A New Transmuted Generalized Lomax Distribution: Properties and Applications to COVID-19 Data

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1. Introduction

Many generators have been studied in recent years by expanding some effective classical distributions. Many applied fields, including dependability, demographics, engineering, economics, actuarial sciences, biological research, hydrology, insurance, medicine, and finance, have employed such created families of distributions for modeling and evaluating lifetime data. However, there are still a lot of real-world data occurrences that do not fit into any of the statistical distributions. Shaw and Buckley [1] introduced a new class of distributions known as transmuted distributions with cumulative distribution function (CDF) as

\[ F_T(x) = h(x)[1 + \beta - \beta h(x)]; \quad |\beta| \leq 1, \quad (1) \]

By differentiating equation (1), we get the probability density function (pdf) as follows:

\[ f_T(x) = R(x)[1 + \beta - 2\beta h(x)], \quad (2) \]

where \( R(x) \) and \( h(x) \) are the base distribution’s pdf and CDF.

There are various transmuted distributions suggested. Aryal and Tsokos [2] proposed the transmuted Weibull distribution as a new generalization of the Weibull distribution. Merovci [3] devised and explored the varied structural properties of the transmuted Rayleigh distribution. Khan and King [4] obtained the transmuted modified Weibull distribution. The transmuted Lomax distribution was presented by Ashour and Eltehiwy [5]. Transmuted Pareto distribution is introduced by Merovci and Puka [6].
The transmuted generalized linear exponential distribution was introduced by Elbatal et al. [7]; among others. Pobokvá and Michalková [8] proposed a transmuted Weibull distribution. Ali and Athar [9] have created a new generalized transmuted family of distributions (TD). They utilized Weibull distribution to generalized transmuted families of distributions (TWDn).

The Lomax distribution is a heavy-tail pdf popular in business, economics, and actuarial modeling. In some cases, it is also known as the Pareto Type-II distribution. In the event of a business failure, Lomax used it to fit data. It is essentially a Pareto distribution with a 0-support level. The pdf is as follows:

\[ f(x; \vartheta, \pi) = 1 - e^{\vartheta \left[ 1 - \left(1 - h(x)\right)\right]^{-\pi}}; \quad x \geq 0, \pi > 0, \]

where \( \vartheta \) and \( \pi \) are extra form parameters that change the tail weights. \( h(x) \) is the parent (or baseline) distribution's pdf.

Now, if the density from (3) and (4) is replaced into (5) and (6), Oguntunde et al. [11] introduced a novel generalization of the Lomax distribution known as the Gompertz Lomax distribution (GoLom) with vector parameters \( \lambda \) where \( \lambda = (\delta, \gamma, \vartheta, \pi) \); the CDF and pdf are

\[ F(x; \lambda) = h(x; \lambda) = 1 - e^{\vartheta \left[ 1 - \left(1 + \gamma x\right)\right]^{-\pi}}; \quad x \geq 0, \pi, \delta, \gamma > 0. \]

The corresponding pdf is created by inserting the densities from (3) and (4) into (6) in the following order:

\[ f(x; \lambda) = h(x; \lambda) = \vartheta \delta \gamma (1 + \gamma x)^{\delta - 1} e^{\vartheta \left[ 1 - \left(1 + \gamma x\right)\right]^{-\pi}}; \quad x \geq 0, \pi, \delta, \gamma > 0. \]

have been about 6.8 million cases of COVID-19 in the United Kingdom as of September 2021, with over 132740 deaths. Therefore, we decided to find the best mathematical-statistical model for modeling the data of the countries of France and the United Kingdom. There were also many researchers who worked on finding a model for these data, such as Almetwally [12], Almetwally [13], Almetwally [14], and others.

The following is a representation of how this article is structured. In Section 2, we define the new distribution. The new distribution's structural features are discussed in Section 3. The maximum likelihood estimators (MLEs) of parameters under complete and Type-II censored samples are investigated in Section 4. Section 5 describes the various bootstrap confidence intervals. Section 6 describes a Monte-Carlo simulation analysis using entire sample sizes and
Type-II censored samples to estimate point and interval estimation of TGL distribution parameters. In Section 7, two real-world data sets are introduced, and at the end of the article, there is a conclusion.

2. Transmuted Generalized Lomax Model

The TGL distribution and its submodels are shown here. The CDF of the TGL distribution with vector parameters \( Z = (\delta, \gamma, \theta, \pi, \beta) \) can be derived by substituting (7) and (8) in (1) and (2) as

\[
F(x; Z) = \left[ 1 - e^{\delta x^{\gamma}} \right] \left[ 1 + \beta e^{\delta x^{\gamma}} \right],
\]

and its pdf is as follows:

\[
f(x; Z) = \delta \gamma (1 + \gamma x)^{\delta - 1} e^{\gamma x} \left[ 1 - \beta + 2\beta e^{\delta x^{\gamma}} \right];
\]

\[x > 0, \pi, \beta, \delta, \gamma \text{ and } \theta > 0.\]

As a result, the pdf (10) is defined as \( X \sim \text{TGL} \) \((\delta, \gamma, \theta, \pi, \beta)\). Table 1 lists the TGL distribution’s special submodels.

The survival (reliability) function \( F(x; Z) \) and the hazard function \( h(x; Z) \) have the following definitions:

\[
F(x; Z) = e^{\delta x^{\gamma}} \left[ 1 + \beta e^{\delta x^{\gamma}} \right],
\]

\[
h(x; Z) = \frac{\delta \gamma (1 + \gamma x)^{\delta - 1} e^{\gamma x}}{1 - \beta + 2\beta e^{\delta x^{\gamma}}},
\]

(11)

For specific parameters selections, the pdf of TGL model is shown in Figure 1.

We can deduce from Figure 1 plots of the TGL distribution’s pdf can be unimodal, normal, or right-skewed.

We may derive from Figure 2 that the TGL distribution’s hazard function can take the form of a decreasing, increasing, or upside-down shape.

Lemma 1. The TGL density function’s limit is provided as

\[
\lim_{x \to 0} f(x; Z) = 0,
\]

\[
\lim_{x \to \infty} f(x; Z) = 0.
\]

Proof. The density function’s conclusion is easy to illustrate (10).

Table 1: The TGL distribution’s special submodels.

| No. | Distributions | \( \gamma \) | \( \theta \) | \( \pi \) | \( \beta \) | Author |
|-----|---------------|-------------|-------------|-------------|-------------|--------|
| 1   | LOM           | 1           | \( \delta \) | 0           | 1           | Lomax [15] |
| 2   | GOLOM         | 1           | \( \delta \) | 0           | \( \pi \)   | Oguntunde et al. [11] |

Figure 1: Plots of the TGL distribution’s pdf.

Furthermore, the TGL hazard function’s limit as \( x \to 0 \) is 0 and \( x \to \infty \) is \( \infty \) as shown as follows:

\[
\lim_{x \to 0} h(x; Z) = 0,
\]

\[
\lim_{x \to \infty} h(x; Z) = \infty.
\]

This statement is simple to demonstrate.

3. Statistical Properties

The statistical aspects of the TGL distribution are examined in the following subsections, including moments, mode, quantile function, Rényi entropy, and order statistics.

3.1. Moments. The \( r \)-th instant near 0 is calculated. We can write as follows from (10).

\[
\mu_r = \delta \gamma \int_0^{\infty} x^r (1 + \gamma x)^{\delta - 1} e^{\gamma x} \left[ 1 - \beta + 2\beta e^{\delta x^{\gamma}} \right] dx = I_1 + I_2.
\]
First, to obtain $I_1$, as a result, we use binomial expansion.

$$I_1 = y^{-r}(1 - \beta) \int_0^{\infty} \left[ \left( \frac{\pi}{\delta} y + 1 \right)^{1/\delta} \pi - 1 \right] e^{-y} dy. \tag{15}$$

So, $I_1$ is given by

$$I_1 = y^{-r}(1 - \beta) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{\pi}{\delta} \right)^{(1/\delta)(r-i) - j} \left( \frac{\pi}{\delta} \right)^{-j} \frac{1}{\delta^{i}(r-i) - j + 1}. \tag{16}$$

In a similar way, $I_2$ is as follows:

$$I_2 = \beta y^{-r} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{1}{\delta^{i}(r-i)} \right) \left( \frac{\pi}{2\delta} \right)^{(1/\delta)(r-i) - j} \Gamma\left( \frac{1}{\delta^{i}(r-i) - j + 1} \right). \tag{17}$$

Then, $\mu'$ can be expressed as
3.2. Mode. The mode of the TGL distribution was obtained in this subsection. The \( \ln f(x) \) is as follows:
\[
\ln f(x) = \ln(\delta \gamma \theta) + (\delta \pi - 1) \ln(1 + \gamma x) + \theta/\pi [1 - (1 + \gamma x)^{\theta}] + \ln[1 - \beta + 2\theta e^{(\theta/\pi)[1 - (1 + \gamma x)^{\theta}]}].
\]
The mode is the solution of the following equation:
\[
2\beta(\delta \gamma - 1)e^{\theta/\pi[1 - (1 + \gamma x)^{\theta}]} + [(\delta \pi - 1) - \delta \theta(1 + \gamma x)^{\delta \theta}](1 - \beta) = 0.
\]
(19)

3.3. Quantile and Median. By inverting \( \text{cdf}(9) \) as follows, the TGL distribution can be easily simulated: on (0, 1), if \( U \) follows a uniform distribution, then
\[
q = \left(1 - e^{\theta/\pi[1 - (1 + \gamma x)^{\theta}]}ight)
\left[1 + \beta e^{\theta/\pi[1 - (1 + \gamma x)^{\theta}]}ight].
\]
(20)

3.4. Rényi Entropy. The variance of the uncertainty is measured by the Rényi entropy of a pdf \( f(x) \) with random variable \( X \) of TGL distribution. The Rényi entropy is defined as for any real parameter \( \psi > 0 \) and \( \psi \neq 1 \):
\[
I_R(\psi) = \frac{1}{\psi - 1} \log \int f^\psi(x)dx, \quad \psi > 0 \text{ and } \psi \neq 1.
\]
(22)

We can extract the integrated component using the density function (10) as follows:
\[
\int f^\psi(x)dx = (\delta \gamma)^\psi \int_0^\infty \left[1 + \gamma x\right]^\psi(\delta \pi - 1)e^{(\theta/\pi)\psi[1 - (1 + \gamma x)^{\theta}]}
\left[1 - \beta + 2\theta e^{(\theta/\pi)[1 - (1 + \gamma x)^{\theta}]}\right]^\psi dx.
\]
(23)

The binomial expansion is then used as follows:
\[
\int f^\psi(x)dx = \sum_{i,j=0}^\infty \sum_{\ell=0}^i \left(\psi \atop i\right) \left(\psi(a - 1) + bi \atop j\right) \left(j \atop 1\right) (-1)^{i+j} \gamma^{j+2\psi} [a(1 + \beta)]^{\psi-i} [\beta(a + b)]^j
\times \int_0^\infty x^\psi J^{j+\ell} e^{-\gamma x(\psi+1)} dx,
\]
(24)

As a result, the TGL distribution’s Rényi entropy is
\[
I_R(\psi) = \frac{(\delta \gamma)^{\psi-1}}{\psi} \sum_{i=0}^\infty \sum_{j=0}^\infty \left(\psi \atop i\right) \left(1 - \psi \atop \frac{1}{\delta \pi} - 1\right) \left(1 - \beta\right)^{\psi-i} (2\theta)^i (\pi/\theta)^j (1 + i)^{j+1} \Gamma(j+1).
\]
(25)

3.5. Order Statistics. In this part, we will look at single-order statistics for the TGL distribution. Let us say \( x_1, \ldots, x_n \); there are \( n \) TGL random variables that are both independent and identically distributed. Let \( x_{(1)}, x_{(2)}, \ldots, x_{(n)} \) stand for the order statistics derived from these \( n \) variables. The pdf of the \( r^{th} \) order statistic, say \( f_{r:n}(x) \), is then calculated as
\[ f_{r,n}(x) = C_{r,n}[F(x)]^{n-r}f(x)[1-F(x)]^{r}, \quad (26) \]

where \( C_{r,n} = n!(r-1)!(n-r)! \) The binomial expansion is used in this case; then, \( r^{th} \) order statistic of TGL distribution is

\[
f_{r,n}(x) = \frac{n!}{(r-1)!(n-r)!} \left[ 1 - e^\theta [1-(1+y\lambda)\hat{r}] \right]^{r-1} \delta \theta \hat{y} (1 + y\lambda)^{\delta - 1}e^{\theta \hat{y}[1-(1+y\lambda)\hat{r}]} \left[ 1 - \beta + 2\beta e^{\theta \hat{y}[1-(1+y\lambda)\hat{r}]} \right]^{n-r}. \quad (27)\]

For the TGL distribution, the \( k^{th} \) moments of \( r^{th} \) order statistics are

\[
\mu_{r,n}^{(k)} = \frac{\delta \theta n!}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} \sum_{j=0}^{n-r} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \binom{r-1}{i} \binom{n-r}{j} \binom{k}{m} \left( \delta \pi \right)_{t} (-1)^{i+m} \left( \frac{\pi}{\delta} \right)^{t} \cdot (1-\beta)^{n-r-j} \cdot j^{i+1} \cdot y^{k-1} \cdot \left[ 1 + \frac{\pi}{\delta} \right]^{1/\delta \pi} \cdot \left[ 1 + \frac{\pi}{\delta} \right]^{1-(1/\delta \pi)} \cdot \left[ 1 - \beta + 2\beta e^{-\gamma} \right] \cdot \frac{1}{\delta \theta \gamma} \frac{1}{1+\pi/\delta}^{1/\delta \pi-1} \cdot dy. \quad (28) \]

By using the binomial expansion,

\[
\mu_{r,n}^{(k)} = \frac{n! y^{-k}}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} \sum_{j=0}^{n-r} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \binom{r-1}{i} \binom{n-r}{j} \binom{k}{m} \left( \delta \pi \right)_{t} (-1)^{i+m} \left( \frac{\pi}{\delta} \right)^{t} \cdot (1-\beta)^{n-r-j} \cdot j^{i+1} \cdot x^{k-1} \cdot \left[ 1 + \frac{\pi}{\delta} \right]^{1/\delta \pi} \cdot \left[ 1 + \frac{\pi}{\delta} \right]^{1-(1/\delta \pi)} \cdot \left[ 1 - \beta + 2\beta e^{-\gamma} \right] \cdot \frac{1}{\delta \theta \gamma} \frac{1}{1+\pi/\delta}^{1/\delta \pi-1} \cdot dy. \quad (29) \]

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\[
\times \left[ 1 - \beta \right] \left[ \frac{1}{(n-r+i+j+l+1)^{t+1}} + \frac{2\beta}{(n-r+i+j+l+2)^{t+2}} \right] \Gamma(t+1). \]
4. Parameter Estimation

The MLEs of the parameters \( Z = (\delta, \gamma, \vartheta, \pi, \beta) \) under complete and Type-II censored samples are investigated in this section. Approximate confidence intervals (ACIs) for unknown values are also calculated using the Fisher information matrix.

4.1. MLEs under Complete Sample. In the statistical literature, various approaches for parameter estimation have been given, with the MLEs method being the most extensively employed. We explore applying MLEs to estimate the parameters of the TGL distribution with a complete sample. If \( x_1, x_2, \ldots, x_n \) is a random sample of this distribution of size \( n \) with a set of parameter vectors \( Z = (\delta, \gamma, \vartheta, \pi, \beta) \), then the log-likelihood function, say \( \ell_1(Z) \), may be stated as

\[
\ell_1(Z) = n[\ln(\delta) + \ln(\gamma)] + (\delta \pi - 1)\sum_{i=1}^{n} \ln(1 + \gamma x_i) + \frac{\delta}{\pi} \sum_{i=1}^{n} \psi_i + \sum_{i=1}^{n} \ln[1 - \beta + 2\beta e^{(\delta/\pi)\psi_i}],
\]

where \( \psi_i = [1 - (1 + \gamma x_i)^{\delta/\pi}] \). The partial differential equations \( \ell_1(Z) \) are calculated as follows:

\[
\frac{\partial \ell_1(Z)}{\partial \delta} = \frac{n}{\delta} + \pi \sum_{i=1}^{n} \ln(1 + \gamma x_i) - \delta \sum_{i=1}^{n} (1 + \gamma x_i)^{\delta/\pi} \ln(1 + \gamma x_i) - \frac{n}{\delta} \sum_{i=1}^{n} \left[ 2\beta e^{(\delta/\pi)\psi_i} \right],
\]

\[
\frac{\partial \ell_1(Z)}{\partial \gamma} = \frac{n}{\gamma} + (\delta \pi - 1) \sum_{i=1}^{n} \frac{x_i}{1 + \gamma x_i} - \delta \sum_{i=1}^{n} x_i (1 + \gamma x_i)^{\delta/\pi - 1} - \sum_{i=1}^{n} \frac{2\beta \delta e^{(\delta/\pi)\psi_i} (1 + \gamma x_i)^{\delta/\pi - 1}}{1 - \beta + 2\beta e^{(\delta/\pi)\psi_i}},
\]

\[
\frac{\partial \ell_1(Z)}{\partial \pi} = \delta \sum_{i=1}^{n} \ln(1 + \gamma x_i) - \frac{\delta}{\pi} \sum_{i=1}^{n} (1 + \gamma x_i)^{\delta/\pi} \ln(1 + \gamma x_i) + \psi_i \]

\[
- \sum_{i=1}^{n} \frac{2\beta e^{(\delta/\pi)\psi_i} (1 + \gamma x_i)^{\delta/\pi - 1} \ln(1 + \gamma x_i) - \psi_i}{1 - \beta + 2\beta e^{(\delta/\pi)\psi_i}},
\]

\[
\frac{\partial \ell_1(Z)}{\partial \beta} = \sum_{i=1}^{n} \frac{2e^{(\delta/\pi)\psi_i} - 1}{1 - \beta + 2\beta e^{(\delta/\pi)\psi_i}}.
\]

The nonlinear equations are numerically solved to determine ML estimators as \( \frac{\partial \ell_1(Z)}{\partial \delta} = 0, \frac{\partial \ell_1(Z)}{\partial \gamma} = 0, \frac{\partial \ell_1(Z)}{\partial \pi} = 0, \text{and} \frac{\partial \ell_1(Z)}{\partial \beta} = 0 \) using an iterative technique.

4.2. MLEs under Type-II Censored Sample. The MLEs of parameters for TGL distribution based on Type-II censored samples are investigated in this subsection. The Fisher information matrix for Type-II censored model is also used to calculate the approximate confidence intervals for the unknown parameters \( Z = (\delta, \gamma, \vartheta, \pi, \beta) \). Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \), and we only look at the first \( k \-)th order statistics based on the Type-II censored sample. Likelihood function in this scenario is of the kind

\[
L_2(Z) = \prod_{i=1}^{k} f(x_{i:k:n}) \left[ 1 - F(x_{i:k:n}) \right]^{n-k}, \quad x_{1:k:n} \leq x_{2:k:n} \leq \cdots \leq x_{k:k:n},
\]

where \( C \) is a constant and \( x_{1:k:n}, x_{2:k:n}, \ldots, x_{k:k:n} \) is the data that has been censored. The log-likelihood function \( \ell_2(Z) \) is possibly written as follows without constant term from (15).
\[\ell_2(Z) = k[\ln(\theta) + \ln(\delta) + \ln(\gamma)] + (\delta \pi - 1) \sum_{i=1}^{k} \ln(1 + \gamma x_i) + \frac{\theta}{\pi} \sum_{i=1}^{k} \psi_i + \frac{\delta}{\pi} \sum_{i=1}^{k} \psi_i \ln[1 - 1 - e^{(\theta/\pi)\psi_i}] \]
\[
+ (n - k) \sum_{i=1}^{k} \ln[1 - 1 - e^{(\theta/\pi)\psi_i}]\left[1 + \beta e^{(\theta/\pi)\psi_i}\right],
\]

where \(x_i = x_i, \quad i = 1, 2, \ldots, k\), denotes the time of the k-th failure and \(x_k\) denotes the time of the k-th failure. The MLEs \(Z = (\delta, \gamma, \theta, \pi, \beta)\) are the solutions to the following five equations:

\[
\frac{\partial \ell_2(Z)}{\partial \theta} = k \frac{\theta}{\pi} + \sum_{i=1}^{k} \psi_i + \sum_{i=1}^{k} \left(\psi_i/\pi\right) e^{(\theta/\pi)\psi_i} + (n - k) \sum_{i=1}^{k} \left(\beta\psi_i/\pi\right) e^{(\theta/\pi)\psi_i} + \left(\psi_i/\pi\right) e^{(\theta/\pi)\psi_i} + (2\beta\psi_i/\pi) e^{(2\theta/\pi)\psi_i} = 0,
\]

\[
\frac{\partial \ell_2(Z)}{\partial \delta} = \frac{k}{\delta} + \frac{\pi}{\delta} \sum_{i=1}^{k} (1 + \gamma x_i) - \frac{\pi}{\delta} \sum_{i=1}^{k} (1 + \gamma x_i) \delta\pi \ln(1 + \gamma x_i) - \sum_{i=1}^{k} 2\beta\theta (1 + \gamma x_i) \delta\pi \ln(1 + \gamma x_i) e^{(\theta/\pi)\psi_i} = 0,
\]

\[
\frac{\partial \ell_2(Z)}{\partial \pi} = \frac{\delta}{\pi} \sum_{i=1}^{k} (1 + \gamma x_i) - \frac{\theta}{\pi} \sum_{i=1}^{k} \pi\delta (1 + \gamma x_i) \delta\pi \ln(1 + \gamma x_i) + \psi_i - \sum_{i=1}^{k} 2\beta\psi_i (1 + \gamma x_i) \delta\pi \ln(1 + \gamma x_i) e^{(\theta/\pi)\psi_i} = 0,
\]

\[
\frac{\partial \ell_2(Z)}{\partial \gamma} = \frac{k}{\gamma} + (\delta \pi - 1) \sum_{i=1}^{k} (1 + \gamma x_i) - \delta\pi \sum_{i=1}^{k} x_i (1 + \gamma x_i) \delta\pi - \sum_{i=1}^{k} 2\beta\delta\psi_i (1 + \gamma x_i) \delta\pi = 0,
\]

\[
\frac{\partial \ell_2(Z)}{\partial \beta} = k \sum_{i=1}^{k} 2\beta e^{(\theta/\pi)\psi_i} - 1 + (n - k) \sum_{i=1}^{k} e^{(\theta/\pi)\psi_i} - 1 = 0.
\]

It is to be noted that equation (16) cannot be solved explicitly. To obtain the MLEs \(Z = (\delta, \gamma, \theta, \pi, \beta)\), a numerical approach is required, and a numerical technique is needed. We obtain the observed Fisher information matrix since its expectation requires numerical integration. The \(5 \times 5\) observed information or Hessian matrix \(H(Z)\) is

\[
H(Z) = \begin{bmatrix}
H_{\delta\delta} & H_{\theta\delta} & H_{\delta\theta} & H_{\delta\pi} & H_{\delta\beta} \\
H_{\delta\gamma} & H_{\theta\gamma} & H_{\delta\gamma} & H_{\delta\beta} & H_{\delta\beta} \\
H_{\delta\psi} & H_{\theta\psi} & H_{\delta\psi} & H_{\delta\beta} & H_{\delta\beta} \\
H_{\delta\delta} & H_{\theta\delta} & H_{\delta\theta} & H_{\delta\pi} & H_{\delta\beta} \\
H_{\delta\gamma} & H_{\theta\gamma} & H_{\delta\gamma} & H_{\delta\beta} & H_{\delta\beta}
\end{bmatrix}
\]
The Fisher information matrix $H(Z)$ is given by the negative expected of second partial derivatives of (15) for the unknown parameters $\tilde{Z} = (\delta, \gamma, \theta, \pi, \beta)$ locally at $(\tilde{\delta}, \tilde{\gamma}, \tilde{\theta}, \tilde{\pi}, \tilde{\beta})$ given in (16). Under some regularity conditions, $(\tilde{\delta}, \tilde{\gamma}, \tilde{\theta}, \tilde{\pi}, \tilde{\beta})$ is approximately normal with mean $(\delta, \gamma, \theta, \pi, \beta)$ and covariance matrix $H^{-1}(Z)$. Practically, we estimate $H^{-1}(Z)$ by $H^{-1}_{\tilde{Z}}(\delta, \gamma, \theta, \pi, \beta)$.

$$
I^{-1}_{\tilde{Z}}(\delta, \gamma, \theta, \pi, \beta) = 
\begin{bmatrix}
H_{\delta\delta} & H_{\delta\gamma} & H_{\delta\theta} & H_{\delta\pi} & H_{\delta\beta} \\
H_{\gamma\delta} & H_{\gamma\gamma} & H_{\gamma\theta} & H_{\gamma\pi} & H_{\gamma\beta} \\
H_{\theta\delta} & H_{\theta\gamma} & H_{\theta\theta} & H_{\theta\pi} & H_{\theta\beta} \\
H_{\pi\delta} & H_{\pi\gamma} & H_{\pi\theta} & H_{\pi\pi} & H_{\pi\beta} \\
H_{\beta\delta} & H_{\beta\gamma} & H_{\beta\theta} & H_{\beta\pi} & H_{\beta\beta}
\end{bmatrix}
^{-1}
$$

5. Bootstrap Confidence Interval

We create two parametric bootstrap confidence intervals (CI) $Z = (\delta, \gamma, \theta, \pi, \beta)$ in this section as follows.

5.1. Percentile Bootstrap (Boot-P)

1. Compute the MLE of $Z = (\delta, \gamma, \theta, \pi, \beta)$ based on complete and censored samples
2. Generate a bootstrap sample using $Z$ to obtain the bootstrap estimate of $Z$, say $Z^b$, using the bootstrap sample
3. Repeat step 2 B times to get $(Z^{b(1)}, Z^{b(2)}, \ldots, Z^{b(B)})$
4. Arrange $(Z^{b(1)}, Z^{b(2)}, \ldots, Z^{b(B)})$ in order of ascending as $(Z^{b(1)}, Z^{b(2)}, \ldots, Z^{b(B)})$.

$$
\tilde{\delta} \pm z_{1-0.5\alpha} \sqrt{\text{Var}(\tilde{\delta})},
\tilde{\gamma} \pm z_{1-0.5\alpha} \sqrt{\text{Var}(\tilde{\gamma})},
\tilde{\theta} \pm z_{1-0.5\alpha} \sqrt{\text{Var}(\tilde{\theta})},
\tilde{\pi} \pm z_{1-0.5\alpha} \sqrt{\text{Var}(\tilde{\pi})},
\tilde{\beta} \pm z_{1-0.5\alpha} \sqrt{\text{Var}(\tilde{\beta})},
$$

where $z_\alpha$ is the 100 $\alpha$ – th percentile of a standard normal distribution.

5.2. Bootstrap-t (Boot-t).

1. The same steps as (1-2) in Boot-p
2. Compute the t-statistic of
$$
Z = (\delta, \gamma, \theta, \pi, \beta)T = (\tilde{Z} - Z)/\sqrt{\text{Var}(\tilde{Z})}
$$
3. Repeat steps 2-3 B times and obtain $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$
4. Arrange $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$ in ascending order as $T^{(1)}, T^{(2)}, \ldots, T^{(B)}$
5. A two-side 100$(1 - \gamma)$% Boot-t CI for the unknown parameters $Z = (\delta, \gamma, \theta, \pi, \beta)$ is given by
$$
\{Z + T^{(B(\gamma/2))}/\sqrt{\text{Var}(Z)} + Z + T^{(B(1-\gamma/2))}/\sqrt{\text{Var}(Z)}\}.
$$

6. Simulation Study

In this section, we discuss the Monte Carlo simulation study to estimate point and interval estimation of parameters of TGL distribution based on complete sample sizes and Type-II censored samples. The simulation results are in Tables 2-5, and concluding remarks of simulation results are obtained in this section. A Monte Carlo simulation is an initial task for studying different properties of parameters of
the TGL model based on different sample schemes; we can use these steps:

1. Generate random sample from a uniform distribution with intervals 0 and 1 and sample size \(n\) as 30, 50, and 100

2. Determine different actual values of parameters of TGL distribution as

   Case 1: \(\pi = 0.3, \beta = 0.3, \delta = 0.5, \gamma = 0.6, \vartheta = 0.5\)

   Case 2: \(\pi = 1.2, \beta = 0.4, \delta = 0.8, \gamma = 0.75, \vartheta = 1.9\)

### Table 2: Various measures for parameters of TGL distribution based on different schemes of samples: Case 1.

| \(N\) | \(M\) | Bias | MSE | LACI | LBPCI | LBTCI | CP (%) |
|------|------|------|-----|------|-------|-------|-------|
| 10   | 0.0046 | 0.0499 | 0.8572 | 0.0992 | 0.0979 | 96.40 |
| 30   | 0.0190 | 0.0392 | 0.7731 | 0.0815 | 0.0812 | 94.60 |
| \(\delta\) | 0.0241 | 0.0211 | 0.5615 | 0.0551 | 0.0547 | 97.80 |
| \(\gamma\) | 0.3174 | 0.5989 | 2.7682 | 0.2530 | 0.2498 | 95.80 |
| \(\vartheta\) | -0.0138 | 0.0392 | 0.7619 | 0.0830 | 0.0834 | 96.20 |
| \(\beta\) | 0.0050 | 0.1381 | 1.4572 | 0.1461 | 0.1466 | 98.50 |
| \(\pi\) | 0.0752 | 0.0515 | 0.8396 | 0.0841 | 0.0846 | 96.30 |

### Table 3: Various measures for parameters of TGL distribution based on different schemes of samples: Case 2.

| \(N\) | \(m\) | Bias | MSE | LACI | LBPCI | LBTCI | CP (%) |
|------|------|------|-----|------|-------|-------|-------|
| 10   | 0.0109 | 0.1883 | 1.6472 | 0.1585 | 0.1567 | 94.60 |
| 30   | -0.3184 | 2.3256 | 4.2623 | 0.3338 | 0.3204 | 99.10 |
| \(\delta\) | 0.1568 | 0.1692 | 1.4916 | 0.1371 | 0.1378 | 95.20 |
| \(\gamma\) | 0.1809 | 0.6919 | 3.0693 | 0.2352 | 0.2353 | 97.10 |
| \(\vartheta\) | -0.0090 | 0.5051 | 0.8773 | 0.0941 | 0.0941 | 94.60 |
| \(\beta\) | 0.0316 | 0.5118 | 2.8058 | 0.2225 | 0.2225 | 92.90 |
| \(\pi\) | -0.4821 | 4.0856 | 7.6986 | 0.4347 | 0.4347 | 93.00 |

### Case 3

- \(\pi = 0.35, \beta = -0.4, \delta = 0.45, \gamma = 0.75, \vartheta = 0.85\)

- \(\pi = 1.5, \beta = -0.6, \delta = 1.4, \gamma = 2.7, \vartheta = 1.8\)

(3) We use the inverse CDF method to transform the CDF in terms of \(u\) and get the sample of TGL distribution

(4) Sort sample and select the first \(m\) failures as 20 and 25 where \(n = 30, 40, 50\) and \(m = 50, 70, 90\) where \(n = 100\)
### Table 4: Various measures for parameters of TGL distribution based on different schemes of samples: Case 3.

| N | M | Bias | MSE | LACI | LBPCI | LBTCI | CP (%) |
|---|---|------|-----|------|-------|-------|--------|
| π | 0.0937 | 0.1825 | 1.6349 | 0.1569 | 0.1576 | 99.89 |
| β | -0.9293 | 1.0697 | 4.0882 | 1.2977 | 1.3166 | 93.69 |
| γ | 0.5007 | 0.0593 | 0.9339 | 0.0523 | 0.0519 | 97.01 |
| δ | 0.7467 | 2.3515 | 5.2533 | 0.7690 | 0.7595 | 95.90 |
| θ | -0.0886 | 0.1389 | 1.4201 | 0.1467 | 0.1464 | 98.45 |
| π | 0.0675 | 0.2530 | 1.9547 | 0.2090 | 0.2062 | 98.60 |
| β | -1.0851 | 2.8773 | 4.3929 | 2.0501 | 1.9617 | 95.10 |
| γ | 0.0922 | 0.0619 | 0.9060 | 0.0849 | 0.0850 | 94.80 |
| δ | 0.8103 | 2.5817 | 5.4756 | 0.4411 | 0.4389 | 96.50 |
| θ | -0.0643 | 0.1778 | 1.6342 | 0.1544 | 0.1553 | 97.20 |
| π | 0.0563 | 0.3745 | 2.3900 | 0.2358 | 0.2358 | 98.50 |
| β | -1.4870 | 3.2350 | 5.2192 | 1.9231 | 1.9803 | 94.50 |
| γ | 0.1298 | 0.0896 | 1.0575 | 0.1038 | 0.1037 | 95.30 |
| δ | 0.9029 | 2.7732 | 5.4879 | 0.5950 | 0.5956 | 95.20 |
| θ | -0.0555 | 0.2429 | 1.9205 | 0.1908 | 0.1920 | 97.00 |
| π | 0.0748 | 0.1549 | 1.5154 | 0.1457 | 0.1476 | 99.60 |
| β | -0.3842 | 0.9235 | 3.6210 | 0.5458 | 0.5468 | 94.40 |
| γ | 0.0510 | 0.0486 | 0.8415 | 0.0621 | 0.0621 | 65.30 |
| δ | 0.4395 | 0.9975 | 3.5173 | 0.3065 | 0.2983 | 94.70 |
| θ | -0.0925 | 0.1053 | 1.2198 | 0.1239 | 0.1238 | 98.50 |
| π | 0.0557 | 0.2313 | 1.8735 | 0.1751 | 0.1735 | 98.20 |
| β | -0.9798 | 1.1929 | 4.0658 | 1.3002 | 1.2948 | 94.80 |
| γ | 0.0855 | 0.0769 | 1.0348 | 0.0723 | 0.0728 | 95.70 |
| δ | 0.6323 | 1.8210 | 4.6755 | 0.3789 | 0.3788 | 95.60 |
| θ | -0.0104 | 0.1360 | 1.3908 | 0.1445 | 0.1457 | 98.10 |
| π | 0.0307 | 0.3267 | 2.2384 | 0.2249 | 0.2275 | 98.60 |
| β | -1.0355 | 3.2609 | 4.9372 | 2.1615 | 2.1135 | 97.00 |
| γ | 0.1278 | 0.0792 | 0.9260 | 0.0974 | 0.0974 | 95.50 |
| δ | 0.6536 | 1.9676 | 4.8675 | 0.5197 | 0.5227 | 94.30 |
| θ | -0.0765 | 0.1560 | 1.5196 | 0.1469 | 0.1522 | 97.00 |

(5) By using different programs as Mathcad, R-software, Mable, and Mathematica, we can obtain the results of the simulation.

(6) We use 10000 iterations in the summation generator.

(7) In point estimation, we obtain bias and the mean squared error (MSE) of parameters of TGL distribution.

(8) In intervals estimation, we obtain a length of CI for ACI denoted as LACI, length of percentile bootstrap CI can be denoted as LBPCI, length of bootstrap-t CI can be denoted as LBTCI, and coverage probability (CP) of ACI.

### Table 5: Various measures for parameters of TGL distribution based on different schemes of samples: Case 4.

| N | M | Bias | MSE | LACI | LBPCI | LBTCI | CP (%) |
|---|---|------|-----|------|-------|-------|--------|
| π | 0.0457 | 0.8716 | 3.6572 | 0.3452 | 0.3467 | 93.10 |
| β | -0.7394 | 2.9165 | 3.1121 | 0.1054 | 0.1049 | 95.00 |
| γ | 0.0200 | 0.1219 | 1.1113 | 0.1042 | 0.1047 | 93.80 |
| δ | 0.0972 | 1.1947 | 4.2698 | 0.3181 | 0.3179 | 94.70 |
| θ | -0.0600 | 0.5720 | 2.9403 | 0.2416 | 0.2414 | 94.00 |
| π | -0.0546 | 1.4718 | 4.7532 | 0.4506 | 0.4508 | 91.50 |
| β | -0.8497 | 3.0780 | 3.2331 | 1.1783 | 1.1561 | 93.00 |
| γ | 0.2697 | 0.3005 | 1.8717 | 0.1570 | 0.1558 | 96.80 |
| δ | 0.2684 | 1.5735 | 4.8057 | 0.4081 | 0.4142 | 92.60 |
| θ | -0.0657 | 0.9053 | 3.7228 | 0.3539 | 0.3501 | 95.40 |
| π | -0.1183 | 1.8618 | 5.3313 | 0.5021 | 0.5052 | 95.60 |
| β | -0.9115 | 3.5046 | 3.9777 | 1.1938 | 1.1806 | 93.60 |
| γ | 0.3144 | 0.3758 | 2.0640 | 0.2992 | 0.2867 | 96.30 |
| δ | 0.3536 | 2.0095 | 5.3839 | 0.5360 | 0.5328 | 93.50 |
| θ | -0.0967 | 1.1029 | 4.1143 | 0.3862 | 0.3746 | 96.30 |

7. Concluding Remarks of Simulation Results

Tables 2–5 show the simulation results of point and interval estimates of TGL distribution parameters using Type-II censored samples and entire sample sizes. Based on these Tables, the following concluding remarks are noticed.
Table 6: The TGL model’s and other competing models’ analytical results using COVID-19 data of France.

| Model          | Estimates | SE    | $\pi$     | $\beta$   | $\delta$ | $\gamma$ | $\theta$ | KS     | $P$ value | $W^*$ | $A^*$ |
|----------------|-----------|-------|-----------|-----------|----------|----------|----------|--------|-----------|-------|-------|
| TGL            | 8.3588    | 3.3853| 0.9078    | 0.2296    | 106.5058 | 0.1131   | 0.0660   | 0.7348 | 0.0795    | 0.4964|
| KEBXII         | 1.5895    | 1.1360| 0.7072    | 52.8189   | 1.4050   | 1.7894   | 0.0719   | 0.6327 | 0.1013    | 0.6654|
| WL             | 2.8707    | 20.4928| 1.5466    | 1.0213    | 0.1104   | —        | 0.0711   | 0.6466 | 0.0968    | 0.6432|
| OEPIV          | 3.7914    | 22.6007| 0.6310    | 0.8663    | 0.1134   | —        | 0.0721   | 0.6281 | 0.1024    | 0.6726|
| MOAPIW         | 3171.2439 | 1410.3818| 0.6122    | 0.0026    | 0.1791   | —        | 0.1100   | 0.1462 | 0.4416    | 2.7501|
| MOAPW          | 7.7493    | 26.4113| 1.0168    | 2.1276    | 0.0244   | —        | 0.0663   | 0.7292 | 0.0909    | 0.5547|
| MOAPEW         | 3.2759    | 6.4510| 0.8594    | 6.2503    | 18.8847  | 23.6501  | 0.0673   | 0.7118 | 0.0944    | 0.5600|
| MOAPL          | 1.0858    | 4.9184| 29.2003   | 7.7492    | 0.5695   | —        | 0.0667   | 0.7223 | 0.0941    | 0.5544|
| GOLOM          | 0.6174    | 0.3285| —         | 6.5887    | 5.1836   | 0.3340   | 0.0912   | 0.3299 | 0.0798    | 0.5392|

8.1. Data Set (1): COVID-19 of France. The COVID-19 data in question is from France, and it covers a period of 108 days, from March 1 to June 16, 2021. This data was formed by using daily new deaths (ND), daily cumulative cases (CC), and daily cumulative deaths (CD) as follows:

$$x_i = \frac{ND_i}{CC_i - CD_{i-1}} \times 1000.$$  \hspace{1cm} (42)

The data are as follows:

0.0045 0.0062 0.0109 0.0113 0.0123 0.0126 0.0129
0.0130 0.0139 0.0152 0.0160 0.0160 0.0161 0.0164 0.0169
0.0173 0.0174 0.0188 0.0219 0.0225 0.0226 0.0248 0.0260
0.0284 0.0303 0.0315 0.0318 0.0320 0.0323 0.0327 0.0329
0.0332 0.0343 0.0345 0.0346 0.0346 0.0347 0.0347 0.0352
0.0359 0.0365 0.0366 0.0370 0.0371 0.0376 0.0384 0.0392
0.0396 0.0419 0.0421 0.0443 0.0445 0.0462 0.0492 0.0506

(i) As sample size increases and fixed other values of the model, the various measures for the parameter of TGL distribution estimates decrease

(ii) As the number of units to be failed increases and fixed other values of the model, the various measures for the parameter of TGL distribution Type-II censored samples estimates decrease

(iii) The bootstrap is the shortest length of CI for interval estimation of parameter TGL distribution.

8. Real Data Analysis

The relevance and potentiality of the TGL distribution are demonstrated in this section through the application of two real data sets.
In Table 6, the TGL distribution is fitted to COVID-19 of France country. The TGL model is compared with other competitive models as Mead and Afify [16] proposed the Burr-XII model (KEBXII) with Kumaraswamy exponentiated, Weibull-Lomax (WL) distribution, Odds Exponential-Pareto IV (OEPIV) distribution proposed by Baharith et al. [17], Marshall–Olkin Alpha power Weibull (MOAPW) by Almetwally et al. [18], Marshall–Olkin Alpha power extended Weibull (MOAPEW) by Almetwally [19], Marshall–Olkin alpha power inverse Weibull (MOAPIW) by Basheer et al. [20], Marshall–Olkin alpha power Lomax (MOAPL) by Almongy et al. [21], and Gompertz Lomax (GOLOM) distribution by Oguntunde et al. [11]. According to this result, we note that the estimate of TGL has the best measure where it has the smallest value of Cramer-von Mises ($W^*$), Anderson-Darling ($A^*$), and Kolmogorov- Smirnov (KS) statistic along with its $P$ value. The fitted TGL, pdf, CDF, and PP-plot of the data set are displayed in Figure 3.

The COVID-19 data in question is from the United Kingdom and spans 82 days, from May 1 to July 16, 2021. This data is formed by using daily ND, daily CC, and daily CD as follows:

In Table 6, the TGL distribution is fitted to COVID-19 of France country. The TGL model is compared with other competitive models as Mead and Afify [16] proposed the Burr-XII model (KEBXII) with Kumaraswamy exponentiated, Weibull-Lomax (WL) distribution, Odds Exponential-Pareto IV (OEPIV) distribution proposed by Baharith et al. [17], Marshall–Olkin Alpha power Weibull (MOAPW) by Almetwally et al. [18], Marshall–Olkin Alpha power extended Weibull (MOAPEW) by Almetwally [19], Marshall–Olkin alpha power inverse Weibull (MOAPIW) by Basheer et al. [20], Marshall–Olkin alpha power Lomax (MOAPL) by Almongy et al. [21], and Gompertz Lomax (GOLOM) distribution by Oguntunde et al. [11]. According to this result, we note that the estimate of TGL has the best measure where it has the smallest value of Cramer-von Mises ($W^*$), Anderson-Darling ($A^*$), and Kolmogorov- Smirnov (KS) statistic along with its $P$ value. The fitted TGL, pdf, CDF, and PP-plot of the data set are displayed in Figure 3.

The COVID-19 data in question is from the United Kingdom and spans 82 days, from May 1 to July 16, 2021. This data is formed by using daily ND, daily CC, and daily CD as follows:

In Table 7, the TGL distribution is fitted to COVID-19 of The United Kingdom country. The TGL model is compared with other competitive models as, KEBXII, WL, OEPIV, MOAPW, MOAPEW, and GOLOM distributions. According to this result, we note that the estimate of TGL
has the best measure where it has the smallest value of $W^*$, $A^*$, and KS statistic along with its $P$ value. The fitted TGL, pdf, CDF, and PP-plot of the data set are displayed in Figure 4.

9. Conclusion

We investigate the so-called five-parameter transmuted generalized Lomax distribution in this study. Lomax and Gompertz Lomax (GoLom) distributions are included in the TGL model. The TGL distribution's structural properties are deduced. The maximum likelihood approach is used to estimate the population parameters based on complete and Type-II censored samples. We discussed the Monte Carlo simulation study to estimate point and interval estimation of parameters of TGL distribution based on complete sample sizes and Type-II censored samples. The proposed distribution was applied to two COVID-19 real-world data sets from France and United Kingdom. We compared a new transmuted generalization of the Lomax distribution (TGL) with KEBXII, WL, OEPIV, MOAPW, MOAPEW, and GOLOM distributions. It was shown to provide a better fit than several other models. We hope that the presented model will be used in a variety of fields, including engineering, survival and lifetime data, meteorology, biology, hydrology, economics (income disparity), and others.

Data Availability

All data used to support the findings of the study are available within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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