The One-Loop One-Mass Hexagon Integral in $D = 6$ Dimensions

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**Abstract:** We evaluate analytically the one-loop one-mass hexagon in six dimensions. The result is given in terms of standard polylogarithms of uniform transcendental weight three.

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1. Introduction

One-loop scalar $n$-point integrals in $D > 4$ dimensions are interesting objects. Via dimensional shifting, they occur in the $O(\epsilon)$ part of the one-loop $n$-point MHV amplitude in $D = 4 - 2\epsilon$ dimensions [1, 2, 3], and connect scalar integrals in $(D + 2)$ dimensions to tensor integrals in $D$ dimensions [4]. Furthermore, it has been noted that they can be related to higher-loop integrals [5, 6]. If the number of dimensions matches the number of points, they feature dual conformal invariance [7], which strongly constrains their form. An example is given by the recently computed one-loop scalar massless hexagon integral in $D = 6$ dimensions [8, 9], whose structure is strikingly similar to the one of the remainder function of two-loop amplitudes and Wilson loops [10, 11, 12, 13, 14]. In this contribution, we evaluate analytically the one-loop one-mass hexagon integral in $D = 6$ dimensions. The computation is made possible by use of the symbol map [14], a certain tensor calculus that allows us to resolve the functional identities among polylogarithms. Using the algorithm of Ref. [15], the symbol of the one-loop one-mass hexagon integral in $D = 6$ dimensions is then integrated to obtain the analytic expression for the integral.

2. The one-loop one-mass hexagon integral

Let us consider a scalar one-loop one-mass integral in $D = 6$ dimensions,

$$I^{D=6}_{0,m} = \int \frac{d^6k}{i\pi^3} \prod_{i=0}^{5} \frac{1}{D_i},$$

with

$$D_0 = k^2 \quad \text{and} \quad D_i = \left( k + \sum_{j=1}^{i} p_j \right)^2, \quad \text{for} \quad i = 1, \ldots, 5,$$

where we have chosen the first momentum as spacelike. Then the mass shell conditions are $p_1^2 = m^2 < 0$, and $p_i^2 = 0$, with $i = 2, \ldots, 6$. The momenta are taken all ingoing, such that momentum conservation reads

$$\sum_{i=1}^{6} p_i = 0.$$  \hspace{1cm} (2.3)

We consider the integral in Euclidean kinematics where all Mandelstam invariants are taken to be negative, $(p_1 + \ldots + p_7)^2 < 0$, and the integral is real. The one-mass hexagon integral is finite in $D = 6$ dimension, so that no regularization is required. We introduce dual coordinates [7, 16, 17],

$$p_i = x_i - x_{i+1},$$

with $x_7 = x_1$, due to momentum conservation.

Since the integration measure in Eq. (2.1) is translation invariant, we can define $k = x_0 - x_1$ and the integral can be rewritten in terms of dual coordinates,

$$I^{D=6}_{0,m} = \int \frac{d^6x_0}{i\pi^3} \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2 x_{05}^2 x_{06}^2},$$

(2.5)
with $x^2_{ij} = (x_i - x_j)^2 = (p_i + \ldots + p_{j-1})^2$. The mass shell conditions become $x^2_{12} = m^2$ and $x^2_{23} = x^2_{34} = x^2_{45} = x^2_{56} = x^2_{61} = 0$. The integral (2.5) is invariant under a $\mathbb{Z}_2$ symmetry that maps the dual variables as follows,

$$x_1 \leftrightarrow x_2 , \quad x_3 \leftrightarrow x_6 , \quad x_4 \leftrightarrow x_5 .$$

(2.6)

In Ref. [7] the notion of dual conformal invariance was introduced, i.e., the action of the conformal group on the dual coordinates $x^i$. The integral (2.5) transforms covariantly under dual conformal transformations. A direct consequence of the dual conformal covariance is that $I^{D=6}_{6,m}$ can only depend on dual conformal cross ratios, up to an overall prefactor which carries the conformal weights. For the one-mass six-point kinematics, there are four independent cross ratios, given in terms of dual coordinates by

$$u_1 = \frac{x^2_{26} x^2_{35}}{x^2_{25} x^2_{36}} , \quad u_2 = \frac{x^2_{13} x^2_{46}}{x^2_{26} x^2_{14}} , \quad u_3 = \frac{x^2_{15} x^2_{24}}{x^2_{14} x^2_{25}} , \quad u_4 = \frac{x^2_{12} x^2_{36}}{x^2_{13} x^2_{26}} .$$

(2.7)

Under the $\mathbb{Z}_2$ symmetry (2.6), the cross ratios $u_1$ and $u_2$ are exchanged, while $u_3$ and $u_4$ stay invariant. In terms of the cross ratios (2.7), $I^{D=6}_{6,m}$ can be written as

$$I^{D=6}_{6,m} = \frac{1}{x^2_{14} x^2_{25} x^2_{36}} \mathcal{I}_{6,m}(u_1, u_2, u_3, u_4) ,$$

(2.8)

where the function $\mathcal{I}_{6,m}$ is manifestly dual conformal invariant,

$$\mathcal{I}_{6,m}(u_1, u_2, u_3, u_4) = \frac{1}{\sqrt{\Delta}} C(u_1, u_2, u_3, u_4) ,$$

(2.9)

with

$$\Delta = (u_1 + u_2 + u_3 - u_1 u_2 u_4 - 1)^2 - 4u_1 u_2 u_3 (1 - u_4) .$$

(2.10)

Note that $u_4$ vanishes in the massless limit $x^2_{12} \to 0$, and $\mathcal{I}_{6,m}$ is reduced to the massless function $\mathcal{I}_6$ defined in Refs. [8, 9].

It is easy to derive a Feynman parameter representation for the one-loop one-mass hexagon integral in six dimensions,

$$I^{D=6}_{6,m} = \int_0^\infty \left( \prod_{i=1}^6 d\alpha_i \right) \delta \left( 1 - \sum_{k \in S} \alpha_k \right) \frac{2}{\mathcal{F}_{6,m}(\alpha_1, \ldots, \alpha_6)^3} ,$$

(2.11)

where $\mathcal{F}_{6,m}$ is defined as

$$\mathcal{F}_{6,m}(\alpha_1, \ldots, \alpha_6) = \sum_{i,j=1 \atop i < j}^6 \alpha_i \alpha_j (-x^2_{ij}) ,$$

(2.12)

and $S$ denotes a subset of $\{1, \ldots, 6\}$. A theorem by Cheng and Wu [18] then guarantees that the Feynman integral is independent of the choice of $S$. In the following we choose $S = \{6\}$, i.e., we freeze the integration variable $\alpha_6$ to 1. The integrations over $\alpha_4$ and
\(\alpha_5\) are now trivially performed, leaving us only with a conformally invariant integral to compute,

\[
\mathcal{I}_{6,m}(u_1, u_2, u_3, u_4) = \int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(u_2 + \alpha_1 + \alpha_2)(u_4 \alpha_1 + u_1 \alpha_3 + \alpha_2)(u_4 \alpha_1 \alpha_2 + \alpha_2 + \alpha_1 \alpha_3 + \alpha_3)}.
\]

(2.13)

This integral can easily be performed in terms of multiple polylogarithms, leaving us with a rather complicated combination of multiple polylogarithms of weight three. The remarkable simplicity of the massless one-loop hexagon integral in six dimensions [8, 9] however suggests that it should be possible to rewrite the result in a much simpler form, a form hidden behind a plethora of complicated functional identities among multiple polylogarithms. These functional identities can be resolved by using the symbol map [14] which we review in the next section.

3. The symbol map

The cornerstone of the simplification of the two-loop six-point remainder function [10, 11, 12, 13] is the introduction of the symbol map [14], a linear map \(S\) that associates a certain tensor to an iterated integral, and thus to a multiple polylogarithm. As an example, the tensor associated to the classical polylogarithm \(\text{Li}_n(x)\) is,

\[
S(\text{Li}_n(x)) = -(1 - x) \otimes x \otimes \ldots \otimes x.
\]

(3.1)

Furthermore, the tensor maps products that appear inside the tensor product to a sum of tensors,

\[
\ldots \otimes (x \cdot y) \otimes \ldots = \ldots \otimes x \otimes \ldots + \ldots \otimes y \otimes \ldots.
\]

(3.2)

It is conjectured that all the functional identities among (multiple) polylogarithms are mapped under the symbol map \(S\) to algebraic relations among the tensors. Hence, if the symbol map is applied to our expression for \(\mathcal{I}_{6,m}(u_1, u_2, u_3)\), it should capture and resolve all the functional identities among the polylogarithms, and therefore allow us to rewrite the result in a simpler form.

Even though deriving the symbol of the one-loop one-mass hexagon is a rather simple exercise, integrating the symbol back to a function can be much more involved. This can however be achieved by using the algorithm developed in Ref. [15], which, after a suitable choice has been made for the functions that should appear in the answer, allows us to reduce the problem of integrating the symbol to a problem of linear algebra. However, in order to apply this algorithm it is important that all the arguments that enter the tensor be multiplicatively independent. As in our case the arguments of the polylogarithms involve square roots of \(\Delta\), this requirement would not be fulfilled. We may remedy this situation by parametrizing the cross ratios (2.7) as

\[
u_1 = \frac{1}{1 - y}, \quad u_2 = \frac{v}{v - u}, \quad u_3 = \frac{(1 - u)(y - x)}{(1 - y)(u - v)}, \quad u_4 = \frac{v - x}{v}.
\]

(3.3)
such that
\[ \Delta = \frac{(ux - y)^2}{(1-y)^2(u-v)^2}. \]  
(3.4)

We note in passing that the Jacobian of the parametrization (3.3) is non zero for generic values of the parameters.

In a nutshell, the algorithm of Ref. [15] proceeds in two steps:

1. Given the symbol \( S(\mathcal{C}) \) of the one-loop one-mass hexagon, construct a set of rational functions \( \{ R_i(u,v,x,y) \} \) such that, e.g., symbols of the form \( S(\text{Li}_n(R_i(u,v,x,y))) \) span the vector space which \( \mathcal{S}(\mathcal{C}) \) is an element of.

2. Once a suitable set of rational functions has been obtained, make an ansatz

\[
\bar{C}(u,v,x,y) = \sum_i c_i \text{Li}_3(R_i(u,v,x,y)) + \sum_{i,j} c_{ij} \text{Li}_2(R_i(u,v,x,y)) \ln R_j(u,v,x,y) \\
+ \sum_{i,j,k} c_{ijk} \ln R_i(u,v,x,y) \ln R_j(u,v,x,y) \ln R_k(u,v,x,y),
\]
(3.5)

where the \( c_i, c_{ij} \) and \( c_{ijk} \) are rational numbers to be determined such that

\[ \mathcal{S}(\bar{C}) = \mathcal{S}(\mathcal{C}). \]
(3.6)

As the objects appearing in this last equation are tensors (i.e., elements of a vector space), the coefficients \( c_i, c_{ij} \) and \( c_{ijk} \) can equally well be seen as coordinates in a vector space, and the problem of finding the coefficients reduces to a problem of linear algebra.

We have implemented the algorithm of Ref. [15] into a MATHEMATICA code, which we have applied to the function \( C(u,v,x,y) \). The result is discussed in the next section.

4. The one-mass hexagon revealed

We have found that in the regions where \( \Delta \) is negative or where all the \( u \)'s are smaller than 1, we can write the function (2.9) as

\[
\mathcal{I}_{6,m}(u_1,u_2,u_3,u_4) = \frac{1}{\sqrt{\Delta}} \left[ -\frac{8}{3} \sum_{i,j=1}^{2} \left( L_3(x_{ij}^+,x_{ij}^-) - \frac{1}{6} \ell_1(x_{ij}^+,x_{ij}^-)^3 - \frac{\pi^2}{6} \ell_1(x_{ij}^+,x_{ij}^-) \right) \\
+ \frac{1}{3} \left( \ell_1(x_{2,1}^+,x_{2,1}^-) + \ell_1(x_{2,2}^+,x_{2,2}^-) \right) \left( 2\ell_1(x_{1,1}^+,x_{1,1}^-) \ell_1(x_{1,2}^+,x_{1,2}^-) + \ell_1(x_{1,1}^+,x_{1,2}^-) \ell_1(x_{3,1}^+,x_{3,1}^-) + \ell_1(x_{1,2}^+,x_{1,1}^-) \ell_1(x_{3,1}^+,x_{3,2}^-) + \ell_1(x_{1,2}^+,x_{1,1}^-) \ell_1(x_{3,2}^+,x_{3,1}^-) \right) \\
+ \ell_1(x_{1,2}^+,x_{1,1}^-) \ell_1(x_{3,2}^+,x_{3,2}^-) + 2\ell_1(x_{3,1}^+,x_{3,1}^-) \ell_1(x_{3,2}^+,x_{3,2}^-) \right]\],
\]
(4.1)
where
\[
L_3(x^+, x^-) = \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!!} \ln^k(x^+ x^-) \left( \ell_{3-k}(x^+) - \ell_{3-k}(x^-) \right),
\]
\[
\ell_n(x) = \frac{1}{2} \left( \text{Li}_n(x) - (-1)^n \text{Li}_n(1/x) \right),
\]
and
\[
\bar{\ell}_n(x^+, x^-) = \ell_n(x^+) - \ell_n(x^-).
\]
In order to define the arguments of the logarithmic functions, we introduce the variables
\[
x_{1m}^\pm = \frac{u_1 + u_2 + u_3 - u_1 u_2 u_4 - 1 \pm \sqrt{\Delta}}{2u_1 u_2 u_3 (1 - u_4)},
\]
\[
\chi^\pm = 2u_1 u_2 u_3 (1 - u_4) x_{1m}^\pm.
\]
As functions of the cross ratios, the arguments of the logarithmic functions are,
\[
x_{1,1}^\pm(u_1, u_2, u_3, u_4) = u_3 x_{1m}^\pm,
\]
\[
x_{2,1}^\pm(u_1, u_2, u_3, u_4) = \frac{(1 - u_3) \chi^\pm - 2u_1 u_2 u_3 u_4}{2u_3 u_3 (1 - u_3 - u_1 u_4)},
\]
\[
x_{3,1}^\pm(u_1, u_2, u_3, u_4) = \frac{\chi^\pm}{2u_2 u_3},
\]
\[
x_{4,1}^\pm(u_1, u_2, u_3, u_4) = \frac{u_4 (\chi^\pm (1 - u_1 u_4) - 2u_1 u_3 (1 - u_4))}{2(1 - u_4)(1 - u_3 - u_1 u_4)},
\]
\[
x_{5,1}^\pm(u_1, u_2, u_3, u_4) = \frac{\chi^\pm - 2u_1 (1 - u_4)}{2u_1 u_4 (1 - u_2)},
\]
\[
x_{6,1}^\pm(u_1, u_2, u_3, u_4) = \frac{\chi^\pm - 2u_1 u_2 (1 - u_4)}{2u_1 (1 - u_2)(1 - u_4)},
\]
\[
x_{7,1}^\pm(u_1, u_2, u_3, u_4) = \frac{(1 - u_1 u_4) \chi^\pm - 2u_1 u_3 (1 - u_4)}{2u_3 (1 - u_1)},
\]
\[
x_{8,1}^\pm(u_1, u_2, u_3, u_4) = \frac{\chi^\pm - 2u_1 u_3}{2u_1 (1 - u_3 - u_2 u_4)},
\]
and \(x_{i,2}^\pm(u_1, u_2, u_3, u_4)\) are defined from \(x_{i,1}^\pm(u_1, u_2, u_3, u_4)\) by exchanging \(u_1\) and \(u_2\),
\[
x_{i,2}^\pm(u_1, u_2, u_3, u_4) = x_{i,1}^\pm(u_2, u_1, u_3, u_4), \quad i = 1, \ldots, 8.
\]
Hence, under the \(\mathbb{Z}_2\) symmetry, \(x_{i,1}^+ \leftrightarrow x_{i,2}^-\), thus making Eq. (4.1) manifestly symmetric. Furthermore, in the massless limit \(u_4 \to 0\), we obtain
\[
x_{i,j}^\pm \to x_1^\pm, \quad x_{2,1}^\pm, x_{3,1}^\pm \to x_2^\pm, \quad x_{2,2}, x_{3,2} \to x_3^\pm,
\]
\[
x_{4,j} \to 0, \quad x_{6,j}^\pm \to \infty, \quad x_{6,1}^\pm \to 1/x_{6,2}^\pm, \quad x_{7,j}^\pm \to 1/x_{8,j}^\pm,
\]
with \(j = 1, 2\), and where the massless hexagon variables are
\[
x_i^\pm = u_i x_{0m}^\pm, \quad i = 1, 2, 3, \quad x_{0m}^\pm = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta_0}}{2u_1 u_2 u_3},
\]
with $\Delta_0 = (u_1 + u_2 + u_3 - 1)^2 - 4u_1 u_2 u_3$. Thus the terms which depend only on $x_{i,j}^\pm$, with $i = 1, 2, 3$, immediately reproduce the massless hexagon. Therefore the contributions from the other variables must vanish. To see that this is the case, we note that

$$L_3(x^+, x^-) = -L_3(x^-, x^+) = L_3\left(\frac{1}{x^+}, \frac{1}{x^-}\right),$$

$$\ell_n(x) = (-1)^{n+1} \ell_n\left(\frac{1}{x}\right). \quad (4.9)$$

Let us take for example the terms in Eq. (4.1) which depend only on $x_{6,j}^\pm$,

$$L_3(x_{6,1}^+, x_{6,1}^-) - \frac{1}{6}(\ell_1(x_{6,1}^+) - \ell_1(x_{6,1}^-))^3 - \frac{\pi^2}{6}(\ell_1(x_{6,1}^+) - \ell_1(x_{6,1}^-))$$

$$+ L_3(x_{6,2}^+, x_{6,2}^-) - \frac{1}{6}(\ell_1(x_{6,2}^+) - \ell_1(x_{6,2}^-))^3 - \frac{\pi^2}{6}(\ell_1(x_{6,2}^+) - \ell_1(x_{6,2}^-)) \quad (4.10)$$

In the massless limit, this becomes

$$L_3(x_{6,1}^+, x_{6,1}^-) - \frac{1}{6}(\ell_1(x_{6,1}^+) - \ell_1(x_{6,1}^-))^3 - \frac{\pi^2}{6}(\ell_1(x_{6,1}^+) - \ell_1(x_{6,1}^-))$$

$$+ L_3\left(\frac{1}{x_{6,1}}, \frac{1}{x_{6,1}}\right) - \frac{1}{6} \left(\ell_1\left(\frac{1}{x_{6,1}}\right) - \ell_1\left(\frac{1}{x_{6,1}}\right)\right)^3 - \frac{\pi^2}{6} \left(\ell_1\left(\frac{1}{x_{6,1}}\right) - \ell_1\left(\frac{1}{x_{6,1}}\right)\right) = 0. \quad (4.11)$$

The same reasoning shows that the terms depending on $x_{7,j}^\pm$ and $x_{8,j}^\pm$ cancel each other. The terms depending on $x_{4,j}^\pm$ and $x_{5,j}^\pm$ are slightly more subtle, because the functions $\ell_n(x)$ are divergent when $x$ approaches either 0 or $\infty$. Let us concentrate on $x_{4,j}^\pm$. Using the inversion formulae for the polylogarithms,

$$\text{Li}_1(x) = \text{Li}_1(1/x) + \ln(-x),$$

$$\text{Li}_2(x) = -\text{Li}_2(1/x) - \frac{1}{2} \ln^2(-x) - \frac{\pi^2}{6}, \quad (4.12)$$

$$\text{Li}_3(x) = \text{Li}_3(1/x) - \frac{1}{6} \ln^3(-x) - \frac{\pi^2}{6} \ln(-x),$$

we can write the $\ell_n$ functions in the form,

$$\ell_1(x) = \text{Li}_1(x) + \frac{1}{2} \ln(-x),$$

$$\ell_2(x) = \text{Li}_2(x) + \frac{1}{4} \ln^2(-x) + \frac{\pi^2}{12}, \quad (4.13)$$

$$\ell_3(x) = \text{Li}_3(x) + \frac{1}{12} \ln^3(-x) + \frac{\pi^2}{12} \ln(-x).$$

In the limit $x \to 0$, the $\ell_n$ function splits into two pieces, a polylogarithmic piece that vanishes powerlike and a logarithmically divergent piece. A little algebra then shows that the logarithms conspire in such a way that

$$\lim_{u_i \to 0} \left(L_3(x_{4,j}^+, x_{4,j}^-) - \frac{1}{6}(\ell_1(x_{4,j}^+) - \ell_1(x_{4,j}^-))^3 - \frac{\pi^2}{6}(\ell_1(x_{4,j}^+) - \ell_1(x_{4,j}^-))\right) = 0. \quad (4.14)$$
The same reasoning of course applies to $x_5^\pm_j$. Thus, in the massless limit $u_4 \to 0$, Eq. (4.1) reduces to the massless hexagon [8, 9].

We stress that Eq. (4.1) is valid only in the regions where $\Delta < 0$, or where all the $u$’s are smaller than 1. Outside those regions, the analytic structure of Eq. (4.1) seems to be more complicated. We plan to study that in the near future.

5. Conclusions

In this paper, we have continued the exploration undertaken in Ref. [8], and we have computed analytically the one-loop one-mass hexagon integral in $D = 6$ dimensions. Just like for the massless hexagon integral, the result is given in terms of standard polylogarithms of uniform transcendental weight three, and in the massless limit it reduces manifestly to the massless hexagon. The similarity in structure between the massless and one-mass hexagons, coupled with the similarity between the one-loop massless hexagon in $D = 6$ dimensions and the remainder function of the two-loop hexagon Wilson loops and amplitudes in $D = 4$ dimensions, points to a similar simple structure for hexagons with more masses in $D = 6$ dimensions and for Wilson loops and amplitudes with 7 or more points in $D = 4$ dimensions. This is left for future work.

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References

[1] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, “One loop selfdual and N=4 superYang-Mills,” Phys. Lett. B 394 (1997) 105 [hep-th/9611127].

[2] V. Del Duca, C. Duhr, E. W. Nigel Glover and V. A. Smirnov, “The One-loop pentagon to higher orders in epsilon,” JHEP 1001, 042 (2010) [arXiv:0905.0097 [hep-th]].

[3] B. A. Kniehl and O. V. Tarasov, “Analytic result for the one-loop scalar pentagon integral with massless propagators,” Nucl. Phys. B 833 (2010) 298 [arXiv:1001.3848 [hep-th]].

[4] Z. Bern, L. J. Dixon and D. A. Kosower, “Dimensionally regulated pentagon integrals,” Nucl. Phys. B 412, 751 (1994) [hep-ph/9306240].

[5] J. M. Drummond, J. M. Henn and J. Trnka, “New differential equations for on-shell loop integrals,” JHEP 1104 (2011) 083 [arXiv:1010.3679 [hep-th]].

[6] C. Anastasiou and A. Banfi, “Loop lessons from Wilson loops in N=4 supersymmetric Yang-Mills theory,” JHEP 1102, 064 (2011) [arXiv:1101.4118 [hep-th]].
[7] J. M. Drummond, J. Henn, V. A. Smirnov and E. Sokatchev, “Magic identities for conformal four-point integrals,” JHEP 0701 (2007) 064 [hep-th/0607160].

[8] V. Del Duca, C. Duhr and V. A. Smirnov, “The massless hexagon integral in D = 6 dimensions,” arXiv:1104.2781 [hep-th].

[9] L. J. Dixon, J. M. Drummond and J. M. Henn, “The one-loop six-dimensional hexagon integral and its relation to MHV amplitudes in N=4 SYM,” arXiv:1104.2787 [hep-th].

[10] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, “The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Phys. Rev. D 78, 045007 (2008) [arXiv:0803.1465 [hep-th]].

[11] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, “Hexagon Wilson loop = six-gluon MHV amplitude,” Nucl. Phys. B 815, 142 (2009) [arXiv:0803.1466 [hep-th]].

[12] V. Del Duca, C. Duhr and V. A. Smirnov, “An Analytic Result for the Two-Loop Hexagon Wilson Loop in N = 4 SYM,” JHEP 1003 (2010) 099 [arXiv:0911.5332 [hep-ph]].

[13] V. Del Duca, C. Duhr and V. A. Smirnov, “The Two-Loop Hexagon Wilson Loop in N = 4 SYM,” JHEP 1005 (2010) 084 [arXiv:1003.1702 [hep-th]].

[14] A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, “Classical Polylogarithms for Amplitudes and Wilson Loops,” Phys. Rev. Lett. 105 (2010) 151605 [arXiv:1006.5703 [hep-th]].

[15] C. Duhr, H. Gangl and J. Rhodes, “Symbol calculus for polylogarithms and Feynman integrals,” in preparation.

[16] A. V. Kotikov, “Differential equation method: The Calculation of N point Feynman diagrams,” Phys. Lett. B 267, 123 (1991).

[17] L. F. Alday and J. M. Maldacena, “Gluon scattering amplitudes at strong coupling,” JHEP 0706, 064 (2007) [arXiv:0705.0303 [hep-th]].

[18] H. Cheng and T. T. Wu, “Expanding Protons: Scattering at High-Energies,” Cambridge, USA: MIT-PR. (1987) 285 p.