On automorphic sheaves on Bun\(_G\)

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Abstract. Let \(X\) be a smooth projective connected curve over an algebraically closed field \(k\) of positive characteristic. Let \(G\) be a reductive group over \(k\), \(\gamma\) be a dominant coweight for \(G\), and \(E\) be an \(\ell\)-adic \(\hat{G}\)-local system on \(X\), where \(\hat{G}\) denotes the Langlands dual group (over \(\overline{\mathbb{Q}}_\ell\)). Let Bun\(_G\) be the moduli stack of \(G\)-bundles on \(X\).

Under some conditions on the triple \((G, \gamma, E)\) we propose a conjectural construction of a distinguished \(E\)-Hecke automorphic sheaf on Bun\(_G\). We are motivated by a construction of automorphic forms suggested by Ginzburg, Rallis and Soudry in [6, 7]. We also generalize Laumon's theorem (Theorem 4.1) for our setting. Finally, we formulate an analog of the Vanishing Conjecture of Frenkel, Gaitsgory and Vilonen for Levi subgroups of \(G\).

1. Introduction

Let \(F\) be a number field, \(A\) be its ring of adeles. If \(G\) is one of the groups \(\text{SO}_{2n+1}\), \(\text{Sp}_{2n}\) or \(\text{SO}_{2n}\) then consider standard representation \(\hat{G} \to \hat{H}\) of the Langlands dual group \(\hat{G}\), so here \(\hat{H}\) is \(\text{GL}_{2n}\), \(\text{GL}_{2n+1}\) or \(\text{GL}_{2n}\) respectively. For an irreducible, automorphic, cuspidal representation \(\tau\) of \(H(A)\) satisfying some additional conditions, D. Ginzburg, S. Rallis and D. Soudry have proposed a conjectural construction of an irreducible, automorphic cuspidal representation \(\sigma\) of \(G(A)\) which lifts to \(\tau\) (cf., [6, 7]).

For example, consider \(G = \text{SO}_{2n+1}\). Let \(X\) be a smooth projective absolutely irreducible curve over \(F_q\). Consider the Langlands dual group \(\hat{G} = \text{Sp}_{2n}\) over \(\overline{\mathbb{Q}}_\ell\). Let \(\hat{H} = \text{GL}_{2n}\) over \(\mathbb{F}_q\). The standard representation \(V\) of \(\hat{G}\) is a map \(\hat{G} \to \hat{H} = \text{GL}(V)\).

Let \(E\) be an \(\ell\)-adic \(\hat{G}\)-local system on \(X\), assume that \(V_E\) is irreducible. According to [8], theorem VII.6, irreducibility of \(V_E\) implies that \(\text{End}(V_E)\) is pure of weight zero. It follows that for each closed point \(x \in X\) the local L-function \(L(E_x, s)\) is regular at \(s = 1\), and the corresponding irreducible unramified representation of \(G(F_x)\) is generic. Here \(F_x\) denotes the completion of \(\mathbb{F}_q(X)\) at \(x\), and \(\mathfrak{g} = \text{Lie} \hat{G}\). An analog of D. Ginzburg, S. Rallis and D. Soudry’s conjecture for function field is to predict that in the L-packet of automorphic forms corresponding to \(E\) there exists a unique nonramified cuspidal generic form \(\varphi_E : \text{Bun}_G(F_q) \to \overline{\mathbb{Q}}_\ell\) (cf. [7], Conjecture on p. 809 and [6]).

In loc.cit. an additional condition is required: the L-function \(L(E, \wedge^2 V, s)\) has a pole of order exactly one at \(s = 1\). This condition is satisfied in our situation. Indeed, \(\wedge^2 V = V' \otimes \overline{\mathbb{Q}}_\ell\), where \(V'\) is an irreducible representation of \(\hat{G}\). Since \(V\) is self-dual, \(H^0(X \otimes \mathbb{F}_q, V'_E) = 0\) and the L-function \(L(E, V', s)\) is a polynomial in \(q^{-s}\). The purity argument shows that \(L(E, V', 1) \neq 0\).

In this paper we consider the problem of constructing a geometric counterpart of \(\varphi_E\). Given a reductive group \(G\), a dominant coweight \(\gamma\) and a \(\hat{G}\)-local system \(E\) on \(X\), we impose on these data some conditions similar to the above. Then we propose a conjectural construction of a
distinguished $E$-Hecke eigensheaf on the moduli stack $\Bun_G$ of $G$-bundles on $X$. Our approach applies to root systems $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n$ for all $n$, $\mathfrak{D}_n$ for odd $n$, and also $\mathfrak{E}_6, \mathfrak{E}_7$. For $\GL_n$ our method reduces to the one proposed by Laumon in [12].

The construction is exposed in Sect. 2, 3. In Sect. 4 we study the additional structure on Levi subgroups induced by $\gamma$, and prove a generalization of Laumon’s theorem ([10], Theorem 4.1) for our setting. We discuss its applications to cuspidality and formulate an analog of the Vanishing Conjecture of Frenkel, Gaitsgory and Vilonen for Levi subgroups of $G$.

2. Statements and conjectures

2.1 Notation Throughout, $k$ will denote an algebraically closed field of characteristic $p > 0$. Let $X$ be a smooth projective connected curve over $k$. Fix a prime $\ell \neq p$. For a $k$-scheme (or $k$-stack) $S$ write $D(S)$ for the bounded derived category of $\ell$-adic étale sheaves on $S$.

Let $G$ be a connected reductive group over $k$. Fix a Borel subgroup $B \subset G$. Let $N \subset B$ be its unipotent radical and $T = B/N$ be the “abstract” Cartan. Let $\Lambda$ denote the coweight lattice. The weight lattice is denoted by $\check{\Lambda}$. The semigroup of dominant coweights (resp., weights) is denoted $\Lambda^+$ (resp., $\check{\Lambda}^+$). The set of vertices of the Dynkin diagram of $G$ is denoted by $\mathcal{I}$. To each $i \in \mathcal{I}$ there corresponds a simple root $\check{\alpha}_i$ and a simple coroot $\alpha_i$. By $\check{\rho} \in \check{\Lambda}$ is denoted the half sum of positive roots of $G$ and by $\nu_0$ the longest element of the Weyl group $W$.

Let $\Lambda^{\text{pos}}$ denote $\mathbb{Z}_+$-span of positive coroots. The set $\Lambda^+$ is equipped with the order $\nu_1 \leq \nu_2$ iff $\nu_2 - \nu_1 \in \Lambda^{\text{pos}}$. Similarly, we have an order on $\check{\Lambda}^+$.

To a dominant weight $\lambda$ one attaches the Weil $G$-module $\mathcal{V}^\lambda$ with a fixed highest weight vector $\nu^\lambda \in \mathcal{V}^\lambda$. For any pair $\check{\lambda}, \check{\nu} \in \check{\Lambda}^+$ there is a canonical map $\mathcal{V}^{\check{\lambda}+\check{\nu}} \to \mathcal{V}^\check{\lambda} \otimes \mathcal{V}^\check{\nu}$ sending $\nu^{\check{\lambda}+\check{\nu}}$ to $\nu^\check{\lambda} \otimes \nu^\check{\nu}$.

For $\lambda \in \Lambda^+$ write $V^\lambda$ for the irreducible representation of $\check{G}$ of highest weight $\lambda$.

The trivial $G$-bundle on a scheme is denoted by $\mathcal{F}^0_G$. Recall that for any finite subfield $k' \subset k$ and any non-trivial character $\psi : k' \to \mathbb{Q}_\ell^*$ one can construct the Artin-Shrier sheaf $\mathcal{L}_\psi$ on $\mathcal{G}_{a,k}$. The intersection cohomology sheaves are normalized to be pure of weight zero.

2.2 Additional data and assumptions We say that $\gamma \in \Lambda^+$ is minuscule if $\gamma$ is a minimal element of $\Lambda^+$ and $\gamma \neq 0$. If $\gamma \in \Lambda^+$ is minuscule then, by (Lemma 1.1, [14]), for any root $\check{\alpha}$ we have $\langle \gamma, \check{\alpha} \rangle \in \{0, \pm 1\}$, and the set of weights of $V^\gamma$ coincides with the $W$-orbit of $\gamma$. For example, if $\gamma \neq 0$ is orthogonal to all roots then $\gamma$ is minuscule. One checks that the natural map from the set of minuscule dominant coweights to $\pi_1(G)$ is injective.

Definition 1. We say that $\{\gamma\}$ is a 1-admissible datum if the following conditions hold

- the center $Z(G)$ is a connected 1-dimensional torus;
- $\pi_1(G) \to \mathbb{Z}$;
- $\gamma \in \Lambda^+$ is a minuscule dominant coweight whose image $\theta$ in $\pi_1(G)$ generates $\pi_1(G)$;
- $V^\gamma$ is a faithful representation of $\check{G}$.
Fix a 1-admissible datum \( \gamma \). For \( k \geq 0 \) set \( \Lambda_{G,S}^{+,k\theta} = \{ \mu \in \Lambda^+ \mid \mu \leq k\gamma \} \). Let \( \Lambda_{G,S}^+ \) be the union of \( \Lambda_{G,S}^{+,k\theta} \) for \( k \geq 0 \). Set

\[
\Lambda_S^+ = \{ \hat{\lambda} \in \Lambda^+ \mid \langle w_0(\lambda), \hat{\lambda} \rangle \geq 0 \text{ for any } \lambda \in \Lambda_{G,S}^+ \}
\]

Let \( \hat{\omega}_0 \) be the generator of the group of weights orthogonal to all coroots, we fix \( \hat{\omega}_0 \) by requiring \( \langle \theta, \hat{\omega}_0 \rangle = 1 \). For \( i \in \mathcal{I} \) denote by \( \hat{\omega}_i \in \Lambda^+ \) the fundamental weight corresponding to \( \alpha_i \) that satisfies \( \langle w_0(\gamma), \hat{\omega}_i \rangle = 0 \). Note that \( \hat{\omega}_0, \hat{\omega}_i (i \in \mathcal{I}) \) form a basis of \( \Lambda \).

The following lemma is straightforward.

**Lemma 1.** The semigroup \( \Lambda_S^+ \) is the \( \mathbb{Z}_+ \)-span of \( \hat{\omega}_0, \hat{\omega}_i (i \in \mathcal{I}) \). Besides

\[
\Lambda_{G,S}^+ = \{ \lambda \in \Lambda^+ \mid \langle w_0(\lambda), \hat{\lambda} \rangle \geq 0 \text{ for all } \hat{\lambda} \in \Lambda_S^+ \} \quad \square
\]

Since \( \pi_1(G) \cong \mathbb{Z} \), it follows that \( [G,G] \) is simply-connected. Note that for \( \bar{\mu} \in \Lambda^+, \bar{\lambda} \in \Lambda_S^+ \) the condition \( \bar{\mu} \leq \bar{\lambda} \) implies \( \bar{\mu} \in \Lambda_S^+ \). Note that \( \{-\langle w_0(\gamma) \rangle\} \) is also a 1-admissible datum.

Since \( V^\gamma \) is faithful, the weights of \( V^\gamma \) generate \( \Lambda \) and for each \( i \in \mathcal{I} \) we have \( \langle \gamma, \hat{\omega}_i \rangle > 0 \). For each maximal positive root \( \bar{\alpha} \) we have \( \langle \gamma, \bar{\alpha} \rangle = 1 \). In particular, if the root system of \( G \) is irreducible (so, nonempty) then \( \gamma \) is a fundamental coweight corresponding to some simple root.

Some examples of 1-admissible data are given in the appendix.

2.2.1 Consider the formal disk \( \mathcal{D} = \text{Spf}(k[[t]]) \). Recall that the Affine Grassmanian \( \text{Gr}_G \) is the ind-scheme classifying pairs \( (\mathcal{F}_G, \beta) \), where \( \mathcal{F}_G \) is a \( G \)-bundle on \( \mathcal{D} \) and \( \beta : \mathcal{F}_G \to \mathcal{F}_G^0 \) is a trivialization over the punctured disk \( \mathcal{D}^* = \text{Spec} k((t)) \). Define the positive part \( \text{Gr}_G^+ \subseteq \text{Gr}_G \) of \( \text{Gr}_G \) as a closed subscheme given by the following condition:

\( \mathcal{F}_G \in \text{Gr}_G^+ \) if for every \( \hat{\lambda} \in \Lambda_S^+ \) the map

\[
\beta^\hat{\lambda} : \mathcal{V}_{\mathcal{F}_G}^\hat{\lambda} |_{\mathcal{D}^*} \to \mathcal{V}_{\mathcal{F}_G^0}^\hat{\lambda} |_{\mathcal{D}^*}
\]

is regular on \( \mathcal{D} \). Note that \( \text{Gr}_G^+ \) is invariant under the natural action of \( G(k[[t]]) \).

Recall that for \( \mu \in \Lambda^+ \) one has the closed subscheme \( \overline{\text{Gr}_G^\mu} \subseteq \text{Gr}_G \) (cf. [2], sect. 3.2). One checks that \( \text{Gr}_G^+ \subseteq \text{Gr}_G^\mu \) iff \( \mu \in \Lambda_{G,S}^+ \). Let \( \pi_1^+(G) \subseteq \pi_1(G) \) be the image of \( \Lambda_{G,S}^+ \) under the projection \( \Lambda \to \pi_1(G) \).

For \( \nu \in \pi_1(G) \) the connected component \( \text{Gr}_G^\nu \) of \( \text{Gr}_G \) is given by the condition:

\[
\mathcal{V}_{\mathcal{F}_G}^{\hat{\omega}_0} (-\langle \nu, \hat{\omega}_0 \rangle) \to \mathcal{V}_{\mathcal{F}_G}^{\hat{\omega}_0}
\]

For \( \nu \in \pi_1(G) \) set \( \text{Gr}_G^{+,\nu} = \text{Gr}_G^+ \cap \text{Gr}_G^\nu \).

2.3 Denote by \( \text{Bun}_G \) the moduli stack of \( G \)-bundles on \( X \). Let \( \mathcal{H}_G^+ \) be the corresponding positive part of the Hecke stack, it classifies collections: \( \mathcal{F}_G, \mathcal{F}_G^0 \in \text{Bun}_G \), an effective divisor \( D \) on \( X \), an isomorphism \( \beta : \mathcal{F}_G |_{X-D} \to \mathcal{F}_G^0 |_{X-D} \) such that for each \( \hat{\lambda} \in \Lambda_S^+ \) the map

\[
\beta^\hat{\lambda} : \mathcal{V}_{\mathcal{F}_G}^\hat{\lambda} \to \mathcal{V}_{\mathcal{F}_G}^\hat{\lambda}
\]
extends to an inclusion of coherent sheaves on $X$, and $\mathcal{V}_{\mathcal{F}_G}^{\infty}(D) \to \mathcal{V}_{\mathcal{F}_G}^{\infty}$. For $k \geq 0$ let $\mathcal{H}_G^{+,k} \subset \mathcal{H}_G^+$ be given by $\deg D = k$.

2.4 Version of Laumon’s sheaf

Given a local system $W$ on $X$ and $d \geq 0$, one defines a sheaf $\mathcal{L}_W^d$ on $\mathcal{H}_G^{+,d}$ as follows.

Let $\mathcal{H}_G^{+,d}$ be the stack of collections: $(\mathcal{F}_G^1, \ldots, \mathcal{F}_G^{d+1})$ and $(\beta^i, x_i \in X)_{i=1, \ldots, d}$, where $\mathcal{F}_G^i \in \text{Bun}_G$ and

$$\beta^i : \mathcal{F}_G^i |_{X-x_i} \to \mathcal{F}_G^{i+1} |_{X-x_i}$$

is an isomorphism such that $(\mathcal{F}_G^1, \mathcal{F}_G^{i+1}, \beta^i, x_i) \in \mathcal{H}_G^{+,i}$ for $i = 1, \ldots, d$.

Note that $\mathcal{H}_G^{+,d}$ is smooth, because $\gamma$ is minuscule. We have a convolution diagram

$$\begin{array}{ccc}
\widehat{\mathcal{H}}_{G}^{+,d} & \xrightarrow{\text{supp}} & X^d \\
\downarrow p & & \downarrow \\
\mathcal{H}_{G}^{+,d} & \xrightarrow{\text{supp}} & X^{(d)}
\end{array}$$

where $\text{supp}$ (resp., $p$) sends the above collection to $(x_1, \ldots, x_d)$ (resp., to $(\mathcal{F}_G^1, \mathcal{F}_G^{i+1}, \beta, D)$ with $D = x_1 + \ldots + x_d$). Let $\text{rss} X^{(d)} \subset X^{(d)}$ be the open subscheme classifying reduced divisors. Over $\text{rss} X^{(d)}$, this diagram is cartesian.

The following proposition is an immediate corollary of (Lemma 9.3 [2]).

**Proposition 1.** The map $p$ is representable proper surjective and small. $\square$

Set $\mathcal{S}pr_W^d = p_* \text{supp}^* W^\otimes d [m](\frac{m}{2})$ with $m = \dim \mathcal{H}_G^{+,d}$. This is a perverse sheaf, the Goresky-MacPherson extension from $\text{supp}^{-1} (\text{rss} X^{(d)})$. It is equipped with a canonical action of $S_d$. Define $\mathcal{L}_W^d$ to be the $S_d$-invariants of $\mathcal{S}pr_W^d$. We have $\mathbb{D}(\mathcal{S}pr_W^d) \to \mathcal{S}pr_W^d$ canonically and $\mathbb{D}(\mathcal{L}_W^d) \cong \mathcal{L}_W^d$.

We have a diagram

$$\begin{array}{ccc}
\text{Bun}_G & \xrightarrow{\mathcal{L}_G^d} & \text{Bun}_G \\
\downarrow \text{pr} & & \downarrow \\
\mathcal{L}_W^d & \xrightarrow{q} & \text{Bun}_G
\end{array}$$

where $p$ (resp., $q$) sends $(\mathcal{F}_G, \mathcal{F}_G', \beta)$ to $\mathcal{F}_G$ (resp., $\mathcal{F}_G'$). By (property 3, sect. 5.1.2 [2]), the sheaf $\mathcal{L}_W^d$ is ULA with respect to both projections $p$ and $q$.

Let $r = \dim V^\gamma$. For a partition $\mu = (\mu_1 \geq \ldots \geq \mu_r \geq 0)$ of $d$ define the polynomial functor $W \mapsto W_\mu$ of a $\mathbb{Q}_\ell$-vector space $W$ by

$$W_\mu = (W^\otimes d \otimes U_\mu)^{S_d},$$

where $U_\mu$ stands for the irreducible representation of $S_d$ corresponding to $\mu$. For $d > 0$ let $\ell(\mu)$ be the greatest index $i \leq r$ such that $\mu_i \neq 0$. For $d = 0$ let $\ell(\mu) = 0$. If $\ell(\mu)$ is less or equal to $\dim W$ then $W_\mu$ is the irreducible representation of $GL(W)$ with h.w. $\mu$, otherwise it vanishes.

For $\nu \in \Lambda^+$ let $\mathcal{A}_\nu$ denote the IC-sheaf on $\text{Gr}$. Recall that the category $\text{Sph}(\text{Gr}_G)$ of spherical perverse sheaves on $\text{Gr}_G$ consists of direct sums of $\mathcal{A}_\nu$, as $\nu$ ranges over the set of dominant coweights. We have the Satake equivalence of tensor categories $\text{Loc} : \text{Rep}(\bar{G}) \to \text{Sph}(\text{Gr}_G)$ (cf. Theorem 3.2.8, [2]). In particular, we have $\text{Loc}(V^\nu) = \mathcal{A}_\nu$. 

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Consider a $k$-point $\sum d_k x_k$, $\mathcal{F}_G'$ of $X^{(d)} \times \text{Bun}_G$. The fibre of $\text{supp} \times q : \mathcal{H}^+_{G,d} \to X^{(d)} \times \text{Bun}_G$ over this point identifies with

$$\prod_k \text{Gr}_{d_k}^G$$

Set $d_G = \dim \text{Bun}_G$. The Satake equivalence yields the following description.

**Proposition 2.** For any local system $W$ on $X$ the restriction of $\mathcal{L}_W^d$ to the fibre of $\text{supp} \times q$ identifies with the exterior product $(\bigotimes_k \mathcal{L}_k)[d + d_G](\frac{d + d_G}{2})$, where each $\mathcal{L}_k$ is

$$\mathcal{L}_k = \bigoplus_{\mu} \text{Loc}((V^\gamma)_\mu) \otimes (W_{x_k})_{\mu},$$

the sum being taken over the set of partitions of $d_k$ of length $\leq r$. □

2.5 Given a local system $W$ on $X$, for $d \geq 0$ define a functor $\text{Av}_W^d : \text{D}(\text{Bun}_G) \to \text{D}(\text{Bun}_G)$ by

$$\text{Av}_W^d(K) = q!(p^*K \otimes \mathcal{L}_W^d)[d_G](\frac{d_G}{2})$$

Let also $\text{Av}_W^{-d} : \text{D}(\text{Bun}_G) \to \text{D}(\text{Bun}_G)$ be given by

$$\text{Av}_W^{-d}(K) = p!(q^*K \otimes \mathcal{L}_W^d)[d_G](\frac{d_G}{2})$$

The functors $\text{Av}_W^d$ and $\text{Av}_W^{-d}$ are both left and right adjoint to each other. As in (Proposition 9.5, [4]) one proves

**Proposition 3.** Let $K$ be a Hecke eigensheaf on $\text{Bun}_G$ with respect to a $\tilde{G}$-local system $E$. Then for the diagram

$$\text{Bun}_G \times X^{(d)} \xrightarrow{\text{p} \times \text{supp}} \mathcal{H}^+_{G,d} \xrightarrow{q} \text{Bun}_G$$

and any local system $W$ on $X$ we have

$$(\text{p} \times \text{supp})!(q^*K \otimes \mathcal{L}_W^d)[d_G](\frac{d_G}{2}) \cong K \boxtimes (W \otimes V_{E})^{(d)}[d](\frac{d}{2})$$

2.6.1 For a $T$-torsor $\mathcal{F}_T$ on $X$ denote by $\text{Bun}_N^{\mathcal{F}_T}$ the stack of collections $(\mathcal{F}_G, \kappa)$, where $\mathcal{F}_G \in \text{Bun}_G$ and for $\lambda \in \Lambda^+$

$$\kappa^\lambda : \mathcal{L}_{\mathcal{F}_T}^\lambda \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^\lambda$$

are inclusions of coherent sheaves on $X$ satisfying Plücker relations (as in [3], section 2.1.2).

The open substack $j : \text{Bun}_N^{\mathcal{F}_T} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T}$ is given by the condition that all $\kappa^\lambda$ are maximal embeddings.

Consider the stack of pairs $(\mathcal{F}_T, \tilde{\omega})$, where $\mathcal{F}_T$ is a $T$-torsor on $X$ and $\tilde{\omega}$ is a trivial conductor, that is, $\tilde{\omega}$ is a collection of isomorphisms $\tilde{\omega}_i : \mathcal{L}_{\mathcal{F}_T}^{\tilde{\omega}_i} \cong \Omega$ for each $i \in I$. The exact sequence

$$1 \to Z(G) \to T \to \prod_{i \in I} \mathbb{G}_m \to 1,$$

where the second map is $\prod_{i \in I} \tilde{\omega}_i$, shows that this stack is noncanonically isomorphic to $\text{Bun}_{Z(G)}$ (recall that by our assumption $Z(G)$ is connected).

Fix a section $T \to B$. Then for each pair $(\mathcal{F}_T, \tilde{\omega})$ we have the evaluation map $\text{ev}^{\tilde{\omega}} : \text{Bun}_N^{\mathcal{F}_T} \to \Lambda^1$ (cf. [3], section 4.1.1). Fix a $T$-torsor on $X$ with a trivial conductor $(\mathcal{F}_T, \tilde{\omega})$. 5
Remark 1. If $(\mathcal{F}_T', \tilde{\omega}')$ is another $T$-torsor with trivial conductor on $X$ then there exists a $Z(G)$-torsor $\mathcal{F}_Z(G)$ on $X$ and an isomorphism $\mathcal{F}_T \otimes \mathcal{F}_Z(G) \cong \mathcal{F}_T'$ with the following property. Let $\text{Bun}^F_T \rightarrow \text{Bun}^F_T$ be the isomorphism that sends $\mathcal{F}_B$ to $(\mathcal{F}_B \times \mathcal{F}_Z(G))/Z(G)$, where $Z(G)$ acts diagonally. Then the diagram commutes

\[
\begin{array}{c}
\text{Bun}^F_T \\
\downarrow \text{ev}_\omega \\
\Lambda^1
\end{array} \quad \cong \quad \begin{array}{c}
\text{Bun}^F_T \\
\downarrow \text{ev}_{\tilde{\omega}'}
\end{array}
\]

2.6.2 For $d \geq 0$ let $\mathcal{Y}_d$ be the stack of collections $(\mathcal{F}_G, D \in X^{(d)}, \kappa)$, where $(\mathcal{F}_G, \kappa) \in \text{Bun}^F_T$ with $\mathcal{F}_T = \mathcal{F}_T(w_0(\gamma)D)$. So, for each $\tilde{\lambda} \in \tilde{\Lambda}^+_{\mathbb{Z}}$ we have an embedding of coherent sheaves

\[
\kappa^\lambda : \mathcal{L}_{\mathcal{F}_T}^\lambda \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^\lambda,
\]

and $\kappa_{\tilde{\omega}_0}$ induces an isomorphism $\mathcal{L}_{\mathcal{F}_T}^{\lambda_0}(D) \cong \mathcal{V}_{\mathcal{F}_G}^{\lambda_0}$.

Let $0\mathcal{Y}_d \subset \mathcal{Y}_d$ denote the open substack given by the condition $(\mathcal{F}_G, \kappa) \in \text{Bun}^F_T$.

Consider the fibred product $\mathcal{Y}_d \times_{\text{Bun}_G} \mathcal{H}_G^{+,k}$, where the map $\mathcal{H}_G^{+,k} \rightarrow \text{Bun}_G$ is $p$. For $k \geq 0$ we have a proper representable map

\[
\mathcal{q}_Y : \mathcal{Y}_d \times_{\text{Bun}_G} \mathcal{H}_G^{+,k} \rightarrow \mathcal{Y}_{d+k}
\]

that sends $(\mathcal{F}_G, \kappa, \mathcal{F}_T', \beta : \mathcal{F}_G |_{X-D'} \rightarrow \mathcal{F}_T' |_{X-D'})$ to $(\mathcal{F}_G, \kappa')$, where $k^\lambda$ are the compositions

\[
\mathcal{L}_{\mathcal{F}_T}^\lambda \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^\lambda \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\lambda'}
\]

Given a local system $W$ on $X$, define the sheaf $\mathcal{P}^d_{W,\psi}$ on $\mathcal{Y}_d$ as follows. Consider the open immersion $j : \text{Bun}^F_T \rightarrow \text{Bun}^F_T \Rightarrow \mathcal{Y}_0$. Set

\[
\mathcal{P}^0_{W,\psi} = \mathcal{P}_0^\psi = j_!(\text{ev}_{\tilde{\omega}}^*)^* \mathcal{L}_\psi[d_N](\frac{d_N^2}{2}),
\]

where $d_N = \dim \text{Bun}^F_T$. By (Theorem 2, [3]), this is a perverse sheaf and $\mathbb{D}(\mathcal{P}^0_\psi) \Rightarrow \mathcal{P}^0_{\psi,1}$. For $d > 0$ set

\[
\mathcal{P}^d_{W,\psi} = \mathcal{q}_Y!(\mathcal{P}^0_\psi \boxtimes \mathcal{L}^d_W)[-d_G](\frac{-d_G}{2})
\]

It is easy to see that $\mathcal{Q}_\ell \boxtimes \mathcal{L}^d_W$ is ULA with respect to the projection $\mathcal{Y}_0 \times_{\text{Bun}_G} \mathcal{H}_G^{+,k} \rightarrow \mathcal{Y}_0$. So, by (property 5, sect. 5.1.2 [2]),

\[
\mathbb{D}(\mathcal{P}^0_\psi \boxtimes \mathcal{L}^d_W)[-d_G](\frac{-d_G}{2}) \Rightarrow \mathcal{P}^0_{\psi,1} \boxtimes \mathcal{L}^d_W[-d_G](\frac{-d_G}{2})
\]

Therefore, $\mathbb{D}(\mathcal{P}^d_{W,\psi}) \Rightarrow \mathcal{P}^d_{W,\psi,1}$ canonically.

Let $\text{rss} \mathcal{Y}_d \subset 0\mathcal{Y}_d$ be the preimage of $\text{rss} X^{(d)}$ under the projection $0\mathcal{Y}_d \rightarrow X^{(d)}$. 
Proposition 4. For any local system $W$ on $X$, $\mathcal{P}_{W,\psi}^d$ is a perverse sheaf on $Y_d$, the Goresky-MacPherson extension from $\tau_{ss}Y_d$. If $W$ is irreducible then $\mathcal{P}_{W,\psi}^d$ is irreducible.

The proof is found in Section 3.2.

2.7 Let $\pi^0 : \text{Bun}_{N}^{F_T} \to \text{Bun}_{G}$ be the projection.

Definition 2. Let $K$ be an $E$-Hecke eigensheaf on $\text{Bun}_{G}$. We say that $K$ is generic normalized if it is equipped with an isomorphism

$$\mathbb{R}\Gamma_e(\text{Bun}_{N}^{F_T}, \mathcal{P}_{\psi}^0 \otimes \pi^0 K) \cong \bar{Q}_\ell[dG](\frac{dG}{2})$$

By Remark 1, this property does not depend (up to a tensoring $K$ by a 1-dimensional vector space) on our choice of the pair $(F_T, \tilde{\omega})$.

Write $\text{Bun}_{dG}$ for the connected component of $\text{Bun}_{G}$ given by $\deg F_G = \deg F_T + d\theta$. Let $\pi : Y_d \to \text{Bun}_{dG}$ and $\phi : Y_d \to X^{(d)}$ be the projections. From Proposition 3 one derives

Corollary 1. Let $K$ be a generic normalized $E$-Hecke eigensheaf on $\text{Bun}_{G}$. Let $W$ be any local system on $X$. Then for each $d \geq 0$ one has

$$\phi_!(\pi^* K \otimes \mathcal{P}_{W,\psi}^d) \cong (W \otimes V_{E^*}^\gamma)^{(d)}[d + dG](\frac{d + dG}{2})$$

For a $\tilde{G}$-local system $E$ pick a $\tilde{G}$-local system $E^*$ such that $V_{E^*}^{\lambda} \cong (V_{E}^{\lambda})^*$ for all $\lambda \in \tilde{\Lambda}^+$. Let $K$ be an $E$-Hecke eigensheaf then $\mathbb{D}K$ is a $E^*$-Hecke eigensheaf. Assume that $\mathbb{D}K$ is generic normalized then from Corollary 1 we get an isomorphism

$$\phi_!(\pi^*(\mathbb{D}K) \otimes \mathcal{P}_{W,\psi}^d) \cong (W \otimes V_{E^*}^\gamma)^{(d)}[d + dG](\frac{d + dG}{2})$$

By adjunction, it yields a nonzero map

$$\mathbb{D}K \to \pi_* \mathcal{R}\mathcal{H}\text{om}(\mathcal{P}_{W,\psi}^d, \phi^!(W \otimes V_{E^*}^\gamma)^{(d)}[d + dG](\frac{d + dG}{2}))$$

Dualizing, we see that this is equivalent to providing a nonzero map

$$\pi_!(\mathcal{P}_{W,\psi}^d \otimes \phi^*(W^* \otimes V_{E}^\gamma)^{(d)}) \to K[dG - d](\frac{dG - d}{2})$$ (2)

Set $W = V_{E}^\gamma$, so we have a canonical map $\bar{Q}_\ell \to (W^* \otimes V_{E}^\gamma)^{(d)}$ on $X^{(d)}$. Composing with (2) we get a morphism

$$\pi_!\mathcal{P}_{W,\psi}^d \to K[dG - d](\frac{dG - d}{2})$$ (3)
Conjecture 1 (geometric Langlands). Let $E$ be a $\hat{G}$-local system on $X$. Assume that $W = V \gamma$ is irreducible and satisfies the condition

(A) If $E'$ is a $G$-local system on $X$ such that $V \gamma 
rightarrow V \gamma$ then $E' \nrightarrow E$.

Then there exists $N > 0$ and for each $d \geq N$ a nonempty open substack $U_d \subset \text{Bun}_G^d$ with the following property. There exists a $E$-Hecke eigensheaf $K$ on $\text{Bun}_G$ such that

- both $K$ and $\mathbb{D}K$ are generic normalized;
- for $d \geq N$ the complex $\pi_!P^d_{W,\psi}|_{U_d}$ is placed in perverse degrees $\leq d - d_G$, and the map

\[ \mathcal{H}^{d-d_G}(\pi_!P^d_{W,\psi}) \nrightarrow K|_{U_d} \]

on the top perverse cohomology sheaves;

- $K$ is an irreducible perverse sheaf over each $\text{Bun}_G^d$, which does not vanish over $U_d$.

Remarks.

i) The sheaf $K$ from Conjecture 1 is unique up to an isomorphism if it exists.

ii) For any local system $W$ on $X$ we have $\pi_!P^d_{W,\psi} \nrightarrow \text{Av}_W^d(\pi_!P^0_{\psi})$ naturally.

2.8 INFORMAL MOTIVATION If the ground field $k$ was finite then according to Langlands’ spectral decomposition theorem (13), each function from $L^2(\text{Bun}_G(k))$ would be written as linear combination (more precisely, a direct integral) of Hecke eigenfunctions.

Conjecturally, some version of spectral decomposition should exist for the derived category $\text{D}(\text{Bun}_G)$ itself. We also have an analog of the scalar product of two objects $K_1, K_2 \in \text{D}(\text{Bun}_G)$, which is the cohomology $R\Gamma_c(\text{Bun}_G, K_1 \otimes \mathbb{D}(K_2))$ (we ignore here all convergence questions).

Let $E$ be a $\hat{G}$-local system on $X$ satisfying the assumptions of Conjecture 1. One may hope that to $E$ is associated a $E$-Hecke eigensheaf $K$, which is unique in appropriate sense.

Since $K$ is expected to be generic normalized, the ”scalar product” of $\pi_!P^0_{\psi}$ and $K$ should equal “one”. That is, $K$ should appear in the spectral decomposition of $\pi_!P^0_{\psi}$ with multiplicity one. By Proposition 3 the functor $\text{Av}_W^d$ applied to $\pi_!P^0_{\psi}$ with $d$ large enough, will kill all the terms in the spectral decomposition of $\pi_!P^0_{\psi}$ except $K$ itself. So, roughly speaking, $\text{Av}_W^d(\pi_!P^0_{\psi})$ should equal $K$ tensored by some constant complex.

2.9 STRATIFICATIONS For $\mu \in \Lambda^{pos}$ denote by $X^\mu$ the moduli scheme of $\Lambda^{pos}$-valued divisors of degree $\mu$. If $\mu = \sum_{i \in I} a_i \alpha_i$ then $X^\mu = \prod_{i \in I} X^{(a_i)}$.

For $D \in X^{(d)}$ consider $\mathcal{F}_T = \mathcal{F}_T(w_0(\gamma)D)$. The stack $\mathcal{Y}_d$ is the stack of pairs: $D \in X^{(d)}$ and a point $(\mathcal{F}_G, \kappa) \in \overline{\text{Bun}}^F_{\mathcal{F}_T}$. Recall that $\overline{\text{Bun}}^F_{\mathcal{F}_T}$ is stratified by locally closed substacks $\mu \text{Bun}_N^F$ indexed by $\mu \in \Lambda^{pos}$. Namely, $(\mathcal{F}_G, \kappa)$ lies in $\mu \text{Bun}_N^F$ iff there exists a divisor $D_{pos} \in X^\mu$ such that for all $\lambda \in \Lambda^+$ the meromorphic maps

\[ \mathcal{L}^{\lambda}_{\mathcal{F}_T}((D^\mu, \lambda)) \rightarrow V^{\lambda}_{\mathcal{F}_T} \]
are regular everywhere and maximal. We have a projection
\[ \mu^* \text{Bun}_{\tilde{N}}^F \to X^\mu \]
whose fibre over \( D^\text{pos} \) is isomorphic to \( \text{Bun}_{\tilde{N}}^F \), where \( \tilde{F}_T \to \mathcal{F}_T(w_0(\gamma)D + D^\text{pos}) \).

Denote by \( \mu \mathcal{Y}_d \subset \mathcal{Y}_d \) the locally closed substack given by \((\mathcal{F}_G, \kappa) \in \mu \text{Bun}_{\tilde{N}}^F \). Let
\[ \phi^\mu : \mu \mathcal{Y}_d \to X^{(d)} \times X^\mu \]
be the projection. Let \( X^{d,\mu} \subset X^{(d)} \times X^\mu \) be the closed subscheme given by the condition:
\[ \gamma D + w_0(D^\text{pos}) \] is dominant. This condition ensures that the maps \( \tilde{\omega}_i : \mathcal{L}_{\tilde{F}_T}^{\delta_i} \to \Omega \) are regular for all \( i \in I \). Let \( \mu \mathcal{Y}_d^+ \subset \mu \mathcal{Y}_d \) be the preimage of \( X^{d,\mu} \) under \( \phi^\mu \). So, we have the evaluation map
\[ \text{ev}_\mu : \mu \mathcal{Y}_d^+ \to \mathbb{A}^1 \]

Note that if \((D, D^\text{pos}) \in X^{d,\mu} \) then \( \gamma D + w_0(D^\text{pos}) \) is a \( \Lambda_{G,S}^+ \)-valued divisor of degree \( \gamma D + w_0(\mu) \). In this way \( X^{d,\mu} \) is the moduli scheme of \( \Lambda_{G,S}^+ \)-valued divisors on \( X \) of degree \( \gamma D + w_0(\mu) \).

2.10 For any local system \( W \) on \( X \) let \( \tilde{\mathcal{P}}_{d,w}^{d,\mu} \) be the complex obtained by replacing in the definition of \( \mathcal{P}_{d,\mu}^d \) Laumon’s sheaf by Springer’s sheaf
\[ \tilde{\mathcal{P}}_{d,w}^{d,\mu} = q_!(P^0 \otimes S\text{pr}^d_X)[\frac{-d_G}{2}] \]

**Proposition 5.** 1) For each \( \mu \in \Lambda^\text{pos} \) the restriction of \( \tilde{\mathcal{P}}_{d,w}^{d,\mu} \) to \( \mu \mathcal{Y}_d \) is supported by \( \mu \mathcal{Y}_d^+ \) and is isomorphic to
\[ \phi^\mu_* \tilde{W}^{d,\mu} \otimes \text{ev}_\mu^* \mathcal{L}_\psi \otimes (\mathbb{Q}_\ell(\frac{1}{2}[1]) \otimes d + d_N + (\gamma d - \mu, 2\rho)) \]
for some sheaf \( \tilde{W}^{d,\mu} \) on \( X^{d,\mu} \). Here \( \tilde{W}^{d,\mu} \) is placed in usual degree zero. Let \( W^{d,\mu} \) denote the corresponding sheaves for \( \mathcal{P}_{d,\mu}^d \).

2) If \((D = \sum d_k x_k, D^\text{pos} = \sum \mu_k x_k) \) is a \( k \)-point of \( X^{d,\mu} \) then the fibre of \( W^{d,\mu} \) at this point is the tensor product over all \( x_k \)
\[ \otimes (\otimes \text{Hom}(V_{d_k \gamma + w_0(\mu_k)}, (V^\gamma)_\nu) \otimes (W_{x_k})_\nu), \]
the inside sum being taken over partitions \( \nu \) of \( d_k \) of length \( \leq r \).

The proof is found in Section 3.1. Proposition 5 together with Corollary 1 suggest the following conjecture.

**Conjecture 2.** Let \( D = \sum \lambda_k x_k \) be a divisor on \( X \) with \( \lambda_k \) dominant coweights, let \( \lambda \) be the degree of \( D \). Denote by \( a : \text{Bun}_{\tilde{N}}^F \to \text{Bun}_G \) the projection, where \( \tilde{F}_T = \mathcal{F}_T(w_0(D)) \). Let \( \text{ev}_\lambda : \text{Bun}_{\tilde{N}}^F \to \mathbb{A}^1 \) be the evaluation map given by the conductor data. Let \( E \) be a \( G \)-local system on \( X \), \( K \) be a generic normalized \( E \)-Hecke eigensheaf on \( \text{Bun}_G \). Then
\[ \text{R}\Gamma_c(\text{Bun}_{\tilde{N}}^F, a^* K \otimes \text{ev}_\lambda^* \mathcal{L}_\psi) \otimes (\mathbb{Q}_\ell(\frac{1}{2}[1])^{d_N - d_G + (\lambda, 2\rho)} \otimes \otimes_{k} (V_{E_k}^\lambda))_{x_k} \]
2.11 Consider the diagram

\[ Y_d \times X \xleftarrow{p_Y \times \text{supp}} Y_d \times \text{Bun}_G \mathcal{H}_G^{+,1} \xrightarrow{q_Y} Y_{d+1} \]

where \( p_Y : Y_d \times \text{Bun}_G \mathcal{H}_G^{+,1} \rightarrow Y_d \) is the projection. For any local system \( W \) on \( X \) define a natural map

\[ (p_Y \times \text{supp}) q_Y^* \mathcal{P}^{d+1}_W \otimes \bar{Q}_{\ell}(\frac{1}{2})[1]^{\otimes (\gamma,2\rho)} \rightarrow \mathcal{P}^d_{W,\psi} \boxtimes W \otimes \bar{Q}_{\ell}(\frac{1}{2})[1] \]

as follows. Consider the diagram

\[ Y_d \times \text{Bun}_G \mathcal{H}_G^{+,1} \xleftarrow{p_Y \times \text{supp}} Y_d \times X \]

Clearly, \( \mathcal{L}_W^{d+1} \) is a direct summand of \( \text{conv}_W (\mathcal{L}_W^d \boxtimes \text{supp}^* W) \otimes \bar{Q}_{\ell}(\frac{1}{2})[1]^{\otimes 1+(\gamma,2\rho)} \). From the diagram

\[ Y_0 \times \text{Bun}_G \mathcal{H}_G^{+,d} \xrightarrow{id} Y_0 \times \text{Bun}_G \mathcal{H}_G^{+,d+1} \]

we see that \( \mathcal{P}^d_{W,\psi} \) is a direct summand of

\[ q_Y^* (\mathcal{P}^d_{W,\psi} \boxtimes \text{supp}^* W) \otimes \bar{Q}_{\ell}(\frac{1}{2})[1]^{\otimes 1+(\gamma,2\rho)} \]

This yields a morphism

\[ q_Y^* \mathcal{P}^{d+1}_{W,\psi} \rightarrow (\mathcal{P}^d_{W,\psi} \boxtimes \text{supp}^* W) \otimes \bar{Q}_{\ell}(\frac{1}{2})[1]^{\otimes 1+(\gamma,2\rho)} \]

Since \( p_Y \times \text{supp} : Y_d \times \text{Bun}_G \mathcal{H}_G^{+,1} \rightarrow Y_d \times X \) is smooth of relative dimension \( (\gamma,2\rho) \), we get a map

\[ q_Y^* \mathcal{P}^{d+1}_{W,\psi} \rightarrow (p_Y \times \text{supp})^! (\mathcal{P}^d_{W,\psi} \boxtimes W) \otimes \bar{Q}_{\ell}(\frac{1}{2})[1]^{\otimes 1-(\gamma,2\rho)} \]

and, by adjunction, the desired map \( \text{(4)} \).

**Proposition 6.**

1) The complex

\[ (p_Y \times \text{supp}) q_Y^* \mathcal{P}^{d+1}_{W,\psi} \otimes \bar{Q}_{\ell}(\frac{1}{2})[1]^{\otimes (\gamma,2\rho)} \]

is placed in perverse degrees \( \leq 0 \).

2) For any \( \mu \in \Lambda^{\text{pos}} \), the restriction of \( \text{(5)} \) to \( \mu Y_d \times X \) is supported by \( \mu Y_d^{+,1} \times X \) and is isomorphic to the tensor product of

\[ \text{ev}^*_\mu \mathcal{L}_\psi \otimes (\bar{Q}_{\ell}(\frac{1}{2})[1])^{\otimes d+1+d_N+(\gamma d-\mu,2\rho)} \]

with some sheaves \( W_0^{d,\mu} \) coming from \( X^{d,\mu} \times X \). Here \( W_0^{d,\mu} \) is placed in usual degree zero.
The proof is given in Section 3.3

Remarks. i) One may show that for any $\mu \in \Lambda^\text{pos}$ the restriction of (4) to $\mu \mathcal{Y}_d \times X$ comes from a morphism of sheaves $W_0^{d,\mu} \to W^{d,\mu} \boxtimes W$ on $X^{d,\mu} \times X$. The latter map is an isomorphism over the open substack of $X^{d,\mu} \times X$ classifying triples $(D, D^\text{pos}, x)$ such that $x$ does not appear in $D$. Therefore, (4) is an isomorphism over the locus of $(\mathcal{F}, \kappa, D, x) \in \mathcal{Y}_d \times X$ such that $x$ does not appear in $D$.

ii) In the situation of Conjecture 1 we expect that the map (4) yields the Hecke property of $K$ corresponding to the coweight $\gamma$. Moreover, it should also yield the Hecke properties corresponding to all $\lambda \in \Lambda^+_G, S$ (as it indeed happens for $\text{GL}_n$). Define the Hecke functor $H_Y : D(Y_{d+1}^+) \to D(Y_d \times X)$ by

$$H_Y(F) = (p_Y \times \text{supp})_! q_Y^* F \otimes \overline{\mathbb{Q}}_\ell(\frac{1}{2})[1]^{\otimes (\gamma, 2\hat{\rho})}$$

Note that the cohomological shifts in the definition of the Hecke functor $H : D(\text{Bun}_d G) \to D(\text{Bun}_G \times X)$ corresponding to $\gamma$ and $H_Y$ differ by one! So, the Hecke property of $K$ can not be simply the push-forward of (4) with respect to $\pi : Y_d \to \text{Bun}_G$.

2.12 Let $\omega$ be a generator of the group of coweights orthogonal to all roots. Since the image of $\omega$ in $\pi_1(G)$ is not zero, we assume that this image equals $d_{\omega} \theta$ for some $d_{\omega} > 0$. Since $d_{\omega} \gamma - \omega$ is dominant, we have $d_{\omega} \gamma - \omega \in \Lambda^\text{pos}$ and $\omega \in \Lambda^+_G, S$. Consider the map $q^\omega : \text{Bun}_G \times X \to \text{Bun}_G$ sending $(F_G, x)$ to $F_G'$, where $F_G'$ and $F_G$ are identified over $X-x$ and

$$V^\lambda_{F_G}(\langle \omega, \hat{\lambda} \rangle x) \cong V^\lambda_{F_G'}$$

for each $\hat{\lambda} \in \hat{\Lambda}^+$. Let also

$$q_Y^\omega : \mathcal{Y}_d \times X \to \mathcal{Y}_{d+d_{\omega}}$$

be the map sending $(\mathcal{F}_G, \kappa, x)$ to $(\mathcal{F}_G', \kappa')$, where $\mathcal{F}_G' = q_\omega^*(\mathcal{F}_G, x)$ and $\kappa'^{\hat{\lambda}}$ is the composition

$$L_{\mathcal{F}_G}^\hat{\lambda} \hookrightarrow V^\lambda_{\mathcal{F}_G} \hookrightarrow V^\lambda_{\mathcal{F}_G'}$$

for all $\hat{\lambda} \in \hat{\Lambda}^+_S$. Let $E$ be a $\hat{G}$-local system on $X$ and set $W = V^\gamma_E$. Then there is a natural map

$$(q_Y^\omega)_* \mathcal{P}_{W, \psi}^{d+d_{\omega}} \otimes \overline{\mathbb{Q}}_\ell(\frac{1}{2})[1]^{\otimes 1-d_{\omega}} \to \mathcal{P}_{W, \psi}^d \otimes V^\psi_E \otimes \overline{\mathbb{Q}}_\ell(\frac{1}{2})[1]$$

This is not an isomorphism in general, and one may show that the LHS of (6) is placed in perverse degrees $\leq 0$.

2.13 Whittaker Sheaves Let $K(\mathcal{Y}_d)$ denote the Grothendieck ring of the triangulated category $D(\mathcal{Y}_d)$.

To each $\hat{G}$-local system $E$ on $X$ and $d \geq 0$ we attach the Whittaker sheaf $\mathcal{W}^d_{E, \psi} \in K(\mathcal{Y}_d)$ defined as follows.
Let \( \mu \in \Lambda^{pos} \) be such that \( d\gamma + w_0(\mu) \) is dominant. Let \( \tau \) be a partition of \( d\gamma + w_0(\mu) \), that is, a way to write \( d\gamma + w_0(\mu) = \sum \lambda_i \) with \( \lambda_i \in \Lambda^+_{G,S} \) pairwise different and \( n_i > 0 \). Let \( \tau X \subset \prod X^{(n_i)} \) be the complement to all diagonals. We consider \( \tau X \subset X^{d,\mu} \) as the locally-closed subscheme classifying divisors \( \sum \lambda_k x_k \) of degree \( d\gamma + w_0(\mu) \) with \( x_k \in X \) pairwise different.

Let \( \tau E \) denote the restriction of 

\[
\boxtimes_i (V^\mu_i)^{(n_i)}
\]

under \( \tau X \hookrightarrow \prod X^{(n_i)} \). Let \( \tau Y \subset \mu \gamma^+ \) be the preimage of \( \tau X \) under \( \phi^\mu : \mu \gamma^+ \rightarrow X^{d,\mu} \).

**Definition 3.** Set \( W_{E,\psi}^d \in \mathcal{K}(\mathcal{Y}_d) \) to be the (unique) complex with the following properties. Its \( * \)-restriction to each stratum \( \mu \gamma_d \) is supported by \( \mu \gamma_d^+ \). If \( d\gamma + w_0(\mu) \) is dominant then for any partition \( \tau \) of \( d\gamma + w_0(\mu) \) the \( * \)-restriction of \( W_{E,\psi}^d \) to \( \tau \gamma \) is

\[
\phi^\mu*(\tau E) \otimes \text{ev}^\ast \mathcal{L}_\psi \otimes \hat{Q}_\ell[\ell^2]^{(\gamma,2\rho)}
\]

The sheaf \( W_{E,\psi}^d \) should satisfy the Hecke property, in particular we suggest

**Conjecture 3.** Recall the diagram (cf. Sect. 2.11)

\[
\mathcal{Y}_d \times X \overset{p_Y \times \text{supp}}{\leftarrow} \mathcal{Y}_d \times \text{Bun}_G \mathcal{H}_{G,1}^+ \overset{q_Y}{\rightarrow} \mathcal{Y}_{d+1}
\]

There is a canonical isomorphism in the Grothendieck ring \( \mathcal{K}(\mathcal{Y}_d \times X) \)

\[
(p_Y \times \text{supp}) \mathcal{Q}_d \mathcal{W}_{E,\psi}^{d+1} \otimes \hat{Q}_\ell[\ell^2]^{(\gamma,2\rho)} \rightarrow \mathcal{W}_{E,\psi}^d \otimes V_E^\gamma \otimes \hat{Q}_\ell[\ell^2]^{(\gamma,2\rho)}
\]

2.13.1 We don’t know if \( \Lambda^+_{G,S} \) is a free semigroup in general, however this is the case for our examples \( \text{GL}_n, \text{GSp}_{2n}, \text{GSpin}_{2n+1} \) (cf. appendix).

Assuming that \( \Lambda^+_{G,S} \) is a free semigroup, we can describe \( W_{E,\psi}^d \) more precisely, namely ”glue” the pieces on the strata \( \tau \gamma \) to get a sheaf on \( \mu \gamma_d^+ \). To do so, we will glue the sheaves \( \tau E \) to get a constructible sheaf \( AE^{d,\mu} \) on \( X^{d,\mu} \) (here ‘A’ stands for ‘automorphic’).

Let \( \lambda_1, \ldots, \lambda_m \) be free generators of \( \Lambda^+_{G,S} \) thus yielding \( \Lambda^+_{G,S} \rightarrow (\mathbb{Z}_+)^m \). Given \( d \geq 0 \) and \( \mu \in \Lambda^{pos} \) with \( d\gamma + w_0(\mu) = \sum m a_i \lambda_i \) dominant, we get

\[
X^{d,\mu} \rightarrow \prod X^{(a_i)}
\]

Consider the sheaf

\[
\bigotimes_{i=1}^m (V_E^{\lambda_i})^{(a_i)}
\]

on \( X^{d,\mu} \). Let \( D = \sum \nu_k x_k \) be a \( k \)-point of \( X^{d,\mu} \), where \( x_k \) are pairwise different and \( \nu_k \in \Lambda^+_{G,S} \). Write \( \nu_k = \sum a_{i,k} \lambda_i \) for each \( k \). The fibre of (7) at \( D \) is

\[
\otimes x_k \bigotimes_{i=1}^m \text{Sym}^{a_{i,k}} (V_E^{\lambda_i}) x_k
\]
There is (a unique up to a nonzero multiple) inclusion of $G$-modules $V^{\nu_k} \hookrightarrow \otimes_{i=1}^m \Sym^{a_i,k}(V^\lambda_i)$.
This yields a map
\[
\otimes_{x_k}(V^{\nu_k})_{x_k} \hookrightarrow \otimes_{x_k} \otimes_{i=1}^m \Sym^{a_i,k}(V^\lambda_i)_{x_k}
\] (8)

The following is borrowed from [12].

**Proposition 7.** Assume that $\Lambda^+_{G,S}$ is a free semigroup. There is a unique constructible subsheaf $AE_{d,\mu} \subset \otimes_{i=1}^m (V^\lambda_i)_{E_k}$ whose fibre at any $k$-point $D = \sum_k \nu_k x_k$ of $X^{d,\mu}$ is the image of (8). □

### 3. Some proofs

3.1 Recall that $\Gr_G$ is stratified by locally closed ind-subschemes $S^\mu$ indexed by all coweights $\mu \in \Lambda$. Informally, $S^\mu$ is the $N(\hat{K})$-orbit of the point $\mu(t) \in \Gr_G$, where $\hat{K} = k[[t]]$. We refer the reader to [3], Section 7.1 for the precise definition.

Recall the following notion from loc.cit., section 7.1.4. Set $\hat{O} = k[[t]]$. Let $\hat{\Omega}$ denote the completed module of relative differentials of $\hat{O}$ over $k$ (so, $\hat{\Omega}$ is a free $\hat{O}$-module generated by $dt$). Given a coweight $\eta \in \Lambda$ and isomorphisms $s_i : \hat{O}(\langle \eta, \hat{\alpha}_i \rangle) \sim \hat{\Omega}$ for each $i \in I$, one defines an admissible character $\chi_\eta : N(\hat{K}) \to \Ga$ of conductor $\eta$ as the sum $\chi_\eta = \sum_{i \in I} \chi_i^\eta$, where $\chi_i^\eta : N(\hat{K}) \to \Ga$ is the composition
\[
N(\hat{K}) \to N/[N,N](\hat{K}) \xrightarrow{u_i} \Ga(\hat{K}) \xrightarrow{s_i} \hat{\Omega}(\hat{K}) \xrightarrow{\Res} \Ga
\]

Here $u_i : N/[N,N] \to \Ga$ is the natural coordinate corresponding to the simple root $\hat{\alpha}_i$. By (loc.cit., Lemma 7.1.5), for $\nu \in \Lambda$ there exists a $(\hat{N}(\hat{K}), \chi_\eta)$-equivariant function $\chi_\eta^\nu : S^\nu \to \Ga$ if and only if $\nu + \eta \in \Lambda^+$. In the latter case this function is unique up to an additive constant.

**Proof of Proposition [3]** Let $(\sum d_k x_k, F_G)$ be a $k$-point of $X^{(d)} \times \Bun_G$. The fibre of $\supp \times q : H_{G}^{+,d} \to X^{(d)} \times \Bun_G$ over this point identifies with
\[
\prod_k \Gr_{d_k \gamma} \Ga
\]

The restriction of $S_{pr_W}^{d}$ to this fibre is the tensor product of $(\otimes_k(W_{x_k}^{d_k})) [d + d_G](\frac{d + d_G}{2})$ with
\[
\otimes_k (\oplus_{\nu \leq d_k \gamma} A_\nu \otimes V_{\nu,k})
\]

where the inside sum is taken over dominant coweights $\nu \in \Lambda^+$ such that $\nu \leq d_k \gamma$, and $V_{\nu,k}$ are some vector spaces. Let
\[
0q_Y : \Bun_{F_{\mathbb{F}}} \times_{\Bun_G} H_{G}^{+,d} \to Y_d
\]

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denote the restriction of \( q_Y \) to the open substack \( \text{Bun}^{F_T}_{G} \times_{\text{Bun}_G} \mathcal{H}^{+,d}_G \subset \mathcal{Y}_0 \times_{\text{Bun}_G} \mathcal{H}^{+,d}_G \).

Fix a \( k \)-point \( y \in \mathcal{Y}_d \), it is given by \( (D = \sum d_k x_k, D^{\text{pos}} = \sum \mu_k x_k, \mathcal{F}_G, \kappa) \) such that \( x_k \) are pairwise different (and some of \( d_k \) may be zero). Let \( K_y \) denote the fibre at \( y \) of

\[
0_{\mathcal{Y}}(ev^* \mathcal{L}_\psi \boxtimes \mathcal{S}pr^d_W)[d_N - d_G](\frac{d_N - d_G}{2})
\]

The fibre of \( 0_{\mathcal{Y}} \) over \( y \) identifies with

\[
\prod_k S^{w_0(\gamma)d_k + \mu_k} \cap \mathcal{G}^{d_k \gamma}_G
\]

For each \( k \) set \( \nu_k = -w_0(\gamma)d_k - \mu_k \). An equivariance argument (as in \[3\], Lemma 6.2.8) shows that \( K_y \) vanishes unless all \( \nu_k \) are dominant.

By (Lemma 7.2.7(2), \[3\]) the restriction of the map

\[
\text{Bun}^{F_T}_{G} \times_{\text{Bun}_G} \mathcal{H}^{+,d}_G \rightarrow \text{Bun}^{F_T}_{G} \rightarrow \mathbb{A}^1
\]

to \((0_{\mathcal{Y}})^{-1}(y)\) becomes the sum over all \( k \) of

\[
\chi_{\nu_k}^{-\nu_k} : S^{-\nu_k} \cap \mathcal{G}^{d_k \gamma}_G \rightarrow \mathbb{A}^1
\]

plus \( \text{ev}(\mathcal{F}_G, \kappa, D, D^{\text{pos}}) \in \mathbb{A}^1 \). By (Theorem 1, \[3\]), for any \( \nu \in \Lambda^+ \) such that \( \nu \leq d_k \gamma \) the complex

\[
\mathbb{R}\Gamma_c(S^{-\nu_k} \cap \mathcal{G}^{d_k \gamma}_G, \mathcal{A}_\nu \otimes (\chi_{\nu_k}^{-\nu_k})^* \mathcal{L}_\psi)
\]

vanishes unless \( \nu = -w_0(\nu_k) \). In the latter case, it is canonically \( \mathbb{Q}[2]\{\nu_k, 2\tilde{\mu}\}(\nu_k, \tilde{\mu}) \).

The above equivariance argument shows also that the restriction of \( \mathcal{P}^d_{\mathcal{W}_\psi} \) to \( \mu \mathcal{Y}_d^+ \), after tensoring by \( \text{ev}^* \mathcal{L}_{\psi^{-1}} \), descends with respect to the projection \( \mu \mathcal{Y}_d^+ \rightarrow X^{d,\mu} \). Combining this with Proposition 3.2 one finishes the proof of Proposition 4. \( \square \)

The above proof combined with (Proposition 3.2.6, \[2\]) also gives the following

**Corollary 2.** Over the open substack \( 0_{\mathcal{Y}}_d \subset \mathcal{Y}_d \), the map \( 0_{\mathcal{Y}} : \text{Bun}^{F_T}_{G} \times_{\text{Bun}_G} \mathcal{H}^{+,d}_G \rightarrow \mathcal{Y}_d \) is an isomorphism.

### 3.2

In this subsection we prove Proposition 4.

Given a pair \( d \geq 0, \mu \in \Lambda^{\text{pos}} \) such that \( d\gamma + \omega_0(\mu) \) is dominant, a *partition* \( \tau \) of \( d\gamma + \omega_0(\mu) \) is a presentation of \( d\gamma + \omega_0(\mu) \) as a sum of nonzero elements from \( \Lambda^+_{G,S} \), that is, \( \sum_k (d_k \gamma + \omega_0(\mu_k)) = d\gamma + \omega_0(\mu) \) with \( d_k \geq 0, \mu_k \in \Lambda^{\text{pos}} \) and \( d_k \gamma + \omega_0(\mu_k) \) dominant for all \( k \).

Given a partition \( \tau \) of \( d\gamma + \omega_0(\mu) \), consider the locally closed subscheme \( \tau X \subset X^{d,\mu} \), which is the moduli scheme of divisors \( \sum_k (d_k \gamma + \omega_0(\mu_k)) x_k \) with \( x_k \) pairwise different. Given \( \tau \), if \( k \) runs through the set consisting of \( m \) elements then \( \dim \tau X = m \). Clearly, the schemes \( \tau X \) form a stratification of \( X^{d,\mu} \).
Let $\tau Y \subset \mu Y^+_d$ be the preimage of $\tau X$ in $\mu Y^+_d$. Suppose that $\tau$ is of length $m$, that is, $\dim \tau X = m$. Then from (Lemma 7.2.4, [3]) it follows that
\[
\dim \tau Y = m + d_N + \langle \gamma d, 2\tilde{\rho} \rangle
\]
In particular, we have $\dim \mathcal{Y}_d = d + d_N + \langle d\gamma, 2\tilde{\rho} \rangle$. Now from Proposition 5 we learn that the restriction of $\mathcal{P}_W,\psi$ to $\tau Y$ is placed in perverse degrees $\leq 0$. Moreover, the inequality is strict unless $\mu = 0$ and $m = d$. Since $\mathcal{P}_W,\psi$ is self-dual (up to replacing $W$ by $W^*$ and $\psi$ by $\psi^{-1}$), our assertion follows. □

Remarks . i) As a corollary, note that the restriction of $P_{W,\psi}$ to $0\mathcal{Y}_d$ identifies canonically with
\[
\phi_0^* W(d) \otimes \text{ev}^* \mathcal{L}_{\psi} \otimes \langle \tilde{Q}_{\ell}(\frac{1}{2})[1] \rangle \otimes d + d_N + \langle \gamma d, 2\tilde{\rho} \rangle
\]
ii) For each pair $d \geq 0, \mu \in \Lambda^{\text{pos}}$ such that $d\gamma + w_0(\mu)$ is dominant, the projection $X^{d,\mu} \rightarrow X^{(d)}$ is a finite morphism. Let $W$ and $W'$ be any local systems on $X$. Then the complex
\[
\phi_! (\mathcal{P}_{W,\psi} \otimes \mathcal{P}_{W',\psi^{-1}})
\]
is placed in usual cohomological degree $-2d$. This is seen by calculating this direct image with respect to the stratification of $\mathcal{Y}_d$ by $\mu \mathcal{Y}_d$. For $G = \text{GL}_n$ the complex (9) is a Rankin-Selberg integral considered in [11].

3.3 In this subsection we prove Proposition 6

Lemma 2. Let $\lambda \in \Lambda^+, \mu \in \Lambda$. Let $w \in W$ be any element of the Weil group such that $\mu + w\lambda \in \Lambda^+$. The function $\chi_{\mu,\lambda}^w : S^{w\lambda} \rightarrow A^1$ is constant on $S^{w\lambda} \cap \overline{\text{Gr}^\lambda}$ if and only if $\mu \in \Lambda^+$.

Proof This follows from the description of $S^{w\lambda} \cap \overline{\text{Gr}^\lambda}$ given in ([14], Lemma 5.2). □

Proof of Proposition 7. The fibre of $p \times \text{supp} : \mathcal{H}^{+,1}_G \rightarrow \text{Bun}_G \times X$ over a $k$-point $(F_G, x)$ is identified with $\mathcal{G}_{-w_0(\gamma)}^{G}$. This fibre is stratified by the subschemes
\[
S^{-w_0(\gamma)} \cap \mathcal{G}_{-w_0(\gamma)}^{G}
\]
indexed by $w(\gamma) \in W_G$. By ([14], Lemma 5.2), (10) is an affine space of dimension $\langle \gamma + w(\gamma), \tilde{\rho} \rangle$.

Consider a $k$-point of $\mu \mathcal{Y}_d$ given by $(F_G, \kappa, D, D^{\text{pos}})$. Let $(F_G, F_G', \beta, x \in X) \in \mathcal{H}^{+,1}_G$ correspond to a point of $S^{-w_0(\gamma)} \cap \mathcal{G}_{-w_0(\gamma)}^{G}$ for some $w(\gamma) \in W_G$. The image of this collection under the map
\[
q_{\mathcal{Y}} : \mathcal{Y}_d \times \text{Bun}_G \mathcal{H}^{+,1}_G \rightarrow \mathcal{Y}_{d+1}
\]
is the point of $\mu' \mathcal{Y}_{d+1}$ given by
\[
(F_G', \kappa', D + x, D^{\text{pos}} + w_0 w(\gamma) x - w_0(\gamma) x)
\]

15
with $\mu' = \mu + w_0(\gamma) - w_0(\gamma)$. If $\gamma D + w_0(D^{\text{pos}}) + w(\gamma)x$ is not dominant then, by Proposition 5, the stratum (10) does not contribute to the direct image (5).

Assume $\gamma D + w_0(D^{\text{pos}}) + w(\gamma)x$ dominant. Let $d_x \in \mathbb{Z}_+$ (resp., $\mu_x \in \Lambda^{\text{pos}}$) denote the multiplicity of $x$ in $D$ (resp., in $D^{\text{pos}}$). Then the restriction of $\text{ev}_{\mu'} \circ \mathcal{Q}_y$ to the stratum (10) is $(N(\check{K}), \chi_\nu)$-equivariant for some admissible character $\chi_\nu : N(\check{K}) \rightarrow \mathbb{G}_a$ of conductor $\nu = -w_0(\gamma)d_x - \mu_x$. If $\nu$ is dominant then this restriction is constant and equals $\text{ev}_\mu(\mathcal{F}_G, \kappa, D, D^{\text{pos}})$. If $\nu$ is not dominant then, by Lemma 2, the stratum (10) does not contribute to the direct image (5), because $R\Gamma_c(A^1, \mathcal{L}_\psi) = 0$.

We conclude that the restriction of $[\mathfrak{m}]$ to $\mu \mathcal{Y}_d \times X$ vanishes outside the closed substack $\mu \mathcal{Y}_d^+ \times X$, and is isomorphic to

$$F[d + 1 + d_N + \langle \gamma d - \mu, 2\check{\rho} \rangle]$$

for some sheaf $F$ on $\mu \mathcal{Y}_d^+ \times X$ placed in usual degree zero. An equivariance argument (as in the proof of Proposition 5) assures that $F \otimes \text{ev}_\mu^* \mathcal{L}_\psi^{-1}$ descends with respect to the projection $\mu \mathcal{Y}_d^+ \times X \rightarrow X^{d\mu} \times X$.

Since $\dim(\mu \mathcal{Y}_d^+) \leq d + d_N + \langle \gamma d - \mu, 2\check{\rho} \rangle$ for any $\mu \in \Lambda^{\text{pos}}$, the complex $[\mathfrak{m}]$ is placed in perverse degrees $\leq 0$. Proposition 6 is proved. $\square$

4. Additional structure on Levi subgroups

4.1 Throughout this section, we fix a standard parabolic subgroup $P \subset G$ corresponding to a subset $\mathcal{I}_M \subset \mathcal{I}$. Let $\Lambda_{G,P}$ be the quotient of $\Lambda$ by the $\mathbb{Z}$-span of $\alpha_i, i \in \mathcal{I}_M$. Let

$$\check{\Lambda}_{G,P} = \{ \check{\lambda} \in \check{\Lambda} | \langle \alpha_i, \check{\lambda} \rangle = 0 \text{ for } i \in \mathcal{I}_M \}$$

be the dual lattice. Let $U(P)$ be the unipotent radical of $P$ and $M = P/U(P)$. Write $\Lambda^+_M \subset \Lambda$ for the semigroup of dominant coweights for $M$, and $w_0^M$ for the longest element of the Weil group $W_M$ of $M$. The irreducible $M$-module of h.w. $\lambda$ is denoted $U^\lambda$. Fix a section $M \rightarrow P$ of the projection $P \rightarrow M$.

**Definition 4.** Set $\Lambda^+_M = \{ \lambda \in \Lambda^+_M | \text{ there exists } w \in W \text{ such that } w\lambda \in \Lambda^+_G \}$. Let $\pi_1(M) \subset \pi_1(M) = \Lambda_{G,P}$ denote the image of the projection $\Lambda^+_M \rightarrow \Lambda_{G,P}$. Set

$$\check{\Lambda}^+_M = \{ \check{\lambda} \in \check{\Lambda}^+_M | \text{ there exists } w \in W \text{ such that } w\check{\lambda} \in \check{\Lambda}^+_S \}$$

The following identities are straightforward.

**Lemma 3.** We have

$$\Lambda^+_M = \{ \lambda \in \Lambda^+_M | \langle w(\lambda), \check{\lambda} \rangle \geq 0 \text{ for all } w \in W, \check{\lambda} \in \check{\Lambda}^+_S \}$$

$$\Lambda^+_M = \{ \lambda \in \Lambda^+_M | \langle w_0^M(\lambda), \check{\lambda} \rangle \geq 0 \text{ for all } \check{\lambda} \in \check{\Lambda}^+_M \}$$

$$\check{\Lambda}^+_M = \{ \check{\lambda} \in \check{\Lambda}^+_M | \langle w(\lambda), \check{\lambda} \rangle \geq 0 \text{ for all } w \in W, \lambda \in \Lambda^+_G \}$$
\[ \hat{\Lambda}_{M,S}^+ = \{ \hat{\lambda} \in \hat{\Lambda}_M^+ \mid (w_0^M(\lambda), \hat{\lambda}) \geq 0 \text{ for all } \lambda \in \Lambda_{M,S}^+ \} \]

**Definition 5.** Let \( \text{Gr}_M^+ \subset \text{Gr}_M \) be the closed subscheme given by the condition: \((\mathcal{F}_M, \beta) \in \text{Gr}_M^+\) iff for each \( \hat{\lambda} \in \hat{\Lambda}_{M,S}^+ \) the map

\[ \beta^\hat{\lambda} : U^\hat{\lambda}_{\mathcal{F}_M} \to U^0_{\mathcal{F}_M} \]

is regular on \( \mathcal{D} \). Here \( U^\hat{\lambda} \) stands for the irreducible quotient of the Weil module \( U^\hat{\lambda} \) for \( M \).

Clearly, \( \text{Gr}_M^+ \) is a \( M(\hat{\mathcal{O}}) \)-invariant subscheme of \( \text{Gr}_M \). For \( \nu \in \Lambda_M^+ \) one has the closed subscheme \( \text{Gr}_M^\nu \subset \text{Gr}_M \) (cf. [2], sect. 3.2). For \( \nu \in \Lambda_M^+ \) we have \( \text{Gr}_M^\nu \subset \text{Gr}_M^+ \) if \( \nu \in \Lambda_{M,S}^+ \).

Recall that for \( \mu \in \pi_1(M) \) the connected component \( \text{Gr}_M^\mu \) of \( \text{Gr}_M \) classifies \((\mathcal{F}_M, \beta) \in \text{Gr}_M \) such that for \( \hat{\lambda} \in \hat{\Lambda}_{G,P} \) we have

\[ \mathcal{V}^\lambda_{\mathcal{F}_M}((\mu, \hat{\lambda})) \xrightarrow{\sim} \mathcal{V}^\lambda_{\mathcal{F}_G} \]

For \( \mu \in \pi_1(M) \) set \( \text{Gr}_M^{+;\mu} = \text{Gr}_M^+ \cap \text{Gr}_M^\mu \). So, \( \text{Gr}_M^{+;\mu} \) is nonempty iff \( \mu \in \pi_1^+(M) \). It may be shown that \( \text{Gr}_M^{+;\mu} \) is connected for each \( \mu \in \pi_1^+(M) \).

Recall the following definition from ([2], sect. 4.3.1). For \( \mu \in \Lambda_{G,P} \) let \( S_P^\mu \subset \text{Gr}_G \) denote the locally closed subscheme classifying \((\mathcal{F}_G, \beta : \mathcal{F}^0_G |_{Q^\mu} \xrightarrow{\sim} \mathcal{F}_G |_{Q^\mu}) \in \text{Gr}_G \) such that the composition

\[ \mathcal{L}^{\mu}_{\mathcal{F}_G}((\nu, \hat{\lambda})) \to \mathcal{L}^{\mu}_{\mathcal{F}_P} \to \mathcal{V}^\lambda_{\mathcal{F}_G} \]

has neither pole nor zero over \( \mathcal{D} \) for every \( \hat{\lambda} \in \hat{\Lambda}_{G,P} \cap \hat{\Lambda}^+ \).

For each \( \nu \in \pi_1(G) \) the component \( \text{Gr}_G^\nu \) is stratified by \( S_P^\mu \) indexed by those \( \mu \in \Lambda_{G,P} \) whose image in \( \pi_1(G) \) is \( \nu \). Moreover, we have a natural map \( t^\mu_S : S_P^\mu \to \text{Gr}_M^\nu \).

**Lemma 4.** For each \( \mu \in \Lambda_{G,P} \) the map \( t^\mu_S : \text{Gr}_G^\mu \cap S_P^\mu \to \text{Gr}_M^\mu \) factors through \( \text{Gr}_M^{+;\mu} \to \text{Gr}_M^\mu \), and the induced map \( t^{+;\mu}_S : \text{Gr}_G^\mu \cap S_P^\mu \to \text{Gr}_M^{+;\mu} \) is surjective.

**Proof.** Let \((\mathcal{F}_G, \beta : \mathcal{F}^0_G |_{Q^\mu} \xrightarrow{\sim} \mathcal{F}_G |_{Q^\mu}) \in \text{Gr}_G^\mu \cap S_P^\mu \). So, for any \( \hat{\lambda} \in \hat{\Lambda}_{G,P} \cap \hat{\Lambda}^+ \)

\[ \mathcal{L}^{\mu}_{\mathcal{F}_G}((\nu, \hat{\lambda})) \to \mathcal{V}^\lambda_{\mathcal{F}_G} \]

is a subbundle. There is unique \((\mathcal{F}_P, \beta : \mathcal{F}^0_P |_{Q^\mu} \xrightarrow{\sim} \mathcal{F}_P |_{Q^\mu}) \in \text{Gr}_P \) that induces \((\mathcal{F}_G, \beta) \), and \( t^\mu_S \) sends \((\mathcal{F}_G, \beta) \) to \( \mathcal{F}_M = \mathcal{F}_P \times_P M \). Since for any \( \hat{\lambda} \in \hat{\Lambda}_S^+ \) the maps

\[ \beta^\hat{\lambda} : U^\hat{\lambda}_{\mathcal{F}_P} \to U^0_{\mathcal{F}_P} \]

are regular, the first assertion is reduced to the next sublemma.

**Sublemma 1.** Let \( \hat{\nu} \in \Lambda_{M,S}^+ \). So, there exists \( w \in W \) with \( w \hat{\nu} \in \Lambda_S^+ \). Let \( \text{Res}_{G,P}^P V^{w\hat{\nu}} \) denote \( V^{w\hat{\nu}} \) viewed as a \( P \)-module. There exists a subquotient \( V' \) of \( \text{Res}_{G,P}^P V^{\hat{\nu}} \) on which \( U(P) \) acts trivially and such that

\[ \text{Hom}_M(U^\hat{\nu}, V') \neq 0 \]
Let $\nu \in \Lambda^+_{M,S}$ be such that $\text{Gr}^\nu_M \subset \text{Gr}^{+,\mu}_M$. Recall the notation $\hat{\mathcal{O}} = k[[t]]$. Since $t^{+,\mu}_S$ is $M(\hat{\mathcal{O}})$-invariant, it suffices to show that $\nu(t) \in \text{Gr}^{+,\mu}_M$ lies in the image of $t^{+,\mu}_S$. We know that there exists $w \in \mathcal{W}$ with $w\nu \in \Lambda^+_{G,S}$. Therefore, $\nu(t)\mathcal{G}(\hat{\mathcal{O}})$ defines a point of $\text{Gr}^+_G \cap S^\mu_P$, which is sent by $t^{+,\mu}_S$ to $\nu(t) \in \text{Gr}^{+,\mu}_M$.

\[\square\text{(Lemma 4)}\]

Note as a consequence that for each $\nu \in \pi^+_1(G)$ the scheme $\text{Gr}^{+,\nu}_G$ is stratified by locally closed subschemes $\text{Gr}^{+,\nu}_G \cap S^\mu_P$ indexed by those $\mu \in \pi^+_1(M)$ whose image in $\pi_1(G)$ is $\nu$.

For $d \geq 0$ write $\Lambda^+_{M,S}(\nu)$ for the preimage of $d\theta$ under $\Lambda^+_{M,S} \to \pi^+_1(G)$.

**Lemma 5.** For any $\lambda \in \Lambda^+_{M,S}(\nu)$ there exist $\lambda_1, \ldots, \lambda_d \in \Lambda^{+,\theta}_{M,S}$ such that $\lambda \leq \lambda_1 + \ldots + \lambda_d$.

**Proof.** Pick any $k$-point $(\mathcal{F}_M, \beta)$ of $\text{Gr}^{\mu}_M$. Let $\mu$ be the image of $\lambda$ in $\pi^+_1(M)$. Pick any $k$-point $(\mathcal{F}_G, \beta)$ of $\text{Gr}^{+,\nu}_G \cap S^\mu_P$ whose $t^{+,\nu}_S$-image is $(\mathcal{F}_M, \beta)$. Let $\text{Gr}^{\gamma}_G \times \ldots \times \text{Gr}^{\gamma}_G$ be the scheme classifying collections

$$
(\mathcal{F}^1_G, \ldots, \mathcal{F}^{d+1}_G = \mathcal{F}_G, \beta_i),
$$

where $\mathcal{F}^i_G$ is a $G$-bundle on $\mathcal{D}$, and $\beta_i : \mathcal{F}^i_G |_{\mathcal{D}^i} \to \mathcal{F}^{i+1}_G |_{\mathcal{D}^i}$ is an isomorphism such that $\mathcal{F}^i_G$ is in the position $\gamma$ with respect to $\mathcal{F}^{i+1}_G$ for $i = 1, \ldots, d$.

Pick a $k$-point $(\mathcal{F}_G, \beta)$ whose image under the convolution map

$$
\text{Gr}^{\gamma}_G \times \ldots \times \text{Gr}^{\gamma}_G \to \overline{\text{Gr}^{\gamma}_G},
$$

is $(\mathcal{F}_G, \beta)$. There exist a unique collection

$$(\mathcal{F}^1_M, \ldots, \mathcal{F}^d_M, \mathcal{F}^{d+1}_M = \mathcal{F}_M, \beta_i),$$

where $\mathcal{F}^i_M$ is a $P$-torsor on $\mathcal{D}$ and $\beta_i : \mathcal{F}^i_M |_{\mathcal{D}^i} \to \mathcal{F}^{i+1}_M |_{\mathcal{D}^i}$, that induces $(\mathcal{F}_G, \beta)$ by extension of scalars from $P$ to $G$. Extending the scalars from $P$ to $M$, one gets a collection

$$(\mathcal{F}^1_M, \ldots, \mathcal{F}^d_M, \mathcal{F}^{d+1}_M = \mathcal{F}_M, \beta_i),$$

where $\mathcal{F}^1_M = \mathcal{F}_M$. For $i = 1, \ldots, d$ let $\lambda_i \in \Lambda^+_{M,S}(\nu)$ be such that $\mathcal{F}^i_M$ is in the position $\lambda_i$ with respect to $\mathcal{F}^{i+1}_M$. By Lemma 4, $\lambda_i \in \Lambda^{+,\theta}_{M,S}$ for all $i$. Let

$$
\text{Gr}^{\lambda_1}_M \times \ldots \times \text{Gr}^{\lambda_d}_M
$$

be the scheme classifying $(\mathcal{F}^1_M, \ldots, \mathcal{F}^{d+1}_M = \mathcal{F}_M, \beta)$, where $\mathcal{F}^i_M$ is a $M$-torsor on $\mathcal{D}$ and $\beta_i : \mathcal{F}^i_M |_{\mathcal{D}^i} \to \mathcal{F}^{i+1}_M |_{\mathcal{D}^i}$ is an isomorphism such that $\mathcal{F}^i_M$ is in the position $\lambda_i$ with respect to $\mathcal{F}^{i+1}_M$ for $i = 1, \ldots, d$. We learn that $(\mathcal{F}_M, \beta)$ lies in the image of the convolution map

$$
\text{Gr}^{\lambda_1}_M \times \ldots \times \text{Gr}^{\lambda_d}_M \to \text{Gr}^{+,\mu}_M
$$

But the image of the latter map is contained in $\overline{\text{Gr}^{\lambda_1+\ldots+\lambda_d}_M}$, so $\text{Gr}^{\lambda}_M \subset \overline{\text{Gr}^{\lambda_1+\ldots+\lambda_d}_M}$. \[\square\]

Denote by $\pi^\theta_1(M)$ the image of $\Lambda^+_{M,S}(\nu) \to \pi^+_1(M)$. By the above lemma, $\pi^\theta_1(M)$ generates $\pi^+_1(M)$ as a semigroup. Since $V^\gamma$ is faithful, $\pi^\theta_1(M)$ generates $\pi_1(M)$ as a group.
Lemma 6. 1) Each $\lambda \in \Lambda_{M,S}^{+,\theta}$ is a minuscule dominant coweight for $M$.
2) The natural map $\Lambda_{M,S}^{+,\theta} \rightarrow \pi_1(M)$ is bijective.

Proof 1) For $\lambda \in \Lambda_{M,S}^{+,\theta}$ we have $\text{Hom}_M(U^\lambda, V^\gamma) \neq 0$. Let $\lambda' \in \Lambda_{M}^{+}$ and $\lambda' \leq \lambda$. Then $\lambda'$ is a weight of $V^\gamma$, so that $\lambda' = w \gamma$ for some $w \in W$, and $\lambda = w \gamma + \alpha$, where $\alpha$ is a sum of positive coroots for $M$. However, if $\alpha \neq 0$ then the length of $\lambda$ is strictly bigger than the length of $\gamma$, which implies that $\lambda$ is not a weight of $V^\gamma$. This contradiction shows that $\lambda' = \lambda$.

2) follows from 1). □

4.2 For $d \geq 0$ consider the stack

$$\mathcal{H}^{+,d}_{G} \times_{\text{Bun}_G} \text{Bun}_P,$$

(12)

where we used the projection $q : \mathcal{H}^{+,d}_{G} \rightarrow \text{Bun}_G$ in the fibred product.

For $\mu \in \Lambda_{G,P}$ let $\mathcal{H}^{+,\mu}_{P}$ be the locally closed substack of (12) classifying

$$(D \in X^{(d)}, F_G, \beta : F_G |_{X-D} \rightarrow F'_{P} |_{X-D}, F'_P \times_P G \rightarrow F'_G)$$

for which there exists a $\Lambda_{G,P}$-valued divisor $D^\mu$ on $X$ of degree $\mu$ with the property: for all $\lambda \in \Lambda_{G,P} \cap \Lambda^+$ the meromorphic maps

$$L^\lambda_{F_{M/[M,M]}} \rightarrow V^\lambda_{F_G} \rightarrow V^\lambda_{F_G}$$

realise $L^\lambda_{F_{M/[M,M]}}$ as a subbundle of $V^\lambda_{F_G}(D^\mu, \lambda)$. Here we have denoted $F'_M = F'_P \times_P M$.

Clearly, we have $\langle D^\mu, \omega_0 \rangle = D$ and $\langle D^\mu, \omega_i \rangle \geq 0$ for $i \in I - I_M$. Let $D_i = \langle D^\mu, \omega_i \rangle$ for $i \in I - I_M$ then we have an equality of $\pi_1(M)$-valued divisors on $X$

$$D^\mu = w_0(\gamma)D + \sum_{i \in I - I_M} D_i \alpha_i$$

By Lemma 4 $\mathcal{H}^{+,\mu}_{P}$ is non empty iff $\mu \in \pi_1^+(M)$ and actually $D^\mu$ is a $\pi_1^+(M)$-valued divisor on $X$. So, for each $d \geq 0$ the stack (12) is stratified by locally closed substacks $\mathcal{H}^{+,\mu}_{P}$ indexed by those $\mu \in \pi_1^+(M)$ whose image in $\pi_1(G)$ is $d \theta$.

For $\mu \in \pi_1^+(M)$ let $d = \langle \mu, \omega_0 \rangle$ and $d_i = \langle \mu, \omega_i \rangle$ for $i \in I - I_M$ and let $X^\mu_M$ denote the scheme image of the projection

$$\mathcal{H}^{+,\mu}_{P} \rightarrow X^{(d)} \times \prod_{i \in I - I_M} X^{(d_i)}$$

We will think of $X^\mu_M$ as the moduli scheme of $\pi_1^+(M)$-valued divisors on $X$ of degree $\mu$. As we will see, $X^\mu_M$ need not be irreducible. For $\mu \in \pi_1^+(M)$ we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^{+,\mu}_{P} & \hookrightarrow & \mathcal{H}^{+,d}_{G} \times_{\text{Bun}_G} \text{Bun}_P \\
\downarrow \text{supp}_P & & \downarrow \\
X^\mu_M & \xrightarrow{s_M} & X^{(d)}
\end{array}$$

where we have denoted by $\text{supp}_P$ and $s_M$ the natural projections.
Definition 6. For \( \mu \in \pi_1^+(M) \) let \( \mathcal{H}_M^{+\mu} \) be the \( \mathcal{H}_M^{+\mu} \) be the stack of collections \((\mathcal{F}_M, \mathcal{F}_M', D^\mu \in X_M^\mu, \beta)\), where \( \mathcal{F}_M, \mathcal{F}_M' \in \text{Bun}_M \) and
\[
\beta : \mathcal{F}_M |_{X - D} \cong \mathcal{F}_M' |_{X - D}
\]
is an isomorphism of \( M \)-torsors with \( D = s_M(D^\mu) \) such that for each \( \tilde{\lambda} \in \tilde{\Lambda}_{M,S}^+ \) the map
\[
\beta^{\tilde{\lambda}} : U^{\tilde{\lambda}}_{\mathcal{F}_M} \hookrightarrow U^{\tilde{\lambda}}_{\mathcal{F}_M'}
\]
extends to an inclusion of coherent sheaves on \( X \), and
\[
\mathcal{L}^{\tilde{\lambda}}_{\mathcal{F}_M/M,M}(\langle D^\mu, \tilde{\lambda} \rangle) \cong \mathcal{L}^{\tilde{\lambda}}_{\mathcal{F}_M/M,M}
\]
for each \( \tilde{\lambda} \in \tilde{\Lambda}_{G,P} \).

We have a diagram
\[
\begin{array}{ccc}
\text{Bun}_M & \xleftarrow{\text{p}_M} & \mathcal{H}_M^{+\mu} & \xrightarrow{\text{q}_M} & \text{Bun}_M \\
\downarrow & & \downarrow \text{supp}_M & & \\
X_M^\mu & & & & \\
\end{array}
\]
where \( \text{p}_M \) (resp., \( \text{q}_M \)) sends a point of \( \mathcal{H}_M^{+\mu} \) to \( \mathcal{F}_M \) (resp., to \( \mathcal{F}_M' \)), and \( \text{supp}_M \) stands for the projection. If \( (\mathcal{F}_M', D^\mu = \sum_k \mu_k x_k) \) is a \( k \)-point of \( \text{Bun}_M \times X_M^\mu \), then the fibre of \( \text{q}_M \times \text{supp}_M : \mathcal{H}_M^{+\mu} \to \text{Bun}_M \times X_M^\mu \)
over it identifies with \( \prod_k \text{Gr}_M^{+\mu_k} \).

4.3 Given \( \mu \in \pi_1^+(M) \), we will denote by \( \mathfrak{A}(\mu) \) the elements of the set of decompositions of \( \mu \) as a sum of non-zero elements of \( \pi_1^+(M) \). More precisely, \( \mathfrak{A}(\mu) \) is a way to write \( \mu = \sum_k n_k \mu_k \),
where all \( n_k > 0 \) and \( \mu_k \in \pi_1^+(M) - \{0\} \) are pairwise distinct.

For \( \mathfrak{A}(\mu) \) we set \( X^{\mathfrak{A}(\mu)} = \prod_k X^{(n_k)} \). We have a natural map \( s^{\mathfrak{A}(\mu)} : X^{\mathfrak{A}(\mu)} \to X_M^\mu \)
sending \( \{D_k\} \) to \( \sum_k \mu_k D_k \). Let \( \tilde{X}^{\mathfrak{A}(\mu)} \subset X^{\mathfrak{A}(\mu)} \) be the complement to all diagonals. The composition
\[
\tilde{X}^{\mathfrak{A}(\mu)} \hookrightarrow X^{\mathfrak{A}(\mu)} \to X_M^\mu
\]
is a locally closed embedding, and in this way \( X_M^\mu \) is stratified by subschemes \( \tilde{X}^{\mathfrak{A}(\mu)} \).

We say that \( \mathfrak{A}(\mu) \) is in general position if \( \sum_k n_k = d \). Write \( \text{rss} X_M^\mu \) for the preimage of \( \text{rss} X(d) \) under \( s_M : X_M^\mu \to X(d) \). The connected components of \( \text{rss} X_M^\mu \) are exactly \( \tilde{X}^{\mathfrak{A}(\mu)} \) indexed by \( \mathfrak{A}(\mu) \) in general position.

Definition 7. Given a local system \( W \) on \( X \), for each \( \mu \in \pi_1^+(M) \) define Laumon’s sheaf \( \mathcal{L}_W^\mu \) on \( \mathcal{H}_M^{+\mu} \) as follows. Recall the diagram
\[
\mathcal{H}_M^{+\mu} \xrightarrow{\text{supp}_M} X_M^\mu \xrightarrow{s_M} X(d)
\]
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Let \( rss \mathcal{H}_M^{+\mu} \) be the preimage of \( rss X^{(d)} \) under \( s_M \circ \text{supp}_M \). The stack \( rss \mathcal{H}_M^{+\mu} \) is smooth and over it we let

\[
\mathcal{L}_W^\mu = \text{supp}_M s_M^* W^{(d)}[a](\frac{q}{2}),
\]

where \( a \) denotes the dimension of the corresponding connected component of \( rss \mathcal{H}_M^{+\mu} \). Then we extend this perverse sheaf by Goresky-MacPherson to \( \mathcal{H}_M^{+\mu} \).

By definition, \( \mathbb{D}(\mathcal{L}_W^\mu) \cong \mathcal{L}_W^\mu \), and \( \mathcal{L}_W^\mu \cong \mathcal{L}_W^\mu \otimes \mathcal{L}_W^{\mathfrak{A}(\mu)} \) is a direct sum of perverse sheaves indexed by \( \mathfrak{A}(\mu) \) in general position. Set \( d_M = \dim \text{Bun}_M \).

For \( \mu \in \pi_1^+(M) \) whose image in \( \pi_1(G) \) is \( d \theta \) consider the diagram

\[
\mathcal{H}_M^{+\mu} \times_{\text{Bun}_M} \text{Bun}_P \xrightarrow{q_M} \mathcal{H}_P^{+\mu} \xrightarrow{f_M} \mathcal{H}_G^{+\dot{d}},
\]

where we used \( q_M : \mathcal{H}_M^{+\mu} \to \text{Bun}_M \) in the fibred product, \( f_M \) is the natural map, and \( q_M \) sends \( (\mathcal{F}_P, \mathcal{F}'_P, D^\mu, \beta) \) to \( (\mathcal{F}_M, \mathcal{F}'_M, D^\mu, \mathcal{F}'_P \times P M \to \mathcal{F}_M, \beta) \). We also have

\[
\mathcal{H}_M^{+\mu} \times_{\text{Bun}_M} \text{Bun}_P \xleftarrow{p_M} \mathcal{H}_P^{+\mu} \xrightarrow{f_M} \mathcal{H}_G^{+\dot{d}},
\]

where now we used \( p_M : \mathcal{H}_M^{+\mu} \to \text{Bun}_M \) in the fibred product, and \( p_M \) sends \( (\mathcal{F}_P, \mathcal{F}'_P, D^\mu, \beta) \) to \( (\mathcal{F}'_M, \mathcal{F}_P, D^\mu, \mathcal{F}'_P \times P M \to \mathcal{F}_M, \beta) \).

Here is a generalization of Laumon’s theorem ([10], Theorem 4.1).

**Proposition 8.** Let \( W \) be a local system on \( X \). Let \( \mu \in \pi_1^+(M) \) with image \( d \theta \) in \( \pi_1^+(G) \). The complex \( q_M f_M^* \mathcal{L}_W^d \) is canonically isomorphic to the inverse image of

\[
\mathcal{L}_W^\mu \otimes \mathbb{Q}_\ell[1](\frac{1}{2})^\otimes d_G - d_M + (\mu, 2\rho_M - 2\check{\rho})
\]

under the projection \( \mathcal{H}_M^{+\mu} \times_{\text{Bun}_M} \text{Bun}_P \to \mathcal{H}_M^{+\mu} \). (We have used the fact that \( \check{\rho}_M - \check{\rho} \in \check{\Lambda}_{G,P} \).

The complex \( p_M f_M^* \mathcal{L}_W^d \) is canonically isomorphic to the inverse image of

\[
\mathcal{L}_W^\mu \otimes \mathbb{Q}_\ell[1](\frac{1}{2})^\otimes d_G - d_M - (\mu, 2\rho_M - 2\check{\rho})
\]

under the projection \( \mathcal{H}_M^{+\mu} \times_{\text{Bun}_M} \text{Bun}_P \to \mathcal{H}_M^{+\mu} \).

The proof is given in Sections 4.4-4.5.

4.4 Let \( J = \{ i \in I \mid \langle \gamma, \check{\alpha}_i \rangle = 0 \} \). Let \( W_J \subset W \) be the subgroup generated by the reflection corresponding to \( i \in J \). Using Bruhat decomposition, one checks that the map \( W/W_J \to W \gamma \) sending \( w \) to \( w \gamma \) is a bijection.

Fix a section \( T \to B \). Let \( P_\gamma \) denote the parabolic of \( G \) generated by \( T \) and \( U_\check{\alpha} \) for all roots \( \check{\alpha} \) such that \( \langle \gamma, \check{\alpha} \rangle \leq 0 \). So, \( P_\gamma \) contains the opposite Borel. We have a bijection \( \Lambda_{M,S}^{+,\check{\theta}} \to W_M \backslash W/W_J \) sending \( w \gamma \in \Lambda_{M,S}^{+,\check{\theta}} \) to the coset \( W_M w W_J \).
The map $G/P_\gamma \to \text{Gr}^\gamma_G$ sending $g \in G(k) \subset G(\hat{\mathcal{O}})$ to $g\gamma(t)G(\hat{\mathcal{O}})$ is an isomorphism. The scheme $\text{Gr}^\gamma_G$ is stratified by $\text{Gr}^\gamma_G \cap S^\lambda_P$ indexed by $\lambda \in \Lambda^+_{M,S}$. The above isomorphism transforms this stratification into the stratification of $G/P_\gamma$ by $P$-orbits. We have a disjoint decomposition

$$G = \bigsqcup_{w \in W_M \setminus W/J} PwP_\gamma$$

So, we have

$$\text{Gr}^\gamma_G \cap S^\lambda_P \cong PwP_\gamma/P_\gamma \cong P/P \cap wP_\gamma w^{-1}$$

Similarly, for $\lambda \in \Lambda^+_{M,S}$ let $P_\lambda(M)$ be the parabolic of $M$ generated by $T$ and $U_\alpha$, where $\alpha$ runs through those roots of $M$ for which $(\lambda, \alpha) \leq 0$. Then the map $M/P_\lambda(M) \to \text{Gr}^\lambda_M$ sending $m \in M(k)$ to $m\lambda(t)M(\hat{\mathcal{O}})$ is an isomorphism. So, the map

$$\text{Gr}^\gamma_G \cap S^{w_\gamma}_P \to \text{Gr}^{w_\gamma}_M$$

(13)

is nothing else but the map $P/P \cap wP_\gamma w^{-1} \to M/P_{w\gamma}(M)$ sending $p$ to $p \mod M$. The correctness is due to

**Lemma 7.** We have $P_{w\gamma}(M) = M \cap wP_\gamma w^{-1}$.

**Proof** The inclusion $P_{w\gamma}(M) \subset M \cap wP_\gamma w^{-1}$ follows from definitions. Further, $M \cap wP_\gamma w^{-1}$ contains the opposite Borel of $M$, hence is a parabolic subgroup of $M$ (in particular, connected). The assertion follows now from: for a root $\alpha$ of $M$ we have $U_\alpha \subset P_{w\gamma}(M)$ if and only if $U_\alpha \subset M \cap wP_\gamma w^{-1}$. □

We see that $U(P)$ acts transitively on the fibres of $13$. So, $13$ is a fibration with fibre isomorphic to an affine space of dimension $\langle \gamma + w\gamma, \hat{\rho} \rangle - \langle w\gamma, 2\hat{\rho}_M \rangle$.

**Remark 2.** For $w\gamma \in \Lambda^+_{M}$ one can calculate the dimension of $\text{Gr}^\gamma_G \cap S^{w_\gamma}_P$ as follows. Stratify it by $B$-orbits, that is, by the schemes $\text{Gr}^\gamma_G \cap S^{w_1w_\gamma}_P$ with $w_1 \in W_M$. Then $\text{Gr}^\gamma_G \cap S^{w_1w_\gamma}_P$ is an affine space of dimension $\langle \gamma + w_1w\gamma, \hat{\rho} \rangle$. The maximum of these numbers, as $w_1$ ranges through $W_M$ is $\langle \gamma + w\gamma, \hat{\rho} \rangle$.

4.5 Consider a collection $\tilde{\mu} = (\mu_1, \ldots, \mu_d)$ with $\mu = \mu_1 + \cdots + \mu_d$ and $\mu_i \in \pi^\theta_1(M)$. Let $\mathcal{H}^{+\tilde{\mu}}_M$ be the stack of collections

$$(\mathcal{F}^i_M, \mathcal{F}^{i+1}_M, x_1, \ldots, x_d \in X, \beta^i),$$

(14)

where $\beta^i : \mathcal{F}^i_M |_{x-x_i} \cong \mathcal{F}^{i+1}_M |_{x-x_i}$ is an isomorphism such that $(\mathcal{F}^i_M, \mathcal{F}^{i+1}_M, \beta^i, x_i) \in \mathcal{H}^{+,\mu_i}_M$ for $i = 1, \ldots, d$.

If we denote by $\lambda_i$ the element of $\Lambda_{M,S}^{+,\theta}$ that maps to $\mu_i$ then $\mathcal{F}^i_M$ is in the position $\lambda_i$ with respect to $\mathcal{F}^{i+1}_M$ at $x_i$. We have a convolution map

$$\text{conv}^{\tilde{\mu}} : \mathcal{H}^{+\tilde{\mu}}_M \to \mathcal{H}^{+,\mu}_M$$

sending $14$ to $(\mathcal{F}^1_M, \mathcal{F}^{i+1}_M, \beta, D^\mu)$, where $D^\mu = \sum_i \mu_i x_i$ and $\beta : \mathcal{F}^1_M |_{x-x} \cong \mathcal{F}^{i+1}_M |_{x-D}$ with $D = s_M(D^\mu)$.

Denote by $s^{\tilde{\mu}} : \mathcal{H}^{+,\tilde{\mu}}_M \to X^d$ the map sending $14$ to $(x_1, \ldots, x_d)$. From (Lemma 9.3, 14) one derives

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Lemma 8. i) The map \( \text{conv}^\mu \) is representable, proper and small over its image. Besides, the perverse sheaf

\[
\text{conv}^\mu (s^\mu)^* W^{\mathbb{R}^d} [a](\frac{a}{2})
\]

(15)
is the Goresky-MacPherson extension from \( rss \mathcal{H}^+_{M,\mu} \). Here \( a = \dim \mathcal{H}^+_{M,\mu} \).

ii) The \( d \)-tuple \( \tilde{\mu} \) gives rise to \( \mathcal{A}(\mu) \) in general position, say \( \mu = \sum_k n_k \nu_k \). The group \( \prod_k S_{n_k} \) acts naturally on \( \tilde{\mathcal{L}}_M \), and the sheaf of \( \prod_k S_{n_k} \)-invariants is canonically isomorphic to the direct summand of \( \mathcal{L}_M \) corresponding to \( \mathcal{A}(\mu) \). □

Remark 3. From Lemma 8 it follows that for any \( \mu \in \pi^+_1(M) \) the complex \( \mathcal{L}^\mu_W \) is ULA with respect to both projections \( p_M, q_M : \mathcal{H}^+_{M,\mu} \rightarrow \text{Bun}_M \).

Proof of Proposition 8
1) Consider the diagram

\[
\begin{array}{c}
\mathcal{H}^+_{P,\mu} \xrightarrow{q_M} \mathcal{H}^+_{M,\mu} \times_{\text{Bun}_M} \text{Bun}_P \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Bun}_P \times X^\mu_M
\end{array}
\]

For any \( k \)-point of \( \text{Bun}_P \times X^\mu_M \) given by \( (F_P, D^\mu = \sum_k \mu_k x_k) \), where \( x_k \) are pairwise distinct, over \( (F_P, D^\mu = \sum_k \mu_k x_k) \) the map \( q_M \) becomes the product

\[
\prod_k \text{Gr}^+ \cap S^\mu_P \rightarrow \prod_k \text{Gr}^+ \mu_k
\]

of the maps \( \iota^+_{S^\mu_k} \). In particular, \( q_M \) is surjective.

Let \( \mathcal{A}(\mu) \) in general position be given by \( \mu = \sum_k n_k \nu_k \). Let \( U \subset rss \mathcal{H}^+_{M,\mu} \) be the preimage of the corresponding connected component of \( rss X^\mu_M \). Over the open substack \( U \times \text{Bun}_M \text{Bun}_P \), the map \( q_M \) is a fibration with fibre isomorphic to an affine space of dimension

\[
a(\mathcal{A}(\mu)) = \langle d\gamma, \hat{\rho} \rangle + \sum_k (n_k \lambda_k, \hat{\rho} - 2 \hat{\rho}_M),
\]

where \( \lambda_k \in \Lambda^+_{M,S} \) maps to \( \mu_k \in \pi^0(M) \).

The restriction of \( f_M^* \mathcal{L}^d_W \) to \( q_M^*(U \times \text{Bun}_M \text{Bun}_P) \) comes from \( rss X^d \). So, over \( U \times \text{Bun}_M \text{Bun}_P \), we get the desired isomorphism. Now it suffices to show that, up to a shift, \( q_M f_M^* \mathcal{L}^d_W \) is a perverse sheaf, the Goresky-MacPherson extension from \( rss \mathcal{H}^+_{M,\mu} \times_{\text{Bun}_M} \text{Bun}_P \).

For a \( d \)-tuple \( \tilde{\mu} = (\mu_1, \ldots, \mu_d) \) with \( \mu = \mu_1 + \ldots + \mu_d \) and \( \mu_i \in \pi^0(M) \), let \( \mathcal{H}^+_{P,\tilde{\mu}} \) be the stack of collections

\[
(F^1_P, \ldots, F^{d+1}_P, x_1, \ldots, x_d, \beta^i),
\]

where \( x_i \in X \) and \( (F^i_P, F^{i+1}_P, x_i, \beta^i) \in \mathcal{H}^+_{P,\mu_i} \) for \( i = 1, \ldots, d \). The stack

\[
\mathcal{H}^+_{P,\mu} \times_{(\mathcal{H}^+_{G,\text{Bun}_G \text{Bun}_P}) (\mathcal{H}^+_{G,\mu} \times_{\text{Bun}_G} \text{Bun}_P)}
\]

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is stratified by locally closed substacks \( \mathcal{H}_{P}^{+} \) indexed by such tuples \( \tilde{\mu} \).

We have a diagram

\[
X^{(d)} \xrightarrow{\phi} \mathcal{H}_{P}^{+} \xrightarrow{\alpha} \mathcal{H}_{M}^{+} \times \text{Bun}_{P} \to \mathcal{H}_{M}^{+} \times \text{Bun}_{M} \text{Bun}_{P},
\]

where \( \alpha \) sends \( \mathbf{0} \) to \((F_{1}^{d}, \ldots, F_{d}^{d+1}, x_{1}, \ldots, x_{d}, \beta)\) with \( F_{i}^{j} = F_{P}^{j} \times_{P} M \), and the last map is \( \text{conv}^{\tilde{\mu}} \times \text{id} \). It suffices to show that for each \( \mu \) as above,

\[
(\text{conv}^{\tilde{\mu}} \times \text{id})_{\alpha}((s^{\tilde{\mu}})^{*}W(d))
\]

is a perverse sheaf (up to a shift), the Goresky-MacPherson extension from \( r_{s\theta}^{*}\mathcal{H}_{M}^{+} \times \text{Bun}_{M} \text{Bun}_{P} \). Since \( \alpha \) is a composition of affine fibrations, our statement about \( q_{M!}f_{M}^{*}L_{W}^{d} \) follows from Lemma 3 i).

2) Applying 1) to the 1-admissible data \( \{-w_{0}(\gamma)\} \), one gets the formula for \( p_{M!}f_{M}^{*}L_{W}^{d} \). \( \square \)

### 4.6.1 Averaging functors for Levi subgroups

Recall for any \( \mu \in \pi_{1}^{+}(M) \) the diagram \( \text{Bun}_{M} \xrightarrow{p_{M}} \mathcal{H}_{M}^{+} \xrightarrow{q_{M}^{*}} \text{Bun}_{M} \). For a local system \( W \) on \( X \) denote by \( \text{Av}_{W}^{\mu} : \text{D}(\text{Bun}_{M}) \to \text{D}(\text{Bun}_{M}) \) the functor

\[
\text{Av}_{W}^{\mu}(K) = (q_{M})!(p_{M}^{*}K \otimes L_{W}^{d})[-dM][\frac{-dM}{2}]
\]

Let also \( \text{Av}_{W}^{-\mu} : \text{D}(\text{Bun}_{M}) \to \text{D}(\text{Bun}_{M}) \) be given by

\[
\text{Av}_{W}^{-\mu}(K) = (p_{M})!(q_{M}^{*}K \otimes L_{W}^{d})[-dM][\frac{-dM}{2}]
\]

By Remark 3 we have \( \mathbb{D} \circ \text{Av}_{W}^{\mu} \cong \text{Av}_{W}^{-\mu} \circ \mathbb{D} \) and \( \mathbb{D} \circ \text{Av}_{W}^{-\mu} \cong \text{Av}_{W}^{-\mu} \circ \mathbb{D} \) naturally.

The proof of the following result is completely analogous to that of Proposition 3.

**Proposition 9.** Let \( \mu \in \pi_{1}^{+}(M) \). Let \( \mathfrak{A}(\mu) \) in general position be given by \( \mu = \sum_{k}n_{k}\mu_{k} \). Recall the map \( s_{\mathfrak{A}(\mu)} : X_{\mathfrak{A}(\mu)} \to X_{M}^{\mu} \) (cf. sect. 4.3). Let \( \lambda_{k} \in \Lambda_{M,S}^{\mu,\theta} \) be the element that maps to \( \mu_{k} \in \pi_{1}^{\theta}(M) \). Let \( W \) be any local system on \( X \). Let \( E \) be a \( M \)-local system on \( X \), \( K \) be \( E \)-Hecke eigensheaf on \( \text{Bun}_{M} \). Then for the diagram

\[
\text{Bun}_{M} \xrightarrow{p_{M}} \mathcal{H}_{M}^{+} \xrightarrow{q_{M}^{*} \times \text{supp}_{M}} \text{Bun}_{M} \times X_{M}^{\mu}
\]

there is a canonical isomorphism

\[
(q_{M} \times \text{supp}_{M})! (p_{M}^{*}K \otimes L_{W}^{d})[-dM][\frac{-dM}{2}] \cong K \boxtimes s_{\mathfrak{A}(\mu)}^{*} (K \otimes U_{E}^{\lambda_{k}}(n_{k}))[d][\frac{d}{2}]
\]

**Corollary 3.** For each standard proper parabolic \( P \) of \( G \) there exists a constant \( c(P) \) with the following property. Let \( E \) be any \( M \)-local system on \( X \), \( K \) be a \( E \)-Hecke eigensheaf on \( \text{Bun}_{M} \). Let \( W \) be an irreducible local system on \( X \) of rank \( r = \dim V \). For any \( \mu \in \pi_{1}^{+}(M) \) whose image in \( \pi_{1}(G) \) is \( d\theta \) with \( d > c(P) \) we have \( \text{Av}_{W}^{\mu}(K) = 0 \).
The functor $\mathbb{A}_W^\mu = \oplus \mathbb{A}_W^\mu \mathcal{Z}$ is a direct sum of functors indexed by $\mathfrak{A}(\mu)$ in general position. In the notation of Proposition 9 we have

$$\mathbb{A}_W^\mu (K) \to K \otimes (\otimes_k \text{Sym}^n_k \Gamma(X, W \otimes U^\lambda_k)) [d] \left( \frac{d}{2} \right)$$

(17)

Here $d = \sum n_k$, and $k$ runs through the finite set $\pi_1^\theta(M)$. For $d$ large enough at least one of $n_k$ will satisfy $n_k > r(2g - 2) \dim U^\lambda_k$, and the RHS of (17) will vanish. □

Generalizing the Vanishing Conjecture of Frenkel, Gaitsgory and Vilonen ([4]), we suggest

**Conjecture 4.** Let $\mathcal{W}$ be an irreducible local system on $\mathcal{X}$ of rank $r = \dim V^\gamma$. Assume that $\mathcal{P}$ is a standard proper parabolic of $\mathcal{G}$. Then for all $\mu \in \pi_1^+(M)$ whose image in $\pi_1^1(G)$ equals $d\theta$, the functor $\mathbb{A}_W^\mu$ vanishes identically.

4.6.2 Consider the diagram $\text{Bun}_G^\alpha \mathcal{P} \leftarrow \text{Bun}_G^\beta \mathcal{P} \to \text{Bun}_M$. The constant term functor $\text{CT}_\mathcal{P} : D(\text{Bun}_G) \to D(\text{Bun}_M)$ is defined by $\text{CT}_\mathcal{P}(K) = \beta_\mathcal{P}! \alpha_\mathcal{P}^* P(K)$.

The following is a generalization of Lemma 9.8, [4].

**Lemma 9.** Let $\mathcal{W}$ be any local system on $\mathcal{X}$. For any $K \in D(\text{Bun}_G)$ and $d \geq 0$ the complex $\text{CT}_\mathcal{P} \circ \mathbb{A}_W^d(K)$ in $D(\text{Bun}_M)$ has a canonical filtration by complexes

$$\mathbb{A}_W^\mu \circ \text{CT}_\mathcal{P}(K) \otimes \mathcal{Q}_d[1] \left( \frac{1}{2} \right)^{\otimes (\mu, 2\mathfrak{p} - 2\mathfrak{p}M)}$$

(18)

indexed by those $\mu \in \pi_1^+(M)$ whose image in $\pi_1(G)$ is $d\theta$.

**Proof** Consider the stack $\mathcal{H}_G^{+,d} \times_{\text{Bun}_G} \text{Bun}_P$, where we used $q : \mathcal{H}_G^{+,d} \to \text{Bun}_G$ from the fibred product. The complex $\text{CT}_\mathcal{P} \circ \mathbb{A}_W^d(K)$ is the direct image with respect to the natural map

$$\mathcal{H}_G^{+,d} \times_{\text{Bun}_G} \text{Bun}_P \to \text{Bun}_M$$

(19)

Recall that $\mathcal{H}_G^{+,d} \times_{\text{Bun}_G} \text{Bun}_P$ is stratified by locally closed substacks $\mathcal{H}_P^{+,\mu}$ indexed by those $\mu \in \pi_1^+(M)$ whose image in $\pi_1(G)$ is $d\theta$. This gives a filtration on $\text{CT}_\mathcal{P} \circ \mathbb{A}_W^d(K)$.

The restriction of the map (19) to the stratum $\mathcal{H}_P^{+,\mu}$ can be written as a composition

$$\mathcal{H}_P^{+,\mu} \overset{\mathcal{P}^\to}{\to} \mathcal{H}_M^{+,\mu} \times_{\text{Bun}_M} \text{Bun}_P \to \text{Bun}_M$$

So, by Proposition 9 the contribution of the stratum $\mathcal{H}_P^{+,\mu}$ to the direct image in question is exactly (18). □

**Corollary 4.** Assume that Conjecture 4 holds. Then

1) Let $d$ satisfy $d > c(P)$ for any standard proper parabolic of $G$. Then for any $K \in D(\text{Bun}_G)$ and any irreducible local system $\mathcal{W}$ on $\mathcal{X}$ of rank $r = \dim V^\gamma$ the complex $\mathbb{A}_W^d(K)$ is cuspidal.

2) Let $\mathcal{E}$ be $G$-local system on $\mathcal{X}$ and $\mathcal{K}$ be a $E$-Hecke eigensheaf on $\text{Bun}_G$. If $V_E^\gamma$ is irreducible then $K$ is cuspidal.
Proof 1) is clear.
2) The argument given in ([H, Theorem 9.2]) applies in our setting. Namely, pick \(d\) such that \(d > c(P)\) for any standard proper parabolic of \(G\). Set \(W = (V_E^\vee)^*\). By Proposition 8,

\[
\CT_P \circ \Av^d_{\tilde{H}}(K) \cong \CT_P(K) \otimes \RG(X(d), (W \otimes V_E^\vee)\langle d \rangle \otimes \tilde{\Lambda} \otimes \tilde{\Lambda})
\]

The LHS vanishes by Lemma 9. Since \(\RG(X(d), (W \otimes V_E^\vee)\langle d \rangle)\) is not zero, \(\CT_P(K) = 0\). □

Remark 4. For \(G = \GL_n\) Conjecture \([\text{H}]\) is proved by D. Gaitsgory ([\text{M}]). For \(G = \GSp_4\) (example 2 in the appendix) Conjecture \([\text{M}]\) also holds, it is easily reduced to the result of loc.cit. for \(\GL_2\). So, for \(G = \GSp_4\) Corolary \([\text{M}]\) is unconditional.

**APPENDIX. 1-Admissible groups**

**Definition 8.** Let \(H\) be a connected, semi-simple and simply-connected group (over \(k\)). Assume that the center \(Z(H)\) is cyclic of order \(h\) and fix an isomorphism \(\zeta : \mu_h \cong Z(H)\). Assume that the characteristic of \(k\) does not divide \(h\). Denote by \(G\) the quotient of \(H \times \mathbb{G}_m\) by the diagonally embedded \(\mu_h\). Call a reductive group \(G\) over \(k\) 1-admissible, if it is obtained in this way.

Let \(H\) be a connected, semi-simple and simply-connected group (over \(k\)). Let \(T_H\) be a maximal torus of \(H\). Write \(\tilde{\Lambda}_H\) (resp., \(\Lambda_H\)) for the weight (resp., coweight) lattice of \(T_H\). Let \(Q_H \subset \tilde{\Lambda}_H\) be the root lattice. Set

\[
Q_H = \{ \lambda \in \Lambda_H \otimes \mathbb{Q} \mid \langle \lambda, \tilde{\lambda} \rangle \in \mathbb{Z} \text{ for any } \tilde{\lambda} \in Q_H \}
\]

We have a natural pairing \(\Lambda_H/\tilde{Q}_H \times Q_H/\Lambda_H \to (\frac{1}{h}\mathbb{Z})/\mathbb{Z}\). Therefore, any isomorphism

\[
\tau : Q_H/\Lambda_H \cong (\frac{1}{h}\mathbb{Z})/\mathbb{Z}
\]

yields an isomorphism \(\check{\tau} : \tilde{\Lambda}_H/\tilde{Q}_H \cong \mathbb{Z}/h\mathbb{Z}\). Since the characteristic of \(k\) does not divide \(h\), \(\text{Hom}(Z(H), k^*) \cong \Lambda_H/\tilde{Q}_H\) canonically. So, \(\check{\tau}\) yields \(\zeta : \mu_h \cong Z(H)\) such that for \(x \in \mu_h, \tilde{\lambda} \in \tilde{\Lambda}_H\) we have

\[
\tilde{\lambda}(\zeta(x)) = x^{\check{\tau}(\tilde{\lambda})}
\]

So, \(\tau\) gives rise to a 1-admissible group \(G = (H \times \mathbb{G}_m)/\mu_h\).

For the maximal torus \(T = (T_H \times \mathbb{G}_m)/\mu_h\) in \(G\) the weight lattice is

\[
\Lambda = \{ (\tilde{\lambda}, a) \in \tilde{\Lambda}_H \times \mathbb{Z} \mid \check{\tau}(\tilde{\lambda}) + a = 0 \mod h \}
\]

and the coweight lattice is

\[
\Lambda = \{ (\lambda, b) \in Q_H \times (\frac{1}{h}\mathbb{Z}) \mid \tau(\lambda) - b \in \mathbb{Z} \}
\]

It is understood that the pairing \(\Lambda \times \tilde{\Lambda} \to \mathbb{Z}\) sends \((\lambda, b), (\tilde{\lambda}, a)\) to \(\langle \lambda, \tilde{\lambda} \rangle + ab\). The map \((\lambda, b) \mapsto b\) yields an isomorphism \(\pi_1(G) \cong \Lambda/\Lambda_H \cong \frac{1}{h}\mathbb{Z}\). Note also that \(\pi_1(G) \cong \tilde{\Lambda}/\tilde{Q}_H \cong \mathbb{Z}\).

The next result follows from definitions.
Lemma 10. Let $\gamma_H \in Q_H$ be a dominant coweight for $H_{ad} = H/Z(H)$. Assume that

- either $\gamma_H$ is minuscule or ($H = 1$ and $\gamma_H = 0$);
- $\gamma_H$ generates $Q_H/\Lambda_H$;
- the irreducible representation $V^{\gamma_H}$ of $(H_{ad})$ is faithful.

Let $\tau : Q_H/\Lambda_H \cong (\frac{1}{h}(\mathbb{Z})/\mathbb{Z}$ be the isomorphism sending $\gamma_H$ to $\frac{1}{h}$, and $G$ be the corresponding 1-admissible group. Set $\gamma = (\gamma_H, \frac{1}{h}) \in \Lambda$. Then $\{\gamma\}$ is a 1-admissible datum for $G$. □

Examples of 1-admissible data

The examples below are produced using Lemma 10.

1. The case $G = GL_n$. In the standard notation $\Lambda = \mathbb{Z}^n$, $\check{\Lambda} = \mathbb{Z}^n$. For $1 \leq i < n$ take $\check{\omega}_i = (1, \ldots, 1, 0, \ldots, 0)$ where 1 appears $i$ times, and $\check{\omega}_0 = (1, \ldots, 1)$. All the conditions are satisfied and $\gamma = (1, 0, \ldots, 0)$.

2. The case $G = Sp_{2n}$, $n \geq 1$. The group $G$ is a quotient of $\mathbb{G}_m \times Sp_{2n}$ by the diagonally embedded $\{\pm 1\}$. Realise $G$ as the subgroup of $GL(k^{2n})$ preserving up to a scalar the bilinear form given by the matrix

$$
\begin{pmatrix}
0 & E_n \\
-E_n & 0
\end{pmatrix},
$$

where $E_n$ is the unit matrix of $GL_n$.

The maximal torus $T$ of $G$ is $\{(y_1, \ldots, y_{2n}) \mid y_i y_{n+i} \text{ does not depend on } i\}$. Let $\check{\epsilon}_i \in \check{\Lambda}$ be the character that sends a point of $T$ to $y_i$. The roots are

$$
\check{R} = \{\pm \check{\alpha}_{ij} \mid i < j \leq 1, \ldots, n\}, \pm \check{\beta}_{ij} \mid i \leq j \leq 1, \ldots, n\},
$$

where $\check{\alpha}_{ij} = \check{\epsilon}_i - \check{\epsilon}_j$ and $\check{\beta}_{ij} = \check{\epsilon}_i - \check{\epsilon}_{n+j}$.

We have $\Lambda = \{(a_1, \ldots, a_{2n}) \mid a_i + a_{n+i} \text{ does not depend on } i\}$. The weight lattice is

$$
\check{\Lambda} = \mathbb{Z}^{2n}/\{\check{\epsilon}_i + \check{\epsilon}_{n+i} - \check{\epsilon}_j - \check{\epsilon}_{n+j}, i < j\}.
$$
Let $e_i$ denote the standard basis of $\mathbb{Z}^{2n}$. The coroots are

$$R = \{ \pm \alpha_{ij} \ (i < j \in 1, \ldots, n), \ \pm \beta_{ij} \ (i \leq j \in 1, \ldots, n) \},$$

where $\beta_{ij} = e_i + e_j - e_{n+i} - e_{n+j}$ for $i < j$ and $\beta_{ii} = e_i - e_{n+i}$. Besides, $\alpha_{ij} = e_i + e_{n+j} - e_j - e_{n+i}$.

Fix positive roots $\tilde{R}^+ = \{ \tilde{\alpha}_{ij} \ (i < j \in 1, \ldots, n), \ \tilde{\beta}_{ij} \ (i \leq j \in 1, \ldots, n) \}$.

Then the simple roots are $\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{n-1,n}$ and $\tilde{\beta}_{n,n}$.

For $1 \leq i < n$ pick the fundamental weight $\tilde{\omega}_i$ corresponding to the simple coroot $\alpha_{i,i+1}$ to be $\tilde{\omega}_i = (1, \ldots, 1, 0, \ldots, 0)$, where $1$ appears $i$ times, and $0$ appears $2n-i$ times. Let the fundamental weight $\tilde{\omega}_n$ corresponding to $\beta_{n,n}$ be $\tilde{\omega}_n = (1, \ldots, 1, 0, \ldots, 0)$, where $1$ appears $n$ times. The orthogonal to the coroot lattice is the subgroup $\mathbb{Z}^{n-\omega}$ with $\tilde{\omega}_0 = (1, 0, \ldots, 0; 1, 0, \ldots, 0)$.

All our conditions are satisfied and $\gamma = (1, \ldots, 1; 0, \ldots, 0) \ (\text{here } 1 \text{ appears } n \text{ times})$.

For $1 \leq i < n$ let $\gamma_i = (2, \ldots, 2, 1, \ldots, 1; 0, \ldots, 0, 1, \ldots, 1)$, where $2$ appears $i$ times then $1$ appears $n-i$ times then $0$ appears $i$ times and finally $1$ appears $n-i$ times. The element $\omega = (1, \ldots, 1) \in \Lambda$ generates the group of coweights orthogonal to all roots. The semigroup $\Lambda^+_G$ is the $\mathbb{Z}_+$-span of $\gamma, \gamma_1, \ldots, \gamma_{n-1}, \omega$. In fact, these $n+1$ elements are linearly independent in $\Lambda$, so $\Lambda^+_G \supset (\mathbb{Z}_+)^{n+1}$.

Note that $V^\gamma$ is the spinor representation of $\tilde{G} \cong \tilde{\text{GSpin}}_{2n+1}$ of dimension $2^n$. We have

$$V^\gamma \otimes V^\gamma \cong V^{2\gamma} \oplus V^\omega \oplus \sum_{i=1}^{n-1} V^{\gamma_i}$$

and $\dim V^{\gamma_i} = 2n+1$. Besides, $\wedge^i V^{\gamma_i} \cong V^{\gamma_i+(i-1)\omega}$ for $i = 1, \ldots, n-1$ and $\wedge^n V^{\gamma_i} \cong V^{2\gamma+(n-1)\omega}$. There is an exact sequence $1 \to G_m \to \tilde{G} \to \text{SO}_{2n+1} \to 1$, and $V^{\gamma_i-\omega}$ comes from the standard representation of $SO_{2n+1}$.

3. The case $G = \text{GSpin}_{2n+1}$, $n \geq 1$. The group $G$ is the quotient of $G_m \times \text{Spin}_{2n+1}$ by the diagonally embedded $\{ \pm 1 \}$. We have $\tilde{G} \cong \text{GSpin}_{2n}$, the root data for $G$ is dual to that of example 2. Interchanging the role of objects and coobjects in example 2, we get

$$\Lambda = \mathbb{Z}^{2n}/\{ \epsilon_i + \epsilon_{n+i} - \epsilon_j - \epsilon_{n+j}, \ i < j \}$$

where $(\epsilon_i)$ is the standard basis of $\mathbb{Z}^{2n}$. The weight lattice is

$$\tilde{\Lambda} = \{(a_1, \ldots, a_{2n}) \in \mathbb{Z}^{2n} \mid a_i + a_{n+i} \text{ does not depend on } i \}$$

Define $\tilde{\omega}_i$ as follows. For $1 \leq i < n$ let

$$\tilde{\omega}_i = (2, \ldots, 2, 1, \ldots, 1; 0, \ldots, 0, 1, \ldots, 1) \in \tilde{\Lambda}$$

where $2$ appears $i$ times then $1$ appears $n-i$ times then $0$ appears $i$ times and finally $1$ appears $n-i$ times. Let $\tilde{\omega}_n = (1, \ldots, 1; 0, \ldots, 0)$ where $1$ appears $n$ times. Let $\tilde{\omega}_0 = (1, \ldots, 1)$. 28
All the conditions are satisfied and $\gamma = (1, 0, \ldots, 0) \in \Lambda$.

For $1 \leq i \leq n$ let $\gamma_i = (1, \ldots, 1, 0, \ldots, 0) \in \Lambda$, where 1 appears $i$ times and 0 appears $2n - i$ times. The group of coweights orthogonal to all roots is $\mathbb{Z}\omega$ with $\omega = (1, 0, \ldots, 0; 1, 0, \ldots, 0)$, here 1 appears on the first and $(n + 1)$-th places. The semigroup $\Lambda_{G,S}^+$ is the $\mathbb{Z}_+$-span of $\gamma_1, \ldots, \gamma_n, \omega$. These $n + 1$ elements are linearly independent in $\Lambda$, so $\Lambda_{G,S}^+ \to (\mathbb{Z}_+)^{n+1}$.

We have $\wedge^2 V^\gamma = V^{\gamma_2} \oplus V^\omega$. To assure condition (A) of Conjecture 1 it suffices to require that $V_{E}^\gamma$ is irreducible and $V_{E}^{\gamma_2}$ has no local subsystems of rank one.

4. The case $H = \text{Spin}_{2n}$, $n \geq 3$ odd. We have

$$\hat{\Lambda}_H = \{(a_1, \ldots, a_n) \in (\frac{1}{2}\mathbb{Z})^n \mid a_i - a_j \in \mathbb{Z}\}$$

and $\Lambda_H = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 + \ldots + a_n \text{ is even}\}$. The roots are

$$\tilde{R} = \{\pm \tilde{\alpha}_{ij}, \pm \tilde{\beta}_{ij} \mid 1 \leq i < j \leq n\}$$

with $\tilde{\alpha}_{ij} = \tilde{e}_i - \tilde{e}_j$ and $\tilde{\beta}_{ij} = \tilde{e}_i + \tilde{e}_j$. The coroots are

$$R = \{\pm \alpha_{ij}, \pm \beta_{ij} \mid 1 \leq i < j \leq n\}$$

with $\alpha_{ij} = e_i - e_j$ and $\beta_{ij} = e_i + e_j$. Pick positive roots $\hat{R} = \{\hat{\alpha}_{ij}, \hat{\beta}_{ij} \mid 1 \leq i < j \leq n\}$. Then simple roots are

$$\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{n-1,n}, \tilde{\beta}_{n-1,n}$$

We have $Q_H = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 + \ldots + a_n \text{ is even}\}$,

$$Q_H = \{(a_1, \ldots, a_n) \in (\frac{1}{2}\mathbb{Z})^n \mid a_i - a_j \in \mathbb{Z}\}$$

There are two possible choices for $\gamma_H$, namely $\gamma_H = (\frac{1}{2}, \ldots, \frac{1}{2})$ or $\gamma_H = (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$, the fundamental coweights corresponding to $\tilde{\beta}_{n-1,n}$ or to $\tilde{\alpha}_{n-1,n}$ respectively.

5. The case $E_6$. So, $H$ is the simply-connected group corresponding to $E_6$ root system. There are two possible choices for $\gamma_H$, namely $\omega_1$ or $\omega_6$, the fundamental coweights corresponding to the simple roots $\tilde{\alpha}_1$ and $\tilde{\alpha}_6$ in the Bourbaki table ([1], Ch. 6, no. 4.12, p.219).

6. The case $E_7$. So, $H$ is the simply-connected group corresponding to $E_7$ root system. Take $\gamma_H$ to be the fundamental coweight $\omega_7$ corresponding to the root $\tilde{\alpha}_7$ in the Bourbaki table ([1], Ch. 6, no. 4.11, p. 217).

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