UNIFICATION of SPINS AND CHARGES
in GRASSMANN SPACE and
in SPACE of DIFFERENTIAL FORMS

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and

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Abstract

Polynomials in Grassmann space can be used to describe all the internal degrees of freedom of spinors, scalars and vectors, that is their spins and charges \[5\][6]. It was shown\[6\] that Kähler spinors\[4\], which are polynomials of differential forms, can be generalized to describe not only spins of spinors but also spins of vectors as well as spins and charges of scalars, vectors and spinors. If the space (ordinary and noncommutative) has 14 dimensions or more, the appropriate spontaneous break of symmetry leads gravity in \(d\) dimensions to manifest in four dimensional subspace as ordinary gravity and all needed gauge fields as well as the Yukawa couplings. Both approaches, the Kähler’s one (if generalized) and our, manifest four generations of massless fermions, which are left handed \(SU(2)\) doublets and right handed \(SU(2)\) singlets. In this talk a possible way of spontaneously broken symmetries is pointed out on the level of canonical momentum.

1 Introduction.

Our world has besides the ordinary space-time the internal space of spins and charges. Without the internal space, no matter would exist and accordingly no complexity, which is needed for the life to exist. We have shown\[5\] how a space of anticommuting coordinates can be used to describe spins and charges of not only fermions but also of bosons, unifying spins and

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\(^1\)The invited talk, presented on the International workshop Bled99/BSM, "What comes beyond the Standard model" and on the International Conference on Clifford Algebra and their Application in Mathematical Physics, Ixtapa, 27 June-4 July, 1999, Mexico, Dept. of Physics, University of Ljubljana, Jadranska 19, and J. Stefan Institute, Jamova 39, Ljubljana, 1111, Slovenia, and Primorska Institute of Natural Sciences and Technology, C. Marežanskega upora 2, 6000 Koper E-mail: norma.s.mankoc@ijs.si
charges for either fermions or for bosons and that gravity in $d$ dimensions manifests after appropriate break of symmetry in $d = 4$ dimensional subspace as ordinary gravity and all known gauge fields. Kähler has shown how to use differential forms to describe the spin of fermions. In the present talk we point out the analogy and nice relations between the two different ways of achieving the appearance of spin one half degrees of freedom when starting from pure vectors and tensors. We comment the necessity of appearance of four copies of Dirac fermions in both approaches. This work was done together with H. B. Nielsen. Comparing carefully the two approaches we generalize the Kähler approach to describe also integer spins as well as charges for either spinors or vectors, unifying spins and charges.

We present the possible Lagrange function for a free particle and the canonical quantization of anticommuting coordinates. Introducing vielbeins and spin connections, we demonstrate how the spontaneous break of symmetry may lead to the symmetries of the Standard model. In this part of the talk (it has been done together with A. Borštnik), we follow, how the break of symmetries from $SO(1,13)$ to symmetries of the Standard model manifests on canonical momentum. We show how the symmetry of the group $SO(1,13)$ breaks to $SO(1,7)$ (leading to multiplets with left handed $SU(2)$ doublets and right handed $SU(2)$ singlets) and $SO(6)$, which then leads to the $SO(1,3) \times SU(2) \times U(1) \times SU(3) \times U(1)$. The two $U(1)$ symmetries enable besides the hypercharge, needed in the Standard model, additional hypercharge, which is nonzero for right handed $SU(2)$ singlet neutrino. For the pedagogical reasons we comment the break of symmetry on the canonical momentum for the Standard model, that is from $SU(2) \times U(1)$ to $U(1)$ as well.

2 Dirac equations in Grassmann space.

What we call quantum mechanics in Grassmann space is the model for going beyond the Standard Model with extra dimensions of ordinary and anticommuting coordinates, describing spins and charges of either fermions or bosons in an unique way. In a $d$-dimensional space-time the internal degrees of freedom of either spinors or vectors and scalars come from the odd Grassmannian variables $\theta^a, \ a \in \{0, 1, 2, 3, 5, \ldots, d\}$.

We write wave functions describing either spinors or vectors in the form

$$<\theta^a|\Phi> = \sum_{i=0,1,\ldots,3,5,\ldots,d} \sum_{\{a_1<a_2<\ldots<a_i\}\in\{0,1,\ldots,3,5,\ldots,d\}} \alpha_{a_1,a_2,\ldots,a_i} \theta^a \theta^{a_2} \cdots \theta^{a_i},$$

where the coefficients $\alpha_{a_1,a_2,\ldots,a_i}$ depend on commuting coordinates $x^a, \ a \in \{0, 1, 2, 3, 5, \ldots, d\}$. The wave function space spanned over Grassmannian coordinate space has the dimension $2^d$.

Completely analogously to usual quantum mechanics we have the operator for the conjugate variable $\theta^a$ to be

$$p^a_\theta = -i \bar{\partial} \theta_a.$$

The right arrow tells, that the derivation has to be performed from the left hand side. These operators then obey the odd Heisenberg algebra, which written by means of the generalized
commutators

\[ \{A, B\} := AB - (-1)^{n_{AB}}BA, \]

where

\[ n_{AB} = \begin{cases} +1, & \text{if } A \text{ and } B \text{ have Grassmann odd character} \\ 0, & \text{otherwise}, \end{cases} \]

takes the form

\[ \{p^a, p^b\} = 0 = \{\theta^a, \theta^b\}, \quad \{p^a, \theta^b\} = -i\eta^{ab}. \]

Here \( \eta^{ab} \) is the flat metric \( \eta = \text{diag}\{1, -1, -1, \ldots\} \).

We may define the operators

\[ \tilde{a}^a := i(p^a - i\theta^a), \quad \tilde{\tilde{a}}^a := -(p^a + i\theta^a), \]

for which we can show that the \( \tilde{a}^a \)'s among themselves fulfill the Clifford algebra as do also the \( \tilde{\tilde{a}}^a \)'s, while they mutually anticommute:

\[ \{\tilde{a}^a, \tilde{a}^b\} = 2\eta^{ab} = \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}, \quad \{\tilde{a}^a, \tilde{\tilde{a}}^b\} = 0. \]

We could recognize formally

either \( \tilde{a}^a p_a |\Phi\rangle = 0 \), or \( \tilde{\tilde{a}}^a p_a |\Phi\rangle = 0 \)

as the Dirac-like equation, because of the above generalized commutation relations. Applying either the operator \( \tilde{a}^a p_a \) or \( \tilde{\tilde{a}}^a p_a \) on the two equations we get the Klein-Gordon equation

\[ p^a p_a |\Phi\rangle = 0, \]

where we define \( p_a = i\frac{\partial}{\partial x^a} \).

One can check that none of the two equations \( \text{8} \) have solutions which would transform as spinors with respect to the generators of the Lorentz transformations, when taken in analogy with the generators of the Lorentz transformations in ordinary space \( (L^{ab} = x^a p^b - x^b p^a) \)

\[ \tilde{S}^{ab} := \theta^a p^b - \theta^b p^a. \]

But we can write these generators as the sum

\[ \tilde{S}^{ab} = \tilde{\tilde{S}}^{ab} + \tilde{\tilde{S}}^{ab}, \quad \tilde{\tilde{S}}^{ab} := -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b], \quad \tilde{\tilde{S}}^{ab} := -\frac{i}{4}[\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b], \]

with \( [A, B] := AB - BA \) and recognize that the solutions of the two equations \( \text{8} \) now transform as spinors with respect to either \( \tilde{\tilde{S}}^{ab} \) or \( \tilde{\tilde{S}}^{ab} \).

One also can easily see that the untilded, the single tilded and the double tilded \( S^{ab} \) obey the \( d \)-dimensional Lorentz generator algebra

\[ \{M^{ab}, M^{cd}\} = -i(M^{ad} \eta^{bc} + M^{bc} \eta^{ad} - M^{ac} \eta^{bd} - M^{bd} \eta^{ac}), \]

when inserted for \( M^{ab} \).

We shall present our approach in more details in section \( 4 \) when pointing out the similarities between this approach and the Kähler approach and generalizing the Kähler approach. In section \( 6 \) we shall present the Lagrange function, which leads after canonical quantization in both spaces, the ordinary one and the space of anticommuting coordinates, to operators and equations presented in this section.
3 Kähler formulation of spinors.

Kähler formulated spinors in terms of wave functions which are superpositions of the p-forms in the $d = 4$ - dimensional space. The 0-forms are scalars, the 1-forms are defined as dual vectors to the (local) tangent spaces, the higher p-forms are defined as antisymmetrized Cartesian (exterior ($\wedge$)) products of the one-form spaces. A general linear combination of forms is then written

$$u = u_0 + u_1 + \ldots + u_d, \quad u_p = \sum_{i_1 < i_2 \ldots < i_p} a_{i_1 i_2 \ldots i_p} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \ldots \wedge dx^{i_p}. \quad (12)$$

The exterior product has the property of making the product of a p-form and a q-form to be a (p+q)-form, if a p-form and a q-form have no common differentials. One can define also the Clifford product ($\vee$) among the forms. The Clifford product $dx^a \vee$ on a p-form is either a $p+1$ form, if a p-form does not include a one form $dx^a$, or a $p-1$ form, if a one form $dx^a$ is included in a p-form.

Kähler found how the Dirac equation could be written in terms of differential forms

$$-i\delta u = m \vee u, \quad \delta u = \sum_{i=1}^{3} \frac{\partial u}{\partial x^i} \vee dt - dt \vee \frac{\partial u}{\partial t}. \quad (13)$$

with $u$ defined in Eq.(12). The symbol $\delta$ denotes the inner differentiation, $a \in \{0, 1, 2, 3\}$ and $m$ means the electron mass.

For a free massless particle living in a d dimensional space-time Eq.(13) can be rewritten in the form

$$dx^a \vee p_a \quad u = 0, \quad a = 0, 1, 2, 3, 5, \ldots, d. \quad (14)$$

The wave function describing the state of the spin one half particle is packed into the exterior algebra function $u$.

4 Parallelism between the two approaches.

We demonstrate the parallelism between the Kähler and our approach in steps, first paying attention on spin $\frac{1}{2}$ only, as Kähler did. Using simple and transparent definitions of the exterior and interior product in Grassmann space, we generalize the Kähler approach first by defining the two kinds of $\delta$ (Eq.(13)) operators on the space of p-forms and accordingly three kinds of the generators of the Lorentz transformations, two of the spinorial and one of the vectorial character. We try to put clearly forward how the spinorial degrees of freedom emerge out of vector objects like the 1-forms or $\theta^a$'s. We then generalize the p-forms to describe not only spins but also charges of spin $\frac{1}{2}$ and spin 0 and 1 objects, unifying also in the space of forms spins and charges, separately for fermions and separately for bosons.
4.1 Dirac-Kähler equation and Dirac equation in Grassmann space for massless particles.

We present here, side by side, the operators in the space of differential forms and in Grassmann space: the "exterior" product

\[ dx^a \wedge dx^b \wedge \ldots, \quad \theta^a \theta^b \ldots, \]

the operator of "differentiation"

\[ -i e^a, \quad p^{\theta a} = -i \frac{\partial}{\partial \theta^a}, \]

and the two superpositions

\[ dx^a \tilde{\wedge} := dx^a \wedge + e^a, \quad \tilde{a}^a := i (p^{\theta a} - i \theta^a), \]
\[ dx^a \tilde{\wedge} := i (dx^a \wedge - e^a), \quad \tilde{\tilde{a}}^a := -(p^{\theta a} + i \theta^a). \]

The superposition, which we signed by \( \tilde{\cdot} \) is the one used by Kähler (Eqs.(13)).

One easily finds (see Eqs.(6,7)) the commutation relations, understood in the generalized sense of Eq.(3)

\[ \{ dx^a \tilde{\wedge}, dx^b \tilde{\wedge} \} = 2 \eta^{ab}, \quad \{ \tilde{a}^a, \tilde{b}^b \} = 2 \eta^{ab}, \]
\[ \{ dx^a \tilde{\wedge}, dx^b \tilde{\wedge} \} = 2 \eta^{ab}, \quad \{ \tilde{\tilde{a}}^a, \tilde{\tilde{b}}^b \} = 2 \eta^{ab}. \]

Since \( \{ e^a, dx^b \wedge \} = \eta^{ab} \) and \( \{ e^a, e^b \} = 0 = \{ dx^a \wedge, dx^b \wedge \} \), while \( \{-i p^{\theta a}, \theta^b \} = \eta^{ab} \) and \( \{ p^\theta a, p^\theta b \} = 0 = \{ \theta^a, \theta^b \} \), it is obvious that \( e^a \) plays in the p-form formalism the role of the derivative with respect to a differential 1-form, similarly as \( ip^{\theta a} \) does with respect to a Grassmann coordinate.

We find for both approaches the Dirac-like equations:

\[ dx^a \tilde{\wedge} p_a u = 0, \quad \tilde{a}^a p_a \Phi(\theta^a) = 0, \]
\[ dx^a \tilde{\wedge} p_a u = 0, \quad \tilde{\tilde{a}}^a p_a \Phi(\theta^a) = 0. \]

Taking into account the above definitions it follows that

\[ dx^a \tilde{\wedge} p_a \quad dx^b \tilde{\wedge} p_b u = p^a p_a u = 0, \quad \tilde{a}^a p_a \tilde{b}^b p_b \Phi(\theta^b) = p^a p_a \Phi(\theta^b) = 0. \]

We see that either \( dx^a \tilde{\wedge} p_a \quad u = 0 \) or \( dx^a \tilde{\wedge} p_a \quad u = 0 \), similarly as either \( \tilde{a}^a p_a \Phi(\theta^a) = 0 \) or \( \tilde{\tilde{a}}^a p_a \Phi(\theta^a) = 0 \) can represent the Dirac-like equation.

Both, \( dx^a \tilde{\wedge} \) and \( dx^a \tilde{\wedge} \) define the algebra of the \( \gamma^a \) matrices and so do both \( \tilde{a}^a \) and \( \tilde{\tilde{a}}^a \). One would thus be tempted to identify

\[ \gamma^a_{\text{naive}} := dx^a \tilde{\wedge}, \quad \text{or} \quad \gamma^a_{\text{naive}} := \tilde{a}^a. \]
But there is a large freedom in defining what to identify with the gamma-matrices, because except when using $\gamma^0$ as a parity operation, one has an even number of gamma matrices occurring in the physical applications such as construction of currents $\bar{\psi}\gamma^a\psi$ or for the Lorentz generators on spinors $\frac{i}{4}[\gamma^a, \gamma^b]$. Then all the gamma matrices can be multiplied by some factor provided it does disturb neither their algebra nor their even products. This freedom might be used to solve, what seems a problem:

Having an odd Grassmann character, neither $\tilde{a}^a$ nor $\tilde{\tilde{a}}^a$ and similarly neither $dx^a \tilde{\nu}$ nor $dx^a \tilde{\tilde{\nu}}$ should be recognized as the Dirac $\gamma^a$ operators, since they would change, when operating on polynomials of $\theta^a$ or on superpositions of p-form, objects of an odd Grassmann character to objects of an even Grassmann character. One would, however, expect - since Grassmann odd fields second quantize to fermions, while Grassmann even fields second quantize to bosons - that the $\gamma^a$ operators do not change the Grassmann character of wave functions so that the canonical quantization of Grassmann odd fields then automatically assures the anticommuting relations between the operators of the fermionic fields.

We may propose that accordingly

\[
\text{either } \tilde{\gamma}^a := i dx^0 \tilde{\nu} dx^a \tilde{\nu}, \quad \text{or } \tilde{\gamma}^a = i \tilde{\tilde{a}}^0 \tilde{a}^a
\]  

(22)

are recognized as the Dirac $\gamma^a$ operators operating on the space of p-forms or polynomials of $\theta^a$'s, respectively, since they both have an even Grassmann character and they both fulfill the Clifford algebra $\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab}$. ( The role of $\tilde{\nu}$ and $\tilde{\tilde{\nu}}$ can in either the Kähler case or the case of polynomials in Grassmann space, be exchanged. )

The two definitions of gamma-matrices ((22), (21)) make only a difference when $\gamma^0$-matrix is used alone. This $\gamma^0$-matrix has to simulate the parity reflection which is

\[
\text{either } \tilde{dx} \rightarrow -\tilde{dx}, \quad \text{or } \tilde{\theta} \rightarrow -\tilde{\theta}.
\]  

(23)

The "ugly" gamma-matrix identifications (22) indeed perform this operation.

Kähler did not connect evenness and oddness of the forms with the statistics. He used the "naive" gamma-matrix identifications (21). The same can be said for the Becher-Joos ([1]) paper.

### 4.2 Generators of Lorentz transformations.

We are presenting the generators of the Lorentz transformations of spinors for both approaches

\[
M^{ab} = L^{ab} + S^{ab}, \quad L^{ab} = x^a p^b - x^b p^a.
\]  

(24)

The two approaches differ in the definition of the generators of the Lorentz transformations in the internal space $S^{ab}$. While Kähler suggested the definition for spin $\frac{1}{2}$ particles

\[
S^{ab} = dx^a \wedge dx^b, \quad S^{ab}u = \frac{1}{2}((dx^a \wedge dx^b) \lor u - u \lor (dx^a \wedge dx^b)),
\]  

(25)
in the Grassmann case the two kinds of the operators $S^{ab}$ for spinors can be defined, presented in Eqs. (11), with the properties

$$[\tilde{S}^{ab}, \tilde{a}^c] = i(\gamma^a \tilde{a}^b - \gamma^b \tilde{a}^a), \quad [\tilde{\tilde{S}}^{ab}, \tilde{a}^c] = i(\gamma^a \tilde{\tilde{a}}^b - \gamma^b \tilde{\tilde{a}}^a), \quad [\tilde{S}^{ab}, \tilde{\tilde{S}}^{ab}] = 0 = [\tilde{\tilde{S}}^{ab}, \tilde{a}^c].$$ (26)

Following the approach in Grassmann space one can also in the Kähler case define two kinds of the Lorentz generators for spinors, which (both) simplify Eq. (25)

$$\tilde{S}^{ab} = -\frac{i}{4}[dx^a \wedge + e^a, \ dx^b \wedge + e^b], \quad \tilde{\tilde{S}}^{ab} = \frac{i}{4}[dx^a \wedge - e^a, \ dx^b \wedge - e^b],$$

$$\tilde{S}^{ab} = -\frac{i}{4}[\gamma^a, \gamma^b].$$ (27)

The above definition enables us to define also in the Kähler case the generators of the Lorentz transformations of the vectorial character

$$S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab} = -i(dx^a \wedge e^b - dx^b \wedge e^a), \quad S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a}. \quad (28)$$

The operator $S^{ab} = -i(dx^a \wedge e^b - dx^b \wedge e^a)$, being applied on differential $p$-forms, transforms vectors into vectors.

### 4.3 Scalar product.

In our approach the scalar product between the two functions $\langle \theta^a | \Phi_1 \rangle$ and $\langle \theta^a | \Phi_2 \rangle$ is defined

$$\langle \Phi_1 | \Phi_2 \rangle = \int d^d \theta \ (\omega \ <\theta^a | \Phi_1 > \ <\theta^a | \Phi_2 >)$$ (29)

and $\omega$ is a weight function

$$\omega = \prod_{i=0,1,..,d} (\theta^i + \bar{\theta}^i),$$ (30)

which operates on only the first function $\langle \theta^a | \Phi_1 \rangle$ and

$$\int d\theta^a = 0, \quad \int d^d \theta \theta^0 \theta^1 .. \theta^d = 1, \quad d^d \theta = \theta^d .. \theta^1 \theta^0. \quad (31)$$

According to the above definition and Eq. (11) it follows

$$\langle \Phi^{(1)} | \Phi^{(2)} \rangle = \sum_{0, \alpha_1<\alpha_2<..<\alpha_d} \sum_{\alpha_1 .. \alpha_i} \alpha^{(1)*}_{\alpha_1 .. \alpha_i} \alpha^{(2)}_{\alpha_1 .. \alpha_i}$$ (32)

in complete analogy with the usual definition of the scalar product in ordinary space. Kähler defined the scalar product of two $p$-forms (Eq. (12)) as

$$\langle u^{(1)} | u^{(2)} \rangle = \sum_{0, \alpha_1<\alpha_2<..<\alpha_d} \sum_{\alpha_1 .. \alpha_i} \alpha^{(1)*}_{\alpha_1 .. \alpha_i} \alpha^{(2)}_{\alpha_1 .. \alpha_i},$$ (33)

which agrees with Eq. (32).
4.4 Four copies of Weyl bi-spinors in Kähler or in approach in Grassmann space and vector representations.

In the case of $d = 4$ one may arrange the space of $2^d$ vectors into four copies of two Weyl spinors, one left ($\langle \tilde{\Gamma}^{(4)} \rangle = -1$, $\Gamma^{(4)} = i \frac{(-2)^2}{4!} \epsilon_{abcd} S^{ab} S^{cd}$) and one right ($\langle \tilde{\Gamma}^{(4)} \rangle = 1$) handed (we have made a choice of $\tilde{\Gamma}$), in such a way that they are at the same time the eigen vectors of the operators $\tilde{S}^{12}$ and the $\tilde{S}^{03}$ and have either an odd or an even Grassmann character. These vectors are in the Kähler approach the superpositions of p-forms and in our\footnote{3} approach the polynomials of $\theta^m$'s, $m \in \{0, 1, 2, 3\}$. The two Weyl vectors of one copy of the Weyl bi-spinors are connected by the $\tilde{\gamma}^m$ (Eq. (22)) operators, while the two copies of different Grassmann character are connected by $\tilde{\alpha}^a$ or $dx^a \check{\nu}$, respectively. The two copies of an even Grassmann character are connected by the (a kind of a time reversal operation) $\theta^0 \rightarrow -\theta^0$ or equivalently $dx^0 \rightarrow -dx^0$.

We present in Table I four copies of the Weyl two spinors as polynomials of $\theta^a$. Replacing $\theta^a$'s by $dx^a \wedge$ the presentation for differential forms follow. Eigenstates are orthonormalized according to the scalar product of Eq. (32).
| a  | i  | $< \theta| a \Phi_i >$ | $\tilde{S}^{12}$ | $\tilde{S}^{03}$ | $\tilde{\Gamma}^{(4)}$ | family | Grass. cha. |
|----|----|---------------------|----------------|----------------|----------------|--------|------------|
| 1  | 1  | $\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | -1 | | even |
| 1  | 2  | $-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | -1 | | |
| 2  | 1  | $\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$ | $\frac{1}{2}$ | $\frac{i}{2}$ | 1 | | |
| 2  | 2  | $-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | 1 | | |
| 3  | 1  | $\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$ | $\frac{1}{2}$ | $\frac{i}{2}$ | 1 | | |
| 3  | 2  | $-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | 1 | | |
| 4  | 1  | $\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | -1 | | |
| 4  | 2  | $-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | -1 | | |
| 5  | 1  | $\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | -1 | | |
| 5  | 2  | $-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | -1 | | |
| 6  | 1  | $\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$ | $\frac{1}{2}$ | $\frac{i}{2}$ | 1 | | |
| 6  | 2  | $-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | 1 | | |
| 7  | 1  | $\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$ | $\frac{1}{2}$ | $\frac{i}{2}$ | 1 | | |
| 7  | 2  | $-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | 1 | | |
| 8  | 1  | $\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | -1 | | |
| 8  | 2  | $-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | -1 | | |

Table I: The polynomials of $\theta^m$, representing the four times two Weyl spinors, are written. For each state the eigenvalues of $\tilde{S}^{12}, \tilde{S}^{03}, \tilde{\Gamma}^{(4)} := i\tilde{a}^0\tilde{a}^1\tilde{a}^2\tilde{a}^3$ are written. The Roman numerals tell the possible family number. We use the relation $\tilde{a}^a|0> = \theta^a$.

Analyzing the irreducible representations of the group $SO(1, 3)$ with respect to the generator of the Lorentz transformations of the vectorial type (Eqs. (28)) one finds for $d = 4$ two scalars ( a scalar and a pseudo scalar), two three vectors (in the $SU(2) \times SU(2)$ representation of $SO(1, 3)$ denoted by $(1, 0)$ and $(0, 1)$ representation, respectively, with $<\Gamma^{(4)}> = \pm 1$) and two four vectors. One can find the polynomial representation for this case in ref. [3].
4.5 Generalization to extra dimensions.

It has been suggested\[5\] that the Lorentz transformations in the space of \(\theta^a\)'s in \(d-4\) dimensions manifest themselves as generators for charges observable for the four dimensional particles. Since both the extra dimension spin degrees of freedom and the ordinary spin degrees of freedom originate from the \(\theta^a\)'s or the forms we have a unification of these internal degrees of freedom.

Let us take as an example the model\[5\] which has \(d=14\) and at first - at the high energy level - \(SO(1,13)\) Lorentz group, but which should be broken (in two steps) to first \(SO(1,7) \times SO(6)\) and then to \(SO(1,3) \times SU(3) \times SU(2)\). We shall comment on this model in section 8.

5 Appearance of spinors.

One of course wonders about how it is at all possible that the Dirac equation appears for a spinor field out of models with only scalar, vector and tensor objects! It only can be done by exchanging the Lorentz generators \(S^{ab}\) by the \(\tilde{S}^{ab}\) say (or the \(\tilde{\tilde{S}}^{ab}\) if we choose them instead), see equations (10, 27). This indeed means that one of the two kinds of operators fulfilling the Clifford algebra and anticommuting with the other kind - it has been made a choice of \(dx^a \tilde{\nabla}\) in the Kähler case and \(\tilde{a}^a\) in our approach - are put to zero in the operators of the Lorentz transformations; as well as in all the operators representing physical quantities. The use of \(dx^a \tilde{\nabla}\) or \(\tilde{a}^0\) in the operator \(\tilde{\gamma}^0\) is the exception, only used to simulate the Grassmann even parity operation \(dx^a \rightarrow -dx^a\) and \(\tilde{\theta} \rightarrow -\tilde{\theta}\), respectively.

We shall argue away (\[5\]) the \(\tilde{a}^a\)'s in section 6 on the ground of the action.

6 Lagrange function for a free massless particle in ordinary and Grassmann space and canonical quantization.

We present in this section the Lagrange function for a particle which lives in a d-dimensional ordinary space of commuting coordinates and in a d-dimensional Grassmann space of anticommuting coordinates \(X^a \equiv \{x^a, \theta^a\}\) and has its geodesics parametrized by an ordinary Grassmann even parameter \((\tau)\) and a Grassmann odd parameter\((\xi)\). We derive the Hamilton function and the corresponding Poisson brackets and perform the canonical quantization, which leads to the Dirac equation with operators presented in sections 2, 4.

\(X^a = X^a(x^a, \theta^a, \tau, \xi)\) are called supercoordinates. We define the dynamics of a particle by choosing the action \([4, 3]\) \(I = \frac{1}{2} \int d\tau d\xi E^i_A \partial_i X^a E^j_B \partial_j \xi \eta_{ab} \eta^{AB}\), where \(\partial_i := (\partial_\tau, \partial_\xi)\), \(\tau^i \equiv (\tau, \xi)\), while \(E^i_A\) determines a metric on a two dimensional superspace \(\tau^i\), \(E = \det(E^i_A)\). We choose \(\eta_{AA} = 0, \eta_{12} = 1 = \eta_{21}\), while \(\eta_{ab}\) is the Minkowski metric with the diagonal elements \((1, -1, -1, -1, ..., -1)\). The action is invariant under the Lorentz transformations

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of supercoordinates: \( X'^a = \Lambda^a_b X^b \). Since a supermatrix \( E^i_A \) transforms as a vector in a two-dimensional superspace \( \tau \) under general coordinate transformations of \( \tau \), \( E^i_A \) is invariant under such transformations and so is \( d^2 \tau E \). The action is locally supersymmetric. The inverse matrix \( E^A_i \) is defined as follows: \( E^i_A E^A_i = \delta^i_B \).

Taking into account that either \( x^a \) or \( \theta^a \) depend on an ordinary time parameter \( \tau \) and that \( \xi^2 = 0 \), the geodesics can be described as a polynomial of \( \xi \) as follows:

\[
X^a = x^a + \varepsilon \xi \theta^a.
\]

We choose \( \varepsilon^2 \) to be equal either to \(+i\) or to \(-i\) so that it defines two possible combinations of supercoordinates. Accordingly we also choose the metric \( E^i_A \):

\[
E^{11} = 1, \quad E^{12} = -\varepsilon M, \quad E^{21} = \xi, \quad E^{22} = N - \varepsilon M,
\]

with \( N \) and \( M \) Grassmann even and odd parameters, respectively. We write \( \dot{A} = \frac{d}{d \tau} A \), for any \( A \).

If we integrate the above action over the Grassmann odd coordinate \( d\xi \), the action for a superparticle follows:

\[
\int d\tau \left( \frac{1}{N} \dot{x}^a \dot{x}_a + \varepsilon^2 \dot{\theta}^a \theta_a - \frac{2 \varepsilon^2 M}{N} \dot{x}^a \theta_a \right).
\]

(34)

Defining the two momenta

\[
p_\theta^a := \overrightarrow{\partial L}{\partial \dot{\theta}}^a = \varepsilon^2 \theta^a, \quad p_a := \frac{\partial L}{\partial \dot{x}^a} = \frac{2}{N}(\dot{x}_a - M p_\theta a),
\]

(35)

the two Euler-Lagrange equations follow:

\[
\frac{dp_\theta^a}{d\tau} = 0, \quad \frac{dp_a}{d\tau} = \varepsilon^2 M \frac{p^\theta_a}{2 p^a}.
\]

(36)

Variation of the action (Eq.(34)) with respect to \( M \) and \( N \) gives the two constraints

\[
\chi^1 := p_\theta^a a_\theta^a = 0, \quad \chi^2 := p^a p_a = 0, \quad a_\theta^a := i p_\theta^a + \varepsilon^2 \theta_a,
\]

(37)

while \( \chi^3_a := -p_\theta^a + \varepsilon^2 \theta_a = 0 \) (Eq.(35)) is the third type of constraints of the action(34). For \( \varepsilon^2 = -i \) we find that \( a_\theta^a = \tilde{a}_a \), which agrees with Eq.(3), while \( \chi^3_a = \tilde{a}_a = 0 \), which makes a choice between \( \tilde{a}_a \) and \( \tilde{a}_a \).

We find the generators of the Lorentz transformations for the action(34) to be

\[
M^{ab} = L^{ab} + S^{ab}, \quad L^{ab} = x^a p^b - x^b p^a, \quad S^{ab} = \theta^a p^b - \theta^b p^a = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab},
\]

(38)

which agree with definitions in Eq.(11) and show that parameters of the Lorentz transformations are the same in both spaces.

We define the Hamilton function:

\[
H := \dot{x}^a p_a + \dot{\theta}^a p_\theta a - L = \frac{1}{4} N p_\theta^a p_a + \frac{1}{2} M p^a (\tilde{a}_a + i \tilde{\tilde{a}}_a)
\]

(39)

and the corresponding Poisson brackets

\[
\{A, B\}_p = \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a} \frac{\partial B}{\partial x^a} + \frac{\partial \tilde{A}}{\partial \theta^a} \frac{\partial \tilde{B}}{\partial p_\theta a} + \frac{\partial \tilde{\tilde{A}}}{\partial p_\theta a} \frac{\partial \tilde{\tilde{B}}}{\partial \theta^a},
\]

(40)
which fulfill the algebra of the generalized commutators of Eq. (3).

If we take into account the constraint $\chi^3_a = \tilde{a}_a = 0$ in the Hamilton function (which just means that instead of $H$ the Hamilton function $H + \sum \alpha^i \chi^i + \sum_a \alpha^3_a \chi^3_a$ is taken, with parameters $\alpha^i, i = 1, 2$ and $\alpha^3_a = -\frac{M_p}{\tau} p_a$, $a = 0, 1, 2, 3, 5, \ldots$, chosen on such a way that the Dirac brackets of the three types of constraints with the new Hamilton function are equal to zero) and in all dynamical quantities, we find:

\begin{equation}
H = \frac{1}{4} N p^a p_a + \frac{1}{2} M p^a \tilde{a}_a,
\chi^1 = p^a p_a = 0,
\chi^2 = p^a \tilde{a}_a = 0,
\end{equation}

which agrees with the Euler Lagrange equations (46).

We further find

\begin{equation}
\chi^i = \{H, \chi^i\}_P = 0, \quad i = 1, 2, \quad \chi^3_a = \{H, \chi^3_a\}_P = 0, \quad a = 0, 1, 2, 3, 5, \ldots, 
\end{equation}

which guarantees that the three constraints will not change with the time parameter $\tau$ and that $\tilde{M}^{ab} = 0$, with $\tilde{M}^{ab} = L^{ab} + S^{ab}$, saying that $\tilde{M}^{ab}$ is the constant of motion.

The Dirac brackets, which can be obtained from the Poisson brackets of Eq.(40) by adding to these brackets on the right hand side a term $-\{A, \tilde{a}_a\}_P$: $\{\tilde{a}_a, B\}_P$, give for the dynamical quantities, which are observables, the same results as the Poisson brackets. This is true also for $\tilde{a}_a$, $\{\tilde{a}_a, \tilde{b}_b\}_P = i \eta^{ab} = \{\tilde{a}_a, \tilde{a}_b\}_P$, which is the dynamical quantity but not an observable since its odd Grassmann character causes supersymmetric transformations. We also find that $\{\tilde{a}_a, \tilde{b}_b\}_P = 0 = \{\tilde{a}_a, \tilde{b}_b\}_P$. The Dirac brackets give different results only for the quantities $\theta^a$ and $p^b$ and for $\tilde{a}_a$ among themselves: $\{\theta^a, p^b\}_P = \gamma^{ab}, \{\theta^a, p^b\}_D = \frac{1}{2} \eta^{ab}, \{\tilde{a}_a, \tilde{b}_b\}_P = 2 i \eta^{ab}, \{\tilde{a}_a, \tilde{b}_b\}_D = 0$. According to the above properties of the Poisson brackets, we suggested that in the quantization procedure the Poisson brackets (41) rather than the Dirac brackets are used, so that variables $\tilde{a}_a$, which are removed from all dynamical quantities, stay as operators. Then $\tilde{a}_a$ and $\tilde{a}_a$ are expressible with $\theta^a$ and $p^b$ (Eq.(41)) and the algebra of linear operators introduced in sections 3, 4 can be used. We shall show, that suggested quantization procedure leads to the Dirac equation, which is the differential equation in ordinary and Grassmann space and has all desired properties.

In the proposed quantization procedure $-i\{A, B\}_P$ goes to either a commutator or to an anticommutator, according to the Poisson brackets (41). The operators $\theta^a, p^b$ (in the coordinate representation they become $\theta^a \rightarrow \theta^a, p^a \rightarrow \frac{\partial}{\partial \theta^a}$) fulfill the Grassmann odd Heisenberg algebra, while the operators $\tilde{a}_a$ and $\tilde{a}_a$ fulfill the Clifford algebra (Eq.(41)).

The constraints (Eqs.(37)) lead to the Weyl-like and the Klein-Gordon equations

\begin{equation}
p^a \tilde{a}_a |\tilde{\Phi} > = 0, \quad p^a p_a |\tilde{\Phi} > = 0, \quad \text{with} \quad p^a \tilde{a}_a p^b \tilde{a}_b = p^a p_a.
\end{equation}

Trying to solve the eigenvalue problem $\tilde{a}_a |\tilde{\Phi} > = 0, \quad a = (0, 1, 2, 3, 5, \ldots, d)$, we find that no solution of this eigenvalue problem exists, which means that the third constraint $\tilde{a}_a = 0$ can’t
be fulfilled in the operator form (although we take it into account in the operators for all dynamical variables in order that operator equations would agree with classical equations). We can only take it into account in the expectation value form

\[ \langle \tilde{\Phi} | \tilde{a}^a | \tilde{\Phi} \rangle = 0. \]  \hspace{1cm} (45)

Since \( \tilde{a}^a \) are Grassmann odd operators, they change monomials (Eq.(1)) of an Grassmann odd character into monomials of an Grassmann even character and opposite, which is the supersymmetry transformation. It means that Eq.(45) is fulfilled for monomials of either odd or even Grassmann character and that superpositions of the Grassmann odd and the Grassmann even monomials are not solutions for this system.

We define the projectors

\[ P_{\pm} = \frac{1}{2} (1 \pm \sqrt{(-)^{\frac{d}{2}} \tilde{\Upsilon} \tilde{\Upsilon} - \tilde{\Upsilon} \tilde{\Upsilon}}), \]  \hspace{1cm} (46)

where \( \tilde{\Upsilon} \) and \( \tilde{\Upsilon} \) are the two operators defined for any dimension \( d \) as follows

\[ \tilde{\Upsilon} = i^a \prod_{a=0,1,2,3,5,...,d} \tilde{a}^a \sqrt{\eta^{aa}}; \hspace{1cm} \tilde{\Upsilon} = i^a \prod_{a=0,1,2,3,5,...,d} \tilde{a}^a \sqrt{\eta^{aa}}, \]  

with \( \alpha \) equal either to \( \frac{d}{2} \) or to \( \frac{d-1}{2} \) for even and odd dimension \( d \) of the space, respectively. It can be checked that \( (\tilde{\Upsilon})^2 = 1 = (\tilde{\Upsilon})^2 \).

We can use the projector \( P_{\pm} \) of Eq.(46) to project out of monomials either the Grassmann odd or the Grassmann even part. Since this projector commutes with the Hamilton function \( \{ P_{\pm}, H \} = 0 \), it means that eigenfunctions of \( H \), which fulfil the Eq.(45), have either an odd or an even Grassmann character. In order that in the second quantization procedure fields \( | \tilde{\Phi} \rangle \) would describe fermions, it is meaningful to accept in the fermion case Grassmann odd monomials only. (See discussions in ref.\([6]\).)

### 7 Particles in gauge fields.

The dynamics of a point particle in gauge fields, the gravitational in \( d \)-dimensions, which then, as we shall show, manifests in the subspace \( d = 4 \) as ordinary gravity and all the Yang-Mills fields, can be obtained by transforming vectors from a freely falling to an external coordinate system \([9]\). To do this, supervielbeins \( e^a_{\mu} \) have to be introduced, which in our case depend on ordinary and on Grassmann coordinates, as well as on two types of parameters \( \tau^i = (\tau, \xi) \). The index \( a \) refers to a freely falling coordinate system (a Lorentz index), the index \( \mu \) refers to an external coordinate system (an Einstein index).

We write the transformation of vectors as follows

\[ \partial_i X^a = e^a_{\mu} \partial_i X^\mu, \hspace{0.5cm} \partial_i X^\mu = f^\mu_{\nu a} \partial_i X^a, \hspace{0.5cm} \partial_i = (\partial_\tau, \partial_\xi). \]

From here it follows that

\[ e^a_{\mu} f^\mu_{\nu b} = \delta^a_{b}, \hspace{0.5cm} f^\mu_{a} e^a_{\nu} = \delta^\mu_{\nu}. \]

Again we make a Taylor expansion of vielbeins with respect to \( \xi \),

\[ e^a_{\mu} = e^a_{\mu} + \varepsilon \xi \theta^b e^a_{\mu b}, \hspace{0.5cm} f^\mu_{a} = f^\mu_{a} - \varepsilon^2 \xi \theta^b f^\mu_{ab}. \]

Both expansion coefficients again depend on ordinary and on Grassmann coordinates. Having an even Grassmann character \( e^a_{\mu} \) will describe the spin 2 part of a gravitational field. The coefficients \( e^a_{\mu b} \) define the spin connections \([5]\).
It follows that \( e^a \mu f^\mu_b = \delta^a_b \), \( f^\mu_a e^a_\nu = \delta^\mu_\nu \), \( e^a\mu f^\mu_c = e^a_\mu f^\mu_{cb} \).

We find the metric tensor \( g_{\mu\nu} = e^a_\mu e^a_\nu \), \( g^{\mu\nu} = f^\mu_a f^{\nu}_a \).

Rewriting the action from section 6 in terms of an external coordinate system, using the Taylor expansion of supercoordinates \( X^\mu \) and superfields \( e^a_\mu \) and integrating the action over the Grassmann odd parameter \( \xi \), the action follows

\[
I = \int d\tau \left\{ \frac{1}{N} g^{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{\varepsilon^2}{2M} \theta_a e^a_\mu \dot{x}^\mu + \varepsilon^2 \left( \dot{\theta}^a \theta_a - \theta_a \dot{\theta}^a \right) e^a_\mu + \right.
\]

\[
+ \varepsilon^2 \frac{1}{2} \left( \theta^a \theta_a - \theta_a \theta^a \right) e^a_\mu \dot{x}^\mu \right\},
\]

which defines the two momenta of the system \( p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = p_{0\mu} + \frac{1}{2} \tilde{S}^{ab} e_{a\mu b} \), \( p^0_\mu = -i \theta_a e^a_\mu \) ( \( \varepsilon^2 = -i \)). Here \( p_{0\mu} \) are the canonical (covariant) momenta of a particle. For \( p^0_\mu = p^0_\mu f^\mu_a \), it follows that \( p^0_\mu \) is proportional to \( \theta_a \). Then \( \tilde{a}_a = i(p^0_a - i \theta_a) \), while \( \tilde{a}_a = 0 \). We may further write

\[
p_{0\mu} = p_\mu - \frac{1}{2} \tilde{S}^{ab} e_{a\mu b} = p_\mu - \frac{1}{2} \tilde{S}^{ab} \omega_{a\mu b} , \quad \omega_{a\mu b} = \frac{1}{2} ( e_{a\mu b} - e_{b\mu a} ),
\]

which is the usual expression for the covariant momenta in gauge gravitational fields. One can find the two constraints

\[
p^0_\mu p_{0\mu} = 0 = p_{0\mu} f^\mu_a \tilde{a}_a \).
\]

We shall comment on the break of symmetries which leads in \( d = 4 \) dimensional subspace as ordinary gravity and all the gauge field in section 8.

8 Breaking \( SO(1,13) \) through \( SO(1,7) \times SO(6) \) to \( SO(1,3) \times SU(2) \times U(1) \times SU(3) \).

In this section, we shall first discuss a possible break of symmetry, which leads from the unified theory of only spins and gravity in \( d \) dimensions to spins and charges and to the symmetries and assumptions of the Standard model, on the algebraic level. We shall then comment on the break of symmetries on the level of canonical momentum, first for the Standard model case, to only demonstrate the way of the break, and then for the general case, that is for the particle in the presence of the gravitational field.

We shall present as well the possible explanation for that postulate of the Standard model, which requires that only left handed weak charged massless doublets and right handed weak charged massless singlets exist, and accordingly connect spins and charges of fermions.

8.1 Algebraic considerations of symmetries.

The algebra of the group \( SO(1, d-1) \) or \( SO(d) \) contains \( n \) subalgebras defined by operators \( \tau^{Ai}, A = 1, n; i = 1, n_A \), where \( n_A \) is the number of elements of each subalgebra, with the properties

\[
[\tau^{Ai}, \tau^{Bj}] = i \delta^{AB} f^{Ai} \dot{r}^{Ak},
\]
if operators $\tau^{Ai}$ can be expressed as linear superpositions of operators $M^{ab}$

$$
\tau^{Ai} = c^{Ai}_{ab} M^{ab}, \quad c^{Ai}_{ab} = -c^{Ai}_{ba}, \quad A = 1, n, \quad i = 1, n_A, \quad a, b = 1, d.
$$

(51)

Here $f^{Aijk}$ are structure constants of the $(A)$ subgroup with $n_A$ operators. According to the three kinds of operators $S^{ab}$, two of spinorial and one of vectorial character, there are three kinds of operators $\tau^{Ai}$ defining subalgebras of spinorial and vectorial character, respectively, those of spinorial types being expressed with either $\tilde{S}^{ab}$ or $\check{S}^{ab}$ and those of vectorial type being expressed by $S^{ab}$. All three kinds of operators are, according to Eq. (50), defined by the same coefficients $c^{Ai}_{ab}$ and the same structure constants $f^{Aijk}$. From Eq. (50), the following relations among constants $c^{Ai}_{ab}$ follow

$$
-4 c^{Ai}_{ab} c^{Bjbc} - \delta^{AB} f^{Aijk} c^{Akac} = 0.
$$

(52)

When we look for coefficients $c^{Ai}_{ab}$ which express operators $\tau^{Ai}$, forming a subalgebra $SU(n)$ of an algebra $SO(2n)$ in terms of $M^{ab}$, the procedure is rather simple [2, 5]. We find:

$$
\tau^{Am} = \frac{i}{2} (\tilde{\sigma}^{Am})_{jk} \{ M^{(2j-1)(2k-1)} + M^{(2j)(2k)} + iM^{(2j)(2k-1)} - iM^{(2j-1)(2k)} \}. 
$$

(53)

Here $(\tilde{\sigma}^{Am})_{jk}$ are the traceless matrices which form the algebra of $SU(n)$. One can easily prove that operators $\tau^{Am}$ fulfill the algebra of the group $SU(n)$ for any of three choices for operators $M^{ab} : S^{ab}, \tilde{S}^{ab}, \check{S}^{ab}$.

While the coefficients are the same for all three kinds of operators, the representations depend on the operators $M^{ab}$. After solving the eigenvalue problem for invariants of subgroups, the representations can be presented as polynomials of coordinates $\theta^a$, or $dx^a \wedge$, $a = 0, 1, 2, 3, 5, ... , 14$. The operators of spinorial character define the fundamental representations of the group and the subgroups, while the operators of vectorial character define the adjoint representations of the groups. We shall from now on, for the sake of simplicity, refer to the polynomials of Grassmann coordinates only.

We first analyze the space of $2^d$ vectors for $d = 14$ with respect to commuting operators (Casimirs) of subgroups $SO(1, 7)$ and $SO(6)$, so that polynomials of $\theta^0, \theta^1, \theta^2, \theta^3, \theta^5, \theta^6, \theta^7$ and $\theta^8$ are used to describe states of the group $SO(1, 7)$ and then polynomials of $\theta^9, \theta^{10}, \theta^{11}, \theta^{12}, \theta^{13}$ and $\theta^{14}$ further to describe states of the group $SO(6)$. The group $SO(1, 13)$ has the rank equal to $r = 7$, since it has 7 commuting operators (namely for example $S^{01}, S^{12}, S^{35}, ... , S^{13 14}$), while the ranks of the subgroups $SO(1, 7)$ and $SO(6)$ are accordingly $r = 4$ and $r = 3$, respectively. We may further decide to arrange the basic states in the space of polynomials of $\theta^0, ... , \theta^8$ as eigenstates of 4 Casimirs of the subgroups $SO(1, 3), SU(2)$, and $U(1)$ (the first has $r = 2$, the second and the third have $r = 1$) of the group $SO(1, 7)$, and the basic states in the space of polynomials of $\theta^9, ... , \theta^{14}$ as eigenstates of $r = 3$ Casimirs of subgroups $SU(3)$ and $U(1)$ ( with $r = 2$ and $r = 1$, respectively) of the group $SO(6)$.

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We presented in Table I the eight Weyl spinors, two by two - one left ( \( \tilde{\Gamma}^{(4)} = -1 \)) and one right ( \( \tilde{\Gamma}^{(4)} = 1 \)) handed - connected by \( \tilde{\gamma}^m \), \( m = 0, 1, 2, 3 \) into Weyl bi-spinors. Half of vectors have Grassmann odd (odd products of \( \theta^m \)) and half Grassmann even character. The two four vectors of the same Grassmann character are connected by the discrete time reversal operation \( \theta^0 \rightarrow -\theta^0 \) (ref.\((\mathbb{[6]}))\), while the two four vectors, which differ in Grassmann character, are connected by the operation of \( \tilde{a}^a \).

According to Eqs.\((\mathbb{[50,51,52]}))\), one can express the generators of the subgroups \( SU(2) \) and \( U(1) \) of the group \( SO(1,7) \) in terms of the generators \( S^{ab} \).

We find (since the indices 0, 1, 2, 3 are reserved for the subgroup \( SO(1,3) \))

\[
\tau^{31} := \frac{1}{2}(S^{58} - S^{67}), \quad \tau^{32} := \frac{1}{2}(S^{57} + S^{68}), \quad \tau^{33} := \frac{1}{2}(S^{56} - S^{78}).
\] (54)

One also finds

\[
\tau^{41} := \frac{1}{2}(S^{56} + S^{78}).
\] (55)

The algebra of Eq.\((\mathbb{[50]}))\) follows\(^2\)

\[
\{\tau^{3i}, \tau^{3j}\} = i\epsilon_{ijk} \tau^{3k}, \quad \{\tau^{41}, \tau^{3i}\} = 0.
\] (56)

One notices that \( \tau^{51} := \frac{1}{2}(S^{58} + S^{67}) \) and \( \tau^{52} := \frac{1}{2}(S^{57} - S^{68}) \) together with \( \tau^{41} \) form the algebra of the group \( SU(2) \) and that the generators of this group commute with \( \tau^{3i} \).

We present in Table II the eigenvectors of the operators \( \tilde{\tau}^{33} \) and \( (\tilde{\tau}^3)^2 = (\tilde{\tau}^{31})^2 + (\tilde{\tau}^{32})^2 + (\tilde{\tau}^{33})^2 \), which are at the same time the eigenvectors of \( \tilde{\tau}^{41} \), for spinors. We find, with respect to the group \( SU(2) \), two doublets and four singlets of an even and another two doublets and four singlets of an odd Grassmann character.

\(^2\)Since the operators \( \tau^{41} \) have an even Grassmann character, the generalized commutation relations agree with the usual commutators, denoted by \([\ , \ ]\).
Table II: The eigenstates of the operators $\tilde{\tau}^{33}, \tilde{\tau}^{41}$ are presented. We find two doublets and four singlets of an even Grassmann character and two doublets and four singlets of an odd Grassmann character. One sees that complex conjugation transforms one doublet of either odd or even Grassmann character into another of the same Grassmann character changing the signum of the value of $\tilde{\tau}^{33}$, while it transforms one singlet into another singlet of the same Grassmann character and of the opposite value of $\tilde{\tau}^{41}$. One can check that $\tilde{a}^h, \ h \in (5, 6, 7, 8)$, transforms the doublets of an even Grassmann character into singlets of an odd Grassmann character.

One sees that $\tilde{\tau}^{5i}, \ i = 1, 2$, transform doublets into singlets (which can easily be understood if taking into account that $\tilde{\tau}^{5i}$ close together with $\tau^{14}$ the algebra of $SU(2)$ and that the two $SU(2)$ groups are isomorphic to the group $SO(4)$).

One also sees the following very important property of representations of the group $SO(1, 7)$: If applying the operators $\tilde{S}^{ab}, a, b = 0, 1, 2, 3, 5, 6, 7, 8$ on the direct product of

| a | i | $< \theta | \Phi^{a,i} >$ | $\tilde{\tau}^{33}$ | $\tilde{\tau}^{41}$ | Grassmann character |
|---|---|------------------|-----------------|----------------|---------------------|
| 1 | 1 | $\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(1 + i\tilde{a}^7\tilde{a}^8)$ | $-\frac{1}{2}$ | 0 | even |
| 2 | 2 | $\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$ | $\frac{1}{2}$ | 0 | |
| 2 | 1 | $\frac{1}{2}(1 + i\tilde{a}^5\tilde{a}^6)(1 - i\tilde{a}^7\tilde{a}^8)$ | $\frac{1}{2}$ | 0 | even |
| 2 | 2 | $\frac{1}{2}(\tilde{a}^5 - i\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$ | $-\frac{1}{2}$ | 0 | |
| 3 | 1 | $\frac{1}{2}(1 + i\tilde{a}^5\tilde{a}^6)(1 + i\tilde{a}^7\tilde{a}^8)$ | 0 | $\frac{1}{2}$ | |
| 4 | 1 | $\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$ | 0 | $\frac{1}{2}$ | |
| 5 | 1 | $\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(1 - i\tilde{a}^7\tilde{a}^8)$ | 0 | $-\frac{1}{2}$ | |
| 6 | 1 | $\frac{1}{2}(\tilde{a}^5 - i\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$ | 0 | $-\frac{1}{2}$ | |
| 7 | 1 | $\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$ | $\frac{1}{2}$ | 0 | even |
| 7 | 2 | $-\frac{1}{2}(\tilde{a}^5 - i\tilde{a}^6)(1 + \tilde{a}^7\tilde{a}^8)$ | $\frac{1}{2}$ | 0 | |
| 8 | 1 | $\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$ | $-\frac{1}{2}$ | 0 | odd |
| 8 | 2 | $\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(1 - i\tilde{a}^7\tilde{a}^8)$ | $\frac{1}{2}$ | 0 | |
| 9 | 1 | $\frac{1}{2}(1 - i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 - i\tilde{a}^8)$ | 0 | $-\frac{1}{2}$ | |
| 10 | 1 | $\frac{1}{2}(\tilde{a}^5 + i\tilde{a}^6)(1 + \tilde{a}^7\tilde{a}^8)$ | 0 | $-\frac{1}{2}$ | |
| 11 | 1 | $\frac{1}{2}(1 + i\tilde{a}^5\tilde{a}^6)(\tilde{a}^7 + i\tilde{a}^8)$ | 0 | $-\frac{1}{2}$ | |
| 12 | 1 | $\frac{1}{2}(\tilde{a}^5 - i\tilde{a}^6)(1 - \tilde{a}^7\tilde{a}^8)$ | 0 | $-\frac{1}{2}$ | |
polynomials of Table I and Table II, which forms the representations of the group $SO(1,7)$, one finds that a multiplet of $SO(1,7)$ exists, which contains left handed $SU(2)$ doublets and right handed $SU(2)$ singlets. It exists also another multiplet which contains left handed $SU(2)$ singlets and right handed $SU(2)$ doublets. It turns out that the operators $\tilde{S}^{mh}$, with $m = 0,1,2,3$ and $h = 5,6,7,8$, although having an even Grassmann character, change the Grassmann character of that part of the polynomials which belong to Table I and Table II, respectively, keeping the Grassmann character of the products of the two types of polynomials unchanged. This can be understood if taking into account that $\tilde{S}^{mh} = -\frac{i}{2} \tilde{a}^{m} \tilde{a}^{h}$ and that the operator $\tilde{a}^{m}$ changes the polynomials of an odd Grassmann character of Table I, into an even polynomial, transforming a left handed Weyl spinor of one family into a right handed Weyl spinor of another family, while $\tilde{a}^{h}$ changes simultaneously the $SU(2)$ doublet of an even Grassmann character into a singlet of an odd Grassmann character.

The symmetry, called the mirror symmetry, presented in this approach, is not broken, as none of the symmetry is broken. We only have arranged basic states to demonstrate possible symmetries.

We can express the generators of subgroups $SU(3)$ and $U(1)$ of the group $SO(6)$ in terms of the generators $S^{ab}$ (according to Eq.(51)).

We find (since the indices 9,10,11,12,13,14 are reserved for the subgroup $SO(6)$)

\begin{align}
\tau^{61} &:= \frac{1}{2}(S^{9\ 12} - S^{10\ 11}), \\
\tau^{62} &:= \frac{1}{2}(S^{9\ 11} + S^{10\ 12}), \\
\tau^{63} &:= \frac{1}{2}(S^{10\ 10} - S^{11\ 12}), \\
\tau^{64} &:= \frac{1}{2}(S^{9\ 14} - S^{10\ 13}), \\
\tau^{65} &:= \frac{1}{2}(S^{9\ 13} + S^{10\ 14}), \\
\tau^{66} &:= \frac{1}{2}(S^{11\ 14} - S^{12\ 13}), \\
\tau^{67} &:= \frac{1}{2}(S^{11\ 13} + S^{12\ 14}), \\
\tau^{68} &:= \frac{1}{2\sqrt{3}}(S^{9\ 10} + S^{11\ 12} - 2S^{13\ 14}).
\end{align}

(57)

(58)

One finds in addition

\[ \tau^{71} := -\frac{1}{3}(S^{9\ 10} + S^{11\ 12} + S^{13\ 14}). \]

(59)

The algebra for the subgroups $SU(3)$ and $U(1)$ follows from the algebra of the Lorentz group $SO(1,13)$

\[ \{\tau^{6i}, \tau^{6j}\} = if_{ijk}\tau^{6k}, \quad \{\tau^{71}, \tau^{6i}\} = 0, \text{ for each } i. \]

(60)

The coefficients $f_{ijk}$ are the structure constants of the group $SU(3)$.

We can find the eigenvectors of the Casimirs of the groups $SU(3)$ and $U(1)$ for spinors as polynomials of $\theta^{h}$, $h = 9,...,14$. The eigenvectors, which are polynomials of an even Grassmann character, can be found in ref.\cite{5}. We shall present here only not yet published \cite{7} polynomials of an odd Grassmann character.

18
| a | i | \(<\theta|\Phi^a_i>\) |
|---|---|---|
| 1 | 1 | \(\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 - i\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 1 | 2 | \(\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(\bar{a}^{11} - i\bar{a}^{12})\) |
| 1 | 3 | \(-\frac{1}{\sqrt{2}}(\bar{a}^{13} - i\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 2 | 1 | \(\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(1 - i\bar{a}^9\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 2 | 2 | \(\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
| 2 | 3 | \(-\frac{1}{\sqrt{2}}(\bar{a}^{13} - i\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 3 | 1 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(\bar{a}^9 - i\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 3 | 2 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
| 3 | 3 | \(-\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 4 | 1 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(1 - i\bar{a}^9\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 4 | 2 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(\bar{a}^{11} - i\bar{a}^{12})\) |
| 4 | 3 | \(-\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 5 | 1 | \(\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
| 5 | 2 | \(\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 5 | 3 | \(-\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
| 6 | 1 | \(\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(\bar{a}^{11} - i\bar{a}^{12})\) |
| 6 | 2 | \(\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 - i\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 6 | 3 | \(-\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(\bar{a}^9 - i\bar{a}^{10})(\bar{a}^{11} - i\bar{a}^{12})\) |
| 7 | 1 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} - i\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(\bar{a}^{11} - i\bar{a}^{12})\) |
| 7 | 2 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} - i\bar{a}^{14})(1 - i\bar{a}^9\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 7 | 3 | \(-\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(1 - +i\bar{a}^9\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 8 | 1 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} - i\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
| 8 | 2 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} - i\bar{a}^{14})(\bar{a}^9 - i\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 8 | 3 | \(-\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 - i\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
| 9 | 1 | \(\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 10 | 1 | \(\frac{1}{\sqrt{2}}(1 + i\bar{a}^{13}\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 11 | 1 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(1 + i\bar{a}^9\bar{a}^{10})(1 + i\bar{a}^{11}\bar{a}^{12})\) |
| 12 | 1 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} + i\bar{a}^{14})(\bar{a}^9 + i\bar{a}^{10})(\bar{a}^{11} + i\bar{a}^{12})\) |
| 13 | 1 | \(\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(\bar{a}^9 - i\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
| 14 | 1 | \(\frac{1}{\sqrt{2}}(1 - i\bar{a}^{13}\bar{a}^{14})(1 - i\bar{a}^9\bar{a}^{10})(\bar{a}^{11} - i\bar{a}^{12})\) |
| 15 | 1 | \(\frac{1}{\sqrt{2}}(\bar{a}^{13} - i\bar{a}^{14})(1 - i\bar{a}^9\bar{a}^{10})(1 - i\bar{a}^{11}\bar{a}^{12})\) |
Table III: The eigenstates of the operators $\tilde{\tau}^{63}, \tilde{\tau}^{68}, \tilde{\tau}^{71}$ are presented for odd Grassmann character polynomials. We find four triplets, four antitriplets and eight singlets. One sees that complex conjugation transforms one triplet into antitriplet, while $\tilde{\tau}^{8i}$ transform triplets into antitriplets or singlets.

One finds four triplets and four antitriplets as well as eight singlets. Besides the eigenvalues of the commuting operators $\tilde{\tau}^{63}$ and $\tilde{\tau}^{68}$ of the group $SU(3)$ also the eigenvalue of $\tilde{\tau}^{71}$ forming $U(1)$, is presented. The operators $\tilde{\tau}^{81} := \frac{1}{2}(\tilde{S}^{9 \ 12} + \tilde{S}^{10 \ 11}), \ \tilde{\tau}^{82} := \frac{1}{2}(\tilde{S}^{9 \ 11} - \tilde{S}^{10 \ 12}), \ \tilde{\tau}^{83} := \frac{1}{2}(\tilde{S}^{9 \ 14} + \tilde{S}^{10 \ 13}), \ \tilde{\tau}^{84} := \frac{1}{2}(\tilde{S}^{9 \ 13} - \tilde{S}^{10 \ 14}), \ \tilde{\tau}^{85} := \frac{1}{2}(\tilde{S}^{11 \ 14} + \tilde{S}^{12 \ 13}), \ \tilde{\tau}^{86} := \frac{1}{2}(\tilde{S}^{11 \ 13} - \tilde{S}^{12 \ 14}), \ \tilde{\tau}^{87} := \frac{1}{2}(\tilde{S}^{9 \ 13} - \tilde{S}^{10 \ 14})$, which transform triplets of the group $SU(3)$ into antitriplets and singlets with respect to the group $SU(3)$.

The spinorial representations of the group $SO(1, 13)$ are the direct product of polynomials of Table I, Table II and Table III.

We can find all the members of a spinorial multiplet of the group $SO(1, 13)$ by applying $\tilde{S}^{ab}$ on any initial Grassmann odd product of polynomials, if one polynomial is taken from Table I, another from Table II and the third from Table III. In the same multiplet there are triplets, singlets and antitriplets with respect to $SU(3)$, which are doublets or singlets with respect to $SU(2)$, and are left and right handed with respect to $SO(1, 3)$.

We can arrange in the same sense also eigenstates of operators of vectorial character, with bosonic character. In this paper we shall not do that.

8.2 Dynamical arrangement of representations of $SO(1, 13)$ with respect to subgroups $SO(1, 7)$ and $SO(6)$.

To see how Yang-Mills fields enter into the theory, we shall rewrite the Weyl-like equation in the presence of the gravitational field $(49)$ in terms of components of fields which determine gravitation in the four dimensional subspace and of those which determine gravitation in higher dimensions, assuming that the coordinates of ordinary space with indices higher than four stay compacted to unmeasurable small dimensions (or can not at all be noticed for some other reason). Since Grassmann space only manifests itself through average values of observables, compactification of a part of Grassmann space has no meaning. However, since parameters of Lorentz transformations in a freely falling coordinate system for both spaces have to be the same, no transformations to the fifth or higher coordinates may occur at measurable energies. Therefore, at low energies, the four dimensional subspace of Grassmann space with the generators defining the Lorentz group $SO(1, 3)$ is (almost) decomposed from the rest of the Grassmann space with the generators forming the (compact) group $SO(d - 4)$, because of the decomposition of ordinary space. This is valid on the classical level only.

According to the previous subsection, the break of symmetry of $SO(1, 13)$ should, however, appears in steps, first through $SO(1, 7) \times SO(6)$ and later to the final symmetry, which is needed in the Standard model for massless particles.

We shall comment on possible ways of spontaneously broken symmetries by studying the Weyl equation in the presence of gravitational fields in d dimensions for massless particles.
\begin{align}
\tilde{\gamma}^a p_{0a} = 0, \quad p_{0a} &= f^\mu_a p_{0\mu}, \quad p_{0\mu} = p_\mu - \frac{1}{2} \tilde{S}^{ab} \omega_{ab\mu}. \tag{61}
\end{align}

\section*{8.2.1 Standard model case.}

To make discussions more transparent we shall first comment on the well known case of the Standard model. Before the break of the symmetry \(SU(3) \times SU(2) \times U(1)\) into \(SU(3) \times U(1)\), the canonical momentum \(p_{0\alpha}(\alpha = 0, 1, 2, 3 \text{ and } d = 4)\) includes the gauge fields, connected with the groups \(SU(3)\), \(SU(2)\) and \(U(1)\). We shall pay attention on only the groups \(SU(2)\) and \(U(1)\), which are involved in the break of symmetry

\begin{equation}
p_{0\alpha} = p_\alpha - g \tau^i A^i_\alpha - g' Y B_\alpha, \tag{62}
\end{equation}

where \(g\) and \(g'\) are the two coupling constants. Introducing \(\tau^\pm = \tau^1 \pm i \tau^2\), the superposition follows \(A^\pm_\alpha = A^1_\alpha \mp i A^2_\alpha\). If defining \(A^3_\alpha = g/g' \sqrt{1 + (g/g')^2} Z_\alpha + 1/\sqrt{1 + (g/g')^2} A_\alpha\) and \(B_\alpha = -g' \sqrt{1 + (g/g')^2} Z_\alpha + g/g' \sqrt{1 + (g/g')^2} A_\alpha\), so that the transformation is orthonormalized, one can easily rewrite Eq.(62) as follows

\begin{equation}
p_{0\alpha} = p_\alpha - \frac{g}{2} (\tau^+ A^+_\alpha + \tau^- A^-_\alpha) + \frac{g g'}{\sqrt{g^2 + g'^2}} QA_\alpha + \frac{g^2}{\sqrt{g^2 + g'^2}} Q' Z_\alpha. \tag{63}
\end{equation}

with

\begin{equation}
Q = \tau^3 + Y, \quad Q' = \tau^3 - (\frac{g'}{g})^2 Y. \tag{64}
\end{equation}

In the Standard model \(\langle Q \rangle\) is conserved quantity and \(\langle Q' \rangle\) is not, due to the fact that \(\langle Q \rangle\) is zero for the Higgs fields in the ground state, while \(\langle Q' \rangle\) is nonzero (\(\langle Q' \rangle = -\frac{1}{2} (1 + (\frac{g'}{g})^2)\)).

We further see that in the case that \(g = g'\), it follows that \(Q = \tau^3 + Y\) and \(Q' = \tau^3 - Y\). If no symmetry is spontaneously broken, that is if no Higgs breaks symmetry by making a choice for his ground state symmetry, the only thing which has been done by introducing linear superpositions of fields, is the rearrangement of fields, which always can be done without any consequence, except that it may help to better see the symmetries.

Spontaneously broken symmetries cause the nonconservation of quantum numbers, as well as massive clusters of fields.

\section*{8.2.2 Spin connections and gauge fields leading to the Standard model.}

We shall rewrite the canonical momentum of Eq.(61) to manifest possible ways of breaking symmetries of \(SO(1, 13)\) down to the symmetries of the Standard model. We first write

\begin{equation}
\tilde{\gamma}^a p_{0a} = 0 = \tilde{\gamma}^a f^\mu_a p_{0\mu} = (\tilde{\gamma}^m f^\alpha_m + \tilde{\gamma}^h f^\alpha_h)p_{0\alpha} + (\tilde{\gamma}^m f^\sigma_m + \tilde{\gamma}^h f^\sigma_h)p_{0\sigma}, \tag{65}
\end{equation}

\section*{21}
with $\alpha, m \in \{0, 1, 2, 3\}$ and $\sigma, h \in \{5, \ldots, 14\}$ to separate the $d = 4$ dimensional subspace out of $d = 14$ dimensional space. We may further rearrange the canonical momentum $p_\mu$

$$p_\mu = p_\mu - \frac{1}{2} \tilde{S}^{h_1 h_2} \omega_{h_1 h_2 \mu} - \frac{1}{2} \tilde{S}^{k_1 k_2} \omega_{k_1 k_2 \mu} - \frac{1}{2} \tilde{S}^{h_1 k_1} \omega_{h_1 k_1 \mu},$$

with $h_1 \in \{0, 1, \ldots, 8\}$ and $k_1 \in \{9, \ldots, 14\}$ so that $\tilde{S}^{h_1 h_2}$ define the algebra of the subgroup $SO(1, 7)$, while $\tilde{S}^{k_1 k_2}$ define the algebra of the subgroup $SO(6)$. The generators $\tilde{S}^{h_1 k_1}$ rotate states of a multiplet of the group $SO(1, 13)$ into each other.

Taking into account subsection [5.1] we may rewrite the generators $\tilde{S}^{a_b}$ in terms of the corresponding generators of subgroups $\tilde{A}^i$ and accordingly, similarly to the Standard model case, introduce new fields (see subsection [6.2]), which are superpositions of the old ones

$$g A^{31} \mu = \frac{1}{2}(\omega_{58} - \omega_{67} \mu), \quad g A^{32} \mu = \frac{1}{2}(\omega_{57} + \omega_{68} \mu), \quad g A^{33} \mu = \frac{1}{2}(\omega_{56} - \omega_{78} \mu), \quad g A^{41} \mu = \frac{1}{2}(\omega_{56} + \omega_{78} \mu),$$
$$g A^{51} \mu = \frac{1}{2}(\omega_{58} + \omega_{67} \mu), \quad g A^{52} \mu = \frac{1}{2}(\omega_{57} - \omega_{68} \mu).$$

It follows then

$$\frac{1}{2} \tilde{S}^{h_1 h_2} \omega_{h_1 h_2 \mu} = g \tilde{A}^i A^i \mu,$$

where for $A = 3$, $i = 1, 2, 3$, for $A = 4$, $i = 1$ and for $A = 5$ $i = 1, 2$. Accordingly, the fields $A^i \mu$ are the gauge fields of the group $SU(2)$, if $A = 3$ and of $U(1)$ if $A = 4$. Since $\tilde{A}^1$ and $\tilde{A}^5$ form the group $SU(2)$ as well, the corresponding fields could be the gauge fields of this group. The break of symmetry should make a choice between the gauge groups $U(1)$ and $SU(2)$.

We leave the notation for spin connection fields in the case that $h_i \in \{0, 1, 2, 3\}$ unchanged. We also leave unchanged the spin connection fields for the case, that $h_1 = 0, 1, 2, 3$ and $h_2 = 5, 6, 7, 8$ as well as for the case, that $h_1 \in \{0, 1, \ldots, 8\}$ and $k_1 \in \{9, \ldots, 14\}$, while we arrange terms with $k_i \in \{8, \ldots, 14\}$ to demonstrate the symmetry $SU(3)$ and $U(1)$

$$g A^{61} \mu = \frac{1}{2}(\omega_{9} \mu_{10} - \omega_{10} 11 \mu), \quad g A^{62} \mu = \frac{1}{2}(\omega_{9} 11 \mu + \omega_{10} 12 \mu), \quad g A^{63} \mu = \frac{1}{2}(\omega_{9} 10 \mu - \omega_{11} 12 \mu),$$
$$g A^{64} \mu = \frac{1}{2}(\omega_{9} 14 \mu - \omega_{10} 13 \mu), \quad g A^{65} \mu = \frac{1}{2}(\omega_{9} 13 \mu + \omega_{10} 14 \mu), \quad g A^{66} \mu = \frac{1}{2}(\omega_{9} 14 \mu - \omega_{12} 13 \mu),$$
$$g A^{67} \mu = \frac{1}{2}(\omega_{11} 13 \mu + \omega_{12} 14 \mu), \quad g A^{68} \mu = \frac{1}{2}(\omega_{9} 10 \mu + \omega_{11} 12 \mu - 2\omega_{13} 14 \mu),$$
$$g A^{71} \mu = -\frac{1}{2}(\omega_{9} 10 \mu + \omega_{11} 12 \mu + \omega_{13} 14 \mu).$$

We may accordingly define fields $g A^{81} \mu = \frac{1}{2}(\omega_{9} 12 \mu - \omega_{10} 11 \mu), \quad g A^{82} \mu = \frac{1}{2}(\omega_{9} 11 \mu - \omega_{10} 12 \mu), \quad g A^{83} \mu = \frac{1}{2}(\omega_{9} 14 \mu + \omega_{10} 13 \mu), \quad g A^{84} \mu = \frac{1}{2}(\omega_{9} 13 \mu - \omega_{10} 14 \mu), \quad g A^{85} \mu = \frac{1}{2}(\omega_{11} 14 \mu +...
\( \omega_{1213\mu} \), \( g A_{\mu}^{86} = \frac{1}{2}(\omega_{1113\mu} - \omega_{1214\mu}) \), so that it follows

\[
\frac{1}{2} \delta^{k_1 k_2} \omega_{k_1 k_2 \mu} = g \tilde{\tau}^{A_i} A^A_i \mu,
\]

(72)

with \( A = 6, 7, 8 \). While \( A_{\mu}^{6i} \), \( i \in \{1, \ldots, 8\} \), form the gauge field of the group \( SU(3) \) and \( A_{\mu}^{71} \) corresponds to the gauge group \( U(1) \), terms \( g \tilde{\tau}^{71} A_{\mu}^{71} \) transform \( SU(3) \) triplets into singlets and antitriplets. Again, without additional requirements, all the coupling constants \( g \) are equal. To be in agreement with what the Standard model needs as an input, we further rearrange the gauge fields belonging to the two \( U(1) \) fields, one coming from the subgroup \( SO(1,7) \) the other from the subgroup \( SO(6) \). We therefore define

\[
Y_1 = (\tau^{41} + \tau^{71}), \quad Y_2 = -(\tau^{41} - \tau^{71})
\]

(73)

and accordingly similarly to the Standard model case of subsection 8.2.1

\[
A_{\mu}^1 = \frac{1}{2}(A_{\mu}^{41} + A_{\mu}^{71}), \quad A_{\mu}^2 = -\frac{1}{2}(A_{\mu}^{41} - A_{\mu}^{71}).
\]

(74)

The rearrangement of fields demonstrates all the symmetries of the massless particles of the Standard model and more.

Taking into account Tables I, II and III one finds for the quantum numbers of spinors, which belong to a multiplet of \( SO(1,7) \) with left handed \( SU(2) \) doublets and right handed \( SU(2) \) singlets and which are triplets or singlets with respect to \( SU(3) \), the ones, presented on Table IV. We use the names of the Standard model to denote triplets and singlets with respect to \( SU(3) \) and \( SU(2) \).

| SU(3) triplets | SU(2) doublets | SU(2) singlets |
|-----------------|-----------------|-----------------|
| \( \tilde{\tau}^{63} = (\frac{1}{2}, -\frac{1}{2}, 0) \) | \( u_i \) | \( \tilde{\tau}^{33} \) |
| \( \tilde{\tau}^{68} = (-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}) \) | \( d_i \) | \( \tilde{\tau}^{41} \) |

| SU(3) singlets | SU(2) doublets | SU(2) singlets |
|-----------------|-----------------|-----------------|
| \( \tilde{\tau}^{63} = 0 \) | \( \nu_i \) | \( \tilde{\tau}^{33} \) |
| \( \tilde{\tau}^{68} = 0 \) | \( e_i \) | \( \tilde{\tau}^{41} \) |

Table IV: Expectation values for generators \( \tilde{\tau}^{63} \) and \( \tilde{\tau}^{68} \) of the group \( SU(3) \) and the generator \( \tilde{\tau}^{71} \) of the group \( U(1) \), the two groups are subgroups of the group \( SO(6) \), and of generators
o \tilde{\tau}^{33} of the group \textit{SU}(2), \tilde{\tau}^{41} of the group \textit{U}(1) and \tilde{\Gamma}^{(4)} of the group \textit{SO}(1,3), the three groups are subgroups of the group \textit{SO}(1,7) for the multiplet (with respect to \textit{SO}(1,7)), which contains left handed (\langle \Gamma^{(4)} \rangle = -1) \textit{SU}(2) doublets and right handed (\langle \Gamma^{(4)} \rangle = 1) \text{SO}(2) singlets. In addition, values for \tilde{Y}_1 and \tilde{Y}_2 are also presented. Index i of \textit{u}_i, \textit{d}_i, \nu_i and \textit{e}_i runs over four families presented in Table I.

We see that, besides \tilde{Y}_2, this are just the quantum numbers needed for massless fermions of the Standard model. The value for the additional hyper charge \tilde{Y}_2 is nonzero for the right handed neutrinos, as well as for other states, except right handed electrons.

Since no symmetry is broken yet, all the gauge fields are of the same strength. To come to the symmetries of massless fields of the Standard model, surplus symmetries should be broken so that all the coupling constants connected with the fields \omega_{a\mu} which do not determine the fields \textit{A}^A_{\mu}, \textit{A} = 3, 6 (Eqs.(67,70)) and \textit{A}^1_{\mu} (Eq.(74)) should be small and yet the coupling constants of these three fields should not be equal. Accordingly also the operator \tilde{Y}_2 could, similarly to the case of Eq.(64), depend on the coupling constants.

The mirror symmetry should be broken so that multiplets of \textit{SO}(1,7) with right handed \textit{SU}(2) doublets and left handed \textit{SU}(2) singlets become very massive. All the surplus multiplets, either bosonic or fermionic should become of large enough masses not to be measurable yet.

The proposed approach predicts four rather than three families of fermions.

Although in this paper, we shall not discuss possible ways of appearance of spontaneously broken symmetries, bringing the symmetries of the group \textit{SO}(1,13) down to symmetries of the Standard model, we still would like to know, whether there are terms in the Weyl equation (Eq.65) which may behave like the Yukawa couplings. We see that indeed the term \tilde{\gamma}^h f^\sigma h p_{\sigma}, with \textit{h} \in \{5,6,7,8\} and \textit{\sigma} \in \{5,6,..\} really may, if operating on a right handed \textit{SU}(2) singlet transform it to a left handed \textit{SU}(2) doublet. We also can find among scalars the terms with quantum numbers of Higgs bosons (which are \textit{SU}(2) doublets with respect to operators of the vectorial character.) All this is in preparation and not yet finished or fully understood.

9 Concluding remarks.

In this paper, we demonstrated that if assuming that the space has \textit{d} commuting and \textit{d} anticommuting coordinates, then, for \textit{d} \geq 14, all spins in \textit{d} dimensions, described in the vector space spanned over the space of anticommuting coordinates, demonstrate in four dimensional subspace as spins and all charges, unifying spins and charges of fermions and bosons independently, although the supersymmetry, which means the same number of fermions and bosons, is a manifesting symmetry. The anticommuting coordinates can be represented by either Grassmann coordinates or by the Kähler differential forms.

We demonstrated that either our approach or the approach of differential forms suggest four families of quarks and leptons, rather than three.

We have shown that starting (in any of the two approaches) with the Lorentz symmetry in the tangent space in \textit{d} \geq 14, spins degrees of freedom (described by dynamics in the
space of anticommuting coordinates) manifests in four dimensional subspace as spins and colour, weak and hyper charges, with one additional hyper charge, in a way that only left handed weak charge doublets together with right handed weak charge singlets appear, if the symmetry is spontaneously broken from $SO(1,13)$ first to $SO(1,7)$ and $SO(6)$, so that a multiplet of $SO(1,7)$ with only left handed $SU(2)$ doublets and right handed $SU(2)$ singlets survive, while the mirror symmetry is broken, and then to $SO(1,3), SU(2), SU(3)$ and $U(1)$.

We have demonstrated that the gravity in D dimensions manifests as ordinary gravity and all gauge fields in four dimensional subspace, after the break of symmetry and the accordingly changed coupling constant. We also have shown that there are terms in the Weyl equations, which in four dimensional subspace manifest as yukawa couplings.

The two approaches, the Kähler one after the generalization, which we have been suggested, and our, lead to the same results.

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