Second-Order Coding Rates for Conditional Rate-Distortion

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Abstract

This paper characterizes the second-order coding rates for lossy source coding with side information available at both the encoder and the decoder. We first provide non-asymptotic bounds for this problem and then specialize the non-asymptotic bounds for three different scenarios: discrete memoryless sources, Gaussian sources, and Markov sources. We obtain the second-order coding rates for these settings. It is interesting to observe that the second-order coding rate for Gaussian source coding with Gaussian side information available at both the encoder and the decoder is the same as that for Gaussian source coding without side information. Furthermore, regardless of the variance of the side information, the dispersion is 1/2 nats squared per source symbol.

I. INTRODUCTION

In almost lossless source coding, the Shannon entropy of a source is, on average, the minimum number of bits required to represent a given source [1]. In lossy source coding, the rate-distortion function (which, in this paper, is more specifically called the rate-distortion function without side information) plays the role of the Shannon entropy [2]. The rate-distortion function without side information is the minimum number of bits per symbol required to reconstruct a given source with the probability of excess distortion being asymptotically small, or with an average distortion that does not exceed a specified upper bound.

The class of source coding problems with side information is important as it can model many practical problems. Consider a scenario when a source wants to transmit a high-resolution image to a receiver who happens to have a low-resolution version of the same image. In another example, the source may be a piece of music contaminated by a background noise source and the intended receiver has already had observations of the background noise. The rate-distortion problem without side information can be extended to the case when the side information is available at both the encoder and the decoder [3], [4], only causally available at the decoder [5], or non-causally available at the decoder (i.e., Wyner-Ziv problem) [6]. The rate-distortion function for stationary-ergodic sources with side information was found in [7]. The rate-distortion function for mixed types of side information (i.e., a mixture of some side information known at both the encoder and the decoder and some known only at the decoder) was evaluated in [8]. For memoryless sources, delayed side information at the decoder does not improve the rate-distortion function. However, this is not the case for sources with memory [9]. The authors of [10] considered source coding with side information, and with distortion measures as functions of side information.

All the results shown above hold provided the blocklength, i.e., the number of source symbols, is allowed to grow without bound. However, some applications are required to operate with short blocklengths due to delay or complexity constraints at the destination. Thus, it is of high interest to characterize the finite blocklength rate-distortion function, i.e., the minimum number of bits per symbol that is required to reconstruct a source at a given fixed blocklength. This is, in general, a difficult task, and thus, we focus on approximating this quantity.

A. Related Works

Strassen [11] obtained the second-order coding rate for almost lossless source coding without side information. Recently, Hayashi [12] considered second-order coding rate for fixed-length source coding and showed that the outputs of fixed-length source codes are not uniformly distributed (debunking Han’s folklore theorem [13] in the second-order sense). Kostina and Verdú [14] and Ingber and Kochman [15] characterized the dispersion of lossy...
source coding problem without side information. When the source is stationary and memoryless, they showed that the finite blocklength rate-distortion function without side information $R_{\text{noSI}}(n, D, \epsilon)$ can be approximated as

$$R_{\text{noSI}}(n, D, \epsilon) = R_{\text{noSI}}(D) + \frac{\sqrt{V_{\text{noSI}}(D)}}{n} Q^{-1}(\epsilon) + O\left(\frac{\log n}{n}\right),$$

(1)

where $R_{\text{noSI}}(D)$ is the rate-distortion function without side information, $V_{\text{noSI}}(D)$ is the dispersion that characterizes the convergence rate to the Shannon limit $R_{\text{noSI}}(D)$, $n$ is the blocklength, $D$ is the excess distortion threshold, and $\epsilon$ is the upper bound on the probability that the distortion exceeds $D$. The rate-distortion problem may also be studied from the moderate deviations perspective [16] and the fundamental limit there is also dependent on $V_{\text{noSI}}(D)$. Achievable second-order coding rates for the Wyner-Ahlswede-Korner problem of almost-lossless source coding with rate-limited side-information, the Wyner-Ziv problem of lossy source coding with side-information at the decoder and the Gelfand-Pinsker problem of channel coding with non-causal state information available at the decoder were established in [17]. The paper [18] studied second-order coding rates for the fixed-to-variable lossless compression. For other related works in the study of fixed error asymptotics, the reader is referred to [19].

B. Main Contributions

This paper focuses on the analysis and approximation of the finite blocklength rate-distortion function for source coding with side information available at both the encoder and the decoder. The contributions of this paper are stated below.

- A non-asymptotic achievability bound is established for the problem of lossy source coding with side information available at both the encoder and the decoder.
- We establish the second-order coding rate for the discrete memoryless source with a side information variable taking values in a finite alphabet. As a corollary, we obtain the second-order coding rate for the case when the source alphabet, the reconstruction alphabet and the side information alphabet are finite and the distortion measure is the Hamming distance.
- We establish the second-order coding rate for Gaussian source with Gaussian side information and the squared-error distortion measure. Somewhat interestingly, the dispersion does not depend on the variance of the side-information and is $1/2$ squared nats per source symbol.
- When the source has memory, we establish the second-order coding rate for the case where the sequence of source and side information variables jointly forms a time-homogeneous Markov chain.

C. Paper Outline

The paper is organized as follows. In section II we formulate the problem, and define important concepts which are used throughout the paper. In section III we present non-asymptotic bounds for the source coding problems with side information available at both the encoder and the decoder. These so-called one-shot bounds hold for any blocklength. Based on the bounds established in section III, we establish the second-order coding rates for the discrete memoryless source, the Gaussian source and the Markov source in sections IV, V and VI respectively. Technical proofs are presented in section VIII.

II. PROBLEM FORMULATION AND DEFINITIONS

Let $\mathcal{X}$ be the source alphabet, let $\mathcal{Y}$ be the reproduction alphabet, and let $\mathcal{S}$ be the side information alphabet. The random variables $X, Y$ and $S$ follow the distribution

$$P_{YXS}(yxs) = P_{Y|XS}(y|x,s)P_{X|S}(x|s)P_{S}(s).$$

(2)

We use a single-letter fidelity criterion to measure the distortion between the source sequence $x^n$ and the reproducing sequence $y^n$, i.e.,

$$d(x^n, y^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i),$$

(3)

where $d : \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{R}_+$, for $n \in \mathbb{N}$, is a bounded real-valued non-negative distortion function.
Definition 1. An \((M_n, n, D, \epsilon_n)\)-code for the source coding system with side information (see Figure 1) consists of an encoding function
\[
\phi_n : \mathcal{X}^n \times \mathcal{S}^n \rightarrow \mathcal{M}_n \triangleq \{1, 2, \ldots, M_n\},
\]
and a decoding function
\[
\psi_n : \mathcal{M}_n \times \mathcal{S}^n \rightarrow \mathcal{Y}^n,
\]
such that the probability of excess distortion satisfies
\[
\Pr\{d[X^n, \psi_n(\phi_n(X^n, S^n), S^n)] > D\} \leq \epsilon_n.
\]

An \((M_n, n, D, \epsilon_n)\)-code, which is defined as shown above, is called a \textit{D-semifaithful} code in the rate-distortion literature [20], [21].

Definition 2. A rate \(R\) is defined to be \((\epsilon, D)\)-achievable if there exists a sequence of \((M_n, n, D, \epsilon_n)\)-codes satisfying
\[
\limsup_{n \to \infty} \frac{1}{n} \log M_n \leq R,
\]
\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon.
\]

In contrast to the above definition, the following definition is non-asymptotic.

Definition 3. A rate \(R\) is defined to be \((\epsilon, D, n)\)-achievable if there exists a \((\lfloor \exp(nR) \rfloor, n, D, \epsilon_n)\)-code. The \((\epsilon, D, n)\) finite blocklength rate-distortion function \(R(\epsilon, D, n)\) is defined as the infimum of the set of all \((\epsilon, D, n)\)-achievable rates.

The following definition defines the quantity of interest in this paper.

Definition 4. A number \(L \in \mathbb{R}\) is defined to be second-order \((\epsilon, D, \kappa)\)-achievable if there exists a sequence of \((M_n, n, D, \epsilon_n)\)-codes satisfying
\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} (\log M_n - n\kappa) \leq L,
\]
\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon.
\]
The \((\epsilon, D, \kappa)\) second-order rate-distortion function \(L^*(\epsilon, D, \kappa)\) is defined as the infimum of the set of all second-order \((\epsilon, D, \kappa)\)-achievable rates.

The aim of this paper is to characterize the \((\epsilon, D, \kappa)\) second-order rate-distortion function \(L^*(\epsilon, D, \kappa)\) for source coding with side information available at both the encoder and the decoder.

Before presenting the main result, we state some definitions that will be used throughout this paper.

Definition 5. Fix the distribution of \(XS\) as \(P_{XS}\). Define the rate-distortion function with side information as
\[
R(X; D|S) = \min_{P_{Y|XS}} I(X; Y|S),
\]
where the minimum is taken over the set of all marginal conditional distributions \( P_{Y|XS} \) satisfying
\[
P_{Y|XS}(y|x,s) \geq 0 \quad \text{for all } (y, x, s), \\
\sum_{y \in \mathcal{Y}} P_{Y|XS}(y|x,s) = 1, \\
\sum_{s \in S, x \in \mathcal{X}, y \in \mathcal{Y}} P_{Y|XS}(y|x,s)P_{X|S}(x|s)P_S(s)d(x, y) \leq D.
\]

To make the dependence on the distribution \( P_{XS} \) explicit, we sometimes also denote \( R(X; D|S) \) as \( R(P_{X|S}, D|P_S) \). Assume the distribution that achieves the minimum in (11) is unique. When there is no side information, i.e., \( S = \emptyset \), we recover the rate-distortion function without side information denoted as \( R(X; D) \) or \( R(P_X, D) \).

When the excess distortion criterion is employed, we have the following first-order result for the source coding problem with side information [3] (i.e., the conditional rate-distortion problem [4]),
\[
\lim_{\epsilon \to 0} \liminf_{n \to \infty} R(\epsilon, D, n) = R(X; D|S).
\]

In order to characterize the second-order rate-distortion function, we state the following definitions. The notion of information densities will play an important role in characterizing the second-order rate-distortion function. In fact, in order to deal with the constraints inherent in the rate-distortion problem, the concept of \( D \)-tilted information densities, which was introduced in [22], is useful.

**Definition 6.** Define the conditional information densities as follows:
\[
i_{X,Y|S}(x; y|s) \triangleq \log \frac{P_{XY|S}(x,y|s)}{P_{Y|S}(y|s)P_{X|S}(x|s)}, \quad \text{and} \\
i_{X|S}(x|s) \triangleq i_{X,Y|S}(x; y|s).
\]

Note that \( i_{X|S} \) is also known as the conditional self-information.

**Definition 7.** Define the conditional \( D \)-tilted information density as follows:
\[
j_{X|S}(x, D|s) \triangleq \log \frac{1}{\mathbb{E}[\exp\{\lambda^* D - \lambda^*d(X, Y^*)\}]|S = s} \\
\]

where \( P_{Y^*|XS} \) is the distribution that achieves the minimum in (5), the expectation is taken with respect to the induced output distribution \( P_{Y^*|S}(y|s) = \sum_x P_{Y^*|XS}(y|x,s)P_{X|S}(x|s) \), and \( \lambda^* \) is defined as
\[
\lambda^* \triangleq \frac{dR(P_{X|S}, D|P_S)}{dD}.
\]

**Remark 1.** In this definition, the conditional \( D \)-tilted information density has a built-in feature which takes the distortion constraint into consideration.

The conditional \( D \)-tilted information density \( j_{X|S}(x, D|s) \) has some important properties which can be found in [22]. We review them here.

**Lemma 1.** The conditional \( D \)-tilted information density \( j_{X|S}(x, D|s) \) has the following properties.
1. \( j_{X|S}(x, D|s) = i_{X,Y^*|S}(x; y|s) + \lambda^*d(x, y) - \lambda^* D. \)
2. \( R(X; D|S) = \mathbb{E}[j_{X|S}(X, D|S)]. \)
3. For any \( P_{Y|S} \) where \( X \to S \to Y \), we have \( \mathbb{E}[\exp\{\lambda^* d - \lambda^*d(X, Y) + j_{X|S}(X, D|S)\}] \leq 1. \)

In the achievability proof of the conditional rate-distortion problem, the following concept is important.

**Definition 8.** Given a source sequence \( x^n \in \mathcal{X}^n \), define the \( D \)-ball \( B_D(x^n) \) around this sequence as
\[
B_D(x^n) \triangleq \{ y^n \in \mathcal{Y}^n | d(x^n, y^n) \leq D \}.
\]
The following is the cumulative distribution function of a standard Gaussian distribution
\[
\Phi(t) \triangleq \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) \, du.
\] (21)

The complementary cumulative distribution function is \(Q(t) \triangleq 1 - \Phi(t)\). Since these functions are monotonic, they admit inverses, which we will denote as \(\Phi^{-1}\) and \(Q^{-1}\).

III. NON-ASYMPTOTIC BOUNDS

In this section, we first present a non-asymptotic achievability bound.

**Lemma 2** (Achievability). For every \(P_{Y^n|S^n}\), there exists an \((M_n, D, n, \epsilon_n)\)-code such that
\[
\epsilon_n \leq \mathbb{E}\{1 - P_{Y^n|S^n}(B_D(x^n)|S^n)^M\}
\] (22)
where we have
\[
P_{Y^n|S^n} = P_{Y^n|S^n}P_{X^n|S^n}P_{S^n}.
\] (23)

*Proof:* Given each side information sequence \(S^n = s^n\), we construct a reconstruction codebook \(C(s^n)\), which consists of \(M\) random reconstruction sequences \(\{Y^n(m, s^n)\}_{m=1}^{M}\). Each of the sequence \(Y^n(m, s^n)\), for \(m \in M \triangleq \{1, 2, \ldots, M\}\), is generated independently according to an arbitrary distribution \(P_{Y^n|S^n=s^n}\), which satisfies equation (23). Choose a sub-code \((\phi_n, \psi_n)\), the encoder and decoder of which are defined as
\[
\phi_n(x^n, s^n) = \arg\min_{m \in M} d(x^n, Y^n(m, s^n)),
\] (24)
\[
\psi_n(m, s^n) = Y^n(m, s^n).
\] (25)

The average probability of error of this sub-code is given by
\[
\bar{\epsilon}(s^n) = \mathbb{E}\{1 \{ \min_{m \in M} d(X^n, Y^n(m, s^n)) > D \} | S^n = s^n \}
\] (26)
\[
= \mathbb{E}\left[ \prod_{m=1}^{M} 1\{d(X^n, Y^n(m, s^n)) > D \} | S^n = s^n \right]
\] (27)
\[
= \mathbb{E}\left[ \prod_{m=1}^{M} 1\{d(X^n, Y^n(m, s^n)) > D \} | X^n \right] | S^n = s^n
\] (28)
\[
= \mathbb{E}\left[ \prod_{m=1}^{M} \mathbb{E}[1\{d(X^n, \tilde{Y}^n) > D \} | X^n ] | S^n = s^n \right]
\] (29)
\[
= \mathbb{E}\left[ 1 - P_{Y^n|S^n}(B_D(X^n)) | S^n = s^n \right]^{M}
\] (30)

where equation (29) follows from the independence of reconstruction sequences.

Taking the average over all sub-codes, we have the average probability of error is
\[
\bar{\epsilon} = \sum_{s^n \in S^n} P_{S^n}(s^n) \bar{\epsilon}(s^n)
\] (31)
\[
= \mathbb{E}\{1 - P_{Y^n|S^n}(B_D(X^n)) | S^n \}^{M} \}.
\] (32)

By the random coding argument, there exists an \((M_n, D, n, \epsilon_n)\)-code such that
\[
\epsilon_n \leq \mathbb{E}\{1 - P_{Y^n|S^n}(B_D(X^n)) | S^n \}^{M} \}.
\] (33)

This concludes the proof.

Next, we relax the bound in Lemma 2 to obtain the following lemma, which turns out to be more amenable to asymptotic evaluations.
Lemma 3. For any $\gamma_n, \beta_n,$ and $\delta_n,$ there exists an $(M_n, D, n, \epsilon_n)$-code such that
\[
\epsilon_n \leq \Pr[j_{X^n|S^n}(X^n, D|S^n) > \log \gamma_n - \log \beta_n - \lambda_n^n \delta_n] \\
+ E[\min(1 - \beta_n, \Pr[D - \delta_n \leq d(X^n, Y^n) \leq D|X^n]) | S^n]] \\
+ e^{-\epsilon_n/\beta_n} E[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D|S^n)) | S^n)],
\] (34)
where $P_{Y^n|X^n}$ achieves the minimum in (5), and $P_{Y^n|X^n}=P_{S^n}$ is the $n$-th order product distribution of $P_{Y^n|X^n}$.

This lemma is proved in section VIII-A.

The following lemma, which plays an important part in the converse, was derived in [22].

Lemma 4. Any $(M_n, D, \epsilon_n)$-code for the lossy source coding system with side information satisfies
\[
\epsilon_n \geq \sup_{\gamma > 0} \{\Pr[j_{X^n|S^n}(X^n, D|S^n) \geq \log M_n + \gamma] - \exp(-\gamma)\}.
\] (35)

IV. DISCRETE MEMORYLESS SOURCE WITH I.I.D. SIDE INFORMATION

In this section, we consider the discrete memoryless source. Assume that the source alphabet $\mathcal{X},$ the reproduction alphabet $\mathcal{Y},$ and the side information alphabet $\mathcal{S}$ are finite. The source coding system is memoryless and stationary in the sense that
\[
P_{X^n S^n}(x^n s^n) = \prod_{i=1}^{n} P_{X S}(x_i s_i).
\] (36)

Before presenting the main results of this section, we define an important quantity.

Definition 9. Define the variance $V$ of the $D$-tilted information density $j_{X|S}(X, D|S)$ with respect to $P_{XS}$ as
\[
V \equiv \var(j_{X|S}(X, D|S)) = \sum_{x \in \mathcal{X}, s \in \mathcal{S}} P_{XS}(x s) [j_{X|S}(x, D|s)]^2 - [R(X; D|S)]^2.
\] (37)

Next, we present the first main result of this paper.

Theorem 1. The second-order rate-distortion function $L^*(\epsilon, D, R(X; D|S))$ for the discrete memoryless source coding with side information is given by
\[
L^*(\epsilon, D, R(X; D|S)) = \sqrt{V} Q^{-1}(\epsilon).
\] (39)

Let us mention that the dispersion [14] is an operational quantity that is closely related to the second-order coding rate. It characterizes the speed at which the rate of optimal codes converge to the first-order fundamental limit. For conditional rate-distortion, we may define the dispersion $V_{dps}$ as
\[
V_{dps} \equiv \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{n \to \infty} \left(\frac{\sqrt{n}(R(\epsilon, D, n) - R(X; D|S))}{Q^{-1}(\epsilon)}\right)^2.
\] (40)

From Theorem 1, we observe that the operational quantity $V_{dps}$ is equal to the information quantity $V.$

Let $V_s \equiv \var(j_{X|S}(X, D|S) | S = s)$ be the dispersion$^1$ of the source $X_s \sim P_{X|S}(\cdot | s).$ Now notice that by the law of total variance, $V$ can be decomposed as
\[
V = \mathbb{E}[\var(j_{X|S}(X, D|S) | S)] + \var[\mathbb{E}(j_{X|S}(X, D|S) | S)]
= \mathbb{E}[V_S] + \var[\mathbb{E}(j_{X|S}(X, D|S) | S)].
\] (41)

The first term represents the randomness of the source weighted by the probability mass function of the side information, while the second term represents the randomness of the side information in terms of the constituent rate-distortion functions.

$^1$Note that term dispersion [14] here refers to the unconditional rate-distortion problem. This should not cause any confusion in the sequel.
Theorem 1 is proved in subsection VIII-B. One of the key ideas in the achievability proof of Theorem 1 is to apply the random coding bound (Lemma 2) in the asymptotic evaluation. The key idea in the converse proof of Theorem 1 is to make use of the non-asymptotic converse bound (Lemma 4) in the asymptotic evaluation.

We illustrate this theorem through an example.

**Example 1.** Consider the case when the source alphabet $X$, the reconstruction alphabet $Y$ and the side information alphabet $S$ are binary $\{0, 1\}$. The distortion function is the Hamming distance function $d(x, y) = 1\{x \neq y\}$. Assume $P_S(1) = a$, $P_S(0) = 1 - a$, $P_X(1) = b$ and $P_X(0) = 1 - b$, for $0 < a, b < 1$. Assume $P_{X|S}(1|0) = P_{X|S}(1|1) = c$, and $P_{X|S}(0|0) = P_{X|S}(0|1) = 1 - c$, for $0 < c < \frac{1}{2}$. It can shown that

$$j_{X|S}(x, D|s) = i_{X|S}(x|s) - H(D)$$

if $0 < D < c$, and $0$ if $D \geq c$. Note that the conditional $D$-tilted information density in this case is independent of the marginal distributions $P_X$ and $P_S$. Next, we have

$$R(X; D|S) = H(X|S) - H(D)$$

$$= H(c) - H(D)$$

if $0 < D < c$, and $0$ if $D \geq c$. Here $H(D)$ is the entropy of a Bernoulli($D$) source.

In general, we have the following corollary.

**Corollary 1.** The second-order rate-distortion function $L^*(\epsilon, D, R(X; D|S))$ for the binary source with binary side information and Hamming distortion function is given by

$$L^*(\epsilon, D, R(X; D|S)) = \sqrt{\text{var}[i_{X|S}(X|S)]}Q^{-1}(\epsilon).$$

**A. Remarks concerning Theorem 1**

1) In fact, it is also straightforward to characterize $L^*(\epsilon, D, \kappa)$ when $\kappa \neq R(X; D|S)$. We have

$$L^*(\epsilon, D, \kappa) = \begin{cases} +\infty & \kappa < R(X; D|S) \\ \sqrt{V}Q^{-1}(\epsilon) & \kappa = R(X; D|S) \\ -\infty & \kappa > R(X; D|S) \end{cases}$$

The first statement above (for the case $\kappa < R(X; D|S)$) implies the strong converse for conditional rate-distortion. The strong converse for unconditional rate-distortion for discrete memoryless sources is already well known (e.g., [23, Chapter 7]).

2) From Theorem 1, we can deduce that there exists a sequence of $(M_n, n, D, \epsilon_n)$-codes for the source coding system with side information such that its rate is

$$\frac{1}{n}\log M_n = R(X; D|S) + \frac{1}{n}Q^{-1}(\epsilon) + o\left(\frac{1}{\sqrt{n}}\right)$$

and its asymptotic probability of excess distortion satisfies

$$\epsilon_n \leq \epsilon + o(1).$$

It is observed that $V$ characterizes the rate of convergence to the first-order rate-distortion function $R(X; D|S)$.

3) In order to compute $V$, it is noted that the gradient of $R(X; D|S)$ plays an important role.

**Definition 10.** For each $a \in X$, $b \in S$, define

$$R'(P_{X|S}(a|b), D|P_S(b)) \triangleq \frac{dR(P_{X|S}(D|P_S))}{dP_{X|S}(ab)|_{P_{X|S}=P_{X|S}}}.$$
The function $R(P_{X|S}, D|P_S)$ can be thought of as that of $|\mathcal{X}||S|$ variables. By stacking up $|\mathcal{X}||S|$ partial derivatives as defined in Definition 10, we form the gradient $\nabla R(P_{X|S})$ of $R(P_{X|S}, D|P_S)$ evaluated at $P_{XS}$. The joint distribution $P_{XS}$ can be regarded as a length-$|\mathcal{X}||S|$ vector that sums to one.

Even though the conditional $D$-tilted information density $j_{X|S}(X, D|S)$ is useful in characterizing the second-order rate-distortion function, it is not easy to compute. The task of computing $V$ is made easier by the following lemma.

**Lemma 5.** For any $a \in \mathcal{X}$ and $b \in S$, we have

$$j_{X|S}(a, D|b) = R'(P_{X|S}(a|b), D|P_S(b)).$$

**Proof:** We have

$$R'(P_{X|S}(a|b), D|P_S(b)) = \frac{dR(P_{X|S}, D|P_S)}{dP_{XS}(ab)}|_{P_{XS}=P_{XS}} = \frac{dE[j_{X|S}(X, D|S)]}{dP_{XS}(ab)}|_{P_{XS}=P_{XS}} = \frac{d\left[\sum_{x,s} P_{XS}(xs)j_{X|S}(x, D|s)\right]}{dP_{XS}(ab)}|_{P_{XS}=P_{XS}} = j_{X|S}(a, D|b) + \frac{dE[j_{X|S}(X, D|S)]}{dP_{XS}(ab)}|_{P_{XS}=P_{XS}},$$

Using part 1) of Lemma 1, it is evident that

$$\frac{dE[j_{X|S}(X, D|S)]}{dP_{XS}(ab)}|_{P_{XS}=P_{XS}} = 0.$$ \hfill (57)

This completes the proof of the lemma. \hfill \blacksquare

Let us remark that according to [24, Theorem 2.2], the $D$-tilted information density for the source coding without side information is given by

$$j_X(a, D) = R'(P_X(a), D) - \log e = \frac{dR(P_X, D)}{dP_X(a)}|_{P_X=P_X} - \log e.$$ \hfill (58)

This is because

$$\frac{dR(P_X, D)}{dP_X(a)}|_{P_X=P_X} = \frac{dE[j_X(X, D)]}{dP_X(a)}|_{P_X=P_X} = j_X(a, D) + \frac{dE[j_X(X, D)]}{dP_X(a)}|_{P_X=P_X},$$ \hfill (59)

and in this case we have

$$\frac{dE[j_X(X, D)]}{dP_X(a)}|_{P_X=P_X} = - \log e.$$ \hfill (60)

Observe that the term $- \log e$ is present in the no-side information setting (58) but not in the side information setting (52). This is due to (61).

As a consequence of Lemma 5, the variance of the conditional $D$-tilted information $V$, defined in (37)–(38), can be alternatively expressed as the variance of the gradient $\nabla R(P_{XS})$ with respect to $P_{XS}$, i.e.,

$$V = \text{var}(\nabla R(P_{XS}))$$ \hfill (62)

$$= \sum_{a \in \mathcal{X}} \sum_{b \in S} P_{XS}(ab)[R'(P_{X|S}(a|b), D|P_S(b))]^2 - \left[\sum_{a \in \mathcal{X}} \sum_{b \in S} P_{XS}(ab)R'(P_{X|S}(a|b), D|P_S(b))\right]^2.$$ \hfill (63)

4) The relationship between the side-information dependent rate-distortion function $R(P_{X|S}(\cdot|s), D)$ and the conditional rate-distortion function $R(P_{X|S}, D|P_S)$ is given by the following lemma [4].
Lemma 6. We have
\[ R(P_X|S, D|P_S) = \inf_{\{d_s\}_{s \in D}} \sum_{s \in S} P_S(s) R(P_X|S(\cdot|s), d_s), \]  
where the set \( D \) is defined as
\[ D = \left\{ \{d_s\}_{s \in S} \bigg| \sum_{s \in S} P_S(s) d_s = D, d_s \geq 0 \right\}. \]

Intuitively, any achievable code for the conditional rate-distortion problem can be thought of as a combination of sub-codes for sub-channels with the side information \( S = s \) and the excess distortion \( d_s \). The total distortion \( D \) is the \( P_S \)-convex combination of the constituent excess distortions \( d_s \). Note that Ingber-Kochman [15] used the method of types (similarly to the technique used in Marton’s covering lemma [25]) to perform a second-order (dispersion) analysis for the rate-distortion problem without side information. We attempted to adapt their technique for our setting but it was not straightforward to generalize their method to the conditional rate-distortion problem at hand. This is because Lemma 5 intuitively suggests to treat \( X \) and \( S \) jointly to obtain the second-order rate-distortion function \( L^*(\epsilon, D, R(P_X|S, D|P_S)) \). However, if the method of types is used, the relationship in Lemma 6 restricts us to treat \( X \) conditioning on \( S = s \) first, in the achievability proof, in order to obtain the first-order term. However, this method leads to a different (and, in fact, inferior) second-order term. The beauty in the random coding bound in Lemma 3 is that it allows us to treat \( X \) and \( S \) jointly.

V. GAUSSIAN MEMORYLESS SOURCE WITH I.I.D. SIDE INFORMATION

In this section, we consider the i.i.d. Gaussian source. More specifically,
\[ X_i \sim \mathcal{N}(0, \sigma^2_X). \]  

The side information is given by
\[ S_i = X_i + Z_i \]  
where \( i = 1, 2, ..., n \),
\[ Z_i \sim \mathcal{N}(0, \sigma^2_Z) \]
and \( Z_i \) is independent of \( X_i \). We consider the squared-error distortion function, i.e.,
\[ d(x^n, y^n) \triangleq \sum_{i=1}^{n} (x_i - y_i)^2. \]  

Define the conditional variance as
\[ \sigma^2_{X|S} \triangleq \frac{\sigma^2_X \sigma^2_Z}{\sigma^2_X + \sigma^2_Z}. \]  

The case where \( D \geq \sigma^2_{X|S} \) is trivial as \( R(X; D|S) = 0 \). It is assumed that \( 0 < D < \sigma^2_{X|S} \). In this case, it is well-known that [4] the conditional rate-distortion function is given by
\[ R(X; D|S) = \frac{1}{2} \log \frac{\sigma^2_{X|S}}{D}. \]  

The second-order rate-distortion function in this case is given by the following theorem.

Theorem 2. The second-order rate-distortion function \( L^*(\epsilon, D, R(X; D|S)) \) for Gaussian source coding with side information is given by
\[ L^*(\epsilon, D, R(X; D|S)) = \sqrt{T} Q^{-1}(\epsilon) \log e. \]  

This theorem is proved in subsection VIII-C.
A. Remarks concerning Theorem 2

1) From Theorem 2, we observe that the dispersion for Gaussian source coding with side information is $1/2$ nats squared per source symbol. In other words, the second-order rate-distortion function for Gaussian source coding with side information is the same as that for Gaussian source coding without side information \[14\] even though the rate-distortion functions for both coding problems are different in general. The presence of side information at both the encoder and the decoder does not affect the second-order coding rate. Intuitively, given the side information $s^n$, the encoder and the decoder can adapt to it and design a second-order optimal sub-code for each source-encoding sub-test channel (indexed by $s^n$). The second-order coding rate for each sub-test channel is basically the same as that for the source coding system without side information. The second-order rate-distortion function for Gaussian source coding with side information is the average of all second-order coding rates for sub-test channels, when the average is taken with respect to the side information random variable. Thus, this explains the observation.

2) It would be interesting to investigate if the statement mentioned in the previous item still holds when the side information is available at either only the decoder or only the encoder. Of course, the rate-distortion functions for the cases where the side information is known at both terminals and at the decoder only are identical in the Gaussian case \[26, Chapter 11\]. Thus one wonders whether the dispersion remains at $1/2$ nats$^2$ per source symbol for the Gaussian Wyner-Ziv problem \[6\].

3) Scarlett \[27\] showed that the dispersion for dirty paper coding (Gaussian Gel’fand-Pinsker) is the same as that when there is no interference. Furthermore, he showed that the same holds true even if the interference is not Gaussian but satisfies some mild concentration conditions. It would be interesting to investigate if the same is true in the lossy compression with (encoder and decoder) side information scenario.

VI. MARKOV SOURCE WITH MARKOV SIDE INFORMATION

So far, we have considered only memoryless sources. In this section, we consider the system in which the source and side information jointly forms an irreducible, ergodic and time-homogeneous Markov chain, i.e.,

$$X_1S_1 \rightarrow X_2S_2 \rightarrow \ldots \rightarrow X_nS_n.$$ \hspace{1cm} (73)

We further assume that the source alphabet $\mathcal{X}$ and the side information alphabet $\mathcal{S}$ are both finite. Denote the stationary distribution of this Markov chain as $\pi_{XS}$. Assume that this Markov chain starts from the stationary distribution, i.e.,

$$P_{X_1S_1} = \pi_{XS}.$$ \hspace{1cm} (74)

Under the assumption in (74), all the marginals $P_{X_iS_i}$ for $i \geq 1$ are equal to $\pi_{XS}$.

First, we define a few relevant quantities.

**Definition 11.** Define

$$\mu \triangleq R(X; D|S)|_{P_{XS}=\pi_{XS}},$$ \hspace{1cm} (75)

$$V_n \triangleq \text{var} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} j_{X_i}|S_i}(X_i, D|S_i) \right).$$ \hspace{1cm} (76)

We have the following important lemma.

**Lemma 7.** For the Markov chains considered above, the following limit exists

$$\lim_{n \to \infty} V_n$$ \hspace{1cm} (77)

and is equal to

$$V_\infty \triangleq \text{var}[j_{X_1}|S(X, D|S)]|_{P_{XS}=\pi_{XS}} + 2 \sum_{i=1}^{\infty} \text{cov}[j_{X_1}|S_1}(X_1, D|S_1), j_{X_{1+i}}|S_{1+i}(X_{1+i}, D|S_{1+i})].$$ \hspace{1cm} (78)
Proof: The lemma follows from the fact that

\[
V_n = \frac{1}{n} \operatorname{var} \left( \sum_{i=1}^{n} j_{X_i|S_i}(X_i, D|S_i) \right) = \frac{1}{n} \sum_{k,j=1}^{n} \operatorname{cov} \left[ j_{X_k|S_k}(X_k, D|S_k), j_{X_j|S_j}(X_j, D|S_j) \right] = \frac{1}{n} \sum_{j=1}^{n} (n-j) \operatorname{cov} \left[ j_{X_j|S_j}(X_j, D|S_j), j_{X_{j+i}|S_{j+i}}(X_{j+i}, D|S_{j+i}) \right].
\]  

(79)

The equality in (80) follows from the time-homogeneity of the chain and simple rearrangements. Now, since the covariance \( \operatorname{cov} \left( j_{X_j|S_j}(X_j, D|S_j), j_{X_{j+i}|S_{j+i}}(X_{j+i}, D|S_{j+i}) \right) \) decays exponentially fast in the lag \( j \) for this class of Markov chains,

\[
\lim_{n \to \infty} \sum_{j=1}^{n} j \cdot \operatorname{cov} \left[ j_{X_j|S_j}(X_j, D|S_j), j_{X_{j+i}|S_{j+i}}(X_{j+i}, D|S_{j+i}) \right] = 0,
\]

(81)

and thus

\[
\lim_{n \to \infty} V_n = \frac{1}{n} \operatorname{var} \left[ j(X, D|S) \right]_{P_{X,S} = \pi_{X,S}} + 2 \sum_{j=1}^{\infty} \operatorname{cov} \left[ j_{X_j|S_j}(X_j, D|S_j), j_{X_{j+i}|S_{j+i}}(X_{j+i}, D|S_{j+i}) \right].
\]

(82)

The right-hand-side is exactly \( V_\infty \) as desired. \( \blacksquare \)

The second-order rate-distortion function for the Markov sequence is given by the following theorem.

**Theorem 3.** The second-order rate-distortion function \( L^*(\epsilon, D, \mu) \) for the Markov source with side information is given by

\[
L^*(\epsilon, D, \mu) = \sqrt{V_\infty Q^{-1}(\epsilon)}.
\]

(83)

This theorem is proved in subsection VIII-D and it uses a Markov generalization of the Berry-Esséen theorem due to Tikhomirov [28].

**A. Remarks concerning Theorem 3**

1) Notice that the second-order coding rate for the Markov case consists of two parts:

\[
A \triangleq \frac{1}{n} \operatorname{var} \left[ j(X, D|S) \right]_{P_{X,S} = \pi_{X,S}}, \quad \text{and} \quad B \triangleq \sum_{i=1}^{\infty} \operatorname{cov} \left[ j_{X_i|S_i}(X_i, D|S_i), j_{X_{i+i}|S_{i+i}}(X_{i+i}, D|S_{i+i}) \right].
\]

(84)

When the sequence of random variables \( \{X_iS_i\}_{i=1}^{\infty} \) is independent and identically distributed, the second part \( B \) in (84) vanishes and we recover the result in section IV. Thus, the infinite sum in the definition of \( V_\infty \) in (78) quantifies the effect that the mixing of the Markov chain \( \{X_iS_i\}_{i=1}^{\infty} \) has on rate of convergence the finite blocklength rate-distortion function to the Shannon limit. The faster the mixing is, the faster the convergence to the Shannon limit is.

2) Denote \( \Xi \) as transitional matrix of the Markov chain \( X_1S_1 \to X_2S_2 \to \ldots \to X_nS_n \). If \( \Xi \) is diagonalizable, we can compute \( V_\infty \) using the following lemma.

**Lemma 8.** Assume \( \Xi = U \operatorname{diag}(1, \lambda_2, \ldots, \lambda_{|X||S|}) U^\dagger \). We have

\[
V_\infty = \left[ \operatorname{cov} \left[ j(X, D|S), j(X', D'|S') \right] \right]_{P_{X,S,X',S'} = \pi_{X,S}P_{X',S'|X,S}} = \left[ \operatorname{cov} \left[ j(X, D|S), j(X', D'|S') \right] \right]_{P_{X,S,X',S'} = \pi_{X,S}P_{X',S'|X,S}}
\]

(85)

where

\[
P_{X',S'|X,S}(x's'|xs) = \left[ U \operatorname{diag} \left[ \frac{1}{1 - \lambda_2}, \ldots, \frac{1 + \lambda_{|X||S|}}{1 - \lambda_{|X||S|}} \right] U^\dagger \right]_{x's'|xs}.
\]

(86)

This lemma can be proved using techniques presented by Tomamichel and Tan in [29, Appendix A]. Briefly, we make use of the fact that the Markov chain \( \{X_iS_i\}_{i=1}^{\infty} \) is time-homogeneous and starts from the stationary distribution. Secondly, in the diagonalization of the transition matrix \( \Xi \), except for eigenvalue \( \lambda_1 \), all the rest of the eigenvalues satisfy \( |\lambda_i| < 1 \). Thus, we have \( \sum_{k=1}^{\infty} \lambda_k^s = \frac{\lambda_1}{1 - \lambda_1} \) for all but the leading eigenvalue.
VII. CONCLUSION

In this paper, the second-order coding rates for the source coding problem with side information available at both the encoder and the decoder are characterized for three different kinds of sources: discrete memoryless sources, Gaussian memoryless sources and Markov sources. The conditional $D$-tilted information density is found to play a key role in our second-order analysis.

One of the interesting findings from our work is that the second order rate-distortion functions are same for both Gaussian source coding without side information and with side information (at the encoder and decoder). The means that the dispersion for both problems is the same and equal to $1/2$ nats$^2$ per source symbol. An intriguing open problem emanating from this work is whether the dispersion of the Gaussian Wyner-Ziv system [6] is also $1/2$ nats$^2$ per source symbol.

VIII. APPENDIX

A. Proof of Lemma 3

Lemma 3 is a corollary of Lemma 2. From Lemma 2, we can show the existence of an $(M_n, D, n, \epsilon_n)$-code such that

$$\epsilon_n \leq \mathbb{E}\{[1 - P_{Y^n | S^n}(B_D(x^n))]|S^n = s^n]\}$$

$$= \sum_{s^n} P_{S^n}(s^n)\mathbb{E}\{[1 - P_{Y^n | S^n}(B_D(x^n))]|S^n = s^n]\}. \quad (87)$$

Using techniques from [24, Corollary 2.20], we can show that for every $s^n$,

$$\mathbb{E}\{[1 - P_{Y^n | S^n}(B_D(x^n))]|S^n = s^n]\} \leq \Pr[j_{X^n,S^n}(X^n, D|S^n) > \log \gamma_n - \log \beta_n - \lambda_n^* \delta_n | S^n = s^n]$$

$$+ \mathbb{E}[1 - \beta_n \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n]|S^n = s^n]$$

$$+ e^{-M \frac{1}{n}} \mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n,S^n}(X^n, D, S^n))|S^n = s^n)], \quad (89)$$

for any $\gamma_n, \beta_n, \delta_n, \lambda_n^*$.

Taking the average of both sides of inequality (89) over all sequences $s^n$ completes the proof of this lemma.

B. Proof of Theorem 1

In this subsection, we prove Theorem 1. The proof makes use of the Berry-Esséen Theorem [30, Theorem 2, Chapter XVI. 5]. This theorem is stated as follows.

Theorem 4 (Berry-Esséen Theorem). Let $X_k, k = 1, 2, \ldots, n$ be independent random variables with $\mu_k = \mathbb{E}[X_k]$, $\sigma_k^2 = \text{var}[X_k]$, $t_k = \mathbb{E}[|X_k - \mu_k|^{3}]$, $\sigma^2 = \sum_{k=1}^{n} \sigma_k^2$, and $T = \sum_{k=1}^{n} t_k$. Then for any $\lambda \in \mathbb{R}$, we have

$$\Pr\left[\sum_{k=1}^{n} (X_k - \mu_k) \geq \lambda \sigma \right] - Q(\lambda) \leq \frac{6T}{\sigma^3}. \quad (90)$$

1) Achievability proof of Theorem 1: In this part, we prove that, for any $\delta > 0$, $\sqrt{V}Q^{-1}(\epsilon) + \delta$ is second-order $(\epsilon, D, \kappa)$-achievable when $\kappa = R(X; D|S)$.

We apply Lemma 3 to construct a sequence of $(M_n, D, n, \epsilon_n)$-codes as follows. Choose $\delta_n = \frac{D}{100}$. Similar to the proof in [14, Lemma 4], it can be proved that

$$\Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n = x^n] \geq \frac{C}{\sqrt{n}}, \quad (91)$$

when $n$ is sufficiently large, for some constant $C$. Intuitively, this is because $\mathbb{E}[d(X_i, Y_i^*)]$ has mean $D$, finite variance and finite absolute third-order moment. Thus, we can apply Theorem 4 here.

Choose $\beta_n = \frac{\sqrt{n}}{C^2}$. We have

$$\mathbb{E}[1 - \beta_n \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n]^{+}|S^n]] = 0, \quad (92)$$
when \( n \) is sufficiently large.

Choose \( \gamma_n = \frac{M}{\sqrt{n}} \). We have

\[
e^{-\frac{M}{\gamma_n}} \mathbb{E}\{\mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n | S^n} (X^n, D, S^n))) | S^n]\}
= e^{-\sqrt{n}} \mathbb{E}\{\mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n | S^n} (X^n, D, S^n))) | S^n]\}
\leq e^{-\sqrt{n}} \mathbb{E}\{\mathbb{E}[1 | S^n]\}
= e^{-\sqrt{n}}.
\]

Choose

\[
\log M_n = nR(X; D|S) + \sqrt{n}Q^{-1}(\hat{\epsilon}_n) + \log \sqrt{n} + \lambda_n^* \frac{D}{100} + \log \frac{\sqrt{n}}{C},
\]

where

\[
\hat{\epsilon}_n \triangleq \epsilon - \frac{B_n}{\sqrt{n}} - e^{-\sqrt{n}} \quad (97)
\]

\[
B_n \triangleq \frac{6T_n}{\sqrt{3/2}} \quad (98)
\]

\[
T_n \triangleq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[j_{X|S}(X_i, D|S) - R(X; D|S)]^2. \quad (99)
\]

Applying Lemma 3, for \( n \) sufficiently large, we have

\[
\epsilon_n \leq \Pr\left[ j_{X^n | S^n}(X^n, D|S^n) > nR(X; D|S) + \sqrt{n}Q^{-1}(\hat{\epsilon}_n) \right] + e^{-\sqrt{n}} \quad (100)
\]

\[
\leq \Pr\left[ \sum_{i=1}^{n} j_{X|S}(X_i, D|S_i) > nR(X; D|S) + \sqrt{n}Q^{-1}(\hat{\epsilon}_n) \right] + e^{-\sqrt{n}} \quad (101)
\]

\[
\leq \epsilon \quad (102)
\]

where equation (102) follows from Theorem 4.

Therefore, we have constructed a sequence of \((M_n, D, n, \epsilon_n)\)-codes satisfying

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} (\log M_n - nR(X; D|S)) = \sqrt{n}Q^{-1}(\epsilon)
\]

\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon. \quad (104)
\]

2) Converse proof of Theorem 1: Let \( L \) be a second-order \((\epsilon, D, R(X; D|S))\)-achievable. We want to show \( Q^{-1}(\epsilon)\sqrt{n} \leq L + \delta \), for any \( \delta > 0 \).

Since \( L \) is second-order \((\epsilon, D, R(X; D|S))\)-achievable, by definition, there exists a sequence of \((M_n, n, D, \epsilon_n)\)-codes satisfying

\[
\log M_n \leq nR(X; D|S) + \sqrt{n}(L + \delta),
\]

\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon, \quad (106)
\]

when \( n \) is sufficiently large.

Using Lemma 4 for \( M_n \) satisfying equation (105) and \( \gamma = \log \sqrt{n} \), we have

\[
\epsilon_n \geq \Pr[j_{X^n | S^n}(X^n, D|S^n) \geq \log M_n + \log \sqrt{n}] - \frac{1}{\sqrt{n}} \quad (107)
\]

\[
= \Pr\left[ \sum_{i=1}^{n} j_{X|S}(X_i, D|S_i) \geq \log M_n + \log \sqrt{n} \right] - \frac{1}{\sqrt{n}} \quad (108)
\]

\[
\geq \Pr\left[ \sum_{i=1}^{n} j_{X|S}(X_i, D|S_i) \geq nR(X; D|S) + \sqrt{n}(L + \delta) + \log \sqrt{n} \right] - \frac{1}{\sqrt{n}} \quad (109)
\]
By the weak law of large numbers, we observe that the second term and the third term become vanishingly small

\[ \Pr \left[ \sum_{i=1}^{n} j_{X_i}X_i, D_i, S_i \right] - nR(X; D) \geq \sqrt{n}V \left( \frac{L + \delta + \log \sqrt{n}}{\sqrt{V}} \right) - \frac{1}{\sqrt{n}} \]  

(110)

\[ \geq Q \left( \frac{L + \delta + \log \sqrt{n}}{\sqrt{V}} \right) - B_n \frac{1}{\sqrt{n}} \]  

(111)

\[ = Q \left( \frac{L + \delta}{\sqrt{V}} \right) + O \left( \frac{\log \sqrt{n}}{\sqrt{n}} \right) - B_n + 1 \]  

(112)

where equation (111) follows from Theorem 4 and in this equation \( B_n \) is defined in (98), and (112) follows from the continuity of \( Q(\cdot) \) and Taylor expansion.

Combining (112) and (106), we have

\[ \epsilon \geq \limsup_{n \to \infty} \epsilon_n \]  

(113)

\[ = Q \left( \frac{L + \delta}{\sqrt{V}} \right). \]  

(114)

Thus, all second-order achievable rates \( L \) must satisfy \( L \geq Q^{-1}(\epsilon)\sqrt{V} - \delta \). Taking \( \delta \downarrow 0 \), we complete the proof of the converse.

C. Proof of Theorem 2

Define the correlation coefficient \( \rho \) between \( X_i \) and \( S_i \), for \( i = 1, 2, \ldots, n \) as

\[ \rho \triangleq \frac{\mathbb{E}[XS]}{\sqrt{\mathbb{E}[X^2]\mathbb{E}[S^2]}} = \frac{\sigma_X}{\sqrt{\sigma_Z^2 + \sigma_X^2}}. \]  

(115)

Next, we define the conditional mean of \( X \) given \( S = s \) as

\[ \mu(s) \triangleq \rho \cdot \frac{\sigma_X}{\sigma_S} = \rho^2 \cdot s = \frac{\sigma_X^2}{\sigma_Z^2 + \sigma_X^2} \cdot s. \]  

(116)

This is simply the minimum mean squared estimate of \( X \) given \( S = s \).

1) Achievability proof of Theorem 2: In this part, we prove that, for any \( \delta > 0 \), \( \sqrt{\frac{1}{2}Q^{-1}(\epsilon)\log(e)} + \delta \) is second-order \((\epsilon, D, \frac{1}{2} \log \frac{\sigma_X^2}{\sigma_Z^2})\)-achievable. We apply Lemma 2 to construct a sequence of \((M_n, D, n, \epsilon_n)\)-codes as follows. For each \( s^n \), choose the distribution \( P_{Y^n|S^n}(\cdot|S^n = s^n) \) in equation (22) as the uniform distribution on the surface of the \( n \)-dimensional sphere, with radius \( r_n \triangleq \sqrt{n(\sigma_X^2|S - D)} \) and centre at

\[ \mu(s^n) \triangleq (\mu(s_1), \mu(s_2), \ldots, \mu(s_n)). \]  

(117)

Observe that \( P_{Y^n|S^n}(B_D(x^n))|S^n = s^n \) = 0 if

\[ |x^n - \mu(s^n)| < \sqrt{n(\sigma_X^2|S - D)} - \sqrt{nD} \triangleq r_1 \]  

(118)

or

\[ |x^n - \mu(s^n)| > \sqrt{n(\sigma_X^2|S - D)} + \sqrt{nD} \triangleq r_2. \]  

(119)

Therefore, we have a sequence of \((M_n, D, n, \epsilon_n)\)-codes that satisfies

\[ \epsilon_n \leq \mathbb{E}\{\mathbb{E}[\left(1 - P_{Y^n|S^n}(B_D(X^n))|S^n\right)^M_n]\} \]  

(120)

\[ \leq \mathbb{E}\{\mathbb{E}[\left(1 - P_{Y^n|S^n}(B_D(X^n))\right)^M_n, \Pr(r_1 \leq |x^n - \mu(s^n)| \leq r_2)|S^n]\} + \mathbb{E}\{\Pr(r_1 \leq |x^n - \mu(s^n)|)|S^n\} \]  

\[ + \mathbb{E}\{\Pr(r_2 > |x^n - \mu(s^n)|)|S^n\}. \]  

(121)

By the weak law of large numbers, we observe that the second term and the third term become vanishingly small as \( n \to \infty \). Now, we analyze the first term.
\[ \theta(s^n) = \cos^{-1}\left( \frac{|x^n - \mu(s^n)|^2 + r_0^2 - nD}{2|x^n - \mu(s^n)|r_0} \right). \tag{122} \]

We have

\[
E\left\{ \mathbb{E}\left[ (1 - P_{\chi_n^2|S^n}(B_D(X^n)))^{M_n} \right] . \Pr\left( r_1 \leq |x^n - \mu(S^n)| \leq r_2 \right| S^n \right) \right\}
= E\left\{ \left( 1 - \frac{A_n(r_0)}{A_n(r_0, \theta(s^n))} \right)^{M_n} . \Pr\left( r_1 \leq |x^n - \mu(S^n)| \leq r_2 \right| S^n \right) \right\}
\leq E\left\{ \left( 1 - \frac{\Gamma(\frac{n}{2} + 1)}{\sqrt{\pi n \Gamma(\frac{n-1}{2} + 1)}} \frac{1}{\sin(\theta(s^n)))^{n-1}} \right)^{M_n} . \Pr\left( r_1 \leq |x^n - \mu(S^n)| \leq r_2 \right| S^n \right) \right\}
\leq E\left\{ n \int_0^\infty (1 - f(n, z))^{M_n} 1\{r_1 \leq z \leq r_2\} P_{\chi_n^2}(nz)dz \right| S^n \right) \right\}
= n \int_0^\infty (1 - f(n, z))^{M_n} 1\{r_1 \leq z \leq r_2\} P_{\chi_n^2}(nz)dz
\tag{127}
\]

where

- (124) comes from geometry,
- (125) comes from a lower bound on \( A_n(r_0, \theta(s^n)) \) \[31\], and
- in (126), the function \( f(n, z) \) is defined as

\[
f(n, z) \triangleq \frac{\Gamma(\frac{n}{2} + 1)}{\sqrt{\pi n \Gamma(\frac{n-1}{2} + 1)}} \left( 1 - \frac{\left( 1 + z - 2z\frac{D}{\sigma_{X|S}}} \right)^2}{4 \left( 1 - \frac{D}{\sigma_{X|S}} \right) z} \right)^{\frac{n-1}{2}}
\tag{128}\]

and \( P_{\chi_n^2} \) is the central \( \chi_n^2 \) probability density function.
Next, we choose the sequence $M_n$ such that
\[
\frac{\log M_n}{n} = \frac{1}{2} \log \frac{\sigma_X^2}{D} + \sqrt{\frac{1}{2n} Q^{-1}(\epsilon) \log e + \frac{\log n}{2n} + \log \log n} + O \left( \frac{1}{n} \right). \tag{129}
\]
We can check that
\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \left( \log M_n - \frac{n}{2} \log \frac{\sigma_X^2}{D} \right) = \sqrt{\frac{1}{2} Q^{-1}(\epsilon) \log e}. \tag{130}
\]
Using similar techniques as in [14, Appendix K], we can show that the bound in (127) can be analyzed using the Gaussian approximation to yield
\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon. \tag{131}
\]

2) Converse proof of Theorem 2: The conditional $D$-tilted information in the jointly Gaussian case is
\[
j_{X^n|S^n}(x^n, D|s^n) = \frac{n}{2} \log \frac{\sigma_X^2}{D} + \left| \frac{x^n - \frac{\sigma_X^2}{\sigma_X^2 + \sigma_{S}^2} s^n}{2\sigma_X^2} \right|^2 \log e - \frac{n}{2} \log e. \tag{132}
\]
For each $i \in \{1, 2, \ldots, n\}$, we have
\[
\mathbb{E}[j_{X_i|S_i}(X_i, D|S_i)] = \frac{1}{2} \log \frac{\sigma_X^2}{D}, \tag{133}
\]
and
\[
\text{var}[j_{X_i|S_i}(X_i, D|S_i)] = \mathbb{E} \left[ \frac{|X_i - \mu(S_i)|^2}{2\sigma_X^2} \log e - \frac{\log e}{2} \right]^2 \tag{134}
= \mathbb{E} \left[ \mathbb{E} \left[ \left( \frac{|X_i - \mu(S_i)|^2}{2\sigma_X^2} \log e - \frac{\log e}{2} \right)^2 \bigg| S_i \right] \right] \tag{135}
= (\log e)^2 \mathbb{E} \left[ \mathbb{E} \left[ \left( \frac{|X_i - \mu(S_i)|^4}{4\sigma_X^4} - \frac{|X_i - \mu(S_i)|^2}{2\sigma_X^2} + \frac{1}{4} \right) \bigg| S_i \right] \right] \tag{136}
= \frac{1}{2} (\log e)^2. \tag{137}
\]
Let $L$ be second-order $(\epsilon, D, \frac{1}{2} \log \frac{\sigma_X^2}{D})$-achievable. We want to show that $Q^{-1}(\epsilon) \sqrt{\frac{1}{2} \log e} \leq L + \delta$, for any $\delta > 0$.
Since $L$ is second-order $(\epsilon, D, \frac{1}{2} \log \frac{\sigma_X^2}{D})$-achievable, there exists a sequence of $(M_n, n, D, \epsilon_n)$-codes satisfying
\[
\log M_n \leq \frac{n}{2} \log \frac{\sigma_X^2}{D} + \sqrt{n}(L + \delta), \tag{138}
\]
\[
\limsup_{n \to \infty} \epsilon_n \leq \epsilon, \tag{139}
\]
where (138) holds for all $n$ sufficiently large.
Using Lemma 4 for $M_n$ satisfying equation (138) and $\gamma = \log \sqrt{n}$, we have
\[
\epsilon_n \geq \text{Pr}[j_{X^n|S^n}(X^n, D|S^n) \geq \log M_n + \log \sqrt{n} - \frac{1}{\sqrt{n}}] \tag{140}
\geq Q \left( \frac{L + \delta}{\sqrt{\frac{1}{2} \log e}} + \frac{\log \sqrt{n}}{\sqrt{\frac{1}{2} \log e}} \right) - \frac{B_n}{\sqrt{n}} - \frac{1}{\sqrt{n}} \tag{141}
\]
Assume that the strong mixing coefficient is exponentially decaying, i.e.,

\[ F_n = \log \sqrt{n} \]  

where equation (141) follows from Theorem 4 and in this equation \( B_n \) is the constant in Theorem 4, and (142) follows from the continuous differentiability of \( Q(\cdot) \) and Taylor expansion.

Combining (142) and (139), we have

\[ \epsilon \geq \limsup_{n \to \infty} \epsilon_n \]  

(143)

\[ = Q \left( \frac{L + \delta}{\sqrt{\frac{1}{2} \log e}} \right). \]  

(144)

This completes the proof of the converse upon taking \( \delta \downarrow 0 \).

\section{Proof of Theorem 3}

To prove Theorem 3, we use a variant of Berry-Esséen Theorem [28] to deal with a sequence of random variables that forms a Markov chain. This theorem is stated as follows.

\textbf{Theorem 5.} Consider a stationary process \( \{X_k : k \geq 1\} \), with \( \mathbb{E}X_1 = 0 \) and finite variance. Define the strong mixing coefficient \( \alpha(n) \) as

\[ \alpha(n) \triangleq \sup \{ |\Pr(A \cap B) - \Pr(A) \Pr(B)| : A \in \mathcal{F}_a^k, B \in \mathcal{F}_b^k, k \in \mathbb{Z} \}, \]  

(145)

where \( \mathcal{F}_a^b = \sigma(X_i : i \in [a, b] \cap \mathbb{Z}) \) is the \( \sigma \)-field generated by \( \{X_i : i \in [a, b] \cap \mathbb{Z}\} \). In this equation (142) follows from Theorem 4 and in this equation \( B_n \) is the constant in Theorem 4, and (143) follows from the continuous differentiability of \( Q(\cdot) \) and Taylor expansion.

Assume that the strong mixing coefficient is exponentially decaying, i.e., \( \alpha(n) \leq K e^{-\kappa n} \) for some \( K \) and \( \kappa \) and all \( n \geq 1 \). Assume \( \mathbb{E}[X_1^{2+\gamma}] < \infty \) for some \( \gamma, 1 \geq \gamma > 0 \). Then, there is a constant \( B(K, \kappa, \gamma) > 0 \) such that, for all \( n \in \mathbb{N} \),

\[ \sup_{x \in \mathbb{R}} \left| \Pr \left[ \frac{1}{\sigma_n} \sum_{k=1}^{n} X_k \leq \lambda \right] - \Phi(\lambda) \right| \leq \frac{B(K, \kappa, \gamma) (\log n)^{1+\frac{\gamma}{2}}}{n^\frac{\gamma}{2}}. \]  

(147)

Note that the strong mixing coefficient of a time-homogeneous, irreducible and ergodic Markov chain decays to zero and, in fact, vanishes exponentially fast [32, Theorem 3.1].

In this proof, we make use of the following lemma.

\textbf{Lemma 9.} If the sequence \( X_1S_1 \to X_2S_2 \to X_3S_3 \to \ldots \) forms a Markov chain, then the sequence of conditionally \( D \)-tilted information densities \( \{j_{X_i|S_i}(X_i, D|S_i)\}_{i=1}^{\infty} \) also forms a Markov chain.

This lemma is proved in section VIII-E

1) Achievability proof of Theorem 3: In this part, we prove that, for any \( \delta > 0 \), \( \sqrt{\log Q^{-1}(\epsilon)} + \delta \) is second-order \((\epsilon, D, \mu)\)-achievable.

We apply Lemma 3 to construct a sequence of \((M_n, D, n, \epsilon_n)\)-codes as follows. Choose \( \delta_n = \frac{D}{1000} \).

Similar to the proof in [14, Lemma 4], it can be proved that

\[ \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n = x^n] \geq \frac{C}{\sqrt{n}}, \]  

(148)

when \( n \) is sufficiently large, for some constant \( C \). Intuitively, this is because \( \mathbb{E}[d(X_i, Y_i^*)] \) has mean \( D \), finite variance. Thus, we can apply Theorem 5 for a sum of weakly dependent variables.

Choose \( \beta_n = \frac{\sqrt{n}}{C} \). We have

\[ \mathbb{E} \mathbb{E}[1 - \beta_n \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n| \uparrow |S^n]] = 0, \]  

(149)
when $n$ is sufficiently large.

Choose $\gamma_n = \frac{M}{\sqrt{n}}$. We have

$$e^{-\frac{M}{\gamma_n}} \mathbb{E}\left\{ \mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D, S^n)))|S^n]\right\}$$

$$= e^{-\sqrt{\pi}} \mathbb{E}\left\{ \mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D, S^n)))|S^n]\right\}$$

$$\leq e^{-\sqrt{\pi}} \mathbb{E}\left\{ \mathbb{E}[1|S^n]\right\}$$

$$= e^{-\sqrt{\pi}}. \tag{152}$$

Choose

$$\log M_n = n\mu + \sqrt{n\nu_n}Q^{-1}(\hat{\epsilon}_n) + \log \sqrt{n} + \lambda_{n}^{\epsilon} \frac{D}{100} + \log \frac{\sqrt{n}}{C}, \tag{153}$$

where

$$\hat{\epsilon}_n \triangleq \epsilon - \frac{B(K, \kappa_1, \gamma)(\log n)^{1+\frac{2}{\gamma}}}{n^2} - e^{-\sqrt{\pi}} \tag{154}$$

and $B(K, \kappa_1, \gamma)$ is found in Theorem 5.

Applying Lemma 3, for $n$ sufficiently large, we have

$$\epsilon_n \leq \Pr \left[ j_{X^n|S^n}(X^n, D|S^n) > n\mu + \sqrt{n\nu_n}Q^{-1}(\hat{\epsilon}_n) \right] + e^{-\sqrt{\pi}} \tag{155}$$

$$\leq \Pr \left[ \sum_{i=1}^{n} j_{X_i|S_i}(X_i, D|S_i) > n\mu + \sqrt{n\nu_n}Q^{-1}(\hat{\epsilon}_n) \right] + e^{-\sqrt{\pi}} \tag{156}$$

$$\leq \epsilon \tag{157}$$

where equation (157) follows from Theorem 5.

Therefore, we have constructed a sequence of $(M_n, D, n, \epsilon_n)$-codes satisfying

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}}(\log M_n - n\mu) = \sqrt{V_{\infty}}Q^{-1}(\epsilon) \tag{158}$$

$$\lim_{n \to \infty} \epsilon_n \leq \epsilon. \tag{159}$$

2) Converse proof of Theorem 3: Let $L$ be second-order $(\epsilon, D, \mu)$-achievable. In this part, we want to show that $Q^{-1}(\epsilon)\sqrt{V_{\infty}} \leq L + \delta$, for any $\delta > 0$.

Since $L$ is $(\epsilon, D, \mu)$-second-order achievable there exists a sequence of $(M_n, n, D, \epsilon_n)$-codes satisfying

$$\log M_n \leq n\mu + \sqrt{n}(L + \delta), \tag{160}$$

$$\limsup_{n \to \infty} \epsilon_n \leq \epsilon, \tag{161}$$

when $n$ is sufficiently large.

Using Lemma 4 for $M_n$ satisfying equation (160) and $\gamma = \log \sqrt{n}$, we have

$$\epsilon_n \geq \Pr \left[ j_{X^n|S^n}(X^n, D|S^n) \geq \log M_n + \log \sqrt{n} \right] - \frac{1}{\sqrt{n}} \tag{162}$$

$$= \Pr \left[ \sum_{i=1}^{n} j_{X_i|S_i}(X_i, D|S_i) \geq \log M_n + \log \sqrt{n} \right] - \frac{1}{\sqrt{n}} \tag{163}$$

$$\geq \Pr \left[ \sum_{i=1}^{n} j_{X_i|S_i}(X_i, D|S_i) \geq n\mu + \sqrt{n}(L + \delta) + \log \sqrt{n} \right] - \frac{1}{\sqrt{n}} \tag{164}$$

$$\geq \Pr \left[ \sum_{i=1}^{n} j_{X_i|S_i}(X_i, D|S_i) - n\mu \geq \sqrt{n} \left( \frac{L + \delta}{\sqrt{n}} + \frac{\log \sqrt{n}}{\sqrt{n}V_{n}} \right) \right] - \frac{1}{\sqrt{n}} \tag{165}$$
\[ Q \left( \frac{L + \delta}{\sqrt{V_n}} + \frac{\log \sqrt{n}}{nV_n} \right) - \frac{B(K, \kappa_1, \gamma)(\log n)^{1+\frac{\gamma}{2}}}{n^\frac{\gamma}{2}} - \frac{1}{\sqrt{n}} \geq (166) \]

\[ = Q \left( \frac{L + \delta}{\sqrt{V_n}} \right) + O \left( \frac{\log \sqrt{n}}{nV_n} \right) - \frac{B(K, \kappa_1, \gamma)(\log n)^{1+\frac{\gamma}{2}}}{n^\frac{\gamma}{2}} - \frac{1}{\sqrt{n}} (167) \]

where equation (166) follows from Theorem 5 and in this equation \( B(K, \kappa_1, \gamma) \) is defined in Theorem 5, and (167) follows from the continuity of \( Q(\cdot) \) and Taylor expansion.

Combining (167) and (161), we have

\[ \epsilon \geq \lim \sup_{n \to \infty} \epsilon_n \]

\[ = Q \left( \frac{L + \delta}{\sqrt{V_n}} \right) \]  

where in (169), we use the fact that \( V_n \to V_\infty \).

E. Proof of Lemma 9

In the proof of this lemma, we make use of the following lemma.

Lemma 10. Let \( \{ A_i \}_{i=1}^{\infty} \) be a Markov chain in state space \( A \). Consider the sequence \( \{ B_i = f(X_i) \}_{i=1}^{\infty} \), where \( f : A \to B \) is a function from \( A \) to \( B \). Suppose that there exists a function \( g : B \times B \to \mathbb{R} \) such that

\[ \Pr(B_{i+1} = b|X_i = a) = g(f(a), b) \]

for any \( a \in A \) and \( b \in B \). Then the sequence \( \{ B_i \}_{i=1}^{\infty} \) forms a Markov chain.

The proof of this lemma can be found in [33, Lemma 13]. Note that if \( f \) is one-to-one, then it is obvious that the sequence generated by \( f \) acting on a Markov chain is also a Markov chain.

Here, \( j_{X|S} \) is a composition of several functions \( \log, \frac{1}{t} \) for \( t \neq 0 \), \( \exp \), summation and \( d(\cdot, \cdot) \). So, Lemma 9 follows from Lemma 10.

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References

[1] C. E. Shannon. A mathematical theory of communication. Bell System Technical Journal, pages 379–423, 1948.
[2] C. E. Shannon. Coding theorems for a discrete source with a fidelity criterion. IRE Nat. Conv. Rec., pages 142–163, 1959.
[3] T. Berger. Rate Distortion Theory: A Mathematical Basis for Data Compression. Prentice-Hall, 1971.
[4] R. M. Gray. Conditional rate-distortion theory. Technical Report, Stanford University, AD-753260, Oct. 1972.
[5] T. Weissman and A. El Gamal. Source coding with limited-look-ahead side information at the decoder. IEEE Transactions on Information Theory, 52(12):5218–5239, Dec. 2006.
[6] A. D. Wyner and J. Ziv. The rate-distortion function for source coding with side information at the decoder. IEEE Transactions on Information Theory, 22(1):1–10, Jan. 1976.
[7] B. M. Leiner and R. M. Gray. Rate-distortion theory for ergodic sources with side information. IEEE Transactions on Information Theory, 20(5):672–675, Sep. 1974.
[8] M. Fleming and M. Effros. On rate-distortion with mixed types of side information. IEEE Transactions on Information Theory, 52(4):1698–1705, Apr. 2006.
[9] O. Simeone and H. H. Permuter. Source coding when the side information may be delayed. IEEE Transactions on Information Theory, 59(6):3607–3618, June. 2013.
[10] T. Linder, R. Zamir, and K. Zeger. On source coding with side-information-dependent distortion measures. IEEE Transactions on Information Theory, 46(7):2697–2704, Jul. 2000.
[11] V. Strassen. Asymptotische abschatzungen in shannon’s informationstheorie. Trans. Third Prague Conf. Information Theory, pages 689–723, 1962.
[12] M. Hayashi. Second-order asymptotics in fixed-length source coding and intrinsic randomness. IEEE Transactions on Information Theory, 54(10):4619–4637, Oct. 2008.
[13] T. S. Han. Folklore in source coding: Information-spectrum approach. IEEE Transactions on Information Theory, 51(2):747–753, 2005.
[14] V. Kostina and S. Verdú. Fixed-length lossy compression in the finite blocklength regime. IEEE Transactions on Information Theory, 58(6):3309–3338, Jun. 2012.
[15] A. Ingber and Y. Kochman. The dispersion of lossy source coding. In Proc. Data Compression Conference, pages 53–62, 2011.
[16] V. Y. F. Tan. Moderate-deviations of lossy source coding for discrete and Gaussian sources. In Proc. International Symposium on Information Theory, pages 920–924, Cambridge, MA, Jul 2012.
[17] S. Watanabe, S. Kuzuoka, and V. Y. F. Tan. Non-asymptotic and second-order achievability bounds for coding with side-information, 2013. arXiv:1301.6467.
[18] I. Kontoyiannis and S. Verdú. Optimal lossless data compression: Non-asymptotics and asymptotics. IEEE Transactions on Information Theory, 60(2):777–795, Feb 2014.
[19] V. Y. F. Tan. Asymptotic estimates in information theory with non-vanishing error probabilities. Foundations and Trends® in Communications and Information Theory, 11(1–2):1–184, 2014.
[20] B. Yu and T. P. Speed. A rate of convergence result for a universal d-semifaithful code. IEEE Transactions on Information Theory, 39(3):813–820, May 1993.
[21] Z. Zhang, E. h. Yang, and V. K. Wei. The redundancy of source coding with fidelity criterion-part one: Known statistics. IEEE Transactions on Information Theory, 43(1):71–91, Jan. 1997.
[22] V. Kostina and S. Verdú. A new converse in rate distortion theory. In Proc. Annual Conference on Information Sciences and Systems, volume 46, Princeton, NJ, 2012.
[23] I. Csiszar and J. Korner. Information Theory: Coding Theorems for Discrete Memoryless Systems. Cambridge University Press, 2nd edition, 2011.
[24] V. Kostina. Lossy Data Compression: Nonasymptotic Fundamental Limits. PhD thesis, Department of Electrical Engineering, Princeton, 2013.
[25] K. Marton. Error exponent for source coding with a fidelity criterion. IEEE Transactions on Information Theory, 20(2):197–199, 1974.
[26] A. El Gamal and Y.-H. Kim. Network Information Theory. Cambridge University Press, Cambridge, U.K., 2012.
[27] J. Scarlett. On the dispersion of dirty paper coding. In Proc. IEEE International Symposium on Information Theory, pages 2282–2286, Honolulu, HI, Jul 2014. arXiv:1309.6200 [cs.IT].
[28] A. N. Tikhomirov. On the convergence rate in the central limit theorem for weakly dependent random variables. Theory of Probability and its Applications, 25(4):790–809, 1980.
[29] M. Tomamichel and V. Y. F. Tan. Second-order coding rates for channels with state. IEEE Transactions on Information Theory, 60(8):4427–4448, Aug. 2014.
[30] W. Feller. An Introduction to Probability Theory and Its application, volume II. John Wiley and Sons, 2nd edition, 1971.
[31] D. Sakrison. A geometric treatment of the source encoding of a Gaussian random variable. IEEE Transactions on Information Theory, 14(3):481–486, May 1968.
[32] R. C. Bradley. Basic properties of strong mixing conditions. A survey and some open questions. Probability Surveys, 2:107–144, 2005.
[33] T. Konstantopoulos. Introductory lecture notes on Markov chains and random walks. 2009. Available at http://www2.math.uu.se/~takis/L/McRw/mcrw.pdf.