On the quantitative variation of congruence ideals and integral periods of modular forms

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Abstract

We prove the conjecture of Pollack and Weston on the quantitative analysis of the level lowering congruence à la Ribet for modular forms of higher weight. It was formulated and studied in the context of the integral Jacquet–Langlands correspondence and anticyclotomic Iwasawa theory for modular forms of weight two and square-free level for the first time. We use a completely different method based on the \( R = T \) theorem established by Diamond–Flach–Guo and Dimitrov and an explicit comparison of adjoint \( L \)-values. We briefly discuss arithmetic applications of our main result at the end.

Keywords: Modular forms, Modularity lifting theorems, Congruence ideals, Shimura curves, Level lowering, Tamagawa exponents

Mathematics Subject Classification: 11F33 (Primary), 11F67; 11R23 (Secondary)

1 Introduction

1.1 Overview

1.1.1 Main result

The goal of this article is to prove Theorem 1.1, which is the higher-weight generalization of the conjecture of Pollack and Weston [38, Conjecture 1.4].

Let \( p \geq 5 \) be a prime and fix embeddings \( \iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p \) and \( \iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \). Let \( f = \sum_{n \geq 1} a_n(f)q^n \in S_k(\Gamma_0(N)) \) be a newform and \( \mathbb{Q}_f \) the Hecke field of \( f \). Let \( \lambda \) be the prime of \( \mathbb{Q}_f \) lying above \( p \) induced from \( \iota_p \). Let \( E = \mathbb{Q}_f^{\overline{\mathbb{Q}}_\lambda} \) be the completion at \( \lambda \), \( O = O_E \), and \( \mathbb{F} = O/\lambda O \) the residue field. Following Deligne’s construction, there exists a continuous \( \lambda \)-adic Galois representation arising from \( f \) (and \( \iota_p \))

\[
\rho_f : G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}_E(V_f) \simeq \text{GL}_2(E)
\]

which is unramified outside \( Np \) and is characterized by \( \text{tr}(\rho_f(\text{Frob}_\ell)) = \iota_p(a_\ell(f)) \) for each prime \( \ell \) not dividing \( Np \) where \( \text{Frob}_\ell \) is a geometric Frobenius at \( \ell \).

By choosing a Galois stable \( O \)-lattice of \( V_f \), the mod \( \lambda^n \) representation \( \rho_n := \rho_f \pmod{\lambda^n} \) and the residual representation \( \overline{\rho} := \overline{\rho_f} := \rho_1 \) are defined, and these are unique up to scalars if \( \overline{\rho} \) is irreducible. The Tamagawa exponent \( t_f(q) \) of \( f \) at a prime \( q \) dividing \( N \)
is defined by the largest integer such that $\rho_{tf}(q)$ is unramified at $q$ and $\rho_{tf}(q)+1$ is ramified at $q$. See §2.2 for more details.

**Theorem 1.1** (Main Theorem) Decompose the level $N$ of $f$ by $N = N^+ N^-$ and assume the following conditions:

1. $(N,p) = 1$,
2. the restriction $\mathcal{P} f|_{\text{Gal}(\overline{\mathbb{Q}}(\sqrt{p^*}))}$ is absolutely irreducible, where $p^* = (-1)^{\frac{p-1}{2}} p$,
3. $2 \leq k \leq p - 1$,
4. $(N^+, N^-) = 1$,
5. $N^-$ is square-free, and
6. if a prime $q \equiv \pm 1 \pmod{p}$ and $q$ divides $N^-$, then $\rho_{tf}$ is ramified at $q$.

Then, we have

$$\text{ord}_\lambda \eta_f(N) = \text{ord}_\lambda \eta_f(N^+, N^-) + \sum_{q|N^-} t_f(q)$$

(1.1)

where $\eta_f(N)$ is the congruence ideal of $f$ in $S_k(\Gamma_0(N))$ and $\eta_f(N^+, N^-)$ is the congruence ideal of $f$ in the $N^-$-new subspace $S_k(\Gamma_0(N))^N^-$ of $S_k(\Gamma_0(N))$ (reviewed in §2.1).

**Remark 1.2**

1. The formula (1.1) quantifies the level lowering congruences [39], while Wiles’ numerical criterion [48, Theorem 2.17] quantifies the level raising congruences (cf. [8, Theorem 5.3]).

2. When $k = 2$ and $N$ is square-free, (1.1) is first formulated in the context of anticyclotomic Iwasawa theory for modular forms, and is proved if $\mathcal{P} f$ is ramified at least two primes [38]. The approach of Pollack and Weston is based on the work of Ribet–Takahashi [40,43] on the comparison among the parametrizations by modular and Shimura curves, and it is unclear how to generalize this geometric approach when $k > 2$ or $N$ is not square-free. In the higher-weight case, by using an $R = \mathbb{T}$ theorem for quaternion algebras, Chida–Hsieh [5] proved (1.1), assuming $\mathcal{P} f$ is ramified at every prime $q | N^-$, i.e. $\sum_{q|N^-} t_f(q) = 0$. Our contribution is to weaken all these assumptions of the previous works on weight, level, and the ramification.

1.1.2 Our approach

Since the comparison among various congruence ideals is closely related to the freeness of the (quaternionic) Hecke modules over the associated Hecke algebras, it seems inevitable to use the $R = \mathbb{T}$ technique to obtain the equality (1.1). However, when $\sum_{q|N^-} t_f(q) \neq 0$, the standard $R = \mathbb{T}$ approach is not strong enough since it is difficult to write down $\eta_f(N^+, N^-)$ in terms of the size of a certain adjoint Selmer group. In particular, it is unclear how to figure out the correct local (deformation) condition of the adjoint Selmer group at a prime $q$ dividing $N^-$ under our setting.

Our approach consists of two steps. We first use standard $R = \mathbb{T}$ arguments and Galois cohomology calculations to relate a certain congruence ideal and the size of an adjoint Selmer group with relaxed local condition at primes dividing $N^-$. We also compute the size of an adjoint Selmer group with “new” local (deformation) condition at primes dividing $N^-$ by establishing a slightly refined $R = \mathbb{T}$ theorem. Unfortunately, the comparison of the sizes of these two adjoint Selmer groups does not give us the equality (1.1) exactly. We call this comparison the Selmer computation.
In order to remove the difference between (1.1) and the Selmer computation, we interpret the difference in terms of Euler factors of the adjoint $L$-values, which is the analytic aspect of congruence ideals. We call this process the $L$-value computation. It is unexpected that we do not fix any quaternion algebra in the argument and we only use classical $R = T$ theorems established in [9,13].

1.1.3 Applications
In §6, we briefly discuss arithmetic applications of Theorem 1.1. It includes the comparison between Hida’s canonical periods and Gross periods (e.g. [47]), the $\mu$-part of the anticyclotomic main conjecture for modular forms, the relation between the periods of modular and (indefinite!) Shimura curves, and more.

1.1.4 Related results
After the completion of the first version of this paper, Fakhruddin–Khare–Ramakrishna released a preprint (since published as [19]) in which they study the quantitative aspect of level lowering congruences in a different context. Their work mainly is concerned with producing optimal $\text{mod } p^n$ congruent modular forms (of weight two and square-free level), but our work is concerned with describing the difference between two congruence ideals in terms of purely local numerical invariants, Tamagawa exponents.

1.2 Organization
We describe the background material to give a precise description of Theorem 1.1 and explain the idea of the proof of Theorem 1.1 in §2. We review deformation theory of Galois representations in §3. Especially, all the local deformation conditions we use are precisely described and a presentation of the deformation ring is discussed. We prove a refined $R = T$ theorem (Theorem 3.14) at the end. In §4, we compute the difference of two adjoint Selmer groups with different local conditions in terms of purely local invariants by using Theorem 3.14. This completes the Selmer computation (Theorem 2.13). In §5, we study congruence ideals in terms of adjoint $L$-values and the $L$-value computation (Proposition 2.15) is proved. Thus, we obtain a proof of Theorem 1.1. Arithmetic applications are discussed in §6.

2 Congruence ideals and Tamagawa exponents
In this section, we first introduce some notation on congruence ideals and review the $R = T$ theorem. After that, we precisely state two key results (Theorem 2.13 and Proposition 2.15), whose proofs are given in the following sections.

2.1 Modularity lifting theorem and congruence ideals
We review deformation rings and Hecke algebras and introduce two auxiliary level structures. Although these level structures are complicated at first, each level structure is needed for the Selmer computation and the $L$-value computation, respectively. Then, we recall the notion of congruence ideals and the $R = T$ theorem and define the $N^\sim$-new variant of congruence ideals.
2.1.1 Deformation rings and Hecke algebras

Let

\[ \bar{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}) \]

be an odd, absolutely irreducible, and continuous Galois representation where \( \mathbb{F} \) is a finite extension of \( \mathbb{F}_p \). Denote by \( N(\bar{\rho}) \) the conductor of \( \bar{\rho} \). Then, \( \bar{\rho} \) is modular by [31] and [28].

Following [9,13], we assume the following conditions throughout this article.

Assumption 2.1 (FL) There exists a newform \( f = \sum_{n \geq 1} a_n(f)q^n \in S_k(\Gamma_0(N)) \) such that

\[ 2 \leq k \leq p - 1, \quad p \nmid N, \quad \text{and} \quad \bar{\rho}_f \simeq \bar{\rho} \]

where \( \bar{\rho}_f \) is the residual Galois representation associated to \( f \).

(TW) The restriction \( \bar{\rho}|_{G_{\mathbb{Q}(\sqrt{p})}} \) is absolutely irreducible where \( p^* = (-1)^{p-1} p \).

Let \( E \) be a finite totally ramified extension of \( \text{Frac}(W(\mathbb{F})) \) and write \( \mathcal{O} := \mathcal{O}_E \) as before, and \( E \) plays the role of the coefficient field and can be enlarged if necessary.

Following [9, Page 717] and [13, Definition 4.6], we briefly recall the notion of \( \Sigma \)-ramified deformationsof \( \bar{\rho} \). See § 3 for the detailed description of local deformation conditions.

Let \( \Sigma \) be a finite set of primes \( \ell \) not equal to \( p \). For a field \( F \), \( G_F \) denotes the absolute Galois group of \( F \). A deformation \( \rho \) of \( \bar{\rho} \) to a complete Noetherian local \( \mathcal{O} \)-algebra \( R \) with residue field \( \mathbb{F} \) is \( \Sigma \)-ramified if \( \rho : G_{\mathbb{Q}} \to \text{GL}_2(R) \) is a continuous representation such that

\[ \begin{align*}
\cdot & \quad \rho|_{G_{\mathbb{Q}_p}} \text{ is a low-weight crystalline deformation in the sense of Definition 3.3}, \\
\cdot & \quad \rho|_{G_{\mathbb{Q}_\ell}} \text{ is minimally ramified at } \ell \notin \Sigma \text{ in the sense of Definition 3.5}, \text{ and} \\
\cdot & \quad \det(\rho) = \chi_1^{1-k} \otimes_{\mathcal{O}} R
\end{align*} \]

where \( \chi_1 \) is the \( p \)-adic cyclotomic character.\footnote{The Hodge–Tate weight of the \( p \)-adic cyclotomic character is \(-1\).} A \( \emptyset \)-ramified deformation is called minimally ramified (in the global sense). Denote by \( \mathcal{T}_\Sigma \) the \( \Sigma \)-ramified deformation ring of \( \bar{\rho} \).

Let \( S \) be a large finite set of primes containing \( \Sigma \) and we define the \( \Sigma \)-ramified Hecke algebra \( \mathcal{T}_\Sigma \) by the \( \mathcal{O} \)-subalgebra of \( \prod_f \mathcal{O} \) generated by \( (\iota_p(a_{\ell}(f)))_{\ell \notin S} \) where \( f \) runs over newforms of fixed weight \( k \) such that the associated \( p \)-adic Galois representation \( \rho_f \) is a \( \Sigma \)-ramified deformation of \( \bar{\rho} \). (e.g. [8, §3.3]) Note that \( \mathcal{T}_\Sigma \) is independent of \( S \). Following [13, Definition 4.2], we recall

\[ P_{\bar{\rho}} := \left\{ \ell \text{ primes : } \ell | N(\bar{\rho}), \quad \bar{\rho}|_{G_{\mathbb{Q}_\ell}} \text{ is irreducible, } \bar{\rho}|_{I_{\ell}} \text{ is reducible, } \ell \equiv -1 \pmod{p} \right\} \]

where \( I_{\ell} \subseteq G_{\mathbb{Q}_\ell} \) is the inertia subgroup.

Remark 2.2 Due to the argument in [9, §1.7.1], we may assume that all the newforms in this article have minimal conductor among its twist when we work with \( \text{ad}^0(f) \). (cf. [13, §4.1].)

2.1.2 The level structure \( N_{\bar{\rho}, \Sigma} \) for the Selmer computation

Following [13, (21)], we introduce the level structure depending on \( \bar{\rho} \) and \( \Sigma \). For \( \ell \neq p \), we put \( c(\ell) := \text{ord}_p(N(\bar{\rho})), d(\ell) := \dim_{\mathbb{F}} H^0(I_{\ell}, \bar{\rho}), \) and \( c(p) = d(p) = 0 \). For \( \bar{\rho} \) and \( \Sigma \), we
write

\[ N_{\rho, \Sigma} := r \cdot \prod_{\ell \in \Sigma} \ell^{c(\ell)+d(\ell)} \cdot \prod_{\ell \notin \Sigma} \ell^{c(\ell)} = r \cdot N(\overline{\rho}) \cdot \prod_{\ell \in \Sigma} \ell^{d(\ell)} \]  

(2.2)

where \( r > 3 \) is an auxiliary prime such that \( r \not\equiv 1 \pmod{p} \), \( \overline{\rho} \) is unramified at \( r \), and \((tr(\overline{\rho}(Frob_r)))^2 \not\equiv r^{k-2}(1 + r)^2 \) in \( \mathbb{F} \). Here, \( Frob_r \) is a geometric Frobenius at \( r \). The prime \( r \) is needed to have neat levels. It is known that there exist infinitely many such \( r \) (e.g., [15, Lemma 11], [14, Lemma 2 (when \( p = 3 \)]).

Enlarge \( S \) enough to contain \( \Sigma \), primes dividing \( N(\overline{\rho}) \), and \( r \) as in [13, §5.3]. Let \( T^S = \mathcal{O}[T_\ell, S_\ell : \ell \notin S] \) be the abstract Hecke algebra over \( \mathcal{O} \) generated by standard Hecke operators \( T_\ell \) and \( S_\ell \) for every prime \( \ell \) not in \( S \) (”\( S \)-anemic”). Denote by \( m_{\overline{\rho}} = (\lambda, T_\ell - \text{tr}(\overline{\rho}(Frob_\ell)), S_\ell - \ell^{-1} \cdot \det(\overline{\rho}(Frob_\ell)) : \ell \notin S) \) the maximal ideal of \( T^S \) corresponding to \( \overline{\rho} \).

Let \( \mathcal{F}^k_p = \text{Sym}^{k-2}(R^1S_*\mathbb{Z}_p) \) be the \( p \)-adic local system where \( s \) is the map from the universal elliptic curve to the modular curve with the \( \Gamma_1(N) \)-level structure over \( \text{Spec}(\mathbb{Z}[1/N]) \) with \( k \geq 2, N \geq 3 \) following [9, §1.2.3]. See also [13, (2) in §2.1].

We recall the Hecke modules and Hecke algebras following [13, (35)].

For an integer \( N \geq 1 \), denote by \( T(N) \) the full Hecke algebra over \( \mathcal{O} \) faithfully acting on the Hecke module \( H^1_{\text{et,c}}(Y_1(N), \mathcal{F}^k_p) \langle (-) - 1 \rangle \) where \( H^1_{\text{et,c}} \) means the compactly supported étale cohomology, and \( M[(-) - 1] \) is the Hecke submodule of \( M \) on which the diamond operator acts trivially.

Let \( m_{\Sigma} \subseteq T(N_{\rho, \Sigma}) \) be the maximal ideal generated by \( m_{\rho}, U_r - \alpha_r \), and, \( U_q \) for \( q \in \Sigma \) where \( \alpha_r \) is a chosen eigenvalue of \( \overline{\rho}(Frob_r) \).

Denote by \( T(N_{\rho, \Sigma})_{m_{\Sigma}} \) the localization of \( T(N_{\rho, \Sigma}) \) at \( m_{\Sigma} \). As in [13, §5.3], \( T(N_{\rho, \Sigma})_{m_{\Sigma}} \) can be identified with the image of \( T^S \) in the ring of \( \mathcal{O} \)-endomorphisms of

\[ M_{N_{\rho, \Sigma}} := H^1_{\text{et,c}}(Y_1(N_{\rho, \Sigma}), \mathcal{F}^k_p) \langle (-) - 1 \rangle_{m_{\Sigma}}. \]

See [13, Lemma 5.4.(iii)] and [17, Proposition 2.4.2] for details.

Remark 2.3 Denote by \( S_k(\Gamma_0(N), \mathcal{O}) \) the \( \mathcal{O} \)-module generated by normalized eigenforms in \( S_k(\Gamma_0(N)) \). Although it is easy to observe that \( M_{N_{\rho, \Sigma}} \) is isomorphic to two copies of \( S_k(\Gamma_0(N_{\rho, \Sigma}), \mathcal{O})_{m_{\Sigma}} \) via the integral Eichler–Shimura isomorphism [46, (3) in §1.2] under Assumption 2.1, we do not use the \( \Gamma_0(N_{\rho, \Sigma}) \)-level structure for \( M_{N_{\rho, \Sigma}} \) directly in order to avoid the issues on the moduli problem and the smoothness. Notably, \( T(N_{\rho, \Sigma})_{m_{\Sigma}} \) acts faithfully on \( S_k(\Gamma_0(N_{\rho, \Sigma}))_{m_{\Sigma}} \).

Lemma 2.4 If \( \Sigma \) contains \( P_{\overline{\rho}} \), then there exists a unique isomorphism of \( T^S \)-algebras

\[ T(N_{\rho, \Sigma})_{m_{\Sigma}} \cong T_{\Sigma}. \]

Proof As in [13, Lemma 5.4], it is enough to show that there exists a unique isomorphism of \( T^S \otimes_\mathcal{O} \mathbb{C} \)-algebras \( T(N_{\rho, \Sigma}) \otimes_{\mathcal{O}} \mathbb{C} \cong T_{\Sigma} \otimes_{\mathcal{O}} \mathbb{C} \), where we fix an embedding \( E \rightarrow \mathbb{C} \). Since our \( M_{N_{\rho, \Sigma}} \otimes \mathbb{C} \) is isomorphic to \( M_{\Sigma} \otimes \mathbb{C} \) in [13] with \( F = \mathbb{Q} \). Hence, by [13, Lemma 6.4.(i)], we complete the proof.

\[ \square \]

Remark 2.5 Note that \( P_{\overline{\rho}} \subseteq \Sigma \) is not assumed in [8, Proposition 4.7] since this case is excluded there. See [11, Remark 3.7 and §7.2] for detail. It corresponds to the type \( \text{V} \) in [11, §2]. In [9], it is bypassed by using [9, Lemma 1.5]. See also the comment right after [13, Proposition 6.5].
2.1.3 The level structure \( N_f^\Sigma \) for the \( L \)-value computation

Following [9, §1.7.3], we introduce another level structure depending on a newform \( f \) of level \( N_f \) and \( \Sigma \). Let \( d_0(\ell) := \dim_E H^0(U_{\ell}, V_f) \) and define

\[
N_f^\Sigma := N_f \cdot \prod_{\ell \in \Sigma} \rho_d(\ell).
\]

(23)

For a newform \( f = \sum a_n q^n \) of level \( N_f \) and \( \Sigma \), we define the \( \Sigma \)-imprimitive eigenform \( f^\Sigma = \sum b_n q^n \) of level \( N_f^\Sigma \) by

\[
b_n = \begin{cases} 
0 & \text{if } n \text{ is divisible by a prime in } \Sigma, \\
 a_n & \text{otherwise.}
\end{cases}
\]

Let \( \mathcal{T}(N_f^\Sigma) \) be the full Hecke algebra over \( \mathcal{O} \) faithfully acting on \( H^1_{\text{ét}}(Y_1(N_f^\Sigma), \mathcal{F}_q)[(-1) - 1] \), and \( m^\Sigma \subseteq \mathcal{T}(N_f^\Sigma) \) be the maximal ideal generated by \( m_f \) and \( U_q \) for \( q \in \Sigma \). Denote by \( \mathcal{T}(N_f^\Sigma)_{m^\Sigma} \) the localization of \( \mathcal{T}(N_f^\Sigma) \) at \( m^\Sigma \), and it is isomorphic to the image of \( \mathcal{T}^S \) in \( \text{End}_{\mathcal{O}}(H^1_{\text{ét}}(Y_1(N_f^\Sigma), \mathcal{F}_q)[(-1) - 1]) \) as before.

2.1.4 Congruence ideals and the \( R = \mathbb{T} \) theorem

Definition 2.6 (Congruence ideals; [9,10,13])

(1) For a newform \( f \) which arises as a \( \Sigma \)-ramified deformation of \( \overline{\rho} \), let \( \pi_{f,\Sigma} : T_{\Sigma} \to \mathcal{O} \) be the projection to the \( f \)-component. Then, the \( \Sigma \)-ramified congruence ideal of \( f \) is defined by

\[
\eta_{f,\Sigma} := \pi_{f,\Sigma} \left( \text{Ann}_{T_{\Sigma}}(\ker \pi_{f,\Sigma}) \right).
\]

(2) For a newform \( f \in S_k(\Gamma_0(N)) \) and \( \Sigma \), consider the \( \Sigma \)-imprimitive eigenform \( f^\Sigma \) of level \( N_f^\Sigma \) associated to \( f \) and define \( \pi_{f,\Sigma} : \mathcal{T}(N_f^\Sigma)_{m^\Sigma} \to \mathcal{O} \) by \( T_{\ell} \mapsto a_\ell(f) \) for all \( \ell \nmid N_f^\Sigma \). Then the congruence ideal of \( f^\Sigma \) is defined by

\[
\eta_{f,\Sigma}(N_f^\Sigma) := \pi_{f,\Sigma} \left( \text{Ann}_{\mathcal{T}(N_f^\Sigma)_{m^\Sigma}}(\ker \pi_{f,\Sigma}) \right).
\]

We also define the eigenform \( f_{\Sigma,\alpha_r}^\Sigma \) of level \( N_{\pi,\Sigma} \) by the \( r \)-stabilization of \( f^\Sigma \) with \( U_r \)-eigenvalue \( \alpha_r \).

Lemma 2.7

(1) If we choose \( \Sigma \) by the set of primes dividing \( N_f/N(\overline{\rho}) \), then \( N_{\pi,\Sigma} = r \cdot N_f^\Sigma \).

(2) If \( \Sigma \) contains \( P_r \) and \( N_{\pi,\Sigma} = r \cdot N_f^\Sigma \), then we have \( \eta_{f,\Sigma}(N_f^\Sigma) = \eta_{f,\Sigma,\alpha_r}(N_{\pi,\Sigma}) = \eta_{f,\Sigma} \) for a newform \( f \) of level \( N \) dividing \( N_{\pi,\Sigma} \).

Proof Since the first statement immediately follows from the definitions, we focus on the second one. Since the second equality immediately follows from Lemma 2.4, it suffices to check the first equality. By a basic property of the congruence ideals (cf. [8, (5.2.2)]), we have \( \eta_{f,\Sigma,\alpha_r}(r^2 \cdot N_f^\Sigma) \subseteq \eta_{f,\Sigma,\alpha_r}(r \cdot N_f^\Sigma) \subseteq \eta_{f,\Sigma}(N_f^\Sigma) \). It implies that \( \mathcal{O}/\eta_{f,\Sigma,\alpha_r}(r^2 \cdot N_f^\Sigma) \geq \mathcal{O}/\eta_{f,\Sigma,\alpha_r}(r \cdot N_f^\Sigma) \geq \mathcal{O}/\eta_{f,\Sigma}(N_f^\Sigma) \). The properties of \( r \) directly imply that the Euler factor of the adjoint \( L \)-function of \( f \) at \( s = 1 \) is a unit. By the freeness of the Hecke modules (e.g. §5.2), [13, Proposition 6.3 and Proof of Theorem 6.6,(1)], and the property of \( r \) above, we have \( \mathcal{O}/\eta_{f,\Sigma,\alpha_r}(r^2 \cdot N_f^\Sigma) = \mathcal{O}/\eta_{f,\Sigma}(N_f^\Sigma) \). Therefore, the conclusion follows. \( \Box \)

Remark 2.8

The level \( N_{\pi,\Sigma} \) in (2.2) is useful to work with \( R = \mathbb{T} \) theorem and the level \( N_f^\Sigma \) in (2.3) is convenient when we work with adjoint \( L \)-values.
We are now ready to recall the $R = \mathbb{T}$ theorem which we use later.

**Theorem 2.9** ([Diamond–Flach–Guo [9], Dimitrov [13]]) Let $\overline{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F})$ be an odd, absolutely irreducible, and continuous Galois representation satisfying Assumption 2.1. Assume that $\Sigma$ contains $P_{\overline{\rho}}$. Then, the natural surjective map

$$\pi_\Sigma : \mathcal{R}_\Sigma \to T_\Sigma$$

is an isomorphism of finite flat complete intersections over $\mathcal{O}$ and $\mathcal{M}_{\mathcal{N}_{\Sigma}}$ is free of rank two over $T_\Sigma$. In particular, all $\Sigma$-ramified deformations of $\overline{\rho}$ are modular.

Moreover, for all newforms $f$ such that $\rho_f$ is a $\Sigma$-ramified deformation of $\overline{\rho}$, we have

$$\# \text{Sel}_\Sigma(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O}) = \# \mathcal{O}/\eta_f, \Sigma < \infty$$

where $\text{Sel}_\Sigma(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})$ is the $\Sigma$-imprimitive adjoint Selmer group defined in Definition 3.11.

**Remark 2.10** The notion of cohomology congruence ideals is used in the original statement of [9]. See §5.2 for detail. The term was coined by Diamond in [10]. Careful readers will notice that the modularity lifting theorems in [9] and [13] have slightly different conditions on the image of the residual representation. However, it is easy to check either one is enough to obtain the same result when the base field is $\mathbb{Q}$.

Let $f \in S_k(\Gamma_0(N))$ be a normalized eigenform whose residual representation is isomorphic to $\overline{\rho}$. When we write $\mathbb{T}(N)_{m_f,}$, it is the localization of the full Hecke algebra $\mathbb{T}(N)$ at the maximal ideal $m_f$ generated by the Hecke eigensystem of $f$ and $\lambda$. As before, when $S$ contains $\Sigma$ and all prime divisors of $N$, $\mathbb{T}(N)_{m_f}$ can be identified with the image of $\mathbb{T}^{S}$ in $\text{End}_{\mathcal{O}} \left(H^2_{\text{et}}(Y_1(N), \mathcal{F})((-1)^{-1})_{m_f}\right)$. Note that $m_f$ is generated by $\overline{\rho}$, $U_q - \alpha_q(f)$ for $q$ exacty dividing $N$, and $U_q$ for $q$ dividing $N$ more than twice where $\alpha_q$ is the $q$-th Fourier coefficient of $f$. Since $\lambda \in m_f$, the choice of a congruent eigenform does not change the maximal ideal $m_f$.

We decompose $N = N^+ : N^-$ with $(N^+, N^-) = 1$ and $N^-$ square-free. Assume that $f$ is new at all primes dividing $N^-$. Then the map $\pi_f : \mathbb{T}(N^+ N^-)_{m_f} \to \mathcal{O}$ defined by $T_\ell \mapsto \alpha_\ell(f)$ for all $\ell \mid N^+ N^-$ factors through the $N^-$-new quotient $\mathbb{T}(N^+ N^-)_{m_f}^{N^-}$ of $\mathbb{T}(N^+ N^-)_{m_f}$. Thus, we naturally have the surjective map $\pi_f^{N^-} : \mathbb{T}(N^+ N^-)_{m_f}^{N^-} \to \mathcal{O}$.

**Definition 2.11** The $N^-$-new congruence ideal of $f$ is defined by

$$\eta_f(N^+, N^-) := \pi_f^{N^-} \left( \text{Ann}_{\mathbb{T}(N^+ N^-)_{m_f}^{N^-}}(\ker \pi_f^{N^-}) \right).$$

Note that $\mathbb{T}(N^+ N^-)_{m_f}^{N^-}$ acts faithfully on the $N^-$-new subspace $S_k(\Gamma_0(N))_{m_f}^{N^-}$ of $S_k(\Gamma_0(N))_{m_f}$.

### 2.2 Tamagawa exponents

Since we assume the image of $\overline{\rho}$ is irreducible (Assumption 2.1.(TW)), a Galois stable $\mathcal{O}$-lattice $T_f$ is uniquely determined up to scalars, so $\rho_n$ and $\overline{\rho}$ also have the same uniqueness property. We define $A_f := V_f / T_f$ and then have $A_f[\lambda^n] \simeq T_f / \lambda^n T_f$ for all $n \geq 1$. Following [38, Definition 3.3], Tamagawa exponents are defined as follows.
**Definition 2.12** The Tamagawa exponent $t_f(q)$ for $f$ is defined by the largest integer $t$ such that $A_f[\lambda^t]$ is unramified at $q$ and $A_f[\lambda^{t+1}]$ is ramified at $q$.

Note that $t_f(q)$ is finite if $q$ divides $N$, and $t_f(q) = 0$ if $q$ divides $N/p$. It is known that $(\lambda^t_f(q)) = \mathbb{F}_{q^t} \left( \left( H^1(I_{q^t}, T_f) \right)^{\text{tors}} \right)$ when $q$ divides $N^r$ (c.f. [21, 1.4.2.2]). Here, $\mathbb{Q}_q^{\text{tor}}$ is the unramified quadratic extension of $\mathbb{Q}_q$, and $I_{q^t}$ is the inertia subgroup of $G_{\mathbb{Q}_q^{\text{tor}}}$. In the case of elliptic curves, the Tamagawa exponent also coincides with the $p$-exponent of the local Tamagawa factor at $q$ of the elliptic curve over $\mathbb{Q}_q$. See [38, Page 1354], [4, Corollary 2 and Corollary 6.15], and [50, §6.3].

### 2.3 The idea of proof

In the proof of the Bloch–Kato conjecture for adjoint motives of modular forms [9], the following connections are established:

- The connection between adjoint $L$-values and cohomology congruence ideals is due to Hida's formula [24]. See also [8].
- The connection between cohomology congruence ideals and congruence ideals follows from the freeness of the Hecke module over the associated Hecke algebra. Such freeness results can be obtained in two ways: the method of Mazur, Ribet, Wiles, and Faltings–Jordan [18] (cf. [46, Theorem 1.13]) based on the $q$-expansion principle and the crystalline comparison isomorphism (Fontaine's $C_{\text{cris}}$) and the Diamond's improvement of the Taylor–Wiles system argument [12]. In Diamond's method, the level of the Hecke module should be of the form $N_{\Sigma, \Sigma'}$ for some $\Sigma$ [12, Theorem 2.4].
- The connection between congruence ideals and adjoint Selmer groups follows from the Taylor–Wiles system argument.

Our main theorem (Theorem 1.1) precisely measures the difference of two congruence ideals $\eta_f(N)$ and $\eta_f(N^+, N^-)$, but making such a connection is not straightforward at all. The following diagram summarizes how the connection is made.

The proof of Theorem 1.1 consists of two parts. The first part (the Selmer computation) is an approximation of the main theorem (Theorem 2.13) whose proof is based on the $R = \mathbb{T}$ argument and the computation of Galois cohomology. However, it does not give the exact formula but a slightly different formula. The second one (the $L$-value computation) removes the difference (Proposition 2.15) whose proof is based on the explicit comparison among adjoint $L$-values. Theorem 1.1 immediately follows from these two results.
2.3.1 The Selmer computation

Let $\Sigma$ be the set of primes dividing $N/N(\overline{\rho})$, $\Sigma^+$ the subset of $\Sigma$ consisting of primes not dividing $N^-$, and $\Sigma^- := \Sigma \setminus \Sigma^+$. We also decompose $N(\overline{\rho}) = N(\overline{\rho})^+ \cdot N(\overline{\rho})^-$ following the decomposition $N = N^+ \cdot N^-$ in Theorem 1.1 such that $N(\overline{\rho})^\pm \mid N^\pm$, respectively. For a prime $\ell$ dividing $N^+$, we put $c^+(\ell) := \text{ord}_\ell(N(\overline{\rho})^+)$ and $d(\ell) = \dim_{\mathbb{F}_\ell} H^0(I_\rho, \overline{\rho})$. We define

$$N^+_{\overline{\rho}, \Sigma^+} := r \cdot \prod_{\ell \in \Sigma^+} \ell^{c^+(\ell)+d(\ell)} \prod_{\ell \notin \Sigma^+, \ell \mid N(\overline{\rho})^+} \ell^{c^+(\ell)}$$

(2.4)

where $r$ is the same one in (2.2). Then, we have $N^+ \mid N^+_{\overline{\rho}, \Sigma^+}$. Note that $\ell \in \Sigma^+$ implies $\ell^2 \mid N^+_{\overline{\rho}, \Sigma^+}$. Compared with $N(\overline{\rho})^+$ and $N^+$, we have

$$N^+_{\overline{\rho}, \Sigma^+} = r \cdot N(\overline{\rho})^+ \cdot \prod_{\ell \in \Sigma^+} \ell^{d(\ell)} = r \cdot N^+ \cdot \prod_{\ell \in \Sigma^+} \ell^{d_0(\ell)}.$$

How much $N^+_{\overline{\rho}, \Sigma^+}$ and $N^+$ differ at primes in $\Sigma^+$? For $\ell \in \Sigma^+$, we observe the following:

- $d_0(\ell) \neq 2$ since $\Sigma^+$ is (minimally) chosen;
- $d_0(\ell) = 0$ if $\ell^2$ divides $N^+$;
- $d_0(\ell) = 1$ if $\ell$ divides $N^+$ exactly.

Note that $\text{ord}_{\ell}(N^+_{\overline{\rho}, \Sigma^+}/N^+)$ does not imply $\ell \mid N(\overline{\rho})$.

**Theorem 2.13** (Selmer computation) Let $f \in S_k(\Gamma_0(N))$ be a newform with $\overline{\rho}_f \simeq \overline{\rho}$, and $\Sigma$ the set of primes dividing $N/N(\overline{\rho})$. Suppose that $\overline{\rho}$ satisfies Assumption 2.1. We assume the following conditions:

- $2 \leq k \leq p - 1$.
- $\Sigma$ contains $P_\rho$ (defined in (2.1)).
- $p$, $N^+$, and $N^-$ are pairwise relatively prime.
- $N^-$ is square-free.
- For a prime divisor $q$ of $N^-$, if $q \equiv 1 \pmod{p}$, then $\overline{\rho}$ is ramified at $q$.

Then, we have

$$\text{ord}_{\lambda}(N^+_{\overline{\rho}, \Sigma^+}) = \text{ord}_{\lambda}(N^+_{\overline{\rho}, \Sigma^+}/r, N^-) + \sum_{q \mid N^-} t_f(q).$$

(2.5)

We prove Theorem 2.13 in §3 and §4. In §3, we review the deformation theory of Galois representations, study a presentation of the Galois deformation ring, and prove a refined $R = \mathbb{T}$ theorem. In §4, we review the standard facts of Galois cohomology and compute the difference between adjoint Selmer groups with different local conditions.

**Remark 2.14** (1) Lemma 2.7 is used in (2.5).

(2) The disadvantage of the Selmer computation is the rigidity of the level structure $N^+_{\overline{\rho}, \Sigma^+}/r$. If $\ell \in \Sigma^+$, then $\ell^2$ must divide $N^+_{\overline{\rho}, \Sigma^+}$. For example, the Selmer computation does not imply

$$\text{ord}_{\lambda}(N^+_{\overline{\rho}, \Sigma^+}) = \text{ord}_{\lambda}(N^+_{\overline{\rho}, \Sigma^+}/r \cdot \ell, N^-/\ell) + \sum_{q \mid N^-/\ell} t_f(q).$$
for any prime $\ell$ dividing $N^-$ at which $\overline{\rho}$ is unramified (cf. [33]).

In order to obtain Theorem 1.1 from Theorem 2.13, we will show the following implications

\[
\text{ord}_{\lambda} \eta_{/\Sigma}(N^\Sigma_f) = \text{ord}_{\lambda} \eta_{/\Sigma}(N^+_{p,\Sigma}/r, N^-) + \sum_{q | N^-} t_f(q)
\]

\[
\Rightarrow \text{ord}_{\lambda} \eta_{/\Sigma}(N^+_{p,\Sigma}/r \cdot N^-) = \text{ord}_{\lambda} \eta_{/\Sigma}(N^+_{p,\Sigma}/r, N^-) + \sum_{q | N^-} t_f(q)
\]

\[
\Rightarrow \text{ord}_{\lambda} \eta_{/\Sigma}(N^+ \cdot N^-) = \text{ord}_{\lambda} \eta_{/\Sigma}(N^+, N^-) + \sum_{q | N^-} t_f(q)
\]

by using the $L$-value computation.

### 2.3.2 The $L$-value computation

We assume the freeness result described in §5.2 to identify the congruence ideals and the cohomology congruence ideals here. The $L$-value computation is the following proposition.

**Proposition 2.15** (L-value computation) We keep the assumptions of Theorem 2.13.

1. Let $\Sigma^-$ be the set of primes dividing $N^-$ where $\overline{\rho}$ is unramified. If any prime in $\Sigma^-$ is not congruent to $\pm 1$ modulo $p$, then

\[
\text{ord}_{\lambda} \eta_{/\Sigma}(N^\Sigma_f) = \text{ord}_{\lambda} \eta_{/\Sigma}(N^+_{p,\Sigma}/r \cdot N^-).
\]

2. We have

\[
\text{ord}_{\lambda} \eta_{/\Sigma}(N^+_{p,\Sigma}/r \cdot N^-) - \text{ord}_{\lambda} \eta_{/\Sigma}(N^+_{p,\Sigma}/r, N^-) - \text{ord}_{\lambda} \eta_{/\Sigma}(N^+, N^-).
\]

We prove Proposition 2.15 in §5. In §5, we recall cohomology congruence ideals and study their interpretation as the adjoint $L$-values. Theorem 1.1 immediately follows from Theorem 2.13 and Proposition 2.15.

### 3 Deformation theory and a refined $R = \mathbb{T}$ theorem

We recall the relevant deformation theory of Galois representations and prove a refined $R = \mathbb{T}$ theorem (Theorem 3.14).

Let $\overline{\rho}$ be the residual representation fixed in §2.1. Then, $\overline{\rho}$ factors through $G_{\mathbb{Q}_S} = \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$ where $\mathbb{Q}_S$ is the maximal extension of $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$. From now on, we regard $\overline{\rho}$ as a representation of $G_{\mathbb{Q}_S}$

\[
\overline{\rho} : G_{\mathbb{Q}_S} \to \text{GL}_2(\mathbb{F}).
\]

We fix the determinant of all the liftings and deformations of $\overline{\rho}$ to be $\chi^{-k}_{\text{cyc}}$ throughout this article.

#### 3.1 A local–global principle of deformation functors

Let $\text{CNL}_\mathcal{O}$ be the category of complete Noetherian local (CNL) $\mathcal{O}$-algebras with residue field $\mathbb{F}$ whose morphisms are local $\mathcal{O}$-algebra morphism inducing the identity map on $\mathbb{F}$. Let $\mathcal{D}_{G_{\mathbb{Q}_S}} : \text{CNL}_\mathcal{O} \to \text{Sets}$ be the functor such that for $R \in \text{CNL}_\mathcal{O}$, $\mathcal{D}_{G_{\mathbb{Q}_S}}(R)$ is the set of
deformations $\rho : G_{Q,S} \to \text{GL}_2(R)$ of $\bar{\rho}$ such that $\det(\rho) = \chi_{cyc}^{1-k} \otimes_{O} R$. In particular, $\rho$ is unramified outside $S \cup \{\infty\}$.

Under Assumption 2.1.(TW), $\mathcal{D}_{G_{Q,S}}$ is representable and represented by the universal deformation ring $R_{G_{Q,S}}$ (with fixed determinant $\chi_{cyc}^{1-k}$).

If we consider $\Sigma$-ramified deformations, we impose unrestricted deformation conditions at primes in $\Sigma$.

We impose certain local conditions at $\ell \in S$ to cut out irrelevant deformations in $R_{G_{Q,S}}$. Imposing these local conditions can be interpreted as defining relatively representable subfunctors $\mathcal{D}(\ell)$ of the universal deformation functor $\mathcal{D}_{G_{Q,S}}$ which classifies the deformations of $\bar{\rho}|_{G_{Q,\ell}}$ with determinant $\chi_{cyc}^{1-k}$ (see [35, §21]) and [1, §3] for the details on relatively representable functors, noting that $\mathcal{C}_O$ in [1] denotes our $\text{CNL}_O$). We note that since $\bar{\rho}|_{G_{Q,\ell}}$ may not satisfy $\text{End}_{\bar{\rho}}(\bar{\rho}|_{G_{Q,\ell}}) = \mathbb{F}_\ell$, we will only have relatively representable functors for local deformations in general. Given $\mathcal{D}(\ell) \subseteq \mathcal{D}_{G_{Q,\ell}}$ as above for $\ell \in S$, a subfunctor $\mathcal{D}(S) : \text{CNL}_O \to \text{Sets}$ of $\mathcal{D}_{G_{Q,S}}$ is defined by the Cartesian diagram of functors

$$
\xymatrix{ \mathcal{D}(S) & \prod_{\ell \in S} \mathcal{D}(\ell) \ar[d] \ar[l] \ar[r] & \prod_{\ell \in S} \mathcal{D}_{G_{Q,\ell}} \ar[d] \\
\mathcal{D}_{G_{Q,S}} & \prod_{\ell \in S} \mathcal{D}_{G_{Q,\ell}}, \text{ "res"}}$
$$

where “res” is the natural transformation arising from the restriction of $G_{Q,S} \to \text{GL}_2(\mathbb{F})$ to $G_{Q,\ell} \to \text{GL}_2(\mathbb{F})$. More explicitly, for $R \in \text{CNL}_O$, $\mathcal{D}(S)(R)$ is the set of deformations $\rho : G_{Q,S} \to \text{GL}_2(R)$ contained in $\mathcal{D}_{G_{Q,S}}(R)$ such that the restriction $\rho|_{G_{Q,\ell}} : G_{Q,\ell} \to \text{GL}_2(R)$ satisfies the local deformation condition imposed by $\mathcal{D}(\ell)(R)$ for all $\ell \in S$ [35, §23]. By [1, Proposition 3.4], $\mathcal{D}(S)$ is also representable, and the deformation ring corresponding to $\mathcal{D}(S)$ is denoted by $R(S)$.

The tangent space of the deformation functors can be described in terms of adjoint Selmer groups. For $\ell \in S$, let $\rho_\ell \in \mathcal{D}_{G_{Q,\ell}}(R)$ be a deformation of $\bar{\rho}|_{G_{Q,\ell}}$ to a finite-length ring $R \in \text{CNL}_O$. Then, the **tangent space** $t_{\rho_\ell}$ of $\mathcal{D}_{G_{Q,\ell}}$ at $\rho_\ell \in \mathcal{D}_{G_{Q,\ell}}(R)$ is defined by the inverse image of $[\rho_\ell]$ under the map $\mathcal{D}_{G_{Q,\ell}}(R[\epsilon]/(\epsilon^2)) \to \mathcal{D}_{G_{Q,\ell}}(R)$ induced by $\epsilon \mapsto 0$. Then, it naturally has a structure of $R$-module, and there exists a natural isomorphism

$$
t_{\rho_\ell} \cong \text{H}^1(\mathbb{Q}_\ell, \text{ad}^0(\rho_\ell)) \quad (3.1)
$$

(cf. [35, §21]). Thus, $\mathcal{D}(\ell)$ gives a submodule $\mathcal{D}(\ell)(R[\epsilon]/(\epsilon^2)) \cap t_{\rho_\ell}$ of $t_{\rho_\ell}$, and via (3.1) it also give a subspace of $\text{H}^1(\mathbb{Q}_\ell, \text{ad}^0(\rho_\ell))$ which depends on $\rho_\ell$.

Let $\rho : G_{Q,S} \to \text{GL}_2(O) \subseteq \mathcal{D}(S)(O)$, and let $\pi_S : R(S) \to O$ be the corresponding specialization with kernel denoted by $\varphi_{\pi_S}$. Let $\rho_\ell : G_{Q,\ell} \to \text{GL}_2(O/\lambda^n)$ be the mod $\lambda^n$ reduction of $\rho$. The restriction of $\rho_\ell$ to $G_{Q,\ell}$ yields a subgroup $L_{\ell}\subsetneq \text{H}^1(\mathbb{Q}_\ell, \text{ad}^0(\rho_\ell))$ as explained above, and the direct limit defines $L_{\ell} \subseteq \text{H}^1(\mathbb{Q}_\ell, \text{ad}^0(\rho) \otimes E/O)$. Then, $\mathcal{L} := (L_{\ell})_{\ell \in S}$ becomes a Selmer structure for $S$ and $\text{ad}^0(\rho) \otimes E/O$ and we have an isomorphism

$$
\text{Hom}_O(\varphi_{\pi_S}, \varphi_{\pi_S}^2 \otimes E/O) \cong \text{Sel}_C(G_{Q,S}, \text{ad}^0(\rho) \otimes E/O) \quad (3.2)
$$

as in [48, Proposition 1.2] and [35, §28].

Let $\mathcal{C}_O$ be the category whose objects are pairs $(R, \pi_R)$ where $R$ is an object of $\text{CNL}_O$ and $\pi_R : R \to O$ is a surjective local $O$-algebra homomorphism, and whose morphisms are morphisms in $\text{CNL}_O$ which commute with $\pi_R$’s. For a pair $(R, \pi_R)$ in $\mathcal{C}_O$, write $\varphi_{\pi_R} := \ker(\pi_R)$. As indicated in (3.2), the **cotangent space** to $\text{Spec } R$ at $\varphi_{\pi_R}$ is defined by

$$
\Phi_{\pi_R} := \varphi_{\pi_R}/\varphi_{\pi_R}^2
$$
and the congruence ideal of \((R, \pi_R)\) by \(\eta_{R} := \pi_R (\text{Ann}_R(\varphi_{\pi_R})) \subseteq \mathcal{O}.

The relation between the cotangent space and the congruence ideal is now well-known as follows:

**Proposition 3.1.** Let \((R, \pi_R) \in \mathcal{C}_\mathcal{O}\) such that \(R\) is a finite flat \(\mathcal{O}\)-algebra and suppose that \(\eta_R \neq 0\). Then, \(R\) is a complete intersection if and only if \(\text{Fitt}_\mathcal{O}(\Phi_{\pi_R}) = \eta_{R} \).

**Proof.** See [32, Corollary 10]. \(\square\)

**Remark 3.2.** Since \(\text{Fitt}_\mathcal{O}(\Phi_{\pi_R}) = \left(\lambda \text{length}_\mathcal{O}(\Phi_{\pi_R})\right), \text{ord}_\lambda(\eta_{R})\) can be written in terms of length_{\mathcal{O}}(\Phi_{\pi_R}).

### 3.2 Local deformation conditions

For a prime \(\ell\), let \(\mathcal{R}(\ell)\) be the local versal deformation ring of \(\overline{\mathcal{F}}|_{\mathcal{G}_{\ell}}\) relatively representing the deformation functor \(\mathcal{O}(\ell)\) corresponding to the local Selmer condition \(L_{\ell,1} \subseteq H^1(\mathbb{Q}_\ell, \text{ad}^0(\overline{\mathcal{F}}))\). The existence of versal deformation rings follows from [42, Theorem 2.11] even without the \(\text{End}(\mathcal{F})|_{\mathcal{G}_{\ell}} = \mathcal{F}\) condition, and they are determined only up to non-canonical isomorphisms. Thanks to [1, Theorem 1.2.(i)], \(\mathcal{R}(\ell)\) admits presentation

\[
\mathcal{R}(\ell) \simeq \mathcal{O}[x_1, \ldots, x_r]/a_\ell
\]

where \(r = \dim_{\mathbb{Q}} L_{\ell,1}\). It is known that the number of generators of \(a_\ell\) is bounded by the dimension of \(H^2(\mathbb{Q}_\ell, \text{ad}^0(\overline{\mathcal{F}}))\). Denote by \(\text{gen}(a_\ell)\) the (minimal) number of generators of \(a_\ell\).

We quickly review the useful local deformation problems (cf. [44, P1-P7], [7, Definition 2.2.2]) and discuss vanishing of \(a_\ell\). In [7], the deformation problems are considered without fixing determinants, but it does not cause any problem in our setting since \(p > 2\).

Although we explicitly write down the local Selmer conditions only for \(\text{ad}^0(\rho_n)\) in this section, they easily generalize to \(\text{ad}^0(\rho_n)\) for all \(n \geq 1\).

#### 3.2.1 Low-weight crystalline

We first recall some of the Fontaine–Laffaille theory [20], [9, §1.1.2]. Let \(\mathcal{M}\mathcal{F}\) be the category of filtered \(\varphi\)-modules whose objects are finitely generated \(\mathcal{O}\)-modules \(M\) equipped with

- a decreasing filtration such that \(\text{Fil}^a M = M\) and \(\text{Fil}^b M = 0\) for some \(a, b \in \mathbb{Z}\), and for each \(i \in \mathbb{Z}\), \(\text{Fil}^i M\) is a direct summand of \(M\);
- \(\mathcal{O}\)-linear maps \(\varphi^i : \text{Fil}^i M \to M\) for \(i \in \mathbb{Z}\) satisfying \(\varphi^i|_{\text{Fil}^{i+1} M} = p \cdot \varphi^{i+1}\) and \(M = \sum_i \text{Im}(\varphi^i)\).

It is known that \(\mathcal{M}\mathcal{F}\) is an abelian category. Denote by \(\mathcal{M}\mathcal{F}^a\) the full subcategory of \(\mathcal{M}\mathcal{F}\) consisting of objects \(M\) satisfying \(\text{Fil}^a M = M\) and \(\text{Fil}^{a+p} M = 0\) and having no non-trivial quotients \(M'\) with \(\text{Fil}^{a+p} M' = M'\). Denote by \(\mathcal{M}\mathcal{F}^a_{\text{tor}}\) the full subcategory of \(\mathcal{M}\mathcal{F}^a\) consisting of objects of finite length. Also, \(\mathcal{M}\mathcal{F}^a\) and \(\mathcal{M}\mathcal{F}^a_{\text{tor}}\) are abelian categories and stable under taking subobjects, quotients, direct products, and extensions in \(\mathcal{M}\mathcal{F}\).

Fontaine and Laffaille constructed the fully faithful contravariant functor from \(\mathcal{M}\mathcal{F}^0_{\text{tor}}\) to the category of finite \(\mathcal{O}[\mathcal{G}_{\mathbb{Q}_p}]\)-modules, which is called the Fontaine–Laffaille functor.

Let \(R \in \mathcal{C}_\mathcal{O}\) with maximal ideal \(\mathfrak{m}_R\) and \(\rho|_{\mathcal{G}_{\mathbb{Q}_p}} : \mathcal{G}_{\mathbb{Q}_p} \to \text{Aut}_R(M) \simeq \text{GL}_2(R)\) be a deformation of \(\overline{\rho}\), so \(M\) is uniquely determined up to isomorphisms.
Definition 3.3  (Low-weight crystalline deformation) A deformation $\rho|_{G_{Q_p}} : G_{Q_p} \rightarrow \text{Aut}_R(M) \simeq \text{GL}_2(R)$ is a low-weight crystalline deformation of $\overline{\rho}|_{G_{Q_p}}$ if for every $n \geq 1$, $M/m_n^2M$ lies in the image of the Fontaine–Laffaille functor on $\mathcal{M}_F^0$. Let $H^1_{\text{GQ}}(\mathbf{Q}_p, \text{ad}^0(\overline{\rho})) := L_{p,1} \subseteq H^1(\mathbf{Q}_p, \text{ad}^0(\overline{\rho}))$ be the local condition at $p$ corresponding to the low-weight crystalline deformations via isomorphism (3.1). We omit the precise definition of $L_{p,1}$. See [9, §2.1], [7, §2.4.1] for example. Following [9, Corollary 2.3], we have $\dim_{\mathbb{F}} L_{p,1} = H^0(\mathbf{Q}_p, \text{ad}^0(\overline{\rho})) + 1$.

Furthermore, $a_p = 0$ by [7, Lemma 2.4.1] ("liftable").

3.2.2 Unramified
Let $\ell \neq p$ and assume that $\overline{\rho}|_{G_{Q_\ell}}$ is unramified; thus, $p \nmid \#(I_{Q_\ell})$. Let $L_\ell$ be the local condition at $\ell$ corresponding to the deformation functor parametrizing all unramified deformations of $\overline{\rho}|_{G_{Q_\ell}}$. Then

$$H^1_{\text{GQ}}(\mathbf{Q}_\ell, \text{ad}^0(\overline{\rho})) := L_{\ell,1} = H^1(G_{Q_\ell}/I_\ell, \text{ad}^0(\overline{\rho})^{I_\ell}).$$

By [44, E1], $H^2(G_{Q_\ell}/(I_\ell \cap \ker(\overline{\rho})), \text{ad}^0(\overline{\rho})) = 0$. It follows from [1, Theorem 1.2.(iv)] that $a_\ell = 0$.

Lemma 3.4  Let $\ell$ be any prime (including $p$). Let $M$ be a discrete $O$-module endowed with continuous $G_{Q_\ell}$-action. For all $n \geq 1$,

$$\#H^1(G_{Q_\ell}/I_\ell, M[\lambda^n]) = \#H^0(\mathbf{Q}_\ell, M[\lambda^n]).$$

Taking the direct limit, we have

$$\text{cork}_O H^1(G_{Q_\ell}/I_\ell, M[I_\ell]) = \text{cork}_O H^0(\mathbf{Q}_\ell, M).$$

Proof Comparing the kernel and the cokernel of $\text{Frob}_\ell - 1 : M[\lambda^n]^{I_\ell} \rightarrow M[\lambda^n]^{I_\ell}$, we obtain the conclusion. $\square$

3.2.3 Unrestricted
Let $\ell$ be a prime not dividing $p$. It is easy to see that $L_{\ell,1} = H^1(G_{Q_\ell}/I_\ell, \text{ad}^0(\overline{\rho}))$ corresponds to the unrestricted deformations of $\overline{\rho}|_{G_{Q_\ell}}$. By the computation in [1, Example 5.1.(i)], we have

$$\dim_{\mathbb{F}} L_{\ell,1} = \dim_{\mathbb{F}} H^1(G_{Q_\ell}/I_\ell, \text{ad}^0(\overline{\rho})) - \text{gen}(a_\ell) \geq 0.$$

3.2.4 Minimally ramified
Let $\ell$ be a prime dividing $N(\overline{\rho})$. Following [9, §3.1] and [33, §3.2], we have the following definition.

Definition 3.5  A deformation $\rho|_{G_{Q_\ell}}$ of $\overline{\rho}|_{G_{Q_\ell}}$ to $R \in \text{CN}_O$ is minimally ramified if it satisfies:

1. If $p \nmid \#(I_\ell)$, then $\#(I_\ell) = \#(I_\ell)$.
2. If $p \mid \#(I_\ell)$, then $\rho|_{I_\ell}$ has unramified quotient of rank one.

The corresponding Selmer local condition at $\ell$ is denoted by $H^1_{\text{min}}(\mathbf{Q}_\ell, \text{ad}^0(\overline{\rho})) := L_{\ell,1}$. 

Suppose that \( p \nmid \#p(I) \). Then, the minimally ramified liftings coincide with unramified liftings by [7, Lemma 2.4.22]. By applying Lemma 3.4, \( \dim_{\mathbb{F}} L_{\ell,1} = \dim_{\mathbb{F}} H^1(Q_{\ell}, \text{ad}^0(\overline{\rho})) \). By applying [44, E1] again, we have \( a_{\ell} = 0 \).

Suppose that \( p \mid \#p(I) \). Then, the minimally ramified liftings coincide with [44, E3]. The relevant definition and computation are given in the \( \ell \)-new deformation case.

### 3.2.5 New

We closely follow [44, E3]. Let \( \ell \) be a prime not equal to \( p \).

**Assumption 3.6** We assume one of the following conditions:

1. \( \ell \not\equiv 1 \pmod{p} \), or
2. \( p \mid \#p(G_{Q_{\ell}}) \)

**Definition 3.7**

1. The first case corresponds to the type \( \textbf{P} \) in [11, §2] and its deformation following [44, E3] is called an \( \ell \)-new deformation.

2. The second case corresponds to the type \( \textbf{S} \) in [11, §2] and its deformation following [44, E3] is called a minimally ramified deformation (of type (2) in Definition 3.5).

In both cases, following [11, §2] and [26, §4.2.5], there exists a choice of basis of \( \mathbb{F}^2 \) such that

\[
\overline{\rho}|_{G_{Q_{\ell}}} \sim \begin{pmatrix} 1 & \xi \\ 0 & \chi_{\text{cyc}} \end{pmatrix} \otimes \chi_{\text{cyc}}^{1-k/2} \text{ where } \xi \in \mathbb{Z}^1(Q_{\ell}, \mathbb{F}(1-k/2)).
\]

Since the universal deformation rings of equivalent representations up to a character twist are canonically isomorphic [34, Proposition 1], we may assume \( \overline{\rho}|_{G_{Q_{\ell}}} \sim \begin{pmatrix} 1 & \xi \\ 0 & \chi_{\text{cyc}} \end{pmatrix} \) without loss of generality.

We consider the collection of the deformations \( \rho|_{G_{Q_{\ell}}} \) of the following form \( \begin{pmatrix} 1 & \xi \\ 0 & \chi_{\text{cyc}} \end{pmatrix} \). In order to define the local condition \( L_{\ell,1} \subseteq H^1(Q_{\ell}, \text{ad}^0(\overline{\rho})) \) corresponding to the deformations, we first explicitly describe how \( \text{ad}^0(\rho)|_{G_{Q_{\ell}}} \) looks like. A straightforward computation shows that

\[
\text{ad}^0(\rho)|_{G_{Q_{\ell}}} \sim \begin{pmatrix} \chi_{\text{cyc}} & -2\xi \chi_{\text{cyc}} & -\xi^2 \chi_{\text{cyc}} \\ 0 & 1 & \xi \\ 0 & 0 & \chi_{\text{cyc}}^{-1} \end{pmatrix}.
\]

(3.3)

Let \( M = \text{ad}^0(\rho)|_{G_{Q_{\ell}}} \otimes E/\mathcal{O} \) be the discrete \( G_{Q_{\ell}} \)-module corresponding to the matrix form above and consider \( G_{Q_{\ell}} \)-stable filtration

\[
0 \subset M_2 \subset M_1 \subset M_0 = M
\]

induced by (3.3).

Let \( \rho_{n}|_{G_{Q_{\ell}}} \) be the mod \( \lambda^n \) reduction of \( \rho|_{G_{Q_{\ell}}} \) and \( M[\lambda^n] \) be the \( \lambda^n \)-torsion of \( M \). Then, we have \( \text{ad}^0(\rho_{n}|_{G_{Q_{\ell}}} \otimes E/\mathcal{O} \simeq \text{ad}^0(\rho|_{G_{Q_{\ell}}} \otimes \lambda^{-n}\mathcal{O}/\mathcal{O}) \). Note that the ramification of \( \rho_{n}|_{G_{Q_{\ell}}} \) is completely controlled by the 1-cocycle \( \xi \pmod{\lambda^n} \). More explicitly, considering the equation

\[
\begin{pmatrix} \chi_{\text{cyc}} & -2\xi \chi_{\text{cyc}} & -\xi^2 \chi_{\text{cyc}} \\ 0 & 1 & \xi \\ 0 & 0 & \chi_{\text{cyc}}^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \chi_{\text{cyc}} \cdot a - 2\xi \chi_{\text{cyc}} \cdot b - \xi^2 \chi_{\text{cyc}} \cdot c \\ b + \xi \cdot c \\ \chi_{\text{cyc}}^{-1} \cdot c \end{pmatrix},
\]

(3.4)

we can explicitly observe that \( M_2 \) is generated by \( ^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( M_1 \) is generated by \( ^t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( ^t \begin{pmatrix} 0 & 1 \end{pmatrix} \). Thus, we have \( H^1(Q_{\ell}, M_2) \simeq H^1(Q_{\ell}, E/\mathcal{O}(1)) \). The local Selmer condition
at $\ell$ corresponding to the deformations above is defined by

$$H^1_{\text{new}}(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho})) = L_{\ell,1} := \text{Im} \left( H^1(\mathbb{Q}_\ell, M_2[\lambda]) \to H^1(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho})) \right) \subseteq H^1(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho}))$$

when $\overline{\rho}$ is unramified at $\ell$. Even when $\overline{\rho}$ is ramified at $\ell$, we also denote it by $H^1_{\text{new}}(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho})) = H^1_{\text{min}}(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho}))$ for convenience.

**Lemma 3.8** Suppose that $L_{\ell,1}$ is the local Selmer condition corresponding to $\ell$-new or minimally ramified deformations as in Definition 3.7. Then,

1. $\dim_{\mathbb{F}_\ell} L_{\ell,1} = \dim_{\mathbb{F}_\ell} H^1(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho}))$, and
2. the versal deformation ring of $\overline{\rho}|_{G_{\mathbb{Q}_\ell}}$ corresponding to $L_{\ell,1}$ is smooth over $\mathcal{O}$, i.e. unobstructed.

**Proof** See [44, E3].

### 3.3 The deformation ring and the adjoint Selmer group

Putting all the local deformation conditions discussed together, we define the following deformation ring.

Let $\Sigma^+$ and $\Sigma^-$ be two finite sets of primes $\ell(\neq p)$ such that $\Sigma^+ \cap \Sigma^- = \emptyset$.

**Definition 3.9** ($\Sigma^+$-ramified $\Sigma^-$-new deformation rings) Let $\mathcal{D}_{\Sigma^-}^{\Sigma^+}$ be the deformation functor satisfying the following local deformation conditions:

1. The local deformation at $p$ is a low-weight crystalline deformation (Definition 3.3);
2. At $\ell \notin \Sigma^+ \cup \Sigma^-$, the local deformation at $\ell$ is minimally ramified;
3. At $\ell \in \Sigma^+$, the local deformation at $\ell$ is unrestricted;
4. At $\ell \in \Sigma^-$, the local deformation at $\ell$ is new if $\overline{\rho}$ is unramified at $\ell$;
5. At $\ell \in \Sigma^-$, the local deformation at $\ell$ is minimally ramified if $\overline{\rho}$ is ramified at $\ell$.

The deformation ring representing the functor $\mathcal{D}_{\Sigma^-}^{\Sigma^+}$ is called the $\Sigma^+$-ramified $\Sigma^-$-new deformation ring and denoted by $\mathcal{R}_{\Sigma^-}^{\Sigma^+}$, and denote by $\rho_{\Sigma^-}^{\Sigma^+}$ the corresponding representation. If $\Sigma^- = \emptyset$, we write $\mathcal{R}_{\Sigma^+} = \mathcal{R}_{\Sigma^+}^{\emptyset}$ and $\rho_{\Sigma^+} = \rho_{\Sigma^+}^{\emptyset}$.

**Remark 3.10** The representability of $\mathcal{D}_{\Sigma^-}^{\Sigma^+}$ is ensured by the argument described in §3.1. See also [27,30] (depending on [40]), and [49] for another description.

Let $f$ be a newform such that $\rho_f$ is a $\Sigma^+$-ramified $\Sigma^-$-new deformation of $\overline{\rho}$. Then we define the adjoint Selmer group of $f$ corresponding to the deformation problem following (3.2) as follows. For notational convenience, write $M = \text{ad}^0(f) \otimes E/\mathcal{O}$.

**Definition 3.11** The $\Sigma^+$-imprimitive $\Sigma^-$-new adjoint Selmer group $\text{Sel}_{\Sigma^-}^{\Sigma^+}(\mathbb{Q}, M)$ of $f$ is defined by the kernel of the natural restriction map

$$\phi_{\Sigma^-}^{\Sigma^+} : H^1(\mathbb{Q}_{\Sigma \cup \Sigma^+}/\mathbb{Q}, M) \to \begin{array}{c} \text{H}^1(\mathbb{Q}_p, M) \\ H^1_f(\mathbb{Q}_p, M) \end{array} \oplus \bigoplus_{\ell \in \Sigma \setminus \Sigma^+} \begin{array}{c} \text{H}^1(\mathbb{Q}_\ell, M) \\ \text{H}^1_{\text{min}}(\mathbb{Q}_\ell, M) \end{array} \oplus \bigoplus_{\ell \in \Sigma^-} \begin{array}{c} \text{H}^1(\mathbb{Q}_\ell, M) \\ \text{H}^1_{\text{new}}(\mathbb{Q}_\ell, M) \end{array}.$$

### 3.4 A presentation of the deformation ring

We quickly summarize [1, §5]. It turns out that many naturally defined local deformation conditions satisfy the following properties.
(1) If \( \ell \) does not divide \( p \), then
\[
\dim \mathbb{F}_{L,1} - \dim \mathbb{F}_{H_0(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho}))} - \text{gen}(a_\ell) \geq 0.
\]

(2) If we impose a suitable semi-stable assumption on deformations at \( p \), then
\[
\sum_{v | p} \left( \dim \mathbb{F}_{L,p} - \dim \mathbb{F}_{H_0(\mathbb{Q}_p, \text{ad}^0(\overline{\rho}))} - \text{gen}(a_p) \right) \geq 0.
\]

Note that the condition at the infinite place is automatic since \( \overline{\rho} \) is odd. If we further assume that \( \text{gen}(a_p) \leq 1 \), then the global deformation ring with these local constraints is a complete intersection. The precise statement is as follows.

**Theorem 3.12** (Böckle) Let \( \overline{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_p) \) be an odd, continuous, and absolutely irreducible Galois representation. We assume the following conditions.

1. If \( \ell \) does not divide \( p \), then
\[
\dim \mathbb{F}_{L,1} - \dim \mathbb{F}_{H_0(\mathbb{Q}_\ell, \text{ad}^0(\overline{\rho}))} - \text{gen}(a_\ell) \geq 0.
\]

2. If \( \ell = p \), then \( \overline{\rho}|_{G_{\mathbb{Q}_p}} \) is low-weight crystalline and
\[
\dim \mathbb{F}_{L,p} - \dim \mathbb{F}_{H_0(\mathbb{Q}_p, \text{ad}^0(\overline{\rho}))} - \text{gen}(a_p) \geq 0.
\]

3. \( \overline{\rho} \) satisfies Assumption 2.1.(TW).

Then, the corresponding deformation ring \( R \) is a complete intersection over \( \mathcal{O} \), i.e.
\[
R \simeq \mathcal{O}[[X_1, \ldots, X_n]]/(f_1, \ldots, f_n)
\]
for suitable \( f_i \in \mathcal{O}[[X_1, \ldots, X_n]] \).

**Proof** See [1, Corollary 4.3 and Theorem 5.8]. Note that \( H^0(G_{\mathbb{Q}, S}, \text{ad}^0(\overline{\rho})^\vee) = 0 \) where \( M^\vee = \text{Hom}_{\mathbb{F}}(M, \mathbb{F})(1) \) since \( \overline{\rho} \) satisfies Assumption 2.1.(TW). \( \square \)

**Corollary 3.13** The deformation ring \( R_{\Sigma^+}^- \) is a complete intersection; thus, we have
\[
R_{\Sigma^+}^- \simeq \mathcal{O}[[X_1, \ldots, X_n]]/(f_1, \ldots, f_n)
\]

**Proof** Considering all the local deformation conditions in §3.2, the conclusion immediately follows. \( \square \)

### 3.5 A refined \( R = \mathbb{T} \) theorem

The goal of this section is to prove the following theorem.

**Theorem 3.14** Keep Assumption 2.1. Let \( N \geq 1 \) be an integer such that \( N(\overline{\rho}) \) divides \( N \). Let \( \Sigma^+ \) and \( \Sigma^- \) be two finite sets of primes not equal to \( p \) such that \( P_{\overline{\rho}} \subseteq \Sigma \). Write \( \Sigma^- = \Sigma \setminus \Sigma^+ \). There exists an isomorphism
\[
\pi_{\Sigma^+}^\Sigma^- : R_{\Sigma^+}^\Sigma^- \simeq \mathbb{T}(N_{\overline{\rho}, \Sigma^+}^- N_{\Sigma^-}^-)_{m_{\Sigma^+}^-}
\]
of reduced finite flat complete intersections over \( \mathcal{O} \).

Recall that \( \mathbb{T}(N_{\overline{\rho}, \Sigma^+}^- N_{\Sigma^-}^-)_{m_{\Sigma^+}^-} \) is isomorphic to the image of the abstract \( S \)-anemic Hecke algebra \( \mathbb{T}^S \) in \( \text{End}_{\mathbb{O}}(H^1_{\text{et}, S}(Y_1(N_{\overline{\rho}, \Sigma^+}^- N_{\Sigma^-}^-)_{m_{\Sigma^+}^-} F_p^p) \langle -1 \rangle m_{\Sigma^+}^-) \) with \( \Sigma \subseteq S \) and \( \mathbb{T}(N_{\overline{\rho}, \Sigma^+}^- N_{\Sigma^-}^-)_{m_{\Sigma^+}^-} \) is the \( N^- \)-new quotient of \( \mathbb{T}(N_{\overline{\rho}, \Sigma^+}^- N_{\Sigma^-}^-)_{m_{\Sigma^+}^-} \).
Before giving a proof, we recall two lemmas.

**Lemma 3.15** Suppose that $R \simeq O landfill X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ is a finite $O$-algebra with $m \leq n$. Then $m = n$ and $R$ is a finite flat complete intersection.

**Proof** See [33, Lemma 2.3]. \(\square\)

**Lemma 3.16** Suppose that $R$ is a finite flat reduced $O$-algebra. Let $R/\alpha$ be a quotient of $R$ which is a finite flat $O$-algebra. Then, $R/\alpha$ is reduced.

**Proof** See [33, Lemma 2.4]. \(\square\)

**Proof of Theorem 3.14** By Theorem 2.9, we have an isomorphism

$$\pi : R_\Sigma \simeq T_\Sigma$$

of reduced finite flat complete intersections over $O$. Note that the reduced and finite flat properties come from $T_\Sigma$ (cf. [13, §1.2 and Theorem 6.2]). Since $R_\Sigma$ is a quotient of $R$, $R_\Sigma$ is finite over $O$. By Corollary 3.13, $R_\Sigma$ is a complete intersection. Furthermore, the flatness and the reduceness of $R_\Sigma$ follows from Lemma 3.15 and Lemma 3.16, respectively. We explicitly construct the map

$$\pi : R_\Sigma \rightarrow T(N_{p, \Sigma}^+, N^-)^{N^+}_{m^+}$$

satisfying the commutative diagram

$$\begin{array}{ccc}
R_\Sigma & \xrightarrow{\pi} & T_\Sigma \\
\downarrow_{\text{Thm. 2.9}} & & \downarrow_{\text{Lem. 2.4}} \\
R_\Sigma^+ & \xrightarrow{\pi^+} & T(N_{p, \Sigma}^+, N^-)^{N^+}_{m^+} \\
\end{array}$$

and show that it is an isomorphism.

Suppose that there is an $O$-algebra morphism $\alpha : R_\Sigma^+ \rightarrow O'$ where $O'$ is a domain of characteristic zero and the corresponding deformation is denoted by $\rho'$. By making the following composition

$$\begin{array}{ccc}
T(N_{p, \Sigma}^+, N^-)^{N^+}_{m^+} & \xrightarrow{\pi^1} & R_\Sigma \\
\downarrow & & \downarrow \alpha \\
O' & & O'
\end{array}$$

there exists a newform $g$ of level dividing $N_{p, \Sigma}$ and $\rho'$ and $\rho_\ell$ are obviously equivalent.

Let $\ell$ be a prime dividing $N^-$ not dividing $N(\overline{p})$. Since $\rho'|_{G_{\ell}}$ is a $\ell$-new deformation at $\ell$, we have

$$\text{tr} \rho'(\text{Frob}_\ell) = \pm \ell^{k-2}(\ell + 1).$$

By considering the Ramanujan–Petersson bound (at good primes), it is easy to see that $\rho'$ is ramified at all primes dividing $N^-$. Thus, $g$ is new at all primes dividing $N^-$ and the map (3.6) factors through $T(N_{p, \Sigma}^+, N^-)^{N^+}_{m^+}$ and we obtain a map $\beta : T(N_{p, \Sigma}^+, N^-)^{N^+}_{m^+} \rightarrow O'$.

The universality of $R_\Sigma^+$ induces the surjective map

$$\pi : R_\Sigma^+ \rightarrow T(N_{p, \Sigma}^+, N^-)^{N^+}_{m^+}$$

such that $\alpha = \beta \circ \pi$. Since $\alpha = \beta \circ \pi^+$, we have the bijection between
• the characteristic zero minimal prime ideals of $R_{\Sigma_1^-}^-$ and
• the characteristic zero minimal prime ideals of $T(N_{\rho_{\Sigma_1}}^+ N^-)_{m_{\Sigma_1^+}}$.

We now claim the injectivity of $\pi_{\Sigma_1^-}^-$. Due to the bijection between minimal primes above, the kernel of $\pi_{\Sigma_1^-}^-$ is contained in the intersection of all the characteristic zero minimal prime ideals of $R_{\Sigma_1^-}^-$. In other words,

$$\ker(\pi_{\Sigma_1^-}^-) \subseteq \bigcap_{\text{char } 0} \wp.$$ 

Here, $\bigcap_{\text{char } 0}$ means that ideal $\wp$ runs over the set of minimal ideals of $R_{\Sigma_1^-}^-$ of characteristic zero. Since $O \to R_{\Sigma_1^+}^-$ is finite flat, a uniformizer $\lambda$ of $O$ maps to a nonzero divisor of $R_{\Sigma_1^-}^-$. Thus, any minimal prime $\wp$ does not contain $\lambda$. Thus, we obtain the conclusion due to the reduced property of $R_{\Sigma_1^+}^-$. 

**Remark 3.17** In (3.5), one may expect the existence of the deformation ring $"R_{\Sigma_1^-}^-$-ss" isomorphic to $T(N_{\rho_{\Sigma_1}}^+ N^-)^{m_{\Sigma_1^+}}$. If so, the following diagram would commute:

$$\begin{array}{cccc}
R_{\Sigma} & \xrightarrow{\pi_{\Sigma}} & T_{\Sigma} & \xrightarrow{=} & T(N_{\rho_{\Sigma}})_{m_{\Sigma}} \\
\downarrow \text{Thm. 2.9} & & \downarrow \text{Lem. 2.4} & & \downarrow \\
"R_{\Sigma_1^-}^-\text{-ss}" & \xrightarrow{=} & T(N_{\rho_{\Sigma_1}}^+ N^-)_{m_{\Sigma_1^+}} \\
\downarrow & & \downarrow & & \downarrow \\
R_{\Sigma_1^-}^- & \xrightarrow{\pi_{\Sigma_1^-}^-} & T(N_{\rho_{\Sigma_1}}^+ N^-)^{N_{\Sigma_1^+}^+}_{m_{\Sigma_1^+}^+}.
\end{array}$$

However, it looks difficult to impose the right local deformation condition at primes dividing $N^-$ (unless $\overline{\rho}$ is ramified at all primes dividing $N^-$) since the local deformation condition at unramified primes dividing $N^-$ should include both unramified and new deformations. This is also pointed out in [16, §9].

**4 Relative computation of adjoint Selmer groups**

The goal of this section is to prove Theorem 2.13 (the Selmer computation).

**4.1 Preliminaries on Galois cohomology**

Let $T$ be a free $O$-module of rank $d$ endowed with continuous action of $G_{\mathbb{Q}}$ and $S$ a finite set of places of $\mathbb{Q}$ containing $p$, $\infty$, and the ramified primes for $T$. In other words, we have a continuous $d$-dimensional integral Galois representation

$$\rho : G_{\mathbb{Q},S} \to \text{GL}_d(O) \simeq \text{Aut}_O(T).$$

Let $A = T \otimes_O E/\mathcal{O}$ be the associated discrete Galois module. For a Selmer structure $L = (L_\ell)_{\ell \in S}$ with $L_\ell \subseteq H^1(\mathbb{Q}_\ell, A)$, we define the **discrete Selmer group of $A$ with respect to $L$** by

$$\text{Sel}_L(G_{\mathbb{Q},S}, A) := \ker \left( \phi_L : H^1(G_{\mathbb{Q},S}, A) \to \prod_{\ell \in S} \frac{H^1(\mathbb{Q}_\ell, A)}{L_\ell} \right)$$

The Tate local duality gives us the non-degenerate pairing

$$H^1(\mathbb{Q}_\ell, A) \times H^1(\mathbb{Q}_\ell, T^*) \to E/\mathcal{O}$$
where $T^* := \text{Hom}(A,E/O(1))$. For a Selmer structure $L$ for $S$ and $A$, we define the dual Selmer structure $L^* = (L^*_{\ell})_{\ell \in S}$ for $S$ and $T^*$ by $L^*_\ell := L^*_\ell$ under the pairing. Then, we define the **dual (compact) Selmer group of $T^*$ with respect to $L^*$** by

$$\text{Sel}_{L^*}(G_{Q,S}, T^*) := \ker \left( \phi_{L^*} : H^1(G_{Q,S}, T^*) \to \prod_{\ell \in S} \frac{H^1(Q_\ell, T^*)}{L^*_\ell} \right).$$

The comparison between two Selmer groups sometimes reduces to the comparison of local conditions under the surjectivity of the global-to-local map defining the smaller Selmer group.

**Proposition 4.1** Let $L$ and $N$ be two Selmer structures for $S$ and $A$. If $L_{\ell} \subseteq N_{\ell}$ for all $\ell \in S$, then

$$\text{Sel}_{L}(G_{Q,S}, A) \subseteq \text{Sel}_{N}(G_{Q,S}, A).$$

If we further assume that $\phi_L$ is surjective, then we have

$$\frac{\text{Sel}_{N}(G_{Q,S}, A)}{\text{Sel}_{L}(G_{Q,S}, A)} \cong \prod_{\ell \in S} \frac{N_{\ell}}{L_{\ell}}.$$

**Proof** It immediately follows from the surjectivity of $\phi_L$. \hfill $\Box$

The following proposition is the direct limit version of the formula of Greenberg–Wiles [48, Proposition 1.6], which is an application of the Poitou–Tate exact sequence with Selmer structures. See also [33, Lemma 2.6].

**Proposition 4.2** Let $L$ be a Selmer structure for $S$ and $A$. Then, $\text{Sel}_L(G_{Q,S}, A)$ is cofinitely generated and $\text{Sel}_{L^*}(G_{Q,S}, T^*)$ is finitely generated as $O$-modules. Moreover, we have an equality

$$\text{cork}_O \text{Sel}_L(G_{Q,S}, A) - \text{rk}_O \text{Sel}_{L^*}(G_{Q,S}, T^*)$$

$$= \text{cork}_O H^0(G_{Q,S}, A) - \text{rk}_O H^0(G_{Q,S}, T^*) + \sum_{\ell \in S} (\text{cork}_O L_{\ell} - \text{cork}_O H^0(Q_\ell, A)).$$

In order to have the surjectivity of the global-to-local map defining Selmer groups, the local conditions should be “well-balanced” as follows. This is [33, Proposition 2.7], and we include the proof for the completeness.

**Proposition 4.3** Assume that $A[\lambda]$ and $T^*/\lambda T^*$ are irreducible as $G_{Q,S}$-modules. If $\text{Sel}_L(G_{Q,S}, A)$ is finite, and

$$\sum_{\ell \in S} \text{cork}_O L_{\ell} = \sum_{\ell \in S} \text{cork}_O H^0(Q_\ell, A),$$

then the global-to-local map $\phi_L$ is surjective.

**Proof** By Proposition 4.2, the finiteness of $\text{Sel}_L(G_{Q,S}, A)$ implies $\text{rk}_O \text{Sel}_{L^*}(G_{Q,S}, T^*) = 0$, so $\text{Sel}_{L^*}(G_{Q,S}, T^*)$ is also finite. Thus, $\text{Sel}_{L^*}(G_{Q,S}, T^*)$ is contained in the $O$-torsion of $H^1(G_{Q,S}, T^*)$. Due to the irreducibility assumption, $H^1(G_{Q,S}, T^*)$ is torsion free as an $O$-module following the argument in [22, §2.2]. Thus, $\text{Sel}_{L^*}(G_{Q,S}, T^*) = 0$. By [22, Proposition 3.1.1], $\phi_L$ is surjective. \hfill $\Box$
4.2 Local computation

We quickly recall some materials in §3.2.5. Let \( f \) be a new form such that \( \rho_f \simeq \rho \) such that \( \rho|_{G_{Q_\ell}} \) is an \( \ell \)-new deformation of \( \overline{\rho}|_{G_{Q_\ell}} \). More explicitly, \( \overline{\rho}|_{G_{Q_\ell}} \) is unramified but \( \rho|_{G_{Q_\ell}} \) ramified and is twist-equivalent to

\[
\begin{pmatrix}
1 & \xi \\
0 & \chi_{cyc}
\end{pmatrix}
\]

where \( \xi \in \mathbb{Z}^1(Q_\ell, \mathcal{O}(1 - k/2)) \). Then, \( \text{ad}^0(\rho)|_{G_{Q_\ell}} \) is equivalent to

\[
\begin{pmatrix}
\chi_{cyc} & -2\xi \chi_{cyc} & -\xi^2 \chi_{cyc} \\
0 & 1 & \xi \\
0 & 0 & \chi_{cyc}^{-1}
\end{pmatrix}
\]

Let \( M = \text{ad}^0(\rho)|_{G_{Q_\ell}} \otimes E/\mathcal{O} \) with \( G_{Q_\ell} \)-stable filtration

\[
0 \subseteq M_2 \subseteq M_1 \subseteq M_0.
\]

More explicitly, recall the equation (3.4)

\[
\begin{pmatrix}
\chi_{cyc} & a \\
0 & b
\end{pmatrix}
\begin{pmatrix}
\chi_{cyc} & \cdot a - 2\xi \chi_{cyc} \cdot b - \xi^2 \chi_{cyc} \cdot c \\
0 & b + \xi \cdot c
\end{pmatrix}
\begin{pmatrix}
1 \\
\chi_{cyc}^{-1} \cdot c
\end{pmatrix}
= 
\begin{pmatrix}
\chi_{cyc} & \cdot a \\
0 & b
\end{pmatrix}
\begin{pmatrix}
\chi_{cyc} & a \\
0 & b
\end{pmatrix}
\begin{pmatrix}
\chi_{cyc} & \cdot a - 2\xi \chi_{cyc} \cdot b - \xi^2 \chi_{cyc} \cdot c \\
0 & b + \xi \cdot c
\end{pmatrix}.
\]

Since \( M_2 \) is generated by \( t \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \) and \( M_1 \) is generated by \( t \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \), we have the following computation

\[
H^0(Q_\ell, M_2[\lambda^n]) \simeq \lambda^{-t} \mathcal{O}/\mathcal{O}
\]

\[
H^0(Q_\ell, M_0[\lambda^n]) \simeq \lambda^{-s} \mathcal{O}/\mathcal{O} \oplus \lambda^{-t} \mathcal{O}/\mathcal{O} \oplus \lambda^{-\min(s,t)} \mathcal{O}/\mathcal{O}
\]

\[
H^0(Q_\ell, M_0[\lambda^n]/M_2[\lambda^n]) \simeq \lambda^{-n} \mathcal{O}/\mathcal{O} \oplus \lambda^{-\min(s,t)} \mathcal{O}/\mathcal{O}
\]

for \( n \gg 0 \) where

- \( s \) is the largest integer such that \( \chi_{cyc}(g)c \equiv c \pmod{\lambda^s} \), i.e. \( s = \text{ord}_\ell(\ell - 1) \), and
- \( t \) is the largest integer such that \( \rho_t|_{G_{Q_\ell}} \) is semi-simple, i.e. \( t = t_f(\ell) \) (Definition 2.12).

Since the cohomological dimension of \( G_{Q_{\ell}}/I_{\ell} \) is one, the inflation-restriction sequence yields the exact sequence

\[
0 \rightarrow H^1(G_{Q_{\ell}}/I_{\ell}, M_2^{I_{\ell}}) \rightarrow H^1(Q_\ell, M_2) \rightarrow H^1(I_{\ell}, M_2)^{G_{Q_{\ell}}/I_{\ell}} \rightarrow 0.
\]

Here, we have

\[
H^1(G_{Q_{\ell}}/I_{\ell}, M_2^{I_{\ell}}) \simeq M_2^{I_{\ell}}/(\text{Frob}_{\ell} - 1)M_2^{I_{\ell}}
\]

\[
\simeq M_2/(\text{Frob}_{\ell} - 1)M_2
\]

\[
\simeq 0
\]

where the first isomorphism follows from that \( G_{Q_{\ell}}/I_{\ell} \) is topologically generated by \( \text{Frob}_{\ell}, \)
the second isomorphism follows from \( M_2^{I_{\ell}} = M_2 \) (since \( \ell \neq p \)), and the last isomorphism follows from that \( \text{Frob}_{\ell} - 1 \) acts on \( M_2 \) as multiplication by \( \ell - 1 \) and \( M_2 \) is divisible.
Thus, we have $\text{H}^1(\mathbb{Q}_\ell, M_2) \simeq \text{H}^1(I_\ell, M_2)^{G_{\mathbb{Q}_\ell} / I_\ell}$. Since $I_\ell$ acts on $M_2$ trivially, we have

\begin{equation}
\begin{aligned}
\text{H}^1(\mathbb{Q}_\ell, M_2) &\simeq \text{H}^1(I_\ell, M_2)^{G_{\mathbb{Q}_\ell} / I_\ell} \\
&\simeq \text{Hom}_{G_{\mathbb{Q}_\ell} / I_\ell}(I_\ell, M_2) \\
&\simeq \text{Hom}_{G_{\mathbb{Q}_\ell} / I_\ell}(\mathbb{Z}_p(1), E / \mathcal{O}(1)) \\
&\simeq \text{Hom}_{G_{\mathbb{Q}_\ell} / I_\ell}(\mathcal{O}(1), E / \mathcal{O}(1)) \\
&\simeq \text{Hom}_\mathcal{O}(E, E / \mathcal{O}) \\
&\simeq E / \mathcal{O}
\end{aligned}
\end{equation}

(4.2)

From the short exact sequence

$$
0 \longrightarrow M_2 \longrightarrow M_0 \longrightarrow M_0 / M_2 \longrightarrow 0,
$$

we obtain the exact sequence

$$
0 \longrightarrow \text{H}^0(\mathbb{Q}_\ell, M_2) \longrightarrow \text{H}^0(\mathbb{Q}_\ell, M_0) \longrightarrow \text{H}^0(\mathbb{Q}_\ell, M_0 / M_2) \longrightarrow \text{H}^1(\mathbb{Q}_\ell, M_2) \longrightarrow \text{H}^1(\mathbb{Q}_\ell, M_0).
$$

By counting the size of each term as in (4.1) and (4.2), the connecting map $\delta$ becomes the multiplication by $\lambda^t$ on $E / \mathcal{O}$ (up to a unit), which is surjective. Thus, the image of $\text{H}^1(\mathbb{Q}_\ell, M_2)$ in $\text{H}^1(\mathbb{Q}_\ell, M_0)$ vanishes. It proves Proposition 4.4.(1).

Recall the inflation-restriction sequence

$$
0 \longrightarrow \text{H}^1(\mathcal{G}_{\mathbb{Q}_\ell} / I_\ell, M_0^{I_\ell}) \longrightarrow \text{H}^1(\mathbb{Q}_\ell, M_0) \longrightarrow \text{H}^1(I_\ell, M_0) / \text{H}^1(\mathbb{Q}_\ell, M_0).
$$

Since the action of $I_\ell$ on $M_0$ factors through the tame quotient $I_\ell^t$, we have

$$
\text{H}^1(I_\ell, M_0) \simeq \text{H}^1(I_\ell^t, M_0^{I_\ell}) \simeq M_0 / (\tau - 1) M_0
$$

where $\tau$ is a topological generator of $I_\ell^t$. By using Equation (3.4), we check $(\tau - 1) M_0 = M_1$. Therefore, $\text{H}^1(I_\ell, M_0) \simeq M_2 / M_1 \simeq E / \mathcal{O}(-1)$. Finally, we have $\text{H}^1(I_\ell, M_0)^{G_{\mathbb{Q}_\ell} / I_\ell} \simeq \lambda^{-1} \mathcal{O} / \mathcal{O}$.

Note that $\text{H}^1(\mathcal{G}_{\mathbb{Q}_\ell} / I_\ell, M_0^{I_\ell}) \simeq M_0^{I_\ell} / (\text{Frob}_\ell - 1) M_0^{I_\ell}$. Then by using Equation (4.1) again, we have

$$
M_0^{I_\ell} \simeq E / \mathcal{O}(1) \oplus \lambda^{-t} \mathcal{O} / \mathcal{O} \oplus \lambda^{-t} \mathcal{O} / \mathcal{O}(-1)
$$

as $\mathcal{G}_{\mathbb{Q}_\ell} / I_\ell$-modules. Thus, we also have

$$(\text{Frob}_\ell - 1) M_0^{I_\ell} \simeq (\ell - 1) E / \mathcal{O} \oplus 0 \oplus (\ell - 1) \lambda^{-t} \mathcal{O} / \mathcal{O}$$

$$
\simeq E / \mathcal{O} \oplus 0 \oplus \lambda^{-t} \mathcal{O} / \mathcal{O}
$$

as $\mathcal{O}$-modules. Thus, $\text{H}^1(\mathcal{G}_{\mathbb{Q}_\ell} / I_\ell, M_0^{I_\ell}) \simeq \lambda^{-t} \mathcal{O} / \mathcal{O} \oplus \lambda^{-\min(s,t)} \mathcal{O} / \mathcal{O}$ as $\mathcal{O}$-modules, so Proposition 4.4.(2) follows.

**Proposition 4.4** Let $f$ be a newform such that $\rho_f \simeq \rho$ and $\rho|_{G_{\mathbb{Q}_\ell}}$ is an $\ell$-new deformation of $\overline{\rho}|_{G_{\mathbb{Q}_\ell}}$. Then, we have the following statements:

1. **The local condition**

   $$
   L_\ell = \text{Im} \left( \text{H}^1(\mathbb{Q}_\ell, M_2) \to \text{H}^1(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E / \mathcal{O}) \right)
   $$

   vanishes.
(2) \(H^1(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E/\mathcal{O}))\) is a non-trivial finite \(\mathcal{O}\)-module and
\[
\text{length}_{\mathcal{O}} H^1(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E/\mathcal{O}) = \text{ord}_p(\ell - 1) + \text{tr}(\ell) + \min(\text{ord}_p(\ell - 1), \text{tr}(\ell)).
\]

**Corollary 4.5** Keep all the assumptions of Proposition 4.4. If we further assume that \(\ell \equiv 1 \pmod{p}\) for all primes \(\ell\) dividing \(N^-/N(\mathfrak{N})^-\), then
\[
\text{length}_{\mathcal{O}} H^1(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E/\mathcal{O})) = \text{tr}(\ell).
\]

### 4.3 Global computation

Let \(\pi_{f^\Sigma} : \mathbb{T}(N_{\mathfrak{P},x}/r)_{\mathfrak{m}_x} \rightarrow \mathcal{O}\) be the map associated to \(f^\Sigma\) as defined in §2.1. Let \(\varphi_{f,x} = \ker(\pi_{f,x})\), and denote by \(\Phi_{f,x}\) the cotangent space at \(\varphi_{f,x}\) and by \(\eta_{f,x}(N_{\mathfrak{P},x}/r)\) the congruence ideal. Then we have
\[
\text{ord}_p \eta_{f,x}(N_{\mathfrak{P},x}/r) = \text{length}_{\mathcal{O}} \Phi_{f,x} = \text{length}_{\mathcal{O}} \text{Sel}_x(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O})
\]

where the first equality follows from Theorem 2.9, Proposition 3.1, and Remark 3.2 and the second equality follows from (3.2).

Let \(\pi_{f,x}^{N^-} : \mathbb{T}(N_{\mathfrak{P},x}^+/r)_{\mathfrak{m}_x} \rightarrow \mathcal{O}\) be the \(N^-\)-new quotient map and \(f^\Sigma^+\) as defined in proof of Theorem 3.14. Let \(\varphi_{f,x}^{N^-} = \ker(\pi_{f,x}^{N^-})\), and denote by \(\Phi_{f,x}^{N^-}\) the cotangent space at \(\varphi_{f,x}^{N^-}\) and by \(\eta_{f,x}^{N^-}(N_{\mathfrak{P},x}^+/r)\) the congruence ideal. Then, we have
\[
\text{ord}_p \eta_{f,x}^{N^-}(N_{\mathfrak{P},x}^+/r) = \text{length}_{\mathcal{O}} \Phi_{f,x}^{N^-} = \text{length}_{\mathcal{O}} \text{Sel}_x^{N^-}(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O})
\]

where the first equality follows from Proposition 3.1, Remark 3.2, and Theorem 3.14 and the second equality follows from (3.2).

Now it suffices to compute the difference between \(\text{Sel}_x^{N^-}(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O})\) and \(\text{Sel}_x(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O})\). Note that
\[
\text{Sel}_x^{N^-}(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O}) \subseteq \text{Sel}_x(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O}).
\]

Consider the following “small” Selmer group \(\text{Sel}_x(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O})\) satisfying
\[
\begin{align*}
L_p & = H^1_{u_0}(\mathbb{Q}_p, \text{ad}^0(f) \otimes E/\mathcal{O}), \\
L_\ell & = H^1_{\text{min}}(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E/\mathcal{O}) \text{ if } \ell \notin \Sigma^+ \cup \Sigma^- \cup \{p\}, \\
L_\ell & = H^1_{\text{new}}(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E/\mathcal{O}) \text{ if } \ell \in \Sigma^-, \text{ and} \\
L_\ell & = H^1_{u_0}(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E/\mathcal{O}) \text{ if } \ell \in \Sigma^+.
\end{align*}
\]

Then, we have
\[
\text{Sel}_x(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O}) \subseteq \text{Sel}_x^{N^-}(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O}) \subseteq \text{Sel}_x(\mathbb{Q}_x, \text{ad}^0(f) \otimes E/\mathcal{O}).
\]
and $\phi_C$ is surjective by Proposition 4.3 and the properties of local deformation conditions in §3.2. By Proposition 4.1, we have

$$\frac{\text{Sel}_{\Sigma}(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})}{\text{Sel}_{E}(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})} \cong H^1_{ur} \times \prod_{\ell \in \Sigma^+} H^1_{ur},$$

$$\frac{\text{Sel}_{\Sigma}(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})}{\text{Sel}_{E}(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})} \cong H^1_{ur} \times \prod_{\ell \in \Sigma^+} H^1_{ur} \times \prod_{\ell \in \Sigma^-} H^1_{\text{new}}$$

where $H^1_{ur}$, $H^1_{ur}$, and $H^1_{\text{new}}$ are obvious abbreviations. Thus, we have

$$\frac{\text{Sel}_{\Sigma}(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})}{\text{Sel}_{E}(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})} \cong \prod_{\ell \in \Sigma^-} H^1_{\text{new}} \cong \prod_{\ell \in \Sigma^-} H^1$$

where the last isomorphism follows from Proposition 4.4.(1). Combining all the results, we have

$$\text{ord}_\lambda \eta_{\Sigma^+}(N_{\Sigma^+}^+/R) - \text{ord}_\lambda \eta_{\Sigma^-}(N_{\Sigma^-}^+ / R) = \text{length}_\Sigma \text{Sel}_{\Sigma}(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O}) - \text{length}_\Sigma \text{Sel}_{\Sigma}^-(\mathbb{Q}, \text{ad}^0(f) \otimes E/\mathcal{O})$$

$$= \sum_{\ell \in \Sigma^-} \text{length}_\Sigma H^1(\mathbb{Q}_\ell, \text{ad}^0(f) \otimes E/\mathcal{O})$$

and Theorem 2.13 follows from Proposition 4.4.(2) and Corollary 4.5.

### 5 Congruence ideals and adjoint $L$-values

The goal of this section is to prove Proposition 2.15 (the $L$-value computation).

#### 5.1 Adjoint $L$-functions

Let $f \in S_k(\Gamma_0(N))$ be a newform. Then, the $L$-function of $f$ is defined by

$$L(f, s) := \prod_{\ell \mid N} (1 - a_\ell(f)p^{-s} + p^{k-1-2s})^{-1} \cdot \prod_{\ell \nmid N} (1 - a_\ell(f)p^{-s})^{-1}$$

and note that the Euler factor $L_\ell(f, s)$ at $\ell \nmid N$ is

$$(1 - a_\ell(f)p^{-s} + p^{k-1-2s})^{-1} = (1 - a_\ell(f)p^{-s})^{-1} \cdot (1 - \beta_\ell(f)p^{-s})^{-1}.$$

For a prime $\ell \neq p$, we recall

$$c_0(\ell) := \text{ord}_\ell(N), \quad d_0(\ell) := \dim_E H^1(l_\ell, V_f).$$

Then, we have

$$d_0(\ell) = \begin{cases} 2 & \text{if } \ell \nmid N, \\ 1 & \text{if } \ell \mid N \text{ and } a_\ell(f) \neq 0, \\ 0 & \text{if } \ell \mid N \text{ and } a_\ell(f) = 0. \end{cases}$$

If $d_0(\ell) = 2$, then

$$L_\ell(\text{ad}^0(f), s) := \left( 1 - \frac{a_\ell(f)}{\beta_\ell(f)}p^{-s} \right)^{-1} \cdot (1 - \ell^{-s})^{-1} \cdot \left( 1 - \frac{\beta_\ell(f)}{a_\ell(f)}p^{-s} \right)^{-1}.$$

If $d_0(\ell) = 1$, then

$$L_\ell(\text{ad}^0(f), s) := \begin{cases} (1 - \ell^{-1-s})^{-1} & \text{if } \pi_\ell(f) \text{ is special}, \\ (1 - \ell^{-s})^{-1} & \text{if } \pi_\ell(f) \text{ is principal series}. \end{cases}$$
where \( \pi_\ell(f) \) is the local component of the automorphic representation attached to \( f \) at \( \ell \).

We omit the definition of \( L_\ell^m(\text{ad}^0(f), s) \) when \( d_0(\ell) = 0 \), but note that it may not be 1 in general (“the exceptional set for \( f \)” cf. [9, §1.7.2 and §1.8.1]). For the exact calculation of congruence ideals, we modify Euler factors as follows:

\[
L_\ell^m(\text{ad}^0(f), s) := \begin{cases} 
L_\ell(\text{ad}^0(f), s) & \text{if } d_0(\ell) > 0, \\
1 & \text{if } d_0(\ell) = 0.
\end{cases}
\]

5.2 Cohomology congruence ideals and their variants

We quickly review cohomology congruence ideals following [8, §4.4], [10], and [9, §1.7.3].

Let \( k \) be an integer with \( 2 \leq k \leq p - 1 \). Let \( \rho \) be the fixed irreducible residual representation and \( f \in S_k(\Gamma_0(N)) \) an eigenform with \( \rho_f \cong \rho \). Let \( T(N)_{m_f} \) be the full Hecke algebra localized at the non-Eisenstein maximal ideal \( m_f \). As in §2.1.2, it follows from [13, Lemma 5.4.(iii)] that \( T(N)_{m_f} \) is isomorphic to the image of \( T^S \) in \( \text{End}_\mathcal{O}(H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{F}_{k,G})(-1)_{m_f}) \).

We write \( M_N = H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{F}_{k,G})_{m_f} \). We do not care too much about the compactly supported or parabolic cohomologies since \( m_f \) is non-Eisenstein.

The cup product and Poincaré duality on Betti cohomology and the comparison between Betti and étale cohomologies imply that there exists an alternating and non-degenerate pairing

\[
\delta_f(N) : = \wedge^2 M_N[\ell_f] \subseteq \mathcal{O}.
\]

In order to identify congruence ideals and cohomology congruence ideals, we need the following freeness result.

**Proposition 5.1** Suppose that \( m_f \) is non-Eisenstein. Then,

1. \( T(N)_{m_f} \) is Gorenstein, and
2. \( M_N \) is free over \( T(N)_{m_f} \) of rank two.

**Proof** See [13, Proposition 5.5] (cf. [18, Theorem 2.1], [46, Theorem 1.13]). \( \square \)

**Remark 5.2** The freeness obtained from Diamond’s Taylor–Wiles system argument requires that the level should be of the form \( N_{\rho, \Sigma} \) for some \( \Sigma \) [12, Theorem 3.4].

Let \( a^{N^\perp} = \ker(T(N)_{m_f} \to T(N)_{m_f}^{N^\perp}) \) and we define \( M_N^{N^\perp} : = M_N[a^{N^\perp}] \).

**Lemma 5.3** Suppose that \( m_f \) is non-Eisenstein. The quotient \( M_N/ \left( M_N[a^{N^\perp}] \right) \) is torsion-free.

**Proof** It is straightforward from Proposition 5.1 and the torsion-freeness of \( \mathcal{O} \). \( \square \)
Corollary 5.4 Suppose that $m_{f}$ is non-Eisenstein and $f$ is now at all primes dividing $N^{-}$. Then, $\mathcal{M}^{N^{-}_f}$ is free over $\mathbb{T}(N)^{N^{-}_f}$ of rank two and $\mathbb{T}(N)^{N^{-}_f}$ is Gorenstein.

Proof By Lemma 5.3, we have

$$\mathcal{M}^{N^{-}_f} \otimes_{\mathcal{O}} F \subseteq \mathcal{M} \otimes_{\mathcal{O}} F.$$ 

Since $S_{k}(\Gamma_{0}(N))^{N^{-}_f} \neq 0$, we have

$$0 \neq \left( \mathcal{M}^{N^{-}_f} \otimes_{\mathcal{O}} F \right) [m_{f}] \subseteq (\mathcal{M} \otimes_{\mathcal{O}} F) [m_{f}] \simeq F^{\oplus 2}$$

and all are even dimensional. Thus, we have $(\mathcal{M}^{N^{-}_f} \otimes_{\mathcal{O}} F) [m_{f}] \simeq F^{\oplus 2}$. Note that $\mathcal{M}^{N^{-}_f}$ is free over $\mathcal{O}$ and $\dim_{E} \left( \mathcal{M}^{N^{-}_f} \otimes_{\mathcal{O}} E \right) = 2 \cdot \dim_{E} \left( \mathbb{T}(N)^{N^{-}_f} \otimes_{\mathcal{O}} E \right)$. Thus, $\mathcal{M}^{N^{-}_f}$ is free of rank two over $\mathbb{T}(N)^{N^{-}_f}$. The Gorenstein property immediately follows from the freeness.

\[ \Box \]

Remark 5.5 Since we work in the subspace of classical modular forms, we do not impose any condition on $N^{-}$ here. If we use modular forms on Shimura curves or Hida varieties attached to the quaternion algebra of discriminant $N^{-}$ via the Jacquet–Langlands correspondence, then we should assume $\ell \neq \pm 1 \pmod{p}$ for all residually unramified prime $\ell$ dividing $N^{-}$ at least (e.g. Assumption 6.3).

From the perfect pairing (5.1), we have a perfect pairing

$$\mathcal{M}_{N}[a^{N^{-}}] \times \left( \mathcal{M}_{N} \otimes_{\mathbb{T}(N)m_{f}} \mathbb{T}(N)^{N^{-}_f} \right) \to \mathcal{O}.$$ 

By Corollary 5.4, the perfect pairing can be written as

$$\mathcal{M}^{N^{-}_N} \times \mathcal{M}^{N^{-}_f} \to \mathcal{L}^{N^{-}}$$

where $\mathcal{L}^{N^{-}} := \wedge^{2} \mathcal{M}^{N^{-}_N} = \lambda^{t(N^{+},N^{-})} \mathcal{O}$ for some $t(N^{+},N^{-})$ (comparing with $\mathcal{L} = \mathcal{O}$). Then, we define the $N^{-}$-new cohomology congruence ideal of $f$ by

$$\delta_{f}(N^{+},N^{-}) := \lambda^{-t(N^{+},N^{-})} \cdot \delta_{f}(N).$$

Consider the map

$$\gamma : \mathcal{M}_{N} \to \mathcal{M}^{N^{-}_{f}}$$

defined in [9, §1.7.3]. Under the comparison between de Rham and étale cohomologies, this map corresponds to the map $f \mapsto f^{\Sigma}$ in the de Rham side. Then, we define the cohomology congruence ideal of $f^{\Sigma}$ by

$$\delta_{f^{\Sigma}}(N^{\Sigma}_{f}) := \wedge^{2} \mathcal{M}^{N^{-}_{f}}(\gamma_{f^{\Sigma}}) \subseteq \mathcal{L} := \wedge^{2} \mathcal{M}^{N^{-}_{f}} = \mathcal{O}$$

where the last identification is the normalization of $\mathcal{L}$ as before.

Proposition 5.6 Suppose that $\overline{\rho}$ is irreducible. Then,

$$\delta_{f^{\Sigma}}(N^{\Sigma}_{f}) = \delta_{f}(N) \cdot \prod_{\ell \in \Sigma} I_{\ell}^{nv} \cdot (\text{ad}^{0}(f), 1)^{-1}.$$

Proof See [9, Proposition 1.4.(c)].

Decompose $N = N^{+} \cdot N^{-}$ and suppose that $\Sigma^{+} = \Sigma$ does not contain any divisor of $N^{-}$. Consider the restriction of $\gamma$ to the $N^{-}$-new parts

$$\gamma^{N^{-}} : \mathcal{M}^{N^{-}_N} \to \mathcal{M}^{N^{-}_{f}, N^{-}}.$$
Let $\mathcal{L}^{N_{-}^+} := \wedge^2 M^{N_{-}^+} = \kappa^{t(N_{p_{\Sigma}}^+ / r N_{-})} \mathcal{O} \subseteq \mathcal{L}_{\Sigma} = \mathcal{O}$ for some $t(N_{p_{\Sigma}}^+ / r N_{-})$. In the same manner, we define the $N_{-}^+$-new cohomology congruence ideal of $f_{\Sigma}^+$ by

$$\delta_{\Sigma}^+(N_{p_{\Sigma}}^+ / r N_{-}) := \kappa^{-t(N_{p_{\Sigma}}^+ / r N_{-})} \cdot \delta_{\Sigma}^+(N_{p_{\Sigma}}^+ / r N_{-}).$$

**Corollary 5.7** Suppose that $\bar{p}$ is irreducible. Then,

$$\delta_{\Sigma}^+(N_{p_{\Sigma}}^+ / r N_{-}) = \delta_{\Sigma}(N_{p_{\Sigma}}^+, N_{-}) \cdot \prod_{\ell \in \Sigma^+} L^{\nu_{\ell}}_{\ell}(ad^0(f), 1)^{-1}.$$  

Thus, $t(N_{p_{\Sigma}}^+ / r N_{-}) = t(N_{p_{\Sigma}}^+, N_{-})$.

**Proof** The same proof as in [9, Proposition 1.4.(c)] directly works. □

**Remark 5.8**

1. Indeed, $t(N_{p_{\Sigma}}^+, N_{-})$ becomes $\sum_{q \mid N_{-}} t_f(q)$ at the end under our assumptions. Thus, $t(N_{p_{\Sigma}}^+ / r N_{-}) = t(N_{p_{\Sigma}}^+, N_{-})$ can be expected easily.

2. Cohomology congruence ideals are stable under base change [9, §1.7.3].

**Corollary 5.9** Suppose that $m_f$ is non-Eisenstein.

1. $\delta_f(N) = \eta_f(N)$, and

2. $\delta_f(N_{p_{\Sigma}}^+, N_{-}) = \eta_f(N_{p_{\Sigma}}^+, N_{-})$.

**Proof** See [8, Lemma 4.17] (cf. [10, Lemma 3.1]). □

Thus, Proposition 2.15 immediately follows.

### 6 Arithmetic applications

In this section, we briefly describe various arithmetic applications of Theorem 1.1.

In §6.1 and §6.2, we generalize the work of Pollack–Weston [38] and Chida–Hsieh [5] on the comparison between Hida’s canonical periods and Gross periods to higher-weight modular forms (Corollary 6.7). We also extend the $\mu$-part of the anticyclotomic main conjecture for modular forms of higher weight to Greenberg Selmer groups (Corollary 6.11).

In §6.3, we study the integral periods arising from Shimura curves. Under certain assumptions, we show that if the canonical periods arising from modular curves and Shimura curves differ by a $p$-adic unit, then Prasanna’s conjecture holds. See Corollary 6.16 for the precise statement.

#### 6.1 Comparison with Gross periods

We recall [5, §6] with some modifications. Let $f \in S_k(\Gamma_0(N))$ be a newform as in Theorem 1.1. We keep the following assumption in this subsection.

**Assumption 6.1** Assume that $N_{-}$ is the square-free product of an odd number of primes.

#### 6.1.1 Quaternionic modular forms

Let $B$ be the definite quaternion algebra over $\mathbb{Q}$ of discriminant $N_{-}$ and $R$ an Eichler order of level $N_{p_{\Sigma}}^+$. Let $f_B$ be a Jacquet–Langlands transfer of $f$, i.e. a continuous function

$$f_B : B^* \setminus \widetilde{B}^* / \widetilde{R}_{(p)^\infty} \rightarrow \text{Sym}^{k-2}(E^2).$$
such that \( f_B(a \cdot g \cdot r) = r^{-1} \circ f_B(g) \) for \( a \in B^\times \) and \( r \in R_2^\times \simeq \text{GL}_2(\mathbb{Z}_p) \) and the Hecke eigenvalues of \( f \) and \( f_B \) are same at all primes not dividing \( N^- \). Denote by

\[
S_k^{N^-}(R, E)
= \left\{ f : B^\times \backslash \hat{B}^\times / \hat{R}^\times \rightarrow \text{Sym}^{k-2}(E^2) : r(a \cdot g \cdot r) = r^{-1} \circ f(g), f \text{ is not constant} \right\}.
\]

We recall the following simple form of the Jacquet–Langlands correspondence.

**Theorem 6.2** There exists a non-canonical isomorphism

\[
S_k(\Gamma_0(N), \mathbb{C})^{N^-} \simeq S_k^{N^-}(R, \mathbb{C})
\]

of Hecke modules with identification their Hecke algebras over \( \mathbb{C} \).

**Proof** See [25, Theorem 2.30 in §2.3.6] for this form of the correspondence. Indeed, the isomorphism should be understood as the one-to-one correspondence between Hecke eigensystems. \( \Box \)

By using fixed embeddings \( \iota_p \) and \( \iota_\infty \), we integrally normalize each quaternionic modular form \( f_B \) by its mod \( p \) non-vanishing of the values of \( f_B \) as in [5, §4.1]. The integral normalization of classical modular forms is given by the \( q \)-expansion. Using these two integral structures, we are able to identify the Hecke modules

\[
S_k(\Gamma_0(N), \mathcal{O})^{N^-} \simeq S_k^{N^-}(N, \mathcal{O})
\]

with identification of their Hecke algebras over \( \mathcal{O} \); however, this identification itself is ad hoc. It will have a precise meaning after establishing the freeness of the quaternionic Hecke modules (Theorem 6.5).

### 6.1.2 Quaternionic congruence ideals and Gross periods

Consider the perfect pairing

\[
\langle -,- \rangle_N : S_k^{N^-}(N, \mathcal{O}) \times S_k^{N^-}(N, \mathcal{O}) \rightarrow \mathcal{O}
\]

defined in [5, (6.1)];

\[
[f_B, g_B]_N := \sum_{[b]} \langle f_B(b), g_B(bw) \rangle_k \cdot \text{det}(\#(B^\times \cap b\hat{R}^\times b^{-1}\hat{\mathbb{Q}}^\times) / \hat{\mathbb{Q}}^\times)^{-1}
\]

where \( w \) is the Atkin–Lehner operator for level \( N^+ \), \([b] \) runs over \( B^\times \backslash \hat{B}^\times / \hat{R}^\times \hat{\mathbb{Q}}^\times \), and \( \langle -,- \rangle_k : \text{Sym}^{k-2}(\mathcal{O}) \times \text{Sym}^{k-2}(\mathcal{O}) \rightarrow \mathcal{O} \) is the perfect pairing defined in [5, §2.3].

Let \( \xi_{f_B}(N^+, N^-) \) be the quaternionic analogue of the cohomology congruence ideal for \( f_B \) using the above pairing (6.1) as in [38, §2.1] and [5, (3.9) and (4.3)].

**Hida’s canonical period for** \( f \) **is defined by**

\[
\Omega_f := \frac{(4\pi)^k \langle f, f \rangle_{\Gamma_0(N)}}{\eta_f(N)}
\]

and the **Gross period for** \( f \) **is defined by**

\[
\Omega_f^{N^-} := \frac{(4\pi)^k \langle f, f \rangle_{\Gamma_0(N)}}{\xi_{f_B}(N^+, N^-)}
\]

where \( \langle -,- \rangle_{\Gamma_0(N)} \) is the Petersson inner product.
6.1.3 The freeness of quaternionic Hecke modules and the comparison of periods

**Assumption 6.3** (CR$^+$) Suppose that $\pi$ satisfies Assumption 2.1. Let $f \in S_k(\Gamma_0(N))$ be a newform with decomposition $N = N^+ \cdot N^-$ such that

1. $\overline{\rho}_f \simeq \overline{\rho}$,
2. $2 \leq k < p - 1$ and $p \nmid N$,
3. $p$, $N^+$, and $N^-$ are pairwisely relatively prime,
4. $N^-$ is square-free and the product of an odd number of primes,
5. If $q$ divides $N^-$ and $q \equiv \pm 1 \pmod{p}$, then $q$ divides $N(\overline{\rho})$,
6. If $q$ divides $N^+$ exactly and $q \equiv 1 \pmod{p}$, then $q$ divides $N(\overline{\rho})$, and
7. $(N(\overline{\rho}), N / N(\overline{\rho})) = 1$.

**Remark 6.4** Assumption 6.3 is more strict than the assumptions in Theorem 1.1.

The following theorem follows from the standard Taylor–Wiles system argument.

**Theorem 6.5** Under Assumption 6.3, $S_k^{N^-}(N, \mathcal{O})_{m_f}$ is a free $\mathcal{T}(N)_{m_f}$-module of rank one.

**Proof** See [5, Proposition 6.8]. \(\square\)

**Remark 6.6** (1) In [5, (D3) in §6.2], the ordinary deformation condition at $p$ is considered only; however, the replacement of the ordinary deformation condition by the low-weight crystalline deformation condition does not affect any of result in [5, §6]. It is well known that the low-weight crystalline deformation condition fits well with the Taylor–Wiles system argument [9, 13].

(2) Considering the local deformation condition at a prime $q$ exactly dividing $N / N(\overline{\rho})$ as in [5, (D4) in §6.2], the tame level $N^+_{\overline{\rho}, \Sigma^+}$ in the Taylor–Wiles system argument can be directly chosen as $N^+$ under Assumption 6.3.(6).

We obtain the following comparison of two different periods.

**Corollary 6.7** Under Assumption 6.3, the following statements are valid.

1. $\eta_f(N^+, N^-) = \xi_{f_0}(N^+, N^-)$.
2. $\text{ord}_q(\Omega_f^{N^-} / \Omega_f) = \text{ord}_q \left( \frac{\eta_f(N)}{\eta_f(N^+, N^-)} \right) = \sum_{q \mid N^-} t_f(q)$.

**Proof** The first statement follows from Theorem 6.2 and [8, Lemma 4.17]. The proof of [5, Proposition 6.1] applies to the second statement in the exactly same manner. \(\square\)

6.2 Anticyclotomic $\mu$-invariants of modular forms

The goal of this subsection is to prove the $\mu$-part of the anticyclotomic main conjecture for modular forms of higher weight as an application of Corollary 6.7 (so of Theorem 1.1). We only give a sketch of the argument here. See [4, 5, 38] for details.

We keep Assumption 6.1 and Assumption 6.3 in this subsection. Let $K$ be an imaginary quadratic field such that $(\text{disc}(K), Np) = 1$ such that

- if a prime $\ell$ divides $N^-$, then $\ell$ is inert in $K$, and
- if a prime $\ell$ divides $N^+$, then $\ell$ splits in $K$.

**Assumption 6.8** Assume that $f$ is ordinary at $p$ and $a_p(f) \neq \pm 1 \pmod{p}$. 

Following [5,38], we are able to define two slightly different anticyclotomic $p$-adic $L$-functions $L_p(K_\infty,f)$ and $L_p(K_\infty,\mathcal{f})$ of $(f,K_\infty/K)$ in $\Lambda = \mathcal{O}[\text{Gal}(K_\infty/K)]$ relative to the Gross period and the Hida’s canonical period, respectively. Then, by Corollary 6.7 (cf. [38, §2.2]), we have

$$L_p(K_\infty,f) = L_p(K_\infty,\mathcal{f}) \cdot \eta_f(N) \cdot \xi_f(N^+,N^-).$$

Let $A_f^*$ be the central critical twist of $A_f$. The minimal Selmer group $\text{Sel}(K_\infty,A_f^*)$ is defined as the kernel of the map

$$H^1(K_\infty,A_f^*) \to \prod_{w \nmid p} H^1(K_\infty,w,A_f^*) \times \prod_{w | p} H^1_\text{ord}(K_\infty,w,A_f^*)$$

and the Greenberg Selmer group $\mathcal{S}\text{el}(K_\infty,A_f^*)$ is defined as the kernel of the map

$$H^1(K_\infty,A_f^*) \to \prod_{w \nmid p} H^1_\text{ord}(K_\infty,w,A_f^*) \times \prod_{w | p} H^1_\text{ord}(K_\infty,w,A_f^*)$$

where $w$ runs over all places of $K_\infty$, $I_\infty,w$ is the inertia subgroup at $w$ and $H^1_\text{ord}$ is the standard ordinary condition.

For $\Lambda$-module $M = \text{Sel}(K_\infty,A_f^*)$ or $\mathcal{S}\text{el}(K_\infty,A_f^*)$, we denote by $\mu(M)$ and $\lambda(M)$ by the $\mu$-invariant and the $\lambda$-invariant of the characteristic ideal of the Pontryagin dual of $M$, respectively.

**Proposition 6.9** (Pollack–Weston) *Under the assumptions in Theorem 1.1, $k = 2$, and Assumption 6.8, we have

$$\lambda(\text{Sel}(K_\infty,A_f^*)) = \lambda(\mathcal{S}\text{el}(K_\infty,A_f^*)), \quad \mu(\mathcal{S}\text{el}(K_\infty,A_f^*)) = \mu(\text{Sel}(K_\infty,A_f^*)) + \sum_{q | N^-} t_f(q).$$

**Proof** This is [38, Corollary 5.2].

**Theorem 6.10** (Chida–Hsieh) *Under Assumption 6.3 and Assumption 6.8, we have

$$\mu(L_p(K_\infty,f))) = \mu(\text{Sel}(K_\infty,A_f^*)) = 0.$$

**Proof** See [5, Theorem C] and [4, Corollary 1] with an enhancement by [29].

We extend Theorem 6.10 to Greenberg Selmer groups and anticyclotomic $p$-adic $L$-functions relative to Hida’s canonical periods.

**Corollary 6.11** *Under Assumption 6.3 and Assumption 6.8, we have

$$\mu(L_p(K_\infty,f))) = \mu(\mathcal{S}\text{el}(K_\infty,A_f^*)) = \sum_{q | N^-} t_f(q).$$

**Proof** It is immediate from Corollary 6.7, Proposition 6.9, and Theorem 6.10.

Corollary 6.11 completes a higher-weight generalization of the $\mu$-part of the anticyclotomic main conjecture [38, Theorem 6.9]. It seems possible to prove the supersingular analogue of Corollary 6.11 when $a_p(f) = 0$ as in [38], but we omit it.
6.3 Comparison with integral periods of Shimura curves

Let \( f \in S_k(\Gamma_0(N)) \) be a newform as in Theorem 1.1.

**Assumption 6.12** Assume that \( N^- \) is the square-free product of an even number of primes.

Let \( B \) be the indefinite quaternion algebra over \( \mathbb{Q} \) of discriminant \( N^- \) and \( f_B \) the Jacquet–Langlands transfer of \( f \).

### 6.3.1 Integrality of automorphic forms and the freeness of the Hecke modules

An integral normalization of \( f_B \) is much more delicate than the definite case since the geometry of Shimura curves is substantially involved and the normalization via the \( q \)-expansion is not available due to the lack of cusps. In [36], \( f_B \) is \( p \)-integrally normalized by considering the minimal regular model of the corresponding Shimura curve over \( \mathbb{Z}_p \).

It can also be checked by considering the values of \( f_B \) at CM points via [36, Proposition 2.9]. We assume the integrality in this subsection.

**Assumption 6.13** The Jacquet–Langlands transfer \( f_B \) of \( f \) is integrally normalized in the sense of [36].

Let \( X^{N^-}(N^+) \) be the Shimura curve of level \( N^+ \) and discriminant \( N^- \) over \( \mathbb{Q} \) and

\[
\mathcal{M}'_{N^+} := H^1_{\text{et}}(X^{N^-}(N^+), \mathcal{F}_B^\bullet)_{\mathfrak{m}_B}
\]

the cohomology of \( X^{N^-}(N^+) \) localized at the non-Eisenstein maximal ideal \( \mathfrak{m}_B \).

We make the following freeness assumption.

**Assumption 6.14** The Hecke module \( \mathcal{M}'_{N^+} \) is free of rank two over \( \mathbb{T}(N)_{\mathfrak{m}_B}^{-N^-} \).

**Remark 6.15** We discuss the known freeness result of \( \mathcal{M}'_{N^+} \) in §6.3.3. Since it is not established in full generality, we keep it as an assumption.

It is possible to pin down the canonical periods of \( f_B \) via the Eichler–Shimura isomorphism for Shimura curves (e.g. [41, (3.6)]) under Assumption 6.14. Using the Poincaré duality and the Betti-étale comparison as explained in §5.2, there exists a (surjective) perfect pairing

\[
\langle - , - \rangle' : \mathcal{M}'_{N^+} \times \mathcal{M}'_{N^+} \to \mathcal{O}.
\]

and we are able to define the following analogue of the cohomology congruence ideals for Shimura curves by

\[
\xi_{f_B}(N^+, N^-) := \wedge^2 \mathcal{M}'_{N^+}[\mathfrak{m}_B] \subseteq \mathcal{O}.
\]

Let \( S_k(\Gamma_0(N), \mathcal{O})_{\mathfrak{m}_B}^{-N^-} \subseteq S_k(\Gamma_0(N), \mathcal{O})_{\mathfrak{m}_B} \) be the \( \mathcal{O} \)-submodule generated by normalized eigenforms in \( S_k(\Gamma_0(N)) \) which are \( N^- \)-new and congruent to \( f \) modulo \( \lambda \). Let \( \mathcal{M}'_{N^+} \subseteq \mathcal{M}_N \) be the Hecke-stable submodule generated by the image of the two copies of \( S_k(\Gamma_0(N), \mathcal{O})_{\mathfrak{m}_B}^{-N^-} \) under the integral Eichler–Shimura isomorphism (Remark 2.3). Then the Jacquet–Langlands correspondence (e.g. [37, Remark 2.2]) identifies \( \mathcal{M}'_{N^+} \) with \( \mathcal{M}_{N^-} \). Under the freeness of the (identified) Hecke module over \( \mathbb{T}(N)_{\mathfrak{m}_B}^{-N^-} \) (Assumption 6.14), [8, Lemma 4.17] identifies

\[
\xi_{f_B}(N^+, N^-) = \eta_f(N^+, N^-).
\] (6.2)
See also [37, Remark 2.1]. We define the canonical period for $f_B$ by
\[
\Omega^{N^-}_f := \frac{(4\pi)^k \langle f_B, f_B \rangle_{\Gamma}}{\eta_f(N^+, N^-)}
\]
where $\langle f_B, f_B \rangle_{\Gamma}$ is the Petersson inner product on $X^{N^-} (N^+)(\mathbb{C})$.

### 6.3.2 Prasanna’s conjecture

In [37, Remark 2.2 and Example 2.3], it is expected that $\text{ord}_\lambda (\Omega^{N^-}_f / \Omega_f)$ is a $p$-adic unit if $\mathfrak{p}$ is irreducible. Note that it is a different phenomenon from the definite case (cf. Corollary 6.7). In the indefinite case, it expected that the difference between congruence ideals is encoded in the ratio of Petersson inner products, not the ratio of canonical periods. See [37, Conjecture 4.2] for the statement of the conjecture over totally real fields.

Combining all the contents in this section, the following statement immediately follows.

**Corollary 6.16** Let $f$ be a newform of weight $k$ and level $\Gamma_0(N)$ with decomposition $N = N^+ N^-$ such that

1. the restriction $\overline{\rho}_f |_{G_{\mathbb{Q}(\sqrt{p})}}$ is absolutely irreducible where $p^* = (-1)^{a+1} p$,
2. $2 \leq k \leq p - 1$,
3. $p, N^+, \text{ and } N^-$ are pairwise relatively prime,
4. $N^-$ is square-free, and
5. if a prime $q \equiv \pm 1 \pmod{p}$ and $q$ divides $N^-$, then $\overline{\rho}_f$ is ramified at $q$.

We further assume the following statements:

(a) $N^-$ is the square-free product of an even number of primes (Assumption 6.12).
(b) The Jacquet–Langlands transfer $f_B$ of $f$ is integrally normalized in the sense of [36] (Assumption 6.13).
(c) The Hecke module $M'_N$ is free of rank two over $\mathcal{T}(N)_{m_f}$ (Assumption 6.14).
(d) $\text{ord}_\lambda (\Omega^{N^-}_f / \Omega_f)$ is a $\lambda$-adic unit.

Then,
\[
\text{ord}_\lambda \left( \frac{\langle f_B, f_B \rangle_{\Gamma}}{\langle f, f \rangle_{\Gamma}} \right) = \sum_{q \mid N^-} t_q(f).
\]

**Proof** Since $\text{ord}_\lambda (\Omega^{N^-}_f / \Omega_f)$ is a $\lambda$-adic unit, $\text{ord}_\lambda \left( \frac{\langle f_B, f_B \rangle_{\Gamma}}{\langle f, f \rangle_{\Gamma}} \right)$ becomes the ratio between the congruence ideal and the $N^-$-new congruence ideal (with help of (6.2)). Then the conclusion follows from Theorem 1.1. \qed

**Remark 6.17**

1. In the case of elliptic curves, Assumption (b) on the integrality can be removed by the geometric method of Ribet–Takahashi. See [36, §2.2.1].
2. In the case of weight two forms, Assumption (c) follows from [23, Corollary 8.11 and Remark 8.12] under the tame level assumption (5). In [23], although the level is assumed to be square-free, it seems easy to be removed via Ihara’s lemma for Shimura curves over $\mathbb{Q}$.
3. In the case of elliptic curves, Assumption (d) on the ratio of the canonical periods can be removed by using Faltings’ isogeny theorem, but we need it for the higher-weight case. See [37, Example 2.3].
6.3.3 Remarks on the freeness of the Hecke modules for Shimura curves

We briefly review the development of the freeness of the higher-weight Hecke modules for Shimura curves over $\mathbb{Q}$ based on [3,6]. Let $\Sigma^+$ be the primes in $\Sigma$ not dividing $N^-$ as defined in §2.3.1.

**Theorem 6.18** (Cheng, Cheng–Fu) We assume the following conditions:

- Assumption 2.1.(TW).
- $\overline{\rho}$ occurs in $\mathcal{M}_{N^+}(\overline{\rho})$.
- $N(\overline{\rho})$ is square-free.
- If $\ell$ divides $N^-$ and $\ell^2 \equiv 1 \pmod{p}$, then $\ell$ divides $N(\overline{\rho})$.
- $\text{End}_{\mathcal{F}_p[G_{\mathbb{Q}p}]}([\overline{\rho}]|_{G_{\mathbb{Q}p}} \otimes \mathcal{F}_p) = \mathcal{F}_p$.

Then, $\mathcal{M}_{N^+}^{\Sigma^+}$ is free of rank two over $\mathbb{T}(N^+_{\overline{\rho}}, \Sigma^+ \cdot N^-_{\overline{\rho}})^{\mathbb{Z}^{\Sigma^+}}$.

**Proof** See [6, Theorem 5.14] for the $N^+_{\overline{\rho}} = N(\overline{\rho})$ case, i.e. $\Sigma^+ = \emptyset$. For general $\Sigma^+$, see [3, Theorem 3.11], which is based on [12] and Ihara’s lemma for Shimura curves [15].

\[\Box\]

**Remark 6.19**

1. The square-freeness condition of $N(\overline{\rho})$ is imposed in [6, Page 420].
2. The condition $\text{End}_{\mathcal{F}_p[G_{\mathbb{Q}p}]}([\overline{\rho}]|_{G_{\mathbb{Q}p}} \otimes \mathcal{F}_p) = \mathcal{F}_p$ is imposed in [6, (3.2)]. It seems that it comes from [45, (9), Page 30] in order to have the representability of potentially Barsotti–Tate deformation rings (cf. [2]). We expect that the replacement of this condition by the low-weight crystalline condition would not cause any problem. Indeed, this modification is already spelled out in the global deformation problem in [6, Definition 5.12].

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Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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