POISSON STRUCTURES ON 
COTANGENT BUNDLES

by

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ABSTRACT. We make a study of Poisson structures of $T^*M$ which are graded 
structures when restricted to the fiberwise polynomial algebra, and give ex-
amples. A class of more general graded bivector fields which induce a given 
Poisson structure $w$ on the base manifold $M$ is constructed. In particular, the 
horizontal lifting of a Poisson structure from $M$ to $T^*M$ via connections gives 
such bivector fields and we discuss the conditions for these lifts to be Poisson 
bivector fields and their compatibility with the canonical Poisson structure on 
$T^*M$. Finally, for a 2–form $\omega$ on a Riemannian manifold, we study the con-
ditions for some associated 2–forms of $\omega$ on $T^*M$ to define Poisson structures 
on cotangent bundles.

1. Graded Poisson structures on cotangent bundles

Let $M$ be an $n$–dimensional differentiable manifold and $\pi : T^*M \to M$ its cotangent bundle. If $(x^i)$, $(i = 1, ..., n)$, are local coordinates on $M$, we 
denote by $(p_i)$ the covector coordinates with respect to the cobasis $(dx^i)$. (We 
assume that everything is $C^\infty$ in this paper).

In this section we discuss graded Poisson structures $W$ on the cotangent 
bundle $T^*M$ obtained as lifts of Poisson structures $w$ on the base manifold

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\[ M, \text{ in the sense that the canonical projection } \pi \text{ is a Poisson mapping (see [6]).} \]

Denote by \( S_k(TM) \) the space of \( k \)-contravariant symmetric tensor fields on \( M \) and by \( \odot \) the symmetric tensor product on the algebra \( \bigoplus_{k \geq 0} S_k(TM) \). The spaces of fiberwise homogeneous \( k \)-polynomials

\[ \mathcal{HP}_k(T^*M) := \{ \tilde{Q} = Q^{i_1 \ldots i_k} p_{i_1} \ldots p_{i_k} / \]

\[ Q = Q^{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \odot \ldots \odot \frac{\partial}{\partial x^{i_k}} \in S_k(TM) \} \]

are interesting subspaces of the function space \( C^\infty(T^*M) \), and will play an important role in this paper. (\( := \) denotes a definition).

The map

\[ \sim : (S(TM), \odot) \longrightarrow (P(T^*M), \cdot) , \quad \sim Q := \tilde{Q} , \]

where \( P(T^*M) := \bigoplus_k \mathcal{HP}_k(T^*M) \) is the polynomial algebra and the dot denotes the usual multiplication, is an isomorphism of algebras.

On \( T^*M \) we also have the spaces of (fiberwise) non homogeneous polynomials of degree \( \leq k \)

\[ \mathcal{P}_k(T^*M) := \bigoplus_{h=0}^k \mathcal{HP}_h . \]

For \( k = 1 \), \( \mathcal{A}(T^*M) := \mathcal{P}_1(T^*M) \) is the space of affine functions, having the elements of the form

\[ a(x,p) = f(x) + m(X) , \]

where \( f \in C^\infty(M) \), \( X \in \chi(M) \) (the space of vector fields on \( M \)) and \( m(X) := \sim X \) is the momentum of \( X \). \( m(X) \) is \( X \) regarded as a function on \( T^*M \).

The elements of the space \( \mathcal{P}_2(T^*M) \) of non-homogeneous quadratic polynomials are

\[ t(x,p) = f(x) + m(X) + s(Q) , \]

where \( Q = Q^{ij}(\partial/\partial x^i) \odot (\partial/\partial x^j) \) is a symmetric contravariant tensor field on \( M \) and \( s(Q) := \sim Q \).

Hereafter, by a polynomial on \( T^*M \) we always mean a fiberwise polynomial. Also, we write \( f \) for both \( f \) on \( M \), and \( f \circ \pi \) on \( T^*M \).
DEFINITION 1.1. A Poisson structure $W$ on $T^*M$ is called polynomially graded if $\forall Q, R \in \mathcal{P}(T^*M)$

(1.3) \[ Q \in \mathcal{P}_h, \ R \in \mathcal{P}_k \implies \{Q, R\}_W \in \mathcal{P}_{h+k}. \]

PROPOSITION 1.2. A polynomially graded Poisson structure $W$ on $T^*M$ induces a Poisson structure $w$ on the base manifold $M$, such that the projection $\pi : (T^*M, W) \rightarrow (M, w)$ is a Poisson mapping.

Proof. Any function $f$ on $M$ is a polynomial $(f \circ \pi) \in \mathcal{P}_0(T^*M)$. By (1.3), $\forall f, g \in C^\infty(M)$, $\{f \circ \pi, g \circ \pi\}_W \in C^\infty(M)$ and

(1.4) \[ \{f, g\}_w := \{f \circ \pi, g \circ \pi\}_W, \]

defines a Poisson structure $w$ on $M$. \[ \square \]

Hereafter, the bracket $\{\ , \ \}_w$ will be denoted simply by $\{ \ , \ \}$.

If the local coordinate expression of the Poisson structure $w$ introduced by Proposition 1.2 is

(1.5) \[ w = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \]

Definition 1.1 tells us that $W$ must have the local coordinate expression

(1.6) \[ W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^j} + p_a A_i^a(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} + \]

\[ + \frac{1}{2} (\eta_{ij}(x) + p_a B_i^{ab}(x) + p_b p_C C_{ij}^{ab}(x)) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}, \]

where $w, \varphi, \eta, A, B, C$ are local functions on $M$.

The Poisson structure $W$ is completely determined by the brackets $\{f, g\}$, $\{m(X), f\}$ and $\{m(X), m(Y)\}$, where $f, g \in C^\infty(M)$ and $X, Y \in \chi(M)$, since

the local coordinates $x^i$ and $p_i$ are functions of this type ($p_i = m(\partial/\partial x^i)$).

By (1.3), the bracket $\{m(X), f\}$ is in $\mathcal{P}_1(T^*M)$, i.e.,

(1.7) \[ \{m(X), f\} = Z_X f + m(\gamma_X f), \]

where $Z_X f \in C^\infty(M)$ and $\gamma_X f \in \chi(M)$.

$\{m(X), \cdot\}$ is a derivation of $C^\infty(M)$. Hence, $Z_X$ is a vector field on $M$, and the mapping $\gamma_X : C^\infty(M) \rightarrow \chi(M)$ also is a derivation. Therefore, $\gamma_X f$ depends only on $df$. 

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From the Leibniz rule we get that \( Z_{hX} = hZ_X \) \((h \in C^\infty(M))\) and \( \gamma \) must satisfy

\[
\gamma_{hX} f = h\gamma_X f + (X_h^w f)X.
\]

The bracket of two affine functions has an expression of the form

\[
\{m(X), m(Y)\} = \beta(X, Y) + m(V(X, Y)) + s(\Psi(X, Y)),
\]

where \( \beta(X, Y) \in C^\infty(M), V(X, Y) \in \chi(M) \) and \( \Psi(X, Y) \in S_2(TM) \) are skew-symmetric operators. If we replace \( Y \) by \( fY \) in (1.9), the Leibniz rule gives that \( \beta \) is a 2-form on \( M \) and

\[
\beta(X, Y) = \sum_{a,b} p_a p_b C_{ab}^{ij}(x) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}.
\]

As in [6], a bivector field \( W \) on \( T^*M \) which is locally of the form (1.6) (respectively (1.11)) is called a polynomially graded (respectively graded) bivector field.

**Proposition 1.4.** If \( W \) is a graded bivector field on \( T^*M \) which is \( \pi \)-related with a Poisson structure \( w \) on \( M \), there exists a contravariant connection \( D \) on the Poisson manifold \((M, w)\) such that

\[
\{m(X), f\} = -m(D_{df} X), \quad X \in \chi(M), \ f \in C^\infty(M).
\]

Moreover, if \( W \) is a graded Poisson structure on \( T^*M \) then the connection \( D \) is flat.

**Proof.** A contravariant connection on \((M, w)\) is a contravariant derivative on \( TM \) with respect to the Poisson structure [10].

DEFINITION 1.3. A polynomially graded Poisson structure \( W \) on \( T^*M \) is said to be a **graded structure** if \( \forall Q \in \mathcal{HP}_h, \forall R \in \mathcal{HP}_k, \{Q, R\}_W \in \mathcal{HP}_{h+k} \).

Remark that a polynomially graded structure on \( T^*M \) is graded iff \( Z_X = 0, \beta = 0 \) and \( V = 0 \). In this case (1.6) reduces to

\[
W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \sum_{a,b} p_a A_{ja}^{(a)}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} + \sum_{a,b} p_a p_b C_{ij}^{ab}(x) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}.
\]
The required connection is defined by

\[ D_{df} X := -\gamma_X \alpha f . \]  

That we really get a connection, which is flat in the Poisson case, follows in exactly the same way as in [6]. □

PROPOSITION 1.5. If \( Q \) is a symmetric contravariant tensor field on \( M \) and \( \tilde{Q} \) is its corresponding polynomial then, for any graded Poisson bivector field \( W \) on \( T^*M \), one has

\[ \{ \tilde{Q}, f \}_W = -\tilde{D}_{df} Q . \]  

Proof. \( D_{df} \) of (1.14) is extended to \( S(TM) \) by

\[ (D_{df} Q)(\alpha_1, ..., \alpha_k) = X^w_f(Q(\alpha_1, ..., \alpha_k)) - \sum_{i=1}^k Q(\alpha_1, ..., D_{df} \alpha_i, ..., \alpha_k) , \]  

where \( \alpha_1, ..., \alpha_k \in \Omega^1(M) \), and \( D_{df} \alpha \) is defined by

\[ < D_{df} \alpha, X > = X^w_f < \alpha, X > - < \alpha, D_{df} X >, \quad X \in \chi(M) . \]  

We put

\[ D_{dx^i} \frac{\partial}{\partial x^j} = -\Gamma^i_{jk} \frac{\partial}{\partial x^k} , \]  

and by a straightforward computation we get for \( \{ \tilde{Q}, f \} \) and \( -(\tilde{D}_{df} Q) \) the same local coordinate expression. (See [6] for the complete proof in the case of a symmetric covariant tensor field on \( M \).) □

In order to discuss the next two Jacobi identities, let us make some remarks concerning the operator \( \Psi \) of (1.9), which is given in the case of a graded Poisson structure on \( T^*M \) by

\[ \{ m(X), m(Y) \} = s(\Psi(X, Y)) , \quad X, Y \in \chi(M) . \]  

With (1.13), the second relation (1.10) becomes

\[ \Psi(X, fY) = f\Psi(X, Y) - \frac{1}{2}(D_{df} X \otimes Y + Y \otimes D_{df} X) . \]  

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and this allows us to derive the local coordinate expression of $\Psi$. If $X = X^i(\partial/\partial x^i)$ and $Y = Y^j(\partial/\partial x^j)$, we obtain

\[(1.18) \quad \Psi(X, Y) = X^i Y^j \Psi \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) + \left( X^h \frac{\partial Y^j}{\partial x^k} \Gamma^k_i_h - Y^h \frac{\partial X^i}{\partial x^k} \Gamma^k_i_h \right) \frac{\partial}{\partial x^i} \circ \frac{\partial}{\partial x^j} + w^{kh} \frac{\partial X^i}{\partial x^k} \frac{\partial Y^j}{\partial x^h} \frac{\partial}{\partial x^i} \circ \frac{\partial}{\partial x^j}. \]

Remark that $\Psi : TM \times TM \rightarrow \odot^2 TM$ is a bidifferential operator of the first order.

**PROPOSITION 1.6.** If we define an operator $D_{df}$ which acts on $\Psi$ by

\[(1.19) \quad (D_{df} \Psi)(X, Y) := D_{df}(\Psi(X, Y)) - \Psi(D_{df}X, Y) - \Psi(X, D_{df}Y), \]

the Jacobi identity

\[(1.20) \quad \{\{m(X), m(Y)\}, f\} + \{\{m(Y), f\}, m(X)\} + \{\{f, m(X)\}, m(Y)\} = 0 \]

has the equivalent form

\[(1.21) \quad (D_{df} \Psi)(X, Y) = 0, \quad \forall X, Y \in \chi(M). \]

**Proof.** Using (1.12), (1.14) and (1.16) for $Q = \Psi(X, Y)$, (1.20) becomes (1.21). □

We also find

\[(1.22) \quad (D_{df} \Psi)(X, hY) = h(D_{df} \Psi)(X, Y) - [C_D(df, dh)X] \circ Y \]

and hence, we see that (1.21) is invariant by $X \mapsto fX, \ Y \mapsto gY$ ($f, g \in C^\infty(M)$) iff the curvature $C_D = 0$.

Concerning the Jacobi identity

\[(1.23) \quad \sum_{(X,Y,Z)} \{\{m(X), m(Y)\}, m(Z)\} = 0, \]

(putting indices between parentheses denotes that summation is on cyclic permutations of these indices) remark that one must have an operator $\Theta$ such that

\[(1.24) \quad \{s(G), m(X)\} = \Theta(G, X), \quad X \in \chi(M), \ G \in S_2(M), \]
and $\Theta(G,X)$ is a symmetric 3-contravariant tensor field on $M$.

We get the formula

\[(1.25) \quad \Theta(fG, hX) = fh\Theta(G, X) - f(D_{dh}G)\circ X + hG\circ D_{df}X + \{f, h\}_uG\circ X,\]

and then, the local coordinate expression

\[(1.26) \quad \Theta(G, X) = G^{ij}X^k\Theta\left(\frac{\partial}{\partial x^i} \circ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + \frac{1}{3} \sum_{(i,j,k)}(G^{h}_{ij}\frac{\partial X^k}{\partial x^a}\Gamma^a_h)
\]

\[+ G^{ai}\frac{\partial X^k}{\partial x^a}\Gamma^a_i - \frac{\partial G^{ij}}{\partial x^a}X^h\Gamma_{ah}^k + w^{ab}\frac{\partial G^{ij}}{\partial x^a}\frac{\partial X^k}{\partial x^b}\frac{\partial}{\partial x^i} \circ \frac{\partial}{\partial x^j} \circ \frac{\partial}{\partial x^k}.\]

Using the operator $\Theta$, the Jacobi identity (1.23) becomes

\[(1.27) \quad \sum_{(X,Y,Z)} \Theta(\Psi(X, Y), Z) = 0,\]

and we may summarize our analysis concerning the graded Poisson structures on $T^*M$ in

PROPOSITION 1.7. A graded Poisson structure $W$ on $T^*M$ with the bracket $\{\cdot,\cdot\}$ is defined by

a) a Poisson structure $w$ on the base manifold $M$ such that

\[\{f, g\}_w = \{f, g\}_w, \quad f, g \in C^\infty(M).\]

b) a flat contravariant connection $D$ on $(M, w)$ such that

\[\{m(X), f\} = -m(D_{df}X), \quad X \in C^\infty(M).\]

c) an operator $\Psi : TM \times TM \to \odot^2 TM$ such that

\[\{m(X), m(Y)\} = s(\Psi(X, Y)), \quad X, Y \in \chi(M)\]

and formula (1.21) holds.

d) an operator $\Theta$ defined by (1.24), satisfying (1.27).

To give examples, we consider the following situation, similar to [3].

Let $(M, w)$ be an $n$-dimensional Poisson manifold and suppose that its symplectic foliation $S$ is contained in a regular foliation $F$ on $M$, such that $TF$ is a foliated bundle i.e., there are local bases $\{Y_u\}$ $(u = 1, \ldots, p, p =$
Consider a decomposition
\[(1.28)\quad TM = T\mathcal{F} \oplus \nu\mathcal{F},\]
where \(\nu\mathcal{F}\) is a complementary subbundle of \(T\mathcal{F}\), and \(\mathcal{F}\)-adapted local coordinates \((x^a, y^u)\) \((a = 1, \ldots, n - p)\) on \(M\).

The Poisson bivector \(w\) has the form
\[(1.29)\quad w = \frac{1}{2} w^{uv}(x, y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v}, \quad (w^{vu} = -w^{uv}),\]
since \(S \subseteq \mathcal{F}\).

If \(\{\beta^u\}, \{\tilde{\beta}^u\}\) \((u, v = 1, \ldots, p)\) are the dual cobases of \(\{Y_u\}, \{\tilde{Y}_v\}\) \((\beta^u(Y_v) = \delta_v^u)\) then their transition functions are constant along the leaves of \(\mathcal{F}\).

Now, \(\forall \alpha \in T^*M, \alpha = \zeta_a dx^a + \varepsilon_u \beta^u\) and we may consider \((x^a, y^u, \zeta, \varepsilon)\) as distinguished local coordinates on \(T^*M\). The transition function are
\[(1.30)\quad \tilde{x}^a = \tilde{x}^a(x), \quad \tilde{y}^u = \tilde{y}^u(x, y), \quad \tilde{\zeta}_u = \frac{\partial x^a}{\partial \tilde{x}^a} \zeta_a, \quad \tilde{\varepsilon}_u = a^u_v(x)\varepsilon_v.\]

**PROPOSITION 1.8.** Under the previous hypotheses, \(W\) given with respect to the distinguished local coordinates by
\[(1.31)\quad W = \frac{1}{2} w^{uv}(x, y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v}\]
defines a graded Poisson bivector on \(T^*M\).

**Proof.** From (1.30) it follows that \(W\) of (1.31) is a global tensor field on \(T^*M\). The Schouten-Nijenhuis bracket \([W, W]\) has the same expression as \([w, w]\) on \(M\) and thus, the Poisson condition \([W, W] = 0\) holds.

To prove that \(W\) is graded, we also consider natural coordinates and show that the expression of \(W\) with respect to these coordinates becomes of the form (1.11) (see \([3]\)).

There are some interesting particular cases of Proposition 1.8:

a) \(w\) is a regular Poisson structure, and the bundle \(TS\) is a foliated bundle; in this case we may take \(\mathcal{F} = S\).

b) \(S\) is contained in a regular foliation \(\mathcal{F}\) which admits adapted local coordinates \((x^a, y^u)\) with local transition functions
\[\tilde{y}^u = p^u_v(x) y^v + q^u(x).\]
(\mathcal{F} \text{ is a leaf-wise, locally affine, regular foliation.}) \text{ In this case } (\partial/\partial y^u) = \sum_w a^u_v(x)(\partial/\partial \tilde{y}^v) \text{ and we may use the local vector fields } Y_u = \partial/\partial y^u.

c) There exists a flat linear connection \nabla \text{ (possibly with torsion) on the Poisson manifold } (M, w). \text{ In this case we may consider as leaves of } \mathcal{F} \text{ the connected components of } M, \text{ and the local } \nabla-\text{parallel vector fields have constant transition functions along these leaves. Therefore, we may take them as } Y_i \text{ (} i = 1, ..., n \text{).}

In particular, we have the result of c) for a locally affine manifold } M \text{ (where } \nabla \text{ has no torsion), using as } Y_i \text{ local } \nabla-\text{parallel vector fields, and also for a parallelizable manifold } M \text{ (where we have global vector fields } Y_i \text{).}

As a consequence, Proposition 2.8 holds for the Lie-Poisson structure \[10\] of any dual } \mathcal{G}^* \text{ of a Lie algebra } \mathcal{G}, \text{ the graded Poisson structure being defined on } T^* \mathcal{G}^* = \mathcal{G}^* \times \mathcal{G}.

2. Graded bivector fields on cotangent bundles

In this section we will discuss graded bivector fields on a cotangent bundle } T^* M, \text{ which may be seen as lifts of a given Poisson structure } w \text{ on } M, \text{ that satisfy less restrictive existence conditions than in the case of graded Poisson structures.}

Recall the following definition from [8]. \text{ Let } \mathcal{F} \text{ be an arbitrary regular foliation, with } p-\text{dimensional leaves, on an } n-\text{dimensional manifold } N. \text{ We denote by } C_{\text{fol}}^\infty(N) \text{ the space of foliated functions (the functions on } N \text{ which are constant along the leaves of } \mathcal{F}). \text{ A transversal Poisson structure of } (N, \mathcal{F}) \text{ is a bivector field } w \text{ on } N \text{ such that}

\begin{equation}
\{f, g\} := w(df, dg), \quad f, g \in C_{\text{fol}}^\infty(N)
\end{equation}

is a Lie algebra bracket on } C_{\text{fol}}^\infty(N). \text{ A bivector field } w \text{ on } N \text{ defines a transversal Poisson structure of } (N, \mathcal{F}) \text{ iff [8]}

\begin{equation}
(\mathcal{L}_Y w)|_{\text{Ann } T\mathcal{F}} = 0, \quad [w, w]|_{\text{Ann } T\mathcal{F}} = 0,
\end{equation}
for all $Y \in \Gamma(T\mathcal{F})$ (the space of global cross sections of $T\mathcal{F}$), where $Ann\ T\mathcal{F} \subseteq \Omega^1(N)$ is the annihilator space of $T\mathcal{F}$. ($\Omega^1(N)$ denotes the space of Pfaff forms on $N$.)

The cotangent bundle $T^*M$ of any manifold $M$ has the vertical foliation $\mathcal{F}$ by fibers with the tangent distribution $V := T\mathcal{F}$.

Obviously, the set of foliated functions on $T^*M$ may be identified with $C^\infty(M)$.

**Proposition 2.1.** Any polynomially graded bivector field $W$ on $T^*M$, which is $\pi$-related with a Poisson structure of $M$ is a transversal Poisson structure of $(T^*M, V)$.

**Proof.** The local coordinate expression of $W$ is of the form (1.6), and $W$ is $\pi$-related with the bivector field $w$ defined on $M$ by the first term of (1.6). Then, (2.2) holds, because $w$ is a Poisson bivector on $M$. □

**Definition 2.2.** A transversal Poisson structure of the vertical foliation of $T^*M$ will be called a semi-Poisson structure on $T^*M$.

**Remark 2.3.** The structures $W$ of Proposition 2.1 are polynomially graded semi-Poisson structures on $T^*M$.

In what follows, we will discuss some interesting classes of graded semi-Poisson structures of $T^*M$. Then, we give a method to construct all the graded semi-Poisson bivector fields on $T^*M$ which induce the same Poisson structure $w$ on the base manifold $M$.

Let $D$ be a contravariant derivative on a Poisson manifold $(M, w)$. First, $\forall Q \in S_k(TM)$, define $^sDQ \in S_{k+1}(TM)$ by

\[(^sDQ)(\alpha_1, \ldots, \alpha_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} (D_{\alpha_i}Q)(\alpha_1, \ldots, \hat{\alpha_i} \ldots \alpha_{k+1}),\]

where $\alpha_1, \ldots, \alpha_{k+1} \in \Omega^1(M)$, and the hat denotes the absence of the corresponding factor.

If $X = X^i(\partial/\partial x^i) \in \chi(M)$ then $DX$, defined by $(DX)(\alpha_1, \alpha_2) = (D_{\alpha_1}X)\alpha_2$, is a 2-contravariant tensor field on $M$, and

\[(DX) = D^iX^j \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},\]

where $D^iX^j = (D_{dx^i}X)dx^j = D_{dx^i}X^j - X(D_{dx^i}dx^j)$. According to (1.15) we must have

\[(D_{dx^i}dx^j) = \Gamma^i_k dx^k,\]
and obtain

\[(2.6) \quad D^i X^j = (dx^i)^{\ast} X^j - \Gamma^i_k dx^k = \{x^i, X^j\}_w - \Gamma^i_k X^k.\]

Then

\[s DX = \frac{1}{2} (D^i X^j + D^j X^i) \frac{\partial}{\partial x^i} \circ \frac{\partial}{\partial x^j}\]

and we get

\[(2.7) \quad s DX = \frac{1}{2} \{x^i, X^j\}_w + \{x^j, X^i\}_w - \Gamma^i_k X^k - \Gamma^j_i X^k] \frac{\partial}{\partial x^i} \circ \frac{\partial}{\partial x^j}.\]

**PROPOSITION 2.4.** Let \((M, w)\) be a Poisson manifold and \(D\) a contravariant derivative of \((M, w)\). The bivector field \(W_1\) on \(T^*M\), of bracket \(\{ , \}_W\) defined by the conditions

\[(2.8) \quad \{f, g\}_W := \{f, g\}_w,\]

\[(2.9) \quad \{m(X), f\}_W := -m(DgX),\]

\[(2.10) \quad \{m(X), m(Y)\}_W = \frac{1}{2} [s]^* D < X, Y >

\[ - < s DX, Y > - < X, s DY >\]

where \(f, g \in C^\infty(M), \ X, Y \in \chi(M)\) and \(\langle , \rangle\) is the Schouten-Nijenhuis bracket of symmetric tensor fields (defined by the natural Lie algebroid of \(M\)) [4, 5] defines a graded semi-Poisson structure on \(T^*M\) which is \(\pi\)-related with \(w\).

**Proof.** If the local coordinate expression of \(w\) is (1.5), using (2.7) and the properties of \(\langle , \rangle\) we get

\[(2.11) \quad W_1 = \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} - p_a \Gamma^i_{ja} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j}

\[ - \frac{1}{4} p_a p_b \left[ \frac{\partial}{\partial x^i} (\Gamma^a_i + \Gamma^b_a) - \frac{\partial}{\partial x^i} (\Gamma^a_j + \Gamma^b_j) \right] \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}. \]

**REMARK 2.5.** The relation (2.10) provides us the expression of the operator \(\Psi_{W_1}\) associated to \(W_1\) (see (1.16)):

\[(2.12) \quad \Psi_{W_1}(X, Y) = \frac{1}{2} [s]^* D < X, Y > - < s DX, Y > - < X, s DY >.\]
Now, instead of $D$ we consider a linear connection $\nabla$ on a Poisson manifold $(M, w)$ and define the vector field $K$ on $T^*M$ by
\begin{equation}
K(\alpha) = (\natural w \alpha)^H, \quad \alpha \in T^*M,
\end{equation}
where $\natural w : T^*M \rightarrow TM$ is defined by $\beta(\alpha^\sharp) = w(\alpha, \beta), \forall \beta \in \Omega^1(M)$, and the upper index $H$ denotes the horizontal lift with respect to $\nabla$ ([3] [4]). In local coordinates we get
\begin{equation}
K = p_a w^{ai} \frac{\partial}{\partial x^i} + \frac{1}{2} p_a p_b (w^{ak} \Gamma^b_{ki} + w^{bk} \Gamma^a_{ki}) \frac{\partial}{\partial p_i}.
\end{equation}

On $T^*M$ we have the canonical symplectic form $\omega = d\lambda = dp_i \wedge dx^i$ where $\lambda = p_i dx^i$ is the Liouville form, and the vector bundle isomorphism
\[\natural \omega : T^*M \rightarrow TM, \quad i_X \omega \in T^*M \mapsto X \in TM\]
leads to the canonical Poisson bivector $W_0 := \natural \omega \omega$ on $T^*M$. It follows
\begin{equation}
W_0(dF, dG) = \omega(\natural(dF), \natural(dG)) , \quad F, G \in C^\infty(T^*M),
\end{equation}
and locally one has
\begin{equation}
W_0 = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i}.
\end{equation}

**Proposition 2.6.** If $(M, w)$ is a Poisson manifold then the bivector field
\begin{equation}
W_2 = \frac{1}{2} \mathcal{L}_K W_0,
\end{equation}
defines a graded semi-Poisson structure on $T^*M$ which is $\pi-$related with $w$.

**Proof.** We get
\begin{equation}
W_2 = \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{1}{2} p_a (\nabla_j w^{ai} + 2w^{ik} \Gamma^a_{kj}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} + \frac{1}{4} p_a p_b \left[ \frac{\partial}{\partial x^j} (w^{ak} \Gamma^b_{ki} + w^{bk} \Gamma^a_{ki}) - \frac{\partial}{\partial x^i} (w^{ak} \Gamma^b_{kj} + w^{bk} \Gamma^a_{kj}) \right] \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j},
\end{equation}
where $\nabla_j w^{ai}$ are the components of the $(2,1)$–tensor field on $M$ defined by $X \mapsto \nabla_X w, \ X \in \chi(M)$. \qed
We will say that $W_2$ of (2.17) is the graded $\nabla - $lift of the Poisson structure $w$ of $M$.

Using local coordinates and the notation of (1.2) we get

\[(2.19) \quad \mathcal{L}_K \tilde{Q} = (^sD)Q,\]

where $D$ is the contravariant derivative induced by the linear connection $\nabla$, defined by $D_{df} = \nabla_{(df)^2}$ (see [10]).

From (2.17) we have

\[(2.20) \quad \{F_1, F_2\}_{W_2} := W_2(dF_1, dF_2)\]

\[= \frac{1}{2}(\mathcal{L}_K \{F_1, F_2\}_w) - \{\mathcal{L}_K F_1, F_2\}_w - \{F_1, \mathcal{L}_K F_2\}_w,\]

where $F_1, F_2 \in C^\infty(T^*M)$.

If $Q_1, Q_2 \in S(TM)$, using (2.19) and the relation

\[(2.21) \quad \{\tilde{Q}, \tilde{H}\}_{W_0} := \langle \tilde{Q}, \tilde{H} \rangle, \quad Q, H \in S(TM)\]

(see [1, 4]) we get the explicit formula

\[(2.22) \quad \{\tilde{Q}_1, \tilde{Q}_2\}_{W_2} = \frac{1}{2} \sim \langle ^sD Q_1, Q_2 \rangle - \langle ^sD Q_1, Q_2 \rangle - \langle Q_1, ^sD Q_2 \rangle.\]

**PROPOSITION 2.7.** The graded $\nabla - $lift $W_2$ of $w$ is characterized by:

i) the Poisson structure induced on $M$ by $W_2$ is $w$, i.e.

\[(2.23) \quad \{f, g\}_{W_2} = \{f, g\}_w, \quad \forall f, g \in C^\infty(M);\]

ii) for every $f \in C^\infty(M)$ and $X \in \chi(M)$

\[(2.24) \quad \{m(X), f\}_{W_2} = -m(\tilde{D}_g X),\]

where $\tilde{D}$ is the contravariant derivative of $(M, w)$ defined by

\[(2.25) \quad \tilde{D}_a \beta = D_a \beta + \frac{1}{2}(\nabla w)(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(M),\]

the contravariant derivative $D$ is induced by $\nabla$ and $(\nabla w)(\alpha, \beta)$ is the 1-form $X \mapsto (\nabla_X w)(\alpha, \beta)$. 

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iii) for any vector fields $X$ and $Y$ of $M$ we have

\begin{align}
\{m(X), m(Y)\}_{W_2} &= \frac{1}{2}s(sD < X, Y > \\
&\quad - < s DX, Y > - < X, sDY >) .
\end{align}

Proof. i) If $f \in C^\infty(M)$ then $Df = -X^w_f$ and from (2.20), (2.21) and the formula

$$< Q, f > = i(df)Q , \quad f \in C^\infty(M) , \quad Q \in S_p(TM) ,$$

we get

$$\{f, g\}_{W_2} = \frac{1}{2}(< Df, g > + < f, Dg >) = \frac{1}{2}(X^w_f g - X^w_f g) = \{f, g\}_w .$$

ii) As $W_2$ is graded, the bracket $\{m(X), f\}_{W_2}$ must be of the form (2.24).

Denoting

$$D_{dx^i} dx^j = \bar{\Gamma}^{ij}_k dx^k$$

(2.18) give us

\begin{align}
\bar{\Gamma}^{ij}_k &= \Gamma^{ij}_k + \frac{1}{2}\nabla^k w^{ij} ,
\end{align}

where

\begin{align}
\Gamma^{ij}_k &= -w^{jk} \Gamma^{ij}_{hk} ,
\end{align}

($\Gamma^{ij}_{jk}$ are the coefficients of the linear connection $\nabla$) and hence (2.25).

iii) (2.26) is a direct consequence of (2.22). \[ \square \]

Notice from (2.26) that the operator $\Psi_{W_2}$ associated to $W_2$ has the same expression as $\Psi_{W_1}$ of (2.12), but in the case of $W_1$ the contravariant derivative $D$ is induced by a linear connection $\nabla$ on $M$.

PROPOSITION 2.8. If the graded semi-Poisson structure $W_1$ is defined by a linear connection on $(M, w)$ then it coincides with $W_2$ iff $w$ is $\nabla$-parallel.

Proof. Compare the characteristic conditions of Propositions 2.4 and 2.7 (or the coefficients of $(\partial/\partial x^i) \wedge (\partial/\partial p_j)$ of (2.11) and of (2.18), using (2.28)). \[ \square \]

We will prove now
PROPOSITION 2.9. Let \((M, w)\) be a Poisson manifold and \(\pi : T^*M \longrightarrow M\) its cotangent bundle. The graded semi-Poisson structures \(W\) on \(T^*M\) which are \(\pi\)--related with \(w\) are defined by the relations

\[
\{f, g\}_W = \{f, g\}_w, \quad \{m(X), f\}_W = -m(DfX) \quad \text{and} \quad \{m(X), m(Y)\}_W = s(\Psi(X, Y)),
\]

where \(D\) is an arbitrary contravariant connection of \((M, w)\) and the operator \(\Psi\) is given by

\[
(2.29) \quad \Psi = \Psi_0 + A + T,
\]

where \(\Psi_0\) is the operator \(\Psi\) of a fixed graded semi-Poisson structure, \(A : TM \times TM \longrightarrow \odot^2 TM\) is a skew-symmetric, first order, bidifferential operator such that

\[
(2.30) \quad A(X, fY) = fA(X, Y) - \tau(df, X) \odot Y,
\]

where \(\tau\) is a \((2, 1)\)--tensor field on \(M\), and \(T\) is a \((2, 2)\)--tensor field on \(M\) with the properties \(T(Y, X) = -T(X, Y)\) and \(T(X, Y) \in S_2(TM), \forall X, Y \in \chi(M)\).

Proof. If two graded semi-Poisson bivector fields, \(\pi\)--related with \(w\), have associated the same contravariant connection \(D\), it follows from (1.17) that the difference \(\Psi' - \Psi\) is a tensor field \(T\), as in Proposition. To change \(D\) means to pass to a contravariant connection \(D' = D + \tau\), where \(\tau\) is a \((2, 1)\)--tensor field on \(M\) and from (1.17) again, it follows that \(A = \Psi' - \Psi\) becomes a bidifferential operator with the property (2.29). \(\Box\)

3. Horizontal lifts of Poisson structures

In this section we define and study an interesting class of semi-Poisson structures on \(T^*M\) which are produced by a process of horizontal lifting of Poisson structures from \(M\) to \(T^*M\) via connections.

On \(T^*M\) we distinguish the vertical distribution \(V\), tangent to the fibers of the projection \(\pi\) and, by complementing \(V\) by a distribution \(H\), called horizontal, we define a nonlinear connection on \(T^*M\) \([7, 8]\).
We have \textit{(adapted)} bases of the form
\begin{equation}
V = \text{span} \left\{ \frac{\partial}{\partial p_i} \right\}, \quad H = \text{span} \left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{ij} \frac{\partial}{\partial p_j} \right\},
\end{equation}
and $N_{ij}$ are the \textit{coefficients of the connection} defined by $H$.

Equivalently, a nonlinear connection may be seen as an almost product structure $\Gamma$ on $T^*M$ such that the eigendistribution corresponding to the eigenvalue $-1$ is the vertical distribution $V$.\footnote{\cite{7}}

We assume that the nonlinear connection above is symmetric, i.e., $N_{ji} = N_{ij}$. This condition is independent \footnote{\cite{7}} on the local coordinates.

The complete integrability of $H$, in the sense of the Frobenius theorem, is equivalent to the vanishing of the curvature tensor field
\begin{equation}
R = R_{kij} dx^i \wedge dx^j \otimes \frac{\partial}{\partial p_k}, \quad R_{kij} = \frac{\delta N_{kj}}{\delta x^i} - \frac{\delta N_{ki}}{\delta x^j}.
\end{equation}

For a later utilization, we also notice the formulas \footnote{\cite{4,8}}
\begin{equation}
\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = -R_{kij} \frac{\partial}{\partial p_k}, \quad \left[ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_j} \right] = -\Phi^j_{ik} \frac{\partial}{\partial p_k}, \quad \Phi^j_{ik} = -\frac{\partial N_{ik}}{\partial p_j}.
\end{equation}

Let $w$ be a bivector on $M$, with the local coordinate expression (1.5).

**DEFINITION 3.1.** The \textit{horizontal lift} of $w$ to the cotangent bundle $T^*M$ is the (global) bivector field $w^H$ defined by
\begin{equation}
w^H = \frac{1}{2} w_{ij}(x) \frac{\delta}{\delta x^i} \wedge \frac{\delta}{\delta x^j}.
\end{equation}

**PROPOSITION 3.2.** Let $(M, w)$ be a Poisson manifold. If the connection $\Gamma$ on $T^*M$ is defined by a linear connection $\nabla$ on $M$, the bivector $w^H$ defines a graded semi-Poisson structure on $T^*M$.

**Proof.** In this case the coefficients of $\Gamma$ are
\begin{equation}
N_{ij} = -p_k \Gamma_{ij}^k,
\end{equation}
where $\Gamma_{ij}^k$ are the coefficients of $\nabla$ and, with respect to the bases \{\partial/\partial x^i, \partial/\partial p_j\}, the local expression of $w^H$ becomes
\begin{equation}
W = \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + w^{ik} \Gamma_{kj}^a \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j}.
\end{equation}
PROPOSITION 3.3. The horizontal lift \( w^H \) is a Poisson bivector on the cotangent bundle \( T^*M \) iff \( w \) is a Poisson bivector on the base manifold \( M \) and

\[
R(X^H_f, X^H_g) = 0, \quad \forall f, g \in C^\infty(M),
\]

where \( X^H_f \) denotes the usual horizontal lift \([3, 11]\), from \( M \) to \( T^*M \), of the \( w \)--Hamiltonian vector field \( X_f \) on \( M \).

In this case, the projection \( \pi : (T^*M, w^H) \to (M, w) \) is a Poisson mapping.

Proof. We compute the bracket \([w^H, w^H]\) with respect to the bases (3.1) and get that the Poisson condition \([w^H, w^H] = 0\) is equivalent with the pair of conditions

\[
\sum_{(i,j,k)} w^{hk} \frac{\partial w^{ij}}{\partial p^h} = 0, \quad w^{il} w^{jh} R_{klh} = 0.
\]

(Putting indices between parentheses denotes that summation is on cyclic permutations of these indices.)

The first condition (3.8) is equivalent to \([w, w] = 0\) and the second is the local coordinate expression of (3.7).

Notice that the condition (3.7) has the equivalent form

\[
R((\sharp \alpha)^H, (\sharp \beta)^H) = 0, \quad \forall \alpha, \beta \in \Omega^1(M).
\]

REMARK 3.4. If \( w \) is defined by a symplectic form on \( M \), condition (3.8) becomes \( R = 0 \).

COROLLARY 3.5. If \( (M, w) \) is a Poisson manifold and the connection \( \Gamma \) on \( T^*M \) is defined by a linear connection \( \nabla \) on \( M \), the bivector \( w^H \) defines a Poisson structure on \( T^*M \) iff the curvature \( C_D \) of the contravariant connection induced by \( \nabla \) on \( TM \) vanishes.

Proof. If \( R^h_{kij} \) are the components of the curvature \( R_\nabla \) then

\[
R^h_{kij} = -p^h R^h_{kij}
\]

and (3.9) becomes

\[
R_\nabla(\sharp \alpha, \sharp \beta)Z = 0, \quad \forall \alpha, \beta \in \Omega^1(M), \quad \forall Z \in \chi(M),
\]
(or, equivalently)

\[(3.11') \quad R_\nabla(X_f, X_g)Z = 0, \forall f, g \in C^\infty(M), \forall Z \in \chi(M).\]

This is equivalent to \(C_D = 0. \square\)

In the case where \(w^H\) is a Poisson bivector, it is interesting to study its compatibility with the canonical Poisson structure \(W_0\) of (2.15).

**PROPOSITION 3.6.** If \(w^H\) is a Poisson bivector, then it is compatible with \(W_0\) iff

\[(3.12) \quad \frac{\partial w^{ij}}{\partial x^k} + w^{ih}\Phi^j_{hk} - w^{jh}\Phi^i_{hk} = 0, \quad w^{ih}R_{hjk} = 0.\]

**Proof.** By a straightforward computation we get that the compatibility condition \([w^H, W] = 0\) is equivalent to (3.12). \(\square\)

The Bianchi identity [7]

\[(3.13) \quad R_{kij} + R_{ijk} + R_{jki} = 0,\]

shows that the second relation (3.12) implies (3.7). Then

**COROLLARY 3.7.** If \((M, w)\) is a Poisson manifold and the cotangent bundle \(T^*M\) is endowed with a symmetric nonlinear connection, then \(w^H\) is a Poisson bivector on \(T^*M\) compatible with \(W_0\) iff conditions (3.12) hold.

**REMARK 3.8.** Considering the isomorphism

\[\Psi : V_u \rightarrow H_u^\ast, \quad \Psi(X_k\partial/\partial p_k) = X_kdq^k,\]

where \(u \in T^*M\) and \(H_u^\ast\) is the dual space of \(H_u\), the second condition (3.12) may be written in the equivalent form

\[(3.14) \quad [\Psi(R(X, Y))](\sharp_w\alpha)^H = 0, \forall X, Y \in \chi(T^*M), \forall \alpha \in \Omega^1(M).\]

We recall that a symmetric linear connection \(\nabla\) on a Poisson manifold \((M, w)\) is called a Poisson connection if \(\nabla w = 0\). Such connections exists iff \(w\) is regular, i.e. \(\text{rank } w = \text{const}\) (see [11]).

**PROPOSITION 3.9.** Let \((M, w)\) be a regular Poisson manifold with a Poisson connection \(\nabla\). Then, the bivector \(w^H\), defined with respect to \(\nabla\), is a Poisson structure on \(T^*M\) compatible with the canonical Poisson structure \(W_0\) iff the 2–form

\[(3.15) \quad (X, Y) \rightarrow R_\nabla(X, Y)(\sharp_w\alpha), \quad X, Y \in \chi(M)\]
vanishes for every Pfaff form $\alpha$ on $M$.

Proof. With (3.5), the first condition (3.12) becomes $\nabla w = 0$, which we took as a hypotheses. The second condition (3.12) becomes

$$w^{ih} R^l_{hjk} = 0,$$

and we get the required conditions. $\Box$

REMARK 3.10. If $w$ is defined by a symplectic structure of $M$ then (3.15) means $R^\nabla = 0$.

4. Poisson structures derived from differential forms

If $\omega$ is a 2–form on a Riemannian manifold $(M, g)$ we associate with it a 2–form $\Theta(\omega)$ on the cotangent bundle $\pi : T^*M \rightarrow M$, and considering (pseudo-) Riemannian metrics on $T^*M$ related to $g$, we study the conditions for $\Theta(\omega)$ to produce a Poisson structure on this bundle.

Let $(M, g)$ be a $n$–dimensional manifold and $\nabla$ its Levi-Civita connection. If $\Gamma^k_{ij}$ are the local coefficients of $\nabla$, a connection $\Gamma$ with the coefficients (3.5) is obtained on $T^*M$.

The system of local 1–forms $(dx^i, \delta p_i)$, $(i = 1, ..., n)$, where

$$\delta p_i := dp_i + N^{ij} dx^j$$

defines the dual bases of the bases $\{\delta/\delta x^i, \partial/\partial p_i\}$.

The components of the curvature form are given by (3.2). Since the connection is symmetric, the Bianchi identity (3.13) holds. The elements $\Phi^k_{ij}$ of (3.3) are

$$\Phi^k_{ij} = \Gamma^k_{ij}.$$

The Riemannian metric $g$ provides the "musical" isomorphism $\sharp_g : T^*M \rightarrow TM$ and the codifferential

$$\delta_g : \Omega^k(M) \rightarrow \Omega^{k-1}(M), \quad (\delta_g \alpha)_{i_1...i_{k-1}} = -g^{st} \nabla_t \alpha_{si_1...i_{k-1}}.$$
where $k \geq 1$,
\[
\alpha = \frac{1}{k!} \alpha_{i_1...i_k} dx^{i_1} \wedge ... \wedge dx^{i_k} \in \Omega^k(M)
\]
and $(g^{st})$ are the entries of the inverse of the matrix $(g_{ij})$.

Let
\[
\omega = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j, \quad \omega_{ji} = -\omega_{ij},
\]
be a 2–form on $M$.

**DEFINITION 4.1.** The 2–form $\Theta(\omega)$ on $T^*M$ given by
\[(4.4) \quad \Theta(\omega) = \pi^*\omega - d\lambda ,
\]
where $\lambda$ is the Liouville form, is said to be the associated 2–form of $\omega$.

With respect to the cobases $(dx^i, \delta p_i)$ we get
\[(4.5) \quad \Theta(\omega) = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j + dx^i \wedge \delta p_i .
\]

Now, we consider two (pseudo-) Riemannian metrics $G_1$ and $G_2$ on $T^*M$ and study the conditions for the bivectors $W_i = \sharp G_i \Theta(\omega)$, $(i = 1, 2)$ to define Poisson structures on $T^*M$. The Poisson condition $[W_i, W_i] = 0$, $i = 1, 2$ is equivalent to
\[(4.6) \quad \delta_{G_i}(\Theta(\omega) \wedge \Theta(\omega)) = 2\Theta(\omega) \wedge \delta_{G_i}\Theta(\omega) , \quad i = 1, 2.
\]

First, consider [7, 8] the pseudo-Riemannian metric $G_1$ of signature $(n, n)$
\[(4.7) \quad G_1 = 2\delta p_i \odot dx^i .
\]

To find the condition which ensure that (4.6) holds, we need the local expression of the codifferential $\delta_{G_1}$ of $G_1$. Denote by $\tilde{\nabla}$ the Levi-Civita connection of $G_1$, and for simplicity we put
\[(4.8) \quad \tilde{\nabla}_i = \tilde{\nabla}_{\frac{\partial}{\partial x^i}}, \quad \tilde{\nabla}^i = \tilde{\nabla}_{\frac{\partial}{\partial p_i}} .
\]
$\tilde{\nabla}$ is defined by [7]
\[(4.9) \quad \tilde{\nabla}^i \frac{\partial}{\partial p_j} = 0 , \quad \tilde{\nabla}_i \frac{\partial}{\partial p_j} = -\Gamma^j_{ik} \frac{\partial}{\partial p_k} , \quad \tilde{\nabla}^i \frac{\partial}{\partial q^j} = 0 , \quad \tilde{\nabla}_i \frac{\partial}{\partial q^j} = \Gamma^k_{ij} \frac{\partial}{\partial q^k} - p_h R^h_{ijk} \frac{\partial}{\partial p_k} .
\]
PROPOSITION 4.2. The bivector $\sharp G_1 \Theta(\omega)$ defines a Poisson structure on the cotangent bundle $T^*M$ iff $\omega$ is a closed 2-form on $M$ and $\Gamma^a_{ai} = 0$, $\forall i = 1, ..., n$. In this case $\Theta(\omega)$ is a symplectic form.

Proof. The proof is by a long computation in local coordinates. After computing the exterior product $\Theta(\omega) \wedge \Theta(\omega)$ we get

$$\delta_{G_1}(\Theta(\omega) \wedge \Theta(\omega)) = \frac{2}{3!} \sum_{(i,j,k)} \nabla_i \omega_{jk} dx^i \wedge dx^j \wedge dx^k.$$ 

Then, we compute $\delta_{G_1} \Theta(\omega)$ and obtain

$$\Theta(\omega) \wedge \delta_{G_1} \Theta(\omega) = \frac{2}{3!} \sum_{(i,j,k)} \omega_{ij} \Gamma^a_{ak} dx^i \wedge dx^j \wedge dx^k + (\delta^k_j \Gamma^a_{ai} - \delta^k_i \Gamma^a_{aj}) dx^i \wedge dx^j \wedge \delta p_k.$$ 

(4.6) implies

$$\delta^k_j \Gamma^a_{ai} - \delta^k_i \Gamma^a_{aj} = 0, \quad \forall i, j, k = 1, ..., n.$$ 

Making the contraction $k = j$ it follows that $\Gamma^a_{ai} = 0$. Conversely, if $\Gamma^a_{ai} = 0$ then (4.12) holds. Also, since $\nabla$ is symmetric, we get

$$\sum_{(i,j,k)} \frac{\partial \omega_{jk}}{\partial x^i} = \sum_{(i,j,k)} \nabla_i \omega_{jk}.$$ 

Therefore, the condition $\sum_{(i,j,k)} \nabla_i \omega_{jk} = 0$ is equivalent to $d \omega = 0$. \qed

Let us consider now the Riemannian metric of Sasaki type

$$G_2 = g_{ij} dx^i \odot dx^j + g^{ij} \delta p_i \odot \delta p_j,$$

(see [2] for the Sasaki metric).

LEMMA 4.3. The local coordinate expression of the Levi-Civita connection $\nabla$ of $G_2$ is

$$\nabla^i \frac{\partial}{\partial p_j} = 0, \quad \nabla^i \frac{\partial}{\partial p_j} = -\frac{1}{2} R^{jk}_{i} \delta \delta q^k - \Gamma^j_{ik} \frac{\partial}{\partial p_k},$$

$$\nabla^i \frac{\delta}{\delta q^j} = \frac{1}{2} R^i_{jk} \delta \delta q^k, \quad \nabla^i \frac{\delta}{\delta q^j} = \Gamma^k_{ij} \delta \delta q^k - \frac{1}{2} R_{klij} \frac{\partial}{\partial p_k},$$

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where we used again the notations of (4.8) and $R^i_j$ (also $R^i_j$) are obtained from $R_{kij}$ by the operation of lifting the indices, i.e.

$$
R^i_j = g^{ia} g^{kb} R_{abi} , \quad R^i_j = g^{ia} g^{kb} R_{ajb} .
$$

Proof. The result is proved by a straightforward computation. ∎

PROPOSITION 4.4. The bivector $\delta G_2 (\Theta (\omega))$ defines a Poisson structure on the cotangent bundle $T^* M$ iff

$$(4.15) \quad \nabla \omega = 0 , \quad g^{ab} R^k_{abi} = 0 , \quad \omega^a R^k_{iab} = 0 ,$$

where $\omega_{ij} = g^{ai} g^{bj} \omega_{ij}$ are the components of the bivector $w = \sharp g \omega$ on $M$.

Proof: By a new long computation again, we get

$$
\frac{1}{2} \delta G_2 (\Theta (\omega) \wedge \Theta (\omega)) = \frac{1}{3!} g^{ab} \nabla_a (\sum_{(i,j,k)} \omega_{ij} \omega_{kb}) dx^i \wedge dx^j \wedge dx^k -
$$

$$-(g^{ab} \sum_{(i,j,k)} (\nabla_a \omega_{ij} \delta^k_i) dx^i \wedge dx^j \wedge \delta p_k + \frac{1}{2} \omega_{ab} (R_{abi} R^k_{j} - R_{abj} R^k_{i}) dx^i \wedge \delta p_j \wedge \delta p_k$$

and

$$
\Theta (\omega) \wedge \delta G_2 (\Theta (\omega)) = \frac{1}{3!} (\delta G_2 (\Theta (\omega)))_k dx^i \wedge dx^j \wedge dx^k +
$$

$$+ \frac{1}{2!} [\delta^k_i (\delta G_2 (\Theta (\omega)))_j - \delta^k_j (\delta G_2 (\Theta (\omega)))_i] dx^i \wedge dx^j \wedge \delta p_k ,$$

where

$$
\delta G_2 (\Theta (\omega)) = (\delta G_2 (\Theta (\omega)))_k dx^k = g^{ab} (\nabla_a \omega_{kb} - \frac{1}{2} R_{abk}) dx^k .
$$

Identifying the coefficients, the Poisson conditions (4.6) for $W_2$ becomes:

$$(4.16) \quad g^{ab} \sum_{(i,j,k)} \omega_{ij} R^h_{iab} = 0 , \quad g^{ab} \sum_{(i,j,k)} (\nabla_a \omega_{ij}) \omega_{kb} = 0 ,$$

$$(4.17) \quad \nabla \omega = 0 , \quad g^{ab} R^k_{abi} = 0 ,$$

and

$$(4.18) \quad \omega^a R^k_{iab} = 0 .$$
Let us remark that the conditions (4.17) implies (4.16), because, if $\nabla \omega = 0$ then $\nabla_a \omega_{ij} = 0$, and $g^{ab} R^k_{ab} = 0$ implies $g^{ab} \omega_{ij} R^b_{ab} = 0$.

**Remark 4.5.** If the bivector $\sharp G_2 \Theta(\omega)$ defines a Poisson structure on $T^* M$ then $w = \sharp g \omega$ defines a Poisson structure on $M$, as the second condition (4.16) is equivalent to the Poisson condition $\sum_{(i,j,k)} w^{ia} \nabla_a w^{jk} = 0$.

(The local coordinate expression of $w$ is (1.5).)

**Corollary 4.6.** If $\sharp G_2 \Theta(\omega)$ is a Poisson bivector on $T^* M$, then the scalar curvature $r$ of $(M, g)$ vanishes.

**Proof:** The expression of $r$ is $r = g^{ab} R_{ab}$, where $R_{ba} = R^k_{akb} = R_{ab}$ are the components of the Ricci tensor, and if we make the contraction $k = i$ in the second relation (4.15) we get $g^{ab} R^k_{akb} = 0$, and whence $r = 0$.

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