The Dirac equation vs. the Dirac type tensor equation

N.G. Marchuk

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Abstract

We discuss a connection between the Dirac equation for an electron and the Dirac type tensor equation with U(1) gauge symmetry.

In the previous paper [2], using results of P. Dirac [4], D. Ivanenko and L. Landau [5], E. Kähler [6], F. Gürsey [7], D. Hestenes [8], we present the, so-called, Dirac type tensor equation. In this paper, developing results of [2], we concentrate our attention on a connection between the Dirac equation for an electron and the Dirac type tensor equation with U(1) gauge symmetry (see formulas (6,7,8)).

Let $\mathcal{R}^{1,3}$ be the Minkowski space with coordinates $x^\mu$, with the metric tensor $\|g_{\mu\nu}\| = \text{diag}(1, -1, -1, -1)$, $g = \det \|g_{\mu\nu}\| = -1$, with basis coordinate vectors $e\mu$, and with basis covectors $e^\mu = g^{\mu\nu}e_\nu$. Greek indices run over 0, 1, 2, 3, Latin indices run over 1, 2, 3, 4, and the summation convention over repeating indices is assumed. Consider a covariant antisymmetric tensor field of rank $k$ in $\mathcal{R}^{1,3}$

$$ u_{\mu_1...\mu_k} = u_{[\mu_1...\mu_k]}, $$

where square brackets denote the operation of alternation (with division by $k!$) and $u_{\mu_1...\mu_k} = u_{\mu_1...\mu_k}(x)$ are smooth real valued functions $\mathcal{R}^{1,3} \to \mathcal{R}$. It is suitable to write this tensor field as the exterior form

$$ \frac{1}{k!} u_{\mu_1...\mu_k} e^{\mu_1} \wedge \ldots \wedge e^{\mu_k}. \quad (1) $$
Under a linear nondegenerate change of coordinates

\[ \dot{x}^\mu = p^\mu_\nu x^\nu, \]

where \( p^\mu_\nu \) are real constants, the components of tensor field transform as

\[ \dot{u}^{\nu_1...\nu_k} = q^{\mu_1}_{\nu_1} \cdots q^{\mu_k}_{\nu_k} u^{\mu_1...\mu_k}, \]

and aggregates \( e^{\mu_1} \wedge \ldots \wedge e^{\mu_k} \) transform as components of contravariant tensor field of rank \( k \)

\[ \dot{e}^{\nu_1} \wedge \ldots \wedge \dot{e}^{\nu_k} = p^{\nu_1}_{\mu_1} \cdots p^{\nu_k}_{\mu_k} e^{\mu_1} \wedge \ldots \wedge e^{\mu_k}, \]

where

\[ p^{\nu}_{\mu} q^{\lambda}_{\nu} = \delta^{\lambda}_{\mu}, \quad p^{\nu}_{\mu} q^{\mu}_{\lambda} = \delta^{\nu}_{\lambda} \]

and \( \delta^\mu_\mu = 1, \delta^\mu_\nu = 0 \) for \( \mu \neq \nu \). That means the exterior form \( [I] \) is an invariant

\[ \frac{1}{k!} u^{\mu_1...\mu_k} e^{\mu_1} \wedge \ldots \wedge e^{\mu_k} = \frac{1}{k!} \dot{u}^{\nu_1...\nu_k} e^{\nu_1} \wedge \ldots \wedge e^{\nu_k}. \]

In this paper we admit changes of coordinates from the proper orthohroneous Lorentz group \( \text{SO}^+(1, 3) \) only, i.e.,

\[ P^T g P = g, \quad \det P = 1, \quad p^0_0 > 0, \]

where \( P = \|p^\mu_\nu\| \).

Also we consider nonhomogeneous exterior forms

\[ U = \sum_{k=0}^{4} \frac{1}{k!} u^{\mu_1...\mu_k} e^{\mu_1} \wedge \ldots \wedge e^{\mu_k}. \]

Denote by \( \Lambda \) the set of all such exterior forms and by \( \Lambda_k \) the sets of exterior forms of rank \( k \). The set \( \Lambda_0 \) is identified with the set of smooth scalar functions \( \mathcal{R}^{1,3} \rightarrow \mathbb{R} \). We have

\[ \Lambda = \Lambda_0 \oplus \cdots \oplus \Lambda_4 = \Lambda_{\text{ev}} \oplus \Lambda_{\text{od}}, \]

\[ \Lambda_{\text{ev}} = \Lambda_0 \oplus \Lambda_2 \oplus \Lambda_4, \quad \Lambda_{\text{od}} = \Lambda_1 \oplus \Lambda_3. \]

Elements of \( \Lambda_{\text{ev}} \) and \( \Lambda_{\text{od}} \) are called even and odd exterior forms respectively. At any point \( x \in \mathcal{R}^{1,3} \) we may consider the sets \( \Lambda, \Lambda_{\text{ev}}, \Lambda_{\text{od}}, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \).
as linear spaces of dimensions $16, 8, 8, 1, 4, 6, 4, 1$ respectively. Basis elements of $\Lambda$ are
\[ 1, e^\mu, e^{\mu_1} \wedge \ldots \wedge e^{\mu_k}, \quad \mu_1 < \cdots < \mu_k, \quad k = 2, 3, 4. \]
Also we deal with sets $\Lambda_c, \Lambda_{c_{ev}}, \Lambda_{c_{od}}, \Lambda_0^c, \Lambda_1^c, \Lambda_2^c, \Lambda_3^c, \Lambda_4^c$ of complex valued exterior forms.

The exterior product $U, V \rightarrow U \wedge V$ of exterior forms is defined in the usual way and
\[ U \wedge V = (-1)^{rs} V \wedge U \quad \text{for} \quad U \in \Lambda_r, V \in \Lambda_s. \]

Consider a Hodge star operator $\star : \Lambda_k \rightarrow \Lambda_{4-k}$. By definition, put
\[ \star U = \frac{1}{k!(4-k)!} \epsilon_{\mu_1 \ldots \mu_4} u^{\mu_1 \ldots \mu_k} e^{\mu_{k+1}} \wedge \ldots \wedge e^{\mu_4}, \]
where $U \in \Lambda_k$ is from (1), $\epsilon_{\mu_1 \ldots \mu_4}$ is the totally antisymmetric tensor ($\epsilon_{0123} = 1$) and $u^{\mu_1 \ldots \mu_k} = g^{\mu_1 \nu_1} \ldots g^{\mu_k \nu_k} u_{\nu_1} \ldots u_{\nu_k}$. It can be checked that
\[ \star \star U = (-1)^{k+1} U. \]

Let us define a central product of exterior forms $U, V \rightarrow UV$ by the following rules:

1. For $U, V, W \in \Lambda, \alpha \in \Lambda_0$
   \[ 1U = U1 = U, \]
   \[ (\alpha U)V = U(\alpha V) = \alpha(UV), \]
   \[ U(VW) = (UV)W, \]
   \[ (U + V)W = UW + VW; \]

2. $e^\mu e^\nu = e^\mu \wedge e^\nu + g^{\mu \nu}$;

3. $e^{\mu_1} \ldots e^{\mu_k} = e^{\mu_1} \wedge \ldots \wedge e^{\mu_k}$ for $\mu_1 < \cdots < \mu_k$.

Note that from the second rule we get the equalities $e^\mu e^\nu + e^\nu e^\mu = 2g^{\mu \nu}$, which appear in the Clifford algebra.

\footnote{In a special case the central product was invented by H. Grassmann \cite{Grassmann} in 1877 as an attempt to unify the exterior calculus (the Grassmann algebra) with the quaternion calculus. A discussion on that matter see in \cite{Baez}. In some papers the central product is called a Clifford product.}
Theorem 1. (I, p.1278). The central product of exterior forms is an exterior form.

In other words, the operation of central product maps $\Lambda \times \Lambda$ to $\Lambda$.

It can be checked that

$$UV = U \wedge V - \star(U \wedge \star V)$$

(2)

for $U \in \Lambda_1$, $V \in \Lambda$. If we formally substitute $U = e^\mu \partial_\mu$ into (3), then we get

$$e^\mu \partial_\mu V = dV - \star d \star V = (d - \delta)V,$$

where $d : \Lambda_k \to \Lambda_{k+1}$ is called a differential of exterior form ($d^2 = 0$), and

$\delta = \star d \star : \Lambda_k \to \Lambda_{k-1}$ is called a codifferential of exterior form ($\delta^2 = 0$).

By $\ell$ denote a volume form

$$\ell = \sqrt{-g} e^0 \wedge \ldots \wedge e^3 = e^0 \wedge \ldots \wedge e^3.$$

The volume form commutes with all even exterior forms and anticommutes with all odd exterior forms with respect to (w.r.t.) the central product.

Suppose that exterior forms $H \in \Lambda_1; I, K \in \Lambda_2$ are such that

$$H^2 = 1, \quad I^2 = K^2 = -1, \quad [H, I] = [H, K] = \{I, K\} = 0,$$

$$\partial_\mu H = \partial_\mu I = \partial_\mu K = 0,$$

(3)

where $[H, I] = HI - IH$, $\{I, K\} = IK + KI$. Then the exterior forms $\ell, H, I, K$ are said to be invariant generators of $\Lambda$. In particular, in fixed coordinates the exterior forms

$$\hat{H} = e^0, \quad \hat{I} = -e^1 \wedge e^2, \quad \hat{K} = -e^1 \wedge e^3$$

(4)

satisfy (3) that means $\ell, \hat{H}, \hat{I}, \hat{K}$ are invariant generators of $\Lambda$.

The 16 exterior forms

$$1 \in \Lambda_0; \quad H, \ell H I, \ell H K, \ell H I K \in \Lambda_1;$$

$$I, K, I K, \ell I, \ell K, \ell I K \in \Lambda_2;$$

$$H I, H K, H I K, \ell H \in \Lambda_3; \quad \ell \in \Lambda_4$$

are linear independent at any point $x \in \mathcal{R}^{1,3}$ and they can be used as basis exterior forms of $\Lambda$. 

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Let us take the exterior form
\[ t = \frac{1}{4}(1 + H)(1 - iI) \in \Lambda^C \]
such that
\[ t^2 = t, \quad Ht = t, \quad It = it. \]
The equality \( t^2 = t \) means that \( t \) is an idempotent and we may consider the left ideal
\[ \mathcal{I}(t) = \{ Ut : U \in \Lambda^C \} \subset \Lambda^C. \]
It can be checked that complex dimension of this left ideal is equal to four. The exterior forms \( t_k = F_k t, \quad k = 1, 2, 3, 4 \), where
\[ F_1 = 1, \quad F_2 = K, \quad F_3 = -\ell I, \quad F_4 = -KI\ell, \quad \text{(5)} \]
are linear independent and can be considered as basis elements of \( \mathcal{I}(t) \).

Let us define operations of conjugation \( * \) and Hermitian conjugation \( \dagger \), which map \( \Lambda_k \to \Lambda_k \) or \( \Lambda^C_k \to \Lambda^C_k \). For \( U \in \Lambda^C_k \)
\[ U^* := (-1)^{\frac{k(k-1)}{2}}\bar{U}, \]
\[ U^\dagger := HU^*H, \]
where \( \bar{U} \) is the exterior form with complex conjugated components (if \( U \in \Lambda \), then \( \bar{U} = U \)). We see that
\[ (UV)^* = V^*U^*, \quad U^{**} = U, \quad (UV)^\dagger = V^\dagger U^\dagger, \quad U^{\dagger\dagger} = U \]
for \( U, V \in \Lambda \) or \( \Lambda^C \).

Let us define a trace of exterior form as the linear operation \( \text{Tr} : \Lambda \to \Lambda_0 \)
such that \( \text{Tr}(1) = 1 \) and \( \text{Tr}(e^{\mu_1} \wedge \ldots \wedge e^{\mu_k}) = 0 \) for \( k = 1, 2, 3, 4 \). It is easy to prove that
\[ \text{Tr}(UV - VU) = 0 \quad \text{for} \quad U, V \in \Lambda. \]

Now we may define an operation \( (\cdot, \cdot) : \mathcal{I}(t) \times \mathcal{I}(t) \to \Lambda^C_0 \) by the formula
\[ (U, V) = 4 \text{Tr}(UV^\dagger) \quad \text{for} \quad U, V \in \mathcal{I}(t). \]
This operation has all properties of Hermitian scalar product

\[(\alpha U, V) = \alpha(U, V), \quad (U, V) = (\overline{V}, \overline{U}), \quad (U + W, V) = (U, V) + (W, V), \quad (U, U) > 0 \quad \text{for} \quad U \neq 0,\]

where \(U, V, W \in \mathcal{I}(t), \alpha \in \Lambda_0^c\). This scalar product converts the left ideal \(\mathcal{I}(t)\) into the four dimensional unitary space with orthonormal basis \(t_k = t^k, (k = 1, 2, 3, 4)\)

\[(t_k, t^n) = \delta_k^n.\]

**Theorem 2** ([2], Theorem 4). If \(\Phi \in \mathcal{I}(t)\) is given and \(\Psi \in \Lambda_{ev}\) is unknown even exterior form, then the equation

\[\Psi t = \Phi\]

has a unique solution

\[\Psi = F_k(\alpha^k + \beta^k I),\]

where \(\Phi\) has the form

\[\Phi = (\alpha^k + i\beta^k) t_k\]

and \(F_k\) are defined in (3).

This theorem establishes the one-to-one correspondence between \(\Lambda_{ev}\) and \(\mathcal{I}(t)\).

Let \(\mathcal{M}(4, \mathbb{C})\) be the algebra of 4×4-matrices with complex valued elements. We define a map \(\gamma : \Lambda \rightarrow \mathcal{M}(4, \mathbb{C})\) with the aid of equalities

\[Ut_k = \gamma(U)_{n}^{k} t_n, \quad k = 1, 2, 3, 4; \quad U \in \Lambda.\]

Here \(\gamma(U)^{n}_k\) are elements of a four dimensional square matrix \(\gamma(U)\) (an upper index enumerate rows and a lower index enumerate columns of a matrix). If \(\partial_p U = 0\), then elements of matrix \(\gamma(U)\) are constants. Otherwise they are smooth functions \(\mathcal{R}^{1,3} \rightarrow \mathbb{C}\). It is easily shown that

\[\gamma(UV) = \gamma(U)\gamma(V), \quad \gamma(U + V) = \gamma(U) + \gamma(V), \quad \gamma(\alpha U) = \alpha \gamma(U)\]

for \(U, V \in \Lambda, \alpha \in \Lambda_0\). Hence the map \(\gamma\) is a matrix representation of \(\Lambda\) such that a central product of exterior forms \(UV\) corresponds to the product of matrices \(\gamma(U)\gamma(V)\). This map depends on invariant generators \(\ell, H, I, K\). In
and using the relations $H_t(\text{tuting} \gamma$ defined with the aid of the map

Proof (Formula (7) gives one-to-one correspondence between solu-

Theorem 3. Formula (7) gives one-to-one correspondence between solu-

particular, if we take invariant generators (4), then we get the following well

nonnegative constant (the electron mass). Let us take matrices $\gamma^\mu = \gamma(e^\mu)$,

defined with the aid of the map $\gamma$, and scalar functions $\psi^k : \mathbb{R}^{1,3} \to \mathbb{C}$

Now we may write down an equation, which we call a Dirac type tensor

equation

$$\begin{align*} (d - \delta)\Psi + A\Psi I + m\Psi HI &= 0, \\
$$

where an even exterior form $\Psi \in \Lambda_{ev}$ is interpreted as a wave function of

an electron, a 1-form $A = a_\mu e^\mu \in \Lambda_1$ is identified with a potential of elec-

 tromagnetic field, exterior forms $H, I$ are defined in (3), and $m$ is a real nonnegative constant (the electron mass). Let us take matrices $\gamma^\mu = \gamma(e^\mu)$,

defined with the aid of the map $\gamma$, and scalar functions $\psi^k : \mathbb{R}^{1,3} \to \mathbb{C}$

of the Dirac type tensor equation (6) and solutions $\Psi = (\psi^1 \psi^2 \psi^3 \psi^4)^T$ of the Dirac equation

$$\begin{align*} \gamma^\mu(\partial_\mu \psi + ia_\mu \psi) + im\psi &= 0. \\
$$

Theorem 3. Formula (7) gives one-to-one correspondence between solu-

sions $\Psi \in \Lambda_{ev}$ of the Dirac type tensor equation (6) and solutions $\psi = (\psi^1 \psi^2 \psi^3 \psi^4)^T$ of the Dirac equation

$$\begin{align*} \gamma^\mu(\partial_\mu \psi + ia_\mu \psi) + im\psi &= 0. \\
$$

Proof. Let $\Psi \in \Lambda_{ev}$ be a solution of the Dirac type tensor equation. Substitu-

ting $e^\mu \partial_\mu$ for $d - \delta$ in (6), multiplying both sides of (6) from right by $Ht$,

and using the relations $Ht = t, It = it, \partial_\mu t = 0$, we get

$$\begin{align*} 0 &= (e^\mu(\partial_\mu \Psi + a_\mu \Psi I) + m\Psi HI)Ht \\
&= e^\mu(\partial_\mu(\Psi t) + a_\mu(\Psi t)i) + m(\Psi t)i \\
&= e^\mu(\partial_\mu(\psi^k t_k) + a_\mu(\psi^k t_k)i) + m(\psi^k t_k)i \\
&= (e^\mu t_k)(\partial_\mu\psi^k + a_\mu\psi^k i) + m(\psi^k t_k)i \\
&= \gamma(e^\mu t_n)(\partial_\mu\psi^k + a_\mu\psi^k i) + m(\psi^k t_n)i \\
&= (\gamma(e^\mu) t_n)(\partial_\mu\psi^k + a_\mu\psi^k i) + m(\psi^k t_n)i. \\
$$

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Since \( \{t_n\} \) is an orthonormal basis of \( \mathcal{I}(t) \), it follows that
\[
\gamma(e^\mu)_k^n(\partial_\mu \psi^k + a_\mu \psi^k i) + m\psi^n i = 0, \quad n = 1, 2, 3, 4.
\]
These equalities are equivalent to \( \mathcal{F} \). Hence \( \psi = (\psi^1 \psi^2 \psi^3 \psi^4)^T \), where \( T \) denote transposition, is a solution of the Dirac equation.

Conversely, suppose that scalar complex valued functions \( \psi^k \) are such that the column \( \psi = (\psi^1 \psi^2 \psi^3 \psi^4)^T \) satisfies equation \( \mathcal{F} \). By Theorem 2 there exists a unique solution \( \Psi \in \Lambda_{ev} \) of the equation \( \mathcal{G} \). Arguing as above but in inverse order, we see that the exterior form
\[
\Omega = (e^\mu(\partial_\mu \Psi + a_\mu \Psi I) + m\Psi HI)H \in \Lambda_{ev}
\]
satisfy equality
\[
\Omega t = 0.
\]
By Theorem 2 we get \( \Omega = 0 \). This means that the exterior form \( \Psi \in \Lambda_{ev} \) satisfies the Dirac type tensor equation \( \mathcal{H} \). This completes the proof.

For the sequel we need a set of even exterior forms
\[
\text{Spin}(1, 3) = \{ S \in \Lambda_{ev} : S^* S = 1, \partial_\mu S = 0 \}.
\]
This set can be considered as a group w.r.t. the central product. It can be shown that if \( U \in \Lambda_k \) and \( S \in \text{Spin}(1, 3) \), then \( S^* US \in \Lambda_k \). In particular,
\[
S^* e^\mu S = p^\mu_\nu e^\nu,
\]
where \( p^\mu_\nu \) are real constants and the matrix \( P = \|p^\mu_\nu\| \) is such that
\[
P^T gP = g, \quad \det P = 1, \quad p^0_0 > 0. \tag{9}
\]
Now we may consider a change of coordinates
\[
x^\mu \to \hat{x}^\mu = p^\mu_\nu x^\nu, \quad e^\mu \to \hat{e}^\mu = p^\mu_\nu e^\nu, \tag{10}
\]
which associated with the exterior form \( S \in \text{Spin}(1, 3) \). According to the formulas \( \mathcal{I} \), this change of coordinates is from the group \( \text{SO}^+(1, 3) \).

Conversely, if we take any change of coordinates \( \mathcal{J} \) from the group \( \text{SO}^+(1, 3) \), i.e., \( p^\mu_\nu \) satisfy \( \mathcal{I} \), then there exists a unique pair of exterior forms \( \pm S \in \text{Spin}(1, 3) \) such that
\[
p^\mu_\nu e^\nu = S^* e^\mu S.
\]
We claim that correspondence (7) between the Dirac type tensor equation (6) and the Dirac equation (8) is the same in any coordinates $\dot{x}^\mu$ such that transformation $x^\mu \rightarrow \dot{x}^\mu$ is from the proper orthohroneous Lorentz group $SO^+(1,3)$. Indeed, consider a change of coordinates (10) from the group $SO^+(1,3)$, which associated with the exterior form $S \in \text{Spin}(1,3)$. The exterior forms $\Psi, A, H, I$ from (6) are invariants and the operators $d, \delta$ are invariant under this change of coordinates. Therefore the Dirac type tensor equation has the same form in coordinates $\dot{x}^\mu$. Consider relation (7). As $\Psi, t, t_k$ are exterior forms, i.e., invariants, the functions $\psi_k : R^{1,3} \rightarrow C$ must be invariants too. Let us write the Dirac equation (8) in coordinates $\dot{x}^\mu$

$$\dot{\gamma}^\mu (\dot{\partial}_\mu \psi + i \dot{a}_\mu \psi) + im\psi = 0, \quad (11)$$

where

$$\dot{\partial}_\mu = \frac{\partial}{\partial \dot{x}^\mu} = q_\mu^\nu \partial_\nu, \quad \dot{a}_\mu = q_\mu^\nu a_\nu, \quad q_\mu^\nu q_\nu^\lambda = \delta_\mu^\lambda,$$

$$\dot{\gamma}^\mu = \gamma(\dot{e}^\mu) = \gamma(p_\mu^\nu e^\nu) = \gamma(S^* e^\mu S) = \gamma(S^*) \gamma(e^\mu) \gamma(S) = R^{-1} \gamma^\mu R,$$

$$R = \gamma(S).$$

Substituting $R^{-1} \gamma^\mu R$ for $\dot{\gamma}^\mu$ in (11), we get

$$R^{-1} \gamma^\mu R(\dot{\partial}_\mu \psi + i \dot{a}_\mu \psi) + im\psi = 0, \quad (12)$$

or, equivalently,

$$\gamma^\mu (\dot{\partial}_\mu (R\psi) + i \dot{a}_\mu (R\psi)) + im(R\psi) = 0. \quad (13)$$

Thus, postulating relation (7) between the Dirac type tensor equation and the Dirac equation, we arrive at the conventional transformation rule for the Dirac equation (13) under changes of coordinates from the group $SO^+(1,3)$. Let us recall that in the first paper on a theory of electron \[1\] P. A. M. Dirac proves covariance of his equation (8) assuming transformation rule (12), i.e., the column $\psi$ is invariant and $\gamma$-matrices are transform according to the rule $\gamma^\mu \rightarrow R^{-1} \gamma^\mu R$. Later it became conventional to prove covariance of the Dirac equation assuming transformation rule (13), i.e., $\gamma$-matrices are invariant and the column $\psi$ transforms according to the rule $\psi \rightarrow R\psi$. Columns $\psi$ with such transformation property are called Dirac spinors or bispinors.
Further, let us write the Dirac type tensor equation together with the Maxwell equations (Quantum Electrodynamics equations)

\[ H^2 = 1, \quad I^2 = -1, \quad [H, I] = 0, \quad \partial_\mu H = \partial_\mu I = 0, \quad (14) \]

\[ (d - \delta)\Psi + A\Psi I + m\Psi HI = 0, \quad (15) \]

\[ dA = F, \quad (16) \]

\[ \delta F = \alpha J, \quad (17) \]

\[ J = \Psi H\Psi^*, \quad (18) \]

where \( H \in \Lambda_1, \quad I \in \Lambda_2, \quad \Psi \in \Lambda_{ev}, \quad A \in \Lambda_1, \quad F \in \Lambda_2, \quad J \in \Lambda_1, \quad m \) and \( \alpha \) are constants. These values have the following physical interpretation: \( \Psi \) is the tensor wave function of electron, \( A \) and \( F \) are the potential and strength of electromagnetic field respectively, \( J \) is the electric current generated by the electron, \( m \) is the electron mass, and \( \alpha \) is a real constant dependent on physical units (the speed of light is equal to 1).

It can be shown (see [3]) that if an even exterior form \( \Psi \) satisfies (15), then the 1-form \( J = \Psi H\Psi^* \) satisfies the equality

\[ \delta J = 0, \quad (19) \]

which is called a charge conservation law for the Dirac type tensor equation. Taking into account the identity \( \delta^2 = 0 \), we see that (19) is consistent with equation (17).

**Theorem 4.** System of equations (14-18) is invariant under the following gauge transformation

\[ \Psi \rightarrow \Psi' = \Psi \exp(\lambda I), \]

\[ A \rightarrow A' = A - d\lambda, \]

where \( \lambda = \lambda(x) \) is a smooth scalar function, i.e., \( \lambda \in \Lambda_0 \) and \( \exp(\lambda I) = \cos \lambda + I\sin \lambda \).

**Proof.** Let us multiply equation (15) from the right by \( \exp(\lambda I) \). Then, using the relations \( [H, I] = 0 \) and \( d - \delta = e^\mu \partial_\mu \), we obtain

\[ (d - \delta)\Psi' + A'\Psi'I + m\Psi'H I = 0. \]

The identities

\[ \exp(\lambda I)^* = \exp(-\lambda I) = \exp(\lambda I)^{-1} \quad (20) \]
give the invariance of the equality
\[ J = \Psi H \Psi^* = \Psi' H (\Psi')^*. \]

This completes the proof.

Identities (20), together with the identity
\[ \exp(\lambda I)^* = \exp(\lambda I)^\dagger, \]
show that the set of even exterior forms
\[ \{\exp(\lambda I) : \lambda \in \Lambda_0\}, \]
considered at any point \( x \in \mathbb{R}^{1,3} \), is isomorphic to the unitary group \( U(1) \). Theorem 4 corresponds to the well known fact that the Dirac equation is invariant under the gauge transformations from the group \( U(1) \).

Finally, consider the Lagrangian (Lagrange density)
\[ \mathcal{L} = \frac{1}{4} \text{Tr}(H(\Psi^*Q + Q^*\Psi)), \]
where
\[ Q = (d - \delta)\Psi I - A\Psi - m\Psi H. \]

Variating the Lagrangian \( \mathcal{L} \) w.r.t. components of the exterior form \( \Psi \), we may derive the Dirac type tensor equation.

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