Lower bound on the communication cost of simulating bipartite quantum correlations

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(Dated: October 19, 2009)

Suppose Alice and Bob share a maximally entangled state of any finite dimension and each perform two-outcome measurements on the respective part of the state. It is known, due to the recent result of Regev and Toner, that if a classical model is augmented with two bits of communication then all the quantum correlations arising from these measurements can be reproduced. Here we show that two bits of communication is in fact necessary for the perfect simulation. In particular, we prove that a pair of maximally entangled four-dimensional quantum systems cannot be simulated by a classical model augmented by only one bit of communication.

PACS numbers: 03.65.Ud, 03.67.-a

I. INTRODUCTION

Let us assume two spatially separated parties, Alice and Bob, who receive local inputs and produce subsequently outputs. As pointed out by John Bell [1], joint correlations of Alice’s and Bob’s outputs resulting from quantum theory cannot be reproduced classically if communication is not taken place between them. Actually, his proof consists a setup of a gedanken experiment where contradiction arises if only local resources are used by the two parties. This is exemplified by the violation of the so-called CHSH inequalities [2], which has been demonstrated experimentally several times up to some technical loopholes [3]. Considering that local resources are not sufficient to simulate all the correlations of quantum mechanics one may ask whether Alice and Bob would benefit from sending some bits of communication in order to reproduce all the set of quantum correlations. This problem has been addressed by Maudlin [4] and later independently by Brassard et al. [5] and Steiner [6]. However, the approach they used in defining the communication cost of the simulation is slightly different. Brassard et al. took the worst case communication, i.e., the maximal number of bits to be sent in the worst case to simulate quantum correlations, while in Steiner’s and Maudlin’s model the average amount of communication is considered. In this work we apply the worst case scenario, however, there are several works discussing the cost of average communication for simulation (see for example, [7, 8, 9]). In this respect we also wish to highlight two more recent works. One is by Gavinsky [10], who present a nonlocality game, which requires a large amount of communication so as to be simulated classically. In an other work by Roland and Szegedy [11], an asymptotic lower bound is established on the cost of simulating quantum correlations.

In Ref. [8] the intriguing result has been found that one can simulate all quantum correlations arising from two-outcome projective measurements on the two-qubit maximally entangled state, provided a finite (eight) number of bits of communication is added to local resources. This result has been slightly improved later by Csirik to six bits [12]. Then six bits could be further improved even to one bit by the protocol of Toner and Bacon [14] in 2003. One may assume that this result is due to the fact that the Hilbert space of each party is restricted to two dimensions. So the question was natural to ask that how many bits are needed to simulate correlations arising from two-outcome projective measurements without restricting the size of the Hilbert space. First, it has been exhibited a protocol by Toner and Regev (see [15]) which solves this problem in any Hilbert space dimension with five bits, moreover this result was subsequently improved to two bits [13]. This is the best upper bound up to now, and an open question is whether two bits of communication are needed at all, or the minimal amount of one bit is enough to simulate all two-outcome projective measurements on a maximally entangled state of any finite dimension. We wish to address this problem and show that two bits are indeed necessary, as conjectured by Regev and Toner [15], by exhibiting a pair of four-dimensional quantum systems and measurement settings on Alice’s and Bob’s part so that the corresponding quantum correlations cannot be simulated with a classical model using one bit of communication. Note, that for more parties even the question of finite communication is not settled yet. In this respect, Broadbent et al. [16] proved that at least $n \log n - 2n$ bits are needed to simulate classically an $n$-party GHZ state improving on an earlier result of Buhrman et al. [17].

In the two-party two-outcome case, on the other hand, Bacon and Toner [18] have shown that for three measurements per party one bit of communication suffices to solve this task. We will show in this work that for an infinite number of measurements the exchange of one bit is not sufficient. We believe that this result holds for a finite number of settings as well, but we leave it as an open problem. Note that to the best of our knowledge there is no two-party protocol in the literature even for the general $d \geq 2$-outcome case where it is proved that
quantum measurements cannot be simulated classically with one bit of information.

In the quantum two-party two-outcome scenario Alice and Bob share an entangled state \( \rho \) and on the respective parts they perform projective measurements described by the observables \( \mathbf{A}(a) \) and \( \mathbf{B}(b) \) with eigenvalues \( \pm 1 \) for Alice and Bob, respectively. Here \( a \) and \( b \) label the observables. Then Alice and Bob each output two bits, \( \alpha \) and \( \beta \). In the quantum setting the correlation \( E[\alpha\beta|ab] \) satisfies \( E[\alpha\beta|ab] = \text{Tr}(\mathbf{A}(a) \otimes \mathbf{B}(b)\rho) \).

As Tsirelson has shown, there is an equivalent formulation of this problem by means of unit vectors \[19\]: Alice receives as an input the unit vector \( a \in \mathbb{R}^n \) and outputs a bit \( \alpha \in \pm 1 \), also Bob receives as an input the unit vector \( b \in \mathbb{R}^n \) and outputs a bit \( \beta \in \pm 1 \). Their goal is to produce a correlation \( E[\alpha\beta|ab] \) which satisfies \( E[\alpha\beta|ab] = \mathbf{A}(a) \cdot \mathbf{B}(b) \), where \( \mathbf{A}(a) \) and \( \mathbf{B}(b) \) are some unit vectors in \( \mathbb{R}^n \) and \( \cdot \) denotes dot product. Then, due to Tsirelson’s construction \[19\], these correlations can be realized as \( \pm 1 \)-valued observables \( \mathbf{A}(a) \) and \( \mathbf{B}(b) \) on a maximally entangled state of local dimension \( D = 2^{[n/2]} \).

Such as in the context of Bell inequalities, let us now take a linear function of the correlations
\[
B(M) = \int_{a,b \in S^{n-1}} M(a,b)A(a) \cdot B(b)d\sigma(a)d\sigma(b),
\]
where we have the functions \( A, B : S^{n-1} \rightarrow S^{m-1} \). Thus \( B(M) \) can be interpreted as a two-party two-outcome Bell expression with a continuous number of measurement settings. Let us fix \( M(a,b) = a \cdot b \), which is the famous example studied by Grothendieck \[20\], and was also a subject of recent studies \[21, 22\] related to dimension witnesses \[21\]. We wish to find the following values:
\[
Q(n) = \max_{A,B: S^{n-1} \rightarrow S^{m-1}} B(M),
\]
\[
L(n) = \max_{A,B: S^{n-1} \rightarrow \pm 1} B(M),
\]
where \( Q(n) \) is the maximum value of the Bell expression \( B(M) \) achievable by means of quantum systems (without imposing bound on the value of \( m \)), while \( L(n) \) corresponds to the maximum \( B(M) \) attainable by local hidden variables systems.

In Sec. II A we give a lower bound to \( Q(n) \) by setting particularly \( A(a) = a \) and \( B(b) = b \), which bound is known to be the exact maximum value as well (see \[21, 22\]). Then, in Sec. II B the local bound \( L(n) \) is given, which was also considered by Grothendieck himself. Next, in Sec. II C we evaluate the local bound augmented with one bit of communication, which we denote by \( C(n) \). Since \( M(a,b) = a \cdot b \) is symmetric in the vectors \( a \) and \( b \), it is enough to treat the case where Alice may transmit one bit of information to Bob. In this case Bob’s outcome may depend on the input Alice received. Thus, by allowing one bit of communication, the \( \pm 1 \) valued function \( B(b) \) of a locally classical model becomes \( B(b, f(a)) \). However, the function \( f(a) \) may depend only on a bipartition of the set \( a \), since a bipartition carries just one bit of information. Then substituting \( B(b, f(a)) \) in place of \( B(b) \) in Eq. (11), one gets
\[
C(S', S'', n) = \max_{A,S'^{-1}, S''^{-1} \rightarrow \pm 1} \int (a \cdot b)A(a) \cdot B'(b)d\sigma(a)d\sigma(b) + \max_{A,S'^{-1}, S''^{-1} \rightarrow \pm 1} \int (a \cdot b)A(a) \cdot B''(b)d\sigma(a)d\sigma(b),
\]
where \( S'^{-1} = S' \cup S'' \). Then the maximum value achievable by a local model plus one bit of communication is
\[
C(n) = \max_{S'/S''} C(S', S'', n),
\]
that is, we have to maximize with respect to all possible bi-partitions of the unit sphere \( S^{n-1} \).

The main result of this paper is the proof that \( C(n) < Q(n) \) for \( n \geq 5 \), where \( Q \) is a lower bound to \( Q(n) \) (however, due to \[21, 22\] this bound is tight). This implies by the construction of Tsirelson, that measurements on maximally entangled four-dimensional systems (ququarts) cannot be simulated by a local classical model allowing only one bit of communication.

II. CALCULATION OF THE LIMITS

A. Quantum bound

A lower bound to the value of \( Q(n) \) can be given by substituting \( A(a) = a \) and \( B(b) = b \) into the definition \[9\], where the Bell expression \( B(M) \) is defined by \[11\]. Then \( m = n \) and a lower bound to \( Q(n) \) is given by
\[
\hat{Q}(n) = \int_{a,b \in S^{n-1}} |a \cdot b|^2 d\sigma(a)d\sigma(b) \leq Q(n),
\]
where \( \int d\sigma(a) = \int d\sigma(b) = 1 \) is the normalized Haar measure. Due to rotational invariance we can assume \( b = (1,0,\ldots,0) \), thus we can further write \[9\] to obtain \( \hat{Q}(n) = |b|^2 d\sigma(b) = (1/n) \int \sum_i |b_i|^2 d\sigma(b) = 1/n \), where we used that all \( b_i, i = 1,\ldots,n \) are equal owing to symmetry arguments. As it has been shown \[21, 22\], \( \hat{Q}(n) \) is equal to \( Q(n) \), but in our proof we do not have to use this fact.

B. Local bound

Here we evaluate the classical limit \( L(n) \). According to \[12\] and using the explicit form \( M(a,b) = a \cdot b \), this is
\[
L(n) = \max_{A,B: S^{n-1} \rightarrow \pm 1} \int (a \cdot b)A(a)B(b)d\sigma(a)d\sigma(b),
\]
where now \( A(a) \) and \( B(b) \) are \( \pm 1 \) valued functions.
Let us define \( h(a) = \int (a \cdot b) B(b) d\sigma(b) = a \cdot \int bB(b) d\sigma(b) = a \cdot Z = k(a \cdot z) \), where \( z \) is a unit vector and \( k = h(z) \) is the length of the vector \( Z \). Therefore, \( h(a) = h(z)a \cdot z \) and according to (11)

\[
L(n) = \max_{A : B \in S^{n-1} \pm \pm} \int A(h) a d\sigma(a)
\]

\[
= \max_{A : B \in S^{n-1} \pm \pm} h(z) \int (z \cdot a) A(a) d\sigma(a), \tag{8}
\]

where \( z \) in the second line is defined through

\[
Z = \lambda z = \int_{b \in S^{n-1}} bB(b) d\sigma(b)
\]

(9)

according to the formulae below Eq. (11). Then (8) above can be further written as

\[
L(n) = \max_{A : B \in S^{n-1} \pm \pm} \left( \int_{a \in S^{n-1}} |z \cdot a| d\sigma(a) \right)^2, \tag{10}
\]

where the second line follows from the substitutions \( A(a) = \text{sgn}(z \cdot a) \) and \( B(b) = \text{sgn}(z \cdot b) \), which choices are incidentally consistent with the definition (10) for \( z \). Due to symmetry arguments one can choose \( z = (1,0,\ldots,0) \) and then we have \( L(n) = \left[ \int |a_1| d\sigma(a) \right]^2 \). By an explicit calculation one obtains \( L(n) = s_{n-1}/(ns_n) \) (see also [21, 22]), where \( s_n = \int_0^\pi \sin^n \theta d\theta = \sqrt{n} \Gamma((n + 1)/2)/(\Gamma((n + 2)/2)) \). This formula can be evaluated analytically either for small values of \( n \) or for the continuum limit \( n \to \infty \). We have collected the ratios \( \hat{Q}(n)/L(n) \) for some small values of \( n \) and also for \( n \to \infty \) in Table I (remembering that \( \hat{Q}(n) = 1/n \) for \( n > 1 \)).

C. Local plus one bit bound

Let us now calculate an upper bound on the value of \( C(n) \) defined by Eq (5). We use the same arguments to separate the terms involving the vectors \( a \) and \( b \), which lead from (7) to (10). Then, the first double integral in (4) can be upper bounded as

\[
\max_{A : B \in S^{n-1} \pm \pm} \int (a \cdot b) A(a) B'(b) d\sigma(a) d\sigma(b) \leq \max_{z'}
\]

\[
\left\{ \max_{A : B \in S^{n-1} \pm \pm} \left( \int (z' \cdot a) A(a) d\sigma(a) \right) \max_{B : B' \in S^{n-1} \pm \pm} \left( \int (z' \cdot b) B'(b) d\sigma(b) \right) \right\}
\]

\[
= \max_{z'} \left\{ \int_{a \in S^n} |z' \cdot a| d\sigma(a) \int_{b \in S^{n-1}} |z' \cdot b| d\sigma(b) \right\}, \tag{11}
\]

where \( z' \) is defined by the normalized value of \( Z' = \int_{b \in S^{n-1}} bB'(b) d\sigma(b) \). The second term in (11) can be written similarly to (11) but replacing the terms (’) by (’’). In order to arrive at the third line we used the substitutions \( A(a) = \text{sgn}(z' \cdot a) \) and \( B'(b) = \text{sgn}(z'' \cdot b) \), \( a \in S' \), \( b \in S' \) and likewise for case (’’) we will have \( A(a) = \text{sgn}(z'' \cdot a) \) and \( B''(b) = \text{sgn}(z'' \cdot b) \), \( a \in S'' \), \( b \in S'' \). By the virtue of the definition of \( Z' \) (and \( Z'' \)) above, for any alignment of the vectors \( z' \) and \( z'' \), the functions \( B' \) and \( B'' \) are defined consistently. On the other hand, due to rotational invariance, the last integral term in the third line of (11) does not depend on the vector \( z' \). By exploiting this fact also in the expression (’’) analog with (11) and by using the the definitions in (4), we can write

\[
C(n) \leq \int_{b \in S^{n-1}} |b_1| d\sigma(b)
\]

\[
\max_{z' \in S'/S''} \left\{ \int_{a \in S'} |z' \cdot a| d\sigma(a) + \int_{a \in S''} |z'' \cdot a| d\sigma(a) \right\}
\]

\[
= \int_{b \in S^{n-1}} |b_1| d\sigma(b) \max_{z', z''} \int_{a \in S^{n-1}} \max \{|z' \cdot a|, |z'' \cdot a|\} d\sigma(a).
\]

(12)

Since only the angle between \( z' \) and \( z'' \) enters above, without loss of generality we can choose them as

\[
z' = (\sin \theta, \cos \theta, 0, \ldots, 0),
\]

\[
z'' = (-\sin \theta, \cos \theta, 0, \ldots, 0).
\]

(13)

Now, taking into account Eq. (10) we have

\[
C(n) \leq \sqrt{L(n)} \max_{z', z''} \int_{a \in S^{n-1}} \max \{|z' \cdot a|, |z'' \cdot a|\} d\sigma(a),
\]

(14)

which readily depends only on the angle \( 2\theta \) between \( z' \) and \( z'' \) in (13). Also, the RHS of (14) above is invariant under the sign changes of \( z' \), \( z'' \) and \( a \). Thus, it is enough to consider the angle \( 0 \leq \theta \leq \pi/2 \) and \( a \) being located in the first quadrant of the first two coordinates. However, in this case \( |z' \cdot a| > |z'' \cdot a| \), and the integral in (14) becomes \( 4 \max_\theta \int_{a \in S^{n-1}} |z' \cdot a| d\sigma(a) \), where the integral is performed on the sphere \( S^{n-1} \) except in the first two coordinates, where integration is only over the first quadrant (the range of \( a \) designated by \( S^{n-1}_+ \)).

Let us next evaluate an upper bound on the ratio \( C(n)/L(n) \), i.e., on the ratio of the local bound with one bit of communication to the local bound without communication. Applying Eqs. (5), (10), (11) and (14), we have

\[
\frac{C(n)}{L(n)} \leq \frac{4 \max_\theta \int_{a \in S^{n-1}} |z' \cdot a| d\sigma(a)}{\int_{a \in S^{n-1}} |a| d\sigma(a)}
\]

\[
= \max_{\theta} \frac{\pi/2}{\sin^2(\theta) + \sin^2(\theta)}
\]

\[
= \max_{\theta} \{\cos \theta + \sin \theta\} = \sqrt{2} \sim 1.41421,
\]

(15)

the maximum taken up by \( \theta = \pi/4 \), where we have taken in the integration the explicit forms of \( z' \) and \( z'' \) from (13). Notice, that this upper bound to the ratio is independent of dimension \( n \). However, a lower bound on the quantum per local bound can be seen in Table I. This shows clearly that for \( n = 5 \) the ratio exceeds the value
\(\sqrt{2}\), indicating that quantum mechanical correlations of bipartite quantum systems cannot be simulated by local models augmented with one bit of communication.

Due to the work of Tsirelson, for \(n = 5\) one can construct the measurement operators and states in the local four-dimensional Hilbert spaces. In the following, we exhibit the explicit form of them. Let the quantum state be the maximally entangled pair of ququarts, \(|\psi^+\rangle = (1/2) \sum_{i=1}^{4} |ii\rangle\). Then, the respective observables of Alice and Bob are

\[
\begin{align*}
A(a) &= \sum_{k=1}^{5} A^{(k)}(a) \gamma_k \\
B(b) &= \sum_{k=1}^{5} B^{(k)}(b) \gamma_k,
\end{align*}
\]

where \(A^{(k)}(a), B^{(k)}(b)\) are the components of the vectors \(A(a)\) and \(B(b)\) and the five anticommuting, traceless \(\gamma\) matrices are

\[
\begin{align*}
\gamma_1 &= \sigma_x \otimes \mathbb{1} \\
\gamma_2 &= \sigma_y \otimes \mathbb{1} \\
\gamma_3 &= \sigma_z \otimes \sigma_x \\
\gamma_4 &= \sigma_z \otimes \sigma_y \\
\gamma_5 &= \sigma_x \otimes \sigma_z.
\end{align*}
\]

It can be checked that with this, one has indeed \(\langle\psi^+|A(a) \otimes B(b)|\psi^+\rangle = A(a) \cdot B(b)\) as required.

**III. SUMMARY**

In this paper we have shown that two bits of communication are necessary for perfectly simulating classically the correlations of measurement outcomes carried out by two distant parties. In particular, we proved that two-outcome projective measurements on a pair of maximally entangled four-dimensional quantum systems cannot be simulated by a classical model augmented by only one bit of communication. In order to prove it, our scenario involved an infinite number of measurements. We pose it as an open question whether a finite number of measurements would suffice for the proof as well. In this respect we mention that in a recent work Briët et al. \[21\] could discretize the Bell expression \[1\] with \(M = a \cdot b\) to involve only finite number of measurement settings and obtained bounds on the maximum quantum values depending on the dimension. Their result would probably help in the one bit communication problem, discussed here, as well.

Finally, we list some interesting open questions related to recent results in the literature. Let us restrict to the case of two parties and binary outputs. We know that in this case measurements can be simulated classically with one bit (two bits) of communication performed on maximally entangled qubits \[13\] (qudits \[15\]). Recently, N. Gisin posed the question \[24\], whether there exist measurements on a pair of partial entangled qubit states which cannot be simulated by a single bit of communication. Similarly, it would be interesting to know how hard to simulate partially entangled qudits, e.g. whether it would require exchanging more than two bits. Also, it is known \[24\] that a hypothetical non-local machine, the so-called PR-box \[26\] is a strictly weaker resource than one bit of communication. Nevertheless, it can simulate two-outcome projective measurements on a maximally entangled qubit pair \[25\]. On the other hand, it has been recently proved that projective measurements on a maximally entangled pair of qudits with two outcomes can be simulated by three PR-boxes \[27\]. In light of our result that no 1-bit communication model exists for simulating measurements on maximally entangled qudits, no 1 PR-box model would exist either. Then it would be interesting to find out whether 2 PR-boxes were enough to simulate two-outcome projective measurements on maximally entangled qudits or not.

**Acknowledgments**

T.V. has been supported by a János Bolyai Programme of the Hungarian Academy of Sciences. We wish to thank Ben Toner, Oded Regev and an anonymous Referee for filling gaps in the proof of Sec. IIC. We also thank Nicolas Brunner for pointing out the lower bound result on PR-boxes.

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