamental solution of the heat equation in the half line with Dirichlet boundary condition at zero, as expected. Of course, once one has the formula (1.7) as a candidate, verifying that it is indeed a fundamental solution for the (1.5) is an elementary task. Aside of the formula itself, our contribution here consists on providing a probabilistic way to deduce (1.7).

2. Probabilistic Deduction

At a first moment, we will additionally assume that \( g(0) = 0 \), which is known in the PDE literature as a compatibility condition (between the initial condition and the boundary condition). Such assumption is in general rather technical than physical. For the moment, admit that \( g \) is smooth and decays to zero as \( u \) goes to infinity.

Denote by \( B_t \) the standard Brownian motion starting at some point \( u > 0 \), and let \( \mathbb{P}_u \) and \( \mathbb{E}_u \) be the corresponding probability and expectation, respectively. For \( \alpha > 0 \), denote \( M_t = B_t + \alpha t \), which is the Brownian motion with drift to the right. Let \( \tau = \inf\{t \geq 0 : M_t = 0\} \) the hitting time of zero and let \( X_t = M_t \wedge \tau \) be the Brownian motion with drift to the right and absorbed at the origin. Denote by \( C_0[0, +\infty) \) the set of continuous functions on \([0, +\infty)\) which vanish at infinity.

The infinitesimal generator of the Markov process \( X_t \) is defined as the operator \( \mathcal{L} : \mathcal{D}(\mathcal{L}) \to C_0[0, +\infty) \) given by
\[ \mathcal{L}f(u) = \lim_{t \searrow 0} \frac{\mathbb{E}_u[f(X_t)] - f(u)}{t}, \quad \text{for all } u \geq 0, \]
being the domain \( \mathcal{D}(\mathcal{L}) \) defined as the subset of functions in \( C_0[0, +\infty) \) for which the limit above exists and it is a function in \( C_0[0, +\infty) \), see [6, Chapter VII].

It can be checked directly that \( \mathcal{D}(\mathcal{L}) \) contains the space
\[ \widetilde{\mathcal{D}}(\mathcal{L}) := \left\{ f \in C_0[0, +\infty) : f(0) = 0, \frac{df}{du} \in C_0[0, +\infty) \text{ and } \frac{d^2f}{du^2} \in C_0[0, +\infty) \right\}, \]
and that for \( f \in \widetilde{\mathcal{D}}(\mathcal{L}) \) we have \( \mathcal{L}f = \frac{1}{2} \frac{d^2f}{du^2} + \alpha \frac{df}{du} \). Under the additional assumption that \( g \in \mathcal{D}(\mathcal{L}) \), we expect that
\[ \phi(t, u) := \mathbb{E}_u[g(X_t)] \] (2.1)
is a solution of (1.5). Indeed, by [6, page 282, Prop. 1.2], we have that
\[ \partial_t \phi(t, u) = L \left( \mathbb{E}_u \left[ g(X_t) \right] \right), \]
and if additionally \( \phi(t, \cdot) \in \widetilde{G}(L) \), then \( L \phi(t, u) = \frac{1}{2} \partial^2_{uu} g(u) + \alpha \partial_u g(u) \). Besides, the compatibility condition implies the Dirichlet boundary condition, since \( \phi(t, 0) = \mathbb{E}_0 \left[ g(X_t) \right] = g(0) = 0 \). And we have immediately that
\[ \phi(0, u) = \mathbb{E}_u \left[ g(X_0) \right] = \mathbb{E}_u \left[ g(u) \right] = g(u). \]

We proceed now to evaluate the expectation (2.1). Note that
\[ \mathbb{E}_u \left[ g(X_t) \right] = \mathbb{E}_u \left[ g(X_t) \mathbb{1}_{\{ \inf_{s \leq t} X_s > 0 \}} \right] + \mathbb{E}_u \left[ g(X_t) \mathbb{1}_{\{ \inf_{s \leq t} X_s = 0 \}} \right]. \tag{2.2} \]

Since \( X_t \) is absorbed at zero and we are assuming that \( g(0) = 0 \), the second expectation on the right hand side of above is null. The fact that the process \( X_t \) has a drift forbids us to apply directly the Reflection Principle on the first expectation on the right hand side of (2.2). For that reason, the next two propositions will be necessary. The first one is a consequence of Girsanov’s Theorem:

**Proposition 2.1.** If \( \{ B_s : s \leq t \} \) is a standard Brownian motion (starting from zero) defined in some probability space \( (\Omega, \mathcal{F}, \mathbb{P}_0) \), then, for any fixed \( \alpha \in \mathbb{R} \), \( \{ B_s - \alpha s : s \leq t \} \) is a Brownian motion with respect to the measure \( \mathbb{Q}_t \), defined by
\[ d\mathbb{Q}_t := \exp \left\{ \alpha B_t - \frac{\alpha^2 t}{2} \right\} d\mathbb{P}_0. \tag{2.3} \]

Note that if \( \alpha \neq 0 \) the result above holds only for a finite time horizon \( t > 0 \) because the law of \( \{ B_s - \alpha s : s \geq 0 \} \) is singular with respect to that of \( \{ B_s : s \geq 0 \} \). Nevertheless, this will be enough for our purposes. For Girsanov’s Theorem and how to arrive at the proposition above, see [5, page 196, Section C] for example. Moreover, we will need also the following consequence of the Reflection Principle:

**Proposition 2.2.** If \( \{ B_s : s \leq t \} \) is a standard Brownian motion (starting from zero) defined in some probability space \( (\Omega, \mathcal{F}, \mathbb{P}_0) \), then the joint distribution of \( (B_t, \sup_{s \leq t} B_s) \) has a density with respect to the Lebesgue measure given by
\[ \mathbb{P}_0 \left[ B_t \in du, \sup_{s \leq t} B_s \in dv \right] = \frac{2(2v-u)}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{(2v-u)^2}{2t} \right\} du dv, \]
on \{ v \geq 0, u \leq v \}, and zero otherwise.

For a proof of Proposition 2.2, see [6, page 110, Exercise (3.14)] for instance. Let us go back to (2.2). Performing a translation, we have that
\[ \mathbb{E}_u \left[ g(X_t) \mathbb{1}_{\{ \inf_{s \leq t} X_s > 0 \}} \right] = \mathbb{E}_u \left[ g(B_t + \alpha t) \mathbb{1}_{\{ \inf_{s \leq t} (B_s + \alpha s) > 0 \}} \right] \]
where

$$\mathbb{E}_0 \left[ g(B_t + at + u) \mathbb{1}_{\inf\{B_s + as \} > -u} \right].$$

Since $-B_t$ is also a Brownian motion, the last expectation above is equal to

$$\mathbb{E}_0 \left[ g(u - (B_t - at)) \mathbb{1}_{\sup\{B_s - as \} < u} \right],$$

which we can rewrite as

$$\mathbb{E}_{\mathbb{Q}_t} \left[ g(u - (B_t - at)) \mathbb{1}_{\sup\{B_s - as \} < u} \exp \left\{ -\alpha B_t + \frac{\alpha^2 t}{2} \right\} \right],$$

where $\mathbb{E}_{\mathbb{Q}_t}$ is the expectation with respect to the measure $\mathbb{Q}_t$ mentioned in (2.3). By Proposition 2.1, up to time $t$, the process $\tilde{B}_t := B_t - \alpha s$ is a Brownian motion with respect to $\mathbb{Q}_t$. We therefore rewrite the previous expectation as

$$\mathbb{E}_{\mathbb{Q}_t} \left[ g(u - \tilde{B}_t) \mathbb{1}_{\sup\tilde{B}_s < u} \exp \left\{ -\alpha \tilde{B}_t - \frac{\alpha^2 t}{2} \right\} \right]$$

$$= e^{-\alpha^2 t/2} \mathbb{E}_{\mathbb{Q}} \left[ g(u - \tilde{B}_t) \mathbb{1}_{\sup\tilde{B}_s < u} \exp \left\{ -\alpha \tilde{B}_t \right\} \right]$$

$$= e^{-\alpha^2 t/2} \int_0^u \int_{-\infty}^v g(u-w) \frac{2(2v-w)}{\sqrt{2\pi t^3}} \exp \left\{ -aw - \frac{(2v-w)^2}{2t} \right\} dw dv,$$

where the last equality follows from Proposition 2.2. Putting these facts together, we obtain

$$\mathbb{E}_0[g(X_t)] = e^{-\alpha^2 t/2} \int_0^u \int_{-\infty}^v g(u-w) \frac{2(2v-w)}{\sqrt{2\pi t^3}} \exp \left\{ -aw - \frac{(2v-w)^2}{2t} \right\} dw dv.$$

Performing the change of variables $u - w \mapsto w$, we get

$$\mathbb{E}_u[g(X_t)] =$$

$$e^{-\alpha^2 t/2} \int_0^u \int_{u-v}^\infty g(w) \frac{2(2v+w-u)}{\sqrt{2\pi t^3}} \exp \left\{ \alpha(w-u) - \frac{(2v+w-u)^2}{2t} \right\} dw dv. \quad (2.4)$$

Now, we will look for the fundamental solution of the PDE, that is, a function $p_t(u,v)$ such that the expression on the right hand side of (2.4) is equal to $\int_0^\infty p_t(u,v) g(v) dv$. To simplify notation, make the change of variables $u - v \mapsto z$ in (2.4), which leads to

$$e^{-\alpha^2 t/2} \int_0^u \int_z^\infty g(w) \frac{2(2u-z+w-u)}{\sqrt{2\pi t^3}} \exp \left\{ \alpha(w-u) - \frac{(2u-z+w-u)^2}{2t} \right\} dw dz$$

$$= \int_0^\infty g(w) e^{-\alpha^2 t/2} \int_0^{u+w-2z} \frac{2(u+w-2z)}{\sqrt{2\pi t^3}} \exp \left\{ \alpha(w-u) - \frac{(u+w-2z)^2}{2t} \right\} dz dw,$$
so that

\[ p_t(u, v) = e^{-\alpha^2 t/2} \int_0^{u \wedge v} \frac{2(u + v - 2z)}{\sqrt{2 \pi t}} \exp \left\{ \alpha(v - u) - \frac{(u + v - 2z)^2}{2t} \right\} dz, \]

if \( v, u \geq 0 \) and \( p_t(u, v) = 0 \) otherwise. We solve the integral above by performing the change of variables \( u + v - 2z \mapsto u' \), which after some calculations gives us

\[ p_t(u, v) = \exp \left\{ \frac{-\alpha^2 t + \alpha(v - u)}{2 \sqrt{2 \pi t}} \right\} \left[ \exp \left\{ -\frac{(v - u)^2}{2t} \right\} - \exp \left\{ -\frac{(v + u)^2}{2t} \right\} \right]. \]

Up to here, it has been assumed that \( g : \mathbb{R}_+ \to \mathbb{R} \) satisfies the conditions:

- \( g(0) = 0 \),
- \( g \) is twice continuously differentiable,
- \( g \) converges to zero as \( u \) goes to infinity.

Nevertheless, under the assumptions of the statement, the formula (1.6) which has been obtained still gives us the solution of the PDE, as one can verify by noticing that \( p_t(0, v) = 0 \) and verifying that \( \partial_t p_t = \frac{1}{2} \partial^2_{uu} p_t(u, v) + \alpha \partial_u p_t(u, v) \).

It remains to check that expression (1.6) satisfies the initial condition of the Cauchy problem (1.5). In other words, we need to prove the pointwise convergence \( \lim_{t \downarrow 0} T_t g(u) = g(u) \) for any \( u > 0 \). Recall that \( T_t g(u) = \mathbb{E}_u[ g(X_t) ] \), where \( X_t \) is the absorbed Brownian motion with drift. Under \( \mathbb{E}_u \), the probability distribution of \( X_t \) converges in the weak sense to \( \delta_u \) as \( t \downarrow 0 \) (see [1] on weak convergence of probability measures). Hence, since \( g \) is a bounded continuous function,

\[ T_t g(u) = \mathbb{E}_u[ g(X_t) ] \xrightarrow{t \downarrow 0} \mathbb{E}_u[ g(X_0) ] = g(u), \]

finishing the argument.

As a final remark, the assumption on boundedness and continuity of \( g \) can be replaced by much weaker hypotheses, whilst the convergence towards the initial condition can be much strengthened. In possession of the explicit fundamental solution this is a standard issue, and we do not enter into details.

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REFERENCES

[1] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[2] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[3] R. J. Iorio Jr. and V. de M. Iorio. *Fourier analysis and partial differential equations*, volume 70 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2001.

[4] J. D. Jackson. *Classical Electrodynamics*. John Wiley and Sons, 3rd edition, 1999.

[5] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[6] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

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