Space-Time Uncertainty Principle and Conformal Symmetry in D-Particle Dynamics

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Abstract

Motivated by the space-time uncertainty principle, we establish a conformal symmetry in the dynamics of D-particles. The conformal symmetry, combined with the supersymmetric non-renormalization theorem, uniquely determines the classical form of the effective action for a probe D-particle in the background of a heavy D-particle source, previously constructed by Becker-Becker-Polchinski-Tseytlin. Our results strengthen the conjecture proposed by Maldacena on the correspondence, in the case of D-particles, between the supergravity and the supersymmetric Yang-Mills matrix models in the large $N$-limit, the latter being the boundary conformal field theory of the former in the classical D-particle background in the near horizon limit.

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1. Introduction

The dynamics of D-branes at least in the low-energy regime is described by the supersymmetric Yang-Mills theories. The interpretation of the Yang-Mills theories is entirely different from the usual applications to the unified theories of particle interactions. The Higgs fields which emerge from a part of the components of the gauge fields as a result of dimensional reduction are now identified as describing the transverse coordinates of D-branes and the associated open string degrees of freedom. The gauge symmetry is regarded as a symmetry which generalizes the statistics symmetry for ordinary particles in quantum mechanics. Through the correspondence between open and closed string theories, the Yang-Mills theories so interpreted are supposed to describe even the gravity which is necessarily included in the closed string theories. However, it is fair to say that we do not yet have appropriate understanding on the fundamental principles which explain why such Yang-Mills theories describe the gravity.

Recently, based on studies of D-brane interactions, a remarkable conjecture has emerged relating the Yang-Mills theory in the large $N$-limit and supergravity. In particular, the $N = 4$ super symmetric Yang-Mills theory in $3+1$ dimensions in a strong-coupling regime should be described by the type IIB supergravity in the anti-de Sitter background $\text{AdS}_5 \times \text{S}^5$ corresponding to the near horizon geometry of the $p=3$ extremal black hole solution. A more concrete formulation of this conjecture has been given in \cite{3,4}. Basically, the effective Yang-Mills theory of D3-branes is identified as the boundary conformal field theory of the 4+1 dimensional anti-de Sitter space-time. The $\text{SO}(4,2)$ symmetry of the latter turns into the conformal symmetry of the Yang-Mills theory defined at the 3+1 dimensional asymptotic boundary of the anti-de Sitter space-time. The existence of such a boundary field theory is natural, since all legitimate observables in general relativistic quantum theory must be defined asymptotically.

On the other hand, one of the most promising proposals related to the D-brane dynamics is the so-called Matrix theory, which interprets the 0+1 dimensional N=16 super symmetric Yang-Mills theory as a non-perturbative formulation of the M-theory, assuming the D-particles as the fundamental degrees of freedom behind the type IIA strings. A justification for such a description comes from the assumption of the infinite Lorentz
boost which is expected to ensure the small velocity of D-particles along the transverse directions and the decoupling of the higher open string modes [6].

In the case of Maldacena's conjecture, the decoupling of the higher open string modes is ensured by taking the $\alpha' \to 0$ limit with the energy $U \equiv \frac{r}{\alpha'}$ of the open strings stretched between the D3-branes kept fixed, where $r$ is the transverse distance between the D3-branes. The large $N$ limit with large fixed $g_s N$ is assumed in order to have small curvature $\sim \frac{1}{g_{YM}^2 N}$ in the string unit with small string coupling constant $g_s \propto g_{YM}^2$.

Then a natural question arises: Is it possible to apply the conjecture to D-particle quantum mechanics? In view of the role played by the conformal symmetry in the case of D3-branes, one of the crucial questions would then be whether there exists any symmetry which takes place of the conformal symmetry of the D3 case. The purpose of the present paper is to give an affirmative answer to the last question. We will argue that there indeed exists in both sides, the type IIA supergravity and 0+1 dimensional Yang-Mills matrix model, some extended conformal symmetry. This suggests that the D-particle quantum mechanics may also be interpreted as a boundary “conformal” field theory in the background of the classical D-particle solution.

Since at its root our work is strongly motivated by the space-time uncertainty principle [7, 8], which has been proposed by one of the present authors as a possible principle characterizing the short-distance space-time structure of the string theory, let us start in the next section by explaining the connection between the space-time uncertainty relation and the conformal symmetry. For a recent review of the space-time uncertainty principle including its application to D-branes, we refer the reader to [9]. We can regard the conformal symmetry as a mathematical structure characterizing the space-time uncertainty principle.
2. Space-time uncertainty principle and conformal symmetries

The conformal symmetry of the effective Yang-Mills theory of the D3-brane comes basically from the symmetry under the scale transformation

\[ X_i(x_a) \rightarrow X'_i(x'_a) = \lambda X_i(x_a), \quad (2.1) \]

\[ x_a \rightarrow x'_a = \lambda^{-1} x_a, \quad (2.2) \]

where \( X_i(x_a) (i = 1, \ldots, 6) \) are the Higgs fields representing the space-time coordinates and open string fields which are transverse to the D3-branes and \( x_a (a = 0, 1, 2, 3) \) are the world-volume coordinates including the time \( x_0 = t \). The above scalings which are opposite\(^2\) for the transverse and longitudinal (including time) directions can be regarded as a signature of the space-time uncertainty relation

\[ \Delta T \Delta X \sim a'. \quad (2.3) \]

Here, \( \Delta T \) and \( \Delta X \) are qualitative measures for the effective longitudinal (i.e. along the world volume) and transverse space-time distances, respectively. Note that because of the choice of the static gauge, the longitudinal distance is directly related to the distance in the target space-time. The uncertainty relation for D-branes in general says that the long distance phenomena in the (transverse) target space is dual to the short distance phenomena in the world volume and viceversa. From the view point of the space-time uncertainty relation, the transformations (2.1) and (2.2) are the simplest transformation which leaves the uncertainty relation invariant, and it is natural to interpret the full (super) conformal symmetry as constituting a set of more general symmetry transformations which keep the uncertainty relation invariant and hence characterize the possible mathematical structure behind the uncertainty principle.

That the conformal symmetry and the space-time uncertainty relation are related in this way is remarkable, if one remembers that the original proposal of the space-time uncertainty relation also came out as a simple reinterpretation, in terms of the space-time language, of the world sheet conformal invariance, or more precisely the \( s-t \) duality.

\[^2\]In the usual field theory language, this is nothing but the trivial fact that the dimensionality of the field is negative with respect to the length dimension of the base space.
which includes the open-closed string dualities, in perturbative string theories. These two conformal symmetries are in a sense dual to each other, and may be regarded as yet another example of the dual roles of various symmetries in the world sheet and target space-time in string theory. In contrast to the world sheet conformal symmetry, the conformal symmetry for D-branes involves the target coordinates explicitly, reflecting the fact that the s-t duality operates for the open fundamental strings stretched between the D-branes. In the case of fundamental strings, the uncertainty relation is valid for the effective space-time distances measured along the time ($\Delta T$) and spatial ($\Delta X$) directions on the world sheet.

Once the conformal symmetry is related to the space-time uncertainty principle, we should expect that the symmetry is not restricted to D3-brane. We can indeed consider the transformation of the same nature as above for D-particles in the form

$$X_i(t) \rightarrow X'_i(t') = \lambda X_i(t), \quad t \rightarrow t' = \lambda^{-1} t$$

(2.4)

where now the index $i$ runs from $i = 1$ to $i = 9$. Note that for D-particles, the “longitudinal” distance $\Delta T$ literally refers to the time as in ref. [11]. In this case, the effective super Yang-Mills theory is invariant provided we simultaneously make the transformation of the string coupling constant as

$$g_s \rightarrow g'_s = \lambda^3 g_s.$$  

(2.5)

This scaling property, which is a special case of the scaling $g_s \rightarrow g'_s = \lambda^{3-p} g_s$ for general D$p$-branes, is equivalent to the fact that the characteristic spatial and temporal length scales of D-particle dynamics are fixed by the string coupling as $\Delta X \sim g^{1/3}_s \sqrt{\alpha'}$ [10] and $\Delta T \sim g^{-1/3}_s \sqrt{\alpha'}$, respectively. Given that the mass of a D-particle is proportional to $1/g_s \sqrt{\alpha'}$, this is a direct consequence of the space-time uncertainty relation as shown in [11]. Since, as is well known, the string coupling constant $g_s$ can in principle be treated as a dynamical variable corresponding to the vacuum expectation value of the dilaton, it is reasonable to regard this property as a symmetry in the D-particle dynamics. Alternatively, we can regard the string coupling as a part of the background fields, and the symmetry requires to change the background and the manifest dynamical degrees of freedom simultaneously. In principle, there should be some mechanism which allows us
to eliminate the string coupling in terms of other genuine dynamical degrees of freedom. In the present formulation of the Matrix theory, this is not however manifest.

We also note that the D3-brane \( (p = 3) \) is special in that the string coupling is inert under the scale transformation. This means that the dynamics of D3-brane involves all the scales in both the target and world volume, keeping the dual nature of them. For other “dilatonic” branes, on the other hand, the effective scales are fixed by the vacuum expectation values of the dilaton. The space-time uncertainty relation itself is valid, however, irrespective of such specialities for general D-branes.

Motivated by these considerations, we are led to ask ourselves whether it is possible to generalize the transformations (2.4) and (2.5) into the 0+1 dimensional analogue of the full conformal symmetry of the 3+1 dimensional super Yang-Mills theory.

The action of the supersymmetric Yang-Mills matrix quantum mechanics is

\[
S = \int dt \, \text{Tr} \left( \frac{1}{2g_s \ell_s} D_t X_i D_t X_i + i\theta D_t \theta + \frac{1}{4g_s \ell_s^5} [X_i, X_j]^2 - \frac{1}{\ell_s^2} \theta \Gamma_i[\theta, X_i] \right)
\]  

(2.6)

where the covariant derivative is defined by \( D_t X = \frac{\partial}{\partial t} X + [A, X] \) and \( \ell_s \) is the string length constant (\( \ell_s \propto \sqrt{\alpha'} \)). This is obviously invariant under the scale transformations (2.4) and (2.5) provided that the gauge field \( A \) is transformed as \( A(t) \rightarrow \lambda A(t') \) and the time translation \( t \rightarrow t' = t + c, X_i(t) \rightarrow X_i(t'), A(t) \rightarrow A(t'), g_s \rightarrow g'_s = g_s \). The corresponding infinitesimal transformations are

\[
\delta_D X_i = X_i, \ \delta_D A = A, \ \delta_D t = -t, \ \delta_D g_s = 3g_s,
\]

(2.7)

and

\[
\delta_H X_i = 0, \ \delta_H A = 0, \ \delta_H t = 1, \ \delta_H g_s = 0,
\]

(2.8)

respectively. In addition to these trivial symmetries, the action is also invariant (up to a total derivative as usual) under the special conformal transformation given by

\[
\delta_K X_i = 2t X_i, \ \delta_K A = 2t A, \ \delta_K t = -t^2, \ \delta_K g_s = 6tg_s.
\]

(2.9)

In all these symmetry transformations, the fermionic variable \( \theta \) is assumed to be invariant (namely, as zero-dimensional scalar with respect to the conformal transformation). Note that in the above expressions for the infinitesimal transformations we have suppressed
the infinitesimal parameters and also that time derivatives of $X_i(t)$ do not appear since we have defined the variations of the field as $\delta X(t) \equiv X'(t') - X(t)$ to compare the transformation rule of the classical solution in type IIA string theory to be discussed in the next section. The transformations (2.7), (2.8) and (2.9) together form an SU(1,1) algebra.

\[ [\delta_D, \delta_H] = \delta_H, \quad [\delta_D, \delta_K] = -\delta_K, \quad [\delta_H, \delta_K] = 2\delta_D. \]  

(2.10)

After the special conformal transformation the string coupling constant is no more constant. As remarked before, this is acceptable since we regard the string coupling as a part of dynamical variables playing the role of a background field.

The operator forms of these generators in the Hamiltonian formalism are given as

\[ \tilde{H} = \text{Tr}\left( \frac{1}{2} \tilde{\Pi}^2 - \frac{g_s}{4}[\tilde{X}_i, \tilde{X}_j]^2 + \sqrt{g_s} \theta \gamma \cdot [\tilde{X}, \theta] \right) + \frac{9}{c} g_s^{7/3} \frac{\partial^2}{\partial g_s^2}, \]  

(2.11)

\[ \tilde{K} = -\text{Tr} \tilde{X}^2 + c g_s^{-1/3}, \]  

(2.12)

\[ \tilde{D} = -\frac{1}{4} \text{Tr} (\tilde{X} \tilde{\Pi} + \tilde{\Pi} \tilde{X}) - i3(g_s \frac{\partial}{\partial g_s}). \]  

(2.13)

The field dependent parts of these operators can easily be inferred from the total derivative terms of the action under the transformations (2.7) ~ (2.9). Here, $c$ is an arbitrary constant and

\[ (g_s \frac{\partial}{\partial g_s}) \equiv \frac{1}{2} g_s^{7/3}(\frac{\partial}{\partial g_s} g_s^{-1/3} + g_s^{-1/3} \frac{\partial}{\partial g_s}). \]  

(2.14)

For notational simplicity, we have set $\ell_s = 1$ and chosen a particular ordering of $g_s$ and $\frac{\partial}{\partial g_s}$. The ordering is however not unique. In the present paper, we will not elaborate on this point, since the ordering is not important for the following discussions.

In constructing the closed operator algebra, we adopted a special frame for the coordinate fields defined by

\[ X_i = \sqrt{g_s} \bar{X}_i, \]  

(2.15)

\[ \Pi_i = \frac{1}{\sqrt{g_s}} \bar{\Pi}_i = \frac{1}{g_s} D_t X_i = \frac{1}{\sqrt{g_s}} D_t \bar{X}_i, \]  

(2.16)

and treated the canonical variables with tilde and $g_s$ as independent variables. Note that in terms of these variables, the space-time scaling transformations (2.4) and (2.5) are

\[ \bar{X} \to \lambda^{-1/2} \bar{X}, \]  

(2.17)
\[ \Pi \rightarrow \lambda^{1/2}\Pi, \quad (2.18) \]
\[ g_s \rightarrow \lambda^3 g_s. \quad (2.19) \]

The algebra is
\[ i[\tilde{D}, \tilde{H}] = \tilde{H}, \quad (2.20) \]
\[ i[\tilde{D}, \tilde{K}] = -\tilde{K}, \quad (2.21) \]
\[ i[\tilde{H}, \tilde{K}] = 2\tilde{D}. \quad (2.22) \]

The transformations of fields induced by these generators can be regarded as the transformations at \( t = 0 \). For example, the special conformal transformation can be expressed as
\[ \delta_K \tilde{X}_i|_{t=0} = i[\tilde{K}, \tilde{X}] = 0, \quad (2.23) \]
\[ \delta_K \Pi_i|_{t=0} = i[\tilde{K}, \Pi], = 2\Pi_i \quad (2.24) \]
\[ \delta g_s|_{t=0} = i[\tilde{K}, g_s] = 0, \quad (2.25) \]
\[ \delta p_g|_{t=0} = i[\tilde{K}, p_g] = -6g_s, \quad (2.26) \]

where
\[ p_g \equiv i \frac{18}{c} g_s^{7/3} \frac{\partial}{\partial g_s}. \quad (2.27) \]

These results coincide with the transformations (2.19) since \( \Pi_i|_{t=0} = i[\tilde{H}, \Pi_i] = \frac{d}{dt}\tilde{X}_i \) and \( p_g|_{t=0} = i[\tilde{H}, g_s] = \frac{d}{dt}g_s \).

One of the curious features of the above operators is that the time translation generator acquires a kinetic term for the string coupling. It is not clear whether this means that the time development of the string coupling should really be taken into account in the dynamics of D-particles. For our present purposes, it is suffice to regard these operators just as the infinitesimal generators for the conformal transformations which are the symmetry of the action (2.6) without the kinetic term for the string coupling as explained in the beginning of the section 2. From this point of view, the time dependence of the transformed string coupling only means that the particular time dependence generated by this operator can be eliminated by making the conformal transformation of the fields \( X_i \).
3. Conformal symmetry of the classical D-particle background in the near horizon limit

As is well known, the classical background solution in the type IIA superstring theory corresponding to the D-particle can be obtained by the dimensional reduction from the 11 dimensional plane wave solution [12]. The solution rewritten in a form which is appropriate for 10 dimensions [13] is

\[
 ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} (dx_{11} - A_0 dt)^2
\]  

(3.1)

with

\[
 A_0 = -\frac{1}{g_s} \left( \frac{1}{1 + \frac{q}{r^7}} - 1 \right),
\]

(3.2)

where as usual our convention is to use \(x_{11}\) for the 10th spatial coordinate, and the indices \(i\) runs only through the transverse directions from 1 to 9 (\(r = \sqrt{x_i^2}\)). The dilaton field is given by

\[
 e^\phi = g_s e^{\tilde{\phi}}
\]

(3.3)

with

\[
 e^{\tilde{\phi}} = (1 + \frac{q}{r^7})^{3/4}
\]

(3.4)

and the charge \(q\) is given by

\[
 q = 60\pi^3 (\alpha')^{7/2} g_s N
\]

(3.5)

for \(N\) coincident D-particles. Here, we have put \(\alpha'\) explicitly. The 10 dimensional string frame metric \(ds_{10}^2\) is

\[
 ds_{10}^2 = -e^{-2\tilde{\phi}/3} dt^2 + e^{2\tilde{\phi}/3} dx_i^2.
\]

(3.6)

Following [2][14], we now consider the near horizon limit \(\alpha' \to 0\), keeping fixed the energy of open strings between the D-particles and the probe measuring the metric and also the Yang-Mills coupling constant,

\[
 U \equiv \frac{r}{\alpha'}, \quad 4\pi^2 g_Y^2 M \equiv \frac{g_s}{\alpha'^3/2}.
\]

(3.7)

In this limit, the 10 dimensional metric, the dilaton, and the \(U(1)\) gauge field \(A_\mu\) which is identified with the Ramond-Ramond 1-form gauge field, becomes

\[
 ds_{10}^2 = \alpha' \left( -\frac{U^{7/2}}{\sqrt{Q}} dt^2 + \frac{\sqrt{Q}}{U^{7/2}} (dU^2 + U^2 d\Omega_8^2) \right),
\]

(3.8)
\( e^\phi = g_s \left( \frac{q}{\alpha'^7 U^7} \right)^{3/4} = g_{YM}^2 \left( \frac{Q}{U^7} \right)^{3/4}, \quad (3.9) \)

\[ A_0 = \frac{\sqrt{\alpha'} U^7}{g_{YM}^2 Q}, \quad (3.10) \]

respectively, where the charge \( Q \) is now redefined as

\[ Q = 60 \pi^3 (\alpha')^{-3/2} g_s N = 240 \pi^5 g_{YM}^2 N. \quad (3.11) \]

Since the solution is static, the spatial components of the RR gauge field are zero.

The 11 dimensional metric in the near horizon limit is invariant under the same scale transformation as before

\[ U \rightarrow \lambda U, \quad (3.12) \]
\[ t \rightarrow \lambda^{-1} t, \quad (3.13) \]
\[ g_s \rightarrow \lambda^3 g_s, \quad (3.14) \]
if we assume that the 11th coordinate \( x_{11} \) is invariant.

Furthermore, the 10 dimensional metric (3.8) and the dilaton (3.9) are invariant under the special coordinate transformation whose infinitesimal form is

\[ \delta_K t = -(t^2 + k g_{YM}^2 U^5), \quad (3.15) \]
\[ \delta_K U = 2tU, \quad (3.16) \]
\[ \delta_K g_s = 6tg_s, \quad (3.17) \]
where \( k \) is a constant independent of the string coupling \( k = 96 \pi^5 N \). Together with the trivial time translation \( \delta_H t = 1, \delta_H U = 0, \delta_H g_s = 0 \) and the scaling \( \delta_D t = -t, \delta_D U = U, \delta_D g_s = 3g_s \), we again have the SU(1,1) algebra (2.10). The gauge field \( A_0 \) transforms as a conformal field of dimension 1.

Since at the asymptotic boundary \( U \rightarrow \infty \), the extra \( U \) dependent part of the special conformal transformation (3.13) vanishes, it is consistent to interpret the transformations (2.7)\( \sim (2.9) \) of the Yang-Mills quantum mechanics as the symmetry corresponding to the transformation at the boundary of the near horizon geometry described by the background fields (3.8)\( \sim (3.10) \). The situation is similar to the 3+1 dimension Yang-Mills theory.
It should be kept in mind that the space-time geometry of the 10 dimensional metric (3.6) is not the AdS$_2 \times S^8$, since the would-be radius of the anti-de Sitter space $\propto (g_{YM}^2 N/U^3)^{1/4}$ is not a constant. The above result, however, strongly indicates that the Yang-Mills matrix model may be interpreted as the boundary conformal field theory of the type IIA supergravity, in essentially the same sense as for the 3+1 dimensional Yang-Mills theory and the ADS$_5 \times S^5$ space-time, provided we properly take into account the dilaton which is now space-time dependent. As discussed in [14], the correspondence is expected to be valid in the region

$$g_{YM}^{2/3} N^{1/7} \ll U \ll g_{YM}^{2/3} N^{1/3} \quad (3.18)$$

where the first and second inequalities come from the weak coupling condition $\epsilon^0 \ll 1$ and the small curvature condition $R \sim (g_s N)^{-1/2} U^{3/2} (\alpha')^{-1/4} \ll (\alpha')^{-1}$, respectively. In terms of the original coordinate $r$ and the string coupling, the condition is

$$\sqrt{\alpha'} g_s^{1/3} N^{1/7} \ll r \ll \sqrt{\alpha'} (g_s N)^{1/3}. \quad (3.19)$$

The near-horizon condition requires $r \ll \sqrt{\alpha'} (g_s N)^{1/7}$. For sufficiently large $N$ and small $g_s$ with large $g_s N$, there is an overlap region for the validity of these conditions. Note that the conditions (3.18) (or (3.19)) are dilatation invariant, while the near horizon condition is not. However, the latter is automatically satisfied in the Maldacena limit.

On the other hand, it should be noted that the range for the validity of the naive loop expansion in the matrix model is $U > g_{YM}^{2/3} N^{1/3}$. Thus, there seems to be no overlap with the region where we can trust the results of naive loop expansion. However, if we are only interested in the “classical” contribution expanded in the special combination $G_{11} N r^{-7}(\frac{dr}{dt})^2$, where $G_{11} \propto g_s^{3/2} \ell_s^9$, $R_{11} = g_s \ell_s$ are the 11 dimensional Newton constant and 11 dimensional compactification radius, respectively, the loop expansion can be compared to the supergravity in the limit of very small velocity $v^2 \ll \frac{(\alpha')^{7/2} r^7}{g_s N}$.

4. The conformal symmetry and D-particle interactions

We next show that the conformal symmetry found above puts a strong constraint on the effective action for D-particle interactions. In the case of D3-brane, it was argued
in [2] that the anti-de Sitter conformal symmetry, combined with a supersymmetric non-renormalization theorem, determines the bosonic part of the Born-Infeld action of a D3-brane in the AdS background. What we will establish in the following is the counterpart of this result for the case of D-particle.

Let us consider the scattering of a probe D-particle in the background of the source system with a large number \( N \) of coincident D-particles. Let the distance between the probe and the source be \( U(t) \) using the same convention as in the last section. If we neglect the possible acceleration dependent terms in the effective action and consider only the motion along the radial direction \( U \) for simplicity, the time translation and the scaling symmetry \((3.12) \sim (3.14)\) restricts the form of the bosonic effective action into

\[
S_{\text{eff}} = \int dt \frac{1}{2g_s} \left( \frac{dU}{dt} \right)^2 F \left( \frac{g_s}{U^3}, \frac{1}{U^4} \left( \frac{dU}{dt} \right)^2 \right) \tag{4.1}
\]

where we have assumed invariance under time inversion too. This form has been already known from the work [17]. Now under the special conformal transformation, the field \( U(t) \) and the string coupling \( g_s \) are transformed as

\[
U(t) \to U'(t) = (1 + 2\epsilon t)U(t'),
\]

\[
t' = t + \epsilon (t^2 + \frac{a}{U^5}),
\]

\[
g_s \to (1 + 6\epsilon t)g_s,
\]

to the first order with respect to the infinitesimal parameter \( \epsilon \). Note that the sign in \((4.3)\) which is due to our use of \( t \), instead of \( t' \), in \( U'(t) \) and \( dt' = (1 + 2\epsilon t - \epsilon \frac{5a}{U^6})dt \). Here we abbreviated as \( a \equiv kg^2_{YM} = 24\pi^3 g_s (\alpha')^{-3/2}N \). The transformation law of the velocity is thus

\[
\left( \frac{dU(t)}{dt} \right) \to (1 + 4\epsilon t) \frac{dU(t')}{dt'} + 2\epsilon U(t') - \epsilon \frac{5a}{U'(t')^6} \left( \frac{dU(t')}{dt'} \right)^2, \tag{4.5}
\]

\[
\left( \frac{dU(t)}{dt} \right)^2 \to (1 + 8\epsilon t) \left( \frac{dU(t')}{dt'} \right)^2 + 4\epsilon U(t') \frac{dU(t')}{dt'} - \epsilon \frac{10a}{U'(t')^6} \left( \frac{dU(t')}{dt'} \right)^3, \tag{4.6}
\]

\[\text{....etc.}\]

This shows that while the combination \( g_s/U^3 \) is invariant under the special conformal transformation, \( \frac{1}{U^4} \left( \frac{dU}{dt} \right)^2 \) is not invariant. In particular, if we expand the function \( F \) in
the double expansion with respect to these two combinations, the velocity expansion must
appear as a power series with respect to the special combination

\[ w \equiv \frac{g_s}{U^7} \left( \frac{dU}{dt} \right)^2 \]  

(4.7)

and then the coefficients of the expansion are uniquely determined by the coefficient of
the first term. The reason for this is that, by the transformation rule (4.5), \( w^n \) generates
\( w^{n-1}U(dU/dt) \sim U^{-7(n-1)+1}(dU/dt)^2(n-1)+1 \) and \( w^{n-1}U^{-6}(dU/dt)^3 \sim U^{-7(n-1)-6} \)
\times (dU/dt)^2(n-1)+3. This shows that the power of the squared velocity can increase only
with the power of \( U^{-7} \), and further allows us to fix the coefficients recursively from lower
to higher \( n \) up to an arbitrary function of \( g_s/U^3 \) which is common to all the coefficients.

Thus, the effective lagrangian must take the form

\[ L = \frac{1}{2g_s} \left( \frac{dU}{dt} \right)^2 f \left( \frac{g_s}{U^3} \right) \sum_{n=0}^{\infty} c_n w^n. \]  

(4.8)

Note that we automatically get a factorized form. However, the supersymmetric non-
renormalization theorem for supersymmetric particle mechanics§ tells us that the coefficient
of the first term \( (dU/dt)^2 \) should be the same as the classical one, namely a constant
independent of \( U \). We can thus set \( f = \text{constant} \). The first few coefficients determined by
applying the transformation (4.5) are

\[ c_0 = 1, \quad c_1 = \frac{15}{16} N \left( \frac{\alpha'}{4\pi^2} \right)^{-3/2}, \quad c_2 = \frac{225}{64} N^2 \left( \frac{\alpha'}{4\pi^2} \right)^{-3}, \ldots. \]  

(4.9)

In this form, as discussed in the end of the last section, the result can be compared with
the loop expansion at least for sufficiently small velocity and it is known that the results
agree up to 2 loops [16]. The effective action can be rewritten in the closed form which
was derived in [17]

\[ S_{\text{eff}} = - \int dt \frac{1}{R} \frac{\sqrt{1 - h_{--}(r)} \left( \frac{dr}{dt} \right)^2 - 1}{h_{--}(r)} \]  

(4.10)

with

\[ h_{--}(r) = \frac{Q(\alpha')^5}{r^4}. \]  

(4.11)

§For a recent discussion on the constraints from the super symmetry, see [15] where the often stated
folklore theorem about the \( v^2 \) term is explicitly proven.
where we have returned to the original coordinate $r$ instead of the energy $U$ of open strings. By comparing with the convention of ref. [17], $h_{--}(r) = \frac{15N}{2R^2M^{1/3}r}$ where $R = g_s\ell_s$ and $M^{-1} = g_s^{1/3}\ell_s$ are the compactification radius along the light-like 11th direction and the 11 dimensional Planck mass, respectively, we have $\ell_s = (2\pi)^{3/7}\sqrt{\alpha'}$ in our convention.

It is easy to directly check that the effective action is indeed invariant up to a total derivative under the special conformal transformation (4.2) $\sim$ (4.4). We also note that although we have only considered the motion along the radial direction for simplicity, the final form of the effective action can be extended to the general case by making the replacement $(dr/dt)^2 \rightarrow v^2 = (dx_i/dt)^2$ because of rotation invariance in the transverse space.

The above result is not trivial, since the 11 dimensional metric itself is not invariant under the conformal transformation. Only after taking the near horizon limit for the 10 dimensional metric, we have the conformal invariance. On the other hand, in the original derivation [17] of the above effective action, the choice of the light-cone frame was essential. We interpret this result as an indication that the Maldacena limit for D-particles in 10 dimensions has the same effect with respect to the conformal symmetry as caused by going to the light-cone frame for the massless plane wave in 11 dimensions. Remember that, if the compactification were performed in the space-like 11th direction instead of the light-like direction, we would have obtained a different action in which $h_{--}$ is replace by $1 + h_{--}$. This may be regarded as evidence for the fact that the Maldacena limit and the discrete light-cone prescription [18] have a common range of validity and can be smoothly connected to each other. This also suggests that the conformal symmetry is present even in the region $U < g_{\text{YM}}^{2/3}N^{1/9}$ of Matrix black holes as discussed in [14] where the plane-wave description of the 11D classical solution can no more be trusted.

5. Concluding remarks

First we note that our results partially explain, as a consequence of the conformal symmetry, why the BFSS matrix model gives the classical effective action of the same form as derived from the supergravity theory in 11 dimensions. However, it should be emphasized that our discussion does not prove that we must have the complete agreement to all
orders, since we have not shown explicitly that the field dependent transformation of the form (3.15) is realized for the probe D-particle in the matrix model. It is possible that, in the matrix model, the field dependent transformation for the probe D-particle might become more complicated due to the $\alpha'$ and quantum-gravity corrections, if the Matrix theory conjecture is correct, and hence the form of the effective action might be subject to further corrections.

It should in principle be possible to derive the field dependent special conformal transformation from the linear transformation law (2.9). For example, we can separate the operator $\tilde{K}$ into the diagonal and off-diagonal contributions.

$$\tilde{K} = \tilde{K}_{\text{diagonal}} + \tilde{K}_{\text{off-diagonal}}$$  \hspace{1cm} (5.12)

where

$$\tilde{K}_{\text{off-diagonal}} = -\frac{1}{g_s} \sum_{a \neq b} |X_{ab}|^2.$$  \hspace{1cm} (5.13)

The one-loop expectation value of this quantity in the background with a large number of coincident source D-particles and a probe D-particle at the distance $U$ is expanded in the form

$$\langle \sum_{a \neq b} |X_{ab}|^2 \rangle \propto \sum_n c_n \frac{g_s}{U} \left( \frac{1}{U^4} \left( \frac{dU}{dt} \right)^2 \right)^n.$$  \hspace{1cm} (5.14)

We can see that the $n = 1$ term is just consistent with the field dependent transformation of the form (4.3) or (4.5). The question is then whether or not the whole off-diagonal contributions can be represented by this term $U^{-5} \left( \frac{dU}{dt} \right)^2$ in the operator formalism. We have no clear answer at present. We emphasize that there are similar questions for the D3-brane, but answer is not known. It would be extremely interesting if there is a systematic way of determining the field dependent symmetry transformations for arbitrary background from the Yang-Mills matrix models alone. It would amount to the proof of the Maldacena conjecture and would be of great help for investigating the dynamics of D-branes in the matrix models beyond the usual loop computations in the semi-classical approximation.

Finally, we mention other remaining problems:

1. Extension to other D$p$-branes ($p \neq 0, 3$): From the discussions in section 2, it should be more or less clear that the idea behind the conformal symmetry, namely
the space-time uncertainty principle, is rather general and the discussions of the present paper are not restricted to D-particles. Most of the results can be extended to other cases without much difficulty, except for the problem of non-renormalizable (perturbatively, at least) world-volume theories for $p \geq 4$.

2. Extension to superconformal symmetry: In view of the result of the last section, we may hope that the supersymmetric extension of the conformal symmetry will put very powerful constraints on the D-particle dynamics. In particular, we may have generalized non-renormalization theorems by using the super-conformal Ward identities.

3. Geometrical interpretation of the conformal transformation: Although the near-horizon geometry is not AdS, our results suggest that there might be some geometrical structure if one properly takes into account the nontrivial behavior of the dilaton and string coupling constant.

4. Comparison of Green functions and spectrum between the supergravity and matrix model: Since the dilaton is not constant, computation will be more complicated than the case of D3-brane and other non-dilatonic branes [1, 2, 3]. Such study is expected to be useful for understanding the non-perturbative structure of the large $N$ dynamics of the Matrix theory.

5. Dynamics of the dilaton and the string coupling: The time translation generator of our conformal transformations includes a kinetic term for the string coupling. Therefore, if we adopt it as the Hamiltonian for time evolution of the system, there is a nontrivial dynamics for the ground-state value of the dilaton which is coupled to the dynamics of D-particles. It is interesting to see whether or not the extended Hamiltonian really captures the dynamics of the dilaton. For example, the dilaton kinetic term might be related to the zero mode of light-cone formulation.

6. Elimination of string coupling constant: Another question related to the dilaton is whether it is possible to eliminate the string-coupling constant from the formalism. For example, in the string field theory formalism, we can indeed eliminate the string
coupling from the action by making a shift for the string field as discussed in [19]. If we can reformulate the matrix theory in a similar way, we would not have to perform the conformal transformation for the string coupling explicitly, and would be able to give a more satisfactory formulation of the conformal symmetry as a realization of the space-time uncertainty principle. This question is closely related to the problem of background (in)dependence of the Yang-Mills matrix theory.

Some of these issues will be discussed in a forthcoming paper [20].

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