INTERNAL CHARACTERIZATION OF BREZIS – LIEB SPACES

E. Y. EMELYANOV\textsuperscript{1,2} AND M. A. A. MARABEH\textsuperscript{3}

Abstract. In order to find an extension of Brezis – Lieb’s lemma to the case of nets, we replace the almost everywhere convergence by the unbounded order convergence and introduce the Brezis – Lieb property in normed lattices. Then we identify a wide class of Banach lattices in which the Brezis – Lieb lemma holds true. Among other things, it gives an extension of the Brezis – Lieb lemma for nets in $L^p$ for $p \in [1, \infty)$.

1. Introduction

Let $(\Omega, \Sigma, \mu)$ be a measure space in which, for every set $A \in \Sigma$, $\mu(A) > 0$, there exists $\Sigma \ni A_0 \subseteq A$, such that $0 < \mu(A_0) < \infty$. Given $p \in (0, \infty)$, denote by $L^p = \{f : \int |f|^p \mu < \infty\}$ the vector space of $p$-integrable functions from $\Omega$ into $\mathbb{C}$. The Brezis – Lieb lemma \cite{3} Thm.1 is known as the following useful refinement of the Fatou lemma.

\textbf{Theorem 1} (Brezis – Lieb’s lemma for $L^p$ $(0 < p < \infty)$). Suppose $f_n \xrightarrow{\text{a.e.}} f$ and $\int |f_n|^p d\mu \leq C < \infty$ for all $n$ and some $p \in (0, \infty)$. Then

\begin{equation}
\lim_{n \to \infty} \int_{\Omega} (|f_n|^p - |f_n - f|^p) d\mu = \int_{\Omega} |f|^p d\mu.
\end{equation}

As the following example shows, Theorem \textbullet\ does not have a reasonable direct generalization for nets.

\textbf{Example 1.} Consider $[0,1] \subset \mathbb{R}$ with the Lebesgue measure $\mu$. Let $\Delta$ be the family of all finite subsets of $[0,1]$ ordered by inclusion, and $\mathbb{F}$ be the
indicator function of $F \in \Delta$. Then $\|F\|_{\infty} \rightarrow \|\cdot\|_{[0,1]}$ and $\int_0^1 |I_F|d\mu = 0$, however

$$
\lim_{F \rightarrow \infty} \int_0^1 (|I_F| - |\cdot|_{[0,1]})d\mu = \lim_{F \rightarrow \infty} \int_0^1 (-|\cdot|_{[0,1]})d\mu = -1 \neq 1 = \int_0^1 |\cdot|_{[0,1]}d\mu.
$$

In order to avoid the collision, we restate Theorem 1 in the case of $1 \leq p < \infty$ in terms of the Banach space $L^p$ of equivalence classes of functions from $L^p(\mu)$ w.r. to $\mu$ (cf. [8, Thm.2]).

**Theorem 2** (Brezis – Lieb’s lemma for $L^p$ $(1 \leq p < \infty)$). Let $f_n \rightharpoonup f$ in $L^p(\mu)$ and $\|f_n\|_p \rightarrow \|f\|_p$, where $\|f_n\|_p := \left(\int_{\Omega}|f_n|^p d\mu\right)^{1/p}$ with $f_n \in L^p(\mu)$ and $f \in L^p(\mu)$. Then $\|f_n - f\|_p \rightarrow 0$.

Although in Theorem 2 we still have a.e. convergent sequences in $L^p$, it is possible now (e.g. due to [5, Prop.3.1]) to replace the a.e. convergence by the $\text{uo}$ convergence and restate Theorem 1 once more (cf. also [4, Prop.2.2] and [7, Prop.1.5]) as follows.

**Theorem 3** (Brezis – Lieb’s lemma for $\text{uo}$ convergent sequences in $L^p$). Let $x_n \text{uo} \rightarrow x$ in $L^p$, where $p \in [1, \infty)$. If $\|x_n\|_p \rightarrow \|x\|_p$ then $\|x_n - x\|_p \rightarrow 0$.

Notice that Theorem 3 is a result of the Banach space theory which does not involve the measure theory directly. This observation motivates us to investigate those Banach lattices in which the statement of Theorem 3 holds true. We call them by the $\sigma$-Brezis – Lieb spaces in Definition 1. After introducing a geometrical property of normed lattices in Definition 2 we prove Theorem 4 which is the main result of the present paper. Theorem 4 gives an internal geometric characterization of $\sigma$-Brezis – Lieb’s spaces and implies immediately the following result.

**Proposition 1.** Let $f_\alpha \text{uo} \rightarrow f$ in $L^p(\mu)$ $(1 \leq p < \infty)$, and $\|f_\alpha\|_p \rightarrow \|f\|_p$. Then $\|f_\alpha - f\|_p \rightarrow 0$.

It is worth mentioning that Proposition 1 may serve as a net-extension of the Brezis – Lieb lemma (in its form of Theorem 3).

### 2. Brezis – Lieb spaces

In this section we consider normed lattices over the complex field $\mathbb{C}$. A vector space $E$ over $\mathbb{C}$ is said to be a normed lattice if $E$ is a complexification of $
a uniformly complete real normed lattice $F$ (see e.g. [1, Def.3.17]). More precisely, the modulus of $z = x + iy \in E = F \oplus iF$ is defined by

$$|z| = \sup_{\theta \in [0, 2\pi]} [x \cos \theta + y \sin \theta],$$

and its norm is defined by $\|z\| = \|z\|_E := \|z\|_F$ (cf. [1, p.104]). We also adopt notations $E_+ = F_+$, $z = [z]_r + i[z]_i$, $x = \Re[z]$, and $y = \Im[z]$ for $z = x + iy$ in $E$.

A net $v_\alpha$ in a vector lattice $E$ is said to be $u_0$–convergent to $v \in E$ whenever, for every $u \in E_+$, the net $|v_\alpha - v| \wedge u$ converges in order to 0.

For the further theory of vector lattices, we refer to [2, 1] and, for the unbounded order convergence, to [3, 5].

**Definition 1.** A normed lattice $(E, \| \cdot \|)$ is said to be a Brezis – Lieb space (shortly, $BL$–space) (resp. $\sigma$-Brezis – Lieb space ($\sigma$-$BL$–space)) if, for any net $x_\alpha$ (resp. for any sequence $x_n$) in $X$ such that $\|x_\alpha\| \to \|x_0\|$ (resp. $\|x_n\| \to \|x_0\|$) and $x_\alpha \to_0 x_0$ (resp. $x_n \to_0 x_0$), we have $\|x_\alpha - x_0\| \to 0$ (resp. $\|x_n - x_0\| \to 0$).

Trivially, any normed BL-space is a $\sigma$-BL-space, and any finite-dimensional normed lattice is a $BL$-space. Furthermore, by [5, Thm.3.2], any regular sublattice $F$ of any normed BL-space ($\sigma$-BL-space) $E$ is itself a BL-space ($\sigma$-BL-space). Taking into account the fact that the a.e.–convergence for sequences in $L^p$ coincides with the $u_0$–convergence [5, Prop.3.1], Theorem 3 says exactly that $L^p$ is a $\sigma$-BL-space for $1 \leq p < \infty$.

Now, we consider examples of Banach lattices which are not $\sigma$–Brezis – Lieb spaces.

**Example 2.** The Banach lattice $(c_0, \| \cdot \|_\infty)$ is not a $\sigma$-$BL$–space. To see this, take $x_n = e_{2n} + \sum_{k=1}^n \frac{1}{k} e_k$ and $x = \sum_{k=1}^\infty \frac{1}{k} e_k$ in $c_0$. Clearly, $\|x\| = \|x_n\| = 1$ for all $n$ and $x_n \to_0 x$, however $1 = \|x - x_n\|$ does not converge to 0.

We do not know whether or not for an arbitrary lattice norm $\| \cdot \|$ in $c_0$, which is equivalent to $\| \cdot \|_\infty$, the Banach lattice $(c_0, \| \cdot \|)$ is not a $\sigma$-$BL$–space.

**Example 3.** Since $c_0$ is an order ideal in $c$ and in $\ell^\infty$, $c_0$ is regular there, and hence, both Banach lattices $(c, \| \cdot \|_\infty)$ and $(\ell^\infty, \| \cdot \|_\infty)$ are not $\sigma$-$BL$–spaces. Accordingly to the fact, that $c_0$ is a regular sublattice of $c$ and to the last sentence of Example 2, it is also unknown whether or not the Banach lattice $(c, \| \cdot \|)$ is not a $\sigma$-$BL$–space for an arbitrary lattice norm $\| \cdot \|$ that is equivalent to $\| \cdot \|_\infty$. 

In opposite to $c$, the Banach lattice $\ell^\infty$ is Dedekind complete. Let $\| \cdot \|$ be any lattice norm in $\ell^\infty$ that is equivalent to $\| \cdot \|_\infty$. Clearly, the norm $\| \cdot \|$ is not order continuous. Therefore, by Theorem 3, $(\ell^\infty, \| \cdot \|)$ is not a $\sigma$-BL-space.

A slight change of an infinite-dimensional BL-space can turn it into a normed lattice which is not even a $\sigma$-BL-space.

**Example 4.** Let $E$ be a normed lattice, $\dim(E) = \infty$. Let $F = C \oplus_\infty E$. Take any disjoint sequence $y_n$ in $E$ such that $\|y_n\|_E = 1$ for all $n$. Then $y_n \to 0$ in $E$ [2, Cor.3.6]. Let $x_n = (1, y_n) \in F$. Then $\|x_n\|_F = \sup(1, \|y_n\|_E) = 1$ and $x_n = (1, y_n) \to (1, 0) =: x$ in $F$, however $\|x_n - x\|_F = \|(0, y_n)\|_F = \|y_n\|_E = 1$, and so $x_n$ does not converge to $x$ in $(F, \| \cdot \|_F)$. Therefore $F = C \oplus_\infty E$ is not a $\sigma$-BL-space.

In order to characterize BL-spaces, we introduce the following definition.

**Definition 2.** A normed lattice $(E, \| \cdot \|)$ is said to have the Brezis–Lieb property (shortly, BL-property), whenever $\limsup_{n \to \infty} \|u_0 + u_n\| > \|u_0\|$ for any disjoint normalized sequence $(u_n)_{n=1}^{\infty}$ in $E_+$ and for any $u_0 \in E_+$. Every finite dimensional normed lattice $E$ has the BL-property. It is easy to see that the Banach lattices $c_0$, $c$, and $\ell^\infty$ w.r. to the supremum norm $\| \cdot \|_\infty$ do not have the BL-property. The modification of the norm in an infinite-dimensional Banach lattice $E$ with the BL-property, as in Example 4, turns it into the Banach lattice $F = C \oplus_\infty E$ without the BL-property. Indeed, take a disjoint normalized sequence $(y_n)_{n=1}^{\infty}$ in $E_+$. Let $u_0 = (1, 0)$ and $u_n = (0, y_n)$ for $n \geq 1$. Then $(u_n)_{n=0}^{\infty}$ is a disjoint normalized sequence in $F_+$ with $\limsup_{n \to \infty} \|u_0 + u_n\| = 1 = \|u_0\|$. Remarkably, it is not a coincidence. The following theorem identifies BL-spaces among $\sigma$-Dedekind complete Banach lattices.

**Theorem 4.** For a $\sigma$-Dedekind complete Banach lattice $E$, the following conditions are equivalent:

1. $E$ is a Brezis–Lieb space;
2. $E$ is a $\sigma$-Brezis–Lieb space;
3. $E$ has the Brezis–Lieb property, and the norm in $E$ is order continuous.

**Proof.** (1) $\Rightarrow$ (2) It is trivial.
(2) $\Rightarrow$ (3) We show first that $E$ has the BL-property. Notice that, in this part of the proof, the $\sigma$-Dedekind completeness of $E$ will not be used. Suppose that there exist a disjoint normalized sequence $(u_n)_{n=1}^{\infty}$ in $E_+$ and
$u_0 \in E_+$ with $\limsup_{n \to \infty} ||u_0 + u_n|| = ||u_0||$. Since $||u_0 + u_n|| \geq ||u_0||$, then
\lim_{n \to \infty} ||u_0 + u_n|| = ||u_0||$. Denote $v_n := u_0 + u_n$. By [3 Cor.3.6], $u_n \rightharpoonup u_0$ and hence $v_n \rightharpoonup u_0$. Since $E$ is a $\sigma$-BL-space and $\lim_{n \to \infty} ||v_n|| = ||u_0||$, then $||v_n - u_0|| \to 0$, which is impossible in view of $||v_n - u_0|| = ||u_0 + u_n - u_0|| = ||u_n|| = 1$.

Assume that the norm in $E$ is not order continuous. Then, by the Fremlin–Meyer-Nieberg theorem (see e.g. [2 Thm.4.14]) there exist $y \in E_+$ and a disjoint sequence $e_k \in [0, y]$ such that $||e_k|| \rightharpoonup 0$. Without lost of generality, we may assume $||e_k|| = 1$ for all $k \in \mathbb{N}$. By the $\sigma$–Dedekind completeness of $E$, for any sequence $\alpha_n \in \mathbb{R}_+$, there exist the following vectors
\begin{equation}
(2.1) \quad x_0 = \sum_{k=1}^{\infty} e_k, \quad x_n = \alpha_{2n} e_{2n} + \sum_{k=1,n \neq n,k \neq 2n}^{\infty} e_k \quad (\forall n \in \mathbb{N}).
\end{equation}

Now, we choose $\alpha_{2n} \geq 1$ in (2.1) such that $||x_n|| = ||x_0||$ for all $n \in \mathbb{N}$. Clearly, $x_n \rightharpoonup x_0$. Since $E$ is a $\sigma$-BL-space, then $||x_n - x_0|| \to 0$, violating

$$||x_n - x_0|| = ||(\alpha_{2n} - 1)e_{2n} - e_n|| = ||(\alpha_{2n} - 1)e_{2n} + e_n|| \geq ||e_n|| = 1.$$ 

The obtained contradiction shows that the norm in $E$ is order continuous.

(3) ⇒ (1) If $E$ is not a Brezis–Lieb space, then there exists a net $(x_\alpha)_{\alpha \in A}$ in $E$ such that $x_\alpha \rightharpoonup x$ and $||x_\alpha|| \to ||x||$, but $||x_\alpha - x|| \rightharpoonup 0$. Then $||x_\alpha|| \to ||x||$ and $||x_\alpha|| \to ||x||$.

Notice that $||x_\alpha|| - ||x|| \rightharpoonup 0$. Indeed, if $||x_\alpha|| - ||x|| \to 0$, then, for any $\varepsilon > 0$, $(x_\alpha)_{\alpha \in A}$ is eventually in $[-||x||, ||x||] + \varepsilon B_E$. Thus $((x_\alpha)_{\alpha \in A}$, and hence $(\Re[x_\alpha])_{\alpha \in A}$ and $(\Im[x_\alpha])_{\alpha \in A}$ are both almost order bounded. Since $E$ is order continuous and $x_\alpha \rightharpoonup x$, then $\Re[x_\alpha] \rightharpoonup \Re[x]$ and $\Im[x_\alpha] \rightharpoonup \Im[x]$. By [5 Pop.3.7.], $||\Re[x_\alpha - x]|| \to 0$ and $||\Im[x_\alpha - x]|| \to 0$, and hence $||x_\alpha - x|| \to 0$, that is impossible. Therefore, without lost of generality, we may assume $x_\alpha \in E_+$ and, by normalizing, also $||x_\alpha|| = ||x|| = 1$ for all $\alpha$.

Passing to a subnet, denoted by $x_\alpha$ again, we may assume
\begin{equation}
(2.2) \quad ||x_\alpha - x|| > C > 0 \quad (\forall \alpha \in A).
\end{equation}

Notice that $x \geq (x - x_\alpha)^+ = (x_\alpha - x)^- \rightharpoonup 0$, and hence $(x_\alpha - x)^- \rightharpoonup 0$. The order continuity of the norm ensures
\begin{equation}
(2.3) \quad ||(x_\alpha - x)^-|| \to 0.
\end{equation}

Denoting $w_\alpha = (x_\alpha - x)^+$ and using (2.2) and (2.3), we may also assume
\begin{equation}
(2.4) \quad ||w_\alpha|| = ||(x_\alpha - x)^+|| > C \quad (\forall \alpha \in A).
\end{equation}
In view of (2.4), we obtain

\[ 2 = \|x_\alpha\| + \|x\| \geq \|(x_\alpha - x)\| = \|w_\alpha\| > C \quad (\forall \alpha \in A). \]

Since \( w_\alpha \xrightarrow{\omega} (x - x)^+ = 0 \), then, for any fixed \( \beta_1, \beta_2, \ldots, \beta_n \),

\[ 0 \leq w_\alpha \land (w_\beta_1 + w_\beta_2 + \ldots + w_\beta_n) \xrightarrow{\omega} 0 \quad (\alpha \to \infty). \]

Since \( x_\alpha \xrightarrow{\omega} x \), then \( x_\alpha \land x \xrightarrow{\omega} x \land x = x \) and so \( x_\alpha \land x \xrightarrow{\omega} x \). By the order continuity of the norm, there is an increasing sequence of indices \( \alpha_n \) in \( A \) with

\[ \|x - x_\alpha \land x\| \leq 2^{-n} \quad (\forall \alpha \geq \alpha_n). \]

Furthermore, by (2.6), we may also suppose that

\[ \|w_\alpha \land (w_{\alpha_1} + w_{\alpha_2} + \ldots + w_{\alpha_n})\| \leq 2^{-n} \quad (\forall \alpha \geq \alpha_{n+1}). \]

Since

\[ \sum_{k=1, k \neq n}^{\infty} \|w_{\alpha_n} \land w_{\alpha_k}\| \leq \sum_{k=1}^{n-1} \|w_{\alpha_n} \land (w_{\alpha_1} + \ldots + w_{\alpha_{n-1}})\| + \]

\[ \sum_{k=n+1}^{\infty} \|w_{\alpha_n} \land (w_{\alpha_1} + \ldots + w_{\alpha_{k-1}})\| \leq (n - 1) \cdot 2^{-n+1} + \sum_{k=n+1}^{\infty} 2^{-k+1} = n 2^{-n+1}, \]

the series \( \sum_{k=1, k \neq n}^{\infty} w_{\alpha_n} \land w_{\alpha_k} \) converges absolutely and hence in norm for any \( n \in \mathbb{N} \). Take

\[ \omega_{\alpha_n} := \left( w_{\alpha_n} - \sum_{k=1, k \neq n}^{\infty} w_{\alpha_n} \land w_{\alpha_k} \right)^+ \quad (\forall n \in \mathbb{N}). \]

First, we show that the sequence \( (\omega_{\alpha_n})_{n=1}^{\infty} \) is disjoint. Let \( m \neq p \), then

\[ \omega_{\alpha_m} \land \omega_{\alpha_p} = \left( w_{\alpha_m} - \sum_{k=1, k \neq m}^{\infty} w_{\alpha_m} \land w_{\alpha_k} \right)^+ \land \left( w_{\alpha_p} - \sum_{k=1, k \neq p}^{\infty} w_{\alpha_p} \land w_{\alpha_k} \right)^+ \leq \]

\[ (w_{\alpha_m} - w_{\alpha_m} \land w_{\alpha_p})^+ \land (w_{\alpha_p} - w_{\alpha_p} \land w_{\alpha_m})^+ = \]

\[ (w_{\alpha_m} - w_{\alpha_m} \land w_{\alpha_p}) \land (w_{\alpha_p} - w_{\alpha_m} \land w_{\alpha_p}) = 0. \]

By (2.9),

\[ \|w_{\alpha_n} - \omega_{\alpha_n}\| = \|w_{\alpha_n} - \left( w_{\alpha_n} - \sum_{k=1, k \neq n}^{\infty} w_{\alpha_n} \land w_{\alpha_k} \right)^+\| = \]

\[ \left\|w_{\alpha_n} - \left( w_{\alpha_n} - w_{\alpha_n} \land \sum_{k=1, k \neq n}^{\infty} w_{\alpha_n} \land w_{\alpha_k}\right)\right\| = \left\|w_{\alpha_n} - \sum_{k=1, k \neq n}^{\infty} w_{\alpha_n} \land w_{\alpha_k}\right\| \leq \]
\[ \| \sum_{k=1,k \neq n}^{\infty} w_{\alpha_k} \| \leq n2^{-n+1} \quad (\forall n \in \mathbb{N}). \quad (2.11) \]

Combining (2.11) with (2.5) gives
\[ 2 \geq \| w_{\alpha_n} \| \geq \| \omega_{\alpha_n} \| \geq C - n2^{-n+1} \quad (\forall n \in \mathbb{N}). \quad (2.12) \]

Passing to further increasing sequence of indices, we may assume that
\[ \| w_{\alpha_n} \| \to M \in [C, 2] \quad (n \to \infty). \]

Now
\[
\lim_{n \to \infty} \| M^{-1}x + \| \omega_{\alpha_n} \|^{-1} \omega_{\alpha_n} \| = M^{-1} \lim_{n \to \infty} \| x + \omega_{\alpha_n} \| = [\text{by (2.11)}] = M^{-1} \lim_{n \to \infty} \| x + \omega_{\alpha_n} \| = [\text{by (2.3)}] = M^{-1} \lim_{n \to \infty} \| x + (x_{\alpha_n} - x) \| = M^{-1} \lim_{n \to \infty} \| x_{\alpha_n} \| = M^{-1} = \| M^{-1}x \|,
\]
violating the Brezis – Lieb property for \( u_0 = M^{-1}x \) and \( u_n = \| \omega_{\alpha_n} \|^{-1} \omega_{\alpha_n} \), \( n \geq 1 \). The obtained contradiction completes the proof. \( \square \)

Since every order continuous Banach lattice is Dedekind complete, the following result is a direct consequence of Theorem 4.

**Corollary 1.** For an order continuous Banach lattice \( E \), the following conditions are equivalent:
1. \( E \) is a BL-space;
2. \( E \) is a \( \sigma \)-BL-space;
3. \( E \) has the BL-property.

Corollary 1 applied to the order continuous Banach lattices \( L^p \) \((1 \leq p < \infty)\) gives Proposition 1.

We do not know where or not implication (2) \( \Rightarrow \) (3) of Theorem 4 holds true without the assumption that the Banach lattice \( E \) is \( \sigma \)-Dedekind complete.

More precisely:

**Question 1.** Does every \( \sigma \)-Brezis – Lieb Banach lattice have an order continuous norm?

In the proof of (2) \( \Rightarrow \) (3) of Theorem 4 the \( \sigma \)-Dedekind completeness of \( E \) has been used only for showing that \( E \) has an order continuous norm. So, any \( \sigma \)-Brezis – Lieb Banach lattice has the Brezis – Lieb property. Therefore, for answering in positive the question of possibility to drop \( \sigma \)-Dedekind completeness assumption in Theorem 4 it suffices to answer in positive the following question.
Question 2. Does the BL-property imply order continuity of the norm in the underlying Banach lattice?

In the end of the paper, we mention one more question closely related to the question in the last sentence of Example 2.

Question 3. Does the BL-property of a Banach lattice $E$ ensure that $E$ is a KB-space?

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1 Middle East Technical University, 06800 Ankara, Turkey
E-mail address: eduard@metu.edu.tr

2 Sobolev Institute of Mathematics, 630090 Novosibirsk, Russia
E-mail address: emelanov@math.nsc.ru

3 Department of Applied Mathematics, College of Sciences and Arts, Palestine Technical University-Kadoorie, Tulkarem, Palestine
E-mail address: mohammad.marabeh@ptuk.edu.ps, m.maraabeh@gmail.com