Ehrenfest’s Principle and the Problem of Time in Quantum Gravity

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Abstract

We elaborate on a proposal made by Greensite and others to solve the problem of time in quantum gravity. The proposal states that a viable concept of time and a sensible inner product can be found from the demand for the Ehrenfest equations to hold in quantum gravity. We derive and discuss in detail exact consistency conditions from both Ehrenfest equations as well as from the semiclassical approximation. We also discuss consistency conditions arising from the full field theory. We find that only a very restricted class of solutions to the Wheeler-DeWitt equation fulfills all consistency conditions. We conclude that therefore this proposal must either be abandoned as a means to solve the problem of time or, alternatively, be used as an additional boundary condition to select physical solutions from the Wheeler-DeWitt equation.

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1 Introduction

Despite many attempts, a viable quantum theory of gravity is still elusive. Apart from the absence of experimental hints, a major difficulty is the lack of any obvious physical principle, which would enable one to find such a theory, in analogy to the role that the equivalence principle played in the construction of general relativity.

One candidate for such a guiding principle may be the “problem of time in quantum gravity” which gained considerable interest recently (see, for example, [1], [2], and Chap.6 of [3]). This problem occurs in all systems whose classical version is invariant under reparametrisations of the time parameter, which leads to the absence of this parameter at the quantum level. The formal question is how to handle the classical Hamiltonian constraint, $H \approx 0$, in the quantum theory. Connected with the problem of time is the “Hilbert space problem” [1, 2] – it is not at all obvious which inner product of states one has to use in quantum gravity, and whether there is a need for such a structure at all. It is thus also not clear whether there is any sensible notion of unitary evolution of quantum states.

Basically, the approaches to address the problem of time can be classified as to whether an appropriate time variable is identifiable already at the classical level, or only after quantisation. The former try, for example, to cast the Hamiltonian constraint through an appropriate canonical transformation into the reduced form $P_T + h \approx 0$, where $P_T$ denotes the momentum conjugate to the time variable $T$. If this were possible, the problems of time and Hilbert space would be solved, since the new form of the constraints would be transformed into a (functional) Schrödinger equation upon quantisation, and the standard inner product could be used. This has been shown to work in special examples [2, 4], but it is far from clear to which extent it works in the general case. It is, however, known that it cannot work for the full configuration space [3].

Attempts to isolate a concept of time at the quantum level can be subdivided into many possibilities [1, 2], but have the common feature that time – if it exists at all at the most fundamental level – has to be searched for amongst the dynamical variables of the theory. Most approaches implement the classical constraint à la Dirac as a condition on physically allowed wave functionals, $\hat{H}\Psi_{\text{phys}} = 0$, the Wheeler-DeWitt equation. One may then, motivated by the indefinite kinetic term in $\hat{H}$, use a Klein-Gordon type of inner product [4]. Since this inner product is not positive definite, however, many problems arise which have provoked some authors to invoke a “third quantisation” of the theory (see the review in [1, 2]). A more recent attempt consists in constructing a positive definite physical Hilbert space from some auxiliary Hilbert space via some “spectral analysis proposal” [4].
A necessary requirement for all approaches is of course the ability to recover a sensible notion of semiclassical time from the full theory. This is mostly done in the context of some Born-Oppenheimer type of approximation scheme which employs an expansion of the full wave functional in powers of the Planck mass (see, e.g., the review in [8]). A notion of semiclassical time, or “WKB time”, emerges thereby from the phase of the wave functional in the first order of approximation. Higher orders modify the definition of WKB time through the back reaction of the quantum degrees of freedom onto the semiclassical ones.

A semiclassical approximation of a somewhat different kind can also be made for the approach in [7], where an explicit map between the physical Hilbert space of the full theory and the standard “external Hilbert space” of the Schrödinger equation can be made through the use of “almost ideal clocks” [9].

An interesting mixture of some of the above ideas has been suggested by Greensite [10, 11] (see also similar attempts in [12, 13]). Contrary to the semiclassical approximation, time is there defined by the exact phase of the full wave functional, and it is insisted on the validity of an exact Schrödinger-type of inner product. Since it would be inconsistent to demand the validity of a Schrödinger equation for the full theory, the weaker condition of Ehrenfest equations for appropriate expectation values is imposed. The idea then is to define time by the validity of these equations. We call this notion “Ehrenfest time” and reserve the label “phase time” for the more general concept of defining time from the full wave functional without further conditions. The idea implicit in [10-13] is that a notion of “phase time” automatically implies an “Ehrenfest time”.

The purpose of our work is to investigate in detail how far the notion of Ehrenfest time can be used as a viable candidate for a concept of time in quantum gravity. We shall find that, contrary to what has been thought previously, only a small subset of solutions to $H\Psi = 0$ allows the definition of an Ehrenfest time. Moreover, only a small subset of these have a sensible semiclassical limit. Our overall conclusion will thus be that one has either to reject this proposal as a way to solve the problem of time in quantum gravity or, alternatively, to use it as an additional boundary condition to select sensible wavefunctions from all solutions to the Wheeler-DeWitt equation.

Our paper is organised as follows. In Section 2 we start with a review of the proposal put forward in [10, 11]. We then explore further properties of the first Ehrenfest equation (the one associated with the configuration variables) and demonstrate, in particular, that a phase time does not necessarily possess the properties of an Ehrenfest time. We then derive an exact condition for the validity of the second Ehrenfest equation (the one associated with the canonical momentum).

Section 3 is devoted to a detailed comparison with the semiclassical approximation. We shall show, first, that the solutions which lead to an Ehrenfest time
do not, in general, allow a sensible semiclassical limit. We then study the connection with the back reaction-corrected WKB time in those cases where a sensible semiclassical limit does hold.

Section 4 investigates particular issues which appear in the full, infinite-dimensional, theory. If a solution to the Wheeler-DeWitt equation admits a (local) Ehrenfest time, we show that it can be mapped to a scalar function on superspace (the configuration space of three-metrics modulo diffeomorphisms). We can also conclude that this result must hold for all orders of the semiclassical approximation scheme described in $\S$.

Section 5 presents our conclusion, in particular a discussion of the above proposal in the light of the “problems of time” in $\S$.

Some lengthy calculations are relegated to an appendix.

## 2 Ehrenfest’s equations in quantum gravity

In this section we investigate the consequences which arise from the demand for the validity of Ehrenfest’s equations in quantum gravity. We start with a review of the proposal in $\cite{10,11}$ and then proceed to work out further properties.

The coordinates in configuration space shall be denoted by \{q^a\}. In quantum general relativity, these are the coordinates on full superspace. However, in practice it is more convenient to work with the (redundant) variables before the diffeomorphism group has been factored out – the three-metric and matter fields. Moreover, we restrict ourselves here to finite-dimensional models (thus, $\alpha$ runs from 1, ..., $N$) and discuss some peculiarities of infinite dimension in Section 4.

The proposal is to find a coordinate transformation \{q^a\} $\rightarrow$ \{t, $\eta^i$\}, such that for an “observable” $A(q^a, p_\beta)$ Ehrenfest’s condition holds:

$$\frac{d}{dt} <A> = \int D\eta M(t, \eta) \Psi^*(t, \eta) i[H, A] \Psi(t, \eta) + \frac{\partial A}{\partial t},$$

where

$$< A > \equiv \int D\eta M(t, \eta) \Psi^*(t, \eta) A \Psi(t, \eta)$$

is the standard expression for an expectation value in the Schrödinger-type inner product. It is assumed therein that the full Hamilton operator, $H$, of the Wheeler-DeWitt equation $H \Psi = 0$ can be used as the “dynamical operator” describing the evolution of expectation values. The important point is that a time variable is singled out by this prescription, since the integration is over the $\eta$-variables only. The wave function $\Psi$ in these expressions is assumed to be a given solution to the Wheeler-DeWitt equation; $M(t, \eta)$ denotes a measure which is specified below.
Note that the “observables” $A$ are not assumed to commute with the full Hamiltonian, since otherwise the content of (1) would be trivial. They are “observables” in the sense of [14] (in contrast to “perennials”) – quantities which commute with the diffeomorphism constraints of general relativity, but not necessarily with the Hamiltonian constraint.

The Hamiltonian $H$ can formally be written as (we use $\eta$ with greek indices if $t$ is adjoined to $\eta^i, \eta^0 \equiv t$)

$$H = " G^{\alpha\beta} \frac{\partial}{\partial \eta^\alpha} \frac{\partial}{\partial \eta^\beta} " + \xi R + V(t, \eta) ,$$

where the quotations in the kinetic term emphasise that the factor-ordering problem has not been addressed yet, and where an additional factor-ordering ambiguity proportional to the Ricci scalar of configuration space has been taken into account. The supermetric with respect to the $(t, \eta)$-coordinates is labeled with a prime.

The task is now to explicitly determine the above coordinate transformation on configuration space. Consider first the Ehrenfest equation with respect to the configuration variables, i.e., choose $A = \eta^i$. We shall refer to this as the “first Ehrenfest equation”. One finds

$$[H, \eta^i] \Psi = -2i G^{\alpha\beta} \delta^i_\alpha \frac{\partial}{\partial \eta^\beta} \Psi .$$

The demand for $H$ to be hermitean requires that $G^{0i} = 0$; otherwise one would be left with $t$-derivatives which cannot be transferred from $\Psi$ to $\Psi^*$ upon partial integration in the above inner product. The condition $G^{0i} = 0$ means that the time direction runs orthogonally to hypersurfaces of constant time. If we demand, in addition, the invariance of the first Ehrenfest equation with respect to transformations of the variables $\{\eta^i\}$ on $t = \text{const.}$, the measure in (2) is fixed to be $M(t, \eta) = \sqrt{G'}$, where $G'$ denotes the determinant of the metric $G'_{\alpha\beta}$, and the kinetic term in (3) is the Laplace-Beltrami-operator. The Hamiltonian (3) thus can be written as

$$H = -\frac{1}{2} \left[ \frac{1}{\sqrt{G'}} \frac{\partial}{\partial t} \sqrt{G'} G^{ij} \frac{\partial}{\partial \eta^j} + \frac{1}{\sqrt{G'}} \frac{\partial}{\partial t} \sqrt{G'} G^{00} \frac{\partial}{\partial \eta^0} \right] + \xi R + V(t, \eta)

=: -\frac{1}{2} D^2 - \frac{1}{2} D_0^2 + \xi R + V(t, \eta) .$$

It was shown in [11] that for the first Ehrenfest equation to be fulfilled, $\Psi$ must be of the form

$$i \frac{\partial \Psi}{\partial t} = \left( -\frac{1}{2} D^2 + \tilde{V} - i \frac{\partial \ln \sqrt{G'}}{\partial t} \right) \Psi .$$
Despite its formal similarity, this is not a Schrödinger equation, since $\Psi$ is here assumed to be a given solution to the Wheeler-DeWitt equation. Decomposing $\Psi$ into its real and imaginary parts and using $H\Psi = 0$, one finds from (3) that

$$V = \frac{1}{\Psi^*\Psi} \left[ \frac{1}{2} Re(\Psi^* D^2 \Psi) - Im(\Psi^* \partial_t \Psi) \right],$$

(7)

$$\partial_t \ln \sqrt{G'} = \frac{1}{\Psi^*\Psi} \left[ Im(\Psi^* D^2 \Psi) - \partial_t (\Psi^* \Psi) \right].$$

(8)

Writing $\Psi$ as

$$\Psi(t, \eta) = \varrho(t, \eta)e^{i\theta(t, \eta)},$$

(9)

one finds from (8)

$$G'_{00} \partial_t \theta = 1 + \frac{f(\eta)}{\sqrt{G' \varrho^2}} \equiv \kappa(t, \eta)$$

(10)

as the condition which must be satisfied after employing the coordinate transformation $\{q^\alpha\} \rightarrow \{t, \eta^i\}$. Note that $f = 0$ was chosen in [11]. Note also that the validity of the first Ehrenfest equation includes conservation of probability, since $\frac{d}{dt} <\Psi|\Psi> = 0$ follows for $A \equiv \mathbb{1}$. As can be immediately recognised from (10), the proposal works only for complex wave functions, i.e. for $\theta \neq 0$.

It is instructive to make a connection between Ehrenfest time and derivatives with respect to the old coordinates $\{q^\alpha\}$. Writing

$$\frac{\partial}{\partial t} = T^\alpha \frac{\partial}{\partial q^\alpha},$$

we find with the help of (11)

$$\frac{\partial \theta}{\partial t} = T^\alpha \frac{\partial \theta}{\partial q^\alpha} = \kappa \ G'_{00} = \kappa \ G_{\alpha\beta} \frac{\partial q^\alpha}{\partial t} \frac{\partial q^\beta}{\partial t} = \kappa \ G_{\alpha\beta} T^\alpha T^\beta,$$

where $G_{\alpha\beta}$ denotes the components of the configuration space metric with respect to $\{q^\alpha\}$. Since this yields $T^\beta = \kappa^{-1} G^{\alpha\beta} \partial \theta / \partial q^\alpha$, one has

$$\frac{\partial}{\partial t} = \kappa^{-1} G^{\alpha\beta} \frac{\partial \theta}{\partial q^\alpha} \frac{\partial}{\partial q^\beta}. $$

(11)

This expression shows explicitly that Ehrenfest time is constructed from the phase of $\Psi$, i.e., that it is proportional to phase time. This also resembles the definition of the WKB time in the semiclassical approximation (see below). From $G'_{0i} = 0$ and (11) one finds that $\partial \theta / \partial \eta^i = 0$ and, thus, $\theta$ is a functional of $t$ only.

Decomposing now the Wheeler-DeWitt equation into its real and imaginary parts, one then finds (a dot denoting a derivative with respect to $t$):

Real part:

$$\frac{1}{2} G^{00} \dot{\varrho}^2 + \frac{1}{\varrho} H \varrho = 0,$$

(12)
Imaginary part:

\[ \frac{1}{2} \frac{1}{\sqrt{G'}} \left( \partial_t \sqrt{G'} G'^{00} \right) \dot{\theta} + G'^{00} \frac{\dot{\phi}}{\phi} + \frac{1}{2} G'^{00} \ddot{\phi} = 0 \]

\[ \Leftrightarrow \frac{1}{2} \partial_t \ln \sqrt{G'} \left[ 1 + \frac{f(\eta)}{\sqrt{G'} g^2} \right] + \frac{1}{2} \partial_t \left[ 1 + \frac{f(\eta)}{\sqrt{G'} g^2} \right] + \frac{\dot{\phi}}{\phi} \left[ 1 + \frac{f(\eta)}{\sqrt{G'} g^2} \right] = 0 \]

\[ \Leftrightarrow \partial_t (\sqrt{G'} g^2) = 0 . \]  

(13)

Equation (13) means that the solution \( \Psi \) is stationary with respect to Ehrenfest time, i.e., the integrand in \( < \Psi | \Psi > \), which is given by \( \sqrt{G'} \Psi^* \Psi \), is itself time independent. Therefore, \( G'^{00} \dot{\phi} = \kappa(\eta) \) from (10). Consequently,

\[ \frac{d}{dt} < \eta^i > = \frac{d}{dt} \int D\eta \sqrt{G'} g^2 \eta^i = 0 , \]

and thus

\[ \frac{d}{dt} < A(t, \eta) > = < \frac{\partial}{\partial t} A(t, \eta) > , \]

(14)

i.e., the time dependence in \( < A(q^i) > \) arises solely from the explicit dependence of \( A \) on time. Since therefore \( 0 = \frac{d}{dt} < \eta^i >= < i[H, \eta^i] > \), one could call the variables \( \eta^i \) perennials \( [14] \) with respect to the scalar product used here.

We note that in the semiclassical approximation (see Sect.3) equation (13) just corresponds to the “prefactor equation” in the highest order of approximation when written in “comoving coordinates” \( [2, 8, 15] \). The above derivation shows that \( \Psi \) has the form \( \Psi = G'(t, \eta)^{-1/4} g(\eta) e^{i \theta(t)} \) with some function \( g \) depending on \( \eta \) only.

We now address the conditions which are necessary for the second Ehrenfest equation (see equation (16) below) to be fulfilled. In \( [11] \) an approximate validity was shown in the case where the phase is rapidly varying. Here we shall present an exact condition.

As in quantum field theory on curved backgrounds we shall use the momentum \( p_i \) which is hermitean with respect to the measure \( \sqrt{G'} \) in the inner product. It reads

\[ p_i = -i G'^{-1/4} \partial_i G'^{1/4} = -i \partial_i - \frac{i}{2} \left( \partial_i \ln \sqrt{G'} \right) . \]

(15)

We demand that

\[ \frac{d}{dt} < p_i > = i < [H, p_i] > + < \frac{\partial p_i}{\partial t} > . \]

(16)

The calculations are straightforward, but somewhat lengthy, and have thus been relegated to the appendix. The result is that (16) can only hold if the last term vanishes explicitly, i.e., if

\[ < \frac{\partial p_i}{\partial t} > = 0 = < \partial_i \partial_t \ln \sqrt{G'} > . \]

(17)
This yields an additional restriction on allowed physical states. Note that – in contrast to \( <\eta' > \) – stationarity of \( \Psi \) does not lead to \( \frac{d}{dt} < p_i > = 0 \).

We conclude this section with a simple example which demonstrates that not every “phase time” satisfies the condition for an Ehrenfest time.

Consider the model of an indefinite harmonic oscillator (arising, e.g., from a Friedmann model with a conformally coupled scalar field \([16]\)). The Wheeler-DeWitt equation reads

\[
\left[ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - x^2 + y^2 \right] \Psi(x,y) = 0,
\]

and we shall choose in particular the simple solution

\( \Psi = \exp(i xy) \).

From \((\Pi)\) with \( \kappa = 1 \) (this choice is justified in Sect.3) we have

\[
\frac{\partial}{\partial t} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y},
\]

yielding \( \dot{x} = -y \), \( \dot{y} = x \), and \( \dot{\theta} = x^2 - y^2 \). Since \( \theta \) must not depend on \( \eta \) (see above), we find from the condition \( \partial_\eta \theta = 0 \), \( \partial_\eta \dot{\theta} = 0 \) the equations (a prime denotes a derivative with respect to \( \eta \))

\[
yx' + xy' = 0, \quad xx' - yy' = 0,
\]

which allow only the trivial solution \( x' = y' = 0 \). Thus, there does not exist any coordinate transformation from \((x, y)\) to \((t, \eta)\), such that \( t \) has the properties of an Ehrenfest time (independent of the validity of the second Ehrenfest equation). The demand for the Ehrenfest equation to hold is thus much more restrictive than was originally assumed \([10-13]\).

On the other hand, if one reversed, e.g., the sign of the potential in this example and considered a solution of the type \( \Psi = \exp(\frac{i}{2}(x^2 - y^2)) \), the Ehrenfest conditions could be fulfilled.

### 3 Ehrenfest time and the semiclassical approximation

In this section we make a detailed comparison of the Ehrenfest time with the standard semiclassical approximation to quantum gravity and the approximate notion of WKB time \([8]\).
First, we note from (11) that in order to be able to recover the momentum in the semiclassical limit through \( p_\alpha = \partial \theta / \partial q^\alpha \), we must choose \( \kappa = \text{const} \) (for simplicity we choose \( \kappa = 1 \)). Next, using (11) with \( \kappa = 1 \), we write the “Ehrenfest” condition (10) in the form

\[
\frac{1}{2} G^{\alpha\beta} \frac{\partial \theta}{\partial q^\alpha} \frac{\partial \theta}{\partial q^\beta} - \frac{1}{2 G^{00}} = 0 . \tag{18}
\]

If a semiclassical approximation were valid, \( \theta \) would also obey the Hamilton-Jacobi equation

\[
\frac{1}{2} G^{\alpha\beta} \frac{\partial \theta}{\partial q^\alpha} \frac{\partial \theta}{\partial q^\beta} + V \approx 0 . \tag{19}
\]

Since \( G^{00} \) can be a function of \( t \) only (see (10) with \( \theta = \theta(t) \) and \( \kappa = 1 \)), the last two equations would only be compatible if \( V \approx V(t) \). Since \( V \) is a given function, this is of course a strong restriction on the class of allowed semiclassical states.

To discuss the connection with the semiclassical approximation reviewed in [8], we start from the Wheeler-DeWitt equation

\[
\left( -\frac{1}{2M} \frac{1}{\sqrt{G}} \frac{\partial}{\partial h^a} \sqrt{G} G^{ab} \frac{\partial}{\partial h^b} + MV + H_m \right) \Psi = 0 \tag{20}
\]

with

\[
H_m = \frac{1}{2} \left( -\frac{1}{\sqrt{h}} \frac{\partial^2}{\partial \phi^2} + \sqrt{h} (m^2 \phi^2 + U(\phi)) \right) \tag{21}
\]

being the Hamiltonian for a homogeneous scalar field. The variables \( \{ h^a \} \) denote the components of the three-dimensional metric, and \( h \) is its determinant.

The important point to note is that the degrees of freedom \( \{ q^a \} \) have been divided into some “heavy ones” with large “mass” \( M \) (the gravitational degrees of freedom) and some “light” degree of freedom, the scalar field \( \phi \) (see [8]). Equation (18) then reads

\[
\frac{1}{2M} G^{ab} \frac{\partial \theta}{\partial h^a} \frac{\partial \theta}{\partial h^b} + \frac{1}{2 \sqrt{h}} \left( \frac{\partial \theta}{\partial \phi} \right)^2 - \frac{1}{2 G^{00}} = 0 . \tag{22}
\]

Expanding now the phase \( \theta \) into inverse powers of \( M \),

\[
\theta = MS_0 + Re(S_1) + M^{-1} Re(S_2) + ... , \tag{23}
\]

Equation (22) yields the following equations at consecutive orders of \( M \):

\[
O(M^2):
\frac{1}{\sqrt{h}} \left( \frac{\partial S_0}{\partial \phi} \right)^2 - \frac{1}{G^{00}} \bigg|_{M^2} = 0 , \tag{24}
\]
\[ O(M^1): \]
\[
G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial S_0}{\partial h^b} + \frac{2}{\sqrt{\hbar}} \frac{\partial S_0}{\partial \phi} \frac{\partial \text{Re}(S_1)}{\partial \phi} - \frac{1}{G^{\alpha\alpha}} \bigg|_{M^1} = 0 ,
\]
(25)

\[ O(M^0): \]
\[
2G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial \text{Re}(S_1)}{\partial h^b} + \frac{1}{\sqrt{\hbar}} \left( \frac{\partial \text{Re}(S_1)}{\partial \phi} \right)^2 + \frac{2}{\sqrt{\hbar}} \frac{\partial S_0}{\partial \phi} \frac{\partial \text{Re}(S_2)}{\partial \phi} - \frac{1}{G^{\alpha\alpha}} \bigg|_{M^0} = 0 .
\]
(26)

On the other hand, the standard Born-Oppenheimer type of expansion yields \[8\]

\[ O(M^2): \]
\[
\frac{\partial S_0}{\partial \phi} = 0 ,
\]
(27)

\[ O(M^1): \]
\[
\frac{1}{2} G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial S_0}{\partial h^b} + V = 0 ,
\]
(28)

\[ O(M^0): \]
\[
G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial D}{\partial h^b} = \frac{1}{2} G^{ab} \frac{\partial^2 S_0}{\partial h^a \partial h^b} D ,
\]
(29)

\[
i G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial \chi}{\partial h^b} \equiv i \frac{\partial \chi}{\partial t_{WKB}} = \mathcal{H}_m \chi .
\]
(30)

Here we have introduced
\[
\chi(h^a, \phi) \equiv D(h^a) \exp\{iS_1(h^a, \phi)\} \equiv \rho(h^a, \phi) \exp\{i\text{Re}(S_1)\}
\]
(31)

and chosen for \(D(h^a)\) the usual prefactor equation \[8, 15\]. We can thus write (31) in the form
\[
G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial \text{Re}(S_1)}{\partial h^b} = i G^{ab} \frac{1}{\rho} \frac{\partial S_0}{\partial h^a} \frac{\partial \rho}{\partial h^b} - \frac{1}{\chi} \mathcal{H}_m \chi .
\]
(32)

Up to order \(M^0\), the full wave function thus reads
\[
\Psi = \frac{1}{D} e^{iMS_0} \chi = \frac{\rho}{D} e^{i(MS_0 + \text{Re}(S_1))} .
\]
(33)

Comparison of the two expansion schemes then yields
\[
\left. \frac{1}{2G^{\alpha\alpha}} \right|_{M^2} = 0 ,
\]
\[
\left. \frac{1}{2G^{\alpha\alpha}} \right|_{M^1} = -V ,
\]
\[
\left. \frac{1}{2G^{\alpha\alpha}} \right|_{M^0} = i G^{ab} \frac{1}{\rho} \frac{\partial S_0}{\partial h^a} \frac{\partial \rho}{\partial h^b} - \frac{1}{\chi} \mathcal{H}_m \chi + \frac{1}{2} \frac{1}{\sqrt{\hbar}} \left( \frac{\partial \text{Re}(S_1)}{\partial \phi} \right)^2 .
\]
Using these results in (22), one gets

\[
\frac{G^{ab}}{2M} \left( M \frac{\partial S_0}{\partial h^a} + \frac{\partial \text{Re}(S_1)}{\partial h^a} \right) \left( M \frac{\partial S_0}{\partial h^b} + \frac{\partial \text{Re}(S_1)}{\partial h^b} \right) + MV + \frac{H_m \chi}{\chi} \tag{34}
\]

\[-i \frac{G^{ab}}{\rho} \frac{\partial S_0}{\partial h^a} \frac{\partial \rho}{\partial h^b} + O(\frac{1}{M}) = 0.\]

Multiplying this equation with \( \chi^* \) and integrating over \( \phi \) yields

\[
\frac{G^{ab}}{2M} \left( M \frac{\partial S_0}{\partial h^a} + \langle \chi | \frac{\partial \text{Re}(S_1)}{\partial h^a} \chi \rangle \phi \right) \left( M \frac{\partial S_0}{\partial h^b} + \langle \chi | \frac{\partial \text{Re}(S_1)}{\partial h^b} \chi \rangle \phi \right) + MV \tag{35}
\]

\[+ \langle \chi | H_m \chi \rangle \phi + O(\frac{1}{M}) = 0,\]

where \( \langle \psi | \varphi \rangle_\phi \equiv \int D\phi \, \psi^* \varphi \). In the derivation of (35) we have made use of the fact that \( \chi \), which is a solution to the Schrödinger equation (30), can be normalised, and thus

\[
\langle \chi | \frac{1}{\rho} G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial \rho}{\partial h^b} \chi \rangle \phi = \int d\phi \, e^{-i\text{Re}(S_1)} G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial \rho}{\partial h^b} \rho e^{i\text{Re}(S_1)}
\]

\[= G^{ab} \frac{\partial S_0}{\partial h^a} \frac{\partial}{\partial h^b} \int d\phi \frac{1}{2} \rho^2
\]

\[= 0.\]

Equation (35) is just the “back reaction corrected” Hamilton-Jacobi equation in this order of approximation \([8]\), which we have here shown to be consistent with a semiclassical approximation of the “Ehrenfest condition” (22). Note that, in contrast to the spirit of the standard approach, the concept of time is here fixed once and for all by the Ehrenfest condition, whereas in the standard approach a concept of time emerges at \( O(M^0) \) (the WKB time introduced in (30)) and is modified at higher orders through back reaction effects \([8]\). This modification – which in the standard approach is made by hand – follows here automatically.

The scalar product which we have been using in (35) is not the one of the full theory, see (2). How are these inner products related?

In the highest order of the semiclassical approximation, Ehrenfest time agrees with WKB time, and the integration in the total inner product is thus over \( D\tilde{\eta}D\phi \), where \( \tilde{\eta} \) denotes the part of the three-metric orthogonally to the flow generated by WKB time. It was shown in \([15]\) that the choice of a prefactor \( D \) (see (29)) which is sharply peaked in \( \tilde{\eta} \) leads to

\[
\int D\tilde{\eta} \, D\phi \, \Psi^* \Psi \approx \left( \int D\tilde{\eta} \, D^{-2} \right) \left( \int D\phi \, \chi^* \chi \right) + O(M^{-1}).
\]

10
Since the left-hand side agrees with the full inner product $\int D\eta \Psi^* \Psi$ up to order $O(M^{-1})$, the use of the $\phi$-inner product in (33) to find an effective equation for the gravitational field is justified.

We also note that one can somewhat modify the standard semiclassical expansion to incorporate back reaction directly into the gravitational part of the Wheeler-DeWitt equation [17]. Simple minisuperspace models may be used to explicitly compute the various concepts of time [18].

In a ($k = -1$) Friedmann model with a scalar field, for example, the Wheeler-DeWitt equation reads

$$\left[ \frac{1}{a} \frac{1}{M} \frac{\partial^2}{\partial a^2} + \frac{1}{a^2} \frac{1}{M} \frac{\partial}{\partial a} - \frac{1}{a^3} \frac{\partial^2}{\partial \phi^2} + Ma \right] \Psi(a, \phi) = 0 ,$$

(36)

where $a$ denotes the scale factor. We choose the following solution at $O(M^0)$:

$$\Psi(a, \phi) \approx a \exp \left(-i \frac{Ma^2}{2}\right) \chi(a, \phi) ,$$

where

$$\chi(a, \phi) = \frac{1}{\pi^{1/4}} \frac{\sigma^{1/2}}{(\sigma^2 + i/2a^2)^{1/2}} \exp \left(-\frac{\phi^2}{2\sigma^2 + i/a^2}\right) ,$$

and $\sigma$ is an arbitrary constant. The WKB time is then given by

$$\frac{\partial}{\partial t_{WKB}} = G^{\phi_0} \frac{\partial S_0}{\partial a} \frac{\partial}{\partial a} = \frac{\partial}{\partial a} ,$$

while for the “back reaction corrected” WKB time $\tilde{t}$ (which agrees with the Ehrenfest time in this order of approximation after the $\phi$-field has been integrated out) one has [18]

$$\frac{\partial}{\partial \tilde{t}} = \left(1 - \frac{1}{4M\sigma^2 a^4}\right) \frac{\partial}{\partial a} .$$

Choosing an exact solution which reduces to the above semiclassical solution for $a \to \infty$, one can find the following approximate “Ehrenfest time”

$$\frac{\partial}{\partial t} = \left[1 - \frac{2}{Ma^4} \left(\frac{1}{4\sigma^2} - \frac{\phi^2}{4\sigma^4} - \frac{1}{M}\right)\right] \frac{\partial}{\partial a} + \frac{\phi}{2a^5\sigma^4} \frac{\partial}{\partial \phi}$$

which of course, in contrast to $t_{WKB}$ and $\tilde{t}$, is defined through all variables of the theory.

Another example of a WKB and a phase time – there called Ehrenfest time in the spirit of [11], but in our sense to be understood as a candidate for an Ehrenfest time – can be found in [19].

11
4 Consistency conditions from field theory

Up to now, we have only considered models with a finite number of degrees of freedom. Since the Ehrenfest proposal was intended to apply for full quantum gravity, it must be investigated to which extent the above conditions can be generalised to the field theoretic case. We shall proceed analogously to [20], where consistency conditions were discussed for the semiclassical approximation. While there it was found that the WKB time cannot exist as a scalar function on the space of three-metrics, but only on superspace, we shall here find an analogous result for the Ehrenfest time. We shall thereby also be led to a characteristic property of transformation to the Ehrenfest coordinates \((t, \eta)\).

The local form \(\tau(x)\) of the phase time \([11]\) would read (recall that \(\kappa = 1\))

\[
\frac{\delta}{\delta \tau(x)} \equiv \xi_x = G^{\alpha\beta} \frac{\delta \theta}{\delta q^\alpha(x)} \frac{\delta}{\delta q^\beta(x)} .
\] (37)

This can of course only be consistently done, if

\[
[\xi_x, \xi_y] = 0.
\] (38)

As in [20] it turns out to be convenient to work with the “smeared out” quantities

\[
\xi^N_x = \int dx N(x) G^{\alpha\beta}(x) \frac{\delta \theta}{\delta q^\alpha(x)} \frac{\delta}{\delta q^\beta(x)}
\] (39)

with some arbitrary “test function” \(N(x)\). We thus get for the commutator

\[
[\xi^N_x, \xi^M_y] = \int_x \int_y N(x) M(y) G^{\alpha\beta}(x) \frac{\delta \theta}{\delta q^\alpha(x)} \frac{\delta}{\delta q^\beta(x)} G^{\kappa\lambda}(y) \frac{\delta \theta}{\delta q^\kappa(y)} \frac{\delta}{\delta q^\lambda(y)}
\]

\[
- \int_x \int_y M(x) N(y) G^{\alpha\beta}(x) \frac{\delta \theta}{\delta q^\alpha(x)} \frac{\delta}{\delta q^\beta(x)} G^{\kappa\lambda}(y) \frac{\delta \theta}{\delta q^\kappa(y)} \frac{\delta}{\delta q^\lambda(y)}
\]

\[
= \int_x \int_y (N(x) M(y) - N(y) M(x)) G^{\alpha\beta}(x) \frac{\delta \theta}{\delta q^\beta(x)} \frac{\delta^2 \theta}{\delta q^\lambda(y) \delta q^\kappa(x)}
\]

\[
\times G^{\kappa\lambda}(y) \frac{\delta}{\delta q^\kappa(y)} .
\] (40)

This commutator can only vanish if the second functional derivative of \(\theta\) in \([10]\) is proportional to \(\delta(x - y)\).

To calculate this quantity we consider the functional version of \([12]\), which reads

\[
E_x := \frac{1}{2} G^{\alpha\beta}(x) \frac{\delta \theta}{\delta q^\alpha(x)} \frac{\delta \theta}{\delta q^\beta(x)} + \frac{1}{\varrho} H \varrho = 0 .
\] (41)
Differentiating $E_x$ with respect to $q^\lambda(y)$ yields

$$
0 = \frac{\delta E_x}{\delta q^\lambda(y)} + \int dz \frac{\delta E_x}{\delta \left(\frac{\delta}{\delta q^\alpha(z)}(z)\right)} \frac{\delta^2 \theta}{\delta q^\alpha(x)\delta q^\lambda(y)} + \int dz \delta E_x \frac{\delta \theta}{\delta q^\alpha(x)} \frac{\delta^2 \theta}{\delta q^\lambda(y)} .
$$

(42)

For the first term on the right-hand side we have

$$
\frac{\delta E_x}{\delta q^\lambda(y)} = F_\lambda \delta(x - y) + \frac{\delta}{\delta q^\lambda(y)} \frac{1}{\rho} H \rho ,
$$

(43)

where the explicit form of the function in front of $\delta(x - y)$ is not needed below and has therefore been abbreviated by $F_\lambda$.

The Hamiltonian density $H$ is given explicitly by

$$
H = -\frac{1}{2M} G_{abcd} \frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - 2M \sqrt{h}(R - 2\Lambda) + \mathcal{H}_m ,
$$

(44)

where

$$
\mathcal{H}_m = \frac{1}{2} \left( -\frac{1}{\sqrt{h}} \frac{\delta^2}{\delta \phi^2} + \sqrt{h} h^{ab} \phi_{,a} \phi_{,b} + \sqrt{h} \left( m^2 \phi^2 + U(\phi) \right) \right)
$$

(45)

is the Hamiltonian density for a scalar field. Then,

$$
\frac{\delta}{\delta q^\lambda(y)} \left( \frac{1}{\rho} H \rho \right) = M_\lambda(x) \delta(x - y) + \frac{\delta V(x)}{\delta q^\lambda(y)}
$$

$$
= M_\lambda(x) \delta(x - y) + 2G^{ijab} \delta_{,ij}(x - y) \delta^{{h_{ab}}} + \sqrt{h} h^{ab} \phi_{,a} \delta_{,b}(x - y) \delta^{\{h_{ab}\}}
$$

(46)

where here $V$ is given by

$$
V = -2\sqrt{h}(R - 2\Lambda) + \sqrt{h} h^{ab} \phi_{,a} \phi_{,b} + \sqrt{h} \left( m^2 \phi^2 + U(\phi) \right) ,
$$

(47)

and $\delta^{\{h_{ab}\}} = 1$ ($\delta_{, \phi} = 1$) if $q^\lambda \in \{h_{ab}\}$ ($q^\lambda = \phi$) and otherwise zero. The second functional derivatives of the $\delta$-function in (46) arise from the Ricci scalar $R$ in (14). With the result (16) we know the second derivatives of $\theta$ in (12), which in turn give the following expression for the commutator (10)

$$
[\xi^N_x, \xi^M_y] = \int_x \int_y (N(x)M(y) - N(y)M(x))
$$

$$
\times G^{\alpha\beta}(x) \frac{\delta \theta}{\delta q^\beta(x)} \frac{\delta^2 \theta}{\delta q^\lambda(y)\delta q^\alpha(x)} G^\mu\lambda(y) \frac{\delta}{\delta q^\mu(y)}
$$

$$
= \int_x \int_y (N(y)M(x) - N(x)M(y))
$$

$$
\times \left[ 2G^{ijab}(y) G_{abcd}(y) \delta_{,ij}(x - y) \frac{\delta}{\delta h_{cd}(y)} + h^{ab} \phi_{,a} \delta_{,b}(x - y) \frac{\delta}{\delta \phi(y)} \right] .
$$
After some partial integrations this yields
\[ \left[ \xi_x^N, \xi_y^M \right] = -2 \int_x (N \partial_a M - M \partial_a N) \left( \frac{\delta}{\delta h_{ab}} \right)_{|b} - \int_x (N \partial_a M - M \partial_a N) h^a b \frac{\delta}{\delta \phi} \]
\[ = \int dx \left( \mathcal{L}_K h_{ab} \right) \frac{\delta}{\delta h_{ab}} - \int dx \left( \mathcal{L}_K \phi \right) \frac{\delta}{\delta \phi} \neq 0 , \quad (48) \]

where
\[ K^a := (NM_{ib} - MN_{ib}) h_{ab} . \]

The expressions on the right-hand side of (48) are just the diffeomorphism constraints of general relativity in their quantised form. Therefore, there is in analogy to the semiclassical case [20] no time function \( \tau(x) \) available on the space of three-metrics (since (48) does not vanish), but such a function is available on the space of all three-\textit{geometries}, i.e., after the diffeomorphism constraints are divided out. This of course makes sense, since it is assumed that in the inner product which is used for the Ehrenfest equations all unphysical variables are eliminated.

Note that for our result only the special structure of the time derivative (37) and the functional version of equation (12) as a condition for the phase of the wave function are important. For this reason every description by “time vector fields” with a representation (48) has this property. From this point of view the analogous result for the WKB time (30) in [20] comes out naturally. We can also conclude immediately that the same result must hold for every order of the semiclassical approximation scheme described in [8], if the higher orders are understood to describe the influence of back reaction on \( t_{WKB} \) by corrections to the phase.

After this test of consistence, we finally show a characteristic property of the transformation from the old variables (the three-metric and matter fields, collectively denoted by \( \{q^\alpha\} \) ) to the Ehrenfest variables.

For this purpose we insert into (41) the functional form of \( G^{00} \hat{\theta} = 1 \) (cf.(10)), which reads \( G^{\alpha \beta} \frac{\delta \hat{\theta}}{\delta q^\alpha} \frac{\delta \hat{\theta}}{\delta q^\beta} = G'_{00} \) and find
\[ G'_{00} = -\frac{2}{\hat{\theta}} H \hat{\theta} . \quad (49) \]

Assuming that the metric depends on the new coordinates ultralocally (i.e., that it contains no spatial derivatives of the coordinate), partial differentiation of (44) leads to
\[ \frac{\delta G'_{00}(x)}{\delta q^a(y)} = \int dz \frac{\delta \hat{\eta}^a(z)}{\delta q^a(y)} \frac{\delta G'_{00}(x)}{\delta \hat{\eta}^a(z)} = \frac{\delta \hat{\eta}^a(x)}{\delta q^a(y)} K_\sigma(x) . \]

Comparing this with (40) yields the following expressions for the coordinate transformation
\[ \frac{\delta \hat{\eta}^a(x)}{\delta h_{ab}(y)} K_\sigma(x) = -4 G^{ijab} \delta_{ij} (x - y) + \ldots \delta(x - y) , \]
\[
\frac{\delta \eta^\sigma(x)}{\delta \phi(y)} K_\sigma(x) = -2\sqrt{\hbar} h^{ab} \phi_a \delta_{b}(x - y) + \ldots \delta(x - y).
\]

We recognise from these expressions that the transformation from the old coordinates to Ehrenfest coordinates cannot be ultralocal.

5 Summary and Conclusion

In this paper we have studied various consequences which arise in using the Ehrenfest equations as a way to solve the problem of time in quantum gravity [10-13]. Is the Ehrenfest time a viable candidate for a concept of time in quantum gravity?

In the following we investigate this concept in view of the “problems of time” which are listed in [4].

Existence problem: This may in fact be the major problem. As we have shown in Section 2, only very few solutions of the Wheeler-DeWitt equation allow the validity of the Ehrenfest condition (10) (with \(\kappa = 1\)) and (17). Although it was of course clear that the proposal does not work for real solutions (such as, e.g., the Hartle-Hawking wave function), this drastic restriction does not seem to have been considered in [10-13]. We emphasise that the existence problem already occurs locally in configuration space, independent of possible global obstructions which one would expect to arise anyway.

Hilbert space problem: This concerns the question of the correct inner product in quantum gravity. If the Ehrenfest equations hold, this problem is solved by the choice made in (2). Moreover, through equation (1) it is possible to get sensible answers for expectation values of observables from the “wave function of the universe”.

Uniqueness problem: In case of existence, one can uniquely construct the corresponding phase time from a given solution to the Wheeler-DeWitt equation after an initial hypersurface \(t_0 = \text{const.}\) has been specified.

The spacetime problem as well as the sandwich problem do not play any role in this approach, since they only arise if embeddings of hypersurfaces into spacetime are considered. But there is a new problem, the semiclassical problem: Only very few solutions which allow an Ehrenfest time do possess a sensible semiclassical limit, as was shown in Section 3.

Thus, in summary, the alternatives are either to reject this proposal as a solution to the problem of time in quantum gravity, or to interpret it as an additional boundary condition to extract a sensible solution from the Wheeler-DeWitt equation. It is important in this respect to note that the Ehrenfest proposal does not respect the superposition principle, i.e., the sum of two “Ehrenfest solutions” is not an Ehrenfest solution again. Whether such an “Ehrenfest boundary condition” turns out to be successful is an issue which has not yet been explored.
A Appendix: Calculation of the second Ehrenfest condition

In this appendix we shall present the necessary steps to derive Eq. (17). For this purpose we first calculate ˜\(V\), cf. (7),

\[ ˜V = \frac{1}{\Psi^*\Psi} \left[ \frac{1}{2} Re(\Psi^*D^2\Psi) - Im(\Psi^*\partial_t\Psi) \right] \]

\[ = \frac{1}{\varrho^2} \left[ -\frac{1}{2} Re(\Psi^*D_0^2\Psi) + V\varrho^2 + \xi R - \varrho^2\dot{\theta} \right]. \] (50)

Inserting \(D_0^2\) and calculating the real part leads to

\[ V = ˜V - \xi \dot{\theta} - \frac{1}{2} G''00\dot{\theta}^2 + \frac{1}{2} \frac{\varrho}{\varrho^2} \partial_t (\sqrt{G'}G''00 \dot{\theta}) \].

The first step in our derivation is given by the following

**Proposition:**

\[ \frac{d}{dt} <p_i> = i <[\tilde{H}, p_i]> , \]

where \(p_i\) is given in (15), and \(\tilde{H} \equiv -\frac{1}{2} D^2 + \tilde{V}\).

**Proof:**

\[ <i[\tilde{H}, p_i]> \]

\[ = i \int D\eta \sqrt{G'} \left[ (\tilde{H}\Psi)^* p_i \Psi - \Psi^* p_i \tilde{H}\Psi \right] \]

\[ = \int D\eta \sqrt{G'} \left[ \frac{1}{2} (\partial_t \ln \sqrt{G'}) \Psi^* p_i \Psi + \partial_t \Psi^* p_i \Psi + \Psi^* p_i \partial_t \Psi + \frac{1}{2} \Psi^* p_i (\partial_t \ln \sqrt{G'}) \Psi \right] \]

\[ = \int D\eta \sqrt{G'} \left[ \partial_t \ln \sqrt{G'} \right] \Psi^* p_i \Psi + \partial_t \Psi^* p_i \Psi + \Psi^* p_i \partial_t \Psi - \Psi^* i \frac{\partial}{\partial t} \Psi \right] \]

\[ = \int D\eta \sqrt{G'} \left[ \partial_t \ln \sqrt{G'} \right] \Psi^* p_i \Psi + \partial_t \Psi^* p_i \Psi + \Psi^* p_i \partial_t \Psi + \Psi^* \frac{\partial p_i}{\partial t} \Psi \right]. \]

The last equality follows from

\[ \frac{\partial p_i}{\partial t} = -i \frac{1}{2} \partial_t (\partial_t + \frac{\partial_t \sqrt{G'}}{\sqrt{G'}}) = -i \frac{1}{2} \partial_t (2\partial_t + (\partial_t \ln \sqrt{G'})) = -i \frac{1}{2} (\partial_t \partial_t \ln \sqrt{G'}). \] (51)

Thus, our proposition has been proven.

Since the expectation value of the explicit time dependence of \(p_i\) must be real, it is clear from (51) that

\[ <\frac{\partial p_i}{\partial t}> = 0 = <\partial_t \partial_t \ln \sqrt{G'}> \] (52)
We first determine $H$ must hold. The second Ehrenfest equation is thus fulfilled if
\[ <i[H, p_i]> = <i[H, p_i]> . \]
To show this we first insert the expressions for $H$, $\tilde{H}$ and $\tilde{V}$ into this equation:
\[ <i[-\frac{1}{2}D_0^2 + \dot{\theta} - \frac{1}{2}G''\dot{\theta}^2 + \frac{1}{2} \frac{1}{\partial \sqrt{G'}} \partial_t (\sqrt{G'}G''\dot{\theta}), p_i]> = 0. \quad (53) \]

We first determine
\[
2i \left[ D_0^2, p_i \right] \Psi = \left[ \frac{1}{\sqrt{G'}} \partial_t \sqrt{G'} G'' \partial_t \partial_t + \frac{1}{\sqrt{G'}} \partial_t \right] \Psi
\]
\[
= \frac{2}{\sqrt{G'}} \partial_t \sqrt{G'} G'' \partial_t \partial_t \Psi - \frac{2}{\sqrt{G'}} \partial_t \partial_t \sqrt{G'} G'' \partial_t \Psi
\]
\[
+ \frac{1}{\sqrt{G'}} (\partial_t \ln \sqrt{G'}) \partial_t \sqrt{G'} G'' \partial_t \Psi
\]
\[
+ \frac{1}{\sqrt{G'}} \partial_t \sqrt{G'} G'' \partial_t (\partial_t \ln \sqrt{G'}) \Psi
\]
\[
+ \frac{1}{\sqrt{G'}} \partial_t \sqrt{G'} G'' \partial_t \partial_t \Psi
\]
\[
= 2G'' \partial_t^2 \partial_t \Psi + \frac{2}{\sqrt{G'}} (\partial_t \sqrt{G'} G'' \partial_t \partial_t \Psi
\]
\[
- \frac{2}{\sqrt{G'}} \partial_t \sqrt{G'} G'' \partial_t^2 \Psi - \frac{2}{\sqrt{G'}} \partial_t (\partial_t \sqrt{G'} G'' \partial_t \Psi
\]
\[
+ (\partial_t \ln \sqrt{G'}) G'' \partial_t^2 \Psi + \frac{1}{\sqrt{G'}} (\partial_t \ln \sqrt{G'}) (\partial_t \sqrt{G'} G'' \partial_t \Psi
\]
\[
+ G'' (\partial_t \partial_t \ln \sqrt{G'}) \partial_t \Psi + \frac{1}{\sqrt{G'}} (\partial_t \sqrt{G'} G'' (\partial_t \partial_t \ln \sqrt{G'}) \Psi
\]
\[
+ G'' (\partial_t \ln \sqrt{G'}) \partial_t^2 \Psi + \frac{1}{\sqrt{G'}} (\partial_t \sqrt{G'} G'' (\partial_t \ln \sqrt{G'}) \partial_t \Psi
\]
\[
= -2(\partial_t G'' \partial_t^2 \Psi - \frac{2}{\sqrt{G'}} (\partial_t \sqrt{G'} (\partial_t G'') \partial_t \Psi
\]
\[
+ \frac{1}{\sqrt{G'}} (\partial_t \sqrt{G'} G'' (\partial_t \partial_t \ln \sqrt{G'}) \Psi
\]
\[
= -2(\partial_t G'' [\frac{\dot{\theta}}{\dot{\theta}} + 2i \frac{\dot{\theta}}{\dot{\theta}} + i \dot{\theta} - \dot{\theta}^2] \Psi
\]
\[
- \frac{2}{\sqrt{G'}} (\partial_t \sqrt{G'} (\partial_t G'') \frac{\dot{\theta}}{\dot{\theta}} + i \dot{\theta}] \Psi
\]
\[
+ \frac{1}{\sqrt{G'}} (\partial_t \sqrt{G'} G'' (\partial_t \partial_t \ln \sqrt{G'}) \Psi .
\]

Then we calculate
\[
2i[V + \xi \mathcal{R} - \tilde{V}, p_i] \Psi = -2(\partial_t \dot{\theta}) \Psi + (\partial_t G'' \dot{\theta}^2) \Psi - (\partial_t \frac{(\partial_t \sqrt{G'} G'' \dot{\theta})}{\dot{\theta}} \Psi.
\]
Inserting these results into (53) and performing a decomposition into real and imaginary part yields

\[
0 = \int D\eta \sqrt{G'} g^2 \left[ -\frac{(\sqrt{G'} G') \hat{\dot{\theta}}}{\sqrt{G'}} + \frac{\hat{\dot{\theta}}}{2\sqrt{G'}} \left( \frac{\hat{\dot{\theta}}}{g} \right) + \hat{\dot{\theta}} \frac{G'''}{G'} \right],
\]

\[
0 = \int D\eta \partial_t \left( \sqrt{G'} g^2 (\partial_t G') \hat{\dot{\theta}} \right).
\]

We now make use in these equations of the relations \( \theta = \theta(t), G'' = \kappa(\eta) \) and \( \partial_t (\sqrt{G'} g^2) = 0 \), which follow from the first Ehrenfest equation. One immediately recognises that the imaginary part vanishes. The vanishing of the real part can be explicitly checked as follows:

\[
-\frac{(\sqrt{G'} G') \hat{\dot{\theta}}}{\sqrt{G'}} + \frac{\hat{\dot{\theta}}}{2\sqrt{G'}} \left( \frac{\hat{\dot{\theta}}}{g} \right) + \hat{\dot{\theta}} \frac{G'''}{G'} = -g (\partial_t G''') \frac{\hat{\dot{\theta}}}{g^2} + \frac{1}{2} g^2 (\partial_t G''') \frac{\hat{\dot{\theta}}}{g^2} - \partial_t (g \partial_t G''') \frac{\hat{\dot{\theta}}}{g^2}
\]

\[
= -\partial_t (g (\partial_t G''')) \frac{\hat{\dot{\theta}}}{g^2} + (\partial_t G''') \frac{\hat{\dot{\theta}}}{g^2} - \partial_t (G''' \partial_t \frac{\hat{\dot{\theta}}}{g^2})
\]

\[
= \partial_t (G''' \partial_t \frac{\hat{\dot{\theta}}}{g^2}) - G''' \partial_t (\frac{\hat{\dot{\theta}}}{g^2}) - \partial_t (G''' \partial_t \frac{\hat{\dot{\theta}}}{g^2}) + 2 \hat{\dot{\theta}} \frac{G''}{g^2}
\]

\[
= 0.
\]

Thus, (53) holds, and the only condition from the second Ehrenfest equation is (52). We note that the factor ordering term \( \xi \mathcal{R} \) did not play any role in the derivation of (52). Furthermore, the exact form of \( \kappa(\eta) \) (which we have fixed to be \( \kappa \equiv 1 \) from semiclassical considerations) does not enter these calculations.

That the imaginary part of (53) must vanish follows of course immediately from the fact that \( \hat{H} \) and \( p_i \) are both hermitean and that thus \( < i \hat{H} - \hat{H}, p_i > \) must be real.
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