HYPOCOERCIVITY OF LANGEVIN-TYPE DYNAMICS ON ABSTRACT SMOOTH MANIFOLDS

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Abstract. In this article we investigate hypocoercivity of Langevin-type dynamics in nonlinear smooth geometries. The main result stating exponential decay to an equilibrium state with explicitly computable rate of convergence is rooted in an appealing Hilbert space strategy by Dolbeault, Mouhot and Schmeiser. This strategy was extended in [GS14] to Kolmogorov backward evolution equations in contrast to the dual Fokker-Planck framework. We use this mathematically complete elaboration to investigate wide ranging classes of Langevin-type SDEs in an abstract manifold setting, i.e. (at least) the position variables obey certain smooth side conditions. Such equations occur e.g. as fibre lay-down processes in industrial applications. We contribute the Lagrangian-type formulation of such geometric Langevin dynamics in terms of (semi-)sprays and point to the necessity of fibre bundle measure spaces to specify the model Hilbert space.

1. Introduction

A huge amount of research is going on in the area of hypocoercivity, hypoellipticity and diverse analytic methods to study long-time behaviour of degenerated stochastically perturbed systems. Herein, we concentrate on a hypocoercivity method applied to (geometric) Langevin equations, see [CKW12] for some background of these equations and physical or chemical applications. For the hypocoercivity approach we think of Langevin equations as evolution equations of Kolmogorov backward type. Formulated as an abstract Cauchy problem in a Hilbert space a clever choice of an entropy functional gives rise of a certain norm on this Hilbert space measuring the desired exponential decay towards an equilibrium. This is the fundamental idea by J. Dolbeault, C. Mouhot and C. Schmeiser for a hypocoercivity strategy, see [DMS15]. However, we use the Kolmogorov backward (hypocoercivity) setting developed in [GS14], because our focus lies on SDEs. It’s by no means clear how to apply their Hypocoercivity Theorem in case of stochastically perturbed mechanics on an abstract position manifold, since there are various approaches and terminologies: Y. Gliklikh discusses Langevin equations on manifolds of Itô-type, see [Gli97, Section 17]; in his opinion the Itô formulation is the most natural one. However, he rarely talks about generators whereas by [WI81, Theorem V.1.2] one easily gets the infinitesimal generators of certain Stratonovich SDEs. Besides, in [WI81, Section V.4] it is explained how the theory of diffusions on manifolds in terms of Stratonovich SDEs as motion in the frame bundle is strongly connected to Itô’s stochastic parallel displacement. V. Kolokoltsov uses a notion of stochastic Hamiltonian systems to study a ‘curvilinear Ornstein-Uhlenbeck processes’ on the cotangent space, see [Kol00, Chapter 4]. Using other ideas from classical mechanics one could investigate critical points of the stochastic Hamilton-
Pontryagin action integral see e. g. [BO09], in particular [BO09, Theorem 3.2]. This might be interesting from a computational point of view and could be linked to the other formulations via local Lagrangian vector fields. But Kolokoltsov’s approach is directly based on a certain generator – another approach of this kind can be found in [Sol95]. Kolokoltsov prefers local coordinate forms, however he is aware of [Jo78] providing a construction of an ‘Ornstein-Uhlenbeck process’ in the tangent space in invariant form. E. Jørgensen uses the McKean-Gangolli injection scheme, see [JM69] and [Gan64], to construct his process projecting a process in the frame bundle which is a slightly inconvenient state space for real life applications.

The term (classical) Langevin equation refers in the purely Euclidean setting to the following system of equations:

\[
\begin{align*}
    dx_t &= v_t \, dt \\
    dv_t &= -\nabla \Psi(x_t) \, dt + \sigma \circ dW_t - \alpha \cdot v_t \, dt, 
\end{align*}
\]

where \( x_t \in \mathbb{R}^d \) are positions in a space \( \mathbb{R}^d \) and \( v_t \in \mathbb{R}^d \) are velocities for all times \( t \in [0, \infty) \) respectively – the spaces \( \mathbb{R}^d \) and \( \mathbb{R}^d \) are thought as independent copies of \( \mathbb{R}^d \). The model parameter \( \alpha \) is interpreted as a friction parameter, similarly \( \sigma \) as a diffusion parameter – both are nonnegative. The potential \( \Psi: \mathbb{R}^d \to \mathbb{R} \) satisfies certain (weak) regularity properties and \( W = (W_t)_{t \in [0, \infty)} \) is a \( d \)-dimensional Wiener process.

In fibre lay-down applications, where the Langevin equation is used as a surrogate model, one additionally would assume that the (Euclidean) norm of the velocities is 1 constantly, i.e. \( v_t \in S^{d-1} \subseteq \mathbb{R}^d \) for all times \( t \).

We follow the philosophy of describing a stochastic dynamic via its Kolmogorov backwards generator in invariant form. In Section 3, we stay rather close to classical Lagrangian mechanics, since the ‘space of velocities \( v' \) \( Q = TM \) serves as configuration manifold\(^1\) over the ‘space of positions \( x' \) namely \( M \), and we consider an evolution in the velocity phase space \( TQ = TTM \), so in a (double) tangent space.

Together with the fact that Langevin-type equations are second order differential equations this leads quite naturally to Ehresmann connections and semisprays; both concepts are closely related on a purely geometric level. We have been imbued with these ideas during the reading of works by I. Bucataru e.g. [Buc12, BCD11]. In turn, those are rooted in results e.g. by M. Crampin, J. Grifone or R. Miron. Later on, in Section 4 we talk about fibre lay-down models and demonstrate how to geometrically implement an algebraic side condition like normalised velocities. For sake of completeness, we also mention [Bis15] wherein J.-M. Bismut talks about Langevin processes in terms of hypoelliptic Laplacians, i.e. as a diffusion interpolating Brownian motion and geodesic flow.

A key achievement of this paper is that we explicitly don’t need parallelisability of \( M \). It’s a wide spread mistake to work in tangent bundles treating them like trivial bundles, even though it’s very well-known that most of the spheres are not parallelisable – e.g. \( S^2 \) as the Hairy Ball Theorem shows. Indeed, since the works [Ker58] by M. Kervaire and independently [BM58] by R. Bott and J. Milnor it’s known that \( S^{d-1} \) is parallelisable exactly for \( d \in \{1, 2, 4, 8\} \). E.g. E. Jørgensen was sensitised to the issue as the last remark in [Jo78, Section 1] shows. If our model fails to capture the geometry of general spheres, we didn’t find a reasonable model. For that reason, we abolish notation like \( \langle x, v \rangle \in TM \) with position \( x \) and velocity component \( v \) completely. Instead we will be very careful to always emphasise the bundle structure: Throughout this paper, we always denote by \( \pi_0 \) a tangent bundle projection no matter what the base manifold is and extract the information on

\(^1\)E.g. in [MR99], this is also called configuration space, however we reserve this term for the configuration space as a model of multiparticle systems. The configuration space formalism might become handy, if one studies fibre lay-down models of multiple filaments at once.
the position from a tangent vector \( v \) via \( x = \pi_0(v) \) – i.e. \( \pi_0 \) serves as an accessor or ‘getter’ method. In the absence of a product structure of tangent spaces, we rely on the ‘almost product’ structure induced by the universal property of local trivialisation. As the example of spheres illustrates, this is not just a technicality to be wiped out crudely with embeddings into larger Euclidean spaces.

Note that our configuration manifold is going to be \( Q = TM - \) or some subfibre bundle, see Section 4. Therefore, we don’t need \( M \) itself to be orientable. The commonly used integration of differential forms on manifolds instead of functions is extended by integration wrt. so-called 1-densities, where the latter concept does not need orientability. We won’t discuss this in Section 2, but refer to [Fo99, Section 11.4] and [Nic96, §3.4.1] as well as to [BGV92, p. 34] for the existence and to [Nic96, Example 3.4.2] for elementary properties of the canonical 1-density associated to the Riemannian metric. Such a notion of integration on \( M \) is absolutely sufficient for our purposes. The integration by parts formula, which enables us to use techniques related to generalised Dirichlet form theory in the first place, has to hold not for integration over the position manifold \( M \), but for integration over the configuration manifold \( Q \) which will automatically be orientable. Hence, we will just talk about a ‘Riemannian volume measure \( \lambda_m \)’ on the Riemannian manifold \((M, m)\): either there is some orientation and this measure is induced by the canonical volume form \( d\lambda_m \), or there is none and the measure is induced by the canonical 1-density \( |d\lambda_m|\).

In Section 5 we first prove existence of Markov processes solving the SDEs (3.1) and (4.2) treated in the preceding sections. By finding appropriate cores for the corresponding generators \( (L, D(L)) \) and proving m-dissipativity in Section 3 or Section 4 respectively, we have a strongly continuous semigroup \((T_t)_{t \in [0,\infty)} \) generated by \( (L, D(L)) \) on the model Hilbert space \( H = L^2(Q; \mu) \), where \( \mu \) is the equilibrium distribution. For a suitable test function \( g : Q \to \mathbb{R} \) the assignment \( u(t, \cdot) = T_tg \) yields a solution to the abstract Cauchy problem \( \dot{u} = Lu \) on the model Hilbert space. Using the theory of generalised Dirichlet forms we show that there is a \( Q \)-valued Markov process \((X_t)_{t \in [0,\infty)} \) which solves the \( L \)-martingale problem, has \( \mu \) as invariant measure, is conservative, and most importantly it’s properly associated to \((L, D(L)) \) in the resolvent sense. The latter means that the transition resolvent \( \int_{(0,\infty)} \exp(-as)E_{t\mu}(g(X_t)) \lambda(ds) \) is a quasi-continuous \( \mu \)-version of the resolvent \( \int_{(0,\infty)} \exp(-as)T_tg \lambda(ds) \) for all \( a \in (0,\infty) \) and functions \( g \in H \). Second, we show that these Markov processes are \( L^2 \)-exponentially ergodic in the sense of [CG10].

We summarise our main results as follows:

- We heavily make use of rather geometric concepts like semisprays and Ehresmann connections to formulate analytic problems and objects in invariant form. Thereby, we choose a quite accessible Lagrangian-type approach to higher-order SDEs on manifolds. Even though the geometric tools themselves are well-known, they have not been used to treat higher-order SDEs on abstract manifolds systematically.

- The applications of the general Hilbert space hypoocoercivity strategy presented in [GS14] and [GS16] are extended to the case of a quite abstract position manifold. During this course we see that strong mixing with exponential convergence to equilibrium of the corresponding semigroups are features of Langevin-type equations all across (finite-dimensional) smooth geometry with just view natural geometric assumptions. The main theorems are Theorem 3.2 and Theorem 4.1.

- Among possible applications of Langevin equations on manifolds we spotlight their usage as surrogate model of fibre lay-down processes in industrial
production of nonwovens. See Section 4 and in particular Theorem 4.1. Compare these results to e.g. [GKMW07, KMW12, GS13].

• In Section 5 we first prove in Theorem 5.1 the existence of a Markov process solving the martingale problem for our Langevin-type SDEs, namely (3.1) and (4.2). Afterwards we deduce that these solutions are \( L^2 \)-exponentially ergodic in the sense of [CG10], meaning that the rate of ergodicity corresponds to exponential convergence of the semigroups. In the end, we argue that the \( L^2 \)-exponential rate is even optimal. See Corollary 5.2 and the subsequent remark.

2. Preliminaries

Before we give a brief recap on the general hypocoercivity method and establish several geometrical tools, we fix the assumptions on the position manifold.

Condition 2.1 (Position manifold (M)).

(M1) general geometry: Let \((M, m)\) be a real, finite dimensional, connected Riemannian manifold with \( n := \dim(M) \geq 2 \).

(M2) completeness: Let \( M \) endowed with the intrinsic metric be a complete metric space\(^2\).

Note that following a common way of speaking, the manifold as described in (M1) has trivial boundary \( \partial M = \emptyset \). In this paper, we do not address boundary problems for sake of simplicity. From an analytic point of view, assumption (M2) is plausible in its own right and necessary e.g. for a discussion of Sobolev spaces on noncompact manifolds, see [Heb99, Chapter 3]. By the Hopf-Rinow Theorem, we have that (M2) is equivalent to geodesic completeness of \( M \). If we additionally assume an orientation on a manifold which satisfies (M1) and (M2), then it is well-known that the usual Laplace-Beltrami operator on smooth test functions is essentially self-adjoint. See [Gaf51, Gaf54a, Gaf54b] for generalisations of the standard case on compact manifolds as well as [jLM89, Section II.5] for general statements on so-called Dirac operators on spinors. Moreover, we also want to recommend [Wol73]. If we would assume nontrivial boundary, then essentially self-adjointness of the Laplace-Beltrami operator is rather delicate. This challenging problem has been tackled in [Pe13] and the noteworthy paper [ILP15].

2.1. General hypocoercivity method. At this point, we want to give an almost criminally brief overview of the hypocoercivity method in the abstract Hilbert space setting. If the reader is familiar with this topic, this section just clarifies some notation.

Originally, the method was algebraically developed by J. Dolbeault, C. Mouhot and C. Schmeiser in [DMS09] and [DMS15] studying linear kinetic equations – algebraically in the sense that issues of operator domains have been neglected. Substantial contributions on the general hypocoercivity approach can be found in [Vil07] and [Vil09] – e.g. the oncoming condition (P3) on the potentials Hessian appears in [Vil09, Theorem 35] first. Our main reference is [GS14] for two reasons. On conceptual side, the authors really tackled the long-time behaviour of solutions of the SDE via the Kolmogorov backward setting instead of investigating the Fokker-Planck equation. In the case of Euclidean position space, normalised velocity, and the invariant measure having a density wrt. Lebesgue measure there is an isometric isomorphism the Kolmogorov backward to the Fokker-Planck setting, see [GMS12].

\(^2\)The intrinsic metric induces the same topology as given on \( M \) originally. This is a consequence of the very definition of a (topological) manifold as in [Lee12] which we choose to use here.
However, in a general manifold it is not clear how to map between these two settings. On technical side, to gain a rigorous proof of the Hypocoercivity Theorem and to check its assumptions in applications several domain issues have to be taken into account, which are often times just omitted. For broader discussion and more detailed explanations we also refer to [Sti14]. The main result, we aim to apply to our fibre lay-down model, tells us what microstructure we have to expect in the fleece: The faster the convergence to the equilibrium state the more uniform the nonwoven material will appear. The formal result reads as follows.

**Theorem 2.2** (Hypocoercivity Theorem). Assume the conditions (D) as well as (H) (as given below) and denote by \((T_t)_{t \in [0, \infty)}\) the operator semigroup generated by \(L\) on the Hilbert space \(H\).

Then, there exist constants \(\kappa_1, \kappa_2 \in (0, \infty)\) computable in terms of the constants \(\Lambda_m, \Lambda_M, c_1\) and \(c_2\) appearing in the assumptions such that it holds

\[
\|T_t g - (g, 1)_H\|_H \leq \kappa_1 e^{-\kappa_2 t}\|g - (g, 1)_H\|_H \quad \text{for all } t \geq 0
\]

and for all \(g \in H\).

**Proof.** See [GS14, Theorem 2.18]; in its proof one learns how to compute the constants \(\kappa_1\) and \(\kappa_2\). \(\square\)

Now, we are going to explain what the sets of conditions (D) and (H) are. We just mention as a fact that there are more general data assumptions such that the Hypocoercivity Theorem is still valid, see [Sti14, Section 2.2.3].

**Condition 2.3** (Data conditions (D)).

(D1) **model Hilbert space:** Let \((E, \mathcal{C}, \mu)\) be a probability space and define the Hilbert space \(H\) to be \(L^2(E; \mu) = L^2(\mu)\).

(D2) **Strongly continuous semigroup and its infinitesimal generator:** Let \((L, D(L))\) be a linear operator on \(H\) and \((T_t)_{t \in [0, \infty)}\) be the strongly continuous semigroup generated by \(L\), i.e. \(T_0 = 1_{Id_H}\) and \(T_t f \to f\) as \(t \downarrow 0\) for all \(f \in H\).

(D3) **Core property:** Let \(D \subseteq D(L)\) be dense in \(H\) and an operator core of \((L, D(L))\), i.e. the closure of \((L, D)\) coincides with \((L, D(L))\).

(D4) **SAD-Decomposition of generator into symmetric and antisymmetric part:** Let \((S, D(S))\) be symmetric and let \((A, D(A))\) be closed and antisymmetric on \(H\) s.t. \(D \subseteq D(S) \cap D(A)\) and the restriction of \(L\) to the core can be decomposed as \(L|_D = S - A\).

(D5) **Projection:** Let \(P : H \to H\) be an orthogonal projection such that \(P(H) \subseteq D(S)\) and \(SP = 0\) as well as \(P(D) \subseteq D(A)\) and \(AP(D) \subseteq D(A)\). Define

\[
P_S : H \to H, \ f \mapsto Pf + (f, 1)_H.
\]

(D6) **Invariant measure:** Let \(\mu\) be invariant for \((L, D)\) in the sense that

\[
(Lf, 1)_H = \int_E Lf \, d\mu = 0\quad \text{for all } f \in D.
\]

(D7) **semigroup conservativity:** Let \(1 \in D(L)\) and \(L1 = 0\).

**Condition 2.4** (Hypocoercivity conditions (H)).

(H1) **algebraic relation:**

\[
PAP|_D = 0
\]

(H2) **microscopic coercivity:**

\[
\exists \Lambda_m \in (0, \infty) \forall f \in D : \Lambda_m \| (\text{Id} - P_S)f \|_H^2 \leq -(Sp, f)_H
\]

(H3) **macroscopic coercivity:**

\[
\exists \Lambda_M \in (0, \infty) \forall f \in D((AP)^*(AP)) : \Lambda_M \| Pf \|_H^2 \leq \| APf \|_H^2
\]
(H4) boundedness of auxiliary operators:
\[ \exists c_1, c_2 \in (0, \infty) \forall f \in D: ||Bf||_H \leq c_1 ||(\text{Id} - P_j)f||_H \]
\[ \land ||B(\text{Id} - P)f||_H \leq c_2 ||(\text{Id} - P_j)f||_H \]
for \( B := (\text{Id} - (AP)(AP)^*)(AP)^* \) on \( D((AP)^*) \) and projections \( P_j \in \{P, P_S\}, j \in \{1, 2\} \).

2.2. Bundle measures and weighted bundles. We start with defining bundle measures: Thinking of measures on some manifold \( \mathcal{B} \) we should search for induced measures in bundles over \( \mathcal{B} \), i.e. natural measures on the respective total spaces. Proceeding on the measure theoretic path in the direction of Radon-Nikodým derivatives, the well-known notion of weighted manifolds is revisited. The concept of multiplying a density weight function to a volume measure is not to be confused with conformal transformations of manifolds. Indeed, conformal transformations affect the geometry; the difference is the definition of the divergence and thus the integration by parts formula (formations affect the geometry; the difference is the definition of the divergence and thus the integration by parts formula (2.1) below. As far as we know, there is very little literature on measure theory paired with the more geometric concept of fibre bundles. After developing a notion of bundle measures on our own, we found the seemingly unknown paper [Goe99], where the author concedes more degrees of topological freedom. Also, there is [Nic96, Section 3.4.5] wherein the author describes a 'fibred calculus' just for smoothly indexed families of manifolds. We give these references for sake of completeness, even though we stay with our bespoke construction of bundle measures.

Throughout this section, let \( \pi : \mathcal{E} \rightarrow \mathcal{B} \) be a smooth fibre bundle with standard fibre \( F \) and define \( \mathcal{E}_b := \pi^{-1}(\{b\}) \) for all \( b \in \mathcal{B} \). Furthermore, we suppose that the fibre is actually a measurable space \( (F, \mathcal{F}) \). The property of local trivialisation yields a natural \( \sigma \)-algebra \( \mathfrak{B}(\mathcal{B}) \otimes_{\text{loc}} \mathcal{F} := \sigma(\mathfrak{G}) \) on \( \mathcal{E} \) with generator
\[
\mathfrak{G} := \left\{ \varphi^{-1}(U \times U_F) \mid \begin{array}{l}
 b \in \mathcal{B}, U_F \in \mathcal{F}, W \text{ chart domain at } b, \\
 U_b \subseteq \mathcal{B} \text{ open neighbourhood of } b \text{ s.t.} \\
 \text{diffeomorphism } \varphi \text{ renders the diagram in Figure 1 commutative,} \\
 U := U_b \cap W.
\end{array} \right\}
\]
One might say that \( \mathfrak{B}(\mathcal{B}) \otimes_{\text{loc}} \mathcal{F} \) is the canonical \( \sigma \)-algebra on \( \mathcal{E} \), however we will refer to it as the local product-\( \sigma \)-algebra. If \( \mathcal{F} = \mathfrak{B}(F) \), then \( \mathfrak{B}(\mathcal{B}) \otimes_{\text{loc}} \mathcal{F} = \mathfrak{B}(\mathcal{E}) \).

\[ \pi^{-1}(U_b) \xrightarrow{\varphi} U_b \times F \]
\[ \pi \]
\[ U_b \]
\[ \text{pr}_1 \]

**Figure 1.** Local trivialisation of a fibre bundle

In the next definition, we restrict ourselves to probability measures as this is the sole situation of interest in the present paper.

**Definition 2.5** (bundle measure). Let \( \mu_B \) and \( \nu_F \) be probability measures on \( (\mathcal{B}, \mathfrak{B}(\mathcal{B})) \) and \( (F, \mathcal{F}) \) respectively. The (fibre) bundle measure on the total space \( \mathcal{E} \) is the pullback measure \( \mu_E := \pi^* \mu_B \) of \( \mu_B \) wrt. \( \pi \) supplemented with the fibre measure \( \nu_F \). That is the unique probability measure on \( \mathcal{E} \) satisfying
\[
\int_{\mathcal{E}} f \, d\mu_E = \int_{\mathcal{B}} \int_{\pi^{-1}(\{b\})} f|_{\pi^{-1}(\{b\})}(e) \, d\nu_F(e) \, d\mu_B(b)
\]
for all bounded \( \mathfrak{B}(\mathcal{B}) \otimes_{\text{loc}} \mathcal{F} - \mathfrak{B}(\mathbb{R}) \)-measurable functions \( f : \mathcal{E} \rightarrow \mathbb{R} \).
Remark 2.6. In Definition 2.5, the measure \(\nu_F\) on the fibre \(\pi^{-1}(b)\) is thought as an independent copy of \(\nu_F\) defined on \(F\). By definition a bundle measure yields a disintegration, but obviously the former concept is motivated by the local product structure of fibre bundles and tries to find an analogue of product measures respecting this structure, whereas the latter is kind of the ‘factorisation’ of measures that are not necessarily product measures. In all generality, it’s a difficult problem to give conditions for a disintegration to exists; in our situation it appears as a byproduct.

The following lemma is stated as an equivalent characterisation of bundle measures, however it could be formulated as an existence statement, since its proof is constructive.

Lemma 2.7 (bundle measures locally are product measures). Let \(\mu_B\) and \(\nu_F\) be probability measures on \(\mathcal{B}\) and \(F\) respectively. Denote by \(\mu_E := \pi^*\mu_B\) the (fibre) bundle measure on \(E\) supplemented with fibre measure \(\nu_F\). Then, \(\mu_E\) is the unique measure on \(E\) which locally trivialises to the product measure \(\mu_B \otimes \nu_F\) in the following sense: By definition of fibre bundles, for any \(b \in \mathcal{B}\) there is a neighbourhood \(U_b \subseteq \mathcal{B}\) of \(b\) as well as a diffeomorphism \(\varphi\) which renders the diagram in Figure 1 commutative. Let \(V \in \mathcal{B}(\mathcal{B}) \otimes_{\text{loc}} \mathcal{B}\), \(W\) a chart domain at \(b\) and define \(U := W \cap U_b\) as well as \(V_U := V \cap \pi^{-1}(U)\). Then, \(\mu_E\) obeys the transformation rule

\[
\mu_E = (\varphi^{-1})_*(\mu_B \otimes \nu_F) \quad \text{on } V_U.
\]

We say that \(\mu_E\) is a local-product measure and introduce \(\mu_E = \mu_B \otimes_{\text{loc}} \nu_F\) as the corresponding notation.

Proof. First, we construct the bundle measure just from the local transformation rule – basically, that’s the proof of existence of such a measure. Note that measurable sets of the same form as \(V_U\) generate the local product-\(\sigma\)-algebra on \(E\), that they are linked to cylinder sets of the product-\(\sigma\)-algebra on \(U \times F\) via \(\varphi\), and that they form a family stable wrt. intersections, when the empty set is included of course. Since \(\varphi\) is continuously invertible, its inverse is measurable. As the pushforward measure of \(\mu_E\) wrt. \(\varphi\) should be \(\mu_B \otimes \nu_F\), we declare \(\mu_E\) to be the pushforward measure of \(\mu_B \otimes \nu_F\) wrt. \(\varphi^{-1}\). It satisfies the desired transformation rule by construction. By Fubini-Tonellis this is the particular instance of the integral equation

\[
\int_E f \, d\mu_E = \int_B \int_{\pi^{-1}(b)} f \, |\det(d\varphi^{-1})| \, d\nu_F(e) \, d\mu_B(b)
\]

with \(f = \mathbb{1}_{V_U}\). We use this to obtain the general equation via approximating bounded measurable functions with simple functions and apply Lebesgue dominated convergence. The uniqueness assertion is fulfilled by the general uniqueness theorem for measures which finishes the proof. \(\square\)

Example 2.8 (Möbius strip). A Möbius strip can be thought as fibre bundle with base manifold \(\mathcal{B} = S^1\) and standard fibre \(F\) being an open interval of finite length. The natural volume measure on the strip coincides on \(\Theta\) with a bundle measure where the base measure \(\mu_B\) is the volume measure on the circle and the fibre measure is the Lebesgue measure on an interval: \(\nu_F = \lambda\). Since \(\Theta \cup \{\emptyset\}\) is a generator stable wrt. intersections, the volume measure on the Möbius strip coincides with the Lebesgue-type bundle measure. For a set \(V_U := V \cap \pi^{-1}(U)\) we have that

\[
\mu_E(V_U) = \int_{\pi(V_U) \times \text{pr}_2 \circ \varphi(V_U)} |\det(d\varphi^{-1})| \, d\mu_B \otimes \nu_F,
\]

where \(\text{pr}_2\) denotes projection to the second component.
Remark 2.9. At this point, it should be clear that $L^2(\mathbb{E}; \mu_\mathcal{B} \otimes_{\text{loc}} \nu_F)$ is not isomorphic to the Hilbert space tensor product $L^2(\mathbb{B}; \mu_\mathcal{B}) \otimes L^2(F; \nu_F)$. Anyway, there might be nice spaces of test functions that are dense in $L^2$-spaces for bundle measures. For sake of simplicity, suppose that the base measure is absolutely continuous wrt. a given Riemannian volume measure and moreover that $\mu$ and $\rho$ are base weight and fibre weight. Therefore, we introduce the corresponding notation $(\text{existence and uniqueness of bundle weightings})$

Definition 2.10 (weighted (fibre) bundles). Consider functions $\rho_\mathcal{B} : \mathbb{B} \to [0, \infty)$ and $\rho_F : F \to [0, \infty)$. A function $\rho : \mathbb{E} \to [0, \infty)$ is called bundle weighting with base weight $\rho_\mathcal{B}$ and fibre weight $\rho_F$ if it locally trivialises to the product function $\rho_\mathcal{B} \cdot \rho_F$ in the following sense: Consider the local trivialisation over an open neighbourhood $U_b \subseteq \mathbb{B}$ at $b$ with diffeomorphism $\varphi$ rendering the diagram in Figure 1 commutative. Then, $\rho$ satisfies

$$\rho(v) = \rho_\mathcal{B}(\pi(v)) \cdot \rho_F(\varphi_2 \circ \varphi(v)) \quad \text{for all } v \in \pi^{-1}(U_b).$$

A fibre bundle together with a bundle weight is called a weighted (fibre) bundle.

Lemma 2.11 (existence and uniqueness of bundle weightings). In the situation of Definition 2.10 a bundle weight $\rho$ exists and is uniquely determined by base weight and fibre weight. Therefore, we introduce the corresponding notation $\rho = \rho_\mathcal{B} \otimes_{\text{loc}} \rho_F$.

Proof. In the local trivialisation the only possible weight function is given via

$$\rho = ((\rho_\mathcal{B} \circ \text{pr}_1) \cdot (\rho_F \circ \text{pr}_2)) \circ \varphi \quad \text{on } \pi^{-1}(U_b).$$

Fix an open cover of $\mathbb{B}$ by chart domains, then the preimages form an open cover of $\mathbb{E}$. With a partition of unity subordinate to the latter cover, we can glue the definitions of $\rho$ in the respective local trivialisations together which finishes the proof.

Remark 2.12 (local Radon-Nikodým derivatives). Suppose some bundle measure $\mu_\mathcal{E} = \mu_\mathcal{B} \otimes_{\text{loc}} \nu_F$ such that $\mu_\mathcal{B}$ has a Radon-Nikodým derivative $\rho_\mathcal{B} := \frac{d\mu_\mathcal{B}}{d\nu_F}$ wrt. another measure $\mu_\mathcal{B}$ on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$ and also $\nu_F$ has a Radon-Nikodým derivative $\rho_F := \frac{d\nu_F}{d\mu_\mathcal{E}}$. Then, we think of the induced bundle weighting $\rho = \rho_\mathcal{B} \otimes_{\text{loc}} \rho_F$ as a local Radon-Nikodým derivative in view of it holds

$$\mu_\mathcal{E} = \mu_\mathcal{B} \otimes_{\text{loc}} \nu_F = (\rho_\mathcal{B} \mu_\mathcal{B}) \otimes_{\text{loc}} (\rho_F \nu_F)$$

$$= (\rho_\mathcal{B} \otimes_{\text{loc}} \rho_F)(\mu_\mathcal{B} \otimes_{\text{loc}} \mu_\mathcal{E}) = \rho(m_\mathcal{B} \otimes_{\text{loc}} m_\mathcal{E}).$$

For sake of brev’ty, we call $\rho$ loc-density of $\mu_\mathcal{E}$.

Remark 2.13 (Ehresmann connection and weighting). We emphasise again that weighting does not effect geometry: The Ehresmann connection induced by the metric $m$ on $\mathcal{M}$, see Section 2.3 below, does not change in the course of the weighting procedure. Indeed, the corresponding Levi-Civita connection $\nabla^m$ is not affected by reweighting as one can easily check using the Leibniz rule when checking the metric compatibility condition. Also several geometric objects associated to this connection just depend on the original metric $m$ and not on the weight function.
Example 2.14 (trivial bundles). If the fibre bundle is trivial, then bundle measures are pushforwards of product measures wrt. the trivialisation isomorphism. Loc-densities basically are products of the densities for the respective measures.

One particularly simple example arises with the standard fibre being a singleton. Then, every fibre measure is absolutely continuous wrt. the Dirac measure for the single point in $F$ and fibre weightings reduce to multiplication with a constant factor.

Example 2.15 (weighted manifolds). Consider an orientable Riemannian manifold $(B, b)$ and a strictly positive, nonconstant, smooth\(^3\) weight function $\rho_B \in C^\infty(B)$. The Riemannian metric weighted by $\rho_B$ or just $\rho_B$-weighted metric on $B$ is given as

$$b(v, w) := b(\rho_B v, \rho_B w)$$

for all $v, w \in \mathbb{T}B$ with $\pi_0(v) = \pi_0(w)$. We do not keep the usual definition of gradients for this weighted metric, but in fact choose the $\rho_B$-weighted gradient as $\nabla_b := \frac{1}{\rho_B} \nabla_b$. I.e. for all smooth vector fields $\mathcal{X}$ on $B$ and functions $f \in C^\infty(B)$ holds $b(\mathcal{X}, \nabla_b f) = \rho_B \cdot \partial_X f$. The $\rho_B$-weighted divergence is to be defined as

$$\text{div}_b := \frac{1}{\rho_B} \text{div}_b(\rho_B^2 \cdot \text{Id}),$$

since the $\rho_B$-weighted Laplace-Beltrami operator

$$\Delta_b := \text{div}_b(\nabla_b) = \frac{1}{\rho_B} \text{div}_b(\rho_B \nabla_b)$$

shall obey the following integration by parts formula:

$$\int_B \Delta_b f \cdot g \, d\lambda_B = - \int_B b(\nabla_b f, \nabla_b g) \, d\lambda_B = - \int_B b(\nabla_b f, \nabla_b g) \, d\lambda_B = \int_B f \cdot \Delta_b g \, d\lambda_B$$

for all $f, g \in C^\infty(M)$, where $\lambda_b$ and $\lambda_B = \rho_B \lambda_b$ refer to Riemannian volume measures. If the weight function is constant, then the definition of the gradient is not to be changed: $\nabla_B = \nabla_b$.

\(\text{Notation 2.16.}\) For sake of readability, we write $L^p(B; \rho_B)$ instead of $L^p(B; \lambda_B)$. Similar notation is used for other spaces depending on a weighted Riemannian volume measure.

Remark 2.17 (adjoint vector fields wrt. weighted metric). Let $(B, b)$ be an orientable Riemannian manifold. The general form of an adjoint vector field is given via the Divergence Theorem: If $\mathcal{X}$ is a vector field on $B$, then its adjoint wrt. the Riemannian metric, i.e. wrt. the $L^2(B; \rho_B)$-scalar product, is $\mathcal{X}^\ast = -\mathcal{X} - \text{div}_b(\mathcal{X})$. Thus, solenoidal vector fields could be viewed as antisymmetric operators. Introducing a smooth, nonconstant weight function $\rho_B$ on $B$ as above yields

$$\int_B \mathcal{X} f \cdot g \, d\lambda_B = \int_B - f \mathcal{X}(\rho_B g) - \text{div}_b(\mathcal{X})\rho_B f g \, d\lambda_B$$

$$= \int_B - \rho_B f \mathcal{X} g - fg \mathcal{X} - \text{div}_b(\mathcal{X})\rho_B f g \, d\lambda_B$$

$$= \int_B - f \mathcal{X} g - fg \frac{1}{\rho_B} \mathcal{X} \rho_B - \text{div}_b(\mathcal{X})f g \, d\lambda_B$$

\(^3\)Smoothness is actually not required, we could use e.g. a loc-Lipschitzian weight function instead. But in order to have Equation (2.1) we need that at least locally a weak gradient of $\rho_B$ exists and a weak version of Stokes Theorem.
for all \( f, g \in C^\infty_c(\mathbb{B}) \). Hence, the adjoint of \( \mathcal{X} \) wrt. \( b \) reads as

\[
\mathcal{X}^* = -\mathcal{X} - \text{div}_b(\mathcal{X}) - \frac{1}{\rho_b} \mathcal{X}\rho_b.
\]

Therefore, we would correct the differential operator \( \mathcal{X} \) for a vector field \( \mathcal{X} \) solenoidal wrt. \( b \) by the logarithmic derivative of \( \rho \) along \( \mathcal{X} \) to become antisymmetric again, i.e. replace \( \mathcal{X} \) by \( \mathcal{X} + \frac{1}{\rho_b} \mathcal{X}\rho_b \).

**Lemma 2.18** (weighted Laplace-Beltrami in terms of logarithmic derivative). Again, let \((\mathbb{B}, b)\) be an orientable Riemannian manifold weighted by \( \rho_b \) strictly positive, nonconstant and smooth. The weighted Laplace-Beltrami is written as

\[
\Delta_b = \Delta_b + \frac{1}{\rho_b} b(\nabla_b \rho_b, \nabla_b) = \Delta_b + \frac{1}{\rho_b} \nabla_b \rho_b.
\]

We call the second summand the logarithmic derivative of \( \rho_b \). In particular, for \( \rho_b = \exp(-\psi) \) with \( \psi \in C^\infty(\mathbb{B}) \) we obtain

\[
\Delta_b = \Delta_b - \nabla_b \psi.
\]

**Proof.** Substantially, the proof looks like in the Euclidean case as we need just the Leibniz rule, Stokes Theorem and the defining characterisation of the gradient. \( \square \)

Later on, we will have to talk about Poincaré inequalities. For this purpose among others, we shall fix some notions concerning sections in general. In a nutshell, we are relying on the assumption that smooth vector fields viewed as first order differential operators could equivalently seen as smooth sections. This is true in all applications we are interested in, but we are aware of counterexamples like noncommutative tori, see [Ros13]. In this paper, we just leave issues of noncommutative geometry aside and won’t mention them again.

**Notation 2.19** (space of sections). We denote by \( \Gamma(\mathbb{B}; E) \) the space of measurable sections. If no confusion is possible, we omit the base manifold writing just \( \Gamma(\mathbb{E}) \). Moreover, we denote the space of \( m \)-times continuously differentiable sections by \( \Gamma^m(\mathbb{B}; E) \) or just \( \Gamma^m(\mathbb{E}) \). As usual the differentiability parameter equals the regularity of the differential structure, thus it is \( m = \infty \) in our context for sake of simplicity.

For the rest of this subsection, we consider \( F \) to be a Banach space, i.e. the fibre bundle is a Banach bundle, see e.g. [Lan95, Chapter III]. This gives us a section \( |\cdot|_\mathbb{E} \) in the bundle of functions \( \mathbb{E} \to \mathbb{R} \) over \( \mathbb{B} \) such that \( |\cdot|_{\mathbb{E}, b} := |\cdot|_{\mathbb{E}}(b) : \mathbb{E}_b \to \mathbb{R} \) is a norm and \( (\mathbb{E}_b, |\cdot|_{\mathbb{E}, b}) \) is a Banach space for all \( b \in \mathbb{B} \). One might think of \( |\cdot|_\mathbb{E} \) as a ‘Riemannian norm’, but should be careful since it is not necessarily related to a Riemannian metric. Furthermore, we assume a \( \sigma \)-finite measure \( \mu_b \) on \((\mathbb{B}, \mathcal{B}(\mathbb{B}))\).

**Definition 2.20** (Integrable sections). Let \( p \in [1, \infty) \) and \( \mathcal{X} \in \Gamma(\mathbb{E}) \). We call \( \mathcal{X} \) \( p \)-integrable wrt. \( \mu_b \), if the integral \( \int_{\mathbb{B}} |\mathcal{X}(b)|_{\mathbb{E}, b}^p \, d\mu_b(b) \) is finite. By \( L^p(\mathbb{B} \to \mathbb{E}; \mu_b) \) we denote the set of equivalence classes of \( p \)-integrable sections wrt. equality \( \mu_b \)-almost everywhere. Clearly, we endow \( L^p(\mathbb{B} \to \mathbb{E}; \mu_b) \) with the norm

\[
\|\cdot\|_{L^p(\mathbb{B} \to \mathbb{E}; \mu_b)} := \left( \int_{\mathbb{B}} |\cdot|_{\mathbb{E}, b}^p \, d\mu_b(b) \right)^{1/p}.
\]

As usual for the case \( p = \infty \), we define \( L^\infty(\mathbb{B} \to \mathbb{E}; \mu_b) \) as the space of measurable sections \( \mathcal{X} \in \Gamma(\mathbb{E}) \) which are bounded almost everywhere, i.e. there is a \( c \in (0, \infty) \) such that \( |\mathcal{X}(b)|_{\mathbb{E}, b} \leq c \) for \( \mu_b \)-almost all \( b \in \mathbb{B} \). The norm of such a \( \mathcal{X} \in L^\infty(\mathbb{B} \to \mathbb{E}; \mu_b) \) is the infinimum of all such bounds \( c \).
Completeness of these spaces of integrable sections is shown like in the Fischer-Riesz Theorem for usual \(L^p\)-spaces. One might say that we defined \(L^p(B \to \mathcal{E}; \mu_b)\) as a ‘direct integral of Banach spaces’. This is merely a verbalisation of a more general concept of integrating homomous fibres wrt. some \((\sigma\text{-finite})\) measure on the base space. This belongs to mathematical folklore and we can not discuss this construction here in exhausting detail.

**Example 2.21** (direct integral of Hilbert space fibres). Let \(\mathcal{B}\) be a \(\mathfrak{d}\)-dimensional Riemannian manifold, the standard fibre \(\calF\) of the Riemannian metric \(|\cdot|\) be \(\mathbb{R}^d\) for some natural number \(d\) and let a section \(\mathfrak{b}\) in the bundle of symmetric bilinear forms \(\text{Sym}(\mathcal{E}) \to \mathcal{B}\) such that it is pointwise positive semidefinite. Then, \(\mathfrak{b}\) induces a section of norms via
\[
|e|_{\mathfrak{b}} := b_{\pi(e)}(e, e) \quad \text{for all } e \in \mathcal{E}
\]
and all fibres are Hilbert spaces, since all norms on \(\mathbb{R}^d\) are equivalent. Thus, we can describe the space of square-integrable sections as the direct integral \(\int_{\mathcal{B}} \mathcal{E}_b \, d\mu_b(b)\) wrt. \(\mu_b\). This gives us a Hilbert space again. Of course, \((\mathcal{B}, \mathfrak{b})\) being a Riemannian manifold would be the most interesting case, i.e. \(\mathcal{E} = \mathcal{B}\) with \(F = \mathbb{R}^d\) and the section \(\mathfrak{b}\) being a Riemannian metric. Then, embracing Notation 2.16 we have that
\[
L^2(\mathcal{B} \to \mathcal{T}\mathcal{B}; \mathfrak{b}) = \int_{\mathcal{B}} \mathcal{T}_b\mathcal{B} \, d\lambda_b(b).
\]

In principle, a function \(u \in L^1_{\text{loc}}(\mathcal{B}; \mathfrak{b})\) on a Riemannian manifold \((\mathcal{B}, \mathfrak{b})\) is weakly differentiable if the composition \(u \circ \mathfrak{b}\) is weakly differentiable for every chart \(\mathfrak{b}\). For the more formal (topological) definition of Sobolev spaces \(H^{m,p}(\mathcal{B})\) on \(\mathcal{B}\) we refer to [Heb99, Section 2.2].

As in the case of Sobolev spaces, we are not content with characterising a weak gradient in charts as vector field with locally integrable components, but from the existence proof for Sobolev spaces as function spaces we know the coordinate-free description of weak gradients: A function \(u\) is weakly differentiable if there is an element \(\mathcal{U} \in L^1_{\text{loc}}(\mathcal{B} \to \mathcal{T}\mathcal{B}; \mathfrak{b})\) satisfying
\[
\int_{\mathcal{B}} b(X, \mathcal{U}) \, d\lambda_b = -\int_{\mathcal{B}} \text{div}_b(X) \cdot u \, d\lambda_b \quad \text{for all } X \in \Gamma^\infty_c(T\mathcal{B}).
\]

We denote such a weak gradient as \(\mathcal{U} = \nabla_b u\) in analogy to the usual gradient. In case of \(u \in H^{1,p}(\mathcal{B})\) we have that \(\nabla_b u \in L^p(\mathcal{B} \to \mathcal{T}\mathcal{B}; \mathfrak{b})\). The previous characterisation instances to
\[
\int_{\mathcal{B}} \mathcal{U}\varphi \, d\lambda_b = \int_{\mathcal{B}} b(\nabla_b\varphi, \mathcal{U}) \, d\lambda_b = -\int_{\mathcal{B}} u \nabla_b \varphi \, d\lambda_b \quad \text{for all } \varphi \in C^\infty_c(\mathcal{B}).
\]

If \(\int_{\mathcal{B}} u \Delta_b \varphi \, d\lambda_b = 0\) for all test functions \(\varphi \in C^\infty_c(\mathcal{B})\), then we call \(u\) weakly harmonic. Using this definition we bypass the problem to extend the divergence operator – which is taking the trace of the covariant derivative – with domain \(\Gamma^\infty(T\mathcal{B})\) to all the possible weak gradients in the Banach space \(L^p(\mathcal{B} \to \mathcal{T}\mathcal{B})\). In view of Weyl’s Lemma characterising weakly harmonic functions those functions can be thought as harmonic functions, since we don’t have boundaries in this paper.

**Example 2.22** (loc-Lipschitz potentials). We know that if \(\psi\) is Lipschitzian\(^4\) with compact support, then \(\psi \in H^{1,p}(\mathcal{B})\) for all \(p \in [1, \infty]\) meaning that \(\nabla_b \psi\) exists as a weak gradient, see [Heb99, Proposition 2.4]. Indeed, if \(\mathcal{B}\) is a compact manifold, then

\(^4\)We say that functions \(f \colon \mathcal{B} \to \mathbb{R}\) are Lipschitzian if there is a positive constant \(C\) such that \(|f(x) - f(y)| \leq Cd_b(x, y)\) for all \(x, y \in \mathcal{B}\), where we denote by \(d_b\) the intrinsic metric induced by the Riemannian metric \(b\). As a consequence, if we carelessly embed \(\mathcal{B}\) into some \(\mathbb{R}^d\) large enough, the embedding might change the structure of the metric space and thus the class of Lipschitz functions.
the proposition applies to any Lipschitzian $\psi$; in general, we have not a compact manifold, but consider arbitrary compact subsets instead.

2.3. Ehresmann connections and Sasaki metric. In this subsection, we introduce Ehresmann connections as a decomposition of the double tangent bundle in terms of a Whitney sum as well as the Sasaki metric on the tangent space as the Riemannian metric respecting the entire Ehresmann connection.

**Definition 2.23** (vertical bundle). Let $\pi: E \to \mathcal{B}$ be a (smooth) fibre bundle. The space of vertical (tangent) vectors is $V^E := \text{Null}(d\pi)$, the nullspace of the differential $d\pi: TE \to TB$. Vertical vectors are thought as being tangent to the fibres of $\pi$. This yields the so-called vertical bundle $\pi_0: V^E \to E$. As $d_0\pi$ is surjective for all $v \in E$, the vertical bundle is a smooth subbundle. Smooth sections in this bundle are called vertical.

Additionally, if $\pi: E \to \mathcal{B}$ happens to be a vector bundle, than we can define the vertical lift at $v$ $vl_v: E_{\pi(v)} \to T_vE$ for $v \in E$ fixed via the action on arbitrary test functions $f \in C^\infty(E_{\pi(v)})$ as $(vl_v(w), d_vf) = \frac{d}{dt}f(v + tw)|_{t=0}$. If $E = TB$, then determines the lift uniquely. The smooth section $\mathcal{V} \in \Gamma^\infty(E; V^E)$ given as $v \mapsto \mathcal{V}(v) := vl_v(v)$ is called canonical vector field. Furthermore, the vertical projection $\text{vpr}$ is given at $v \in E$ as the projection mapping $T_vE \to V_vE$.

Now, we restrict ourselves to the case of vector bundles, but a definition of Ehresmann connections for fibre bundles can be found e.g. in [KMS93, Section 9].

**Definition 2.24** (Ehresmann connection). Let $\pi: E \to \mathcal{B}$ be a (smooth) vector bundle. A (smooth) subbundle $HE \subseteq TE$ is called Ehresmann connection or horizontal (tangent) bundle if

$$T_vE = V_vE \oplus H_vE \quad \text{for all } v \in E.$$  

For sake of readability we just write $TE = V^E \oplus HE$ in the sense of a Whitney sum.

The horizontal lift at $v$ of $w \in E$ is the unique vector $hl_v(w) \in HE$ such that $w = \langle hl_v(w), d\pi \rangle$.

Finally, the projection of tangent vectors to their horizontal parts is denoted by $\text{hpr}$.

Compare this usage of the terms ‘vertical’ and ‘horizontal’ to the usage in stochastic analysis as e.g. in [W181, Section V.4] or [Hsu02, Chapter 2]. Luckily, we escaped the frame bundle via the McKean-Gangolli injection scheme as described by E. Jørgensen.

Henceforth, we consider $\mathcal{B} = \mathcal{M}$ with the properties (M), since this is the sole instance of interest for this paper. Furthermore, we just take the tangent space $E = TM$, but note that we are interested in $SM \subseteq TM$ in case of the fibre lay-down model. Moreover, let us specify the one Ehresmann connection we always will consider without further mentioning: the Riemannian horizontal bundle.

**Definition 2.25** (connector map and Riemannian horizontal bundle). Let $U \subseteq \mathcal{M}$ be a neighbourhood of $o \in \mathcal{M}$ with preimage $V := \pi_0^{-1}(U) \subseteq TM$ such that the exponential $\exp_o: TM \to \mathcal{M}$ maps a 0-neighbourhood to $V$ diffeomorphically. Let $\tau: V \to T_o\mathcal{M}$ denote parallel transport of $v \in V$ along the unique geodesic arc connecting $\tau(v)$ and $o$. Let $r_u: T_o\mathcal{M} \to T_u\mathcal{M}$ be the translation $w \mapsto w - u$ by the vector $u \in T_o\mathcal{M}$. Now, consider the mapping

$$\kappa: V \to \mathcal{M}, \quad v \mapsto (\exp_o \circ r_{-u} \circ \tau)(v).$$
The dependency on the chart vanishes when passing to the differential
\[ d_u \kappa : T_u TM \to T_{\pi_0(u)} M, \quad a \mapsto \langle a, d(\exp_{\pi_0(u)} \circ r_u \circ \tau) \rangle \]
which is called the connector map, cf. [Dom62, Section 2]. Also see [Sak96, Section II.4] for explanation in terms of local coordinates. Via the assignment \( \text{HE} := \text{Null}(dk) \) we gain an Ehresmann connection which we call the Riemannian horizontal bundle, see e.g. [Dom62, Appendix (ii)]. As the exponential map depends on the given Riemannian metric, so does this horizontal bundle.

Some authors call \( dk \) the vertical projection and \( d\tau_0 \) the horizontal projection. E.g. in Notation 2.26 we use the mappings in a way that would justify such a naming. However, we do not recommend this terminology and introduced the notions ‘vpr’ and ‘hpr’ to avoid confusion. Concerning the vertical or horizontal lift of functions there is a well-established consent what it should be.

\textbf{Notation 2.26 (vertical/horizontal lift of functions).} Let \( f_0 \) be a real-valued function with domain in \( M \). We call the pullback of \( f_0 \) wrt. the tangent bundle projection \( \pi_0 \) the vertical lift of \( f_0 \). For sake of brev'ty we define \( f_0^v := \pi_0^* f_0 \) whenever it’s defined.

A direct analogy would be that the horizontal lift of \( f_0 \) is the pullback wrt. the connector map and we declare \( f_0^h := \kappa^* f_0 \) whenever it’s defined. However, this is not straight forward, since \( \kappa \) depends on the choice of base point \( o \) of the exponential and the translation vector \( u \). At first, the horizontal lift of a smooth function \( f_0 \) is a function
\[ (u, v) \mapsto \left( f_0 \circ \exp_{\pi_0(u)} \circ r_u \circ \tau \right)(v) \]
or a bevy of functions indexed by \( u \); we should exclude the case that \( u \) and \( v \) are in the same fibre as this would yield the vertical lift again. We do not worry too much about well-definedness and differentiability: Due to assumption (M2) and the Hopf-Rinow Theorem the exponential mapping at \( \pi_0(u) \) is defined everywhere on the tangent space, furthermore it is almost everywhere a diffeomorphism by [Sak96, Lemma III.4.4]. But as aforementioned earlier we can project from the double tangent space into the tangent space via \( d\tau_0 \) and \( ds \) and get \( df_0(d\tau_0 + ds) = df_0^v + df_0 d\kappa \). The horizontal lift of \( f_0 \) is a function \( f_0^h \) such that \( \langle a, df_0^h \rangle = \langle a, df_0 d\kappa \rangle \) for all \( a \in TT M - \) algebraically this makes perfectly sense and is unique up to constant offsets. A more analytic intuition would be \( f_0^h(v) = \int_{T_x TM} df_0 d\kappa d\lambda \), where we consider the Lebesgue measure on the standard fibre \( F = \mathbb{R}^{2n} \) of the double tangent bundle.

\textbf{Lemma 2.27.} Let \( f_0 \in C^\infty(M) \) and \( \mathcal{X} \in \Gamma^\infty(TM) \). Then, the following relations hold
\[ \text{vl} \langle \mathcal{X}, f_0^v \rangle = 0 \quad \text{and} \quad \text{hl} \langle \mathcal{X}, f_0^v \rangle = \langle \mathcal{X}, f_0 \rangle^v, \]
\[ \text{vl} \langle \mathcal{X}, f_0^h \rangle = \langle \mathcal{X}, f_0 \rangle^v \quad \text{and} \quad \text{hl} \langle \mathcal{X}, f_0^h \rangle = 0. \]

\textbf{Proof.} Elementary. \qed

Next, we define the Sasaki metric. It’s original definition was given in the famous paper [Sas58] containing many other results of general interest. Sasaki refereed just to the Riemannian horizontal bundle and so do we, but it’s clear that the following definition can easily be adapted to a situation without connector map.

\textbf{Definition 2.28 (Sasaki metric).} The Sasaki metric \( s \) on \( TM \) is uniquely characterised as the Riemannian metric respecting inner products in the given Ehresmann connection, i.e. the metric \( s \) is natural in the sense of [GK02] meaning that
\[ s(\text{hl}(\mathcal{X}), \text{hl}(\mathcal{Y})) = m(\mathcal{X}, \mathcal{Y}) \circ \pi_0 \quad \text{and} \quad s(\text{hl}(\mathcal{X}), \text{vl}(\mathcal{Y})) = 0 \]
for all $X, Y \in \Gamma^\infty(TM)$, and additionally the metric respects the inner product under vertical lifting in the sense that

$$s(\nu(X), \nu(Y)) = m(X, Y) \circ \pi_0$$

for all $X, Y \in \Gamma^\infty(TM)$. This metric is explicitly given as the sum of pullbacks of the metric tensor of the base manifold:

$$s(a, b) := \kappa^*m(a, b) + \pi_0^*m[a, b] = m(\langle a, d\pi_0 \rangle, \langle b, d\pi_0 \rangle)$$

for all $a, b \in TTM$. We call the component $h := \pi_0^*m$ the horizontal (Sasaki) metric. Mutatis mutandis, the vertical (Sasaki) metric $v$ is $v := \kappa^*m$. This yields $s = v + h$.

**Notation 2.29.** If we consider the $\rho_M$-weighted manifold $(M, m)$, then the weighting procedure is naturally reflected in the horizontal bundle via the vertical lift. Explicitly, we think of $(TM, h)$ as a $\rho_M^*$-weighted Riemannian manifold, where $h$ is the weighted version of the horizontal Sasaki metric $h$. In the first place, the vertical bundle is not affected. If we weight the fibre $F = \mathbb{R}^n$ by $\rho_F$, then the vertical metric changes and the horizontal one does not. The weighted vertical Sasaki metric is denoted by $v$. If and only if a bundle weighting is specified and no confusion possible, we denote the corresponding weighted Sasaki metric by $s$ and the weighted tangent bundle just as $(TM, s)$.

It turns out that the volume form corresponding to the Sasaki metric – it exists independently of orientability of $M$ as tangent spaces always are orientable – induces a volume measure with very neat loc-product structure.

**Lemma 2.30 (Sasakian volume measure).** The volume measure $\lambda_s$ wrt. the Sasaki metric coincides with the bundle measure on $TM$ supplemented with Lebesgue fibre measure, i.e. it holds $\lambda_s = \lambda_m \otimes_{loc} \lambda$, where we abbreviated the n-fold Lebesgue measure $\lambda^{\otimes n}$ just by $\lambda$.

**Proof.** Form Definition 2.28 we know that the Sasaki metric can be written as the sum $s = v + h$ of vertical and horizontal metric. We encounter this situation when considering a product of Riemannian manifolds and thus, we know that the Sasakian volume measure is a product measure basically. Indeed, let a chart $(\nu^j)_{j=1}^{2n}$ with domain $V \subseteq TM$ respecting the Ehresmann connection in the sense that $(\partial\nu^j)_{j=1}^{2n}$ provides a basis for either the vertical or the horizontal vector fields and $(\partial\nu^j)_{j=n+1}^{2n}$ provides a basis for the complementary type of vector fields; the matrix representation of the Sasaki metric is of block diagonal form with zero matrix at south west and north east position. Therefore, the Sasakian volume form $d\lambda_s$ reads in those coordinates as

$$d\lambda_s = \sqrt{\pi_0^*m} \sqrt{\kappa^*m} \bigwedge_{j=1}^{2n} d\nu^j.$$  

Wlog. $V$ is preimage of a chart domain in $M$, i.e. $V = \pi_0^{-1}(U)$, such that there is a diffeomorphism $\varphi$ rendering the diagram in Figure 1 commutative. Hence the pushforward measure $\varphi_*\lambda_s$ has to coincide with product volume measure on $U \times F$, where the fibre $F = \mathbb{R}^n$ is naturally endowed with the Lebesgue measure. But Lemma 2.7 uniquely determines the bundle measure on $TM$ supplemented the fibre measure $\nu_F = \lambda$, so the Sasakian volume measure has to be this bundle measure. □

Note that by Lemma 2.27 we already know that the gradients $\nabla v$ and $\nabla h$ wrt. vertical and horizontal Sasaki metric respectively satisfy $\nabla v f_0^a = 0$ and $\nabla h f_0^a = 0$. 

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In the next lemma we give more insight into the three basic operators induced by the metrics from the Ehresmann connection.

**Lemma 2.31** (Sasaki gradient, divergence and Laplacian). The gradient, divergence and Laplace-Beltrami operators corresponding to the vertical Sasaki metric are characterised by

\[ \nabla_v f_0^h = \text{vl}(\nabla_m f_0 \circ \pi_0), \quad \text{div}_v (\text{vl} \mathcal{X}) = (\text{div}_m \mathcal{X}) \circ \pi_0 \quad \text{and} \quad \Delta_v f_0^h = (\Delta_m f_0) \circ \pi_0 \]

for all \( f_0 \in C^\infty(\mathcal{M}) \) and \( \mathcal{X} \in \Gamma^\infty(T\mathcal{M}) \). Similarly, for the case of the horizontal Sasaki metric we have that

\[ \nabla_h f_0^h = \text{hl}(\nabla_m f_0 \circ \pi_0), \quad \text{div}_h (\text{hl} \mathcal{X}) = (\text{div}_m \mathcal{X}) \circ \pi_0 \quad \text{and} \quad \Delta_h f_0^h = (\Delta_m f_0) \circ \pi_0 \]

for all \( f_0 \in C^\infty(\mathcal{M}) \) and \( \mathcal{X} \in \Gamma^\infty(T\mathcal{M}) \).

Eventually, we have

\[ \text{div}_s(\mathcal{Y}) = \text{div}_v(\text{vpr}(\mathcal{Y})) + \text{div}_h(\text{hpr}(\mathcal{Y})) \quad \text{for all} \quad \mathcal{Y} \in \Gamma^\infty(T\mathcal{M};TT\mathcal{M}). \]

**Proof.** We restrict ourselves to the vertical case, since the other statements follow analogously. Let \( f_0 \in C^\infty(\mathcal{M}) \) and \( \mathcal{X} \in \Gamma^\infty(T\mathcal{M}) \) arbitrary be fixed. Then, by definition and from Lemma 2.27 we know that the following two equations hold simultaneously which characterises the vertical gradient:

\[ \nu(\text{vl} \mathcal{X}, \nabla_v f_0^h) = (\mathcal{X} f_0) \circ \pi_0 \quad \text{and} \quad \nu(\text{vl} \mathcal{X}, \text{vl}(\nabla_m f_0)) = m(\mathcal{X}, \nabla_m f_0) = (\mathcal{X} f_0) \circ \pi_0. \]

Regarding the divergence, we use the definition in terms of the Lie derivative, which is mostly preferred in manifold theory, and apply Cartan’s magical formula. We have for the Riemannian metrics \( m \) and \( \nu \) the defining equations of divergence:

\[ \mathcal{L}_\mathcal{X} \lambda_m = \text{div}_m(\mathcal{X}) \cdot \lambda_m \quad \text{and} \quad \mathcal{L}_{\text{vl} \mathcal{X}} \lambda_\nu = \text{div}_v(\text{vl} \mathcal{X}) \cdot \lambda_\nu, \]

where \( \mathcal{L} \) denotes the Lie derivative. Again, the symbol \( \lambda_m \) is to be interpreted as either volume form or 1-density. Now, in case of a volume form Cartan’s formula yields the equation

\[ \mathcal{L}_{\text{vl} \mathcal{X}} \lambda_\nu = d(\text{vl} \mathcal{X}) \wedge \kappa_m \lambda_m + \text{vl}(\mathcal{X}) \wedge d\kappa_m \lambda_m = (d\mathcal{X} \wedge \lambda_m) \circ \pi_0 + \text{vl}(\mathcal{X}) \wedge d\kappa_m \lambda_m = (d\mathcal{X} \wedge \lambda_m) \circ \pi_0 \quad \text{for all} \quad \mathcal{X} \in \Gamma^\infty(T\mathcal{M}). \]

Thus, the statement follows; this can locally be reused for the case of a 1-density, cf. the proof of [Nic96, Proposition 3.4.3]. Combining the results for gradient and divergence we get the equation for the Laplace-Beltrami operator.

Finally, let \( \mathcal{Y} \in \Gamma^\infty(T\mathcal{M};TT\mathcal{M}) \). Recall that the Sasaki volume measure is realised by a volume form \( d\lambda_0 \) which could be written as wedge product of a vertical and horizontal volume form: \( d\lambda_s = d\lambda_v \wedge d\lambda_h \). Then, the divergence wrt. Sasaki metric is characterised by

\[ \mathcal{L}_\mathcal{Y} d\lambda_s = \mathcal{L}_{\text{vpr}(\mathcal{Y})} d\lambda_v \wedge d\lambda_h + \mathcal{L}_{\text{hpr}(\mathcal{Y})} d\lambda_h \]

\[ = (\text{div}_v(\text{vpr}(\mathcal{Y})) + \text{div}_h(\text{hpr}(\mathcal{Y}))) \cdot d\lambda_v \wedge d\lambda_h = \text{div}_s(\mathcal{Y}) \cdot d\lambda_s, \]

where we used the local coordinate description of Lie derivatives in line (2.2). \( \square \)

**Remark 2.32** (Sasaki gradient of loc-density). Basically, the Sasaki gradient of a tangent bundle weighting \( \rho = \rho_M \otimes \text{loc} \rho_F \) decomposes as

\[ \nabla_s \rho = \rho_M \cdot \nabla_v \rho_F \oplus \rho_F \cdot \nabla_h \rho_M \]

reminiscent of the Leibniz rule. Here, we think of \( \rho_F \) as an independent copy defined on the pointwise tangent spaces. Actually, the symbol \( \nabla_v \rho_F \) := \text{vpr}(\nabla_s \rho)
is an intuitive shorthand for the vertical component of the local vector field \( \varphi_2^{-1} (\text{Id}, \nabla_{\text{euc}} \rho \circ \text{pr}_2 \circ \varphi_1) \), where the diffeomorphisms \( \varphi_1, \varphi_2 \) render the diagram in Figure 2 commutative.

\[
\begin{array}{ccc}
\pi_1^{-1}(U) & \xrightarrow{\varphi_2} & \pi_0^{-1}(U \times \mathbb{R}^n) \\
\pi_0 \downarrow & & \pi_0 \downarrow \\
\pi_0^{-1}(U) & \xrightarrow{\varphi_1} & U \times \mathbb{R}^n \\
\pi_0 \downarrow & & \pi_0 \downarrow \\
U & \xrightarrow{\text{pr}_1} & \mathbb{R}^n \\
\end{array}
\]

**Figure 2.** Local trivialisation reflected in the double tangent bundle

We define the basic concept of semisprays which arises in the abstract study of second order ordinary differential equations – particularly, of Newton’s (second) law on manifolds, see [Gli97, Section 4] –, and also encodes geometric extra structure in several interesting ways. As a consequence of the discussion, the direct sum \( V \oplus H \) of canonical vector field and semispray \( H \) is understood as the diagonal mapping \( T \mathcal{M} \to T(T \mathcal{M}), \; v \mapsto v \text{l}(v) \oplus h \text{l}(v) \).

**Definition 2.33** (Semispray). A section \( H \in \Gamma^\infty(T \mathcal{M}; TT \mathcal{M}) \) is a semispray if it satisfies \( \langle H, d\pi_0 \rangle = \text{Id}_{T \mathcal{M}} \) or equivalently if any integral curve \( s : I \to T \mathcal{M} \) takes the form \( s = (\pi_0 \circ s)' \). A curve \( c : I \to \mathcal{M} \) is called geodesic of the semispray \( H \) if there is an integral curve \( s : I \to T \mathcal{M} \) such that \( c = \pi_0 \circ s \). Equivalently, \( c \) is geodesic if \( H \circ c' = c'' \). For a local coordinate form of semisprays see the Appendix A.

**Example 2.34** (Semispray associated to an Ehresmann connection). Consider an Ehresmann connection \( TT \mathcal{M} = VT \mathcal{M} \oplus HT \mathcal{M} \). Then, given \( v \in T \mathcal{M} \) there is a unique horizontal vector \( a \) – namely the horizontal lift of \( v \) – such that \( v = \langle a, d\pi_0 \rangle \). Furthermore, \( a \) depends on \( v \) smoothly. Hence, there is \( H \in \Gamma^\infty(T \mathcal{M}; HT \mathcal{M}) \) such that \( \text{Id}_{T \mathcal{M}} = \langle H, d\pi_0 \rangle \). Then, \( H \) is a semispray and we say that it is the semispray associated to the Ehresmann connection. In particular, there is the semispray \( H_m \) associated to the Riemann metric \( m \) via the corresponding Ehresmann connection. We call it Riemannian semispray.

Another common name is geodesic spray, since it can be constructed just in terms of geodesics corresponding to the Riemannian metric, see [Car92, Lemma III.2.3]. So, the geodesics of this semispray are just classical geodesics. In fact, this semispray satisfies a homogeneity condition making it a (full) spray whence the name – we will use the spray structure a few times indeed.

**Remark 2.35.** In a nutshell, choosing an Ehresmann connection adds the same geometric information as choosing a covariant derivative or a semispray. See [Buc12] and various references therein. We think that this observation might be a good starting point for generalisations even to rough geometries, as it translates well to Lagrange and Finsler spaces.

**Example 2.36** (Semisprays induced by Lagrangians). Let the Lagrangian \( L = \frac{1}{2} |\cdot|^2 \). Then, the Lagrangian vector field \( H_L \) is a semispray, see [MR99, Section 7.5]. As outlined in [Lan95, Section VII.6] this relation can be translated to general Riemannian manifolds modelled on some Hilbert space. Note that existence

\[5\] In fact, it is the geodesic spray again.
of such a vector field is due to the more general result [Lan95, Proposition VII.5.9]: Let $Q := TM$, then the cotangent space of $Q$ naturally yields the structure of a symplectic manifold $(T^*Q, \Omega)$. Furthermore, we make $Q$ into a Riemannian manifold endowing it with a ‘natural metric’ in the sense of Gudmundsson-Kappos. This specifies the musical isomorphisms, in particular $\sharp$. Thus, there is a unique section $H_L \in \Gamma^\infty(TQ)$ such that
\[
(\Omega^2 \circ H_L)_w(a) := \Omega^2_w(H_L(w), a) = d_wL(a) \quad \text{for all } w \in Q, a \in T_wQ,
\]
where $\Omega^2$ denotes the pullback of $\Omega$ wrt. $\sharp$. With that, we can say that $H_L = (dL)^2 = \nabla L$, where the gradient is taken wrt. the ‘natural’ metric chosen before. See [Buc06, Section 3] for another short discussion of semisprays on general Lagrange spaces.

Remark 2.37 (Revisiting Example 2.34). We shall characterise the Riemannian semispray using [Lan95, Proposition VII.5.9] similar to the previous example. Again, let $Q := TM$ and the symplectic manifold $(T^*Q, \Omega)$. Let $h = \pi^*\Omega$ be the horizontal metric, which is the simplest ‘natural’ metric, and denote the pullback of the canonical 2-form $\Omega$ wrt. $\Omega^h$. Locally we can think of $\alpha', \beta' \in T_0^\infty Q$, $w \in Q$, as tuples $\alpha' = (u_1, u_2)$ and $\beta' = (v_1, v_2)$ with $u_1, u_2, v_1, v_2 \in Q$ and $u_2' = m_x(\cdot, u_2)$, $v_2' = m_x(\cdot, v_2) \in Q^\ast$, where $x := \pi_0(w)$. Then, the form $\Omega^h$ reads as
\[
\Omega^h_w((u_1, u_2), (v_1, v_2)) = m_x(u_1, v_2) - m_x(v_1, u_2) = \langle u_1, v_2' \rangle - \langle v_1, u_2' \rangle = \Omega_w(\alpha', \beta').
\]
There is a 1-form $\omega$ which reads in such local regimes as $\omega_w(b) = \omega_w(v_1, v_2) = m_{\pi_0(w)}(w, v_1 - v_2)$. The unique section $H \in \Gamma^\infty(TQ)$ such that $\Omega^h \circ H = \omega$ is the Riemannian semispray.

Example 2.38 (Euclidean case). The Riemannian semispray $H_m$ acts on vertically lifted functions $f = f^0_0$ with $f_0 \in C^\infty(M)$ as
\[
H_mf = h(H_m, \nabla_hf^0_0) = m_{\pi_0}(\text{Id}_{TM}, \nabla_mf_0 \circ \pi_0).
\]
In case of $M = \mathbb{R}^n$ with standard Riemannian metric this action is written as $H_{\text{euc}}f(x, v) = (v, \nabla_xf_0(x))_{\text{euc}}$ for smooth functions $f: \mathbb{R}^n_x \times \mathbb{R}^n_v \to \mathbb{R}$, $(x, v) \mapsto f_0(x)$ and $x, v \in \mathbb{R}^n$.

By analogy, the canonical vector field $\mathcal{V}$ acts on horizontally lifted functions $f = f^0_0$ with $f_0 \in C^\infty(M)$ as
\[
\mathcal{V}f = v(\mathcal{V}, \nabla_vf^0_0) = m_{\pi_0}(\text{Id}_{TM}, \nabla_mf_0 \circ \pi_0).
\]
In the Euclidean case, this action is written as $\mathcal{V}f(x, v) = (v, \nabla_vf_0(v))_{\text{euc}}$ for smooth functions $f: \mathbb{R}^n_x \times \mathbb{R}^n_v \to \mathbb{R}$, $(x, v) \mapsto f_0(v)$ and $x, v \in \mathbb{R}^n$.

The following theorem is well-known, but comes in a few quite different formulations e.g.: The geodesic flow preserves the volume of $TM$. We just stick to the formulation below, since it tells us that the Riemannian semispray is an antisymmetric operator wrt. $L^2(TM; s)$-scalar product.

Theorem 2.39 (Liouville’s Theorem). The semispray $H_m$ is solenoidal wrt. Sasaki metric $s$.

Proof. A proof could be done via explicitly calculating the Sasakiian volume form in normal coordinates. This approach is taken from the next book by M. do Carmo, cf. [Car92, Exercise 3.14].

---

6Up to sign conventions: Let $\Theta$ denote the canonical 1-form, then we define the 2-form $\Omega$ by $\Omega = -d\Theta$.
The final lemma of this section characterises the test functions on the tangent space exploiting again the ‘almost product’ structure due to local trivialisation. This result will be very important arguing for a reasonable set of test functions being a core of the (Langevin/fibre lay-down) generator.

**Lemma 2.40 (Tensor product of lifted test functions).** Define the tensor product space of pulled back test functions

$$D_0 := \pi_0^* C^\infty_c(M) \otimes \kappa^* C^\infty_c(M)$$

$$= \text{span} \{ \pi_0^* f_0 \otimes \kappa^* g_0 := \pi_0^* f_0 \cdot \kappa^* g_0 \mid f_0, g_0 \in C^\infty_c(M) \}$$

$$= \text{span} \{ f_0^* \otimes g_0^h \mid f_0, g_0 \in C^\infty_c(M) \} .$$

Then, $D_0$ is dense\(^7\) in $C^\infty_c(TM)$.

**Proof.** Consider a local trivialisation in $U := U_x \cap W$ for a chart domain $W \subseteq M$ at $x \in M$ with $V := \pi_0^{-1}(U)$ and the diffeomorphism $\varphi$ rendering the diagram in Figure 1 commutative. Then, we immediately conclude the relations

$$\pi_0^* C^\infty_c(U) = \varphi^*(C^\infty_c(U) \otimes \{1\}) \quad \text{and} \quad \kappa^* C^\infty_c(U) = \varphi^*(\{1\} \otimes C^\infty_c(\mathbb{R}^n)).$$

Thus, by [Hor66, Proposition 4.8.1] the tensor product $\pi_0^* C^\infty_c(U) \otimes \kappa^* C^\infty_c(U)$ is dense in $C^\infty_c(V) = \varphi^* C^\infty_c(U \times \mathbb{R}^n)$ and the proof is finished via a partition of unity argument. \hfill \Box

3. Hypocoercivity for geometric Langevin dynamics

In this section we apply the abstract Hilbert space hypocoercivity method to the Langevin equation with some Riemannian manifold as position space, see Theorem 3.2 below. That is the direct generalisation of the situation of [GS16]. The techniques of proving we learned from [GS14]. Our interest arose from industrial fibre laydown applications and qualitative analysis of the nonwoven. In this context, the position manifold $M$ could e.g. reflect a sagging conveyor belt or a belt moving over a cylindrical roller. As we mentioned in the introduction, Langevin dynamics have wide ranging applications and our approach offers the freedom to include any ‘smooth’ side condition on the position variable.

Consider the following Stratonovich SDE in TM:

$$d\eta = \mathcal{H}_m \, dt + v_\eta (-\nabla_m \Psi) \, dt + \sigma \cdot v_\eta \left( \sum_{j=1}^n \frac{\partial}{\partial x_j} \right) \circ dW_t - \alpha \cdot \mathcal{V} \, dt , \quad (3.1)$$

where $\eta : \mathbb{I} \to TM$ is a curve with time interval $\mathbb{I}$ and $(x^1, x^2, \ldots, x^n)$ is a chart at $\pi_0(\eta)$ providing normal coordinates. Recall that $\mathcal{V}$ denotes the canonical vector field. Bellow we specify certain assumptions on the potential $\Psi : M \to \mathbb{R}$. The nonnegative model parameters $\alpha$ and $\sigma$ are related by $\sigma = \sqrt{2\alpha/\beta}$, where $\beta$ is a nonnegative rescaling of the potential as $\Phi = \beta \Psi$. Recall that the horizontal motion in Equation (3.1), i.e. $hpr(d\eta) = \mathcal{H}_m \, dt$, reflects the natural requirement of $(\pi_0 \circ \eta)' = \dot{\eta}$ which is the form that any integral curve of $\mathcal{H}_m$ attains by definition. We call Equation (3.1) the *Langevin equation on $M$* or just *geometric Langevin equation*.

\(^7\)Dense wrt. the usual locally convex topology induced by appropriate seminorms, which implies uniform convergence of all derivatives on compacts; for the well-known Euclidean case see [Hor66, Example 2.4.10] and for the generalised geometric case see [GKOS01, Section 3.1.3]. In common notation for test function spaces with this topology, the result reads as $\pi_0^* \mathcal{D}(M) \otimes \kappa^* \mathcal{D}(M)$ is dense in $\mathcal{D}(TM)$. By the way, the authors of [GKOS01] not only discuss generalised distribution spaces rigorously, they shed some light on integration of 1-densities from a very analytic perspective.
Using [W181, Theorem V.1.2] the Kolmogorov generator $L$ is given as
\[ L = \mathcal{H}_m - \frac{1}{\beta} \nabla_v \Phi \beta + \frac{\alpha}{\beta} \Delta_v - \alpha \mathcal{L}. \tag{3.2} \]

This operator is defined for all smooth functions on $TM$, but we consider the domain $D := C_0^\infty(TM)$ of smooth test functions with compact support. Note that $(L, D)$ is densely defined by Remark 2.9. We usually call $L$ as in Equation (3.2) the Langevin generator. Compare this generator to [GS16, Equation (1.2)] taking Example 2.38 into account. As we will see in Lemma 3.4, the Langevin generator basically decomposes into two components: the vertical diffusion and the (not entirely horizontal) component liaising the appropriate notion of second order differential equations. Compare this decomposition to so-called hypoelliptic Laplacians; via this concept J.-M. Bismut links Brownian motion on manifolds and geodesic flow in order to find a Langevin process in [Bis15].

Before we start checking the data and hypocoercivity conditions, we shall fix conditions on the potential in the geometric Langevin equation, and thus on the base weight $\rho_m = \exp(-\Phi) = \exp(-\beta \Psi)$.

**Condition 3.1 (Potential conditions (P)).**

(P1) **General regularity and boundedness:** Let $\Phi = \beta \Psi$ a loc-Lipschitzian potential which is bounded from below and such that $\lambda_m = \rho_m \lambda = \exp(-\Phi) \lambda_m$ is a probability measure on $(M, \mathcal{B}(M))$.

(P2) **Poincaré inequality:** The weighted Riemannian measure $\lambda_m$ satisfies the Poincaré inequality
\[ \|\nabla_m f_0\|_{L^2(M, \lambda_m)}^2 \geq \Lambda \|f_0 - (f_0, 1)_{L^2(M, \lambda_m)}\|_{L^2(M, \lambda_m)}^2 \tag{3.3} \]
for all $f_0 \in C_0^\infty(M)$ and some $\Lambda \in (0, \infty)$.

(P3) **Hessian dominated by gradient:** Assume $\Phi \in C^2(M)$. There is a constant $c \in (0, \infty)$ such that
\[ |\text{Hess}_m(\Phi)(x)| \leq c (1 + |\nabla_m \Phi(x)|_m) \]
holds for all $x \in M$.

Here, $'\text{Hess}_m'(\cdot)'$ denotes the Hessian wrt. the given Riemannian metric and the norm $'|\text{Hess}_m(\cdot)|'$ of the Hessian is the Frobenius tensor norm induced by the Riemannian metric.

As explained in [GS14, Remark 3.16] the condition (P3) as above can be weakened. Clearly, the Poincaré inequality (3.3) is the most restrictive condition wrt. the geometry of the weighted position manifold $(M, m)$. It’s satisfied for compact manifolds, see [Heb99, Theorems 2.10, 2.11]. For the noncompact case we refer to [Heb99, Lemma 3.1], where the necessary geometric assumption is that the Ricci curvature is bounded from below by some multiple of $m$. Furthermore, we point out that in Lemma 3.4 the nasty assumption of a weakly harmonic potential appears. In view of Weyl’s theorem it would be quite a restrictive assumption. That’s why in Proposition 3.7 we remove this assumption for the general setting of this section. Note that by Lemma B.1 the Riemannian manifold $(M, m)$ weighted by a potential as above is complete again.

We formulate the main theorem of this section:

**Theorem 3.2** (Hypocoercivity of the geometric Langevin dynamic).

Let $\alpha, \beta \in (0, \infty)$ and $(M, m)$ be a Riemannian manifold satisfying $(M)$. We assume that the potential $\Phi: M \to \mathbb{R}$ fulfils the conditions (P) above. Denote by $\nu$ the zero-mean Gaussian measure with covariance matrix $\beta^{-1} \text{Id}$ and define $\mu := \lambda_m \otimes \text{loc} \nu$. 
Then, the Langevin operator
\[
(L, C^\infty_c(TM)) = \left( \frac{\alpha}{\beta} \Delta + \alpha V + \mathcal{H}_m - \frac{1}{\beta} \nabla^b \Phi, C^\infty_c(TM) \right)
\]
is closable in $H := L^2(TM; \mu)$. Moreover, its closure $(L, D(L))$ generates a strongly continuous contraction semigroup $(T_t)_{t \in [0, \infty)}$. Finally, there are constants $\kappa_1, \kappa_2 \in (0, \infty)$ such that the Liouville’s Theorem holds for all $g \in H$ and times $t \in [0, \infty)$ holds
\[
\| T_t g - (g, 1)_H \|_H \leq \kappa_1 e^{-\kappa_2 t} \| g - (g, 1)_H \|_H.
\]

Clearly, we are going to prove it by applying the Hypocoercivity Theorem.

### 3.1. Data conditions.

**Definition 3.3** (model Hilbert space (D1)). Consider the probability space
\[
(E, \mathcal{E}, \mu) = (TM, \mathcal{B}(TM), \lambda_m),
\]
where $\lambda_m = \lambda \otimes \nu$ is the weighted Sasaki volume measure with $\mu$ is weighted by $\rho_M := \exp(-\Phi) = \exp(-\beta \Psi)$ with $\beta \in (0, \infty)$ such that $\lambda_m$ is a probability measure on $(M, \mathcal{B}(M))$, and $\nu = N(0; \beta^{-1} \text{Id}_m)$ is the zero-mean normal distribution on the fibre $F = \mathbb{R}^n$ with covariance matrix $\beta^{-1} \text{Id}_n$. In other words, $\lambda_m$ has the loc-density $\exp(-\Phi) \otimes \nu$ such that for all $f \in H$ holds
\[
\| T_t g - (g, 1)_H \|_H \leq \kappa_1 e^{-\kappa_2 t} \| g - (g, 1)_H \|_H.
\]

Clearly, we are going to prove it by applying the Hypocoercivity Theorem.

**Lemma 3.4** (SAD-decomposition (D3), (D4), (D6)). Let condition (P1) hold and let $\Phi$ be weakly harmonic. Consider the SAD-decomposition $L = S - A$ on $D$ with
\[
Sf := \frac{\alpha}{\beta} \Delta f = \frac{\alpha}{\beta} \Delta f - \alpha V f
\]
and
\[
Af = -\mathcal{H}_m f := -\mathcal{H}_m f + \frac{1}{\beta} \left( \nabla^b \Phi \right)(f)
\]
for all $f \in D$. Then, the following assertions hold:

(i) $(S, D)$ is symmetric and negative semidefinite.

(ii) $(A, D)$ is antisymmetric.

(iii) For all $f \in D$ we have that $Lf \in L^1(TM; \mu)$ and $\int_{TM} Lf \, d\mu = 0$.

**Proof**

(i) Combining the form of $\nabla^b \rho$ from Remark 2.32 with the result on the weighted Laplace-Beltrami, see Lemma 2.18, we know that we have to look at $\frac{1}{\rho_F} \nabla^b \rho_F$. Condensing notation a bit we calculate that
\[
\frac{1}{\rho_F} \nabla^b \rho_F (f) = \nabla_{\text{euc}} \left( -\beta \langle \text{Id}_F, d\nu \rangle \nabla^b \Phi \right)(f) = -\beta \langle \text{Id}_F, d\nu \rangle \nabla^b \Phi f
\]
holds for all $f = f_0^\rho \in \kappa^* C^\infty_c(M)$. Hence, it follows
\[
(Sf, g)_H = -\int_{TM} \nabla^b \Phi f \nabla^b \Phi g \, d\mu \quad \text{for all } f, g \in D_0,
\]
and therefore $(S, D_0)$ is symmetric and nonpositive definite. Since $S$ is well-defined on $D$ and $D_0$ is dense in $D$ wrt. graph norm, part (i) follows directly.

(ii) By Liouville’s Theorem and similar reasoning as in Remark 2.17 we know the adjoint of the Riemannian semispray wrt. $L^2(TM; \mu)$-scalar product:
\[
\mathcal{H}_m^* = -\mathcal{H}_m - \frac{1}{\rho} \mathcal{H}_m \rho = -\mathcal{H}_m + \frac{1}{\rho_F} \beta \rho_F \cdot \mathcal{H}_m \Psi^c
\]
Remark 2.32

Lemma 3.4

\[ f, g \] for all \( \rho \) denote by \( \sim \) Via integration by parts one can show that \( \sim \) operator

Example 3.6.

Hence, \((A, D)\) is antisymmetric. Since \((A, D)\) is densely defined on \(H\), it is closable.

(iii) The integrability statement is clear: Let \( h \in D \) arbitrary, then we conclude that \( \int_{TM} Sh \, d\mu = 0 \) by part (i) and \( \int_{TM} Ah \, d\mu = 0 \) by part (ii). We finish the proof pointing out that \((L, D)\) is densely defined on \( H \) and dissipative, thus it is closable.

□

Notation 3.5. Due Lemma 3.4 \((S, D)\), \((A, D)\) and \((L, D)\) are closable. The closures we denote by \((S, D(S))\), \((A, D(A))\) and \((L, D(L))\) respectively.

From the easiest example of Euclidean space we learn that one actually doesn't need a weakly harmonic potential. To see this, we make use of Poisson manifolds.

Example 3.6. Let \( M := \mathbb{R}^n_+ \). Then, the inverse \( \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix} \) of the symplectic matrix yields an almost complex structure \( J \) on \( \mathbb{T}_M^c \cong \mathbb{R}^n_+ \times \mathbb{R}^n_- \). It's the same as the one constructed in [Dom62, Paragraph 5] or later in [TO62], by our convention of listing the vertical component first and the horizontal one second. Moreover, it is compatible with the Euclidean metric and the canonical symplectic form \( \Omega \) on \( T^\ast \mathbb{R}^n_\ast \cong \mathbb{R}^n_+ \times (\mathbb{R}^n_-)^\ast \) in the sense that:

\[ (v, w)_{\text{euc}} = \Omega(v, \|w\|) \quad \text{and} \quad \Omega(v, w) = (\|v\|, w)_{\text{euc}} \quad \text{for all } v, w \in \mathbb{R}^{2n}, \]

cf. [MR99, Exercise 2.2.1]. The symplectic form gives rise to a Poisson bracket \( \{\cdot, \cdot\} \) such that the corresponding Poisson tensor reads as:

\[ \{f, g\}(x, v) = -\langle \nabla_{\text{euc}} f(x, v), \nabla_{\text{euc}} g(x, v) \rangle_{\text{euc}} = \Omega(x, v)(df, dg) \quad \text{for all } f, g \in C^\infty(\mathbb{R}^{2n}) \text{ and } (x, v) \in \mathbb{R}^n_+ \times \mathbb{R}^n_- \]

In this terminology, the proof of [GS16, Lemma 3.4 part ii)] relies on linking the operator \((A, D)\) to the antisymmetric bilinear form \((A, D)\) of integrating minus the Poisson bracket wrt. \( \mu \): \( A(f, g) := \int_{\mathbb{R}^{2n}} \{f, g\} \, d\mu \) for all \( f, g \in D = C^\infty_c(\mathbb{R}^{2n}) \). Via integration by parts one can show that \( (A f, g)_{L^2(\mu)} = A(f, g) \) holds for all \( f, g \in C^\infty(\mathbb{R}^{2n}) \). This can be done without the assumption of a weakly harmonic potential, since the vector field action can be represented in terms of the Hamiltonian vector fields. Indeed: Denote by \( H_f \) the Hamiltonian vector field of \( f \in C^\infty(\mathbb{R}^{2n}) \), i.e. it fulfills \( H_f(g) = \{f, g\} \) for all \( g \in C^\infty(\mathbb{R}^{2n}) \). We get the explicit formula \( -H_f = (\nabla_v f, \nabla_x)_{\text{euc}} - (\nabla_x f, \nabla_v)_{\text{euc}} \) and conclude that \( H_f \) is solenoidal by Schwarz's theorem. Hence, the following equality is true:

\[ A(f, g) = \int_{\mathbb{R}^{2n}} H_f(\rho) \cdot g \, d\lambda = (A f, g)_{L^2(\mu)} \quad \text{for all } f, g \in D, \]
where \( \rho := \frac{\mu}{\lambda} \) is the (loc-)density of \( \mu \).

**Proposition 3.7** (SAD-decomposition (2nd version)). The assertions of Lemma 3.4 are true without the assumption of \( \Psi \) being weakly harmonic.

**Proof.** We enhance the proof of Lemma 3.4 part (ii) via the technique discussed in the previous example which works due to the particular choices of the Gaussian fibre measure and the Sasaki metric wrt. the most natural Ehremsmann connection on \( TM \).

Let’s abbreviate by \( \rho := \frac{\mu}{\lambda} \) the (loc-)density of \( \mu \). Denote by \( \mathcal{J} \) minus the almost complex structure on \( TM \) constructed in [Dom62], and let \( \Omega \) be the canonical symplectic form on \( T^*M \). By construction, the Sasaki metric, \( \mathcal{J} \) and \( \Omega \) are compatible, cf. [MR99, page 341]. Hence, they define the same Poisson bracket \( \{\cdot, \cdot\} \) on \( TM \) via the assignments

\[
\Omega_\nu (df, dg) := \{ f, g \}(v) := -s_\nu (\nabla_s f(v), J\nabla_s g(v))
\]

for all \( f, g \in C^\infty(TM) \) and \( v \in TM \). Thus, for any fixed \( f \in C^\infty(TM) \) there is a unique Hamilton vector field \( H_f \) by [MR99, Proposition 10.2.1]. Let a chart \((v^j)_{j=1}^{2n}\) that gives normal coordinates and respects the Ehremsmann connection such that \((\partial v^j)_{j=1}^{2n}\) provides local basis for vertical vector fields and \((\partial v^k+n)_{k=1}^{n}\) provides a basis for the horizontal vector fields; we find that in these coordinates the Hamiltonian vector fields attain the form

\[
-H_f = \sum_{k=1}^{n} \partial_{u_k} f \cdot \partial_{u_k+n} - \sum_{i=1}^{n} \partial_{u_i+n} f \cdot \partial_{u_i}.
\]

In any such coordinates we easily compute that \( \text{div}_s (H_f) = 0 \) using Schwarz’s Theorem, in other words all \( H_f \) are solenoidal.

Define the antisymmetric bilinear form \( (A,D) \) by \( A(f, g) := \int_{TM} \{ f, g \} \, d\mu \) for all \( f, g \in D = C^\infty_c(TM) \). From the Divergence Theorem it follows that

\[
A(f, g) = \int_{TM} -H_f(g) \, d\lambda_s = \int_{TM} H_f(\rho) \cdot g \, d\lambda_s.
\]

Using our comments on the Sasaki gradient of a loc-density, see Remark 2.32, we can infer that \( H_f(\rho) = -\rho \cdot H_m f \). In a nutshell, we localise in the support of \( f \) via a partition of unity argument where the corresponding open cover is formed by charts \((v^j)_{j=1}^{2n}\) that are respecting the Ehremsmann connection and also are restricted to domains of local trivialisation; therein, we can use the local coordinate form of \( H_f \) and that the loc-density \( \rho \) trivialises to a product of exponential-type densities. Hence, we gain that \( (Af, g)_\mu = A(f, g) \) for all \( f, g \in D \) which finishes the proof. \( \Box \)

We are going to construct the projection \( P \) mentioned in (D5). For every \( f \in H \) we call the mapping

\[
E_\nu [f] : \mathcal{M} \to \mathbb{R}, \ x \mapsto \int_{T_x \mathcal{M}} f \, d\nu
\]

the fibrewise average of \( f \). Clearly, the operator \( E_\nu \) acts trivially on vertically lifted functions, since they are fibrewise constant: \( E_\nu [f_0] = f_0 \) for all \( f_0 \in L^2(\mathcal{M}; \mathcal{M}) \). We define \( P_S f := E_\nu [f] \circ \pi_0 \) assigning to an element of the tangent space the average of \( f \) over the fibre corresponding to this element. Thinking in a local trivialisation the vertical lift of the fibrewise average erases the dependency of \( f \) on the velocity component, since \( f \) can be thought as a bivariate function of position and velocity in this localisation. The range of \( P_S \) is precisely \( P_S(H) = \pi_0^\nu (L^2(\mathcal{M}; \mathcal{M})) \), the set of vertically lifted \( L^2 \)-functions on the weighted position manifold, and \( P_S \) is a projection.
Moreover, we have that
\[
\int_{TM} (Ps f)^2 \, d\mu = \int_{TM} (E_w[f] \circ \pi_0)^2 \, d\lambda_m \otimes \nu = \int_{TM} (E_w[f])^2 \circ \pi_0 \, d\lambda_m \otimes \nu
\]
for all \( f \in H \). This implies \( \|Ps f\|_H = \|E_w[f]\|_{L^2(M,m)} \leq \|f\|_H \) meaning that \( P_S \) is continuous with norm 1. Last but not least, \( P_S \) also is an orthogonal projection, cf. [Con90, Proposition 3.3]. Finally, we define \( P := P_S - (1,1)_H \).

**Lemma 3.8** (Properties of projection \( P \) and semigroup conservativity (D5), (D7)).
Let condition (P1) hold. Then, we have \( P(H) \subseteq D(S), \ SP = 0, \ P(D) \subseteq D(A) \) and \( AP(D) \subseteq D(A) \). Furthermore, \( 1 \in D(L) \) and \( L1 = 0 \).

**Proof.** We follow the lines of the proof of [GS16, Lemma 3.4 (iv)] for the Euclidean case.

First of all, we start with the statements related to \( S \). Let \( \varphi \in C_c^\infty(M;[0,1]) \) be a cut-off function\(^8\) such that \( \varphi = 1 \) on \( U(o,1) \) and \( \varphi = 0 \) on \( U(o,2) \). Define \( \varphi_n := \varphi(\cdot/\ell_n) \) for all \( n \in \mathbb{N} \setminus \{0\} \). Note that there is some constant \( c \in (0,\infty) \) such that 
\[
|\nabla_m \varphi_n(x)|_m \leq \frac{c}{n} \quad \text{and} \quad |\text{Hess}_m \varphi_n(x)|_\infty \leq \frac{c}{n^2}
\]
for all \( x \in M \) and \( n \in \mathbb{N} \setminus \{0\} \), cf. [GS16, Definition 3.3]. Clearly, \( (\varphi_n)_{n \in \mathbb{N}} \) pointwisely converges to the constant function 1 as \( n \to \infty \). Let \( f_0 \in C_c^\infty(M) \) and define
\[
f_n := f_0^\varphi \otimes \varphi_n^h = f_0^\varphi \cdot \varphi_n^h \quad \text{for all } n \in \mathbb{N} \setminus \{0\}.
\]
This yields a sequence in \( D \) converging to \( f_0^\varphi \) in \( H \). From dominated convergence we can conclude that
\[
S f_n = \frac{\alpha}{\beta} f_0^\varphi \cdot \Delta_v \varphi_n^h - \alpha f_0^\varphi \cdot \nabla v \varphi_n^h \longrightarrow 0 \quad \text{in } H \text{ as } n \to \infty,
\]
where we use that the function \( TM \to \mathbb{R}, v \mapsto |\nabla(v)|_\nu \) is in \( H \) as on each fibre \( T_x M \cong \mathbb{R}^n \) the function \( |Id_{T^*v}|_m \) is in \( H \) and \( \text{Hess}_m \varphi_n(x)|_\infty \leq \frac{c}{n^2} \).

Next, we turn to the statements involving \( A \). Let \( f_0 \in C_c^\infty(M) \) and define the sequence \((f_n)_{n \in \mathbb{N} \setminus \{0\}}\) as in (3.4) approximating \( f_0^\varphi \) in \( H \). Using \( |H_m|_s = |IdTM|_m \in H \) as above and that \( \nabla_m \Phi \in L^2(M \to TM; m) \) we can infer via dominated convergence that
\[
Af_n = -\varphi_n^h \cdot \text{H}_m f_0^\varphi + \frac{1}{\beta} f_0^\varphi \cdot (\nabla_v \Phi^h) \varphi_n^h = -\varphi_n^h \cdot \text{H}_m f_0^\varphi + \frac{1}{\beta} f_0^\varphi \cdot m(\nabla_m \Phi \cdot \nabla_v \varphi_n^h)
\]
\(^8\)An explicit choice of \( \varphi \) is given as follows: Let \( h \) be a chart at \( o \) – wlog. the ball \( U(o,2) \) is contained in the chart domain. Define the auxiliary ‘mountain’ function
\[
m: \mathbb{R} \to \mathbb{R}, \ t \mapsto \exp(-\frac{1}{n(1-t)}) \cdot 1_{(0,1)}(t).
\]
We transform the ‘mountain’ to a ‘table mountain’ by the assignment
\[
\tau: \mathbb{R}^n \to [0,1], \ y \mapsto \int_0^{2-n|y|^2} m(t) \, dt.
\]
Then, the choice \( \varphi := \tau \circ h \cdot 1_U(o,2) \) yields a smooth function with the desired properties.
\[ \rightarrow - \mathcal{H}_m f_n^0 + \frac{1}{\beta} f_n^0 \cdot 0 \quad \text{in } H \text{ as } n \rightarrow \infty. \]

Closedness of \((A, D(A))\) implies \(f_0^0 \in D(A)\) as well as \(A f_0^0 = - \mathcal{H}_m f_0^0\). For the inclusion \(P(D) \subseteq D(A)\) it’s enough to show that \(1 \in D(A)\) and \(A1 = 0\). Similar as before the function \(v \rightarrow |\mathcal{H}_m(v)|_s = |v|_m\) is in \(H\), and thus dominated convergence gives us

\[ A \varphi_n^\alpha = - \mathcal{H}_m \varphi_n^\alpha \rightarrow 0 \quad \text{in } H \text{ as } n \rightarrow \infty. \]

In order to prove now that \(AP(D) \subseteq D(A)\) we adhere to our approximation strategy and define \(h_n := \varphi_n^h \cdot \mathcal{H}_m f_0^0\) for all \(n \in \mathbb{N} \setminus \{0\}\). The sequence \((h_n)_{n \in \mathbb{N}}\) converges to \(\mathcal{H}_m f_0^0\) both pointwisely and in \(H\). We note that the function \(\mathcal{H}_m f_0^0 = \mathcal{H}_m(\mathcal{H}_m f_0^0)\) is dominated by

\[ \|\mathcal{H}_m\|_{L^2(TM \rightarrow TTM; \mu)} \cdot \|\mathcal{H}_m f_0^0\|_H \leq \|\mathcal{H}_m\|^2_{L^2(TM \rightarrow TTM; \mu)} \cdot \|f_0^0\|_H = \|\text{Id}_{TM}|_m\|^2_H \cdot \|f_0^0\|_H \]

due to the Cauchy-Bunyakovsky-Schwarz inequality applied twice. This dominating function is in \(H\) as on each fibre \(T, M \simeq \mathbb{R}^n\) the function \(|\text{Id}_{TM}|_m|^2\) is in \(L^2(\mathbb{R}^n; \nu)\). Togethet with afore-mentioned facts that \(|\text{Id}_{TM}|_m \in H\) and \(\nabla_{m} \Phi \in L^2(M \rightarrow TM; m)\) this yields

\[ Ah_n = - \varphi_n^h \cdot \mathcal{H}_m f_0^0 + \frac{1}{\beta} \cdot (\nabla_{\nu} \Phi^h) h_n \]

\[ = - \varphi_n^h \cdot \mathcal{H}_m f_0^0 + \frac{1}{\beta} \cdot ((\mathcal{H}_m f_0^0) \cdot (\nabla_{\nu} \Phi^h)) \varphi_n^h + \varphi_n^h \cdot (\nabla_{\nu} \Phi^h)(H_m f_0^0) \]

\[ \rightarrow - \mathcal{H}_m^2 f_0^0 + 0 + \frac{1}{\beta} \cdot (\nabla_{\nu} \Phi^h)(H_m f_0^0) \quad \text{in } H \text{ as } n \rightarrow \infty \]

by dominated convergence. Since \((A, D(A))\) is closed, the function \(\mathcal{H}_m f_0^0\) is an element of \(D(A)\).

Finally, the statements on \(L\) follow the very same way: Let \(f_0 \in C_0^\infty(M)\) and define \((f_n)_{n \in \mathbb{N} \setminus \{0\}}\) as in (3.4); repeat the previous steps and conclude from closedness of \((L, D(L))\) that \(f = f_0^0 \in D(L)\) and \(L f_0^0 = - A f_0^0\). In particular, the sequence \((L \varphi_n^0)_{n \in \mathbb{N}}\) converges in \(H\) to 0 as \(n \rightarrow \infty\), and by closedness it follows that \(f \in D(L)\) with \(L1 = 0\).

**Remark 3.9.** As we deduced \(Pf \in D(A)\) for all \(f \in D\), we can calculate that

\[ APf = - \mathcal{H}_m(\mathcal{E}_0[f]) = - \langle \mathcal{H}_m, \mathcal{E}_0[f] \circ d\pi_0 \rangle \]

\[ = - \langle \text{Id}_{TM}, \mathcal{E}_0[f] \rangle = - d\mathcal{E}_0[f]. \quad (3.5) \]

However, one can give this formula a more intuitive form:

\[ APf = - \mathcal{H}_m(Pf) = - \mathcal{H}_m(\mathcal{E}_0[f]) = - h(\mathcal{H}_m, \nabla_0(P^2 f)) \]

\[ = - h(\mathcal{H}_m, h(\nabla_0 \mathcal{E}_0[f])) = - m_{\pi_0}(|\text{Id}_{TM}|_m, \nabla_0 \mathcal{E}_0[f] \circ \pi_0). \quad (3.6) \]

Unlike Equation (3.5), we recognise that Equation (3.6) is in perfect correspondence to [GS16, Equation (3.12)].

Turning to the remaining condition (D2), i.e. the question whether \((L, D(L))\) generates a strongly continuous semigroup, we first look into the case of a smooth potential \(\Psi \in C^\infty(M)\). The proof relies on methods from [HN05] as explained in [GS14, Section 4]. We briefly quote a consequence of the Hörmander Theorem, namely [HN05, Proposition A.1]. Let \(T\) be a second order differential operator on a Riemannian manifold \((\mathbb{B}, b)\) of the form \(T = c + \lambda_0 + \sum_{k=1}^\ell \lambda_k\) with \(c \in C^\infty(\mathbb{B})\) and \(\lambda_k \in \Gamma^\infty(T\mathbb{B})\) for all \(k \in \{0, \ldots, \ell\}\). We say that \(T\) satisfies the **Hörmander condition** if at any point \(b \in \mathbb{B}\) holds

\[ \dim(\text{Lie}_b(\mathcal{X}_0, \ldots, \mathcal{X}_\ell)) = \dim(T_b \mathbb{B}) = \dim(\mathbb{B}), \]
where \( \text{Lie}(X_0, \ldots, X_t) \) denotes the generated Lie algebra.

**Proposition 3.10.** Let \( T \) satisfy the Hörmander condition and let \( f \in L^1_{\text{loc}}(\mathbb{B}; b) \) such that
\[
\int_{\mathbb{B}} f \cdot T \psi \, d\lambda = 0 \quad \text{for all } \psi \in C^\infty_c(\mathbb{B}).
\]
Then, \( f \) has a smooth representative.

**Proof.** The proof of this proposition works as for [GS14, Proposition A.1]: It is done in chart domains, and within these domains it’s perfectly fine to consider \( X_j = \partial x^j = \frac{\partial}{\partial x^j} \), where \( (x^j)_j=1^{\dim(\mathbb{B})} \) denotes local coordinates provided by the chart.

Then, we have smooth representatives in chart domains serving as starting point for a partition of unity argument. Thus, even if \( T \) is just available in local coordinate form, we can apply the previous proposition as soon as the respective chart domains form an open cover. \( \blacksquare \)

**Lemma 3.11** (Hörmander condition for the Langevin generator). Consider a smooth potential \( \Psi \in C^\infty(\mathbb{M}) \). Then, the Langevin generator satisfies the Hörmander condition. More precisely, let \( v \in T\mathbb{M} \) and \( (x^i)_i=1^{\dim(\mathbb{M})} \) be local coordinates corresponding to a chart at \( \pi_\mathbb{M}(v) \). Then, we have that
\[
\dim(\text{Lie}_v(\mathcal{H}_m, \text{vl}(\nabla_m \Psi), \text{vl}(\partial x^1), \ldots, \text{vl}(\partial x^m), \mathcal{V})) = \dim(T_v T\mathbb{M}) = \dim(T\mathbb{M}) = 2n.
\]

**Proof.** Obviously, neither \( \text{vl}(\nabla_m \Psi) \) nor \( \mathcal{V} \) contribute anything to the generated Lie algebra. From [GK02, Proposition 5.1] we know explicit forms for the Lie brackets of vertically and horizontally lifted vector fields in any combination. With that said, we have the equations
\[
[\text{vl}(\mathcal{X}_v), \text{vl}(\mathcal{Y}_v)] = 0 \quad \text{and} \quad [\text{hl}(\mathcal{Y}_v), \text{vl}(\mathcal{X}_v)] = \text{vl}(\nabla^m_m \mathcal{X}_v)
\]
for all \( \mathcal{X}, \mathcal{Y} \in \Gamma^\infty(T\mathbb{M}) \). Thus, the only nontrivial pairing of the seeding vector fields \( \mathcal{X}, \mathcal{Y} \in \{\partial x^1, \ldots, \partial x^m\} \) is \( [\text{hl}(\mathcal{Y}_v), \text{vl}(\mathcal{X}_v)] = \text{vl}(\nabla^m_m \mathcal{X}_v) \), but clearly all vector fields of this form are linear dependent of the vertical vector fields \( \{\text{vl}(\partial x^1), \ldots, \text{vl}(\partial x^m)\} \) generating the Lie algebra. This just shows that the collection \( \{\partial x^1, \ldots, \partial x^m\} \) might not serve our purpose even if we would build our generator with both kinds of liftings.

The statement concerning Lie brackets of one vertically and one horizontally lifted vector fields does not apply to \( \mathcal{H}_m \), as a semispray cannot arise as horizontal lift of a vector field. However, in Lemma A.1 we proved that \( [\mathcal{H}_m, \text{vl}(\partial x^i)] = \text{hl}(\partial x^i) - \sum_{i=1}^m N^j_i \cdot \text{vl}(\partial x^j) \) for certain functions \( N^j_i \). This yields \( m \) many linear independent horizontal vector fields in the generated Lie algebra which finishes the proof. \( \blacksquare \)

**Theorem 3.12** (**(D2)**) for smooth potentials using a hypoellipticity startegy). Let \( \Psi \in C^\infty(\mathbb{M}) \) be a smooth potential. Then, \( (L, D) \) is essentially \( m \)-dissipative. Thus, its closure \( (L, D(L)) \) generates a strongly continuous semigroup.

**Proof.** From Lemma 3.4 we know that \( (L, D) \) is dissipative. In view of the Lumer-Philips Theorem, we have to show that the range \( (\text{Id}_H - L)(D) \) is dense in \( H \). Let \( f \in H \) fixed such that
\[
((\text{Id}_H - L)u, f)_H = 0 \quad \text{for all } u \in D. \tag{3.7}
\]
We claim that \( f = 0 \).

Due to the choice of \( f \) we have that \( \exp(-\Phi^\nu)f \in L^1_{\text{loc}}(TM; \lambda_m \otimes_{\text{loc}} \nu) \) and by Lemma 3.11 we can assume that \( \exp(-\Phi^\nu)f \) is smooth. Let \( (\varphi_n)_{n \in \mathbb{N}} \) be a sequence of cut-off functions in \( C^\infty_c(\mathbb{M}; [0, 1]) \) with \( \varphi_n \to 1 \) pointwise as \( n \to \infty \).
and such that \( \| \nabla_m \varphi_n \|^2_{L^\infty(M \to T^*M)} \leq \frac{1}{n} C \) holds for all \( n \in \mathbb{N} \setminus \{0\} \) and some \( C \in (0, \infty) \). Now, define \( u_n := (\varphi_n^2)^2 f \) for all \( n \in \mathbb{N} \). It’s clear that

\[
(u_n, f)_H = (Lu_n, f)_H = (Su_n, f)_H - (Au_n, f)_H.
\]

Due to the fact that \((S, D)\) is nonpositive definite and multiplying with \( \varphi \) commutes with the action of \( S \) we see that \((Su_n, f)_H = (S(\varphi_n^2 f), \varphi_n^2 f)_H \leq 0 \). Similarly, using antisymmetry we get that

\[
(Au_n, f)_H = (A(\varphi_n^2) \varphi_n^2 f, f)_H + (A(\varphi_n^2 f), \varphi_n^2 f)_H = (A(\varphi_n^2) \varphi_n^2 f, f)_H.
\]

Altogether, this yields the estimate

\[
(u_n, f)_H = \int_{TM} (\varphi_n^2)^2 f^2 \, \d \mu \leq \frac{1}{n} C \int_{TM} \varphi_n^2 f^2 \, \d \mu \leq \frac{1}{n} C \| f \|_H^2
\]

using (3.6). By dominated convergence this implies that \( \| f \|_H^2 \leq 0 \), thus \( f = 0 \). \( \Box \)

For the case of a \( \text{loc} \)-Lipschitzian potential, we leave the base weight aside for a moment, but keep the fibre weight. In other words, we could think of \( TM \) endowed with bundle weight \( 1 \otimes \text{loc} \rho_F \). Let \( \Psi \in L^1_{\text{loc}}(\mathcal{M}; m) \), and \( L_0 := \frac{D}{\Psi} \Delta_v + H_m \) be defined on the very same set \( D_0 \) as before – it should be clear at this point that \( L_0 \) has to consist of the weighted diffusion and the non-corrected semispray instead of \( H_m \).

**Lemma 3.13.** \((L_0, D_0)\) is essentially \( m \)-dissipative on \( L^2(TM; \lambda_m \otimes \text{loc} \nu) \).

**Proof.** From **Theorem 3.12** applied to the smooth case of the zero potential, we know that \((L_0, D)\) is essentially \( m \)-dissipative on \( L^2(TM; \lambda_m \otimes \text{loc} \nu) \). We have to show that \((L_0, D)\) is contained in the closure of \((L_0, D_0)\).

For any \( f \in D \) there is an approximating sequence \((f_n)_{n \in \mathbb{N}}\) in \( D_0 \) wrt. usual locally convex topology implying uniform convergence of all derivatives on compacts cf. **Lemma 2.40**. Furthermore, there is a common compact set in \( TM \) large enough containing \( \supp(f) \) and \( \supp(f_n) \) for all \( n \in \mathbb{N} \). Therefore, we have that \( \sup_{v \in TM} |L_0 f_n(v) - L_0 f(v)| \to 0 \) as \( n \to \infty \). Thus, \( f_n \to f \) in \( L^2(TM; \lambda_m \otimes \text{loc} \nu) \) as \( n \to \infty \). \( \Box \)

Denote by \( H_0^{1, \infty}(\mathcal{M}) \) the closure of \( C_c^\infty(\mathcal{M}) \) wrt. \( H^{1, \infty} \)-norm. Define

\[
D_1 := \pi_0^* H_0^{1, \infty}(\mathcal{M}) \otimes \kappa^* C_c^\infty(\mathcal{M}).
\]

Considering the operator \((L, D_1)\) we realise that **Lemma 3.4** still does apply – the proof has to be adapted just slightly. Further on, we define the unitary isomorphism \( U : H = L^2(TM; \lambda_m \otimes \text{loc} \nu) \to L^2(TM; \lambda_m \otimes \text{loc} \nu), f \mapsto \exp(-\Phi/2)^v \cdot f \).

Note that \( U(D_1) = D_1 \). Thus, we define the operator \( \tilde{L} := U L U^{-1} \) on \( D_1 \). One directly observes that

\[
\tilde{L} = \tilde{S} - \tilde{A} := U \Delta_v U^{-1} - UAU^{-1}
\]

holds on \( D_1 \) with operators \((\tilde{S}, D_1)\) symmetric, negative semidefinite and \((\tilde{A}, D_1)\) antisymmetric both on \( L^2(TM; \lambda_m \otimes \text{loc} \nu) \), as well as

\[
\tilde{A} f_0^v = -U(f_0^v \cdot m_{\pi_0}(\text{Id}_{TM}, \nabla_m \exp(\Phi/2) \circ \pi_0) + \exp(\Phi/2)^v \cdot m_{\pi_0}(\text{Id}_{TM}, \nabla_m f_0 \circ \pi_0))
\]

\[
-\frac{1}{2} f_0^v \cdot m_{\pi_0}(\text{Id}_{TM}, \nabla_m \Phi \circ \pi_0) - m_{\pi_0}(\text{Id}_{TM}, \nabla_m f_0 \circ \pi_0)
\]

\[
-\frac{1}{2} f_0^v \cdot H_m \Phi^v - H_m f_0^v \quad \text{for all } f_0 \in H_0^{1, \infty}(\mathcal{M}).
\]
For $f = f_0^\gamma \otimes g_0^h \in D_1$ we get that
\[
\tilde{A} f = \frac{1}{2} f_0^\gamma \cdot \mathcal{H}_m \Phi^\nu - g_0^h \cdot \mathcal{H}_m f_0^\gamma + f_0^\gamma \cdot \frac{1}{\beta} (\nabla, \Phi^h) (g_0^h)
\]
\[
= \frac{1}{2} f \cdot \mathcal{H}_m \Phi^\nu - \mathcal{H}_m f + \frac{1}{\beta} (\nabla, \Phi^h) (f).
\]

In straight argument to [GS14, Section 4], the proof of the next lemma, which deals with the globally Lipschitzian case, is based on a perturbation theorem for essentially m-dissipative operators. We present it here for sake of completeness.

**Theorem 3.14** (Kato perturbation of an essentially m-dissipative operator). Let an essentially m-dissipative operator $Z$ and a dissipative operator $T$ have common domain in some given Hilbert space with norm $\| \cdot \| := \sqrt{\langle \cdot , \cdot \rangle}$. Assume that there are constants $c_1 \in \mathbb{R}$ and $c_2 \in (0, \infty)$ such that
\[
\| Tf \|^2 \leq c_1 (Z f , f) + c_2 \| f \|^2
\]
holds for all $f$ from the common domain. Then, the perturbation $Z + T$ of $Z$ by $T$ defined on the common domain is essentially m-dissipative.

**Proof.** See [Dav80, Corollary 3.8, Lemma 3.9 and Problem 3.10].

**Lemma 3.15** (Essential m-dissipativity in case of globally Lipschitzian potentials). Assume that $\Psi$ is globally Lipschitzian. Then, $(L, D_1)$ is essentially m-dissipative on $L^2(\mathbb{M}; \lambda_{m} \otimes \nu)$. Hence, $(L, D_1)$ is essentially m-dissipative on the space $H = L^2(\mathbb{M}; \lambda_{m} \otimes \nu)$.

**Proof.** Define $Z := L_0$ on $D_1$. Then, $(Z, D_1)$ is a dissipative extension of $(Z, D_0)$. Thus, $(Z, D_1)$ is essentially m-dissipative on $L^2(\mathbb{M}; \lambda_{m} \otimes \nu)$ by Lemma 3.13.

Define the perturbation
\[
T f := -\frac{1}{\beta} (\nabla, \Phi^h) (f) + \frac{1}{2} f \cdot \mathcal{H}_m \Phi^\nu
\]
for all $f = f_0^\gamma \otimes g_0^h \in D_1$. Since by Liouville’s Theorem $(\mathcal{H}_m, D_1)$ is antisymmetric and also $(\tilde{A}, D_1)$ is antisymmetric in $L^2(\mathbb{M}; \lambda_{m} \otimes \nu)$, $(T, D_1)$ is antisymmetric as well. Thus, $(T, D_1)$ is dissipative.

Choose $g$ such that $f = U g$. Using the Cauchy-Bunyakovsky-Schwarz inequality we get that
\[
\int_{\mathbb{M}} (\nabla \cdot (\Phi^h \cdot f))^2 \, d\lambda_{m} \otimes \nu
\]
\[
= \int_{\mathbb{M}} |\nabla \cdot (\Phi_h \cdot f)|^2 \, d\lambda_{m} \otimes \nu = \int_{\mathbb{M}} |\nabla \cdot (\Phi_h \cdot g)|^2 \, d\lambda_{m} \otimes \nu
\]
\[
\leq \| \nabla \cdot (\Phi_h \cdot f) \|^2_{L^2(\mathbb{M} \to \mathbb{M}; \lambda_{m} \otimes \nu)} \cdot \int_{\mathbb{M}} |\nabla \cdot g|^2 \, d\lambda_{m} \otimes \nu
\]
\[
= \| \nabla \cdot (\Phi_h \cdot f) \|^2_{L^2(\mathbb{M} \to \mathbb{M}; \lambda_{m} \otimes \nu)} \cdot \left( -\Delta_g g, g \right)_{L^2(\mathbb{M}; \lambda_{m} \otimes \nu)}
\]
\[
= \| \nabla \cdot (\Phi_h \cdot f) \|^2_{L^2(\mathbb{M} \to \mathbb{M}; \lambda_{m} \otimes \nu)} \cdot \left( -\tilde{S} f, f \right)_{L^2(\mathbb{M}; \lambda_{m} \otimes \nu)},
\]
by the integration by parts formula (2.1). Abbreviate $C_{\Phi} := \| \nabla \cdot (\Phi_h \cdot f) \|^2_{L^2(\mathbb{M} \to \mathbb{M}; \lambda_{m} \otimes \nu)}^{\infty}$. Then, we immediately conclude
\[
\| Tf \|^2_{L^2(\lambda_{m} \otimes \nu)} \leq \frac{1}{\beta^2} C_{\Phi} \cdot \left( -\tilde{S} f, f \right)_{L^2(\lambda_{m} \otimes \nu)} + \frac{1}{4} C_{\Phi} \cdot \| f \|^2_{L^2(\lambda_{m} \otimes \nu)}
\]
\[
= c_1 (Z f , f)_{L^2(\lambda_{m} \otimes \nu)} + c_2 \| f \|^2_{L^2(\lambda_{m} \otimes \nu)},
\]
with $c_1 := \frac{\beta}{2} C_{\Phi}$ and $c_2 := \frac{1}{4} C_{\Phi}$, since we know $(\mathcal{H}_m f , f)_{L^2(\lambda_{m} \otimes \nu)} = 0$. Finally, the claim follows applying Theorem 3.14 to $(Z + T, D_1)$.
Corollary 3.16 (\textit{D2} for globally Lipschitzian potentials). Assume that $\Psi$ is a globally Lipschitzian potential. Then, $(L, D)$ is essentially $m$-dissipative on $H$.

Proof. Note that $(L, D)$ is a dissipative extension of $(L, D_0)$. Thus, we show that $(L, D_1)$ is contained in the closure of $(L, D_0)$ and then apply Lemma 3.15.

Let $f = f_n \in \pi_0^1 H^1(M)$, $g \in \kappa^* C_c^\infty(M)$ and a sequence $(f_n)_{n \in \mathbb{N} \setminus \{0\}}$ in $C_c^\infty(M)$ such that its vertical lifting approximates $f$ in $H^{1,2}$-sense, i.e.

\begin{align*}
(1) \quad f_n & \to f_0 \text{ as } n \to \infty \text{ in } L^2(M;m) \text{-sense and } \\
(2) \quad \frac{\partial f_n}{\partial t} & \to \frac{\partial f_0}{\partial t} \text{ as } n \to \infty \text{ in } L^2(M;m) \text{-sense for any chart } x = (x^i)_{j=1}^n.
\end{align*}

This convergence is maintained under passing to $\pi_0^1 L^2(M;m)$, i.e. weighting the manifold. Finally, we conclude that
\[ L(f_n \otimes g) \to L(f_0 \otimes g) = L(f \otimes g) \text{ in } H \text{ as } n \to \infty. \]

\hfill $\square$

The final prove of this section basically is the same as in [GS14, Theorem 4.7] as we have taken geometric effects into account before.

Theorem 3.17 (\textit{D2} for locally Lipschitzian potentials). Let $\Psi$ be a loc-Lipschitzian potential bounded from below. Then, $(L, D)$ is essentially $m$-dissipative on $H$.

Proof. Wlog. we assume that $\Psi \geq 0$. Let $\varepsilon \in (0, \infty)$ and fix some $g \in D \setminus \{0\}$. Choose $\varphi, \psi \in D$ such that
\[ \varphi|_{\text{supp}(g)} = 1, \quad \psi|_{\text{supp}(\varphi)} = 1 \quad \text{and} \quad 0 \leq \varphi \leq \psi \leq 1. \]

Let $f \in D$ arbitrary. Throughout the proof, we add to the generators and invariant measures a subscript to indicate the corresponding potential, e.g. $\mu_0 = \lambda_s$ in case of the zero potential. By construction and using dissipativity of $(L, D)$ on $L^2(\text{TM}|\mu_{\psi})$ we get that
\begin{align*}
&\|(\text{Id} - L_{\Psi})(\varphi f) - g\|_{L^2(\mu_{\psi})} \\
\leq &\|\varphi((\text{Id} - L_{\Psi})f - g)\|_{L^2(\mu_{\psi})} + \|f\|_{L^2(\mu_{\psi})} \cdot \|\nabla m \varphi\|_{L^\infty(\lambda_m)} \\
\leq &\|(\text{Id} - L_{\Psi})f - g\|_{L^2(\mu_{\psi})} + \|(\text{Id} - L_{\Psi})f\|_{L^2(\mu_{\psi})} \cdot \|\nabla m \varphi\|_{L^\infty(\lambda_m)}.
\end{align*}

Now, we tighten the requirements on $\varphi$ via additionally demanding that
\[ \|\nabla m \varphi\|_{L^\infty(\text{TM}|m)} = \|\nabla m \varphi\|_{L^\infty(m)} \leq \frac{\varepsilon}{4} \cdot \|g\|_{L^2(\mu_0)}^{-1} = \frac{\varepsilon}{4} \cdot \|g\|_{L^2(\mu_0)}^{-1}. \]

Due to Corollary 3.16 $(L_{\psi}, D)$ is essentially $m$-dissipative on $L^2(\text{TM}|\mu_{\psi})$, hence as a consequence of the Lumer-Philips Theorem, there is $f \in D$ such that simultaneously hold
\[ \|(\text{Id} - L_{\Psi})f - g\|_{L^2(\mu_{\psi})} \leq \frac{\varepsilon}{2} \quad \text{and} \quad \|(\text{Id} - L_{\Psi})f\|_{L^2(\mu_{\psi})} \leq 2\|g\|_{L^2(\mu_{\psi})}. \]

For such an $f$ we end up with $\|(\text{Id} - L_{\Psi})(\varphi f) - g\|_{L^2(\mu_{\psi})} < \varepsilon$. In conclusion, we proved that $(\text{Id} - L_{\Psi})(D)$ is dense in $H = L^2(\text{TM};\mu_{\psi})$. \hfill $\square$

3.2. Hypocoercivity conditions.

Lemma 3.18 (algebraic relation (H1)). Let condition (P1) hold. Then, we have $P A P' |_D = 0$.

Proof. Recall Equation (3.6) from Remark 3.9. Furthermore, we are going to apply the formula for Gaussian integrals from [GS14, Lemma 3.1]: Let $f \in D$ and consider polar coordinates in the fibre at $x \in \mathbb{M}$ using [For12, Satz 14.8] which is an application of the transformation formula and Fubini. We get that
\[ \int_{T_x \mathbb{M}} A P f \ d\nu \]
Lemma 3.18 is fulfilled with $\text{Bec}_89$ for the second. With this ansatz we calculate applying $\text{GS}_14$ relies on $\text{(P1)}$.

Thus, $\nu$ is the diffeomorphism corresponding to $(0, \infty) \times S^{n-1} \to \mathbb{T}_m\mathbb{M}$, $(r, u) \mapsto v(r, u)$ is the diffeomorphism corresponding to $(0, \infty) \times S^{n-1} \to F = \mathbb{R}^n$, $(r, u) \mapsto ru$. We point out that this argument just works as $\nu$ is invariant wrt. rotations.

We have seen that $P_S AP$ is trivial an $D$, so we can use orthogonality of $P_S$ to obtain

$$0 = (P_S APf , 1)_{L^2(\mathbb{M}, \nu)} = (APf , 1)_H$$

for all $f \in D$. Thus, $PAP|_D = 0$. ☐

**Lemma 3.19** (microscopic hypocoercivity (H2)). Let condition $\text{(P1)}$ hold.

Then, condition $(H2)$ is fulfilled with $\lambda_m = \alpha$.

**Proof.** Let $f \in D$. Using the Poincaré inequality for Gaussian measures, see [Bec89], we deduce that

$$\langle -Sf , f \rangle_H = \frac{\alpha}{\beta} (\langle \nabla v f , \nabla v f \rangle)_{L^2(\mathbb{T}_m\mathbb{M}, \nu)} = \frac{\alpha}{\beta} \| \nabla v f \|_{L^2(\mathbb{T}_m\mathbb{M}, \nu)}^2$$

$$\geq \alpha \| f - (f , 1) \|_{L^2(\mathbb{T}_m\mathbb{M}, \nu)}^2 = \alpha \| (\text{Id}_H - P_S) f \|_H^2$$

and the claim follows. ☐

The strategy for proving condition $(H3)$ relies on [GS14, Corollary 2.13]. Most importantly, we have to prove that $(\text{Id}_H - PA^2P, D)$ is essentially $m$-dissipative. To do so, we characterise $\text{Id}_H - PA^2P$ on $D$ starting from Equation (3.6) and show that the range $(\text{Id}_H - PA^2P)(D)$ is dense in $H$.

Let $s$: $(-\delta, \delta) \to \mathbb{T}_m\mathbb{M}$ be a curve such that $s(0) = v$ and $s'(0) = H_m(v)$ for $v \in \mathbb{T}_m\mathbb{M}$ fixed and some small $\delta \in (0, \infty)$. Let $x := \pi_0(v)$. The following computation relies on $\pi_0 \circ s$ being a geodesic of $H_m$ and the characterisation of the directional derivative in terms of parallel transport ‘pt’ along $s$ given by the Levi-Civita connection:

$$-H_m APf(v) \stackrel{(3.6)}{=} -H_m(-m_{\pi_0}(\text{Id}_{\mathbb{T}_m\mathbb{M}} , \nabla_{\nu} E_v[f]) \circ \pi_0)(v)$$

$$= \left( \lim_{t \to 0} \frac{1}{t} \left( pt_{s(t)}^{\nu}(\text{Id}_{\mathbb{T}_m\mathbb{M}} \circ s(t)) - pt_{s(t)}^{\nu}(\text{Id}_{\mathbb{T}_m\mathbb{M}} \circ s(0)) \right) \right) \cdot \nabla_h P_S f(v)$$

$$+ h_v(\nabla_h P_S f(v))$$

$$= h_v(\nabla_{\nu} E_v[f])(v) + m_x(v , \nabla_{\nu} E_v[f](v))$$

For the second equality we also used the metric compatibility of the Levi-Civita connection $\nabla^\ast$. The last but one line is obtained using Lemma A.2 for the second summand. Now, we transform into polar coordinates in the fibre $\mathbb{T}_m\mathbb{M}$ similar as in the proof of Lemma 3.18. With this ansatz we calculate applying $\text{GS}_14,
Lemma 3.1] twice that
\[ \int_{T_xM} \mathcal{H}_m(-m \pi_0(\text{Id}_TM, \nabla_m E_\nu[f] \circ \pi_0)) \, d\nu = \int_{T_xM} m_x(v, \nabla_m E_\nu[f](x)) \, d\nu(v) + \int_{T_xM} m_x(v, \nabla_m^m(\nabla_m E_\nu[f](x))) \, d\nu(v) \]

\[ = \int_{(0,\infty)} \int_{S^{n-1}} m_x(v, \nabla_m E_\nu[f](x)) \, d\nu(v) \]

\[ = \frac{1}{\beta} \int_{(0,\infty)} r^2 \cdot \int_{S^n} \frac{1}{S(S^{\nu-1})} \int_{S^{\nu-1}} \frac{d\nu(v(r, u))}{d\nu(v)} \cdot \frac{d\nu(v(u))}{d\nu(v)} \cdot dS(u) \cdot dr \]

\[ = \frac{1}{\beta} \Delta_m E_\nu[f](x) \cdot 1, \]

(3.8)

where we have taken $u_0 \in S^{\nu-1}$ arbitrary, since $\nu$ is invariant wrt. rotations. In order to arrive at the last but one line, we consider some chart $\nu$ where we have taken $\nu_0$ arbitrary, since $\nu$ is invariant wrt. rotations.

For all $\nu, \nu' \in \Gamma^\infty(TM)$ with local coordinate expression $\mathcal{X} = \sum_{i=1}^{n} X^i \cdot \partial x^j$ and $\mathcal{Y} = \sum_{j=1}^{n} Y^j \cdot \partial x^j$. Thus, we can understand the mapping

\[ S^{\nu-1} \rightarrow S^{\nu-1}, \ u \mapsto (\nabla_m^m \nabla_m E_\nu[f])(x) \]

as the matrix in [GS14, Lemma 3.1]. The last step of Equation (3.8) is due to the fact that the mean of a chi-squared distribution equals the number of degrees of freedom, i.e. $n$ in the present case.

With the proof of part (ii) of Lemma 3.4 and with [GS14, Lemma 3.1] we similarly get that for every $v \in TM$ with $x := \pi_0(v)$ holds

\[ P_S \left( \frac{1}{\beta} \nabla_v \Phi^h(\text{AP}f) \right)(v) = \int_{T_xM} \mathcal{H}_m \Phi^v \cdot \text{AP}f \, d\nu \]

\[ = - \int_{T_xM} h(\mathcal{H}_m, \nabla_h \Phi^v) \cdot h(\mathcal{H}_m, \nabla_h (P_Sf)) \, d\nu \]

\[ = - \int_{T_xM} m_x(\text{Id}_TM, \nabla_m \Phi(x)) \cdot m_x(\text{Id}_TM, \nabla_m E_\nu[f](x)) \, d\nu \]

\[ = - \frac{1}{\beta} m_x(\nabla_m \Phi(x), \nabla_m E_\nu[f](x)) = - \frac{1}{\beta} \nabla_{\nabla_m \Phi}(E_\nu[f])(x). \]

(3.9)

Hence, we proved combining Equation (3.8) and Equation (3.9) that

\[ PA^2 Pf = P_S A^2 Pf = \frac{1}{\beta} \cdot (\Delta_m E_\nu[f] \circ \pi_0 - \nabla_{\nabla_m \Phi} E_\nu[f] \circ \pi_0) \]

\[ = \frac{1}{\beta} \cdot \Delta_h(E_\nu[f])^\nu = \frac{1}{\beta} \cdot \Delta_h(P_Sf) \]

(3.10)

for all $f \in D$. Compare our result to [GS16, Equation (3.16)]. These preparations give shape to the following corollary, cf. [GS16, Proposition 3.9].
Corollary 3.20 (PA²P is essentially m-dissipative). Let condition (P1) hold. Then, the range \((\text{Id}_H - PA²P)(D)\) is dense in \(H\), thus \(PA²P\) is essentially m-dissipative on \(D\).

Proof. Right away, we know that \((PA²P, D) \stackrel{\text{def}}{=} (\frac{1}{\beta} \Delta_h \circ P_S, D)\) is essentially m-dissipative on \(P_S(H) = \pi^*_M L^2(\mathbb{M}; \mathfrak{m})\), as \(P_S(D) = \pi^*_M C^\infty_0(\mathbb{M})\) and \((\Delta_h, D)\) is essentially self-adjoint in \(H\). The later is true, since \((\Delta_h, D_0)\) is essentially self-adjoint in \(H\) — as so is the Laplace-Beltrami on \((\mathbb{M}, h)\), cf. the beginning of Section 2 — and the fact that \(D_0\) is dense in \(D\).

Let \(g \in H\) such that

\[
(\text{Id}_H - PA²P) f , g \rangle_H = 0
\]

for all \(f \in D\) and we claim that \(g = 0\). Our assumption immediately implies that

\[
0 = (\text{Id}_H - PA²P) f_0 , P_S g \rangle_H = \left( f_0 - \frac{1}{\beta} \Delta_h f_0 , P_S g \right)_{\pi^*_M L^2(\mathbb{M}; \mathfrak{m})}
\]

for all \(f_0 \in C^\infty_0(\mathbb{M})\). Thus, \(P_S g = 0\), since the range \((\text{Id}_H - \frac{1}{\beta} \Delta_h)(\pi^*_M C^\infty_0(\mathbb{M}))\) is dense in \(\pi^*_M L^2(\mathbb{M}; \mathfrak{m})\). Ultimately, this means that

\[
(f , g)_{\pi^*_M L^2(\mathbb{M}; \mathfrak{m})} = 0
\]

for all \(f \in D\), which implies \((f, g)_H = 0\) for all \(f \in D\), hence \(g = 0\) as claimed. \(\square\)

Proposition 3.21 (macroscopic hypocoercivity (H3)). Let the conditions (P1) and (P2) hold. Then, condition (H3) is fulfilled with \(\Lambda_M = \frac{1}{\beta} \Lambda\).

Proof. Let \(f \in D\). Since \((PA²P, D) \stackrel{\text{def}}{=} (\frac{1}{\beta} \Delta_h P_S, D)\) pregenerates a variant of the weighted horizontal gradient form in the sense that

\[
(PA²P f , g)_{\pi^*_M L^2(\mathbb{M}; \mathfrak{m})} = -\frac{1}{\beta} \int_{\mathbb{T}_\mathbb{M}} h(\nabla_h P_S f , \nabla_h P_S g) \, d\mu
\]

for all \(f, g \in D\), we easily compute that

\[
\|AP f\|_{\pi^*_M L^2(\mathbb{M}; \mathfrak{m})}^2 = \frac{1}{\beta} \int_{\mathbb{T}_\mathbb{M}} |\nabla_h P_S f(x)|_{\mathfrak{m}}^2 \, d\mu(x) = \frac{1}{\beta} \int_{\mathbb{M}} |\nabla_m \mathcal{E}_\nu[f](x)|_{\mathfrak{m}}^2 \, d\lambda_m(x)
\]

\[
= \frac{1}{\beta} \|\nabla_m \mathcal{E}_\nu[f]\|_{L^2(\mathbb{M}; \mathfrak{m})}^2 \geq \frac{1}{\beta} \Lambda \|\mathcal{E}_\nu[f] - (\mathcal{E}_\nu[f], 1)_{L^2(\mathbb{M}; \mathfrak{m})}\|_{L^2(\mathbb{M}; \mathfrak{m})}^2
\]

by Poincaré inequality. Combining this estimates with the previous corollary, then [GS14, Corollary 2.13] finishes the proof. \(\square\)

The remaining hypocoercivity condition (H4) is checked via a standard procedure relying on [GS14, Lemma 2.14] and [GS14, Proposition 2.15] cf. also [GS16, Proposition 3.11].

Lemma 3.22 (boundedness of \((BS, D)\), first part of \((H4)\)). Let condition (P1) hold. Then, with \(c_1 := \frac{1}{\beta} \alpha\) it holds that

\[
\|BS f\|_H \leq c_1 \|(\text{Id}_H - P_j) f\|_H
\]

and \(P_j \in \{P, P_S\}, j \in \{1, 2\}\).

Proof. First, we show that \(SAP = \alpha AP\) on \(D\). For the first time, it will become important here that \(\mathcal{H}_m\) is not just a semispray, but actually a spray, i.e. additionally we have that \([\mathcal{V}, \mathcal{H}_m] = \mathcal{H}_m\). This is due to the fact that \(\mathcal{H}_m\) was chosen in correspondence to the Levi-Civita connection which is an affine connection. See [APS60].

Let \(f \in D\) be fixed. Then, we immediately get that

\[
SAP f = SAP f = -\alpha \mathcal{V}(-\mathcal{H}_m(P_S f)) = \alpha \mathcal{V}(\mathcal{H}_m(\mathcal{E}_\nu[f]))
\]
since we know from the proof of Lemma 3.4 that $(A_h, 1)_H = 0$ for all $h \in D$. Using the Koszul formula we calculate that for all $\mathcal{X} \in \Gamma^\infty(TM)$ holds

$$Vh(\mathcal{H}_m, h\mathcal{X}) = 0 − h(\mathcal{V}, \mathcal{H}_m), h\mathcal{X} + h(\mathcal{V}, h\mathcal{X}) \cdot \mathcal{H}_m).$$

Similar to [GK02, Proposition 5.1] mentioned before in Lemma 3.11, one could use local coordinates for $\mathcal{V}$ in order to show that $[\mathcal{V}, h\mathcal{X}]$ is purely vertical. As $\mathcal{H}_m$ even is a spray, we gain that

$$ SAPf = αVh(\mathcal{H}_m, \nabla_h(E_v[f])^\nu) = −αh(\mathcal{H}_m, \nabla_h(E_v[f])^\nu)$$

$$ = −α\mathcal{H}_m(E_v[f])^\nu = α(−\mathcal{H}_m(E_v[f])^\nu) = αAPf$$

Setting $c_1 := \frac{2}{\alpha}$ the claim follows with [GS14, Lemma 2.14].

**Lemma 3.23** (boundedness of $(BA(\text{Id}_H − P), D)$, second part of (H4)). Let the potential conditions (P) hold. Then, there exists a constant $c_2 \in (0, \infty)$ such that

$$\|BA(\text{Id}_H − P)f\|_H \leq c_2\|(\text{Id}_H − P_j)f\|_H \quad \text{for all } f \in D$$

and $P_j \in \{P, P_2\}, j \in \{1, 2\}$.

**Proof.** Let $f \in D$ and $g := (\text{Id}_H − \text{PA}^2P)f$. We know that $g \in D((BA)^*)$ with $(BA)^*g = −\text{A}^2Pf$, cf. [GS14, Proposition 2.15]. Using our knowledge from the proof of part (ii) of Lemma 3.4, furthermore Equation (3.6) and the Cauchy-Bunyakovsky-Schwarz inequality we estimate that

$$\|(BA)^*g\|_H \leq \|v \mapsto \mathcal{H}_m^2(E_v[f])^\nu(v)\|_H + \|v \mapsto \frac{1}{\beta} \nabla_v\Phi^h(APf)(v)\|_H$$

$$\leq \|\mathcal{H}_m^2(E_v[f])^\nu\|_H + \|\mathcal{H}_m\Phi^h \cdot [\mathcal{H}_m(E_v[f])^\nu]\|_H$$

$$\leq \|v \mapsto |v|^2_H \cdot (\|\text{Hess}_m(E_v[f])^\nu\|_H + \|\nabla_v \Phi^h \cdot |\nabla_h E_v[f]\|_h\|_H)$$

$$= \frac{n}{\beta}(\|\text{Hess}_m(E_v[f])^\nu\|_H + \|\nabla_m \Phi^h \cdot |\nabla_m E_v[f]\|_m\|_H).$$

Due to the form of $\text{PA}^2P$ we derived in Equation (3.10), we know that $u := Pf$ solves the elliptic equation

$$u − \frac{1}{\beta} \Delta_h P_su = g$$

in

$$\{u \in π_0^sL^2(M; \mathfrak{m}) \mid \exists f_0 \in C^\infty_c(M) : u = f_0^\circ − (f_0^\circ, 1)_H\}.$$ 

As we assumed the necessary potential conditions, the a priori estimates of Dolbeault, Mouhot and Schmeiser, cf. [GS14, Appendix], yield existence of a constant $c_2 \in (0, \infty)$ independent of $Pf$ and $g$ such that

$$\|(BA)^*g\|_H \leq c_2 \cdot \|Pg\|_H \leq c_2 \cdot \|g\|_H.$$ 

Now, [GS14, Propositions 2.15] does apply which finishes the proof. □

Collecting the individual results of Section 3.1 and Section 3.2 we can infer Theorem 3.2 using the Hypocoercivity Theorem.

### 4. Hypocoercivity for the fibre lay-down model

In this section we demonstrate that the hypocoercivity method does apply to the fibre lay-down model with the position space being a Riemannian manifold. In other words, we generalise the results from [GS14] to the case of higher-order SDAEs with abstract position manifolds. The statement, which is an application of the Hypocoercivity Theorem, reads as follows:
Theorem 4.1 (Hydrocoercivity of the geometric fibre lay-down dynamic).

Let $\sigma \in (0, \infty)$ and $(M, m)$ be a Riemannian manifold satisfying $(M)$. Furthermore, let the potential conditions $(P)$ hold. We assume that $\mathcal{H}_m|_{[0, \infty)} \Psi^\nu = (n - 1) \Psi^h$ holds. Denote by $\nu$ the normalised surface measure on $(S^{n-1}, \mathcal{B}(S^{n-1}))$ and define $\mu := \lambda_m \otimes \nu$.

Then, the fibre lay-down operator

$$(L, C_c^\infty(SM)) = \left( \frac{\sigma^2}{2} \Delta_S + \mathcal{H}_m - \frac{1}{n-1} \nabla_S(\mathcal{H}_m \Psi^\nu), C_c^\infty(SM) \right)$$

is closable in $H := L^2(TSM; \mu)$. Moreover, its closure $(L, D(L))$ generates a strongly continuous contraction semigroup $(T_t)_{t \in [0, \infty)}$. Finally, there are constants $\kappa_1, \kappa_2 \in (0, \infty)$ computable in terms of $n = \dim(M)$, $\Lambda$, $c$ and $\sigma$ such that for all $g \in H$ and times $t \in [0, \infty)$ holds

$$\|T_t g - (g, 1)_H \|_H \leq \kappa_1 e^{-\kappa_2 t} \|g - (g, 1)_H\|_H.$$

Langevin-type models serve as surrogate models for fibre dynamics under a turbulent regime, since the model including all physical details, see [MW07], leads to enormous computational effort. In view of the production process, it’s a reasonable model assumption that velocities should be normalised, as fibre filaments are extruded with constant speed, see e.g. [GS13, Sti14] and various references therein. Hence, in contrast to Section 3 we look at an SDAE on $TM$ incorporating the ‘polynomial-type’ normalisation assumption on the velocities. In distinction from fibre lay-down applications with a different focus like in [LMS+17], we subsume possible side conditions on the position variables just in the position manifold $M$. Whenever we speak of an SDAE in this paper, there is no ambiguity wrt. the nature of the algebraic side condition.

We implement the algebraic side condition geometrically via replacing the standard fibre $F = \mathbb{R}^q$ by $F = S^{n-1}$, thus the tangent bundle over $M$ by the unit tangent bundle $\pi_{BS}: SM \to M$. This modification has several side effects, as the unit tangent bundle is a submanifold $Q := SM \leq TM$ and consequently we consider the SDAE as an SDE in the sub-fibre bundle $TSM \leq TT M$. Our choice of the Sasaki metric on $TM$ is important here, as the normal bundle of $SM$, which is just the quotient bundle $TTM|_{SM}/TSM$ in the first place, can be realised as the orthogonal complement $NSM := TSM^\perp$ of $TSM$ wrt. the chosen metric. This gives us the normal bundle really as a sub-fibre bundle $NSM \leq TTM$ and the Whitney sum $TTM|_{SM} = TSM \oplus NSM$. It should not be too surprising at this point that we can just restrict the horizontal bundle as $HSM := HTM|_{SM}$, but are forced to replace the vertical lift by another lifting procedure. We think that the problem has been explained very well in [FyE01, Abschnitt 2.1]: Let’s consider $v \in SM$ and $a \in T_SM$. Then, there exists a $w \in T_{\pi_S(v)}M$ such that $w$ is vertically lifted to $a$ at $v$, i.e. $vl_a(w) = a$. However, we do not find a vector field $X \in \Gamma^\infty(TM)$ – not even a local vector field – such that $vl_a(X) = a$ and $vl(X)|_{SM} \in \Gamma^\infty(TSM)$ hold. In other words, the vertical lift of a vector field does not need to be tangent to the unit tangent bundle. Thus, we adapt the vertical lift slightly in order to guarantee the lift of elements in $T_{\pi_S(v)}M$ are elements of $T_SM$. For the next definition see also [BVA97, BV01].

Definition 4.2 (tangential lift). Let $x \in M$, $v \in T_xM$ and $u \in SM$. The tangential lift of $v$ is defined as

$$tl_a(v) := vl_u(v - m(v, u) \cdot u) = vl_u(v) - m(v, u) \cdot N_u(u),$$

where the unit normal vector field $N_u \in \Gamma^\infty(NSM)$ has the following properties:

$$\langle N_u, d\pi_0 \rangle = 0 \quad \text{and} \quad \langle N_u, ds \rangle = \text{Id}_{SM}.$$
We call $\mathcal{N}_f$ the Sasaki normal vector field, since this normal vector field depends on our choice of the Sasaki metric on $TM$ as outlined above.

The tangential lift enables us to write $TSM = \text{tl}(SM) \oplus HSM$ as an Ehresmann connection and furthermore the decomposition

$$TTM|_{SM} = \text{tl}(SM) \oplus HSM \oplus NSM.$$ 

That said, starting from Section 4.1 we might write $f_0^\ast$ for given $f_0 \in C_c^\infty(M)$ and from the context it should be clear that we mean $\pi_{0|BS}^\ast f_0 = f_0^\ast|_{SM} \in C_c^\infty(SM)$. Similarly, we just write $\mathcal{H}_m$ instead of $\mathcal{H}_m|_{SM}$ et cetera. Be that as it may, one should pay attention to read e.g. $\nabla_h f_0 = \text{hl}|_{SM}(\nabla_m f_0)$ with $\text{hl}|_{SM}$ being the restriction of the horizontal lift to SM. Moreover, the horizontal lift of $f_0$ is up to constants characterised by $\langle a, df_0 \rangle = \langle a, df_0 \circ d\kappa \rangle$ for all $a \in TSM$, and one may think it as $f_0^\ast(v) = f_{T,SM} df_0 \circ d\kappa.\lambda$.

Alas, the term ‘tangential’ lift used in the literature is inherently flawed even more than the terms ‘vertical’ and ‘horizontal’. As terms like ‘tangential gradient’ might cause serious confusion, we depart from our naming scheme as follows.

**Definition 4.3 (spherical lift, gradient, divergence and Laplacian).**

(i) For all $X \in \Gamma_c^\infty(TM)$ we call $\text{tl}(X) \in \Gamma_c^\infty(TM; TSM)$ the spherical lift of $X$.

(ii) The spherical gradient $\nabla_S$ is defined by the relation

$$\nabla_S f \in \Gamma_c^\infty(\text{tl}(SM)) \quad \text{and} \quad s(\nabla_S f, \text{tl}(X)) = \text{tl}(X)f$$

for $f \in C_c^\infty(SM)$ and $X \in \Gamma_c^\infty(TM)$ arbitrary.

(iii) The spherical divergence $\text{div}_S$ is defined via

$$\mathcal{L}_X \nabla_S |_{TSM} = \text{div}_S(\text{tl}(X)) \cdot v|_{TSM}$$

for $X \in \Gamma_c^\infty(TM)$ arbitrary, where $\mathcal{L}$ again denotes the Lie derivative.

(iv) The spherical Laplace-Beltrami operator $\Delta_S$ is defined by $\Delta_S := \text{div}_S(\nabla_S)$, as usual.

Basically, these are just the natural modifications of the elementary objects within our calculus. E.g. it’s easily verified that $\nabla_S f_0 = \text{tl}(\nabla_m f_0 \circ \pi_{0|BS})$ holds for all $f_0 \in C_c^\infty(M)$. Other statements from Lemma 2.3.1 can be translated similarly. In this paper we only consider the normalised surface measure on the sphere, therefore weighted vertical gradients etc. do not appear. We propose the notation $\nabla_{\text{tl}} := \nabla_S$ for that we can use a boldfaced ‘v’ to signify the weighted vertical gradient $\nabla_{\text{tl}}$ in case of a nonconstant fibre weight.

The algebraic side condition yields some more interesting effects. We want to fix a very important consequence of the relation

$$\Delta_F \text{Id}_F = -(n - 1)\text{Id}_F \quad \text{for } F = S^{n-1}, \quad (4.1)$$

where the Laplacian is taken componentwise in standard Euclidean coordinates. See [GS14, Lemma 3.2] and for a general proof on eigenvalues of the spherical Laplace-Beltrami we refer e.g. to [DX13, Theorem 1.4.5].

**Lemma 4.4.** It holds $\Delta_S \mathcal{H}_m = -(n - 1)\mathcal{H}_m$, where we also denote by $\Delta_S$ the spherical tensor Laplacian and think of the vector field $\mathcal{H}_m$ as a $(1,0)$-tensor field.

**Proof.** For fixed $w \in SM$ with $x := \pi_{0|BS}(w)$ we get that

$$h_w(\Delta_S \mathcal{H}_m|_{S_xM}, \text{hl}(w)) = \Delta_S h_w(\mathcal{H}_m|_{S_xM}, \text{hl}(w)) = \Delta_S m_w(\text{Id}_{S_xM}, w) = m_w(\Delta_F \text{Id}_F, w).$$

Use Equation (4.1). \qed
Consider the following Stratonovich SDE in SM:

\[ \mathrm{d} \eta = \mathcal{H}_m \, \mathrm{d} t + t \eta \cdot (-\nabla_m \Psi) \, \mathrm{d} t + \sigma \cdot t \eta \left( \sum_{j=1}^n \frac{\partial}{\partial x_j} \right) \circ \mathrm{d} W_t, \quad (4.2) \]

where the chart \((x_1, x_2, \ldots, x_n)\) at \(\pi_0S(\eta)\) provides normal coordinates and \(\sigma\) is a nonnegative diffusion parameter. Note that neither we rescale the potential nor we incorporate a friction term. We call Equation (4.2) the fibre lay-down equation on \(M\) or just geometric fibre lay-down model. The corresponding Kolmogorov generator attains the form

\[ L = \mathcal{H}_m - t \eta \cdot (\nabla_m \Psi) + \frac{\sigma^2}{2} \Delta_\nu. \quad (4.3) \]

We call \(L\) as in Equation (4.3) the fibre lay-down generator. Per se this operator is defined for all smooth functions on the tangent space, whilst the obvious choice for the domain of test functions is \(D := C^\infty_c(M)\). Modifying the proof of Lemma 2.40 slightly we get that

\[ D_{0|S} := \pi_0S^*C^\infty_c(M) \otimes \kappa^*C^\infty_c(M) \]

is dense in \(D\) again.

In the next sections we restrict ourselves to computations substantially different from Section 3. Briefly speaking, the differences occur due to the change of the fibre measure space and affect some of the constants.

### 4.1. Data conditions.

**Definition 4.5** (model Hilbert space (D1)). Consider the probability space

\((E, \mathcal{C}, \mu) = (SM, \mathcal{B}(SM), \lambda_\nu),\)

where \(\lambda_\nu = \lambda_m \otimes \text{loc} \nu\) is the weighted Sasaki volume measure with \(\mathfrak{m}\) weighted by \(\rho_\mathfrak{m} := \exp(-\Psi)\) such that \(\lambda_\mathfrak{m}\) is a probability measure on \((M, \mathcal{B}(M))\), and \(\nu\) is the normalised surface measure on \((F, \mathcal{F}) = (S^{n-1}, \mathcal{B}(S^{n-1}))\), thus the fibre weight is a constant factor. The model Hilbert space is \(H := L^2(E; \mu) = L^2(SM; s)\).

Note that our choice of \(\nu\) is the only possible for a probability measure on the measurable fibre space with a density that is invariant wrt. rotations. Furthermore, we point out that up to slight modifications we could keep the set of conditions (P) of Section 3. The assumption of a weakly harmonic potential in Lemma 3.4, which in the end was not necessary, turns into another condition that can not be overcome so easily. Specifically, we require the potential to satisfy the relation

\[ \mathcal{H}_m \Psi^\nu = (n - 1) \Psi^b \quad \text{on SM} \quad (4.4) \]

up to an additional constant summand. This will become evident during the proof of the oncoming lemma. Indeed, the fibre lay-down generator attains the form analogous to [GS14, Equation 3.18] under this assumption. Later on in Lemma 4.8 we seemingly get rid of Assumption (4.4) using the Poisson bracket again. But it turns out that this result doesn’t fit our purposes and we want to add Assumption (4.4) to the set of conditions on the potential.

**Example 4.6** (Assumption (4.4) for Euclidean position space). Let \(M = \mathbb{R}^n\) be endowed with standard Euclidean metric \(\nu = ((\cdot, \cdot)_{\text{euc}})_{x \in \mathbb{R}^n}\). Then, the Riemannian semispray \(\mathcal{H}_{\text{euc}}\) effectively is just the identity mapping. The interested reader easily verifies this in local coordinates. More formally, every \(a' \in T^*TM \simeq (\mathbb{R}^n_x \times \mathbb{R}^n_v)^*\) is identified with an \(a = (a_x, a_v) \in \mathbb{R}^{2n}\) via \(a'(x, v) = (v, a_x)_{\text{euc}} + (x, a_v)_{\text{euc}}\) for all \((x, v) \in \mathbb{R}^n_x \times \mathbb{R}^n_v\). Then, the semispray \(\mathcal{H}_{\text{euc}}\) is characterised by

\[ (\Omega^b \circ \mathcal{H}_{\text{euc}})_{\omega}(a) = (w, a_x - a_v)_{\text{euc}} \quad \text{for all } (x, w) \in \mathbb{R}^n_x \times \mathbb{R}^n_v, a \in \mathbb{R}^{2n} \]
We have chosen the notation \( \alpha = (a_v, a_x)^\top \) for sake of readability in view of this remark. Hence, we can think of the mapping \( \mathcal{H}_{euc}(w) \) as the gradient of \( z \mapsto U_w(z) := (w, z)_{euc} \). When restricting the semispray to \( Q = S\mathbb{R}^n_x, \) i.e. \( w \in S^{n-1} \), we calculate via usual integration by parts that
\[
\int_{S,\mathbb{R}^n} \mathcal{H}_m f \, d\nu = \int_{S,\mathbb{R}^n} \left( \nabla_{S} U_w(z), \nabla_{S} f(v) \right)_{euc} \, d\nu(v)
\]
for all \( f \in C_0^\infty(S\mathbb{R}^n_x), z \in \mathbb{R}^n \). Cf. [GS14, Lemma 3.3].

Now, let \( f_0 \in C_0^\infty(\mathbb{R}^n_x) \) and \( \alpha = (a_v, a_x)^\top \in TQ \) with \( d\pi_0(a) = a_x \) as well as \( dc(a) = a_v = v := \pi_0(a) \). In this situation we have that
\[
\langle a, a_0^h \rangle = \langle (a_v, a_x), df \circ dc \rangle = \langle v, d\pi_0 |\mathbb{R}^n \rangle = \left( v, \frac{\nabla_{euc} f_0}{|\nabla_{euc} f_0|} \circ \pi_0 z \right)_{euc}.
\]
Thus, \( f_0^h(z) = U_{\nabla_{euc} f_0/|\nabla_{euc} f_0|}(z) \) for all \( z \in S,\mathbb{R}^n_x \). Both results can be combined as follows:
\[
\int_{S,\mathbb{R}^n} \mathcal{H}_{euc} \Psi^v \cdot g_0^h \, d\nu = \int_{S,\mathbb{R}^n} \mathcal{H}_{euc}(\Psi^v \cdot g_0^h) \, d\nu
\]
\[= (n-1) \int_{S,\mathbb{R}^n} \left( v, \frac{\nabla_{euc} \Psi(x)}{|\nabla_{euc} \Psi(x)|} \right)_{euc} \, \Psi^v(v) g_0^h(v) \, d\nu(v)
\]
holds for all \( g_0 \in C^\infty(\mathbb{R}^n_x) \) with the particular choice of \( z = \frac{1}{\Psi(x)} \nabla_{euc} \Psi(x) \). Note that we assume \( \Psi > 0 \) wlog. in view of (P1). Hence, Assumption (4.4) always is fulfilled.

**Lemma 4.7** (SAD-decomposition (D3), (D4), (D6)). Let the potential \( \Psi \) be locally Lipschitzian such that Assumption (4.4) is fulfilled. Consider the SAD-decomposition \( L = S - A \) on \( D \) with
\[
Sf := \frac{\sigma^2}{2} \Delta_S f
\]
and
\[
Af = -\mathcal{H}_m f := -\mathcal{H}_m f + \frac{1}{n-1} \nabla_{S}(\mathcal{H}_m \Psi^v)
\]
for all \( f \in D \).

Then, the following assertions hold:
(i) \( (S, D) \) is symmetric and negative semidefinite.
(ii) \( (A, D) \) is antisymmetric.
(iii) For all \( f \in D \) we have that \( Lf \in L^1(SM; \mu) \) and \( \int_{SM} Lf \, d\mu = 0 \).

**Proof.**
(i) Using integration by parts we see that \( (S, D) \) pregenerates the weighted spherical gradient form on \( SM \). Cf. the proof of part (i) of Lemma 3.4.

(ii) The adjoint of \( \nabla_{S} \Psi^h \) \( (4.4) \) \( \frac{1}{n-1} \nabla_{S}(\mathcal{H}_m \Psi^v) \) wrt. \( L^2(SM; s) \)-scalar product is computed using Lemma 4.4 as
\[
\left( \frac{1}{n-1} \nabla_{S}(\mathcal{H}_m \Psi^v) \right)^* = -\frac{1}{n-1} \nabla_{S}(\mathcal{H}_m \Psi^v) - \frac{1}{n-1} \Delta_{S}(\mathcal{H}_m \Psi^v)
\]
\[= -\frac{1}{n-1} \nabla_{S}(\mathcal{H}_m \Psi^v) + \mathcal{H}_m \Psi^v.
\]

The rest follows as in the proof of part (ii) of Lemma 3.4.

(iii) Follows with the parts (i) and (ii).
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Given up on the form of \((A, D)\) as in [GS14, Equation (3.22)] and turning to a more intuitive one in view of Equation (4.3), we gain weaker assumptions on the potential in general.

**Lemma 4.8** (SAD-decomposition (2nd version)). If we define \(\mathcal{H}_m := \mathcal{H}_m - \text{tl}(\nabla_m \Psi)\), then the assertions of Lemma 4.7 are true without the Assumption (4.4) on \(\Psi\).

**Proof.** Indeed, we can copy the proof of Proposition 3.7 and it’s enough that the configuration manifold \(Q\) is a submanifold of \(TM\).

First, we notice that both \(\Omega\) and \(J\) restricted to \(T^*Q\) or \(TQ\) respectively are still a symplectic form and an almost complex structure respectively. They are compatible with the restricted Sasaki metric, thus generate the same Poisson bracket on \(Q\) which in turn defines Hamiltonian vector fields \(H_f \in \Gamma^\infty(TQ)\) for all \(f \in C^\infty(Q)\).

Second, we find that \(H_f(\rho) = -\rho \cdot (\mathcal{H}_m f - \text{tl}(\nabla_m \Psi)(f))\) for all \(f \in D = C^\infty(Q)\), by investigating the action of the Hamiltonian vector field in local coordinates for \(Q\) that respect the Ehresmann connection and also provide a local trivialisation. □

In contrast to Proposition 3.7 the statement of Lemma 4.8 is not of much use for an application of the hypocoercivity method, even though the operator \((A, D) = (-\mathcal{H}_m, D)\) there is the more natural formulation. The simple reason is that the two competing definitions of \(\mathcal{H}_m\) not necessarily coincide on \(D\). But we need the operator as in Lemma 4.7 during the characterisation of \((PA^2P, D)\) specifically in Equation (4.8). However, this calculation is part of checking the hypocoercivity assumptions, whereas condition (D2), the existence of a nice semigroup, can be checked following the same steps as in Section 3.1 for the antisymmetric operator of Lemma 4.8.

Moving on, the fibrewise average is defined the very same way as in Section 3 just with ‘\(S, \mathcal{M}\)’ instead of ‘\(T, \mathcal{M}\)’. Also, the form of the operator \((AP, D)\) given in Equation (3.6) just changes marginally:

\[
APf = -m_{\pi_0/S}(\text{Id}_{\mathcal{M}}, \nabla_m E_v[f] \circ \pi_{0/S}) \quad \text{for all } f \in D. \tag{4.5}
\]

Indeed, the other statements concerning data conditions translate to the fibre lay-down model on \(\mathcal{M}\) with minor modifications. This is a bit different when it comes to the hypocoercivity conditions in the next section. Nevertheless, we want to draw the readers attention to the fact the reasoning for essential m-dissipativity of \((L, D)\) under the assumption of loc-Lipschitzian potentials does barely depend on the fibre measure space. Arguments gleaned in [GS14, Section 4] on the notoriously subtle question of core property and \(L\) generating a semigroup stay valid.

Since the standard fibre is compact now, we can simplify the proof Lemma 3.8 a bit, similar to [GS14, Lemma 3.8].

**Lemma 4.9.** Let condition (P1) hold. Then, we have \(P(H) \subseteq D(S), \ SP = 0, \ P(D) \subseteq D(A)\) and \(AP(D) \subseteq D(A)\). Furthermore, \(1 \in D(L)\) and \(L1 = 0\).

**Proof.** The range \(P(H)\) is identified with a subset of \(L^2(M; m)\) via the vertical lift. For any \(f_0 \in L^2(M; m)\) there is an \(L^2\)-approximating sequence \((f_n)_{n \in \mathbb{N} \setminus \{0\}}\) in \(C^0_c(M)\). Since the standard fibre \(F = S^{n-1}\) is compact, it holds \(f_n \rightarrow f_0\) in \(D\) and \(S(f_n) = 0\) for all \(n \in \mathbb{N} \setminus \{0\}\). We conclude that \(f_n \in D(S)\) and \(f_n \in \text{Null}(S)\) as \(f_n \rightarrow f_0\) in \(H\) as \(n \rightarrow \infty\) and \((S, D(S))\) is closed.

We fix an \(f \in D\). Choose \(o \in \mathcal{M}\) and an open ball \(U(o, r)\) centred at \(o\) with radius \(r \in (0, \infty)\) wrt. the intrinsic metric on \((\mathcal{M}, m)\) such that the support of \(f\) is completely contained in \(\pi_{0/S}^{-1}(U(o, r)) \subseteq SM\). Then, the support of \(E_v[f]\) is contained in \(U(o, r)\). Thus, \(P_S f \in D\). Therefore, \(P(D) \subseteq D \subseteq D(A)\).
Besides, we calculate via chain rule that
\[ APf = -\mathcal{H}_m(E_v[f]^\nu) = -\langle \mathcal{H}_m , dE_v[f] \circ d\pi_0 \rangle \\
= -\langle \text{Id}_{TM}, dE_v[f] \rangle = -dE_v[f]. \] (4.6)
Consequently, \( AP(D) \subseteq D \subseteq D(A) \), as the right-hand side of Equation (4.6) is smooth with compact support. Let \( \varphi \in C^\infty_c(\mathbb{M}; [0,1]) \) be a cut-off function such that \( \varphi = 1 \) on \( U(\alpha,1) \) and \( \varphi = 0 \) on \( U(\alpha,2) \). Define \( \varphi_n := \varphi(\text{Id}/n) \) for all \( n \in \mathbb{N} \setminus \{0\} \). Note that \( |\nabla_m \varphi_n(x)|_m \leq \frac{1}{2} ||\nabla_m \varphi||_{L^\infty(m)} \) for all \( x \in M \) and \( n \in \mathbb{N} \setminus \{0\} \). By construction, we have that
\[ AP\varphi_n^\nu = -\Delta_m \varphi_n^\nu = \langle \mathcal{H}_m , d\varphi_n \circ d\pi_0 \rangle = -d\varphi_n \rightarrow 0 \quad \text{as } n \rightarrow \infty \]
pointwise and in \( L^2 \)-sense. Since \( (A,D(A)) \) is closed, we have \( 1 \in D(A) \) and \( A1 = 0 \).

Since we know form Lemma 4.7 part (i) that \( S\varphi_n^\nu = 0 \) for all \( n \in \mathbb{N} \), we have \( L\varphi_n^\nu = -A\varphi_n^\nu \) for all \( n \in \mathbb{N} \). The sequence \( (L\varphi_n^\nu)_{n \in \mathbb{N}} \) converges in \( H \) to 0 as \( n \rightarrow \infty \). \( \square \)

4.2. Hypocoercivity conditions.

Lemma 4.10 (algebraic relation (H1)). Let \( \Psi \) be loc-Lipschitzian such that \( \lambda_m = \exp(-\Psi) \lambda_m \) is a probability measure on \( (\mathbb{M}, \mathcal{B}(\mathbb{M})) \). Then, we have \( PAP |_{D} = 0 \).

Proof. Using only [GS14, Lemma 3.1] and Equation (4.5) we calculate that
\[ \int_{S,M} APf \, d\nu \overset{(4.5)}{=} \int_{S,M} -m_x(\text{Id}_{SM}, \nabla_m E_v[f](x)) \, d\nu = 0 \]
holds for all \( f \in D \) and \( x \in M \). The rest of the proof works as in Lemma 3.18. \( \square \)

Lemma 4.11 (microscopic coercivity (H2)). Let \( \Psi \) loc-Lipschitzian such that \( \lambda_m = \exp(-\Psi) \lambda_m \) is a probability measure on \( (\mathbb{M}, \mathcal{B}(\mathbb{M})) \). Then, condition (H2) holds with \( \Lambda_m = (n-1)\frac{2}{\gamma} \).

Proof. The proof works the same way as in Lemma 3.3 using the Poincaré inequality for the spherical measure, see [Bec89, Theorem 2]. \( \square \)

For proving condition (H3), we want to characterise the operator \( (PA^2P \circ D) \) as a weighted horizontal Laplace-Beltrami composed with fibrewise average again. Then, we get essential m-dissipativity of this operator as in Corollary 3.20. Mirroring the computations in Equation (3.8) we calculate that
\[ \int_{S,M} \mathcal{H}_m(-m_x(\text{Id}_{SM}, \nabla_m E_v[f] \circ \pi_0|_S)) \, d\nu \\
\overset{(4.5)}{=} \int_{S,M} m_x(v, \nabla_m E_v[f](x)) \, d\nu(v) + \int_{S,M} m_x(v, \nabla_m E_v[f](x)) \, d\nu(v) \]
\[ = \frac{1}{n} \Delta_m E_v[f](x) \]
for all \( v \in SM \) with \( x := \pi_0|_S(v) \).

As for Equation (3.9) we use the proof of part (ii) of Lemma 4.7 to get that
\[ PS \left( \frac{1}{n} \nabla \Psi \right)(H_m \Psi \nu)(APf)(v) \overset{\text{Equation (3.9)}}{=} \int_{S,M} \mathcal{H}_m \Psi \nu \cdot APf \, d\nu \\
= -\frac{1}{n} m_x(\nabla_m \Psi(x), \nabla_m E_v[f](x)) = -\frac{1}{n} \nabla \Psi \nu(E_v[f](x)) \]
for all \( v \in SM \) with \( x := \pi_0|_S(v) \).
Together Equation (3.8) and Equation (3.9) imply the relation
\[ PA^2f = PS^2A^2f = \frac{1}{n} \cdot (\Delta_m E_v[f] \circ \pi_0S - \nabla_{m\Psi} E_v[f] \circ \pi_0S) \]
\[ = \frac{1}{n} \cdot \Delta_h(E_v[f]^\vee) = \frac{1}{n} \cdot \Delta_h(PSf) \]
(4.9)
for all \( f \in D \). Compare this to [GS14, Equation (3.27)].

**Proposition 4.12** (macroscopic coercivity (H3)). Let \( \Psi \) loc-Lipschitzian such that \( \lambda_m = \exp(-\Psi) \lambda_m \) is a probability measure on \((M, \mathcal{B}(M))\) satisfying the Poincaré inequality (3.3). Then, condition (H3) is fulfilled with \( \Lambda_M = \frac{1}{m} \Lambda \). Cf. [GS14, Proposition 3.14].

**Proof.** As before, we compute that for all \( f \in D \) holds
\[ \|APf\|^2_H = \int_M \int_{S,M} (APf)^2 |_{S,M} \, d\nu d\lambda_m(x) \]
\[ = \int_M \int_{S,M} m_x(v, \nabla_m E_v[f](x))^2 \, d\nu d\lambda_m(x) \]
\[ = \frac{1}{n} \int_M |\nabla_m E_v[f](x)|^2_m \, d\lambda_m(x) = \frac{1}{n} \|\nabla_m E_v[f]\|^2_{L^2(M \to TM, m)} \]
\[ \geq \frac{1}{n} \Lambda \|E_v[f] - (E_v[f], 1)_{L^2(M, m)}\|^2_{L^2(M, m)} \]
using the Poincaré inequality of the weighted base measure. The claim follows with [GS14, Corollary 2.13], since \((PA^2P, D)\) is essentially m-dissipative due to the modification of Corollary 3.20 to the case of \( Q = SM \). \( \square \)

This time, we do not even need \( \mathcal{H}_m \) to be a spray when checking the first part of condition (H4). Compare the following Lemmas 4.13 and 4.14 as well as their proofs to [GS14, Proposition 3.15].

**Lemma 4.13** (boundedness of \((BS, D)\), first part of (H4)). Let \( \Psi \) be loc-Lipschitzian such that \( \lambda_m = \exp(-\Phi) \lambda_m \) is a probability measure on \((M, \mathcal{B}(M))\). Then, with \( c_1 := (n - 1)\frac{\alpha^2}{2} \) it holds that
\[ \|BSf\|_H \leq c_1 \|(Id - P_j)f\|_H \quad \text{for all } f \in D \]
and \( P_j \in \{P, Ps\} \), \( j \in \{1, 2\} \).

**Proof.** Let \( f \in D \) be fixed. Then, we observe that
\[ SAPf = SAP_Sf = \frac{\alpha^2}{2} \Delta_{S,m}(P_Sf) \]
\[ = \frac{\alpha^2}{2} (n - 1) \cdot \mathcal{H}_m(P_Sf) = -\frac{\alpha^2}{2} (n - 1)APf \]
by Lemma 4.4, and since \((Ah, 1)_{H} = 0\) holds for all \( h \in D \) by Lemma 4.7 part (ii). \( \square \)

**Lemma 4.14** (boundedness of \((BA(Id - P), D)\), second part of (H4)). Let all the conditions of Theorem 4.1 on the potential hold. Then, there exists a constant \( c_2 \in (0, \infty) \) such that
\[ \|BA(Id - P)f\|_H \leq c_2 \|(Id - P_j)f\|_H \quad \text{for all } f \in D \]
and \( P_j \in \{P, Ps\} \), \( j \in \{1, 2\} \).

**Proof.** Let \( f \in D \) and \( g := (Id - PA^2P)f \) as in the proof of Lemma 3.23. Now, the relevant estimate reads as
\[ \|(BA)^*g\|_H \leq \|\text{Hess}_m(E_v[f])\|_H + \frac{1}{n} \|\nabla_m \Phi_m\|_{\lambda_m} \cdot |\nabla_m E_v[f]|_{\lambda_m} \|_H. \]
In view of Equation (4.9) we have the solution \( u := Pf \) of the elliptic equation
\[
\frac{1}{n} \Delta_h P_S u = g
\]
in \( \{ u \in \pi_{05}^* L^2(\mathcal{M}; \mathbf{m}) \mid \exists f_0 \in C_c^\infty(\mathcal{M}) : u = f_0 - (f_0', 1)_H \} \).

The proof is completed as in Lemma 3.23. \( \square \)

Combining the results of Section 4.1 and Section 4.2 our main theorem, Theorem 4.1, follows from the Hypocoercivity Theorem.

5. Existence of martingale solutions and \( L^2 \)-exponential ergodicity

Finally, we show existence of \( L \)-martingale solutions to the SDEs investigated in this article. The strong mixing of the corresponding semigroups with exponential rate of convergence then implies their \( L^2 \)-exponential ergodicity. Let the configuration manifold \( \mathcal{Q} \in \{ \mathcal{T} \mathcal{M}, \mathcal{S} \mathcal{M} \} \) be the state space \( \mathcal{E} = \mathcal{Q} \) as mentioned in the end of Section 1. For basic notions used in the following theorem we refer to [Sta99], [Tru00] and [Tru03].

**Theorem 5.1** (existence of martingale solutions). If \( \mathcal{Q} = \mathcal{T} \mathcal{M} \), let the assumptions of Theorem 3.2 hold. If \( \mathcal{Q} = \mathcal{S} \mathcal{M} \), let the assumptions of Theorem 4.1 hold. Then, there is a Hunt process
\[
\text{HP} = \left( \Omega, \mathfrak{A}, \mathbb{F} = (\mathbb{F}_t)_{t \in [0, \infty)}, \eta = (\eta_t)_{t \in [0, \infty)}, (\mathbb{P}_v)_{v \in \mathcal{Q}} \right)
\]
properly associated in the resolvent sense with \( (L, D(L)) \) having infinite life-time and continuous paths \( \mathbb{P}_v \)-almost surely for all \( v \in \mathcal{Q} \). I.e. if \( (G_a)_{a \in (0, \infty)} \) denotes the resolvent corresponding to \( (L, D(L)) \), then the transition resolvent \( R_a f \) of \( G_a f \) for all \( f \in L^2(\mathcal{Q}; \mu) \) and \( a \in (0, \infty) \), where \( R_a f(v) = \int (0, \infty) \exp(-as) E_v[f(\eta_s)] \lambda(ds) \).

Moreover, for quasi every initial point \( v \in \mathcal{Q} \) the probability measure \( \mathbb{P}_v \) solves the martingale problem for \( (L, C_c^\infty(\mathcal{Q})) \), i.e. HP is a martingale solution to either (3.1) if \( \mathcal{Q} = \mathcal{T} \mathcal{M} \) or to (4.2) if \( \mathcal{Q} = \mathcal{S} \mathcal{M} \) for quasi every initial point \( v \in \mathcal{Q} \).

**Proof.** As we have seen before, \( D = C_c^\infty(\mathcal{Q}) \) is a core of \( (L, D(L)) \), see Theorem 3.17 and the explanations on page 37. Observe that \( D \) also is an algebra which separates the points of \( \mathcal{Q} \). Thus, \( (L, D(L)) \) defines a generalised Dirichlet form fulfilling the assumptions of [Sta99, Theorem IV.2.2]. This theorem provides a special standard process HP properly associated with \( (L, D(L)) \) in the resolvent sense. Now, infinite life-time follows from (D7), i.e. conservativity, together with [Sta99, Theorem IV.3.8 (ii)]. Moreover, continuous paths are obtained via [Tru03, Theorem 3.3]. Summarising, HP is a Hunt process indeed. For the statement concerning the martingale problem see [CG08, Corollary 1] and its proof. Note that there are even some finer statements on the martingale problem, cf. [Tru00]. \( \square \)

Now, we turn to the matter of ergodicity. Consider the probability measure \( \mathbb{P} \) on \( (\Omega, \mathfrak{A}) \) be given as
\[
\mathbb{P}(A) := \int_{\mathcal{Q}} \mathbb{P}_v(A) \mu(dv) \quad \text{for all } A \in \mathfrak{A}.
\]
We estimate for all \( g \in L^2(\mathcal{Q}; \mu) \) with \( E_\mu[g] = 0 \) and \( t \in (0, \infty) \) that
\[
\left\| \frac{1}{t} \int_{[0,t]} g(\eta_s) \lambda(ds) \right\|_{L^2(\mathbb{P})}^2 = \int_{\Omega} \frac{1}{t^2} \int_{[0,t]^2} g(\eta_s)g(\eta_u) \lambda(ds, du) \ d\mathbb{P}
\]
\[ \frac{1}{L^2} \int_{[0,t]^2} \mathbb{E}[g(\eta_s)g(\eta_u)] \lambda(ds, du) = \frac{2}{L^2} \int_{[0,t]} \int_{[0,s]} \mathbb{E}[g(\eta_s)g(\eta_u)] \lambda(du) \lambda(ds) \]

(5.1)

\[ \frac{2}{L^2} \int_{[0,t]} \int_{[0,s]} (T_{s-u}g \cdot g)_{L^2(\mu)} \lambda(du) \lambda(ds) \]

(5.2)

\[ \frac{4}{L^2} \int_{[0,2t]} \int_{[0,t]} (T_g \cdot g)_{L^2(\mu)} \lambda(dv) \lambda(du) = 8 \int_{[0,t]} (T_g \cdot g)_{L^2(\mu)} \lambda(dv) \]

(5.3)

\[ \leq \frac{8}{L} \|g\|_{L^2(\mu)} \int_{[0,t]} \|T_g\|_{L^2(\mu)} \lambda(dv). \]

At step (5.1) we use Fubini. Afterwards at step (5.2), we can ensure \( u < s \) for symmetry reasons and transform expectation wrt. \( P \) using the (weak) Markov property. Then, at step (5.3) we apply the 2D-transformation formula with \( v = s - u \) and \( w = s + u \), and finish with the Cauchy-Bunyakovsky-Schwarz inequality.

The previous estimate shows that using our main theorems – depending on \( Q \) either Theorem 3.2 or Theorem 4.1 – we not only can infer convergence to 0 as \( t \to \infty \), but also the rate of convergence is explicitly computable. Indeed, with \( g \) as before we gain

\[ \left\| \frac{1}{t} \int_{[0,t]} g(\eta_s) \lambda(ds) \right\|_{L^2(P)}^2 \leq \frac{8}{L} \|g\|_{L^2(\mu)} \int_{[0,t]} \|T_g\|_{L^2(\mu)} \lambda(dv) \]

\[ \leq \frac{8}{L} \|g\|_{L^2(\mu)}^2 \int_{[0,t]} \kappa_1 e^{-\kappa_2 t} \lambda(dv) \]

\[ = \frac{8}{L} \frac{\kappa_1}{\kappa_2} (1 - e^{-\kappa_2 t}) \cdot \|g\|_{L^2(\mu)}^2. \]

Thus, we proved the following corollary after reducing everything to zero-mean functions wrt. \( \mu \).

**Corollary 5.2** \((L^2\text{-exponentially ergodic with optimal rate and explicit constants})\). If \( Q = TM \), let the assumptions of Theorem 3.2 hold. If \( Q = SM \), let the assumptions of Theorem 4.1 hold. Moreover, let \( \kappa_1 \) and \( \kappa_2 \) be the constants form these respective theorems. Then, we have

\[ \left\| \frac{1}{t} \int_{[0,t]} f(\eta_s) \lambda(ds) - \mathbb{E}_\mu[f] \right\|_{L^2(P)} \leq \frac{2}{\sqrt{L}} \sqrt{\frac{2\kappa_1}{\kappa_2} (1 - e^{-\kappa_2 t}) \cdot \|f - \mathbb{E}_\mu[f]\|_{L^2(\mu)}} \]

for all \( t \in (0, \infty) \).

**Remark 5.3.** In the title of Corollary 5.2 we claim that the rate \( t^{-1/2} \) is optimal. This is obvious in the case that the spectrum of the generator \((L, D(L))\) has, apart from the eigenvalue zero, the largest element \(-\kappa < 0\) which is an eigenvalue of \((L, D(L))\). Evidently, all the inequality in the estimates prior to Corollary 5.2 are equalities when choosing the function \( g \) there as the eigenvector corresponding to \(-\kappa\). Hence, the rate of convergence in Corollary 5.2 is sharp with \( \kappa_1 = 1 \) and \( \kappa_2 = \kappa \).

In situations where \((L, D(L))\) can be controlled by a Lyapunov function, see e.g. [HM19] in case of purely Euclidean setting, one obtains also exponential rates of convergence for the corresponding semigroups; even in (weighted) total variation distance. This implies pointwise convergence of the semigroup applied to test functions at an exponential rate. But even this convergence with an exponential rate would not give a better rate as the one in Corollary 5.2.

As in [CG10], we call the martingale solutions to the SDEs investigated in this article \(L^2\text{-exponential ergodic}, \) i.e. ergodic with a rate that corresponds to exponential convergence of the corresponding semigroups.
APPENDIX A. SOME EXPRESSIONS IN LOCAL COORDINATES

A chart \( x = (x^i)_{j=1}^n \) with domain \( U \subseteq M \) induces local coordinates \( (v^k)_{k=1}^{2n} \) for the preimage \( V := \pi^{-1}_0(U) \) in a natural way:

\[
v^j := x^j \circ \pi_0 \quad \text{and} \quad v^{\alpha+j} := (dx^j)^{\alpha} \quad \text{for} \ j \in \{1, \ldots , n\}.
\]

We might write \( \pi_0^*x = x \circ \pi_0 = (v^j)_{j=1}^n \) and \( dx = (v^{\alpha+j})_{j=1}^n \), where the latter shorthand doesn’t lead to confusion as we denote the Riemannian volume form by \( d\lambda_m \).

From [GK02, Lemma 4.1] we know some particular vertical and horizontal lifts:

\[
\text{vl} \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial v^{\alpha+i}} \quad \text{and} \quad \text{hl} \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial v^i} - \sum_{j \in \{1, \ldots , n\}} (\Gamma^j_{ij} \circ \pi_0) v^{\alpha+j} \frac{\partial}{\partial v^{\alpha+i}}
\]

for all \( i \in \{1, \ldots , n\} \). Next, the following expression of semisprays has been taken from [Buc06, Section 1]: A semispray \( \mathcal{H} \) reads in local coordinate form as

\[
\mathcal{H} = \sum_{j=1}^n v^{\alpha+j} \partial v^j - 2G^j(\pi_0^*x, dx) \partial v^{\alpha+j}.
\]

The family \((G^j)_{j=1}^n\) of functions on \( V \) is characterised by functions \( N^j_i = \frac{\partial G^j}{\partial v^{\alpha+i}} \), \( i, j \in \{1, \ldots , n\} \), which are given pointwisely by

\[
\text{hl} \left( \frac{\partial}{\partial x^i} \right) |_u = \frac{\partial}{\partial v^i} |_u - \sum_{j=1}^n N^j_i(u) \frac{\partial}{\partial v^{\alpha+i}} |_u
\]

for \( u \in TM \) arbitrary. For instance, [Buc06, Equation (19)] yields coefficients \((G^j)\) of a semispray corresponding to a given Lagrangian \( L \). Recall Example 2.36.

In the following lemmas, we prove some formulæ used in Section 3.

**Lemma A.1.** For all \( k \in \{1, \ldots , n\} \) holds that

\[
[\mathcal{H} , \text{vl}(\partial x^k)] = \text{hl}(\partial x^k) - \sum_{j=1}^n N^j_k \cdot \text{vl}(\partial x^j).
\]

**Proof.** Let \( k \in \{1, \ldots , n\} \). Then, we calculate that

\[
[\mathcal{H} , \text{vl}(\partial x^k)] = [\mathcal{H} , \partial v^{\alpha+k}] = \sum_{j=1}^n [v^{\alpha+j} \partial v^j , \partial v^{\alpha+k}] - 2[G^j \partial v^{\alpha+j} , \partial v^{\alpha+k}]
\]

\[
= \sum_{j=1}^n \frac{\partial v^{\alpha+j}}{\partial v^{\alpha+k}} \cdot \partial v^j + v^{\alpha+j} \left[ \frac{\partial v^j}{\partial v^{\alpha+k}} \right]_{\alpha=0}
\]

\[
- 2 \left( \frac{\partial G^j}{\partial v^{\alpha+k}} \cdot \partial v^{\alpha+j} + G_j \left[ \frac{\partial v^{\alpha+j}}{\partial v^{\alpha+k}} \right]_{\alpha=0} \right)
\]

\[
= \sum_{j=1}^n \delta_{jk} \cdot \partial v^j - 2N^j_k \cdot \partial v^{\alpha+j} = \partial v^k - 2 \sum_{j=1}^n N^j_k \cdot \partial v^{\alpha+j}
\]

\[
= \text{hl}(\partial x^k) - \sum_{j=1}^n N^j_k \cdot \partial v^{\alpha+j} = \text{hl}(\partial x^k) - \sum_{j=1}^n N^j_k \cdot \text{vl}(\partial x^j).
\]

\[\square\]

The next formula appears to be rather intuitive:
Lemma A.2. For all $\mathcal{X} \in \Gamma^\infty(TM)$ holds that
\[ h(\nabla^s h(\mathcal{X}) , H) = m_{\pi_0}(\nabla^s_{Id_{TM}}(\mathcal{X} \circ \pi_0) , Id_{TM}). \]

Proof. Let $\mathcal{X} \in \Gamma^\infty(TM)$. Then, the Koszul formula describing the Levi-Civita connection $\nabla^s$ wrt. $s$ uniquely reads as in our special instance as
\[ 2h(\nabla^s h(\mathcal{X}) , H) = H(h(h(\mathcal{X}) , H)) + h(h(\mathcal{X})(h(H , H))) - h(h(\mathcal{X}) , [H , H]) - h(H, [h(h(\mathcal{X}) , H)) + h(H , [H, h(h(\mathcal{X})] \]) = h(h(\mathcal{X}) , \nabla^s h(h(h(\mathcal{X}) , H))) + 2h(H , [H , h(h(\mathcal{X}))]) = h(h(\mathcal{X}) , \nabla^s h(h(h(h(\mathcal{X}) , H))) + 2h(H , [H , h(h(\mathcal{X}))]). \]

First, we note that
\[ d\pi_0[H , h(h(\mathcal{X})] = d\pi_0[H , h(h(\mathcal{X})] = [Id_{TM} , \mathcal{X} \circ \pi_0] = \nabla^s_{Id_{TM}}(\mathcal{X} \circ \pi_0). \]

Second, the value of $|Id_{TM}|^2_{m}$ does not specifically depend on the current position and therefore it could be approximated just by functions from $\kappa^* C^\infty(M)$. In other words, this function is a horizontal lift and the horizontal gradient of a horizontal lift equals 0 always. Hence, the claim is proven. \hfill \Box

Appendix B. Miscellaneous

The following lemma is pretty elementary and rather an intuitive statement on weighted metric spaces. Obviously, its assumptions are fulfilled by the exponential type weight as chosen in this article.

Lemma B.1. Let some given base weight $\rho_M$ strictly positive and loc-Lipschitzian. Denote by $d_m$ and $d_M$ the metrics wrt. the nonweighted and weighted Riemannian metric respectively. Then, those metrics induce equivalent topologies. In particular, by (M2) the weighted manifold $\langle M, m \rangle$ is complete as a metric space.

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a $d_m$-Cauchy sequence. Since $M$ is finite dimensional, this sequence is contained in a compactum $K \subseteq M$. On $K$ the weight function is continuous, thus it attains minimum and maximum. Let $\varepsilon \in (0, \infty)$. Then, we have for some $N_\varepsilon \in \mathbb{N}$ and all $n_1, n_2 \geq N_\varepsilon$ that
\[ \frac{\varepsilon}{2} > d_m(x_{n_1}, x_{n_2}) \geq \inf_{y \in K} \rho_M(y)^2 \cdot d_m(x_{n_1}, x_{n_2}). \]

Hence, $(x_n)_{n \in \mathbb{N}}$ is a $d_m$-Cauchy sequence and by (M2) it converges to $x \in M$. This $d_m$-limit also is the $d_m$-limit:
\[ d_m(x_{n_1}, x) \leq d_m(x_{n_1}, x_{n_2}) + \|\rho_M\|_{L^\infty(K, \lambda_M)} \cdot d_m(x_{n_2}, x) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon \]

for $n_2 \geq N_\varepsilon$ large enough.

If we start with a $d_m$-Cauchy sequence, then it is a $d_m$-Cauchy sequence by a similar estimate. If the $d_m$-limit exists, it’s easily verified that it also is the $d_m$-limit. \hfill \Box
References

[APS60] W. Ambrose, R. S. Palais, and I. M. Singer, Sprays., Anais Acad. Brasil. Ci. 32 (1960), 163–178 (English).

[BCD11] Ioan Bucataru, Oana Constantinescu, and Matias F. Dahl, A geometric setting for systems of ordinary differential equations, International Journal of Geometric Methods in Modern Physics 08 (2011), no. 06, 1291–1327.

[Bec89] William Beckner, A generalized Poincaré inequality for Gaussian measures., Proc. Am. Math. Soc. 105 (1989), no. 2, 397–400 (English).

[BG92] Nicole Berline and Ezra Getzler and Michèle Vergne, Heat kernels and Dirac operators, Springer, 1992.

[Bis15] Jean-Michel Bismut, Hypoelliptic Laplacian and probability., J. Math. Soc. Japan 67 (2015), no. 4, 1317–1357 (English).

[BM58] Raoul Bott and John W. Milnor, On the parallelizability of the spheres., Bull. Am. Math. Soc. 64 (1958), 87–89 (English).

[BO09] Nawaf Bou-Rabee and Houman Owhadi, Stochastic variational integrators, IMA JNA 29 (2009), no. 2, 421–443.

[Buc06] Ioan Bucataru, Metric nonlinear connections., Differential Geometry and its Applications 25 (2006), no. 3, 335–343 (English).

[Buc12] Geometric structures for differential equations fields., 2012, habilitation thesis elaborated at Universitatea "Alexandru Ioan Cuza" din Iași.

[BV01] E. Boeckx and L. Vanhecke, Unit Tangent Sphere Bundles with Constant Scalar Curvature., Czechoslovak Mathematical Journal 51 (2001), no. 3, 523–544.

[BVA97] E. Boeckx, L. Vanhecke, and G. Auchmuty, Characteristic reflections on unit tangent sphere bundles., Houston J. Math. 23 (1997), no. 3, 427–448.

[Car92] Manfredo P. do Carmo, Riemannian geometry, Birkhäuser, Boston, Mass. u.a., 1992 (English).

[CG08] Florian Conrad and Martin Grothaus, Construction of n-particle Langevin dynamics for $H^1$-potentials via generalized Dirichlet forms., Potential Anal. 28 (2008), no. 3, 261–282 (English).

[CG10] Construction, ergodicity and rate of convergence of N-particle Langevin dynamics with singular potentials., J. Evol. Equ. 10 (2010), no. 3, 623–662 (English).

[CKW12] William T. Coffey, Yuri P. Kalmykov, and John T. Waldron, The Langevin equation. With applications to stochastic problems in physics, chemistry and electrical engineering. 3rd ed., 3rd ed. ed., vol. 27, Hackensack, NJ: World Scientific, 2012 (English).

[Con90] John B. Conway, A course in functional analysis. 2nd ed., 2nd ed. ed., vol. 96, New York etc.: Springer-Verlag, 1990 (English).

[Dav80] Edward Brian Davies, One-parameter semigroups., London Mathematical Society, Monographs, No. 15. London etc.: Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers. VIII, 1980, p. 230.

[DM90] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser, Hypocoercivity for kinetic equations with linear relaxation terms, Comptes Rendus Mathematique 347 (2009), no. 9, 511 – 516.

[DM95] Hypocoercivity for linear kinetic equations conserving mass, Trans. Amer. Math. Soc. 367 (2015), no. 6, 3807–3828. MR 3324910

[Dom62] Peter Dombrowski, On the geometry of the tangent bundle, J. Reine Angew. Math. 210 (1962), 73–88 (English).

[DX13] Feng Dai and Yuan Xu, Approximation theory and harmonic analysis on spheres and balls., New York, NY: Springer, 2013 (English).

[For99] Gerald B. Folland, Real Analysis: Modern Techniques and Their Applications, Wiley, 1999 (English).

[For12] Otto Forster, Analysis 3: Mäß- und Integrationstheorie, Integralsätze im IRn und Anwendungen, Vieweg+Teubner Verlag, 2012 (German).

[Fy01] Johannes Ulrich Führer y Escartílar, Die Sasaki-Metrík auf dem Tangential- und dem Sphärenbündel, diploma thesis, University of Cologne, 2001.

[Gaf51] Matthew P. Gaffney, The harmonic operator for exterior differential forms., Proc. Natl. Acad. Sci. USA 37 (1951), 48–50 (English).

[Gaf54a] A special Stoke’s theorem for complete Riemannian manifolds., Ann. Math. (2) 60 (1954), 140–145 (English).

[Gaf54b] The heat equation method of Milgram and Rosenbloom for open Riemannian manifolds., Ann. Math. (2) 60 (1954), 458–466 (English).

[Gar64] R. Gangolli, On the construction of certain diffusions on a differentiable manifold., Z. Wahrscheinlichkeitstheor. Verw. Geb. 2 (1964), 406–419 (English).
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[HK02] Sigmundur Gudmundsson and Elias Kappa, On the geometry of tangent bundles., Expo. Math. 20 (2002), no. 1, 1–41 (English).

[GKMW07] T. Götz, A. Klar, N. Marheineke, and R. Wegener, A stochastic model and associated Fokker-Planck equation for the fiber lay-down process in nonwoven production processes., SIAM J. Appl. Math. 67 (2007), no. 6, 1704–1717 (English).

[GKOS01] Michael Grosser, Michael Kunzinger, Michael Oberguggenberger, and Roland Steinbauer, Geometric theory of generalized functions with applications to general relativity, Mathematics and its Applications, vol. 537, Kluwer Academic Publishers, Dordrecht, 2001. MR 1883263

[Gli97] Yuri Gliklikh, Global Analysis in Mathematical Physics. Geometric and Stochastic Methods, Springer-Verlag New York, 1997 (English).

[GMS12] Martin Grothaus, Johannes Maringer, and Patrik Stilgenbauer, Geometry, mixing properties and hypocoercivity of a degenerate diffusion arising in technical textile industry, preprint at arXiv:1203.4502v1 [math.PR], 2012.

[KMW12] Axel Klar, Johannes Maringer, and Raimund Wege-}

[Go59] Abraham Goetz, On measures in fibre bundles., Colloq. Math. 7 (1959), 11–18. MR 0151576

[GS13] Martin Grothaus and Patrik Stilgenbauer, Geometric langevin equations on submanifolds and applications to the stochastic melt-spinning process of nonwovens and biology, Stochastics and Dynamics 13 (2013), no. 04, 1350001.

[GS14] M. Grothaus and P. Stilgenbauer, Hypocoercivity for Kolmogorov backward evolution equations and applications, JFA 267 (2014), no. 10, 3515–3556.

[GS16] Martin Grothaus and Patrik Stilgenbauer, Hilbert space hypocoercivity for the Langevin dynamics revisited, Methods Funct. Anal. Topology 22 (2016), no. 2, 152–168 (English). MR 3522857

[Heb99] Emmanuel Hebey, Nonlinear Analysis on Manifolds. Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, vol. 5, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. MR 1688256

[HM19] David P. Herzog and Jonathan C. Mattingly, Ergodicity and Lyapunov functions for Langevin dynamics with singular potentials., Commun. Pure Appl. Math. 72 (2019), no. 10, 2231–2255 (English).

[HN05] Bernard Helffer and Francis Nier, Hypoellipticity estimates and spectral theory for Fokker-Planck operators and Witten Laplacians., vol. 1862, Berlin: Springer, 2005 (English).

[Hor66] John Horvath, Topological vector spaces and distributions. Vol. I., Addison-Wesley Series in Mathematics.) Reading, Mass.-Palo Alto-London-Don Mills, Ontario: Addison-Wesley Publishing Company. XII, 449 p. (1966)., 1966.

[Hsu02] Elton P. Hsu, Stochastic analysis on manifolds., vol. 38, Providence, RI: American Mathematical Society (AMS), 2002 (English).

[ILP15] Alberto Ibort Latre, Fernando Lledó Macau, and Juan Manuel Pérez Pardo, Self-adjoint extensions of the Laplace–Beltrami operator and unitaries at the boundary., JFA 268 (2015), no. 3, 634 – 670.

[LM89] H. Blaine jun. Lawson and Marie-Louise Michelsohn, Spin geometry., Princeton University Press, 1989 (English).

[McK69] H. P. jun. McKean, Stochastic integrals., New York-London: Academic Press XIII, 140 p. (1969)., 1969.

[Jr78] J. Jorgensen, Construction of the Brownian motion and the Ornstein-Uhlenbeck process in a Riemannian manifold on basis of the Gangolli-McKean injection scheme., Z. Wahrscheinlichkeitstheor. Verw. Geb. 44 (1978), 71–87 (English).

[Ker58] Michel A. Kervaire, Non-parallelizability of the n-sphere for n > 7., Proc. Natl. Acad. Sci. USA 44 (1958), 280–283 (English).

[KMS93] Ivan Kolář, Peter W. Michor, and Jan Slovák, Natural Operations in differential geometry, Springer-Verlag Berlin Heidelberg, 1993 (English).

[KMW12] Axel Klar, Johannes Maringer, and Raimund Wegener, A 3D model for fiber lay-down in nonwoven production processes., Math. Models Methods Appl. Sci. 22 (2012), no. 9, 1250020, 18 (English).

[Kol00] V. N. Kolokoltsov, Semiclassical analysis for diffusions and stochastic processes., vol. 1724, Berlin: Springer, 2000 (English).

[Lang5] Serge Lang, Differential and Riemannian manifolds, 3rd ed.; Repr. of the orig. 1972., 3rd ed.; repr. of the orig. 1972 ed., vol. 160, Heidelberg: Springer-Verlag, 1995 (English).

[Lee12] John Lee, Introduction to Smooth Manifolds, Springer, 2012 (English).

[LMS+17] Felix Lindner, Nicole Marheineke, Holger Stroot, Alexander Vibe, and Raimund Wegener, Stochastic dynamics for inextensible fibers in a spatially semi-discrete setting., Stoch. Dyn. 17 (2017), no. 2, 29 (English).
[MR99] Jerrold Marsden and Tudor Ratiu, *Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems*, Springer-Verlag New York, 1999 (English).

[MW07] Nicole Marheineke and Raimund Wegener, *Fiber dynamics in turbulent flows: specific Taylor drag*, SIAM J. Appl. Math. 68 (2007), no. 1, 1–23 (English).

[Nic96] Liviu I. Nicolaescu, *Lectures On The Geometry Of Manifolds*, World Scientific Publishing, 1996 (English).

[Pe13] Juan Manuel Pérez Pardo, *On the Theory of Self-adjoint Extensions of the Laplace-Beltrami Operator, Quadratic Forms and Symmetry*, Ph.D. thesis, Universidad Carlos III de Madrid, 2013.

[Ros13] Jonathan Rosenberg, *Levi-Civita’s Theorem for Noncommutative Tori*, SIGMA 9 (2013), 9 (English).

[RS80] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I: Functional analysis. Rev. and enl. ed.*, New York etc.: Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers, XV, 400 p. $ 24.00 (1980), 1980.

[Sak96] Takashi Sakai, *Riemannian geometry. Transl. from the Japanese by Takashi Sakai.*, vol. 149, Providence, RI: AMS, American Mathematical Society, 1996 (English).

[Sas58] Shigeo Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds.*, Tohoku Math. J. (2) 10 (1958), 338–354 (English).

[So95] M. R. Soloveitchik, *Fokker-Planck equation on a manifold. Effective diffusion and spectrum.*, Potential Anal. 4 (1995), no. 6, 571–593 (English).

[Sta99] Wilhelm Stannat, *The theory of generalized Dirichlet forms and its applications in analysis and stochastics.*, vol. 678, Providence, RI: American Mathematical Society (AMS), 1999 (English).

[Sti14] Patrik Stilgenbauer, *The Stochastic Analysis of Fiber Lay-Down Models*, Ph.D. thesis, Technische Universität Kaiserslautern (TUK), 2014.

[TO62] Shunichi Tachibana and M. Okumura, *On the almost-complex structure of tangent bundles of Riemannian spaces.*, Tohoku Math. J. (2) 14 (1962), 156–161 (English).

[Tru00] Gerald Trutnau, *Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions.*, Osaka J. Math. 37 (2000), no. 2, 315–343 (English).

[Tru03] _______, *On a class of non-symmetric diffusions containing fully non-symmetric distorted Brownian motions.*, Forum Math. 15 (2003), no. 3, 409–437 (English).

[Vil07] Cédric Villani, *Hypocoercive diffusion operators.*, Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8) 10 (2007), no. 2, 257–275 (English).

[Vil09] _______, *Hypocoercivity.*, Mem. Am. Math. Soc. 950 (2009), 1–141 (English).

[W81] Shinzō Watanabe and Nobuyuki Ikeda, *Stochastic Differential Equations and Diffusion Processes*, North Holland, 1981 (English).

[Wol73] Joseph A. Wolf, *Essential self adjointness for the Dirac operator and its square.*, Indiana Univ. Math. J. 22 (1973), 611–640 (English).

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