A THEORY OF AVERAGE RESPONSE TO LARGE JUMP PERTURBATIONS

RAFAIL V. ABRAMOV

Abstract. A key feature of the classical Fluctuation Dissipation theorem is its ability to approximate the average response of a dynamical system to a sufficiently small external perturbation from an appropriate time autocorrelation function of the unperturbed dynamics of this system. In the present work, we examine the situation where the state of a nonlinear dynamical system is perturbed by a finitely large, instantaneous external perturbation (jump). Such jump can be either deterministic or stochastic, and in the case of a stochastic jump its randomness can be spatial, or temporal, or both. We show that, even for large instantaneous jumps, the average response of the system can be expressed in the form of a suitable time autocorrelation function of the corresponding unperturbed dynamics. For stochastic jumps, we consider two situations: one where a single spatially random jump of a system state occurs at a predetermined time, and another where jumps occur randomly in time with small space-time dependent statistical intensity. For all studied configurations, we compute the corresponding average response formulas in the form of suitable time autocorrelation functions of the unperturbed dynamics. Some efficiently computable approximations are derived for practical modeling scenarios.

1. Introduction

The classical Fluctuation Dissipation theorem (FDT) [49, 50, 62, 70] provides a leading order approximation to the statistical response of a dynamical system to a small deterministic external perturbation via statistical correlations of the unperturbed dynamics. The FDT offered more insight into statistical properties of dynamical processes near equilibrium in various scientific applications, such as statistical mechanics of identical particles [23, 30, 51], Ornstein–Uhlenbeck Brownian motion [49, 50, 63, 73], motion of electric charges [62, 74], turbulence [47, 48], quantum field theory [24, 32, 34, 75], chemical physics [71, 72], and physical chemistry [60]. In geophysical science, the FDT was proposed as a sensible approximation for appropriate variables in a complex climate system [42, 52, 53] despite the absence of a classical Gaussian equilibrium state of the traditional statistical mechanics. This observation spurred a series of works [21, 25, 26, 30, 36–40, 44, 51, 52, 56, 57, 61, 65], where various applications of the FDT in the weather and climate modeling were proposed. In the author’s past works [1–3, 6, 8, 13–15], a computational framework predicting the average response of both deterministic and stochastic dynamical systems to a small deterministic or stochastic external perturbation was developed, studied, and used in a new method for the parameterization of unresolved processes in reduced models of multiscale dynamics [4, 5, 7, 9].
Thus far, the extensively studied types of external perturbations were largely limited to small bounded forcing perturbations – either deterministic, or in the form of a Brownian motion. However, in many practical applications the external perturbations are, first, not necessarily small, and, second, not necessarily in the form of a bounded external forcing. For example, a physical system may experience an external “impulse forcing” – that is, the external forcing in the form of a delta-function, which instantaneously changes the state of the system. Moreover, this change can be finitely large – that is, the delta-function of the impulse forcing may not be necessarily scaled by a small parameter.

An ubiquitous example of such a global event is known as the Permian–Triassic extinction \([29, 45, 76]\). Due to an unknown cause, catastrophic global climate changes occurred on Earth some 250 million years ago between Permian and Triassic periods. As a result, about 90% of all living species has become extinct \([41, 67–69]\). One of the accepted scientific hypotheses is that this event was triggered by a large meteorite impact \([19, 20]\). It is strikingly obvious that any mathematically and physically adequate explanation or model of this event cannot possibly assume that this impact was “small” in any reasonable sense – although one can likely reasonably assume that this event was “instantaneous”, at least relative to the time scale of the subsequent global climate change. In addition, another reasonable assumption is that such events tend to occur randomly in time, although they are statistically unlikely to occur frequently.

In the current work, we focus on the scenario where the state of a nonlinear dynamical system is perturbed by a finitely large, instantaneous external perturbation, or “jump”. For generality, here we assume that this jump can be either deterministic or stochastic, and for a stochastic jump we further assume that its randomness can manifest in the spatial configuration of the jump, or its temporal frequency, or both. We find that the average response of the system can be expressed in the form of a suitable time autocorrelation function of the corresponding unperturbed dynamics even for large instantaneous jumps. In the case of jumps which incorporate a random component, we examine two scenarios. The first one is where a single spatially random jump of a system state occurs at a predetermined time, while the second scenario is the one where jumps occur randomly in time with small space-time dependent statistical intensity. We also find that, for all studied configurations, the corresponding average response formulas are computable in the form of suitable time autocorrelation functions of the unperturbed dynamics, just like in the classical FDT formulation. For practical modeling scenarios, we also derive suitable approximations of the general response formulas which can be efficiently computed in the context of a numerical simulation.

The work is organized as follows. In Section 2 we state the general form of the underlying, unperturbed nonlinear dynamical system, together with its corresponding forward Kolmogorov (or Fokker–Planck) equation \([18, 64, 70]\). In Section 3 we derive the explicit formula of the average response of the system to a deterministic instantaneous jump perturbation. In Section 4 we extend the previously derived average response formula on the case of a random instantaneous perturbation, which nonetheless occurs at a prescribed time. In Section 5 we derive the average response formula for the most general perturbation scenario, where spatially random jump perturbations also occur at random times. In Section 6 we discuss the results and suggest future research directions.
2. Unperturbed dynamics

In this section, we need to specify the general form of an unperturbed dynamical system for a typical application in natural sciences. From what is to be presented below, the basic requirements on the form of the unperturbed dynamical system can be summarized as follows. First, the unperturbed dynamical system may incorporate a deterministic vector field, as well as stochastic effects. We will, however, restrict the form of the unperturbed dynamical system so that its solutions are continuous, as it is implied that the discontinuities will be introduced via the jump perturbations. Additionally, we require that the unperturbed system has an explicitly formulated forward Kolmogorov equation \([64, 70]\) with a differentiable stationary solution. The corresponding perturbed system must also have a forward Kolmogorov equation, although we are not going to consider its stationary solutions, if any.

Other than the points listed above, there do not appear to be any other fundamental constraints which would impose further unmitigable issues. In particular, the presence of the stochastic component of the dynamics is not necessarily required for the stationary solution of the forward Kolmogorov equation to be differentiable – for example, if the deterministic vector field preserves a quadratic energy, the uniform distribution on a constant energy surface is usually both the stationary statistical state for the system and a smooth distribution \([10–12]\).

Thus, throughout the work, we will assume that the unperturbed dynamical system is described, in general, via the following Itô stochastic differential equation:

\[
(2.1) \quad x(t) = x_0 + \int_0^t f(x(s))\,ds + \int_0^t G(x(s))\,dW(s).
\]

Here, \(x \in \mathbb{R}^K\) is the state vector of the system, \(W(t)\) is a \(K\)-dimensional Wiener process, while \(f : \mathbb{R}^K \to \mathbb{R}^K\) and \(G : \mathbb{R}^K \to \mathbb{R}^{K \times K}\) are smooth vector fields. As motivated above, the form of (2.1) is chosen so that it is the most general form of the Lévy-type Feller process \([31]\) whose solutions are almost surely continuous, and which, under a random jump perturbation, retains the form of the infinitesimal generator compatible with Courrège’s theorem \([27]\), so that the forward Kolmogorov equation of the jump-perturbed process can be obtained in an explicit manner \([18]\).

Let \(\psi : \mathbb{R}^K \to \mathbb{R}\) be a twice differentiable function with bounded second derivatives. Let \(E_{t,t_0}[\psi](x)\) denote the conditional expectation of \(\psi\) at time \(t\), provided that the initial state of the system at time \(t_0\) is \(x\). Then, the infinitesimal generator \([18, 35, 64]\) of \(E_{t,t_0}[\psi](x)\) is given via:

\[
(2.2) \quad \frac{\partial}{\partial t} E_{t,t_0}[\psi] \bigg|_{t=t_0} = \mathcal{L}[\psi] = f \cdot \frac{\partial \psi}{\partial x} + \frac{1}{2} GG^T : \frac{\partial^2 \psi}{\partial x^2},
\]

where “:\:" denotes the Frobenius matrix product.

Let \(p(t, x)\) denote the probability distribution of solutions of (2.1). Then, we can relate the \(p\)-average of \(\psi\) at time \(t + s\) to the probability density \(p(t, x)\) via

\[
(2.3) \quad \langle \psi \rangle(t + s) = \int_{\mathbb{R}^K} \psi(x)p(t + s, x)\,dx = \int_{\mathbb{R}^K} E_{t+s,t}[\psi](x)p(t, x)\,dx.
\]
The forward partial differential equation for \( p(t, x) \) is known either as the forward Kolmogorov equation [18,35,64] or as the Fokker-Planck equation [70], and is given via

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \cdot (pf) = \frac{1}{2} \frac{\partial^2}{\partial x^2} : (pGG^T).
\]

In what follows, we will assume that (2.4) has a stationary solution \( p_0(x) \),

\[
\frac{\partial}{\partial x} \cdot (p_0f) = \frac{1}{2} \frac{\partial^2}{\partial x^2} : (p_0GG^T).
\]

If the union of all solutions to (2.5) consists of multiple ergodic components, we will assume below that \( p_0(x) \), whose support we denote as \( \mathcal{A} \), is the indecomposable ergodic component which is “physically relevant” – that is, a generic initial condition to (2.1) almost certainly falls into \( \mathcal{A} \), and the subsequent jump perturbations are such that the state of the system in (2.1) never leaves \( \mathcal{A} \). This assumption is necessitated by the need to use the Birkhoff–Khinchin theorem [22,46] for the practical computation of the average response further below.

2.1. Conditional probability density. In what follows, it is convenient to introduce the conditional probability density \( P_{t,t_0}(x|y) \), defined via

\[
\mathbb{E}_{t,t_0}[\psi](x) = \int_{\mathbb{R}^k} \psi(y) P_{t,t_0}(y|x) dy.
\]

From (2.3), and observing that \( \psi \) is arbitrary, it follows that

\[
p(t + s, x) = \int_{\mathbb{R}^k} P_{t+s,s}(x|y)p(s, y) dy.
\]

Substituting the above equation into (2.4) and stripping the integral over \( p(y)dy \), we obtain the equation for \( P \):

\[
\frac{\partial}{\partial t} P_{t,t_0}(x|y) + \frac{\partial}{\partial x} \cdot (P_{t,t_0}(x|y)f) = \frac{1}{2} \frac{\partial^2}{\partial x^2} : (P_{t,t_0}(x|y)GG^T), \quad P_{t_0,t_0}(x|y) = \delta(x - y).
\]

Observing that the above equation is autonomous with respect to \( t \), it is clear that \( P_{t,t_0} = P_{t-t_0} \), that is, the conditional probability density is a function of difference between the starting and ending times. Therefore, without loss of generality, we can assume that \( t_0 = 0 \), which yields

\[
\frac{\partial}{\partial t} P_{t}(x|y) + \frac{\partial}{\partial x} \cdot (P_{t}(x|y)f) = \frac{1}{2} \frac{\partial^2}{\partial x^2} : (P_{t}(x|y)GG^T), \quad P_{0}(x|y) = \delta(x - y).
\]

Consequently, according to (2.6), the conditional expectation is also a function of the time difference: \( \mathbb{E}_{t,t_0}[\psi](x) = \mathbb{E}_{t-0}[\psi](x) \).

2.2. Special case: the Ornstein–Uhlenbeck process. Here we consider a special case of the unperturbed dynamical system in (2.1), which has the convenience of being exactly solvable. Consider the following centered Ornstein–Uhlenbeck process [73]:

\[
x(t) = x_0 - \int_0^t Lx(s) ds + \int_0^t GdW(s).
\]
Above, $L$ is a constant positive definite $K \times K$ matrix, while $G$ is a constant $K \times K$ matrix, with $GG^T$ being positive definite. The solution to the Ornstein–Uhlenbeck process above is given via Duhamel’s principle:

\begin{equation}
    x(t) = e^{-tL}x_0 + \int_0^t e^{(s-t)L}GdW(s).
\end{equation}

Via the properties of the Itô integral \([35,43,64]\), it can be readily seen that the expectation of the state variable $x(t)$ satisfies

\begin{equation}
    E_t[x](x_0) = e^{-tL}x_0.
\end{equation}

The stationary probability density $p_0(x)$ of the Ornstein–Uhlenbeck process in (2.10) is given via

\begin{equation}
    p_{OU}^0(x) = \frac{1}{(2\pi)^{K/2}\sqrt{\det C}} \exp \left(-\frac{1}{2}x^T C^{-1}x\right),
\end{equation}

where $C$ is the covariance matrix of the process, given via the equation

\begin{equation}
    LC + CL^T = GG^T.
\end{equation}

More details on this topic can be found in \[70\].

### 3. A large deterministic jump perturbation

In this work, the definition of the average response is identical to that in our earlier works on this topic \[1–3,6,8,13–15,57\]. Namely, assume that we have a large statistical ensemble of solutions of (2.1). This ensemble is initially distributed according to $p_0(x)$. At the prescribed time $t_0$, the following instantaneous jump perturbation is applied to the states of all ensemble members:

\begin{equation}
    x(t_0) = x(t_0-) + h(x(t_0-)),
\end{equation}

Above in (3.1), $h : \mathbb{R}^K \rightarrow \mathbb{R}^K$ is a continuous function, and $x(t-)$ denotes the left-limit at $t$. Observe that the action of $h(x)$ can be interpreted as the instantaneous jump perturbation of a given trajectory, which, generally, depends on the pre-jump state (but not on the time of the jump). As noted above, here we assume that for any $x \in \mathcal{A}$, $h(x) \in \mathcal{A}$ – that is, the state of the system does not leave the support $\mathcal{A}$ of the physically relevant ergodic component $p_0(x)$ of (2.1) as a result of the jump.

The resulting statistical discrepancy between the perturbed ensemble and the unperturbed ensemble, as a function of time past the moment of the perturbation, is what we refer to as the “average response”. The key difference between the previous studies and the current work is that in our previous works a time-dependent forcing was applied at the initial time and past that, whereas here the perturbed trajectories continue according to (2.1) once the jump perturbation has been applied.

The average response of the statistical ensemble is measured via the difference between the perturbed and unperturbed conditional expectations $E_t[\psi]$ of a test function $\psi(x)$, averaged over the equilibrium probability density $p_0(x)$:

\begin{equation}
    \Delta \langle \psi \rangle (t - t_0) = \int_{\mathcal{A}} (E_{t-t_0}[\psi](x + h(x)) - E_{t-t_0}[\psi](x)) p_0(x)dx,
\end{equation}
where \( t_0 \) is the time when the external perturbation has occurred.

3.1. An exactly solvable example. As an example, we can easily compute the mean state response of the Ornstein–Uhlenbeck process in (2.10), with help of (2.12):

\[
\Delta \langle x \rangle_{OU}(t - t_0) = e^{(t_0 - t)L} \int_{\mathbb{R}^k} h(x)p_0^{OU}(x)dx, \tag{3.3}
\]

where we observe that \( p_0^{OU} \) is indecomposable and supported on the whole space, and thus \( \mathcal{A} \) is the same as \( \mathbb{R}^k \). Above, we only need to be able to evaluate the integral in terms of elementary functions, which, given the form of \( p_0^{OU}(x) \) in (2.13), leaves a broad choice for \( h(x) \) (it can be, for example, a polynomial in \( x \)).

3.2. A practical average response formula for computational modeling. It is also a possibility that the dynamical system in (2.1) is not exactly solvable, although can be numerically simulated or modeled. In this case, we need a suitable adaptation of the response formula in (3.2) to the typical practical restrictions of numerical modeling. Here, we will assume that, first, there exists an approximation to the stationary probability distribution \( p_0(x) \) in terms of elementary functions, and, second, there does not exist a similar approximation to \( \mathbb{E}_{t}[\psi](x) \). These assumptions are reasonable for a range of prototype nonlinear dynamical systems, such as the Lorenz 96 model [54, 55, 57], the barotropic model of Earth atmosphere [15, 17, 33], or the quasigeostrophic 1.5-layer wind driven double gyre ocean circulation model [16, 58, 59].

In such a situation, let \( q : \mathbb{R}^K \to \mathbb{R}^K \) be the inverse of \( x + h(x) \). Clearly, \( q(x) \) satisfies

\[
q(x) + h(q(x)) = x, \quad \text{for all} \ x. \tag{3.4}
\]

Then, in the perturbed expectation integral, we can change the variables \( x \to q(x) \) as follows:

\[
\int_{\mathcal{A}} \mathbb{E}_{t-t_0}[\psi](x + h(x))p_0(x)dx = \int_{\mathcal{A}} \mathbb{E}_{t-t_0}[\psi](q(x))p_0(q(x)) \left| \frac{\partial q}{\partial x} \right| dx, \tag{3.5}
\]

where \( |\partial q/\partial x| \) is the Jacobian of \( q(x) \). The average response formula in (3.2) can thus be written in the form

\[
\Delta \langle \psi \rangle(t - t_0) = \int_{\mathcal{A}} \mathbb{E}_{t-t_0}[\psi](q(x)) \left( p_0(q(x)) \left| \frac{\partial q}{\partial x} \right| - p_0(x) \right) dx = \int_{\mathcal{A}} \mathbb{E}_{t-t_0}[\psi](q(x)) \left( \frac{p_0(q(x))}{p_0(x)} \left| \frac{\partial q}{\partial x} \right| - 1 \right) p_0(x)dx, \tag{3.6}
\]

where we can divide by \( p_0(x) \) since it is nonzero in \( \mathcal{A} \). Finally, for the purposes of practical computation, we use the ergodicity property of \( p_0(x) \) in \( \mathcal{A} \) and replace, with help of the Birkhoff–Khintchine theorem, the measure average with the following time autocorrelation function over the long-term trajectory of the unperturbed system in (2.1):

\[
\Delta \langle \psi \rangle(t - t_0) = \lim_{r \to \infty} \frac{1}{r} \int_0^r \psi(x(t - t_0 + s)) \left( \frac{p_0(q(x(s)))}{p_0(x(s))} \left| \frac{\partial q}{\partial x}(x(s)) \right| - 1 \right) ds, \tag{3.7}
\]

where the initial condition for the time series is presumed to be taken in \( \mathcal{A} \). The practical computational constraint here (aside from the necessity for an accurate approximation
for \( p_0(x) \) is the ability to invert the function \( x + h(x) \). While, obviously, a large variety of possible forms of the jump function \( h(x) \) is available in general, here we point out a simple form of \( h(x) \) which is explicitly invertible and is likely broad enough for the majority of practical applications:

\[
(3.8) \quad h(x) = h + Hx,
\]

where \( h \) is a constant \( K \)-vector, and \( H \) is a constant \( K \times K \) matrix. For such a form of \( h(x) \), we obtain, explicitly,

\[
(3.9) \quad q(x) = (I + H)^{-1}(x - h), \quad \frac{\partial q}{\partial x} \bigg|_{x} = \frac{1}{|\det(I + H)|}.
\]

Finally, we would like to emphasize that none of the average response formulas above rely on a “small parameter” of any kind – any imprecisions of the average response computation above will manifest due to, for example, inaccuracy of the approximation for \( p_0(x) \), or the statistical undersampling of the long-term trajectory of (2.1).

3.3. The quasi-Gaussian approximation. The quasi-Gaussian approximation [13–15,57] for the average response formula in (3.7) emerges when the stationary distribution \( p_0(x) \) is replaced by the corresponding Gaussian distribution with the same mean state and covariance matrix. More precisely, let \( m \) be the mean state of \( p_0(x) \), and let \( C \) be its covariance matrix. Then, the quasi-Gaussian approximation for (3.7) is given via

\[
(3.10) \quad \Delta \langle \psi \rangle(t - t_0) = \lim_{r \to \infty} \frac{1}{r} \int_0^r \psi(x(t - t_0 + s)) \left( \frac{p_{m,C}(x(s))}{p_{m,C}(x(s))} \left| \frac{\partial q}{\partial x}(x(s)) \right| - 1 \right) ds,
\]

where \( p_{m,C}(x) \) is given explicitly via

\[
(3.11) \quad p_{m,C}^G(x) = \frac{1}{(2\pi)^{K/2}\sqrt{\det C}} \exp \left( -\frac{1}{2}(x - m)^T C^{-1}(x - m) \right).
\]

The above expression, together with (3.9), renders the integrand of the time autocorrelation function in (3.10) explicitly computable for a given (computed or observed) time series \( x(t) \), and thus the average response in (3.10) can be computed numerically using standard methods [13–15,57].

It may seem that, with the replacement of \( p_0(x) \) with \( p_{m,C}^G(x) \) in (3.10), one may lose the information of the relevant ergodic component of \( p_0(x) \) (that is, if the latter is not ergodic on the whole space \( \mathbb{R}^K \)). However, observe that it is not that case – due to the fact that the time average in (3.10) is computed over the exact time series of the original dynamical system in (2.1), the statistical sampling of the integrand of (3.10) still occurs on \( A \) with the distribution \( p_0(x) \). Moreover, the most remarkable property of the time autocorrelation function in (3.10) is that an observed or historically recorded time series \( x(t) \) can be used for its evaluation, without any need to numerically simulate the dynamical process. This suggests that the average response could be computed with sufficient degree of accuracy even for complex geophysical phenomena, as long as the recorded time series are sufficiently detailed.
4. A LARGEx RANDOM JUMP PERTURBATION

An interesting generalization of the previous scenario is the randomization of the jump $h(x)$ (which still occurs at the time $t_0$). Namely, let $z : \Omega \to \mathbb{R}^d$ be a random variable with the distribution measure $\nu$, and let $h(x, z) : \mathbb{R}^{K+d} \to \mathbb{R}^K$ be a continuous function of $x$, and Borel measurable of $z$. Then, it is natural to define the average response of $\psi$ not only by averaging over the states of the system (that is, over $p_0(x)$), but also over the possible jumps (that is, over $\nu$):

$$\Delta \langle \psi \rangle (t - t_0) = \int_A \int_{\mathbb{R}^d} (\mathbb{E}_{t-t_0}[\psi](x + h(x, z)) - \mathbb{E}_{t-t_0}[\psi](x)) \nu(dz) p_0(x) dx.$$

Observe that the only difference between (3.2) and (4.1) is the additional average over $\nu$, which allows to easily extend the average response formulas in (3.3) and (3.7) onto the random perturbation $h(x, z)$ in a straightforward fashion.

4.1. An exactly solvable example. The corresponding exactly solvable average response of the mean state of the Ornstein–Uhlenbeck process is given via

$$\Delta \langle x \rangle_{OU}(t - t_0) = e^{(t_0-t)L} \int_{\mathbb{R}^K} \int_{\mathbb{R}^d} h(x, z) \nu(dz) p_{0,OU}^0(x) dx,$$

which can be easily seen by taking $\psi(x) = x$ and recalling the formula for the expectation in (2.12). The practical computability of the mean state response formula in (4.2) now depends, in addition to the choice of $h(x, z)$, also on the choice of the intensity measure $\nu$. In the case of $h(x, z)$ being of the form

$$h(x, z) = \sum_i \xi_i(z) h_i(x),$$

we find that the double integral becomes the sum of the product of single integrals:

$$\Delta \langle x \rangle_{OU}(t - t_0) = \sum_i e^{(t_0-t)L} \left( \int_{\mathbb{R}^d} \xi_i(z) \nu(dz) \right) \left( \int_{\mathbb{R}^K} h_i(x) p_{0,OU}^0(x) dx \right).$$

Here, observe that, for each term in the sum, we obtained the formula for the mean state response of an Ornstein–Uhlenbeck process to a deterministic perturbation in (3.3), additionally multiplied by the integral over $\xi_i(z) \nu(dz)$. The computability of this integral depends entirely on the choice of $\xi_i(z)$ and $\nu$, and is independent of the rest of the set-up.

4.2. A practical average response formula for computational modeling. In order to obtain an analog of the time-averaged response formula in (3.7) for the response to the random perturbation in (4.1), we follow the same steps as in the previous section. We let $q : \mathbb{R}^{K+d} \to \mathbb{R}^K$ be the $x$-inverse of $x + h(x, z)$ for a given $z$:

$$q(x, z) + h(q(x, z)) = x, \quad \text{for all } x, z.$$
Then, we can again replace $x \to q(x)$ in the perturbed integral and write

\begin{equation}
\int_A \int_{\mathbb{R}^d} E_{t-t_0}[\psi](x + h(x, z)) \nu(dz) p_0(x) dx =
\int_A \int_{\mathbb{R}^d} E_{t-t_0}[\psi](x) \int_{\mathbb{R}^d} p_0(q(x, z)) \left| \frac{\partial q(x, z)}{\partial x} \right| \nu(dz) dx,
\end{equation}

where in the unperturbed integral we used the fact that $\nu$ is normalized to 1, and, as before, $|\partial q/\partial x|$ is the Jacobian of $q(x, z)$ in the $x$-variable. The average response formula in (4.1) can thus be written in the form

\begin{equation}
\Delta \langle \psi \rangle(t-t_0) = \int_A E_{t-t_0}[\psi](x) \left( \frac{J(x)}{p_0(x)} - 1 \right) p_0(x) dx,
\end{equation}

where

\begin{equation}
J(x) = \int_{\mathbb{R}^d} p_0(q(x, z)) \left| \frac{\partial q(x, z)}{\partial x} \right| \nu(dz).
\end{equation}

Upon the application of the Birkhoff–Khinchin theorem, the average response formula becomes

\begin{equation}
\Delta \langle \psi \rangle(t-t_0) = \lim_{r \to \infty} \frac{1}{r} \int_0^r \psi(x(t-t_0 + s)) \left( \frac{J(x(s))}{p_0(x(s))} - 1 \right) ds.
\end{equation}

Here, again observe that the average response formula above does not rely on a small parameter of any kind. Any imprecisions of the average response computation above will manifest due to, for example, inaccuracy of the approximation for $p_0(x)$, or a possible approximation for $J(x)$, or the statistical undersampling of the long-term trajectory of (2.1).

However, note that for an efficient numerical computation of the time autocorrelation function in (4.9), the term $J(x)$ in (4.8) must be expressed in terms of elementary functions. This requirement, obviously, places restrictions on the choice of $\nu$, $h(x, z)$, and the approximation for $p_0(x)$. Below, we consider a few special cases where the $\nu$-integral above is explicitly computable.

### 4.3. Special case 1: a set of spatially pre-determined jumps

As a simplest example where the $\nu$-integral is explicitly computable, we consider the following special case. Assume that, instead of a single deterministic jump, several different, yet spatially pre-determined jumps may occur randomly with prescribed probabilities. In other words, the random variable $z : \Omega \to \mathbb{R}^d$ may return only a finite set of values $\{z_j\}$, each with probability $\gamma_j$, $j = 1 \ldots Q$. In this case, we can define

\begin{equation}
\nu(dz) = \sum_{j=1}^Q \gamma_j \delta(z - z_j)dz, \quad \sum_{j=1}^Q \gamma_j = 1,
\end{equation}
and, therefore,

\[ J(x) = \sum_{j=1}^{Q} \gamma_j p_0(q(x, z_j)) \left| \frac{\partial q(x, z_j)}{\partial x} \right|. \]  

Observe that, as long as \( p_0(x) \) and \( q(x, z) \) are available in the form of explicit formulas, \( J(x) \) is also an explicit function of \( x \), which allows to compute the time autocorrelation function in (4.9) efficiently. For a practically computable example of \( h(x, z) \), we can generalize (3.8) to include the \( z \)-dependence:

\[ h(x, z) = h(z) + H(z)x, \]

where the \( K \)-vector \( h(z) \) and \( K \times K \) matrix \( H(z) \) are known, explicit functions of \( z \). This leads to

\[ q(x, z) = (I + H(z))^{-1}(x - h(z)), \quad \left| \frac{\partial q(x, z)}{\partial x} \right| = \frac{1}{|\det(I + H(z))|}, \]

and, subsequently,

\[ J(x) = \sum_{j=1}^{Q} \gamma_j \frac{p_0((I + H(z_j))^{-1}(x - h(z_j)))}{|\det(I + H(z_j))|}. \]

4.4. Special case 2: a Gaussian jump distribution. As a practical example, let us consider the scenario where \( \nu(dz) \) has a Gaussian density, with its own mean state vector \( m_\nu \) and covariance matrix \( C_\nu \):

\[ \nu(dz) = p_{m_\nu, C_\nu}^G(z)dz = \frac{1}{(2\pi)^{d/2}\sqrt{|\det C_\nu|}} \exp \left( -\frac{1}{2}(z - m_\nu)^T C_\nu^{-1}(z - m_\nu) \right) dz. \]

For practical computation, we restrict the \( z \)-dependent jump function \( h(x, z) \) in (4.12) to

\[ h(x, z) = h + Hx + H_\ast z, \]

with \( h, H \) and \( H_\ast \) being the constant \( K \)-vector, \( K \times K \) matrix and \( K \times d \) matrix, respectively. This yields \( q(x, z) \) in the form

\[ q(x, z) = (I + H)^{-1}(x - h - H_\ast z), \quad \left| \frac{\partial q(x, z)}{\partial x} \right| = \frac{1}{|\det(I + H)|}. \]

The form of \( q(x, z) \) in (4.17) leads to the following expression for \( J(x) \) in (4.8):

\[ J(x) = \frac{1}{|\det(I + H)|} \int_{\mathbb{R}^d} p_0 \left( (I + H)^{-1}(x - h - H_\ast z) \right) \nu(dz). \]

To be able to compute the \( \nu \)-integral above in the form of an explicit formula, we will assume that \( p_0(x) \) is also given by its quasi-Gaussian approximation \( p_{m_\nu, C}^G(x) \) in (3.11). This leads to

\[ J(x) = \frac{1}{(2\pi)^{K/2}|\det(I + H)|\sqrt{|\det C|\det(C_\nu A)}} \exp \left( \frac{1}{2}(b^T(x)A^{-1}b(x) - c(x)) \right), \]

where the terms \( A, b(x) \) and \( c(x) \) are given via

\[ A = ((I + H)^{-1}H_\ast)^T C^{-1}(I + H)^{-1}H_\ast + C_\nu^{-1}, \]
The details of the computation are given in the Appendix A.

Observe that the whole expression in (4.19) is an explicit function of \( x \) (\( A \) is constant, \( b \) is linear in \( x \), and \( c \) is quadratic in \( x \)), which means that it can be evaluated efficiently for the computation of the time autocorrelation function in (4.9).

4.5. Special case 3: a linear combination of Gaussian densities. The natural generalization of the above formula can be easily derived for a scenario where both \( p_0(x) \) and \( \nu(dz) \) are linear combinations of Gaussian densities:

\[
(4.21a) \quad p_0(x) = \sum_{i=1}^{P} \beta_i p_{m_i}^{G}(x), \quad \sum_{i} \beta_i = 1,
\]

\[
(4.21b) \quad \nu(dz) = \sum_{j=1}^{Q} p_{m_{v,j}}^{G}(z)dz, \quad \sum_{j} \gamma_j = 1,
\]

for some integers \( P > 0, Q > 0 \). It is easy to verify that the corresponding integral \( J(x) \) in (4.8) becomes

\[
(4.22) \quad J(x) = \frac{1}{(2\pi)^{K/2} |\det(I + H)|} \sum_{i,j=1}^{P,Q} \frac{\beta_i \gamma_j \exp \left( \frac{1}{2} \left( b_{i,j}^{T}(x) A_{i,j}^{-1} b_{i,j}(x) - c_{i,j}(x) \right) \right)}{\sqrt{\det C_i \det(C_{v,j} A_{i,j})}},
\]

with

\[
(4.23a) \quad A_{i,j} = ((I + H)^{-1}H_s)^T C_i^{-1}(I + H)^{-1} H_s + C_{v,j}^{-1},
\]

\[
(4.23b) \quad b_{i,j}(x) = ((I + H)^{-1}H_s)^T C_i^{-1}(m_i + (I + H)^{-1}(h - x)) - C_{v,j}^{-1} m_{v,j},
\]

\[
(4.23c) \quad c_{i,j}(x) = (m_i + (I + H)^{-1}(h - x))^T C_i^{-1}(m_i + (I + H)^{-1}(h - x)) + m_{v,j}^T C_{v,j}^{-1} m_{v,j},
\]

where \( m_i \) and \( C_i \) are the mean state and covariance matrix of \( p_i^{G}(x) \), respectively. Again, observe that \( A_{i,j} \) are constants, while \( b_{i,j}(x) \) and \( c_{i,j}(x) \) are explicit functions of \( x \). This means that the measure integral can be evaluated efficiently in the numerical computation of the time autocorrelation function in (4.9).
5. A sequence of large random jump perturbations at random times

An important generalization of the previously studied random jump process is the extension of the randomness of the perturbation onto time. In what follows, we will assume that not only the perturbation is spatially distributed in a random fashion, but that there is also randomness in the time when it may happen. We will assume that the spatial distribution of the jump is still governed by a distribution measure \( \nu \) and a suitable jump function \( h(x, z) \), just as in the previous section. However, rather than having a single jump at a prescribed time \( t_0 \), here we are going to assume that the jumps are going to occur randomly and independently in time, with the average intensity of jumps \( \lambda \) being a function of both time and the pre-jump spatial state of the system:

\[
\lambda = \alpha \eta(t) g(x(t-)).
\]

Above, \( \eta : \mathbb{R} \to \mathbb{R}_{>0} \) is a bounded Riemann-integrable function on a real line, and \( g : \mathbb{R}^K \to \mathbb{R}_{>0} \) is a bounded Borel-measurable function on \( \mathbb{R}^K \). The small constant scaling parameter \( 0 < \alpha \ll 1 \) signifies that the intensity of jumps must be small.

In order to define such perturbations, we need a suitable random process which will “trigger” the jumps in the state of the system in (2.1), and the framework of random jump processes naturally lends itself to this application. Here, we start by introducing a compound Poisson process \( n : T \times \Omega \to \mathbb{R}^d \) with intensity measure \( \nu \). Since \( \nu \) is normalized to 1 (as it is the same measure that was used to define a single random jump in the previous section), the average intensity of jumps of \( n(t) \) is one jump per unit of time – that is, the jumps of \( n(t) \) are “spread” in time statistically uniformly.

In order to adjust the intensity of jumps of \( n(t) \) to (5.1), let us define the following time-inhomogeneous Poisson counting process \( m \):

\[
m(t) = n(\tau(t)),
\]

where \( \tau(t) \) is given via

\[
\tau(t) = \int_0^t \lambda(s) ds = \alpha \int_0^t \eta(s) g(x(s-)) ds.
\]

Above, \( \tau(t) \) is a non-decreasing, bounded and continuous function of \( t \). For \( m(t) \) to indeed be a Poisson process, \( \tau(t) \) must also be adapted to \( \mathcal{F}(t) \) [66], where the latter is the natural filtration of \( m(t) \). Below we will show that it is indeed the case.

At this point, let us introduce the jump process \( \Delta m(t) \) via

\[
\Delta m(t) = m(t) - m(t-).
\]

Let \( M \) denote the corresponding Poisson random measure of \( \Delta m(t) \):

\[
M([t_0, t_1], A) = \text{number of values of } \Delta m(t) \in A \subset \mathbb{R}^d \setminus \{0\} \text{ for } t_0 < t \leq t_1.
\]

We now define the perturbed dynamical system via the following Lévy-type Feller process [18,27,31]:

\[
x(t) = x_0 + \int_0^t f(x(s)) ds + \int_0^t G(x(s)) dW(s) + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(x(s-), z) M(ds, dz).
\]
At this point, observe that if $x(t)$ is a solution of (5.6), then it is indeed adapted to $F(t)$, and so are (5.1) and (5.3), which is what is required in [66] for $m(t)$ in (5.2) to qualify for a time-inhomogeneous compound Poisson process. Moreover, $\tau'(t)$ is also $F(t)$-adapted, which we will make use of below.

5.1. The formulation of the average response to external perturbation. In what follows, we study the same scenario as in our previous works [1–3, 6, 8, 9, 13–16, 57], except that the external perturbation here is of a completely new type. Namely, assume that $p_0$ is the invariant measure for (2.1), the statistical properties of such an ensemble are stationary in time (even though the trajectories themselves are not constant in time).

Next, at time $t_0$, we replace the unperturbed system in (2.1) with the perturbed one in (5.6), and use the perturbed system to propagate this statistical ensemble for $t > t_0$. Now that the underlying dynamics is perturbed, the distribution $p_0(x)$, according to which the statistical ensemble is distributed at $t_0$, is no longer invariant. The question is: how do the statistical properties of the ensemble are going to respond to the introduced perturbation? More precisely, given a suitable test function $\psi : \mathbb{R}^K \rightarrow \mathbb{R}$, we are interested in the difference between the statistical averages

\begin{equation}
\Delta(\psi)(t, t_0) = \int_A \left( E_{t, t_0}^* [\psi](x) - E_{t-t_0} [\psi](x) \right) p_0(x) dx = \int_A \psi(x) (p(t, x) - p_0(x)) dx,
\end{equation}

where the asterisk signifies that the underlying dynamics of the conditional expectation are given by the perturbed system in (5.6), and $p(t, x)$ is the distribution of the perturbed ensemble trajectories at time $t$ under the condition that it was $p_0(x)$ at $t_0$. Observe that, unlike in (3.2) and (4.1), the average response in (5.7) is not necessarily a function of the time differences $(t - t_0)$ – the reason is that, in the perturbed system (5.6), the intensity of jumps depends on time explicitly.

The first step in computing the formula above in (5.7) is to obtain the infinitesimal generator of (5.6).

5.2. The infinitesimal generator. In order to obtain the infinitesimal generator of (5.6), we need to transform the stochastic jump integral in (5.6) into an integral of a time-inhomogeneous Poisson process, so that the standard Itô formula [18] for Lévy-type Feller processes [31] could be used. With (5.2), we can write the stochastic $M$-integral in the perturbed process (5.6) as

\begin{equation}
\int_0^t \int_{\mathbb{R}^d - \{0\}} h(x(s^-), z) M(ds, dz) = \sum_{0 < s \leq t, \Delta m(s) \neq 0} h(x(s^-), \Delta m(s)) = \sum_{0 < s \leq t, \Delta n(\tau(s)) \neq 0} h(x(s^-), \Delta n(\tau(s))) = \sum_{\tau(0) < s \leq \tau(t), \Delta n(s) \neq 0} h(x(\tau^{-1}(s^-)), \Delta n(s)) = \int_{\tau(0)}^{\tau(t)} \int_{\mathbb{R}^d - \{0\}} h(x(\tau^{-1}(s^-)), z) N(ds, dz),
\end{equation}
where $\Delta n(t)$ is the corresponding jump process of $n(t)$, and $N$ is its Poisson random measure. We note that the intensity measure of $N$ is given via $\nu$:

\begin{equation}
\mathbb{E}N(1, \cdot) = \nu(\cdot).
\end{equation}

Now that the stochastic jump integral in the right-hand side of (5.6) has been expressed via a time-homogeneous compound Poisson process, we can obtain the infinitesimal generator of (5.6) using the Itô formula in [18]. For a differentiable test function $\psi : \mathbb{R}^K \to \mathbb{R}$ we have the following Itô formula:

\begin{equation}
\psi(x(t + \varepsilon)) - \psi(x(t)) = \mathcal{I}[t, t + \varepsilon] + \int_{\tau(t)}^{\tau(t+\varepsilon)} \int_{\mathbb{R}^d - \{0\}} r(\tau^{-1}(s), z)N(ds, dz),
\end{equation}

where $\mathcal{I}[t, t + \varepsilon]$ denotes the usual Itô integral from $t$ to $t + \varepsilon$ for the drift-diffusion process in (2.1), while $r(t, z)$ denotes the following expression:

\begin{equation}
r(t, z) = \psi(x(t-)) + h(x(t-), z) - \psi(x(t-)).
\end{equation}

In order to obtain the infinitesimal generator, we need to compute the expectation of the above integral, conditioned on $\mathcal{F}(t)$. The expectation of $\mathcal{I}[t, t + \varepsilon]$ is, of course, given via the usual Itô isometry for drift-diffusion processes. For the stochastic integral over the Poisson random measure $N$, we observe that $n(t)$ is time-stationary with independent increments, which leads to the following expression (keep in mind that $\tau'(t)$ is $\mathcal{F}(t)$-adapted):

\begin{equation}
\mathbb{E} \int_{\tau(t)}^{\tau(t+\varepsilon)} \int_{\mathbb{R}^d - \{0\}} r(\tau^{-1}(s), z)N(ds, dz) = \\
= \mathbb{E} \int_{0}^{\tau(t+\varepsilon) - \tau(t)} \int_{\mathbb{R}^d - \{0\}} r(\tau^{-1}(\tau(t) + s), z)N(ds, dz) = \\
= \int_{0}^{\tau'(t)} \mathbb{E} \int_{\mathbb{R}^d - \{0\}} r(\tau^{-1}(\tau(t) + s), z)N(ds, dz) + o(\varepsilon) = \\
= \int_{0}^{\tau'(t)} \int_{\mathbb{R}^d - \{0\}} r(\tau^{-1}(\tau(t) + s), z)\mathbb{E}N(ds, dz) + o(\varepsilon) = \\
= \tau'(t) \int_{\mathbb{R}^d} r(t, z)\nu(dz) + o(\varepsilon).
\end{equation}

Recalling (5.3), we obtain the following infinitesimal generator for (5.6):

\begin{equation}
\frac{\partial}{\partial \varepsilon} \mathbb{E}[\psi(x(t + \varepsilon))] \bigg|_{\varepsilon = 0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}[\psi(x(t + \varepsilon)) - \psi(x(t))] = \frac{\partial}{\partial \varepsilon} \mathbb{E}[\mathcal{I}[t, t + \varepsilon]] \bigg|_{\varepsilon = 0} + \\
+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \int_{\tau(t)}^{\tau(t+\varepsilon)} \int_{\mathbb{R}^d - \{0\}} r(\tau^{-1}(s), z)N(ds, dz) = \\
= L\psi + \alpha \eta(t)g(x) \int_{\mathbb{R}^d} [\psi(x + h(x, z)) - \psi(x)]\nu(dz).
\end{equation}

Above, $x = x(t)$, and $L$ denotes the infinitesimal generator of the unperturbed system in (2.2).
5.3. The forward Kolmogorov equation. Observe that the only difference between the unperturbed (2.2) and perturbed (5.13) infinitesimal generators is that the latter has an additional term corresponding to the random jump perturbation. Therefore, it is easy to extend the unperturbed Kolmogorov equation in (2.4) onto the perturbed dynamics, as long as the intensity integral in (5.13) can be integrated by parts against \( p \delta x \). Indeed, omitting \( \alpha \eta(t) \) for convenience (as this factor is independent of \( x \)), we have

\[
\int_{\mathbb{R}^k} g(x) \left( \int_{\mathbb{R}^d} (\psi(x + h(x,z)) - \psi(x)) v(\delta z) \right) p(x) dx =
\]

\[
= \int_{\mathbb{R}^k} g(x) \left( \int_{\mathbb{R}^d} \psi(x + h(x,z)) v(\delta z) \right) p(x) dx - \int_{\mathbb{R}^k} g(x) \psi(x) p(x) dx,
\]

where in the second term we use the fact that \( v \) is normalized to 1.

In the second term above, the integral in \( \psi \delta x \) can be readily stripped, however, the first term requires an appropriate change of variables. With help of \( q(x,z) \), we change the variables \( x \rightarrow q(x,z) \) in the first integral in the right-hand side above:

\[
\int_{\mathbb{R}^k} g(x) \left( \int_{\mathbb{R}^d} \psi(x + h(x,z)) v(\delta z) \right) p(x) dx =
\]

\[
= \int_{\mathbb{R}^k} \left( \int_{\mathbb{R}^d} g(q(x,z)) p(q(x,z)) \frac{\partial q(x,z)}{\partial x} \right) v(\delta z) \psi(x) dx.
\]

Combining the terms, stripping the integrals in \( \psi \delta x \), and multiplying by \( \alpha \eta(t) \), we obtain the perturbed Kolmogorov equation in the form

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \cdot (pf) = \frac{1}{2} \frac{\partial^2}{\partial x^2} : (pgg^T) +
\]

\[
+ \alpha \eta(t) \left( \int_{\mathbb{R}^d} g(q(x,z)) p(q(x,z)) \frac{\partial q(x,z)}{\partial x} \right) v(\delta z) - b(x)p(x)
\]

5.4. The leading order response formula. We now can look for a solution of (5.16) around the stationary solution \( p_0(x) \) in the power series form

\[
p(t,x) = p_0(x) + \alpha p_1(t,x) + O(\alpha^2),
\]

which, upon substitution, yields for \( p_1 \)

\[
\frac{\partial p_1}{\partial t} + \frac{\partial}{\partial x} \cdot (p_1 f) = \frac{1}{2} \frac{\partial^2}{\partial x^2} : (p_1 g g^T) + \eta(t) \left( I_g(x) - g(x)p_0(x) \right),
\]

where by \( I_g(x) \) we denote the expression

\[
I_g(x) = \int_{\mathbb{R}^d} g(q(x,z)) p_0(q(x,z)) \left| \frac{\partial q(x,z)}{\partial x} \right| v(\delta z).
\]

Assuming that the initial condition is given via \( p_1(0,x) = 0 \), it is clear that \( p_1(t,x) \) can be found via Duhamel’s principle:

\[
p_1(t,x) = \int_{t_0}^t ds \eta(s) \int_A P_{t-s} (x|y) \left( I_g(y) - g(y)p_0(y) \right) dy,
\]
where \( P_t(x|y) \) is the conditional distribution of the unperturbed Kolmogorov equation in (2.9). Here, the terms under the \( \nu \)-integral are known. On the other hand, the transition probability \( P_t(x|y) \) is usually unavailable in the form of an explicit formula.

However, we can circumvent the direct computation of \( P_t(x|y) \) if the average response of a test function \( \psi \) is needed. Indeed, observe that, according to (5.7), we can express

\[
\Delta \langle \psi \rangle (t) = \int_A \psi(x) (p(t,x) - p_0(x)) \, dx = \alpha \int_A \psi(x) p_1(t,x) \, dx + o(\alpha).
\]

In turn, the integral of (5.22) can be expressed via

\[
\int_A \psi(x) p_1(t,x) \, dx = \int_A \psi(x) \int_{t_0}^t \eta(s) \int_A P_{t-s}(x|y) (J'_\eta(y) - g(y)p_0(y)) \, dy =
\]

\[
= \int_{t_0}^t \eta(s) \int_A \left( \int_{\mathbb{R}^\kappa} \psi(y) P_{t-s}(y|x) \, dy \right) (J'_\eta(x) - g(x)p_0(x)) \, dx = \int_{t_0}^t R_\psi(t-s) \eta(s) \, ds,
\]

where we denote the \textit{average response operator} \( R_\psi \) via

\[
R_\psi(t) = \int_A \left( \int_{\mathbb{R}^\kappa} \psi(y) P_t(y|x) \, dy \right) (J'_\eta(x) - g(x)p_0(x)) \, dx = \int_A \mathbb{E}_t[\psi](x) (J'_\eta(x) - g(x)p_0(x)) \, dx.
\]

5.5. \textbf{An exactly solvable example.} In order to obtain the exact formula for the mean state response of the Ornstein–Uhlenbeck process, we, first, observe that the part of the integral in (5.23) with \( J'_\eta(x) \) can be written as

\[
\int_A \mathbb{E}_t[\psi](x) J'_\eta(x) \, dx = \int_A \int_{\mathbb{R}^\kappa} \mathbb{E}_t[\psi](x) g(q(x,z)) p_0(q(x,z)) \left| \frac{\partial q(x,z)}{\partial x} \right| \nu(dz) \, dx =
\]

\[
= \int_A \left( \int_{\mathbb{R}^\kappa} \mathbb{E}_t[\psi](x + h(x,z)) \nu(dz) \right) g(x)p_0(x) \, dx,
\]

which leads to \( R_\psi \) in the form

\[
R_\psi(t) = \int_A \left( \int_{\mathbb{R}^\kappa} \mathbb{E}_t[\psi](x + h(x,z)) \nu(dz) - \mathbb{E}_t[\psi](x) \right) g(x)p_0(x) \, dx.
\]

Now, for \( \psi(x) = x \) and the Ornstein–Uhlenbeck process (2.10), we have

\[
R_x(t) = e^{-tL} \int_{\mathbb{R}^\kappa} \left( \int_{\mathbb{R}^\kappa} h(x,z) \nu(dz) \right) g(x)p_0^{OU}(x) \, dx.
\]

For the jump function \( h(x,z) \) of the form in (4.3), we have

\[
R_x(t) = \sum_i e^{-tL} \left( \int_{\mathbb{R}^\kappa} \zeta_i(z) \nu(dz) \right) \left( \int_{\mathbb{R}^\kappa} h_i(x) g(x)p_0^{OU}(x) \, dx \right).
\]
Observe that the only difference between (4.4) and the formula above is the presence of $g(x)$ in the integral over $dx$. As the invariant measure $p_0^{OU}$ in (2.13) is Gaussian, certain forms of $h_i(x)$ and $g(x)$ allow to compute the integral explicitly – for example, if $h_i(x)$ is a polynomial, and $g(x)$ is a Gaussian function itself (remember that $g(x) > 0$, since it is part of the statistical intensity of random jumps).

5.6. A practical computational formula for the average response. To compute (5.23) in practice, one can employ the Birkhoff–Khinchin theorem and replace the integral over $p_0(x)dx$ with the long-term average over time series $x(t)$ of the unperturbed system in (2.1), which leads to the following time autocorrelation function:

$$R_\psi(t) = \lim_{r \to \infty} \frac{1}{r} \int_0^r \psi(x(t + s)) \left( \frac{J_g(x(s))}{p_0(x(s))} - g(x(s)) \right) ds.$$  

(5.28)

For an efficient computation of the time autocorrelation function above in (5.28), it is necessary for the term $\int_0^r J_g(x) ds$ to be in the form of an explicit formula. Below we consider a few special cases where the corresponding integral in (5.19) is explicitly computable.

5.7. Special case 1: a set of spatially pre-determined jumps. As a simplest example where the $\nu$-integral in (5.19) is explicitly computable, we consider the following special case. Assume that the time-homogeneous compound Poisson process $n(t)$ (and, subsequently, the time-inhomogeneous compound Poisson process $m(t)$) produce a finite set of jumps – that is, the jump process $\Delta n(t)$ may assume only a finite set of values $\{z_j\}$, each with probability $\gamma_j, j = 1 \ldots Q$. In this case, the intensity measure $\nu$ of $\Delta n(t)$ is the same as in (4.10), and, therefore,

$$J_g(x) = \sum_{j=1}^Q \gamma_j g(q(x, z_j)) p_0(q(x, z_j)) \left| \frac{\partial q(x, z_j)}{\partial x} \right|.$$  

(5.29)

Observe that, as long as $p_0(x)$ and $q(x, z)$ are available in the form of explicit formulas, $J_g(x)$ is also an explicit function of $x$, which allows to compute the time autocorrelation function in (4.9) efficiently. For a practically computable example, we can take $h(x, z)$ of the form (4.12), which leads to

$$J(x) = \sum_{j=1}^Q \gamma_j g \left( (I + H(z_j))^{-1} (x - h(z_j)) \right) \frac{p_0((I + H(z_j))^{-1} (x - h(z_j)))}{\det(I + H(z_j))}.$$  

(5.30)

5.8. Special case 2: a Gaussian jump distribution. As another practical example, let us consider the scenario where $\nu(dz)$ has a Gaussian density, with its own mean state vector $m_\nu$ and covariance matrix $C_\nu$, as above in (4.15). In addition, we choose the same form for jump function $h(x, z)$ as above in (4.16), which yields the same $q(x, z)$ as in (4.17). This form of $\nu$ and $q(x, z)$ leads to the following expression for $J_g(x)$ in (5.19):

$$J_g(x) = \frac{1}{\det(I + H)} \int_{\mathbb{R}^d} g \left( (I + H)^{-1} (x - H z) \right) p_0((I + H)^{-1} (x - H z)) p_m^{G, C}(z) dz.$$  

(5.31)
Here we will further assume that \( p_0(x) \) is also given by the Gaussian density \( p_{m,C}^G(x) \) in (3.11) (that is, we use the quasi-Gaussian approximation for \( p_0 \)).

At this point, the main difference between the current set-up and the previously examined scenario in (4.19) is the additional presence of the function \( g(x) \), whose range must lie above zero. Clearly, the most simple choice of \( g(x) = \text{const} > 0 \) leads back to (4.19). Here, we are going to assume that \( g(x) \) is given via the Gaussian function of the form

\[
g(x) = \exp \left( -\frac{1}{2}(x - m_g)\mathbf{C}_g^{-1}(x - m_g) \right) = (2\pi)^{K/2} \sqrt{\det \mathbf{C}_g} p_{m_g,C_g}^G(x),
\]

with the \( K \)-vector \( m_g \) and symmetric positive definite \( K \times K \) matrix \( \mathbf{C}_g \) being constant parameters. Observe that such form of \( g(x) \) maximizes the statistical intensity of jumps in the vicinity of the state \( x = m_g \), with the “width” of the intensity bump described via \( \mathbf{C}_g \). This leads to \( J_g(x) \) of the form

\[
J_g(x) = \frac{1}{(2\pi)^{K/2} |\det(I + H)| \sqrt{\det \mathbf{C} \det(\mathbf{C}_v \mathbf{A}_g)}} \exp \left( \frac{1}{2} (b_g^T(x) \mathbf{A}_g^{-1} b_g(x) - c_g(x)) \right),
\]

with \( \mathbf{A}_g, b_g(x) \) and \( c_g(x) \) given via

\[
\begin{align*}
\mathbf{A}_g &= ((I + H)^{-1}H_*)^T(\mathbf{C}^{-1} + \mathbf{C}_g^{-1})(I + H)^{-1}H_* + \mathbf{C}_v^{-1}, \\
b_g(x) &= ((I + H)^{-1}H_*)^T((\mathbf{C}^{-1} + \mathbf{C}_g^{-1})(I + H)^{-1}(h - x) + \mathbf{C}_v^{-1}m + \mathbf{C}_g^{-1}m_g) - \mathbf{C}_v^{-1}m_v, \\
c_g(x) &= ((I + H)^{-1}(h - x))^T((\mathbf{C}^{-1} + \mathbf{C}_g^{-1})(I + H)^{-1}(h - x)) + \\
&+ 2((I + H)^{-1}(h - x))^T(\mathbf{C}^{-1}m + \mathbf{C}_g^{-1}m_g) + m^T \mathbf{C}^{-1}m + m_v^T \mathbf{C}_v^{-1}m_v + m_v^T \mathbf{C}^{-1}m_v.
\end{align*}
\]

The details of the computation are given in the Appendix A.

As before, observe that the whole expression above is an explicit function of \( x (\mathbf{A}_g \) is constant, \( b_g \) is linear in \( x \), and \( c_g \) is quadratic in \( x \)\), which means that it can be evaluated efficiently for the computation of the time autocorrelation function in (5.28).

### 5.9. Special case 3: a linear combination of Gaussian densities.

The natural generalization of the above formula can be easily derived for a scenario where \( p_0(x), \nu(dz) \) and \( g(x) \) are linear combinations of Gaussian densities:

\[
\begin{align*}
p_0(x) &= \sum_{i=1}^{P} \beta_i p_{m_i,C_i}^G(x), \quad \sum_{i} \beta_i = 1, \\
\nu(dz) &= \sum_{j=1}^{Q} \gamma_j p_{m_{v,j},C_{v,j}}^G(z)dz, \quad \sum_{j} \gamma_j = 1, \\
g(x) &= \sum_{k=1}^{T} (2\pi)^{K/2} \sqrt{\det \mathbf{C}_{g,k} \mathbf{p}_{m_{g,k},C_{g,k}}^G(x)},
\end{align*}
\]
for some integers $P > 0$, $Q > 0$, $T > 0$. It is easy to verify that the corresponding integral $Jg(x)$ in (5.19) becomes

$$J(x) = \frac{1}{(2\pi)^{K/2} \det(I + H)} \sum_{i=1}^{P} \sum_{j=1}^{Q} \sum_{k=1}^{T} \frac{\beta_i \gamma_j \xi_k}{\sqrt{\det C_i \det(C_{ij}A_{g,ijk})}} \exp \left( \frac{1}{2} \left( b_{g,ijk}^T(x)A_{g,ijk}^{-1}b_{g,ijk}(x) - c_{g,ijk}(x) \right) \right),$$

with

$$A_{g,ijk} = ((I + H)^{-1}H_i)^T(C_i^{-1} + C_{g,k}^{-1})(I + H)^{-1}H_i + C_{v,ij},$$

$$b_{g,ijk}(x) = ((I + H)^{-1}H_i)^T \left( (C_i^{-1} + C_{g,k}^{-1})(I + H)^{-1}(h - x) + C_i^{-1}m_i + C_{g,k}^{-1}m_{g,k} \right) - C_{v,ij}m_{v,j},$$

$$c_{g,ijk}(x) = ((I + H)^{-1}(h - x))^T(C_i^{-1} + C_{g,k}^{-1})((I + H)^{-1}(h - x)) + 2((I + H)^{-1}(h - x))^T(C_i^{-1}m_i + C_{g,k}^{-1}m_{g,k}) + m_i^T(C_i^{-1}m_i + m_{g,k}^T(C_{g,k}^{-1}m_{g,k}) + m_{v,ij}^T(C_{v,ij}^{-1}m_{v,j}).$$

where $m_i$ and $C_i$ are the mean state and covariance matrix of $p_i^G(x)$, respectively. Again, observe that $A_{g,ijk}$ are constants, while $b_{g,ijk}(x)$ and $c_{g,ijk}(x)$ are explicit functions of $x$. This means that the measure integral can be evaluated efficiently in the numerical computation of the time autocorrelation function in (5.28).

6. Discussion

In the current work, we develop a theory for the average response of a general nonlinear, possibly stochastic, dynamical system to instantaneous external jump perturbations of its state. In real-world physical processes, such perturbations could be a result of an “impulse forcing” – that is, an external forcing in the form of a delta-function. We consider three distinct scenarios of the jump perturbation; the first is where a deterministic jump perturbation is applied at a prescribed time, the second is where a spatially random jump perturbation is applied at a prescribed time, and the third is where the jump perturbation is spatially random, and also occurs at random times (although statistically infrequently).

In all scenarios, we derive the formulas for the average response of a chosen general test function through suitable time autocorrelation functions of the unperturbed dynamics. Throughout the study, we never assume that these jump perturbations are “small” in any reasonable or perceived sense; on the contrary, these perturbations are presumed to be finitely large in all studied scenarios. Below we compare and contrast the developed formulas for the average response to jump perturbations with the response formulas of the classical Fluctuation Dissipation theorem.
6.1. **Similarities to the classical FDT.** We find that, in the same manner as with the classical Fluctuation Dissipation theorem, the average response of a chosen test function to a finitely large jump perturbation can be computed in terms of a suitable autocorrelation function of the time series of the unperturbed dynamics – in particular, no explicit knowledge of the unperturbed dynamical system is required as long as a sufficiently detailed long-term historical record of its time series is available. Thus, in both frameworks, one could theoretically use the observed time series of the corresponding unperturbed real-world process directly to compute the average response, potentially reducing the need in direct numerical simulations of the (likely complex, large scale and computationally expensive) real-world dynamics.

Also, just like with the classical FDT, a suitable approximation of the probability density of the invariant statistical state of the unperturbed dynamics is necessary to compute the time autocorrelation function. In particular, this results directly in the analog of the quasi-Gaussian FDT formula \([13-15,57]\) for the jump perturbations.

6.2. **Differences from the classical FDT.** First, recall that the classical Fluctuation Dissipation theorem furnishes a leading order response formula \([8,57,70]\) to an external perturbation, which, in turn, is accurate only if the magnitude of the perturbation is small enough. In the response framework for jump perturbations, developed here, the accuracy of the average response formulas does not depend on the magnitude of the perturbation jumps. The only leading order approximation used above is with respect to the statistical intensity of randomly triggered jumps – in order for the response formula to be accurate, the jumps should occur statistically infrequently.

Second, in context of the classical Fluctuation Dissipation theorem, the average response is computable for a largely arbitrary additive external perturbation. In the jump-response framework, the average response formula is not necessarily available for an arbitrary jump function – instead, the jump function must satisfy the inverse condition in \((3.4)\) or \((4.5)\), and the inverse needs to be known explicitly. Also, in the case of random jumps, the invariant probability density of the unperturbed dynamics, the statistical intensity of external jump perturbations, and the spatial distribution of the random jumps must all be such that corresponding intensity integral in the average response formula is computable in terms of elementary functions. The latter requirement is necessary for the efficient numerical computation of the time autocorrelation function for the average response.

At the same time, recall that the average response formula for the classical FDT involves the gradient of the invariant probability density of the unperturbed dynamics, which means that any approximation to the latter must also approximate the gradient sufficiently well. In contrast, no differentiation of the invariant probability density is present in any of the jump-response formulas developed here. Potentially, this may mean that less accurate approximations for the invariant probability density could be used without compromising the precision of the predicted response to a considerable extent.

One of the “nice” properties of the classical FDT response formula is that it is linear with respect to the external forcing – usually it is a matrix of certain time autocorrelation functions multiplying the vector of external perturbations. This makes the classical FDT
particularly suitable for the “inverse problem”, that is, the computation of the perturbation vector from the known or measured response. Observe, however, that all of the average response formulas, developed above for the external jump perturbations, are nonlinear with respect to the jump function. As a result, the computation of the jump perturbation which caused a known response – for example, the computation of the approximate location and magnitude of the meteorite impact from the measured climate response – is naturally a more challenging and interesting problem.

6.3. Future research directions. Given the introductory nature of the present work, here we feel compelled to point out a few relevant directions of future research.

The basic research directions are those driven by the applicability of the present framework to real-world situations. Arguably, the most limiting restriction in the presented theory of the average response is the need of the jump function to have an explicit inverse (whereas there is no similar requirement in the classical FDT framework). While the linear jump function used in the present work is likely a viable option in a variety of scenarios, there are situations where such a jump function does not work. As an example, one can consider a scenario where the jumps must occur between the states of equal energy in an otherwise energy-conserving system – clearly, the jump function must be nonlinear to ensure such a property. Thus, the study and categorization of possible nonlinear (for example, quadratic) jump functions with explicit inverses should likely be of high priority.

Another question of importance is the already mentioned “inverse problem”, that is, the reconstruction of the perturbation from a known response. While in the classical FDT framework the inverse problem is technically trivial due to the linearity of the response with respect to the external forcing, here we can see that the average response even to a simplest form of the jump perturbation – a constant deterministic jump which occurred at a prescribed time – is inherently nonlinear. Thus, recovering the jump function from the measured response is clearly a more interesting and challenging problem in the present framework.

Unlike the classical FDT framework, where the response is obtained by taking the time-convolution of the linear response operator with the external forcing, here observe that, in the scenario with a single jump perturbation at a deterministic time the average response is the time autocorrelation function itself. Naturally, one expects the response to such a perturbation to decay in time, so that the dynamical system eventually returns to its statistical equilibrium state. However, recall that, unlike the classical linear response, here the time autocorrelation function is the exact average response to the external perturbation, provided that the invariant probability state is known exactly. Thus, if the jump-response time autocorrelation function decays to zero, it means that the system relaxes back toward its equilibrium state, and vice versa.

If a dynamical system is strongly mixing (in addition to being ergodic), then the decay of its time autocorrelation functions is guaranteed, and so is the response to a single jump perturbation. However, for an ergodic (and even chaotic) nonlinear dynamical system, being additionally strongly mixing is not a necessary requirement. Moreover, there are empirical examples of chaotic systems which do not appear to be strongly mixing – for example, the Lorenz 96 system [54, 55, 57] in weakly chaotic regimes does not exhibit
noticeable decay of time autocorrelation functions at linearly unstable wavenumbers. In such a case, it could be possible that the corresponding dynamical system never “settles down” after a single jump perturbation.

Thus, the question is: are there real-world scenarios in which the geophysical dynamics do not rapidly return back to statistical equilibrium after being subjected to an external jump perturbation? In particular, could the event, which caused the Permian–Triassic extinction, be an example of such a perturbation, such that the occasional planetary climate shifts due to that event continue to occur to this day? The long-time response of the planetary climate dynamics to jump perturbations is clearly a very interesting problem in multiple aspects.

Appendix A. Computation of the response for the Gaussian distribution of jumps

For (4.17), we have

\[
(A.1) \quad p_{m, C}^G(q(x, z)) = p_{m, C}^G \left( (I + H)^{-1}(x - h - H_*z) \right) = \frac{1}{(2\pi)^{K/2} \sqrt{\det C}} \exp \left( -\frac{1}{2} ((I + H)^{-1}(x - h - H_*z) - m)^T C^{-1}((I + H)^{-1}(x - h - H_*z) - m) \right).
\]

To obtain (4.19), we first multiply the above expression by (4.15). The resulting form of \( J(x) \) is given via

\[
(A.2) \quad J(x) = \frac{1}{|\det(I + H)|} \int_{\mathbb{R}^d} p_{m, C}^G \left( (I + H)^{-1}(x - h - H_*z) \right) p_{m', C'}^G(z) \, dz.
\]

Multiplying the Gaussian densities under the integral, in the argument of the resulting exponential we have

\[
(A.3) \quad ((I + H)^{-1}(x - h - H_*z) - m)^T C^{-1}((I + H)^{-1}(x - h - H_*z) - m) + (z - m_v)^T C_v^{-1}(z - m_v) = z^T A z + 2b^T z + c = (z + A^{-1}b)^T (z + A^{-1}b) + c - b^T A^{-1}b,
\]

where \( A, b \) and \( c \) are given via (4.20). Observing that the integral

\[
(A.4) \quad \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} ((z + A^{-1}b)^T A(z + A^{-1}b) + c - b^T A^{-1}b) \right) \, dz = \exp \left( -\frac{1}{2} (c - b^T A^{-1}b) \right) \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} (z + A^{-1}b)^T A(z + A^{-1}b) \right) \, dz = \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \exp \left( \frac{1}{2} (b^T A^{-1}b - c) \right),
\]
we arrive at

\begin{equation}
(A.5) \quad \int_{\mathbb{R}^d} p_{m,T}^G \left( (I + H)^{-1}(x - h - H_s z) \right) v(\text{d}z) = \frac{1}{(2\pi)^{K/2} \sqrt{\det C \det(C_T A)}} \exp \left( \frac{1}{2} \left( b^T(x) A^{-1} b(x) - c(x) \right) \right),
\end{equation}

and, subsequently, to (4.19).

To obtain (5.33), we first multiply \( g(q(x, z)) \) in (5.32) by (4.15) and (A.1). The resulting form of \( J_g(x) \) is given via

\begin{equation}
(A.6) \quad J_g(x) = \frac{(2\pi)^{K/2} \sqrt{\det C_g}}{\det(I + H)} \int_{\mathbb{R}^d} p_{m_g,T_g}^G \left( (I + H)^{-1}(x - h - H_s z) \right) p_{m_s,T_s}^G \left( (I + H)^{-1}(x - h - H_s z) \right) p_{m_v,T_v}^G (z) \text{d}z.
\end{equation}

Multiplying the Gaussian densities under the integral, in the argument of the resulting exponential we have

\begin{equation}
(A.7) \quad ( (I + H)^{-1}(x - h - H_s z) - m)^T C^{-1} ((I + H)^{-1}(x - h - H_s z) - m) + ((I + H)^{-1}(x - h - H_s z) - m_g)^T C_g^{-1} ((I + H)^{-1}(x - h - H_s z) - m_g) + (z - m_v)^T C_v^{-1} (z - m_v) = z^T A_g z + 2b_s^T z + c_g = (z + A_g^{-1} b_g)^T A_g (z + A_g^{-1} b_g) + c_g - b_s^T A_s^{-1} b_s,
\end{equation}

where \( A_g, b_g(x) \) and \( c_g(x) \) are given via (5.34). Observing that the integral

\begin{equation}
(A.8) \quad \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} \left( z + A_g^{-1} b_g \right)^T A_g (z + A_g^{-1} b_g) + c_g - b_s^T A_s^{-1} b_s \right) \text{d}z = \exp \left( -\frac{1}{2} (c_g - b_s^T A_s^{-1} b_s) \right) \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} (z + A_g^{-1} b_g)^T A_g (z + A_g^{-1} b_g) \right) \text{d}z = \frac{(2\pi)^{d/2}}{\sqrt{\det A_g}} \exp \left( \frac{1}{2} (b_s^T A_s^{-1} b_s - c_g) \right),
\end{equation}

we arrive at

\begin{equation}
(A.9) \quad \int_{\mathbb{R}^d} p_{m_g,T_g}^G \left( (I + H)^{-1}(x - h - H_s z) \right) p_{m_s,T_s}^G \left( (I + H)^{-1}(x - h - H_s z) \right) p_{m_v,T_v}^G (z) \text{d}z = \frac{1}{(2\pi)^{K/2} \sqrt{\det C_g}} \frac{1}{(2\pi)^{K/2} \sqrt{\det C \det(C_T A)}} \exp \left( \frac{1}{2} (b_s^T(x) A_s^{-1} b_s(x) - c_g(x)) \right),
\end{equation}

and, subsequently, to (5.33).
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