Inverse Approach in Ordinary Differential Equations: Applications to Lagrangian and Hamiltonian Mechanics

Jaume Llibre · Rafael Ramírez · Natalia Sadovskaia

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Abstract This paper is on the so called inverse problem of ordinary differential equations, i.e. the problem of determining the differential system satisfying a set of given properties. More precisely we characterize under very general assumptions the ordinary differential equations in $\mathbb{R}^N$ which have a given set of either $M$ partial integrals, or $M < N$ first integral, or $M < N$ partial and first integrals. Moreover, for such systems we determine the necessary and sufficient conditions for the existence of $N - 1$ independent first integrals. We give two relevant applications of the solutions of these inverse problem to constrained Lagrangian and Hamiltonian systems respectively. Additionally we provide the general solution of the inverse problem in dynamics.

Keywords Algebraic limit circles · Polynomial planar differential system · Polynomial vector fields · Invariant circles · Invariant algebraic circles · Darboux integrability · 16th Hilbert’s problem

Mathematics Subject Classification 34C07
1 Introduction

In the theory of ordinary differential equations we can find two fundamental problems. The direct problem which consists in a broad sense in to find the solutions of a given ordinary differential equation, and the inverse problem. An inverse problem of ordinary differential equations is to find the more general differential system satisfying a set of given properties. For instance what are the differential systems in $\mathbb{R}^N$ having a given set of invariant hypersurfaces, or of first integrals?

Probably the first inverse problem appeared in Celestial Mechanics, it was stated and solved by Newton in [22], and it concerns with the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely the Kepler’s laws.

Bertrand in [4] proved that the expression for Newton’s force of attraction can be obtained directly from the Kepler first law. He stated also a more general problem of determining a positional force, under which a particle describes a conic section under any initial conditions. Bertrand’s ideas were developed in particular in the works [6, 8, 11, 14, 25, 28, 32].

In the modern scientific literature the importance of the inverse problem in Celestial Mechanics was already recognized by Szebehely (see [35]).

Clearly in view of the second Newton law, acceleration is equal to force, it follows that the mentioned inverse problems are equivalent to determine second order differential equations from the given properties on the right hand side of the equation.

The first statement of an inverse problem as the problem of finding the more general differential systems of first order satisfying a set of given properties was stated by Erugin [9] in dimension 2 and developed by Galiullin in [11].

The new approach of an inverse problem which we propose uses as an essential tool the Nambu bracket. We deduce new properties of this bracket which plays a very important role in the proof of all the results of this work and in its applications. We observe that the applications of the Nambu bracket which we will give in this paper are original and represent a new direction in order to develop the Nambu ideas.

In the first section we present two different kind of results. First under very general assumptions we characterize the ordinary differential equations in $\mathbb{R}^N$ which have a given set of either $M$ partial integrals, or $M < N$ first integrals, or $M \leq N$ partial and first integrals. Second in $\mathbb{R}^N$ we provide necessary and sufficient conditions on the integrability, in the sense that the characterized differential equations admit $N - 1$ independent first integrals.

In the third section applying results of the first section, we state and solve the inverse problem for the constrained Lagrangian Mechanics: For a given natural mechanical system with $N$ degrees of freedom determine the most general field of force depending only on the position of the system satisfying a given set of constraints which are linear in the velocity. One of the main objectives in this inverse problem is to study the behavior of the constrained Lagrangian systems with linear constraints with respect to the velocity in a way different from the classical approach deduced from the d’Alembert-Lagrange principle.

The section four is devoted to study the inverse problem for constrained Hamiltonian systems. That is, for a given submanifold $\mathcal{M}$ of a symplectic manifold $\mathbb{M}$ we determine the differential systems having the submanifold $\mathcal{M}$ invariant by their flow.
2 The Nambu Bracket. New Properties

In the seventies Nambu in [19] proposed a new approach to the classical dynamics based on an $N$ dimensional Nambu–Poisson manifold replacing the even dimensional Poisson manifold and on $N - 1$ Hamiltonian $H_1, \ldots, H_{N-1}$ instead of a single Hamiltonian $H$. In the canonical Hamiltonian formulation, the equation of motion (Hamilton equations) are defined via the Poisson bracket. In Nambu’s formulation, the Poisson bracket is replaced by the Nambu bracket. Nambu had originally considered the case $N = 3$.

Although the Nambu formalism is a generalization of the Hamiltonian formalism its real applications are not as rich as the applications of this last one.

Let $D$ be an open subset of $\mathbb{R}^N$. Let $h_j = h_j(x)$ for $j = 1, 2, \ldots, M$ with $M \leq N$ be $C^1$ functions $h_j : D \rightarrow \mathbb{R}$. In all this article the functions be at least of class $C^1$. We define the matrix

$$S_{M,N} = \begin{pmatrix} dh_1(\partial_1) & \cdots & dh_1(\partial_N) \\ \vdots & \ddots & \vdots \\ dh_M(\partial_1) & \cdots & dh_M(\partial_N) \end{pmatrix} = \begin{pmatrix} \partial_1 h_1 & \cdots & \partial_N h_1 \\ \vdots & \ddots & \vdots \\ \partial_1 h_M & \cdots & \partial_N h_M \end{pmatrix},$$

where $\partial_j h = \frac{\partial h}{\partial x_j}$ and $dh = \sum_{j=1}^N \partial_j h \, dx_j$. The matrix $S_{M,N}$ is also denoted by

$$\frac{\partial(h_1, \ldots, h_M)}{\partial(x_1, \ldots, x_N)}.$$

We say that the functions $h_j$ for $j = 1, \ldots, M \leq N$ are independent if the rank of the matrix $S_{M,N}$ is $M$ for all $x \in D$, except perhaps in a subset of $D$ of zero Lebesgue measure.

If $M \geq N$ we define the matrix $S = S_{N,N}$. We note that $S$ is the Jacobian matrix of the function $(h_1, \ldots, h_N)$. The Jacobian of $S$, i.e. the determinant of $S$, is denoted by

$$|S| = \left| \frac{\partial(h_1, \ldots, h_N)}{\partial(x_1, \ldots, x_N)} \right| = \left| \begin{array}{ccc} dh_1(\partial_1) & \cdots & dh_1(\partial_N) \\ \vdots & \ddots & \vdots \\ dh_N(\partial_1) & \cdots & dh_N(\partial_N) \end{array} \right| = \{h_1, \ldots, h_N\}.$$

This last bracket is known in the literature as the Nambu bracket [3,17,19,36].

The objective of this section is to provide properties of the Nambu bracket, some of them are new. These properties will play an important role in some of the proofs of the main results of this paper.

The Nambu bracket $\{h_1, \ldots, h_N\}$ satisfies the following known properties.

(i) It is a skew symmetry bracket, i.e.

$$\{h_{\sigma(1)}, \ldots, h_{\sigma(N)}\} = (-1)^{|\sigma|}\{h_1, \ldots, h_N\},$$

for arbitrary functions $h_1, \ldots, h_N$ and arbitrary permutation $\sigma$ of $(1, 2, \ldots, N)$. Here $|\sigma|$ is the order of $\sigma$.

(ii) It is a derivation, i.e. it satisfies the Leibniz rule

$$\{h_1, \ldots, v\lambda\} = \{h_1, \ldots, v\}\lambda + v\{h_1, \ldots, \lambda\}.$$
(iii) It satisfies the fundamental identity (Filippov Identity)

\[
F (f_1, \ldots, f_{N-1}, g_1, \ldots, g_N) := \{ f_1, \ldots, f_{N-1}, \{ g_1, \ldots, g_N \} \} - \sum_{n=1}^{N} \{ g_1, \ldots, g_{n-1}, \{ f_1, \ldots, f_{N-1}, g_n \}, g_{n+1}, \ldots, g_N \} = 0,
\]

where \( f_1, f_2, \ldots, f_{N-1}, g_1, \ldots, g_N \) are arbitrary functions. For more details see [10, 17, 19, 36]. The property (i) follows directly from the properties of the determinants. The property (ii) is obtained using the properties of the derivative plus the properties of the determinants. The property (iii) will be the property (ix) with \( \lambda = 1 \), and we shall prove it in Proposition 2.

Unfortunately the properties listed above of the Nambu bracket are not sufficient for solving some of the problems that we shall consider.

We observe that the applications of the Nambu bracket that we will do are original and represent a new direction in order to develop the Nambu ideas.

We shall need the next results (new properties of the Nambu bracket).

**Proposition 1** The following four identities hold.

(iv) \( \sum_{j=1}^{N} \frac{\partial f}{\partial x_j} \{ g_1, \ldots, g_{n-1}, x_j, g_{n+1}, \ldots, g_N \} = \{ g_1, \ldots, g_{n-1}, f, g_{n+1}, \ldots, g_N \} \),

(v) \( \frac{\partial f}{\partial x_n} = \{ x_1, \ldots, x_{n-1}, f, x_{n+1}, \ldots, x_N \} \),

(vi) \( K_n^N := \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \{ g_1, \ldots, g_{n-1}, x_j, g_{n+1}, \ldots, g_N \} = 0 \), for \( n = 1, 2, \ldots, N \), and

(vii)

\[
\frac{\partial f_1}{\partial x_N} \left| \frac{\partial (G, f_2, \ldots, f_N)}{\partial (y_1, \ldots, y_N)} \right| + \ldots + \frac{\partial f_N}{\partial x_N} \left| \frac{\partial (f_1, \ldots, f_{N-1}, G)}{\partial (y_1, \ldots, y_N)} \right| = \frac{\partial G}{\partial y_1} \left| \frac{\partial (f_1, \ldots, f_N)}{\partial (x_N, y_2, \ldots, y_N)} \right| + \ldots + \frac{\partial G}{\partial y_N} \left| \frac{\partial (f_1, \ldots, f_N)}{\partial (y_1, \ldots, y_{N-1}, x_N)} \right|.
\]

Here the functions \( g_1, \ldots, g_N, f_1, \ldots, f_N, G \) and \( f \) are arbitrary.
Proof The proof of (iv) is

\[
\{g_1, \ldots, g_{n-1}, f, g_{n+1}, \ldots, g_N\} = \begin{vmatrix}
\partial_1 g_1 & \ldots & \partial_N g_1 \\
\vdots & \ddots & \vdots \\
\partial_1 g_{n-1} & \ldots & \partial_N g_{n-1} \\
\partial_1 f & \ldots & \partial_N f \\
\partial_1 g_{n+1} & \ldots & \partial_N g_{n+1} \\
\vdots & \ddots & \vdots \\
\partial_1 g_N & \ldots & \partial_N g_N \\
\end{vmatrix}
\]

\[
= \partial_1 f
\]

\[
\begin{vmatrix}
\partial_1 g_1 & \partial_2 g_1 & \ldots & \partial_N g_1 \\
\vdots & \ddots & \vdots & \vdots \\
\partial_1 g_{n-1} & \partial_2 g_{n-1} & \ldots & \partial_N g_{n-1} \\
0 & 0 & \ldots & 1 \\
\partial_1 g_{n+1} & \partial_2 g_{n+1} & \ldots & \partial_N g_{n+1} \\
\vdots & \ddots & \vdots & \vdots \\
\partial_1 g_N & \partial_2 g_N & \ldots & \partial_N g_N \\
\end{vmatrix}
\]

\[+ \partial_N f
\]

\[= \{g_1, \ldots, g_{n-1}, x_1, g_{n+1}, \ldots, g_N\}\partial_1 f + \cdots
\]

\[+ \{g_1, \ldots, g_{n-1}, x_N, g_{n+1}, \ldots, g_N\}\partial_N f.
\]

The proof of (v) follows easily from the definition of the Nambu bracket.

The proof of (vi) is done by using the mathematical induction. Without loss of generality we shall prove that

\[
K_1^N = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \{x_j, g_2, \ldots, g_N\} = 0. \tag{2}
\]

For \(N = 2\) we obtain

\[
K_1^2 = \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \{x_j, g_2\} = \frac{\partial}{\partial x_1} \left( \frac{\partial g_2}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left( \frac{\partial g_2}{\partial x_1} \right) = 0.
\]

Now we suppose that

\[
K_1^{N-1} = \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j} \{x_j, g_2, \ldots, g_{N-1}\} = 0
\]
holds and we shall prove (2). Indeed, by considering that
\[\{x_j, g_2, \ldots, g_N\} = \sum_{k=2}^{N} (-1)^{N+k+1} \frac{\partial g_k}{\partial x_N} \{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\]
for \( j = 1, \ldots, N - 1, \)
\[\{x_N, g_2, \ldots, g_N\} = (-1)^{N+1} \{g_2, \ldots, g_N\},\]
we deduce that
\[K_N^{[N]} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \{x_j, g_2, \ldots, g_N\}\]
\[= \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j} \left( \sum_{k=2}^{N} (-1)^{N+k+1} \frac{\partial g_k}{\partial x_N} \{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\} \right)\]
\[+ \frac{\partial}{\partial x_N} \{x_N, g_2, \ldots, g_N\}\]
\[= \sum_{k=2}^{N} (-1)^{N+k+1} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j} \left( \frac{\partial g_k}{\partial x_N} \right) \{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\]
\[+ \sum_{k=2}^{N} (-1)^{N+k+1} \frac{\partial g_k}{\partial x_N} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j} \{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\]
\[+ (-1)^{N+1} \frac{\partial}{\partial x_N} \{g_2, \ldots, g_N\}\]
\[= \sum_{k=2}^{N} (-1)^{N+k+1} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j} \left( \frac{\partial g_k}{\partial x_N} \right) \{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\]
\[+ (-1)^{N+1} \frac{\partial}{\partial x_N} \{g_2, \ldots, g_N\}.\]

Here we apply the assumption that \(K_N^{[N-1]} = 0\) with the functions \((g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N)\) instead of \((g_2, \ldots, g_{k-1}, g_k, g_{k+1}, \ldots, g_{N-1})\).

In view of the property (iv) we obtain that
\[\sum_{k=2}^{N} (-1)^{N+k+1} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j} \left( \frac{\partial g_k}{\partial x_N} \right) \{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\]
\[= \sum_{k=2}^{N} (-1)^{N} \{(-1)^{k+1} \frac{\partial g_k}{\partial x_N}, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\]
\[= \sum_{k=2}^{N} (-1)^{N} \{g_2, \ldots, g_{k-1}, \frac{\partial g_k}{\partial x_N}, g_{k+1}, \ldots, g_N\}\]
\[= (-1)^{N} \frac{\partial}{\partial x_N} \{g_2, \ldots, g_N\}.\]
Hence
\[ K^N_1 = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \{x_j, g_2, \ldots, g_N \} \]
\[ = (-1)^N \frac{\partial}{\partial x_N} \{g_2, \ldots, g_N \} + (-1)^{N+1} \frac{\partial}{\partial x_N} \{g_2, \ldots, g_N \} = 0, \]
consequently the property (vi) is proved.

The proof of (vii) is easy to obtain by considering that the value of determinant
\[
\begin{vmatrix}
\frac{\partial f_1}{\partial y_1} & \ldots & \frac{\partial f_1}{\partial y_N} & \frac{\partial f_1}{\partial x_N} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial f_N}{\partial y_1} & \ldots & \frac{\partial f_N}{\partial y_N} & \frac{\partial f_N}{\partial x_N} \\
\frac{\partial G}{\partial y_1} & \ldots & \frac{\partial G}{\partial y_N} & 0
\end{vmatrix}
\]
can be obtained by developing by the last row and by the last column. \(\square\)

**Proposition 2** We define
\[ \Omega (f_1, \ldots, f_{N-1}, g_1, \ldots, g_N, G) := -\{f_1, \ldots, f_{N-1}, G\}\{g_1, \ldots, g_N\} \]
\[ + \sum_{n=1}^{N} \{f_1, \ldots, f_{N-1}, g_n\}\{g_1, \ldots, g_{n-1}, G, g_{n+1}, \ldots, g_N\}, \]
and
\[ F_{\lambda} (f_1, \ldots, f_{N-1}, g_1, \ldots, g_N) := -\{f_1, \ldots, f_{N-1}, \lambda\}\{g_1, \ldots, g_N\} \]
\[ + \sum_{n=1}^{N} \{g_1, \ldots, g_{n-1}, \lambda\}\{f_1, \ldots, f_{N-1}, g_n\}, \ldots, g_N, \lambda. \]

Then the Nambu bracket satisfies the following two identities:

(iii) \( \Omega (f_1, \ldots, f_{N-1}, g_1, \ldots, g_N, G) = 0, \) and

(ix) \( F_{\lambda} (f_1, \ldots, f_{N-1}, g_1, \ldots, g_N) = 0. \) Note this identity is a generalization of the fundamental identity (1) which is obtained when \( \lambda = 1. \)

**Proof** Indeed \( \Omega := \Omega (f_1, \ldots, f_{N-1}, g_1, \ldots, g_N, G) \) can be written as
\[
\Omega = - \begin{vmatrix}
dg_1(\partial_1) & \ldots & dg_1(\partial_N) & \{f_1, \ldots, f_{N-1}, g_1\} \\
: & \ddots & \vdots & \vdots \\
dg_N(\partial_1) & \ldots & dg_N(\partial_N) & \{f_1, \ldots, f_{N-1}, g_N\} \\
dG(\partial_1) & \ldots & dG(\partial_N) & \{f_1, \ldots, f_{N-1}, G\}
\end{vmatrix}. \]
and using (iv) we obtain
\[
\Omega = - \left| \begin{array}{ccc}
dg_1(\partial_1) & \ldots & dg_1(\partial_N) \\
\vdots & \ddots & \vdots \\
dg_N(\partial_1) & \ldots & dg_N(\partial_N)
\end{array} \right| \sum_{j=1}^{N} \{f_1, \ldots, f_{N-1}, x_j\} dg_1(\partial_j) \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^{N} \{f_1, \ldots, f_{N-1}, x_j\} dg_N(\partial_j)
\right| \\
\sum_{j=1}^{N} \{f_1, \ldots, f_{N-1}, x_j\} dG(\partial_j) \\
\right| = 0.
\]

This proves identity (viii).

The proof of (ix) is as follows. Taking \( G = x_j \) in the identity of (viii) and multiplying it by \( \lambda \) we obtain
\[
\lambda \Omega \left( f_1, \ldots, f_{N-1}, g_1, \ldots, g_N, x_j \right) \\
\lambda \{f_1, \ldots, f_{N-1}, g_1\} \{x_j, g_2, \ldots, g_N \} + \ldots + \lambda \{f_1, \ldots, f_{N-1}, g_N\} \{g_1, \ldots, g_{N-1}, x_j \} \\
\ldots - \lambda \{f_1, \ldots, f_{N-1}, x_j\} \{g_1, \ldots, g_N \} = 0.
\]

Using (vi) from the last expression we have
\[
0 = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( \lambda \Omega \left( f_1, \ldots, f_{N-1}, g_1, \ldots, g_N, x_j \right) \right) \\
= \sum_{j=1}^{N} \{x_j, g_2, \ldots, g_N\} \frac{\partial}{\partial x_j} \left( \lambda \{f_1, \ldots, f_{N-1}, g_1\} \right) + \ldots \\
+ \sum_{j=1}^{N} \{g_1, \ldots, g_{N-1}, x_j\} \frac{\partial}{\partial x_j} \left( \lambda \{f_1, \ldots, f_{N-1}, g_N\} \right) \\
- \sum_{j=1}^{N} \{f_1, \ldots, f_{N-1}, x_j\} \frac{\partial}{\partial x_j} \left( \lambda \{g_1, \ldots, g_N\} \right).
\]

Now using (iv) the previous expression becomes
\[
\{\lambda \{f_1, \ldots, f_{N-1}, g_1\}, g_2, \ldots, g_N\} + \ldots + \{g_1, g_2, \ldots, g_{N-1}, \lambda \{f_1, \ldots, f_{N-1}, g_N\} \} \\
+ \ldots - \{f_1, \ldots, f_{N-1}, \lambda \{g_1, \ldots, g_N\} \} = F_\lambda \left( f_1, \ldots, f_{N-1}, g_1, \ldots, g_N \right) = 0.
\]

This complete the proof of the identity (ix). \( \square \)

Remark 3 We note that (ix) has obtained from (viii). So in some sense (viii) is more basic. In fact from the proof of (ix) we obtain
\[
F_\lambda \left( f_1, \ldots, f_{N-1}, g_1, \ldots, g_N \right) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( \lambda \Omega \left( f_1, \ldots, f_{N-1}, g_1, \ldots, g_N, x_j \right) \right).
\]
Now we establish the relationship between the Nambu bracket and the classical Poisson bracket. We suppose that \( N = 2n \), and \( x_j = x_j \) and \( x_{j+n} = y_j \) for \( j = 1, \ldots, n \). The Poisson bracket \( \{ H, F \}^* \) of the functions \( H \) and \( F \) is defined as

\[
\{ H, F \}^* := \sum_{j=1}^{n} \left( \frac{\partial H}{\partial x_j} \frac{\partial F}{\partial y_j} - \frac{\partial F}{\partial x_j} \frac{\partial H}{\partial y_j} \right).
\]

**Proposition 4** Between the Poisson bracket and the Nambu bracket the following two equalities hold for arbitrary functions \( H, f, G, f_1, \ldots, f_{2n} \):

\[ (x) \sum_{j=1}^{n} \{ x_1, \ldots, x_{j-1}, H, x_j+1, \ldots, x_{n+j-1}, f, x_{n+j+1}, \ldots, x_{2n} \} = \{ H, f \}^*, \]

\[ (xi) \sum_{j=1}^{2n} \{ H, f_j \}^* \{ f_1, \ldots, f_{j-1}, G, f_{j+1}, \ldots, f_{2n} \} = \{ H, G \}^* \{ f_1, \ldots, f_{2n} \}. \]

**Proof** The identity (x) is obtained by using the property (iv). Indeed, in view of (iv) we get

\[
\sum_{j=1}^{n} \{ x_1, \ldots, x_{j-1}, H, x_j+1, \ldots, x_{n+j-1}, f, x_{n+j+1}, \ldots, x_{2n} \}
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{2n} \frac{\partial H}{\partial x_k} \{ x_1, \ldots, x_{j-1}, x_k, x_j+1, \ldots, x_{n+j-1}, f, x_{n+j+1}, \ldots, x_{2n} \}
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{2n} \frac{\partial H}{\partial x_k} \frac{\partial f}{\partial x_m} \{ x_1, \ldots, x_{j-1}, x_k, x_j+1, \ldots, x_{n+j-1}, x_m, x_{n+j+1}, \ldots, x_{2n} \},
\]

in the previous equalities we consider that \( x_0 = x_1 \). Since

\[
\{ x_1, \ldots, x_{j-1}, x_k, x_j+1, \ldots, x_{n+j-1}, x_m, x_{n+j}, \ldots, x_{2n} \}
\]

\[
= \begin{cases} 
1 & \text{if } k = j, \ m = n + j, \\
-1 & \text{if } k = n + j, \ m = j, \\
0 & \text{otherwise,}
\end{cases}
\]

we obtain

\[
\sum_{j=1}^{n} \{ x_1, \ldots, x_{j-1}, H, x_j+1, \ldots, x_n, x_{n+1}, \ldots, x_{n+j-1}, f, x_{n+j+1}, \ldots, x_{2n} \}
\]

\[
= \sum_{j=1}^{n} \left( \frac{\partial H}{\partial x_j} \frac{\partial f}{\partial y_j} - \frac{\partial f}{\partial x_j} \frac{\partial H}{\partial y_j} \right) = \{ H, f \}^*.
\]

Thus the property (x) is proved.

Now we prove (xi), i.e.

\[
\sum_{k=1}^{2n} \{ H, f_k \}^* \{ f_1, \ldots, f_{k-1}, G, f_{k+1}, \ldots, f_{2n} \}
\]

\[
= \sum_{k=1}^{2n} \left( \sum_{j=1}^{n} \{ x_1, \ldots, x_{j-1}, H, x_j+1, \ldots, x_n, x_{n+1}, \ldots, x_{n+j-1}, f_k, x_{n+j+1}, \ldots, x_{2n} \} \cdot \{ f_1, \ldots, f_{k-1}, G, f_{k+1}, \ldots, f_{2n} \} \right).
\]
here we have used (x). Now we have

$$
\sum_{j=1}^{n} \left( \sum_{k=1}^{2n} \left[ x_1, \ldots, x_{j-1}, H, x_{j+1}, \ldots, x_{n+j-1}, f_k, x_{n+j+1}, \ldots, x_{2n} \right] \right).
$$

$$
\{ f_1, \ldots, f_{k-1}, G, f_{k+1}, \ldots, f_{2n} \}
$$

$$
= \sum_{j=1}^{n} \{ x_1, \ldots, x_{j-1}, H, x_{j+1}, \ldots, x_{n+j-1}, G, x_{n+j+1}, \ldots, x_{2n} \} \{ f_1, \ldots, f_{2n} \}
$$

$$
= \{ H, G \}^* \{ f_1, \ldots, f_{2n} \}.
$$

Here in the first equality we have used (viii), and in the second equality we have used (x). □

3 Ordinary Differential Equations in $\mathbb{R}^N$ with $M \leq N$ Partial Integrals

Let $D$ be an open subset of $\mathbb{R}^N$. By definition an autonomous differential system is a system of the form

$$
\dot{x} = X(x), \quad x \in D,
$$

(3)

where the dependent variables $x = (x_1, \ldots, x_N)$ are real, the independent variable (the time $t$) is real and the $C^1$ function $X(x) = (X_1(x), \ldots, X_N(x))$ is defined in the open set $D$.

The $C^1$ function $g : D \to \mathbb{R}$ and the set $\{ x \in D : g = g(x) = 0 \}$ are called partial integral and invariant hypersurface of the vector field $X$ respectively, if $X(g)|_{g=0} = 0$.

In this section we construct the most general autonomous differential system in $D \subset \mathbb{R}^N$ having the set of partial integrals $g_j$ for $j = 1, 2, \ldots, M$, with $M \leq N$.

Our first result characterizes the differential systems (3) having a given set of $M$ partial integrals with $M \leq N$.

**Theorem 5** Let $g_j = g_j(x)$ for $j = 1, 2, \ldots, M$ with $M \leq N$ be a given set of independent functions defined in an open set $D \subset \mathbb{R}^N$. Then any differential system defined in $D$ which admits the set of partial integrals $g_j$ for $j = 1, 2, \ldots, M$ can be written as

$$
\dot{x}_j = \sum_{k=1}^{M} \Phi_k \left\{ g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N \right\} \frac{\{ g_1, g_2, \ldots, g_N \}}{\{ g_1, g_2, \ldots, g_N \}}
$$

$$
+ \sum_{k=M+1}^{N} \lambda_k \left\{ g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N \right\} \frac{\{ g_1, g_2, \ldots, g_N \}}{\{ g_1, g_2, \ldots, g_N \}} = X(x_j)
$$

(4)

where $g_{M+j} = g_{M+j}(x)$ for $j = 1, \ldots, N - M$, are arbitrary functions defined in $D$ which we choose in such a way that the Jacobian $|S| = \{ g_1, \ldots, g_N \} \neq 0$ in the set $D$ and the functions $\Phi_j = \Phi_j(x)$, for $j = 1, 2, \ldots, M$ and $\lambda_{M+k} = \lambda_{M+k}(x)$ for $k = 1, 2, \ldots, N - M$ are arbitrary functions such that

$$
\Phi_j|_{g_j=0} = 0, \quad \text{for} \quad j = 1, \ldots, M,
$$

if $|S|$ is nonzero in $D$ and

$$
\Phi_j = \{ g_1, \ldots, g_N \} \mu_j \quad \text{for} \quad j = 1, \ldots, M,
$$

$$
\lambda_{M+k} = \{ g_1, \ldots, g_N \} v_{M+k} \quad \text{for} \quad k = 1, \ldots, N - M,
$$

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if $|S|$ is vanishes in some zero Lebesgue measure set $\Delta \subset D$. Then from (4) we get the following differential system

$$
\dot{x}_j = \sum_{k=1}^{M} \mu_k \{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\} \\
+ \sum_{k=M+1}^{N} \nu_k \{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}
$$

(5)

where $\mu_1, \ldots, \mu_M$ and $\nu_1, \ldots, \nu_M$ are arbitrary functions defined in $D$ such that

$$
\mu_j |_{g_j=0} = 0 \text{ for } j = 1, \ldots, M \leq N.
$$

Proof of Theorem 5. We consider the vector field

$$
X = -\frac{1}{\{g_1, \ldots, g_N\}} \begin{vmatrix}
        dg_1(\partial_1) & \ldots & dg_1(\partial_N) & \Phi_1 \\
        dg_2(\partial_1) & \ldots & dg_2(\partial_N) & \Phi_2 \\
        \vdots & \ddots & \vdots & \vdots \\
        dg_M(\partial_1) & \ldots & dg_M(\partial_N) & \Phi_M \\
        dg_{M+1}(\partial_1) & \ldots & dg_{M+1}(\partial_N) & \lambda_{M+1} \\
        \vdots & \ddots & \vdots & \vdots \\
        dg_N(\partial_1) & \ldots & dg_N(\partial_N) & \lambda_N \\
        \partial_1 & \ldots & \partial_N & 0 
\end{vmatrix}
$$

$$
= \sum_{k,j=1}^{N} \frac{S_{jk} P_j}{|S|} \partial_k = \{S^{-1} P, \partial_0\},
$$

(6)

where $|S| \neq 0$, $S_{jk}$ for $k, j = 1, \ldots, N$ is the determinant of the adjoint of the matrix $S$ after removing the row $j$ and the column $k$, $S^{-1}$ is the inverse matrix of $S$, and $P = (P_1, \ldots, P_N)^T = (\Phi_1, \ldots, \Phi_M, \lambda_{M+1}, \ldots, \lambda_N)^T$.

From (6) by developing the determinant with respect to the last column and denote by $\{g_1, \ldots, g_{k-1}, *, g_{k+1}, \ldots, g_N\}$ the following vector field

$$
\begin{vmatrix}
        \partial_1 g_1 & \ldots & \partial_N g_1 \\
        \vdots & \ddots & \vdots \\
        \partial_1 g_{k-1} & \ldots & \partial_N g_{k-1} \\
        \partial_1 * & \ldots & \partial_N * \\
        \partial_1 g_{k+1} & \ldots & \partial_N g_{k+1} \\
        \vdots & \ddots & \vdots \\
        \partial_1 g_N & \ldots & \partial_N g_N 
\end{vmatrix}
$$
we obtain the following expression for the vector field $X$

$$X(\ast) = \Phi_1 \left\{ g_1, g_2, \ldots, g_N \right\} + \ldots$$

$$+ \Phi_M \left\{ g_1, \ldots, g_{M-1}, \ast, g_{M+1}, \ldots, g_N \right\}$$

$$+ \lambda_{M+1} \left\{ g_1, \ldots, g_M, g_{M+1}, \ldots, g_N \right\} + \ldots$$

$$+ \lambda_N \left\{ g_1, \ldots, g_{N-1}, \ast \right\} \left\{ g_1, \ldots, g_{N-1}, g_N \right\}.$$  \hfill (7)

Then it is easy to obtain the relationship

$$X(g_j) = \Phi_j \left\{ g_j, g_2, \ldots, g_N \right\} + \ldots + \Phi_M \left\{ g_1, \ldots, g_{M-1}, g_j, g_{M+1}, \ldots, g_N \right\}$$

$$+ \lambda_{M+1} \left\{ g_1, \ldots, g_M, g_j, g_{M+2}, \ldots, g_N \right\} + \ldots + \lambda_N \left\{ g_1, \ldots, g_{N-1}, g_j \right\}$$

$$\left\{ g_1, \ldots, g_{N-1}, g_N \right\}$$

$$= \begin{cases} \Phi_j & \text{for } 1 \leq j \leq M, \\ \lambda_j & \text{for } M + 1 \leq j \leq N. \end{cases}$$

Thus

$$X(g_j) = \Phi_j, \quad X(g_{M+k}) = \lambda_{M+k},$$

for $j = 1, 2, \ldots, M$, and $k = 1, \ldots, N - M$. Since $\Phi_j|_{g_j=0} = 0$ we obtain that the $g_j = 0$ for $j = 1, 2, \ldots, M$ are invariant hypersurfaces of the vector field $X$.

The vector field $X$ given in (6) already was used in [26, 28].

Now we shall prove that system (4) is the most general differential system which admits the given set of independent partial integrals. Indeed let

$$\dot{x} = \tilde{X}(x) = \left( \tilde{X}_1(x), \ldots, \tilde{X}_N(x) \right)$$

be another differential system having $g_1, g_2, \ldots, g_M$ as partial integrals, that is $\tilde{X}(g_j)|_{g_j=0} = 0$ for $j = 1, 2, \ldots, M$. Then taking

$$\Phi_j = \tilde{X}(g_j) = \sum_{l=1}^{N} \tilde{X}_l \partial_l g_j = \sum_{l=1}^{N} \tilde{X}_l \{ x_1, \ldots, x_{l-1}, g_j, x_{l+1}, \ldots, x_N \},$$

for $j = 1, 2, \ldots, M$, and

$$\lambda_{M+k} = \tilde{X}(g_{M+k}) = \sum_{l=1}^{N} \tilde{X}_l \partial_l g_{M+k} = \sum_{l=1}^{N} \tilde{X}_l \{ x_1, \ldots, x_{l-1}, g_{M+k}, x_{l+1}, \ldots, x_N \}.$$
for \( k = 1, \ldots, N - M \) (here we use the identity (v)) and substituting \( \Phi_j \) and \( \lambda_{M+k} \) into formula (7) we get for an arbitrary function \( F \)

\[
X(F) = \sum_{l=1}^{N} \Phi_j \left\{ g_l, \ldots, g_{j-1}, F, g_{j+1}, \ldots, g_M, \ldots, g_N \right\} / \left\{ g_1, g_2, \ldots, g_N \right\} \\
+ \sum_{j=M+1}^{N} \lambda_{M+j} \left\{ g_l, \ldots, g_M, g_{M+1}, \ldots, g_{j-1}, F, g_{j+1}, \ldots, g_N \right\} / \left\{ g_1, g_2, \ldots, g_N \right\} \\
= \sum_{j=1}^{N} \sum_{l=1}^{N} \tilde{X}_l \{ x_1, \ldots, x_{l-1}, g_j, x_{l+1}, \ldots, x_N \} \left\{ g_l, \ldots, g_{j-1}, F, g_{j+1}, \ldots, g_N \right\} / \left\{ g_1, g_2, \ldots, g_N \right\} \\
= \sum_{l=1}^{N} \tilde{X}_l \{ x_1, \ldots, x_{l-1}, F, x_{l+1}, \ldots, x_N \} = \tilde{X}(F),
\]

Here we have used the identity (iv) and the identity (ix). Hence, in view of the arbitrariness of \( F \) the theorem has been proved. \( \square \)

**Corollary 6** Under the assumptions of Theorem 5 if \( M = N \), then system (4) and (5) takes the form

\[
\dot{x}_j = \Phi_1 \left\{ x_j, g_1, g_2, \ldots, g_{N-1}, g_N \right\} / \left\{ g_1, g_2, \ldots, g_{N-1}, g_N \right\} + \ldots + \Phi_N \left\{ g_1, g_2, \ldots, g_N \right\} / \left\{ g_1, g_2, \ldots, g_N \right\},
\]

(8)

for \( j = 1, 2, \ldots, N \), where \( \Phi_1, \ldots, \Phi_N \) are arbitrary functions such that if

\( \{ g_1, \ldots, g_N \} \neq 0 \) in \( D \), then \( \Phi_j|_{g_j=0} = 0 \), for \( j = 1, \ldots, N \),

and if we choose

\( \Phi_j = \{ g_1, \ldots, g_N \} \mu_j \), being \( \mu_j|_{g_j=0} = 0 \) for \( j = 1, \ldots, N \),

from (8) we get the following differential system

\[
\dot{x}_j = \mu_1 \{ x_j, g_1, g_2, \ldots, g_{N-1}, g_N \} + \ldots + \mu_N \{ g_1, g_2, \ldots, g_{N-1}, x_j \},
\]

(9)

for \( j = 1, 2, \ldots, N \), respectively.

In particular if \( M = N = 2 \) then differential system (8) and (9) takes the form

\[
\dot{x}_j = \Phi_1 \left\{ x_j, g_2 \right\} / \left\{ g_1, g_2 \right\} + \Phi_2 \left\{ g_1, x_j \right\} / \left\{ g_1, g_2 \right\},
\]

(10)

for \( j = 1, 2 \), where \( \Phi_1, \Phi_2 \) are arbitrary functions such that if

\( \{ g_1, g_2 \} \neq 0 \) in \( D \), then \( \Phi_j|_{g_j=0} = 0 \), for \( j = 1, 2 \),

and if

\( \Phi_j = \{ g_1, g_2 \} \mu_j \), being \( \{ g_1, g_2 \} \mu_j|_{g_j=0} = 0 \) for \( j = 1, 2 \),

from (10) we get the following differential system

\[
\dot{x}_j = \mu_1 \{ x_j, g_2 \} + \mu_2 \{ g_1, x_j \},
\]
for \( j = 1, 2 \), respectively.

4 Differential Equations with Partial and First Integrals

In this section we construct the most general autonomous differential system in the open set \( D \subset \mathbb{R}^N \) having the set of partial integrals \( g_j \) for \( j = 1, 2, \ldots, M_1 \), and the given set of first integrals \( f_k \) for \( k = 1, 2, \ldots, M_2 \), with \( M = M_1 + M_2 \leq N \).

The non-locally constant function \( H = H(x) \) defined in an open subset \( D_1 \) of \( D \) such that its closure coincides with \( D \) is called a first integral if it is constant on the solutions of system (3) contained in \( D_1 \), i.e. \( X(H)|_{D_1} = 0 \).

Our second main result characterizes the differential systems (3) having a given set of \( M_1 \) partial integrals and \( M_2 \) first integrals with \( 1 \leq M_2 < N \) and \( M_1 + M_2 \leq N \).

**Theorem 7** Let \( g_l = g_l(x) \) for \( l = 1, 2, \ldots, M_1 \) and \( f_k = f_k(x) \) for \( k = 1, 2, \ldots, M_2 < N \) with \( M_1 + M_2 = M \leq N \) be independent functions defined in the open set \( D \subset \mathbb{R}^N \). Then the most general differential systems in \( D \) which admit the partial integrals \( g_l \) for \( j = 1, \ldots, M_1 \) and the first integrals \( f_k \) for \( k = 1, \ldots, M_2 \) are

\[
\dot{x}_j = \sum_{k=1}^{M_1} \Phi_k \left\{ \frac{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1} \ldots g_N}{g_1, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, \ldots, g_N} \right\}
+ \sum_{k=M_1+1}^{N} \lambda_k \left\{ \frac{g_1, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1} \ldots g_N}{g_1, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, \ldots, g_N} \right\}
\tag{11}
\]

for \( j = 1, 2, \ldots, N \), where \( g_{M+j} \) for \( j = 1, \ldots, N - M \) are arbitrary functions satisfying that \( |S| = \{ g_1, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, \ldots, g_N \} \neq 0 \) in the set \( D \); and the functions \( \Phi_j = \Phi_j(x) \), for \( j = 1, 2, \ldots, M_1 \) and \( \lambda_{M+k} = \lambda_k(x) \) for \( k = M_1 + 1, 2, \ldots, N \) are arbitrary functions such that

\[
\Phi_j|_{g_j=0} = 0, \quad \text{for} \quad j = 1, \ldots, M,
\tag{12}
\]

if \( |S| \) is nonzero in \( D \), and

\[
\Phi_j = |S|\mu_j \quad \text{being} \quad |S|\mu_j|_{g_j=0} = 0 \quad \text{for} \quad j = 1, \ldots, M_1,
\]

\[
\lambda_k = |S|\nu_k \quad \text{for} \quad k = 1, \ldots, M_2,
\]

if \( |S| \) is vanishes in some zero Lebesgue measure set \( \Delta \subset D \).

**Proof** Let \( X \) be the vector field

\[
X = \frac{-1}{|S|} \begin{vmatrix}
g_1(\partial_1) & \ldots & g_1(\partial_N) & \Phi_1 \\
\vdots & & \vdots & \vdots \\
g_{M_1}(\partial_1) & \ldots & g_{M_1}(\partial_N) & \Phi_{M_1} \\
df_1(\partial_1) & \ldots & df_1(\partial_N) & 0 \\
\vdots & & \vdots & \vdots \\
df_{M_2}(\partial_1) & \ldots & df_{M_2}(\partial_N) & 0 \\
g_{M+1}(\partial_1) & \ldots & g_{M+1}(\partial_N) & \lambda_{M+1} \\
\vdots & & \vdots & \vdots \\
g_N(\partial_1) & \ldots & g_N(\partial_N) & \lambda_N \\
\partial_1 & \ldots & \partial_N & 0
\end{vmatrix} = \langle S^{-1}\mathbf{P}, \partial_x \rangle,
\]
where $\mathbf{P} = (P_1, \ldots, P_N)^T = (\Phi_1, \ldots, \Phi_M, 0, \ldots, 0, \lambda_{M+1}, \ldots, \lambda_N)^T$, which is the vector field associated to differential system (11). Clearly this vector field is well defined in view of the assumptions.

From $\mathbf{X}(g_j) = \Phi_j$, $\Phi_j|_{g_j=0} = 0$, for $j = 1, \ldots, M_1$ we deduce that $g_j$ are partial integrals of vector field $\mathbf{X}$, and $\mathbf{X}(f_k) = 0$ for $k = 1, \ldots, M_2$ we obtain that $f_k$ are first integrals of vector field $\mathbf{X}$.

Now we prove that system (11) is the most general differential system admitting the partial integrals $g_j$ and the first integrals $f_k$. Indeed let $\tilde{\mathbf{x}} = \tilde{\mathbf{X}}(x)$ be another differential system which admits $g_j$ for $j = 1, \ldots, M_1$ partial integrals and $f_k$ for $k = 1, \ldots, M_2$ first integrals with $M_1 + M_2 \leq N$, i.e. $\tilde{\mathbf{X}}(g_j)|_{g_j=0} = 0$ for $j = 1, \ldots, M_1$ and $\tilde{\mathbf{X}}(f_k) = 0$ for $k = 1, \ldots, M_2$. Then taking $\Phi_j = \tilde{\Phi}(g_j)$ for $j = 1, \ldots, M_1$ and $\lambda_{M+k} = \tilde{\Phi}(g_k)$ for $k = M_1 + 1, \ldots, N$ and doing an analogous proof to the Proof of Theorem 5 we deduce that the vector field $\tilde{\mathbf{X}}$ is a particular case of the vector field $\mathbf{X}$. Thus the theorem is proved. \qed

The next two results follow easily from the Proof of Theorem 7.

**Corollary 8** Under the assumptions of Theorem 7 but without partial integrals, i.e. if $M_1 = 0$, and $M_2 = M < N$, then the most general differential system in $D$ which admit the first integrals $f_k$ for $k = 1, \ldots, M_2$ is

$$
\dot{x}_j = \sum_{k=M+1}^{N} \lambda_k \frac{\{f_1, \ldots, f_M, g_{M+1}, \ldots, g_{k-1}, \dot{x}_j, g_{k+1}, \ldots, g_N\}}{\{f_1, \ldots, f_M, g_{M+1}, g_{M+2}, \ldots, g_N\}}, \quad (13)
$$

for $j = 1, 2, \ldots, N$.

**Corollary 9** Under the assumptions of Theorem 7 and if $M_1 + M_2 = M = N$, then the differential system (11) takes the form

$$
\dot{x}_j = \sum_{k=1}^{M_1} \phi_k \frac{\{g_1, \ldots, g_{k-1}, \dot{x}_j, g_{k+1}, \ldots, g_M, f_1, \ldots, f_{M_2}\}}{\{g_1, g_2, \ldots, g_M, f_1, \ldots, f_{M_2}\}} \quad (14)
$$

for $j = 1, 2, \ldots, N$.

**Corollary 10** Under the assumptions of Theorem 7 the following statements hold.

(a) If $M_2 = N - 1$ and $M_1 = 1$, then the differential system (14) takes the form

$$
\dot{x}_j = \Phi_N \frac{\{f_1, \ldots, f_{N-1}, \dot{x}_j\}}{\{f_1, \ldots, f_{N-1}, g_N\}}, \quad \text{with} \quad \Phi_N|_{g_N=0} = 0.
$$

(b) If $M_2 = N - 1$ and $M_1 = 0$, then the differential system (13) takes the form

$$
\dot{x}_j = \lambda_N \frac{\{f_1, \ldots, f_{N-1}, \dot{x}_j\}}{\{f_1, \ldots, f_{N-1}, g_N\}}
$$

where $\lambda_N$ and $g_N$ are arbitrary functions such that $\{f_1, \ldots, f_{N-1}, g_N\} \neq 0$.

Consequently the vector field $\mathbf{X}$ in both cases admits the representation

$$
\mathbf{X}(\ast) = \mu \{f_1, \ldots, f_{N-1}, \ast\}, \quad (15)
$$

where $\mu$ is an arbitrary function.
5 Differential Equations and Integrability

In what follows we study the integrability of systems (4).

We say that system (3) is integrable if it admits \( N - 1 \) independent first integrals.

**Theorem 11** Under the assumptions of Theorem 5 differential system (4) is integrable if and only if \( \Phi_l = \mu\{F_1, \ldots, F_{N-1}, g_l\}, \) \( \lambda_k = \mu\{F_1, \ldots, F_{N-1}, g_k\} \) for \( l = 1, \ldots, M \) and \( k = M + 1, \ldots, N \), where \( \mu, F_1, \ldots, F_{N-1} \) are convenient functions such that \( F_1, \ldots, F_{N-1} \) are independent in \( D \) and

\[
\mu\{F_1, \ldots, F_{N-1}, g_l\}|_{g_l=0} = 0.
\]

**Proof** Assume that the vector field \( X \) associated to differential system (4) is integrable, i.e. admit \( N - 1 \) independent first integrals \( F_1, \ldots, F_{N-1} \), without loss of generality we suppose that

\[
\begin{vmatrix}
\partial_1 F_1 & \cdots & \partial_{N-1} F_1 \\
\vdots & \ddots & \vdots \\
\partial_1 F_{N-1} & \cdots & \partial_{N-1} F_{N-1}
\end{vmatrix} = \{F_1, \ldots, F_{N-1}, x_N\} \neq 0.
\]

Thus from the equations \( X(F_k) = \sum_{j=1}^{N} \frac{\partial F_k}{\partial x_j} X_j(x) = 0 \) for \( k = 1, \ldots, N - 1 \) or, equivalent

\[
\sum_{j=1}^{N-1} \frac{\partial F_k}{\partial x_j} X_j(x) = -X_N(x) \frac{\partial F_k}{\partial x_N},
\]

we obtain

\[
X_j(x) = X_N(x) \frac{\{F_1, \ldots, F_{j-1}, x_j, F_{j+1}, \ldots, F_{N-1}\}}{\{F_1, \ldots, F_{N-1}, x_N\}}.
\]

Taking \( X_N(x) = \mu\{F_1, \ldots, F_{N-1}, x_N\} \), we obtain the representation

\[
X(\ast) = \mu\{F_1, \ldots, F_{N-1}, \ast\},
\]

where \( \mu \) is an arbitrary function. Thus

\[
X(g_l) = \Phi_l = \mu\{F_1, \ldots, F_{N-1}, g_l\}, \quad X(g_k) = \lambda_k = \mu\{F_1, \ldots, F_{N-1}, g_k\},
\]

for \( l = 1, \ldots, M \) and \( k = M + 1, \ldots, N \). So the “if” part of the theorem follows. Now we shall prove the “only if” part.

We suppose that \( \Phi_l = \mu\{F_1, \ldots, F_{N-1}, g_l\} \), and \( \lambda_k = \mu\{F_1, \ldots, F_{N-1}, g_k\} \). Thus the vector field associated to differential system (4) takes the form

\[
X(x_j) = \sum_{n=1}^{M} \Phi_n \frac{\{g_1, \ldots, g_{n-1}, x_j, g_{n+1}, \ldots, g_M, \ldots, g_N\}}{\{g_1, \ldots, g_N\}}
\]

\[
+ \sum_{n=M+1}^{N} \lambda_n \frac{\{g_1, \ldots, g_M, g_{M+1}, \ldots, g_{n-1}, x_j, g_{n+1}, \ldots, g_N\}}{\{g_1, \ldots, g_N\}}
\]

\[
= \mu \sum_{n=1}^{N} \{F_1, \ldots, F_{N-1}, g_n\} \frac{\{g_1, \ldots, g_{n-1}, x_j, g_{n+1}, \ldots, g_N\}}{\{g_1, \ldots, g_N\}}.
\]
In view of the identity (ix) we have that

\[ X(x_j) = \mu\{F_1, \ldots, F_{N-1}, x_j\} \{g_1, \ldots, g_N\} = \mu\{F_1, \ldots, F_{N-1}, x_j\}. \]

Thus functions \(F_1, \ldots, F_{N-1}\) are first integrals of \(X\). Hence the vector field is integrable. \(\square\)

### 6 Ordinary Differential Equations in \(\mathbb{R}^N\) with \(M > N\) Partial Integrals

In this section we determine the differential systems (3) having a given set of \(M\) partial integrals with \(M > N\).

**Theorem 12** Let \(g_j = g_j(x)\) for \(j = 1, 2, \ldots, M\) with \(M > N\), be a set of functions defined in the open set \(D \subset \mathbb{R}^N\) such that at least \(N\) of them are independent at points of the set \(D\), i.e. without loss of generality we can assume that \(\{g_1, \ldots, g_N\} \neq 0\) in the set \(D\). Then the most general differential systems in \(D\) which admit the partial integrals \(g_j\) for \(j = 1, 2, \ldots, M\) are

\[
\dot{x}_j = \sum_{j_1, \ldots, j_{N-1}=1}^{M+N} G_{j_1, \ldots, j_{N-1}}(g_{j_1}, \ldots, g_{j_{N-1}}, x_j),
\]

for \(j = 1, 2, \ldots, N\), where \(G_{j_1, \ldots, j_{N-1}} = G_{j_1, \ldots, j_{N-1}}(x)\) are arbitrary functions satisfying

\[
\dot{g}_j|_{g_j=0} = \left( \sum_{j_1, \ldots, j_{N-1}=1}^{M+N} G_{j_1, \ldots, j_{N-1}}(g_{j_1}, \ldots, g_{j_{N-1}}, g_j) \right)|_{g_j=0} = 0,
\]

for \(j = 1, 2, \ldots, M\), and \(g_{M+j} = x_j\) for \(j = 1, 2, \ldots, N\).

**Proof of Theorem** First of all we determine the differential systems having the \(N\) independent partial integrals \(g_j = g_j(x)\) for \(j = 1, 2, \ldots, N\). Thus we obtain system (8). Clearly this differential system admits additional partial integrals \(g_j\) for \(j = N+1, \ldots, M\) if and only if \(X(g_v) = \Phi_v\), \(\Phi_v|_{g_v=0} = 0\), for \(v = N+1, \ldots, M\), or equivalently,

\[
\Phi_1\{g_v, g_2, \ldots, g_N\} + \ldots + \Phi_N\{g_1, \ldots, g_{N-1}, g_v\} - \Phi_v\{g_1, \ldots, g_{N-1}, g_N\} = 0.
\]

Now we prove that

\[
\Phi_v = \sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} G_{\alpha_1, \ldots, \alpha_{N-1}}(g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_v)
\]

is a solution of (18) for \(v = 1, 2, \ldots, M \geq N\), where \(G_{\alpha_1, \ldots, \alpha_{N-1}} = G_{\alpha_1, \ldots, \alpha_{N-1}}(x)\) are arbitrary functions satisfying (17).

Indeed, in view of (18) and (19) we obtain

\[
\sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} G_{\alpha_1, \ldots, \alpha_{N-1}}(\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_v\} + \ldots + \{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_N\})
\]

\[= \sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} \Omega(g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_1, \ldots, g_\alpha, g_v) = 0,\]

\(\square\)
which is identically zero by the identity (viii).

Inserting (19) into (8) and from the identity (v) we obtain from the equation

\[
\dot{x}_v = \Phi_1 \left\{ x_v, g_2 \cdots g_N \right\} + \cdots + \Phi_N \left\{ g_1, \cdots, g_{N-1}, x_v \right\}
\]

\[
= \sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} \frac{G_{\alpha_1, \ldots, \alpha_{N-1}}}{\{g_1, \ldots, g_N\}} \sum_{n=1}^N \{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_n\}\{g_1, \ldots, g_{N-1}, x_v, g_{N+1}, \ldots, g_N\}
\]

\[
= \sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} G_{\alpha_1, \ldots, \alpha_{N-1}} \{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, x_v\}
\]

for \( j = 1, 2, \ldots, N \). Now we prove that this differential system which coincides with (16) is the most general one. Indeed using that \( g_{M+j} = x_{j+1}, x_{N+1} = x_1 \), for \( j = 1, \ldots, N \), system (16) admits the representation

\[
\begin{align*}
\dot{x}_1 &= \sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} \frac{G_{\alpha_1, \ldots, \alpha_{N-1}}}{\{g_1, \ldots, g_{N-1}\} \neq (M+2, \ldots, N)} \sum_{j=1} G_{M+2, M+3, \ldots, M+N} \{x_2, \ldots, x_N, x_1\}, \\
\vdots & \quad \vdots \\
\dot{x}_N &= \sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} \frac{G_{\alpha_1, \ldots, \alpha_{N-1}}}{\{g_1, \ldots, g_{N-1}\} \neq (M+1, \ldots, N-1)} G_{M+1, M+2, \ldots, M+N-1} \{x_1, \ldots, x_{N-1}, x_N\}.
\end{align*}
\]

Note that \( \{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N, x_j\} \in \{-1, 1\} \). Therefore if \( \dot{x}_j = \tilde{X}_j \) for \( j = 1, \ldots, N \) is another differential system having the given set of partial integrals, then by choosing conveniently functions \( G_{M+2, M+3, \ldots, M+N}, \ldots, G_{M+1, M+2, \ldots, M+N-1} \) we deduce that the vector field (20) contains the vector field \( \tilde{X} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_N) \). So the Proof of Theorem 12 follows.

\[ \square \]

The following result is proved in a similar way to the Proof of Theorem 11.

**Theorem 13** Under the assumptions of Theorem 12 the differential system (16) is integrable if and only if

\[
\Phi_l = \sum_{\alpha_1, \ldots, \alpha_{N-1}=1}^{M+N} G_{\alpha_1, \ldots, \alpha_{N-1}} \{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_l\} = \mu \{F_1, \ldots, F_{N-1}, g_l\},
\]

for \( l = 1, \ldots, M > N \), where \( \mu, F_1, \ldots, F_{N-1} \) are convenient functions such that \( F_1, \ldots, F_{N-1} \) are independent in \( D \) and \( \mu \{F_1, \ldots, F_{N-1}, g_l\}\}_{g_l=0} = 0. \]
Corollary 14 Under the assumptions of Theorem 12 for \( N = 2 \) system (16) takes the form
\[
\dot{x} = \sum_{j=1}^{M} G_j \{g_j, x\} + G_{M+1}\{y, x\} = \sum_{j=1}^{M} G_j \{g_j, x\} - G_{M+1},
\]
\[
\dot{y} = \sum_{j=1}^{M} G_j \{g_j, y\} + G_{M+2}\{x, y\} = \sum_{j=1}^{M} G_j \{g_j, y\} + G_{M+2},
\]
where \( G_j = G_j(x, y) \) for \( j = 1, 2, \ldots M + 2 \) are arbitrary functions satisfying (17).
Moreover (17) becomes
\[
\left( \sum_{j=1}^{M} G_j \{g_j, g_k\} + G_{M+1}\{y, g_k\} + G_{M+2}\{x, g_k\} \right)_{g_k=0} = 0,
\]
for \( k = 1, 2, \ldots M \).

Proof It follows easily from Theorem 12.

Remark 15 We note that conditions (22) hold identically if
\[
G_j = \lambda_j \prod_{m=1}^{M} g_m
\]
where \( \lambda_j = \lambda_j(x, y) \) for \( j = 1, \ldots, M + 2 \) are arbitrary functions.
Inserting (23) into (21) we obtain the following differential system
\[
\dot{x} = -\lambda_{M+2} \prod_{m=1}^{M} g_m + \lambda \sum_{j=1}^{M} \{g_j, x\} \prod_{m=1}^{M} g_m,
\]
\[
\dot{y} = \lambda_{M+1} \prod_{m=1}^{M} g_m + \lambda \sum_{j=1}^{M} \{g_j, y\} \prod_{m=1}^{M} g_m.
\]
In particular if in (24) we assume that \( \lambda_j = \lambda \) for \( j = 1, \ldots, M \), then we obtain the following differential system
\[
\dot{x} = -\lambda_{M+2} \prod_{m=1}^{M} g_m + \lambda \sum_{j=1}^{M} \{g_j, x\} \prod_{m=1}^{M} g_m,
\]
\[
\dot{y} = \lambda_{M+1} \prod_{m=1}^{M} g_m + \lambda \sum_{j=1}^{M} \{g_j, y\} \prod_{m=1}^{M} g_m.
\]
By introducing the function \( g = \prod_{m=1}^{M} g_m \), we get the differential system
\[
\dot{x} = -g\lambda_{M+1} - \lambda \frac{\partial g}{\partial y}, \quad \dot{y} = g\lambda_{M+2} + \lambda \frac{\partial g}{\partial x}.
\]
Indeed by considering that
\[
\frac{\partial g}{\partial y} = \sum_{j=1}^{M} \{g_j, x\} \prod_{m=1 \atop m \neq j}^{M} g_m, \quad \frac{\partial g}{\partial x} = \sum_{j=1}^{M} \{g_j, y\} \prod_{m=1 \atop m \neq j}^{M} g_m,
\]
we easily obtain from (24) differential system (25).

7 Inverse Problem for Constrained Lagrangian Systems

The aim of this section is to provide a solution of the inverse problem of the constrained Lagrangian Mechanics which can be defined as follows: Determine for a given natural mechanical system with \(N\) degrees of freedom the most general field of forces depending only on the positions and satisfying a given set of constraints with are linear in the velocities.

The statement of the inverse problem for constrained Lagrangian systems is new.

As we observe from Sect. 1 (see for instance Theorem 5) the solutions of the inverse problem in ordinary differential equations have a very high arbitrariness due to the undetermined functions which appear in them. To obtain more exact solutions we need additional conditions for reducing this arbitrariness. In this section we will obtain these additional conditions for getting the equations of motion for the constrained Lagrangian mechanics.

One of the main objectives in this inverse problem is to study the behavior of the nonholonomic systems with linear constraints with respect to the velocity in a way different of the classical approach deduced from the d’Alembert-Lagrange principle or from the vakonomic approach, see for instance [24]. We explain this in more detail in the Remark 20.

We shall introduce the notations and definitions that we need for presenting our applications of Theorem 5.

We shall denote by \(Q\) an \(N\)-dimensional smooth manifold and by \(TQ\) the tangent bundle of \(Q\) with local coordinates \(x = (x_1, \ldots, x_N)\), and \((x, \dot{x}) = (x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N)\) respectively (see for instance [13]).

The following definitions can be found in [2].

A Lagrangian system is a pair \((Q, \tilde{L})\) consisting of a smooth manifold \(Q\), a function \(\tilde{L} : TQ \rightarrow \mathbb{R}\). The point \(x \in Q\) denotes the position of the system and we call each tangent vector \(\dot{x} \in T_xQ\) the velocity of the system at the point \(x\). A pair \((x, \dot{x})\) is called a state of the system. In Lagrangian mechanics it is usual to call \(Q\), the configuration space, the tangent bundle \(TQ\) is called the phase space, \(\tilde{L}\) is the Lagrange function or Lagrangian, and the dimension \(N\) of \(Q\) is the number of degrees of freedom.

The equations
\[
h_j = h_j(x, \dot{x}) = 0, \quad \text{for} \quad j = 1, \ldots, M \leq N,
\]
with rank \(\left( \frac{\partial (h_1, \ldots, h_M)}{\partial (\dot{x}_1, \ldots, \dot{x}_N)} \right) = M\), in all the points of \(Q\), except in a zero Lebesgue measure set, define \(M\) independent constraints for the Lagrangian systems \((Q, \tilde{L})\), i.e. we want that the orbits \((x(t), \dot{x}(t))\) of the mechanical system satisfy (26).

Let \(M^*\) be the submanifold of \(TQ\) defined by the Eq. (26), i.e.
\[
M^* = \{(x, \dot{x}) \in TQ : h_j(x, \dot{x}) = 0, \quad \text{for} \quad j = 1, \ldots, M \leq N\}
\]
A constrained Lagrangian system is a triplet \((Q, \tilde{L}, M^*)\).

We call the inverse problem for the constrained Lagrangian system the problem of determining for a given constrained Lagrangian system \((Q, \tilde{L}, M^*)\), the field of force
\( F = F(x) = (F_1(x), \ldots, F_N(x)) \) in such a way that the given submanifold \( M^* \) is invariant by the flow of the second order differential equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = F_j(x) \quad \text{for} \quad j = 1, \ldots, N.
\]

We shall study the case when the constraints are linear in the velocities in \( M^* \), i.e.

\[
h_j(x, \dot{x}) = \sum_{k=1}^{N} a_{jk}(x) \dot{x}_k + \alpha_j(x) = 0, \quad \text{for} \quad j = 1, \ldots, M.
\]

Our first main result provides the equations of motion of a constrained mechanical system with Lagrangian function

\[
\bar{L} = T = \frac{1}{2} \sum_{n,j=1}^{N} G_{jn}(x) \dot{x}_j \dot{x}_n := \frac{1}{2} \langle \dot{x}, \dot{x} \rangle = \frac{1}{2} ||\dot{x}||^2.
\]  

(27)

where \( T \) is a Riemannian metric on \( Q \) (the kinetic energy of the system), and \( M = N \) linear constraints given by

\[
g_j = \sum_{n=1}^{N} G_{jn}(x) (\dot{x}_n - v_n(x)) = 0 \quad \text{for} \quad j = 1, \ldots, N,
\]  

(28)

where \( v(x) = (v_1(x), \ldots, v_N(x)) \) is a given vector field.

**Theorem 16** Let \( \Sigma \) be a constrained Lagrangian mechanical system with configuration space \( Q \), kinetic energy \( T \) given in (27), and constraints given by (28). The equations of motion of \( \Sigma \) are the Lagrange differential equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = 0 \quad \text{for} \quad j = 1, \ldots, N,
\]  

(29)

with \( L = \frac{1}{2} ||\dot{x} - v||^2 = T - \langle \dot{x}, v \rangle + \frac{1}{2} ||v||^2 \), which are equivalently to

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{1}{2} ||v||^2 \right) + \sum_{n=1}^{N} \dot{x}_n \left( \frac{\partial p_j}{\partial x_n} - \frac{\partial p_n}{\partial x_j} \right)
\]

(30)

\[
= \frac{\partial}{\partial x_j} \left( \frac{1}{2} ||v||^2 \right) + \sum_{n=1}^{N} v_n \left( \frac{\partial p_j}{\partial x_n} - \frac{\partial p_n}{\partial x_j} \right),
\]

where

\[
p_j = \sum_{n=1}^{N} G_{jn} v_n, \quad \text{for} \quad j = 1, 2, \ldots, N.
\]  

(31)

**Proof** We consider the differential system (4) with \( N \) as \( 2N \) and with invariant hypersurfaces \( g_j(x_1, \ldots, x_{2N}) = 0 \) for \( j = 1, \ldots, N_1 \leq N \). Taking the functions \( g_m \) for \( m = N_1, \ldots, 2N \) as follows \( g_\alpha = g_\alpha(x_1, \ldots, x_{2N}) \), \( g_{N+j} = x_j \), for \( \alpha = N_1 + 1, \ldots, N \) if \( N_1 < N \) and \( j = 1, \ldots, N \). We assume that \( \{g_1, g_2, \ldots, g_N, x_1, \ldots, x_N\} \neq 0 \). Hence system (4) takes
the form
\[
\dot{x}_j = \lambda_{N+j},
\]
\[
\dot{x}_{j+N} = \sum_{k=1}^{N_1} \Phi_k \frac{\{g_1, \ldots, g_{k-1}, x_{j+N}, g_{k+1}, \ldots, g_N, x_1, \ldots, x_N\}}{\{g_1, g_2, \ldots, g_N, x_1, \ldots, x_N\}} + \ldots
\]
\[
+ \sum_{k=N_1+1}^{2N} \lambda_k \frac{\{g_1, \ldots, g_{N_1+1}, \ldots, g_{k-1}, x_{j+N}, g_{k+1}, \ldots, g_N, x_1, \ldots, x_N\}}{\{g_1, \ldots, g_N, x_1, \ldots, x_N\}}, \tag{32}
\]
for \( j = 1, \ldots, K \).

In particular if we take \( g_j = x_{N+j} - p_j(x_1, \ldots, x_N) = 0 \), where \( p_j = p_j(x_1, \ldots, x_N) \) are convenient functions for \( j = 1, \ldots, N \), then from (32) we obtain
\[
\dot{x}_j = \lambda_{N+j}, \quad \dot{x}_{N+j} = \Phi_j + \sum_{n=1}^{N} \lambda_{N+n} \frac{\partial p_j}{\partial x_n},
\]
thus
\[
\dot{x}_j = \lambda_{N+j}, \quad \frac{d}{dt} (x_{N+j} - p_j) = \Phi_j. \tag{33}
\]

Taking the arbitrary functions \( \lambda_{N+j} \) and \( \Phi_j \) as follows
\[
\lambda_{N+j} = \sum_{n=1}^{N} \tilde{G}_{jn} x_{N+n}, \quad \Phi_j = \frac{\partial L}{\partial x_j},
\]
for \( j = 1, \ldots, N \), where \( \tilde{G}_{jn} = \tilde{G}_{jn}(x_1, \ldots, x_N) \) are elements of a symmetric definite positive matrix \( \tilde{G} \), and
\[
L = \frac{1}{2} \sum_{n,j=1}^{N} G_{jn}(\dot{x}_j - v_j)(\dot{x}_n - v_n) = \frac{1}{2} ||\dot{x} - v||^2 = \frac{1}{2} ||x||^2 - \langle \dot{x}, v \rangle + \frac{1}{2} ||v||^2
\]
\[
= T - \langle \dot{x}, v \rangle + \frac{1}{2} ||v||^2,
\]
where \( G = (G_{jk}) \) is the inverse matrix of \( \tilde{G} = (\tilde{G}_{jk}) \).

We can write \( g_j \) as
\[
g_j = x_{j+N} - p_j = \sum_{n=1}^{N} G_{jn}(\dot{x}_n - v_n) = 0,
\]
for \( j = 1, \ldots, N \). Then, \( g_j = 0 \) if and only if \( \dot{x}_1 - v_1 = \ldots = \dot{x}_N - v_N = 0 \). Since
\[
\Phi_j = \frac{\partial L}{\partial x_j} = -\left( \dot{x} - v, \frac{\partial v}{\partial x_j} \right) \Rightarrow \Phi_j|_{g_j=0} = \frac{\partial L}{\partial x_j}|_{x=v} = 0,
\]
for \( j = 1, \ldots, N \).

On the other hand in view of the relations
\[
g_j = x_{j+N} - p_j = \sum_{n=1}^{N} G_{jn}(\dot{x}_n - v_n) = \frac{\partial L}{\partial \dot{x}_j},
\]
we finally deduce that Eq. (33) can be written as the Lagrangian differential equations
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = 0, \quad \text{for} \quad j = 1, \ldots, N. \]

After computation and in view of the constraints (28) we finally obtain differential system (30). This completes the proof of the theorem. \[\square\]

In view of the second Newton law: _acceleration is equal to force_ (see for instance [34]), we obtain that the right hand side of the equations of motion (30) are the generalized forces acting on the mechanical system which depends only on its position. Consequently the field of force \( F \) with components
\[ F_j = \frac{\partial}{\partial x_j} \left( \frac{1}{2} ||v||^2 \right) + \sum_{n=1}^{N} v_n \left( \frac{\partial p_j}{\partial x_n} - \frac{\partial p_n}{\partial x_j} \right) \]
is the most general field of force depending only on the position of the _natural mechanical system_ which is constrained to move on the \( N \) dimensional subset of the phase space given by (28). In short the equations of motion (30) provide a complete answer to the inverse problem (i) when the constraints are given in the form (28).

Now we want to solve the inverse problem (i) for the classical constraints
\[ \sum_{n=1}^{N} a_{jn}(x) \dot{x}_n = 0, \quad \text{for} \quad j = 1, \ldots, M. \quad (34) \]

We recall that the equations of motion of a constrained Lagrangian system with Lagrangian \( \tilde{L} = \frac{1}{2} ||\dot{x}||^2 - U(x) \), and constrains given by (34) but with a field of forces \( \tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_N) \) depending on positions and velocities are the _Lagrange differential equations with multipliers_
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_k} \right) - \frac{\partial T}{\partial x_k} = \tilde{F}_k(x, \dot{x}) = -\frac{\partial U}{\partial x_k} + \sum_{j=1}^{M} \mu_j a_{jk}, \quad \text{for} \quad k = 1, \ldots, N, \]
\[ \sum_{n=1}^{N} a_{jn}(x) \dot{x}_n = 0, \quad \text{for} \quad j = 1, \ldots, M, \quad (35) \]
where \( \mu_j = \mu_j(x, \dot{x}) \) are the _Lagrangian multipliers_. As we can observe the forces \( \tilde{F} \) are composed by the potential forces with components \( -\frac{\partial U}{\partial x_k} \) and the reactive forces generated by constraints with components \( \sum_{j=1}^{M} \mu_j a_{jk} \) for \( k = 1, \ldots, N \). For more details see [2].

In short we have two equations of motions: the ones given in (29), or what is the same (30) for constraints of type (28), and the classical ones given in (35) for the constraints (34). In order to solve the problem (i) for the constraints (34) we establish the relationship between these two sets of equations. For doing this we shall choose conveniently the vector field \( v \) which appear in (28).

In view of that the constraints (28) are equivalently to the constraints \( \dot{x}_j = v_j(x) \) for \( j = 1, \ldots, N \). On the other hand from (34) we obtain that \( \langle a_j, v \rangle = \sum_{n=1}^{N} a_{jn} v_n = 0 \), thus \( v \) must be orthogonal to the independent vectors \( a_j = (a_{j1}, \ldots, a_{jN}) \) for \( j = 1, \ldots, M \). So
we introduce the \( N \) independent 1-forms, the first \( M \) of these 1-forms are associated to the \( M \) constraints (34), i.e.

\[
\Omega_j = \sum_{n=1}^{N} a_{jn}(x) dx_n \quad \text{for} \quad j = 1, \ldots, M,
\]

and we choose the 1-forms \( \Omega_j \) for \( j = M + 1, \ldots, N \) arbitrarily, but satisfying that the determinant \( |\Upsilon| \) of the matrix \( \Upsilon = (a_{jk}) \):

\[
\Upsilon = \begin{pmatrix}
\Omega_1(\partial_1) & \cdots & \Omega_1(\partial_N) \\
\vdots & \ddots & \vdots \\
\Omega_N(\partial_1) & \cdots & \Omega_N(\partial_N)
\end{pmatrix} = \begin{pmatrix}
a_{11} & \cdots & a_{N1} \\
\vdots & \ddots & \vdots \\
a_{N1} & \cdots & a_{NN}
\end{pmatrix},
\]

is nonzero. The ideal case would be when this determinant is constant. In other words the \( N \) 1-forms \( \Omega_j \) for \( j = 1, \ldots, N \) are independent. Now we define the vector field \( v \) as

\[
v = -\frac{1}{|\Upsilon|} \begin{pmatrix}
\Omega_1(\partial_1) & \cdots & \Omega_1(\partial_N) & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\Omega_M(\partial_1) & \cdots & \Omega_M(\partial_N) & 0 \\
\Omega_{M+1}(\partial_1) & \cdots & \Omega_{M+1}(\partial_N) & v_{M+1} \\
\vdots & \ddots & \vdots & \vdots \\
\Omega_N(\partial_1) & \cdots & \Omega_N(\partial_N) & v_N \\
\partial_1 & \cdots & \partial_N & 0
\end{pmatrix} = \langle \Upsilon^{-1}P, \partial_x \rangle,
\]

where \( P = (0, \ldots, 0, v_{M+1}, \ldots, v_N)^T \), the functions \( v_j = v_j(x) \) are nonzero arbitrary functions due to the arbitrariness of \( \Omega_j \) for \( j = M + 1, \ldots, N \).

**Proposition 17** The vector field (37) is the most general vector field satisfying the constraints (34), i.e. \( \Omega_j(v) = 0 \) for \( j = 1, \ldots, M \), where the \( \Omega_j \) are given in (36).

**Proof** First we prove that the vector field (37) is such that

\[
\sum_{n=1}^{N} \Omega_j(\partial_n) v_n = \Omega_j(v) = 0 \quad \text{for} \quad j = 1, \ldots, M,
\]

\[
\sum_{n=1}^{N} \Omega_{M+k}(\partial_n) v_n = \Omega_{M+k}(v) = v_{M+k} \quad \text{for} \quad k = M + 1, \ldots, N.
\]

Indeed, from the relation \( v(x) = S^{-1}P \) we get that

\[
\Upsilon v(x) = (\Omega_1(v), \ldots, \Omega_M(v), \Omega_{M+1}(v), \ldots, \Omega_N(v))^T = P = (0, \ldots, 0, v_{M+1}, \ldots, v_N)^T.
\]

Thus we obtain (38). Consequently the vector field \( v \) satisfies the constraints.

Now we show that vector field \( v \) is the most general vector field satisfying these constraints. Let \( \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_N) \) be another vector field satisfying the constraints, i.e. \( \sum_{n=1}^{N} \Omega_j(\partial_n) \tilde{v}_n = \Omega_j(\tilde{v}) = 0 \) for \( j = 1, \ldots, M \). Taking the arbitrary functions

\[\text{ Springer}\]
Thus Proposition 17 is proved. 

\[ v_{M+1}, \ldots, v_N \text{ as follows } v_{M+k} = \sum_{n=1}^{N} \Omega_{M+j}(\partial_n) \tilde{v}_n \] 

we obtain from (37) the relations

\[
v = - \frac{1}{|T|} \sum_{n=1}^{N} \tilde{v}_n \begin{vmatrix}
\Omega_1(\partial_1) & \cdots & \Omega_1(\partial_N) & \sum_{n=1}^{N} \Omega_1(\partial_n) \tilde{v}_n \\
\vdots & \ddots & \vdots & \vdots \\
\Omega_M(\partial_1) & \cdots & \Omega_M(\partial_N) & \sum_{n=1}^{N} \Omega_M(\partial_n) \tilde{v}_n \\
\Omega_{M+1}(\partial_1) & \cdots & \Omega_{M+1}(\partial_N) & \sum_{n=1}^{N} \Omega_{M+1}(\partial_n) \tilde{v}_n \\
\vdots & \ddots & \vdots & \vdots \\
\Omega_N(\partial_1) & \cdots & \Omega_N(\partial_N) & \sum_{n=1}^{N} \Omega_N(\partial_n) \tilde{v}_n \\
\partial_1 & \cdots & \partial_N & 0 
\end{vmatrix} = - \frac{1}{|T|} \sum_{n=1}^{N} \tilde{v}_n \partial_n = \tilde{v}.
\]

Thus Proposition 17 is proved.

We define

\[
\Lambda = \Lambda(x) = (\Lambda_1(x), \ldots, \Lambda_N(x))^T = (\Upsilon^T)^{-1} H v(x) = A p,
\]

where \( A = (A_{jk}) \) is an \( N \times N \) skew symmetric matrix such that

\[
A = \left( \Upsilon^T \right)^{-1} H \Upsilon^{-1}, \quad H = (H_{jn}) = \left( \frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right).
\]

**Theorem 18** Let \( \Sigma \) be a constrained Lagrangian mechanical system with configuration space \( Q \), kinetic energy \( T \) given in (27), and constraints given by (28) with \( v = (v_1, \ldots, v_N)^T \) given by (37).

The equations of motion of \( \Sigma \) are

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = F_j(x) = \frac{\partial}{\partial x_j} \left( \frac{1}{2} ||v||^2 \right) + \sum_{k=1}^{M} \Lambda_k a_{kj},
\]
for \( j = 1, \ldots, N \), where the \( \Lambda_k \)'s are defined in (39) with

\[
\Lambda_k = \sum_{j=1}^{M} A_{kj} v_j = 0 \quad \text{for} \quad k = M + 1, \ldots, N.
\]  

(42)

Proof Let \( \sigma \) be the 1-form associated to the vector field \( \mathbf{v} \), i.e.

\[
\sigma = \langle \mathbf{v}, \, d\mathbf{x} \rangle = \sum_{j,k=1}^{N} G_{jk} v_j dx_k = \sum_{n=1}^{N} p_n dx_n.
\]  

(43)

Then the 2-form \( d\sigma \) admits the development

\[
d\sigma = \sum_{n,j=1}^{N} \left( \frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right) dx_j \wedge dx_n = \frac{1}{2} \sum_{n,j=1}^{N} A_{nj} \Omega_n \wedge \Omega_j.
\]  

(44)

Here we have used that the 1-forms \( \Omega_1, \ldots, \Omega_N \) are independent, and consequently they form a basis of the 1-form space. Hence \( \Omega_k \wedge \Omega_n \) for \( k, n = 1, \ldots, N \) form a basis of the 2-form space. From (44) we have that the inner product of vector field \( \mathbf{v} \) and \( d\sigma \), i.e. \( \iota_v d\sigma \) is such that

\[
\iota_v d\sigma = \sum_{n,j=1}^{N} v_n \left( \frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right) dx_j = \langle H \mathbf{v}, \, d\mathbf{x} \rangle,
\]  

(45)

where the matrix \( H \) is

\[
\left( \frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right).
\]

Again from (44) we have that

\[
\iota_v d\sigma(\ast) = d\sigma(\mathbf{v}, \ast) = \frac{1}{2} \sum_{n,j=1}^{N} b_{nj} \Omega_n \wedge \Omega_j(\mathbf{v}, \ast)
\]

\[
= \frac{1}{2} \sum_{n,j=1}^{N} A_{nj} (\Omega_n(\mathbf{v}) \Omega_j(\ast) - \Omega_j(\mathbf{v}) \Omega_n(\ast))
\]

\[
= \frac{1}{2} \sum_{n,j=1}^{N} A_{nj} \Omega_n(\mathbf{v}) \Omega_j(\ast) - \frac{1}{2} \sum_{n,j=1}^{N} A_{jn} \Omega_n(\mathbf{v}) \Omega_j(\ast)
\]

(46)

\[
= \frac{1}{2} \sum_{n,j=1}^{N} (A_{nj} - A_{jn}) \Omega_n(\mathbf{v}) \Omega_j(\ast) = \sum_{n,j=1}^{N} A_{nj} \Omega_n(\mathbf{v}) \Omega_j(\ast)
\]

Now from the last equality and (44) we have

\[
\iota_v d\sigma(\partial_j) = \sum_{n=1}^{N} \Lambda_n \Omega_n(\partial_j) = \sum_{n,j=1}^{N} v_n \left( \frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right).
\]  

(47)

Clearly from these relations it follows that \( H \mathbf{v}(\mathbf{x}) = \Upsilon^T \Lambda \), hence

\[
\Lambda = (\Upsilon^T)^{-1} H \mathbf{v}(\mathbf{x}) = (\Upsilon^T)^{-1} H \Upsilon^{-1} \mathbf{P} = \Lambda \mathbf{P},
\]

here we have used the equality \( \mathbf{v}(\mathbf{x}) = \Upsilon^{-1} \mathbf{P} \).
From (47) and (30) we obtain
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{1}{2} ||v||^2 \right) + \sum_{j=1}^{N} \Lambda_j \Omega_j (\dot{\kappa}_k),
\] (48)
for \( k = 1, \ldots, N \). From (48), (36) and (42) we get (41). In short Theorem 18 is proved. \( \square \)

Remark 19 Equation (42) define a system of first order partial differential equations with unknown functions \( v_{M+1}, \ldots, v_N \) (see (37), (40) and (42)).

We observe that equations (42) can be rewritten as follows
\[
\tilde{A} \mathbf{b} = \mathbf{0} \quad \text{with} \quad \mathbf{b} = (v_{M+1}, \ldots, v_N)^T.
\]
where \( \tilde{A} \) is an \((N - M) \times (N - M)\) skew symmetric matrix. Thus if \( N - M \) is even then, from (37), it follows that the vector \( \mathbf{b} \) is nonzero, consequently the determinant of the matrix \(|\tilde{A}| = \mu_{N,M}^2\) must be zero, i.e. \( \mu_{N,M} = 0 \). If \( N - M \) is odd then \(|\tilde{A}|\) is always zero. If in this case \( \text{rank}(\tilde{A}) = r \), then without loss of generality we can assume that (42) takes the form
\[
\sum_{j=M+1}^{N} A_{kj} v_j = 0 \quad \text{for} \quad k = M + 1, \ldots, M + r.
\]
In particular for \( M = 1, N = 3, \quad M = 2, N = 4 \) we obtain respectively
\[
\mu_{3,1} = a_1 H_{23} + a_2 H_{31} + a_3 H_{12} = 0,
\]
\[
\mu_{4,2} = (a_{32} a_{31} - a_{32} a_{41}) H_{12} + (a_{41} a_{22} - a_{21} a_{42}) H_{13}
+ (a_{21} a_{32} - a_{31} a_{22}) H_{14} + (a_{42} a_{11} - a_{12} a_{41}) H_{23}
+ (a_{12} a_{31} - a_{32} a_{11}) H_{24} + (a_{22} a_{11} - a_{12} a_{21}) H_{34} = 0.
\] (49)

Remark 20 Equation (41) can be interpreted as the equations of motion of the constrained Lagrangian system with Lagrangian \( \tilde{L} = T + \frac{1}{2} ||v||^2 \) and constraints (34). The field of force with components
\[
F_j (\mathbf{x}) = \frac{\partial}{\partial x_j} \left( \frac{1}{2} ||v||^2 \right) + \sum_{k=1}^{M} \Lambda_k a_{kj},
\] (50)
for \( j = 1, \ldots, N \), has the same structure than the field of forces determine in (35), but there are three important differences. First the potential and reactive components in (50) are related through the vector field \( \mathbf{v} \) (which itself is determined by the constraints), while in (35) the potential \( U \) is completely independent of the reactive forces with components \( \sum_{k=1}^{M} \mu_k a_{kj} \). Second the multipliers \( \Lambda_1, \ldots, \Lambda_M \) in (50) depend only on the position of the mechanical system, while in (35) the Lagrangian multipliers \( \mu_j \) depend on the position and velocity, and finally system (41) was deduced from Lagrangian differential system (29), while system (35) in general has no relations with the Lagrangian equations.

In the applications of Theorem 18 we will determine the functions \( v_{M+1}, \ldots, v_N \) as solutions of (42) together with the condition
\[
U = -\frac{1}{2} ||v||^2 + h,
\] (51)
where \( h \) is a constant. Under the potential (51) we obtain that between the fields of force \( \mathbf{\tilde{F}} \) given in (35) and \( \mathbf{F} \) given in (50) the only difference consists in the coefficients which determine the reactive forces.
The following two questions arise: Do exist solutions of Eqs. (42) and (51) in such a way that the solutions of the differential system
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = -\frac{\partial U}{\partial x_j} + \sum_{k=1}^{M} \Lambda_k a_{kj},
\]
for \( j = 1, \ldots, N \), where the \( \Lambda_k \)'s are defined in (39), coincide with the solutions of (35)?

If the answer to the previous question is always positive, then there are equations of motion with field of forces only depending on the positions (41) equivalent to the Lagrangian equations of motions with constraints (35). In short, we would have a new model to describe the behavior of the mechanical systems with linear constraints with respect to the velocity.

The second question is: What is the mechanical meaning of the differential equations generated by the vector field (37), i.e.
\[
\dot{x} = v(x) = Y^{-1}P,
\]
under the conditions (42) and of the differential equations
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = \frac{1}{2} ||v||^2 + \sum_{k=1}^{M} \mu_k a_{kj}?
\]

Partial answer to these questions are given in the examples of the next section.

8 Examples

In this section we illustrate in some particular cases the relation between three mathematical models:

(i) the classical model deduced from the d’Alembert-Lagrange principle (see (35)),
(ii) the model deduced from the Lagrangian equations (29) (see (41)), and
(iii) the model obtained from the first order differential Eq. (52) under the conditions (42).

Example 21 Suslov problem. In this example we study the classical problem of nonholonomic dynamics formulated by Suslov [33]. We consider the rotational motion of a rigid body around a fixed point and subject to the nonholonomic constraint \(<\tilde{a}, \omega> = 0\) where \(\omega = (\omega_1, \omega_2, \omega_3)\) is the angular velocity of the body, \(\tilde{a}\) is a constant vector, and \(<, >\) is the scalar product. Suppose that the body rotates in a force field with potential \(U(y) = U(y_1, y_2, y_3)\). Applying the method of Lagrange multipliers we write the equations of motion (35) in the form
\[
I \dot{\omega} = I \omega \wedge \omega + \gamma \wedge \frac{\partial U}{\partial \gamma} + \mu \tilde{a}, \quad \dot{\gamma} = \gamma \wedge \omega, \quad <\tilde{a}, \omega> = 0,
\]
where
\[
\gamma = (y_1, y_2, y_3) = (\sin z \sin x, \sin z \cos x, \cos z),
\]
\((x, y, z) = (\varphi, \psi, \theta)\) are the Euler angles, and \(I\) is the tensor of inertia.

We observe that the equations \(\dot{\gamma} = \gamma \wedge \omega\) is called the Poisson equations.
Using the constraint equation \(< \mathbf{a}, \omega > = 0\), the Lagrange multiplier \(\mu\) can be expressed as a function of \(\omega\) and \(\gamma\) as follows

\[
\mu = -\frac{\left[ \mathbf{a}, I \omega \wedge \omega + \gamma \wedge \frac{\partial U}{\partial \gamma} \right]}{[\mathbf{a}, I^{-1} \mathbf{a}]}.
\]

We shall suppose that the vector \(\mathbf{a} = (0, 0, 1)\), consequently the constraint takes the form

\[
\omega_3 = \dot{x} + \dot{y} \cos z = 0.
\]

To determine the vector field \(\mathbf{v}\) we suppose that the manifold \(Q\) is the special orthogonal group of rotations of \(\mathbb{R}^3\), i.e. \(Q = SO(3)\), with the Riemann metric \(G\) given by

\[
\begin{pmatrix}
I_3 & I_3 \cos z & 0 \\
I_3 \cos z & (I_1 \sin^2 x + I_2 \cos^2 x) \sin^2 z + I_3 \cos^2 z & (I_1 - I_2) \sin x \cos x \sin z \\
0 & (I_1 - I_2) \sin x \cos x \sin z & I_1 \cos^2 x + I_2 \sin^2 x
\end{pmatrix},
\]

with determinant \(|G| = I_1 I_2 I_3 \sin^2 z\).

By choosing the 1-form \(\Omega_j\) for \(j = 1, 2, 3\) as follows \(\Omega_1 = dx + \cos z dy\), \(\Omega_2 = dy\), \(\Omega_3 = dz\). we obtain that \(|\mathbf{Y}| = 1\). Thus the vector field \(\mathbf{v}\) takes the form

\[
\mathbf{v} = v_2 \cos z \frac{\partial}{\partial x} - v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z}.
\]

Hence the differential system (52) can be written as

\[
\dot{x} = v_2 \cos z, \quad \dot{y} = -v_2, \quad \dot{z} = -v_3.
\] (53)

From (31) we compute

\[
p_1 = 0, \quad p_2 = (I_1 \sin^2 x + I_2 \cos^2 x) v_2 \sin^2 z + (I_2 - I_1) v_3 \cos x \sin x \sin z, \quad p_3 = -v_3 (I_2 \sin^2 x + I_1 \cos^2 x) + (I_2 - I_1) v_2 \sin x \cos x \sin z.
\]

Changing \(v_1\) and \(v_2\) by \(\mu_1\) and \(\mu_2\) as

\[
\mu_1 = I_2 (v_3 \sin x - v_2 \sin z \cos x), \quad \mu_2 = I_1 (v_3 \cos x + v_2 \sin z \sin x),
\]

we obtain

\[
p_1 = 0, \quad p_2 = \mu_1 \sin z \cos x - \mu_2 \sin z \sin x, \quad p_3 = \sin x \mu_1 + \cos x \mu_2.
\]

Now the first of condition (49) takes the form

\[
\mu_{3,1} = a_1 H_{23} + a_2 H_{31} + a_3 H_{12} = \partial_x p_2 - \partial_y p_3 + \cos z \partial_x p_3 = 0.\] (54)

After the change \(\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z\) the system (53) by considering the constraints and condition (54) can be written as

\[
\begin{align*}
\dot{\gamma}_1 &= \frac{1}{I_2} \mu_1 \gamma_3, \\
\dot{\gamma}_2 &= \frac{1}{I_1} \mu_2 \gamma_3, \\
\dot{\gamma}_3 &= -\frac{1}{I_1 I_2} (I_1 \mu_1 \gamma_1 + I_2 \mu_2 \gamma_2) \\
\sin z \left( \gamma_3 \left( \frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1} \right) - \gamma_2 \frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1 \frac{\partial \mu_2}{\partial \gamma_3} \right) - \cos x \partial_z \mu_2 - \sin x \partial_y \mu_1 &= 0.
\end{align*}
\] (56)
Clearly if \( \mu_j = \mu_j(x, z, K_1, K_4) \) for \( j = 1, 2 \), where \( K_1 \) and \( K_4 \) are arbitrary constants, then equation (56) takes the form

\[
\gamma_3 \left( \frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1} \right) - \gamma_2 \frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1 \frac{\partial \mu_2}{\partial \gamma_3} = 0,
\]

By comparing the Poisson’s differential system under the condition \( \omega_3 = 0 \) with (55) we obtain that

\[
\omega_1 = -\frac{\mu_2}{I_1}, \quad \omega_2 = \frac{\mu_1}{I_2}.
\]

For more details in the Suslov’s problem see [23].

**Example 22** Nonholonomic Chaplygin systems. We illustrated Theorem 18 in the nonholonomic Chaplygin systems.

In many nonholonomic systems the generalized coordinates \( x_1, \ldots, x_N \) can be choosen in such a way that the equations of the non–integrable constraints can be written in the form

\[
\dot{x}_j = \sum_{k=M+1}^{N} \hat{a}_{jk}(x_{M+1}, \ldots, x_N)\dot{x}_k, \quad \text{for} \quad j = 1, 2, \ldots, M, \quad (57)
\]

A *constrained Chaplygin mechanical system* is the mechanical system with Lagrangian \( \tilde{L} = \tilde{L}(x_{M+1}, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N) \), subject to \( M \) linear nonholonomic constraints (57) (see [20]).

We shall solve the inverse problem for this constrained system when the Lagrangian function is

\[
\tilde{L} = T = \frac{1}{2} \sum_{n,j=1}^{N} G_{jn}(x_{M+1}, \ldots, x_N)\dot{x}_j\dot{x}_n \quad (58)
\]

We determine the vector field (37) and the differential system (41) for constrained Chaplygin-Lagrangian mechanical system with Lagrangian (58).

First we determine the 1-forms \( \Omega_j \) for \( j = 1, \ldots, N \). Taking

\[
\Omega_j = dx_j - \sum_{k=M+1}^{N} \hat{a}_{jk}(x_{M+1}, \ldots, x_N)dx_k, \quad \text{for} \quad j = 1, 2, \ldots, M,
\]

\[
\Omega_k = dx_k \quad \text{for} \quad k = M + 1, \ldots, N,
\]

we obtain that

\[
\Upsilon = \begin{pmatrix}
1 & 0 & \cdots & 0 & -\hat{a}_{1M+1} & \cdots & -\hat{a}_{1N} \\
0 & 1 & \cdots & 0 & -\hat{a}_{2M+1} & \cdots & -\hat{a}_{2N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 & -\hat{a}_{M,M+1} & \cdots & -\hat{a}_{MN} \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\end{pmatrix}.
\]
Thus $|\Upsilon| = 1$ and consequently

$$
\Upsilon^{-1} = \begin{pmatrix}
1 & 0 & \cdots & 0 & \hat{a}_{1, M+1} & \cdots & \hat{a}_{1, N} \\
0 & 1 & \cdots & 0 & \hat{a}_{2, M+1} & \cdots & \hat{a}_{2, N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \hat{a}_{M, M+1} & \cdots & \hat{a}_{M, N} \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 
\end{pmatrix}.
$$

Then the vector field (37) in this case generates the following differential equations

$$
\dot{x}_j = \sum_{n=M+1}^N \hat{a}_{jn} v_n \quad \dot{x}_k = v_k \quad \text{for} \quad j = 1, \ldots, M, \quad k = M + 1, \ldots, N.
$$

Differential system (41) in this case can be written as

$$
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_k} \right) = \frac{\partial}{\partial x_k} \left( \frac{1}{2} |v|^2 \right) + \Lambda_k,
$$

$$
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{1}{2} |v|^2 \right) - \sum_{k=1}^M \Lambda_k \hat{a}_{kj},
$$

for $j = M + 1, \ldots, N, \ k = 1, \ldots, M$, where $\Lambda_1, \ldots, \Lambda_M$ are determined by the formula (39), (40) and (42).

We observe that system (59) coincides with the Chaplygin system. Indeed, excluding $\Lambda_k$ from the first of the equations of (59) and denoting by $L^*$ the expression in which the velocities $\dot{x}_1, \ldots, \dot{x}_M$, have been eliminated by means of the constraints Eq. (57) we get

$$
L^* = \left. L \right|_{\dot{x}_j = \sum_{k=M+1}^N \hat{a}_{jk} \dot{x}_k} = (T + \frac{1}{2} |v|^2) \left|_{\dot{x}_j = \sum_{k=M+1}^N \hat{a}_{jk} \dot{x}_k} \right.
$$

Therefore we obtain

$$
\frac{\partial L^*}{\partial \dot{x}_j} = \frac{\partial L}{\partial \dot{x}_j} + \sum_{\alpha=1}^M \frac{\partial L}{\partial \dot{x}_\alpha} \hat{a}_{\alpha j}, \quad \frac{\partial L^*}{\partial x_j} = \frac{\partial L}{\partial x_j} + \sum_{\alpha=1}^M \sum_{m=M+1}^N \frac{\partial L}{\partial x_\alpha} \hat{x}_m \frac{\partial \hat{a}_{em}}{\partial x_j},
$$

for $j = M + 1, \ldots, N$.

From these relations we have

$$
\frac{d}{dt} \left( \frac{\partial L^*}{\partial \dot{x}_j} \right) - \frac{\partial L^*}{\partial x_j} = \sum_{m=M+1}^N \sum_{l=1}^M \left( \frac{\partial \hat{a}_{lj}}{\partial x_m} - \frac{\partial \hat{a}_{lm}}{\partial x_j} \right) \hat{x}_m \frac{\partial L}{\partial \dot{x}_l},
$$

for $j = M + 1, \ldots, N, \ k = 1, \ldots, M$, which are the equations which Chaplygin published in the Proceedings of the Society of the Friends of Natural Science in 1897.

**Example 23** We shall illustrate this case in the following system which we call Gantmacher’s system (see for more details [12]).

Two material points $m_1$ and $m_2$ with equal masses are linked by a metal rod with fixed length $l$ and small mass. The system can move only in the vertical plane and so the speed of
the midpoint of the rod is directed along the rod. It is necessary to determine the trajectories of the material points \( m_1 \) and \( m_2 \).

Let \((q_1, r_1)\) and \((q_2, r_2)\) be the coordinates of the points \( m_1 \) and \( m_2 \). Introducing the following change of coordinates: \( x_1 = \frac{q_2 - q_1}{2}, \ x_2 = \frac{r_1 - r_2}{2}, \ x_3 = \frac{r_2 + r_1}{2}, \ x_4 = \frac{q_1 + q_2}{2} \), we obtain the mechanical system with configuration space \( Q = \mathbb{R}^4 \).

Lagrangian function \( L = \frac{1}{2} \sum_{j=1}^{4} \dot{x}_j^2 - g x_3 \), where as usual \( g \) denotes the gravitational acceleration and the constraints are

\[
x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0, \quad x_1 \dot{x}_3 - x_2 \dot{x}_4 = 0.
\]

The equations of motion (35) obtained from the d’Alembert-Lagrange principle are

\[
\dot{x}_1 = \mu_1 x_1, \quad \dot{x}_2 = \mu_1 x_2, \quad \dot{x}_3 = -g + \mu_2 x_1, \quad \dot{x}_4 = -\mu_2 x_2,
\]

where \( \mu_1, \mu_2 \) are the Lagrangian multipliers which we determine as follows

\[
\mu_1 = -\frac{\dot{x}_1^2 + \dot{x}_2^2}{x_1^2 + x_2^2}, \quad \mu_2 = \frac{\dot{x}_2 \dot{x}_4 - \dot{x}_1 \dot{x}_3 + g \mu_1}{x_1^2 + x_2^2}.
\]

After the integration of (60) we obtain (for more details see [12])

\[
\dot{x}_1 = -\varphi x_2, \quad \dot{x}_2 = \varphi x_1, \quad \dot{x}_3 = \frac{f}{r} x_2, \quad \dot{x}_4 = \frac{f}{r} x_1,
\]

where \((\varphi, r)\) are the polar coordinates: \( x_1 = r \cos \varphi \), \( x_2 = r \sin \varphi \), and \( f \) is a solution of the differential equation \( \dot{f} = -\frac{2g}{r} x_2 \).

To construct the differential systems (52) and (37) we determine the 1-forms \( \Omega_j \) for \( j = 1, 2, 3, 4 \) as follow (see [28])

\[
\Omega_1 = x_1 dx_1 + x_2 dx_2, \quad \Omega_2 = x_1 dx_3 - x_2 dx_4, \\
\Omega_3 = -x_1 dx_2 + x_2 dx_1, \quad \Omega_4 = x_2 dx_3 + x_1 dx_4.
\]

Here \( \Omega_1 \) and \( \Omega_2 \) are given by the constraints, and \( \Omega_3 \) and \( \Omega_4 \) are chosen in order that the determinant \( |\Gamma| \) becomes nonzero. Hence we obtain that \( |\Gamma| = -(x_1^2 + x_2^2)^2 = -\frac{l^2}{4} \neq 0 \).

By considering that in this case \( N = 4 \) and \( M = 2 \), then from (49) we obtain

\[
\mu_{4,2} = x_2 \partial_{x_3} v_3 - x_1 \partial_{x_4} v_3 + x_2 \partial_{x_1} v_4 + x_1 \partial_{x_2} v_4 = 0.
\]

Differential system (52) takes the form

\[
\dot{x}_1 = -v_3 x_2, \quad \dot{x}_2 = v_3 x_1, \quad \dot{x}_3 = v_4 x_2, \quad \dot{x}_4 = v_4 x_1.
\]

It is easy to show that the functions \( v_3 \) and \( v_4 \) are

\[
v_3 = g_3(x_1^2 + x_2^2), \quad v_4 = \sqrt{\frac{2(-gx_3 + h)}{(x_1^2 + x_2^2)} - g_3^2(x_1^2 + x_2^2)},
\]

where \( h \) is an arbitrary constant and \( g_3 \) is an arbitrary function in the variable \( x_1^2 + x_2^2 \), are solutions of (63) as a consequence from the relation (51) we have

\[
2||v||^2 = (x_1^2 + x_2^2)(v_3^2 + v_4^2) = 2(-g x_3 + h) = 2(-U + h).
\]
The solutions of (64) with $v_3$ and $v_4$ given in (65) are

$$
x_1 = r \cos \alpha, \quad x_2 = r \sin \alpha, \quad \alpha = \alpha_0 + g_3(r)t, \\
x_3 = u_0^3 + \frac{g}{2g_3(r)} - \frac{g}{4g_3^2(r)} \sin 2\alpha - \sqrt{\frac{2g}{g_3(r)}} \cos \alpha, \\
x_4 = -h + \frac{r^2 g_3^2(r)}{2g} + \left( \frac{\sqrt{g}}{\sqrt{2}g_3(r)} \sin \alpha + C \right)^2,
$$

(66)

where $C$, $r$, $\alpha_0$, $u_0^3$ are arbitrary constants and $g_3$ is an arbitrary function of $r^2$.

To compare these solutions with the solutions obtained from (62) we observe that they coincide. We note that we have obtained the trajectories of the masses $m_1$ and $m_2$ solving the differential equations of first order (64) with the functions (65).

Finally we observe that for the Gantmacher system the system (41) takes the form

$$
\ddot{x}_1 = \Lambda_1 x_1, \quad \ddot{x}_2 = \Lambda_2 x_2, \quad \ddot{x}_3 = -g + \Lambda_2 x_1, \quad \ddot{x}_4 = \Lambda_2 x_2,
$$

(67)

and admits as solutions the ones given in (66) (see Remark 20).

Remark 24 From these examples we give a partial answer to the questions stated in Remark 20. Differential equations generated by the vector field (37) under the conditions (42) can be applied to study the behavior of the nonholonomic systems with linear constraints with respect to the velocity (at least for certain class of such systems. Is it possible to apply this mathematical model to describe the behavior of the nonholonomic systems with linear constraints with respect to velocity in general? For the moment we have no answer to this question.

9 Inverse Problem in Dynamics. Generalized Dainelli Inverse Problems

Now we consider a mechanical system with configuration space $Q$ of the dimension $N$ and kinetic energy $T$ given by (27). The problem of determining the most general field of force depending only on the position of the system, for which the curves defined by

$$
f_j - c_j := f_j(x) - c_j = 0 \in \mathbb{R} \quad \text{for} \quad j = 1, \ldots, N - 1,
$$

(68)

are formed by orbits of the mechanical system, is called as the generalized Dainelli’s inverse problem in dynamics. If we assume that the given family of curves (68) admits the family of orthogonal hypersurfaces $S = S(x) = c_N$, then this problem is called the generalized Dainelli–Joukovski’s inverse problem.

If the field of force is potential in the generalized Dainelli inverse problems, then such problems coincide with the Suslov’s inverse problem, or the inverse problem in Celestial Mechanics and the generalized Dainelli–Joukovski’s inverse problem coincides with the Joukowsky problem (for more details see [28]).

The solutions of the generalized Dainelli’s problem for $N = 2$, and of the Joukovski’s problems for $N = 2, 3$ can be found in [6,11,14,37]. A complete solution of the Suslov problem can be found in [32], but this solution in general is complicated to implement.

The following result provides a solution of these inverse problems.

Theorem 25 Under the assumptions of Theorem 18 if the given $M = N - 1$ 1-forms (36) are closed, i.e. $\Omega_j = df_j$ for $j = 1, \ldots, N - 1$, then the following statements hold.
(a) System (41) takes the form
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = \frac{1}{2} \left| \left| v \right| \right|^2 + v_N \sum_{k=1}^{N-1} A_{Nk} \frac{\partial f_k}{\partial x_j} =: F_j,
\] (69)
for \( j = 1, \ldots, N \), where \( v_N = v_N(x) \) is an arbitrary function. Clearly \( F_j \) are the components of the most general field of force that depends only on the position under which a given \( N - 1 \) parametric family of curves (68) can be described as orbits of the mechanical system.

(b) If
\[
v_N \sum_{k=1}^{N-1} A_{Nk} \frac{\partial f_k}{\partial x_j} = - \frac{\partial h}{\partial x_j},
\] (70)
for \( j = 1, \ldots, N - 1 \), where \( h = h(f_1, \ldots, f_{N-1}) \), then the family of curves (68) can be freely described by a mechanical system under the influence of forces derived from the potential function \( V = -U = \frac{1}{2} \left| \left| v \right| \right|^2 - h(f_1, \ldots, f_{N-1}) \).

(c) If we assume that the given family of curves (68) admits the family of orthogonal hypersurface \( S = S(x) = c_N \) defined by
\[
\left\langle \frac{\partial S}{\partial x}, \frac{\partial f_j}{\partial x} \right\rangle = 0 \quad \text{for} \quad j = 1, \ldots, N - 1,
\] (71)
then the most general field of force that depends only on the position of the system under which the given family of curves are formed by orbits of (69) is
\[
F = \frac{\partial}{\partial x} \left( \frac{\nu}{\sqrt{2}} \left\| \frac{\partial S}{\partial x} \right\|^2 \right) + \left( \frac{\partial}{\partial x} \left( \frac{\nu^2}{2} \right) \right) \frac{\partial S}{\partial x} - \left\| \frac{\partial S}{\partial x} \right\|^2 \frac{\partial}{\partial x} \left( \frac{\nu^2}{2} \right),
\] (72)
where \( \nu = \nu(x) \) is an arbitrary function on \( Q \). If we choose \( \nu \) and \( h = h(f_1, \ldots, f_{N-1}) \) satisfying the first order partial differential equation
\[
\left\langle \frac{\partial}{\partial x} \left( \frac{\nu^2}{2} \right), \frac{\partial S}{\partial x} \right\rangle \frac{\partial S}{\partial x} - \left\| \frac{\partial S}{\partial x} \right\|^2 \frac{\partial}{\partial x} \left( \frac{\nu^2}{2} \right) = - \frac{\partial h}{\partial x},
\] (73)
then the field of force \( F \) is given by the potential
\[
V = \frac{\nu^2}{2} \left\| \frac{\partial S}{\partial x} \right\|^2 - h(f_1, \ldots, f_{N-1}).
\] (74)
If (68) is such that \( f_j = x_j \) for \( j = 1, \ldots, N - 1 \), then (74) takes the form
\[
V = \frac{\nu^2 |\tilde{G}|}{2\Delta} \left( \frac{\partial S}{\partial x_N} \right)^2 - h(x_1, \ldots, x_{N-1}),
\] (75)
where \( \tilde{G} = (\tilde{G}_{nm}) \) is the inverse matrix of the matrix \( G \) and
\[
\Delta = \begin{vmatrix} \tilde{G}_{11} & \ldots & \tilde{G}_{1,N-1} \\ \vdots & \ddots & \vdots \\ \tilde{G}_{1,N-1} & \ldots & \tilde{G}_{N-1,N-1} \end{vmatrix}.
\]
Clearly (73) holds in particular if \( \nu = \nu(S) \) and \( h \) is a constant.
(d) Under the assumption (b) we have that \( \int_{g_\nu^t(\gamma)} \sigma = \text{const.} \), where \( \sigma = \langle v, d\mathbf{x} \rangle \) is the 1-form associated to vector field \( v \), \( g_\nu^t \) is the flow of \( v \), and \( \gamma \) is an arbitrary closed curve on \( Q \).

We note that statement (a) of Theorem 25 provides the answer to the generalized Dainelli’s inverse problem, which before was only solved for \( N = 2 \) by Dainelli. Statement (b) of Theorem 25 gives a simpler solution to the Suslov’s inverse problem, already solved by the same Suslov. Statement (c) of Theorem 25 provides the answer to the generalized Dainelli-Joukovski’s problem solved by Joukovski for the case when the field of force is potential and \( N = 2, 3 \). Finally statement (d) of Theorem 25 is the well known Thomson’s Theorem (see [15]) in our context.

Proof of Theorem 25. In this case we obtain that the vector field (37) is

\[
v = -\frac{1}{|Y|} \begin{vmatrix} df_1(\partial_1) & \cdots & df_1(\partial_N) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ df_{N-1}(\partial_1) & \cdots & df_{N-1}(\partial_N) & 0 \\ df_N(\partial_1) & \cdots & df_N(\partial_N) & v_N \\ \partial_1 & \cdots & \partial_N & 0 \end{vmatrix}
\]

\[
= \frac{v_N}{|Y|} \begin{vmatrix} df_1(\partial_1) & \cdots & df_1(\partial_N) \\ \vdots & \ddots & \vdots \\ df_{N-1}(\partial_1) & \cdots & df_{N-1}(\partial_N) \\ \partial_1 & \cdots & \partial_N \end{vmatrix} = \tilde{v}\{f_1, \ldots, f_{N-1}, \Psi\}.
\]

Condition (42) in this case takes the form \( \Lambda_N = A_{NN} v_N = 0 \). Since the matrix \( A \) is skew symmetric, then \( A_{NN} = 0 \). On the other hand from \( \Lambda_j = A_{Nj} v_N \), for \( j = 1, \ldots, N - 1 \), we deduce that system (41) takes the form

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_j} - \frac{\partial T}{\partial x_j} = F_j = \frac{\partial}{\partial x_j} \left( \frac{1}{2}||v||^2 \right) + \sum_{k=1}^{N-1} \Lambda_k df_k(\partial_j)
\]

\[
= \frac{\partial}{\partial x_j} \left( \frac{1}{2}||v||^2 \right) + v_N \sum_{k=1}^{N-1} A_{Nk} df_k(\partial_j).
\]

From these relations we obtain the proof of statement (a) of the theorem.

The statement (b) follows trivially from the previous result.

The proof of statement (c) follows by considering that under the assumption (71) we have

\[
\begin{pmatrix} \partial S \\ \partial \Psi \end{pmatrix} = \varrho \begin{vmatrix} df_1(\partial_1) & \cdots & df_1(\partial_N) \\ \vdots & \ddots & \vdots \\ df_{N-1}(\partial_1) & \cdots & df_{N-1}(\partial_N) \\ d\Psi(\partial_1) & \cdots & d\Psi(\partial_N) \end{vmatrix} = \varrho\{f_1, \ldots, f_{N-1}, \Psi\},
\]

where \( \Psi \) and \( \varrho = \varrho(x_1, \ldots, x_N) \) are arbitrary functions. Hence the 1-form associated to the vector field \( v \) is \( \sigma = \langle v, d\mathbf{x} \rangle = \left\langle v \frac{\partial S}{\partial \mathbf{x}}, d\mathbf{x} \right\rangle = v dS \) where \( v = \frac{\dot{v}}{\varrho} \) (see (43)). Thus
\[ d\sigma = dv \land dS \] and consequently from (45) we have

\[ \iota_v d\sigma = \sum_{n,j=1}^N v_n \left( \frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right) dx_j = dv(v) dS - dS(v) dv \]

After some computations we deduce that the field of force \( \mathbf{F} \) which from (47) can be written as

\[ F_j = \frac{\partial}{\partial x_j} \left( \frac{1}{2} ||v||^2 \right) + \iota_v d\sigma (\partial_j). \]

Hence we obtain (72).

If the curve is given by intersection of the hyperplanes \( x_j - c_j = 0 \) for \( j = 1, \ldots, N-1 \), then the condition (71) takes the form

\[ \sum_{k=1}^N \tilde{G}_{ak} \frac{\partial S}{\partial x_k} = 0, \quad \alpha = 1, \ldots, N-1, \tag{77} \]

where \( \tilde{G} \) is the inverse matrix of the matrix \( G \).

By solving these equations with respect to \( \frac{\partial S}{\partial x_k} \) for \( k = 1, \ldots, N-1 \), we obtain that \( \frac{\partial S}{\partial x_k} \) is equal to

\[
\begin{bmatrix}
\tilde{G}_{11} & \ldots & \tilde{G}_{1,k-1} & -\tilde{G}_{1N} & \tilde{G}_{1,k+1} & \ldots & \tilde{G}_{1,N-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\tilde{G}_{N-1,1} & \ldots & -\tilde{G}_{N-1,k-1} & \tilde{G}_{N-1,N} & -\tilde{G}_{N-1,k+1} & \ldots & \tilde{G}_{N-1,N-1}
\end{bmatrix} \frac{\partial S}{\partial x_N} = L_k \frac{\partial S}{\partial x_N}.
\]

By using these relations and in view of (77), after some computations by considering that

\[ \sum_{n=1}^N L_n \tilde{G}_{NN} = |\tilde{G}|, \]

we deduce that

\[ \begin{bmatrix} \frac{\partial S}{\partial x} & \frac{\partial F}{\partial x} \end{bmatrix} := \sum_{j,k=1}^N \tilde{G}_{jk} \frac{\partial S}{\partial x_k} \frac{\partial F}{\partial x_j} = \sum_{j=1}^N \tilde{G}_{Nk} \frac{\partial S}{\partial x_k} \frac{\partial F}{\partial x_N} = \frac{|\tilde{G}|}{\Delta} \frac{\partial S}{\partial x_N} \frac{\partial F}{\partial x_N}. \tag{78} \]

Consequently we obtain the following expression for the Eq. (73)

\[ \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x} \left( \frac{v^2}{2} \right) & \frac{\partial S}{\partial x} & \frac{\partial F}{\partial x} \end{bmatrix} - \begin{bmatrix} \frac{\partial S}{\partial x} \frac{\partial S}{\partial x} \frac{\partial F}{\partial x} \frac{\partial F}{\partial x} \end{bmatrix} \]

\[ = \frac{|\tilde{G}|}{\Delta} \frac{\partial S}{\partial x_N} \left( \frac{\partial}{\partial x_N} \left( \frac{v^2}{2} \right) \frac{\partial S}{\partial x_N} - \left( \frac{\partial S}{\partial x_N} \right) \frac{\partial F}{\partial x_N} \right). \tag{79} \]

From (78) we obtain that the potential function \( V \) takes the form

\[ V = \frac{v^2}{2} \left( \frac{\partial S}{\partial x} \right)^2 - h(f_1, \ldots, f_{N-1}) = \frac{v^2}{2} \frac{|\tilde{G}|}{\Delta} \left( \frac{\partial S}{\partial x_N} \right)^2 - h(x_1, \ldots, x_{N-1}). \]

We observe that if \( \tilde{G}_{\alpha N} = 0 \) for \( \alpha = 1, \ldots, N-1 \), then \( |\tilde{G}| = \Delta \tilde{G}_{NN} \) and \( S_N = x_N = c_N \) is an orthogonal family of the hyperplane to \( x_j - c_j = 0 \) for \( j = 1, \ldots, N-1 \). After
integrating (79) we obtain that
\[
V = \frac{1}{2} \tilde{G}_{NN} v^2 - h = \left( g(x_N) - \sum_{j=1}^{N-1} \int h(x_1, \ldots, x_{N-1}) \frac{\partial}{\partial x_j} \left( \frac{1}{\tilde{G}_{NN}} \right) dx_j \right) \tilde{G}_{NN},
\]
where \( g = g(x_N) \) and \( h = h(x_1, \ldots, x_{N-1}) \) are arbitrary functions.

Clearly if \( \nu = \nu(S) \). Then \( \sigma = d\Phi(S) \) where \( \Phi = \int \nu(S) dS \). Therefore \( d\sigma = 0 \). So \( \iota_v d\sigma = 0 \). The proof of statement (c) follows.

Now we prove statement (d). We use the homotopy formula \( L_v = \iota_v d + dt_v \), see [13]. The condition (70) in view of (46) is equivalent to
\[
\iota_v d\sigma = \sum_{j=1}^{N-1} \Lambda_j df_j \nu_N \sum_{j=1}^{N-1} A_{Nj} df_j = -dh.
\]
Thus \( L_v \sigma = \iota_v d\sigma + dt_v \sigma = -dh + d\sigma(v) = -dh + d||v||^2 = d \left(||v||^2 - h\right) \). here we use the relation \( \sigma(v) = < v, v > = ||v||^2 \). Hence, if \( g'_v \) is the flow of \( v \) and \( \gamma \) is a closed curve on \( Q \), then the integral \( I = \int_{g'_v(\gamma)} \sigma \) is a function of \( t \). In view of the well-known formula (see [15]) \( \dot{I} = \int_{g'_v(\gamma)} L_v \sigma \), we obtain that \( \dot{I} = 0 \). In short Theorem 25 is proved.

In the two following sections we illustrate the statements (b) and (c) of Theorem 25.

10 Generalized Inverse Bertrand’s Problem

For a particle with kinetic energy \( T = \frac{1}{2}(t^2 + \dot{y}^2) \) we determine the most general field of force \( F = (F_x, F_y) \) capable of generating the family of planar orbits \( f(x, y) = \text{const} \).

From (69) we obtain for \( N = 2 \) the equation
\[
F = \frac{\partial}{\partial x} \left( \frac{1}{2} ||v||^2 \right) + v a_{21} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{2} v^2 ((\partial_x f)^2 + (\partial_y f)^2) \right) - v ((\partial_x (v \partial_x f)) + (\partial_y (v \partial_y f))) \frac{\partial f}{\partial x}.
\]

This field of force coincides with the solutions of Dainelli’s problem given in [37].

Clearly if the arbitrary function \( v \) which appears in the expression of \( F \) is chosen as a solution of the equation
\[
v ((\partial_x (v \partial_x f)) + (\partial_y (v \partial_y f))) = \frac{\partial h(f)}{\partial f}, \quad (80)
\]

then the vector field \( F \) is potential with potential function
\[
U = \frac{1}{2} v^2 ((\partial_x f)^2 + (\partial_y f)^2) - h(f).
\]

In order to apply this result we prove that the potential-energy function \( U \) capable of generating an one-parameter family of conics \( r + bx = c \), where \( r = \sqrt{x^2 + y^2} \) is the function
\[ U = a_{-1} H_{-1}(\tau) + K_{-1} \log (r(1+b\tau)) \]
\[ + \sum_{j \in \mathbb{Z} \setminus \{-1\}} a_j r^{j+1} \left( H_j(\tau) + K_j \frac{(1+b\tau)^{j+1}}{j+1} \right) , \]
when \( b \neq 0 \), where \( a_j \) and \( K_j \) are real constants and \( H_j \), for \( j \in \mathbb{Z} \) are functions
\[ H_j(\tau) = M_j(\tau) \left( C_j - \frac{2K_j}{j+1} \int \frac{(1+b\tau)^j}{(1-\tau^2)M_j(\tau)} d\tau, \right) \]
\[ \Psi(\tau) = (1-\tau)^{\frac{j+1}{2b}} (\tau+1)^{\frac{j+1}{2b}} , \]
where \( C_j \) are arbitrary constants, \( \tau = \cos \theta, ' = \frac{d}{d\tau} \). We have that
\[ U = \frac{\Psi(\tau)}{r^2} - \frac{2}{r^2} \int h(r) dr, \]
when \( b = 0 \), where \( \Psi = \Psi(\tau) \) and \( h = h(r) \) are arbitrary functions (see [28]).
Indeed, from (80) it follows that the require potential field of force exists if and only if
\[ \left( \frac{x}{\sqrt{x^2+y^2}} + b \right) \frac{\partial \nu^2}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \frac{\partial \nu^2}{\partial y} + 2 \frac{\nu^2}{r} = 2 \frac{\partial h}{\partial f}. \]
This equation in polar coordinates \( x = r \cos \theta, y = r \sin \theta \), takes the form
\[ (1+b \cos \theta) \frac{\partial \nu^2}{\partial r} - \frac{b \sin \theta}{r} \frac{\partial \nu^2}{\partial \theta} + 2 \frac{\nu^2}{r} = 2 \frac{\partial h}{\partial f}. \]
or equivalently
\[ (1+b\tau) \frac{\partial \nu^2}{\partial r} + \frac{b(1-\tau^2)}{r} \frac{\partial \nu^2}{\partial \tau} + 2 \frac{\nu^2}{r} = 2 \frac{\partial h}{\partial f}, \]
for \( j \in \mathbb{Z} \).
Now we shall study the case when \( b \neq 0 \) and \( h \) is such that
\[ h(f) = -a_{-1} K_{-1} \ln |f| - \sum_{j \in \mathbb{Z}} a_j K_j \frac{f^{j+1}}{j+1}, \]
where \( v_j \) for \( j \in \mathbb{Z} \), are real constants, and \( \lambda \) is determined in such a way that
\[ \nu^2 = \sum_{j \in \mathbb{Z}} a_j r^{j+1} H_j(\tau). \]
It is clear that we are assuming that the series (83) and (84) converge.
By inserting (83) and (84) into (82) we obtain
\[ b(1-\tau^2) H'_j(\tau) + ((j+1)b\tau + j+3) H_j(\tau) + 2K_j(1+b\tau)^j = 0, \]
for \( j \in \mathbb{Z} \).
The general solution of these equations are the functions (81).

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Consequently the required potential function $U$ must be
\[ U(r, \tau) = \frac{1}{2} \lambda^2 (1 + b^2 + 2b\tau) - h(f) = \sum_{j \in \mathbb{Z}} a_j U_j(r, \tau), \]
where
\[ U_j(r, \tau) = \frac{1}{2} r^{j+1} H_j(\tau)(1 + b^2 + 2b\tau) + \frac{K_j}{j + 1} f^{j+1}, \]
for $j \neq -1$ and
\[ U_{-1}(r, \tau) = \frac{1}{2} H_{-1}(\tau)(1 + b^2 + 2b\tau) + K_{-1} \ln |f|. \]
We will study the subcase when $b = 1$ separately from the subcase when $b \neq 1$.

If $b = 1$, then
\[ U(r, \tau) = \lambda^2 (1 + \tau) - h(f) = \sum_{j \in \mathbb{Z}} a_j U_j(r, \tau), \]
where
\[ U_j(r, \tau) = r^{j+1} (1 - \tau)^{j+2} \left( C_j - 2K_j \int \frac{(1 + \tau)^j}{(1 - \tau)^{j+3}} d\tau \right) + \frac{K_j}{j + 1} f^{j+1}, \]
for $j \neq -1$ and
\[ U_{-1}(r, \tau) = (1 - \tau) \left( C_{-1} - 2K_{-1} \int \frac{d\tau}{(1 - \tau)^2(1 + \tau)} \right) + K_{-1} \ln |f|. \]

Easily verifies that
\[ U_{-2} = \frac{C_{-2}}{r} - 2 \frac{K_{-2} r}{(1 + \tau)^2(1 - \tau)} \left( \int \frac{d\tau}{(1 + \tau)^2(1 - \tau)} + \frac{1}{1 + \tau} \right), \]
where $g(\tau) = \log \sqrt{\frac{1 - \tau}{1 + \tau}}$. Therefore, if $b = 1$ then
\[ U(r, \tau) = \frac{a_{-2} C_{-2}}{r} + \frac{a_{-2} K_{-2} g(\tau)}{r} + \sum_{j \in \mathbb{Z}, j \neq -2} a_j U_j(r, \tau). \]

If $b \neq 1$, and $b \neq 0$, it is easy to prove that
\[ H_{-2}(\tau) = \frac{(1 - \tau)}{1 + b} \frac{2b}{C_{-2} - \frac{2K_{-2}}{(b\tau + 1)(1 - b^2)}}, \]
\[ U_{-2}(r, \tau) = \frac{H_{-2}}{2r} (1 + b^2 + 2b\tau) - \frac{K_{-2} r}{r(b\tau + 1)} = \frac{2K_{-2}}{r(b^2 - 1)} + \frac{C_{-2}}{r} G(\tau), \]
where
\[ G(\tau) = \frac{1}{2} \left( \frac{1 - \tau}{1 + \tau} \right)^\frac{1}{b} \frac{1}{1 - \tau^2}. \]
Under these conditions the potential function $U$ takes the form

$$U(r, \tau) = \frac{a - 2C}{r} G(\tau) + \frac{2aK}{r(b^2 - 1)} + \sum_{j \in \mathbb{Z}, j \neq -2} a_j U_j(r, \tau).$$

Summarizing the above computations we deduce that if $b \neq 0$ the function $U$ becomes

$$U(r, \tau) = \alpha + \frac{\beta(\tau)}{r} + \sum_{j \in \mathbb{Z}, j \neq -2} a_j U_j(r, \tau),$$

where $\alpha$ is a constant and $\beta = \beta(\tau)$ is a convenient function.

If $b = 0$, then $f = r$ and condition (82) takes the form

$$\partial_r \lambda^2 + 2 \frac{\lambda^2}{r} = 2 \partial_f h(f).$$

Therefore

$$r^2 \lambda^2 = 2 \int r^2 \partial_r h(r)dr + 2 \Psi(\tau),$$

or equivalently

$$\lambda^2 = \frac{2}{r^2} \int r^2 \partial_r h(r)dr + \frac{2 \Psi(\tau)}{r^2} = 2h(r) - \frac{4}{r^2} \int h(r)rdr + \frac{2 \Psi(\tau)}{r^2},$$

where $\Psi$ is an arbitrary function.

Hence

$$U(r, \tau) = \frac{\Psi(\tau)}{r^2} - \frac{2}{r^2} \int h(r)dr$$

when $b = 0$.

11 Inverse Stäckel’s problem

Let

$$f_j - c_j = f_j(x) - c_j = \sum_{k=1}^{N} \int \frac{\varphi_{kj}(x_k)}{\sqrt{K_k(x_k)}} dx_k - c_j = 0, \quad j = 1, 2, \ldots, N - 1, \quad (85)$$

be a given $N - 1$-parametric family of orbits in the configuration space $Q$ of the mechanical system with $N$ degrees of freedom and kinetic energy

$$T = \frac{1}{2} \sum_{j=1}^{N} \frac{x_j^2}{A_j}, \quad (86)$$

where $K_k(x_k) = 2\Psi_k(x_k) + 2 \sum_{j=1}^{N} \alpha_j \varphi_{kj}(x_k), \quad \alpha_k$ for $k = 1, 2, \ldots, N$ are constants, $\Psi_k = \Psi_k(x_k)$ are arbitrary functions and $A_j = A_j(x)$ such that

$$\{\varphi_1, \ldots, \varphi_{N-1}, x_j\} = A_j, \quad (87)$$
for $j = 1, 2, \ldots, N$. Here $d\phi_\alpha = \sum_{k=1}^{N} \phi_\alpha(x_k) dx_k$, and $\phi_{\alpha} = \phi_{\alpha}(x_k)$, are arbitrary functions for $k = 1, \ldots, N$, and $\alpha = 1, \ldots, N$.

From (86) it follows that the metric $G$ is diagonal with $G_{jj} = \frac{1}{A_j}$.

The inverse Stäckel problem is the problem of determining the potential field of force that under which any curve of the family (85) is a trajectory of the mechanical system. The solution is as follows (see [28]).

**Proposition 26** For a mechanical system with a configuration space $Q$ and kinetic energy (86), the potential field of force $F = \frac{\partial V}{\partial x}$, for which the family of curves (85) are trajectories is

$$V = -U = v^2(S) \left( \{\phi_1, \ldots, \phi_{N-1}, \Psi\} + \alpha_N \right) - h_0,$$

where $S = \int \sum_{j=1}^{N} \sqrt{\Psi_j(x_k) + \sum_{k=1}^{N} \alpha_j \phi_{kj}(x_k) dx_k} = \int \sum_{k=1}^{N} \frac{dx_k}{q_k(x_k)}$ is a function such that the hypersurface $S - c_N = 0$ is orthogonal to the given hypersurfaces $f_j - c_j = 0$.

We observe that from (87) and (88) it follows that the metric $G$ and potential function $U$ can be determined from the given functions (85).

**Proof** After some tedious computations we get the equality

$$\left\{ f_1, \ldots, f_{N-1}, * \right\} \{ f_1, \ldots, f_{N-1}, f_N \} = \begin{vmatrix} q_1 d\phi_1(\partial_1) & \ldots & q_N d\phi_1(\partial_N) \\ \vdots & \ddots & \vdots \\ q_1 d\phi_{N-1}(\partial_1) & \ldots & q_N d\phi_{N-1}(\partial_N) \\ \partial_1 & \ldots & \partial_N \end{vmatrix} \prod_{j=1}^{N} q_j \{\phi_1, \ldots, \phi_N\} = \sum_{j=1}^{N} \left( A_j q_j \right) = \sum_{j=1}^{N} \left( A_j \frac{\partial S}{\partial x_j} \right),$$

From (76) we have $v(x) = vG^{-1} \frac{\partial S}{\partial x}$, hence in view of the identity (viii) of the Nambu bracket we obtain

$$\left\langle \frac{\partial S}{\partial x}, \frac{\partial f_j}{\partial x} \right\rangle = \sum_{k=1}^{N} A_k \phi_{kj} = \sum_{k=1}^{N} A_k \frac{\partial \phi_j}{\partial x_k} = \{\phi_1, \ldots, \phi_{N-1}, \phi_j\} = 0.$$
for $j = 1, \ldots, N - 1$, thus it follows the orthogonality of the surfaces. On the other hand the following equalities hold

$$\norm{\mathbf{v}}^2 = v^2 \sum_{k=1}^{N} A_k^2(K_k(x_k))^2 = v^2 \sum_{k=1}^{N} \alpha_j \varphi_{kj}(x_k)$$

$$= 2v^2 \sum_{k=1}^{N} A_k \Psi_k(x_k) + 2v^2 \sum_{j=1}^{N} \alpha_j A_k \varphi_{kj}(x_k)$$

$$= 2v^2 \left( \frac{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_j\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} + \sum_{j=1}^{N} \frac{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_j\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} \right)$$

$$= 2v^2 \left( \frac{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_j\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} + \alpha_N \right).$$

Here we used the identity (iv) of the Nambu bracket, where $d\Psi = \sum_{j=1}^{N} \Psi_k(x_k)dx_k$. We observe that if we choose $v = v(S)$, then from (73) we obtain that the field of force which generates the given family of orbits (85) is potential with potential function given by (88). In particular if $v = 1$ and $h_0 = \alpha_N$ then we obtain the classical St"{u}ckel potential (see [5]).

Example 27 Joukovski’s example We shall study a mechanical systems with three degrees of freedom. If we denote by $x_1 = p$, $x_2 = q$, $x_3 = r$, then we consider the mechanical system with kinetic energy

$$T = \frac{1}{2r^2} \left( p^2 - 2p \dot{p} \dot{r} + q^2 - 2q \dot{q} \dot{r} + \left( \frac{p^2 + q^2}{r^2} + r^2 \right) r^2 \right).$$

Consequently the matrix $\tilde{G}$ is such that

$$\tilde{G} = \begin{pmatrix} \frac{p^2 + q^2}{r^2} & \frac{pq}{r} & \frac{p}{r} \\ \frac{pq}{r} & \frac{q^2 + r^4}{r^2} & \frac{q}{r} \\ \frac{p}{r} & \frac{q}{r} & 1 \end{pmatrix}.$$

Then we get $|\tilde{G}| = r^4$, $\Delta = p^2 + q^2 + r^4$. We determine the field of force derived from the potential-energy function (75) in such a way that the family of curves $p = c_1$, $q = c_2$ can be freely described by a particle with kinetic energy $T$.

In this case Eq. (77) are

$$\frac{\partial S}{\partial p} + \frac{\partial S}{\partial q} + \frac{\partial S}{\partial r} = \frac{p^2 + q^2}{r^2} \frac{\partial S}{\partial p} + \frac{pq}{r^2} \frac{\partial S}{\partial q} + \frac{p}{r^2} \frac{\partial S}{\partial r} = 0,$$

$$\frac{\partial S}{\partial p} + \frac{\partial S}{\partial q} + \frac{\partial S}{\partial r} = \frac{q^2 + r^4}{r^2} \frac{\partial S}{\partial q} + \frac{pq}{r^2} \frac{\partial S}{\partial p} + \frac{q}{r^2} \frac{\partial S}{\partial r} = 0.$$

The solutions of these partial differential equations are $S = S \left( \frac{p^2 + q^2}{r^2} - r^2 \right)$ where $S$ is an arbitrary function in the variable $\frac{p^2 + q^2}{r^2} - r^2$. 

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Without loss of generality below we consider that \( S = \frac{p^2 + q^2}{r^2} - r^2 \). Hence after some computations we obtain that conditions \((79)\) take the form

\[
\begin{align*}
\frac{\partial h}{\partial p} &= 2p \frac{\partial v^2}{\partial r} + \frac{(p^2 + q^2 + r^4)}{r^2} \frac{\partial v^2}{\partial p}, \\
\frac{\partial h}{\partial q} &= 2q \frac{\partial v^2}{\partial r} + \frac{(p^2 + q^2 + r^4)}{r^2} \frac{\partial v^2}{\partial q}.
\end{align*}
\]

From the compatibility conditions of these equations we obtain that \( h = h(p^2 + q^2), \ v = v(p^2 + q^2, r) \). In the coordinates \( \xi = p^2 + q^2, \ r = r \) the conditions \((89)\) write

\[
\frac{\partial h}{\partial \xi} = \frac{1}{r^2} \left( r \frac{\partial v^2}{\partial r} + 2(\xi + r^4) \frac{\partial v^2}{\partial \xi} \right).
\]

Thus, from \((75)\), the potential function takes the form

\[
V = \frac{1}{2} v^2(\xi, r) \left( \frac{\xi}{r^2} + r^2 \right) - h(\xi),
\]

where \( v = v(\xi, r) \) and \( h = h(\xi) \) are solutions of \((90)\).

We shall look for the solution \( h = h(\xi) \) of \((90)\) when the function \( v^2 \) is given by

\[
v^2 = \Psi \left( \frac{\xi}{r^2} - r^2 \right) + \sum_{j=-\infty}^{+\infty} a_j(\xi)r^j.
\]

where we assume that Laurent series converges, and \( \Psi = \Phi(\frac{\xi}{r^2} - r^2) \) is an arbitrary function.

By inserting \( v^2 \) in \((90)\) we obtain

\[
\sum_{j=-\infty}^{+\infty} \left( ja_j + 2\xi \frac{da_j}{d\xi} + 2 \frac{da_{j-4}}{d\xi} \right) r^j = \frac{r^2}{2} \frac{dh}{d\xi}.
\]

We choose the coefficients \( a_j \) satisfying

\[
ja_j + 2\xi \frac{da_j}{d\xi} + 2 \frac{da_{j-4}}{d\xi} = 0 \iff (j - 2)a_j + \frac{d}{d\xi} \left( 2\xi a_j + 2a_{j-4} \right), \text{ for } j \neq 2,
\]

\[
2a_2 + 2\xi \frac{da_2}{d\xi} + 2 \frac{da_{-2}}{d\xi} = \frac{dh}{d\xi} \iff \frac{d}{d\xi} \left( 2\xi a_2 + 2a_{-2} - \frac{h}{2} \right) = 0.
\]

Consequently the potential function \((91)\) takes the form

\[
V = 4 \left( \Psi \left( \frac{\xi}{r^2} - r^2 \right) + \sum_{j=-\infty}^{+\infty} a_j(\xi)r^j \right) \left( \frac{\xi}{r^2} + r^2 \right) - 4\xi a_2 - 4a_{-2} - h_0.
\]

If we change \( p = xz, \ q = yz, \ r = z \) where \( x, y, z \) are the cartesian coordinates, then in these coordinates the kinetic and potential functions take the form

\[
T = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2),
\]

\[
V = 4 \left( \Psi \left( x^2 + y^2 - z^2 \right) + \sum_{j=-\infty}^{+\infty} a_j(z^2(x^2 + y^2))z^j \right) (x^2 + y^2 + z^2)
\]

\[
- 4z^2(x^2 + y^2) - 4a_{-2}(z^2(x^2 + y^2)) - h_0.
\]
respectively. Clearly if \( a_j = 0 \) for \( j \in \mathbb{Z} \), then we get the potential
\[
V = \Psi \left( x^2 + y^2 - z^2 \right) \left( x^2 + y^2 + z^2 \right) - h_0,
\]
also obtained by Joukovski in [14]. On the other hand, if \( \Psi = 0 \), \( a_j = 0 \) for \( j \in \mathbb{Z} \setminus 2 \) and \( 4a_2 = a \), then we obtain the potentials \( V = az^4 - h_0 \) given in [28].

12 Inverse Problem for Constrained Hamiltonian Systems

Constrained Hamiltonian systems arise in many fields, for instance in multi-body dynamics or in molecular dynamics. The theory of such systems was mainly developed by Dirac (see for instance [7]). See general references for constrained dynamics in [30].

The inverse problem for constrained Hamiltonian systems can be stated as follows: for a given submanifold \( M \) of a symplectic manifold \( \mathbb{M} \) we must determine the differential systems having \( \mathcal{M} \) invariant by their flows.

We find the equations of motion of a constrained Hamiltonian system in the following cases:

(i) The given constraints are \( l \) first integrals with \( l \in [\dim \mathbb{M}/2, \dim \mathbb{M}) \). In particular the differential equations obtained solving this inverse problem are Hamiltonian only if the first integrals are in involution.

(ii) The given constraints are \( M < \dim \mathbb{M}/2 \) partial integrals. We deduce the differential equations which can be interpreted as a normal form of the equations of motion of a nonholonomic system with nonlinear constraints with respect to the momenta.

We observe that these two statements of the inverse problem for constrained Hamiltonian systems are new.

Now we consider \( \mathbb{M} \) a \( 2N \)-dimensional smooth manifold with local coordinates \((x, y) = (x_1, \ldots, x_N, y_1, \ldots, y_N)\). Let \( \Omega^2 \) be a closed non-degenerate 2-form, then \((\mathbb{M}, \Omega^2)\) is a symplectic manifold. Let \( H : \mathbb{M} \rightarrow \mathbb{R} \) be a smooth function, and let \( \mathcal{M} \) be a submanifold of \( \mathbb{M} \).

The quaternary \((\mathbb{M}, \Omega^2, \mathcal{M}, H)\) is called a constrained Hamiltonian system (see [2]). We essentially study two inverse problems for the constrained Hamiltonian systems, for the first the submanifolds \( \mathcal{M} \) are obtained fixing the values of the given first integrals, and for the second these submanifolds are defined by the hypersurfaces given by partial integrals.

Now we can formulate the inverse problem for constrained Hamiltonian systems with equations, i.e. we want to determine the vector fields \( W \) with components \((W_1, \ldots, W_{2N})\), where \( W_j = W_j(x, y) \), satisfying that the submanifold \( \mathcal{M} \) is invariant by the flow of the differential system
\[
\begin{align*}
\dot{x}_k &= \{H, x_k\}^* + W_k, \\
\dot{y}_k &= \{H, y_k\}^* + W_{N+k},
\end{align*}
\]
for \( k = 1, \ldots, N \), where
\[
\{H, G\}^* = \sum_{k=1}^{N} \left( \frac{\partial H}{\partial y_k} \frac{\partial G}{\partial x_k} - \frac{\partial H}{\partial x_k} \frac{\partial G}{\partial y_k} \right),
\]
is the Poisson bracket. In this chapter we solve this inverse problem.

We note that if \( W_k = 0 \) for \( k = 1, \ldots, N \), then the Eq. (92) are the standard Hamiltonian equations for a mechanical system which is under the action of a external force with components \( W_{N+1}, \ldots, W_{2N} \).
13 Hamiltonian System with Given First Integrals

We have the following result.

**Theorem 28** Let \((\mathcal{M}, \Omega^2, \mathcal{M}_1, H)\) be a constrained Hamiltonian system and let \(f_j = f_j(x, y)\) for \(j = 1, \ldots, N\) be a given set of independent functions defined in \(\mathcal{M}\).

Assume that

(i) \[
\{f_1, \ldots, f_N, x_1, \ldots, x_N\} \neq 0 \text{ in } \mathcal{M},
\]

then the manifold

\[
\mathcal{M}_1 = \{(x, y) \in \mathcal{M} : f_j(x, y) = c_j \in \mathbb{R} \text{ for } j = 1, \ldots, N\},
\]

where \(c_j\) for \(j = 1, \ldots, N\) are arbitrary constants, is invariant by the flow of the differential system

\[
\dot{x}_k = \{H, x_k\}^*,
\]

\[
\dot{y}_k = \{H, y_k\}^* - \sum_{j=1}^{N} \frac{\{H, f_j\}^* \{f_1, \ldots, f_{j-1}, y_k, f_{j+1}, \ldots, f_N, x_1, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}} \quad (94)
\]

for \(k = 1, \ldots, N\).

(ii) Under the assumptions

\[
\{f_1, \ldots, f_N, x_1, \ldots, x_N\} = 0 \text{ and } \{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\} \neq 0,
\]

the submanifold \(\mathcal{M}_1\) is invariant by the flow of the differential system

\[
\dot{x}_k = \{H, x_k\}^*, \text{ for } k = 1, \ldots, N-1,
\]

\[
\dot{x}_N = \{H, x_N\}^* - \sum_{j=1}^{N} \frac{\{H, f_j\}^* \{f_1, \ldots, f_{j-1}, x_N, f_{j+1}, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}}
\]

\[
= \{H, x_N\}^* + W_N,
\]

\[
\dot{y}_1 = \{H, y_1\}^* + \lambda \{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}
\]

\[
= \{H, y_1\}^* + W_{1+N},
\]

\[
\dot{y}_k = \{H, y_k\}^* - \sum_{j=1}^{N} \frac{\{H, f_j\}^* \{f_1, \ldots, f_{j-1}, y_k, f_{j+1}, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}}
\]

\[
+ \lambda \{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_k\}
\]

\[
= \{H, y_k\}^* + W_{k+N}, \text{ for } k = 2, \ldots, N,
\]

\[(96)\]

where \(\lambda = \lambda(x, y)\) is an arbitrary function.

We observe that the solution (94) of the inverse problem in constrained Hamiltonian systems for the case when the first integrals are pairwise in involution, and \(H = H(f_1, \ldots, f_N)\) becomes into the Hamiltonian system \(\dot{x}_k = \{H, x_k\}^*\). \(\dot{y}_k = \{H, y_k\}^*\). Additionally system (96), when the first integrals are pairwise in involution satisfying (95) and \(H = H(f_1, \ldots, f_N)\), becomes into the differential system

\[
\dot{x}_k = \{H, x_k\}^*, \quad \dot{y}_k = \{H, y_k\}^* + \lambda \{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_k\},
\]
for \( k = 1, \ldots, N \). These equations are the equations of motion of the mechanical system with the constraints \( \{ f_1, \ldots, f_N, x_1, \ldots, x_N \} = 0 \).

**Proof of Theorem 28.** Under the assumptions of Corollary 8 taking the \( N \) of the corollary as \( 2N \), introducing the notations \( y_j = x_{N+j} \), and choosing \( g_{N+j} = x_j \) for \( j = 1, \ldots, N \), we obtain that the differential systems (13) takes the form

\[
\dot{x}_j = \lambda_{N+j}, \quad \dot{y}_j = \sum_{k=1}^{N} \lambda_{N+k} \frac{\{ f_1, \ldots, f_N, x_1, \ldots, x_k-1, y_j, x_k+1, \ldots, x_N \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N \}},
\]

(97)

for \( j = 1, 2, \ldots, N \). These equations are the most general differential equations which admits \( N \) independent first integrals and satisfy the condition \( \{ f_1, \ldots, f_N, x_1, \ldots, x_N \} \neq 0 \).

The Proof of Theorem 28 is obtained by choosing the arbitrary functions \( \lambda_{N+j} \) as follows \( \lambda_{N+j} = (H, x_j)^* \), where \( H \) is the Hamiltonian function for \( j = 1, \ldots, N \). From the identity (xi) of the Nambu bracket with \( G = y_k \), \( f_{N+j} = x_j \) for \( j = 1, \ldots, N \), we obtain that differential system (97) can be rewritten as

\[
\begin{align*}
\dot{x}_j &= \{ H, x_j \}^*, \\
\dot{y}_j &= \sum_{k=1}^{N} \{ (H, x_k)^* \} \frac{\{ f_1, \ldots, f_N, x_1, \ldots, x_k-1, y_j, x_k+1, \ldots, x_N \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N \}} \\
&= \{ H, y_j \}^* - \sum_{k=1}^{N} \{ (H, x_k)^* \} \frac{\{ f_1, \ldots, f_{k-1}, y_j, f_k+1, \ldots, f_N, x_1, \ldots, x_N \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N \}} \\
&\quad + \sum_{k=1}^{N} W_j \frac{\{ f_1, \ldots, f_N, x_1, \ldots, x_k-1, y_j, x_k+1, \ldots, x_N \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N \}}.
\end{align*}
\]

Clearly if the first integrals are in involution and \( W_j = 0 \), then we obtain that the Hamiltonian system with Hamiltonian \( H = H(f_1, \ldots, f_N) \) is integrable by quadratures.

Now we shall prove the Eq. (96). Since \( \{ f_1, \ldots, f_N, x_1, \ldots, x_N \} = 0 \) and \( \{ f_1, \ldots, f_N, x_1, \ldots, x_N-1, y_1 \} \neq 0 \). Taking \( W_j = 0 \) for \( j = 1, \ldots, N-1 \) and \( \lambda_{N+j} = \frac{\partial H}{\partial y_j} = (H, x_j)^* \), for \( j = 1, \ldots, N-1 \), where \( H \) is the Hamiltonian function and in view of the identity (xi) with \( G = x_N \), \( f_{N+j} = x_j \) for \( j = 1, \ldots, N-1 \), \( f_{2N} = y_1 \), and \( G = y_j \), \( f_{N+j} = x_j \) for \( j = 1, \ldots, N-1 \), \( f_{2N} = y_1 \), we obtain that differential system (97) can be rewritten as

\[
\begin{align*}
\dot{x}_j &= \{ H, x_j \}^*, \quad \text{for } j = 1, \ldots, N-1, \\
\dot{x}_N &= \sum_{k=1}^{N-1} \{ (H, x_k)^* \} \frac{\{ f_1, \ldots, f_N, x_1, \ldots, x_k-1, x_N, x_k+1, \ldots, y_1 \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N-1, y_1 \}} \\
&\quad + \lambda_{2N} \frac{\{ f_1, \ldots, f_N, x_1, \ldots, x_N-1, x_N \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N-1, y_1 \}} \\
&= \{ H, x_N \}^* - \sum_{k=1}^{N} \{ (H, f_j)^* \} \frac{\{ f_1, \ldots, f_{k-1}, x_N, f_k+1, \ldots, f_N, x_1, \ldots, y_1 \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N-1, y_1 \}} \\
&\quad + (\lambda_{2N} - \{ H, y_1 \}^*) \frac{\{ f_1, \ldots, f_N, x_1, \ldots, x_N-1, x_N \}}{\{ f_1, \ldots, f_N, x_1, \ldots, x_N-1, y_1 \}}.
\end{align*}
\]
\[ \dot{y}_1 = \lambda_{2N}, \]
\[ \dot{y}_j = \sum_{k=1}^{N-1} \{H, x_k\} \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}} + \lambda_{2N} \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_j\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}} \]
\[ = \{H, y_j\} - \sum_{k=1}^{N} \{H, f_k\} \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}} + (\lambda_{2N} - \{H, y_1\}) \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_k\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}}. \]

Therefore by choosing \( \lambda_{2N} = \{H, y_1\} + \lambda \{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\} \), we get the differential system (96).

In view of the identity (vii) with \( G = f_j \) from (96) we obtain the relations
\[ f_k = \sum_{j=1}^{N} \frac{\partial f_k}{\partial y_j} \{f_1, \ldots, f_N, x_1, \ldots, y, \ldots, y_j, \ldots, x_N\} = 0. \]

Differential system (96) when \( \{H, f_j\} = 0 \) for \( j = 1, \ldots, N \) is the standard Hamiltonian system with the constraints \( \{f_1, \ldots, f_N, x_1, \ldots, x_N\} = 0. \)

**Example 29** Neumann–Moser integrable systems. We shall illustrate these theorems in the Neumann–Moser’s integrable system. Now we study the case when we have \( N \) independent involutive first integrals of the form
\[ f_v = (Ax_v + By_v)^2 + C \sum_{j \neq v} \frac{(x_v y_j - x_j y_v)^2}{a_v - a_j}, \quad (98) \]
for \( v = 1, \ldots, N \), where \( A, B \) and \( C \) are constants such that \( C(A^2 + B^2) \neq 0 \). Thus we study the constrained Hamiltonian system \( (\mathbb{R}^{2N}, \Omega^2, \mathbb{M}, H) \).

The case when \( A = 0, B = 1, C = 1 \) and \( A = 1, B = 0, C = 1 \) was studied in particular in [18]. The case when \( AB \neq 0 \) was introduced in [27]). In particular if \( C = (A + B)^2 \) then from (98) we obtain that \( f_v = A^2 f_v^{(1)} + B^2 f_v^{(2)} + 2AB f_v^{(3)} \) where
\[ f_v^{(1)} = x_v^2 + \sum_{j \neq v} \frac{(x_v y_j - x_j y_v)^2}{a_v - a_j}, \]
\[ f_v^{(2)} = y_v^2 + \sum_{j \neq v} \frac{(x_v y_j - x_j y_v)^2}{a_v - a_j}, \]
\[ f_v^{(3)} = x_v y_v + \sum_{j \neq v} \frac{(x_v y_j - x_j y_v)^2}{a_v - a_j}. \]

It is easy to show that the following relations hold \( \{f_k^{(a)}, f_m^{(a)}\}^* = 0 \), for \( a = 1, 2, 3, \ldots, m \), and \( k = 1, \ldots, N \), i.e. are in involution.

After some computations we obtain that \( \{f_1, \ldots, f_N, x_1, \ldots, x_N\} \neq 0 \) if \( B \neq 0 \). Then taking in (94) \( H = H(f_1, \ldots, f_N) \), and \( W_j = 0 \) for \( j = 1, \ldots, N \) we obtain a completely integrable Hamiltonian system \( \dot{x}_j = \{H, x_j\}^*, \quad \dot{y}_j = \{H, y_j\}^* \).
If \( B = 0 \) then \( \{f_1, \ldots, f_N, x_1, \ldots, x_N\} = 0 \). So taking in (96) \( H = H(f_1, \ldots, f_N), W_j = 0 \) for \( j = 1, \ldots, N \) and in view of the relations

\[
\{f_1, \ldots, f_N, x_1, \ldots, x_N - 1, x_j\} = \varrho(x) x_j
\]

for \( j = 1, \ldots, N \), for convenient function \( \varrho = \varrho(x) \), we obtain the differential system

\[
\dot{x} = \{H, x\}^*, \quad \dot{y} = \{H, y\}^* + \tilde{x},
\]

(99)

where \( \tilde{\lambda} = \varrho \lambda \). In particular for \( N = 3 \) we deduced that

\[
\{f_1, f_2, f_3, x_1, x_2, x_3\} = 0, \quad \{f_1, f_2, f_3, x_1, x_2, x_1\} = \frac{K}{\Delta} x_3 x_1,
\]

\[
\{f_1, f_2, f_3, x_1, x_2, x_2\} = \frac{K}{\Delta} x_3 x_2, \quad \{f_1, f_2, f_3, x_1, x_2, x_3\} = \frac{K}{\Delta} x_3 x_3,
\]

where \( \Delta = (a_1 - a_2)(a_2 - a_3)(a_1 - a_3) \), and \( K \) is a convenient function. Thus the differential system (99) with \( \varrho = \frac{K x_3}{\Delta} \) describes the behavior of the particle with Hamiltonian

\[
H = H(f_1, f_2, f_3)
\]

and constrained to move on the sphere \( x_1^2 + x_2^2 + x_3^2 = 1 \). If we take

\[
H = \frac{1}{2} (a_1 f_1 + a_2 f_2 + a_3 f_3) = \frac{1}{2} (||x||^2 ||y||^2 - \langle x, y \rangle^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2)
\]

and \( \lambda = \Psi(x_1^2 + x_2^2 + x_3^2) \), then from Eq. (99) we deduce that the equations of motion of a particle on a 3-dimensional sphere, with an anisotropic harmonic potential (Neumann’s problem). This system is one of the best understood integrable systems of classical mechanics.

**Theorem 30** Let \((\mathbb{M}, \Omega^2, \tilde{\mathcal{M}}_1, H)\) be a constrained Hamiltonian system and let \( f_j = f_j(x, y) \) for \( j = 1, \ldots, N + r \), with \( r < N \) be a given set of independent functions defined in \( \mathbb{M} \) and such that \( \{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\} \neq 0 \). Then the manifold

\[
\tilde{\mathcal{M}}_1 = \{ (x, y) \in \mathbb{M} : f_j(x, y) = c_j \in \mathbb{R} \text{ for } j = 1, \ldots, N + r \},
\]

where \( c_j \) are arbitrary constants, is invariant by the flow of the differential system

\[
\begin{align*}
\dot{x}_k &= \{H, x_k\}^*, \\
\dot{y}_n &= \{H, y_n\}^* - \sum_{j=1}^{N+r} \frac{\{H, f_j\}^* \{f_1, \ldots, f_{j-1}, x_n, f_{j+1}, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}, \\
\dot{y}_m &= \{H, y_m\}^* - \sum_{j=1}^{N+r} \frac{\{H, f_j\}^* \{f_1, \ldots, f_{j-1}, y_m, f_{j+1}, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}, \\
\end{align*}
\]

(100)

for \( k = 1, \ldots, N - r, n = N - r + 1, \ldots, N, m = 1, \ldots, N \).
Proof of Theorem 30. The differential systems (13) under the assumptions of Theorem 30 takes the form
\[
\begin{align*}
\dot{x}_j &= \lambda_{N+j}, \quad \text{for } j = 1, 2, \ldots, N - r, \\
\dot{x}_n &= \sum_{k=N+1}^{2N} \lambda_k \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_{k-1}, x_n, x_{k+1}, \ldots, x_N \right\} / \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_N \right\}, \\
&\quad \text{for } n = N - r + 1, \ldots, N, \\
\dot{y}_m &= \sum_{k=N+1}^{2N} \lambda_k \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_{k-1}, y_m, x_{k+1}, \ldots, x_N \right\} / \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_N \right\}, \\
&\quad \text{for } m = 1, 2, \ldots, N.
\end{align*}
\]

These equations are the most general differential equations which admits \(N + r\) first integrals which satisfies the condition \(\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N - r} \} \neq 0\).

By choosing in (92) the arbitrary functions \(W_j = 0\) and \(\lambda_{N+j} = \{ H, x_j \}^*\) for \(j = 1, \ldots, N - r\), where \(H\) is the Hamiltonian and by using the identity (xi) of the Nambu bracket with \(G = x_k, \quad f_{N+r+j} = x_j\) for \(j = 1, \ldots, N - r\), and \(G = y_k, \quad f_{N+r+j} = x_j\) for \(j = 1, \ldots, N - r\), we obtain that differential system (101) can be rewritten as
\[
\begin{align*}
\dot{x}_j &= \{ H, x_j \}^* \quad \text{for } j = 1, 2, \ldots, N - r, \\
\dot{x}_k &= \sum_{j=1}^{N-r} \{ H, x_j \}^* \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_{j-1}, x_k, x_{j+1}, \ldots, x_{N-r} \right\} / \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r} \right\}, \\
&\quad \text{for } k = N - r + 1, \ldots, N, \\
\dot{y}_j &= \sum_{k=1}^{N-r} \{ H, x_k \}^* \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_k, y_j, x_{k+1}, \ldots, x_N \right\} / \left\{ f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r} \right\}, \\
&\quad \text{for } j = 1, 2, \ldots, N. \quad \text{Hence we get the differential system (100).}
\end{align*}
\]

Remark 31 With respect to Theorems 28 and 30 we observe the following. If we assume that \(\{ f_1, \ldots, f_N, x_1, \ldots, x_N \} \neq 0\), in \(\mathbb{M}\), and \(H = H(f_1, \ldots, f_N)\) then the system of equations \(f_j(x, y) = c_j, \quad \text{for } j = 1, \ldots, N\) can be solved locally with respect to \(y\), (momenta) i.e. \(y_j = u_j(x, e)\), \quad \text{for } j = 1, \ldots, N\) where \(e = (c_1, \ldots, c_N)\). If the given first integrals are pairwise in involution, i.e. \(\{ f_j, f_k \} = 0\), then \(\sum_{j=1}^{N} u_j(x, e)dx_j = dS(x)\). Consequently from the Liouville’s theorem:

**Theorem 32** If a Hamiltonian system has \(N\) independent first integrals in involution, which can be solved with respect to the momenta, then its motion can be obtained with quadratures, that is, the equation of motion can be solved simply by evaluating integrals.

In general the given set of first integrals is not necessarily in involution. The solution of the inverse problem in constrained Hamiltonian system shows that in this case the differential
equations which have as invariant the submanifold $\mathcal{M}_1$ is in general not Hamiltonian. The origin of the theory on noncommutative integration is the Nekhoroshev’s Theorem (see [21]). The following result holds (see [15]).

**Theorem 33** If a Hamiltonian system with $N$ degrees of freedom has $N+r$ independent first integrals $f_j$ for $j = 1, \ldots, N+r$, such that the $f_1, \ldots, f_{N-r}$ first integrals are in involution with all integrals $f_1, \ldots, f_{N+r}$. Then the Hamiltonian system is integrable by quadratures.

If $f_1, f_2, \ldots, f_{N-r}$ are the first integrals which are in involution with all the first integrals and $H = H(f_1, f_2, \ldots, f_{N-r})$, then the differential system (100) is Hamiltonian and is integrable by quadratures.

### 14 Hamiltonian System with Given Partial Integrals

In this section we prove the following theorem.

**Theorem 34** Let $(\mathbb{M}, \Omega^2, \mathcal{M}_2, H)$ be a constrained Hamiltonian system and let $g_j : \mathbb{M} \to \mathbb{R}$ for $j = 1, \ldots, M < N$ be given independent functions in $\mathbb{M}$, where

$$
\mathcal{M}_2 = \{ (x, y) \in \mathbb{M} : g_j(x, y) = 0 \text{ for } j = 1, \ldots, M < N \}.
$$

We choose the arbitrary functions $g_m$ for $m = M + 1, \ldots, 2N$ in such a way that the determinant \( \{ g_1, \ldots, g_M, g_{M+1}, \ldots, g_{2N} \} \neq 0 \) in $\mathbb{M}$.

We shall study only the case when $\{ g_1, \ldots, g_M, g_{M+1}, \ldots, g_N, x_1, \ldots, x_N \} \neq 0$. Then the submanifold $\mathcal{M}_2$ is an invariant manifold by the flow of the differential system

\[
\dot{x}_k = [H, x_k]^*, \quad \dot{y}_k = [H, y_k]^* \quad + \sum_{j=1}^{M} \frac{\Phi_j - [H, g_j]^*}{\{g_1, \ldots, g_{N}, x_1, \ldots, x_N\}} \{g_1, \ldots, g_{N}, x_1, \ldots, x_N\} \\
+ \sum_{j=M+1}^{N} \frac{\lambda_j - [H, g_j]^*}{\{g_1, \ldots, g_{2N+1}, g_{j-1}, y_k, g_j, g_{j+1}, \ldots, g_N, x_1, \ldots, x_N\}} \{g_1, \ldots, g_{2N+1}, g_{j-1}, y_k, g_j, g_{j+1}, \ldots, g_N, x_1, \ldots, x_N\} \\
= \{H, y_k\}^* + W_{k+N},
\]

(102)

for $k = 1, \ldots, N$, where $\lambda_j$ for $j = M + 1, \ldots, N$, and $\Phi_j$ are arbitrary functions satisfying $\Phi_j|_{g_j=0} = 0$ for $j = 1, \ldots, M$.

Equation (102) on the submanifold $\mathcal{M}_2$ when the arbitrary functions $\lambda_k$ are $\lambda_k = \{H, g_k\}^*$ become

\[
\dot{x}_j = \{H, x_j\}^*, \quad \dot{y}_j = \{H, y_j\}^* - \sum_{k=1}^{M} \frac{\{H, g_k\}^* \{g_1, \ldots, g_{k-1}, y_j, g_{k+1}, \ldots, g_{N}, x_1, \ldots, x_N\}}{\{g_1, \ldots, g_{N}, x_1, \ldots, x_N\}},
\]

(103)

for $j = 1, \ldots, N$. This system can be interpreted as the equations of motion of the constrained mechanical system with Hamiltonian $H$ under the action of the external forces with
Proof of Theorem 34. Analogously to the Proof of Theorem 12 from formula (32), denoting by \( \{ \tilde{\lambda}, x_j \}^* \) for \( j = 1, \ldots, N \), where \( \tilde{\lambda} \) is the Hamiltonian function, from identity (xi) with \( f_j = g_j \), \( f_{N+j} = x_j \), \( G = y_j \), for \( j = 1, \ldots, N \), we obtain the differential system (102). This completes the proof of the theorem.

Example 35 Gantmacher system We shall illustrate Theorem 34 in the nonholonomic system study example 23. Thus we shall study the constrained Hamiltonian system \( \mathcal{M}_2 = \{ g_1 = x_1 y_1 + x_2 y_2 = 0, \ g_2 = x_1 y_3 - x_2 y_4 = 0 \} \). We choose the arbitrary functions \( g_j \) for \( j = 3, \ldots, 8 \) as follows

\[
g_3 = x_1 y_2 - x_2 y_1, \quad g_4 = x_2 y_3 + x_1 y_4, \quad g_{j+4} = x_j, \quad \text{for} \quad j = 1, 2, 3, 4.
\]

We apply Theorem 34. In view of the relations

\[
\begin{align*}
\{g_1, g_2, g_3, g_4, x_1, \ldots, x_4\} &= -(x_1^2 + x_2^2)^2, \\
\{y_1, g_2, g_3, g_4, x_1, \ldots, x_4\} &= -x_1(x_1^2 + x_2^2), \\
\{g_1, y_1, g_3, g_4, x_1, \ldots, x_4\} &= 0, \\
\{g_1, g_2, y_1, g_4, x_1, \ldots, x_4\} &= x_2(x_1^2 + x_2^2), \\
\{g_1, g_2, g_3, y_1, x_1, \ldots, x_4\} &= 0, \\
\{g_1, g_2, g_3, g_4, y_1, x_2, x_3, x_4\} &= (x_1 y_1 - x_2 y_2)(x_1^2 + x_2^2), \\
\{g_1, g_2, g_3, g_4, x_1, y_1, x_3, x_4\} &= (x_1 y_2 + x_2 y_1)(x_1^2 + x_2^2), \\
\{g_1, g_2, g_3, g_4, x_1, x_2, y_1, x_4\} &= 0, \\
\{g_1, g_2, g_3, g_4, x_1, x_2, x_3, y_4\} &= 0.
\end{align*}
\]

In a similar form we can obtain the remain determinant. Thus system (102) takes the form

\[
\begin{align*}
\dot{x}_j &= \{\tilde{H}, x_j\}^*, \quad \text{for} \quad j = 1, 2, 3, 4 \\
\dot{y}_1 &= \{\tilde{H}, y_1\}^* - \frac{x_1 \{H, g_1\}^*}{x_1^2 + x_2^2} - (\lambda_3 - \{H, g_3\}^*) \frac{x_2}{x_1^2 + x_2^2}, \\
\dot{y}_2 &= \{\tilde{H}, y_2\}^* - \frac{x_2 \{H, g_1\}^*}{x_1^2 + x_2^2} + (\lambda_3 - \{H, g_3\}^*) \frac{x_1}{x_1^2 + x_2^2}, \\
\dot{y}_3 &= \{\tilde{H}, y_3\}^* - \frac{x_1 \{H, g_2\}^*}{x_1^2 + x_2^2} + (\lambda_4 - \{H, g_4\}^*) \frac{x_2}{x_1^2 + x_2^2}, \\
\dot{y}_4 &= \{\tilde{H}, y_4\}^* + \frac{x_2 \{H, g_2\}^*}{x_1^2 + x_2^2} + (\lambda_4 - \{H, g_4\}^*) \frac{x_1}{x_1^2 + x_2^2}.
\end{align*}
\]

In particular, taking \( \lambda_3 = \{H, g_3\}^*, \lambda_4 = \{H, g_4\}^* \) and \( H = \frac{1}{2} (y_1^2 + y_2^2 + y_3^2 + y_4^2) - gx_3 \), thus in view of (61) we obtain

\[
\{H, g_1\}^* = y_1^2 + y_2^2 = -\mu_1(x_1^2 + x_2^2), \quad \{H, g_2\}^* = y_1 y_3 - y_2 y_4 + gx_1 = -\mu_2(x_1^2 + x_2^2).
\]
Consequently differential Eq. (103) take the form

\[
\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2, \quad \dot{x}_3 = y_3, \quad \dot{x}_4 = y_4
\]
\[
\dot{y}_1 = x_1 \mu_1, \quad \dot{y}_2 = x_2 \mu_1, \quad \dot{y}_3 = -g + x_1 \mu_2, \quad \dot{y}_4 = -x_2 \mu_2,
\]

which coincide with the Hamiltonian form of Eq. (67).

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