RIESZ MEANS OF FOURIER SERIES AND INTEGRALS: STRONG SUMMABILITY AT THE CRITICAL INDEX

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ABSTRACT. We consider spherical Riesz means of multiple Fourier series and some generalizations. While almost everywhere convergence of Riesz means at the critical index $(d-1)/2$ may fail for functions in the Hardy space $h^1(\mathbb{T}^d)$, we prove sharp positive results for strong summability almost everywhere. For functions in $L^p(\mathbb{T}^d)$, $1 < p < 2$, we consider Riesz means at the critical index $d(1/p - 1/2) - 1/2$ and prove an almost sharp theorem on strong summability. The results follow via transference from corresponding results for Fourier integrals. We include an endpoint bound on maximal operators associated with generalized Riesz means on Hardy spaces $H^p(\mathbb{R}^d)$ for $0 < p < 1$.

1. Introduction

We consider multiple Fourier series of functions on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For $\ell \in \mathbb{Z}^d$ let $e_\ell(x) = e^{2\pi i (x,\ell)}$ and define the Fourier coefficients of $f \in L^1(\mathbb{T}^d)$ by $\langle f, e_\ell \rangle = \int_{\mathbb{T}^d} f(y)e^{-2\pi i (y,\ell)}dy$. We shall examine the pointwise behavior of (generalized) Riesz means of the Fourier series. Fix a homogeneous distance function $\rho$, continuous on $\mathbb{R}^d$, positive and $C^\infty$ on $\mathbb{R}^d \setminus \{0\}$, and satisfying, for some $b > 0$, $\rho(t^b \xi) = t\rho(\xi)$ for all $\xi \in \mathbb{R}^d$. For $f \in L^1(\mathbb{T}^d)$ define the Riesz means of index $\lambda$ with respect to $\rho$, by

$$\mathcal{R}_\lambda^f = \sum\limits_{\ell \in \mathbb{Z}^d, \rho(\ell/\lambda) \leq 1} \frac{1}{1 - \rho(\ell/\lambda)^{1/\lambda}} \langle f, e_\ell \rangle e_\ell.$$

The classical Riesz means are recovered for $\rho(\xi) = |\xi|$, and when in addition $\lambda = 1$ we obtain the Fejér means. The Bochner-Riesz means are covered with $b = 1/2$ by taking $\rho(\xi) = |\xi|^2$.

It is well known via classical results for Fourier integrals ([34], [40], [31]) and transference ([25], [20], [1]) that for $\lambda > \frac{d-1}{2}$ and $f \in L^1(\mathbb{T}^d)$ we have $\lim_{t \to \infty} \mathcal{R}_\lambda^t f = f$, both with respect to the $L^1$ norm and also almost everywhere. For the critical index $\lambda = \frac{d-1}{2}$, it is known that the Riesz means are of weak type $(1, 1)$ and one has convergence in measure ([8], [10]) but Stein [35] showed early that a.e. convergence may fail (see also [40]). Indeed, extending ideas of Bochner, he proved the existence of an $L^1(\mathbb{T}^d)$ function for which the Bochner-Riesz means at index $\frac{d-1}{2}$ diverge almost everywhere, as $t \to \infty$. Stein’s theorem can be seen as an analogue of the theorem by Kolmogorov [23] on the failure of a.e. convergence for Fourier series in $L^1(\mathbb{T})$, see [48, ch. VIII-4]. Later, Stein [37] proved a stronger result showing that even for some functions in the subspace $h^1(\mathbb{T}^d)$ (the local Hardy space) the Bochner-Riesz means at the critical index may diverge almost everywhere. It is then natural to ask what happens if we replace almost everywhere convergence with the weaker notion of strong convergence a.e. (also known as strong summability a.e.) which goes back to Hardy and Littlewood [18].

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Definition. Let $0 < q < \infty$. Given a measurable function $g : (0, \infty) \to \mathbb{C}$ we say that $g(t)$ converges $q$-strongly to $a$, as $t \to \infty$, if

$$\lim_{T \to \infty} \left( \frac{1}{T} \int_0^T |g(t) - a|^q \, dt \right)^{1/q} = 0.$$ 

If $g(t)$ refers to the partial sum of a series then one also says that the series is strongly $H_q$ summable. Clearly if $\lim_{t \to \infty} g(t) = a$ then $g(t)$ converges $q$-strongly to $a$ for all $q < \infty$. Vice versa if $g(t)$ converges $q$-strongly to $a$ for some $q > 0$ then $g(t)$ is almost convergent to $a$ as $t \to \infty$. That is, there is a (density one) subset $E \subset [0, \infty)$ satisfying

$$\lim_{t \to \infty} \frac{|E \cap [0, T]|}{T} = 1 \quad \text{and} \quad \lim_{t \to \infty} g(t) = a.$$ 

See [48, ch.XIII, (7.2)] and also Corollary 7.3 below.

For the classical case of a Fourier series of an $L^1(\mathbb{T})$ function, Zygmund [47] proved that the partial sum $\sum_{n \leq T} (f, e_n) e_n(x)$ converges $q$-strongly to $f(x)$ as $t \to \infty$ a.e. for all $q < \infty$, extending an earlier result by Marcinkiewicz [28] for $q = 2$. Zygmund used complex methods, but in more recent papers one can find alternative approaches with stronger results and some weaker extensions to rectangular partial sums of multiple Fourier series; see, e.g., [30] and [46] and references therein. See also [24] for an overview of recent developments on topics related to the convergence of Fourier series.

Regarding spherical partial sums of multiple Fourier series, $q$-strong convergence results have been available for $L^p(\mathbb{T}^d)$ functions for the Bochner-Riesz means of index $\lambda > \lambda(p)$ when $p \leq 2$, $q = 2$, where $\lambda(p) = d(\frac{1}{p} - \frac{1}{2}) - \frac{d}{2}$ is the critical index (cf. [34], [42]). The question of strong convergence a.e. for the Bochner-Riesz means at the critical index $\lambda(1) = \frac{d-1}{2}$, for either $f \in L^1(\mathbb{T}^d)$ or $f \in h^1(\mathbb{T}^d)$ had been left open and was posed by S. Lu in the survey article [27]. We answer this question in the affirmative for $f \in h^1(\mathbb{T}^d)$ for generalized Riesz means with any distance function $\rho$ under consideration.

Theorem 1.1. Let $q < \infty$ and $\lambda(1) = \frac{d-1}{2}$. Then, for all $f \in h^1(\mathbb{T}^d)$ the following statements hold.

(i) There is a constant $C$ such that for all $\alpha > 0$,

$$\text{meas}\left( \left\{ x : \sup_{T > 0} \left( \frac{1}{T} \int_0^T |\mathcal{R}_{t}^{\lambda(1)} f(x)\|^q \, dt \right)^{1/q} > \alpha \right\} \right) \leq C \alpha^{-1} \|f\|_{h^1}.$$ 

(ii) 

$$\lim_{T \to \infty} \left( \frac{1}{T} \int_0^T |\mathcal{R}_{t}^{\lambda(1)} f(x) - f(x)|^q \, dt \right)^{1/q} = 0 \quad \text{for almost every} \ x \in \mathbb{T}^d.$$ 

We remark that for the classical Riesz means (or generalized Riesz means assuming finite type conditions on the cosphere $\Sigma_\rho = \{ \xi : \rho(\xi) = 1 \}$), Theorem 1.1 for the range $q \leq 2$ could have been extracted from [32], although that result is not explicitly stated there. The full range $q < \infty$ obtained here seems to be new. Regarding the question posed for $f \in L^1(\mathbb{T}^d)$, in Section 7, we derive some weaker results including $q$-strong convergence up to passing to a subsequence.

We now address the question of strong convergence of Riesz means for $L^p(\mathbb{T}^d)$ functions at the critical index $\lambda = \lambda(p)$. In this case, $q$-strong convergence results may fail for large $q$. Our next result identifies nearly sharp range of $q$ for which $\mathcal{R}_{t}^{\lambda(p)} f(x)$ converges $q$-strongly to $f(x)$ almost everywhere for any $f \in L^p(\mathbb{T}^d)$. We denote by $p' = \frac{p}{p-1}$ the exponent dual to $p$. 
Theorem 1.2. Let $1 < p < 2$, $q < p'$ and $\lambda(p) = d\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$. Then, for all $f \in L^p(\mathbb{T}^d)$ the following statements hold.

(i) There is a constant $C$ such that for all $\alpha > 0$,
\[
\text{meas}\left(\{ x \in \mathbb{T}^d : \sup_{T > 0} \left(\frac{1}{T} \int_0^T |R_t^\lambda(f(x))|^{\frac{q}{p}} dt\right)^{\frac{1}{q}} > \alpha \} \right) \leq C\alpha^{-p} \|f\|_{L^p(\mathbb{T}^d)}.
\]

(ii) \[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |R_t^\lambda(f(x) - f(x)|^{\frac{q}{p}} dt\right)^{\frac{1}{q}} = 0 \text{ for almost every } x \in \mathbb{T}^d.
\]

(iii) For suitable $f \in L^p(\mathbb{T}^d)$ statements (i), (ii) fail when $q > p'$.

Part (ii) in both theorems follow by a standard argument from the respective part (i), using the fact that pointwise (in fact uniform) convergence holds for Schwartz functions. We note that Theorem 1.1 is sharp in view of the above mentioned example by Stein. Moreover, part (iii) of Theorem 1.2 shows that the result is essentially sharp for all $p \in (1, 2)$, but the case $q = p'$ remains open.

We state a special case of Theorem 1.2 for $\lambda(p) = 0$, i.e., for the case of generalized spherical partial sums of Fourier series as a corollary.

Corollary 1.3. Let $d \geq 2$, $q < \frac{2d}{d-1}$ and $f \in L^{\frac{2d}{d-1}}(\mathbb{T}^d)$. Then
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \sum_{\rho(\xi/t) \leq 1} \langle f, e_\xi \rangle e_\xi(x) - f(x) \right|^q dt \right)^{\frac{1}{q}} = 0 \text{ for almost every } x \in \mathbb{T}^d.
\]

In particular, for almost every $x \in \mathbb{T}^d$, the partial sums $\sum_{\rho(\xi/t) \leq 1} \langle f, e_\xi \rangle e_\xi(x)$ are almost convergent to $f(x)$ as $t \to \infty$, in the sense of (1.2).

We remark that there are analogues of above results for generalized Riesz means of Fourier integral in $\mathbb{R}^d$:
\[
(1.3) \quad R_t^\lambda f(x) = \int_{\rho(\xi/t) \leq 1} (1 - \rho(\xi/t))^\lambda \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.
\]
See §2. Indeed, we derive Theorems 1.1 and 1.2 from corresponding theorems for Fourier integrals in $\mathbb{R}^d$ using transference arguments. Our proof uses somewhat technical arguments on atomic decomposition and Calderón-Zygmund theory. Unlike the proofs of the $L^p$ boundedness of Bochner-Riesz means (such as in [38], [5] and the references therein), our proof does not rely on Fourier restriction theory thanks to the averaging over the dilation parameter $t$. In particular, the curvature of the cosphere $\Sigma_\rho = \{ \xi : \rho(\xi) = 1 \}$ does not play a role in the argument (cf. [10], [11]), which allows us to work with generalized Riesz means with respect to any smooth homogeneous distance function.

Notation. Given two quantities $A, B$ we use the notation $A \lesssim B$ to mean that there is a constant $C$ such that $A \leq CB$. We use $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

This paper. In §2 we formulate Theorems 2.1 and 2.2 on strong convergence for Riesz means of critical index in $\mathbb{R}^d$ and reduce their proof to the main weak type inequality stated in Theorem 2.3. Some preliminary estimates are contained in §3. The proof of the main Theorem 2.3 is given in §4. In §5 we use transference arguments to prove the positive results in Theorems 1.1 and 1.2. In §7 we discuss a weaker result for $L^1$ functions. In §6 we show the essential sharpness of our $L^p$ results, namely that Theorems 1.2 and 2.2 require the condition $q \leq p'$ (the failure of the maximal theorems for $h^1$ already follows from Stein's example [37]). In §8 we include the proof of an extension of a theorem by Stein, Taibleson and Weiss ([39]), namely an $H^p \to L^{p, \infty}$
estimate for the maximal function \( \sup_{t>0} |R_t^{\lambda(p)} f(x)| \) associated with generalized Riesz means in Hardy spaces \( H^p \) with \( p < 1 \). Finally, we discuss some open problems in §9.

2. The main weak type estimate

We state results on \( \mathbb{R}^d \) which are analogous to Theorems 1.1 and 1.2 and reduce them to a crucial inequality for a vector-valued operator stated in Theorem 2.3. Let \( \rho \) be as in the introduction. Recall the definition of Riesz means \( R_t^\lambda \) for Fourier integrals from (1.3).

**Theorem 2.1.** Let \( q < \infty \) and \( \lambda(1) = \frac{d-1}{2} \). Then, for all \( f \in H^1(\mathbb{R}^d) \), for all \( \alpha > 0 \),

\[
\text{meas}\left( \left\{ x \in \mathbb{R}^d : \sup_{T>0} \left( \frac{1}{T} \int_0^T |R_t^{\lambda(1)} f(x)|^q dt \right)^{1/q} > \alpha \right\} \right) \leq C \alpha^{-1} \| f \|_{H^1(\mathbb{R}^d)}.
\]

**Theorem 2.2.** Let \( 1 < p < 2 \), \( q < p' \) and \( \lambda(p) = d\left(\frac{1}{p'} - \frac{1}{2}\right) \). Then, for all \( f \in L^p(\mathbb{R}^d) \), for all \( \alpha > 0 \),

\[
\text{meas}\left( \left\{ x \in \mathbb{R}^d : \sup_{T>0} \left( \frac{1}{T} \int_0^T |R_t^{\lambda(p)} f(x)|^q dt \right)^{1/q} > \alpha \right\} \right) \leq C \alpha^{-p} \| f \|_{L^p(\mathbb{R}^d)}.
\]

As a consequence of these estimates we obtain

\[
\lim_{T \to 0} \left( \frac{1}{T} \int_0^T |R_t^{\lambda(p)} f(x) - f(x)|^q dt \right)^{1/q} = 0
\]

for almost every \( x \in \mathbb{R}^d \), for every \( f \in L^p(\mathbb{R}^d) \) when \( 1 < p < 2 \) and \( f \in h^1(\mathbb{R}^d) \) or \( H^1(\mathbb{R}^d) \) when \( p = 1 \).

We start the reduction of Theorems 2.1 and 2.2 to Theorem 2.3 by replacing the multipliers for the Riesz means \( R_t^\lambda \) with similar multipliers supported away from the origin, see (2.2) below.

2.1. **Contribution near the origin.** Let \( v_0 \in C^\infty(\mathbb{R}) \) so that \( v_0(\rho) = 1 \) for \( \rho \leq 4/5 \) and \( v_0(\rho) = 0 \) for \( \rho \geq 9/10 \). It is then standard that the maximal function \( \sup_{t>0} |\mathcal{F}^{-1}[v_0(\rho(\cdot/t))(1-\rho(\cdot/t))^{\lambda/\tilde{f}}]| \) defines an operator of weak type \((1,1)\) and bounded on \( L^p \) for all \( p > 1 \). A small complication occurs if \( \rho \) is not sufficiently smooth at the origin. We address this complication as follows.

Define, for \( N > 0 \), the functions \( u, u_N \) with domain \((0,\infty)\) by \( u(\tau) = v_0(\tau)(1-\tau)^\lambda \) and \( u_N(s) = u(s^{1/N}) \). It is then straightforward to check that for all \( M \)

\[
\int_0^\infty s^M |u_N^{(M+1)}(s)| ds < \infty
\]

and we have the subordination formula ([45])

\[
(2.1) \quad u(\rho(\xi)) = u_N(\rho^N(\xi)) = \frac{(-1)^{M+1}}{M!} \int_0^\infty \left( 1 - \frac{\rho(\xi)}{s} \right)^N s^{M-1} u_N^{(M+1)}(s) ds
\]

which is proved by integration by parts. Given any \( m > 0 \) one has \( |\mathcal{F}^{-1}[(1-\rho^N)^M]_+(\xi)| \lesssim_m (1+|\xi|)^{-m} \) provided \( M \) and \( N \) are large enough. This is used to show that \( \sup_{t>0} |\mathcal{F}^{-1}[u \circ \rho(\cdot/t)](\xi)| \) is dominated by a constant times the Hardy-Littlewood maximal function of \( f \) (see also Lemma 8.2).

We can now replace the operator \( R_t^\lambda \) in Theorems 2.1 and 2.2 by \( S_t^\lambda \) defined by

\[
(2.2) \quad S_t^\lambda f(\xi) = (1 - v_0(\rho(\xi/t)))(1 - \rho(\xi/t))^{\lambda/\tilde{f}}(\xi).
\]
2.2. Further decompositions. We first recall standard dyadic decompositions on the frequency side. Let \( \eta \in C^\infty_c(\mathbb{R}^d \setminus \{0\}) \) such that \( \eta \) is nonnegative,
\[
\eta(\xi) = 1 \text{ on } \{ \xi : \rho(\xi/t) \in [1/4, 4], \ 1/2 \leq t \leq 2 \}.
\]
Define \( L_k f \) by \( \overline{L}_k f(\xi) = \eta(2^{-k}\xi)\hat{f}(\xi) \).

We use the nontangential version of the Peetre maximal operators
\[
\mathcal{M}_k f(x) = \sup_{|h| \leq 2^{-k+10d}} |L_k f(x+h)|
\]
and the associated square function
\[
\mathcal{S} f(x) = \left( \sum_{k \in \mathbb{Z}} |\mathcal{M}_k f(x)|^2 \right)^{1/2}.
\]
Then
\[
\|\mathcal{S} f\|_{L^1} \leq C\|f\|_{H^1},
\]
and
\[
\|\mathcal{S} f\|_{L^p} \leq C_p\|f\|_{L^p}, \quad 1 < p < \infty.
\]
see (Peetre [29]).

The inequalities in Theorems 2.1 and 2.2 follow from
\[
\left\| \sup_{T>0} \left( \frac{1}{T} \int_0^T |S^\lambda_t f|^q dt \right)^{1/q} \right\|_{L^{p,\infty}} \lesssim \|\mathcal{S} f\|_p
\]
for \( 1 \leq p < 2, \ q < p' \). Here \( L^{p,\infty} \) is the weak type Lorentz space and the expression \( \|g\|_{L^{p,\infty}} = \sup_{\alpha>0} \alpha(\text{meas}(\{x : |g(x)| > \alpha\}))^{1/p} \) is the standard quasi-norm on \( L^{p,\infty} \). We may, by Hölder’s inequality, assume that \( 2 \leq q < p' \). We can then use
\[
\sup_{T>0} \left( \frac{1}{T} \int_0^T |S^\lambda_t f(x)|^q dt \right)^{1/q} \leq 2^{1/q} \left( \sum_{k \in \mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |S^\lambda_t f(x)|^q dt \right)^{1/q}.
\]
We now use the standard idea to decompose the multiplier \( (1 - \nu_0 \circ \rho)(1 - \rho)\lambda_\cdot \) into pieces supported where \( \rho(\xi) \in [1 - 2^{-j}, 1 - 2^{-j-2}] \). Generalizing slightly we assume that we are given \( C^\infty \) functions \( \varphi_j \) supported in \( [1 - 2^{-j}, 1 - 2^{-j-2}] \) and satisfying
\[
\|\partial^n \varphi_j\|_\infty \leq C_n 2^{jn}.
\]
for \( n = 0, 1, 2, \ldots \). Let \( I : = [1, 2] \). For \( t \in I, \ k \in \mathbb{Z} \) define
\[
T^k_\cdot f(\xi, t) = \varphi_j(\rho(2^{-k}t^{-1}\xi))\hat{f}(\xi)
\]
We may decompose \( S^\lambda_{2^{-j}t} f = \sum_{j \geq 1} 2^{-j\lambda}T^k_\cdot f(x, t) \), with \( T^j_k \) of the form in (2.7). The asserted estimates for \( S^\lambda_t f \) follow now from weak type bounds for the expression on the right hand side of (2.6). By (2.3) we have \( \eta(2^{-k}\xi) = 1 \) whenever \( \rho(2^{-k}\xi/t) \in \text{supp}(\varphi_j) \) for any \( t \in I \). Thus after changing variables the desired estimate can be restated as
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \int_I \left( \sum_{j=1}^{\infty} 2^{-j\lambda}T^k_\cdot L_k f(\cdot, t) \right)^{q} dt \right)^{1/q} \right\|_{L^{p,\infty}} \lesssim \|\mathcal{S} f\|_p.
\]
Since \( \ell^2 \subset \ell^q \) for \( q \geq 2 \) this follows from the following stronger statement, our main estimate.
Theorem 2.3. For $1 \leq p < 2$, $\lambda(p) = d(1/p - 1/2) - 1/2$, $q < p'$,
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{t}^{\infty} \left| \sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_k^j L_k f(\cdot, t) \right|^q dt \right)^{2/q} \right)^{1/2} \right\|_{L^{p, \infty}(\mathbb{R}^d)} \lesssim \| \mathcal{G} f \|_{L^p(\mathbb{R}^d)}.
\]

The theorem will be proved in §4. Some preparatory material is contained in §3.

3. Preliminary estimates

We gather elementary estimates for the operators $T^k_j$ defined in (2.7).

Lemma 3.1. For $2 \leq q \leq \infty$, 
\[
\left\| \left( \int_{1}^{2} |T^k_j f(\cdot, t)|^q dt \right)^{1/q} \right\|_2 \lesssim 2^{-j/q} \| f \|_2.
\]

Proof. Use the convexity inequality, $\| \gamma \|_q \leq \| \gamma \|_2^{2/q} \| \gamma \|_\infty^{1-2/q}$, for $\gamma \in L^q([1, 2])$, and for $\gamma \in C^1$ we have $\| \gamma \|_\infty \lesssim \| \gamma \|_2^{1/2} (\| \gamma \|_2 + \| \gamma' \|_2)^{1/2}$ and hence
\[
(3.1) \quad \left( \int_{1}^{2} |\gamma(t)|^q dt \right)^{1/q} \lesssim \left( \int_{1}^{2} |\gamma(t)|^2 dt \right)^{1/2} + \left( \int_{1}^{2} |\gamma(t)|^2 dt \right)^{1/2} \left( \int_{1}^{2} |\gamma'(t)|^2 dt \right)^{1/(2-q)}.
\]

We obtain after some standard estimations
\[
\left\| \left( \int_{1}^{2} |T^k_j f(\cdot, t)|^2 dt \right)^{1/2} \right\|_2 + 2^{-j} \left\| \int_{1}^{2} \frac{d}{dt} T^k_j f(\cdot, t)^2 dt \right\|_2 \lesssim 2^{-j/2} \| f \|_2
\]
and then the assertion of the lemma follows from (3.1) applied to $\gamma(t) = T^k_j f(x, t)$, followed by Hölder’s inequality in $x$. \hfill \Box

To prove the $L^1$ estimate we rely on a spherical decomposition introduced in [12]. For each fixed $j \geq 1$, we use a $C^\infty$ partition of unity $\{\chi_{j,\nu}\}_{\nu \in \mathcal{Z}_j}$ for an index set $\mathcal{Z}_j$ with $\# \mathcal{Z}_j = O(2^{j(d-1)/2})$, which has the following properties; each $\chi_{j,\nu}$ is homogeneous of degree $0$, the restriction of the support of $\chi_{j,\nu}$ to the sphere $\{\xi : |\xi| = 1\}$ is supported in a set of diameter $2^{-j/2}$ and each unit vector is contained in the supports of $\chi_{j,\nu}$ for $O(1)$ indices $\nu$. We may choose the index set $\mathcal{Z}_j$ so that for every $\nu$, there is a unit vector $\xi_{j,\nu} \in \text{supp}(\chi_{j,\nu})$ so that $\text{dist}(\xi_{j,\nu}, \xi_{j,\nu'}) \geq c2^{-j/2}$ for $\nu \neq \nu'$. We assume that the $\chi_{j,\nu}$ satisfy the natural differential estimates, i.e. $\partial^\beta \chi_{j,\nu}(\xi) = O(2^j(\beta_1 + \ldots + \beta_d))$. Define $T^k_{j,\nu}$ by
\[
(3.2) \quad T^k_{j,\nu} f(\xi, t) = \chi_{j,\nu}(\xi) \varphi_j(\rho(2^{-k} t^{-1} \xi)) \widehat{f}.
\]
Let $K_j = F^{-1}[\varphi_j(\rho(\cdot))]$, and $K_{j,\nu} = F^{-1}[\varphi_j(\rho(\cdot))\chi_{j,\nu}]$. Let $\Phi_0 \in C_c^\infty(\mathbb{R}^d)$ supported in $\{ x : |x| \leq 1 \}$ so that $\Phi_0(x) = 1$ for $|x| \leq 1/2$ and, for $n \geq 1$, let $\Phi_n(x) = \Phi_0(2^{-n} x) - \Phi_0(2^{1-n} x)$. Define, for $n = 0, 1, 2, \ldots$,
\[
K^0_j(x) = K_j(x) \Phi_0(2^{-j} x)
\]
\[
K^n_{j,\nu}(x) = K_{j,\nu}(x) \Phi_n(2^{-j} x)
\]
and
\[
T^{n,k}_j f(x, t) = (2^k t)^d K^n_j(2^k t) \ast f,
\]
\[
T^{n,k}_{j,\nu} f(x, t) = (2^k t)^d K^n_{j,\nu}(2^k t) \ast f.
\]
Then
\[
(3.3) \quad T^k_j f = \sum_{\nu \in \mathcal{Z}_j} T^k_{j,\nu} f = \sum_{n=0}^{\infty} T^{n,k}_j f = \sum_{n=0}^{\infty} \sum_{\nu \in \mathcal{Z}_j} T^{n,k}_{j,\nu} f.
\]
Lemma 3.2. Let $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$. Then
\[
|\hat{K}_j^n(\xi)| \leq C_{M_0,M_1}2^{-nM_0}(1 + 2^j \text{dist}(\xi,\Sigma_\rho))^{-M_1}.
\]

Sketch of Proof. Let $\Psi(x) = \Phi_0(x/2) - \Phi_0(x)$. Then, for $n \geq 1$, we may use that $\Psi$ has vanishing moments and write
\[
\hat{K}_j^n(\xi) = \int \varphi_j(\rho(\xi - y))2^{j+n\delta}\hat{\Psi}(2^{j+n}y)dy
\]
(3.4)
\[
= \int_0^1 \frac{(1-s)^{N-1}}{(N-1)!} \int \langle y, \nabla \rangle^N [\varphi_j \circ \rho](\xi - sy)2^{j+n\delta}\hat{\Psi}(2^{j+n}y)dy ds
\]
by Taylor’s formula. The estimate is now straightforward. When $n = 0$ we just use the first line in (3.4) with $\Psi$ replaced by $\Phi_0$. \hfill \Box

For each $\nu$ choose $\xi_{j,\nu}$ such that $\rho(\xi_{j,\nu}) = 1$ and $\xi_{j,\nu} \in \text{supp}(\chi_{j,\nu})$. Take $e_{j,\nu} = \frac{\nabla \rho(\xi_{j,\nu})}{|\nabla \rho(\xi_{j,\nu})|}$ and let $P_{j,\nu}$ be the orthogonal projection to $e_{j,\nu}^\perp$, i.e.
\[
P_{j,\nu}h = h - \langle h, e_{j,\nu} \rangle e_{j,\nu}.
\]

Lemma 3.3. For every $M \geq 0$,
\[
\sup_{t \in I} |t^d K_{j,\nu}(tx)| \leq C(M) \frac{2^{-j(d+1)/2}}{(1 + 2^{-j}|x|)^M (1 + 2^{-j/2}|P_{j,\nu}(x)|)^M}.
\]

Proof. This is standard (and follows after integration by parts), see, e.g., [12], [11], or [31]. \hfill \Box

Lemma 3.4. (i) For $k \in \mathbb{Z}$,
\[
\| \sup_{t \in I} |T_{j,j}^{n,k} f(\cdot, t)| \|_1 \leq C_N 2^{j(d-1)/2} 2^{-nN}\|f\|_1.
\]
(ii) For $1 < p \leq 2$, $q \leq p'$ and $k \in \mathbb{Z}$,
\[
\left\| \left( \int_I |T_{j,j}^{n,k} f(\cdot, t)|^q dt \right)^{1/q} \right\|_p \leq C_N 2^{j(d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2})} 2^{-nN}\|f\|_p.
\]
(iii) For $2 \leq q \leq \infty$,
\[
\left\| \left( \int_1^2 |T_{j,j}^{n,k} f(\cdot, t)|^q dt \right)^{1/q} \right\|_2 \lesssim 2^{-j/q} 2^{-nN}\|f\|_2.
\]

Proof. Lemma 3.3 easily implies $\| \sup_{t \in I} |T_{j,j}^{n,k} f(\cdot, t)| \|_1 \leq C_N 2^{-nN}\|f\|_1$ and part (i) follows after summing in $\nu$. Using Lemma 3.2 we see that the proof of Lemma 3.1 also gives
\[
\left\| \left( \int_1^2 |T_{j,j}^{n,k} f(\cdot, t)|^2 dt \right)^{1/2} \right\|_2 \lesssim_N 2^{-nN} 2^{-j/2}\|f\|_2.
\]
Part (ii) now follows by complex interpolation.

Part (iii) for $q = 2$ is just the previous displayed inequality. For $q > 2$ it follows by the argument in Lemma 3.1 (cf. (3.1)) applied to $T_{j,j}^{n,k}$ in place of $T_{j,j}^k$, in conjunction with Lemma 3.2. \hfill \Box
4. Proof of Theorem 2.3

The proof combines ideas that were used in the proof of weak type inequalities for Bochner-Riesz means and other radial multipliers, and elsewhere ([15], [8], [9], [10], [32]). It combines atomic decompositions with Calderón-Zygmund estimates using $L^r$-bounds for $r > p$ in the complement of suitable exceptional sets together with analytic interpolation arguments inspired by [9].

In this section we fix a Schwartz function $f$ whose Fourier transform has compact support in $\mathbb{R}^d \setminus \{0\}$. Observe that then $\mathcal{L}_k f = 0$ for all but a finite number of indices $k$ (depending on $f$). This assumption together with the Schwartz bounds can be used to justify the a priori finiteness of various expressions showing up in the arguments below, but they do not enter quantitatively in the estimates.

We need to prove the inequality

\[
\text{meas}\left\{ x \in \mathbb{R}^d : \left( \sum_{k} \left| \int \left[ \sum_{j=1}^{\infty} 2^{-j\lambda(p)} T_j^k \mathcal{L}_k f(x,t) \right]^{q/2} dt \right|^{2/q} \right)^{1/2} > \alpha \right\} \lesssim \alpha^{-p} \|\mathcal{S} f\|^p_p,
\]

for arbitrary but fixed $\alpha > 0$. The implicit constant does not depend on $\alpha$ and the choice of $f$.

4.1. Preliminaries on atomic decompositions. Let $\mathcal{R}_k$ be the set of dyadic cubes of side length $2^{-k}$ so that each $R \in \mathcal{R}_k$ is of the form $\prod_{i=1}^d [n_i 2^{-k}, (n_i + 1)2^{-k})$ for some $n \in \mathbb{Z}^d$. For $\mu \in \mathbb{Z}$

\[
\Omega_{\mu} = \{ x : |\mathcal{S} f(x)| > 2^\mu \}
\]

and let $\mathcal{R}_k^\mu$ be the set of dyadic cubes of length $2^{-k}$ with the property that $|R \cap \Omega_{\mu}| \geq |R|/2$ and $|R \cap \Omega_{\mu+1}| < |R|/2$.

Clearly if $\mathcal{S} f \in L^p$ then every dyadic cube in $\mathcal{R}_k$ belongs to exactly one of the sets $\mathcal{R}_k^\mu$. We then have ([7])

\[
\sum_{k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_k^\mu} \int_R |\mathcal{L}_k f|^2 dx \lesssim 2^{2\mu} \text{meas}(\Omega_{\mu}).
\]

For completeness we give the argument. Observe that

\[
|\mathcal{L}_k f(x)| \leq M_k f(z), \quad \text{for } x, z \in R, \ R \in \mathcal{R}_k^\mu.
\]

Let

\[
\overline{\Omega}_{\mu} = \{ x : M_{HL} \mathbb{1}_{\Omega_{\mu}} > 10^{-d} \},
\]

where $M_{HL}$ denotes the Hardy-Littlewood maximal operator. Then

\[
\text{meas}(\overline{\Omega}_{\mu}) \lesssim \text{meas}(\Omega_{\mu})
\]

and we have $\cup_{k \in \mathbb{Z}} \cup_{R \in \mathcal{R}_k^\mu} R \subset \overline{\Omega}_{\mu}$. Now

\[
\sum_{k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_k^\mu} \|\mathbb{1}_R \mathcal{L}_k f\|^2 \leq \sum_{k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_k^\mu} 2 \int_{R \setminus \Omega_{\mu+1}} |M_k f(x)|^2 dx
\]

\[
\leq 2 \int_{\Omega_{\mu} \setminus \Omega_{\mu+1}} \sum_{k \in \mathbb{Z}} |M_k f(x)|^2 dx \leq 2^{2\mu+1} \text{meas}(\overline{\Omega}_{\mu}) \leq C 2^{2\mu} \text{meas}(\Omega_{\mu})
\]

which yields (4.2).

Next we work with a Whitney decomposition of the open set $\overline{\Omega}_{\mu}$, which is a disjoint union of dyadic cubes $W$, such that

\[
\text{diam}(W) \leq \text{dist}(W, \overline{\Omega}_{\mu}) \leq 4 \text{diam}(W).
\]
See [36, ch. VI.1]. We denote by $\mathfrak{M}^\mu$ the collection of these Whitney cubes. Each $R \in \mathcal{R}_k^\mu$ is contained in a unique $W(R) \in \mathfrak{M}^\mu$. For each $W$ define

$$\mathcal{R}_k^\mu(W) = \{ R \in \mathcal{R}_k^\mu : R \subset W \}$$

and

$$\gamma_{W,\mu} = \left( \frac{1}{|W|} \sum_k \sum_{R \in \mathcal{R}_k^\mu(W)} \int_R |L_k f(y)|^2 dy \right)^{1/2}.$$ 

Define

$$U(x) = \sum_\mu \sum_{W \in \mathfrak{M}^\mu} \gamma_{W,\mu}^p 1_W(x).$$

Observe that

$$\|U\|_1 = \sum_\mu \sum_{W \in \mathfrak{M}^\mu} |W| \gamma_{W,\mu}^p = \sum_\mu \sum_{W \in \mathfrak{M}^\mu} |W|^{1-p/2} (|W|^{1/2} \gamma_{\mu,W})^p$$

$$\leq \sum_\mu \left( \sum_{W \in \mathfrak{M}^\mu} |W| \right)^{1-p/2} \left( \sum_{W \in \mathfrak{M}^\mu} |W| \gamma_{\mu,W}^2 \right)^{p/2}$$

$$\leq \sum_\mu |\Omega_\mu|^{1-p/2} \left( \sum_k \sum_{W \in \mathfrak{M}^\mu} \sum_{R \in \mathcal{R}_k^\mu(W)} \|1_R L_k f\|_2^2 \right)^{p/2}$$

$$\leq \sum_\mu |\Omega_\mu|^{1-p/2} (2^{2\mu} |\Omega_\mu|)^{p/2} \lesssim \sum_\mu 2^{\mu p} |\Omega_\mu|,$$

by (4.2), and thus

$$\sum_\mu \sum_{W \in \mathfrak{M}^\mu} |W| \gamma_{W,\mu}^p = \|U\|_1 \lesssim \|S f\|_p^p.$$  

For $\alpha > 0$ let

$$O_\alpha = \{ x : M_{HL} U > \alpha^p \}$$

and

$$\tilde{O}_\alpha = \{ x : M_{HL} 1_{O_\alpha}(x) > (10d)^{-d} \}$$

so that $O_\alpha \subset \tilde{O}_\alpha$ and

$$\text{meas}(\tilde{O}_\alpha) \lesssim \text{meas}(O_\alpha) \lesssim \alpha^{-p} \|S f\|_p^p.$$  

Let $\Omega_\alpha = \{ Q \}$ be the collection of Whitney cubes for the set $\tilde{O}_\alpha$ (cf. [36, ch. VI]) so that

$$\text{diam}(Q) \leq \text{dist}(Q, \tilde{O}_\alpha^c) \leq 4 \text{diam}(Q).$$

In analogy to the usual terminology of “good” and “bad” functions in Calderón-Zygmund theory we split, for fixed $\alpha$, the collection $\mathfrak{M}^\mu$ into two subcollections $\mathfrak{M}_\mu^{\text{good}} \equiv \mathfrak{M}_\mu^{\text{good}}(\alpha)$ and $\mathfrak{M}_\mu^{\text{bad}} \equiv \mathfrak{M}_\mu^{\text{bad}}(\alpha)$ by setting

$$\mathfrak{M}_\mu^{\text{bad}} = \{ W \in \mathfrak{M}^\mu : \gamma_{W,\mu} > \alpha \},$$

$$\mathfrak{M}_\mu^{\text{good}} = \{ W \in \mathfrak{M}^\mu : \gamma_{W,\mu} \leq \alpha \}.$$ 

We relate the collection $\mathfrak{M}_\mu^{\text{bad}}$ with the collection of Whitney cubes $\Omega_\alpha$ for the set $\tilde{O}_\alpha$.

**Lemma 4.1.** Let $W \in \mathfrak{M}_\mu^{\text{bad}}$. Then $W \subset O_\alpha$. Moreover, there is a unique cube $Q = Q(W) \in \Omega_\alpha$ containing $W$. 
Proof. For the first statement, assume otherwise that there is \( x \in W \cap \mathcal{O}_n^c \) for some \( W \in \mathcal{M}_{\mu}^{\mathcal{G}} \). Then \( U(x) \leq \alpha^p \) and therefore \( \gamma^p_{W, \mu} \leq \alpha^p \), which is a contradiction.

For the second statement, we first claim that \( W^* \subset \tilde{\mathcal{O}}_n \), where \( W^* \) is the \( 10d^{1/2} \)-dilate of \( W \) (with same center). The claim follows because for all \( y \in W^* \) we have \( M_{HL} 1_{\mathcal{O}_n}(y) \geq |W^*|/|W^*| = (10\sqrt{d})^{-d} \) by the first statement. Let \( x_W \) be the center of \( W \). Then by the claim

\[
\text{dist}(x_W, (\tilde{\mathcal{O}}_n)^c) \geq \text{dist}(x_W, (W^*)^c) = \frac{\text{diam}(W^*)}{2\sqrt{d}} = 5 \text{diam}(W).
\]

Let \( Q \in \mathcal{Q}_n \) such that \( x_W \in Q \). Then the last displayed inequality implies

\[
5 \text{diam}(W) \leq \text{dist}(x_W, (\tilde{\mathcal{O}}_n)^c) \leq \text{diam}(Q) + \text{dist}(Q, (\tilde{\mathcal{O}}_n)^c) \leq 5 \text{diam}(Q)
\]

and hence \( \text{diam}(Q) \geq \text{diam}(W) \). Since both \( W, Q \) are dyadic cubes containing \( x_W \) this implies \( W \subset Q \). Uniqueness of \( Q \) follows since the cubes in \( \mathcal{Q}_n \) have disjoint interior. \( \square \)

In light of Lemma 4.1, we also set, for a dyadic cube \( Q \in \mathcal{Q}_n \),

\[
(4.11) \quad \mathcal{M}^\mu(Q) = \{ W \in \mathcal{M}_{\mu}^{\mathcal{G}} : W \subset Q \}.
\]

Lemma 4.2. Let \( Q \in \mathcal{Q}_n \). Then

\[
\sum_\mu \sum_{W \in \mathcal{M}^\mu(Q)} |W| \gamma^p_{W, \mu} \leq 10^d \alpha^p |Q|.
\]

Proof. Since \( Q \) is a Whitney cube for the set \( \tilde{\mathcal{O}}_n \), there is \( \tilde{x} \in \tilde{\mathcal{O}}_n^c \subset \mathcal{O}_n^c \) such that \( \text{dist}(\tilde{x}, Q) \leq 4 \text{diam}(Q) \). If \( Q^* \) denotes the cube centered at \( \tilde{x} \) with diameter equal to \( 10 \text{diam}(Q) \) then \( Q \subset Q^* \). Since \( \tilde{x} \in \mathcal{O}_n^c \) we have \( M_{HL} U(\tilde{x}) \leq \alpha^p \). Hence \( \int_Q U \leq \int_{Q^*} U \leq \alpha^p |Q^*| = 10^d \alpha^p |Q| \) and the assertion follows. \( \square \)

4.2. Outline of the proof of the weak type inequalities. For \( R \in \mathcal{R}_k \) let

\[
(4.12) \quad e_R(x) = 1_R(x) \mathcal{L}_k f(x)
\]

and as in (4.10) split \( \mathcal{L}_k f = g^k + b^k \), where

\[
(4.13) \quad g^k = \sum_\mu \sum_{W \in \mathcal{M}^\mu_{\text{good}}(R)} \sum_{R \in \mathcal{R}_k(W)} e_R,
\]

\[
(4.14) \quad b^k = \sum_\mu \sum_{W \in \mathcal{M}^\mu_{\text{bad}}(R)} \sum_{R \in \mathcal{R}_k(W)} e_R.
\]

In view of (4.9) it suffices to show, for \( 2 \leq q < \infty \),

\[
(4.15) \quad \text{meas}\left\{ x : \left( \sum_k \left( \int_I \left| \sum_j 2^{-j \lambda(p)} T_j^{k} g^k(x,t) \right|^q dt \right)^{2/q} \right)^{1/2} > \alpha/2 \right\} \lesssim \alpha^{-p} \| \mathcal{S} f \|_p^p
\]

and

\[
(4.16) \quad \text{meas}\left\{ x \in \tilde{\mathcal{O}}_n^c : \left( \sum_k \left( \int_I \left| \sum_j 2^{-j \lambda(p)} T_j^{k} b^k(x,t) \right|^q dt \right)^{2/q} \right)^{1/2} > \alpha/2 \right\} \lesssim \alpha^{-p} \| \mathcal{S} f \|_p^p.
\]
Since \( \lambda(p) \geq 1/p - 1 > -1/q \) we can use Lemma 3.1 to bound
\[
\left\| \left( \sum_k \left( \int \left( \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \left| T_j g^k(x) \right|^q dt \right)^{2/q} \right)^{1/2} \right\|_2
\leq \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \left\| \left( \sum_k \left( \int \left| T_j g^k(x) \right|^q dt \right)^{2/q} \right)^{1/2} \right\|_2
\leq \sum_{j=1}^{\infty} 2^{-j\lambda(p) + \frac{1}{q}} \left( \sum_k \|g^k\|_2^2 \right)^{1/2} \lesssim \left( \sum_k \|g^k\|_2^2 \right)^{1/2}.
\]
Hence, by Tshebyshev’s inequality, the left hand side of (4.15) is bounded by
\[
4\alpha^{-2} \left\| \left( \sum_k \left( \int \left( \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \left| T_j g^k(x) \right|^q dt \right)^{2/q} \right)^{1/2} \right\|_2^2 \lesssim ^{-2} \sum_k \|g^k\|_2^2.
\]
Now
\[
\sum_k \|g^k\|_2^2 \leq \sum_k \left\| \sum_{W \in \mathfrak{m}_\mu} \sum_{R \in \mathcal{R}_k^\mu} e_R \right\|_2^2 \leq \sum_k \sum_{W \in \mathfrak{m}_\mu} \sum_{R \in \mathcal{R}_k^\mu} \|e_R\|_2^2
= \sum_{W \in \mathfrak{m}_\mu} \left| W \right| \gamma_{W,\mu} \lesssim \alpha^{-p} \sum_{W \in \mathfrak{m}_\mu} \left| W \right| \gamma_{W,\mu} \lesssim \alpha^{-p} \|\mathcal{S}f\|_p^p,
\]
where we have used \( \gamma_{W,\mu} \leq \alpha \) for \( W \in \mathfrak{m}^\mu_{\text{good}} \). (4.15) follows.

We turn to (4.16). We write \( L(Q) = m \) if the side length of \( Q \) is \( 2^m \). Define, for \( m \geq -k \),
\[
B^k_m = \sum_{Q \in \Omega_\alpha} \sum_{W \in \mathfrak{m}_\mu} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R
\]
so that \( b^k = \sum_{m \geq -k} B^k_m \).

Note that for \( R \in \mathcal{R}_k^\mu(W) \), we have \( L(W) \geq -k \). Then
\[
b^k = \sum_{m \geq -k} B^k_m = \sum_{m \geq -k} \sum_{\sigma \geq 0} B^k_{m,\sigma},
\]
where
\[
B^k_{m,\sigma} = \sum_{Q \in \Omega_\alpha} \sum_{W \in \mathfrak{m}_\mu(Q)} \sum_{L(W) = -k + \sigma} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R.
\]
We handle the case of the contributions \( T_j B^k_{m,\sigma} \) with \( m \leq j - k \) differently from those with \( m > j - k \). Moreover we distinguish the cases where \( |j - k - m| \geq \sigma \) and \( |j - k - m| < \sigma \). If we use Tshebyshev’s inequality and take into account (4.9) we see that in order to establish (4.16) it suffices to show the following three inequalities, assuming \( 2 \leq q < p' \) (and hence \( p < q' \leq 2 \):
\[
\left\| \left( \sum_k \left( \int \left( \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \left| T_j B^k_{m,\sigma}(\cdot, t) \right|^q dt \right)^{q'/q} \right)^{1/q'} \right\|_{L^{q'}(\mathbb{R}^d)} \lesssim \alpha^{q'-p} \|\mathcal{S}f\|_p^p,
\]
\[
\left\| \left( \sum_k \left( \int \left( \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \left| T_j B^k_{m,\sigma}(\cdot, t) \right|^q dt \right)^{q'/q} \right)^{1/p'} \right\|_{L^p(\mathbb{R}^d )} \lesssim \|\mathcal{S}f\|_p^p,
\]
and
\begin{equation}
\left\| \sum_{k} \left[ \int_{I} \left( \sum_{j=1}^{\infty} 2^{-j\lambda(p)} \sum_{m,\sigma} J_{j}^{k} \mathbb{B}_{m,\sigma}^{k}(\cdot, t) \big| q \ dt \right) \right]^{2/\alpha} \right\|_{L^{p}(\mathbb{R}^{d} \setminus \partial_{a})}^{1/2} \lesssim \| \mathcal{G} f \|_{p}^{p}.
\end{equation}

The proofs will be given in Sections 4.4, 4.5, and 4.6. We shall handle the cases \( p = 1 \), \( 2 \leq q < \infty \), and \( 1 < p < 2 \), \( 2 \leq q < p' \), in a unified way but will need an additional analytic families interpolation argument for \( 1 < p < 2 \).

### 4.3. Analytic families

Fix \( p, \alpha \) and consider for \( 0 \leq \text{Re} \,(z) \leq 1 \) the family of functions
\begin{equation}
b_{Q,\sigma}^{k,z} = \sum_{\mu} \sum_{W \in \mathcal{M}^{\mu}(Q)} \gamma_{W,\mu,z} \sum_{R \in \mathcal{R}_{k}^{\mu}(W)} e_{R},
\end{equation}
where for \( W \in \mathcal{M}_{\text{bad}}^{\mu} \)
\[ \gamma_{W,\mu,z} = \gamma_{W,\mu}^{p(1-z/2)-1} \]
and \( Q \) belongs to \( \Omega_{\alpha} \). Observe that \( b_{Q,\sigma}^{k,z} \) is supported in \( Q \). Notice that \( z \mapsto \gamma_{W,\mu,z} \) is an entire function for \( W \in \mathcal{M}_{\text{bad}}^{\mu} \). We also set
\begin{equation}
B_{m,\sigma}^{k,z} = \sum_{Q \in \Omega_{\alpha}} b_{Q,\sigma}^{k,z}
\end{equation}
and, for \( 0 \leq \text{Re} \,(z) \leq 1 \), define \( p_{z} \) and \( \lambda(p_{z}) \) by
\begin{equation}
\frac{1}{p_{z}} = 1 - z + \frac{z}{2}, \quad \lambda(p_{z}) = \frac{d(1 - z) - 1}{2}.
\end{equation}

If \( 1 < p < 2 \) then we set \( \vartheta = 2 - 2/p \) so that
\[ p_{\vartheta} = p, \quad \lambda(p_{\vartheta}) = d\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}, \quad B_{m,\sigma}^{k,\vartheta} = B_{m,\sigma}^{k}. \]

For \( \text{Re} \,(z) = 1 \) we have

**Lemma 4.3.** *For fixed \( k, m \geq -k \) let \( \mathcal{N}_{k,m} \subset \mathbb{Z} \). Then*
\[ \sum_{k \in \mathbb{Z}} \sum_{m \geq -k} \left\| \sum_{\sigma \in \mathcal{N}_{k,m}} B_{m,\sigma}^{k,z} \right\|_{2}^{2} \lesssim \| \mathcal{G} f \|_{p}^{p}, \quad \text{Re} \,(z) = 1. \]

**Proof.** The left hand side is equal to
\[ \sum_{k \in \mathbb{Z}} \sum_{m \geq -k} \left\| \sum_{Q \in \Omega_{\alpha}} \sum_{\mu} \sum_{\sigma \in \mathcal{N}_{k,m}} \sum_{W \in \mathcal{M}^{\mu}(Q)} \lambda_{W,\mu}^{p(1-z/2)-1} \right\|_{2}^{2} \sum_{R \in \mathcal{R}_{k}^{\mu}(W)} e_{R} \|_{2}^{2}. \]

Let for each \( W, Q(W) \) be the unique cube in \( \Omega_{\alpha} \) such that \( W \subset Q \). We use that for fixed \( k \) the supports of the functions \( e_{R}, R \in \mathcal{R}_{k}^{\mu} \) have disjoint interior and dominate for \( \text{Re} \,(z) = 1 \) the last display by
\[ \sum_{k \in \mathbb{Z}} \sum_{m \geq -k} \sum_{Q \in \Omega_{\alpha}} \sum_{\mu} \sum_{W \in \mathcal{M}^{\mu}(Q)} \lambda_{W,\mu}^{p(1-z/2)-1} \sum_{R \in \mathcal{R}_{k}^{\mu}(W)} \| e_{R} \|_{2}^{2} \lesssim \sum_{\mu} \sum_{W \in \mathcal{M}^{\mu}_{\text{bad}}} \lambda_{W,\mu}^{p-2} \sum_{k \in \mathcal{N}_{k,m}} \| e_{R} \|_{2}^{2} \lesssim \sum_{\mu} \sum_{W \in \mathcal{M}^{\mu}} \lambda_{W,\mu}^{p} \| W \| \lesssim \| \mathcal{G} f \|_{p}^{p}. \]
Lemma 4.4. There exists a universal constant $C$ dependent only on the dimension such that for every $Q \in \Omega_\alpha$ and every $\mathcal{N} \subset \mathbb{N} \cup \{0\}$

$$
\int \left( \sum_k \left| \sum_{\sigma \in \mathcal{N}} i_{Q,\sigma}^{k,z}(x) \right|^2 \right)^{1/2} \, dx \leq C \alpha^p |Q|, \quad \text{if } \text{Re}(z) = 0.
$$

Proof. For each $W \in \mathbb{M}^\mu$ let $W_*$ be its double. By Minkowski’s inequality the left hand side is dominated by

$$
\sum_\mu \sum_{W \in \mathbb{M}^\mu} \gamma_{W,\mu}^{-1} \int \left( \sum_{k+L(W) \in \mathcal{N}} e_R(x) \right)^2 \, dx
\lesssim \sum_\mu \sum_{W \in \mathbb{M}^\mu} \gamma_{W,\mu}^{-1} \int_{W_*} \left( \sum_k e_R(x) \right)^2 \, dx
$$

which by the Cauchy-Schwarz inequality can be estimated by

$$
\sum_\mu \sum_{W \in \mathbb{M}^\mu} \gamma_{W,\mu}^{-1} \left( \sum_k |R_k|^2 \right)^{1/2} |W_*|^{1/2} \lesssim \sum_\mu \sum_{W \in \mathbb{M}^\mu} |W_*|^2 \gamma_{W,\mu} \lesssim \alpha^p |Q|.
$$

Here we have used Lemma 4.2. \hfill \square

4.4. Proof of (4.19). Let $1 \leq p < 2$ and $2 \leq q < p'$. The asserted inequality follows from

(4.25a) $\left\| \left( \sum_k \left( \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p)} T_j^{k,s} B_{j-k-s,\sigma}(\cdot, t) \right|^{q'/q} \, dt \right)^{q'/q} \right\|_{q'}$

$$
\lesssim (1+s)^{1-\frac{2}{q}-s/2} 2^{-s(d-1)(\frac{1}{p}-\frac{1}{q})} \alpha^{p(\frac{1}{p}-\frac{1}{q})} \|\mathcal{G}f\|_{p'/q'}
$$

and,

(4.25b) $\left\| \left( \sum_k \left( \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p)} T_j^{k,s} B_{j-k-s,\sigma}(\cdot, t) \right|^{q'/q} \, dt \right)^{q'/q} \right\|_{q'}$

$$
\lesssim (1+j)^{1-\frac{2}{q}-j(\frac{1}{p}-\frac{1}{q})} \alpha^{p(\frac{1}{p}-\frac{1}{q})} \|\mathcal{G}f\|_{p'/q'}, \quad j \leq s \leq j.
$$

If in addition $p > 1$ we use a complex interpolation argument, embedding $B_{m,\sigma}^k$ in an analytic family of functions, see (4.23).

Define $r$ by

$$
\frac{1}{r} = \left( \frac{1}{p} - \frac{1}{q} \right) / \left( \frac{2}{p} - 1 \right),
$$

so that $1 < r \leq 2$ and for $\vartheta = 2 - 2/p$ we have $(1 - \vartheta)(1, \frac{1}{p}) + \vartheta(\frac{1}{2}, 1) = (\frac{1}{p}, \frac{1}{q})$. Then by complex interpolation (i.e. the three lines lemma and duality) we deduce (4.25a), (4.25b) from

(4.27a) $\left\| \left( \sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p,j)} T_j^{k,s} B_{j-k-s,\sigma}(\cdot, t) \right|^2 \, dt \right|^{1/2} \right\|_{2} \lesssim \|\mathcal{G}f\|^\vartheta_{p/2}, \quad \text{Re}(z) = 1,$

(4.27b) $\left\| \left( \sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p,j)} T_j^{k,s} B_{j-k-s,\sigma}(\cdot, t) \right|^2 \, dt \right|^{1/2} \right\|_{2} \lesssim \|\mathcal{G}f\|^\vartheta_{p/2}, \quad j \leq s \leq j, \quad \text{Re}(z) = 1.$
and

\[
(4.28a) \quad \left\| \left( \sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_x)} T_j B_{j-k-s,\sigma}^k(z, t) \right|^r \right\|_r^{1/r} \right\|_r \lesssim (1 + s)^{1/2} \left( 1 - \frac{1}{2} \right) \alpha^{p/2}\| \mathcal{G} f \|_p^{p/r}, \quad \text{Re}(z) = 0,
\]

\[
(4.28b) \quad \left\| \left( \sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_x)} T_j B_{j-k-s,\sigma}^{k,z}(z, t) \right|^r \right\|_r^{1/r} \right\|_r \lesssim (1 + j)^{-1/2} - j \alpha^{p/2}\| \mathcal{G} f \|_p^{p/r}, \quad \frac{j}{2} \leq s \leq j, \quad \text{Re}(z) = 0.
\]

We note that for the special case \( p = 1 \) inequalities (4.28a), (4.28b) with \( r = q' \) and \( z = 0 \) imply inequalities (4.25a), (4.25b) with \( p = 1 \).

The proof of (4.27a), (4.27b) is straightforward, using orthogonality, i.e. the fact that for each \( k, t, \xi \) there are at most five \( j \) for which \( \varphi_j(\rho(2^{-k}t^{-1}\xi)) \neq 0 \). Therefore we get for \( \text{Re}(z) = 1 \) (and \( \text{Re}(\lambda(p_x)) = -1/2 \))

\[
\left\| \left( \sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_x)} T_j B_{j-k-s,\sigma}^k(z, t) \right|^2 \right\|_2 \right\|_2^{1/2} \lesssim \sum_k \int_I \sum_{j \geq 2s} 2j \left| \phi_j(\rho(2^{-k}t^{-1}\xi)) \right|^2 \sum_{0 \leq \sigma \leq s} B_{j-k-s,\sigma}^{k,z}(\xi) \right|^2 d\xi dt \lesssim \sum_k \sum_{j \geq 2s} \left\| \sum_{0 \leq \sigma \leq s} B_{j-k-s,\sigma}^{k,z} \right\|_2^2 \lesssim \| \mathcal{G} f \|_p^p,
\]

by Lemma 4.3. Similarly, for fixed \( j \)

\[
\left\| \left( \sum_k \int_I \left| \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_x)} T_j B_{j-k-s,\sigma}^{k,z}(z, t) \right|^2 \right\|_2 \right\|_2^{1/2} \lesssim \sum_k \left\| \sum_{0 \leq \sigma \leq s} B_{j-k-s,\sigma}^{k,z} \right\|_2^2 \lesssim \| \mathcal{G} f \|_p^p.
\]

This concludes the proof of (4.27a) and (4.27b).

We now come to the main part of the proof, namely the inequalities (4.28a), (4.28b) when \( 1 \leq p < 2 \) and \( \text{Re}(z) = 0 \). We fix \( z \) with \( \text{Re}(z) = 0 \) and then use another interpolation inequality based on

\[
[L^1(\ell^1(L^\infty)), L^2(\ell^2(L^2))]_{\theta} = L^\theta(\ell^\theta(L^\prime)) \quad \text{for} \quad \theta = 2 - \frac{2}{p},
\]

where Calderón’s complex interpolation method is applied to vector-valued \( L^p \) spaces (see [2, Theorem 5.1.2]. As a consequence we have

\[
\| \cdot \|_{L^\theta(\ell^\theta(L^\prime))} \lesssim \| \cdot \|_{L^1(\ell^1(L^\infty))}^{2-1} \| \cdot \|_{L^2(\ell^2(L^2))}^{2-\frac{2}{p}}.
\]

Assuming \( 1 \leq p < 2 \), (4.28a), (4.28b) follow from

\[
(4.29a) \quad \left\| \left( \sum_k \int_I \left| \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_x)} T_j B_{j-k-s,\sigma}^k(z, t) \right|^2 \right\|_2 \right\|_2^{1/2} \lesssim 2^{-s/2} \alpha^{p/2}\| \mathcal{G} f \|_p^{p/2}, \quad \text{Re}(z) = 0,
\]
\[
(4.29b) \left\| \left( \sum_{k} \left| \sum_{j, \nu} 2^{-j \lambda(p_s)} T^k_{j} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right|^2 \right) dt \right\|^{1/2}_2 \leq 2^{-j \frac{d+1}{4}} \alpha^{p/2} \| \mathcal{G} f \|^{p/2}_p, \quad \frac{j}{2} \leq s \leq j, \quad \text{Re} (z) = 0,
\]
and
\[
(4.30a) \left\| \sum_{k} \sup_{t \in I} \left| \sum_{j, \nu} 2^{-j \lambda(p_s)} T^k_{j} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right| \right\|_1 \leq (1 + s) \| \mathcal{G} f \|^{p/2}_p, \quad \text{Re} (z) = 0,
\]
\[
(4.30b) \left\| \sum_{k} \sup_{t \in I} \left| \sum_{j, \nu} 2^{-j \lambda(p_s)} T^k_{j} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right| \right\|_1 \leq (1 + j) \| \mathcal{G} f \|^{p/2}_p, \quad \frac{j}{2} \leq s \leq j, \quad \text{Re} (z) = 0.
\]

This proof of (4.29a), (4.29b) is inspired by the work of Christ and Sogge [10], [11]. We use the decomposition (3.3) and orthogonality, first in the \( j \)-sum and then, for each \( j \), also in the \( \nu \) sums, where \( \nu \in \mathcal{Z}_j \). We then see that
\[
\left\| \left( \sum_{k} \int_I \left| \sum_{j \geq 2s, \nu} 2^{-j \lambda(p_s)} T^k_{j} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right|^2 dt \right) \right\|^{1/2}_2 \leq \sum_{k} \sum_{j \geq 2s} \sum_{\nu} 2^{-j(d-1)} \int_I \left\| \sum_{\nu} T^k_{j, \nu} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right\|_2^2 dt
\]
\[
= \sum_{m} \sum_{k \geq 2s} \sum_{\nu} 2^{-(k+m)(d-1)} \int_I \left\| \sum_{\nu} T^k_{k+m, \nu} B_{m-s, \sigma}^{k,z}(\cdot, t) \right\|_2^2 dt.
\]

We use
\[
\int_I \| T^k_{j, \nu} g \|_2^2 dt = \int_I \int \int (2^k t)^d h_{j, \nu}(2^k t(x-y)) g(y) \overline{g(x)} dy \ dx \ dt,
\]
where \( h_{j, \nu}(x) = F^{-1}[|\chi_{j, \nu} \varphi_j(\rho(\cdot))|^2](x) \). The kernel \( h_{j, \nu} \) satisfies kernel estimates which are analogous to the right hand side of (3.6), i.e.
\[
\sup_{t \in I} |t^d h_{j, \nu}(t(x))| \lesssim_N \frac{2^{-j \frac{d+1}{2}}}{(1 + 2^{-j} |x|)^N (1 + 2^{-j/2} |P_{j, \nu}(x)|)^N}.
\]

Using \( j = k + m \) we can then estimate, for \( t \in I \)
\[
2^{-(k+m)(d-1)} \left\| \sum_{\nu} T^k_{k+m, \nu} B_{m-s, \sigma}^{k,z}(\cdot, t) \right\|_2^2 \leq C_N \times
\]
\[
\int \int 2^{-(k+m)\frac{d+1}{2}} \frac{2^{-md}}{(1 + 2^{-m} |x-y|)^N} \frac{1}{(1 + 2^{-m} \sum_{\nu} |P_{k+m, \nu}(x-y)|)^N} |\rho^{k,z}_{m, \sigma}(y)| \ dy \ |\rho^{k,z}_{m, \sigma}(x)| \ dx,
\]
with
\[
(4.32) \rho^{k,z}_{m, \sigma} := \sum_{\nu} B_{m-s, \sigma}^{k,z}.
\]

Consider a maximal set \( \mathcal{Z}^s \) of \( c2^{-s} \) separated unit vectors \( \eta_k \), and let \( P^s_\zeta \) be the orthogonal projection to the orthogonal complement of \( \nabla \rho(\eta_k) \). Notice that for each \( s \) there are \( \approx 2^{(d-1)\left(\frac{d}{2}-s\right)} \) of the vectors \( \xi_\nu \) with \( \nu \in \mathcal{Z}_j \) which are of distance \( \leq C2^{-s} \) to \( \eta_k \). For those \( \nu \) we then have
\[
\left| \frac{\nabla \rho(x)}{|\nabla \rho(x)|} - \frac{\nabla \rho(y)}{|\nabla \rho(y)|} \right| = O(2^{-s}). \]
Consequently, for those \( \nu \), and \( j = k + m \geq 2s \)
\[
\frac{2^{-md}}{(1 + 2^{-m}|x - y|)^N} \frac{1}{(1 + 2^{-m+s}|P^s \nu(x - y)|)^N} \lesssim \frac{2^{-md}}{(1 + 2^{-m}|x - y|)^N} \frac{1}{(1 + 2^{-m+s}|P^s \nu(x - y)|)^N}
\]
and there are \( O(2^{d-1}) \) indices \( \nu \in \mathbb{Z}_{k+m} \) for which we may use this inequality. Then, setting
\[
A_{k,m,\varsigma}(x) = \int 2^{-s(d-1)} \frac{2^{-md}}{(1 + 2^{-m}|x - y|)^N} \frac{1}{(1 + 2^{-m+s}|P^s \nu(x - y)|)^N} |\beta_{S_{k,m,\varsigma}}(y)| \, dy,
\]
we get by the above considerations
\[
(4.33) \quad A_{k,m,\varsigma}(x) = \int 2^{-s(d-1)} \frac{2^{-md}}{(1 + 2^{-m}|x - y|)^N} \frac{1}{(1 + 2^{-m+s}|P^s \nu(x - y)|)^N} |\beta_{S_{k,m,\varsigma}}(y)| \, dy,
\]
we get by the above considerations
\[
(4.31) \lesssim \sum_{\varsigma \in Z^s} \sum_m \sum_{k \geq 2s-m} \int A_{k,m,\varsigma}(x) |\beta_{S_{k,m,\varsigma}}(x)| \, dx
\]
\[
(4.32) \quad \lesssim \sum_{\varsigma \in Z^s} \sum_m \int \left( \sum_{k \geq 2s-m} |A_{k,m,\varsigma}(x)|^2 \right)^{1/2} \left( \sum_k |\beta_{S_{k,m,\varsigma}}(x)|^2 \right)^{1/2} \, dx.
\]
We first establish that
\[
(4.34) \quad \sup_m \sum_{\varsigma \in Z^s} \left\| \left( \sum_{k \geq 2s-m} |A_{k,m,\varsigma}|^2 \right)^{1/2} \right\|_{\infty} \lesssim \alpha^p 2^{-s(d-1)}.
\]
For each dyadic cube \( Q \) let \( y_Q \) be the center of \( Q \). Using (4.33) we estimate for fixed \( x \in \mathbb{R}^d \)
\[
\left( \sum_{k \geq 2s-m} |A_{k,m,\varsigma}(x)|^2 \right)^{1/2} \lesssim 2^{-s(d-1)} \times \sum_{Q: L(Q) = 2^{m-s}} \frac{2^{-md}}{(1 + 2^{-m}|x - y_Q|)^N} \frac{1}{(1 + 2^{-m+s}|P^s \nu(x - y_Q)|)^N} \int \left( \sum_k |b_{S_{k,m,\varsigma}}(y)|^2 \right)^{1/2} \, dy
\]
and using Lemma 4.4 we bound this expression by
\[
2^{-s(d-1)} \sum_{Q: L(Q) = 2^{m-s}} \frac{2^{-md}}{(1 + 2^{-m}|x - y_Q|)^N} \frac{1}{(1 + 2^{-m+s}|P^s \nu(x - y_Q)|)^N} \alpha^p |Q| \lesssim \alpha^p 2^{-s(d-1)} \int \frac{2^{-md}}{(1 + 2^{-m}|x - w|)^N} \frac{1}{(1 + 2^{-m+s}|P^s \nu(x - w)|)^N} \, dw \lesssim \alpha^p 2^{-2s(d-1)}.
\]
We sum over \( \varsigma \in Z^s \) and use that \#\( Z^s = O(2^{s(d-1)}) \) to obtain (4.35).
Combining (4.35) and (4.34) we obtain
\[
(4.31) \lesssim 2^{-s(d-1)} \alpha^p \sum_m \sum_{Q \in \Omega_m} \left\| \left( \sum_k \left( \sum_{0 \leq \varsigma \leq s} |b_{S_{k,m,\varsigma}}|^2 \right)^{1/2} \right) \right\|_{1}.
\]
Finally, by Lemma 4.4 again
\[
\sum_m \sum_{Q \in \Omega_m} \left\| \left( \sum_k \left( \sum_{0 \leq \varsigma \leq s} |b_{S_{k,m,\varsigma}}|^2 \right)^{1/2} \right) \right\|_{1} \lesssim \sum_{Q \in \Omega_m} |Q| \alpha^p \lesssim \alpha^p |\tilde{Q}_m| \lesssim \|\mathcal{F}f\|_{p}^p,
\]
by (4.9). This finishes the proof of (4.29a).
The proof of (4.29b) uses the same idea. We estimate for fixed \( j \in [s/2, s] \), \( \text{Re}(z) = 0 \),

\[
(4.36) \quad \left\| \left( \sum_{k} \int_{I} \left( \sum_{0 \leq \sigma \leq s} 2^{-j\lambda(p_{\nu})} T_{j}^{k} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right)^{2} dt \right)^{1/2} \right\|_{2}^{2}
\]

\[
\lesssim 2^{-j(d-1)} \sum_{k} \int_{I} \left\| T_{j}^{k} \beta_{j-k-s, \sigma}^{k,z}(\cdot, t) \right\|_{2}^{2} \lesssim 2^{-j(d-1)} \sum_{\nu \in Z} \int_{I} \left\| T_{j}^{k} \beta_{j-k-s, \sigma}^{k,z}(\cdot, t) \right\|_{2}^{2} dt
\]

\[
\lesssim 2^{-j(d-1)} \sum_{\nu \in Z} \int \mathcal{A}_{k,j,\nu}(x) \left| \beta_{j-k,s}^{k,z}(x) \right| dx,
\]

where again \( \beta_{m,s}^{k,z} \) is as in (4.32) and

\[
\mathcal{A}_{k,j,\nu}(x) := \int \frac{2^{kd} \left( 1 + 2^{-k-j} |x-y| \right)^{N} \left( 1 + 2^{-k-j} |P_{j,\nu}(x-y)| \right)^{N} \left| \beta_{j-k,s}^{k,z}(y) \right|}{\left( 1 + 2^{k-j} |x-w| \right)^{N} \left( 1 + 2^{k-j} |P_{j,\nu}(x-w)| \right)^{N}} \, dy.
\]

Now \( \mathcal{A}_{k,j,\nu}(x) \lesssim \int \frac{2^{kd} \left( 1 + 2^{-k-j} |x-w| \right)^{N} \left( 1 + 2^{k-j} |P_{j,\nu}(x-w)| \right)^{N} \left| \beta_{j-k,s}^{k,z}(y) \right|}{\left| |x-w| \right| \, dy}
\]

which is bounded by \( C\alpha^{p} \). Consequently

\[
(4.36) \lesssim 2^{-j(d-1)} \alpha^{p} \sum_{\nu \in Z} \sum_{\nu \in \mathcal{Q}_{\alpha}} \left\| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z} \right\|_{1}
\]

\[
\lesssim 2^{-j(d-1)/2} \alpha^{p} \sum_{\nu \in \mathcal{Q}_{\alpha}} \left\| \sum_{0 \leq \sigma \leq s} b_{Q,\sigma}^{k,z} \right\|_{1}
\]

\[
\lesssim 2^{-j(d-1)/2} \alpha^{p} \left( \left\| \sum_{k} \right\|_{2}^{2} \right)^{1/2} \lesssim 2^{-j(d-1)/2} \alpha^{p} \left\| \mathcal{G} f \right\|_{p}
\]

by Lemma 4.4.

We now turn to the proof of (4.30a), (4.30b), here still \( \text{Re}(z) = 0 \). We estimate the left hand side of (4.30a) using Lemma 3.4 by

\[
\sum_{k} \sum_{j \geq 2s} 2^{-j(d-1)} \sum_{0 \leq \sigma \leq s} \left\| \sup_{t \in I} \left| T_{j}^{k} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right| \right\|_{1}
\]

\[
\leq \sum_{k} \sum_{j \geq 2s} \sum_{0 \leq \sigma \leq s} \left\| B_{j-k-s, \sigma}^{k,z} \right\|_{1}
\]

and the right hand side of the last display is dominated by

\[
\sum_{0 \leq \sigma \leq s} \sum_{k} \sum_{\mu} \sum_{Q \in \mathcal{Q}_{\alpha}} \sum_{W \in \mathcal{W}(Q)} \sum_{L(Q) = j-k-s} \sum_{L(W) = -k+\sigma} \gamma_{W,\mu}^{p-1} \left| W \right|^{1/2} \left( \sum_{R \in \mathcal{R}_{k}^{p}(W)} \left\| e_{R} \right\|^{2} \right)^{1/2}
\]

\[
\lesssim \sum_{0 \leq \sigma \leq s} \sum_{k} \sum_{\mu} \sum_{Q \in \mathcal{Q}_{\alpha}} \sum_{W \in \mathcal{W}(Q)} \sum_{L(Q) = j-k-s} \sum_{L(W) = -k+\sigma} \gamma_{W,\mu}^{p} \left| W \right| \lesssim (1 + s) \left\| \mathcal{G} f \right\|_{p}.
\]

The left hand side of (4.30b) is estimated for fixed \( j \in [s, 2s] \) by

\[
2^{-j(d-1)/2} \sum_{k} \sum_{0 \leq \sigma \leq s} \left\| \sup_{t \in I} \left| T_{j}^{k} B_{j-k-s, \sigma}^{k,z}(\cdot, t) \right| \right\|_{1}
\]

\[
\leq \sum_{0 \leq \sigma \leq s} \sum_{k} \left\| B_{j-k-s, \sigma}^{k,z} \right\|_{1}
\]
and the subsequent estimation is as for (4.30a). This concludes the proof of (4.19).

\[ \square \]

4.5. **Proof of (4.20).** It suffices to show, assuming \( 1 \leq p < 2 \), \( q = p' \) that for some \( a(p, q) > 0 \) and \( s \geq 0 \)

\[
\left\| \left( \sum_k \left[ \int_t^\infty \sum_{j=1}^\infty 2^{-j\lambda(p)} \sum_{0 \leq \sigma \leq s} T_j B_{j-k+s, \sigma}^z (\cdot, t)^q dt \right]^{p/q} \right)^{1/p} \right\|_{L_p(\mathbb{R}^d \setminus \tilde{O}_\alpha)} \leq 2^{-a(p, q)s} \| \mathcal{S} f \|_p.
\]

When \( p > 1 \) we use the analytic family of functions in (4.23). It suffices to prove the inequalities (4.37)

\[
\left\| \left( \sum_k \int_t^\infty \sum_{j=1}^\infty 2^{-j\lambda(p_z)} \sum_{0 \leq \sigma \leq s} T_j B_{j-k+s, \sigma}^z (\cdot, t)^2 dt \right)^{1/2} \right\|_{L^2(\mathbb{R}^d \setminus \tilde{O}_\alpha)} \lesssim \| \mathcal{S} f \|_{p/2}, \quad \text{Re} \ (z) = 1,
\]

and

\[
\sum_k \sup_t \left\| \sum_{j=1}^\infty 2^{-j\lambda(p_z)} \sum_{0 \leq \sigma \leq s} T_j B_{j-k+s, \sigma}^z (\cdot, t) \right\|_{L^1(\mathbb{R}^d \setminus \tilde{O}_\alpha)} \lesssim 2^{-\varepsilon s} \| \mathcal{S} f \|_p, \quad \text{Re} \ (z) = 0, \quad 0 \leq s \leq 0.
\]

For some \( \varepsilon > 0 \).

To show (4.37) we replace the \( L^2(\mathbb{R}^d \setminus \tilde{O}_\alpha) \) norm by the \( L^2(\mathbb{R}^d) \) norm and argue exactly as in the proof of (4.27a), using Lemma 4.3.

To show (4.38) it suffices to prove, after Minkowski’s inequality for the \( \sigma \)-summation (involving \( O(1 + s) \) terms),

\[
\sum_k \sup_t \left\| \sum_{j=1}^\infty 2^{-j\lambda(p_z)} T_j B_{j-k+s, \sigma}^z (\cdot, t) \right\|_{L^1(\mathbb{R}^d \setminus \tilde{O}_\alpha)} \lesssim 2^{-\varepsilon s} \| \mathcal{S} f \|_p, \quad \text{Re} \ (z) = 0, \quad 0 \leq s \leq 0.
\]

For the proof observe that, for \( t \in I \), \(
T_j^{n,k} B_{j-k+s, \sigma}^z (\cdot, t)
\) is supported in \( \tilde{O}_\alpha \) when \( n \leq s \) and thus does not contribute to the \( L^1(\mathbb{R}^d \setminus \tilde{O}_\alpha) \) norm. We then use the simple \( L^1 \) estimate in Lemma 3.4, part (i), for \( n > s \) and \( \text{Re} \ (\lambda(p_z)) = (d - 1)/2 \) to estimate the left hand side of (4.39) by a constant times

\[
2^{-sN} \sum_k \sum_j \| B_{j-k+s, \sigma}^z \|_1 \lesssim 2^{-sN} \sum_k \sum_j \sum_{Q \in \Omega_k} \sum_{L(Q) = j-k+s} \sum_{\mu} \sum_{W \in \mathfrak{M}^p(Q)} \sum_{L(W) = -k+\sigma} \gamma_{W, \mu}^{p-1} \| \mathcal{S} f \|_p \left\| \mathcal{S} f \right\|_p.
\]

We interchange the sums and note that each \( W \) is contained in a unique cube \( Q \in \Omega_\alpha \), and thus because of the disjointness of the cubes in \( \Omega_\alpha \) the \( (j, Q) \) sums corresponding to a fixed \( W \) collapses to a single term. Hence we can bound the previous expression by \( C_N \) times

\[
2^{-sN} \sum_k \sum_{\mu} \sum_{W \in \mathfrak{M}^p} \gamma_{W, \mu}^{p-1} |W|^{1/2} \left( \sum_{R \in \mathcal{R}_W^p(W)} \| \mathcal{S} f \|_p \right)^{1/2} \lesssim 2^{-sN} \sum_{\mu} \sum_{W \in \mathfrak{M}^p} \gamma_{W, \mu}^{p-1} |W| \lesssim 2^{-sN} \| \mathcal{S} f \|_p.
\]

This completes the proof of (4.20).

\[ \square \]
4.6. Proof of (4.21). The estimate follows from the inequalities

\[(4.40) \left\| \left( \sum_k \left( \int_I \left( \sum_{j \geq 1} 2^{-j\lambda(p)} \sum_{m,\sigma : \sigma > |m+k-j|, \sigma \geq j} T_j^k B_{m,\sigma}^k(\cdot, t) \right)^q dt \right) \right)^{2/q} \right\|_{L^p(\mathbb{R}^d \setminus \hat{O}_\alpha)} \lesssim \| f \|_p \]

and

\[(4.41) \left\| \left( \sum_k \left( \int_I \left( \sum_{j \geq 1} 2^{-j\lambda(p)} \sum_{m,\sigma : \sigma > |m+k-j|, \sigma < j} T_j^k B_{m,\sigma}^k(\cdot, t) \right)^q dt \right) \right)^{p/q} \right\|_{L^p(\mathbb{R}^d)} \lesssim \| f \|_p. \]

4.6.1. Proof of (4.40). We use the decomposition \( T_j^k = \sum_{n \geq 0} T_j^{n,k} \) and use Minkowski’s inequality for the \( j \) and \( n \) sums. When \( j + n \leq \sigma \) the support of \( T_j^{k,n} B_{m,\sigma}^k(\cdot, t) \) is contained in \( \hat{O}_\alpha \), for all \( t \in I \). Thus in (4.40) we only need to consider the terms with \( |m + k - j| < \sigma \) and \( j \leq \sigma \leq j + n \). Since \( \lambda(p) + 1/q > 0 \) it suffices to show for fixed \( j \geq 1 \), that

\[(4.42) \left\| \left( \sum_k \left( \int_I \sum_{m,\sigma : \sigma > |m+k-j|, j \leq \sigma \leq j+n} T_j^{n,k} B_{m,\sigma}^k(\cdot, t) \right)^q dt \right) \right\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-n} 2^{-j/q} \| f \|_p. \]

This follows from

\[(4.43) \left\| \left( \sum_k \left( \int_I \sum_{m,\sigma : \sigma > |m+k-j|, j \leq \sigma \leq j+n} T_j^{n,k} B_{m,\sigma}^k(\cdot, t) \right)^q dt \right) \right\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-n} 2^{-j/q} \| f \|_p^{2/q}, \ \text{Re} (z) = 1, \]

and

\[(4.44) \left\| \left( \sum_k \left( \int_I \sum_{m,\sigma : \sigma > |m+k-j|, j \leq \sigma \leq j+n} T_j^{n,k} B_{m,\sigma}^k(\cdot, t) \right)^q dt \right) \right\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-n} 2^{-j/q} \| f \|_p, \ \text{Re} (z) = 0. \]

By Lemma 3.4, part (iii), the left hand side of (4.43) is

\[\left( \sum_k \left( \int_I \sum_{m,\sigma : \sigma > |m+k-j|, j \leq \sigma \leq j+n} T_j^{n,k} B_{m,\sigma}^k(\cdot, t) \right)^q dt \right)^{1/q} \lesssim 2^{-n-j/q} \left( \sum_k \left( \sum_{m,\sigma : \sigma > |m+k-j|, j \leq \sigma \leq j+n} B_{m,\sigma}^k \right)^2 \right)^{1/2}.\]

Recall that

\[\text{supp}(B_{m,\sigma}^k) \subset \bigcup_{Q \in \Omega_\alpha} Q.\]

Therefore, for \( \text{Re} (z) = 1 \) we have

\[\left( \sum_k \left( \sum_{m,\sigma : \sigma > |m+k-j|, j \leq \sigma \leq j+n} B_{m,\sigma}^k \right)^2 \right)^{1/2} \lesssim \| f \|_p \]

by Lemma 4.3. Hence (4.43) follows.
We now turn to the proof of (4.44), where \( \text{Re} (z) = 0 \). For \( W \in \mathfrak{M}_{\text{bad}}^\mu \) let \( Q(W) \) be the unique cube in \( \Omega_\alpha \) containing \( W \). We can split

\[
B_{m,\sigma}^{k,z} = \sum_{\mu \in \mathbb{Z}} \sum_{W \in \mathfrak{M}_{\text{bad}}^\mu} \tilde{B}_{m,\sigma,\mu,W}^{k,z},
\]

where

\[
\tilde{B}_{m,\sigma,\mu,W}^{k,z} = \begin{cases} \gamma_{W,\mu,z} \sum_{R \in R_k^\mu (W)} e_R, & \text{if } L(Q(W)) = m \text{ and } L(W) = -k + \sigma, \\ 0 & \text{if either } L(Q(W)) \neq m \text{ or } L(W) \neq -k + \sigma. \end{cases}
\]

Observe that for \( j \leq \sigma \), \( L(W) = -k + \sigma \), the function \( T_j^{n,k} \tilde{B}_{m,\sigma,\mu,W}^{k,z} \) is supported in a \( 2^{n+3} \)-dilate of \( W \) (with respect to its center). Hence, by the Minkowski and Cauchy-Schwarz inequalities we estimate for fixed \( j, n \)

\[
\left\| \left( \sum_k \left[ \int \left| \sum_{m,z} T_j^{n,k} B_{m,\sigma}^{k,z} (\cdot, t) \right|^q dt \right]^{2/q} \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \\
\lesssim \sum_{\mu} \sum_{W \in \mathfrak{M}_{\text{bad}}^\mu} 2^{nd/2} |W|^{1/2} \left( \sum_{k:|L(Q(W))|+k-j < L(W)+k \leq j+n} \left[ \int I_k^{n,k} B_{L(Q(W)),L(W)+k,\mu,W}^{k,z} (\cdot, t) \right|^q dt \right)^{2/q} \right)^{1/2}
\]

which by an application of Lemma 3.4 is bounded by

\[
C_N 2^{-n(N-d/2)2-j/q} \sum_{\mu} \sum_{W \in \mathfrak{M}_{\text{bad}}^\mu} |W|^{1/2} \left( \sum_{k:|L(Q(W))|+k-j < L(W)+k \leq j+n} \gamma_{W,\mu,z} \sum_{R \in R_k^\mu (W)} e_R \right)^{2/2}^{1/2} \\
\lesssim 2^{-n-j/q} \sum_{\mu} \sum_{W \in \mathfrak{M}_{\text{bad}}^\mu} |W|^{1/2} \gamma_{W,\mu} \left( \sum_{k} \sum_{R \in R_k^\mu (W)} \| e_R \|_2^2 \right)^{1/2} \\
\lesssim 2^{-n-j/q} \sum_{\mu} \sum_{W \in \mathfrak{M}_{\text{bad}}^\mu} |W| \gamma_{W,\mu} \lesssim 2^{-n-j/q} \| \mathcal{S} f \|_p^p.
\]

4.6.2. Proof of (4.41). By Minkowski’s inequality (4.41) follows if we can prove for fixed \( \sigma > 0 \),

\[
(4.45) \left\| \left( \sum_k \left[ \int_1 \left| \sum_{j > \sigma} 2^{-j \lambda (p)} \sum_{m:|m+k-j|} T_j^{k} B_{m,\sigma}^{k,z} (\cdot, t) \right|^q dt \right]^{p/q} \right)^{1/p} \right\|_{L^p(\mathbb{R}^d)} \\
\lesssim (1 + \sigma)^{1/p} 2^{-\sigma d \left( \frac{1}{q} - \frac{1}{p} \right)} \| \mathcal{S} f \|_p.
\]

When \( p > 1 \) we use complex interpolation to deduce this from

\[
(4.46) \left\| \left( \sum_k \int_1 \left| \sum_{j > \sigma} 2^{-j \lambda (p_2)} \sum_{m:|m+k-j|} T_j^{k} B_{m,\sigma}^{k,z} (\cdot, t) \right|^2 dt \right)^{1/2} \right\|_{L^2(\mathbb{R}^d)} \\
\lesssim (1 + \sigma)^{1/2} \| \mathcal{S} f \|_p^{p/2}, \quad \text{Re} (z) = 1,
\]
and, with $\frac{1}{q_0} = (\frac{1}{p} - \frac{1}{q})/(\frac{2}{p} - 1)$,

$$\left\| \sum_k \left( \int_I \left\| \sum_{j>\sigma} 2^{-j\lambda(p_s)} \sum_{\sigma|m+k-j} T_j^k B_{m,\sigma}^{k,z}(\cdot,t) \right\|^{q_0} dt \right\|^{1/q_0}_{L^1(\mathbb{R}^d)} \right\| \lesssim (1+\sigma)^{-\sigma d/q_0} \|\mathcal{S}f\|_p^p, \quad \Re(z) = 0.$$  

Note that $1/q_0 = 1 - 1/r$ where $r$ is as in (4.26), and we have $(1-\vartheta)(1, \frac{1}{q_0}) + \vartheta(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{p}, \frac{1}{q})$ for $\vartheta = 2/p'$.

We first consider the inequality for $\Re(z) = 1$. We can use the orthogonality of the functions $\varphi_j(\rho(t))$ to estimate

$$\left\| \sum_k \left( \int_I \left\| \sum_{j>\sigma} 2^{-j\lambda(p_s)} \sum_{\sigma|m+k-j} T_j^k B_{m,\sigma}^{k,z}(\cdot,t) \right\|^{q_0} dt \right\|^{1/q_0}_{L^1(\mathbb{R}^d)} \right\| \lesssim \left( \sum_k \left( \sum_{j>\sigma} \left\| \sum_{\sigma|m+k-j} T_j^k B_{m,\sigma}^{k,z}(\cdot,t) \right\|^{q_0} dt \right\|^{1/2} \right)^2 \lesssim \left( \sum_k \left( \sum_{j>\sigma} \sum_{\sigma|m+k-j} B_{m,\sigma}^{k,z} \right) \right)^{1/2}. $$

We use the disjointness of the cubes in $\Omega_{\alpha}$ and then interchange the $m, j$ summations. Using that for fixed $m, k$ there are $O(1+\sigma)$ terms in the $j$ summation, we bound the last expression by

$$\left( \sum_k \left( \sum_{j>\sigma} \sum_{\sigma|m+k-j} \left\| B_{m,\sigma}^{k,z} \right\|^{q_0} \right)^{1/2} \right)^{1/2} \lesssim (1+\sigma)^{1/2} \left( \sum_k \sum_{m \geq -k} \left\| B_{m,\sigma}^{k,z} \right\|^{q_0} \right)^{1/2} \lesssim (1+\sigma)^{1/2} \|\mathcal{S}f\|_p^{p/2},$$

where in the last line we have applied Lemma 4.3 to conclude (4.46).

We now turn to (4.47). We split $T_j^k = \sum_{n=0}^\infty T_{j,n}^k$, set

$$b_{W,\mu,z}^k = \gamma_{W,\mu,z} \sum_{R \in \mathcal{R}_k^\mu(W)} e_R$$

and estimate the left hand side of (4.47) by

$$\left( \sum_k \sum_{n \geq 0} \sum_{j>\sigma} 2^{-j\lambda(p_s)} \sum_{\sigma|m+k-j} Q_{\in \Omega_{\alpha}} \sum_{L(Q) = m} \sum_{W \in \mathfrak{M}^p(Q)} \mathfrak{M}^p(W) = -k+\sigma \right) \left( \int_I \left( \sum_k \sum_{j>\sigma} T_{j,n}^k b_{W,\mu,z}^k(\cdot,t) \right)^{q_0} dt \right)^{1/q_0} \right\|_1. $$

We bound for fixed $W$, with $L(W) = -k + \sigma$,

$$\left\| \left( \int_I \left( \sum_k \sum_{j>\sigma} T_{j,n}^k b_{W,\mu,z}^k(\cdot,t) \right)^{q_0} dt \right)^{1/q_0} \right\| \lesssim 2^{-(k+j+n)d/q_0} \left\| \left( \int_I \left( \sum_k \sum_{j>\sigma} T_{j,n}^k b_{W,\mu,z}^k(\cdot,t) \right)^{q_0} dt \right)^{1/q_0} \right\|_{q_0'} \lesssim 2^{-(k+j+n)d/q_0} 2^{j(d(\frac{1}{q_0'} - \frac{1}{q_0}) - \frac{1}{2})} 2^{-nN} \|b_{W,\mu,z}^k\|_{q_0'},$$

by Lemma 3.4, part (ii). Hence after summing in $n$

$$\left(4.48\right) \lesssim \sum_k \sum_{j>\sigma} \sum_{\sigma|m+k-j} \sum_{Q_{\in \Omega_{\alpha}}} \sum_{L(Q) = m} \sum_{W \in \mathfrak{M}^p(Q)} \sum_{L(W) = -k+\sigma} \sum_\mu \sum_{W \in \mathfrak{M}^p(Q)} \|b_{W,\mu,z}^k\|_{q_0'}.$$
Observe that for \( L(W) + k = \sigma \),
\[
2^{-kd/q_0} \| \hat{b}^k_{W,\mu,z} \|_{q_0} \leq 2^{-kd/q_0} |W|^{1/q_0 - 1/2} \| \hat{b}^k_{W,\mu,z} \|_2 \leq 2^{-\sigma d/q_0} |W|^{1/2} \left( \sum_{R \in R^W_k(W)} \| e_R \|_2^2 \right)^{1/2} \leq 2^{-\sigma d/q_0} |W|^{\gamma^p_{W,\mu}}.
\]
We interchange summations and use that, for fixed \( W \in \mathcal{M}_{\text{bad}} \),
\[
\# \{ j \geq \sigma : |L(Q(W)) + \sigma - L(W) - j | < \sigma \} = O(1 + \sigma).
\]
We then obtain
\[
(4.48) \lesssim 2^{-\sigma d/q_0} (1 + \sigma) \sum_{\mu} \sum_{W \in \mathcal{M}_\mu} |W|^{\gamma^p_{W,\mu}} \lesssim 2^{-\sigma d/q_0} (1 + \sigma) \| \mathcal{G} f \|_p^p.
\]
This completes the proof of (4.41), and then (4.21) and finally the proof of Theorem 2.3.

5. Proofs of Theorems 1.1 and 1.2

In this section we use Theorems 2.1 and 2.2 proved in \( \mathbb{R}^d \) and transference argument to establish the corresponding versions for periodic functions. Such transference arguments go back to De Leeuw [25]. See also [20] for transference of maximal operators and [26], [14] inequalities in Hardy spaces on \( \mathbb{T}^d \). In our presentation we rely on the method in [14].

5.1. The \( h^1(\mathbb{T}^d) \rightarrow L^{1,\infty}(\mathbb{T}^d) \) bound. We identify functions \( f \) on \( \mathbb{T}^d \) with functions on \( \mathbb{R}^d \) satisfying \( f(x + n) = f(x) \) for all \( n \in \mathbb{Z}^d \). Let \( Q^0 = [-\frac{1}{2}, \frac{1}{2}]^d \).

Let
\[
h_\lambda(s) = (1 - \nu_0(s))(1 - s)^\lambda_+
\]
and \( S_\lambda f = \sum_{\ell \in \mathbb{Z}^d} h(\rho(\ell/t)) \langle f, e_\ell \rangle e_\ell \). Let \( \lambda(1) = \frac{d-1}{2} \). After a reduction analogous to the one in §2.1 we need to prove the bound
\[
\left\| \left( \sum_{k>0} \int_I |S_\lambda^{(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{T}^d)} \lesssim \| f \|_{h^1(\mathbb{T}^d)}.
\]
By normalizing we may assume that \( \| f \|_{h^1(\mathbb{T}^d)} = 1 \).

By the atomic decomposition for periodic functions ([17], [14]) we may assume that
\[
f = f_0 + \sum_{Q \in Q} c_Q a_Q,
\]
where \( f_0 \in L^2 \), \( \| f_0 \|_2 \lesssim 1 \), where \( Q \) is a collection of cubes of sidelength at most \( 1/4 \) which intersect the fundamental cube \( Q^0 \) and where \( a_Q \) is periodic and supported in \( Q + \mathbb{Z}^d \), satisfying \( \| a_Q \|_{L^2(Q^0)} \leq |Q|^{-1/2} \) and
\[
(5.1) \quad \int_Q a_Q(x) P(x) dx = 0
\]
for all polynomials of degree at most \( 2d \). Moreover
\[
(5.2) \quad \| f \|_{h^1} > \| f_0 \|_2 + \sum_{Q \in Q} |c_Q| \approx 1.
\]
The contribution acting on \( f_0 \) is taken care of by standard \( L^2 \) estimates.

Now let \( \gamma = (\gamma_1, \ldots , \gamma_d) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}^d =: \Gamma \) and let \( Q^\gamma = \gamma + Q^0 \). We can then split the family of cubes \( Q \) into \( 3^d \) disjoint families \( Q_\gamma \) so that each cube \( Q \in Q_\gamma \) has the property that
As in [14] we use the following formula, which is valid at least for measurable \( \meas \) functions.

\[
\sup_{\alpha > 0} \meas \left\{ x \in Q^\gamma : \left( \sum_{k \in N'} \int_I \left| S_{2kt}^{(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q a_Q \right] \right|^q dt \right)^{1/q} > \alpha \right\} \lesssim 1.
\]

It suffices to show for every finite subset \( \mathcal{F}^\gamma \) of \( Q^\gamma \)

\[
\sup_{\alpha > 0} \meas \left\{ x \in Q^\gamma : \left( \sum_{k \in N'} \int_I \left| S_{2kt}^{(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q a_Q \right] \right|^q dt \right)^{1/q} > \alpha \right\} \lesssim \sum_{Q \in \mathcal{F}^\gamma} |c_Q|,
\]

where the implicit constant is independent of \( \mathcal{F}^\gamma \). To see the reduction we split \( Q^\gamma = \bigcup_{n=0}^\infty \mathcal{F}^\gamma_n \), where \( \mathcal{F}^\gamma_n \) is finite and \( \sum_{Q \in \mathcal{F}^\gamma_n} |c_Q| \leq 2^{-n} \). By using the result of Stein and N. Weiss on adding \( L^1 \) functions [41, Lemma 2.3] the left hand side in (5.3) is bounded by \( C \sum_{n=0}^\infty (1+n)2^{-n} \lesssim 1 \), as claimed.

In order to prove (5.4) we can renormalize again, replacing \( c_Q \) with \( c_Q (\sum_{Q' \in \mathcal{F}^\gamma} |c_{Q'}|)^{-1} \) and \( \alpha \) with \( \alpha (\sum_{Q' \in \mathcal{F}^\gamma} |c_{Q'}|)^{-1} \). It therefore remains to prove for every finite subset \( \mathcal{F}^\gamma \) of \( Q^\gamma \), and for \( \sum_{Q \in \mathcal{F}^\gamma} |c_Q| = 1 \), that

\[
\sup_{\alpha > 0} \meas \left\{ x \in Q^\gamma : \left( \sum_{k \in N'} \int_I \left| S_{2kt}^{(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q a_Q \right] \right|^q dt \right)^{1/q} > \alpha \right\} \lesssim 1,
\]

where the implicit constant is independent of \( \mathcal{F}^\gamma \).

Now fix \( \alpha > 0 \). Let \( \phi \in C_\infty \) supported in \( \{ x : |x| \leq 1 \} \) such that \( \int \phi(x) dx = 1 \) and let \( \phi_\varepsilon = \varepsilon^{-d} \phi(\varepsilon^{-1} \cdot) \). Choose \( \varepsilon_0 \) be small, less than one tenth of the sidelength of \( Q \) so that in addition \( \| \phi_\varepsilon(Q) \ast a_Q - a_Q \|_2 \lesssim \alpha^{1/2} \). Let \( \tilde{a}_Q = \phi_\varepsilon(Q) \ast a_Q \). Then by Chebyshev’s inequality and standard \( L^2 \) estimates (such as in §3)

\[
\meas \left\{ x \in Q^\gamma : \left( \sum_{k \in N'} \int_I \left| S_{2kt}^{(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q (a_Q - \tilde{a}_Q) \right] \right|^q dt \right)^{1/q} > \alpha \right\}
\]

\[
\lesssim \alpha^{-2} \left\| \left( \sum_{k \in N'} \int_I \left| S_{2kt}^{(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q (a_Q - \tilde{a}_Q) \right] \right|^q dt \right)^{1/q} \right\|_2^2
\]

\[
\lesssim \alpha^{-2} \left( \sum_{Q \in \mathcal{F}^\gamma} |c_Q| \| a_Q - \tilde{a}_Q \|_2 \right)^2 \lesssim \alpha^{-1} \left( \sum_{Q \in \mathcal{F}^\gamma} |c_Q| \right)^2 \lesssim \alpha^{-1};
\]

here we have used the normalization \( \sum_Q |c_Q| \leq 1 \).

It suffices to show that

\[
\meas \left\{ x \in Q^\gamma : \left( \sum_{k \in N'} \int_I \left| S_{2kt}^{(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right] \right|^q dt \right)^{1/q} > \alpha \right\} \lesssim \alpha^{-1}.
\]

We shall now follow the argument in [14] and set

\[
\Psi(x) = \prod_{i=1}^d (1 - x_i^2/4)_, \quad \Psi_N^\gamma(x) = \Psi(N^{-1}(x - \gamma)).
\]

As in [14] we use the following formula, which is valid at least for \( g \) in the Schwartz space of \( \mathbb{T}^d \), for all \( x \in \mathbb{R}^d \).

\[
\Psi_N^\gamma(x)S_{2kt}^\lambda g(x) - S_{2kt}^\lambda [\Psi_N^\gamma g](x) = \sum_{t \in \mathbb{Z}^d} \langle g, e_t \rangle e_t(x) \int \left[ h_\lambda \left( \rho \left( \frac{\ell}{2kt} \right) \right) - h_\lambda \left( \rho \left( \frac{\ell + N^{-1} \xi}{2kt} \right) \right) \right] \Psi(\xi) e^{2\pi i(x - \gamma, N^{-1} \xi)} d\xi.
\]
As the Fourier coefficients $\langle g, e_t \rangle$ decay rapidly, $\tilde{\Psi} \in L^1$ and $h_\lambda$ is Hölder continuous for $\lambda > 0$ this implies

$$
(5.9) \quad \lim_{N \to \infty} \sup_{t \in I} \sup_{x \in \mathbb{R}^d} |\Psi_N^\gamma(x)S_{2^t}^{\lambda(1)}g(x) - S_{2^t}^{\lambda(1)}[\Psi_N^\gamma g](x)| = 0,
$$

for $k \in N$.

Next we observe that $\Psi_N^\gamma(x) \geq (3/4)^d$ for all $x \in m + Q^\gamma$, when $-N \leq m_i \leq N$ for $i = 1, \ldots, d$. Using periodicity we see that the left hand side of (5.6) is equal to

$$
(2N + 1)^{-d} \sum_{-N \leq m_i \leq N} \text{meas}\left( \left\{ x \in m + Q^\gamma : \left( \sum_{k \in N} \int_I \left| \Psi_N^\gamma(x)S_{2^t}^{\lambda(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right](x) \right|^q dt \right)^{1/q} > \alpha \right\} \right)
$$

$$
\leq (2N + 1)^{-d} \text{meas}\left( \left\{ x \in \mathbb{R}^d : \left( \sum_{k \in N} \int_I \left| \Psi_N^\gamma(x)S_{2^t}^{\lambda(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right](x) \right|^q dt \right)^{1/q} > (3/4)^d \alpha / 2 \right\} \right).
$$

Consider the periodic $C^\infty$ function $g = \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q$ and apply (5.9). Hence there is $N_0 = N_0(g, \alpha, N)$ such that for every $x \in \mathbb{R}^d$ and $N > N_0$,

$$
\left( \sum_{k \in N} \int_I \left| \Psi_N^\gamma(x)S_{2^t}^{\lambda(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right](x) \right|^q dt \right)^{1/q} < (3/4)^d \alpha / 2.
$$

Assuming $N > N_0$ in what follows we see that it suffices to bound

$$
(5.10) \quad (2N + 1)^{-d} \text{meas}\left( \left\{ x \in \mathbb{R}^d : \left( \sum_{k \in N} \int_I \left| S_{2^t}^{\lambda(1)} \left[ \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right](x) \right|^q dt \right)^{1/q} > (3/4)^d \alpha / 2 \right\} \right).
$$

Define for $Q \in \mathcal{F}^\gamma$, $m \in \mathbb{Z}^d$,

$$a_{Q,m}(y) = \mathbbm{1}_{m + Q^\gamma}(y) \Psi_N^\gamma(y) \tilde{a}_Q(y).
$$

Then the support of $a_{Q,m}$ is in the interior of $m + Q^\gamma$ and $\Psi_N^\gamma$ coincides on the support of $a_{Q,m}$ with a bounded polynomial of degree $2d$. Hence $a_{Q,m}$ is an $L^2$ function supported on the double of $Q$, such that $\int a_{Q,m}(y) dy = 0$ and such that $\|a_{Q,m}\|_2 \lesssim |Q|^{1/2}$. Moreover $a_{Q,m}$ is nontrivial only when $|m_i| \leq 2N$ for $i = 1, \ldots, d$. This implies

$$
\left\| \Psi_N^\gamma \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right\|_{H^1(\mathbb{R}^d)} \leq \sum_{-2N \leq m_i \leq 2N} \sum_{Q \in \mathcal{F}^\gamma} |c_Q| \|a_{Q,m}\|_{H^1(\mathbb{R}^d)} \lesssim (4N + 1)^d.
$$

We now apply Theorem 2.1 to see that the left hand side of (5.10) is bounded by

$$
C \alpha^{-1} (2N + 1)^{-d} \left\| \Psi_N^\gamma \sum_{Q \in \mathcal{F}^\gamma} c_Q \tilde{a}_Q \right\|_{H^1(\mathbb{R}^d)} \lesssim (2N + 1)^{-d}(4N + 1)^d \alpha^{-1} \lesssim \alpha^{-1}
$$

which finishes the proof of the theorem. \hfill \square

5.2. The $L^p \to L^{p,\infty}$ bound. The proof is similar (but more straightforward), therefore we will be brief. Now $\lambda(p)$ can be negative, but we have $\lambda > -1/q$. The limiting relation (5.9) is now replaced by

$$
(5.11) \quad \lim_{N \to \infty} \sup_{x \in \mathbb{R}^d} \left( \int_I \left| \Psi_N^\gamma(x)S_{2^t}^{\lambda(p)}g(x) - S_{2^t}^{\lambda(p)}[\Psi_N^\gamma g](x) \right|^q dt \right)^{1/q} = 0, \quad k \in N.
$$

Here, we consider $g \in \mathcal{S}(\mathbb{T}^d)$. We sketch a proof of (5.11), based on (5.8).

We start by observing that

$$
(5.12) \quad \int_I |h_\lambda(\rho(\zeta/t))|^q dt \leq C,
$$
uniformly in $\zeta \in \mathbb{R}^d$. To see this, note that $\rho(\zeta/t) = \rho(\zeta) t^{-1/b}$ and we may assume that $\rho(\zeta) \sim 1$ due to the support of $h_\lambda$. Therefore, (5.12) follows by a change of variable. From this observation, we may reduce (5.11) to

$$
(5.13) \quad \lim_{N \to \infty} \left( \int_I |h_\lambda(\rho(\frac{\ell}{2^k t})) - h_\lambda(\rho(\frac{\ell + N^{-1} \xi}{2^k t})))|^q dt \right)^{1/q} = 0
$$

for fixed $l, k, \xi$ using (5.8), Minkowski’s inequality and the dominated convergence theorem.

For (5.13), we argue as follows. Let $h \in L^q(J)$ for a compact subinterval $J$ of $(0, \infty)$. Then for any $a > 0$

$$
\lim_{\delta \to 0} \left( \int_J |h(as) - h((a + \delta)s)|^q ds \right)^{1/q} = 0
$$

and the limit is uniform if $a$ is taken from a compact subset of $(0, \infty)$. This is easily seen for smooth $h$ and follows for general $h \in L^q(J)$ by an approximation argument. Changing variables $s = t^{-1/b}$ we obtain that for any compact subinterval $I \subset (0, \infty)$

$$
(5.14) \quad \lim_{\delta \to 0} \left( \int_I |h(at^{-1/b}) - h((a + \delta)t^{-1/b})|^q dt \right)^{1/q} = 0.
$$

Then (5.13) follows from (5.14) with $h = h_\lambda$, $\delta = \rho((\ell + N^{-1} \xi)/2^k) - \rho(\ell/2^k)$ and $a = \rho(\ell/2^k)$ using the homogeneity and continuity of $\rho$.

Finally, using (5.11) we get, for sufficiently large $N$,

$$
\text{meas} \left( \left\{ x \in Q^0 : \left( \sum_{k \in \mathbb{N}} \int_I |S_{2^k t}^\lambda(p) |^q dt \right)^{1/q} > \alpha \right\} \right)
$$

$$
\lesssim (2N + 1)^{-d} \text{meas} \left( \left\{ x \in \mathbb{R}^d : \left( \sum_{k \in \mathbb{N}} \int_I |S_{2^k t}^\lambda(p) |^q dt \right)^{1/q} > \left(3/4\right)^d \alpha^2 \right\} \right)
$$

and by Theorem 2.2 we bound the right hand side by

$$
C(2N + 1)^{-d} \alpha^{-p} \|\Psi_N g\|_{L^p(\mathbb{R}^d)}^{p} \lesssim \alpha^{-p} \|g\|_{L^p(\mathbb{T}^d)}^{p}.
$$

Remark. It is also possible to build a proof of Theorem 1.2 from Theorem 2.2 using modifications of a duality argument by deLeeuw [25], see also [40] and [20].

6. Sharpness

In this section we show that Theorems 1.2 and 2.2 fail for $q > p'$. We shall first reduce the argument for Fourier series to the one for Fourier integrals by a familiar transplantation method and then modify an argument that was used by Tao to obtain necessary conditions for the Bochner-Riesz maximal operator, see [43, sect.5], and also the work by Carbery and Soria [6] where a related argument appears in the context of localization results for Fourier series. Note that the almost everywhere convergence assertion in part (ii) of Theorem 1.2 also fails for $q > p'$, by Stein-Nikishin theory ([35]).

6.1. Fourier series. We have for $f \in L^p(\mathbb{T}^d)$

$$
(6.1) \quad \left\| \sup_{T > 0} \left( \frac{1}{T} \int_0^T |R^\lambda f|^q dt \right)^{1/q} \right\|_{L^{p, \infty}(\mathbb{T}^d)} \geq \sup_{T > 0} \left\| \left( \frac{1}{T} \int_0^T |R^\lambda f|^q dt \right)^{1/q} \right\|_{L^{p, \infty}(\mathbb{T}^d)}
$$

and our necessary condition will follow from Proposition 6.2 below and the following result.
Lemma 6.1. Let \( 1 \leq p \leq 2 \). Suppose that for some \( C > 0 \)
\[
\|f\|_{L^p(T^d)} \leq C.
\]
Then also
\[
\|f\|_{L^p(\mathbb{R}^d)} \leq C.
\]

Proof. By scaling, density of \( C^\infty_c \) functions in \( L^p \) and the monotone convergence theorem it suffices to show for all \( f \in C^\infty_c(\mathbb{R}^d) \), all compact sets \( K \), all \( \delta \in (0,1) \), all \( \varepsilon \in (0,1) \) and all \( \alpha > 0 \)
\[
\text{meas}\left( \{ x \in K : \left( \int_\delta^1 |R_1 f(x)|^q dt \right)^{1/q} > \alpha \} \right) \leq C^p (1 - \varepsilon)^{-p} \alpha^p \|f\|_p^p.
\]
Fix such \( f, \alpha, \delta, \varepsilon \) and \( K \). For large \( L \in \mathbb{N} \) define
\[
V_{L,t}^\lambda f(x) = \sum_{\ell \in \mathbb{Z}^d} L^{-d} \hat{f}(L^{-1} \ell) (1 - \rho(t^{-1} L^{-1} \ell))^\lambda e^{2\pi i L^{-1} (x, \ell)}.
\]
Then \( V_{L,t}^\lambda f(x) \) is a Riemann sum for the integral representing \( R_1 f(x) \). Hence we have
\[
\lim_{L \to \infty} V_{L,t}^\lambda f(x) = R_1 f(x)
\]
with the limit uniform in \( t \in [\delta, 1], x \in K \). We may therefore choose \( L \) such that
\[
\text{supp}(f(L \cdot)) \subset \{ x : |x| < 1/4 \} \text{ and } K \subset LQ^0
\]
with \( Q^0 = [-1, 2, 1/2]^d \), and
\[
\sup_{\delta \leq \ell \leq K} \sup_{x \in K} |R_1 f(x) - V_{L,t}^\lambda f(x)| < \alpha \varepsilon.
\]
It remains to show
\[
\text{meas}\left( \{ x \in K : \left( \int_\delta^1 |V_{L,t}^\lambda f(x)|^q dt \right)^{1/q} > \alpha (1 - \varepsilon) \} \right) \leq C^p (1 - \varepsilon)^{-p} \alpha^p \|f\|_p^p.
\]
Observe that for \( w \in Q^0 \)
\[
\left( \int_\delta^1 |V_{L,t}^\lambda f(Lw)|^q dt \right)^{1/q} = \left( \frac{1}{L} \int_{\delta L}^L \left| \sum_{\ell \in \mathbb{Z}^d} L^{-d} \hat{f}(L^{-1} \ell) (1 - \rho(\ell/s))^\lambda e^{2\pi i w \ell} \right|^q ds \right)^{1/q}.
\]
Let \( f^\text{per}_L(w) = \sum_{\kappa \in \mathbb{Z}^d} f(L(w + \kappa)) \). Then from the Poisson summation formula the Fourier coefficients of the periodic function \( f^\text{per}_L \) are given by \( \langle f^\text{per}_L, e_\ell \rangle = L^{-d} \hat{f}(L^{-1} \ell) \). Hence the expression on the right hand side of the last display is equal to \( (L^{-1} \int_{\delta L}^L |R_1 f^\text{per}_L|^q dt)^{1/q} \). Replacing \( K \) by the larger set \( LQ_0 \) and then changing variables \( x = Lw \) we see that the expression on the left hand side of (6.4) is dominated by
\[
L^d \text{meas}\left( \{ w \in Q^0 : \left( \frac{1}{L} \int_{\delta L}^L |R_1 f^\text{per}_L(w)|^q dt \right)^{1/q} > \alpha (1 - \varepsilon) \} \right) \leq L^d C^p \alpha^{-p} (1 - \varepsilon)^{-p} \int_{Q_0} |f^\text{per}_L(w)|^p dw,
\]
where the last inequality follows from assumption (6.2). Since the support of \( f(L \cdot) \) is contained in \( Q_0 \) one immediately gets
\[
\|f^\text{per}_L\|_{L^p(Q^0)} = \|f(L \cdot)\|_{L^p(\mathbb{R}^d)} = L^{-d} \|f\|_{L^p(\mathbb{R}^d)}^p.
\]
This shows (6.4) and concludes the proof. \( \Box \)
6.2. Fourier integrals. Using the $\mathbb{R}^d$ analogue of (6.1) we reduce the sharpness claim in Theorem 2.2 to the following proposition.

Proposition 6.2. Let $1 < p \leq 2$, and $\lambda > -1/2$. Assume that there is a constant $C > 0$ such that

$$\label{6.5} \sup_{T > 1} \left\| \frac{1}{T} \int_0^T |R_t^\lambda f|^q dt \right\|_{L_p,\infty(\mathbb{R}^d)}^{1/q} \leq C \|f\|_{L_p(\mathbb{R}^d)}$$

for all Schwartz functions $f$. Then

$$\lambda \geq \lambda(p) + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right).$$

In particular, if (6.5) holds for $\lambda = \lambda(p)$, then $q \leq p'$.

Proof. We note that the inequality with a given $\rho$ is equivalent to the inequality with $\rho \circ A$ where $A$ is any rotation.

Let $\xi^0 \in \Sigma_\rho$ such that $|\xi^0|$ is maximal. Then the Gaussian curvature does not vanish at $\xi^0$. Choose small neighborhoods $U_1, U_0$ of $\xi^0$ in $\Sigma_\rho$ such that $U_1 \subset U_0$, the Gauss map is injective in a neighborhood of $\partial U_0$ and the curvature is bounded below on $U_0$. Let $\gamma$ be homogeneous of degree zero, $\gamma(\xi) \neq 0$ for $\xi \in U_1$ with $\gamma$ supported on the closure of the cone generated by $U_0$. Let $n(\xi_0) = \nabla_{\rho(\xi)}n(\xi_0)$ be the outer normal at $\xi_0$, let $\Gamma_\varepsilon = \{ x \in \mathbb{R}^d : |\xi_0 - n(\xi_0)| \leq 2\varepsilon \}$, with $\varepsilon$ so small that $\Gamma_\varepsilon$ is contained in the cone generated by the normal vectors $\nabla \rho(\xi)$ with $\xi \in U_1$. Let, for $R >> 1$, $\Gamma_{\varepsilon,R} = \{ x \in \Gamma_\varepsilon : |x| \geq R \}$. By the choice of $\varepsilon$ there is, for each $x \in \Gamma_\varepsilon$, a unique $\Xi(x) \in \Sigma_\rho$, so that $\gamma(\Xi(x)) \neq 0$ and so that $x$ is normal to $\Sigma_\rho$ at $\Xi(x)$. Clearly $x \mapsto \Xi(x)$ is homogeneous of degree zero on $\Gamma$, smooth away from the origin. By a rotation we may assume (6.6)

$$n(\xi^0) = (0, \ldots, 0, 1).$$

By §2.1 inequality (6.5) also implies the similar inequality where $R_t^\lambda f$ is replaced with $S_t^\lambda f$ and $S_t^\lambda$ is as in (2.2). Let $h_\lambda(s) = (1 - v_0(s))(1 - s)^\lambda$ and

$$K_{\lambda,t}(x) = t^d \mathcal{F}^{-1} [\gamma h_\lambda \rho](tx).$$

Observe that $K_{\lambda,t} \ast f = S_t^\lambda f_{\gamma}$ with $\hat{f}_{\gamma} = \gamma \hat{f}$. By the Hörmander multiplier theorem $\gamma$ is a Fourier multiplier of $L^p$ and we see that (6.5) implies that

$$\sup_{T > 0} \left\| \frac{1}{T} \int_0^T |K_{\lambda,t} \ast f|^q dt \right\|_{L_p,\infty(\mathbb{R}^d)}^{1/q} \leq C \|f\|_{L_p(\mathbb{R}^d)}.$$

We now derive an asymptotic expansion for $K_{\lambda,1}(x)$ when $x \in \Gamma_{\varepsilon,R}$. Recall that $\rho$ is homogeneous of degree $1/b$, i.e. $\rho(t^b \xi) = t\rho(\xi)$. We use generalized polar coordinates $\xi = \rho^b \xi(\omega)$ where $\omega \to \xi(\omega)$ is a parametrization of $\Sigma_\rho$ in a neighborhood of $U_0$. Then

$$d\xi = \rho^{db-1}d\rho \langle \xi(\omega), n(\xi(\omega)) \rangle \left( \det \frac{\partial \xi}{\partial \omega} \right)^{1/2} d\omega = \rho^{db-1}d\rho |\nabla \rho(\xi')|^{-1} d\sigma(\xi'), \quad \xi' = \xi(\omega).$$

Here we have used Euler’s homogeneity relation $b(\xi, \nabla \rho(\xi)) = \rho(\xi)$ for vectors on $\Sigma_\rho$. Then

$$K_{\lambda,1}(x) = \int_0^\infty h_\lambda(\rho) \rho^{db-1} \int_{\Sigma_\rho} \gamma(\xi') \rho^{2\pi i \rho(\xi') x} |\nabla \rho(\xi')| d\sigma(\xi').$$

We use the method of stationary phase and get for $x \in \Gamma_{\varepsilon,R}$

$$K_{\lambda,1}(x) = I(x) + \sum_{j=1}^N II_j(x) + III(x),$$

where
where
\[I(x) = c \int_0^\infty h_\lambda(\rho) \rho^{d-1} e^{2\pi \rho\langle x, y \rangle} \frac{\gamma(\Xi(x)) |\nabla \rho(\Xi(x))|^{-1}}{(\Xi(x), x)^{d-1}} \frac{|\nabla \rho(\Xi(x))|^{-1}}{|\text{curv}(\Xi(x))|^{1/2}},\]
where curv(\Xi(x)) is the Gaussian curvature at \Xi(x) and \( c \neq 0 \), and
\[II_j(x) = c_j \int_0^\infty h_\lambda(\rho) \rho^{d-1} e^{2\pi \rho\langle x, y \rangle} \frac{\gamma_j(\Xi(x))}{(\Xi(x), x)^{d-1} + |\text{curv}(\Xi(x))|^{1/2}},\]
where \( \gamma_j \) is smooth. For the remainder term we get
\[|III(x)| \lesssim_N \|h\|_1 |x|^{-N}, \quad x \in \Gamma_{\varepsilon, R}.\]

In the resulting \( \rho \) integrals we use asymptotics for the one-dimensional Fourier transform of \( h_\lambda \), cf. [13, §2.8], and see that for \( x \in \Gamma_{\varepsilon, R} \):
\[\int_0^\infty h_\lambda(\rho) \rho^{d-1} e^{2\pi \rho\langle x, y \rangle} \frac{\gamma(\Xi(x)) |\nabla \rho(\Xi(x))|^{-1}}{(\Xi(x), x)^{d-1}} \frac{|\nabla \rho(\Xi(x))|^{-1}}{|\text{curv}(\Xi(x))|^{1/2}} \]
with similar asymptotics for the \( \rho \)-integrals in the terms \( II_j \).

Now let \( x \in \Gamma_{\varepsilon, R} \), \( H(x) = \langle \Xi(x), x \rangle \) and use Euler’s homogeneity relation to write
\[H(x) = |x| \langle \Xi(x), \frac{\nabla \rho(\Xi(x))}{b|\nabla \rho(\Xi(x))|} \rangle = |x| \frac{\rho(\Xi(x))}{b|\nabla \rho(\Xi(x))|} = \frac{|x|}{b|\nabla \rho(\Xi(x))|},\]
If \( \varepsilon \) is small we then have for \( t|x| \gg R \),
\[K_{\lambda,t}(x) = A_{\lambda}(x, t) + B_{\lambda}(x, t), \quad |x'| \leq \varepsilon^2 |x_d|,\]
where
\[A_{\lambda}(x, t) = C(\lambda) t^{d-\frac{d+1}{2} - \lambda} G(x) e^{2\pi \varepsilon t H(x)},\]
(6.10)
where \( G(x) = H(x)^{-\frac{d+1}{2} - \lambda} \frac{\gamma(\Xi(x)) |\nabla \rho(\Xi(x))|^{-1}}{|\text{curv}(\Xi(x))|^{1/2}} \]
and
\[B_{\lambda}(x, t) \lesssim t^{d-\frac{d+1}{2} - \lambda} H(x)^{-\frac{d+1}{2} - \lambda}.\]

Recall (6.6) and split \( y = (y', y_d) \). We now let
\[P(T, \varepsilon) = \{ y : |y'| \leq T^{-1} \varepsilon, \ |y_d| \leq T^{-1/2} \varepsilon \}\]
and define
\[f_T(y) = 1_{P(T, \varepsilon)}(y) e^{2\pi \varepsilon t y_d}.\]
Then
(6.11)
\[\|f_T\|_p \lesssim T^{\frac{1}{2} - \frac{d}{4}}.\]
We examine the integrals \( K_{\lambda,t} * f_T(x) \) for \( |x| \approx 1 \) and \( R \ll t \approx \varepsilon T \). We may obtain a lower bound for the absolute value of this integral if we can choose \( t \) for given \( x \) such that
\[2\pi(\varepsilon T y_d + t H(x - y) - t H(x)) \in (-\frac{\pi}{4}, \frac{\pi}{4}) \text{ for all } y \in \text{supp}(f_T).\]
As the Gauss map is invertible near \( \varepsilon^0 \) we observe that \( H \) is smooth and homogeneous of degree 1. We have \( \nabla H(x) = \varepsilon^0 + O(\varepsilon) \) and thus \( \partial_{x_d} H(x) \geq c > 0 \). Now
(6.13)
\[\varepsilon T y_d + t H(x - y) - t H(x) = -t \sum_{i=1}^{d-1} y_i \partial_{x_i} H(x) + y_d(\varepsilon T - t \partial_{x_d} H(x)) + t \sum_{i,j=1}^d y_i y_j \int_0^1 (1 - s) \partial_{x_i,x_j}^2 H(x - sy) ds.\]
The first and the third term on the right hand side are $O(\varepsilon)$ when $y \in \text{supp}(f_T)$. We choose $t$ in the interval

$$I_{x,T} = \left[ \frac{\varepsilon T}{\partial_x H(x)} - \varepsilon T^{1/2}, \frac{\varepsilon T}{\partial_x H(x)} + \varepsilon T^{1/2} \right].$$

We assume that $\varepsilon$ is chosen so small that $I_{x,T} \subset [0, T]$. If $t \in I_{x,T}$ the second term on the right hand side of (6.13) will be $O(\varepsilon)$ as well so that (6.12) is satisfied.

We now split

$$K_{\lambda,t} \ast f_T(x) = \mathcal{I}_1(x,t) + \mathcal{I}_2(x,t) + \mathcal{I}_3(x,t)$$

with

$$\mathcal{I}_1(x,t) = C(\lambda)G(x)e^{2\pi itH(x)}t^{\frac{d-1}{2}} - \lambda \int e^{2\pi i(T \varepsilon y + tH(x-y) - tH(x))} \mathbb{1}_{B(T,\varepsilon)}(y) dy,$$

$$\mathcal{I}_2(x,t) = C(\lambda)t^{\frac{d-1}{2}} - \lambda \int e^{2\pi i(T \varepsilon y + tH(x-y))(G(x) - G(y))} \mathbb{1}_{B(T,\varepsilon)}(y) dy,$$

$$\mathcal{I}_3(x,t) = \int B_{\lambda}(x-y,t) f_T(y) dy.$$

We estimate these terms for

$$x \in \Omega := \{x : |x'| \leq \varepsilon^2|x_d|, \ 1/2 \leq |x_d| \leq 1\}, \ t \in I_{x,T}.$$  

Then by (6.12) the real part of the integrand in the definition of $\mathcal{I}_1(x,t)$ is bounded below by $2^{-1/2} \mathbb{1}_{B(T,\varepsilon)}(y)$ and therefore, for $x \in \Omega$,

$$|\mathcal{I}_1(x,t)| \geq CG(x)t^{\frac{d-1}{2}} - \lambda \int \mathbb{1}_{B(T,\varepsilon)}(y) dy\geq ct^{\frac{d-1}{2}} - \lambda T^{\frac{1}{2}} - d,$$

Moreover,

$$|\mathcal{I}_2(x,t)| \leq |t^{\frac{d-1}{2}} - \lambda T^{-\frac{1}{2}} T^{\frac{1}{2}} - d$$

Hence for small $\varepsilon$ and $t|x| \gg R$, $t \in I_{x,T}$ the term $|\mathcal{I}_1(x,t)|$ is significantly larger than the terms $|\mathcal{I}_2(x,t)|$ and $|\mathcal{I}_3(x,t)|$. Consequently, by $|I_{x,T}| \geq \varepsilon T^{1/2}$, and assuming (6.15) we get

$$\left( \frac{1}{T} \int_0^T |K_{\lambda,t} \ast f_T(x)|^q dt \right)^{1/q} \geq \left( \frac{1}{T} \int_{I_{x,T}} |K_{\lambda,t} \ast f_T(x)|^q dt \right)^{1/q}\geq c \varepsilon^{1/2} T^{-1/2} (\varepsilon T)^{\frac{d-1}{2}} - \lambda T^{1/2} - d = c \varepsilon T^{-\frac{d-1}{2}} - \lambda T^{-\frac{1}{2}}$$

and thus

$$\left\| \left( \frac{1}{T} \int_0^T |K_{\lambda,t} \ast f_T|^q dt \right)^{1/q} \right\|_{L^p,\infty} \geq \varepsilon T^{-\frac{d-1}{2}} - \lambda T^{\frac{d-1}{2}} - \frac{1}{q} \|f_T\|_p$$

which for $T \to \infty$ implies $\lambda \geq \lambda(p) + \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q})$.

7. An $L^1$ result

We currently do not have an analogue of Theorem 1.1 for general functions in $L^1(\mathbb{T}^d)$. We formulate a weaker result which is essentially a consequence of Theorem 1.1.
**Theorem 7.1.** (i) Let $f \in L^1(\mathbb{T}^d)$. Then for all $q < \infty$, and $\lambda(1) = \frac{d-1}{2}$,

$$\lim_{T \to \infty} \left\| \left( \frac{1}{T} \int_0^T |R_t^{\lambda(1)} f - f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{T}^d)} = 0.$$ 

(ii) The analogous statement holds on $L^1(\mathbb{R}^d)$ with $R_t^{\lambda(1)} f$ in place of $R_t^{\lambda(1)} f$.

**Proof.** Since the convergence holds for Schwartz function one can by a standard approximation argument reduce the proof of (ii) to the inequality

$$\sup_{T>0} \left\| \left( \frac{1}{T} \int_0^T |R_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}.$$ 

Similarly, the proof of (i) is reduced to a corresponding inequality on $\mathbb{T}^d$, with the supremum in $T$ extended over $T \geq 1$. The weak type $(1,1)$ inequality in the $\mathbb{T}^d$ case follows from the $\mathbb{R}^d$ case by the transference arguments of §5. Therefore, it suffices to show (7.1).

By the maximal estimate in §2.1 it remains to prove

$$\left\| \left( \frac{1}{T} \int_0^T |S_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)},$$

where $S_t^{\lambda(1)}$ is as in (2.2). We may assume $q \geq 2$. Now

$$\left( \frac{1}{T} \int_0^T |S_t^{\lambda(1)} f(x)|^q dt \right)^{1/q} \leq \sum_{l=0}^{\infty} 2^{-l/q} \left( \frac{1}{T 2^{-l}} \int_{T 2^{-l}}^{T 2^{-l-1}} |S_t^{\lambda(1)} f(x)|^q dt \right)^{1/q}$$

and we claim the inequality

$$\sup_{A>0} \left\| \left( \frac{1}{A} \int_A^{2A} |S_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}} \leq C_q \|f\|_1.$$ 

Assuming that (7.3) is verified we can deduce that the left hand side of (7.2) is bounded by $C_q \sum_{l>0} (1 + l)^{2 - l/q} \|f\|_1 \lesssim \|f\|_1$, by the theorem of Stein and N. Weiss [41, Lemma 2.3] on summing $L^{1,\infty}$ functions.

Let $\eta$ be as in (2.3). Then our main result, Theorem 2.3, yields for all $A > 0$

$$\left\| \left( \frac{1}{A} \int_A^{2A} |S_t^{\lambda(1)} f|^q dt \right)^{1/q} \right\|_{L^{1,\infty}} \leq C_q \|\mathcal{F}^{-1}[\eta(A^{-1} \cdot)] \ast f\|_{H^1(\mathbb{R}^d)}.$$

Since $\eta$ is $C^\infty$ and compactly supported away from the origin we have

$$\|\mathcal{F}^{-1}[\eta(A^{-1} \cdot)] \ast f\|_{H^1(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}$$

uniformly in $A$. This yields (7.3) and concludes the proof of (7.2). \hfill \Box

As an immediate consequence of Theorem 7.1 we get

**Corollary 7.2.** Let $f \in L^1(\mathbb{T}^d)$. There is a subsequence $T_j \to \infty$ such that

$$\lim_{j \to \infty} \left( \frac{1}{T_j} \int_0^{T_j} |R_t^{\lambda(1)} f(x) - f(x)|^q dt \right)^{1/q} = 0 \ a.e.$$ 

Arguing as in [48, ch. XIII.7] or [46, §4] we get

**Corollary 7.3.** Let $f \in L^1(\mathbb{T}^d)$. For almost every $x \in \mathbb{T}^d$ there is a measurable set $E = E(f, x)$ of upper density one, i.e. satisfying

$$\limsup_{T \to \infty} \frac{|E \cap [0, T]|}{T} = 1.$$
such that

\[
\lim_{t \to \infty} \mathcal{R}_t^{\lambda(1)} f(x) = f(x).
\]

For convenience of the reader we give a proof.

**Proof.** Fix \(x\) such that (7.4) in Corollary 7.2 holds and let \(g(t) = |\mathcal{R}_t^{\lambda(1)} f(x) - f(x)|\). We may assume that \(T_j\) is increasing in \(j\). For \(m = 1, 2, \ldots\) let \(E_m = \{t : g(t) \leq 1/m\}\). By Tshebyshov’s inequality we have

\[
\frac{|E_m \cap [0, T_j]|}{T_j} \leq m^q \frac{1}{T_j} \int_0^{T_j} g(t)^q dt
\]

which by assumption tends to 0 as \(j \to \infty\). Hence \(\lim_{j \to \infty} T_j^{-1}|E_m \cap [0, T_j]| = 1\). Thus we may choose a strictly increasing sequence \(j_m\) of positive integers such that \(T_j^{-1}|E_m \cap [0, T_j]| > 1 - m^{-1}\) for \(j \geq j_m\). Let \(E = [0, T_{j_1}] \cup \bigcup_{m=1}^{\infty} (E_m \cap [T_{j_m}, T_{j_m+1}])\). Since the sets \(E_m\) are decreasing we have

\[
|E \cap [0, T_{j_m+1}]| \geq |E_m \cap [0, T_{j_m+1}]| \geq (1 - m^{-1})T_{j_m+1}
\]

and hence \(\lim\sup_{T \to \infty} T^{-1}|E \cap [0, T]| = 1\). Now \(E \cap [T_{j_m}, \infty] \subset E_m\) and thus \(g(t) \leq m^{-1}\) on this set. It follows that \(g(t) \to 0\) as \(t \to \infty\) within \(E\). 

It would be desirable to replace the \(\lim\sup\) in (7.5) by the \(\lim\inf\). The proof of the corollary shows that this would require the existence a.e. of the limit in (7.4) for all sequences \(T_j \to \infty\). We can currently prove this only for functions in \(h^1\).

### 8. Maximal functions on \(H^p(\mathbb{R}^d)\) for \(p < 1\)

We now consider the maximal operator associated with the generalized Riesz means when they act on functions or distributions in the Hardy space \(H^p(\mathbb{R}^d)\) for \(p < 1\). The following result generalizes one by Stein, Taibleson and Weiss [39] for the standard Bochner-Riesz means. Other generalizations for specific rough \(\rho\) were considered in [19] and the references therein.

Let \(R_t^\lambda\) be as in (1.3).

**Theorem 8.1.** For \(0 < p < 1\), \(\lambda(p) = d(1/p - 1/2) - 1/2\) we have for all \(f \in H^p(\mathbb{R}^d)\)

\[
\|\sup_{t > 0} |R_t^{\lambda(p)} f|\|_{L^{p, \infty}(\mathbb{R}^d)} \lesssim \|f\|_{H^p(\mathbb{R}^d)}.
\]

We use the same reductions as in §2. Write, for \(t > 0\)

\[
R_t^\lambda f(x) = \mathcal{F}^{-1}[u(\rho(\cdot/t))\hat{f}](x) + \sum_{j=1}^{\infty} 2^{-j\lambda} T_j f(x, t),
\]

where \(u\) is as in §2.1 and \(\hat{T_j f}(\xi, t) = \varphi_j(\rho(\xi/t))\hat{f}((\xi)\) with \(\phi_j\) as in §2.2. This is similar to (2.7) (except that now \(t\) ranges over \((0, \infty)\)). The functions \(u, \varphi_j\) depend on \(\lambda\) but satisfy uniform estimates as \(\lambda\) is taken over a compact subset of \(\mathbb{R}\). Let

\[
\mathcal{M}_0 f(x) = \sup_{t > 0} |\mathcal{F}^{-1}[u(\rho(\cdot/t))\hat{f}](x)
\]

and for \(j \geq 1\),

\[
\mathcal{M}_j f(x) = \sup_{t > 0} |T_j f(x, t)|.
\]

We then have

\[
(8.1) \quad \sup_{t > 0} |R_t^{\lambda(p)} f(x)| \leq \mathcal{M}_0 f(x) + \sum_{j \geq 1} 2^{-j\lambda(p)} \mathcal{M}_j f(x),
\]

where
and we shall derive a weak type inequality on $H^p$ for the right hand side in (8.1). The ingredients are $H^p \to L^p$ bounds for the maximal operators $M_0$ and $M_j$.

Let $M$ be a nonnegative integer. We recall that a function $a$ supported on a ball $B$ is a $(p, M)$ atom if $\|a\|_\infty \leq \text{vol}(B)^{-1/p}$ and $\int a(x)P(x)dx = 0$ for all polynomials of degree at most $M$. By the atomic decomposition it suffices to check the $H^p \to L^p$ bounds on $(p, M)$ atoms for every non-negative integer $M > d(p^{-1} - 1) - 1$. The bound for $M_0a$ is straightforward:

**Lemma 8.2.** Let $M + 1 > d(p^{-1} - 1)$ and let $a$ be a $(p, M)$-atom. For $0 < p \leq 1$ we have

$$\|M_0a\|_p \lesssim 1.$$  

**Proof.** This follows by a variant of the argument in §2.1. Define

$$\mathfrak{M}_{N_1,N_2}f = \sup_{\tau > 0}|\mathcal{F}^{-1}[(1 - \rho(\cdot / \tau)^{N_1})\hat{f}]|.$$ 

Let $N_1$, $N_2$ be large so that $\mathfrak{M}_{N_1,N_2}$ maps $H^p$ to $L^p$. By the subordination formula (2.1) we have

$$\sup_{t > 0} |\mathcal{F}^{-1}[u(\rho(\cdot/t))\hat{a}](x)| \lesssim |\mathfrak{M}_{N_1,N_2}a(x)| \frac{1}{N_2!} \int_0^\infty \hat{s}^{N_2}|u(N_1+1)(s)|ds$$

and the integral is finite. Hence we get the desired $L^p$ bound for $M_0a$. □

### 8.1. The main $H^p \to L^p$ bound.

**Proposition 8.3.** Let $0 < p \leq 1$, $j \geq 1$, $\nu \in Z_j$. Let $M + 1 > d(p^{-1} - 1)$ and let $a$ be a $(p, M)$-atom. Then

$$\|M_ja\|_p \lesssim 2^{j(d(\frac{d}{2} - \frac{1}{2}) - \frac{d}{2})}.$$ 

We further decompose $T_jf(x,t) = \sum_{\nu \in Z_j} T_{j,\nu}f(x,t)$ where we use the homogeneous partition of unity as in (3.2). Let for $\nu \in Z_j$,

$$M_{j,\nu}f(x) = \sup_{t > 0} |T_{j,\nu}f(x,t)|.$$ 

Then $M_jf(x) \leq \sum_{\nu \in Z_j} M_{j,\nu}f(x)$. Since $\#Z_j = O(2^{j(d-1)/2})$ we can use the triangle inequality in $L^p$, $p \leq 1$, to see that the proposition follows from

$$\|M_{j,\nu}a\|_p \lesssim 2^{j\frac{d+1}{2}(\frac{1}{p} - 1)}.$$ 

We proceed with the proof of (8.3).

By translation and scaling, we may assume that $a$ is supported in the ball $B$ of radius 1 centered at the origin, $\|a\|_\infty \leq 1$ and $\int a(x)P(x)dx = 0$ for all polynomials of degree $\leq M$. By a rotation we may also assume that $\nabla \rho(\xi_{j,\nu})$ is parallel to $(1,0, \ldots,0)$ and thus writing $x = (x_1, x')$ we have

$$|\partial^\alpha K_{j,\nu}(x)| \leq C_{N_1,N_2,\alpha} \frac{2^{-j(d+1)/2}}{(1 + 2^{-j}|x_1|)^{N_1}(1 + 2^{-j/2}|x'|)^{N_2}}$$ 

for all multiindices $\alpha \in \mathbb{N}_0^d$ and all $N_1, N_2 \geq 0$; cf. [11] or [31]. Let

$$\mathcal{D} = \{(x_1, x') \in \mathbb{R}^d : |x_1| \leq 5 \cdot 2^j, |x'| \leq 5 \cdot 2^{j/2}\}.$$ 

In the following subsections, we estimates the $L^p$-quasi-norm of $M_{j,\nu}a(x)$ over $\mathcal{D}$ and $\mathcal{D}^c$, respectively, using the cancellation condition for the atom when $x \in \mathcal{D}^c$. 
8.1.1. Estimation over $\mathcal{D}$. Let

\[ \mathcal{D}_0 = \{ (x_1, x') \in \mathbb{R}^d : |x_1| \leq 5, |x'| \leq 5 \} \]
\[ \mathcal{D}_1 = \{ (x_1, x') \in \mathbb{R}^d : |x_1| \leq 5 \cdot 2^{j/2}, |x'| \leq 5 \} \]

and

\[ E = \{ (x_1, x') \in \mathbb{R}^d : |x'| \geq 2^{-j/2}|x_1| \} . \]

We derive the following pointwise estimates

\[ \mathcal{M}_{j, \nu} a(x) \lesssim \begin{cases} 
1 & \text{if } x \in \mathcal{D}_0, \\
2^{j/2}|x_1|^{-(1+\epsilon)} & \text{if } x \in \mathcal{D}_1 \setminus \mathcal{D}_0, \\
2^{-j/2}|x'|^{-d} & \text{if } x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E, \\
2^{j-1/2}|x_1|^{-d} & \text{if } x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap \mathcal{E}. 
\end{cases} \]

If we use this for $0 < \epsilon < \frac{1}{p} - 1$ then straightforward integrations give the desired bound

\[ \| \mathcal{M}_{j, \nu} a \|_{L^p(\mathbb{D})} \lesssim 2^{j(d+1)/(p-1)} . \]

To verify (8.6) first observe the pointwise bound \[ |t^d K_{j, \nu}(t) \ast a(x)| \lesssim t^d 2^{-j(d+1)/2} (2^{-j/2} t |x_1|)^{-(1+\epsilon)} \int_{|y'| \leq 1} (2^{-j/2} t |x'| - |y'|)^{-(d-1-\epsilon)} dy' \lesssim 2^{j/2} |x_1|^{-(1+\epsilon)} \]

for all $x \in \mathcal{D}_1 \setminus \mathcal{D}_0$.

Assume that $x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E$. Then $|x'| \geq 5$ and thus $|x' - y'| \geq c|x'|$ for some $c > 0$ for all $|y'| \leq 1$. Using (8.4) with $N_1 = 1 + \epsilon$ and $N_2 = d - 1 - \epsilon$, we have

\[ |t^d K_{j, \nu}(t) \ast a(x)| \lesssim t^d 2^{-j(d+1)/2} (2^{-j/2} t |x'|)^{-d} = 2^{-j/2} |x'|^{-d} . \]

Finally, when $x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap \mathcal{E}$, we have $|x_1 - y_1| \geq c|x_1|$ and necessarily $|x_1| \geq 2^{-j/2}$. If we put $N_1 = d$, $N_2 = 0$ in (8.4) we get

\[ |t^d K_{j, \nu}(t) \ast a(x)| \lesssim t^d 2^{-j(d+1)/2} (2^{-j/2} t |x_1|)^{-d} = 2^{j(d-1)/2} |x_1|^{-d} . \]

This concludes the proof of the pointwise estimate (8.6) which implies (8.7).

8.1.2. Estimation over $\mathcal{E}$. When $x \in \mathcal{E}$ we use the cancellation of the atom and Taylor’s formula to write

\[ t^d K_{j, \nu}(t) \ast a(x) = t^d \int \left( K_{j, \nu}(tx - ty) - \sum_{n=1}^{M} \frac{(-ty, \nabla)^n K_{j, \nu}(tx)}{n!} \right) a(y) dy \\
= \frac{(-1)^{M+1}}{M!} t^{d+M+1} \int_0^1 (1-s)^M \int \langle y, \nabla \rangle^{M+1} K_{j, \nu}(tx - sty) a(y) dy ds . \]

We now use (8.4) for the derivatives of order $M + 1$. Also notice that if $E$ is as in (8.5) we have $|x'| \geq 5 \cdot 2^{j/2}$ for $x \in \mathcal{E} \cap E$ and $|x_1| \geq 5 \cdot 2^j$ for $x \in \mathcal{E} \cap \mathcal{E}$. We obtain

\[ \mathcal{M}_{j, \nu} a(x) \lesssim \begin{cases} 
2^{jM/2}|x'|^{-d} & \text{if } x \in \mathcal{E} \cap E, \\
2^{j(M+1)} |x_1|^{-d} & \text{if } x \in \mathcal{E} \cap \mathcal{E} . 
\end{cases} \]
where for $x \in D^c \cap E$ we took $N_1 = 0$, $N_2 = d + M + 1$ in (8.4) and for $x \in D^c \cap E^c$ we took $N_1 = d + M + 1$ and $N_2 = 0$. Hence

$$\|M_{j,\nu}a\|_{L^p(D^c \cap E)} \lesssim 2^{jM/2} \left( \int_{|x'| \geq 2^{j/2}} |x'|^{-(d+1+M)p} \int_{|x| \leq 2^{j/2}} dx' dx \right)^{1/p}$$

and

$$\|M_{j,\nu}a\|_{L^p(D^c \cap E^c)} \lesssim 2^{j(M + d/p)} \left( \int_{|x_1| \geq 2^j} |x_1|^{-(d+1+M)p} \int_{|x'| \leq 2^{-j/2}} dx' dx_1 \right)^{1/p}.$$ 

Both integrals are $\lesssim 2^{j+d/2(p-1)}$ provided $p > \frac{d}{d+1+M}$, which is the hypothesis on $p$ and $M$. This concludes the proof of (8.3).

8.2. Proof of Theorem 8.1, conclusion. As a crucial ingredient we shall use the generalized triangle inequality for $L^{p,\infty}$, namely

$$\left\| \sum_{j} f_j \right\|_{L^{p,\infty}} \lesssim A_p \left( \sum_{j} \left\| f_j \right\|_{L^{p,\infty}}^p \right)^{1/p}, \tag{8.8}$$

which holds with $A_p = O((1 - p)^{-1/p})$ for $0 < p < 1$. See either the paper by Kalton [21] or the paper by Stein-Taibleson-Weiss [39]. By Lemma 8.2 it suffices to prove

$$\left\| \sum_{j \geq 1} 2^{-j\lambda(p)} M_{j,f} \right\|_{L^{p,\infty}} \lesssim \|f\|_{L^p}, \tag{8.9}$$

and by (8.8) and the atomic decomposition we may assume that $f$ is a $(p,M)$-atom $a$, with $M+1 > d(p^{-1} - 1)$. By dilation and translation invariance we may assume that $a$ is function supported in $\{x: |x| \leq 1\}$ such that $\|a\|_{\infty} \leq 1$ and $\int a(x) P(x) dx = 0$ for all polynomials of degree $\leq M$. Because of this normalization we notice that (up to a harmless constant) the function $a$ is also a $(p_1,M)$ and a $(p_0,M)$ atom where $p_1 < p < p_0 < 1$ and we pick $p_1$ sufficiently close to $p$ such that $M+1 > d(p_1^{-1} - 1)$.

We need to verify for all $\alpha > 0$

$$\text{meas}\left( \left\{ x: \sum_{j \geq 1} 2^{-j\lambda(p)} M_{j,a} > \alpha \right\} \right) \lesssim \alpha^{-p}. \tag{8.10}$$

By Proposition 8.3 we have for every $j \geq 1$

$$\left\| 2^{-j\lambda(p)} M_{j,a} \right\|_{p_i} \lesssim 2^{j(\lambda(p_i) - \lambda(p))}. \tag{8.11}$$

We employ a variant of an interpolation argument in [4] to estimate

$$\text{meas}\left( \left\{ x: \sum_{j \geq 1} 2^{-j\lambda(p)} M_{j,a} > \alpha \right\} \right) \leq I + II,$$

where $I$ is the measure of the set on which $\sum_{2^j \leq \alpha^{-p/d}} 2^{-j\lambda(p)} M_{j,a} > \alpha/2$ and $II$ is the measure of the set on which $\sum_{2^j > \alpha^{-p/d}} 2^{-j\lambda(p)} M_{j,a} > \alpha/2$. By Chebyshev’s inequality

$$I \lesssim (2/\alpha)^{p_1} \left\| \sum_{j \in N_1: 2^j \leq \alpha^{-p/d}} 2^{-j\lambda(p)} M_{j,a} \right\|_{p_1}^{p_1},$$

$$II \lesssim (2/\alpha)^{p_0} \left\| \sum_{j \in N_1: 2^j > \alpha^{-p/d}} 2^{-j\lambda(p)} M_{j,a} \right\|_{p_0}^{p_0}. $$
Apply (8.11) to obtain
\[
I + II \lesssim \alpha^{-p_1} \sum_{2^j \leq \alpha^{-p/d}} 2^j(\lambda(p_1)-\lambda(p))p_1 + \alpha^{-p_0} \sum_{2^j > \alpha^{-p/d}} 2^j(\lambda(p_0)-\lambda(p))p_0
\]
\[
= \alpha^{-p_1} \sum_{2^j \leq \alpha^{-p}} 2^jd(1-\frac{p_1}{p}) + \alpha^{-p_0} \sum_{2^j > \alpha^{-p}} 2^jd(1-\frac{p_0}{p}) \lesssim \alpha^{-p}.
\]
This yields (8.10) and concludes the proof. \(\square\)

**Remark.** Versions of the Fan-Wu transference argument in §5.1 for maximal functions and \(h^p\) for \(p < 1\) can be used to prove a theorem for Riesz means of Fourier series analogous to Theorem 8.1, i.e. the maximal function \(\sup_{t > 0} |R^\alpha_t(R^{(1/p-1/2)-1/2}) f|\) defines an operator that maps \(h^p(\mathbb{T}^d)\) to \(L^{p,\infty}(\mathbb{T}^d)\) when \(p < 1\).

### 9. Open Problems

**9.1. Spaces near \(L^1\).** For \(f \in L^1(\mathbb{T}^d)\) it remains open whether the Riesz means \(R^\alpha_t f(x)\) converge \(q\)-strongly a.e. for any \(q < \infty\). In particular can one upgrade in Corollary 7.3 the conclusion of upper density one of \(E(f, x)\) to density one?

It may also be interesting to investigate strong convergence a.e. for spaces intermediate between \(L^1\) and \(L \log L\).

**9.2. The case \(q = p\).** For \(f \in L^p(\mathbb{T}^d)\), \(1 < p < 2\), prove or disprove that \(R^\alpha_t f(x)\) converges \(q\)-strongly a.e. when \(q = p\). For \(f \in L^1(\mathbb{T}^d)\), is there a version of Rodin’s theorem [30] in one dimension, that applies to Riesz means at the critical index \(\lambda(1) = \frac{d-1}{2}\) where the \(L^q\)-average norm in \(t\)-variable is replaced by a \(BMO\)-average?

**9.3. Problems involving nonisotropic dilations.** One can ask the same questions for quasi-radial Riesz means when the isotropic dilation group is replaced by a nonisotropic dilation group \(t^P\) where \(P\) is a matrix with positive eigenvalues and \(\rho\) satisfies \(\rho(t^P \xi) = t\rho(\xi)\). It turns out that the results depend on the geometry of the surface in relation to the eigenvectors of \(P\). In the case that \(\Sigma_\rho = \{ \xi : \rho(\xi) = 1\}\) has nonvanishing curvature everywhere one has almost everywhere convergence for \(\lambda > \frac{d-1}{2}\), but there are other examples where a.e. convergence fails for \(\lambda < d/2\), see [22] for details. Even in the case of nonvanishing curvature we have currently no endpoint results for strong convergence of \(R^\lambda_t f\), for the critical \(\lambda = \lambda(p)\) when the dilations are nonisotropic.

**9.4. Almost everywhere convergence.** For \(1 < p < 2\) the problem of a.e. convergence, and the critical \(q\) for strong summability for \(\lambda > \lambda(p)\) is wide open. Optimal results for the maximal operators are currently known only for the subspace \(L^p_{rad}\) of radial \(L^p\) functions, see [16]. For general \(L^p\) functions results that improve on Stein’s classical theorem for a.e. convergence of Riesz means of index \(> (d-1)(1/p - 1/2)\) are currently only known in two dimensions, see Tao’s paper [44].

### References

[1] Nakhlé Asmar, Earl Berkson, Jean Bourgain, Restrictions from \(\mathbb{R}^n\) to \(\mathbb{Z}^n\) of weak type \((1,1)\) multipliers. Studia Math. 108 (1994), no. 3, 291–299.

[2] Jörn Bergh, Jörgen Löfström, *Interpolation spaces*. Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[3] S. Bochner, Summation of multiple Fourier series by spherical means. Trans. Amer. Math. Soc. 40 (1936), no. 2, 175–207.

[4] Jean Bourgain, Estimations de certaines fonctions maximales. C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), no. 10, 499–502.
References

[5] Jean Bourgain, Lawrence Guth, Bounds on oscillatory integral operators based on multilinear estimates. Geom. Funct. Anal. 21 (2011), no. 6, 1239–1295.

[6] Anthony Carbery, Fernando Soria, Pointwise Fourier inversion and localisation in $\mathbb{R}^n$. J. Fourier Anal. Appl., 3, Supplement 1 (1997), 847–858.

[7] Sun-Yung A. Chang, Robert Fefferman, A continuous version of duality of $H^1$ and $BMO$ on the bidisc. Ann. of Math. 112 (1980), 179–201.

[8] Michael Christ, Weak type $(1,1)$ bounds for rough operators. Ann. of Math. (2) 128 (1988), no. 1, 19–42.

[9] Weak type endpoint bounds for Bochner-Riesz multipliers. Rev. Mat. Iberoamericana 3 (1987), no. 1, 25–31.

[10] Michael Christ, Christopher D. Sogge, The weak type $L^1$ convergence of eigenfunction expansions for pseudodifferential operators. Invent. Math. 94 (1988), no. 2, 421–453.

[11] On the $L^1$ behavior of eigenfunction expansions and singular integral operators. Miniconferences on harmonic analysis and operator algebras (Canberra, 1987), 29–50, Proc. Centre Math. Anal. Austral. Nat. Univ., 16, Austral. Nat. Univ., Canberra, 1988.

[12] Antonio Córdoba, A note on Bochner-Riesz operators. Duke Math. J. 46 (1979), no. 3, 505–511.

[13] A. Erdélyi, Asymptotic expansions. Dover Publications, Inc., New York, 1956.

[14] Da Shan Fan, Zhi Jian Wu, Transference of maximal multipliers on Hardy spaces. Proc. Amer. Math. Soc. 123 (1995), no. 10, 3169–3174.

[15] Charles Fefferman, Inequalities for strongly singular convolution operators. Acta Math. 124 (1970) 9–36.

[16] Gustavo Garrigós, Andreas Seeger, A note on maximal operators associated with Hankel multipliers, Revista de la Unión Matemática Argentina, 50, No.2 (2009), 137-148.

[17] David Goldberg, A local version of real Hardy spaces. Duke Math. J. 46 (1979), no. 1, 27–42.

[18] G.H. Hardy, J.E. Littlewood, Sur la séries de Fourier d’une fonction à carré sommable, Comptes Rendus (Paris) 156 (1913), 1307–1309.

[19] Sunggeum Hong, Joonil Kim, Chan Woo Yang, Riesz means associated with certain product type convex domain. J. Math. Anal. Appl. 380 (2011), no. 2, 585–606.

[20] Carlos E. Kenig, Peter A. Tomas, Maximal operators defined by Fourier multipliers. Studia Math. 68 (1980), no. 1, 79–83.

[21] Nigel J. Kalton, Linear operators on $L^p$ for $0 < p < 1$. Trans. Amer. Math. Soc. 259 (1980), no. 2, 319–355.

[22] Y. Kim, A. Seeger, A note on pointwise convergence of quasiradial Riesz means. Acta Sci. Math. (Szeged) 62 (1996), no. 1-2, 187–199.

[23] A.N. Kolmogorov, Une série de Fourier-Lebesgue divergente presque partout, Fund. Math. 4 (1923), 324-328.

[24] Sergey V. Konyagin, Almost everywhere convergence and divergence of Fourier series, International Congress of Mathematicians. Vol. II 1393–1403, Eur. Math. Soc., Zürich, 2006.

[25] Karel De Leeuw, On $L^p$ multipliers, Ann. of Math. 91 (1965), 364–379.

[26] Zhi Xin Liu, Shan Zhen Lu, Transference and restriction of maximal multiplier operators on Hardy spaces, Studia Math. 105 (1993), no. 6, 121–134.

[27] Shan Zhen Lu, Conjectures and problems on Bochner-Riesz means. Front. Math. China 8 (2013), no. 6, 1237–1251.

[28] J. Marcinkiewicz, Sur la sommabilité forte de séries de Fourier. J. Lond. Math. Soc. 14 (1939), 162–168.

[29] Jaak Peetre, On spaces of Triebel-Lizorkin type. Ark. Mat. 13 (1975), 123–130.

[30] V. A. Rodin, The space $BMO$ and strong means of Fourier series, Anal. Math. 16 (1990), no. 4, 291–302.

[31] Andreas Seeger, Estimates near $L^1$ for Fourier multipliers and maximal functions. Arch. Math. (Basel) 53 (1989), no. 2, 188–193.

[32] Endpoint estimates for multiplier transformations on compact manifolds, Indiana Univ. Math. J., 40 (1991), 471–533.

[33] Endpoint inequalities for Bochner-Riesz multipliers in the plane. Pacific J. Math. 174 (1996), no. 2, 543–553.

[34] Elias M. Stein, Localization and summability of multiple Fourier series. Acta Math. 100 (1958), 93–147.

[35] On limits of sequences of operators. Ann. of Math. (2) 74 (1961), 140–170.

[36] Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[37] An $H^1$ function with nonsummable Fourier expansion. Harmonic analysis (Cortona, 1982), 193–200, Lecture Notes in Math., 992, Springer, Berlin, 1983.

[38] Oscillatory integrals in Fourier analysis, Beijing Lectures in Harmonic Analysis, Annals Math. St. 112, Princeton UP (1986).

[39] Elias M. Stein, Mitchell H. Taibleson, Guido Weiss, Weak type estimates for maximal operators on certain $H^p$ classes. Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980). Rend. Circ. Mat. Palermo (2) 1981, suppl. 1, 81–97.
[40] Elias M. Stein, Guido Weiss, \textit{Introduction to Fourier analysis on Euclidean spaces}. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.

[41] Elias M. Stein, Norman J. Weiss, On the convergence of Poisson integrals. Trans. Amer. Math. Soc. 140 (1969) 35–54.

[42] Gen-ichirô Sunouchi, On the Littlewood-Paley function $g^*$ of multiple Fourier integrals and Hankel multiplier transformations. Tôhoku Math. J. (2) 19 (1967), 496–511.

[43] Terence Tao, The weak-type endpoint Bochner-Riesz conjecture and related topics. Indiana Univ. Math. J. 47 (1998), no. 3, 1097–1124.

[44] , On the maximal Bochner-Riesz conjecture in the plane for $p < 2$. Trans. Amer. Math. Soc. 354 (2002), no. 5, 1947–1959.

[45] Walter Trebels, Some Fourier multiplier criteria and the spherical Bochner-Riesz kernel, Rev. Roumaine Math. Pures Appl. 20 (1975), 1173-1185.

[46] Bobby Wilson, On almost everywhere convergence of strong arithmetic means of Fourier series. Trans. Amer. Math. Soc., 367 (2015), 1467–1500.

[47] A. Zygmund, On the convergence and summability of power series on the circle of convergence, II, Proc. Lond. Math. Soc. 47 (1941), 326-50.

[48] , \textit{Trigonometric series}, second ed., vol. 1,2, Cambridge University Press, 1959.

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