The hydrogen atom in $D = 3 - 2\epsilon$ dimensions

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The nonrelativistic hydrogen atom in $D = 3 - 2\epsilon$ dimensions is the reference system for perturbative schemes used in dimensionally regularized nonrelativistic effective field theories to describe hydrogen-like atoms. Solutions to the $D$-dimensional Schrödinger-Coulomb equation are given in the form of a double power series. Energies and normalization integrals are obtained numerically and also perturbatively in terms of $\epsilon$. The utility of the series expansion is demonstrated by the calculation of the divergent expectation value $\langle (V')^2 \rangle$.

For over a century the hydrogen atom has been a touchstone of fundamental physics. From the earliest days of quantum physics, the challenge to understand the structure and behavior of hydrogen has been a driver of new developments and has stimulated the craft of countless innovators in the field. One can mention the Bohr Model, Sommerfeld’s relativistic hydrogen atom, wave and matrix mechanics at the birth of quantum mechanics, the development of QED to explain the Lamb Shift and the electron’s anomalous moment as observed at contact in terms of the small parameter $\epsilon = (3 - D)/2$. For energies, results through $O(\epsilon^2)$ are obtained with estimates for the $O(\epsilon^3)$ terms, while for the wave functions at contact results through $O(\epsilon)$ are obtained with estimates for the $O(\epsilon^2)$ terms.

Much of the earlier work on hydrogen in $D$ dimensions made use of a strict $1/r$ potential instead of the physical $1/r^{D-2}$ potential implied by Gauss’ Law, or was restricted to an integer number of dimensions. Quantum mechanics with the physical potential in non-integral dimension has been studied by Andrew and Supplee, Morales, and is reviewed in with additional references.

The $D$-dimensional Schrödinger-Coulomb equation is

$$-rac{1}{2m}\vec{\nabla}^2 \psi(\vec{x}) + \vec{V}(\vec{r})\psi(\vec{x}) = E\psi(\vec{x}),$$

where we use the bar to signify $D$-dimensional quantities, with no bar for 3-dimensional ones. The potential energy

$$\bar{V}(r) = -\frac{\Gamma(D/2 - 1)\mu^2\alpha}{\pi^{D/2 - 1}r^{D - 2}}$$

arises as the $D$-dimensional Fourier transform of the momentum-space Coulomb interaction term $-4\pi\alpha\frac{\mu^2}{r^2}$. It follows that [1] is the lowest approximation for the study of hydrogen-like atoms in NRQED/QCD (where $\hbar = 1$, $\alpha$ is the fine structure constant, and $Z$ is the nuclear charge in units of the electron charge magnitude). The potential $\bar{V}(r)$ can also be deduced from the requirement that the electric field derived from it satisfy Gauss’s law in $D$ dimensions, or equivalently that the potential satisfy the $D$-dimensional Poisson equation with a point charge source. The mass scale $\mu$ has been introduced to ensure that $Z\alpha$
remains dimensionless in $D$ dimensions, and $\bar{\mu}$ (with $\bar{\mu}^2 \equiv \mu^2 e^{1/2}/(4\pi)$) is the corresponding $\overline{\mathrm{MS}}$ scale. It is convenient to separate variables in the Schrödinger equation using spherical coordinates. The $D$-dimensional Laplacian can be written as

$$\nabla^2 = \partial_r^2 + \frac{D - 1}{r} \partial_r - \frac{L^2}{r^2}$$

(3)

where $r = (\sum_i x_i^2)^{1/2}$ is the usual radius and $L^2 = \sum_{i<j} \bar{L}_{ij}$ with $\bar{L}_{ij} = -i(x_i \partial_j - x_j \partial_i)$ is the angular momentum squared. We separate variables in the wave function according to $\tilde{\psi}(\vec{x}) = \tilde{R}(r) Y(\hat{x})$ where $\hat{x} = \vec{x}/r$. The angular functions $Y(\hat{x})$ are eigenstates of $L^2$ [13,14]:

$$L^2 Y_l(\hat{x}) = \ell(\ell + D - 2) Y_l(\hat{x}),$$

(4)

where the allowed quantum numbers $\ell$ are 0, 1, 2, $\cdots$, just as in an integral numbers of dimensions. The $D$-dimensional angular functions have $(2\ell + D - 2)(\ell + D - 3)!/\ell!(D - 2)!$ independent components. An explicit representation is given by the symmetric traceless harmonic polynomials. For example, the lowest few are $Y_0(\hat{x}) = A_0$, $Y_1(\hat{x}) = A_1 \hat{x}$, $Y_2(\hat{x}) = A_2 (\hat{x}^2 - \hat{x} \hat{\delta}_i \hat{\delta}_j / D)$, containing 1, $D$, and $(D - 1)(D + 2)/2$ independent components, where the $A_i$ are appropriate normalization factors.

The radial equation is the object of our main concern. It is

$$\frac{1}{2m} \left\{ -\partial_r^2 - \frac{D - 1}{r} \partial_r + \frac{\ell(\ell + D - 2)}{r^2} \right\} \tilde{R}_{\ell m}(r) + \tilde{V}(r) \tilde{R}_{\ell m}(r) = \tilde{E}_{\ell m} \tilde{R}_{\ell m}(r).$$

(5)

We follow the usual steps of first working out the leading short and long distance behavior of $\tilde{R}$. The $r \to 0$ limit of (3) shows that $\tilde{R}(r) \to r^\ell$ for small $r$. We also find $\tilde{R}(r) \to e^{-\gamma r}$ for large $r$ where $\tilde{E} = -\gamma^2/(2m)$. We define a new function $L_{\ell m}(\rho)$ according to

$$\tilde{R}_{\ell m}(r) = \tilde{\phi}_{\ell m} \Omega_D^{1/2} \left( \frac{(n + \ell)!}{(n - \ell - 1)!} \right)^{1/2} \frac{\rho^{\ell - 1} e^{-\rho/2}}{(2\ell + 1)!} L_{\ell m}(\rho)$$

(6)

where $\rho \equiv 2\gamma r$ is dimensionless and $\Omega_n \equiv 2^n \Gamma(n + 1/2)/\Gamma(n + 1)$ is the surface area of a unit $n$-sphere. We normalize $L_{\ell m}(\rho)$ so that $L_{\ell m}(0) = 1$. We see that $\tilde{\phi}_{\ell m} \equiv \lim_{r \to 0} \tilde{\psi}_{\ell m}(r)$ is the S-state wave function at the origin (at “contact”), and generally $\tilde{\phi}_{\ell m}$ is proportional to $\lim_{r \to 0} \tilde{R}_{\ell m}(r)/r^{\ell/2}$. When expressed in terms of $\rho$ and $L_{\ell m}(\rho)$, the radial equation becomes

$$\left\{ \frac{\partial^2}{\rho^2} + \left( \frac{2(\ell + 1 - \epsilon)}{\rho} - \frac{\ell + 1 - \epsilon}{\rho} \right) \partial_{\rho} - \frac{\ell + 1 - \epsilon}{\rho} \right\} \bar{L}_{\ell m}(\rho) = 0$$

(7)

where

$$\bar{n} \equiv m Z \alpha \Gamma(1/2 - \epsilon) / \pi^{1/2 - \epsilon} \left( \frac{\bar{\mu}}{2\gamma} \right)^{2\epsilon}.$$ 

(8)

In three dimensions, $\bar{n}$ would be the principal quantum number $n$ and $\bar{\gamma}$ would be the momentum scale factor $m Z \alpha / n$.

We intend to find a series solution for (7) about the origin. Since $\rho^{\ell - 1}$ is not analytic in a region containing the origin for most values of $\epsilon$, the usual type of series solution won’t work. We require the more general form

$$L_{\ell m}(\rho) = \sum_{j=0}^{\infty} \sum_{k=0}^{n} a_{jk} \bar{n}^k \rho^{j + 2k}.$$ 

(9)

Using (8) in (7) and assuming that all powers $\rho^{j + 2k}$ are independent, we obtain the recursion relation:

$$a_{jk} = a_{j-1,k} \frac{(j + \ell + \epsilon(2k - 1)) - a_{j-1,k-1}}{(j + 2k)(j + 2\ell + 1 + 2\epsilon(k - 1))}.$$ 

(10)

Using (10) and the initial condition $a_{00} = 1$, it is easy to calculate as many coefficients $a_{j k}$ as desired and obtain a convergent series solution near $\rho = 0$. (When $\epsilon \to 0$, the solution for $a_{j k}$ is $(-1)^j s_\ell (\ell + 1)(j,k)/(j!2^\ell 2^j)$, where the $s_\ell (j,k)$ are “non-central Stirling numbers of the first kind” as defined by Koutras [15] and $n^j$ is the rising factorial $n^j = n(n + 1) \cdots (n + j - 1)$. In this limit the $L_{\ell m}$ reduce to the usual associated Laguerre polynomials.) We use the series to find $L_{\ell m}(\rho)$ in a small region ($0 \leq \rho \leq \rho_0$) around the origin and extend that region to $0 < \rho < \infty$ using standard numerical methods to solve (7). We developed a procedure to home in on acceptable values of $\bar{n}$ for which $L_{\ell m}(\rho)$ can be normalized (as in the integral (11) below). For each value of $\ell$ we labeled these solutions by the “radial quantum number” $n_r$ taking values 0, 1, 2, $\cdots$. We also define the standard principal quantum number $n$ with $n_r = n - \ell - 1$, which takes positive integer values starting with $\ell + 1$ for each value of $\ell$. The acceptable values of $\bar{n}$ with $\epsilon = 0.001$ for the low-lying states are shown in Table II as $n^{DE}$. We used the numerical solutions to compute values for the integrals

$$I_{\ell m} \equiv \frac{(n + \ell)!}{2\pi^{n+1} \Gamma(1/2 - \epsilon)} \int_0^\infty d\rho \rho^{D - 1 + 2\ell} e^{-\rho} \left[ L_{\ell m}(\rho) \right]^2.$$ 

(11)

that are related to the normalization of the corresponding states. These appear in the table as $I^{DE}$. Were $D = 3$, the $I_{\ell m}$ integrals would all be one.

As a complement to the numerical solutions obtained above we have also worked out results for $\bar{n}$ and $I$ using perturbation theory in the small parameter $\epsilon$. This was done in order to confirm the consistency of the whole $D$-dimensional procedure and for use in the evaluation of coordinate-space matrix elements. The zeroth-order problem for this perturbative calculation is also $D$-dimensional, but with a potential $\tilde{V}(r) = -Z \alpha / r$. It is essential that the zeroth-order problem be $D$-dimensional, as the two Hamiltonians and the perturbation must be hermitian in the same space. Fortunately
where the general formula for the problem can be expressed as

$$E = R_{\text{pert.}}(\ell) + R_{\text{DE}} - R_{\text{pert.}}(\ell)$$

In the zeroth-order problem has an exact solution \([16, 17]\) as described by Nieto. The radial equation in this case is identical to \([7]\) except that the potential term \(\tilde{n} / \rho^\gamma / \rho \) is replaced by \(\tilde{n} / \rho \). The exact solution to this zeroth-order problem can be expressed as

$$\tilde{R}_{\text{pert.}}(r) = N(n, \ell) e^{-\rho/\rho} L_{\ell+1}^{2+1-2\ell}(\rho)$$

where \(\rho = 2\tilde{n} r\), \(n = n - \ell\), \(\tilde{n} = mZ\alpha / \tilde{n}\). The bound state energy is \(E_n = -\frac{\tilde{n}^2}{2(2\ell + 1)}\), and the normalization constant is given in \([17]\). The associated Laguerre polynomials are defined in the standard way: \(L_n^\ell(x) = \sum_{j=0}^n \frac{(n+j)!(x-j)!}{j!(n-j)!}\). The perturbation is

$$H' = V(r) - \tilde{V}(r) = -\frac{2Z\alpha}{r} \left(\log(\mu/r) + \gamma_E\right) + O(\epsilon^2).$$

It is straightforward to work out the first energy correction:

$$\Delta E_n^{(1)} = 4E_n (L + H_{n+\ell}) \epsilon,$$

where \(E_n = -\frac{\gamma^2}{2(2\ell + 1)}\) is the standard Bohr energy, \(\gamma = mZ\alpha / \tilde{n}\), \(L = \log(\mu/[2\gamma])\), and \(H_n = \sum_{j=1}^n 1/j\) is the \(n\)th harmonic number. My calculation of the second order energy correction makes use of the form for the reduced Schrödinger-Coulomb-Green’s function \(g_{n\ell}\) given by Johnson and Hirschfelder \([18]\). I was not able to obtain a general formula for the \(O(\epsilon^2)\) energy correction. For any particular state I was able to obtain the \(O(\epsilon^2)\) correction in terms of \(\kappa_{\text{nl}}\) where \(E_n \kappa_{\text{nl}} = \langle V \ln(2\gamma) \rangle g_{n\ell} \ln(2\gamma) V\rangle\), for which I could only obtain numerical results. (In the calculation of \(\kappa_{\text{nl}}\) it was adequate to use the \(\epsilon \to 0\) limit of \(H'\) and standard 3-dimensional expressions for the states and the reduced Green’s function.) For instance, the ground state energy has the expansion

$$E_{10} = 1 + \epsilon \{4L + 6\} + \epsilon^2 \left\{8L^2 + 16L - 4\gamma_E^2 \right\}$$

$$\quad + 15 - \gamma_E(2 + 4\kappa_{10}) + O(\epsilon^3),$$

where \(\gamma_E\) is the Euler-Mascheroni constant. From the energies, we can obtain the series for \(\gamma = (-2mE)^{1/2}\), then \(\tilde{n}\) using \([8]\), and finally \(\xi = \tilde{n} / n = 1 + \xi^{[1]} \epsilon + \xi^{[2]} \epsilon^2 + \xi^{[3]} \epsilon^3 + \cdots\).

The exact result for \(\xi^{[1]}\) is \(\xi^{[1]} = 2\gamma_E - 2H_{n+\ell} - 1/n\), and for the ground state one finds

$$\xi_{10} = 1 + \epsilon \left\{2\gamma_E - 3\right\} + \epsilon^2 \left\{4\gamma_E^2 - 6\gamma_E + 2\gamma_E(2 - 2\kappa_{10}) + O(\epsilon^3).$$

Table II contains numerical results for \(\xi^{[2]}\) as calculated using perturbation theory as well as estimates for \(\xi^{[3]}\) obtained by a numerical exploration of the difference between \(\tilde{n}^{\text{DE}}\) and the truncated series \(\tilde{n}^{\text{pert.}} = n + 1 + \xi^{[1]} \epsilon + \xi^{[2]} \epsilon^2\) for various small values of \(\epsilon\). The series for \(\xi\) seems well-behaved at least through \(O(\epsilon^2)\).

Now we work out the perturbative result for \(\tilde{\phi}_{n\ell}\) describing the short-distance behavior of the wave function and the related result for the normalization integral \([11]\) for the radial functions \(L_{n\ell}(\rho)\). We can calculate \(\tilde{\phi}_{n\ell}\) from \([10]\) as the short-distance limit

$$\tilde{\phi}_{n\ell} = \Omega_D^{-1/2} \left(\frac{mn_{\ell}!}{(n+\ell)!}\right)^{1/2} \left(\frac{2\ell + 1)!}{(2\gamma)!}\right)^{1/2} \lim_{r \to 0} \frac{1}{r} \tilde{R}_{n\ell}(r).$$

We use first-order perturbation theory based on the exact solution of the \(D\)-dimensional 1/r problem to find the \(O(\epsilon)\) correction to the wave function and then to \(\tilde{\phi}_{n\ell}\). Since the perturbation is purely radial, we can factor out
the angular momentum dependence and write
\[ \tilde{R}_{nl}(r) = \hat{R}_{nl}(r) + \int dr_1 r_1^{D-1} \hat{y}_{nl}(r,r_1)H'(r_1)\tilde{R}_{nl}(r_1) + O(\epsilon^2), \] (18)
where \( \hat{y}_{nl}(r,r_1) \) is the component of the reduced Green’s function for the \( D \)-dimensional \( 1/r \) problem having angular momentum \( \ell \). The \( O(\epsilon) \) correction here contains an explicit factor of \( \epsilon \) in \( H' \), so in order to get just the first order correction we can take \( \epsilon \to 0 \) in the rest and simply use the regular 3-dimensional reduced Green’s function and radial wave function. The result for the expansion of \( \tilde{\phi}_{nl} \) is
\[ \tilde{\phi}_{nl} = \left( \frac{\alpha}{\pi} \right)^{1/2} \left\{ 1 + \epsilon \left[ 3L + 2n \text{diH}_+(n + \ell, -n_r) 
- n \left( \frac{H^2}{2} - H_{n+\ell}^{(2)} \right) + n \left( \frac{H^2}{2} + H_{n+\ell}^{(2)} \right) \right] \right\} + \frac{1}{\ell^2} + O(\epsilon^2). \] (19)

Here \( H_n^{(2)} = \sum_{j=1}^{\ell} 1/j^2 \) is a generalized harmonic number. We have found it convenient to define as well the “diharmonic” numbers
\[ \text{diH}_\pm(n,m) = \sum_{i=1}^{n} \frac{H_{m+\pm i}}{i} = \sum_{i=1}^{n} \sum_{j=1}^{m+\pm i} \frac{1}{ij}, \] (20)
in terms of which it is possible to express any diharmonic sum (a sum of \( 1/(ij) \) for positive integers \( i, j \)) over a region of the \( i, j \) lattice having a boundary that includes vertical, horizontal, and diagonal sides of angle \( \pm 45^\circ \) only. The normalization integral \( I_{nl} \) is connected to \( \tilde{\phi}_{nl} \) because the radial wave function is normalized in \( D \)-dimensional space: 1 = \( \int_0^\infty dr r^{D-1} \tilde{R}_{nl}^2(r) = \tilde{\phi}_{nl}^2 \Omega D/2 \tilde{\gamma}_D^D \), so the normalization integral \( I_{nl} \) must have the value
\[ I_{nl} = \frac{2^{D-2} \Gamma(D/2)}{\pi^{D/2-1}} \left( \frac{\gamma_D}{\pi \tilde{\phi}_{nl}} \right). \] (21)

We use our earlier result for \( \tilde{\gamma} \) and (19) for \( \tilde{\phi}_{nl} \) to write
\[ I_{nl} = 1 + 2 \epsilon \left[ n \left( \frac{H^2}{2} - H_{n+\ell}^{(2)} \right) - n \left( \frac{H^2}{2} + H_{n+\ell}^{(2)} \right) 
- 2n \text{diH}_+(n + \ell, -n_r) + 2n\zeta(2) + H_{n+\ell} 
- 2H_{2\ell+1} + \frac{1}{2n} + \gamma_E \right] + O(\epsilon^2). \] (22)

This result for \( I_{nl} \) truncated at \( O(\epsilon) \) is given in Table II as \( I_{nl}^{pert,th.} \). By comparison with the numerical result \( I_{nl}^{DE} \) for various small values of \( \epsilon \), an estimate for the \( O(\epsilon^2) \) term \( I_{nl}^{[2]} \) was also obtained and displayed in the table.

In this work we can see that \( \bar{n} \) and \( I \) play the role of scale invariant quantities independent of \( \mu \) since they are determined directly from the scale-free differential equation (3). On the other hand, the energy \( E \), momentum scale \( \bar{\gamma} \), and the short distance wave function factor \( \tilde{\phi} \) all depend on \( \mu \), as can be seen directly from their definitions and from the logarithms present in their series expansions.

The numerical approach developed here gives precise results for all \( D \) in the range 2 < \( D \) < 4. This range is bordered by \( D = 2 \), where the potential becomes logarithmic and the spectrum takes a qualitatively different form \( [19, 20] \); and by \( D = 4 \), where the potential and centrifugal terms merge and stable solutions do not exist (as reviewed in \([21]\)). We can compare our numerical results for the ground state energy to previous numerical results by Andrew and Supplee \([8]\), who obtained a numerical solution to the Schrödinger equation directly, Morales \([10]\), who used the “shifted 1/d method”, and Waldstein \([22]\), who used the variational method with trial function \( r^n e^{-br} \). The results are given in Table II using the same units \( m = \bar{\mu} = 1 \) and \( 4\pi Z\alpha\mu^2/\Omega_D/1 = 1 \) in \( [3, 10] \).

As an example of the utility of knowing the series expansion for the wave function, we will evaluate \( \langle (V')^2 \rangle_{n0} \), an expectation needed when working out energy corrections for hydrogenic systems at \( O(ma^3) \). This expectation value is easy to evaluate for \( \ell > 0 \) but is divergent, containing a \( 1/\epsilon \), when \( \ell = 0 \), in which case
\[ \langle (V')^2 \rangle_{n0} = \int_0^\infty dr r^{D-1} [V'(r)]^2 \tilde{R}_{n0}^2(r). \] (23)

We write \( \tilde{R}_{n0}(r) = \tilde{\phi}_{n0}^{1/2} \tilde{\gamma} \tilde{R}_{n0}(r) \) and use the series expansions \( L_{n0}(r) = \tilde{L}_{n0}(r) + [L_{n0}(r) - \tilde{L}_{n0}(r)] \) where \( \tilde{L}_{n0}(r) = 1 + \rho/2 - n\rho^{1+2\epsilon}/(2[1 + 2\epsilon]) \). The only divergence comes from the part of (23) containing \( \tilde{L}_{n0}^2(r) \)—the rest is finite and is relatively easy to evaluate for all \( n \). We find
\[ \langle (V')^2 \rangle_{n0} = \pi m(Z\alpha)^3 \tilde{\phi}_{n0}^2 \tilde{\gamma}^2 \times \left\{ -2 - 8L + 8H_n + \frac{4}{3n^2} - \frac{4}{n} - \frac{16}{3} \right\}. \] (24)
This result for $(V')^2_{\alpha 0}$ can also be obtained by a momentum space calculation. Other dimensionally regularized expectation values needed at $O(m\alpha^6)$ coming from second order perturbation theory, not easily accessible to momentum space calculations, can also be obtained by means of the series expansion for the wave function. The detailed information about the short distance behavior of the wave function contained in the double series expansion (9) allows for calculations involving arbitrary values of $n$ to be achieved completely in dimensional regularization.

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[1] J. S. Rigden, Am. J. Phys. 50, 299 (1982).
[2] J. S. Rigden, Hydrogen: The Essential Element, (Harvard University Press, Cambridge, Massachusetts, 2002).
[3] W. E. Caswell and G. P. Lepage, Phys. Lett. B 167, 437 (1986).
[4] G. T. Bodwin, E. Braaten, G. P. Lepage, Phys. Rev. D 51, 1125 (1995).
[5] A. H. Hoang, Heavy Quarkonium Dynamics, Ch. 37 of At the Frontier of Particle Physics: Handbook of QCD, vol. 4, ed. by M. A. Shifman (World Scientific, Singapore, 2002).
[6] A. A. Petrov and A. E. Blechman, Effective Field Theories (World Scientific, Singapore, 2015).
[7] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. A 59, 4316 (1999).
[8] U. D. Jentschura, A. Czarnecki, and K. Pachucki, Phys. Rev. A 72, 062102 (2005).
[9] K. Andrew and J. Supplee, Am. J. Phys. 58, 1177 (1990).
[10] D. A. Morales, Int. J. Quant. Chem. 57, 7 (1996).
[11] S.-H. Dong, Wave Equations in Higher Dimensions, (Springer, Dordrecht, The Netherlands, 2011).
[12] J. D. Louck, J. Mol. Spectroscopy 4, 298 (1960).
[13] J. Avery, Hyperspherical Harmonics: Applications in Quantum Theory, (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1989).
[14] J. S. Avery, J. Comput. Appl. Math. 233, 1366 (2010).
[15] M. Koutras, Discrete Math. 42, 73 (1982).
[16] S. P. Alliluev, J. Exptl. Theor. Phys. 33, 200 (1957) [Sov. Phys. JETP 6, 156 (1958)].
[17] M. M. Nieto, Am. J. Phys. 47, 1067 (1979).
[18] B. R. Johnson and J. O. Hirschfelder, J. Math. Phys. 20, 2484 (1979).
[19] F. J. Asturias and S. R. Aragón, Am. J. Phys. 53, 893 (1985).
[20] K. Eveker, D. Grow, B. Jost, C. E. Monfort III, K. W. Nelson, C. Stroh, and R. C. Witt, Am. J. Phys. 58, 1183 (1990).
[21] M. Bureš, Quantum Physics with Extra Dimensions, (Doctoral Thesis, Masaryk University, 2015).
[22] I. Waldstein, Quantum Mechanics in d-Dimensions, (Franklin & Marshall College Senior Thesis, 2008, unpublished).