Supersymmetric String Solitons

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Abstract

These notes are based on lectures given by C. Callan and J. Harvey at the 1991 Trieste Spring School on String Theory and Quantum Gravity. The subject is the construction of supersymmetric soliton solutions to superstring theory. A brief review of solitons and instantons in supersymmetric theories is presented. Yang-Mills instantons are then used to construct soliton solutions to heterotic string theory of various types. The structure of these solutions is discussed using low-energy field theory, sigma-model arguments, and in one case an exact construction of the underlying superconformal field theory.

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1. Introduction

The theme of most of the lectures given at this school has been the study of simplified or “toy” models of string theory. The motivation of course is to better understand what a fundamental and non-perturbative formulation of string theory should look like through the use of simple models. These lectures describe a different, but hopefully complementary, approach to the study of non-perturbative string theory through the development of semi-classical techniques.

There are several reasons why this is an interesting enterprise. First, in realistic unified string theories there is little hope that we will obtain exact non-perturbative results. As in most field theories, probably the best we can hope for is to obtain approximate non-perturbative results through the use of semi-classical techniques. Still, even achieving this limited goal is bound to teach us many interesting things about the structure of string theory, as it has in field theory. Second, there are a number of fascinating technical issues in conformal field theory which must be resolved before this program can be carried out. These include the proper treatment of collective coordinates and the proper definition of the mass or action in string theory. Related issues may also arise in the treatment of back reaction in the exact black hole solution [1] discussed in the lectures of H. Verlinde at this school. Third, the solitons are in many cases the endpoints of black hole (or black p-brane) evaporation, and they are therefore an essential ingredient in resolving the fascinating issues surrounding Hawking radiation and coherence loss in string theory. Finally, we may learn something dramatically new about string theory through the study of semi-classical solutions. For example, the conjecture [2] that strongly coupled string theory is dual to weakly coupled fivebranes would have fascinating consequences if true.

The study of semi-classical string theory is still in its infancy. We do not yet know how to classify semi-classical solutions directly through stringy topological invariants nor do we have all the techniques necessary for extracting physical quantities by expanding string theory about these solutions. What has been accomplished recently is to find solutions to string theory, some of which can be studied directly as (super) conformal field theories, which at large distances are solutions to the low-energy effective string field theory equations of motion and which have the properties of solitons and instantons. One goal of these lectures is to discuss these solutions and to indicate how they might be used to study some of the issues raised above.

In order to make these lectures reasonably self-contained we will start in Section 2 by reviewing certain features of solitons in field theory that will be important in what follows. Section 3 reviews aspects of Yang-Mills instantons in supersymmetric theories. The Montonen-Olive conjecture of a weak-strong coupling duality in $N = 4$ Yang-Mills is described in Section 4, along with comments on its possible relevance to string theory. Section 5 discusses solitons and instantons in low-energy heterotic string theory and the special role played by fivebrane solitons. The geometry and charges of these solutions is explained in Section 6. Symmetries, non-renormalization theorems, and other aspects of the underlying worldsheet sigma model are discussed in Section 7. An algebraic construction of certain symmetric solutions is given in Section 8, and the appearance of an exotic $N = 4$ algebra is described. Section 9 briefly reviews some additional supersymmetric solitons. Concluding comments are made in Section 10.
2. Soliton Review

Roughly speaking, solitons are static solutions of classical field equations in $D$ space-time dimensions which are localized in $(D - 1) - d$ spatial coordinates and independent of the $d$ other spatial coordinates. The usual case is $d = 0$ and the soliton then has many characteristics of a point particle. For arbitrary $d$ the solution is called a $d$-brane and describes an extended object. In four spacetime dimensions $d = 1$ corresponds to a string and $d = 2$ to a membrane or domain wall. Such objects are of interest in various cosmological scenarios.

Solitons are usually characterized by the following properties [3]. First, they are non-perturbative. They are solutions to non-linear field equations which cannot be found by perturbation of the linearized field equations. In addition, their mass (or mass per unit $d$-volume) is inversely proportional to some power of a dimensionless coupling constant. As a result, they become arbitrarily massive compared to the perturbative spectrum at weak coupling. Thus quantum effects due to exchange of solitons will be non-perturbative effects which vanish to all orders in perturbation theory. Second, solitons are characterized by a topological rather than a Noether charge. Finally, soliton solutions typically depend on a finite number of parameters called moduli which act as coordinates on the moduli space of soliton solutions of fixed topological charge.

A simple and standard example which illustrates these features is the “kink” solution in $D = 1 + 1$ spacetime dimensions. Start with the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi)$$  \hspace{1cm} (2.1)

with the potential given by $U(\phi) = \lambda(\phi^2 - m^2/\lambda)^2/4$. The dimensionless coupling in this example is $g \equiv \lambda/m^2$. The conserved topological charge is

$$Q = \int_{-\infty}^{+\infty} dx j_0$$  \hspace{1cm} (2.2)

where $j_\mu = (\sqrt{g}/2) \epsilon_{\mu\nu} \partial^\nu \phi$ is the conserved topological current. Thus

$$Q = \frac{\sqrt{g}}{2} (\phi(+\infty) - \phi(-\infty))$$  \hspace{1cm} (2.3)

and is $\pm 1$ for a kink (anti-kink) in which $\phi$ varies from the minimum of $U$ at $\phi = \mp 1/\sqrt{g}$ at $x = +\infty$ to the minimum at $\phi = \pm 1/\sqrt{g}$ at $x = -\infty$.

To find the explicit form of the solution we consider the classical equation of motion for a static configuration

$$\phi'' = \frac{\partial U}{\partial \phi}.$$  \hspace{1cm} (2.4)

For a solution with $U$ and $\phi'$ vanishing at infinity we can integrate this once to find

$$\frac{1}{2} (\phi')^2 = U(\phi).$$  \hspace{1cm} (2.5)
Integrating this equation with the previous choice of $U$ yields the kink (anti-kink) solution

$$
\phi_K(x) = \pm \frac{m}{\sqrt{\lambda}} \tanh\left[ m(x-x_0)/\sqrt{2} \right].
$$

(2.6)

The energy (rest mass) of this configuration is

$$
E = \int dx \frac{1}{2} (\phi')^2 + U(\phi) = \frac{2\sqrt{2} m}{3 g}
$$

(2.7)

so that the kink mass divided by the mass of the elementary scalar is proportional to $1/g$ showing the non-perturbative nature of the solution.

Also, in accordance with the earlier discussion, the solution for fixed $Q$ and mass depends on a parameter $x_0$, the center of mass coordinate of the soliton. In this example the existence of $x_0$ follows from translational invariance of the underlying field theory. The presence of $x_0$ introduces an important problem into the quantization of the theory expanded about $\phi_K$. This is due to the fact that the quadratic fluctuation operator

$$
O_2 = \frac{\delta^2 S}{\delta \phi^2}|_{\phi=\phi_K} = \partial^2 + m^2 - 3\lambda \phi_K^2
$$

(2.8)

has a zero mode $\eta_0$ given by an infinitesimal translation of $\phi_K$, $\eta_0 = \partial \phi_K/\partial x_0$. We can think of $\eta_0$ as a tangent vector to the curve in configuration space given by translation of $\phi_K$. A zero mode of $O_2$ will lead to a divergence in perturbation theory about $\phi_K$ of the form $\det^{-1/2} O_2 = \infty$.

The solution is to separate out the dependence on $x_0$ by a change of coordinates in field space through the introduction of a collective coordinate. Naively we would expand $\phi$ in the kink sector as

$$
\phi(x,t) = \phi_K(x-x_0,t) + \sum_{n=0}^{\infty} c_n(t) \eta_n(x-x_0)
$$

(2.9)

where the $\eta_n$ are a complete set of eigenfunctions of $O_2$ and the $c_n(t)$ are time dependent coefficients which act as coordinates in configuration space. The collective coordinate method involves a change of coordinates from the $\{c_n, n = 0\ldots\infty\}$ to $\{x_0(t), c_n(t), n = 1\ldots\infty\}$ which promotes the modulus $x_0$ to a time-dependent coordinate. The expansion of $\phi$ is then given by

$$
\phi(x,t) = \phi_K(x-x_0(t)) + \sum_{n=1}^{\infty} c_n(t) \eta_n(x-x_0(t)).
$$

(2.10)

It is then possible to separate out explicitly the dependence on $x_0(t)$ and to quantize this as the center of mass coordinate of the soliton and to be left with a well defined perturbation theory for the non-zero modes.
In the more complicated examples we consider later there will be moduli resulting from symmetries of the underlying theory (translation invariance, scale symmetry, gauge symmetry, supersymmetry) as well as moduli which follow from index theorems but are not required by the underlying symmetries.

Two trivial modifications of this theory lead to solutions with somewhat different interpretations. We can add some spatial dimensions, say two, to obtain a static solution to 3 + 1 dimensional field equations which is independent of two of the coordinates, \( \phi(x_1, x_2, x_3, t) = \phi_K(x_3) \). Such a solution describes a two-brane or what would be called in cosmology a domain wall. Alternatively we can remove the time dimension and reinterpret the solution \( \phi_K(x) \) as a solution in Euclidean time of a 0 + 1-dimensional theory (i.e. quantum mechanics) \( \phi_K(x) \sim x(t_E) \). The solution is then an instanton describing tunneling between two degenerate minima.

A far less trivial modification is to introduce fermions into the theory. Since we are interested in solitons in supersymmetric string theories, we introduce a supersymmetric coupling to fermions by taking the Lagrangian to be

\[
L = -\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} \bar{\psi} i \gamma^\mu \partial_\mu \psi - \frac{1}{2} V^2(\phi) - \frac{1}{2} V'(\phi) \bar{\psi} \psi
\]  

(2.11)

where \( \psi \) is a Majorana fermion and \( V \) is a function chosen so that the potential \( V^2 \) has two degenerate minima. For concreteness we may take \( V = \lambda(\phi^2 - a^2) \). This theory has two chiral supercharges given by

\[
Q_{\pm} = \int dx (\dot{\phi} \pm \phi') \psi_{\pm} \mp V(\phi) \psi_{\mp}
\]  

(2.12)

where \( \psi_{\pm} \) are the left- and right-handed components of \( \psi \).

It was shown in [4] that a careful treatment of boundary terms leads to the supersymmetry algebra

\[
Q_+^2 = P_+, \quad Q_-^2 = P_-, \quad \{Q_+, Q_-\} = T
\]  

(2.13)

with \( P_\pm = P_0 \pm P_1 \), and where the central term \( T \) is equivalent to the topological charge defined earlier, although its precise functional form is different. The existence of this central term in the supersymmetry algebra in the soliton sector has important consequences. First, the relation

\[
P_+ + P_- = (Q_+ + Q_-)^2 - T = (Q_+ - Q_-)^2 + T
\]  

(2.14)

implies a Bogomolnyi bound, \( M \geq T/2 \), where \( M \) is the rest mass. This bound is saturated precisely for those states \( |s\rangle \) for which \( (Q_+ \pm Q_-)|s\rangle = 0 \), i.e. for states annihilated by some combination of the supersymmetry charges. Classically one can check that this does hold for the kink solution using (2.12) and (2.5). In fact, (2.5), or rather its square root could have been derived by demanding that the kink solution preserve a linear combination of supersymmetries. The fact that demanding unbroken supersymmetry leads to a sort of “square root” of the equations of motion will be put to great use later. Also, the fact that the kink solution is annihilated by one combination of the supercharges implies saturation of the Bogomolnyi bound. The other combination of supercharges does not annihilate
the kink state but instead produces a fermion zero mode in the kink background. This is because a supersymmetry variation of a solution to the bosonic equations of motion if it is nonzero produces a solution to the fermionic equations of motion in the bosonic background.

This simple example provides a paradigm for more complicated examples. In most known examples of solitons in supersymmetric theories one finds that half of the supercharges annihilate the classical solution leading to saturation of a Bogomolnyi bound, while the other half acting on the soliton produce fermion zero modes in the soliton background. In addition, searching for configurations which preserve some of the supercharges provides a shortcut to solving the full equations of motion since the resulting equations are typically first order as compared to the second order equations of motion.

In the kink example there is a single fermion zero mode. It also has interesting consequences for the spectrum of the theory. The mode expansion of the fermion field in the soliton sector will have the form

$$\psi = b_0 f_0 + \sum_{n \neq 0} b_n f_n$$

where $f_0$ is the zero mode. Canonical quantization implies that $b_0$ obeys $\{b_0, b_0\} = 1$ which in turn implies that the vacuum must consist of two degenerate states, $|+\rangle$ and $|-\rangle$ with $|\pm\rangle = b_0 |\mp\rangle$. We can think of the bosonic and fermionic zero modes $(x_0, b_0)$ as providing coordinates on the supermoduli space of zero energy perturbations of the soliton.

### 3. Yang-Mills Instantons

After this brief review of solitons we turn our attention to instantons in Yang-Mills theory. These will provide the starting point for the construction of soliton solutions to string theory in ten dimensions, so we will quickly review some of the features of instanton solutions which are important in the later constructions.

We start with the Euclidean Yang-Mills action

$$S_E = \frac{1}{2g^2} \int d^4 x \text{Tr} F_{\mu\nu} F^{\mu\nu}$$

and look for solutions of the equations of motion which approach pure gauge at infinity,

$$A_\mu \to g^{-1} \partial_\mu g.$$  \hspace{1cm} (3.2)

Such configurations are labelled by the winding number of the map from $S^3$ into the gauge group $G$ provided by the gauge function $g$ of (3.2) evaluated at infinity. For simplicity we

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1 Although this is not always true. A fascinating counterexample will be discussed in Section 9.
take $G = SU(2)$. It is simple to show that the action obeys a Bogomolnyi bound in this theory, $S_E \geq 8\pi^2 k/g^2$ where

$$k = \frac{1}{16\pi^2} \int d^4x \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

(3.3)

is the winding number and $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$. The bound is saturated for (anti) self-dual gauge fields obeying

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}.$$  

(3.4)

The interpretation of these solutions is that they correspond to tunneling between degenerate minima labelled by elements of $\pi_3(SU(2))$. Applications of Yang-Mills instantons are described in [6] and [3].

To study the bosonic zero modes (moduli) of a solution to the self-dual equations we consider a fixed solution $A^0_\mu$ of (3.4) and look for perturbations $\delta A_\mu$ which preserve (3.4) with a plus sign. Writing $A_\mu = A^0_\mu + \delta A_\mu$ gives $F_{\mu\nu} = F^0_{\mu\nu} + D_\mu \delta A_\nu - D_\nu \delta A_\mu$ where the covariant derivative $D$ is evaluated using $A^0$. Equation (3.4) then requires that

$$D_\mu \delta A_\nu - D_\nu \delta A_\mu - \epsilon_{\mu\nu\lambda\rho} D^\lambda \delta A^\rho = 0.$$  

(3.5)

We are interested in solutions of (3.5) which are not of the form $\delta A_\mu = D_\mu \Lambda$, that is not pure gauge. The number of such solutions can be determined through the use of the Atiyah-Singer index theorem as discussed in [7]. Briefly, one considers the following elliptic complex

$$0 \xrightarrow{D^0} V^0 \xrightarrow{D^1} V^1 \xrightarrow{D^2_+} V^2_+ \xrightarrow{D^2_-} 0$$

(3.6)

where $V^0$ is the space of zero-forms (scalars), $V^1$ is the space of one-forms (deformations of the instanton), and $V^2_\pm$ is the space of self-dual two-forms. $D^0$ and $D^3$ are the identity map and the projection map onto anti-self dual forms respectively. $D^1 = d + A^0$ and $D^2_\pm$ is the projection onto the self-dual part of the covariant exterior derivative (as in (3.3)). It is easy to check that $D^2_+ D^1 = 0$ iff $F = \tilde{F}$. We then define spaces

$$H^i = \frac{\text{Ker} D^{i+1}}{\text{Im} D^i}$$

(3.7)

and set $h^i = \dim H^i$. $H^1$ is the moduli space we are after, that is the space of solutions of (3.5) modulo solutions which are pure gauge. The Atiyah-Singer index theorem then gives the alternating sum of the $h^i$ in terms of topological invariants. For $G = SU(2)$ one has $h^0 - h^1 + h^2 = -8k + 3$ for base manifold $S^4$ while for $R^4$ this sum is simply $8k$. For $G = SU(2)$ in these cases it is easy to show that $h^0 = 0$, that is there are no non-trivial solutions of $D_\mu \Phi = 0$ (For other groups and/or manifolds this is not necessarily true. Gauge connections for which $h^0 \neq 0$ are called reducible connections.) It is also possible to show using a simple positivity argument that $h^2 = 0$ so that on $R^4$ the dimension of the moduli space is just $8k$. In the later constructions it will be important that we are working on $R^4$ and not $S^4$ and that the total number of moduli is a multiple of four.
The construction of an Ansatz which contains all $8k$ parameters is a difficult and subtle mathematical problem whose solution can be found in [8]. For our purposes this will not be necessary. There is however a construction which yields $5k$ of the parameters and which plays an important role in the later constructions.

This ansatz involves writing the gauge field in terms of the derivative of a scalar field $f$ as

$$A_\mu = \Sigma^\nu_\mu \partial_\nu \ln f$$

where the $SU(2)$ indices have been suppressed and with $\Sigma_{\mu\nu}$ a two by two anti-symmetric matrix which is anti-self-dual on the $\mu\nu$ indices. A simple construction of $\Sigma$ which will be used later arises from embedding the $SU(2)$ gauge group in $SO(4)$. Letting $m, n = 1 \cdots 4$ be $SO(4)$ indices we have

$$\Sigma_{mn}^{\mu\nu} = \frac{1}{2} \left( \delta_{mn}^{\mu\nu} - \frac{1}{2} \epsilon_{mn}^{\mu\nu} \right).$$

which is anti-self-dual both on the $\mu, \nu$ indices and on $m, n$. This latter fact shows that the $\Sigma$ lie in a $SU(2)$ subgroup of $SO(4)$.

Substituting this ansatz into the equation (3.4) results in a simple equation for $f$

$$\frac{1}{f} \Box f = 0.$$  

(3.10)

If $f$ is non-singular then it is constant and hence $A_\mu = 0$. On the other hand, $f$ can be singular and still solve (3.10) everywhere if the singularities in $\Box f$ are cancelled by the $1/f$ factor. The general solution of this form is

$$f(x) = 1 + \sum_{i=1}^{k} \frac{\rho_i^2}{(x - x_i^0)^2}.$$  

(3.11)

and depends on $5k$ parameters $(x_i^0, \rho_i)$ which give the location and the scale size of the instanton. The $3k$ parameters which are missing correspond roughly to the relative $SU(2)$ orientations of the $k$ instantons plus an overall global $SU(2)$ orientation.

For $k = 1$ the solution (3.11) yields the gauge field

$$A_\mu = -2\rho^2 \Sigma_{\mu\nu} \frac{(x - x^0)^\nu}{(x - x^0)^2((x - x^0)^2 + \rho^2)}.$$  

(3.12)

The gauge field (3.12) is singular as $x \to x^0$ and does not approach pure gauge at infinity. However, by making a large gauge transformation one obtains a gauge field

$$A'_\mu = -2\Sigma_{\mu\nu} \frac{(x - x^0)^\nu}{(x - x^0)^2 + \rho^2},$$  

(3.13)

which does approach pure gauge at infinity with winding number one and is non-singular. Here $\Sigma_{\mu\nu}$ is the self-dual analog of $\Sigma_{\mu\nu}$. 



We also want to consider the generalization of these solutions to super Yang-Mills theory. In 3 + 1 dimensions the super Yang-Mills action in component form is obtained by adding the fermion action
\[ \mathcal{L}_F = \text{Tr}(\bar{\chi}i\gamma^\mu D_\mu \chi) \] (3.14)
to the Minkowski version of (3.1) where \( \chi \) is a Majorana fermion in the adjoint representation of the gauge group. The Euclidean continuation of this action is not straightforward because of the absence of Majorana fermions in Euclidean space. This can be dealt with in various ways. Luckily these subtleties will not concern us because we will eventually embed the four-dimensional Euclidean theory in a ten-dimensional Minkowski theory which does have Majorana fermions. We can thus proceed as usual except that we must remember that left and right-handed fermions are no longer related by complex conjugation.

To determine whether the instanton solution (3.13) is supersymmetric we need to know whether the supersymmetric variation of this solution is non-zero. The supersymmetry variation of a configuration with vanishing fermion fields is
\[ \delta \chi = F_{\mu\nu} \gamma^{\mu\nu} \epsilon \] (3.15)
where \( \epsilon \) is the parameter of the supersymmetry variation. Under \( SO(4) \) \( \epsilon \) can be decomposed into irreducible representations \( \epsilon = \epsilon_L + \epsilon_R \) with \( \epsilon_L \sim (2, 1) \) and \( \epsilon_R \sim (1, 2) \) under \( SO(4) \equiv SU(2)_L \times SU(2)_R \).

For an instanton with \( F_{\mu\nu} = \tilde{F}_{\mu\nu} \) we have
\[ \delta \chi = F_{\mu\nu} \gamma^{\mu\nu}(1 + \gamma^5) \epsilon \] (3.16)
which is vanishing for \( \epsilon_L \) but not for \( \epsilon_R \). Just as in the kink example, half of the supersymmetries are broken by the solution while the other half give rise to fermion zero modes. Closer examination shows that of the four fermion zero modes predicted by the Atiyah-Singer index theorem for \( k = 1 \), only two arise from supersymmetry transformations corresponding to the broken supersymmetries as described above. However, at the classical level, this theory is also conformally invariant and this leads to superconformal transformations of the instanton background which produce the two remaining fermion zero modes.

For general \( k \) the index theorem predicts \( 4k \) zero modes, only 4 of which can be constructed as above. There is however an important pairing of fermion zero modes with the bosonic moduli which holds for arbitrary \( k \). Given a variation \( \delta A_\mu \in H^1 \) we can construct a fermion zero mode of the form
\[ \chi_0 = \delta A_\mu \gamma^\mu \epsilon_L \] (3.17)
There is a redundancy in this pairing so that two bosonic zero modes are associated to every fermion zero mode. For details see [9].

4. Magnetic Monopoles and the Montonen-Olive Conjecture

Our final field theory example before discussing string theory has to do with magnetic monopole solutions in Yang-Mills theory and in particular with the properties of monopoles in \( N = 4 \) super Yang-Mills theory.
The action for a Yang-Mills Higgs theory with gauge group $SU(2)$ and a Higgs field $\Phi$ in the adjoint representation is given by

$$S = \int -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \text{Tr} D_\mu \Phi D^\mu \Phi - V(\Phi).$$ \hspace{1cm} (4.1)

We assume that $V$ has a minimum in which $\langle \text{Tr} \Phi^2 \rangle = v^2$ which breaks $SU(2)$ down to $U(1)$. For a static configuration the energy is given by

$$E = \int d^3 x \text{Tr}(B^i \pm D^i \Phi)^2 + \int d^3 x V(\Phi) + 4\pi v|Q_M|$$ \hspace{1cm} (4.2)

where $v$ is the asymptotic vacuum expectation value of $\Phi$. The energy thus satisfies a Bogomolnyi bound

$$E \geq 4\pi v|Q_M|$$ \hspace{1cm} (4.3)

where

$$Q_M = \int \vec{B} \cdot d\vec{S}$$ \hspace{1cm} (4.4)

is the magnetic charge and $\vec{B}$ above is the asymptotic value of the gauge invariant $U(1)$ field strength. The magnetic charge $Q_M$ is also related through the equations of motion to the element of $\pi_2(SU(2)/U(1)) = Z$ which labels the topological charge carried by the Higgs field configuration. The Bogomolnyi bound is saturated iff $V(\phi) \equiv 0$ and $B^i = \pm D^i \Phi$.

If we work in this limit of vanishing potential then we are free to impose as a boundary condition that the vacuum expectation value of $\Phi$ approach an arbitrary constant $v$ at spatial infinity. Of course quantum mechanically we expect a potential to be generated by renormalization even if it is absent classically.

Nonetheless let us work in this limit for the time being. Then the first order equations $B^i = \pm D^i \Phi$ can be integrated to give explicit monopole solutions. The charge one solution can be found in [10], multi-monopole solutions are discussed in [11]. In this limit there is a universal formula for the classical mass of the particles of the theory given by

$$M^2 = (4\pi)^2 v^2 (Q_E^2 + Q_M^2)$$ \hspace{1cm} (4.5)

where $Q_E$ and $Q_M$ are the electric and magnetic charges of the particle respectively. For monopoles this is just the statement that the Bogomolnyi bound is saturated. For the massive gauge bosons it is just the usual relation between the gauge boson mass and the Higgs vacuum expectation value. The photon is of course neutral and massless as is the remaining physical Higgs boson due to the vanishing potential.

There are some other curious features of this limit. One is the fact that the static force between two monopoles of like charge or between two gauge bosons of like charge vanishes. This is due to a cancellation between a repulsion due to photon exchange and an attraction due to massless Higgs boson exchange [12].

These facts as well as some others led Montonen and Olive [13], following ideas of [14] to conjecture the existence of a “dual” formulation of the theory in which electric and magnetic charges are exchanged and in which the the magnetic monopoles become the
gauge bosons and the gauge bosons arise as solitons of the dual theory. In two spacetime dimensions the Thirring model – sine-Gordon duality provides an example where topological and Noether charges are exchanged in the theory and its dual, but this duality relies on the peculiar properties of two dimensions.

One stumbling block to this more ambitious conjecture in 3 + 1 dimensions is the fact that monopoles appear to have spin zero while gauge bosons of course have spin one. In addition, the vanishing of the potential is not natural from the quantum point of view so there is no reason to expect the mass formula to be exact when quantum corrections are included.

This second objection can be overcome by embedding the theory in $N = 2$ super Yang-Mills theory. Then the charges $Q_E$ and $Q_M$ appear as central charges in the supersymmetry algebra as in the simpler kink example and the mass formula (13) is exact for supersymmetric states [4]. However an inspection of the fermion zero modes and the resulting monopole spectrum in this theory shows that the monopole states fill out a matter supermultiplet consisting of spin zero and spin one-half states. In order to construct monopoles with integer spin it is necessary to extend the supersymmetry to $N = 4$, the maximal allowable global supersymmetry in 3 + 1 dimensions. This theory has a number of remarkable features. First, the structure of the fermion zero modes is such that the monopole supermultiplet now coincides with the gauge supermultiplet and includes states of spin 1, 1/2, and 0 [13]. Second, the scalar potential has exact flat directions due to supersymmetry and again the mass formula is exact. Finally, this theory is finite with vanishing beta-function so that a duality which relates $g \rightarrow 1/g$ can make sense quantum mechanically at all scales. Thus in this special theory all of the simple objections to the existence of the sort of duality suggested by Montonen and Olive disappear! Of course this is a far cry from showing that such a duality actually holds, but the evidence is suggestive enough that the idea is well worth pursuing.

Finally, it is perhaps worth mentioning that the $N = 4$ theory and the Montonen-Olive conjecture may have some ties to ten-dimensional physics and hence to string theory. For one thing, the $N = 4$ theory can be obtained by dimensional reduction of $N = 1$ super Yang-Mills in ten dimensions. Second, the Montonen-Olive conjecture for more general gauge groups says that the dual gauge group should have a weight lattice dual to the weight lattice of the original group. Self-dual lattices of course play a crucial role in ten-dimensional heterotic string theory with the gauge groups $SO(32)/Z_2$ and $E_8 \times E_8$ with self-dual lattices being singled out by anomaly cancellation. Finally, a stringy analog of the Montonen-Olive conjecture [2] will be briefly discussed in the following section.

5. Low-Energy Heterotic String Theory

Having discussed soliton and instanton solutions of various supersymmetric field theories we would like to generalize these considerations to string theory. Let us first discuss the problem of finding string solitons via the “strings in background fields” spacetime approach. The beta functions for strings propagating in a background of massless fields are the equations of motion of a certain master spacetime action which can be computed as an expansion in the string tension $\alpha'$. For the heterotic string, the leading terms in this


action are identical to the $D = 10, N = 1$ supergravity and super Yang-Mills action. The bosonic part of this action reads

$$S = \frac{1}{\alpha'^4} \int d^{10}x \sqrt{-g} e^{-2\phi} \left( R + 4(\nabla \phi)^2 - \frac{1}{3} H^2 - \frac{\alpha'}{30} \text{Tr} F^2 \right), \quad (5.1)$$

where the three-form antisymmetric tensor field strength is related to the two-form potential by the familiar anomaly equation [16]

$$H = dB + \alpha' \left( \omega_3^L(\Omega_+) - \frac{1}{30} \omega_3^M(A) \right) + \ldots \quad (5.2)$$

(where $\omega_3$ is the Chern-Simons three-form) so that

$$dH = \alpha'(trR \wedge R - \frac{1}{30} Tr F \wedge F). \quad (5.3)$$

The trace is conventionally normalized so that $Tr F \wedge F = \sum_i F^i \wedge F^i$ with $i$ an adjoint gauge group index. An important, and potentially confusing, point is that the connection $\Omega_\pm$ appearing in (5.2) is a non-Riemannian connection related to the usual spin connection $\omega$ by

$$\Omega_{\pm M}^{AB} = \omega_{M}^{AB} \pm H_{M}^{AB}. \quad (5.4)$$

Since the antisymmetric tensor field plays a crucial role in all of our solutions, this subtlety will be crucial.

Rather than directly solve the equations of motion for this action, it is much more convenient to look for bosonic backgrounds which are annihilated by some of the $N=1$ supersymmetry transformations (only the vacuum is annihilated by all the the supersymmetries). Both the kink solution (2.6) and the self-dual equation (3.4) could have been found in this manner. The Fermi field supersymmetry transformation laws which follow from (5.1) are

$$\delta \chi = F_{MN} \gamma^{MN} \epsilon$$

$$\delta \lambda = (\gamma^M \partial_M \phi - \frac{1}{6} H_{MNP} \gamma^{MNP}) \epsilon$$

$$\delta \psi_M = (\partial_M + \frac{1}{4} \Omega_{\pm M}^{AB} \gamma_{AB}) \epsilon, \quad (5.5)$$

and it is apparent that to find backgrounds for which all of (5.5) vanish, it is only necessary to solve first-order equations, rather than the more complicated second-order equations which follow from varying the action. We will shortly construct a simple ansatz for the bosonic fields which does just this.

Although there are a variety of soliton or instanton solutions of (5.5) we could consider, it is useful to first study the simplest and most natural solutions. In realistic theories we would consider a spacetime of the form $M \times K$ with $M$ four-dimensional Minkowski or Euclidean space and $K$ a six-dimensional compact space which solves the equations of motion following from (5.1). Soliton and instanton solutions involving gauge fields on
such spaces have been studied and a general classification has been discussed in [17]. However these solutions are either related to the usual Yang-Mills instantons or involve the structure of $K$ in various ways. A simpler but less realistic starting point would be to ask for solutions to string theory in ten-dimensional Minkowski space (or Euclidean space in the case of instantons). We will discuss solutions of this form both because they allow us to discuss solitons without the complication of discussing compactification and because the solitons we find may have implications for fundamental string theory.

The solutions of (5.5) that we will consider are fivebrane solitons. The motivation for making such an apparently arbitrary choice is based on an analogy between these solitons in string theory and Dirac monopoles in electromagnetism. In $3 + 1$ dimensions a point particle naturally couples to a one-form potential $A$ via

$$\int_L A \equiv \int d\tau \frac{dx^\mu}{d\tau} A_\mu$$

(5.6)

where $L$ indicates the world-line of the particle, determined by $x^\mu(\tau)$ which gives the particle location in spacetime as a function of a time-like parameter $\tau$. This coupling results in an “electric” source term in the equation of motion for the field strength $F = dA$

$$d^* F = j_E.$$  

(5.7)

Using Stoke’s theorem the electric charge of the point particle is

$$Q_E = \int_{S^{2}_\infty} *F$$

(5.8)

where $S^{2}_\infty$ is a two-sphere at spatial infinity.

Dirac conjectured that there might exist magnetic monopoles which would obey a dual version of these equations

$$dF = 0 \implies dF = \tilde{j}_M$$

(5.9)

$$d^* F = j_E \implies d^* F = 0$$

(5.10)

$$Q_E = \int_{S^{2}_\infty} *F \implies Q_M = \int_{S^{2}_\infty} F$$

(5.11)

and showed that quantum consistency imposes a quantization condition on the product of the electric and magnetic charges $Q_E Q_M = 2\pi n$ with $n$ an integer.

There is a construction analogous to this in string theory. Since we now want to couple to the world-sheet of a string rather than the world-line of a particle we introduce a two-form potential $B$, called a Kalb-Ramond field, which couples to the string world-sheet $S$ via $\int_S B$ and has a three-form field strength $H = dB$. This is the same $H$ as in (5.2) except that for the moment we ignore the Chern-Simons terms in Equation (5.2). The analogs of the previous equations and their dual forms are then (in D spacetime dimensions)

$$dH = 0 \implies dH = j_M$$

(5.12)
\[ d^* H = j_E \implies d^* H = 0 \]  
(5.13)

\[ Q_E = \int_{S^{D-3}_\infty} * H \implies Q_M = \int_{S^3_\infty} H. \]  
(5.14)

Because \(*H\) is a \(D-3\) form, it must be integrated over a \(D-3\) dimensional manifold. This fits perfectly with our string picture since a static infinite string in \(D\) spacetime dimensions has a \(D-3\) sphere at spatial infinity. However this also means that the dual object has a charge measured by integrating \(H\) over a three-dimensional manifold. So, unless \(D = 6\), the string and its dual object do not have the same dimension. For example, consider \(D = 4\). It is well known that the two form \(B\) is equivalent to a massless scalar field in four dimensions and that this scalar field has couplings analogous to the usual axion field. The “electric” charge, measured by integrating \(*H\) over the \(S^1\) at infinity around an infinite string, is just the axion charge of the string. The dual object has a “magnetic” charge obtained by integrating \(H\) over a three-sphere at infinity. But this means that the dual object must be localized in both space and time, i.e. it is an instanton! Thus we learn the important fact that in four dimensions strings are dual, in the Dirac sense, to instantons. If we move up to \(D = 10\) as required by superstring theory, then the dual object is a fivebrane, that is an object with extent in five spatial dimensions which sweeps out a six-dimensional world-volume as it evolves in time.

This fivebrane is also intimately related to Yang-Mills instantons in four dimensions. One way to see this connection is to note that in string theory we no longer have \(H = dB\) but rather

\[ H = dB - \frac{\alpha'}{30} \omega_{YM} \]  
(5.15)

(the additional gravitational contributions which will be discussed shortly). We thus have the equations

\[ dH = -\frac{\alpha'}{30} Tr(F \wedge F) \]  
(5.16)

\[ d^* H = 0 \]  
(5.17)

in a region free of electric sources. Comparison with the previous equations clearly shows that the Yang-Mills topological charge density acts as a magnetic source term for \(H\).

This duality argument doesn’t guarantee that the dual objects actually exist, of course, but we shall see that in fact they do. The existence of objects which are geometrically dual to strings is quite interesting and raises the question of whether they could also be dynamically dual as in the conjecture of Montonen and Olive. This possibility was raised in [4], some issues associated with and evidence for this conjecture are discussed in [18]. Although the idea is a tantalizing one, proving or disproving it seems beyond our current ability.

With this motivation let us attempt to construct a fivebrane solution to (5.5). The supersymmetry variations are determined by a positive chirality Majorana-Weyl \(SO(9,1)\) spinor \(\epsilon\). Because of the fivebrane structure, it is useful to note that \(\epsilon\) decomposes under \(SO(9,1) \supset SO(5,1) \otimes SO(4)\) as

\[ 16 \rightarrow (4_+, 2_+) \oplus (4_-, 2_-) \]  
(5.18)
where the $\pm$ subscripts denote the chirality of the representations. Denote world indices of the four-dimensional space transverse to the fivebrane by $\mu, \nu = 6 \ldots 9$ and the corresponding tangent space indices by $m, n = 6 \ldots 9$. We assume that no fields depend on the longitudinal coordinates (those with indices $M = 0 \ldots 5$) and that the nontrivial tensor fields in the solution have only transverse indices. Then the gamma matrix terms in (5.5) are sensitive only to the $SO(4)$ part of $\epsilon$ and, in particular, to its $SO(4)$ chirality.

One immediately sees how to make the gaugino variation vanish (in what follows we treat $\epsilon$ as an $SO(4)$ spinor and let all indices be four-dimensional): As a consequence of the four-dimensional gamma-matrix identity

$$\gamma^{mn}\epsilon_\pm = \mp \frac{1}{2} \epsilon^{mnrst} \gamma^{rs} \epsilon_\pm,$$

one has

$$F_{mn} \gamma^{mn} \epsilon_\pm = \mp \tilde{F}_{mn} \gamma^{mn} \epsilon_\pm,$$

where the dual field strength is defined by

$$\tilde{F}_{mn} = \frac{1}{2} \epsilon^{mnrst} F^{rs}.$$

Therefore, $\delta \chi$ vanishes if $F_{mn} = \pm \tilde{F}_{mn}$ and $\epsilon = (4_+, 2_\pm)$ which is to say that if the gauge field is taken to be an instanton, then $\delta \chi$ vanishes for all supersymmetries with positive $SO(4)$ chirality. This is just the argument of (3.16) except that the supersymmetry parameter now transforms under $SO(5,1)$ as well.

To deal with the other supersymmetry variations, we must adopt an ansatz [22] for the non-trivial behavior of the metric and antisymmetric tensor fields in the four dimensions transverse to the fivebrane. For the metric tensor we write

$$g_{\mu\nu} = e^{-2\phi} \delta_{\mu\nu}, \quad \mu, \nu = 6 \ldots 9$$

and for the antisymmetric tensor field strength

$$H_{\mu\nu\lambda} = -\epsilon_{\mu\nu\lambda} \sigma \partial_\sigma \phi,$$

where $\phi$ is to be identified with the dilaton field. With this ansatz and the rather obvious vierbein choice $e^m_\mu = \delta^m_\mu e^{+\phi}$, we can also calculate the generalized spin connection (5.4) which appears in (5.5) and (5.2):

$$\Omega_{\pm m\mu} = \delta_{m\mu} \partial_n \phi - \delta_{n\mu} \partial_m \phi \mp \epsilon_{m\mu n} \partial_\rho \phi .$$

Now consider the $\delta \lambda$ term in (5.5). Because of the ansatz, both terms are linear in $\partial \phi$. By standard four-dimensional gamma-matrix algebra, the relative sign of the two terms is proportional to the $SO(4)$ chirality of the spinor $\epsilon$. We have chosen the sign and normalization of the ansatz for $H$ so that $\delta \lambda$ vanishes for $\epsilon \in (4_+, 2_\pm)$. Finally, consider the gravitino variation in (5.5). A crucial fact, following from (5.21), is that while $\Omega_\pm$ would in general be an $SO(4)$ connection, with the chosen ansatz it is actually pure $SU(2)$. To be precise,

$$\Omega_{\pm m\mu} \gamma^{mn} \epsilon_\eta = 2(\gamma^{nm} \partial_\rho \phi)(1 \mp \eta 1) \epsilon_\eta ,$$

(where $\eta = \pm$) so that $\Omega_{\pm}$ annihilates the $(4_+, 2_\pm)$ spinor. Since (5.22) involves only $\Omega_-$, it suffices to take $\epsilon$ to be a constant $(4_+, 2_\pm)$ spinor to make the gravitino variation vanish. It can be shown that all supersymmetric fivebrane configurations are of the form given by the ansatz (5.19), (5.20).

Putting all this together, we see that if we choose the gauge field to be any instanton and fix the metric and antisymmetric tensor in terms of the dilaton according to the
above ansatz, then the state is annihilated by all supersymmetry variations generated by a spacetime constant \((4_+,2_+)\) spinor. Thus, half of the supersymmetries are unbroken, and the other half, by standard reasoning, will be associated with fermionic zero-modes bound to the soliton.

The one unresolved question concerns the functional form of the dilaton field. Notice that the ansatz for the antisymmetric tensor was given in terms of its three-form field strength \(H_{\mu\nu\rho}\), rather than its two-form potential \(B_{\mu\nu}\). This is potentially inconsistent, since the curl of the field strength must satisfy the anomalous Bianchi identity (5.3). Within the ansatz (5.20), the curl of \(H\) is given by

\[
dH = -\frac{1}{2} \Box e^{-2\phi} = \alpha' (\text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F) .
\]

where \(\Box\) is the flat space laplacian and \(*\) is the four-dimensional Hodge dual. This equation can be solved perturbatively in \(\alpha'\) beginning with the instanton solution for \(F\). Equation (5.23) then implies that \(\partial \phi \sim O(\alpha')\) which in turn implies that \(R \sim O(\alpha')\). Therefore, to leading order in \(\alpha'\), one is entitled to drop the \(R \wedge R\) term in (5.23). Substituting the explicit gauge field strength for an instanton of scale size \(\rho\), one obtains the following dilaton solution:

\[
e^{+2\phi} = e^{+2\phi_0} + 8\alpha' \left(\frac{x^2 + 2\rho^2}{(x^2 + \rho^2)^2}\right) + O(\alpha'^2) .
\]

(5.24)

The metric and antisymmetric tensor fields are built out of this dilaton field according to the spacetime-supersymmetric ansatz of (5.19) and (5.20). One can examine higher-order in \(\alpha'\) corrections to the beta functions and verify that the solution must receive corrections. At the same time, one can examine subleading corrections to the supersymmetry transformations [23] and verify that it is possible to maintain spacetime supersymmetry in the \(\alpha'\)-corrected solution. However one would like to know if there is an exact solution of string theory which agrees with this one at long distances. In fact the existence of such an exact solution can be demonstrated using low-energy spacetime supersymmetry. As these notes focus on the worldsheet point of view, we refer the reader to the original literature [18] for details.

While (5.24) is in some sense the most obvious supersymmetric fivebrane solution, there is in fact a different solution with more worldsheet supersymmetry. This extra worldsheet supersymmetry will be seen to have powerful consequences: the solution is exact without any \(\alpha'\) corrections. This symmetric solution is characterized, from the spacetime point of view, by \(dH = 0\). This condition requires, according to (5.3), that the curvature \(R(\Omega_-)\) should cancel against the instanton Yang-Mills field \(F\). We will take the instanton to be embedded in an \(SU(2)\) subgroup of the gauge group (this is always the lowest-action instanton), so what is needed is that \(\Omega_-\) be a self-dual \(SU(2)\) connection. In fact \(\Omega_-\) is manifestly an \(SU(2)\) connection, so the only issue is self-duality. Given the special ansatz and coordinate system of (5.19), it is easy to calculate the curvature of \(\Omega_\pm\).

It is then a simple arithmetical exercise to show that, in four dimensions and under the condition that \(\Box e^{2\phi} = 0\), \(\Omega\) is self-dual:

\[
R(\Omega_\pm)_{\mu\nu}^{mn} = \frac{1}{2} \epsilon_{\mu\nu}^{\lambda\sigma} R(\Omega_\pm)_{\lambda\sigma}^{mn} .
\]

(5.25)
Since a self-dual SU(2) connection is an instanton connection, it will be possible to choose a gauge instanton which exactly matches the “metric” instanton $\Omega_-$ and makes the r.h.s of (5.3) vanish, thus making the whole solution self-consistent. In the next section, we will explore the qualitative properties of the solutions which we have constructed in the above rather roundabout manner.

A feature of the above development which could cause confusion is the intricate interplay of the two non-Riemannian connections $\Omega(\pm)$. To refresh the reader’s memory, we will summarize the essentials of this phenomenon (we denote the $(4_+, 2_+)$ spinor by $\epsilon_+$): The gravitino supersymmetry variation equation boils down to $\Omega^{ab}_{\mu} \gamma_{ab} \epsilon_+ = 0$, which in turn implies that $R_{\mu\nu}(\Omega_-)^{ab}_{\gamma} \epsilon_+ = 0$. The index-pair interchange symmetry $R(\Omega_+)_{ab,cd} = R(\Omega_-)_{cd,ab}$, allows us to convert the previous condition for $\epsilon_+$ to $R_{\mu\nu}(\Omega_+)^{ab}_{\gamma} \epsilon_+ = 0$. If we then make the identification $F_{\mu\nu}^{ab} \sim R_{\mu\nu}(\Omega_+)^{ab}$, we see that we have reproduced the gaugino supersymmetry variation equation. This is simply to emphasize that, because of the crucial role of the antisymmetric tensor in these solutions, the precise way in which the $\Omega_+$’s and $\Omega_-$’s appear in the various equations we deal with is tightly constrained and quite critical.

6. Development and Interpretation of the Solutions

Now we will work out the geometry and physical interpretation of the solutions described in the previous section. We begin with a discussion of the “gauge” solution described by (5.24).

There are two charges one can associated with this solution. These are the instanton winding number

$$\nu = \frac{1}{480\pi^2} \int \text{Tr} F \wedge F,$$

(6.1)

where the integral is over a four-dimensional cross section, and the axion charge

$$Q = -\frac{1}{2\pi^2} \int H,$$

(6.2)

where the integral is over an asymptotic $S^3$ surrounding the fivebrane. The instanton winding number is quantized as usual and the minimal value is $\nu = 1$. The axion charge $Q$ is also quantized in integer multiples of $\alpha'$. The point is that, if the flux of $H$ through a non-trivial $S^3$ is non-zero, then there cannot be a unique two-form potential $B$ covering the whole sphere. The best one can do is to have two sections, $B^\pm$, covering the upper and lower halves of the $S_3$ and related to each other by a gauge transformation in an overlap region which is topologically an $S_2$. Since the sigma model action involves $B$, not $H$, the non-uniqueness of $B$ could lead to an ill-defined sigma model path integral. It is possible to show that, with our definitions of the sigma model action, this danger is avoided if and only if the flux of $H$ is an integral multiple of $\alpha'$. The details of this argument can be found in [24]. Although the minimal allowed value of $Q$ is $\alpha'$, the gauge solution in fact has $Q = 8\alpha'$ as can be verified by explicit calculation.
The form of the solution (5.24) was determined by a “low-energy” expansion with the expansion parameter being $\alpha'/\rho^2$. Solutions of scale size less than $\sqrt{\alpha'}$ can be shown to exist using arguments based on low-energy supersymmetry, but are not well approximated by the solution of (5.24).

We now turn to the “symmetric” solution. In the previous section we found that the following is a solution of the low-energy spacetime effective action of the heterotic string:

$$ds^2 = e^{2\phi(x)} \delta_{\mu\nu} dx^\mu dx^\nu + \eta_{\alpha\beta} dy^\alpha dy^\beta$$

$$H_{\mu\nu\lambda} = - \epsilon_{\mu\nu\lambda} \partial_\sigma \phi$$

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} = R_{\mu\nu}(\Omega_-)^{\mu\nu},$$

where $\mu, \nu = 6...9$, $\alpha, \beta = 0...5$ and $\eta_{\alpha\beta}$ is the Minkowski metric. The last equation expresses the fact that the gauge field is a self-dual instant on with moduli chosen so that it coincides (up to gauge transformations of course) with the curvature of the generalized connection of the theory. The consistency condition for all this is just $\Box e^{2\phi} = 0$.

The solution of the consistency condition on $\phi$ is just a constant plus a sum of poles:

$$e^{2\phi} = e^{2\phi_0} + \sum_{i=1}^{N} \frac{Q_i}{(x - x_i)^2}$$

which should be compared with the analogous expression (3.11) which appeared in the 'tHooft ansatz. The constant term is fixed by the (arbitrary) asymptotic value of the dilaton field, $\phi_0$. In string theory, $e^\phi$ is identified with the local value of the string loop coupling constant, $g_{str}$. For the solution described by (6.4), $g_{str}$ goes to a constant at spatial infinity and goes to infinity at the locations of the poles! We shall worry about the physical interpretation of this fact in due course. Now, the metric of our solution is conformally flat with conformal factor given by (6.4). Since $\phi$ goes to a constant at infinity, the geometry is asymptotically flat, which is precisely what we want for a soliton interpretation. In the neighborhood of a singularity, we can replace $e^{2\phi}$ by a simple pole $Q/r^2$ and obtain the approximate line element

$$ds^2 \sim \frac{Q}{r^2} (dr^2 + r^2 d\Omega_3^2)$$

$$= dt^2 + Q d\Omega_3^2$$

where $d\Omega_3^2$ is the line element on the unit three-sphere and we have introduced a new radial coordinate $t = \sqrt{\Omega} \log(r/\sqrt{\Omega})$. This expression becomes more and more accurate as $t \to -\infty$. In this same limit, the other fields are given by

$$\phi = -t/\sqrt{Q} \quad H = -Q \epsilon_3,$$

where $\epsilon_3$ is the volume form on the three sphere. In Sect. 5 we will see that the linear behavior of the dilaton field plays a crucial role in the underlying exact conformal field
theory. The geometry described by (6.3) is a cylinder whose cross-section is a threesphere of constant area $2\pi^2 Q$. The global geometry is that of a collection of semi-infinite cylinders, or semi-wormholes (one for each pole in $e^{+2\phi}$), glued into asymptotically flat four-dimensional space. The semi-wormholes are semi-infinite since the approximation of (6.3) becomes better and better as $t \to -\infty$ and breaks down as $t \to +\infty$. It is these semi-wormholes which we propose to interpret as solitons.

A further crucial fact is that the residues, $Q_i$, are quantized. Consider an $S_3$ which surrounds a single pole, of residue $Q$, in $e^{+2\phi}$. The net flux of $H$ through this $S_3$ is entirely due to the enclosed pole and can easily be calculated:

$$\int_{S_3} H = -2\pi^2 Q$$

The consequence for us is that the residues of the poles in $e^{2\phi}$ are discretely quantized: $Q_i = n_i\alpha'$. As a result, the cross-sectional areas of the individual semi-wormholes are quantized in units of $2\pi^2\alpha'$ and there is thus a minimal transverse scale size of the fivebrane. (This fact may be useful in future attempts to quantize the transverse fluctuations of the fivebrane.)

Finally, we want to characterize the instanton component of this solution. The key point is that, when the dilaton field satisfies $e^{2\phi} = 0$, we can construct a self-dual $SU(2)$ connection out of the scalar field $\phi$ and we want to identify this connection with the gauge instanton.

But this is simple in terms of the ansatz (3.8) and the corresponding scalar field (3.11). An important point is that the total instanton number of the solution built on $f$ of (3.11) is $N$, the number of poles. The gauge potential which follows from taking $f$ to have a single pole is

$$A_\mu = -2\rho^2 \Sigma_{\mu\nu} \frac{x^\nu}{x^2(x^2 + \rho^2)} ,$$

an expression which one immediately recognizes as the singular gauge instanton of scale size $\rho$ centered at $x = 0$. The only way this can match our construction of a self-dual generalized connection is if we make the identification

$$f(x) = e^{-2\phi_0} e^{2\phi} = 1 + \sum_{i=1}^{N} \frac{e^{-2\phi_0} Q_i}{(x - x_i)^2} .$$

Thus, given the solution (6.4) for the dilaton field, we can assert that the associated instanton has instanton number $N$, with instantons of scale size $\rho_i^2 = e^{-2\phi_0} Q_i\alpha'$ localized at positions $x_i$. Since the $Q_i$ are quantized, so are the instanton scale sizes. The only free parameters (moduli) are the $4N$ center locations of the instantons. In ten dimensions, the multiple instanton solution corresponds to multiple fivebranes with the locations in the transverse four-dimensional space of the individual fivebranes given by the center coordinates of the individual instantons.

An important fact about the solution we have just constructed is that it is not perturbative in $\alpha'$. As we saw in the discussion following (6.3), the semi-wormhole associated
with a pole of residue \( Q = n \alpha' \) has a cross-section which is a sphere of area \( 2\pi^2 Q \) and therefore has curvature \( R \sim 1/Q \). In the perturbative sigma model approach to strings in background fields, one finds that the sigma model expansion parameter is \( \alpha' R \). In the case at hand, this becomes \( \alpha'/Q = 1/n \), which is obviously not small for the elementary fivebrane, which has \( n = 1 \). Since our solution has been constructed by solving the leading-order-in-\( \alpha' \) beta function equations, ignoring all higher-order corrections, one can legitimately worry whether it makes any sense. In the next two sections we will present evidence that all corrections to this particular solution actually vanish, and the leading-order solution is exact.

There is another perturbation theory issue which should be mentioned here. String theory has two expansion parameters: the string tension \( \alpha' \) and the string loop coupling constant \( g_{\text{str}} \sim e^{\phi_0} \). The latter is the quantum expansion parameter of string theory and, in this paper, we are working to zeroth-order in an expansion in \( g_{\text{str}} \). In effect, we are producing an exact solution of classical string field theory. However, as we have already pointed out, our solution has the unusual feature that \( g_{\text{str}} \) grows without limit down the throat of a semi-wormhole so that there is, strictly speaking, no reliable classical limit! Since virtually nothing is known about non-perturbative-in-\( g_{\text{str}} \) physics, we don’t know what this means for the ultimate validity of this sort of solution. Similar issues arise in the matrix model/Liouville theory approach to two-dimensional quantum gravity, and we hope eventually to gain some insight from that source.

Another interesting point concerns what happens when we lift the requirement of spacetime supersymmetry and look for solutions of the beta function equations rather than the condition that some supersymmetry charges annihilate the solution. Our solutions have the property that the mass (the ADM mass, to be precise) per unit fivebrane area is proportional to the axion charge: \( M_5 = Q \). This equality can be understood via a Bogomolnyi bound: any solution of the leading-order field equations with the fivebrane topology must satisfy the inequality \( M_5 \geq Q \) and our solution saturates the inequality. One can easily imagine a process in which mass, but not axion charge, is increased by sending a dilaton wave down one of the semi-wormhole throats. Since the semi-wormhole throat is semi-infinite, this wave need not be reflected back: It can continue to propagate down the throat forever, leaving an exterior solution for which \( M_5 > Q \). Such solutions of the leading-order beta function equations have indeed been found [25] and they resemble the familiar Reissner-Nordstrom family of charged black holes: they have an event horizon and a singularity, but the singularity retreats to infinity as the mass is decreased to the extremal value that saturates the Bogomolny bound. Perhaps not surprisingly, the non-extremal solutions are not annihilated by any spacetime supersymmetries. Nevertheless it is possible in some cases to give exact conformal field theory constructions of these solutions [26]. These developments should eventually allow us to make progress on understanding the string physics of black holes, Hawking radiation and the like.

In the rest of these lectures, we will pursue the much more limited goal of showing that the symmetric solution is an exact solution of string theory using world-sheet arguments.

7. Worldsheets Sigma Model Approach

To show conclusively that a given spacetime configuration is a solution of string theory,
we must show that it derives from an appropriate superconformal worldsheet sigma model. In this section we will show that the worldsheet sigma models corresponding to the five-branes constructed in section 5 possess extended worldsheet supersymmetry of type $(4,4)$.

The notation derives from the fact that in a conformal field theory, the left-moving fields (functions of $z$) and the right-moving fields (functions of $\bar{z}$) are dynamically independent. It is therefore possible to have different numbers of right- and left-moving supercharges $Q^I_\pm$. The general case, referred to as $(p,q)$ supersymmetry, is described by the algebra

\[
\{Q^I_+, Q^J_-\} = 2\delta^{IJ} P_+ \quad I, J = 1 \ldots p \\
\{Q^I_-, Q^J_+\} = 2\delta^{IJ} P_- \quad I, J = 1 \ldots q \\
\{Q^I_+, Q^J_-\} = 0.
\] (7.1)

The minimal possibility, corresponding to a generic solution of the heterotic string, has $(1,0)$ supersymmetry. Any left-right-symmetric, and therefore non-anomalous theory, will have $(p,p)$ supersymmetry (this is sometimes referred to as $N = p$ supersymmetry). The maximal possibility is $(4,4)$ which, it turns out, is what is realized in our fivebrane solution. We will argue that, in the $(4,4)$ case, there is a nonrenormalization theorem which makes the lowest-order in $\alpha'$ solution for the spacetime fields exact. The latter issue is closely related to the question of finiteness of sigma models with torsion and with extended supersymmetry [27,28] and the results we find are slightly at variance with the conventional wisdom, at least as we understand it. We will comment upon this at the appropriate point.

First we digress to explain why we expect four-fold extended supersymmetry in this problem. The models of interest to us are structurally equivalent to a compactification of ten-dimensional spacetime down to six dimensions: there are six flat dimensions (along the fivebrane) described by a free field theory and four ‘compactified’ dimensions (transverse to the fivebrane) described by a nontrivial field theory. The fact that the ‘compactified’ space is not really compact has no bearing on the supersymmetry issue. The defining property of all the fivebranes of section 5 is that they are annihilated by the generators of a six-dimensional $N=1$ spacetime supersymmetry. That is, they provide a compactification to six dimensions which maintains $N=1$ spacetime supersymmetry. Now, it is well-known that in compactifications to four dimensions, the sigma model describing the six compactified dimensions must possess $(2,0)$ worldsheet supersymmetry in order for the theory to possess $N=1$ four-dimensional spacetime supersymmetry [29]. Roughly speaking, the conserved $U(1)$ current of the $(2,0)$ superconformal algebra defines a free boson which is used to construct the spacetime supersymmetry charges. It is also known that, if one wants to impose $N=2$ four-dimensional spacetime supersymmetry, the compactification sigma model must have $(4,0)$ supersymmetry [30]. The conserved $SU(2)$ currents of the $(4,0)$ superconformal algebra are precisely what are needed to construct the larger set of $N=2$ spacetime supersymmetry charges. Since, by dimensional reduction, $N=1$ supersymmetry in six dimensions is equivalent to $N=2$ in four dimensions, the above line of argument implies that spacetime supersymmetric compactifications to six dimensions (including our fivebrane) require a compactification sigma model with at least $(4,0)$ worldsheet supersymmetry. Since our solution is constructed to cancel the anomaly, it will be left-right symmetric and therefore automatically of type $(4,4)$. 

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Now we turn to a study of string sigma models. The generic sigma model underlying the heterotic string describes the dynamics of D worldsheet bosons $X^M$ and D right-moving worldsheet fermions $\psi_R^M$ (where D, typically ten, is the dimension of spacetime) plus left-moving worldsheet fermions $\lambda^a_L$ which lie in a representation of the gauge group $G$ (typically $SO(32)$ or $E_8 \otimes E_8$). The generic Lagrangian for this sigma model is written in terms of coupling functions $G_{MN}$, $B_{MN}$ and $A_M$ which eventually get interpreted as spacetime metric, antisymmetric tensor and Yang-Mills gauge fields. This Lagrangian has the explicit form

$$\frac{1}{4\pi\alpha'} \int d^2 \sigma \{ G_{MN}(X) \partial_+ X^M \partial_- X^N + 2B_{MN}(X) \partial_+ X^M \partial_- X^N$$

$$+ iG_{MN} \psi_R^M D_- \psi_R^N + i\delta_{ab} \lambda^a_L D_+ \lambda^b_L + \frac{1}{2}(F_{MN})_{ab} \psi_R^M \psi_R^N \lambda^a_L \lambda^b_L \}$$

(7.2)

where $H = dB$. In this expression, the covariant derivatives on the left-moving fermions are defined in terms of the Yang-Mills connection, while the covariant derivatives on the right-moving fermions are defined in terms of a non-riemannian connection involving the torsion (which already appeared in section 5):

$$D_- \psi_R^A = \partial_- \psi_R^A + \Omega^{-A}_{\phantom{-}B} \partial_- X^N \psi_R^B,$$

$$D_+ \lambda^a_L = \partial_+ \lambda^a_L + A^a_{\phantom{a}b} \partial_+ X^N \lambda^b_L.$$

(7.3)

We use indices of type $M$ for coordinate space indices, type $A$ for the tangent space and type $a$ for the gauge group. An absolutely crucial feature of this action is that the connection appearing in the covariant derivative of the right-moving fermions is the generalized connection $\Omega_-$, not the Christoffel connection. This action has a naive $(1,0)$ worldsheet supersymmetry and can be written in terms of $(1,0)$ superfields. Superconformal invariance is broken by anomalies of various kinds unless the coupling functions satisfy certain ‘beta function’ conditions which are equivalent to the spacetime field equations discussed in section 5. The dilaton enters these equations in a rather roundabout, but by now well-understood, way.

To proceed further, we must construct the specific sigma models corresponding to the fivebrane solutions. For the generic fivebrane, (7.2) undergoes a split into a nontrivial four-dimensional theory and a free six-dimensional theory: the sigma model metric (as opposed to the canonical general relativity metric) then describes a flat six-dimensional spacetime times four curved dimensions. The right-moving fermions couple via the kinetic term to the generalized connection $\Omega_-$, which acts only on the four right-movers lying in the tangent space orthogonal to the fivebrane. The other six right-movers are free (we momentarily ignore the four-fermi coupling) so there is a six-four split of the right-movers as well. The left-moving fermions couple to an instanton gauge field which may or may not be identified with the other generalized connection, $\Omega_+$. In all the cases of interest to us, the gauge connection is an instanton connection and acts only in some $SU(2)$ subgroup of the full gauge group, so that four of the left-movers couple nontrivially, while the other 28 are free. Finally, the four-fermion interaction term couples together precisely those left- and right-movers which couple to the nontrivial gauge and $\Omega_-$ connections and is therefore
consistent with the six-four split defined by the kinetic terms. The remaining variables can be regarded as defining a heterotic, but free, theory (6 \( X \), 6 \( \psi_R \) and 28 \( \lambda_L \)) living in the six ‘uncompactified’ dimensions along the fivebrane. From now on, we focus our attention on the nontrivial piece of (7.2) referring to the four-dimensional part of the split. For string theory consistency, it must have a central charge of 6, which would be trivially true if the connections were all flat, but is far from obvious for a fivebrane.

Now let us further specialize to the sigma model underlying the left-right symmetric (and therefore non-anomalous) fivebrane solution of section 5. It is constructed by identifying the gauge connection with the ‘other’ generalized connection \( \Omega^+ \) and making that connection self-dual by imposing the condition \( \Box e^{2\phi} = 0 \) on the metric conformal factor. The result of this is that the four bosonic coordinates transverse to the fivebrane and the four nontrivially-coupled left- and right-moving fermions are governed by the worldsheet action

\[
\frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ G_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu + 2B_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu \\
+iG_{\mu\nu} \psi_R^\mu D_- \psi_R^\nu + iG_{\mu\nu} \lambda_L^\mu D_+ \lambda_L^\nu + \frac{1}{2} R(\Omega^+)^{\mu\nu\lambda\rho} \psi_R^\mu \psi_R^\nu \lambda_L^\lambda \lambda_L^\rho \right\}
\]

(7.4)

where \( D_\pm \) are the covariant derivatives built out of the generalized connections \( \Omega^\pm \). In fact, as long as the \( H \) appearing in \( \Omega^\pm \) is given by \( dB \), (7.4) is identical to the basic left-right symmetric, \((1,1)\) supersymmetric nonlinear sigma model with torsion \([31]\). Despite the apparent asymmetry of the coupling of \( \lambda_L \) to \( \Omega^+ \) and \( \psi_R \) to \( \Omega^- \), the theory nonetheless has an overall left-right symmetry (under which \( B \rightarrow -B \)) and is non-anomalous. To exchange the roles of \( \psi_R \) and \( \lambda_L \) one has to replace the curvature of \( \Omega^- \) by that of \( \Omega^+ \). This exchange symmetry property relies on the non-riemannian relation

\[
R(\Omega^+)^{\mu\nu\lambda\rho} = R(\Omega^-)^{\lambda\rho\mu\nu}
\]

(7.5)

which indeed holds for the generalized connection (5.4) when \( dB = 0 \). To summarize, we have shown that the heterotic sigma model describing the nontrivial four-dimensional geometry of the fivebrane is actually an example of a left-right symmetric sigma model with at least \((1,1)\) supersymmetry. As we will now show, it actually has \((4,4)\) worldsheet supersymmetry.

We now turn to the question of extended supersymmetry. The basic worldsheet supersymmetry of a \((1,1)\) model like (7.4) is

\[
\delta X^M = \epsilon_L \psi_R^M + \epsilon_R \psi_L^M \\
\delta \psi_L^A + \Omega^A_{+M} B \delta X^M \psi_L^B = \partial X^A \epsilon_R + \ldots \\
\delta \psi_R^A + \Omega^A_{-M} B \delta X^M \psi_R^B = \partial X^A \epsilon_L + \ldots
\]

(7.6)

The worldsheet supersymmetry of the \((1,0)\) model is obtained by dropping the contributions of \( \epsilon_R \) and \( \psi_L \). The general structure of a possible second supersymmetry transformation is

\[
\hat{\delta} X^M = \epsilon_L f_R(X)^M_N \psi_R^N + \epsilon_R f_L(X)^M_N \psi_L^N \\
\hat{\delta} \psi_L^A + \Omega^A_{+M} B \hat{\delta} X^M \psi_L^B = - f_L(X)^A_B \partial X^B \epsilon_R + \ldots \\
\hat{\delta} \psi_R^A + \Omega^A_{-M} B \hat{\delta} X^M \psi_R^B = - f_R(X)^A_B \partial X^B \epsilon_L + \ldots
\]

(7.7)
The function $f$ is normalized and fully defined by the requirements that \{\hat{\delta}, \delta\} = 0 and that \hat{\delta} anticommute with itself to give ordinary translations as in (7.1). The question is, what conditions must $f$ satisfy in order for \hat{\delta} to be a symmetry and how many can exist?

This question was first addressed in [33] for the case of left-right symmetric theories without torsion (i.e. without an antisymmetric tensor coupling term). The more complex case of left-right symmetry with torsion was subsequently dealt with in [34, 27, 28]. The basic result is that the pair of tensors $f_{R,L}$ must be complex structures, covariantly constant with respect to the appropriate connection:

$$f^2_{\pm} = -1$$

$$D_A^\pm f_{\pm}^B = \partial_A f_{\pm}^B + \Omega^{(\pm)}_{AD} f_{\pm}^D - \Omega^{(\pm)}_{AC} f_{\pm}^B = 0,$$  \hspace{1cm} (7.8)

where the $\pm$ notation is equivalent to the $L, R$ notation. The tensors in (7.8) are written in tangent space indices which is why the generalized spin connections $\Omega^{(\pm)}$ appear in the covariant derivative. The equation could, of course, also have been written in coordinate indices. In general, it is not obvious that such a pair of complex structures can be found, but, if one can, we know that the sigma model actually possesses $(2, 2)$ worldsheet supersymmetry. A further question is whether multiple pairs $f^{(r)}_{\pm}$ of such complex structures can be found. If we can find $p - 1$ of them, then the sigma model has $(p, p)$ supersymmetry. It turns out that the only consistent possibility for multiple complex structures is that there be three of them [33] and that they satisfy the Clifford algebra

$$f^{(r)}_{\pm} f^{(s)}_{\pm} = -\delta_{rs} + \epsilon_{rst} f^{(t)}_{\pm}.$$  \hspace{1cm} (7.9)

This corresponds to the case of $(4, 4)$ supersymmetry. It is worth noting that each complex structure leads to a conserved (chiral) current:

$$J^{(r)}_{\pm} = \psi^{A}_{\pm} (f^{(r)}_{\pm})_{AB} \psi^{B}_{\pm}.$$  \hspace{1cm} (7.10)

This yields a $U(1)$ symmetry in the $(2, 2)$ case and an $SU(2)$ symmetry in the $(4, 4)$ case.

The question of left-right asymmetric theories, such as those which underlie the “gauge” fivebranes discussed in section 5, is more delicate. According to [27], a heterotic sigma model will have $(p, 0)$ supersymmetry if there are $p - 1$ complex structures $f^{(r)}_{\pm}$ which are covariantly constant under the connection which couples to the right-moving fermions (those which do not couple to the gauge field) and if the gauge field (which affects the left-moving fermions) satisfies a condition which reduces, for a four-dimensional base space, to self-duality. The latter condition is met for all of the fivebranes of interest to us since they are all built on instanton gauge fields. Thus, in all cases, the essential issue is the existence of complex structures.

To count complex structures, we will use the connection between complex structures and covariantly constant spinors (a nice pedagogical discussion can be found in [35]). We start with a spinor $\eta$ (in our case four-dimensional) of definite chirality ($\gamma_5 \eta = +\eta$, say) and unit normalized ($\eta^{\dagger} \eta = 1$). Then we define a tensor

$$J_{AB} = -i \eta^{\dagger} \gamma_{AB} \eta.$$  \hspace{1cm} (7.11)
which we will identify as a complex structure tensor (in tangent space indices and with indices raised and lowered by the identity metric). It is then automatic that if the spinor is covariantly constant with respect to some connection, so is \( J_{AB} \). A simple Fierz identity argument then shows that \( J \) squares to \(-1 \) \((J_A^B J_B^C = -\delta^C_A)\) and is indeed a complex structure.

We are now ready to construct the explicit complex structures. As was explained in the discussion after (5.22), on the fivebrane, constant spinors of definite four-dimensional chirality are covariantly constant. Using the Weyl representation for the four-dimensional gamma matrices, one has the following solutions of the two covariant constancy conditions:

\[
\mathcal{D}_\mu (\Omega_+) \epsilon_+ = 0 \Rightarrow \epsilon_+ = \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \tag{7.12}
\]

\[
\mathcal{D}_\mu (\Omega_-) \epsilon_- = 0 \Rightarrow \epsilon_- = \begin{pmatrix} 0 \\ \chi \end{pmatrix},
\]

where \( \chi \) is any constant two-spinor (which we might as well unit normalize). Since there are three parameters needed to specify the general normalized two-spinor, there should be three independent choices for the two-spinor \( \chi \) and therefore three choices for both \( \epsilon_+ \) and \( \epsilon_- \). We will define the independent \( \chi_r \) \((r = 1, 2, 3)\) as those which give expectation values of the spin operator along the three coordinate axes:

\[
\chi^\dagger_r \sigma^i \chi_r = \delta^i_r. \tag{7.13}
\]

This finally leads, with the help of (7.11), to the following set of three right- and left-handed complex structures:

\[
\begin{align*}
J^+_1 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, & J^-_1 &= \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \\
J^+_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & J^-_2 &= \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \\
J^+_3 &= \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, & J^-_3 &= \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \tag{7.14}
\end{align*}
\]

It is trivial to show that the \( J^+ \) commute with all the \( J^- \) and that they satisfy the Clifford algebra (7.9). These are precisely the conditions needed to generate (4, 4) supersymmetry in a left-right symmetric theory (or (4, 0) supersymmetry in a heterotic theory). The complex structures are thus extremely simple indeed.

Finally, we come to the questions of finiteness and need for higher-order or non-perturbative in \( \alpha' \) corrections to our solutions. It is rather firmly established that two-dimensional nonlinear sigma models with (4, 4) supersymmetry without torsion \((B_{\mu\nu} = 0)\) are in fact finite. The general proof was given quite some time ago by Alvarez-Gaume and Freedman and assumes that (4, 4) supersymmetry is not explicitly broken at the quantum level. They then show that no (4, 4)-invariant counterterms - perturbative or non-perturbative- of the needed dimension can be constructed. If the theory is finite,
the beta-functions get no higher-order corrections and the choice of background fields which made the beta functions vanish at leading order must continue to make them vanish at all orders in $\alpha'$. A similar result was shown in \cite{36} to hold, relying heavily on the results of \cite{34,27,28} for (4,4) models with torsion. (These arguments are backed up at the perturbative level by superfield non-renormalization theorems\cite{27}.) The functional form of the action must satisfy certain conditions in order to have (4,4) supersymmetry and one can see that the most general solution of these conditions corresponds precisely to our special multi-fivebrane solution.

As an aside, we mention that it has been argued that one really only needs (4,0) supersymmetry to achieve finiteness \cite{27}. This would apply to variations on the solution described in Sect. 2 in which, for example, the gauge instanton scale size did not match the semi-wormhole throat transverse scale size or to the original instanton solution (5.24). In the discussion given earlier in this section, we recall that the existence and properties of the right-moving complex structures $f_i^{(+)}$ have nothing to do with the properties of the gauge field (which governs the left-moving complex structures). So, if we keep the same metric then we should have the same $f_i^{(+)}$ and thus at least a (4,0) supersymmetry. In this case there will be corrections to the beta functions so that the theory is not finite, but may be constructible order by order, as was shown in \cite{18} for the solution (5.24) using spacetime methods. This subject has yet to be explored in any detail from the worldsheet point of view.

8. Algebraic CFT Approach

It is one thing to show that a sigma model is a superconformal field theory, as we have done in the previous section, and quite another to be able to classify its primary field content and calculate n-point functions of its vertex operators. Indeed, in order to answer all the interesting questions about string solitons, it would be desirable to have as detailed an algebraic understanding of the underlying conformal field theory as we already have for, say, the minimal models. We are far from having such an understanding, but in this section we will see that in some case useful progress can be made.

Recall from section 6 that the (four-dimensional part of the) metric of the symmetric solution has the form

$$ds^2 = e^{2\phi} dx^2$$

(8.1)

where $dx^2$ is the flat metric on $R^4$ and

$$e^{2\phi(x)} = e^{2\phi_0} + \sum_{i=1}^{n} \frac{Q_i}{(x-x_i)^2}.$$  

(8.2)

The singularities in $e^{2\phi}$ are associated with the semi-wormholes. Taking $n = 1$ and the limit $e^{2\phi_0} \to 0$ gives

$$e^{2\phi} = \frac{Q}{x^2},$$

(8.3)

which is the solution corresponding to the semi-wormhole throat itself. Using spherical coordinates centered on the singularity, and defining a logarithmic radial coordinate by
\[ t = \sqrt{Q} \ln \sqrt{x^2/Q}, \]  

the metric, dilaton and axion field strength of the throat may be written in the form

\[ ds^2 = dt^2 + Q d\Omega_3^2, \]
\[ \phi = -t/\sqrt{Q}, \]
\[ H = -Q \epsilon, \] 

(8.4)

where \( d\Omega_3^2 \) is the line element and \( \epsilon \) the volume form of the unit 3-sphere obeying \( \int \epsilon = 2\pi^2 \).

The geometry of the throat is thus a 3-sphere of radius \( \sqrt{Q} \) times the open line \( R^1 \) and the dilaton is linear in the coordinate of the \( R^1 \). Remarkably, these metric and antisymmetric tensor fields are such that the curvatures constructed from the generalized connections, defined in (5.3), are identically zero, reflecting the parallelizability of \( S^3 \). The axion charge \( Q \) is integrally quantized. So, since \( Q \) appears in the metric, the radius of the \( S^3 \) is quantized as well.

The sigma model defined by these background fields is an interesting variant of the Wess-Zumino-Witten model and the underlying conformal field theory can, it turns out, be analyzed in complete detail. The basic observation along these lines was made in [37] in the lorentzian context and euclideanized in [38,39]: the \( S^3 \) and the antisymmetric tensor field are equivalent to the \( O(3) \) Wess-Zumino-Witten model of level

\[ k = Q/\alpha', \] 

(8.5)

while the \( R^1 \) and the linear dilaton define a Feigin-Fuks-like free field theory with a background charge induced by the linear dilaton. Both systems are conformal field theories of known central charges:

\[ c_{wzw} = \frac{3k}{k+2}, \quad c_{ff} = 1 + \frac{6}{k}. \] 

(8.6)

The shift of the \( R^1 \) central charge away from unity is a familiar background charge effect which has been exploited in constructions of the minimal models [40] and in cosmological solutions [37].

For the combined theory to make sense, the net central charge must be four. Let us for the moment consider the bosonic string. If we expand \( c_{wzw} \) in powers of \( k^{-1} \) (this corresponds to the usual perturbative expansion in powers of \( \alpha' \)), we see instead that

\[ c_{tot} = c_{wzw} + c_{ff} = 4 + O(k^{-2} \sim \alpha'^2). \] 

(8.7)

But, we should not have expected to do any better: the field equations we solved in section 5 to get this solution are only the leading order in \( \alpha' \) approximation to the full bosonic string theory field equations and we must expect higher-order corrections to the fields and central charges. In fact, this issue can be studied in detail and it can be shown [41] that the metric and antisymmetric tensor fields are not modified and that the only modification of the dilaton is to adjust the background charge of the \( R^1 \) (i.e. the coefficient of the linear term in \( \phi \)) so as to maintain \( c_{tot} \) exactly equal to four.

While this is quite interesting, we are really interested in the superstring case. The leading-order-in-\( \alpha' \) metric, dilaton etc. fields are the same as in the bosonic case (and,
because of the non-renormalization theorems, we expect no corrections to them) but various fermionic terms are added to the previous purely bosonic sigma model. The structure is that of the (1,1) worldsheet supersymmetric sigma model (7.4) discussed in section 7. There is still an $S^3 \times R^1$ split, but the component theories are supersymmetrized versions of Wess-Zumino-Witten and Feigin-Fuks. The Feigin-Fuks theory is still essentially free. In the supersymmetric WZW theory, the four-fermi terms vanish identically because, as pointed out above, the generalized curvature vanishes for this background. As a consequence, the generalized connections are locally pure gauge and can be eliminated from the fermion kinetic terms by a gauge rotation of the frame field. Since the fermions are effectively free, they make a trivial addition to the central charges of both the $S^3$ and the $R^1$ models:

$$c_{wzw} = \frac{3k}{k+2} + \frac{3}{2}, \quad c_{ff} = 1 + \frac{6}{k} + \frac{1}{2}.$$  \hspace{1cm} (8.8)

There is, however, a small subtlety: the gauge rotation which decouples the fermions is chiral, and therefore anomalous, because the left- and right-moving fermions couple to two different pure gauge generalized connections, $\Omega_+$ and $\Omega_-$. The entire effect of this anomaly on the central charge turns out to be the replacement in $c_{wzw}$ of $k$ by $k - 2$ (the details can be found in [12]) with the result that

$$c_{wzw} = \frac{3(k - 2)}{k} + \frac{3}{2}, \quad c_{tot} = c_{wzw} + c_{ff} = 6.$$  \hspace{1cm} (8.9)

Six is, of course, exactly the value we want for the central charge. The remarkable fact is that, in the supersymmetric theory, the expansion of $c_{wzw}$ in powers of $k^{-1}$ terminates at first non-trivial order and no modification of the dilaton field is needed to maintain the desired central charge of six. These results are consistent with the non-renormalization theorems discussed in section 7, but are not tied to perturbation theory, since they derive from exactly-solved conformal field theories. On the other hand, since the present discussion makes no reference to the (4,4) supersymmetry which was crucial in proving the perturbative non-renormalization theorems of section 7, an important element is still missing.

This is a good point to remind the reader of the hierarchy of superconformal algebras. Much of what we know about conformal field theory comes from studying the representation theory of these algebras. The basic N=1 superconformal algebra contains an energy-momentum tensor $T(z)$ and its superpartner $G(z)$. The essential information is contained in the algebra obeyed by their Laurent coefficients $L_n$ and $G_r$:

$$[L_m, L_n] = \frac{\hat{c}}{8} m(m^2 - 1) \delta_{m+n,0} + (m-n)L_{m+n},$$

$$\{G_r, G_s\} = \frac{\hat{c}}{2} (r^2 - \frac{1}{4}) \delta_{r+s,0} + 2L_{r+s},$$

$$[L_m, G_r] = (\frac{m}{2} - r) G_{m+r}$$ \hspace{1cm} (8.10)

with $\hat{c} = 2c/3$ in terms of the usual conformal anomaly. All superstring theories have at least this much worldsheet supersymmetry. The N=2 superconformal algebras differ from
this by having a conserved current $J(z)$ and *two* supercharges $G^\pm(z)$ distinguished by the value $(\pm 1)$ of their charge with respect to the current $J(z)$. This charge also plays a key role in the GSO projection which rids the theory of tachyons. The important new algebraic relations are contained in the commutation relations

\[
[J_m, G^\pm_r] = \pm G^\pm_r,
\]

\[
\{G^+_r, G^+_s\} = \{G^-_r, G^-_s\} = 0,
\]

\[
\{G^+_r, G^-_s\} = L_{r+s} + \frac{1}{2}(r-s)J_{r+s} + \frac{\hat{c}}{4}(r^2 - \frac{1}{4})\delta_{r+s}
\]

There is an $N=1$ subalgebra generated by $T(z)$ and $G(z) = \frac{1}{\sqrt{2}}(G^+(z) + G^-(z))$. The ‘practical’ utility of the $N=2$ algebra is that the conserved current defines a free field $H$ by the relation $J(z) = i\sqrt{\frac{c}{3}}\partial_z H(z)$ and this free field can be used to construct the $N=1$ *space-time* supersymmetry charge in a compactification to four dimensions [29]. One further extension, to four supercharges, turns out to be possible. There are now three conserved currents $J^i$ which generate an $SU(2)$ Kac-Moody algebra and the supercharges $G^\alpha_r(z), \bar{G}^\alpha_r(z)$ are in $I = 1/2$ representations of the conserved $SU(2)$. The relevant (anti-) commutation relations are

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{k}{2}m(m^2 - 1)\delta_{m+n,0},
\]

\[
\{G^\alpha_r, G^\beta_s\} = \{\bar{G}^\alpha_r, \bar{G}^\beta_s\} = 0,
\]

\[
\{G^\alpha_r, G^\beta_s\} = 2\delta^{\alpha\beta} L_{r+s} - 2(r-s)\sigma^\alpha J^i_{r+s} + \frac{k}{2} (4r^2 - 1)\delta_{r+s,0},
\]

\[
[J^i_m, J^j_n] = i\epsilon^{ijk} J^k_{m+n} + \frac{1}{2}km\delta_{m+n,0}\delta_{ij},
\]

\[
[J^i_m, G^\alpha_r] = -\frac{1}{2}\sigma^i_{\alpha\beta} G^\beta_{m+r},
\]

\[
[J^i_m, \bar{G}^\alpha_r] = \frac{1}{2}\bar{\sigma}^i_{\alpha\beta} \bar{G}^\beta_{m+r}
\]

with $\sigma^i$ the usual Pauli matrices and $\bar{\sigma}^i$ their complex conjugates.

The triplet of conserved charges is what is needed to construct the larger space-time supersymmetry algebra associated with a compactification down to six, rather than four, dimensions. The $SU(2)$ Kac-Moody algebra is of arbitrary level $k$, but we can see by comparison with (8.10) that the central charge $c$ is constrained to be $6k$. Since the level is constrained by unitarity to be integer, the only allowed values of the central charge are $6, 12, \ldots$. Fortunately, $c = 6$ is just what we need, and this suggests that the $N=4$ algebra will be important for us.

We will now show that a closer examination of the algebraic structure of the throat conformal field theory reveals the existence of just the right extended supersymmetry. An important clue to understanding the structure of the $(4, 4)$ superconformal symmetry comes from the fact that there must be *two* $SU(2)$ Kac-Moody symmetries: The first is part of the standard $N=4$ superalgebra. This algebra contains the energy-momentum
tensor $T(z)$, four supercurrents $G^a(z)$ and three currents $J^i(z)$ of conformal weight 1, which generate an $SU(2)$ Kac-Moody algebra of a level tied to the conformal anomaly (in our case, level one). The second is the $SU(2)$ Kac-Moody algebra of the Wess-Zumino-Witten part of the throat conformal field theory. It has a general level $n$, related to the area of the throat cross-section (or, equivalently, its axion charge) and is clearly distinct from the N=4 $SU(2)$ Kac-Moody. Since the superconformal algebra is quite tightly constrained, it is not a priori obvious that such an $SU(2) \otimes SU(2)$ Kac-Moody is compatible with N=4 supersymmetry and useful information, such as restrictions on allowed values of the central charge, might be obtained by explicitly constructing the algebra (assuming a consistent one to exist). Quite remarkably, precisely the algebra we need has already been constructed by Schoutens in [43]. A closely related version of this algebra was presented in Sevrin et. al. [44], as an alternate N=4 superalgebra, containing an $SU(2) \otimes SU(2) \otimes U(1)$ Kac-Moody algebra, which had been missed in previous attempts at a general classification of extended superalgebras. Similar results also appear in [45]. In what follows* we briefly summarize enough of this work to explain its significance for the throat problem and, in particular, to verify the assertions made in section 7 about the rôle of (4,4) supersymmetry. In addition to establishing the presence of a (4,4) superconformal symmetry, this construction is a useful starting point for studying the correspondence between the instanton moduli space and perturbations of the superconformal field theory.

The construction discovered by Sevrin et al. goes as follows: Start with the bosonic WZW model for an $SU(2) \otimes U(1)$ group manifold (this is the geometry of the throat if we let the radius of the $U(1)$ be infinite). The conformal model contains four dimension-one Kac-Moody currents, $J^i \ i = 1, 2, 3$ which satisfy the Kac-Moody algebra of $SU(2)$ with level $n$ and an additional $U(1)$ current $J^0$ which satisfies the $U(1)$ algebra with level one. This is supersymmetrized by adding a set of four dimension-1/2 fields $\psi^a$ satisfying the free fermion algebra (this is motivated by the arguments given earlier in this section that the fermions in a supersymmetric WZW model are, modulo anomalies, free).

As usual, the Sugawara construction provides an energy-momentum tensor

$$T(z) = -J^0 J^0 - \frac{1}{n+2} J^i J^i - \partial \psi^a \psi^a$$

with respect to which the fields $J^a \ (\psi^a)$ are primaries of weight 1 (1/2) and which has the expected $S_{wzw}$ conformal anomaly

$$c_{s_{wzw}} = \frac{3n}{n+2} + 3 = 6(n+1)/(n+2).$$

There is also a Sugawara-like construction of four real supersymmetry charges $G^a$, with $a = 0,..,3$:

$$G^0 = 2[J^0 \psi^0 + (1/\sqrt{n+2}) J^i \psi^i + (2/\sqrt{n+2}) \psi^1 \psi^2 \psi^3]$$

$$G^1 = 2[J^0 \psi^1 + (1/\sqrt{n+2})(-J^1 \psi^0 + J^2 \psi^3 - J^3 \psi^2) - (2/\sqrt{n+2}) \psi^0 \psi^2 \psi^3]$$

* This discussion was developed in collaboration with E. Martinec.
These supercharges could have been packaged as a complex $I = 1/2$ multiplet, as in (8.12). The operator product expansion of these supercharges with themselves reads

$$G^a(z)G^b(w) = 4 \frac{(n + 1)}{(n + 2)} \delta^{ab}(z - w)^{-3} + 2 \delta^{ab} T(w)(z - w)^{-1}$$

$$- 8 \left[ \frac{1}{n + 2} \alpha_{ab}^{+i} A_i^+(w) + \frac{n + 1}{n + 2} \alpha_{ab}^{-i} A_i^-(w) \right] (z - w)^{-2}$$

$$- 4 \left[ \frac{1}{n + 2} \alpha_{ab}^{+i} \partial A_i^+(w) + \frac{n + 1}{n + 2} \alpha_{ab}^{-i} \partial A_i^-(w) \right] (z - w)^{-1}$$

where

$$\alpha_{ab}^{\pm i} = \pm \delta_{[a} \delta_{b]}^0 + \frac{1}{2} \epsilon_{iab}$$

and

$$A_i^- = \psi^0 \psi^i + \epsilon_{ijk} \psi^j \psi^k \quad A_i^+ = - \psi^0 \psi^i + \epsilon_{ijk} \psi^j \psi^k + J^i$$

are commuting $SU(2)$ Kac-Moody algebras of levels 1 and $n+1$, respectively. The c-number term (the central charge) and the term involving $T(z)$ are obligatory in any higher-N superalgebra, while the terms involving dimension 1 operators are what differentiate the various possible extended superalgebras. With further effort, one shows that the $G \cdot A^\pm$ OPE generates combinations of $G^a$ and $\psi^a$ while the $G \cdot \psi$ OPE yields $A^\pm$ and $J^0$. No new operators appear in further iterations, so the complete algebra generated by the supercharges contains just $T$ (dimension 2), $G^a$ (dimension 3/2), $A_i^\pm$ and $J^0$ (dimension 1) and $\psi^a$ (dimension 1/2). The Kac-Moody algebra defined by the dimension 1 operators is evidently $SU(2) \times SU(2) \times U(1)$, which accords with our expectations derived from the throat geometry.

The superalgebra whose construction we have outlined above is a particular example of a one-parameter family of N=4 algebras dubbed the $A_\gamma$ algebras. The only problem with it is that the sigma model analysis of extended supersymmetry (see for example [28]) makes quite clear that the canonically defined supercharges and energy-momentum tensor must satisfy the standard N=4 algebra, which closes on $T$, $G^a$ and a single level-one $SU(2)$ Kac-Moody algebra $J^i$. The supercharges defined above obviously do not have that property. However, if we ‘improve’ them as follows

$$\tilde{T} = T - \frac{1}{\sqrt{n + 2}} \partial J^0 \quad \tilde{G}^a = G^a - \frac{1}{\sqrt{n + 2}} \partial \psi^a$$

we can show that $\tilde{T}$, $\tilde{G}^a$ and $A_i^\pm$ (the level-one Kac-Moody current) close on themselves and enjoy precisely the standard N=4 superalgebra. This says that the full algebra has the standard algebra as a subalgebra, perhaps no great surprise.

This improvement has a simple physical interpretation: $J^0$ generates a $U(1)$ symmetry which can be regarded as a translation in a free coordinate $\rho$ (that is, we can write $J(z) \sim \partial_z \rho(z)$ where $\rho$ is a free scalar field). The original algebra (8.16) makes no reference to the dilaton and corresponds physically to a constant dilaton field. It is well-known
that, if one turns on a dilaton which is linear in a free coordinate $\rho$, this has the effect of adding a term proportional to $\partial_z^2 \phi \sim \partial_z J^0(z)$ to $T(z)$ and shifting the central charge of the superconformal algebra by a constant. With a little care we can show that the linear dilaton implicit in (8.1) is precisely what we obtained earlier in this section in our discussion of the WZW-Feigin-Fuks conformal field theory of the throat. This is a further piece of evidence that the improved energy-momentum tensor $\tilde{T}$ is the physically relevant one. Now comes the miracle: $T$ is, in any event, not physically acceptable because it has a central charge of $6(n+1)/(n+2)$. The central charge of $\tilde{T}$, however, can easily be shown to be 6, precisely the required value!

This shows that there is an exact conformal field theory of just the right central charge associated with the throat geometry and verifies the key rôle of N=4 extended supersymmetry in establishing the physics of the model. There are many fivebrane applications of this exact throat conformal field theory which are just beginning to be worked out. Perhaps the most interesting concern the vertex operators of excitations about the wormhole throat, among which one must find the moduli of the exact solutions. In any event, this line of argument shows that the dramatic consequences of (4,4) superconformal symmetry, which we first extracted from perturbative considerations, seem to have nonperturbative status.

9. Other Supersymmetric Solitons

For completeness we will briefly mention in this section some of the other supersymmetric string solitons which have been discussed in the literature. These solitons are mainly understood from a spacetime point of view; a worldsheet analysis of the type discussed in this paper remains to be performed.

In [46,47], supersymmetric one-brane solutions were found. The construction of this solution involves an intriguing interplay between spacetime and worldsheet methods. These “solutions” do not solve the equations of motion following from (5.1) in the usual sense, rather there is a singularity at the origin corresponding to the presence of a zero-thickness fundamental string. The form of this singularity is then precisely dictated by the known coupling of spacetime vertex operators to the string worldsheet.

In [5] a one-brane solution of heterotic string theory was found which is an everywhere smooth solution of the equations of motion of (5.1). The construction of this solution involves crucially the properties of octonions. One of the many bizarre features of this soliton is that it breaks 15 of the 16 supersymmetries of ten-dimensional Minkowski space, in contrast to previously known examples of supersymmetric solitons which all break half of the supersymmetries. Clearly this is the odd duck in the family of supersymmetric string solitons. The fact that this solution does not have finite energy per unit length further complicates its physical implications.

The fivebrane solutions discussed in these lectures are characterized by a nonvanishing integral of the three-form field strength $H$, while the abovementioned string solutions carry a non-zero integral of the dual of $H$. In type II strings, there are a variety of $d$-form field strengths. These fields are not present in the heterotic string because they are created by Ramond-Ramond vertex operators. For each field there exists a $p$-brane solution.
carrying the corresponding charge or its dual. A conformal field theoretic description of these solutions is a challenging and interesting problem because of the non-trivial Ramond-Ramond backgrounds.

The possibility has often been contemplated that our universe is in fact a three-brane embedded in a higher-dimensional space. The difficulty with this idea is the opposite of the difficulty associated with Kaluza-Klein compactification: there are too few solutions rather than too many, and none with the right properties have been found. In particular, although it is easy to obtain scalar and spin 1/2 zero modes on such a membrane, it has not been possible until recently to find theories with higher spin zero modes. In [18] it was shown that there do exist five-brane solutions of type II string theory which possess not only spin zero and spin 1/2 zero modes but also spin one zero modes. Unfortunately, the construction presented there also suggests that spin two zero modes cannot be found by the same mechanism. As a perhaps more realistic example, the $p$-brane solutions of [25] do in fact contain a three-brane solution [18]. It is associated with the self-dual five-form of the chiral IIb theory. The effective four-dimensional theory has extended $N = 4$ supersymmetry. This may be a useful model for exploring this alternate method of getting rid of extra dimensions.

Finally there is the issue of solitons in the context of string compactification to four dimensions [17,19]. This is a rich subject–too rich to review here. One hope is that ‘realistic’ string solitons may have peculiar enough properties that their detection could serve as a ‘smoking gun’ for string theory. For example in [50] it is argued that the fractional charge carried by certain string solitons could serve this purpose. Of course direct detection of string solitons is generally difficult because they are typically ultramassive, but perhaps there are surprises in store.

10. Conclusion

In these lectures, we have constructed a special set of conformal field theories which have the interpretation of soliton solutions of heterotic string theory. We first constructed them as solutions of the leading order in $\alpha'$ beta function conditions and then showed that, owing to an extended worldsheet supersymmetry, the associated nonlinear sigma model is an exact conformal field theory. It is the existence of an explicit and exact conformal field theory associated with the soliton solution which distinguishes the solution described here from previous attempts to construct string theory solitons. There are several lines of inquiry which can be pursued now that “exact” string solitons exist. One issue concerns the mass of the soliton. In all previous discussions of string solitons, the mass has been computed using the lowest-order spacetime effective action (5.1). It would obviously be desirable to have a purely conformal field theoretic definition of the mass—perhaps in terms of some correlation functions. Our exact soliton conformal field theory should provide a useful context for addressing this question. A second issue is the question of stringy collective coordinates and their semiclassical quantization. It should be an instructive challenge to translate the well-known standard field theory physics of collective coordinates into the string theory context. This is a nontrivial matter because motion in collective coordinate space becomes motion in a space of conformal field theories and it
is a nontrivial matter to find the action associated with such motions (and knowing the exact underlying conformal field theories should help). Yet another question to pursue is that of stringy black hole physics. We noted that our solitons were similar to the extreme Reissner-Nordstrom black holes in the sense that, while they have no singularity or event horizon, if one increases their mass by any finite amount (while keeping the axion charge fixed), an event horizon and a singularity (lying at a finite geodesic distance from any finite point) will appear. Such black hole solitons can easily be created by scattering some external particle on an extremal soliton and, by studying stringy scattering theory about the extremal soliton, one should be able to explore, in a controlled way, how a stringy black hole Hawking radiates and the nature of the final state it approaches. These are quite difficult questions, but having precise control of the underlying conformal field theory may allow us to make progress on them. Perhaps it will be possible to report on progress along these lines at the next Trieste School.

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