ON THE AUTOMORPHISM GROUP OF RATIONAL MANIFOLDS

TURGAY BAYRAKTAR

ABSTRACT. In this note, we prove that every automorphism of a rational manifold which is obtained from $\mathbb{P}^k$ by a finite sequence blow-ups along smooth centers of dimension at most $r$ with $k > 2r + 2$ has zero topological entropy.

1. INTRODUCTION

A holomorphic automorphism of a compact Kähler manifold has positive topological entropy if and only if absolute value of one of the eigenvalues of $f^*$ on the cohomology $H^*(X, \mathbb{C})$ is larger than one. It follows from the results of Cantat [Can01, Can99] that a compact complex surface admit an automorphism with positive entropy if it is Kähler and bimeromorphic to one of the following: a rational surface, a torus, a $K3$ surface or an Enriques surface. In particular, if $X$ is a rational surface admitting an automorphism with positive entropy then $X$ is obtained from $\mathbb{P}^2$ by blowing up a finite sequence of at least ten points [Nag61]. Examples of rational surface automorphisms with positive entropy were given by [BK06, BK10, McM07]. On the other hand, in higher dimensions the question if one can obtain automorphisms with interesting dynamics by blowing up certain subvarieties of $\mathbb{P}^k$ remained open. Our first result partially addresses this question:

Theorem 1.1. Let $X$ be a rational manifold such that $X = X_m$ and $\pi_i : X_{i+1} \to X_i$ is obtained by blowing up a smooth irreducible subvariety of dimension at most $r$ with $k > 2r + 2$ in $X_i$ where $X_0 = \mathbb{P}^k$. If $f : X \to X$ is a holomorphic automorphism then $f$ has zero topological entropy.

In particular, if $X$ is obtained from $\mathbb{P}^k$ with $k \geq 3$ by blowing up a finite sequence of points then every holomorphic automorphism of $X$ has zero topological entropy. This was observed in [Tru12] when $k = 3$.

A cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ is called numerically effective (nef in short) if $\alpha$ lies in the closure of classes of Kähler forms. Following [Kaw85] we define numerical dimension of a nef class as

$$\nu(\alpha) := \max\{p \in \mathbb{N} : \alpha^p := \alpha \wedge \cdots \wedge \alpha \neq 0 \text{ in } H^{p,p}(X, \mathbb{R})\}. $$

Next, we prove that if an automorphism of a compact Kähler manifold preserves a numerically-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ with large numerical dimension then it has zero entropy.

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Theorem 1.2. Let $X$ be a compact Kähler manifold with $f \in \text{Aut}(X)$. If there exists a nef class $\alpha \in H^{1,1}(X, \mathbb{R})$ such that $\nu(\alpha) \geq \dim X - 1$ and $f^* \alpha = \alpha$ then $f$ has zero topological entropy.

In some sense Theorem 1.2 can be considered a generalization of Liberman’s result [Lie78] (see also [Zha09] for big and nef case) which asserts that if $X$ is a compact Kähler manifold and $f \in \text{Aut}(X)$ preserves a Kähler class then an iterate of $f$ belongs to $\text{Aut}_0(X)$, the connected component of the identity, and hence $f$ has zero topological entropy.

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2. Preliminaries

Let $X$ be a compact Kähler manifold. We denote the de Rham (respectively Dolbeault) cohomology groups by $H^{2p}(X, \mathbb{R})$ (respectively $H^{p,p}(X, \mathbb{C})$) and define

$$H^{p,p}(X, \mathbb{R}) := H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{R}).$$

Note that $H^{2p}(X, \mathbb{R})$, $H^{p,p}(X, \mathbb{C})$ are finite dimensional and one can identify $H^{p,p}(X, \mathbb{R})$ with a real subspace of $H^{p,p}(X, \mathbb{C})$. In the sequel, we implicitly use the fact that the cohomology classes can be defined in terms of smooth forms or currents. We refer the reader to [CH] for basic results in Hodge theory. A cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ is called numerically effective (nef in short) if $\alpha$ lies in the closure of classes of Kähler forms. The set of nef classes $H^{1,1}_{\text{nef}}(X, \mathbb{R})$ forms a closed convex cone which is strict that is $H^{1,1}_{\text{nef}}(X, \mathbb{R}) \cap -H^{1,1}_{\text{nef}}(X, \mathbb{R}) = \{0\}$.

We let $\text{Pic}(X)$ denote the Picard group of $X$ that is isomorphism classes of line bundles with the group operation tensor product and denote the Chern map by

$$c_1 : \text{Pic}(X) \to H^2(X, \mathbb{Z}).$$

By a slight abuse of notation we will write $c_1(L) \in H^2(X, \mathbb{R})$ where we consider the image of $c_1(L)$ under the inclusion $i : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$. The Neron-Severi group of $X$ is defined by $NS(X) = c_1(\text{Pic}(X)) \subset H^2(X, \mathbb{R})$ that is the Chern classes of line bundles on $X$. It follows from Lefschetz theorem on $(1, 1)$ classes that

$$NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R}).$$

We also let $NS_{\mathbb{R}}(X)$ be the real vector space $NS_{\mathbb{R}}(X) = NS(X) \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$.

A holomorphic line bundle $L$ is called numerically effective (nef) if

$$L \cdot C = \int_C c_1(L) \geq 0$$

for every curve $C \subset X$. It follows from [Dem92] that $L$ is nef if and only if $c_1(L) \in H^{1,1}_{\text{nef}}(X, \mathbb{R})$. A line bundle $L$ is said to be big if $\kappa(L) = \dim X$ where $\kappa(L)$ denotes the Kodaira-Iitaka dimension of $L$. It is well-known that a nef line bundle $L$ is big if and only if $L^k := \int_X c_1(L)^k > 0$. 

2.1. Dynamics of automorphisms of compact Kähler manifolds. Let $X$ be a compact Kähler manifold of dimension $k$ and $\omega$ be a fixed Kähler form on $X$. We let $\text{Aut}(X)$ denote the set of holomorphic automorphisms of $X$. Every $f \in \text{Aut}(X)$ induces a linear action

$$f^* : H^{p,p}(X, \mathbb{R}) \to H^{p,p}(X, \mathbb{R})$$

$$f^* \{ \theta \} := \{ f^* \theta \}$$

where $\{ \theta \}$ denotes the class of the smooth $(p,p)$ form $\theta$ in $H^{p,p}(X, \mathbb{R})$.

For $f \in \text{Aut}(X)$ the $i$th dynamical degree of $f$ is defined by

$$\lambda_i(f) := \limsup_{m \to \infty} \left( \int_X (f^n)^* \omega^i \wedge \omega^{k-i} \right)^{\frac{1}{n}}.$$

Since $X$ is compact this definition is independent of $\omega$. The following properties of dynamical degrees are well known [DS04a].

Lemma 2.1. Let $f : X \to X$ be an automorphism. Then

(i) $1 \leq \lambda_i$ is the spectral radius of $f^*|_{H^{i,i}(X, \mathbb{R})}$.

(ii) $i \to \log \lambda_i(f)$ is concave on $\{0, 1, \ldots, k\}$. 

(iii) $\lambda_1(f)^i \geq \lambda_1(f)$ and $\lambda_i(f)^i \geq \lambda_1(f)$ for $1 \leq i \leq k-1$.

(iv) $\lambda_i(f) = \lambda_{k-i}(f^{-1})$ for $i \in \{0, 1, \ldots, k\}$.

Theorem 2.2 (Gromov and Yomdin). Let $f \in \text{Aut}(X)$ then the topological entropy of $f$ is given by

$$h_{\text{top}}(f) = \max_{0 \leq i \leq k} \log \lambda_i(f).$$

The next proposition will be useful in the proof of Theorem 1.2.

Proposition 2.3. [DS04a] Let $\alpha, \alpha', \alpha_1, \ldots, \alpha_r \in H^{1,1}(X, \mathbb{R})$ be nef classes where $r \leq k-2$.

1. If $\alpha \wedge \alpha' = 0$ then $\alpha$ and $\alpha'$ are colinear.

2. If $\alpha \wedge \alpha' \wedge \alpha_1 \cdots \wedge \alpha_r = 0$ in $H^{r+2,r+2}(X, \mathbb{R})$ then there exists real numbers $(a, b) \neq (0, 0)$ such that

$$(a \alpha + ba') \wedge \alpha_1 \cdots \wedge \alpha_r = 0.$$ 

Furthermore, if $\alpha' \wedge \alpha_1 \cdots \wedge \alpha_r \neq 0$ then the pair $(a, b)$ is unique up to a multiplicative constant.

3. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Let $X = X_m$ be a rational manifold where $\pi_i : X_{i+1} \to X_i$ is obtained by blowing up a smooth irreducible subvariety $Y_i \subset X_i$ of dimension at most $r$ and $X_0 = \mathbb{P}^k$ with $k > 2r + 2$.

By Lemma 2.1 we have $\lambda_1(f)^{k-r-1} \geq \lambda_{k-r-1}(f)$. First, we will show that

$$\lambda_1^{k-r-1}(f) = \lambda_{k-r-1}(f).$$

Indeed, assuming otherwise $\lambda_1^{k-r-1}(f) > \lambda_{k-r-1}(f)$ we will derive a contradiction. Since $f^*$ preserves the nef cone it follows from a version of Perron-Frobenius theorem [Bir67] that there exists a nef class $\alpha \in NS_R(X)$ such that $f^* \alpha = \lambda_1(f) \alpha$. Now, as $f^*$ preserves the intersection product and $\lambda_1(f)^{k-r-1} > \lambda_{k-r-1}(f)$ we see that $\alpha^{k-r-1} = 0$ in $H^{k-r-1,k-r-1}(X, \mathbb{R})$. Therefore, $\nu(\alpha) \leq k-r-2$. Thus, the assertion follows from the next lemma:
Lemma 3.1. Let $X = X_m$ be a rational manifold where $\pi_i : X_{i+1} \to X_i$ is obtained by blowing up a smooth irreducible subvariety $Y_i \subset X_i$ of dimension at most $r$ and $X_0 = \mathbb{P}^k$ with $k \geq r + 2$. If $\alpha \in H^{1,1}_{\text{nef}}(X, \mathbb{R})$ is non-zero then $\nu(\alpha) \geq k - r - 1$.

Proof. It follows from [Kaw85] that

$$\nu(\alpha) = \max\{p : \alpha^p \cdot A^{k-p} \neq 0\}$$

where $A$ is any ample divisor. Therefore, it is enough to show that there exists a divisor $D$ such that $\alpha^{k-r-1} \cdot D^{r+1} \neq 0$.

It is classical that [GH] the Picard group $\text{Pic}(X)$ is generated by the classes $H_X, E_1, \ldots, E_m$ where

$$\pi = \pi_{m-1} \circ \pi_{m-2} \circ \cdots \circ \pi_1 : X \to X_0 = \mathbb{P}^k$$

$$H_X := \pi^*(H)$$

$H$ is the class of a generic hyperplane in $\mathbb{P}^k$ and $E_i$ is the exceptional divisor of the blow up $\pi_i : X_{i+1} \to X_i$ and

$$E_{i-1} := \pi_i^{-1}(E_{i-1} - Y_i)$$

is the class of the proper transform in $X_i$ of the exceptional divisor $E_{i-1}$. Then we can represent the class $\alpha$ as

$$\alpha = aH_X + \sum c_iE_i.$$ 

where $a, c_i \in \mathbb{R}$. Since $\pi(E_i) \subset \mathbb{P}^k$ has codimension at least 2, a generic line in $\mathbb{P}^k$ does not intersect $\pi(E_i)$. Then by the projection formula [Ful98] we have

$$E_i \cdot H_X^{k-1} = 0.$$ 

Since $\alpha$ is nef and nonzero this implies that $\alpha \cdot H_X^{k-1} = a > 0$.

Now, since the dimension of $Y_i$ is at most $r$, a generic subvariety of $\mathbb{P}^k$ of codimension $r + 1$ does not intersect $\pi(E_i)$. This in turn implies that $E_i^{k-r-1} \cdot H_X^{r+1} = 0$ hence,

$$\alpha^{k-r-1} \cdot H_X^{r+1} = a^{k-r-1} > 0.$$ 

Hence, we deduce that $\alpha^j \neq 0$ in $H^{j,j}(X, \mathbb{R})$ and $\lambda_1(f^j) = \lambda_j(f)$ for all $1 \leq j \leq k - r - 1$. Therefore, applying the same argument to $f^{-1}$ and using Lemma 2.1 we conclude that

$$\lambda_1(f)(k-r-1)^2 = \lambda_{k-r-1}(f)^{k-r-1} = \lambda_{r+1}(f^{-1})^{k-r-1} = \lambda_1(f^{-1})(k-r-1)(r+1)$$

$$= \lambda_{k-r-1}(f^{-1})^{r+1} = \lambda_{r+1}(f)^{r+1} = \lambda_1(f)^{(r+1)^2}$$

since $k > 2 + 2r$ this contradicts $\lambda_1(f) > 1$. 

Proof of Theorem 1.2. Let $f \in \text{Aut}(X)$ and assume that the first dynamical degree, $\lambda_1 > 1$ we will derive a contradiction. Since $f^*$ preserves the nef cone there exists a class $\beta \in H^{1,1}_{\text{nef}}(X, \mathbb{R})$ such that $f^*\beta = \lambda_1 \beta$.

Now, $\nu(\alpha) \geq k-1$ implies that $\alpha^{k-2} \neq 0$ in $H^{k-2,k-2}(X, \mathbb{R})$. On the other hand, since $f$ is an automorphism it preserves the cup product hence

$$f^*(\alpha^{k-1} \wedge \beta) = \lambda_1 \alpha^{k-1} \wedge \beta.$$
Since the topological degree $\lambda_k = 1$ we must have
$$\alpha^{k-1} \wedge \beta = 0.$$ 
Then, by Proposition 2.3 there exists (up to a scalar multiple) unique real numbers $(a, b) \neq (0, 0)$ such that
$$(a\beta + ba) \wedge \alpha^{k-2} = 0.$$ 
Pulling-back this equation by $f$, we obtain
$$(a\lambda_1\beta + ba) \wedge \alpha^{k-2} = 0.$$ 
Since $\lambda_1 > 1$, we see that $b = 0$. Thus,
$$\beta \wedge \alpha^{k-2} = 0.$$ 
Applying the same argument repeatedly we obtain that
$$\beta \wedge \alpha = 0.$$ 
Then Proposition 2.3 implies that $\beta = c\alpha$ for some $c \in \mathbb{R}_+$ but this contradicts $\lambda_1 > 1$. □

The following result is an immediate corollary of Theorem 1.2 and [Kaw85, Proposition 2.2]:

**Corollary 3.2.** Let $X$ be a projective manifold and $f \in \text{Aut}(X)$. If there exists a nef $\mathbb{R}$-divisor $L$ such that $\kappa(L) \geq k - 1$ and $f^*L \cong L$ then $h_{\text{top}}(f) = 0$.

Recall that a compact complex manifold is called Fano if the anti-canonical bundle $-K_X$ is ample. It follows from Kodaira embedding theorem that a Fano manifold is projective. More generally, a compact complex manifold is called weak Fano if the anti-canonical bundle $-K_X$ is big and nef. The following immediate corollary is well-known [Zha09]:

**Corollary 3.3.** Let $X$ be a projective weak Fano manifold and $f \in \text{Aut}(X)$ then $h_{\text{top}}(f) = 0$.

**Proof.** Note the $f$ preserves the divisor class $-K_X$ which is big and nef by definition of $X$. Thus, the assertion follows from the Corollary 3.2. □

Blanc and Lamy [BL] recently proved that blow up of the complex projective space $\mathbb{P}^3$ along a curve $C$ of degree $d$ and genus $g$ lying on a smooth quadric gives a weak Fano manifold if $4d - 30 \leq g \leq 14$ or $(g, d) = (19, 12)$ or there is no 5-secant line, 9-secant conic, nor 13-secant twisted cubic to $C$. In particular, Corollary 3.3 implies that blowing up $\mathbb{P}^3$ along such curves does not give rise to a rational manifold admitting an automorphism with positive entropy. More generally, it was observed in [Tru12] that if a rational manifold is obtained from $\mathbb{P}^3$ by blowing up finitely many curves with no common intersection then any automorphism of $X$ is of zero entropy.

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Mathematics Department, Johns Hopkins University 21218 Maryland, USA
E-mail address: bayraktar@jhu.edu