On a class of rational cuspidal plane curves *

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Abstract

We obtain new examples and the complete list of the rational cuspidal plane curves \( C \) with at least three cusps, one of which has multiplicity \( \deg C - 2 \). It occurs that these curves are projectively rigid. We also discuss the general problem of projective rigidity of rational cuspidal plane curves.

A curve \( C \subset \mathbb{P}^2 \) is called cuspidal if all its singular points are cusps. By a cusp we mean a locally irreducible singular point. Here we are interested in rational cuspidal plane curves. While there is a variety of such curves with one or two cusps [Y1-4; Sa; Ts], there are only very few known examples with three or more cusps. The simplest one is the three cuspidal Steiner quartic. In degree five, there are two rational cuspidal quintics with three cusps and another one with four cusps (see [Na]).

For a rational cuspidal curve \( C \) the inequality \( d < 3m \) holds, where \( d = \deg C \) and \( m \) is the maximal multiplicity of the singular points of \( C \) [MaSa]. By Bezout’s theorem, \( m \leq d - 2 \) if \( C \) has at least two cusps.

In this paper we will give new examples and the complete list of rational cuspidal plane curves with at least three cusps and with \( m = d - 2 \) (see Theorem 3.5 below). It contains all those mentioned above. Up to projective equivalence, for any \( d \geq 4 \) there are exactly \( \left\lfloor \frac{d-1}{2} \right\rfloor \) curves of this class. Therefore, they are all projectively rigid. We also discuss the general problem of projective rigidity of rational cuspidal plane curves.

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1 On multiplicity sequences

1.1. Definition. Let \((C, P) \subset (\mathbb{C}^2, P)\) be an irreducible analytic plane curve germ, and let

\[
\mathbb{C}^2 = V_0 \xleftarrow{\sigma_1} V_1 \xleftarrow{\sigma_2} \ldots \xleftarrow{\sigma_n} V_n
\]

be the sequence of blow ups over \(P\) that yields the minimal embedded resolution of singularity of \(C\) at \(P\). Thus, the complete preimage of \(C\) in \(V_n\) is a simple normal crossing divisor \(D = E + C_n\), where \(E\) is the exceptional divisor of the whole resolution and \(C_n\) is the proper preimage of \(C\) in \(V_n\). Denote by \(E_n\) the only \(-1\)-component of \(E\), so that \(E_n \cdot (D_{\text{red}} - E_n) \geq 3\).

Let \(E_i \subset V_i\) be the exceptional divisor of the blow up \(\sigma_i\), \(C_i \subset V_i\) be the proper transform of \(C\) at \(V_i\), and let \(P_{i-1} = \sigma_i(E_i) \in E_{i-1} \cap C_{i-1}\) be the centrum of \(\sigma_i\). Thus, \(C = C_0 \subset V_0\) and \(P = P_0 \in C_0\).

Let \(m_i\) denote the multiplicity of the point \(P_i \in C_i\). The sequence \(\bar{m}_P = (m_0, m_1, \ldots, m_n)\), where \(m_0 \geq m_1 \geq \ldots \geq m_n = 1\), is called the multiplicity sequence of \((C, P)\). We have

\[
\mu = 2\delta = \sum_{i=0}^{n} m_i (m_i - 1),
\]

where \(\mu\) is the Milnor number of \((C, P)\) and \(\delta\) is the virtual number of double points of \(C\) at \(P\) [Mil].

The following proposition gives a characterization of the multiplicity sequences.

1.2. Proposition. The multiplicity sequence \(\bar{m}_P = (m_0, m_1, \ldots, m_n)\) has the following two properties:

i) for each \(i = 1, \ldots, n\) there exists \(k \geq 0\) such that

\[
m_{i-1} = m_i + \ldots + m_{i+k},
\]

where

\[
m_i = m_{i+1} = \ldots = m_{i+k-1},
\]

and
ii) if
\[ m_{n-r} > m_{n-r+1} = \ldots = m_n = 1, \]
then \( m_{n-r} = r - 1. \)

Conversely, if \( \bar{m} = (m_0, m_1, \ldots, m_n) \) is a non-increasing sequence of positive integers satisfying conditions i) and ii), then \( \bar{m} = \bar{m}_P \) for some irreducible plane curve germ \( (C, P) \).

The proof is based on the following lemma.

1.3. Lemma. Let \( \bar{m}_P = (m_0, m_1, \ldots, m_n) \) be the multiplicity sequence of an irreducible plane curve singularity \( (C, P) \). Denote by \( E_i^{(k)} \) the proper transform of the exceptional divisor \( E_i \) of \( \sigma_i \) at the surface \( V_{i+k} \), so that, in particular, \( E_i = E_i^{(0)} \). Then the following hold.

a) \( E_i C_i = m_{i-1} \) and
\[ E_i^{(k)} C_{i+k} = \max \{0, m_{i-1} - m_i - \ldots - m_{i+k-1}\}, \quad k > 0. \]
In particular, \( E_i^{(1)} C_{i+1} = m_{i-1} - m_i \).

b) If
\[ m_{i-1} > m_i + \ldots + m_{i+k-1}, \]
then
\[ m_i = m_{i+1} = \ldots = m_{i+k-1} \]
and
\[ m_{i-1} \geq m_i + \ldots + m_{i+k}. \]

Proof. a) From the equalities \( C_i^{*} := \sigma_i^*(C_{i-1}) = C_i + m_{i-1}E_i, E_i^2 = -1 \) and \( C_i^{*}E_i = 0 \) it follows that \( C_i E_i = m_{i-1} \). Assume by induction that a) holds for \( k \leq r - 1 \), where \( r \geq 1 \). If \( C_i^{(r)}E_i^{(r)} > 0 \), then \( C_i^{(r-1)}E_i^{(r-1)} > 0 \) and \( P_{i+r-1} \in C_{i+r-1} \cap E_i^{(r-1)} \). Therefore, by induction hypothesis we have
\[ C_i^{(r-1)}E_i^{(r-1)} = m_{i-1} - m_i - \ldots - m_{i+r-2} > 0, \]
\[ C_{i+r} = C^{*}_{i+r-1} - m_{i+r-1}E_{i+r} \quad \text{and} \quad E_i^{(r)} \cdot E_{i+r} = 1. \] Hence, \[ E_i^{(r)}C_{i+r} = E_i^{(r)}C^{*}_{i+r-1} - m_{i+r-1}E_i^{(r)} = E_i^{(r-1)}C_{i+r-1} - m_{i+r-1} = m_{i-1} - m_i - \ldots - m_{i+r-1}. \] This proves (a), and also proves that \[ m_{i-1} \geq m_i + \ldots + m_{i+r-1} \]

if \[ m_{i-1} > m_i + \ldots + m_{i+r-2}, \]
which is the second assertion of (b).

To prove the first assertion of (b), note that \( E_i^{(r-1)} \) is tangent to \( C_{i+r-1} \) at the point \( P_{i+r-1} \) iff \( E_i^{(r-1)}C_{i+r-1} > m_{i+r-1} \). As it was done in the proof of (a), one can easily show that the latter is equivalent to the inequality

\[ E_i^{(r)}C_{i+r} = m_{i-1} - m_i - \ldots - m_{i+r-1} > 0, \]

and it implies in turn that \( E_i^{(k)} \) is tangent to \( C_{i+k} \) for each \( k = 0, \ldots, r - 1 \). Since by (a) \( C_{i+k}E_{i+k} = m_{i+k-1} \), the inequality \( m_{i+k-1} > m_{i+k} \), where \( 1 \leq k \leq r - 1 \), would mean that the curve \( E_{i+k} \) is tangent to \( C_{i+k} \) at \( P_{i+k} \), which is impossible, since it is transversal to \( E_i^{(k)} \). Therefore, \( m_{i+k-1} = m_{i+k} \) for all \( k = 1, \ldots, r - 1 \). \( \square \)

**Proof of Proposition 1.2.** Let \( \bar{m}_P = (m_0, m_1, \ldots, m_n) \) be the multiplicity sequence of an irreducible plane curve singularity \((C, P)\). Write \( m_{i-1} = k_i m_i + r_i \) with \( 0 \leq r_i < m_i \). It follows from Lemma 1.3(b) that

356
Thus, if \( r_i = 0 \), then the condition i) is fulfilled. If \( r_i > 0 \), then \( m_{i-1} > k_i m_i = m_i + \ldots + m_{i+k_i-1} \), so that by Lemma 1.3(b) we have

\[ m_{i-1} \geq k_i m_i + m_{i+k_i}, \]

and whence \( r_i \geq m_{i+k_i} \). But \( r_i > m_{i+k_i} \) would imply that

\[ m_{i-1} > m_i + \ldots + m_{i+k_i}, \]

which in turn implies by Lemma 1.3(b) that

\[ m_i = \ldots = m_{i+k_i} < r_i, \]
which is a contradiction. Therefore, in this case $m_{i+k_i} = r_i$, and so

$$m_{i-1} = m_i + \ldots + m_{i+k_i-1} + m_{i+k_i},$$

where

$$m_i = \ldots = m_{i+k_i-1}.$$

The proof of (ii) is easy, and so it is omitted.

To prove the converse, we need the following lemma. For the moment we change the convention and define the multiplicity sequences to be infinite, setting $m_\nu = 1$ for all $\nu \geq n$. Thus, the sequence $(1, 1, \ldots)$ serves as multiplicity sequence of a smooth germ.

**1.4. Lemma.** Let $(C, P)$ be an irreducible plane curve germ with multiplicity sequence $\bar{m}_P = (m_0, m_1, \ldots, m_n, \ldots)$. Then there exists a germ of a smooth curve $(\Gamma, P)$ through $P$ with $(\Gamma C)_P = k$ iff $k$ satisfies the condition

$$(*) \quad k = m_0 + m_1 + \ldots + m_s \quad \text{for some} \quad s > 0 \quad \text{with} \quad m_0 = m_1 = \ldots = m_{s-1}.$$

**Proof.** We proceed by induction on the number of $m_\nu$ which are bigger than 1. If it is equal to zero, i.e. if $(C, P)$ is a smooth germ, then our statement is evidently true.

Let $(\Gamma, P) \subset (V_0, P)$ be a smooth curve germ through $P$, and let $\Gamma' \subset V_1$ be the proper transform of $\Gamma$. Then $C^* = C_1 + m_0 E_1$, and so

$$k = (\Gamma C)_P = \Gamma'C_1 + m_0 \Gamma'E_1 = \Gamma'C_1 + m_0.$$

If $\Gamma' C_1 = 0$, then we are done. If not, then by induction hypothesis (applied to $C_1$) we have

$$\Gamma'C_1 = m_1 + \ldots + m_s$$

for some $s > 0$ and $m_1 = \ldots = m_{s-1}$. If $s = 1$ then this proves the Lemma. If $s > 1$, i.e. $k = m_0 + m_1 + m_2 + \ldots$, then we have to show that $m_0 = m_1$. Denote by $\Gamma''$ the proper transform of $\Gamma'$ on $V_2$. We have, as above,

$$k - m_0 = \Gamma'C_1 = \Gamma''C_2 + m_1,$$
which yields that \( \Gamma''C_2 = k - m_0 - m_1 > 0 \), i.e. \( \Gamma'' \) meets \( C_2 \). Moreover, since \( \Gamma' C_1 = m_1 + m_2 + \ldots > m_1 \), \( \Gamma' \) is tangent to \( C_1 \) at \( P_1 \in C_1 \), and hence \( P_2 \in \Gamma'' \). Since \( \Gamma' \) meets \( E_1 \) transversally, \( \Gamma'' \) does not meet the proper transform \( E_1^{(1)} \) of \( E_1 \) in \( V_2 \). This means that \( \Gamma'' \) and \( E_1^{(1)} \) meet \( E_2 \) in different points, and therefore \( E_1^{(1)}C_2 = 0 \). By Lemma 1.3(a) we have \( E_1^{(1)}C_2 = m_0 - m_1 \); thus, \( m_0 = m_1 \). This completes the proof in one direction.

Conversely, assume that \( k \) satisfies (*). Then \( k - m_0 \) satisfies (*) with respect to \((C_1, P_1)\). If \( k = m_0 \), then any generic smooth curve \( \Gamma \) through \( P = P_0 \) satisfies the condition \((\Gamma C)_P = k = m_0 \). If \( k - m_0 > 0 \), then by inductive hypothesis there is a smooth curve germ \( \Gamma' \subset V_1 \) through \( P_1 \) with \( \Gamma' C_1 = k - m_0 \). Let \( \Gamma \) be the image of \( \Gamma' \) in \( V \). Then \( \Gamma C = \Gamma' C_1 + m_0 \Gamma' E_1 \). If \( k - m_0 = m_1 \), then \( \Gamma' \) can be chosen generically, so transversally to \( E_1 \), and thus we have \( \Gamma C = k \). If \( k - m_0 > m_1 \), then as above \( \Gamma''C_2 = k - m_0 - m_1 > 0 \) and so \( \Gamma'' E_1^{(1)} = 0 \), which implies that \( \Gamma' E_1 = 1 \). Hence, \( \Gamma C = k \) also in this case. The lemma is proven. \( \Box \)

Returning to the proof of Proposition 1.2, fix a non-increasing sequence \( \bar{m} = (m_0, m_1, \ldots, m_n) \) that satisfies (i) and (ii). Note that the sequence \( \bar{m}' := (m_1, \ldots, m_n) \) satisfies the same assumptions. Let \( \sigma_1 : V_1 \to V_0 = \mathbb{C}^2 \) be the blow up at the point \( P \in \mathbb{C}^2 \). Fix a point \( P_1 \in E_1 = \sigma_1^{-1}(P) \subset V_1 \). Consider first the case when \( m_1 > 1 \). We may assume by induction that there exists an irreducible plane curve germ \((C_1, P_1)\) with multiplicity sequence \( \bar{m}_{P_1} = \bar{m}' = (m_1, \ldots, m_n) \). Since \( \bar{m} \) satisfies (i) and (ii), from Lemma 1.4 it easily follows that there is an embedding \((C_1, P_1) \hookrightarrow (V_1, P_1)\) such that \((E_1C_1)_{P_1} = m_0 \). Then obviously \( C := \sigma_1(C_1) \subset \mathbb{C}^2 \) is a plane curve singularity with multiplicity sequence \( \bar{m}_P = \bar{m} = (m_0, m_1, \ldots, m_n) \). Finally, assume that \( m_1 = 1 \). Choose \( C_1 \subset V_1 \) to be a smooth curve with \((C_1E_1)_{P_1} = m_0 \). Then again \( C := \sigma_1(C_1) \subset \mathbb{C}^2 \) has multiplicity sequence \( \bar{m}_P = \bar{m} = (m_0, m_1, \ldots, m_n) \), as desired. This proves Proposition 1.2. \( \Box \)

1.5. Remark. It is well known that the multiplicity sequence carries the same information as the Puiseux characteristic sequence, i.e. each of them can be computed in terms of the other [MaSa]. Moreover, the multiplicity sequence determines the weighted dual graph of the embedded resolution of the cusp and vice versa. This easily follows from the proofs of (1.2) and (1.3), see also [EiNe] or [OZ1,2].
1.6. Let $f : \mathcal{X} \to S$ be a flat family of irreducible plane curve singularities, i.e. there is a diagram

\[
\begin{array}{ccc}
\mathcal{X} & \hookrightarrow & \mathbb{C}^2 \times S \\
f & & \downarrow \text{pr} \\
S & & \\
\end{array}
\]

and a subvariety $\Sigma \subset \mathcal{X}$ such that $f|\Sigma : \Sigma \to S$ is (set theoretically) bijective, $f|\mathcal{X} \setminus \Sigma : \mathcal{X} \setminus \Sigma \to S$ is smooth and the fibre $X_s := f^{-1}(s)$ has a cusp at the point $\{x_s\} = X_s \cap \Sigma$. We say that the family $f$ is \textit{equisingular} if it possesses a simultaneous resolution, i.e. there is a diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{X}} & \hookrightarrow & \mathcal{Z} \\
\pi & & \downarrow \pi \\
\mathcal{X} & \hookrightarrow & \mathbb{C}^2 \times S \\
f & & \downarrow \text{pr} \\
S & & \\
\end{array}
\]

where $\mathcal{Z}$ is smooth over $S$ and for each $s \in S$ the induced diagram of the fibres

\[
\begin{array}{ccc}
\tilde{X}_s & \hookrightarrow & Z_s \\
\pi & & \downarrow \pi \\
X_s & \hookrightarrow & \mathbb{C}^2 \\
\end{array}
\]

yields an embedded resolution of $X_s$ in such a way that the weighted dual graphs of $\pi^{-1}(X_s)$ are all the same.

Observe that if the family $f$ is equisingular, then all the cusps $(X_s, x_s)$ have the same multiplicity sequence, see (1.5). Vice versa, we have the following simple lemma, which will be useful in the next section.

1.7. Lemma. Let $f : \mathcal{X} \to S$ be a flat family of irreducible plane curve singularities. Assume that $S$ is normal and all the cusps $(X_s, x_s)$, $s \in \Sigma$, have the same
multiplicity sequence. Then the family $f$ is equisingular.

**Proof.** Note that $\Sigma$ is necessarily normal and $f|\Sigma : \Sigma \to S$ is an isomorphism. Blowing up $\Sigma$ gives a morphism $\pi_1 : Z_1 \to \mathbb{C}^2 \times S$ whose restriction to the fibre over $s$ yields the blowing up of $\mathbb{C}^2$ at $x_s$. Then the proper transform $X_1$ of $X$ in $Z_1$ is the blowing up $\pi_1 : X_1 \to \mathcal{X}$ along $\Sigma$. The singular set of the induced map $X_1 \to S$ is a subvariety $\Sigma_1$ mapped one–to–one onto $S$. Repeating the procedure and using the fact that all multiplicity sequences of the cusps $(X_s, x_s)$ are the same, leads to a simultaneous resolution of $f$ as above.  

\[ \Box \]

### 2 Computation of deformation invariants in terms of multiplicity sequences

**2.1. On the Rigidity Problem.** Consider a minimal smooth completion $V$ of an open surface $X = V \setminus D$ by a simple normal crossing (SNC for short) divisor $D$. Let $\Theta_V(D)$ be the logarithmic tangent bundle. By [FZ] the groups $H^i(\Theta_V(D))$ control the deformations of the pair $(V, D)$; more precisely, $H^0(\Theta_V(D))$ is the space of its infinitesimal automorphisms, $H^1(\Theta_V(D))$ is the space of infinitesimal deformations and $H^2(\Theta_V(D))$ gives the obstructions for extending infinitesimal deformations. In [FZ, Lemma 1.3] we proved that if $X$ is a $\mathbb{Q}$–acyclic surface, i.e. $H_i(X; \mathbb{Q}) = 0$, $i > 0$, then the Euler characteristic of $\Theta_V(D)$ is equal to $K_V(K_V + D)$. If, in addition, $X$ is of log–general type, i.e. its log–Kodaira dimension $\tilde{k}(X) = 2$, then $h^0(\Theta_V(D)) = 0$ (indeed, by Iitaka’s theorem [Ii, Theorem 6] the automorphism group of a surface $X$ of log–general type is finite). We conjectured in [FZ] that such surfaces are rigid and have unobstructed deformations, i.e. that for them

$$h^1(\Theta_V(D)) = h^2(\Theta_V(D)) = 0,$$

and thus also

$$\chi(\Theta_V(D)) = 0.$$  

This, indeed, is true in all examples that we know [FZ].

Let now $X = \mathbb{P}^2 \setminus C = V \setminus D$, where $C$ is an irreducible plane curve and $V \to \mathbb{P}^2$ is the minimal embedded resolution of singularities of $C$, so that the total transform
\(D\) of \(C\) in \(V\) is an SNC–divisor. In view of (1.6) and (1.7) the deformations of \((V, D)\) correspond to equisingular embedded deformations of the curve \(C\) in \(\mathbb{P}^2\). We say shortly that \(C\) is \textit{projectively rigid} (resp. \textit{(projectively) unobstructed}) if the pair \((V, D)\) has no infinitesimal deformations, i.e. \(h^1(\Theta_V \langle D \rangle) = 0\) (resp. \(h^2(\Theta_V \langle D \rangle) = 0\)).

Observe that \(C \subset \mathbb{P}^2\) is projectively rigid iff the only equisingular deformations of \(C\) as a plane curve are those obtained via the action of the automorphism group \(\text{PGL}(3, \mathbb{C})\) on \(\mathbb{P}^2\). Indeed, suppose that \(C_t \subset \mathbb{P}^2, t \in T,\) is a family of deformations of \(C_0 = C\) such that all the members \(C_t\) have at the corresponding singular points the same multiplicity sequence. Then the singularities can be resolved simultaneously at a family of surfaces \((V_t, D_t), t \in T,\) see (1.6), (1.7). In view of the rigidity, there is a local isomorphism with the trivial family \((V_0, D_0) \times T,\) and so by blowing down this leads to a family of projective isomorphisms \(C_t \xrightarrow{\phi_t} C_0.\) The converse is evidently true.

It is easily seen that if \(C\) is a rational cuspidal curve, then the complement \(X = \mathbb{P}^2 \setminus C\) is \(\mathbb{Q}\)–acyclic. If, in addition, \(C\) has at least three cusps, then \(X\) is also of log–general type [Wak]. Thus, the rigidity conjecture of [FZ] says that such a curve \(C\) should be projectively rigid and unobstructed. Here we compute the deformation invariants of \(X\) in terms of multiplicity sequences of the cusps of \(C\). In the next section we apply these computations to check the above rigidity conjecture for the complements of rational cuspidal curves considered there (see Lemma 3.3; cf. also section 4).

\[2.2.\text{ Definition (cf. [MaSa, FZ])}.\] Let the notation be as in Definition 1.1. The blowing up \(\sigma_{i+1}, i \geq 1,\) of \(V_i\) at the point \(P_i \in C_i\) is called \textit{inner} (or \textit{subdivisional}) if \(P_i \in E_i \cap E^{(k)}_{i-k}\) for some \(k > 0\), and it is called \textit{outer} (or \textit{sprouting}) in the opposite case. Note that \(\sigma_1\) is neither inner nor outer. Moreover, \(\sigma_2\) is always outer, and so \(\rho \geq 1,\) where \(\omega = \omega_p\) resp. \(\rho = \rho_p\) denotes the number of inner resp. outer blowing ups. Denote also by \(k = k_p\) the total number of blow ups, i.e. the length of the multiplicity sequence \(\bar{m}_p = (m_0, m_1, \ldots, m_{k_p})\) minus one. Clearly, \(\omega + \rho = k - 1.\)

By \([a]\) we denote the smallest integer \(\geq a.\)
2.3. Lemma. 

\[ \omega_P = \sum_{i=1}^{k_P} \left( \left\lfloor \frac{m_i-1}{m_i} \right\rfloor - 1 \right) \]

**Proof.** It is clear that the total number of exceptional curves \( E_i^{(j)} \subset V_{i+j} \), where \( 1 \leq i + j < k \), passing through the centers \( P_{i+j} \) of the blow ups \( \sigma_{i+j+1} \) is \( 2\omega + \rho \). If \( m_{i-1} = sm_i \), then by Lemma 1.3 \( P_{i+j} \in E_i^{(j)} \) for \( j = 0, 1, \ldots, s-1 \), i.e. exactly \( s \) times, except in the case when \( i = k_P \). If \( m_{i-1} = sm_i + r \), where \( 0 < r < m_i \), then this happens for \( j = 0, 1, \ldots, s \), so \( (s+1) \) times. In any case, this happens \( \left\lceil \frac{m_i-1}{m_i} \right\rceil \) times, with the only exception when \( i = k_P \). Therefore,

\[ 2\omega + \rho = \sum_{i=1}^{k} \left( \left\lfloor \frac{m_i-1}{m_i} \right\rfloor - 1 \right) = \sum_{i=1}^{k} \left( \left\lfloor \frac{m_i-1}{m_i} \right\rfloor - 1 \right) + (k-1). \]

Since \( \omega + \rho = k - 1 \), we have the desired result. \( \square \)

2.4. Proposition. Let \( V_0 \) be a smooth compact complex surface, \( C \subset V_0 \) be an irreducible cuspidal curve, and \( V \to V_0 \) be the embedded resolution of singularities of \( C \). Denote by \( K_V \) resp. \( K_{V_0} \) the canonical divisor of \( V \) resp. \( V_0 \), by \( D \) the reduced total preimage of \( C \) at \( V \), and by \( \tilde{m}_P = (m_{P,0}, m_{P,1}, \ldots, m_{P,k_P}) \) the multiplicity sequence at \( P \in \text{Sing} C \). Let, as before, \( \omega_P \) be the number of inner blow ups over \( P \). Set

\[ \eta_P = \sum_{i=0}^{k_P} (m_{P,i} - 1). \]

Then

\[ K_V(K_V + D) = K_{V_0}(K_{V_0} + C) + \sum_{P \in \text{Sing} C} (\eta_P + \omega_P - 1). \]

**Proof.** Let \( \sigma_{i+1} : V_{i+1} \to V_i \) be a step in the resolution of singularities of \( C \). Put \( K_i = K_{V_i} \) and let \( D_i \) be the reduced total preimage of \( C \) at \( V_i \). We have

\[ K_{i+1} = K_i^* + E_{i+1} \quad \text{and} \quad D_i^* = \sigma_{i+1}^*(D_i) = D_{i+1} + (m_i - 1)E_{i+1} + \delta_i E_{i+1}, \]

where

\[ \delta_i = \begin{cases} 
0 & \text{if } \sigma_{i+1} \text{ is neither inner nor outer} \\
1 & \text{if } \sigma_{i+1} \text{ is outer} \\
2 & \text{if } \sigma_{i+1} \text{ is inner} 
\end{cases} \]
It follows that
\[ K_i(K_i + D_i) = K_{i+1}(K_i^* + D_i^*) = K_{i+1}(K_{i+1} + D_{i+1} + (m_i + \delta_i - 2) E_{i+1}) \]
\[ = K_{i+1}(K_{i+1} + D_{i+1}) - (m_i + \delta_i - 2). \]
Thus,
\[ K_{i+1}(K_{i+1} + D_{i+1}) = K_i(K_i + D_i) + (m_i - 1) + (\delta_i - 1). \]
Now the desired equality easily follows. \( \square \)

2.5. Corollary. Let \( C \subset \mathbb{P}^2 \) be a plane cuspidal curve of degree \( d \geq 3 \), and let \( \pi : V \to \mathbb{P}^2 \) be the embedded resolution of singularities of \( C \), \( D \) be the reduced total preimage of \( C \) in \( V \) and \( K = K_V \) be the canonical divisor. Then
\[ \chi(\Theta_V(D)) = K(K + D) = -3(d - 3) + \sum_{P \in \text{Sing} C} (\eta_P + \omega_P - 1). \tag{1} \]

2.6. Remark. In view of (2.5), in the case when \( C \subset \mathbb{P}^2 \) is a rational cuspidal curve with at least three cusps, the rigidity conjecture mentioned in (2.1) in particular yields the identity
\[ \sum_{P \in \text{Sing} C} (\eta_P + \omega_P - 1) = 3(d - 3), \]
which, indeed, is true in all examples that we know (see e.g. Lemma 3.3 below).

3 Rational cuspidal plane curves of degree \( d \) with a cusp of multiplicity \( d - 2 \)

3.1. Lemma. Let \( C \subset \mathbb{P}^2 \) be a rational cuspidal curve of degree \( d \) with a cusp \( P \in C \) of multiplicity \( m_P \) with multiplicity sequence \( \bar{m}_P = (m_{P,0}, \ldots, m_{P,k_P}) \). Then the projection \( \pi_P : C \to \mathbb{P}^1 \) from \( P \) has at most \( 2(d - m - 1) \) branching points. Furthermore, if \( Q_1, \ldots, Q_s \) are the other cusps of \( C \) with multiplicities \( m_1, \ldots, m_s \), then
\[ \sum_{j=1}^s (m_j - 1) + (m_{P,1} - 1) \leq 2(d - m - 1). \]
Proof. By the Riemann–Hurwitz formula, applied to the composition \( \tilde{\pi}_P : \mathbb{P}^1 = \tilde{C} \to \mathbb{P}^1 \) of the normalization map \( \tilde{C} \to C \) and the projection \( \pi_P \), which has degree \( d - m \), we obtain that

\[
2(d - m) = 2 + \sum_{Q \in C} (v_Q - 1),
\]

where \( v_Q \) is the ramification index of \( \tilde{\pi}_P \) at \( Q \). The singular point \( Q_i \) of \( C \) gives rise to a branching point with ramification index \( \geq m_i \), and after blowing up at \( P \in C \) the first infinitesimal point to \( P \) gives rise to a branching point with ramification index \( \geq m_{P,1} \). This proves the lemma. \( \square \)

Denote by \( (m_a) \), where \( m > 1 \), the following multiplicity sequence:

\[
(m_a) = (m, \ldots, m, 1, \ldots, 1).
\]

We write simply \( (m) \) instead of \( (m_1) \) for \( a = 1 \). Notice that \( (2^k) \) is the multiplicity sequence of a simple plane curve singularity of type \( A_{2k} \left( x^2 + y^{2k+1} = 0 \right) \); thus, \( (2) \) corresponds to an ordinary cusp \( x^2 + y^3 = 0 \).

### 3.2. Lemma

Let \( C \subset \mathbb{P}^2 \) be a rational cuspidal curve of degree \( d \) with a cusp \( P \in C \) of multiplicity \( d - 2 \). Then \( C \) has at most three cusps. Assume further that \( C \) has three cusps. Then they are not on a line and have multiplicity sequences resp. \( [(d - 2), (2_a), (2_b)] \), where \( a + b = d - 2 \). Each of these cusps has only one Puiseux characteristic pair; they are, respectively, \( (d - 1, d - 2) \), \( (2a + 1, 2) \), \( (2b + 1, 2) \).

Proof. The projection \( C \to \mathbb{P}^1 \) from \( P \in C \) being 2–sheeted, by the preceding Lemma it has at most two ramification points. Thus, by Bezout’s Theorem the multiplicities of other singular points are at most two and there are at most two of them. Moreover, it follows from Lemma 3.1 that in the case when there are two more singular points, the multiplicity sequence at \( P \) should be \( (d - 2) \). Hence, the only multiplicity sequences in the case of three cusps are \( [(d - 2), (2_a), (2_b)] \). By the genus formula we have

\[
\binom{d - 2}{2} + a + b = \binom{d - 1}{2},
\]

and thus \( a + b = d - 2 \).
That the three cusps do not lie on a line follows from Bezout’s theorem.

\[ \blacksquare \]

3.3. Lemma. Let \( C \subset \mathbf{P}^2 \) satisfies the assumptions of Lemma 3.2. Then \( C \) is projectively rigid and unobstructed.

Proof. Let \( (V, D) \to (\mathbf{P}^2, C) \) be the minimal embedded resolution of singularities of \( C \). Then, first of all, the Euler characteristic of the holomorphic tangent bundle \( \chi = \chi(\Theta_V(D)) \) vanishes. This follows from (1). Indeed, if \( P \) has multiplicity sequence \( \bar{m}_P = (m) \), then
\[
\eta_P + \omega_P - 1 = 2m - 3,
\]
whereas for the multiplicity sequence \( (2a) \) this quantity equals \( a \). Thus, under the assumptions of Lemma 3.2 we have
\[
\chi = 9 - 3d + (a + b) + 2(d - 2) - 3 = 0.
\]

Furthermore, the projection from the point \( P \in C \) of multiplicity \( d - 2 \) yields a morphism \( \pi_P : V \to \mathbf{P}^1 \), which is a \( \mathbf{P}^1 \)-ruling. Its restriction to \( D \) is 3-sheeted. Moreover, \( X = V \setminus D = \mathbf{P}^2 \setminus C \) is a \( \mathbf{Q} \)-acyclic affine surface, i.e. \( H_i(X; \mathbf{Q}) = 0 \), \( i > 1 \). By Proposition 6.2 from [FZ] it follows that \( h^2(\Theta_V(D)) = 0 \), and so \( C \) is unobstructed. Since \( \tilde{k} (V \setminus D) = 2 \) [Wak], due to Theorem 6 from [Ii] we also have \( h^0(\Theta_V(D)) = 0 \). Therefore, \( h^1(\Theta_V(D)) = 0 \), that means that \( (V, D) \) is a rigid pair, and hence \( C \) is projectively rigid (see (2.1)).

\[ \blacksquare \]

3.4. Lemma. Let \( (C, 0) \subset (\mathbf{C}^2, 0) \) be a plane curve germ given parametrically by
\[
t \mapsto (f(t), g(t)) = (t^m, \sum_{\nu=1}^{\infty} c_{\nu} t^{\nu}).
\]

Then the multiplicity sequence of \( (C, 0) \) has the form
\[
(m, \ldots, m, \ldots)
\]
iff \( (** \) \( c_i = 0 \) for all \( i \) with \( i < mr \) such that \( m \not| i \).

\[ ^2 \text{see (2.1) for the definitions.} \]
Furthermore, $(C, 0)$ has multiplicity sequence $(2r)$ iff $m = 2$, the first $r$ odd coefficients vanish: $c_1 = c_3 = \ldots = c_{2r-1} = 0$ and, moreover, $c_{2r+1} \neq 0$.

Proof. After coordinate change of type $(f(t), g(t)) \mapsto (f(t), g(t) - p(f(t)))$, where $p \in \mathbb{C}[z]$, we may assume that $c_m = c_{2m} = \ldots = c_{rm} = 0$. Then

$$g(t) = c_s t^s + \text{higher order terms},$$

with $c_s \neq 0$ and either $s > rm$ or $m \not| s$.

First of all, we show that if $(C, 0)$ has multiplicity sequence $(m, \ldots, m, \ldots)$, then $s > mr$, which is equivalent to (**). Let $s = \rho m + s_1$, where $0 \leq s_1 < m$. If $\rho < r$, then after blowing up $\rho$ times we obtain the parametrized curve germ

$$(f(t), g(t)/t^m),$$

which still has multiplicity $m$. But since $g(t)/t^m$ has multiplicity $s - \rho m = s_1$, this contradicts the assumption that $s_1 < m$. Thus, if $(C, 0)$ has multiplicity sequence $(m, \ldots, m, \ldots)$, then the condition (**)) is satisfied. The converse is clear.

Finally, assume that $m = 2$, $c_1 = c_3 = \ldots = c_{2r-1} = 0$ and $c_{2r+1} \neq 0$. Then after the above coordinate change we have $(f(t), g(t)) = (t^2, c_{2r+1} t^{2r+1} + \ldots)$, and so due to the above criterion $(C, 0)$ has multiplicity sequence $(2r)$. Once again, the converse is clear. \qed

3.5. Theorem. For any $d \geq 4$, $a \geq b \geq 1$ with $a + b = d - 2$ there is a unique, up to projective equivalence, rational cuspidal curve $C = C_{d, a} \subset \mathbb{P}^2$ of degree $d$ with three cusps with multiplicity sequences $[(d - 2), (2a), (2b)]$.

In appropriate coordinates this curve can be parametrized as

$$C_{d, a} = (P : Q : R) = \langle s^2(s - t)^{d-2} : t^2(s - t)^{d-2} : s^2 t^2 q_{d, a}(s, t) \rangle,$$

where $q_{d, a}(s, t) = \sum_{i=0}^{d-4} c_i s^i t^{d-4-i}$ and the polynomial $\tilde{q}_{d, a}(T) = \sum_{i=0}^{d-4} c_i T^i$ is defined as

$$\tilde{q}_{d, a}(T) = \frac{f_{d, a}(T^2) + T^{2a-1}}{(1 + T)^{d-2}}.$$
Here \( f_{d,a}(T) \) is a polynomial of degree \( d - 3 \) uniquely defined by the divisibility condition \( (1 + T)^{d-2} | (f_{d,a}(T^2) + T^{2a-1}) \).

**Proof.** Suppose that \( C \subset \mathbb{P}^2 \) is such a curve. Since by Lemma 3.2 its three cusps are not at a line, up to projective transformation we may assume that \( C \) has cusps at the points \((0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)\) with multiplicity sequences resp. \((d - 2), (2a), (2b)\). Let \( h = (P : Q : R) : \mathbb{P}^1 \to C \hookrightarrow \mathbb{P}^2 \) be the normalization of \( C \), where \((P : Q : R)\) is a triple of binary forms of degree \( d \) without common zero such that

\[
\begin{align*}
    h(1 : 1) &= (0 : 0 : 1) \\
    h(0 : 1) &= (0 : 1 : 0) \\
    h(1 : 0) &= (1 : 0 : 0).
\end{align*}
\]

Since \( C \) is required to have cusps of multiplicity \( d - 2 \) at \( h(1 : 1) \) and of multiplicity \( 2 \) at \( h(0 : 1) \) and at \( h(1 : 0) \), up to multiplication by constant factors we may write

\[
\begin{align*}
    P(s, t) &= (s - t)^{d-2} s^2 \\
    Q(s, t) &= (s - t)^{d-2} t^2 \\
    R(s, t) &= s^2 t^2 q(s, t),
\end{align*}
\]

where

\[
q(s, t) = \sum_{i=0}^{d-4} c_i s^i t^{d-4-i} \quad \text{and} \quad c_0 \neq 0, \quad c_{d-4} \neq 0, \quad q(1, 1) \neq 0.
\]

We will show that under our assumptions \( q \) is uniquely defined.

To impose the conditions that there is a cusp of type \( (2a) \) at the point \( h(0 : 1) = (0 : 1 : 0) \) resp. of type \( (2b) \) at the point \( h(1 : 0) = (1 : 0 : 0) \), we rewrite the above parametrization in appropriate affine coordinates at the corresponding points.

At \((0 : 1)\) we set \( \xi = s/t \) and we have

\[
\tilde{f}(\xi) = \frac{P}{Q} = \frac{s^2}{t^2} = \xi^2
\]

\(^3\)For the explicit equations, see Proposition 3.9 below.
\[
\tilde{g}(\xi) = \frac{R}{Q} = \frac{s^2q(s, t)}{(s - t)^{d-2}} = \frac{\xi^2\tilde{q}(\xi)}{(\xi - 1)^{d-2}},
\]

where
\[
\tilde{q}(\xi) = \sum_{i=0}^{d-4} c_i \xi^i.
\]

By Lemma 3.4 \(C\) has a cusp of type \((2a)\) at \(h(0 : 1) = (0 : 1 : 0)\) iff the odd coefficients of \(\xi^i\) of the function \(\frac{R}{\xi^2Q} = \frac{\tilde{q}(\xi)}{(\xi - 1)^{d-2}}\) vanish up to order \(2a - 3\) (this imposes \((a - 1)\) conditions) and the coefficient of \(\xi^{2a-1}\) does not vanish.

At \((1 : 0)\) we set \(\tau = t/s\) and we have
\[
\tilde{f}(\tau) = \frac{Q}{P} = \frac{t^2}{s^2} = \tau^2
\]
\[
\tilde{g}(\tau) = \frac{R}{P} = \frac{\tau^2\tilde{q}(\tau)}{(1 - \tau)^{d-2}},
\]

where
\[
\tilde{q}(\tau) = \sum_{i=0}^{d-4} c_i \tau^{d-4-i}.
\]

By Lemma 3.4 \(C\) has a cusp of type \((2b)\) at \(h(1 : 0) = (1 : 0 : 0)\) iff the odd coefficients of \(\tau^i\) of the function \(\frac{R}{\tau^2P} = \frac{\tilde{q}(\tau)}{(1 - \tau)^{d-2}}\) vanish up to order \(2b - 3\) (this imposes \((b - 1)\) conditions) and the coefficient of \(\tau^{2b+1}\) does not vanish.

Note that the coefficients \(\tilde{c}_i\) of \(\xi_i\) in \(\tilde{g}(\xi)/\xi^2\) and those \(\tilde{c}_i\) of \(\tau_i\) in \(\tilde{g}(\tau)/\tau^2\) are linear functions in \(c_0, \ldots, c_{d-4}\). We must show that the system
\[
\tilde{c}_1 = \tilde{c}_3 = \ldots = \tilde{c}_{2a-3} = 0, \quad \tilde{c}_{2a-1} = 1
\]
\[
\tilde{c}_1 = \ldots \tilde{c}_{2b-3} = 0
\]
has the unique solution. Indeed, by symmetry then also the coefficient \(\tilde{c}_{2b-1}\) is uniquely defined and non-zero. This follows from the fact that the associate homogeneous system
\[
\tilde{c}_1 = \tilde{c}_3 = \ldots = \tilde{c}_{2a-3} = \tilde{c}_{2a-1} = 0
\]
\[
\tilde{c}_1 = \ldots \tilde{c}_{2b-3} = 0
\]
has the unique solution, which corresponds to \(q \equiv 0\). Observe that it has
\[
(a - 1) + (b - 1) + 1 = d - 3
\]

16
equations and the same number of variables. To show the uniqueness we need the following lemma. Its proof is easy and can be omitted.

3.6. Lemma. Let

\[ h(T) = \sum_{\nu \geq 0} a_\nu T^\nu \in \mathbb{C}[T] \]

and

\[ \tilde{h}(T) = h(T)(1 + T^2 u(T^2)) \]

for some power series \( u \in \mathbb{C}[[T]] \). Set \( \tilde{h}(T) = \sum_{\nu \geq 0} \tilde{a}_\nu T^\nu \). Then

\[ \tilde{a}_1 = \tilde{a}_3 = \ldots = \tilde{a}_{2k+1} = 0 \]

iff

\[ a_1 = a_3 = \ldots = a_{2k+1} = 0. \]

Returning to the proof of the theorem, put \( n = d - 4 \) and

\[ F(T) = \tilde{q}(T)(1 + T)^{n+2} = \frac{\tilde{q}(T)}{(1-T)^{n+2}} (1 - T^2)^{n+2} \]

\[ G(T) = \tilde{q}(T)(1 + T)^{n+2} = \frac{\tilde{q}(T)}{(1-T)^{n+2}} (1 - T^2)^{n+2}. \]

By Lemma 3.6 the first \( a \) (resp. \( b-1 \)) odd coefficients of \( F(T) \) (resp. of \( G(T) \)) vanish iff the same is true for \( \frac{\tilde{q}(T)}{(1-T)^{n+2}} \) (resp. for \( \frac{\tilde{q}(T)}{(1-T)^{n+2}} \)). Note that by definition \( \tilde{q}(T) = \tilde{q}(\frac{1}{T})T^n \). Thus, we have that \( \text{deg } F = 2n + 2 \) is even and

\[ F\left(\frac{1}{T}\right)T^{2n+2} = \tilde{q}\left(\frac{1}{T}\right)T^n (1 + \frac{1}{T})^{n+2} T^{n+2} = \tilde{q}(T)(1 + T)^{n+2} = G(T). \]

Therefore, the conditions that the first \( a \) odd coefficients of \( F \) and the first \( b-1 \) odd coefficients of \( G \) vanish are equivalent to \( F \) being an even function: \( F(T) = F(-T) \). Indeed, since \( a + b - 1 = d - 3 = n + 1 \), the above conditions mean that all odd coefficients of \( F \) vanish. Now we use the following elementary facts.

3.7. Lemma. Assume that \( p \in \mathbb{C}[T] \) and \( (1+T)^k p(T) \) is even. Then \( (1-T)^k \mid p(T) \).
**Proof.** By the condition we have \((1 + T)^k p(T) = (1 - T)^k p(-T)\), as the product is even. Thus \((1 - T)^k | p(T)\). \(\Box\)

From this lemma immediately follows

**3.8. Corollary.** If \(\deg p \leq n\) and \((1 + T)^{n+2} p(T)\) is even, then \(p \equiv 0\).

Being applied to \(p = \tilde{q}\) and \(F(T) = (1 + T)^{n+2} \tilde{q}(T)\), Corollary 3.8 implies that \(\tilde{q} \equiv 0\) and so \(q \equiv 0\), i.e. the above homogeneous system has a unique solution. This completes the proof of the first part of Theorem 3.5.

As for the second one, we must prove the explicit presentation of \(\tilde{q} = \tilde{q}_{d,a}\). As above, it follows from the assumptions that the first \((a - 1)\) and the last \((b - 1)\) odd coefficients of \(F(T)\) vanish, while the coefficient of \(T^{2a-1}\) is non-zero. Therefore, \(F(T) = f(T^2) + T^{2a-1}\) with \(f\) being a polynomial of degree \(d - 3\). Hence

\[
\tilde{q}(T) = \frac{f(T^2) + T^{2a-1}}{(1 + T)^{d-2}}.
\]

From the equality \(F(T) = (1 + T)^{d-2} \tilde{q}(T)\) we have that

\[
F(-1) = F'(-1) = \ldots = F^{(d-3)}(-1) = 0.
\]

These equations uniquely define the derivatives of the polynomial \(f(\xi)\) at \(\xi = 1\) up to order \((d - 3)\), and therefore \(f_{d,a}(\xi) = f(\xi) = \sum_{k=0}^{d-3} \frac{a_k}{k!} (\xi - 1)^k\) is determined in a unique way. This completes the proof of Theorem 3.5. \(\Box\)

**3.9. Proposition.** a) The polynomial \(f = f_{d,a}\) in Theorem 3.5 can be given as

\[
f(T) = \sum_{k=0}^{d-3} \frac{a_k}{k!} (T - 1)^k,
\]

where \(a_0 = 1\), \(a_1 = a - \frac{1}{2}\) and

\[
a_k = \frac{1}{2^k} (2a-1)(2a-3) \ldots (2a-(2k-1)) = a_1(a_1-1) \ldots (a_1-(k-1)), \ k = 1, \ldots, d-3,
\]

i.e. it coincides with the corresponding partial sum of the Taylor expansion at \(T = 1\) of (the positive branch of) the function \(T^{a_1}\).
b) In the affine chart \((X = x/z, Y = y/z)\) the curve \(C_{d, a}\) as in Theorem 3.5 can be given by the equation \(p(X, Y) = 0\), where \(p = p_{d, a} \in \mathbb{Q}[X, Y]\) is defined as follows:

\[
p(X, Y) = \frac{X^{2a+1}Y^{2b+1} - ((X - Y)^{d-2} - XY \hat{f}(X, Y))^2}{(X - Y)^{d-2}},
\]

and where \(\hat{f}(X, Y) = Y^{d-3}f(\frac{X}{Y})\) is the homogeneous polynomial which corresponds to \(f(T)\).

**Proof.** We start with the proof of b). In the notation of Theorem 3.5 in the affine chart \(\xi = s/t\) in \(\mathbb{P}^1\) we have

\[
\frac{X}{Y} = \frac{P}{Q} = \xi^2
\]

and

\[
X = \frac{(\xi - 1)^{d-2}}{\hat{q}(\xi)},
\]

where

\[
\hat{q}(\xi) = \hat{q}_{d, a}(\xi) = \sum_{i=0}^{d-4} c_i \xi^i
\]

is as above. Thus,

\[
(X^2 - 1)^{d-2} = X\hat{q}(\xi)(\xi - 1)^{d-2} = X(f_{d, a}(\xi^2) + \xi^{2a-1})
\]

by the definition of \(\hat{q}(\xi)\). Plugging here \(\xi^2 = X/Y\) we obtain

\[
(X - Y)^{d-2} = XY(Y^{d-3}f(\frac{X}{Y}) + \xi X^{a-1}Y^b) = XY\hat{f}(X, Y) + \xi X^aY^{b+1}.
\]

Hence,

\[
\xi = \frac{(X - Y)^{d-2} - XY \hat{f}(X, Y)}{X^aY^{b+1}}
\]

and so

\[
\xi^2 = \frac{X}{Y} = \frac{(X - Y)^{d-2} - XY \hat{f}(X, Y))^2}{X^{2a}Y^{2b+2}}.
\]

Therefore, the curve \(C_{d, a}\) in the affine chart \((X, Y)\) satisfies the equation \(\tilde{p} = 0\), where

\[
\tilde{p}(X, Y) = X^{2a+1}Y^{2b+1} - ((X - Y)^{d-2} - XY \hat{f}(X, Y))^2.
\]

Since \(C_{d, a}\) is an irreducible curve of degree \(d\), b) follows from the next lemma.
3.10. Lemma.

\((X - Y)^{d-2} \mid \tilde{p}(X, Y)\).

**Proof.** We have

\[
\tilde{p}(X, Y) \equiv \psi(X, Y) \mod (X - Y)^{d-2},
\]

where

\[
\psi(X, Y) := X^{2a+1}Y^{2b+1} - X^2Y^2\hat{f}^2(X, Y).
\]

The polynomial \(\psi\) is homogeneous of degree \(2d - 2\), and thus it is enough to show that

\[
(X - 1)^{d-2} \mid \psi(X, 1),
\]

or equivalently, that

\[
(X^2 - 1)^{d-2} \mid \psi(X^2, 1).
\]

Since \(\psi(X^2, 1)\) is an even polynomial and \((X^2 - 1)^{d-2} = (X - 1)^{d-2}(X + 1)^{d-2}\), by (3.7) it is sufficient to check that

\[
(X + 1)^{d-2} \mid \psi(X^2, 1).
\]

But

\[
\psi(X^2, 1) = X^{4a+2} - X^4\hat{f}^2(X^2, 1) \equiv 0 \mod (X + 1)^{d-2},
\]

because by definition,

\[
\hat{f}(X^2, 1) \equiv -X^{2a-1} \mod (X + 1)^{d-2}.
\]

\(\square\)

**Proof of Proposition 3.9, a).** From (2) it follows that

\[
f^2(T) - T^{2a-1} = (f(T) - T^{a_1})(f(T) + T^{a_1}) \equiv 0 \mod (T - 1)^{d-2},
\]

where by \(T^{a_1}\) we mean those branch of the square root of \(T^{2a-1}\) which is positive at \(T = 1\). Since \((T - 1)^{d-2}\) does not divide the second factor, we have

\[
f(T) - T^{a_1} \equiv 0 \mod (T - 1)^{d-2}.
\]
Thus, indeed, $f(T)$ is the $(d-3)$-th partial sum of the Taylor series of the function $T^{a_1} = T^{2a-1}$ at the point $T = 1$, and a) follows. This proves the Proposition. □

3.11. Remark. By the way, it follows that any rational cuspidal plane curve $C$ with at least three cusps, one of which has multiplicity $\deg C - 2$, can be defined over $\mathbb{Q}$.

3.12. Examples. Here we present the affine equations $p_{d,a} = 0$ of the curves $C_{d,a}$ for $4 \leq d \leq 7$.

$d = 4$ and $a = 1$ (Steiner’s quartic)

$$p_{4,1}(X, Y) = \frac{Y^2X^2}{4} - (X - Y)^2 + XY(Y + X)$$

$d = 5$ and $a = 2$

$$p_{5,2}(X, Y) = \frac{Y^3X^2}{64} - \frac{9Y^2X^3}{64} - (X - Y)^3 + XY\left(\frac{3YX}{2} - \frac{Y^2}{4} + \frac{3X^2}{4}\right)$$

$d = 6$ and $a = 2$

$$p_{6,2}(X, Y) = \frac{7Y^3X^3}{128} - \frac{Y^2X^4}{256} - \frac{Y^4X^2}{256} - (X - Y)^4$$

$$+ XY\left(\frac{9Y^2X}{8} - \frac{Y^3}{8} + \frac{9YX^2}{8} - \frac{X^3}{8}\right)$$

$d = 6$ and $a = 3$

$$p_{6,3}(X, Y) = \frac{3Y^3X^3}{128} - \frac{25Y^2X^4}{256} - \frac{Y^4X^2}{256} - (X - Y)^4$$

$$+ XY\left(\frac{Y^3}{8} - \frac{5Y^2X}{8} + \frac{15YX^2}{8} + \frac{5X^3}{8}\right)$$

$d = 7$ and $a = 3$

$$p_{7,3}(X, Y) = \frac{475Y^3X^4}{16384} - \frac{25Y^2X^5}{16384} - \frac{75Y^4X^3}{16384} + \frac{9Y^5X^2}{16384} - (X - Y)^5$$

$$+ XY\left(\frac{3Y^4}{64} - \frac{5Y^3X}{16} + \frac{45Y^2X^2}{32} + \frac{15YX^3}{16} - \frac{5X^4}{64}\right)$$

\[4\] they were found with "Maple".
\( d = 7 \) and \( a = 4 \)

\[
p_{7,4}(X, Y) = \frac{459 Y^3 X^4}{16384} - \frac{1225 Y^2 X^5}{16384} - \frac{155 Y^4 X^3}{16384} + \frac{25 Y^5 X^2}{16384} - (X - Y)^5
\]

\[
+ XY \left( \frac{7 Y^3 X}{16} - \frac{5 Y^4}{64} - \frac{35 Y^2 X^2}{32} + \frac{35 Y X^3}{16} + \frac{35 X^4}{64} \right).
\]

**3.13. Remark.** The weighted dual graph of the resolution of a cusp with multiplicity sequence \((m)\) looks like

![Diagram](attachment://diagram.png)

while the dual resolution graph of a cusp \((2a) = A_{2a}\) looks like

![Diagram](attachment://diagram2.png)

Therefore, the dual graph of the total transform \( D = D_{d,a} \) of \( C_{d,a} \) in its minimal embedded resolution \( V \to \mathbb{P}^2 \) looks as follows:

![Diagram](attachment://diagram3.png)
where \( b = d - a - 2 \) and boxes mean the corresponding local resolution trees, as above.

3.14. Remark \[1\]. Here we show that each curve \( C_{d,a} \) can be birationally transformed into a line. More precisely, let \( P_0, P_a, P_b \) be the cusps of \( C = C_{d,a} \) with multiplicity sequences \( (d - 2), (2a), (2b) \). Let \( l_0 = \{ x = 0 \}, l_\infty = \{ y = 0 \} \) be the lines through \( P_0, P_a \), resp \( P_0, P_b \), and \( l_1 = \{ x - y = 0 \} \) be the cuspidal tangent line to \( C \) at \( P_0 \). We will show that there exist three other rational cuspidal curves \( C_1, C_2, C_3 \), which meet \( C \) only at the cusps of \( C \), such that the curve \( T = C \cup l_0 \cup l_1 \cup l_\infty \cup C_1 \cup C_2 \cup C_3 \) can be transformed into a configuration \( T' \) of 7 lines in \( \mathbb{P}^2 \) by means of a birational transformation \( \alpha : \mathbb{P}^2 \to \mathbb{P}^2 \) which is biregular on the complements \( \mathbb{P}^2 \setminus T \) and \( \mathbb{P}^2 \setminus T' \). In fact, \( \alpha \) consists of several birational transformations composed via the following procedure.

1) Blowing up at \( P_0 \), we obtain the Hirzebruch surface \( \pi : \Sigma(1) \to \mathbb{P}^1 \) together with a two–sheeted section \( C' \) (the proper preimage of \( C \)), the exceptional divisor \( E \) (which is a section of \( \pi \)) and with three fibres \( F_0 = l'_0, F_1 = l'_1, F_\infty = l'_\infty \) through three points of \( C' \) which we still denote resp. as \( P_a, P_0, P_b \). Observe that \( C' \) is smooth at \( P_0 \) and by (1.3, a) \( i(C', E; P_0) = d - 2 \).

2) Perform \( a \) resp. \( b \) elementary transformations at \( P_a \in C' \cap F_0 \) resp. \( P_b \in C' \cap F_\infty \), first blowing up at this point and then blowing down the proper preimage of the fibre \( F_0 \) resp. \( F_\infty \). We arrive at another Hirzebruch surface \( \Sigma(N) \) equipped with a smooth two–sheeted section \( C'' \), which is tangent to the fibres \( F_0 \) and \( F_\infty \) and to the section \( E' \), where now \( E'^2 = d - 3 \).

3) Performing further \( d - 2 \) elementary transformations at \( P_0 = E' \cap C'' \cap F_1 \), we return back at \( \Sigma(1) \) with \( E^2 = -1 \), this time the image \( C''' \) of \( C'' \) being a smooth two-sheeted section which does not meet \( E \).

4) Contract \( E \) back to a point \( P_0 \in \mathbb{P}^2 \). Then the image \( \hat{C} \) of \( C''' \) is a conic in \( \mathbb{P}^2 \), and the images of the fibres \( F_0, F_1, F_\infty \) are resp. the lines \( l_0, l_1, l_\infty \) through \( P_0 \not\in \hat{C} \).

This remark is due to a discussion with T. tom Dieck, who constructed examples of cuspidal plane curves starting from certain plane line arrangements, and with E. Artal Bartolo. We are grateful to both of them.
where $l_0, l_\infty$ are tangent to $\hat{C}$ resp. at the points $P_a, P_b \in \hat{C}$, and $l_1$ is a secant line passing, say, through a point $A \in \hat{C}$.

5) Performing the Cremona transformation with centers at the points $A, P_a, P_b \in \hat{C}$, we obtain an arrangement $T'$ of 7 lines in $\mathbb{P}^2$ with 6 triple points. It can be described (in an affine chart) as a triangle together with its three medians and one more line through the middle points of two sides. It is easily seen that such a configuration $T'$ is projectively rigid.

The $\mathbb{Q}$–acyclic surface $\mathbb{P}^2 \setminus C$ can be reconstructed starting from the arrangement $T'$ by reversing the above procedure. In the tom Dieck-Petrie classification [tDP, Theorem D] this line configurations is denoted as $L(4)$.

3.15. Remark. E. Artal Bartolo has computed the fundamental groups $\pi_1(\mathbb{P}^2 \setminus C_{d,a})$.

Let, as always, $a + b = d - 2$, where $a \geq b \geq 1$. Set $2n + 1 = \gcd(2a + 1, 2b + 1)$. Then $\pi_1(\mathbb{P}^2 \setminus C_{d,a}) \cong G_{d,n}$, where $G_{d,n}$ is the group with presentation

$$G_{d,n} = \langle u, v \mid u( vu)^n = (vu)^n v, (vu)^{d-1} = v^{d-2} \rangle.$$  

In particular, $G_{d,n}$ is abelian iff $n = 0$, i.e. $\gcd(2a + 1, 2b + 1) = 1$. Furthermore, among the non–abelian groups $G_{d,n}$ only $G_{4,1}$ and $G_{7,1}$ are finite. Note that, being non–isomorphic, the curves $C_{13,7}$ and $C_{13,10}$ have isomorphic fundamental groups of the complements, which are both infinite non–abelian groups isomorphic to $G_{13,1}$. Evidently, there are infinitely many such pairs.

4 Miscelleneous

Let $C \subset \mathbb{P}^2$ be an irreducible plane curve, $V \to \mathbb{P}^2$ the minimal embedded resolution of singularities of $C$, $\tilde{C} \subset V$ the proper transform of $C$ and $K = K_V$ the canonical divisor of $V$. Let also $D \subset V$ be the reduced total transform of $C$. Recall (see (2.1)) that $C$ being unobstructed simply means that $h^2(\Theta_V(D)) = 0$. In the next lemma we give a sufficient condition for a plane curve to be unobstructed.

4.1. Lemma. Let the notation be as above.

a) If $K\tilde{C} < 0$, then $H^2(\Theta_V(D)) = 0$. 

24
b) Assume that $C$ is a cuspidal curve with cusps $P_1, \ldots, P_s$ having multiplicity sequences

$$m_{P_\sigma} = (m_{\sigma 1}, \ldots, m_{\sigma r_\sigma}, \overbrace{1, \ldots, 1}^{m_{\sigma r_\sigma} + 1}),$$

where $m_{\sigma r_\sigma} \geq 2$. If

$$K \tilde{C} < \sum_{\sigma=1}^s m_{\sigma r_\sigma},$$

then $H^2(\Theta_V\langle D \rangle) = 0$.

Proof. a) Fix $\omega \in H^0(\Omega^1_V\langle D \rangle \otimes \omega_V)$. Then we have $\text{Res}_{\tilde{C}}(\omega) = 0 \in H^0(O_{\tilde{C}}(K \tilde{C}))$, since by assumption the degree of $O_{\tilde{C}}(K \tilde{C})$ is negative. Regarding $\omega$ as a meromorphic section in $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2} \otimes \omega_{\mathbb{P}^2})$ it follows that $\omega$ is holomorphic outside the cusps of $C$. Therefore, $\omega$ extends to a section in $\Omega^1_{\mathbb{P}^2} \otimes \omega_{\mathbb{P}^2}$, and hence $\omega = 0$. Thus, $H^0(\Omega^1_V\langle D \rangle \otimes \omega_V) = 0$. Now the result follows by Serre duality.

For the proof of b) consider a factorization of the embedded resolution as

$$V \to V' \to \mathbb{P}^2$$

such that $V' \to \mathbb{P}^2$ yields the minimal resolution of $C$ in the following sense:

(i) The proper transform, say $C'$, of $C$ in $V'$ is smooth, and

(ii) $C$ can not be resolved by fewer blowing ups.

It is easily seen that

$$K_{V'} C' = K_{V'} \tilde{C} - \sum_{\sigma=1}^s m_{\sigma r_\sigma}$$

(cf. the proof of (4.3, b) below). By the above arguments, if $K_{V'} C' < 0$, then $H^0(\Omega^1_{V'}\langle D' \rangle \otimes \omega_{V'}) = 0$, where $D'$ is the reduced total transform of $C$ in $V'$. Hence also $H^0(\Omega^1_V\langle D \rangle \otimes \omega_V) = 0$. \hfill $\square$

4.2. Corollary. With the notation as in (4.1, b), assume that $C$ is a rational cuspidal curve with $\kappa(\mathbb{P}^2 \setminus C) = 2$. If

$$\sum_{\sigma=1}^s \sum_{j=1}^{r_\sigma} m_{\sigma j} < 3d,$$

then

$$\chi(\Theta_V\langle D \rangle) = K(K + D) = -h^1(\Theta_V\langle D \rangle) \leq 0.$$

25
Proof. From Lemma 4.3,a) below it follows that
\[
\tilde{C}^2 + \sum_{\sigma=1}^{s} m_{\sigma r_\sigma} = 3d - 2 - \sum_{\sigma=1}^{s} \sum_{j=1}^{r_\sigma} m_{\sigma j}.
\]
Therefore, (3) is equivalent to the inequality
\[
\tilde{C}^2 + \sum_{\sigma=1}^{s} m_{\sigma r_\sigma} \geq -1.
\]
Thus, we have
\[
K\tilde{C} = -\tilde{C}^2 - 2 < \sum_{\sigma=1}^{s} m_{\sigma r_\sigma},
\]
and hence by (4.1, b) \( h^2(\Theta_V(\langle D \rangle)) = 0 \). Since \( \bar{k}(\mathbb{P}^2 \setminus C) = 2 \), then also \( h^0(\Theta_V(\langle D \rangle)) = 0 \) (see [Ii, Theorem 6]), and the statement follows. \( \square \)

Note that in our examples, i.e. for \( C = C_{d,a} \) being as in section 3, we have \( K_V C = d - 4 \) (see (4.3, b)) and \( \sum_{\sigma} m_{\sigma r_\sigma} = d + 2 \); furthermore, \( \sum_{\sigma} \sum_{j=1}^{r_\sigma} m_{\sigma j} = 3(d - 2) < 3d \). Thus, (4.1) or (4.2) gives another proof of unobstructedness of \( C_{d,a} \) (cf. (3.3)).

4.3. Lemma. Let \( C \subset \mathbb{P}^2 \) be a rational cuspidal curve, with cusps \( P_1, \ldots, P_s \) having multiplicity sequences \( \bar{m}_{P_\sigma} = (m_{\sigma 1}, \ldots, m_{\sigma k_\sigma}) \). Then
a) in the minimal embedded resolution \( V \to \mathbb{P}^2 \) of singularities of \( C \) the proper transform \( \tilde{C} \) of \( C \) has selfintersection
\[
\tilde{C}^2 = 3d + s - 2 \sum_{i,j} m_{ij} = 3d - 2 - \sum_{\sigma=1}^{s} \sum_{j=1}^{r_\sigma} m_{\sigma j} - \sum_{\sigma=1}^{s} m_{\sigma r_\sigma}.
\]
b) Furthermore, if \( K = K_V \) is the canonical divisor, then
\[
K\tilde{C} = -3d - s + \sum_{i,j} m_{ij}.
\]

Proof. a) Clearly,
\[
\tilde{C}^2 = C^2 - \sum_{i,j} m_{ij}^2 + s = d^2 + s - \sum_{i,j} m_{ij}^2.
\]
The genus formula yields
\[(d - 1)(d - 2) = \sum_{i,j} m_{ij}(m_{ij} - 1).\]

Thus
\[d^2 - \sum_{i,j} m_{ij}^2 = 3d - 2 - \sum_{i,j} m_{ij},\]

and (a) follows.

b) follows from (a) and the equality \(K\tilde{C} + \tilde{C}^2 = -2\). An alternative proof: we proceed by induction on the number of blow ups. First of all, for \(K = K_{\mathbb{P}^2}\) and \(C \subset \mathbb{P}^2\) we have \(KC = -3d\). Furthermore, let \(C \subset V\) be a curve on a surface \(V\) and \(K = K_V\) be the canonical divisor, \(\sigma : V' \to V\) be the blow up at a cusp of \(C\) of multiplicity \(m\) and \(K' = K_{V'}\), \(C' \subset V'\) be the proper preimage of \(C\). We have:

\[KC = K'C^* = (C' + mE)K' = C'K' + mEK' = C'K' + m(E(K' + E) - E^2) = C'K' + m(-2 + 1) = K'C' - m,\]

hence \(K'C' = KC + m\). This completes the proof. \(\square\)

4.4. Remark. Let \(E_P \subset V\) be the reduced exceptional divisor of the blow ups over \(P \in \text{Sing } C\). Then by Lemma 2 in [MaSa]

\[E_P^2 = -\omega_P - 1.\]

If \(D = \tilde{C} + \sum_{P \in \text{Sing } C} E_P \subset V\) is the reduced total transform of \(C\) in \(V\), then we have (cf. [MaSa, Lemma 4])

\[D^2 = \tilde{C}^2 + 2\text{card}(\text{Sing } C) + \sum_{P \in \text{Sing } C} E_P^2 = \tilde{C}^2 - \sum_{P \in \text{Sing } C} (\omega_P - 1)\]

\[= 3d - 2 - \sum_{P \in \text{Sing } C} \left(\sum_{j=0}^{k_i} m_{P,j} + \omega_P - 1\right).\]

4.5. Remark. In [OZ2, Proposition 4] the following observation is done.
A projectively rigid rational cuspidal curve $C \subset \mathbb{P}^2$ cannot have more than 9 cusps.

The reason is quite simple. Denote by $\kappa$ the number of cusps of $C$. Assuming that $\kappa \geq 3$ we will have $\bar{k}(\mathbb{P}^2 \setminus C) = 2$ [Wak], and therefore due to Theorem 6 in [Ii], $h^0 = 0$, where $h^i := h^i(\Theta_V \langle D \rangle)$, $i = 0, 1, 2$. Let $K + D = H + N$ be the Zariski decomposition in the minimal embedded resolution $V \to \mathbb{P}^2$ of singularities of $C$. It can be shown that $N^2 = \sum_{P \in \text{Sing} C} N^2_P$, where the local ingredient $N^2_P$ over a cusp $P \in \text{Sing} C$ has estimate $-N^2_P > 1/2$. Thus,

$$\kappa < 2 \sum_{P \in \text{Sing} C} (-N^2_P) = -2N^2. \quad (4)$$

We also have

$$(K+D)^2 = H^2+N^2 \quad \text{and} \quad (K+D)^2 = K(K+D)+D(K+D) = K(K+D) - 2, \quad (5)$$

where [FZ, (1.3)]

$$K(K+D) = \chi(\Theta_V \langle D \rangle) = h^2 - h^1. \quad (6)$$

From (4)–(6) and the logarithmic Bogomolov-Miyaoka-Yau inequality $H^2 \leq 3$ [KoNaSa] we obtain

$$\kappa < -2N^2 = -2(K+D)^2 + 2H^2 \leq 6 - 2(K+D)^2 = 10 - 2K(K+D) = 10 - 2h^2 + 2h^1.$$

Therefore,

$$\kappa < 10$$

as soon as $h^1 = 0$, i.e. for a projectively rigid curve $C$.

Hence, once one constructs a rational cuspidal plane curve with 10 cusps or more, we know that it is not projectively rigid. The latter means that such a curve is a member of an equisingular family of rational cuspidal plane curves, generically pairwise projectively non–isomorphic (see (2.1)).

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6i.e. with cusps of the same type.
7i.e. non–equivalent under the action of the automorphism group PGL(3, C) on $\mathbb{P}^2$. 

28
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