Semigroup algebras of submonoids of polycyclic-by-finite groups and maximal orders∗

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Dedicated to Fred Van Oystaeyen, on the occasion of his sixtieth birthday

Abstract

Necessary and sufficient conditions are given for a prime Noetherian algebra $K[S]$ of a submonoid $S$ of a polycyclic-by-finite group $G$ to be a maximal order. These conditions are entirely in terms of the monoid $S$. This extends earlier results of Brown concerned with the group ring case and of the authors for the case where $K[S]$ satisfies a polynomial identity.

1 Introduction

In this paper we continue our investigations in [9, 10, 12] on semigroup algebras $K[S]$ that are prime Noetherian maximal orders (for a survey we refer the reader to [14]). There are two main issues to be dealt with. First, when such algebras are Noetherian and second when they are a maximal order. We briefly give some background. Recall that group algebras of polycyclic-by-finite groups are the only known examples of Noetherian group algebras. In [3, 4], K.A. Brown characterized when such group algebras are prime maximal orders. In the search for more classes of prime Noetherian maximal orders, it is thus natural to consider subalgebras of Noetherian group algebras. In [13] it is proved that the semigroup algebra $K[S]$ of a submonoid $S$ of a polycyclic-by-finite group is right Noetherian if and only if $S$ has a group of right quotients $G = SS^{-1}$, with normal subgroups $F$ and $N$ such that $F \subseteq S \cap N$, $G/N$ is finite, $N/F$ is abelian and $S \cap N$ is finitely generated. In particular, in this situation, $S$ is finitely generated and, if the unit group $U(S)$ is trivial then $K[S]$ satisfies a polynomial identity. Furthermore, it follows that such a semigroup algebra is right Noetherian if and only if it is left Noetherian. We simply call such algebras Noetherian. In [10], the authors determined conditions under which the semigroup algebra of a submonoid $S$ of a finitely generated abelian-by-finite

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group is a prime Noetherian maximal order. It turns out that the action of the
group of quotients on the minimal primes of some abelian submonoid of $S$ is
very important (some invariance condition is crucial).

In this paper we deal with the general case, provided such an invariance
condition holds. Crucial for our investigations is Theorem 1.1 in [10] that says
that the height one prime ideals $P$ of a prime Noetherian algebra $K[S]$ with
$P \cap S \neq \emptyset$, where $S$ is a submonoid of a polycyclic-by-finite group, are precisely
the ideals of the form $K[Q]$ with $Q$ a minimal prime ideal of $S$ (recall that the
other height one prime ideals are contractions of height one prime ideals of the
group algebra $K[SS^{-1}]$). The set of all minimal primes of $S$ will be denoted by
$X^1(S)$.

For an arbitrary abelian monoid $A$, Anderson [1, 2] (see also [7, 8]) proved
that $K[A]$ is a prime Noetherian maximal order if and only if $A$ is a finitely
generated submonoid of a torsion free abelian group, so that $A$ is a maximal
order in its group of quotients. In that case, these monoids $A$ are precisely
the finitely generated abelian monoids $A$ so that $A = U(A) \times A_1$ where $A_1 =
A_1A_1^{-1} \cap F^+$ with $F^+$ a positive cone of a free abelian group $F$ that contains
the group of quotients $A_1A_1^{-1}$ of a submonoid $A_1$ of $A$.

2 Algebras of submonoids of polycyclic-by-finite
groups

Let $S$ be a submonoid of a polycyclic-by-finite group such that the semigroup
algebra $K[S]$ is Noetherian. Hence, $S$ has a group of quotients $G = SS^{-1}$ with
normal subgroups $F$ and $N$ such that $F \subseteq S \cap N$, $N/F$ is abelian, $G/N$ is finite
and $S \cap N$ is finitely generated. Without loss of generality we may assume that
the groups $N$ and $N/F$ are torsion free. By $\sim_F$ we denote the congruence on
$S$ defined by: $s \sim_F t$ if and only if $s = ft$ for some $f \in F$. The set of $\rho$-classes
$S/\rho$ has a natural semigroup structure inherited from $S$ and we denote this by
$S/F$. Because of the natural bijection between the minimal primes of $S$ and the
minimal primes of $S/F$, it is easily verified that $S$ is a maximal order in its
group of quotients $G$ if and only if the semigroup $S/F$ is a maximal order
in its group of quotients $G/F$. Throughout this paper we will freely use all
this notation. Recall from [14] Lemma 4.1.3 that also $K[S \cap N]$ is Noetherian.
Moreover $K[S \cap N]$ is a domain as $N$ is torsion free (see [14] Theorem 37.5)).

The following notation will be used. For an element $\alpha = \sum_{s \in S} k_s s \in K[S]$,
with each $k_s \in K$, we put $\text{supp}(\alpha) = \{s \in S \mid k_s \neq 0\}$, the support of $\alpha$. By
$Q_d(R)$ we denote the classical ring of quotients of a prime Noetherian ring $R$.
Recall that $R$ is said to be a maximal order if the following property holds for
every subring $T$ of $Q_d(R)$ with $R \subseteq T$: if there exist regular elements $r_1, r_2 \in R$
so that $r_1 Tr_2 \subseteq R$ then $R = T$. Equivalently, $(I :_R I) = (I :_R I) = R$ for every
(fractional) ideal $I$ of $R$; here we put $(I :_R I) = \{q \in Q_d(R) \mid Iq \subseteq I\}$ and
similarly one defines $(I :_R I)$. For more information and details we refer the
reader to [14] Section 3.6]. In order to prove the main theorem we need the
following proposition.

**Proposition 2.1** Let $S$ be a submonoid of a polycyclic-by-finite group $G$ such that the semigroup algebra $K[S]$ is Noetherian. Then the semigroup $S \cap N$ is a maximal order in its group of quotients if and only if the semigroup algebra $K[S \cap N]$ is a maximal order in its classical ring of quotients (with $N$ a torsion free subgroup of finite index in $G = SS^{-1}$, as above).

**Proof.** If $K[S \cap N]$ is a prime maximal order, then it is well known and easy to prove that the semigroup $S \cap N$ is a maximal order. Conversely, assume $S \cap N$ is a maximal order in its group of quotients $N$. Because $N$ is torsion free, we know that $K[S \cap N]$ is a Noetherian domain. To prove that $K[S \cap N]$ is a maximal order, let $I$ be a non-zero ideal of $K[S \cap N]$ and let $0 \neq q \in Q_{cl}(K[S \cap N])$ be such that $qI \subseteq I$. Then $qIK[N] \subseteq IK[N]$. Note that $IK[N]$ is a two-sided ideal of $K[N]$ (see for example [11, Theorem 9.20]).

Because $N$ is a torsion free polycyclic-by-finite group, we know from Brown’s result [3] that $K[N]$ is a maximal order. Hence, it follows that $q \in K[N]$. Now, since $K[N] = K[F] \ast (N/F)$, a crossed product of the finitely generated torsion free abelian group $N/F$ over the group algebra $K[F]$, we have that $q = \sum_{i=1}^{n} \alpha_{i}q_{i}$, with $\alpha_{i} \in K[F]$ and all $q_{i}$ are in a transversal of $F$ in $N$. The image of $q_{i}$ in $N/F$ we denote by $\overline{q_{i}}$. Let $\preceq$ denote an ordering on the group $N/F$. Then, we may assume that $\overline{q_{1}} < \cdots < \overline{q_{n}}$. Every nonzero element of $I$ can be written in the form $\beta + \alpha$ for some $t \in S \cap N$, $0 \neq \beta \in K[F]$ and some $\alpha \in K[S \cap N]$ such that $\overline{s} < \overline{t}$ for all $s \in \text{supp}(\alpha)$ (if $\alpha \neq 0$). Let $h(I)$ denote the set consisting of all such possible elements $t \in S \cap N$. Then $h(I)$ is an ideal of $S \cap N$. Since $qI \subseteq I$, $N/F$ is ordered and $K[N]$ is a domain, we get (using a standard graded algebra argument) that $q_{\alpha}(h(I)) \subseteq h(I)$. As $S \cap N$ is a maximal order, this implies that $q_{\alpha} \in S \cap N$ and thus $\alpha_{i}q_{\alpha} \in (I : I) \cap K[S \cap N]$. So, $q - \alpha_{i}q_{\alpha} \in (I : I)$ and $|\text{supp}(q - \alpha_{i}q_{\alpha})| < |\text{supp}(q)|$. Hence, by an induction argument, we may assume that $q - \alpha_{i}q_{\alpha} \in K[S \cap N]$. So, $q \in K[S \cap N]$, as desired.

Similarly, one shows that $(I : I) = K[S \cap N]$.

In order to state the main result we need some more notation. By $\Delta^{+}(G)$ we denote the torsion subgroup of the finite conjugacy center $\Delta(G)$ of a group $G$. It is well known that $K[G]$ is prime if and only if $\Delta^{+}(G) = \{1\}$ (see [10, Theorem 5.5]). The following terminology is used in [3]. A group $G$ is said to be dihedral-free if the normalizer of any subgroup $H$ isomorphic with the infinite dihedral group is of infinite index in $G$.

**Theorem 2.2** Let $S$ be a submonoid of a polycyclic-by-finite group such that the semigroup algebra $K[S]$ is Noetherian, i.e., there exist normal subgroups $F$ and $N$ of $G = SS^{-1}$ such that $F \subseteq S \cap N$, $N/F$ is abelian, $G/N$ is finite and $S \cap N$ is finitely generated. Suppose that for every minimal prime $P$ of $S$ the intersection $P \cap N$ is $G$-invariant.

Then, the semigroup algebra $K[S]$ is a prime maximal order if and only if the monoid $S$ is a maximal order in its group of quotients $G$, the group $G$ is dihedral-free and $\Delta^{+}(G) = \{1\}$.
Proof. First note that the $G$-invariance of $P \cap N$, for every minimal prime ideal $P$ of $S$, is inherited on $P \cap M = (P \cap N) \cap M$, for any normal subgroup $M$ of $G$ with $M \subseteq N$ and $N/M$ finite. In particular, we may for the remainder assume that $N$ is torsion free (and $N/F$ is torsion free).

If $K[S]$ is a prime maximal order then (as before) $S$ is a maximal order in $G$. Furthermore, because $K[G]$ is a localization of $K[S]$, we know that $K[G]$ is a prime maximal order as well. Hence, by Brown’s result [3], it follows that the group $G$ is dihedral-free and $\Delta^+(G) = \{1\}$. For the converse implication, suppose that $S$ is a maximal order in its group of quotients $G$, the group $G$ is dihedral-free and $\Delta^+(G) = \{1\}$. Hence, $K[G]$ and therefore also $K[S]$ is prime. Again, by Brown’s result, $K[G]$ is a maximal order.

Let $P$ be a minimal prime ideal of $S$. Then, by [10] Theorem 1.1] (see the introduction), $K[P]$ is a height one prime of $K[S]$. Clearly, $K[S]$ has a natural $G/N$-gradation with homogeneous component of degree $e$ (the identity of $G/N$) the algebra $K[S \cap N]$. So, from [17] Theorem 17.9], it then follows that

$$\bar{P}(N) = K[P] \cap K[S \cap N] = K[P \cap N] = Q_1 \cap \cdots \cap Q_n,$$

with each $Q_i$ a height one prime ideal of $K[S \cap N]$; and these are all the height one primes of $K[S \cap N]$ containing $\bar{P}(N)$. Because of the assumption on the invariance of $P \cap N$, it easily is verified that the set $\{Q_1, \ldots, Q_n\}$ is a full orbit (under the conjugation action) of height one primes in $K[S \cap N]$. Clearly $Q_i \cap (S \cap N) \neq \emptyset$. So, again by the result mentioned in the introduction, $Q_i = K[Q_i \cap (S \cap N)]$; moreover, $Q_i \cap (S \cap N)$ is a minimal prime ideal of $S \cap N$ and these are all the minimal primes of $S \cap N$ containing $P \cap N$. Because $K[S \cap N]$ is Noetherian and $N$ is a polycyclic-by-finite group, we also know (see for example [14] Corollary 4.4.12]) that each $Q_i$ contains a normal element $n_i$, that is an element such that $(S \cap N)n_i = n_i(S \cap N)$. Furthermore, because $S$ is a maximal order, we get that $S/F$ is a maximal order in its group of quotients $G/F$, in which $N/F$ is abelian and of finite index. Hence, it follows from [14] Lemma 7.1.1] that $(S \cap N)/F$ is a maximal order as well. Consequently, $S \cap N$ is a maximal order. So, by Proposition 2.1] the Noetherian algebra $K[S \cap N]$ is a maximal order. Since each $Q_i$ is a height one prime containing a divisorial ideal (namely $K[(S \cap N)n_i]$), it therefore follows that it is a divisorial height one prime ideal. It then follows from Proposition 1.9 and Proposition 1.10 in [6] that each $Q_i$ is localizable (the localization will be denoted $K[S \cap N]_{Q_i}$) and hence also that $\bar{P}(N)$ is a localizable semiprime ideal of $K[S \cap N]$. Furthermore, $K[S \cap N]_{\bar{P}(N)} = K[S \cap N]_{Q_1} \cap \cdots \cap K[S \cap N]_{Q_n}$. Here we denote by $K[S \cap N]_{\bar{P}(N)}$ the localization of $K[S \cap N]$ with respect to the set $C_N(P) = \{c \in K[S \cap N] \mid c + K[P \cap N] \text{ is a regular element of the ring } K[S \cap N]/K[P \cap N]\}$. Moreover (see for example [16] Lemma 13.3.5)], $C_N(P)$ is an Ore set of regular elements of $K[S]$ and thus an element $c \in K[S \cap N]$ belongs to $C_N(P)$ if and only if $c + K[P]$ is regular in $K[S]/K[P]$. We begin by showing that the localized ring $K[S]_{C_N(P)}$ is a maximal order. To do so, we show that $K[S]_{C_N(P)}$ is a local ring with unique maximal ideal $PK[S]_{C_N(P)}$ and so that $PK[S]_{C_N(P)}$ is invertible and every proper non-zero ideal of $K[S]_{C_N(P)}$ is of the form $(PK[S]_{C_N(P)})^k$ for
some positive integer \( k \).

From [5 Théorème 4.1.6] we know that \( K[S \cap N]_{\tilde{P}(N)} \) is a semi-local maximal order that is a principal left and right ideal ring. In particular, it is an Asano order, it has dimension one and its Jacobson radical is equal to \( \tilde{P}(N)K[S \cap N]_{\tilde{P}(N)} \).

Since this ring is the component of degree \( e \) of the \( G/N \)-graded ring \( K[S]_{C_N(P)} \), it follows from [17 Theorem 17.9] that \( K[S]_{C_N(P)} \) also has dimension one. By the above, it follows that the non-zero prime ideals of \( K[S \cap N]_{\tilde{P}(N)} \) are precisely the \( n \) prime ideals \( Q_iK[S \cap N]_{\tilde{P}(N)} = K[Q_i \cap S \cap N]K[S \cap N]_{\tilde{P}(N)} \). Consequently, the height one primes of \( K[S \cap N]_{\tilde{P}(N)} \) are the form \( K[S \cap N]_{\tilde{P}(N)}I \), where \( I \) is an ideal of \( S \cap N \).

From [10 Proposition 1.3], it follows that \( S \cap N \) is a minimal prime ideal of \( K[S]_{C_N(P)} \) and \( S \cap N \) is invertible. In particular, it is an Asano order, all its non-zero ideals are products of height one prime ideals. Hence all non-zero ideals of \( K[S \cap N]_{\tilde{P}(N)} \) are of the form \( K[S \cap N]_{\tilde{P}(N)}I \), where \( I \) is an ideal of \( S \cap N \). We now show that \( PK[S]_{C_N(P)} \) is the only height one prime of \( K[S]_{C_N(P)} \). So, let \( Q \) be a height one prime ideal of \( K[S]_{C_N(P)} \). Again, because of the \( G/N \)-gradation, \( \tilde{P}(N) = Q \cap K[S \cap N]_{\tilde{P}(N)} = K[S \cap N \cap Q]_{\tilde{P}(N)} \) is a non-zero semiprime ideal of \( K[S \cap N]_{\tilde{P}(N)} \). Hence, as explained above, non-zero ideals of \( K[S \cap N]_{\tilde{P}(N)} \) are generated by their intersection with \( S \cap N \), and \( (Q \cap S) \cap N = I \cap (S \cap N) \) is an intersection of some of the \( Q_i \cap (S \cap N) \). In particular, \( Q \cap S \neq \emptyset \). Clearly, \( Q \cap K[S] \) is a height one prime ideal of \( K[S] \) and thus \( Q \cap K[S] = K[Q \cap S] \) and \( Q \cap S \) is a minimal prime ideal of \( S \).

The assumptions therefore imply that \( Q \cap (S \cap N) \) is \( G \)-invariant. Since \( Q \cap (S \cap N) \subseteq Q_i \), for some \( i \), we thus obtain that

\[
Q \cap (S \cap N) \subseteq \bigcap_{i=1}^{n} Q_i = K[P \cap N].
\]

From [10 Proposition 1.3], it follows that

\[
P = B(S((Q_1 \cap (S \cap N)) \cap \cdots \cap (Q_n \cap (S \cap N)))S) = Q \cap S,
\]

where \( B(J) \) denotes the prime radical of an ideal \( J \) of \( S \). It follows that \( Q = PK[S]_{C_N(P)} \), as desired.

As, by assumption, \( S \) is a maximal order, it follows that \( P(S : P) \) is an ideal of \( S \) that is not contained in \( P \). Hence \( PK[S]_{C_N(P)}(S : P) \) is an ideal of \( K[S]_{C_N(P)} \) that is not contained in \( PK[S]_{C_N(P)} \). Consequently, \( PK[S]_{C_N(P)}(S : P) = K[S]_{C_N(P)} \), i.e. \( PK[S]_{C_N(P)} \) is invertible. In particular, by [15 Proposition 4.2.6] this ideal satisfies the Artin-Rees property. So, by a result of P. Smith (see [16 Theorem 11.2.13]), \( \bigcap_{k} (PK[S]_{C_N(P)})^k = \{0\} \). It then easily follows (and it is well known) that every proper non-zero ideal of \( K[S]_{C_N(P)} \) is of the form \( (PK[S]_{C_N(P)})^k \), for some unique positive integer \( k \). So, each non-zero ideal of \( K[S]_{C_N(P)} \) is invertible. This proves the desired properties of \( K[S]_{C_N(P)} \) and thus \( K[S]_{C_N(P)} \) is a maximal order.

Next we will prove that \( \bigcap_{P \in \mathcal{X}_1(S)} K[S]_{C_N(P)} \cap G = S \). For this, suppose \( g \in \bigcap_{P \in \mathcal{X}_1(S)} K[S]_{C_N(P)} \cap G \). Then, for every minimal prime \( P \) of \( S \), there exists an element \( \beta_P \in C_N(P) \) such that \( \beta_P g \in K[S] \). We can assume that the image
\( \overline{\beta} \) of \( \beta \) is central in \( K[G/F] \). Indeed, since \( \overline{P}(N) \) is \( G \)-invariant, it follows that \( \prod_{g \in T} \beta^g \in C_N(P) \) (product in any fixed order) for some finite transversal \( T \) for \( N \) in \( G \). Since \( N/F \) is abelian, it follows that \( \prod_{g \in T} \beta^g \) is central in \( K[G/F] \) and we can replace \( \overline{\beta} \) by this product. Furthermore, \( \overline{\beta} \beta^g \subseteq K[S/F] = K[S] \) and hence \( \overline{S} \supp(\overline{\beta} \beta^g) \subseteq \overline{S} \). The union \( \bigcup_p \overline{S} \supp(\overline{\beta} \beta^g) \), over all the minimal primes \( P \) of \( S \), is an ideal of \( \overline{S} \) that is not contained in any minimal prime \( \overline{P} \) of \( \overline{S} \). Since also \( \bigcup_p \overline{S} \supp(\overline{\beta} \beta^g) \subseteq \overline{S} \), and, since \( \overline{S} \) is a maximal order by the comment in the beginning of this section, it follows that \( \overline{g} \in \overline{S} \). Hence \( g \in S \), as desired.

Because of the remark stated in the beginning of the proof, the previous holds for any normal subgroup \( M \) of \( G \) with \( M \subseteq N \) and \( N/M \) finite.

Finally, we prove the following claim: \( K[S] = \bigcap_{P \in \chi(S), M} K[S]_{C_M(P)} \cap K[G] \), with \( M \) running through all torsion free normal subgroups of \( G \) with \( M \subseteq N \) and \( N/M \) finite. Note that this claim implies the result, i.e., \( K[S] \) is a maximal order. Indeed, let \( I \) be a non-zero ideal of \( K[S] \) and suppose that \( q \in Q_{\alpha}(K[S]) \) is such that \( qI \subseteq I \). Let \( P \) be a minimal prime of \( S \) and let \( M \) be a subgroup as described. Since \( qI \subseteq I \), we get \( qIK[S]_{C_M(P)} \subseteq IK[S]_{C_M(P)} \), with \( IK[S]_{C_M(P)} \) a two-sided ideal of \( K[S]_{C_M(P)} \) by [11, Theorem 9.20]. As \( K[S]_{C_M(P)} \) is a maximal order, this yields \( q \in K[S]_{C_M(P)} \). On the other hand, as also \( qIK[G] \subseteq IK[G] \) and as \( K[G] \) is a maximal order, we get that \( q \in K[G] \). Hence the claim implies that \( q \in K[S] \). So we have shown that \( I : qI = K[S] \).

Similarly, \( (I : qI) = K[S] \), and thus indeed \( K[S] \) is a maximal order.

So, to prove the claim, let \( q = \sum_{i=1}^n k_i g_i \in \bigcap_{P \in \chi(S), M} K[S]_{C_M(P)} \cap K[G] \), where \( k_i \neq 0 \in K \) and \( g_i \in G \) for each \( 1 \leq i \leq n \) and \( g_i \neq g_j \) for \( i \neq j \). It is enough to show that \( q \in K[S] \). We prove this by induction on \( n \). If \( n = 1 \) then \( q = kg \) with \( g \in \bigcap_{P \in \chi(S), M} K[S]_{C_M(P)} \cap G \) and it follows from the above that \( g \in S \), as desired. Hence assume \( n > 1 \).

Because \( G \) is residually finite, there exists a normal subgroup of finite index \( M_0 \) in \( G \) such that \( M_0 \subseteq N \) and \( g_i g_j^{-1} \notin M_0 \) for all \( i \neq j \). Note that \( C_{M_1 \cap M_2}(N) \subseteq C_{M_1}(N) \subseteq C_N(P) \) for any two normal subgroups \( M_1, M_2 \) of \( G \) so that \( M_1, M_2 \subseteq N \) and each \( N/M_i \) is finite. Hence, in the intersection \( \bigcap_{P \in \chi(S), M} K[S]_{C_M(P)} \cap K[G] \) we may assume that \( M \) runs through all normal subgroups of \( G \) with \( M \subseteq M_0 \) and \( M_0/M \) finite. In other words we may replace \( N \) by \( M_0 \) in the intersection. It follows that the intersection \( \bigcap_{P \in \chi(S), M} K[S]_{C_M(P)} \cap K[G] \) is a \( G/M_0 \)-graded ring. Hence, the induction hypothesis yields that we may assume that \( q \) is \( G/M_0 \)-homogeneous, that is, each \( g_i g_j^{-1} \in M_0 \). Consequently, \( n = 1 \) and thus by the above we get \( q \in K[S] \). This ends the proof.

Suppose that in the previous theorem one also assumes that the group \( G = SS^{-1} \) is abelian-by-finite. Then, in [10], it is shown that the condition “for every minimal prime \( P \) of \( S \) the intersection \( P \cap N \) is \( G \)-invariant” is necessary for \( K[S] \) to be a maximal order. It is unknown whether this necessity holds in general, nor it is known whether this condition is redundant. That is, no example of a maximal order \( S \) in a polycyclic-by-finite group \( G \) (with \( \Delta^+(G) = \{1\} \) and \( G \)
dihedral-free) is known so that \( K[S] \) is not a maximal order. Proposition 2.1 shows that if such a monoid \( S \) exists then \( G \) does not contain a normal subgroup \( F \) so that \( F \subseteq U(S) \) and \( G/F \) is torsion free abelian.

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