Polya-Ostrowski Group and Unit Index in Real Biquadratic Fields

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Abstract: Pólya group of a Galois number field $K$ is the subgroup of the ideal class group of $K$ generated by all strongly ambiguous ideal classes. In this paper, using Galois cohomology and some results in [14,15], we give an explicit relation between order of Pólya groups and Hasse unit indices in real biquadratic fields. As an application, we correct some minor mistakes for computing unit indices in Hayashi-type number fields [5].

Keywords: Pólya fields, Pólya groups, Biquadratic fields, Hasse unit index, Galois cohomology.

Notations. For a number field $K$, we denote by $Cl(K)$, $h_K$ and $U_K$ the ideal class group, class number and group of units of $K$, respectively.

1. Introduction

Let $K$ be a number field with ring of integers $\mathcal{O}_K$. Study of the ring of integer-valued polynomials on $\mathcal{O}_K$, denoted by $\text{Int}(\mathcal{O}_K)$, was initiated by Pólya [13] and Ostrowski [12]. Regarding $\text{Int}(\mathcal{O}_K)$ as a free $\mathcal{O}_K$-module [13 Section 2], $K$ is called a Pólya field whenever $\text{Int}(\mathcal{O}_K)$ admits a regular basis, that is a basis with exactly one polynomial from each degree.

Definition 1.1. [12] For every integer $q \geq 2$, the Ostrowski ideal $\Pi_q(K)$ of $K$ is defined as follows:

$$
\Pi_q(K) = \prod_{m \in \text{Max}(\mathcal{O}_K)} m.
$$

By convention, if $K$ has no ideal with norm $q$, we put $\Pi_q(K) = \mathcal{O}_K$.

Definition 1.2. [3] § II.3] The Pólya-Ostrowski group or Pólya group $Po(K)$ of $K$, is the following subgroup of $Cl(K)$:

$$
Po(K) = \langle [\Pi_q(K)] : q \in \mathbb{N} > .
$$

Indeed, $Po(K)$ can be seen as an “obstruction measure” for $K$ to be Pólya:

Proposition 1.3. [3] § II.4] A number field $K$ is a Pólya field if and only if $Po(K)$ is trivial.

Obviously every number field of class number one is a Pólya field, but not conversely, see e.g. Proposition 1.4 below.
1.1. Pólya group in Galois extensions. For a Galois extension \( K/\mathbb{Q} \), Pólya group coincides with the subgroup of strongly ambiguous ideal classes [2 Section 2]. Thanks to Hilbert’s studies on ambiguous ideals in quadratic fields, one can easily describe Pólya groups in quadratic fields:

**Proposition 1.4.** [3 Theorem 105-106] Let \( K \) be a quadratic field and denote the number of ramified primes in \( K/\mathbb{Q} \) by \( r_K \). Then:

\[
\#Po(K) = \begin{cases} 2^{r_K-2} & \text{ if } K \text{ is real with no negative norm unit} \\ 2^{r_K-1} & \text{ otherwise} \end{cases}
\]

The above “restriction on ramification” in Pólya quadratic fields, has been generalized by Zantema for arbitrary Galois number fields (a generalization can be found in [4, 11]):

**Proposition 1.5.** [15, Proposition 3.1] If \( K/\mathbb{Q} \) is Galois, the following sequence of Abelian groups is exact:

\[
\{0\} \rightarrow H^1(Gal(K/\mathbb{Q}), U_K) \rightarrow \bigoplus_{p \text{ prime}} \mathbb{Z}/e_p\mathbb{Z} \rightarrow Po(K) \rightarrow \{0\},
\]

where \( e_p \) denotes the ramification index of a prime \( p \) in \( K \).

The main aim of the present paper, is to relate Pólya groups and Hasse unit indices in real biquadratic fields, and the above exact sequence plays an important role, see Theorem (3.3) below. The reader is referred to [3, 4, 6, 7, 9, 10, 11] for some results on Pólya fields and Pólya groups.

2. Units of positive norm in real quadratic fields

In this section, we give a brief introduction to positive norm units in real quadratic fields to facilitate some group isomorphisms introduced in [14], see Proposition [3.1] below.

For \( d > 1 \) a squarefree integer and a real quadratic field \( k = \mathbb{Q}(\sqrt{d}) \) with Galois group \( G = \{1, \sigma\} \), by Hilbert’s Theorem 90 [3], for \( u > 1 \) the generator of positive norm units in \( k \), there exists an \( \alpha_u \in \mathcal{O}_k \), unique up to sign, such that \( \alpha_u \) is not divisible by any rational prime, i.e. \( \alpha_u \) is reduced, and \( u = \sigma(\alpha_u)/\alpha_u \).

Since \( \sigma(\alpha_u) = \alpha_u \cdot u \), the ideal \( (\alpha_u) \) is an ambiguous ideal which implies that \( (\alpha_u) \) is a product of distinct ramified prime ideals (Note that \( \alpha_u \) is reduced).

For \( n_{\alpha_u} := N_{k/\mathbb{Q}}(\alpha_u) = \alpha_u \cdot \sigma(\alpha_u) \), one has \( n_{\alpha_u} \) is a positive squarefree divisor of \( \text{disc}(k) \). Indeed, from

\[
\frac{(\sigma(\alpha_u) + \alpha_u)^2}{n_{\alpha_u}} = \frac{\sigma(\alpha_u)}{\alpha_u} + \frac{\alpha_u}{\sigma(\alpha_u)} + 2 = (u + 1)(\sigma(u) + 1) = N_{k/\mathbb{Q}}(u + 1),
\]

we find that \( n_{\alpha_u} \) is the squarefree part of \( N_{k/\mathbb{Q}}(u + 1) \). Note that \( n_{\alpha_u} \cdot u = \sigma(\alpha_u)^2 \) is a square in \( k^* \) which implies that for a non-square unit \( u \) we have \( n_{\alpha_u} \neq 1, d \).

Summing up the above arguments, we have:

**Proposition 2.1.** Let \( k = \mathbb{Q}(\sqrt{d}) \) be a real quadratic field and \( \epsilon > 1 \) be the generator of positive norm units in \( k \). Assume that \( n_k \) is the squarefree part of \( N_{k/\mathbb{Q}}(\epsilon + 1) \). Then \( n_k \notin \{1, d\} \), \( n_k \) divides the discriminant of \( k \), \( n_k \) is the norm of an integer in \( k \), and \( n_k \cdot \epsilon \) is a square in \( k \).
3. Setzer Isomorphisms and Main Result

In this section, summarizing some Setzer’s results [14] for real biquadratic fields \( K \), we find an explicit relation between \( \#Po(K) \) and Hasse unit index of \( U_K \).

**Convention.** For the rest of the paper, we will use the following notations:

For a real biquadratic field \( K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \) \((d_1, d_2 > 1\) are distinct square-free positive integers), we specify the usual order for quadratic subfields of \( K \) as \( K_1 = \mathbb{Q}(\sqrt{d_1}), K_2 = \mathbb{Q}(\sqrt{d_2}) \) and \( K_3 = \mathbb{Q}(\sqrt{d_1d_2}) \). We identify \( \text{Gal}(K/\mathbb{Q}) \) as \( G = \{1, S_1, S_2, S_3\} \) where \( k_i \) is the fix field of \( S_i \). For \( i \in \{1, 2, 3\} \), \( \epsilon_i > 1 \) denotes the fundamental unit of \( k_i \). Further, let \( H \) be \( H^1(G, U_K) \) and \( H[2] \) be the 2-torsion subgroup of \( H \). Finally, define:

\[
G = \{(u_1, u_2, u_3) \in U_{k_1} \times U_{k_2} \times U_{k_3} \mid N_{k_1/\mathbb{Q}}(u_1) = N_{k_2/\mathbb{Q}}(u_2) = N_{k_3/\mathbb{Q}}(u_3) = \text{sign}(u_1u_2u_3)\}.
\]

**Proposition 3.1.** [14] With the notations of this section, the following assertions hold:

(i) \([14]\) Theorem 4 \((H : H[2]) \leq 2\). Moreover \((H : H[2]) = 2\) if and only if 2 ramifies totally in \( K/\mathbb{Q} \) and all \( k_i \)'s contain elements with the same norm \( \pm 2 \) (either 2 or \(-2\)).

(ii) \([14], \S\, 4.5\) Denote by \( t \) the number of quadratic subfields of \( K \) whose fundamental units have negative norm. Then

\[
(U_K : U_{k_1}U_{k_2}U_{k_3}) \cdot \#H[2] = \begin{cases} 2^{5-t} : t = 0, 1, 2 \\ 2^3 : t = 3 \end{cases}
\]

Moreover for \( t = 3 \), \( H[2] \) contains a subgroup whose order is 4.

(iii) \([14], \text{Lemma A}\) Define

\[
\alpha_i := \begin{cases} \epsilon_i^2 & : \text{if } N_{k_i/\mathbb{Q}}(\epsilon_i) = -1 \\ \epsilon_i & : \text{if } k_i \text{ has no unit of negative norm} \end{cases}
\]

Then a set of generators for \( G \) is given by

\[
\{(\alpha_1, 1, 1), (1, \alpha_2, 1), (1, 1, \alpha_3), (1, -1, -1), (-1, 1, -1), (-\epsilon_1, -\epsilon_2, -\epsilon_3)\} : t = 3 \\
\{(\alpha_1, 1, 1), (1, \alpha_2, 1), (1, 1, \alpha_3), (1, -1, -1), (-1, 1, -1)\} : \text{Otherwise}
\]

(iv) \([14], \S\, 2\) Define the group homomorphism \( \varphi \) as:

\[
\varphi : U_K \to U_{k_1} \times U_{k_2} \times U_{k_3} \\
u \mapsto (N_{K/k_1}(u), N_{K/k_2}(u), N_{K/k_3}(u))
\]

Then \( \varphi(U_K) \subseteq G \) and \( \overline{\varphi(U_K)} \) is a 2-torsion group.

(v) \([14], \S\, 2\) The following map \( \psi \) is a group isomorphism:

\[
\psi : \overline{\varphi(U_K)} \to H[2] \\
[(\alpha_1, 1, 1)] \mapsto [\sigma_{\alpha_1}] \\
[(1, \alpha_2, 1)] \mapsto [\sigma_{\alpha_2}] \\
[(1, 1, \alpha_3)] \mapsto [\sigma_{\alpha_3}] \\
[(1, -1, -1)] \mapsto [\sigma_{(1, -1, -1)}] \\
[(-1, 1, -1)] \mapsto [\sigma_{(-1, 1, -1)}] \\
[(-\epsilon_1, -\epsilon_2, -\epsilon_3)] \mapsto [\sigma_{(\epsilon_1, \epsilon_2, \epsilon_3)}]
\]
Corollary 3.2. With the notations of this section:

\[ \sigma_{\alpha_1}(1) = \sigma_{\alpha_1}(S_i) = 1, \quad \sigma_{\alpha_1}(S_j) = \frac{1}{\alpha_i} \text{ (for } j \neq i) , \]

\[ \sigma_{(1,-1,-1)}(1) = \sigma_{(1,-1,-1)}(S_1) = 1, \quad \sigma_{(1,-1,-1)}(S_j) = -1 \text{ (for } j \neq 1) , \]

\[ \sigma_{(-1,1,-1)}(1) = \sigma_{(-1,1,-1)}(S_2) = 1, \quad \sigma_{(-1,1,-1)}(S_j) = -1 \text{ (for } j \neq 2) , \]

\[ \sigma_{(\epsilon_1,\epsilon_2,\epsilon_3)}(1) = 1, \quad \sigma_{(\epsilon_1,\epsilon_2,\epsilon_3)}(S_i) = \frac{-1}{\epsilon_j} \text{ (where } \{i,j,k\} = \{1,2,3\} ) . \]

(Note that the last 1-cocyle, namely \( \sigma_{(\epsilon_1,\epsilon_2,\epsilon_3)} \), is defined iff all \( k_i \)'s have units of negative norm.)

\[ (vi) \text{ [14] Theorem 5] If at least one of } k_i \text{'s has no unit of negative norm, then the following map } \rho \text{ is a group monomorphism:} \]

\[
\rho : \frac{G}{\varphi(U_K)} \to \begin{cases} 
\mathbb{Q}^* \\
\mathbb{Q}^{*2} \end{cases} \\
[(\alpha_1,1,1)] \mapsto [N_{k_i/\mathbb{Q}}(\alpha_1 + 1)] \\
[(1,\alpha_2,1)] \mapsto [N_{k_2/\mathbb{Q}}(\alpha_2 + 1)] \\
[(1,1,\alpha_3)] \mapsto [N_{k_3/\mathbb{Q}}(\alpha_3 + 1)] \\
[(1,-1,-1)] \mapsto [d_1] \\
[(-1,1,-1)] \mapsto [d_2] 
\]

As an immediate consequence, we have:

**Corollary 3.2.** With the notations of this section:

(i) If at least one of \( k_i \)'s has no unit of negative norm, then the following map \( \Theta \) is a group isomorphism:

\[
\Theta : H[2] \to \rho \left( \frac{G}{\varphi(U_K)} \right) \\
[\sigma_{\alpha_1} ] \mapsto [ N_{k_i/\mathbb{Q}}(\alpha_1 + 1) ] \quad (i = 1, 2, 3) \\
[\sigma_{(1,-1,-1)} ] \mapsto [d_1] \\
[\sigma_{(-1,1,-1)} ] \mapsto [d_2] 
\]

(ii) If \( k_1 = \mathbb{Q}(\sqrt{d_1}) \) (resp. \( k_2 = \mathbb{Q}(\sqrt{d_2}) \)) has some units of negative norm, then the 1-cocyles \( \sigma_{\alpha_1} \) and \( \sigma_{(1,-1,-1)} \) (resp. \( \sigma_{\alpha_2} \) and \( \sigma_{(-1,1,-1)} \)), defined in Proposition (3.1), have the same class in \( H := H^1(\text{Gal}(K/\mathbb{Q}), U_K) \).

**Proof.**

(i) Immediately follows from parts (v) and (vi) of Proposition (3.1).

(ii) Let \( \epsilon_1 \) be the fundamental unit of \( k_1 = \mathbb{Q}(\sqrt{d_1}) \) with norm \( N_{k_1/\mathbb{Q}}(\epsilon_1) = -1 \).

It is straightforward to check that for every \( g \in \text{Gal}(K/\mathbb{Q}) \) one has:

\[
(\sigma_{\alpha_1}(g)) \cdot (\sigma_{(1,-1,-1)}(g))^{-1} = (\epsilon_1)^g \cdot (\epsilon_1)^{-1},
\]

where \( (\epsilon_1)^g \) denotes the Galois action of \( g \) on \( \epsilon_1 \).

\[ \Box \]

Note that the first cohomology group \( H^1(G, U_K) \) can be seen as a “common term” in the above Zantema and Setzer results. This point of view leads us to relate order of Pólya group with the Hasse unit index:
Theorem 3.3. Let \( K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \) be a real biquadratic field with quadratic subfields \( k_i = \mathbb{Q}(\sqrt{d_i}) \) (\( d_i \) is the squarefree part of \( d_1 d_2 \)). Denote by \( t \) the number of quadratic subfields of \( K \) whose fundamental unit have negative norm.

(i) If 2 ramifies totally in \( K/\mathbb{Q} \) and all \( k_i \)'s contain elements with the same norm either 2 or \(-2\), then:

\[
\#Po(K) = \begin{cases} 
\frac{(U_K:U_{k_1}U_{k_2}U_{k_3}) \cdot \prod_{p \mid \text{disc}(K)} e(p)}{(U_K:U_{k_1}U_{k_2}U_{k_3}) \cdot \prod_{p \mid \text{disc}(K)} e(p)} : t = 0, 1, 2 \\
\frac{2^t}{2^3} : t = 3 
\end{cases}
\]

(ii) Otherwise:

\[
\#Po(K) = \begin{cases} 
\frac{(U_K:U_{k_1}U_{k_2}U_{k_3}) \cdot \prod_{p \mid \text{disc}(K)} e(p)}{(U_K:U_{k_1}U_{k_2}U_{k_3}) \cdot \prod_{p \mid \text{disc}(K)} e(p)} : t = 0, 1, 2 \\
\frac{2^t}{2^3} : t = 3 
\end{cases}
\]

In particular, for \( t \in \{2, 3\} \) Pólya groups have the same order.

Proof. Combining parts (i)–(ii) in Proposition 3.1 with Zantema's exact sequence in Proposition 1.3, proves the assertion.

\[ \square \]

4. Improving Hayashi’s results

Hayashi \cite{Hayashi} found sufficient conditions for real biquadratic fields to have odd class numbers and also gave their corresponding Hasse unit indices. Using some results in the previous sections, we improve these results of Hayashi and correct some minor mistakes in his computation of unit indices (Recall that, we will use the same notations as in Section 3).

Proposition 4.1. \cite{Hayashi} Let \( K \) be a real biquadratic field whose class number is odd. Then \( K \) has one of the following forms, where \( p_i \)'s (resp. \( q_i \)'s) denote distinct odd prime numbers congruent to 1 (resp. 3) modulo 4:

(i) \( K = \mathbb{Q}(\sqrt{2}, \sqrt{q_1}), K = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2}), K = \mathbb{Q}(\sqrt{q_1 q_2}, \sqrt{q_1 q_3}), \\
K = \mathbb{Q}(\sqrt{2q_1}, 2q_2); \)

(ii) \( K = \mathbb{Q}(\sqrt{2}, \sqrt{p_1}), K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}); \)

(iii)

(iii-1) \( K = \mathbb{Q}(\sqrt{p_1}, \sqrt{2q_2}), \)

(iii-2) \( K = \mathbb{Q}(\sqrt{p_1}, \sqrt{q_1}), \)

(iii-3) \( K = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2 q_3}), K = \mathbb{Q}(\sqrt{2q_1}, \sqrt{q_2 q_3}), K = \mathbb{Q}(\sqrt{q_1 q_2}, \sqrt{q_3 q_4}); \)

(iv) \( K = \mathbb{Q}(\sqrt{2}, \sqrt{q_1 q_2}), K = \mathbb{Q}(\sqrt{p_1}, \sqrt{2q_3}), K = \mathbb{Q}(\sqrt{p_1}, \sqrt{q_1 q_2}); \)

(v) \( K = \mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2}), K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2 q_3}), K = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}); \)

(vi) \( K = \mathbb{Q}(\sqrt{2}, \sqrt{p_1 q_1}), K = \mathbb{Q}(\sqrt{q_1}, \sqrt{p_1 q_2}), K = \mathbb{Q}(\sqrt{q_1}, \sqrt{2p_1}); \)

(vii) \( K = \mathbb{Q}(\sqrt{p_1 p_2}, \sqrt{q_1 q_2}), K = \mathbb{Q}(\sqrt{2p_1 q_1}, \sqrt{q_1 q_2}), K = \mathbb{Q}(\sqrt{p_1 q_1 q_2}, \sqrt{q_1 q_3}), \)

\( K = \mathbb{Q}(\sqrt{q_1 q_2}, \sqrt{2q_1 q_3}), K = \mathbb{Q}(\sqrt{2p_1 q_1}, \sqrt{q_1 q_3}), K = \mathbb{Q}(\sqrt{2p_1 q_1}, \sqrt{2q_2}). \)

Moreover, Hasse unit indices for all cases are given in Table 1 below (Corrections are needed, see Table 3).
4.1. Improving Case (ii). By the next theorem, one can see that case (ii) of the Proposition (4.1) can be improved:

**Theorem 4.2.** For $p_1 \equiv p_2 \equiv 1 \pmod{4}$ distinct prime numbers, the families

1. $K = \mathbb{Q}(\sqrt{2}, \sqrt{p_1})$,  
2. $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2})$,

are Pólya fields and have Hasse unit index 2.

**Proof.** We only prove the first case and the other is proved similarly:

For $K = \mathbb{Q}(\sqrt{2}, \sqrt{p_1})$, let $k_1 = \mathbb{Q}(\sqrt{2})$, $k_2 = \mathbb{Q}(\sqrt{p_1})$, $k_3 = \mathbb{Q}(\sqrt{2p_1})$ and denote the fundamental unit of $k_i$ by $\epsilon_i$, respectively. Since $N_{k_i}/\mathbb{Q}(\epsilon_i) = N_{k_j}/\mathbb{Q}(\epsilon_j) = -1$, by Theorem (3.3):

\[2 \cdot \text{Po}(K) = (U_K : U_{k_1}U_{k_2}U_{k_3})\]

Let $H = H^1(Gal(K/\mathbb{Q}), U_K)$. Since 2 doesn’t ramify totally in $K/\mathbb{Q}$, part (i) in Proposition (3.1) implies that $H = H^2$. Using the same notations as in Section (3), we have:

- If $N_{k_3}/\mathbb{Q}(\epsilon_3) = +1$, by Proposition (1.5) and Corollary (3.2):
  \[\langle 2, p_1 \rangle (\text{mod } \mathbb{Q}^*) \subseteq \rho \left( \frac{G}{\varphi(U_K)} \right) \simeq H[2] = H = \mathbb{Z}/2Z \oplus \mathbb{Z}/2Z;\]

- If $N_{k_3}/\mathbb{Q}(\epsilon_3) = -1$, by Propositions (1.5) and (3.1):
  \[\mathbb{Z}/2Z \oplus \mathbb{Z}/2Z \subseteq H[2] = H = \mathbb{Z}/2Z \oplus \mathbb{Z}/2Z.

Hence $H \simeq \mathbb{Z}/2Z \oplus \mathbb{Z}/2Z$ and by Zantema’s exact sequence (1.1), $K$ is Pólya. Equation (4.1) completes the proof. $\square$

**Remark 4.3.** Leriche’s result in [9, Proposition 5.2] also shows that the families of real biquadratic fields $K$ in Theorem (4.2) are Pólya.

4.2. Improving Case (v). Now we consider the three classes of real biquadratic fields in case (v) of Proposition (4.1).

**Theorem 4.4.** For $p_1 \equiv p_2 \equiv p_3 \equiv 1 \pmod{4}$ distinct prime numbers, and the families

1. $K = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2})$;  
2. $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2p_3})$;  
3. $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2})$,

we have (Note the order for the convention in Section (3)):

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Case} & (i) & (ii) & (iii) & (iv) & (v) & (vi) & (vii) \\
\hline
(U_K : U_{k_1}U_{k_2}U_{k_3}) & 4 & 2 & 1, 2 & 4 & 2 & 2 & 2, 4 \\
\hline
\end{array}
$$

**Table 1.** Hayashi’s computation of unit indices for biquadratic fields in Proposition (4.1). Bold non-Italic cases will be corrected in Table (3).
Proof. We only prove case (4.2) and will be dealt with in Theorem (4.7).

(1) If $N_{k_2/Q}(\epsilon_2) = +1$ and $N_{k_3/Q}(\epsilon_3) = -1$ (resp. $N_{k_3/Q}(\epsilon_3) = +1$), then $(U_K : U_{k_1}U_{k_2}U_{k_3}) = 1$ (resp. $(U_K : U_{k_1}U_{k_2}U_{k_3}) = 2$) and $K$ is Pólya;

(2) If $N_{k_2/Q}(\epsilon_2) = -1$, then $\# P^o(K) = (U_K : U_{k_1}U_{k_2}U_{k_3}) \in \{1, 2\}$.

Remark 4.5. Note that all the families of real biquadratic fields $K$ in case (2) of Theorem (4.4), have only one Pólya quadratic subfield. Hence Leriche’s result in § Proposition 5.2 is not applicable here.

4.3. Improving Case (vii). In Proposition (4.1), all families of real biquadratic fields in the case (vii), have only one Pólya quadratic subfield; all except one family, have four ramifications which are addressed in Theorem (4.6) below. The exception one has three ramifications and will be dealt with in Theorem (4.7).

Theorem 4.6. Let $p_1 \equiv 1 \pmod{4}$ and $q_1 \equiv q_2 \equiv q_3 \equiv 3 \pmod{4}$ be distinct prime numbers. For the following real biquadratic fields:

1. $K = \mathbb{Q}(\sqrt{p_1 q_1}, \sqrt{q_1 q_2})$;
2. $K = \mathbb{Q}(\sqrt{2p_1 q_1}, \sqrt{q_1 q_2})$;

Using Relation (4.2), Proposition (3.1) and Corollary (3.2) we have:

$(U_K : U_{k_1}U_{k_2}U_{k_3}) \in \{1, 2\}$, part (i) of Proposition (3.1) implies that

$s \leftarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$.

By Relations (4.4) and (4.5) we have $\# H = 2^3$ and Proposition (4.3) implies that $K$ is Pólya. Relation (4.3) completes the proof.

(2) Let $N_{k_2/Q}(\epsilon_2) = -1$. Since 2 doesn’t ramify totally in $K/Q$, by Theorem (3.3) we have $\# P^o(K) = (U_K : U_{k_1}U_{k_2}U_{k_3})$. Moreover:

- for $N_{k_3/Q}(\epsilon_3) = +1$, by Corollary (3.2):

$(2, p_1p_2) \pmod{Q^{*2}} \subseteq \rho \left( \frac{G}{\phi(U_K)} \right) \simeq H[2] = H$.

- for $N_{k_3/Q}(\epsilon_3) = -1$, by part (ii) of Proposition (3.1):

$s \leftarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \subseteq H[2] = H$.

Therefore $2^2 \mid \# H$ and Zantema’s exact sequence (1.1) yields the claim. \hfill \Box
(3) \( K = \mathbb{Q}(\sqrt{p_1q_1q_2}, \sqrt{q_1q_3}) \);
(4) \( K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{2q_1q_3}) \);
(5) \( K = \mathbb{Q}(\sqrt{2p_1q_1}, \sqrt{2q_2}) \),

we have
\[
(U_K : U_{k_1}U_{k_2}U_{k_3}) = 2.\#P_0(K) \in \{2, 4\}.
\]
Moreover, in Table 2 below some sufficient conditions are given under which the corresponding unit index is as shown in the third column.

| \( K \)                                | Conditions                                      | \( (U_K : U_{k_1}U_{k_2}U_{k_3}) \) |
|----------------------------------------|------------------------------------------------|-------------------------------------|
| \( \mathbb{Q}(\sqrt{2^\alpha p_1q_1}, \sqrt{q_1q_2}) \): \( \alpha = 0, 1 \) | \( (\frac{q_1}{p_1}) = -1 \) or \( (\frac{q_1}{p_1}) = -1 \) | 2                                   |
| \( \mathbb{Q}(\sqrt{2^\alpha p_1q_1}, \sqrt{q_1q_2}) \): \( \alpha = 0, 1 \) | \( (\frac{q_1}{p_1}) = 1 \), \( p_1 \equiv 5 \) (mod 8) | 4                                   |
| \( \mathbb{Q}(\sqrt{p_1q_1q_2}, \sqrt{q_1q_3}) \) | \( (\frac{q_1}{p_1}) = -1 \) or \( (\frac{q_1}{p_1}) = -1 \) | 2                                   |
| \( \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{2q_1q_3}) \) | \( q_1 \equiv 3 \) (mod 8) or \( q_3 \equiv 3 \) (mod 8) | 2                                   |
| \( \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{2q_1q_3}) \) | \( q_1 \equiv q_2 \equiv 7 \) (mod 8), \( q_3 \equiv 3 \) (mod 8) | 4                                   |
| \( \mathbb{Q}(\sqrt{2p_1q_1}, \sqrt{2q_2}) \) | \( (\frac{q_1}{p_1}) = -1 \) or \( p_1 \equiv 5 \) (mod 8) | 2                                   |
| \( \mathbb{Q}(\sqrt{2p_1q_1}, \sqrt{2q_2}) \) | \( (\frac{q_1}{p_1}) = 1 \), \( p_1 \equiv 1 \) (mod 8) | 4                                   |

**TABLE 2.**

**Proof.** We only prove case (1), other cases can be proved similarly:

For \( K = \mathbb{Q}(\sqrt{p_1q_1}, \sqrt{q_2}) \) there are four ramified primes and 2 doesn’t ramify totally. The congruence conditions on \( q_i \)'s guarantee that none of \( k_i \)'s have negative norm units. Hence by Theorem 3.3 we get:

\[
(4.7) \quad (U_K : U_{k_1}U_{k_2}U_{k_3}) = 2.\#P_0(K).
\]

Let \( H = H^1(\text{Gal}(K/\mathbb{Q}), U_K) \), \( k_1 = \mathbb{Q}(\sqrt{p_1q_1}) \), \( k_2 = \mathbb{Q}(\sqrt{q_1q_2}) \), \( k_3 = \mathbb{Q}(\sqrt{p_1q_2}) \) and denote the fundamental unit of \( k_i \) by \( \epsilon_i \). By Propositions 2.1, 3.1 and Corollary 3.2 we have:

\[
(4.8) \quad < p_1q_1, N_{k_2/\mathbb{Q}}(\epsilon_2 + 1), q_1q_2 > (\bmod \mathbb{Q}^2) \subseteq \rho \left( \frac{G}{\varphi(U_K)} \right) \simeq H[2] = H,
\]
and one can conclude that \( 2^3 \mid \#H \). By Zantema’s exact sequence (1.1):

\[
(4.9) \quad \#H.\#P_0(K) = 2^4,
\]
which implies that \( \#P_0(K)/2 \). By Relation 4.7, the main assertion is proved.

For verifying the given indices in first two rows of Table 2 (for \( \alpha = 0 \)):

- If \( (\frac{q_1}{p_1}) = -1 \) (without loss of generality), then the Diophantine equations
  \[ x^2 - p_1q_1y^2 = \pm p_1 \]
  and
  \[ x^2 - p_1q_1y^2 = \pm q_1 \]
  have no integer solutions and using Proposition 2.1, we have

\[
(4.10) \quad 2 \mid N_{k_1/\mathbb{Q}}(\epsilon_1 + 1).
\]

In this case by Relation 4.8:

\[ < 2, p_1, q_1, q_2 > (\bmod \mathbb{Q}^2) \leftrightarrow H, \]
and Relations 4.7, 4.9 yield the claim.
As usual, let

\[\text{Proof.} \]

\[\text{Theorem 4.7.} \]

Using Proposition (2.1) and Corollary (3.2) we have:

\[2 \mid N_{k_i/q}(\epsilon_1 + 1).N_{k_3/q}(\epsilon_3 + 1),\]

since 2 cannot be a quadratic residue modulo \(p_1\). Therefore by Proposition (3.1) (part (i)), Corollary (3.2) and Relation (4.8) we find:

\[H = H^2 \simeq \frac{\mathcal{O}(U_K)}{\varphi(U_K)} = <p_1, q_1, q_2 > (\bmod \mathbb{Q}^2).\]

By Relations (4.7) and (4.9), we have \((U_K : U_{k_1}U_{k_2}U_{k_3}) = 2.\#Po(K) = 4.\]

Finally, for the only family of real biquadratic fields with three ramifications in case (vii) of Proposition (4.1), we have:

**Theorem 4.7.** Let \(K = \mathbb{Q}(\sqrt{2p_1}, \sqrt{2q_1})\), where \(p_1 \equiv 1(\mod 4)\) and \(q_1 \equiv 3(\mod 4)\) are prime numbers. Let \(k_1 = \mathbb{Q}(\sqrt{2p_1})\), \(k_2 = \mathbb{Q}(\sqrt{2q_1})\), \(k_3 = \mathbb{Q}(\sqrt{p_1q_1})\) and denote the fundamental unit of \(k_i\) by \(\epsilon_i\) (for \(i = 1, 2, 3\)).

1. If \(N_{k_i/q}(\epsilon_i) = 1\), then \((U_K : U_{k_1}U_{k_2}U_{k_3}) = \#Po(K) = 2.\)
2. If \(N_{k_i/q}(\epsilon_i) = -1\), then \((U_K : U_{k_1}U_{k_2}U_{k_3}) = 4.\) Moreover, \(K\) is Pólya if and only if \(\Pi_2(k_3)\) is principal.

**Proof.** As usual, let \(H = H^1(\text{Gal}(K/\mathbb{Q}), U_K)\). Since \(N_{k_3/q}(\epsilon_2) = N_{k_3/q}(\epsilon_3) = 1\), using Proposition (2.1) and Corollary (3.2) we have:

\[H^2 \simeq <2p_1, 2q_1, N_{k_1/q}(\epsilon_1 + 1), N_{k_2/q}(\epsilon_2 + 1), N_{k_3/q}(\epsilon_3 + 1) > (\bmod \mathbb{Q}^2),\]

where \(\alpha_1 = \epsilon_1\) or \(\epsilon_1^2\) depending on whether \(N_{k_1/q}(\epsilon_1) = +1\) or \(-1\), respectively (see Proposition (3.1), part (iii)). Therefore

\[H^2 \simeq <2, p_1, q_1 > (\bmod \mathbb{Q}^2) \implies \#H^2 = 2^3.\]

Hence by part (ii) of Proposition (3.1), we find:

\[(U_K : U_{k_1}U_{k_2}U_{k_3}).\#H^2 = \begin{cases} 2^4 & : N_{k_1/q}(\epsilon_1) = -1 \\ 2^5 & : N_{k_1/q}(\epsilon_1) = +1 \end{cases}\]

By Relations (4.12) and (4.13), the desired unit index in both cases is obtained.

For computing \(\#Po(K)\):

1. For \(N_{k_1/q}(\epsilon_1) = -1\) by Proposition (1.4), \(k_1 = \mathbb{Q}(\sqrt{2p_1})\) is not Pólya. In this case if \(\Pi_2(k_1)\) is principal, then so is \(\Pi_{p_1}(k_1)\) (since \(\Pi_2(k_1) = (\sqrt{2p_1})\mathcal{O}_{k_1}\)), and we reach a contradiction. Hence \(k_1\) doesn’t contain an integer with norm \(\pm 2\) and using Theorem (3.3) we find:

\[\#Po(K) = (U_K : U_{k_1}U_{k_2}U_{k_3}) = 2.\]

2. For \(N_{k_1/q}(\epsilon_1) = -1\) by Proposition (1.4), \(k_1 = \mathbb{Q}(\sqrt{2p_1})\) is Pólya. On the other hand, \(k_2 = \mathbb{Q}(\sqrt{2q_1})\) is also Pólya, while \(k_3\) is not. Since in this case \((U_K : U_{k_1}U_{k_2}U_{k_3}) = 4, \) by Theorem (3.3) we have:

\[\#Po(K) = \begin{cases} 1 & : \Pi_2(k_3)\text{ is principal} \\ 2 & : \text{Otherwise} \end{cases}\]

\[\square\]
Remark 4.8. With the notations of Theorem 4.7, one can find some sufficient conditions to guarantee that in case (2), $\Pi_2(k_3)$ is principal. For instance, let $(\frac{2}{p_1}) = -1$. Since $N_{k_1/Q}(\epsilon_1) = +1$, we must have $p_1 \equiv 1(\text{mod } 8)$, see [11 Theorem 11.5.6]. Using Proposition (2.1) one can easily see that $N_{k_3/Q}(\epsilon_3 + 1) \in \{2, 2p_1q_1\}$, which implies that $\Pi_2(k_3)$ is principal.

Summing up the above results, we have:

Corollary 4.9. The corrected unit indices for real biquadratic fields in Proposition 4.1 are given in Table 3 below:

| Case | (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) |
|------|-----|------|-------|------|-----|------|-------|
| $(\bar{U}_K : \bar{U}_{k_1}\bar{U}_{k_2}\bar{U}_{k_3})$ | 4   | 2    | 4     | 2    | 2   | 1,2  | 2,4   | 2,4   |

TABLE 3. Corrected Unit indices for biquadratic fields in Proposition 4.1

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