Information Design in Crowdfunding under Thresholding Policies

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Abstract

In crowdfunding, an entrepreneur often has to decide how to disclose the campaign status in order to collect as many contributions as possible. We propose information design as a tool to help the entrepreneur to improve revenue by influencing backers' beliefs. We introduce a heuristic algorithm to dynamically compute information-disclosure policies for the entrepreneur, followed by an empirical evaluation to demonstrate its competitiveness over the widely-adopted immediate-disclosure policy. Our work sheds light on information design in a dynamic setting where agents follow thresholding policies.

Introduction

Crowdfunding reinvents the way that entrepreneurs raise external funding for implementing creative ideas or projects. In crowdfunding, the contributions made by early backers largely determine whether a campaign (see Figure 1 for a view of the procedure) will get funded or not (Mollick 2014; Solomon, Ma, and Wash 2015; Hu, Li, and Shi 2015). However, backers usually have high uncertainty about the fate of the campaign and are likely to wait (Mollick 2014; Alaei, Malekian, and Mostagir 2016). Their reluctance usually undermines later backers' confidence in the success of the campaign and is the major factor that leads the project to failure (Mollick 2014; Solomon, Ma, and Wash 2015). To get the projects funded, entrepreneurs must take measures to increase early backers' likelihood to contribute.

Figure 1: A typical crowdfunding campaign.

To coordinate backers’ actions, an entrepreneur needs to have prior knowledge about the backers’ arrival process, valuation of the reward, and the estimate process of the probability that the campaign will get funded (i.e., Probability of Success, or PoS). However, none of this information is perfectly known to the entrepreneur. Thus, it is challenging for the entrepreneur to figure out what actions will make backers, especially early backers be more willing to contribute. If conditions permit, the entrepreneur can manipulate backers’ payoffs by offering appealing discounts to early backers that face high uncertainty (Ellman and Hurkens 2015; Strausz 2016). The problem of devising allocation and payment schemes falls into the field of mechanism design (Nisan and Ronen 1999). While illuminating, it requires additional budgets and thus diminishes the entrepreneur’s revenue (Ellman and Hurkens 2015; Strausz 2017). Absent from sophisticated or even unrealistic assumptions of the backers’ private types (e.g., valuation, arrival time, departure time), it is rather difficult or even unfeasible for the entrepreneur to implement effective mechanisms (Ellman and Hurkens 2015; Strausz 2017). This is particularly the case in online settings where the entrepreneur has little knowledge about how the backers make their projections of the campaign’s PoS.

Alternatively, the entrepreneur can improve backers’ beliefs of the campaign’s PoS by choosing the information that they observe. In particular, the entrepreneur can and is permitted to voluntarily disclose the project status (i.e., how many contributions have been collected up to a given timestamp), a critical factor that influences backers’ beliefs of the campaign’s PoS (Alaei, Malekian, and Mostagir 2016). The problem of determining which pieces of information are disclosed to whom is called information design (Taneva 2015).

In this paper, we formally define the information design problem in which the entrepreneur voluntarily reveals the project status to backers to influence their beliefs of the project’s probability of success. We introduce the concept of vertical information to characterize the order of two project status reports. We prove that excessive information disclosure weakly shrinks the entrepreneur’s revenue. We show that immediate disclosure is optimal if the funding goal has been achieved and if the project status increases monotonically by at least one contribution each time. We further in-
introduce a heuristic algorithm to identify the project status revealed to the backers each time. Our results demonstrate the promise of dynamic information design when backers follow thresholding policies.

Background

This section discusses the rationale behind our choice of the decision model for the backers, followed by an overview of information design.

Consumer Behavior Studies on consumer purchasing behavior show that buyers usually follow thresholding policies to decide whether to purchase goods or not (Kau and Hill 1972; Kahneman 2003; Zhou, Fan, and Cho 2005). They will buy the products when the prices are no more than their reserved values. This is particularly the case when consumers are facing high degrees of uncertainty and have little knowledge about the environment or the future, as frequently observed in clinical decision making (Pauker and Kassirer 1980), airline ticket sales (Zhou, Fan, and Cho 2005), online shopping (Lee and Lin 2005), management science (Su 2007) and crowdfunding (Mollick 2014; Alaei, Malekian, and Mostagir 2016). Under certain circumstances, thresholding policies are optimal policies and hence represent rational behavior (Ohannessian et al. 2014).

In this paper, we consider a dynamic setting in which backers follow thresholding policies when they decide whether to contribute to a project or not. Future studies on information design should address the cases when agents use other decision models.

Information Design The information design problem in our work falls into a broad category of scenarios called Signaling Games (Cho and Kreps 1987; Banks and Sobel 1987), or Disclosure Games (Newman and Sansing 1993; Brocas and Carrillo 2007) in particular. The simplest form of the game considers the strategic interaction between an information sender and a receiver only (Rayo and Segal 2010; Kamenica and Gentzkow 2011).

Our work is closely related to the dynamic information disclosure problem (Au 2015) and the information design problem (Janeva 2015) studied in economics. It, nevertheless, differs from these prior works in three major aspects. First, we consider a dynamic environment where a sender (i.e., the entrepreneur) interacts with multiple receivers (i.e., the backers) who arrive sequentially. Second, the information that the sender provides has both vertical and horizontal information components (Aoki 1986; Au 2015). Third, the information receivers use thresholding policies to make decisions due to high uncertainty about the environment. We focus on this dynamic setting because it is challenging, and has a broad range of real-world applications, including online shopping (Lee and Lin 2005), intelligent transportation systems (Shen, Lopes, and Crandall 2016), job markets (Bergemann and Morris 2016), smart grids (Ipakchi and Albuyeh 2009), and politics (Banks 1991) as well.

Decision Problems in Crowdfunding

In this section, we formally define the decision problems for both the backers and the entrepreneur.

Preliminaries We consider discrete time \( t \in T = \{1, 2, 3, ..., T\} \), where \( T \) is the deadline for the campaign. Before launching the campaign, the entrepreneur must determine a fundraising goal \( G \) to get funded, a deadline \( T \) for reaching the goal, the number of rewards \( N \), the minimal amount of contributions for a reward \( P \), and a detailed description of the project such as motivation, product, milestones, and profiles of the team. All this information is fixed and is disclosed to all the backers.

After the campaign starts to accept contributions, only high-value backers (i.e., \( v_i \geq P \), where \( v_i \) denotes backer \( i \)'s valuation of a reward) arrive at the campaign since low-value backers never contribute even if they are sure that the campaign will succeed. For simplicity, we assume that high-value backers arrive at the campaign sequentially with at most one each time. This is without loss of generality because batch arrivals can be viewed as a special case where the time interval is minimal (Shen, Lopes, and Crandall 2016; Alaei, Malekian, and Mostagir 2016). Let \( b(t) \in \{0, 1\} \) denote the number of arrivals at time \( t \in T \).

At the beginning of time \( t \), the entrepreneur discloses the state of the campaign (i.e., project status) \( s(k) \) to each backer \( i \) that is in the campaign. Here, \( s(k) = (s(k), k) \) where \( s(k) \) refers to the percentage of funds that have been raised up to time \( k \leq t \) (\( k \) not included), with respect to the fundraising goal \( G \). We denote the entrepreneur’s decision on information disclosure for backer \( i \) at time \( t \) by:

\[
d(i, t) = (s(k), t) \quad \text{s.t. } k \in T, k \leq t, |s(1)| = 0. \quad (1)
\]

The disclosed project status \( s(k) \) must reflect the true state of the project at time \( k \), which is enforced by the crowdfunding platform. In real-world crowdfunding campaigns, entrepreneurs are allowed to voluntarily disclose truthful project status (Solomon, Ma, and Wash 2015; Alaei, Malekian, and Mostagir 2016). In our work, we assume that any information about the project status observed by the backers is directly revealed by the entrepreneur. Future work should address the scenarios when backers have exogenous information due to information contagion (Arthur and Lane 1993).

The Backers’ Problem Let \( r_i(t, d(i, t)) \in [0, 1] \) represent high-value backer \( i \)'s estimate of the campaign’s PoS given the report \( d(i, t) \), and \( \phi_i \in (0, 1) \) be his threshold on \( r_i(\cdot) \) to contribute. We denote backer \( i \)'s decision on whether to contribute or not at time \( t \) by \( \alpha_i(t) \in \{0, 1\} \), where \( 0 \) indicates Not Pledging, and \( 1 \) represents Pledging. Backer \( i \)'s
expected utility $u_i$ is determined as follows:

$$u_i(t, \alpha_i(t), d(i, t)) = \left\{ \begin{array}{ll} c_i \cdot \alpha_i(t) & \text{if } r_i(t, d(i, t)) \geq \phi_i; \\ 0 & \text{otherwise}. \end{array} \right.$$  

(2)

Here, $c_i > 0$ is backer $i$‘s expected utility if he contributes (i.e., $\alpha_i(t) = 1$) when his estimate of PoS is no less than the threshold $\phi_i$. Note that $c_i$, $r_i(t)$ and $\phi_i$ are all private information known to backer $i$ only, while his arrival and voting behavior are observed by the entrepreneur through the platform. Without loss of generality, we assume that each contributing backer pledges the same amount ($P$) of fund to the project for a reward.

In practice, a backer might lose (i.e., $u_i < 0$) if he had contributed to a campaign whose crowdfunding goal was not achieved in the end. There was an opportunity cost for him even if he receives a refund upon the campaign’s failure. He would have used the money to invest other activities that could produce profits for him if he were sure that the project would not be likely to succeed. Moreover, the money might be subject to discounting. We do not explicitly identify these complex factors in our model. Instead, we capture them by assuming that a backer only contributes if his utility reaches a cut-off point privately known to himself only. This assumption allows us to preserve essential ingredients of backers’ decision-making, while avoiding dealing with the infinite hierarchies of beliefs (ELY and Peski 2006).

Backer $i$ stays at the campaign for at most $l_i \in \{1, 2, 3, ..., L\}$ periods, where $l_i$ is known to backer $i$ only. This makes use of generality because although backers might dynamically enter and exit the system and check the progress, these situations can be viewed as a case that the backers stay in the system for a sufficient period.

Let $I(t)$ denote the group of backers who have arrived at the campaign before or at time $t$, have at least one time period to leave and have not yet claimed a contribution. At time $t$, for each backer $i \in I(t)$, his objective function is $B_i(t) = \max_{\alpha_i(t)} u_i(t, \alpha_i(t), d(i, t)) \cdot G, T, N, P, l_i$, where $u_i$ is determined by Equation (2). At time $t$, backer $i$ will leave the campaign either if he claims a contribution (i.e., $\alpha_i(t) = 1$) or his own deadline $l_i$ is reached.

### The Entrepreneur’s Problem

Let $M(t)$ denote the funds that the entrepreneur has raised up to time $t$ (t included) when she uses the disclosure policy $DP(t)$, we have $M(t) = \sum_{t'=1}^{t} \sum_{i \in I(t')} \alpha_i(t') \cdot P$, s.t. $G, T, N, P, DP(t)$ is determined by Equation (2). At time $t$, backer $i$ will leave the campaign either if he claims a contribution (i.e., $\alpha_i(t) = 1$) or his own deadline $l_i$ is reached.

**Definition 1** (Optimal Information Design). An optimal information design in crowdfunding is to find a disclosure policy $DP_{opt}(T)$, such that $M(T)$ is maximized, i.e., $DP_{opt} = \text{argmax} M(T)$.

Due to the dynamic nature of crowdfunding, the design of disclosure policy $DP(t)$ cannot be based on backers’ later decisions $(\alpha_j(t)_{j \in I(t)})_{t > t}$, or use later project status $(s(t))_{t > t}$. This constraint is called No Clairvoyance.

### Optimal Information Design

After introducing vertical and horizontal information, we show that excessive information weakly shrinks revenue. We further identify conditions under which immediate disclosure is optimal in crowdfunding.

### Vertical and Horizontal Information

Given two pieces of information, if one is always (weakly) preferred whatever the information receivers’ types are, then they are vertical information (see Definition 2). If the two pieces of information are not comparable without prior knowledge about the receivers’ types, then the information is horizontal (definition omitted since it complements vertical information).

**Definition 2** (Vertical Information). Given two pieces of information $\xi_1$ and $\xi_2$, where $\xi_1 \not\equiv \xi_2$, if $\forall i \in I, u_i(\xi_1) \geq u_i(\xi_2)$, then $\xi_1 \succ \xi_2$, where $I$ denotes a set of agents and $\succ$, indicating preferred or indifferent to, is independent of agent $i$’s private type. If $\xi_1 \equiv \xi_2$, the information is vertical.

In crowdfunding, backer $i$‘s estimate of the campaign’s PoS (i.e., $r_i(t, d(i, t))$) is both time and state-dependent. Both the amount of funding (measured by $|s(k)|$) raised, and the time of the project status (denoted by $k$) are important. We identify three scenarios of vertical information. First, a higher state of project status is always more favorable if the time of the state is the same (e.g., $20\% > 5\%$), which is obvious. Second, the earlier report of project status is always (weakly) preferred if the project status of the two reports are the same (see Proposition 1). Third, the later report of project status is always (weakly) preferred if the revenue increases by more than $P$ each time between the timestamps of the two statuses. (see Proposition 2).

**Proposition 1.** Given project status $s(k_1)$ and $s(k_2)$, where $k_1 < k_2$, $|s(k_1)| = |s(k_2)|$, we have $\forall t \geq k_2, \forall i \in I(t): (s(k_1), t) \succ (s(k_2), t)$.

**Proof.** By condition, $|s(k_1)| = |s(k_2)|$. That means the revenue does not increase between time $k_1$ to time $k_2$. Since $k_1 < k_2$, we have $T - k_2 < T - k_1$, which means less time is left to achieve the fundraising goal $G$. Thus, the campaign’s PoS decreases or stays the same, regardless of backers’ arrivals from time $k_1$ to $k_2$. That is, $r(t, s(k_1), t) \geq r(t, s(k_2), t)$. By Equation (2), the utility of all the subsequently arriving backers weakly decreases, i.e., $\forall t \geq k_2, \forall i \in I(t): u_i(s(k_1), t) \geq u_i(s(k_2), t)$. By Definition 2 we have $(s(k_1), t) \succ (s(k_2), t)$ for all $t \geq k_2$, and for all $i \in I(t)$.

**Proposition 2.** Given status reports $\varepsilon_1 : (s(k_1), t_1)$ and $\varepsilon_2 : (s(k_2), t_2)$, where $k_1 < k_2, t_1 < t_2$ and $|s(k_2)| = |s(k_1)| \geq (k_2 - k_1) \cdot P/G$, we have $\forall t \geq k_2, \forall i \in I(t): \varepsilon_2 \succ \varepsilon_1$.

**Proof.** By assumption $\forall t \in T, b(t) \in \{0, 1\}$, we have that the number of arrivals from time $k_1$ to $k_2$ is: $\sum_{k_1}^{k_2} b(t') \leq k_2 - k_1$. Since $|s(k_2)| = |s(k_1)| \geq (k_2 - k_1) \cdot P/G$,
we have that at least \((k_2 - k_1)\) backers have contributed from time \(k_1\) to \(k_2\). Without loss of generality, one can assume that each time from \(k_1\) to \(k_2\), at least one backer arrives at the campaign and makes a contribution. In other words, the revenue of the campaign grows faster than or equal to the arrival of backers between \(k_1\) and \(k_2\). Thus, the campaign’s PoS increases or stays the same. That is, \(r(t, (s(k_2), t)) \geq r(t, (s(k_1), t))\). By Equation 2, the utility of all the subsequently arriving backers weakly increases, i.e., \(\forall t \geq k_2, \forall \epsilon \in \mathcal{I}(t) : u_i(s(k_2), t_2) \geq u_i((s(k_1), t_1))\).

By Definition 2, we have that at least \(|s(15)| = 30\%\) of all the subsequently arriving backers weakly increases, \(|s(15)| = 40\%\), without prior knowledge about backer \(i\)’s projection of PoS (i.e., \(r_i(\cdot)\)), it is unclear which project status is more favorable by \(i\). This is because \(|s(15)| = 0.1 < 0.1 \cdot (15 - 10)\).

**Excessive Information Disclosure Shrinks Revenue**

Given two project status reports, if their partial order can be identified according to Proposition 1 and 2, then the entrepreneur only needs to disclose the one with higher order. This is because revealing the low-order report does not increase the chance that backers contribute to the campaign (see Lemma 1). If the partial order of the two reports cannot be identified, the entrepreneur should also refrain from disclosing additional information. Depending on how backers estimate the campaign’s PoS, revealing more information than necessary can decrease the revenue. The reason is that excessive information disclosure can decrease backers’ projections of the PoS (See Lemma 2). In our model of crowdfunding, excessive information disclosure weakly diminishes a backer’s willingness to contribute (see Theorem 1) and thus weakly shrinks the revenue as well as the entrepreneur’s ability to implement optimal information-disclosure policies. To collect as many contributions as possible, the entrepreneur should not disclose more information about the project status than necessary.

**Lemma 1.** Given project status reports \(\epsilon_1 : (s(k_1), t_1)\) and \(\epsilon_2 : (s(k_2), t_2)\), if \(\forall t \geq \max\{k_1, k_2\}, \mathcal{I}(t) : \epsilon_2 \succ \epsilon_1\), we have \(E(\alpha_i = 1|\epsilon_2) \geq E(\alpha_i = 1|\epsilon_1))\), where \(E(\alpha_i = 1|\cdot)\) denotes the expectation that backer \(i\) contributes to the campaign given the information \(\epsilon\).

**Proof.** By condition, we have \(\epsilon_2 \succ \epsilon_1\). By relation of preferences and utility (Chambers and Echenique 2016), \(\epsilon_2 \succ \epsilon_1 \Leftrightarrow u_i(\epsilon_2, t) \geq u_i(\epsilon_1, t)\). According to backers’ utility function (Equation 2), \(r_i(\epsilon_2, t) \geq r_i(\epsilon_1, t)\). Depending on the order of the backer \(i\)’s threshold \(\phi_i\), his belief \(r_i(\epsilon_1, t)\) when given report \(\epsilon_1\) and the belief \(r_i(\epsilon_2, t)\) given report \(\epsilon_2\), there are the following three cases:

- \(r_i(\epsilon_1, t) \leq r_i(\epsilon_2, t)\) and \(\phi_i \leq r_i(\epsilon_2, t)\): given report \(\epsilon_2\), backer \(i\) will contribute to the campaign and leave the system. An additional signal about the project status cannot improve his possibility of pledging, i.e., \(E(\alpha_i = 1|\epsilon_2) \geq E(\alpha_i = 1|\epsilon_1, \epsilon_2)\).
- \(\phi_i < r_i(\epsilon_1, t) \leq r_i(\epsilon_2, t)\): under this condition, backer \(i\) will contribute to the campaign and leave the system given either of the two reports. That is, \(E(\alpha_i = 1|\epsilon_2) = E(\alpha_i = 1|\epsilon_1, \epsilon_2) = 1\).
- \(r_i(\epsilon_1, t) > r_i(\epsilon_2, t)\): in this case, backer \(i\) will not contribute to the campaign given either \(\epsilon_2\) or \(\epsilon_1, \epsilon_2\).

Thus, \(E(\alpha_i = 1|\epsilon_2) = 1\).

**Lemma 2.** Given project status reports \(\epsilon_1 : (s(k_1), t_1)\) and \(\epsilon_2 : (s(k_2), t_2)\) where \(k_1 \neq k_2\), if the partial order of the two cannot be identified by the entrepreneur, then \(\forall t \geq \max\{k_1, k_2\}, \mathcal{I}(t) : r_i(\epsilon_1, t) \leq r_i(\epsilon_2, t)\).

**Proof.** We prove it by contradiction. Assume, to the contrary, that \(\exists t \geq \max\{k_1, k_2\}, j \in \mathcal{I}(t) : r_j(\epsilon_1, t) > r_j(\epsilon_2, t)\). Depending on the order of backer \(j\)’s threshold \(\phi_j\), \(r_j(\epsilon_1, t)\), \(r_j(\epsilon_2, t)\), and \(\max\{r_j(\epsilon_1, t), r_j(\epsilon_2, t)\}\), we have the following three cases:

- \(r_j(\epsilon_1, t) > \max\{r_j(\epsilon_1, t), r_j(\epsilon_2, t)\}\): if \(r_j(\epsilon_1) > r_j(\epsilon_2)\), we have \(r_j(\epsilon_1, t) > r_j(\epsilon_1, t)\). By utility function (Equation 2), \(u_j(\epsilon_1, t) = u_j(\epsilon_1) = r_j(\epsilon_1, t)\). By the relation of preferences and utility (Chambers and Echenique 2016), \((\epsilon_1, t) \sim (\epsilon_1, t)\) for \(\epsilon_1\), where \(\sim\) denotes indifferent to. This contradicts that \(r_j(\epsilon_1, t) > r_j(\epsilon_1, t)\). A similar contradiction occurs if \(r_j(\epsilon_2, t) > r_j(\epsilon_1, t)\).
- \(\phi_j > r_j(\epsilon_1, t)\) or \(r_j(\epsilon_1, t) > \max\{r_j(\epsilon_1, t), r_j(\epsilon_2, t)\}\): in this case, by Equation 2, \(u_j(\epsilon_1, t) = u_j(\epsilon_2) = 0\). By the relation of preferences and utility (Chambers and Echenique 2016), \((\epsilon_1, t) \sim (\epsilon_2, t)\) for \(\epsilon_1\), where \(\sim\) denotes indifferent to. This contradicts that \(r_j(\epsilon_1, t) > r_j(\epsilon_2, t)\).
- \(r_j(\epsilon_1, t) > \max\{r_j(\epsilon_1, t), r_j(\epsilon_2, t)\}\): by Equation 2, we have \(u_j(\epsilon_1, t) = c_j \cdot \alpha_i(t)\) and \(u_j(\epsilon_2) = u_j(\epsilon_2) = 0\). By the relation of preferences and utility (Chambers and Echenique 2016), \((\epsilon_1, t) \succ (\epsilon_2, t)\) for \(\epsilon_1\), where \(\succ\) denotes strictly preferred to. If \(\epsilon_1 \sim \epsilon_2\), then \((\epsilon_1, t) \sim (\epsilon_2, t)\). This contradicts that \(\epsilon_1 \sim \epsilon_2\). Thus, \(r_i(\epsilon_1, t) \leq \max\{r_i(\epsilon_1, t), r_i(\epsilon_2, t)\}\).

**Theorem 1.** Given two project status reports \(\epsilon_1 : (s(k_1), t_1)\) and \(\epsilon_2 : (s(k_2), t_2)\) where \(k_1 \neq k_2\), let \(\alpha_i\) denote backer \(i\)'s decision on pledging if given report \(\epsilon_1\) or \(\epsilon_2\), and \(\alpha_i'\) denote his decision on contribution if given \(\epsilon_1, \epsilon_2\). \(\forall t \geq \max\{k_1, k_2\}, \mathcal{I}(t) : E(\alpha_i = 1) \geq E(\alpha_i') = 1\).

**Proof.** If the information is vertical, by Lemma 1, we have \(E(\alpha_i = 1) \geq E(\alpha_i') = 1\). If the information is horizontal,
by Lemma 2, \( r_i(t, (\varepsilon_1, \varepsilon_2)) \leq \max\{r_i(t, (\varepsilon_1)), r_i(t, (\varepsilon_2))\} \).

By Equation 2, there are three cases:

- \( \phi_i \leq r_i(t, (\varepsilon_1, \varepsilon_2)) \leq \max\{r_i(t, (\varepsilon_1)), r_i(t, (\varepsilon_2))\} \): in this case, backer \( i \) will contribute for both conditions, \( E(\alpha_i' = 1) = E(\alpha_i'' = 1) = 1 \).
- \( r_i(t, (\varepsilon_1, \varepsilon_2)) \leq \max\{r_i(t, (\varepsilon_1)), r_i(t, (\varepsilon_2))\} \): in this case, backer \( i \) will not contribute for both conditions, \( E(\alpha_i' = 1) = E(\alpha_i'' = 1) = 0 \).
- \( r_i(t, (\varepsilon_1, \varepsilon_2)) \leq \phi_i \): in this case, backer \( i \) will not contribute if given \( (\varepsilon_1, \varepsilon_2) \), i.e., \( E(\alpha_i' = 1) = 0 \). If given either \( \varepsilon_1 \) or \( \varepsilon_2 \), backer \( i \) will either contribute or not contribute, i.e., \( E(\alpha_i' = 1) \geq 0 \).

Therefore, \( E(\alpha_i' = 1) \geq E(\alpha_i'' = 1) \).

**Immediate Disclosure is not Always Optimal**

The immediate-disclosure policy (see Definition 5) is widely adopted by entrepreneurs on major crowdfunding platforms (e.g., Kickstarter, Indiegogo) due to its ease of implementation (Alaei, Malekian, and Mostagir 2016). It is thus important to investigate if immediate disclosure is optimal.

**Definition 3 (Immediate Disclosure).** An immediate disclosure (denoted by \( DP_{im} \)) is: \( DP_{im}(t) = (((d(j, t'))_{j \in T(t')})_{t' \leq t}) \text{ s.t. } d(j, t') = (s(t'), t') \).

If the entrepreneur and the backers have identical information, immediate disclosure is optimal (Rayo and Segal 2010; Kamenica and Gentzkow 2011; Au 2015). It still holds if the entrepreneur has some unique information provided that such information does not affect the backers’ decisions. Unfortunately, in our model, all information about the campaign (e.g., \( G, T, P, N \)) except the project status is known to both the entrepreneur and the backers. Backer \( i \)’s estimate of PoS (i.e., \( r_i(t, d(i, t)) \)) is influenced by the project status \( s(k) \) that the entrepreneur reveals. Thus, it is critical for the entrepreneur to identify conditions when immediate disclosure is optimal.

From Proposition 2, we see that to improve backer \( i \)’s belief of the campaign’s PoS, the entrepreneur should always disclose the project status that is preferred by all the backers if available. Otherwise, the information design is not optimal. With this intuition in mind, we show that before the campaign reaches the fundraising goal, immediate disclosure is optimal if and only if the project status increases monotonically in time by at least one contribution each time (see Lemma 3). This condition characterizes the relationship between the growth rates of the revenue and the maximum possible arrival rate of the backers. Though the entrepreneur does not have prior knowledge of the backers’ types (e.g., beliefs, thresholds), she can observe the progress of the project and determine if immediate disclosure is optimal given the tracking record of project status.

After successfully reaching the funding objective, it is certain that the campaign will get funded, so immediate disclosure is optimal (see Lemma 4).

**Lemma 3.** If \( M(t) < G \), then we have: \( DP_{im}(t) = \arg \max_{DP(t)} M(t) \iff \forall t' \leq t: |s(t')| - |s(t' - 1)| \geq P/G \).

**Proof.** \( \iff \) If the right side holds for all \( t' \leq t, i \in T(t) \), then \( \forall t' \leq t: |s(t')| - |s(t' - 1)| \geq P/G \), then \( |s(t')| \geq P/G \) for all \( t' \leq t \), according to Proposition 4. \( \forall i \in T(t): \exists s(t'), t \succ (s(t' - 1), t) \), where \( t \geq t' \). That is, immediate disclosure is always preferred by all the backers in the campaign.

\( \implies \) We prove it by contradiction. Assume, to the contrary, that \( |s(t')| - |s(t' - 1)| < P/G \) such that \( DP_{im}(t) \) is optimal for some \( t, t' \). Without loss of generality, we let \( |s(t')| = |s(t' - 1)| + |\Delta s|, |\Delta s| < P/G \). Since \( |s(t')| = |s(t' - 1)| + \sum_{i \in T(t')} \alpha_i(t') P/G, \) where \( \alpha_i \in \{0, 1\} \), we have \( |\Delta s| = \sum_{i \in T(t')} \alpha_i(t') \cdot P/G < P/G \). Therefore, \( |\Delta s| = 0 \), which indicates that \( |s(t')| = |s(t' - 1)| \). According to Proposition 1, \( t' \geq t: (s(t' - 1), t) \succ (s(t'), t) \), which contradicts the supposition that \( DP_{im} \) is optimal.

**Lemma 4.** If \( M(t) \geq G \), then we have: \( DP_{im} = \arg \max_{DP(t)} M(t) \).

**Proof.** We prove it by contradiction. Assume, to the contrary, that \( DP_{im} \) is not optimal for some \( d(i, t') = (s(k'), t') \). This indicates that at time \( t' \geq t \), the entrepreneur could possibly profit by delaying information disclosure. Without loss of generality, we assume that \( d(i, t')_{opt} = (s(k'), t') \) is the disclosure strategy used in the optimal disclosure policy for backer \( i \in T(t') \), where \( k' < t' \). Depending on the relation between \( t' \) and \( k' \), there are two cases (since \( t' \geq t \)): (i) \( t < k' < t' \); (ii) \( k' < t < t' \).

- \( t < k' < t' \): at time \( t \), the project has already succeeded, which means there is no uncertainty for backer \( i \) since the campaign’s PoS is 1. Therefore, delaying disclosure does not increase backer \( i \)’s estimate on PoS, which contradicts the proposition that \( d(i, t') = (s(k'), t') \) is not optimal.
- \( k' < t < t' \): at time \( t \), backer \( i \)’s estimate \( r_i(t', d(i, t')_{opt}) \leq 1 \), while \( r_i(t', d(i, t')) = 1 \). That is, \( d(i, t') \succ d(i, t')_{opt} \forall i \in T(t') \), which contradicts the proposition that \( DP_{im} \) is not optimal.

**Dynamic Information Design**

Instead of restricting our attention to optimal information design, we introduce a heuristic algorithm, called Dynamic Shrinkage with Heuristic Selection (DHS), to help the entrepreneur make decisions on information disclosure.

**The DHS Algorithm**

DHS treats the two conditions separately: before and after project success (see Algorithm 1). Before the campaign reaches the funding goal, the algorithm determines...
the disclosure policy according to two processes: dynamic shrinkage and heuristic selection. In the dynamic-shrinkage process, the algorithm ranks all the available choices \(H_i(t)\), and removes the least promising choices which are less preferred by the backers according to Propositions 1 and 2. By doing so, the entrepreneur avoids excessive information disclosure that weakly shrinks revenue. In the heuristic-selection process, the remaining candidates are horizontal. After the campaign reaches the fundraising goal (if it happens), the algorithm discloses information immediately. DSHS is highly flexible in the sense that it allows the entrepreneur to easily customize both the shrinkage process and the selection process with different methods.

### Algorithm 1 DSHS

| Input: \(t, s(t), I(t)\) | Output: \(\{d(i, t)\}_{i \in I(t)}\) |
|------------------------|----------------------------------|
| 1: if \(t \leq T\) then |
| 2: for \(i \in I(t)\) do |
| 3: if \(M(t) < G\) then \(\triangleright\) before success |
| 4: \(H_i(t) \leftarrow \{s(k)\}_{k \in \{k_0, k_0+1, \ldots, t\}}\) s.t. |
| 5: \(d(i, t') = (s(k_0), t')\), where \(t' \in \{1, 2, \ldots, t\}\) |
| 6: \(H_i(t) \leftarrow \text{SORT}(H_i(t))\) |
| 7: \(s(k_{\text{sel}}) \leftarrow \text{SELECT}(H_i(t))\) |
| 8: \(d(i, t) \leftarrow (s(k_{\text{sel}}), t)\) |
| 9: else \(\triangleright\) after success |
| 10: \(d(i, t) \leftarrow (s(t), t)\) |
| 11: end if |
| 12: \(M(t + 1) = M(t) + \alpha_i(t) \cdot P\) \(\triangleright\) record |
| 13: \(s(t + 1) = M(t + 1)/G\) |
| 14: end for |
| 15: end if |

### Dynamic Shrinkage

To avoid excessive information disclosure, this process removes the least promising candidates — the project status disclosures that are least preferred by the backers based on Propositions 1 and 2.

Initially, the DSHS algorithm includes all the project status disclosures \(s(k)\) since the last disclosure for backer \(i\) into a set \(H_i(t)\) (see line 4, Algorithm 1). It then sorts \(H_i(t)\) in ascending order of \(|s(k)|\) by calling the function SORT (see line 5). This sorting problem can be easily solved by Quicksort (Hoare 1962) with time complexity \(O(|H_i(t)| \log |H_i(t)|)\). Since \(|H_i(t)| \leq T\), the worst-case complexity for the function is \(O(T \log T)\).

The DSHS algorithm then removes the least promising candidates through the function SHRINK (see Algorithm 2). While there are at least two choices available, the SHRINK algorithm removes the project status with later time if the two status has the same progress (see line 4, Algorithm 1) according to Proposition 1. This process is equivalent to removing duplicates in a sorted array, which can be solved in \(O(T)\) time. Given two project status, if they satisfy the relation in Proposition 2, then the algorithm removes the project status with the earlier time (see line 7, Algorithm 1). This step takes \(O(T \log T)\) time in the worst case. The algorithm does nothing if only one disclosure strategy exists.

### Algorithm 2 SHRINK

| Input: \(H\) |
| Output: \(H'\) |
| 1: if \(|H| \geq 2\) then |
| 2: while \(s(k_1), s(k_2) \in H, k_1 < k_2\) do |
| 3: if \(|s(k_1)| = |s(k_2)|\) then |
| 4: \(H \leftarrow H \setminus \{s(k_2)\}\) |
| 5: end if |
| 6: if \(|s(k_2)| - |s(k_1)| \geq (k_2 - k_1) \cdot P/G\) then |
| 7: \(H \leftarrow H \setminus \{s(k_1)\}\) |
| 8: end if |
| 9: end while |
| 10: end if |
| 11: \(H' \leftarrow H\). |

### Heuristic Selection

After the shrinkage process, if there are still at least two choices available (i.e., \(|H_i(t)| \geq 2\), then the remaining set \(H_i(t)\) is horizontal. The entrepreneur has to select some \(s(k_{\text{sel}}) \in H_i(t)\) to attract as many contributions as possible. This optimization problem is similar with the renowned restless bandit problem (Whittle 1988), which is not solvable due to incomplete information. However, simple heuristics such as random selection, greedy selection, \(\epsilon\)-greedy exploration, and softmax exploration can be used to produce acceptable results. See the appendix for details of these heuristics.

We further introduce a meta algorithm (See Algorithm 3). The intuition is that the algorithm can improve the quality of decisions by only using the experts that have a satisficing performance for producing the final results (Crandall 2014). Besides, we take an ensemble approach to calculate the final selection instead of directly applying the results produced by the selected experts. The benefit is that the algorithm can further reduce potential performance loss due to biases of a single individual expert (Kuncheva and Whitaker 2003).

Before describing the meta algorithm, we first introduce the notations used. Let \(X\) denote the set of experts, where \(x \in X\) is one of the four heuristics. We write \(z^*(x)\) for expert \(x\)’s expected revenue at time \(t\) and write \(w^*(x)\) for expert \(x\)’s revenue prospect. Here, \(z^*(x)\) is computed by: \(z^*(x) = \sum_{s(k) \in H'} \sum_{i \in I(t)} \rho^x_k(s(k)) \cdot \gamma_x(s(k), i, t)\), where \(\rho^x_k(s(k))\) denotes the probability that \(s(k)\) is selected as the targeted project status by expert \(x\) at time \(t\), and \(\gamma_x(s(k), i, t)\) is the entrepreneur’s expected increase of revenue given \((s(k), t)\) for backer \(i \in I(t)\) by using expert algorithm \(x\). Details of computing \(\gamma_x(s(k), i, t)\) for each expert \(x\) is described in the appendix. Initially, \(w^* = \max_{x \in X} \{z^*(x)\}\).

Each time the algorithm selects a subset \(X'\) of experts whose expected revenue is higher than a learned threshold—the minimum learned prospect \(w^*(x)\). This step eliminates the experts that fail to produce better expected revenue than the threshold. The algorithm then performs a majority vote from the results generated by each expert \(x \in X'\). The selected project status \(s(k_{\text{sel}})\) is the one with the most votes. Ties are broken by choosing the result generated by the expert with the highest \(z^*(x)\). This step aims to improve the robustness of selection by reducing the performance loss.
caused by biases of a single expert.

When a new expert algorithm is selected, the prospect for the expert is updated by
\[ q^t(x) = (1 - \sigma) q^t(x) + \sigma w^t(x). \]

Here, \( \delta \) is the number of periods that expert \( x \) has been used, and \( \sigma \in [0, 1] \) is the learning rate (\( \sigma = 0.9 \) in our paper).

\( q^t(x) \) is the entrepreneur’s average revenue gain per time by using expert algorithm \( x \) in the last \( \delta \) periods. It is calculated by:
\[ q^t(x) = \sum_{t'=t-\delta}^{t} \sum_{j \in X(t')} \frac{\alpha_j(t')}{\delta}. \]

Algorithm 3 META

```
Input: H', X, T
Output: s(ksel)
1: Compute z'(x) for x ∈ X
2: Initialize w'(x) = \max_{x \in X} z'(x)
3: while t < T do
4:   X' = \{x : z'(x) ≥ min w'(x)\}
5:   Perform a majority vote for s(ksel) as the s(k) with the majority rule
6:   Select s(ksel) as the s(k) with the majority rule
7:   Update w'(x) = (1 - \sigma) q'(x) + \sigma w'(x) if a new expert x is selected
8:   Update q'(x) and z'(x) for each x ∈ X
9: end while
```

Empirical Evaluation

This section describes the experimental settings and the results.

Experimental Setup We collected the campaign data from a randomly selected subset of Kickstarter projects using a web crawler. The dataset contains 1569 projects from 07/15/2016 to 10/15/2016. Each campaign lasted for exactly 1440 hours and includes hourly project status, the fundraising goal, the deadline, the minimal amount of contribution, and the number of contributions every hour. The data samples allow us to mimic the operation of real-world crowdfunding projects when the underlying factors and correlations that impact them are yet to be identified (Mollick 2014, Alaei, Malekian, and Mostagir 2016).

We simulated backers’ arrivals using Poisson (Ross and others 1996) distribution for each project. The arrival process was independently and identically distributed with a mean \( \bar{\nu}(t) \) across time, where \( t \in T \). The mean of \( \bar{\nu}(t) \) was 0.1 (consistent with the empirical arrival rates of backers in crowdfunding projects (Marwell 2015)). We used the anticipating random walk (Alaei, Malekian, and Mostagir 2016) model for simulating backers’ projections of PoS because it is tailored for computing backers’ estimates of PoS in crowdfunding. For each project, the backers’ valuation of a reward was Gaussian (Ross and others 1996) distributed with a mean equivalent to the value of the reward \( P \).

We performed six groups of experiments: immediate disclosure, DSIS with five heuristics (random, greedy, e-greedy, softmax, and meta). Each group was run 30 times with the same 2.9GHz quad-core machine.

Results Figure 2a shows the average revenue (normalized by the highest revenue achieved in all experiments) obtained in the end by each group. The actual revenue excluded the projects that failed (\( M(T) < G \)), while the expected revenue included all the projects regardless of whether they succeeded to meet the funding goal or not. Not surprisingly, the expected revenue of each group was significantly higher than their respective actual revenue. This is because the majority of the campaigns failed due to not having met the funding goal within the deadline (see Figure 2c). Among the six groups, the meta group scored the best for both the expected revenue (mean = 0.7435, std = 0.0244) and the actual revenue (mean = 0.3722, std = 0.0092), followed by the softmax group, and the e-greedy group. The greedy group and the immediate group performed better than the immediate group in the expected revenue, but not in the actual revenue due to a lower success rate. The random group received the lowest scores in terms of both the expected revenue (mean = 0.3210, std = 0.0273) and the actual revenue (mean = 0.1004, std = 0.0329).

At the beginning, the immediate, the meta, the greedy, and e-greedy groups performed better than the other two (see Figure 2a). As time progressed, the meta and the e-greedy groups continued to lead the way until the later left behind the former at around \( t = 400 \). The e-greedy group kept the second until at time \( t = 900 \) that it was surpassed by the softmax group. One explanation is that the e-greedy algorithm initially encouraged exploration to a higher degree than the softmax algorithm. However, it acted more greedily than the softmax exploration over time, which was not favorable since better choices were rarely explored. The meta group used a set of refined policies to produce more robust decisions than the others. The random group performed the worst possibly because the random algorithm completely ignored the history of backers’ responses.

The meta algorithm took the most computation time (mean = 0.2300 std = 0.0290), while the random method required the least time (mean = 0.1037, std = 0.0034) (see Figure 2b). Immediate disclosure required no additional time.

In summary, although it required the most computation time, the meta group performed consistently the best among all the groups in terms of both actual and expected revenue. This echoes our previous findings that immediate disclosure is not always optimal in crowdfunding.

Conclusion

In this paper, we demonstrate that the entrepreneur can benefit from dynamic information disclosure. To the best of our knowledge, our work is the very first study on information design where a sender interacts with multiple receivers under thresholding policies.

While our analysis is in the context of crowdfunding, extensions to other domains (e.g., transportation systems, and smart grid, online shopping) are straightforward. A natural direction is to extend our model into a more generic setting where there are multiple senders and multiple receivers (mirroring the dynamics of two-sided markets). Another exciting path is to study how information design and mechanism design can jointly influence agents’ decisions.
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s(k) ∈ H_i(t):
\[ \Upsilon_{old}(s(k), i, t) = \frac{1}{n_k(t)} \sum_{t'=1}^{t-1} \sum_{j \in I(t')} \frac{\alpha_j(t')}{\eta_j(t')/\eta(t)} , \]

where \( n_k(t) \) denotes the times that \( s(k) \) has been revealed to backers up to time \( t \), \( \alpha_j(t') \) is backer \( j \)'s action given disclosure strategy \( (s(k), t') \), \( \eta_j(t') = |s(t')|/t' \), and \( \eta_j(t) = |s(t)|/t \) are the revenue growth rates up to time \( t' \) and \( t \), respectively. Note that \( \Upsilon_{old} = 0 \) if \( n_k(t) = 0 \) or \( t = 1 \).

The historical belief represents the entrepreneur’s estimate of the average revenue she receives by revealing project status \( s(k) \) to the backers, with discounting of the revenue growth rates. It is a rough estimate of revenue increase that \( d(i, j) = (s(k), t) \) brings.

At time \( t \), the entrepreneur’s temporal belief \( \Upsilon_{temp} \) of the expected increase in the revenue given the disclosure decision \( d(i, t) = (s(k), t) \), is determined as follows:
\[ \Upsilon_{temp}(s(k), i, t) = \sum_{j \in I(t-1)} \frac{\alpha_j(t-1)}{|I(t-1)|} , \]

where \( |I(t-1)| \) denotes the set of backers in the campaign at time \( t-1 \) and \( \alpha_j(t-1) \) is backer \( j \)'s action at time \( t-1 \). If \( |I(t-1)| = 0 \) or \( t = 1 \), then \( \Upsilon_{temp} = 0 \). The temporal belief captures the latest decisions of the backers that will most probably stay in the campaign at time \( t \).

The entrepreneur estimates the belief of the expected increase of revenue, given \( d(i, t) = (s(k), t) \) for backer \( i \in I(t) \), with the following equation:
\[ \Upsilon(s(k), i, t) = (1-\lambda) \Upsilon_{old}(s(k), i, t) + \lambda \Upsilon_{temp}(s(k), i, t) , \]

where \( \lambda \in [0, 1] \) is the learning rate (we use \( \lambda = 0.1 \)).

The greedy algorithm then selects the project status \( s(k_{sel}) \) by using the following equation:
\[ s(k_{sel}) = \arg\max_{s(k) \in H} \Upsilon(s(k), i, t) . \]

The probability for selecting the project status \( s(k_{sel}) \) is 1 and 0 for others.

**C. \( \epsilon \)-Greedy Exploration** In \( \epsilon \)-greedy exploration, with probability \( \epsilon \) the algorithm selects a random choice \( s(k) \). Otherwise, with probability \( 1 - \epsilon \) it selects the greedy choice determined in Equation (6)
\[ Pr(s(k)) = \begin{cases} 1 - \epsilon + \frac{\epsilon}{n_k(t)} , & \text{if Equation (6)} \\ \frac{\epsilon}{n_k(t)} , & \text{otherwise} \end{cases} \]

where \( n_k(t) \) is the times that \( s(k) \) has been reveled to backers up to time \( t \), \( \epsilon = c/n_k(t) \), and \( c \in [0, 1] \) is a constant. Note that \( \epsilon = c \) if \( n_k(t) = 0 \). Initially, the algorithm encourages exploration. As time progresses, it becomes more greedy.

**D. Softmax Exploration** Softmax exploration selects the choice using a Boltzmann distribution (Ross and others 1996). At time \( t \), the algorithm selects choice \( s(k) \) with the probability \( Prob(s(k)) \):
\[ Prob(s(k)) = \frac{e^{\Upsilon(s(k), i, t)/\tau}}{\sum_{s(k) \in H_i(t)} e^{\Upsilon(s(k), i, t)/\tau}} \cdot (8) \]
where \( \tau \) is the temperature parameter given by \( \tau = \max\{\mu, C^t / \log n_k(t)\} \). where, \( \tau = 1 \) when \( n_k(t) = 0 \), \( \mu = 0.0001 \) and \( C^t \) is determined by:

\[
C^t = \max_{s(k_1), s(k_2) \in H_i(t)} |\Upsilon(s(k_1), i, t) - \Upsilon(s(k_2), i, t)| . \tag{9}
\]

The softmax exploration selects each choice with a probability that is proportional to the average \( \Upsilon \).