Existence of covers with fixed ramification in positive characteristic

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Abstract

We discuss two elementary constructions for covers with fixed ramification in positive characteristic. As an application, we compute the number of certain classes of covers between projective lines branched at 4 points and obtain information on the structure of the Hurwitz curve parametrizing these covers.

1 Introduction

In this note, we consider the question of determining the number of covers between projective lines in positive characteristic with specified ramification data and fixed branch points. The ramification data considered are the degree of the cover, together with a list of the ramification indices in the fibers of the branch points. Over an algebraically closed field of characteristic zero, it is in principal possible to solve this problem by Riemann’s Existence Theorem. Namely, the number of covers can be expressed as the cardinality of a finite set, which can be explicitly constructed in concrete cases. In particular, this approach shows that the number of covers is finite and does not depend on the position of the branch points.

In positive characteristic, the situation is drastically different. For example, the number of covers with fixed ramification depends on the position of the branch points. Moreover, if the characteristic $p$ divides one of the ramification indices, the number of covers is in general infinite. There are only few general results on the number of covers in this situation (we refer to [3] for an overview).

The work of Osserman ([6], [7], [5]) suggests that a particularly nice case to look at is that of covers $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d$ which are ramified at $r$ points $x_1, \ldots, x_r$ with $f(x_i)$ pairwise distinct (the so-called single-cycle case). We write $h(d; e_1, e_2, e_3, \ldots, e_r)$ for the number of single-cycle covers with fixed branch locus over $\mathbb{C}$, where $e_i$ is the ramification index of $x_i$; this number is called the Hurwitz number.

Let $k$ be an algebraically closed field of positive characteristic $p$. We only consider covers $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ in the tame and single-cycle case. We denote by $h_p(d; e_1, e_2, e_3, \ldots, e_r)$ the maximal number of covers with fixed branch locus,
where the maximum is taken over all possible branch loci. This number is called the \(p\)-Hurwitz number. Since \(p \nmid e_i\) for all \(i\), this number is finite and does not depend on \(k\). It can be shown that there the maximum is attained if the branch locus belongs to a dense open subset \(U \subset (\mathbb{P}^1_k)^r \setminus \Delta\). Here \(\Delta\) is the fat diagonal.

We start by summarizing the results on the number of covers with fixed branch locus in the single-cycle case for \(r \in \{3, 4\}\). In [5], F. Liu and B. Osserman give a closed formula for the number of such covers in characteristic zero. In [6] and [7], B. Osserman determines the \(p\)-Hurwitz number \(h_p(d; e_1, e_2, e_3)\) using linear series. In [3] the number \(h_p(p; e_1, e_2, e_3, e_4)\) is computed. This last case is substantially more difficult and the proof relies on the theory of stable reduction of covers.

In this note, we also consider covers \(f : \mathbb{P}^1_k \to \mathbb{P}^1_k\) of ramification type \((d; e_1, e_2, e_3, e_4)\). In contrast with the situation in [3], the degree \(d\) is not fixed. We consider two elementary constructions, which yield previously unknown results on some \(p\)-Hurwitz numbers \(h_p(d; e_1, e_2, e_3, e_4)\). Both constructions were known before and can be found for example in [6]. However, the implications for the \(p\)-Hurwitz numbers have not been fully exploited. As an additional result, we obtain rather complete information on the structure of the Hurwitz curve, parameterizing covers of the type considered, in positive characteristic. These are the first such results.

The first result deals with the case \(1 < e_1 < p\) and \(e_4 = p - 1\). In this situation, we compute the \(p\)-Hurwitz number \(h_p(d; e_1, e_2, e_3, e_4)\). We can even obtain something stronger, namely an explicit description of the Hurwitz curve \(H_p(d; e_1, e_2, e_3, e_4)\) parameterizing all covers of type \((d; e_1, e_2, e_3, e_4)\). This yields, in particular, not only a formula for all covers with generic branch locus, but also exactly describes the values for which the number of covers drops. As far as we know, this is the first nontrivial example of a complete description of Hurwitz curves in positive characteristic (e.g., [1], [2] and [3]) do not yield such description. We refer to §3 for the precise statement of the result.

The second result considers the case \(e_1 > p\) and \(2 < e_2, e_3, e_4 < p\). Here, we relate the \(p\)-Hurwitz numbers \(h_p(d; e_1, e_2, e_3, e_4)\) and \(h_p(d; e_1, e_2, e_3, e_4)\). In general none of these two numbers is known. However, we illustrate in a concrete example the kind of results obtained using this method.

## 2 Notation and basic results

Let \(k\) be an algebraically closed field. We consider tamely ramified covers \(f : X \to \mathbb{P}^1_k\) between smooth projective curves. Let \(x = (x_1 = 0, x_2 = 1, x_3 = \infty, x_4, \ldots, x_r)\) be the branch points of \(f\), which we consider to be ordered. We first associate to \(f\) a ramification invariant.

**Definition 2.1** Let \(f : X \to \mathbb{P}^1_k\) be a tamely ramified cover as above. Denote by \(d\) the degree of \(f\). For every \(i\), let \((e_{i,1}, \ldots, e_{i,n_i})\) be the partition of \(d\) corresponding to the ramification indices of the points in the fiber \(f^{-1}(x_i)\). This
partition defines a conjugacy class $C_i$ of the symmetric group $S_d$. The ramification type of $f$ is defined as the datum $C = (d; C_1, \ldots, C_r)$. If $C_i$ is the class of a single cycle (i.e., exactly one $e_{i,j}$ is different from 1) we simply write $C_i = e_i$, where $e_i$ is the length of the cycle. If $p$ is a prime number, we say that $C$ is $p$-tame if all $e_{i,j}$ are relatively prime to $p$.

Since we assume that $f$ is tamely ramified, the Riemann–Hurwitz formula states that

$$
\sum_{i,j} e_{i,j} - \sum_i n_i = 2g(X) - 2 + 2d.
$$

Since the genus $g(X)$ of $X$ only depends on the ramification type of the cover, we sometimes denote it by $g(C)$ and refer to it as the genus of the ramification type. A datum $(d; C_1, \ldots, C_r)$, where $C_i = (e_{i,1}, \ldots, e_{i,n_i})$ is a partition of $d$ and $g(C)$ an integer, is called a ramification type. A ramification type such that $g(C) = 0$ is called a genus-0 type.

There exists several variants of these definitions. For example, in some cases it makes sense to include the Galois group of the Galois closure of $f$ into the definition. In this note, we mostly consider the case that each $C_i = e_i$ is the conjugacy class of a single cycle. In this case, the Galois group is typically $S_d$ or $A_d$, with few exceptions in small degree.

Two covers $f_i : X_i \to \mathbb{P}^1$ are considered isomorphic if there exists an isomorphism $\iota : X_1 \to X_2$ making

\[
\begin{array}{ccc}
X_1 & \sim \quad \iota \quad \sim \quad \iota & X_2 \\
\downarrow \quad \sim & \downarrow & \downarrow \\
\mathbb{P}^1 & \quad \sim & \mathbb{P}^1
\end{array}
\]

commutative. In particular, both covers have the same branch locus and the same ramification type.

The number of isomorphism classes of covers with a given ramification type $(d; C_1, \ldots, C_r)$ and fixed branch points $x$ is finite, since we only consider tame ramification. We first consider the classical characteristic zero case, for which the number of isomorphism classes of covers does not depend on the position of the branch points. It follows from Riemann’s Existence Theorem that this number, called the Hurwitz number and denoted by $h(d; C_1, \ldots, C_r)$, is the cardinality of the set

$$\left\{ (g_1, \ldots, g_r) \in C_1 \times \cdots \times C_r \mid \langle g_i \rangle \subset S_d \text{ transitive, } \prod_i g_i = e \right\} / \sim,$$

where $\sim$ denotes uniform conjugacy by the group $S_d$. The condition that the $g_i$ generate a transitive subgroup of $S_d$ guarantees that the corresponding cover $f$ is connected.
Let \( C := (d; C_1, \ldots, C_r) \) be a ramification type. We denote by \( \mathcal{H}_k(C) \) the Hurwitz space parameterizing isomorphism classes of covers \( f : Y \to \mathbb{P}^1_k \) with ramification type \( C \) defined over \( k \).

Let \( \pi : \mathcal{H}_k(C) \to (\mathbb{P}^1_k)^{r-3} \setminus \Delta \) be the natural map \([f] \mapsto x\) which sends the class of a cover to its branch locus. Here, \( \Delta := \{ x \mid x_i = x_j \text{ for some } i \neq j \} \) is the fat diagonal. In characteristic zero, this map is finite and flat. Its degree is exactly the Hurwitz number \( h(d; C_1, \ldots, C_r) \). Moreover, the map \( \pi \) is unramified. We remark that the Hurwitz space may not be connected.

Now assume that the characteristic \( p \) of \( k \) is positive. Then the number of covers of given ramification type may depend on the position of the branch points. The \( p \)-Hurwitz number \( h_p(C) \) is defined as the number of isomorphism classes of covers of ramification type \( C \) for which the branch locus \( x \) is generic, in the sense that it corresponds to the generic point of \((\mathbb{P}^1_k)^{r-3} \setminus \Delta\). The \( p \)-Hurwitz number is also the maximum number of covers of given type as the branch locus \( x \) varies.

The following well-known lemma gives some information on the Hurwitz number in this context. Part (a) follows from the fact that every tame cover in characteristic \( p \) lifts to characteristic zero. Part (b) is a consequence of the isomorphism between the prime-to-\( p \) part of the fundamental group \( \pi^0(\mathbb{P}^1_k \setminus x, \ast) \) and the prime-to-\( p \) part of the fundamental group of the complement of \( r \) points on \( \mathbb{P}^1 \) in characteristic zero ([4]).

**Lemma 2.2** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \).

(a) The \( p \)-Hurwitz number \( h_p(C) \) only depends on \( p \), and not on the field \( k \).

(b) We have \( h_p(C) \leq h(C) \) with equality if \( d < p \).

Note that the difference \( h(C) - h_p(C) \) corresponds to the number of covers in characteristic zero which have generic branch locus and bad reduction to characteristic \( p \). We call this number sometimes the bad degree of the ramification type \( C \) if \( p \) is clear from the context.

Suppose that \( C = (d; C_1, \ldots, C_r) \) is a genus-0 ramification type. In this case the degree \( d \) of a cover of type \( C \) is determined by the conjugacy classes \( C_i \) via the Riemann–Hurwitz formula (1). For convenience, we may therefore drop \( d \) from the notation. We write \( h^0(C_1, \ldots, C_r) = h(d; C_1, \ldots, C_r) \) and \( h^0_p(C_1, \ldots, C_r) = h_p(d; C_1, \ldots, C_r) \) for the Hurwitz numbers in the genus-0 case.

The following lemma describes the Hurwitz numbers in the 3-point single-cycle case under the assumption that the genus of the ramification type is zero.

**Lemma 2.3** Let \( C = (d; e_1, e_2, e_3) \) be a ramification type with \( g(C) = 0 \). Then

(a) \( h^0(e_1, e_2, e_3) = 1 \).

(b) Assume additionally that \( e_1, e_2 < p \). Then \( h^0_p(e_1, e_2, e_3) = 1 \) if and only if \( d < p \) and \( 0 \) otherwise.
Proof: Part (a) is an elementary calculation using the combinatorial description of the Hurwitz number given above (see for example [5, Lemma 2.1]). Part (b) is proved by B. Osserman ([7, Cor 2.5]) using linear series.

B. Osserman proves a stronger version of Lemma 2.3, calculating $h_p(d; e_1, e_2, e_3)$ in the situation of Lemma 2.3, but without the assumption that $e_1, e_2 < p$. We do not recall the full result here, since it its formulation if quite involved. The following proposition, proved in [5], is a generalization of Lemma 2.3.(a) to the case of 4 branch points.

Proposition 2.4 Let $C = (d; e_1, e_2, e_3, e_4)$ be a ramification type with $g(C) = 0$. Then

(a) $h^0(C) = \min_i(e_i(d + 1 - e_i)).$

(b) The Hurwitz curve $H_C(C)$ is connected.

Remark 2.5 Assume that $C = (d; e_1, \ldots, e_r)$ is a single-cycle ramification type with genus $g(C) = 0$. Write $x = (x_1 = 0, x_2 = 1, x_3 = \infty, x_4, \ldots, x_r)$ for the branch points and $y = (y_1, \ldots, y_r)$ for the ramification points, where $f(y_i) = x_i$. Up to isomorphism, the associated cover $f : Y \simeq \mathbb{P}^1_k \rightarrow X \simeq \mathbb{P}^1_k$ may be normalized such that $y_1 = 0, y_2 = 1$ and $y_3 = \infty$. If this is the case, we say that $f$ is normalized. Note that any isomorphism class contains a unique normalized representative. Assuming $f$ is normalized, we may therefore regard $f$ as element of $k(T)$, where $x$ is a coordinate on $Y \simeq \mathbb{P}^1_k$ with $T(i) = i$ for $i \in \{0, 1, \infty\}.$

3 An elementary construction

We start by recalling an elementary construction due to B. Osserman [6, Lemma 5.2]. Let $1 < e_1, \ldots, e_r < p$. Osserman’s result establishes a bijection between maps $f : \mathbb{P}^1_k \rightarrow \mathbb{P}^1_k$ with ramification indices $(e_i)$ and maps $h : \mathbb{P}^1_k \rightarrow \mathbb{P}^1_k$ with ramification indices $p - e_1, e_2, \ldots, e_{r-1}, p - e_r$. The maps $f$ and $h$ need not have the same degree. Although B. Osserman does not explicitly consider this, the construction also works when either $e_1$ or $e_r$ equals $p - 1$ in which case the number of ramification points of $f$ differs from that of $h$.

In the rest of this section, we assume that $r = 4$. The following lemma is a normalized version of the result of Osserman. In our version, we make sure that the ramification points map to distinct points. This allows to deduce a statement on Hurwitz numbers in characteristic $p > 0$.

Let $k$ be an algebraically closed field of characteristic $p > 0$. We fix a genus-0 ramification type $C = (d; e_1, e_2, e_3, p-1)$ and a branch locus $x = (x_1 = 0, x_2 = 1, x_3 = \infty, x_4 = \lambda)$. We assume that $1 < e_i < p$ for all $i$. Note that the degree of a cover of type $C$ is equal to $d = (e_1 + e_2 + e_3 + p - 3)/2$.

Lemma 3.1 (a) Let $f : \mathbb{P}^1_k \rightarrow \mathbb{P}^1_k$ be a normalized cover of type $C = (d; e_1, e_2, e_3, p-1)$ and branch locus $x = (x_1 = 0, x_2 = 1, x_3 = \infty, x_4 = \lambda)$.
We denote the unique ramification point of $f$ above $\lambda$ by $\mu$. Then the cover $h : \mathbb{P}^1_k \to \mathbb{P}^1_k$ defined by
\[ g(y) = \frac{(y - \mu)^p}{f - \lambda}, \quad h(y) := \frac{g - g(0)}{g(1) - g(0)} \]
is a normalized cover of type $\tilde{C} := (\tilde{d}; e_1, e_2, p - e_3)$ and branch locus $\tilde{x} = (x_1 = 0, x_2 = 1, x_3 = \infty)$. Here $\tilde{d} = d + 1 - e_3 = (e_1 + e_2 - e_3 + p - 1)/2$.

(b) Conversely, choose an element $\mu \in \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ with $h(\mu) \neq 0, 1, \infty, \mu^p$. Suppose given a normalized cover $h : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of ramification type $C := (d; e_1, e_2, p - e_3)$. Then the cover $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ defined by
\[ w(y) = \frac{(y - \mu)^p}{h - h(\mu)}, \quad f(y) = \frac{w - w(0)}{w(1) - w(0)} \]
is a normalized cover of type $C^* = (d; e_1, e_2, p - e_3)$ with branch locus $x = (0, 1, \infty, \lambda := \mu^p(1 - h(\mu)/(\mu^p - h(\mu)))$.

(c) The constructions of (a) and (b) are inverse to each other.

**Proof:** Let $f$ be as in (a). The statement on the ramification indices of $h$ follows immediately from the definition of $h$. To prove the statement of the ramification type, it remains to show that $h(0), h(1), h(\infty)$ are pairwise distinct. The condition $h(0) = h(1)$ is equivalent to $\mu = \lambda$. If this were the case, the cover $h$ would only have 2 branch points, with ramification type $C^* = (\tilde{d}; e_1, e_2, 1 \cdots 1, p - e_3 - 1 \cdots 1)$. It is well-known that such a cover does not exist. This shows that $h$ has the claimed ramification type. The statement on the degree follows from the observation that $h$ is a genus-0 cover.

Let $h$ be as in (b), and write $h(y) = y^{e_1}h_1/h_2$, where $\deg(h_1) = \tilde{d} - e_1$ and $\deg(h_2) = \tilde{d} - p + e_3$. Note that the definition of $w$ may be rewritten as
\[ w(y) = \frac{(y - \mu)^{p-1}h_2}{h_3}, \]
where $h_3 := (y^{e_1}h_1 - h(\mu)h_2)/(y - \mu)$ is a polynomial of degree $\tilde{d} - 1$. It follows that $f$ maps the ramification points $0, 1, \infty$ to the points $0, 1, \infty$, respectively.

Define $\lambda := f(\mu)$. An easy computation leads to the identity
\[ \lambda = f(\mu) = \frac{-w(0)}{w(1) - w(0)} = \frac{\mu^p(1 - h(\mu))}{\mu^p - h(\mu)}. \]
The choice of $\mu$ implies that $f(0), f(1), f(\infty), f(\mu)$ are pairwise distinct. Therefore (b) follows similarly to (a).

Part (c) is an easy verification. We prove one direction and leave the other as an exercise. Let $h : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a normalized cover of type $\tilde{C}$, and define $f$ as in the statement of (b). The definition of $\lambda$ implies that
\[ f - \lambda = \frac{w}{c} = \frac{(y - \mu)^p}{c(h - h(\mu))}, \]
where have set $c := w(1) - w(0)$. The cover $g$ associated with $f$ in (a) satisfies $g = c(h - h(\mu))$. Since $h$ is normalized, we conclude that the unique normalized polynomial associated with $g$ is again $h$.

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**Proposition 3.2** Let $p$ be an odd prime number and $1 < e_1, e_2, e_3 < p$ integers such that $E := e_1 + e_2 + e_3$ is even and $e_3 \neq p - 1$. Put $d = (E + p - 3)/2$, i.e. $C := (d; e_1, e_2, e_3, p - 1)$ is a genus-0 ramification type.

(a) The Hurwitz number $h_p^0(C)$ is positive if and only if

$$p + 1 \leq E \leq p - 1 + 2 \min(e_i).$$

(b) If the equivalent conditions of (a) are satisfied, we have that

$$h_p^0(C) = \frac{1}{2}(3p - 1 - E).$$

**Proof:** Write $\tilde{C} = (\tilde{d}; e_1, e_2, p - e_3)$, where $\tilde{d} = d = 1 - e_3$. Lemma 3.1 implies that $h_p^0(C)$ is positive if and only if $h_p^0(\tilde{C})$ is positive. A necessary and sufficient condition for $\tilde{C}$ to be a ramification type with three branch points is that

$$1 < e_1, e_2, p - e_3 \leq \tilde{d}.$$  

In our situation, the conditions (3) can be rewritten as

$$\begin{align*}
e_1 + e_2 & > 1, \\
e_3 & < p - 1, \\
p + 1 & \leq e_1 + e_2 + e_3, \\
e_1 + e_2 + e_3 & \leq p - 1 + 2e_1, \\
e_1 + e_2 + e_3 & \leq p - 1 + 2e_2.
\end{align*}$$

Assume that these conditions are satisfied. Lemma 2.3.(b) implies that $h_p^0(e_1, e_2, p - e_3)$ is positive if and only if

$$\tilde{d} = (e_1 + e_2 - e_3 + p - 1)/2 < p.$$  

Combining these inequalities immediately yields (a).

To prove (b), we assume that $h_p^0(e_1, e_2, p - e_3)$ is positive, i.e. that (2) holds. Lemma 2.2.(b) implies that $h_p^0(e_1, e_2, p - e_3)$ is positive as well, hence $h_p^0(e_1, e_2, p - e_3) = 1$ (cf. Lemma 2.3.(a)). Applying Lemma 2.2.(b) and using the fact that $\tilde{d} < p$, we therefore obtain the identity $h_p^0(e_1, e_2, p - e_3) = 1$.

Denote by $h : \mathbb{P}^1_k \rightarrow \mathbb{P}^1_k$ the unique normalized cover of type $(e_1, e_2, p - e_3)$. Choose $\mu$ as in Lemma 3.1.(b), and define $f$ and $\lambda$ as the statement of Lemma 3.1.(b). Consider the birational map

$$\mu \mapsto \lambda(\mu) = \frac{\mu^p(1 - h(\mu))}{\mu^p - h(\mu)}, \quad \mathbb{P}^1_k \rightarrow \mathbb{P}^1_k.$$
Obviously, the degree $\deg(\lambda)$ of this map equals the $p$-Hurwitz number $h_p^0(e_1, e_2, p-e_3)$. We compute $\deg(\lambda)$ by computing the divisor of this map.

Since $h$ is normalized, it follows that

$$\text{ord}_{\mu=0}(\lambda) = p - e_1, \quad \text{ord}_{\mu=1}(\lambda) = e_2 - e_2 = 0, \quad \text{ord}_{\mu=\infty}(\lambda) = -(p - e_3) < 0.$$  

Moreover, $\lambda$ has $\tilde{d} - e_2$ simple zeros, which are different from $\mu = 0, 1, \infty$. Note that $\mu^p - h(\mu)$ has $\tilde{d} - p + e_3$ poles different from $\mu = \infty$, which all have multiplicity one. However, these are exactly the (simple) poles of $1 - h(\mu)$ which are different from 1, hence these do not yield zeros of $\lambda$. We conclude that

$$\deg(\lambda) = p - e_1 + \tilde{d} - e_2 = (3p - 1 - (e_1 + e_2 + e_3))/2,$$

which proves (b). \hfill \Box

**Remark 3.3** B. Osserman gives a similar statement ([6, Cor. 8.1]), counting covers with 4 ramification points. His result is more general, since it does not require one of the ramification indices to be $p - 1$. However, he fixes the ramification points of the cover, rather than the branch points. Therefore his result does not compute Hurwitz numbers in characteristic $p > 0$.

Computing $p$-Hurwitz numbers is in general more difficult than counting covers with fixed ramification. Beside the classical result from Lemma 2.2.(b), the only general result on $p$-Hurwitz numbers is the main result of [3], which computes $h_p(p; e_1, e_2, e_3, e_4)$. That result relies on subtle and deep results on the stable reduction of Galois covers.

The following corollary translates the statement of Proposition 3.2 into a statement on the Hurwitz curve $\mathcal{H}_p(C)$.

**Corollary 3.4** Let $C = (d; e_1, e_2, e_3, p-1)$ be as in the statement of Proposition 3.2. The Hurwitz curve $\mathcal{H}_p(C)$ is connected.

**Proof:** The statement immediately follows from the proposition. Let $\pi_p : \mathcal{H}_p(C) \to \mathbb{P}^1$ be the natural map which sends a cover of type $C$ to the branch point $\lambda$. Then $\pi$ is birationally equivalent to the map $\mu \mapsto \lambda$ described in the proof of that proposition. \hfill \Box

Let $C := (d; e_1, e_2, e_3, p-1)$ be a ramification type satisfying the equivalent conditions of Proposition 3.2.(a). Put $\tilde{C} = (d; e_1, e_2, p-e_3)$ and let $h : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the unique cover of type $\tilde{C}$ (compare with the proof of Proposition 3.2.(a)). We may write $h(y) = h_1(y)/h_2(y)$, where the $h_i \in k[y]$ are relatively prime and satisfy the relations $\deg(h_1) = d - e_1$ and $\deg(h_2) = d - (p - e_3) = (e_1 + e_2 + e_3 - p - 1)/2$.

It follows from Lemma 2.2 that there exist finitely many values $\lambda \in \mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$ for which the number of covers of ramification type $C$ and branch locus $(0, 1, \infty, \lambda)$ is strictly less than $h_p(C)$. We let $\Sigma(C) \subset \mathbb{P}_\lambda^1 \setminus \{0, 1, \infty\}$ be this exceptional set and call it the supersingular locus.
Corollary 3.5 With the above notation, we have

$$\Sigma(C) = \{y \in \mathbb{P}^1_k \setminus \{0,1,\infty\} \mid h_2(y) = 0\}.$$ 

**Proof:** We recall that the construction of Lemma 3.1.(b) works if and only if \(h(\mu) \neq 0, 1, \infty, \mu^p\). Moreover, equation (5) gives an expression of the fourth branch point \(\lambda\) of \(f\) as function of \(\mu\).

Assume that \(h(\mu) = 0\). Then (5) implies that either \(\mu = 0\) or \(h(\mu) = 1\). By definition \(0, 1 \notin \Sigma(C)\). Therefore it suffices to consider the solutions of \(h(\mu) = 1\) with \(\mu \neq 1\). We may write \(h(\mu) = 1 = \mu^p \varphi(\mu)\), where \(\varphi(1) \neq 1\). Substituting this in (5) yields

$$\lambda(\mu) = \frac{\mu^p \varphi(\mu)}{(\mu-1)^{p-\epsilon_2} - \varphi(\mu)}.$$ 

In particular, it follows that \(\lambda(\mu) = 0\) if \(\mu\) is a zero of \(\varphi\). Hence these zeroes are not contained in \(\Sigma(C)\). Similarly it follows that the solutions of \(h(\mu) = 1\) don’t belong to \(\Sigma(C)\).

Assume that \(h(\mu) = \infty\) and \(\mu \neq \infty\), i.e. \(h_2(\mu) = 0\) according to the notation introduced above the statement of the corollary. We then have the identity \(\lambda(\mu) = \mu^p\). Therefore \(\mu \in \Sigma(C)\), since \(\mu \neq 0, 1, \infty\).

Finally, assume that \(\mu = \mu^p\) and \(\mu \notin \{0,1,\infty\}\). Then \(\lambda = \infty\), hence this does not yield any new value. \(\square\)

Example 3.6 We illustrate the results of this section with two concrete examples.

(a) Let \(p \geq 5\) be a prime, and consider the genus-0 ramification type \(C = (d; 2, 2, p-3, p-1)\). Note that the condition of Proposition 3.2.(a) is satisfied. Hence Proposition 3.2.(b) implies that \(h_p(C) = p - 1\) and Proposition 2.4.(a) leads to \(h(C) = \min(3(p-3), 2(p-2), p-1) = p - 1\). We therefore find the equality \(h(C) = h_p(C)\), so that all covers of this type with generic branch locus have good reduction.

The unique normalized cover \(h : \mathbb{P}^1 \to \mathbb{P}^1\) of type \((3; 2, 2, 3)\) is given by

\[h(y) = 3y^3 - 2y^2.\]

Therefore

\[\lambda(\mu) = \frac{\mu^p(1 + 2\mu^3 - 3\mu^2)}{\mu^p + 2\mu^3 - 3\mu^2} = \frac{\mu^p(1 + 2\mu^3 - 3\mu^2)}{\sum_{i=1}^{p-3} i\mu^{p-3-i}}.\]

This confirms that the degree \(\deg(\lambda)\) equals \((3p - (e_1 + e_2 + e_3))/2 = p - 1\).

We have already remarked that all covers with generic branch locus have good reduction to characteristic \(p > 0\). Arguing as in the proof of [2, Theorem 4.2], one may deduce from this observation that the map \(\pi_p : \mathcal{H}_p(C) \to \mathbb{P}^1_k \setminus \{0,1,\infty\}\) is finite. However, in this concrete example the finiteness of \(\pi_p\) immediately follows from Corollary 3.5.

(b) Next, we consider the case \(C = (p; 3, 2, p-2, p-1)\) and \(\tilde{C} = (3; 3, 2, 2)\), again assuming that \(p \geq 5\). The unique cover \(h : \mathbb{P}^1_k \to \mathbb{P}^1_k\) of type \(\tilde{C}\) is given
by \( h(y) = y^3/(3y - 2) \) and a direct computation leads to the expression

\[
\lambda(\mu) = \frac{\mu^{p-3}(-\mu^3 + 3\mu - 2)}{3\mu^{p-2} - 2\mu^{p-3} - 1}.
\]

Dividing the numerator and the denominator by \((\mu - 1)^2\), we find \( \deg(\lambda) = p - 2 \), which confirms Proposition 3.2.(b). As in Corollary 3.5, the supersingular values are the poles of \( h \) different from \( \infty \). In this concrete example, we find a unique value, namely \( \mu = 2/3 \). The case \( d = p \) has been considered in [3].

Fix a genus-0 ramification type \( C = (d; e_1, e_2, e_3, p - 1) \) which is \( p \)-tame, i.e. \( p \nmid e_i \). Recall that \( h(C) - h_p(C) \) denotes the “bad degree” of the ramification type. This is the number of covers with generic branch locus which have bad reduction to characteristic \( p \).

**Proposition 3.7** The notation being as above, assume that the minimum \( \min_{i \in \{1, 2, 3\}} e_i(d + 1 - e_i) \) is attained for \( e_1 \). This is not a restriction, since we may permute the branch points. Then, the bad degree \( h(C) - h_p(C) \) is given by

\[
\begin{align*}
0 & \quad \text{if } d \leq p - 1, \\
p(d + 1) & \quad \text{if } p \leq d \leq p - 2 + e_1, \\
\lambda(C) = e_1(d + 1 - e_1) & \quad \text{otherwise}.
\end{align*}
\]

**Proof:** The first case immediately follows from Lemma 2.2. Assume that \( p \leq d \leq p - 2 + e_1 \). In this case, Propositions 2.4.(a) and 3.2.(a) assert that \( h(C) = (p - 1)(d + 2 - p) \) and \( h_p(C) \neq 0 \). Statement (b) therefore follows from Proposition 3.2.(b).

For \( d > p - 2 + e_1 \), Proposition 2.4.(a) implies that \( h(C) = e_1(d + 1 - e_1) \). Since \( d > p - 2 + \min e_i = p - 2 + e_1 \) by assumption, we conclude from Proposition 3.2.(a) that \( h_p(C) = 0 \) and statement (c) follows. \( \square \)

In the second case of Proposition 3.7, some covers have good reduction while others have bad reduction. In the first (resp. third) case all covers have good (resp. bad) reduction to characteristic \( p > 0 \). The following corollary therefore follows from Proposition 3.7 and its proof. A similar phenomenon occurs in the situation of [2, Section 4].

**Corollary 3.8** Let \( C \) be as in Proposition 3.7, and assume that \( h(C) \neq h_p(C) \neq 0 \). Then the bad degree \( h(C) - h_p(C) \) is divisible by \( p \).
4 A variant

In this section, we present a variant of the construction of Section 3. This construction and the idea of the proof of the following lemma has been taken from [6, Prop. 5.4]. We fix integers $e_1 > p$ and $1 < e_2, e_3, e_4 < p$ with $\gcd(e_1, p) = 1$ such that $e_1 + e_2 + e_3 + e_4$ is even and $e_3 + e_4 \leq d := (e_1 + e_2 + e_3 + e_4 - 2)/2$.

Lemma 4.1 Let $k$ be an algebraically closed field of positive characteristic $p$.

(a) Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a cover with ramification type $\mathcal{C} = (d; e_1, e_2, e_3 - e_4)$. We assume that the ramification points are $x = \infty, 0, 1, \rho$ and that $f(\infty) = \infty, f(0) = 0$ and $f(1) = f(\rho) = 1$. Then for every $c \in \mathbb{P}^1_k \setminus \{0, -1, -\rho^{-p}, \infty\}$ the rational function

$$f_c = f(x) + cx^p$$

defines a cover of ramification type $\mathcal{C} = (d; e_1, e_2, e_3)$.

(b) Conversely, assume that $g$ is a cover of ramification type $\mathcal{C} = (d; e_1, e_2, e_3, e_4)$. Then there exists a $c \in k$ such that $g + cx^p$ has ramification type $\mathcal{C}' = (d; e_1, e_2, e_3 - e_4)$.

Proof: (a) Let $f$ and $f_c$ be as in the statement of the lemma. We may write

$$f(x) = \frac{x^{e_2}f_1}{f_2},$$

where $\deg(f_1) = d - e_2$ and $\deg(f_2) = d - e_1$. Moreover, the polynomials $f_1$ and $f_2$ have simple zeros and are relatively prime. This implies that

$$f_c(x) = \frac{x^{e_2}(f_1 + cx^p - e_2 f_2)}{f_2}.$$  

Therefore, the ramification index of $f_c$ in $x = 0$ is $e_2$. Since $e_1 > p$ it follows also that the ramification index of $f_c$ in $x = \infty$ is $e_1$. Similarly, the ramification indices of $f_c$ in $x = 1, \rho$ are $e_2, e_3$, respectively.

The equality

$$\frac{\partial f_c}{\partial x} = \frac{\partial f}{\partial x}$$

implies that $f_c$ is unramified outside $x = 0, 1, \rho, \infty$. The assumption on $c$ implies that the image of $x = 0, 1, \rho, \infty$ under $f_c$ are all distinct and the statement of the lemma follows.

(b) Let $g$ be as in the statement of the lemma. Define $g_c = g + cx^p$. Since $\partial g_c/\partial x = \partial g/\partial x \neq 0$ it follows that $g_c$ is separable. Moreover, for all $c$ such that the image under $g_c$ of the ramification points are pairwise distinct, the ramification type of $g_c$ is still $\mathcal{C} = (d; e_1, e_2, e_3, e_4)$. Assume that two of the ramification points, for example $x_3$ and $x_4$, have the same image under $g_c$. Then the ramification type is $\mathcal{C}' = (d; e_1, e_2, e_3 - e_4)$. The connectedness of the Hurwitz curve $\mathcal{H}_p(\mathcal{C})$ (Proposition 2.4.(b)) implies that there exists a $c$ such that $g_c(x_3) = g_c(x_4)$, which proves (b). \qed

The following proposition is a direct consequence of Lemma 4.1.
Proposition 4.2 The assumptions being as above, assume additionally that \( e_3 \neq e_4 \).

(a) We then have the equality

\[
h_p(d, e_1, e_2, e_3, e_4) = h_p(d; e_1, e_2, e_3-e_4).
\]

(b) If \( h_p(d; e_1, e_2, e_3-e_4) > 0 \) then the Hurwitz curve \( \mathcal{H}_p(d, e_1, e_2, e_3, e_4) \) contains \( h_p(d; e_1, e_2, e_3-e_4) \) irreducible components of genus 0. Moreover, the restriction of the natural map \( \pi : \mathcal{H}_p(d, e_1, e_2, e_3, e_4) \to \mathbb{P}^1_k \) which sends \([f]\) to its fourth branch point has degree 1 on each of these components.

Proof: To prove (a), it is sufficient to show that nonisomorphic covers \( f_i \) of type \((d, e_1, e_2, e_3, e_4)\) give rise to nonisomorphic covers under the construction of Lemma 4.1.

Let \( f^i : \mathbb{P}^1_k \to \mathbb{P}^1_k \) be two nonisomorphic covers of type \((d, e_1, e_2, e_3-e_4)\), and assume they are normalized as in the statement of Lemma 4.1. The branch points of \( f^i \) are \( \infty, 0, 1 + c, 1 + cp^e \). Normalizing the third branch point to 1 yields the normalized cover \( g^i_c(x) := f^i_c(x)/(1 + c) \) with branch points \( \infty, 0, 1, (1 + cp^e)/(1 + c) =: \lambda_i \). The assertion that the \( g^i_c \) are nonisomorphic follows immediately from the assumption that \( e_3 \neq e_4 \).

Statement (b) follows immediately from the explicit expression for the cover \( f_c \) given in the proof of Lemma 4.1. \( \square \)

In the rest of this section, we discuss a concrete application of this result to Hurwitz curves in positive characteristic.

Lemma 4.3 Let \( p \) be a prime, and choose \( e_1 = p + 2, e_2 = 3, 2 \leq e_3 < e_4 < p \) with \( e_3 + e_4 = p + 1 \). Put \( d = (e_1 + e_2 + e_3 + e_4 - 2)/2 = p + 2 \). We then have the inequality

\[
h_p(d; e_1, e_2, e_3-e_4) \geq 1.
\]

Proof: Let \( f : \mathbb{P}^1_k \to \mathbb{P}^1_k \) be a cover of type \((d; e_1, e_2, e_3-e_4)\), and assume that \( f \) is normalized as in Lemma 4.1. Then

\[
f - 1 = c(x - 1)^{e_3}(x - \rho)^{e_4}(x - a),
\]

for some \( a \in \mathbb{P}^1_k \setminus \{0, 1, \rho, \infty\} \).

Let \( f \) be as in (6). We want to determine which covers of this form define a cover of ramification type \( C = (d; e_1, e_2, e_3-e_4) \).

The cover \( f \) being ramified of order 3 at \( x = 0 \), we obtain the relations

\[
e_3a^2 + 2e_3a + 2 - e_3 = 0, \quad \rho = \frac{3a + 2 - e}{e + 1}.
\]

The coefficient \( c \) is uniquely determined by \( a \) and \( \rho \) and the condition \( f(0) = 0 \). A polynomial \( f \) satisfying these conditions defines a cover of ramification type \((d; e_1, e_2, e_3-e_4)\) if and only if the values \( 0, 1, \infty, a \) and \( \rho \) are pairwise distinct.

Our assumptions on the \( e_i \) imply in particular that \( e_3 \not\equiv -1 \pmod{p} \) and \( d \equiv 2 \).
It then easily follows that $0, 1, \infty, a,$ and $\rho$ are pairwise distinct if and only if
\begin{equation}
\tag{7}
a \notin \{(e_3 - 2)/2, (2e_3 - 1)/3, -1\}.
\end{equation}

If $a$ satisfies (7), then the ramification points $0, 1, \infty, a,$ and $\rho$ are pairwise distinct for all $a$ satisfying $e_3a^2 + 2e_3a + 2 - e_3 = 0$. Hence in this case we have $h_p(d; e_1, e_2, e_3-e_4) = 2$.

Note that the integers $(e_3 - 2)/2, (2e_3 - 1)/3$, and $-1$ are pairwise distinct, since $e_3 \not\equiv -1 \pmod{p}$ by assumption. Therefore if $a \in \{(e_3 - 2)/2, (2e_3 - 1)/3, -1\}$ the equation $e_3a^2 + 2e_3a + 2 - e_3 = 0$ has exactly one solution $a$ for which the ramification points $0, 1, \infty, a,$ and $\rho$ are pairwise distinct. Therefore, in this case we find $h_p(d; e_1, e_2, e_3-e_4) = 1$.

The following result immediately follows from Proposition 4.2 and Lemma 4.3.

**Corollary 4.4** Let $p > 3$ be a prime, and choose $e_1 = p + 2, e_2 = 3, 2 \leq e_3 < e_4 < p$ with $e_3 + e_4 = p + 1$. Setting $d = (e_1 + e_2 + e_3 + e_4 - 2)/2 = p + 2$, we then have the inequality
\[h_p(d; e_1, e_2, e_3, e_4) \geq 1.\]

**Remark 4.5** The result of Proposition 4.2.(a) may also be deduced from Lemma 4.1 by using deformation of admissible covers (see [3, §2] and the references therein). However, this argument does not yield the information on the Hurwitz curve from Proposition 4.2.(b).

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