Fundamental and vortex solitons in a two-dimensional optical lattice

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Fundamental and vortex solitons in a two-dimensional optically induced waveguide array are reported. In the strong localization regime, the fundamental soliton is largely confined to one lattice site, while the vortex state comprises of four fundamental modes superimposed in a square configuration with a phase structure that is topologically equivalent to the conventional vortex. However, in the weak localization regime, both the fundamental and vortex solitons spread over many lattice sites. We further show that fundamental and vortex solitons are stable against small perturbations in the strong localization regime. \textit{OCIS codes: 190.0190.}

Discrete optical spatial solitons are localized modes of nonlinear waveguide arrays that form when discrete diffraction is balanced by nonlinearity \cite{1}. When a low intensity beam is focused into a waveguide array, the propagating field spreads over many sites (due to optical tunneling) exhibiting a typical discrete diffraction pattern with the intensity mainly concentrated in the outer lobes \cite{2}. However, at sufficiently high power, the beam self-traps to form a localized state.

Discrete solitons in waveguide arrays were first predicted to exist as solutions to the discrete nonlinear Schrödinger equation \cite{3} and later observed in AlGaAs waveguide arrays \cite{2}. This experimental observation stimulated much new research such as diffraction management \cite{4,5}, soliton interaction and beam steering \cite{6}, and discrete solitons in a two-dimensional (2D) photorefractive optical lattice \cite{7–9}. The experiments in \cite{7–9} are particularly interesting since the 2D waveguide array there is formed optically, thus it is very versatile and easily tunable. This allowed a host of 2D localization phenomena to be observed in \cite{7–9}. However, many questions on discrete solitons in 2D optical lattices still remain open. For instance, stability properties of such solitons have not been carefully studied. In addition, new localization states such as those analogous to vortex solitons in a homogeneous waveguide have not been explored.

In this Letter, fundamental and vortex solitons in a 2D optical lattice are found and their stability analysed. In the strong localization regime, the fundamental soliton is confined largely on one lattice site with a uniform phase. The vortex soliton comprises of four fundamental modes located at the bottoms of the optical potential in a square configuration with a phase structure that is topologically equivalent to the conventional vortex. By winding around the zero intensity position along any simple closed curve, the phase of the vortex state acquires $2\pi$ increment. We call this structure a vortex cell. When the localization is weak, both fundamental solitons and vortex-cells spread over many lattice sites. In the strong localization regime, we show that the fundamental soliton as well as the vortex cell are both linearly and nonlinearly stable under weak perturbations.

We start with the 2D nonlinear Schrödinger equation

\begin{equation}
 i\frac{\partial \psi}{\partial z} + \left( \frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} \right) \psi - V\psi + |\psi|^2\psi = 0 ,
\end{equation}

where $V = V_0 (\cos^2 X + \cos^2 Y)$ is the optical lattice potential, and $V_0$ is its intensity. Such a potential can be obtained by optically interfering two pairs of laser beams \cite{9}. Transverse distances $X$ and $Y$ in (1) have been normalized by the lattice period $D$, and distance $z$ normalized by $4n_0D^2/\pi\lambda$, where $n_0$ is the refractive index, and $\lambda$ is the wavelength of the beam \cite{9}. For typical values $D = 9\mu m$, $n_0 = 2.3$ and $\lambda = 0.5\mu m$ \cite{8,9}, the unit distance $z$ in (1) corresponds to $0.5\mu m$ in physical distance.

Without the lattice potential, solitons would suffer collapse under small perturbations \cite{12}. However, as we shall show here, optical lattices can suppress the collapse of fundamental solitons and vortex-cells.

Eq. (1) conserves two quantities: the power $P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi|^2 dX dY$, and the energy $E$:

\begin{equation}
 E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ |\nabla \psi|^2 - \frac{1}{2} |\psi|^4 + V|\psi|^2 \right\} dX dY .
\end{equation}

We look for stationary solutions of the form $\psi(X,Y,z) = e^{-i\mu z}u(X,Y)$ where $\mu$ being the propagation constant of the soliton. Then $u(X,Y)$ satisfies

\begin{equation}
 \frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} - Vu + |u|^2 u = -\mu u .
\end{equation}

Solutions to this equation can be obtained by a Fourier-iteration method as was suggested by Petviashvili \cite{14}.

\textbf{Fundamental solitons.} A fundamental soliton of Eq. (3) has a single main hump sitting at a bottom of the potential, say $(X,Y) = (\pi/2,\pi/2)$. Two examples corresponding to propagation constants $\mu = 0$ and $0.88$ with $V_0 = 1$ are displayed in Fig. 1(c,d). We see that for small $\mu$ (Fig. 1c), the beam is largely confined on one lattice site, while at higher $\mu$ (Fig. 1d), it spreads over many lattice sites. To quantify these solitons, we calculate the dependence of the normalized power $P$ on the propagation constant $\mu$ for $V_0 = 1$ as displayed in Fig. 1a. When $\mu \rightarrow -\infty$, $P$ approaches a constant 11.70. This is apparently because in this limit, the fundamental...
soliton is highly localized, thus it approaches the lattice-free fundamental-soliton state, which has critical power \( P_c \approx 11.70 \). As \( \mu \) goes to a cut-off value which is approximately 0.95, \( P \) appears to go to infinity. In this limit, the fundamental state becomes uniformly distributed in space. Thus, this cut-off value should be the boundary of the band gap in the linear-wave spectrum [10]. When \( \mu \approx 0.72 \), \( P \) is minimal. When \( \mu > 0.72 \), \( dP/d\mu > 0 \). In this region, the Vakhitov-Kolokolov (VK) theorem suggests that the soliton is linearly unstable due to the presence of a purely-real unstable eigenvalue \( \sigma \) in the linearized equation [11]. We have confirmed this instability by numerically simulating the linearized version of (1) around the above soliton. The results for unstable eigenvalues are shown in Fig. 1b. On the other hand, when \( \mu < 0.72 \) (\( V_0 = 1 \)), the VK instability is absent, and the soliton state is linearly stable.

For the two-dimensional self-focusing case, collapse is an important issue. The above linear stability analysis does not guarantee that the fundamental soliton will not collapse under small perturbations. In the study of collapse, the energy \( E \) plays an important role. In the absence of the lattice potential [12] or when the potential is harmonic [13], the soliton collapses if its energy is negative. In case the energy is positive, however, the soliton collapses only if it is strongly perturbed. For the optical lattice, we have calculated the energy \( E \) of fundamental solitons at various values of \( \mu \) and plotted the results in Fig. 1(a). The energy is found to be always positive. Thus we can expect that this state is able to withstand small perturbations without collapse. To confirm the above expectations on stability, we numerically study the nonlinear evolution of the fundamental soliton under small perturbations by directly simulating Eq. (1) with initial condition

\[
\psi(X, Y, z = 0) = u(X, Y)[1 + \epsilon u_p(X, Y)],
\]

where \( \epsilon \ll 1 \), and \( u_p(X, Y) \) is the initial perturbation. We first take \( u_p \) to be white noise. A large number of simulations with small \( \epsilon \) and various realizations of random noise perturbations have been performed, and we have found that for \( V_0 = 1 \), if \( \mu < 0.72 \), the fundamental soliton is indeed stable against white noise perturbations; when \( \mu > 0.72 \), the soliton is unstable. To study the nonlinear evolution process, we now take \( u_p = 1 \). For \( V_0 = 1 \), \( \mu = 0 \) and \( \epsilon = \pm 0.01 \), nonlinear evolutions are plotted in Fig. 2a. We see that the perturbed soliton only oscillates weakly around the fundamental-soliton state, meaning that the soliton is both linearly and nonlinearly stable. On the other hand, at \( V_0 = 1 \) and \( \mu = 0.88 \) (where the soliton is linearly unstable), the dynamics is different as two scenarios are identified: (i) at higher input power \( \epsilon > 0 \), the perturbed state relaxes into a \( z \)-periodic bound state (Fig.2b,c); (ii) at lower input power \( \epsilon < 0 \), the perturbed state decays into linear Bloch waves (Fig. 2b,d) (similar scenarios can be found in [15] for a different system). Collapse is not observed in either scenarios.

![Image](image1.png)

**FIG. 1.** (a) Normalized power \( P \) and energy \( E \) of fundamental solitons versus \( \mu \) for \( V_0 = 1 \). (b) unstable eigenvalues \( \sigma \) of these solitons for \( V_0 = 1 \); (c, d) profiles of fundamental solitons at \( \mu = 0 \) and 0.88 \( (V_0 = 1) \) respectively.

![Image](image2.png)

**FIG. 2.** Nonlinear evolutions of fundamental solitons under perturbations (4) for \( V_0 = 1 \), \( \epsilon = \pm 0.01 \), \( u_p = 1 \) and \( \mu = 0 \) (a), 0.88 (b). Snapshots of the soliton intensity corresponding to the dynamics depicted in (b) for \( \epsilon = 0.01 \), \( z = 80 \) (c) and \( \epsilon = -0.01 \), \( z = 60 \) (d).

**Vortex cell.** In addition to the fundamental solitons, we have numerically found vortex solitons as well. Two examples with \( V_0 = 1 \), \( \mu = 0 \) and 0.82 are shown in Fig. 3(b,c,d). At \( \mu = 0 \) (strong localization regime), the vortex state comprises of four fundamental solitons superimposed in a square configuration with a phase structure which is topologically equivalent to a conventional vortex [see Fig. 3b,c]. By winding around the center along any closed curve, the phase of the vortex acquires \( 2\pi \) increment, thus, we name it a vortex cell. At \( \mu = 0.82 \) (weak localization regime), the vortex cell spreads out to more lattice sites and becomes more intricate, as can be seen in Fig. 3d. But its phase structure is almost the same as with \( \mu = 0 \). We should point out that these vortex cells are different from conventional vortices without optical lattice on a major aspect: their densities and phases.
depend on both $r$ and $\theta$. The normalized power and energy diagrams versus propagation constant for these vortex cells at $V_0 = 1$ are shown in Fig. 3(a). This figure is similar to that for fundamental solitons [Fig. 1(a)], except that both $P$ and $E$ here are about four times larger. Unstable eigenvalues $\sigma$ of vortex-cells are determined by simulating the linearized equation (1) around vortex-cells. The results are shown in Fig. 4a for $V_0 = 1$. We see that vortex cells experience an oscillatory instability for $\mu > -7.8$ and become stable for $\mu < -7.8$. These vortex cells also suffer the VK instability in the region $\mu > 0.73$ where $dP/d\mu > 0$ [see Fig. 3a]. However, the oscillatory instability is much stronger as it occurs over a wider region and has a higher growth rate. Fig. 3(a) shows that vortex cells also have positive energy. Thus, if a vortex cell is linearly stable, it should be able to resist collapse under small perturbations [13]. However, if it suffers the linear oscillatory instability discussed above, this instability could result in power exchange from one part of the cell to another so that the intensity at some small spots becomes high, triggering local collapse to occur. We have observed this scenario numerically. An example is shown in Fig. 4(b,c,d) which displays the development of a vortex cell with $\mu = 0$ and $V_0 = 1$ when it is amplified by 1% initially.

![FIG. 3. (a) Normalized power $P$ and energy $E$ of vortex cells versus $\mu$ for $V_0 = 1$. (b,d) Intensity plots of vortex cells with $V_0 = 1$, $\mu = 0$ and 0.82 respectively. (c) the phase plot of the vortex cell in (b).](image)

We next discuss the effect of varying the potential strength $V_0$ on the formation and stability of fundamental and vortex solitons. For this purpose, we have chosen $V_0 = 1.5$, and repeated most of the above calculations. The results are summarized as follows: (i) fundamental solitons and vortex-cells exist at higher values of $\mu$ (up to approximately 1.35); (ii) at the same value of $\mu$, both the fundamental and vortex solitons have lower power ($P$) and higher values of energy ($E$); (iii) the oscillatory instability suffered by vortex cells is reduced (it could even be completely suppressed when $V_0$ becomes even larger). Thus, we conclude that higher lattice potentials stabilize both fundamental and vortex solitons.

![FIG. 4. (a): Unstable eigenvalues of vortex cells with $V_0 = 1$; (b,c,d) instability development of the vortex with $V_0 = 1$ and $\mu = 0$ [see Fig. 3(b,c)] when it is initially amplified by 1%; intensity plots at $z = 50, 51.5$ and 52 are shown respectively.](image)

In conclusion, we have studied new types of fundamental and vortex solitons in a two-dimensional optical lattice potential, and shown that both solitons are stable in the strong localization regime.

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