QUADRATIC AND SYMPLECTIC STRUCTURES ON 3-(HOM)-ρ-LIE ALGEBRAS

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ABSTRACT. Our purpose in this paper is the generalization of the notions of quadratic and symplectic structures to the case of 3-(Hom)-ρ-Lie algebras. We describe some properties of them by expressing the related lemmas and theorems. Also, we introduce the concept of 3-pre-(Hom)-ρ-Lie algebras and define their representation.

1. Introduction

At the first, Hartwig, Larsson and Silvestrov in [24] generalized Lie algebras and constructed a new class of Lie algebras in the name of Hom-Lie algebras. These algebras very soon were attracted considerable attentions and many algebraic and geometric structures were introduced on them. In 1994, the other concept was emerged as "ρ-Lie algebra" or "Lie color algebra" by P.J. Bongaarts [16] and then in 1998, Scheunert and Zhang introduced a new notion which is called a cohomology theory of Lie color algebras [32]. Also, the notion of Hom-Lie color algebras, as an extension of Hom-Lie superalgebras to $G$-graded algebras, was introduced by Yuan [38]. These Lie algebras were applied by the researchers and in 2015, Abdaoui, Ammar and Makhlouf defined representations and a cohomology of Hom-Lie color algebras in [1]. After two years, in 2017, B. Sun et al. studied some concepts such as $T^*$-extension and abelian extension on Hom-Lie color algebras [35].

The notion of pre-Lie algebra was given by Gerstenhaber in [22] in the study of deformations and the cohomology theory of associative algebras. These algebras in the other research articles appear with titles such as left-symmetric algebras or right-symmetric algebras. Also, the notion of Hom-pre-Lie algebra was introduced in [28]. This kind of algebras in the construction of Hom-Lie 2-algebras has an important role (see [33]). Representation of a Hom-pre-Lie algebra was constructed in [34], in the study of Hom-pre-Lie bialgebras. The geometrization and the bialgebra theory of these algebras was studied in [34] [40]. For the commutator bracket $[,]_c$, every Hom-pre-Lie algebra $(A, \cdot, \alpha)$ gives rise to a Hom-Lie algebra $(A,[,]_c,\alpha)$. This Lie algebra is called the subadjacent Hom-Lie algebra and denoted by $A^c$. Actually, we can say the Hom-pre-Lie algebras have a interconnected with the Hom-Lie algebras.

The concept of $n$-Lie algebra was introduced by Filippov in 1985. These Lie algebras are known by different names such as Filippov algebras, Nambu-Lie algebras, Lie $n$-algebras. The special case of these algebras are 3-Lie algebras and in the Hom case 3-Hom-Lie algebras, which extracted from 3-Lie algebras base on Hom-Lie algebras. These algebras were defined and checked by H. Ataguema et al. in [7]. Y. Liu et al. took into consideration these notions and studied the representations and module-extensions on them in [27]. Later, the new notions such as 3-Lie color algebra, 3-Hom-Lie color algebra and 3-pre-Lie
algebra were constructed and studied in [12, 31, 39]. In 1996, Y. Daletskii and V. Kushnirevich introduced the notion of n-Lie superalgebra, as a natural generalization of n-Lie algebra ([18]). When n = 3, the 3-Lie superalgebras are a particular case of 3-\(\rho\)-Lie algebras when the abelian group is \(\mathbb{Z}_2\) (see for more details [2, 3, 23]).

Ternary Lie algebras have vast applications in various realms of modern physics and mechanics such as Nambu, field and Hamiltonian theories. For example, in Nambu consideration for a case that the algebra of functions in a manifold \(M\) is ternary, one can construct a bracket defined by a trivector on \(M\) that is famous as Nambu bracket in literature. In this way, the dynamics of \(M\) is controlled by two Hamiltonians in return of classical case. Indeed, 3-ary bracket in ternary Lie algebra is a suitable alternative for Poisson alternative in Hamiltonian mechanics. To study how Ternary Lie algebras can model this frameworks, see [21, 29, 36]. Nahm equations describing some facts in string and superstring theories can be formulated using ternary Lie algebras also. In [13] it is described the benefits of replacing ternary Lie algebras instead of using simple Lie algebras in lifted Nahm equations and in [9] the authors for the lifted Nahm equations replaced the Lie algebra in the Nahm equation by a 3-Lie algebra. In the framework of Bagger-Lambert Gustavsson model of multiple \(M2\) brains, it is possible to revisit the super-symmetric alternative of world-volume theory of \(M\)-theory membrane in special cases such \(N = 2\). See [9, 11, 25, 30] for more details in the way.

In 2006, quadratic Lie algebras were discussed by I. Bajo et al. in [11]. A quadratic Lie algebra comes equipped with a non-degenerate, symmetric and invariant bilinear form. Quadratic Lie algebras are important in Mathematics and Physics specially in conformal field theory [20]. S. Benayadi and A. Makhlouf, in 2010 studied the quadratic Hom-Lie algebras [15] as a generalization of the classical case. This process continued until a generalization to the case of quadratic (even quadratic) color Lie algebras was obtained in [37]. Also, quadratic color Hom-Lie algebras were introduced and studied by F. Ammar et al. in [5]. Also, the quadratic-algebraic structures have been studied on the class of \(\Gamma\)-graded Lie algebras specially in the case where \(\Gamma = \mathbb{Z}_2\). These algebras are called homogeneous (even or odd) quadratic Lie superalgebras (see [4, 8, 14, 32]).

In this paper, we discuss the quadratic and symplectic structures on 3-(Hom)-\(\rho\)-Lie algebras and study the representations of them. The other aim is to introduce the notion of 3-pre-(Hom)-\(\rho\)-Lie algebras and check their phase spaces. Also, we study the relationship between the 3-pre-(Hom)-\(\rho\)-Lie algebras and 3-(Hom)-\(\rho\)-Lie algebras.

This paper is arranged as follows: In Section 2, we give a brief introduction to 3-\(\rho\)-Lie algebras and study the quadratic and symplectic structures on 3-\(\rho\)-Lie algebras. Also, we study the metrics and derivations on 3-\(\rho\)-Lie algebras and we explore the relationship between the symplectic structure and the invertible derivation which is antisymmetric with respect to a metric. Section 3 is devoted to study of representation theory on 3-\(\rho\)-Lie algebras and some classification results will be given. In Section 4, we introduce the notion of 3-pre-\(\rho\)-Lie algebras and discuss some basic properties of 3-pre-\(\rho\)-Lie algebras including representation theory and some classification results. Also, we define \(\rho\)-O-operator associated to the representation \((V, \mu)\) of a 3-\(\rho\)-Lie algebra and construct a 3-\(\rho\)-pre-Lie algebra structure on the representation \(V\) by a \(\rho\)-O-operator and bilinear map \(\mu\). This section also contains the notion of a phase space of a 3-\(\rho\)-Lie algebra and it will be shown that a 3-\(\rho\)-Lie algebra has a phase space if and only if it is sub-adjacent to a 3-\(\rho\)-pre-Lie algebra. In Section 5, we study the constructions of Section 2, such as quadratic, symplectic structures, 3-associative Lie algebras in the case of 3-Hom-\(\rho\)-lie algebras. The
representation theory of 3-Hom-$\rho$-lie algebras is also included in this section. Section 6 is devoted to the 3-pre-Hom-$\rho$-Lie algebras, which are studied similar to the classical case in Section 4. Actually, we give the similar results to Section 4 in the case of 3-pre-Hom-$\rho$-Lie algebras.

2. QUADRATIC 3-$\rho$-LIE ALGEBRAS

We begin this section by describing the class of $\rho$-Lie algebras, which includes the special multiplication that is known as bracket and a two-cycle. This Lie algebras is called 3-$\rho$-Lie algebra. We summarize some definitions and introduce the notions such as quadratic structure, symplectic structure $\omega$, metric $\varphi$, derivation $D$ and $\varphi$-antisymmetric derivation $D$ on 3-$\rho$-Lie algebras and show that a symplectic structure $\omega$ may be defined on a 3-$\rho$-Lie algebra if and only if there exists a $\varphi$-antisymmetric invertible derivation $D$. Also, We give a brief exposition of 3-associative algebras and give the constructions of direct product 3-$\rho$-Lie algebra structures from 3-associative algebras.

Let us consider $(G, +)$ as an abelian group over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). A map $\rho : G \times G \rightarrow \mathbb{K}^*$ with the properties

$$\rho(a, b) = \rho(b, a)^{-1}, \quad a, b \in G,$$

$$\rho(a + b, c) = \rho(a, c)\rho(b, c), \quad a, b, c \in G,$$

is called a two-cycle. From two above properties we can deduce $\rho(a, b) \neq 0$, $\rho(0, b) = 1$ and $\rho(c, c) = \pm 1$ for all $a, b, c \in G$, $c \neq 0$. Note that if $B$ is a $G$-graded vector space ($B = \oplus_{a \in G} B_a$), an element $f \in B_a$ is called homogeneous of $G$-degree $a$. We denote by $Hg(B)$ the set of homogeneous elements in $B$ and by $|f|$ the $G$-degree of $f \in Hg(B)$.

A $\rho$-Lie algebra is a triple $(B, [, ]_B, \rho)$, where

i. $B$ is a $G$-graded vector space,

ii. $\rho$ is a two-cycle,

iii. $[,]_B$ is a bilinear map on $B$ such that

- $|[f, g]|_B = |f| + |g|,$
- $[f, g]_B = -\rho(|f|, |g|)[g, f]_B,$
- $\rho(|h|, |f|)[f, [g, h]]_B + \rho(|g|, [h |f, g]]_B + \rho(|f|, |g|)[g, [h, f]]_B = 0.$

The third condition is equivalent to

$$[f, [g, h]]_B = [[f, g]_B, h]_B + \rho(|f|, |g|)[g, [f, h]]_B.$$

In the next, for simplicity we use $\rho(f, g)$ instead of $\rho(|f|, |g|)$.

Definition 2.1. A triple $(B, [, ]_B, \rho)$ is called a 3-$\rho$-Lie algebra if $B$ is a $G$-graded vector space, $[,]_B : B \times B \times B \rightarrow B$ is a 3-linear map, $\rho$ is a two-cycle and moreover the following conditions are satisfied

1. $|[f_1, f_2, f_3]|_B = |f_1| + |f_2| + |f_3|,$

2. $[f_1, f_2, [g_1, g_2, g_3]|_B = [[f_1, f_2, g_1]|_B, g_2, g_3]|_B + \rho(f_1 + f_2, g_1)[g_1, [f_1, f_2, g_2]|_B, g_3]|_B + \rho(f_1 + f_2, g_1 + g_2)[g_1, g_2, [f_1, f_2, g_3]|_B, \rho$-fundamental identity)

3. $[f_1, f_2, f_3]_B = \rho(f_1, f_2)[f_2, f_1, f_3]|_B = \rho(f_2, f_3)[f_1, f_3, f_2]|_B.$ (\rho$-skew-symmetry property)
We denote the 3-$\rho$-Lie algebra $(\mathcal{B},[\ldots,]_{\mathcal{B}},\rho)$ by $\mathcal{B}_\rho$.

**Example 2.2.** One of the most important case of 3-$\rho$-Lie algebras are 3-ary Lie superalgebras when abelian group $G$ is $\mathbb{Z}_2$.

In the following, we extend to 3-$\rho$-Lie algebras the study of quadratic color Hom-Lie algebras introduced in [5] and in the next of this paper we investigate Hom version. We apply a non-degenerate bilinear form on 3-$\rho$-Lie algebra $\mathcal{B}_\rho$ to define a quadratic structure.

**Definition 2.3.** Let $\chi : \mathcal{B}_\rho \times \mathcal{B}_\rho \rightarrow \mathbb{K}$ be a non-degenerate bilinear form on $\mathcal{B}_\rho$. Then $\chi$ is said to be a quadratic structure if for any $f_1,f_2,f_3,f_4 \in H_{\mathcal{B}}(\mathcal{B}_\rho)$, the following conditions are satisfied

i. $\chi(f_1,f_2) = \rho(f_1,f_2)\chi(f_2,f_1)$ (\rho-symmetric),

ii. $\chi([f_1,f_2,f_3]_{\mathcal{B}},f_4) = \chi(f_1,[f_2,f_3,f_4]_{\mathcal{B}})$ (invariant).

Also $(\mathcal{B},[\ldots,]_{\mathcal{B}},\rho,\chi)$ is said to be a quadratic 3-$\rho$-Lie algebra if $\chi$ is a quadratic structure of $\mathcal{B}$. We use the symbol $\mathcal{B}_{\rho,\chi}$ to represent any quadratic 3-$\rho$-Lie algebra $(\mathcal{B},[\ldots,]_{\mathcal{B}},\rho,\chi)$.

**Definition 2.4.** Let $\text{End}(\mathcal{B}_\rho)$ be the endomorphism algebra of $\mathcal{B}_\rho$. A vector subspace $\text{Cent}(\mathcal{B}_\rho)$ of $\text{End}(\mathcal{B}_\rho)$ is defined by

$$\text{Cent}(\mathcal{B}_\rho) = \{\psi \in \text{End}(\mathcal{B}_\rho) : \psi([f,g,h]_{\mathcal{B}}) = [\psi(f),g,h]_{\mathcal{B}} = \rho(\psi,f)[f,\psi(g),h]_{\mathcal{B}} = \rho(\psi,f+g)[f,g,\psi(h)]_{\mathcal{B}}\}.$$  (2.3)

Cent$(\mathcal{B}_\rho)$ is a subalgebra of $\text{End}(\mathcal{B}_\rho)$ and it is called the centroid of $\mathcal{B}_\rho$.

In the following proposition, we construct 3-$\rho$-Lie algebras from a 3-$\rho$-Lie algebra and an even element in its centroid.

**Proposition 2.5.** Let $\psi \in \text{Cent}(\mathcal{B}_\rho)$ be an even element. If we define two multiplications $[\cdot,\cdot,\cdot]_{\psi}$ and $[\cdot,\cdot,\cdot]_{\psi}^\psi$, on $\mathcal{B}$ by

$$[f,g,h]_{\psi} = [\psi(f),g,h]_{\mathcal{B}}, \quad [f,g,h]_{\psi}^\psi = [\psi(f),\psi(g),\psi(h)]_{\mathcal{B}}, \quad \forall f,g,h \in \mathcal{B}_\rho,$$

then we have two 3-$\rho$-Lie algebras $(\mathcal{B},[\cdot,\cdot,\cdot]_{\psi},\rho)$ and $(\mathcal{B},[\cdot,\cdot,\cdot]_{\psi}^\psi,\rho)$.

**Proof.** The definition of the bracket $[\cdot,\cdot,\cdot]_{\psi}$ and this fact that $\psi$ is an even element in $\text{Cent}(\mathcal{B}_\rho)$, lead to

(a) $[f_1,f_2,[g_1,g_2,g_3]_{\mathcal{B}}]_{\psi} = \psi^2[f_1,f_2,[g_1,g_2,g_3]_{\mathcal{B}}]_{\mathcal{B}},$

(b) $[[f_1,f_2,g_1]_{\mathcal{B}},g_2,g_3]_{\psi} = \psi^2[[f_1,f_2,g_1]_{\mathcal{B}},g_2,g_3]_{\mathcal{B}},$

(c) $\rho(f_1+f_2,g_1)[g_1,[f_1,f_2,g_2]_{\mathcal{B}},g_3]_{\psi} = \rho(f_1+f_2,g_1)\psi^2[g_1,[f_1,f_2,g_2]_{\mathcal{B}},g_3]_{\mathcal{B}},$

(d) $\rho(f_1+f_2,g_1+g_2)[g_1,[f_1,f_2,g_2]_{\mathcal{B}},g_3]_{\psi} = \rho(f_1+f_2,g_1+g_2)\psi^2[g_1,[f_1,f_2,g_2]_{\mathcal{B}},g_3]_{\mathcal{B}}.$

According to the above equalities and this fact that $\mathcal{B}_\rho$ is a 3-$\rho$-Lie algebra, it is easy to show that $[\cdot,\cdot,\cdot]_{\psi}$ is a 3-$\rho$-Lie algebra structure. Also, by the similar way, we can show that $(\mathcal{B},[\cdot,\cdot,\cdot]_{\psi}^\psi,\rho)$ is a 3-$\rho$-Lie algebra.

**Proposition 2.6.** Consider $\mathcal{B}_{\rho,\chi}$ and an even invertible $\chi$-symmetric element $\psi \in \text{Cent}(\mathcal{B}_\rho)$. If we define the even bilinear form $\chi_{\psi}$ by $\chi_{\psi}(f,g) = \chi(\psi(f),g)$, then we have two quadratic 3-$\rho$-Lie algebras $(\mathcal{B},[\ldots,\ldots]_{\psi},\rho,\chi_{\psi})$ and $(\mathcal{B},[\ldots,\ldots]_{\psi}^\psi,\rho,\chi_{\psi})$. □
Proof. \( \chi \psi \) is a non-degenerate and \( \rho \)-symmetric bilinear form, since \( \psi \) is invertible and \( \chi \) is a non-degenerate, \( \rho \)-symmetric bilinear form on \( B \). Then, it is enough to show that \( \chi \psi \) is invariant. For this, we have

\[
\chi \psi([f_1, f_2, f_3]_\psi, f_4) = \chi(\psi(f_1), f_2, [f_3, f_4]_A) = \chi([\psi(f_1), f_2, f_3], \psi(f_4)) = \chi(\psi(f_1), [f_2, f_3, \psi(f_4)]_A) = \chi \psi(f_1, [f_2, f_3, f_4]_\psi).
\]

Also, we can show that \( \chi \psi([f_1, f_2, f_3]_\psi, f_4) = \chi \psi(f_1, [f_2, f_3, f_4]_\psi) \), where \( f_1, f_2, f_3, f_4 \) are homogeneous elements of \( B \).

\[\Box\]

**Definition 2.7.** A 3-associative algebra is a pair \((g, \mu)\) consisting of a \( G \)-graded vector space \( g \) and an even 3-linear map \( \mu : g \times g \times g \to g \) (i.e. \( \mu(g_a, g_b, g_c) \subseteq g_{a+b+c} \)), such that for all \( a, b, c, d, e \in g \)

\[\mu(a, b, \mu(c, d, e)) = \mu(a, \mu(b, c), d, e).\]

If \( \rho \) is a two-cycle on \( G \) and the linear map \( \mu \) is the \( \rho \)-symmetry with respect to the displacement of every two elements, the 3-associative algebra \((g, \mu)\) is called commutative and denoted by \( g_{\rho, C} \).

**Definition 2.8.** A quadratic 3-associative algebra is a 3-associative algebra \((g, \mu)\) equipped with a \( \rho \)-symmetric, invariant and non-degenerate bilinear form \( \chi_g \). We denote a quadratic 3-associative algebra \((g, \mu, \chi_g)\) and a commutative quadratic 3-associative algebra respectively by \( g_{\chi_g} \) and \( g_{\rho, C, \chi_g} \).

Note that if \( A \) and \( A' \) are two \( G \)-graded vector spaces, then \( A \otimes A' \) is also a \( G \)-graded vector space such that

\[ A \otimes A' = \bigoplus_{\gamma \in G}(A \otimes A')_{\gamma}, \]

where \( (A \otimes A')_{\gamma} = A_a \otimes A'_{a'} \) and so \( f \otimes g \in (A \otimes A')_{\gamma} \) has the degree \( |f \otimes g| = \gamma = a + a' \).

According to the above point, we have the following theorem, which gives us a quadratic 3-\( \rho \)-Lie algebra structure starting from a quadratic 3-\( \rho \)-Lie algebra and a quadratic commutative 3-associative algebra.

**Theorem 2.9.** Consider \( B_{\rho, \chi} \) and \( g_{\rho, C, \chi_g} \). The tensor product \((E = B \otimes g, [\cdot, \cdot, \cdot]_E, \rho, \chi_E)\) is a quadratic 3-\( \rho \)-Lie algebra, such that \([\cdot, \cdot, \cdot]_E\) and \( \chi_E \) are given by

\[
[f \otimes a, g \otimes b, h \otimes c]_E = \rho(a, g + h)\rho(b, h)[f, g, h]_B \otimes \mu(a, b, c), \tag{2.4}
\]

\[
\chi_E(f \otimes a, g \otimes b) = \rho(a, g)\chi(f, g)\chi_g(a, b), \tag{2.5}
\]

for all \( f, g, h \in Hg(B_{\rho, \chi}) \) and \( a, b, c \in Hg(g_{\rho, \chi}) \).
Proof. Using [2,4] and the commutative property of product $\mu$, we get

\[
[f \otimes a, g \otimes b, [h_1 \otimes c, h_2 \otimes d, h_3 \otimes e]_E]_E = \rho(a + b, h_1 + h_2 + h_3)\rho(c, h_2 + h_3)\rho(d, h_3)\rho(a, g)
\]

\[
[f, g, [h_1, h_2, h_3]_B]_E \otimes \mu(a, b, \mu(c, d, e)),
\]

\[
[[f \otimes a, g \otimes b, h_1 \otimes c]_E, h_2 \otimes d, h_3 \otimes e]_E = \rho(a + b, h_1 + h_2 + h_3)\rho(c, h_2 + h_3)\rho(d, h_3)\rho(a, g)
\]

\[
[[f, g, h_1]_B, h_2, h_3]_B \otimes \mu(a, b, \mu(c, d, e)),
\]

\[
\rho(f + a + g + b, h_1 + c)[h_1 \otimes c, [f \otimes a, g \otimes b, h_2 \otimes d]_E, h_3 \otimes e]_E = \rho(a + b, h_1 + h_2 + h_3)\rho(c, h_2 + h_3)
\]

\[
\rho(d, h_3)\rho(a, g)\rho(f + g, h_1 + h_2 + h_3)[h_1, [f, g, h_2]_B, h_3]_B \otimes \mu(a, b, \mu(c, d, e)),
\]

\[
\rho(f + a + g + b, h_1 + c + h_2 + d)[h_1 \otimes c, h_2 \otimes d, [f \otimes a, g \otimes b, h_3 \otimes e]_E]_E = \rho(a + b, h_1 + h_2 + h_3)
\]

\[
\rho(c, h_2 + h_3)\rho(d, h_3)\rho(a, g)\rho(f + g, h_1 + h_2)[h_1, h_2, [f, g, h_3]_B]_B \otimes \mu(a, b, \mu(c, d, e)),
\]

where $f, g, h_1, h_2, h_3 \in Hg(B_{p,\chi})$ and $a, b, c, d, e \in Hg(g_{p,\chi,c})$. So $[\ldots, \ldots]_E$ is a 3-$\rho$-Lie algebra structure on $E$. Also, we have

\[
\chi_E(f \otimes a, g \otimes b) = \rho(f + g, g + b)\chi_E(g \otimes b, f \otimes a),
\]

and

\[
\chi_E([f \otimes a, g \otimes b, h \otimes c]_E, h_1 \otimes d) = \rho(a, g + h)\rho(b, h)(([f, g, h]_A \otimes \mu(a, b, c, h_1 \otimes d) = \rho(a, g + h)\rho(b, h)\rho(a + b + c, h_2)\chi([f, g, h]_A, h_1)\chi_p(\mu(a, b, c, d))
\]

\[
= \rho(a, g + h)\rho(b, h)\rho(a + b + c, h_2)\chi(f, [g, h, h_1]_B)\chi_g(a, \mu(b, c, d))
\]

\[
= \rho(b, h)\rho(b + c, h_1)\chi_E(f \otimes a, [g, h, h_1] \otimes \mu(b, c, d))
\]

\[
= \chi_E(f \otimes a, [g \otimes b, h \otimes c, h_1 \otimes d]_E),
\]

which imply $\chi_E$ is a quadratic structure. \hfill \Box

2.1. Ideal of 3-$\rho$-Lie algebras. In this part, we introduce the definitions of subalgebra and ideal of $B_{\rho}$ and give some properties related to ideals of $B_{\rho}$.

Definition 2.10. A subalgebra of $B_{\rho}$ is a sub-vector space $I \subseteq B_{\rho}$ such that $[I, I, I]_B \subseteq I$. $I$ is also called an ideal of $B_{\rho}$ if $[I, B_{\rho}, B_{\rho}]_B \subseteq I$.

Definition 2.11. Let $I$ be an ideal of $B_{\rho,\chi}$.

(i) $I$ is said to be non-degenerate if $\chi|_{I \times I}$ is non-degenerate.

(ii) The orthogonal $I^\perp$ of $I$, with respect to $\chi$, is defined by

\[
I^\perp = \{ f \in B_{\rho}, \chi(f, g) = 0 \ \forall g \in I \}. \quad (2.6)
\]

Lemma 2.12. Let $Z(B_{\rho})$ be the center of $B_{\rho}$ and defined by

\[
Z(B_{\rho}) = \{ f \in B_{\rho} | [f, g, h]_B = 0 \ \forall g, h \in B_{\rho} \}. \quad (2.7)
\]

Then $Z(B_{\rho})$ is an ideal of $B_{\rho}$.

Proof. It is clear that $[Z(B_{\rho}), B_{\rho}, B_{\rho}] = \{0\} \subseteq Z(B_{\rho})$. \hfill \Box

Lemma 2.13. Let $I$ be an ideal of $B_{\rho,\chi}$. Then $I^\perp$ with respect to $\chi$ is an ideal of $B_{\rho,\chi}$. 
Definition 2.16. For any $f, g, h$ given in [19],

\[
\rho(\rho([f, g, h]_B, s)) = \chi(f, [g, h, s]_B) = \rho(g + h, s)\chi(f, [s, g, h]) = 0,
\]
since $f \in I^\perp$ and $I$ is an ideal of $B_{\rho, \chi}$.

\[\square\]

**Symplectic Structure:** In the following, we introduce the notions like symplectic structure $\omega$, metric $\varphi$, derivation $D$ and $\varphi$-antisymmetric derivation $D$ on $B_\rho$ and show that a symplectic structure $\omega$ may be defined on $B_\rho$ if and only if there exists a $\varphi$-antisymmetric invertible derivation $D$ of $B_\rho$.

Definition 2.14. A non-degenerate and $\rho$-skew-symmetric bilinear form $\omega \in \wedge^2 B^*$ is called a symplectic structure on $B_\rho$ if

\[
\omega([f_1, f_2, f_3]_B, f_4) - \rho(f_1, f_2 + f_3 + f_4)\omega([f_2, f_3, f_4]_B, f_1) + \rho(f_1 + f_2, f_3 + f_4)\omega([f_3, f_4, f_1]_B, f_2) - \rho(f_1 + f_2 + f_3, f_4)\omega([f_4, f_1, f_2]_B, f_3) = 0.
\]

We will say that $(B, [,..,], B_\rho, \chi, \omega)$ is a quadratic symplectic 3-$\rho$-Lie algebra if $(B, \chi)$ is quadratic and $(B, \omega)$ is symplectic.

In the next example, we are going to check the 4-dimensional 3-$\rho$-Lie algebra, which it’s classical case is given in [19].

Example 2.15. Let us consider $B$ as a $G = \mathbb{Z}_3^2$-graded vector space with the basis $\{l_1, l_2, l_3, l_4\}$. Then $B = B_{(1,0,0)} \oplus B_{(0,1,0)} \oplus B_{(0,0,1)} \oplus B_{(1,1,1)}$ with

\[
l_1 \in B_{(1,0,0)}, \quad l_2 \in B_{(0,1,0)}, \quad l_3 \in B_{(0,0,1)}, \quad l_4 \in B_{(1,1,1)}.
\]

Also, $\rho : G \times G \to \mathbb{C}^*$ is defined by the matrix

\[
\begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 \\
-1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix},
\]

where $(a, b) \in \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\} \times \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$. The 3-$\rho$-Lie algebra bracket associate to the basis $\{l_1, l_2, l_3, l_4\}$ is given by

\[
[l_1, l_2, l_3]_B = l_4, \quad [l_1, l_2, l_4]_B = l_3, \\
[l_2, l_3, l_4]_B = l_1, \quad [l_1, l_3, l_4]_B = l_2.
\]

It is clear that any non-degenerate $\rho$-skew symmetric bilinear form is a symplectic structure on $B$. For example

\[
\omega_1 = l_3^* \wedge l_1^* + l_4^* \wedge l_2^*, \quad \omega_2 = l_2^* \wedge l_1^* + l_4^* \wedge l_3^*, \quad \omega_3 = l_2^* \wedge l_3^* + l_4^* \wedge l_1^*,
\]

\[
\omega_4 = l_3^* \wedge l_2^* + l_4^* \wedge l_3^*, \quad \omega_5 = l_2^* \wedge l_1^* + l_3^* \wedge l_4^*, \quad \omega_6 = l_1^* \wedge l_3^* + l_2^* \wedge l_4^*,
\]

where $\{l_1^*, l_2^*, l_3^*, l_4^*\}$ is the dual basis of $\{l_1, l_2, l_3, l_4\}$.

Definition 2.16. A derivation on $B_\rho$ is a linear map $D : B_\rho \to B_\rho$ such that

\[
D[f, g, h]_B = [D(f), g, h] + \rho(D, f)[f, D(g), h] + \rho(D, f + g)[f, g, D(h)],
\]

for any $f, g, h \in Hg(B_\rho)$. Let us denote by $\text{Der}(B_\rho)$ the space of all derivations of $B_\rho$. 
Definition 2.17. A metric on ρ-Lie algebra $B$ is a non-degenerate ρ-symmetric bilinear form $\varphi : B \times B \rightarrow \mathbb{K}$ such that

$$\varphi([f,g],h) = -\rho(f,g)\varphi(g,[f,h]).$$

Also, non-degenerate ρ-symmetric bilinear form $\varphi$ is said to be a metric on $B_\rho$ if

$$\varphi([f,g,h],z) + \rho(f + g,h)\varphi(h,[f,g,z]) = 0, \quad \forall f,g,h,z \in Hg(B_\rho).$$

In this case, $(B_\rho,\varphi)$ is called a metric 3-ρ-Lie algebra and we denote it by $B_{\rho,\varphi}$.

Definition 2.18. Consider $B_{\rho,\chi}$ ($(B_\rho,\varphi)$ ) and let $D$ be an endomorphism or a derivation of $B_\rho$. $D$ is called χ-antisymmetric (ϕ-antisymmetric) if

$$\chi(D(f),g) = -\rho(D,f)\chi(f,D(g)) \quad (\varphi(D(f),g) = -\rho(D,f)\varphi(f,D(g))),$$

for all $f,g \in Hg(B_\rho)$. Denote by End$_\chi(B_\rho)$ (End$_\varphi(B_\rho)$) or Der$_\chi(B_\rho)$ (Der$_\varphi(B_\rho)$) the space of χ-antisymmetric (ϕ-antisymmetric) endomorphism or derivation of $B_\rho$.

Proposition 2.19. Consider the 3-ρ-Lie algebra $B_\rho$. The following assertions are equivalent:

i. The existence of a symplectic structure on $B_\rho$.

ii. The existence of an invertible derivation $D \in \text{Der}_\varphi(B_\rho)$.

Proof: At first, we assume that there is an invertible derivation $D$ on $B_\rho$ such that $\varphi(D(f),g) = -\rho(D,f)\varphi(f,D(g))$. For $f, g \in B_\rho$, define $\omega(f,g) = \varphi(D(f),g)$. Thus $\omega$ is a non-degenerate and ρ-symmetric bilinear form. Now, we show that it is a symplectic structure. For homogeneous elements $f, g, h, z \in B_\rho$, we deduce

$$\omega([f_1,f_2,f_3]_B,f_4) = -\rho(f_1 + f_2 + f_3 + f_4)\omega([f_2,f_3,f_4]_B,f_1) + \rho(f_1 + f_2,f_3 + f_4)\omega([f_3,f_4,f_1]_B,f_2)$$

$$\quad - \rho(f_1 + f_2 + f_3,f_4)\omega([f_4,f_1,f_2]_B,f_3)$$

$$= \varphi(D[f_1,f_2,f_3]_B,f_4) - \rho(f_1 + f_2 + f_3 + f_4)\varphi(D[f_2,f_3,f_4]_B,f_1)$$

$$\quad + \rho(f_1 + f_2,f_3 + f_4)\varphi(D[f_3,f_4,f_1]_B,f_2) - \rho(f_1 + f_2 + f_3,f_4)\varphi(D[f_4,f_1,f_2]_B,f_3)$$

$$= \varphi(D[f_1,f_2,f_3]_B,f_4) - \varphi(D[f_1,f_2,f_3]_B,f_4) = 0.$$

Conversely is similar. $\square$

Consider the quadratic 3-ρ-Lie algebra $B_{\rho,\chi}$. For any symmetric bilinear form $\chi'$ on $B_\rho$, there is an associated map $D : B_\rho \rightarrow B_\rho$ satisfying

$$\chi'(f,g) = \chi(D(f),g), \quad \forall f,g \in B_\rho.$$  

(2.9)

Since $\chi$ and $\chi'$ are symmetric, then $D$ is symmetric with respect to $\chi'$, i.e.,

$$\chi(D(f),g) = \rho(D,f)\chi'(f,D(g)), \quad \forall f,g \in Hg(B_\rho).$$

(2.10)

Lemma 2.20. Let $\chi'$ be defined by (2.9) on $B_\rho$. Then

i. $\chi'$ is invariant if and only if $D$ satisfies

$$D([f,g,h]_B) = [D(f),g,h]_B = \rho(D,f)[f,D(g),h]_B = \rho(D,f + g)[f,g,D(h)]_B.$$  

(2.10)

ii. $\chi'$ is non-degenerate if and only if $D$ is invertible.
Proof. (i) Assuming \( f, g, h, s \in \mathcal{B}_\rho \), we have
\[
\chi'([f, g, h]_{\mathcal{B}}, s) = \chi(D([f, g, h]_{\mathcal{B}}), s) \quad \text{and} \quad \chi'(f, [g, h, s]_{\mathcal{B}}) = \chi(D(f), [g, h, s]_{\mathcal{B}}).
\]
Since \( \chi \) is invariant, so \( \chi'(f, [g, h, s]_{\mathcal{B}}) = \chi([D(f), g, h]_{\mathcal{B}}, s) \). On the other hand, since \( \chi \) is non-degenerate, therefore \( \chi' \) is invariant if and only if \( D([f, g, h]_{\mathcal{B}}) = [D(f), g, h]_{\mathcal{B}} \). Also, since 3-Lie algebra structure \([\ldots, \cdot]_{\mathcal{B}}\) is anticommutative with respect to displacement of every two elements, then \( D([f, g, h]_{\mathcal{B}}) = \rho(D, f)[f, D(g), h]_{\mathcal{B}} = \rho(D, f + g)[f, g, D(h)]_{\mathcal{B}} \).

(ii) We assume that \( \chi' \) is non-degenerate and \( f \in \mathcal{B}_\rho \) such that \( D(f) = 0 \). Then \( \chi(D(f), \mathcal{B}_\rho) = 0 \) and so \( \chi'(f, \mathcal{B}_\rho) = 0 \). Therefore \( f = 0 \) and \( D \) is invertible. Conversely, assume that \( D \) is invertible and \( \chi'(f, \mathcal{B}_\rho) = 0 \). So \( \chi(D(f), \mathcal{B}_\rho) = 0 \). Since \( \chi \) is non-degenerate, thus \( D(f) = 0 \). \( D \) is invertible, therefore \( f = 0 \) and \( \chi' \) is non-degenerate. \( \square \)

**Definition 2.21.** A \( \rho \)-symmetric map \( D : \mathcal{B}_\rho \rightarrow \mathcal{B}_\rho \) satisfying \( (2.10) \) is called a centromorphism of \( \mathcal{B}_\rho \). The space of centromorphisms of \( \mathcal{B}_\rho \) is denote by \( \mathcal{C}(\mathcal{B}_\rho) \).

**Proposition 2.22.** Let \( \delta \in \text{Der}_\chi(\mathcal{B}_\rho) \) and \( D \in \mathcal{C}(\mathcal{B}_\rho) \) such that \( D \circ \delta = \delta \circ D \). Then \( D \circ \delta \) is also a \( \chi \)-antisymmetric derivation of \( \mathcal{B}_\rho \).

**Proof.** We have
\[
D \circ \delta[f, g, h] = D[\delta(f), g, h]_{\mathcal{B}} + \rho(\delta, f)D[f, \delta(g), h]_{\mathcal{B}} + \rho(\delta, f + g)[f, g, \delta(h)]_{\mathcal{B}}
\]
\[
= [D \circ \delta(f), g, h]_{\mathcal{B}} + \rho(D + \delta, f)D[f, D \circ \delta(g), h]_{\mathcal{B}} + \rho(\delta, f + g)[f, g, D \circ \delta(h)]_{\mathcal{B}},
\]
and \( B(D \circ \delta(f), g) = \rho(D + \delta, f)B(f, \delta \circ D(g)) = -\rho(D + \delta, f)B(f, D \circ \delta(g)) \) for all \( f, g \in HG(\mathcal{B}_\rho) \). So we deduce that \( D \circ \delta \in \text{Der}_\chi(\mathcal{B}_\rho) \). \( \square \)

### 3. Representation of 3-\( \rho \)-Lie algebras

In this section we extend the representation theory of 3-Lie algebras introduced in [12] and [26] to color Lie version. We introduce a representation of 3-\( \rho \)-Lie algebras and discuss the cases of adjoint and coadjoint representations for 3-\( \rho \)-Lie algebras.

Consider \( \mathcal{L} = \wedge^2 \mathcal{B}_\rho \) and call it the fundamental set. Defining
\[
[(f_1, f_2), (g_1, g_2)]_{\mathcal{L}} = ([f_1, f_2, g_1], g_2) + \rho(f_1 + f_2, g_1)(g_1, [f_1, f_2, g_2]), \quad (3.1)
\]
we construct a \( \rho \)-Lie algebra structure on \( \mathcal{L} \).

**Example 3.1.** Let \((\mathcal{B}, [\ldots, \cdot]_{\mathcal{B}}, \rho, \varphi)\) be a metric 3-\( \rho \)-Lie algebra. Then \((\mathcal{L}, [\ldots, \cdot]_{\mathcal{L}}, \varphi_{\mathcal{L}})\) is a metric \( \rho \)-Lie algebra, where
\[
\varphi_{\mathcal{L}}((f_1, f_2), (g_1, g_2)) = \varphi([f_1, f_2, g_1]_{\mathcal{B}}, g_2).
\]
Definition 3.2. The pair \((V, \mu)\) is called a representation of \(\mathcal{B}_\rho\) if the following conditions are satisfied

\[
\mu[(f_1, f_2), (g_1, g_2)] = \mu(f_1, f_2)\mu(g_1, g_2) - \rho(f_1 + f_2, g_1 + g_2)\mu(g_1, g_2)\mu(f_1, f_2),
\]

\[
\mu([g_1, g_2, g_3]_B, f) = \mu(g_1, g_2)\mu(g_3, f) + \rho(g_1, g_2 + g_3)\mu(g_2, g_3)\mu(g_1, f) + \rho(g_1 + g_2, g_3)\mu(g_3, g_1)\mu(g_2, f),
\]

\[
\mu(g, [f_1, f_2, f_3]_B) = \rho(g, f_1 + f_2)\mu(f_1, f_2)\mu(g, f_3) + \rho(g, f_2 + f_3)\mu(f_1, f_2 + f_3)\mu(f_2, f_3)\mu(g, f_1) + \rho(g, f_1 + f_3)\mu(f_1 + f_2, f_3)\mu(g, f_2),
\]

where \(V\) is a \(G\)-graded vector space and \(\mu\) is a linear map from \(\mathcal{L} = \wedge^2 \mathcal{B}_\rho\) to \(\mathfrak{gl}(V)\).

One of the important examples of the \(3-\rho\)-Lie algebra representation is the adjoint representation, which is a bilinear map \(\text{ad} : \mathcal{B}_\rho \times \mathcal{B}_\rho \rightarrow \mathfrak{gl}(\mathcal{B}_\rho)\) defined by \(\text{ad}(f_1, f_2)(f_3) = [f_1, f_2, f_3]_B\), that is a representation of \(\mathcal{B}_\rho\) on itself.

Lemma 3.3. For the representation \((V, \mu)\) of \(\mathcal{B}_\rho\), we have

\[
0 = \rho(f_1 + f_2, g_1)\mu(g_1, [f_1, f_2, g_2]_B) + \mu([f_1, f_2, g_1]_B, g_2)
+ \rho(f_1 + f_2, g_1 + g_2)\rho(g_1 + g_2, f_1)\mu(f_1, [g_1, g_2, f_2]_B) + \rho(f_1 + f_2, g_1 + g_2)\mu([g_1, g_2, f_1]_B, f_2).
\]

Proof. Using Definition 3.2, we obtain

\[
\rho(f_1 + f_2, g_1)\mu(g_1, [f_1, f_2, g_2]_B) + \mu([f_1, f_2, g_1]_B, g_2)
+ \rho(f_1 + f_2, g_1 + g_2)\mu([g_1, g_2, f_1]_B, f_2)
+ \rho(f_1 + f_2, g_1 + g_2)\rho(g_1 + g_2, f_1)\mu(f_1, [g_1, g_2, f_2]_B)
= \mu(f_1, f_2)\mu(g_1, g_2) - \rho(f_1 + f_2, g_1 + g_2)\mu(g_1, g_2)\mu(f_1, f_2)
+ \rho(f_1 + f_2, g_1 + g_2)\mu(g_1, g_2)\mu(f_1, f_2) - \mu(f_1, f_2)\mu(g_1, g_2) = 0.
\]

Consider the pair \((\mathcal{B}_\rho, V)\), where \(V\) is a \(G\)-graded vector space. The direct sum \(\mathcal{B} \oplus V\) is also graded and \((\mathcal{B} \oplus V)_a = \mathcal{B}_a \oplus V_a\). Furthermore, a homogeneous element of \(\mathcal{B} \oplus V\) has the form \(f + v\) such that \(f \in \mathcal{B}_\rho\) and \(v \in V\) and \(|f + v| = |f| = |v|\).

In the following, we discuss some properties of 3-\(\rho\)-Lie algebras representations. Actually, the following theorem examines the relationship between the 3-\(\rho\)-Lie algebra representation and the existence of 3-\(\rho\)-Lie algebra structure on the graded vector space \(\mathcal{B} \oplus V\):

Proposition 3.4. The pair \((V, \mu)\) is a representation of \(\mathcal{B}_\rho\) if and only if \((\mathcal{B} \oplus V, [\ldots, \cdot, \cdot]_{\mathcal{B} \oplus V})\) is a 3-\(\rho\)-Lie algebra, where the structure \([\ldots, \cdot, \cdot]_{\mathcal{B} \oplus V}\) is given by

\[
[f_1 + v_1, f_2 + v_2, f_3 + v_3]_{\mathcal{B} \oplus V} = [f_1, f_2, f_3]_B + \rho(f_1, f_2)\mu(f_1, f_2)v_3 + \rho(f_1, f_2 + f_3)\mu(f_2, f_3)v_1 + \rho(f_1 + f_2, f_3)\mu(f_3, f_1)v_2,
\]

for homogeneous elements \(f_1, f_2, f_3 \in \mathcal{B}_\rho\) and \(v_1, v_2, v_3 \in V\).

Proof. The proof of this proposition follows by a direct computation and the definition of the representation. □
In the following, we describe the dual representations and coadjoint representations of 3-\(\rho\)-Lie algebras such as the adjoint representations. At first, we recall the dual space of a graded vector space and explore the graded space of direct sum of a graded vector space and its dual.

Let \(B = \oplus_{a \in G} B_a\) be a \(G\)-graded space. Then \(B^*\), the dual space of \(B\), is also \(G\)-graded space and

\[ B^*_a = \{ \alpha \in B^*| \alpha(f) = 0, \ \forall f: |f| \neq -a \}. \]

Furthermore, the direct sum \(B \oplus B^*\) is \(G\)-graded, where

\[ B \oplus B^* = \oplus_{a \in G}(B \oplus B^*)_a = \oplus_{a \in G}(B_a \oplus B_a^*), \]

A homogeneous element of \(B \oplus B^*\) has the form \(f + \alpha\) such that \(f \in B\) and \(\alpha \in B^*\) and \(|f + \alpha| = |f| = |\alpha|\).

**Proposition 3.5.** Let \((V, \mu)\) be a \(\rho\)-skew-symmetric representation of \(B_\rho\) and let \(V^*\) be the dual vector space of \(V\). Defining the linear map \(\mu^*: B_\rho \times B_\rho \rightarrow \text{gl}(V^*)\) by \(\mu^*(f_1, f_2)(g) = -\rho(f_1 + f_2, g) \circ \mu(f_1, f_2)\), where \(f_1, f_2 \in B, g \in V^*\), \((V^*, \mu^*)\) defines a representation of \(B_\rho\), which is called the dual representation.

**Proof.** By invoking Definition 4.2 and Lemma 3.3 we can prove this proposition by direct computations. \(\square\)

**Remark 3.6.** An example of the dual representation is the linear map \(\text{ad}^*: B_\rho \times B_\rho \rightarrow \text{gl}(B_\rho^*)\) defined by

\[ \text{ad}^*(f_1, f_2)(g)(f_3) = -\rho(f_1 + f_2, g)[f_1, f_2, f_3]_B = -\rho(f_1 + f_2, g)\text{ad}(f_1, f_2)(f_3), \]

for \(f_1, f_2, f_3 \in Hg(B_\rho)\) and \(g \in B_\rho^*\), which is a representation of \(B\) on \(B_\rho^*\) (\(\text{ad}^*\) is called coadjoint representation). Since \(\text{ad}^*\) is a representation of \(B_\rho\), then by Proposition 4.4 \(B \oplus B^*\) is a 3-\(\rho\)-Lie algebra by the following structure

\[ [f_1 + (\alpha_1, f_2 + \alpha_2, f_3 + \alpha_3), (\alpha_1, \alpha_2, \alpha_3)]_{B \oplus B^*} = [f_1, f_2, f_3]_B + \text{ad}^*(f_1, f_2)\alpha_3 \]

\[ + \rho(f_1, f_2 + f_3)\text{ad}^*(f_2, f_3)\alpha_1 + \rho(f_1 + f_2, f_3)\text{ad}^*(f_3, f_1)\alpha_2. \]

Considering \(\varphi: B \oplus B^* \times B \oplus B^* \rightarrow \mathbb{K}\) by \(\varphi(f + \alpha, g + \beta) = \alpha(g) + \rho(f, g)\beta(f)\), we have a metric 3-\(\rho\)-Lie algebra \((B \oplus B^*, [\cdot, \cdot], \text{ad}^*, \rho, \varphi)\).

4. 3-pre-\(\rho\)-Lie Algebra

In \cite{12} the notion of 3-pre-Lie algebras was introduced as a generalization of pre-Lie algebras. In this section, we introduce the notion of 3-pre-\(\rho\)-Lie algebras and investigate their properties. Also, the 3-pre-\(\rho\)-Lie algebras representations and the phase spaces of 3-pre-\(\rho\)-Lie algebras will be examined.

**Definition 4.1.** A triple \((B, \{\cdot, \cdot, \cdot\}, \rho)\) is called a 3-pre-\(\rho\)-Lie algebra if the following equalities hold

\[ |\{f_1, f_2, f_3\}| = |f_1| + |f_2| + |f_3|, \]

\[ \{f_1, f_2, f_3\} = -\rho(f_1, f_2)\{f_2, f_1, f_3\}, \]

\[ \{f_1, f_2, \{g_1, g_2, g_3\}\} = \{\{f_1, f_2, g_1\}, g_2, g_3\} + \rho(f_1 + f_2, g_1)\{g_1, f_1, f_2, g_2\}, \]

\[ + \rho(f_1 + f_2, g_1 + g_2)\{g_1, g_2, f_1, f_2, g_3\}, \]

\[ \{f_1, f_2, f_3\}, g_1, g_2\} = \{f_1, f_2, \{f_3, g_1, g_2\}\} + \rho(f_1 + f_2, f_3)\{f_2, f_3, f_1, g_1, g_2\}, \]

\[ + \rho(f_1 + f_2, f_3)\{f_3, f_1, f_2, g_1, g_2\}. \]
where $\mathcal{B}$ is a $G$-graded vector space, $\{.,.,.\} : \otimes^3 \mathcal{B} \rightarrow \mathcal{B}$ is a trilinear map, $\rho$ is a two-cycle and

$$[f_1, f_2, f_3]_c = \{f_1, f_2, f_3\} + \rho(f_1 + f_2 + f_3)\{f_2, f_3, f_1\} + \rho(f_1 + f_2, f_3)\{f_1, f_3, f_2\}.$$ 

To simplify notation, we write $\mathcal{B}_\{\}$ instead of $(\mathcal{B}, \{.,.,.\}, \rho)$.

**Proposition 4.2.** $([.,.,.], \rho)$ give a $3$-$\rho$-Lie algebra structure, when $(\mathcal{B}, \{.,.,.\}, \rho)$ is a $3$-$\rho$-Lie algebra. $(\mathcal{B}, [.,.,.], \rho)$ is called the sub-adjacent $3$-$\rho$-Lie algebra of $\mathcal{B}$ and is denoted by $\mathcal{B}_\rho^c$.

**Proof.** Using the definition of $\{.,.,.\}_c$, (4.2) and (4.3), we have

$$[f_1, f_2, [f_3, f_4, f_5]_c]_c - [f_1, f_2, f_3]_c f_4 - \rho(f_1 + f_2, f_3)\{f_1, f_2, f_4\} f_5 - \rho(f_1 + f_2, g_1 + g_2)\{f_3, f_4, [f_1, f_2, f_5]_c\} = 0.$$ 

$\square$

Note that, $\mathcal{B}_\{\}$ is called the compatible $3$-$\rho$-pre-Lie algebra structure on the $3$-$\rho$-Lie algebra $\mathcal{B}_\rho^c$.

The following lemma gives a representation of $\mathcal{B}_\rho^c$ by the left multiplication:

**Lemma 4.3.** Defining the left multiplication $L : \wedge^2 \mathcal{B}_\{\} \rightarrow gl(\mathcal{B}_\{\})$ by $L(f, g)h = \{f, g, h\}$, $(\mathcal{B}, L)$ is a representation of $\mathcal{B}_\rho^c$.

**Proof.** By Definition 3.2 we have

$$L([f_1, f_2], (g_1, g_2))_L(h) = L([f_1, f_2, g_1]_c, g_2)(h) + \rho(f_1 + f_2, g_1) L(g_1, [f_1, f_2, g_2]_c)(h)$$

$$= \{f_1, f_2, g_1\} g_2 + \rho(f_1 + f_2, g_1)\{g_1, [f_1, f_2, g_2]_c\}$$

$$= \{f_1, f_2, [g_1, g_2, h]_c\} - \rho(f_1 + f_2, g_1 + g_2)\{g_1, g_2, [f_1, f_2, h]\}$$

$$= L(f_1, f_2)L(g_1, g_2)(h) - \rho(f_1 + f_2, g_1 + g_2)L(g_1, g_2)L(f_1, f_2)(h).$$

The other ones prove in a similar way. $\square$

**Representation of 3-$\rho$-pre-Lie algebras:** Here, we are going to represent the representation theory of 3-$\rho$-pre-Lie algebras. Also, the dual representation will be given.
*Definition 4.4.* A pair $(\mu, \tilde{\mu})$ is called a representation of $B_\{\}$ on $G$-graded vector space $V$ if for all homogeneous elements $f_1, f_2, f_3, f_4 \in Hg(B)$, the following equalities hold

\[
\tilde{\mu}(f_1, \{f_2, f_3, f_4\}) = \rho(f_1, f_2 + f_3)\mu(f_2, f_3)\tilde{\mu}(f_1, f_4) + \rho(f_1, f_3 + f_4)\rho(f_2, f_3 + f_4)\tilde{\mu}(f_3, f_4)\tilde{\mu}(f_1, f_2) \\
- \rho(f_1, f_2 + f_4)\rho(f_3, f_4)\tilde{\mu}(f_2, f_4)\tilde{\mu}(f_1, f_3) + \rho(f_1 + f_2, f_3 + f_4)\rho(f_3, f_4)\tilde{\mu}(f_3, f_4)\mu(f_1, f_2) \\
+ \rho(f_1, f_2 + f_3 + f_4)\rho(f_3, f_4)\tilde{\mu}(f_2, f_4)\tilde{\mu}(f_3, f_1) \\
- \rho(f_1, f_2 + f_4)\rho(f_3, f_4)\rho(f_2, f_4)\mu(f_3, f_1) \\
- \rho(f_2, f_3 + f_4)\rho(f_1, f_2 + f_3 + f_4)\rho(f_3, f_4)\tilde{\mu}(f_3, f_4)\tilde{\mu}(f_2, f_1),
\]

\[
\mu(f_1, f_2)\tilde{\mu}(f_3, f_4) = \rho(f_1 + f_2, f_3 + f_4)\tilde{\mu}(f_3, f_4)\mu(f_1, f_2) - \rho(f_1 + f_2, f_3 + f_4)\rho(f_1, f_2)\tilde{\mu}(f_3, f_4)\tilde{\mu}(f_2, f_1) \\
+ \rho(f_1 + f_2, f_3 + f_4)\tilde{\mu}(f_3, f_4)\tilde{\mu}(f_1, f_2) + \tilde{\mu}([f_1, f_2, f_3], f_4) \\
+ \rho(f_1 + f_2, f_3)\tilde{\mu}(f_3, \{f_1, f_2, f_3\}),
\]

\[
\tilde{\mu}([f_1, f_2, f_3], f_4) = \mu(f_1, f_2)\tilde{\mu}(f_3, f_4) + \rho(f_1, f_2 + f_3)\mu(f_2, f_3)\tilde{\mu}(f_1, f_4) + \rho(f_1 + f_2, f_3)\mu(f_3, f_1)\tilde{\mu}(f_2, f_4),
\]

\[
\tilde{\mu}(f_3, f_4)\mu(f_1, f_2) = \rho(f_1, f_2)\tilde{\mu}(f_3, f_4)\mu(f_1, f_2) - \tilde{\mu}(f_3, f_4)\tilde{\mu}(f_1, f_2) + \rho(f_3, f_4, f_1 + f_2)\mu(f_1, f_2)\tilde{\mu}(f_3, f_4) \\
- \rho(f_3 + f_4, f_1 + f_2)\rho(f_1, f_2)\tilde{\mu}(f_2, \{f_1, f_3, f_4\}) + \rho(f_3 + f_4, f_1 + f_2)\tilde{\mu}(f_1, \{f_2, f_3, f_4\}),
\]

where $\mu : \wedge^2 B_\rho^c \rightarrow gl(V)$ is a representation of the 3-pre-$\rho$-Lie algebra $B_\rho^c$ and $\tilde{\mu} : \otimes^2 B \rightarrow gl(V)$ is a bilinear map.

In the following lemma, we construct a representation of $B_\{\}$ by the left and right multiplication. The left and right multiplication are two bilinear maps of the form $L, R : \otimes^2 B_\{\} \rightarrow gl(B_\{\})$ which are defined by $L(f, g)h = \{f, g, h\}$ and $R(f, g)h = \rho(f + g, h)\{h, f, g\}$, respectively.

*Lemma 4.5.* Let $B_\{\}$ be a 3-pre-$\rho$-Lie algebra. Then the pair $(L, R)$ is a representation of $B_\{\}$ on itself (this representation is called the regular representation).

*Proof.* For example, let us examine one of the properties of the representation. For this, we have

\[
L(f_1, f_2)R(f_3, f_4)f_5 + \rho(f_1, f_2 + f_3)L(f_2, f_3)R(f_1, f_4)f_5 + \rho(f_1 + f_2, f_3)L(f_3, f_1)R(f_2, f_4)f_5 \\
= \rho(f_3 + f_4, f_5)\{f_1, f_2, \{f_3, f_4, f_5\}\} + \rho(f_1 + f_2 + f_3)\rho(f_1 + f_4, f_5)\{f_2, f_3, \{f_5, f_1, f_4\}\} \\
+ \rho(f_1 + f_2, f_3)f_2, f_3, \{f_5, f_2, f_4\}) \\
= -\rho(f_4, f_5)\{[f_1, f_2, f_3], f_5, f_4\} = \rho(f_1 + f_2 + f_3 + f_4, f_5)\{f_5, [f_1, f_2, f_3], f_4\} \\
= R([f_1, f_2, f_3], f_4)f_5.
\]

$\square$

Consider $(B, \{\ldots, \}, \rho)$ and $(V, \mu)$ as a 3-pre-$\rho$-Lie algebra and a representation of $B_\rho^c$, respectively. Then $(V, \mu, 0)$ is a representation of $B_\{\}$. 
Proposition 4.6. Let \((V, \mu, \bar{\mu})\) be a representation of \(B_1\). Then \((B \oplus V, \{\ldots\}_{\mu, \bar{\mu}}, \rho)\) is a 3-pre-\(\rho\)-Lie algebra, where the operation \ \{\cdot, \cdot, \cdot\}_{\mu, \bar{\mu}} : \otimes^3(B \oplus V) \rightarrow B \oplus V\) is defined by
\[
\{f_1 + v_1, f_2 + v_2, f_3 + v_3\}_{\mu, \bar{\mu}} = \{f_1, f_2, f_3\} + \mu(f_1, f_2)v_3 + \rho(f_1, f_2 + f_3)\bar{\mu}(f_2, f_3)v_1 - \rho(f_2, f_3)\bar{\mu}(f_1, f_3)v_2,
\]
and \(f_1, f_2, f_3 \in Hg(B)\) and \(v_1, v_2, v_3 \in Hg(V)\).

Proof. Using Definitions 4.1 and 4.4 and a direct computation, we get the result.

Let \(V\) be a graded vector space. Define the operator \(\zeta : \otimes^2V \rightarrow \otimes^2V\) by \(\zeta(f_1 \otimes f_2) = f_2 \otimes f_1\), where \(f_1 \otimes f_2 \in \otimes^2V\).

Proposition 4.7. Let \((V, \mu, \bar{\mu})\) be a representation of \(B_1\). If we define the bilinear map \(\nu : \wedge^2B \rightarrow gl(V)\) by \(\nu(f_1, f_2) = (\mu - \rho(f_1, f_2)\bar{\mu}\xi + \bar{\mu})(f_1, f_2)\) for all \(f_1, f_2 \in Hg(B)\), then \((V, \nu)\) is a representation of \(B^c_\rho\).

Proof. Proposition 4.6 implies that we have the 3-pre-\(\rho\)-Lie algebra \((B \oplus V, \{\ldots\}_{\mu, \bar{\mu}}, \rho)\). Also, Proposition 4.2 implies that \((B^c \oplus V, \{\ldots\}_{\mu, \bar{\mu}}, \rho)\) is the sub-adjacent 3-\(\rho\)-Lie algebra, where
\[
[f_1 + v_1, f_2 + v_2, f_3 + v_3]_c = \{f_1 + v_1, f_2 + v_2, f_3 + v_3\}_{\mu, \bar{\mu}} + \rho(f_1, f_2 + f_3)\{f_2 + v_2, f_3 + v_3, f_1 + v_1\}_{\mu, \bar{\mu}}
\]
and \(f_1, f_2, f_3 \in Hg(B)\) and \(v_1, v_2, v_3 \in Hg(V)\).

By using 4.4, we get
\[
[f_1 + v_1, f_2 + v_2, f_3 + v_3]_c = [f_1, f_2, f_3]_c + (\mu - \rho(f_1, f_2)\bar{\mu}\xi + \bar{\mu})(f_1, f_2)v_3
\]
\[
+ \rho(f_1, f_2 + f_3)(\mu - \rho(f_2, f_3)\bar{\mu}\xi + \bar{\mu})(f_2, f_3)v_1
\]
\[
+ \rho(f_1 + f_2, f_3)(\mu - \rho(f_3, f_1)\bar{\mu}\xi + \bar{\mu})(f_3, f_1)v_2
\]
\[
= [f_1, f_2, f_3]_c + \nu(f_1, f_2)v_3 + \rho(f_1, f_2 + f_3)\nu(f_2, f_3)v_1
\]
\[
+ \rho(f_1 + f_2, f_3)\nu(f_3, f_1)v_2.
\]

Finally, by Proposition 3.3 we deduce that \((V, \nu)\) is a representation of \(B^c_\rho\).

According to the above proposition, we can deduce that if \((B, \{\ldots\}, \rho)\) is a 3-pre-\(\rho\)-Lie algebra with the representation \((V, \mu, \bar{\mu})\), then the 3-pre-\(\rho\)-Lie algebras \((B \oplus V, \{\ldots\}_{\mu, \bar{\mu}}, \rho)\) and \((B \oplus V, \{\ldots\}_{\nu, \bar{\nu}}, \rho)\) have the same sub-adjacent 3-\(\rho\)-Lie algebra given by \((4.5)\).

In the next proposition, we find the dual of the representation \((V, \mu, \bar{\mu})\) of 3-pre-\(\rho\)-Lie algebra \((B, \{\ldots\}, \rho)\). Let us define \(\nu^* : \mu^* - \rho(f_1, f_2)\bar{\mu}^*\xi + \bar{\mu}^*(f_1, f_2)\).

Proposition 4.8. Consider \(B_1\) equipped with a \(\rho\)-skew-symmetric representation \((V, \mu, \bar{\mu})\). Then \((V^*, \nu^*, -\bar{\mu}^*)\) is a representation of \(B_1\), which is called the dual representation of the representation \((V, \mu, \bar{\mu})\).

Proof. By Propositions 8.8, 4.7 and Definition 4.4, we get the result.

If \(B_1\) is a 3-pre-\(\rho\)-Lie algebra with the \(\rho\)-skew-symmetric representation \((V, \mu, \bar{\mu})\), then the 3-pre-\(\rho\)-Lie algebras \((B \oplus V^*, \{\ldots\}_{\mu^*, \bar{\mu}^*}, \rho)\) and \((B \oplus V^*, \{\ldots\}_{\nu^*, -\bar{\mu}^*}, \rho)\) have the same sub-adjacent 3-\(\rho\)-Lie algebra \((B^c \oplus V^*, \{\ldots\}_{\mu^*, \bar{\mu}^*}, \rho)\).
\( \rho \)-\( O \)-operator. In the following, we define the notion of \( \rho \)-\( O \)-operator associated to the representation \((V, \mu)\) of \( B_\rho \), then we construct a 3-\( \rho \)-pre-Lie algebra structure on the representation \( V \) by a \( \rho \)-\( O \)-operator \( T \) and bilinear map \( \mu \). In the next, we investigate the relationship between compatible 3-\( \rho \)-pre-Lie algebra structure and an invertible \( \rho \)-\( O \)-operator of \( B_\rho \) (for the classical case refer to \([12]\)).

**Definition 4.9.** Let \( B \) be a 3-\( \rho \)-Lie algebra with the representation \((V, \mu)\). An even linear operator \( T : V \to B \) is called a \( \rho \)-\( O \)-operator associated to \((V, \mu)\) if \( T \) satisfies

\[
[Tx, Ty, Tz]_B = T(\mu(Tx, Ty)z + \rho(x, y + z)(\mu(Ty, Tz)x + \rho(x + y, z)\mu(Tz, Tx)y), \quad \forall x, y, z \in V.
\]

**Lemma 4.10.** Consider the following three assumptions

i. \( B_\rho \) is a 3-\( \rho \)-Lie algebra,

ii. \((V, \mu)\) is a \( \rho \)-skew-symmetric representation of \( B_\rho \),

iii. \( T \) is a \( \rho \)-\( O \)-operator associated to \((V, \mu)\).

If we define a new multiplication on \( V \) by

\[
\{x, y, z\}_V = \mu(Tx, Ty)z, \quad \forall x, y, z \in V,
\]

then \((V, \{\ldots\}_V)\) is a 3-\( \rho \)-\( O \)-Lie algebra.

**Proof.** Suppose that \( x, y, z \in Hg(V) \). It is easy to see that

\[
\{x, y, z\} = -\rho(x, y)\{y, x, z\}, \quad T[x, y, z] = [Tx, Ty, Tz].
\]

Also, for \( x_1, x_2, x_3, x_4, x_5 \in Hg(V) \), we have

\[
\{x_1, x_2, \{x_3, x_4, x_5\}\} = \mu(Tx_1, Tx_2)\mu(Tx_3, Tx_4)x_5, \quad (4.7)
\]

\[
\{[x_1, x_2, x_3], x_4, x_5\} = \mu([Tx_1, Tx_2, Tx_3], Tx_4)x_5, \quad (4.8)
\]

\[
\rho(x_1 + x_2, x_3)\{x_3, \{x_1, x_2, x_4\}, x_5\} = \rho(x_1 + x_2, x_3)\mu(Tx_3, [Tx_1, Tx_2, Tx_4])x_5, \quad (4.9)
\]

\[
\rho(x_1 + x_2, x_3 + x_4)\{x_3, x_4, \{x_1, x_2, x_5\}\} = \rho(x_1 + x_2, x_3 + x_4)\mu(Tx_3, Tx_4)\mu(Tx_1, Tx_2)x_5. \quad (4.10)
\]

By Definition \(3.2\) and \(4.7-4.10\), we get the result. \( \Box \)

**Corollary 4.11.** By the above proposition, we deduce that \((V, [\cdot, \cdot, \cdot]_C)\) is a 3-\( \rho \)-Lie algebra as the sub-adjacent of the 3-\( \rho \)-pre-Lie algebra \((V, \{\cdot, \cdot, \cdot\})\). Furthermore, \( T(V) = \{Tv | v \in V\} \subset A \) is a subalgebra of \( B_\rho \) and there is an induced 3-\( \rho \)-pre-Lie algebra structure on \( T(V) \) given by

\[
\{Tu, Tv, Tw\}_{T(V)} := T\{u, v, w\}, \quad \forall u, v, w \in V.
\]

**Lemma 4.12.** Consider 3-\( \rho \)-Lie algebra \( B_\rho \). Then the following assertions are equivalent

i. existence of a compatible 3-\( \rho \)-pre-Lie algebra structure,

ii. existence of an invertible \( \rho \)-\( O \)-operator.

**Proof.** Let \( T \) be an invertible \( \rho \)-\( O \)-operator of \( B_\rho \) associated to \((V, \mu)\). BY \(4.6\), \(4.11\) and this fact that \( T \) is a \( \rho \)-\( O \)-operator, we can easily deduce that \((A = T(V), \{\cdot, \cdot, \cdot\}_{T(V)})\) is a compatible 3-\( \rho \)-pre-Lie algebra. Conversely, the identity map id is an \( \rho \)-\( O \)-operator of the sub-adjacent 3-\( \rho \)-Lie algebra of a 3-\( \rho \)-pre-Lie algebra associated to the representation \((B, L)\). \( \Box \)
4.1. Phase spaces of 3-$\rho$-Lie algebras. In this section, the notion of a phase space of a 3-$\rho$-Lie algebra will be introduced and we show that a 3-$\rho$-Lie algebra has a phase space if and only if it is sub-adjacent to a 3-$\rho$-pre-Lie algebra.

Proposition 4.13. Consider the symplectic 3-$\rho$-Lie algebra $(B, [\cdot,\cdot,\cdot]_B, \rho, \omega)$. Then the following structure defines a compatible 3-pre-$\rho$-Lie algebra structure on $B$.

$$\omega(\{f, g, h\}, s) = -\rho(f + g, h)\omega(h, [f, g, s]_B), \quad \forall f, g, h, s \in B.$$  \hspace{1cm} (4.12)

Proof. Let us define a linear map $T : B^* \rightarrow B$ by $\omega(f, g) = (T^{-1}(f))(g)$, or equivalently, $\omega(T\alpha, g) = \alpha(g)$ for all $f, g \in Hg(B)$ and $\alpha \in B^*$. Since $\omega$ is a symplectic structure, we deduce that $T$ is an invertible $\rho$-$O$-operator associated to $(B^*, Ad^*)$. Also, by (4.6) and (4.11), there exists a compatible 3-$\rho$-pre-Lie algebra on $B$ given by $\{f, g, h\} = T(ad^*(f, g)T^{-1}(h))$. So, (4.12) holds. \hfill $\square$

Definition 4.14. $(B \oplus B^*, [\cdot,\cdot,\cdot]_{B \oplus B^*}, \omega)$ is called a phase space of $B_\rho$, if

i. $[\cdot,\cdot,\cdot]_{B \oplus B^*}$ is a 3-$\rho$-Lie algebra structure on the vector space $B \oplus B^*$,

ii. $\omega$ is a natural non-degenerate $\rho$-skew-symmetric bilinear form on $B \oplus B^*$ given by

$$\omega(f + \alpha, g + \beta) = \alpha(g) - \rho(f, g)\beta(f),$$  \hspace{1cm} (4.13)

such that $(B \oplus B^*, [\cdot,\cdot,\cdot]_{B \oplus B^*}, \rho, \omega)$ is a symplectic 3-$\rho$-Lie algebra,

iii. $(B, [\cdot,\cdot,\cdot]_B, \rho)$ and $(B^*, [\cdot,\cdot,\cdot]_{B^*}, \rho)$ are 3-$\rho$-Lie subalgebras of $(B \oplus B^*, [\cdot,\cdot,\cdot]_{B \oplus B^*}, \rho)$.

Theorem 4.15. A 3-$\rho$-Lie algebra has a phase space if and only if it is sub-adjacent to a 3-pre-$\rho$-Lie algebra.

Proof. Let $B_{\{\}}$ be a 3-pre-$\rho$-Lie algebra. By Propositions 3.3, 3.5, 4.13 and Lemma 4.3 we have the 3-$\rho$-Lie algebra $(B^* \oplus B^*, [\cdot,\cdot,\cdot]_{B^* \oplus B^*}, \rho)$. Moreover, for all $f_1, f_2, f_3, f_4 \in Hg(B)$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in Hg(B^*)$, by
\[\rho(f_1 + f_2 + f_3 + f_4)\omega(\{f_2 + \alpha_2, f_3 + \alpha_3, f_4 + \alpha_4\}_{B^c \oplus B^*}, f_1 + \alpha_1)\]

\[= -\rho(f_1, f_3 + f_4 + f_2)\alpha_2\{f_3, f_4, f_1\}\]

\[+ \rho(f_1, f_4 + f_1)\rho(f_3, f_4)\alpha_2\{f_4, f_1, f_3\}\]

\[+ \rho(f_2, f_3 + f_4)\alpha_2\{f_1, f_3, f_4\}\]

\[+ \rho(f_2, f_3 + f_4)\alpha_2\{f_3, f_1, f_2\}\]

\[+ \rho(f_2, f_3 + f_4)\rho(f_3 + f_1, f_4)\alpha_2\{f_1, f_3, f_2\}\]

\[\rho(f_1 + f_2 + f_3 + f_4)\omega(\{f_4 + \alpha_4, f_1 + \alpha_1, f_2 + \alpha_2\}_{B^c \oplus B^*}, f_3 + \alpha_3)\]

\[= -\rho(f_1 + f_2, f_3 + f_4)\rho(f_4 + f_1, f_2)\alpha_3\{f_1, f_2, f_3\}\]

\[+ \rho(f_1 + f_2, f_3 + f_4)\rho(f_4 + f_1, f_2)\alpha_3\{f_2, f_4, f_1\}\]

\[+ \rho(f_1 + f_2, f_3 + f_4)\rho(f_4 + f_1, f_2)\alpha_3\{f_3, f_2, f_4\}\]

By the above relations, we deduce that \(\omega\) is a symplectic structure on \(B^c \oplus B^*\). Moreover, \(B^c\) and \(B^*\) are the subalgebra and the abelian subalgebra of \(B^c \oplus B^*\), respectively. Thus, the symplectic 3-\(\rho\)-Lie algebra \((B^c \oplus B^*, [\ldots]_{B^c \oplus B^*}, \rho, \omega)\) is a phase space of \(B^c_{\rho}\). Conversely, let \((B \oplus B^*, [\ldots]_{B \oplus B^*}, \rho, \omega)\) be a phase space of \(B^c_{\rho}\). By Proposition 1.13 there is a compatible 3-pre-\(\rho\)-Lie algebra structure \(\{\ldots\}\) on \(B \oplus B^*\). Thus

\[\omega(\{f, g, h\}, z) = -\rho(f + g, h)\omega(h, [f, g, z]_{B \oplus B^*}) = -\rho(f + g, h)\omega(h, [f, g, z]_{B^c \oplus B^*}) = 0 \quad \forall f, g, h, z \in Hg(B).\]

The last equality holds, since \(B^c_{\rho}\) is a subalgebra of \((B \oplus B^*, [\ldots]_{B \oplus B^*}, \rho)\). Therefore, \(\{f, g, h\} \in B^c_{\rho}\) that says \((B, \{\ldots\}_{B \oplus B^*}, \rho)\) is a subalgebra of the 3-pre-\(\rho\)-Lie algebra \((B \oplus B^*, [\ldots]_{B \oplus B^*}, \rho)\) and so its sub-adjacent 3-\(\rho\)-Lie algebra \(B^c_{\rho}\) is the 3-pre-\(\rho\)-Lie algebra \(B^c_{\rho}\).

**Corollary 4.16.** Let \((B \oplus B^*, [\ldots]_{B \oplus B^*}, \rho, \omega)\) be a phase space of \(B^c_{\rho}\) and \((B \oplus B^*, [\ldots]_{B \oplus B^*}, \rho)\) be the associated 3-pre-\(\rho\)-Lie algebra. Then \((B, \{\ldots\}_{B}, \rho)\) and \((B^*, \{\ldots\}_{B^*}, \rho)\) are subalgebras of the 3-pre-\(\rho\)-Lie algebra \((B \oplus B^*, [\ldots]_{B \oplus B^*}, \rho)\).
Lemma 4.17. Let $(V, \mu)$ be a representation of $B_\rho$ and $(B \oplus B^*, \ldots, B^*, \rho, \omega)$ be a phase space of it. Then there exists a 3-pre-$\rho$-Lie algebra structure on $B$ given by

$$\{f, g, h\} = \mu(f, g)h, \quad \forall f, g, h \in B.$$ 

Proof. Assume that $f, g, h \in Hg(B)$ and $\alpha \in Hg(B^*)$. Then, we have

$$\rho(f + g + h, \alpha)(\{f, g, h\}) = -\omega(\{f, g, h\}, \alpha) = \rho(f + g, h)\omega(h, [f, g, \alpha]_{B\oplus B^*}) = \rho(f + g, h)\omega(h, \mu^*(f, g)\alpha)$$

$$= -\rho(f + g, h)(\rho(h, f + g + \alpha)\mu^*(f, g)\alpha(h) = \rho(f + g + h, \alpha)\alpha\mu(f, g)h.$$ 

\[\square\]

5. QUADRATIC 3-HOM-$\rho$-LIE ALGEBRAS

In this section, we study 3-Hom-$\rho$-Lie algebras, which 3-$\rho$-lie algebras are special case of these Lie algebras with respect to the map $Id$. The constructions of Section 2, such as quadratic, symplectic structures, 3-associative Lie algebras and etc, will be investigated in this section to color Hom-Lie case. In this section, the representation theory of 3-Hom-$\rho$-lie algebras is also included.

Definition 5.1. Multiple $(B, \ldots, B, \rho, \phi)$ is called a 2-Hom-$\rho$-Lie algebra (or simply a Hom-$\rho$-Lie algebra) if

i. $B$ is a $G$-graded vector space, where $G$ is an abelian group,

ii. $\rho$ is a two-cycle,

iii. $\phi : B \rightarrow B$ is an even linear map,

iv. $\ldots, B$ is a bilinear map on $B$, satisfying the conditions

- $|\{f, g\}B| = |f| + |g|$, 
- $[f, g]B = -\rho(f, g)[g, f]B$,
- $\rho(h, f)[\phi(f), [g, h]]B + \rho(g, h)[\phi(h), [f, g]]B + \rho(f, g)[\phi(g), [h, f]]B = 0 \quad \forall f, g, h \in B$.

The third condition is equivalent to

$$[\phi(f), [g, h]]B = [[f, g]B, \phi(h)]B + \rho(f, g)[\phi(g), [h, f]]B.$$ 

Definition 5.2. Let

i. $B$ be a $G$-graded vector space,

ii. $\phi : B \rightarrow B$ be an even linear map,

iii. $\rho$ be a two-cycle,

iv. $\ldots, B$ be a trilinear map on $B$ satisfying

$$|[f_1, f_2, f_3]B| = |f_1| + |f_2| + |f_3|$$

$$[\phi(f_1), \phi(f_2), [g_1, g_2, g_3]]B = [[f_1, f_2, g_1]B, \phi(g_2), \phi(g_3)]B + \rho(f_1 + f_2, g_1)[\phi(g_1), [f_1, f_2, g_2]B, \phi(g_3)]B$$

$$+ \rho(f_1 + f_2, g_1 + g_2)[\phi(g_1), \phi(g_2), [f_1, f_2, g_3]]B,$$

$$[f_1, f_2, f_3]B = -\rho(f_1, f_2)[f_2, f_1, f_3]B = -\rho(f_2, f_3)[f_1, f_3, f_2]B,$$

for any $f_1, f_2, f_3, g_1, g_2 \in Hg(B)$. Then $(B, \ldots, B, \rho, \phi)$ is called a 3-$\rho$-Lie algebra. Let us denote the 3-Hom-$\rho$-Lie algebra $(B, \ldots, B, \rho, \phi)$ by $B_{\rho, \phi}$. 
Definition 5.3. Let \((\mathcal{B}, [\cdot, \cdot]_\mathcal{B}, \phi)\) and \((\mathcal{A}, [\cdot, \cdot]_\mathcal{A}, \psi)\) be two 3-Hom-\(\rho\)-Lie algebras. A linear map \(\alpha : \mathcal{B} \rightarrow \mathcal{A}\) is said to be a morphism of 3-Hom-\(\rho\)-Lie algebras if for all \(f, g, h \in \mathcal{B}\)

\[ \alpha[f, g, h]_\mathcal{B} = [\alpha(f), \alpha(g), \alpha(h)]_\mathcal{A}, \]

and

\[ \alpha \circ \phi = \psi \circ \alpha. \]

Definition 5.4. \(\mathcal{B}_{\rho, \phi}\) is said to be

1. multiplicative if \(\phi\) is a Lie algebra morphism, i.e. for any \(f, g, h \in \mathcal{B}\), \(\phi[f, g, h]_\mathcal{B} = [\phi(f), \phi(g), \phi(h)]_\mathcal{B}\).
2. regular if \(\phi\) is an automorphism for \([\cdot, \cdot]_\mathcal{B}\),
3. involutive if \(\phi^2 = \text{Id}_\mathcal{B}\).

In the following theorem, a new 3-Hom-\(\rho\)-Lie algebra structure will be constructed by combining a 3-Hom-\(\rho\)-Lie algebra structure and an even 3-\(\rho\)-Lie algebra endomorphism.

Theorem 5.5. Consider \(\mathcal{B}_{\rho, \phi}\) and let \(\beta : \mathcal{B}_{\rho, \phi} \rightarrow \mathcal{B}_{\rho, \phi}\) be an even 3-\(\rho\)-Lie algebra endomorphism. Then \((\mathcal{B}, [\cdot, \cdot]_\mathcal{B} = \beta \circ [\cdot, \cdot]_\mathcal{B}, \beta \circ \phi)\) is a 3-Hom-\(\rho\)-Lie algebra.

Proof. It is easy to see that \([\cdot, \cdot]_\mathcal{B}\) is \(\rho\)-skew symmetric with respect to the displacement of every two elements. Now, we check the \(\rho\)-fundamental identity:

(a) \([\beta \circ \phi(f_1), \beta \circ \phi(f_2), [f_3, f_4, f_5]_\mathcal{B}]_\mathcal{B} = [\beta \circ \phi(f_1), \beta \circ \phi(f_2), \beta \circ [f_3, f_4, f_5]_\mathcal{B}]_\mathcal{B} = \beta^3[\phi(f_1), \phi(f_2), [f_3, f_4, f_5]_\mathcal{B}]_\mathcal{B},

(b) \([f_1, f_2, f_3]_\mathcal{B}, \beta \circ \phi(f_4), \beta \circ \phi(f_5)]_\mathcal{B} = \beta^3[[f_1, f_2, f_3]_\mathcal{B}, \phi(f_4), \phi(f_5)]_\mathcal{B},

(c) \(\rho(f_1 + f_2, f_3)[\beta \circ \phi(f_4), [f_1, f_2, f_4]_\mathcal{B}, \beta \circ [f_3, f_4, f_5]_\mathcal{B}]_\mathcal{B} = \rho(f_1 + f_2, f_3)[\beta \circ \phi(f_4), [f_1, f_2, f_4]_\mathcal{B}, \beta \circ [f_3, f_4, f_5]_\mathcal{B}]_\mathcal{B} = \beta^3[\phi(f_3), [f_1, f_2, f_4]_\mathcal{B}, \phi(f_5)]_\mathcal{B},

(d) \(\rho(f_1 + f_2, f_3 + f_4)[\beta \circ \phi(f_5), [f_1, f_2, f_5]_\mathcal{B}, \beta \circ [f_3, f_4, f_5]_\mathcal{B}]_\mathcal{B} = \rho(f_1 + f_2, f_3 + f_4)[\beta \circ \phi(f_5), [f_1, f_2, f_5]_\mathcal{B}, \beta \circ [f_3, f_4, f_5]_\mathcal{B}]_\mathcal{B} = \beta^3[\phi(f_5), [f_1, f_2, f_5]_\mathcal{B}, \phi(f_4)]_\mathcal{B}.

According to the above four items and this fact that \(\mathcal{B}\) is a 3-Hom-\(\rho\)-Lie algebra, the proof is complete. \(\square\)

Example 5.6. Consider \(\mathcal{B}_{\rho, \phi}\) and let \(\phi\) be a 3-\(\rho\)-Lie algebra morphism. Then \((\mathcal{B}, [\cdot, \cdot]_\mathcal{B} = \phi \circ [\cdot, \cdot]_\mathcal{B}, \rho, \phi)\) is a multiplicative 3-Hom-\(\rho\)-Lie algebra.

Definition 5.7. A non-degenerate bilinear form \(\chi\) on \(\mathcal{B}_{\rho, \phi}\) satisfying

\[ \chi(f_1, f_2) = \rho(f_1, f_2)\chi(f_2, f_1) \forall f_1, f_2 \in Hg(\mathcal{B}) \text{ (\(\rho\)-symmetric)}, \]

\[ \chi([f_1, f_2, f_3]_\mathcal{B}, f_4) = \chi(f_1, [f_2, f_3, f_4]_\mathcal{B}) \forall f_1, f_2, f_3, f_4 \in \mathcal{B} \text{ (invariant)}, \]

\[ \chi(\phi(f_1), f_2) = \chi(f_1, \phi(f_2)) \text{ (\(\phi\) is \(\chi\)-symmetric).} \]

is called a quadratic structure on \(\mathcal{B}_{\rho, \phi}\). A quadratic 3-Hom-\(\rho\)-Lie algebra is a 3-Hom-\(\rho\)-Lie algebra equipped with a quadratic structure. We denote a quadratic 3-Hom-\(\rho\)-Lie algebra by \(\mathcal{B}_{\rho, \phi, \chi}\). Also, We recover quadratic 3-\(\rho\)-Lie algebras when \(\phi = \text{Id}_\mathcal{B}\).

Consider the subalgebra \(\text{Cent}(\mathcal{B}_\rho)\) of \(\text{End}(\mathcal{B}_\rho)\) defined by \(\ref{2.8}\) we have the following:

Proposition 5.8. Let \(\psi\) be an even element in \(\text{Cent}(\mathcal{B}_{\rho, \phi})\). For any \(f, g, h \in Hg(\mathcal{B})\), we define two following structures

\[ [f, g, h]_\psi = [\psi(f), g, h]_\mathcal{B}, \quad [f, g, h]^\psi = [\psi(f), \psi(g), \psi(h)]_\mathcal{B}. \]
Then, we have the following 3-Hom-$\rho$-Lie algebras

$$ (\mathcal{B}[., ., .], \rho, \psi \circ \phi), \quad (\mathcal{B}[., ., .]^\psi, \rho, \phi), \quad (\mathcal{B}[., ., .]^\psi, \rho, \psi \circ \phi), $$

$$(\mathcal{B}[., ., .]^\psi, \rho, \phi), \quad (\mathcal{B}[., ., .]^\psi, \rho, \psi \circ \phi).$$

**Proof.** This proposition will be proved in the similar way of Proposition 2.5. \(\square\)

**Proposition 5.9.** Let $\psi$ be an even invertible and $\chi$-symmetric element of the centroid of $\mathcal{B}_{\rho, \phi, \chi}$ such that $\psi \circ \phi = \phi \circ \psi$. Then $\mathcal{B}$ together with two structures $[., ., .]^\psi, [. , ., .]^\psi$ and the even bilinear form $\chi_\psi$ defined by $\chi_\psi(f, g) = \chi(\psi(f), g)$ makes two quadratic 3-Hom-$\rho$-Lie algebras.

**Proof.** According to the proof of Proposition 2.6 it is enough to check that $\chi_\psi(\phi(f), g) = \chi_\psi(f, \phi(g))$. For this, we have

$$\chi_\psi(\phi(f), g) = \chi(\psi \circ \phi(f), g) = \chi(\phi \circ \psi(f), g) = \chi(\psi(f), \phi(g)) = \chi_\psi(f, \phi(g)).$$ \(\square\)

**Definition 5.10.** Let $g$ be a $G$-graded vector space, $\mu : g \times g \times g \rightarrow g$ be an even 3-linear map (i.e. $\mu(a, b, c) \in g_{\alpha(b) + c}$) and $\alpha : g \rightarrow g$ be an even homomorphism such that for elements $a, b, c, d, e \in g$

$$\mu(a, b, c) = \mu(a, b, c), \mu(c, d, e) = \mu(c, d, e),$$

then $(g, \mu, \alpha)$ is called a 3-Hom-associative algebra. If the bilinear map $\mu$ is the $\rho$-symmetry with respect to the displacement of every two elements, the 3-Hom-associative algebra $(g, \mu, \alpha)$ is called commutative. We denote this algebra by $g_{\rho, \phi, \chi}$.

**Definition 5.11.** A quadratic structure on 3-Hom-associative algebra $g$ is a $\rho$-symmetric, invariant and non-degenerate bilinear form $\chi_g$ on $g$ such that $\alpha$ is $\chi_g$-symmetric. Let us denoted by $g_{\rho, \chi}$ a quadratic 3-Hom-associative algebra.

**Theorem 5.12.** Consider $g_{\rho, \phi, \chi}$ and $\mathcal{B}_{\rho, \phi, \chi}$. Then the tensor product $E = \mathcal{B} \otimes g$ equipped with the structures

$$[f \otimes a, g \otimes b, h \otimes c]_E = \rho(a, g + h)\rho(b, h)[f, g, h]_E \otimes \mu(a, b, c),$$

$$\chi_E(f \otimes a, g \otimes b) = \rho(a, g)\chi(f, g)\chi_g(a, b),$$

$$\phi_E(f \otimes a, g \otimes b) = \phi(f) \otimes \alpha(a), \quad \forall f, g, h \in Hg(\mathcal{B}), a, b, c \in Hg(g),$$

is a quadratic 3-Hom-$\rho$-Lie algebra.

**Proof.** By the proof of Theorem 2.6, we can easily show that $(E, [., ., .]_E; \rho, \phi_E)$ is a 3-Hom-$\rho$-Lie algebra and $\chi_E$ is $\rho$-symmetric and invariant. It is enough to check that $\phi_E$ is $\chi_E$-symmetric, which completes the proof. For this, we have

$$\chi_E(\phi_E(f \otimes a), g \otimes b) = \chi_E(\phi(f) \otimes \alpha(a), g \otimes b) = \rho(a, g)\chi(f, g)\chi_g(a, b)$$

$$= \rho(a, g)\chi(f, g)\chi_g(a, \alpha(b)) = \chi_E(f \otimes a, \phi_E(g \otimes b)).$$ \(\square\)

**Ideals of 3-Hom-$\rho$-Lie algebras:** In this part, we introduce the definitions of subalgebra and ideal of $\mathcal{B}_{\rho, \phi}$ and give some properties related to ideals of $\mathcal{B}_{\rho, \phi}$.
Definition 5.13. A Hom-subalgebra of \( B_{\rho,\phi} \) is defined as a sub-vector space \( I \subseteq B_{\rho,\phi} \) with the property \([I, I]_B \subseteq I\) and \( \phi(I) \subseteq I\). I also is called a Hom ideal of \( B \) if \( \phi(I) \subseteq I \) and \([I, B_{\rho,\phi}, B_{\rho,\phi}]_B \subseteq I\).

Lemma 5.14. \( Z(B_{\rho,\phi}) \), the center of multiplicative quadratic 3-Hom-\( \rho \)-Lie algebra \( B_{\rho,\phi,\chi} \), is an Hom-ideal of \( B_{\rho,\phi} \).

Proof. It is easy to see that \([Z(B_{\rho,\phi}), B_{\rho,\phi}, B_{\rho,\phi}] \subseteq B_{\rho,\phi} \). We assume that \( f \in Z(B_{\rho,\phi}) \) and \( g, h, m \in B \), so
\[
\chi(\phi(f), g, h)_B, m) = \chi(\phi(f), [g, h, m]_B) = \chi(f, [g, h, m]_B)
\]
\[
= \chi(f, [\phi(g), \phi(h), \phi(m)]_B) = \chi([f, \phi(g), \phi(h)]_B, \phi(m)) = 0.
\]
The last equality holds, since \( f \in Z(B_{\rho,\phi}) \). On the other hand, since \( \chi \) is non-degenerate, thus the above relation gives \([\phi(f), g, h]_B \) for any \( g, h \in B \). This implies that \( \phi(f) \in Z(B_{\rho,\phi}) \) and therefore \( Z(B_{\rho,\phi}) \) is an Hom-ideal of \( B_{\rho,\phi} \). \( \square \)

Lemma 5.15. The orthogonal \( I^\perp \) of \( I \) (I is a Hom-ideal of \( B_{\rho,\phi,\chi} \)) with respect to \( \chi \) is a Hom-ideal of \( B_{\rho,\phi} \).

Proof. It is clear that \([I^\perp, B_{\rho,\phi}, B_{\rho,\phi}] \subseteq I^\perp \). Also, for \( f \in I^\perp, g \in I \), we have \( B(\phi(f), g) = B(f, \phi(g)) = 0 \), since \( \phi(I) \subseteq I \) and \( f \in I^\perp \). \( \square \)

Symplectic structure of 3-Hom-\( \rho \)-Lie algebras: This part is devoted to study of symplectic structure, metrics and their properties. Also, we show that a symplectic structure \( \omega \) may be defined on \( B_{\rho,\phi} \) if and only if there exists an invertible derivation of \( B_{\rho,\phi} \) that is antisymmetric with respect to metric.

Definition 5.16. If \( \omega \in \wedge^2 B^* \) is a non-degenerate and \( \rho \)-skew-symmetric bilinear form such that
\[
\omega([f_1, f_2, f_3]_B, \phi(f_4)) - \rho(f_1, f_2 + f_3 + f_4)\omega([f_2, f_3, f_4]_B, \phi(f_1))
\]
\[
+ \rho(f_1 + f_2, f_3 + f_4)\omega([f_3, f_4, f_1]_B, \phi(f_2)) - \rho(f_1 + f_2 + f_3, f_4)\omega([f_4, f_1, f_2]_B, \phi(f_3)) = 0,
\]
then \( \omega \) is called a symplectic structure and \((B, [\ldots, ]_B, \rho, \phi, \omega)\) is said to be a symplectic 3-Hom-\( \rho \)-Lie algebra.

We will say that \((B, [\ldots, ]_B, \rho, \phi, \chi, \omega)\) is a quadratic symplectic 3-Hom-\( \rho \)-Lie algebra if \((B, \chi)\) is quadratic and \((B, \omega)\) is symplectic.

Example 5.17. Let \((B, [\ldots, ]_B, \rho, \omega)\) be the 4-dimensional symplectic 3-\( \rho \)-Lie algebra provided in Example 2.17. We consider the multiplex \((B, [\ldots, ]_\phi, \rho, \phi)\) defining a 3-\( \rho \)-Hom-Lie algebra, according to Theorem 2.3, such that the bracket \([\ldots, ]_\phi\) and the even linear map \( \phi \) are defined by
\[
[l_1, l_2, l_3]_\phi = l_4, \quad \phi(l_4) = l_4,
\]
\[
[l_1, l_2, l_4]_\phi = l_3, \quad \phi(l_3) = -l_3,
\]
\[
[l_1, l_3, l_4]_\phi = l_2, \quad \phi(l_2) = -l_2,
\]
\[
[l_2, l_3, l_4]_\phi = l_1, \quad \phi(l_1) = -l_1.
\]

Moreover, \((B, [\ldots, ]_\phi, \rho, \phi)\) together with the bilinear map \( \omega \) is given in Example 2.17 is a 4-dimensional symplectic 3-\( \rho \)-Lie algebra.
In the following, we define a derivation of $B_{ρ,φ}$ and examine the relationship between symplectic structures and $φ$-symmetric invertible derivations.

**Definition 5.18.** A linear map $D$ on $B_{ρ,φ}$ satisfying

\[D \circ φ = φ \circ D,\]

\[D[f, g, h]_B = [D(f), g, h]_B + ρ(D, f)[f, D(g), h] + ρ(D, f + g)[f, g, D(h)],\]

is called a derivation of $B_{ρ,φ}$. We denote by $\text{Der}(B_{ρ,φ})$ the space of all derivations of $B_{ρ,φ}$.

**Definition 5.19.** $(B, [\cdot, \cdot]_B, ρ, φ, χ, K)$ is called a metric 3-Hom-$ρ$-Lie algebra, if

i. $K$ is a field ($C$ or $R$),

ii. $(B, [\cdot, \cdot]_B, ρ, φ)$ is a 3-Hom-$ρ$-Lie algebra,

iii. $φ : B \times B \rightarrow K$ is a non-degenerate $ρ$-symmetric bilinear form on $B$ satisfying the following condition

\[φ([f, g, h]_B, φ(z)) + ρ(f + g, h)φ(φ(h), [f, g, z]_B) = 0, \quad ∀f, g, h, z \in Hg(B).\]

Let us denote by $B_{ρ,φ,φ}$ the metric 3-Hom-$ρ$-Lie algebra $(B, [\cdot, \cdot]_B, ρ, φ, χ, K)$.

**Proposition 5.20.** There exists a symplectic structure on $B_{ρ,φ}$ if and only if there exists an invertible derivation $D \in \text{Der}_φ(B_{ρ,φ})$.

*Proof.* It is enough to define $ω(f, g) = φ(D(f), g)$. \[\square\]

For any symmetric bilinear form $χ'$ on $B_{ρ,φ,χ}$, there is an associated map $D : B \rightarrow B$ satisfying

\[χ'(f, g) = χ(D(f), g), \quad ∀f, g \in Hg(B).\] (5.1)

Since $χ$ and $χ'$ are symmetric, then $D$ is symmetric with respect to $χ$, i.e.,

\[χ(D(f), g) = ρ(D, f)χ(f, D(g)), \quad ∀f, g \in Hg(B).\]

**Lemma 5.21.** Let $(B, [\cdot, \cdot]_B, ρ, φ, χ)$ be a quadratic 3-Hom-$ρ$-Lie algebra and $χ'$ defined by (5.1). Then $χ'$ is invariant if and only if

\[D([f, g, h]_B) = [D(f), g, h]_B + ρ(D, f)[f, D(g), h] + ρ(D, f + g)[f, g, D(h)]_B.\] (5.2)

Also, $χ'$ is non-degenerate if and only if $D$ is invertible and $φ$ is $χ'$-symmetric if and only if $D φ = φ D$.

*Proof.* By the proof of Lemma 2.20, it is enough to show that $χ'(φ(f), g) = χ'(f, φ(g))$. For this, we have

\[χ'(φ(f), g) = χ(D φ(f), g) = χ(φ D(f), φ(g)) = χ(D(f), φ(g)) = χ'(f, φ(g)).\] \[\square\]

**Definition 5.22.** A $ρ$-symmetric map $D : B_{ρ,φ} \rightarrow B_{ρ,φ}$ satisfying (2.10) is called a centromorphism of $B_{1,φ}$. We denote by $C(B_{ρ,φ})$ the space of all centromorphisms of $B_{ρ,φ}$.

**Proposition 5.23.** Let $δ \in \text{Der}_χ(B_{ρ,φ})$ and $D \in C(B_{ρ,φ})$ such that $D \circ δ = δ \circ D$ and $D \circ φ = φ \circ D$. Then $D \circ δ$ is also a $χ$-antisymmetric derivation of $B_{ρ,φ}$. 
Lemma 5.25. For the proof refer to Lemma 3.3.

Proposition 5.26. Consider the 3-Hom-$\rho$-Lie algebra $\mathcal{B}_{\rho,\phi}$. The fundamental set $L = \wedge^2 \mathcal{B}_{\rho,\phi}$ together with operation

$$[(f_1, f_2), (g_1, g_2)]_L = ([f_1, f_2, g_1], \phi(g_2)) + \rho(f_1 + f_2, g_1)(\phi(g_1), [f_1, f_2, g_2]),$$

and the even linear map $\phi : L \rightarrow L$ defined by $\phi(f_1, f_2) = (\phi(f_1), \phi(f_2))$ is the multiplicative Hom-$\rho$-Lie algebra.

Definition 5.24. $\lambda$ bilinear map $\mu : L \rightarrow \mathfrak{gl}(V)$ is called a representation of $\mathcal{B}_{\rho,\phi}$ on vector space $V$ with respect to $\beta \in \mathfrak{gl}(V)$ if the following equalities are satisfied

$$\mu((f_1, f_2), (g_1, g_2))_L \circ \beta = \mu(\phi_1(f_1, f_2)) \mu(g_1, g_2) - \rho(f_1 + f_2, g_1 + g_2) \mu(\phi_1(g_1, g_2)) \mu(f_1, f_2),$$

(5.4)

$$\mu((g_1, g_2, g_3), \phi(f))_L \circ \beta = \mu(\phi_1(g_1, g_2)) \mu(g_1, f_3) + \rho(g_1, g_2 + g_3) \mu(\phi_1(g_2, g_3)) \mu(g_1, f)$$

(5.5)

$$+ \rho(g_1 + g_2, g_3) \mu(\phi_1(g_3, g_1)) \mu(g_2, f_3),$$

$$\mu(\phi(g), [f_1, f_2, f_3])_L \circ \beta = \rho(g, f_1 + f_2) \mu(\phi_1(f_1, f_2)) \mu(g, f_3) + \rho(g, f_2 + f_3) \rho(f_1, f_2 + f_3) \mu(\phi_1(f_2, f_3)) \mu(g, f_1)$$

(5.6)

$$+ \rho(g, f_1 + f_3) \rho(f_1 + f_2, f_3) \mu(\phi_1(f_3, f_1)) \mu(g, f_2).$$

If $\mathcal{B}_{\rho,\phi}$ is multiplicative, in addition to the above conditions, the following condition must be hold too

$$\mu(\phi(f), \phi(g))_L \circ \beta = \beta \circ \mu(f, g).$$

Lemma 5.25. Let $(V, \mu, \beta)$ be a representation of $\mathcal{B}_{\rho,\phi}$. Then, we have

$$0 = \rho(f_1 + f_2, g_1) \mu(\phi(g_1), [f_1, f_2, g_2]) \circ \beta + \mu([f_1, f_2, g_1]) \circ \beta$$

+ $\rho(f_1 + f_2, g_1 + g_2) \rho(g_1 + g_2, f_1) \mu(\phi(f_1), [g_1, g_2, f_2]) \circ \beta$

+ $\rho(f_1 + f_2, g_1 + g_2) \mu([g_1, g_2, f_1]) \circ \beta$.

Proof. For the proof refer to Lemma 3.3.

If $V = \mathcal{B}_{\rho,\phi}$ and $\phi = \beta \in \mathfrak{gl}(\mathcal{B}_{\rho,\phi})$, then the bilinear map $ad : \mathcal{B} \times \mathcal{B} \rightarrow \mathfrak{gl}(\mathcal{B})$ defined by $ad(f_1, f_2)(f_3) = [f_1, f_2, f_3]$ is a representation of $\mathcal{B}_{\rho,\phi}$ with respect to $\beta = \phi$.

Proposition 5.26. Consider the 3-Hom-$\rho$-Lie algebra $\mathcal{B}_{\rho,\phi}$. Let $V$ be a graded vector space, $\beta \in \mathfrak{gl}(V)$ and $\mu : \wedge^2 \mathcal{B} \rightarrow \mathfrak{gl}(V)$ be a bilinear map. Then $(V, \mu, \beta)$ is a representation of $\mathcal{B}_{\rho,\phi}$ if and only if $\mathcal{B} \oplus V$ is a 3-Hom-$\rho$-Lie algebra with the following structures

$$[f_1 + v_1, f_2 + v_2, f_3 + v_3]_{\mathcal{B} \oplus V}^{\mu} = [f_1, f_2, f_3]_\mathcal{B} + \mu(f_1, f_2) v_3$$

+ $\rho(f_1 + f_2, f_3) \mu(f_2, f_3) v_1 + \rho(f_1 + f_2, f_3) \mu(f_3, f_1) v_2$,

$$\psi(f_1 + v_1) = \phi(f_1) + \beta(v_1).$$
where $f_1, f_2, f_3 \in H^g(B)$ and $v_1, v_2, v_3 \in H^g(V)$.

Consider the representation $(V, \mu, \beta)$ of $B_{\rho, \phi}$ and $V^*$ as the dual of vector space $V$. We define bilinear map $\tilde{\mu} : B \times B \rightarrow \text{End}(V^*)$ by $\tilde{\mu}(f_1, f_2)(g) = -\rho(f_1 + f_2, g)g \circ \mu(f_1, f_2)$, where $f_1, f_2 \in H^g(B)$, $g \in V^*$ and set $\tilde{\beta}(g) = g \circ \beta$.

Note that by Definition 3.5 and Lemma 5.25, we can deduce that if $(V, \mu, \beta)$ is a $\rho$-skew symmetric representation, then $(V^*, \tilde{\mu}, \tilde{\beta})$ is a representation of $B_{\rho, \phi}$.

By abuse of the above notation, $(B^*, ad^*)$ with respect to $\tilde{\beta}(g) = g \circ \phi$ is a representation of $B_{\rho, \phi}$. So by Proposition 5.26, $B \oplus B^*$ is a 3-Hom-$\rho$-Lie algebra together with the following structure

$$[f_1 + \alpha_1, f_2 + \alpha_2, f_3 + \alpha_3]_{B \oplus B^*} = [f_1, f_2, f_3]_B + ad^*(f_1, f_2)\alpha_3$$

$$+ \rho(f_1, f_2 + f_3)ad^*(f_2, f_3)\alpha_1 + \rho(f_1 + f_2, f_3)ad^*(f_3, f_1)\alpha_2,$$

$$(\phi + \phi^*)(f + \alpha) = \phi(f) + \alpha \circ \phi.$$

Now, consider the bilinear map $\varphi(f + \alpha, g + \beta) = \alpha(g) + \rho(f, g)\beta(f)$. Then $(B \oplus B^*, [...], B_{B \oplus B^*}, \rho, \phi + \phi^*, \varphi)$ is a metric 3-Hom-$\rho$-Lie algebra.

6. 3-PRE-HOM-$\rho$-LIE ALGEBRAS

This part is devoted to the 3-pre-Hom-$\rho$-Lie algebras, which are studied similar to the classical case in Section 4. Actually, we give the similar results to Section 4 in the case of 3-pre-Hom-$\rho$-Lie algebras.

Definition 6.1. $(\mathcal{B}, [...], \rho, \phi)$ is called a 3-pre-Hom-$\rho$-Lie algebra if

i. $\mathcal{B}$ is a $G$-graded vector space,

ii. $\phi : \mathcal{B} \rightarrow \mathcal{B}$ is an even linear map,

iii. $\{..., ...\} : \otimes^3 \mathcal{B} \rightarrow \mathcal{B}$ is a trilinear map satisfying the following relations

$$\{|f_1, f_2, f_3|| = |f_1| + |f_2| + |f_3|,$$

$$\{f_1, f_2, f_3\} = -\rho(f_1, f_2)\{f_2, f_1, f_3\},$$

$$\{\phi(f_1), \phi(f_2), \{g_1, g_2, g_3\}\} = \{\{f_1, f_2, g_1\}_c, \phi(g_2), \phi(g_3)\} + \rho(f_1 + f_2, g_1)\{\phi(g_1), \phi(f_1), \phi(f_2), \phi(g_3)\}$$

$$+ \rho(f_1 + f_2, g_1 + g_2)\{\phi(g_1), \phi(g_2), \phi(f_1), \phi(f_2), \phi(g_3)\},$$

$$\{\{f_1, f_2, f_3\}_c, \phi(g_1), \phi(g_2)\} = \{\phi(f_1), \phi(f_2), \{f_3, g_1, g_2\}\} + \rho(f_1 + f_2, f_3)\{\phi(f_1), \phi(f_2), \phi(g_1), \phi(g_2)\}$$

$$+ \rho(f_1 + f_2, f_3)\{\phi(f_3), \phi(f_1), \{f_2, g_1, g_2\}\},$$

where

$$\{f_1, f_2, f_3\}_c = \{f_1, f_2, f_3\} + \rho(f_1, f_2 + f_3)\{f_2, f_3, f_1\} + \rho(f_1 + f_2, f_3)\{f_3, f_1, f_2\}.$$

We denoted by $B(\{1, \phi\}$ the 3-pre-Hom-$\rho$-Lie algebra $(\mathcal{B}, [...], \rho, \phi)$.

For a 3-pre-Hom-$\rho$-Lie algebra $(\mathcal{B}, [...], \rho, \phi)$, it is easy to see that $(\mathbb{B}, [...], \rho, \phi)$ is a 3-Hom-$\rho$-Lie algebra. We denote it by $B^{(\rho, \phi)}_{\mathcal{B}}$.

Let us define two multiplications $L : \wedge^2 \mathcal{B} \rightarrow gl(B)$ and $R : \otimes^2 \mathcal{B} \rightarrow gl(B)$ on $B(\{1, \phi\}$, which is called respectively the left and right multiplications, by $L(f, g)h = \{f, g, h\}$ and $R(f, g)h = \rho(f + g, h)\{h, f, g\}$. Then $(B, L)$ is a representation of $B_{\rho, \phi}$ with respect to $\beta = \phi$. 
Definition 6.2. The triple \((V, \mu, \tilde{\mu})\) consisting of a \(G\)-graded vector space \(V\), a representation \(\mu\) of \(\mathfrak{B}_{1,\phi}^c\) and a bilinear map \(\tilde{\mu} : \otimes^2 \mathfrak{B} \to \mathfrak{gl}(V)\) is called a representation of \(\mathfrak{B}_{1,\phi}^c\) with respect to \(\beta \in \mathfrak{gl}(V)\) if for all homogeneous elements \(f_1, f_2, f_3, f_4 \in Hg(\mathfrak{B})\), the following equalities hold

\[
\tilde{\mu}(\phi(f_1), \{f_2, f_3, f_4\}) \circ \beta = \rho(f_1, f_2 + f_3)\mu(\phi(f_2), \phi(f_3))\tilde{\mu}(f_1, f_4) \\
+ \rho(f_1, f_3 + f_4)\rho(f_2, f_3 + f_4)\tilde{\mu}(\phi(f_3), \phi(f_4))\tilde{\mu}(f_1, f_2) \\
- \rho(f_1, f_2 + f_4)\rho(f_3, f_4)\tilde{\mu}(\phi(f_2), \phi(f_4))\tilde{\mu}(f_1, f_3) \\
+ \rho(f_1 + f_2, f_3 + f_4)\tilde{\mu}(\phi(f_3), \phi(f_4))\mu(f_1, f_2) \\
+ \rho(f_1, f_2 + f_3 + f_4)\tilde{\mu}(\phi(f_2), \phi(f_4))\tilde{\mu}(f_1, f_3) \\
- \rho(f_1, f_2 + f_4)\rho(f_3, f_4)\tilde{\mu}(\phi(f_2), \phi(f_4))\tilde{\mu}(f_1, f_2). 
\]

\[
\mu(\phi(f_1), \phi(f_2))\tilde{\mu}(f_3, f_4) = \rho(f_1 + f_2, f_3 + f_4)\tilde{\mu}(\phi(f_3), \phi(f_4))\mu(f_1, f_2) \\
- \rho(f_1 + f_2, f_3 + f_4)\rho(f_1, f_2)\tilde{\mu}(\phi(f_3), \phi(f_4))\tilde{\mu}(f_1, f_2) \\
+ \rho(f_1 + f_2, f_3 + f_4)\tilde{\mu}(\phi(f_3), \phi(f_4))\mu(f_1, f_2) \\
+ \rho(f_1 + f_2, f_3 + f_4)\tilde{\mu}(\phi(f_3), \phi(f_4))\tilde{\mu}(f_1, f_2). 
\]

\[
\tilde{\mu}([f_1, f_2, f_3]_B, \phi(f_4)) \circ \beta = \mu(\phi(f_1), \phi(f_2))\tilde{\mu}(f_3, f_4) + \rho(\phi(f_1), \phi(f_2))\phi(f_3)\tilde{\mu}(f_1, f_4) \\
+ \rho(f_1 + f_2, f_3)\phi(\phi(f_4))\tilde{\mu}(\phi(f_3), \phi(f_4))\tilde{\mu}(f_1, f_4). 
\]

For example \((\mathfrak{B}, L, R, \phi)\) is a representation of \(\mathfrak{B}_{1,\phi}^c\). Let us to define the operation \(\{\ldots\}_\mu, \tilde{\mu} : \otimes^2 (\mathfrak{B} \oplus V) \to \mathfrak{B} \oplus V\) by

\[
\{f_1 + v_1, f_2 + v_2, f_3 + v_3\}_{\mu, \tilde{\mu}} = \{f_1, f_2, f_3\} + \mu(\phi(f_1, f_2))v_3 + \rho(\phi(f_1, f_2))v_1 - \rho(f_2, f_3)\tilde{\mu}(f_1, f_3)v_2, \\
\psi(f_1 + v_1) = \phi(f_1) + \beta(v_1),
\]

for all \(f_1, f_2, f_3 \in Hg(\mathfrak{B})\) and \(v_1, v_2, v_3 \in V\). Then \((\mathfrak{B} \oplus V, \{\ldots\}_{\mu, \tilde{\mu}}, \rho, \psi)\) is a 3-pre-Hom-\(\rho\)-Lie algebra.

Consider the operator \(\zeta : \otimes^2 V \to \otimes^2 V\) given by \(\zeta(f_1 \otimes f_2) = f_2 \otimes f_1\), where \(V\) is a graded vector space and \(f_1 \otimes f_2 \in \otimes^2 V\).

Using the above notation, in the following proposition, we give a representation of \(\mathfrak{B}_{1,\phi}^c\):

Proposition 6.3. Let \((V, \mu, \tilde{\mu}, \beta)\) be a representation of \(\mathfrak{B}_{1,\phi}^c\). Then there exists a representation \(\nu : \wedge^2 \mathfrak{B} \to \mathfrak{gl}(V)\) of \(\mathfrak{B}_{1,\phi}^c\) on \(V\) that is given by \(\nu(f_1, f_2) = (\mu - \rho(f_1, f_2)\mu_\xi)\tilde{\mu}(f_1, f_2)\) for all \(f_1, f_2 \in Hg(\mathfrak{B})\).
The above proposition conclude that if \((V, \mu, \tilde{\mu})\) is a representation of \(B_{\{1, \phi\}}\), then the 3-pre-Hom-\(\rho\)-Lie algebras \((B \oplus V, \{1, \ldots, 1\}_\mu, \tilde{\mu}, \rho, \psi)\) and \((B \oplus V, \{1, \ldots, 1\}_\phi, \rho, \psi)\) have the same sub-adjacent 3-Hom-\(\rho\)-Lie algebra given by \((6.5)\).

Let us define the dual of the representation \((V, \nu)\) by \(\nu^* (f_1, f_2) = \mu^* - \rho(f_1, f_2) \tilde{\mu}^* \xi + \tilde{\mu}^* (f_1, f_2)\). In the following proposition, we will talk about the dual of the representation \((V, \mu, \tilde{\mu}):\)

**Proposition 6.4.** Consider \(B_{\{1, \phi\}}\) equipped with the \(\rho\)-skew-symmetric representation \((V, \mu, \tilde{\mu}, \beta)\). Then \((V^*, \nu^*, -\tilde{\mu}^*, \beta)\) is a representation of \(B_{\{1, \phi\}}\), which is called the dual representation of \((V, \mu, \tilde{\mu}):\)

**Remark 6.5.** Consider \(B_{\{1, \phi\}}\) as a 3-pre-Hom-\(\rho\)-Lie algebra with the \(\rho\)-skew-symmetric representation \((V, \mu, \tilde{\mu}, \beta)\). Then the 3-pre-Hom-\(\rho\)-Lie algebras \((B \oplus V^*, \{1, \ldots, 1\}_\mu^*, \nu^*, \tilde{\mu}^*, \rho, \psi)\) and \((B \oplus V^*, \{1, \ldots, 1\}_\phi^*, \nu^*, -\tilde{\mu}^*, \rho, \psi)\) have the same sub-adjacent 3-Hom-\(\rho\)-Lie algebra \((B^* \oplus V^*, \{1, \ldots, 1\}_\mu^*, \rho, \phi)\) with the representation \((V^*, \mu^*, \bar{\beta})\).

In the following definition, we define the \(\rho\)-Hom-\(\mathcal{O}\)-operator associated to representation \((V, \mu, \beta)\). The importance of defining this operator is that it can be used to construct a 3-pre-Hom-\(\rho\)-Lie algebra structure on representation \(V\).

**Definition 6.6.** Let \((V, \mu, \beta)\) be a representation of \(B_{\rho, \phi}\). If an even linear operator \(T : V \rightarrow B_{\rho, \phi}\) satisfies

\[
T \circ \beta = \phi \circ T,
\]

\[
[Tx, Ty, Tz]_B = T(\mu(Tx, Ty)z + \rho(x, y + z)(\mu(Ty, Tz)x + \rho(x + y, z))\mu(Tz, Tx)y), \quad \forall x, y, z \in V,
\]

it is called a \(\rho\)-Hom-\(\mathcal{O}\)-operator associated to representation \((V, \mu, \beta)\).

**Lemma 6.7.** Let \((V, \mu, \beta)\) be a \(\rho\)-skew-symmetric representation of \(B_{\rho, \phi}\) and \(T : V \rightarrow B_{\rho, \phi}\) is a \(\rho\)-Hom-\(\mathcal{O}\)-operator associated to \((V, \mu, \beta)\). Then \((V, \{1, \ldots, 1\}_\mu, \rho, \psi)\) is a 3-pre-Hom-\(\rho\)-Lie algebra, where

\[
\psi = \beta, \quad \{x, y, z\} = \mu(Tx, Ty)z, \quad \forall x, y, z \in V. \tag{6.1}
\]

**Proof.** Suppose that \(x, y, z \in Hg(V)\). It is easy to see that

\[
\{x, y, z\} = -\rho(x, y)\{y, x, z\}, \quad T[x, y, z]_c = [Tx, Ty, Tz].
\]

Also, for \(x_1, x_2, x_3, x_4, x_5 \in Hg(V)\), we have

\[
\{\psi(x_1), \psi(x_2), \{x_3, x_4, x_5\}\} = \mu(T \circ \psi(x_1), T \circ \psi(x_2))\mu(Tx_3, Tx_4)x_5 = \mu(\phi \circ T(x_1), \phi \circ T(x_2))\mu(Tx_3, Tx_4)x_5, \tag{6.3}
\]

\[
\{[x_1, x_2, x_3]_c, \psi(x_4), \psi(x_5)\} = \mu([Tx_1, Tx_2, Tx_3], T \circ \psi(x_4))x_5 = \mu([Tx_1, Tx_2, Tx_3], \phi \circ T(x_4))x_5, \tag{6.4}
\]

\[
\rho(x_1 + x_2, x_3)\{\psi(x_3), \{x_1, x_2, x_4\}, \psi(x_5)\} = \rho(x_1 + x_2, x_3)\mu(T \circ \psi(x_3), [Tx_1, Tx_2, Tx_4])x_5 = \rho(x_1 + x_2, x_3)\mu(\phi \circ T(x_3), [Tx_1, Tx_2, Tx_4])x_5, \quad \tag{6.5}
\]

\[
\rho(x_1 + x_2, x_3 + x_4)\{\psi(x_3), \psi(x_4), \{x_1, x_2, x_5\}\} = \rho(x_1 + x_2, x_3 + x_4)\mu(T \circ \psi(x_3), T \circ \psi(x_4))\mu(Tx_1, Tx_2)x_5 = \rho(x_1 + x_2, x_3 + x_4)\mu(\phi \circ T(x_3), \phi \circ T(x_4))\mu(Tx_1, Tx_2)x_5. \tag{6.6}
\]

By Definition 5.24 and the equalities (6.3)–(6.6), we get the result. \(\Box\)
**Lemma 6.8.** Consider $\mathcal{B}_{\rho,\phi}$ as a 3-Hom-$\rho$-Lie algebra. Then the following assertions are equivalent

i. existence of a compatible 3-pre-Hom-$\rho$-Lie algebra structure,

ii. existence of an invertible $\rho$-$\mathcal{O}$-operator.

**Proof.** Refer to the proof of Lemma 4.12 □

**Phase Space:** In the following, we extend the study of phase space of a 3-$\rho$-Lie algebra to 3-Hom-$\rho$-Lie algebras and show that a 3-Hom-$\rho$-Lie algebra has a phase space if and only if it is sub-adjacent to a 3-pre-Hom-$\rho$-Lie algebra.

**Proposition 6.9.** Consider the symplectic 3-Hom-$\rho$-Lie algebra $(\mathcal{B},[.,.,.]_{\mathcal{B}},\rho,\phi,\omega)$. Then the following structure defines a compatible 3-pre-Hom-$\rho$-Lie algebra structure on $\mathcal{B}$

$$\omega([f,g,h],\phi(s)) = -\rho(f + g, h)\omega(\phi(h),[f,g,s]_{\mathcal{B}}), \ \forall f,g,h,s \in Hg(\mathcal{B}).$$

**Proof.** Let us define a linear map $T : \mathcal{B}^* \rightarrow \mathcal{B}$ by $\omega(\phi(f),g) = \omega(f,\phi(g)) = (T^{-1}(f))(g)$, or equivalently, $\omega(T\alpha,\phi(g))) = \alpha(g)$ for all $f,g \in Hg(\mathcal{B})$ and $\alpha \in \mathcal{B}^*$. Since $\omega$ is a symplectic structure, we deduce that $T$ is an invertible $\rho$-$\mathcal{O}$-operator associated to $(\mathcal{B}^*,Ad^*)$. Also, by (4.6) and (6.2), there there exists a compatible 3-$\rho$-pre-Lie algebra on $B$ given by $\{f,g,h\} = T(ad^*(f,g)T^{-1}(h))$. So

$$\omega([f,g,h],\phi(s)) = \omega(T(ad^*(f,g)T^{-1}(h)),\phi(s)) = ad^*(f,g)T^{-1}(h)(s)$$

$$= -\rho(f + g, h)T^{-1}(h)[f,g,s]_{\mathcal{B}} = -\rho(f + g, h)\omega(\phi(h),[f,g,s]_{\mathcal{B}}).$$

□

**Definition 6.10.** $(\mathcal{B} \oplus \mathcal{B}^*,[.,.,.]_{\mathcal{B} \oplus \mathcal{B}^*},\rho,\phi + \phi^*,\omega)$ is called a phase space of the 3-Hom-$\rho$-Lie algebra $\mathcal{B}$, if

i. $(\mathcal{B} \oplus \mathcal{B}^*,[.,.,.]_{\mathcal{B} \oplus \mathcal{B}^*},\rho,\phi + \phi^*)$ is a 3-Hom-$\rho$-Lie algebra,

ii. $\omega$ is a natural non-degenerate skew-symmetric bilinear form on $\mathcal{B} \oplus \mathcal{B}^*$ given by

$$\omega(f + \alpha, g + \beta) = \alpha(g) - \rho(f,g)\beta(f),$$

such that $(\mathcal{B} \oplus \mathcal{B}^*,[.,.,.]_{\mathcal{B} \oplus \mathcal{B}^*},\rho,\phi + \phi^*,\omega)$ is a symplectic 3-Hom-$\rho$-Lie algebra,

iii. $(\mathcal{B},[.,.,.]_{\mathcal{B}},\rho,\phi)$ and $(\mathcal{B}^*,[.,.,.]_{\mathcal{B}^*},\rho,\phi^*)$ are 3-Hom-$\rho$-Lie subalgebras of $(\mathcal{B} \oplus \mathcal{B}^*,[.,.,.]_{\mathcal{B} \oplus \mathcal{B}^*},\rho,\phi + \phi^*)$.

**Theorem 6.11.** A 3-Hom-$\rho$-Lie algebra has a phase space if and only if it is sub-adjacent to a 3-pre-Hom-$\rho$-Lie algebra.

**Proof.** By Theorem 4.15 it is easy to get the result. □

**Lemma 6.12.** Let $(V,\mu,\beta)$ be a representation of $\mathcal{B}_{\rho,\phi}$ and $(\mathcal{B} \oplus \mathcal{B}^*,[.,.,.]_{\mathcal{B} \oplus \mathcal{B}^*},\rho,\phi + \phi^*,\omega)$ be a phase space of it. Then the relation $\{f,g,h\} = \mu(f,g)h$ constructs a 3-pre-Hom-$\rho$-Lie algebra structure on $\mathcal{B}$ if $\beta = \phi$. #
Proof. Assume that \( f, g, h \in Hg(\mathcal{B}) \) and \( \alpha \in Hg(\mathcal{B}^*) \). Then, we have
\[
\rho(f + g + h, \alpha) \circ \phi(\{f, g, h\}) = -\omega(\{f, g, h\}, \alpha \circ \phi) = \rho(f + g, h)\omega(\phi(h), [f, g, \alpha]_{B \oplus B^*})
\]
\[
= \rho(f + g, h)\omega(\phi(h), \mu^*(f, g)\alpha)
\]
\[
= -\rho(f + g, h)\rho(h, f + g + \alpha)\mu^*(f, g)\alpha(\phi(h))
\]
\[
= \rho(f + g + h, \alpha) \circ \phi(f, g)h.
\]

Data Availability Statements. The data that support the findings of this study are available from the corresponding author upon reasonable request.

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