WELL-POSEDNESS OF DISPERSION MANAGED NONLINEAR SCHRÖDINGER EQUATIONS

MI-RAN CHOI, DIRK HUNDERTMARK, YOUNG-RAN LEE

Abstract. We prove local and global well-posedness results for the Gabitov–Turitsyn or dispersion managed nonlinear Schrödinger equation with a large class of nonlinearities and arbitrary average dispersion on $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ for zero and non-zero average dispersions, respectively. Moreover, when the average dispersion is non-negative, we show that the set of ground states is orbitally stable. This covers the case of non-saturated and saturated nonlinear polarizations and yields, for saturated nonlinearities, the first proof of orbital stability.

Contents

1. Introduction 1
2. Nonlinear estimates 9
3. Local existence 14
4. Continuous dependence on the initial data 17
5. Mass and energy conservation 20
6. Global existence 23
7. Orbital stability for (non-)saturated nonlinearities 26
References 32

1. Introduction

1.1. The Cauchy problem. We prove local and global existence results for the initial value problem for a dispersion managed nonlinear Schrödinger equation (NLS)

\[
\begin{cases}
    i\partial_t u + d_{av} \partial^2_x u + \int_{\mathbb{R}} T_r^{-1}(P(T_r u)) \psi(r) dr = 0, \\
    u(x, 0) = u_0(x),
\end{cases}
\]

for a large class of nonlinear polarizations $P$ where $u = u(x, t)$, $x, t \in \mathbb{R}$, is a complex-valued function, $d_{av} \in \mathbb{R}$, $\psi \geq 0$ is in $L^q(\mathbb{R})$ for suitable $q \geq 1$, and $T_r = e^{ir\partial^2_x}$ is the solution operator for the free Schrödinger equation, that is, $w(x, r) = (T_r f)(x)$ solves the initial value problem

\[
\begin{cases}
    i\partial_r w + \partial^2_r w = 0, \\
    w(x, 0) = f(x).
\end{cases}
\]

Date: December 15, 2022.

Key words and phrases. nonlocal NLS, dispersion management, well-posedness, orbital stability.

MSC2020 classification. 35Q55, 35Q60, 35A01, 35B35, 35B30.

©2022 by the authors. Faithful reproduction of this article, in its entirety, by any means is permitted for non-commercial purposes.
The case \( d_{av} = 0 \) is a singular limit. Positive average dispersion, \( d_{av} > 0 \), corresponds to a focusing nonlinearity, while \( d_{av} < 0 \) corresponds to a defocusing nonlinearity, see Remark 1.11. In the application of (1.1) in nonlinear optics, \( t \) corresponds to the distance along the fiber and \( x \) denotes the (retarded) time. The name “dispersion management” refers to the fact that the equation (1.1) models the propagation of signals through glass–fiber cables where the local dispersive properties vary periodically between strongly positive and strongly negative dispersion, with some small average dispersion \( d_{av} \), along the cable. It is an effective equation describing the electromagnetic wave propagation in optical fibers in the so–called strong dispersion management regime. See Section 1.2 for a short discussion on how the probability density \( \psi \) is determined from the local dispersion profile in dispersion managed glass fiber cables.

The technique of dispersion management was invented to balance the competing effects of nonlinearity and dispersion. It has led to new type of glass–fiber cables for ultra–high speed data transfer through optical fiber over long distances. The dispersion managed NLS has intensively been studied, mainly on a non–rigorous level starting with [1, 13, 14], see also the survey [32] and references therein. There are much fewer rigorous results available, e.g., [9, 12, 16, 18, 19, 25, 29, 33]. The Kerr–type nonlinearity, i.e., the case when \( P(z) = |z|^2z \), was originally studied by Gabitov and Turitsyn in [13, 14] and is also assumed in much of the rigorous and non–rigorous work. It corresponds to taking a Taylor series expansion of the polarization \( P \) in the glass–fiber cable and keeping the only the first non–trivial term. We will not make this simplifying assumption in our paper but consider a rather large class of nonlinearities, instead.

We assume that the nonlinearity \( P : \mathbb{C} \to \mathbb{C} \) in (1.1) is of the form of \( P(z) = h(|z|)z \).

Our main assumptions on \( h : [0, \infty) \to \mathbb{R} \) are:

A1) Local and global well–posedness, zero average dispersion: \( h \) is continuous on \([0, \infty)\) and continuously differentiable on \((0, \infty)\) with \( \lim_{a \to 0} h'(a) a = 0 \). There exists \( 0 \leq p \leq 4 \) such that

\[
|h(a)| \lesssim 1 + a^p \quad \text{for all } a \geq 0,
|h'(a)| \lesssim a^{-1} + a^{p-1} \quad \text{for all } a > 0.
\]  

(1.2)

A2) Local well–posedness, non–zero average dispersion: \( h \) is continuous on \([0, \infty)\) and continuously differentiable on \((0, \infty)\) with \( \lim_{a \to 0} h'(a) a = 0 \). There exist increasing functions \( J_1, J_2 : [0, \infty) \to [0, \infty) \) such that

\[
|h(a)| \leq J_1(a) \quad \text{for all } a \geq 0,
|h'(a)| \leq J_2(a)(1 + a^{-1}) \quad \text{for all } a > 0.
\]  

(1.3)

For global well–posedness, when the average dispersion is non–zero, we need to assume in addition

A3) Global well–posedness, non–zero average dispersion: For an increasing function \( \tilde{J} : [0, \infty) \to [0, \infty) \) with

\[
\lim_{a \to \infty} \frac{\tilde{J}(a)}{a^4} = 0
\]  

(1.4)

and for some \( 0 \leq p \leq 4 \) the nonlinearity \( h \) satisfies

\[
h(a) \leq \tilde{J}(a)(1 + a^p) \quad \text{for all } a \geq 0, \quad \text{when } d_{av} > 0,
h(a) \geq -\tilde{J}(a)(1 + a^p) \quad \text{for all } a \geq 0, \quad \text{when } d_{av} < 0.
\]  

(1.5)

Above, we use the convention \( f \lesssim g \), if there exists a finite constant \( C > 0 \) such that \( f \leq Cg \).
Remarks 1.1. (i) In assumption A1, the growth condition (1.2) on $h$ is consistent with the fact that the nonlocal nonlinearity in (1.1) is bounded on $L^2(\mathbb{R})$ for $P(z) = |z|^p z$ and $\psi \in L^{\frac{4}{p}}$ only for $0 \leq p \leq 4$, see Lemma 2.5. However, the assumption on $h'$ in (1.2) is rather weak, allowing a blow-up of $h'$ for small $a$. For example, our assumptions cover even highly oscillating nonlinearities of the form

$$h(a) = a^\delta \sin \left( \frac{1}{a^\kappa} \right) \quad (1.6)$$

with $h(0) = 0$ and $0 < \kappa < \delta \leq 4$. Also sign changing polarizations of the form

$$P(z) = c_1 |z|^{p_1} z - c_2 |z|^{p_2} z \quad (1.7)$$

for $c_1, c_2 > 0$ and exponents $0 \leq p_1 \leq p_2 \leq 4$ are covered by assumption A1.

(ii) Assumption A2 is even weaker: $h$ only has to be locally bounded, without any growth condition at infinity and the possibility of large oscillations of $h'(a)$ for small and large values of $a$. The example (1.6) satisfies assumption A2 for all $0 < \kappa < \delta$ and also assumption A3 for all $0 < \kappa < \delta < 8$. The example (1.7) satisfies assumption A2 for all $0 \leq p_1 \leq p_2 < \infty$ and A3 for all $0 \leq p_1 \leq p_2 < 8$. Polarizations given by the power law $P(z) = |z|^p z$ satisfy assumption A2 for all $p \geq 0$. Since $h(a) = a^\delta$ is bounded from below in this case, assumption A3 is also satisfied for all $p > 0$ when $d_{av} > 0$. However, to satisfy assumption A3, when $d_{av} > 0$, we need to restrict to $p < 8$ for power law polarizations.

(iii) Moreover, assumptions A1, A2, and A3 allow for saturated nonlinearities. For example, the polarization is allowed to be of the form

$$P(z) = \frac{|z|^2 z}{1 + \sigma |z|^2},$$

with $\sigma > 0$. In this case, we have $h(a) = a^2/(1 + \sigma a^2)$, i.e., $h$ is asymptotically constant for large values of $a$.

Before presenting our main results, we make the notion of a solution more precise, see [7, 31]: Let $X_1, X_2$ be Banach spaces. A function $u : \mathbb{R} \times [-M_-, M_+] \to \mathbb{C}$, for some positive $M_\pm$, is called a (local) strong solution of (1.1) if $u \in C([-M_-, M_+], X_1) \cap C^1((-M_-, M_+), X_2)$ satisfies the equation

$$i \partial_t u + d_{av} \partial_x^2 u + Q(u) = 0$$

and $u(\cdot, 0) = u_0$, where the nonlocal nonlinearity $Q$ is given by

$$Q(u(t)) := \int_{\mathbb{R}} T^{-1}_r(P(T_r(u(t)))) \psi(r) dr.$$  

If $d_{av} \neq 0$, we take $X_1 = H^1(\mathbb{R})$ and $X_2 = H^{-1}(\mathbb{R})$, for a definition of the scale of Sobolev spaces $H^s(\mathbb{R})$ see the next section. This is motivated by the fact that under suitable conditions on the nonlinearity, see Lemma 2.6, $Q$ maps $H^1(\mathbb{R})$ into itself and thus, if $u(t) \in H^1(\mathbb{R})$ solves (1.1), then $\partial_t u(t) \in H^{-1}(\mathbb{R})$. If $d_{av} = 0$, then we take $X_1 = X_2 = L^2(\mathbb{R})$, since, under suitable conditions on the nonlinearity, $Q$ maps $L^2(\mathbb{R})$ into itself, see Lemma 2.5.

It is well–known that $u$ is a strong solution of (1.1) with initial datum $u_0$ if and only if $u \in C([-M_-, M_+], H^1(\mathbb{R}))$ for some positive $M_\pm$ and $u$ fulfills the Duhamel formula

$$u(t) = e^{it d_{av} \partial_x^2} u_0 + i \int_0^t e^{i(t-t') d_{av} \partial_x^2} Q(u(t')) dt'.$$  \quad (1.8)
for all $t \in [-M_-, M_+]$, see [7, Proposition 3.1.3] and, also, [6, 31]. It is a global strong solution, if $[M_-, M_+]$ can be replaced by $\mathbb{R}$. In the following, we will mainly work with the integral version (1.8) instead of (1.1).

The Cauchy problem (1.1), or better the integral equation (1.8), is \textit{locally well–posed} in $H^1(\mathbb{R})$ for $d_{av} \neq 0$ if, for any initial data $u_0 \in H^1(\mathbb{R})$, there exist times $M_+ > 0$ and a ball $B$ in $H^1(\mathbb{R})$ containing $u_0$ that for each $\phi \in B$ there exist a unique strong solution $u \in C([-M_-, M_+], H^1(\mathbb{R}))$ of (1.8) with initial datum $\phi$ and the map $\phi \mapsto u$ is continuous from $B$ to $C([-M_-, M_+], H^1(\mathbb{R}))$. It is \textit{globally well–posed} in $H^1(\mathbb{R})$ if we can take $M_+$ arbitrary large. For $d_{av} = 0$, we replace $H^1(\mathbb{R})$ by $L^2(\mathbb{R})$.

For zero average dispersion, we get global well–posedness just assuming A1.

\textbf{Theorem 1.2} (Global well–posedness in $L^2(\mathbb{R})$ for $d_{av} = 0$). Let $h$ satisfy assumption A1 and $\psi \in L^1(\mathbb{R}) \cap L^{4/3}(\mathbb{R})$. Then the Cauchy problem (1.1) is globally well–posed in $L^2(\mathbb{R})$ and the mass is conserved.

For non–zero average dispersion we get local well–posedness under assumption A2.

\textbf{Theorem 1.3} (Local well–posedness in $H^1(\mathbb{R})$ for $d_{av} \neq 0$). Let $h$ satisfy assumption A2 and $\psi \in L^1(\mathbb{R})$. Then the Cauchy problem (1.1) is locally well–posed in $H^1(\mathbb{R})$ and the mass and energy are conserved.

When the average dispersion is not zero we need assumptions A2 and A3 for global existence.

\textbf{Theorem 1.4} (Global well–posedness in $H^1(\mathbb{R})$ for $d_{av} \neq 0$). Let $h$ satisfy assumptions A2, A3 and $\psi \in L^1(\mathbb{R}) \cap L^{4/3}(\mathbb{R})$. Then the Cauchy problem (1.1) is globally well–posed in $H^1(\mathbb{R})$ and the mass and energy are conserved.

The mass, which in nonlinear optics is the power of the pulse, is given by

$$m(u(t)) := \|u(t)\|_{L^2}^2.$$  (1.9)

The energy is given by

$$E(u(t)) := \frac{d_{av}}{2}\|\partial_x u(t)\|_{L^2}^2 - \int \int_{\mathbb{R}^2} V(|T_r u(t)|) dx \psi(r) dr,$$  (1.10)

where $V(a) = \int_0^a P(s) \, ds = \int_0^a h(s)s \, ds$ for $a \geq 0$.

\textbf{Remark 1.5}. For zero–average dispersion, any solution of (1.8) in the space $C(\mathbb{R}, L^2(\mathbb{R}))$ is continuously differentiable, i.e., in $C^1(\mathbb{R}, L^2(\mathbb{R}))$. Similarly, for $d_{av} \neq 0$, any solution of (1.8) in $C(\mathbb{R}, H^1(\mathbb{R}))$ is in the space $C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}))$. Thus our results also show what is known as \textit{unconditional uniqueness} in the literature. For a Kerr–type polarization $P(z) = |z|^2 z$, global well–posedness was proven in [2, 33]. The assumptions of [2] on the local dispersion profile imply that $\psi \in L^\infty(\mathbb{R})$ and has compact support. Our Theorems 1.2, 1.3, and 1.4 show that local as well as global well–posedness still holds for a much larger class of nonlinearities with minimal smoothness assumptions on the nonlinearity. We can also allow for a larger class of dispersion profiles in the dispersion managed NLS.

While we need assumption A3 in the proof of Theorem 1.4, we also have a global well–posedness result for \textit{small initial data} just assuming A2.

\textbf{Theorem 1.6} (Small data global well–posedness). Let $d_{av} \neq 0$ and $h$ satisfy assumption A2.

(i) For any initial datum $u_0 \in H^1(\mathbb{R})$ with small enough $H^1$-norm, the Cauchy problem (1.1) is globally well–posed when $\psi \in L^1(\mathbb{R})$. 

(ii) If \( J_1(a) \leq 1 + a^8 \) for \( a \geq 0 \), then the Cauchy problem (1.1) is globally well–posed for initial conditions \( u_0 \in H^1(\mathbb{R}) \) with \( \|u_0\|_{L^2} \) small enough when \( \psi \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \).

(iii) If \( \lim_{a \to 0} J_1(a)/a^4 = 0 \) then the Cauchy problem (1.1) is globally well–posed for initial conditions \( u_0 \in H^1(\mathbb{R}) \) with \( \|u'_0\| \) small enough (depending on \( \|u_0\| \)) when \( \psi \in L^1(\mathbb{R}) \).

**Remark 1.7.** We note that in \( L^2(\mathbb{R}) \) there are initial conditions \( u_0 \in H^1(\mathbb{R}) \) with \( \|u_0\|_{L^2} \) large and \( \|u'_0\|_{L^2} \) arbitrarily small. An example is given by \( u_0(x) = C(3\delta^2/2)^{1/2}(1 - \delta|\xi|)_+ \) for small \( \delta > 0 \) and \( C > 0 \). Then \( \|u_0\|_{L^2} = C \) and \( \|u_0'\|_{L^2} = \sqrt{3}\delta \to 0 \) as \( \delta \to 0 \). Thus the last statement in Theorem 1.6 is not empty.

For orbital stability, we consider ground states of (1.1), that is, stationary, or standing wave, solutions of (1.1) in the form \( u(x, t) = e^{-i\omega t} f(x) \) with minimal energy. These are given by minimizers of the nonlocal nonlinear constrained variational problem

\[
E^\text{dav}_\lambda = \inf \{ E(f) : f \in X, \|f\|_{L^2}^2 = \lambda \}
\]

where \( \lambda > 0 \), \( d_{\text{av}} \geq 0 \), and \( X = L^2(\mathbb{R}) \) for \( d_{\text{av}} = 0 \); \( X = H^1(\mathbb{R}) \) for \( d_{\text{av}} > 0 \). When \( d_{\text{av}} < 0 \) the ‘kinetic energy’ term in the energy is non–positive, so one should maximize the ‘energy’ when \( d_{\text{av}} < 0 \). Equivalently, one could keep the sign of \( d_{\text{av}} \) positive and flip the sign of the nonlinearity. Thus the case \( d_{\text{av}} < 0 \) corresponds to ‘defocusing nonlinearities’, where, at least for Kerr–type nonlinearities one does not expect to have stationary solutions, see Remark 1.10. Thus, for orbital stability, we only consider \( d_{\text{av}} \geq 0 \). If \( d_{\text{av}} > 0 \), assumption A3 implies that the energy is coercive, see (6.4). Every nonlinear ground state \( f \) weekly solves the equation

\[
\omega f = -d_{\text{av}} f'' - \int_{\mathbb{R}} T_1^{-1}(P(T_1 f))\psi(r)dr
\]

for some Lagrange multiplier \( \omega \).

We denote by \( S^\text{dav}_\lambda \) the set of all ground states

\[
S^\text{dav}_\lambda = \{ f \in X : E(f) = E^\text{dav}_\lambda, \|f\|_{L^2}^2 = \lambda \},
\]

for \( \lambda > 0 \) and \( d_{\text{av}} \geq 0 \). To get orbital stability of \( S^\text{dav}_\lambda \), even to guarantee that \( S^\text{dav}_\lambda \neq \emptyset \), see Theorem 7.1, we need additional assumptions. We distinguish between saturated and non–saturated nonlinearities.

**A4) Non–saturated nonlinearity:** There exists a constant \( p_0 > 2 \) with

\[
h(a)a^2 \geq p_0 \int_0^a h(s)s \, ds \text{ for all } a > 0. \tag{1.12}
\]

**A5) Saturated nonlinearity:** There exists a continuous function \( p : [0, \infty) \to (2, \infty) \), where we allow \( \lim_{a \to \infty} p(a) = 2 \), such that

\[
h(a)a^2 \geq p(a) \int_0^a h(s)s \, ds \text{ for all } a > 0. \tag{1.13}
\]

**A6) There exists** \( a_0 > 0 \) with \( \int_0^{a_0} h(s)s \, ds > 0 \).

**Remarks 1.8.** (i) In terms of \( V(a) = \int_0^a P(s) \, ds = \int_0^a h(s)s \, ds \) the condition (1.12) is equivalent to \( V'(a)a \geq p_0 V(a) \) for \( a > 0 \). This is the well-known Ambrosetti–Rabinowitz condition from the calculus of variations [3]. Condition (1.13) is a weakened version of the classical Ambrosetti–Rabinowitz condition which allows for saturating nonlinearities. That the variational approach for constructing nonlinear ground states also works under such a weaker condition is less known, see [20] for the case of dispersion management solitons.
Condition A6 together with A5 (or A4) guarantees that the nonlinearity is positive for large $a$. Indeed, let $V(a) = \int_0^a h(s) ds$. Since $V$ is continuous and $V(a_0) > 0$, there exists $a_0 < d \leq \infty$ such that $V(a) > 0$ for $a_0 \leq a < d$ and $V(d) = 0$ if $d < \infty$. The assumption A5 is equivalent to
\[
\frac{V''(a)}{V(a)} \geq p(a)a^{-1} \quad \text{for } a_0 < a < d
\]
and integrating this from $a_0$ to $a$ shows
\[
V(a) \geq V(a_0) \exp \left( \int_{a_0}^a p(s) \frac{ds}{s} \right) > V(a_0) \left( \frac{a}{a_0} \right)^2
\]
(1.14)
since $p(a) > 2$. Thus $V$ grows at least quadratically near infinity, hence $d = \infty$.

Moreover, (1.13) shows
\[
h(a)a^2 \geq p(a)V(a) > 2V(a_0) \left( \frac{a}{a_0} \right)^2
\]
so $h(a) > 2V(a_0)a_0^{-2} > 0$ for all $a \geq a_0$.

Note that when $\lim_{a \to \infty} p(a) = 2$ the condition A5 allows the nonlinearity to saturate. The bound (1.14) is the best one can get, since $V(a)$ will asymptotically quadratically when $p(a) \to 2$ fast enough as $a \to \infty$. In this case, the nonlinear polarization $P(z) = h(|z|)z$ is asymptotically linear for $|z|$ large.

Under the stronger condition A4, which does not allow for saturation, one has the lower bound
\[
V(a) \geq V(a_0) \left( \frac{a}{a_0} \right)^{p_0}
\]
for all $a \geq a_0$. In this case $h(a) \geq V(a_0)a_0^{-p_0}a^{p_0-2}$ for $a \geq a_0$, so $h$ grows at least with exponent $p_0 - 2$ when the nonlinearity does not saturate.

(ii) If one prefers to have a local condition on the nonlinearity $h$, a suitable substitute for (1.12) is
\[
h'(a)a \geq (p_0 - 2)h(a) \quad \text{for all } a > 0 \quad (1.15)
\]
and
\[
h'(a)a \geq (p(a) - 2)h(a) \quad \text{for all } a > 0 \quad (1.16)
\]
for (1.13). In fact, (1.15) is equivalent to $(h(a)a^2)' \geq p_0 h(a)a$, and integrating this, one gets (1.12). Similarly (1.16) implies (1.13), replacing $p(a)$ with $\inf_{0 \leq s \leq a} p(s)$.

To state the last theorem, we need one more notation: Given $r \geq 1$ we say that $\psi \in L^{r+\delta}$ if $\psi \in L^{r+8}$ for some $\delta > 0$. We also set $L^{\infty+} = L^\infty$.

**Theorem 1.9.** Suppose that the nonlinearity $h$ satisfies assumption A6 and either of the following:

(i) **Zero average dispersion, non–saturated nonlinearity:** The nonlinearity $h$ satisfies assumption A1 for its derivative $h'$, except that the growth condition on $h$ is slightly strengthened to $|h(a)| \lesssim a^{p_1} + a^{p_2}$ for some $0 < p_1 < p_2 < 4$. It also satisfies assumption A4, and the density $\psi \in L^{4\frac{4}{p_2+}}(\mathbb{R})$ has compact support.

(ii) **Zero average dispersion, saturated nonlinearity:** The nonlinearity $h$ satisfies assumption A1 for its derivative $h'$, except that the growth condition on $h$ is strengthened to $|h(a)| \lesssim a^{p_1} + a^{p_2}$ for some $1 \leq p_1 < p_2 < 3$. It also satisfies assumption A5, and the density $\psi \in L^{\frac{4}{4\frac{4}{p_2+}}}(\mathbb{R})$ has compact support.
(iii) **Positive average dispersion, saturated and non–saturated nonlinearities:**

The nonlinearity \( h(s) \) satisfies assumption \( A2 \) and the bound \( |h(a)| \lesssim a^{p_1} + a^{p_2} \) for some \( 0 < p_1 \leq p_2 < 8 \). The density \( \psi \in L^{\frac{4}{p+2}}(\mathbb{R}) \) has compact support. Moreover, \( h \) satisfies either assumption \( A4 \) or assumption \( A5 \).

Then there exists a critical threshold \( 0 \leq \lambda_{cr}^{\text{av}} < \infty \) such that if \( \lambda > \lambda_{cr}^{\text{av}} \) then \( S_{\lambda}^{\text{av}} \neq \emptyset \). Moreover, the set of ground states \( S_{\lambda}^{\text{av}} \) is orbitally stable in the sense that for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( u_0 \in X \) with

\[
\inf_{f \in S_{\lambda}^{\text{av}}} \|u_0 - f\|_X < \delta,
\]

then the solution \( u \) with the initial datum \( u_0 \) satisfies

\[
\inf_{f \in S_{\lambda}^{\text{av}}} \|u(\cdot, t) - f\|_X < \varepsilon
\]

for all \( t \in \mathbb{R} \), where \( X = H^1(\mathbb{R}) \) if \( d_{\text{av}} > 0 \) and \( X = L^2(\mathbb{R}) \) if \( d_{\text{av}} = 0 \).

In addition, if \( d_{\text{av}} > 0 \) and \( 0 < \lambda < \lambda_{cr}^{\text{av}} \) then \( S_{\lambda}^{\text{av}} = \emptyset \).

**Remarks 1.10.**

(i) In the third assumption of the above theorem, the condition \( |h(a)| \lesssim a^{p_1} + a^{p_2} \) for some \( 0 < p_1 \leq p_2 < 8 \) and all \( a > 0 \) clearly implies assumption \( A3 \).

(ii) For the existence of a critical threshold \( \lambda_{cr} \) for which the set of ground states \( S_{\lambda}^{\text{av}} \) is not empty when \( \lambda > \lambda_{cr} \), slightly weaker assumptions suffice, see Theorem 7.1. For saturating nonlinearities we need to restrict the range of \( p \) from \( 0 < p < 4 \) to \( 1 \leq p < 3 \) in order to have \( S_{\lambda} \neq \emptyset \). The additional assumptions are needed to ensure global existence of solutions.

(iii) If the average dispersion is negative, \( d_{\text{av}} < 0 \), the nonlinearity in (1.1) is defocusing, at least when it is given by the Kerr approximation. For the local NLS it is known that there are no stationary solutions, i.e., solitons, in this case. For the dispersion managed NLS this is not known. While there are some numerical simulations, which show stable propagation of pulses for negative average dispersion \( d_{\text{av}} < 0 \) with \( |d_{\text{av}}| \) small, it seems that these pulses lose energy over time by radiation. Thus they are not expected to be true stationary solutions, see [33, Remark 3.2]. However, this has not been shown rigorously.

**1.2. The connection to nonlinear optics.** Equation (1.1) is an averaged version of the local, but non–autonomous dispersion managed NLS

\[
i \partial_t w = -d_{\text{loc}}(t) \partial_x^2 w - P(w),
\]

where the dispersion \( d_{\text{loc}}(t) \) is parametrically modulated and \( P \) is the nonlinear interaction due to the polarizability of the glass–fiber cable. The constant \( d_{\text{av}} \) is the average dispersion over one period along the cable and the function \( \psi \) is the density of a probability measure related to the mean–zero periodic part of the local dispersion profile,

\[
d_{\text{loc}}(t) = d_{\text{av}} + d_{\text{per}}(t).
\]

Local well–posedness of the non–averaged equation (1.17) has been shown for power–law type nonlinearities in [5, 10, 11], for example. The question of global existence versus finite time blowup of solutions of (1.17) has been investigated in [5], see also [26] for related results.

In the case of strong dispersion management, one assumes that the mean zero periodic part \( d_{\text{per}} \) is give by

\[
d_{\text{per}}(t) = \varepsilon^{-1} d_0(t/\varepsilon)
\]
with $d_0$ periodic, of period $L > 0$ and zero mean, and $\varepsilon > 0$ small. Since (1.17) is non-autonomous with a highly oscillating periodic local dispersion, Gabitov and Turitsyn [13, 14] found an approximation which is good for small $\varepsilon$, i.e., in the regime of strong dispersion management. Roughly, the idea is as follows: Let $T_r = e^{ir\partial_t^2}$, $D(t) = \int_0^t d_0(s) \, ds$, and make the ansatz

$$w(x, t) = T_{D(t/\varepsilon)} v(\cdot, t)(x).$$

Then (1.17) is equivalent to

$$i\partial_t v = -d_{aw} \partial_x^2 v - T^{-1}_{D(t/\varepsilon)} \left[P(T_{D(t/\varepsilon)} v)\right]$$

which now contains the fast oscillating term $T_{D(t/\varepsilon)}$ in the nonlinearity, but the linear part is constant in $t$; since $d_0$ has mean zero and period $L$, the cumulative dispersion $D(t/\varepsilon)$ is periodic with period $\varepsilon L$. The idea of Gabitov and Turitsyn, for the special case of a Kerr nonlinearity, is to average the fast oscillating nonlinear terms containing $T_{D(t/\varepsilon)}$ over one period in $t$, which yields the dispersion managed NLS

$$i\partial_t u = -d_{aw} \partial_x^2 u - \frac{1}{\varepsilon L} \int_0^{\varepsilon L} T_{D(s/\varepsilon)}^{-1} \left[P(T_{D(s/\varepsilon)} u)\right] \, ds$$

(1.21)

where $u$ now is the average profile of the pulse $v$. This is analogous to Kapitza’s treatment of the unstable pendulum, which is stabilized by fast oscillations of the pivot, see [23].

Proofs of the averaging theorem were given in [33] and [10] for Kerr type nonlinearity and [11] for power-law type nonlinearities. In the case of fast dispersion management, where the periodic mean–zero part of the dispersion profile is given by $d_{per}(t) = d_0(t/\varepsilon)$, an averaging theorem is proven in [5].

We prefer to rewrite (1.21) a bit: Introduce a probability measure $\mu$ on the Borel sets of $\mathbb{R}$ by $\mu(B) := \frac{1}{L} \int_0^L \mathbf{1}_B(D(s)) \, ds$ and make to change of variables $r = D(s)$ to see that (1.21) is equivalent to

$$i\partial_t u = -d_{aw} \partial_x^2 u - \int_{\mathbb{R}} T_{r}^{-1} \left[P(T_r u)\right] \mu(\, dr)$$

which is equivalent to (1.1) when $\mu$ has density $\psi$.

Note that since the local mean zero periodic dispersion profile $d_0$ is locally integrable, its integrated version $D$ is bounded, hence the probability measure $\mu$ has compact support. In particular, its density, once it exists, has compact support in all physically interesting cases. The existence and suitable $L^p$ properties of the density $\psi$ follow from physically natural conditions on the local mean zero periodic dispersion profile $d_0$.

The model case, which is usually assumed, is a two step local dispersion profile $d_0 = d_{\text{model}}$ with $d_{\text{model}}(t) = +1$ if $0 \leq t \leq 1$ and $d_{\text{model}}(t) = -1$ if $1 < t < 2$, extended periodically to $t \in \mathbb{R}$. For such a model case the probability density $\psi$ is given by

$$\psi_{\text{model}} = \mathbf{1}_{[0,1]},$$

the characteristic function of the interval $[0,1]$. This simplifying assumption is often made but we will not make it here. We refer to [19, Section 1.2] or [9, Section 1.2] for a detailed discussion how the probability density $\psi$ is connected to the local periodic dispersion profile, see [19, Lemma 1.4]. Most important for us is the criterion that, if $d_0$ stays away from zero and changes its sign finitely many times over one period, then

$$\psi \in L^q \text{ for } q > 1 \text{ whenever } \int_0^L |d_{\text{per}}(s)|^{1-q} \, ds < \infty,$$

(1.22)
see [19, Lemma 1.4]. In particular, all the $L^q$–type conditions on $\psi$ are fulfilled for all physically reasonable local dispersion profiles $d_{\text{per}}$.

**Remark 1.11.** The well–known local NLS in one space dimension is often written in the form

$$i \partial_t u = -\partial_x^2 u - \lambda P(u)$$

and a coupling constant $\lambda \in \mathbb{R}$. In this case $\lambda > 0$ is called a focusing and $\lambda < 0$ is called a defocusing nonlinearity. Thus for the dispersion managed NLS (1.1), $d_{\text{av}} > 0$ corresponds to the focusing, and $d_{\text{av}} < 0$ to the defocusing, case of the usual local NLS, at least when $h$ is nonnegative, where the nonlinearity is given by $P(u) = h(|u|)u$.

This paper is organized as follows. In Section 2 we gather the necessary nonlinear bounds. Due to the nonlocality of the nonlinearity, these are quite different from what is usually used in the study of NLS. Local existence is done in Section 3. Since our assumptions on the nonlinearity are rather weak, the existence proof does not immediately yield continuous dependence on the initial data, at least when $d_{\text{av}} \neq 0$. This local well–posedness is done in Section 4. Global existence is based on mass and energy conservation. Due to our rather weak differentiability assumptions on the nonlinearity, the usual approach to prove conservation of energy and mass is not applicable in our case, see the discussion in the beginning of Section 5. We use a twisting trick to avoid the usual approximation arguments. Our argument directly proves differentiability of the mass and energy and allows for low regularity solutions. The proof of global existence is finished in Section 6 and in Section 7 we give the proof of orbital stability of the set of ground states for non–negative average dispersion.

## 2. Nonlinear estimates

Before we collect the estimates we need, let us introduce some notations. $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$ and $H^s(\mathbb{R})$, $s \in \mathbb{R}$, are the usual Lebesgue and Sobolev spaces with norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^s}$, respectively. That is, $L^p(\mathbb{R})$ is the space of (equivalence classes of) functions $f$ for which

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p} < \infty .$$

For $f \in L^2(\mathbb{R})$, we will simply write $\|f\|_{L^2} = \|f\|$. The Sobolev space is given by

$$H^s(\mathbb{R}) = \left\{ f \in S^* : \int_{\mathbb{R}} \langle \eta \rangle^{2s} |\hat{f}(\eta)|^2 \, d\eta < \infty \right\}$$

with the norm $\|f\|_{H^s} = \|\langle \eta \rangle^s \hat{f}\|$, where $S^* = S^*(\mathbb{R})$ denotes the tempered distributions on $\mathbb{R}$, $\langle \eta \rangle := (1 + \eta^2)^{1/2}$, and $\hat{f}$ is the Fourier transform of $f$, defined by

$$\hat{f}(\eta) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\eta} f(x) \, dx$$

for $f \in S$, the Schwartz space of infinitely smooth, rapidly decreasing functions, and extended by duality to the space of tempered distributions $S^*$.

We denote by $L^q_t(J, L^p_x(I))$, for $1 \leq p, q < \infty$ and intervals $I, J$, the space of all functions $u$ for which

$$\|u\|_{L^q_t(J, L^p_x(I))} = \left( \int_J \left( \int_I |u(x,t)|^p \, dx \right)^\frac{q}{p} \, dt \right)^\frac{1}{q}$$
is finite. If \( p = \infty \) or \( q = \infty \), use the essential supremum instead. For notational simplicity, we write \( L^q(L^p) \) for \( L^q_t(L^p_x(\mathbb{R})) \). For a Banach space \( X \) with norm \( \| \cdot \|_X \) and an interval \( J, C(J,X) \) is the space of all continuous functions \( u : J \to X \). When \( J \) is compact, it is a Banach space with norm

\[
\| u \|_{C(J,X)} = \sup_{t \in J} \| u(t) \|_X
\]

and \( C^1(J,X) \) is the Banach space of all continuously differentiable functions \( u : J \to X \).

Now we gather some properties of the solution operator \( T_r = e^{i r \partial_x^2} \) for the free Schrödinger equation in spatial dimension one. It is a unitary operator on \( L^2(\mathbb{R}) \) and, also, on \( H^1(\mathbb{R}) \) and therefore for every \( r \in \mathbb{R} \)

\[
\| T_r f \| = \| f \| \quad \text{and} \quad \| T_r f \|_{H^1} = \| f \|_{H^1}.
\]

The following is the one-dimensional Strichartz estimate in the form that we need.

**Lemma 2.1 (One-dimensional Strichartz estimates).**

(i) Let \( 2 \leq p \leq \infty \) so that

\[
\frac{1}{p} + \frac{2}{q} = \frac{1}{2}.
\]

If \( f \in L^2(\mathbb{R}) \), then the map \( r \mapsto T_r f \) belongs to \( L^q(L^p) \cap C(\mathbb{R},L^2) \) and

\[
\| T_r f \|_{L^q(L^p)} \lesssim \| f \|,
\]

where the implicit constant depends only on \( p \). Moreover, if \( f \in H^1(\mathbb{R}) \), then the map \( r \mapsto T_r f \) is in \( C(\mathbb{R},H^1) \).

(ii) Let \( J \) be a bounded interval containing zero. If \( F \in L^1(J,L^2) \), then the map

\[
r \mapsto \Psi_F(r) := \int_0^r T_{(r-r')} F(\cdot,r') dr'
\]

belongs to \( L^\infty(J,L^2) \cap C(J,L^2) \) and

\[
\| \Psi_F \|_{L^\infty(J,L^2)} \lesssim \| F \|_{L^1(J,L^2)}.
\]

Moreover, if \( F \in L^1(J,H^1) \), then \( \Psi_F \) is in \( C(J,H^1) \).

The Strichartz inequalities have a long history. The first proof by Strichartz [30], valid in all dimensions, was for the special case \( p = q \). It was then later extended by several authors, see, for example, [15, 22]. The above formulation is from [7] for the case of one space dimension.

To state the space time bounds we need, which are based on Strichartz type estimates, we introduce one more notation. For a non-negative function \( \psi \) on \( \mathbb{R} \) we denote by \( L^q(\mathbb{R}^2, dx \psi dr) \), \( 1 \leq q < \infty \), the Banach space of all functions with the weighted norm

\[
\| u \|_{L^q(\mathbb{R}^2, dx \psi dr)} = \left( \iint_{\mathbb{R}^2} |u(x,r)|^q dx \psi(r) dr \right)^{1/q}.
\]

**Lemma 2.2.** Let \( 2 \leq q \leq 6 \) and \( \psi \in L^{\frac{1}{6-q}}(\mathbb{R}) \). Then for all \( f \in L^2(\mathbb{R}) \),

\[
\| T_r f \|_{L^q(\mathbb{R}^2, dx \psi dr)} \lesssim \| f \|^q,
\]

where the implicit constant depends only on the \( L^{\frac{1}{6-q}} \) norm of \( \psi \).

**Proof.** The bound (2.2) is exactly the same as provided by Lemma 2.1 in [9]. We give a simpler proof. Using Hölder’s inequality with exponents \( \frac{1}{q-2} \) and \( \frac{1}{6-q} \) in the \( r \)-integral and then Strichartz inequality from Lemma 2.1 one obtains

\[
\iint_{\mathbb{R}^2} |T_r f|^q dx \psi(r) dr \leq \| T_r f \|^q_{L^{4q/(q-2)}(L^q)} \| \psi \|^q_{L^{4/(6-q)}} \lesssim \| f \|^q \| \psi \|^q_{L^{4/(6-q)}}.
\]
Similar to Proposition 2.15 in [9], one can extend the bound (2.2) to \( q > 6 \) for \( f \in H^1(\mathbb{R}) \). In the following we use \( a_+ = \max(a, 0) \) for the positive part of \( a \in \mathbb{R} \).

**Lemma 2.3.** Let \( 2 \leq q < \infty \) and \( \psi \geq 0 \) in \( L^{\frac{4}{q-\kappa}}(\mathbb{R}) \) for some \( (q-6)_+ \leq \kappa \leq q-2 \). Then for all \( f \in H^1(\mathbb{R}) \)
\[
\|T_r f\|_{L^q(\mathbb{R}^2, dx \psi dr)} \lesssim \|f\|_{L^\infty}^{q}\|f\|_{L^2}^{q-\frac{2q}{q-\kappa}},
\]
where the implicit constant depends only the \( L^{\frac{4}{q-\kappa}} \) norm of \( \psi \).

**Proof.** This can be found in the proof of Proposition 2.15 in [9]. For the reader’s convenience, we give the short proof: Since \( 2 \leq q - \kappa \leq 6 \) and \( \psi \in L^{\frac{4}{q-\kappa}}(\mathbb{R}) \), applying Lemma 2.2, we get
\[
\int \int_{\mathbb{R}^2} |T_r f|^q dx \psi(r) dr \leq \sup_{r \in \mathbb{R}} ||T_r f||_{L^\infty}^q \int \int_{\mathbb{R}^2} |T_r f|^q dx \psi(r) dr \lesssim \sup_{r \in \mathbb{R}} ||T_r f||_{L^\infty}^q \|f\|^{q-\kappa}.
\]
Now using the well–known bound
\[
\|g\|_{L^\infty}^2 \leq \|g'\| \|g\|,
\]
which follows easily from
\[
|g(x)|^2 = 2\text{Re} \int_{-\infty}^{x} g(t)g'(t) dt = -2\text{Re} \int_{x}^{\infty} g(t)g'(t) dt
\]
for all \( g \in H^1(\mathbb{R}) \) and \( x \in \mathbb{R} \), we obtain
\[
\sup_{r \in \mathbb{R}} ||T_r f||_{L^\infty}^2 \leq \sup_{r \in \mathbb{R}} \|\partial_x (T_r f)\| \|T_r f\| = \|f'\| \|f\|, \quad (2.4)
\]
where we used the fact that \( \partial_x \) and \( T_r = e^{i r \partial_x^2} \) commute and \( T_r \) is unitary on \( L^2(\mathbb{R}) \). Combining (2.3) and (2.4) completes the proof.

**Remark 2.4.** It immediately follows from Lemma 2.3 that
\[
||T_r f||_{L^q(\mathbb{R}^2, dx \psi dr)} \lesssim \|f\|_{H^1}
\]
under the conditions in Lemma 2.3. A similar argument shows that if \( 2 \leq q \leq \infty \) and \( f \in H^1(\mathbb{R}) \), then
\[
||T_r f||_{L^q} \lesssim \|f\|_{H^1}
\]
for arbitrary \( r \in \mathbb{R} \). Indeed, this bound clearly holds due to (2.4) when \( q = \infty \). For \( 2 \leq q < \infty \), we have
\[
\int_{\mathbb{R}} |T_r f|^q dx \leq ||T_r f||_{L^\infty}^{q-2} \int_{\mathbb{R}} |T_r f|^2 dx \leq \|f\|_{L^\infty}^{q-2} \|f'\|_{L^\infty}^{q-2} \leq \|f\|_{H^1}^q.
\]
We denote the nonlocal nonlinearity in (1.1) by
\[
Q(f) := \int_{\mathbb{R}} T_r^{-1}(P(T_r f)) \psi(r) dr \quad (2.5)
\]
for \( f \) in either \( L^2(\mathbb{R}) \) or \( H^1(\mathbb{R}) \). Then the map \( f \mapsto Q(f) \) is bounded and locally Lipschitz continuous as in the following two lemmas.
Lemma 2.5. Suppose that \( h \) satisfies assumption A1 and \( 0 \leq \psi \in L^1(\mathbb{R}) \cap L^{4/p}(\mathbb{R}) \). Then for all \( f, g \in L^2(\mathbb{R}) \) we have
\[
\|Q(f)\| \lesssim \|f\| + \|f\|^{p+1}
\] (2.6)
and
\[
\|Q(f) - Q(g)\| \lesssim (1 + \|f\|^{p} + \|g\|^{p}) \|f - g\|,
\] (2.7)
where the implicit constants depend only on \( p \) and the \( L^1, L^{4/p} \) norms of \( \psi \).

Proof. Since \( P(z) = h(|z|)z \), the triangle inequality for integrals implies
\[
\|Q(f)\| = \int_{\mathbb{R}}\|T_r^{-1}(P(T_rf))\|\psi(r)dr \lesssim \int_{\mathbb{R}}(\|T_r f\| + \|T_r f\|^{p+1})\|\psi(r)\|dr,
\]
where we used assumption A1. For the first term, we note that \( \|T_r f\| = \|f\| \), since \( T_r \) is unitary on \( L^2(\mathbb{R}) \). For the second term, we use Hölder’s inequality with exponents \( \frac{4}{p} \) and \( \frac{4}{4-p} \) in \( r \) to get
\[
\int_{\mathbb{R}}\|T_r f\|^{p+1}\|\psi(r)\|dr = \int_{\mathbb{R}}\|T_r f\|^{\frac{p+1}{2(p+1)}}\|\psi(r)\|^\frac{2}{2(p+1)}dr
\] \[
\leq \left(\int_{\mathbb{R}}\|T_r f\|^{\frac{d(p+1)}{2(p+1)}}dr\right)^\frac{2}{d(p+1)}\left(\int_{\mathbb{R}}|\psi(r)|^{\frac{4}{4-p}}dr\right)^\frac{4-p}{4}.
\]
Thus (2.6) follows from the Strichartz estimate (2.1).

For the second bound, we again use the triangle inequality and the unitarity of \( T_r \) on \( L^2(\mathbb{R}) \) to see that
\[
\|Q(f) - Q(g)\| \leq \int_{\mathbb{R}}\|P(T_rf) - P(T_rg)\|\psi(r)dr.
\]
Let \( w, z \in \mathbb{C} \). From assumption A1 one gets for any \( 0 \leq s \leq 1 \)
\[
\left| \frac{d}{ds}P(w+s(z-w)) \right| = \left| \frac{d}{ds}[h(|w+s(z-w)|)(w+s(z-w))] \right|
\] \[
\leq |h'(|w+s(z-w)|)|w+s(z-w)||z-w| + |h(|w+s(z-w)|)||z-w|
\] \[
\lesssim (1 + |w+s(z-w)|^{p})|z-w| \leq (1 + \max(|w|,|z|^{p})|z-w|)
\]
and the fundamental theorem of calculus gives for all \( z, w \in \mathbb{C} \)
\[
|P(z) - P(w)| = \left| \int_{0}^{1} \frac{d}{ds}(P(w+s(z-w)))ds \right| \lesssim (1 + \max(|w|,|z|^{p})|z-w|).
\] (2.8)

Therefore
\[
\|Q(f) - Q(g)\| \lesssim \int_{\mathbb{R}}\left(1 + \max(|T_r f|,|T_r g|)^{p}\right)\|T_r(f - g)\|\psi(r)dr
\] \[
\leq \int_{\mathbb{R}}\|T_r(f - g)\|\psi(r)dr + \int_{\mathbb{R}}\|T_r f\|^{p} + \|T_r g\|^{p}\|T_r(f - g)\|\psi(r)dr.
\] (2.9)

Note that the first term equals \( \|f - g\|\|\psi\|_{L^1} \). If \( p = 0 \), the second term is bounded in the same way. So to control the second term, it is enough to assume that \( 0 < p \leq 4 \). Use Hölder’s inequality with \( \alpha \) and \( \frac{2\alpha}{\alpha - 2} \) in \( x \) to get
\[
\|T_r f\|^{p}T_r(f - g)\| \lesssim \|T_r f\|^{p}\|T_r(f - g)\|_{L^{\frac{2\alpha}{\alpha - 2}}} = \|T_r f\|^{p}\|T_r(f - g)\|_{L^{\frac{2\alpha}{\alpha - 2}}}.
\]
Note that $\frac{2\alpha}{\alpha - 2} > 2$ for any $\alpha > 2$ and one can always choose $\alpha > 2$ such that also $\alpha p > 2$. Fix such an $\alpha > 2$ and use Hölder’s inequality with three exponents $\frac{4\alpha}{\alpha p - 2}$, $2\alpha$ and $\frac{4}{4 - p}$ in $r$ to obtain

$$\int_{\mathbb{R}} \|T_r f\|^{4\alpha} \|T_r(f - g)\| \psi(r) dr \leq \int_{\mathbb{R}} \|T_r f\|^{4\alpha} \|T_r(f - g)\| \frac{4\alpha}{\alpha p - 2} \psi(r) dr$$

$$\leq \left( \int_{\mathbb{R}} \|T_r f\|^{4\alpha} \frac{4\alpha}{\alpha p - 2} dr \right)^{\frac{\alpha p - 2}{4\alpha}} \left( \int_{\mathbb{R}} \|T_r(f - g)\|^{2\alpha} \psi(r) dr \right)^{\frac{\alpha}{2\alpha}} \left( \int_{\mathbb{R}} |\psi(r)|^{-\frac{1}{4-p}} dr \right)^{\frac{4-p}{4}}$$

$$\lesssim \|f\|^p \|f - g\| \|\psi\| \frac{4}{4 - p},$$

where we used the Strichartz estimate for the first two factors. Using this in (2.9) proves the second part of the lemma.

**Lemma 2.6.** Suppose that $h$ satisfies assumption A2 and $\psi \geq 0$ in $L^1(\mathbb{R})$. Then for all $f, g \in H^1(\mathbb{R})$ we have

$$\|Q(f)\|_{H^1} \lesssim \left[ J_1(\|f\|_{H^1}) + J_2(\|f\|_{H^1})(1 + \|f\|_{H^1}) \right] \|f\|_{H^1}$$

and with $a \vee b = \max(a, b)$ for real numbers $a$ and $b$

$$\|Q(f) - Q(g)\| \lesssim \left[ J_1(\|f\|_{H^1} \vee \|g\|_{H^1}) + J_2(\|f\|_{H^1} \vee \|g\|_{H^1})(1 + \|f\|_{H^1} \vee \|g\|_{H^1}) \right] \|f - g\|,$$

(2.10)

where the implicit constants depend only on the $L^1$ norm of $\psi$.

**Proof.** Let $f \in H^1(\mathbb{R})$. We first show

$$\|Q(f)\| \leq J_1(\|f\|_{H^1}) \|f\|_{H^1}.$$ 

Use the triangle inequality, the unitarity of $T_r$ on $L^2(\mathbb{R})$, and assumption A2 to get

$$\|Q(f)\| \leq \int_{\mathbb{R}} \|P(T_r f)\| \psi(r) dr \leq \int_{\mathbb{R}} \|J_1(\|T_r f\|)\|_{L^\infty} \|T_r f\| \psi(r) dr$$

$$\leq \int_{\mathbb{R}} J_1(\|T_r f\|_{L^\infty}) \|T_r f\| \psi(r) dr \leq J_1(\|T_r f\|_{H^1}) \|\psi\|_{L^1} \|f\|_{H^1},$$

(2.11)

where we also used the assumption that $J_1$ is increasing and $\|T_r f\|_{L^\infty} \leq \|T_r f\|_{H^1} = \|f\|_{H^1}$.

For any $g \in H^1(\mathbb{R})$

$$|\partial_x P(g)| = |\partial_x h(|g|)g| = |h(|g|)g' + h'(|g|)\text{Re} \left( \frac{\overline{g}}{|g|} g' \right) |g|$$

$$\leq |h(|g|)g'| + |h'(|g|)||g'|||g| \leq \left[ J_1(|g|) + J_2(|g|)(1 + |g|) \right] |g'|,$$

where we used assumption A2. Since $J_1$ and $J_2$ are increasing, we get

$$\|\partial_x (P(T_r f))\| \leq \|J_1(\|T_r f\|) + J_2(\|T_r f\|)(1 + |T_r f|)\|_{L^\infty} \|\partial_x T_r f\|$$

$$\leq \left[ J_1(\|f\|_{H^1}) + J_2(\|f\|_{H^1})(1 + \|f\|_{H^1}) \right] \|f'\|.$$

From this we obtain

$$\|\partial_x Q(f)\| \leq \int_{\mathbb{R}} \|\partial_x P(T_r f)\| \psi(r) dr$$

$$\leq \left[ J_1(\|f\|_{H^1}) + J_2(\|f\|_{H^1})(1 + \|f\|_{H^1}) \right] \|f'\| \|\psi\|_{L^1},$$

which together with (2.11) proves the first bound of the lemma.
Next, we prove the second bound. Arguing similarly as in the derivation of (2.8), we have for \( z, w \in \mathbb{C} \)
\[
|P(z) - P(w)| = |h(|z|)z - h(|w|)w|
\leq |z - w| \int_{0}^{1} \left[ |h'(|w + s(z - w)|)||w + s(z - w)| + |h(|w + s(z - w)|)| \right] ds
\leq |z - w| \left[ J_1(|z| \vee |w|) + J_2(|z| \vee |w|)(1 + |z| \vee |w|) \right],
\]
where we used assumption **A2** for \( h \) in the last bound. This implies
\[
\|Q(f) - Q(g)\| \leq \int_{\mathbb{R}} \|P(T_rf) - P(T_r g)\| \psi(r) dr
\leq \int_{\mathbb{R}} \| J_1(|T_rf| \vee |T_r g|) + J_2(|T_rf| \vee |T_r g|)(1 + |T_rf| \vee |T_r g|)\|_{L^\infty} \|T_r(f - g)\| \psi(r) dr.
\]
This proves (2.10), since \( J_1 \) and \( J_2 \) are increasing, \( \|T_rf\|_{L^\infty} \leq \|f\|_{H^1}, \) and \( T_r \) is unitary. \( \blacksquare \)

### 3. Local existence

In this section, we prove the existence of local strong solutions of (1.1), equivalently, local solutions of (1.8). This can be proven with by now standard arguments (see, for example, [7, 21]). However, since, in particular in the \( H^1 \) setting, we want to impose rather weak differentiability conditions on the nonlinearity, the proofs are somewhat technical and we prefer to give the proofs in detail for the reader’s convenience. Here and below, we use \( C \) to denote various constants. First, we show the existence of local solutions of (1.8) in the case of vanishing average dispersion.

**Proposition 3.1.** Let \( d_{av} = 0 \). Suppose that \( h \) satisfies assumption **A1** and \( \psi \in L^1(\mathbb{R}) \cap L^{\frac{4}{1 + \rho}}(\mathbb{R}) \). Then there exists a unique local solution of (1.8). More precisely, for any \( K > 0 \) there exist positive numbers \( M_\pm \), depending also on \( p \) and the \( L^1, L^{\frac{4}{1 + \rho}} \) norms of \( \psi \), such that for any initial condition \( u_0 \in L^2(\mathbb{R}) \) with \( \|u_0\| \leq K \) there exists a unique solution \( u \in C([-M_-, M_+], L^2) \) of (1.8). Moreover,
\[
\|u(t)\| \leq 2K \quad \text{for all} \quad t \in [-M_-, M_+]. \quad (3.1)
\]

An immediate consequence is

**Corollary 3.2.** Let \( d_{av} = 0 \). Suppose that \( h \) satisfies assumption **A1** and \( \psi \in L^1(\mathbb{R}) \cap L^{\frac{4}{1 + \rho}}(\mathbb{R}) \). For any initial datum \( u_0 \in L^2(\mathbb{R}) \) there exists maximal life times \( T_\pm \in (0, \infty] \) such that there is a unique solution \( u \in C([-T_-, T_+], L^2) \) of (1.8). Moreover, the blow-up alternative for solutions holds: If \( T_+ < \infty \) then
\[
\lim_{t \to T_+} \|u(t)\| = \infty
\]
and similarly, if \( T_- < \infty \), then
\[
\lim_{t \to T_-} \|u(t)\| = \infty.
\]

**Remark 3.3.** Due to mass conservation given in (5.2), Corollary 3.2 immediately yields a unique global solution when \( d_{av} = 0 \).

**Proof of Proposition 3.1.** We will prove the existence of local solutions for positive times only since the case of negative times is done similarly. Fix \( u_0 \in L^2(\mathbb{R}) \) and for each \( M > 0 \)
define the map \( \Phi \) on \( C([0, M], L^2) \) by

\[
\Phi(u)(t) = u_0 + i \int_0^t Q(u(t')) dt',
\]
where \( Q \) is defined in (2.5). It is easy to see that \( \Phi(u) \in C([0, M], L^2) \).

For each \( R > 0 \), define the ball

\[
B_{M,R} = \{ u \in C([0, M], L^2) : \|u\|_{C([0, M], L^2)} \leq R \},
\]
equipped with the distance

\[
d(u, v) = \|u - v\|_{C([0, M], L^2)}.
\]

For appropriate values of \( R \) and \( M \), the map \( \Phi \) is a contraction on \( B_{M,R} \) with respect to the metric \( d \). Indeed, Lemma 2.5 shows that there exists a constant \( C \) depending only on \( p \) and the \( L^1, L^{\frac{4}{p}} \) norms of \( \psi \) such that for all \( f, g \in L^2(\mathbb{R}) \),

\[
\|Q(f)\| \leq C(\|f\| + \|f\|^{p+1})
\]
and

\[
\|Q(f) - Q(g)\| \leq C(1 + \|f\|^p + \|g\|^p) \|f - g\|.
\]

Thus, if \( u, v \in C([0, M], L^2) \), then

\[
\|\Phi(u)(t)\| \leq \|u_0\| + \int_0^t \|Q(u(t'))\| dt'
\]
\[
\leq \|u_0\| + C \int_0^t \|u(t')\| + \|u(t')\|^{p+1} dt'
\]
and

\[
\|\Phi(u)(t) - \Phi(v)(t)\| \leq \int_0^t \|Q(u(t')) - Q(v(t'))\| dt'
\]
\[
\leq C \int_0^t (1 + \|u(t')\|^p + \|v(t')\|^p) \|u(t') - v(t')\| dt'.
\]

Therefore, for all \( u, v \in B_{M,R} \),

\[
\|\Phi(u)\|_{C([0, M], L^2)} \leq \|u_0\| + CM(R + R^{p+1}) \quad (3.2)
\]
and

\[
d(\Phi(u), \Phi(v)) \leq CM(1 + 2R^p) d(u, v). \quad (3.3)
\]

Now assume that \( \|u_0\| \leq K \), set \( R = 2K \), and choose \( M_+ > 0 \) satisfying

\[
CM_+(1 + (2K)^p) < \frac{1}{2}.
\]

Then using (3.2) and (3.3), we conclude that \( \Phi \) is a contraction from \( B_{M_+, 2K} \) into itself and since \( B_{M_+, 2K} \) is complete, Banach’s contraction mapping theorem shows that there exists a unique solution \( u \) of (1.8) in \( B_{M_+, 2K} \). This also proves (3.1).

**Remark 3.4.** By standard arguments, the contraction mapping also yields that on compact time intervals the solution depends continuously on the initial condition. A more quantitative bound is derivable with the help of a Gronwall argument, see Proposition 4.1.
Proof of Corollary 3.2. Given an initial datum $u_0 \in L^2(\mathbb{R})$, let
\[ T_+ = T_+(u_0) = \sup \{ M : \exists \text{ unique solution } u \in C([0, M], L^2) \text{ with } u(0) = u_0 \}. \]
Proposition 3.1 shows $T_+ > 0$ and that $u$ is the unique solution of (1.8) with initial datum $u_0$ for all $t \in [0, T_+)$. To see the blow-up alternative, assume that $T_+ < \infty$, but
\[ K := \liminf_{t \to T_+} \|u(t)\| + 1 < \infty. \]
Then there exists a sequence of times $t_n \to T_+$, as $n \to \infty$, with $\|u(t_n)\| < K$.

By simply shifting in time, the already proven local existence result from Proposition 3.1 shows that there is a time $\Delta T$, depending only on $p$ and the $L^1$, $L^{\frac{4}{3}}$ norms of $\psi$, and $K$, such that there is a unique solution $\tilde{u} \in C([t_n, t_n + \Delta T], L^2)$ of (1.8). This solution agrees with $u$ on the time interval $[t_n, T_+]$ and thus concatenating these two unique solutions one gets, for all $n \in \mathbb{N}$, a unique solution $u$ in $C([0, t_n + \Delta T], L^2)$ for the given initial condition $u_0$ at time $t = 0$. Since $t_n + \Delta T > T_+$ for large enough $n$, this contradicts the maximality of the life time interval $(0, T_+)$. Thus, if $0 < T_+ < \infty$ we must have $\lim_{t \to T_+} \|u(t)\| = \infty$. The case of negative times is done similarly.

Next, we present the local existence result in $H^1(\mathbb{R})$ when the average dispersion does not vanish.

Proposition 3.5. Let $d_{av} \neq 0$. If $h$ satisfies assumption A2 and $\psi \in L^1(\mathbb{R})$, then there exists a unique local solution of (1.8). More precisely, for any $K > 0$ there exist positive numbers $M_{\pm}$, depending also on the $L^1$ norm of $\psi$ and $J_1, J_2$ from assumption A2, such that for any initial condition $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{H^1} \leq K$, there exists a unique solution $u \in C([-M_-, M_+], H^1)$ of (1.8). Moreover,
\[ \|u(t)\|_{H^1} \leq 2K \quad \text{for all } t \in [-M_-, M_+]. \] (3.4)

Remark 3.6. The proof of Proposition 3.5 follows a strategy due to Kato [21], see also [7]. It yields existence and uniqueness, but falls short of proving continuous dependence on the initial datum, i.e., it does not yield well–posedness. This is done in Proposition 4.3.

As for the case of vanishing average dispersion, an immediate consequence is

Corollary 3.7. Let $d_{av} \neq 0$ and $h$ satisfy assumption A2 and $\psi \in L^1(\mathbb{R})$. For any initial datum $u_0 \in H^1(\mathbb{R})$ there exist maximal life times $T_\pm \in (0, \infty]$ such that there is a unique solution $u \in C([-T_-, T_+], H^1)$ of (1.8). Moreover, the blow–up alternative for solutions holds: If $T_+ < \infty$ then
\[ \lim_{t \to T_+} \|u(t)\|_{H^1} = \infty \]
and similarly, if $T_- < \infty$, then
\[ \lim_{t \to T_-} \|u(t)\|_{H^1} = \infty. \]

Given Proposition 3.5, the proof of Corollary 3.7 is a straightforward copy of the proof of Corollary 3.2. So it is enough to give the

Proof of Proposition 3.5. As before, we consider only the case of positive times. For each $M > 0$ and $R > 0$, let
\[ B_{M, R} = \{ u \in L^\infty([0, M], H^1) : \|u\|_{L^\infty([0, M], H^1)} \leq R \} \]
be equipped with the distance
\[ d(u, v) = \|u - v\|_{L^\infty([0, M], L^2)}. \]
It is easy to see that boundedness in $H^1$ and convergence in $L^2$ imply convergence in $H^1$. Thus $(B_{M,R},d)$ is a complete metric space, even though the distance $d$ is measured in $L^2$. Let $K > 0$ and $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{H^1} \leq K$ be fixed. Define the map $\Phi$ on $B_{M,R}$ by

$$\Phi(u)(t) = e^{itd_\psi^2}u_0 + i \int_0^t e^{i(t-t')d_\psi^2}Q(u(t'))dt'.$$

We can apply the same argument in the proof of Proposition 3.1, using Lemma 2.6 instead of Lemma 2.5. Then we see that, for all $u,v \in B_{M,R}$,

$$\|\Phi(u)\|_{L^\infty([0,M],H^1)} \leq K + CM\left(J_1(R) + J_2(R)(1 + R)\right)R$$

and

$$d(\Phi(u),\Phi(v)) \leq CM\left(J_1(R) + J_2(R)(1 + R)\right)d(u,v).$$

Now set $R = 2K$ and choose $M_+ > 0$ satisfying

$$CM_+\left(J_1(2K) + J_2(2K)(1 + 2K)\right) < \frac{1}{2},$$

then we obtain that $\Phi$ is a contraction from $B_{M_+,2K}$ into itself, so $\|u\|_{L^\infty([0,M_+],H^1)} \leq 2K$, which shows (3.4). Moreover, the second part of Lemma 2.1 shows that $u$ is even in $C([0,M_+],H^1)$. \hfill \Box

4. Continuous dependence on the initial data

To complete the proof of Theorem 1.3 and the local well–posedness part of Theorem 1.2, we need to show that the solution depends continuously on the initial datum. To do this for zero average dispersion, we prove that the map $u_0 \mapsto u(t)$ is locally Lipschitz continuous on $L^2(\mathbb{R})$ by a Gronwall argument.

**Proposition 4.1.** Let $d_\psi = 0$, $h$ satisfy assumption A1 and $\psi \in L^1(\mathbb{R}) \cap L^{\frac{4}{1-p}}(\mathbb{R})$. Then, for every $K > 0$, there exists a positive constant $C$ depending only on $K,p$, and the $L^1$ and $L^{\frac{4}{1-p}}$ norms of $\psi$ such that for all initial data $u_0,v_0 \in L^2(\mathbb{R})$ with $\|u_0\|,\|v_0\| \leq K$ we have

$$\|u - v\|_{C([-M_-,M_+],L^2)} \leq e^{C\max(M_-,M_+)}\|u_0 - v_0\|,$$

where $u$ and $v$ are the corresponding local strong solutions of (1.1) with initial data $u_0,v_0$ on the time interval $[-M_-,M_+]$ of existence, guaranteed by Proposition 3.1.

**Proof.** Without loss of generality, we assume that $t \geq 0$. From (3.1) we know that $\|u(t)\|,\|v(t)\| \leq 2K$ for $0 \leq t \leq M_+$. Since

$$u(t) - v(t) = u_0 - v_0 + i \int_0^t \left(Q(u(t')) - Q(v(t'))\right)dt',$$

we can use (2.7) and the triangle inequality for norms and integrals to obtain

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\| + \int_0^t \|Q(u(t')) - Q(v(t'))\|dt'
\leq \|u_0 - v_0\| + C_1 \int_0^t \left(1 + \|u(t')\|^p + \|v(t')\|^p\right)\|u(t') - v(t')\|dt'
\leq \|u_0 - v_0\| + C_1(1 + 2^{p+1}K^p) \int_0^t \|u(t') - v(t')\|dt'$$

for $0 \leq t \leq M_+$. \hfill \Box
for $0 \leq t \leq M_+$. Therefore, setting $C = C_1(1 + 2^{p+1}K^p)$, it follows from Gronwall’s inequality that if $0 \leq t \leq M_+$, then

$$\|u(t) - v(t)\| \leq e^{Ct}\|u_0 - v_0\|$$

which completes the proof.

**Remark 4.2.** Using that for zero average dispersion one has mass conservation, see the beginning of Section 5, the local solutions are global and the above proof yields

$$\|u(t) - v(t)\| \leq e^{C_1(1+\|u_0\|^p+\|v_0\|^p)t}\|u_0 - v_0\|$$

for all $t$, where $C_1$ depends only on $p$ and the $L^1$, $L^{\frac{4}{p-2}}$ norms of $\psi$.

It remains to show continuous dependence on the initial datum when $d_{av} \neq 0$.

**Proposition 4.3.** Let $d_{av} \neq 0$, $h$ satisfy assumption A2, and $\psi \in L^1(\mathbb{R})$. Then the local solution of the Cauchy problem (1.1) depends continuously on the initial datum. More precisely, if $\varphi_n \in H^1(\mathbb{R})$ with $\varphi_n \to \varphi$ in $H^1(\mathbb{R})$ as $n \to \infty$, then there exists a common time interval $[-M_-, M_+]$ for which the strong solutions $u$, respectively $u_n$, of the Cauchy problem (1.1) with initial data $\varphi$, respectively $\varphi_n$, exist and

$$u_n \to u \quad \text{in} \quad C([-M_-, M_+], H^1) \cap C^1((-M_-, M_+), H^{-1}) \quad \text{as} \quad n \to \infty.$$

**Proof.** Choose a positive $K$ such that $\|\varphi\|_{H^1}$, $\|\varphi_n\|_{H^1} \leq K$ for all $n \in \mathbb{N}$. It is enough to consider only positive times. Using Proposition 3.5 we then know there exists $M_+ > 0$ such that on $[0, M_+]$ the solutions $u, u_n$ of (1.1) with initial data $\varphi, \varphi_n$ exist for all $n$ and

$$\|u(t)\|_{H^1}, \|u(t)\|_{H^1} \leq 2K$$

for all $0 \leq t \leq M_+$.

It suffices to prove

$$u_n \to u \quad \text{in} \quad C([0, M_+], H^1)$$

as $n \to \infty$, since then $Q(u_n)$ converges to $Q(u)$ in $C([0, M_+], L^2)$ by (2.10) and $\partial_x^2 u_n$ converges to $\partial_x^2 u$ in $C([0, M_+], H^{-1})$. Hence

$$\partial_t u_n = id_{av}\partial_x^2 u_n + iQ(u_n) \to \partial_t u \quad \text{in} \quad C([0, M_+], H^{-1}) \quad \text{as} \quad n \to \infty.$$

Furthermore, since $u, u_n \in C([0, M_+], H^1)$ for all $n \in \mathbb{N}$, it is enough to show

$$u_n \to u \quad \text{in} \quad L^\infty([0, M_+], H^1).$$

Using

$$u_n(t) - u(t) = e^{idd_{av}\partial_x^2 (\varphi_n - \varphi)} + i \int_0^t e^{i(t-t')}d_{av}\partial_x^2 (Q(u_n(t')) - Q(u(t'))) dt'$$

and similar arguments as in the proof of Proposition 3.5, we obtain

$$\|u_n - u\|_{L^\infty([0, M_+], L^2)} \leq \|\varphi_n - \varphi\| + CM_+ \left( J_1(2K) + J_2(2K)(1 + 2K) \right) \|u_n - u\|_{L^\infty([0, M_+], L^2)}$$

$$\leq \|\varphi_n - \varphi\| + \frac{1}{2} \|u_n - u\|_{L^\infty([0, M_+], L^2)};$$

which yields

$$\|u_n - u\|_{L^\infty([0, M_+], L^2)} \leq 2 \|\varphi_n - \varphi\|. \quad (4.3)$$
It remains to get a similar bound on \( \| \partial_x (u_n - u) \|_{L^\infty([0,M_+],L^2)} \). Using (4.2) we also get

\[
\| \partial_x (u_n - u) (t) \| \leq \| \varphi'_n - \varphi' \| + \int_0^t \| \partial_x (Q(u_n(t')) - Q(u(t'))) \| dt' \\
\leq \| \varphi'_n - \varphi' \| + \int_0^t \int_{\mathbb{R}} \| \partial_x (P(T_r u_n(t')) - P(T_r u(t'))) \| \| \psi(r) \| dr dt'.
\]

(4.4)

Note that, for any differentiable complex-valued functions \( f \) and \( g \) on \( \mathbb{R} \),

\[
\frac{d}{dx} (h(|f|) f - h(|g|) g) \\
= h(|f|) f' - h(|g|) g' + \frac{1}{2} [h'(|f|)|f'|f' - h'(|g|)|g'|g'] + \frac{1}{2} [h'(|f|)|f|^{-1} f^2 \text{Re} f' - h'(|g|)|g|^{-1} g^2 \text{Re} g'] \\
= h(|f|) (f' - g') + (h(|f|) - h(|g|)) g' + \frac{1}{2} h'(|f|)|f|(f' - g') + \frac{1}{2} (h'(|f|)|f| - h'(|g|)|g|) g' \\
+ \frac{1}{2} h'(|f|)|f|^{-1} f^2 (f' - g') + \frac{1}{2} (h'(|f|)|f|^{-1} f^2 - (h'(|g|)|g|^{-1} g^2) \text{Re} g'.
\]

Thus

\[
\left| \frac{d}{dx} [h(|f|) f - h(|g|) g] \right| \leq \left( |h(|f|)| + |h'(|f|) f| \right) |f' - g'| + |h(|f|) - h(|g|)| |g'| \\
+ |h'(|f|)|f| - h'(|g|)|g| |g'| + |h'(|f|)|f|^{-1} f^2 - h'(|g|)|g|^{-1} g^2 |g'|.
\]

We apply this in (4.4) to get

\[
\| \partial_x (u_n - u) (t) \| \\
\leq \| \varphi'_n - \varphi' \| \\
+ \int_0^t \int_{\mathbb{R}} \left| [h(|T_r u_n(t')|) + |h'(|T_r u_n(t')|) T_r u_n(t')] \partial_x (T_r u_n - T_r u)(t') \right| \| \psi(r) \| dr dt' \\
+ \int_{\mathbb{R}} \left| [h(|T_r u_n|) - h(|T_r u|)] \partial_x (T_r u) \right| \| \psi(r) \| dr \\
+ \int_{\mathbb{R}} \left| [h'(|T_r u_n|)|T_r u_n| - h'(|T_r u|)|T_r u|] \partial_x (T_r u) \right| \| \psi(r) \| dr \\
+ \int_{\mathbb{R}} \| [h'(|T_r u_n|)|T_r u_n|^{-1}(T_r u_n)^2 - h'(|T_r u|)|T_r u|^{-1}(T_r u)^2] \partial_x (T_r u) \| \| \psi(r) \| dr.
\]

(4.5)

Note

\[
\| T_r u_n \|_{L^\infty([0,M_+],L^\infty)} \leq \| T_r u_n \|_{L^\infty([0,M_+],[H^1])} = \| u_n \|_{L^\infty([0,M_+],[H^1])} \leq 2K
\]

(4.6)

for all \( n \) and \( r \in \mathbb{R} \) because of (4.1). So

\[
\| h(|T_r u_n|) + |h'(|T_r u_n|) T_r u_n \|_{L^\infty([0,M_+],L^\infty)} \leq J_1(2K) + J_2(2K)(1 + 2K),
\]

hence the first integral in (4.5) is bounded:

\[
\int_0^t \int_{\mathbb{R}} \left| [h(|T_r u_n(t')|) + |h'(|T_r u_n(t')|) T_r u_n(t')] \partial_x (T_r u_n - T_r u)(t') \right| \| \psi(r) \| dr dt' \\
\leq M_+ \left( J_1(2K) + J_2(2K)(1 + 2K) \right) \| \psi \|_{L^1} \| \partial_x (u_n - u) \|_{L^\infty([0,M_+],L^2)}.
\]

For the second integral in (4.5), use (4.6) to obtain

\[
\left| [h(|T_r u_n|) - h(|T_r u|)] \partial_x (T_r u) \right| \leq \left( J_1(|T_r u_n|) + J_1(|T_r u|) \right) |\partial_x T_r u| \leq 2J_1(2K) |\partial_x T_r u|.
\]

(4.7)
Then
\[
\left\| h([T_r u_n]) - h([T_r u]) \right\|_{L^1([0, M_+], L^2)} \lesssim \left\| \partial_x (T_r u) \right\|_{L^1([0, M_+], L^2)} = \left\| \partial_x u \right\|_{L^1([0, M_+], L^2)} \leq 2K M_+ .
\]

Thus, since \( \psi \in L^1(\mathbb{R}) \), by the dominated convergence theorem, it is enough to show
\[
\lim_{n \to \infty} \left\| h([T_r u_n]) - h([T_r u]) \right\|_{L^1([0, M_+], L^2)} = 0 (4.8)
\]
for almost every \( r \in \mathbb{R} \) to conclude that the third integral in (4.5) converges to zero as \( n \to \infty \).

Fix \( r \in \mathbb{R} \). Then
\[
\left\| [T_r u_n] - T_r u \right\|_{L^\infty([0, M_+], L^2)} = \left\| u_n - u \right\|_{L^\infty([0, M_+], L^2)} \leq 2\| \varphi_n - \varphi \|,
\]
where we used (4.3). Therefore, for almost all \((x, t) \in \mathbb{R} \times [0, M_+]\), \( T_r u_n \to T_r u \) as \( n \to \infty \). Hence \( h([T_r u_n]) \to h([T_r u]) \to 0 \) as \( n \to \infty \), since \( h \) is continuous. Thus, because of (4.7) we can use the dominated convergence theorem again to see that (4.8) holds.

This shows
\[
\lim_{n \to \infty} \int_\mathbb{R} \left\| h([T_r u_n]) - h([T_r u]) \right\|_{L^1([0, M_+], L^2)} \left| \psi(r) \right| dr = 0 .
\]
To show that the last two integrals in (4.5) converge to zero as \( n \to \infty \), note that the maps \( z \mapsto h'(|z|)z \) and \( z \mapsto h'(|z|)^2 \) extended by zero to \( z = 0 \), are continuous on the complex plane, by assumption. Moreover,
\[
\left\| h'(T_r u_n) |T_r u_n| - h'(T_r u) |T_r u| \partial_x (T_r u) \right\| \leq J_2([T_r u_n])(1 + |T_r u_n|) + J_2([T_r u])(1 + |T_r u|) \left| \partial_x T_r u \right|
\]
and
\[
\left\| [h'(T_r u_n)]^{-1} (T_r u_n)^2 - h'(T_r u) |T_r u|^{-2} (T_r u)^2 \right\| \partial_x (T_r u) \right\| \leq J_2(2K)(2 + 4K) \left| \partial_x T_r u \right|
\]
for almost all \((x, t) \in \mathbb{R} \times [0, M_+]\). Thus we can use the same argument as for the third integral in (4.5) to show that the last two integrals in (4.5) converge to zero as \( n \to \infty \).

Thus we end up with
\[
\left\| \partial_x (u_n - u) (t) \right\|_{L^\infty([0, M_+], L^2)} \lesssim \left\| \varphi_n' - \varphi' \right\| + M_+ \left\| \partial_x (u_n - u) (t) \right\|_{L^\infty([0, M_+], L^2)} + o_n(1),
\]
where \( o_n(1) \) denotes terms which go to zero in the limit \( n \to \infty \). Choosing \( M_+ \) small enough, we conclude
\[
\left\| \partial_x (u_n - u) (t) \right\|_{L^\infty([0, M_+], L^2)} \lesssim \left\| \varphi_n' - \varphi' \right\| + o_n(1). \tag{5.1}
\]

5. MASS AND ENERGY CONSERVATION

The usual approach to prove global existence from local existence on \( L^2(\mathbb{R}) \) is to show that the mass
\[
m(u(t)) = \left\| u(t) \right\|^2
\]
is conserved. This is easy when the average dispersion vanishes since then
\[
\dot{u}(t) := \partial_t u(t) = iQ(u(t)) \in L^2(\mathbb{R})
\]
for any strong solution \( u \) of (1.1). Thus \( \left\| u(t) \right\|^2 = \langle u(t), u(t) \rangle \) is differentiable in \( t \) and
\[
\frac{d}{dt} \left\| u(t) \right\|^2 = 2\Re \langle u, \dot{u} \rangle = 2\Re(i \langle u, Q(u) \rangle) = 0 \tag{5.2}
\]
Thus \( \|u(t)\|^2 \) is constant, i.e., the mass is conserved. The last equality in (5.2) follows from

\[
\langle u, Q(u) \rangle = \int_{\mathbb{R}} \langle u, T_r^{-1}(P(u)) \rangle \psi(r) dr = \int_{\mathbb{R}} \langle T_r u, P(u) \rangle \psi(r) dr = \int_{\mathbb{R}} \int_{\mathbb{R}^2} h(|T_r u|)|T_r u|^2 dx \psi(r) dr
\]

(5.3)

which shows that \( \langle u, Q(u) \rangle \) is real.

The conservation of mass when \( d_{av} \neq 0 \) is a little bit trickier: In order to calculate the derivative of the mass one would like to argue that

\[
\frac{d}{dt} \|u(t)\|^2 = 2 \text{Re} \langle u, \dot{u} \rangle = 2 \text{Re} \left( i \langle u, d_{av} \partial_x^2 u \rangle + i \langle u, Q(u) \rangle \right) = 2 \text{Re} \left( -i d_{av} \langle \partial_x u, \partial_x u \rangle + i \langle u, Q(u) \rangle \right) = 0
\]

since both \( \langle \partial_x u, \partial_x u \rangle \) and \( \langle u, Q(u) \rangle \) are real. This argument misses, however, that \( \partial_t u \in H^{-1}(\mathbb{R}) \), so \( \langle u, \partial_t u \rangle \) is not defined.

While the above argument can be saved using that \( u \in H^1(\mathbb{R}) \), so the pairing of \( u \) and \( \partial_x^2 u \) is well–defined, the problem is much more pronounced, when one tries to prove conservation of the energy

\[
E(u(t)) = \frac{d_{av}}{2} \|\partial_x u(t)\|^2 - \int_{\mathbb{R}^2} V(|T_r u(t)|) dx \psi(r) dr
\]

(5.4)

as a first step in order to from local existence to global existence. Here, \( V \) is the antiderivative of the nonlinearity \( P \) with \( V(0) = 0 \), i.e., \( V(a) = \int_0^a P(s) ds \) for \( a \in \mathbb{R}_+ \).

In this case, the derivative of the kinetic energy of \( u \) is not well–defined since, informally

\[
\frac{d}{dt} \|\partial_x u(t)\|^2 = 2 \text{Re} \langle \partial_x u, \partial_x \partial_t u \rangle = 2 \text{Re} \left( i d_{av} \langle \partial_x u, \partial_x \partial_x^2 u \rangle + i \langle \partial_x u, \partial_x Q(u) \rangle \right).
\]

However, since \( \partial_x u \in L^2(\mathbb{R}) \) and \( \partial_x^3 u \in H^{-2}(\mathbb{R}) \), the scalar product \( \langle \partial_x u, \partial_x \partial_x^2 u \rangle \) is not defined anymore.

In order to circumvent this problem, one usually approximates the solution \( u \) by smooth ones and uses an approximation argument. Following this route, one has to study solutions of (1.1) for initial condition in Sobolev spaces \( H^s(\mathbb{R}) \) with high enough regularity \( s > 1 \). This poses additional conditions on the nonlinearity, in particular, high enough differentiability of \( h \), which we need to avoid. To circumvent this problem we will use the twisting trick from [4, 27], which goes back to Dirac’s interaction picture in quantum mechanics.

As a warm up, we use the twisting trick to give a simple proof of mass conservation.

**Proposition 5.1 (Mass conservation).** Any solution \( u \in C([-M_-, M_+], H^1) \) for \( d_{av} \neq 0 \), or \( u \in C([-M_-, M_+], L^2) \) for \( d_{av} = 0 \), of the integral equation (1.8) has conserved mass,

\[
\|u(t)\|^2 = \|u_0\|^2 \quad \text{for all} \quad t \in [-M_-, M_+].
\]

(5.5)

**Proof.** In order to rigorously show conservation of mass and energy when \( d_{av} \neq 0 \), we twist the solution \( u \). In physics this is known as Dyson’s interacting picture. Given \( u \) let \( v(t) := e^{-itd_{av}\partial_x^2} u(t) \). Then since \( u \) solves (1.8), \( v \) solves

\[
v(t) = u_0 + i \int_0^t e^{-it'd_{av}\partial_x^2} \mathcal{Q}(u(t')) dt'.
\]

(5.6)

Under the assumptions on the nonlinearity, \( Q \) maps \( L^2(\mathbb{R}) \) boundedly into \( L^2(\mathbb{R}) \) for \( d_{av} = 0 \), respectively, \( H^1(\mathbb{R}) \) boundedly into \( H^1(\mathbb{R}) \) when \( d_{av} \neq 0 \). Thus (5.6) shows that \( v \) is differentiable with respect to \( t \) and

\[
\dot{v}(t) = \partial_t v(t) = ie^{-itd_{av}\partial_x^2} \mathcal{Q}(u(t))
\]

(5.7)
where we used that in\(^2\) is unitary on \(L^2(\mathbb{R})\), we have \(\|u(t)\| = \|v(t)\|\) for all \(t\), hence
\[
\frac{d}{dt}\|u(t)\|^2 = \frac{d}{dt}\|v(t)\|^2 = 2\text{Re}\langle v(t), \dot{v}(t)\rangle = 2\text{Re}\langle v(t), ie^{-itd_{av}\partial_x^2}Q(u(t))\rangle
\]
\[
= 2\text{Re}(i(e^{itd_{av}\partial_x^2}v(t), Q(u(t)))) = 2\text{Re}(i\langle u(t), Q(u(t))\rangle) = 0
\]
since (5.3) shows that \(\langle u(t), Q(u(t))\rangle\) is real. Hence the \(L^2\) norm of the strong solution \(u\) is constant in \(t\).

In the following we abbreviate the nonlinear energy by
\[
N(f) = \iint_{\mathbb{R}^2} V(|T_r f|) dx\psi(r)dr.
\]
(5.8)
Then the energy of \(u\) is given by
\[
E(u(t)) = \frac{d_{av}}{2}\|\partial_x u(t)\|^2 - N(u(t)).
\]
Proposition 5.2 (Energy conservation, \(d_{av} \neq 0\)). Any solution \(u \in \mathcal{C}([-M_-, M_+], H^1)\) of the integral equation (1.8) has conserved energy,
\[
E(u(t)) = E(u_0) \quad \text{for all } t \in [-M_-, M_+].
\]
(5.9)
Proof. We use again the twisted solution \(v(t) = e^{-itd_{av}\partial_x^2}u(t)\). Since \(e^{-itd_{av}\partial_x^2}\) commutes with \(\partial_x\), we have
\[
E(u(t)) = \frac{d_{av}}{2}\|\partial_x u(t)\|^2 - N(u(t)) = \frac{d_{av}}{2}\|\partial_x v(t)\|^2 - N(u(t)).
\]
Using again \(\dot{v}(t) = \partial_t v(t) = ie^{-itd_{av}\partial_x^2}Q(u(t))\), one sees that the first term is differentiable in \(t\) with
\[
\frac{d}{dt}\|\partial_x v(t)\|^2 = 2\text{Re}\langle \partial_x v(t), \partial_x \dot{v}(t)\rangle = 2\text{Re}\langle \partial_x v(t), i\partial_x e^{-itd_{av}\partial_x^2}Q(u(t))\rangle
\]
\[
= -2\text{Im}(\partial_x u(t), \partial_x Q(u(t))).
\]
(5.10)
To compute the derivative of the second term, let \(w \in \mathcal{C}^1([-M_-, M_+], H^1)\) and consider \(N(w(t))\). The chain rule yields
\[
\partial_t N(w(t)) = DN(w(t))[\dot{w}(t)] = \iint_{\mathbb{R}^2} V'(|T_r w(t)|)\text{Re}\left(\frac{T_r w(t)}{|T_r w(t)|} T_r \dot{w}(t)\right) dx\psi(r)dr
\]
\[
= \iint_{\mathbb{R}^2} P(|T_r w(t)|)\text{Re}\left(\frac{T_r w(t)}{|T_r w(t)|} T_r \dot{w}(t)\right) dx\psi(r)dr = \text{Re}\langle \dot{w}(t), Q(w(t))\rangle
\]
\[
= \text{Re}\langle (1 - \partial_x)^{-1} \dot{w}(t), (1 + \partial_x)Q(w(t))\rangle.
\]
(5.11)
where we used that \(\partial_x\) is skew adjoint, so \(1 - \partial_x : H^1(\mathbb{R}) \to L^2(\mathbb{R})\) is invertible.

The right hand side of (5.11) extends to \(w \in \mathcal{C}([-M_-, M_+], H^1) \cap \mathcal{C}^1([-M_-, M_+], H^{-1})\), by the usual density arguments: If \(\dot{w}(t) \in H^{-1}(\mathbb{R})\), then \((1 - \partial_x)^{-1} \dot{w}(t) \in L^2(\mathbb{R})\) and \(Q(w(t)) \in H^1(\mathbb{R})\), so \((1 + \partial_x)Q(w(t)) \in L^2(\mathbb{R})\). This shows that \(N(w(t))\) is differentiable in \(t\) with derivative given by the last line of (5.11) for any \(w \in \mathcal{C}([-M_-, M_+], H^1) \cap \mathcal{C}^1([-M_-, M_+], H^{-1})\).

Any solution \(u \in \mathcal{C}([-M_-, M_+], H^1)\) of the integral equation (1.8) has derivative
\[
\dot{u}(t) = \partial_t u(t) = id_{av}\partial_x^2 u(t) + iQ(u(t)) \in H^{-1}(\mathbb{R})
\]
and the right hand side above is continuous in \( t \) with values in \( H^{-1}(\mathbb{R}) \). So (5.11) applies to \( u \) and shows that for any solution \( u \in C([-M_-, M_+], H^1) \) of (1.8), \( N(u(t)) \) is differentiable in \( t \) with derivative
\[
\partial_t N(u(t)) = \text{Re} \langle (1 - \partial_x)^{-1} \dot{u}(t), (1 + \partial_x)Q(u(t)) \rangle \\
= \text{Re} \langle (1 - \partial_x)^{-1}(d_{av}\partial_x^2 u(t) + Q(u(t)), (1 + \partial_x)Q(u(t)) \rangle \\
= \text{Im} \langle (1 - \partial_x)^{-1}(d_{av}\partial_x^2 u(t) + Q(u(t)), (1 + \partial_x)Q(u(t)) \rangle.
\]

Note that
\[
\langle (1 - \partial_x)^{-1}Q(u(t)), (1 + \partial_x)Q(u(t)) \rangle = \langle Q(u(t)), Q(u(t)) \rangle \in \mathbb{R}
\]
and, since \(-\partial_x(1 + \partial_x)^{-1} \) is the adjoint of \((1 - \partial_x)^{-1}\partial_x \) and bounded on \( L^2(\mathbb{R}) \),
\[
\langle (1 - \partial_x)^{-1}\partial_x^2 u(t), (1 + \partial_x)Q(u(t)) \rangle = \langle \partial_x u(t), -\partial_x(1 + \partial_x)^{-1}(1 + \partial_x)Q(u(t)) \rangle \\
= -\langle \partial_x u(t), \partial_x Q(u(t)) \rangle.
\]
Thus
\[
\partial_t N(u(t)) = -d_{av}\text{Im}\langle \partial_x u(t), \partial_x Q(u(t)) \rangle,
\]
which together with (5.10) proves that the energy \( E(u(t)) \) is differentiable and
\[
\frac{d}{dt} E(u(t)) = 0
\]
for all \( t \in [-M_-, M_+] \). Hence the energy is constant.

\[\]  

Remark 5.3. In the case of zero average dispersion, which is a kind of singular limit, the energy is given by \( E(f) = -N(f) \) with \( N \) defined in (5.8). In this case, any strong solution of the dispersion management equation (1.1), or of (1.8), satisfies energy conservation. This is easy to see: In this case \( \dot{u}(t) = \partial_t u(t) = iQ(u(t)) \in L^2 \) and a calculation leading to (5.11), but now simpler, shows
\[
\partial_t N(u(t)) = \text{Re} \langle \dot{u}(t), Q(u(t)) \rangle = \text{Im} \langle Q(u(t)), Q(u(t)) \rangle = 0.
\]

6. Global existence

In this section, we finish the proof of global well–posedness of the dispersion managed NLS (1.1). We only have to show that solutions exist globally, since the local existence and uniqueness results tother with the continuous dependence on the data proved in Section 4 apply to all times for which the solution exists. For \( d_{av} = 0 \), assume that \( h \) satisfies assumption A1 and \( \psi \in L^1(\mathbb{R}) \cap L^{\frac{4}{p}}(\mathbb{R}) \), then it follows from the mass conservation that local solutions for (1.8) obtained in Proposition 3.1 are bounded in \( L^2(\mathbb{R}) \). Hence the blow–up alternative from Corollary 3.2 shows that the solution is global in \( t \). This finishes the proof of Theorem 1.2.

Proposition 6.1. Let \( d_{av} \neq 0 \), \( h \) satisfy assumptions A2, A3, and \( \psi \in L^1(\mathbb{R}) \cap L^{\frac{4}{p}}(\mathbb{R}) \). Then the Cauchy problem (1.1) has a unique global strong solution \( u \) in \( C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1}) \) for any initial datum \( u_0 \in H^1(\mathbb{R}) \).

Remark 6.2. In particular, for negative average dispersion we have a global existence result for nonlinearities for which \( h \) satisfies assumption A2 and is bounded from below. In applications, the polarization \( P(a) = h(a)a \), \( a \geq 0 \), is usually non–negative, so the requirement that \( h \) is bounded from below is a rather weak additional condition on the nonlinearity.
If $h$ fulfills a growth condition of the form

$$|h(a)| \lesssim 1 + a^\beta \quad \text{for } a > 0$$

then it is easy to see that the condition A3 is fulfilled when $0 \leq \beta < 8$. Other growth conditions such as

$$|h(a)| \lesssim 1 + a^8(\ln(2 + a))^{-1} \quad \text{for } a > 0$$

also yield global existence.

**Proof.** Since the mass is conserved by Proposition 5.1 it is enough to bound $\|\partial_x u(t)\|$ in order to control the $H^1$ norm of the solution $u$. In order to be able to use energy conservation in Proposition 5.2, we need to control the nonlinearity in a first step.

Recall that the nonlocal nonlinearity is given by

$$N(f) = \int_{\mathbb{R}^2} V(|T_r f|) dx \psi(r) dr,$$

where $V(a) = \int_0^a P(s) ds = \int_0^a h(s) s ds$ for all $a > 0$. Assume $h(a) \leq \tilde{J}(a)(1 + a^\alpha)$, then

$$V(a) \lesssim \tilde{J}(a)(a^2 + a^{p+2}) \quad \text{for all } a \geq 0,$$

and

$$N(f) \lesssim \tilde{J}((\|f\|\|f\|)^{1/2}) \int_{\mathbb{R}^2} (|T_r f|^2 + |T_r f|^{p+2}) dx \psi(r) dr$$

since $\tilde{J}$ is increasing and $|T_r f| \leq (\|\partial_x T_r f\|\|T_r f\|)^{1/2} = (\|\partial_x f\|\|f\|)^{1/2}$. Also, since $T_r$ is unitary on $L^2(\mathbb{R})$ we have $\int_{\mathbb{R}^2}|T_r f|^2 dx \psi(r) dr = \|f\|^2\|\psi\|_{L^1}$. For the last term we use Lemma 2.2 to obtain

$$\int_{\mathbb{R}^2}|T_r f|^{p+2} dx \psi(r) dr \lesssim \|f\|^{p+2}\|\psi\|_{L^1/(1-p)}.$$  

Thus

$$N(f) \lesssim \tilde{J}((\|f\|\|f\|)^{1/2}) (\|f\|^2 + \|f\|^{p+2}). \quad (6.1)$$

In case that $h(a) \geq -\tilde{J}(a)(1 + a^\alpha)$, we get similarly

$$N(f) \gtrsim -\tilde{J}((\|f\|\|f\|)^{1/2}) (\|f\|^2 + \|f\|^{p+2}). \quad (6.2)$$

Let $d_{av} \neq 0$, then Corollary 3.7 tells us that there exist $T_+ > 0$ depending only on $\|u_0\|_{H^1}$ and the $L^1$ norm of $\psi$ such that a unique solution $u$ for (1.8) exists in $C((-T_-, T_+), H^1)$ with initial data $u_0 \in H^1(\mathbb{R})$. Moreover, if $T_+ < \infty$, the $H^1$ norm of the solution must blow up as $t \to T_+$ and similarly for $T_-.$

The energy conservation (5.9) shows

$$\|\partial_x u(t)\|^2 = \frac{2E(u_0)}{d_{av}} + \frac{2N(u(t))}{d_{av}} \leq \frac{2E(u_0)}{d_{av}} + C\tilde{J}(\|\partial_x u(t)\||u_0|)^{1/2})(\|u_0\|^2 + \|u_0\|^{p+2})$$

for some finite positive constant $C$, due to (6.1) when $d_{av} > 0$, respectively (6.2) when $d_{av} < 0$, and $\|u(t)\| = \|u_0\|$ by conservation of mass (5.5).

The bound (6.3) is clearly equivalent to

$$\|\partial_x u(t)\|^2 \left(1 - C\tilde{J}(\|\partial_x u(t)\||u_0|)^{1/2})(\|u_0\|^2 + \|u_0\|^{p+2})\right) \lesssim 1$$
for all $t \in (-T_-, T_+)$ for some maybe different constant $C$. Due to the growth condition (1.4) on $J$ this shows that $\|\partial_x u(t)\|$ cannot blow up as $t \to T_+$ or $t \to -T_-$. By mass conservation, this shows that the $H^1$ norm of the solution does not blow up. Hence the blow–up alternative from Corollary 3.7 implies that the solution exists globally. □

**Remark 6.3.** The above proof shows that under assumption A3 we have for $d_{av} > 0$

$$E(f) \geq \frac{d_{av}}{2} \|f'\|^2 - C\tilde{J}\left(\left\|\|f'\|\|f\|\right\|^{1/2}\right)\left(\|f\|^2 + \|f\|^{p+2}\right).$$

Thus the energy is coercive: for any sequence $f_n \in H^1(\mathbb{R})$ with $\|f_n\|$ bounded and $\|f'_n\| \to \infty$ as $n \to \infty$ one has

$$\lim_{n \to \infty} E(f_n) = \infty. \quad (6.4)$$

This will be useful for the proof of orbital stability in Section 7.

Now we come to the proof of Theorem 1.6, which for the convenience of the reader, we recall.

**Proposition 6.4 (= Theorem 1.6).** Let $d_{av} \neq 0$ and $h$ satisfy assumption A2.

(i) For any initial datum $u_0 \in H^1(\mathbb{R})$ with small enough $H^1$ norm, the Cauchy problem (1.1) is globally well–posed when $\psi \in L^1(\mathbb{R})$.

(ii) If $J_1(a) \leq 1 + a^8$ for $a \geq 0$, then the Cauchy problem (1.1) is globally well–posed for initial conditions $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|$ small enough when $\psi \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$.

(iii) If $\lim_{a \to 0} J_1(a)/a^4 = 0$ then the Cauchy problem (1.1) is globally well–posed for initial conditions $u_0 \in H^1(\mathbb{R})$ with $\|u'_0\|$ small enough (depending on $\|u_0\|$) when $\psi \in L^1(\mathbb{R})$.

**Proof.** Since we proved local well–posedness in Sections 3 and 4, we only have to prove global existence of solutions.

Since $V(a) = \int_0^a h(s) s \, ds \leq J_1(a) a^2/2$ we can argue similarly to the derivation of (6.1) to see that

$$|N(f)| \leq \frac{1}{2} J_1\left(\left\|\|f'\|\|f\|\right\|^{1/2}\right) \int_{\mathbb{R}} |T_r f|^2 \, dx \psi(r) \, dr = \frac{1}{2} J_1\left(\left\|\|f'\|\|f\|\right\|^{1/2}\right) \|f\|^2 \|\psi\|_{L^1}. \quad (6.5)$$

Thus energy and mass conservation again yields

$$\left|\frac{2E(u_0)}{d_{av}}\right| \geq \frac{2E(u_0)}{d_{av}} = \|\partial_x u(t)\|^2 - \frac{2N(u(t))}{d_{av}} \geq \|\partial_x u(t)\|^2 - \frac{2|N(u(t))|}{d_{av}} \geq \|\partial_x u(t)\|^2 - \frac{d_{av}}{\alpha} J_1\left(\left\|\|f'\|\|f\|\right\|^{1/2}\right) \|u_0\|^2 \|\psi\|_{L^1}. \quad (6.6)$$

Given $\alpha, s \geq 0$, let $G_\alpha(s) = s^2 - |d_{av}|^{-1} \|\psi\|_{L^1} J_1(\alpha s)^2 \alpha^2$. Then the above shows

$$G_{\|u_0\|}(\|\partial_x u(t)\|) \leq \frac{2E(u_0)}{d_{av}}. \quad (6.6)$$

for all $t$ for which the solution $u$ exists.

We will show shortly that the bound (6.6) forces $\|\partial_x u(t)\|$ to stay bounded when the $H^1$ norm of the initial condition $u_0$ is small enough. Together with the blow–up alternative from Corollary 3.7 this shows that the solution is global.

Note that $G_\alpha(s)$ is decreasing in $\alpha \geq 0$ for fixed $s \geq 0$. In addition, when $\alpha_0 > 0$ is small enough, we have

$$c_{\alpha_0} = \inf_{0 \leq \alpha \leq \alpha_0} \sup_{s \geq 0} G_\alpha(s) > 0. \quad (6.7)$$

If $0 < c < c_{\alpha_0}$ then there exist $0 < a < b$ such that $G_\alpha(s) > c$ for all $a < s < b$ and all $0 \leq \alpha \leq \alpha_0$. Thus $\alpha_0 < 0$ small enough, $0 \leq \alpha \leq \alpha_0$, and $G_\alpha(s) \leq c$ implies $0 \leq s \leq a$ or
s ≥ b. Hence if the initial condition \( u_0 \) is such that \( \| u_0 \| \leq \alpha_0 \) and \( 2E(u_0)/d_{av} \leq c \), then (6.6) implies \( \| \partial_x u(t) \| \leq a \) or \( \| \partial_x u(t) \| \geq b \) for all times for which the solution exists.

Due to (6.5), we can make \( |2E(u_0)/d_{av}| \leq c \) by choosing \( \| u_0 \| \) and \( \| u_0' \| \) small enough. Making them even smaller, if necessary, we can also assume that \( \| u_0 \| \leq \alpha_0 \) and \( \| u'(0) \| \leq a \). Since either \( \| \partial_x u(t) \| \leq a \) or \( \| \partial_x u(t) \| \geq b \) and \( t \mapsto \| \partial_x u(t) \| \) is continuous, this shows that \( \| \partial_x u(t) \| \leq a \) for all times for which the solution exists. This finishes the proof of the first part of the proposition.

For the second part, we note that using Lemma 2.3 with \( \kappa = 4 \) and \( q = 10 \) we have

\[
|N(f)| \lesssim \iint_{\mathbb{R}^2} (\|T_r f\|^2 + \|T_r f\|^{10})\, dx \psi(r)\, dr \lesssim \|f\|^2 + \|f'\|^2\|f\|^8.
\]

Together with energy and mass conservation this implies

\[
\| \partial_x u(t) \|^2 (1 - C\|u_0\|^8) \leq \frac{2E(u_0)}{d_{av}} + C\|u_0\|^2
\]

similarly as for (6.6). Hence, as soon as \( \| u_0 \| \) is small enough, the kinetic energy \( \| \partial_x u(t) \| \) stays bounded, so the blow–up alternative from Corollary 3.7 applies again.

For the proof of the third part, we note that the assumption \( \lim_{a \to 0} J_1(a)/a^4 = 0 \) implies that the constant \( c_{aa} \) given by (6.7) is now positive for all \( \alpha_0 > 0 \). Moreover, for fixed \( L^2 \) norm of the initial condition \( u_0 \) the bound (6.5) together with \( \lim_{a \to 0} J_1(a) = 0 \) shows that we can make its energy \( E(u_0) \) as small as we like by having \( \| u_0' \| \) small.

Thus, with some small and straightforward modifications, the proof of the first case now shows that \( \| \partial_x u(t) \| \) stays bounded for all times for which the solution \( u \) exists, as long as \( \| u'_0 \| \) is small enough, depending only on how large \( \| u_0 \| \) is. Together with energy and mass conservation the blow–up alternative from Corollary 3.7 again shows that the solution \( u \) is global.

Once one knows global existence, a natural next step is a more detailed investigation of the long time behavior of the solutions, such as scattering or the stability of solitary solutions under small perturbations. Scattering results are easier to prove in higher dimensions, since the wave has more directions at its disposal in order to move to infinity. It can disperse more easily in higher dimensions than in one dimension, where it can only move to the left or right and the dispersive effects are the weakest. Modified scattering for one dimensional dispersion managed NLS with a cubic nonlinearity was shown for small, well–localized initial data in [25], see the review paper, [24], for NLS in one dimension. In the next section we show orbital stability for the focussing case, which in our notation means \( d_{av} > 0 \), and also vanishing average dispersion \( d_{av} = 0 \), including saturating nonlinearities for both cases.

7. Orbital stability for (non–)saturated nonlinearities

In this section, we prove Theorem 1.9. We consider only non–negative average dispersion, \( d_{av} \geq 0 \), since the set of ground states is ill-defined when \( d_{av} < 0 \). Recall the set of ground states

\[
S^{d_{av}}_{\lambda} = \{ f \in X : E(f) = E^{d_{av}}_{\lambda}, \| f \|^2 = \lambda \}
\]

for each \( \lambda > 0 \) and \( d_{av} \geq 0 \) and

\[
E^{d_{av}}_{\lambda} = \inf \{ E(f) = \frac{d_{av}}{2}\| f' \|^2 - N(f) : f \in X, \| f \|^2 = \lambda \}.
\]
Here, $X = H^1(\mathbb{R})$ for $d_{av} > 0$ and $X = L^2(\mathbb{R})$ for $d_{av} = 0$.  
Recall that the nonlocal nonlinearity functional is given by 

$$N(f) = \int_{\mathbb{R}^2} V(|T_{r} f|) dx \psi(r) dr,$$

where $V(a) = \int_0^a P(s) ds$, for $a \geq 0$, the antiderivative of $P$, and the nonlinearity $P$ is given by $P(z) = h(|z|)z$ for $z \in \mathbb{C}$.

Recall also the additional assumptions for the orbital stability of $S^{d_{av}}_\lambda$:

**A4)** There exists $p_0 > 0$ with

$$h(a)a^2 \geq p_0 \int_0^a h(s)s ds \text{ for all } a > 0,$$

(7.1)

**A5)** There exists a continuous decreasing function $p : [0, \infty) \to (2, \infty)$ such that

$$h(a)a^2 \geq p(a) \int_0^a h(s)s ds \text{ for all } a > 0.$$

(7.2)

**A6)** There exists $a_0 > 0$ with $\int_0^{a_0} h(s)s ds > 0$.

Our first result concerns the question whether $S^{d_{av}}_\lambda$ is empty or not.

**Theorem 7.1** (Existence of thresholds for $S^{d_{av}}_\lambda \neq \emptyset$). Suppose that the nonlinearity $h$ satisfies assumption **A6** and either of the following:

(i) **Zero average dispersion, non–saturated nonlinearity**: The nonlinearity $h$ satisfies assumption **A4** and the bound $|h(a)| \lesssim a^{p_1} + a^{p_2}$ for some $0 < p_1 \leq p_2 < 4$. The density $\psi \in L^{\frac{4}{3-p_2}}(\mathbb{R})$ has compact support.

(ii) **Zero average dispersion, saturated nonlinearity**: The nonlinearity $h$ satisfies assumption **A5** and the bound $|h(a)| \lesssim a^{p_1} + a^{p_2}$ for some $1 \leq p_1 \leq p_2 < 3$. The density $\psi \in L^{\frac{4}{3-p_2}}(\mathbb{R})$ has compact support.

(iii) **Positive average dispersion, saturated and non–saturated nonlinearities**: The nonlinearity $h$ satisfies the bound $|h(a)| \lesssim a^{p_1} + a^{p_2}$ for some $0 < p_1 \leq p_2 < 8$. The density $\psi \in L^{\frac{4}{3-p_2}}(\mathbb{R})$ has compact support. Moreover, $h$ satisfies either assumption **A4** or assumption **A5**.

Then there exists a critical threshold $0 \leq \lambda_{cr} < \infty$ such that if $\lambda > \lambda_{cr}^{d_{av}}$ then $E^{d_{av}}_\lambda < 0$ and the set $S^{d_{av}}_\lambda$ is not empty. Moreover, if there exists $\varepsilon > 0$ such that $h(a) > 0$ for $0 < a \leq \varepsilon$ when $d_{av} = 0$ or that $h(a) \gtrsim a^q$ for $0 < q < 4$ when $d_{av} > 0$, then $\lambda_{cr}^{d_{av}} = 0$.

If $\lambda_{cr}^{d_{av}} > 0$ and $d_{av} > 0$, then $S^{d_{av}}_\lambda = \emptyset$ for all $0 < \lambda < \lambda_{cr}^{d_{av}}$.

**Proof.** These results can be found for non–saturating nonlinearities in [9] and for saturating nonlinearities in [20]. If the average dispersion is zero and the nonlinearity is saturating, the condition $1 \leq p < 3$ is needed to guarantee that $S^0_\lambda$ is not empty, at least for large enough $\lambda$. $lacksquare$

**Remarks 7.2.** (i) The requirement that $\psi$ has compact support is very natural from the point of view of applications for dispersion managed NLS, see Section 1.2.

(ii) The condition on $V(a) = \int_0^a h(s)s ds$ used in [9, 20] is $|V'(a)| \lesssim a^{\gamma_1-1} + a^{\gamma_2-1}$ for some $2 \leq \gamma_1 \leq \gamma_2 < 10$ for $d_{av} > 0$, and $2 < \gamma_1 \leq \gamma_2 < 6$ for $d_{av} = 0$ in [9]. The conditions in Theorem 7.1 are a bit more general than the assumptions used in [9, 20]. However, the proofs carry over to our more general situation: The main tools for the proofs in [9, 20] that $S^{d_{av}}_\lambda$ is not empty are tightness results, modulo translations, for energy minimizing sequences, see [9, Proposition 4.4 and 4.6]. These bounds follow from the
strict subadditivity of the energy and the splitting bounds for the nonlocal nonlinearity in [9, Section 2.2]. This strict subadditivity is shown in [9] under assumption A4 and in [20] under assumption A5. The splitting bounds relied on the pointwise bounds for $V$ from [9, Lemma 2.14]. Under the conditions of Theorem 7.1 suitable replacements, which are sufficient for us, still holds. For example, we have

$$|V(|z + w|) - V(|z|) - V(|w|)| \leq 4J_1(|z| + |w|)|z||w|$$

(7.3)

for all $z, w \in \mathbb{C}$, which is a suitable replacement for the bound in equation (2.17) from [9, Lemma 2.14].

To prove (7.3) just argue as in the proof of [9, Lemma 2.14] using now

$$|V(|z + w|) - V(|z|)| = \left| \int_{|z|}^{|z+w|} h(s)s \, ds \right| \leq J_1(|z| + |w|)(|z| + |w|)|w|.$$ 

To prove Theorem 1.9, we need one more result which is the continuity of the nonlinear functional $N$ similar to [9, Lemma 4.7].

**Lemma 7.3.** (i) Assume that $h$ satisfies $|h(a)| \lesssim 1 + a^p$ for all $a \geq 0$ and some $0 \leq p \leq 4$ and $\psi \geq 0$ in $L^1(\mathbb{R}) \cap L^{\frac{4}{4-p}}(\mathbb{R})$. Then the nonlinear nonlocal functional $N : L^2(\mathbb{R}) \to \mathbb{R}$ given by

$$L^2(\mathbb{R}) \ni f \mapsto N(f) = \iint_{\mathbb{R}^2} V(|T_r f|) dx \psi(r) \, dr$$

is locally Lipshitz continuous on $L^2(\mathbb{R})$ in the sense that

$$|N(f_1) - N(f_2)| \lesssim (\|f_1\| + \|f_2\| + \|f_1\|^{p+1} + \|f_2\|^{p+1}) \|f_1 - f_2\|,$$

(7.4)

where the implicit constant depends only on $p$ and the $L^1, L^{\frac{4}{4-p}}$ norms of $\psi$.

(ii) Assume that $h$ satisfies $|h(a)| \lesssim J_1(a)$ for all $a \geq 0$ and some increasing function $J_1 \geq 0$ and $\psi \geq 0$ in $L^1(\mathbb{R})$. Then the nonlinear nonlocal functional $N : H^1(\mathbb{R}) \to \mathbb{R}$ given by

$$H^1(\mathbb{R}) \ni f \mapsto N(f) = \iint_{\mathbb{R}^2} V(|T_r f|) dx \psi(r) \, dr$$

is locally Lipschitz continuous in the sense that

$$|N(f_1) - N(f_2)| \lesssim J_1(\|f_1\|_{H^1} \vee \|f_2\|_{H^1})(\|f_1\| + \|f_2\|) \|f_1 - f_2\|,$$

(7.5)

where the implicit constant depends only on the $L^1$ norm of $\psi$.

**Remark 7.4.** Note that the second part of Lemma 7.3 shows a somewhat surprising result: while the Lipschitz constant of $N$ on $H^1(\mathbb{R})$ clearly depends on the $H^1$ norm, the difference $N(f_1) - N(f_2)$ is small whenever $f_1$ is close to $f_2$ in the weaker $L^2$ norm as soon as $f_1$ and $f_2$ are bounded in $H^1(\mathbb{R})$.

**Proof.** Recall the notation $a \vee b = \max(a, b)$, we also use $a \wedge b = \min(a, b)$ in the following. To prove the first part recall $V'(a) = \int_a^\infty P(s) \, ds = \int_a^\infty h(s)s \, ds$. Thus

$$|V(|z|) - V(|w|)| \leq \int_{|z|}^{\infty} h(s)s \, ds \leq \int_{|z|}^{\infty} (1 + s^p)s \, ds$$

$$\lesssim (|z| + |w| + |z|^{p+1} + |w|^{p+1}) |z - w|$$

for all $z, w \in \mathbb{C}$. Thus,

$$|N(f_1) - N(f_2)| \leq \iint_{\mathbb{R}^2} |V(|T_r f_1|) - V(|T_r f_2|)| dx \psi(r) \, dr$$
which proves (for all $n$),

\[
\sup_{r \in \mathbb{R}} \left\| \int_{\mathbb{R}^2} (|T_r f_1| + |T_r f_2| + |T_r f_1|^{p+1} + |T_r f_2|^{p+1}) |T_r f_1 - f_2| dx \psi(r) dr \right\|_L^1 \leq \left( \left\| f_1 \right\| + \left\| f_2 \right\| \right) \left\| f_1 - f_2 \right\| \left\| \psi \right\|_{L^1}.
\]

Substituting the last two bounds in (7.6) proves (7.4).

To prove the second part note that now

\[
|V(|z|) - V(|w|)| \leq \int_{|w|}^{\max(|z|, |w|)} h(s) ds \leq J_1(|z|) \left| z - w \right| \leq J_1(|z|) \left| z - w \right|
\]

and using this in (7.6) together with the Cauchy–Schwartz inequality yields

\[
\left| N(f_1) - N(f_2) \right| \leq \sup_{r \in \mathbb{R}} J_1(\left| T_r f_1 \right|_{L^\infty} \vee \left| T_r f_2 \right|_{L^\infty}) \int_{\mathbb{R}^2} (|T_r f_1| + |T_r f_2|) |T_r f_1 - T_r f_2| dx \psi(r) dr \leq J_1(\left| f_1 \right|_{H^1} \vee \left| f_2 \right|_{H^1}) \left( \left| f_1 \right| + \left| f_2 \right\| \right) \left| f_1 - f_2 \right| \left\| \psi \right\|_{L^1},
\]

which proves (7.5).

**Proof of Theorem 1.9.** We show the stability of the set of ground states adapting a proof from [17], see also [8]. We will first prove the positive average dispersion case.

Arguing by contradiction, assume that $S^d_{\chi}$ is not stable. Then there exist $\varepsilon_0 > 0$, a sequence $(\phi_n)_n$ in $H^1(\mathbb{R})$ with

\[
d(\phi_n, S^d_{\chi}) := \inf_{f \in S^d_{\chi}} \left\| \phi_n - f \right\|_{H^1} < \frac{\varepsilon_0}{n} \quad \text{for } n \in \mathbb{N}, \tag{7.7}
\]

and a sequence $(t_n)_n$ of times such that

\[
d(u_n(\cdot, t_n), S^d_{\chi}) \geq \varepsilon_0 \tag{7.8}
\]

for all $n$, where $u_n$ are solutions of (1.1) with the initial data $\phi_n$.

We can then choose a sequence $(f_n)_n \in S^d_{\chi}$ such that $\left\| \phi_n - f_n \right\|_{H^1} < \frac{\varepsilon_0}{n}$ for all $n \in \mathbb{N}$. Since $\|f_n\|^2 = \lambda$ and $E(f_n) = E^d_{\chi}$, the coercivity of the energy (6.4) shows that $\|f_n\|$ is bounded, hence $(\phi_n)_n$ is a bounded sequence in $H^1(\mathbb{R})$. In addition,

\[
\left| \left\| \phi_n \right\| - \lambda^{1/2} \right| = \left| \left\| \phi_n \right\| - \left\| f_n \right\| \right| \leq \left\| \phi_n - f_n \right\|_{H^1} \to 0 \quad \text{as } n \to \infty,
\]

Using the Cauchy–Schwarz inequality in the $x$-integral and $T_r$ being unitary on $L^2$ for all $r \in \mathbb{R}$ one has

\[
\int_{\mathbb{R}^2} (|T_r f_1| + |T_r f_2|) |T_r (f_1 - f_2)| dx \psi(r) dr \leq \int_{\mathbb{R}} (\left\| T_r f_1 \right\| + \left\| T_r f_2 \right\|) \left\| T_r (f_1 - f_2) \right\| \psi(r) dr = (\left\| f_1 \right\| + \left\| f_2 \right\|) \left\| f_1 - f_2 \right\| \left\| \psi \right\|_{L^1}.
\]
so \(\|\phi_n\|^2 \to \lambda\) as \(n \to \infty\). By mass conservation we also have \(\|u_n(t)\|^2 \to \lambda\) as \(n \to \infty\)

unformly in \(t \in \mathbb{R}\).

Moreover, \(E(\phi_n) \to E^\text{dav}_\lambda\) as \(n \to \infty\). Indeed, we have

\[
|E(\phi_n) - E^\text{dav}_\lambda| = |E(\phi_n) - E(f_n)| \leq \frac{d_{\text{av}}}{2} \|\phi_n'\|^2 - \|f_n'\|^2 + |N(\phi_n) - N(f_n)|.
\]

Using the reverse triangle inequality, we obtain

\[
\|\phi_n'\|^2 - \|f_n'\|^2 = (\|\phi_n'\| + \|f_n'\|) \|\phi_n'\| - \|f_n'\| = (\|\phi_n'\| + \|f_n'\|) \|\phi_n - f_n\|
\]

which together with \((\phi_n)_n\) and \((f_n)_n\) being bounded in \(H^1(\mathbb{R})\) and (7.5) shows

\[
|E(\phi_n) - E^\text{dav}_\lambda| \lesssim (\|\phi_n'\| + \|f_n'\|) \|(\phi_n - f_n)'\| + J_1(\|\phi_n\|_{H^1} \vee \|f_n\|_{H^1}) (\|\phi_n\|_{H^1} + \|f_n\|_{H^1}) \|\phi_n - f_n\| \lesssim \|(\phi_n - f_n)'\| + \|\phi_n - f_n\| \to 0
\]
as \(n \to \infty\).

By energy conservation we also have \(E(u_n(\cdot, t_n)) = E(\phi_n) \to E^\text{dav}_\lambda\), i.e., it is an energy minimizing sequence, except that it might not have the correct \(L^2\) norm. Since \((u_n(\cdot, t_n))_n\)

is energy minimizing and its \(L^2\) norm is bounded, the coercivity of the energy (6.4) implies that \((u_n(\cdot, t_n))_n\) is bounded in \(H^1(\mathbb{R})\).

To normalize the \(L^2\) norm of \(u_n(\cdot, t_n)\) let \(\alpha_n := \lambda^{1/2}\|\phi_n\|^{-1}\) and set \(g_n := \alpha_n u_n(\cdot, t_n)\).

From mass conservation it is clear that

\[
\|g_n\|^2 = \alpha_n^2 \|u_n(\cdot, t_n)\|^2 = \alpha_n^2 \|\phi_n\|^2 = \lambda.
\]

Moreover, \(\alpha_n \to 1\) as \(n \to \infty\), since \(\|\phi_n\|^2 \to \lambda\), so by (7.5) we also have, similarly as above,

\[
|E(g_n) - E(u_n(\cdot, t_n))| \leq \frac{d_{\text{av}}}{2} \|g_n'\|^2 - \|u_n(\cdot, t_n)'\|^2 + |N(g_n) - N(u_n(\cdot, t_n))| \\
\lesssim \|(g_n - u_n(\cdot, t_n))'\| + \|g_n - u_n(\cdot, t_n)\| \\
\lesssim |\alpha_n - 1| \to 0\quad\text{as}\quad n \to \infty
\]
since \((g_n)_n\) and \((u_n(\cdot, t_n))_n\) are bounded in \(H^1(\mathbb{R})\).

Hence \((g_n)_n\in\mathbb{N}\) is a proper energy minimizing sequence. The tightness result of [9, Proposition 4.5], more precisely, its extension to our more general setting, see the second part of Remark 7.2, shows us that there exists \(K < \infty\) such that, for any \(L > 0\),

\[
\sup_{n \in \mathbb{N}} \int_{|\eta| > L} |\hat{g}_n(\eta)|^2 d\eta \leq \frac{K}{L^2}
\]

where \(\hat{g}_n\) is the Fourier transform of \(g_n\), and that there exist shifts \(y_n\) such that

\[
\lim_{R \to \infty} \sup_{n \in \mathbb{N}} \int_{|x| > R} |g_n(x - y_n)|^2 dx = 0.
\]

Of course, the shifted sequence \(\tilde{g}_n = g_n(\cdot - y_n)\) is again a minimizing sequence and thanks to the above bounds for \(g_n\) it is tight in the sense of measures. Since it is also bounded in \(H^1(\mathbb{R})\), there exist a subsequence, we still denote by \(\tilde{g}_n\) which converges weakly in \(H^1(\mathbb{R})\) to some \(\tilde{g} \in H^1(\mathbb{R})\), hence also weakly in \(L^2(\mathbb{R})\). The tightness bounds above then imply that this subsequence also converges strongly in \(L^2(\mathbb{R})\), see, for example, [19, Lemma A.1] or [28]. Thus \(\|\tilde{g}\|^2 = \lambda > 0\) and since \(\tilde{g}_n\) is bounded in \(H^1(\mathbb{R})\) the inequality (7.4) shows \(N(\tilde{g}_n)\) converges to \(N(\tilde{g})\). Moreover, by weak convergence in \(H^1(\mathbb{R})\) we have \(\liminf_{n \to \infty} \|\tilde{g}_n'\|^2 \geq \|\tilde{g}'\|^2\), i.e., the energy is lower semi–continuous under weak convergence. Since \(\tilde{g}_n\) is an energy minimizing sequence, this yields \(E(\tilde{g}) = \lim_{n \to \infty} E(\tilde{g}_n) = E^\text{dav}_\lambda\). Thus \(\lim_{n \to \infty} \|\tilde{g}_n'\|^2 = \|\tilde{g}'\|^2\) and so \(\tilde{g}_n\) converges strongly in \(H^1(\mathbb{R})\) to \(\tilde{g}\).
Let \( k_n = \bar{g}(\cdot + y_n) \). Then clearly \( k_n \in S^d_{\lambda} \) and
\[
\| u_n(\cdot, t_n) - k_n \|_{H^1} \leq \| u_n(\cdot, t_n) - g_n \|_{H^1} + \| g_n - k_n \|_{H^1} = |1 - \alpha_n| \| u_n \|_{H^1} + \| \bar{g}_n - \bar{g} \|_{H^1} \to 0
\]
as \( n \to \infty \), which contradicts (7.8). Thus \( S^d_{\lambda} \) is orbitally stable if \( d_{av} > 0 \).

Now we come to the proof of orbital stability of \( S^0_{\lambda} \) for saturated nonlinearities, i.e., under assumption A5, when the average dispersion is zero. We again argue by contradiction and assume that there exist \( \epsilon > 0 \) and a sequence \( (\phi_n)_{n \in \mathbb{N}} \) such that (7.7) and (7.8) hold. We then choose a sequence \( f_n \in S^0_{\lambda} \) with \( \lim_{n \to \infty} \| \phi_n - f_n \| = 0 \). Arguing as in the case of \( d_{av} > 0 \) we also have \( \lim_{n \to \infty} \| \phi_n \|^2 = \lambda \) and, by mass conservation, \( \| u_n(t) \|^2 \to \lambda \) as \( n \to \infty \) uniformly in \( t \in \mathbb{R} \). Moreover, since \( |h(a)| \lesssim a^{p_1} + a^{p_2} \) with \( 1 \leq p_1 \leq p_2 < 3 \) the bound (7.4) still applies. Hence
\[
|E(\phi_n) - E^0_n| = |E(\phi_n) - E(f_n)| = |N(\phi_n) - N(f_n)| \lesssim \| \phi_n - f_n \| \to 0
\]
as \( n \to \infty \), since \( (\phi_n)_n \) and \( (f_n)_n \) are bounded in \( L^2(\mathbb{R}) \). Thus \( E(\phi_n) \to E^0 \) as \( n \to \infty \) and \( E(u_n(\cdot, t_n)) = E(\phi_n) \to E^0 \), again by energy conservation. So as before \( (u_n(t_n)) \) is an energy minimizing sequence, except that it might have the proper normalization only in the limit of large \( n \). We again normalize this sequence, setting \( g_n := \alpha_n u_n(\cdot, t_n) \), with \( \alpha_n := \lambda^{1/2} \| \phi_n \|^{-1} \), which converges to 1 in the limit \( n \to \infty \). We follow the previous arguments, in the case \( d_{av} > 0 \), to see that \( (g_n)_{n \in \mathbb{N}} \) is again an energy minimizing sequence which is properly normalized in \( L^2 \).

However, at this stage we have to deviate from the arguments for \( d_{av} > 0 \), because we only know that the sequence \( (g_n)_n \) is normalized in \( L^2 \), and, unlike the case \( d_{av} > 0 \), we do not have any additional information about \( (g_n)_n \) at this stage. In particular, we do not know whether \( \| g_n \|_{L^\infty} \leq C \) uniformly in \( n \) for some constant \( C < \infty \), which allowed us to use a modification of the tightness results from [9] when \( d_{av} > 0 \), see Remark 7.2.ii.

To get around this dilemma, we note that [20, Lemma 3.10] shows that given the minimizing sequence \( (g_n)_n \) there exists another minimizing sequence \( (h_n)_{n \in \mathbb{N}} \subset L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with
\[
\sup_{r \in \text{supp}(\psi)} \| T_r h_n \|_{L^\infty} \leq C_{\lambda}.
\]
From the construction of this modified minimizing sequence, see the proof of Lemma 3.10 in [20], we also know that
\[
\| g_n - h_n \| \to 0
\]
as \( n \to \infty \). This allows us to apply a modification of the tightness result [9, Proposition 4.6] for the sequence \( (h_n)_n \), see Remark 7.2.ii. This yields shifts \( y_n \) and boosts \( \xi_n \) such that
\[
\lim_{R \to \infty} \sup_{n \in \mathbb{N}} \int_{|x| > R} |h_n(x - y_n)|^2 \, dx = 0
\]
and
\[
\lim_{L \to \infty} \sup_{n \in \mathbb{N}} \int_{|\eta| > L} |\hat{h}_n(\eta - \xi_n)|^2 \, d\eta = 0
\]
where \( \hat{h}_n \) is the Fourier transform of \( h_n \).

Let \( \tilde{h}_n = e^{i\xi_n} h_n(\cdot - y_n) \), \( n \in \mathbb{N} \) be the shifted and boosted sequence. Then \( (\tilde{h}_n)_n \) is again a minimizing sequence with \( \| \tilde{h}_n \|^2_{L^2} = \lambda \), which is bounded in \( L^\infty \) and it is tight, i.e., the bounds (7.10) and (7.11) hold with \( y_n = \xi_n = 0 \) and \( h \) replaced by \( \tilde{h} \). Using a weakly convergent subsequence, also denoted by \( \tilde{h}_n \), the tightness of \( \tilde{h}_n \) yields strong convergence
of this subsequence, see, e.g., [19, Lemma A.1] or [28], to some $\tilde{h}$. Using $L^2$ continuity of the energy when $d_{av} = 0$, this function $\tilde{h}$ is a minimizer of the energy, i.e., $\tilde{h} \in S_0^0$, and so are the shifted and boosted functions $k_n = e^{-i\xi_n (\cdot + y_n)}\tilde{h}(\cdot + y_n)$, where we unravel the boosts and shifts which lead from $h_n$ to $\tilde{h}_n$. By construction, $\|h_n - k_n\| \to 0$ for $n \to \infty$.

Using that $0 < \varepsilon_0 \leq \|u_n(\cdot, t_n) - f\|$ for any $f \in S_0^0$ we get the contradiction

$$0 < \varepsilon_0 \leq \|u_n(\cdot, t_n) - k_n\| \leq \|u_n(\cdot, t_n) - g_n\| + \|g_n - h_n\| + \|h_n - k_n\| = (1 - \alpha_n)\|u_n\| + \|g_n - h_n\| + \|h_n - k_n\| \to 0 \text{ for } n \to \infty.$$ 

Hence $S_0^0$ is orbitally stable, even for saturating nonlinearities.

For non–saturated nonlinearities, the proof for the zero average dispersion case is analogous to that for saturating nonlinearities, except that one can directly use the tightness result from Proposition 4.6 in [9] and does not have to modify the minimizing sequence to make it bounded in $L^\infty$.

**Acknowledgements:** Young–Ran Lee and Mi–Ran Choi are supported by the National Research Foundation of Korea (NRF) grants funded by the Korean government (MSIT) NRF-2020R1A2C1A01010735 and (MOE) NRF-2021R1I1A1A01045900. Dirk Hundertmark is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

**References**

[1] M. J. Ablowitz and G. Biondini, *Multiscale pulse dynamics in communication systems with strong dispersion management*. Optics Letters 23 (1998), 1668–1670.

[2] J. Albert and E. Kahlil, *On the well–posedness of the Cauchy problem for some nonlocal nonlinear Schrödinger equations*. Nonlinearity 30 (2017), 2308–2333.

[3] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*. J. Funct. Anal. 14 (1973), 349–381.

[4] I. Anapolitanos, M. Hott, and D. Hundertmark, *Derivation of the Hartree equation for compound Bose gases in the mean field limit*. Rev. Math. Phys. 29 (2017), no. 7, 1750022, 28 pp.

[5] P. Antonelli, J. C. Saut, and C. Sparber, *Well-posedness and averaging of NLS with time-periodic dispersion management*. Adv. Differential Equations 18 (2013), 49–68.

[6] J. M. Ball, *Strongly Continuous Semigroups, Weak Solutions, and the Variation of Constants Formula*. Proc. Amer. Math. Soc. 63 (1977), no. 2, 370–373.

[7] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp.

[8] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger Equations*. Comm. Math. Phys. 85 (1982), 549–561.

[9] M.–R. Choi, D. Hundertmark, and Y.–R. Lee, *Thresholds for existence of dispersion management solitons for general nonlinearities*. SIAM J. Math. Anal. 49 (2017), no. 2, 1519–1569.

[10] M.–R. Choi, Y. Kang, and Y.–R. Lee, *On dispersion managed nonlinear Schrödinger equations with lumped amplification*. J. Math. Phys. 62 (2021), no. 7, 071506, 1–16.

[11] M.–R. Choi and Y.–R. Lee. *Averaging of dispersion managed nonlinear schrödinger equations*. Nonlinearity 35 (2022), no. 4, 2121–2133.

[12] M. B. Erdoğan, D. Hundertmark, and Y.–R. Lee, *Exponential decay of dispersion managed solitons for vanishing average dispersion*. Math. Res. Lett. 18 (2011), no. 1, 13–26.

[13] I. Gabitov and S. K. Turitsyn, *Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation*. Opt. Lett. 21 (1996), 327–329.
[14] I. Gabitov and S. K. Turitsyn, *Breathing solitons in optical fiber links*. JETP Lett. 63 (1996), 861.

[15] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*. Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985), 309–327.

[16] W. Green and D. Hundertmark, *Exponential Decay for dispersion managed solitons for general dispersion profiles*. Lett. Math. Phys. 106 (2016), no. 2, 221–249.

[17] D. Hundertmark, P. Kunstmann, and R. Schnaubelt, *Stability of dispersion managed solitons for vanishing average dispersion*. Arch. Math. 104 (2015), no. 3, 283–288.

[18] D. Hundertmark and Y.–R. Lee, *Decay estimates and smoothness for solutions of the dispersion managed non-linear Schrödinger equation*. Comm. Math. Phys. 286 (2009), no. 3, 851–873.

[19] D. Hundertmark and Y.–R. Lee, *On non-local variational problems with lack of compactness related to non-linear optics*. J. Nonlinear Sci. 22 (2012), 1–38.

[20] D. Hundertmark, Y.–R. Lee, T. Ried, and V. Zharnitsky, *Solitary waves in nonlocal NLS with dispersion averaged saturated nonlinearities*. J. Differential Equations 265 (2018), no. 8, 3311–3338.

[21] T. Kato, *On nonlinear Schrödinger equations*. Ann. Inst. H. Poincare Phys. Theor. 46 (1987), 113–129.

[22] M. Keel and T. Tao, *Endpoint Strichartz estimates*. Amer. J. Math. 120 (1998), no 5, 955–980.

[23] L. D. Landau and E. M. Lifshitz, *Course of theoretical physics. Vol. 1. Mechanics*. Third edition. Butterworth-Heinemann, Oxford-New York-Toronto, Ont., 1976.

[24] J. Murphy, *A review of modified scattering for the 1d cubic NLS*. Harmonic analysis and nonlinear partial differential equations, RIMS Kôkyûroku Bessatsu B88 (2021), 119–146.

[25] J. Murphy and T. V. Hoose, *Modified scattering for a dispersion–managed nonlinear Schrödinger equation*. NoDEA Nonlinear Differential Equations Appl. 29 (2022), no. 1, 1–11

[26] J. Murphy and T. V. Hoose, *Well–posedness and blowup for the dispersion-managed non-linear Schrödinger equation*. arXiv:2110.08372

[27] T. Ozawa, *Remarks on proofs of conservation laws for nonlinear Schrödinger equations*. Calc. Var. Partial Differential Equations 25 (2006), 403–408.

[28] R. L. Pego, *Compactness in $L^2$ and the Fourier transform*. Proc. Amer. Math. Soc. 95 (1985), no. 2, 252–254.

[29] M. Stanislavova, *Regularity of groundstate solutions of dispersion managed nonlinear schrödinger equations*. J. Differential Equations 210 (2005), no. 1, 87–105.

[30] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*. Duke Math. J. 44 (1977), 705–714.

[31] T. Tao, *Nonlinear Dispersive Equations. Local and Global Analysis*. CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.

[32] S. K. Turitsyn, B. Bale, and M. P. Fedoruk, *Dispersion-managed solitons in fibre systems and lasers*. Physics Reports, 521 (2012), no. 4, 135–203.

[33] V. Zharnitsky, E. Grenier, C. K. R. T. Jones, and S. K. Turitsyn, *Stabilizing effects of dispersion management*. Phys. D. 152-153 (2001), 794–817.

DEPARTMENT OF MATHEMATICS, SOGAN UNIVERSITY, 35 BAEBBEOM-RO (SINSU–DONG), MAPO-GU, SEOUL, 04107, SOUTH KOREA.

Email address: mrcho@sogang.ac.kr

DEPARTMENT OF MATHEMATICS, INSTITUTE FOR ANALYSIS, KARLSRUHE INSTITUTE OF TECHNOLOGY, 76128 KARLSRUHE, GERMANY, AND DEPARTMENT OF MATHEMATICS UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN 1409 W. GREEN STREET URBANA, ILLINOIS 61801-2975

Email address: dirk.hundertmark@kit.edu
DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, 35 BAEKBEOM-RO (SINSU-DONG), MAPO-GU, SEOUL, 04107, SOUTH KOREA.

Email address: younglee@sogang.ac.kr