Qualitative properties of systems of 2 complex homogeneous ODE’s: a connection to polygonal billiards

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(Dated: 9 September 2020)

A correspondence between the orbits of a system of 2 complex, homogeneous, polynomial ordinary differential equations with real coefficients and those of a polygonal billiard is displayed. This correspondence is general, in the sense that it applies to an open set of systems of ordinary differential equations of the specified kind. This allows to transfer results well-known from the theory of polygonal billiards, such as ergodicity, the existence of periodic orbits, the absence of exponential divergence, the existence of additional conservation laws, and the presence of discontinuities in the dynamics, to the corresponding systems of ordinary differential equations. It also shows that the considerable intricacy known to exist for polygonal billiards, also attends these apparently simpler systems of ordinary differential equations.

I. INTRODUCTION

In the following, we study a special case of the system of 4 real ordinary differential equations with homogeneous right-hand sides:

\[
\begin{align*}
\dot{y}_1 &= p_i(y_1,y_2,y_3,y_4) & (1 \leq i \leq 4), & (1) \\
\end{align*}
\]

where the \(p_i\) are homogeneous polynomials of degree \(r\) in the 4 variables \(y_i\). Such equations are a bit anomalous due to the absence of linear terms, nevertheless they are studied to a considerable extent in various fields, such as reaction kinetics and ecology.

The case we shall study is special in 2 respects

1. First, and most importantly, the system (1) is assumed to arise from a complex system of 2 ODE’s:

\[
\begin{align*}
\dot{x}_1 &= p_r(x_1,x_2) & (2a) \\
\dot{x}_2 &= q_r(x_1,x_2). & (2b) \\
\end{align*}
\]

Here \(p_r(x,y)\) and \(q_r(x,y)\) are both homogeneous polynomials of degree \(r\).

2. Second, the coefficients of the polynomials \(p_r(x_1,x_2)\) and \(q_r(x_1,x_2)\) are real.

Clearly, the most important restriction is the first one. The second can certainly be significantly generalised. However, the first one is essential, and is responsible for the highly atypical behaviour we identify for such systems.

As we shall see, these systems can be completely understood, once a corresponding polygonal billiard problem is solved: each orbit of (2) can be brought uniquely into correspondence with the orbit of a given polygonal billiard, the characteristics of which do not depend on the initial conditions, but only on the system (2) itself.

In particular, we find the following properties, which differ rather strongly from the generic properties of such ODE’s: first and foremost, the dynamics is regular: that is, with probability one, the orbits are defined and finite for all times. As follows, for example, from within the sets of quadratic systems, there is an open set with the property that the time for a solution to diverge is finite for an open set of initial conditions. On the other hand, for systems of the type (2), solutions are non-singular for all times with probability one. Note that this does not mean that the orbit remains bounded: in a rather general case, as we shall see, the orbit never diverges, but with probability one assumes arbitrarily large values over small time intervals.

In a polygonal billiard, the singular orbits (that hit a corner) have an important characteristic: in their vicinity, the regular orbits vary discontinuously: Figure 1 shows how this occurs for an orbit hitting a \(2\pi/3\) corner, but the effect occurs generally, except for angles of the form \(\pi/n\). In the ODE system, this is reflected in the fact that the orbits hitting a corner diverge, so that the theorems on continuous dependence of the solution of an ODE on initial conditions, fail in their vicinity.

Another striking feature of these systems is the fact that their Lyapunov exponents are always zero. This means that the dynamics can actually be predicted efficiently over large times, though, as we shall see, the dynamics can indeed be quite complicated.

We thus show that a correspondence exists between any given system of ODE’s of the type (2) and the dynamics of a free particle bouncing elastically within a polygon, the shape of which is uniquely and elementarily determined by the system (2). Since many results on the existence and nature of periodic orbits, on ergodicity, Lyapunov exponents and several other properties, are known for polygonal billiards, it turns out that they trivially translate into corresponding properties for the system of ODE’s (2). This is the essence of this paper.

In Section III we show how the system (2) can be solved by quadratures. Since the integrals involved are complicated, inverting them is not a trivial task, and an understanding of the orbit requires an additional remark. In Section III, we establish in detail the correspondence, in Section IV we establish the general consequences of this correspondence. The
In the following, we present a solution by quadratures of this system. This is not new, but has been explicitly formulated by Garnier. This approach has also been used in [2] to identify particularly simple special cases of (2) for \( r = 2 \).

We define the following quantities

\[
\begin{align*}
  u &= x_1/x_2, \\
  x_1 &= uR(u), \\
  x_2 &= R(u).
\end{align*}
\]

From (5) follows the following equation for \( u \)

\[
\ddot{u} = -x_2^{-1} [uQ_r(u) - P_r(u)] =: -x_2^{-1}S_{r+1}(u)
\]

where the final equation defines \( S_{r+1}(u) \), which is a monic polynomial of degree \( r + 1 \). Note the use we have made of the normalisation of \( Q_r(u) \) as a monic polynomial.

From (5) and (4c), one obtains

\[
\ddot{u} = -R(u)^{-1}S_{r+1}(u).
\]

Now from (5b) and (5c) one finds

\[
\begin{align*}
  \dot{x}_2 &= R'(u)\dot{u} \\
  &= -R(u)^{-1}R'(u)S_{r+1}(u) \\
  &= R(u)^{-1}Q_r(u).
\end{align*}
\]

This in turn leads to a separable equation for \( R(u) \):

\[
\frac{R'(u)}{R(u)} = \frac{Q_r(u)}{S_{r+1}(u)}.
\]

Let \( u_\alpha, 0 \leq \alpha \leq r \) be the zeros of \( S_{r+1}(u) \), that is:

\[
S_{r+1}(u) = \prod_{\alpha=0}^{r} (u - u_\alpha).
\]

We decompose the right-hand side of (8) in partial fractions, assuming that none of the \( u_\alpha \) are double zeros:

\[
\frac{Q_r(u)}{S_{r+1}(u)} = \frac{1}{r - 1} \sum_{\alpha=0}^{r} \frac{\mu_\alpha}{u - u_\alpha},
\]

where the \( 1/(r-1) \) prefactor is introduced for future convenience. Matching the \( u \rightarrow \infty \) behaviours of both sides of (10), remembering that both \( Q_r(u) \) and \( S_{r+1}(u) \) are monic, we obtain

\[
\sum_{\alpha=0}^{r} \mu_\alpha = r - 1.
\]

Note that the \( \mu_\alpha \) and the \( u_\alpha \) are altogether independent of the initial conditions and instead characterise the system (2) itself. (8) is now immediately integrated to yield

\[
R(u) = C \prod_{\alpha=0}^{r} (u - u_\alpha)^{-\mu_\alpha/(r-1)}.
\]

II. SOLUTION BY QUADRATURES

We consider the system (2) of ordinary differential equations. These are viewed as complex equations, that is, viewed as a system involving real quantities, they correspond to a system of 4 equations in 4 unknowns, corresponding to the real and imaginary parts of \( x_1 \) and \( x_2 \). We rewrite these equations as follows:

\[
\begin{align*}
  \dot{x}_1 &= x_2^rP_r(x_1/x_2), \\
  \dot{x}_2 &= x_2^rQ_r(x_1/x_2).
\end{align*}
\]

Here \( P_r(u) \) and \( Q_r(u) \) are both polynomials of degree \( r \). There is a minor loss of generality in this description, as we assume that there exists in both equations a term \( x_2^r \). This can always be reached by a linear transformation of the dependent variables. By an appropriate scaling of \( x_2 \) we may further choose \( Q_r(u) \) to be a monic polynomial.
Here \( C \) is an integration constant determined by the relation
\[
x_2(0)^2 = C \prod_{\alpha=0}^{r} [x_1(0) - u_\alpha x_2(0)]^{-\mu_\alpha/(r-1)}.
\] (13)

We now proceed to a final normalisation step: the solutions of (2) can always be scaled by a fixed real factor \( \lambda \), which corresponds to a scaling of \( t \) by the factor \( \lambda^{-r} \). We may hence, without loss of generality, scale the initial conditions accordingly and therefore fix the norm of \( C \). We thus set \(|C| = 1\) and
\[
C = -e^{i\mathcal{Z}_0}.
\] (14)

We may now determine the time-dependence of \( u \) using (6):
\[
\dot{u} = e^{i\mathcal{Z}_0} \prod_{\alpha=0}^{r} (u - u_\alpha)^{-\mu_\alpha+1}.
\] (15)

which leads to the expression via quadratures
\[
t = e^{-i\mathcal{Z}_0} \int_{u(0)}^{u} \prod_{\alpha=0}^{r} (u' - u_\alpha)^{\mu_\alpha-1} du'.
\] (16)

where \( u(0) = x_1(0)/x_2(0) \).

### III. Correspondence Between the ODE’s and Polygonal Billiards

We now limit ourselves to the subclass of systems in which the coefficients of the polynomials \( P_r(u) \) and \( Q_r(u) \) are all real. Under these conditions, the fact that all \( u_\alpha \) should be real and simple, is no more exceptional. Indeed, given a system with that property, all other systems that are sufficiently close also have this property, so that we are in a generic case.

The essential observation we now make is the following: if all \( u_\alpha \in \mathbb{R} \), all \( \mu_\alpha \in \mathbb{R} \) as well. It then follows that, for appropriate values of the \( \mu_\alpha \), specifically for \( 0 \leq \mu_\alpha \leq 1 \), the transformation defined by (15) is the conformal map from the upper half-plane to a finite, convex polygon \( \mathcal{P} \), having \( r+1 \) sides and interior angles \( \mu_\alpha \pi \), the well-known Schwarz–Christoffel transformation. (8–10) It follows immediately from (11) that the sum of the interior angles of the polygon is, as it must be, equal to \((r-1)\pi \). Further note that the polygon’s shape, which is the main object of our consideration, is determined both by the interior angles given by the \( \mu_\alpha \), and by the relative lengths of the sides, determined by \( r-3 \) values of the \( u_\alpha \). We denote by \( v_\alpha \) the vertices of \( \mathcal{P} \) corresponding to \( u_\alpha \), by \( \mathcal{S}_\alpha \) the side of \( \mathcal{P} \) connecting \( v_\alpha \) to \( v_{\alpha+1} \), where \( \alpha+1 \) is computed modulo \( r+1 \). To \( \mathcal{S}_\alpha \) corresponds in the boundary of the upper half-plane, the interval \( I_\alpha = [u_\alpha, u_{\alpha+1}] \).

Note in passing that the task we face here is different from, and in many ways easier than, the one usually solved by the Schwarz–Christoffel transformation: normally one is given a polygonal domain and looks for a conformal transformation. In that case, the determination of the \( u_\alpha \) can be challenging. In our case, we are given the \( u_\alpha \) and the angles \( \mu_\alpha \), and our task is merely to determine the image of the upper half-plane under this transformation.

For definiteness’s sake, let us assume the initial condition \( u(0) \) to be in the upper half-plane (the opposite case is similar). The map
\[
\Phi(u) = \int_{u(0)}^{u} \prod_{\alpha=0}^{r} (u' - u_\alpha)^{\mu_\alpha-1}.
\] (17)

maps the upper half-plane onto the inside of a convex \( r \)-sided polygon \( \mathcal{P} \) containing the origin. We have in particular
\[
\Phi(u_\alpha) = v_\alpha \quad (0 \leq \alpha \leq r).
\] (18)

The equation (16) means that the straight line \( \mathcal{L} \) defined by \( e^{i\mathcal{Z}_0 t} \), for all real \( t \), is the image of the orbit \( u(t) \) under the map \( \Phi \).

We must therefore determine the inverse image of \( \mathcal{L} \) under \( \Phi \). For the segment of \( \mathcal{L} \) that lies entirely in \( \mathcal{P} \), the corresponding part of the trajectory lies wholly in the upper half-plane. As the line leaves \( \mathcal{P} \) by the side \( \mathcal{S}_\alpha \) corresponding to the real interval \( I_\alpha = [u_\alpha, u_{\alpha+1}] \), the corresponding orbit of \( u \) leaves the upper half-plane by the interval \( I_\alpha \). Due to the Schwarz reflection principle, the image under \( \Phi \) of the sheet which the orbit \( u(t) \) enters, is the polygon described by the reflection of \( \mathcal{P} \) on the side \( \mathcal{S}_\alpha \). We may therefore keep the orbit of \( u \) inside the upper half-plane by specularly reflecting it with respect to the real axis, and correspondingly keep the line \( \mathcal{L} \) inside the polygon \( \mathcal{P} \), also by reflection. Indefinite repetition of this procedure, for both positive and negative times, leads to a billiard orbit inside \( \mathcal{P} \). For an illustration of the way this proceeds, see Figure 3. Note that this construction is rather similar to one used in [12] for a somewhat related problem. If we now take the inverse image under \( \Phi \) of this billiard orbit, we obtain the orbit \( u(t) \) specularly reflected each time it crosses the real axis, from which the actual \( u(t) \) orbit is readily reconstructed.

At this stage let us define some additional notation: the initial segment of the billiard trajectory is the straight line segment \( e^{i\mathcal{Z}_0 t} \). After \( n \) bounces we define the corresponding straight line segment to be \( e^{i\mathcal{Z}_0 (t - \tau_n)} \). Here \( \tau_n \) is the time at which the orbit hits \( \mathcal{P} \) and begins the \( n \)-th bounce. As the billiard orbit is successively followed, the connection between \( t \) and \( u \) given by (16) is modified to
\[
t - \tau_n = e^{-i\mathcal{Z}_0} \int_{u(0)}^{u} \prod_{\alpha=0}^{r} (u' - u_\alpha)^{\mu_\alpha-1} du'.
\] (19)

We therefore see that the inverse image under \( \Phi \) of a billiard orbit of \( \mathcal{P} \) is the orbit of \( u \) reflected back into the upper half-plane each time it hits the real axis. Since \( \Phi \) is not an easily determined map, this does not represent an exact solution, but remembering the many results known about polygonal billiards, the correspondence yields several non-trivial results concerning the solutions’ qualitative behaviour.

Before we proceed to describe these, however, it is of some importance to extend the validity of the correspondence as far as possible. In the case \( 0 \leq \mu_\alpha < 1 \), the image of the upper half-plane is an \( r \)-sided convex polygon. If we generalise this to \( 0 \leq \mu_\alpha < 2 \), we obtain arbitrary bounded polygons, whether convex or not: the polygon’s interior angles are then \( \mu_\alpha \pi \), and
they still add up to \((r - 1)\pi\). The extension to negative values of \(\mu_\alpha\) leads to unbounded polygons. Due to (11), negative values of \(\mu_\alpha\) must always coexist with positive ones. If \(\mu_\alpha < 0\), the integral describing \(\Phi\) diverges as \(u \to u_\alpha\) on \(\mathbb{R}\), both from the right and the left. The polygon’s boundary thus contains two lines diverging to infinity and forming an angle \(\mu_\alpha\pi\). We may thus draw a polygon corresponding to all real values of \(\mu_\alpha\) satisfying both (11) and \(-2 < \mu_\alpha < 2\).

Further extensions, whether to values of \(\mu_\alpha\) with \(|\mu_\alpha| \geq 2\) or complex values of \(\mu_\alpha\) may well be possible, but it is not obvious how to extend the above construction to such cases, and more generally speaking, how to obtain meaningful results from them.

IV. CONSEQUENCES OF THE CORRESPONDENCE

A. General results

For arbitrary shapes of the polygon, the following remarks hold: generically the orbits \(x_{1,2}(t)\) remain finite, since divergence could only arise if the orbit \(u(t)\) hits \(u_\alpha\), which does not happen generically. Another remarkable feature of such systems is a very sensitive dependence on the parameters characterising the system. Indeed, the properties of polygonal billiards with rational and irrational angles are very different, so that the corresponding systems of homogeneous ODE’s also show such dependence.

Another general feature is the structure of periodic orbits. In generic systems, in particular in chaotic systems, periodic orbits are isolated. In the presence of a conservation law, the orbits are isolated once the system is reduced to a surface where the conserved quantity takes a fixed value. However, for polygonal billiards, periodic orbits of even period always appear in one-parameter families, even though no conservation law may exist, as is the case, for instance, in irrational billiards. Again, this feature translates into the systems of homogeneous ODE’s discussed here.

Finally the central role played by discontinuities in the dynamics, both in polygonal billiards and in the homogeneous systems of ODE’s we are considering here, should be emphasized. Whenever the orbit of a polygonal billiard hits a corner, it cannot be continued. However, as an orbit is continuously moved through a corner, the orbits undergo a discontinuous variation, unless the angle of the corner is equal to an angle of the form \(\pi / n\) for \(n \in \mathbb{N}\), see Figure [1] for the case of a 2\(\pi / 3\) corner. In the corresponding systems of homogeneous ODE’s, hitting a corner corresponds to divergence of the \(x_{1,2}(t)\), beyond which the orbit cannot be continued, and similarly, the orbits in the vicinity of such a divergence also show a discontinuous variation.

B. Scattering systems

Here we consider the case in which one or more of the quantities \(\mu_\alpha\) are negative or zero. In this case the the polygon extends to infinity, either with straight lines that diverge at a strictly positive angle, or, if \(\mu_\alpha = 0\), two parallel sides extending to infinity. The billiard orbit is then a scattering orbit in the strict sense, that is, it comes from infinity, bounces a finite number of times on the sides of the polygon, and then goes back to infinity.

The first issue we address is whether, during the scattering event, the orbit \(u(t)\) may diverge. As is readily seen, the function \(\Phi(u)\) has a well-defined finite value \(\Phi_\infty\) for \(u \to \infty\). If the billiard orbit hits this value, \(u\) will diverge for this specific value of \(t\). This, of course, will generically not happen, but well it may occur that an orbit passes close to \(t_\infty\), in which case \(u\) becomes anomalously large. Indeed, for \(u \to \infty\)

\[
\Phi(u) = \int_{u(0)}^{u} \frac{du'}{u'} \prod_{\alpha=0}^{r-1} \left(1 - \frac{u_\alpha}{u}\right)^{-1} = \Phi_\infty - \frac{1}{u} \left[1 + O(u^{-1})\right].
\]

Let us now assume that \(\Phi_\infty\) lies close the piece of the billiard orbit defined by \(e^{it_n} (t_n)\). In other words, there exists \(t_\infty \in \mathbb{C}\) such that \(\Phi_\infty = e^{it_n} (t_\infty - t_n)\) and such that \(|t_\infty - t_n| \ll 1\) on the \(n\)-th bounce; \(u\) therefore diverges if \(t \to t_\infty\). From (20) follows that, for \(\Phi(u)\) close to \(\Phi_\infty\),

\[
(t - t_\infty)u = e^{-it_n} \left[1 + O(t - t_\infty)\right]
\]
It thus follows that a scattering orbit that passes through $t_\infty$ has a simple pole singularity in $u$. Using \[11\] and \[10\], we see that, as $t \to t_\infty$,

\[
x_2(t) = R(u) = -e^{i\omega_0} \frac{1}{u} [1 + O(u^{-1})]
\]

and

\[
x_2 = e^{i\omega_0} \frac{1}{u} \prod_{\alpha=0,\alpha \neq \alpha_i} (u_{\alpha_i} - u)^{1 - \mu_{\alpha_i}} (u - u_{\alpha_i})^{\mu_{\alpha_i}}.
\]

so that $x_2(t)$ has a simple zero, whereas $x_1(t)$ is regular at $t = t_\infty$. This divergence is therefore not a sign of singular behaviour of the solution of \[2\].

We now turn to the asymptotic behaviour of the scattering orbits for large times. It is known that, for a large class of unbounded polygonal billiards, \textit{almost all} orbits eventually go to infinity, and are therefore asymptotically in free motion. Note that this statement, while it may at first appear obvious, is in fact quite non-trivial: see Appendix A for details and references to the literature. The class of polygons for which it holds includes among others, all polygons such that $\mu_\alpha \in \mathbb{Q}$ for all $0 \leq \alpha \leq r$, but also a set of irrational polygons large in the sense of category $\mathcal{F}$, strictly speaking a denumerable intersection of dense open sets. On the other hand, the stronger statement that \textit{all} orbits eventually go to infinity is obviously wrong, as shown in Figure 3. Note that this example shows that the corresponding equations \[2\] can have a family of periodic orbits depending on one real parameter, as described in the caption of Fig. 1.

Translated into the corresponding language for the $u$ orbit, we see that, for the class of polygons described above, one has almost certainly

\[
u(t) \to u_{\alpha_i} \quad (t \to \pm \infty)
\]

In the following, we limit ourselves to the behaviour as $t \to \infty$, but the formulae for $t \to -\infty$ are entirely similar.

Assuming that the piece of the billiard orbit that escapes to infinity corresponds to the $n$th bounce, we obtain from \[19\] that

\[
\tag{23}
\]

In the limit $t \to \infty$ and correspondingly $u \to u_{\alpha_i}$, the first summand in \[24\] remains bounded, and is of the order $O((u - u_{\alpha_i})^{\mu_{\alpha_i}})$, whereas the second diverges. Asymptotically we therefore find

\[
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\[
\tag{25a}
\]

Inverting we get, since $t \to \infty$ and $\tau_n$ remains constant

\[
\tag{25b}
\]

This means, see \[46\], that, for large times, $x_1(t)$ goes as $u_{\alpha_i} x_2(t)$ and that $x_2(t)$ is given by the asymptotic expression.

\[
\tag{26}
\]

This holds includes among others, all polygons such that $\mu_\alpha \in \mathbb{Q}$ for all $0 \leq \alpha \leq r$, but also a set of irrational polygons large in the sense of category $\mathcal{F}$, strictly speaking a denumerable intersection of dense open sets. On the other hand, the stronger statement that \textit{all} orbits eventually go to infinity is obviously wrong, as shown in Figure 3. Note that this example shows that the corresponding equations \[2\] can have a family of periodic orbits depending on one real parameter, as described in the caption of Fig. 1.

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Assuming that the piece of the billiard orbit that escapes to infinity corresponds to the $n$th bounce, we obtain from \[19\] that

\[
\tag{23}
\]
The connection between $u(t)$ remains bounded apart from the possible divergences linked to passing through the point $\Phi_\alpha$ as well as when the billiard orbit hits a corner of the polygon $\mathcal{P}$. However, as we have noted before, the former divergence does not correspond to a singularity of the dynamics (2), so we may say that $x_1$ and $x_2$ remain bounded unless the orbit hits, or comes close to, a corner. If it hits a corner, we cannot proceed further. On the other hand, in the vicinity of a corner, we must determine the behaviour of the solution of (2).

Being in the vicinity of a corner is equivalent, for the orbit of $u$, to being in the vicinity of an $u_\alpha$, say $u_\alpha_0$. It follows that, if the orbit in the $n$-th bounce hits the corner $v_\alpha_0$ corresponding to $u_\alpha_0$ at a time $t_\alpha_0 \in \mathbb{C}$, that is, if

$$e^{i\tau_n}(t_\alpha_0 - \tau_n) = \Phi(u_\alpha_0) = v_\alpha_0,$$

we may again derive (24) with $\alpha_+$ replaced by $\alpha_0$. As above, one obtains

$$t - t_\alpha_0 = [K^{-1}(u - u_\alpha_0)]^{\mu_{\alpha_0}} [1 + O(u - u_\alpha_0)] \quad (30a)$$

$$K = -e^{i\tau_0} |\mu_{\alpha_0}| \prod_{\alpha = 0, \alpha \neq \alpha_0} (u_\alpha_0 - u_\alpha)^{1 - \mu_\alpha} \quad (30b)$$

Inverting, it follows that

$$u - u_\alpha_0 = (K |t - t_\alpha_0|)^{1/\mu_{\alpha_0}} [1 + O(|t - t_\alpha_0|^{1/\mu_{\alpha_0}})]. \quad (31)$$

The connection between $t$ and $u$ near a corner is thus given by the power $1/\mu_{\alpha_0}$. For $x_2(t)$ one finds

$$x_2(t) = -e^{i\tau_0} (u - u_\alpha_0)^{-\mu_{\alpha_0}/(r-1)} \prod_{\alpha = 0, \alpha \neq \alpha_0} (u - u_\alpha)^{-\mu_\alpha/(r-1)}$$

$$= K' |t - t_\alpha_0|^{-1/(r-1)} \left[ 1 + O\left(|t - t_\alpha_0|^{1/\mu_{\alpha_0}}\right) \right]. \quad (32)$$

Here $K'$ is another constant. Since $u(t)$ is close to the finite value $u_\alpha_0$, it follows that in the vicinity of the corner, $x_1(t)$ behaves as $u_\alpha_0 x_2(t)$, so that the 2 variables have similar qualitative behaviour, unless, of course $u_\alpha_0 = 0$, in which case the leading behaviour of $x_1(t)$ is the subleading behaviour of $x_2(t)$.

The behaviour upon hitting a corner corresponds to the case in which $t_\alpha_0 \in \mathbb{R}$, so that the values of $x_{1,2}(t)$ diverge. On the other hand, coming close to a corner means that $\Im t_\alpha_0$ is small. In this case, the values of $x_{1,2}(t)$ become large, and their subsequent behaviour depends on the sign of the imaginary part, a fact which corresponds to the discontinuity of the billiard dynamics near a corner.

We may now apply well-known results for polygonal billiards to obtain corresponding results for the dynamics of (2).

For arbitrary polygons, we cannot say very much. The most important generally valid result is the existence of singular orbits and the fact that they have measure zero. Indeed, whenever an orbit in $\mathcal{P}$ hits a corner $v_\alpha$ such that $\mu_\alpha \neq 1/n$, with $n \in \mathbb{N}$, the orbit cannot be continued continuously: this arises from the fact that an orbit which comes arbitrarily close to $v_\alpha$ but hits first the side $S_{\alpha-1}$ and then $S_\alpha$ comes out at a different direction from a similar orbit which first hits $S_\alpha$ and then $S_{\alpha-1}$. Only in the specific case $\mu_\alpha = 1/n$, for $n \in \mathbb{N}$, does continuity hold; the dynamics can then be meaningfully defined after hitting a corner. However, there are only a finite number of cases in which all $\mu_\alpha$ satisfy this condition, and these correspond to integrable cases. However, the dynamics remains well-defined, since the orbits hitting a corner form a set of zero measure.

The discontinuities are nevertheless important, and may be said to structure the entire set of orbits. In the case of ergodic billiards, see below, almost all orbits pass arbitrarily close to a corner, and two arbitrarily close orbits may be eventually separated by hitting a corner on different sides.

If, on the other hand, two orbits differ initially by an infinitesimal amount, the rate of divergence of the distance between the two orbits is linear, that is, the Lyapunov exponents are all zero.

Another universally valid remark is the following: whereas it is not known whether any given irrational polygon has a periodic orbit, it is known that when it does, the orbits with a primitive period consisting of an even number of bounces all appear in one-parameter parallel families: that is, if the orbit is shifted by a sufficiently small amount without changing its direction of motion, the orbit remains periodic. As the orbit is shifted further, it will typically disappear by hitting a corner. This is, of course, in clear contrast to the behaviour of generic or chaotic systems.

It should be added that proving results for arbitrary polygons is quite difficult. Numerical work thus provides impor-
tant additional indications. For valuable results obtained in this manner, see in particular.

On the other hand, if the $\mu_\alpha$ are rational, we can additionally say the following:

1. The angle at which the orbit hits the boundary can only take finitely many different values. Since the map $\Phi$ is conformal, we see that, whenever $u(t)$ crosses the real axis, $\arg \hat{u}(t)$ can only take finitely many values.

2. The polygonal billiard has a dense set of periodic orbits: this again translates into the corresponding statement for the $u(t)$ orbit.

3. With probability one, an initial direction is ergodic, in the sense that the points where the orbit hits $\mathcal{P}$ are uniformly distributed on $\mathcal{P}$. This implies that the intersections of the orbit $u(t)$ with the real axis are uniformly distributed with respect to the density obtained from the uniform measure by taking the inverse image of $\mathcal{P}$ under $\Phi$.

4. An interesting remark also follows from the theorem that any orbit which hits one of the sides of a rational polygon at a right angle, is periodic, see Appendix A for details. Since the Schwarz–Christoffel map is conformal, this translates into the statement that any solution which hits the real axis perpendicularly is periodic. This means in particular that whenever the $\mu_\alpha \in \mathbb{Q}$ lead to a bounded $\mathcal{P}$, the initial condition $\chi_0 = \pi/2$ leads to a periodic orbit for all values of $u(0)$ for which the orbit is non-singular, and hence for almost all values of $u(0)$.

Finally let us discuss the issue of the orbit’s boundedness. When the billiard is bounded, then clearly so is the billiard orbit. The inverse image of the triangle via the map $\Phi$ yields $u$ and $R(u)$ yields $x_2(t)$. The only way in which $x_2(t)$ can diverge, is if $u$ takes the values $u_\alpha$, $0 \leq \alpha \leq r$, which themselves correspond to the corners of the triangle. Whenever the triangle is ergodic, that is, if the triangle is irrational, or if it belongs to the large set of ergodic irrational triangles, then almost every orbit passes arbitrarily close to a corner. More specifically, for almost every orbit we may state that the average fraction of time spent within a distance $\varepsilon$ of a corner is itself proportional to the area of the $\varepsilon$-neighbourhood of the corner, that is $\varepsilon^2$. On the other hand, passing within a distance $\varepsilon$ of a corner means that $x_{1,2}(t)$ are of order $\varepsilon^{-1/(r-1)}$. Thus, for any $B$ sufficiently large, the average fraction of time such that $|x_{1,2}(t)| > B$ goes as $B^{-2(r-1)}$. Qualitatively, this means that sudden sharp peaks of the solution will occur rather frequently, and that the probability of $x_{1,2}(t)$ taking large values decays as a power-law. The appearance of sharp peaks is indeed frequently observed in numerical work, see for example the periodic orbit in Figure 4 which were not selected for the purpose.

V. NUMERICAL ILLUSTRATIONS

In the following we illustrate using numerical simulations some of the findings described in Section IV. All the simulations are performed directly on the system (2), without using the results of Section II.

The system is constructed from the given data $\mu_\alpha$, $1 \leq \alpha \leq r+1$, which vary from system to system, as follows: the $u_\alpha$ are always conventionally taken to be

$$u_\alpha = \alpha - 1/2 - \left[ \frac{r}{2} \right]$$

(33)

and the polynomial $S_{r+1}(u)$ is computed from (9), from which $Q_i(u)$ and from that eventually $P_r(u)$ are computed using (10). The initial conditions are taken with a random, or generic, value of $\chi_0$ and, if not otherwise stated, with a value of $u(0) = 1/4$ always different from $u_\alpha$. Since $|C| = 1$, we can fully determine the initial conditions. If not stated otherwise, we shall always be dealing with the case $r = 2$.

First let us show periodic orbits. As we saw, whenever the $u_\alpha$ are real and $\chi_0 = \pi/2$, the resulting orbit is almost surely periodic. Further, they vary continuously as $u(0)$ varies, apart from an obvious discontinuity when $u(0)$ crosses a $u_\alpha$, since this corresponds to a corner. We show this in Figure 4 where we look at a a rational case $\mu_0 = 1/2, \mu_1 = 3/7$ and $\mu_2 = 3/14$.
which yields the equations
\[ \dot{x}_1 = \frac{3}{14} x_1^2 + \frac{5}{14} x_1 x_2 - \frac{3}{8} x_2^2, \quad (34) \]
\[ \dot{x}_2 = x_1^2 - \frac{9}{7} x_1 x_2 + \frac{3}{28} x_2^2. \quad (35) \]

We proceed to display ergodicity. If the triangle’s angles, that is the \( \mu_\alpha \), are rational, then almost every direction is ergodic. The places where the orbit is reflected on the triangle \( \mathcal{P} \) are thus uniformly distributed. Translating this to the \( u \) variables, this means that the values of \( u \) where the orbit crosses the real axis have the probability distribution
\[ p(u) = \frac{1}{\mathcal{N}} \prod_{k=0}^{r} (u-u_\alpha)^{\mu_\alpha-1}, \quad (36) \]
where \( \mathcal{N} \) is the normalisation. An example of the histogram for the \( u \) values of the real crossings of a single orbit over a time of \( 5 \cdot 10^4 \) is given, together with the predicted distribution \( (36) \). We see a good agreement in the case described by Figure 5 in the rational case shown in (36). A similarly good agreement (not reported) is found for the irrational case \( (35) \). We see a good agreement in the case described by Figure 6.

On the other hand, in Figure 6, we display evidence for one of the basic differences between rational and irrational angles: in Figure 6, we display the values of \( x_2(t) \) on the complex plane at those times in which \( u(t) = x_1(t)/x_2(t) \) crosses the real axis for one single long orbit. Indeed, an arbitrary orbit lies in the 3-dimensional subspace of possible values of \( x_1 \) and \( x_2 \) defined by the equation \( |C| = 1 \). The intersection of the orbit with the 2-dimensional space defined by imposing the additional condition \( \text{Im}(x_1(x_2(t)) = 0 \) are thus isolated points. Apart from this we have no further indication. In the two plots of this nature shown in Figure 6, the lower one, corresponding to the case of irrational values of \( \mu_\alpha \), indeed shows a set of points more or less randomly scattered on the plane. However, this is definitely not the case for the upper diagram, which corresponds to simple rational values of the \( \mu_\alpha \). There the set of points is essentially a set of curves, in other words, it is one-dimensional. This corresponds, of course, to the fact that in the corresponding orbit in the triangular billiard, the direction of the orbit can only take a finite number of values\(^{16} \).

Finally, let us verify the validity of the remarks made at the end of Section VI concerning large peaks in the values of \( x_{1,2}(t) \). We consider the case \( r = 2 \), and the rational case discussed above. We find as a histogram of the absolute value of \( x_1(t) \) taken at unit time intervals for an orbit of duration \( 5 \cdot 10^4 \). The existence of a power-law is undeniable, and the agreement with the theoretical prediction of an exponent \( -3 \) is fairly convincing.

VI. CONCLUSIONS

We identify a class of systems of 2 complex ODE’s with remarkable properties which follow from the fact that we can associate to every orbit of the system a unique orbit of a corresponding bounded polygonal billiard. From this identification follow various remarkable qualitative properties: the Lyapunov exponents are all zero, the motion almost surely never diverges and remains bounded by a constant \( B \) for a fraction of the time that goes to one as \( B \to \infty \).

While the results are rather special, being limited to systems of 4 real homogeneous ODE’s derived from a complex analytic system, they can be significantly extended: since the properties here described are qualitative in nature, they extend to every system that can be obtained from (2) via a change of variables. As a trivial example, using real linear transformations, it is possible to obtain homogeneous systems of 4 ODE’s for which the analyticity property is hidden. Similarly, all non-linear transformations which preserve the homogeneity property can also be used to extend the relevant class.

Similarly, starting from the complex Newtonian equation
\[ \ddot{z} = \frac{z}{\bar{z}} \]
we obtain by the transformation\(^2 \)
\[ x_1 = \frac{z^{(k-1)/2}}{x_1}, \quad (38) \]
the set of complex equations
\[ \dot{x}_1 = x_1 x_2, \quad (40) \]
\[ \dot{x}_2 = x_1^2 + \frac{2}{1-k^2} x_2. \quad (41) \]

These belong to our class for all real values of \( k \neq 1 \), so the various results derived above, concerning the existence of periodic orbits, the vanishing of the Lyapunov exponent and so
FIG. 6. Poincaré plots for rational (above) and irrational (below) values of $\mu_\alpha$. Specifically, the points represent the complex values of $x_2(t)$ at the times when $u(t) = x_1(t)/x_2(t)$ crosses the real axis, where one single orbit of duration $5 \cdot 10^4$ starting with $u(0) = 1/4$ and $\chi_0 = 2.51558$.

FIG. 7. Histogram of the absolute values of $x_1(t)$ taken over an orbit of duration $5 \cdot 10^4$. Note a clear power-law decay, due to the large values of $x_{1,2}(t)$ arising when the corresponding billiard orbit approaches a corner. The continuous curve corresponds to the $B^{-3}$ decay predicted by theory.

on, all follow for this Newtonian equation. Note that, in this case, the existence of a solution in terms of quadratures follows trivially from energy conservation, but this solution leads to hyperelliptic integrals for which it is not straightforward to obtain the various results stated above.

Other extensions are possible. In particular, it is possible to extend the solution by quadratures to the case of a set of 2 homogeneous complex ODE’s with a linear term of the following form

$$\dot{x}_1 = -\alpha x_1 + p_r(x_1, x_2) \quad (42a)$$

$$\dot{x}_2 = -\alpha x_2 + q_r(x_1, x_2). \quad (42b)$$

However, the nature of the billiard motion is significantly different and its study is left for future work.

ACKNOWLEDGEMENTS

I would like to acknowledge financial support by UNAM PAPIIT-DGAPA-IN113620 as well as by CONACyT Ciencias Básicas 254515.

AIP PUBLISHING DATA SHARING POLICY

Data sharing not applicable—no new data generated.

Appendix A: Known results on triangular billiards

Here we summarise the results known on polygonal billiards which we use in this paper. To avoid unnecessary complications, we define a polygonal billiard to be a particle moving with unit velocity inside a bounded polygon, and being specularly reflected whenever the trajectory hits a side of the polygon. We limit ourselves to simply connected polygons (no “holes”) as these are the only ones generated by the Schwarz–Christoffel transformation as we use it.

As an aside, note that polygonal billiards are different in one important respect from ordinary dynamical system: there exist orbits for which no continuation is possible past a given point, namely when they hit one of the polygon’s vertices. Additionally, two orbits that are initially close to each other, but hit a vertex on different sides, are in general separated by a finite amount afterwards: in other words, the dynamics is discontinuous. However, the set of singular orbits, namely those which encounter one or two vertices in their course, is denumerable, and hence of measure zero.

We divide this in two essentially different parts: the results concerning rational billiards, that is, billiards such that all their interior angles are rational multiples of $\pi$ and those concerning arbitrary billiards, which will generally be assumed not to be rational. Note that for an unbounded rational polygonal billiard, we include a requirement that all the angles at infinity be rational.

The basic property distinguishing rational billiards from others is the existence of an additional conservation law: any
orbit on a rational billiard can only assume a finite number of different velocities, or said differently, it can only go in a finite number of different directions. The main results are the following:

1. Periodic orbits always exist, and the set of directions corresponding to periodic orbits is denumerable\textsuperscript{15,16}.

2. The set of directions for which the intersections with the polygon $\mathcal{P}$ are not dense in $\mathcal{P}$ is denumerable\textsuperscript{16}. Note, however, that to each direction there may correspond an interval of parallel orbits.

3. The set of directions $\theta$ for which the directional dynamics is not ergodic, has measure zero\textsuperscript{16,17}. By a direction $\theta$ being ergodic we mean the following: let $f(s)$ be an arbitrary continuous function of the arclength of $\mathcal{P}$, and the total length of $\mathcal{P}$ be normalised to 1. If $x_k$, $1 \leq n < \infty$ are the successive points at which an arbitrary orbit having direction $\theta$, intersects $\mathcal{P}$, then ergodicity implies

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{\mathcal{P}} f(s) ds.$$  \hspace{1cm} (A1)

4. Any orbit that hits any side at a right angle is periodic. Indeed, it can be shown that, an orbit with such an initial condition will eventually return to the same side, again hitting it perpendicularly. This then automatically leads to the orbit backtracking on itself, thus becoming eventually periodic. Note that this result may well hold under less restrictive conditions: it holds, for example, for arbitrary right triangles, whether rational or not, as well as for all polygonal billiards, the sides of which are all parallel to one or the other of two directions and has numerically been found to hold generally\textsuperscript{18}.

Note finally that the above results easily imply the claim that, for all unbounded rational billiards, almost all orbits will go to infinity: indeed, we may seal off all the unbounded channels by adding one wall that separates each channel from an inside finite region. The additional separating wall can additionally be put in such a way that, whenever a particle crosses the wall, it necessarily gets into the channel with no possibility of returning to the finite region, see Fig. 8 for an illustration. An orbit that remains forever in the finite region can never hit these sides of the finite region that separate it from the infinite channels. By Property 3 the set of corresponding directions is denumerable, and thus the set of such orbits has measure zero.

For more general polygons, the results are very different. In particular, it is no more true that each orbit only goes in a finite number of directions. The main result is then that the set of ergodic billiards, where ergodic is now taken, as usual, to refer both to velocity and position, is a large set, in the sense of being the countable intersection of dense open subsets of the space of all $n$-sided polygons. To define the latter, we assume that the set of such polygons is normalised so that all polygons have perimeter one. The polygons are then determined by a finite number of parameters (angles and sides) all of which remain bounded. The set of all $n$-sided polygons is thus an open bounded set in a finite dimensional space, so that topological concepts can be defined. It is not known at present whether this set has positive measure.

A general, rather obvious property of polygonal billiards, is that their Lyapunov exponent is zero\textsuperscript{16}. Concerning the existence of periodic orbits, rather little is known. Whereas it is assumed that all triangles have periodic orbits, this is only known with certainty for triangles, the largest angle of which is less than or equal to 100 degrees\textsuperscript{19,20}. Additionally, it is shown in\textsuperscript{19} that the minimal number of bounces for a periodic orbit is not continuous as a function of the angles of the triangle: indeed, it diverges in the vicinity of the right triangle with angles $(\pi/2, \pi/3, \pi/6)$. The problem is therefore unexpectedly difficult.

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