Entropy flux and Lagrange multipliers: information theory and thermodynamics

Raquel Domínguez-Cascante and David Jou
Departament de Física
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona), Spain

Abstract
We analyze the relation between the nonequilibrium Lagrange multipliers used in information theory and those used in Liu’s technique to exploit the entropy inequality. In particular, we deal with some of the subtleties found in the analysis of the entropy flux.

1 Introduction
Lagrange multipliers play a central role in information theory and statistical mechanics, not only as a mathematical tool to introduce a set of restrictions on the distribution function, but also because of their general physical meaning as intensive parameters. When one tries to use this technique in nonequilibrium situations, the mathematical extension is not so complicated, in contrast with the subtle conceptual problems related to the physical interpretation of the nonequilibrium Lagrange multipliers.

Furthermore, Lagrange multipliers are also used in nonequilibrium thermodynamics in the elegant method proposed by Liu[1] to take into account the restrictions placed on the thermodynamic fields by the balance equations. Some years ago, Dreyer[2] showed that, under given conditions, the Lagrange multipliers used in statistical mechanics and those appearing in the formulation of nonequilibrium thermodynamics could be identified. This is of course an important question in order to relate microscopic and macroscopic results.

Here, we deal with some subtleties which arise in the analysis of the entropy flux. Some of these subtleties, related to the role of (nonequilibrium) absolute

---

1 This paper is dedicated to Prof. J. Casas-Vázquez in his 60th birthday
2 Electronic address: raquel@ulises.uab.es
3 Also at Institut d’Estudis Catalans, C/ Carme, 47, Barcelona
temperature \([3, 4]\) in the definition of the entropy flux, were considered in \([5]\), both from the macroscopic point of view of Extended Irreversible Thermodynamics (EIT) \([7]-[12]\) and considering an information theoretical formalism. In the first case, an entropy flux of the form

\[
\vec{J}_S = \vec{q} \theta
\]  

was assumed, where \(\vec{q}\) is the heat flux and \(\theta\) the nonequilibrium temperature defined as the derivative of the generalized, flux-dependent specific entropy \(s(u, \vec{q})\) considered by EIT, with respect to the internal energy \(u\). When this expression for the entropy flux is considered, one is led to an entropy production \(\sigma^S\) of the form

\[
\sigma^S = \vec{q} \cdot \nabla \left( \frac{1}{\theta} \right) - \frac{\vec{\alpha}}{\theta} \cdot \vec{q} \geq 0,
\]

where \(\vec{\alpha} := -\left(\theta/v\right) \left(\partial s/\partial \vec{q}\right)\) and \(v = 1/\rho\) is the specific volume of the system whose density is \(\rho\). The simplest evolution equation for the heat flux verifying the inequality (2), would be a generalized Maxwell-Cattaneo equation, namely

\[
\mu \vec{q} = \nabla \left( \frac{1}{\theta} \right) - \frac{\alpha}{\theta} \cdot \vec{q}, \quad \mu > 0,
\]

with \(\vec{\alpha} =: \alpha \vec{q}\) (this should hold according to the representation theorem, as \(\vec{q}\) is the only vector among the independent variables and we should require that the equations reflect the isotropy of the system).

However, one could add new terms to equation (1), what would merely modify the expression for the entropy production and, within this formalism, the allowed evolution equations for the heat flux verifying the second law of Thermodynamics. For instance, by using an information theoretical formalism a generalized entropy flux has been obtained in \([5]\), namely,

\[
\vec{J}_S = \frac{\vec{q}}{\theta} - \frac{\vec{\alpha}}{\theta} \cdot \vec{Q},
\]

where \(\vec{Q}\) is the flux of heat flux and \(\vec{\alpha}\) and \(\theta\) are related to the Lagrange multipliers introduced in the formalism. When this entropy flux is considered, the entropy production arising from the evolution equation for the specific entropy would be

\[
\sigma^S = \rho \dot{s} + \nabla \vec{J}_S = \vec{q} \cdot \nabla \left( \frac{1}{\theta} \right) - \frac{\vec{\alpha}}{\theta} \cdot \vec{q} - \nabla \left( \frac{\vec{\alpha} \cdot \vec{Q}}{\theta} \right) \geq 0.
\]

On the other hand, one can consider an evolution equation for the heat flux of the form

\[
\dot{\vec{q}} + \nabla \cdot \vec{Q} = \vec{\sigma}^q.
\]

In this case, the entropy production would be

\[
\sigma^S = \vec{q} \cdot \nabla \left( \frac{1}{\theta} \right) - \frac{\vec{\alpha}}{\theta} \cdot \vec{\sigma}^q - \nabla \left( \frac{\vec{\alpha}}{\theta} \right) \cdot \vec{Q} \geq 0.
\]
Let us note that (3) can be written in the form (6), if it is assumed that \( Q = C \), being \( C \) a constant tensor and
\[
\vec{\sigma}^q = \frac{\partial}{\partial \alpha} \nabla \left( \frac{1}{\theta} \right) - \frac{\mu \theta}{\alpha} \vec{q}.
\] (8)

(what will prove not to be convenient as it implies the assumption that \( \nabla \theta = \nabla \theta(u, \vec{q}) \) instead of \( \theta = \theta(u, \vec{q}) \)) or, otherwise, that
\[
Q = \frac{1}{\alpha} U,
\]
\[
\vec{\sigma}^q = -\frac{\mu \theta}{\alpha} \vec{q}.
\] (9)

(but now we should assume that the product \( \alpha \theta \) is a constant independent of \( u \) and \( \vec{q} \)). However, in a general situation, it is not possible to write (3) in the form (6).

Now, we try to generalize the previous results by applying the technique developed by Liu [1] to exploit the entropy inequality. Let us note that, from this new point of view, we start by knowing the proper evolution equation for the heat flux (which can be derived, for instance, from kinetic a theory) and try to find restrictions to the form the specific entropy and the entropy flux may take. We will, thus, observe under what assumptions the previous results may be recovered. In addition, we should note that the information theory results obtained in [5] and Liu’s technique are seen not to be equivalent in this situation, in spite of the results in [2]. The reason will become evident in the following section.

The plan of the paper is as follows: In Sec. 2, we briefly introduce the fundamentals of Liu’s technique, while in Sec. 3 we apply such a technique to derive the form of the entropy flux and entropy production when a Maxwell-Cattaneo-like evolution equation for the heat flux is assumed. We will observe under what further assumptions the macroscopic results in [5] are recovered. In Sec. 4, however, we will consider an evolution equation of the form (6) instead, in order to compare it with the results obtained by using information theory. Sec. 5 is devoted to the conclusions of this paper.

2 Liu’s technique to exploit the entropy inequality

The aim of thermodynamics is the determination of the fields \( F_i \) (\( i = 1..N \)) that constitute the state space \( Z \) of the system and are governed by balance equations of the form
\[
\rho \frac{\partial F_i}{\partial t} + \nabla \cdot \vec{J}^{F_i} = \sigma^{F_i},
\] (11)
where $\vec{J}^{i}$ and $\sigma^{i}$ are, respectively, the flux and production of $F_i$ and are functions of the state space $Z$. Such balance equations should be supplemented with constitutive equations relating the fluxes and the productions to the variables spanning the state space. If the constitutive equations were known, one would be able to explicitly solve the balance equations and obtain the thermodynamic processes.

However, such constitutive equations are not known, so we need a method that may help us to obtain them or, at least, to limit their form. Liu’s technique accomplishes this by the exploitation of the entropy principle in order to obtain expressions for the constitutive equations (or, at least, to reduce their generality). In fact, the constitutive equations must be of such form that they verify the following principles:

1. the entropy principle
2. the requirements of convexity and causality, which demand that the field equations be symmetric hyperbolic, and
3. the principle of relativity.

The entropy principle is a local expression for the second law of thermodynamics, written as an evolution equation for the specific entropy $s(Z)$

$$\rho \frac{\partial s}{\partial t} + \nabla \cdot \vec{J}^{S} = \sigma^{S} \geq 0. \quad (12)$$

The entropy production $\sigma^{S}$ must be positive for any solution of the balance equations. These requirements may be taken into account by introducing a set of Lagrange multipliers $\lambda_i(Z)$, such that

$$\rho \frac{\partial s}{\partial F_i} + \nabla \cdot \vec{J}^{S} - \sum_{i=1}^{N} \lambda_i \left( \rho \frac{\partial F_i}{\partial t} + \nabla \cdot \vec{J}^{i} - \sigma^{i} \right) \geq 0, \quad (13)$$

for all continuous differentiable fields $F_i$. By making use of the chain rule, equation (13) may be seen to adopt the form

$$\sum_{i=1}^{N} \rho \left( \frac{\partial s}{\partial F_i} - \lambda_i \right) \frac{\partial F_i}{\partial t} + \sum_{i=1}^{N} \left( \frac{\partial \vec{J}^{S}}{\partial F_i} - \lambda_i \frac{\partial \vec{J}^{i}}{\partial F_i} \right) \cdot \nabla F_i + \sum_{i=1}^{N} \lambda_i \sigma^{i} \geq 0, \quad (14)$$

and, as this expression must hold for any field, the terms in brackets must vanish so that

$$\frac{\partial s}{\partial F_i} = \lambda_i, \quad \frac{\partial \vec{J}^{S}}{\partial F_i} = \lambda_i \frac{\partial \vec{J}^{i}}{\partial F_i}, \quad \forall i. \quad (15)$$

Thus we can write

$$ds = \sum_{i=1}^{N} \lambda_i dF_i, \quad d\vec{J}^{S} = \sum_{i=1}^{N} \lambda_i d\vec{J}^{i} \quad (16)$$
and the residual inequality reads

$$\sum_{i=1}^{N} \lambda_i \sigma F_i \geq 0. \quad (17)$$

We should compare (16) with

$$\vec{J}^S = \sum_{i=1}^{N} \lambda_i \vec{J} F_i \quad (18)$$

in [3]. Both expressions differ, while according to [2], information theory is equivalent to Liu’s technique. The reason for such differences relies in the definition given to entropy. While in [5], entropy is defined according to Boltzmann’s expression, namely

$$S = -\frac{k_B}{N h^3 N!} \int f \ln f \, d\Gamma_N, \quad (19)$$

in [4] it is considered that

$$S = -k_B \int \left[ f \ln \left( \frac{f}{y} \right) + \frac{y}{a} \left( 1 - \frac{a}{y} f \right) \ln \left( 1 - \frac{a}{y} f \right) \right] d\Gamma_N, \quad (20)$$

where \( a = 1, -1 \) for fermions and bosons respectively and \( y \) is the degeneracy of the state. Let us note that from this expression we can easily recover (19) if \( a = 0 \) is chosen. The entropy flux is thus given in [4] by

$$\vec{J}^S = -k_B \int \left[ f \ln \left( \frac{f}{y} \right) + \frac{y}{a} \left( 1 - \frac{a}{y} f \right) \ln \left( 1 - \frac{a}{y} f \right) \right] \sum \vec{C}_i d\Gamma_{N-1} d\vec{c}_N, \quad (21)$$

where \( \vec{C}_i = \vec{c}_i - \vec{v} \) is the peculiar velocity of particle \( i \), and if we take into account that \( f \) is a generalized canonical distribution function, namely

$$f = \frac{y}{\exp(\sum_i \lambda_i A_i) + a}, \quad (22)$$

we can obtain

$$S = \sum_i \lambda_i < A_i > + k_B \frac{y}{a} \int \ln \left( 1 + a \exp \left( - \sum_i \lambda_i A_i \right) \right) d\Gamma_N, \quad (23)$$

and

$$\vec{J}^S = \sum_i \lambda_i \vec{J} A_i + k_B \frac{y}{a} \int \ln \left( 1 + a \exp \left( - \sum_i \lambda_i A_i \right) \right) \sum_i \vec{C}_i d\Gamma_{N-1} d\vec{c}_N. \quad (24)$$

Note that if we take \( a = 0 \) in (24), we recover equation (18). However, if we evaluate the differential form

$$d \vec{J}^S = \sum_i \lambda_i d \vec{J} A_i + \sum_i d \lambda_i \vec{J} A_i$$
\[ + k_B \frac{\mu}{a} \sum_i d\lambda_i \frac{\partial}{\partial \lambda_i} \int \ln \left( 1 + a \exp \left( - \sum_j \lambda_j A_j \right) \right) \sum_k \tilde{C}_k d\Gamma_{N-1} d\bar{c}_N, \]  

the last term is easily seen to be equal to \(- \sum_i d\lambda_i A_i\), so that

\[ d\bar{J} = \sum_i \lambda_i d\bar{J}_i, \]  

independently of the value of \(a\), as in \[2\]. Thus, we observe that, although in the case of bosons and fermions the results obtained from information theory are coincident with those obtained by the exploitation of the entropy principle, one must be careful if classical particles are considered. This question should be further investigated as it is not convenient, neither can be guaranteed that

\[ \sum_i d\lambda_i A_i = 0 \]  

for classical particles.

### 3 Heat flux determined by a Maxwell-Cattaneo evolution equation

Let us consider a system at rest submitted to a heat flux and verifying a Maxwell-Cattaneo evolution equation. Thus, the two evolution equations for the variables \(u\) and \(\vec{q}\) spanning the space state are

\[ \rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{q} = 0, \quad \frac{\partial \vec{q}}{\partial t} + \frac{1}{\tau} (\vec{q} + \lambda \nabla \Theta) = 0 \]  

where \(\tau\) is the relaxation time of heat pulses, \(\lambda\) the thermal conductivity of the material and \(\Theta\) the temperature of the system (we do not specify yet whether this temperature is the nonequilibrium temperature, as assumed in \[5\] or the local-equilibrium one, as usually done in linear EIT: we will consider both assumptions in the following). \(2\) coincides with \(1\) as long as we identify

\[ \theta = \Theta, \quad \tau = \frac{\alpha}{\mu \Theta}, \quad \lambda = \frac{1}{\mu \Theta^2}. \]  

The last two equalities are mere definitions of parameters as long as the first identification is performed. Thus, \(2\) is more general than \(1\), because we have not already assumed what temperature \(\Theta\) is. In addition, let us remark that, as commented above, \(2\) cannot be written in the form \(1\), unless we assume that \(\nabla \Theta\) is a function of the state variables. If such assumption is taken,
we can obtain
\[
\rho \left( \frac{\partial s}{\partial u} - \lambda u \right) \frac{\partial u}{\partial t} + \rho \left( \frac{\partial s}{\partial \vec{q}} - \vec{\lambda}_q \right) \cdot \frac{\partial \vec{q}}{\partial t} + \frac{\partial \vec{J}^S}{\partial u} \cdot \nabla u \\
+ \left( \frac{\partial \vec{J}^S}{\partial \vec{q}} - \lambda u \vec{U} \right) : \nabla \vec{q} - \vec{\lambda}_q \rho \frac{\partial \Theta}{\partial u} \geq 0,
\]
and thus
\[
\frac{\partial s}{\partial u} = \lambda u, \quad \frac{\partial s}{\partial \vec{q}} = \vec{\lambda}_q, \\
\frac{\partial \vec{J}^S}{\partial u} = 0, \quad \frac{\partial \vec{J}^S}{\partial \vec{q}} = \lambda u \vec{U}, \\
- \vec{\lambda}_q \cdot \rho \frac{\partial \Theta}{\partial u} \geq 0.
\]
In agreement with the representation theorem, as \( \vec{q} \) is the only vectorial variable in the state space, we can write:
\[
\vec{J}^S(u, \vec{q}) = \varphi(u, q^2) \vec{q}, \quad \vec{\lambda}_q = \Lambda(u, q^2) \vec{q}, \\
s = s(u, q^2), \quad \lambda_u = \lambda_u(u, q^2),
\]
so the previous equations may be rewritten as
\[
\frac{\partial s}{\partial u} = \lambda_u, \quad \frac{\partial s}{\partial q^2} = \frac{\Lambda}{2}, \\
\frac{\partial \varphi}{\partial u} = 0, \quad \frac{\partial \varphi}{\partial q^2} 2\vec{q} \vec{q} + \varphi \vec{U} = \lambda_u \vec{U},
\]
and it can easily be seen that
\[
\varphi = \lambda_u
\]
is a constant, independent of the state variables. Thus we have
\[
s = \lambda_u u + F(q^2), \quad \vec{J}^S = \lambda_u \vec{q}
\]
However, a constant temperature is indeed unphysical and furthermore it is not seen to be related with the temperature \( \Theta \) appearing in the Maxwell-Cattaneo evolution equation, so the previous assumption that \( \nabla \Theta \) is a state function should not be considered. However, if we assume that \( \Theta = \Theta(Z) \), we can write
\[
\rho \left( \frac{\partial s}{\partial u} - \lambda u \right) \frac{\partial u}{\partial t} + \rho \left( \frac{\partial s}{\partial \vec{q}} - \vec{\lambda}_q \right) \cdot \frac{\partial \vec{q}}{\partial t} + \left( \frac{\partial \vec{J}^S}{\partial u} - \vec{\lambda}_q \rho \frac{\partial \Theta}{\partial u} \right) \cdot \nabla u \\
+ \left( \frac{\partial \vec{J}^S}{\partial \vec{q}} - \lambda u \vec{U} - \vec{\lambda}_q \rho \frac{\partial \Theta}{\partial u} \right) : \nabla \vec{q} - \vec{\lambda}_q \rho \frac{\partial \Theta}{\partial u} \geq 0,
\]
so that, if we define \( \xi := \rho \lambda / \tau \), we obtain

\[
\frac{\partial s}{\partial u} = \lambda_u, \quad \frac{\partial s}{\partial q} = \bar{\lambda}_q, \quad (41)
\]

\[
\frac{\partial \bar{J}^S}{\partial u} = \bar{\lambda}_q \xi \frac{\partial \Theta}{\partial u}, \quad \frac{\partial \bar{J}^S}{\partial q} = \lambda_u U + \bar{\lambda}_q \xi \frac{\partial \Theta}{\partial q}, \quad (42)
\]

\[-\bar{\lambda}_q \frac{\rho}{\tau} \bar{q} \geq 0. \quad (43)\]

Again we can make use of the representation theorem

\[
\bar{J}^S(u, \bar{q}) = \varphi(u, q^2) \bar{q}, \quad \bar{\lambda}_q = \Lambda(u, q^2) \bar{q}, \quad (44)
\]

\[
s = s(u, q^2), \quad \lambda_u = \lambda_u(u, q^2), \quad \Theta = \Theta(u, q^2), \quad (45)
\]

and obtain

\[
\frac{\partial s}{\partial u} = \lambda_u, \quad \frac{\partial s}{\partial q^2} = \frac{\Lambda}{2}, \quad (46)
\]

\[
\frac{\partial \varphi}{\partial u} = \Lambda \xi \frac{\partial \Theta}{\partial u}, \quad \frac{\partial \varphi}{\partial q^2} 2 \bar{q} \bar{q} + \varphi U = \lambda_u U + \Lambda \xi \frac{\partial \Theta}{\partial q^2} 2 \bar{q} \bar{q}. \quad (47)
\]

The last equation splits into

\[
\varphi = \lambda_u, \quad \frac{\partial \varphi}{\partial q^2} = \Lambda \xi \frac{\partial \Theta}{\partial q^2}. \quad (48)
\]

Thus, the entropy flux is seen to adopt the form predicted in (1), that is \( \lambda_u \equiv 1/\theta \),

\[
\bar{J}^S = \frac{\bar{q}}{\theta} \quad (49)
\]

From (47) \(_1\) and (48) \(_2\), we observe that \( \varphi \) is function of \( \Theta \) only, i.e.

\[
\frac{d \varphi}{d \Theta} = \Lambda \xi \quad (50)
\]

and thus, also \( \Lambda \xi \). To proceed, we should make further assumptions concerning the temperature \( \Theta \).

• If we assume that \( \Theta = \theta \), that is \( \varphi = 1/\Theta \), we can write

\[
\Lambda \xi = -\frac{1}{\theta^2} \quad \Rightarrow \quad \Lambda = -\frac{1}{\xi \theta^2} = -\frac{\tau}{\rho \lambda \theta^2}, \quad (51)
\]

so we recover EIT’s generalized specific entropy:

\[
ds = \frac{du}{\theta} - \frac{\tau}{\rho \lambda \theta^2} \bar{q} \cdot d \bar{q}. \quad (52)
\]
The verification of the residual inequality, namely,

\[ \sigma^S = -\vec{\lambda}_q \rho \cdot \vec{q} \geq 0 \]  

(53)

is guaranteed by the negativity of \( \Lambda = -\frac{\tau}{\rho \lambda \theta^2} \).

Note that no approximation has been performed and therefore, (52) is only limited by the validity of equation (28). Thus our procedure has been quite different from that adopted in usual EIT, where one departs from a generalized entropy and obtains a Maxwell-Cattaneo-like equation under the assumptions that a linear relation between fluxes and forces exists and that \( \vec{J}^S = \vec{q}/\theta \). Now, we have seen that if a Maxwell-Cattaneo evolution equation for the heat flux holds, the generalized entropy of EIT is the only possible one and the entropy flux must be given by equation (5).

• If we consider that \( \Theta \) is the local-equilibrium temperature \( T \), it must be independent of the flux \( \vec{q} \), so also must be \( \theta \), and, of course, \( \theta = T \). Thus we should obtain an specific entropy of the form

\[ s = s_{eq}(u) + F(q^2) \]  

(54)

and \( \Lambda = \Lambda (q^2) \). If no further assumptions are made, \( \xi = \xi (u, q^2) \) and the residual inequality implies, as before that \( \Lambda \leq 0 \). Thus, we observe that we can demand that the "physical" temperature appearing in the Maxwell-Cattaneo equation be the local-equilibrium one and thus, as suggested by Banach in [13], the correction on entropy due to nonequilibrium situations must be additive and independent of the equilibrium variables. Let us note, however, that our requirement is much more restrictive than his. In fact, one of the problems of developing a extension of CIT by spanning the state space is determining which variables should be considered. One can always choose the nonequilibrium variables in order that the corrections to entropy do not depend on the equilibrium quantities. However, if equation (28) holds and we have already chosen the heat flux \( \vec{q} \) as the proper nonequilibrium variable, we have proven that (54) must hold. On the other hand, in [14] it has also been proposed, within the context of discrete systems, that the nonequilibrium entropy should take the form

\[ S = \frac{U}{\Theta} + F(\Theta, \Theta^+, \Theta^-, \dot{\Theta}, \dot{\Theta}^+, \dot{\Theta}^-), \]  

(55)

i.e. that out of equilibrium one can consider the generalized contact temperature as an independent variable (Note that the existence of three temperatures \( \Theta, \Theta^+ \) and \( \Theta^- \) is a consequence of considering a discrete system and replace the temperature field). In this sense, the correction to the entropy does also depend only on nonequilibrium variables, although the first term also differs from its equilibrium counterpart, where \( \Theta \) becomes a function of \( U \).
4 Heat flux determined by a general evolution equation

If we now assume an evolution equation for the heat flux as given in equation (6), (note that such evolution equation arises for instance, when one integrates the Boltzmann equation) and considers that $Q = Q(u, \vec{q})$ and $\sigma^q = \sigma^q(u, \vec{q})$, we may write

$$\rho \left( \frac{\partial s}{\partial u} - \lambda_u \right) \frac{\partial u}{\partial t} + \rho \left( \frac{\partial s}{\partial \vec{q}} - \vec{\lambda}_q \right) \cdot \frac{\partial \vec{q}}{\partial t} + \left( \frac{\partial \vec{J}^S}{\partial u} - \vec{\lambda}_q \frac{\partial Q}{\partial u} \right) \cdot \nabla u$$

$$+ \left( \frac{\partial \vec{J}^S}{\partial \vec{q}} - \lambda_u U - \vec{\lambda}_q \cdot \frac{\partial Q}{\partial \vec{q}} \right) : \nabla \vec{q} + \vec{\lambda}_q \cdot \sigma^q \geq 0,$$

so we now have

$$\frac{\partial s}{\partial u} = \lambda_u, \quad \frac{\partial s}{\partial \vec{q}} = \vec{\lambda}_q, \quad (57)$$

$$\frac{\partial \vec{J}^S}{\partial u} = \vec{\lambda}_q \frac{\partial \vec{q}}{\partial \vec{q}}, \quad \frac{\partial \vec{J}^S}{\partial \vec{q}} = \lambda_u U + \vec{\lambda}_q \cdot \frac{\partial \vec{q}}{\partial \vec{q}}, \quad (58)$$

$$- \vec{\lambda}_q \cdot \sigma^q \geq 0. \quad (59)$$

Once more can we make use of a representation theorem to write

$$\vec{J}^S(u, \vec{q}) = \varphi(u, q^2)\vec{q}, \quad \vec{\lambda}_q = \Lambda(u, q^2)\vec{q}, \quad \sigma^q = \sigma(u, q^2)\vec{q}, \quad (60)$$

$$s = s(u, q^2), \quad \lambda_u = \lambda_u(u, q^2), \quad Q = a(u, q^2)U + b(u, q^2)\vec{q} \vec{q}, \quad (61)$$

so the previous equations may be simplified and one obtains

$$\frac{\partial s}{\partial u} = \lambda_u, \quad \frac{\partial s}{\partial \vec{q}^2} = \frac{\Lambda}{2}, \quad (62)$$

$$\frac{\partial \varphi}{\partial u} = \lambda_u \left( \frac{\partial a}{\partial u} + \frac{\partial b}{\partial u} q^2 \right), \quad \frac{\partial \varphi}{\partial \vec{q}^2} = \Lambda \left[ \frac{\partial a}{\partial q^2} + \frac{\partial b}{\partial q^2} q^2 + \frac{b}{2} \right], \quad (63)$$

$$\varphi = \lambda_u + \Lambda b q^2. \quad (64)$$

Let us observe that the entropy flux we obtain is given by (we write $\lambda_u = 1/\theta$):

$$\vec{J}^S = \frac{\vec{q}}{\theta} + \Lambda b q^2 \vec{q}, \quad (65)$$

while in [5] we had obtained

$$\vec{J}^S = \frac{\vec{q}}{\theta} + \Lambda \left( a + b q^2 \right) \vec{q}. \quad (66)$$
Thus we can observe that in this concrete example equation (27) does not hold. Note, in addition that, in equilibrium, $Q$ is not null but isotropic, thus this latter expression will also differ from the usual local-equilibrium one, namely, $\vec{J}^S = \vec{q}/T$ and thus is not convenient. If we define

$$m := a + bq^2, \quad x := \frac{q^2}{2}, \quad \Lambda b := f,$$

equations (62) and (63) may be simplified and yield

$$\frac{\partial s}{\partial u} = \varphi - 2xf, \quad \frac{\partial s}{\partial x} = \Lambda,$$  

(68)

$$\frac{\partial \varphi}{\partial u} = \Lambda \frac{\partial m}{\partial u}, \quad \frac{\partial \varphi}{\partial x} = \Lambda \frac{\partial m}{\partial x} - f.$$  

(69)

In order to obtain a concrete expression for the entropy of the system, we must solve this set of equations, taking into account that we must recover the equilibrium result in the case of null heat flux together with the convexity requirement for the entropy. Such requirement, namely, that $\delta^2 s \geq 0$ yields the following inequalities:

$$\frac{\partial \varphi}{\partial u} - 2x \frac{\partial f}{\partial u} \leq 0, \quad \Lambda \leq 0, \quad \Lambda + \frac{\partial \Lambda}{\partial x} 2x \leq 0,$$

(70)

$$\left(\frac{\partial \varphi}{\partial u} - 2x \frac{\partial f}{\partial u}\right) \left(\Lambda + \frac{\partial \Lambda}{\partial x} 2x\right) - \left(\frac{\partial \Lambda}{\partial u}\right)^2 2x \geq 0.$$  

If we restrict ourselves up to second order in the heat flux $\vec{q}$, as both $\Lambda$ and $b$ vanish in local-equilibrium, we may take $f = 0$, so equations (68) and (69) reduce to

$$\frac{\partial s}{\partial u} = \varphi, \quad \frac{\partial s}{\partial x} = \Lambda,$$  

(71)

$$\frac{\partial \varphi}{\partial u} = \Lambda \frac{\partial m}{\partial u}, \quad \frac{\partial \varphi}{\partial x} = \Lambda \frac{\partial m}{\partial x},$$  

(72)

and within this approximation the entropy flux reduces to (1) and $m$ is a function of $\varphi$.

5 Concluding remarks

In this paper we have applied Liu’s technique to the simple situation considered in [4], where EIT and Information Theory are applied to a system submitted to a heat flux $\vec{q}$. Such a simplification allows to go beyond a linear approximation. We observe how by using Liu’s technique, one can recover the results in usual EIT, but the procedure is quite different. As shown in [4], in EIT one assumes a flux-dependent specific entropy and a concrete form for the entropy flux, namely,
\( \vec{J}^S = \vec{q}/\theta \), and thus one is led to a hyperbolic evolution equation for the heat flux. Now, the viewpoint is the opposite: we depart from a given evolution equation for the heat flux, and are thus led to a concrete form for the specific entropy and the entropy flux.

On the other hand, we have considered a general evolution equation for the heat flux, namely

\[
\frac{\partial \vec{q}}{\partial t} + \nabla \cdot \vec{Q} = \vec{\sigma},
\]

and derived the corresponding form for the entropy flux and the set of partial differential equations whose solution allows the determination of the generalized nonequilibrium entropy of the system. We have also observed that our results differ from those obtained in [3], due to the definition employed for the entropy and entropy flux. Our present result, namely,

\[
\vec{J}^S = \vec{q}/\theta + \Lambda b q^2 \vec{q},
\]

which reduces to \( \vec{J}^S = \vec{q}/T \) in local-equilibrium, is thus much more convenient.

Hence, we observe that there is not a complete equivalence between Liu’s technique and information theory. This lack of equivalence is also observed when discrete systems are considered [4]. For such systems, one cannot usually write evolution equations for the fluxes, so by applying Liu’s technique it is possible to prove that entropy cannot depend on them. On the other hand, if one assumes that the state of the system is described by the total internal energy \( U \) and the heat flux \( \dot{Q} \), information theory yields to an entropy which depends on both \( U \) and \( \dot{Q} \), in spite of the fact that no evolution equations neither for \( U \) nor for \( \dot{Q} \) need to be included. This is due to the fact that we deal with such restrictions by maximizing

\[
- k_B \int f \ln f d\vec{c} - \lambda_U \int f H d\vec{c} - \lambda_Q \int f \dot{Q} d\vec{c}.
\]

Thus, we see that there may be some subtle differences between the use of Lagrange multipliers in usual information theory and in Liu’s procedure.

**Acknowledgments**

Stimulating discussions with Prof. J. Casas-Vázquez and J. Faraudo are acknowledged. One of the authors (R.D-C) is supported by a doctoral scholarship from the Programa de formació d’investigadors of the Generalitat de Catalunya under grant FI/94-2.009. We also acknowledge partial financial support from the Dirección General de Investigación of the Spanish Ministry of Education and Science (grant PB94-0718).
References

[1] Liu, I.S., Method of Lagrange multipliers for exploitation of the entropy principle, Arch. Rat. Mech. Anal. 46, (1972), 131.

[2] Dreyer, W., Maximisation of the entropy in non-equilibrium, J. Phys. A: Math. Gen. 20, (1987), 6505-6517.

[3] Jou, D., Casas-Vázquez, J., Possible experiment to check the reality of a nonequilibrium absolute temperature, Phys. Rev. A, 45 (1992), 8371-8373.

[4] Casas-Vázquez, J., Jou, D., Nonequilibrium temperature versus local equilibrium temperature, Phys. Rev. E, 49 (1994), 1040-1048.

[5] Domínguez-Cascante, R., Jou, D., Entropy flux and absolute temperature in Extended Irreversible Thermodynamics, J. Non-Equilib. Thermodyn. 20 (1995), 263-273.

[6] Verhas, J., On the entropy current, J. Non-Equilib. Thermodyn., 8 (1983), 201-206.

[7] Jou, D., Casas-Vázquez, J., Lebon, G., Extended Irreversible Thermodynamics, Springer, Berlin, 1996.

[8] Müller, I., Ruggeri, T., Extended Irreversible Thermodynamics, Springer, New York, 1993.

[9] Salamon, P., Sieniutycz, S., Eds., Extended Irreversible Thermodynamics, Taylor and Francis, New York, 1992.

[10] Sieniutycz, S., Conservation Laws in Variational Thermo-hydrodynamics, Kluwer Academic Publishers, Dordrecht, 1994.

[11] García-Colín, L.S., Extended Non-equilibrium Thermodynamics, scope and limitations, Revista Mexicana de Física (Mexico) 34, (1988), 344-366.

[12] Nettleton, R., Relaxation thermal theory of conduction in liquids, Phys. Fluids, 3 (1960), 216.

[13] Banach, Z., Piekarski, S., Thermodynamic potentials and extremum principles for a Boltzmann gas, Arch. Mech. 48, (1996), 791-812.

[14] Muschik, W., Domínguez-Cascante, R., On extended thermodynamics of discrete systems, Physica A 233, (1996), 523-550.