A bootstrapped test of covariance stationarity based on orthonormal transformations

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We propose a covariance stationarity test for an otherwise dependent and possibly globally non-stationary time series. We work in a generalized version of the new setting in Jin, Wang and Wang (2015), who exploit Walsh (1923) functions in order to compare sub-sample covariances with the full sample counterpart. They impose strict stationarity under the null, only consider linear processes under either hypothesis in order to achieve a parametric estimator for an inverted high dimensional asymptotic covariance matrix, and do not consider any other orthonormal basis. Conversely, we work with a general orthonormal basis under mild conditions that include Haar wavelet and Walsh functions, and we allow for linear or nonlinear processes with possibly non-iid innovations. This is important in macroeconomics and finance where nonlinear feedback and random volatility occur in many settings. We completely sidestep asymptotic covariance matrix estimation and inversion by bootstrapping a max-correlation difference statistic, where the maximum is taken over the correlation lag $h$ and basis generated sub-sample counter $U_1$. Of particular note, our test is capable of detecting breaks in variance, and distant, or very mild, deviations from stationarity.

Keywords: Covariance stationarity; max-correlation test; multiplier bootstrap; orthonormal basis; Walsh functions

1. Introduction

Assume $\{X_t : t \in \mathbb{Z}\}$ is a possibly non-stationary time series process in $L_2$. We want to test whether $X_t$ is covariance stationary, without explicitly assuming stationarity under the null hypothesis, allowing for linear or nonlinear processes with a possibly non-iid innovation, and a general memory property. Such generality is important in macroeconomics and finance where nonlinear feedback and non-iid innovations occur in many settings due to asymmetries and random volatility, including exchange rates, bonds, interest rates, commodities, and asset return levels and volatility. Popular models for such time series include symmetric and asymmetric GARCH, Stochastic Volatility, nonlinear ARMA-GARCH, and switching models like smooth transition autoregression. See, e.g., Teräsvirta (1994), Gray (1996) and Francq and Zakoïan (2019).

Evidence for nonstationarity, whether generally or in the variance or autocovariances, has been suggested for many economic time series, where breaks in variance and model parameters are well known (e.g. Busset and Taylor, 2003, Gianetto and Raissi, 2015, Hendry and Massmann, 2007, Perron, 2006). Knowing whether a time series is globally nonstationary has large implications for how analysts approach estimation and inference. Indeed, it effects whether conventional parametric and semi-(non)parametric model specifications are correct. Pretesting for deviations from global stationarity therefore has important practical value.

There are many tests in the literature on covariance stationarity, and concerning locally stationary processes. Tests for stationarity based on spectral or second order dependence properties have a long history, where pioneering work is due to Priestley and Subba Rao (1969). Spectrum-based tests with $L_2$-distance components have many versions. Paparoditis (2010) uses a rolling window method...
to compare subsample local periodograms against a full sample version. The maximum is taken over the $L_2$-distance between periodograms over all time points. An asymptotic theory for the max-statistic, however, is not provided, although an approximation theory is (see their Lemmas 1 and 3). Furthermore, conforming with many offerings in the literature, under the null $X_t$ is a linear process with iid Gaussian innovations. Dette, Preuß and Vetter (2011) study locally stationary processes, and impose linearity with iid Gaussian innovations. Their statistic is based on the minimum $L_2$-distance between a spectral density and its version under stationarity, and local power is non-trivial against $T^{1/4}$-alternatives. Aue et al. (2009) propose a nonparametric test for a break in covariance for multivariate time series based on a version of a cumulative sum statistic.

Wavelet methods have arisen in various forms recently. von Sachs and Neumann (2000), using technical wavelet decomposition components from Neumann and von Sachs (1997), propose a Haar wavelet based localized periodogram test of covariance stationarity for locally stationary processes (cf. Dahlhaus, 1997, 2009), but neglect to characterize power. Haar wavelet functions form an orthonormal basis on $L_2([0,1])$, but the proposed frequency domain tests are complicated, a local power analysis is not feasible, and empirical power may be weak (see simulation evidence from Jin, Wang and Wang (2015)).

Dwivedi and Subba Rao (2011) and Jentsch and Rao (2015) use the discrete Fourier transform [DFT] $J_{rT} (\omega_k) = (2\pi T)^{-1/2} \sum_{t=1}^{T} X_t \exp \{it\omega_k\}$ at canonical frequencies $\omega_k = 2\pi k/T$ and $1 \leq k \leq T$. Dwivedi and Subba Rao (2011) generate a portmanteau statistic from a normalized sample DFT covariance, exploiting the fact that an uncorrelated DFT implies second order stationarity. Nason (2013) presents a covariance stationarity test based on Haar wavelet coefficients of the wavelet periodogram, they assume linear local stationarity, and do not treat local power. See also Nason, von Sachs and Kroisandt (2000).

In a promising offering in the wavelet literature, Jin, Wang and Wang (2015) (JWW) exploit so-called Walsh functions (akin to “global square waves” although not truly wavelets; cf. Walsh (1923) and their implied systematic samples for comparing sub-sample covariances with the full sample one. They utilize a sample-size dependent maximum lag $H_T$ and maximum systematic sample counter $K_T$, and show their Wald test exhibits non-negligible local power against $\sqrt{T}$-alternatives. They do not consider any other orthonormal transformation because Walsh functions, they argue, have “desirable properties” based primarily on simulation evidence, asymptotic independence of a sub-sample and sample covariance difference $(\sqrt{T} (\hat{\gamma}_h^{(k_1)} - \hat{\gamma}_h), \sqrt{T} (\hat{\gamma}_h^{(k_2)} - \hat{\gamma}_h))$ across systematic samples $k_1 \neq k_2$, and joint asymptotic normality (JWW, p. 897). It seems, however, that such theoretical properties are available irrespective of the orthonormal basis used, although we do not provide a proof. See Section 2.1, below, for definitions and notation. We do, however, find in the sequel that the Walsh basis has superlative properties vis-à-vis a Haar wavelet basis.

JWW’s asymptotic analysis is driven by local stationarity and linearity $X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$, with zero mean iid $Z_t$, and $E|Z_t|^{4+\delta} < \infty$, $\delta > 0$, which expedites characterizing a parametric asymptotic covariance matrix estimator. The iid and linearity assumptions, however, rule out many important processes, including nonlinear models like regime switching, and random coefficient processes, and any process with a non-iid error (e.g. nonlinear ARMA-GARCH). JWW’s Wald-type test statistic requires an inverted parametric variance estimator that itself requires five tuning parameters and choice of two kernels.\(^1\) Indeed, most of the tuning parameters only make sense under linearity given how they approach asymptotic covariance matrix estimation.

\(^1\)One tuning parameter $\lambda \in (0,5)$ governs the number $Q_T = \lfloor T^{1/4} \rfloor$ of sample covariances that enter the asymptotic covariance matrix estimator (see their p. 899). The remaining four $(c_1, c_2, \bar{\epsilon}, \bar{\epsilon}_2)$ are used for kernel bandwidths $b_j = c_j T^{-\bar{\epsilon}_j}$, $j = 1, 2$, for computing the kurtosis of the iid process $Z_t$ under linearity (see p. 902-903). The authors set $c_j$ equal to 1.2 times a so-called \textit{"crude scale estimate"} which is nowhere defined.
Now define the lag \( h \) autocovariance coefficient at time \( t \):

\[
\gamma_h(t) \equiv E \left[ (X_t - E [X_t]) (X_{t-h} - E [X_{t-h}]) \right], \ h = 0, 1, \ldots
\]

The hypotheses are:

\[
H_0 : \gamma_h(s) = \gamma_h(t) = \gamma_h \quad \forall s, t, \forall h = 0, 1, \ldots \quad \text{(cov. stationary)} \tag{1.1}
\]

\[
H_1 : \gamma_h(s) \neq \gamma_h(t) \text{ for some } s \neq t \text{ and } h = 0, 1, \ldots \quad \text{(cov. nonstationary)}
\]

Under \( H_0 \), \( X_t \) is second order stationary, and the alternative is any deviation from the null: the autocovariance differs across time at some lag, allowing for a (lag zero) break in variance. The null hypothesis otherwise accepts the possibility of global nonstationarity.

In this paper we do away with parametric assumptions on \( X_t \), and impose either a mixing or physical dependence property that allows us to bound the number of usable covariance lags \( \mathcal{H}_T \) and systematic samples \( \mathcal{K}_T \). The conditions allow for global nonstationarity under either hypothesis, allowing us to focus the null hypothesis on only second order stationarity. We show that use of the physical dependence construct in \( \text{Wu (2005)} \) is a boon for bounding \( \mathcal{H}_T \) since it allows for slower than geometric memory decay, and ultimately need only hold uniformly over \((h, k)\). The mixing condition imposed, however, requires geometric decay, and must ultimately hold jointly over all lags \( h = 1, \ldots, \mathcal{H}_T \) (\( h \) is the covariance lag, and \( k \) is a systematic subsample counter). The latter leads to a much diminished upper bound on the maximum lag growth \( \mathcal{H}_T \rightarrow \infty \). This may be of independent interest given the recent rise of high dimensional central limit theorems under weak dependence (e.g. \( \text{Chang, Chen and Wu, 2024, Chang, Jiang and Shao, 2023, Zhang and Wu, 2017} \)).

Rather than operate on a Wald statistic constructed from a specific orthonormal transformation of covariances, our statistic is the maximum generic orthonormal transformed sample correlation coefficient, where the maximum is taken over \((h, k)\) with increasing upper bounds \((\mathcal{H}_T, \mathcal{K}_T)\). By working in a generic setting we are able to make direct comparisons, and combine bases for possible power improvements.

We provide examples of Haar wavelet and Walsh functions in Sections 2.1 and 2.2, and show how they yield different systematic samples. This suggests a power improvement may be available by using multiple orthonormal transforms. As JWW (p. 897) note, however, clearly other orthonormal transformations are feasible, although simulation evidence agrees with their suggestion that the Walsh basis works quite well.

We use a dependent wild bootstrap for the resulting test statistic, allowing us to sidestep asymptotic covariance matrix estimation, a challenge considering we do not assume a parametric form, and the null hypothesis requires us to look over a large set of \((h, k)\). We sidestep all of JWW’s tuning parameters, and require just one governing the block size for the bootstrap. We ultimately achieve a significantly better upper bound on the rate of increase for \((\mathcal{H}_T, \mathcal{K}_T)\) than JWW. Penalized and weighted versions of our test statistic are also possible, as in JWW and \( \text{Hill and Motegi (2020)} \) respectively. There is, though, no compelling theory to justify penalties on \((h, k)\) in our setting, and overall a non-penalized and unweighted test statistic works best in practice.

Note that \( \text{Hill and Motegi (2020)} \) study the max-correlation statistic for a white noise test, and only show their limit theory applies for some increasing maximum lag \( \mathcal{H}_T \), but do not derive an upper bound. In the present paper we use a different asymptotic theory, derive upper bounds for \( \mathcal{H}_T \) and \( \mathcal{K}_T \), and of course do not require a white noise property under \( H_0 \).

\( \text{Jin, Wang and Wang (2015, Section 2.6)} \) rule out the use of autocorrelations because, they claim, if the sample variance were included, i.e. \( h \geq 0 \), then consistency may not hold because the limit theory neglects the joint distribution of \( \hat{\gamma}_h \) and the correlation differences. We show for our proposed test that
the difference between full sample and systematic sample autocorrelations at lag zero asymptotically reveals whether \( E[X_i^2] \) is time dependent. Further, our test is consistent whether non-stationarity is caused by variances, or covariances, or both. See Section 3.3 and Example 3.5. Our proposed test is consistent against a general (nonparametric) alternative, and exhibits nontrivial power against a sequence of \( \sqrt{T} \)-local alternatives.

The max-correlation difference is particularly adept at revealing subtle deviations from covariance stationarity, similar to results revealed in Hill and Motegi (2020). Consider a distant form of a model treated in Paparoditis (2010, Model I) and Jin, Wang and Wang (2015, Section 3.2: models NVI, NVII), \( X_t = .08 \cos(1.5 - \cos(4\pi t/T))\xi_{t-d} + \epsilon_t \) with large \( d \) (JWW use \( d = 1 \) or 6). JWW’s test exhibits trivial power when \( d \geq 20 \), while the max-correlation difference is able to detect this deviation from the null even when \( d \geq 50 \). The reason is the same as that provided in Hill and Motegi (2020): the max-correlation difference operates on the single most useful statistic, while Wald and portmanteau statistics congregate many standardized covariances that generally provide little relevant information under a weak signal.

In Section 2 we develop the test statistic. Sections 3 and 4 present asymptotic theory and the bootstrap method and theory. We then perform a Monte Carlo study in Section 5, and conclude with Section 6. The supplemental material Hill and Li (2024) contains all proofs, an empirical study concerning international interest rates, and complete simulation results.

We use the following notation. \( \lfloor z \rceil \) rounds \( z \) to the nearest integer. \( L_2 \) is the space of square integrable random variables; \( L_2(a,b) \) is the class of square integrable functions on \( (a,b) \). \( \| \cdot \|_p \) and \( \| \cdot \| \) are the \( L_p \) and \( L_2 \) norms respectively, \( p \geq 1 \). Let \( \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \) and \( \mathbb{N} = \{ 0, 1, 2, \ldots \} \). \( K > 0 \) is a finite constant whose value may be different in different places. \( \text{a} \text{w} \text{p} \text{1} \) denotes “asymptotically with probability approaching one”. Write \( \max_{\mathcal{H}_T} = \max_{0 \leq h \leq \mathcal{H}_T} \). \( \max_{K_{T}}, \max_{K_{T} \cap \mathcal{H}_T} = \max_{0 \leq h \leq \mathcal{H}_T}, 1 \leq k \leq \mathcal{K}_T \). Similarly, \( \max_{\mathcal{H}_T} a(h,h) = \max_{0 \leq h \leq \mathcal{H}_T} a(h,h) \), etc. \( |d|_+ \equiv a \vee 0 \).

2. Max-correlation with orthonormal transformation

Our test statistic is the maximum of an orthonormal transformed sample covariance. In order to build intuition, we first derive the test statistic under Walsh function and Haar wavelet-based bases. We then set up a general environment, and present the main results.

In order to reduce notation, assume here \( \mu = E[X_i] = 0 \) is known. In practice this is enforced by using \( X_t - \bar{X} \) where \( \bar{X} \equiv 1/T \sum_{t=1}^{T} X_t \). In Hill and Li (2024, Lemmas B.3 and B.3”) we prove using \( X_t - \bar{X} \) or \( X_t - \mu \) leads to identical results asymptotically. Thus in proofs of the main results we simply assume \( \mu = 0 \).

2.1. Walsh functions

The following class of Walsh functions \( \{ W_i(x) \} \equiv \{ W_i(x) : i = 0, 1, 2, \ldots \} \) define a complete orthonormal basis in \( L_2(0,1) \). The functions \( W_i(x) \) are defined recursively (see, e.g., Ahmed and Rao, 1975, Stoffer, 1987, 1991, Walsh, 1923):

\[
W_0(x) = 1 \text{ for } x \in [0, 1) \text{ and } W_1(x) = \begin{cases} 1, & x \in [0, .5) \\ -1, & x \in [.5, 1) \end{cases},
\]

and for any \( i = 1, 2, \ldots, \)

\[
W_{2i}(x) = \begin{cases} W_i(2x), & x \in [0, .5) \\ (-1)^i W_i(2x - 1), & x \in [.5, 1) \end{cases} \quad \text{and} \quad W_{2i+1}(x) = \begin{cases} W_i(2x), & x \in [0, .5) \\ (-1)^{i+1} W_i(2x - 1), & x \in [.5, 1) \end{cases}.
\]
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In the \{-1,1\}-valued sequence \{\mathcal{W}_i(x) : i = 0, 1, 2, \ldots\}, \(i\) indexes the number of zero crossings, yielding a square shaped wave-form. See Figure 1, below, and see Stoffer (1991, Figure 5) and Jin, Wang and Wang (2015, Figure 1) and their references. The \(k^{th}\) discrete Walsh functions used in this paper are then for \(t = 1, \ldots, T\):

\[
\{\mathcal{W}_k^{(1)}, \ldots, \mathcal{W}_k^{(T)}\} \text{ where } \mathcal{W}_k^{(t)} = \mathcal{W}_k \left(\frac{(t-1)}{T}\right).
\]

Now define the covariance coefficient for a covariance stationary time series, \(\gamma_h \equiv E \{X_t X_{t-h}\}\), and denote the usual (co)variance estimator \(\hat{\gamma}_h \equiv 1/T \sum_{t=1}^{T-h} X_t X_{t+h}, h \in \mathbb{N}\). JWW use \{\mathcal{W}_i(x)\} to construct a set of discrete Walsh covariance transformations: for some integer \(K \geq 1\),

\[
\hat{\gamma}_h^{(k)} \equiv \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} \left\{1 + (-1)^{k-1} \mathcal{W}_k(t)\right\}, \ h = 0, 1, \ldots, T-1, \text{ and } k = 1, 2, \ldots, K. \tag{2.1}
\]

As they point out, a sequence of systematic (sub)samples \(T^W_k : k = 1, 2, \ldots, K\) in the time domain can be defined on the basis of Walsh functions:

\[
T^W_k \equiv \left\{t \in T : (-1)^{k-1} \mathcal{W}_k(t) = 1\right\}.
\]

Now let \(N_k\) be the smallest power of 2 that is at least \(k\). The first systematic sample is the first half of the sample time domain \(T^W_1 = \{1, \ldots, [T/2]\}\); the second is the middle half \(T^W_2 = \{[T/4], [T/4] + 1, \ldots, [3T/4]\}\); the third \(T_3\) is the first and third time blocks, and so on. Notice \(T^W_k\) consists of \((k + 1)/2\) blocks with at least \([T/N_k]\) elements. Thus, when \(h < T/N_k\) then \(\hat{\gamma}_h^{(k)}\) is just an estimate of \(\gamma_h\) on the \(k^{th}\) systematic sample:

\[
\hat{\gamma}_h^{(k)} = \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} \left\{1 + (-1)^{k-1} \mathcal{W}_k(t)\right\} = \frac{2}{T} \sum_{t \in T_k} X_t X_{t+h}.
\]

The condition \(h < T/N_k\) holds asymptotically in the Section 4 bootstrap setting.

The difference between the \(k^{th}\) systematic sample and full sample estimators is:

\[
\hat{\gamma}_h^{(k)} - \hat{\gamma}_h = (-1)^{k-1} \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} \mathcal{W}_k(t).
\]

Notice the \(-1,1\)-valued nature of \(\mathcal{W}_k(t)\) yields a sub-sample comparison: \(\hat{\gamma}_h^{(k)} - \hat{\gamma}_h = 1/T \sum_{t \in T_k} X_t X_{t+h} - 1/T \sum_{t \in T} X_t X_{t+h}\). Our test is based on the maximum \(|\hat{\gamma}_h^{(k)} - \hat{\gamma}_h|\), in which case the multiple \((-1)^{k-1}\) is irrelevant. We therefore drop it everywhere. Under the null hypothesis and mild assumptions this difference is \(O_p(1/\sqrt{T})\) at all lags \(h\) and for all systematic samples \(k\). Thus, a test statistic can be constructed from \(\sqrt{T}(|\hat{\gamma}_h^{(k)} - \hat{\gamma}_h|)\).

2.2. Haar wavelet functions

Define the usual Haar wavelet functions \(\psi_{k,m}(x) \equiv 2^{k/2} \psi(2^k x - m)\) with \(x \in \mathbb{R}\), where \(0 \leq k \leq K_F\) for some integer sequence \(\{K_F\}, 0 \leq m \leq 2^k - 1\), and mother wavelet (Haar, 1910):

\[
\psi(x) = \begin{cases} 
1, & x \in [0,.5) \\
-1, & x \in [.5,1) \\
0, & \text{otherwise}
\end{cases}
\]
Haar functions \( \{ \psi_{k,m}(x) \} \) form a complete orthonormal basis in \( \mathcal{L}[0,1] \). The discretized version is:
\[
\Psi_{k,m}(t) = \psi_{k,m}\left(\frac{(t - 1)}{T}\right) = 2^{k/2} \psi\left(2^k \frac{(t - 1)}{T - m}\right).
\]
Systematic samples derived from \( \{ \Psi_{k,m}(t) \} \) are generally too “local”: \( 1/T \sum_{t=1}^{T-h} X_t X_{t+h} \Psi_{2,m}(t) \), for example, compares just the first eighth to the second eighth subsample \((m = 0)\); the third eighth to the fourth eighth subsample \((m = 1)\); and so on.

In order to yield a test statistic that compares sub-sample complements, comparable to Walsh functions, we compile \((\psi_{k,m}(x), \Psi_{k,m}(t)) \) over \( 0 \leq m \leq 2^k - 1 \). Set \( \psi_0(x) = I(0 \leq x \leq 1) \), and for \( k = 0, 1, \ldots \)
\[
\psi_{k+1}(x) \equiv \frac{1}{2^{k/2}} \sum_{m=0}^{2^k-1} \psi_{k,m}(x) = \sum_{m=0}^{2^k-1} \psi(2^k x - m)
\]
\[
\Psi_{k+1}(t) \equiv \frac{1}{2^{k/2}} \sum_{m=0}^{2^k-1} \Psi_{k,m}(t) = \sum_{m=0}^{2^k-1} \psi(2^k (t - 1)/T - m).
\]

We set \( \psi_0(x) = I(0 \leq x \leq 1) \) in order to unify the local alternative analysis below, similar to the Walsh basis. It can be shown that \( \psi_k(x) \in \{-1, 1\} \), and \( \{ \psi_k(x) : 1 \leq k \leq K_T \} \) forms a complete orthonormal basis: see Lemma B.1 in Hill and Li (2024) for this and other properties. Now, in the same manner as (2.1), define for \( k = 1, 2, \ldots, K \):
\[
\hat{\gamma}_h^{(k)}(k) = \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} (1 + \Psi_{k}(t)) , \ h = 0, 1, \ldots, T - 1.
\]

The discretized Haar functions \( \Psi_{k,m}(t) \) generate systematic samples \( T_k^H = \{ t \in T : \bigcup_{m=0}^{2^k-1} (\Psi_{k,m}(t) = 1) \} \). This yields the first sample half \( T_0^H = \{1, \ldots, [T/2]\} \); the first and third quarter subsamples \( T_1^H = \{1, \ldots, [T/4]; 1 + [T/2], \ldots, [3T/4]\} \); the first, third, fifth and seventh eights \( T_2^H \); etc. See Figure 1 for plots of Walsh and composite Haar functions \( W_k(x) \) and \( \psi_k(x) \), \( k = 1, \ldots, 6 \).

Walsh and Haar systematic samples are quite different for \( k \geq 2 \). \( T_k^W \) involves fewer interspersed subsample segments, in some cases of varying lengths, while \( T_k^H \) have \( 2^k \) segments of equal length \( [T/2^k] \) in all cases (ignoring truncation due to the lag \( h \)). Haar sub-samples are therefore non-redundant only when \( T/2^K \geq 1 \), hence we need \( K_T \leq \ln(T)/\ln(2) \).

Indeed, it can be shown that the two bases coincide in the sense that \( W_{k_1}(x) = \psi_{k_2}(x) \) for all \( x \) and only \( (k_1, k_2) \in \{(1, 1), (3, 2), (3, 7)\} \), or in all other cases for \( x \) on a subset of \([0,1]\) with Lebesgue measure 1/2. Roughly speaking, only 50% of the data points in \( \hat{\gamma}_h^{(k_1)} - \hat{\gamma}_h \) are the same as those in \( \hat{\gamma}_h^{(k_2)} - \hat{\gamma}_h \) for nearly all systematic samples \((k_1, k_2)\). Thus the two bases are intrinsically different, suggesting potential advantages and weaknesses against certain deviations from the null.

### 2.3. Max-correlation orthonormal transforms

Now let \( \{ B_k(x) : 0 \leq k \leq K \} \) denote a \([-1, 1]\)-valued orthonormal basis on \( \mathcal{L}[0,1] \), \( B_0(x) = I(0 \leq x \leq 1) \), let \( B_k(t) = B_k((t - 1)/T) \) be the discretized version, and define the usual subsample covariance for this generic discrete basis \( \hat{\gamma}_h^{(k)} = 1/T \sum_{t=1}^{T-h} X_t X_{t+h} (1 + B_k(t)) \).

Define the sample correlation coefficient:
\[
\hat{\rho}_h = \frac{\hat{\gamma}_h}{\hat{\gamma}_0}.
\]
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Figure 1: Orthonormal bases

(a) Walsh functions \( \{W_k(x)\}_{k=1}^6 \)

(b) Composite Haar \( \{\psi_k(x)\}_{k=1}^6 \)

and a set of discrete orthonormal correlation transformations, over systematic sample \( k \):

\[
\hat{\rho}_{h,1}^{(k)} = \frac{\hat{\gamma}_h^{(k)}}{\hat{\gamma}_0} = \frac{1}{\hat{\gamma}_0} \times \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} \{1 + B_k(t)\}, \quad k = 1, 2, \ldots, K.
\]

Thus, the difference between systematic sample and full sample estimators is:

\[
\hat{\rho}_{h,1}^{(k)} - \hat{\rho}_h = \frac{1}{\hat{\gamma}_0} \times \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) = \frac{\hat{\gamma}_h^{(k)} - \hat{\gamma}_h}{\hat{\gamma}_0}.
\]  

(2.2)

The correlation difference \( \hat{\rho}_{h,1}^{(k)} - \hat{\rho}_h \) is sensible even at lag 0, considering

\[
\hat{\rho}_{0,1}^{(k)} - \hat{\rho}_0 = \frac{\hat{\gamma}_0^{(k)}}{\hat{\gamma}_0} - 1.
\]

Thus, under nonstationarity \( \hat{\rho}_{0,1}^{(k)} \overset{P}{\not\rightarrow} 1 \) for some systematic sample \( k \) when \( \hat{\gamma}_0^{(k)}/\hat{\gamma}_0 \overset{P}{\not\rightarrow} 1 \); that is, when the second moment \( E[X_t^2] \) is not constant over \( t \).

Alternatively, we may incorporate the systematic sample variance estimators \( \hat{\gamma}_0^{(k)} \). The autocorrelation estimator in that case becomes, for example:

\[
\hat{\rho}_{h,2}^{(k)} = \frac{1}{\hat{\gamma}_0} \times \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} + \frac{1}{\hat{\gamma}_0} \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t)
\]
\[ \hat{\rho}_h + \frac{1}{\hat{\gamma}_0} \sum_{t=1}^{T-h} \frac{1}{T} X_t X_{t+h} B_k(t), \]

hence

\[ \hat{\rho}_{h,2} - \hat{\rho}_h = \frac{1}{\hat{\gamma}_0} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \]

At lag 0 notice:

\[ \hat{\rho}_{0,2} - \hat{\rho}_0 = \frac{1}{\hat{\gamma}_0} \sum_{t=1}^{T} X_t^2 B_k(t) = \frac{\hat{\gamma}_0 - \gamma_0}{\hat{\gamma}_0} = 1 - \frac{\hat{\gamma}_0}{\gamma_0}. \]

Compare this to \( \hat{\rho}_{0,1} - \hat{\rho}_0 = \gamma_0 / \gamma_0 - 1 \). Thus, again \( \hat{\rho}_{0,2} \sim 1 \) for some systematic sample \( k \) when \( E[X_k^2] \) is not constant over \( t \).

Asymptotically \( \hat{\rho}_{h,1} \) are identical in probability. Indeed,

\[ \hat{\rho}_{h,1} - \hat{\rho}_{h,2} = \left( \frac{1}{\gamma_0} - \frac{1}{\hat{\gamma}_0} \right) \frac{1}{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \]

\[ = \left( \frac{1}{\hat{\gamma}_0} - \frac{1}{\gamma_0} \right) \left( \hat{\gamma}_0 - \gamma_0 \right). \]

This reveals \( \hat{\rho}_{h,1} - \hat{\rho}_{h,2} \) for each \( h \geq 0 \) simultaneously captures systematic sample differences in variance and covariance. Under \( H_0 \) and general conditions presented in Section 3, \( \max_{H_1,K} |\hat{\gamma}_h - \gamma_h| \) and \( |\gamma_0 - \gamma_0| \) are \( O_p(1/\sqrt{T}) \), where \( \{H_1,K_T\} \) are sequences defined below with \( H_1 \to \infty \) and \( K_T \to \infty \). Thus:

\[ \max_{H_1,K_T} \sqrt{T} \left( \hat{\rho}_{h,1} - \hat{\rho}_{h,2} \right) = O_p(1/\sqrt{T}). \]

Under \( H_1 \), however, it holds that \( \sqrt{T} \max_{H_1,K_T} |\hat{\gamma}_h - \gamma_h| \to \infty \) if and only if \( E[X_k^2] \) and \( E[X_t X_{t-h}] \) for some \( h \geq 1 \) are time dependent. This suggests \( D_T \equiv \max_{H_1,K_T} \sqrt{T} \left( \hat{\rho}_{h,1} - \hat{\rho}_{h,2} \right) \) could be used as a third test statistic: \( D_T \) will reject \( H_0 \) asymptotically with power approaching one when \( X_t \) is non-stationary in variance and autocovariance at some lag \( h \geq 1 \). Conversely, either \( \max_{H_1,K_T} \sqrt{T} \left( \hat{\rho}_{h,i} - \gamma_h \right) \) is consistent against \( H_1 \) in general: power is one asymptotically if \( E[X_k^2] \) and/or some \( E[X_t X_{t-h}] \) are time dependent.

In order to focus ideas we only consider \( \hat{\rho}_{h,1} \), so put:

\[ \hat{\rho}_{h} \equiv \hat{\rho}_{h,1}. \]

The proposed test statistic is therefore the maximum normalized \( \hat{\rho}_h - \hat{\rho}_0 \) over \( (h,k) \):

\[ M_T \equiv \max_{H_1,K_T} \left| \hat{\rho}_h - \gamma_0 \right| = \frac{1}{\gamma_0} \max_{H_1,K_T} \left| \sqrt{T} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t) \right|. \]
By construction $M_T$ uses the most informative systematic sample correlation difference. Notice we search over all lags $h \in \{0, \ldots, H_T\}$.

A penalized version is also possible:

$$M_T^{(p)} \equiv \max_{\mathcal{H}_T, \mathcal{K}_T} \left\{ \sqrt{T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| - \mathcal{P}(h, k) \right\},$$

where $\mathcal{P}(h, k)$ is a non-random, positive, strictly monotonically increasing function of $h$ and $k$. JWW use $\mathcal{P}(h, k) = p_h + q_h$ with AIC-like lag penalty $p_h = 2h$ in an order selection-type Wald statistic. This is sensible considering the Wald statistic is pointwise asymptotically chi-squared with mean 2

$$\sqrt{k-1}$$

where $k$ is a non-random, positive, strictly monotonically increasing function of $h$.

In our non-Wald setting a similar reasoning for choosing $\mathcal{P}(h, k) = p_h + q_h$ does not apply, nor do we have any comparable requirements for penalizing $k$. Indeed, a compelling reason for “penalizing” $M_T$ at all would be to counter the loss of observations at higher lags or to control for lag specific heterogeneity, but that historically is ameliorated with weights, for example $\max_{\mathcal{H}_T, \mathcal{K}_T} \left\{ \sqrt{T} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| \right\}$, where $\mathbb{W}_T^{(k)}$ are possibly stochastic, $\lim_{T \to \infty} \min_{\mathcal{H}_T, \mathcal{K}_T} \mathbb{W}_T^{(k)} > 0$ a.s., and $\max_{\mathcal{H}_T, \mathcal{K}_T} \left\{ \sqrt{T} \left| \hat{\rho}_h^{(k)} - \mathbb{W}_T^{(k)} \right| \right\} \to 0$ where non-stochastic $\mathbb{W}_h^{(k)}$ satisfy $\min_{\mathcal{H}_T, \mathcal{K}_T} \mathbb{W}_h^{(k)} > 0$. Choices include Ljung-Box type weights, or an inverted non-parametric standard deviation estimator, cf. Hill and Motegi (2020).

Consider the latter, and define a sample covariance function $\hat{\nu}_T(i; h, k) \equiv 1/T \sum_{t=1}^{T-h-i} \hat{z}_t(i, h, k)$ where

$$\hat{z}_t(i, h, k) \equiv \left\{ X_i X_{t+h} B_k(t) - \frac{1}{T} \sum_{t=1}^{T-h} X_i X_{t+h} B_k(t) \right\}.$$

Under fourth order stationarity under the null, and because $\hat{\gamma}_0$ only operates as a scale asymptotically, cf. Theorem 3.2 below, the weights are $\mathbb{W}_T^{(k)} = 1/\hat{\nu}_T(h, k)$ where, e.g.,

$$\hat{\nu}_T^2(h, k) = \hat{\gamma}_0^{-2} \left\{ \hat{\nu}_T(0; h, k) + 2 \sum_{i=1}^{T-h-1} |\mathcal{K}(i/\beta_T)| \hat{\nu}_T(i; h, k) \right\}$$

with symmetric, square integrable kernel function $\mathcal{K} : \mathbb{R} \to [-1, 1]$ satisfying $\mathcal{K}(0) = 1, 2$ and bandwidth $\beta_T \to \infty$ where $\beta_T = o(T)$.

A penalized weighted version is thus:

$$M_T^{(w,p)} \equiv \max_{0 \leq h \leq H_T, 1 \leq k \leq K_T} \left\{ \sqrt{T} \mathbb{W}_T^{(k)} \left| \hat{\rho}_h^{(k)} - \hat{\rho}_h \right| - \mathcal{P}(h, k) \right\}.$$

In Monte Carlo work we study $M_T$, $M_T^{(p)}$, and $M_T^{(w,p)}$ with Walsh or Haar bases, various penalties, and/or an inverted standard deviation weight or Ljung-Box weight. We find $\mathcal{P}(h, k) = p_h + q_k$ where $p_h = (h + 1)^{\alpha}/2$ and $q_k = k^{\alpha}/2$ with $\alpha = [1/8, 1/2]$, or $\mathcal{P}(h, k) = \sqrt{(h + 1)k}$, promote accurate empirical size but generally does not lead to dominant power, and may lead to decreased power in some cases. Conversely, inverted standard error weights $\mathbb{W}_T^{(k)}$ generally lead to over-sized tests, and Ljung-Box weights do not offer an advantage under either hypothesis. Overall a non-penalized and non-weighted statistic dominates.

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*See, for example, class $\mathcal{R}_2$ in Andrews (1991), or class $\mathcal{R}$ in de Jong and Davidson (2000, Assumption 1).*
Finally, a power improvement may be yielded by combining bases with uniquely defined systematic samples. Let $\mathcal{M}_T(\mathcal{B}_j)$ be max-statistics based on $\mathcal{F} \in \mathbb{N}$ orthonormal bases $\mathcal{B}_{j,k}(x)$, $j = 1, \ldots, \mathcal{F}$. Then, ignoring penalties and weights to ease notation, define a so-called “max-max-statistic”:

$$\tilde{M}_T \equiv \max_{1 \leq j \leq \mathcal{F}} \{ \mathcal{M}_T(\mathcal{B}_j) \}.$$  \hfill (2.6)

We study $\tilde{M}_T \equiv \max \{ M_T(W), M_T(\Psi) \}$ in simulation work, where $M_T(W)$ and $M_T(\Psi)$ use Walsh and composite Haar bases. An asymptotic theory for $\tilde{M}_T$ and its bootstrapped $p$-value follow directly from results given below and the mapping theorem since $\mathcal{F}$ is a finite constant. Other basis combinations are clearly feasible. Consider discretized bases $\mathcal{B}_{j,k}(t)$ and the set $\{ \tilde{B}_k(t) \}_{k=1}^K = \{ \mathcal{B}_{j,k}(t) : j \in \mathcal{J}^*; k \in \mathcal{K}^* \}$ where $\mathcal{J}^*$ and $\mathcal{K}^*$ are index subsets of $\{1, \ldots, \mathcal{J}\}$ and $\{1, \ldots, \mathcal{K}\}$ yielding unique $\mathcal{B}_{j,k}(x)$ $\forall x$. Test statistics can then be derived from $\{ \tilde{B}_k(t) \}_{k=1}^K$.

3. Asymptotic theory

Write

$$z_t(h,k) \equiv X_t X_{t+h} B_k(t) - E [X_t X_{t+h}] B_k(t) \quad (3.1)$$

and define a variance function $\sigma^2_T(h,k) \equiv E [Z_T^2(h,k)]$. In the general case $E[X_t] \in \mathbb{R}$ replace $X_t$ with $X_t - E[X_t]$.

The main contributions of this section deliver a class of sequences $\{ \mathcal{H}_T, \mathcal{K}_T \}$, and an array of random variables $\{Z_T(h,k) : T \in \mathbb{N}\}_{h \geq 0,k \geq 1}$ normally distributed $Z_T(h,k) \sim N(0,\sigma^2_T(h,k))$, such that the Kolmogorov distance

$$\rho_T \equiv \sup_{\varepsilon \geq 0} \left| P \left( \max_{\mathcal{H}_T,\mathcal{K}_T} |Z_T(h,k)| \leq \varepsilon \right) - P \left( \max_{\mathcal{H}_T,\mathcal{K}_T} |Z_T(h,k)| \leq \varepsilon \right) \right| \rightarrow 0. \quad (3.3)$$

We work under mixing or physical dependence settings. The approximation does not require standardized $Z_T$ and $Z_{\tilde{T}}$ in view of non-degeneracy Assumption 1.c below. We then apply the approximation to the max-correlation difference statistic.

3.1. Mixing

Define $\sigma$-fields $\mathcal{F}_{T,\tau}^\infty \equiv \sigma \left( X_t : \tau \geq t \right)$ and $\mathcal{F}_{T,\tau}^t \equiv \sigma \left( X_t : \tau \leq t \right)$, and $\alpha$-mixing coefficients (Rosenblatt, 1956), $\alpha(t) \equiv \sup_{\varepsilon \in \mathbb{R}} \sup_{A \in \mathcal{F}_{T,\tau}^\infty, B \in \mathcal{F}_{T,\tau}^t} |P(A \cap B) - P(A) P(B)|$, for $l > 0$. We work in the setting of Chang, Jiang and Shao (2023), cf. Chang, Chen and Wu (2024), who deliver high-dimensional central limit theorems for possibly non-stationary mixing sequences or under a physical dependence setting similar to Zhang and Wu (2017). Chernozhukov, Chetverikov and Kato (2014, Appendix B), cf. Chernozhukov, Chetverikov and Kato (2019, Supplemental Appendix), allow for almost surely bounded stationary $\beta$-mixing processes, while Zhang and Wu (2017) extend results in Chernozhukov, Chetverikov and Kato (2013) to a large class of dependent stationary processes. Stationarity is not suitable here because even under the null we want to allow for global non-stationarity, and boundedness is typically too restrictive for many financial and macroeconomic time series.
Assumption 1.

a. (weak dependence): $\alpha(l) \leq K_1 \exp(-K_2 t^\phi)$ for some universal constants $\phi, K_1, K_2 > 0$.

b. (subexponential tails): $\max_{1 \leq t \leq T} P(|X_t| > c) \leq \theta_1 \exp(-\theta_2 t^{\varphi})$ for some universal constants $\varphi, \theta_1, \theta_2 > 0$.

c. (nondegeneracy): $\liminf_{T \to \infty} E[Z_t^2(h,k)] > 0 \forall (h,k)$.

d. (orthonormal basis): $\{B_k(x) : 0 \leq k \leq K\}$ forms a complete orthonormal basis on $\mathcal{L}(0,1)$; $B_k(x) \in \{-1,1\}$ on $[0,1]$; $\sum_{t=1}^T B_k(t) = O(n(k))$ for some positive strictly monotonic function $n : \mathbb{R}_+ \to \mathbb{R}_+$, $n(k)$ grows as $k \to \infty$.

Remark 1. A version of (a)-(c) are imposed in Chang, Jiang and Shao (2023, Conditions AS1-AS3). Under (a) their high dimensional mixing condition AS2 applies to $\{z_t(h,k)\}_{t=1,k=0}^{\infty}$; (b) trivially generalizes their AS1, which sets $\theta_1 \geq 1$ and $\varphi \in (0,1]$ for notational convenience. It covers traditional sub-exponential tails, and slower decay such that a moment generating function may not exist (Vershynin, 2018, Proposition 2.7.1), while permitting general forms of global nonstationarity. (c) yields AS3.

Remark 2. Nondegenerate (c) is common in the time series literature (e.g. Doukhan, 1994, Theorem 1), in particular for non-standardized statistics involving nonstationary sequences. It classically rules out degenerate dispersion and deviant negative co-dependence within the sequence $\{X_i, X_{i+h} - E[X_i, X_{i+h}]\}_{i=1}^{T-h}$. Simply note $E[Z_t^2(h,k)] = ((T-h)/T) \times E[(\lambda'X_{T-h}X_{T-h})^2]$ where $X_{T-h} = (X_1X_{i+h} - E[X_1X_{i+h}])_{i=1}^{T-h}$ and $\lambda = (T-h)^{-1/2}(B_k(1),...,B_k(T-h))$. Notice $\lambda'X_{T-h}X_{T-h} = 1$ since $B_k^2(i) = 1$. Thus (c) is satisfied by a classic positive definiteness property: $\inf_{r',t=1} E[(\lambda r'X_{T-h})^2] > 0 \forall (h,k)$ and $\forall T \geq T$ and some $T \in \mathbb{N}$. For example, impose fourth order stationarity (and therefore the null), and white noise $E[X_iX_{i+h}] = 0 \forall i \geq 1$ to reduce notation. Now define fourth order correlation coefficients $\rho(a,b,c,d) \equiv E[X_{i,h}X_{i+k},X_{i+h}] / E[X_{i,(i+h)}^2].$ Then by expanding $E[(\lambda r'X_{T-h})^2]$, (c) holds under pointwise non-degeneracy $E[X_{i,h}X_{i+h}] > 0$, and $\inf_{r',t=1} E[1 + \sum_{i=1}^{T-h-l} \lambda_i \lambda_{t-i}] > 0 \forall (h,k)$, $\forall T \geq T$, ruling out deviant negative linear dependence. See also the discussion in Chang, Chen and Wu (2024, p. 4-5). We cannot, however, impose fourth order stationarity broadly, and thus the preceding sufficient conditions, because that rules out an asymptotic analysis under local or global alternatives, cf. Section 3.3.

Remark 3. Assumption 1 reveals a trade-off vis-à-vis JWW. We allow for nonlinear $\{X_t\}$ with possibly non-iid errors, and possibly global nonstationarity under the null, but $X_t$ must have exponentially decaying tails and geometric dependence. The former rules out conventional GARCH processes (which lack higher moments), but includes GARCH-type processes with errors that have bounded support. JWW focus exclusively on linear processes $X_t = \sum_{i=0}^\infty \psi_i Z_{t-i}$ with iid $Z_t$, where $E[Z_t]^{4v} < \infty$ for some $v > 1$, excluding important nonlinear and conditionally heteroscedastic processes. They impose $\psi_i = O(1/(i(i^k))^{1/k})$ for some $k > 0$ and strict stationarity under the null, yielding $\sum_{h=1}^\infty |\gamma_h| < \infty$. Thus JWW allow for hyperbolic and geometric memory decay and the possible nonexistence of higher moments. Under physical dependence Assumption 1.a below, however, we allow for hyperbolic memory.

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3See, e.g., de Jong (1997, Assumption 2.a), but also see Billingsley (1999, Theorem 19.1).
Remark 4. The bound $|\sum_{t=1}^{T} B_k(t)| = O(\eta(k))$ (in $d$) is generally driven by the number of zero crossings on $[0, 1)$ in the underlying smooth basis function $B_k(x)$. Indeed, by Lemma 3 in HWW, Walsh $\mathcal{W}_k(t)$ exhibit up to $k$ zero crossings, and $|\sum_{t=1}^{T} \mathcal{W}_k(t)| \leq k + 1$ hence $\eta(k) = k$. Conversely, Haar composite $\Psi_k(t)$ exhibit up to $2^k$ zero crossings, and $|\sum_{t=1}^{T} \Psi_k(t)| = O(2^k)$ by Lemma B.1 in Hill and Li (2024), hence $\eta(k) = 2^k$.

Lemma 3.1. Under Assumption 1 we have $pr \lesssim \mathcal{H}_T^{1/2} (\ln(\mathcal{H}_T))^{7/6}/T^{1/9} \to 0$ for any sequences $\{\mathcal{H}_T, \mathcal{K}_T\}$ with $0 \leq \mathcal{H}_T \leq T - 1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^*)$ for some finite $K > 0$, and $\eta(\mathcal{K}_T) = O(\sqrt{T})$ where $\eta(\cdot)$ is the Assumption 1.d discrete basis summand bound. In this case $\max_{\mathcal{H}_T, \mathcal{K}_T} |\mathbf{Z}_T(h, k)| \rightarrow_{d} \max_{h, k \in \mathbb{N}} |\mathbf{Z}(h, k)|$ where $\mathbf{Z}(h, k) \sim N(0, \lim_{T \to \infty} \sigma^2_T(h, k))$ and $\lim_{T \to \infty} \sigma^2_T(h, k) < \infty$.

Remark 5. We require the orthonormal basis $\mathcal{B}(x)$ bound function $\eta(\cdot)$ and systematic sample counter $\mathcal{K}_T$ to satisfy $\eta(\mathcal{K}_T) = o(\sqrt{T})$ to ensure the mean summation $S_T^{(k)}(h) \equiv 1/\sqrt{T} \sum_{i=1}^{T-h} E[X_i, X_{i+h}] B_k(i)$ is negligible in the proof of Theorem 3.2 below. Simply note that under $H_0$ and Assumption 1.d, $|S_T^{(k)}(h)| \leq \gamma_h(1/\sqrt{T}) \sum_{i=1}^{T-h} B_k(i) \leq \gamma_h \eta(k)/\sqrt{T}$. Thus $\max_{1 \leq k \leq \mathcal{K}_T} |S_T^{(k)}| \leq \gamma_h(\eta(\mathcal{K}_T))/\sqrt{T} \to 0$ when $\eta(\mathcal{K}_T) = o(\sqrt{T})$. In a time series setting $\mathcal{H}_T = o(T)$ must hold to ensure consistency of sample autocovariances. Specifically $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$ and $\mathcal{K}_T = o(T^*)$ for any $K > 0$ yield $pr \lesssim \mathcal{H}_T^{1/2} (\ln(\mathcal{H}_T))^{7/6}/T^{1/9}$. $\mathcal{K}_T = o(T^*)$ is implied by $\eta(\mathcal{K}_T) = o(\sqrt{T})$ for Walsh and Haar functions: see below. $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$ is necessary due to high dimensional lagging: $[z_{T+h, h}]_{h=1,k=0}^{\mathcal{H}_T,\mathcal{K}_T}$ is $\sigma(X_\tau : \tau \leq t + \mathcal{H}_T)$-measurable and therefore has mixing coefficients $\alpha(l - \mathcal{H}_T^*)$, greatly impacting feasible $\{\mathcal{H}_T\}$ (cf. Chang, Jiang and Shao, 2023, Proposition 3).

Remark 6. Walsh functions $\mathcal{W}_k(t)$ have $\eta(k) = k$ hence $\mathcal{K}_T = o(\sqrt{T})$, while Haar composite $\Psi_k(t)$ have $\eta(k) = 2^k$ hence $\mathcal{K}_T = o(\sqrt{T})$, yielding $\mathcal{K}_T = o(T^*)$ respectively for some, or any, $K > 0$.

Now define $\sigma^2(h, k) = \lim_{T \to \infty} \sigma^2_T(h, k)$. Under $H_0$ and Assumption 1, $\sigma^2(h, k) \in (0, \infty)$. We now have a limit theory for the max-correlation difference.

Theorem 3.2. Let $H_0$ and Assumption 1 hold, and let $\mathcal{H}_T, \mathcal{K}_T \to \infty$. Let $\{\mathbf{Z}(h, k) : h, k \in \mathbb{N}\}$ be a zero mean Gaussian process with $\mathbf{Z}(h, k) \sim N(0, \sigma^2(h, k))$. Then it holds that $\mathbf{M}_T \rightarrow_{d} \gamma_0^{-1} \max_{h, k \in \mathbb{N}} |\mathbf{Z}(h, k)|$ for any $\{\mathcal{H}_T, \mathcal{K}_T\}$ with $0 \leq \mathcal{H}_T \leq T - 1$, $\mathcal{H}_T = O(T^{1/9}(\ln(T))^{1/3})$, $\mathcal{K}_T = o(T^*)$ for some finite $K > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$.

Remark 7. Consider the weighted/penalized version $\mathcal{M}_T^{(w, p)}$ in (2.5), and assume the weights satisfy $\lim_{T \to \infty} \inf_{\mathcal{H}_T, \mathcal{K}_T} \mathbf{M}_T^{(w, p)}(h, k) > 0$ a.s., and $\max_{\mathcal{H}_T, \mathcal{K}_T} |\mathbf{M}_T^{(w, p)}(h, k) - \mathbf{M}_h^{(w, p)}| \rightarrow_{p} 0$ where non-stochastic $\mathbf{M}_h^{(w, p)}$ satisfy $\lim_{h, k \in \mathbb{N}} |\mathbf{M}_h^{(w, p)}(h, k)| > 0$. The penalty functions $(p_v, p_w)$ are positive, monotonically increasing and bounded on compact sets. Then from arguments used to prove Theorem 3.2, it follows:

$$\mathcal{M}_T^{(w, p)} \rightarrow_{d} \gamma_0^{-1} \max_{h, k \in \mathbb{N}} \left[ \max_{h, k \in \mathbb{N}} \mathbf{M}_h^{(w, p)}(h, k) - \rho_h \right]$$

Now suppose we standardize with $\mathbf{M}_T^{(w, p)} = 1/\sqrt{\hat{V}_T(h, k)}$ with HAC estimator $\hat{V}_T^{(2)}(h, k)$ in (2.4), and kernel function $K(\cdot)$ belonging to class $R$ in de Jong and Davidson (2000, Assumption 1), or class $R_2$. 

The correlation difference expands to:

\[
\sqrt{T} (\hat{\rho}_h^{(k)} - \bar{\rho}_h) = \frac{1}{\gamma_0} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} (X_{t+h} - E[X_{t+h}]) B_h(t) + \frac{1}{\gamma_0} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} E[X_{t+h}] B_h(t). \tag{3.4}
\]

Under either hypothesis \(1/\sqrt{T} \sum_{t=1}^{T-h} (X_{t+h} - E[X_{t+h}]) B_h(t)\) is asymptotically normal. For the sample variance, we similarly have under either hypothesis and Assumption 1:

\[
\sqrt{T} \left( \gamma_0 - \frac{1}{T} \sum_{t=1}^{T} E[X_t^2] \right) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (X_t^2 - E[X_t^2]) = O_p(1).
\]
Hence, $\hat{g}_0 = g_0 + O_p(1/\sqrt{T})$ assuming existence of $g_0 \equiv \lim_{T \to \infty} 1/T \Sigma_{t=1}^T E[|X_t^2|]$. See below for derivations of $g_0$ under local and global alternatives.

In order to handle $1/\sqrt{T} \Sigma_{t=1}^T E[X_t X_{t+h}] B_k(t)$ in (3.4), we need a representation of a non-stationary covariance for fixed and local alternatives. Let $\gamma_h(u)$ be the time varying autocovariance function on $[0, 1]$. In the framework of locally stationary processes (cf. Dahlhaus, 1997, 2009), we may state the global alternative hypothesis as

$$H_1: \int_0^1 \left( \gamma_h(u) - \int_0^1 \gamma_h(v) dv \right)^2 du > 0 \text{ for some } h \geq 0.$$  

(3.5)

Thus under $H_1$ there exists a lag $h$ and subset $S_h \subset [0, 1]$ with positive Lebesgue measure such that $\gamma_h(u) \neq \int_0^1 \gamma_h(v) dv$ on $S_h$; hence $\gamma_h(u)$ is not almost everywhere constant on $[0, 1]$.

Now, by completeness of $\{B_k(u) : 0 \leq k \leq K\}$ under Assumption 1.d, we may write $\gamma_h(u) = \sum_{k=0}^\infty \omega_{h,k} B_k(u) = \omega_{h,0} + \sum_{k=1}^\infty \omega_{h,k} B_k(u)$, where $\omega_{h,k} = \int_0^1 \gamma_h(u) B_k(u)$ by orthonormality. Hence, under $H_1$ and orthonormality, for some $h \geq 0$,

$$\int_0^1 \left( \gamma_h(u) - \int_0^1 \gamma_h(v) dv \right)^2 du = \int_0^1 \left( \sum_{k=1}^\infty \omega_{h,k} B_k(u) \right)^2 du = \sum_{k=1}^\infty \omega_{h,k}^2 > 0,$$

which yields $\max_{h,k \in \mathbb{N}} \int_0^1 \gamma_h(u) B_k(u) du > 0$ under $H_1$.

A sequence of local alternatives with $\sqrt{T}$-drift logically follows:

$$H_1^L: E[X_i X_{i+h}] = \gamma_h + c_h(t/T)/\sqrt{T},$$

(3.6)

where $\gamma_h$ is a constant for each $h$, $\max_{h \in \mathbb{N}} |\gamma_h| \leq K < \infty$, and $c_h : [0, 1] \to \mathbb{R}$ are integrable functions on $[0, 1]$ uniformly over $h$ (i.e. $\sup_{h \in \mathbb{N}} \int_0^1 |c_h(u) du| < \infty$), that satisfy (3.5). Thus, under local alternative (3.6), by the preceding discussion:

$$\lim_{T \to \infty} \max_{h,k \in \mathbb{N}} \left| \int_0^1 c_h(u) B_k(u) du \right| > 0.$$  

(3.7)

In order to ensure $\min_{t \in \mathbb{Z}} E[X_t^2] > 0$, assume $\gamma_0 > 0$ and $c_0(u) \geq 0$ almost everywhere. Notice $\lim_{T \to \infty} |T^{-1} \Sigma_{t=1}^T c_0(t/T)| = |\int_0^1 c_0(u) du| < \infty$ yields under $H_1^L$:

$$g_0 \equiv \lim_{T \to \infty} \frac{1}{T} \Sigma_{t=1}^T E[X_t^2] = \gamma_0 + \lim_{T \to \infty} \frac{1}{\sqrt{T}} \frac{1}{T} \Sigma_{t=1}^T c_0(t/T) = \gamma_0.$$

Under Assumption 1.d $|\Sigma_{t=1}^T B_k(t)| = O(\eta(k))$, and $\max_{h \in \mathbb{N}} |\gamma_h| \leq K$ and $\eta(K_T) = o(\sqrt{T})$ by supposition. Hence $\max_{K_T} |\gamma_h| = o(1)$. Thus under $H_1^L$:

$$\frac{1}{\sqrt{T}} \Sigma_{t=1}^{T-h} E[X_t X_{t+h}] B_k(t) = \gamma_h \frac{1}{\sqrt{T}} \Sigma_{t=1}^{T-h} B_k(t) + \frac{1}{T} \Sigma_{t=1}^{T-h} c_h(t/T) B_k(t)$$

(3.8)

$$= o(1) + \frac{1}{T} \Sigma_{t=1}^{T-h} c_h(t/T) B_k(t) \to \int_0^1 c_h(u) B_k(u) du,$$
where here and below $o(1)$, and all subsequent $O_p(\cdot)$ and $o_p(\cdot)$ terms, do not depend on $(h,k)$.

Asymptotics rest on a uniform limit theory over $(h,k)$, which here needs to extend to the limit in (3.8). We therefore enhance local alternative (3.6) by assuming $c_h(\cdot)$ satisfies for any $\{\mathcal{H}_T, \mathcal{K}_T\}$:

$$
\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \frac{1}{T} \sum_{t=1}^{T-h} c_h(t/T) B_k(t) - \int_0^1 c_h(u) B_k(u) du \right| \to 0. \tag{3.9}
$$

Now define:

$$
C(h,k) = \int_0^1 c_h(u) B_k(u) du. \tag{3.10}
$$

Then under $H^L_1$, $\liminf_{T \to \infty} \max_{h,k \in \mathbb{N}} |C(h,k)| > 0$ in view of (3.7). Use arguments in the proof of Theorem 3.2 to yield under $H^L_1$:

$$
\sqrt{T} (\hat{\rho}^{(k)}_h - \hat{\rho}_h) = \frac{1}{g_0} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} (X_t X_{t+h} - E [X_t X_{t+h}]) B_k(t) + \frac{1}{g_0} \left( \int_0^1 c_h(u) B_k(u) du + o(1) \right) + O_p \left( \frac{1}{\sqrt{T}} \right).
$$

Hence, by Lemma 3.1 for any $\{\mathcal{H}_T, \mathcal{K}_T\}$ with $\mathcal{H}_T = O(T^{1/9} \ln(T))^{1/3}$, $\mathcal{K}_T = o(T^\epsilon)$ for some finite $\epsilon > 0$, and $\eta(\mathcal{K}_T) = o(\sqrt{T})$:

$$
\max_{\mathcal{H}_T, \mathcal{K}_T} \left| \sqrt{T} (\hat{\rho}^{(k)}_h - \hat{\rho}_h) \right| \xrightarrow{d} \frac{1}{g_0} \max_{h,k \in \mathbb{N}} |Z(h,k) + C(h,k)|. \tag{3.11}
$$

Thus, the proposed test has non-negligible power under the sequence of $\sqrt{T}$-local alternatives (3.6) when $c_h(\cdot)$ satisfy (3.5), for any complete orthonormal basis in view of (3.7). Notice under $H_0$ we have $g_0 = \gamma_0$, and $c_h(u) = 0 \forall u,h$ so that $C(h,k) = 0 \forall h,k$, yielding Theorem 3.2.

As a global generalization of $H^L_1$, we may write $H_1$ in discrete form as

$$
H_1 : E [X_t X_{t+h}] = \gamma_h + c_h(t/T), \tag{3.12}
$$

where as above $c_h(\cdot)$ satisfies (3.5). In this case $g_0 = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E [X_t^2]$ is identically:

$$
g_0 = \gamma_0 + \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} c_0(t/T) = \gamma_0 + \int_0^1 c_0(u) du > 0.
$$

Repeating the above derivations, we find similar to (3.8)

$$
\sqrt{T} (\hat{\rho}^{(k)}_h - \hat{\rho}_h) = \frac{1}{g_0} \frac{1}{\sqrt{T}} \sum_{t=1}^{T-h} (X_t X_{t+h} - E [X_t X_{t+h}]) B_k(t) + \frac{1}{g_0} \left( \int_0^1 c_h(u) B_k(u) du + o(1) \right) + O_p \left( \frac{1}{\sqrt{T}} \right).
$$
Thus $\max_{H_T, K_T} |\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h)| \overset{P}{\to} \infty$ given $\lim_{T \to \infty} \max_{H_T, K_T} \left| \int_0^1 c_h(u) B_k(u) du \right| > 0$. Identical arguments hold under physical dependence by replacing Lemma 3.1 and Theorem 3.2 with Theorem 3.3 (cf. Lemma 3.1* and Theorem 3.2* in Hill and Li (2024)).

The next result summarizes the preceding discussion.

**Theorem 3.4.** Let Assumption 1,b,c,d hold, and let $\{H_T, K_T\}$ satisfy $0 \leq H_T \leq T - 1$, $H_T \to \infty$, $K_T \to \infty$, and $H_T = O(T^{1/9}(\ln(T))^{1/3})$ under Assumption 1.a, or $H_T = o(T)$ under Assumption 1.a*.

a. Under $H_1$, (3.11) holds for non-zero $C(h, k)$ in (3.10), and any sequence $\{K_T\}$ with $K_T = o(T^\kappa)$ for some finite $\kappa > 0$ and $\eta(K_T) = o(\sqrt{T})$.

b. Under $H_1$, $\max_{H_T, K_T} |\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h)| \overset{P}{\to} \infty$ for any $\{K_T\}$.

In the following example we study a simple break in variance in order to show how the max-test behaves asymptotically.

**Example 3.5 (Structural Break in Variance).** Assume covariances do not depend on time: $E[X_tX_{t+h}] = \gamma_h$ for every $h \geq 1$, and there is a structural break in variance at mid-sample, cf. Perron (2006):

$$E[X_t^2] = g_{1,t} \text{ for } t = 1, \ldots, [T/2] \text{ and } E[X_t^2] = g_{2,T} \text{ for } t = [T/2] + 1, \ldots, T$$

for some strictly positive finite sequences $\{g_{1,t}, g_{2,T}\}$, $g_{1,t} \neq g_{2,T}$. In terms of Walsh or composite Haar systematic samples and $H_1$, this translates to $c_0(u) = c_{0,1} > 0$ for $u \in [0, 1/2)$, and $c_0(u) = c_{0,2} > 0$ for $u \in [1/2, 1]$, where $c_{0,1} \neq c_{0,2}$. All other $c_h(u) = 0$ on $[0, 1]$, $h \geq 1$. Hence, by construction of the first Walsh function $W_1(u)$ (or Haar composite $\psi_1(u)$):

$$\int_0^1 c_0(u) W_1(u) du = \int_0^{1/2} c_0(u) du - \int_{1/2}^1 c_0(u) du = \frac{c_{0,1} - c_{0,2}}{2} \neq 0.$$

Furthermore:

$$g_0 \equiv \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T E[X_t^2] = \gamma_0 + \int_0^1 c_0(u) du = \gamma_0 + \frac{c_{0,1} + c_{0,2}}{2}.$$  

The normalized correlation difference therefore satisfies for $h \geq 1$,

$$\sqrt{T}(\hat{\rho}_h^{(k)} - \hat{\rho}_h) = \frac{1}{g_0 \sqrt{T}} \sum_{t=1}^{T-h} (X_t X_{t+h} - E[X_t, X_{t+h}]) B_k(t) + o_p(1).$$

Under $H_1$ at lag $h = 0$ and $k = 1$ we then have:

$$\sqrt{T}(\hat{\rho}_0^{(1)} - \hat{\rho}_0) = \frac{1}{g_0 \sqrt{T}} \sum_{t=1}^T \left(X_t^2 - E[X_t^2]\right) W_1(t) + \sqrt{T} \left(\frac{c_{0,1} - c_{0,2}}{2g_0}\right) + o_p(1) = Z_T + C_T + o_p(1),$$

say. In view of asymptotic normality of $Z_T$, and $|C_T| \to \infty$, the max-correlation difference test is consistent when only the variance $E[X_t^2]$ exhibits a break given $\max_{H_T, K_T} \sqrt{T}|\hat{\rho}_h^{(k)} - \hat{\rho}_h| \geq \sqrt{T}|\hat{\rho}_0^{(1)} - \hat{\rho}_0| = \sqrt{T}|Z_T + C_T| \to \infty.$
4. Dependent wild bootstrap

We exploit a blockwise wild (multiplier) bootstrap for p-value approximation (cf. Liu, 1988). The method appears in various places as a multiplier bootstrap extension of block-based bootstrap methods (e.g. Künch, 1989). Shao (2010) presents a general nonoverlapping dependent wild bootstrap, exploiting a class of kernel smoothing weights that omits the truncated kernel, and uses only “big” blocks of data (“little” block size is effectively zero). Shao (2011) uses the same method exclusively with a truncated kernel for a white noise test for a stationary process that is a measurable function of an iid sequence. In both cases a sequence \( \{X_t\}_{t=1}^T \) is decomposed into \([T/b_T]\) blocks of size \(1 \leq b_T < T, b_T \to \infty\) and \(b_T = o(T)\).

Chernozhukov, Chetverikov and Kato (2019) exploit a Bernstein-like “big” and “little” block multiplier bootstrap for high dimensional sample means of stationary, dependent and bounded sequences, expanding that method for stationary processes by using two mutually independent iid sequences, one each for big and small blocks.

We expand ideas in Shao (2011) to non-stationary sequences. The use of only one set of “big” blocks and a truncated kernel eases technical arguments and notation, but a more general use of smoothing kernels and big/little blocks is readily supported by the theory presented here.

Set a block size \(b_T\) such that \(1 \leq b_T < T\), \(b_T/T^1 \to \infty\) and \(b_T/T^{1-\epsilon} \to 0\) for some tiny \(\epsilon > 0\). The number of blocks is \(N_T = [T/b_T]\). Denote index blocks \(\mathcal{B}_s = \{s-1)b_T+1, \ldots, sb_T\}\) for \(s = 1, \ldots, N_T\), and \(\mathcal{B}_{N_T+1} = \{N_Tb_T, \ldots, T\}\). Generate iid random numbers \(\{\xi_1, \ldots, \xi_{N_T}\}\) with \(E[\xi_i] = 0, E[\xi_i^2] = 1\), and \(E[\xi_i^4] < \infty\). Typically \(\xi_i\) is iid \(N(0, 1)\) in practice, and we make that assumption here to shorten proofs of weak convergence, cf. Lemmas B.4 and B.4* in Hill and Li (2024). See the proof of Lemma B.4 for further comments on \(\xi_i\).

Define an auxiliary variable \(\tilde{\varphi}_t = \xi_t\) if \(t \in \mathcal{B}_s\), and let \(\Delta \tilde{g}_T^{(dw)}(h, k)\) be a centered) bootstrapped version of \(\hat{\gamma}_T(h, k) - \gamma_h = T^{-1} \sum_{t=1}^{T-h} X_t X_{t+h} B_k(t)\):

\[
\Delta \tilde{g}_T^{(dw)}(h, k) \equiv T^{-1} \sum_{t=1}^{T-h} \tilde{\varphi}_t X_t X_{t+h} B_k(t) - T^{-1} \sum_{s=1}^{T-h} X_s X_{s+h} B_k(s).
\]

An asymptotically equivalent technique centers only on \(X_t X_{t+h}\), the key stochastic term:

\[
\Delta \tilde{g}_T^{(dw)}(h, k) \equiv T^{-1} \sum_{t=1}^{T-h} \tilde{\varphi}_t X_t X_{t+h} - T^{-1} \sum_{s=1}^{T-h} X_s X_{s+h} B_k(t).
\]

The bootstrapped test statistic is then \(M_T^{(dw)} = \hat{\gamma}_0^{-1} \max_{H_T, N_T} |\sqrt{T} \Delta \tilde{g}_T^{(dw)}(h, k)|\). Repeat \(M\) times. As a by-product of the main result below, conditional on the sample \(\{X_t\}_{t=1}^T\) this results in a sequence \(\{M_T^{(dw)}\}_{t=1}^M\) of iid draws \(M_T^{(dw)}\) from the limit null distribution of \(M_T\) as \(T \to \infty\) asymptotically with probability approaching one. The approximate p-value is \(p_T^{(dw)} = 1/M \sum_{l=1}^M 1(M_T^{(dw)} \geq M_T^{(dw)})\). The bootstrap test rejects \(H_0\) at significance level \(\alpha\) when \(p_T^{(dw)} < \alpha\).

The multiplier bootstrap has been studied in many contexts. Consult, e.g., Liu (1988), Shao (2010), and Shao (2011) to name a few. Centering \(X_t X_{t+h} B_k(t) - 1/T \sum_{s=1}^{T-h} X_s X_{s+h} B_k(s)\) is required because we use \(\{X_t X_{t+h} B_k(t)\}\) to approximate the null distribution, whether it is true or not, and \(E[X_t X_{t+h} B_k(t)] \neq 0\) for some \((h, k)\) under \(H_1\). The block-wise independent zero-mean Gaussian multiplier \(\tilde{\varphi}_t\) serves the purpose that \(\tilde{\varphi}_t (X_t X_{t+h} B_k(t) - 1/T \sum_{s=1}^{T-h} X_s X_{s+h} B_k(s))\), conditioned on the sample \(X_T = \{X_t\}_{t=1}^T\), is zero mean normally distributed; indeed, \(\Delta \tilde{g}_T^{(dw)}(h, k) X_T \sim N(0, \mathcal{V}_T(h, k))\).
for some \( \mathcal{V}_T(h, k) > 0 \ a.s. \). The blocks are constructed such that the dispersion term \( \mathcal{V}_T(h, k) \) well approximates the null limiting variance under general dependence, that is \( \mathcal{V}_T(h, k) \overset{p}{\rightarrow} \sigma^2(h, k). \) Thus, in the jargon of Gini and Zinn (1990, Section 3), \( \Delta \hat{g}_T^{(dw)}(h, k) | \mathcal{X}_T \overset{d}{\rightarrow} Z(h, k) \sim N(0, \sigma^2(h, k)) \) in probability, ensuring the bootstrapped process yields the null distribution, irrespective of whether \( H_0 \) holds or not.

Recall \( z_t(h, k) \) in (3.1) and \( Z_T(h, k) = 1/\sqrt{T} \sum_{t=1}^{T-h} z_t(h, k). \) Write \( \tilde{g}_T(h, k) \equiv 1/(T - h) \sum_{u=1}^{T-h} E \left[ X_u X_{u+h} | B_k(u) \right] \) and

\[
\mathcal{X}_{T,l}(h, k) \equiv \sum_{t=(l-1)b_T+1}^{lb_T} \{ X_t X_{t+h} B_k(t) - \tilde{g}_T(h, k) \},
\]

and define pre-asymptotic and asymptotic long run covariance functions \( s_T^T(h, k; \tilde{h}, \tilde{k}) \equiv 1/T \sum_{l=1}^{T-h/(b_T - 1)} E \left[ \mathcal{X}_{T,l}(h, k) \mathcal{X}_{T,l/(b_T - 1)}(\tilde{h}, \tilde{k}) \right] \) and \( s^T(h, k; \tilde{h}, \tilde{k}) \equiv \lim_{T \to \infty} s_T^T(h, k; \tilde{h}, \tilde{k}). \)

**Assumption 2.**

\( a. \) (i) \( \lim \inf_{T \to \infty} s^T(h, k; \tilde{h}, \tilde{k}) > 0 \ \forall(h, \tilde{h}, k, \tilde{k}); \) (ii) \( \max_{H_T, \mathcal{K}_T} | s^T(h, k; \tilde{h}, \tilde{k}) - s^2(h, k; \tilde{h}, \tilde{k}) | = O(T^{-\iota}) \) for some infinitesimal \( \iota > 0. \)

\( b. \) \( b_T/T^4 \to \infty \) and \( b_T = o(T^{1/2 - \iota}) \) for some infinitesimal \( \iota > 0. \)

**Remark 10.** (\( a. i \)) is the fourth order block bootstrap version of Assumption 1.c, used to ensure a high dimensional central limit theory extends to a long run bootstrap variance, cf. Chernozhukov, Chetverikov and Kato (2013, Lemma 3.1). (\( a. ii \)) seems unavoidable, and is required to link covariance functions for a high dimensional bootstrap theory, cf. Chernozhukov, Chetverikov and Kato (2013, Lemma 3.1) and Chernozhukov, Chetverikov and Kato (2015, Theorem 2, Proposition 1). The property is trivial under stationary geometric mixing or physical dependence, and otherwise restricts the degree of allowed heterogeneity. (\( b \)) simplifies a bootstrap weak convergence proof, but can be weakened at the cost of added notation, e.g. \( b_T/(\ln(T))^a \to \infty \) and \( b_T = o(T^{1/2} / (\ln(T))^b) \) for some \( a, b > 0. \)

The blockwise wild bootstrap is valid asymptotically under mixing or physical dependence.

**Theorem 4.1.** Let Assumptions 1.b,c,d and 2 hold, let \( \mathcal{H}_T, \mathcal{K}_T \to \infty, \) and let the number of bootstrap samples \( M = M_T \to \infty \) as \( T \to \infty. \) Let \( \{ b_T, \mathcal{H}_T \} \) satisfy \( b_T \to \infty \) and \( b_T = O(T^{1/2 - \iota}), \) \( 0 \leq \mathcal{H}_T \leq T - 1, \) and under Assumption 1.a \( \mathcal{H}_T = O(T^{1/9} / (\ln(T))^{1/3}), \) or \( \mathcal{H}_T = O(T^{1 - \iota} / b_T) \) for tiny \( \iota > 0 \) under Assumption 1.a*. Under \( H_0, P(\hat{p}^{(dw)}_{T,M} < \alpha) \to \alpha \) for any sequence \( \{ \mathcal{K}_T \} \) satisfying \( \mathcal{K}_T = o(T^k) \) for some finite \( k > 0 \) and \( \eta(\mathcal{K}_T) = o(\sqrt{T}). \) Under \( H_1 \) in (3.12) where \( c_h(\cdot) \) satisfy (3.7), \( P(\hat{p}^{(dw)}_{T,M} < \alpha) \to 1 \) for any \( \{ \mathcal{K}_T \}. \)

**5. Monte carlo study**

We now study the proposed bootstrap test in a controlled environment. We generate 1000 independently drawn samples from various models, with sample sizes \( T \in \{ 64, 128, 256, 512 \}. \) The models under the null and alternative hypotheses are detailed below.
5.1. Empirical size

We use four models of covariance stationary processes: MA(1), AR(1), Self Exciting Threshold AR(1) [SETAR], and GARCH (1,1):

- null-1 MA(1) \( X_t = \varepsilon_t \)
- null-2 AR(1) \( X_t = .5X_{t-1} + \varepsilon_t \)
- null-3 SETAR \( X_t = .7X_{t-1} - 1.4X_{t-1}I(X_{t-1} > 0) + \varepsilon_t \)
- null-4 GARCH(1,1) \( X_t = \sigma_t z_t, \quad z_t \stackrel{iid}{\sim} N(0,1), \quad \sigma_t^2 = 1 + .3\varepsilon_{t-1}^2 + .6\sigma_{t-1}^2 \)

Models #1-#3 have an iid error \( \varepsilon_t \) distributed \( N(0,1) \) or Student’s-t with 5 degrees of freedom (\( \tau_5 \)); or \( \varepsilon_t \) is stationary GARCH(1,1) \( \varepsilon_t = \sigma_t z_t, \quad z_t \stackrel{iid}{\sim} N(0,1), \quad \sigma_t^2 = 1 + .3\varepsilon_{t-1}^2 + .6\sigma_{t-1}^2 \), with iteration \( \sigma_t^2 = 1 + .3\varepsilon_{t-1}^2 + .6\sigma_{t-1}^2 \) for \( t = 2, ..., T \). The SETAR model switches between AR(1) regimes with correlations .7 and -.7. GARCH and SETAR models, and any model with GARCH errors, do not have a linear form \( X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i} \), with iid \( Z_t \) and non-random \( \psi_i \). We simulate 2\( T \) observations for each model and retain the latter \( T \) observations for analysis. Test results in GARCH cases should be viewed with caution: max-test asymptotics have only been established under sub-exponential tails while GARCH processes have regularly varying (i.e., “power”) tails, and JWW’s test requires a linear model with an iid error.

5.2. Empirical power

We study empirical power by using models similar to those in Paparoditis (2010); Dette, Preuß and Vetter (2011), Preuß, Vetter and Dette (2013) and JWW, with the addition of allowing for non-iid errors and non-stationarity in variance. The models are as follows:

- alt-1 (NI) \( X_t = 1.1 \cos(1.5 - \cos(4\pi t/T)) \varepsilon_{t-1} + \varepsilon_t \)
- alt-2 (NVI) \( X_t = .8 \cos(1.5 - \cos(4\pi t/T)) \varepsilon_{t-6} + \varepsilon_t \)
- alt-3 (NII) \( X_t = .6 \times \sin(4\pi t/T) X_{t-1} + \varepsilon_t \)
- alt-4 (NIII) \( X_t = \begin{cases} .5X_{t-1} + \varepsilon_t \quad \text{for } 1 \leq t \leq T/4 \cup \{3T/4 < t \leq T \} \\ -0.5X_{t-1} + \varepsilon_t \quad \text{for } T/4 < t \leq 3T/4 \end{cases} \)
- alt-5 (NVI) \( X_t = \begin{cases} .5X_{t-1} + \varepsilon_t \quad \text{for } 1 \leq t \leq T/2 \\ -0.5X_{t-1} + \varepsilon_t \quad \text{for } T/2 < t \leq T \end{cases} \)
- alt-6 (eq. (16)) \( X_t = 2\varepsilon_t - \{1 + .5\cos(2\pi t/T)\} \varepsilon_{t-1} \)
- alt-7 (NV) \( X_t = -.9 \sqrt{(t/T)} X_{t-1} + \varepsilon_t \)
- alt-8 \( X_t = .5X_{t-1} + \psi_t; \quad \psi_t = \begin{cases} \varepsilon_t \quad \text{for } 1 \leq t \leq T/3 \\ 2\varepsilon_t \quad \text{for } 3T/4 < t \leq T \end{cases} \)
- alt-9 \( X_t = .8 \cos(1.5 - \cos(4\pi t/T)) \varepsilon_{t-25} + \varepsilon_t \)

Models 1-7 are used in JWW: we display parenthetically their corresponding model/equation number. Models 1, 2, 4 are considered in Paparoditis (2010); Dette, Preuß and Vetter (2011) use models 1, 2, 4, and 6; Preuß, Vetter and Dette (2013) study 2, 5, and 7. Alt-8 presents a structural change in variance only, and alt-9 is a distant version of alt-2 and therefore more difficult to detect (lag 25 as opposed to lag 6). As above, we use either iid standard normal, iid \( \tau_5 \), or GARCH(1,1) \( \varepsilon_t \).

5.3. Tests

5.3.1. Max-test

We perform the bootstrapped max-correlation difference test with \( M_T \) and \( M_T^{(p)} \). The latter has penalties \( p_h = (h + 1)^{1/4}/2 \) and \( q_h = k^{1/4}/2 \). More severe penalties, e.g., \( q_h = k^{1/2}/2 \), do not improve test
Table 1. \(H_T, K_T\) Combinations by Basis

|                      | Walsh Basis \(\{W_k(t)\}\) | Haar Basis \(\{\Psi_k(t)\}\) |
|----------------------|-----------------------------|-----------------------------|
| Case 1 (JWW)         | Case 2                      | Case 1 (JWW)                | Case 2                      |
| \(T\)                | \(H_T\) | \(K_T\) | \(H_T\) | \(K_T\) | \(H_T\) | \(K_T\) | \(H_T\) | \(K_T\) |
| \(\log_2(T)^{-99} - 3\) | \(T^{1/3}\) | \(2T^{-49}\) | \(5T^{-49}\) | \(\log_2(T)^{-99} - 3\) | \(\log_2(T)^{-99} - 3\) | \(\log_2(T)^{-99} - 3\) | \(\log_2(T)^{-99} - 3\) |
| 64                   | 2 | 4 | 14 | 3 | 2 | 4 | 14 | 4 |
| 128                  | 3 | 5 | 20 | 5 | 3 | 5 | 20 | 5 |
| 256                  | 4 | 6 | 30 | 7 | 4 | 5 | 30 | 5 |
| 512                  | 5 | 8 | 42 | 10 | 5 | 6 | 42 | 6 |

performance. A weighted version of the test with HAC estimator (2.4) leads to competitive size but generally lower power, hence we focus only on \(M_T\) and \(M_T^{(p)}\). We use Walsh or Haar functions for two max-tests, and a third combined max-max-statistic shown below (2.6). We only report results based on Walsh functions because (i) the Haar-based tests (max-test, and JWW’s test detailed below) yielded far lower power across most alternatives studied here; hence (ii) the max-max test performed essentially on par with, or was slightly trumped by, the Walsh-based test.

We use 500 bootstrap samples with multiplier iid variable \(\xi_t \sim N(0, 1)\). Theorem 4.1 requires a block size bound \(b_T = o(T^{1/2-\tau})\) for some tiny \(\tau > 0\), hence we use \(b_T = [T^{1/2-\eta}]\) where \(\eta = 10^{-10}\). Similar block sizes, e.g. \(b_T = [bT^{1/2-\eta}]\) with \(b \in [5, 2]\) lead to similar results.4

Theorem 4.1 also requires \(H_T = O(T^{1-\kappa}/b_T)\), \(K_T = o(T^{\kappa})\) for some \(\kappa > 0\), and \(\eta(K_T) = o(\sqrt{T})\). In the Walsh case \(\eta(k) = k\) hence \(K_T = o(\sqrt{T})\); in the Haar case \(\eta(k) = 2^k\) hence \(K_T = o(ln(T))\). In the Walsh case, we used two pairings of sequences \(\{H_T, K_T\}\). The first \(H_T = [\log_2(T)^{-99} - 3]\) and \(K_T = [T^{1/3}]\) is used in JWW. The second \(H_T = [2T^{-49}]\) and \(K_T = [.5T^{-49}]\) satisfies our assumptions but are not valid in JWW. The latter \((H_T, K_T)\) are generally larger, where \(H_T\) is larger by an order of \(\times 7\). This will lead to higher power for large \(T\) in theory, but in small samples obviously a larger \(h\) results in fewer observations for computation, and therefore a loss in sharpness in probability. In the Haar case we use either \(H_T\) above, and \(K_T = [(ln(T))^{-99}]\). Refer to Table 1.

5.3.2. JWW test

Write \(\hat{\gamma}_h \equiv [\hat{\gamma}_1, ..., \hat{\gamma}_h]^{'}\) and \(\hat{\gamma}_h^{(k)} \equiv [\hat{\gamma}_1^{(k)}, ..., \hat{\gamma}_h^{(k)}]^{'}\). The test statistic is

\[
\hat{D}_T \equiv \max_{1 \leq k \leq K_T, \max_{1 \leq h \leq H_T}} \left\{ T \left( \hat{\gamma}_h^{(k)} - \hat{\gamma}_h \right) \left( \hat{\Gamma}_h^{(k)} \right)^{-1} \left( \hat{\gamma}_h^{(k)} - \hat{\gamma}_h \right) - 2h \right\} - \sqrt{k-1}
\]

where \(\hat{\Gamma}_h^{(k)}\) is an estimator of the \(h \times h\) asymptotic covariance matrix of \(\sqrt{T} \left( \hat{\gamma}_h^{(k)} - \hat{\gamma}_h \right)\). See Jin, Wang and Wang (2015, Sections 2.3-2.5) for details on computing \(\hat{\Gamma}_h^{(k)}\) (under the assumption

4Shao (2011) uses \(b_T = [bT^{1/2}]\) with \(b \in [5, 1, 2]\), leading to qualitatively similar results. Hill and Motegi (2020) also use \(b = 1\), but find qualitatively similar results for values \(b \in [5, 1, 2]\).
of linearity $X_t = \sum_{i=0}^{\infty} \theta_i Z_{t-i}$ with an iid $Z_t$.\(^5\) We use both Walsh and Haar bases, the same tuning parameters that JWW use for covariance matrix estimation, and the same $\{H_T, K_T\}$ described above.\(^6\)

We perform the test both based on a simulated critical values (denoted $\mathcal{D}_T^{(s)}$), and bootstrapped p-values ($\mathcal{D}_T^{(b)}$) in order to make a direct comparison with the method developed here. We simulate critical values for each basis and each pair $(H_T, K_T)$ by running a separate simulation with 200,000 independently drawn samples of size $T$ of iid $N(0,1)$ distributed random variables $X_t$, and use the true excess kurtosis value $0$ in the covariance estimator $\hat{\Gamma}_h$. The bootstrap is performed by replacing $\hat{\gamma}^{(k)}_h - \hat{\gamma}_h$ in $\mathcal{D}_T$ with $\Delta\hat{\delta}_T^{(bw)}(h,k)$ from (4.1). We do not prove asymptotic validity of the bootstrapped p-value, but once uniform consistency of $\hat{\gamma}^{(k)}_h$ is established, it follows identically from arguments given in the proof of Theorem 4.1. Indeed, the bootstrap is valid for linear and nonlinear processes with iid or non-iid innovations, and covering the nonstationary processes under $H_1$. The simulated critical values, however, are suitable in theory only for linear processes with iid innovations since they rely on the specific form of $\hat{\gamma}^{(k)}_h$ used here, and a pivotal Gaussian null limit distribution, cf. Jin, Wang and Wang (2015, Sections 2.3-2.5).

### 5.4. Results

Tables A.3-A.6 in Hill and Li (2024, Appendix D) present rejection frequencies at (1%, 5%, 10%) significance levels when a Walsh basis is used. The penalized max-test does not perform better than the non-penalized test, and generally performs worse under the alternative. Indeed, as discussed above, there is no theory driven reason for adding penalties for a max-test. In the sequel we therefore only discuss the non-penalized test.

Similarly, the bootstrapped JWW test is generally over-sized, and massively over-sized at small $T$ under $(H, K)$ Case 1, the only valid case in this study. We suspect the cause is the estimated variance matrix due to its many components and tuning parameters. We henceforth only discuss results based on simulated critical values.

#### 5.4.1. Null

Both tests are comparable for MA and AR models with iid Gaussian or $t_5$ errors, with fairly accurate empirical size. The max-test has accurate size in many cases, and is otherwise conservative. JWW’s test tends to be over-sized in the AR model with GARCH errors under both $(H, K)$ cases, and is over-sized in the AR model with $t_5$ errors under Case 2 when $T \leq 128$. Recall $H_T$ is much larger under Case 2, which will be a hindrance at smaller $T$ for test statistics that simultaneously incorporate a set of autocovariances (e.g. Wald or portmanteau statistics).

In the SETAR case JWW’s test is largely over-sized, while the max-test is slightly under-sized with improvement under $(H, K)$ Case 2. JWW’s test is over-sized for small $T$ with the GARCH model, but otherwise works well.

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\(^5\)There is a typo in Jin, Wang and Wang (2015, Theorem 2) concerning their covariance matrix and therefore its estimator. A parameter $k_2$, referred to as the kurtosis of the iid $Z_t$, is in fact the excess kurtosis ($\text{kurtosis} - 3$). See Proposition 7.3.1 in Brockwell and Davis (1991), in particular eq. (7.3.5), cf. Jin, Wang and Wang (2015, p. 915).

\(^6\)The bandwidth parameter $\lambda$ in $[T^{-1}]$, the number of sample covariances that enter the asymptote covariance matrix estimator, is set to $\lambda = .4$ based on a private communication with the authors. In order to compute the (excess) kurtosis of iid $Z_t$ under linearity, similar to Jin, Wang and Wang (2015, eq. (15)) we use an estimator in Kreiss and Paparoditis (2015), with two bandwidths $b_j = c_j T^{-1/3}$ where each $c_j = 1.25 \times \hat{\gamma}(0)$ (see Jin, Wang and Wang, 2015, p. 903).
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We give a simple summary of which test generally dominates for each model and case based on similar endeavors the question of whether other bases may yet perform better than the Walsh basis for a test of basis based tests direct power toward alternatives implied by basis-specific systematic samples. Thus, that allows for nonlinearity and random volatility, and heterogeneity under either hypothesis. Our test we present a max-correlation difference test for testing covariance stationarity in a general setting.

6. Conclusion

We present a max-correlation difference test for testing covariance stationarity in a general setting that allows for nonlinearity and random volatility, and heterogeneity under either hypothesis. Our test exploits a generic orthonormal basis under mild conditions, with Walsh and Haar wavelet function examples. We do not require estimation of an asymptotic covariance matrix, our test can detect a break in variance, and we deliver an asymptotically valid dependent wild bootstrapped p-value. Orthonormal basis based tests direct power toward alternatives implied by basis-specific systematic samples. Thus, by combining bases a power improvement may be achievable. In controlled experiments, however, we find the Walsh basis yields superior results compared to a composite Haar basis. We leave for future endeavors the question of whether other bases may yet perform better than the Walsh basis for a test of covariance stationarity.

Furthermore, the max-test dominates JWW’s in some case, while JWW’s dominates in others. The max-test is best capable of delivering sharp empirical size for a nonlinear process and when errors are

| H1 \ H2 | N(0, 1) t5 GARCH | N(0, 1) t5 GARCH |
|--------|------------------|------------------|
| alt-1  | D_T small n D_T small n D_T | D_T small n D_T small n D_T |
| alt-2  | M_T M_T M_T | M_T M_T M_T |
| alt-3  | D_T small n D_T small n D_T | D_T small n D_T small n D_T |
| alt-4  | D_T small n D_T small n D_T | D_T small n D_T small n D_T |
| alt-5  | M_T M_T M_T | M_T M_T M_T |
| alt-6  | M_T M_T M_T | M_T M_T M_T |
| alt-7  | D_T large n D_T large n D_T | D_T large n D_T large n D_T |
| alt-8  | M_T M_T M_T | M_T M_T M_T |
| alt-9  | M_T M_T M_T | M_T M_T M_T |

Each cell dictates which test performed best (in certain cases). For example “D_T small n” implies D_T dominates for smaller sample sizes, and for other n the two tests are comparable. “M_T” implies M_T dominates across sample sizes.

5.4.2. Alternative

In Table 2 we give a simple summary of which test generally dominates for each model and case based on the complete simulation results. In brief, each test dominates for certain models, and in some cases they are comparable. JWW’s test generally dominates in models 1, 3, and 4, and for model 7 for larger sample sizes. This applies across error cases, including GARCH errors.

The max-test dominates in models 2, 6, 8 and 9, with strong domination for model 8 (break in variance), and models 2 and 9 (distant nonstationarity). Indeed, JWW’s test has only negligible power for models 2, 8 and 9: by construction it cannot detect a break in variance (model 8), and seems incapable of detecting a distant (model 9), or even semi-distant (model 2), form of covariance nonstationarity.

Overall, both tests clearly have merit, and seem to complement each other based on the different cases in which they each excel. See Hill and Li (2024, Appendix C) for an application of both tests to international exchange rates.
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non-iid, and is particularly suited for detecting distant (large lag) forms of covariance non-stationarity, and a break in variance. The former corroborates findings in Hill and Motegi (2020), who find a max-correlation white noise test strongly dominates Wald and portmanteau tests when there is a distant non-zero correlation. We conjecture this will carry over to other nonstationary models with distant breaks in covariance, but leave this idea for future consideration.

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Supplementary material

Additional results, omitted proofs and complete simulation results.

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