Stability of combinatorial polynomials and its applications *

Ming-Jian Ding\textsuperscript{a} and Bao-Xuan Zhu\textsuperscript{b}

\textsuperscript{a}School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China
\textsuperscript{b}School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, PR China

Abstract

Many important problems are closely related to the zeros of certain polynomials derived from combinatorial objects. The aim of this paper is to make a systematical study on the stability of polynomials in combinatorics.

Applying the characterizations of Borcea and Brändén concerning linear operators preserving stability, we present criteria for real stability and Hurwitz stability of recursive polynomials. We also give a criterion for Hurwitz stability of the Turán expressions of recursive polynomials. As applications of these criteria, we derive some stability results occurred in the literature in a unified manner. In addition, we obtain the Hurwitz stability of Turán expressions for alternating runs polynomials of types $A$ and $B$ and solve a conjecture concerning Hurwitz stability of alternating runs polynomials defined on a dual set of Stirling permutations.

Furthermore, we prove that the Hurwitz stability of any symmetric polynomial implies its semi-$\gamma$-positivity. We study a class of symmetric polynomials and derive many nice properties including Hurwitz stability, semi-$\gamma$-positivity, non-$\gamma$-positivity, unimodality, strong $q$-log-convexity, the Jacobi continued fraction expansion and the relation with derivative polynomials. In particular, these properties of the alternating descents polynomials of types $A$ and $B$ can be obtained in a unified approach.

Finally, based on the $h$-polynomials from combinatorial geometry, we use real stability to prove a criterion for zeros interlacing between a polynomial and its reciprocal polynomial, which in particular implies the alternatingly increasing property of the original polynomial. This criterion extends a result of Brändén and Solus and unifies such properties for many combinatorial polynomials, including ascent polynomials for $k$-ary words, descent polynomials on signed Stirling permutations and colored permutations and $q$-analog of descent polynomials on colored permutations, and so on. Furthermore, we also obtain a recurrence relation and zeros interlacing of $q$-analog of descent polynomials on colored permutations that extend some results of Brändén and Brenti. In addition, as an application of Hurwitz stability, we prove the alternatingly increasing property and zeros interlacing for two kinds of peak polynomials on the dual set of Stirling permutations.

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Email address: ding-mj@hotmail.com (M.-J. Ding), bxzhu@jsnu.edu.cn (B.-X. Zhu)
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1 Introduction

The analytic theory of polynomials plays a significant role in different fields, such as analysis, combinatorics, probability, optimization, real algebraic geometry, automatic control theory and statistical physics, see the monograph [59]. In particular, the theory of multivariate stable polynomials recently displays more and more power to solve some hard problems [11, 12, 13, 14, 17, 74]. The problems center in the analytic theory of polynomials is the study of the zeros or coefficients. The zeros of a polynomial can often reveal a variety of information. In addition, many important problems can be transformed to the distribution of zeros of polynomials, such as the four color problem [8], the Riemann hypothesis [37], the Lee-Yang program on phase transitions in equilibrium statistical mechanics [45, 76], and the construction of Ramanujan graphs [58]. In combinatorics, the zeros of polynomials are often used to determine the (combinatorial) information of the coefficients, such as asymptotical normality, unimodality, log-concavity, q-log-convexity, γ-positivity, Pólya frequency, total positivity, alternatingly increasing property, see [18, 22, 69].

The differential operators often arise in analysis. Many classical orthogonal polynomials can be generated from different differential operators, such as Legendre polynomials \( L_n(x) = \frac{1}{2^n n!} D_x^n (x^2 - 1)^n \), Laguerre polynomials \( L_n(x) = e^x D_x^n (x^n e^{-x}) \), Hermite polynomials \( H_n(x) = (-1)^n e^{x^2} D_x^n e^{-x^2} \), where \( D_x = d/dx \). In addition, orthogonal polynomials often satisfy certain differential recursive relations, for example, the Jacobi polynomial \( P_n^{(\alpha,\beta)}(x) \) satisfies
\[
2n P_n^{(\alpha,\beta)}(x) = [\alpha - \beta + x(\alpha + \beta + 2)] P_{n-1}^{(\alpha+1,\beta+1)}(x) - (x^2 - 1) D_x P_{n-1}^{(\alpha+1,\beta+1)}(x).
\]

The combinatorial polynomials often also have such property. For example,
\[
(xD_x)^n \frac{1}{1-x} = \frac{x A_n(x)}{(1-x)^{n+1}},
\]
where \( A_n(x) \) is the classical Eulerian polynomial. We refer the reader to [1, 9, 25] for more combinatorial polynomials generated in this way. On the other hand, the classical Eulerian polynomial \( A_n(x) \) satisfies the recurrence relation
\[
A_n(x) = [(n-1)x+1] A_{n-1}(x) + x(1-x) D_x A_{n-1}(x), \tag{1.1}
\]
where \( A_0(x) = 1 \). In fact, for some combinatorial sequences, their recurrence relations are very nice feature, which are a useful way to study many properties. In this paper, we will mainly consider the zeros distribution of the polynomial \( T_n(x) \) satisfying the following generalized recurrence relation:
\[
T_{n+1}(x) = (\alpha_n x^2 + \beta_n x + \gamma_n) T_n(x) + (\mu_n x^3 + \nu_n x^2 + \varphi_n x + \psi_n) D_x T_n(x), \tag{1.2}
\]
where all \( \alpha_n, \beta_n, \gamma_n, \mu_n, \nu_n, \varphi_n, \psi_n \) are real sequences in \( \mathbb{R} \).

In Section 2, with the help of the characterizations of Borcea and Brändén concerning the linear operator preserving stability [11], we present criteria for the real stability of \( T_n(x) \) (see Theorem 2.5) and the Hurwitz stability of \( T_n(x) \) for \( \nu_n = \psi_n = 0 \) (see Theorem 2.15). These criteria can be applied to a large number of combinatorial polynomials, such
as the generalized Eulerian polynomials, the Stirling-Whitney-Riordan polynomials, and deal with those known results occurred in the literature [38, 75, 77, 82, 83] in a unified approach. In particular, we obtain the Hurwitz stability of alternating runs polynomials defined on a dual set of Stirling permutations, which solves a conjecture in [55]. Furthermore, we give a criterion for Hurwitz stability of certain linear combination of $T_n(x)$ for $\alpha_n = \mu_n = \psi_n = 0$ (see Proposition 2.17), which extends a corresponding result for $A_n(x)$ due to Zhang and Yang [77].

In Section 3, we prove a result for the Hurwitz stability of a nonlinear operator on polynomials called the Turán expression (see Theorem 3.2). It unifies plenty of known results in [26, 80, 82, 83]. In addition, we also prove the Hurwitz stability of Turán expressions for alternating runs polynomials of types $A$ and $B$, up-down runs polynomials, and so on. In particular, Hurwitz stability of these Turán expressions implies $q$-log-convexity of the original polynomial sequence, respectively.

The symmetric polynomials often have more nice properties. In Section 4, we prove that Hurwitz stability of any symmetric polynomial implies its semi-$\gamma$-positivity (see Theorem 4.3), which is similar to that real rootedness of any symmetric polynomial implies the $\gamma$-positivity (see Brändén [15]). Moreover, we demonstrate the Hurwitz stability and semi-$\gamma$-positivity for a class of symmetric polynomials $T_n(x)$ for $\alpha_n = -m_n \mu_n$, $\gamma_n = \beta_n + m_n \nu_n$, $\varphi_n = -\nu_n$ and $\psi_n = -\mu_n$, where $m_n = \deg(T_n(x))$ (see Theorem 4.6). We also derive many other nice properties including unimodality, non-$\gamma$-positivity, strong $q$-log-convexity, the Jacobi continued fraction expansion and the relation with derivative polynomials. In particular, these properties of the alternating descents polynomials of types $A$ and $B$ can be obtained in a unified approach.

In Section 5, based on the $h$-polynomials from combinatorial geometry, we present a criterion for zeros interlacing between a polynomial and its reciprocal polynomial, which in particular implies the alternatingly increasing property of the original polynomial (see Theorem 5.3). This criterion extends a result of Brändén and Solus [21] and unifies such properties for many combinatorial polynomials, including ascent polynomials for $k$-ary words, descent polynomials on signed Stirling permutations and colored permutations and $q$-analog of descent polynomials on colored permutations, and so on. On the other hand, we obtain a recurrence relation and zeros interlacing of $q$-analog of descent polynomials on colored permutations that extend some results of Brändén [16] and Brenti [23]. Finally, using our results for Hurwitz stability, we show the alternatingly increasing property and zeros interlacing for two kinds of peak polynomials on the dual set of Stirling permutations.

The next is the definition of some notations. Denote $\mathbb{N}^+$, $\mathbb{N}$, $\mathbb{R}^>$, $\mathbb{R}^{\geq0}$, $\mathbb{R}$ and $\mathbb{C}$ be the positive integers, nonnegative integers, positive real numbers, nonnegative real numbers, real numbers and complex numbers, respectively. Let $\mathbb{R}[x]$ (resp., $\mathbb{C}[x]$) denote the set of polynomials over $\mathbb{R}$ (resp., $\mathbb{C}$) and $\mathbb{R}_n[x]$ (resp., $\mathbb{C}_n[x]$) denote the set of polynomials with degree at most $n$ over $\mathbb{R}$ (resp., $\mathbb{C}$). Let $S_n$ represent the symmetric group on $[n] = \{1, 2, \ldots, n\}$. 



2 Stability of polynomials

2.1 Definitions of stability

Let $H \subset \mathbb{C}$ be an open half-plane whose boundary contains the origin, namely $H = \{ z \in \mathbb{C} \mid \Im(e^{i\theta}z) > 0 \}$ for $\theta \in \mathbb{R}$, where $\Im(z)$ is the image part of $z$ for $z \in \mathbb{C}$. We say that $f \in \mathbb{C}[z_1, \ldots, z_n]$ is $H$-stable if it is either identically zero or nonvanishing whenever $z_i \in H$ for any $i \in [n]$. In particular, $f$ is called stable if $H$ is the upper half-plane ($\theta = 0$), and $f$ is real stable if all coefficients of $f$ are real. Clearly, a univariate polynomial $f$ is real stable if and only if $f$ has only real zeros. Similarly, $f$ is called Hurwitz stable if $H$ is the right half-plane ($\theta = \pi/2$). We will consider the real stability and Hurwitz stability of the polynomials in this paper.

Let $f, g \in \mathbb{R}[x]$ be real-rooted with zeros $\{r_i\}$ and $\{s_j\}$, respectively. We say that $g$ interlaces $f$ if $\deg(f) = \deg(g) + 1 = n$ and

$$r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1,$$

and that $g$ alternates left of $f$ if $\deg(f) = \deg(g) = n$ and

$$s_n \leq r_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1.$$

Denote either $g$ interlaces $f$ or $g$ alternates left of $f$ by $g \preceq f$. If no equality sign occurs in (2.1) and (2.2), then we say that $g$ strictly interlaces $f$ and $g$ strictly alternates left of $f$, respectively, denoted $g \prec f$. Here, we denote $g \ll f$ if $g \preceq f$ and the leading coefficients of $f, g$ have same sign or $f \preceq g$ and the leading coefficients of $f, g$ have opposite sign. The following Hermite-Biehler Theorem (see [62, Theorem 6.3.4]), which is a very classical result in geometry of polynomials, characterizes two zeros-interlacing polynomials.

**Theorem 2.1** (Hermite-Biehler Theorem). Let $\{f(x), g(x)\} \subseteq \mathbb{R}[x]$. Then $g(x) \ll f(x)$ if and only if $f(x) + ig(x)$ is stable.

Following Theorem 2.1, we state an important result obtained by Borcea and Brändén as follows.

**Proposition 2.2.** [13, Lemma 2.6] Let $f(x)$ be a real-rooted polynomial that is not identically zero. The sets

$$\{g(x) \in \mathbb{R}[x] : g(x) \ll f(x)\} \quad \text{and} \quad \{g(x) \in \mathbb{R}[x] : f(x) \ll g(x)\}$$

are convex cones.

In addition, for Theorem 2.1, Borcea and Brändén [11] gave an equivalent result: For $f(x), g(x) \in \mathbb{R}[x]$, the stability of $f(x) + ig(x)$ is equivalent to that of the bivariate polynomial $f(x) + yg(x)$. Thus, in order to show the alternating property of zeros of two polynomials, the real stability of bivariate polynomials is very useful.

For a linear operator $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$, we define its algebraic symbol in $\mathbb{R}[z, w]$ by

$$G_T(z + w) := T[(z + w)^n] = \sum_{k \leq n} \binom{n}{k} T(z^k) w^{n-k}.$$

The following result for linear operators preserving real stability of multivariate polynomials is a powerful tool to study real stability.
Theorem 2.3. [11, Theorem 2.2] For $n \in \mathbb{N}$, let $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ be a linear operator. Then $T$ preserves stability if and only if either

(a) $T$ has range of dimension at most two and is of the form

$$T(f) = \alpha(f)P + \beta(f)Q,$$

where $\alpha, \beta : \mathbb{R}_n[z] \to \mathbb{R}$ are linear functional and $P, Q$ are real stable polynomial such that $P \ll Q$, or

(b) the bivariate polynomial $G_T(z + w)$ is stable, or

(c) the bivariate polynomial $G_T(z - w)$ is stable.

2.2 Real stability

For the recurrence relation (1.2), for brevity, let $\alpha$ (resp., $\beta, \gamma, \mu, \nu, \varphi, \psi$) denote $\alpha_n$ (resp., $\beta_n, \gamma_n, \mu_n, \nu_n, \varphi_n, \psi_n$). Then we can rewrite (1.2) as

$$T_{n+1}(x) = (\alpha x^2 + \beta x + \gamma)T_n(x) + (\mu x^3 + \nu x^2 + \varphi x + \psi)D_x T_n(x). \quad (2.3)$$

Let $\deg(T_n(x)) = n$ and define $F(x)$ and $G(x)$ by

$$
\begin{cases}
F(x) = \alpha x^2 + \beta x + \gamma, \\
G(x) = (\alpha + m_n \mu)x^3 + (\beta + m_n \nu)x^2 + (\gamma + m_n \varphi)x + m_n \psi.
\end{cases} \quad (2.4)
$$

For the recurrence relation (2.3), it can be generated from a linear operator $T$ defined by

$$T := (\alpha x^2 + \beta x + \gamma)I + (\mu x^3 + \nu x^2 + \varphi x + \psi)D_x, \quad (2.5)$$

where $I$ is the identity operator and $D_x$ is the differential operator $d/dx$. We present one of the main results concerning stability as follows.

Theorem 2.4. The operator $T$ defined by (2.5) preserves real stability if $F(x) \ll G(x)$.

Proof. According to (2.5) and Theorem 2.3, it suffices to show that $T(x + y)^{m_n}$ is real stable. Note that we have

$$T(x + y)^{m_n} = (x + y)^{m_n-1} [(\alpha x^2 + \beta x + \gamma)(x + y) + m_n (\mu x^3 + \nu x^2 + \varphi x + \psi)]$$

$$= (x + y)^{m_n-1} [(\alpha + m_n \mu)x^3 + (\beta + m_n \nu)x^2 + (\gamma + m_n \varphi)x + m_n \psi]$$

$$+ (\alpha x^2 + \beta x + \gamma)y$$

$$= (x + y)^{m_n-1} [G(x) + F(x)y].$$

Obviously, $(x + y)^{m_n-1}$ is real stable. Then we only need to show that $G(x) + F(x)y$ is real stable. By Theorem 2.1, $G(x) + F(x)y$ is real stable if and only if $F(x) \ll G(x)$. This completes the proof. $\square$

Next, we will give the sufficient conditions for operator $T$ defined by (2.5) preserving real stability according to the degree conditions of $F(x)$ and $G(x)$. 
Theorem 2.5. Assume that both the leading coefficients of $F(x)$ and $G(x)$ are positive and $0 \leq \deg(G(x)) - \deg(F(x)) \leq 1$. If $T_{n_0}(x)$ is real stable, then so is $T_n(x)$ in (2.3) for $n \geq n_0$ under any of the following conditions:

1. $\deg(F(x)) \leq 1$ and $\beta\gamma\varphi - \gamma^2\nu - \beta^2\psi \geq 0$,
2. $\deg(F(x)) = \deg(G(x)) = 2$, $\psi = 0$ and $m_n(\beta + m_n\nu)(\beta\varphi - \gamma\nu) - \alpha(\gamma + m_n\varphi)^2 \geq 0$.

Proof. We will prove that $T_n(x)$ is real stable by induction on $n$. By the assumption, $T_{n_0}(x)$ is real stable. It follows from Theorem 2.4 that $T_n(x)$ for $n \geq n_0$ is real stable if $F(x) \ll G(x)$. Thus, we will prove that both conditions in (1) and (2) imply that $F(x) \ll G(x)$.

For (1), because $0 \leq \deg(G(x)) - \deg(F(x)) \leq 1$, we divide its proof into the following three cases in terms of the degree conditions.

Case 1: $\deg(F(x)) = 0$ and $\deg(G(x)) \leq 1$. Obviously, we have $\alpha = \beta = \nu = 0$. This implies $\beta\gamma\varphi - \gamma^2\nu - \beta^2\psi = 0$. By the assumption that the leading coefficients of $F(x)$ and $G(x)$ are positive, we have $\gamma > 0$ and $\gamma + m_n\varphi > 0$. Then the bivariate polynomial $G(x) + F(x)y$ is reduced to

$$(\gamma + m_n\varphi)x + m_n\psi + \gamma y,$$

which is clearly real stable.

Case 2: $\deg(F(x)) = \deg(G(x)) = 1$. We have $\alpha = \beta + m_n\nu = 0$. By the assumption that the leading coefficients of $F(x)$ and $G(x)$ are positive, we have $\beta > 0$ and $\gamma + m_n\varphi > 0$. Thus the condition $\beta\gamma\varphi - \gamma^2\nu - \beta^2\psi \geq 0$ implies

$$\gamma^2 + m_n\gamma\varphi - m_n\beta\psi \geq 0. \quad (2.6)$$

Then $F(x) \ll G(x)$ is reduced to

$$\beta x + \gamma \ll (\gamma + m_n\varphi)x + m_n\psi.$$

This is equivalent to

$$-\frac{\gamma}{\beta} \leq -\frac{m_n\psi}{\gamma + m_n\varphi},$$

which follows from the inequality (2.6).

Case 3: $\deg(F(x)) = 1$ and $\deg(G(x)) = 2$. By the assumption that the leading coefficients of $F(x)$ and $G(x)$ are positive, we have $\beta > 0$ and $\beta + m_n\nu > 0$. So $F(x) \ll G(x)$ is reduced to

$$\beta x + \gamma \ll (\beta + m_n\nu)x^2 + (\gamma + m_n\varphi)x + m_n\psi.$$

Obviously, the interlacing follows from

$$(\beta + m_n\nu)\left(\frac{\gamma}{\beta}\right)^2 - (\gamma + m_n\varphi)\frac{\gamma}{\beta} + m_n\psi \leq 0. \quad (2.7)$$

By calculation, the inequality (2.7) is equivalent to the known condition

$$\beta\gamma\varphi - \gamma^2\nu - \beta^2\psi \geq 0.$$
So we complete the proof of (1).

For (2), \(\deg(F(x)) = \deg(G(x)) = 2\). By the assumption that the leading coefficients of \(F(x)\) and \(G(x)\) are positive, we have \(\alpha > 0\) and \(\beta + \gamma > 0\). Hence for \(\psi = 0\), \(F(x) \leq G(x)\) is reduced to

\[
\alpha x^2 + \beta x + \gamma \leq (\beta + \gamma) x^2 + (\gamma + \alpha) x.
\]

The interlacing is implied by the next inequality

\[
\alpha \left(\frac{\gamma + \alpha}{\beta + \gamma}\right)^{\beta + \gamma} - \beta \frac{\gamma + \alpha}{\beta + \gamma} + \gamma \leq 0,
\]

that is

\[
m_n(\beta + \gamma) (\beta - \gamma) - \alpha (\gamma + \alpha)^2 \geq 0.
\]

Thus we complete the proof. \(\square\)

**Remark 2.6.** Generally speaking, we mainly consider the polynomial \(T_n(x)\) defined by (2.3) with nonnegative coefficients and the positive leading coefficients of corresponding \(F(x)\) and \(G(x)\). Then the stronger result than Theorem 2.4 is that the linear operator \(T\) defined by (2.5) preserves real stability if and only if \(F(x)\) and \(G(x)\) have interlacing zeros. In fact, the proof for sufficiency is similar to Theorem 2.4 by the (b) and (c) of Theorem 2.3 and the proof for necessity can be verified by the linear operator \(T\) acting \((x + w)^m\) for any \(w \in \mathbb{R}\).

In terms of the recurrence relation (2.3), it is well known that many combinatorial polynomials can be viewed as the special case of \(T_n(x)\). In what follows, we will apply Theorem 2.5 to the real stability of some combinatorial polynomials.

Let \(a_i, b_i \in \mathbb{R}\) for \(i \in [3]\). Define a nonnegative triangular array \([A_{n,k}]_{n,k \geq 0}\) by

\[
A_{n,k} = (a_1n + a_2k + a_3)A_{n-1,k} + (b_1n + b_2k + b_3)A_{n-1,k-1}
\]

for \(n \geq 1\), where \(A_{0,0} = 1\) and \(A_{n,k} = 0\) unless \(0 \leq k \leq n\). For example, \(A_{n,k}\) is the signless Stirling number of the first kind for \(a_1 = -a_3 = b_3 = 1\) and the others are zero and the Stirling number of the second kind for \(a_2 = b_3 = 1\) and the others are zero, see [75] for more examples. In terms of the nonnegativity of \([A_{n,k}]_{n,k \geq 0}\), it is natural to let \(a_1n + a_2k + a_3 \geq 0\) for \(n > k \geq 0\), which is equivalent to

\[
a_1 \geq 0, \quad a_1 + a_2 \geq 0, \quad a_1 + a_3 \geq 0.
\]

Let the row-generating function \(A_n(x) = \sum_{k=0}^{n} A_{n,k} x^k\). Then we have

\[
A_{n+1}(x) = [(b_1n + b_1 + b_2 + b_3)x + a_1n + a_1 + a_3]A_n(x) + (b_2x^2 + a_2x)D_x A_n(x),
\]

where \(\deg(A_n(x)) = n\). Hence, by Theorem 2.5, we immediately get the following result due to Wang and Yeh [75].

**Corollary 2.7.** [75] Let \([A_{n,k}]_{n,k \geq 0}\) be defined by (2.8). If \(a_1b_2 \leq a_2b_1\) and \((a_1 + a_3)b_2 \leq (b_1 + b_2 + b_3)a_2\), then the row-generating function \(A_n(x)\) has only real zeros for \(n \in \mathbb{N}\).
Proof. Note that $A_n(x)$ satisfies the recurrence relation (2.9). For the real rootedness of $A_n(x)$, taking $\beta = b_1 n + b_1 + b_2 + b_3$, $\gamma = a_1 n + a_1 + a_3$, $\varphi = a_2$, $\nu = b_2$ and $\psi = 0$ in (1) of Theorem 2.5, it suffices to prove for $n \geq 0$ that

$$(b_1 n + b_1 + b_2 + b_3)(a_1 n + a_1 + a_3)a_2 - (a_1 n + a_1 + a_3)^2 b_2 \geq 0,$$

which is obvious from the conditions $a_1 b_2 \leq a_2 b_1$ and $(a_1 + a_3) b_2 \leq (b_1 + b_2 + b_3) a_2$. \hfill $\square$

In terms of the recurrence relation (2.9), we define an operator $A$ by

$$A := [(b_1 n + b_1 + b_2 + b_3)x + a_1 n + a_1 + a_3] I + (b_2 x^2 + a_2 x) D_n. \quad (2.10)$$

By Theorem 2.4 and Remark 2.6, we know that the condition in (1) of Theorem 2.5 is actually equivalent to that the operator $A$ preserves real stability. Thus, for the operator $A$, we have the following stronger result.

**Proposition 2.8.** Let $U = b_1 a_2 - a_1 b_2$ and $V = (b_1 + b_2 + b_3)a_2 - (a_1 + a_3) b_2$. The operator $A$ defined by (2.10) preserves real stability if and only if $V + n U \geq 0$.

**Remark 2.9.** Proposition 2.8 implies [38, Theorem 3.3]. In fact, in [38, Theorem 3.3], Hao et al. assumed that

$$b_1 \geq 0, \quad b_1 + b_2 \geq 0, \quad b_1 + b_2 + b_3 \geq 0.$$

The following example indicates that we can drop the restrict condition $b_1 + b_2 \geq 0$.

**Example 2.10 (André Polynomials).** Let $d_{n,k}$ denote the number of the augmented André permutations in $S_n$ with $k - 1$ left peaks. Let

$$D_n(x) = \sum_{k \geq 1} d_{n,k} x^k.$$

It is known that

$$d_{n+1,k} = k d_{n,k} + (n - 2k + 3) d_{n,k-1},$$

where $d_{1,1} = 1$, see Foata and Scützenberger [33] and [68, A094503] for instance. Note that

$$D_{n+1}(x) = (n + 1) x D_n(x) + x (1 - 2x) D_n(x) D_n(x)$$

and the degree of $D_n(x)$ is $\lceil n/2 \rceil$. Taking $\beta = n + 1, \gamma = 0, \nu = -2, \varphi = 1$ and $\psi = 0$ in (1) of Theorem 2.5, we have that the operator

$$D := (n + 1) x I + x (1 - 2x) D_n$$

preserves real stability, which implies the real-rootedness of $D_n(x)$.

As a generalization of the Stirling triangle of the second kind, the Whitney triangle of the second kind and one triangle of Riordan, the Stirling-Whitney-Riordan triangle $[\mathcal{S}_{n,k}]_{n,k \geq 0}$ satisfies the recurrence relation

$$[\mathcal{S}_{n,k}] = (b_k + b_2) \mathcal{S}_{n-1,k-1} + [(2 \lambda b_1 + a_1) k + \lambda (b_1 + b_2) + a_2] \mathcal{S}_{n-1,k} + \lambda (a_1 + \lambda b_1)(k + 1) \mathcal{S}_{n-1,k+1} \quad (2.11)$$
where \( \mathcal{J}_{0,0} = 1 \) and \( \mathcal{J}_{n,k} = 0 \) unless \( 0 \leq k \leq n \), see [83]. For its row-generating function \( \mathcal{J}_n(x) = \sum_{k=0}^{n} \mathcal{J}_{n,k} x^k \), it satisfies the recurrence relation

\[
\mathcal{J}_n(x) = [a_2 + (b_1 + b_2)(x + \lambda)] \mathcal{J}_{n-1}(x) + (x + \lambda) [a_1 + b_1(x + \lambda)] D_x \mathcal{J}_{n-1}(x), \tag{2.12}
\]

where \( \deg(\mathcal{J}_n(x)) = n \). By Theorem 2.5, we get the following result [83, Theorem 3.2].

**Corollary 2.11.** [83, Theorem 3.2] Let \( a_1, a_2, b_1, b_2, \lambda \) be nonnegative. If \( a_1(b_1 + b_2) \geq a_2 b_1 \), then \( \mathcal{J}_n(x) \) defined by (2.12) has only real zeros.

**Proof.** By (2.12), we have \( F(x) \) and \( G(x) \) corresponding to (2.4) as follows

\[
\begin{align*}
F(x) &= (b_1 + b_2)x + \lambda(b_1 + b_2) + a_2, \\
G(x) &= (b_1 n + b_2)x^2 + [(2n - 1)\lambda b_1 + \lambda b_2 + (n - 1)\lambda a_1 + a_2]x + (n - 1)\lambda(a_1 + \lambda b_1).
\end{align*}
\]

For the real rootedness of \( \mathcal{J}_n(x) \), taking \( \beta = b_1 + b_2, \gamma = \lambda(b_1 + b_2) + a_2, \nu = b_1, \varphi = a_1 + 2\lambda b_1 \) and \( \psi = \lambda(a_1 + \lambda b_1) \) in (1) of Theorem 2.5, it suffices to prove for \( n \geq 0 \) that

\[(b_1 + b_2)[\lambda(b_1 + b_2) + a_2] - [\lambda(b_1 + b_2) + a_2]^2 b_1 - (b_1 + b_2)^2 \lambda (a_1 + \lambda b_1) \geq 0.\]

This inequality is equivalent to \( a_1(b_1 + b_2) \geq a_2 b_1 \). \(\square\)

Based on the classical Eulerian triangle and various triangular arrays from staircase tableaux, tree-like tableaux and segmented permutations, Zhu [82] considered a generalized Eulerian triangle \([\mathcal{T}_{n,k}]_{n,k \geq 0}\), which satisfies the recurrence relation:

\[
\mathcal{T}_{n,k} = \lambda(a_1 k + a_2) \mathcal{T}_{n-1,k} + [(b_1 - d a_1)n - (b_1 - 2d a_1)k + b_2 - d(a_1 - a_2)] \mathcal{T}_{n-1,k-1} + \frac{d(b_1 - da_1)}{\lambda}(n-k+1)\mathcal{T}_{n-1,k-2}, \tag{2.13}
\]

where \( \mathcal{T}_{0,0} = 1 \) and \( \mathcal{T}_{n,k} = 0 \) unless \( 0 \leq k \leq n \). In particular, (2.13) can reduce to some combinatorial sequences, such as the classical Eulerian numbers by taking \( b_2 = d = 0 \) and \( a_1 = a_2 = b_1 = \lambda = 1 \) and the numbers enumerating in symmetric tableaux by taking \( b_2 = d = 0, a_1 = a_2 = \lambda = 1 \) and \( b_1 = 2 \) (see [68, A109062]). We refer the reader to [82] for more examples.

We can rewrite (2.13) by its row-generating function as follows:

\[
\mathcal{T}_n(x) = p_n(x) \mathcal{T}_{n-1}(x) + q_n(x) D_x \mathcal{T}_{n-1}(x), \tag{2.14}
\]

where

\[
\begin{align*}
p_n(x) &= \frac{(n-1)d(b_1-da_1)}{\lambda}x^2 + [(n-1)(b_1-da_1)+b_2+da_2]x + \lambda a_2, \\
q_n(x) &= -\frac{d(b_1-da_1)}{\lambda}x^3 - (b_1-2da_1)x^2 + \lambda a_1x.
\end{align*}
\]

and \( \deg(\mathcal{T}_n(x)) = n \).

The following result for real rootedness of \( \mathcal{T}_n(x) \) proved in [82] can easily follow from Theorem 2.5.
Corollary 2.12. [82, Theorem 2.16] Let \( a_1, b_1, \lambda \) be positive and \( a_2, b_2, d \) be nonnegative. If \( a_2 + b_2 > 0 \) and \( b_1 - da_1 \geq 0 \), then the row-generating function \( T_n(x) \) of \([T_{n,k}]_{n,k}\) in (2.13) has only real zeros.

Proof. By (2.14), we have \( F(x) \) and \( G(x) \) corresponding to (2.4) as follows:

\[
\begin{align*}
F(x) &= \frac{(n-1)d}{\lambda}(b_1 - da_1)x^2 + [(n-1)(b_1 - da_1) + b_2 + da_2]x + \lambda a_2, \\
G(x) &= [(n-1)da_1 + b_2 + da_2]x^2 + \lambda[(n-1)a_1 + a_2]x.
\end{align*}
\]

Next, we will consider two different cases in terms of \( \text{deg}(F(x)) \).

Case 1: If \( \text{deg}(F(x)) \leq 1 \), then \( d(b_1 - da_1) = 0 \). Furthermore, by \( a_1 > 0 \) and \( a_2 + b_2 > 0 \), we have \( 0 \leq \text{deg}(G(x)) - \text{deg}(F(x)) \leq 1 \). For the real-rootedness of \( T_n(x) \), taking \( \beta = (n-1)(b_1 - da_1) + b_2 + da_2, \gamma = \lambda a_2, \nu = 2da_1 - b_1, \varphi = \lambda a_1, \psi = 0 \) in (1) of Theorem 2.5, and it suffices to show

\[
\lambda^2 a_1 a_2 [(n-1)(b_1 - da_1) + b_2 + da_2] - (\lambda a_2)^2 (2da_1 - b_1) \geq 0,
\]

which is equivalent to

\[
(n-1)a_1 (b_1 - da_1) + a_1 b_2 + a_2 (b_1 - da_1) \geq 0.
\]

This is obvious from \( a_1 > 0, b_2 \geq 0 \) and \( b_1 - da_1 \geq 0 \).

Case 2: If \( \text{deg}(F(x)) = 2 \), then \( \text{deg}(G(x)) = 2 \). Similarly, taking \( \alpha = (n-1)d(b_1 - da_1)/\lambda, \beta = (n-1)(b_1 - da_1) + b_2 + da_2, \gamma = \lambda a_2, \nu = 2da_1 - b_1, \varphi = \lambda a_1, \psi = 0 \) and \( m_n = n - 1 \) in (2) of Theorem 2.5. It suffices to show that

\[
(n-1)[(n-1)da_1 + b_2 + da_2][\lambda a_1 [(n-1)(b_1 - da_1) + b_2 + da_2] + \lambda a_2 (b_1 - 2da_1)]
- (n-1)d(b_1 - da_1)[\lambda a_2 + (n-1)\lambda a_1]^2/\lambda \geq 0.
\]

This is equivalent to

\[
(n-1)\lambda b_2 [a_1 b_2 + a_2 b_1 + (n-1)a_1 b_1] \geq 0.
\]

This inequality follows from nonnegativity of \( a_i, b_i \) and \( \lambda \). \( \square \)

2.3 Hurwitz stability

As we know that many combinatorial polynomials have only real zeros. However, for some other combinatorial polynomials, they don’t always have only real zeros. In this case, they often have all zeros in the left half-plane, i.e., they are Hurwitz stable. For any univariate Hurwitz stable polynomial, a nice property is that if its leading coefficient is positive, then so are all coefficients (see [62, Proposition 11.4.2]). This is also a useful approach to verifying the positivity of coefficients of a polynomial.

Let

\[
r(x) = \sqrt{\frac{1+x}{1-x}}.
\]
By induction, one can get

\[(xD)^n(r(x)) = \frac{\mathcal{R}_n(x)}{(1 - x)^n(1 + x)^{n-1}\sqrt{1 - x^2}},\]

where \(\mathcal{R}_n(x) = \sum_{k=0}^{2n-1} \mathcal{R}(n, k)x^k\). It is easy to know that the polynomial \(\mathcal{R}_n(x)\) satisfies the recurrence relation

\[\mathcal{R}_{n+1}(x) = (2nx + 1)x\mathcal{R}_n(x) + x(1 - x^2)D_x\mathcal{R}_n(x)\]  

(2.15)

for \(n \geq 0\), \(\mathcal{R}_0(x) = 1\) and \(\mathcal{R}_1(x) = x\). For the coefficient \(\mathcal{R}(n, k)\), it counts the number of a dual set of Stirling permutations of order \(n\) with \(k\) alternating runs, see [55]. In addition, in [55], it was found that this polynomial \(\mathcal{R}_n(x)\) does not have only real zeros and proposed the following conjecture.

**Conjecture 2.13.** [55, Conjecture 4.1] The polynomial \(\mathcal{R}_n(x)\) in (2.15) is Hurwitz stable for \(n \in \mathbb{N}\).

It is natural to study the Hurwitz stability of combinatorial polynomials. In the following, we will consider the Hurwitz stability of \(T_n(x)\) in (2.3) with \(\nu = \psi = 0\), i.e., satisfying the following recurrence relation:

\[T_{n+1}(x) = (\alpha x^2 + \beta x + \gamma)T_n(x) + (\mu x^3 + \varphi x)D_xT_n(x),\]  

(2.16)

where all \(\alpha, \beta, \gamma, \mu, \varphi\) are real sequences in \(\mathbb{R}\). In order to show the Hurwitz stability of \(T_n(x)\), we need the following characterization of linear operators preserving Hurwitz stability of multivariate polynomials, see Borcea and Brändén [11, Remark 7.1].

**Theorem 2.14.** For \(n \in \mathbb{N}\), let \(\mathbb{T} : \mathbb{C}_n[z] \to \mathbb{C}[z]\) be a linear operator. Then \(\mathbb{T}\) preserves Hurwitz stability if and only if either

(a) \(\mathbb{T}\) has range of dimension at most one and is of the form \(\mathbb{T}(f) = \alpha(f)P\), where \(\alpha\) is a linear functional on \(\mathbb{C}_n[z]\) and \(P\) is a Hurwitz stable polynomial, or

(b) the bivariate polynomial

\[\mathbb{T}[(1 + zw)^n] := \sum_{k \leq n} \binom{n}{k} \mathbb{T}(z^k)w^k\]

is Hurwitz stable.

Our result for Hurwitz stability can be presented as follows.

**Theorem 2.15.** Let \(T_n(x)\) be defined by (2.16) with all \(\beta, \gamma, \varphi \geq 0\) and \(T_{n_0}(x)\) be Hurwitz stable. If one of the followings is true,

(1) \(\deg(T_n(x)) = n\) and \(\alpha = -n\mu \geq 0\),

(2) \(\deg(T_n(x)) = m_n \) (\(m_n \neq n\)) and \(\alpha \geq -m_n\mu \geq 0\),

then \(T_n(x)\) is Hurwitz stable for \(n \geq n_0\).
Proof. We will present the proof by induction on \( n \). By the Hurwitz stable assumption of \( T_{n_0}(x) \), then the statement holds for \( n = n_0 \). Let \( T = (\alpha x^2 + \beta x + \gamma)I + (\mu x^3 + \varphi x)D \). The statement for \( n \geq n_0 + 1 \) is immediate if the operator \( T \) preserves Hurwitz stability. In what follows, we will prove that \( T \) preserves Hurwitz stability according to two different cases of \( \text{deg}(T_n(x)) \).

(1) If \( \text{deg}(T_n(x)) = n \), then by (2.16), we have \( \alpha + n\mu = 0 \). By Theorem 2.14, it suffices to show that

\[
T(1 + xy)^n = (1 + xy)^{n-1} \left[ (\alpha x^2 + (\beta x + \gamma)(1 + xy) + n\varphi xy \right]
\]

\[
= (1 + xy)^n \left( \beta + \frac{\gamma}{x} + \frac{\alpha x}{1 + xy} + \frac{n\varphi y}{1 + xy} \right)
\]

is Hurwitz stable. Since \( (1 + xy)^n x \) is Hurwitz stable by definition, we need to prove that

\[
\beta + \frac{\gamma}{x} + \frac{\alpha x}{1 + xy} + \frac{n\varphi y}{1 + xy}
\]

is Hurwitz stable. Let \( \Re(z) \) denote the real part of \( z \), where \( z \in \mathbb{C} \). Note that

\[
\Re \left( \beta + \frac{\gamma}{x} + \frac{\alpha x}{1 + xy} + \frac{n\varphi y}{1 + xy} \right) = \beta + \Re \left( \frac{\gamma}{x} \right) + \Re \left( \frac{\alpha}{x + y} \right) + \Re \left( \frac{n\varphi}{x + \frac{1}{y}} \right).
\]

Whenever \( \Re(x) > 0 \) and \( \Re(y) > 0 \), we have \( \Re \left( \frac{1}{x} \right) > 0 \) and \( \Re \left( \frac{1}{y} \right) > 0 \). In consequence, it is obvious that

\[
\Re \left( \frac{\gamma}{x} \right) \geq 0, \quad \Re \left( \frac{\alpha}{x + y} \right) \geq 0, \quad \Re \left( \frac{n\varphi}{x + \frac{1}{y}} \right) \geq 0,
\]

since all \( \alpha, \gamma, \varphi \geq 0 \). Hence, by \( \beta \geq 0 \), the function in (2.17) does not have zeros in the right half-plane, and thus \( T(1 + xy)^n \) is Hurwitz stable. In consequence, \( T \) preserves Hurwitz stability.

(2) It is similar to (1). We have

\[
T(1 + xy)^{mn} = (1 + xy)^{mn-1} \left[ (\alpha x^2 + (\beta x + \gamma)(1 + xy) + m_n\mu x^3 y + m_n\varphi xy \right]
\]

\[
= (1 + xy)^{mn} \left[ (\alpha + m_n\mu) x + \beta + \frac{\gamma}{x} + \frac{m_n\varphi y}{1 + xy} - \frac{m_n\mu x}{1 + xy} \right]
\]

\[
= (1 + xy)^{mn} \left[ (\alpha + m_n\mu) x + \beta + \frac{\gamma}{x} + \frac{m_n\varphi}{x + \frac{1}{y}} - \frac{m_n\mu}{x + y} \right]
\]

is Hurwitz stable in terms of the nonnegativity of \( \beta, \gamma, -\mu, \varphi \) and \( \alpha + m_n\mu \). Hence \( T \) preserves Hurwitz stability.

As an immediate application of Theorem 2.15, we verify Conjecture 2.13 as follows.

Proposition 2.16. The polynomial \( \mathcal{R}_n(x) \) in (2.15) is Hurwitz stable for \( n \in \mathbb{N} \).

Proof. Obviously, \( \mathcal{R}_0(x) = 1 \) is Hurwitz stable and \( \text{deg}(\mathcal{R}_n(x)) = 2n - 1 \) due to (2.15). Taking \( \alpha = 2n, \beta = 1, \gamma = 0, \mu = -1, \varphi = 1 \) and \( m_n = 2n - 1 \) in (2) of Theorem 2.15, we get that the polynomial \( \mathcal{R}_n(x) \) is Hurwitz stable. \( \square \)
For the classical Eulerian polynomial $A_n(x)$ in (1.1), it is well known that $A_n(x)$ has only real zeros and $A_{n-1}(x) \ll A_n(x)$. Furthermore, it is an interesting problem to consider the distribution of zeros for some linear combinations of $A_{n-1}(x)$ and $A_n(x)$. In particular, Yang and Zhang [77] proved that the following linear combination:

$$(x + 1)A_{n-1}(x) + kxA_{n-2}(x)$$

is Hurwitz stable for $n \geq 2$ and $k \geq -n$. In addition, in [77], this Hurwitz stability result played an important role in proving the interlacing property of the Eulerian polynomials of between type $D$ and affine type $B$, and a conjecture about the half Eulerian polynomials of type $B$ and type $D$ proposed by Hyatt in [41]. As an extension, we will consider the Hurwitz stability of the next linear combination:

$$(\varphi - \nu x)\rho T_{n+1}(x) + [\varphi \eta x + (\varphi - \gamma)\varphi \rho]T_n(x)$$

(2.18)

for $T_n(x)$ in (2.3) with $\alpha = \mu = \psi = 0$, i.e., satisfying the recurrence relation

$$T_{n+1}(x) = (\beta x + \gamma)T_n(x) + (\nu x^2 + \varphi x)D_x T_n(x).$$

(2.19)

Here $\rho$ and $\eta$ are abbreviated notation for real sequences in $\mathbb{R}$. As a consequence of Theorem 2.15, we present the Hurwitz stability for the linear combination in (2.18) as follows.

**Proposition 2.17.** Let $\text{deg}(T_n(x)) = m_n$ and both $\varphi$ and $\rho$ be nonnegative sequences. If $(\beta + m_n \nu)\nu \leq 0$ and $(\beta \varphi - \gamma \nu)\rho + \varphi \eta \geq 0$, then the linear combination in (2.18) is Hurwitz stable for any $n \in \mathbb{N}$.

**Proof.** By (2.19), for the linear combination in (2.18), we have

$$x [(\varphi - \nu x)\rho T_{n+1}(x) + [\varphi \eta x + (\varphi - \gamma)\varphi \rho]T_n(x)]$$

$$= [\nu - \beta + \varphi \rho x^2 + ((\beta \varphi - \gamma \nu)\rho + \varphi \eta)x]xT_n(x) + (-\nu^2 \rho x^3 + \varphi \rho x)D_x(xT_n(x)).$$

Then according to the assumption, the Hurwitz stability for the linear combination in (2.18) follows from Theorem 2.15. \hfill \Box

**Example 2.18 (Flower triangle).** It is known that the flower triangle $[F_{n,k}]_{n,k \geq 0}$ satisfies the following recurrence relation (see [68, A156920]):

$$F_{n,k} = (1 + k)F_{n-1,k} + (2n - 2k + 1)F_{n-1,k-1},$$

where $F_{0,0} = 1$ and $F_{n,k} = 0$ unless $0 \leq k \leq n$. Then the row-generating function $F_n(x)$ satisfies

$$F_{n+1}(x) = [(2n + 1)x + 1]F_n(x) + x(1 - 2x)D_x F_n(x).$$

Taking $m_n = n$, $\beta = 2n+1$, $\gamma = 1$, $\nu = -2$, and $\varphi = 1$ in (2.19). If both $\rho$ and $(2n+3)\rho + \eta$ are nonnegative sequences, then

$$\rho(2x + 1)F_{n+1}(x) + \eta x F_n(x)$$

is Hurwitz stable for any $n \in \mathbb{N}$. 


3 The Hurwitz stability of Turán expressions

In the end of the former section, we consider the Hurwitz stability of certain linear combination. In this section, we mainly consider the Hurwitz stability of a non-linear operator.

Given a polynomial sequence \( P = (P_n(x))_{n \geq 0} \) with \( \deg(P_n(x)) = n \), we denote the \( n \)th Turán expression by

\[
I_n(P; x) := (P_{n+1}(x))^2 - P_{n+2}(x)P_n(x).
\]

The concept of Turán expression owed to Turán [73] who found Turán’s inequalities concerning Legendre polynomial sequence \( P \): \( I_n(P; x) \geq 0 \) for \( x \in [-1, 1] \) and \( n \in \mathbb{N} \). However, it was first published by Szegö [72]. We refer the reader to [26, 80] and references therein for more information about Turán’s inequalities. We say that \( (P_n(q))_{n \geq 0} \) is \( q \)-log-concave (resp., \( q \)-log-convex) if all coefficients of \( I_n(P; q) \) (resp., \( -I_n(P; q) \)) are nonnegative. The definition of the \( q \)-log-concavity was first suggested by Stanley and that of the \( q \)-log-convexity was first introduced Liu and Wang. Note the fact that if a univariate polynomial is Hurwitz stable, then the signs of its all coefficients are same. Thus the Hurwitz stability of a Turán expression implies that the original polynomial sequence is either \( q \)-log-concave or \( q \)-log-convex.

It is known that both the classical Eulerian polynomials and Bell polynomials are \( q \)-log-convex [49]. Moreover, their Turán expressions are Hurwitz stable. For many other combinatorial polynomials, including the Eulerian polynomials of types \( B \), Lah polynomials, descent polynomials on segmented permutations, and so on, their Turán expressions are also Hurwitz stable, see [26, 31, 80, 82, 83]. In this section, we will derive a new criterion for the Hurwitz stability of Turán expression. Then we apply this criterion to many combinatorial polynomials in a unified manner. The following result for two interlacing polynomials plays an important role in our proof.

**Lemma 3.1.** [31, Lemma 1.20] Let both \( f(x) \) and \( g(x) \) be standard real polynomials with only real zeros. Assume that \( \deg(f(x)) = n \) and all real zeros of \( f(x) \) are \( r_1, \ldots, r_n \). If \( \deg(g) = n - 1 \) and we write

\[
g(x) = \sum_{i=1}^{n} c_i f \left( \frac{x}{x - r_i} \right),
\]

then \( g \ll f \) if and only if all \( c_i \) are nonnegative.

Let \( (P_n(x))_{n \geq 0} \) be a sequence of polynomials with nonnegative coefficients and satisfy the recurrence relation

\[
P_{n+1}(x) = p_n(x)P_n(x) + q(x)D_xP_n(x),
\]

where \( \deg(P_n(x)) = \deg(P_{n-1}(x)) + 1 \). Denote by \( \{r_k\}_{k=1}^{n} \) all zeros of \( P_n(x) \) and define

\[
(x - r_k) [p_n(x) - p_{n-1}(x)] + q(x) := h_n(x) \sum_{i=0}^{3} a_k (x - r_k)^i
\]

for \( 1 \leq k \leq n \), where \( h_n(x) \) is a polynomial.

The main result of this section can be stated as follows.
Theorem 3.2. Let \( P_n(x) \) be defined by (3.1) and \( P_n(x) \ll P_{n+1}(x) \). Assume that \( h_n(x) \) is Hurwitz stable for each \( n \). If all elements of \( \bigcup_{k=1}^{n} \{-a_{k_1}, a_{k_1}, a_{k_0}\} \) have same sign, and the right side of (3.2) has same sign for \( 1 \leq k \leq n \) and \( x > 0 \), then \( J_n(P; x) \) is Hurwitz stable for each \( n \).

Proof. In terms of the hypothesis \( P_n(x) \ll P_{n+1}(x) \), all the zeros \( r_k \) of \( P_n(x) \) are real and non-positive and by Lemma 3.1 we can write

\[
\frac{P_{n-1}(x)}{P_n(x)} = \sum_{i=1}^{n} \frac{t_i}{x-r_i}, \quad (3.3)
\]

where all \( t_i \) are nonnegative. Furthermore, we have

\[
D_x \left( \frac{P_{n-1}(x)}{P_n(x)} \right) = \sum_{i=1}^{n} \frac{-t_i}{(x-r_i)^2}. \quad (3.4)
\]

By (3.1)-(3.4), we get

\[
J_n(P; x) = [p_n(x)P_n(x) + q(x)D_xP_n(x)] \frac{P_{n-1}(x)}{P_n(x)} - P_n(x) [p_{n-1}(x)P_{n-1}(x) + q(x)D_xP_{n-1}(x)]
\]

\[
= [p_n(x) - p_{n-1}(x)] P_n(x) \frac{P_{n-1}(x)}{P_n(x)} + q(x) \left[ \frac{P_{n-1}(x)}{P_n(x)} D_xP_n(x) - P_n(x) D_xP_{n-1}(x) \right]
\]

\[
= P^2_n(x) \left[ [p_n(x) - p_{n-1}(x)] \frac{P_{n-1}(x)}{P_n(x)} - q(x) D_x \left( \frac{P_{n-1}(x)}{P_n(x)} \right) \right]
\]

\[
= P^2_n(x) \sum_{k=0}^{n} t_k \left[ \frac{(x-r_k)(p_n(x) - p_{n-1}(x)) + q(x)}{(x-r_k)^2} \right]
\]

\[
= P^2_n(x) h_n(x) \sum_{k=0}^{n} t_k \left[ a_{k_3}(x-r_k) + a_{k_2} + \frac{a_{k_1}}{x-r_k} + \frac{a_{k_0}}{(x-r_k)^2} \right].
\]

Obviously, \( P_n(x) \) and \( h_n(x) \) are Hurwitz stable. Thus we will consider the following function:

\[
a_{k_3}(x-r_k) + a_{k_2} + \frac{a_{k_1}}{x-r_k} + \frac{a_{k_0}}{(x-r_k)^2}. \quad (3.5)
\]

Without loss of generality, we assume that \( \bigcup_{k=1}^{n} \{-a_{k_3}, a_{k_1}, a_{k_0}\} \) has positive (resp., negative) sign. If \( \Re(x) > 0 \) and \( \Im(x) \neq 0 \), then, obviously, for the image part of (3.5), we derive

\[
\Im(x) \Im \left( a_{k_3}(x-r_k) + a_{k_2} + \frac{a_{k_1}}{x-r_k} + \frac{a_{k_0}}{(x-r_k)^2} \right) < 0 \quad \text{(resp., > 0)}
\]

for all \( k \in [n] \). Hence \( \Im \left( \sum_{k=1}^{n} t_k \left[ a_{k_3}(x-r_k) + a_{k_2} + \frac{a_{k_1}}{x-r_k} + \frac{a_{k_0}}{(x-r_k)^2} \right] \right) \neq 0 \) for \( \Re(x) > 0 \) and \( \Im(x) \neq 0 \).

If \( \Re(x) > 0 \) and \( \Im(x) = 0 \), then, by hypothesis, we have that all signs of

\[
a_{k_3}(x-r_k) + a_{k_2} + \frac{a_{k_1}}{x-r_k} + \frac{a_{k_0}}{(x-r_k)^2}
\]

are same for all \( k \in [n] \). In consequence, \( \sum_{k=1}^{n} t_k \left[ a_{k_3}(x-r_k) + a_{k_2} + \frac{a_{k_1}}{x-r_k} + \frac{a_{k_0}}{(x-r_k)^2} \right] \neq 0 \).

Hence, from above two cases, we get that \( J_n(P; x) \) is nonzero when \( \Re(x) > 0 \). Namely \( J_n(P; x) \) is Hurwitz stable for each \( n \).

\[\square\]
Remark 3.3. Obviously, the conclusion for $I_n(P; x)$ in Theorem 3.2 can be extended to that $I_n(P; x + z_n)$ is Hurwitz stable if $z_n$ is not less than the largest zero of $P_n(x)$ for all nonnegative integers $n$.

3.1 The generalized Eulerian polynomials

Fisk showed that the Turán expressions of Eulerian polynomials are Hurwitz stable in his unfinished book (see [31, Lemma 21.91]), but his proof is incorrect. In [80], Zhu again proved the Hurwitz stability of Eulerian polynomials. And here, we will give a generalized result.

For $r \geq 1$, Riordan [64] defined the $r$-Eulerian polynomial

$$E_{n,r}(x) = \sum_{\pi \in S_n} x^{\text{exc}_r(\pi)},$$

where $\text{exc}_r(\pi)$, the number of $r$-excedances of $\pi$, is defined by

$$\text{exc}_r(\pi) = |\{i \in [n] : \pi_i \geq i + r\}|.$$

And then, Riordan [64, p. 214] got the following recurrence relation:

$$E_{n,r}(x) = [(n-r)x + r]E_{n-1,r}(x) + x(1-x)D_xE_{n-1,r}(x), \quad (3.6)$$

where $E_{r,r}(x) = r!$ for $n \geq r$. Note that whenever $r = 1$, $E_{n,1}(x)$ is the classical Eulerian polynomial.

In addition, to study the volume of the usual permutohedron, Postnikov [61] introduced the mixed Eulerian numbers $A_{a_1,\ldots,a_r}$, where $a_i \geq 0$ and $a_1 + \cdots + a_r = r$. In terms of mixed Eulerian numbers, Berget et al. [7] defined the polynomial

$$A_{a_1,\ldots,a_r}(x) := \sum_{i=0}^{n-r} A_{0^{a_r},a_1,\ldots,a_{r-1},0^{n-r-i}} x^i$$

for $a_i \geq 1$ and $a_1 + \cdots + a_r = n$. And then, they gave the recurrence relation of the polynomial $A_{a_1,\ldots,a_r}(x)$ as follows:

$$A_{a_1,\ldots,a_r+1}(x) = [(n-r+1)x + r]A_{a_1,\ldots,a_r}(x) + x(1-x)D_xA_{a_1,\ldots,a_r}(x). \quad (3.7)$$

Note that $A_{a_1,\ldots,a_r}(x)$ is the $r$-Eulerian polynomial $E_{n,r}(x)$ whenever $a_i = 1$ for $i \in [r-1]$ and $a_r = n - r + 1$. In particular, it follows from the recurrence relation of $A_{a_1,\ldots,a_r}(x)$ that $A_{a_1,\ldots,a_r}(1) = n!$ which was conjectured by Stanley and proved by Postnikov (see [61, Theorem 16.4]). On the other hand, by using the method of zeros interlacing, it is easy to know that $A_{a_1,\ldots,a_r}(x)$ has only non-positive zeros, moreover, zeros of $A_{a_1,\ldots,a_r}(x)$ interlace those of $A_{a_1,\ldots,a_r+1}(x)$. Thus, the coefficients of $A_{a_1,\ldots,a_r}(x)$ are unimodal and log-concave.

In addition, we define the following polynomial

$$J_{n,r}^{a_1,\ldots,a_r}(x) := \frac{x^{n-r}A_{a_1,\ldots,a_r}(1/x)}{r!}. \quad (3.8)$$
Combining (3.7) and (3.8), it is easy to know that $\mathcal{J}_{n,r}^{a_1,\ldots,a_r}(x)$ satisfies the following relation

$$
\mathcal{J}_{n,r}^{a_1,\ldots,a_r}(x) = [(n-1)x + 1]\mathcal{J}_{n-1,r}^{a_1,\ldots,a_r}(x) + x(1-x)D_x\mathcal{J}_{n-1,r}^{a_1,\ldots,a_r}(x),
$$

where $\mathcal{J}_{n,r}^{a_1,\ldots,a_r}(1) = n!/r!$.

Let $\mathcal{J}_{n,r}$ be the set of injections $\pi : [n - r] \to [n]$. Based on the set of images of $\pi$, define the polynomial

$$
\mathcal{J}_{n,r}(x) = \sum_{\pi \in \mathcal{J}_{n,r}} x^{\text{exc}(\pi)},
$$

where $\text{exc}(\pi) = \text{exc}_1(\pi)$. Then, a relation between $E_{n,r}(x)$ and $\mathcal{J}_{n,r}(x)$ was proved in [64] as follows:

$$
\mathcal{J}_{n,r}(x) = \frac{x^{n-r}E_{n,r}(1/x)}{r!}.
$$

Combining (3.6) and (3.10), Elizalde [30] gave

$$
\mathcal{J}_{n,r}(x) = [(n-1)x + 1]\mathcal{J}_{n-1,r}(x) + x(1-x)D_x\mathcal{J}_{n-1,r}(x)
$$

for $n > r$, where $\mathcal{J}_{r,r}(x) = 1$. Note that $\mathcal{J}_{n,r}(x) = \mathcal{J}_{n-1,1}^{1,\ldots,n-r+1}(x)$. This recurrence relation is the same as that of the classical Eulerian polynomials, but the initial condition is different. For $r \in \{2, 3, 4, 5\}$, the reader can be referred to [68, A144696-A144699]. Obviously, $\mathcal{J}_{n+r,r}(x)$ is a special case of the generalized Eulerian polynomial $\mathcal{T}_n(x)$ in (2.14) by taking $d = 0$ and $\lambda = 1$.

Archer et.al [3] introduced the quasi-Stirling permutations $\bar{Q}_n$, which is a set of $\pi = \pi_1 \pi_2 \cdots \pi_n$ in the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ avoiding 1212 and 2121, i.e., there does not exist $i < j < k < \ell$ such that $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$ for any $\pi \in \bar{Q}_n$. Elizalde [30] defined the quasi-Stirling polynomial

$$
\bar{Q}_n(x) = \sum_{\pi \in \bar{Q}_n} x^{\text{des}(\pi)}
$$

and he got $\bar{Q}_n(x) = \mathcal{J}_{2n,n+1}(x)$.

These different kinds of Eulerian polynomials can be obtained by the transformation of the special cases of the generalized Eulerian polynomial $\mathcal{T}_n(x)$ in (2.14) by taking $d = 0, \lambda = 1$ and different initial conditions. Applying Theorem 3.2 to the generalized Eulerian polynomial $\mathcal{T}_n(x)$, we get the following result proved by Zhu [82].

**Corollary 3.4.** [82, Theorem 2.16] Let $\mathcal{T} = (\mathcal{T}_n(x))_{n \geq 0}$, where $\mathcal{T}_n(x)$ is the $n$-row generating function of the generalized Eulerian triangle in (2.13). If $\{a_1, b_1, \lambda\} \subseteq \mathbb{R}^{>0}$ and $\{a_2, b_2, d\} \subseteq \mathbb{R}^{\geq 0}$ with $a_2 + b_2 > 0$, then $\mathcal{T}_n(\mathcal{T}; x)$ is Hurwitz stable for all $n$.

**Proof.** It was proved for $n \in \mathbb{N}$ that $\mathcal{T}_n(x) \ll \mathcal{T}_{n+1}(x)$ and all zeros of $\mathcal{T}_n(x)$ are in $[-\lambda/d, 0]$ in [82]. By the recurrence relation (2.14), we derive the corresponding (3.2) as follows:

$$(x - r_k)[p_n(x) - p_{n-1}(x)] + q(x) = \frac{x}{\lambda}(dx + \lambda)[(\lambda + dr_k)a_1 - b_1 r_k].$$

We take $h_n(x) = x(dx + \lambda)/\lambda$ and $a_{k_0} = (\lambda + dr_k)a_1 - b_1 r_k$. By the assumption conditions and $r_k \in [-\lambda/d, 0]$, then the desired result is immediate by Theorem 3.2.
Remark 3.5. By (3.10), the Turán expressions of polynomial sequence \((E_{n+r,r}(x))_{n\geq 0}\) are also Hurwitz stable. So are those of \((A_{a_1,\ldots,a_r+n}(x))_{n\geq 0}\) since \(\mathcal{J}_{a_1,\ldots,a_r}(x)\) is the special case of the generalized Eulerian polynomial \(\mathcal{E}_n(x)\) in (2.14) by taking \(d = 0\) and \(\lambda = 1\).

Note that the exponential generating function of the mixed Eulerian numbers \(A_{a_1,\ldots,a_r}\) is the volume \(\text{Vol} P_{r+1}\) of a permutohedron \(P_{r+1}\) (see [61, Section 16] for details). In fact, \(\text{Vol} P_{r+1}\) is a Lorentzian polynomial by using the conclusion in [19]. Thus, \(\text{Vol} P_{r+1}\) has the corresponding properties, such as the M-convexity of \(\text{supp}(\text{Vol} P_{r+1})\) and discrete log-concavity of the mixed Eulerian numbers \(A_{a_1,\ldots,a_r}\).

3.2 The generalized Bell polynomials

For the Bell polynomial \(B_n(x)\), it satisfies the recurrence relation

\[
B_{n+1}(x) = xB_n(x) + xD_xB_n(x), \quad \text{where} \quad B_0(x) = 1.
\]

Fisk showed that Bell polynomials are Hurwitz stable in his unfinished book (see [31, Lemma 21.92]). But Chasse et al. pointed out that Fisk’s proof is incorrect and reproved that of Bell polynomials in [26]. In fact, Chasse et al. proved the Hurwitz stability of Turán expression for the generalized Bell polynomials in the following result, which follows from Theorem 3.2.

Corollary 3.6. [26, Theorem 1.1] Let \(\mathcal{B} = (\mathcal{B}_n(x))_{n\geq 0}\) be a real polynomial sequence with \(\deg(\mathcal{B}_n(x)) = n\). If \(\mathcal{B}_n(x)\) satisfies

\[
\mathcal{B}_{n+1}(x) = a(x + b)(c_n + D_x)\mathcal{B}_n(x), \tag{3.11}
\]

where \(a \neq 0, b \geq 0\) and \(c_{n+1} \geq c_n > 0\) for all \(n \in \mathbb{N}\), then \(\mathcal{J}_n(\mathcal{B}; x - b)\) is Hurwitz stable for all \(n \in \mathbb{N}\).

Proof. Without loss of generality, we assume \(a > 0\) and \(\mathcal{B}_0(x) > 0\). Obviously (3.11) implies that all coefficients of \(\mathcal{B}_n(x)\) are real and nonnegative for \(n \in \mathbb{N}\). By induction on \(n\), we can show that \(\mathcal{B}_n(x)\) is real-rooted with all zeros \(r_k \leq -b\) for \(k \in [n]\) and \(\mathcal{B}_n(x) \ll \mathcal{B}_{n+1}(x)\) for all \(n \in \mathbb{N}\).

Then the corresponding (3.2) for \(\mathcal{B}_n(x)\) is

\[
(x - r_k)(p_n(x) - p_{n-1}(x)) + q(x) = a(x + b) \left[ (c_n - c_{n-1})(x - r_k) + 1 \right].
\]

We can take \(h_n(x) = a(x + b), a_{k_0} = 1\) and \(a_{k_1} = c_n - c_{n-1}\). Hence \(\mathcal{J}_n(\mathcal{B}; x)\) is Hurwitz stable by Theorem 3.2. So is \(\mathcal{J}_n(\mathcal{B}; x - b)\) by Remark 3.3. \(\square\)

3.3 The Stirling-Whitney-Riordan polynomials

For the Turán expression of the row-generating function of the Stirling-Whitney-Riordan triangle (2.11), Zhu [83] proved the following result concerning its Hurwitz stability. It can also be looked as a corollary of Theorem 3.2.

Corollary 3.7. [83, Theorem 3.2] Let \(\mathcal{S} = (\mathcal{J}_n(x))_{n\geq 0}\), where \(\mathcal{J}_n(x)\) is the \(n\)-th row-generating function of the Stirling-Whitney-Riordan triangle in (2.11). If \(\{\lambda, a_1, a_2, b_1, b_2\} \subseteq \mathbb{R}_{\geq 0}\) and \(a_1(b_1 + b_2) \geq a_2b_1\), then \(\mathcal{J}_n(\mathcal{S}; x - \lambda)\) is Hurwitz stable for all \(n\).
Proof. Note that it was proved in [83] that all zeros of $\mathcal{S}_n(x)$ are in $(-\lambda - a_1/b_1, -\lambda)$ and $\mathcal{S}_{n-1}(x) \ll \mathcal{S}_n(x)$ for all $n \in \mathbb{N}$. Thus, by Remark 3.3, it suffices to show that $\mathcal{I}_n(S; x)$ is Hurwitz stable for all $n$.

By (2.12), we get the corresponding (3.2) for $\mathcal{S}_n(x)$ as follows:

$$(x - r_k)(p_n(x) - p_{n-1}(x)) + q(x) = (x + \lambda) [a_1 + b_1(x + \lambda)].$$

By taking $h_n(x) = (x + \lambda) [a_1 + b_1(x + \lambda)]$ and $a_{k_0} = 1$, the desired result concerning Hurwitz stability is immediate by Theorem 3.2.

There exists some combinatorial polynomials such that Corollaries 3.4, 3.6 and 3.7 can not be used. But our Theorem 3.2 is still valid. Some such examples are given in the following.

### 3.4 Alternating runs of type $A$

We say that $\pi \in S_n$ changes direction at position $i$ if either $\pi_{i-1} < \pi_i > \pi_{i+1}$ or $\pi_{i-1} > \pi_i < \pi_{i+1}$ for $i \in \{2, \ldots, n-1\}$. Let $R(n, k)$ be the number of $\pi \in S_n$ having $k$ alternating runs, namely there are $k - 1$ indices $i$ such that $\pi$ changes direction at these positions. For example, let $\pi = 31264875$ and its alternating runs are 312, 264, 648. André [2] gave the recurrence relation as follows:

$$R(n, k) = kR(n-1, k) + 2R(n-1, k-1) + (n-k)R(n-1, k-2)$$

(3.12)

for $n, k \geq 1$, where $R(1, 0) = 1$ and $R(1, k) = 0$ for $k \geq 1$. Let the row-generating function $R_n(x) = \sum_{k=0}^{n} R(n, k)x^k$. Then the recurrence relation (3.12) implies

$$R_{n+2}(x) = x(nx + 2)R_{n+1}(x) + x(1 - x^2)D_x R_{n+1}(x)$$

with $R_1(x) = 1$ and $R_2(x) = 2x$. Zhu [80] proved the q-log-convexity of $R_n(q)$, which is immediate from the following stronger result.

**Proposition 3.8.** The Turán expressions of $(R_n(x))_{n \geq 0}$ are Hurwitz stable.

**Proof.** We know that all zeros of $R_n(x)$ are in $[-1, 0]$ and $R_n(x) \ll R_{n+1}(x)$ (see Ma and Wang [56] for the details). Then the corresponding (3.2) for $R_n(x)$ is

$$(x - r_k)(p_n(x) - p_{n-1}(x)) + q(x) = x[-r_k(x - r_k) + 1 - r_k^2].$$

Taking $h_n(x) = x, a_{k_0} = 1 - r_k^2$ and $a_{k_1} = -r_k$. The desired result follows from Theorem 3.2 since $r_k \in [-1, 0]$.

### 3.5 The longest alternating subsequences and up-down runs

Let $\tilde{\pi} = \pi_{i_1} \cdots \pi_{i_k}$ be a subsequence of $\pi \in S_n$. We say $\tilde{\pi}$ is an alternating subsequence of $\pi$ if $\tilde{\pi}$ satisfies

$$\pi_{i_1} > \pi_{i_2} < \pi_{i_3} > \cdots \pi_{i_k}.$$
Denote by $a(n, k)$ the number of $\pi \in S_n$, where the length of the longest alternating subsequence of $\pi$ is $k$. Bóna [10, Section 1.3.2] showed that the row-generating function $t_n(x) = \sum_{k=0}^{n} a(n, k)x^k$ satisfies the following identity:

$$t_n(x) = \frac{1}{2}(1 + x)R_n(x)$$

for $n \geq 2$. In addition, $t_0(x) = 1$ and $t_1(x) = x$.

Note that $t_n(x)$ coincides with the up-down runs polynomial, see [68, A186370]. In addition, $t_n(x)$ is closely related to two kinds of peak polynomials $W_n(x)$ and $\tilde{W}_n(x)$, which are defined by

$$W_n(x) = \sum_{\pi \in S_n} x^{lpk(\pi)} = \sum_{k \geq 0} W_{n,k}x^k,$$

$$\tilde{W}_n(x) = \sum_{\pi \in S_n} x^{lpk(\pi)} = \sum_{k \geq 0} \tilde{W}_{n,k}x^k,$$

where $W_1(x) = 1$, $\tilde{W}_0(x) = 1$ and $lpk(\pi)$ and $lpk(\pi)$ denote the number of interior peaks and left peaks of $\pi \in S_n$, respectively, see Petersen [60], Stembridge [71] and [68, A008303, A008971] for instance.

Based on these, Ma [50] defined the polynomials $M_n(x)$ by

$$M_n(x) = xW_n(x^2) + \tilde{W}_n(x^2),$$

where $M_1(x) = 1 + x$. In fact, the coefficients of $M_n(x)$ arise in expansion of $n$-th derivative of $\tan(x) + \sec(x)$, see [68, A198895]. It is known that $M_n(x)$ satisfies the recurrence relation

$$M_{n+1}(x) = (nx^2 + 1)M_n(x) + x(1 - x^2)D_xM_n(x).$$

It was shown that all zeros of $M_n(x)$ are in $[-1, 0]$ and $M_n(x) \ll M_{n+1}(x)$ in [50]. Thus, by Proposition 3.8 or Theorem 3.2, we immediately have the following result, which in particular implies $q$-log-convexity of $(t_n(q))_{n \geq 0}$ and $(M_n(q))_{n \geq 0}$ [80].

**Proposition 3.9.** The Turán expressions of $(t_n(x))_{n \geq 0}$ and $(M_n(x))_{n \geq 0}$ are both Hurwitz stable.

### 3.6 Alternating runs of type B

Now, we consider the alternating runs of type $B$. Let $B_n$ be all signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all $i \in [n]$, where $\pm[n] = \{\pm1, \pm2, \ldots, \pm n\}$. We say that $\pi \in B_n$ is a alternating run if $\pi_{i-1} < \pi_i > \pi_{i+1}$ or $\pi_{i-1} > \pi_i < \pi_{i+1}$ for $i \in [n-1]$ in the order $\cdots < \bar{2} < \bar{1} < 0 < 1 < 2 < \cdots$, where $\pi_0 = 0$. Taking the subset $B^n_n \subseteq B_n$, which satisfies $\pi_1 > 0$ whenever $\pi \in B^n_n$. We call $B^n_n$ the up signed permutations. For example, taking $\pi = 31264875$, whose alternating runs is $\{31, 264, 487, 875\}$. Let $Z(n, k)$ denote the number of up signed permutations $\pi \in B^n_n$ having $k - 1$ alternating runs. Zhao [78] got the following recurrence relation:

$$Z(n, k) = (2k - 1)Z(n - 1, k) + 3Z(n - 1, k - 1) + (2n - 2k + 2)Z(n - 1, k - 2)$$

(3.15)
for \( n \geq 2 \) and \( k \in [n] \), where \( Z(1, 1) = 1 \) and \( Z(1, k) = 0 \) for \( k > 1 \).

Let the row-generating function \( Z_n(x) = \sum_{k=1}^{n} Z(n, k)x^k \). Then the recurrence relation (3.15) implies

\[
Z_n(x) = [(2n - 2)x^2 + 3x - 1]Z_{n-1}(x) + 2x(1 - x^2)D_xZ_{n-1}(x),
\]

where \( Z_1(x) = x \) and \( Z_2(x) = x + 3x^2 \). It was proved in [80] that \( (Z_n(q))_{n \geq 1} \) is \( q \)-log-convex, which is also immediate from the following stronger result.

**Proposition 3.10.** The Turán expressions of \( (Z_n(x))_{n \geq 1} \) are Hurwitz stable.

The proof is similar to that of Proposition 3.8, thus we omit it for brevity.

### 4 Semi-\( \gamma \)-positivity and Hurwitz stability

The location of zeros of polynomials implies much information. For example, the well-known Newton inequalities say that if all zeros of a polynomial are real and nonpositive, then its coefficients are log-concave and unimodal. Moreover, Brändén in [15] proved that if all zeros of a symmetric polynomial are real and nonpositive, then the polynomial has \( \gamma \)-positivity. In this section, we will demonstrate a similar result concerning Hurwitz stability and semi-\( \gamma \)-positivity.

For \( f(x) = \sum_{k=0}^{n} f_k x^k \in \mathbb{R}[x] \), we say \( f(x) \) is unimodal if there exists \( m \) such that \( f_0 \leq f_1 \leq \cdots \leq f_m \geq \cdots \geq f_n \) and is symmetric if \( f_k = f_{n-k} \) for \( 0 \leq k \leq n \). Clearly, \( f(x) \) is symmetric if and only if \( f(x) = x^n f(1/x) \). We know that any symmetric polynomial \( f(x) \) has the following decomposition:

\[
f(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} g_k x^k (1 + x)^{n-2k}.
\]

If \( g_k \geq 0 \) for all \( 0 \leq k \leq n \), then we say that \( f(x) \) is \( \gamma \)-positive. In particular, \( \gamma \)-positivity implies unimodality. Furthermore, in terms of parity of \( n \), one can write \( f(x) \) as

\[
f(x) = (1 + x)^{\chi(n \bmod 2)} \sum_{k=0}^{\lfloor n/2 \rfloor} g_k x^k (1 + x^2)^{\lfloor n/2 \rfloor - k}.
\]

Based on these, Ma et al. [53] introduced the following semi-\( \gamma \)-positivity.

**Definition 4.1.** Let \( \nu = 0 \) or 1. If a polynomial

\[
f(x) = (1 + x)^{\nu} \sum_{k=0}^{n} g_k x^k (1 + x^2)^{n-k}
\]

and \( g_k \geq 0 \) for all \( 0 \leq k \leq n \), then we say that \( f(x) \) is semi-\( \gamma \)-positive.

Corresponding to \( f(x) \), define a polynomial \( g(x) \) by

\[
g(x) = \sum_{k=0}^{n} g_k x^k.
\]

In order to show the \( \gamma \)-positivity of \( f(x) \), it is a useful approach to verifying whether all zeros of \( g(x) \) are nonpositive. The reason is from the next result (see [34, Remark 3.1.1])
Proposition 4.2. Let \( f(x) \in \mathbb{R}[x] \) with symmetric coefficients. Then \( f(x) \) has nonnegative coefficients and only real zeros if and only if so does \( g(x) \).

For a symmetric polynomial with nonnegative coefficients, in analogy to this relation between \( \gamma \)-positivity and real-rootedness, we give a criterion for semi-\( \gamma \)-positivity and Hurwitz stability as follows.

Theorem 4.3. Let \( f(x) \) and \( g(x) \) be defined as (4.1) and (4.2), respectively. Then \( f(x) \) is Hurwitz stable if and only if so is \( g(x) \). In particular, if \( f(x) \) is Hurwitz stable and its leading coefficient is positive, then \( f(x) \) is semi-\( \gamma \)-positive.

Proof. By (4.1), we have

\[
    f(x) = (1 + x)^n (1 + x^2) g \left( \frac{x}{1 + x^2} \right) = (1 + x)^n (1 + x^2) g \left( \frac{1}{x + \frac{1}{x}} \right).
\]

Let \( z = \frac{1}{x + \frac{1}{x}} \). Obviously, \( \Re(z) \Re(x) > 0 \). In consequence, we immediately get that the Hurwitz stability of \( f(x) \) is equivalent to that of \( g(x) \).

In particular, if \( f(x) \) is Hurwitz stable and its leading coefficient is positive, then so is \( g(x) \). Thus \( g_k \geq 0 \) for all \( k \). That is to say that \( f(x) \) is semi-\( \gamma \)-positive. \( \square \)

Generally speaking, \( \gamma \)-positivity is stronger than semi-\( \gamma \)-positivity. Thus, for a symmetric polynomial \( f(x) \), it may have the semi-\( \gamma \)-positivity when it is not \( \gamma \)-positive. Some such examples will be arranged as follows.

4.1 Alternating runs of Stirling permutations

For the generating function \( \mathcal{R}_n(x) \) in (2.15) of the number of a dual set of Stirling permutations of order \( n \) with \( k \) alternating runs, Ma et al. [53, Theorem 19] proved the following result concerning semi-\( \gamma \)-positivity by using context-free grammars. Obviously, it is immediate from our Proposition 2.16 and Theorem 4.3.

Corollary 4.4. The polynomial \( \mathcal{R}_n(x) \) is semi-\( \gamma \)-positive.

4.2 A class of symmetric polynomials

Recall (2.3) as follows:

\[
    T_{n+1}(x) = (\alpha x^2 + \beta x + \gamma)T_n(x) + (\mu x^3 + \nu x^2 + \varphi x + \psi)D_x T_n(x).
\]

It is nature to consider the question when is the polynomial \( T_n(x) \) symmetric. Note the fact that a symmetric polynomial \( f(x) \) with degree \( n \) has the following relation:

\[
    x^{n+1}D_x f \left( \frac{1}{x} \right) = -nf(x) + xD_x f(x). \tag{4.3}
\]

With the help of (4.3), we obtain a class of symmetric polynomial \( T_n(x) \) satisfying

\[
    T_{n+1}(x) = (-m_n \mu x^2 + \beta x + \beta + m_n \nu)T_n(x) + (\mu x^3 + \nu x^2 - \nu x - \mu)D_x T_n(x), \tag{4.4}
\]
where \( \deg(T_n(x)) = m_n \) and \( \deg(T_{n+1}(x)) = \deg(T_n(x)) + 1 \). In the subsection, we assume \( \mu \leq 0 \leq \nu \).

In what follows, we will prove that \( T_n(x) \) in (4.4) is Hurwitz stable and semi-\( \gamma \)-positive. Before it, we need one criterion for real stability of polynomials.

For multivariate polynomials with real coefficients of degree at most one, Brändén gave a criterion about their real stability (see [17, Theorem 5.6]). Furthermore, Leake [44] extended it to general polynomials with real coefficients by Walsh’s coincidence Theorem (see [62, Theorem 3.4.1.b]). We state it as follows.

**Lemma 4.5.** Let \( f \in \mathbb{R}^k[X] \). Then \( f \) is real stable if and only if for all \( i \neq j \) we have

\[
\Delta_{x_i,x_j} = D_{x_i}f \cdot D_{x_j}f - f \cdot D_{x_i}D_{x_j}f \geq 0
\]

and for all \( i \) we have

\[
\Delta_{x_i,x_i} = (1 - k_i^{-1})(D_{x_i}f)^2 - f \cdot D_{x_i}^2f \geq 0
\]

everywhere in \( \mathbb{R}^n \), where \( k_i \) is the degree of \( x_i \) in \( f \).

Now, we give the result for Hurwitz stability of \( T_n(x) \) as follows.

**Theorem 4.6.** Let \( T_n(x) \) be defined by (4.4) and \( \deg(T_n(x)) = m_n \). Assume that \( T_0(x) \) is Hurwitz stable. If \( \mu + \nu \leq 0 \) and \( 2\beta + m_n(\mu + \nu) \geq 0 \), then \( T_n(x) \) is Hurwitz stable and semi-\( \gamma \)-positive.

**Proof.** We will prove that \( T_n(x) \) is Hurwitz stable by induction on \( n \), by Theorem 4.3, which implies that \( T_n(x) \) is semi-\( \gamma \)-positive. Let

\[
T := (-m_n\mu x^2 + \beta x + \beta + m_n \nu)I + (\mu x^3 + \nu x^2 - \nu x - \mu)D_x.
\]

We only need to prove that \( T \) preserves Hurwitz stability. By Theorem 2.14, it is equivalent to prove that the following polynomial

\[
T(1 + xy)^{m_n} = (1 + xy)^{m_n-1}(-m_n\mu x^2 + \beta x + \beta + m_n \nu)(1 + xy) + m_n(\mu x^3 + \nu x^2 - \nu x - \mu)y
\]

\[
= -m_n\mu(1 + xy)^{m_n-1} \left\{ \left( x^2 - \frac{\beta}{m_n\mu} x - \frac{\beta}{m_n\mu} - \frac{\nu}{\mu} \right) (1 + xy) + (1 - x) \left[ 1 + (1 + \frac{\nu}{\mu})x + x^2 \right] y \right\}
\]

\[
= \left\{ \left[ (1 - x)^2 + \frac{3\mu + \nu}{\mu - \nu} (1 + x)^2 \right] (1 + y) - \frac{4[\beta + m_n(\mu + \nu)]}{m_n(\mu - \nu)}(1 + x)(1 + xy) \right\} \times
\]

\[
\frac{m_n(\nu - \mu)(1 + xy)^{m_n-1}}{4}
\]

is Hurwitz stable. This is immediate from the next claim.

**Claim 1.** For any \( r \geq 1 \) and \( r + s \geq 1 \), the bivariate polynomial

\[
[(1 - x)^2 + r(1 + x)^2](1 + y) + s(1 + x)(1 + xy)
\]

is Hurwitz stable.
Proof. Let \( x = -ix, y = -iy \). That is equivalent to show that the right hand side of the below equality

\[
[(1 + ix)^2 + r(1 - ix)^2](1 - iy) + s(1 - ix)(1 - xy)
= (r + 1)(1 - x^2) - 2(r - 1)xy + s(1 - xy) - [(r - 3)x + (r + 1)y + (r + s + 1)x(1 - xy)]i
\]

is stable. By Theorem 2.1, it is enough to prove that

\[
(r + 1)(1 - x^2) - 2(r - 1)xy + s(1 - xy) - [(r - 3)x + (r + 1)y + (r + s + 1)x(1 - xy)]z
\]

is real stable. By computing, we get

\[
\Delta_{xy} = s(r + s + 1)(1 - xz)^2 + s(r + 1)(x - z)^2 + 2(r - 1)(r + s + 1)(1 + x^2 z^2) + 2(r + 1)(x^2 + z^2) \geq 0,
\]

\[
\Delta_{xz} = s(r + s + 1)(1 - xy)^2 + s(r + 1)(x - y)^2 + 2(r - 1)(r + s + 1)(1 + x^2 y^2) + 2(r + 1)(x^2 + y^2) \geq 0,
\]

\[
\Delta_{yz} = (r + 1)(r + s + 1)(1 - x^2)^2 + 4(r - 1)(r + s - 1)x^2 \geq 0,
\]

\[
\Delta_{xx} = 2(2r + s - 2)(y^2 + z^2) + 8(r + 1)(r + s + 1)(1 + y^2 z^2) - 4(s^2 + 8s + 16r)yz \\
\geq 32(r - 1)(r + s - 1)|yz| \\
\geq 0
\]

for any \( x, y, z \in \mathbb{R} \) and \( r \geq 1, r + s \geq 1 \). Hence, according to Lemma 4.5, which confirms the claim.

Therefore, we complete the proof. \( \square \)

A polynomial sequence \((f_n(q))_{n \geq 0}\) is called strongly q-log-convex if

\[
f_{n+1}(q)f_{m-1}(q) - f_n(q)f_m(q)
\]

has only nonnegative coefficients for any \( n \geq m \geq 1 \). See [28, 79] for the details concerning the development of strong q-log-convexity. In the following context, we assume \( \delta \in \mathbb{N}^+ \).

**Theorem 4.7.** Let \( T_n(x) \) be defined by (4.4), where all \( \beta, \mu, \nu \) are real numbers and \( m_n = n - \delta + 1 \). If \( \mu + \nu = 0 \), then we have

(i) its exponential generating function is

\[
\sum_{n \geq 0} T_{n+\delta-1}(x) \frac{t^n}{n!} = \frac{(1 - x)^{3/\nu}}{[(1 - x) \cos(\nu(1 - x)t) - (1 + x) \sin(\nu(1 - x)t)]^{3/\nu}},
\]

(ii) its ordinary generating function has the Jacobi continued fraction expansion

\[
\sum_{n=0}^\infty T_{n+\delta-1}(x)t^n = \frac{1}{1 - r_0 t - \frac{s_1 t^2}{1 - r_1 t - \frac{s_2 t^2}{1 - r_2 t - \ldots}}},
\]

where \( r_i = (2\nu i + \beta)(1 + x) \) and \( s_i = 2\nu i[\beta + \nu(i - 1)](1 + x^2) \) for \( i \geq 0 \);
(iii) the polynomial sequence \((T_n(q))_{n \geq 0}\) is strongly \(q\)-log-convex for \(\beta \geq 0\);

(iv) the polynomial \(T_n(x)\) is not \(\gamma\)-positive for \(n \geq \delta + 2\) and \(\beta \geq 0\).

Proof. For (i), define a polynomial \(g_n(x)\) for \(n \geq 0\) by the following relation:

\[
g_n(x) := \frac{\delta^n}{2^n} (1 + x)^n T_{n+\delta-1} \left( \frac{x - 1}{x + 1} \right).
\]

By (4.4), then we have a recurrence relation for \(g_n(x)\) as follows:

\[
g_{n+1}(x) = \beta \delta x g_n(x) + \nu \delta (1 + x^2) D_x g_n(x).
\]

Our aim is to get the exponential generating function of \(g_n(x)\). We first have the following general result.

Claim 2. Let \(\{r, s\} \subseteq \mathbb{R}\) and \(\{u, v\} \subseteq \mathbb{R}^\geq 0\). Assume that a polynomial sequence \((f_n(x))_{n \geq 0}\) satisfies the following recurrence relation:

\[
f_{n+1}(x) = rsxf_n(x) - s(u + vx^2)D_x f_n(x),
\]

where \(f_0(x) = 1\). Then the exponential generating function of \(f_n(x)\) is

\[
\sum_{n \geq 0} f_n(x) \frac{t^n}{n!} = \left[ \cos(s\sqrt{uv}t) + \sqrt{v/ux} \sin(s\sqrt{uv}t) \right]^{(r/v)}.
\]

Proof. Let the exponential generating function

\[
\mathcal{F}(x, t) := \sum_{n \geq 0} f_n(x) \frac{t^n}{n!}.
\]

Then, by (4.7), we have the next partial differential equation:

\[
\mathcal{F}_t(x, t) = rsx \mathcal{F}(x, t) - s(u + vx^2) \mathcal{F}_x(x, t)
\]

with the initial condition \(\mathcal{F}(x, 0) = 1\). It is routine to check that

\[
\mathcal{F}(x, t) = \left[ \cos(st\sqrt{uv}) + \sqrt{v/ux} \sin(st\sqrt{uv}) \right]^{(r/v)}
\]

is a solution of (4.8) with the initial condition. \(\square\)

In consequence, taking \(r = -\beta \delta, s = -1\) and \(u = v = \nu \delta\) in (4.7), then we have the exponential generating function of \(g_n(x)\):

\[
\sum_{n \geq 0} g_n(x) \frac{t^n}{n!} = \frac{1}{\cos(\nu \delta t) - x \sin(\nu \delta t)}^{\beta/v}.
\]

In addition, it follows from (4.5) that we have

\[
T_{n+\delta-1}(x) = \frac{(1 - x)^n}{\delta^n} g_n \left( \frac{1 + x}{1 - x} \right).
\]
Combining (4.9) and (4.10) gives (i).

For (ii), if let

\[ T_{n+\delta-1}(x) = (1 + x)^n h_n \left( \frac{-2x}{(1 + x)^2} \right) \tag{4.11} \]

for \( n \geq 0 \), then combining (4.4) and (4.11) derives the recurrence relation of \( h_n(x) \) as follows:

\[ h_{n+1}(x) = [2n\nu(x + 1) + \beta]h_n(x) - 2\nu(x + 1)(2x + 1)D_x h_n(x). \]

Let

\[ S_n(x) = h_n(x - 1), \tag{4.12} \]

where \( \deg(S_n(x)) = \lfloor n/2 \rfloor \). Then \( S_n(x) \) satisfies the following recurrence relation:

\[ S_n(x) = [2(n - 1)\nu x + \beta]S_{n-1}(x) - 2\nu(1 - 2x)D_x S_{n-1}(x). \]

That is to say, the coefficients \( S_{n,k} \) of \( S_n(x) \) satisfy

\[ S_{n,k} = (2\nu k + \beta)S_{n-1,k} + 2\nu(n - 2k + 1)S_{n-1,k-1}, \tag{4.13} \]

where \( S_{n,k} = 0 \) unless \( 0 \leq k \leq n \) with initial conditions \( S_{0,0} = 1 \). Then by [84, (4.10)] we have the Jacobi continued fraction expansion

\[ \sum_{n=0}^{\infty} S_n(x) t^n = \frac{1}{1 - s_0 t^2}, \tag{4.14} \]

where \( r_i = 2\nu i + \beta \) and \( s_i = 2\nu [\nu(i - 1) + \beta]x \) for \( i \geq 0 \).

Then by taking \( x \to 1 + x \) in (4.14), we get

\[ \sum_{n=0}^{\infty} h_n(x) t^n = \frac{1}{1 - s_0 t^2}, \tag{4.15} \]

where \( r_i = 2\nu i + \beta \) and \( s_i = 2\nu [\nu(i - 1) + \beta]x \) for \( i \geq 0 \). Moreover, by taking \( t \to (1 + x)t \) and \( x \to \frac{-2x}{(1 + x)^2} \) in (4.15), then we get

\[ \sum_{n=0}^{\infty} T_{n+\delta-1}(x) t^n = \frac{1}{1 - s_0 t^2}, \]

where \( r_i = (2\nu i + \beta)(1 + x) \) and \( s_i = 2\nu [\nu(i - 1) + \beta](1 + x^2) \) for \( i \geq 0 \).

For (iii), note the following criterion for the strong \( q \)-log-convexity [79]:
Let
\[
\sum_{n=0}^{\infty} F_n(q)t^n = \frac{1}{1 - r_0(q)t - \frac{s_1(q)t^2}{1 - r_1(q)t - \frac{s_2(q)t^2}{1 - r_2(q)t - \ldots}}},
\]
where both \(r_n(q)\) and \(s_{n+1}(q)\) are polynomials with nonnegative coefficients for \(n \geq 0\). If all coefficients of \(r_i(q) - s_{i+1}(q)\) are nonnegative for all \(i \geq 0\), then \((F_n(q))_{n \geq 0}\) is strongly \(q\)-log-convex. For \(T_n(q)\), it is obvious that
\[
r_i(q) - s_{i+1}(q)
= (2\nu i + \beta)(2\nu i + 2\nu + \beta)(1 + q^2) - 2\nu(i + 1)(\nu i + \beta)(1 + q^2)
= [2\nu^2 i^2 + 2\nu(\nu + \beta)i + \beta^2](1 + q^2) + 2(2\nu i + \beta)[2\nu(i + 1) + \beta]q
\]
has only nonnegative coefficients for \(i, \beta \geq 0\). Hence \((T_n(q))_{n \geq 0}\) is strongly \(q\)-log-convex for \(\beta \geq 0\).

For (iv), by (4.12), we have
\[
h_{n,k} = \sum_{i \geq 0} S_{n,i} \binom{i}{k},
\]
where \(S_{n,i}\) satisfies the recurrence relation (4.13). In addition, by (4.13), it is easy to prove that \(S_{n,i}\) is nonnegative for \(\beta \geq 0\) and \(\nu > 0\). In consequence, we obtain the expansion of \(T_n(x)\) in the gamma basis
\[
\left\{ x^k(1 + x)^{n-\delta+1-2k} | 0 \leq k \leq \left\lfloor \frac{n - \delta + 1}{2} \right\rfloor \right\}
\]
as follows:
\[
T_n(x) = (1 + x)^{n-\delta+1}h_{n-\delta+1} \left( \frac{-2x}{(1 + x)^2} \right)
= \sum_{k \geq 0} h_{n-\delta+1,k}(-2)^k x^k (1 + x)^{n-\delta+1-2k}
= \sum_{k \geq 0} (-2)^k \left( \sum_{i \geq 0} S_{n-\delta+1,i} \binom{i}{k} \right) x^k (1 + x)^{n-\delta+1-2k}.
\]
Then, the result is desired. This completes the proof.

**Remark 4.8.** Flajolet [32] gave a general combinatorial interpretation in terms of weighted Motzkin paths for a Jacobi continued fraction expansion. From this, by (4.14), we can also obtain that \(S_n(x)\) has only nonnegative coefficients in \(x\) for \(\beta \geq 0\) and \(\nu > 0\), see [83, Remark 5.6] for instance.

It is known that \(\gamma\)-positivity is stronger than unimodality. Though Theorem 4.7 (iv) says that \(T_n(x)\) is not \(\gamma\)-positivity, it may still be unimodal.
Theorem 4.9. If \( \beta = 1 \) and \( \nu = -\mu = 1/\delta \), then \( T_n(x) \) be defined by (4.4) with \( m_n = n - \delta + 1 \) is unimodal for any \( n \geq \delta + 2 \).

Proof. We will prove it by induction on \( n \). Whenever \( n = \delta + 2 \), we have

\[
\delta T_{\delta+2}(x) = (4 + 6\delta + \delta^2)(1 + x^3) + (4 + 6\delta + 3\delta^2)(x + x^2),
\]

which is unimodal. Assume that \( T_i(x) \) is unimodal for \( i = n > \delta + 2 \geq 3 \). By induction hypothesis, whenever \( i = n + 1 \), we need to verify \( \delta(T_{n+1,k} - T_{n+1,k-1}) \geq 0 \) for \( 1 \leq k \leq [(n - \delta + 2)/2] \).

By (4.4), the coefficients of \( T_{n+1}(x) \) satisfy the recurrence relation

\[
\delta T_{n+1,k} = (k + 1)T_{n,k+1} + (n - k + 1)T_{n,k} + (k + \delta - 1)T_{n,k-1} + (n - \delta - k + 3)T_{n,k-2}.
\]

It helps us to get

\[
\delta(T_{n+1,k} - T_{n+1,k-1}) = (k + 1)T_{n,k+1} + (\delta - 2)T_{n,k} + (n - \delta - 2k + 3)(T_{n,k} - T_{n,k-1})
\]

\[
+ (n - 2\delta - 2k + 5)T_{n,k-2} - (n - \delta - k + 4)T_{n,k-3}.
\]  

(4.16)

For \( 1 \leq k < [(n - \delta + 2)/2] \), \( T_{n,i} \) is increasing as \( i \) from 0 to \( [(n - \delta + 1)/2] \) by assumption. Note that the sum of the coefficients in right hand side of (4.16) is 0, which implies

\[
\delta(T_{n+1,k} - T_{n+1,k-1}) \geq 0.
\]

For \( k = [(n - \delta + 2)/2] \), we will consider two cases in terms of parity of \( n - \delta + 2 \).

Case 1: \( n - \delta + 2 = 2\ell + 1 \) and \( k = \ell \). Then \( T_{n,\ell+1} = T_{n,\ell-1} \) and

\[
\delta(T_{n+1,\ell} - T_{n+1,\ell-1}) = (\ell + 1)T_{n,\ell+1} + \delta T_{n,\ell} - 2T_{n,\ell-1} - (\delta - 4)T_{n,\ell-2} - (\ell + 3)T_{n,\ell-3}
\]

\[
= \delta T_{n,\ell} + (\ell - 1)T_{n,\ell-1} - (\delta - 4)T_{n,\ell-2} - (\ell + 3)T_{n,\ell-3}
\]

\[
\geq \delta T_{n,\ell-2} + (\ell - 1)T_{n,\ell-2} - (\delta - 4)T_{n,\ell-2} - (\ell + 3)T_{n,\ell-2}
\]

\[
= 0
\]

because \( T_{n,i} \) is increasing as \( i \) from 0 to \( \ell \).

Case 2: \( n - \delta + 2 = 2\ell \) and \( k = \ell \). Then \( \ell \geq 3, T_{n,\ell+1} = T_{n,\ell-2} \) and \( T_{n,\ell} = T_{n,\ell-1} \). Thus we have

\[
\delta(T_{n+1,\ell} - T_{n+1,\ell-1}) = (\ell + 1)T_{n,\ell+1} + (\delta - 1)T_{n,\ell} - T_{n,\ell-1} - (\delta - 3)T_{n,\ell-2} - (\ell + 2)T_{n,\ell-3}
\]

\[
= (\delta - 2)T_{n,\ell-1} + (\ell - \delta + 4)T_{n,\ell-2} - (\ell + 2)T_{n,\ell-3}.
\]  

(4.17)

If \( \delta \geq 2 \), then

\[
\delta(T_{n+1,\ell} - T_{n+1,\ell-1}) = (\delta - 2)T_{n,\ell-1} + (\ell - \delta + 4)T_{n,\ell-2} - (\ell + 2)T_{n,\ell-3}
\]

\[
\geq (\delta - 2)T_{n,\ell-2} + (\ell - \delta + 4)T_{n,\ell-2} - (\ell + 2)T_{n,\ell-3}
\]

\[
\geq (\ell + 2)T_{n,\ell-2} - (\ell + 2)T_{n,\ell-3}
\]

\[
\geq 0
\]

because \( T_{n,i} \) is increasing as \( i \) from 0 to \( \ell \).
If $\delta = 1$, then (4.17) becomes to
\[
T_{n+1,\ell} - T_{n+1,\ell-1} \\
= (\ell^2 + \ell - 3)T_{n-1,\ell-1} + (2\ell + 8)T_{n-1,\ell-2} - (4\ell + 11)T_{n-1,\ell-3} \\
+ (6\ell + 12)T_{n-1,\ell-4} - (\ell^2 + 5\ell + 6)T_{n-1,\ell-5} \\
= (\ell^2 - \ell - 6)(T_{n-1,\ell-1} - T_{n-1,\ell-5}) + (2\ell + 3)(T_{n-1,\ell-1} - T_{n-1,\ell-3}) \\
+ (2\ell + 8)(T_{n-1,\ell-2} - T_{n-1,\ell-3}) + (6\ell + 12)(T_{n-1,\ell-4} - T_{n-1,\ell-5}) \\
\geq 0
\]
for $\ell \geq 3$. This completes all proof.

4.3 A relation with the derivative polynomials

The polynomial $T_n(x)$ has a close relation with the derivative polynomials. Knuth and Buckholtz [43] introduced the derivative polynomials to compute the tangent and secant numbers, where the derivative polynomial for secant defined by
\[
D_n^\theta \sec \theta = \sec \theta \cdot Q_n^{(\theta)}(\tan \theta).
\]
Based on this, Hoffman [40] studied the exponential generating functions and the combinatorial interpretation of the coefficients for those polynomials. In addition, he also studied the Springer and Shanks numbers in terms of the Eulerian polynomials. Josuat-Vergès [42] defined the generalized derivative polynomials for secant as follows:
\[
D_n^\delta \sec \delta \theta = \sec \delta \theta \cdot Q_n^{(\delta)}(\tan \theta),
\]
where $Q_n^{(\delta)}(x)$ satisfies the following recurrence relation:
\[
Q_{n+1}^{(\delta)}(x) = \delta xQ_n^{(\delta)}(x) + (1 + x^2)D_xQ_n^{(\delta)}(x)
\]
with the initial condition $Q_0^{(\delta)}(x) = 1$. For the generalized derivative polynomials, Josuat-Vergès studied the ordinary (resp., exponential) generating functions in terms of the Jacobi continued fraction expansion (resp., trigonometric functions). We refer the reader to [39, 40, 42] and references therein for more details.

Combining (4.6), (4.18), (4.19) and (ii) of Theorem 4.7 gives the following result. It not only gives a relation between $T_n(x)$ and $Q_n^{(\delta)}(x)$, but also implies some properties of $Q_n^{(\delta)}(x)$.

**Proposition 4.10.** Let $Q_n^{(\delta)}(x)$ be defined by (4.18). If $\beta = 1$ and $\nu = -\mu = 1/\delta$, then

(i) it has the relation with the derivative polynomial
\[
Q_n^{(\delta)}(x) = \frac{\delta^n(1 + x)^n}{2^n}T_{n+\delta-1} \left( \frac{x - 1}{x + 1} \right); \quad (4.19)
\]

(ii) its exponential generating function is
\[
\sum_{n \geq 0} Q_n^{(\delta)}(x) \frac{t^n}{n!} = \frac{1}{(\cos t - x \sin t)^\delta};
\]
(iii) its ordinary generating function has the Jacobi continued fraction expansion

\[ \sum_{n=0}^{\infty} Q_n^{(\delta)}(x)t^n = \frac{1}{1 - r_0 t - \frac{s_1 t^2}{1 - r_1 t - \frac{s_2 t^2}{1 - r_2 t - \ldots}}}, \]

where \( r_i = (2i + \delta)x \) and \( s_i = i(i + \delta - 1)(1 + x^2) \) for \( i \geq 0 \).

**Remark 4.11.** The (ii) and (iii) in Proposition 4.10 were also proved by Josuat-Vergèsit [42] using the different method.

In addition, we also give a convolutional relation among the polynomial \( T_n(x) \) in (4.4) for different \( \delta \). For convenience, denote \( T_n^{(\delta)}(x) = T_n(x) \) for \( m_n = n - \delta + 1 \). Then, we have the following result.

**Proposition 4.12.** If \( \beta = 1 \) and \( \nu = -\mu = 1/\delta \), then we have

\[ (\delta_1 + \delta_2)^n T_{n+\delta_1+\delta_2-1}(x) = \sum_{k \geq 0} \binom{n}{k} \delta_1^k \delta_2^{n-k} T_{k+\delta_1-1}(x) T_{n-k+\delta_2-1}(x) \]

for \( \delta_1, \delta_2 \in \mathbb{N} \).

**Proof.** By (ii) of Proposition 4.10, we have the following the relation

\[ Q_n^{(\delta_1+\delta_2)}(x) = \sum_{k \geq 0} \binom{n}{k} Q_k^{(\delta_1)}(x) Q_{n-k}^{(\delta_2)}(x). \]  

(4.20)

Combining (4.19) and (4.20) derives the desired result. \( \square \)

**Remark 4.13.** In [29], we also obtain some similar results for \( q \)-analog of Theorem 4.7, Theorem 4.9 and Proposition 4.10.

### 4.4 Alternating descents of permutations

The number of alternating descents of a permutation \( \pi \in S_n \) is defined by

\[ \text{altdes}_A(\pi) = |\{ 2i : \pi(2i) < \pi(2i + 1) \} \cup \{ 2i + 1 : \pi(2i + 1) > \pi(2i + 2) \} |. \]

Define the alternating Eulerian polynomial \( \widehat{A}_n(x) \) as follows:

\[ \widehat{A}_n(x) = \sum_{\pi \in S_n} x^{|\text{altdes}_A(\pi)|} = \sum_{k=0}^{n} \widehat{A}(n, k)x^k, \]

where \( \widehat{A}(n, k) \) is called the alternating Eulerian number.

In recent years, several authors paid attention to the polynomial \( \widehat{A}_n(x) \). For example, Chebikin [27] studied the exponential generating function. Remmel [63] computed a
generating function for the joint distribution of the alternating descent statistic and the alternating major statistic over $S_n$. Moreover, Gessel and Zhuang [36] extended some results in [27, 63] by using noncommutative symmetric functions. For $n \geq 1$, Ma and Yeh [57] gave the explicit formula and the recurrence relation

$$2\hat{A}_{n+1}(x) = [(n-1)x^2 + 2x + n+1]\hat{A}_n(x) + (1-x)(1+x^2)D_x\hat{A}_n(x)$$

with initial conditions $\hat{A}_1(x) = 1$ and $\hat{A}_2(x) = 1 + x$. We sum up some other known properties for $\hat{A}_n(x)$ in the following result, which is immediate from Theorems 4.6, 4.7 and 4.9 by taking $\delta = 2$.

**Theorem 4.14.** Let $\hat{A}_n(x)$ be the alternating Eulerian polynomial of type $A$. Then

(i) it has the relation with derivative polynomials

$$\hat{A}_{n+1}(x) = \frac{(1-x)^n}{2^n}Q_n^{(2)}(\frac{1+x}{1-x});$$

(ii) it is symmetric and unimodal for any $n \in \mathbb{N}$;

(iii) its exponential generating function is

$$\sum_{n \geq 0} \hat{A}_{n+1}(x)\frac{t^n}{n!} = \frac{(1-x)^2}{[(1-x)\cos((1-x)t/2) - (1+x)\sin((1-x)t/2)]^2}; \quad (4.21)$$

(iv) its ordinary generating function has the Jacobi continued fraction expansion

$$\sum_{n=0}^{\infty} \hat{A}_{n+1}(x)t^n = \frac{1}{1 - r_0 t - \frac{s_1 t^2}{1 - r_1 t - \frac{s_2 t^2}{1 - r_2 t - \ldots}}},$$

where $r_i = (i+1)(1+x)$ and $s_i = i(i+1)(1+x^2)/2$ for $i \geq 0$;

(v) it is strongly $q$-log-convex;

(vi) it is Hurwitz stable and semi-$\gamma$-positive for $n \geq 1$;

(vii) it has the following decomposition

$$\hat{A}_n(x) = \sum_{k \geq 0} (-2)^k \left[ \sum_{i \geq 0} S_{n-1,i}(i^k) \right] x^k(1+x)^{n-1-2k},$$

see $S_{n,i}$ in [68, A094503, A113897]. Moreover, it is not $\gamma$ positive for $n \geq 3$.

**Remark 4.15.** We refer the reader to [47, 57, 81] for the corresponding different proof for Theorem 4.14. Integrating with respect to (4.21) in $t$, we recover the exponential generating function of $\hat{A}_n(x)$ occurred in [27, Theorem 4.2] as follows:

$$\sum_{n \geq 0} \hat{A}_n(x)\frac{t^n}{n!} = \frac{\sec(1-x)t + \tan(1-x)t - 1}{1 - x(\sec(1-x)t + \tan(1-x)t)},$$

since the left part of (4.21) is equivalent to 1 whenever $t = 0$.  

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### 4.5 Alternating descents of signed permutations

Similarly, the number of alternating descents of a permutation $\pi \in B_n$ is defined by

$$\text{altdes}_B(\pi) = | \{ 2i : \pi(2i) < \pi(2i + 1) \} \cup \{ 2i + 1 : \pi(2i + 1) > \pi(2i + 2) \} |,$$

where $i \geq 0$ and $\pi(0) = 0$. We call $\pi(2i) < \pi(2i + 1)$ (resp., $\pi(2i) > \pi(2i + 1)$) be the even alternating descent (resp., ascent) space and $\pi(2i + 1) > \pi(2i + 2)$ (resp., $\pi(2i + 1) < \pi(2i + 2)$) be the odd alternating descent (resp., ascent) space. Define the alternating Eulerian polynomial of type $B$ be

$$\hat{B}_n(x) = \sum_{\pi \in B_n} x^{\text{altdes}_B(\pi)} = \sum_{k=0}^{n} \hat{B}(n,k)x^k,$$

where $\hat{B}(n,k)$ is called the alternating Eulerian number of type $B$.

We list the first few terms as follows:

$$\begin{align*}
\hat{B}_0(x) &= 1, \\
\hat{B}_1(x) &= 1 + x, \\
\hat{B}_2(x) &= 3 + 2x + 3x^2, \\
\hat{B}_3(x) &= 11 + 13x + 13x^2 + 11x^3, \\
\hat{B}_4(x) &= 57 + 76x + 118x^2 + 76x^3 + 57x^4.
\end{align*}$$

Using the similar combinatorial interpretation of alternating descent numbers of type $A$ [47], we have the next recurrence relation

$$\hat{B}_{n+1,k} = (n - k + 2)\hat{B}_{n,k-2} + k\hat{B}_{n,k-1} + (n - k + 1)\hat{B}_{n,k} + (k + 1)\hat{B}_{n,k+1},$$

which implies

$$\hat{B}_{n+1}(x) = (nx^2 + x + n + 1)\hat{B}_n(x) + (1 - x)(1 + x^2)D_x\hat{B}_n(x).$$

We refer the reader to [51] for a different proof from the context-free grammar.

In analogy to $A_n(x)$, we sum up the other properties of $\hat{B}_n(x)$ as follows, which is immediate by Theorems 4.6, 4.7 and 4.9 with $\delta = 1$.

**Theorem 4.16.** Let $\hat{B}_n(x)$ be the alternating Eulerian polynomial of type $B$. Then

(i) it has the relation with derivative polynomials

$$\hat{B}_n(x) = (1 - x)^nQ_n^{(1)}\frac{1 + x}{1 - x};$$

(ii) it is symmetric and unimodal for any $n \geq 3$;

(iii) its exponential generating function is

$$\sum_{n \geq 0} \hat{B}_n(x)\frac{t^n}{n!} = \frac{1 - x}{(1 - x)\cos(1 - x)t - (1 + x)\sin(1 - x)t}.$$
(iv) its ordinary generating function has the Jacobi continued fraction expansion

\[
\sum_{n=0}^{\infty} \hat{B}_n(x)t^n = \frac{1}{1 - r_0 t - \frac{s_1 t^2}{1 - r_1 t - \frac{s_2 t^2}{1 - r_2 t - \ldots}}},
\]

where \( r_i = (2i + 1)(1 + x) \) and \( s_i = 2i^2(1 + x^2) \) for \( i \geq 0 \);

(v) the polynomial sequence \((\hat{B}_n(q))_{n \geq 0}\) is strongly \( q \)-log-convex;

(vi) it is Hurwitz stable and semi-\( \gamma \)-positive for \( n \geq 1 \);

(vii) it has the following decomposition

\[
\hat{B}_n(x) = \sum_{k \geq 0} (-4)^k \left[ \sum_{i \geq 0} \tilde{W}_{n,i} \binom{i}{k} \right] x^k (1 + x)^{n-2k},
\]

where \( \tilde{W}_{n,i} \) is the left peaks in (3.13). Moreover, it is not \( \gamma \) positive for \( n \geq 2 \).

Remark 4.17. For (i)-(iii) and (vii) of Theorem 4.16, they were recently proved by Ma et al [51] using the different method.

In particular, taking \( \delta_1 = \delta_2 = 1 \) in Proposition 4.12, we get a result for the alternating Eulerian polynomials of types \( A \) and \( B \) as follows.

Proposition 4.18. The alternating Eulerian polynomials of types \( A \) and \( B \) have following relation:

\[
2^n \hat{A}_{n+1}(x) = \sum_{k \geq 0} \binom{n}{k} \hat{B}_k(x) \hat{B}_{n-k}(x).
\]

5 The alternatingly increasing property

Let the polynomial \( p = \sum_{k=0}^{n} p_k x^k \in \mathbb{R}[x] \). We call \( p \) alternatingly increasing if the coefficients of \( p \) satisfy

\[
0 \leq p_0 \leq p_n \leq p_1 \leq p_{n-1} \leq \ldots \leq p_{\lfloor (n+1)/2 \rfloor}.
\]

It is obvious that the alternatingly increasing property implies unimodality, that is to say, it is an approach to proving unimodality of combinatorial sequences. The unimodality problems have been extensively investigated in many branches of mathematics, see [18, 22, 69] for details concerning the development of unimodality.

The alternatingly increasing property of a polynomial \( p \) has a close relation with the symmetric decomposition of the polynomial \( p \). It is known that every polynomial \( p \) of degree at most \( n \) can be uniquely decomposed as \( p = a + xb \) where \( a \) and \( b \) are symmetric with respect to \( n \) and \( n - 1 \), respectively. We call the ordered pair of polynomial \((a, b)\) the (symmetric) \( \mathcal{I}_n \)-decomposition of the polynomial \( p \). Beck et al. pointed out that a
polynomial \( p \) is alternatingly increasing if and only if both \( a \) and \( b \) have only nonnegative coefficients and are unimodal (see [6, Lemma 2.1]).

Recently, some authors paid attention to the alternatingly increasing property that raised combinatorics and geometry. Schepers and Van Langenhoven [67] proved that the coefficients of the \( h^* \)-polynomial for a lattice parallelepiped are alternatingly increasing. Moreover, Beck et al. [6] extended these results in [67] and proved that the \( h^* \)-polynomial for centrally symmetric lattice zonotopes and coloop-free lattice zonotopes are alternatingly increasing. Athanasiadis [4] proved that \( r \)-color Eulerian polynomials, \( r \)-color derangement polynomials and binomial Eulerian polynomials are alternatingly increasing by \( \gamma \)-positivity decomposition. Brändén and Solus [21] developed the symmetric decomposition method to prove the alternatingly increasing property of some polynomials, such as \( r \)-color Eulerian polynomials and \( r \)-color derangement polynomials. We refer the reader to [4, 21, 52, 67] and references therein for more examples.

In this section, based on the relation between a polynomial and its reciprocal polynomial, we extend a result of Brändén and Solus [21]. Therefore, we get the alternatingly increasing property of some polynomials, such as two kinds of peak polynomials on 2-Stirling permutations, descent polynomials on signed permutations of the 2-multiset and colored permutations and ascent polynomials for \( k \)-ary words. In addition, we also obtain a recurrence relation and zeros interlacing of the \( q \)-analog of descent polynomials on colored permutations that extend some results of Brändén and Brenti. Moreover, we get the alternatingly increasing property of this polynomials. Finally, we show the alternatingly increasing property and zeros interlacing for two kinds of peak polynomials on the dual set of Stirling permutations by using our result for Hurwitz stability.

5.1 \( h \)-polynomials

A polynomial \( h(x) \in \mathbb{R}[x] \) is called as \( h \)-polynomial if it satisfies the following relation:

\[
\sum_{m \in \mathbb{N}} i(m)x^m = \frac{h(x)}{(1-x)^{n+1}} \tag{5.1}
\]

with \( i(x) \in \mathbb{R}[x] \) and \( \deg(i(x)) = n \). And a polynomial \( f(x) \) satisfying the following transformation:

\[
f(h; x) = (1 + x)^n h \left( \frac{x}{1 + x} \right) \tag{5.2}
\]

is called as \( f \)-polynomial of the polynomial \( h(x) \) with respect to \( n \). Following the transformation, we know that if \( h(x) \) with nonnegative coefficients has only real zeros, then \( f(h; x) \) has all zeros in \([-1, 0]\) and nonnegative coefficients. By (5.2), the following relation is immediate

\[
h(x) = (1 - x)^n f \left( \frac{x}{1 - x} \right). \tag{5.3}
\]

Moreover, if both \( h_1(x) \) and \( h_2(x) \) with degree \( n \) have only nonnegative coefficients and real zeros, then we have the following equivalent relation:

\[
h_1(x) \ll h_2(x) \iff f(h_1; x) \ll f(h_2; x),
\]

which provides a choice to study their properties in an easier way.
For a polynomial $p \in \mathbb{R}[x]$ with degree at most $n$, we denote

$$\mathcal{I}_n(p(x)) := x^n p(1/x) \quad \text{and} \quad \mathcal{R}_n(p(x)) := (-1)^n p(-1 - x)$$

Then we know that there exists unique pair polynomials $\tilde{a} \in \mathbb{R}[x]$ and $\tilde{b} \in \mathbb{R}[x]$ such that $p = \tilde{a} + x\tilde{b}$, where $\mathcal{R}_n(\tilde{a}) = \tilde{a}$ and $\mathcal{R}_{n-1}(\tilde{b}) = \tilde{b}$. We call the ordered pair of polynomials $(\tilde{a}, \tilde{b})$ the (symmetric) $\mathcal{R}_n$-decomposition of the polynomial $p$. In fact, $\tilde{a}$ (resp., $\tilde{b}$) is the $f$-polynomial of $a$ (resp., $b$) for the (symmetric) $\mathcal{I}_n$-decomposition of a polynomial $p$ and $f(\mathcal{I}_n(p); x) = \mathcal{R}_n(f(p; x))$ by [21, Lemma 2.3]. Recently, Brändén and Solus gave several equivalent forms for the interlacing condition of $a$ and $b$ as follows.

**Lemma 5.1.** [21] Let $p \in \mathbb{R}[x]$ have degree at most $n$ and $\mathcal{I}_n$-decomposition $(a, b)$, for which both $a$ and $b$ have only nonnegative coefficients. Then the following are equivalent:
1. $b \ll a$,
2. $a \ll p$,
3. $b \ll p$,
4. $\mathcal{I}_n(p) \ll p$.

Note that $p$ has only nonnegative coefficients and $\mathcal{I}_n(p) \ll p$, which implies that both $a$ and $b$ have nonnegative coefficients and interlacing zeros. However, for the general $\mathcal{I}_n$-decomposition $(a, b)$, the zeros of $a$ do not interlace those of $b$. Define the subdivision operator $\varepsilon : \mathbb{R}[x] \to \mathbb{R}[x]$ by

$$\varepsilon \left( \binom{x}{k} \right) = x^k$$

for all $k \geq 0$, where $\binom{x}{k} = x(x-1) \cdots (x-k+1)/k!$. It is known that the relation between polynomials $i(x)$ and $h(x)$ in (5.1) is $\varepsilon(i(x)) = f(h; x)$ by [21, Lemma 2.7]. Thus, the study about $\mathcal{I}_n$-decomposition of $h(x)$ can be transformed to this about $\mathcal{R}_n$-decomposition of $i(x)$.

It is known that the $r$-color Eulerian polynomial $A^r_n(x)$ have the following identity relation by Steingrímsson [70]:

$$\sum_{m \geq 0} (rm + 1)^n x^m = \frac{A^r_n(x)}{(1 - x)^{n+1}}.$$  \hspace{1cm} (5.4)

Define a refined polynomial $A^r_{n,k}(x)$ by the relation

$$\sum_{m \geq 0} (rm)^k (rm + 1)^{n-k} x^m = \frac{A^r_{n,k}(x)}{(1 - x)^{n+1}}.$$  

Obviously, for $k = 0$, $A^r_{n,0}(x)$ is the $r$-colored Eulerian polynomial of order $n$. Based on this, Brändén and Solus got the following general result to show that $A^r_n(x)$ is alternatingly increasing for $n \in \mathbb{N}$ and fixed $r \in \mathbb{N}$.

**Theorem 5.2.** [21, Theorem 3.1] Let a polynomial $p$ be defined by

$$p = \sum_{r \geq 2} \sum_{k=0}^n c_{r,k} A^r_{n,k}(x)$$

for some $c_{r,k} \geq 0$. Then $\mathcal{I}_n(p) \ll p$ for $\deg(p) = n$. In particular, $p$ is real-rooted and alternatingly increasing.
Now, we consider a more general situation that \( i(x) \) is a nonnegative combination of some polynomials which have only zeros in \([-1, 0]\). We will give a condition making sure the alternatingly increasing property of the polynomial \( h(x) \). For fixed \( k \in \mathbb{N}^+ \), assume \( 0 \leq r_{k_1} \leq r_{k_2} \leq \cdots \leq r_{k_n} \leq 1 \) for any \( n \in \mathbb{N} \), and we let

\[
\sum_{m \geq 0} \prod_{i=1}^{n} (m + r_{k_i})x^m = \frac{h_{n,k}(x)}{(1-x)^{n+1}}.
\]  

(5.5)

The next more general result in particular implies Theorem 5.2 by taking \( r_{k_i} \in \{0, 1/r\} \) and \( c_k = r^n \) for \( r \geq 2 \).

**Theorem 5.3.** Let \( h_{n,k}(x) \) be defined in (5.5). Assume that \( p \in \mathbb{R}[x] \) and has the expression

\[
p = \sum_{k \geq 1} c_k h_{n,k}(x)
\]

for all \( c_k \geq 0 \) and the \( \mathcal{I}_n \)-decomposition \((a,b)\). If \( 0 \leq r_{k_i} + r_{k_{i+1}} \leq 1 \) for any \( k, i \in \mathbb{N}^+ \), then \( \mathcal{I}_n(p) \ll p \) for \( \deg(p) = n \). In particular, \( b \ll a \) and \( p \) is alternatingly increasing.

**Proof.** Let

\[
i_k(x) = \prod_{i=1}^{k}(x + r_{k_i}) \quad \text{and} \quad i(x) = \sum_{k \geq 1} c_k i_k(x).
\]

Note that both \( \varepsilon \) and \( \mathcal{R} \) are linear operators, then

\[
\sum_{k \geq 1} c_k \varepsilon(i_k(x)) = \varepsilon(i(x))
\]

Taking \( \{(x_k)^n\}_{k=0}^n \) as a set of basis of \( \mathbb{R}[x]_n \), it is easy to verify that the operator \( \mathcal{R} \) and \( \varepsilon \) have commutativity on this basis. Thus

\[
\mathcal{R}_n(\varepsilon(i_k(x))) = \varepsilon(\mathcal{R}_n(i_k(x))) = \varepsilon(\prod_{i=1}^{n}(x+1-r_{k_i})).
\]

Note that the fact (see [16, Theorem 4.6]): Assume that two standard polynomials \( f \) and \( g \) both have only real zeros \( \alpha_n \leq \cdots \leq \alpha_2 \leq \alpha_1 \) and \( \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \), respectively. If all these zeros are in the interval \([-1, 0]\) and the above fact, we derive \( \mathcal{R}_n(\varepsilon(i_k(x))) \ll \varepsilon(i(x)) \) for any \( k, i \in \mathbb{N}^+ \). By Proposition 2.2 and \( c_k \geq 0 \) for any \( k \in \mathbb{N}^+ \), then we obtain \( \mathcal{R}_n(\varepsilon(i_k(x))) \ll \varepsilon(i(x)) \). In addition, \( p \) has only nonnegative coefficients and real zeros by (5.3) since \( \varepsilon(i(x)) = f(p; x) \). Combining \( \mathcal{R}_n(f(p; x)) = f(\mathcal{I}_n(p); x) \) and \( \mathcal{R}_n(\varepsilon(i(x))) \ll \varepsilon(i(x)) \) derives \( f(\mathcal{I}_n(p); x) \ll f(p; x) \), thus \( \mathcal{I}_n(p) \ll p \). The alternatingly increasing property of \( p(x) \) and \( b \ll a \) are immediate by Lemma 5.1. \(\square\)

**Remark 5.4.** Define the linear map \( \mathcal{D} : \mathbb{R}[x] \rightarrow \mathbb{R}[x] \) by

\[
\mathcal{D}(x^k) = d_k(x)
\]

for all \( k \geq 0 \), where \( d_k(x) \) is the \( k \)-th derangement polynomial. Then, we have an analogous result to Theorem 5.3. Taking \( p = \sum_{k \geq 1} c_k h_{n,k}(x) \), where \( h_{n,k}(x) \) is defined by (5.5). If \( c_k \geq 0 \) for all \( k \in [n] \), then \( \mathcal{D}(p) \ll \mathcal{I}_n(\mathcal{D}(p)) \) for \( \deg(p) = n \). The proof is similar to Corollary 3.7 in [21], so we omit it here for brevity. In fact, it is more general than Corollary 3.7 in [21], which can be used to prove \( \mathcal{I}_n(d_{n,r}) \ll d_{n,r} \), where \( d_{n,r} \) is the \( n \)-th \( r \)-color derangement polynomial.
5.2 Ascent polynomials for $k$-ary words

Let $w \in S = \{0, 1, \ldots, k - 1\}^n$ be a $k$-ary words of length $n$. We assume $w_0 = 0$ for the convention. Let $\text{asc}(w)$ denote the number of $w_i < w_{i+1}$ for $i \in [n-1] \cup \{0\}$. Then the $n$-th ascent polynomial for $k$-ary words is defined by

$$
A^k_n(x) = \sum_{w \in S} x^{\text{asc}(w)}.
$$

(5.6)

It is known that $A^k_n(x)$ has the following relation (see [65, Corollary 8]):

$$
\sum_{m \geq 0} \frac{(n+km)}{n} x^m = \frac{A^k_n(x)}{(1-x)^{n+1}}.
$$

That is to say,

$$
\sum_{m \geq 0} \frac{k^n}{n!} \prod_{i=1}^{n} \left( m + \frac{i}{k} \right) x^m = \frac{A^k_n(x)}{(1-x)^{n+1}}.
$$

Taking $r_i = i/k$ for $i \in [n]$, $c_1 = k^n/n!$ and the others to be zero in Theorem 5.3, we get the following result.

**Proposition 5.5.** Let the ascent polynomial $A^k_n(x)$ be defined by (5.6) and $(a,b)$ be its $\mathcal{I}_n$-decomposition. If $k > n$, then $\mathcal{I}_n(A^k_n) \ll A^k_n$ for $\deg(A^k_n(x)) = n$. In particular, $A^k_n(x)$ is alternatingly increasing and $b \ll a$.

5.3 Descent polynomials on signed permutations of the 2-multiset

Recently, Lin [46] considered the descent polynomials on signed permutations of the general multiset $M_s := \{1^{s_1}, 2^{s_2}, \ldots, n^{s_n}\}$ for each vector $s := (s_1, s_2, \ldots, s_n)$. Let $s = s_1 + s_2 + \cdots + s_n$ and $\pi_0 = 0$. Define $p^\pm_s(x)$ by

$$
p^\pm_s(x) = \sum_{\pi \in p^\pm_s} x^{\text{des}_\pi},
$$

where $p^\pm_s$ is the set of all permutations $\pi = \pm\pi_1 \pm \pi_2 \cdots \pm \pi_s$ with $\pi_1\pi_2\cdots\pi_s$ be a permutation on the multiset $M_s$ and $\text{des}_\pi$ is the descent number of $\pi$. Moreover, Lin got the following relationship:

$$
\sum_{m \geq 0} \prod_{r=1}^{n} \frac{(2m+1)(2m+2) \cdots (2m + s_r)}{s_j!} x^m = \frac{p^\pm_s(x)}{(1-x)^{s+1}}.
$$

In particular, let $p_s(x) = p^\pm_s(x)$ whenever $s_j \in \{1, 2\}$ for all $j \in [n]$, namely,

$$
\sum_{m \geq 0} (m+1)^{s-n}(2m+1)^nx^m = \frac{p_s(x)}{(1-x)^{s+1}}.
$$

(5.7)

For the polynomial $p_s(x)$, we have the following result.
**Proposition 5.6.** Let \( p_s(x) \) satisfy (5.7) and \((a, b)\) be its \(I\)-decomposition. Then \( I_{s-1}(p_s) \ll p_s \). In particular, \( p_s(x) \) is alternatingly increasing and \( b \ll a \).

**Proof.** Let \( i(x) = (x + 1)^{s-n}(2x + 1)^n \). We obtain \( \varepsilon(i(x)) = f(p_s; x) \). In consequence, we have

\[
\varepsilon(R_n(i(x))) = R_n(\varepsilon(i(x))) = R_n(f(I_s(I_s(p_s)); x)) = f(I_s(p_s); x).
\]

That is to say, \( I_s(p) \) satisfies the following relation:

\[
\sum_{m \geq 0} 2^n m^{s-n} \left( m + \frac{1}{2} \right)^n x^m = \frac{I_s(p_s)}{(1 - x)^{s+1}}.
\]

Taking \( r_i \in \{0, 1/2\} \), \( c_1 = 2^n \) and the others to be zero in Theorem 5.3, we have \( p_s \ll I_s(p_s) \). Note that \( \deg(p_s) = s - 1 \), thus \( I_{s-1}(p_s) \ll p_s \). Both the alternatingly increasing property of \( p_s(x) \) and \( b \ll a \) are immediate by Theorem 5.3. \( \square \)

**Remark 5.7.** For \( s_j = 2 \) with all \( j \in [n] \), the alternatingly increasing property of \( p_s(x) \) was also proved by Ma et al. [52, Theorem 11] in a different method.

## 5.4 Descent polynomials on \( r \)-colored permutations

The half Eulerian polynomials of type B are given by

\[
B^+_n(x) = \sum_{\pi \in B^+_n} x^{\text{des}_B \pi} \quad \text{and} \quad B^-_n(x) = \sum_{\pi \in B^-_n} x^{\text{des}_B \pi},
\]

where \( B^+_n \) (resp., \( B^-_n \)) is the Coxeter group of type B of rank \( n \) with \( \pi_n > 0 \) (resp., \( \pi_n < 0 \)). By bijection from \( B^+_n \) to \( B^-_n \), it is easy to know that \( B^-_n(x) = I_n(B^+_n(x)) \) (see [5, Lemma 7.1]) since \( \deg(B^+_n(x)) = n - 1 \). And by [5, (7.5)], we have

\[
\sum_{m \geq 0} [(2m + 1)^n - (2m)^n] x^m = \frac{B^+_n(x)}{(1 - x)^n},
\]

\[
\sum_{m \geq 0} [(2m)^n - (2m - 1)^n] x^m = \frac{B^-_n(x)}{(1 - x)^n}.
\]

The wreath product group \( \mathbb{Z}_r \wr S_n \) consists of all permutations \( \pi \in [0, r - 1] \times [n] \). Namely, the element in \( \mathbb{Z}_r \wr S_n \) is thought of as \( \pi = \xi_1 \pi_1 \xi_2 \pi_2 \cdots \xi_n \pi_n \), where \( e_i \in [0, r - 1] \) and \( \pi \in S_n \). Define the following total order relation on the elements of \( \mathbb{Z}_r \wr S_n \):

\[
\xi^{-1} n < \cdots < \xi n < \cdots < \xi^{-1} 2 < \cdots < \xi 2 < \xi^{-1} 1 < \cdots < \xi 1 < 0 < \xi 0 1 < \cdots < \xi 0 n.
\]

Assume that \((\mathbb{Z}_r \wr S_n)^+\) is the set of colored permutations \( \pi \in \mathbb{Z}_r \wr S_n \) with first coordinate of zero color and \( \text{des} (\pi) \) is the descent number of \( \pi \). Athanasiadis [4] defined the following polynomial

\[
A^+_{r,n}(x) = \sum_{\pi \in (\mathbb{Z}_r \wr S_n)^+} x^{\text{des}(\pi)}, \quad (5.8)
\]
The first three terms are listed as follows:

\[
A^+_{r,1}(x) = 1, \\
A^+_{r,2}(x) = 1 + (2r - 1)x, \\
A^+_{r,3}(x) = 1 + (3r^2 + 3r - 2)x + (3r^2 - 3r + 1)x^2.
\]

Athanasiadis showed that \(A^+_{r,n}(x)\) can be interpreted as the \(h^*\)-polynomial of a lattice polyhedral complex and got the following expression:

\[
\sum_{m \geq 0} [(rm + 1)^n - (rm)^n] x^m = \frac{A^+_{r,n}(x)}{(1 - x)^n}.
\] (5.9)

Obviously, \(A^+_{r,n}(x)\) can be looked as a generalization of \(B^+_{n}(x)\) because \(A^+_{2,n}(x) = B^+_{n}(x)\). Note that \(\deg(A^+_{r,n}(x)) = n - 1\), thus we have the following result.

**Proposition 5.8.** Let \(A^+_{r,n}(x)\) be defined by (5.9). Then \(\mathcal{I}_{n-1}(A^+_{r,n}) \ll A^+_{r,n}\). In particular, \(A^+_{r,n}(x)\) is alternatingly increasing for \(r \geq 2\) and \(n \in \mathbb{N}^+\).

**Proof.** At first, we have the following decomposition:

\[
(rm + 1)^n - (rm)^n = \sum_{k=0}^{n-1} r^{n-1} m^k \left( m + \frac{1}{r} \right)^{n-1-k}.
\]

Taking \(r_k \in \{0,1/r\}\) and \(c_k = r^{n-1}\), then the desired result is immediate by Theorem 5.3. \(\square\)

Note that we have \(B^+_{n}(x) \ll B^+_{n}(x)\) whenever \(r = 2\). It can be used to prove the real rootedness of the Eulerian polynomials of type B that was proved by Hyatt [41] using compatible polynomials and Yang and Zhang [77] in terms of Hurwitz stability.

**Remark 5.9.** Athanasiadis [4] gave the explanation of \(A^+_{r,n}(x)\) by Ehrhart theory. Namely, \((rm + 1)^n - (rm)^n\) is equal to the number of lattice points in the \(m\)th dilate of the union of the \(n\) facets of \(P\) which do not contain the origin, where \(P\) is the \(r\)th dilate of the standard unit \(n\)-dimensional cube. Define

\[
A^-_{r,n}(x) = \sum_{\pi \in (\mathbb{Z}_r \wr S_n)^-} x^{\text{des}(\pi)},
\]

where \((\mathbb{Z}_r \wr S_n)^-\) is the set of colored permutations \(\pi \in \mathbb{Z}_r \wr S_n\) with first coordinate of non-zero color. By (5.4) and (5.9), we can get the following equality:

\[
\sum_{m \geq 0} [(rm)^n - (rm - r + 1)^n] x^m = \frac{A^-_{r,n}(x)}{(1 - x)^n}.
\] (5.10)

We will give an explanation of \(A^-_{r,n}(x)\) by Ehrhart theory. Let \(P\) be the \(r\)th dilate of the standard unit \(n\)-dimensional cube. Then \((rm)^n - (rm - r + 1)^n\) is equal to the number of lattice points in the \(m\)th dilate of the union of the lattice point that is \(i \in [r(r - 1)]\) units away from the \(n\) facets of \(P\) which do not contain the origin. That is to say, \(A^-_{r,n}(x)\) is the \(h^*\)-polynomial of a lattice polyhedral complex, namely the collection of all faces of the facet that is \(i \in [r - 1]\) units away from \(n\) facets of \(P\) which do not contain the origin.
In [20], Brändén and Leander considered the $q$-analog of the $r$-colored Eulerian polynomials

$$A_r^n(x; q_1, q_2, \ldots, q_n) := \sum_{\pi \in S_r} x^{des}(\pi) q_1^{c_1(\pi)} q_2^{c_2(\pi)} \cdots q_n^{c_n(\pi)}, \quad (5.11)$$

where $e_i(\pi) = e_i$. For example, $\pi = \xi^1 \xi^3 \xi^1 \xi^0 \xi^2 \xi^4 \xi^4$, the responding term in the polynomial $A_r^n(x; q_1, q_2, \ldots, q_n)$ is $x^4 q_1 q_2 q_3 q_4 q_4$. For $r \in \mathbb{N}^+$ and $q \geq 0$, denote $[r]_q := 1 + q + q^2 + \cdots + q^{r-1}$. We have the following result.

**Proposition 5.10.** For $n \in \mathbb{N}$ and $r \in \mathbb{N}^+$, let $A_r^n(x; q_1, q_2, \ldots, q_n)$ be defined by (5.11). Then we have

(i) its recurrence relation is

$$A_r^n(x; q_1, q_2, \ldots, q_n) = \left(\left(n[q]_q \right) - 1\right)x + 1,$$

where $A_r^n(x; q_1, q_2, \ldots, q_n) = ([r]_{q_i} - 1)x + 1$;

(ii) $A_r^n(x; q_1, q_2, \ldots, q_n) \ll A_{r+1}^n(x; q_1, q_2, \ldots, q_n)$ for $q_i \geq 0$;

(iii) $\mathcal{I}_n(A_r^n(x; q_1, q_2, \ldots, q_n)) \ll A_r^n(x; q_1, q_2, \ldots, q_n)$ whenever $r \geq 2$, $q_i \geq 0$ and $0 \leq [r]_{q_i} + [r]_{q_{i-1}} \leq [r]_{q_i} [r]_{q_{i-1}}$ for any $i \in [n]$;

(iv) the polynomial $A_r^n(x; q_1, q_2, \ldots, q_n)$ is alternatingly increasing for $r \geq 2$ and $q_i \geq 0$.

**Proof.** For (i), Brändén and Leander in [20] used $s$-lecture hall $P$-partitions to get the following identity

$$\sum_{m \geq 0} \prod_{i=1}^{n} [r]_{q_i} m + 1)x = \frac{A_r^n(x; q_1, q_2, \ldots, q_n)}{(1 - x)^{n+1}}. \quad (5.13)$$

It is easy to check that the recurrence relation (5.12) satisfies the identify (5.13) with initial condition $A_r^n(x; q_1, q_2, \ldots, q_n) = ([r]_{q_i} - 1)x + 1$, we omit the proceed here.

For (ii), the result is immediate by using the method of zeros interlacing (see [48] for details).

For (iii) and (iv), we rewrite (5.13) as

$$\sum_{m \geq 0} \prod_{i=1}^{n} [r]_{q_i} \prod_{i=1}^{n} \left( m + \frac{1}{[r]_{q_i}} \right) x = \frac{A_r^n(x; q_1, q_2, \ldots, q_n)}{(1 - x)^{n+1}}. \quad (5.14)$$

Taking $r_i = 1/[r]_{q_i}$, $c_1 = \prod_{i=1}^{n} [r]_{q_i}$ and the others to be zero in Theorem 5.3 whenever $r \geq 2$ and $q_i \geq 0$, we get the desired results. \hfill \Box

**Remark 5.11.** In particular, the polynomial $A_r^n(x; q_1, q_2, \ldots, q_n)$ is the $q$-analog of Eulerian polynomial type of $B$ whenever $r = 2$ and $q_i = q_j$ for $i, j \in [n]$ and is the $r$-colored Eulerian polynomial whenever $q_i = 1$ for $i \in [n]$, whose alternatingly increasing property was obtained in [21]. In addition, Proposition 5.10 can be looked as the further generalization of Theorem 6.4 in [16] and Theorem 3.4 in [23].
5.5 Peak polynomials on dual set of 2-Stirling permutations

Denote $i^j = i, i, \ldots, i$ for $i, j \geq 1$. Stirling permutations were defined by Gessel and Stanley [35]. A Stirling permutation of order $n$ is a permutation $\pi$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that $\pi_s > \pi_k$ for all $k < s < \ell$ whenever $\pi_k = \pi_\ell$. Moreover, we say that a permutation of the multiset $\{1^r, 2^r, \ldots, n^r\}$ is a $r$-Stirling permutation of order $n$, denoted as $Q_{n,r}$, if $\pi_s \geq \pi_k$ for all $k < s < \ell$ whenever $\pi_k = \pi_\ell$.

In this subsection, we will consider the peak polynomials on the generalization of $r$-Stirling permutations, which extend the dual set of 2-Stirling permutations in [54]. Let $\pi = \pi_1\pi_2\ldots\pi_{rn} \in Q_{n,r}$ and define $\Phi_r$ be the injection which maps each $\ell$-th occurrence of entry $i$ in $\pi$ to $ri - \ell + 1$. For example, $\Phi_3(111233322) = (321698754)$ whenever $n = 3, r = 3$. Define the $r$-multiple set $\Phi_r(Q_{n,r})$ of $Q_{n,r}$ as follows:

$$\Phi_r(Q_{n,r}) = \{\pi : \sigma \in Q_{n,r}, \Phi_r(\sigma) = \pi\}.$$

The statistics interior peak and left peak in $\pi \in Q_{n,r}$ were defined by

$$ipk(\pi) = |\{i \in [rn - r + 1] \setminus \{1\} : \pi_{i-1} < \pi_i > \pi_{i+1} > \cdots > \pi_{i+r-1}\}|,$$

$$lpk(\pi) = |\{i \in [rn - r + 1] : \pi_{i-1} < \pi_i > \pi_{i+1} > \cdots > \pi_{i+r-1}\}|,$$

where $\pi_0 = 0$. Thus we can define the peak polynomials on $\Phi_r(Q_{n,r})$ as follows:

$$M_{n,r}(x) = \sum_{\pi \in \Phi_r(Q_{n,r})} x^{ipk(\pi)},$$

$$\widetilde{M}_{n,r}(x) = \sum_{\pi \in \Phi_r(Q_{n,r})} x^{lpk(\pi)}.$$

Let $M_{n,r,k}$ denote the number of $\pi \in \Phi_r(Q_{n,r})$ with $k$ interior peaks, which can be obtained from $\Phi_r(Q_{n-1,r})$ by the following two cases:

1. For $i \in ipk(\pi)$ and $j \in \{-1, 0\} \cup [r - 2]$, inserting $(rn)(rn - 1) \cdots (rn - r + 1)$ into the right-hand side of $\pi_{i+j}$ will preserve the number of $ipk(\pi)$. In addition, inserting $(rn)(rn - 1) \cdots (rn - r + 1)$ into the left-hand side of $\pi_1$ also preserves the number of $ipk(\pi)$. Thus, if $ipk(\pi) = k$, then there are $rk + 1$ ways to obtain a permutation in $\Phi_r(Q_{n,r})$ with $k$ interior peaks.

2. For $i \notin \{\ell + j : \ell \in ipk(\pi) \land j \in \{-1, 0\} \cup [r - 2]\}$, inserting $(rn)(rn - 1) \cdots (rn - r + 1)$ into the right-hand side of $\pi_i$ will increase the number of $ipk(\pi)$ by 1. Thus, if $ipk(\pi) = k - 1$, then there are $r(n - 1) - r(k - 1) = r(n - k)$ ways to obtain a permutation in $\Phi_r(Q_{n,r})$ with $k$ interior peaks.

Then we can get the following recurrence relation for $M_{n,r,k}$:

$$M_{n,r,k} = (rk + 1)M_{n-1,r,k} + r(n - k)M_{n-1,r,k-1}. \quad (5.15)$$

By (5.15), $M_{n,r}(x)$ satisfies the recurrence relation:

$$\begin{cases} M_{n,r}(x) = [(rn - r)x + 1] M_{n-1,r}(x) + rx(1 - x) D_1 M_{n-1,r}(x), \\
M_{1,r}(x) = 1, M_{2,r}(x) = 1 + rx. \quad (5.16)\end{cases}$$
In fact, \( M_{n,r}(x) \) is equivalent to the \( 1/r \)-Eulerian polynomial \( \mathcal{A}_n^r(x) \) because

\[
\mathcal{A}_n^r(x) = [(rn - r)x + 1] \mathcal{A}_{n-1}^r(x) + rx(1 - x)D_x \mathcal{A}_{n-1}^r(x)
\]

with \( \mathcal{A}_n^r(1) = 1 \), see [24, 66].

Similarly, \( \widetilde{M}_{n,r}(x) \) satisfies the recurrence relation:

\[
\begin{align*}
\widetilde{M}_{n,r}(x) &= (rn - r + 1)x \widetilde{M}_{n-1,r}(x) + rx(1 - x)D_x \widetilde{M}_{n-1,r}(x), \\
\widetilde{M}_{0,r}(x) &= 1, \widetilde{M}_{1,r}(x) = x.
\end{align*}
\]

By (5.16) and (5.17), we obtain \( M_{n,r}(x) = I_n(\widetilde{M}_{n,r}(x)) \). Obviously, \( \widetilde{M}_{n,r}(x) \) is a special case of the generalized Eulerian polynomial \( J_n(x) \) in (2.14) by taking \( d = 0 \) and \( \lambda = 1 \).

By Corollary 3.4, the following result is immediate.

**Proposition 5.12.** Let \( (M_{n,r}(x))_{n \geq 0} \) and \( (\widetilde{M}_{n,r}(x))_{n \geq 0} \) be defined by (5.16) and (5.17), respectively. Then the Turán expressions of \( (M_{n,r}(x))_{n \geq 0} \) and \( (\widetilde{M}_{n,r}(x))_{n \geq 0} \) are Hurwitz stable for all \( r \geq 2 \).

**Remark 5.13.** Obviously, Proposition 5.12 implies that all \( (M_{n,r}(q))_{n \geq 0} \), \( (\widetilde{M}_{n,r}(q))_{n \geq 0} \), and \( (\mathcal{A}_n^r(q))_{n \geq 0} \) are \( q \)-log-convex for any \( r \geq 2 \). In fact, they are all \( q \)-Stieltjes moment by Theorem [84, Theorem 1.3], i.e., all minors of their Hankel matrices are polynomials with nonnegative coefficients.

Constructing a new polynomial sequence \( (T_{n,r}(x))_{n \geq 0} \) as follows:

\[
(1 + x)T_{n,r}(x) := xM_{n,r}(x^2) + \widetilde{M}_{n,r}(x^2).
\]

By (5.16)-(5.18), we get the recurrence relation of \( T_{n,r}(x) \) as follows:

\[
T_{n+1,r}(x) = \left( rnx^2 + \frac{rx - r + 2}{2} \right) T_{n,r}(x) + \frac{rx}{2}(1 - x^2)D_x T_{n,r}(x) + \frac{r - 2}{2}(1 - x)\widetilde{M}_{n,r}(x^2).
\]

Based on empirical evidence and computer’s arithmetic for \( T_{n,r}(x) \), we propose the following conjecture.

**Conjecture 5.14.** Let \( T_{n,r}(x) \) satisfy (5.18). Then \( T_{n,r}(x) \) is Hurwitz stable for all \( r \geq 2 \) and \( n \in \mathbb{N} \).

Note \( M_{n,r}(x) = I_n(\widetilde{M}_{n,r}(x)) \). Thus if this conjecture is true, then it implies that both \( M_{n,r}(x) \) and \( \widetilde{M}_{n,r}(x) \) are alternately increasing for all \( r \geq 2 \) and \( n \in \mathbb{N} \). In the following, we will prove this conjecture for \( r = 2 \). Before it, we need a criterion for two zeros-interlacing polynomials.

Suppose that

\[
f(z) = \sum_{k=0}^{n} a_k z^k.
\]

Let

\[
f^E(z) = \sum_{k=0}^{[n/2]} a_{2k} z^k \quad \text{and} \quad f^O(z) = \sum_{k=0}^{[(n-1)/2]} a_{2k+1} z^k.
\]

Then, the following result is an equivalent form of Hermite-Biehler Theorem.
**Theorem 5.15.** [62, Theorem 6.3.4] Let \( f(z) = z f^O(z^2) + f^E(z^2) \) be a polynomial with real coefficients. Suppose that \( f^E(z) f^O(z) \neq 0 \). Then \( f(z) \) is Hurwitz stable if and only if \( f^E(z) \) and \( f^O(z) \) have only real and non-positive zeros, and \( f^O(z) \ll f^E(z) \).

Thus, we have the following result.

**Proposition 5.16.** Let \((a,b)\) be the (symmetric) \(I_n\)-decomposition of \(\tilde{M}_{n,2}(x)\). Then \(T_{n,2}(x)\) is Hurwitz stable for \(n \in \mathbb{N}\) and \(b \ll a\). In particular, \(M_{n,2}(x)\) and \(M_{n,2}(x)\) are alternatingly increasing for \(n \in \mathbb{N}\).

**Proof.** By (5.18), for \(r = 2\), we get

\[
(1 + x)T_{n,2}(x) = xM_{n,2}(x^2) + \tilde{M}_{n,2}(x^2).
\]

Moreover, we have

\[
T_{n+1,2}(x) = (2nx^2 + x)T_{n,2}(x) + x(1 - x^2)D_xT_{n,2}(x),
\]

where \(T_{0,2}(x) = 1\) and \(T_{1,2}(x) = x\). This coincides with (2.15). Thus Proposition 2.16 implies that \(T_{n,2}(x)\) is Hurwitz stable. By Theorem 5.15, we have

\[
I_n(\tilde{M}_{n,2}(x)) = M_{n,2}(x) \ll \tilde{M}_{n,2}(x).
\]

It is equivalent that \(b \ll a\) by Theorem 5.1. And thus, \(\tilde{M}_{n,2}(x)\) is alternatingly increasing for all \(n \in \mathbb{N}\).

Let \((\tilde{a},\tilde{b})\) be the \(I_{n-1}\)-decomposition of \(M_{n,2}(x)\). Note that \(M_{n,2}(x) = I_n(\tilde{M}_{n,2}(x))\) and the degree of \(M_{n,2}(x)\) is \(n - 1\), then

\[
M_{n,2}(x) \ll \tilde{M}_{n,2}(x) = I_n(M_{n,2}(x)) = xI_{n-1}(M_{n,2}(x)).
\]

That is to say, \(I_{n-1}(M_{n,2}(x)) \ll M_{n,2}(x)\), i.e., \(\tilde{b} \ll \tilde{a}\). Thus, \(M_{n,2}(x)\) is alternatingly increasing for all \(n \in \mathbb{N}\).

\[\square\]

**Remark 5.17.** The alternatingly increasing property of \(M_{n,2}(x)\) and \(\tilde{M}_{n,2}(x)\) was also proved in [54, Theorem 12] in a different way. Here our Proposition 5.16 gives a stronger result than the alternatingly increasing property.

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