Mass Spectrum of Exotic $X(5568)$ State via Artificial Neural Network

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In this paper, we study mass spectrum of $X(5568)$ resonance by Artificial Neural Network method. Two scenarios, diquark-antidiquark and molecule, are taken into account. We have solved Schrödinger equation in four regions as a function of $r$ and found that diquark-antidiquark scenario is more suitable for the mass of $X(5568)$.

I. INTRODUCTION

With the discovery of the exotic states, i.e., states that cannot be interpreted by the conventional meson (quark-antiquark) and baryon (three quark) pictures, hadron spectroscopy may have a different direction apart from the Standard Model.

The milestone of exotic states was the observation of charmoniumlike resonance $X(3872)$ by Belle collaboration in 2003 [1]. It was later confirmed by D0 [2], CDF II [3] and BABAR [4] collaborations. This observation opened a new era in understanding of our nature and as well as Standard Model framework. Many new charmoniumlike states were observed and this new particle zoo is named $XYZ$ particles.

The lack of conventional picture of mesons to interpret the underlying structure of the exotic states paved the way for new theoretical approaches [5–9]. These states are out of conventional meson picture ($qq$). Therefore they must be attributed as four-quark states or hybrids. Although there is no consensus about the internal structure of exotic states, some properties of these states are accommodated by tetraquark models, molecular models or updated potential models. Among these approaches, tetraquark model and molecular model got more attention. In tetraquark model, two heavy and two light quarks come in together or mix. These quarks may cluster into the colored diquark-antidiquark, $(Qq)+(\bar{Q}\bar{q})$ doublet. In the molecular model, the exotic particle is thought to be a bound state of two color-singlet mesons, $(Q\bar{q})+(\bar{Q}q)$.

In 2016, a resonance named $X(5568)$ was reported by the D0 collaboration at the $B_s\pi^\pm$ invariant mass spectrum with the mass and width, respectively [10]

\[ M = 5567.8 \pm 2.9^{+0.9}_{-1.9} \text{ MeV}, \]
\[ \Gamma = 21.9 \pm 6.4^{+5.0}_{-2.5} \text{ MeV}. \]

As it is mentioned in [10], it is the first observation of a hadronic state with four different flavor quarks. The reason for that is that, as the decay rate of $X(5568) \rightarrow B_s\pi^\pm$ is much larger than the weak interaction prediction, one can conclude that strong interaction is responsible for this decay. Since strong interactions do not touch the flavor, in the final state there are four different quarks of $B_s = \bar{b}s$ and $\pi^+ = ud$ which are cannot be created by the vacuum [11].

Beside other puzzling features, $X(5568)$ was not observed in $X \rightarrow B_s\pi^\pm$ channel as reported by LHCb collaboration [12], the CMS collaboration [13], the CDF collaboration at Fermilab [14] and ATLAS collaboration of LHC [15]. In 2018, the D0 collaboration announced that they had confirmed the existence of $X(5568)$ from the decay $X(5568) \rightarrow B_s\pi^\pm$ via a sequent semileptonic decay $B_s^0 \rightarrow \mu^\pm D_s^\mp$ [16] and the results were in consistent except the width is shifted to

\[ \Gamma = 18.6^{+7.9}_{-6.3}(\text{stat})^{\pm3.5}_{-3.8}(\text{syst}) \text{ MeV}. \]

The clear discrepancy between D0 collaboration results and other experimental groups fired a dispute. Since $X(5568)$ may be the first observed exotic state with four different flavors, both theoretical and experimental studies on it can enlighten our realization of quark model. There are different approaches related to $X(5568)$ which calculate masses, widths, decay constants, decay channels and argue about internal structure [17–40]. All these studies conclude the mysterious and curious case of $X(5568)$ resonance.

In the present study, we adopt diquark-antidiquark and molecular pictures of $X(5568)$ and calculate mass spectra by artificial neural network for the first time. Artificial neural networks (ANNs) are being used since two decades to solve both ordinary and partial differential equations. They maintain many attractive features compared to known existing semi-analytical and numerical techniques. One of the main advantage of ANNs to solve differential equations is they require less number of model parameters than any other technique. Besides that, machine learning which is nowadays a hot topic in physics is provided via using ANNs.

The outline of paper is as follows. In Section II we introduce ANN formalism and the necessary details for application to quantum mechanics. In Section III we give our results and discuss. In Section IV we summarize our findings.

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II. FORMALISM OF ARTIFICIAL NEURAL NETWORK

Artificial neural networks are computer systems which are capable of deriving and creating new information and also discovering them through learning which is one of the features of human brain. Neural networks are mimicking versions of biological nervous systems. They are parallel and distributed information processing elements. These elements have their own memory and are connected to each other via weighted connections.

The fundamental ingredient of an artificial neural network is neuron (perceptron in computerized systems) and it is the processing element in a neural network. Figure 2 represents a single artificial neuron.

![Figure 1: A model of single neuron](image1)

![Figure 2: Multilayer neural network](image2)

In this study, we consider a feed forward neural network with one input layer, one hidden layer and one output layer. In feed forward neural networks, the information moves in only one direction, from input nodes to the hidden and output nodes. Fig. 1 is an example of feed forward neural network.

A. Mathematical Model of an Artificial Neural Network

The general method for solving differential equations is presented in [41] with the quantum mechanical applications in [42–45]. The relationship of the input-output of the layers can be written as follows:

\[
\begin{align*}
    o_i &= \sigma(n_i), \\
    o_j &= \sigma(n_j), \\
    o_k &= \sigma(n_k),
\end{align*}
\]

where \(i\) is for input, \(j\) is for hidden and \(k\) is for output layers. Input to the perceptrons are given as

\[
\begin{align*}
    n_i &= (\text{Input signal to the NN}), \\
    n_j &= \sum_{i=1}^{N_i} \omega_{ij} o_i + \theta_j, \\
    n_k &= \sum_{i=1}^{N_j} \omega_{jk} o_j + \theta_k,
\end{align*}
\]

where \(N_i\) and \(N_j\) represents the numbers of the units belonging to input and hidden layers, \(\omega_{ij}\) is the synaptic weight parameter which connects the neurons \(i\) and \(j\) and \(\theta_j\) represents threshold parameter for the neuron \(j\) and \(\theta_j\) is the threshold parameter [45]. The output of the network can be written as

\[
o_k = \sum_{j=1}^{b_{\text{prev}}} \omega_{jk} \sigma \left( \sum_{i=1}^{a_{\text{in}}} \omega_{ij} n_i + \theta_j \right) + \theta_k.
\]
Derivative of this function is needed in further evaluation of error function. This can be obtained as
\[
\frac{\partial \phi}{\partial \omega_{ij}} = \omega_{jk} \sigma^{(1)}(n_j)n_i, \tag{8}
\]
\[
\frac{\partial \phi}{\partial \omega_{jk}} = \sigma(n_j) \delta_{kk'}, \tag{9}
\]
\[
\frac{\partial \phi}{\partial \theta_j} = \omega_{jk} \sigma^{(1)}(n_j), \tag{10}
\]
\[
\frac{\partial \phi}{\partial \theta_k} = \delta_{kk'}. \tag{11}
\]

In this work we use a sigmoid function
\[
\sigma(z) = \frac{1}{1 + e^{-z}} \tag{12}
\]
as an activation function since it is possible to derive all the derivatives of \(\sigma(z)\) in terms of itself. This differentiability is an important aspect for the Schrödinger equation.

### B. Application to Quantum Mechanics

Following the work of [42], let us consider the following differential equation:
\[
H \Psi(r) = f(r) \tag{13}
\]
where \(H\) is a linear operator, \(f(r)\) is a known function and \(\Psi(r) = 0\) at the boundaries. In order to solve this differential equation, a trial function
\[
\Psi_t(r) = A(r) + B(r, \lambda)N(r, p) \tag{14}
\]
of the form can be written which uses a feed forward neural network with parameter vector \(p\) and \(\lambda\) to be adjusted. The parameter \(p\) refers to the weights and biases of the neural network. The functions \(A(r)\) and \(B(r, \lambda)\) should be specified in a convenient way so that \(\Psi_t(r)\) satisfies the boundary conditions regardless of the \(p\) and \(\lambda\) values. To obtain a solution for Eqn. (13), the collocation method can be used and the differential equation can be transformed into a minimization problem
\[
\min_{p, \lambda} \sum_i \left[ H\Psi_t(r_i) - f(r_i) \right]^2. \tag{15}
\]

For Schrödinger equation Eqn. (13) takes the form
\[
H \Psi(r) = \epsilon \Psi(r) \tag{16}
\]
with the boundary condition, \(\Psi(r) = 0\). In this case, the trial solution can be written as
\[
\Psi_t(r) = B(r, \lambda)N(r, p) \tag{17}
\]
where \(B(r, \lambda) = 0\) at boundary conditions for a range of \(\lambda\) values. By discretizing the domain of the problem, it is transformed into a minimization problem with respect to the parameters \(p\) and \(\lambda\)
\[
E(p, \lambda) = \sum_i \left[ H\Psi_t(r_i, p, \lambda) - \epsilon \Psi_t(r_i, p, \lambda) \right]^2 \tag{18}
\]
where \(E\) is the error function and \(\epsilon\) can be computed as
\[
\epsilon = \int |\Psi_t|^2 dr. \tag{19}
\]

We used a trial wave function as
\[
\phi_t(x) = e^{-\beta z^2} N(x, u, w, v), \quad \beta \geq 0 \tag{20}
\]
with \(N\) being a feed forward neural network with one hidden layer and \(m\) sigmoid hidden units
\[
N(x, u, w, v) = \sum_{j=1}^{m} v_j \sigma(w_j x + u_j). \tag{21}
\]

By employing this approach it is possible to obtain energy eigenvalues of the Schrödinger equation. We trained the network with 200 equidistance points in the intervals of \(0 < r < 0.5, \ 0 < r < 1, \ 0 < r < 1.5\) and \(0 < r < 2\) (for simplicity we denote these intervals as Region A, Region B, Region C and Region D, respectively) with \(m = 8\) and solved the Schrödinger equation.

### III. NUMERICAL RESULTS AND DISCUSSION

We consider the radial Schrödinger equation with Cornell potential
\[
\frac{d^2 \phi(r)}{dr^2} + 2 \mu (E + V(r)) \phi(r) = 0 \tag{22}
\]
where
\[
V(r) = -\frac{4 \alpha_s}{3} + \sigma r, \tag{23}
\]
with \(\alpha_s = 0.354\) and \(\sigma = 0.472\) GeV².

We used the quark model parameters as [28]:
\[
\begin{align*}
    m_u &= 313 \text{ MeV} \\
    m_s &= 536 \text{ MeV} \\
    m_c &= 1728 \text{ MeV} \\
    m_b &= 5512 \text{ MeV}.
\end{align*}
\]
and the experimental values as [46]:
\[
\begin{align*}
    B^0_s &= 5366 \text{ MeV} \\
    \pi &= 139 \text{ MeV} \\
    B^0 &= 5415 \text{ MeV} \\
    \rho &= 770 \text{ MeV} \\
    B^+ &= 5279 \text{ MeV} \\
    K^0 &= 497 \text{ MeV} \\
    B^{++} &= 5325 \text{ MeV} \\
    K^* &= 892 \text{ MeV}
\end{align*}
\]
The results are given in Table I for regions of A, B, C and D.

| $r$ | $B_s^0\pi$ | $B_s^0\rho$ | $B^+K^0$ | $B^{++}K^{*-}$ |
|-----|-------------|-------------|-----------|---------------|
| Region A | 5519 | 5518 | 5526 | 5515 | 5512 |
| Region B | 5580 | 5561 | 5573 | 5570 | 5573 |
| Region C | 5719 | 5717 | 5704 | 5718 | 5717 |
| Region D | 8722 | 7846 | 5739 | 5739 | 5737 |

In Ref. [47], the authors calculated mass of the $X(5568)$ as an exotic axial-vector state and obtained $M = 5864 \pm 158$ MeV whereas in Ref. [48], it is obtained as $M = 5584 \pm 137$ MeV in diquark-antidiquark picture. Our results of regions A, B and C agree within the error bars with these results in diquark-antidiquark picture. In Ref. [28], the authors calculated $X(5568)$ mass in view of molecule model. They obtained $M = 5507$ MeV for $B^0\pi$, $M = 6182$ MeV for $B^0\rho$, $M = 5774$ MeV for $B^+K^0$, and $M = 6233$ MeV for $B^{++}K^{*-}$. Regions A, B and C of $B^0\pi$ and regions A, B, C and D of $B^+K^0$ agree good with the results of [28]. In Ref. [21], the authors assumed $X(5568)$ as molecule of $B^+K^0$ and obtained mass as $M = 5757 \pm 145$ MeV. The average value of all regions for $B^+\pi^0$ which is $M = 5636$ MeV agree well within the error bars of that calculation.

The size of a molecular state is controlled by its binding energy-better to say corresponding binding momentum. In particular one can estimate the size of a molecular state by

$$ r = \frac{1}{\sqrt{2\mu E_b}} \quad (24) $$

where $\mu$ denotes the reduced mass for the two constituents that form the molecular state and $E_b$ is the positive binding energy [49]. If the binding energy is small then it is natural to think that the state has a molecular structure. For example in the case of $X(3872)$ the binding energy is

$$ E_b = M_{D^0} + M_{\bar{D}^*0} - M_X = (0.12 \pm 0.26) \text{ MeV}, \quad (25) $$

despite it is natural to assume to be a molecular state. This consideration seems to be inappropriate for the molecular structure of $X(5568)$ since the mass is hundreds MeV below for any such threshold. The related binding energies are

$$ E_B = M_{B_s^0} + M_\pi - M_X(5568) = -63 \text{ MeV}, \quad (26) $$

$$ E_B = M_{B^*} + M_\rho - M_X(5568) = 617 \text{ MeV}, \quad (27) $$

$$ E_B = M_{B^+} + M_{K^0} - M_X(5568) = 208 \text{ MeV}, \quad (28) $$

$$ E_B = M_{B^{*+}} + M_{K^{*-}} - M_X(5568) = 649 \text{ MeV} \quad (29) $$

which were evaluated in view of molecular picture. From this consideration it can be deduced that molecular picture is inappropriate for $X(5568)$. Our results also support this result since in the long range distance of $r > 1 \text{ fm}$, obtained mass values are a few hundreds above than the $X(5568)$ mass. Note that if $X(5568)$ is a molecule of $B^0\pi$, which indeed was observed in $B^0\pi^\pm$ invariant-mass spectrum, both the binding energy and output results in the regions of C and D do not support this assumption.

On the otherside a tetraquark is thought to be made of colored diquark-antidiquark-accordingly they are subject to confinement and their size should be controlled by the confinement radius which is $< 1 \text{ fm}$. From Table I it can be seen that in the tetraquark model, our results for the regions of A and B are in nice agreement with the mass of $X(5568)$. While the result of region C is a few hundreds MeV above than the mass of $X(5568)$, the result of region D is three thousands MeV above. This can be considered as a consequence of tetraquark model and being outside of the confinement region.

**IV. SUMMARY**

In this work, we obtained mass of $X(5568)$ resonance under different assumptions, diquark-antidiquark and molecule pictures. The prominent feature of this resonance is that it is the first state that contains four different flavors of quark. Although the other collaborations LHCb, CMS, ATLAS and CDF have not confirmed the existence of this state up to now, the statistical significance of $5.1 \sigma$ in the $B^0\pi^\pm$ invariant-mass spectrum challenges our understanding of the quark model. In the original quark model framework such exotic states or better to say multi-quark states were predicted by Gell-Mann. Therefore it should be not surprising the existence of four-quark state with all different flavors.

It is interesting to note that by using a potential which does not include spin-spin and spin-orbit interactions or relativistic corrections, artificial neural network framework produced mass of the $X(5568)$ when it was sufficiently trained. It is difficult to handle four-body problem with interactions among the system. Tetraquark and molecule scenarios make this effort more plausible. Due to some advantages provided by artificial neural networks such as continuity of solution over all the domain of integration and not increasing of computational complexity when the sampling points and number of dimensions involved, such elaborations can be made more safely.

Our results support the tetraquark scenario of $X(5568)$. The mass value calculation alone itself does not corroborate the internal structure of any state, exotic or not but gives an idea about the validation of the models. Further theoretical and experimental studies would clarify the status of $X(5568)$. 

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TABLE I: Mass values in the different intervals for diquark-antidiquark and molecule scenarios. The units are in MeV.

| $r$ | $|s\bar{s}| [\text{bd}]$ | $B_s^0\pi$ | $B_s^0\rho$ | $B^+K^0$ | $B^{++}K^{*-}$ |
|-----|----------------|-------------|-------------|-----------|---------------|
| Region A | 5519 | 5518 | 5526 | 5515 | 5512 |
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