The Born rule

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We deduce the Born rule. No use is required of quantum postulates. One exploits only rudimentary quantum mathematics—a linear, not Hilbert’s, vector space—and empirical notion of the statistical length of a state. Its statistical nature comes from experimental micro-events being formalized into the abstract quantum clicks.

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The connection between quantum theory and physical experiment begins with the famous ‘square modulus’ formula $|\psi|^2$. This intuitive guess by Max Born [3]—“an intuition without a precise justification” (A. Cabello)—determines the statistical interpretation of quantum wave function and, presently, no violation of the ‘rule of squares’ has ever been discovered. Its purposeful experimental testing, however, came into implementation relatively recently. The pioneering works of U. Sinha et al [22, 23] have demonstrated, in a 3-slit interference laser experiment, the null-effect within an accuracy $10^{-2} \pm 10^{-3}$. The rule is considered as one of the cornerstone of the theory, although many researchers have long pointed out [2, 7, 9, 13, 20, 21, 27–29], and it seems to be a majority opinion, that this Born formula is not a fundamental ‘mantra’ and can be derived, rather than being proved.

Gleason [10] proved an advanced equivalent of Born’s result as a statement about abstract measures in Hilbert’s spaces (see also [5] for a more comprehensive variant of this result), and Everett [9], in the framework of his famous treatment of QM [21], considered specifically the rule. In 1999, D. Deutsch [7] revived Everett–DeWitt’s approach and initiated a new one, which relates the QM-theory with the representation theorems of classical decision theory through the characteristic terminology: strategies of a rational agent, bets, weigh/utility functions (attributed to experimental outcomes), game theory, etc. Deutsch’s ideas were refined by Wallace [25], [21, p. 227–263] and Saunders [20] in the 2000’s; see also Ch. 3 in the book [21] and bibliography therein. W. Zurek, by the “fine/coarse-graining” technique [27–29], developed a different—envariance/decoherence—strategy for deriving the rule. Graham [11] and Hartle [13], in the 1960–70’s, have proposed the frequency-operator method. There are other ways of looking at the problem [21, Ch. 5–6], [1, 26]. These references are by no means complete; say, arXiv yields hundreds items with mentioning the ‘Born rule’ in abstracts.

All approaches—for extended bibliography see [17]—have been subject to mutual criticism [21, Ch. 4, 28, p. 25], [2, 6, 14, 17, 18, 24]. In particular, most if not all of the derivations appeal to unitary $t$-evolution and tensor products, whereas neither of these concepts has been present in Gleason’s theorem. One of the typical objections voiced against the alternative ideas is circular reasoning [6, 8, 11, 21, 29]. This is a criticism made not just by proponents of one approach towards another, but one that is sometimes admitted by the authors of the ideas themselves [21, p. 415]. Most of the known approaches, including the Everettian one, have undergone revisions and refinements [21, Ch. 5], [8, 29]. These points reflect the long-standing problem with quantum foundations—linguistic self-referentiality in their substantiating. Thus the situation seems to be one whereby the numerous attempts to rationalize the ‘square’ preserve the status quo; none of the approaches have been widely accepted to date. The Born rule is continuing to exist as an arduous task, especially considering that the formula should be derived, rather than being proved.

In this work we exhibit a straightforward deducing the mod-squared dependence. In doing so, it is suffice to rely not on the canonical axiomatics (of a Hilbert space) but on a formulation of QM-foundations as a theory of micro-events (clicks) [4]. One uses only the most primitive property of the quantum-state set: to be a linear vector space (LVS). The primary idea of derivation—separation of the number entities—was in effect announced in sects. 9.1–2 of the work [4]. These two sections, including some illustrative (counter)examples therein, may be considered as an extended introduction to the present work and we reproduce the ideology here very briefly.

DOCTRINE OF NUMBERS
IN QUANTUM THEORY, REVISITED

Theory begins with a number and intuitive perception of this object is always accompanied by the notion of a physical unit [4, sects. 7.1–3 and Remark 16]. This is the interpretation of the number in terms of the ‘quantity of something real’: metres, Stücke, sheep, etc. The re-leasing the number from such units—mathematization—turns it into an abstract operator $\hat{n}$ and, then, into an abstract element $n$ of the abstract set $\mathbb{R}$ with arithmetic
The abstracta themselves, the process of abstracting, its naturalness and inevitability are the subject matter of a comprehensive discussion in sects. 9.2-3 of [4].

operations \{+, \times\}. Thereupon there arises a \(\mathbb{C}\)-structure of the complex numbers \(a := n + im\) equipped with the binary operations \(\{\oplus, \odot\}\) and unary involutions

\[
(n + im) \mapsto (n - im), \quad (n + im) \mapsto (m + in). \tag{1}
\]

With specification of the number conceptions missing, recall [4, II-nd principium of QM], the exegesis of 'everything the quantum' acquires the character of a circular argument. The last step is a creation of the other kind number entities: non-abstract, reified quantities per se. It constitutes a mathematical realization of what we have been calling statistics, means, sizes, spectra, and other observable quantities.

Now, the quantum mathematics, in its rudimentary form, is but an abstract\* algebra of a linear space \(H\) over the \(\mathbb{C}\)-number objects and \(\{\alpha\}\)-expansions

\[
a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots \in H \tag{2}
\]

with respect to the eigen-vectors \(|\alpha_i\rangle\) of an instrument \(\mathcal{A}\). Because the notion of 'observable' is initially absent not only in nature but in theory as well,

\* the numerical values of that which is associated with the term 'observable quantity' may arise only as a supplement to the \(\mathbb{C}\)-algebra: the extra rules for manipulating the symbols \(\{a, |\alpha_i\rangle, \ast, +, \ast, -\}\) in the construct (1)-(2).

These rules must constitute the mathematical maps—math add-on's over \(H\) —into the ordered continuum equipped with arithmetic \(\{+, \ast\}\): the \(\mathbb{R}\)-numbers for short. This continuum is ordered due to the language's notion 'greater/smaller'. Such a scheme furnishes the only means of formalizing anything that accompanies the low-level quantum mathematics in the form of notions that we portray in terms of natural language. These are usually referred to as physical quantities.

For example, statistics of \(\underline{a}\)-clicks [4, sect. 2.5–6], i.e., the relative frequencies \((\nu_1, \nu_2, \ldots)\) may come only from \(a\)-coefficients in (2):

\[
(a_1, a_2, \ldots) \mapsto \nu_j \iff \nu_j = \sum_j (a_1, a_2, \ldots). \tag{3}
\]

However, the \(\nu\)-numbers are not the primordial empirical entities. In experiments—colliders, ion traps, interferometers, or any other quantum machine, we are dealing not with quantities that are subject to 'rather specific' constraints \(0 \leq \nu_j \leq 1\) —a theoretical act that does not follow from QM-empiricism—but with gathering the registered micro-events \(\underline{a}\). It has been just these (additive) accumulations, being formalized into \(a_j\)-coefficients, which are to be turned into the \(\mathbb{R}\)-numbers mentioned above, because it is in this way that the number tokens arise in theory at all [4, sect. 7.2]. Therefore, what is taken as a primary mathematical map must be not (3) but what we shall call statistical length of an \(|\alpha\rangle\)-representation:

**StatLength** of (2).

Inasmuch as mathematics of \(\underline{a}\)-clicks implies the endlessness of quantum-click ensembles, the StatLength should be created as a mathematical equivalent to the empirical wording ‘the quantity of micro-events’ having regard to—also empirical—\(\Sigma\)-postulate about infinity of the event number [4, sect. 2.5]. In other words, we rely on the following underlying semantics:

\[
(\text{infinite}) \text{ number of } \underline{a}\text{-clicks} = \text{StatLength} \times \infty. \tag{4}
\]

Notice that the integer-valued domain \(\mathbb{Z}\), as such, does not appear in quantum theory. The discrete infinity \(\aleph_0\), upon applying the \(\Sigma\)-postulate, disappears and yields to continuum \(2^{\aleph_0}\). There arises the sequence of infinities \(\aleph_0 \mapsto 2^{\aleph_0} \mapsto \mathbb{R} \mapsto (\mathbb{R} \times \mathbb{R}) := \mathbb{C}\) [4, sect. 4]. It is the quantum ensembles that give birth to the state-vector \(\psi\) itself. The function StatLength is thus understood further to be the \(\mathbb{R}\)-numeric one.

**STATISTICAL LENGTH**

First and foremost, the StatLength is associated only with (2) because quantum “empiricism” . . . yields not *states* and their superpositions but *|\alpha\rangle*-representations” [4, sect. 8.3]. For example, the writing StatLength\(|\Psi\rangle + |\Phi\rangle\) lacks meaning—or rather, in no way determinable—unless the \(|\psi\rangle\), \(|\Phi\rangle\) are indicators of certain eigen-elements. At the same time, the writing StatLength\(|\Psi\rangle\) is admissible since any element \(|\Psi\rangle = 1 \cdot |\Psi\rangle\in H\) may serve as the eigen one for a certain instrument \(\mathcal{B}\). What are the empirical definienda (linguistic semantics) for the conception StatLength?

Each of \(\underline{a}\)-clicks, in accord with their (\(\infty\))-distinguishability, corresponds to a certain ket \(|\alpha_i\rangle \leftrightarrow \underline{a}\). Consequently, the need for frequencies (3) means that the partial lengths StatLength\(|\alpha_i\rangle\) should come into play. Certainly, these lengths must correlate with the total StatLength of (2). Besides, the numeric values of all the StatLength’s appear to be compatible with each other, for any statistical \(\mathcal{A}\)-representative

\[
a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots = \cdots
\]

is re-developable with respect to other instrument \(\mathcal{B}\):

\[
\cdots = b_1 \cdot |\beta_1\rangle + b_2 \cdot |\beta_2\rangle + \cdots \tag{5}
\]

(device-independence). What is more, even the very formal \(|\Psi\rangle\)-object cannot be constructed without matching...
the two instruments \( \mathcal{A} \), \( \mathcal{B} \) [4, sect. 5.4]. Let us take a closer look at the situation, in order to ascertain properties of the function \( \text{StatLength} \).

If the two events \( \alpha_1 \) and \( \alpha_2 \) are distinguishable by the \( \mathcal{A} \)-instrument \( \alpha_1 \neq \alpha_2 \) then the statistical length of a \((+)-sum\) of two statistical \( |\alpha\rangle \)-representatives is invariant to involutions (1):

\[
\text{StatLength}(a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle) = \text{StatLength}(a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle) = \cdots ,
\]

by the very nature of ‘the number of clicks’ and of ‘the mutual exclusivity of \( \alpha \)-s’, is split into the numeric sum of the partial lengths:

\[
\cdots = \text{StatLength}(a_1 \cdot |\alpha_1\rangle) + \text{StatLength}(a_2 \cdot |\alpha_2\rangle). \tag{6}
\]

This property determines translation (homomorphism) of the ‘abstract’ \((+)-operation\) on \( \mathbb{R} \)-vectors into the ‘concrete arithmetical plus \(+\)’ between the \( \mathbb{R} \)-numbers. Of course, this is a peculiarity of \( |\alpha\rangle \)-bases, not of the arbitrary \( \ell^2 \)-spaces’ ones.

Meantime, there is yet another operation with the \( \mathbb{H} \)-space’s vectors—the unary multiplication \( |\alpha\rangle \mapsto \epsilon \cdot |\alpha\rangle \)—and it should also be carried over to the arithmetic of the \( \text{StatLength} \)-numbers:

\[
\text{StatLength}(|\alpha\rangle) \mapsto \text{StatLength}(\epsilon \cdot |\alpha\rangle) = ? \tag{11}
\]

Clearly, \( \text{StatLength}(\epsilon \cdot |\alpha\rangle) \) is a certain function of the \( \text{StatLength} \) of \( |\alpha\rangle \). Therefore, simplifying notation \( \text{StatLength} \mapsto \mathcal{N} \), we have to find a \( \mathcal{C} \)-function:

\[
\mathcal{N}(\epsilon \cdot |\alpha\rangle) = \mathcal{C}(\mathcal{N}(|\alpha\rangle)). \tag{7}
\]

On the other hand, \( |\alpha\rangle \)-objects are elements of \( \ell^2 \)-spaces. This means that the \( \mathcal{N} \)-function must respect its axioms.

In particular, the distributivity

\[
\epsilon \cdot (|\alpha\rangle + |\beta\rangle) = \epsilon \cdot |\alpha\rangle + \epsilon \cdot |\beta\rangle \tag{8}
\]

events when \( |\alpha\rangle \) and \( |\beta\rangle \) are distinguishable \( (\alpha \neq \beta) \), the additivity (6) entails a translation \((+ \mapsto +)\) on the right:

\[
\mathcal{N}(\epsilon \cdot (|\alpha\rangle + |\beta\rangle)) = \mathcal{N}(\epsilon \cdot |\alpha\rangle) + \mathcal{N}(\epsilon \cdot |\beta\rangle) .
\]

All the \( \mathcal{N} \)-functions here are the ones of \( \epsilon \cdot (\cdots) \). Hence,

\[
\mathcal{C}(\mathcal{N}(|\alpha\rangle + |\beta\rangle)) = \mathcal{C}(\mathcal{N}(|\alpha\rangle)) + \mathcal{C}(\mathcal{N}(|\beta\rangle))
\]

and, applying additivity (6), now on the left, we obtain

\[
\mathcal{C}(\mathcal{N}(|\alpha\rangle) + \mathcal{N}(|\beta\rangle)) = \mathcal{C}(\mathcal{N}(|\alpha\rangle)) + \mathcal{C}(\mathcal{N}(|\beta\rangle)).
\]

The \((+)-abstractum\) disappears and we arrive at the standard functional equation for the linear (real-valued, continuous) numeric function [16, pp. 128–129]:

\[
\mathcal{C}(x + y) = \mathcal{C}(x) + \mathcal{C}(y) \Rightarrow \mathcal{C}(x) = \text{const} \cdot x. \tag{9}
\]

Thus, the abstract (\(\cdot\))-sign in (7) has been converted into the numeric \(\times\). Summing up, we introduce the function \( \mathcal{N} \) by a definition, which will suffice to derive the rule.

**Definition (axioms of \( \text{StatLength} \)).** The \( \mathbb{R} \)-valued function \( \mathcal{N} \) (homomorphically) formalizes the statistical-length conception by the rules of carrying the abstracta \((+, \cdot)\) over to the arithmetic \((+, \times)\):

\[
(+) \mapsto (+) : \quad \mathcal{N}(a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots) = \mathcal{N}(a_1 \cdot |\alpha_1\rangle) + \mathcal{N}(a_2 \cdot |\alpha_2\rangle) + \cdots, \tag{10}
\]

\[
(\cdot) \mapsto (\times) : \quad \mathcal{N}(\epsilon \cdot (a \cdot |\alpha\rangle)) = \text{const}(\epsilon) \cdot \mathcal{N}(a \cdot |\alpha\rangle). \tag{11}
\]

The total \( \text{StatLength} \) is device-independent (meaningfulness of the \( \mathcal{N} \)-number):

\[
a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots = b_1 \cdot |\beta_1\rangle + b_2 \cdot |\beta_2\rangle + \cdots \tag{12}
\]

\[
\downarrow \quad \mathcal{N}(a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots) = \mathcal{N}(b_1 \cdot |\beta_1\rangle + b_2 \cdot |\beta_2\rangle + \cdots), \tag{13}
\]

and the function \( \mathcal{N} \) is invariant to involutions (1):

\[
\mathcal{N}(a^* \cdot |\alpha\rangle) = \mathcal{N}(a \cdot |\alpha\rangle) = \mathcal{N}(\bar{a} \cdot |\alpha\rangle). \tag{14}
\]

Property (11) is actually not an axiom because the sequencing between formulas (7) and (9) is a derivation of (11); nor is (14) an axiom [4]. Beyond that, the (\(\times\))-multiplicativity (11) may be postulated even purely semanti-
cally. Indeed, an operator characterization of the number \[4, \text{sect. 7.2}\]—no matter, real/complex—entails the replication of quantum ensembles. The replication means that ‘observable quantity’ \(\text{StatLength}(\Psi)\), upon action of the ‘\(\epsilon\)-operator’ on \(\Psi\), is multiplied by a certain \(\mathbb{R}\)-\text{const}(\epsilon)\). Speaking more loosely, we are dealing with a kind of homomorphism

\[(\sim)\text{-replication} = \{ \text{to be multiplied by } \cdots \}.
\]

But it is just this mechanism—a group with operator automorphisms—that is realized in the axiomatic structure which has been calling ‘the LVS’; in particular, axiom (8). See a selected thesis following Remark 16 in the work [4]. As far as the axiom (10) is concerned, symbols \(\dagger\) and \(\ast\) are inherited from the ‘ensemble-accumulation theory’ by means of the union operation \(\cup\) [4, sect. 5].

A shorter way to put all the said above is that the language usage of the notion \((\bullet)\) is not something that is conceived of or characterized by various words, but precisely/emdash.cyr and this we stress with emphasis/emdash.cyr serialization of quantum ensembles. The replication means that the ‘\(\epsilon\)-operator’ on \(\Psi\), is merely multiplied by a certain \(\mathbb{R}\)-\text{const}(\epsilon)\). Here, as always in the sequel, we have adopted a bar notation for the complex conjugation \(a^\ast = \bar{a}\) and the standard convention for the addition/multiplication symbols \(\{\circ, \circ\}\) and \(\{+,-\}\) between both the \(\mathbb{C}\)- and \(\mathbb{R}\)-numbers.

Let us consider the \(N\)-representation of the scalability property (11):

\[N_r(a, \overline{a}) = \text{const}(c) \times N_r(a, \overline{a}).\]  

This identity, upon substitution (16), reads as follows

\[\sum_{\ell, p} \gamma_{\ell p} (ca + c\overline{a})^\ell (c\overline{a}c\overline{a})^p = \text{const}(c) \times \sum_{\ell, p} \gamma_{\ell p} (a + \overline{a})^\ell (a\overline{a}a\overline{a})^p.\]

Since \(c\) is arbitrary in axiom (11), put \(c = r \in \mathbb{R}\) for a moment. One obtains

\[\sum_{\ell, p} \gamma_{\ell p} \left\{ r^{2p+\ell} - \text{const}(r) \right\} (a + \overline{a})^\ell (a\overline{a}a\overline{a})^p = 0 \quad \forall r, a, \overline{a}\]

and, hence, nontrivial solutions for \(\text{const}(r)\) is possible only if \(2p + \ell\) is a fixed (external) integer; denote it \(K\).

Therefore, the sum (16) becomes the one of finitely many terms and all of them are homogeneous in \(a, \overline{a}\):

\[N_r(a, \overline{a}) = \sum_{\ell, p} \gamma_{\ell p} (a + \overline{a})^\ell (a\overline{a}a\overline{a})^p \bigg|_{2p+\ell = K} = \sum_{\ell, p} \gamma_{2p+\ell, \ell} (a + \overline{a})^{K-2p} (a\overline{a}a\overline{a})^p.\]  

When \(K = 1, 3, 5, \ldots\) we have only the odd \((K - 2p)\)-powers \((a + \overline{a})^1, (a + \overline{a})^3, \ldots\) in the p-sum (18). In such a case, \(N_r(a, \overline{a}) \sim (a + \overline{a})\) and, hence, \(N(a) = 0\) at \(a = i\mathbb{R} \neq 0\). That \(K\)-case must be discarded because \(N(a) = 0\) only if \(a = 0\) by the very statistical nature of the \(\epsilon\)-coordinates. More formally, suppose the contrary, i.e., that there exists some ‘specific’ \(a'\) \neq 0 such that \(N(a') = 0\). From (11) and (15) there follows

\[\forall c: N_r(c a') = \text{const}(c) \times N_r(a') = \text{const}(c) \times 0 = 0 \implies N_r(a') = 0 \iff N(c') = 0 \quad \forall c';\]

the trivial solution.

Thus, only the even \(K = 0, 2, 4, \ldots\) and even powers \((K - 2p) \in \{0, K, K - 2, \ldots\}\) of \((a + \overline{a})\) are allowed in (18):

\[N_r(a, \overline{a}) = \gamma_0 (a + \overline{a})^0 (a\overline{a}a\overline{a})^p + \gamma_2 (a + \overline{a})^2 (a\overline{a}a\overline{a})^{p-1} + \cdots\]
we renormalized \( \gamma \)'s). Then the scaling \( a \mapsto ca \) amounts to the change \( (\varrho, \kappa) \mapsto (c\varrho, \kappa + t) \) in the latter expression. One gets, instead of (17),

\[
\epsilon = c \varrho \epsilon
\]

where all the variables \( (\varrho, \kappa; r, t) \) are understood to be independent and equal in rights. It is immediately seen that there is only one possibility here:

\[
\text{const}(r, t) = \text{const}'(r) \equiv r^2 \varrho, \quad \gamma_0 = \text{free},
\]

and \( \gamma_2 = \gamma_4 = \cdots = 0; \) put, for example, \( \kappa = 0 \). As a result, only one term survives in sum (16):

\[
N(a) = \gamma_0 \times (a\overline{a})^p
\]

with yet free \( p = 1, 2, 3, \ldots \). It is not difficult to verify that the \((-\cdots)^p\) invariance (1) is satisfied automatically, and derivation in terms of symmetrical polynomials \( \{a + \overline{a}, a\overline{a}\} \) would yield (an exercise with \( a = i\overline{a} \) the same answer:

\[
N(a, \overline{a}_1) \equiv a_2 \cdot (a_2^2 + \cdots ) = \text{const} \times \left( |a_1|^{2p} + |a_2|^{2p} + \cdots \right), \quad (19)
\]

Getting ahead of ourselves, we could claim \( p = 1 \) right here because none of the values \( p = 2, 3, \ldots \) may be preferable to any other (the ‘world constant’ \( p = 2？\)) while \( p = 1 \) is minimal in this series. And yet, we address the device-independence (12) because it implies a changing of instruments \( \mathcal{A} \equiv \mathcal{B} \) and so the change of eigen-state bases: \( \{\alpha_1, \alpha_2, \ldots \} \equiv \{\beta_1, \beta_2, \ldots \} \).

When the family of \( \mathcal{A}' \)-distinguishable clicks coincides with the family of the \( \mathcal{B} \)-distinguishable ones \( \{\alpha_1, \alpha_2, \ldots \} \equiv \{\beta_1, \beta_2, \ldots \} \), we have actually one and the same instrument: \( \mathcal{A} = \mathcal{B} \). In the \( \{\alpha\} \)-language, this means

\[
\{\alpha_1, \alpha_2, \ldots \} \equiv \{\beta_1, \beta_2, \ldots \}.
\]

and the scale transformations \( \{\alpha\} \mapsto c \cdot \{\alpha\} \) may be disregarded here since the eigen-states themselves are defined up to multiplicative constants. We then have to declare transformations like \( |\alpha_1\rangle \mapsto |\beta_1\rangle \sim |\alpha_2\rangle \) as \textit{trivial} permutations.

It is clear that the arbitrary permutation is formed from transpositions like \( \{\alpha_1, \alpha_2, \alpha_3\} \mapsto \{\alpha_2, \alpha_1, \alpha_3\} \). Therefore it will suffice to consider the 2-dimensional changes and to exclude the trivial diagonal (identical) and anti-diagonal ones:

\[
\left(\begin{array}{c} |\beta_1\rangle \\ |\beta_2\rangle \end{array}\right) = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \left(\begin{array}{c} |\alpha_1\rangle \\ |\alpha_2\rangle \end{array}\right) = \left(\begin{array}{c} \langle 0 1 | \\ 1 0 \langle \end{array}\right) \left(\begin{array}{c} |\alpha_1\rangle \\ |\alpha_2\rangle \end{array}\right).
\]

The nontrivial basis-changes, say, the simplest ones

\[
\{\alpha_1, |\alpha_2\rangle; |\alpha_3\rangle, \ldots \}_\alpha \equiv \{|\beta_1\rangle, |\beta_2\rangle; |\alpha_3\rangle, \ldots \}_\beta,
\]

(20)

correspond to observations by ‘non-commuting devices’ \( \mathcal{A} \neq \mathcal{B} \) and the latter do, without fail, exist in quantum theory [4, III-nd principal of QM]. We now have to pass to the ‘erasing’ the \( |\kappa\rangle\rangle\)-symbols from (13) because (15) and the formal applying (19) to (13) ignore the down-arrow (12) and thereby any relationships (20) between \( |\alpha\rangle \)'s and \( |\beta\rangle \)'s:

\[
(a_1 \overline{a}_1)^p + (a_2 \overline{a}_2)^p = (b_1 \overline{b}_1)^p + (b_2 \overline{b}_2)^p. \quad (21)
\]

Inasmuch as we have dealt with an L.V.S.-basis change (20), the coordinate representative \( (a_1, a_2, \ldots) \) of (one and the same) \( |\kappa\rangle\rangle \)-vector (5) undergoes an associated \textit{linear} transformation \( U \). Consequently, there must exist the numeric changes

\[
\left(\begin{array}{c} a_1 \\ a_2 \end{array}\right) \overset{U}{\mapsto} \left(\begin{array}{c} b_1 \\ b_2 \end{array}\right) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\begin{array}{c} a_1 \\ a_2 \end{array}\right) \quad (22)
\]

and their (anti)diagonal subclass

\[
U = \left(\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right) \quad \text{or} \quad U = \left(\begin{array}{cc} 0 & b \\ a & 0 \end{array}\right)
\]

should also be thought of as the trivial changes. Apart from the obvious det \( U \neq 0 \), this yields the nontriviality condition for (22):

\[
ab \neq 0 \neq cd. \quad (23)
\]

That said, equality (21) should be supplemented with (22) and obeyed under all \( a \)'s. Simplifying notation \( (a_1, a_2, \ldots) \mapsto (x, y) \), we require

\[
(x \overline{y})^p + (y \overline{x})^p = (ax + by)^p + (cx + dy)^p (\overline{ax + by})^p
\]

for all \( (x, y, \overline{x}, \overline{y}) \), which are understood to be independent variables. By expanding, some binomial expansions arise (\( p \geq 2 \)):

\[
x^p \overline{y}^p + y^p \overline{x}^p = (a^p \overline{a}^p + c^p \overline{c}^p) \cdot x^p \overline{x}^p + \cdots
\]

\[
\left\{ (ax)^p - (by)^p \right\} \cdot \left(\overline{x^p ay^p} \right) + \cdots
\]

\[
+ (cx)^p - (dy)^p \cdot (\overline{cx}^p - \overline{dy}^p) \left(\overline{x\overline{y}^p} \right) + \cdots
\]

where only one cross \((x \overline{y} \cdot \overline{y})\)-term has been displayed. Collecting in \( x \overline{y} \) and \( y \overline{y} \), one gets (among other terms)
\[ \cdots + p^2 \cdot \left\{ |a|^2 |b|^2 + |c|^2 d^2 \right\} \times (x^2 y^2)^{-1} (xy) + \cdots . \]

Clearly, such expressions have always been present in the sum and the wavy-embellished term must be zero. Hence,

\[ |a|^2 - |b|^2 + |c|^2 d^2 = 0 \quad \Rightarrow \quad \{ab = 0 = cd\} . \]

This contradicts (23). Only trivial permutations are allowed under \( p > 2 \). Thus, \( p = 1 \) and (19) is refined:

\[ \text{StatLength}(a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots ) = \text{const} \times \left( |a_1|^2 + |a_2|^2 + \cdots \right) , \quad (24) \]

where \( \text{const} \) must be a common, while free, constant for all the \( |\alpha\rangle \)-representations. The \( U \)-matrices (22) preserve the sum of squares (24), and that’s the point where the concept of unitarity comes into quantum theory.

We now return to the task (3). The semantics (4) suggests the only way of harmonizing the ‘theoretical infinity \( \infty \)’ (\( \Sigma \)-postulate) with finite quantities coming from experiment; their \( \mathbb{R}^n \)-numerical images, to be precise. Namely, we introduce by definition the concept (it was not so far) of the micro-events’ relative frequencies:

\[ \nu_k := \frac{\text{StatLength}_k \times \infty}{\sum_k (\text{StatLength}_k \times \infty)} . \]

Finally, the completed formulation of Born’s result has not been exhausted by the squares’ formula.

- **The 2-nd theorem of quantum empiricism**

1) Basis-independence: the ‘sum of squares’ (24) is the only rule that is compatible with the StatLength-additivity and the ‘device non-commutativity’ \( \{ |\alpha_1\rangle \} \neq \{ |\beta_1\rangle \} \).

2) Unitary equivalence of bases: the changing of observational instruments \( \mathcal{A} \rightleftharpoons \mathcal{B} \) is represented in \( \mathbb{H} \) by unitary transformation \( \{ |\alpha_1\rangle \} \rightleftharpoons \{ |\beta_1\rangle \} \) between their eigen-states.

3) The \( \mathcal{A} \)-events’ statistics for representation (2) is approximated according to the Born rule

\[ \nu_k := \frac{|a_k|^2}{|a_1|^2 + |a_2|^2 + \cdots } . \quad (25) \]

4) No use is required of the Hilbertian/tensor/orthogonality/projector/operator/spectra/…/unitarity structures when deducing the rule.

The rudimentary physics at the moment is just the click collections. Therefore the rule (25) does not require—it should also be emphasized—any physical terminology: interactions, dynamics, evolution, measuring processes, observables, apparatus, etc. Nor does the derivation address such categories as space/time/causality (e.g., in EPR-controversy), (non)relativity, gravity*, and (non)inertial reference frames; to say nothing of the moot and debatable \([18, 21, 25]\) conceptions such as quantum collapse, ‘the world(s)/mind(s)’, the MWI-bifurcations of the universe \([9, 21]\), (classical/objective) reality, or subjective/anthropic \([8, \text{p. 155–165}]\) notions like rational belief/preferences \([21]\). In essence, we have made do only with the two obvious premises: (10) and (13). These are obligatory requirements, which is why the word StatLength may be formally even cast away from the theorem. The quadratic dependence above is, roughly speaking, a mathematical statement concerning the correctly defined—invariance (13)—function on \( \mathbb{H} \) with \( |\alpha\rangle \)-additivity (10). An additive property, in one form or another, is present almost in all works on derivation of the rule \([5, 7, 8, 10, 12, 13, 20, 21, 26]\).

**DISCUSSION**

**How would derivation look in the orthodoxy?**

Let us forget the theorem about linearity of quantum superposition, i.e., about ‘accumulation of clicks into coefficients \( a_j \)’ \([4]\). What points should be introduced into the minimal QM-axiomatics in order to derive (25)?

First of all, we should accept the statistical treatment of the \( a_j \)-coordinates. It is widely known as early as the 1926 works by Born himself \([3]\). The words “Statistik/statistischen” appear at the very end of the first brief communication \([3]\) (and do not appear in the second of the works \([3]\)); though in the context of the particle-collision processes, not of the abstract micro-events.

The (relative) frequency view of the state-rays in a Hilbert space—the multiplicatively statistical reading of the equivalence \( |\Psi\rangle \approx c \cdot |\Psi| \) suggests to give up the structures like ‘up to a constant’ or ‘inaccessible phases’ and to deal with the non-normalized \( (+) \)-sums** (2), i.e., without constraint \( ||\Psi|| = 1 \); cf. \([8, \text{p. 185}]\). Therefore, a certain notion of the additive ‘quantifying/sizing’ must be introduced \([26, \text{p. 1296}]\). Such an additivity manifests in the well-known orthogonality and distinguishability of eigen-states. See, e.g., \([10, 28]\) and also a concept of the orthogonal additivity in \([12, \text{sect. 5.2}]\). The \( \mathcal{A} \)-\( |\alpha\rangle \)-distinguishability is thus of fundamental importance when deducing both the LVA and StatLength structures.

* In particular, the binding the rule to unitarity or \( t \)-dynamics would entail a grave problem of reconciliation with the well-known issues in quantum gravity \([15]\): the problem with the very Hilbert (and Fock) space, with the dynamical background and the concepts of particles and their number, of time, etc.

** Whether this idea has been expressed in the literature, the author is not aware. I would be grateful for an information in this regard.
Further, the difference between an abstract state (or an abstract sum $a \cdot |\Psi\rangle + b \cdot |\Phi\rangle$) and its (empirically C-numerical) basis-dependent $|\alpha\rangle$-representation (in a reference frame for the ‘observer $\alpha'$) does of course not go away and remains the conceptual point [4, sect. 8.3]. No $\alpha'$-instrument is exclusive because any $|\alpha\rangle$-preference—e.g., privileged observables or pointer states [21] in some views of the ‘measurement problem’—runs counter to the basic principle of the representation invariance of physical theories and of QM-mathematics (12) in particular; “democracy of bases” (J. Barrett).

We should also declare what the complex $(\ast)$-conjugation does in QM-mathematics; except for a scalar product axiom. The declaration is this: a $(\ast)$-invariance of the $\nu$-statistics. Subsequent actions, including the device non-commutativity $A \neq B$, do not then require any postulations and have been described in the previous section. The sequence $(7) \rightarrow (8) \rightarrow (9)$ and point 4) in the theorem remain in force.

Informally, to disclose ‘Born’s square’ by manipulating the $\{|\alpha\rangle\}$-symbols like $|\alpha\rangle\langle\alpha|\psi\rangle$ is a rather non-efficient way, to say the least. The Born rule is a statement not about (in/out) $|\phi\rangle$-kets and projectors $\hat{P}$’s but about numbers. That is, about C-numerical representatives $a_1, a_2, \ldots$ irrespective of their calculation method $a = \langle\alpha|\psi\rangle$, because $a_\alpha$-coefficients of the LVS-vectors are not ‘aware’ of the inner-product structure, of orthogonality, etc [19].

A. Gleason (with his famous representation theorem [10]) and H. Everett [9] were perhaps the first to attempt at vindicating the rule in the framework of the orthodox axiomatics. Everettian approach came under criticism of many authors [2, 18, 21] and later N. Graham [11] and J. Hartle [13] reconsidered Everett’s conclusions through the frequency operator as an observable; see, however, [24].

Remark 2. When deducing the rule, Everett [9] freely changes the function arguments, puts $\mathcal{M}(a_\alpha) = \mathcal{M}(\sqrt{a_\alpha^* a_\alpha})$ and does “impose the additivity requirement”, then restricts the “choice of $\mathcal{M}$ to the square amplitude” and puts “$\mathcal{M}(a_\alpha) = a_\alpha^* a_\alpha$,” does “replace the $a_\alpha$ by their amplitudes $\mu_i = |a_i|^2$,” defines “a new function $g(\mu) = \mathcal{M}(\sqrt{\mu})$, etc, etc. [9, p. 71]. Finally, on p. 72, he draws a conclusion that “the only choice … is the square amplitude measure”. Put roughly, by use of the fact that square of a coefficient is a sum of other squares (Hilbert), one infers a rule of squares. Clearly, in no way is this any proof [11, p. 236], [8, pp. 163, 185], however, its ‘refinements and justifications’ have got even into textbooks [30, sect. 8.4.1, “Everett’s theorem”].

It is also not clear, what would be changed in reasoning on pp. 71–72 of [9], if the two and $\sqrt{\cdot}$ would be substituted for $\rho$ and $\sqrt{\cdot}$. Expressed another way, why and which the $L^p$-norms are relevant to the quantum state-space?

Math-rigors: topology, continuity, and the like

The latter question was fully considered by S. Aaronson in the work [1] wherein the exclusive-ness of an $L^2$-norm was justified. His analysis, besides other important questions, is extended even to non-integer $p$‘s, and realization of device-independence $A \equiv B$ by the $U$-matrices above fits completely Aaronson’s idea of the (power dependence) norm’s preservation under linear transformation. In this quantum context, the Pythagorean theorem Aaronson mentions [1, pp. 2, 4] should be thought of as just the only possible way of introducing the very first numeric ‘observable/beable’ in QM-theory: the function (24).

On the other hand, the state-space $\mathbb{H}$ is a ‘bare’ LVS at the moment. It is neither a normed or a topological space, because construction of (continuous) maps from $\mathbb{H}$—no matter where—does not yet arise as a task. Inasmuch as the states themselves are not observable entities (whatever that means [4, sect. 10]) and are not yet comparable with each other, the low level quantum ‘$\mathbb{H}$-mathematics’ does not care questions like ‘whether we need a construction $\rightarrow$ with axioms of a norm?—the triangle inequality, etc’. The more so as there is a (topological) equivalence relation on norms of the finite-dimensional LVS [19]; e.g., the $L^2$-norm is equivalent to the $L^p$-norm.

The QM-empiricism in turn does not yet give grounds to introduce any functions on $\mathbb{H}$, other than StatLength (24). We thus draw a conclusion that if such a function is exclusive, is that how it will induce the topology on the abstract $\mathbb{H}$-vectors through the numeric $N$-function (24) of their $|\alpha\rangle$-representatives. This does precede the Hilbert space and Born statistics, and not the other way round. The quantum state-space can thus be turned into an $L^2$-normed vector space whose topology conforms to the $C$- and $\mathbb{R}$-field topology of numbers $\alpha_j$.

This R-topology has already been used when deriving the C-function (9)*. On the other part, ansatz (16) should be understood not as the (infinite) series in $a$, $a^*$ but just as a finite (purely algebraic) symmetrical sum. Otherwise, if this were the ‘infinity’-case, we would deal with a non-motivated non-algebraic extension of the ‘pure’ $\mathbb{H}$-algebra and thereby with some extra-topological requirements that do not follow from empiricism. However, the restriction on such an ‘implied infinity’ is not a loss of generality because, in any case, homogeneity (17)–(18) extracts the only term from (16).

Yet a further aspect of function $N$ concerns the very statement of the problem. Every LVS has infinitely many bases. However, as the space $\mathbb{H}$ was arising alongside the bases of observables $\mathcal{A}$—eigen-vectors $|\alpha_i\rangle$, let us ask ourselves the question: What is the way in which the basis of an observable stands out from the other abstract bases, which are as good as any one? Quantum empiricism tells us that all one has to do this is to invoke some

* All the other solutions to this equation are “pretty ‘weird’” (J. Aczel–J. Dhombres). They are globally/locally irregular [16, pp. 129–130] and their graphs are everywhere dense in $\mathbb{R}^2$.
statistical considerations. These will boil down to a certain numeric function on \(\mathbb{H}\), which reflects the natural notion of the accumulating—additivity—the distinguishable (= mutual exclusivity) micro-events. The existence (or non) of such a function—a new math add-on over \(\mathbb{H}\)—will determine these ‘good bases’. Therefore the mathematics surrounding the quantum statistics—motivation and the Definition itself—can be restated as the question of special bases of LVS and has the quite minimalistic formalization without reference to physics:

- Given an abstract (and ‘bare’ as above) LVS over \(\tilde{C}^*\), define the \(\mathcal{A}\)-base(s)—due to QM-non-commutativity, it should be not a single one—by the following requirement. Basis \(\{\alpha_j\}\) is referred to as basis of an observable \(\mathcal{A}\) if there exists a well-defined nonzero function \(N\) on \(\mathbb{H}\), which satisfies the properties of \(\{\alpha\}\)-additivity (10) and of involutory invariance (14).

Is this definition consistent? What is the function? How is it derived/calculated? Whether it exists and is unique? What are relationships between different \(\mathcal{A}\)-bases? Where does unitarity come from? The answers to these questions are in derivation of the theorem. All the other bases remain the abstract ones in LVS. Parenthetically, the same method provides a tool of deriving the ‘topological \(\mathcal{N}\)-function’ for other linear manifolds: different numeric fields, different involutions, etc.

Summing up, the questions of topology on the \(\mathbb{H}\)-space (and on numbers) are, strictly speaking, to be solved simultaneously with the construction of function \(\mathcal{N}\), which, in turn, comes from quantum empiricism as the Stat-Length.

Of course, the reasoning given above is not quite rigorous arguments and is merely a mathematical ideology. However we believe that the entire quantum foundations, and not just their algebraic LVS-constituent, admit a considerable strengthening the mathematical rigor—a proposal for the mathematics experts—even to the extent of pedantic justification of all the topologies, of what is ordered and what isn’t, the \(R\)-domain for StatLength or just \(\mathbb{R}^+\), the (general quantum) case \(\dim \mathbb{H} = \infty\), propositional logic, and the like. In the first place, this fully applies to the work [4]. The more so as the mathematical grounds to the semantic notions of continuity, connectivity, and the physical (numeric) lexicon of approximations, infinitesimal \(\varepsilon\)’s, convergence, etc have long been formalized in topology [19].

A word on the physical (3+1)-space-time. This topic bears on the full \((x, t)\)-representation—the continuous \((x, t)\)-parameters of automorphism—of the invariant quantum mathematics of \(\{|\psi\rangle\}\) and of \(\{\alpha\}\)-bases (2), because the abstract states themselves are not to be related to the notions of space-time, causality, etc. Here, there is no way to bypass the matters of principle. Among them:

- a state-space separability, precise definition of the \(\mathbb{H}\)-representatives to observables in QM/QFT and of Hilbert’s space itself—why/where the binary (?) inner (?) product (?) comes from (realization of the Hilbert space in quantum gravity is not a ‘t-constant’, as with the elementary QM). We should also ascertain the nature of observables in quantum gravity [15] and comprehension of coordinates on manifolds (the equivalence principle), locality, degrees of freedom, dimension \(D = 3 + 1\) (?), and other data. In particular, we encounter non-rhetorical question about bringing formula (25) into correlation with non-discrete constructions like

\[
\left|\psi(x)\right|^2 dx, \quad \left|\psi(x, t)\right|^2 dx (?).
\]

These matters call for special consideration, and will be treated at length elsewhere. The absence of the word ‘probability’ in the present work is not an accident [4, sect. 11.2]. As we have seen, the micro-events supplemented with the LVS-structure—superposition principle—do not require such a concept.

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