ON SOME NONLINEAR PROBLEM
FOR THE THERMOPLATE EQUATIONS

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Abstract. In this paper, we prove the local and global well-posedness of some nonlinear thermoelastic plate equations with Dirichlet boundary conditions. The main tool for proving the local well-posedness is the maximal L_p-L_q regularity theorem for the linearized equations, and the main tool for proving the global well-posedness is the exponential stability of C_0 analytic semigroup associated with linear thermoelastic plate equations with Dirichlet boundary conditions.

1. Introduction.

1.1. Previous research. In this paper, we consider a nonlinear thermoelastic plate equation given by

\[ \begin{align*}
    u_{tt} + \Delta^2 u + \Delta \theta + b \Delta (\Delta u)^3 &= f_1 \quad \text{in } \Omega \times (0, T), \\
    \theta_t - \Delta \theta - \Delta u_t &= f_2 \quad \text{in } \Omega \times (0, T),
\end{align*} \]

with positive constants \( b \) and \( T \), subject to the initial condition

\[ \begin{align*}
    u|_{t=0} &= u_0 \quad \text{in } \Omega, \\
    u_t|_{t=0} &= u_1 \quad \text{in } \Omega, \\
    \theta|_{t=0} &= \theta_0 \quad \text{in } \Omega,
\end{align*} \]

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and the Dirichlet boundary condition
\[ u|_\Gamma = \theta|_\Gamma = \partial_\nu u|_\Gamma = 0. \] (3)

In equations (1), \( u = u(x,t) \) denotes a vertical displacement of the plate and \( \theta = \theta(x,t) \) describes the temperature relative to a constant reference temperature at time \( t \in (0, T) \) and point \( x = (x_1, \ldots, x_n) \in \Omega \) (see e.g. [14]). In (3), \( \nu \) is the unit outer normal to \( \Gamma \). Here, \( \Omega \) is a domain in \( \mathbb{R}^N (N \geq 2) \) whose boundary \( \Gamma \) is a \( C^4 \)-hypersurface and \( \Delta \) stands for the Laplace operator in \( \Omega \). The nonlinear term appearing in (1) represents the nature of magnetoelastic material due to a nonlinear dependence between the tension of deformation and stress.

Numerous studies have attempted to prove the existence and uniqueness of solutions of the thermoelastic plate equations in the area of both linear and nonlinear. For linearized problem \( (b = 0) \), Lasiecka and Triggiani in [16] provided in-depth analysis on physically relevant boundary conditions. For a bounded domain \( \Omega \), exponential stability of the associated semigroup in \( L^2 \) have been studied by Kim [12], Munos Rivera and Racke [25], Liu and Zheng [24], Lasiecka and Triaggini [16, 17, 18], and Shibata [28]. The study on general von Karman evolution equations was carried out by Chuesov and Lacieska [2]. Moreover, the analyticity of the associated semigroup is the important aspect of the equation (1). Although the first equation in (1) is a simply dispersive equation (the product of two Schrödinger equations) with respect to \( u \), the heat equations in \( \theta \) has vigorous effect to have the analyticity of the whole system. This fact was addressed by Liu and Renardy [23], Liu and Liu [21], Liu and Young [22] in the \( L^2 \)-setting.

On the other hand, the \( L^p \)-approach is appropriate to treat the equations with low regularity of the initial data. Therefore, it is essential to analyze the problem (1) to (3) in the \( L^p \)-setting. Concerning \( L^p \) framework, Denk and Racke [4] found the analyticity of the generated semigroup of the linearized problem in the whole-space. After that, Naito and Shibata [27] and Naito [26] studied the linearized equation (1) - (3) in the half-space case and proved the generation of \( C_0 \)-analytic semigroup. Additionally, Denk, Racke and Shibata [5] proved some decay estimates in bounded domains and exterior domains.

Recently, Lasiecka, Maad and Amol [20] and Lasiecka and Wilke [19] provided a rather complete analysis of the equation (1) with the boundary condition \( u = \Delta u = 0 \). In [19], the maximal regularity of the system was proved by using an abstract operator-theoretic idea, since the operator \( \Delta^2 \) appearing in the first equation in (1) can be defined as a square of Dirichlet-Laplace operator due to the boundary condition: \( u = \Delta u = 0 \). In the case of general domain, Denk and Shibata [8, 9] proved the maximal \( L^p-L_q \) regularity of the linearized system with free boundary conditions:
\[
\begin{align*}
\Delta u - (1 - \beta)\Delta' u + \theta &= g_1 & &\text{on } \Gamma \times (0, T), \\
\partial_\nu (\Delta u - (1 - \beta)\Delta' u + \theta) &= g_2 & &\text{on } \Gamma \times (0, T), \\
\partial_\nu \theta &= g_2 & &\text{on } \Gamma \times (0, T),
\end{align*}
\]
where \( \Delta' \) is the Laplace-Beltrami operator on \( \Gamma \). In that paper, some localization technique was used to handle the free boundary conditions because an abstract approach similar to that in [19] was seemingly not available. Regarding to the free boundary conditions, analyticity properties of the associated semigroup in a Hilbert setting has been shown by Lasiecka and Triaggini in [15] and [17]. More recently, Inna [11] proved the maximal \( L^p-L_q \) regularity of the linearized system (1) to (3). The generation of \( C_0 \) analytic semigroup was also shown in [11].
The object of this paper is to prove the local and global in time solvability of the problem (1) to (3). Let $C^3(\mathbb{R})$ be a set of $C^2$ functions whose second derivatives are Lipschitz continuous, and let $\phi \in C^3(\mathbb{R})$ with

$$
\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi''(0) = 0.
$$

We consider more general version of (1) by replacing the cubic nonlinearity by $b\Delta \phi(\Delta u)$ with a positive constant $b$. Namely, we consider

$$
\begin{align*}
&u_{tt} + \Delta^2 u + \Delta \theta + b\Delta(\phi(\Delta u)) = f_1, \quad \text{in } \Omega \times (0, T), \\
&\theta_t - \Delta \theta - \Delta u_t = f_2, \quad \text{in } \Omega \times (0, T),
\end{align*}
$$

subject to the initial and boundary conditions (2) and (3) respectively.

To solve the problem (4), (2) and (3) without smallness assumption, we use the following linearized principle. Setting $v = u_t$ and rewriting $\Delta \phi(\Delta u)$ as

$$
\Delta \phi(\Delta u) = \phi''(\Delta u)|\nabla \Delta u|^2 + \phi'(\Delta u) \Delta^2 u
$$

we transform problem (4), (2) and (3) to the following first order system of equations:

$$
U_t - A_\alpha(D)U = F(U) \quad \text{in } \Omega \times (0, T), \quad B(D)U|_{\Gamma} = 0 \quad U|_{t=0} = U_0, \quad (5)
$$

with

$$
A_\alpha(D) = \begin{pmatrix}
0 & 1 & 0 \\
-\alpha \Delta^2 & 0 & -\Delta \\
0 & \Delta & \Delta
\end{pmatrix}, \quad B(D) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \partial_\nu & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

$$
F(U) = \begin{pmatrix}
f_1 - b(\phi'(\Delta u_0) - \phi'(\Delta u)|\nabla \Delta u|^2) \\
0 \\
f_2
\end{pmatrix}
$$

$U = (u, v, \theta)^T$, $U_0 = (u_0, v_0, \theta_0)^T$, $\alpha = \alpha(x) = 1 + b\phi'(\Delta u_0(x))$, where $f_1, f_2 \in L_p((0, T), L_q(\Omega))$ are given functions. Here and in the following, $M^T$ stands for the transposed of $M$.

To prove the global in time unique existence theorem, we assume that initial data and right members are small, and so the linearization principle is the following:

$$
U_t - A(D)U = G(U) \quad \text{in } \Omega \times \mathbb{R}_+, \quad B(D)U|_{\Gamma} = 0 \quad U|_{t=0} = U_0, \quad (6)
$$

with

$$
A(D) = \begin{pmatrix}
0 & 1 & 0 \\
-\Delta^2 & 0 & -\Delta \\
0 & \Delta & \Delta
\end{pmatrix}, \quad G(U) = \begin{pmatrix}
f_1 - b(\phi'(\Delta u)|\nabla \Delta u |^2 + \phi''(\Delta u)|\nabla \Delta u|^2) \\
0 \\
f_2
\end{pmatrix}
$$

$U = (u, v, \theta)^T$, and $U_0 = (u_0, v_0, \theta_0)^T$. Here and in the following, $\mathbb{R}_+ = (0, \infty)$.

1.2. Notations and main results. To state our results, we first introduce several symbols used throughout the paper. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural numbers, real numbers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any scalar function $f$ and vector-valued function $\mathbf{g}$

$$
\nabla f = (\partial_1 f, \ldots, \partial_N f), \quad \nabla^\ell f = (\partial_\alpha^\ell f : |\alpha| = \ell),
$$

$$
\nabla \mathbf{g} = (\partial_1 g_1, \ldots, \partial_N g_N), \quad \nabla^\ell \mathbf{g} = (\nabla^\ell g_1, \ldots, \nabla^\ell g_N),
$$

where $\partial_\alpha^\ell = \partial_1^{\alpha_1} \ldots \partial_N^{\alpha_N}$, $\partial_\alpha = \partial / \partial x_\alpha$, and $|\alpha| = \alpha_1 + \ldots + \alpha_N$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$. For any domain $G$, we denote the Lebesgue space, Sobolev space
and Besov space by $L^q_q(G)$, $H^m_q(G)$ and $B^s_{q,p}(G)$, respectively, and their norms are written by $\| \cdot \|_{L^q_q(G)}$, $\| \cdot \|_{H^m_q(G)}$ and $\| \cdot \|_{B^s_{q,p}(G)}$, respectively. Here, $H^m_q(G)$ is $L^q_q(G)$. In particular, we use the following symbols:

\[ H^2_{q,D}(G) = \{ f \in H^2_q(G) \mid f|_{\partial G} = \nu_{G} \cdot \nabla f|_{\partial G} = 0 \}, \]

\[ H^2_{q,0}(G) = \{ f \in H^2_q(G) \mid f|_{\partial G} = 0 \}, \]

\[ H^0_q(G) = \{ (f, g, h)^\top \mid f \in H^2_{q,D}(G), \ g, h \in L^q_q(G) \}, \]

\[ H^1_q(G) = \{ (f, g, h)^\top \in H^0_q(G) \mid f \in H^2_q(G), \ g, h \in H^1_{q,0}(G) \}, \]

\[ H^2_q(G) = \{ (f, g, h)^\top \in H^1_q(G) \mid f \in H^2_q(G), \ g, h \in H^2_{q,0}(G) \}, \]

where $\partial G$ is the boundary of $G$ and $\nu_{G}$ the unit outer normal to $\partial G$. Let

\[ \|(f, g, h)^\top\|_{\mathbf{H}^2_{q,G}(G)} = \|f\|_{H^2_{q+2}(G)} + \|(g, h)\|_{H^1_{q,G}(G)} \]

for $F = (f, g, h)^\top \in H^0_q(G)$ ($j = 0, 1, 2$). Let $X$ and $Y$ be Banach spaces with the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, and let $\mathcal{L}(X, Y)$ be the space of all bounded linear operators from $X$ to $Y$ and $\mathcal{L}(X, X)$ is written simply by $\mathcal{L}(X)$. Let

\[ X \times Y^2 = \{ (f, g) \mid f \in X, \ g \in Y \}, \quad \|(f, g)^\top\|_{X \times Y^2} = \|f\|_X + \|(g, h)\|_Y. \]

For scalar functions $f$ and $g$, we set $(f, g)_\Omega = \int_\Omega f(x)g(x)\, dx$, where $g(x)$ denotes the complex conjugate of $g(x)$. For vector valued functions $U = (u_1, u_2, u_3)^\top$ and $V = (v_1, v_2, v_3)^\top$, we set $(U, V)_\Omega = \sum_{j=1}^3 (u_j, v_j)_\Omega$. Let $T \in (0, \infty)$ or $T = \infty$. For time interval $J = (0, T)$, we denote the $X$-valued Lebesgue space and the $X$-valued Sobolev space by $L^p(J, X)$ and $H^m_p(J, X)$ ($m \in \mathbb{N}$) and their norms by $\| \cdot \|_{L^p(J, X)}$ and $\| \cdot \|_{H^m_p(J, X)}$, respectively. For any domain $V$ in $\mathbb{C}$, we denote the set of all $X$-valued functions $f = f(\lambda)$, defined for $\lambda = \eta + i\tau \in V$, that are continuously differentiable with respect to $\tau$ when $\lambda \in V$ by $\mathrm{Hol}(\lambda, X)$, and set $\Sigma_\eta = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid \arg \lambda < \eta \}$ for $\eta \in (0, \pi)$ and let $\Sigma_{\eta, \lambda_0} = \{ \lambda \in \Sigma_\eta \mid |\lambda| \geq \lambda_0 \}$ for $\lambda_0 \geq 0$. For any complex number $\lambda$, $\tau$ denotes the imaginary part of $\lambda$, that is $\lambda = \eta + i\tau \in \mathbb{C}$. The letter $C$ denotes generic constants. $C_{a,b,\cdots}$ denotes a constant depending on the quantities $a, b, \cdots$. The values of $C$ and $C_{a,b,\cdots}$ may change from line to line.

Secondly, we make a definition.

**Definition 1.1.** A domain $\Omega$ is called a uniformly $C^4$-domain if there exist positive constants $K, L_1, L_2$ such that for any $x_0 \in \Gamma$ there exists a coordinate number $j$ and $C^4$-function $h(x')$ defined on $B_{L_1}(x'_0)$ such that $\|h\|_{H^4_4(B_{L_1}(x'_0))} \leq K$ and

\[ \Omega \cap B_{L_2}(x_0) = \{ x \in \mathbb{R}^N \mid x_j > h(x') (x' \in B_{L_1}(x'_0)) \cap B_{L_2}(x_0), \]

\[ \Gamma \cap B_{L_2}(x_0) = \{ x \in \mathbb{R}^N \mid x_j = h(x') (x' \in B_{L_1}(x'_0)) \cap B_{L_2}(x_0), \]

where $x' = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_N)$ for $x = (x_1, \cdots, x_N)$, $B_a(x'_0) = \{ x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < a \}$, and $B_b(x_0) = \{ x \in \mathbb{R}^N \mid |x - x_0| < b \}$.

We now state the main results of this paper.

**Theorem 1.2.** Let $N < q < \infty$, $2 < p < \infty$, $T_1 > 0$ and $R > 0$. Assume that $\Omega$ is a uniform $C^4$-domain in $\mathbb{R}^N$. If the initial data $U_0 = (u_0, v_0, \theta_0)^\top$ and functions $f_1, f_2 \in L_p((0, T_1), L_q(\Omega))$ of the right side of (4) satisfy

\[ \|U_0\|_{H^2_4(\Omega) \times B^1_{4,p}(\Omega)^2} + \|(f_1, f_2)\|_{L_p((0, T_1), L_q(\Omega)^2)} \leq R \]

and satisfy compatibility conditions:

\[ u_0|_\Gamma = \theta_0|_\Gamma = \partial_\nu u_0|_\Gamma = 0 \]
and a positive condition:

$$A_1 \leq 1 + b\phi'(\Delta u_0) \leq A_2$$  \hspace{1cm} (9)$$

for some positive numbers $A_1$ and $A_2$, then there exists a time $T \in (0, T_1)$ depending on $R$, $b$, $A_1$ and $A_2$ such that problem (5) admits a unique solution $U$ with

$$U \in L_p((0, T), H^2_q(\Omega)) \cap H^1_p((0, T), H^1_q(\Omega))$$

possessing the estimate:

$$\|U\|_{L_p((0, T), H^2_q(\Omega))} + \|\partial_t U\|_{L_p((0, T), H^1_q(\Omega))} \leq CR$$

with some constants $C$ depending on $R$, $A_1$ and $A_2$.

**Theorem 1.3.** Let $N < q < \infty$, $2 < p < \infty$. Assume that $\Omega$ is a bounded domain whose boundary $\Gamma$ is a compact $C^4$ hypersurface. Then, there exist small positive numbers $\epsilon$ and $\eta$ such that if the initial data $U_0 = (u_0, v_0, \theta_0)^T$ and functions $f_1$ and $f_2 \in L_p(\mathbb{R}_+, L_q(\Omega))$ of the right side of (4) satisfy

$$\|U_0\|_{B^{2(1-1/p)}_q(\Omega) \times B^{2(1-1/p)}_p(\Omega)^2} + \|f_1, f_2\|_{L_p(\mathbb{R}_+, L_q(\Omega))} \leq \epsilon$$  \hspace{1cm} (10)$$

and satisfy compatibility conditions

$$u_0|\Gamma = \theta_0|\Gamma = \partial_\nu u_0|\Gamma = 0,$$

then the problem (6) with $T = \infty$ admits a unique solution $U$ with

$$U \in L_p(\mathbb{R}_+, H^2_q(\Omega)) \cap H^1_p(\mathbb{R}_+, H^1_q(\Omega))$$

possessing the estimate:

$$\|e^{\eta t} U\|_{L_p(\mathbb{R}_+, H^2_q(\Omega))} + \|e^{\eta t} \partial_t U\|_{L_p(\mathbb{R}_+, H^1_q(\Omega))} \leq CE. $$

To prove Theorem 1.2, the main step is to prove the $L_p-L_q$ maximal regularity for the following linearized problem:

$$U_t - A_\alpha(D)U = F \quad \text{in} \quad \Omega \times (0, T), \quad B(D)U|\Gamma = 0 \quad U|_{t=0} = U_0,$$  \hspace{1cm} (11)$$

with

$$A_\alpha(D) = \begin{pmatrix} 0 & 1 & 0 \\ -\alpha \Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{pmatrix},$$

where $\alpha$ is a uniformly continuous function with respect to $x \in \bar{\Omega}$ and satisfies

$$A_1 \leq \alpha(x) \leq A_2$$  \hspace{1cm} (12)$$

for positive constants $A_1$ and $A_2$. Namely, we prove

**Theorem 1.4.** Let $1 < p, q < \infty$ and $T > 0$. Assume that $\Omega$ is a uniform $C^4$-domain in $\mathbb{R}^N$. Let

$$D_{q,p}(\Omega) = (H^q_q(\Omega), H^2_q(\Omega))_{1-1/p, p},$$  \hspace{1cm} (13)$$

where $(\cdot, \cdot)_{q,p}$ denotes the real interpolation functor. Then, there exists a positive number $\lambda_1$ such that for any initial data $U_0 = (u_0, v_0, \theta_0)^T \in D_{q,p}(\Omega)$ and $F = (f, g, h)^T \in L_p((0, T), H^q_q(\Omega))$, Eq. (11) admits a unique solution $U$ with

$$U \in L_p((0, T), H^2_q(\Omega)) \cap H^1_p((0, T), H^1_q(\Omega))$$

possessing the estimate:

$$\|U\|_{L_p((0, T), H^2_q(\Omega))} + \|\partial_t U\|_{L_p((0, T), H^1_q(\Omega))} \leq CE^{\lambda_1 T}(\|U_0\|_{B^{2(1-1/p)}_q(\Omega) \times B^{2(1-1/p)}_p(\Omega)^2} + \|F\|_{L_p((0, T), H^q_q(\Omega) \times L_q(\Omega)^2)})$$. 


Remark 1. Note that $D_{q,p}(\Omega) \subset B_{q,p}^{2+2(1-1/p)}(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^2$. Moreover, if $(u_0, v_0, \theta_0) \in D_{q,p}(\Omega)$, we have

\[
\begin{align*}
    u_0 &= \partial_\nu u_0 = v_0 = \theta_0 = 0 \quad \text{on } \Gamma, \quad \text{provided } 1 > 2/p + 1/q, \\
    u_0 &= \partial_\nu u_0 = 0 \quad \text{on } \Gamma, \quad \text{provided } 3 > 2/p + 1/q > 1, \\
    u_0 &= 0 \quad \text{on } \Gamma, \quad \text{provided } 4 > 2/p + 1/q > 3.
\end{align*}
\]

On the other hand, $D_{q,p}(\Omega) = B_{q,p}^{2+2(1-1/p)}(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^2$ provided $2/p + 1/q > 4$.

To prove Theorem 1.3, the main step is to prove the exponential stability of solutions of the following linearized equations:

\[ U_t - A(D)U = G \quad \text{in } \Omega \times \mathbb{R}_+, \quad B(D)U|_{\Gamma} = 0, \quad U|_{t=0} = U_0. \quad (14) \]

**Theorem 1.5.** Let $1 < p, q < \infty$ Assume that $\Omega$ is a bounded domain whose boundary $\Gamma$ is a compact $C^4$ hypersurface. Let $D_{q,p}(\Omega)$ be the same set as in Theorem 1.4. Then, there exists a positive number $\eta$ such that for any initial data $U_0 = (u_0, v_0, \theta_0)^T \in D_{q,p}(\Omega)$ and right-hand side $G = (f, g, h)^T$ with $e^{\eta t}G \in L_p(\mathbb{R}_+, H^4_q(\Omega))$, Eq. (14) admits a unique solution $U$ with $e^{\eta t}U \in L_p((\mathbb{R}_+, H^2_q(\Omega)) \cap H^1_p((0, T), H^3_q(\Omega)))$ possessing the estimate:

\[
\begin{align*}
    \|e^{\eta t}U\|_{L_p(\mathbb{R}_+, H^2_q(\Omega))} + \|e^{\eta t}\partial_t U\|_{L_p(\mathbb{R}_+, H^4_q(\Omega))} \\
    &\leq C(\|U_0\|_{B_{q,p}^{2+2(1-1/p)}(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^2} + \|e^{\eta t}G\|_{L_p([0,T) \times L_q(\Omega))}).
\end{align*}
\]

Here and in the following, we write

\[
\|e^{\eta t}f\|_{L_p(\mathbb{R}_+, X)} = \left(\int_0^\infty (e^{\eta t}\|f(\cdot, t)\|X)^p dt\right)^{1/p}.
\]

2. **$\mathcal{R}$-bounded solution operators.** To prove the maximal $L_p-L_q$-regularity of problem (11), we show the existence of an $\mathcal{R}$-bounded solution operator of the associated resolvent problem:

\[
\lambda U - A_\alpha(D)U = F \quad \text{in } \Omega, \quad B(D)U|_{\Gamma} = 0, \quad U|_{t=0} = U_0 \quad (15)
\]

with $\alpha$ satisfying (12). Let $\vartheta_0 (\pi/2 < \vartheta_0 < \pi)$ be a constant given in (20) below.

In this section, we shall prove the following theorem.

**Theorem 2.1.** Let $1 < q < \infty$. Let $\alpha(x)$ be a real valued uniformly continuous function defined on $\Omega$ satisfying (12). Then, there exist a positive number $\lambda_0$ and an operator family $S_\beta(\lambda)$ with $S_\beta(\lambda) \in \text{Hol}(\Sigma_{\vartheta_0, \lambda_0}, \mathcal{L}(H^0_{\rho}(\Omega), H^2_q(\Omega)))$ such that for any $\lambda \in \Sigma_{\vartheta_0, \lambda_0}$ and $F \in H^0_{\rho}(\Omega)$, $U = S_\beta(\lambda)F$ is a unique solution of Eq. (15), and there hold the estimates:

\[
\mathcal{R}_{\mathcal{L}(H^0_{\rho}(\Omega), H^2_q(\Omega)), \Sigma_{\vartheta_0, \lambda_0}^{s-k}}{(\lambda^{s-k/2}S_\beta(\lambda) | \lambda \in \Sigma_{\vartheta_0, \lambda_0})} \leq \gamma
\]

for $s = 0, 1$ and $k = 0, 1, 2$. Here, $\gamma$ is a constant independent of $\alpha$.

Our analysis is based on the results due to Inna [11] (cf. Naito and Shibata [27] and Denk and Shibata [8]). Before proving Theorem 2.1, we introduce the definition and some fundamental properties of $\mathcal{R}$-bounded operators and Bourgain’s results concerning Fourier multiplier theorems with scalar multiplier.
Definition 2.2. A family $\mathcal{T} \subset \mathcal{L}(X,Y)$ of operators is called $\mathcal{R}$-bounded if there exists a constant $C$ such that for all $m \in \mathbb{N}$, $(T_k)_{k=1,\ldots,m} \subset \mathcal{T}$, and $(x_k)_{k=1,\ldots,m} \subset X$, we have
\[
\left\| \sum_{k=1}^{m} r_k T_k x_k \right\|_{L_p([0,1],Y)} \leq C \left\| \sum_{k=1}^{m} r_k x_k \right\|_{L_p([0,1],Y)}.
\]
Here the Rademacher function $r_k, k \in \mathbb{N}$, are given by $r_k : [0,1] \to \{-1,1\}, t \mapsto \text{sign}(2^k \pi t))$. The smallest such $C$ is called the $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$ which is written by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$ in what follows.

Proposition 1. Let $1 < p, q < \infty$.
(a) Let $X$ and $Y$ be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$. Then, $\mathcal{T} + \mathcal{S} = \{ T + S \mid T \in \mathcal{T}, S \in \mathcal{S} \}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Y)$ and
\[
\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).
\]
(b) Let $X$, $Y$ and $Z$ be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then, $\mathcal{ST} = \{ ST \mid T \in \mathcal{T}, S \in \mathcal{S} \}$ also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Z)$ and
\[
\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{ST}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).
\]
(c) Let $D$ be a domain in $\mathbb{R}^N$. Let $m = m(\lambda)$ be a bounded function defined on a subset $\Lambda \subset \mathbb{C}$ and let $M_m(\lambda)$ be a map defined by $M_m(\lambda) f = m(\lambda) f$ for any $f \in L_q(D)$. Then, $\mathcal{R}_{\mathcal{L}(L_q(D))}(\{ M_m(\lambda) \mid \lambda \in \Lambda \}) \leq C_{N,q,D} \| m \|_{L_\infty(\Lambda)}$.
(d) Let $n = n(\lambda)$ be a $C^1$-function defined on $\mathbb{R} \setminus \{0\}$ that satisfies the conditions $|n(\lambda)| \leq \gamma$ and $|\tau n'(\tau)| \leq \gamma$ with some constant $\gamma > 0$ for any $\tau \in \mathbb{R} \setminus \{0\}$. Let $T_n$ be the scalar-valued Fourier multiplier defined by $T_n f = F^{-1}(n \hat{f})$ for any $f$ with $F[f] \in \mathcal{D}(\mathbb{R}, L_q(D))$. Then, $T_n$ is extended to a bounded linear operator from $L_p(\mathbb{R}, L_q(D))$ into itself. Moreover, denoting this extension also by $T_n$, we have
\[
\| T_n \|_{\mathcal{L}(L_p(\mathbb{R}, L_q(D)))} \leq C_{p,q,D} \gamma.
\]
Here, $\mathcal{D}(\mathbb{R}, L_q(D))$ denotes the set of all $L_q(D)$-valued $C^\infty$-functions on $\mathbb{R}$ with compact support.

2.1. Analysis in the whole space. Let $A_1$ and $A_2$ be two positive numbers given in (12). In this section, we assume that $\alpha$ is a positive number such that $A_1 \leq \alpha \leq A_2$. We consider the resolvent problem in the whole space:
\[
\lambda U - A_\alpha(D) U = F \quad \text{in} \; \mathbb{R}^N, \tag{16}
\]
with
\[
U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \quad A_\alpha(D) = \begin{pmatrix} 0 & 1 & 0 \\ -\alpha \Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f_1 \\ f_2 \end{pmatrix}.
\]
Let
\[
\hat{A}_\alpha(\xi) = \begin{pmatrix} 0 & 1 & 0 \\ -\alpha|\xi|^4 & 0 & |\xi|^2 \\ 0 & -|\xi|^2 & -|\xi|^2 \end{pmatrix},
\]
and then by using the Fourier transform, we transform Eq. (16) to the following linear equation:
\[
(\lambda I - \hat{A}_\alpha(\xi)) U = \hat{F} \quad \text{in} \; \mathbb{R}^N, \tag{17}
\]
so that
\[
\hat{U}(\xi) = (\lambda I - \hat{A}_\alpha(\xi))^{-1} \hat{F}.
\]
Here, the Fourier transform of a function $g$ is defined by

$$\hat{g}(\xi) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} g(x) \, dx.$$ 

The determinant of $\lambda I - \hat{A}_\alpha(\xi)$ is given by

$$\det(\lambda I - \hat{A}_\alpha(\xi)) = (\lambda + \gamma_1|\xi|^2)(\lambda + \gamma_2|\xi|^2)(\lambda + \gamma_3|\xi|^2),$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are numbers such that

$$f(s) = s^3 + s^2 + (\alpha + 1)s + \alpha = (s + \gamma_1)(s + \gamma_2)(s + \gamma_3).$$  \hspace{1cm} (18)

Let $\gamma_1$ be a real number, and then $\gamma_1 \in (0, 1)$. Moreover, since $f'(s) > 0$, $\gamma_2$ and $\gamma_3$ are complex numbers such that $\gamma_3 = \frac{\pi}{2}$ and $\Re \gamma_2 = \Re \gamma_3$. Since $\gamma_1$ is a continuous function with respect to $\alpha$,

$$\max_{A_1 \leq \alpha \leq A_2} |\Im \gamma_i| = \gamma_{max} < \pi/2$$  \hspace{1cm} (19)

because $\Re \gamma_2 = \Re \gamma_3 \in (0, 1/2)$. Let $\vartheta_0$ be a number such that $\pi/2 < \vartheta_0 < \pi$ and $\vartheta_0 + \gamma_{max} = \vartheta_1 < \pi$. Then, we have

$$\lambda \gamma_i^{-1} \in \Sigma_{\vartheta_i} \quad \text{provided} \quad \lambda \in \Sigma_{\vartheta_0}.$$  \hspace{1cm} (20)

Employing the same argumentation as in Naito and Shibata [27] in $\mathbb{R}^N$ (cf. also Denk and Shibata [8], Inna [11]), we have the following theorem.

**Theorem 2.3.** Let $\alpha$ be a positive number in $[A_1, A_2]$. Let $1 < q < \infty$ and $0 < \lambda_0 < \infty$. Then, there exists an operator family $A_{\alpha}(\lambda)$ with

$$A_{\alpha}(\lambda) \in \text{Hol}(\Sigma_{\vartheta_0, \lambda_0}, \mathcal{L}(H_q^0(\mathbb{R}^N), H_q^2(\mathbb{R}^N)))$$

such that for any $\lambda \in \Sigma_{\vartheta_0, \lambda_0}$ and $F \in H_q^0(\mathbb{R}^N)$, $U = A_{\alpha}(\lambda)F$ is a unique solution of Eq. (16), and

$$\mathcal{R}_{L(H^0_q(\mathbb{R}^N), H^{2-k}_q(\mathbb{R}^N))}(\{((\tau \partial_\tau)^s(\lambda^{k/2} \partial_\tau^k A_{\alpha}(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0, \lambda_0}\}) \leq \gamma_{\alpha, \lambda_0}$$

for $s = 0, 1$ and $(\kappa, k) \in \mathbb{N}_0^{N+1}$ with $|\kappa| + k = 2$ and $k = 0, 1, 2$. Here, $\gamma_{\alpha, \lambda_0}$ is a constant depending on $\alpha$ and $\lambda_0$. Moreover, we have set

$$H_q^j(D) = \{(f, g, h)^T \mid f \in H_q^{2+j}(D), g, h \in H_q^j(D)\} \quad (j = 0, 1, 2),$$

where $D$ is a domain in $\mathbb{R}^N$.

We next show that there exists a $\gamma_0$ depending on $\lambda_0$ such that

$$\max_{A_1 \leq \alpha \leq A_2} \gamma_{\alpha, \lambda_0} = \gamma_0.$$  \hspace{1cm} (21)

Let $\alpha$ and $\beta$ be two positive numbers in $[A_1, A_2]$, and we consider the equation:

$$\lambda U - A_{\beta} U = F \quad \text{in} \quad \mathbb{R}^N.$$  \hspace{1cm} (22)

Then, we have

$$\lambda U - A_{\alpha} U = (\lambda U - A_{\beta} U) + (A_{\beta} U - A_{\alpha} U) = F + (A_{\beta} - A_{\alpha}) U \quad \text{in} \quad \mathbb{R}^N.$$

Note that

$$A_{\beta} - A_{\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ - (\beta - \alpha) \Delta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Let $U = A_{\alpha}(\lambda)F$, and then

$$\lambda U - A_{\alpha} U = F + (A_{\beta} - A_{\alpha}) A_{\alpha}(\lambda) F \quad \text{in} \quad \mathbb{R}^N.$$
From the estimate:

\[
\|(A_\beta - A_\alpha)A_\alpha(\lambda)F\|_{H^s_0(\mathbb{R}^N)} \leq |\beta - \alpha|\|A_\alpha(\lambda)F\|_{H^s_0(\mathbb{R}^N)} \\
\leq \gamma_{0}\alpha|\beta - \alpha|\|F\|_{H^s_0(\mathbb{R}^N)},
\]

it follows that

\[
\mathcal{R}_{\mathcal{L}(H^s_0(\mathbb{R}^N))}(\{((\tau\partial_\tau)^s((A_\beta - A_\alpha)A_\alpha(\lambda)) \mid \lambda \in \Sigma_{\beta_0,\lambda_0}\}) \leq \gamma_{\alpha,\lambda_0}|\beta - \alpha|
\]

for \(s = 0, 1\). Thus, choosing \(\beta\) in such a way that \(|\beta - \alpha|\gamma_{\alpha,\lambda_0} \leq 1/2\), we have

\[
U_\alpha(\lambda) = (I + (A_\beta - A_\alpha)A_\alpha(\lambda))^{-1}
\]

and

\[
\mathcal{R}_{\mathcal{L}(H^s_0(\mathbb{R}^N))}(\{((\tau\partial_\tau)^sU_\alpha(\lambda) \mid \lambda \in \Sigma_{\beta_0,\lambda_0}\}) \leq 2.
\]

For each \(\alpha \in [A_1, A_2]\), let

\[
B_\alpha = \{\beta \in [A_1, A_2] \mid (I + (A_\beta - A_\alpha)A_\alpha(\lambda))^{-1} \text{ exists}\}.
\]

Obviously, \([A_1, A_2] \subset \bigcup_{\alpha \in [A_1, A_2]} B_\alpha\). Since \([A_1, A_2]\) is compact, there exist a finite number of \(\alpha_i\) (\(i = 1, \cdots, n\)) such that \([A_1, A_2] = \bigcup_{i=1}^{n} B_{\alpha_i}\). Thus, letting \(\max_{i=1,\cdots,n} \gamma_{\alpha_i,\lambda_0} = \gamma_{\lambda_0}\), we see that \(A_\beta(\lambda) = A_\alpha(\lambda)U_\alpha(\lambda)\) furnishes that

\[
\mathcal{R}_{\mathcal{L}(H^s_0(\mathbb{R}^N),H^2_0(\mathbb{R}^N))}(\{((\tau\partial_\tau)^s(\lambda^{j/2}A_\alpha(\lambda)) \mid \lambda \in \Sigma_{\lambda_0,\lambda_0}\}) \leq 1
\]

for \(s = 0, 1\) and \(j = 0, 1, 2\). Moreover, for any \(\lambda \in \Sigma_{\lambda_0,\lambda_0}\) and \(F \in H^s_0(\mathbb{R}^N)\),

\[
U = A_\beta(\lambda)F
\]

is a unique solution of Eq. (22). Summing up, we have proved the following theorem.

**Theorem 2.4.** Let \(\alpha\) be any positive number in \([A_1, A_2]\). Let \(1 < q < \infty\) and \(0 < \lambda_0 < \infty\). Then, there exists an operator family \(A_\alpha(\lambda)\) with

\[
A_\alpha(\lambda) \in \text{Hol}(\Sigma_{\lambda_0,\lambda_0},\mathcal{L}(H^s_0(\mathbb{R}^N),H^2_0(\mathbb{R}^N)))
\]

such that for any \(\lambda \in \Sigma_{\lambda_0,\lambda_0}\) and \(F \in H^s_0(\mathbb{R}^N)\),

\[
U = A_\alpha(\lambda)F
\]

is a unique solution of Eq. (16), and

\[
\mathcal{R}_{\mathcal{L}(H^s_0(\mathbb{R}^N),H^2_0(\mathbb{R}^N))}(\{((\tau\partial_\tau)^s(\lambda^{k/2}A_\alpha(\lambda)) \mid \lambda \in \Sigma_{\lambda_0,\lambda_0}\}) \leq \gamma_0
\]

for \(s = 0, 1\) and \(k = 0, 1, 2\). Here, \(\gamma_0\) is a constant independent of \(\alpha \in [A_1, A_2]\) but depends on \(\lambda_0\).

**2.2. Analysis of a perturbed problem in \(\mathbb{R}^N\).** Let \(\alpha(x)\) be a real valued continuous function satisfying the assumption (12). Let \(x_0\) be any point in \(\Omega\) and let \(M_1\) be any number in \((0, 1)\). Let \(d_0 > 0\) be a small positive number such that

\[
|\alpha(x) - \alpha(x_0)| \leq M_1 \quad \text{for any } x \in B_{d_0}(x_0).
\]

We may assume that \(B_{d_0}(x_0) \subset \Omega\). Let \(\varphi(x)\) be a function in \(C_{0}^\infty(\mathbb{R}^N)\) which equals one in \(B_{d_0/2}(x_0)\) and zero outside of \(B_{d_0}(x_0)\). Let

\[
\beta(x) = \varphi(x)\alpha(x) + (1 - \varphi(x))\alpha(x_0).
\]

In this subsection, we consider the resolvent problem:

\[
\lambda U - A_\beta(D)U = F \quad \text{in } \mathbb{R}^N.
\]

We shall prove the following theorem.
Theorem 2.5. Let $1 < q < \infty$. Then, there exist an $M_1 \in (0, 1)$ and an operator family $\tilde{B}_\beta(\lambda) \in \text{Hol}(\Sigma_{\theta_0,1}, \mathcal{L}(H^q_0(\mathbb{R}^N), H^q_2(\mathbb{R}^N)))$ such that for any $\lambda \in \Sigma_{\theta_0,1}$ and $F \in H^q_0(\mathbb{R}^N)$, $U = \tilde{B}_\beta(\lambda)F$ is a unique solution of Eq. (24) and

$$
\mathcal{R}_\mathcal{L}(H^q_0(\mathbb{R}^N), H^q_2(\mathbb{R}^N)) \{(\partial_{x_j})^s(\lambda^{1/2}B_\beta(\lambda)) \mid \lambda \in \Sigma_{\theta_0,1}\} \leq \tilde{\gamma}_0 
$$

(25)

for $s = 0, 1$ and $j = 0, 1, 2$. Here, $\tilde{\gamma}_0$ is a constant independent of $M_1$.

Proof. Let $B_{\alpha(x_0)}(\lambda)$ be a solution operator in $\text{Hol}(\Sigma_{\theta_0,1}, \mathcal{L}(H^q_0(\mathbb{R}^N), H^q_2(\mathbb{R}^N)))$ of the equation:

$$
\lambda U - A_{\alpha(x_0)}(D)U = F \quad \text{in} \mathbb{R}^N,
$$

and there holds the estimate:

$$
\mathcal{R}_\mathcal{L}(H^q_0(\mathbb{R}^N), H^q_2(\mathbb{R}^N)) \{(\partial_{x_j})^s(\lambda^{1/2}B_{\alpha(x_0)}(\lambda)) \mid \lambda \in \Sigma_{\theta_0,1}\} \leq \gamma_0 
$$

(27)

for $s = 0, 1$ and $j = 0, 1, 2$. Here, $\gamma_0$ is a constant independent of $\alpha(x_0)$.

Such an operator family is obtained in Theorem 2.4 with $\lambda_0 = 1$. Inserting the formula $U = B_{\alpha(x_0)}(\lambda)F$ into Eq. (24), we have

$$
\lambda U - A_{\alpha(x_0)}(D)U = R(D)U = F - R(D)U \quad \text{in} \mathbb{R}^N.
$$

Here,

$$
R(D)U = (0, \varphi(x)(\alpha(x) - \alpha(x_0))\Delta^2 u, 0)^T
$$

for $U = (u, v, \theta)^T$. Thus, we set

$$
\mathcal{R}(\lambda)F = R(D)B_{\alpha(x_0)}(\lambda)F,
$$

and then Eq. (24) reads

$$
\lambda U - A_\beta U = F - \mathcal{R}(\lambda)F \quad \text{in} \mathbb{R}^N.
$$

(28)

By (23) and (27), we have

$$
\mathcal{R}_\mathcal{L}(H^q_0(\mathbb{R}^N)) \{(\partial_{x_j})^s(\mathcal{R}(\lambda)) \mid \lambda \in \Sigma_{\theta_0,1}\} \leq M_1 \gamma_0
$$

for $s = 0, 1$, choosing $M_1 \in (0, 1)$ so small that $M_1 \gamma_0 < 1/2$, we have

$$
\mathcal{R}_\mathcal{L}(H^q_0(\mathbb{R}^N)) \{(\partial_{x_j})^s(\mathcal{R}(\lambda)) \mid \lambda \in \Sigma_{\theta_0,1}\} \leq 1/2
$$

and so $(I - \mathcal{R}(\lambda))^{-1} = \sum_{j=0}^{\infty} \mathcal{R}(\lambda)^j$ exists in $\text{Hol}(\Sigma_{\theta_0,1}, \mathcal{L}(H^q_0(\mathbb{R}^N)))$ and

$$
\mathcal{R}_\mathcal{L}(H^q_0(\mathbb{R}^N)) \{(\partial_{x_j})^s(I + \mathcal{R}(\lambda))^{-1} \mid \lambda \in \Sigma_{\theta_0,1}\} \leq 2
$$

(29)

for $s = 0, 1$. Let $\tilde{B}_\alpha(\lambda) = B_{\alpha(x_0)}(I + \mathcal{R}(\lambda))^{-1}$, and then by (28) we see that for any $\lambda \in \Sigma_{\theta_0,1}$ and $F \in H^q_0(\mathbb{R}^N)$, $U = \tilde{B}_\alpha(\lambda)F$ is a solution of Eq. (24). Moreover, putting (27) and (29) together gives

$$
\mathcal{R}_\mathcal{L}(H^q_0(\mathbb{R}^N), H^q_2(\mathbb{R}^N)) \{(\partial_{x_j})^s(\lambda^{1/2}\tilde{B}_\alpha(\lambda)) \mid \lambda \in \Sigma_{\theta_0,1}\} \leq 2\gamma_0
$$

for $s = 0, 1$ and $j = 0, 1, 2$. This completes the proof of the existence part of Theorem 2.5.

To prove the uniqueness, let $U \in H^q_2(\mathbb{R}^N)$ satisfy the homogeneous equation:

$$
\lambda U - A_\beta(D)U = 0 \quad \text{in} \mathbb{R}^N,
$$

which reads as

$$
\lambda U - A_{\alpha(x_0)}U = R(D)U \quad \text{in} \mathbb{R}^N.
$$

Thus, $U = B_{\alpha(x_0)}R(D)U$. By (27) and (23), we have $\|U\|_{H^q_2(\mathbb{R}^N)} \leq \gamma_0 M_1 \|U\|_{H^q_2(\mathbb{R}^N)}$. Since $\gamma_0 M_1 < 1/2$, we have $U = 0$. This shows the uniqueness, and therefore the proof is completed.

\[ \square \]
2.3. Analysis in the half space. Let $A_1$ and $A_2$ be two positive numbers such that $A_1 < A_2$. Let $\alpha$ be a positive number such that $A_1 \leq \alpha \leq A_2$. We consider the resolvent problem:

$$(\lambda I - A_\alpha(D))U = F \quad \text{in } \mathbb{R}_+^N, \quad u|_{x_N=0} = \frac{\partial u}{\partial x_N}|_{x_N=0} = \theta|_{x_N=0} = 0 \quad (30)$$

in the half-space $\mathbb{R}_+^N = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}$. The following theorem is the main result of this section.

**Theorem 2.6.** Let $A_1$ and $A_2$ be positive numbers with $A_1 < A_2$ and $\alpha \in [A_1, A_2]$. Let $1 < q < \infty$ and $0 < \lambda_0 < \infty$. Then, there exists an operator family $B_\alpha(\lambda) \in \text{Hol}(\Sigma_{\theta_0, \lambda_0}, L(H_q^0(\mathbb{R}_+^N), H_q^0(\mathbb{R}_+^N)))$ such that for any $\lambda \in \Sigma_{\theta_0, \lambda_0}$ and $F \in H_q^0(\mathbb{R}_+^N)$, $U = B_\alpha(\lambda)F$ is a unique solution of Eq. (30), and

$$\mathcal{R}_{L(H_q^0(\mathbb{R}_+^N), H_q^{2-s}(\mathbb{R}_+^N))} \{((\tau \partial_\tau)\lambda^{k/2} B_\alpha(\lambda)) \mid \lambda \in \Sigma_{\theta_0, \lambda_0}\} \leq \gamma_1$$

for $s = 0, 1$ and $k = 0, 1, 2$. Here, $\gamma_1$ is a constant depending only on $\lambda_0$.

In the following, we prove Theorem 2.6. Given $k$ defined on $\mathbb{R}_+^N$, let $k^0$ be the odd extension of $k$, which is defined by

$$k^0(x) = \begin{cases} k(x', x_N) & (x_N > 0), \\ -k(x', -x_N) & (x_N < 0), \end{cases} \quad (31)$$

where $x' = (x_1, \ldots, x_{N-1})$. Let $A_\alpha(\lambda)$ be the solution operator given in the Theorem 2.4, and let

$$A_\alpha(\lambda)F = A_\alpha(\lambda)F^0 \quad (32)$$

for any $F \in H_q^0(\mathbb{R}_+^N)$. Let $U^0 = A_\alpha(\lambda)F$, and then

$$\lambda U^0 - A_\alpha(D)U^0 = F \quad \text{in } \mathbb{R}_+^N. \quad (33)$$

Moreover, by Theorem 2.4, we have

$$\mathcal{R}_{L(H_q^0(\mathbb{R}_+^N), H_q^{2-s}(\mathbb{R}_+^N))} \{((\tau \partial_\tau)\lambda^{k/2} A_\alpha(\lambda)) \mid \lambda \in \Sigma_{\theta_0, \lambda_0}\} \leq 2\gamma_0$$

for $s = 0, 1$ and $j = 0, 1, 2$, because $\|F^0\|_{H_q^0(\mathbb{R}_+^N)} = 2\|F\|_{H_q^0(\mathbb{R}_+^N)}$. Let $(u^0, v^0, \theta^0) = A_\alpha(\lambda)F$, and then

$$u^0|_{x_N=0} = \theta^0|_{x_N=0} = 0. \quad (35)$$

Thus, we will construct a solution $U^1 = (u^1, v^1, \theta^1)^T$ such that

$$\lambda U^1 - A_\alpha(D)U^1 = 0 \quad \text{in } \mathbb{R}_+^N, \quad u^1 = \theta^1 = 0, \quad \frac{\partial u^1}{\partial x_N} = \frac{\partial u^0}{\partial x_N} \quad \text{on } \mathbb{R}_0^N, \quad (36)$$

where

$$\mathbb{R}_0^N = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}.$$ 

And then, $U = U^0 - U^1$ is a solution of Eq. (30).

Let $w = u^1$, $\lambda w = v^1$, $\tau = \theta^1$ and $W = \frac{\partial w}{\partial x_N}$, and then

$$\lambda^2 w + \alpha \Delta^2 w + \Delta \tau = 0 \quad \text{in } \mathbb{R}_+^N,$$ 

$$\lambda \tau - \Delta \tau - \lambda \Delta w = 0 \quad \text{in } \mathbb{R}_+^N,$$ 

$$w = \tau = 0, \quad \frac{\partial w}{\partial x_N} = W \quad \text{on } \mathbb{R}_0^N. \quad (37)$$
To solve (37), we apply the partial Fourier transform, defined by
\[ \tilde{u}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} u(x', x_N) e^{-ix' \xi'} dx', \]
to (37), where \( \xi' = (\xi_1, \ldots, \xi_{N-1}) \), and then we obtain a system of ordinary differential equations:
\[
\begin{align*}
\lambda^2 \tilde{w} + \alpha (\partial_N^2 - |\xi'|^2) \tilde{w} + (\partial_N^2 - |\xi'|^2) \tilde{r} &= 0 & (x_N > 0), \\
\lambda \tilde{w} - (\partial_N^2 - |\xi'|^2) \tilde{r} - \lambda (\partial_N^2 - |\xi'|^2) \tilde{w} &= 0 & (x_N > 0), \\
\tilde{w}|_{x_N=0} = \tilde{r}|_{x_N=0} = 0, \quad \frac{\partial \tilde{w}}{\partial x_N} \bigg|_{x_N=0} = \tilde{W}|_{x_N=0}.
\end{align*}
\]

Following Naito and Shibata in [27, Sec. 3 and Sec. 4], we will construct a solution formula of Eq. (38). It follows from [27, (3.15)–(3.19)] that the \( \tilde{w} \) and \( \tilde{r} \) are given by
\[
\begin{align*}
\tilde{w}(\lambda, \xi', x_N) &= \frac{1}{\det(M(\lambda, \xi'))} e^{-A_j(\lambda, \xi') x_N} \tilde{W}(\xi', 0), \\
\tilde{r}(\lambda, \xi', x_N) &= -\sum_{j=1}^3 \lambda(\gamma_j^2 + \alpha + 1) \frac{\delta_j}{\det(M(\lambda, \xi'))} e^{-A_j(\lambda, \xi') x_N} \tilde{W}(\xi', 0),
\end{align*}
\]
with
\[
\delta_1 = \gamma_2^2 - \gamma_3^2, \quad \delta_2 = \gamma_3^2 - \gamma_1^2, \quad \delta_3 = \gamma_1^2 - \gamma_2^2, \tag{40}
\]
where \( \gamma_i \) \((i = 1, 2, 3)\) are the numbers given in (18), while
\[
A_k(\lambda, \xi') = \sqrt{\lambda \gamma_k^{-1} + |\xi'|^2} \quad (k = 1, 2, 3) \tag{41}
\]
and
\[
M(\lambda, \xi') := \begin{pmatrix} 1 & 1 & 1 \\
0 & -A_1 & -A_2 \\
\gamma_1^2 + \alpha + 1 & \gamma_2^2 + \alpha + 1 & \gamma_3^2 + \alpha + 1 \end{pmatrix}. \tag{42}
\]
We have
\[
\det(M(\lambda, \xi')) = -\sum_{j=1}^3 \delta_j A_j. \tag{43}
\]

Employing the same argument as in Naito and Shibata [27] (cf. also Denk and Shibata [8], Inna [11]), we have the following theorem.

**Theorem 2.7.** Let \( A_1 \) and \( A_2 \) be positive numbers with \( A_1 < A_2 \) and \( \alpha \in [A_1, A_2] \). Let \( 1 < q < \infty \) and \( 0 < \lambda_0 < \infty \). Then, there exists an operator family \( B_0(\lambda) \in \text{Hol}(\Sigma_{\theta_0, \lambda_0}, \mathcal{L}(\mathcal{H}_1^0(\mathbb{R}_+) \oplus \mathcal{H}_2^0(\mathbb{R}_+))) \) such that for any \( \lambda \in \Sigma_{\theta_0, \lambda_0} \) and \( F \in \mathcal{H}_1^0(\mathbb{R}_+) \), \( U = B_0(\lambda)F \) is a unique solution of Eq. (30), and
\[
\mathcal{R}_L(\mathcal{H}_1^0(\mathbb{R}_+), \mathcal{H}_2^{s-k}(\mathbb{R}_+)) \{ (\tau \partial_\tau)^s(\lambda^{k/2} B_0(\lambda)) \mid \lambda \in \Sigma_{\theta_0, \lambda_0} \} \leq \gamma_{1, \alpha}
\]
for \( s = 0, 1 \) and \( k = 0, 1, 2 \). Here, \( \gamma_{1, \alpha} \) is a constant depending on \( \alpha \) and \( \lambda_0 \).

Employing the same argument as in Subsec 2.1 and using Theorem 2.7, we have Theorem 2.6.
2.4. Analysis in a bent half space. Let \( \Phi \) be a diffeomorphism of class \( H^s_\infty \) on \( \mathbb{R}^N \) and \( \Phi^{-1} \) the inverse of \( \Phi \). Let \( \nabla \Phi(x) = A + B(x) \) and \( \nabla \Phi^{-1}(y) = A^{-1} + B_{-1}(y) \), where \( A \) and \( A^{-1} \) are orthonormal matrices with constant coefficients and \( B(x) \) and \( B_{-1}(y) \) are matrices of \( H^s_\infty(\mathbb{R}^N) \)-functions satisfying the conditions:

\[
\| (B, B_{-1}) \|_{L_\infty(\mathbb{R}^N)} \leq M_1, \quad \| \nabla (B, B_{-1}) \|_{L_\infty(\mathbb{R}^N)} \leq C_K, \quad \| \nabla^2 (B, B_{-1}) \|_{H^s_\infty(\mathbb{R}^N)} \leq M_2. \tag{44}
\]

Here, \( C_K \) is a constant depending on the constants \( K, L_1 \) and \( L_2 \) appearing in Definition 1.1 but independent of \( M_1 \). We choose \( M_1 \) small enough eventually, so that we may assume that \( 0 < M_1 \leq 1 \leq M_2 \) without loss of generality. Let \( \Omega^+ = \Phi(\mathbb{R}^N_+) = \{ y \in \mathbb{R}^N \mid y = \Phi(x), x \in \mathbb{R}^N_+ \} \) and \( \Gamma^+ = \Phi(\mathbb{R}^N_+ \cap \mathbb{R}^N_0) = \{ y \in \mathbb{R}^N \mid y = \Phi(x), x \in \mathbb{R}^N_0 \} \). Let \( \nu^+ \) be the unit outer normal to \( \Gamma^+ \) and let \( \partial_{\nu^+} = \nu^+ \cdot \nabla \).

Let \( \alpha \) be a real valued function satisfying the condition (12). Let \( y_0 \) be any point on \( \Gamma^+ \) and we assume in this subsection that there exists a positive number \( d_0 \) such that

\[
| \alpha(y) - \alpha(y_0) | \leq M_1 \quad \text{for any } y \in B_{d_0}(y_0) \cap \Omega^+. \tag{45}
\]

Let \( \varphi(y) \) be a function in \( C^\infty_0(\mathbb{R}^N) \) which equals one for \( x \in B_{d_0/2}(y_0) \) and zero outside of \( B_{d_0}(y_0) \). Let

\[
\beta(y) = \varphi(y)\alpha(y) + (1 - \varphi(y))\alpha(y_0). \tag{46}
\]

In this section, consider the following resolvent problem in a bent half space:

\[
\lambda U - A_{ij}(\lambda)U = F \quad \text{in } \Omega^+, \quad u = \partial_{\nu^+}u = \theta = 0 \quad \text{on } \Gamma^+. \tag{47}
\]

For \( U = (u, v, \theta)^T \). We shall prove the following theorem.

**Theorem 2.8.** Let \( 1 < q < \infty \). Then, there exist a small positive number \( M_1 \) in (44) and a constant \( \lambda_0 > 0 \) depending on \( M_1 \) and \( M_2 \), and an operator family \( C_\beta(\lambda) \) with

\[
C_\beta(\lambda) \in \text{Hol}(\Sigma_{\theta_0, \lambda_0}, \mathcal{L}(H^0_q(\Omega^+), H^2_q(\Omega^+)))
\]

such that for any \( \lambda \in \Sigma_{\theta_0, \lambda_0} \) and \( F \in H^0_q(\Omega^+) \), \( U = C_\beta(\lambda)F \) is a unique solution of Eq. (46), and there hold the estimates:

\[
\mathcal{R}_{\mathcal{E}(H^0_q(\Omega^+), H^{s-2}(\Omega^+))]\{ (\tau \partial_\tau)^s (\lambda^{k/2}C_\beta(\lambda)) \mid \lambda \in \Sigma_{\theta_0, \lambda_0} \} \leq \gamma_+ \tag{48}
\]

for \( s = 0, 1 \) and \( k = 0, 1, 2 \). Here, \( \gamma_+ \) is a constant independent of \( M_1 \) and \( M_2 \).

**Proof.** Following the argument as in [11], we transform Eq. (46) to the equations in the half-space case by changing variables: \( y = \Phi(x) \). Let

\[
\frac{\partial x_j}{\partial y_k}(\Phi(x)) = A_{jk} + B_{jk}(x) \tag{49}
\]

and then, by (44)

\[
\sum_{j=1}^N A_{jk}A_{jl} = \sum_{j=1}^N A_{kj}A_{lj} = \delta_{kl}, \quad \| B_{jk} \|_{L_\infty(\mathbb{R}^N)} \leq M_1, \quad \| \nabla B_{jk} \|_{L_\infty(\mathbb{R}^N)} \leq C_K, \quad \| \nabla^2 B_{jk} \|_{H^s_\infty(\mathbb{R}^N)} \leq M_2. \tag{48}
\]

Since \( \Gamma^+ \) is represented by \( x_N = \Phi^+_N(y) = 0 \) with \( \Phi^{-1} = (\Phi_1^-, \cdots, \Phi_N^-) \), we have

\[
\nu^+(y) = -d^{-1}(\frac{\partial x_N}{\partial y_1}(y), \cdots, \frac{\partial x_N}{\partial y_N}(y)) \quad (y \in \Gamma^+) \tag{49}
\]
with \( d = \sqrt{\sum_{j=1}^{N}(\partial x_N)/\partial y_j}^2 \). By (49), we may assume that \( \nu_+ \) is defined in \( \mathbb{R}^N \). By (47), we have
\[
\frac{\partial}{\partial y_j} = \sum_{k=1}^{N} \frac{\partial x_k}{\partial y_j} \frac{\partial}{\partial x_k} = \sum_{k=1}^{N} (A_{kj} + B_{kj}(x)) \frac{\partial}{\partial x_k},
\]
and therefore,
\[
\Delta_y = \Delta_x + E^1(D), \quad \Delta_y^2 = \Delta_x^2 + E^2(D)
\]
with
\[
E^1(D) = \sum_{1 \leq |\alpha| \leq 2} e_\alpha^1(x) \partial_x^\alpha, \quad E^2(D) = \sum_{1 \leq |\alpha| \leq 4} e_\alpha^2(x) \partial_x^\alpha,
\]
where \( e_\alpha^1 \) and \( e_\alpha^2 \) satisfy the estimate
\[
||e_\alpha^1||_{L_\infty(\mathbb{R}^N)} \leq CM_1 (|\alpha| = 2), \quad ||e_\alpha^2||_{L_\infty(\mathbb{R}^N)} \leq CM_1 (|\alpha| = 4),
\]
\[
||\nabla e_\alpha^1||_{L_\infty(\mathbb{H}^N(\mathbb{R}^N))} \leq CM_2 (|\alpha| = 2), \quad ||\nabla e_\alpha^2||_{L_\infty(\mathbb{H}^N(\mathbb{R}^N))} \leq CM_2 (|\alpha| = 4),
\]
\[
||e_\alpha^1||_{\mathbb{H}^2(\mathbb{R}^N)} \leq CM_3 (|\alpha| = 1), \quad ||e_\alpha^2||_{\mathbb{H}^2(\mathbb{R}^N)} \leq CM_3 (1 \leq |\alpha| \leq 3).\tag{52}
\]
Let \( U = (u, v, \theta)^T \) be a solution of Eq. (46), and set \( \tilde{u}(x) = u(\Phi(x)), \tilde{v}(x) = v(\Phi(x)), \tilde{\theta}(x) = \theta(\Phi(x)) \), and \( \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\theta})^T \). From the boundary conditions: \( u = \theta = 0 \) on \( \Gamma_+ \) it follows that \( \tilde{u} = \tilde{\theta} = 0 \) on \( \mathbb{R}^N_0 \), i.e. \( \tilde{u}(x', 0) = \tilde{\theta}(x', 0) = 0 \). In particular,
\[
\frac{\partial \tilde{u}}{\partial x_j}|_{x_N = 0} = 0 \quad (j = 1, \ldots, N - 1).\tag{53}
\]
Thus, the boundary condition: \( u|_{\Gamma_+} = \partial_{\nu_+} u|_{\Gamma_+} = 0 \) implies that \( \tilde{u}|_{x_N = 0} = \partial_N \tilde{u}|_{x_N = 0} = 0 \). Hence, the problem (46) is transformed to
\[
\lambda \tilde{U} - A\tilde{u}(x_0)(D)\tilde{U} - R(D)\tilde{U} = \tilde{F} \quad \text{in} \quad \mathbb{R}^N_+, \quad \tilde{u}|_{x_N = 0} = \partial_N \tilde{u}|_{x_N = 0} = \tilde{\theta}|_{x_N = 0} = 0 \tag{54}
\]
with \( \tilde{F}(x) = F'(\Phi(x)) = (\tilde{f}(x), \tilde{g}(x), \tilde{h}(x))^T \) and \( \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\theta}) \). Moreover, we set \( \tilde{\alpha}(x) = \alpha(\Phi(x)), \tilde{\beta}(x) = \beta(\Phi(x)), \tilde{\varphi}(x) = \varphi(\Phi(x)) \), and
\[
\tilde{R}(D)\tilde{U} = \begin{pmatrix} 0 \\ \tilde{\varphi}(x)(\tilde{\alpha}(x) - \tilde{\alpha}(x_0))\Delta^2 \tilde{u} - \tilde{\beta}(x) E^2(D) \tilde{u} - E^1(D) \tilde{\theta} \end{pmatrix}.
\]
Let \( \tilde{B}_{\tilde{\alpha}(x_0)}(\lambda) \) be the solution operator constructed in Theorem 2.6 with \( \lambda_0 = 1 \) and there holds the estimate
\[
\mathcal{R}_{L(\mathbb{H}_q^1(\mathbb{R}^N), \mathbb{H}_q^2(\mathbb{R}^N))}(\left\{ (\tau \partial_x)^s (\lambda^{k/2} \tilde{B}_{\tilde{\alpha}(x_0)}(\lambda)) \right\} | \lambda \in \Sigma_{\kappa, 1}) \leq \gamma_1 \tag{55}
\]
for \( s = 0, 1 \) and \( k = 0, 1, 2 \). But since \( \gamma_1 \) is given in Theorem 2.6 with \( \lambda_0 = 1 \). And so, \( \gamma_1 \) is independent of \( M_1 \) and \( M_2 \). Inserting the formula: \( \tilde{U} = \tilde{B}_{\tilde{\alpha}(x_0)}(\lambda)\tilde{F} \) in Eq. (54), we have
\[
\lambda \tilde{U} - A\tilde{u}(x_0)(D)\tilde{U} - \tilde{R}(D)\tilde{U} = \tilde{F} - \tilde{R}(D)\tilde{B}_{\tilde{\alpha}(x_0)}(\lambda)\tilde{F} \quad \text{in} \quad \mathbb{R}^N_+, \quad \tilde{u}|_{x_N = 0} = \partial_N \tilde{u}|_{x_N = 0} = \tilde{\theta}|_{x_N = 0} = 0.\tag{56}
\]
By (52) and (45),
\[
||\tilde{R}(D)\tilde{U}||_{\mathbb{H}_q^1(\mathbb{R}^N)} = M_1 ||\nabla^2 \tilde{u}||_{L_q(\mathbb{R}^N)} + ||\nabla E^2(D) \tilde{u}||_{L_q(\mathbb{R}^N)} + ||E^1(D) \tilde{u}||_{L_q(\mathbb{R}^N)} \leq CM_1 A_2 ||\tilde{U}||_{\mathbb{H}_q^2(\mathbb{R}^N)} + CM_2 ||\tilde{U}||_{\mathbb{H}_q^1(\mathbb{R}^N)}.
\]
Here and in the following, $C$ denotes generic constants independent of $M_1$ and $M_2$ and $C_{M_2}$ denotes generic constants depending on $M_2$. Let $\lambda_0$ be any number $\geq 1$. For any $m \in \mathbb{N}$, $\{\lambda_k\}_{k=1}^m \subset \Sigma_{\vartheta_0, \lambda_0}$, and $\{F_k\}_{k=1}^m \subset \mathbf{H}_q^1(\mathbb{R}_+^N)^m$, we have

$$\int_0^1 \|\sum_{k=1}^m r_k(u)\tilde{R}(D)\tilde{B}_{\tilde{\alpha}(x_0)}(\lambda_k)F_k\|_{\mathbf{H}^1_q(\mathbb{R}_+^N)}\,du$$

$$= \int_0^1 \|\tilde{R}(D)(\sum_{k=1}^m r_k(u)\tilde{B}_{\tilde{\alpha}(x_0)}(\lambda_k)F_k)\|_{\mathbf{H}^1_q(\mathbb{R}_+^N)}\,du$$

$$\leq C M_1 A_2 \int_0^1 \|\sum_{k=1}^m r_k(u)\tilde{B}_{\tilde{\alpha}(x_0)}(\lambda_k)F_k\|_{\mathbf{H}^1_q(\mathbb{R}_+^N)}\,du$$

$$+ C_{M_2} \int_0^1 \|\sum_{k=1}^m r_k(u)\tilde{B}_{\tilde{\alpha}(x_0)}(\lambda_k)F_k\|_{\mathbf{H}^1_q(\mathbb{R}_+^N)}\,du.$$
for \( s = 0, 1 \). Let \( \tilde{C}_\beta(\lambda) = \tilde{B}_{\alpha}(\lambda) (I - \tilde{R}(D)\tilde{B}_{\alpha}(\lambda))^{-1} \), and then by (56) \( \tilde{U} = \tilde{C}_\beta(\lambda)\tilde{F} \) is a solution of (54), and moreover by (59) and Theorem 2.6,
\[
\mathcal{R}_{C(H_0^{q}(\Omega), H_0^{2-q}(\Omega))}(\{(\tau\partial_r)^s\lambda^{1/2}\tilde{C}_\beta(\lambda) \mid \lambda \in \Sigma_{\theta_0, \lambda_0}\}) \leq s\gamma_1 \tag{60}
\]
for \( s = 0, 1 \) and \( j = 0, 1, 2 \). For \( F \in H_0^{q}(\Omega) \), we define an operator family \( C_{\beta}(\lambda) \) acting on \( F \) by letting
\[
C_{\beta}(\lambda)F = (\tilde{C}_\beta(\lambda)\tilde{F}) \circ \Phi^{-1}.
\]
Obviously, \( C_{\beta}(\lambda) \) is a solution operator of Eq. (46). Finally, we shall estimate the \( R \) bound of \( C_{\beta}(\lambda) \). We may assume that \( 0 < M_1 \leq 1 \). Moreover, \( |A| \leq 1 \) and \( |A^{-1}| \leq 1 \), because \( AA^T = I \) and \( A^{-1}(A^{-1})^T = I \). Thus, by (44), we have the following estimate for the composite functions
\[
\begin{align*}
\|g \circ \Phi^{-1}\|_{H_0^{q}(\Omega)} & \leq C_q(\|\nabla^2 g\|_{L_q(\Omega)} + \|\nabla^2 g\|_{L_q(\Omega)_{x_{r,1}}}) + C_{q,M_2}\|g\|_{H_0^{q}(\Omega)}, \\
\|g \circ \Phi^{-1}\|_{H_0^{q}(\Omega)} & \leq C_q(\|\nabla^2 g\|_{L_q(\Omega)} + \|\nabla^2 g\|_{L_q(\Omega)_{x_{r,1}}}) + C_{q,M_2}\|g\|_{H_0^{q}(\Omega)}, \\
\|g \circ \Phi^{-1}\|_{H_0^{q}(\Omega)} & \leq C_q\|g\|_{H_0^{q}(\Omega)} \quad (k = 0, 1, 2), \\
\|F \circ \Phi\|_{H_0^{q}(\Omega)} & \leq C_q\|F\|_{H_0^{q}(\Omega)}.
\end{align*}
\]
Let \( \tilde{\lambda}_0 \) be any number \( \geq \lambda_0 \). For any \( m \in \mathbb{N} \), \( \{\lambda_k\}_{k=1}^m \subset \Sigma_{\theta_0, \lambda_0}, \{F_k\}_{k=1}^m \subset H_0^{q}(\Omega)^m \), we have by (60) and Proposition 1 (b) and (c),
\[
\begin{align*}
\int_0^1 \|\sum_{k=1}^m r_k(\lambda)C_{\beta}(\lambda)F_k\|_{H_0^{q}(\Omega)} \, du & \leq C_q\int_0^1 \|\sum_{k=1}^m r_k(\lambda)C_{\beta}(\lambda)F_k\|_{H_0^{q}(\Omega)} \, du \\
& \quad + C_{q,M_2}\int_0^1 \|\sum_{k=1}^m r_k(\lambda)C_{\beta}(\lambda)F_k\|_{H_0^{q}(\Omega)} \, du \\
& \leq C_q\int_0^1 \|\sum_{k=1}^m r_k(\lambda)C_{\beta}(\lambda)F_k\|_{H_0^{q}(\Omega)} \, du \\
& \quad + C_{q,M_2}\sum_{k=1}^{\tilde{\lambda}_0^{-1}}\int_0^1 \|\sum_{k=1}^m r_k(\lambda)\lambda_kC_{\beta}(\lambda)F_k\|_{H_0^{q}(\Omega)} \, du \\
& \leq 8(C_q + C_{q,M_2}\tilde{\lambda}_0^{-1})\gamma_1\int_0^1 \|\sum_{k=1}^m r_k(\lambda)F_k\circ \Phi\|_{H_0^{q}(\Omega)} \, du \\
& \leq 8C_q(C_q + C_{q,M_2}\tilde{\lambda}_0^{-1})\gamma_1\int_0^1 \|\sum_{k=1}^m r_k(\lambda)F_k\|_{H_0^{q}(\Omega)} \, du.
\end{align*}
\]
We choose \( \tilde{\lambda}_0 \) so large that \( C_{q,M_2}\tilde{\lambda}_0^{-1} \leq 1 \), and then we have shown that there exists a constant \( \lambda_1 \geq \lambda_0 \) depending on \( M_1 \) and \( M_2 \) such that
\[
\mathcal{R}_{C(H_0^{q}(\Omega), H_0^{2-q}(\Omega))}(\{(\tau\partial_r)^s(\lambda^{k/2}C_{\beta}(\lambda)) \mid \lambda \in \Sigma_{\theta_0, \lambda_1}\}) \leq \gamma_+
\]
for \( s = 0, 1 \) and \( k = 0, 1, 2 \). Here, \( \gamma_+ \) is a constant independent of \( M_1 \) and \( M_2 \).

The uniqueness can be shown by a priori estimates of the solutions of homogeneous equations. The argument is the same as in the proof of Theorem 2.5, and so we may omit the proof of the uniqueness. This completes the proof of Theorem 2.8. \( \square \)
2.5. **Analysis in a general Domain.** To prove Theorem 2.1, we use several properties of uniform $C^4$ domain, which are stated in the following proposition.

**Proposition 2.** Let $\Omega$ be a uniform $C^4$-domain in $\mathbb{R}^N$ with boundary $\Gamma$. Then, for any positive constant $M_1$, there exist constants $M_2 > 0$, $d^0, d^1 \in (0,1)$, at most countably many functions $\Phi_j \in C^4(\mathbb{R}^N)$ and points $x^0_j \in \Gamma$ and $x^1_j \in \Omega$ such that the following assertions hold:

1. For every $j \in \mathbb{N}$, the map $\Omega \ni x \to \Phi_j(x) \in \mathbb{R}^N$ is bijective.
2. $\Omega = \left( \bigcup_{j=1}^\infty \Phi_j(\mathbb{R}^N) \right) \cap B_{d^0}(x^0_j) \cup \left( \bigcup_{j=1}^\infty B_d(x^1_j) \right) \subset \Omega \cap \Phi_j(\mathbb{R}^N) \cap B_{d^0}(x^0_j) = \Omega \cap B_{d^0}(x^0_j)$, and $\Phi_j(\mathbb{R}^N) \cap B_{d^0}(x^0_j) = \Gamma \cap B_{d^0}(x^0_j)$.
3. There exist $C^\infty$ functions $\zeta_j^i$ $(j \in \mathbb{N}, i = 0, 1)$ such that $0 \leq \zeta_j^i, \tilde{\zeta}_j^i \leq 1$, $\text{supp} \zeta_j^i \subset B_d(x^1_j)$, $\|\zeta_j^i, \tilde{\zeta}_j^i\|_{L^\infty(\mathbb{R}^N)} \leq 1$, $\|\nabla \zeta_j^i, \nabla \tilde{\zeta}_j^i\|_{L^\infty(\mathbb{R}^N)} \leq c_0$, $\|\nabla^2 \zeta_j^i, \nabla^2 \tilde{\zeta}_j^i\|_{L^\infty(\mathbb{R}^N)} \leq c_0$, $\zeta_j^i = 0$ on supp $\tilde{\zeta}_j^i$, $\sum_{i=0}^\infty \sum_{j=0}^\infty \zeta_j^i = 1$ on $\Omega$, $\sum_{j=1}^\infty \zeta_j = 1$ on $\Gamma$. Here, $c_0$ is a constant which depends on $M_2$ and $N$, but is independent of $j \in \mathbb{N}$.
4. There exist constant $N \times N$ orthogonal matrices $R_j^i$ and $R_j^i$, and $N \times N$ matrices of $H^3_\infty$-functions $R_j^i$ and $R_j^i$, defined on $\mathbb{R}^N$, such that $\nabla \Phi_j = R_j^i + R_j^i, \nabla (\Phi_j)^{-1} = R_j^i + R_j^i$. Moreover, $R_j^i$ and $R_j^i$ satisfy the conditions:
   $\| (R_j^i, R_j^i) \|_{L^\infty(\mathbb{R}^N)} \leq M_1$, $\| (R_j^i, R_j^i) \|_{L^\infty(\mathbb{R}^N)} \leq C_K$,
   $\| (R_j^i, R_j^i) \|_{L^\infty(\mathbb{R}^N)} \leq M_2$
   for any $j \in \mathbb{N}, i = 0, 1$. Here, $C_K$ is a constant depending on the constants $K, L_1$ and $L_2$ appearing in Definition 1.1 but independent of $j \in \mathbb{N}, M_1$ and $M_2$.
5. There exists a natural number $L > 2$ such that any $L + 1$ distinct sets of $\{B_d(x^j_i) \mid i = 0, 1, j \in \mathbb{N}\}$ have an empty intersection.

In what follows, we write $\Omega_0 = \mathbb{R}^N, \Omega_1 = \Phi_1(\mathbb{R}^N)$ and $\Omega_\ell = \Phi_\ell(\mathbb{R}^N)$ for $\ell \in \mathbb{N}$. Furthermore, we simply write $V_\ell^i = \Omega_\ell \cap B_d(x^1_j)$ ($\ell \in \mathbb{N}, i = 0, 1$). Moreover, $\zeta_j^i$ and $\tilde{\zeta}_j^i$ are functions given in Proposition 2.

From Proposition 2 (5), we have

$$\left( \sum_{i=0}^\infty \sum_{j=1}^\infty \| f \|_{L_q(V_\ell^i)}^q \right)^{1/q} \leq C_q \| f \|_{L_q(\Omega)}$$

(61)

for any $f \in L_q(\Omega)$ and $1 \leq q < \infty$. Then by (61) we have the following lemma [11, Lemma 5.1].

**Lemma 2.9.** Let $\{f_j\}_{j=1}^\infty$ be a sequence of functions in $L_q(\Omega)$ such that $\text{supp} f_j \subset \Omega \cap B_d(x^i_j) (j \in \mathbb{N})$ and $\sum_{j=1}^\infty \| f_j \|_{L_q(\Omega)}^q < \infty$. Then, $\sum_{j=1}^\infty f_j \in L_q(\Omega)$ and

$$\| \sum_{j=0}^\infty f_j \|_{L_q(\Omega)} \leq C_q \left( \sum_{j=1}^\infty \| f_j \|_{L_q(\Omega)}^q \right)^{1/q}. $$

Since $\alpha$ is uniformly continuous in $\bar{\Omega}$, choosing $d^i > 0$ smaller, we may assume that $|\alpha(x) - \alpha(x^i_j)| \leq M_1$ for any $x \in B_d^i, i = 0, 1, j \in \mathbb{N}$. Moreover, after choosing $M_2$ and $d^i$ according to $M_1$ in Proposition 2, we choose $M_2$ again so large that $A_2 \leq M_2$.

Let

$$\beta_j^i(x) = \tilde{\zeta}_j^i(x) \alpha(x) + (1 - \tilde{\zeta}_j^i(x)) \alpha(x^i_j).$$
Since $\zeta_j^i(x) = 1$ on $\text{supp} \tilde{\zeta}_j$, we have
$$\zeta_j^i A_u U = \zeta_j^i A_{\beta_j^i} U. \quad (62)$$

By Theorem 2.5 and Theorem 2.8, we can find $\lambda_1 \geq 1$ depending on $M_2$ but independent of $j \in \mathbb{N}$ and operator families $C_{\alpha,\ell}^i(\lambda)$ with
$$C_{\alpha,\ell}^i \in \text{Hol}(\Sigma_{\varrho_0,\lambda_1}, \mathcal{L}(\mathbf{H}_q^0(\Omega_{\ell}^j), \mathbf{H}_q^2(\Omega_{\ell}^j)))$$
such that for any $F_\ell^j \in \mathbf{H}_q^0(\Omega_{\ell}^j)$ and $\lambda \in \Sigma_{\varrho_0,\lambda_1}$, $U_\ell^j = (u_{\ell}^i, v_{\ell}^i, \theta_{\ell}^i)^T = C_{\alpha,\ell}^i(\lambda) F_\ell^j$ are solutions of the equations:
$$\lambda U_\ell^j - A_{\beta_j^i}(D) U_\ell^j = F_\ell^j \quad \text{in } \mathbb{R}^N, \quad (63)$$
$$\lambda U_\ell^j - A_{\beta_j^i}(D) U_\ell^j = F_\ell^j \quad \text{in } \Omega_{\ell}, \quad u_{\ell}^j = \partial_{\nu_\ell} u_{\ell}^j = \theta_{\ell}^j = 0 \quad \text{on } \Gamma_{\ell}, \quad (64)$$
where each $\nu_\ell$ is the unit outer normal to $\Gamma_{\ell}$, and
$$\mathcal{R}_C(\mathbf{H}_q^0(\mathbb{R}^N), \mathbf{H}_q^{2-\epsilon}(\Omega)) \{ (\tau \Delta)^{s/2} C_{\alpha,\ell}(\lambda) \} \leq \gamma_2 \quad (65)$$
for $s = 0, 1$, $j = 0, 1, 2$, $i = 0, 1$ and $\ell \in \mathbb{N}$. Here, $\gamma_2$ is independent of $M_1$. Let
$$T_\alpha(\lambda) F = \sum_{i=0}^1 \sum_{j=1}^\infty \zeta_j^i C_{\alpha,j}^i(\lambda) \tilde{\zeta}_j^i F, \quad (66)$$
and then by Lemma 2.9, (63), and (64) we have
$$\mathcal{R}_C(\mathbf{H}_q^0(\Omega), \mathbf{H}_q^{2-\epsilon}(\Omega)) \{ (\tau \Delta)^{s/2} T_\alpha(\lambda) \} \leq C_M \gamma_2 \quad (67)$$
for $s = 0, 1$ and $\ell = 0, 1, 2$. In fact, for any $m \in \mathbb{N}$, $\{ \lambda_k \}_{k=1}^m \in \Sigma_{\varrho_0,\lambda_1}^m$, and $\{ F_k \}_{k=1}^m \in \mathbf{H}_q^0(\Omega)^m$, we have
$$\int_0^1 \left\| \sum_{k=1}^m r_k(u) \lambda_k^{s/2} T(\lambda_k) F_k \right\|_{\mathbf{H}_q^{2-\epsilon}(\Omega)}^q \, du$$
$$\leq \sum_{i=0}^1 \sum_{j=0}^\infty \int_0^1 \left\| \sum_{k=1}^m r_k(u) \lambda_k^{s/2} C_{\alpha,j}^i(\lambda_k) \tilde{\zeta}_j^i F_k \right\|_{\mathbf{H}_q^{2-\epsilon}(\Omega)}^q \, du$$
$$\leq \sum_{i=0}^1 \sum_{j=0}^\infty \sum_{i=0}^1 \sum_{j=0}^\infty \int_0^1 \left\| \sum_{k=1}^m r_k(u) \lambda_k^{s/2} C_{\alpha,j}^i(\lambda_k) \tilde{\zeta}_j^i F_k \right\|_{\mathbf{H}_q^{2-\epsilon}(\Omega)}^q \, du$$
$$\leq \sum_{i=0}^1 \sum_{j=0}^\infty \int_0^1 \left\| \sum_{k=1}^m r_k(u) \tilde{\zeta}_j^i F_k \right\|_{\mathbf{H}_q^0(\Omega_j^j)}^q \, du$$
$$\leq 2C_q \sum_{i=0}^1 \sum_{j=0}^\infty \int_0^1 \left\| \sum_{k=1}^m r_k(u) F_k \right\|_{\mathbf{H}_q^0(\Omega_j^j)}^q \, du,$$
which gives (67).

Let
$$D_2(\varphi) u := \Delta(\varphi u) - \varphi \Delta u = 2(\nabla \varphi) \cdot (\nabla u) + (\Delta \varphi) u,$$
$$D_4(\varphi) u := \{ \Delta^2(\varphi u) - \varphi \Delta^2 u \}$$
$$= \{ 4(\nabla \varphi) \cdot (\nabla \Delta u) + 2(\Delta \varphi) \Delta u + 4(\nabla^2 \varphi) : (\nabla^2 u) + 4(\nabla \Delta \varphi) \cdot (\nabla u) + (\Delta^2 \varphi) u \},$$
and then
\[ A_\alpha(\varphi U) - \varphi A_\alpha U = G_\alpha(\varphi U) \]
with
\[ G_\alpha(\varphi U) := \begin{pmatrix} 0 \\ -\alpha D_4(\varphi) u - D_2(\varphi) \theta \end{pmatrix} \quad (U = (u, v, \theta)^T). \]

Using (62), \( \sum_{i=0}^{1} \sum_{j=1}^{\infty} \zeta_j^i \tilde{\gamma}^i_j = 1 \), (63) and (64), we have
\[ \lambda T_\alpha(\lambda) F - A_\alpha T_\alpha(\lambda) F = F - \mathcal{R}(\lambda) F \quad \text{in } \Omega, \]
with
\[ \mathcal{R}(\lambda) F = \sum_{i=0}^{1} \sum_{j=1}^{\infty} G_\alpha(\zeta_j^i \tilde{C}_j^i(\lambda) \tilde{\gamma}_j^i F. \]

We now estimate \( \mathcal{R}(\lambda) F \). Let \( \tilde{\lambda}_2 \) be any number \( \geq \tilde{\lambda}_1 \). For any \( m \in \mathbb{N} \), \( \{\lambda_k\}_{k=1}^{m} \in \Sigma_{\tilde{\theta}_0, \tilde{\lambda}_2} \) and \( \{F_k\}_{k=1}^{m} \in H_q^1(\Omega)^m \), by (61), Lemma 2.9, Proposition 1, and (65),
\[ \int_{0}^{1} \left\| \sum_{k=1}^{m} r_k(u) \mathcal{R}(\lambda_k) F_k \right\|_{H_q^1(\Omega)}^q \, du \]
\[ \leq C_q \sum_{i=0}^{1} \sum_{j=0}^{\infty} \int_{0}^{1} \left\| \sum_{k=1}^{m} r_k(u) G_\alpha(\zeta_j^i \tilde{C}_j^i(\lambda_k) \tilde{\gamma}_j^i F_k \right\|_{H_q^1(\Omega)}^q \, du \]
\[ \leq C_q M_{\gamma}^q c^q_0 \sum_{i=0}^{1} \sum_{j=0}^{\infty} \int_{0}^{1} \left\| \sum_{k=1}^{m} r_k(u) C_{\mu, j}(\lambda_k) \tilde{\gamma}_j^i F_k \right\|_{H_q^1(\Omega)}^q \, du \]
\[ \leq C_q M_{\gamma}^q c^q_0 \sum_{i=0}^{1} \sum_{j=0}^{\infty} \int_{0}^{1} \left\| \sum_{k=1}^{m} r_k(u) C_{\mu, j}(\lambda_k) \tilde{\gamma}_j^i F_k \right\|_{H_q^1(\Omega)}^q \, du \]
\[ \leq C_q M_{\gamma}^q c^q_0 \sum_{i=0}^{1} \sum_{j=0}^{\infty} \int_{0}^{1} \left\| \sum_{k=1}^{m} r_k(u) C_{\mu, j}(\lambda_k) \tilde{\gamma}_j^i F_k \right\|_{H_q^1(\Omega)}^q \, du \]
\[ \leq C_q M_{\gamma}^q c^q_0 \sum_{i=0}^{1} \sum_{j=0}^{\infty} \int_{0}^{1} \left\| \sum_{k=1}^{m} r_k(u) C_{\mu, j}(\lambda_k) \tilde{\gamma}_j^i F_k \right\|_{L_q(\Omega)}^q \, du \]
\[ \leq C_q M_{\gamma}^q c^q_0 \sum_{i=0}^{1} \sum_{j=0}^{\infty} \int_{0}^{1} \left\| \sum_{k=1}^{m} r_k(u) F_k \right\|_{H_q^1(\Omega)}^q \, du. \]

Choosing \( \tilde{\lambda}_2 \geq \tilde{\lambda}_1 \) so large that \((C_q)^{1/q} M_{\gamma}^q c^q_0 \tilde{\lambda}_2^{-1/2} \gamma_2 \leq 1/2\), we have
\[ \mathcal{R}_L(H_q^1(\Omega)) \{((\tau_0)^s \mathcal{R}(\lambda) | \lambda \in \Sigma_{\tilde{\theta}_0, \tilde{\lambda}_2}) \} \leq 1/2 \]
for \( s = 0, 1 \). Thus, \((I - \mathcal{R}(\lambda))^{-1} = I + \sum_{j=1}^{\infty} \mathcal{R}(\lambda)^j\) exists and
\[ \mathcal{R}_L(H_q^1(\Omega)) \{((\tau_0)^s (I - \mathcal{R}(\lambda))^{-1} | \lambda \in \Sigma_{\tilde{\theta}_0, \tilde{\lambda}_2}) \} \leq 4, \]
for \( s = 0, 1 \).

We next consider the boundary conditions. Writing \( T_\alpha(\lambda) F = (u, v, \theta)^T \) and \( \tilde{C}_{\alpha, j}(\lambda) \tilde{\gamma}_j^i F = (u_j^i, v_j^i, \theta_j^i)^T \), we have \( u = \sum_{i=0}^{1} \sum_{j=0}^{\infty} \zeta_j^i u_j^i, v = \sum_{i=0}^{1} \sum_{j=0}^{\infty} \zeta_j^i v_j^i \) and
\( \theta = \sum_{i=0}^{1} \sum_{j=0}^{\infty} \zeta_{i} \theta_{j} \). Since \( \Gamma_{j} \cap B_{j}^{i} = \Gamma \cap B_{j}^{i} \), we have \( u|_{\Gamma} = \theta|_{\Gamma} = 0 \), because \( u_{j} \big|_{\Gamma_{j}} = \theta_{j} \big|_{\Gamma_{j}} = 0 \). Notice that \( \nu = \nu_{j} \) on \( \Gamma_{j} \cap \partial_{r}(x_{j}) \). Thus,

\[
\partial_{\nu} u|_{\Gamma} = \sum_{j=1}^{\infty} \zeta_{j} \partial_{\nu} u_{j}|_{\Gamma_{j}} = \sum_{j=1}^{\infty} \zeta_{j} \partial_{\nu} u_{j}|_{\Gamma_{j}} + \sum_{j=1}^{\infty} \nu_{j} \cdot (\nabla \zeta_{j}) u_{j}|_{\Gamma_{j}} = 0,
\]

that is,

\[
B(D)T_{\alpha}(\lambda) F|_{\Gamma} = 0. \tag{70}
\]

Let

\[
S_{\alpha}(\lambda) F = T_{\alpha}(\lambda)(I - \mathcal{R}(\lambda))^{-1} F,
\]

and then, by (68) and (70), \( U = S_{\alpha}(\lambda) F \) is a solution of Eq. (15) for any \( \lambda \in \Sigma_{0,0,\lambda_{2}} \) and \( F \in H_{0}^{2}(\Omega) \). Moreover, by Proposition 1, (67) and (69),

\[
\mathcal{R}_{L}(H_{0}^{2}(\Omega),H_{0}^{2-n}(\Omega))\{ ((\tau \partial_{\tau}(\lambda^{1/2} \Sigma_{\alpha}(\lambda))) \mid \lambda \in \Sigma_{0,0,\lambda_{2}} \} \leq 4C_{M}2\gamma_{2}.
\]

The uniqueness of solutions follows from the a priori estimates, which can be proved in the same manner as in the proof of Theorem 2.5, and so we have proved Theorem 2.1.

3. Maximal \( L_{p}-L_{q} \) regularity and the local well-posedness.

3.1. A proof of Theorem 1.4. In this section, we shall prove the Theorem 1.4. First of all, we consider the Cauchy problem:

\[
\partial_{t} V - A_{\alpha}(D) V = 0 \quad \text{in} \quad \Omega \times (0,T), \quad B(D) V|_{\Gamma} = 0, \quad V|_{t=0} = U_{0}. \tag{71}
\]

Let \( \mathcal{A} \) be an operator defined by \( \mathcal{A} U = A_{\alpha}(D) U \) for \( U \in H_{q}^{2}(\Omega) \), then by Theorem 2.1 we have \( \Sigma_{0,0,\lambda_{0}} \) is contained in the resolvent set of \( \mathcal{A} \) and \( (\lambda I - \mathcal{A})^{-1} F \in S_{\alpha}(\lambda) F \) for \( F \in H_{0}^{2}(\Omega) \). Moreover, by the definition of \( \mathcal{R} \)-boundedness with \( m = 1 \) in Definition 2.2,

\[
|\lambda|\| (\lambda I - \mathcal{A})^{-1} F \|_{H_{2}^{q}(\Omega)} + \| (\lambda I - \mathcal{A})^{-1} F \|_{H_{3}^{q}(\Omega)} \leq C_{\alpha}\| F \|_{H_{0}^{2}(\Omega)} \tag{72}
\]

for any \( \lambda \in \Sigma_{0,0,\lambda_{0}} \) and \( F \in H_{q}^{2}(\Omega) \) with some constant \( C_{\alpha} > 0 \). Thus, \( \mathcal{A} \) generates a \( C_{0} \) analytic semigroup \( \{ T(t) \}_{t \geq 0} \) on \( H_{0}^{2}(\Omega) \) satisfying the estimate:

\[
\| T(t) F \|_{H_{2}^{q}(\Omega)} \leq C e^{\eta t}\| F \|_{H_{0}^{2}(\Omega)} \tag{73}
\]

with \( \eta_{0} \geq \Lambda_{0} \geq 1 \). Employing the same argument as that in Shibata and Shimizu [30, Proof of Theorem 3.9], we see that for any \( U_{0} = (u_{0},v_{0},\theta_{0})^{T} \in D_{q,p}(\Omega) \), Eq. (71) admits a unique solution \( V \) with

\[
V \in L_{p}((0,T),H_{q}^{2}(\Omega)) \cap H_{p}^{1}((0,T),H_{q}^{1}(\Omega))
\]

possessing the estimate:

\[
\| V \|_{L_{p}((0,T),H_{q}^{2}(\Omega))} + \| \partial_{t} V \|_{L_{p}((0,T),H_{q}^{1}(\Omega))} \leq C e^{\tau T}\| U_{0} \|_{B_{2+2\alpha(\lambda_{1}^{1/2})}(\Omega) \times B_{2+\alpha(\lambda_{2}^{1/2})}(\Omega)^{2}}
\]

with some positive constants \( C \) and \( c \).

The next step is to homogenize the equation with respect to the initial data. For this purpose we introduce the unknown variable \( W = U - V \), so that \( W|_{t=0} = 0 \). Then the desirable solution \( U \) can be stated as \( U = V + W \) where \( W \) is the solution of

\[
\partial_{t} W - A_{\alpha}(D) W = F \quad \text{in} \quad \Omega \times (0,T), \quad B(D) W|_{\Gamma} = 0, \quad W|_{t=0} = 0. \tag{74}
\]

wit \( F = (f_{0},f_{1},f_{2})^{T} \in L_{q}(\Omega)^{3} \).
Let $F_0$ be the zero extension of $F$, that is, $F_0(x, t) = F(x, t)$ for $t \in (0, T)$ and $F_0(x, t) = 0$ for $t \not\in (0, T)$. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ be the Laplace transformation and inverse Laplace transformation defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) \, dt, \quad \mathcal{L}^{-1}[f](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} f(\lambda) \, d\lambda$$

with $\lambda = \eta + i\tau \in \mathbb{C}$. We consider the resolvent problem:

$$\lambda \hat{W} - A_\lambda(D)\hat{W} = \mathcal{L}[F_0] \quad \text{in } \Omega, \quad B(D)\hat{W}|_\Gamma = 0.$$

By Theorem 1.2, $\hat{U}$ is given by $\hat{W} = S_\alpha(\lambda)\mathcal{L}[F_0](\lambda)$. Thus, we define $W$ by $W = \mathcal{L}^{-1}[S_\alpha(\lambda)\mathcal{L}[F_0]](t)$. By Theorem 1.2 and Weis’s operator valued Fourier multiplier theorem [32], there exist constants $\eta_0 > 0$ and $C > 0$ such that

$$\|e^{-\eta t}W\|_{L_p(\mathbb{R}, H^q_\alpha(\Omega))} + \|e^{-\eta t}\partial_t W\|_{L_p(\mathbb{R}, H^{q-1}_\alpha(\Omega))} + \eta \|e^{-\eta t}W\|_{L_p(\mathbb{R}, H^q_\alpha(\Omega))} \leq C \| e^{-\eta t} F_0 \|_{L_p((0,T), H^q_\alpha(\Omega))} \leq C \| F \|_{L_p((0,T), H^q_\alpha(\Omega))}$$

(75) for any $\eta > \eta_0$, where we have set

$$\|e^{-\eta t} f\|_{L_p(I, X)} = \left( \int_I (e^{-\eta t} \| f(t) \|_X)^p \, dt \right)^{1/p}$$

for any time interval $I$ and Banach space $X$ with the norm $\| \cdot \|_X$. By (75), we have

$$\eta \| W \|_{L_p((\mathbb{R}, H^q_\alpha(\Omega))} \leq \eta \| e^{-\eta t} W \|_{L_p(\mathbb{R}, H^{q-1}_\alpha(\Omega))} \leq C \| F \|_{L_p((0,T), H^q_\alpha(\Omega))}$$

for any $\eta > \eta_0$. Letting $\eta \to 0$, we have $\| W \|_{L_p((\mathbb{R}, H^q_\alpha(\Omega))} = 0$, which implies that $W$ vanishes for $t \in (-\infty, 0)$. In particular, $W|_{t=0} = 0$. Thus, we have proved that Eq. (74) admits a solution $U$ with

$$W \in L_p((0,T), H^q_\alpha(\Omega)) \cap H^1_p((0,T), H^q_\alpha(\Omega))$$

possessing the estimate:

$$\| W \|_{L_p((0,T), H^q_\alpha(\Omega))} + \| \partial_t W \|_{L_p((0,T), H^{q-1}_\alpha(\Omega))} \leq C e^{\eta T} \| F \|_{L_p((0,T), H^q_\alpha(\Omega))}$$

We finally prove the uniqueness. Let $U \in H^1_p((0,T), H^q_\alpha(\Omega)) \cap L_p((0,T), H^q_\alpha(\Omega))$ satisfy the homogeneous equitions:

$$\partial_t U - A_\lambda(D) U = 0 \quad \text{in } \Omega \times (0,T), \quad B(D) U|_\Gamma = 0, \quad U|_{t=0} = 0. \quad (76)$$

Let $U_0$ be the zero extension of $U$ to $t < 0$, that is $U_0(\cdot, t) = U(\cdot, t)$ for $t \in (0, T)$ and $U_0(\cdot, t) = 0$ for $t < 0$. Since $U|_{t=0} = 0, U_0 \in H^1_p((-\infty, 0), H^q_\alpha(\Omega)) \cap L_p((-\infty, 0), H^q_\alpha(\Omega))$. Let $V$ be the reflection of $U_0$ at $t = T$, that is $V(\cdot, t) = U_0(\cdot, t)$ for $t < T$ and $V(\cdot, t) = U_0(\cdot, 2T - t)$ for $t > T$. Then, $V \in H^1_p(\mathbb{R}, H^q_\alpha(\Omega)) \cap L_p(\mathbb{R}, H^q_\alpha(\Omega))$. Let $\hat{V}$ be the Laplace transform of $V$, that is

$$\hat{V}(\cdot, \lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} V(\cdot, t) \, dt.$$

Since $V(\cdot, t)$ vanishes for $t \not\in (0, 2T)$, for $\lambda \in \Sigma_{\delta_0, \lambda_0}$ by Hölder’s inequality we have

$$\| \hat{V}(\cdot, \lambda) \|_{H^q_\alpha(\Omega)} \leq \frac{1}{|\text{Re}\lambda|^{1/p}} e^{2|\text{Re}\lambda| T} \| U \|_{L_p((0,T), H^q_\alpha(\Omega))}.$$

Applying the Laplace transform to Eq. (76), we have

$$\lambda \hat{V} - A_\lambda(D) \hat{V} = 0 \quad \text{in } \Omega, \quad B(D) \hat{V}|_\Gamma = 0.$$

Thus, the uniqueness of Eq. (11) implies that $\hat{V} = 0$, and so $U = 0$. This completes the proof of Theorem 1.4.
3.2. A proof of the Theorem 1.2. Let $T \in (0, T_1)$ and $L > 0$ be numbers determined later, and set

$$I_T = \{ U = (u, v, \theta)^T \mid U \in L_p((0, T), H_q^2(\Omega)) \cap H_p^1((0, T), H_q^0(\Omega)) \}$$

$$U|_{t=0} = U_0, \quad ||U||_{E_{p,q,T}} \leq L. \quad (77)$$

Where, we have set

$$||U||_{E_{p,q,T}} = ||U||_{L_p((0, T), H_q^2(\Omega))} + ||\partial_t U||_{L_p((0, T), H_q^0(\Omega))}$$

for the notational simplicity. Given $U = (u, v, \theta)^T \in I_T$, let $W$

$$W \in L_p((0, T), H_q^2(\Omega)) \cap H_p^1((0, T), H_q^0(\Omega))$$

be a unique solution of the equations:

$$\partial_t W - A_\alpha(D)W = F(U) \quad \text{in} \quad \Omega \times (0, T), \quad B(D)U|_{\Gamma} = 0, \quad W|_{t=0} = U_0, \quad (78)$$

where $\alpha = 1 + b\phi'(\Delta u_0)$, and

$$F(U) = \begin{pmatrix} f_1 - b(\phi'(\Delta u_0) - \phi'(\Delta u))\Delta^2 u - \phi''(\Delta u)|\nabla \Delta u|^2 \\ 0 \end{pmatrix}.$$

By real interpolation theorem and (77), we have

$$\sup_{0 \leq t < T} ||u(\cdot, t)||_{B_{p',p}^{2+2(1-1/p)}(\Omega)}$$

$$\leq C(||u||_{B_{q,p}^{2+2(1-1/p)}(\Omega)} + ||u||_{L_p((0, T), H_q^1(\Omega))} + ||\partial_t u||_{L_p((0, T), H_q^0(\Omega))})$$

$$\leq C(R + L), \quad (79)$$

where $C$ is a constant independent of $T > 0$ and $R$ is the number given in Theorem 1.2. Since $p > 2$ and $q > N$, we choose $\epsilon > 0$ so small that $N/q + \epsilon < 1$, and then by Sobolev’s imbedding theorem

$$B_{q,p}^{2+2(1-1/p)}(\Omega) \subset W_q^{2+q/N+\epsilon}(\Omega) \subset H_\infty^2(\Omega).$$

Thus, by (79) we have

$$\sup_{0 \leq t < T} ||\Delta u(\cdot, t)||_{L_\infty(\Omega)} \leq C(R + L). \quad (80)$$

In particular, by (80) we have

$$||\phi''(\Delta u)|\nabla \Delta u|^2||_{L_q(\Omega)} \leq \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) ||\nabla \Delta u||_{L_{2q}(\Omega)}^2.$$

Since $(N(1/q - 1/(2q))) = N/(2q) < 1$, we have

$$||\nabla \Delta u||_{L_{2q}(\Omega)}^2 \leq C \left( ||u||_{H_q^2(\Omega)}^{1-N/(2q)} ||u||_{H_q^0(\Omega)}^{N/(2q)} \right)^2.$$

Since $N/q < 1$, by Hölder’s inequality we have

$$||\phi''(\Delta u)|\nabla \Delta u|^2||_{L_p((0,T),L_q(\Omega))}$$

$$\leq \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) \|u\|_{L_\infty((0,T),H_q^0(\Omega))}^{2-N/q} \left( \int_0^T ||u||_{H_q^2(\Omega)}^{N/q} \right)^{1/p}$$

$$\leq \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) \|u\|_{L_\infty((0,T),H_q^0(\Omega))}^{2-N/q} \|u\|_{L_p((0,T),H_q^0(\Omega))}^{N/q} T^{(q-N)/(pq)}.$$

Since $3 \leq 2 + 2(1-1/p)$ as follows from $p > 2$, by (79) we have

$$||u||_{L_\infty((0,T),H_q^0(\Omega))} \leq C ||u||_{L_\infty((0,T),B_{q,p}^{2+2(1-1/p)}(\Omega))} \leq C(R + L). \quad (81)$$
Summing up, we have proved
\[
\|\phi''(\Delta u)\|_{L_p((0,T), L_q(\Omega))} \leq \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) (B + L)^2 T^{(q-N)/(pq)}, \tag{82}
\]

We next consider \((\phi'(\Delta u_0) - \phi'(\Delta u))\Delta^2 u\). By Sobolev’s imbedding theorem and (79),
\[
\|\Delta u_0 - \Delta u\|_{L_\infty(\Omega)} \leq C\|u - u_0\|_{W^{2,N/q+\epsilon}_q(\Omega)}^\epsilon \\
\leq C\|u - u_0\|_{L^1(\Omega)}^{1-N/q+\epsilon} \|u - u_0\|_{L^q(\Omega)}^{N/q+\epsilon} \\
\leq CT^{(1-N/q+\epsilon)/p'} \|\partial_t u\|_{L^p((0,T), L^q(\Omega))} (L + R)^{N/q+\epsilon}
\]

because \(3 < 2 + 2(1 - 1/p)\) as follows from \(p > 2\). Thus, using (81) and (83), we have
\[
\|\phi'(\Delta u_0) - \phi'(\Delta u)\|_{L_\infty(\Omega)} \\
\leq \int_0^T \|\phi''(\Delta u_0 + \theta \Delta(u - u_0))\|_{L_\infty(\Omega)} d\theta \|\Delta u_0 - \Delta u\|_{L_\infty(\Omega)} \\
\leq C \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) (L + R)T \|\Delta u\|_{L^p((0,T), L^q(\Omega))}^{(1-N/q+\epsilon)/p'},
\]

and so,
\[
\|\phi'(\Delta u_0) - \phi'(\Delta u)\|_{L^2(\Omega)} \leq C \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) (L + R)T \|\Delta u\|_{L^p((0,T), L^q(\Omega))}^{(1-N/q+\epsilon)/p'},
\]

which, combined with (82), leads to
\[
\|F(U)\|_{L_p((0,T), H^q(\Omega))} \\
\leq \|(f_1, f_2)\|_{L_p((0,T), L_q(\Omega))}^2 + C \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) (L + R)^2 T^{(q-N)/(qp)} + T^{(1-N/q+\epsilon)/p'}.
\tag{84}
\]

Applying Theorem 1.4 to Eq. (78) and using (84) gives
\[
\|W\|_{E_{p,q,T}} \leq Ce^{\lambda_1 T} \left( R + b \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) (L + R)^2 T^{(q-N)/(qp)} + T^{(1-N/q+\epsilon)/p'} \right). \tag{85}
\]

Choosing \(L > 0\) so large that \(2CeR = L\) and choosing \(T \in (0,T_1)\) so small that \(\lambda T \leq 1\) and
\[
b \left( \sup_{|s| \leq C(R+L)} |\phi''(s)| \right) (L + R)^2 T^{(q-N)/(qp)} + T^{(1-N/q+\epsilon)/p'} \leq R,
\]

by (85) we have
\[
\|W\|_{E_{p,q,T}} \leq L \tag{86}
\]

with \(L = 2CeR\). Let \(\Phi\) be a map defined by \(\Phi U = W\), and then by (86), \(\Phi\) maps \(\bar{\mathcal{I}}_T\) into itself. For \(U_1, U_2 \in \mathcal{I}_T\), we have \(F(U_1) - F(U_2) = (0, -g(u_1, u_2), 0)^T\) with
\[
g(u_1, u_2) = (\phi'(\Delta u_0) - \phi'(\Delta u_1)) \Delta^2 (u_1 - u_2) + (\phi'(\Delta u_2) - \phi'(\Delta u_1)) \Delta^2 u_2 \\
+ \phi''(\Delta u_1) (\nabla \Delta u_1 + \nabla \Delta u_2) \cdot \nabla \Delta (u_1 - u_2) \\
+ (\phi''(\Delta u_1) - \phi''(\Delta u_2)) |\nabla \Delta u_2|^2.
\]
Employing the same argument as that in proving (86), we have
\[ \|\Phi U_1 - \Phi U_2\|_{\mathcal{L}_{p,q}, T} \leq C_{L,R} T^s \|U_1 - U_2\|_{\mathcal{L}_{p,q}, T} \]  \hspace{1cm} (87)
with some constants \( s > 0 \) and \( C_{L,R} \). Here, \( C_{L,R} \) is a constant depending on \( L \) and \( R \). Choosing \( T > 0 \) so small that \( C_{L,R} T^s \leq 1/2 \), by (87), we see that \( \Phi \) is a contraction map on \( \mathcal{I}_T \), and so by Banach’s fixed point theorem, there exists a unique \( U \in \mathcal{I}_T \) such that \( \Phi U = U \), which is a required unique solution of Eq. (5). This completes the proof of Theorem 1.2.

4. Exponential stability and the global well-posedness.

4.1. A proof of Theorem 1.5. Let \( \{T(t)\}_{t \geq 0} \) be the semi-group associated with Eq. (71) given in Subsec. 3.1. In this subsection, we prove the exponential stability of \( \{T(t)\}_{t \geq 0} \). For this purpose, we consider the resolvent problem:
\[ \lambda U - A(D)U = F \quad \text{in } \Omega, \quad B(D)U|_\Gamma = 0. \] \hspace{1cm} (88)
As was seen in Subsec. 3, there exists a \( \lambda_0 > 0 \) such that for any \( \lambda \in \Sigma_{\theta_0, \lambda_0} \) and \( F \in \mathbf{H}^{0}_q(\Omega) \), Eq. (88) admits a unique solution \( U \) possessing the estimate:
\[ |\lambda| \|U\|_{\mathbf{H}^q_2(\Omega)} + \|U\|_{\mathbf{H}^q_2(\Omega)} \leq C \|F\|_{\mathbf{H}^q_2(\Omega)}. \] \hspace{1cm} (89)
Let
\[ \Lambda_{\lambda_0} = \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \text{ and } |\lambda| \leq \lambda_0 \}. \]
We shall prove the following proposition.

Proposition 3. Let \( 1 < q < \infty \) and \( \lambda_0 > 0 \). Assume that \( \Omega \) is a bounded domain whose boundary \( \Gamma \) is a compact \( C^4 \) hypersurface. Then, for any \( \lambda \in \Lambda_{\lambda_0} \) and \( F \in \mathbf{H}^q_0(\Omega) \), Eq. (88) admits a unique solution \( U \in \mathbf{H}^2_2(\Omega) \) possessing the estimate:
\[ \|U\|_{\mathbf{H}^2_2(\Omega)} \leq C \|F\|_{\mathbf{H}^q_2(\Omega)} \] \hspace{1cm} (90)
for some constant \( C > 0 \) independent of \( \lambda \in \Lambda_{\lambda_0} \).

Proof. Since \( \Omega \) is bounded, to prove the unique existence of solutions of Eq. (88), in view of the Riesz-Schauder theorem, the Fredholm alternative principle, it suffices to prove the uniqueness of Eq. (88). Let \( U \in \mathbf{H}^2_q(\Omega) \) be a solution of the homogeneous equation:
\[ \lambda U - A(D)U = 0 \quad \text{in } \Omega, \quad B(D)U|_\Gamma = 0. \] \hspace{1cm} (91)
Notice that Eq. (91) is written componentwise as
\[ \lambda u = v, \quad \lambda v + \Delta^2 u + \Delta \theta = 0, \quad \lambda \theta - \Delta v - \Delta \theta = 0 \quad \text{in } \Omega, \quad u = \partial_{\nu} u = v = \theta = 0 \quad \text{on } \Gamma. \] \hspace{1cm} (92)
We first consider the case where \( 2 \leq q < \infty \).

Since \( \Omega \) is bounded, \( U \in \mathbf{H}^2_2(\Omega) \subset \mathbf{H}^3_2(\Omega) \), and so multiplying Eq. (91) with \( U \), integrating the resultant formula, and using \( \lambda u = v \), we have
\[ 0 = (\lambda u, u)_\Omega + (\lambda v + \Delta^2 u + \Delta \theta, v)_\Omega + (\lambda \theta - \Delta v - \Delta \theta, \theta)_\Omega \]
\[ = \lambda \|u\|_{L^2_2(\Omega)}^2 + \lambda \|v\|_{L^2_2(\Omega)}^2 + \lambda \|	heta\|_{L^2_2(\Omega)}^2 + \lambda \|
abla(\Delta^2 u + \Delta \theta, u)_\Omega - (\lambda \Delta u + \Delta \theta, \theta)_\Omega \]
\[ = \lambda \|u\|_{L^2_2(\Omega)}^2 + \lambda \|v\|_{L^2_2(\Omega)}^2 + \lambda \|	heta\|_{L^2_2(\Omega)}^2 + \lambda \|
abla u\|_{L^2_2(\Omega)}^2 + \|
abla \theta\|_{L^2_2(\Omega)}^2 \]
\[ + \langle \nabla \theta, u \rangle_{\Omega} - \lambda \langle u, \theta \rangle_{\Omega} \]
Since \( \text{Re} \lambda = \text{Re} \lambda \geq 0 \) and \( \langle \nabla \theta, u \rangle_{\Omega} - \lambda \langle u, \theta \rangle_{\Omega} = 2i \text{Im} \lambda \langle \Delta \theta, u \rangle_{\Omega} \), taking the real part we have \( \|
abla \theta\|_{L^2_2(\Omega)}^2 = 0 \), which yields that \( \theta \) is a constant. But, \( \theta = 0 \) on
\(\Gamma\), and so \(\theta = 0\). We next use the third equation in Eq. (92), and then \(\Delta v = 0\) in \(\Omega\). Since \(v = 0\) on \(\Gamma\), we have \(0 = (\Delta v, v)_{\Omega} = -\|\nabla v\|_{L_2(\Omega)}^2\), which yields that \(v\) is a constant. But, \(v = 0\) on \(\Gamma\), and so \(v = 0\). Finally, by the second equation in (92), we have \(\Delta^2 u = 0\) with \(u = \partial_n u = 0\) on \(\Gamma\), which yields that \(\Delta u = 0\) with \(u = \partial_n u = 0\) on \(\Gamma\). In fact, \(0 = (\Delta^2 u, w)_{\Omega} = \|\Delta u\|_{L_2(\Omega)}^2\). Thus, we have \(u = 0\). Namely, we have the uniqueness, which implies the unique existence theorem of Eq. (88).

We next consider the case where \(1 < q < 2\). We first consider the case where \(\lambda \neq 0\). By (92), \(v = \lambda^{-1} u \in H^4_q(\Omega)\) with \(v = \partial_n v = 0\) on \(\Gamma\) and \(\theta \in H^2_q(\Omega)\) with \(\theta = 0\) on \(\Gamma\), and moreover \(v\) and \(\theta\) satisfy the equations:

\[
\lambda^2 v + \Delta^2 v + \lambda \Delta \theta = 0, \quad \lambda \theta - \Delta v - \Delta \theta = 0 \quad \text{in } \Omega.
\]

Thus, by the divergence theorem of Gauss, for any \(w_1 \in H^4_q(\Omega)\) with \(w_1 = \partial_n w_1 = 0\) on \(\Gamma\) and \(w_3 \in H^2_q(\Omega)\) with \(w_3 = 0\) on \(\Gamma\) we have

\[
0 = (\lambda^2 v + \Delta^2 v + \lambda \Delta \theta, w_1)_{\Omega} - (\lambda \theta - \Delta v - \Delta \theta, w_3)_{\Omega}
\]

\[
= (v, \nabla^2 w_1 + \Delta^2 w_1)_{\Omega} + (\theta, \nabla \Delta w_1)_{\Omega} - (\theta, \nabla w_1 - \Delta w_3)_{\Omega} + (v, \Delta w_3)_{\Omega}
\]

\[
= (v, \nabla^2 w_1 + \Delta^2 w_1 + \Delta w_3)_{\Omega} - (\theta, \nabla w_3 - \nabla \Delta w_1 - \Delta w_2)_{\Omega}.
\]

Given \(f_1, f_2 \in L_q(\Omega)\), let \(W = (w_1, w_2, w_3)^T \in H^2_q(\Omega)\) be a solution of the equations:

\[
\nabla W - A(D)W = (0, f_1, -f_2)^T \quad \text{in } \Omega, \quad B(D)W|_{\Gamma} = 0.
\]

(94)

Since \(2 < q' < \infty\), the existence of such \(W\) is guaranteed. If we write (94) componentwise, we have

\[
\nabla^2 \Delta w_1 + \Delta^2 w_1 + \Delta w_3 = f_1, \quad \nabla w_3 - \nabla \Delta w_1 - \Delta w_3 = -f_2 \quad \text{in } \Omega,
\]

\[
w_1 = \partial_n w_1 = w_3 = 0 \quad \text{on } \Gamma,
\]

because \(\nabla w_1 = w_2\). Combining (93) and (94) gives

\[
(v, f_1)_{\Omega} + (\theta, f_2)_{\Omega} = 0.
\]

Since \(f_1\) and \(f_2\) are chosen arbitrarily, we have \(v = \theta = 0\). Since \(\lambda \neq 0\), \(\lambda u = v\) implies that \(u = 0\). Thus, we have the uniqueness in the case where \(\lambda \neq 0\).

We next consider the case where \(\lambda = 0\). From (92) it follows that \(v = 0\). And then, we have \(\Delta^2 u + \Delta \theta = 0\) and \(-\Delta \theta = 0\). Since \(\Delta \theta = 0\) on \(\Omega\) and \(\theta = 0\) on \(\Gamma\), we have \(\theta = 0\). Thus, \(\Delta^2 u = 0\) in \(\Omega\), which, combined with the boundary conditions \(u = \partial_n u = 0\) on \(\Gamma\), leads to \(u = 0\).

Summing up, we have proved the unique existence theorem of Eq. (88) for each \(\lambda \in \Lambda_{\lambda_0}\). Let \(A\) be an operator defined by letting

\[
D_q = \{U = (u, v, \theta)^T \in H^2_q(\Omega) \mid u = \partial_n u = v = \theta = 0 \quad \text{on } \Gamma\},
\]

\[
AU = A(D)U \quad \text{for } U \in D_q.
\]

Then, we see that for any \(\lambda \in \Lambda_{\lambda_0}\), \((\lambda I - A)^{-1}\) exists. By the Banach fixed point theorem, there exists a constant \(C_\lambda\) for which

\[
\|(\lambda I - A)^{-1}F\|_{H^2_q(\Omega)} \leq C_\lambda \|F\|_{H^2_q(\Omega)}.
\]

And then, employing the same argument as in the proof of Theorem 2.4, we see that there exists a constant \(C\) independent of \(\lambda \in \Lambda_{\lambda_0}\) for which

\[
\|(\lambda I - A)^{-1}F\|_{H^2_q(\Omega)} \leq C \|F\|_{H^2_q(\Omega)}.
\]

This completes the proof of Proposition 3. \(\square\)
Let $\mathcal{A}$ be the operator defined in (95). Combining (89) and Proposition 3, we see that $\{T(t)\}_{t \geq 0}$ is exponentially stable, that is there exist positive constants $C$ and $\eta$ for which

$$\|T(t)F\|_{\mathcal{H}_q^p(\Omega)} \leq Ce^{-\eta t}\|F\|_{\mathcal{H}_q^p(\Omega)}$$  \hspace{1cm} (96)

for any $t > 0$ and $F \in \mathcal{H}_q^p(\Omega)$.

We now consider the evolution equations (14). Replacing $\alpha$ by 1, we know that Theorem 1.4 and Theorem 2.1 holds for the operator $A(D)$. In particular, we know the existence of solution $U$ of Eq. (14) with

$$U \in (L_{p,\text{loc}}((0,T), H_\delta^2(\Omega)) \cap H_\delta^1((0,T), \mathcal{H}_q^0(\Omega)))$$

for any initial data $U_0 \in D_{q,p}(\Omega)$ and $F \in L_p(\mathbb{R}, \mathcal{H}_q^0(\Omega))$. By Duhamel’s principle, the solution $U = U(t)$ of Eq. (14) is written as

$$U(t) = T(t)U_0 + \int_0^t T(t-s)F(s)\,ds,$$  \hspace{1cm} (97)

and so for any $\eta_1 \in (0, \eta)$, by (96) and Hölder’s inequality we have

$$\|e^{\eta_1t}U(t)\|_{\mathcal{H}_q^p(\Omega)} \leq e^{\eta_1t}\|T(t)U_0\|_{\mathcal{H}_q^p(\Omega)} + \int_0^t e^{-(\eta-\eta_1)(t-s)}e^{\eta_1s}\|F(s)\|_{\mathcal{H}_q^p(\Omega)}\,ds$$

$$\leq C\|U_0\|_{\mathcal{H}_q^p(\Omega)}$$

$$+ \left( \int_0^t e^{-(\eta-\eta_1)(t-s)}ds \right)^{1/p'} \left( \int_0^t e^{-(\eta-\eta_1)(t-s)}(e^{\eta_1s}\|F(s)\|_{\mathcal{H}_q^p(\Omega)})^p\,ds \right)^{1/p}.$$  \hspace{1cm} (98)

Thus, by the change of integration order, we have

$$\int_0^\infty \|e^{\eta_1t}U(t)\|_{\mathcal{H}_q^p(\Omega)}^p\,dt \leq 2^p \int_0^\infty e^{-(\eta-\eta_1)\eta_1}\,dt\|U_0\|_{\mathcal{H}_q^p(\Omega)}^p$$

$$+ 2^p(\eta-\eta_1)^{-p/p'} \int_0^\infty e^{\eta_1s}\|F(s)\|_{\mathcal{H}_q^p(\Omega)}^p\,ds \int_s^\infty e^{-(\eta-\eta_1)(t-s)}\,dt$$

$$\leq C_p \left\{ \|U_0\|_{\mathcal{H}_q^p(\Omega)} + \int_0^\infty e^{\eta_1s}\|F(s)\|_{\mathcal{H}_q^p(\Omega)}^p\,ds \right\}.$$  \hspace{1cm} (99)

Let $\lambda_1 > 0$ be a large number, and we consider the evolution equation:

$$\partial_t U + \lambda_1 U - A(D) U = \lambda_1 U + F \quad \text{in } \Omega \times \mathbb{R}_+, \quad B(D)U|_{\Gamma} = 0, \quad U|_{t=0} = U_0. \hspace{1cm} (100)$$

Multiplying (100) with $e^{\eta_1t}$, we have

$$\partial_t(e^{\eta_1t}U) + (\lambda_1 - \eta_1) e^{\eta_1t}U - A(D)e^{\eta_1t}U = \lambda_1 e^{\eta_1t}U + e^{\eta_1t}F \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$B(D)e^{\eta_1t}U|_{\Gamma} = 0, \quad e^{\eta_1t}U|_{t=0} = U_0. \hspace{1cm} (101)$$

In view of Theorem 2.1, choosing $\lambda_1 - \eta_1$ large enough, we have

$$\|e^{\eta_1t}U\|_{L_p(\mathbb{R}_+, \mathcal{H}_q^p(\Omega))} + \|e^{\eta_1t}\partial_t U\|_{L_p(\mathbb{R}_+, \mathcal{H}_q^p(\Omega))}$$

$$\leq C(\|e^{\eta_1t}U\|_{L_p(\mathbb{R}_+, \mathcal{H}_q^p(\Omega))} + \|e^{\eta_1t}F\|_{L_p(\mathbb{R}_+, \mathcal{H}_q^p(\Omega))} + \|U_0\|_{B^{2+2(1-1/p)}_{p,p}(\Omega) \times B^{2(1-1/p)}_{p,p}(\Omega))}).$$  \hspace{1cm} (102)

This completes the proof of Theorem 1.5.
4.2. A proof of Theorem 1.3. To solve Eq. (6), we use the Banach fixed point theorem, and so we introduce the underlying space $I_\varepsilon$ defined by letting

$$I_\varepsilon = \{ (u, v, \theta)^\top \mid U \in L_p(\mathbb{R}^+, H^2_q(\Omega)) \cap H^1_p(\mathbb{R}^+, H^0_q(\Omega)), \quad U|_{t=0} = U_0, \quad \| U\|_{E_{p,q}} \leq \varepsilon \}.$$ (103)

Here, $\varepsilon$ is a small positive number determined latter, and

$$\|U\|_{E_{p,q}} = \| e^{\eta t} U \|_{L_p(\mathbb{R}^+, H^2_q(\Omega))} + \| e^{\eta t} \partial_t U \|_{L_p(\mathbb{R}^+, H^0_q(\Omega))},$$

$\eta$ being the same positive constant as in (102). Let $\sigma > 0$ be also a small positive number and we assume that

$$\|U\|_{E_{p,q}} \leq C_0 \varepsilon.$$ (105)

By real interpolation, we have

$$\|e^{\eta t} U\|_{L_p(\mathbb{R}^+, H^2_q(\Omega))} \leq C_0 \|U\|_{E_{p,q}} \leq C_0 \varepsilon \quad \text{for some constant } C_0 \text{ independent of } \varepsilon.$$ (106)

Since $N < q < \infty$, by Sobolev’s imbedding theorem and (105), we have

$$\|e^{\eta t} U\|_{L_p(\mathbb{R}^+, H^2_q(\Omega))} \leq C_1 \varepsilon.$$ (107)

Thus, by (103) and (107)

$$\|e^{\eta t} \phi'((\Delta u)) \Delta^2 u\|_{L_p(\mathbb{R}^+, L_q(\Omega))} \leq C_2 \left( \sup_{|s| \leq 1} |\phi''(s)| \right) \varepsilon^2.$$ (108)

Therefore, we have

$$\|e^{\eta t} G(U)\|_{L_p(\mathbb{R}^+, H^0_q(\Omega))} \leq \|e^{\eta t} (f_1, f_2)\|_{L_p(\mathbb{R}^+, L_q(\Omega))^2} + C_3 b \left( \sup_{|s| \leq 1} |\phi''(s)| \right) \varepsilon^2 \leq \sigma + C_3 b \left( \sup_{|s| \leq 1} |\phi''(s)| \right) \varepsilon^2.$$ (109)

Applying (102), we have $W \in L_p(\mathbb{R}^+, H^2_q(\Omega)) \cap H^1_p(\mathbb{R}^+, H^0_q(\Omega))$ and

$$\|W\|_{E_{p,q}} \leq C_4 \left( \sup_{|s| \leq 1} |\phi''(s)| \right) \varepsilon^2.$$ (110)
Thus, choosing $\sigma$ and $\epsilon$ so small that $C_6 \sigma \leq \epsilon/2$ and $C_6 b(\sup_{|s| \leq 1} |\phi''(s)|) \epsilon \leq 1/2$, by (108) we have $\|W\|_{L^p,q} \leq \epsilon$. If we define a map $\Phi$ by $\Phi U = W$, then $\Phi$ maps $I_\epsilon$ into $I_\epsilon$.

Let $U_i = (u_i, v_i, \theta_i)^T \in I_\epsilon$, $(i = 1, 2)$, and then
\[ G(U_1) - G(U_2) = (0, -g(u_1, u_2), 0)^T, \]
where we have set
\[ g(u_1, u_2) = \phi'(\Delta u_2)\Delta^2 u_2 + \phi''(\Delta u_2)|\nabla \Delta u_2|^2 - (\phi'(\Delta u_1)\Delta^2 u_1 + \phi''(\Delta u_1)|\nabla \Delta u_1|^2). \]
Writing $g(u_1, u_2)$ as
\[ g(u_1, u_2) = \int_0^1 \phi''(\Delta u_1 + \theta \Delta (u_2 - u_1)) d\theta \Delta (u_2 - u_1) \Delta^2 u_2 \]
\[ + \int_0^1 \phi''(\theta \Delta u_1) d\theta \Delta u_2 \Delta^2 u_1 \]
\[ + (\phi''(\Delta u_2) - \phi''(\Delta u_1))|\nabla \Delta u_2|^2 \]
\[ + \phi''(\Delta u_1)|\nabla (u_2 + u_1)\cdot \nabla (u_2 - u_1), \]
we have by $\phi \in C^3(\mathbb{R})$
\[ \| e^{\eta t}(G(U_1) - G(U_2)) \|_{L^p(\mathbb{R}_+, H^3_q(\Omega))} \]
\[ \leq C_6 b(\sup_{|s| \leq 1} |\phi''(s)| + \phi_L) \epsilon \| e^{\eta t}(u_1 - u_2) \|_{L^p(\mathbb{R}_+, H^3_q(\Omega))}. \]

Here, $\phi_L$ is the Lipschitz constant of $\phi''$. Since $W_i = \Phi U_i$ satisfies the equations:
\[ \partial_t (W_1 - W_2) - A(D)(W_1 - W_2) = G(U_1) - G(U_2) \text{ in } \Omega \times \mathbb{R}_+, \]
\[ B(D)(W_1 - W_2)|_{\Gamma} = 0, \quad W_1 - W_2|_{t=0} = 0, \]
applying (102), we have
\[ \| \Phi U_1 - \Phi U_2 \|_{L^p,q} \leq C_7 b(\sup_{|s| \leq 1} |\phi''(s)| + \phi_L) \epsilon \]
\[ \times \| e^{\eta t}(u_1 - u_2) \|_{L^p(\mathbb{R}_+, H^3_q(\Omega))}. \]

Choosing $\epsilon > 0$ so small that
\[ C_7 b(\sup_{|s| \leq 1} |\phi''(s)| + \phi_L) \epsilon \leq 1/2, \]
by (109) we see that $\Phi$ is a contraction map on $I_\epsilon$. Thus, by the Banach fixed point theorem, there exists a unique $U \in I_\epsilon$ for which $\Phi U = U$. Thus, $U$ is a unique solution of Eq. (6), which completes the proof of Theorem 1.3.

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