Non-commutative graphs and quantum error correction for a two-mode quantum oscillator

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Abstract

An important topic in quantum information is the theory of error correction codes. Practical situations often involve quantum systems with states in an infinite dimensional Hilbert space, for example coherent states. Motivated by these practical needs, we apply the theory of non-commutative graphs, which is a tool to analyze error correction codes, to infinite dimensional Hilbert spaces. As an explicit example, a family of non-commutative graphs associated with the Schrödinger equation describing the dynamics of a two-mode quantum oscillator is constructed and maximal quantum anticliques for these graphs are found.

Keywords: quantum error correction, non-commutative graphs, quantum anticliques, quantum oscillator, two-mode field, coherent states.

1 Introduction

An important topic in quantum information is the theory of error correction codes [1–6] used to protect quantum information from decoherence and other
noises. The information is encoded in the density matrix $\rho$ of the system. Given a quantum channel $\Phi$ (i.e., a completely positive trace preserving map) which describes information transmission from sender to receiver and includes noise and other destructive for information transmission effects, an error correction code is a set of states in $\mathcal{H}$ which can be exactly distinguished after a transmission via the channel $\Phi$. Quantum error correcting codes are actively studied theoretically [7–12] and experimentally [13,14]. They can rely on encoding the information in finite-dimensional states of a quantum system, as was originally considered for example in [2], or in infinite-dimensional states as for example using states of a quantum oscillator [7,12].

Every quantum channel has a Kraus decomposition $\Phi(\rho) = \sum_j V_j \rho V_j^*$ with some operators $V_j$. The crucial role in finding an appropriate error correction code is played by the linear space $\mathcal{V}$ spanned by the operators $V_j^*V_k$. The famous Knill–Laflamme condition claims that a zero error transmission via channel $\Phi$ is possible iff $P\mathcal{V}P = \mathcal{C}P$ for the orthogonal projection $P$ on the subspace generated by an error correction code [3]. Throughout this paper we call $\mathcal{V}$ a non-commutative graph.

In [15–17] the study of non-commutative graphs generated by covariant resolutions of identity was initiated for finite-dimensional Hilbert spaces. The cases of the commutative circle group [16] and the non-commutative Heisenberg–Weyl group [17] were considered. In this work we consider non-commutative graphs for an infinite dimensional space $\mathcal{H}$. An example of a non-commutative operator graph generated by the dynamics of diatomic molecule, or coupled oscillators, is explicitly constructed and its maximal anti-clique is found.

The paper is organized as follows. Section 2 is devoted to the introduction of basic definitions. In Section 3.1 the action of the unitary group used for a generation of graph is obtained in the explicit form. In Section 3.2 a family of graphs generated by this group is constructed and in Section 3.3 their maximal quantum anticliques are found. In the Conclusion Section we give some conclusive remarks.

2 Non-commutative graphs generated by orbits of unitary groups

Definition 1. A non-commutative graph is a linear subspace $\mathcal{V}$ of bounded operators in a Hilbert space $\mathcal{H}$ possessing the properties

- $V \in \mathcal{V}$ implies that $V^* \in \mathcal{V}$;
- $I \in \mathcal{V}$
Such objects were introduced in [18] as operator systems and recently re-defined in quantum information theory under the name of a non-commutative graph [19].

In finite-dimensional space, an operator graph $V$ can be generated by a unitary representation $g \rightarrow U_g$ in a Hilbert space $H$ of a compact group $G$ with the Haar measure $\mu$ as

$$V = \text{span}\{U_gQU_g^*, g \in G\},$$

Here $Q$ is an orthogonal projection such that

$$\int_G U_gQU_g^*d\mu(g) = I$$

(2)

to guarantee that $I \in V$. At the moment, only finite dimensional case $\dim H < +\infty$ has been studied in this framework. Below we consider the infinite dimensional case $\dim H = +\infty$.

Suppose that the dynamics of the quantum system is determined by the Schrödinger equation

$$i\psi_t = H\psi.$$  

(3)

Then instead of (1) and (2), we shall search for a linear subspace $V$ and a set of orthogonal projections $Q_\beta$ with a parameter $\beta$ belonging to some set $\mathcal{B}$ such that

$$V = \text{span}\{U_tQ_\beta U_t^*, t \in \mathbb{R}, \beta \in \mathcal{B}\},$$

(4)

$I \in V$,  

(5)

where $(U_t = e^{-itH}, t \in \mathbb{R})$ is a strong continuous unitary group. Since $H$ has infinite dimension, we should take a closure in (4) of the linear span of $U_tQ_\beta U_t^*$ to guarantee that it generates a graph $V$. In the finite dimensional case (1) taking closure is not necessary.

Following to [21], we give the following definition.

**Definition 2.** An orthogonal projection $P$ such that rank $P \geq 2$ is a quantum anticlique for a non-commutative graph $V$ if it satisfies:

$$\dim PVP = 1.$$  

(6)

Using the spectral order on the set of Hermitian operators ($A > B$ iff $A - B > 0$) we can give the following notion for considering a maximal error correction code.

**Definition 3.** A quantum anticlique $P$ for a non-commutative graph $V$ is called maximal if there does not exist a quantum anticlique $\hat{P}$ for $V$ such that $P < \hat{P}$.
In the next section we consider a two-mode quantum oscillator with the Hilbert space $H = L^2(\mathbb{R}^2) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. We show that the inclusion of the identity operator in $V$ can take place for a two-mode quantum oscillator described by the Hamiltonian

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{(q_2 - q_1)^2}{2},$$  \hspace{1cm} (7)$$

where $q_j$ and $p_j$ are the position and momentum operators for $j$th oscillator, $j = 1, 2$. The Hamiltonian (7) is associated with the dynamics of a diatomic molecule. The corresponding operators $Q$ are found and the associated operator graphs are described in detail. Maximal quantum anticliques for these operator graphs are found. For this system the infinite dimensional projection $P$ is generated by one-dimensional entangled projections, while $Q$ is a linear envelope of separable projections.

## 3 Operator graphs and maximal anticliques for a pair of coupled quantum oscillators

### 3.1 Explicit action of the group $U_t$

We begin by deriving an explicit action of the unitary group $U_t$.

Equation (3) in the coordinate representation has the form

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2} \frac{\partial^2 \psi}{\partial y^2} + \frac{(x - y)^2 \psi}{2}. \hspace{1cm} (8)$$

Making change of the variables $\tilde{x} = x + y$ and $\tilde{y} = x - y$ transforms this equation into

$$i \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial \tilde{x}^2} - \frac{\partial^2 \psi}{\partial \tilde{y}^2} + \frac{\tilde{y}^2 \psi}{2}. \hspace{1cm} (9)$$

Here the operator

$$-\frac{\partial^2}{\partial \tilde{y}^2} + \frac{1}{2} \tilde{y}^2$$

is the Hamiltonian of a one-dimensional oscillator with mass $m = 1/2$ and frequency $\omega = \sqrt{2}$. Its eigenstates and the corresponding eigenvalues are

$$\psi_n(\tilde{y}) = \frac{1}{\sqrt{2^n n!(\sqrt{2} \pi)^{1/4}}} H_n \left( \frac{\tilde{y}}{\sqrt{2}} \right) e^{-\frac{\tilde{y}^2}{2}}, \hspace{1cm} \lambda_n = \sqrt{2} \left( \frac{1}{2} + n \right),$$

where

$$H_n(\tilde{y}) = (-1)^n e^{\tilde{y}^2} \frac{d}{d\tilde{y}^n} \left( e^{-\tilde{y}^2} \right).$$

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are Hermite polynomials, \( n = 0, 1, 2, \ldots \).

By representing a solution to (9) in the form 
\[
\psi(t, \tilde{x}, \tilde{y}) = \sum_{n=0}^{\infty} c_n(t, \tilde{x}) \psi_n(\tilde{y})
\]
we obtain the following Cauchy problem for the coefficients \( c_n(t, \tilde{x}) = \langle \psi_n(\cdot), \psi(t, \tilde{x}, \cdot) \rangle \):
\[
\begin{align*}

i \frac{\partial c_n(t, \tilde{x})}{\partial t} &= - \frac{\partial^2 c_n(t, \tilde{x})}{\partial \tilde{x}^2} + \frac{2n + 1}{\sqrt{2}} c_n(t, \tilde{x}) \\

\end{align*}
\]

Its solution is given by the Fresnel integral
\[
c_n(t, \tilde{x}) = \frac{1}{2\sqrt{i\pi t}} e^{-it(2n+1)/\sqrt{8}} \int_{-\infty}^{+\infty} c_n(0, v) dv
\]

Thus the solution to (8) is given by the formula
\[
(U_t \psi)(x, y) = \sum_{n=0}^{\infty} 2^{n+1/2} n! \sqrt{2 \pi} e^{-ixy/2t} H_n(x/\sqrt{2}) e^{-y^2/2(1+2it)}
\]

**Lemma 1.** The functions \( f_n(y) = \frac{1}{\sqrt{n!2\pi}} H_n(y) e^{-y^2/2} \) satisfy the following equality
\[
\int_{-\infty}^{+\infty} e^{-ixy/\sqrt{2t}} f_n(y) dy = \sqrt{4\pi ti} \int_{-\infty}^{+\infty} e^{-iy+x^2/(2t)} H_n(x/\sqrt{1+2ti}) e^{-2tx^2/(1+2ti)}
\]
It follows that \[ \frac{d}{dx} K = K \frac{d}{dx}. \]

On the other hand, \[
x K(x, y) = -2ti K(x, y) \frac{d}{dy} + K(x, y)y,
\]
\[
K(x, y)y = 2ti \frac{d}{dx} K(x, y) + x K(x, y).
\]

Hence, for the creation operator \( a^+ = \frac{x - \frac{d}{dx}}{\sqrt{2}} \) we get
\[ K a^+ = \frac{1}{\sqrt{2}} \left( 2ti \frac{d}{dx} K + x K - \frac{d}{dx} K \right) = b^+ K, \]
where \[ b^+ = \frac{x + (2ti - 1) \frac{d}{dx}}{\sqrt{2}}. \]

Define \( f_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2). \) Then
\[ (K f_0)(x) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{4\pi ti}} e^{\frac{x^2}{4t}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{4t}} e^{\frac{x y}{\sqrt{2}}} dy \]
\[ = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{4\pi ti}} e^{\frac{x^2}{4t}} \int_{-\infty}^{+\infty} \exp \left( -\frac{(2t - iy + x - \frac{x}{\sqrt{2} - i})^2}{4t} \right) \exp \left( -\frac{x^2}{4t(2t - i)} \right) dy \]
\[ = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{4\pi t(2t - i)}} e^{-\frac{x^2}{4(t + 2ti)}} \int_{-\infty}^{+\infty} \exp \left( -\frac{z^2}{4t} \right) dz = \frac{1}{\pi^{1/4} \sqrt{1 + 2ti}} e^{-\frac{x^2}{2(1 + 2ti)}} \equiv g_0(x) \]

Since
\[ f_n(y) = \frac{1}{\sqrt{n!} 2^n} H_n(y) e^{-\frac{y^2}{2}} = \frac{1}{\sqrt{n!}} (a^+)^n (f_0)(y) \]
we obtain
\[ K(f_n)(x) = K \left( \frac{(a^+)^n f_0}{\sqrt{n!}} \right)(x) = \frac{(b^+)^n}{\sqrt{n!}} K(f_0)(x) = \frac{(b^+)^n}{\sqrt{n!}} (g_0)(x) \]
\[ = \frac{1}{\pi^{1/4} \sqrt{1 + 2ti}} \sqrt{n!} H_n \left( \frac{x}{\sqrt{1 + 2ti}} \right) e^{-\frac{x^2}{2(1 + 2ti)}}. \]

Denote
\[ \psi_{lm}(x, y) = \frac{1}{\sqrt{2\sqrt{\pi 2!}^{2m} m!}} H_l \left( \frac{x - y}{\sqrt{2}} \right) H_m \left( \frac{x + y}{\sqrt{2}} \right) e^{-\frac{(x - y)^2 + (x + y)^2}{2\sqrt{2}}}. \] (12)
Proposition 1. Unitary group $U_t$ acts on $\psi_{lm}$ as

$$ (U_t \psi_{lm})(x, y) = \frac{1}{(2\pi)^{\frac{3}{2}} \sqrt{2l!2m!}(1 + \sqrt{2}it)} e^{-it\left(\frac{x+y}{\sqrt{2}}\right)} $$

$$ \times H_l \left(\frac{x - y}{\sqrt{2}}\right) e^{-\frac{(x-y)^2}{4\sqrt{2}}} H_m \left(\frac{x + y}{\sqrt{2}\sqrt{1 + \sqrt{2}it}}\right) e^{-\frac{(x+y)^2}{2\sqrt{2}(1 + \sqrt{2}it)}} $$

Proof. We prove the Proposition by computing the integral

$$ I = \frac{1}{\sqrt{it}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(x+y-(x+y))} H_n \left(\frac{u - v}{\sqrt{2}}\right) e^{-\frac{(u-v)^2}{2\sqrt{2}}} $$

$$ \times H_l \left(\frac{u - v}{\sqrt{2}}\right) H_m \left(\frac{u + v}{\sqrt{2}}\right) e^{-\frac{(u-v)^2+2(u+v)^2}{2\sqrt{2}}} $$

$$ \times H_m \left(\frac{b}{\sqrt{2}}\right) e^{-z^2} \frac{dbdz}{2\sqrt{2}(1 + \sqrt{2}it)} $$

$$ \frac{\sqrt{2}}{\sqrt{it}} \int_{-\infty}^{+\infty} e^{i\left[(x+y-z)\frac{\pi}{4}\right]} e^{-\frac{z^2}{4\pi}} H_m(z) \int_{-\infty}^{+\infty} H_n(b) H_l(b) e^{-z^2} dbdz = $$

$$ \frac{\sqrt{2}\delta_{nl}}{\sqrt{it}} \int_{-\infty}^{+\infty} e^{i\left[(x+y-z)\frac{\pi}{4}\right]} e^{-\frac{z^2}{4\pi}} H_m(z)dz = $$

$$ \frac{\sqrt{2}\delta_{nl}}{\sqrt{it}} \int_{-\infty}^{+\infty} e^{i\left[(x+y-z)\frac{\pi}{4}\right]} e^{-\frac{z^2}{4\pi}} H_m(z)dz = $$

$$ \frac{2\pi^2 \delta_{nl}}{\sqrt{1 + 2it/\sqrt{2}}} H_m \left(\frac{x + y}{\sqrt{2}\sqrt{1 + \sqrt{2}it}}\right) e^{-\frac{(x+y)^2}{2\sqrt{2}(1 + \sqrt{2}it)}} $$

Below we will use coherent states \cite{22,23}. Given a complex number $\alpha \in \mathbb{C}$, the coherent state $\xi_\alpha \in \mathcal{L}^2(\mathbb{R})$ is an eigenvector of the annihilation operator with eigenvalue $\alpha$. Its explicit form is

$$ \xi_\alpha(u) = \frac{1}{\frac{\pi}{4}} \exp \left(-\frac{|\alpha|^2}{2}\right) \exp \left(-\frac{u^2 - 2\sqrt{2}\alpha u + \alpha^2}{2}\right) $$

(13)

Consider in the tensor product of the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2) \cong \mathcal{L}^2(\mathbb{R}) \otimes \mathcal{L}^2(\mathbb{R})$ the product of two coherent states

$$ \psi_{\alpha\beta}(x, y) = \frac{1}{\sqrt{2}} \xi_\alpha \left(\frac{x + y}{\sqrt{2}}\right) \xi_\beta \left(\frac{x - y}{\sqrt{2}}\right). $$

(14)

As a result we have
Corollary 1.

\[ (U_t \psi_{\alpha\beta})(x, y) = \frac{e^{-\frac{it}{\sqrt{2}}}}{\sqrt{2/\sqrt{1 + \sqrt{2}t}}} \xi_\alpha \left( \frac{x + y}{\sqrt{2/\sqrt{1 + \sqrt{2}t}}} \right) \xi_{e^{-i\sqrt{2}t} \beta} \left( \frac{x - y}{\sqrt{2}} \right). \]

Proof. Taking into account that

\[ \psi_{\alpha\beta}(x, y) = e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{l,m=0}^{+\infty} \frac{\alpha l \beta m}{\sqrt{l}\sqrt{m}} \psi_{lm}(x, y), \]

where \( \psi_{lm} \) are defined by (12), and applying Proposition 1 we get the result.

\[ \square \]

3.2 The graph generated by orbits of \((U_t)\)

It is straightforward to check that

\[ \psi_{\alpha\beta}(x, y) = \frac{1}{\sqrt{2}} \xi_{\alpha + \beta} \left( \frac{x}{\sqrt{2}} \right) \xi_{\alpha - \beta} \left( \frac{y}{\sqrt{2}} \right). \]  

(15)

Given \( \gamma \in \mathbb{C} \) denote

\[ \langle x|\gamma \rangle = \frac{1}{\sqrt{2}} \xi_\gamma \left( \frac{x}{\sqrt{2}} \right) \]

the corresponding squeezed coherent state. Taking an arbitrary \( \beta \in \mathbb{C} \) let us define the operator

\[ Q_\beta = \frac{1}{\pi} \int_{\mathbb{C}} \langle \beta + \alpha, \alpha - \beta \rangle \langle \beta + \alpha, \alpha - \beta \rangle d^2\alpha \]

(16)

Proposition 2. The operator \( Q_\beta \) is an orthogonal projection.

Proof. The operator is clearly positive. It follows from (14) that

\[ Q_\beta \cong \frac{1}{\pi} \int_{\mathbb{C}} |\alpha \rangle \langle \alpha | d^2\alpha \otimes |\beta \rangle \langle \beta |. \]  

(17)

Taking into account that

\[ \frac{1}{\pi} \int_{\mathbb{C}} |\alpha \rangle \langle \alpha | d^2\alpha = I_{L^2(\mathbb{R})}, \]

(18)

we obtain that \( Q_\beta \) is an orthogonal projection to the subspace

\[ \mathcal{H}_\beta = \text{span}\{\psi_{\alpha\beta}, \alpha \in \mathbb{C}\}. \]

(19)
Lemma 2. Fix \( \varphi \in [0, 2\pi) \), then the following inclusion holds

\[
I \in \text{span}\{U_tQ_{re}^\varphi U_t^*, \ t \in \mathbb{R}, \ r \in \mathbb{R}_+\}.
\] (20)

Moreover the right hand side of (20) doesn’t depend on a choice of \( \varphi \).

Proof. It follows from Corollary 1 that

\[
(U_t\psi_{\alpha\beta})(x, y) = \frac{e^{-it\sqrt{2}}}{\sqrt{2}\sqrt{1 + \sqrt{2}t}} \xi_{\alpha} \left( \frac{x - y}{\sqrt{2}\sqrt{1 + \sqrt{2}t}} + \xi_{\beta} \left( \frac{x + y}{\sqrt{2}} \right) \right).
\]

Taking into account (18) we obtain for (19)

\[
U_tH_{\beta} = H_{e^{-i\sqrt{2}t}\xi_{\beta}}
\]

and

\[
U_tQ_{\beta}U_t^* = Q_{e^{i\sqrt{2}t}\xi_{\beta}}.
\]

Hence a convex hull of the set

\[
\{U_tQ_{re}^\varphi U_t^*, \ t \in \mathbb{R}, \ r \in \mathbb{R}_+\}
\]

contains the integral over all squeezed coherent vectors. The result follows.

\[\blacksquare\]

The next statement immediately follows from Lemma 2.

Theorem 1. The operator space

\[
\mathcal{V} = \text{span}\{U_tQ_{re}^\varphi U_t^*, \ t \in \mathbb{R}, \ r \in \mathbb{R}_+\} = \text{span}\{Q_{re(-\varphi)}^\beta, \ t \in \mathbb{R}, \ r \in \mathbb{R}_+\}
\]

is a non-commutative graph.

3.3 Maximal anticlique

Fix in \( L^2(\mathbb{R}) \) a unit vector \( g_0 \) and an orthonormal basis \( \{\varphi_k\}, \ k = 1, 2, 3, \ldots \).

Consider pairwise orthogonal unit vectors in \( L^2(\mathbb{R}^2) \) defined by the formula

\[
\eta_k(x, y) = \varphi_k \left( \frac{x - y}{\sqrt{2}} \right) g_0 \left( \frac{x + y}{\sqrt{2}} \right), \quad k = 1, 2, \ldots
\] (21)

Theorem 2. The projection

\[
P = \sum_{k=1}^{\infty} |\eta_k\rangle \langle \eta_k|
\]

is a maximal quantum anticlique for \( \mathcal{V} \).
**Proof.** Using the unitary equivalence like in (17) we get

\[
P Q_{e^{-i\sqrt{2}t}\beta} P \cong \left( \sum_{k=1}^{\infty} |\varphi_k\rangle \langle g_0| \otimes |g_0\rangle \langle g_0| \right) \left( I \otimes |e^{-i\sqrt{2}t}\beta\rangle \langle e^{-i\sqrt{2}t}\beta| \right) \times \left( \sum_{k=1}^{\infty} |\varphi_k\rangle \langle g_0| \otimes |g_0\rangle \langle g_0| \right) \\
= |\langle e^{-i\sqrt{2}t}\beta|g_0\rangle|^2 (I \otimes |g_0\rangle \langle g_0|) \cong c_{t,\beta} P.
\]

Hence

\[
\dim P \forall P = 1. \quad (22)
\]

Suppose that there exists a quantum anticlique \( \tilde{P} \) such that \( P < \tilde{P} \). Then, it should have the form

\[
\tilde{P} \cong I \otimes \tilde{P},
\]

where \( \tilde{P} > |g_0\rangle \langle g_0| \). To satisfy (22) we need

\[
\tilde{P} |e^{-i\sqrt{2}t}\beta\rangle \langle e^{-i\sqrt{2}t}\beta| \tilde{P} = \tilde{c}_{t,\beta} \tilde{P} \quad (23)
\]

for all \( t \) and \( \beta = re^{i\varphi} \). It immediately follows from (23) that \( \tilde{P} \) is one-dimensional, i.e. \( \tilde{P} = |g_0\rangle \langle g_0| \). \( \square \)

## 4 Conclusion

Motivated by the needs of quantum error correction, we extend the notion of operator graphs to infinite-dimensional Hilbert spaces. Our approach has allowed for the first time to construct operator graphs using the dynamical evolution in an infinite dimensional Hilbert space. As an explicit example, non-commutative operator graphs are constructed which are generated by the orbits of the unitary group given by the solution of the Schrödinger equation describing dynamics of a pair of oscillators. The quantum anticlques for this graph are found and their maximality is proved.

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