Soft morphological filtering using hypergraphs

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Abstract. A new framework of soft mathematical morphology on hypergraph spaces is studied. Application in image processing for filtering objects defined in hypergraph spaces are illustrated using several soft morphological operators- openings, closings, granulometries and ASF acting (a) on the subset of vertex and hyperedge set of a hypergraph and (b) on the subhypergraphs of a hypergraph. Experimental results dealing with the extension of soft morphological operators to grayscale images are presented in this paper. The results obtained are promising and is a better substitute for the prevailing methods.

1. Introduction and Related Works

Mathematical morphology is an emerging area of image processing. In the area where image processing can be applied, mathematical morphology have provided solutions to different tasks. Soft mathematical morphology is another approach to mathematical morphology that was introduced by Koskinen et al. [1]. In this method, weighted order statistics are used instead of minima and maxima [3]. Soft morphological operators show better performance than primitive morphological operators as they are less sensitive to additive noise and to small variations in object shape [4]. Soft mathematical morphology have also been extended to fuzzy sets [3].

Morphological operators can be defined on graphs and hypergraphs. In this paper a new context is introduced that lengthens the perceptions of soft morphological filters into hypergraphs. This work is an extension of our preliminary work [2].

2. Preliminaries

In this section, we see some important definitions and properties that will be needed in the sequel.

2.1. Soft mathematical morphology using hypergraphs [2]

Hypergraph [5], [6] is defined as a pair \( H = (H', H^X) \), where \( H' \) is a set of points called vertices and \( H^X \) is a family of subsets of \( H' \) called hyperedges i.e; \( H^X = \{e_i\}, i \in I \) where \( I \) is a finite set of indices. The sets \( H' \),
H^X and H are the subsets of H', subsets of H^X and subhypergraphs of H respectively. Here H^X and H^* are Boolean lattices. The set H of all sub hypergraphs of H forms a complete lattice [5], [6]. Morphological operators are defined on these lattices. In soft mathematical morphology, the structural element is divided into two subsets - the core and the soft boundary. In the development of the final output, weightage of core is more than the soft boundary. Hence we subdivide the hyperedges (along with the vertices belonging to it) into core B1 and soft boundary B2.

2.1.1. Definition 1
The operators \( \delta^* \), \( \varepsilon^* \) are defined from \( H^X \) into \( H^* \) and the operators \( \delta^X \), \( \varepsilon^X \) are defined from \( H^* \) into \( H^X \) as follows: For any \( X^* \subseteq H^* \) and and \( X^X \subseteq H^X \), where \( X^X = e_i \cup e^i \); \( e_i \in B_1 \) and \( e^i \in B_2 \), \( i \in J \) such that \( J \subseteq I \),

\[
\delta^*: H^X \rightarrow H^* \text{ is defined as } \delta^*(X^X) = k^{th} \text{ largest of } [\bigcup_{j \in J} \{k \circ v(e_j), v(e^j) ; e_j \in B_1, e^j \in B_2\}]
\]

\[
e^X: H^* \rightarrow H^X \text{ is defined as } e^X(X^*) = k^{th} \text{ smallest of } \{k \circ e_i, e^i, i \in I | v(e_i), v(e^i) \subseteq X^*, e_i \in B_1, e^i \in B_2\}
\]

\[
e^*: H^* \rightarrow H^* \text{ is defined as } e^*(X^*) = k^{th} \text{ smallest of } [\bigcap_{j \in J} \{k \circ \overline{v(e_j)}, \overline{v(e^j)} ; e_j \in B_1, e^j \in B_2\}]
\]

Here, \( k \circ x \) is read as \( x \) is repeated \( k \) times. Vertex soft dilation (\( \delta \)) and vertex soft erosion (\( \varepsilon \)) that act on \( H^* \), also hyper edge soft erosion (\( \varepsilon \)), hyper edge soft dilation (\( \delta \)) that act on \( H^X \) are defined as follows:

The operators \( \delta \) and \( \varepsilon \) that act on \( H^* \) are defined by \( \delta = \delta^* \circ \delta^X \) and \( \varepsilon = \varepsilon^* \circ \varepsilon^X \) and the operators \( \Delta \) and \( \varepsilon \) that act on \( H^X \) are defined by \( \Delta = \delta^X \circ \delta^* \) and \( \varepsilon = \varepsilon^X \circ \varepsilon^* \). Furthermore the operators \([\delta, \Delta] \) and \([\varepsilon, \varepsilon^*] \) are defined as

\[
[\delta, \Delta](X) = (\delta(X^*), \Delta(X^X)) \text{ and } [\varepsilon, \varepsilon^*](X) = (\varepsilon(X^*), \varepsilon^*(X^X)) \text{ for any } X \in H.
\]

These operators are called the soft dilation and soft erosion acting on the lattice \((H^*, \subseteq)\)

3. Property
(a) Operators \( \varepsilon^X \) and \( \delta^X \) are dual of each other. Similar duality concept hold for \( \varepsilon^* \) and \( \delta^* \).
(b) \( (\varepsilon^X, \delta^*) \) and \( (\varepsilon^*, \delta^X) \) are adjunctions.
(c) \( \delta^* \) and \( \delta^X \) are soft dilations.
(d) \( \varepsilon^* \) and \( \varepsilon^X \) are soft erosions.

Proof (a): This property is proved in [2].

(b): Let \( X^X \subseteq \subseteq X^X \) (\( Y^* \)). Then,

\[
x \in \delta^*(X^X) \implies x \in k^{th} \text{ largest of } [\bigcup_{j \in J} \{k \circ v(e_j), v(e^j) \}]
\]

\[
\implies x \in v(e_j), x \in v(e^j) \text{ for some } j \in J
\]

\[
\exists e \in X^X \text{ such that } x \in v(e)
\]

\[
\implies e \in e^X(Y^*) \text{ since } X^X \subseteq X^X(Y^*)
\]

\[
\implies e \in k^{th} \text{ smallest of } \{k \circ e_i, e^i, i \in I | v(e_i), v(e^i) \subseteq Y^* \}
\]

\[
\implies v(e) \subseteq Y^*
\]

\[
\implies x \in Y^* \text{ since } x \in v(e)
\]

Thus, \( \delta^*(X^X) \subseteq Y^* \).

Conversely, if \( \delta^*(X^X) \subseteq Y^* \).
Then, \( e \in X \Rightarrow \nu(e) \subseteq \delta^*(X^X) \)
\[ \Rightarrow \nu(e) \subseteq Y^* \quad \{ \text{Since } \delta^*(X^X) \subseteq Y^* \} \]
\[ \Rightarrow e \in e^X (Y^*) \]
Therefore, \( X^X \subseteq e^X (Y^*) \).
Hence, \((e^X, \delta^*)\) is an adjunction.

In a similar manner, \((e^*, \delta^X)\) is an adjunction can be proved using the information that operators \( e^X \) and \( \delta^X \) are dual of each other, also \( e^* \) and \( \delta^* \) are dual of each other and \((e^X, \delta^*)\) is an adjunction. Properties (c) and (d) follows directly using property (b).

4. Soft morphological filters
A morphological filter \([5], [8]\) is an operator \( \delta \) acting on a lattice \( \mathcal{L} \), which is increasing and idempotent. If \((\alpha, \beta)\) is an adjunction \([5], [8]\) then \( \alpha \) is an erosion, \( \beta \) is a dilation. Also, \( \beta \cdot \alpha \) is called an opening and \( \alpha \cdot \beta \) is called a closing on \( \mathcal{L} \). Opening and closing are two commonly used filters.

4.1. Definition 2
- Soft opening \( \gamma_1 = \delta \ast \mathcal{E} \) and closing \( \Phi_1 = \mathcal{E} \ast \delta \).
- Soft half opening \( \gamma_{1/2} = \delta^* \ast e^X \) and half closing \( \Phi_{1/2} = e^* \ast \delta^X \).

5. Soft flat morphological Operators on weighted hypergraphs
In real phase, gray-scale soft morphological operations are difficult to work with. Threshold decomposition and stacking principle can be successfully applied on gray-scale soft morphological operations. This property allows the gray scale signals to be decomposed into binary signals and the results so obtained by processing are combined to obtain the desired gray-scale output \([7]\). By threshold decomposition, \([8]\) the lattice of all subhypergraphs of \( H \) induces a lattice \( \text{Fun}(H^*) \otimes \text{Fun}(H^X) \) of pairs of functions weighting respectively the vertices and the hyperedges of \( H \) such that the simultaneous threshold of these two functions at any given level produces a subhypergraph of \( H \) and the properties for hypergraph operators on the lattices \( H^*, H^X \), or \( \mathcal{H} \) also hold for operators on the lattices \( \text{Fun}(H^*), \text{Fun}(H^X) \) and \( \text{Fun}(H^*) \otimes \text{Fun}(H^X) \). Thus we can define a set of soft operators which are stack analogues to the soft operators \( \mathcal{E}^X, \delta^*, \mathcal{E}^* \) and \( \delta^X \) defined in \([2]\).

5.1. Definition 3.
Let \( F^* \in \text{Fun}(H^*) \) and let \( F^X \in \text{Fun}(H^X) \), we define
\[ \delta^*(F^X)(x) = k^{th} \text{ largest of } \{ k \circ F^X(e_i), F^X(e'_i) \} \begin{cases} e_i \in B_1, e'_i \in B_2, B_1 \cup B_2 = H^X & \forall x \in H^* \end{cases} \]
\[ e^*(F^X)(x) = k^{th} \text{ smallest of } \{ k \circ F^X(x), F^X(x) \} \begin{cases} x \in \nu(e), e \in B_1 \cup B_2 & \forall e \in \mathcal{E} \end{cases} \]
\[ \delta^X(F^*)(e) = k^{th} \text{ smallest of } \{ k \circ F^*(e_i), F^*(e'_i) \} \begin{cases} e_i \in B_1, e'_i \in B_2, B_1 \cup B_2 = H^* & \forall x \in H^* \end{cases} \]
\[ \delta^X(F^*)(e) = k^{th} \text{ smallest of } \{ k \circ F^*(x), F^*(x) \} \begin{cases} x \in \nu(e), e \in B_1 \cup B_2 & \forall e \in \mathcal{E} \end{cases} \]

By using soft flat morphology, it is possible to work with gray scale soft dilation, erosion, opening, closing, granulometries and Alternating Sequential Filters.

6. Experimental Result
To represent the hyperedges we use 4-uniform hypergraph illustrated in Figure 1. A hyperedge together with the 4 vertices belonging to it is taken as core, remaining vertices and hyperedges are taken as soft boundary. Using this simple hypergraph structure, experimental result for \( k=1 \) and \( k=2 \) are tabulated. Dilated and eroded gray scale image results are obtained to get the soft flat morphological operators defined in the previous section. Composition of these operators generates alternating sequential filters(ASF) which
are capable of removing noise effectively from binary and gray scale images. Figure 2 represent the gray scale image taken for the experimental purpose. The noisy image obtained by adding 5% salt and pepper noise is shown in figure 3. PSNR of noised image with original image is 17.24. Figure 4 and figure 5 shows the noise removed images obtained by applying alternating sequential filters (ASF) $\gamma_1^0\Phi_1$ and $\gamma_{1/2}^0\Phi_{1/2}$ for $k=1$. The resultant PSNR of the noise removed images with the original image are respectively 40.53 and 40.23. Similarly figure 6 and figure 7 shows the results obtained after applying $\gamma_1^0\Phi_1$ and $\gamma_{1/2}^0\Phi_{1/2}$ for $k=2$. In this case, the resultant PSNR of the noise removed images with the original image are 39.79 and 41.82 respectively. Experiment results depicts that $\gamma_{1/2}^0\Phi_{1/2}$ for $k = 2$ is giving better approximations than $\gamma_1^0\Phi_1$ and $\gamma_{1/2}^0\Phi_{1/2}$ for $k = 1.$

![Figure 1 Four Uniform Hypergraph structure used to represent an image](image1.png)

| PSNR of noised image with original image | 17.24 |
|-----------------------------------------|-------|

![Figure 2. Original Image](image2.png)  ![Figure 3. Noised Image](image3.png)

(Added Noise 5%)
Noise Removed Images using Alternating Sequential Filters

For $K = 1$

| Figure 4. | $\gamma_1 \cdot \Phi_1$ | PSNR = 40.53 |
| Figure 5. | $\gamma_{1/2} \cdot \Phi_{1/2}$ | PSNR = 40.23 |

For $K = 2$

| Figure 6. | $\gamma_1 \cdot \Phi_1$ | PSNR = 39.79 |
| Figure 7. | $\gamma_{1/2} \cdot \Phi_{1/2}$ | PSNR = 41.82 |
7. Conclusion
The aim of this work is to recognize the prospects of using soft morphological operators under the framework of hypergraph. In our study a four uniform hypergraph structure was taken. It was then separated into core and soft boundary. But, it's a well-known fact that 3-uniform hypergraph structure gives good result for binary, gray scale and colour image filtering applications [8]. So by the choice of different hypergraph structures along with the suitable choice of core and soft boundary, results of the above method can be improved for different and bigger values of k. The initial results are promising and future work is to be done in this regard using different hypergraph structures.

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