INCOMPRESSIBLE FLUIDS WITH SHEAR RATE AND PRESSURE DEPENDENT VISCOSITY: REGULARITY OF STEADY PLANAR FLOWS

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Abstract. We study the regularity of steady planar flow of fluids where the shearing stress may depend on the symmetric part of the velocity vector field and the pressure. For simplicity the periodic boundary conditions are considered. Using Meyers estimates we show that there exists a solution which is smooth. In the case where it is allowed to test weak formulation of the problem with a weak solution we prove regularity of all weak solutions.

1. Introduction. In the most famous mathematical model of fluid mechanics, the Navier-Stokes model for incompressible fluid, the viscosity of fluid is considered to be constant. Interestingly, already G. Stokes [19] was aware of the fact that the viscosity might depend on pressure. However at that time the experiments did not show any substantial dependence of the viscosity on the pressure. At the beginning of 20th century there appeared many experimental works showing that there are indeed fluids which, when subject to significant pressure variation, change their viscosity tremendously. As these fluids at the same time do not considerably change their volume, they can be treated as incompressible. The first mathematical studies of Navies Stokes equations with pressure dependent viscosity appeared in [17] where it was observed that stationary variant of the problem may lose its ellipticity due to dependence of the viscosity on the pressure. The research was continued in works [4, 5]. As the logical consequence of the extensive study of generalized Newtonian fluids at the end of the last century, the new topic was born at the dawn of the new millennium—study of fluids with pressure and shear dependent viscosity. For physical background see [3, 14, 12, 6] and the references therein.

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Here we consider the problem
\[- \text{div}(\mathbf{T}(\mathbf{D}(\mathbf{v}), p)) + \nabla p = \mathbf{f} + (\mathbf{v} \cdot \nabla)\mathbf{v}, \quad (1.1)\]
\[\text{div} \mathbf{v} = 0, \quad (1.2)\]
for unknown velocity \( \mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2 \) and pressure \( p : \mathbb{R}^2 \to \mathbb{R} \) together with periodic boundary conditions
\[\mathbf{v}, p \text{ are periodic with respect to } \Omega := (0, 1)^2. \quad (1.3)\]
Moreover, in order to ensure uniqueness of the pressure, see [17] we assume
\[\int_{\Omega} \mathbf{v} = 0; \quad \int_{\Omega} p = p_0 \quad \text{for given } p_0 \in \mathbb{R}. \quad (1.4)\]
Note that while in the Navier-Stokes model the choice of \( p_0 \) changes only \( p \), here it affects both \( p \) and \( \mathbf{v} \) (for more sophisticated discussion see [3, Section 2]).

Regarding the viscous stress tensor \( \mathbf{T} \) we assume that it depends on the symmetric part of the velocity field gradient \( \mathbf{D}(\mathbf{v}) := (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)/2 \) and the pressure \( p \) and satisfies the following condition
\[\mathbf{T}(\mathbf{D}(\mathbf{v}), p) = \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \quad (1.5)\]
for some generalized viscosity \( \nu : \mathbf{S} \times \mathbb{R} \to \mathbb{R} \), \( \mathbf{S} \) being the set of symmetric matrices \( 2 \times 2 \), \( \nu \) is Lipschitz continuous, and for given \( r \in (1, 2], \ C_1, C_2 \in (0, +\infty) \) and a.a. \( \mathbf{D}, \mathbf{B} \in \mathbf{S} \), \( p \in \mathbb{R} \) there holds
\[
\begin{align*}
(A1) \quad C_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} |\mathbf{B}|^2 & \leq \frac{\partial}{\partial \mathbf{D}_{kl}} \left[ \nu(p, |\mathbf{D}|^2) \mathbf{D}_{ij} \right] \mathbf{B}_{ij} \mathbf{B}_{kl} \leq C_2(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} |\mathbf{B}|^2, \\
(A2) \quad \left| \frac{\partial \nu(p, |\mathbf{D}|^2)}{\partial p} \mathbf{D} \right| & \leq \gamma_0(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}.
\end{align*}
\]
In the following \( \gamma_0 > 0 \) will be chosen small enough in such a way that the system stays elliptic, compare with [3]. Typical example of \( \mathbf{T} \) satisfying (1.5), (A1) and (A2), see [3, 14], is
\[\mathbf{T}(\mathbf{D}, p) := (1 + |\mathbf{D}|^2 + \gamma(p))^{\frac{r-2}{2}} \mathbf{D}.\]
with one of the following functions \( (\alpha, q > 0) \)
\[\gamma(p) := \begin{cases} (1 + \alpha^2 p^2)^{-\frac{q}{2}}, \\ (1 + \exp(\alpha p))^{-q}. \end{cases}\]

The first study of the system (1.1) with (1.5), (A1) and (A2) appeared in [7, 14], where the existence of the global solution to the evolutionary variant of (1.1) even in 3d is shown for \( r \in (9/5, 2] \). The research of evolutionary problem continued recently in [1] where the authors consider similar problem with more complex assumptions on \( \partial_\nu T \) and with Naviers slip boundary condition. In 3d they improved bound to \( r \in (8/5, 2] \) putting however stronger assumptions on \( \partial_\nu T \). Stationary variant of the problem in 2d and 3d was studied in [3], where the existence of the weak solution is obtained if \( r \in (3d/(d + 2), 2] \). Regularity of this weak solution is announced in [15], namely partial \( C^{1,\alpha} \) regularity if \( r \in (3d/(d + 2), 2] \), \( d = 2, 3 \) and \( C^{1,\alpha} \) regularity if \( r \in (3/2, 2] \) in 2d.

Let us now formulate our main result. As the difficulties with the convective term are in some sense standard, we start with the generalized Stokes system.
Theorem 1.1. Let $q_0 > 2$, $f \in L^{q_0}(\Omega)^2$. Let (1.5) and (A1) be fulfilled with some $r \in (1,2]$, $C_1, C_2 \in \mathbb{R}_+$. Then there exists $\gamma_0 = \gamma_0(f, C_1, C_2, r) > 0$, $q > 2$ such that if (A2) holds, the uniquely defined weak solution $(v, p)$ of
\[- \text{div}(T(D(v)), p) + \nabla p = f\] with (1.2), (1.3), (1.4) satisfies
\[v \in W^{2,q}(\Omega)^2, \quad p \in W^{1,q}(\Omega).\] If $r = 2$ then $\gamma_0$ may be chosen independent of $f$.

Theorem 1.2. Let the assumptions of Theorem 1.1 hold.

- If $r \in (3/2, 2]$ then all weak solutions of (1.1), (1.2), (1.3), (1.4) satisfy (1.7).
- If $r \in (4/3, 3/2]$ then there exists a weak solution to (1.1), (1.2), (1.3), (1.4) satisfying (1.7).

We improved known results in two directions. Firstly we proved existence of regular solution to (1.1) for all $r \in (4/3, 2]$ and secondly we obtained not only $C^{1,\alpha}$ regularity but even $W^{2,\alpha}$ regularity which allows to continue in the regularity ladder if data of the problem are smoother. However our results have also some drawbacks. One of them might be that the result is stated only for periodic boundary conditions but this is more likely technical assumption and it is almost clear that we would be able to obtain some results also for other types of boundary conditions. Another questions which naturally arise are: Is it possible to obtain existence and regularity of solution to (1.1) for $r \leq 4/3$ and $r > 2$? or even: Is it possible to allow different estimates of growth from below and from above? This questions we want to address thoroughly in the forthcoming paper. The limitation which cannot be weakened is however the fact that we must work in 2d. It is namely in the essence of the method we use. It is based on works [16, 18]. The keystone is precise regularity result for the Stokes-like system with only measurable and bounded coefficients.

Lemma 1.1. Let $A(x) \in L^\infty(\Omega)^2 \times 2$ be symmetric matrix fulfilling for some positive $\gamma_1, \gamma_2$, for all symmetric $D \in \mathbb{R}^{2 \times 2}$ and for a.a. $x \in \Omega$
\[\gamma_1 |D|^2 \leq (AD, D) \leq \gamma_2 |D|^2.\] Then there exist positive constants $L > 0$, $K \geq 1$, such that for all weak solutions to
\[\text{div } u = 0, \quad - \text{div } (AD(u)) + \nabla \pi = f, \quad \int_\Omega \pi \, dx = 0, \quad \int_\Omega u \, dx = 0\] with periodic boundary conditions (1.3) and for all $q \in \langle 2, 2 + L\gamma_1/\gamma_2 \rangle$ there holds
\[\frac{1}{\gamma_2} \|\pi\|_q + \|u\|_{1,q} \leq \frac{K}{\gamma_1} \|f\|_{(W^{1,q})'} .\]

Estimates of $u$ stated in this lemma are proved in [10, Lemma 2.6] for the problem (1.8) with homogeneous Dirichlet boundary condition and $f = \text{div } G$. compare also [9, Lemma 2.1] where the case of the right hand side in the negative Sobolev space is treated. The proof of estimate in (1.9) would follow cited works line by line. From the estimate of $\|u\|_{1,q}$ also the bound for the pressure $\pi$ easily follows by Nečas’ theorem about distributions with derivatives in negative Sobolev spaces, see for example [13, Theorem 5.1.14]. In the next text, the constants $K, L$ always denote the constants from Lemma 1.1.
We use Lemma 1.1 to (1.6) differentiated with respect to some space variable to improve integrability of $\nabla^2 v$. This is a delicate procedure as the ellipticity constants of $\partial_b T(D(v), p)$ depend on $D(v)$. Consequently, $\gamma_1$ depends on $D(v)$ and in order to finish the proof we need $W^{1,q}(\Omega) \to C^{1,\alpha}(\Omega)$ for all $q > 2$, which is true only in 2d.

The sketched method was successfully used to get regularity for various systems with shear dependent viscosity, see [8, 10, 11]. Although we follow the same scheme of the proof it is not a trivial modification as the presence of pressure in the viscosity brings new difficulties to be solved.

2. Existence of $C^{1,\beta}$-solution for quadratic growth. This section is devoted to models with quadratic growth, i.e., we consider the viscosity satisfying (A1) and (A2) with parameter $r = 2$. For such type of viscosity we can prove the following theorem.

**Theorem 2.1.** Let $\nu$ satisfy (A1) and (A2) with $r = 2$. Let $\gamma_0 < \frac{C_1}{C_1 + C_2}$. Let $f \in (W^{1,2})^*$. Then there exists unique weak solution to (1.6) with (1.2)-(1.4). Moreover, if $f \in L^{q_0}$ for some $q_0 > 2$ and $\gamma_0 < \frac{C_1}{KC_2 + C_1}$ then

$$\|v\|_{2,q} + \frac{1}{C_2}\|p\|_{1,q} \leq \frac{C}{C_1 - \gamma_0(C_1 + KC_2)}\|f\|_q$$

(2.1)

for all $q \in (2, \min(q_0, L^{C_1/C_2}))$ and constant $C$ depends only on $\Omega$.

**Proof.** The existence and uniqueness of weak solution can be obtained repeating the same procedure developed in [2] where the evolutionary case is treated. To prove the relation (2.1), we construct the sequence of approximative solutions with desired properties. Passing to the limit leads to our goal after using the uniqueness of weak solution. The use of an approximation is for us necessary as in (2.5) we need that the right hand side is finite.

**Step 1:** We define an $\varepsilon$-approximation such that

$$\text{div} v^\varepsilon = 0, \quad -\text{div} (\nu(p^\varepsilon, |D(v^\varepsilon)|^2)D(v^\varepsilon)) + \nabla p^\varepsilon = f.$$  

(2.2)

where the symbol $p^\varepsilon$ denotes the usual mollification of $p$. The existence of strong solution can be again shown as in [2] (without mollification of the pressure in viscosity).

Having existence of the second gradient of velocity and the first gradient of pressure we can differentiate (2.2) in the weak sense. We fix $k \in \{1, 2\}$ and after denoting

$$u^\varepsilon := \frac{\partial v^\varepsilon}{\partial x_k}, \quad \pi^\varepsilon := \frac{\partial p^\varepsilon}{\partial x_k},$$

$$A_{kl}^\varepsilon(x) := \frac{\partial\nu(p^\varepsilon, |D(v^\varepsilon)|^2)D_{ij}(v^\varepsilon)}}{\partial D_{kl}},$$

$$B(x) := \frac{\partial\nu(p^\varepsilon, |D(v^\varepsilon)|^2)}{\partial p}D(v^\varepsilon),$$

we get the following equations

$$\text{div} u^\varepsilon = 0,$$

$$-\text{div} (Au^\varepsilon) + \nabla \pi^\varepsilon = \frac{\partial f}{\partial x_k} + \text{div} (B\pi^\varepsilon).$$

(2.4)
At this point we need that the right hand side of (2.4) belongs to \((W^{1,q})^*\) for \(q > 2\) which is however consequence of the approximation. Since the viscosity satisfies the assumption (A1) and \(\int_\Omega \pi^* \, dx = 0\), we see that all assumptions of Lemma 1.1 are fulfilled with \(\gamma_1 := C_1, \gamma_2 := C_2\) and we get that for all \(q \in \langle 2, \min(q_0, LC_1/C_2) \rangle\) (after using (A2))

\[
\frac{1}{C_2} \|\pi^*\|_q + \|D(u^*)\|_q \leq \frac{K}{C_1} (\|f\|_q + \gamma_0\|\pi^*_\|_q).
\]  

(2.5)

Using the properties of regularization we finally get from (2.5) that

\[
\left(1 - \frac{\gamma_0 K}{C_1}\right) \|\pi^*\|_q + \|D(u^*)\|_q \leq \frac{K}{C_1} \|f\|_q.
\]

(2.6)

Using the assumption on \(\gamma_0\) we have that \(1 - \gamma_0 \frac{K}{C_1} > 0\) and the estimate (2.1) for \(u^*\) and \(p^*\) easily follows.

Due to the apriori estimates (2.1) we can pass to the limit \(\epsilon \to 0_+\) in (2.2) and we get the solution to our original problem satisfying (2.1). \(\square\)

3. Existence of \(C^{1,\beta}\)-solution for sub-quadratic growth. In this section we study again the model (1.6), satisfying (A1), (A2), but now with parameter \(r < 2\).

**Proof of Theorem 1.1.**

**Step 1:** We fix an arbitrary \(\lambda > 0\) and define an approximative viscosity \(\nu_\lambda(p, |D|^2)\) such that

\[
\nu_\lambda(p, |D|^2) := \begin{cases} 
\nu(p, |D|^2) & \text{if } \nu(p, |D|^2) \geq C_2(1 + \lambda^2)^{\frac{r-2}{2}}, \\
C_2(1 + \lambda^2)^{\frac{r-2}{2}} & \text{if } \nu(p, |D|^2) < C_2(1 + \lambda^2)^{\frac{r-2}{2}},
\end{cases}
\]

(3.1)

We want to solve the following system of equations

\[
\text{div } v = 0, \\
- \text{div}(\nu_\lambda(p, |D(v)|^2)D(v)) + \nabla p = f,
\]

(3.2)

with (1.3)-(1.4) and apply the theory that has been already developed in the previous section. To do it we must recover that the viscosity \(\nu_\lambda\) satisfies the assumptions of Theorem 2.1.

**Lemma 3.1.** Setting

\[
\gamma_1 := \frac{C_1}{4}(1 + \lambda^2)^{\frac{r-2}{2}}, \quad \gamma_2 := C_2,
\]

(3.3)

the viscosity \(\nu_\lambda\) satisfies for a.a. \(D, B \in S, \ p \in \mathbb{R}\)

\[
\gamma_1 |B|^2 \leq \frac{\partial [\nu_\lambda(p, |D|^2)D_{ij}]}{\partial D_{kl}} B_{ij} B_{kl} \leq \gamma_2 |B|^2.
\]

(3.4)

**Proof.** We denote

\[
I := \frac{\partial [\nu_\lambda(p, |D|^2)D_{ij}]}{\partial D_{kl}} B_{ij} B_{kl}.
\]

First note that because the original viscosity \(\nu\) satisfies (A1) with parameters \(r, C_1, C_2\) we can get (setting in (A1) \(B \in S\) such that \(B \neq 0, \ B_{ij}D_{ij} = 0\))

\[
C_1(1 + |D|^2)^{\frac{r-2}{2}} \leq \nu \leq 4C_2(1 + |D|^2)^{\frac{r-2}{2}}.
\]

(3.5)
To verify (3.4), we first discuss the case when \( \nu = \nu_\lambda \). Thus, if \( \nu \geq C_2(1 + \lambda^2)^{-1/4} \), then there holds

\[
C_1(1 + |D|^2)^{1/2} |B|^2 \leq I \leq C_2(1 + |D|^2)^{1/2} |B|^2. \tag{3.6}
\]

Using (3.5) and the fact that now \( \nu \geq C_2(1 + \lambda^2)^{-1/4} \), we can get

\[
\frac{C_1}{4}(1 + \lambda^2)^{1/2} |B|^2 \leq I \leq C_2 |B|^2. \tag{3.7}
\]

On the set where \( \nu < C_2(1 + \lambda^2)^{-1/4} \), we have that \( \nu_\lambda = C_2(1 + \lambda^2)^{-1/4} \) and

\[
I = C_2(1 + \lambda^2)^{-1/4} |B|^2.
\]

Finally, using that \( C_2 \geq C_1 \), we see that (3.4) holds. \( \square \)

As a simple consequence of the definition of \( \nu_\lambda \) we also see that the assumption (A2) is fulfilled with parameter \( \gamma_0 \).

Assume that \( \lambda \) is chosen such that

\[
(1 + \lambda^2)^{-1/4} \leq \frac{C_1}{16\gamma_0KC_2} \quad \iff \quad \gamma_0 \leq \frac{\gamma_1}{4K\gamma_2} \left( \frac{1}{2} \frac{1}{2K\gamma_2 + \gamma_1} \right). \tag{3.8}
\]

Then we can apply Theorem 2.1 to system (3.2) with (1.3)-(1.4) to get for all \( q \in (2, \min(q_0, L\gamma_1/\gamma_2)) \) that

\[
\|v\|_2 \leq C \frac{\gamma_1}{\gamma_1 - \gamma_0(K\gamma_2 + \gamma_1)} \|f\|_2 \leq \frac{2C}{\gamma_1} \|f\|_q. \tag{3.9}
\]

Now, we derive some estimates that are \( \lambda \) independent. To simplify expressions we will assume that \( \|f\|_2 \geq 1 \), thus we can neglect some small constant including them in \( \|f\|_2 \). First, after testing (3.2) by \( v \) we obtain (after using the definition of \( \nu_\lambda \)) that

\[
\int_{\Omega}(1 + |D|^2)^{-1/4} |D(v)|^2 \, dx \leq C(r, C_1) \|f\|_2 \|D(v)\|_r.
\]

Using the same procedure as in [13, Proof of Theorem 5.2.17] we obtain

\[
\|D(v)\|_r \leq C(r, C_1) \|f\|_2^{-1/4}. \tag{3.10}
\]

If we denote \( \Omega_\lambda := \{ x \in \Omega : \nu_\lambda(p(x), |D(v(x))|^2) = \nu(p(x), |D(v(x))|^2) \} \) and multiply (3.2) by \(-\Delta v \) we get (after integration over \( \Omega \) and by parts)

\[
C_1 \int_{\Omega_\lambda}(1 + |D|^2)^{-1/4} |D(v)|^2 \, dx + C_2(1 + \lambda^2)^{-1/4} \int_{\Omega\setminus\Omega_\lambda} |D(v)|^2 \, dx
\leq 2 \int_{\Omega} |f| |D(v)| \, dx + \gamma_0 \int_{\Omega} |\nabla p|(1 + |D|^2)^{-1/4} |D(v)| \, dx \tag{3.11}
\]

Multiplying the equation (3.2) by \( \nabla \chi_{\Omega_\lambda} \) and integrating the result over \( \Omega \) we get (here \( \chi_{\Omega_\lambda} \) denotes the characteristic function of the set \( \Omega_\lambda \))

\[
\|\nabla p\|_{2,\Omega_\lambda} \leq \frac{1}{1 - \gamma_0} \left( \|f\|_2 + C_2 \left( \int_{\Omega_\lambda}(1 + |D|^2)^{-1/4} |D(v)|^2 \right)^{1/2} \right). \tag{3.12}
\]

We define

\[
I := \int_{\Omega}(1 + |D|^2)^{-1/4} |D(v)|^2 \, dx.
\]
The inequality (3.11) together with the fact that $C_1(1 + |D(v)|^2)^{\frac{1}{r}} \leq C_2(1 + \lambda^2)^{\frac{1}{r}}$ in $\Omega \setminus \Omega_x$ then implies that (after using Hölder inequality)

$$C_1 I \leq I^{1/2} \left( \gamma_0 \|\nabla p\|_{2,\Omega_x} + \|f\|_{q_0} \left( \int_{\Omega} (1 + |D(v)|^2)^{\frac{(2-r)q_0}{(2q_0-2)}} \right)^{\frac{2q_0-2}{q_0}} \right). \quad (3.13)$$

If we define the function $V := (1 + |D(v)|^2)^{1/2}$ then (3.13) reduces to

$$C_1 I^{1/2} \leq \left( \gamma_0 \|\nabla p\|_{2,\Omega_x} + \|f\|_{q_0} \|V\|_{2} \left( \frac{2-r}{(2q_0-2)} q_0 \right) \right). \quad (3.14)$$

Combining (3.12) and (3.14) together with our choice of $\gamma_1$ (which is given by choice of $\lambda$ in (3.8)) then implies that

$$I^{1/2} \leq \frac{2}{C_1} \left( \|f\|_{2} + \|f\|_{q_0} \|V\|_{2} \left( \frac{2-r}{(2q_0-2)} q_0 \right) \right). \quad (3.15)$$

In the rest of this step we derive from (3.15) an estimate of $I^{1/2}$ independent of $v$ and $\lambda$. We distinguish two cases.

If $q_0 \geq r'$ we obtain from (3.15) that

$$I^{1/2} \leq \frac{2}{C_1} \left( \|f\|_{2} + \|f\|_{q_0} \|D(v)\|_{r} \right)^{\frac{2}{r}} \leq C(r, C_1) \left( \|f\|_{2} + \|f\|_{q_0} \|f\|_{2} \right)^{\frac{2-r}{2}}. \quad (3.16)$$

If $q_0 < r'$ the estimate is more delicate. It is a simple consequence of the definition of $V$ that for all $q > 2$

$$\|\nabla V^2\|_2 \leq I^2, \quad (3.17)$$

$$\|\nabla V^2\|_q \leq \|v\|_{2,q}.$$ Thus, combining (3.17), (3.10) and using embedding theorem $W^{1,2} \hookrightarrow L^a$ for all $a \in (2, \infty)$ we obtain

$$\|V^2\|_a \leq C(a)(I^2 + \|f\|_{2}^2). \quad (3.18)$$

Substituting this relation with $a = 2q_0(2-r)/((q_0-2)r)$ into (3.15) we have that

$$I^{1/2} \leq \frac{C(r, q_0)}{C_1} \left( \|f\|_{2} + \|f\|_{q_0} \left( I^{\frac{2}{r}} + \|f\|_{2} \right) \right) \quad (3.19)$$

and after using Young’s inequality we get

$$I^{1/2} \leq C(r, q_0, C_1) \left( \|f\|_{2} + \|f\|_{q_0} \|f\|_{2} + \|f\|_{q_0} \|f\|_{2} \right), \quad (3.20)$$

where the right hand side is bounded independently of $\lambda$.

**Step 2:** In this step we finish the proof of Theorem 1.1 combining estimate (3.9) with (3.16) or (3.20). The key idea is following. In approximation (3.1) $\nu_\lambda$ differs from $\nu$ only if $\nu(p, |D|^2) < C_2(1 + \lambda^2)^{(r-2)/2}$, so it is enough to find such $\lambda > 0$ that for the unique solution $(v, p)$ of the approximating problem (3.2) holds

$$\left( \nu_\lambda(p(x), D(v(x))) \right) \geq \left( C_1(1 + |D(v(x))|^2)^{\frac{r-2}{2}} \geq C_2(1 + \lambda^2)^{\frac{r-2}{2}} \right) \text{ for a. a. } x \in \Omega. \quad (3.21)$$

Then the solution $(v, p)$ of the approximative problem (3.2) is also a solution of (1.6). However, we want to gain the estimate (3.21) from (3.9) (together with (3.16) or (3.20)), which was derived under assumption (3.8). Note that while in (3.21) we require that $\lambda$ is large, in (3.8) we need its smallness. This fact suggests that we can not find suitable $\lambda$ for arbitrary choice of $\gamma_0, C_1$ and $C_2$ but that we
need some additional condition for them. The rest of the proof is devoted to finding this condition and \( \lambda > 0 \) such that (3.21) and (3.8) hold.

Assume that we have \( \lambda \) so large that

\[
q_0 \geq 2 + L \frac{\gamma_1}{\gamma_2} \iff (1 + \lambda^2) \geq \left( \frac{LC_1}{4C_2(q_0 - 2)} \right)^{\frac{2}{\gamma^-}}. \tag{3.22}
\]

Then we can set \( q := 2 + L \gamma_1/\gamma_2 \) in (3.9). For arbitrary \( \alpha \in (0, 1) \) we find \( s > 2 \) satisfying

\[
\frac{1}{s} = \frac{\alpha}{2} + \frac{1 - \alpha}{q}. \tag{3.23}
\]

Using successively embedding theorem, see [20, proof of Theorem 2.4.1] for precise embedding constant, interpolation, (3.17), (3.9) and the fact that \( s, q \in (2, q_0) \) together with (3.23) we have that

\[
\|V^\sharp\|_\infty \leq C \left( \frac{1}{s - 2} \right)^{1 - \frac{1}{\alpha}} \|V^\sharp\|_{1, s} \leq C \left( \frac{1}{s - 2} \right)^{1 - \frac{1}{\alpha}} \|V^\sharp\|_{1, 2}^{\alpha} \|V^\sharp\|_{1, q}^{1 - \alpha}
\]

\[
\leq C \left( \frac{1}{s - 2} \right)^{1 - \frac{1}{\alpha}} \|V^\sharp\|_{1, 2}^{\alpha} \|V^\sharp\|_{1, q}^{1 - \alpha}
\]

Using (3.24) still depends on \( \lambda \) through \( q \) and \( \gamma_1 \). Taking into account definition of \( q \) and (3.3) we obtain that

\[
\|V^\sharp\|_\infty \leq C(C_1, C_2, q_0) \frac{1}{1 - \alpha} \|V^\sharp\|_{1, 2}^{\alpha} \|f\|_{q}^{1 - \alpha}. \tag{3.25}
\]

As (3.21) is equivalent to

\[
\|V^\sharp\|_\infty \leq (1 + \lambda^2)^{\frac{C_1}{C_2}} \tag{3.26}
\]

we see, using (3.25), that to obtain (3.21) it is enough to show that

\[
C(C_1, C_2, q_0, r) \frac{(1 + \lambda^2)^{\frac{C_1}{C_2}}(\frac{\lambda^2}{\gamma^-} - 2 + \frac{1}{\alpha})}{1 - \alpha} \|V^\sharp\|_{1, 2}^{\alpha} \|f\|_{q}^{1 - \alpha} \leq (1 + \lambda^2)^{\frac{C_1}{C_2}}. \tag{3.27}
\]

Next, we use the fact that

\[
(1 + \lambda^2)^{\frac{C_1}{C_2}}(\frac{\lambda^2}{\gamma^-} - 2 + \frac{1}{\alpha}) \leq (1 + \lambda^2)^{\frac{C_1}{C_2}}(\frac{\lambda^2}{\gamma^-} - 2)
\]

and obtain that it is enough to have the validity of the following inequality

\[
C(C_1, C_2, q_0, r) (1 - \alpha)^{-1} \|V^\sharp\|_{1, 2}^{\alpha} \|f\|_{q}^{1 - \alpha} \leq (1 + \lambda^2)^{\frac{C_1}{C_2}}(\frac{\lambda^2}{\gamma^-} - 2). \tag{3.28}
\]

First, if \( r > \frac{\lambda}{\gamma} \) then we can set \( \alpha := 0 \) to get from (3.28) that we need to have fulfilled the following inequality

\[
C(C_1, C_2, q_0, r) \|f\|_{q} \leq (1 + \lambda^2)^{\frac{r \gamma^-}{\gamma^-}}. \tag{3.29}
\]

We also need to have fulfilled (3.22), so combining it with (3.29) we need to set \( \lambda \) so that

\[
(1 + \lambda^2) \geq C(C_1, C_2, q_0, r) \max(\|f\|_{q}^{\frac{1}{\gamma^-}}, (q_0 - 2)^{\frac{2}{\gamma^-}}). \tag{3.30}
\]
On the other hand, we need to know that such $\lambda$ satisfies also (3.8) that is equivalent to

$$(1 + \lambda^2) \leq \left( \frac{C_1}{16\eta_0 K C_2} \right)^{\frac{2}{5-4r}}. \quad (3.31)$$

Hence, we can find $\lambda$ satisfying (3.8) and (3.21), especially (3.30) and (3.31) if

$$\gamma_0 \leq C(C_1, C_2, q_0, r) \min \left( \|f\|_{q}^{\frac{2(4r-7)}{12r-5}}, (q_0 - 2) \right) \quad (3.32)$$

that completes the proof for $r > \frac{8}{5}$.

Second, for $r \leq \frac{8}{5}$ we set in (3.28)

$$\alpha := \frac{7 - 4r}{3(2 - r)} \Rightarrow r + \frac{2 - r}{2} \left( \frac{3\alpha}{2} - 2 \right) = \frac{r - 1}{4}. \quad (3.33)$$

Note that because $r \in (1, \frac{8}{5})$ we get that $\alpha \in (0, 1)$. Substituting (3.33) into (3.28) we get that we need

$$C(C_1, C_2, q_0)(r - 1)^{-1} \left\| V^2 \right\|_{1,2, r}^{\frac{7 - 4r}{12r - 5}} \|f\|_{q}^{-\frac{4r-7}{12r-5}} \leq (1 + \lambda^2)^{-\frac{1}{2}} \quad (3.34)$$

that is equivalent to setting $\lambda$ so large that

$$(1 + \lambda^2) \geq C(C_1, C_2, q_0, r) \left\| V^2 \right\|_{1,2, r}^{\frac{4r-7}{12r-5}} \|f\|_{q}^{-\frac{4r-7}{12r-5}} \quad (3.35)$$

is valid. Moreover, we again need to set such $\lambda$ that (3.8) and (3.22) are valid. If we combine the relation (3.8), (3.22) and (3.35), we see that we can find such $\lambda$ if

$$\gamma_0 \leq C(C_1, C_2, q_0, r) \min \left( \|V^2\|_{1,2, r}^{\frac{2(4r-7)}{12r-5}}, \|f\|_{q}^{\frac{7-4r}{12r-5}}, (q_0 - 2) \right), \quad (3.36)$$

that completes the proof for $r < \frac{8}{5}$ if we realize that $\|V^2\|_{1,2, r} \leq C(f, C_1, C_2, r)$ by (3.10) and (3.17) together with (3.16) or (3.20).

\(\square\)

4. Sketch of proof of Theorem 1.2. This final section is devoted to the existence and regularity of solutions to the full system (1.1). We sketch proof of Theorem 1.2. First, note that the existence of weak solution to (1.1)-(1.4) was established in [3] for $r \in (4/3, 2]$.

Next, for $r \in (3/2, 2]$ it is allowed to test weak formulation of (1.1) with a weak solution and by standard difference method it follows that every weak solution is in $W^{2,r}(\Omega)$. For fixed weak solution to (1.1) the convective term can be moved to the right hand side of (1.1) as $\nu \nabla \nu \in L^{r/(2-r)}$ and Theorem 1.1 can be applied. Due to uniqueness of the solution to (1.6) it follows that the given weak solution of the nonlinear problem (1.1) must satisfy (1.7).

If $r \in (4/3, 2]$ we can get existence of the $C^{1,\alpha}$ solution by the same method that is used in Section 3. The only difference is that we start with getting estimates (3.15) and (3.20). In (3.14) on the right hand side there appears new term that must be estimated, namely $\|\nu \nabla \nu\|_2$. If $r > 4/3$ it holds

$$\|\nu \nabla \nu\|_2 \leq \|\nu \nabla \nu\|_{\frac{7}{3-4r}} \leq \|\nabla \nu\|_r \|\nabla^2 \nu\|_r. \quad (4.1)$$

As $\|\nabla^2 \nu\|_r \leq C(f) \mathcal{T}^{1/2}$ the estimates (3.15) and (3.20) follow. Looking at (4.1) once more we see that $\nu \nabla \nu \in L^{r/(2-r)}$. Since $r/(2-r) > 2$ we can add it to the right hand side and continue in the proof exactly as in Section 3. Finally, we get existence of solution satisfying (1.7).
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