Stochastic Comparisons between hitting times for Markov Chains and words’ occurrences

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Abstract

We develop some sufficient conditions for the stochastic ordering between hitting times, in a fixed state, for two Markov chains. In particular, we focus attention on the so called skip-free case. In the analysis of such a case, we develop a special type of coupling. We also compare different types of relations between two, non-necessarily skip-free, Markov chains on the same state space. Such relations have a natural role in establishing the usual and the asymptotic stochastic ordering between the probability distributions of hitting times. Finally, we present some discussions and examples related with words’ occurrences.

Key Words: Skip-free Markov chains, Coupling, Asymptotic Stochastic Ordering, Spectral-Gap, Word Occurrences, Leading Numbers.

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1 Introduction

We consider a Markov chain $X \equiv \{X_n\}_{n=0,1,\ldots}$ on the state space $E = E_k \equiv \{0,1,\ldots,k\}$ or $E = E_\infty \equiv \{0,1,\ldots\}$. We denote by $T_h$ the stopping times

$$T_h = \inf \{n \in \mathbb{N} : X_n \geq h\}, \quad h = 1,2,\ldots,k. \quad (1)$$

$T_h$ is thus the random time needed to reach or exceed the level $h$. In particular we will consider transition matrices $P = (p_{i,j})_{i,j \in E}$ that are skip-free (on the right), i.e.

$$p_{i,j} = 0 \quad \text{if } 1 \leq i + 1 < j. \quad (2)$$

In such cases, if the Markov chain starts from the state zero, $T_h$ is the hitting time of the state $h$.

Throughout the paper, we will denote by $\Upsilon_k$ the class of transition matrices on the state space $E_k$ satisfying (2). We also say that the Markov chain is skip-free if its transition matrix is in
Besides the theoretical interest, the analysis of the hitting times $T_h$ for this class of Markov chains emerges in the applications of probability to different fields such as reliability, networks, biology, and so on. The literature devoted to these topics is then very wide. See in particular [15, 9, 8, 5, 20, 21] and references cited therein, for results concerning the probability distributions of the hitting times $T_h$, under different assumptions. Skip-free Markov chains are encountered, in a fairly direct way, in the problem of first occurrences of words in random sequences of letters from a finite alphabet. This problem will be briefly recalled in the last section. A very large literature has been devoted to such a field, also, in different frameworks and from different points of view. Typically, attention has been concentrated on different aspects of the exact computation of $E(T_k)$ or of the probability distribution of $T_k$.

In this paper we rather consider, for pairs of Markov chains $X$ and $\tilde{X}$ on the same state space $E_k$, stochastic orderings between the corresponding hitting times $T_k$ and $\tilde{T}_k$. The idea of studying stochastic ordering of hitting times was already considered in the papers [10, 7, 4]. In particular, some of our results are in the same spirit of the paper by Irle and Gani, i.e. [10], who consider also the context of detection of words.

Different notions of stochastic orders might be considered for the $N$-valued random variables $T_k$ and $\tilde{T}_k$ (see e.g. [19, 16]); as natural ones in our context, we consider the usual stochastic order $T_k \preceq_{st} \tilde{T}_k$ and the tail (or asymptotic) stochastic order, that is defined in terms of tail behavior of the distributions. A rather detailed analysis of the stochastic tail order has been offered in the recent work [12].

In our results concerning the usual stochastic order, the assumption that $X$ and $\tilde{X}$ belong to $\Upsilon_k$ will be specifically used. The proof of our results in such direction will be based on a coupling method that takes essentially into account the order structure of the state space $E_k$. More precisely, on a same probability space, we construct two Markov chains (sharing the laws of $X$ and $\tilde{X}$, respectively) in such a way that they are “coupled” only in some instants when they visit the same states. When one of the two chains has a transition to a state “higher” than the other one, it stops and waits for the latter, which has an independent evolution in the meantime. A similar approach had been also developed in [6].

For the results concerning the asymptotic stochastic order, we use different methods of proofs. In such a frame, we introduce a specific order relation between two stochastic matrices of the same size. The circumstance that such a relation is maintained under products will have a relevant role in our derivations. In this part of the paper the condition of skip-free will not be necessary.

Our results may also be used to deal with continuous-time Markov chains. Processes in continuous time with a property analogous to (2) have been called free of positive skips (see [11]).

In the specific cases of word occurrences the probability distributions of $T_h$ may appear rather simple at a first glance. In particular, similarly to the geometric ones, they are completely determined by their expected values. However they manifest several apparently paradoxical aspects (see in particular [2, 3]).

Several results, in the literature concerning waiting times to words’ occurrences, have been based on the notion of leading number associated to a word. Such an analysis can appear, in a sense, alternative to the one based on Markov chains. We will point out that the two different approaches can be usefully compared and combined.
The structure of the paper is as follows. In Section 2, we present our results concerning the stochastic order between hitting times for skip-free Markov chains. In Section 3, we start by analyzing the asymptotic stochastic order for hitting times. We will also show that the obtained results can moreover be applied to establishing the usual stochastic order. In that section, we consider a more general setting, where the skip-free condition is dropped. In Section 4, we discuss and apply the results of previous sections in the context of waiting times to words’ occurrences. We also point out the interest of combining the approaches respectively based on Markov chains and leading numbers.

2 Stochastic comparisons between hitting times for skip-free Markov chains

We consider two skip-free Markov chains $X$ and $\tilde{X}$, with transition matrices $P = (p_{i,j})_{i,j \in E}$, $\tilde{P} = (\tilde{p}_{i,j})_{i,j \in E}$, and initial distributions $\pi_0 = (\pi_0(i))_{i \in E}$, $\tilde{\pi}_0 = (\tilde{\pi}_0(i))_{i \in E}$, respectively. We furthermore consider $T_h$ and $\tilde{T}_h$ where $T_h$ is defined for $X$ in (1) and $\tilde{T}_h$ is the analogue for $\tilde{X}$. For a transition matrix $P = (p_{i,j})_{i,j \in E_k}$, we will use the notation $p_{i,j}^{(n)} = (p_{i,0}^{(n)}, \ldots, p_{i,k}^{(n)})$, where $p_{i,j}^{(n)}$ denotes the transition probability from $i$ to $j$ in $n$ steps. The vector $p_{i,j}^{(n)}$ will be seen as a probability distribution over $E_k$.

As far as probability distributions of hitting times are concerned, several results have been given in [5, 8, 9, 15]. In particular, such distributions have been studied in terms of the eigenvalues of the transition matrix obtained by making the “highest” state $k$ absorbing. In [1, 5, 8] it has been shown that the distribution of $T_k$ is a convolution of geometric distributions when all the eigenvalues are non-negative real numbers.

More than on exact probability distributions, our interest is focused on sufficient conditions for stochastic comparisons between $T_h$ and $\tilde{T}_h$. The afore-mentioned distributional results cannot be directly applied for our purposes, even though they provide complete characterizations.

In this section we obtain sufficient conditions for the usual stochastic comparisons between $T_h$ and $\tilde{T}_h$. For basic definitions and properties about such a stochastic order see [14, 16].

Our first result can be seen as an extension of Theorem 4.1 in [10]. Actually, we obtain a same type of conclusion still under easily-applicable conditions that are however weaker and more flexible. The proof of our result is based on a coupling method, as proposed in [6].

**Theorem 1.** Let $P = (p_{i,j} : i, j = 0, \ldots, k)$ and $\tilde{P} = (\tilde{p}_{i,j} : i, j = 0, \ldots, k)$ be two transition matrices in $\Upsilon_k$. Assume that, for any $i = 0, \ldots, k - 1$, there exists $m(i)$ such that

i) $i + m(i) \leq k$;

ii) $p_{i,j}^{(m(i))} \succeq_{st} \tilde{p}_{i,j}^{(m(i))}$.

Moreover suppose that the initial measures are stochastically ordered $\pi \succeq_{st} \tilde{\pi}$. Then

$$T_k \succeq_{st} \tilde{T}_k.$$  (3)
We defer the proof of Theorem 1 after a couple of remarks.

**Remark 1.** In Theorem 4.1 in [10], the case where \( m(i) = 1 \) for \( i = 1, \ldots, k - 1 \) is considered. Our hypothesis, allowing \( m(i) \) to vary with \( i \), permits to apply our result to a wider class of cases. An instance of such a situation is presented in Example 1 below.

Our thesis only aims to compare \( T_k \) with \( \tilde{T}_k \), where \( k \) is the highest level, which is typically the one of specific interest.

The result in [10] on the contrary permits to compare \( T_h \) with \( \tilde{T}_h \), for any \( h = 0, 1, \ldots, k \). However, we can be actually interested to prove \( T_k \preceq_{st} \tilde{T}_k \), even in situations where \( \tilde{T}_h \preceq_{st} T_h \) for some \( h < k \).

**Remark 2.** The method of proof of Theorem 1 can be also convenient for implementation in computer programs and it is based on the skip-free property of Markov chains. A similar method of coupling had been already implemented in [6] with the aim to simulate such Markov chains and to analyze Theorem 4.1 in [10].

In our context, we also notice that such a method leads to efficient estimates of the difference between expected values of two different hitting times. In fact, as can be easily proven, it is more accurate that one based on the separate estimates of the two expected values. This circumstance turns out to be useful in several situations of interest.

For instance, in the comparison between waiting times to words’ occurrences, where the expected values can be extremely large and their differences relatively small, the estimate of expected values might reveal numerically inaccurate if compared with direct estimation of the difference between them.

**Proof.** Let us fix a particular choice of \( m(1), \ldots, m(k - 1) \) such that i) and ii) hold. Moreover, for future convenience, we conventionally fix \( m(k) = 1 \). We will use a coupling method and we will obtain the proof in a recursive way.

On a same probability space \( (\Omega, \mathcal{F}, P) \), we define a sequence of i.i.d. random variables \( U = \{U_n\}_{n \in \mathbb{N}} \) and an independent array of i.i.d. random variables \( \tilde{U} = \{\tilde{U}_{k,n}\}_{k \in \mathbb{N}, n \in \mathbb{N}_+} \). All these variables have uniform distribution on \([0, 1]\).

By using \( U \), we will construct on \( (\Omega, \mathcal{F}, P) \) a homogeneous Markov chain \( X = (X_n)_{n \in \mathbb{N}} \) having the law given by the initial distribution \( \pi = (\pi_0, \ldots, \pi_k) \) and transition matrix \( P = (p_{i,j})_{i,j \in E_k} \). We will also construct, by using \( U \) and \( \tilde{U} \), a homogeneous Markov chain \( \tilde{X} = (\tilde{X}_n)_{n \in \mathbb{N}} \) having the law given by the initial distribution \( \tilde{\pi} = (\tilde{\pi}_0, \ldots, \tilde{\pi}_k) \) and transition matrix \( \tilde{P} = (\tilde{p}_{i,j})_{i,j \in E_k} \). We will prove that the stopping times \( T_k \) and \( \tilde{T}_k \) corresponding to \( X \) and \( \tilde{X} \) respectively, are ordered in the sense that

\[
T_k(\omega) \leq \tilde{T}_k(\omega),
\]

for each \( \omega \in \Omega \).

First, we define \( X_0 \) and \( \tilde{X}_0 \) with distribution \( \pi \) and \( \tilde{\pi} \), respectively.

We set

\[
X_0(U_0) := \inf \{ i \leq k : \sum_{l=0}^{i} \pi_l \geq U_0 \}, \tag{4}
\]
and analogously
\[
\tilde{X}_0(U_0) := \inf\{i \leq k : \sum_{l=0}^{i} \tilde{\pi}_l \geq U_0\}.
\] (5)

It is immediately seen that \(X_0(U_0) \sim_d \pi\) and \(\tilde{X}_0(U_0) \sim_d \tilde{\pi}\). Furthermore, for each value \(u \in [0, 1]\) \(X_0(u) \geq \tilde{X}_0(u)\), in view of the assumption \(\pi \succeq st \tilde{\pi}\).

Letting \(I(0) = I(0; U_0) := X_0(U_0)\), and recalling the meaning of \(m(1), \ldots, m(k-1), m(k) = 1\), we recursively define, for \(n = 1, 2, \ldots\)
\[
X_{m(I(0)) + \ldots + m(I(n-1))} = X_{m(I(0)) + \ldots + m(I(n-1))}(U_0, \ldots, U_n) := \inf\{s \leq k : \sum_{l=0}^{s} p_{I(I(n-1), l)} \geq U_n\},
\] (6)
\[
I(n) = I(n; U_0, \ldots, U_n) := X_{m(I(0)) + \ldots + m(I(n-1))}.
\] (7)

We notice that, if for a given \(n\), we obtain \(I(n) = k\) then also
\[
I(n + 1) = k \quad \text{and} \quad X_{m(I(0)) + \ldots + m(I(n-1)) + 1} = k, \quad a.s.
\]

We claim that
\[
T_k = \sum_{r=1}^{L} m(I(r)),
\] (8)
where \(L\) is the random index
\[
L := \inf\{l \in \mathbb{N} : I(l) = k\}.
\] (9)

We stipulate that the sum (8) is equal to zero if \(L = 0\). It is clear that, when \(L = 0\), (8) holds. Now we prove that also in the case \(L > 0\) the expression for \(T_k\) given in (8) holds true. In fact, at least the inequality \(T_k \leq \sum_{r=1}^{L} m(I(r))\) holds, since \(I(L) = X_{\sum_{r=1}^{L} m(I(r))} = k\), in view of positions (6) and (7).

We then want to show that for any \(t < \sum_{r=1}^{L} m(I(r))\) one has \(X_t \neq k\). Let us first consider the values \(a_s = \sum_{r=1}^{s} m(I(r))\) with \(s = 1, \ldots, L - 1\). For these values, \(X_{a_s} = k\) would contradict the position (9).

Let us then consider the discrete intervals of the form \(B_s = \{a_s + 1, \ldots, a_{s+1} - 1\}\), with \(s \in \{1, \ldots, L - 1\}\) and such that \(a_s + 1 \leq a_{s+1} - 1\). For \(a \in B_s\), it is impossible that \(X_a = k\). In fact \(a + m(a) \leq k\) for \(a \leq k - 1\) and \(X_c - X_b \leq (c - b)\) for any \(c \geq b\).

We now proceed to construct \(\bar{T}_k\). To this purpose we consider a sequence of independent Markov chains \(\{\bar{Y}^{(r)}\}_{r \in \mathbb{N}}\). For any \(r = 0, 1, \ldots\), the Markov chain \(\bar{Y}^{(r)} = \{\bar{Y}^{(r)}_n\}_{n \in \mathbb{N}}\) will be such that \(\bar{Y}^{(r)}_0 = 0\) with probability one and it will admit \(\bar{P}\) as transition matrix. More precisely \(\bar{Y}^{(r)}\) is constructed in terms of \(\{\bar{U}_{r,1}, \bar{U}_{r,2}, \ldots\}\) as follows: for \(r = 1, 2, \ldots\)
\[
I^{(r)}(n - 1) := \bar{Y}^{(r)}_{n-1}(\bar{U}_{r,1}, \ldots, \bar{U}_{r,n-1}),
\] (10)
\[
\bar{Y}^{(r)}_n(\bar{U}_{r,1}, \ldots, \bar{U}_{r,n}) := \inf\{s \leq k : \sum_{l=0}^{s} \bar{p}_{I^{(r)}(n-1), l} \geq \bar{U}_{r,n}\}.
\] (11)
As a function of $U_0, U_1, \ldots$, we now also define the sequence $\tilde{Y} = \{\tilde{Y}_n\}_{n \in \mathbb{N}}$ as follows:

$$\tilde{Y}_0 = \tilde{Y}_0(U_0) := \tilde{X}_0(U_0)$$

(12)

$$\tilde{Y}_n(U_0, \ldots, U_n) := \inf\{s \leq k : \sum_{l=0}^{s} p_{I(l-1), l} \geq U_n\}.$$  (13)

Notice that the random variables $I(0), I(1), \ldots$ appearing in r.h.s. of (13) have been defined in (7). Furthermore we have, by construction, $\tilde{Y}_n(U_0, \ldots, U_n) \leq I(n; U_0, \ldots, U_n)$ in view of condition ii). The sequences $\tilde{Y}$ and $\{\tilde{Y}^{(r)}\}_{r \in \mathbb{N}}$ are stochastically independent.

Let now, for $r \in \mathbb{N}$,

$$N_1^{(r)} := \inf\{n \in \mathbb{N} : \tilde{Y}_n^{(r)} = \tilde{Y}_r\},$$

(14)

$$N_2^{(r)} := \inf\{n \in \mathbb{N} : \tilde{Y}_n^{(r)} = I(r)\}.$$  (15)

We notice that, for any $r = 0, 1, 2, \ldots$, $N_1^{(r)} \leq N_2^{(r)}$ since the chain $\tilde{Y}^{(r)}$ starts in zero, it increases at most of one unit at any step, and $\tilde{Y}_r \leq I(r)$. For any $r \in \mathbb{N}$, $N_1^{(r)}$ and $N_2^{(r)}$ are two stopping times with respect to the filtration $(\mathcal{F}_n^{(r)})_{n \in \mathbb{N}}$ where $\mathcal{F}_0^{(r)} = \sigma(\tilde{Y}_r, I(r))$ and

$$\mathcal{F}_n^{(r)} = \sigma(\tilde{Y}_r, I(r), U_{r1}, \ldots, U_{rn})$$

for any $n = 1, 2, \ldots$.

Now we consider the random variables

$$Z_r := \sum_{n=0}^{r} (N_2^{(n)} - N_1^{(n)}) + \sum_{n=0}^{r} m(I(n)),$$  (16)

for $r = 1, \ldots, L$ and

$$\tilde{X}_{Z_r} := \tilde{Y}_r.$$  (17)

Notice that, letting $r = L$ in (16), one has

$$Z_L := \sum_{n=0}^{L} (N_2^{(n)} - N_1^{(n)}) + \sum_{n=0}^{L} m(I(n)) = \sum_{n=0}^{L} (N_2^{(n)} - N_1^{(n)}) + T_k,$$ (18)

Furthermore, by recalling definition (14), $\tilde{X}_{Z_r} = \tilde{Y}^{(r)}_{N_1^{(r)}}$.

We now consider the following sequence of random variables:

$$\tilde{Y}_0, \tilde{Y}^{(0)}_{N_1^{(0)}+1}, \ldots, \tilde{Y}^{(0)}_{N_2^{(0)}}, \tilde{Y}_1, \tilde{Y}^{(1)}_{N_1^{(1)}+1}, \ldots, \tilde{Y}^{(1)}_{N_2^{(1)}}, \tilde{Y}_2,$$  (19)

obtained by gluing together the sections of trajectories

$$\tilde{Y}_0; \tilde{Y}^{(0)}_{N_1^{(0)}+1}; \ldots, \tilde{Y}^{(0)}_{N_2^{(0)}}; \tilde{Y}_1; \tilde{Y}^{(1)}_{N_1^{(1)}+1}; \ldots, \tilde{Y}^{(1)}_{N_2^{(1)}}; \tilde{Y}_2; \ldots$$
Notice that some of the sections $\tilde{Y}^{(r)}_{N_1^{(r)}+1}, \ldots, \tilde{Y}^{(r)}_{N_2^{(r)}}$ can be missing. This happens when $\tilde{Y}_r = X_r$.

We now set
\[
\tilde{X}_{Z+r} := \tilde{Y}^{(r)}_{N_1^{(r)}+i}, \quad i = 1, \ldots, N_2^{(r)} - N_1^{(r)}.
\] (20)

In view of the strong Markov property, the joint probability distribution of the random variables $\tilde{X}$’s defined by (17) and (20) coincides, by construction, with a finite dimensional distribution for a Markov chain with initial law $\tilde{\pi}$ and transition matrix $\tilde{P}$. The random variables $\tilde{X}$’s have not been defined for any time $t \in \mathbb{N}$. By Kolmogorov’s existence theorem, we can consider however the entire chain $\tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \ldots$ by suitably adding variables at the missing times. From (9) and (18), we have
\[
\tilde{X}_{T_k + \sum_{r=0}^L (N_2^{(r)} - N_1^{(r)})} = k.
\] (21)

Furthermore, by repeating the same argument used above, we can also obtain
\[
\tilde{X}_{l} < k,
\] (22)

for $l = 0, \ldots, T_k + \sum_{r=0}^L (N_2^{(r)} - N_1^{(r)}) - 1$. Thus
\[
\tilde{T}_k = T_k + \sum_{r=0}^L (N_2^{(r)} - N_1^{(r)}),
\] (23)

therefore $\tilde{T}_k \geq T_k$, whence the stochastic comparison in (3) follows.

\[\square\]

**Remark 3.** The validity of the relation (23), proven above, is much more informative than simply $\tilde{T}_k \preceq T_k$, and it allows in particular to provide different types of inequalities.

For our purposes we reformulate Theorem 4.1 in [10] as follows. Such a result gives a stronger conclusion with respect to Theorem 1 but under much stronger conditions.

**Theorem 2.** Let $X$ and $\tilde{X}$ be skip-free Markov chains with transition matrices $P = (p_{i,j})_{i,j \in E}$, $\tilde{P} = (\tilde{p}_{i,j})_{i,j \in E}$, and initial distributions $\pi_0 = (\pi_0(i))_{i \in E}$, $\tilde{\pi}_0 = (\tilde{\pi}_0(i))_{i \in E}$, respectively. Under the conditions
\[ p_i, \geq_{st} \tilde{p}_i, \text{ for each } i = 0, \ldots, k-1, \] (24)
\[ \pi_0 \geq_{st} \tilde{\pi}_0, \] (25)

one has the stochastic comparison
\[ T_h \preceq_{st} \tilde{T}_h, \text{ for } h = 1, \ldots, k, \] (26)

where $T_h$ is defined for $X$ in (1) and $\tilde{T}_h$ is the analogue for $\tilde{X}$.

We present an example in which the hypothesis of Theorem 1 are satisfied but both the hypothesis and the thesis of Theorem 2 fail.
Example 1. Let us consider
\[ E = \{0, 1, 2, 3\} \] and the transition matrices:
\[ P = \begin{pmatrix}
\frac{1}{2} + \epsilon & \frac{1}{2} - \epsilon & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \tilde{P} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
1 - \epsilon & 0 & \epsilon & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (27)\]
where \( \epsilon \in (0, \frac{1}{2}) \). We assume that the initial measure of both Markov chains is concentrated on the state zero. For no value \( \epsilon \in (0, \frac{1}{2}) \) hypothesis of Theorem 2 is satisfied. The first rows of the matrices \( P \) and \( \tilde{P} \) are respectively given by
\[ p^{(2)}_{0, \cdot} = \left( \frac{1}{2} + \epsilon \right)^2, \frac{1}{2} \right) = \left( 3 - 2\epsilon, 1 - \epsilon \right). \quad \tilde{p}^{(2)}_{0, \cdot} = \left( \frac{3 - 2\epsilon}{4}, \frac{1}{4}, \frac{\epsilon}{2}, 0 \right). \quad (28)\]
Therefore we can take \( m(0) = 2 \) and \( m(1) = m(2) = 1 \) to verify the hypothesis of Theorem 1, when \( \epsilon \) is small enough. This shows that \( T_3 \preceq_{st} \tilde{T}_3 \), but \( \tilde{T}_1 \preceq_{st} T_1 \).

The following result appears, at a first glance, to be similar to Theorem 2. However it offers a much wider range of applications. In Section 4, examples will be presented in the frame of word occurrences. Also the proof of this result can be obtained along the same line of Theorem 1, and will then be omitted.

Theorem 3. Given two transition matrices in the space \( \Upsilon_k \), namely \( P = (p_{i,j} : i, j = 0, \ldots, k) \) and \( \tilde{P} = (\tilde{p}_{i,j} : i, j = 0, \ldots, k) \). Let the initial measures be \( \pi = \tilde{\pi} = \delta_0 \) (both the Markov chains start in zero almost surely). Suppose that there exists an integer \( m \in [1, k - 1] \) such that
\[ (i) \quad \tilde{T}_m \succeq_{st} T_m, \]
\[ (ii) \text{ for each } i \in [m, k - 1] \quad \tilde{p}_{i,i+1} \leq p_{i,i+1} \text{ and } \tilde{p}_{i,0} + \tilde{p}_{i,i+1} = 1. \]
Then \( \tilde{T}_i \succeq_{st} T_i \) for \( i \in [m, k] \).

3 Analysis based on the asymptotic stochastic comparisons

In this section, we consider the tail behaviors of hitting times \( T_k \) and \( \tilde{T}_k \) corresponding to two Markov chains. More precisely, we introduce in our analysis the asymptotic stochastic comparison between \( T_k \) and \( \tilde{T}_k \), according to the following definition.

Definition 1. Given two random variables \( X \) and \( Y \) we write \( X \preceq_{st} Y \) if there exists \( t_0 \in \mathbb{R} \) such that \( P(X > t) \geq P(Y > t) \), for each \( t \geq t_0 \).

Consider the equivalence classes formed by probability distributions over \( \mathbb{R} \), that admit the same right tail. By considering the quotient sets with respect to such equivalence relation the asymptotic stochastic order is a partial order.

Such a notion of ordering can find interesting applications in probability (see also [12] and references therein).
We point out that special conditions, such as the skip-free property, are not needed in this section.

The following definition will be relevant to analyze the condition $T_k \preceq_{a.st.} \tilde{T}_k$. It will moreover provide useful information to check the condition $T_k \preceq_{a.st.} \tilde{T}_k$.

**Definition 2.** Let $A = (a_{i,j} : i, j = 0, \ldots, k)$ and $A' = (a'_{i,j} : i, j = 0, \ldots, k)$ be stochastic matrices. We write $A' \preceq A$ if and only if

\[ a'_{i,j} \preceq_{st} a_{i,j}, \quad \forall i \leq j \leq k. \tag{29} \]

**Remark 4.** The relation $\preceq$ is stronger than (24) and it is transitive as well. However it is not reflexive. In this respect we have that the relation $P \preceq P$ holds if and only if $P$ is stochastically monotone. More in general one can see that $A' \preceq A$ holds if and only if a stochastically monotone matrix $B$ exists such that $A' \preceq B \preceq A$, see [16] for a general treatment of related topics.

In view of our purposes, and for simplicity's sake, we consider in what follows that the state $k$ is absorbing. This assumption however is not restrictive at all. We also assume that the initial measure for all the chains is concentrated on the state zero.

The interest of Definition 2 in the present contest dwells in the next result.

**Theorem 4.** Let $X = (X_n)_{n \in \mathbb{N}}$ and $\tilde{X} = (\tilde{X}_n)_{n \in \mathbb{N}}$ be two Markov chains on $E_k = \{0, \ldots, k\}$, both having initial measure concentrated on zero, with transition matrices $P$ and $\bar{P}$ respectively. Let $k$ be an absorbing state for both the matrices. Furthermore let $\tilde{P}^n \preceq P^n$ for all $n$ large enough then $T_k \preceq_{a.st.} \tilde{T}_k$.

**Proof.** We want to check that $P(T_k > L) \leq P(\tilde{T}_k > L)$ for $L$ large enough. We remark, since the state $k$ is absorbing, that the identity $\{T_k > L\} = \{X_L \neq k\}$ holds. Then $P(T_k > L) = 1 - P(L)_{0,k}$ and similarly $P(\tilde{T}_k > L) = 1 - P(L)_{0,k}$. In fact the condition $\tilde{P}^L \preceq P^L$ in particular implies $P(L)_{0,k} \preceq P(L)_{0,k}$. \hfill $\Box$

We now present some results concerning the condition $P^n \preceq \tilde{P}^n$ for $n$ large enough. First we give a probabilistic characterization of the relation $\preceq$.

**Lemma 1.** Let $A$, $A'$ be stochastic matrices on the state space $E = \{0, \ldots, k\}$. $A \preceq A'$ if and only if a Markov chain $(Z_n)_{n=0,1}$ with $Z_n = (Y_n, Y'_n)$ on the state space $E^2$ exists with the following properties:

i) $(Y_n)_{n=0,1}$ is a Markov chain with transition matrix $A$. $(Y'_n)_{n=0,1}$ is a Markov chain with transition matrix $A'$.

ii) $P(Y_1 \leq Y'_1 | Y_0 = i, Y'_0 = i') = 1$, for $i, i' \in E$, with $i \leq i'$.

**Proof.** Assume $A \preceq A'$. Let us define two functions $\phi, \phi' : [0,1] \rightarrow E$ as follows

\[ \phi(u) := \inf \{ r \in E : \sum_{l=0}^{r} a_{i,l} \geq u \}, \quad \phi'(u) := \inf \{ r \in E : \sum_{l=0}^{r} a'_{i,l} \geq u \}. \tag{30} \]
We now consider the two random variables \( Y_1 = \phi(U) \) and \( Y'_1 = \phi'(U) \), where \( U \) is a uniform r.v. over \([0,1]\). Let \( Y_0, Y'_0 \) be two \( E \)-valued random variables such that any pair in \( E^2 \) is taken with positive probability by \((Y_0, Y'_0)\) and independent of \( U \).

Then, conditionally on \( Y_0 = i \) (resp. \( Y'_0 = i' \)), \( Y_1 \) (resp. \( Y'_1 \)) has the law \( a_{i,}. \) (resp. \( a'_{i,.} \)). Furthermore ii) holds in view of (30).

Viceversa, if i) and ii) hold, then (29) follows by definition of stochastic ordering. \( \square \)

The relation \( \preceq \) is maintained under products of transition matrices. We provide a direct proof based on probabilistic arguments.

**Lemma 2.** If \( A, A', B, B' \) are stochastic matrices of order \( k \) such that \( A \preceq A' \) and \( B \preceq B' \) then \( AB \preceq A'B' \). If \((A_i)_{i=1,\ldots,n}\) and \((B_i)_{i=1,\ldots,n}\) are stochastic matrices such that \( A_i \preceq B_i \), for \( i = 1, \ldots, n \), then \( A_1 A_2 \ldots A_n \preceq B_1 B_2 \ldots B_n \).

**Proof.** We want to construct two non-homogeneous Markov chains \((Y_n)_{n=0,1,2}\), \((Y'_n)_{n=0,1,2}\), with the following properties. Let \( Y_0, Y'_0 \) be two \( E \)-valued random variables such that any pair in \( E^2 \) is taken with positive probability by \((Y_0, Y'_0)\).

Furthermore, the transition matrix of \((Y_n)_{n=0,1,2}\) is \( A \) for the first step and \( B \) for the second step. Analogously for \((Y'_n)_{n=0,1,2}\) with \( A' \) and \( B' \).

On this purpose we consider two independent random variables \( U_1 \) and \( U_2 \) uniformly distributed over \([0,1]\), also independent on \((Y_0, Y'_0)\). Now we set, similarly to the proof of Lemma 1,

\[
Y_1 := \inf\{r \in E : \sum_{l=0}^{r} a_{i,l} \geq U_1 \}, \quad Y'_1 := \inf\{r \in E : \sum_{l=0}^{r} a'_{i,l} \geq U_1 \}, \quad (31)
\]

\[
Y_2 := \inf\{r \in E : \sum_{l=0}^{r} b_{i,l} \geq U_2 \}, \quad Y'_2 := \inf\{r \in E : \sum_{l=0}^{r} b'_{i,l} \geq U_2 \}. \quad (32)
\]

Finally define \( X_0 = Y_0, X_1 = Y_2, X'_0 = Y'_0 \) and \( X'_1 = Y'_2 \). The random variables \((X_n)_{n=0,1}\) form a Markov chain with transition matrix \( AB \); also \((X'_n)_{n=0,1}\) is a Markov chain with transition matrix \( A'B' \). The random pairs \((Z_n)_{n=0,1}\) defined by \( Z_n = (X_n, X'_n) \) can be seen as a Markov chain on the state space \( E^2 \). In view of Lemma 1, the relation \( AB \preceq A'B' \) is proven by checking that

\[
P(X_1 \leq X'_1|X_0 = i, X'_0 = i') = P(Y_2 \leq Y'_2|Y_0 = i, Y'_0 = i') =
\]

\[
\sum_{i_1 \leq i'_1} P(Y_2 \leq Y'_2, Y_1 = i_1, Y'_1 = i'_1|Y_0 = i, Y'_0 = i') =
\]

\[
= \sum_{i_1 \leq i'_1} P(Y_2 \leq Y'_2|Y_1 = i_1, Y'_1 = i'_1)P(Y_1 = i_1, Y'_1 = i'_1|Y_0 = i, Y'_0 = i'). \quad (34)
\]

As to the r.h.s. of (33) we obtain, by the Markov property of \( Z_n \),

\[
\sum_{i_1 \leq i'_1} P(Y_2 \leq Y'_2, Y_1 = i_1, Y'_1 = i'_1|Y_0 = i, Y'_0 = i') =
\]

\[
= \sum_{i_1 \leq i'_1} P(Y_2 \leq Y'_2|Y_1 = i_1, Y'_1 = i'_1)P(Y_1 = i_1, Y'_1 = i'_1|Y_0 = i, Y'_0 = i'). \quad (34)
\]

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We also have \( P(Y_2 \leq Y_2' | Y_1 = i_1, Y_1' = i_1') = 1 \), in view of Lemma 1. Therefore the r.h.s. of (34) becomes \( P(Y_1 \leq Y_1' | Y_0 = i, Y_0' = i') \). The latter term is equal to 1 by (31) and this concludes the proof of the first part. The second part is readily obtained from the first part by induction and by taking into account associativity of the product of matrices.

The next Theorem 5 has an immediate application to our problem. It in fact provides an (apparently weaker) condition sufficient for the hypothesis appearing in Theorem 4. Actually it gives an easily implementable condition to check the asymptotic stochastic comparison between the hitting times for two different Markov chains.

**Theorem 5.** Let \( \tilde{P} \) and \( P \) be stochastic matrices of order \( k \). Assume that there exist two coprime integers \( n_1 \) and \( n_2 \) such that
\[
\tilde{P}^{n_1} \preceq P^{n_1} \quad \text{and} \quad \tilde{P}^{n_2} \preceq P^{n_2},
\]
then
\[
\tilde{P}^n \preceq P^n \quad \text{for } n \geq \hat{n}(n_1, n_2) := \inf \{ r : \forall r' \geq r, \quad r' = an_1 + bn_2 \text{ with } a, b \in \mathbb{N} \}.
\]

**Proof.** For \( n \geq \hat{n}(n_1, n_2) \) we can write, by definition of \( \hat{n}(n_1, n_2) \), \( n = an_1 + bn_2 \) for convenient natural numbers \( a, b \). Thus we can write \( P^n = (P^{n_1})^a(P^{n_2})^b \), \( \tilde{P}^n = (\tilde{P}^{n_1})^a(\tilde{P}^{n_2})^b \). Then (35) is readily obtained by Lemma 2.

The number \( \hat{n}(n_1, n_2) \) can be found using the Euclidean algorithm. We notice that, in any case, \( \hat{n}(n_1, n_2) \leq n_1 n_2 \).

We already mentioned that the results of this section, concerning the asymptotic stochastic ordering, can be used also to check the usual stochastic ordering. A sufficient condition for \( T_k \preceq_{st} \tilde{T}_k \) can be obtained as a simple consequence of Theorem 5.

**Proposition 1.** Let \( X = (X_n)_{n \in \mathbb{N}} \) and \( \tilde{X} = (\tilde{X}_n)_{n \in \mathbb{N}} \) be two Markov chains on \( E_k = \{0, \ldots, k\} \), both having initial measure concentrated on zero, with transition matrices \( P \) and \( \tilde{P} \) respectively. Let \( k \) be an absorbing state for both the matrices. Assume that there exists two coprime \( n_1 \) and \( n_2 \) such that \( \tilde{P}^{n_1} \preceq P^{n_1} \) and \( \tilde{P}^{n_2} \preceq P^{n_2} \), then

a. \( T_k \preceq_{a.st.} \tilde{T}_k \).

If, moreover, \( \tilde{p}_{0,k}^{(n)} \leq p_{0,k}^{(n)} \) for \( n = 1, \ldots, \hat{n}(n_1, n_2) - 1 \) then

b. \( T_k \preceq_{st} \tilde{T}_k \).

**Proof.** The condition \( T_k \preceq_{a.st.} \tilde{T}_k \) is equivalent to \( \tilde{p}_{0,k}^{(n)} \leq p_{0,k}^{(n)} \), for \( n \) large enough, (see also the proof of Theorem 4). We then obtain item a. as a consequence of Theorem 5. In fact the condition \( \tilde{P}^n \preceq P^n \) in particular implies \( \tilde{p}_{0,k}^{(n)} \leq p_{0,k}^{(n)} \).

The condition \( T_k \preceq_{st} \tilde{T}_k \) is equivalent to \( \tilde{p}_{0,k}^{(n)} \leq p_{0,k}^{(n)} \), for \( n = 1, 2, \ldots \). When the hypothesis of Theorem 5 holds, checking \( T_k \preceq_{st} \tilde{T}_k \) only requires that \( \tilde{p}_{0,k}^{(n)} \leq p_{0,k}^{(n)} \), for \( n = 1, 2, \ldots, \hat{n}(n_1, n_2) - 1 \), and then item b. is obtained.
We notice that the conditions given in Theorem 5 can be encountered rather often. A simple sufficient condition for its hypothesis will be presented next. To this purpose we need the following notation.

Given a stochastic matrix \( P = (p_{i,j} : i, j = 0, \ldots, k) \), such that \( p_{i,k} < 1 \) for \( i = 0, \ldots, k - 1 \), denote by \( (k)P = ((k)p_{i,j} : i, j = 0, \ldots, k - 1) \) the matrix obtained from \( P \) by making \( k \) a taboo state: \( (k)p_{i,j} = p_{i,j}/(1 - p_{i,k}) \). In view of our focusing on the hitting time in the state \( k \), it is not restrictive to assume \( p_{k,k} = 1 \).

Let \( \mu = \mu(P) \) the modulus of the second eigenvalue of \( P \) and denote by \( \lambda(P) \) the spectral gap of \( P \) i.e.

\[
\lambda(P) = 1 - \mu(P).
\]

Moreover, for a stochastic matrix \( P \), we use the term *ergodic* to designate the condition that there exists a positive integer \( m \) such that all the elements of \( P^m \) are strictly positive.

The eigenvalues of the transition matrix, admitting \( k \) as an absorbing state, determine the probability distribution of \( T_k \), under different types of conditions such as skip-free or reversibility [1, 5, 8, 9, 15, 21]. Here we are exclusively interested on the asymptotic stochastic ordering between the hitting times for two different Markov chains. This restriction allows us to focus attention on the second eigenvalue and to require a mild condition (ergodicity of \( (k)P \)), only.

**Theorem 6.** Let \( \tilde{P} \) and \( P \) be two stochastic matrices on the state space \( E = \{0, \ldots, k\} \). Suppose that \( \tilde{p}_{k,k} = p_{k,k} = 1 \) and that \( (k)\tilde{P} \) is ergodic. Assume furthermore that \( \lambda(\tilde{P}) < \lambda(P) \). Then there exists \( n_0 \) such that \( \tilde{P}^n \leq P^n \), for \( n \geq n_0 \).

**Proof.** First we notice that \( \lambda(P) \) is larger than zero. This condition guarantees that, from each state \( i \in E \), the Markov chain associated to \( P \) and starting in 0 can reach the state \( k \in E \) in a finite number of steps, almost surely. Actually, we will more precisely prove that there exists \( C > 0 \) such that the following inequality holds for any positive integer \( n \) and \( i, j \in \{0, \ldots, k - 1\} \)

\[
p_{i,j}^{(n)} \leq n^k C(1 - \lambda(P))^n.
\]

Then, by Borel-Cantelli Lemma, the probability is zero of remaining in \( E \setminus \{k\} \) for an infinite number of steps.

Concerning the matrix \( \tilde{P} \) we will prove, on the other hand, that there exists \( \tilde{c} > 0 \) such that for \( n \) large enough

\[
\tilde{p}_{i,j}^{(n)} \geq \tilde{c}(1 - \lambda(\tilde{P}))^n.
\]

If (37) and (38) hold, we get, for \( n \) large enough, the inequalities

\[
\sum_{j=0}^{l} \tilde{p}_{i,j}^{(n)} \geq \sum_{j=0}^{l} p_{i,j}^{(n)}
\]

for any \( l \in \{0, \ldots, k-1\} \) and any pairs \( i, \hat{i} \in \{0, \ldots, k-1\} \). This, in particular, guarantees \( \tilde{P}^n \leq P^n \) for \( n \) large enough and concludes the proof.
In order to get the inequality in (37) we can consider the Jordan representation $P = A^{-1}JA$ for the stochastic matrix $P$. In this representation we arrange in the increasing order the absolute values of eigenvalues on the main diagonal and in particular $(J)_{k,k} = 1$.

Thus we can write $|(J^n)_{i,j}| \leq n^k(1 - \lambda(P))^n$, for $i, j \in \{0, \ldots, k - 1\}$. By $P^n = A^{-1}J^nA$ we obtain (37) in view of the assumption $\lambda(P) > 0$.

In order to show (38) we first notice that there is only one eigenvalue of modulus $(1 - \lambda(\tilde{P}))$ as a consequence of Perron-Frobenius theorem, see [18]. We will denote by $\tilde{\mu}$ such an eigenvalue, which is a positive real number (again as a consequence of Perron-Frobenius theorem).

We start considering the case where $\lambda(\tilde{P}) > 0$. In such a case, similarly to above, we use the Jordan representation of $\tilde{P}$, i.e. $\tilde{P} = \tilde{A}^{-1}J\tilde{A}$. We explicitly write

$$
\tilde{P}^{(n)}_{i,j} = \sum_{l=0}^{k} \sum_{m=0}^{k} (\tilde{A}^{-1})_{i,l} (\tilde{J}^n)_{l,m} (\tilde{A})_{m,j}.
$$

To fix ideas again we consider the case in which $(\tilde{J})_{k-1,k-1} = \tilde{\mu}$ (the second highest eigenvalue) and $(\tilde{J})_{k,k} = 1$. Furthermore, in (39), we are allowed to limit attention only to indexes $0 \leq i \leq k - 1$. In fact, the terms $\tilde{P}^{(n)}_{k,j}$ are zero for $j = 0, \ldots, k - 1$ and one for $j = k$. From (39) we obtain

$$
\tilde{P}^{(n)}_{i,k} = (\tilde{A}^{-1})_{i,k}(\tilde{A})_{k,k} + (\tilde{A}^{-1})_{i,k-1}\tilde{\mu}^n(\tilde{A})_{k-1,k} + o(\tilde{\mu}^n),
$$

where $(\tilde{A}^{-1})_{i,k}(\tilde{A})_{k,k} = 1$, for $i = 1, \ldots, k$, since the condition $\lambda(\tilde{P}) > 0$ guarantees $\lim_{n \to \infty} \tilde{P}^{(n)}_{i,k} = 1$ and

$$
\lim_{n \to \infty} (\tilde{A}^{-1})_{i,k-1}\tilde{\mu}^n(\tilde{A})_{k-1,k} + o(\tilde{\mu}^n) = 0.
$$

We also obtain from (39), for $j = 0, \ldots, k - 1$,

$$
\tilde{P}^{(n)}_{i,j} = (\tilde{A}^{-1})_{i,k}(\tilde{A})_{k,j} + (\tilde{A}^{-1})_{i,k-1}\tilde{\mu}^n(\tilde{A})_{k-1,j} + o(\tilde{\mu}^n),
$$

but $(\tilde{A}^{-1})_{i,k}(\tilde{A})_{k,j} = 0$ because the limit $\lim_{n \to \infty} \tilde{P}^{(n)}_{i,j} = 0$.

In this respect we claim that the products $(\tilde{A}^{-1})_{i,k-1}\tilde{A}_{k-1,j}$ in (41) can not be all equal to zero. In fact, if this were the case, all the terms $\tilde{P}^{(n)}_{i,j}$ would not depend on $\tilde{\mu}$ which is absurd.

Therefore there exists $i, j \in \{0, 1, \ldots, k - 1\}$ such that $(\tilde{A}^{-1})_{i,k-1}\tilde{A}_{k-1,j} \neq 0$. One can also deduce that $(\tilde{A}^{-1})_{i,k-1}\tilde{A}_{k-1,j}$ is a positive real number. In fact, from (41), one obtain that

$$
\tilde{P}^{(n)}_{i,j} = (\tilde{A}^{-1})_{i,k-1}\tilde{A}_{k-1,j}\tilde{\mu}^n + o(\tilde{\mu}^n),
$$

with $i, j \in \{0, 1, \ldots, k - 1\}$. Therefore $(\tilde{A}^{-1})_{i,k-1}\tilde{A}_{k-1,j} > 0$ because $\tilde{\mu}^n > 0$ and $\tilde{P}^{(n)}_{i,j} > 0$. Furthermore, for $n$ large enough, $\tilde{P}^{(n)}_{i,j} > \tilde{c}\tilde{\mu}^n$, where we let $\tilde{c} := \frac{1}{2}(\tilde{A}^{-1})_{i,k-1}\tilde{A}_{k-1,j}$.
Remark 6. = 1

\[ \mu_{\tilde{P}} \]

\[ \alpha, \beta \]

transition matrices

Remark 5.

\[ \text{same state space.} \]

efficient tool to check stochastic orderings between the hitting times of two Markov chains on a same state space.

The hypotheses of Theorem 6 are met rather frequently and then Proposition 1 provides an efficient tool to check stochastic orderings between the hitting times of two Markov chains on a same state space.

\[ \text{Remark 6.} \]

As a consequence of Theorem 6 we obtain, for a single Markov chain with transition matrix \( \tilde{P} \) such that \( (k) \tilde{P} \) is ergodic, the large deviation equality \( \lim_{n \to \infty} \frac{1}{n} \ln(1 - \tilde{p}_{i,k}) = \ln \hat{\mu} \) where \( \hat{\mu} = 1 - \lambda(\tilde{P}) \).

\[ \text{Remark 6.} \]

With obvious meaning of notation, consider the following conditions:

a) \( (k) \tilde{P} \) is ergodic and \( \hat{\mu} < \mu \) (as we have noticed in the proof of Theorem 6 \( \mu = \mu(\tilde{P}) \) and \( \hat{\mu} = \mu(\tilde{P}) \) are positive real numbers when \( \pi_{k,k} = \tilde{p}_{k,k} = 1 \));

b) There exists \( n_0 \) such that for all \( n > n_0 \) then \( \tilde{P}^n \preceq P^n \);

c) \( T_k \preceq_{a.st.} \tilde{T}_k \).

By summarizing Theorem 4 and Theorem 6, we have the implications \( a) \Rightarrow b) \) and \( b) \Rightarrow c) \).

We present two examples to show that the reverse implications fail. First let us consider the transition matrices

\[ \tilde{P} = \begin{pmatrix} \alpha & (1 - \alpha) & 0 \\ 0 & \beta & (1 - \beta) \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \beta & (1 - \beta) & 0 \\ 0 & \alpha & (1 - \alpha) \end{pmatrix}, \]

with \( \alpha, \beta \in (0, 1) \) and \( \alpha \neq \beta \). It is immediately seen that both \( P \) and \( \tilde{P} \) have the eigenvalues \( \alpha, \beta \) and 1. Moreover the hitting times \( T_2 \) and \( \tilde{T}_2 \) have the same distribution. Therefore, in particular, \( \tilde{T}_2 \preceq_{a.st.} T_2 \) and \( T_2 \preceq_{a.st.} \tilde{T}_2 \). In any case, for \( n \in \mathbb{N} \), the relations \( P^n \preceq \tilde{P}^n \) and \( \tilde{P}^n \preceq P^n \) are not true. In fact \( \tilde{p}_{0,0}^{(n)} = \alpha^n, \tilde{p}_{1,1}^{(n)} = \beta^n \) while \( p_{0,0}^{(n)} = \beta^n, p_{1,1}^{(n)} = \alpha^n \).
We also notice that the two Markov chains, associated to the transition matrices in (43), cannot be obtained one from the other by simple permutations of the states. Therefore they remain different also after changing the names of the states.

Thus we have obtained two different Markov chains with same hitting-time distribution but such that the conditions $\tilde{T}_2 \preceq_{\text{a.st.}} T_2$ and $T_2 \preceq_{\text{a.st.}} \tilde{T}_2$ cannot be detected by using the relation $\preceq$. Whence c) does not imply b).

Secondly we show that item b) does not imply item a). Let us consider the transition matrices

$$\tilde{P} = \begin{pmatrix} \alpha & (1-\alpha) & 0 \\ 0 & \beta & (1-\beta) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{P} = \begin{pmatrix} \alpha & (1-\alpha) & 0 \\ 0 & \hat{\beta} & (1-\hat{\beta}) \\ 0 & 0 & 1 \end{pmatrix}, \quad (44)$$

with $\alpha, \beta, \hat{\beta} \in (0, 1)$ and $\hat{\beta} < \beta < \alpha$. In both cases, the second largest eigenvalue is $\alpha$, therefore $\tilde{\mu} = \hat{\mu} = 1 - \alpha$. In any case $\tilde{P} \preceq \hat{P}$ and therefore we obtain that $\tilde{P}^n \preceq \hat{P}^n$, for any $n \in \mathbb{N}$. In particular we obtain that $\tilde{T}_2 \preceq_{\text{a.st.}} \hat{T}_2$. However item a), concerning the two matrices $\tilde{P}$ and $\hat{P}$, does not hold.

We can thus conclude as follows: even if the relation $\preceq$ may appear very restrictive at a first glance, we notice however that it provides a tool, to check $T_k \preceq_{\text{a.st.}} \tilde{T}_k$, more frequently applicable than the comparison between the two spectral gaps.

Moreover the use of $\preceq$ has the following advantages:

- As shown by Theorem 5 the use of the comparison $\preceq$ only requires the computation of powers of transition matrices and the latter operation is generally easier than computing eigenvalues.

- When the entries of the two transition matrices are all rational, the computation of powers can be reduced to the case of integer entries, which allows one to obtain easier and definite answers. We notice that simple cases of matrices with rational entries are for example encountered in the analysis of occurrences of words.

- The relation $\preceq$ may also be used to establish the condition $T_k \preceq_{\text{st.}} \tilde{T}_k$ as shown by Proposition 1.

A few words about the comparison between the hypotheses of Theorem 1 and Theorem 5 are in order. The condition $\tilde{P} \preceq P$ implies (24), in Theorem 2, which in turn is stronger than i) and ii) in Theorem 1. However, as a consequence of Lemma 2, the hypothesis used in Theorem 5 is much weaker than $\tilde{P} \preceq P$.

4 Comparisons for times of occurrences of words and leading numbers

In this section we discuss some applications of the results of Section 2 and Section 3 in the frame of words occurrences. Let $A_N \equiv \{a_1, \ldots, a_N\}$ be the alphabet composed by the $N$ letters $a_1, \ldots, a_N$. 

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An ordered sequence \( \mathbf{w} \equiv w_1 w_2 \ldots w_k \), where each of the elements \( w_j \) belongs to \( \mathcal{A}_N \), is then seen as a \textit{word of length} \( k \) \textit{on} \( \mathcal{A}_N \). We consider the space \( \mathcal{A}_N^k \) of all possible words of length \( k \) on \( \mathcal{A}_N \).

Assume that, at any instant \( n = 1, 2, \ldots \), a letter is drawn at random from \( \mathcal{A}_N \). Drawings are supposed to be independent and uniformly distributed over \( \mathcal{A}_N \). We define the space \( \Omega = \mathcal{A}_N^N \); for \( \omega = (\omega_1, \omega_2, \ldots) \in \Omega \), we refer to \( \omega_n \) as the \textit{letter at time} \( n \in \mathbb{N} \). The probability measure on \( \Omega \) is then the product measure that, at any drawing, assigns probability \( 1/N \) to each letter: \( P(\omega_n = a) = \frac{1}{N}, \ a \in \mathcal{A}_N, \ n \in \mathbb{N} \).

For any word \( \mathbf{w} \equiv w_1 w_2 \ldots w_k, \ w \in \mathcal{A}_N^k \), we consider the stopping time
\[
T_\mathbf{w} := \inf\{n \geq k | \omega_{n-k+1} = w_1, \ldots, \omega_n = w_k\},
\]
i.e. the random time until the \textit{first occurrence} of \( \mathbf{w} \).

This scheme gives rise to an homogeneous Markov chain \( X = \{X_n\}_{n \in \mathbb{N}} \) with state space \( E \equiv \{0, 1, \ldots, k\} \). The Markov chain \( X \) is defined as follows:

\begin{enumerate}
  \item \( X_0 = 0 \).
  \item For \( n \geq 1 \) and \( i \in \{1, \ldots, k \wedge n\} \), one has \( X_n = i \) if
    \begin{enumerate}
      \item \( \omega_{n-i+1} \omega_{n-i+2} \ldots \omega_n = w_1 w_2 \ldots w_i \)
      \item \( \omega_{n-h+1} \omega_{n-h+2} \ldots \omega_n \neq w_1 w_2 \ldots w_h, \forall h = i+1, \ldots, k \wedge n \);
    \end{enumerate}
  \item One has \( X_n = 0 \) if \( \omega_{n-i+1} \omega_{n-i+2} \ldots \omega_n \neq w_1 w_2 \ldots w_i \) for all \( 1 \leq i \leq k \wedge n \).
\end{enumerate}

Under these positions, \( T_\mathbf{w} \) coincides with the time \( T_k \) of first visit to the state \( k \) for the chain. For our purposes we sometimes denote by \( P(\mathbf{w}) = (P_i(\mathbf{w})) \) the transition matrix of such a Markov chain associated to \( \mathbf{w} \in \mathcal{A}_N^k \) and denote by \( \mathcal{A}_\mathbf{w} \) the alphabet formed by all the distinct letters belonging to \( \mathbf{w} \). The alphabet \( \mathcal{A}_\mathbf{w} \) will be called the \textit{minimal alphabet} of \( \mathbf{w} \).

Furthermore, for \( \mathbf{w} \equiv w_1 w_2 \ldots w_k \), we denote by \( \varepsilon_\mathbf{w} \) the \textit{leading number} associated to \( \mathbf{w} \). The latter is defined as the binary vector
\[
\varepsilon_\mathbf{w} := (\varepsilon_\mathbf{w}(1), \varepsilon_\mathbf{w}(2), \ldots, \varepsilon_\mathbf{w}(k))
\]
where each \( \varepsilon_\mathbf{w}(u) \) is equal to 0 or to 1, according to the following position: for \( u = 1, 2, \ldots, k \)
\[
\varepsilon_\mathbf{w}(u) := 1\{w_{k-u+1}=w_1, \ldots, w_k=\ldots=w_u\}. \tag{45}
\]
Leading numbers have been introduced by J. Conway and have been repeatedly used in the applied probability literature (see in particular [13], [17] [2]), to deal with the stochastic framework described above. In particular, the distribution of \( T_\mathbf{w} \) only depends on the leading number \( \varepsilon_\mathbf{w} \) and, as a function of it, the mean value \( \mathbb{E}(T_\mathbf{w}) \) has the explicit expression \( \mathbb{E}(T_\mathbf{w}) = \sum_{u=1}^k N^u \varepsilon_\mathbf{w}(u) \).

In what follows, we rather analyze stochastic comparisons between the times \( T_\mathbf{w} \) and \( T_{\mathbf{w}'} \) of occurrences for two different words \( \mathbf{w} \) and \( \mathbf{w}' \) of the same length \( k \). On this purpose we apply
the results of previous sections. We shall see furthermore that an analysis based on the leading numbers $\varepsilon_w$ and $\varepsilon_{w'}$ can usefully be combined with such results.

Let then $P_A = P_A^{(w)}$ and $P'_A = P_A^{(w')}$ be the transition matrices corresponding to the two words $w, w'$.

For several pairs $w, w'$, it can happen that $P_A$ and $P'_A$ satisfy a condition of the type (24). Theorem 2 then gives us a useful criterion to check $T_w \preceq_{st} T_{w'}$.

The stochastic ordering $\preceq_{st}$ is a partial order on the distributions of the times $T_w$. As a first application of Theorem 2 we now show that such a partial order does admit a maximal element.

For $a \in A_N$, let $a$ be the word belonging to $A_N^k$ and containing all letters equal to $a$.

**Proposition 2.** For any word $w \in A_N^k$ and $a \in A_N^k$, we have $T_w \preceq_{st} T_a$.

**Proof.** The transition probabilities for the chain associated to $a$ are given by

$$p_{i,i+1}^{(a)} = \frac{1}{N}, \quad p_{i,0}^{(a)} = 1 - \frac{1}{N},$$

for $0 \leq i \leq k - 1$. We then see that the proof is immediately obtained from Theorem 2. \qed

Under a simple condition, the following result shows that also a minimal element does exist w.r.t. $\preceq_{st}$. For $N \geq k$, let $\bar{w}$ be the word $a_1a_2\ldots a_k$, made with the first letters of the alphabet.

**Proposition 3.** Let $N \geq k$. For any word $w \in A_N^k$ and for $\bar{w} \in A_N^k$, we have $T_w \succeq_{st} T_{\bar{w}}$.

**Proof.** If the leading number associated to the word $w = w_1w_2\ldots w_{k-1}w_k$ is $(0,0,\ldots,0,1)$, then there is nothing to prove because the distributions of $T_w$ and $T_{\bar{w}}$ are equal. Suppose then that the leading number of the word $w$ contains more than only one 1. In such a case $w$ has a repetition of at least one letter then $A_w$ is strictly contained in $A_N$, in view of the condition $N \geq k$. Let for instance be $a_N \notin A_w$, say. Let us consider the word $\bar{w} = w_1w_2\ldots w_{k-1}a_N$. It is clear that the leading number associated with the word $\bar{w}$ is $(0,0,\ldots,0,1)$, see (45). Therefore the distribution of $T_{\bar{w}}$ is the same as the one of $T_w$. Hence, in order to show the stochastic comparison $T_w \succeq_{st} T_{\bar{w}}$, we prove $T_w \succeq_{st} T_{\bar{w}}$. The associated Markov chains are easy to analyze because for $i = 0, \ldots, k-2$ and $l = 0, \ldots, k$ the transition probabilities verify

$$p_{i,l}^{(w)} = p_{i,l}^{(\bar{w})}.$$ 

As far as the transitions from the state $k-1$ are concerned, we notice that for the index $\bar{l} = \max\{l = 1, \ldots, k-1 : \varepsilon_w(l) = 1\}$ we can write $p_{k-1,\bar{l}}^{(w)} = 0$, since, by (45), it must be $w_\bar{l} = w_k$. On the other hand

$$p_{k-1,\bar{l}}^{(\bar{w})} = \frac{1}{N}.$$ 

In fact, when the Markov chain associated to $\bar{w}$ is in the state $k-1$, it has a transition toward the state $\bar{l}$ if and only if the letter $w_k$ is drawn. Moreover

$$p_{k-1,\bar{l}}^{(w)} = p_{k-1,\bar{l}}^{(\bar{w})}.$$
for \( l \neq 0, \tilde{l} \). Therefore

\[
p_{k-1,0}^{(w)} = p_{k-1,0}^{(\tilde{w})} + \frac{1}{N}.
\]

We then see that the proof is immediately obtained from Theorem 2.

\[\square\]

**Remark 7.** A same string \( w \equiv w_1 w_2 \ldots w_k \) can be seen as a word on different alphabets, and we must keep in mind which is the alphabet \( A_N \) from which the random letters \( \omega_1, \omega_2, \ldots \) are drawn. Normally, such an alphabet does not coincide with the minimal alphabet \( A_w \). The probability distribution of \( T_w \) depends on \( w \) only through the leading number \( \varepsilon_w \) and it depends on the alphabet \( A_N \) only through its cardinality \( N \). For brevity’s sake, such dependence on \( N \) is omitted in our notation; however it cannot be neglected, generally.

In applying Theorem 2 to word occurrences, the following proposition can be of interest.

**Proposition 4.** Let condition (24) hold for \( P_A \) and \( P_A' \). Then condition (24) also holds for \( P_{\hat{A}} \) and \( P_{\hat{A}}' \) for any alphabet \( \hat{A} \supset A_w \cup A_{w'} \).

**Proof.** First we notice that (24) reads

\[
\sum_{i=0}^{k} p_{i,l} \geq \sum_{i=0}^{k} p'_{i,l},
\]

for \( i = 0, \ldots, k \) and \( j = 1, \ldots, k \). Furthermore, for \( j \geq 1 \), \( p_{i,j}, p'_{i,j} \) are equal to \( 1/N \) when are not null, \( N \) being the cardinality of \( A \). When the alphabet \( A \) is replaced by the alphabet \( \hat{A} \) then each \( p_{i,l} \), with \( l > 0 \), is replaced by \( p_{i,l} \hat{N}/N \) where \( \hat{N} \) denotes the cardinality of \( \hat{A} \). Then all the inequalities in (46) are maintained. \[\square\]

Proposition 4 guarantees the following property: once we have proved the inequality \( T_w \preceq_{st} T_{w'} \) by checking the condition (24) for a sampling alphabet, then not only \( T_w \preceq_{st} T_{w'} \) holds for any other compatible alphabet, but also this conclusion stands still on the comparison (24).

In some cases Theorem 3 can be used to compare two words \( w, \tilde{w} \) which cannot be compared by means of Theorem 2. An example follows.

**Example 2.** Let \( w' = (A, A, B, A, A) \) and \( w = (A, B, B, B, A) \) be seen as words on an alphabet with \( N \geq 5 \). First we notice that, for these two words, the hypothesis of Theorem 3 holds true with \( m = 4 \). On the other hand, by using Proposition 3, one obtains \( T_4 \preceq_{st} T_4' \). In fact the sub-word \( (A, B, B, B) \) made with the first four letters of \( w \) has the leading number \((0, 0, 0, 1)\).

Furthermore condition (ii) of Theorem 3 is also satisfied. Then we obtain that \( T_w \preceq_{st} T_{w'} \). Notice that this example can be easily generalized by adding a same number \( \nu \) of letters \( A \) on the left and on the right of the two words and by adding a number \( \mu \) of letters \( B \) in the center. In
other terms we are saying that, by means of Theorem 3 and Proposition 3, one can compare two words \( w' \) and \( w \) whose leading numbers have the form

\[
\varepsilon_w(i) = 1, \text{ for } i = 1, \ldots, h; \quad \text{and} \quad \varepsilon_w(i) = 0, \text{ for } i = h + 1, \ldots, k - 1,
\]

\[
\varepsilon_{w'}(i) = 1, \text{ for } i = 1, \ldots, h + 1; \quad \text{and} \quad \varepsilon_{w'}(i) = 0, \text{ for } i = h + 2, \ldots, k - 1,
\]

with \( N \geq k \geq 2(h + 1) \).

As noticed in Remark 4 of the previous section, the condition \( P' \preceq P \) between two stochastic matrices is stronger than (24), however in order to guarantee asymptotic comparisons, we only need \( P^n \preceq P'^n \), for all \( n \) large enough.

Such a condition is not at all so strong. Actually we checked a large number of pairs of words \( w', w \) with \( E(T_w) < E(T_{w'}) \) and \( (k)P' \) ergodic, and in every case we found that \( P'^n \preceq P^n \), for all \( n \) large enough. We notice that for a transition matrix \( P \) associated to a word \( w \), the condition \( (k)P' \) ergodic just means that the state 0 is reachable from the state \( k - 1 \). This condition certainly holds when the alphabet \( A \) contains the minimal alphabet associated to \( w \), strictly.

**Example 3.** We used a simple program to compute powers of matrices and compare them in the sense \( \preceq \). In particular we compared the matrices \( P_A^{(w)} \) and \( P_A^{(w')} \), for several pairs of words \( w \) and \( w' \) on a same alphabet \( A \). Generally we have been able to check \( T_w \preceq_{st} T_{w'} \) by finding \( n_1 \) and \( n_2 \) as in Theorem 5 and by applying Proposition 1. For instance for \( w = (A, B, A, B, A) \) and \( w' = (A, B, A, A, B) \), considered as two words on the binary alphabet, we found \( n_1 = 3 \) and \( n_2 = 4 \). So that \( \hat{n} = 6 \) and the condition \( p^{(n)}_{0,k} \leq p^{(n)}_{0,k} \) for \( n = 1, \ldots, \hat{n} - 1 \) is immediately verified.

For \( w = (A, B, A, A, B) \) and \( w' = (A, B, A, B, A) \), considered as words on the alphabet \( A = \{A, B, C\} \), we found \( n_1 = 4 \) and \( n_2 = 5 \). Also in this case the condition of Proposition 1 is verified and \( T_w \preceq_{st} T_{w'} \) is proven.

So far in this section we discussed the possibility to use the concept of leading number in the analysis of the condition \( T_w \preceq_{st} T_{w'} \). More precisely, we noticed that the leading number can be combined with the results in Section 2. We point out however that the same concept can be usefully combined also with the results, presented in Section 3, based on comparisons of the form \( P^n \preceq P'^n \), for all \( n \) large enough. In fact when the above comparisons cannot be easily checked, one can try to find out, on the same alphabet, a different pair of words \( z, z' \), with \( \varepsilon_z = \varepsilon_w, \varepsilon_{z'} = \varepsilon_{w'} \), so that the results of Section 3 become applicable. In such a procedure we exploit, once again, the equivalence between the conditions \( \varepsilon_z = \varepsilon_w \), and \( T_z, T_w \) are identically distributed.

**Example 4.** Let us compare the words, on the alphabet \( A = \{A, B\} \), \( w' = (A, B, B, B, A) \) and \( w = (A, A, A, A, B) \), for which \( \varepsilon_{w'} = (1, 0, 0, 0, 1), \varepsilon_w = (0, 0, 0, 0, 1) \) and \( E(T_{w'}) = 34, E(T_w) = 32 \).

At least for \( n \leq 30 \), the powers of the corresponding transition matrices are not comparable in the sense \( \preceq \).

Replace now \( w \) with \( z = (B, A, A, A, A) \). We have again \( \varepsilon_z = (0, 0, 0, 0, 1) \). However, it is readily seen that \( P_A^{(w)} \preceq P_A^{(z)} \), and we are in a position to even apply Proposition 1.
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