Integrable crosscaps in classical sigma models

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ABSTRACT: We study the integrable boundaries and crosscaps of classical sigma models. We show that there exists a classical analog of the integrability condition and KT-relation of the boundary and crosscap states of quantum spin chains. We also classify the integrable crosscaps for various sigma models including examples which are relevant in the AdS/CFT correspondence at strong coupling.

KEYWORDS: Integrable Field Theories, Sigma Models, AdS-CFT Correspondence

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1 Introduction

In recent years, intensive research has been done in the defect versions of AdS/CFT duality. The probe brane defects in the string theory side were introduced in [1] and in the gauge theory side they correspond to defect conformal field theories (dCFTs) [2]. For theories with domain wall defects, the one-point functions at weak coupling can be obtained from an overlap between a boundary state (corresponding to the defect) and the Bethe states (corresponding to the single trace operators) of a spin chain (describing the scalar sector of the CFT) [3–5]. Similar overlaps appear also for the three-point functions which contain one single trace and two determinant type operators [6–8].

It was possible to write these overlaps in closed forms [9–12]. These closed forms were validated only by numerical checks. The reason of the existence such exact overlaps is the underlying integrability of the boundary states. The definition of the integrability for boundary states of spin chains was developed in [13] (based on [14]). We call a boundary state integrable if it preserves the half of the conserved charges of the spin chain. This condition can be written in a compact form

$$\langle B|T(u) = \langle B|T^\beta(-u),$$  \hspace{1cm} (1.1)
where $\langle B \rangle$ is the boundary state and $T(u), T^\beta(u)$ are transfer matrices of the spin chain which are connected by a $\mathbb{Z}_2$ automorphism $\beta$ of the underlying symmetry algebra.\footnote{Typically, $\beta$ is the identity or the charge conjugation.} In [15] a systematic algebraic method was introduced which proves the previously proposed form of the exact overlaps for a wide class of boundary states. The heart of this derivation is the so-called $KT$-relation

$$K_0(u)\langle B|T_0(u)\rangle = \langle B|T^\beta_0(-u)K_0(u)\rangle, \tag{1.2}$$

where $T(u), T^\beta(u)$ are the monodromy matrices and $K(u)$ is the $K$-matrix which acts only on the auxiliary space.

Recently, another type of states called crosscap states [16] was introduced for spin chains which are the analogous versions of the crosscap states of 2d CFT [17]. The difference between the boundary and the crosscap states is the following. While the boundary states identify the neighboring sites of the spin chain, the crosscap identifies the antipodal sites. In [18] the $KT$-relation was generalized for crosscap states and it was used to classify the crosscap states for all $\mathfrak{g}(N)$ symmetric spin chains. The previously proposed formula of [16] for overlaps was also proved for a wide class of crosscap states based on the $KT$-relation. In [19] it was argued that integrable crosscap states appeared in the one-point functions of the $\mathcal{N} = 4$ SYM on the $\mathbb{RP}^4$ spacetime.

As we already mentioned, at strong coupling the defect corresponds to a probe D-brane which can be describe as boundary conditions of the string sigma models. For certain boundary conditions the integrability has been already shown [20–22]. During these derivations the boundaries were put on space therefore it was shown that infinitely many conserved charges exist on the worldsheet of the open strings which are attached on the D-brane. However, in the holographic description of the one-point function we have a closed string (corresponding to the operator) which is annihilated on the D-brane. In this setup the boundary is in time, and intuitively, the integrability means that the boundary condition preserves the half of the worldsheet conserved charges of the closed strings. It is clear that it would be a classical analog of the quantum integrability condition (1.1).

The goal of this paper is twofold. Firstly, we want to show that classical analogs of the integrability conditions (1.1) and $KT$-relations (1.2) exist for classical sigma models with boundaries in time. We also show that these relations are automatically satisfied for the classical reflection matrices of [20–22]. We also generalize the integrability condition and $KT$-relations for the crosscaps of classical sigma models. The second goal is to classify the integrable crosscaps of the sigma models which appear in the AdS/CFT at strong coupling. This paper is not intended to provide a complete holographic description, we will not go beyond the classic sigma model. However, this classification for the sigma models can be a good starting point for more comprehensive future investigations.

The organization of the paper is as follows. In section 2 we give the definition of the crosscaps of sigma models. In section 3 we review the Lax description (Lax-connection, transfer matrix etc.) of the sigma models. In section 4 we derive the integrability condition and $KT$-relation when the boundaries are in time. In section 5 we generalize the integrability condition for crosscaps and classify them for the sigma models with target spaces $\text{SU}(N)$,
In section 6 we classify the integrable crosscaps of the sigma models which appear in the AdS$_5$/CFT$_4$ and AdS$_4$/CFT$_3$ dualities at strong coupling.

2 Crosscaps of sigma models

In this section we define the crosscaps for 2d sigma models. Let $\mathcal{M}$ be a semi-Riemann manifold (target space) with local coordinates $X^I$ where $I = 1, \ldots, \dim \mathcal{M}$ and metric $G_{IJ}(X)$. We also define a 2d manifold (worldsheet) with local coordinates $\sigma^\mu = (\sigma^0, \sigma^1) = (\tau, \sigma)$ and the worldsheet metric $\eta$. The fields in the 2d sigma model are maps between the worldsheet and the target space: $X : \Sigma \to \mathcal{M}$. The dynamics are given by the action

$$ S[X] = \int_{\Sigma} d\tau d\sigma G_{IJ}(X) \partial_\mu X^I \partial^\mu X^J. \tag{2.1} $$

Let $\alpha$ be a $\mathbb{Z}_2$ isometry of $\mathcal{M}$ which acts on the fields as $\alpha : X \to X^\alpha$. It is clear that if we have a solution $X^I(\tau, \sigma)$ of the equations of motion then the transformed fields $X^{\alpha,I}(\tau, \sigma)$ also satisfy them.

At first let us consider an infinite cylinder as worksheet: $\Sigma = \mathbb{R} \times S^1$. We choose the local coordinates as $\tau \in \mathbb{R}$ and $\sigma \in [0, L)$ with the identification $\sigma + L \equiv \sigma$. We define the crosscap by a restriction of the allowed configurations of the fields. We allow only the configurations which are invariant under the following $\mathbb{Z}_2$ transformation

$$ X(\tau, \sigma) = X^\alpha \left(-\tau, \sigma + \frac{L}{2}\right). \tag{2.2} $$

We call (2.2) as crosscap identification and the field configurations which satisfy (2.2) are the crosscap configurations. Let us divide the worldsheet into two regions $\Sigma = \Sigma_+ \cup \Sigma_-$ where $\Sigma_+ = [0, \infty) \times S^1$ and $\Sigma_- = (-\infty, 0] \times S^1$. For the crosscap configurations it is enough to give the fields on $\Sigma_-$ since the crosscap identification (2.2) gives also the fields on $\Sigma_+$. It is clear that we can choose any configuration on almost the full $\Sigma_-$. We get non-trivial conditions only on the intersection $\Sigma_+ \cap \Sigma_-$ which is the $\tau = 0$ circle. Substituting to (2.2) we obtain that

$$ X^I \bigg|_{\tau=0, \sigma=\sigma_0} = X^{\alpha,I} \bigg|_{\tau=0, \sigma=\sigma_0 + \frac{L}{2}}. \tag{2.3} $$

We have another non-trivial smoothness condition for the time derivatives

$$ \left( \partial_\tau X^I \right) \bigg|_{\tau=0, \sigma=\sigma_0} = -\left( \partial_\tau X^{\alpha,I} \right) \bigg|_{\tau=0, \sigma=\sigma_0 + \frac{L}{2}}. \tag{2.4} $$

Choosing any field configuration on $\Sigma_-$ with conditions (2.3) and (2.4) we can uniquely extend it to a crosscap configuration on the full $\Sigma$. In the sections 4, 5 and 6 we concentrate on sigma models on the worldsheet $\Sigma_-$ and we call the conditions (2.3) and (2.4) as crosscap conditions.
3 Lax description

Let us consider an integrable 2 dimensional field theory on the worldsheet $\Sigma$ and a Lax connection $A(\lambda) = A_\tau(\lambda)d\tau + A_\sigma(\lambda)d\sigma \in \Omega_1(\Sigma) \otimes \mathfrak{gl}(N)$ where $\Omega_1(\Sigma)$ are the one-forms on $\Sigma$ and $\lambda \in \mathbb{C}$ is the spectral parameter. The Lax connection satisfies the zero curvature equation

$$dA(\lambda) + A(\lambda) \wedge A(\lambda) = 0. \quad (3.1)$$

This equation is the local manifestation of the path independence of the holonomies of the Lax connection

$$\left[ \overleftarrow{\text{exp}} \int_{\gamma_1} -A(\lambda) \right] = \left[ \overleftarrow{\text{exp}} \int_{\gamma_2} -A(\lambda) \right], \quad (3.2)$$

where $\gamma_1$ and $\gamma_2$ are homotopic curves with same end points. The equation (3.2) is the real heart of the integrability since it guaranties the existence of infinite many integrals of motion. We can define monodromy matrices

$$T(\lambda|\tau) = \overleftarrow{\text{exp}} \int_{\tau_0}^{L} -A_\sigma(\lambda)d\sigma. \quad (3.3)$$

Using the integrability condition (3.2) we obtain that the time evolution of the monodromy matrix can be written as

$$T(\lambda|\tau_2) = U_L(\tau_2, \tau_1)T(\lambda|\tau_1)U_0(\tau_2, \tau_1)^{-1}, \quad (3.4)$$

where

$$U_\sigma(\tau_2, \tau_1) = \overleftarrow{\text{exp}} \int_{\tau=\tau_1}^{\tau_2} -A_\tau(\lambda)d\tau. \quad (3.5)$$

For the periodic boundary condition $\sigma \equiv \sigma + L$ we obtain that

$$U_L(\tau_2, \tau_1) = U_0(\tau_2, \tau_1), \quad (3.6)$$

therefore the time evolution of the monodromy matrix is a similarity transformation

$$T(\lambda|\tau_2) = U_0(\tau_2, \tau_1)T(\lambda|\tau_1)U_0(\tau_2, \tau_1)^{-1}. \quad (3.7)$$

The trace of the monodromy matrix (transfer matrix)

$$T(\lambda|\tau) = \text{Tr}T(\lambda|\tau) \quad (3.8)$$

generates the conserved quantities

$$T(\lambda) := T(\lambda|\tau_1) = T(\lambda|\tau_2). \quad (3.9)$$
Examples. In the following we define four well known examples for the integrable sigma model. For more details see the review [23].

Principal chiral model. Let $G$ and $\mathfrak{g}$ be a Lie group and the corresponding Lie algebra. Defining the target space as $g(\tau, \sigma) \in G$, the current $J = g^{-1}dg \in \Omega_1(\Sigma) \otimes \mathfrak{g}$ is a Lie-algebra valued one-form. The equation of motions

$$d \ast J = 0, \quad dJ + J \wedge J = 0$$  \hspace{1cm} (3.10)

are equivalent to the zero curvature equation (3.1) of the Lax connection

$$A(\lambda) = \frac{1}{1 + \lambda^2}J + \frac{\lambda}{1 + \lambda^2} \ast J,$$  \hspace{1cm} (3.11)

where $\ast$ is the Hodge duality.

$O(N)$ sigma model. For the $O(N)$ sigma model the target space is the $N-1$ dimensional sphere $S^{N-1}$ with radius 1. Let us parameterize this sphere with the coordinates $\phi_i(\tau, \sigma) \in \mathbb{R}$ for which

$$\sum_i \phi_i \phi_i = \phi^t \phi = 1$$  \hspace{1cm} (3.12)

We can introduce a group element $h(\tau, \sigma) \in O(N)$ as

$$h = 1 - 2\phi \phi^t,$$  \hspace{1cm} (3.13)

for which

$$h^2 = (1 - 2\phi \phi^t)(1 - 2\phi \phi^t) = 1,$$  \hspace{1cm} (3.14)

therefore $h^{-1} = h^t$. Introducing the current

$$J = hdh = 2\phi d\phi^t - 2d\phi \phi^t,$$  \hspace{1cm} (3.15)

the equations of motion have the form (3.10) therefore the Lax-connection can be written as (3.11).

Sigma model on the $AdS_N$. The target space is the $N$ dimensional anti de-Sitter $AdS_N$ with radius 1. Let us parameterize this space with the coordinates $X_i(\tau, \sigma) \in \mathbb{R}$ where $i = -1, 0, 1, \ldots, N - 1$ for which

$$-(X_{-1})^2 - (X_0)^2 + \sum_{i=1}^{N-1} (X_i)^2 = \sum_{i,j=-1}^{N-1} \eta_{ij}X_iX_j = X^t \eta X = -1,$$  \hspace{1cm} (3.16)

where $X^t = (X_{-1}, X_0, \ldots, X_{N-1})$ is a row vector and

$$\eta = \text{diag}(-1, -1, 1, 1, \ldots, 1).$$  \hspace{1cm} (3.17)

The equation of motion is

$$\partial^2 X - X \left( \partial_{\mu} X^t \eta \partial^\mu X \right) = 0.$$  \hspace{1cm} (3.18)
We can introduce a group element \( h(\tau, \sigma) \in O(2, N - 1) \) as
\[
h = \eta - 2XX^t,
\]
for which \( h = h^t \) and
\[
h\eta h = (\eta + 2XX^t)\eta(\eta + 2XX^t) = \eta,
\]
therefore \( h^{-1} = \eta h \eta \). Introducing the current
\[
J = h^{-1}dh = 2\eta(dXX^t - XdX^t),
\]
the equations of motion are (3.10) therefore the Lax-connection has the form (3.11). We will also use the Poincaré coordinates
\[
X_{-1} = \frac{z}{2} \left( 1 + \frac{1 + x^2}{z^2} \right),
\]
\[
X_{N-1} = \frac{z}{2} \left( 1 - \frac{1 - x^2}{z^2} \right),
\]
\[
X_i = \frac{x_i}{z}, \quad i = 0, 1, \ldots, N - 2,
\]
where \( x^2 = -(x_0)^2 + \sum_{i=1}^{N-2} (x_i)^2 \) and \( z > 0 \). We can invert the first two equations
\[
z = \frac{1}{X_{-1} - X_{N-1}}, \quad x^2 = \frac{X_{-1}^2 - X_{N-1}^2 - 1}{(X_{-1} - X_{N-1})^2},
\]
therefore this Poincaré patch covers the \( X_{-1} > X_{N-1} \) part of the global AdS\(_N\). In the figure 1 we divide the AdS\(_N\) to four regions. The Poincaré patch (3.22) covers the regions I and II. We can also introduce another Poincaré patch which covers the regions I and III (\( X_{-1} > -X_{N-1} \)) as
\[
X_{-1} = \frac{\tilde{z}}{2} \left( 1 + \frac{1 + \tilde{x}^2}{\tilde{z}^2} \right),
\]
\[
X_{N-1} = -\frac{\tilde{z}}{2} \left( 1 - \frac{1 - \tilde{x}^2}{\tilde{z}^2} \right),
\]
\[
X_i = \frac{\tilde{x}_i}{\tilde{z}}, \quad i = 0, 1, \ldots, N - 2.
\]
One could also introduce two other Poincaré patches which would cover the regions II,IV or III,IV.

**Sigma model on the CP\(^{N-1}\).** For the CP\(^{N-1}\) sigma model we use the action
\[
S = \int d\tau d\sigma (D_\mu Y)\dagger (D^\mu Y),
\]
where \( Y \in \mathbb{C}^N \) with the constraint \( Y\dagger Y = 1 \) and we introduced the covariant derivative as
\[
D_\mu = \partial_\mu - iA_\mu,
\]
Figure 1. Regions of the AdS$_N$. The red lines are $X_{-1} = \pm X_{N-1}$ and they divide the AdS$_N$ to four regions. For example the region I is $X_{-1} > |X_{N-1}|$. For the first Poincaré patch (3.22) $X_{-1} > X_{N-1}$ therefore it covers the regions I and II. For the second Poincaré patch (3.24) $X_{-1} > -X_{N-1}$ therefore it covers the regions I and III.

where $A_{\mu}$ is a U(1) gauge field. The equations of motion are

\[ 0 = D_{\mu}D^{\mu}Y + (D_{\mu}Y)^{(\dagger)}(D^{\mu}Y), \quad (3.27) \]
\[ A_{\mu} = i(\partial_{\mu}Y)^{\dagger}Y = -iY^{\dagger}(\partial_{\mu}Y). \quad (3.28) \]

We can introduce a group element $h(\tau, \sigma) \in U(N)$ as

\[ h = 1 - 2YY^{\dagger}, \quad (3.29) \]

for which

\[ h^2 = (1 - 2YY^{\dagger})(1 - 2YY^{\dagger}) = 1, \quad (3.30) \]

therefore $h^{-1} = h = h^{\dagger}$. Introducing the current

\[ J = hdh = 2YdY^{\dagger} - 2dYY^{\dagger} + 4iAYY^{\dagger}, \quad (3.31) \]

the equations of motion are (3.10) therefore the Lax-connection has the form (3.11).

4 Boundaries in time

We make a detour in this section. Before we move on to the Lax description of the crosscaps, we first consider the case where the usual boundary condition is placed not in space but in time. The integrable boundary conditions in space (the boundary is the $\sigma = 0$ line) have been already analyzed in several papers, e.g. [24–28]. In these situations the so-called double row transfer matrices generate the conserved charges on the half plane or the strip. In the following we analyze what happens when we put the same integrable boundaries to the $\tau = 0$ circle of $\Sigma_{-}$.

We saw that the phenomena of integrability comes from path independent holonomies. In the boundary case we can define holonomies which attach to the boundary

\[ \left[ \mathcal{P} \exp \int_{\gamma_2} -A^2(-\lambda) \right] \kappa(\lambda) \left[ \mathcal{P} \exp \int_{\gamma_1} -A(\lambda) \right], \quad (4.1) \]
Figure 2. Path independent holonomy. The red curve corresponds to the holonomy of $A(\lambda)$ between $A$ and $C$. The blue curve corresponds to the holonomy of $A^\beta(-\lambda)$ between $C$ and $B$.

Figure 3. The $KT$-equation. The red curve corresponds to the holonomy of $A(\lambda)$ and the blue curve corresponds to the holonomy of $A^\beta(-\lambda)$.

where $\kappa$ is a reflection matrix and $\beta$ is an automorphism which leaves the flatness condition invariant i.e.

$$dA^\beta(\lambda) + A^\beta(\lambda) \wedge A^\beta(\lambda) = 0. \quad (4.2)$$

The endpoints of the curves $\gamma_1, \gamma_2$ are $[C, A], [B, C]$ where $A$ and $B$ are fixed and $C$ is a common point which is on the boundary. According to the flatness condition of the Lax connection we can freely deform the curves $\gamma_1, \gamma_2$, and it is natural to say, the boundary condition is integrable if the holonomy (4.1) is independent from the common point $C$, see figure 2.

We can differentiate this condition and we obtain the following constraint for the Lax connection

$$\kappa(\lambda)A_{\sigma}(\lambda) - A^\beta_{\sigma}(-\lambda)\kappa(\lambda) \bigg|_{\tau=0} = \partial_\sigma \kappa(\lambda). \quad (4.3)$$

This is the usual equation of the integrable boundaries, only the space and time coordinates are replaced. Some solutions which are relevant in the AdS/CFT correspondence can be found in [20–22].

Using the above defined boundary flatness condition, we can obtain the following equation (see figure 3)

$$\kappa(\lambda)T(\lambda) = T^\beta(-\lambda)\kappa(\lambda), \quad (4.4)$$

where $T(\lambda)$ is the monodromy matrix (3.3) at $\tau = 0$. The equation (4.4) is the classical analog of the quantum $KT$-relation (1.2) which was introduced in [29]. At this point we define two $\beta$-transformations.

1. $A^\beta = A$ (identity).
2. $A^\beta = -A^t$ (charge conjugation).
For the first case the $KT$-relation is simplified as
\begin{equation}
\kappa(\lambda)T(\lambda) = T(-\lambda)\kappa(\lambda),
\end{equation}
which is the classical analog of the untwisted $KT$-relation of [15]. For the second case let us introduce a new notation for the $\beta$-transformed monodromy matrix
\begin{equation}
\hat{T}(\lambda) := T^\beta(\lambda) = \overline{P}\exp\int_{\sigma=0}^{L} A^\beta_\sigma(\lambda) d\sigma.
\end{equation}
We can see that
\begin{equation}
T(\lambda)\hat{T}(\lambda) = 1.
\end{equation}
In the second case (where $A^\beta = -A^\tau$), the $KT$-relation (4.4) is equivalent to
\begin{equation}
\kappa(\lambda)T(\lambda) = \hat{T}(\lambda)\kappa(\lambda),
\end{equation}
which is the classical analog of the twisted $KT$-relation of [15].

From the $KT$-relation and the definition of the transfer matrix (3.8) we can easily derive the condition
\begin{equation}
T(\lambda) = T^\beta(-\lambda),
\end{equation}
therefore the half of the charges are vanishing which is the classical analog of integrability condition of boundary states [13]. Specifying the $\beta$-transformation we obtain the following conditions
\begin{align}
T(\lambda) &= T(-\lambda), \\
\hat{T}(\lambda) &= \hat{T}(-\lambda),
\end{align}
which are the classical analog of the untwisted and twisted quantum integrability conditions [30].

It is worth to mention some related works. The classical $KT$-relation (4.4) with some minor modifications were already appeared in [31]. This paper investigated the non-linear Schrödinger equation and introduced a dual, equal-space, Poisson bracket which describes a Hamiltonian flow in the space direction. In this dual Hamiltonian flow a usual defect in space can be considered as a defect in time. The Liouville integrability was also proved in this dual picture. These ideas have been developed along various directions and for various key models [32–35].

5 Lax description of crosscaps

In this section we can generalize the argument of the previous section for crosscap conditions (2.3), (2.4).

Now we identify the antipodal points of the time boundary $\tau = 0$ therefore the flatness condition has to be modified in the following way: the holonomy
\begin{equation}
\overline{P}\exp\int_{\gamma_2} -A^\beta(\lambda) \kappa [\overline{P}\exp\int_{\gamma_1} -A(\lambda)]
\end{equation}
is independent of the path, see figure 4. The endpoints of the curves $\gamma_1, \gamma_2$ are $[C, A], [B, C']$ where $A$ and $B$ are fixed and $C = (0, \sigma_0), C' = (0, \sigma_0 + \frac{L}{2})$. The holonomy (5.1) is independent from the space coordinate $\sigma_0$.

The local version of this flatness condition is

$$
\kappa A_\sigma(\lambda) \bigg|_{\tau=0, \sigma=\sigma_0} = A_\sigma^\beta(-\lambda) \kappa \bigg|_{\tau=0, \sigma=\sigma_0+\frac{L}{2}}.
$$

As in the previous section, we distinguish two types of automorphism $\beta$:

1. $A^\beta = A$ (identity).

2. $A^\beta = -A^t$ (charge conjugation).

Let us start with the first case when $\beta = \text{id}$. Applying the crosscap condition twice, we obtain that

$$
A_\sigma(\lambda) \bigg|_{\tau=0, \sigma=\sigma_0} = \kappa^{-1} A_\sigma(-\lambda) \kappa \bigg|_{\tau=0, \sigma=\sigma_0+\frac{L}{2}} = (\kappa^{-1})^2 A_\sigma(\lambda) \kappa^2 \bigg|_{\tau=0, \sigma=\sigma_0},
$$

where we used the periodic boundary condition $\sigma \equiv \sigma + L$. Since we do not want to introduce any local constraint for the fields, the crosscap is consistent if

$$
\kappa^2 = c \mathbf{1},
$$

where $c \in \mathbb{C}$. Since the flatness condition is linear in $\kappa$ we can always choose a normalization where $\kappa^2 = \mathbf{1}$. For the second case ($\beta$ is the charge conjugation) the crosscap condition reads as

$$
\kappa A_\sigma(\lambda) \bigg|_{\tau=0, \sigma=\sigma_0} = -A_\sigma^t(-\lambda) \kappa \bigg|_{\tau=0, \sigma=\sigma_0+\frac{L}{2}}.
$$

Applying this crosscap condition twice, we obtain that

$$
A_\sigma(\lambda) \bigg|_{\tau=0, \sigma=\sigma_0} = -\kappa^{-1} A_\sigma^t(-\lambda) \kappa \bigg|_{\tau=0, \sigma=\sigma_0+\frac{L}{2}} = \kappa^{-1} \kappa^t A_\sigma(\lambda)(\kappa^{-1})^t \kappa \bigg|_{\tau=0, \sigma=\sigma_0}.
$$

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We concentrate on the constant $\kappa$-matrices.
Similarly as in the previous case, we do not want to introduce any local constraint for the
fields therefore the crosscap is consistent if

\[ \kappa^t = \pm \kappa. \]  \hspace{1cm} (5.7)

Using the global (5.1) or the local (5.2) crosscap condition of the Lax connection we can obtain the following equation (see figure 5)

\[ \kappa T^{(1)}(\lambda) = T^{(2),\alpha}(-\lambda)\kappa, \]  \hspace{1cm} (5.8)

where \( T^{(1)}(\lambda), T^{(2)}(\lambda) \) are two new monodromy matrices at \( \tau = 0 \):

\[ T^{(1)}(\lambda) = \overrightarrow{\mathcal{P}} \exp \int_0^{L/2} -A_\sigma(\lambda|\tau = 0, \sigma) d\sigma. \]  \hspace{1cm} (5.9)

\[ T^{(2)}(\lambda) = \overrightarrow{\mathcal{P}} \exp \int_{L/2}^L -A_\sigma(\lambda|\tau = 0, \sigma) d\sigma. \]  \hspace{1cm} (5.10)

It is obvious that the full monodromy matrix is the product of them:

\[ T(\lambda) = T^{(2)}(\lambda)T^{(1)}(\lambda). \]  \hspace{1cm} (5.11)

The equation (5.8) is the classical analog of the quantum \( KT \)-relation which was introduced in [18]. For the first case (\( \beta \) is the identity) the \( KT \)-relation is simplified as

\[ \kappa T^{(1)}(\lambda) = T^{(2)}(-\lambda)\kappa, \]  \hspace{1cm} (5.12)

which is the classical analog of the untwisted \( KT \)-relation of [18]. For the second case (where \( A^\beta = -A^t \)) the equation (5.8) is equivalent to

\[ \kappa T^{(1)}(\lambda) = \hat{T}^{(2)}(-\lambda)\kappa, \]  \hspace{1cm} (5.13)

which is the classical analog of the twisted \( KT \)-relation of [18]. Using the connection (4.7) between the monodromy matrices, we can obtain an equivalent form of the twisted \( KT \)-relation

\[ \kappa^t T^{(2)}(\lambda) = \hat{T}^{(1)}(-\lambda)\kappa^t. \]  \hspace{1cm} (5.14)

Applying the untwisted \( KT \)-relation to the transfer matrix we obtain that

\[ T(\lambda) = \text{Tr} T^{(2)}(\lambda)T^{(1)}(\lambda) = \text{Tr} T^{(1)}(-\lambda)\kappa^{-1}\kappa^{-1}T^{(2)}(-\lambda)\kappa. \]  \hspace{1cm} (5.15)

Using the constraint (5.4) for the untwisted \( \kappa \)-matrix, we just obtained that

\[ T(\lambda) = T(-\lambda), \]  \hspace{1cm} (5.16)

which is the classical analog of the untwisted quantum integrability condition [18].
Applying the twisted KT-relation to the transfer matrix we obtain that
\[ T(\lambda) = \text{Tr} T^{(2)}(\lambda) T^{(1)}(\lambda) = \text{Tr} \left( (\kappa')^{-1} \hat{T}^{(1)}(-\lambda) \kappa' \kappa^{-1} \hat{T}^{(2)}(-\lambda) \kappa \right). \]  
(5.17)

Using the constraint (5.7) for the twisted \( \kappa \)-matrix, we just obtained that
\[ T(\lambda) = \hat{T}(-\lambda), \]
(5.18)
which is the classical analog of the twisted quantum integrability condition [18].

Let us analyze the solutions of the crosscap conditions for the models which are described by the type of Lax pairs (3.11). The crosscap conditions for the currents are
\[ J_{\sigma} \bigg|_{\tau=0, \sigma=\sigma_0} = \kappa^{-1} \beta_{\sigma} \kappa, \quad J_{\sigma}^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_{\sigma}^{(1)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \]
(5.19)
\[ J_{\tau} \bigg|_{\tau=0, \sigma=\sigma_0} = -\kappa^{-1} \beta_{\tau} \kappa, \quad J_{\tau}^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_{\tau}^{(1)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \]
(5.20)
where we introduced the transformation
\[ X^\alpha = \Omega(X) := \kappa^{-1} X \beta \kappa. \]  
(5.21)

We already saw that \( \Omega^2 = \text{id} \) for the consistent crosscaps. We can see that the conditions (5.19), (5.20) are equivalent to the crosscap conditions (2.3), (2.4) therefore the construction of this section indeed describes crosscaps.

Let us decompose the current as \( J_\mu = J_\mu^{(0)} + J_\mu^{(1)} \) where \( \Omega(J_\mu^{(0)}) = J_\mu^{(0)} \) and \( \Omega(J_\mu^{(1)}) = -J_\mu^{(1)} \). Using these notations the crosscap condition simplifies as
\[ J_\sigma^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0} = J_\sigma^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_\sigma^{(1)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_\tau^{(1)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_\tau^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \]
(5.22)
\[ J_\tau^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0} = J_\tau^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_\tau^{(1)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_\sigma^{(0)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \quad J_\sigma^{(1)} \bigg|_{\tau=0, \sigma=\sigma_0+\frac{k}{2}}, \]
(5.23)

**Examples.** In the following we analyze the crosscap equations (5.19), (5.20) for some concrete sigma models which were introduced in section 3. More concretely, we specify the manifestation of these crosscap equations on the concrete parametrization of the target spaces. Since we already show that the conditions (5.19), (5.20) are equivalent to the crosscap conditions (2.3), (2.4) we concentrate on the explicit forms of the \( \alpha \)-isometries and the corresponding residual symmetries.

**Principal chiral fields.** Let us concentrate on the SU(\( N \)) principal chiral field i.e. \( \mathfrak{g} = \text{su}(N) \). For the \( \beta = \text{id} \) case we have \( \kappa^2 = 1 \). Using the global SU(\( N \)) symmetry we can diagonalize the \( \kappa \)-s therefore we have
\[ \kappa = \text{diag}(1,\ldots,1,-1,\ldots,-1). \]  
(5.24)
The $\alpha$-automorphism acts on the fields as $g^\alpha = \kappa g \kappa^{-1}$ and the residual symmetry is $\text{SU}(k) \times \text{SU}(N-k) \times \text{U}(1)$. In the second case ($\beta$ is the charge conjugation) $g^\alpha = \kappa (g^\dagger)^{-1} \kappa^{-1}$ and the residual symmetries are $\text{SO}(N)$ or $\text{Sp}(N)$ for $\kappa = \kappa^t$ or $\kappa = -\kappa^t$, respectively.

This classification of classical crosscaps is completely analogous with the classification of the crosscap states of the quantum $\mathfrak{su}(N)$ symmetric spin chains [18].

$O(N)$ sigma model. For the $O(N)$ sigma model the current is an element of $\mathfrak{so}(N)$ i.e. $-J^t = J$ therefore only the $\beta = \text{id}$ case is relevant. Since the current is anti-symmetric the consistent crosscaps require both conditions $\kappa^2 = c1$ and $\kappa^t = \pm \kappa$ therefore we have two classes of $\kappa$-matrices (up to global $O(N)$ rotations)

$$\kappa = \text{diag}(1, \ldots, 1, -1, \ldots, -1),$$

$$\kappa = \begin{pmatrix} 0 & 1 \frac{N}{2} \\ -1 \frac{N}{2} & 0 \end{pmatrix}.$$  \hspace{1cm} (5.25, 5.26)

The symmetric case has residual symmetry $\text{SO}(k) \times \text{SO}(N-k)$ and the anti-symmetric one has $\text{U}(\frac{N}{2})$. In the symmetric case the solution of the equation (5.19) is

$$\phi^\alpha_i = \phi_i, \quad i = 1, \ldots, k,$$

$$\phi^\alpha_i = -\phi_i, \quad i = k + 1, \ldots, N.$$ \hspace{1cm} (5.27, 5.28)

For the anti-symmetric $\kappa$, the equation (5.19) has the solution

$$\phi^\alpha_i = -\phi_{\frac{N}{2} + i}, \quad i = 1, \ldots, \frac{N}{2},$$

$$\phi^\alpha_{\frac{N}{2} + i} = \phi_i, \quad i = \frac{N}{2} + 1, \ldots, N.$$ \hspace{1cm} (5.29, 5.30)

We can see that this is not a consistent crosscap because the square of this transformation is $-1$.

In summary, for the $O(N)$ sigma model the integrable crosscaps correspond to the residual symmetries $\text{SO}(k) \times \text{SO}(N-k)$ and the concrete identifications are given by the equations (5.27)–(5.28). In the table 1 we enumerate these possibilities explicitly for the $O(6)$ model.

Sigma model on the $\text{AdS}_N$. For the sigma model on the $\text{AdS}_N$ the current is an element of $\mathfrak{so}(2, N-1)$ i.e. $-J^t = \eta J \eta$ therefore only the $\beta = \text{id}$ case is relevant for which we have the condition $\kappa^2 = c1$. From the crosscap equation (5.19) we also obtain another constraint since $\kappa J \kappa^{-1} = \mathfrak{so}(2, N-1)$ i.e., $\kappa^t = \pm \eta \kappa \eta$. Let us start with the symmetric $\kappa$-matrix

$$\kappa = \text{diag}(1, 1, 1, \ldots, 1, -1).$$ \hspace{1cm} (5.31)

This $\kappa$-matrix has $\text{SO}(2, N-2)$ symmetry. The equation (5.19) has the solution

$$X^\alpha_i = X_i, \quad i = -1, \ldots, N-2,$$

$$X^\alpha_{N-1} = -X_{N-1}.$$ \hspace{1cm} (5.32)
Using the Poincaré coordinates (3.22) we obtain that
\[
x^\alpha_i = \frac{x_i}{x^2 + z^2}, \quad i = 0, \ldots, N-2,
\]
\[
z^\alpha = \frac{z}{x^2 + z^2}.
\]
(5.33)

Since \(z\) and \(z^\alpha\) should be positive, this transformation is defined only for \(x^2 > -z^2\). More specifically, at the boundary \(z \to 0\), the transformation is singular at the light cone \(x^2 = 0\). However the transformation is well defined in the global AdS (see (5.32)) therefore the crosscap is ill-defined only on the given Poincaré patch, i.e., the \(\alpha\)-transformation sends the points \(x^2 < -z^2\) to another Poincaré patch. It is clear from the figure 1. This Poincaré coordinates cover the regions I and II which corresponds to the domains \(x^2 > -z^2\) and \(x^2 < -z^2\), respectively. For the transformation (5.32) the region I is closed but region II goes to region III. The region III is covered by the second Poincaré coordinates (3.24).

Using global SO(2, \(N-1\)) isometries we can choose a Poincaré patch which is closed under this transformation. An equivalent form of the \(\kappa\)-matrix is
\[
\kappa = \text{diag}(1, 1, 1, \ldots, 1, -1, 1).
\]
(5.34)

It leads to the crosscap conditions
\[
x^\alpha_i = x_i, \quad i = 0, \ldots, N-2,
\]
\[
x^\alpha_{N-2} = -x_{N-2}, \quad z^\alpha = z.
\]
(5.35)

Since the \(\kappa\)-matrices (5.31) and (5.34) are connected by an isometry of AdS\(_N\), the crosscaps (5.33) and (5.35) are also equivalent up to an isometry. We can see that the second crosscap (5.35) is well defined in this Poincaré patch.

The possible \(\kappa^2 = c\mathbf{1}\) matrices (up to global SO(2, \(N-1\)) rotations) are
\[
\kappa = \text{diag}(1, 1, -1, \ldots, -1, 1, \ldots, 1),
\]
(5.36)
\[
\kappa = \text{diag}(1, -1, -1, \ldots, -1, 1, \ldots, 1),
\]
(5.37)

| Residual symmetry | \((\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)^\alpha\) |
|-------------------|--------------------------------------------------|
| SO(6)             | \((\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)\) |
|                   | \((-\phi_1, -\phi_2, -\phi_3, -\phi_4, -\phi_5, -\phi_6)\) |
| SO(5)             | \((\phi_1, \phi_2, \phi_3, \phi_5, -\phi_6)\) |
|                   | \((\phi_1, -\phi_2, -\phi_3, -\phi_4, -\phi_5, -\phi_6)\) |
| SO(4) \times SO(2)| \((\phi_1, \phi_2, \phi_3, -\phi_5, -\phi_6)\) |
|                   | \((\phi_1, \phi_2, -\phi_3, -\phi_4, -\phi_5, -\phi_6)\) |
| SO(3) \times SO(3)| \((\phi_1, \phi_2, \phi_3, -\phi_4, -\phi_5, -\phi_6)\) |

Table 1. Crosscaps of \(O(6)\) sigma model.
where the first one has symmetry \( \text{SO}(2, N - k - 1) \times \text{SO}(k) \) and the second one has \( \text{SO}(1, k) \times \text{SO}(1, N - k - 1) \), for \( k = 0, 1, \ldots, N - 1 \). We can also get anti-symmetric \( \kappa \) but it leads to inconsistent crosscap just as for the \( O(N) \) sigma model. In the first case the equation (5.19) has the solution

\[
X_\alpha^i = X_i, \quad i = -1, 0 \text{ and } i = k + 1, \ldots, N - 1 \\
X_i^\alpha = -X_i, \quad i = 1, \ldots, k, 
\]

(5.38)

Using the Poincaré coordinates we obtain that

\[
X_0^\alpha = x_0, \quad z^\alpha = z, \\
X_i^\alpha = -x_i, \quad i = 1, \ldots, k, \\
X_i^\alpha = x_i, \quad i = k + 1, \ldots, N - 2, 
\]

(5.39)

for \( k = 0, 1, \ldots, N - 2 \). For \( k = N - 1 \) (symmetry class \( \text{SO}(2) \times \text{SO}(N - 1) \)) there is no Poincaré patch which is closed under the crosscap transformation (5.38). On the region I we obtain that

\[
x_0^\alpha = \frac{x_0}{x^2 + z^2}, \quad z^\alpha = \frac{z}{x^2 + z^2}, \\
x_i^\alpha = \frac{-x_i}{x^2 + z^2}, \quad i = 1, \ldots, N - 2. 
\]

(5.40)

For the second \( \kappa \)-matrix (5.37), we can obtain similar conditions

\[
X_\alpha^i = X_i, \quad i = -1 \text{ and } i = k + 1, \ldots, N - 1 \\
X_i^\alpha = -X_i, \quad i = 0, 1, \ldots, k, 
\]

(5.41)

Using the Poincaré coordinates we obtain that

\[
x_0^\alpha = \frac{-x_0}{x^2 + z^2}, \quad z^\alpha = \frac{z}{x^2 + z^2}, \\
x_i^\alpha = \frac{x_i}{x^2 + z^2}, \quad i = 0, \ldots, k, \\
x_i^\alpha = \frac{-x_i}{x^2 + z^2}, \quad i = k + 1, \ldots, N - 2, \\
z^\alpha = \frac{z}{x^2 + z^2}. 
\]

(5.42)

for \( k = 0, 1, \ldots, N - 2 \). For \( k = N - 1 \) (symmetry class \( \text{SO}(1, N - 1) \)) there is no Poincaré patch which is closed under the crosscap transformation (5.38). On the region I we obtain that

\[
x_i^\alpha = \frac{-x_i}{x^2 + z^2}, \quad i = 0, \ldots, N - 2, \\
z^\alpha = \frac{z}{x^2 + z^2}. 
\]

(5.43)

In summary, for the sigma model on the \( \text{AdS}_N \) the integrable crosscaps correspond to the residual symmetries \( \text{SO}(2, N - k - 1) \times \text{SO}(k) \) or \( \text{SO}(1, k) \times \text{SO}(1, N - k - 1) \). In the tables 2 and 3 we enumerate these possibilities explicitly for the \( \text{AdS}_5 \) and \( \text{AdS}_4 \).
This equation (5.19) is diagonalize the $SU(\Sigma)$ model on the

| Residual symmetry | $(X_{-1}, X_0, X_1, X_2, X_3, X_4)^\alpha$ | $(x_0, x_1, x_2, x_3, z)^\alpha$ |
|------------------|-----------------------------------------------|----------------------------------|
| SO(2, 4)         | $(X_{-1}, X_0, X_1, X_2, X_3, X_4)$           | $(x_0, x_1, x_2, x_3, z)$        |
| SO(2, 3)         | $(X_{-1}, X_0, -X_1, X_2, X_3, X_4)$          | $(x_0, -x_1, x_2, x_3, z)$       |
| SO(2, 2) $\times$ SO(2) | $(X_{-1}, X_0, -X_1, -X_2, X_3, X_4)$ | $(x_0, -x_1, -x_2, x_3, z)$       |
| SO(2, 1) $\times$ SO(3) | $(X_{-1}, X_0, -X_1, -X_2, -X_3, X_4)$ | $(x_0, -x_1, -x_2, -x_3, z)$       |
| SO(2) $\times$ SO(4) | $(X_{-1}, X_0, -X_1, -X_2, -X_3, -X_4)$ | $(x_0, -x_1, -x_2, -x_3, z)$       |
| SO(1, 4)         | $(X_{-1}, -X_0, X_1, X_2, X_3, X_4)$          | $(-x_0, x_1, x_2, x_3, z)$      |
| SO(1, 1) $\times$ SO(1, 3) | $(X_{-1}, -X_0, -X_1, -X_2, -X_3, -X_4)$ | $(-x_0, -x_1, x_2, x_3, z)$       |
| SO(1, 2) $\times$ SO(1, 2) | $(X_{-1}, -X_0, -X_1, -X_2, -X_3, -X_4)$ | $(-x_0, -x_1, -x_2, -x_3, z)$       |

Table 2. Crosscaps of the sigma model on AdS$_5$.

| Residual symmetry | $(X_{-1}, X_0, X_1, X_2, X_3)^\alpha$ | $(x_0, x_1, x_2, z)^\alpha$ |
|------------------|-----------------------------------------------|----------------------------------|
| SO(2, 3)         | $(X_{-1}, X_0, X_1, X_2, X_3)$               | $(x_0, x_1, x_2, z)$             |
| SO(2, 2)         | $(X_{-1}, X_0, -X_1, X_2, X_3)$             | $(x_0, -x_1, x_2, z)$            |
| SO(2, 1) $\times$ SO(2) | $(X_{-1}, X_0, -X_1, -X_2, X_3)$ | $(x_0, -x_1, -x_2, z)$            |
| SO(2) $\times$ SO(3) | $(X_{-1}, X_0, -X_1, -X_2, -X_3)$ | $(-x_0, -x_1, x_2, z)$            |
| SO(1, 3)         | $(X_{-1}, -X_0, X_1, X_2, X_3)$             | $(-x_0, x_1, x_2, z)$            |
| SO(1, 1) $\times$ SO(1, 2) | $(X_{-1}, -X_0, -X_1, -X_2, -X_3)$ | $(-x_0, -x_1, x_2, z)$            |
| SO(1, 2) $\times$ SO(1, 2) | $(X_{-1}, -X_0, -X_1, -X_2, -X_3)$ | $(-x_0, -x_1, -x_2, z)$            |

Table 3. Crosscaps of the sigma model on AdS$_4$.

**Sigma model on the $\mathbb{CP}^{N-1}$.** For the $\mathbb{CP}^{N-1}$ sigma model the current is an element of $su(N)$. For the $\beta = id$ case we have $\kappa^2 = 1$. Using the global $SU(N)$ symmetry we can diagonalize the $\kappa$-s therefore we have

$$
\kappa = \text{diag}(\underbrace{1, \ldots, 1, -1, \ldots, -1}_{k \over N-k}). \quad (5.44)
$$

This $\kappa$-matrix has residual symmetry $SU(k) \times SU(N - k) \times U(1)$. The solution of the equation (5.19) is

$$
Y_i^\alpha = Y_i, \quad i = 1, \ldots, k; 
$$
$$
Y_i^\alpha = -Y_i, \quad i = k + 1, \ldots, N. \quad (5.46)
$$
In the second case (the charge conjugation case when \( \kappa^t = \pm \kappa \)) we have two types of \( \kappa \)-matrices (up to global SU\( (N) \) rotations)

\[
\kappa = 1, \quad (5.47)
\]

\[
\kappa = \begin{pmatrix}
0 & 1_N \\
-1_N & 0
\end{pmatrix}, \quad (5.48)
\]

The symmetric case has residual symmetry SO\( (N) \) and the anti-symmetric one has Sp\( (N) \).

In the symmetric case the solution of the equation (5.19) is

\[
Y^\alpha_i = Y^*_i, \quad i = 1, \ldots, N, \quad (5.49)
\]

where * denotes the complex conjugation. For the anti-symmetric \( \kappa \) the equation (5.19) has the solution

\[
Y^\alpha_i = -Y^*_i, \quad i = 1, \ldots, \frac{N}{2}, \quad (5.50)
\]

\[
Y^\alpha_{\frac{N}{2}+i} = Y^*_i, \quad i = \frac{N}{2} + 1, \ldots, N. \quad (5.51)
\]

We can see that this is not a consistent crosscap because the square of this transformation is \(-1\).

In summary, for the \( \mathbb{C}P^{N-1} \) sigma model the integrable crosscaps correspond to the residual symmetries SU\( (k) \times \text{SU}(N-k) \times \text{U}(1) \) or SO\( (N) \) and the concrete identifications are given by the equations (5.45)–(5.46) or (5.49). In the table 4 we enumerate these possibilities explicitly for the \( \mathbb{C}P^3 \).

### 6 Crosscaps in the AdS/CFT duality

In this section we apply the results of the previous sections to the classical sigma models which are relevant in the AdS\( _5 \)/CFT\(_4 \) and the AdS\( _4 \)/CFT\(_3 \) dualities. The string theory side we have type IIB superstrings on the AdS\( _5 \times S^5 \) and type IIA superstrings on the AdS\( _4 \times \mathbb{C}P^3 \). The dual field theories are the \( \mathcal{N} = 4 \) SYM and the ABJM theories. The isometries of the AdS and the \( S^5 \) (or \( \mathbb{C}P^3 \)) correspond to the conformal symmetries and
the $R$-symmetries of the field theories. In the following we also use the latter names for the isometries of the target spaces.

In the following, we classify the integrable crosscaps of classical sigma models corresponding to these superstrings, but we also make a few physically reasonable restrictions. We concentrate on the half-BPS crosscaps. Furthermore, we require some breaking of the conformal symmetry since otherwise there would not be non-vanishing one-point functions. On the other hand we want the dilation operator to be unbroken, in other words, the residual symmetry should contain a lower-dimensional conformal symmetry group.

6.1 AdS$_5$/CFT$_4$

In this subsection we classify the integrable crosscaps for type IIB strings on the AdS$_5 \times S^5$.

The classical string can be described as a sigma model on the supercoset [36]

$$\frac{\text{PSU}(2,2|4)}{\text{SO}(1,4) \times \text{SO}(5)}$$  \hspace{1cm} (6.1)

We can define the current in the usual way $J = g^{-1}dg$ where $g(\tau, \sigma) \in \text{PSU}(2,2|4)$. The superalgebra psu(2,2|4) has a $\mathbb{Z}_4$ automorphism for which the current decomposes as $J = J(0) + J(1) + J(2) + J(3)$. We can define the fixed frame currents as $j^{(m)} = g J^{(m)} g^{-1}$. We can see that this condition breaks the global PSU(2,2|4) symmetry, the residual symmetry is defined by

$$\bar{g}^{-1}k\bar{g} = k.$$ \hspace{1cm} (6.6)

We saw that, the consistency of the crosscap requires that $k^2 = 1$ therefore the possible $k$-s (up to global rotations) are

$$k = \text{diag}(1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1).$$ \hspace{1cm} (6.7)
Let us forget about the signature for a moment. The \( \kappa \)-matrix breaks the global symmetry \( \mathfrak{gl}(4|4) \) to \( \mathfrak{gl}(k|l) \oplus \mathfrak{gl}(4-k|4-l) \) (up to \( u(1) \) factors). Let us concentrate on the 1/2 BPS configurations. We have three possibilities

\[
\mathfrak{gl}(2|2) \oplus \mathfrak{gl}(2|2), \quad \mathfrak{gl}(2|4) \oplus \mathfrak{gl}(2), \quad \mathfrak{gl}(4|2) \oplus \mathfrak{gl}(2), \quad \mathfrak{gl}(3|2) \oplus \mathfrak{gl}(1|2) \quad (6.8)
\]

The last possibility contains bosonic subalgebra \( u(3) \) but we already show that this symmetry has no consistent crosscap for the sigma models neither on the \( S^5 \) or the \( \text{AdS}_5 \). The \( \mathfrak{gl}(4|2) \oplus \mathfrak{gl}(2) \) case preserves the full conformal symmetry \( \text{SO}(2,4) \) which means there cannot be non-vanishing one-point function in the CFT side therefore we neglect this case. We can see that only the first two possibilities remain.

- For the \( \mathfrak{gl}(2|2) \oplus \mathfrak{gl}(2|2) \) case the conformal symmetry breaks as \( \text{SO}(2,4) \to \text{SO}(2,2) \times \text{SO}(2) \) or \( \text{SO}(2) \times \text{SO}(4) \) and we already derived which are the corresponding crosscap conditions on the Poincaré patch, see table 2. The \( R \)-symmetry breaks as \( \text{SO}(6) \to \text{SO}(2) \times \text{SO}(4) \) and we already derived which are the corresponding crosscap conditions on the sphere, see table 1.

- For the \( \mathfrak{gl}(2|4) \oplus \mathfrak{gl}(2) \) case the conformal symmetry breaks as \( \text{SO}(2,4) \to \text{SO}(2,2) \times \text{SO}(2) \) or \( \text{SO}(2) \times \text{SO}(4) \) and the \( R \)-symmetry is unbroken.

Let us continue with the crosscap conditions (5.2) with the transformation \( A^\beta(\lambda) = -K^{-1}A(\lambda)^{st}K \) where

\[
K = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1_2 \\
0 & -1_2 & 0
\end{pmatrix},
\]

and the super-transposition \( ^{st} \) acts in the usual way

\[
\begin{pmatrix}
m \\
\theta \\
\eta \\
n
\end{pmatrix}^{st} = \begin{pmatrix}
m^t & -\eta^t \\
\theta^t & n^t
\end{pmatrix},
\]

(6.10)

where \( m, n \) and \( \theta, \eta \) are \( 4 \times 4 \) bosonic and fermionic matrices. Using these definitions we can show that the \( \beta \) is a \( \mathbb{Z}_2 \) automorphism of \( \mathfrak{gl}(4|4) \). The crosscap conditions read as

\[
A_\sigma(\lambda) \bigg|_{\tau=0,\sigma=\sigma_0} = -\kappa^{-1}K^{-1}A_\sigma(-\lambda)^{st}K\kappa \bigg|_{\tau=0,\sigma=\sigma_0+\frac{\pi}{2}}.
\]

(6.11)

Applying this transformation twice we obtain that

\[
A_\sigma(\lambda) = \kappa^{-1}K^{-1}(\kappa^{-1}K^{-1}A_\sigma(\lambda)^{st}K\kappa)^{st}K\kappa.
\]

(6.12)

For bosonic \( \kappa \)-matrix, we obtain that

\[
A_\sigma(\lambda) = \kappa^{-1}K^{-1}(K K^{-1}(K^{-1}A_\sigma(\lambda)^{st}K) K^{-1})(K^{-1})^t K\kappa
\]

\[
= (\kappa^{-1}K^{-1}K^{-1}K^{-1}A_\sigma(\lambda))^t (K^{-1}(K^{-1})^t K\kappa),
\]

(6.13)
where we used the identity

$$K^{-1}(K^{-1}A_\sigma(\lambda)^{st}K)^{st}K = A_\sigma(\lambda).$$

(6.14)

For a consistent crosscap we obtained the following constraint for the $\kappa$-matrix

$$\kappa = \pm K^{-1}K.\quad (6.15)$$

We have two types of $\kappa$-matrices (up to global rotations). For the $+$ sign the $\kappa$-matrix reads as

$$\kappa = \mathbf{1}_8,\quad (6.16)$$

and for the $-$ sign the $\kappa$-matrix is

$$\kappa = \begin{pmatrix} 0 & 1_2 & 0 & 0 \\ -1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 1_2 & 0 \end{pmatrix}.\quad (6.17)$$

Applying the transformation (6.3), we obtain that

$$A_\sigma(\lambda)\bigg|_{r=0,\sigma=\sigma_0} = g^{-1}K^{-1}\bar{g}^{-1}\left(g^{-1}\right)^{st}A_\sigma(-\lambda)^{st}\bar{g}^{st}K\kappa\bar{g}\bigg|_{r=0,\sigma=\sigma_0+\frac{L}{2}}.\quad (6.18)$$

We can see that this condition breaks the global $\text{PSU}(2|2)\uparrow\text{PSU}(2|2)\downarrow$ symmetry and the residual symmetry is defined by

$$g^{st}K\kappa\bar{g} = K\kappa.\quad (6.19)$$

We already saw that, for a consistent crosscap we have two types of $\kappa$-matrices. For the symmetric $\kappa$-matrix (6.16) we have

$$K\kappa = K = \begin{pmatrix} 1_4 & 0 & 0 \\ 0 & 0 & 1_2 \\ 0 & -1_2 & 0 \end{pmatrix}.$$\quad (6.20)

for which the residual symmetry is $\mathfrak{osp}(4|4)$. For the anti-symmetric $\kappa$-matrix (6.16) we have

$$K\kappa = \begin{pmatrix} 0 & 1_2 & 0 \\ -1_2 & 0 & 0 \\ 0 & 0 & 1_4 \end{pmatrix}.$$\quad (6.21)

for which the residual symmetry is also $\mathfrak{osp}(4|4)$ but now the first $4 \times 4$ bosonic block breaks to $\mathfrak{sp}(4)$ and second breaks to $\mathfrak{so}(4)$. For the sake of clarity, we denote this residual symmetry as $\mathfrak{spo}(4|4)$.

We can see that these are 1/2 BPS crosscaps. In summary we have two possibilities:

- For the $\mathfrak{osp}(4|4)$ residual symmetry, the conformal symmetry breaks as $\text{SO}(2, 4) \to \text{SO}(2, 1) \times \text{SO}(3)$ and $R$-symmetry breaks as $\text{SO}(6) \to \text{SO}(5)$.

- For the $\mathfrak{spo}(4|4)$ residual symmetry, the conformal symmetry breaks as $\text{SO}(2, 4) \to \text{SO}(2, 3)$ and $R$-symmetry breaks as $\text{SO}(6) \to \text{SO}(3) \times \text{SO}(3)$.

In the table 5 we summarized the crosscaps for the sigma model on the coset (6.1). This table shows the residual symmetries of the possible crosscaps and the concrete realizations (which is given by the $\mathbb{Z}_2$ automorphism) on the $\text{AdS}_5$ and the $\text{S}^5$ are given by the tables 2 and 1.
Residual symmetry | Residual isometries on AdS$_5$ | Residual isometries on $S^5$
---|---|---
$\mathfrak{gl}(2|2) \oplus \mathfrak{gl}(2|2)$ | $\text{SO}(2, 2) \times \text{SO}(2)$ | $\text{SO}(4) \times \text{SO}(2)$
$\mathfrak{gl}(2|4) \oplus \mathfrak{gl}(2)$ | $\text{SO}(2, 2) \times \text{SO}(2)$ | $\text{SO}(6)$
$\mathfrak{osp}(4|4)$ | $\text{SO}(2, 1) \times \text{SO}(3)$ | $\text{SO}(5)$

Table 5. Possible crosscaps in the AdS$_5$/CFT$_4$.

During the analysis of this subsection we concentrated on the embeddings of complex Lie-superalgebras. In the appendix A we show the explicit real forms of the subalgebras which appeared in this subsection.

6.2 AdS$_4$/CFT$_3$

In this subsection we classify the integrable crosscaps for type IIA superstrings on AdS$_4 \times \mathbb{CP}^3$. The classical string can be described as a sigma model on the supercoset $[37]$

$$\frac{\mathfrak{osp}(6|4)}{\text{SO}(1, 3) \times U(3)}.\quad (6.22)$$

We can define the current in the usual way $J = g^{-1}dg$ where $g(\tau, \sigma) \in \mathfrak{osp}(6|4)$. We can also define the fixed frame currents and Lax connect operators in the same way as before (6.2). We saw that the consistency of the crosscap requires that $\kappa^2 = 1$ and now the current has the symmetry $J = -VJ^{st}V^{-1}$ where

$$V = \begin{pmatrix} 1_6 & 0 & 0 \\ 0 & 0 & 1_2 \\ 0 & -1_2 & 0 \end{pmatrix},\quad (6.23)$$

therefore the consistency also requires that $\kappa' = \pm V \kappa V^{-1}$ (the $\kappa$-matrix is bosonic). The theory has global $\mathfrak{osp}(6|4)$ symmetry which act on the group element as $g \rightarrow \tilde{g}g$ where $\tilde{g} \in \mathfrak{osp}(6|4)$ is a constant group element. Repeating the argument of the previous subsection we obtain that the residual symmetry is defined by

$$\tilde{g}^{-1} \kappa \tilde{g} = \kappa.\quad (6.24)$$

The type $\kappa' = +V \kappa V^{-1}$ matrices can be written (up to global rotations) as

$$\kappa = \text{diag}(1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, 1).\quad (6.25)$$

This $\kappa$-matrix breaks the global symmetry $\mathfrak{osp}(6|4)$ to $\mathfrak{osp}(k|l) \oplus \mathfrak{osp}(6|4 - l)$. Let us concentrate on the 1/2 BPS configurations. We have five possibilities

$$\mathfrak{osp}(3|2) \oplus \mathfrak{osp}(3|2), \quad \mathfrak{osp}(6|2) \oplus \mathfrak{sp}(2), \quad \mathfrak{osp}(3|4) \oplus \mathfrak{so}(3),$$

$$\mathfrak{osp}(4|2) \oplus \mathfrak{osp}(2|2), \quad \mathfrak{osp}(5|2) \oplus \mathfrak{osp}(1|2).\quad (6.26)$$
Residual symmetry | Residual isometries of AdS\(_4\) | Residual isometries of CP\(^3\)
--- | --- | ---
osp(3\(|2\)) \(\oplus\) osp(3\(|2\)) | SO(2, 2) | SO(4)
osp(6\(|2\)) \(\oplus\) sp(2) | SO(2, 2) | SU(4)
osp(4\(|2\)) \(\oplus\) osp(2\(|2\)) | SO(2, 2) | SU(2) \(\times\) SU(2) \(\times\) U(1)
gl(3\(|2\)) | SO(2, 1) \(\times\) SO(2) | SU(3) \(\times\) U(1)

**Table 6.** Possible crosscaps in the AdS\(_4\)/CFT\(_3\).  

In the last possibility the \(R\)-symmetry breaks as su(4) \(\rightarrow\) so(5) \(\cong\) sp(4) but we already showed that this symmetry has no consistent crosscap for the sigma model on CP\(^3\). The osp(3\(|4\)) \(\oplus\) so(3) case preserves the full conformal symmetry SO(2, 3) which means there cannot be non-vanishing one-point function in the CFT side therefore we neglect this case. We can see that three possibilities remain.

- For the osp(3\(|2\)) \(\oplus\) osp(3\(|2\)) case the conformal symmetry breaks as SO(2, 3) \(\rightarrow\) SO(2, 2) and we already derived which are the corresponding crosscap conditions on the Poincaré patch, see table 3. The \(R\)-symmetry breaks as SU(4) \(\rightarrow\) SO(4) and we already derived which are the corresponding crosscap conditions on the CP\(^3\), see table 4.
- For the osp(6\(|2\)) \(\oplus\) sp(2) case the conformal symmetry breaks as SO(2, 3) \(\rightarrow\) SO(2, 2) and the \(R\)-symmetry is preserved.
- For the osp(4\(|2\)) \(\oplus\) osp(2\(|2\)) case the conformal symmetry breaks as SO(2, 3) \(\rightarrow\) SO(2, 2) and the \(R\)-symmetry breaks as SU(4) \(\rightarrow\) SU(2) \(\times\) SU(2) \(\times\) U(1).

The type \(\kappa' = -V\kappa V^{-1}\) matrices can be written (up to global rotations) as

\[
\kappa = \begin{pmatrix}
0 & \mathbf{1}_3 & 0 & 0 \\
-\mathbf{1}_3 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1}_2 & 0 \\
0 & 0 & 0 & -\mathbf{1}_2
\end{pmatrix}.
\]

(6.27)

This \(\kappa\)-matrix breaks the global symmetry osp(6\(|4\)) to gl(3\(|2\)) which is 1/2 BPS. The conformal symmetry breaks as SO(2, 3) \(\rightarrow\) SO(2, 1) \(\times\) SO(2) and the \(R\)-symmetry breaks as SU(4) \(\rightarrow\) SU(3) \(\times\) U(1).

In the table 6 we summarized the crosscaps for the sigma model on the coset (6.22). This table shows the residual symmetries of the possible crosscaps and the concrete realizations (which is given by the \(Z_2\) automorphism) on the AdS\(_4\) and the CP\(^3\) are given by the tables 3 and 4.

### 6.3 Crosscaps at weak coupling

In the literature there is another classification of crosscaps which is relevant in the field theory side [18]. This paper contains the classifications of crosscap states of SO(6) and the alternating SU(4) spin chains which describes the \(\mathcal{N} = 4\) SYM and ABJM theories at weak
coupling. In this paper we used such parametrization of the bosonic spaces which can be directly match to the CFT side of the duality. The $S^5$ and $\mathbb{CP}^3$ are parameterized with the coordinates $(\phi_1, \phi_2, \ldots, \phi_6)$ and $(Y_1, Y_2, Y_3, Y_4)$ which corresponds to the scalar field of the $\mathcal{N} = 4$ SYM and ABJM theories. The corresponding $\mathbb{Z}_2$ isometries can be found in the tables 1 and 4. We can compare the proposed crosscaps at strong coupling (result of this paper) and at weak coupling (result of [18]) and we find that they are almost the same. There are two differences. There is no $U(3)$ symmetric crosscap for the $S^5$ sigma model and there is no $Sp(4)$ symmetric crosscap for the $\mathbb{CP}^3$ sigma model.

The spin chain classifications belong only to the scalar sectors and they tell nothing about the effect of the crosscap on the spacetime. The main advantage of the analysis of this paper is the following. The classification of the crosscaps of the sigma models on the supercosets tells us what are the consistent combinations of the crosscaps on the $AdS_5$ (or $AdS_4$) and the $S^5$ (or $\mathbb{CP}^3$), see tables 5 and 6. Therefore we obtained candidates for the crosscaps on the CFT side of the duality. In the tables 2 and 3 we can find the possible crosscaps of $AdS_5$ and $AdS_4$ which are parameterized with the Poincaré coordinates $(z, x_0, x_1, \ldots)$ therefore we can obtain the identifications of the spacetime of the $\mathcal{N} = 4$ SYM and ABJM by taking the limit $z = 0$.

In summary we obtained propositions for the possible integrable crosscaps of $\mathcal{N} = 4$ SYM and ABJM theories. The residual symmetries are listed in the tables 5 and 6. Each symmetry classes define identifications on the spacetime and the scalar fields. For each symmetry classes the corresponding identifications of the spacetime and scalar fields can be found in the tables 1–4 (take the $z = 0$ limit). We emphasize that this classification is a conjecture for the $\mathcal{N} = 4$ SYM and ABJM theories and this paper is not intended to provide a concrete field theory description (if that is even possible).

7 Conclusion

In this paper we generalized the Lax description of the sigma models with boundaries in time. For the usual local boundary conditions we obtained constraints for the classical monodromy and transfer matrices and these are the classical analogs of the $KT$-relations [15] and the integrability conditions [13] of the quantum theories. We also generalized this framework for the crosscaps where the classical analogs of the $KT$-relations [18] and the integrability conditions have also appeared.

Based on this framework, we classified the integrable crosscaps for the sigma models with target spaces $SU(N)$, $S^{N-1}$, $AdS_{N-1}$ and $\mathbb{CP}^{N-1}$. The classification is based on the possible residual symmetries of the crosscaps. We gave the defining isometries of the crosscaps for each residual symmetry classes. We also investigated the supercosets which are relevant in the $AdS_5$/CFT$_4$ and the $AdS_4$/CFT$_3$ dualities. We classified the integrable 1/2 BPS crosscaps based on the residual symmetries (see tables 5 and 6). The corresponding defining isometries of the bosonic subspaces can be found in the tables 1–4. It is important to emphasize that this classification only applies to classical sigma models, and at this point we do not know that which versions have consistent holographic descriptions. This is an interesting question for a future research.
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A Real forms

In this appendix we specialize the real forms of the subalgebras appearing in table 5. In this appendix we concentrate only on the AdS$_5$/CFT$_4$ duality and the real forms of the subalgebras appearing in the AdS$_4$/CFT$_3$ duality (table 6) can be obtain in a similar way.

Let us start with the definitions of the supermatrices. The $\mathfrak{gl}(4|4)$ supermatrices have the following block form

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix},$$

(A.1)

where $m, \theta, \eta, n$ are $4 \times 4$ matrices and the matrices $m, n$ have bosonic and $\theta, \eta$ have fermionic entries. We can define the real form $u(2,2|4)$ with the following constraint

$$M^\dagger H + HM = 0,$$

(A.2)

where $M^\dagger = (M^*)^st$, $^*$ is the complex conjugation and

$$M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}, \quad H = \begin{pmatrix} \Sigma & 0 \\ 0 & i4 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}.$$ 

(A.3)

We can parameterize the $M \in u(2,2|4)$ supermatrices as

$$M = \begin{pmatrix} m & \theta \\ -i\theta^\dagger \Sigma & n \end{pmatrix},$$

(A.4)

where the bosonic matrices satisfy the relations: $m^\dagger \Sigma + \Sigma m = 0$, $n^\dagger + n = 0$, i.e., $m \in u(2,2)$ and $n \in u(4)$. We obtain the super Lie-subalgebra $\mathfrak{su}(2,2|4)$ with the constraint $\text{str}M = \text{tr}m - \text{tr}n$. The bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus u(1).$$

(A.5)

The $u(1)$ is generated by the element $X = i1_8$ and let us denote the generated algebra $\mathfrak{h}_0 = \text{span}_R X \cong u(1)$ which commutes with the full $\mathfrak{su}(2,2|4)$. The quotient algebra is the $\mathfrak{psu}(2,2|4)$ i.e.

$$\mathfrak{psu}(2,2|4) = \mathfrak{su}(2,2|4) \backslash \mathfrak{h}_0.$$ 

(A.6)
For the sake of transparency, let us summarize the bosonic matrices which are used afterwards.

\[ \Omega = \begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\[ K = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \]

\[ \Sigma = \begin{pmatrix} B \\ 0 \\ 0 \\ 1 \\ 4 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 2 \\ i1_2 \\ i1_2 \\ 1_2 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}. \]

(A.7)

### A.1 Untwisted crosscap

In the following we analyze the subalgebras which appear for the untwisted crosscaps. We saw the residual symmetry is defined by the equation (6.6)

\[ \bar{g}^{-1} \kappa \bar{g} = \kappa, \]

where \( \kappa^2 = 1_8 \) and \( \bar{g} \in \text{PSU}(2, 2, | 4) \). On the algebraic level this condition is equivalent to

\[ [M, \kappa] = 0, \]

where \( M \in \text{psu}(2, 2|4) \). In the following we specify the subalgebras which satisfy the relation (A.9) for various \( \kappa \) matrices.

**Example 1.** Let us define the matrix

\[ \kappa = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}. \]

(A.10)

Assuming \( M \in \text{u}(2, 2|4) \) we can substitute (A.4) to (A.9) and we obtain that

\[ \begin{pmatrix} m & \theta \\ -i\theta^\dagger \Sigma n \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} m & \theta \\ -i\theta^\dagger \Sigma n \end{pmatrix}, \]

therefore we have the following conditions

\[ [m, \Sigma] = [n, \Sigma] = [\theta, \Sigma] = 0. \]

(A.12)

Solving these equations we obtain that the matrices \( m, n, \theta \) have the following block diagonal forms:

\[ m = \begin{pmatrix} m^{(1)} \\ 0 \\ m^{(2)} \end{pmatrix}, \quad n = \begin{pmatrix} n^{(1)} \\ 0 \\ n^{(2)} \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta^{(1)} \\ 0 \\ \theta^{(2)} \end{pmatrix}. \]

(A.13)
where \(m^{(j)}, n^{(j)} \in u(2)\) for \(j = 1, 2\). The full supermatrix reads as

\[
M = \begin{pmatrix}
m^{(1)} & 0 & \theta^{(1)} & 0 \\
0 & m^{(2)} & 0 & \theta^{(2)} \\
-i\theta^{(1)\dagger} & 0 & n^{(1)} & 0 \\
0 & i\theta^{(2)\dagger} & 0 & n^{(2)}
\end{pmatrix} \cong M^{(1)} \oplus M^{(2)}, \tag{A.14}
\]

where

\[
M^{(1)} = \begin{pmatrix}
m^{(1)} & \theta^{(1)} \\
-i\theta^{(1)\dagger} & n^{(1)}
\end{pmatrix}, \quad M^{(2)} = \begin{pmatrix}
m^{(2)} & \theta^{(2)} \\
i\theta^{(2)\dagger} & n^{(2)}
\end{pmatrix}, \tag{A.15}
\]

therefore \(M_1, M_2 \in u(2|2)\). We can see that the relation (A.9) defines the embedding

\[
u(2|2) \oplus u(2|2) \subset u(2, 2|4) \quad (A.16)
\]

Let us specialize the embedding to \(su(2, 2|4)\). The condition \(\text{str}M = 0\) equivalent to

\[
\text{str}M^{(1)} = -\text{str}M^{(2)} \quad (A.17)
\]

Let \(s(u(2|2) \oplus u(2|2))\) be the subalgebra of \(u(2|2) \oplus u(2|2)\) for which (A.17) is satisfied. Using this notation we get the following embedding

\[
s(u(2|2) \oplus u(2|2)) \subset su(2, 2|4). \quad (A.18)
\]

Since the identity element \(X = i1_8\) obviously satisfies the relation (A.9), \(X \in s(u(2|2) \oplus u(2|2))\) therefore the residual algebra is the following quotient

\[
s(u(2|2) \oplus u(2|2)) \backslash h_0 \subset psu(2, 2|4). \quad (A.19)
\]

The bosonic subalgebra is

\[
su(2) \oplus su(2) \oplus su(2) \oplus su(2) \oplus u(1) \oplus u(1) = so(4) \oplus so(2) \oplus so(4) \oplus so(2) \subset so(2, 4) \oplus so(6), \quad (A.20)
\]

which agrees with the second row of table 5. The semi-simple part of the bosonic algebra is

\[
\begin{pmatrix}
m_1 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 \\
0 & 0 & n_1 & 0 \\
0 & 0 & 0 & n_2
\end{pmatrix}, \quad (A.21)
\]

where \(m_1, m_2, n_1, n_2 \in su(2)\). The \(u(1)\)-s are generated by

\[
Y^L = i\begin{pmatrix}
\Sigma & 0 \\
0 & 0
\end{pmatrix}, \quad Y^R = i\begin{pmatrix}
0 & 0 \\
0 & \Sigma
\end{pmatrix}. \quad (A.22)
\]
Example 2. Let us continue with the $\kappa$-matrix
\[
\kappa = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix}.
\] (A.23)
Substituting to (A.9) we obtain the following conditions
\[
[m, \Omega] = [n, \Omega] = [\theta, \Omega] = 0,
\] (A.24)
therefore the matrices $m, n, \theta$ have the forms:
\[
m = \begin{pmatrix} m_{1,1}^{(1)} & 0 & m_{1,2}^{(1)} \\ 0 & m_{1,1}^{(2)} & m_{1,2}^{(2)} \\ m_{1,2}^{(1)*} & 0 & m_{1,2}^{(2)*} \end{pmatrix}, \quad n = \begin{pmatrix} n_{1,1}^{(1)} & 0 & n_{1,2}^{(1)} \\ 0 & n_{1,1}^{(2)} & n_{1,2}^{(2)} \\ -n_{1,2}^{(1)*} & 0 & n_{1,2}^{(2)*} \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_{1,1}^{(1)} & 0 & \theta_{1,2}^{(1)} \\ 0 & \theta_{1,1}^{(2)} & \theta_{1,2}^{(2)} \\ \theta_{2,1}^{(1)} & 0 & \theta_{2,2}^{(1)} \end{pmatrix},
\] (A.25)
where $m_{j,j}^{(j)}, n_{j,j}^{(j)} \in i\mathbb{R}$, $m_{1,2}^{(j)}, n_{1,2}^{(j)} \in \mathbb{C}$ and $\theta_{j,k}^{(j)}$ are Grassmann numbers. The full supermatrix reads as
\[
M = \begin{pmatrix} m_{1,1}^{(1)} & 0 & m_{1,2}^{(1)} & \theta_{1,1}^{(1)} & 0 & \theta_{1,2}^{(1)} \\ 0 & m_{1,1}^{(2)} & m_{1,2}^{(2)} & \theta_{1,1}^{(2)} & 0 & \theta_{1,2}^{(2)} \\ m_{1,2}^{(1)*} & 0 & m_{1,2}^{(2)*} & \theta_{2,1}^{(1)} & 0 & \theta_{2,2}^{(1)} \\ 0 & m_{1,2}^{(1)*} & 0 & \theta_{2,1}^{(2)} & 0 & \theta_{2,2}^{(2)} \\ -i\theta_{1,1}^{(1)*} & 0 & i\theta_{2,1}^{(2)*} & 0 & n_{1,1}^{(1)} & 0 & n_{1,2}^{(1)} \\ 0 & -i\theta_{1,1}^{(2)*} & 0 & i\theta_{2,1}^{(1)*} & 0 & n_{1,1}^{(2)} & 0 & n_{1,2}^{(2)} \\ -i\theta_{2,1}^{(1)*} & 0 & i\theta_{2,2}^{(1)*} & 0 & -n_{1,2}^{(1)*} & 0 & n_{1,2}^{(2)} \\ 0 & -i\theta_{2,1}^{(2)*} & 0 & i\theta_{2,2}^{(2)*} & 0 & -n_{1,2}^{(1)*} & 0 & n_{1,2}^{(2)} \end{pmatrix} \cong M^{(1)} \oplus M^{(2)},
\] (A.26)
where
\[
M^{(j)} = \begin{pmatrix} m_{j}^{(j)} & \theta^{(j)} \\ -i\theta^{(j)*}\sigma^2 & n^{(j)} \end{pmatrix}, \quad n^{(j)} = \begin{pmatrix} n_{1,1}^{(j)} & n_{1,2}^{(j)} \\ -n_{1,2}^{(j)*} & n_{1,2}^{(j)*} \end{pmatrix}, \quad \theta^{(j)} = \begin{pmatrix} \theta_{1,1}^{(j)} & \theta_{1,2}^{(j)} \\ \theta_{2,1}^{(j)} & \theta_{2,2}^{(j)} \end{pmatrix},
\] (A.27)
therefore $M^{(1)}, M^{(2)} \in \mathfrak{u}(1,1|2)$. We can see that the relation (A.9) defines the embedding
\[
\mathfrak{u}(1,1|2) \oplus \mathfrak{u}(1,1|2) \subset \mathfrak{u}(2,2|4).
\] (A.29)
Repeating the previous argument we can specialize the embedding to $\mathfrak{psu}(2,2|4)$. The residual algebra is the following quotient
\[
\mathfrak{s}(\mathfrak{u}(1,1|2) \oplus \mathfrak{u}(1,1|2)) \backslash \mathfrak{h}_0 \subset \mathfrak{psu}(2,2|4).
\] (A.30)
The bosonic subalgebra is
\[
\mathfrak{su}(1,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(1,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) = \mathfrak{so}(2,2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2) \subset \mathfrak{so}(2,4) \oplus \mathfrak{so}(6),
\] (A.31)
which agrees with the first row of table 5. The semi-simple part of the bosonic algebra is

\[
\begin{pmatrix}
  \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
 0 & 0 & 0 & n_{11} & n_{12} & 0 \\
 0 & 0 & 0 & 0 & n_{21} & n_{22} \\
 0 & 0 & 0 & n_{11}^* & n_{12}^* & 0 \\
 0 & 0 & 0 & n_{21}^* & n_{22}^* & 0 \\
 0 & 0 & 0 & n_{11} & n_{12} & 0 \\
 0 & 0 & 0 & n_{21} & n_{22} & 0 \\
 0 & 0 & 0 & n_{11}^* & n_{12}^* & 0 \\
 0 & 0 & 0 & n_{21}^* & n_{22}^* & 0
\end{pmatrix}
\]

(A.32)

where \( m_{j,k}, n_{j,k} \) form the subalgebras \( \mathfrak{su}(1,1), \mathfrak{su}(1,1) \) and \( \mathfrak{su}(2), \mathfrak{su}(2) \), respectively. The \( \mathfrak{u}(1) \)-s are generated by

\[
Y^L = i\begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}, \quad Y^R = i\begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix}.
\]

(A.33)

**Example 3.** Let us define the matrix

\[
\kappa = \begin{pmatrix} \Sigma & 0 \\ 0 & 1_4 \end{pmatrix}.
\]

(A.34)

Substituting to (A.9) we obtain the following conditions

\[
[m, \Sigma] = \theta - \Sigma \theta = 0,
\]

(A.35)

therefore the matrices \( m, \theta \) read as:

\[
m = \begin{pmatrix} m_{1} & 0 \\ 0 & m_{2} \end{pmatrix}, \quad n = \begin{pmatrix} n_{11} & n_{12} \\ -n_{12}^\dagger & n_{22} \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_{1} & \theta_{2} \\ 0 & 0 \end{pmatrix},
\]

(A.36)

where \( m_{j}, n_{j} \in \mathfrak{u}(2), n_{1,2} \in \mathfrak{gl}(2) \) and \( \theta_{j} \) are \( 2 \times 2 \) fermionic matrices for \( j = 1, 2 \). The full supermatrix reads as

\[
M = \begin{pmatrix}
  \begin{pmatrix} m_{1} & 0 & \theta_{1} & \theta_{2} \\ 0 & m_{2} & 0 & 0 \end{pmatrix} \\
  -i\theta_{1}^\dagger & 0 & n_{11} & n_{12}^\dagger \\
  -i\theta_{2}^\dagger & 0 & -n_{12} & n_{22}^\dagger
\end{pmatrix}
\cong m_{2} \oplus M_{1},
\]

(A.37)

where

\[
M_{1} = \begin{pmatrix}
  \begin{pmatrix} m_{1} & \theta_{1} & \theta_{2} \\ -i\theta_{1}^\dagger & n_{11} & n_{12}^\dagger \\
  -i\theta_{2}^\dagger & -n_{12} & n_{22}^\dagger
\end{pmatrix}
\end{pmatrix}
\]

(A.38)

therefore \( M_{1} \in \mathfrak{u}(2|4) \). We can see that the relation (A.9) defines the embedding

\[
\mathfrak{u}(2) \oplus \mathfrak{u}(2|4) \subset \mathfrak{u}(2,2|4).
\]

(A.39)
where \( m \) which agrees with the fourth row of table 5. The semi-simple part of the bosonic algebra is
\[
\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) = \mathfrak{so}(2) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(6) \subset \mathfrak{so}(2,4) \oplus \mathfrak{so}(6),
\]
which agrees with the fourth row of table 5. The semi-simple part of the bosonic algebra is
\[
\begin{pmatrix}
  m^{(1)} & 0 & 0 \\
  0 & m^{(2)} & 0 \\
  0 & 0 & n
\end{pmatrix},
\]
(A.42)
where \( m^{(1)}, m^{(2)} \in \mathfrak{su}(2) \) and \( n \in \mathfrak{su}(4) \). The \( u(1) \) is generated by
\[
Y^L = i \begin{pmatrix}
  \Sigma & 0 \\
  0 & 0
\end{pmatrix},
\]
(A.43)
which acts non-trivially only on the AdS_5.

**Example 4.** Let us define the matrix
\[
\kappa = \begin{pmatrix}
  \Omega & 0 \\
  0 & 1_4
\end{pmatrix}.
\]
(A.44)
Substituting to (A.9) we obtain the following conditions
\[
[m, \Omega] = \theta - \Omega \theta = 0,
\]
(A.45)
therefore the matrices \( m, n, \theta \) simplify as:
\[
m = \begin{pmatrix}
  m^{(1)}_{1,1} & 0 & m^{(1)}_{1,2} & 0 \\
  0 & m^{(2)}_{1,1} & 0 & m^{(2)}_{1,2} \\
  m^{(1)\ast}_{1,2} & 0 & m^{(1)\ast}_{2,2} & 0 \\
  0 & m^{(2)\ast}_{1,2} & 0 & m^{(2)\ast}_{2,2}
\end{pmatrix}, \quad n = \begin{pmatrix}
  n_{1,1} & n_{1,2} & n_{1,3} & n_{1,4} \\
  -n_{1,2}^\ast & n_{2,2} & n_{2,3} & n_{2,4} \\
  -n_{1,3}^\ast & -n_{2,3}^\ast & n_{3,3} & n_{3,4} \\
  -n_{1,4}^\ast & -n_{2,4}^\ast & -n_{3,4}^\ast & n_{4,4}
\end{pmatrix}, \quad \theta = \begin{pmatrix}
  \theta_{1,1} & \theta_{1,2} & \theta_{1,3} & \theta_{1,4} \\
  0 & 0 & 0 & 0 \\
  \theta_{3,1} & \theta_{3,2} & \theta_{3,3} & \theta_{3,4} \\
  0 & 0 & 0 & 0
\end{pmatrix},
\]
(A.46)
where \( m^{(k)}_{j,j}, n^{(k)}_{j,j} \in i \mathbb{R} \), \( m^{(k)}_{1,2}, n^{(k)}_{1,2} \in \mathbb{C} \) for \( j < n \) and \( \theta_{j,\ell} \) are Grassmann numbers. The full supermatrix reads as
\[
M = \begin{pmatrix}
  m^{(1)}_{1,1} & 0 & m^{(1)}_{1,2} & 0 & \theta_{1,1} & \theta_{1,2} & \theta_{1,3} & \theta_{1,4} \\
  0 & m^{(2)}_{1,1} & 0 & m^{(2)}_{1,2} & 0 & 0 & 0 & 0 \\
  m^{(1)\ast}_{1,2} & 0 & m^{(1)\ast}_{2,2} & 0 & \theta_{3,1} & \theta_{3,2} & \theta_{3,3} & \theta_{3,4} \\
  0 & m^{(2)\ast}_{1,2} & 0 & m^{(2)\ast}_{2,2} & 0 & 0 & 0 & 0 \\
  -i\theta_{1,1}^\ast & 0 & i\theta_{3,1}^\ast & 0 & n_{1,1} & n_{1,2} & n_{1,3} & n_{1,4} \\
  -i\theta_{1,2}^\ast & 0 & i\theta_{3,2}^\ast & 0 & -n_{1,2}^\ast & n_{2,2} & n_{2,3} & n_{2,4} \\
  -i\theta_{1,3}^\ast & 0 & i\theta_{3,3}^\ast & 0 & -n_{1,3}^\ast & -n_{2,3}^\ast & n_{3,3} & n_{3,4} \\
  -i\theta_{1,4}^\ast & 0 & i\theta_{3,4}^\ast & 0 & -n_{1,4}^\ast & -n_{2,4}^\ast & -n_{3,4}^\ast & n_{4,4}
\end{pmatrix} \cong m^{(2)} \oplus M^{(1)}, \quad (A.47)
\]
where
\[ M^{(1)} = \begin{pmatrix} m^{(1)} & \theta^{(1)} \\ -i \theta^{(1)} & \sigma_n \end{pmatrix}, \quad m^{(i)} = \begin{pmatrix} m^{(i)}_{1,1} & m^{(i)}_{1,2} \\ m^{(i)*}_{1,2} & m^{(i)}_{2,2} \end{pmatrix}, \quad \theta^{(1)} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} & \theta_{1,3} & \theta_{1,4} \\ \theta_{3,1} & \theta_{3,2} & \theta_{3,3} & \theta_{3,4} \end{pmatrix}. \]
(A.48)

therefore \( m^{(2)} \in u(1,1), \ M^{(1)} \in u(1,1|4). \) We can see that the relation (A.9) defines the embedding
\[ u(1,1) \oplus u(1,1|4) \subset u(2,2|4). \] (A.49)

Repeating the previous calculations we can obtain the following embedding
\[ s(u(1,1) \oplus u(1,1|4)) \backslash h_0 \subset psu(2,2|4). \] (A.50)

The bosonic subalgebra is
\[ su(1,1) \oplus su(1,1) \oplus su(4) \oplus u(1) = so(2,2) \oplus so(2) \oplus so(6) \subset so(2,4) \oplus so(6), \] (A.51)

which agrees with the third row of table 5. The semi-simple part of the bosonic algebra is
\[ \begin{pmatrix} m^{(1)}_{1,1} & 0 & m^{(1)}_{1,2} & 0 & 0 & 0 & 0 \\ 0 & m^{(2)}_{1,1} & 0 & m^{(2)}_{1,2} & 0 & 0 & 0 \\ m^{(1)*}_{1,2} & 0 & -m^{(1)}_{1,1} & 0 & 0 & 0 & 0 \\ 0 & m^{(2)*}_{1,2} & 0 & -m^{(2)}_{1,1} & 0 & 0 & 0 \\ n_{1,1} & n_{1,2} & n_{1,3} & n_{1,4} & 0 & 0 & 0 \\ -n^{*}_{1,2} & n_{2,2} & n_{2,3} & n_{2,4} & 0 & 0 & 0 \\ -n_{1,3} & -n_{2,3} & n_{3,3} & n_{3,4} & 0 & 0 & 0 \\ -n^{*}_{1,4} & -n^{*}_{2,4} & -n^{*}_{3,4} & n_{4,4} & 0 & 0 & 0 \end{pmatrix}, \] (A.52)

where \( m^{(1)}_{j,k}, \ m^{(2)}_{j,k} \) and \( n_{j,k} \) form the subalgebras \( su(1,1), su(1,1), su(4), \) respectively (\( \sum_{j=1}^{4} n_{j,j} = 0 \)). The \( u(1) \) is generated by
\[ Y^L = i \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}, \] (A.53)

which acts non-trivially only on the AdS_5.

**A.2 Twisted crosscap**

In the following we analyze the subalgebras which are generated the twisted crosscap (6.19). On the algebra level this relation reads as
\[ M^{st} K \kappa + K \kappa M = 0. \] (A.54)

The bosonic \( \kappa \)-matrices satisfy the condition
\[ K \kappa = \pm \kappa^t K. \] (A.55)

Let us introduce the notation
\[ \tilde{\kappa} = K \kappa. \] (A.56)
Example 1. Let us define the matrices

$$\kappa = \begin{pmatrix} \Xi & 0 \\ 0 & i1_4 \end{pmatrix}, \quad \Rightarrow \quad \tilde{\kappa} = \begin{pmatrix} \Xi & 0 \\ 0 & iJ \end{pmatrix}. \quad (A.57)$$

The subalgebra is defined by the equations

$$M^* \tilde{\kappa} + \tilde{\kappa} M = 0, \quad (A.58)$$
$$M^{\dagger} H + HM = 0. \quad (A.59)$$

Let us apply the following basis transformation $M_S = S^\dagger MS$. Let us substitute to the defining equations:

$$M^*_S \tilde{\kappa} S + S \tilde{\kappa} S M_S = 0 \quad (A.60)$$
$$M^\dagger_S S^\dagger H S + S^\dagger H S M_S = 0 \quad (A.61)$$

Using the following identities

$$S \tilde{\kappa} S = iK, \quad (A.62)$$
$$S^{\dagger} H S = i\tilde{K}, \quad (A.63)$$

we obtain that

$$M^*_S \tilde{K} + K M_S = 0, \quad (A.64)$$
$$M^\dagger_S \tilde{K} + \tilde{K} M_S = 0, \quad (A.65)$$

which are the defining relations of the real Lie-superalgebra $\mathfrak{osp}(4^*|4)$. From the first condition we obtain that $\text{str} M_S = 0$ and the identity element $X = i1_8$ obviously does not satisfy it therefore we obtained the following embedding

$$\mathfrak{osp}(4^*|4) \subset \mathfrak{psu}(2,2|4). \quad (A.66)$$

Let us parameterize the supermatrix as

$$M_S = \begin{pmatrix} m_S & \theta_S \\ \eta_S & n_S \end{pmatrix}. \quad (A.67)$$

For the bosonic subalgebra we obtain the relations

$$m^*_S + m_S = 0, \quad n^*_S J + J n_S = 0, \quad (A.68)$$
$$m^\dagger_S J + J m_S = 0, \quad n^\dagger_S + n_S = 0, \quad (A.69)$$

therefore $m_S \in \mathfrak{so}(4^*) = \mathfrak{so}(2,1) \oplus \mathfrak{so}(3)$ and $n_S \in \mathfrak{usp}(4) = \mathfrak{so}(5)$ i.e., the bosonic subalgebra is

$$\mathfrak{so}(2,1) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(5) \subset \mathfrak{so}(2,4) \oplus \mathfrak{so}(6), \quad (A.70)$$

which agrees with the fifth row of table 5.
Example 2. Let us define the matrices
\[
\kappa = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad \Rightarrow \quad \bar{\kappa} = \bar{K}.
\]  

(A.71)  

The subalgebra is defined by the equations
\[
M^{st} \bar{K} + \bar{K} M = 0, \quad (A.72)
\]
\[
M^{\dagger} H + H M = 0. \quad (A.73)
\]

Let us apply the same basis transformation as before
\[
M^{st} S \bar{K} S + S \bar{K} S M = 0, \quad (A.74)
\]
\[
M^{\dagger} S H S + S^{\dagger} H S M = 0. \quad (A.75)
\]

Using the following identities
\[
S \bar{K} S = \bar{K}, \quad (A.76)
\]
\[
S^{\dagger} H S = i \bar{K}, \quad (A.77)
\]

we obtain that
\[
M^{st} S \bar{K} + \bar{K} M S = 0, \quad (A.78)
\]
\[
M^{\dagger} S \bar{K} + \bar{K} M S = 0. \quad (A.79)
\]

Since
\[
M^{st} S = \Upsilon \left( M^{\dagger} S \right)^{st} \Upsilon = -\Upsilon \bar{K} M^{st} \bar{K} \Upsilon = M_S,
\]

(A.80)

the defining relations (A.78)–(A.79) are equivalent to
\[
M^{st} \bar{K} + \bar{K} M S = 0, \quad (A.81)
\]
\[
M^{\dagger} S \bar{K} + \bar{K} M S = 0, \quad (A.82)
\]

which are the defining relations of the real Lie-superalgebra \( \mathfrak{spo}(4|4, \mathbb{R}) \), i.e.
\[
\mathfrak{spo}(4|4, \mathbb{R}) \subset \mathfrak{psu}(2,2|4). \quad (A.83)
\]

For the bosonic subalgebra we obtain the relations
\[
m_S^{\dagger} J + J m_S = 0, \quad n_S^{\dagger} + n_S = 0, \quad (A.84)
\]
\[
m_S^{\dagger} - m_S = 0, \quad n_S^{\dagger} - n_S = 0, \quad (A.85)
\]

therefore \( m_S \in \mathfrak{sp}(4) = \mathfrak{so}(2,3) \) and \( n_S \in \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \), i.e., the bosonic subalgebra is
\[
\mathfrak{so}(2, 3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(2, 4) \oplus \mathfrak{so}(6), \quad (A.86)
\]

which agrees with the sixth row of table 5.
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