Scheme for demonstrating Bell theorem in tripartite entanglement between atomic ensembles

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We propose an experimentally feasible scheme to demonstrate quantum nonlocality, using Greenberger-Horne-Zeilinger (GHZ) and $W$ entanglement between atomic ensembles generated by a new developed method based on laser manipulation and single-photon detection.

PACS number(s): 03.65.Ud, 03.67.-a, 42.50.Gy

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I. INTRODUCTION

Quantum entanglement, which is so contradictory to our intuition, maybe the most fundamental feature of quantum mechanics. From the paper of Einstein, Podolsky, and Rosen (EPR) in 1935 [1], quantum entanglement has been carefully investigated. It has found wide applications in the demonstration of quantum nonlocality and quantum information processing, such as quantum teleportation [4] and quantum cryptography [3]. Greenberger, Horne, and Zeilinger have shown that quantum mechanical predictions for certain measurement results on three entangled particles are in conflict with local realism in cases where quantum theory makes definite predictions, whereas as for EPR state with two entangled particles, the conflict with local realism only results from statistical predictions [4]. Actually, there are only two classes of inequivalent tripartite entanglements under local operation assisted by classical communication (LOCC) [3]: one is GHZ state, expressed as

\[ \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle); \]  

the other is W state, expressed as

\[ \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle). \]  

The experiments to observe photons’ GHZ state and demonstrate quantum nonlocality by it have been accomplished [6,7]. In these experiments, the subsystem of the entanglement is single photon.

Recently, a new method to generate entanglement between atomic ensembles has been developed [8,9,10,11]. In this paper we adopt this method to generate GHZ and W state and devise experimental schemes to demonstrate quantum nonlocality using the generated entanglement states. In contrast to the existing schemes, the one described here takes advantage of the virtues of entanglement between atomic ensembles, such as long lived time and more robust resilience to realistic imperfection..
II. GHZ STATE

The basic element of our scheme is an ensemble of many identical alkali atoms, which can be experimentally realized as either a room-temperature atomic gas \([12,13]\) or a sample of cold trapped atoms \([14,15]\). The relevant level structure of the atom is shown in Fig. 1.

![Figure 1](image)

A pair of metastable states \(|g⟩\) and \(|s⟩\) can correspond to—for example—hyperfine or Zeeman sublevels of the atom. From the two levels \(|g⟩\) and \(|s⟩\) we can define a collective atomic operator \(S = (1/\sqrt{N_a}) \sum_i |g⟩_i ⟨s|\) where \(N_a \gg 1\) is the total atom number. The ensemble ground state can be expressed as \(|0_a⟩ = \Pi_i |g⟩_i\). Here we use the same symbols as in Ref. \([8]\).

The process of the preparation scheme in Ref. \([8]\) is simply described as follows: The ensembles L and R are initially prepared in the ground state and then excited respectively by a short Raman pulse applied to the transition \(|g⟩ → |e⟩\). The Raman pulse is so weak that the forward–scattered Stokes light from the transition \(|e⟩ → |s⟩\) has a mean number much smaller than 1. The forward–scattered Stokes lights from the two ensembles are then interfered at a beam splitter and further detected by two single-photon detectors. In the cases where only one detector register a click, we can not distinguish from which ensemble this registered photon comes; due to this indistinguishability, the projected state of the ensembles L and R is nearly maximally entangled, with the form

\[
|Ψ_{LR}⟩ = \frac{1}{\sqrt{2}} (S_{L}^† + e^{iφ} S_{R}^†)|vac⟩, \tag{3}
\]

where \(φ\) is an unknown phase difference fixed by the optical channel connecting the L and R ensembles, and \(|vac⟩ = |0_a⟩_L |0_a⟩_R\).

We can generate GHZ state by three pairs of such atomic ensembles. The whole state can be described by

\[
|Ψ⟩ = 3 \prod_{i=1}^3 \frac{1}{\sqrt{2}} (S_{L_i}^† + e^{iφ_i} S_{R_i}^†)|vac⟩, \tag{4}
\]
where \(|\text{vac}\rangle\) denote the vacuum of the whole six ensembles. In the expansion of the state (4), there are only two components which have one excitation in each pair. This component state is given by

\[
|\Psi_{\text{GHZ}}\rangle = (1/\sqrt{2})\left(\prod_{i=1}^{3} S_{L_i}^\dagger + e^{i\phi_r} \prod_{i=1}^{3} S_{R_i}^\dagger\right) |\text{vac}\rangle,
\]

with \(\phi_r = \phi_1 + \phi_2 + \phi_3\), which is exactly the three-party GHZ maximal entangled states in the ‘polarization’ basis. In the system, by applying retrieval pulses of suitable polarization that are near-resonant with the atomic transition \(|s\rangle \rightarrow |e\rangle\), we can simultaneously convert the stored atomic excitations into light, and by using single-bit rotations, such as Hadamard transformations, and the number detection through single-photon detectors, we can generate the GHZ state which is practical for experiment. The method described above is similar to the scheme proposed in Ref. [10] to entangle many atomic ensembles, but there are some differences in detail.

The efficiency of our scheme can be described by the total time needed to register the effective GHZ state. Through the similar analysis to Duan’s [11], we can know the total time for registering the GHZ state is \(T \sim 4t_0/(1 - \eta)^2\), where \(t_0\) is the preparation time of state (3), and \(\eta\) describe the overall loss probability of photon detectors.

Now we can use the entanglement state generated above to test the quantum nonlocality. The scheme we present here is similar to the scheme in Ref. [7] in principle, but because we use a different entanglement source, which is a ‘polarization’ maximal entangled (PME) [8] state between atomic ensembles, we should adjust the measurement manners. The diagram of our set-up is shown in Fig. 2.

**Figure 2**

Analogous to polarization entanglement, we can write the generated photons’ GHZ state as

\[
|\Psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|L_1\rangle|L_2\rangle|L_3\rangle + |R_1\rangle|R_2\rangle|R_3\rangle),
\]

where \(|L_i\rangle\) denotes that a photon is emitted from the atomic ensemble \(L_i\), and \(|R_i\rangle\) denotes that a photon is emitted from the atomic ensemble \(R_{i+1}\) here we assume the notation \(3+1\equiv 1\).
for the subscripts. It is obvious that the three photons are in a quantum superposition of the state \(|L_1\rangle|L_2\rangle|L_3\rangle\) (all three photons are emitted from the atomic ensembles marked by L), and the state \(|R_1\rangle|R_2\rangle|R_3\rangle\) (all three photons are emitted from the atomic ensembles marked by R), so none of the three photons has a well defined state of its own.

We can also consider measurements of other bases which can be expressed as

\[
|L'\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle),
\]

\[
|R'\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle);
\]

\[
|L''\rangle = \frac{1}{\sqrt{2}}(|L\rangle + i|R\rangle),
\]

\[
|R''\rangle = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle).
\]

For convenience we will refer to a measurement of \(L'/R'\) as a \(\sigma_x\) measurement, and one of \(L''/R''\) as a \(\sigma_y\) measurement.

It has been shown that the demonstration of the conflict between quantum mechanics and local realism consists of four experiments, each with three spatially separated ‘polarization’ measurements [7]. First, we perform \(\sigma_y\sigma_y\sigma_x\), \(\sigma_y\sigma_x\sigma_y\) and \(\sigma_x\sigma_y\sigma_y\) experiments. If we obtained the results predicted for the GHZ state, then for a \(\sigma_x\sigma_x\sigma_x\) experiment, our consequent expectations according to local-realism are exactly the opposite of the expectations according to quantum mechanics [4]. In our set-up shown by Fig. 2, when none phase plate is installed we implement a \(\sigma_x\) measurement, and when a \(\frac{\pi}{2}\) phase plate is installed, we implement a \(\sigma_y\) measurement. By observing which detector register a photon, we can know the result of a measurement. So we can test the quantum nonlocality in the same steps as in Ref. [7].

### III. \(W\) State

In the following we propose a similar method to generate \(W\) state, which is also based on the preparation of the state (3). The method is divided into three steps as follows:
First, we prepare three pairs of atomic ensembles in the state (3)

$$|Ψ_i⟩ = \frac{1}{\sqrt{2}}(S^\dagger_{B_i} + e^{iφ_i}S^\dagger_{C_i})|vac⟩ \quad (i = 1, 2, 3).$$ (8)

Second, we prepare non-PME $W$ state between three atomic ensembles denoted by $A_1$, $A_2$ and $A_3$. The set-up is shown in Fig. 3.

![Figure 3](image)

**Figure 3**

All the atoms of the three ensembles are initially prepared in the ground state $|g⟩$. The three ensembles are put in a line, and illuminated by a short, off-resonant laser pulse that induces Raman transition into the state $|s⟩$. Behind the third ensemble, we put a filter which can eliminate the pump laser pulse from the forward-scattered Stokes photon, and a single-photon detector to detect the Stokes photon. The set-up is shown in Fig. 3. Because in the free space the velocity of the Stokes light is close to $c$, so the delay between the Stokes photons emitted from different ensembles is much smaller than the pulse width. Therefore when we register only one click in the detector, we can not distinguish which ensemble this registered photon comes from, so we obtain the state

$$|Ψ_A⟩ = \frac{1}{\sqrt{3}}(S^\dagger_{A_1} + e^{iφ_{A_2}}S^\dagger_{A_2} + e^{iφ_{A_3}}S^\dagger_{A_3})|vac⟩,$$ (9)

where $|vac⟩$ denote the vacuum state of the three ensembles. We should notice that $|Ψ_A⟩$ is entangled in Fock basis which is experimentally hard to do certain single-bit operations. The whole state of the nine ensembles can be described by

$$|Ψ⟩ = |Ψ_1⟩|Ψ_2⟩|Ψ_3⟩|Ψ_A⟩.$$ (10)

Third, in the expansion of the state (10), there are only three components which have one excitation in each pair of $A_i$ and $B_i$. This component state is given by

$$|Ψ_W⟩ = \frac{1}{\sqrt{3}}(e^{iφ_1}S^\dagger_{A_1}S^\dagger_{B_2}S^\dagger_{B_3}S^\dagger_{C_1} + e^{i(φ_{A_2}+φ_2)}S^\dagger_{A_2}S^\dagger_{B_1}S^\dagger_{B_3}S^\dagger_{C_2} + e^{i(φ_{A_3}+φ_3)}S^\dagger_{A_3}S^\dagger_{B_1}S^\dagger_{B_2}S^\dagger_{C_3}).$$ (11)

Using the set-up in Fig. 4, we apply retrieval pulses to ensembles $A_i$ and $B_i$ ($i = 1, 2, 3$) and register only the coincidence of the three party, i.e. there is one and only one click on
each party, through the postselection techniques an experimentally practical $W$ state can be obtained.

**Figure 4**

If the state $|\Psi_i\rangle$ ($i = 1, 2, 3$) and $|\Psi_A\rangle$ can be prepared independently at the same time, the total preparation time of state $|\Psi_W\rangle$ is $4 \max(t_0, t_1)/(1 - \eta)^3$, where $t_0$ is the preparation time of $|\Psi_i\rangle$, and $t_1$ is the preparation time of state $|\Psi_A\rangle$. In fact, we can easily know that when atomic ensembles are illuminated by pumping light, the excitation probability of state $|\Psi_i\rangle$ and state $|\Psi_A\rangle$ are at the same order, as well as $t_0$ and $t_1$.

Now let we see how to demonstrate quantum nonlocality by $W$ state. Here we use the same notations as in Ref. [16]: $z_i$ and $x_i$ will be the results ($-1$ or $1$) of $\sigma_z$ and $\sigma_x$ measurements on the party $i$ ($i = 1, 2, 3$). By applying retrieval pulses, if a Stokes photon was detected by the detector $D^i_A$, we assume $x_i = 1$, and if it is detected by the detector $D^i_B$ ($i = 1, 2, 3$), we assume $x_i = -1$. We can implement a $\sigma_z$ measurement on party $i$ by applying a retrieval pulse to the ensemble $C_i$, and if a Stokes photon is detected, i.e. the ensemble has an excitation, we assume $z_i = 1$, if not we assume $z_i = -1$.

According to the rules described above, we can easily check the following three properties experimentally.

$$P(z_i = -1, z_j = -1) = 1,$$
$$P(x_j = x_k | z_i = -1) = 1,$$
$$P(x_i = x_k | z_j = -1) = 1,$$

where $P(z_i = -1, z_j = -1)$ means the probability of two parties (although we cannot tell which two) giving the result $-1$ when implementing $\sigma_z$ measurements on all three parties, and $P(x_j = x_k | z_i = -1)$ is the conditional probability of two $\sigma_x$ measurements on parties $i$ and $j$ having the same results given that the result of a $\sigma_z$ measurement on party $k$ is $-1$ ($i \neq j \neq k$).

Based on EPR's local realism, we can deduced from the properties (12) that $P(x_i = x_j = x_k) = 1$, but according to quantum mechanics $P(x_i = x_j = x_k) = \frac{3}{4}$, where $P(x_i = x_j = x_k)$
means the probability of three separated $\sigma_z$ measurements giving the same results. So we can demonstrate quantum nonlocality with $W$ state. We can also consider Mermin’s inequality with tripartite entanglement \cite{17}

$$-2 \leq \langle a_1a_2a_3 \rangle - \langle a_1b_2b_3 \rangle - \langle b_1a_2b_3 \rangle - \langle b_1b_2a_3 \rangle \leq 2,$$

where $a_i$ and $b_i$ are observables of qubit $i$. By choosing $a_i = \sigma_{z_i}$ and $b_i = \sigma_{x_i}$, we can observe the violation of the inequality by experiments.

IV. CONCLUSION

In summary we have proposed an experimental scheme to generate GHZ and $W$ state between macroscopic atomic ensembles, and demonstrate quantum nonlocality by the generated entanglement state. With the current technology, we can realize the scheme which benefits much from the important property of built-in entanglement purification, furthermore, all we use in the scheme are linear optical elements, which make it easy to manipulate \cite{8}. In addition, our scheme can be easily generalized to entangle more than three atomic ensembles or only two ensembles.

Acknowledgments

This work was funded by the National Fundamental Research Program (2001CB309300), the Innovation Funds from Chinese Academy of Sciences, and also by the outstanding Ph. D thesis award and the CAS’s talented scientist award rewarded to Lu-Ming Duan.
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**Figure Caption.**

**Figure 1.** The relevant level structure of the atoms in the ensemble, with $|g\rangle$, the ground state, $|s\rangle$, the metastable state for storing a qubit, and $|e\rangle$, the excited state.

**Figure 2.** Set-up for generating GHZ state between atomic ensembles and demonstrating quantum nonlocality.

**Figure 3.** Set-up for generating non-PME $W$ state between atomic ensembles.

**Figure 4.** Set-up for generating PME $W$ state between atomic ensembles and demonstrating quantum nonlocality.
Figure 1
Figure 2
Figure 3
Figure 4