TWISTED CONVOLUTION AND MOYAL STAR PRODUCT OF GENERALIZED FUNCTIONS

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We consider nuclear function spaces on which the Weyl–Heisenberg group acts continuously and study the basic properties of the twisted convolution product of the functions with the dual space elements. The final theorem characterizes the corresponding algebra of convolution multipliers and shows that it contains all sufficiently rapidly decreasing functionals in the dual space. Consequently, we obtain a general description of the Moyal multiplier algebra of the Fourier-transformed space. The results extend the Weyl symbol calculus beyond the traditional framework of tempered distributions.

Keywords: Moyal product, twisted convolution, Weyl symbol, Weyl–Heisenberg group, noncommutative field theory, topological *-algebra, generalized function

1. Introduction

The twisted convolution product of functions $g_1(s)$ and $g_2(s)$ on a linear symplectic space is a noncommutative deformation of the ordinary convolution and is defined by the formula

$$(g_1 ∗_θ g_2)(s) = \int g_1(t)g_2(s-t)e^{i\theta(s,t)/2} dt,$$  

(1)

where $\theta$ is the bilinear skew-symmetric form specifying the symplectic structure.\footnote{The twisted convolution operation is often denoted by $*_\theta$, but we use the symbol $\circ_\theta$ here to avoid confusion with the star product $\star_\theta$.} The Fourier transform converts the twisted convolution into the Weyl–Groenewold–Moyal star product $\star_\theta$, which gives the composition rule for the Weyl symbols of quantum mechanical operators and plays a key role in the Weyl quantization (see [1], [2]). We note that the composition rule for phase-space functions, which corresponds to the composition of operators on a Hilbert space, was originally written by von Neumann [3] just in terms of the twisted convolution. It is customary to define the star multiplication and twisted convolution first for smooth and rapidly decreasing functions in the Schwartz space $S$, which forms an associative topological algebra under either of the two operations. But in practice, depending on the problem under study, we must consider an extension of these operations to one or another subspace of the dual space $S'$ of tempered distributions. Antonets proposed a maximal extension by duality that consisted in constructing the multiplier algebra of the algebra $(S, \star_\theta)$ [4]–[6] or, equivalently, of the algebra $(S, \circ_\theta)$. This extension was later studied in many papers and most thoroughly in [7]–[9] (a detailed review and references can be found in [10]).

Field theory models on noncommutative spaces based on using the Moyal star product have been intensively investigated in the last 15 years (see, e.g., [11] for an introduction to this topic). The interest in noncommutative spaces was stimulated by the in-depth analysis of quantum limitations on the accuracy

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of localization of space–time events in quantum theory including gravity [12], [13] and by the study of the low-energy limit of string theory [14]. There is reason [15]–[17] to believe that the framework of tempered distributions is too narrow for a consistent formulation of the general principles of noncommutative quantum field theory. The Moyal product is nonlocal, and its expansion in powers of the noncommutativity parameter \( \theta \) converges only on analytic test functions whose Fourier transforms decrease at infinity faster than the Gaussian exponential function [18], [19]. An analysis of microcausality violations in the simplest noncommutative models [20]–[22] indicates a possible connection between noncommutative field theory and the previously considered nonlocal theories, which treat quantum fields as operator-valued generalized functions defined on a suitable space of analytic test functions instead of the Schwartz space. The problem of appropriately generalizing the Weyl symbol calculus arises.

In [18], we established a condition under which a nuclear test function space \( E(\mathbb{R}^d) \) with the structure of a topological algebra under ordinary convolution is also an algebra under the twisted convolution and its Fourier-conjugate space is hence an algebra under the Moyal product. This condition can be written as
\[
e^{-i\theta/2} \in M(E \otimes E),
\]
where \( E \otimes E \) is the completed projective tensor product identifiable with \( E(\mathbb{R}^{2d}) \) by the kernel theorem and \( M(E \otimes E) \) is the space of its multipliers with respect to ordinary pointwise multiplication. In the case of Gel’fand–Shilov spaces \( S^\alpha_\beta \) [23] considered in [18], condition (2) leads to the restriction \( \alpha \geq \beta \) on the specifying indices. Palamodov precisely described the set of pointwise multipliers for the spaces \( S^\alpha_\beta \) [24], and their corresponding algebras of Moyal multipliers were constructed in [25].

Here, we show that the problem under discussion can be solved in a general form and a Moyal multiplier algebra can be constructed for any complete, nuclear, barreled\(^2\) space on which the Weyl–Heisenberg group acts continuously. We also describe the elements of these algebras using only well-known facts from the theory of tensor products of nuclear spaces [26], which allows avoiding complicated analytic estimates. The basic observation is that condition (2) implies that
\[
v \odot g \in M(E), \quad g \odot v \in M(E) \quad \text{for all} \ g \in E, \ v \in E',
\]
where \( M(E) \) is the space of pointwise multipliers for \( E \). This allows characterizing those elements of the dual space \( E' \) that are multipliers of the algebra \( (E, \odot) \) sufficiently exactly.

The paper is organized as follows. In Sec. 2, we consider function spaces endowed with a continuous linear action of the Weyl–Heisenberg group and define the twisted convolution of elements of the space with elements of its dual. We also present the noncommutative analogues of some properties of the usual convolution. In Sec. 3, we introduce the basic notion of a twisted convolution multiplier. Section 4 contains the main theorems; there, we show that the implication \( (2) \Rightarrow (3) \) holds under natural assumptions on the spaces under consideration, and using this implication, we obtain a characterization of the algebra of twisted convolution multipliers. Section 5 is devoted to the most interesting case of spaces invariant under the Fourier transformation. In Sec. 6, we illustrate the general construction with concrete examples. The appendix contains the proof of a useful simple criterion for the continuity of the action of topological groups on spaces of the considered type.

### 2. The Weyl–Heisenberg group and the twisted convolution

Let \( E \) be a locally convex space of complex-valued functions on \( \mathbb{R}^d \), and let \( \theta \) be a bilinear symplectic form, i.e., a nondegenerate skew-symmetric inner product on \( \mathbb{R}^d \). We assume that the twisted shift operator
\[
\tau_{\theta,s}: g(t) \to e^{i\theta(s,t)/2} g(t-s), \quad g \in E,
\]
\(^2\)Practically all spaces used in the theory of generalized functions have these properties; see [26] for their definition and role in functional analysis.