ANY MULTI-INDEX SEQUENCE HAS AN INTERPOLATING MEASURE

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Abstract. R. P. Boas showed that any single-index sequence \( \{\beta_i\}_{i=0}^{\infty} \) of real numbers can be represented as \( \beta_i = \int_{0}^{\infty} x^i \, d\mu \) (\( i = 0, 1, 2, \ldots \)), where \( \mu \) is a signed measure. As Boas said his observation seemed to be quite unexpected; however, it is even possible to extend the result to any multi-index sequence of real numbers. In addition, we can also prove that any multi-index finite sequence admits a measure of a similar type.

1. Introduction

We first discuss finite sequences and then introduce the result of infinite sequences. Let \( \beta \equiv \beta^{(m)} = \{\beta_i : i \in \mathbb{Z}^d_+, |i| \leq m\} \), with \( \beta_0 \neq 0 \), be a \( d \)-dimensional multisequence of degree \( m \). It is called a truncated moment sequence. For a closed set \( K \subseteq \mathbb{R}^d \), the truncated \( K \)-moment problem (TKMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure \( \mu \) on \( \mathbb{R}^d \) with \( \text{supp} \, \mu \subseteq K \) such that

\[
\beta_i = \int x^i \, d\mu(x) \quad (i \in \mathbb{Z}^d_+, |i| \leq m),
\]

where \( x \equiv (x_1, \ldots, x_d) \), \( i \equiv (i_1, \ldots, i_d) \in \mathbb{Z}^d_+ \), and \( x^i := x_1^{i_1} \cdots x_d^{i_d} \). The measure \( \mu \) is said to be a \( K \)-representing measure for \( \beta \). For the typical case \( K = \mathbb{R}^d \), the problem is referred to as the truncated real moment problem (TRMP) and \( \mu \) is called simply a representing measure.

In a similar way, we consider the full moment problem for an infinite sequence \( \beta \equiv \beta^{(\infty)} = \{\beta_i : i \in \mathbb{Z}^d_+\} \). As well known by H. L. Hamburger for \( d = 1 \), the sequence has a representing measure supported on \( \mathbb{R} \) if and only if the Hankel matrix, \( [\beta_{i+j}]_{0 \leq i \leq k, \ 0 \leq j \leq k} \), is positive semidefinite (or simply, positive). Furthermore, T. J. Stieltjes showed that the single-index sequence has a representing measure supported in \( [0, \infty) \) if and only if both Hankel matrices, \( [\beta_{i+j}]_{0 \leq i \leq k, \ 0 \leq j \leq k} \) and \( [\beta_{i+j+1}]_{0 \leq i \leq k, \ 0 \leq j \leq k} \) for \( k \geq 0 \), are positive.
When \( m = 2n \), we define a moment matrix \( M_d(n) \) of \( \beta \equiv \beta^{(2n)} \) as

\[
M_d(n) \equiv M_d(n)(\beta) := (\beta_{i+j})_{i, j \in \mathbb{Z}_+^2 : |i|, |j| \leq n}.
\]

Some properties of \( M_d(n) \) have been important factors for the existence of a representing measure for \( \beta \); for example, \( M_d(n) \) is necessarily positive (obviously the positivity of \( M_d(n) \) is sufficient for \( d = 1 \) but not sufficient for \( d \geq 2 \) as well known). R. Curto and L. Fialkow have established many elegant results for various moment problems based on a positive extension of \( M_d(n) \). They also have used the functional calculus in the column space of \( M_d(n) \); to introduce the functional calculus, we label the columns and rows of \( M_d(n) \) with monomials \( X^i := X_1^{i_1} \cdots X_d^{i_d} \) in the degree-lexicographic order. Note that each block with the moments of the same order in \( M_d(n) \) is Hankel and that \( M_d(n) \) is symmetric. In addition, one can define a sesquilinear form: for \( i, j \in \mathbb{Z}_+^d \),

\[
\langle X^i, X^j \rangle_{M_d(n)} := (M_d(n)\hat{X}^i, \hat{X}^j) = \beta_{i+j},
\]

where \( \hat{X}^i \) is the column vector associated to the monomial \( X^i \).

For a motivation of the main result, let us consider the basic Fibonacci sequence. In particular, take the first six moments and write them as a 2-dimensional moment matrix \( d \equiv d(\beta) \) and \( \hat{d} \equiv \beta \). The coefficients in the formula of the measure are called densities and the points are atoms of the measure. This example shows that even though a sequence has no representing measure, it may have a signed measure so that some of the densities might be negative. We define such a measure as an interpolating measure \( \mu \) for \( \beta \) (finite or infinite) as a, not necessarily positive, Borel measure such that \( \beta_i = \int x^i \, d\mu(x), \ 0 \leq i \leq 3 \).

Due to the Jordan decomposition theorem, every interpolating measure \( \mu \) has a decomposition, \( \mu = \mu^+ - \mu^- \) of two positive measures \( \mu^+ \) and \( \mu^- \), at least one of which is finite. Interpolating measures appear in many scientific fields. For example, they are useful to represent electric charge; the moment problem about a signed measure is related to quantum physics as in [9]. Furthermore, there is a possibility that Gauss-Jacobi quadratures would be generalized through moment sequences with a signed measure (see [12]).

For \( d = 1 \), R. P. Boas showed that any single-index “infinite” sequence of real numbers admits an interpolating measure supported in \( [0, \infty) \); that is, one can always find a measure for any sequence of the form \( \mu = \mu^+ - \mu^- \) such that both \( \mu^+ \) and \( \mu^- \) are positive Borel measures supported in \( [0, \infty) \) [2]. Moreover, G. Flessas, K. Burton, and R. R. Whitehead found an algorithm to find such a measure supported in the real line for a “finite” real sequence \( \{s_j\}_{j=0}^{2n-1} \) [9]. As a generalization of these results, we will see that any finite or infinite sequence has an interpolating measure supported in \( \mathbb{R}^d \) for any \( d \geq 2 \). Notice that since moment problems about finite sequences are known to be more general than problems about infinite sequences, we need a solution to each of these two problems.

We conclude this section with another application of the moment problem to the numerical integration. For more details, readers can refer to [10].
Definition 1.1. A quadrature (or cubature) rule of size \( p \) and precision \( m \) is a numerical integration formula which uses \( p \) nodes, is exact for all polynomials of degree at most \( m \), and fails to recover the integral some polynomial of degree \( m+1 \).

Example 1.2. [Gaussian Quadrature; size \( n \), precision \( 2n-1 \)] We would like to find nodes \( t_0, t_1, \ldots, t_{n-1} \) satisfying

\[
\int_{-1}^{1} f(t) \, dt = \sum_{j=0}^{n-1} \rho_j f(t_j)
\]

for every polynomial \( f \) with \( \deg f \leq 2n-1 \). Now, we consider interpolating equations with polynomials and we get

\[
\sum_{j=0}^{n-1} \rho_j t_j^k = \int_{-1}^{1} t^k \, dt = \begin{cases} 0 & k = 1, 3, \ldots, 2n-1; \\ \frac{2}{2k+1} & k = 0, 2, \ldots, 2n-2. \end{cases}
\]

If \( n = 2 \), (2) becomes the system of polynomial equations

\[
\begin{align*}
\rho_0 + \rho_1 &= 2; \\
\rho_0 t_0 + \rho_1 t_1 &= 0; \\
\rho_0 t_0^2 + \rho_1 t_1^2 &= 2/3; \\
\rho_0 t_0^3 + \rho_1 t_1^3 &= 0.
\end{align*}
\]

The solution is \( \rho_0 = \rho_1 = 1 \), \( t_0 = -1/\sqrt{3} \), and \( t_1 = 1/\sqrt{3} \). Thus we easily see

\[
\int_{-1}^{1} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) \, dt = a_0 (\rho_0 + \rho_1) + a_1 (\rho_0 t_0 + \rho_1 t_1) + a_2 (\rho_0 t_0^2 + \rho_1 t_1^2) + a_3 (\rho_0 t_0^3 + \rho_1 t_1^3)
\]

\[
= \int_{-1}^{1} (a_0 + a_1 t + a_2 t^2 + a_3 t^3) \, d\mu,
\]

where \( \mu := \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1} \). This solution in numerical analysis textbooks is usually based on Legendre polynomials. With an approach via the truncated moment problem, we can find an alternative solution as follows: Let \( \beta_0 := 2, \beta_1 := 0, \beta_2 := 2/3, \beta_3 := 0 \) and form a Hankel matrix \( H \) with a parameter \( \alpha \),

\[
H := \begin{pmatrix}
\beta_0 & \beta_1 & \beta_2 \\
\beta_1 & \beta_2 & \beta_3 \\
\beta_2 & \beta_3 & \alpha
\end{pmatrix} = \begin{pmatrix}
2 & 0 & 2/3 \\
0 & 2/3 & 0 \\
2/3 & 0 & \alpha
\end{pmatrix}
\]

For the sake of a minimal number of nodes, we want rank \( H = 2 \); thus, \( \alpha = 2/9 \). After labeling the columns in \( H \) as \( 1, T, T^2 \), the column relation in \( H \) can be written as \( T^2 = (1/3) T \). In \[3\], it is known the roots of the equation \( t^2 = 1/3 \) (that is, \( t_0 = -1/\sqrt{3} \) and \( t_1 = 1/\sqrt{3} \)) are the nodes. We may compute the densities by solving the Vandermonde equation:

\[
\begin{pmatrix}
t_0 & t_1 \\
t_0^2 & t_1^2 \\
t_0^3 & t_1^3
\end{pmatrix} \begin{pmatrix}
\rho_0 \\
\rho_1
\end{pmatrix} = \begin{pmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{pmatrix},
\]

whose solution is obviously \( \rho_0 = \rho_1 = 1 \).
This method seems to provide an economical way to solve a quadrature problem and we will see the main result of this article gives a technique for more general cases, that is, when a signed measure arises in [1].

2. The Consistency and Rank-one Decompositions of Moment Matrices

This Section is designed to introduce some background knowledge for dealing with truncated moment sequences.

2.1. The consistency. We are about to define an algebraic set associated to $M_d(n)$. Let $\mathcal{P} := \mathbb{R}[x_1, \ldots, x_d]$ and let $\mathcal{P}_k := \{ p \in \mathcal{P} : \deg p \leq k \}$. Since we labeled columns in $M_d(n)$ with monomials, a column relation in $M_d(n)$ can be written as $p(X) = 0$ for some $p \in \mathcal{P}_n$. Let $Z(p)$ denote the zero set of a polynomial $p$ and we define the algebraic variety $V_\beta$ of $\beta$ or $M_d(n)$ by

$$V_\beta \equiv V_{M_d(n)} := \bigcap_{p(X) = 0} Z(p).$$

Given $\beta \equiv \beta^{(m)}$, define the Riesz functional $\Lambda \equiv \Lambda_\beta : \mathcal{P}_m \to \mathbb{R}$ by $\Lambda \left( \sum a_i x^i \right) := \sum a_i \beta^i$. We also define a notion which is the key to the main result of this note; $\beta \equiv \beta^{(2n)}$ or $M_d(n) \equiv M_d(n)(\beta^{(2n)})$ is said to be $V$-consistent for a set $V \in \mathbb{R}^d$ if the following holds:

$$p \in \mathcal{P}_{2n}, \ p|_V \equiv 0 \implies \Lambda(p) = 0. \quad (4)$$

This is a property of the moment sequence that guarantees the existence of an interpolating measure. Here is a formal result:

**Lemma 2.1.** [5, Lemma 2.3] Let $L : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional and let $V \subseteq \mathbb{R}^d$. Then the following statements are equivalent:

(i) There exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$ and there exist $w_1, \ldots, w_\ell \in V$ such that for all $p \in \mathcal{P}_{2n}$

$$L(p) = \sum_{k=1}^{\ell} \alpha_k p(w_k). \quad (5)$$

(ii) If $p \in \mathcal{P}_{2n}$ and $p|_V \equiv 0$, then $L(p) = 0$.

If $L$ is the Riesz functional of the moment sequence $\beta$, then Lemma 2.1 gives a concrete solution for $\beta$ to have an interpolating measure. Here is a formal result:

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(ii) If $p \in \mathcal{P}_{2n}$ and $p|_V \equiv 0$, then $L(p) = 0$.
2.2. **Rank-one decompositions.** After rearranging the terms in (5) by the sign of densities, we write a measure $\mu$ for a consistent $M_d(n)$ as

$$\mu = \sum_{k=1}^s \alpha_k \delta_{w_k} - \sum_{k=s+1}^\ell \alpha_k \delta_{w_k}, \quad (6)$$

where $\alpha_k > 0$ for all $k = 1, \ldots, \ell$; we denote the first summand in (6) as $\mu^+$ and the second as $\mu^-$. Due to this fact, a bound of the cardinality of the support of an interpolating measure is established:

**Proposition 2.2.** A minimal interpolating measure for a consistent $M_d(n)$ is at most $(2n + 1)(2n + 2)$-atomic.

**Proof.** If $M_d(n)$ is consistent with a measure $\mu = \mu^+ - \mu^-$ of two positive finitely atomic measures $\mu^+$ and $\mu^-$, we may write $M_d(n) = M[\mu^+] - M[\mu^-]$, where each term is a moment matrix generated by the corresponding measure of the same size as $M_d(n)$. A result [1, Theorem 2] by C. Bayer and J. Teichmann showed that the cardinality of the support of a positive measure is at most $\dim \mathcal{P}_{2n}$ in the presence of a representing measure for a moment matrix associated to a moment sequence of degree $2n$.

Since $M[\mu^+]$ and $M[\mu^-]$ have a positive measure, it follows that a minimal measure for each moment matrix is at most $\dim \mathcal{P}_{2n}$-atomic. Therefore, we conclude that the cardinality of a minimal interpolating measure is at most $2(\dim \mathcal{P}_{2n}) = (2n + 1)(2n + 2)$.

Many solutions of TRMP for a positive measure depend on finding a positive moment matrix extension of $M_d(n)$. However, this approach needs to allow new parameters and constructing an extension is not handy for most cases when $n \geq 3$. Alternatively, R. Curto and the second-author recently have used a decomposition of $M_d(n)$ for the study of TRMP. To introduce the decomposition, we now define some notations: Let $w = (w_1, \ldots, w_d) \in \mathbb{R}^d$ and let

(i) $v(w) := (1 \ w_1 \ \cdots \ w_d \ w_1^2 \ w_1w_2 \ w_1w_3 \ \cdots \ w_{d-1}w_d \ w_1^2 \ \cdots \ w_1^n \ \cdots \ w_d^n)$, which is a row vector corresponding to the monomials $w^i$ in the degree-lexicographic order.

(ii) $P(w) := v(w)^T v(w)$, which is indeed the rank-one moment matrix generated by the measure $\delta_w$.

For example, if $d = n = 2$ and $w = (a, b)$, then

$$P(w) = \begin{pmatrix}
1 & a & b & a^2 & ab & b^2 \\
a & a^2 & ab & a^3 & a^2b & ab^2 \\
b & ab & b^2 & a^3b & ab^2 & b^3 \\
a^2 & a^3 & a^2b & a^4 & a^3b & a^2b^2 \\
ab & a^2b & ab^2 & a^3b & a^2b^2 & ab^3 \\
b^2 & ab^2 & b^3 & a^2b^2 & ab^3 & b^4
\end{pmatrix}.$$

Thus, if $M_d(n)$ has an interpolating measure $\mu$ supported in a set $\{w_1, \ldots, w_\ell\}$, then one should be able to write $M_d(n) = \sum_{k=1}^\ell d_k P(w_k)$ for some $d_1, \ldots, d_\ell \in \mathbb{R} \setminus \{0\}$.
3. Main Result

We will verify that any truncated moment matrix turns out to be \( \mathbb{R}^d \)-consistent after applying proper perturbations, and so it admits an interpolating measure. To prove the main result, we begin with auxiliary results:

**Lemma 3.1.** Assume \( A \) and \( B \) are matrices of the same size. Then \( \text{rank} (A + B) = \text{rank} A + \text{rank} B \) if and only if \( \text{range} A \cap \text{range} B = \{0\} \) and \( \text{range} A^T \cap \text{range} B^T = \{0\} \).

As a special case of Lemma 3.1, one can easily prove:

**Lemma 3.2.** Assume \( A \) and \( B \) are Hermitian matrices of the same size and rank \( B = 1 \). Then \( \text{rank} (A + B) = 1 + \text{rank} A \) if and only if \( \text{range} A \cap \text{range} B = \{0\} \).

We are ready to introduce a crucial lemma:

**Lemma 3.3.** A point \( w \) is in \( \mathcal{V}_{M_d(n)} \) if and only if the vector \( v(w) \) is in range \( M_d(n) \).

**Proof.** Assume that \( \{p_k(X) = \sum a_i^{(k)}X_i\}_{1}^{\ell} \) is the set of polynomials obtained from column relations in \( M_d(n) \). Note that span \( \{\hat{p}_k\}_{k=1}^{\ell} = \ker M_d(n) \). Now observe:

\[
\begin{align*}
\text{w} \in \mathcal{V}_{M_d(n)} & \iff p_k(w) = 0 \quad \text{for } k = 1, \ldots, \ell \\
& \iff \sum a_i^{(k)}w_i = 0 \quad \text{for } k = 1, \ldots, \ell \\
& \iff \langle \hat{p}_k, v(w) \rangle = 0 \quad \text{for } k = 1, \ldots, \ell \\
& \iff \hat{p}_k \perp v(w) \quad \text{for } k = 1, \ldots, \ell \\
& \iff v(w) \in (\ker M_d(n))^\perp = \text{range} M_d(n).
\end{align*}
\]

**Theorem 3.4.** Any truncated moment sequence \( \beta \equiv \beta^{(2n)} \) of degree \( 2n \) has an interpolating measure in \( \mathbb{R}^d \) for any positive \( d \in \mathbb{Z}_+ \).

**Proof.** Pick a point \( w_1 \in \mathbb{R}^d \setminus \beta \). Then we know from Lemma 3.3 that \( v(w_1) \notin \text{range} M_d(n)(\beta) \). Since range \( P(w_1) = \{aw_1 : a \in \mathbb{R}\} \), it holds that range \( M_d(n)(\beta) \cap \text{range} P(w_1) = \{0\} \). Therefore, it follows from Lemma 3.2 that \( \text{rank} (M_d(n)(\beta) + P(w_1)) = 1 + \text{rank} M_d(n)(\beta) \). Next, choose a point \( w_2 \) which not in the algebraic variety of \( M_d(n)(\beta) + P(w_1) \) and we know from the same argument that \( \text{rank} (M_d(n)(\beta) + P(w_1) + P(w_2)) = 2 + \text{rank} M_d(n)(\beta) \). Keep this process until we obtain an invertible matrix \( \bar{M} := M_d(n)(\beta) + \sum_{k=1}^{\ell} P(w_k) \) for some \( \ell \). \( \bar{M} \) is naturally consistent, and so it admits an interpolating measure, say \( \bar{\mu} \). Thus, \( M_d(n)(\beta) \) has an interpolating measure of the form \( \bar{\mu} - \sum_{k=1}^{\ell} \delta_{w_k} \).

**Theorem 3.5.** Any finite sequence has an interpolating measure.

**Proof.** It suffices to cover the cases when the given sequence is not the type of \( \beta^{(2n)} \). Such a sequence cannot fill up the associated moment matrix, so we use new parameters to complete the moment matrix. If it is possible to make the moment matrix invertible, then the extended moment sequence is consistent. Thus, the given sequence has an interpolating measure. Otherwise, one can follow the same process in the proof of Theorem 3.4 and verify that the sequence admits an interpolating
measures. Lastly, if a sequence begins with zero, then one need take a new nonzero initial moment and repeat the process used in the above.

Before we conclude this note, let us discuss how investigate the location of atoms of an interpolating measure. In addition, an algorithmic approach to find an explicit formula of a measure will be presented through a concrete example. Recall that in the presence of a (positive) representing measure $\mu$ for a positive $M_d(n)(\beta)$, Proposition 3.1 in [4] states that

$$\tilde{\mu} \in \ker M_d(n)(\beta) \iff p(X) = 0 \iff \text{supp } \mu \subseteq Z(p).$$

This result provides an evidence that where the atoms of $\mu$ lie for a singular $M_d(n)$; that is, the algebraic variety of $M_d(n)$ must contain the support of a representing measure. However, the following example shows such an argument is no longer valid for the moment problem about an interpolating measure; consider

$$M_2(1) \equiv M_2(1) \left( \beta^{(2)} \right) = \begin{pmatrix} -1 & -16 & -4 \\ -16 & -94 & -10 \\ -4 & -10 & 2 \end{pmatrix}. \quad (7)$$

Note that $M_2(1)$ has a single column relation $X_2 = -(4/3)I + (1/3)X_1$. Indeed, the sequence can be generated by an interpolating measure $\nu = \delta_{(1,1)} - \delta_{(2,1)} - \delta_{(1,2)}$; but, different from the case for a positive measure, $\text{supp } \nu \not\subseteq Z(x_2 + 4/3 - (1/3)x_1) = V_\beta(x)$. In other words, an interpolating measure for the sequence may have atoms outside of the algebraic variety. Nonetheless, one can still find an interpolating measure supported in the algebraic variety of $M_2(1)$ as follows:

**Example 3.6.** We illustrate how to find an interpolating measure of the sequence in (7). To find an interpolating measure supported in the algebraic variety of $M_2(1)$, select a point $(a, \frac{a}{3}, \frac{a^2}{9}) \in Z(x_2 + 4/3 - (1/3)x_1)$ for some $a \in \mathbb{R}$. Using the rank-one decomposition, we write

$$M_2(1) = \sum_{u \in \mathbb{R}} \left( \begin{array}{ccc} 1 & a & a-4 \\ a & a^2 & \frac{a}{3} \\ \frac{a^2}{3} & \frac{a(a-4)}{3} & \frac{(a-4)^2}{9} \end{array} \right) (8)$$

for some $u \in \mathbb{R}$. Note that rank $M_2(1) = 2$ and we are attracted to guess that a minimal interpolating measure is 2-atomic (cf. Lemma 3.2 and 3.3). In order to find such a measure, we impose a condition that rank $M_2(1) = 1$; a calculation shows rank $M_2(1) = 1$ if and only if $u = 162/(a^2 - 32a + 94)$. If we take $u = 162/(a^2 - 32a + 94)$, then

$$M_2(1) = \frac{-(a - 16)^2}{a^2 - 32a + 94} \left( \begin{array}{ccc} 1 & 2(8a - 47) & 2(2a - 5) \\ a - 16 & a - 16 & (a - 16)^2 \\ 2(2a - 5)(8a - 47) & (a - 16)^2 & (a - 16)^2 \end{array} \right)$$

$$+ \frac{162}{a^2 - 32a + 94} \left( \begin{array}{ccc} 1 & a & \frac{a - 4}{3} \\ a & a^2 & \frac{a(a - 4)}{3} \\ \frac{a^2}{3} & \frac{a(a - 4)}{3} & \frac{(a - 4)^2}{9} \end{array} \right).$$

Therefore, we get an interpolating measure $\mu = \frac{(a - 16)^2}{a^2 - 32a + 94} \delta \left( \frac{2(8a - 47)}{a - 16}, \frac{2(2a - 5)}{a - 16} \right) + \frac{162}{a^2 - 32a + 94} \delta \left( \frac{a - 4}{3} \right)$ (with $a^2 - 32a + 94 \neq 0$ and $a \neq 16$), which is supported in $V_{M_2(1)}$. 

\[\]
Example 3.7. Consider a truncated moment sequence $\beta^{(4)}$:

\begin{align*}
\beta_{00} &= 6, & \beta_{10} &= 6, & \beta_{01} &= 20, & \beta_{20} &= 18, & \beta_{11} &= 16, & \beta_{02} &= 68, \\
\beta_{30} &= 30, & \beta_{21} &= 56, & \beta_{12} &= 40, & \beta_{03} &= 236, \\
\beta_{40} &= 66, & \beta_{31} &= 88, & \beta_{22} &= 176, & \beta_{13} &= 88, & \beta_{04} &= 836.
\end{align*}

Construct its moment matrix as follows:

\[
\widetilde{M}_2(2) = \begin{pmatrix}
6 & 6 & 20 & 18 & 16 & 68 \\
6 & 18 & 16 & 30 & 56 & 40 \\
20 & 16 & 68 & 56 & 40 & 236 \\
18 & 30 & 56 & 66 & 88 & 176 \\
16 & 56 & 40 & 88 & 176 & 88 \\
68 & 40 & 236 & 176 & 88 & 836
\end{pmatrix}
\]

It is easy to check that the representing measure is $\mu = 2\delta_{(-1,4)} + 4\delta_{(2,3)}$. Assume that for sufficiently small perturbation we have

\[
M_2(2) = \begin{pmatrix}
5.990000 & 5.995000 & 19.998000 & 17.997500 & 15.999000 & 67.999600 \\
5.995000 & 17.997500 & 15.999000 & 29.998750 & 55.999500 & 39.999800 \\
19.998000 & 15.999000 & 67.999600 & 55.999500 & 39.999800 & 235.999920 \\
17.997500 & 29.998750 & 55.999500 & 65.999375 & 87.999750 & 175.999900 \\
15.999000 & 55.999500 & 39.999800 & 87.999750 & 175.999900 & 87.999960 \\
67.999600 & 39.999800 & 235.999920 & 175.999900 & 87.999960 & 835.999984
\end{pmatrix},
\]

which is not positive semidefinite. So, arbitrarily small perturbations of a given sequence eject one from the cone of positive semidefinite matrices. As a result, this sequence does not have a representing measure. Instead, one can find interpolating measures for the sequence. Concretely, one of them is $\mu = -0.01\delta_{(0,5,0,2)} + 2\delta_{(-1,4)} + 4\delta_{(2,3)}$; here the first term with the negative density can be considered as noise.

Removing noise from the original data is a challenging problem in many different fields. Moment sequences need to be modified since data obtained from physical experiments and phenomena are often corrupt or incomplete. By Theorem 3.5, one can find an interpolating measure $\mu$ for the given data, which is $\mu = \mu_+ - \mu_-$ of two positive measures $\mu_+$ and $\mu_-$. Assuming that $\mu_-$ is generated by the distribution of noise, $\mu_+$ can be a measure for the denosing data in a sense.

Finally, we will see Boas’ result introduced earlier can be extended to any infinite multi-index sequences. It suffices to show that the claim holds for the case of double-index sequences because the argument in the proof also is also valid for any multi-index sequences.

**Theorem 3.8.** Any bivariate (real) full moment sequence has an interpolating measure.

**Proof.** Let $\{\beta_{ij}\}_{i,j=0}^\infty$ be an infinite moment sequence. Then we may write $\beta_{ij} = u_iv_j$ for some $u_i, v_j \in \mathbb{R}$. Then, by Boas’ result, there are signed measures $\mu$ and $\nu$ supported on $[0, \infty)$ such that

\[
u_j = \int_0^\infty y^j \, d\nu.
\]

It follows from the Jordan decomposition theorem that any signed measure can be written as a difference of two positive measure; that is,

\[
\mu = \mu_+ - \mu_- \quad \text{and} \quad \nu = \nu_+ - \nu_-,
\]
where \( \mu_+ , \mu_-, \nu_+ , \) and \( \nu_- \) are positive measures supported on \([0, \infty)\). Now,

\[
\beta_{ij} = u_i v_j = \int_0^\infty x^i \, d\mu \int_0^\infty y^j \, d\nu
\]

\[
= \left[ \int_0^\infty x^i \, d(\mu_+ - \mu_-) \right] \left[ \int_0^\infty y^j \, d(\nu_+ - \nu_-) \right]
\]

\[
= \left( \int_0^\infty x^i \, d\mu_+ - \int_0^\infty x^i \, d\mu_- \right) \left( \int_0^\infty y^j \, d\nu_+ - \int_0^\infty y^j \, d\nu_- \right)
\]

\[
= \left( \int_0^\infty x^i \, d\mu_+ \int_0^\infty y^j \, d\nu_+ + \int_0^\infty x^i \, d\mu_- \int_0^\infty y^j \, d\nu_- \right)
\]

\[
- \left( \int_0^\infty x^i \, d\mu_+ \int_0^\infty y^j \, d\nu_- + \int_0^\infty x^i \, d\mu_- \int_0^\infty y^j \, d\nu_+ \right)
\]

\[
= \int_0^\infty \int_0^\infty x^i y^j \, d(\mu_+ \times \nu_+) + \int_0^\infty \int_0^\infty x^i y^j \, d(\mu_- \times \nu_-)
\]

\[
- \int_0^\infty \int_0^\infty x^i y^j \, d(\mu_+ \times \nu_-) - \int_0^\infty \int_0^\infty x^i y^j \, d(\mu_- \times \nu_+)
\]

\[
= \int_0^\infty \int_0^\infty x^i y^j \, d\tau,
\]

where \( \tau := \mu_+ \times \nu_+ - \mu_- \times \nu_- - \mu_+ \times \nu_- - \mu_- \times \nu_+ \) is a signed measure. Note that the second last identity in the above is true since \( \mu \) and \( \nu \) satisfy the hypothesis in the Fubini’s theorem. Indeed, observe that \( u_0 = \int_0^\infty \, d\mu = \mu([0, \infty)) \in \mathbb{R} \) and \( v_0 = \int_0^\infty \, d\nu = \nu([0, \infty)) \in \mathbb{R} \); thus, \( \mu \) and \( \nu \) are finite measures, which means the two measures are \( \sigma \)-finite. Also, \( x^i y^j \) are nonnegative on \([0, \infty) \times [0, \infty)\) for any \( i, j \in \mathbb{N}_0 \). Therefore, \( \{ \beta_{ij} \} \) has an interpolating measure \( \tau \).

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