Exactly Solvable Model of Inergodic Spin Glass

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Abstract

A mean-field model of Ising spin glass with the Hamiltonian being a sum of the infinite-range ferromagnetic and random antiferromagnetic interactions has been studied. It is shown that this model has phase transition in external magnetic field into inergodic spin glass phase with a number of metastable states. The thermodynamic properties of metastable states are studied at $T = 0$ and near the transition. The relations between the characteristics of slow nonequilibrium processes in spin glass phase (such as hysteresis loop form, thermo-remanent and isothermal remanent magnetizations, field-cooled and zero-field-cooled thermodynamic quantities) and thermodynamic parameters of metastable states are established.

1 Introduction

The possibility of existence of spin glass phases as specific thermodynamic states in solid solutions of ferromagnets and antiferromagnets has been first suggested in pioneering work by Edwards and Anderson [1]. According to [1], these phases are characterized by the appearance of local spontaneous magnetic moments with chaotic orientations, which are determined by a random distribution ferromagnetic and antiferromagnetic interactions throughout the crystal. The first attempt to give the quantitative description of spin glass transition was made by Sherrington and Kirkpatrick (SK) [2], who considered a mean-field model with infinite range random interaction. But the solution they got appeared to be unstable in the glass phase for $T < T_{sg}$, $H < H_{AT} \sim (T_{sg} - T)^{3/2}$ [3]. The further attempts to describe the thermodynamics of spin glass phase below the Almeida-Thouless (AT) line, $H < H_{AT}$, resulted in the construction of ‘replica symmetry breaking’ scheme by Parisi [4] being the procedure of the analytical continuation in the replica method used in the studies of the SK model. Now it is common belief that Parisi solution is stable below AT-line and it is the basic result in the spin glass theory. With some reservations concerning the mathematical foundations of the replica method and the suggested procedure of the analytical continuation, one may consider the Parisi’s solution as the first exact description of the thermodynamics of inergodic spin glass phase.

Later, the other exactly solvable spin glass models have been studied with the use of the replica method (the vector models [5, 6], $p$-spin spherical models [7, 8]) and without it (ordinary spherical spin glass [9], Bethe lattice spin glass [10]). Let us also notice the study of the mean-field equations for the local magnetic moments of the SK model [11]. In these studies the equilibrium thermodynamic parameters averaged over random interactions were obtained in the framework of the standard statistical mechanics. Still these results appear to be insufficient for the description of properties of real spin glasses. The reason lies in the known inergodicity of the spin glass phases [12], i. e. the existence a large number of metastable states in these phases. So the experiments give the physical quantities proper to one of these states, in which crystal comes depending on a regime and a sequence of cooling and application of magnetic field [13, 14]. Meanwhile, the equilibrium quantities refer only to the state with the lowest thermodynamic potential and could be obtained after sufficiently large observation time such that crystal could come to the lowest equilibrium state when field and/or temperature are changed. As the barriers between metastable states are macroscopic (divergent as $N \to \infty$), the corresponding relaxation times are generally astronomically large. Thus crystal can stay in the initial state while his potential becomes larger then some other state.

The situation can be elucidated by considering the uniaxial (Ising) ferromagnet being the simplest inergodic system below $T_c$. In the fields, smaller then coercive one, it has two stable states: equilibrium state with magnetization $m$ parallel to $H$, and metastable one having $m$ antiparallel $H$ and greater
potential. The standard result of the equilibrium statistical mechanics for the dependency $m$ on $H$ in this case is a function with a jump at $H = 0$ corresponding to the equilibrium states, while in the real experiments the hysteresis loop is observed, in which metastable states are also present. Thus the description of real experiments needs also the description of the properties of metastable states. In this example this is trivial at least in the mean-field approximation, but in spin glasses the description of the properties of a large number of metastable states is rather difficult task. In particular, the description of magnetic properties of a spin glass phase must include a set of functions $m(H)$, corresponding to various metastable states, and their lines on the $m - H$ plane would fill some region around the origin. In a rather simplified form, the theoretical problem is to determine the boundaries of this inergodic region and the values of various thermodynamic quantities for all points inside it. Such theoretical results could describe a number of slow nonequilibrium processes in the inergodic spin glass phases.

Meanwhile, all data about metastable states obtained via Parisi’s ansatz come to the probability distribution of the ‘overlaps’ of magnetizations in various metastable states [13]. Now it is not clear how this distribution can describe the real experiments. Also there was not got any information on the properties of metastable states of the SK model from the TAP equations [11], it was only established that their number is exponentially large [14].

The studies of the other spin glass models have not also result in the elucidation of the physical characteristics of metastable states. The only exception is the Ising spin glass on the Bethe lattice for which the numerical study of the internal field distribution has explicitly shown the existence of a number of metastable states at $T = 0$ [10].

Generally speaking, the study of the properties of metastable states is not necessary for a description of experiments in inergodic systems as it could be obtained from the study of nonequilibrium dynamics which incorporates automatically the effects of their existence. Thus one can get the hysteresis loop as a reaction on the large slow varying field in the dynamic treatment of the uniaxial ferromagnet. But in doing this one must eliminate large unobservable times of the relaxation between metastable states. Such elimination procedure has been developed in the study of Langevin dynamics of SK model in zero field [13], which help to establish the difference between the unobservable equilibrium susceptibility and those in the local metastable states measured at real times smaller than that of relaxation over macroscopic barriers. Still now it is not clear how this procedure could be generalized to describe in the SK model the reaction on the alternating field of finite amplitude. Now the inergodic effects in finite fields were described only in the simplest spherical spin glass model having two stable states [10].

To resume we may say that the spin glass models considered until now are not sufficiently simple to obtain the thermodynamic properties of metastable states needed for description of inevitably nonequilibrium processes in spin glasses. Hence it seems worthwhile to find and to study more simple models allowing for more complete description of physical properties in the inergodic glass phases. Here we present one such model which allows the analytical description of thermodynamic characteristics of all metastable states at $T = 0$ and near the transition point.

## 2 Hamiltonian of the model and its properties at $T = 0$

The most simple treatment of metastable states is possible in the framework of mean-field models with infinite range interactions when it comes to the determination of local minima thermodynamic potential. Also the random spin glass interaction should be sufficiently simple in all its realizations to make possible an analytical treatment. In Ising spin glasses the glass transition results from the competition between ferromagnetic and antiferromagnetic interactions so the task is to imitate this competition using simple infinite range interactions.

Let us consider the system having $N$ Ising spins $S_{i\alpha} = \pm 1$, divided on $N_b$ blocks which consist of $N_s$ spins, $N = N_b N_s$. Here index $\alpha$ is the block number and index $i$ is the number of the spin inside the block. So the magnetization of the block is

$$m_\alpha = N_s^{-1} \sum_{i=1}^{N_s} S_{i\alpha}$$  \hspace{2em} (1)

and the total magnetization is

$$m_\alpha = N^{-1} \sum_{\alpha=1}^{N_b} \sum_{i=1}^{N_s} S_{i\alpha} = N_b^{-1} \sum_{\alpha=1}^{N_b} m_\alpha$$  \hspace{2em} (2)
Let us also introduce the 'antiferromagnetic' order parameter
\[ \mu_\alpha = m_\alpha - m \] (3)

The model Hamiltonian is a sum of ferromagnetic and 'antiferromagnetic' interactions and the external field term
\[ \mathcal{H} = -\frac{N}{2} J m^2 - \frac{N_s}{2} J_1 \sum_{\alpha=1}^{N_b} \mu_\alpha^2 - N H m \] (4)

Here \( H \) is the homogeneous external field, \( J > 0 \) - ferromagnetic exchange integral with a fixed value, \( J_1 > 0 \) - random 'antiferromagnetic' exchange integral.

The term proportional to \( J_1 \) imitates the antiferromagnetic bonds distributed throughout a crystal so we assume that \( N_b >> 1 \). Also we assume that \( N_s \) diverges in thermodynamic limit (\( N \to \infty \)) in order the mean-field approximation to be valid.

In general, for large \( N_b \) the 'antiferromagnetic' term in Eq.4 gives rise to a number of various types of ordering with \( m = 0 \), so it would be more correct to call it the 'glass' term. The more so that the block analog of the Edwards-Anderson order parameter
\[ q = N_b^{-1} \sum_{\alpha=1}^{N_b} \delta_{N_s,m_\alpha} \sum_{\alpha} s_{\alpha} \] (5)

can be represented as
\[ q = N_b^{-1} \sum_{\alpha=1}^{N_b} \mu_\alpha^2 \]

Thus we may say that for \( N_b >> 1 \) the model Hamiltonian, Eq.4, describes the competition of ferromagnetic and 'glass' ordering.

For \( N_s \to \infty \) it is easy to find the (nonequilibrium) thermodynamic potential depending on \( m_\alpha \):
\[ F(m) = H(m)/N - TS(m) \]
where \( S(m) \) is the entropy per spin:
\[ S(m) = N^{-1} \ln \left[ \text{Tr} \prod_{\alpha=1}^{N_b} \delta_{N_s,m_\alpha} \sum_i s_{i\alpha} \right] \approx N_b^{-1} \sum_{\alpha=1}^{N_b} \ln \left[ 2 - \frac{1 + m_\alpha}{2} \ln (1 + m_\alpha) - \frac{1 - m_\alpha}{2} \ln (1 - m_\alpha) \right] \] (6)

For \( N_s \to \infty \) the description of the equilibrium thermodynamics comes to the finding of the lowest minimum of \( F(m) \) while the less deep minimums correspond to metastable states. The equations defining the extrema of \( F(m) \) are:
\[ T \text{arctanh} m_\alpha + (J_1 - J) m - J_1 m_\alpha = H \] (7)

The minimums correspond to the solutions of Eqs.7 with the positively defined Hessian
\[ \frac{\partial^2 F(m)}{\partial m_\alpha \partial m_\beta} = \left[ T / (1 - m_\alpha^2) - J_1 \right] \delta_{\alpha\beta} + (J_1 - J) / N \] (8)

It is easy to show that this simple model is inergodic at \( T = 0 \). In this case Eqs.7 become
\[ m_\alpha = \text{sign} [H + (J - J_1) m + J_1 m_\alpha] \] (9)

When \(|H| > \max(J,2J_1 - J)\) Eqs.7 have unique solution \( m_\alpha = \text{sign} H \), while at \(|H| > \max(J,2J_1 - J)\) they also have a number of solutions with arbitrary signs of \( m_\alpha \) limited only by the condition
\[ |H + (J - J_1) m|^2 < J_1^2 \] (10)
Figure 1: Field dependencies of magnetization in metastable states (dashed lines) and stable states (solid lines) at $T=0$.

All these solutions are stable thus corresponding to the metastable states of the model. The total magnetization in these states acquires a set of discrete values

$$m = \frac{2n}{N_b} - 1$$

Here $n$ is the number of blocks with magnetization $m_\alpha = 1$, $n = 0, ..., N_b$. There are $\binom{N_b}{n}$ states with a given $m$ which differs by the permutations of $m_\alpha$. The total number of inhomogeneous metastable states could be up to $2^{N_b} - 2$ for a given $H$.

The energy per spin in these states is determined by their magnetization

$$E = (J - J_1) \frac{m^2}{2} - mH - J_1/2$$

and the entropy, Eq. 6 is zero.

The equilibrium magnetization corresponding to the states with minimal energy is $m_{eq}(H) = \text{sign}H$ for $J > J_1$ and

$$m_{eq}(H) = 2 \sum_{n=1}^{N_b-1} \frac{n}{N_b} \theta \left( N_b^{-2} - \varepsilon_n^2 \right) + \text{sign} \left( \varepsilon_{N_b-1} - N_b^{-1} \right)$$

when $J < J_1$. Here $\varepsilon_n \equiv \frac{H}{N_b} - \frac{2n}{N_b} + 1$, and $\theta$ is the Havside’s step function. The field dependencies of magnetization in the equilibrium and metastable states are shown in Fig.1. The steps of the function $m_{eq}(H)$ for $J < J_1$ demonstrate the existence of the first order phase transitions at field values

$$H_n = \left( \frac{2n + 1}{N_b} - 1 \right) (J_1 - J)$$

$n = 0, ..., N_b$, at which the upturns of the block magnetizations take place. Qualitatively, just this behavior of $m_{eq}(H)$ one may expect in the glass phase of the magnet with equal concentrations of ferromagnetic and antiferromagnetic bonds, while the case $J > J_1$ gives a picture proper for a magnet with domination of ferromagnetic bonds. Further we will consider the most interesting case when model Hamiltonian imitates a spin glass, so we assume that probability distribution for $J_1$, $P(J_1)$, is zero for $J > J_1$ and has the form

$$P(J_1) = \theta (J_1 - J) W (J_1 - J)$$

In general, one may notice that averaging over random interaction is superficial when all thermodynamic parameters can be obtained for every random realization as experimental data are not usually averaged over a number of samples. Still in more complex models it is often possible to find only average equilibrium quantities. So it is interesting to compare them with the corresponding quantities in the metastable states. Thus in present model the magnetization in all states does not depend on the field and magnetic susceptibility in them is zero at $T = 0$. In the same time, averaging of Eq. 11 over $J_1$ gives for $N_b >> 1$

$$\langle m_{eq}(H) \rangle = H \int_{|H|}^{\infty} \frac{dJ'}{|J'|} W (J') + \text{sign}H \int_0^{|H|} dJ' W (J')$$

(13)
The absence of self-averaging of magnetic susceptibilities seems to be the specific property of the present model. Indeed the differentiation of Eq. 12 gives

\[ \chi_{eq} = \frac{\partial m_{eq}(H)}{\partial H} = \frac{2}{N_b(J_1 - J)} \sum_{n=1}^{N_b-1} \delta(N_b^{-1} - \varepsilon_n) \]  

When \( N_b \to \infty \) Eq. 13 becomes in the sense of distributions

\[ \chi_{eq} = \frac{1}{(J_1 - J)} \theta[(J_1 - J)^2 - H^2] \]  

The average value of this expression coincides with Eq. 14. Thus \( \langle \chi_{eq} \rangle \) is generally the unobservable quantity as it describes the changes of \( m \) in the series of transitions which take place only astronomically large time scale of overbarrier relaxation. What is more, there are no traces of the transitions in the observable \( \chi_{eq} \) in the thermodynamic limit. The indication of their presence via delta functions in \( \chi_{eq} \) exists only when this number stays finite in the limit \( N \to \infty \).

Meanwhile in the framework of the present model \( \chi_{eq} \) from Eq. 14 contains some information on the reaction of inergodic system on the slow varying external field. Thus the application of slow AC field with amplitude greater than 2\( J_1 - J \) would give a hysteresis loop and \( \chi_{eq} \) defines its slope. In the same time ordinary measurement of susceptibility in small fields would give zero value. Possibly the qualitatively similar meaning \( \chi_{eq} \) has in the real systems. Yet we must note that \( \chi_{eq} \) in present model is a non-self-averaging quantity in the sense that being constant before averaging it becomes a function of \( H \) after it, see Eq. 14. Still more dramatic effect is caused by the averaging on the nonlinear susceptibilities. It follows from Eq. 13 \( (k > 1) \)

\[ \chi_{eq}^{(k)} \equiv \frac{\partial^k m_{eq}(H)}{\partial H^k} = \frac{1}{J_1 - J} \frac{\partial^{k-2}}{\partial H^{k-2}} \left[ \delta(J_1 - J + H) - \delta(J_1 - J - H) \right] \]

and the averaging of this equation gives

\[ \langle \chi_{eq}^{(k)} \rangle \equiv \left( \frac{\partial^k m_{eq}(H)}{\partial H^k} \right) = - \frac{\partial^{k-2}}{\partial H^{k-2}} \left[ W(H) \right] H \]  

The absence of self-averaging of magnetic susceptibilities seems to be the specific property of the present model in which small fluctuations of \( J_1 \) can cause large deviations in \( m_{eq} \) and could be absent in more realistic spin glass models.

Let us note that singularities of non-averaged susceptibilities at \( H = \pm(J_1 - J) \) correspond to the points of the transitions from the inhomogeneous phase into the homogeneous one, thus \( J_1 - J \) has the meaning of the (non-averaged) Almeida-Thouless field. The corresponding anomalies of the averaged quantities would exist at finite \( H = \pm H_{AT} \) if function \( W(J') \) in Eq. 12 has a bounded support, i.e. when \( W(J') = 0 \) for \( J' > J \) and \( W(J') > 0 \) otherwise. Then \( H_{AT} = J \) and anomalies of \( \langle \chi_{eq}^{(k)} \rangle \) for \( H \to \pm H_{AT} \) will be determined by the behavior of \( W(J') \) at \( J' \to J \). This behavior determines also how the block Edwards-Anderson order parameter, Eq. 12, goes to zero when \( H \to \pm H_{AT} \). For \( N_b \to \infty \) we get from Eq. 12

\[ \langle q_{eq} \rangle = 1 - \langle m_{eq}^2 \rangle = \theta(J - |H|) \int_{|H|}^{J} dJ' W(J') \left( 1 - \frac{H^2}{J'^2} \right) \]

In general case \( (W(J) < \infty) \) the transition into spin glass phase with \( \langle q_{eq} \rangle \neq 0 \) at \( |H| < H_{AT} \) is not accompanied with divergencies of \( \langle \chi_{eq}^{(k)} \rangle \) in contrast with SK model. This is because the upturn of the last block along the field is the first order transition. So the instabilities which could cause such divergencies does not occur in the present model.

Let us also note that there are two ferromagnetic phases: inergodic phase with a number of inhomogeneous metastable states and ergodic one having the unique ferromagnetic state. The average value of the field corresponding the transition point between these phases can be obtained by considering the
boundaries (upper and lower) of the region where metastable states exist which represent also the upper and lower branches of hysteresis loop. They are (see Fig.1(a)):

\[ m_\pm (H) = \frac{H \pm J_1}{J_1 - J} \theta \left[ (J_1 - J)^2 - (H \pm J_1)^2 \right] + \text{sign} \left( H \pm J_1 \right) \theta \left[ (H \pm J_1)^2 - (J_1 - J)^2 \right] \]

The averaging of this expression over \( J_1 \) gives

\[ \langle m_\pm (H) \rangle = \text{sign} \left( H \pm J \right) + \theta \left( \mp H - J \right) \int_{\pm (H - J)/2}^{\infty} dJ' W(J') \left( \frac{H \pm J}{J'} \pm 2 \right) \quad \text{(18)} \]

For \(|H| \) greater some \( H_e \) these branches coincide thus indicating the transition into ergodic phase. The condition defining \( H_e \) is vanishing of the integral in Eq.(18) so \( H_e \) will be finite when \( W(J') \) has a bounded support. In this case

\[ H_e = 2J + J = 2H_{AP} + J \]

There exists a functional relation between \( \langle m_\pm (H) \rangle \) and \( \langle m_{eq} (H) \rangle \), Eq.(13) of the following form:

\[ \langle m_\pm (H) \rangle = \pm \left[ 2\theta (\mp H - J) \left\langle m_{eq} \left( \frac{\mp H - J}{2} \right) \right\rangle - 1 \right] \quad \text{(19)} \]

From Eq.(19) follows also:

\[ \frac{\partial \langle m_\pm (H) \rangle}{\partial H} = \theta (\mp H - J) \left\langle \chi_{eq} \left( \frac{\mp H - J}{2} \right) \right\rangle \quad \text{(20)} \]

These relations are specific for the model under consideration but, probably, some relations between field dependency of the average equilibrium magnetization and hysteresis loop contour exist also in other spin glass models at \( T = 0 \).

For the simple ‘rectangular’ function \( W \)

\[ W(J') = \theta (J' - J) / J \]

we get

\[ \left\langle g_{eq} \right\rangle = \theta \left( J^2 - H^2 \right) \left( 1 - \frac{|H|}{J} \right)^2 \]

\[ \left\langle m_{eq} (H) \right\rangle = \theta \left( J^2 - H^2 \right) \frac{H}{J} \ln \left( \frac{eJ}{|H|} \right) + \theta \left( H^2 - J^2 \right) \]

\[ J \left\langle \chi_{eq} \right\rangle = \theta \left( J^2 - H^2 \right) \ln \left( \frac{J}{|H|} \right) \]

\[ \left\langle \chi_{eq}^{(k)} \right\rangle = (k - 1)! \left( -\frac{H}{J} \right)^{1-k} / J \]

The divergency of magnetic susceptibilities at \( H = 0 \) is a consequence of \( W(0) \neq 0 \). They would be finite at zero field if \( W \) goes to zero as some power of \( J' \) or faster when \( J' \to 0 \). For example, the averaging with the ‘triangle’ function

\[ W(J') = 2J' \theta (J' - J) / J^2 \quad \text{(21)} \]

gives the following results

\[ \left\langle g_{eq} \right\rangle = \theta \left( J^2 - H^2 \right) \left( 1 + \frac{H^2}{J} \ln \frac{H^2}{eJ^2} \right) \]

\[ \left\langle m_{eq} (H) \right\rangle = \theta \left( J^2 - H^2 \right) \frac{H}{J} \left( 2 - \frac{|H|}{J} \right) + \theta \left( H^2 - J^2 \right) \text{sign} H \]

\[ J \left\langle \chi_{eq} \right\rangle = \theta \left( J^2 - H^2 \right) 2 \left( 1 - \frac{|H|}{J} \right) \]

\[ \left\langle \chi_{eq}^{(k)} \right\rangle = 0, k > 1 \]
The averaging of the thermodynamic parameters of metastable states existing inside hysteresis loop is trivial their magnetizations do not depend on $H$ and susceptibilities are zero. Also $q = 1 - m^2$ and $S = 0$ in all states. But we must note that equilibrium entropy is not strictly zero as for a given $m$ there are \( \binom{N_b}{N_b/2} \) states with equal potentials so the configurational entropy term

\[
S_{\text{conf}}(m) = N^{-1} \ln \left( \frac{N_b}{N_b(1 - m)/2} \right)
\]

is added to the expression in Eq.8. But in the thermodynamic limit $S_{\text{conf}}(m)$ goes to zero as $N_s^{-1}$.

3 Thermodynamics near the transition.

The stable inhomogeneous solutions of equations of state, Eq.7, appear at $T < J_1$. So in the case considered here ($J_1 > J$) there is a second order phase transition from homogeneous paramagnetic phase into inhomogeneous ergodic spin glass one at $T = J_1$, $H = 0$. Let us consider the thermodynamics of the model in the vicinity of this transitions assuming

\[
m_{\alpha} < < 1
\]

(22)

In this case Eq.7 acquire the form:

\[
\tau_1 m_{\alpha} + (\tau - \tau_1) m + m_{\alpha}^3/3 = h
\]

(23)

Here $h = H/T, \tau_1 = 1 - J_1/T, \tau = 1 - J/T, \tau_1 > \tau$. Hessian, Eq.8, becomes in this region

\[
T^{-1} \frac{\partial^2 F(m)}{\partial m_\alpha \partial m_\beta} = (\tau_1 + m_\alpha^2) \delta_{\alpha\beta} + (\tau - \tau_1) / N
\]

(24)

It follows from Eq.22 and Eq.23 that

\[
h < < 1, \tau_1 < < 1, \tau < < 1
\]

For these conditions to be fulfilled for every random $J_1$ we must assume that $W(J')$ in Eq.12 has sufficiently narrow bounded support, that is the possible values of $J_1 - J$ must be less than some $J > 0$ obeying the condition

\[
J << J
\]

When $\tau_1 > 0$ Eqs.23 have unique homogeneous solution. Let us denote it as $m_0$. It does not depend on $\tau_1$ and obeys the equation

\[
\tau m_0 + m_0^3/3 = h
\]

(25)

When $\tau_1 < 0$ Eqs.23 have $3^{N_b} - 3$ inhomogeneous solutions beside $m_0$. As all $m_\alpha$ obey the same cubic equation they can acquire only three different values which we denote as $\tilde{m}_s, s = 1, 2, 3$. Then all inhomogeneous solutions can be characterized by the three numbers $n_s \neq N_b$,

\[
\sum s n_s = N_b
\]

which show how many $m_\alpha$ have the value $\tilde{m}_s$. There are $\frac{N_b}{n_1!n_2!n_3!}$ solutions which differ by the permutations of $m_\alpha$ and the total number of solutions is

\[
\sum_{n_s \neq N_b} \frac{N_b}{n_1!n_2!n_3!} = 3^{N_b} - 3
\]

But only $2^{N_b} - 2$ of them could be stable. Indeed, Hessian, Eq.24 has three eigenvalues equal to $\tau_1 + \tilde{m}_s^2$ with degeneracy $n_s - 1$, which correspond to the eigenvectors having a zero sum of components. There are also three non-degenerate eigenvalues which are the solutions of the equation

\[
1 + \frac{\tau - \tau_1}{N_b} \sum_s \frac{n_s}{\tau_1 + \tilde{m}_s^2 - \lambda} = 0
\]

(26)
Using the Viet’s theorem for Eq.23 according to which
\[
\sum_s \tilde{m}_s = 0 \quad (27)
\]
we can get the relation
\[
\sum_{s<s'} \tilde{m}_s \tilde{m}_{s'} = 3 \tau_1 \quad (28)
\]
It shows that all three eigenvalues \( \tau_1 + \tilde{m}_s^2 \) could not be positive simultaneously so the stable solutions must have at least one of the numbers \( n_s \) equal to 0 or 1. But if all \( n_s > 0 \) then one of the solutions of Eq.26 becomes negative for large \( N_b \gg 1 \). Thus the stable solutions must have one of \( n_s \) equal to zero. Further we will consider just these solutions putting \( n_3 = 0 \). The stability condition for them reduces to one inequality
\[
\tau_1 + \tilde{m}_3^2 < 0 \quad (29)
\]
It follows from Eq.27 and Eq.28 that \( \tilde{m}_s \) can be represented in the following form
\[
\begin{align*}
\tilde{m}_1 &= 2 \left(-\tau_1\right)^{1/2} \cos \left(\varphi - \frac{\pi}{6}\right) \\
\tilde{m}_2 &= -2 \left(-\tau_1\right)^{1/2} \cos \left(\varphi + \frac{\pi}{6}\right) \\
\tilde{m}_3 &= -2 \left(-\tau_1\right)^{1/2} \sin \varphi
\end{align*}
\quad (30)
\]
so the stability condition, Eq.29, is equivalent to the inequality
\[
|\varphi| < \pi/6 \quad (31)
\]
It follows from the definition of \( m \):
\[
m = \nu_1 \tilde{m}_1 + \nu_2 \tilde{m}_2
\]
Here
\[
\nu_s = \frac{n_s}{N_b} \quad (32)
\]
\[
\nu_1 + \nu_2 = 1
\]
so
\[
m = (-\tau_1)^{1/2} \left(\sqrt{3} \Delta \cos \varphi + \sin \varphi\right) \quad (33)
\]
\[
\Delta = \nu_1 - \nu_2
\]
Inserting Eq.30 into Eq.28 we get
\[
3 (\tau - \tau_1) m = 3h - 2 \left(-\tau_1\right)^{3/2} \sin 3\varphi
\quad (34)
\]
Excluding \( m \) from Eq.33 and Eq.34 we obtain the equation for \( \varphi \):
\[
2 \left(-\tau_1\right)^{3/2} \sin 3\varphi + 3 (\tau - \tau_1) \left(-\tau_1\right)^{1/2} \left(\sqrt{3} \Delta \cos \varphi + \sin \varphi\right) = 3h
\quad (35)
\]
At all \( \tau > \tau_1 \) the left side of Eq.35 is a monotonously growing function of \( \varphi \) for \( |\varphi| < \pi/6 \). Hence, there is only one stable solution for \( \tilde{m}_s \) at a given \( \Delta \). There are \( \binom{N_b}{n_1} \) metastable states corresponding to this solution which differ by \( m_\alpha \) permutations.
The explicit solution of Eq.35 can be found for \( \Delta = 0 \) when it becomes cubic. In the limiting cases \( \Delta = \pm 1 \) Eq.35 reduces also to a cubic one for \( \tilde{m}_1 \) or \( \tilde{m}_2 \) which coincides with Eq.27.
In general case Eq.33 and Eq.35 (or Eq.34) give a parametric representation of a dependency of the homogeneous magnetization in the metastable states with a given $\Delta$ on $\tau$, $\tau_1$ and $h$. The parameter $\varphi$ can be excluded from these equations with the result

$$
\left[ (3\Delta^2 + 1)^2 \tau - 3 (1 - 3\Delta^2) (1 - \Delta^2) \tau_1 \right] m + \frac{8 (9\Delta^2 - 1)}{3(3\Delta^2 + 1)} m^3 +
$$

\[
+ 2\sqrt{3} \frac{\Delta (1 - \Delta^2)}{3\Delta^2 + 1} \left[ (3\Delta^2 + 1) \tau_1 + 4m^2 \right] \left[ - (3\Delta^2 + 1) \tau_1 - m^2 \right]^{1/2} = (3\Delta^2 + 1)^2 h \tag{36}
\]

From the stability condition, Eq.31, and Eq.34 it follows that solutions of Eq.36 is stable in the region

$$
9 [(\tau - \tau_1) m - h]^2 < -4\tau_1^3 \tag{37}
$$

which is the band on the $m - h$ plane. The magnetization is a monotonously growing function of $h$ and $\Delta$ inside this band so the field dependencies of magnetization can be represented as a set of uncrossing lines bounded from above and below by the $m_0(h)$ line as shown in Fig.2. The other thermodynamic parameters of metastable states can be obtained by differentiation of thermodynamic potential which near transition has the form

$$
12F/T = 6 (\tau - \tau_1) m^2 + 6\tau_1 \sum_s \nu_s \tilde{m}_s^2 + \sum_s \nu_s \tilde{m}_s^4 - 12hm - 12 \ln 2 \tag{38}
$$

Expressed via $\varphi$ these parameters are

$$
q = 3\tau_1 (\Delta^2 - 1) \cos \varphi \\
T^{-1} \chi^{-1} = \tau - \tau_1 \left[ 1 + 2 \cos 3\varphi / (\cos \varphi - \sqrt{3}\Delta \sin \varphi) \right] \\
S = \ln 2 + \tau_1 \left( 2 + \cos 2\varphi + \sqrt{3}\Delta \sin 2\varphi \right) / 2
$$

For the heat capacity we get rather more cumbersome expression

$$
C = \frac{2\tau_1 (\sqrt{3}\Delta \sin \varphi - \cos \varphi - 2 \cos 3\varphi) + 3 (\tau - \tau_1) (1 - \Delta^2) \cos \varphi}{2 (\cos \varphi - \sqrt{3}\Delta \sin \varphi) - 4\tau_1 \cos \varphi}
$$
In spite of the absence of explicit expression for \( \varphi \) as a function of \( h, \tau \) and \( \tau_1 \) the above formulae allow to get some notion about the field and temperature dependencies of these quantities. Thus at the boundaries of stability region, \( |\varphi| = \pm \pi/6 \) or at

\[
h = (\tau - \tau_1)^{1/2} (\tau - \tau_1) (3 \Delta \pm 1) / 2 \pm 2 (\tau_1)^{3/2} / 3
\]

(39)

\( q \) and \( \chi^{-1} \) has the lowest values

\[
\chi^{-1} = J_1 - J
\]

\[
q = 9 \tau_1 (\Delta^2 - 1) / 4
\]

(40)

while the entropy and heat capacity are

\[
S = \ln 2 + 3 \tau_1 (1 \pm \Delta) / 2
\]

\[
C = 3 (1 \pm \Delta) / 2 - \tau_1 / (\tau - \tau_1)
\]

(41)

It follows from Eq. 39 that metastable states exist when

\[
|h| < \tilde{h}_e \equiv 2 (\tau - \tau_1)^{1/2} \left( \tau - \frac{4}{3} \tau_1 \right)
\]

(42)

When \( h \) goes to \( \pm h_e \) the more homogeneous states with \( \Delta \to \pm 1 \) stay stable and their magnetization tends to \( m_0 \left( \pm h_e \right) = \pm 2 (\tau_1)^{1/2} \). However the limiting values of magnetic susceptibility, entropy and heat capacity differ from those in homogeneous state: \( \chi_0^{-1} = J_1 (\tau + m_0^2) \), \( S_0 = \ln 2 - m_0^2 / 2 \), \( C = m_0^2 / (\tau + m_0^2) \).

In the middle of the stability band (at \( \varphi = 0 \) or \( h = \Delta (-3 \tau_1)^{1/2} (\tau - \tau_1) \)) we get: \( q = \tau_1 (\Delta^2 - 1) \), \( \chi^{-1} = J_1 (1 - 2 \tau_1) - J \), \( S = \ln 2 + 3 \tau_1 / 2 \), \( C = \frac{3}{2} \left( 1 - \Delta^2 \frac{\tau}{\tau - \tau_1} \right) \). In this case the diminishing of inhomogeneity when \( h \to \pm \tilde{h}_AT \),

\[
\tilde{h}_AT = (\tau - \tau_1)^{1/2} (\tau - \tau_1)
\]

(43)

\( m, \chi, S \) and \( C \) tend to the corresponding values of the homogeneous phase.

The Almeida-Thouless field \( \tilde{h}_AT \), Eq. 38 determines (to the order \( N_b^{-1} \)) the point of the transition into the homogeneous phase. To show this let us find the values \( \Delta_{eq} \) corresponding to the states with the lowest potential. Differentiating \( F \), Eq. 38 over \( \Delta \) and using Eqs. 23, 27 and Eq. 28 we get

\[
\frac{\partial F}{\partial \Delta} = T \tilde{m}_3 (\tilde{m}_1 - \tilde{m}_2)^3 / 24
\]

\[
\frac{\partial^2 F}{\partial \Delta^2} = T (\tau - \tau_1) (\tilde{m}_1 - \tilde{m}_2)^2 / 8 \left[ 1 + \sum_s \frac{\tilde{m}_s}{\tau + \tilde{m}_s} \right]
\]

Thus the states with \( \tilde{m}_3 = 0 \) or, equivalently, \( \varphi = 0 \) (cf. Eq. 31) have the lowest potential. One can see that Eq. 35 has solution \( \varphi = 0 \) when \( \Delta = \tilde{h} / \tilde{h}_AT \) which is possible at \( h^2 < \tilde{h}_AT^2 \). When \( h^2 > \tilde{h}_AT^2 \) \( F (\Delta) \) has no minima inside the region \( \Delta^2 < 1 \) in which it is defined and the minimal values occur at its boundaries for \( \Delta_{eq} = signH \). So the transition into homogeneous state takes place at \( h = \pm \tilde{h}_AT \).

As \( \Delta \) is a rational number of the form \( 2n/N_b - 1 \) (cf. Eq. 34) it can not be exactly equal to \( h/\tilde{h}_AT \) at all \( h^2 < \tilde{h}_AT^2 \). Hence \( \Delta_{eq} \) is defined so that \( \left| \Delta - h/\tilde{h}_AT \right| \) is minimal and can be represented as

\[
\Delta_{eq} = \sum_{n=1}^{N_b-1} \left( \frac{2n}{N_b - 1} \right) \theta \left( N_b^{-2} - \tilde{\varepsilon}_n^2 \right) + signH \theta \left( \frac{h^2 - \left( \frac{N_b - 1}{N_b} \right) \tilde{h}_AT^2}{N_b} \right)
\]

\[
\tilde{\varepsilon}_n \equiv \frac{h}{\tilde{h}_AT} - \frac{2n}{N_b} + 1
\]

Inserting this \( \Delta_{eq} \) into Eq. 35 we get the corresponding values of \( \varphi_{eq} \) at \( h^2 < \tilde{h}_AT^2 \):

\[
\varphi_{eq} = \sqrt{3} \left( \frac{\tau - \tau_1}{\tau - 3 \tau_1} \right) \sum_{n=1}^{N_b-1} \tilde{\varepsilon}_n \theta \left( N_b^{-2} - \tilde{\varepsilon}_n^2 \right)
\]
Inserting $\Delta_{eq}$ and $\varphi_{eq}$ into the parametric representations of $q$ and $m$ we obtain the equilibrium values of these quantities

$$q_{eq} = -3\tau_1 \left( 1 - \frac{h^2}{\bar{h}_{eq}^2} \right)$$

$$m_{eq} = \frac{h}{\tau - \tau_1} \theta \left[ \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} - h^2 \right] - \frac{2\sqrt{3} (\tau_1 - \tau)}{\tau - 3\tau_1} \sum_{n=1}^{N_b-1} \bar{e}_n \theta \left( N_b^{-2} - \bar{e}_n^2 \right) +$$

$$+ m_0 \theta \left[ h^2 - \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} \right]$$

Differentiating $m_{eq}$ over $h$ we get the equilibrium susceptibility

$$\chi_{eq} = \frac{h}{\tau - 3\tau_1} \theta \left[ \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} - h^2 \right] - \frac{4\tau_1}{N_b (\tau - \tau_1) (\tau - 3\tau_1)} \sum_{n=1}^{N_b-1} \delta \left( N_b^{-1} - \bar{e}_n \right) +$$

$$+ \frac{1}{\tau + m_0^2} \theta \left[ h^2 - \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} \right]$$

The equilibrium entropy can be obtained by the differentiation of the equilibrium potential which to the $\bar{e}_n^2$ order is

$$F_{eq} = F \left( \Delta = h/\bar{h}_{AT} \right) - TS_{conf}$$

where configurational entropy $S_{conf}$ is determined by the logarithm of the number of states with the same potential $F$,

$$S_{conf} = N^{-1} \ln \left( \frac{N_b}{N_b \left( 1 - \Delta_{eq} \right) / 2} \right)$$

As at $T = 0$, $S_{conf}$ is of the order $N_s^{-1}$ and can be neglected so

$$S_{eq} = \ln 2 + \frac{3}{2\tau_1} \theta \left[ \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} - h^2 \right] - \frac{2\sqrt{3} (\tau_1 - \tau)}{\tau - 3\tau_1} \sum_{n=1}^{N_b-1} \bar{e}_n \theta \left( N_b^{-2} - \bar{e}_n^2 \right) -$$

$$- \frac{m_0^2}{2} \theta \left[ h^2 - \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} \right]$$

For the equilibrium heat capacity we get

$$C_{eq} = \left( \frac{3}{2} + \frac{h^2}{\tau_1 (\tau - \tau_1) (\tau - 3\tau_1)} \right) \theta \left[ \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} - h^2 \right] -$$

$$- \frac{2h^2}{N_b \tau_1 (\tau - \tau_1) (\tau - 3\tau_1)} \sum_{n=1}^{N_b-1} \delta \left( N_b^{-1} - \bar{e}_n \right) + \frac{m_0^2}{\tau + m_0^2} \theta \left[ h^2 - \left( \frac{N_b - 1}{N_b} \right) \bar{h}^2_{AT} \right]$$

The averaging of these expressions over $J_1$ gives at large $N_b$

$$\langle q_{eq} \rangle = \int_{H/m_0}^T \frac{dJ'}{J'} W(J') \left( \frac{3J'}{J} - 3\tau - \frac{H^2}{Jf^2} \right)$$

$$\langle m_{eq} \rangle = \int_{H/m_0}^T \frac{dJ'}{J} W(J') + m_0 \int_{0}^H \frac{dJ'}{Jf} W(J')$$

$$\langle \chi_{eq} \rangle = \int_{H/m_0}^T \frac{dJ'}{Jf} W(J') + \frac{1}{J + m_0^2} \int_{0}^H \frac{dJ'}{Jf} W(J')$$

$$\langle S_{eq} \rangle = \ln 2 + \frac{3}{2} \int_{H/m_0}^T \frac{dJ'}{J} W(J') \left( \tau + J' - \frac{H/m_0}{2} \right)$$
$$\langle C_{eq} \rangle = \frac{3}{2} \int_{H/m_0}^{J} dJ W(J') + \frac{m_0^2}{\tau + m_0^2} \int_0^{H/m_0} dJ W(J')$$

In derivation of these expression we have used the equivalence of the condition $h^2 < \tilde{h}_{AT}^2$ and the inequality $m_0^2(h) < m_0^2(h_{AT}) = -3\tau_1$ where $m_0(h)$ is a solution of Eq.25 such as $m_0(h) > 0$. In its turn, from Eq.25 it follows that the last inequality is equivalent to $J_1 - J > H/m_0$.

Evidently, the average equilibrium parameters transfer continuously into corresponding values of homogeneous phase at $H > J m_0$. Thus the average Almeida-Thouless field $H_{AT}$ is defined as a solution of the equation $H_{AT} = J m_0 (H_{AT})$ or its equivalent

$$m_0^2 (H_{AT}) = \sigma$$

(44)

where $\sigma = J/T - \tau = (J + J_{AT})/T - 1$.

Let us remind that from Eq.22 it follows that $J >> J_{AT}$, and for $\tau << T$. The solution of Eq.44 exists when $\sigma > 0$ or $T > T_{sg} = J + J_{AT}$. For $T \rightarrow T_{sg}$, $\sigma << J/J$ we get

$$H_{AT} \approx J (3\sigma)^{1/2}$$

and for $\tau < 0$, $-\tau >> J/J$

$$H_{AT} \approx J (-3\sigma)^{1/2}$$

Let us further consider the average boundaries of inergodic region on $m - h$ plane, i.e. the branches of the average hysteresis loop

$$\langle m_\pm (H) \rangle = m_0 \theta (m_0^2 - 4\sigma) + m_0^2 \theta (-\tau) \theta (\pm h_\pm) \theta \left[ 4\sigma - (m_0^2)^2 \right] \theta (m_0^2) - \sigma + \theta (\pm h_\pm) \theta \left[ \sigma - (m_0^2)^2 \right] A_\pm \left[ \tau + (m_0^2)^2 \right] + \theta (\pm h_\pm) \theta \left[ 4\sigma - (m_0^2)^2 \right] A_\pm \left( \tau + m_0^2/4 \right)$$

(45)

Here $h_\pm = h \pm \theta (-\tau) 2 (-\tau)^{3/2}/3$, 

$$A_\pm (z) = J \int \frac{dx}{z} W(Jx) \left[ h \pm \frac{2}{3} (x - \tau)^{3/2} \right] + J m_0^2 \int_0^z dx W(Jx)$$

and $m_0^+ \text{ and } m_0^-$ are the maximal and minimal solutions of Eq.24 correspondingly which exist at $\tau < 0$, $4\sigma + m_0^2 < 0$. When $\tau > 0$ or $\tau < 0$, $4\sigma + m_0^2 > 0$ then $m_0^+ = m_0$.

Eq.44 shows that the branches of the average hysteresis loop coincide when $H^2 > H_e^2$, $H_e$ being the solution of equation

$$m_0^2 (H_e) = 4\sigma$$

For $T \rightarrow T_{sg}$, $\sigma << J/J$ we get

$$H_e \approx 2J \sigma^{1/2}$$

and for $\tau < 0$, $-\tau >> J/J$, $H_e$ almost coincide with the coercive field for the homogeneous solution

$$H_e \approx 2J (-\tau)^{3/2}/3$$

so the hysteresis loops becomes similar to that of ordinary ferromagnet.

Let us present the explicit expressions for the functions $A_\pm$ in Eq.45 for the 'triangular' function $W$, Eq.21.

$$A_\pm \left[ \tau + (m_0^2)^2 \right] = \frac{J^2}{J} \left[ 2h \sigma \pm \frac{8}{15} \sigma^{5/2} + m_0^2 \left( \tau^2 - \frac{1}{5} (m_0^2)^4 \right) \right]$$

$$A_\pm \left[ \tau + m_0^2/4 \right] = \frac{J^2}{J} \left[ 2h \sigma \pm \frac{8}{15} \sigma^{5/2} + m_0 \left( \tau^2 - \frac{7}{80} m_0^4 \right) \right]$$

For the same $W$ the average equilibrium parameters at $H^2 < H_{AT}^2$ are

$$\langle q_{eq} \rangle = 2\sigma \left[ 1 - \left( \frac{H}{m_0 J} \right)^3 \right] - \tau \left[ 1 - 3 \left( \frac{H}{m_0 J} \right)^2 + 2 \left( \frac{H}{m_0 J} \right)^3 \right] + 2 \left( \frac{H}{J} \right)^2 \ln \left( \frac{H}{m_0 J} \right)$$
\[ \langle m_{eq} \rangle = \frac{2H}{J} - \frac{H^2}{m_0 J^2} \]
\[ \langle x_{eq} \rangle = \frac{2}{J} - \frac{2H}{m_0 J^2} + \frac{H^2}{m_0 J^2 J (\tau + m_0^2)} \]
\[ \langle S_{eq} \rangle = \ln 2 - \sigma \left[ 1 - \left( \frac{H}{m_0 J} \right)^3 \right] + \frac{\tau}{2} \left[ 1 - 3 \left( \frac{H}{m_0 J} \right)^2 + 2 \left( \frac{H}{m_0 J} \right)^3 \right] - \frac{1}{2} \left( \frac{H}{J} \right)^2 \]
\[ \langle C_{eq} \rangle = \frac{3}{2} \left[ 1 - \left( \frac{H}{m_0 J} \right)^2 \right] + \frac{H^2}{J^2 (\tau + m_0^2)} \]

Let us note once more that the average equilibrium parameters are generally unobservable quantities. Probably, the experimental values being rather close to them are obtained after cooling in small external fields (field-cooled (FC) regime) for \( T \) near \( T_{sg} \) \([13, 17]\) when barriers between metastable states are relatively small and system could relax into the lowest (or close to it) state at a sufficiently slow cooling. In zero field cooled (ZFC) regime when field is applied after cooling below \( T_{sg} \) in zero field, the observed thermodynamical parameters would differ from equilibrium ones as the system would at first be trapped in the state with \( \Delta = 0 \) and will stay in it if applied field does not exceed \( h_c = (-\tau_1)^{1/2} (\tau - 4\tau_1/3) / 2 \), cf. Eq.39. Thus at \( h < h_c \) the ZFC parameters are those of \( \Delta = 0 \) metastable states. When applied field \( h > h_c \) the system relaxes into the metastable state at the boundary of stability region (on the lower branch of hysteresis loop) having some \( \Delta > 0 \) which is a solution of Eq.39. Inserting this \( \Delta \) in Eq.40 and Eq.41 we get the values of thermodynamic parameters the observed quantities would relax to in ZFC regime at \( h > h_c \).

Similarly, the parameters of metastable states define the other quantities which are determined in the slow nonequilibrium processes in the spin glass phase, such as thermo-remanent magnetization, \( m_{TRM} \), which remains after FC process and subsequent switching off the field, and isothermal remanent magnetization, \( m_{IRM} \), remaining after ZFC process followed by the application for some time (longer than the intravalley relaxation time) an external field \([13, 17]\). Thus \( m_{IRM} \) is apparently nonzero only at \( h > h_c \) and an equation defining it can be obtained by putting \( h = 0 \) in Eq.38 and inserting in this equation the value of \( \Delta \) we get from Eq.39. The equation for \( m_{IRM} \) can also be obtained from Eq.40 by putting in it \( h = 0 \) and \( \Delta = \min (1, h/h_{AT}) \).

4 Conclusions

The most remarkable feature of the model considered is the possibility to imitate the properties of such complex systems as spin glass with the aid of very simple Hamiltonian. It is common belief that the existence of a number of metastable states in spin glasses is caused by the frustration of random Hamiltonian, that is the absence of unique spin configuration providing the minimal energy \([13]\). In the present model the degeneracy of the ground state results from the permutational symmetry of the Hamiltonian instead of frustration. Nevertheless there also exists the transition into the inergodic phase and its magnetic properties appear to be very similar to those of real spin glasses including a set of transitions between metastable states and inclined hysteresis loop resulting from their presence \([13, 17]\). One may suppose that more realistic random Hamiltonians can also have some approximate permutational symmetry and corresponding quasidegeneracy of ground state more essential, perhaps, than that caused by the frustration.

We may note that present model implies the definite mechanism of the transitions between metastable states in a field, namely, the spin-flop transitions caused by the antiferromagnetic interaction between macroscopically large spin blocks. It seems rather probable that in some Ising short-range models of spin glasses it is possible to distinguish many clusters with the mostly antiferromagnetic interactions at the boundaries and relatively weak interactions inside them. Still it is rather evident that there can be also a many more other bond configurations in which degenerate spin configurations can exist at some field values. Thus it is hard to say to what extent spin-flop transitions are characteristic for real spin glasses and if the Hamiltonian, Eq.4 is a reasonable approximation for some random short-range Hamiltonian with some specific type of disorder. Still such possibility seems to be rather probable in view of similarity of the properties of the model to those of some real disordered magnets \([13, 17]\).

Finally we may state that in spite of the qualitative nature of the model it allows to get some notion about the character of theoretical results relevant for the description of real experiments in inergodic
systems. It shows how these result may look like and how the thermodynamic parameters of metastable states are related to the characteristics of nonequilibrium processes in spin glass phases.

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