Curves of genus two over fields of even characteristic

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Abstract. In this paper we classify curves of genus two over a perfect field $k$ of characteristic two. We find rational models of curves with a given arithmetic structure for the ramification divisor and we give necessary and sufficient conditions for two models of the same type to be $k$-isomorphic. As a consequence, we obtain an explicit formula for the number of $k$-isomorphism classes of curves of genus two over a finite field. Moreover, we prove that the field of moduli of any curve coincides with its field of definition, by exhibiting rational models of curves with any prescribed value of their Igusa invariants. Finally, we use cohomological methods to find, for each rational model, an explicit description of its twists. In this way, we obtain a parameterization of all $k$-isomorphism classes of curves of genus two in terms of geometric and arithmetic invariants.

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Introduction

Let $k$ be a perfect field of even characteristic. Igusa described in [5] the moduli variety for curves of genus two as certain 3-dimensional affine variety, whose $k$-points are in bijection with $k^3$. Moreover, he gave explicit formulas to compute the moduli point of a curve in terms of invariants. In section 2 of this paper we prove that the field of moduli of any curve coincides with its field of definition, so that the Igusa invariants determine a bijection between the set of $k$-isomorphism classes of curves of genus two defined over $k$ and the affine space $k^3$. This is

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known to be false for some fields of odd or zero characteristic (cf. [7]). We prove this result by exhibiting rational models of curves with any prescribed value of their invariants.

Rational models of curves of genus two are studied more generally in section 1, where we obtain an effective classification, up to $k$-isomorphism, of curves of genus two defined over $k$. For each possible arithmetic structure of the ramification divisor, the corresponding curves admit quasi-affine models $y^2 + y = u(x)$, where $u(x) \in k(x)$ is a rational function with a concrete divisor of poles. The $k$-isomorphism classes of these curves are in bijection with the orbits of these rational functions $u(x)$ under a double action by the Artin-Schreier group $\text{AS}(k(x))$ and the projective linear group $\text{PGL}_2(k)$. In paragraph 1.3 we carry out this classification in a very explicit manner. As a by-product, we obtain an explicit computation of the $k$-automorphism group of each curve.

In section 3 we describe all twists of a given curve $C$, in terms of the rational models of section 1. This is achieved by computing $H^1(G_k, \text{Aut}(C))$ for any possible structure of $\text{Aut}(C)$. Together with the result of section 2, this computation furnishes an explicit parameterization of all $k$-isomorphism classes of curves of genus two, each class being determined by a couple of invariants: one geometric (a triple $(j_1, j_2, j_3) \in k^3$ of values of the Igusa invariants) and the other arithmetic (a concrete twist of a chosen curve having $(j_1, j_2, j_3)$ as geometric invariants).

Our initial motivation for this work was to classify curves of genus two over finite fields of even characteristic, in view of their use in Cryptography (cf. [4]). In the finite field case our results provide formulas for the number of curves of genus two with a fixed structure of the ramification divisor (paragraph 1.4) and for the number of curves with a fixed structure of the full automorphism group (paragraph 3.2).

1. Rational models of curves of genus two

We fix once and for all a perfect field $k$ of even characteristic and an algebraic closure $\overline{k}$ of $k$. We denote by $G_k$ the Galois group $\text{Gal}(\overline{k}/k)$ and by $x \mapsto \sqrt{x}$ the automorphism of $k$ inverse of the Frobenius automorphism, $x \mapsto x^2$. Also, $\mathbb{F}_{2^m}$ and $\mu_n$ will denote respectively the unique finite subfield of $\overline{k}$ with $2^m$ elements and the subgroup of $\overline{k}^*$ of the $n$-th roots of unity. Finally, we denote $\mu_n(k) = \mu_n \cap k$.

By a curve of genus two we mean a smooth, projective, geometrically irreducible curve of genus two.
1.1. Hyperelliptic curves

In this paragraph we review well-known results on hyperelliptic curves. Let $C$ be a hyperelliptic curve defined over $k$; that is, $C$ is a smooth, projective and geometrically irreducible curve defined over $k$, of genus $g \geq 2$, admitting a degree $2$ morphism $\pi: C \longrightarrow \mathbb{P}^1$, which is also defined over $k$.

The corresponding extension of function fields, $k(C)/k(\mathbb{P}^1)$, is a cyclic quadratic extension (if it were not separable, then $k(C)$ would be a function field of genus zero). To the non-trivial element of the Galois group it corresponds an involution, $\iota: C \longrightarrow C$, which is called the hyperelliptic involution of $C$. By definition, thus, two points $P, Q \in C(\overline{k})$ have the same image under $\pi$ if and only if $Q = P$ or $Q = \iota(P)$.

We recall some basic properties of $\iota$ based on the crucial fact that it is a canonical involution, independent of the morphism $\pi$.

**THEOREM 1.** Let $\pi_1, \pi_2: C \longrightarrow \mathbb{P}^1$ be two $k$-morphisms of degree $2$. Then, there exists a unique $k$-automorphism $\gamma$ of $\mathbb{P}^1$ such that $\pi_2 = \gamma \circ \pi_1$.

**COROLLARY 2.**

1. The hyperelliptic involution is canonical. The fixed points of $\iota$ are the Weierstrass points of $C$ and they are the ramification points of any morphism of degree $2$ from $C$ to $\mathbb{P}^1$.

2. Any $k$-automorphism $\varphi$ of $C$ fits into a commutative diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{\varphi} & C \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}^1 & \xrightarrow{\gamma} & \mathbb{P}^1
\end{array}
$$

for certain uniquely determined $k$-automorphism $\gamma \in \text{Aut}_k(\mathbb{P}^1)$. The map $\varphi \mapsto \gamma$ is a homomorphism (depending on $\pi$) and we have an exact sequence of groups:

$$
1 \longrightarrow \langle \iota \rangle \longrightarrow \text{Aut}_k(C) \xrightarrow{\pi} \text{Aut}_k(\mathbb{P}^1).
$$

In particular, $\iota$ commutes with every automorphism of $C$.

We recall now some generalities on Artin-Schreier (i.e. cyclic quadratic) extensions of a field $K$ of even characteristic. Let $K^{\text{sep}}$ be a fixed separable closure of $K$ and $G_K = \text{Gal}(K^{\text{sep}}/K)$. The Artin-Schreier homomorphism, $x \mapsto \text{AS}(x) = x + x^2$, fits into an exact sequence:

$$
0 \longrightarrow \mathbb{F}_2 \longrightarrow K^{\text{sep}} \xrightarrow{\text{AS}} K^{\text{sep}} \longrightarrow 0,
$$
leading to an isomorphism $H^1(G_K, \mathbb{F}_2) \simeq K/\text{AS}(K)$. Thus, the group $K/\text{AS}(K)$ classifies all cyclic quadratic (or trivial) extensions of $K$. Any $u \in K - \text{AS}(K)$ determines a separable quadratic extension of $K$ by adjoining to $K$ the roots of the separable irreducible polynomial $Y^2 + Y + u$. An element $u' \in K$ furnishes the same extension than $u$ if and only if $u + u' \in \text{AS}(K)$. Moreover, any cyclic quadratic extension of $K$ can be obtained this way. If $K$ is a finite field, $\text{AS}(K)$ coincides with the subgroup of elements of absolute trace zero and $K/\text{AS}(K)$ has only two elements.

By fixing a point $\infty \in \mathbb{P}^1(k)$, we fix isomorphisms $k(\mathbb{P}^1) \simeq k(x)$ and $\text{Aut}_k(\mathbb{P}^1) \simeq \text{PGL}_2(k)$. The function field $k(C)$ is then identified (via $\pi^*$) to an Artin-Schreier extension of $k(x)$ and it admits an Artin-Schreier generator $y \in k(C)$ satisfying $y^2 + y = u(x)$, for certain $u(x) \in k(x) - \text{AS}(k(x))$.

Conversely, for any $u(x) \in k(x) - \text{AS}(k(x))$, the equation $y^2 + y = u(x)$ determines an Artin-Schreier extension, $L = k(x, y)$, of $k(x)$. Such an equation determines a plane non-singular quasi-affine curve $C^{\text{af}} = C_{u(x)}^{\text{af}}$, defined over $k$. We shall denote by $C = C_{u(x)}$ the projective, smooth curve obtained as the normalization of the projective closure of $C^{\text{af}}$. The projection on the first coordinate, $(x, y) \mapsto x$, lifts to a morphism of degree 2, $\pi: C \longrightarrow \mathbb{P}^1$, implicitly associated to the equation. Clearly, the function field of $C$ is $k$-isomorphic to $L$ and the homomorphism $\pi^*: k(\mathbb{P}^1) \longrightarrow k(C)$ translates into $k(x) \hookrightarrow k(x, y)$ under natural identifications.

Since $C^{\text{af}}$ is non-singular, we can identify its points with an open subset of $C(k)$; this allows us to attach affine coordinates to most of the points of $C$. Hence, to deal with these curves in practice, it is not necessary to find equations for $C$ itself, as soon as one has control on the finite set of points at infinity: $Z_{\infty} := C(k) - C^{\text{af}}(k)$. Note that the ramification points of $\pi$ lie above poles of $u(x)$; hence, they are always points at infinity.

The genus $g$ of $C_{u(x)}$ is easily determined in terms of the poles of odd order of the elements in $u(x) + \text{AS}(k(x))$, and $C_{u(x)}$ is always geometrically irreducible, hence a hyperelliptic curve, when $g \geq 2$ (cf. Proposition 4 below). In affine coordinates, the hyperelliptic involution is then expressed as: $(x, y)^e = (x, y + 1)$. After Theorem 1, it is easy to determine in terms of $u(x)$ the $k$-isomorphism class of $C_{u(x)}$:

**Proposition 3.** Two hyperelliptic curves $C_{u(x)}$, $C_{u'(x)}$ are $k$-isomorphic if and only if

$$u'(x) \equiv u(\gamma(x)) \pmod{\text{AS}(k(x))},$$

for some $\gamma \in \text{PGL}_2(k)$. 
Proof. By Theorem 1 and the structure of Artin-Schreier extensions, any $k$-isomorphism between these two curves must be of the type:

$$(x, y) \mapsto (\gamma(x), y + v(x)),$$

for some $\gamma \in \text{PGL}_2(k)$, $v(x) \in k(x)$, and the identity $u'(x) = u(\gamma(x)) + v(x) + v(x)^2$ must hold. \hfill \Box$

Therefore, the classification of hyperelliptic curves up to $k$-isomorphism amounts to the classification of rational functions $u(x) \in k(x)$ (with sufficiently many poles of odd order) under the double action of $\text{AS}(k(x))$ and $\text{PGL}_2(k)$. In the next paragraph we shall indicate how to carry out this classification, and in paragraph 1.3 we shall apply this procedure for curves of genus two in a very explicit manner.

1.2. Ramification divisor

Given a smooth, projective curve $C$ defined over $k$ and a separable $k$-morphism $\pi: C \to \mathbb{P}^1$, we define the ramification divisor of $\pi$ as the divisor, $\text{Diff}(C/\mathbb{P}^1)$, of $C$ naturally associated to the different of the extension of function fields $k(C)/\pi^*(k(\mathbb{P}^1))$ (cf. [9, ch.III]). For any point $P \in C(k)$, if $e_P$ is the ramification index at $P$ and $d_P$ is the exponent of the different at $P$, then $d_P \geq e_P - 1$, with equality if and only if $e_P$ is odd (cf. [9, III,5.1]); hence, the support of the ramification divisor is the set of ramification points of $\pi$.

We have seen in the last paragraph that any $u(x) \in k(x) - \text{AS}(k(x))$ determines a smooth projective curve $C_{u(x)}$ with quasi-affine model $y^2 + y = u(x)$ and implicitly equipped with a separable morphism of degree 2 to $\mathbb{P}^1$. By a procedure that goes back to Hasse (cf. [3]), one can find a suitable element $v(x) \in k(x)$ such that $u(x) + v(x) + v(x)^2$ has no poles of even order. Once this normalization is achieved, the location and order of the poles of $u(x)$ determine the ramification divisor of $C_{u(x)}$. More precisely:

PROPOSITION 4. [9, III.7.8] Let $u(x) \in k(x) - \text{AS}(k(x))$ be a rational function having no poles of even order. Let $C = C_{u(x)}$ be the smooth, projective curve defined over $k$, with quasi-affine model $y^2 + y = u(x)$. Then,

$$\text{Diff}(C/\mathbb{P}^1) = \sum_{Q \in \mathbb{P}^1(k)} \left( \sum_{P \to Q} (m_Q + 1)P \right),$$

where

$$m_Q = \begin{cases} -1, & \text{if } \text{ord}_Q(u(x)) \geq 0, \\ m, & \text{if } \text{ord}_Q(u(x)) = -m < 0. \end{cases}$$
Moreover,  
\[ k(C) \cap \overline{k} = k \iff u(x) \not\in k + \text{AS}(k(x)) \iff \text{Diff}(C/\mathbb{P}^1) \neq 0, \]
and, when this condition is satisfied, one has  \( \deg(\text{Diff}(C/\mathbb{P}^1)) = 2g + 2 \),  
where  \( g \)  is the genus of  \( C \). In particular, if  \( \deg(\text{Diff}(C/\mathbb{P}^1)) \geq 6 \), then  \( C \)  is a hyperelliptic curve defined over  \( k \).

By Theorem 1, if  \( C \)  is a hyperelliptic curve, the ramification divisor is independent of  \( \pi \) and, for any isomorphism  \( \varphi:C \to C' \), we have  \( \varphi^*(\text{Diff}(C'/\mathbb{P}^1)) = \text{Diff}(C/\mathbb{P}^1) \). From now on we shall denote:

\[ W := \pi_*(\text{Diff}(C/\mathbb{P}^1)), \quad \Gamma_W := \{ \gamma \in \text{PGL}_2(k) \mid \gamma^*(W) = W \}. \]

These two objects depend on  \( \pi \), but the reference to  \( \pi \) is omitted and it will be always implicit in the context. Since any  \( k \)-automorphism  \( \varphi \) of  \( C \) leaves the ramification divisor invariant, the automorphism of  \( \mathbb{P}^1 \) associated to  \( \varphi \) lies in  \( \Gamma_W \); hence, we have actually an exact sequence:

\[ 1 \to \langle \iota \rangle \to \text{Aut}_k(C) \xrightarrow{\pi} \Gamma_W. \]

A natural strategy to classify hyperelliptic curves up to  \( k \)-isomorphism is to determine first, up to the action of  \( \text{PGL}_2(k) \), the possible divisors  \( W \) of  \( \mathbb{P}^1 \) that can appear as the push-forward of ramification divisors of hyperelliptic curves and, afterwards, classify the curves linked to a concrete divisor  \( W \). Proposition 4 takes care of the first step. Let us indicate a procedure to carry out the second step.

Let  \( W \) be a fixed effective divisor of  \( \mathbb{P}^1 \) with even coefficients and let  \( W' \) be the divisor obtained by lowering all positive coefficients by one. Note that the isotropy subgroups of  \( W, W' \) under the action of  \( \text{PGL}_2(k) \) are the same. Let  \( \mathcal{R}_W \subseteq k(x) \) be the set of rational functions having  \( W' \) as divisor of poles and let  \( \mathcal{N} \subseteq \mathcal{R}_W \) be a system of representatives of these functions modulo  \( \text{AS}(k(x)) \). We can define an action of  \( \Gamma_W \) over  \( \mathcal{N} \) on the right. Given  \( u(x) \in \mathcal{N} \) and  \( \gamma \in \Gamma_W \), the divisor of poles of  \( \gamma^*(u(x)) = u(\gamma(x)) \) is again  \( W' \) and we define  \( u^\gamma(x) \) to be the only element of  \( \mathcal{N} \) such that:

\[ u(\gamma(x)) \equiv u^\gamma(x) \pmod{\text{AS}(k(x))}. \]

For any  \( \gamma \in \Gamma_W \) and  \( u(x) \in \mathcal{N} \), denote by  \( v_{u,\gamma}(x) \in k(x) \) any choice of a rational function satisfying:

\[ u(\gamma(x)) = u^\gamma(x) + v_{u,\gamma}(x) + v_{u,\gamma}(x)^2. \]

Let us check that  \( (\gamma, u(x)) \mapsto u^\gamma(x) \) defines an action of  \( \Gamma_W \) on  \( \mathcal{N} \) indeed. For any couple  \( \gamma, \eta \in \Gamma_W \), we have:

\[ (\gamma\eta)^*(u(x)) = u^\gamma(x) + v_{u,\gamma}(x) + v_{u,\gamma}(x)^2, \]
\[ \eta^*(u^\gamma(x)) = (u^\gamma)(x) + v_{u^\gamma,\eta}(x) + v_{u^\gamma,\eta}(x)^2 + \eta^*(v_{u,\gamma}(x)) + \eta^*(v_{u,\gamma}(x))^2. \]
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Since these two elements coincide, the functions $u^\gamma(x), (u^\gamma)'(x)$ must be equal too, since both belong to $N$ and they are congruent modulo $AS(k(x))$.

For any $u(x) \in N$, denote by $\Gamma_{u(x)}$ the isotropy group of $u(x)$:

$$\Gamma_{u(x)} = \{ \gamma \in \Gamma_W | u(x) = u^\gamma(x) \}.$$

**PROPOSITION 5.** Let $W, \Gamma_W, N \subseteq k(x)$ be as above.

1. For any $u(x), u'(x) \in N$, the curves $C_{u(x)}, C_{u'(x)}$ are $k$-isomorphic if and only if there exists $\gamma \in \Gamma_W$ such that $u'(x) = u^\gamma(x)$.

2. For any $u(x) \in N$, we have an exact sequence:

$$1 \rightarrow \langle \iota \rangle \rightarrow \text{Aut}_k(C_{u(x)}) \xrightarrow{\pi} \Gamma_{u(x)} \rightarrow 1.$$

Moreover, if the map $v_u: \Gamma_{u(x)} \rightarrow k(x)$, given by $\gamma \mapsto v_{u,\gamma}(x)$, is a homomorphism, then the exact sequence splits.

**Proof.** The first assertion is an immediate consequence of Proposition 3. Let $\varphi$ be a $k$-automorphism of $C_{u(x)}$. By (1), the automorphism $\gamma = \pi(\varphi)$ satisfies:

$$u(x) \equiv u(\gamma(x)) \pmod{AS(k(x))},$$

so that $u(x) = u^\gamma(x)$ and $\gamma \in \Gamma_{u(x)}$. Conversely, given $\gamma \in \Gamma_{u(x)}$, we can construct $\varphi$ by taking:

$$\varphi^*(x) = \gamma(x), \quad \varphi^*(y) = y + v_{u,\gamma}(x).$$

Clearly, if $v_u$ is a homomorphism, the map $\gamma \mapsto \varphi$ obtained in this way is a homomorphism too.

\[\square\]

1.3. **NORMAL EQUATIONS FOR CURVES OF GENUS TWO**

Let $C$ be a smooth, projective, geometrically irreducible curve of genus two defined over $k$. The canonical morphism, $\pi: C \rightarrow \mathbb{P}^1$, is defined over $k$ and has degree 2, so that $C$ is a hyperelliptic curve. Hence, $C$ admits a quasi-affine model $y^2 + y = u(x)$, for certain $u(x) \in k(x) - AS(k(x))$ without poles of even order. By Proposition 4, the ramification divisor of $C$ has degree 6 and one of the following possible forms:

\[
\text{Diff}(C/\mathbb{P}^1) = \begin{cases} 
2P_1 + 2P_2 + 2P_3, & \text{if } u(x) \text{ has three simple poles,} \\
2P_1 + 4P_2, & \text{if } u(x) \text{ has two poles, of orders 1,3,} \\
6P, & \text{if } u(x) \text{ has only one pole, of order 5.}
\end{cases}
\]
We label these possibilities respectively as: case (1,1,1), case (1,3), and case (5). Since this divisor is defined over \( k \), in cases (1,3) and (5) all points in the support are defined over \( k \) too. However, in case (1,1,1) we have three different arithmetic structures for the ramification divisor: three points defined over \( k \), one point defined over \( k \) and two points conjugate over a quadratic extension of \( k \) or, finally, all three points conjugate over a cubic extension of \( k \). We shall refer to them respectively as cases (1,1,1)-split, quadratic and cubic.

The divisor \( W = \pi_*(\text{Diff}(C/P^1)) \) has support on the poles of \( u(x) \). In cases (1,1,1)-split, (1,3) and (5), by applying a suitable \( k \)-automorphism of \( P^1 \), we can impose that this support is a subset of \( \{\infty, 0, 1\} \). Let us display now a couple of Lemmas that are necessary to fix the support of \( W \) in the cases (1,1,1)-quadratic and cubic.

**Lemma 6.** Let \( S, S' \) be two \( G_k \)-subsets of \( P^1(k) \) with \(|S| = |S'| = 3\). Then, there exists \( \gamma \in \text{PGL}_2(k) \) such that \( \gamma(S) = S' \) if and only if \( S \) and \( S' \) are isomorphic as \( G_k \)-sets.

**Proof.** Clearly, any \( \gamma \in \text{PGL}_2(k) \) respects the \( G_k \)-structure. Conversely, if \( \rho: S \to S' \) is a \( G_k \)-bijection, take \( \gamma \in \text{PGL}_2(k) \) such that it coincides with \( \rho \) over \( S \). Then, for any \( \sigma \in G_k \) and any \( x \in S \), we have:

\[
\gamma(\sigma x) = \rho(\sigma x) = \sigma(\rho(x)) = \sigma(\gamma(x)) = \sigma\gamma(x).
\]

Hence \( \gamma = \sigma\gamma \) for all \( \sigma \in G_k \) and \( \gamma \) belongs to \( \text{PGL}_2(k) \).

**Lemma 7.** For any cubic extension \( K/k \), there exists a generator \( \theta \in K \) with minimal equation \( x^3 + sx + s, \ s \neq 0, \overline{k} \). If \( w, w+1 \in k \) are the two roots of the equation \( s + 1 = w + w^2 \), then the other two roots of the polynomial \( x^3 + sx + s \) are:

\[
\theta' = \theta(\theta + w), \quad \theta'' = \theta(\theta + w + 1).
\]

In particular, the extension \( K/k \) is cyclic if and only if \( s + 1 \in \text{AS}(k) \).

**Proof.** Replacing \( \theta \) by \( \theta + \text{Tr}_{K/k}(\theta) \), we can assume that \( \text{Tr}_{K/k}(\theta) = 0 \).

Suppose that \( x^3 + ax + b \) is the minimal polynomial of \( \theta \) over \( k \). Replacing \( \theta \) by \( \theta + \theta^2 \), if necessary, we can assume moreover that \( a \neq 0 \). Then, the minimal polynomial of \( ab^{-1}\theta \) is \( x^3 + sx + s \), where \( s = a^3b^{-2} \). Finally, the assertions concerning the conjugates of \( \theta \) can be checked by a direct computation.

We proceed now to exhibit families of normal equations describing all \( k \)-isomorphism classes of curves of genus two. For each of the above types of ramification divisor we fix a system of representatives of divisors \( W \) of this type modulo the action of \( \text{PGL}_2(k) \), and for each \( W \) we
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apply the procedure indicated at the end of the last paragraph. We find a suitable family \( \mathcal{N} \) of normal equations and, just by computing the action of \( \Gamma_W \) on \( \mathcal{N} \) and applying Proposition 5, we classify the curves in \( \mathcal{N} \) up to \( k \)-isomorphism and we determine the structure of the \( k \)-automorphism group of each curve. In each case, the rational functions in \( \mathcal{N} \) depend on certain parameters, and we implicitly identify \( \mathcal{N} \) with the set of values of these parameters.

**Case \((1,1,1)\)-split**

Let \( \mathcal{N} = k^* \times k^* \times k^* \times (k/\text{AS}(k)) \). Any curve with ramification divisor of this type is \( k \)-isomorphic to one of the curves \( C_{abcd} \) with quasi-affine model:

\[
y^2 + y = ax + \frac{b}{x} + \frac{c}{x+1} + d, \quad (a, b, c, d) \in \mathcal{N}.
\]

For these curves, \( W = 2[\infty] + 2[0] + 2[1] \), and the set \( Z_\infty \) of points at infinity contains only the three Weierstrass points of the curve, which are all defined over \( k \). The subgroup \( \Gamma := \Gamma_W \) is the isotropy group of \( \{\infty, 0, 1\} \) under the action of \( \text{PGL}_2(k) \) over \( \mathbb{P}^1(k) \):

\[
\Gamma = \{x, 1/x, 1 + x, x/(1 + x), 1/(1 + x), (1 + x)/x\} \simeq S_3.
\]

In order to interpret properly the next result and the analogous Propositions taking care of the other cases, we emphasize that, when comparing two elements of \( \mathcal{N} \), the fourth coordinate has to be understood as an element of \( k/\text{AS}(k) \).

**PROPOSITION 8.**

1. For any \((a, b, c, d), (a', b', c', d') \in \mathcal{N} \), the curves \( C_{a'b'c'd'} \), \( C_{abcd} \) are \( k \)-isomorphic if and only if \((a', b', c', d')\) coincides with one of the following elements in the orbit of \((a, b, c, d)\) under the action of \( \Gamma \):

\[
(a, b, c, d)\gamma = \begin{cases} 
(a, b, c, d), & \text{if } \gamma(x) = x, \\
(b, a, c, d + c), & \text{if } \gamma(x) = 1/x, \\
(a, c, b, d + a), & \text{if } \gamma(x) = 1 + x, \\
(c, b, a, d + c + b + a), & \text{if } \gamma(x) = x/(1 + x), \\
(b, c, a, d + c + b), & \text{if } \gamma(x) = 1/(1 + x), \\
(c, a, b, d + b + a), & \text{if } \gamma(x) = (1 + x)/x.
\end{cases}
\]

2. For any \((a, b, c, d) \in \mathcal{N} \), we have an split exact sequence:

\[
1 \rightarrow \langle \iota \rangle \rightarrow \text{Aut}_k(C_{abcd}) \rightarrow \Gamma_{abcd} \rightarrow 1,
\]
\( \Gamma_{abcd} \) being the isotropy group of \((a, b, c, d)\) under the action of \( \Gamma \):

\[
\Gamma_{abcd} \simeq \begin{cases} 
S_3, & \text{if } a = b = c \in \text{AS}(k), \\
C_3, & \text{if } a = b = c \notin \text{AS}(k), \\
C_2, & \text{if exactly two of the coefficients } a, b, c \text{ coincide and the third belongs to } \text{AS}(k), \\
1, & \text{otherwise.}
\end{cases}
\]

Proof. To any \((a, b, c, d) \in N\), we assign a unique element \( u(x) = ax + \frac{b}{2} + \frac{c}{x+1} + d \in k(x) \), by making a choice of \( d \in k \) inside the class modulo \( \text{AS}(k) \) given by the fourth coordinate. The family of rational functions obtained in this way is a system of representatives modulo \( \text{AS}(k(x)) \) of the functions having \([\infty] + [0] + [1] \) as divisor of poles and we can apply Proposition 5. Only the fact that we can choose the mapping \( v_u \colon \Gamma_{u(x)} \longrightarrow k(x) \) to be a homomorphism deserves some explanation. Given \( u(x) \) as above and \( \gamma \in \Gamma_{u(x)} \), it is easily checked that \( v_{u,\gamma}(x) \) is a constant satisfying, respectively,

\[
v_{u,\gamma} + v_{u,\gamma}^2 = 0, \quad c, a, c + b + a, \quad c, b + a,
\]

according to the different possible values of \( \gamma \). Assume for instance that \( a = b = c \in \text{AS}(k) \) and let \( w \in k \) be such that \( a = w + w^2 \). Then, the following choice for \( v_{u,\gamma} \) determines a homomorphism:

\[
v_{u,\gamma} = \begin{cases} 
0, & \text{if } \gamma \text{ has order 1 or 3}, \\
w, & \text{if } \gamma \text{ has order 2}.
\end{cases}
\]

The other cases are similar and easier. \( \square \)

From the exact sequence above, one can determine the \( k \)-automorphisms of \( C_{abcd} \). For instance, if \( a = b = c \in \text{AS}(k) \) and \( a = w + w^2 \) for certain \( w \in k \), the twelve \( k \)-automorphisms of \( C_{abcd} \) are:

\[(x, y) \xrightarrow{id} (x, y), \quad (x, y) \xrightarrow{i} (x, y + 1), \quad (x, y) \xrightarrow{i} (x, y + 1), \]
\[(x, y) \xrightarrow{U} (x + 1, y + w), \quad (x, y) \xrightarrow{U} (x + 1, y + w + 1), \quad (x, y) \xrightarrow{U} (x + 1, y + w + 1), \]
\[(x, y) \xrightarrow{V} (1/(1 + x), y), \quad (x, y) \xrightarrow{V} (1/(1 + x), y + 1), \quad (x, y) \xrightarrow{V} (1/(1 + x), y + 1), \]
\[(x, y) \xrightarrow{V^2} (1/(1 + x)/x, y), \quad (x, y) \xrightarrow{V^2} (1/(1 + x)/x, y + 1), \]
\[(x, y) \xrightarrow{VU} (1/x, y + w), \quad (x, y) \xrightarrow{VU} (1/x, y + w + 1), \quad (x, y) \xrightarrow{VU} (1/x, y + w + 1), \]
\[(x, y) \xrightarrow{V^2U} (x/(1 + x), y + w), \quad (x, y) \xrightarrow{V^2U} (x/(1 + x), y + w + 1), \quad (x, y) \xrightarrow{V^2U} (x/(1 + x), y + w + 1). \]

Case (1,1,1)-quadratic

We fix a system of representatives \( r \in k \) of all non-trivial classes of \( k/\text{AS}(k) \).
Let $\mathcal{N} = k^* \times ((k \times k) - \{(0, 0)\}) \times (k/\text{AS}(k))$. Any curve with ramification divisor of this type is $k$-isomorphic to one of the curves $C_{abcd}^r$ with quasi-affine model:

$$y^2 + y = ax + \frac{bx + c}{x^2 + x + r} + d, \quad (a, b, c, d) \in \mathcal{N}. \quad (2)$$

For these curves, $W = 2[\infty] + 2[\theta] + 2[\theta']$, where $\theta, \theta' \in \overline{k}$ are the roots of the irreducible polynomial $x^2 + x + r$. The set $Z_{\infty}$ of points at infinity contains only the three Weierstrass points of the curve, one of them defined over $k$, the other two defined over the quadratic extension determined by the class of $r$ modulo $\text{AS}(k)$. The subgroup $\Gamma := \Gamma_W$ is the isotropy group of $\{\infty, \theta, \theta'\}$ under the action of $\text{PGL}_2(k)$ over $\mathbb{P}^1(\overline{k})$:

$$\Gamma = \{x, 1 + x\} \simeq C_2.$$

By Lemma 6, two curves corresponding to different values of $r$ are not $k$-isomorphic. If $k$ is a finite field, there is a single value of $r$: any choice of $r \in k - \text{AS}(k)$.

**Proposition 9.** Let $r \in k - \text{AS}(k)$ be fixed.

1. For any $(a, b, c, d), (a', b', c', d') \in \mathcal{N}$, the curves $C_{a'b'c'd'}^r, C_{abcd}^r$ are $k$-isomorphic if and only if $(a', b', c', d')$ coincides with one of the following elements in the orbit of $(a, b, c, d)$ under the action of $\Gamma$:

$$(a, b, c, d)^\gamma = \begin{cases} (a, b, c, d), & \text{if } \gamma(x) = x, \\ (a, b, b + c, d + a), & \text{if } \gamma(x) = 1 + x. \end{cases}$$

2. For any $(a, b, c, d) \in \mathcal{N}$, we have a split exact sequence:

$$1 \longrightarrow \langle \iota \rangle \longrightarrow \text{Aut}_k(C_{abcd}^r) \longrightarrow \Gamma_{abcd} \longrightarrow 1,$$

$\Gamma_{abcd}$ being the isotropy group of $(a, b, c, d)$ under the action of $\Gamma$:

$$\Gamma_{abcd} \simeq \begin{cases} C_2, & \text{if } b = 0, \ a \in \text{AS}(k), \\ 1, & \text{otherwise}. \end{cases}$$

**Proof.** We can argue as in the proof of Proposition 8. When $b = 0$ and $a = w + w^2$ for some $w \in k$, it is obvious that the choice $v_{u, \gamma} = w$, for $\gamma(x) = 1 + x$, determines a homomorphism from $\Gamma_{u(x)}$ to $k$. \qed
Case \((1,1,1)\)-cubic

We fix a system of representatives, \(\text{Cub}_k\), of cubic extensions of \(k\) modulo isomorphism. By Lemma 7, we can identify \(\text{Cub}_k\) with a subset of \(k\) containing representatives \(s \in k\) of cubic irreducible polynomials of the type \(x^3 + sx + s\).

Let \(\mathcal{N} = ((k \times k \times k) - \{(0,0,0)\}) \times (k/\text{AS}(k))\). Any curve with ramification divisor of this type is \(k\)-isomorphic to one of the curves \(C_{abcd}^s\) with quasi-affine model:

\[
y^2 + y = \frac{ax^2 + bx + c}{x^3 + sx + s} + d, \quad (a, b, c, d) \in \mathcal{N}.
\]

For these curves, \(W = 2[\theta] + 2[\theta'] + 2[\theta'']\), where \(\theta, \theta', \theta'' \in \bar{k}\) are the roots of the irreducible polynomial \(x^3 + sx + s\). The set \(Z_{\infty}\) contains the three Weierstrass points of the curve, which are defined over the splitting field of this polynomial, together with two points, \(P_{\infty}, P'_{\infty}\), conjugate by the hyperelliptic involution. These two points lie over the point with projective coordinates \((x, y, z) = (1, 0, 0)\), which is an ordinary double point with tangents \(y^2 + yz + dz^2 = 0\); thus, they are defined over \(k\) if \(d \in \text{AS}(k)\), otherwise they are defined over the quadratic extension determined by \(d\).

By Lemma 6 two curves corresponding to different values of \(s\) are not \(k\)-isomorphic. If \(k\) is a finite field, there is a single value of \(s\): any choice of \(s \in k\) such that \(x^3 + sx + s\) is irreducible.

If the cubic extension \(k(\theta)/k\) is not cyclic, then the isotropy group \(\Gamma := \Gamma_W\) of the set \(\{\theta, \theta', \theta''\}\) under the action of \(\text{PGL}_2(k)\) is trivial, and we have as an immediate consequence of Proposition 5:

**Proposition 10.** Let \(s \in \text{Cub}_k\) correspond to a non-cyclic cubic extension. Then,

1. For any \((a, b, c, d), (a', b', c', d') \in \mathcal{N}\), the curves \(C_{a'b'c'd'}^s, C_{abcd}^s\) are \(k\)-isomorphic if and only if \((a', b', c', d') = (a, b, c, d)\).
2. For any \((a, b, c, d) \in \mathcal{N}\), we have \(\text{Aut}_k(C_{abcd}^s) = \langle \iota \rangle\).

In the cyclic case, and by Lemma 7, we have:

\[
\Gamma = \{x, \frac{wx + s}{x + w + 1}, \frac{(1 + w)x + s}{x + w} \} \simeq C_3,
\]

where \(s = 1 + w + w^2\) for some \(w \in k\). In order to compute the action of \(\Gamma\) we need the following observation, whose proof is straightforward:
LEMMA 11. For any $w \in k$, let us consider the matrix

$$M(w) = \begin{pmatrix}
  w^2 + w^3 & 1 + w^3 & 1 + w + w^5 & w^2 \\
  1 + w + w^2 & w^2 & 1 + w^2 + w^4 & w \\
  1 + w & w & w^3 & 1 \\
  0 & 0 & 0 & 1
\end{pmatrix}.$$ 

Then, $M(w + 1) = M(w)^2$ and $M(w)^3 = I_3$. Moreover, if we let $M(w)$ operate on the right over $N$ just by matrix multiplication:

$$(a, b, c, d)M(w) := (a, b, c, d)M(w),$$

then, the fixed points of this action are $(a, a, a(w + w^2), d)$, for arbitrary $a \in k^*$ and $d \in k/\text{AS}(k)$.

PROPOSITION 12. Let $s \in k$ be such that $x^3 + sx + s$ is irreducible and $s = 1 + w + w^2$ for some $w \in k$. Then,

1. For any $(a, b, c, d), (a', b', c', d') \in N$, the curves $C_{a'b'c'd'}^s$, $C_{abcd}^s$ are $k$-isomorphic if and only if $(a', b', c', d')$ coincides with one of the following elements in the orbit of $(a, b, c, d)$ under the action of $\Gamma$:

$$(a, b, c, d)\gamma = \begin{cases} (a, b, c, d), & \text{if } \gamma(x) = x, \\
(a, b, c, d)M(w), & \text{if } \gamma(x) = \frac{wx + s}{x + w + 1}, \\
(a, b, c, d)M(w + 1), & \text{if } \gamma(x) = \frac{(1+w)x + s}{x + w}. \end{cases}$$

2. For any $(a, b, c, d) \in N$, we have a split exact sequence:

$$1 \longrightarrow \langle \iota \rangle \longrightarrow \text{Aut}_k(C_{abcd}^s) \longrightarrow \Gamma_{abcd} \longrightarrow 1,$$

$$\Gamma_{abcd}$$ being the isotropy group of $(a, b, c, d)$ under the action of $\Gamma$:

$$\Gamma_{abcd} \simeq \begin{cases} C_3, & \text{if } a = b, c = a(1 + s), \\
1, & \text{otherwise}. \end{cases}$$

Proof. We can argue as in the proof of Proposition 8. The computation of $\Gamma_{abcd}$ is consequence of Lemma 11. When $a = b, c = a(1 + s)$ we can choose $\nu_u: \Gamma_{u(x)} \longrightarrow k$ to be the trivial homomorphism.

For instance, when $a = b, c = a(1 + s)$, the subgroup of $\text{Aut}_k(C_{abcd}^s)$ generated by $(x, y) \mapsto (\frac{wx + s}{x + w + 1}, y)$ is cyclic of order three.

If a cubic extension admits a generator with minimal polynomial over $k$ of the type $x^3 + s$, it might be easier to work with models $C_{abcd}^s$ of the type:

$$y^2 + y = \frac{ax^2 + bx + c}{x^3 + s} + d, \quad (a, b, c, d) \in N.$$ 

If the cubic extension is not cyclic, Proposition 10 is valid for these models as well. If a cyclic cubic field admits a cyclic generator, then $\mathbb{F}_4 \subseteq k$. In this case, Proposition 12 might be substituted by:
PROPOSITION 13. Assume that $F_4 \subseteq k$ and let $s \in k$ be such that $x^3 + s$ is irreducible. Then,

1. For any $(a, b, c, d), (a', b', c', d') \in \mathcal{N}$, the curves $C_{a'b'c'd'}^s$, $C_{abcd}^s$ are $k$-isomorphic if and only if
   
   $$(a', b', c', d') = (e^2 a, eb, c, d)$$
   
   for some $e \in \mu_3$.

2. For any $(a, b, c, d) \in \mathcal{N}$, we have an split exact sequence (3) where,
   
   $\Gamma_{abcd} \simeq \begin{cases} C_3, & \text{if } a = b = 0, \\ 1, & \text{otherwise.} \end{cases}$

Case (1,3)

Let $\mathcal{N} = k^* \times k \times k^* \times (k/ \text{AS}(k))$. Any curve with ramification divisor of this type is $k$-isomorphic to one of the curves $C_{abcd}$ with quasi-affine model:

$$y^2 + y = ax^3 + bx^2 + \frac{c}{x} + d, \quad (a, b, c, d) \in \mathcal{N}.$$ 

For these curves, $W = 4[\infty] + 2[0]$, and $Z_\infty$ contains only the two Weierstrass points of the curve, which are defined over $k$. The subgroup $\Gamma := \Gamma_W$ is, in this case, $\Gamma = \{\lambda x \mid \lambda \in k^*\} \simeq k^*$. 

PROPOSITION 14.

1. For any $(a, b, c, d), (a', b', c', d') \in \mathcal{N}$, the curves $C_{a'b'c'd'}$, $C_{abcd}$ are $k$-isomorphic if and only if $(a', b', c', d')$ coincides with one of the following elements in the orbit of $(a, b, c, d)$ under the action of $\Gamma$:
   
   $$(a, b, c, d)^{\lambda x} = (\lambda^3 a, \lambda b, \lambda^{-1} c, d), \quad \lambda \in k^*.$$ 

2. For any $(a, b, c, d) \in \mathcal{N}$, we have $\text{Aut}_k(C_{abcd}) = \langle i \rangle$.

Case (5)

This case is known as the supersingular case, since the jacobian of these curves splits up to isogeny as the square of a supersingular elliptic curve over $k$. This case has been thoroughly studied by van der Geer and van der Vlugt in the finite field case (cf. [10, 11]).

Let $\mathcal{N} = k^* \times k \times k \times (k/ \text{AS}(k))$. Any curve with ramification divisor of this type is $k$-isomorphic to one of the curves $C_{abcd}$ with quasi-affine model:

$$y^2 + y = ax^5 + bx^4 + cx^3 + d, \quad (a, b, c, d) \in \mathcal{N}.$$  

(4)
For these curves, $W = 6(\infty)$, and $Z_{\infty}$ contains only the Weierstrass point of the curve, which is defined over $k$. The subgroup $\Gamma := \Gamma_W$ is now the affine group,

$$\Gamma = \{ \lambda x + \nu \mid (\lambda, \nu) \in k^* \times k \} \simeq k^* \times k.$$ 

**Lemma 15.** Given $a, b, c \in k$, $a \neq 0$, consider the linear separable polynomial $E_{ac}(x) = a^4 x^{16} + c^4 x^8 + c^2 x^2 + ax \in k[x]$. Then, the subset $\Gamma_{abc} := \{(\lambda, \nu) \in k^* \times k \mid E_{ac}(\nu) = b(1 + \lambda)\}$ is a subgroup of $k^* \times k$ and the map

$$\delta_{abc}: \Gamma_{abc} \longrightarrow k/\text{AS}(k), \quad (\lambda, \nu) \longmapsto a \nu^5 + b \nu^4 + c \nu^3,$$

is a group homomorphism.

**Proof.** Given any $(\lambda, \nu), (\lambda', \nu') \in \Gamma_{abc} \subseteq k^* \times k$, their product is $(\lambda \lambda', \nu + \nu')$. If $c \neq 0$, $\Gamma_{abc}$ is a subgroup by the linearity of $E_{ac}(x)$. If $c = 0$ we have also:

$$E_{ac}(\lambda \nu' + \nu) = \lambda E_{ac}(\nu') + E_{ac}(\nu) = \lambda b(1 + \lambda') + b(1 + \lambda) = b(1 + \lambda \lambda').$$

The assertion concerning $\delta_{abc}$ is left to the reader. $\square$

We note that if $c \neq 0$ or $\mu_5(k) = \{1\}$ then $\text{Ker}(\delta_{abc})$ is isomorphic to a subgroup of $C_2 \times C_2 \times C_2 \times C_2$.

As in the previous cases, Proposition 5 leads to:

**Proposition 16.**

1. For any $(a, b, c, d), (a', b', c', d') \in \mathcal{N}$, the curves $C_{a'b'c'd'}, C_{abcd}$ are $k$-isomorphic if and only if $(a', b', c', d')$ coincides with one of the following elements in the orbit of $(a, b, c, d)$ under the action of $\Gamma$:

$$(a, b, c, d) \lambda x + \nu = \Lambda b + E_{ac}(\nu), \Lambda c, a \nu^5 + b \nu^4 + c \nu^3 + d).$$

2. The isotropy group of any $(a, b, c, d) \in \mathcal{N}$ under the action of $\Gamma$ is $\text{Ker}(\delta_{abc})$. We have an exact sequence (which rarely splits):

$$1 \longrightarrow \langle \iota \rangle \longrightarrow \text{Aut}_k(C_{abcd}) \longrightarrow \text{Ker}(\delta_{abc}) \longrightarrow 1. \quad (5)$$
1.4. NUMBER OF CURVES OF GENUS TWO OVER A FINITE FIELD

Let \( k = \mathbb{F}_q \) be a finite field with \( q = 2^m \) elements. In this paragraph we find an explicit formula for the number of curves of genus two defined over \( k \) up to \( k \)-isomorphism. Actually, we count how many curves there are for each type of ramification divisor. After the results of paragraph 1.3, the number of curves having a concrete type of ramification divisor coincides with the numbers of orbits of the action of certain finite group \( \Gamma := \Gamma_W \) on certain finite set \( \mathcal{N} \). Denoting by \( \Gamma \backslash \mathcal{N} \) the set of these orbits, we can count its cardinality by a well-known formula:

\[
|\Gamma \backslash \mathcal{N}| = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |\mathcal{N}_\gamma| = \sum_{\gamma \in C} \frac{|\mathcal{N}_\gamma|}{|\Gamma_\gamma|},
\]

where \( C \) is a system of representatives of all conjugation classes of \( \Gamma \) and 

\[
\mathcal{N}_\gamma = \{ x \in \mathcal{N} | x^\gamma = x \}, \quad \Gamma_\gamma = \{ \rho \in \Gamma | \rho \gamma = \gamma \rho \}.
\]

From now on, a summand \([a]_{b|m}\) in a formula means “add \( a \) if \( b \) divides \( m \)”.

**THEOREM 17.** There are \( 2q^3 + q^2 + q - 2 + [8]_{4|m} \) \( k \)-isomorphism classes of curves of genus two defined over \( \mathbb{F}_q \). The number of curves with a given type of ramification divisor is given in the following table:

| Type          | Formula                                      |
|---------------|----------------------------------------------|
| \((1,1,1)\)-split | \( \frac{1}{6}q(q-1)(2q-1) \)              |
| \((1,1,1)\)-quadratic | \( \frac{1}{2}(q-1)(2q^2 + q - 4) \)         |
| \((1,1,1)\)-cubic   | \( \frac{1}{3}(2q^3 + 4q - 6) \)            |
| \((1,3)\)          | \( 2q(q-1) \)                                 |
| \((5)\)            | \( 4q - 2 + [8]_{4|m} \)                     |

**Proof.** By (6), we need only to compute \( |\mathcal{N}_\gamma|, |\Gamma_\gamma| \) in all cases.

In the case \((1,1,1)\)-split, \( \Gamma \simeq S_3 \) and we can choose \( C = \{ 1, \tau, \sigma \} \), where \( \tau(x) = 1 + x, \sigma(x) = 1/(1 + x) \). Clearly, \( \Gamma_\tau = \{ 1, \tau \}, \Gamma_\sigma = \{ 1, \tau, \sigma^2 \} \), and

\[
\mathcal{N}_\tau = \{ (a, b, b, d) | a \in \text{AS}(k) - \{ 0 \}, b \in k^*, d \in k/\text{AS}(k) \},
\]

\[
\mathcal{N}_\sigma = \{ (a, a, a, d) | a \in k^*, d \in k/\text{AS}(k) \}.
\]

Hence, \( |\mathcal{N}_\tau| = |\mathcal{N}_\sigma| = 2(q-1)^3 \), \( |\mathcal{N}_\tau| = 2(q-1)(q/2 - 1) \) and \( |\mathcal{N}_\sigma| = 2(q-1) \).

In the case \((1,1,1)\)-quadratic, \( \Gamma = \{ 1, \tau \} \), where \( \tau(x) = 1 + x \); hence,

\[
\mathcal{N}_\tau = \{ (a, 0, c, d) | a \in \text{AS}(k) - \{ 0 \}, c \in k^*, d \in k/\text{AS}(k) \}.
\]
Therefore, $|\mathcal{N}_1| = 2(q - 1)(q^2 - 1)$ and $|\mathcal{N}_\tau| = 2(q - 1)(q/2 - 1)$.

In the case (1,1,1)-cubic, $\Gamma = \{1, \gamma, \gamma^2\} \simeq C_3$ and

$$\mathcal{N}_\gamma = \mathcal{N}_\gamma^2 = \{(a, a, a(1 + s), d) \mid a \in k^*, d \in k/\text{AS}(k)\}.$$ 

Hence, $|\mathcal{N}_1| = 2(q^3 - 1)$ and $|\mathcal{N}_\tau| = |\mathcal{N}_\tau^2| = 2(q - 1)$.

In the case (1,1,1)-cubic, $\Gamma = \{1, \gamma, \gamma^2\} \simeq C_3$ and $N\gamma = N\gamma^2 = \{(a, a, a(1 + s), d) \mid a \in k^*, d \in k/\text{AS}(k)\}$.

In the case (1,3), $\Gamma \simeq k^*$ and for any $\lambda \in k^*$, $\lambda \neq 1$, we have $N\lambda = \emptyset$.

Therefore, only $|\mathcal{N}_1| = 2q^2(q - 1)$ counts.

For any given $a \neq 0$, there are only two values of $c$ such that $E_{ac}(1) = 0$; these are: $c_0 = \sqrt{a + a^2}$, $c_1 = 1 + \sqrt{a + a^2}$. Hence,

$$\mathcal{N}_\tau = \{(a, b, c, d) \mid a \in k^*, c \in \{c_0, c_1\}, b \in a + c + \text{AS}(k), d \in k/\text{AS}(k)\},$$

and $|\mathcal{N}_\tau| = 2q(q - 1)$. Finally, if $\lambda \not\in \mu_5(k)$, we have $\mathcal{N}_{\sigma_\lambda} = \emptyset$, whereas for $\lambda \in \mu_5(k)$, $\lambda \neq 1$, we have $\mathcal{N}_{\sigma_\lambda} = \{(a, 0, 0, d) \mid a \in k^*, d \in k/\text{AS}(k)\}$, with cardinality $2(q - 1)$, whereas $|\mathcal{N}_1| = 2(q - 1)q^2$.

Also, from the results of paragraph 1.3 we can obtain a mass formula for the number of curves of genus two:

**THEOREM 18.** Let $[C]$ run on the $k$-isomorphism classes of smooth projective curves $C$ of genus two defined over $k$. Then,

$$\sum_{[C]} |\text{Aut}_k(C)|^{-1} = q^3.$$

More precisely, the partial weighted sums $\sum_{[C]} |\text{Aut}_k(C)|^{-1}$, letting $[C]$ run on the $k$-isomorphism classes of curves having a fixed type of ramification divisor, are given in the following table:

| Type            | Weight |
|-----------------|--------|
| $(1,1,1)$-split  | $\frac{1}{6}(q - 1)^3$ |
| $(1,1,1)$-quadratic | $\frac{1}{2}(q - 1)(q^2 - 1)$ |
| $(1,1,1)$-cubic   | $\frac{1}{3}(q^3 - 1)$ |
| $(1,3)$            | $q^2 - q$ |
| $(5)$             | $q$ |
**Proof.** Each partial sum can be computed as

\[
\sum_{(a,b,c,d) \in \Gamma \setminus \mathcal{N}} \frac{1}{|\operatorname{Aut}_k(C_{abcd})|} = \sum_{(a,b,c,d) \in \Gamma \setminus \mathcal{N}} \frac{1}{2|\Gamma_{abcd}|} = \sum_{(a,b,c,d) \in \mathcal{N}} \frac{1}{2|\Gamma|} = \frac{|\mathcal{N}|}{2|\Gamma|},
\]

for certain finite set \( \mathcal{N} \) and finite group \( \Gamma \) acting on \( \mathcal{N} \), specified in paragraph 1.3.

Both the total and the weighted number of supersingular curves had been already obtained by van der Geer and van der Vlugt in [11].

### 2. Geometric invariants of curves of genus two

In this section we recall from [5] the definition of invariants that classify curves of genus two over \( \overline{k} \) up to isomorphism and we compute explicitly these invariants in terms of the rational models introduced in section 1. In paragraph 2.2 we deal with rationality questions concerning the curve \( C \), its isomorphism class and the values of their geometric invariants.

A \( \overline{k} \)-isomorphism between two curves defined over \( k \) is simply called an *isomorphism* and we denote by \( \operatorname{Aut}(C) := \operatorname{Aut}_{\overline{k}}(C) \) the full group of automorphisms of a curve \( C \).

#### 2.1. Igusa invariants

Let \( C \) be a curve of genus two defined over \( \overline{k} \). According to the different possibilities for the ramification divisor of \( C \), the curve admits a quasi-affine model of the type:

\[
\begin{align*}
    y^2 + y &= ax + \frac{b}{x} + \frac{c}{x+1}, \quad abc \neq 0, \quad \text{type (1,1,1)}, \\
    y^2 + y &= ax^3 + bx + \frac{c}{x}, \quad ac \neq 0, \quad \text{type (1,3)}, \\
    y^2 + y &= ax^5 + cx^3, \quad a \neq 0, \quad \text{type (5)},
\end{align*}
\]

and we define, respectively,

\[
\begin{align*}
    j_1(C) &:= abc, \quad j_2(C) := ab + bc + ca, \quad j_3(C) := a + b + c, \\
    j_1(C) &:= 0, \quad j_2(C) := ac^3, \quad j_3(C) := bc, \\
    j_1(C) &:= 0, \quad j_2(C) := 0, \quad j_3(C) := c^5 / a^3.
\end{align*}
\]
DEFINITION 19. For any curve $C$ of genus two defined over $\overline{k}$ we define the $j$-invariant of $C$ as:

$$j(C) := (j_1(C), j_2(C), j_3(C)) \in \overline{k}^3.$$ 

By applying the results of paragraph 1.3 to the field $\overline{k}$ we get:

THEOREM 20 (Igusa).

1. The invariant $j(C)$ depends only on the isomorphism class of $C$.

2. Two curves of genus two defined over $\overline{k}$ are isomorphic if and only if they have the same $j$-invariant.

3. The following table sums up the ramification type of $C$ and the possible structures of $\text{Aut}(C)$ in terms of the invariant $j(C) = (j_1, j_2, j_3)$:

| Type   | Condition | $\text{Aut}(C)$ | Condition |
|--------|-----------|------------------|-----------|
| $(1,1,1)$ | $j_1 \neq 0$ | $C_2$ | $j_1 \neq j_2 j_3$ |
|        |           | $C_2 \times C_2$ | $j_1 = j_2 j_3, \; j_1 \neq j_3^3$ |
|        |           | $C_2 \times S_3$ | $j_1 = j_2 j_3, \; j_1 = j_2 j_3$ |
| $(1,3)$ | $j_1 = 0, \; j_2 \neq 0$ | $C_2$ | $-$ |
| $(5)$   | $j_1 = j_2 = 0$ | $M_{32}$ | $j_3 \neq 0$ |
|        |           | $M_{160}$ | $j_3 = 0$ |

The groups $M_{32}, M_{160}$ sit in the middle of the non-split exact sequence (5) and we shall study their structure more closely in section 3. Over an algebraically closed field, the group $\text{Ker}(\delta_{abc})$ of (5) is clearly:

$$\text{Ker}(\delta_{abc}) = \begin{cases} \text{Ker} E_{ac} \simeq C_2 \times C_2 \times C_2 \times C_2, & \text{if } c \neq 0 \\ \mu_5 \times \text{Ker} E_{ac}, & \text{if } c = 0. \end{cases}$$

For $j_1 \neq 0$, the first three conditions of the last column of the above table are respectively equivalent to the fact that the polynomial $x^3 + j_3 x^2 + j_2 x + j_1 \in \overline{k}[x]$ is separable, has a double root or a triple root.

We compute now, in terms of the rational models given in section 1, the $j$-invariant of a curve of genus two defined over $k$.

In all cases, the curve determined by the parameters $(a, b, c, d)$ is isomorphic to the curve determined by $(a, b, c, 0)$; hence, the fourth parameter $d$ may be ignored in the computation of the $j$-invariant. Moreover, in the supersingular case, the curve (4) determined by the parameters $(a, b, c, d)$ is isomorphic to the curve determined by $(a, 0, c, 0)$. After these remarks, it only remains to treat the cases $(1,1,1)$-quadratic or cubic.
LEMMA 21.

1. For a curve $C$ of type $(1,1,1)$-quadratic with quasi-affine model given by equation (2), we have:
   \[ j_1(C) = a(rb^2 + bc + c^2), \quad j_2(C) = ab + rb^2 + bc + c^2, \quad j_3(C) = a + b. \]

2. Let $s, t \in k$ be such that the polynomial $x^3 + tx + s$ is irreducible in $k[x]$. For a curve $C$ of type $(1,1,1)$-cubic with quasi-affine model
   \[ y^2 + y = \frac{ax^2 + bx + c}{x^3 + tx + s}, \]  
   the $j$-invariant of $C$ is given by:
   \[ j_1(C) = \frac{1}{s^3}(a^2(s^2a + stb + t^2c) + bc(tb + sa) + sb^3 + c^3), \]
   \[ j_2(C) = \frac{1}{s^4}((a^2t^2 + abs + c^2)(s^2 + t^3) + cst(tb + sa) + b^2t^4), \]
   \[ j_3(C) = \frac{1}{s}(ta + c). \]

Proof. Assume that $C$ is given by equation (2). If $\theta, \theta'$ are the roots of $x^2 + x + r$, we have that
   \[ \frac{bx + c}{x^2 + x + r} = \frac{b\theta + c}{x + \theta} + \frac{b\theta' + c}{x + \theta'}, \]
   hence, the mapping $(x, y) \mapsto (x + \theta, y)$ is an isomorphism from $C$ to the curve $C'$ given by an split equation $y^2 + y = a' + (b'/x) + (c'/x+1) + d'$, where
   \[ a' = a, \quad b' = b\theta + c, \quad c' = b\theta' + c. \]
   It is easily checked now that $j(C) = j(C')$ has the claimed values.

Assume now that $C$ is given by equation (7). If $\theta, \theta', \theta''$ are the roots of $x^3 + tx + s$, we have that
   \[ \frac{ax^2 + bx + c}{x^3 + tx + s} = \frac{A}{x + \theta} + \frac{A'}{x + \theta'} + \frac{A''}{x + \theta''}, \]
   where $A, A', A'' \in \overline{k}$ are given by
   \[ \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} = \frac{1}{s} \begin{pmatrix} \theta^3 & \theta^2 & \theta \\ \theta^3 & \theta^2 & \theta' \\ \theta^2 & \theta' & \theta'' \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \]
   Hence, the mapping $(x, y) \mapsto \left( \frac{\theta'}{\theta + \theta''}, y \right)$ is an isomorphism from $C$ to the curve $C'$ given by the equation $y^2 + y = a' + (b'/x) + (c'/x+1) + d'$, where
   \[ (a', b', c') = \frac{1}{s} (\theta^2 A, (\theta')^2 A', (\theta'')^2 A''). \]
Therefore, the values of \( j_i(C) = j_i(C') \), \( i = 1, 2, 3 \), are the coefficients of the minimal polynomial of \( (a\theta^5 + b\theta^4 + c\theta^3)/s^2 \) over \( k \).

\[ \square \]

### 2.2. Field of Moduli and Field of Definition

Let \( C \) be a curve of genus two defined over \( \overline{k} \). An intermediate field \( K \) of \( \overline{k}/k \) is said to be a field of moduli for \( C \) if \( C \) is isomorphic to \( \sigma C \) for all \( \sigma \in G_K \). By Theorem 20,

\[ C \cong \sigma C \iff j(C) = j(C) \iff j(C) = \sigma j(C), \]

so that \( K \) is a field of moduli for \( C \) if and only if \( j(C) \in K^3 \).

If the curve \( C \) is defined over \( k \), then clearly \( j(C) \in k^3 \) and \( k \) is a field of moduli for \( C \). We want to see the converse: given a curve of genus two defined over \( \overline{k} \), having \( k \) as a field of moduli, then \( C \) is isomorphic to a curve defined over \( k \). This result is false for curves of genus two over fields of characteristic different from 2, with non-trivial two-torsion in the Brauer group, but the situation is well understood (cf. [7], [2]).

**THEOREM 22.** For any given value of \( j \in k^3 \), there exists a curve of genus two \( C \) defined over \( k \) such that \( j(C) = j \).

Equivalently, we could formulate this result in the following form:

**COROLLARY 23.** Let \( C \) denote the quotient set of all curves of genus two defined over \( k \), classified up to \( \overline{k} \)-isomorphism. Then, the \( j \)-invariant sets a bijection between \( C \) and \( k^3 \).

For \( k \) a finite field, Theorem 22 follows immediately from Theorem 18 and the following result of van der Geer and van der Vlugt (cf. [11, Proposition 5.1]):

\[ \sum_{[C]} |\text{Aut}_k(C)|^{-1} = |j(C)|, \]

where \([C]\) runs over all \( k \)-isomorphism classes of curves of genus two defined over \( k \). We give now a direct proof of Theorem 22, for general \( k \), essentially by exhibiting rational models of curves of genus two with prescribed \( j \)-invariant.

**Proof of Theorem 22.** Let \( j = (j_1, j_2, j_3) \in k^3 \) be given. We find a quasi-affine model for a curve \( C \) defined over \( k \) such that \( j(C) = j \).

If \( j_1 = j_2 = 0 \), the following supersingular curves have \( j(C) = j \):

\[ y^2 + y = \sqrt{j_3} x^5 + \sqrt{j_3} x^3, \quad \text{if} \quad j_3 \neq 0, \]

\[ y^2 + y = x^5, \quad \text{if} \quad j_3 = 0. \]
If $j_1 = 0$, $j_2 \neq 0$, then the following curve of type $(1,3)$ has $j(C) = j$:

$$y^2 + y = j_2x^3 + j_3x + \frac{1}{x}.$$  

If $j_1 \neq 0$, consider the polynomial $F(x) = x^3 + j_3x^2 + j_2x + j_1 \in k[x]$. We proceed in this case according to the arithmetic structure of the set of roots of $F(x)$. If $F(x)$ splits in $k[x]$, $F(x) = (x + a)(x + b)(x + c)$, then the curve $C$ of type $(1,1,1)$-split with quasi-affine model:

$$y^2 + y = ax + \frac{b}{x} + \frac{c}{x + 1},$$

has $j(C) = j$. If $F(x) = (x + a)Q(x)$ for some quadratic irreducible polynomial $Q(x) = x^2 + ux + v$, then the curve $C$ of type $(1,1,1)$-quadratic with quasi-affine model:

$$y^2 + y = ax + \frac{u}{x^2 + x + (v/u^2)},$$

has $j(C) = j$, by Lemma 21. Finally, assume that $F(x)$ is irreducible in $k[x]$; let $\omega \in \overline{k}$ be a root of $F(x)$ and $K = k(\omega)$. The element $\theta = \omega + j_3$ has minimal polynomial $x^3 + tx + s$ over $k$, where $t = j_2 + j_3^2$ and $s = j_1 + j_2j_3$. It is easy to check that $\theta^2, \theta^4, \theta^5$ is a $k$-basis of $K$; hence, there exist unique $a, b, c \in k$ such that

$$a\theta^5 + b\theta^4 + c\theta^3 = s^2 \omega. \quad (8)$$

For these values of $a, b, c$, the curve $C$ of type $(1,1,1)$-cubic with quasi-affine model given by (7) has $j(C) = j$, by the proof of Lemma 21. For the sake of completeness, we give an explicit equation (7) in terms of $j_1, j_2, j_3$ in this last case. If $j_2 = j_3^2$, we can take $t = 0$, $s = j_1 + j_2j_3$ and

$$a = 0, \quad b = j_1 + j_2j_3, \quad c = j_3(j_1 + j_2j_3).$$

If $j_2 \neq j_3^2$, taking $t = s = \frac{(j_2 + j_3^2)^3}{(j_1 + j_2j_3)^2}$, the element $\theta = \frac{j_2 + j_3^2}{j_1 + j_2j_3}(\omega + j_3)$ has minimal polynomial $x^3 + sx + s$ over $k$ (see the proof of Lemma 7). For this choice of $t, s, \theta$, the values:

$$a = b = \frac{(j_1 + j_3^2)(j_2 + j_3^2)^2}{(j_1 + j_2j_3)^2},$$

$$c = \frac{(j_2 + j_3^2)^3(j_1(j_2 + j_3^2)^2 + j_3(j_1 + j_3^2)^2)}{(j_1 + j_2j_3)^4},$$

satisfy (8).
3. Arithmetic invariants of curves of genus two

Let $C$ be a curve of genus two defined over $k$. A twist of $C$ over $k$ is any curve of genus two, defined over $k$, which is isomorphic to $C$. We denote by $\text{Tw}(C/k)$ the quotient set of all twists of $C$ over $k$, classified up to $k$-isomorphism. There is a well-known isomorphism of pointed sets,

$$\text{Tw}(C/k) \sim H^1(G_k, \text{Aut}(C)),$$

(9)

sending any twist $\phi: C' \sim C$ to the 1-cocycle $\{\phi^\sigma \phi^{-1}\}_{\sigma \in G_k}$. For details concerning non-abelian cohomology we address the reader to [8, Ch.VII, Annexe].

In this section we obtain an explicit description of the set $\text{Tw}(C/k)$ of twists of a given curve $C$, in terms of the rational models of section 1. This is achieved by an explicit computation of $H^1(G_k, \text{Aut}(C))$ for any possible structure of $\text{Aut}(C)$. The parameters describing the pointed set $H^1(G_k, \text{Aut}(C))$ are what we call the arithmetic invariants of the curve. In this way, we obtain an explicit parameterization of all $k$-isomorphism classes of curves of genus two, each class being determined by a couple of invariants, one geometric and the other arithmetic. For $k$ a finite field, this provides formulas for the number of curves whose full group of automorphisms has a concrete structure. For finite fields of odd characteristic an analogous result has been obtained in [1].

As a by-product, we are able to implement an algorithm that builds up, almost directly, a faithful and complete system of representatives of $k$-isomorphism classes of curves of genus two over a finite field $k$ of even characteristic. This algorithm was used in [6] to carry out a numerical exploration about the existence of jacobians of curves of genus two in a certain isogeny class of abelian surfaces over $k$.

3.1. Hyperelliptic twists

Let $C$ be a curve of genus two defined over $k$, with hyperelliptic involution $\iota$. By Artin-Schreier theory, we have an isomorphism:

$$H := k/\text{AS}(k) \simeq H^1(G_k, \langle \iota \rangle) = \text{Hom}(G_k, \langle \iota \rangle).$$

(10)

Thus, elements in $H$ can be thought as 1-cocycles of $G_k$ with values in $\text{Aut}(C)$ and they furnish twists of $C$ by (9). We obtain in this way an action of the group $H$ on the set of curves of genus two defined over $k$. The curves in the orbit of $C$ under this action are called hyperelliptic twists of $C$, and we denote by $\text{Tw}^\iota(C/k)$ the quotient set of this orbit modulo $k$-isomorphism.

If $C$ admits a quasi-affine model $y^2 + y = u(x)$ for certain $u(x) \in k(x)$, the hyperelliptic twist of $C$ corresponding to $d \in H$ is the curve $C'$
with model $y^2 + y = u(x) + d$. If $v \in \overline{k}$ satisfies $v^2 + v = d$, the mapping $(x, y) \mapsto (x, y + v)$ is an isomorphism between $C'$ and $C$, defined over the quadratic (or trivial) extension of $k$ determined by $d$.

Any isomorphism, $\phi: C' \isom C$, between two curves of genus two commutes with the respective hyperelliptic involutions: $\iota \phi = \phi \iota$. Therefore, the action of $H$ is compatible with isomorphisms and with $k$-isomorphisms. In particular, we have an induced action of $H$ on the set $\text{Tw}(C/k)$. For any $C' \in \text{Tw}(C/k)$ we shall denote by $H_{C'}$ the isotropy group of $C'$ under this action; note that the orbit of $C'$ can be identified with $\text{Tw}^0(C'/k)$. In order to describe $\text{Tw}(C/k)$ we need to compute the set of orbits and all the isotropy subgroups.

Consider the central exact sequence of $G_k$-groups:

$$1 \longrightarrow \langle \iota \rangle \longrightarrow A \longrightarrow A' \longrightarrow 1,$$

where $A := \text{Aut}(C)$, $A' := A/\langle \iota \rangle$ denote respectively the full group and the reduced group of automorphisms of $C$. Since $H^2(G_k, C_2) = 0$, the cohomology exact sequence of (11) induces an exact sequence of pointed sets,

$$1 \longrightarrow H^1(G_k, \langle \iota \rangle)/\delta((A')^{G_k}) \longrightarrow H^1(G_k, A) \longrightarrow H^1(G_k, A') \longrightarrow 1.$$

The connecting homomorphism $\delta$ sends any $U' \in (A')^{G_k}$ to the homomorphism

$$\delta(U'): G_k \longrightarrow \langle \iota \rangle, \quad \sigma \mapsto U^\sigma U^{-1},$$

where $U \in A$ is any preimage of $U'$. From (12) we get:

**PROPOSITION 24.** The isotropy group $H_C$ of $C$ under the action of $H$ on $\text{Tw}(C/k)$ corresponds to $\delta((A')^{G_k})$ under the isomorphism $H \simeq H^1(G_k, \langle \iota \rangle)$ of (10). Moreover,

$$\text{Tw}(C/k) = \coprod_{\xi \in H^1(G_k, A')} \text{Tw}^0(C_\xi/k),$$

where $\xi \in H^1(G_k, A)$ is any choice of a preimage of $\xi'$ and $C_\xi$ is the twist of $C$ corresponding to $\xi$ by (9).

**Proof.** Since $\iota$ is central in $A$, the group $H^1(G_k, \langle \iota \rangle)$ acts on the set $H^1(G_k, A)$ by multiplication and the orbits are the subsets of classes having the same image in $H^1(G_k, A')$. On the other hand, it is easy to check that the twists corresponding to the orbit of one class $\xi \in H^1(G_k, A)$ are precisely those in $\text{Tw}^0(C_\xi/k)$. \hfill \Box

By the Proposition, the orbits of $\text{Tw}(C/k)$ under the action of $H$ are parameterized by $H^1(G_k, A')$ and the isotropy group of each $C_\xi$ is
$H_{\xi} \simeq \delta(A'_Gk)$, where $A'_G$ is the reduced group of automorphisms of the curve $C_{\xi}$. These curves $C_{\xi}$ may have in principle different isotropy subgroups. If $A$ is abelian, we get a more uniform description of $\text{Tw}(C/k)$, since we have then an isomorphism of pointed sets:

$$H^1(G_k, A) \simeq H^1(G_k, \langle \iota \rangle) / \delta((A')^Gk) \times H^1(G_k, A'),$$

and all curves $C_{\xi}$ have the same isotropy subgroup.

It is easy to describe $H_C$ in terms of a rational model of $C$ by applying the results of section 1:

**PROPOSITION 25.** Let $C$ be a curve of genus two defined over $k$ given by a rational model with parameters $(a, b, c, d)$ as in section 1. Then,

1. If $C$ is supersingular, then $H_C = \delta_{abc}(\Gamma_{abc})$ (see Lemma 15).

2. If $C$ is of type $(1,1,1)$-split, two at least of $a, b, c$ are equal and the third coefficient (equal to the former two or not) does not belong to $\text{AS}(k)$, then $H_C \simeq C_2$ is the subgroup generated by this third coefficient.

3. If $C$ is of type $(1,1,1)$-quadratic, $b = 0$ and $a \notin \text{AS}(k)$, then $H_C \simeq C_2$ is the subgroup generated by $a$.

4. In all other cases, $H_C$ is trivial.

### 3.2. Twists of curves of genus two

By Corollary 23, each triple $j = (j_1, j_2, j_3) \in k^3$ can be identified with a unique element in $C$, the quotient set of all curves of genus two defined over $k$ classified up to isomorphism. By Theorem 20, this set $C$ decomposes,

$$C = C_{C_2} \cup C_{C_2C_2} \cup C_{C_2 \times S_3} \cup C_{M_{32}} \cup C_{M_{160}},$$

as the disjoint union of subfamilies gathering all curves with isomorphic automorphism group. In this paragraph we find an explicit description of these subfamilies and of the twists of each curve in the family.

*Curves with $\text{Aut}(C) \simeq C_2$*

This is the generic case. The set of values of the geometric invariants is:

$$j(C_{C_2}) = J := \{(j_1, j_2, j_3) \in k^3 \mid j_1 \neq j_2j_3\} \cup \{(0, j_2, 0) \in k^3 \mid j_2 \neq 0\}.$$
We build up $C_{C_2}$ by choosing for each $j \in J$ the curve with $j(C) = j$ indicated in the proof of Theorem 22. For each curve $C \in C_{C_2}$, $\text{Tw}(C/k) = \text{Tw}^0(C/k)$ is in bijection with the set of arithmetic invariants $H^1(G_k, A) \simeq k/\text{AS}(k)$.

If $k$ is the finite field with $q$ elements, there are $q^3 - q^2 + q - 1$ values of the geometric invariants and the total number of curves with $\text{Aut}(C) \simeq C_2$ is $2(q^3 - q^2 + q - 1)$.

**Curves with $\text{Aut}(C) \simeq C_2 \times C_2$**

The set of values of the geometric invariants is:

$$j(C_{C_2 \times C_2}) = J := \{(j_1, j_2, j_3) \in k^3 \mid j_1 = j_2j_3, \ j_1 \neq 0, j_2 \neq j_3^2\}.$$  

The condition $(j_1, j_2, j_3) \in J$ is equivalent to:

$$x^3 + j_3x^2 + j_2x + j_1 = (x + a)(x + c)^2, \quad a, c \in k^*, \ a \neq c.$$  

Hence, we can make the following choice for $C_{C_2 \times C_2}$:

$$C_{C_2 \times C_2} = \{y^2 + y = ax + \frac{c}{x} + \frac{c}{x+1} \mid a, c \in k^*, \ a \neq c\}.$$  

Let $C \in C_{C_2 \times C_2}$ be fixed and let $w \in \overline{k}$ be such that $w^2 + w = a$. The automorphism group of $C$ is $A = \{1, \iota, U, iU\}$, where $U$ is the non-hyperelliptic involution, $U(x, y) = (x + 1, y + w)$. If $a \in \text{AS}(k)$, then $A$ has trivial $G_k$-action. If $a \notin \text{AS}(k)$, the automorphisms $U$ and $iU$ are exchanged by the action of any $\sigma \in G_k$ with non-trivial image in $\text{Gal}(k(w)/k)$. In both cases $A'$ has trivial action and $H_C \simeq \delta(A')$ is the subgroup of $k/\text{AS}(k)$ generated by the class of $a$. Since $A$ is abelian,

$$H^1(G_k, A) \simeq$$

$$\simeq H^1(G_k, \langle \iota \rangle)/\delta(A') \times H^1(G_k, A') \simeq \frac{k}{\{0, a\} + \text{AS}(k)} \times \frac{k}{\text{AS}(k)}.$$  

A class $\xi' \in H^1(G_k, A') = \text{Hom}(G_k, A')$ is identified with the element $r \in k/\text{AS}(k)$ representing the quadratic or trivial extension of $k$ through which the homomorphism $\xi'$ factorizes. As a preimage $\xi \in H^1(G_k, A)$ we can choose the class corresponding to the twist:

$$C_\xi : \quad y^2 + y = ax + \frac{c}{x^2 + x + r},$$

which is isomorphic to $C$ via

$$(x, y) \mapsto (x + \theta, y + v), \quad \theta^2 + \theta = r, \ v^2 + v = a\theta. \quad (13)$$
We get the following parameterization of \( \text{Tw}(C/k) \) in terms of the arithmetic invariants:

\[
\text{Tw}(C/k) = \{ y^2 + y = ax + \frac{c}{x^2 + x + r} + d \mid (d, r) \in \frac{k}{\{0, a\} + \text{AS}(k) \times \frac{k}{\text{AS}(k)}} \}.
\]

If \( k \) is the finite field with \( q \) elements, there are \((q - 1)(q - 2)\) values of the geometric invariants and the total number of curves with \( \text{Aut}(C) \cong C_2 \times C_2 \) is:

\[
4\left(\frac{q}{2} - 1\right)(q - 2) + 2\frac{q}{2}(q - 2) = (3q - 4)(q - 2).
\]

**Curves with \( \text{Aut}(C) \cong C_2 \times S_3 \)**

The values of the geometric invariants and a concrete choice for \( C_{C_2 \times S_3} \) are:

\[
\begin{align*}
\text{Tw}(C_{C_2 \times S_3}) & = \{ y^2 + y = ax + \frac{a}{x} + \frac{a}{x + 1} \mid a \in k^* \}, \\
\text{AS}(k) & \cong \text{Cub}_k.
\end{align*}
\]

Let \( C \in C_{C_2 \times S_3} \) be fixed and let \( w \in \overline{k} \) be such that \( w^2 + w = a \). The automorphisms

\[
U(x, y) = (x + 1, y + w), \quad V(x, y) = \left( \frac{1}{x + 1}, y \right),
\]

generate a subgroup \( S \subset A \), isomorphic to \( S_3 \), such that \( A = \langle \iota \rangle \times S \) (see the remark after Proposition 8). Hence, \( A' \cong S_3 \) and it is generated by the images \( U', V' \) of \( U, V \). If \( a \neq \text{AS}(k) \), then \( A \) has trivial \( G_k \)-action. If \( a \in \text{AS}(k) \), only the automorphisms \( U, \iota U \) are not \( G_k \)-invariant; they are exchanged by the action of all \( \sigma \in G_k \) with non-trivial image in \( \text{Gal}(k(w)/k) \). In both cases \( A' \) has trivial \( G_k \)-action, so that:

\[
H^1(G_k, A') = \text{Hom}(G_k, A') \setminus \text{Inn}(A') \cong \frac{k}{\text{AS}(k)} \sqcup \text{Cub}_k.
\]

Any non-trivial \( r \in k/\text{AS}(k) \), associated to a quadratic extension \( K/k \), is identified with the class of the 1-cocycle determined by the isomorphism \( \text{Gal}(K, k) \cong \{1, U'\} \). Any \( s \in \text{Cub}_K \), associated to a cubic extension \( K/k \) with normal closure \( \tilde{K} \), is identified with the class of the 1-cocycle determined by an isomorphism \( \text{Gal}(\tilde{K}, k) \cong \{1, V', (V')^2\} \), or \( \text{Gal}(\tilde{K}, k) \cong A' \). We can respectively choose as twists \( C_\xi \) associated to preimages \( \xi \in H^1(G_k, A) \):

\[
C_\xi: \quad y^2 + y = ax + \frac{a}{x^2 + x + r}, \quad y^2 + y = \frac{asx^2 + asx + as(s + 1)}{x^3 + sx + s}.
\]
The curves $C_\xi$ with quadratic denominator are isomorphic to $C$ via (13), whereas for those with cubic denominator we can take:

$$(x, y) \mapsto \left( \frac{\theta(x + \theta)}{\theta'(x + \theta)} \cdot y + v, \theta^3 + s\theta + s = 0, v^2 + v = aw. \right)$$

By Propositions 24 and 25, $\text{Tw}(C/k)$ is parameterized as the disjoint union:

$$\text{Tw}(C/k) = \begin{cases} 
\{ y^2 + y = ax + \frac{a}{x^2 + x + r} + d \mid (d, r) \in \frac{k}{\{0, a\} + \text{AS}(k) \times \text{AS}(k)} \} \\
\{ y^2 + y = \frac{asx^2 + ax + as(s + 1)}{x^3 + sx + s} + d \mid (d, s) \in \frac{k}{\text{AS}(k) \times \text{Cub}_k} \}. 
\end{cases}$$

As mentioned in section 1, for certain values of $s \in \text{Cub}_k$ one can use the curves, $y^2 + y = \frac{as}{x^3 + s} + d$, instead of the above ones.

If $k$ is the finite field with $q$ elements, there are $q - 1$ values of the geometric invariants and the total number of curves with $\text{Aut}(C) \simeq C_2 \times S_3$ is:

$$2q^2 + 4\left(\frac{q}{2} - 1\right) + 2(q - 1) = 5q - 6.$$ 

Curves with $\text{Aut}(C) \simeq M_{32}$

The values of the geometric invariants and a concrete choice for $C_{C_2 \times M_{32}}$ are:

$$j(C_{M_{32}}) = \{(0, 0, j_3) \in k^3 \mid j_3 \neq 0\},$$

$$C_{M_{32}} = \{ y^2 + y = ax^5 + ax^3 \mid a \in k^* \}.$$
From the exact sequence of abelian $G_k$-groups:

$$1 \rightarrow \text{Ker}(E) \rightarrow \overline{k}^r \xrightarrow{E} \overline{k} \rightarrow 1,$$

we see that $H^1(G_k, \text{Ker}(E)) \simeq k/E(k)$. For any $b \in k/E(k)$, the 1-cocycle associated to $b$ is given by $\{\xi'_\sigma = \beta + \sigma \beta\}_{\sigma \in G_k}$, where $\beta \in \overline{k}$ satisfies $E(\beta) = b$. Thus, we can choose the following twists associated to preimages in $H^1(G_k, A)$ of these cocycles:

$$C_{\xi}: \quad y^2 + y = ax^5 + bx^4 + ax^3, \quad b \in k/E(k).$$

Isomorphisms to $C$ are given by $(x, y) \mapsto (x + \beta, y + t_0 + t_1x + t_2x^2)$, where

$$t_1 = a\beta^2(1 + \beta^2), \quad t_2 = \sqrt{a\beta + b}, \quad t_0^2 + t_0 = a\beta^3(1 + \beta^2) + b\beta^4.$$

By Propositions 24 and 25,

$$\text{Tw}(C/k) = \prod_{b \in k/E(k)} \{y^2 + y = ax^5 + bx^4 + ax^3 + d | d \in \frac{k}{\delta(a\beta)(\Gamma_{aba}) + \text{AS}(k)}\}.$$

If $k$ is the finite field with $q$ elements, there are $q - 1$ values of the geometric invariants and the total number of curves with $\text{Aut}(C) \simeq M_{32}$ is $4q - 5 - [2]_m$.

### Curves with $\text{Aut}(C) \simeq M_{160}$

In this case $C_{M_{160}}$ consists in one single curve $C$, with model $y^2 + y = x^5$ and $j$-invariant $j(C) = (0, 0, 0)$.

For any $a \in k^*$, we denote by $E_a(x) := E_a(x) = a^4x^4 + ax$, the $\mathbb{F}_{16}$-linear separable polynomial considered in Lemma 15. Since $E_a(\lambda x) = \lambda E_a(x)$, for all $\lambda \in \mu_5$, there is a well defined action of the group $\mu_5(k)$ on $k/E(a)(k) \simeq H^1(G_k, \text{Ker}(E_a))$.

For $a = 1$, we have $\text{Ker}(E_1) = \mathbb{F}_{16}$, and the exact sequence (5) is:

$$1 \rightarrow \langle \nu \rangle \rightarrow \text{Aut}(C) \rightarrow \mu_5 \times \mathbb{F}_{16} \rightarrow 1. \quad (15)$$

The two automorphisms of $C$ lifting any $(\lambda, \nu) \in \mu_5 \times \mathbb{F}_{16}$ are

$$\pm U_{(\lambda, \nu)}(x, y) = (\lambda x + \nu, y + t_0 + t_1x + t_2x^2),$$

where

$$t_1 = \lambda \nu^4, \quad t_2 = \lambda^2 \nu^8, \quad t_0^2 + t_0 = \nu^5.$$

The plus or minus sign before $U_{(\lambda, \nu)}$ corresponds to any choice of the two possible values of $t_0$. Thus, (15) is an exact sequence of $G_k$-groups and $A' \simeq \mu_5 \times \mathbb{F}_{16}$ as $G_k$-groups.
From the split exact sequence of $G_k$-groups:
\[ 1 \rightarrow \mathbb{F}_{16} \rightarrow A' \rightarrow \mu_5 \rightarrow 1, \]
we get an exact sequence of pointed sets:
\[ H^1(G_k, \mathbb{F}_{16}) \rightarrow H^1(G_k, A') \rightarrow H^1(G_k, \mu_5) \rightarrow 1. \]  
(16)

Under the identification $H^1(G_k, \mathbb{F}_{16}) \simeq k/E_1(k)$, we have that $i(b) = i(b')$ if and only if there exists $\lambda \in \mu_5(k)$ such that $b' = \lambda b$; hence, we get from (16) an exact sequence of pointed sets:
\[ 1 \rightarrow (k/E_1(k)) \rightarrow H^1(G_k, A') \rightarrow H^1(G_k, \mu_5) \rightarrow 1. \]

In a similar way, for any other value of $a \in k^*$, we can consider the mapping
\[ (k/E_a(k)) \rightarrow H^1(G_k, A'), \]
sending the class of any $b \in k$ to the class of the 1-cocycle:
\[ \{ \xi'_\sigma = (\alpha \sigma^{-1}, \alpha(\beta + \sigma \beta)) \}_{\sigma \in G_k}, \]
where $\alpha, \beta \in \mathbb{F}_k$ satisfy $\alpha^5 = a$ and $E_a(\beta) = b$. It is not difficult to check that this mapping is 1-1 with image $\pi^{-1}(a)$, where $[a] \in k^*/(k^*)^5 \simeq H^1(G_k, \mu_5)$ is the class of $a$. Thus,
\[ H^1(G_k, A') \simeq \prod_{a \in k^*/(k^*)^5} (k/E_a(k)) \].

We can choose as twists associated to preimages in $H^1(G_k, A)$:
\[ C_\xi: \quad y^2 + y = ax^5 + bx^4, \quad a \in k^*/(k^*)^5, \quad b \in (k/E_a(k)) \]

isomorphic to $C$ via $(x, y) \mapsto (\lambda(x + \beta), y + t_0 + t_1 x + t_2 x^2)$, where
\[ \lambda^5 = a, \quad t_1 = a\beta^4, \quad t_2 = a^2 \beta^8, \quad t_0 + t_0 = a\beta^5. \]

By Propositions 24 and 25, we can parameterize Tw$(C/k)$ as:
\[ \text{Tw}(C/k) = \prod_{a \in k^*/(k^*)^5} \prod_{b \in (k/E_a(k)) \mu_5(k)} \{ y^2 + y = ax^5 + bx^4 + d \mid d \in \frac{k}{\delta_{ab0}(T_{ab0}) + AS(k)} \}. \]

If $k$ is the finite field with $q = 2^m$ elements, there is only one value of the geometric invariants and the total number of curves with $\text{Aut}(C) \simeq M_{160}$ is $3 + [2]_2^m + [8]_4^m$. 
In fact, if $4 
mid m$, the groups $k^*/(k^*)^5$, $\mu_5(k)$ vanish and
\[ H^1(G_k, A') \simeq k/E_1(k) = k/\text{AS}^4(k) \]
has 2 or 4 elements according to $m$ odd or even. The twists of $C$ are:
\[ \text{Tw}(C/k) = \{ y^2 + y = x^5 + \epsilon x^4 \mid \epsilon \in F \} \cup \{ y^2 + y = x^5 + x^4 + d_0 \}, \]
where $F = F_2, d_0 = 1$ if $m$ is odd, whereas for $m$ even, $F = F_4$ and $d_0$ is any choice of one of the two elements in $F_4 - F_2$.

If $4|m$, we have $k/E_0(k) = 0$ for all $a \notin (k^*)^5$ and $(k/\text{AS}^4(k))\setminus\mu_5$ has 4 elements. Hence, $H^1(G_k, A')$ has 8 elements and the 13 twists of $C$ are:
\[ \text{Tw}(C/k) = \{ y^2 + y = ax^5 + d, \mid (a, d) \in k^*/(k^*)^5 \times k/\text{AS}(k) \} \cup \\
\cup \{ y^2 + y = x^5 + bx^4, \mid b \in ((k/\text{AS}^4(k)) - \{0\})\setminus\mu_5(k) \}. \]

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