New $E_{7(7)}$ Invariants and Amplitudes

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Abstract

We construct a new class of manifest $E_{7(7)}$ duality invariants, which generalize the Cartan quartic invariant, familiar from studies of the black hole entropy. The new ones, being four-linear, are designed for studies of the four-vector amplitudes and to be used, upon supersymmetrization, as initial sources of deformation for the non-linear higher-derivative generalizations of the linear twisted self-duality condition. We show, however, that the new invariants are inconsistent with the expected UV divergent amplitudes in extended supergravities with non-degenerate duality groups of type $E_7$. When $E_{7(7)}$ degenerates into $U(1)$ the new invariants reproduce the recently discovered source of deformation for the Born-Infeld duality invariant model with higher derivatives relevant to UV divergences of the D3 brane action. These facts may explain the UV properties of perturbative supergravity.

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Introduction

The new wave of the interest in extended supergravity is due to a cancellation, unexpected by most supergravity experts, of the 3-loop and 4-loop UV divergences in $\mathcal{N} = 8$ \cite{1,2} and 3-loop in $\mathcal{N} = 4$ \cite{3,4}. $\mathcal{N} = 8$ supergravity has an $E_{7(7)}$ duality symmetry discovered by Cremmer and Julia \cite{5,6}. The dualities of the $\mathcal{N} > 2$ supergravities were first studied by Gaillard and Zumino from a general point of view in \cite{7} where it was shown that these symmetries require the conservation of the Noether-Gaillard-Zumino (NGZ) current and, equivalently, the fulfillment of the NGZ identities. These identities were studied with respect to $\mathcal{N} = 8$ supergravity in \cite{8}, where it was shown that adding the higher loop counterterms to the action would break the $E_{7(7)}$ NGZ current conservation. It was, however, suggested by Bossard and Nicolai \cite{9} that the problem may be cured by adding some extra terms, in addition to the counterterm, so that the NGZ current conservation is restored.

The original proposal to deform the linear twisted self-duality condition made in \cite{9} was studied in \cite{10} and was shown to be incomplete: it required a significant generalization even to derive the familiar Born-Infeld model by this method. A generalized procedure for deriving new models with duality symmetry was proposed and developed in \cite{10}. It turned out to be useful to construct new duality-invariant models such as a Born-Infeld-type model with higher derivatives \cite{11} and new $\mathcal{N} = 2 U(1)$ gauge theories \cite{12}.

To apply the generalized procedure of deformation of the linear twisted self-duality condition \cite{9,10} to $\mathcal{N} = 8$ supergravity it is necessary to find an initial source of deformation which

1. is manifestly $E_{7(7)}$-invariant. This requires the simultaneous use of the 28 vector fields present in the action and the 28 dual vector fields, associated with the derivative of the action over the vectors, combined in doublets which transform in the fundamental $(56)$ of $E_{7(7)}$.

2. constrained by the linear twisted self-duality condition, should match the candidate UV divergence. In $\mathcal{N} = 8$ supergravity we should compare it with the corresponding 4-vector amplitude \cite{13,14}.

While the construction of initial sources of deformation with these properties is possible at a bosonic level, we will show that such invariants cannot exist in the geometric $\mathcal{N} = 8$ superspace construction of \cite{15,16}: relaxing the linear twisted self-duality condition violates the superspace Bianchi identities because the doubling of vector field degrees of freedom (from 28 to 56) is not possible in the framework of the superspace, where there is only one superfield and its derivatives \cite{15,16}. A more general superspace or component construction which may accommodate this doubling of vectors still has to be produced.

We leave this as an open issue and proceed to do what can be done with the available techniques, namely to construct new bosonic $E_{7(7)}$ invariants from fundamental vector field doublets. Notice that $E_{7(7)}$ invariants constructed only of fundamental vector field doublets are rare! The one we need, actually, had not yet been constructed and we will do it here.

If we were able to find the proper bosonic $E_{7(7)}$ invariants, we would have to supersymmetrize them. But, as we will show, the obstruction for the amplitudes will show up already at
the bosonic level, so there will be no need to look for the supersymmetric version of the new invariants, at least with regard to the UV divergence analysis.

The groups of type $E_7$ \cite{17} are groups of linear transformations leaving invariant two multilinear forms: a skew-symmetric bilinear and a symmetric four-linear form\textsuperscript{3}. $E_{7(7)}$ is the prime example of these groups. By definition, two invariants of the groups of type $E_7$ can always be constructed using only objects in the fundamental representation (fundamentals): the bilinear symplectic, antisymmetric, and the four-linear one, which is symmetric. When all the fundamentals are identical, the would-be quadratic invariant vanishes due to antisymmetry and the four-linear invariant gives a quartic invariant.

The familiar Cartan-Cremmer-Julia quartic invariant \cite{20,5} of $E_{7(7)}$ which plays a significant role in describing $N = 8$ black-hole entropy and its relation to quantum entanglement of qubits\textsuperscript{4} is a particular example of this mechanism. It can be obtained from the four-linear invariant of $E_{7(7)}$, whose structure is based on exceptional Jordan algebra $J_{3,8}^O$ over the split octonions $O_S$ \cite{23, 24}. This four-linear invariant of $E_{7(7)}$ will be used in what follows to construct new $E_{7(7)}$ invariants relevant to amplitudes and counterterms. When the duality group reduces to $U(1)$ these new invariants will reduce to the ones identified in the recently constructed Born-Infeld models with higher derivatives \cite{11}. We will then compare these new $E_{7(7)}$ invariants with the UV counterterms of $N = 8$ supergravity \cite{25}.

In particular, we will compare the answer with the expression for the 4-point vector amplitude in $N = 8$ supergravity. Since it is the MHV amplitude, it predicts the structure of the UV divergences in the 4-vector sector, starting from the 3-loop level, as shown in \cite{13}. The analogous explicit contribution to the 4-vector 3-loop UV divergence was obtained in \cite{14}. All we have to do is to take our new manifest $E_{7(7)}$ invariant defined as a functional of the 4 fundamentals 56 (at each of the 4 momenta $p_I$), compute its value when imposing the linear twisted self-duality constraint and compare with the 4-point MHV vector amplitude. We will find that, as long as we consider a non-degenerate case of groups of type $E_7$, the invariants disagree with the expected UV divergences of the 4-vector amplitudes. Meanwhile, for the case of degenerate groups of type $E_7$, the invariants may agree with the expected UV divergences of the 4-vector amplitudes. Specifically, in the $U(1)$ duality case our reduced invariants do reproduce the first loop UV divergence of the D3 brane action computed in \cite{26} and the 1-loop UV divergence of the $N = 2$ supersymmetric Born-Infeld action computed in \cite{27}.

Various aspects of $E_{7(7)}$ symmetry with regard to perturbative $N = 8$ supergravity were studied before \cite{23} - \cite{29} and some of these results will be used here.

This paper is organized as follows: in Section 1 we introduce the graviphoton field strength of $N = 8$ supergravity, $T_{IJ \mu \nu}$ and we discuss the possible construction of manifestly duality-invariant initial sources of deformation of the linear, twisted, selfduality constraint using $T_{IJ \mu \nu}$ or using the vector field strengths in the fundamental of $E_{7(7)}$. We argue that with the current superspace techniques (the argument is explained in detail in Appendix A), such a construction is only possible using the latter. Section 2 is devoted to the general construction of $E_{7(7)}$ invariants of the electric and magnetic charges (up to fourth order in charges), which also fill a fundamental

\textsuperscript{3}See \cite{18,19} for recent applications of this concept in Physics.

\textsuperscript{4}See, for example, \cite{21} - \cite{22}.
representation of that group. We study the cases of a single charge and of four different charges and discuss the uses of these invariants in (extremal) black-hole physics. This section can actually be understood as a preparation for Section 3 in which, using the $E_7(7)$ invariants just constructed we construct new candidates to $E_7(7)$- and Lorentz-invariant initial sources of deformation and scattering amplitudes, showing that those computed in the literature disagree with them. In Section 4 we study the reduction of the invariants to smaller duality groups. Finally, we discuss our results in Section 5. The Appendices B and C contain complementary information on the Green-Schwarz $t^{(8)}$ tensor and on the split octonions, respectively.

1 The strategy: graviphoton, scalars and vectors

1.1 The graviphotons of $\mathcal{N} = 8$ supergravity

The relevant sector of the Lagrangian of $\mathcal{N} = 8, d = 4$ classical supergravity\footnote{Here we follow the tested conventions of Ref. [30] essentially taken from Ref. [31] and introduce the notation and the main definitions we are going to need.} is

$$L = 2\Im \mathcal{N}_{\Lambda\Sigma} F_{\Lambda \mu \nu}^\Lambda F_{\Sigma}^{\mu \nu} - 2\Re \mathcal{N}_{\Lambda\Sigma} F_{\Lambda \mu \nu} \tilde{F}_{\Sigma}^{\mu \nu},$$

(1.1)

where $\tilde{F}$ is the Hodge dual of $F$, defined in Eq. (B.1), $\mathcal{N}_{\Lambda\Sigma}$ is the scalar-dependent period matrix (whose defining property is given in Eq. (1.6) below) and where the indices $\Lambda, \Sigma = 1, \cdots, 28$. Later on, each of these indices will be replaced by an antisymmetrized pair of indices $i, j = 1, \cdots, 8$.

The electromagnetic dual of $F^\Lambda$ (magnetic field strengths) $G_\Lambda$ are defined by

$$\tilde{G}_{\Lambda \mu \nu} \equiv -\frac{1}{4} \frac{\partial L}{\partial F_{\Lambda \mu \nu}} = -3\Im \mathcal{N}_{\Lambda\Sigma} F_{\Sigma}^{\mu \nu} + \Re \mathcal{N}_{\Lambda\Sigma} \tilde{F}_{\Sigma}^{\mu \nu},$$

(1.2)

or, equivalently,

$$G_\Lambda^+ = \mathcal{N}_{\Lambda\Sigma} F_{\Sigma}^+, \quad (1.3)$$

In most of what follows we will consider $F^\Lambda$ and $G_\Lambda$ as independent variables. In particular, this will mean that the $G_\Lambda$ are independent of the scalars and Eq. (1.3) is not satisfied. The dependence will be reintroduced only after imposing the constraint Eq. (1.3), known as linear twisted self-duality constraint. Observe that, given the definition of the magnetic field strengths Eq. (1.2) this constraint contains information enough to reconstruct the Lagrangian Eq. (1.1).

With these field strengths we can construct a 56-dimensional real symplectic vector of field strengths

$$\mathcal{F} \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix},$$

(1.4)

that transforms in the 56 of $E_7(7) \subset Sp(56, \mathbb{R})$.

The scalars of the theory are described by the symplectic section

$$\mathcal{V}_{IJ} \equiv \begin{pmatrix} f_{IJ}^\Lambda \\ h_{IJ} \end{pmatrix},$$

(1.5)
where $I, J = 1, \ldots, 8$ are an antisymmetric pair of indices that are raised and lowered by complex conjugation. The period matrix is defined by the property

$$h_{\Lambda I J} = N_{\Lambda \Sigma} f^{\Sigma}_{I J}.$$  \hfill (1.6)

The components of the section $\mathcal{V}_{I J}$ are related to the components of the coset representative

$$\mathcal{V} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in E_{7(7)}/SU(8) \subset Sp(56, \mathbb{R}), \Rightarrow \begin{cases} A^T C - C^T A = 0, \\ B^T D - D^T B = 0, \\ A^T D - C^T B = 1_{28 \times 28}, \end{cases}$$  \hfill (1.7)

by

$$f = \frac{1}{\sqrt{2}}(A - iB), \quad h = \frac{1}{\sqrt{2}}(C - iD).$$  \hfill (1.8)

This relation of the components of the section $\mathcal{V}_{I J}$ with the components of the symplectic $E_{7(7)}/SU(8)$ coset representative imply the constraints

$$\langle \mathcal{V}_{I J} | \mathcal{V}_{K L} \rangle = -2i\delta_{I J}^{K L}, \quad \langle \mathcal{V}_{I J} | \mathcal{V}_{K L} \rangle = 0.$$  \hfill (1.10)

It also implies that, when we perform a global $E_{7(7)}$ transformations that acts linearly on the $\Lambda, \Sigma$ indices, we have to act linearly with a $SU(8)$ compensating transformation (sometimes called “local” because they depend on the scalar fields) on the indices $I, J, \ldots$. Thus, the indices $I, J, \ldots$ are referred to as $SU(8)$ indices. There is another, completely different kind of $SU(8)$ indices that will enter the game later on and that will be denoted by indices $A, B, C, \ldots$.

The graviphoton field strength is defined by

$$T_{I J} \equiv \langle \mathcal{V}_{I J} | \mathcal{F} \rangle,$$  \hfill (1.11)

and its self- and anti-selfdual parts, defined in Eq. (B.2) are

$$T_{I J}^\pm \equiv \langle \mathcal{V}_{I J} | \mathcal{F}^\pm \rangle.$$  \hfill (1.12)

They all transform under compensating $SU(8)$ transformations only. Since the $SU(8)$ tensor $T_{I J}$ is complex, we have

$$T_{I J}^{\pm} = \overline{(T_{I J}^\pm)}.$$  \hfill (1.13)

Finally, the linear twisted self-duality constraint Eq. (1.3), is equivalent to the vanishing of some of these objects:

$$T_{I J}^{I J} = \overline{(T_{I J}^{I J})} = 0.$$  \hfill (1.14)

This vanishing has important implications. Using the Maurer-Cartan equations

$$\mathcal{D} \mathcal{V}_{I J} = \frac{1}{2} P_{I J K L} \mathcal{V}^{K L},$$  \hfill (1.15)

for the symplectic product $\langle | \rangle$, we use the convention

$$\langle A | B \rangle \equiv B^\Lambda A_{\Lambda} - B_{\Lambda} A^\Lambda.$$  \hfill (1.9)
where $\mathcal{D}$ is the $SU(8)$-covariant derivative and $\mathcal{P}_{IJKL}$ the Vielbein 1-form on the scalar manifold, and the definition of the graviphoton field strength (1.11) we find

$$\mathcal{D}T_{IJ} = \frac{1}{2} \mathcal{P}_{IJKL} \wedge T^{KL},$$

(1.16)

and its complex conjugate.

1.1.1 Manifestly $E_{7(7)}$-invariant terms from graviphotons

Since, as we have stressed above, the graviphoton field strength $T_{IJ}$ (1.11) only transforms under the induced $SU(8)$ compensating transformations which act linearly on the indices $I, J, \ldots$, it is trivial to construct terms which are manifestly invariant under the full $E_{7(7)}$ group by contracting all the $SU(8)$ indices in the standard way. On the other hand, in classical $\mathcal{N} = 8$ supergravity the linear twisted self-duality constraint Eq. (1.14) is valid by construction and the only available non-vanishing tensors that we can use are $T_{IJ}^{\perp}$ and its complex conjugate $T_{IJ}^{+}$.

In the context of the deformation procedure [9, 10, 11, 12] the linear twisted self-duality condition Eq. (1.14) has to be deformed to accommodate the counterterms. A priori, one may consider two possibilities for the initial source of deformation $\mathcal{I}$ from which the counterterms are derived by an iterative procedure:

1. $\mathcal{I}(F, G; \phi) = \mathcal{I}(T)$ depends on the graviphoton and its $SU(8)$-covariant derivatives. In particular, this is a property of the counterterms constructed in [25]. The initial source of deformation $\mathcal{I}$ must be a manifestly Lorentz- and $E_{7(7)}$-invariant expression depending on the fundamental vector doublet $\mathcal{F} = (F, G)$ via the graviphoton components $T_{IJ}^{\perp}, T_{IJ}^{+}$ which do not vanish at the linear level.

Given that initial source of deformation one defines the deformed twisted self-duality constraint

$$T_{IJ}^{\perp} \mu^\nu = \frac{\delta \mathcal{I}(T)}{\delta T_{IJ}^{\perp} \mu^\nu},$$

(1.17)

which can be solved by iteration. At the lowest (linear) order the r.h.s. of this expression vanishes and one recovers the linear twisted self-duality constraint Eq. (1.14). At higher orders the r.h.s. produces non-linear deformations of that constraint. Just as the linear constraint contains enough information to reconstruct the Lagrangian Eq. (1.1), the non-linear deformations lead to Lagrangians with terms of higher orders in the graviphoton field strength.

2. $\mathcal{I}(F, G)$ depends only on the $E_{7(7)}$ fundamental vector field strength doublet $\mathcal{F} = (F, G)$.

In such case, the deformation procedure may be described as the duality covariant variation of $\mathcal{I}(F, G)$ over the doublet, using the scalar-dependent metric, as proposed in [9].

Our first step is to prove that the first possibility encounters an obstruction. We will find that the supersymmetric version of the linear twisted self-duality condition (1.14) is required for the
Bianchi identities of the superspace $[15, 16]$ where the invariants are constructed. In particular, we will focus on the superspace solution of Bianchi identities\footnote{The $SU(8)$ indices that we denote by $I, J, \ldots$ in this paper are denoted by $i, j, \ldots$ in $[15, 16]$.}  

\begin{equation}
F_{\dot{\alpha}\dot{\beta},IJ}(x,\theta) = -i\epsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta IJ}(x,\theta) - i\epsilon_{\alpha\beta}\overline{N}_{\dot{\alpha}\dot{\beta}IJ}(x,\theta)
\end{equation}

which allows, in principle to have a 56-component vector doublet $(M_{\alpha\beta IJ}, N_{\dot{\alpha}\dot{\beta} IJ})$ and the conjugate one $(\overline{M}^I_{\dot{\alpha}\dot{\beta}}, \overline{N}^I_{\alpha\beta})$, since where there are 28 vectors in $M$ and 28 in $N$. However, the superspace integrability condition for the existence of the supervielbein that relates $E_7(7)$ to $SU(8)$ in the form

\begin{equation}
R^I_{\ beloved}(x,\theta) = -\frac{1}{4}P^{JKLM}(x,\theta) \wedge P_{JKLM}(x,\theta)
\end{equation}

requires that the second part of the vector multiplet, $N_{\dot{\alpha}\dot{\beta} IJ}$ is constrained to be a bilinear of the fermion superfields and cannot be independent.

Therefore, the existence of the scalar field-dependent sections of an $Sp(56, \mathbb{R})$ bundle over the $E_7(7)/SU(8)$ coset space $h_{\Lambda IJ}(\phi)$ and $f^\Lambda_{I,J}(\phi)$ would be inconsistent with supersymmetry if the vectors were doubled, which requires the superfield $N_{\dot{\alpha}\dot{\beta} IJ}$ to be independent. We provide the derivation of this assertion in Appendix A.

The observation above precludes the deformation procedure $[9, 10, 11, 12]$ for making consistent $N = 8$ supergravity\footnote{At least with means available. New constructions may change the situation in the future, if successful.} with counterterms, since the cornerstone of this procedure is the source of deformation depending on unconstrained doublet of $E_7(7)$. There is no such scalar-dependent graviphoton superfield doublet in superspace $[15, 16]$ and the alternatives are also not available, at present.

### 1.2 $E_7(7)$ invariants from fundamentals

Here we will construct the $E_7(7)$ invariants relevant for the amplitudes in $N = 8$ supergravity. Since $E_7(7)$ acts on the doublet linearly and there are no scalars with their shift symmetry, we are just looking at the 4-vector local invariants/amplitudes. The two major ingredients in our construction are:

1. The Günyaydin-Koepsell-Nicolai construction of the four-linear invariant of $E_7(7)$ based on exceptional Jordan algebra $J^O_3$ over the split octonions $O_S$ $[23]$. The symplectic invariant and the triple Jordan product will be used to produce an eighth-rank Lorentz tensor

\begin{equation}
J^{(8)}(F_{\mu_1\nu_1}, F_{\mu_2\nu_2}, F_{\mu_3\nu_3}, F_{\mu_4\nu_4})
\end{equation}

antisymmetric in each pair of indices $\mu_1\nu_1$, with $I = 1, 2, 3, 4$, and symmetric under the exchange of such pairs. Each of the four entries is a fundamental 56 taken at one of the four momenta, $F_{I\mu\nu} \equiv \left( F^{ij}_{\mu\nu}(p_I), G_{ij\mu\nu}(p_I) \right)$ so $J^{(8)}$ must be associated to the 4-vector amplitude depending on 4 momenta $p_I$. 

\end{document}
2. The Green-Schwarz eighth-rank tensor \( t^{(8)} \) (a symmetrized sum of products of four Kronecker \( \delta \)'s [44] which provides the kinematic factor of the open string tree-level 4-point amplitude and has the same index symmetry as \( J^{(8)} \)). We will use it here to construct an \( E_{7(7)} \)- and Lorentz-invariant object from the \( E_{7(7)} \)-invariant tensor \( J^{(8)} \) by contracting both tensors. We will also introduce some function of Mandelstam variables, so our candidate for initial source of deformation takes the general form:

\[
t^{(8)} \cdot J^{(8)} f(s, t, u) .
\]

(1.23)

We will compare this proposal with the expression for the 4-point vector amplitude in \( \mathcal{N} = 8 \) supergravity. Since it is the MHV amplitude, it predicts the structure of the UV divergences in the 4-vector sector, starting from the 3-loop level, as shown in [13]. The analogous explicit contribution to the 4-vector 3-loop UV divergence was obtained in [14]. All we have to do is to take the new manifest \( E_{7(7)} \) invariant defined as a functional of the 4 unconstrained fundamentals \( 56 \) (at each of the 4 momenta \( p_i \) given above, compute its value when imposing the linear twisted self-duality constraint, and compare with the 4-point MHV vector amplitude.

In this comparison we will find a discrepancy. As will review in the next section, the four-linear invariant in the complex \( (SU(8)) \) basis consists of 3 terms. Each of them is manifestly invariant under the \( SU(8) \) subgroup of \( E_{7(7)} \) which acts diagonally in the complex basis. However, the invariance under the off-diagonal part of \( E_{7(7)} \) is only achieved when those 3 terms are combined with very specific coefficients. In the simplest case, when there is only one \( 56 \), \( (F_{AB}, F^{AB}) \), the four-linear invariant reduces to the Cartan-Cremmer-Julia [5] quartic invariant:

\[
\hat{\Phi}(F) = \text{Tr}_{SU(8)}(F^{AB}F_{AB}) - \frac{1}{4} \left[ \text{Tr}_{SU(8)}(F^{AB}F_{AB}) \right]^2 + \frac{1}{4} \text{Pf}_{SU(8)} \| F \| + \frac{1}{4} \text{Pf}_{SU(8)} \| F \| ,
\]

(1.24)

and the specific coefficients are \(-1/4, 1/4, 1/4\). In our case, each of the 4 factors \( F \) is taken at different momentum and we form a Lorentz scalar as a product of two Lorentz eight-tensors, but the structure is analogous to (1.24): we have \( SU(8) \) traces of 4 operators, squares if the traces of 2 operators and Pfaffian.

When we will construct our new manifest \( E_{7(7)} \) invariant designed for the initial source of deformation of the linear twisted self-duality condition, we will first test it when the linear twisted self-duality condition is imposed. We will find that the Pfaffians vanish but the \( \text{Tr}_{SU(8)}(F^{AB}F_{AB}) \) and \( \left[ \text{Tr}_{SU(8)}(F^{AB}F_{AB}) \right]^2 \) terms will appear with a relative factor which is not just \(-1/4\), but a non-trivial function of the Mandelstam variables \( \frac{st}{tu} \). This factor breaks \( E_{7(7)} \) symmetry. This also implies that the counterterms are \( E_{7(7)} \)-invariant only when the linear twisted self-duality condition is valid.

Meanwhile, in case of \( U(1) \) duality, one finds the initial source of the deformation manifestly \( U(1) \) invariant in all cases studied in [10, 11, 12]. One finds that this symmetry takes place.

\[\text{When we use the complex basis, that will be introduced in the next section, it will take the form}\]

\[
t^{(8)} \cdot \hat{\Phi}^{(8)} f(s, t, u) .
\]

(1.22)

\[\text{We ignore here the Lorentz indices of the vector field strengths, which do not play any role in this discussion.}\]
independently as to whether the linear twisted self-duality constraint is imposed or not. This made it possible to develop the proposal of [9] for all $U(1)$ models and it was possible to produce novel models with a consistent NGZ $U(1)$ current conservation, which was done in [10, 11, 12].

Below we will show that the UV divergences in $\mathcal{N} = 8$ supergravity starting from the 3-loop level would break the $E_7(7)$ current conservation. We will show that the procedure proposed in [9], as different from the $U(1)$ models, cannot be improved. It is inconsistent with the structure of the four-linear invariant of $E_7(7)$ based on exceptional Jordan algebra $J_{3}^{\mathbb{O}_S}$ over the split octonions $\mathbb{O}_S$ [23].

2 $E_7(7)$ invariants and $\mathcal{N} = 8$ black holes

2.1 Charges

Here we review, for the sake of completeness and as an introduction to the constructions that we will present later, several well-known results and concepts related to the entropy of the black holes of $\mathcal{N} = 8$ supergravity.

Objects transforming in the fundamental (i.e $56$) representation, such as the fundamental vector doublets $\mathcal{F}$ discussed before, can be written in two different bases. In the first basis, $\mathcal{F}$ consists in a pair of real, antisymmetric, independent tensors $F_{ij}$ and $G_{ij}$, $i, j = 1, \ldots, 8$ and the action of $E_7(7)$ embedded into $Sp(56, \mathbb{R})$ is given by

$$\delta \mathcal{F} = \delta \left( \begin{array}{c} F_{ij} \\ G_{ij} \end{array} \right) = \left( \begin{array}{cc} 2\Lambda_{k}^{[i} \delta_{j]}^{l} & \Sigma_{ijkl}^{ijkl} \\ \Sigma_{ijkl}^{ijkl} & 2\Lambda_{k}^{[i} \delta_{j]}^{l} \end{array} \right) \left( \begin{array}{c} F_{kl} \\ G_{kl} \end{array} \right). \tag{2.1}$$

where the $\Lambda_{ij}$ are infinitesimal transformations of the maximal, (non-compact) subgroup $SL(8, \mathbb{R})$ (i.e. $\Lambda^{ii} = 0$) and where the off-diagonal infinitesimal parameters satisfy

$$\Sigma_{ijkl} = \frac{1}{4!} \epsilon_{ijklmn} \Sigma_{mn} \tag{2.2}$$

$F_{ij}$ and $G_{ij}$ transform separately contravariantly and covariantly, respectively, in the $28$ of $SL(8, \mathbb{R})$. Together, they transform as components of a symplectic vector in the $56$ of $E_7(7) \subset Sp(56, \mathbb{R})$.

In the second basis $\mathcal{F}$ is written as a complex, antisymmetric tensor with components $\mathcal{F}_{AB}$ transforming infinitesimally under $E_7(7)$ as

$$\delta \mathcal{F}^{AB} = +2\Lambda^{[A} C^{B]} \mathcal{F}^{C]} + \Sigma^{ABCD} \mathcal{F}_{AB}, \tag{2.3}$$

$$\delta \mathcal{F}_{AB} = -2\Lambda^{C [A} \mathcal{F}_{C]} B + \Sigma_{ABCD} \mathcal{F}^{AB},$$

Here, the $\Lambda^{A} B$ are infinitesimal transformations of the maximal, compact subgroup $SU(8)$ (i.e. $\Lambda^{i} = 0$) and where the off-diagonal infinitesimal parameters $\Sigma_{ABCD}$ satisfy the complex self-duality condition

$$\Sigma_{ABCD} \equiv \Sigma^{ABCD} = \frac{1}{4!} \epsilon^{ABCDEFGH} \Sigma_{EFGH} \tag{2.4}$$
Thus $F_{AB}$ transforms in the 28 of $SU(8)$.

The $\Lambda$ and $\Sigma$ transformations in one basis are a combination of the $\Lambda$ and $\Sigma$ transformations of the other one, but it is a remarkable fact that, algebraically, they appear in a very similar way in both cases. This means that, if we construct an $E_{7(7)}$ invariant in the $SL(8, \mathbb{R})$ basis, we can immediately write another one (not necessarily equivalent) in the $SU(8)$ basis by formally replacing everywhere the components $G_{ij}$ by $F_{AB}$ and the components $F_{ij}$ by $F^{AB}$.

The relation between the components of $X$ in both bases is

$$F^{AB} = \frac{1}{4\sqrt{2}} \left( F^{ij} - iG_{ij} \right) \Gamma^{ij}_{AB},$$

where the $\Gamma^{ij}_{AB}$ are the $SO(8)$ gamma matrices.

Let us consider the charges of the theory. Associated to the electric and magnetic field strengths $F^{ij}$ and $G_{ij}$ we have as many magnetic and electric charges resp. $p^{ij}$ and $q_{ij}$ defined by

$$p^{ij} \equiv \int_{S^2_\infty} F^{ij}, \quad q_{ij} \equiv \int_{S^2_\infty} G_{ij},$$

that we can combine into a real, symplectic, charge vector

$$Q \equiv \begin{pmatrix} p^{ij} \\ q_{ij} \end{pmatrix},$$

which will transform in the 56 $E_{7(7)} \subset Sp(56, \mathbb{R})$. This object occurs naturally in the $SL(8, \mathbb{R})$ basis, but we can rewrite it in the $SU(8)$ basis using the above formulae

$$Q^{AB} \equiv \frac{1}{4\sqrt{2}} \left( p^{ij} - iq_{ij} \right) \Gamma^{ij}_{AB}.$$  

This object is sometimes written in the literature as $Z^{AB}$, which seems to suggest that it is the central charge of the theory, which it is not since, in particular, it is moduli-independent (hence the change of notation). The central charge of the theory is a moduli-dependent quantity given by

$$Z_{IJ}(\phi, Q) \equiv \langle V_{IJ} | Q \rangle = \frac{1}{2} \left( p^{ij} h_{ij IJ} - q_{ij} f^{ij}_{IJ} \right),$$

and only transforms under induced $SU(8)$ transformations. Its value at spatial infinity is the (magnetic) charge of the graviphoton

$$Z_{IJ} = Z_{IJ}(\phi, Q) = \int_{S^2_\infty} T_{IJ}.$$  

### 2.2 Invariants

Let us now consider the invariants that can be constructed with the charges. The symplectic product of two fundamentals in the real basis

$$\langle Q_1 | Q_2 \rangle = -\frac{1}{2} \left[ \text{Tr}_{SL(8, \mathbb{R})} (p_2 \cdot q_1) - \text{Tr}_{SL(8, \mathbb{R})} (p_1 \cdot q_2) \right],$$

11
where
\[ \text{Tr}_{SL(8, \mathbb{R})}(p \cdot q) \equiv p^{ij} q_{ji}, \quad (2.12) \]
is automatically \( E_{7(7)} \) invariant since \( E_{7(7)} \) acts as a subgroup of \( Sp(56, \mathbb{R}) \) but it vanishes identically for a single charge \( Q_1 = Q_2 \) due to its antisymmetry. There are no other moduli-independent quadratic invariants.

Let us consider the quartic invariants of a single fundamental \( Q \). Cartan’s quartic \( E_{7(7)} \) invariant \( J_4(Q) \) \([20]\) is given in the \( SL(8, \mathbb{R}) \) real basis by
\[ J_4(Q) = \text{Tr}_{SL(8, \mathbb{R})}(p \cdot q \cdot p \cdot q) - \frac{1}{4} \left[ \text{Tr}_{SL(8, \mathbb{R})}(p \cdot q) \right]^2 + \frac{1}{4} \text{Pf} ||q|| + \frac{1}{4} \text{Pf} ||p||, \quad (2.13) \]
where
\[ \text{Tr}_{SL(8, \mathbb{R})}(p \cdot q \cdot p \cdot q) \equiv p^{ij} q_{jk} p^{kl} q_{li}, \quad (2.14) \]
and where Pf stands for the Pfaffian of an antisymmetric matrix of even dimension, which is the square root of the determinant:
\[ \text{Pf}_{SL(8, \mathbb{R})} ||q|| \equiv (\det ||q||)^{1/2} = \frac{1}{4!} \varepsilon^{ijklmnpq} q_{ij} q_{kl} q_{mn} q_{pq}, \quad \text{Pf}_{SL(8, \mathbb{R})} ||p|| \equiv (\det ||p||)^{1/2} = \frac{1}{4!} \varepsilon^{ijklmnpq} p^{ij} p^{kl} p^{mn} p^{pq}. \quad (2.15) \]

The Cartan invariant \( J_4(Q) \) is manifestly invariant under the global \( SL(8, \mathbb{R}) \) subgroup of \( E_{7(7)} \). It is enough to check the invariance under the off-diagonal transformations generated by the selfdual \( \Sigma \) parameters
\[ \delta_{\Sigma} p^{ij} = \Sigma^{ijkl} q_{kl}, \quad \delta_{\Sigma} q_{ij} = \Sigma_{ijkl} p^{kl}, \quad (2.16) \]
to prove the invariance under the full \( E_{7(7)} \).

The Julia-Cremmer quartic invariant \( \Diamond(Q) \) \([5]\) is defined in the complex basis and can be obtained from \( J_4 \) by following the recipe given in the paragraph below Eq. (2.4), which explains why it is so similar to it:
\[ \Diamond(Q) = \text{Tr}_{SU(8)}(Q \overline{Q} Q \overline{Q}) - \frac{1}{4} \left[ \text{Tr}_{SU(8)}(Q \overline{Q}) \right]^2 + \frac{1}{4} \text{Pf}_{SU(8)} ||Q|| + \frac{1}{4} \text{Pf}_{SU(8)} ||\overline{Q}||, \quad (2.17) \]
where
\[ \text{Tr}_{SU(8)}(Q \overline{Q} Q \overline{Q}) \equiv Q_{AB} \overline{Q}^{BC} Q_{CD} \overline{Q}^{DE}, \]
\[ \text{Tr}_{SU(8)}(Q \overline{Q}) \equiv -Q_{AB} \overline{Q}^{AB}, \]
\[ \text{Pf}_{SU(8)} ||Q|| \equiv \frac{1}{4!} \varepsilon^{ABCDEFGH} Q_{AB} Q_{CD} Q_{EF} Q_{GH}, \]
\[ \text{Pf}_{SU(8)} ||\overline{Q}|| \equiv \overline{\text{Pf}}_{SU(8)} ||Q||, \quad (2.18) \]
and \( Q_{AB} \) is the complex combination of the electric and magnetic charges defined in Eq. (2.8). \( \Diamond \) is manifestly invariant under the global \( SU(8) \) subgroup of \( E_{7(7)} \). Again, it is enough to
check the invariance under the off-diagonal transformations generated by the complex self-dual \( \Sigma \) parameters

\[
\delta_\Sigma Q_{AB} = \Sigma_{ABCD} Q^{CD}, \quad \delta_\Sigma \bar{Q}^{AB} = \Sigma^{ABCD} \bar{Q}_{CD},
\]

(2.19) to prove the invariance under the full \( E_{7(7)} \), but this follows from the invariance of \( J_4 \), as observed in the paragraph below Eq. (2.4).

It was argued in [5] that \( J_4(Q) \) and \( \bar{\Diamond}(Q) \) should be proportional. The precise relation was established in [32]

\[
J_4(Q) = -\bar{\Diamond}(Q).
\]

(2.20) The detailed proof was presented in [23].

Now let us consider invariants for the central charge \( Z_{IJ}(\phi, Q) \). In this case it is enough to build manifestly-\( SU(8) \)-invariant quantities. At the quadratic level there is only one

\[
\text{Tr}_{SU(8)}(\bar{Z} Z) = \bar{Z}^{IJ} Z_{JI}.
\]

(2.21) As shown in [31], this invariant gives the black-hole potential of the FGK formalism [33]:

\[
V_{bh}(\phi, Q) = -\text{Tr}_{SU(8)}(\bar{Z} Z) = \frac{1}{2} Q^I M(N) Q,
\]

(2.22) where

\[
M(N) \equiv \begin{pmatrix} (\mathcal{J} + \mathfrak{R}\mathcal{J}^{-1}\mathfrak{R})_{\Lambda\Sigma} & - (\mathfrak{R}\mathcal{J}^{-1})_{\Lambda\Sigma} \\ - (\mathcal{J}^{-1}\mathfrak{R})^{A\Sigma} & (\mathcal{J}^{-1})^{A\Sigma} \end{pmatrix},
\]

(2.23)

\[
\mathfrak{R}_{\Lambda\Sigma} \equiv \Re (N_{\Lambda\Sigma}), \quad \mathcal{J}_{\Lambda\Sigma} \equiv \Im (N_{\Lambda\Sigma}), \quad (\mathcal{J}^{-1})^{\Lambda\Sigma} \mathfrak{J}_{\Sigma\Gamma} = \delta^{\Lambda\Gamma}.
\]

At the quartic level four \( E_{7(7)} \) invariants can be constructed from \( Z_{IJ} \), namely

\[
\text{Tr}_{SU(8)}(Z \bar{Z} Z Z) = [\text{Tr}_{SU(8)}(\bar{Z} Z)]^2, \quad \text{Pf}_{SU(8)} ||Z||,
\]

(2.24) and the complex conjugate of the latter (the other two are real). In this case we are interested in real, moduli-independent, \( E_{7(7)} \) invariants and, if any, they should be linear combinations of those four invariants. To check the moduli-independence we must take the derivative of the linear combination w.r.t. the scalar fields and then use the following identity

\[
\mathcal{D} Z_{IJ} = \frac{1}{2} P_{IJKL} \bar{Z}^{KL},
\]

(2.25) which follows from the Maurer-Cartan equations (1.15). The Vielbein components satisfy the complex self-duality constraint

\[
(P_{IJKL}) = \bar{P}^{IJKL} = \frac{1}{4} \epsilon^{IJKLMN} P_{MNOP}.
\]

(2.26) 11It should be stressed that the \( SU(8) \) indices \( A, B, C, \ldots \) that we have just discussed are different from the \( SU(8) \) indices \( I, J, K, \ldots \) of the \( E_{7(7)}/SU(8) \) coset of scalars: the former transform under \( E_{7(7)} \) as described above and the latter always transform via induced \( SU(8) \) transformations. This means that, for instance, the \( SU(8) \)-invariant quadratic expression \( \bar{Z}^{IJ} Z_{IJ} \) is \( E_{7(7)} \) invariant (but moduli-dependent) while the manifestly \( SU(8) \)-invariant quadratic expression \( \Sigma^{AB} Q_{AB} \) is moduli-independent but not fully \( E_{7(7)} \) invariant.
This constraint should be compared with Eq. (2.4).

Observe that the covariant derivative Eq. (2.25) of $Z_{IJ}$ has the same structure as the $\Sigma$ transformation of $Q_{AB}$ Eq. (2.19) with the complex self-dual parameter $\Sigma_{ABCD}$ replaced by the complex self-dual Vielbein $P_{IJKL}$. This means that, if we construct an $E_{7(7)}$ for $Q$ in the $SL(8, \mathbb{R})$, then we can construct another in the $SU(8)$ basis replacing $g_{ij}$ by $Q_{AB}$ and $p^ij$ by $\overline{Q}^{AB}$ and then, replacing everywhere $Q_{AB}$ by $Z_{IJ}$, with the corresponding change of indices $A, B, C, D, \ldots$ by $I, J, K, L, \ldots$, we automatically obtain a moduli-independent $E_{7(7)}$ invariant.

In particular

\[ \diamond(Z) \equiv \text{Tr}_{SU(8)} \left( Z \overline{Z} \right) - \frac{1}{4} \left[ \text{Tr}_{SU(8)} \left( Z \overline{Z} \right) \right]^2 + \frac{1}{4} \left( \text{Pf}_{SU(8)} ||Z|| + c.c. \right). \]  \hspace{1cm} (2.27)

is manifestly $E_{7(7)}$ invariant because each term is separately invariant, and, according to the above argument, it is also automatically moduli-independent [34]. We have used the same symbol as for the Cremmer-Julia invariant because, this invariant is formally identical, although the transformation rules of the $SU(8)$ indices are completely different.

Since this invariant is quartic in charges and moduli-independent, it has been argued [31] that $\diamond(Z)$ is also proportional to $\diamond(Q)$. The identity can be checked in the vanishing scalars limit.

To this (zeroth) order in scalars, the components of the symplectic section $V_{IJ}$ defined in (1.5) are given by

\[ f_{ijIJ} = \frac{i}{\sqrt{2}} \Gamma^{ijIJ}, \quad h_{ijIJ} = \frac{1}{\sqrt{2}} \Gamma^{ijIJ}, \]  \hspace{1cm} (2.28)

where $\Gamma^{ijIJ}$ are the $SO(8)$ gamma matrices, and, then, the components of the central charge matrix are given by

\[ Z_{IJ} = \frac{1}{4\sqrt{2}} \left( p^{ij} - i q_{ij} \right) \Gamma^{ijIJ}, \]  \hspace{1cm} (2.29)

which is nothing but $\overline{Q}^{AB}$, defined in Eq. (2.8), written with indices $I, J$. Then, the scalar independence of the whole expression extends the identity to arbitrary values of the scalars.

### 2.2.1 “Linearization” of the quartic invariant

In non-associative algebras the process of “linearization” (sometimes called “polarization”) is defined as follows: given a homogeneous polynomial $p(x)$ of degree $n$, the process of “linearization” is designed to create a completely symmetric multilinear polynomial $p'(x_1, \ldots, x_n)$ in $n$ variables such that the original polynomial is recovered when all the variables $x_i$ have the same value $x$, that is $p'(x, \cdots, x) = p(x)$. For example, the full linearization of the cube $x^3 = xxx$ of degree 3 is

\[ \frac{1}{3!} \left( x_1 x_2 x_3 + x_1 x_3 x_2 + x_2 x_1 x_3 + x_2 x_3 x_1 + x_3 x_1 x_2 + x_3 x_2 x_1 \right). \]  \hspace{1cm} (2.30)

Here we are interested in the invariant obtained as the linearization of the quartic invariants. A 4-linear invariant $q(Q_1, Q_2, Q_3, Q_4)$ such that

\[ J_4(Q) = q(Q, Q, Q, Q), \]  \hspace{1cm} (2.31)

\footnote{A convention for products of more than two elements (right $x^3 = xxx \equiv x(xx)$ or left association $x^3 = xxx \equiv (xx)x$ is understood to have been chosen, the particular conventions are irrelevant in our discussion.}
was explicitly constructed in \cite{35} in a form based on Freudenthal triple systems. The linearization of $J_4(Q)$ is obtained by averaging over all the permutations of the four entries:

$$J'_4(Q_1, Q_2, Q_3, Q_4) = \frac{1}{4!} \sum_{\pi \in S^4} q(Q_{1\pi}, Q_{2\pi}, Q_{3\pi}, Q_{4\pi}), \quad (2.32)$$

and satisfies

$$J_4(Q) = J'_4(Q, Q, Q, Q). \quad (2.33)$$

An explicit form of this quartic invariant in the $SL(8, \mathbb{R})$ basis convenient for studies of $\mathcal{N} = 8$ supergravity was given in \cite{33} using the symplectic product and the existence the triple product

$$56 \times 56 \times 56 \rightarrow 56, \quad (2.34)$$

that we can denote by $(Q_1, Q_2, Q_3)$ and whose explicit form can be found in \cite{33}. The resulting 4-linear quartic invariant is given by

$$q(Q_1, Q_2, Q_3, Q_4) = \langle (Q_1, Q_2, Q_3) | Q_4 \rangle, \quad (2.35)$$

and it was shown (indirectly) to agree with the one in \cite{35}. It can be modified and, in particular, symmetrized, by adding products of symplectic products of charge vectors, such as $\langle Q_1 | Q_2 \rangle \langle Q_3 | Q_4 \rangle$. The fully symmetrized invariant can be identified with the linearization $J'_4(Q_1, Q_2, Q_3, Q_4)$ and its explicit form is

$$J'_4(Q_1, Q_2, Q_3, Q_4) \equiv \frac{1}{8} \text{Tr}_{SL(8, \mathbb{R})} \left\{ p_1 \cdot q_2 \cdot p_3 \cdot q_4 + p_1 \cdot q_3 \cdot p_4 \cdot q_2 + p_1 \cdot q_4 \cdot p_2 \cdot q_3 + (p \leftrightarrow q) \right\}$$

$$- \frac{1}{16} \left\{ [Q_1 | Q_2] [Q_3 | Q_4] + [Q_1 | Q_4] [Q_2 | Q_3] + [Q_1 | Q_3] [Q_2 | Q_4] \right\}$$

$$+ \frac{1}{4} \left[ \text{Pf}_{SL(8, \mathbb{R})} ||p_1 p_2 p_3 p_4|| + (p \leftrightarrow q) \right], \quad (2.36)$$

where we have defined, for convenience, the symmetric product

$$[Q_1 | Q_2] \equiv - \frac{1}{2} \text{Tr}_{SL(8, \mathbb{R})} (p_1 \cdot q_2 + (p \leftrightarrow q)) = \frac{1}{2} (p_1^{ij} q_2 + p_2^{ij} q_1), \quad (2.37)$$

and we are using the notation

$$\text{Pf}||p_1 p_2 p_3 p_4|| \equiv \frac{1}{4!} \varepsilon^{ijklmnop} q_1 q_2 q_3 q_4 p_1 p_2 p_3 p_4, \quad (2.38)$$

Observe that the first six terms are the linearization of the first term $\text{Tr}_{SL(8, \mathbb{R})} (p \cdot q \cdot p \cdot q)$ in the Cartan invariant $J_4(Q)$ Eq. (2.13), the next three terms are the linearization of the second term in $J_4(Q)$, $\left[ \text{Tr}_{SL(8, \mathbb{R})} (p \cdot q) \right]^2$, and the last two are the linearization of the Pfaffians. They are automatically symmetric, and only one term is needed for each Pfaffian.

It is important here that each of the four $56$’s appears linearly in this invariant. By construction, each term is manifestly invariant by itself under the 63 $SL(8, \mathbb{R})$ transformations.
parametrized by the $\Lambda_{ij}$ of (2.1). However, the remaining 70 $E_{7(7)}$ transformations parametrized by the off-diagonal $\Sigma_{ijkl}$ mix all the terms in $J'_4(Q_1, Q_2, Q_3, Q_4)$ in a non-trivial way, which makes the contribution of all the terms necessary to have complete $E_{7(7)}$ symmetry.

From the point of view of the $E_{7(7)}$ symmetry, the fact that all these terms have to cooperate to provide an invariant can be established, of course, by a brute force computation, using (2.1). However, since the structure of the invariant has a significant impact on the UV finiteness of $\mathcal{N} = 8$ supergravity, it may be useful to explain the octonionic origin of this $E_{7(7)}$ invariant in the context of the non-Euclidean Jordan algebras, following [23, 24] and [36]. For this purpose we need to change the basis from the Cartan one, used in (2.36), to either the Freudenthal/Jordan or the Fano basis. We present some useful information on this in Appendix C.

It is evident that following the rule explained in the paragraph above Eq. (2.27) we can obtain the Cremmer-Julia invariant $\diamond (Q)$ (2.17) from the Cartan invariant $J'_4(Q_1, Q_2, Q_3, Q_4)$ and using it we can get the linearization of the Cremmer-Julia invariant $\hat{\diamond}'(Q_1, Q_2, Q_3, Q_4)$ from the linearization of the Cartan invariant given in Eq. (2.36):

\[
\hat{\diamond}'(Q_1, Q_2, Q_3, Q_4) \equiv \frac{1}{6} \text{Tr}_{SU(8)} \left\{ \overline{Q}_1 \cdot Q_2 \cdot \overline{Q}_3 \cdot Q_4 + \overline{Q}_1 \cdot Q_3 \cdot \overline{Q}_4 \cdot Q_2 + \overline{Q}_1 \cdot Q_4 \cdot \overline{Q}_2 \cdot Q_3 + \text{c.c.} \right\}
- \frac{1}{12} \left\{ [Q_1 | Q_2][Q_3 | Q_4] + [Q_1 | Q_3][Q_2 | Q_4] + [Q_1 | Q_4][Q_2 | Q_3] \right\}
+ \frac{1}{4} \left[ \text{Pf}_{SU(8)}[\overline{Q}_1 \overline{Q}_2 \overline{Q}_3 \overline{Q}_4] \right] + \text{c.c.},
\]

(2.39)

where the Pfaffians and traces have the obvious expressions and the symmetric products are given by

\[
[Q_1 | Q_2] = -\frac{1}{2} \text{Tr}_{SU(8)} \left\{ \overline{Q}_1 \cdot Q_2 \right\} + \text{c.c.} = \frac{1}{2} \left[ \overline{Q}_1 AB Q_2 AB + \text{c.c.} \right].
\]

(2.40)

Finally, we can also linearize the moduli-independent $E_{7(7)}$ invariant of the central charge (2.27), but the linearization procedure does not necessarily guarantee moduli-independence. However, according to the observation in the paragraph above (2.27), if we replace in the linearization of the Cremmer-Julia invariant (2.39) $Q_{A,B}$ by $Z_{I,J}$, we immediately get a moduli-independent $E_{7(7)}$ invariant expression. Nevertheless, there is an important subtlety we must pay attention to: the contraction of “local” $SU(8)$ indices $I, J, \ldots$ is $SU(8)$ and $E_{7(7)}$ invariant if and only if the scalars that occur in the $SU(8)$ matrices are functions of the same variables. For example

\[
Z_{1J} \overline{Z}_{2J} = Z_{1I}[\phi(x_1), Q_1] \overline{Z}_{2J}^{IJ}[\phi(x_2), Q_2],
\]

(2.41)

will only be invariant if $\phi(x_1) = \phi(x_2)$, which, generically, requires that $x_1 = x_2$.

When we discuss the construction of $E_{7(7)}$ invariants for amplitudes from graviphoton field strengths, this observation will play an important role.

2.3 Black-hole areas

According to the general results of [33], the area of the extremal $\mathcal{N} = 8$ black holes is given by the value of the black-hole potential Eq. (2.22) on the horizon, where the scalars take values that
only depend on the charges. The resulting expression $V_{bh}(\phi_h, Q)$ only depends on the charges and is, by construction, invariant under $E_{7(7)}$ duality transformations.

It was argued in [21] that this area is actually proportional to the square root of the Cartan-Julia-Cremmer quartic invariant

$$A = 4\pi \sqrt{\left\langle \phi \right\rangle},$$

which requires the highly non-trivial identity

$$V_{bh}(\phi_h, Q) = -\text{Tr}_{SU(8)}(\mathcal{Z}_h \mathcal{Z}_h) = \frac{1}{2} Q^T M(N)_h Q = 4\pi \sqrt{\left\langle \phi \right\rangle}.$$

### 2.4 $E_{7(7)}$ invariants in 2-center $N = 8$ solutions

In the context of 2-center extremal black-hole solutions there was a complete analysis in [37] of the possible $E_{7(7)}$ invariants depending on the two fundamentals $Q_1$ and $Q_2$ that characterize each black hole. The analysis takes into account the horizontal symmetry group $SL(2, \mathbb{R})$, which rotates the two fundamentals. Seven of them, which are irreducible, are described below. These invariants help to study different physical properties, such as marginal stability and split attractor flow solutions with two dyonic black-hole charge vectors.

There are two invariants which are antisymmetric in $Q_1$ and $Q_2$. One is the quadratic symplectic invariant $\left\langle Q_1 \mid Q_2 \right\rangle$ and the other one is sextic. The quadratic one is related to the total angular momentum of the 2-center solution: in $N = 2$ theories the angular momentum of 2-center solutions is given by the symplectic product of the charge-vectors $Q_{1,2}$ of the two centers (see, e.g., [38])

$$\vec{J} = \left\langle Q_1 \mid Q_2 \right\rangle \frac{(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|},$$

where $\vec{x}_{1,2}$ are the “positions” of the horizons of the two black holes in the transverse Euclidean 3-dimensional space. The Dirac-Schwinger-Zwanziger quantization condition can be applied immediately to it, implying the quantization of the total angular momentum. The same expression must be valid in $N = 8$ supergravity.

The are also five different $E_{7(7)}$ invariants depending on $Q_1$ and $Q_2$ so that at $Q_1 = Q_2$ they all reduce to the familiar quartic invariant. They are usually described in terms of the so-called $\mathbb{K}$-tensor which is totally symmetric in its 4 symplectic indices

$$\mathbb{K}_{MNPQ} = \mathbb{K}_{(MNPQ)},$$

and can be defined by its contraction with four different fundamentals:

$$\mathbb{K}_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q \equiv J'_4(Q_1, Q_2, Q_3, Q_4).$$

---

13 At least, for the supersymmetric ones.
14 It follows from Eq. (4.40) of [30], which is identical to the equation that appears in $N = 2$.
15 Each symplectic index $M, N, \ldots$ is equivalent to a pair of indices $\Lambda, \Sigma, \ldots$ each of which is equivalent, in turn, to an antisymmetric pair of $SL(8, \mathbb{R})$ indices $[ij]$. 
The five quartic invariants relevant for the 2-center solution listed in [37] are, then
\[
I_{+2} = K_{MNPQ} Q_1^M Q_1^N Q_1^P Q_1^Q = J'_4(Q_1, Q_1, Q_1, Q_1) = J_4(Q_1), \\
I_{+1} = K_{MNPQ} Q_1^M Q_1^N Q_1^P Q_2^Q = J'_4(Q_1, Q_1, Q_1, Q_2), \\
I_0 = K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q = J'_4(Q_1, Q_1, Q_2, Q_2), \\
I_{-1} = K_{MNPQ} Q_2^M Q_2^N Q_2^P Q_2^Q = J'_4(Q_2, Q_2, Q_2, Q_2), \\
I_{-2} = K_{MNPQ} Q_2^M Q_2^N Q_2^P Q_2^Q = J'_4(Q_2, Q_2, Q_2, Q_2) = J_4(Q_2).
\] (2.47)

The explicit expression in (2.36) detailing (2.46) will prove very useful, since it will allow us to deduce the specific properties of the four-linear invariant and to compare with amplitudes. These properties can be traced to the fact that in groups of type $E_7$ the quartic invariant is not a perfect square, whereas in the degenerate groups of type $E_7$ they form a perfect square, see [17] for more details.

3 New four-linear $E_{7(7)}$ invariants and 4-vector amplitudes

For the $E_{7(7)}$-symmetric quartic in vectors local action we need 4 different fundamental 56 depending on 4 different momenta\[\footnote{As discussed above Eq. (2.41), in principle we cannot use graviphoton field strengths depending on different momenta to construct $E_{7(7)}$ invariants. However, in the vanishing scalar limit the expressions that we will obtain can be reinterpreted in terms of graviphoton field strengths, see Eq. (3.12).}]
\[
\mathcal{F}^{AB}_{\mu\nu}(p_I) \equiv \frac{1}{4\sqrt{2}} \left( F^{ij}(p_I) - iG_{ij}(p_I) \right) \Gamma^{ij}_{AB}, \quad I = 1, \ldots, 4, \quad p_1 + p_2 + p_3 + p_4 = 0.
\] (3.1)

where $i, j = 1, \ldots, 8$, $A, B = 1, \ldots, 8$.

Using the linearization of the Cartan invariant (2.36) we can construct an $E_{7(7)}$-invariant Lorentz tensor with 4 pairs of antisymmetric indices depending on the 4 momenta $p_I$:
\[
\hat{\omega}^{(8)} \equiv \hat{\omega}^{(8)}(p_1, p_2, p_3, p_4) \equiv \hat{\omega}^i \left[ F_{\mu_1 \nu_1}(p_I), F_{\mu_2 \nu_2}(p_2), F_{\mu_3 \nu_3}(p_3), F_{\mu_4 \nu_4}(p_4) \right].
\] (3.2)

It remains to make this expression Lorentz invariant.

3.1 Taking care of Lorentz invariance

To make (3.2) Lorentz invariant we must contract the 4 pairs of antisymmetric Lorentz indices with another scalar- and vector-independent Lorentz tensor. Which tensor should we use?

We may view the new duality-invariant Born-Infeld models with higher derivatives [11] as the reduction from $E_{7(7)}$ symmetry to $U(1)$ and, in the corresponding manifestly $U(1)$-invariant initial source of deformation, the 4 pairs of antisymmetric Lorentz indices are contracted with the
$t^{(8)}$ tensor. This tensor, whose definition and properties are described in Appendix B, is totally symmetric in four pairs of antisymmetric indices. This fact suggests that we must use $t^{(8)}$. In such case, the expression which is both Lorentz- as well as $E_{7(7)}$- invariant is

$$I_f \equiv t^{(8)} \cdot \hat{\hat{\phi}}^{(8)} f(s, t, u)$$

(3.3)

where $f(s, t, u)$ is an appropriate function of the Mandelstam variables and

$$t^{(8)} \cdot \hat{\hat{\phi}}^{(8)} \equiv t^{(8)abcd} \hat{\hat{\phi}}^{(8)abcd},$$

(3.4)

where we have replaced each pair of Lorentz indices by a single Latin index $a, b, \ldots$, a notation explained in Appendix B. It is of crucial importance here that the function $f(s, t, u)$ only occurs as a common factor of all the terms in $\hat{\hat{\phi}}^{(8)}$, because, otherwise, the $E_{7(7)}$ symmetry would be broken. Thus, (3.3) is our candidate for to manifestly $E_{7(7)}$-invariant source of deformation.

The detailed form of the invariant (3.3) is complicated. The expression can be simplified using the following property, derived in Appendix B:

$$t^{(8)abcd} A^a B^b C^c D^d = 4 \text{Tr}_L(A + B + C + D) + 4 \text{Tr}_L(A + C) \text{Tr}_L(B - D) + 4 \text{Tr}_L(A + D) \text{Tr}(B - C) + \text{(other terms)}.$$  

(3.5)

where $\text{Tr}_L$ stands for the trace over the Lorentz indices. Observe that only terms with the contraction of two selfdual nd two antiselfdual tensors occur.

3.2 Comparison of the unconstrained $E_{7(7)}$ invariants with the counterto-terms and 4-vector amplitudes

The structure of the UV-divergent 4-vector amplitude is obtained by using the linear twisted self-duality constraint on the manifestly $E_{7(7)}$ invariant source of deformation. In our case it means that we have to use a condition $G_\Lambda^+ = \hat{N}_\Lambda \Sigma F^{\Sigma} +$ presented in Eq. (1.3) and where the 28 $\Lambda \rightarrow [ij]$.

If we are interested only in the 4-vector amplitude, we can linearize the vector kinetic matrix $\hat{N}_\Lambda \Sigma$

$$\hat{N}_\Lambda \Sigma = -i \delta_{\Lambda \Sigma} + O(\phi),$$

(3.6)

and, to the lowest order in scalars,

$$G_{ij}^+ \sim i F^{ij},$$

(3.7)

so, in the complex $SU(8)$ basis (2.5)

$$\overline{F}^{AB} \sim \frac{1}{2\sqrt{2}} F^{ij} + \Gamma^{ij}_{AB} \sim -i \frac{1}{2\sqrt{2}} G_{ij}^+ + \Gamma^{ij}_{AB} = \overline{F}^{AB}_+.$$

(3.8)

This implies that

$$\overline{F}^{AB}_- = F_{AB}^+ = 0,$$

(3.9)
which, in turn, implies that the $E_{7(7)}$ transformations (2.3) reduce to global $SU(8)$ transformations

$$\delta \mathcal{F}^{AB+} = +2\Lambda [A|C\mathcal{F}^{C[B]+}, \quad (3.10)$$

$$\delta \mathcal{F}^{AB-} = -2\Lambda [C|A\mathcal{F}^{C[B]-}].$$

Furthermore, using the expression of the the components of the symplectic section $\mathcal{V}_{IJ}$ in this limit, given in Eq. (2.28), we find

$$T_{IJ} \sim \frac{1}{4\sqrt{2}} \left(F^{ij} - iG_{ij}\right) \Gamma^{ij}_{IJ}, \quad (3.11)$$

which can be compared with (2.5). Then, at this order, taking into account the above results, the “local”, lower $IJ$ and “global” upper $AB$ indices can be identified and, in particular,

$$T_{IJ} \sim \frac{1}{2\sqrt{2}} F^{ij} + \Gamma^{ij}_{AB} \sim \mathcal{F}^{AB+}, \quad (3.12)$$

$$\Rightarrow T_{IJ}^- \rightarrow \mathcal{F}^{AB+}.\quad (3.13)$$

The consistency of this identification is guaranteed by the comparison between the constraints (1.14) and (3.9) and by the fact that, at this order in scalars, $\mathcal{F}^{AB+}$ only transforms under global $SU(8)$ transformations (3.10), just like the graviphoton (to this order in scalars). Observe that this implies that, again to this order, expressions like

$$\text{Tr}_{SU(8)} \left( F_1 F_2 \right) = F_1^{AB} F_2^{AB+},$$

$$\text{Tr}_{SU(8)} \left( F_1 F_2 F_3 F_4 \right) = F_1^{AB} F_2^{BC} + F_3^{CD} F_4^{DA+}, (3.13)$$

etc. are fully $E_{7(7)}$-invariant.

The 4-vector amplitude in $\mathcal{N} = 8$ supergravity, being the MHV amplitude, is proportional to the tree amplitude, up to a polynomial in Mandelstam variables. Actually, according to Eq. (3.5) of [13], the tree-level 4-vector amplitude is given by

$$\langle b^-_{AB} b^-_{CD} b^+_{EF} b^+_{GH} \rangle = -(12)^2 [34]^2 \left[ \frac{1}{t} \delta_{AB}^{EF} \delta_{CD}^{GH} + \frac{1}{u} \delta_{AB}^{GH} \delta_{CD}^{EF} + \frac{1}{s} \delta_{AB}^{FG} \delta_{CD}^{HE} \right].$$

Here $\delta_{AB}^{EF}$ is the antisymmetrized product of gammas with weight one, taking the values 0, 1 and −1. The first two terms correspond to

$$\text{Tr}_{SU(8)}(F_1^- F_3^+) \text{Tr}_{SU(8)}(F_2^- F_4^+) = F_1^{AB} F_3^{AB+} F_2^{CD} F_4^{+CD},$$

and

$$\text{Tr}_{SU(8)}(F_1^- F_4^+) \text{Tr}_{SU(8)}(F_2^+ F_3^+) = F_1^{AB} F_4^{AB+} F_2^{CD} F_3^{+CD},$$

while the third term corresponds to

$$\text{Tr}_{SU(8)} \left( F_1^- F_3^+ F_2^- F_4^+ \right).$$

20
The 3-loop 4-vector part of the counterterm is related to the three-loop on shell amplitude by multiplication by the global factor \( stu \). This is shown in Eq. (5.1) of [13] as

\[
M^3_{\text{UV4vec}}(b^{-}_{AB}, b^{-}_{CD}, b^{+}_{+}, b^{+}_{GH}) = -k^4(12)^2[34]^2 [s u \delta_{AB}^{EF} \delta_{CD}^{GH} + s t \delta_{AB}^{FG} \delta_{CD}^{HE} + t u \delta_{AB}^{EF} \delta_{CD}^{HE}] .
\]

In coordinate space the 4-vector amplitude may be given in the form

\[
2\text{Tr}_{SU(8)} \left( \partial_\mu F^{-\alpha\beta} \bar{F}^{+\dot{\alpha}\dot{\beta}} \right) \text{Tr}_{SU(8)} \left( \partial^\mu F^{-\alpha\beta} \partial^\nu F^{+\dot{\alpha}\dot{\beta}} \right) + \text{c.c.}
\]

\[
= 2\partial_\mu \partial_\nu F^{-AB} \partial^\mu F^{CD} \text{Tr}_L \left( F^{+AB} \partial^\nu F^{CD} \right) + \text{c.c.}
\]

where \( \text{Tr}_L \) stands for the trace in the Lorentz indices that are not shown, which is close to the one given in [14].

It is clear from (3.18) that the terms with different \( SU(8) \) index structure have different dependence on the Mandelstam variables. This is not possible if one uses our candidate for \( E_7(7) \)-invariant source of deformation. As shown in (3.4) and (3.3) the 4-linear \( E_7(7) \) invariant only admits a function of Mandelstam variables that appears as a factor, common to all the \( E_7(7) \) and \( SU(8) \) structures. Therefore the manifestly \( E_7(7) \) invariant candidate for the source of deformation is in disagreement with the counterterm[17]. In other words: the counterterm is \( E_7(7) \) invariant, as discussed above, but it cannot be obtained from the \( E_7(7) \)-invariant initial source of deformation (3.4). This rules out the 4-linear forms built from 4 fundamentals, as a possible source of deformation in \( \mathcal{N} = 8 \) supergravity.

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17If one tries to get the counterterm in position space via the invariants

\[
t^{(8)} \cdot \mathcal{O}_{(8)} (\partial_\mu F, \partial_\nu F, \partial^\mu F, \partial^\nu F) , \quad t^{(8)} \cdot \mathcal{O}_{(8)} (\partial_\mu \partial_\nu F, \partial^\mu F, \partial^\nu F, F) , \quad t^{(8)} \cdot \mathcal{O}_{(8)} (\partial_\mu \partial_\nu F, \partial^\mu \partial^\nu F, F, F) , \quad (3.20)
\]

one immediately sees that one finds additional terms required by invariance under the off-diagonal part of \( E_7(7) \) that are not present in the counterterm.
4 Reduction to smaller, non-degenerate and degenerate groups of type \( E7 \)

Here we would like to study the cases with smaller duality symmetry group, for example the case of \( \mathcal{N} = 4 \) supergravity with \( SL(2, \mathbb{R}) \times SU(4) \) duality, representing the non-degenerate group of type \( E7 \), and the case of Born Infeld model with derivatives with \( U(1) \) duality, without scalars, representing the degenerate case of group of type \( E7 \).

4.1 Reduction to \( SL(2, \mathbb{R}) \times SU(4) \)

In our first example we consider the reduction of the \( E_{7(7)} \) duality group of \( \mathcal{N} = 8 \) supergravity to the \( SL(2, \mathbb{R}) \times SU(4) \) duality group of pure \( \mathcal{N} = 4 \) supergravity. This duality group is of type \( E7 \), a class defined by Brown in [17] and studied in the context of extended supergravities in [18]. The reason for our interest in \( \mathcal{N} = 4 \) supergravity is the recent computation of the 3-loop four-particle amplitude in this theory, which was found to be divergence-free in [3].

To get pure \( \mathcal{N} = 4 \) supergravity from \( \mathcal{N} = 8 \) we restrict ourselves to the \( SU(4) \times SL(2, \mathbb{R}) \) subgroup of \( E_{7(7)} \). By restricting the range of the indices in the 56 of \( E_{7(7)} \) \( (F_{ij}, G_{ij}) \) to \( i, j = 1, \cdots, 4 \) we get the \((6, 2)\) of \( SO(6) \times SL(2, \mathbb{R})\): \( F_{ij} \) and \( G_{ij} \) as \( SO(6) \sim SU(4) \) antisymmetric tensors and the pair \( (F, G) \) transforms as an \( SL(2, \mathbb{R}) \) doublet. We can also use a complex basis \( (4.1) \)

\[
\mathcal{F}^{AB}(p_I) \equiv \frac{1}{4\sqrt{2}} (F^{ij}(p_I) - iG_{ij}(p_I)) \Gamma^{ij}AB,
\]

where now \( A, B = 1, \cdots, 4 \). In this basis, the \( SU(4) \times SL(2, \mathbb{R}) \) invariant is given by Eqs. (3.2), (3.3) and (3.4).

With the same restriction in the range of the indices, the rest of the formulae discussed in the previous sections, and in particular the invariants \( J'_i \) in (2.36) and \( \tilde{\Omega} \) in (2.39), reduce to equivalent formulae for the \( \mathcal{N} = 4 \) case. The Pfaffian terms in the \( \mathcal{N} = 8 \) invariants (2.36) and (2.39) drop since in \( \mathcal{N} = 4 \) model the totally antisymmetric 8-rank tensor required in Eq. (2.38) is absent. However, the trace of four operators in the first and second line and the square of the trace of two operators in the third line (both in (2.36) and (2.39)) remain independent. This is a property of the groups of type \( E7 \) [17, 18]. The quartic invariant is not a square of the quadratic one since the symmetric bilinear form does not exist. Therefore, the term with the trace of four operators needs help from the term with the square of the trace of two operators, to form an \( SU(4) \times SL(2, \mathbb{R}) \) invariant. Each of these terms is separately \( SU(4) \) invariant, but only together, in the combination shown in Eqs. (2.36) or (2.39), they are \( SU(4) \times SL(2, \mathbb{R}) \) invariant.

Having constructed the new \( SU(4) \times SL(2, \mathbb{R}) \) invariant relevant for amplitudes, we may now proceed with the same procedure we used for \( \mathcal{N} = 8 \) amplitudes and described in the previous

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\(^{18}\) In \( d = 6 \) maximal supergravity there is a 3-loop divergence [2]. This requires a separate detailed investigation in the context of the \( SO(5, 5) \) duality symmetry of this theory. However, as a comment to this investigation of \( d = 4 \) extended supergravities, we would like to point out that \( SO(5, 5) \) is not a group of type \( E7 \). It has a symmetric bilinear form since there is a constant \( SO(5, 5) \) metric, as in any orthogonal group. This suggests that the extension of the comparison of duality invariants with tensor field amplitudes made in this paper to the \( d = 6 \) case may yield different results, in line with other models of degenerate groups of type \( E7 \), when the restoration of duality is possible via the deformation procedure of [9].
sections. It is easy to see that all the steps leading to the conclusion that the amplitude disagrees with the invariant in the $\mathcal{N} = 8$ case remains valid for $\mathcal{N} = 4$ supergravity.

### 4.2 Reduction to $U(1)$ with no scalars

The source of deformation used in [11] for the case of the open string or the D3 brane 1-loop corrections [26], depends on $T = F - iG$ and $\overline{T} = F + iG$ so that

\[
\mathcal{I} = \frac{\lambda}{24} t^{(8)}_{abcd} \left[ 2\partial_{\mu} T^{+a} \partial_{\nu} T^{-b} \partial_{\rho} T^{+c} \partial_{\sigma} T^{-d} + \partial_{\mu} T^{+a} \partial_{\nu} T^{-b} \partial_{\rho} T^{+c} \partial_{\sigma} T^{-d} \right]
\]

\[
= \frac{\lambda}{24} t^{(8)}_{abcd} \left[ \partial_{\mu} T^{+a} \partial_{\nu} T^{-b} \partial_{\rho} T^{+c} \partial_{\sigma} T^{-d} + \partial_{\mu} T^{+a} \partial_{\nu} T^{-b} \partial_{\rho} T^{+c} \partial_{\sigma} T^{-d} \right]
\]

\[
+ \partial_{\mu} T^{+a} \partial_{\nu} T^{-b} \partial_{\rho} T^{+c} \partial_{\sigma} T^{-d} \right],
\]

where we have used the full symmetry of $t^{(8)}$ in the $a, b, c, d$ indices. This expression corresponds to

\[
\mathcal{I} = \frac{\lambda}{26} t^{(8)}_{abcd} \left[ T^{+a}(p_1)T^{-b}(p_2)T^{+c}(p_3)T^{-d}(p_4) \right] (s^2 + t^2 + u^2),
\]

in momentum space.

If we start with the $E_7(7)$ invariant

\[
\frac{\lambda}{22} t^{(8)} \cdot \Phi^{(8)} (\partial_{\mu} F, \partial_{\nu} F, \partial_{\rho} F, \partial_{\sigma} F),
\]

and reduce it to $U(1)$, the Pfaffians vanish and all the traces become ordinary products and the invariant reduces to exactly (4.2) after identification of $F$ with $T$, which is the manifestly $U(1)$-invariant source of deformation used in [11]. Using the linear selfduality constraint

\[
T^+ = 0,
\]

the expression collapses into

\[
S^{(1)} = \frac{\lambda}{24} \int d^4 x t^{(8)}_{abcd} \partial_{\mu} F^{+a} \partial_{\nu} F^{+b} \partial_{\rho} F^{+c} \partial_{\sigma} F^{+d}.
\]

This confirms that the source of deformation when linear duality constraint is imposed, reduces to the correct quantum correction to the 4-vector amplitude from open string or D3 branes.

In the $U(1)$ case the amplitude has no internal indices, which brings the expression to the following one

\[
M_{\text{vec}}(b^-, b^-, b^+, b^+) = \lambda \langle 12 \rangle^2 \langle 34 \rangle^2 \left[ s u + s t + t u \right].
\]

This expression is in agreement with the one in [11], (compare with Eq. (4.2) there).

\[\text{In [11] } T \text{ is denoted by } T^+.\]
5 Discussion

Here we will discuss shortly the two prototypes of relations between UV divergences and duality which came to light as a consequence of the studies performed in this paper. The first one has to do with computations in [26] and [27] where certain UV divergences were found, in agreement with duality symmetry. The second one has to do with computations in [1] and in [3] where the UV divergences were not found, in agreement with duality current conservation.

As an example of the first prototype consider the computation of the one-loop corrections to the $\kappa$-symmetric D3 brane action performed in [26]. The action consists of 2 terms, the Born-Infeld one and the Wess-Zumino one. After gauge-fixing the local $\kappa$-symmetry the bosonic part of action, up to 4th order in fields is given by (Eq. (37) in [26].

$$L_{BI+WZ}^4 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{8} (F_{\mu\nu} F^{\nu\lambda} F^\lambda_{\delta\mu} - \frac{1}{4} (F_{\mu\nu} F^{\mu\nu})^2) + \ldots$$  \hspace{1cm} (5.1)

where the terms indicated by the dots depend on 6 scalars, vectors and fermions. Note that the term quartic in vectors is

$$F_{\mu\nu} F^{\nu\alpha} F_{\alpha\delta} F^\delta_{\mu} - \frac{1}{4} (F_{\mu\nu} F^{\mu\nu})^2 \equiv F^4 - \frac{1}{4} (F^2)^2 = \frac{1}{4!} t^{(8)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} F^{\mu_1\nu_1} F^{\mu_2\nu_2} F^{\mu_3\nu_3} F^{\mu_4\nu_4}. $$  \hspace{1cm} (5.2)

The 1-loop UV divergence in $d = 4$ Feynman graphs was computed in [26], with the result

$$- \frac{\alpha^4}{(4\pi)^2} \frac{1}{16 \epsilon} (s^2 + t^2 + u^2) t^{(8)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} F^{\mu_1\nu_1} (p_1) F^{\mu_2\nu_2} (p_2) F^{\mu_3\nu_3} (p_3) F^{\mu_4\nu_4} (p_4).$$  \hspace{1cm} (5.3)

The fact that the expected UV divergence does show up in this computation may be interpreted here as follows:

Whenever the candidate counterterms may violate the NGZ duality current conservation, but the complete deformation procedure of [9, 10] adding higher order terms, restoring duality, can be performed, the expected candidate counterterm is not forbidden and there is a UV divergence.

The particular case of the Born-Infeld model with higher derivatives and $U(1)$ duality was worked out in [11] and the existence of the procedure of reconstructing the Noether-Gaillard-Zumino current conservation was demonstrated when higher order terms were recursively produced.

In a related computation performed in [27] one starts with a manifestly $\mathcal{N} = 2$ supersymmetric $d = 4$ BI action and computes the 1-loop UV divergence using the quantization in terms of $\mathcal{N} = 1$ superfields. The Feynman supergraphs computation yields the expected UV divergence which takes the manifestly $\mathcal{N} = 1$ supersymmetric form

$$\frac{\alpha^4}{(4\pi)^2} \frac{1}{8 \epsilon} (s^2 + t^2) W^2(1, 2) \overline{W}^2(3, 4).$$  \hspace{1cm} (5.4)

20Recently the issue was raised in [39] as to whether duality symmetry has to control the quantum theory.
Its bosonic part, due to the complete symmetry of $F^4 - \frac{1}{4}(F^2)^2$ in $s, t, u$ variables, is shown to be proportional to an expression given in (5.3). The existence of a 1-loop UV divergence in this case, again, is supported by the construction of $U(1)$ duality invariant manifestly supersymmetric actions in [12] where the first quartic term would violate the duality current conservation but higher-order terms produced algorithmically restore the Noether-Gaillard-Zumino identity.

From this perspective our work suggests the following possible interpretation of the fact that $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravities are UV finite at the 3-loop order:

*Whenever the candidate counterterms may violate the duality current conservation, and there is an obstruction for the complete procedure adding higher order terms restoring the Noether-Gaillard-Zumino current conservation, the expected candidate counterterm is forbidden and there is no UV divergence.*

In this paper we have produced evidence that in the supergravities whose duality group is of type $E_7$, with $\mathcal{N} = 8$ and $\mathcal{N} = 4$ as explicit examples, the deformation procedure of [9, 10, 11, 12] required for the restoration of the duality current conservation seems to encounter an obstruction: the existence of a superinvariant initial source of deformation depending on unconstrained doublets of electric and magnetic field strengths, which is the cornerstone of the whole procedure, seems to be incompatible with supersymmetry.

The evidence consists of two separate independent parts.

First, we have used the superspace construction of [15, 16] which provides the candidate counterterms and we have tried to promote them to initial sources of deformation. We have found that relaxing the linear twisted self-duality in superspace violates the integrability condition for the existence of the superspace 56-bein. In such case, the proof that the graviphoton superfield is manifestly $E_7(7)$, invariant becomes invalidated.

Secondly, we have used the fact that $E_7(7)$ is based on exceptional Jordan algebra $J_3^{OS}$ over the split octonions $O_{S}$ [23, 24] to construct new $E_7(7)$ invariants relevant to amplitudes and counterterms. When we compare the new invariants with the candidate divergent supergravity amplitudes, we find an inconsistency: the dependence on Mandelstam variables interferes with the internal structure of a generalized $E_7(7)$ quartic invariants which we have constructed. Namely, the trace of four operators and square of the trace of two operators inside the invariant acquire different dependences on $s, t, u$ in the 4-vector amplitude. This breaks $E_7(7)$ invariance: there is no initial source of deformation with required properties, no consistent deformation procedure and no UV divergence prediction. This agrees with the 3- and 4-loop computations of [1, 2] for $\mathcal{N} = 8$. The same argument works for all groups of type $E7$, as long as they are not degenerate, which includes the duality group of the pure $\mathcal{N} = 4$ theory. This leads to a prediction that there is no UV divergence, in agreement with the the 3-loop computations in [3].

In the reduction to $U(1)$ the new $E_7(7)$ invariants reduce to the ones considered in the Born-Infeld models with higher derivatives of [11]. In that case there is no difference between the internal symmetry trace of four operators and the square of the trace of two operators and the functions of the Mandelstam variables do not introduce any discrepancy. There is a manifest $U(1)$ duality source of deformation with the required properties, there is a consistent deformation procedure and there are UV divergences, as shown in [26] and [27].
In the future, hopefully, there will be more computational examples and theoretical studies of the role of duality symmetries in quantum theories, like [39, 40], to test the ideas of this paper.

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A On $\mathcal{N} = 8$ superspace, counterterms and candidates for initial source of deformation

The on-shell superspace of $\mathcal{N} = 8$ supergravity was constructed in [15, 16]. The non-linear, geometric invariant counterterms, starting from the 8-loop order, were proposed in [25]. For example, for the 8-loop order one has

$$S^8 \sim k^{14} \int d^4x \ d^{32}\theta \Ber \chi_{IJK\alpha}(x, \theta)\chi^{IJK\dot{\alpha}}(x, \theta)\chi_{MNL\dot{\alpha}}(x, \theta)\chi^{MNL}(x, \theta).$$

(A.1)

Here

$$T_{I \dot{\alpha}, J \dot{\beta}, K\alpha}(x, \theta) = \epsilon_{\dot{\alpha} \dot{\beta}} \chi_{IJK\alpha}(x, \theta)$$

(A.2)

is the superspace torsion superfield whose first component is a spinor field $\chi_{IJK\alpha}(x)$. The existence of a generic $L$-loop counterterm where the superspace has a well-defined $SU(8)$- and Lorentz-covariant derivatives $D^I, \overline{D}_{\dot{\alpha}}$ and $D_{\alpha\dot{\alpha}}$, is based on the fact that the superspace construction provides a solution of geometric Bianchi identities. Therefore one can increase the dimension of the counterterm easily by inserting covariant derivatives and using more torsions and curvatures in the superinvariants analogous to (A.1).

To promote this construction to the level of the source of deformation [9, 10] of the linear twisted self-duality constraint, valid in classical theory, we have to relax the linear twisted self-duality constraint. This requires to clarify some subtleties in the superspace construction [15, 16], which were not important in the past.

The basic variables are the (super-) vielbein $E^A_M$ and the $SL(2, \mathbb{C}) \times SU(8)$ connection $\Omega_{MA}^B$ from which the torsion $T_{AB}^C$ and the curvature $R_{ABC}^D$ are constructed in the usual way and satisfy, from their definitions, the following Bianchi identities:

$$DT^A = E^B \wedge R_B^A, \quad DR_B^A = 0$$

(A.3)
Notations and conventions are the same as in [16] except for the $SU(8)$ indices that we denote by capital $I, J, \ldots$ instead of lowercase $i, j, \ldots$. In particular, the tangent space rotation is
\[ \delta E^A = E^B L_A^B, \]  
where the matrix $L_A^B$ is
\[ L_{ab} = -L_{ba}, \quad L_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta} L_{ab}, \quad L_{\alpha\beta} = \delta I L_{\alpha}^J + \delta_{\alpha}^J L_{\beta}^I, \quad \bar{L}_I^J = -L_{IJ}^J, \quad L_{\dot{\alpha}\dot{\beta}} = -(L_{\dot{\alpha}\dot{\beta}}). \]  

The summary of this part in Section 4 of [16] is that a complete solution to the geometrical Bianchi identities can be given in terms of the following set of covariant superfields:
\[ \chi_{\alpha[IJK]}, S_{(IJ)}, N_{(\alpha\beta)}^{[IJ]}, G_{\alpha\beta,J}^I, H_{\dot{\alpha}\dot{\beta},J}. \]  
These superfields appear in the superspace torsion as follows, in addition to (A.2)
\[ T_{\dot{\alpha}\dot{\beta},J,K} = i(\epsilon_{\alpha\beta} M^{JK}_{(\alpha\beta)} + \epsilon_{\dot{\alpha}\dot{\gamma}} N_{\alpha\beta}^{JK} + \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\gamma}} S^{JK}), \]  
where
\[ M_{\alpha\beta J} = \frac{1}{6} D^K_{(\alpha\beta)IJK}, \]  
and
\[ T_{\dot{\alpha}\dot{\beta},J} = -i\delta^J \sigma_{\dot{\alpha}\dot{\beta}}^c. \]  
The components of the super curvature satisfy the Bianchi identities. To identify the scalars and vectors in $\mathcal{N} = 8$ superspace there are two options.

1. In [15] it was suggested to use the “somewhat specious” Bianchi identities as they have additional terms over and above the normal ones. The goal was to make a local $U(1)$ compatible with local $SU(8)$ and global $E_7(7)$ and at this point they introduced a “non-geometric” 1-form $P$ and a 2-form $F$ in their Table 4 on p. 271, so that in the end they provide a 56-bein in superspace.

This reminds the 11-dimensional superspace construction [41] where in addition to geometric torsions and curvatures a 3-form gauge superfield $A$ and the 4-form $F = dA$ supercovariant field strength were introduced, and in the end all component structure of 11-dimensional on-shell supergravity was presented in a superspace via a single superfield.

2. The second option, explained in Section 8 of [16] is to introduce additional 28 complex bosonic coordinates and assume that superfields do not depend on these new coordinates. However, the existence of these coordinates provides new components of torsion and curvatures. This, in turn, gives a geometric interpretation to the 1-form $P$ and the 2-form $F$ and to the identities they have to satisfy. In this second option the identities for the 1-form $P$ and the 2-form $F$ become the standard torsion-curvature Bianchi identities in this extended space. Since this version leads to the same results, we will only discuss the first one.
Thus, we start with the end of Section 4 of [16] where only the geometric Bianchi identities were solved. This leaves us with a solution for the torsions and curvatures expressed via the unconstrained superfields $\chi_{[IJK]a}, S^{[IJ]}, N_{(a\beta)}^{[IJ]}, G_{a\beta J}^{I}, H_{a\beta J}^{I}$.

At this point the theory is not on shell (it is possible to continue with off-shell conformal $\mathcal{N} \leq 4$ supergravity or on shell Poincaré). However, in $\mathcal{N} = 8$ case the scalar fields are not present and $M_{a\beta IJ}, \overline{M}_{a\beta}^{IJ}$ is not yet the field strength for a spin-1 field. We now move on to Section 8 of [16] to explain the origin of scalars and vectors in the superspace.

We identify the scalar curvature in $SU(8)$ direction

$$R_{IJ}^{KL} = 2\delta_{[I}^{[K} R_{]L]}^{]J}, \quad (A.11)$$

where $R_{IK}^{I}$ is one of the components of the $SL(2,\mathbb{C}) \times SU(8)$ curvature defined in the tangent space in Eqs. (A.4), (A.5) and determined in terms of superfields $\chi_{[IJK]a}, S^{[IJ]}, N_{(a\beta)}^{[IJ]}, G_{a\beta J}^{I}, H_{a\beta J}^{I}$ in [16].

We now additionally require that

$$R_{IJ}^{KL} = -P_{IJMN} \wedge P_{KLMN} \quad (A.12)$$

where the 1-form $P_{IJKL}$ is a new object not present in geometry such that

$$DP_{IJKL} = 0. \quad (A.13)$$

These two equations above serve the following purpose: (A.12), (A.13) are integrability conditions for the existence of a $56$-bein in a superspace. A 1-form $\hat{\Omega}$ with the properties

$$\hat{\Omega} = \left( \begin{array}{cc} \Omega_{IJ}^{KL} & -P_{IJKL} \\ -P_{IJKL} & \overline{\Omega}_{IJ}^{KL} \end{array} \right), \quad d\hat{\Omega} + \hat{\Omega} \wedge \hat{\Omega} = 0, \quad (A.14)$$

is postulated to exist. Here the connection 1-form is $\Omega_{IJ}^{KL} = 2\delta_{[I}^{[K} \Omega_{]L]}^{]J}$. If Eq. (A.14) is valid, the existence of a $56 \times 56$-dimensional matrix $S \in E_{7(7)}$ such that

$$\hat{\Omega} = -S^{-1} dS, \quad (A.15)$$

is guaranteed.

$\hat{\Omega}$ is a superspace vielbein, a bridge between the local $SU(8)$ and the global $E_{7(7)}$. The scalar fields are described by $S$.

Then, a 2-form $F_{IJKL}, \overline{F}^{KL}$ not present in the geometry and such that

$$DF_{IJK} = \overline{F}^{KL} \wedge P_{KLIJ}, \quad (A.16)$$

is introduced. The choice to solve this one is [16]

$$F_{a\bar{\alpha}, \bar{\beta} IJ} = -2i \epsilon_{a\alpha} \delta_{[I}^{[KL]} \delta_{J]}^{]J}, \quad (A.17)$$

$$F_{a\alpha, \bar{\beta} J, KL} = \epsilon_{\alpha\beta} \chi_{JKL\alpha}, \quad (A.18)$$

$$F_{a\alpha, \bar{\beta} I, J} = -i \epsilon_{\alpha\beta} M_{a\beta IJ} - i \epsilon_{\alpha\beta} \overline{M}_{a\beta IJ}, \quad (A.19)$$

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and conjugate, the rest vanishes. For example,

\[ \bar{F}_{\alpha\beta}^{IJ} = i\varepsilon_{\alpha\beta}F_{\alpha\beta}^{IJ} + i\varepsilon_{\alpha\beta}N_{\alpha\beta}^{IJ}. \]  

(A.20)

To solve (A.13) and (A.16) one finds according to [16] that

\[ P_{\alpha IJKLM} = 2\delta_{[I}^{J|}[\chi^{KLM}]_{\alpha}, \]

\[ P_{\alpha\beta IJKL} = -\frac{1}{2}D_{\alpha IJKL\beta}, \]  

(A.21)

and

\[ D_{(\alpha}^{IJKLMN\beta)} = M_{\alpha\beta}^{IJKLMN}, \]

\[ D_{(\alpha}^{IJKLMN\beta)} = -5\chi^{[JK}_{\alpha}^{L\beta MN]}\chi^{\alpha}, \]  

(A.22)

where

\[ M_{\alpha\beta}^{IJKLMN} \equiv \epsilon^{IJKLMNPQ}M_{\alpha\beta PQ}, \]

\[ \chi^{IJKLM}\alpha \equiv \epsilon^{IJKLMNPQ}\chi_{NPQ}\alpha. \]  

(A.23)

At this point \( M_{\alpha\beta IJ} \) is identified as a self-dual \( SU(8) \) covariant vector field strength and the anti-self-dual one, \( N_{\alpha\beta IJ} \) is arbitrary. This is what we need since the linear twisted self-duality condition in absence of fermions requires that

\[ N_{\alpha\beta IJ} = 0, \]  

(A.24)

and we have 28 \( M_{\alpha\beta IJ} \) and the conjugate 28 \( \bar{M}_{\alpha\beta}^{IJ} \). Relaxing the linear twisted self-duality condition in absence of fermions simply means that \( N_{\alpha\beta IJ} \) and its conjugate are unconstrained.

When there are fermions present, the linear twisted self-duality condition is reduced to the requirement that the superfield \( N_{\alpha\beta IJ} \) is a particular function of the fermion superfield \( \chi_{IJK\alpha} \), as we will see below.

### A.1 Consistency check

So far, the superfield \( N_{\alpha\beta IJ} \) which appears in the unconstrained \( SU(8) \) vector field strength in Eq. (A.20) was not required to satisfy any constraints. But we have not yet checked Eq. (A.11) which is equivalent to Eq. (A.5). It means that the computation of the curvature \( R_{IJ} \) done in the first (geometric) part (before the scalars and vectors were introduced) in Section 4 of [16], has to be used. In Section 4 all geometric identities (in a smaller space) were satisfied with and arbitrary, unconstrained, \( \bar{N}_{\alpha\beta IJ} \), and all the answers for \( R_{IJ} \) are given. We have to extract the value of \( R_{IJ} \) from there and plug it into the integrability equation for the existence of the superspace 56-bein Eq. (A.12) via Eq. (A.11). This is Eq. (8.17) of [16] and it requires that \( \bar{R}_{IJ} \) computed in Section 4 is compatible with existence of the 1- and 2-forms so that Eq. (A.12) is satisfied. But since \( R_{IJ} \) depends on the unconstrained \( \bar{N}_{\alpha\beta IJ} \) and it is part of an unconstrained vector field strength (A.20), we have a chance to see if it is consistent with identities to keep \( \bar{N}_{\alpha\beta IJ} \) unconstrained, i.e. to relax the linear twisted self-duality constraint.
The vector-vector part given in Eq. (4.42) of [16]

\[ R_{\alpha\beta}^{\cdot \cdot} K_{L} = \epsilon_{\alpha\beta} X_{\alpha\beta}^{\cdot \cdot} K_{L} - \epsilon_{\alpha\beta} \overline{X}_{\alpha\beta}^{\cdot \cdot} K_{L}. \]  
(A.25)

The expression for \( X_{\alpha\beta}^{\cdot \cdot} K_{L} \) and its conjugate can be extracted, in principle, from [16]. It depends on the superfields \( \chi, S, N, G, H \) and their covariant derivatives. This dependence is complicated and it is not clear if one can make it useful. For example, if we tried to find the value of the bosonic part of \( X_{\alpha\beta}^{\cdot \cdot} K_{L} \) with \( N \) unconstrained, we may or may not find a restriction on \( N \).

Meanwhile, in the bosonic case, one simply constructs the bosonic 56-bein from scalars, as in [5, 6]. Therefore, there is no need to provide the integrability conditions for its existence. But this procedure may not be the same as extracting the bosonic terms from the superspace. So, we leave this part for future studies and proceed to study other components of the curvature.

The spinor-vector part is in Eq. (4.26) of [16] and, since it is fermionic, we cannot use the simple approach without fermions. Furthermore, it is rather messy with account of the gravitino superfield. So again, we leave this part unfinished and proceed to other study components of curvature.

The spinor-conjugate spinor part is given Eq. (4.19) of [16]. But it only depends on \( G, H \) and \( N \) do not depend on \( N \). It appears that both \( G \) and \( H \) have to vanish in absence of fermions.

The spinor-spinor part is given in Eq. (4.14) of [16], it is concise and easy to analyze

\[ R^{IJK}_{\alpha\beta} L = \epsilon_{\alpha\beta} (\delta^{I} L S^{JK} - \delta^{J} L S^{IK} - B^{IJL}) + \delta^{I} L N^{JK}_{\alpha\beta} + \delta^{J} L N^{IK}_{\alpha\beta}. \]  
(A.26)

Here the superfields \( S \) and \( N \) are unconstrained, while, due to the geometric Bianchi identities of Section 4 of [16] \( B_{IJL}^{JK} = \frac{1}{4} D_{\alpha}^{JK} \chi_{IJL}^{\alpha} \). In absence of fermions, when \( \chi_{IJL}^{\alpha} \) and \( P_{\alpha}^{JKMN} \wedge P_{\beta}^{JKLM} \) are not available, the r.h.s. of \( R^{IJK}_{\alpha\beta} L \) should vanish, according to Eq. (1.19). Since each term in Eq. (A.26) has different symmetry properties, they all must vanish, namely in absence of fermions

\[ S^{(JK)} = \overline{B}^{IKL} = N^{[JK]}_{\alpha\beta} = 0. \]  
(A.27)

For the superfields the actual constraint is given by (A.26), taken together with (1.19) in the form

\[ R^{IJK}_{\alpha\beta} L = -\frac{1}{3!4!} \epsilon^{E_{,}^{MNQRST}} P_{\alpha}^{QRST} P_{\beta}^{LMNP}, \]  
(A.28)

with account of Eq. (A.21). This leads to the following constraints,

\[ S^{IJK} = 0, \]  
(A.29)

\[ N^{IJK}_{\alpha\beta}(x, \theta) = -\frac{1}{72} \epsilon^{E_{,}^{JKLMNPQ}} \chi_{KL\alpha}^{\gamma} \chi_{NPQ\beta}(x, \theta). \]  
(A.30)
The constraints (A.29) and (A.30) in the on-shell superspace of [15] are imposed from the start to solve the Bianchi identities (A.3) (see Table 2 there). One may have thought that they are necessary for the existence of the one-shell conditions, with which we are not concerned here. However, we may start with the solution of Bianchi identities in Section 4 of [16] with unconstrained $N_{ij}$, and we would like to keep it that way. However, if we do that we find a discrepancy with the integrability condition for the existence of the superspace 56-bein since (A.30) follows from (A.11) and (A.12) and does not admit any possibility to relax it.

In conclusion, the superspace vielbein does not exist if Eq. (A.30) is not satisfied. It is not clear, therefore, how exactly to proceed with the “deformation program” for $\mathcal{N} = 8$ supergravity since the known counterterms cannot be automatically promoted to an initial source of deformation $\mathcal{I}$: their manifest $E_7(7)$ invariance, manifest local supersymmetry and manifest local $SU(8)$ symmetry are only valid when the undeformed constraint Eq. (A.30) is imposed.

In the models [9]–[12] this was not an issue. With a bit of work it was possible to find initial sources of deformation with all required symmetry properties, and then it was possible to produce new models in an algorithmic way. For example, in [12] the manifest $\mathcal{N} = 2$ supersymmetry was not affected by the deformation of the action. In $\mathcal{N} = 8$ supergravity it appears that the supersymmetry rules are affected by deformation of the action and to find an initial source of deformation simultaneously consistent with (deformed) duality and (deformed) supersymmetry is a problem. Knowing how to solve it, one may construct an $\mathcal{N} = 8$ Born-Infeld-type supergravity. However, the absence of a solution is an indication of the UV finiteness of $\mathcal{N} = 8$ supergravity.

**B The $t^{(8)}$ tensor**

The components Hodge dual of a Lorentz 2-form $A = \frac{1}{2} A_{\mu\nu} dx^\mu \wedge dx^\nu$ are defined by

$$\hat{A}_{\mu\nu} \equiv \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} A^{\rho\sigma}, \quad \hat{A} = -A,$$  \hspace{1cm} (B.1)

and those of its self- and anti-self-dual parts are defined by

$$A^\pm = \frac{1}{2} (A \pm i\hat{A}), \quad \hat{A}^\mp = \mp i A^\pm.$$  \hspace{1cm} (B.2)

These definitions imply, for any pair of (real or complex) 2-forms $A$ and $B$:

$$\hat{A}\hat{B} = BA - \frac{1}{2} \text{Tr}_L(AB) \mathbb{1},$$  \hspace{1cm} (B.3)

$$\hat{A}B = -\hat{B}A + \frac{1}{2} \text{Tr}_L(A\hat{B}) \mathbb{1},$$  \hspace{1cm} (B.4)

$$A^\pm B^\mp = -B^\pm A^\pm + \frac{1}{2} \text{Tr}_L(A^\pm B^\mp) \mathbb{1},$$  \hspace{1cm} (B.5)

$$A^\pm B^\mp = B^\mp A^\pm,$$  \hspace{1cm} (B.6)
where \( AB \) stands for \( A^\mu B_\mu \), 1 stands for \( \delta_\mu^\rho \), \( \text{Tr}_L(AB) \) stands for \( A_\mu B_\mu \) etc.

The \( t^{(8)} \) tensor is a tensor totally symmetric in four pairs of antisymmetric indices that can be defined by its contraction with 4 arbitrary Lorentz 2-forms \( A, B, C, D \):

\[
t^{(8)}_{abcd} A^a B^b C^c D^d = 8[\text{Tr}_L(ABCD) + \text{Tr}_L(ACBD) + \text{Tr}_L(ACDB)] - 2[\text{Tr}_L(AB)\text{Tr}_L(CD) + \text{Tr}_L(AC)\text{Tr}_L(BD)] + \text{Tr}_L(AD)\text{Tr}_L(BC).
\]

It is convenient to use only one Latin index \( a, b, c \ldots \) to denote each of these four pairs and write \( t^{(8)}_{abcd} \) instead of \( t^{(8)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} = t^{(8)}_{[\mu_1\nu_1][\mu_2\nu_2][\mu_3\nu_3][\mu_4\nu_4]} \). Then, in terms of these indices, \( t^{(8)} \) is completely symmetric \( t^{(8)}_{abcd} = t^{(8)}_{(abcd)} \).

Using the above relations plus the invariance of the trace under transposition and the antisymmetry of the tensors \( A, B, C, D \), one finds the identity

\[
t^{(8)}_{abcd} A^a B^b C^c D^d = 4\text{Tr}_L(A^+ B^+)\text{Tr}_L(C^- D^-) + 4\text{Tr}_L(A^+ C^+)\text{Tr}_L(B^- D^-) + 4\text{Tr}_L(A^+ D^+)\text{Tr}_L(B^- C^-) + (+ \leftrightarrow -).
\]

That is: the only terms that contribute are those with zero helicity. For the particular choice \( A = B = \partial_\mu F, C = D = \partial_\nu F \) the above identity gives

\[
t^{(8)}_{abcd} \partial_\mu F^a \partial_\nu F^b \partial_\rho F^c \partial_\sigma F^d = 4\text{Tr}_L(\partial_\mu F^+ \partial_\nu F^+)\text{Tr}_L(\partial_\rho F^- \partial_\sigma F^-) + 8\text{Tr}_L(\partial_\mu F^+ \partial_\nu F^+)\text{Tr}_L(\partial_\rho F^- \partial_\sigma F^-) + \text{c.c.}
\]

We also find

\[
t^{(8)}_{abcd} \partial_\mu F^{+a} \partial_\nu F^{-b} \partial_\rho F^{+c} \partial_\sigma F^{-d} = 8\text{Tr}_L(\partial_\mu F^+ \partial_\nu F^-)\text{Tr}_L(\partial_\rho F^- \partial_\sigma F^-),
\]

\[
t^{(8)}_{abcd} \partial_\mu F^{+a} \partial_\nu F^{+b} \partial_\rho F^{-c} \partial_\sigma F^{-d} = 8\text{Tr}_L(\partial_\mu F^+ \partial_\nu F^+)\text{Tr}_L(\partial_\rho F^- \partial_\sigma F^-),
\]

which, combined with the previous identity give

\[
t^{(8)}_{abcd} \partial_\mu F^a \partial_\rho F^b \partial_\nu F^c \partial_\sigma F^d = 2t^{(8)}_{abcd} \partial_\mu F^{+a} \partial_\nu F^{-b} \partial_\rho F^{+c} \partial_\sigma F^{-d}
\]

\[
+ t^{(8)}_{abcd} \partial_\mu F^{+a} \partial_\nu F^{+b} \partial_\rho F^{-c} \partial_\sigma F^{-d},
\]

which are the initial sources of deformation used in [11].

Finally, we can rewrite Eq. (B.9) up to total derivatives and up to equations of motion in the form

\[
t^{(8)}_{abcd} \partial_\mu F^a \partial_\nu F^b \partial_\rho F^c \partial_\sigma F^d = 32\text{Tr}_L(F^+ \partial_\mu F^+)\text{Tr}_L(F^- \partial_\mu \partial_\nu F^-) + 8[\text{Tr}_L(\partial_\mu F^+ \partial_\nu F^+)\text{Tr}_L(F^- \partial_\mu \partial_\nu F^-) + \text{c.c.}],
\]

(32)
which is basically the expression in [14] reduced to $U(1)$.

**C \quad E_7(7) and split octonions**

A digression on commutators, associators and triple products

A product (and the corresponding algebra) is **commutative** if $xy = yx$ for all $x, y$. The **commutator** is defined as

$$[x, y] \equiv xy - yx \quad (C.1)$$

and it measures how far two elements are from commuting: $x$ and $y$ commute iff their commutator is zero, $[x, y] = 0$, the algebra is commutative iff all commutators vanish. The role of the commutators in Physics is well known.

The product (and the algebra) is **associative** if $(xy)z = x(yz)$ for all $x, y, z$, in which case we drop all parentheses and write the product as $xyz$. It is common to introduce the **associator**

$$[x, y, z] \equiv (xy)z - x(yz) \quad (C.2)$$

which measures how far three elements are from associating, $x, y, z$ associate iff their associator is zero.

Some algebras are alternative, for example, the octonions algebra is only **slightly non-associative** is **alternative**: $[x_1, x_2, x_3] = (-)^3[x_{1\pi}, x_{2\pi}, x_{3\pi}]$ where $\pi \in S_3$. It means that whereas the associator does not vanish, in general, it does vanish when two of its variables are equal, for example

$$[x, y, x] = (xy)x - x(yx) = 0, \quad [x, x, z] = (x^2)z - x(xz) = 0 \quad (C.3)$$

The Jordan algebra has a particular associator vanishing (and it is sometimes called **power-associative** for it):

$$[x, y, x^2] = (x \circ y) \circ x^2 - x \circ (y \circ x^2) \quad (C.4)$$

Here a symmetric product is defined as $x \circ y = y \circ x$ and $x^2 = x \circ x$.

Finally, there is a Jordan triple product, defined as

$$\{x, y, z\} = x \circ (y \circ z) + z \circ (y \circ x) - (x \circ z) \circ y \quad (C.5)$$

**Split octonions**

For the split octonions, which are relevant to $E_7(7)$, one defines the general element

$$O^s = o_0 + o_1j_1 + o_2j_2 + o_3j_3 + o_4j_4^s + o_5j_5^s + o_6j_6^s + o_7j_7^s, \quad (C.6)$$

and

$$\overline{O}^s = o_0 - o_1j_1 - o_2j_2 - o_3j_3 - o_4j_4^s - o_5j_5^s - o_6j_6^s - o_7j_7^s, \quad (C.7)$$
where the rules of multiplication of the seven “imaginary” units are

\[ j_l j_k = -\delta_{lk} + \eta_{klm} j_m, \quad j^*_\mu j^*_\nu = \delta_{\mu\nu} - \eta_{\mu\nu\kappa} j_\kappa, \quad j_l j^*_\mu = \eta_{l\mu\nu} j_\nu, \]  

(C.8)

where \( l, k, m = 1, 2, 3 \), \( \mu, \nu = 4, 5, 6, 7 \) and \( \eta_{ABC} \), with \( A, B, C = 1, \cdots, 7 \), is completely antisymmetric and given by

\[ \eta_{ABC} = 1 \quad \text{when} \quad (ABC) = (123), (471), (572), (673), (624), (435), (516). \]  

(C.9)

In particular

\[ j_1^2 = j_2^2 = j_3^2 = -1, \quad (j_4^*)^2 = (j_5^*)^2 = (j_6^*)^2 = (j_7^*)^2 = 1. \]  

(C.10)

For real octonions

\[ O = o_0 + o_1 j_1 + o_2 j_2 + o_3 j_3 + o_4 j_4 + o_5 j_5 + o_6 j_6 + o_7 j_7, \]  

(C.11)

and

\[ \overline{O} = o_0 - o_1 j_1 - o_2 j_2 - o_3 j_3 - o_4 j_4 - o_5 j_5 - o_6 j_6 - o_7 j_7, \]  

(C.12)

all the imaginary units square to minus one

\[ j_A^2 = -1, \quad \forall A = 1, \cdots, 7. \]  

(C.13)

Therefore, for real octonions

\[ O \overline{O} = \sum_{A=0}^{7} O_A^2 \]  

(C.14)

whereas for the split ones

\[ O^+ \overline{O}^t = \sum_{B=0}^{3} O_B^2 - \sum_{C=4}^{7} O_C^2. \]  

(C.15)

**Freudenthal/Jordan basis**

The fundamental 56 of \( E_{7(1)} \) can be repackaged using the Freudenthal triple systems and split octonions, as follows. From 28 vectors and 28 dual vectors we separate the zero direction from the other 27, in agreement with the decomposition of \( E_{7(1)} \) via \( \tilde{E}_{6(6)} \). The decomposition involves the \( \mathfrak{so}(1,1) \) grading

\[ \mathfrak{e}_{7,7} = \mathfrak{e}_{6,6} + \mathfrak{so}(1,1) + \mathfrak{p}, \quad \mathfrak{p} = 27_{-2} + 27'_2, \]  

(C.16)

and \( \mathfrak{p} \) carries the representations of \( \mathfrak{e}_{6,6} + \mathfrak{so}(1,1) \). The 56 of \( E_{7(7)} \) takes the form

\[ X = \begin{pmatrix} F^0 & F^I \\ G_I & G^0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \]  

(C.17)
Here the $3 \times 3$ split octonionic Hermitian matrices $x$ and $y$ are elements of the the split exceptional Jordan algebra of degree three, $J^3_h$. The 27 objects in $x$ and 27 in $y$ are organized in terms of 3 split octonions ($3 \times 8 = 24$) and 3 numbers, each, so that

$$x = \begin{pmatrix} \alpha_1^s & O_1^{-} & \sigma_1^s \\ \sigma_1^s & \beta_1^s & O_3^{-} \\ O_3^{-} & \gamma_1^s & \end{pmatrix},$$

and there is an analogous expression for $y$.

The four-linear form derived in [35] is given by the symplectic product of the triple product

$$J^3_h[X_1, X_2, X_3, X_4] = \langle (X_1, X_2, X_3), X_4 \rangle,$$

where the definition of the Freudenthal triple, $(X_1, X_2, X_3)$, a ternary product, corresponds to multiplication of the 3 elements, taken in the form (C.17), (C.18).

**Fano basis**

The fundamental 56 of $E_{7(7)}$ can be repackaged using the decomposition $E_{7(7)} \supset (SL(2, \mathbb{R}))^7$, [42]. In such case there is a relation between Alice, Bob, Charlie, Daisy, Emma, Fred and George and 7 qubits in quantum information theory. This is also related to the fact that the fundamental 56 of $E_{7(7)}$ can be repackaged into 7 subsets with 8 elements each, as shown in [36], [43]. For example, in the notation of [43] with

$$A = (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7),$$

$$G = (g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7),$$

we can write

$$F^{ij} = \begin{pmatrix} 0 & -a_7 & -b_7 & -c_7 & -d_7 & -e_7 & -f_7 & -g_7 \\ a_7 & 0 & f_1 & d_4 & -c_2 & g_6 & -b_4 & -e_1 \\ b_7 & -f_1 & 0 & g_4 & e_4 & -d_2 & a_2 & -c_1 \\ c_7 & -d_4 & -g_1 & 0 & a_1 & f_4 & -e_2 & b_2 \\ d_7 & c_2 & -e_4 & -a_1 & 0 & b_1 & g_4 & -f_2 \\ e_7 & -g_2 & d_2 & -f_4 & -b_1 & 0 & c_1 & a_4 \\ f_7 & b_4 & -a_2 & e_2 & -g_4 & -c_1 & 0 & d_1 \\ g_7 & e_1 & c_4 & -b_2 & f_2 & a_4 & -d_1 & 0 \end{pmatrix},$$

and

$$G_{ij} = \begin{pmatrix} 0 & -a_0 & -b_0 & -c_0 & -d_0 & -e_0 & -f_0 & -g_0 \\ a_0 & 0 & f_6 & d_3 & -c_5 & g_5 & -b_3 & -e_0 \\ b_0 & -f_6 & 0 & g_6 & e_3 & -d_5 & a_5 & -c_3 \\ c_0 & -d_3 & -g_6 & 0 & a_6 & f_3 & -e_5 & b_5 \\ d_0 & c_5 & -e_3 & -a_6 & 0 & b_6 & g_3 & -f_5 \\ e_0 & -g_5 & d_5 & -f_3 & -b_6 & 0 & c_6 & a_3 \\ f_0 & b_3 & -a_5 & e_5 & -g_3 & -c_6 & 0 & d_6 \\ g_0 & e_6 & c_3 & -b_5 & f_5 & a_3 & -d_6 & 0 \end{pmatrix}.$$
This form of the $E_{7(7)}$ doublet was derived from the Coxeter sub-geometry of the split Cayley hexagon and using the 7-fold symmetry of the Coxeter graph. In this form our 4-linear $E_{7(7)}$ invariant (2.36) consists of a set of products of 4 terms where each of the $F$ and $G$ is represented by $8 \times 8$ matrices above.

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