A Simple Homological Characterization of String Algebras of Finite Representation Type

Mariano Suárez-Álvarez

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Abstract
We prove that among the finite-dimensional algebras of finite representation type those that are string algebras are precisely the ones that have the property that the middle term of an arbitrary extension of indecomposable modules has at most two direct factors. On the other hand, we show that non-domestic string algebras are very far from having that property.

Keywords String algebras · Extensions of modules · Degenerations of modules · Representation type

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1 Introduction
Throughout this paper we denote by $k$ an algebraically closed field, algebras are finite-dimensional $k$-algebras and modules over them finitely generated.

I.M. Gel’fand and V.A. Ponomarev made in [10] the observation that the representation theory of the Lorentz group is closely related to that of a certain algebra, and used this to show that the group has tame representation type and to describe explicitly its finite-dimensional indecomposable modules. Starting from that, and thanks to the work of many authors — C.M. Ringel [14], V.M. Bondarenko [2], P.W. Donovan and M.-R. Freislich [7], M.C.R. Butler and C.M. Ringel [5], among others — the class of finite-dimensional algebras to which the same techniques can be applied was identified, shown to be quite extensive and to contain many examples of interest. Today we have a surprisingly detailed picture of the representation theory of these string algebras, as they are called.

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In memory of Andrzej Skowroński

Mariano Suárez-Álvarez
mariano@dm.uba.ar

1 IMAS-CONICET, Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I. (1428) Ciudad de Buenos Aires, Argentina
In particular, it is well-known and important that the middle term of every almost split short exact sequence in the category of modules over a string algebra has at most two direct factors. In this note I will show that, in fact, this is true of \textit{all} extensions of indecomposable modules provided the algebra has finite representation type and, moreover, that this class of algebras is completely characterized by this property. Precisely, my main result is:

\textbf{Theorem A} \ A finite-dimensional algebra of finite representation type is a string algebra if and only if the middle term of every extension between its indecomposable modules has at most two direct factors.

The proof of the necessity of the condition that I can give is based on the proof of Theorem 1 in Christine Riedtmann’s paper [13] and involves the theory of degenerations of modules. The key observation is the following statement that appears as Proposition 3 in the body of this paper:

\textbf{Theorem B} \ Let \( A \) be a string algebra of finite representation type and let \( M \) and \( N \) be two \( A \)-modules with the same dimension vectors. If \( M \) degenerates to \( N \), then \( N \) has at least as many direct factors as \( M \).

The proof of this depends crucially both on the finiteness of the representation type and on the fact that the middle terms of almost split short exact sequences of modules over string algebras have at most two direct factors.

It is natural to wonder what happens to Theorem A if the hypothesis on the representation type is dropped. What I can prove in that direction is that some control on the representation type is needed if one wants a bound on the number of direct factors of an extension of indecomposable modules: for string algebras we need to be as close to having finite representation type as possible, as the following result shows.

\textbf{Theorem C} \ If a string algebra has non-domestic representation type, then there are extensions of indecomposable modules with arbitrarily many direct factors.

Here, that the string algebra, which certainly has tame representation type, have non-domestic representation type means that the minimal number of 1-parametric families of indecomposable modules needed to describe the indecomposable modules of each dimension \( d \) is not bounded by a scalar independent of \( d \). Proving Theorem C seems to require delving into the combinatorics of bands. I do this in Section 4, with a beautiful lemma of Claus Ringel from [16] as starting point.

Theorems A and C leave open the question of whether one can find a bound for the number of direct factors in an extension of indecomposable modules over domestic string algebras or not. This question seems also to require dealing with the combinatorics of bands, and I expect to treat it elsewhere.

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\textbf{Conventions} \ As mentioned above, throughout this paper \( k \) denotes an algebraically closed field. Our algebras are finite-dimensional \( k \)-algebras except possibly for path algebras, and modules are finite-dimensional right modules. If \( \alpha \beta \) is a path in a quiver, then the arrow \( \alpha \)
ends where the arrow $\beta$ starts. If $M$ is a module, we denote by $|M|$ the number of summands in any decomposition of $M$ as a direct sum of indecomposable submodules.

Except when defining what a string algebra is we will implicitly suppose that all algebras are basic. This is no restriction, as our results are all Morita invariant, and allows us to assume that algebras are given by quivers and relations.

## 2 Middle Terms with at Most Two Direct Factors

Let $Q$ be a finite quiver, let $kQ$ be the corresponding path algebra, and let $I$ be an admissible ideal in $kQ$. We say, as Andrzej Skowroński and Josef Waschbüscher do in [17], that the presentation $(Q, I)$ is **special biserial** if the following two conditions are satisfied.

(S1) Every vertex in $Q$ has in-degree at most 2 and out-degree at most 2.
(S2) If $\alpha$ is an arrow in $Q$, then there is at most one arrow $\beta$ such that $\alpha \beta$ is a path in $Q$ and does not belong to $I$, and there is at most one arrow $\gamma$ such that $\gamma \alpha$ is a path in $Q$ and does not belong to $I$.

If additionally the following third condition is satisfied we say that the presentation $(Q, I)$ is **string**.

(S3) The ideal $I$ is generated by the paths it contains.

A finite-dimensional algebra $A$ is a **string algebra** if it is Morita equivalent to an algebra of the form $kQ/I$ for some finite quiver $Q$ and some admissible ideal $I$ in the path algebra $kQ$ such that the presentation $(Q, I)$ is string.

The first step on proving our main result is the following.

**Proposition 1** Let $Q$ be a finite quiver, let $I$ be an admissible ideal in the path algebra $kQ$ and let $A$ be the quotient algebra $kQ/I$.

(i) If every extension between indecomposable $A$-modules has at most two direct factors, then the presentation $(Q, I)$ satisfies the condition (S1).

(ii) Moreover, if the presentation $(Q, I)$ also satisfies the condition (S2), then it is in fact string.

**Proof** Let us suppose that every extension of indecomposable $A$-modules has at most two direct factors and, to reach a contradiction, that there are in $Q$ three arrows $\alpha$, $\beta$ and $\gamma$ with the same target vertex $i$. We let $j_1$, $j_2$ and $j_3$ be the source vertices of those three arrows; notice that the cardinal of the set $\{i, j_1, j_2, j_3\}$ can be any integer from 1 to 4. In any case, $Q$ contains three arrows that look like

\[
\begin{array}{c}
j_1 \\
j_2 \\
j_3
\end{array}
\]

and we can construct an extension of a 5-dimensional indecomposable module by the simple module at $i$ that, if we draw dimension vectors, looks as follows:

\[
0 \rightarrow 0 \ 0 \ 0 \rightarrow 1 \ 0 \ 0 \oplus 0 \ 1 \ 0 \oplus 0 \ 0 \ 1 \rightarrow 1 \ 1 \ 2 \rightarrow 1 \ 2 \ 1 \rightarrow 0
\]
That there are such $A$-modules is immediate: we certainly have $\mathbb{K}Q$-modules with these descriptions and every path of length 2 acts as zero on them, so they are in fact $A$-modules. Since the existence of this short exact sequence contradicts the hypothesis, we see that every vertex in $Q$ has in-degree at most 2. A dual argument also bounds the out-degree of the vertices of $Q$, so $(Q, I)$ satisfies the condition (S1) in the definition of string presentations. This proves part (i) of the proposition.

In order to prove part (ii) let us now assume the hypothesis that the presentation $(Q, I)$ satisfies the condition (S2), so that it is in fact special biserial. According to the corollary to Lemma 1.1 in [17], the ideal $I$ is generated by paths and binomials, that it, linear combinations of two parallel paths. It follows from this that in order to show that the presentation $(Q, I)$ is string it is enough that we prove that whenever a linear combination of two parallel paths is in $I$ then both paths also belong to $I$.

Let us then suppose that there is in $I$ an element of the form $u - \lambda v$ with $u$ and $v$ different parallel paths that do not belong to $I$ and $\lambda$ a non-zero scalar. Multiplying by $\lambda^{-1}$ if needed we may assume that moreover $u$ is not longer than $v$. Both $u$ and $v$ have length at least 2 because the ideal $I$ is admissible.

Suppose for a moment that both $u$ and $v$ start with the same arrow. Since neither $u$ nor $v$ is in $I$ and $u$ is not longer than $v$, the condition (S2) implies that there is a path $w$, which is necessarily closed, such that $v = uw$. Since $u \neq v$ this path $w$ has positive length and, because the ideal $I$ is admissible, there is a positive integer $k$ such that $w^k \in I$. Now we have that $u(1 - \lambda w) = u - \lambda v \in I$, so that

$$I \ni u(1 - \lambda w) \sum_{t=0}^{k-1} \lambda^t w^t = u - \lambda^k uw^k,$$

and this is absurd since $u \notin I$. A symmetric argument shows, of course, that the paths $u$ and $v$ cannot end in with the same arrow.

It follows from all this and the fact that both $u$ and $v$ have length at least 2 that there are arrows $\alpha, \beta, \gamma$ and $\delta$ in $Q$ and paths $\bar{u}$ and $\bar{v}$, possibly of length 0, such that $\alpha \neq \beta, \gamma \neq \delta$, $u = a\bar{u}\gamma$ and $v = \beta\bar{v}\delta$.

Let us write $i$ and $j$ for the common source and target of $u$ and $v$, respectively. If $\eta$ is an arrow coming out of $j$, then because $\gamma \neq \delta$ we have that one of $uv\eta$ or $v\eta\eta$ is in $I$ and therefore, since $u - \lambda v \in I$, that so is the other. Similarly, we have that $\epsilon u$ and $\epsilon v$ are in $I$ for all arrows $\epsilon$ which have $i$ as target.

Let now $w$ be a path of positive length that starts from $i$, is not in $I$, and such that there is no arrow $\eta$ such that $w\eta \notin I$. The arrows coming out of $i$ are $\alpha$ and $\beta$, so one of them is the first arrow in $w$. Let us suppose, for example, that $\alpha$ is. In view of (S2), one of $u$ or $w$ is a prefix of the other, and since for all arrows $\eta$ we have $u\eta \in I$ and $v\eta \in I$, we see that in fact $w = u$. Similarly, if $\beta$ were the first arrow in $w$ we would have that $w = v$. We thus see that $u$ and $v$ are the only paths starting from $i$, not in $I$, and which cannot be extended on the right: in other words, they span the socle of the indecomposable projective module $e_iA$, which is the projective cover of the simple module $S_i := e_iA / \text{rad} e_iA$. Moreover, since $u - \lambda v \in I$, we see that that socle is of dimension 1, so isomorphic to the simple module $S_j$. A dual argument shows that $u$ and $v$ are the only paths not in $I$ that end in $j$ and that cannot be extended on the left: the top of the injective envelope of the simple $S_j$ is thus $S_i$. It
follows from this that the indecomposable projective module $e_i A$ is in fact also injective$^1$ and we therefore have an almost split sequence

$$0 \longrightarrow \text{rad } e_i A \longrightarrow e_i A \oplus \frac{\text{rad } e_i A}{\text{soc } e_i A} \longrightarrow e_i A \longrightarrow 0$$

The two ends of this extension are indecomposable, because the socle and the top of $e_i A$ are simple, so the middle term has at most two direct factors: it follows that the quotient $\frac{\text{rad } e_i A}{\text{soc } e_i A}$ is zero or indecomposable. This is absurd, since in our situation we have

$$\frac{\text{rad } e_i A}{\text{soc } e_i A} \cong \frac{\alpha A}{u A} \oplus \frac{\beta A}{v A}$$

and both summands are non-zero.

We have written Proposition 1 in the slightly weird form that we have so as to make it explicit that it is condition (S2) that is difficult to satisfy. One way to do that is to restrict ourselves to algebras of finite representation type:

**Proposition 2** Let $Q$ be a finite quiver, let $I$ be an admissible ideal in the path algebra $kQ$, and let $A := kQ/I$. If $A$ has finite representation type and every extension between indecomposable $A$-modules has at most two direct factors, then there is another admissible ideal $I'$ in $kQ$ such that $A \cong kQ/I'$ and the presentation $(Q, I')$ is string.

This gives us half of the Theorem A stated in the introduction.

**Proof** Suppose that $A$ satisfies both hypotheses. Since every extension between indecomposable $A$-modules has at most two direct factors, we have in particular that the middle term of every almost split exact sequence of $A$-modules has at most two direct factors and, since $A$ has finite representation type, Theorem 4.6 in [1] tells us that the radical of every non-uniserial indecomposable projective module is the direct sum of two uniserial modules. Since our hypotheses are left-right symmetric, the same is true of the non-uniserial indecomposable projective left modules, and therefore the algebra $A$ is biserial, as in [9] or [17].

On the other hand, since the algebra $A$ has finite representation type, a classic theorem of James Jans [11] tells us that the lattice of bilateral ideals of $A$ is distributive. It follows then from Lemma 2 in [17] that there is an ideal $I'$ in the path algebra $kQ$ such that $A \cong kQ/I'$ and the presentation $(Q, I')$ is special biserial. In view of Proposition 1, the presentation $(Q, I')$ is moreover string. \qed

### 3 Degenerations and the Number of Direct Factors of a Module

Let us recall from K. Bongartz’s paper [3] the notion of degeneration of modules, deferring to that paper and to Ch. Riedtmann’s [13] for anything else on the subject.

---

$^1$The injective envelope $\text{soc } e_i A \cong S_j \hookrightarrow D(Ae_j)$ factors through a map $f : e_i A \rightarrow D(Ae_j)$, and this map is injective because it does not vanish on the simple socle of $e_i A$. Similarly, the projective cover $e_i A \twoheadrightarrow S_i \cong \text{top } D(Ae_j)$ factors through a map $e_i A \rightarrow D(Ae_j)$, and this map is surjective because its composition with $D(Ae_j) \rightarrow \text{top } D(Ae_j)$ is surjective: it follows from this that the map $f$ is an isomorphism.
Let $A$ be a finite-dimensional algebra over our algebraically closed field $k$. If $V$ is a finite-dimensional vector space, then we consider the set $L_A(V) := \text{Hom}(V \otimes A, V)$ of all linear maps $\rho : V \otimes A \to V$ and its subset $\text{Rep}_A(V)$ consisting of those maps $\rho$ that turn $V$ into an $A$-module $V_\rho$. The set $L_A(V)$ is an affine space over $k$, and it is easy to check that $\text{Rep}_A(V)$ is an affine algebraic variety in it. The group $GL(V)$ of linear automorphisms $V \to V$ acts naturally on $L_A(V)$ by conjugation and that action restricts to one on $\text{Rep}_A(V)$. Two points $\rho$ and $\rho'$ of $\text{Rep}_A(V)$ are in the same $GL(V)$-orbit if and only if there is an isomorphism $V_\rho \cong V_{\rho'}$ of $A$-modules. In particular, a finite-dimensional $A$-module $M$ such that $\dim M = \dim V$ determines a unique orbit $O(M)$ in $\text{Rep}_A(V)$, the one consisting of the points $\rho$ such that there is an isomorphism of $A$-modules $M \cong V_\rho$.

If $M$ and $N$ are two $A$-modules with $\dim M = \dim N = \dim V$, then we say that $M$ degenerates to $N$ if the orbit $O(N)$ is contained in the Zariski closure of the orbit $O(M)$.

If a module $M$ degenerates to another module $N$, then $M$ and $N$ have the same dimension vectors, that is, they have the same composition factors. On the other hand, there is in general very little that one can say about the relation between the numbers of their direct factors $|M|$ and $|N|$. A natural guess is that the inequality $|M| \leq |N|$ should hold, but it does not: Bongartz gives his Example 7.1 in [3] examples that show that over the self-injective 4-dimensional algebra $k[X, Y]/(X^2, XY, Y^2)$ modules with arbitrarily many direct summands degenerate to indecomposable ones. For string algebras of finite representation type a little miracle occurs, though, and the natural guess is on the mark.

**Proposition 3** Let $A$ be a string algebra of finite representation type and let $M$ and $N$ be two $A$-modules with the same dimension vector. If $M$ degenerates to $N$, then $|M| \leq |N|$.

This is the Theorem A stated in the introduction.

**Proof** Let us suppose that $M$ degenerates to $N$ and for each $A$-module $V$ let us write $\delta(V) := \dim \text{Hom}_A(V, N) - \dim \text{Hom}_A(V, M)$.

According to Proposition 2.1 in [13] we have $\delta(V) \geq 0$ for all $A$-modules $V$ and, since $M$ and $N$ have the same dimension vector, we have $\delta(V) = 0$ if the module $V$ is projective and indecomposable. For each non-projective indecomposable $A$-module $V$ we write $\Sigma_V$ for the Auslander–Reiten sequence starting at $V$,

$$
\begin{array}{cccc}
0 & \longrightarrow & \tau V & \longrightarrow & E_V & \longrightarrow & V & \longrightarrow & 0 \\
\end{array}
$$

and consider the short exact sequence

$$
\begin{array}{cccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
\end{array}
$$

that is the direct sum $\bigoplus_V \Sigma_V^{\delta(V)}$ of the short exact sequences $\Sigma_V$ for all non-projective indecomposable modules $V$, each taken with multiplicity $\delta(V)$. This direct sum is finite, of course, because the algebra $A$ has finite representation type. In the proof of Theorem 1.1 in [13] it is established that there is an isomorphism

$$
M \oplus X \oplus Z \cong N \oplus Y
$$

and it follows from this and the Krull–Remak–Schmidt theorem that

$$
|N| - |M| = |X| + |Z| - |Y| = \sum_V \delta(V)(|\tau V| + |V| - |E_V|) = \sum_V \delta(V)(2 - |E_V|).
$$
Since we know that $|E_V| \leq 2$ for all $V$ — for example, from the explicit construction done by Butler and Ringel in [5] of all the Auslander–Reiten sequences for $A$ — we see at once that $|N| - |M| \geq 0$, which is precisely what the proposition claims.

It may be helpful to the reader of [13] to know that the unpublished result of Auslander of which Riedtmann makes use — that two modules $M$ and $N$ over an algebra $\Lambda$ are isomorphic as soon as $\dim \text{Hom}_{\Lambda}(U, M) = \dim \text{Hom}_{\Lambda}(U, N)$ for all indecomposable $\Lambda$-modules $U$ — follows easily from Lemma 1.2 in Bongartz’s paper [3].

Let us state a useful special case of the proposition we have just proved:

**Corollary 4** Let $A$ be a string algebra of finite representation type. If

$$0 \to N \to E \to M \to 0$$

is a short exact sequence of $A$-modules, then $|E| \leq |M| + |N|$.

**Proof** It follows from Lemma 1.1 in [3] or the lemma that appears in the proof of Corollary 2.3 in [13] that in the situation of the lemma the module $E$ degenerates to $M \oplus N$, so the corollary is a direct consequence of the proposition.

This corollary gives us the half of Theorem A that we did not prove in the previous section.

**Proof of Theorem A** Proposition 1 tells us that an algebra that satisfies the condition in the statement of the theorem is a string algebra. Conversely, Corollary 4 above immediately implies us that a string algebra of finite representation type satisfies that condition.

Th. Brüstle, G. Douville, K. Mousand, H. Thomas and E. Yıldırım in [4], on one hand, and İ. Çağlı, D. Pauksztello and S. Schroll in [6], on the other, have constructed bases for the extension spaces $\text{Ext}_A^1(M, N)$ corresponding to all choices of indecomposable modules $M$ and $N$ over a gentle algebra $A$ of finite representation type, and the middle terms of the extensions representing the elements of those bases all have at most two direct factors — as they must, according to Corollary 4 above, since gentle algebras are string algebras. It should be noted that in general it does not follow from this that all extensions between indecomposable modules have middle terms with at most two factors, though. Let us give an example of this — a wild algebra, and certainly an ungentle one, that one can expect, given the piling up of ominous adjectives, to be somewhat brutal on any hope.

**Example 5** We consider the path algebra $A := \mathbb{k}Q$ on the quiver

![Quiver](image)

Representations of this quiver in which the five arrows are represented by injective linear maps can be viewed as configurations of five subspaces inside the vector space at the vertex 6, and this allows us to write them down conveniently. In particular, we let $M$ be the
A module that corresponds to the configuration of five lines — three of which coincide, and of which we give generators — in $k^2$

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} \quad \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} \quad \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

This is an indecomposable module. We will also denote some indecomposable modules by their dimension vectors, but only when they are determined by them.

One can check that there are three extensions of the forms

\[
0 \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow M \longrightarrow 0
\]

\[
0 \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow M \longrightarrow 0
\]

\[
0 \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow M \longrightarrow 0
\]

Their middle terms are direct sums of two indecomposable modules, and their classes freely span the vector space

\[
\text{Ext}^1_A \left( M, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right).
\]

There are, though, classes in this space that correspond to extensions whose middle term is a direct sum of \textit{three} indecomposable modules. One such extension is of the form

\[
0 \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow M \longrightarrow 0
\]

\textit{Remark 6} The proof of Proposition 3 shows that over an algebra of finite representation type if a module $M$ degenerates to another module $N$, then $|M| - |N|$ can be estimated from the knowledge of the number of middle terms of Auslander–Reiten short exact sequences and the function $\delta$. It would be interesting to know if this observation can be put to use.

4 String Algebras of Infinite Representation Type

In this section we will study string algebras of infinite representation type and to be able to talk about them we start by recalling as little as it is possible of the language of bands — for
convenience we will use the version presented in [15] and [16], because our constructions will depend on results presented there using that version of the language.

Let $(Q, I)$ be a string presentation. Let us denote by $Q_0$ and $Q_1$ the sets of vertices and arrows of $Q$, respectively, and by $s, t : Q_1 \to Q_0$ the functions mapping each arrow to its source and its target, respectively.

Let $\hat{Q}$ be the quiver obtained from $Q$ by adding for each arrow $\alpha$ in $Q_1$ a new arrow $\alpha^{-1}$ going in the opposite direction, so that $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. We call the arrows of $\hat{Q}$ letters, and a letter is direct or inverse according to whether it is in $Q_1$ or not. There is an involution $l \in \hat{Q}_1 \mapsto l^{-1} \in \hat{Q}_1$ mapping each direct letter $\alpha$ to its formal inverse $\alpha^{-1}$, which we call inversion.

A walk in $Q$ is a path in $\hat{Q}$, possibly one of those of length zero that correspond to the vertices of $Q$. The inverse of a walk $w = l_1 \cdots l_r$ of positive length is the walk $w^{-1} := l_r^{-1} \cdots l_1^{-1}$, while each walk of length zero is its own inverse. A walk is direct or inverse if all its letters are direct or inverse, respectively, and it is serial if it is either direct or inverse. There is an obvious extension of the source and target functions $s$ and $t$ to the set of all walks in $Q$, and if $u$ and $v$ are walks such that $t(u) = s(v)$ we can construct a new walk $uv$ simply by concatenation; this partially defined operation on the set of walks is associative in the appropriate sense. A walk $u$ is a factor of a walk $w$ if there are walks $v_1$ and $v_2$ such that the product $v_1uv_2$ is defined and equal to $w$, and a prefix if we can choose $v_1$ of length zero.

A walk $w$ is a word if (i) $w$ has no factor of the form $ll^{-1}$ with $l$ a letter and (ii) neither $w$ nor $w^{-1}$ has a factor that belongs to $I$. A word $w$ is cyclic if it is not serial and $w^2$ is also a word, and a cyclic word is primitive if it is not of the form $v^r$ for some cyclic word $v$ and some integer $r \geq 2$.

Let $A := \mathbb{k}Q/I$ be the algebra presented by $(Q, I)$. From a word $w$ we can construct a $A$-module $M(w)$, called the string module corresponding to $w$, and from a primitive cyclic word $w$, a positive integer $n$ and a non-zero scalar $\lambda \in \mathbb{k} \setminus \{0\}$ we can construct an $A$-module $B(w, \lambda, n)$, called the band module corresponding to those parameters. A well-known result —due essentially to Gel’fand and Ponomarev [10]— states that in this way we obtain, up to isomorphism, all the indecomposable $A$-modules, and each of them once except for easily controlled repetitions: if $w$ is a word, then the string modules obtained from $w$ and from $w^{-1}$ are isomorphic, and if $w$ is a primitive cyclic word then the band modules obtained from $w$ and from its $\ll$rotations$\gg$ and their inverses are all isomorphic.

As usual, we say that an algebra $\Lambda$ has tame representation type if it is not of finite representation type and for each $d \in \mathbb{N}$ there exist finitely many $(\mathbb{k}[X], \Lambda)$-bimodules $M_1, \ldots, M_t$ such that, up to isomorphism, all but finitely many indecomposable $\Lambda$-modules of dimension $d$ are isomorphic to a $\Lambda$-module of the form $\mathbb{k}[X]/(X - \lambda) \otimes_{\mathbb{k}[X]} M_i$ for some $\lambda \in \mathbb{k}$ and some $i \in \{1, \ldots, t\}$. When this is the case, we write $\mu(d)$ for the minimal number of such bimodules needed for this. If there exists an integer $N$ such that $\mu(d) \leq N$ for all $d \in \mathbb{N}$ then the algebra $\Lambda$ has domestic representation type.

The result of Gel’fand and Ponomarev mentioned above implies that string algebras are either of finite representation type or of tame representation type, and it is easy to decide which among the latter are domestic. If $\alpha$ is an arrow in $Q$, we denote $\mathcal{N}(\alpha)$ the set of all cyclic words starting with $\alpha$ and ending with an inverse letter — which may well be empty, of course. This set $\mathcal{N}(\alpha)$ is closed under concatenation, so a semigroup, and, in fact, easily seen to be a free one, freely generated by its subset $\mathcal{N}(\alpha) \setminus \mathcal{N}(\alpha)^2$. It follows from Proposition 2 of [15] and observations made in the introduction of [16] that the following holds:
Proposition 7 Let \((Q, I)\) be a string presentation. The following three conditions are equivalent:

(a) The algebra \(A := \mathbb{k}Q/I\) does not have domestic representation type.
(b) There are infinitely many primitive cyclic words in \((Q, I)\).
(c) There is an arrow \(\alpha\) in \(Q\) such that the semigroup \(N(\alpha)\) is neither empty nor cyclic.

The other ingredient that we need in order to do what we want with string algebras of non-domestic type is a nice result of H.J. Fine and H.S. Wolf [8] on the combinatorics of words that we will describe next; an excellent reference for this is the book [12] of M. Lothaire, where the result we want appears as Proposition 1.3.5. We fix a set \(A\), which we call in this context the alphabet, and write \(A^*\) for the free monoid generated by \(A\) — in [12] the elements of \(A^*\) are referred to as words, but we will not do that since we are already using that word (!) for something else. The length of an element \(u\) of \(A^*\) is what one expects, and we write it as \(|u|\). On the other hand, if \(u, v\) and \(w\) are elements of \(A^*\), we say that \(w\) is a common left factor of \(u\) and \(v\) if there exist \(u'\) and \(v'\) in \(A^*\) such that \(u = wu'\) and \(v = wv'\).

Proposition 8 [12, Proposition 1.3.5] Let \(u\) and \(v\) be elements of \(A^*\) of lengths \(n\) and \(m\), respectively, and let \(d := \gcd(m, n)\). If two powers \(u^p\) and \(v^q\) of \(u\) and \(v\) have a common left factor of length at least equal to \(n + m - d\), then \(u\) and \(v\) are powers of the same element of \(A^*\).

We have now everything in place to do the construction that we need to prove the following result:

Proposition 9 Let \((Q, I)\) be a string presentation and let \(A := \mathbb{k}Q/I\). If the algebra \(A\) is not domestic, then there are extensions between indecomposable \(A\)-modules whose middle term has an arbitrarily large number of direct factors.

Proof Let us suppose that the algebra \(A\) is not domestic. Lemma 3 in [16] tells us that there are non-serial words \(x, y, z\) such that \(xyx\) and \(z\) are words, the first and last letters of \(y\) are direct, and the first and last letters of \(z\) are inverse. Let us notice first that

\[xy\text{ and }xz\text{ are not powers of the same word.} \tag{1}\]

Indeed, the last letter of \(xy\) is direct, while that of \(xz\) is inverse.

Next, let us check that

\[\text{for all } n \geq 3 \text{ the words } (xyx)^nxy \text{ and } xz(xy) x^n \text{ are cyclic and primitive.} \tag{2}\]

Suppose, for example, that \(n\) is a positive integer such that \((xyx)^nxy = w^t\) for some primitive cyclic word \(w\) and some integer \(t\) with \(t \geq 2\). We claim that \(xyx\) is not a power of \(w\). Indeed, if we had that \(xyx = w^s\), then it would follow that

\[w^t = (xyx)^nxy = w^{sn}xy,\]

so that \(t \geq sn\) and \(xy = w^{t-sn}\). But then we would also have that

\[w^s = xxyx = w^{t-sn}xz,\]

so that \(s \geq t - sn\) and \(xz = w^{(n+1)s-t}\), allowing us to conclude that \(xz\) and \(xy\) are powers of \(w\), in contradiction to Eq. 1. Our claim thus holds. Now the words \((xyx)^{n+1}\) and \(w^t\)
have \((xyxz)^n\) as a common left factor (in the free monoid on the set of letters) and we have shown that \(xyxz\) and \(w\) are not powers of a word: according to Proposition 8, then, we have that \(n|xyzx| < |xyxz| + |w|\), so that

\[
t(n - 1)|xyxz| < t|w| = n|xyxz| + |xy| < (n + 1)|xyxz|
\]

and \(0 < 2 - (t - 1)(n - 1)\). We thus see that \(2 > (t - 1)(n - 1)\geq n - 1\) and, therefore, that \(n < 3\). This proves Eq. 2.

Let now \(p\) be a prime number different from the characteristic of our ground field \(\mathbb{k}\) and larger than 7, let \(n\) be such that \(p = 2n + 1\) and consider the two cyclic words

\[
u = (xyxz)^nxy, \quad v = xz(xyxz)^n
\]

which are primitive since \(n \geq 3\). Let \(\gamma\) and \(\beta\) be the first and last letters of \(y\), and \(\alpha^{-1}\) and \(\delta^{-1}\) the first and last letters of \(z\). Clearly \(zxy\) is a factor of \(u\), so that so is \(\delta^{-1} xy\). In fact, it has many such factors but we choose one arbitrarily. Similarly, \(yxz\) is a factor of \(v\), and therefore so is \(\beta x \alpha^{-1}\), and we fix an appearance of this factor in \(v\). Since \(x\) has positive length, the special biseriality of \((Q, I)\) implies at once that \(\beta \delta \in I\) and \(\alpha \gamma \in I\).

As \(u\) and \(v\) are primitive cyclic words or, equivalently, bands, we can construct from them the band modules \(B(u) := B(u, 1, 1)\) and \(B(v) := B(v, 1, 1)\) with parameters \(1 \in \mathbb{k}\) and \(1 \in \mathbb{N}\). These are irreducible modules of dimensions equal to the lengths of \(u\) and \(v\), respectively. The module \(B(u)\) has a basis corresponding to the vertices the cyclic word \(u\) visits along its way in \(Q\) and the same is true for \(B(v)\). We name \(i_1\) and \(i_2\) the basis elements of \(B(v)\) which correspond to the source and the target of the arrow \(\beta\) that appears in the factor \(\beta x \alpha^{-1}\) that we have chosen in \(v\). Similarly, we let \(j_1\) and \(j_2\) be the basis vectors of \(B(u)\) corresponding to the source and the target of the arrow \(\delta\) that appears in the factor \(\delta^{-1}xy\) that we have chosen in \(u\). The direct sum \(M := B(u) \oplus B(v)\) can be schematically described by the drawing in Fig. 1 —in which we ignore for now the dashed arrows.

We now construct a \(\mathbb{k}Q\)-module \(M'\). As a vector space, \(M'\) coincides with \(M\). To give the \(\mathbb{k}Q\)-module structure on \(M'\) it is enough to describe how the vertices and the arrows of \(Q\) act on it, and this is what we do.

- We let the vertices of \(Q\) act on \(M'\) as they act on \(M\).
- If \(\eta\) is an arrow in \(Q\) and \(k\) is a basis vector of \(M'\) such that the pair \((\eta, k)\) is neither \((\beta, i_1)\) nor \((\delta, i_2)\), we let \(\eta\) act on \(k\) as it acts in \(M\).
- Finally, we put \(i_1 \cdot \beta = i_2 + j_1\) and \(i_2 \cdot \delta = -j_2\).

If \(\rho\) is a path in \(Q\) which belongs to \(I\), then the straightforward consideration of the few possibilities that exist shows that \(\rho\) acts as zero on \(M'\): this means that \(M'\) is an \(A\)-module. Moreover, it is clear that we have an extension

\[
0 \to B(u) \to M' \to B(v) \to 0
\]

Now it is easy to see that the module \(M'\) is isomorphic to the module that can be constructed from the cyclic word \(uv\) just as band modules are constructed using the parameters \(\lambda = -1\) and \(n = 1\). It is not, though, a band module: indeed, by construction we have that \(uv =
Fig. 1 The module constructed in the proof of Proposition 9

$$(xyz)^{2n+1} = (xyz)^p,$$
so that $uv$ is not a primitive cyclic word. That $M'$ is a direct sum of at least $p$ indecomposable direct summands now follows from the following lemma. □

**Lemma 10** Let $(Q, I)$ be a string presentation and let $A := \mathbb{k}Q/I$. If $u$ is a cyclic word and $p$ is an odd prime coprime to the characteristic of the field $\mathbb{k}$, then the “imprimitive band module” $B(u^p, -1, 1)$ has at least $p$ indecomposable direct summands.

**Proof** We will content ourselves with an example: we will suppose that $u = abc$ is a cyclic word of length 3 and $p = 5$. The construction of the module $B(u^5, -1, 1)$ is succinctly illustrated by the diagram in the left here:

Here the dots represent the elements of a basis of the module, the edges the letters in the word $u^5$ (they are oriented counter- or clockwise according to whether they are direct or inverse, but we have not drawn this) and all edges correspond to multiplication by 1 except the one marked with a $-1$, which denotes, of course, multiplication by $-1$. The edge with
the $-1$ is the first one in the word $u$ in the first factor in $u^5$. We could instead construct a module as in the diagram in the right, letting the first arrow of every one of the 5 factors of $u^5$ act as multiplication by $-1$. As 5 is an odd number, the two modules that we have described are isomorphic, and we will pick the second one as a model for the imprimitive band module $B(u^5, -1, 1)$.

The advantage of this choice is that the rotation of order 5 of its diagram preserves it, so it corresponds to a module automorphism $\phi$ of $B(u^5, -1, 1)$ of order 5, that $\ll$rotates$\gg$ the basis elements. This automorphism $\phi$ has finite order coprime to the characteristic of the field $k$, and the latter is algebraically closed, so $\phi$ is diagonalizable and, moreover, its eigenspaces are submodules. Since 5 is not the characteristic of the field $k$, there are five 5th roots of unity in $k$, and clearly to each of them corresponds a non-zero eigenspace. It follows that when we decompose $B(u^5, -1, 1)$ as a direct sum of indecomposable submodules we have at least 5 summands.

\[ \square \]

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Declarations

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