LARGE DEVIATION, BASIC INFORMATION THEORY FOR WIRELESS SENSOR NETWORKS

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Abstract. In this article, we prove Shannon-MacMillan-Breiman Theorem for Wireless Sensor Networks modelled as coloured geometric random graphs. For large $n$, we show that a Wireless Sensor Network consisting of $n$ sensors in $[0,1]^d$ connected by an average number of links of order $n \log n$ can be coded by about $\left\lfloor n(\log n)^{2} \pi^{d/2}/(d/2)! \right\rfloor H$ bits, where $H$ is an explicitly defined entropy. In the process, we derive a joint large deviation principle (LDP) for the empirical sensor measure and the empirical link measure of coloured random geometric graph models.

1. Introduction

Wireless Sensor Networks (WSN) are now popular among theoretical computer scientists because of its uses as a tool for monitoring and controlling the physical environment. Many researches on finding a good network model for WSN have suggested models that have their origin in classical areas of theoretical computer and applied mathematics: regardless of the radio technology use from the topology point of view, at any instant in time a WSN can be represented as a graph with a set of vertices consisting of nodes of the network and a set of edges consisting of the links between nodes. However, recent studies, see Kenniche [KH10] and the references therein, have shown that a random strategy is the only way to deploy the large number of sensors in inaccessible areas and the random geometric graph or geometric random graph is the most appropriate model. Finding a good coding schemes and approximate pattern matching algorithms will be vital for coding the WSN, and the Shannon-MacMillan-Breiman Theorem (SMBT) or Asymptotic Equipartition Property will be of great help in this regard. See, example Dembo and Kontoyiannis [DK02].

In Information theory, the Shannon-MacMillan-Breiman Theorem (SMBT) or Asymptotic Equipartition Property is the analog of the strong law of large numbers. It is a direct consequences of the weak law of large for appropriately defined statistics of a stochastic data source. It allows us to partition output sequence of a stochastic data source into two sets, the typical set, where the sample entropy is close to the true entropy, and the non-typical set, which contains the other sequence. See, Cover and Thomas [CT91].

In this article we derive the SMBT for WSN modelled as CGRG models, using some of the large deviation techniques developed for studying information theory, see [DA10], for networked data structures.
To be specific we derive joint large principle for the empirical sensor measure and the empirical link measure of the CGRG using [DA10] Theorem 3.3 and the methods developed therein. From this LDP we prove the weak law of large numbers for the empirical sensor measure and the empirical link measure. From the weak law of Large numbers we derive the SMBT for CGRG as a model for the WSN.

1.1 The coloured geometric random graph model. In this subsection we shall describe a more general model of random geometric graphs, the CGRG in which the connectivity radius depends on the type or colour or symbol or spin or sensor of the nodes. The empirical sensor measure and the empirical link measure are our main object of study here.

Given a probability measure \( \nu \) on \( \mathcal{X} \) and a function \( r_n: \mathcal{X} \times \mathcal{X} \rightarrow (0, 1] \) we may define the \textit{randomly coloured geometric random graph} or simply \textit{coloured random geometric graph} \( X \) with \( n \) vertices as follows: Pick sites \( Y_1, \ldots, Y_n \) at random independently according to the uniform distribution on \( [0, 1]^d \). Assign to each site \( Y_j \) an \( r \)-sensor \( X(Y_j) \) independently according to the \textit{sensor law} \( \mu \). Given the sensors, we join any two vertices \( Y_i, Y_j, (i \neq j) \) by a link independently of everything else, if

\[
\|Y_i - Y_j\| \leq r_n[X(Y_i), X(Y_j)].
\]

In this article we shall refer to \( r_n(a, b) \), for \( a, b \in \mathcal{X} \) as a connection radius, and always consider

\[
X = (\{(X(Y_i), X(Y_j)) : i, j = 1, 2, 3, \ldots, n\}, E)
\]

under the joint law of graph and sensor. We interpret \( X \) as CGRG with vertices \( Y_1, \ldots, Y_n \) chosen at random uniformly and independently from the vertices space \( [0, 1]^d \). For the purposes of this study we restrict ourselves to the sparse, intermediate and dense cases i.e. the connection radius \( r_n \) satisfies the condition \( nr_n^d(a, b)/\log n \rightarrow \lambda_{[d]}(a, b) \) for all \( a, b \in \mathcal{X} \), where \( \lambda: \mathcal{X}^2 \rightarrow [0, \infty) \) is a symmetric function, which is not identically equal to zero. The CGRG have been suggested by Cannings and Penman [CP03] as a possible extension to the coloured random graphs introduced in Penman [Pe98].

The distance \( r_n \) plays a role similar to that of \( p_n \) in the coloured random graph model proposed in Penman [Pe98] and studied by Doku-Amponsah [DA06]. Based on one’s choice of \( r_n \), qualitatively, different types of behaviour can be seen. Note that, intuitively, the the average degree scales with \( nr^d \). To be more specific, it can be show that in the classical random geometric graph the ratio of the average degree divided by \( nr^d \) tends to a constant in probability as long as \( n^2 r_n^d \rightarrow \infty \). See, [MM05]. As a result of the interpretation of \( nr_n^d \) as a measure of the average degree, we refer to the case where \( nr_n^d/\log n \rightarrow \lambda_{[d]} = 0 \) as sparse case, the case \( nr_n^d/\log n \rightarrow \lambda_{[d]} \) as the intermediate case(s) and \( nr_n^d/\log n \rightarrow \lambda_{[d]} = \infty \) as the dense case.

We associate with any coloured graph \( X \) a probability measure, the \textit{empirical sensor measure} \( \mathcal{L}^1_X \in \mathcal{M}(\mathcal{X}) \), by

\[
\mathcal{L}^1_X(a) := \frac{1}{n} \sum_{j=1}^{n} \delta_{X(Y_j)}(a), \quad \text{for } a_1 \in \mathcal{X},
\]

and a symmetric finite measure, the \textit{empirical link measure} \( \mathcal{L}^2_X \in \tilde{\mathcal{M}}(\mathcal{X}^2) \), by

\[
\mathcal{L}^2_X(a, b) := \frac{1}{n \log n} \sum_{(i, j) \in E} \left[ \delta((X(Y_i), X(Y_j)) + \delta((X(Y_j), X(Y_i)))](a, b), \quad \text{for } (a, b) \in \mathcal{X}^2.
\]

The total mass \( \|\mathcal{L}^2_X\| \) of the empirical link measure is \( 2|E|/n \log n \).

For any finite or countable set \( \mathcal{X} \) we denote by \( \mathcal{M}(\mathcal{X}) \) the space of probability measures, and by \( \tilde{\mathcal{M}}(\mathcal{X}) \) the space of finite measures on \( \mathcal{X} \), both endowed with the weak topology.
2. Statement of main results

Through out the remaining part of this article we assume \( d \geq 2 \) is finite.

2.1 Asymptotic Equipartition Property the Sparse and Intermediate for WSN

The underlying question is, how many bits are needed to store or transmit the information contained in a Wireless Sensor Network consisting of \( n \) sensors connected by number of links? This question can be answered by the SMBT for Wireless Sensor Networks, see Theorem 2.1. To the SMBT we denote by \( P \) the distribution of the an CGRG. We define the measure \( \lambda_{\lfloor d \rfloor} \omega \otimes \omega \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) \) by

\[
\lambda_{\lfloor d \rfloor} \omega \otimes \omega(a, b) = \lambda_{\lfloor d \rfloor}(a, b) \omega(a) \omega(b), \text{ for } a, b \in \mathcal{X}
\]

and write

\[
\int_{\mathcal{X}^2} \lambda_{\lfloor d \rfloor} \omega \otimes \omega(da, db) =: \sum_{a, b \in \mathcal{X}} \omega(a) \lambda_{\lfloor d \rfloor}(a, b) \omega(b).
\]

**Theorem 2.1.** Suppose that \( X \) is an CGRG with sensor law \( \nu \) and connection radius \( r_n : \mathcal{X}^2 \to [0, 1] \) satisfying \( nr_n^d(a, b)/\log n \to \lambda_{\lfloor d \rfloor}(a, b) \), for some symmetric function \( \lambda_{\lfloor d \rfloor} : \mathcal{X}^2 \to [0, \infty) \) not identical to zero. Then, for every \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \mathbb{P} \left\{ \left| - \frac{1}{n \log n} \log P(X) - \frac{n^{d/2}}{2(d/2)!} \int_{\mathcal{X}^2} \lambda_{\lfloor d \rfloor} \nu \otimes \nu(da, db) \right| \geq \epsilon \right\} = 0.
\]

In other words, in order to transmit an WSN in the given sparse or intermediate regime one needs with high probability, about \( [n(\log n)^2 \pi^{d/2} / (d/2)!] \mathcal{H} \) bits, where \( \mathcal{H} \) is the entropy defined by

\[
\mathcal{H} := \frac{1}{2 \log 2} \int_{\mathcal{X}^2} \lambda_{\lfloor d \rfloor} \nu \otimes \nu(da, db).
\]

For the pair of measures \((\omega, \tilde{\omega})\) we define the near entropy \( \tilde{\mathcal{H}}_{\lambda_{\lfloor d \rfloor}} \) by

\[
\tilde{\mathcal{H}}_{\lambda_{\lfloor d \rfloor}}(\tilde{\omega} \| \omega) := H(\tilde{\omega} \| \rho(d) \lambda_{\lfloor d \rfloor} \omega \otimes \omega) + \rho(d) \| \lambda_{\lfloor d \rfloor} \omega \otimes \omega\| - \| \tilde{\omega} \|,
\]

where \( H(\tilde{\omega} \| \hat{\omega}) \) means the relative entropy of the finite measure \( \tilde{\omega} \) with respect to \( \hat{\omega} \).

2.2 Large-deviation principles in the Sparse and Intermediate CGRG. The following LDP is a key ingredient in the proof of our SMBT, see Theorem 2.1.

**Theorem 2.2.** Suppose that \( X \) is an CGRG with sensor law \( \nu : \mathcal{X} \to (0, 1] \) and connection radius \( r_n : \mathcal{X} \times \mathcal{X} \to [0, 1] \) satisfying \( nr_n^d(a, b)/\log n \to \lambda_{\lfloor d \rfloor}(a, b) \), with \( \lambda_{\lfloor d \rfloor} : \mathcal{X} \times \mathcal{X} \to [0, \infty) \) symmetric. Then, for \( n \to \infty \), the pair \((\mathcal{L}_X^1, \mathcal{L}_X^2)\) satisfies a large deviation principle in \( \mathcal{M}(\mathcal{X}) \times \mathcal{M}_{\#}(\mathcal{X} \times \mathcal{X}) \) with speed

(i) \( n \log n \) and good rate function,

\[
I_{\lfloor d \rfloor}^1(\omega, \tilde{\omega}) = \frac{1}{2} \tilde{\mathcal{H}}_{\lambda_{\lfloor d \rfloor}}(\tilde{\omega} \| \omega),
\]

whilst \( \tilde{\mathcal{H}}_{\lambda_{\lfloor d \rfloor}}(\tilde{\omega} \| \omega) \geq 0 \) and equality holds if and only if \( \tilde{\omega} = \rho(d) \lambda_{\lfloor d \rfloor} \omega \otimes \omega \).

(ii) \( n \) and good rate function,

\[
I_{\lceil d \rceil}^2(\omega, \tilde{\omega}) = \left\{ \begin{array}{ll}
H(\omega \| \nu) & \text{if } \tilde{\omega} = \rho(d) \lambda_{\lfloor d \rfloor} \omega \otimes \omega, \\
\infty & \text{otherwise}.
\end{array} \right.
\]
3. Derivation of Theorems 2.2 and 2.1

For any two points \( U_1 \) and \( U_2 \) uniformly and independently chosen from the space \([0, 1]^d\) write
\[
F(t) := \mathbb{P}\{\|U_1 - U_2\| \leq t\},
\]
where \( F(r_n(a, b)) = \rho(d)r_n^d(a, b) \), \( a, b \in X^2 \) i.e. the volume of a \( d \)-dimensional (hyper)sphere with radius \( r(a, b) \) satisfying \( nr_n^d(a, b)/\log n \to \lambda^d(a, b) \). Let \( p_n(a, b) = F(r_n(a, b)) = \rho(d)r_n^d(a, b) \) and \( C(a, b) = \lambda^d(a, b) \).

Then we have
\[
d\mathbb{P}(X) = \prod_{u \in V} \nu(X(Y_u)) \prod_{(u, v) \in E} F(r_n(X(Y_u), X(Y_v))) \prod_{(u, v) \notin E} 1 - F(r_n(X(Y_u), X(Y_v))) \tag{3.1}
\]
\[
= \prod_{u \in V} \nu(X(Y_u)) \prod_{(u, v) \in E} p_n(X(Y_u), X(Y_v)) \prod_{(u, v) \notin E} 1 - p_n(X(Y_u), X(Y_v)) = d\tilde{\mathbb{P}}(X), \tag{3.2}
\]
where \( \tilde{\mathbb{P}}(X) \) is the law of coloured random graph \( X \) with the geometric plane \([0,1]^d\) ignored.

Hence by the exponential equivalence, see [DZ98, Theorem 4.2.13] and [DA10, Theorem 3.3] we have Theorem 2.1 with rate functions \( I_1^d \) and \( I_2^d \).

3.1 Derivation of Theorems 2.1

**Lemma 3.1.** Suppose that \( X \) is an CGRG with sensor law \( \nu: X \to (0, 1] \) and connection radius \( r_n: X \times X \to [0, 1] \) such that \( nr_n^d(a, b)/\log n \to \lambda^d(a, b) \), for \( \lambda^d : X \times X \to [0, \infty) \) nonzero. Then, for any \( \varepsilon > 0 \) we have
\[
\lim_{n \to \infty} \mathbb{P}\{ \sup_{a \in X} |\mathcal{L}_X^1(a) - \nu(a)| \geq \varepsilon\} = 0
\]
and
\[
\lim_{n \to \infty} \mathbb{P}\{ \sup_{a, b \in X} |\mathcal{L}_X^2(a, b) - \nu(a)\lambda^d(a, b)\nu(b)| \geq \varepsilon\} = 0.
\]

From Theorem 2.2(ii), we prove this lemma. To begin, we define a closed set
\[
F_1 = \{(\omega, \varpi) \in \mathcal{M}(X) \times \tilde{\mathcal{M}}_\nu(X \times X) : \sup_{a, b \in X} |\varpi(a, b) - \nu(a)\lambda(a, b)\nu(b)| \geq \varepsilon\}
\]
and
\[
F_2 = \{(\omega, \varpi) \in \mathcal{M}(X) \times \tilde{\mathcal{M}}_\nu(X \times X) : \sup_{a \in X} |\omega(a) - \nu(a)| \geq \varepsilon\}.
\]

We observe that, by Theorem 2.2(ii),
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\{ (\mathcal{L}_X^1, \mathcal{L}_X^2) \in F\} \leq - \inf_{(\omega, \varpi) \in \mathcal{F}} I_2^d(\omega, \varpi), \tag{3.3}
\]
where \( \mathcal{F} = F_1 \cup F_2 \).

This will be shown by contradiction that the right handside of (3.3) is negative. For this purpose, we suppose that there exists sequence \((\omega_n, \varpi_n)\) in \( \mathcal{F} \) such that \( I_2^d(\omega_n, \varpi_n) \downarrow 0 \). Then, because \( I_2^d \) is a good rate function and its level sets are compact, and by lower semi-continuity of the mapping
$(\omega, \varpi) \mapsto I^d_{[d]}(\omega, \varpi)$, there is a limit point $(\omega, \varpi) \in \mathcal{F}$ with $I^d_{[d]}(\omega, \varpi) = 0$. By Theorem 2.2(i), we have $H(\omega \| \nu) = 0$ and $\mathcal{H}_{\mathcal{C}}(\varpi \| \omega) = 0$. This implies $\omega(a) = \nu(a)$ and $\varpi(a, b) = \lambda_{[d]}(a, b) \omega(a) \omega(b)$, for $a, b \in \mathcal{X}$ which contradicts $(\omega, \varpi) \in \mathcal{F}$. Hence as desired.

Recall that $V$ is a fixed set of $n$ vertices, say $V = \{1, \ldots, n\}$, $\mathcal{G}_n$ is the set of all (simple) graphs with vertex set $V = \{1, \ldots, n\}$ and edge set $E \subseteq \mathcal{E} := \{(u, v) \in V \times V : u < v\}$. Now we compute the distribution $P_n : \mathcal{G}_n(\mathcal{X}) \rightarrow [0,1]$ of $X$, 

$$P(x) = \prod_{u \in V} \nu(x(y_u)) \prod_{(u,v) \in E} F(r_n(x(y_u), x(y_v))) \prod_{(u,v) \notin E} (1 - F(r_n(x(y_u), x(y_v))))$$

$$= \prod_{u \in V} \nu(x(y_u)) \prod_{(u,v) \in E} F(r_n(x(y_u), x(y_v))) \prod_{(u,v) \notin E} (1 - F(r_n(x(y_u), x(y_v))))$$

Therefore, we have in the case of Theorem 2.1

$$- \frac{1}{n(\log(n))} \log P(x) = - \int_{\mathcal{X}} \frac{\log \nu(a)}{(\log n)} L^1_\mathcal{X}(da) - \frac{1}{2} \int_{\mathcal{X}^2} \frac{\log(F(r_n(a,b))/(1-F(r_n(a,b))))}{(\log n)} L^2_\mathcal{X}(da, db)$$

$$- \frac{1}{2} \int_{\mathcal{X}^2} \frac{\log(1-(F(r_n(a,b))))}{(\log n)^2/n} L^1_\mathcal{X}(da, db) - \frac{1}{2} \int_{\mathcal{X}^2} \frac{\log(1-(F(r_n(a,b))))}{(\log n)^2} L^1_\Delta(da, da).$$

Now in the first case the integrands $-\frac{\log \nu(a)}{(\log n)^2}$, $-\frac{\log(1-(F(r_n(a,b))))}{(\log n)/n}$ and $-\frac{\log(1-(F(r_n(a,b))))}{(\log n)^2}$ all converge to zero, while $-\frac{\log((F(r_n(a,b))/(1-(F(r_n(a,b))))}{\log n}) \rightarrow 1$, for all $a, b \in \mathcal{X}$. Hence Theorem 2.1 follows from Theorem 3.1

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