A model of GAM intermittency near critical gradient in toroidal plasmas

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Abstract. We have constructed a four-field minimal model that describes the growing intermittency of turbulence associated with the geodesic acoustic mode (GAM) observed in our toroidal Landau-fluid simulations [K Miki et al. 2007 Phys. Rev. Lett. 99, 145003]. The intermittent dynamics are well reproduced by the model for the reference parameters used in the simulation. The model can also reproduce characteristics of turbulent transport associated with the GAM, such as a single burst leading to a full quench of turbulence and also a steady state turbulence mixed with steady zonal flows and GAMs. Investigating the behaviour of the solution trajectories around the fixed points in four-dimensional phase space, we study the comprehensive properties of the model and identify the bifurcation property between Dimits shift and steady state turbulence regimes, which correspond to different eigen-states.

1. Introduction

The transport near the critical gradient is of great interest in understanding various transition dynamics such as L-H transition and formation of internal transport barriers in magnetic fusion plasmas [1]. It is known that near the critical gradient, multiple states dominated by different physical processes coexist, such as turbulent plasma with high transport and quiescent flowing plasma with high confinement. A typical phenomenon around the critical gradient is so-called critical slowing down, representing a concept that the time scale of phenomena is prolonged in order for the system to determine a state in such multiple states [2]. Therefore, small perturbation and/or dissipation can significantly influence the transition dynamics due to the marginal nature of the system. A nonlinear up-shift of the critical gradient, referred to as the Dimits shift, due to the quench of turbulent fluctuation by self-generated zonal flows is one of typical examples [3]. An intermittent dynamics has been observed near the critical gradient due to a collisional damping of zonal flows in ion temperature gradient (ITG) driven turbulence [4]. A theoretical model which primarily describes the intermittency, has been extensively studied in terms of predator-prey approach [5]. A dynamical systems approach which predicts the Dimits shift is also proposed in slab geometry [6].

Recently, geodesic acoustic mode (GAM) [7], which is an oscillatory counterpart of zonal flow in toroidal plasmas coupled with anisotropic pressure perturbation of \((m, n) = (1, 0)\), has been extensively studied, since it significantly changes the characteristics of zonal flows and then the transport [8,9], where \(m\) and \(n\) denote poloidal and toroidal wave numbers. In our previous studies [10], we found that
the GAM dynamics depends sensitively on the magnetic safety factor q-value, so that GAMS are preferentially excited in high-q edge region, whereas stationary zonal flow dominates in low-q core region. Such a dependency on q-value is consistent with recent experiments in which GAMS are observed near the edge of plasmas [11]. It is noted that those studies are performed only far from the critical gradient region.

An alternative concern is the role of such GAMS on turbulent transport near the critical gradient regime, which is closely related to the trigger of plasma instability and transition dynamics. As a typical phenomenon, the nonlinear upshift (Dimits shift) of linear critical gradient was investigated in plasmas dominated by low frequency zonal flow due to low q-value near the magnetic axis (q = 1). However the effect of high frequency zonal flow, which is equivalent to the GAMS, on the critical gradient phenomena has not been addressed so far, even in our previous studies [10].

Recently we have studied this problem based on the global toroidal Landau-fluid simulations [10] by choosing the temperature gradient, i.e. $R/L_T$, slightly larger than the linear instability threshold [12]. We have found a prominent intermittency associated with GAM dynamics near the critical gradient region, which is qualitatively different from that originated from the collisional damping [4]. In particular, we have found that during the intermittent bursts, stationary zonal flow increases with a slow time scale due to the accumulation of undamped residues. We refer to this phenomenon as growing intermittency. Similar intermittent transport phenomena due to GAMS are also observed in experiments [13,14,15].

We have proposed a four-field model consisting of four-differential equations which govern the temporal evolution of turbulence and zonal flow energies, and also anisotropic pressure perturbation and ion parallel sound velocity in order to understand the complex features as seen in figure 1 [12]. The numerical simulation based on the model successfully reproduced the growing GAM intermittency. However, contrary to the conventional two field model only dealing with the turbulence and zonal flow energy, the characteristics of the four-field model have not been investigated in detail.

In this paper, we investigate the details of the four field model as a dynamical system and explore the intrinsic potential of wider physical processes which involve the dynamics of the single burst leading to a full quench of turbulence and also a steady state turbulence mixed with zonal flows/GAMS observed in the simulation. Here, we introduce a fixed point stability analysis which is one of the basic approaches to investigate nonlinear systems [16]. Since it is not simple to derive the analytical solution directly from the present four-field model, we investigate the solution trajectories around the fixed point incorporated with an analysis on eigen-value and corresponding eigen-vector to get some insights for the complex GAM dynamics near the critical gradient. It is expected that the fixed point analysis may predict the bifurcation of the states such as in Dimits shift regime and in the steady state regime above the critical gradient due to the change of governing parameters.

The remainder of this paper is organized as follows. In section 2, we propose the extended predator-prey model. Some numerical results and analyses are shown in section 3. In section 4, the fixed point
analysis is introduced to investigate detailed properties around specific fixed points in the extended predator-prey model. The transition of eigen-states due to the change of parameters, especially the backflow from ion sound wave is also discussed. In section 5, we make some concluding remarks.

2. An extended predator-prey model

Here, we present a theoretical model to reproduce the various phenomena observed in the simulation and to understand the underlying physical processes. In the framework of a fluid model [10], keeping the total energy conservation in mind, we have obtained the equation system of GAM dynamics with respect to zonal flow velocity \( \langle v_{Ez} \rangle \), anisotropic pressure perturbation \( \langle p \sin \theta \rangle = p_{10} \), which corresponds to the GAM, and anisotropic ion sound velocity \( \langle v_{||} \cos \theta \rangle = v_{i0} \), as follows:

\[
\frac{\partial \langle v_{Ez} \rangle}{\partial t} = -\langle \tilde{v}_{Ez} \tilde{\Omega} \rangle - \frac{2a}{n_{eq} R} \langle p \sin \theta \rangle, \tag{1}
\]

\[
\frac{\partial \langle p \sin \theta \rangle}{\partial t} = -\langle \tilde{p} \sin \theta \rangle + (\Gamma + \tau) p_{eq} a R \langle v_{Ez} \rangle + (\Gamma + \tau) p_{eq} a qR \langle v_{||} \cos \theta \rangle - (\Gamma - 1) n_{eq} \frac{8T_{eq}}{\pi} a qR \langle T_{s} \sin \theta \rangle \tag{2}
\]

\[
\frac{\partial \langle v_{||} \cos \theta \rangle}{\partial t} = -\langle \tilde{\phi} \tilde{v}_{||} \rangle \cos \theta - \frac{a}{n_{eq} qR} \langle p \sin \theta \rangle. \tag{3}
\]

where \( \langle \rangle \) denotes the flux average, and where \( a \) and \( R \) denote the minor and major radii of the tokamak plasma, respectively. Further, \( n_{eq}, p_{eq}, \) and \( T_{eq} \) represent the density, pressure and temperature profiles, which are fixed in the present work. Note that we neglected high \( m \) GAM components in equations (2)-(3). Here, we used the following normalization and definition of the variables:

\[
(t, U, G, V) \rightarrow \left( t/\sqrt{T_{eq}}, \langle v_{Ez} \rangle, \langle p \sin \theta \rangle/\sqrt{T_{eq} p_{eq}}, \langle v_{||} \cos \theta \rangle \right).
\]

Equations (1)-(3) are then written as follows

\[
\frac{\partial U}{\partial t} = -\beta G + \tilde{N}_{\{\phi, \phi\}}, \tag{4}
\]

\[
\frac{\partial G}{\partial t} = \beta' U - \gamma_{LD} G + \beta_{2} V + \tilde{N}_{\{\phi, V\}}, \tag{5}
\]

\[
\frac{\partial V}{\partial t} = -\beta_{2} G + \tilde{N}_{\{\phi, v_{||}\}}, \tag{6}
\]

where \( \beta = 2a / R, \beta' = (\Gamma + \tau) a / R, \beta_{2} = (\Gamma + \tau) a / qR, \beta_{2} = a / qR \), and the Landau damping rate \( \gamma_{LD} = T_{eq} \sqrt{8/\pi(a / qR)} \). The nonlinear terms are given by \( \tilde{N}_{\{\phi, \phi\}} = -\langle \tilde{v}_{Ez} \tilde{\Omega} \rangle, \tilde{N}_{\{\phi, V\}} = -\langle \tilde{\phi} \tilde{p} \sin \theta \rangle / p_{eq} \), and \( \tilde{N}_{\{\phi, v_{||}\}} = -\langle \tilde{\phi} \tilde{v}_{||} \rangle \cos \theta \). \( T_{eq} = 1 \) is assumed for simplicity. At first, we analyze the basic feature of the growing intermittency based on equations (4)-(6). Equations (4)-(6) can be expressed as the following second order differential equation with respect to \( G \):

\[
\frac{\partial^2 G}{\partial t^2} + \gamma_{LD} \frac{\partial G}{\partial t} + \omega_{G}^2 G = 0, \tag{7}
\]
where \( \omega_G = \sqrt{\omega_{GAM}^2 + \omega_{sound}^2} = \sqrt{\beta \beta' + \beta G^2} \). It is noted that the nonlinear terms work as the sources to trigger the GAM dynamics. Meanwhile, instead of considering those nonlinear terms, here we simply introduce the effect of the sources as initial conditions for \((U, G, V)\), e.g., \(U(t=0) = U_0, G(t=0) = G_0\), and \(V(t=0) = V_0\). Then equation (7) is solved as follows:

\[
\begin{align*}
U &= A_U \exp(-\gamma_{LD} t/2) \sin(\omega_U t + \delta_U) + C_U, \quad (8) \\
G &= A_G \exp(-\gamma_{LD} t/2) \sin(\omega_G t + \delta_G), \quad (9) \\
V &= A_V \exp(-\gamma_{LD} t/2) \sin(\omega_V t + \delta_V) + C_V, \quad (10)
\end{align*}
\]

where \( \omega_j = \sqrt{\omega_{GAM}^2 - (\gamma_{LD}/2)^2} \), and \((A_U, A_G, A_V)\) and \((C_U, C_V)\) are constants determined by the initial conditions. Here, \(C_U\) and \(C_V\) are expressed \(C_U = (U_0 - 2qV_0)/(1 + 2q^2)\) and \(C_V = -qC_U\). \(\delta_U\) and \(\delta_G\) are the phase factors also determined by initial conditions and \(\Delta \delta = \delta_U - \delta_G = \pi/2\) is estimated. As seen in equations (8)-(10), each variable exhibits a damping oscillation ascribed from the GAM oscillation and decay due to the Landau damping. It is noticed that the undamped residual part can be survived in \(U\) and \(V\) as a stationary zonal flow and ion parallel velocity after the complete damping of \(G\), which balance in accordance with \( \beta' C_U + \beta_2 C_V = 0 \). Therefore, it is expected that the residual part of the zonal flow accumulates during the repetition of the burst and gradually increases as seen in figure 1.

In order to reproduce the intermittent dynamics in a self-consistent manner, an analysis including the nonlinear terms incorporated with the evolution equation of turbulence field is necessary. By defining the turbulent energy as \(N = \sum_{m,n} |\tilde{\phi}_{(m,n)}|^2\), we construct the following four-field equation system as a predator-prey model to represent the possible GAM dynamics:

\[
\begin{align*}
\frac{\partial N}{\partial t} &= \gamma_{NL} N - \gamma_{NL} N^2 - M_{NL} U^2 N - M_{NL} V^2 N - \frac{\beta}{\beta'} G^2 N - \frac{\beta_2}{\beta_2'} V^2 N, \quad (11) \\
\frac{\partial U}{\partial t} &= \frac{1}{2} \alpha_U U N - \beta G - \frac{1}{2} \gamma_{damp} U, \quad (12) \\
\frac{\partial G}{\partial t} &= \frac{1}{2} \alpha_G G N + \beta' G + \beta_2 G - \gamma_{LD} G, \quad (13) \\
\frac{\partial V}{\partial t} &= \frac{1}{2} \alpha_V V N - \beta_2 V, \quad (14)
\end{align*}
\]

where \( \gamma_{NL} \) and \( \gamma_{NL} \) are the growth rate of the turbulent fluctuation and the nonlinear damping rate, respectively. \( \gamma_{damp} \) is the collisional damping rate of the zonal flow. Note here that contrary to the conventional predator-prey model where the zonal flows are expressed in the unit of energy, e.g. \(U^2\), equations (12)-(14) are not described in terms of \(U^2\), \(G^2\), and \(V^2\), but in terms of \(U\), \(G\), and \(V\), in order to represent the oscillatory characteristics of the GAM and ion parallel sound wave. \((\alpha_u, \alpha_G, \alpha_V)\) represent the strength of nonlinear coupling including the cross-coupling [17] among the zonal flows, GAMs and sound waves as well as the directions of energy flow from the turbulence \(N\) to the zonal flow \(U\), pressure perturbation \(G\), and ion parallel sound velocity \(V\), respectively. In this minimal model, it is assumed that the nonlinear cross-coupling, i.e., the off-diagonal terms in equations (12)-(14) have been effectively absorbed in the diagonal terms for simplicity (see Appendix A). The collisional damping effect on the zonal flow is neglected to manifest the effect of GAM damping. Note that this model is constructed so as to satisfy the total energy conservation in the absence of linear driving source and damping terms. Namely, the energy equation is written as follows:
\[
\frac{\partial}{\partial t} (N + E_U + E_G + E_V) = \gamma_L N - \gamma_{NL} N^2 - \gamma_{damp} E_U - 2\gamma_{LD} E_G,
\]

where \(E_U = U^2\), \(E_G = (\beta / \beta') G^2\), and \(E_V = (\beta_2 / \beta' \beta'_2) V^2\).

### 3. Numerical analysis of four-field equation system

In the analyses of the predator-prey model, we employ the parameters as the following, \(\gamma_L = 0.05\), \(\gamma_{NL} = 0.003\), \(\gamma_{damp} = 0\), and \(q = 2.10\), which roughly follow those used in the simulation [12]. The dynamics is found to show different features depending on the relative value of \(\alpha_1\), \(\alpha_2\) and \(\alpha_3\), which characterizes the nonlinear energy flow from turbulence \((N)\) to zonal flow \((U)\), pressure perturbation \((G)\), and ion parallel sound velocity \((V)\). In the following, we show three cases.

#### 3.1 Case 1 \((\alpha_1 < \alpha_2)\):

Here, we choose \(\alpha_1 = 0.05\), \(\alpha_2 = 0.5\) and \(\alpha_3 = 10^{-4}\), respectively. This physically corresponds to the case in which the Reynolds stress is less dominant than the nonlinear coupling term between turbulence and pressure perturbation. The temporal evolutions of \((N, U, G)\) are shown in figure 2 and basic features of the GAM intermittency observed in figure 1, such as quasi-period bursts, gradual increase of zonal flow and quenching of turbulence, are found to be qualitatively reproduced. Namely, the accumulation of undamped residue discussed in section 2 is confirmed. Figure 3 shows the details in a specific time interval of a burst in figure 2. As turbulence energy \(N\) reaches a critical value that satisfies the relation \(\gamma_L - \gamma_{NL} N = E_\alpha = \alpha_1 E_U + \alpha_2 E_G + \alpha_3 E_V\), the abrupt growth of \(E_U\), \(E_G\), and \(E_V\) and successive damping accompanied with typical GAM oscillation are observed as seen in figure 3(b). The turbulence also suffers damping once the relation \(\gamma_L - \gamma_{NL} N < E_\alpha\) is satisfied at \(t \approx 1190\) as seen in figure 3(a). Followed by the turbulence, GAM damping starts after the relation \(\gamma_{LD} > \alpha_2 N/2\) is satisfied as seen in figure 3(c).

**Figure 2.** Time evolution of \(N\), \(E_U\), and \(E_G\) in the numerical calculation with the extended predator-prey model during the time \((2700 < t < 5000)\) in the case 1.

**Figure 3.** (a): Temporal evolution of \(N\), \(E_U\), and \(E_G\) for the case of the extended minimum model in the time interval of one burst \((1100 < t < 1250)\). (b): Temporal evolution of \((\gamma_L - \gamma_{NL} N)\) and \(E_\alpha\). (c): Temporal evolution of \(\alpha_2 N/2\) and \(\gamma_{LD}\).
The characteristics of the abrupt growth of $G$ are investigated by solving equation (13) under the condition $\beta U + \beta_3 V = 0$. Here it is assumed that $U$ and $V$ are almost balanced, and $N = N_0 \exp(\gamma_L t)$ is satisfied where $N_0$ is a reference amplitude when $\partial G/\partial t = 0$, i.e. $\gamma_L = \alpha_2 N_0/2$ is fulfilled. The solution is approximately given by

$$G = G_0 \exp\left(\frac{\gamma_L}{\gamma_L} e^{\gamma_L t} - \gamma_L t\right) = G_0 \exp(\gamma_L t^2 / 2),$$

showing faster growth than that of linear instability. The time scale of the abrupt growth is evaluated as $\tau_b \sim \sqrt{2/\gamma_L \gamma_{LD}}$. For the present parameters, $\tau_b \sim 17.5$ is measured, which is consistent with that observed in figure 3. It is noted that the typical damping rate of $N$ is characterized by $E_u$ after the condition $\gamma_L - \gamma_{NL} N < E_u$ is satisfied. The damping of $N$ is ceased when $E_u$ crosses again $\gamma_L - \gamma_{NL} N$ as seen in figure 3(b) at $t \sim 1212$, and then $N$ once again starts to grow linearly. During the phase of Landau damping, $E_0$ is quenched while certain equilibrium levels of $E_u$ and $E_\nu$ are sustained, which are described as $C_u$ and $C_\nu$, respectively. This process leads to the accumulation of zonal flows as shown in figure 1.

3.2 Case 2 ($\alpha_1 > \alpha_2$): Contrary to the case 1, here, we choose $\alpha_2 = 0.5$, $\alpha_2 = 0.05$, and $\alpha_3 = 10^4$, respectively. The result is shown in figure 4. In this case, the Reynolds stress is dominant compared with the nonlinear coupling between turbulence and $p_{10}$ pressure perturbation. As seen in figure 4, after the single burst of the turbulence occurs, the residual zonal flow immediately quenches both turbulence and $p_{10}$ pressure perturbation and causes the decoupling of zonal flow and GAM. This may correspond to the single burst nature observed in the fluid simulation [12], in which the turbulence is completely suppressed by the generated zonal flow.

As seen in above cases 1 and 2, when $\alpha_2 > \alpha_1$ is satisfied, multiple bursts are exhibited successively, whereas the dynamics is dominated by a single burst when $\alpha_1 > \alpha_2$. The change of the dynamics between case 1 and case 2 is found to take place at $\alpha_1 \approx \alpha_2$. This suggests that a certain amount of energy flow directly from turbulence to $p_{10}$ pressure perturbation may be necessary to have the growing intermittency as seen in figure 2. The underlying physical mechanism that the relative relation between $\alpha_1$ and $\alpha_2$ provides different dynamical features is considered as follows. As discussed in section 3, the saturation of turbulence takes place when the suppression term $- E_\nu = - (\alpha_1 E_u + \alpha_2 E_G + \alpha_3 E_\nu)$ overwhelms the linear driving term $\gamma_L$ in equation (11) in both case 1 and 2. Namely, both $\alpha_1$ and $\alpha_2$ (and/or $\alpha_1 E_u$ and $\alpha_2 E_G$) decide the saturation level of turbulence $N$, i.e. maximum level of turbulence, as a whole. On the other hand, the contribution of GAM in $E_\nu$, i.e.

![Figure 4. Time evolution of $N$, $E_u$ and $E_G$ in case of $\alpha_1$ is dominant (case 2).](image-url)
Due to this reason, the turbulence grows again after the quench in Case 1 ($\alpha_1 < \alpha_2$) leading to a multiple-burst intermittency since $\alpha_1 E_G$ is damped. On the other hand, in Case 2 ($\alpha_2 > \alpha_1$), the turbulence is suppressed mainly by the undamped zonal flow, i.e. $\alpha_1 E_U$ which does not suffer damping. Namely, in Case 2, once the quench of turbulence takes place, next burst does not happen. This feature is consistent to the analysis performed in section 2 (see equations (8)-(10)) that the residual components $C_U$ and $C_V$ do not depend on the initial condition $G_0$ which is equivalent to the energy inflow from the turbulence to $p_{in}$ pressure perturbation through nonlinear coupling. This supports the observation that $\alpha_2$ does not contribute the accumulation of undamped residues. Thus, it is found that the relative relation between $\alpha_1$ and $\alpha_2$ determine, “the quality of saturation” and subsequent dynamics.

### 3.3 Case 3

This case corresponds to the Case 1 for $\alpha_1$ and $\alpha_2$, but a negative value $\alpha_3 = -0.05$ is chosen. In this case, the turbulence is not completely quenched, but approaches to a steady state level gradually as shown in figure 5(a). The other components such as $U$, $G$, and $V$ show an oscillatory feature which corresponds to GAM oscillation. It is found that the phase shift between $U$ and $V$ is almost the same and that between $U$ and $G$ show the difference nearly by $\pi/2$, which is consistent with the analytical result obtained in equations (8)-(10). Enough energy backflow from ion parallel sound velocity to turbulence can sustain steady state turbulent level against the suppression effect of zonal flow. This state will be investigated in more detail with a help of fixed point analysis in the next section.

### 4. Fixed point analysis

In this section, detailed properties of the extended predator-prey model are investigated by using fixed point analysis. A survey of the eigenvalues and corresponding eigenvectors around the fixed points gives comprehensive properties of solution trajectories in phase space.

Before starting the analysis, we introduce the following normalized variables and parameters:

\[
(\alpha_1, \alpha_2, \alpha_3, \gamma_{damp}, \gamma_L, \gamma_{NL}) \rightarrow (\alpha_1, \gamma_L, \alpha_2, \gamma_L, \alpha_3, \gamma_L, \gamma_{damp}, \gamma_L, \gamma_{NL}, \gamma_L),
\]

\[
(\gamma_L, N, U, \frac{\beta^2}{\beta' G}, \frac{\beta^2}{\beta' G} V) \rightarrow (t, N, U, G, V),
\]

\[
\omega_{GAM} = \sqrt{\beta^2 \gamma_L} \rightarrow \omega_{GAM} / \gamma_L, \omega_{sound} = \sqrt{\beta_2' \gamma_L} \rightarrow \omega_{sound} / \gamma_L.
\]

Then, equations (11)-(14) are rewritten as
\[
\frac{\partial N}{\partial t} = N - \gamma_{NL} N^2 - \alpha_1 U^2 N - \alpha_2 G^2 N - \alpha_3 V^2 N , \tag{18}
\]
\[
\frac{\partial U}{\partial t} = \frac{1}{2} \alpha_1 U N - \omega_{GAM} G - \frac{1}{2} \gamma_{damp} U , \tag{19}
\]
\[
\frac{\partial G}{\partial t} = \frac{1}{2} \alpha_2 G N + \omega_{GAM} U + \omega_{\text{sound}} V - \gamma_{LD} G , \tag{20}
\]
\[
\frac{\partial V}{\partial t} = \frac{1}{2} \alpha_3 V N - \omega_{\text{sound}} G , \tag{21}
\]

Here we set \( \gamma_{damp} = 0 \) first for simplicity. In this model there are four types of fixed points distributed in the four-dimensional phase space given by \((N, U, G, V)\). Those are (i) \((0, 0, 0, 0)\), (ii) \((1/\gamma_{NL}, 0, 0, 0)\), (iii) \((0, \omega_{\text{sound}} \delta, 0, -\omega_{GAM} \delta)\), (iv) \((n, u, g, v)\), where \( \delta \) is arbitrary constant value. In (iv) \( n, u, g, v \) are nonzero constants and expressed as

\[
n = \frac{\gamma_{LD}}{\alpha_2} \left[ 1 \pm \sqrt{1 - \frac{4 \alpha_2}{\gamma_{LD}} \left( \frac{\omega_{GAM}^2}{\alpha_1} + \frac{\omega_{\text{sound}}^2}{\alpha_3} \right) n} \right] , \]
\[
u^2 = \frac{2 \omega_{GAM}^2 (1 - \gamma_{NL} n)}{\alpha_2^2 \gamma_{LD} n} , \quad \omega = \frac{\alpha_3 \mu n}{2 \omega_{GAM}} , \quad \text{and} \quad \nu = \frac{\alpha_3 \omega_{\text{sound}}}{\alpha_2 \omega_{GAM}} u . \tag{22}
\]

Note that the fixed points in the case of (iv) only exist under conditions that \( n \) is a positive and also real value.

Next, we investigate the stability around each fixed-point. Small perturbations \( \vec{\delta f} = (\vec{\delta n}, \vec{\delta u}, \vec{\delta g}, \vec{\delta v})^T \) are assumed around a fixed-point \((n_0, u_0, g_0, v_0)\). Namely, substituting \((N, U, G, V) = (n_0 + \vec{\delta n}, u_0 + \vec{\delta u}, g_0 + \vec{\delta g}, v_0 + \vec{\delta v})\) into equations (18)-(21) and linearizing around the fixed-point, we obtain the following equations:

\[
\frac{\partial}{\partial t} \vec{\delta f} = \mathbf{M} \vec{\delta f} , \tag{23}
\]

where \( \mathbf{M} \) is a matrix defined by

\[
\mathbf{M} = \begin{pmatrix}
\Delta & -2 \alpha_1 n_0 u_0 & -2 \alpha_2 n_0 g_0 & -2 \alpha_3 n_0 v_0 \\
\alpha_1 u_0 / 2 & (\alpha_1 n_0 - \gamma_{damp}) / 2 & -\omega_{GAM} & 0 \\
\alpha_2 g_0 / 2 & \omega_{GAM} & (\alpha_2 n_0 / 2 - \gamma_{LD}) & \omega_{\text{sound}} \\
\alpha_3 v_0 / 2 & 0 & -\omega_{\text{sound}} & \alpha_3 n_0 / 2
\end{pmatrix} , \tag{24}
\]

with \( \Delta = 1 - 2 \gamma_{NL} n_0 - \alpha_1 u_0^2 - \alpha_2 g_0^2 - \alpha_3 v_0^2 . \tag{25} \)

Assuming that the perturbations are expressed as \( \vec{\delta f} = \vec{\delta f} e^{\lambda t} \), then equation (23) can be written as an eigen-equation given by \((\lambda \mathbf{I} - \mathbf{M}) \vec{\delta f} = 0\). Here \( \mathbf{I} \) is the identity matrix. In order for all perturbations to have nontrivial solution, \( |\lambda \mathbf{I} - \mathbf{M}| = 0 \) must be satisfied. Calculating the eigenvalues and corresponding eigenvectors of the matrix \( \mathbf{M} \) described by equation (24), we can investigate stability around the fixed point. If all eigenvalues have negative values in their real part, solution trajectories around the fixed-point direct inward, so that they are stable for any perturbation around the fixed point. On the other hand, if some eigenvalues are positive and others are negative or zero, solution trajectories direct outwards along the eigenvectors corresponding to the positive eigenvalues due to “saddle node” nature of the fixed point. In this case, solutions are stable only when the solution
trajectories are along with stable eigenvectors. However, once unstable eigenvectors are included, the solution trajectories become unstable. In the following, we investigate the dynamics around each fixed point.

4.1. Case with the fixed point (0, 0, 0, 0)
Substituting \( (n_0, u_0, g_0, v_0) = (0, 0, 0, 0) \) into equation (24), the eigenvalue \( \lambda \) of the matrix \( M \) can be calculated by solving the following eigen-equation;

\[
|\lambda I - M| = \lambda(\lambda - 1) \left[ \lambda^2 + \gamma_{LD} \lambda + (\omega_{GAM}^2 + \omega_{sound}^2) \right] = 0. \tag{26}
\]

Then, four eigenvalues are obtained as \( \lambda = (0, 1, \omega_s) \), where \( \omega_s \) are complex values associated with GAM frequency, which are expressed as \( \omega_s = -\gamma_{LD}/2 \pm i\sqrt{(\omega_{GAM}^2 + \omega_{sound}^2) - (\gamma_{LD}/2)^2} \). It is assumed that \( (\omega_{GAM}^2 + \omega_{sound}^2) - (\gamma_{LD}/2)^2 > 0 \) is generally fulfilled. Corresponding eigenvectors \( (u_0, u_1, u_{\omega_s}) \) are obtained as

\[
u_0 = \begin{pmatrix} 0,1,0,-\frac{\omega_{GAM}}{\omega_{sound}} \end{pmatrix}, \quad u_1 = (1,0,0,0),
\]

\[
u_{\omega_s} = \begin{pmatrix} 0,1,-\frac{\gamma_{LD} \pm i\sqrt{4(\omega_{GAM}^2 + \omega_{sound}^2) - (\gamma_{LD})/2}}{2\omega_{GAM}}, \frac{\omega_{sound}}{\omega_{GAM}} \end{pmatrix}. \tag{27}
\]

It is found that perturbation along with \( u_1 \) is unstable and that along with \( u_{\omega_s} \) is stable, so that a perturbation starting from \( (U, G, V) \) plane exhibits a spiral trajectory. This kind of fixed point is categorized in saddle focus. The solution trajectories are focused to the original point due to the GAM oscillation with collisionless damping, whereas the perturbation for turbulent fluctuation is advanced. As a result, the solution trajectories are driven in the direction of \( u_1 \).

4.2. Case with the fixed point \((1/\gamma_{NL}, 0, 0, 0) \)
For the fixed point \( (n_0, u_0, g_0, v_0) = (1/\gamma_{NL}, 0, 0, 0) \), the following eigen-equation can be derived as

\[
(\lambda + 1) \left[ \left( \lambda - \frac{\alpha_1}{2\gamma_{NL}} \right) \left( \lambda - \frac{\alpha_2}{2\gamma_{NL}} + \gamma_{LD} \right) \left( \lambda - \frac{\alpha_3}{2\gamma_{NL}} + \omega_{GAM} \left( \lambda - \frac{\alpha_3}{2\gamma_{NL}} + \omega_{sound} \left( \lambda - \frac{\alpha_1}{2\gamma_{NL}} \right) \right) \right] = 0. \tag{28}
\]

Then the eigenvalues are obtained as \( \lambda = (-1, \lambda_1, \lambda_2, \lambda_3) \), where \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are solutions of the cubic equation obtained from equation (28). The comprehensive property of the solution trajectories around the fixed point is understood as a saddle node nature because at least one root is found to be positive (unstable) as long as the condition \( \gamma_{LD} < (\alpha_2/2)^{1/\gamma_{NL}} + 2^{1/\gamma_{NL}}(\alpha_3/2\gamma_{NL}) + \omega_{sound}^2(\alpha_3/2\gamma_{NL}) \) is satisfied, which means that the stationary level of turbulence energy dominated by the collisionless damping, i.e. \( N_0 = (2\gamma_{LD}/\alpha_2) \), is assumed to be lower than that dominated by the nonlinear damping of turbulence, i.e. \( N_{NL} = (1/\gamma_{NL}) \). Note that \( N_0 \) is obtained from equation (30). Therefore it is found in this case that a small perturbation for \( (U, G, V) \) around the fixed point exponentially increases, leading the solution trajectories to diverging. Note that this fixed point is not essential as long as the level of turbulence. \( 1/\gamma_{NL} \) is higher than that around other fixed points.

4.3. Case with the fixed points \((0, \omega_{sound}, \delta, 0, -\omega_{GAM} \delta) \)
Here we set the level of stationary component of zonal flow and ion sound parallel velocity by using the parameter \( \delta \). Note that any \( \delta \) satisfies the condition of fixed points, hence the group of the fixed
point forms a line in the four-dimensional phase space labeled by $\delta$. This suggests that any states parameterized by $\delta$ where stationary zonal flow balances with ion sound parallel flow without pressure perturbation $G$ can exist. Substituting $(n_0, u_0, g_0, v_0) = (0, \omega_{\text{sound}}, 0, -\omega_{\text{GAM}})$ into equation (24), we obtain the eigenvalues $\lambda = (0, \Delta, \omega_s)$, where the definitions of $\Delta$ and $\omega_s$ are the same as those described in the subsection 4.1. The corresponding eigenvectors are solved as follows:

$$
\mathbf{u}_0 = \left(0, 0, -\frac{\omega_{\text{GAM}}}{\omega_{\text{sound}}}\right), \quad \mathbf{u}_{\omega} = \left(0, 1, -\gamma_{\text{LD}} \pm 4(\omega_{\text{GAM}}^2 + \omega_{\text{sound}}^2) \gamma_{\text{LD}} - \frac{\omega_{\text{sound}}}{\omega_{\text{GAM}}}, \frac{\omega_{\text{sound}}}{\omega_{\text{GAM}}}\right),
$$

$$
\mathbf{u}_{\Delta} = \begin{pmatrix}
1,
\frac{\delta \omega_{\text{sound}}}{2} + \frac{\Delta \gamma_{\text{LD}}}{2} + \frac{\Delta^2}{2} + \frac{\omega_{\text{GAM}}^2 + \omega_{\text{sound}}^2}{2} - \frac{(\alpha_1 - \alpha_2) \omega_{\text{sound}}}{2} - \frac{\Delta^2 + \gamma_{\text{LD}} \Delta + (\omega_{\text{GAM}}^2 + \omega_{\text{sound}}^2)}{2}
\end{pmatrix}
$$

(29)

Now, we analyse the characteristics of eigen-values and eigen-vectors. One of four eigen-values, $\lambda = \Delta$ (eigen-state (I)), is unstable when $\Delta > 0$ is satisfied. Note from equation (25) that $\Delta$ is estimated as $\Delta \sim \gamma_{\text{L}}$, corresponding to the linear growth of turbulence as long as the stationary components of $(U, V, G)$ are small enough. Two eigen-values given by $\lambda = \omega_s$ (eigen-state (II)) manifest stable GAM damping oscillation with the damping rate given by $\gamma_{\text{LD}}$. This is also obvious from the corresponding eigen-vectors $\mathbf{u}_{\omega}$ where zonal flow, anisotropic pressure and parallel ion sound perturbation are incorporated. The other one, i.e. $\lambda = 0$ (eigen-state (III)), is marginal. As is found from the eigenvector $\mathbf{u}_0$, this state is identical to the fixed point with residual stationary zonal flow characterized by the parameter $\delta$. Note that the time scale of eigen-state (I) is shorter than that of the eigen-state (II) by $\gamma_{\text{LD}}/\gamma_{\text{L}}$, whereas the time scale of the eigen-state (III) is much longer than that of (I) and (II) due to its marginal nature.

From these discussions, it is found that each eigen-state corresponds to the each dynamics observed during growing intermittency in figure 1, i.e. trigger of burst (→ state (I)), associated GAM damping (→ state (II)), and accumulation of residual zonal flow (→ state (III)). Note that the dynamics is more precisely represented by linear combination of those eigen-states, $\sum_{i=0,\omega,\Delta} c_i \mathbf{u}_i \exp(\tilde{\lambda}_i)$, with a proper choice of coefficient $c_i$. In order to capture the transition dynamics between them, it is useful to investigate the solution trajectories in four-dimensional phase space formed by eigen-vectors as schematically shown in figure 6. Note here that a two-dimensional plane, which is formed by $\mathbf{u}_{\omega}$ and $\mathbf{u}_0$ due to $\mathbf{u}_{\omega} \cdot \mathbf{u}_0 = 0$ in equation (29), whereas $\mathbf{u}_{\Delta}$ is generally orthogonal neither to $\mathbf{u}_0$ nor to $\mathbf{u}_{\omega}$, i.e. $\mathbf{u}_0 \cdot \mathbf{u}_{\Delta} = 0$ and $\mathbf{u}_{\omega} \cdot \mathbf{u}_{\Delta} = 0$. Here, we consider a fixed point (A) labeled by the parameter $\delta_1$, or corresponding plane orthogonal to the vector $\mathbf{u}_0$. A trigger dynamics, i.e. the eigen-state (I), initiated from the point (A) is shown in figure 6 as a movement of the point along with the direction of $\mathbf{u}_s$ from (A) to the point (B), which places on the plane labeled by $\delta_2$. Note that the point (B) is away from the fixed point (C) on the axis along with the vector $\mathbf{u}_0$, so that it is subjected to the oscillation damping toward the point (C). The difference of $\delta$ during the movement from (A) to (C), i.e. $\Delta \delta = \delta_2 - \delta_1$, corresponds to the residual increase of stationary zonal flow balanced with ion parallel velocity. These dynamics also correspond to the behavior of energies in the Landau-fluid simulations as labeled by (A), (B) and (C) in figure 1(b), and the difference of $\delta$ in figure 6 can be comparable to the accumulation of residual zonal flow energy $\delta E_{\text{ZF}}^{(1)}$ in figure 1(b). Note that the increase of $\delta$ during the bursting process results from the non-orthogonal property $\mathbf{u}_0 \cdot \mathbf{u}_{\Delta} = 0$. The
Recursion of the burst and associated accumulation of the residual zonal flow results from the repetition of this process. We can also predict the termination of growing intermittency and associated maximum amplitude of zonal flow. As the burst is repeated, the value of $\delta$ increases and once $\delta$ surpasses $\delta_{\text{crit}} = \sqrt{1/(\alpha_1 \omega_{\text{sound}}^2 + \alpha_3 \omega_{\text{GAM}}^2)}$, which is estimated from $\Delta = 0$ in equation (25), $\Delta$ becomes negative. Then, the eigen-state (I) becomes stable so that the bursting nature is diminished.

Note that such a bursting process occurs only when the condition $u_0 \cdot u_\Delta > 0$ is fulfilled. Namely, in case of $u_0 \cdot u_\Delta < 0$, the bursting event decreases $\delta$, so that the accumulation does not happen. Then, from the condition $u_0 \cdot u_\Delta > 0$, the growing intermittency associate with GAM is realized for $0 < \alpha_1 \omega_{\text{sound}}^2 + \alpha_3 \omega_{\text{GAM}}^2 < 1$. Note that the condition $\alpha_1 \omega_{\text{sound}}^2 + \alpha_3 \omega_{\text{GAM}}^2 > 1$ leads to $\Delta < 0$. Furthermore, an interesting feature is observed in the case of $\alpha_1 \omega_{\text{sound}}^2 + \alpha_3 \omega_{\text{GAM}}^2 < 0$. That is, if $\alpha_3 < \alpha_{3,\text{crit}} \equiv -\alpha_1 (\omega_{\text{sound}}^2 / \omega_{\text{GAM}}^2)$ is satisfied, the suppression role of zonal flow in turbulence is always cancelled by the backflow due to the ion parallel flow for any value of $\delta$ as found from equation (18). In this case, instead of the accumulation of residual zonal flow, a steady state where turbulence coexists with zonal flow and GAM is realized as seen in figure 5.

![Figure 6. Illustration of the solution trajectory around $(n, u, g, v) = (0, \omega_{\text{sound}} \delta, 0, -\omega_{\text{GAM}} \delta)$. The GAM intermittency is identified by suction by GAM damping, blowing off of turbulent fluctuation, and subsequent suction by another GAM phase plane around new fixed point.](image)

![Figure 7. Power spectra of $N$, $U$, and $G$ in the steady-state phase in the case of $\alpha_3 < \alpha_{3,\text{crit}}$.](image)

4.4. Case with the nonzero fixed point described in equation (22)

The eigenvalues around the fixed point $(n, u, g, v)$ described by equation (22) are numerically investigated. As a reference case, the normalized parameters such as $\alpha_1 = 1.0$, $\alpha_2 = 10$, $\alpha_3 = -0.1$ are used in accordance with equation (17), so that $\omega_{\text{GAM}} = 11.55$, and $\omega_{\text{sound}} = 3.89$. In this case the fixed point is numerically estimated as $(n, u, g, v) = (2.94, 5.44, 0.691, -18.3)$ and the corresponding eigenvalues are solved as $\lambda = 4.82 \pm 18.5i$, $3.67 \pm -7.14 \times 10^3$. Three unstable eigenvalues and one weakly stable eigen-value are obtained here. It indicates that the solution trajectories are unstable for the fixed point as a saddle node. Even in the further search of eigenvalues for variable $\alpha_3$, the property of solution trajectory around this type of fixed point is found to have at least two unstable roots in four
solutions. This type of fixed points is distributed far from realistic region, comparing with the other cases. Then, dynamics around the nonzero fixed point may have little effect on the GAM intermittent dynamics.

4.5. Steady state observed when $\alpha_5 < 0$

It is noted that the steady state solution observed in the case of $\alpha_5 < \alpha_{5\text{crit}}$ in subsection 4.3 or in figure 5 is different from those around the fix point discussed in subsection 4.4. This behavior is not categorized as the previous case such as the motion around some fixed points. In this case the turbulence $N$ is sustained in cooperation with $U$, $G$, and $V$. Figure 7 shows the power spectra of $N$, $U$, and $G$ in the case of $\alpha_1 = 1.0$, $\alpha_2 = 10$, $\alpha_3 = -0.2$, where a steady state of turbulence and GAM is established as figure 5(b). Several peaks are observed at the characteristic GAM with frequency $\omega_G = \sqrt{\omega_{\text{sound}}^2 + \omega_{GAM}^2}$. The power spectra of $U$, $G$, and $V$ are characterized by odd harmonics of the GAM frequency, whereas that of $N$ is characterized by the even harmonics. In the spectrum of $N$ another peak characterized by the intermittency is also observed at $\omega_0 = 1.57\gamma_1$, together with the side bands corresponding to $2\omega_G \pm \omega_0$.

We can estimate the level of the steady state. In terms of the following expansions,

$$
U = \sum_{n=0}^{\infty} U_n e^{i(2\pi n+1)\omega_G t}, \quad G = \sum_{n=0}^{\infty} G_n e^{i(2\pi n+1)\omega_G t}, \quad V = \sum_{n=0}^{\infty} V_n e^{i(2\pi n+1)\omega_G t}, \quad N = \sum_{n=0}^{\infty} N_n e^{i2n\omega_G t},
$$

and orderings $f_0 \sim O(\epsilon^n)$ for $f = (N,U,G,V)$, $\alpha_1,\alpha_3 \sim O(\epsilon)$, $\alpha_2 \sim O(1)$, $\omega_G,\omega_0,\omega_{\text{sound}},\gamma_{LD} \sim O(1)$, we obtain the following results by taking account the lowest order of equations (18)-(21),

$$
N_0 = 2\gamma_{LD}/\alpha_2, \quad U_0 = i(\omega_{GAM}/\omega_G)G_0, \quad G_0 = \sqrt{2}/\alpha_2, \quad V_0 = i(\omega_{\text{sound}}/\omega_G)G_0.
$$

The magnitudes of $N_0$, $U_0$, $G_0$, and $V_0$ are estimated from equation (30) as $N_0 \sim 0.5$, $G_0 \sim 0.45$, $U_0 \sim 0.42$, and $V_0 \sim 0.14$, respectively. These are about 10% difference from those in figure 5(b). The differences are within the ordering parameter, $\epsilon \sim \alpha_1/\alpha_2 \sim 0.1$. The phase lag between $U$ and $V$ is about zero, whereas those between $U$ and $G$, and $G$ and $V$ are about $\pi/2$, which are consistent with the results of linear response of the GAM oscillator, i.e. equations (8)-(10). This is why we have concluded that as the lowest order approximation these results are consistent with those obtained in the numerical calculation.

5. Conclusion.

We have proposed a four-field minimum model that describes the growing intermittency of turbulence associated with the GAM observed in our toroidal Landau-fluid simulations. The model can reproduce the different nature of turbulent transport by controlling the coefficients of nonlinear coupling terms. Energy inflow due to the nonlinear coupling between turbulence and zonal flows, i.e. Reynolds stress, can excite GAMs and simultaneously generate residual zonal flows balancing with ion sound parallel velocity. On the other hand, the flow due to nonlinear coupling between turbulence and $p_{\rho \rho}$ pressure perturbation excites GAMs but generates no residual zonal flows. Multiple/single-burst nature in the GAM intermittency originates from qualitatively difference between the two nonlinear coupling terms. Energy backflow due to the nonlinear coupling between turbulence and ion sound parallel velocity cancels the effect of Reynolds stress. When the backflow always overwhelms the inflow due to the Reynolds stress, steady state of turbulence appears.

We have also investigated the comprehensive properties of solution trajectories around the fixed points incorporated with eigen-value and corresponding eigen-vector. It is found that the coefficient of the nonlinear coupling term between turbulence and ion sound velocity can make a bifurcation between turbulence quench due to the accumulation of zonal flow and steady state of turbulence. This
result may also predict the bifurcation of states such as Dimits shift regime and steady state regime above the critical gradient due to the change of governing parameters.

A predator-prey model we presented reproduces the Dimits shift region where turbulence is quenched through the intermittent bursts or single one. The model also reproduces a quasi-linear state turbulent transport incorporated with zonal flows. Those two different states are obtained by changing the ratio and/or the sign of \((\alpha_1, \alpha_2, \alpha_3)\) in equations (11)-(14). However, it is not straightforward to explicitly relate those parameters to that of the instability strength such as \(\eta\) and/or \(R/L_t\), which is out of scope from our present study. In order to determine the shift quantity or to determine the critical gradient, we need to further improve the model, e.g. by coupling with transport equation [18]. Intermittent transport phenomena which are related to GAM dynamics are observed in experiments [14,15]. Comparison between the growing intermittency observed in our simulations and those experiments is our future work.

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Appendix A: Approximate treatment of the off-diagonal terms in four-field minimal model
In this appendix, we describe as a minimal analytical modelling how we can effectively involve the impacts of nonlinear cross-coupling (i.e., off-diagonal terms) in the diagonal parts of Equations (12)-(14) for simplicity. The nonlinear terms in equations (12)-(14) originate from the convective nonlinearities such as the terms \(\tilde{N}_{[\varphi,p]}\), \(\tilde{N}_{[\varphi,p]}\) and \(\tilde{N}_{[\varphi,\phi]}\). They can be expressed as diagonal and off-diagonal parts based on a quasi-linear description of modulational coupling loop. Taking the GAM equation (13) as an example, the nonlinear coupling term \(\tilde{N}_{[\varphi,p]}\) can be decomposed into four pairs of couplings

\[
\tilde{N}_{[\varphi,p]} \sim \left[ \left( \tilde{\phi}_p, \tilde{p}_p \right) \right] \sin \theta = \left[ \left( \tilde{\phi}_p^m, \tilde{p}_p^m \right) \right] \sin \theta + \left[ \left( \tilde{\phi}_p^{\text{SB}}, \tilde{p}_p^{\text{SB}} \right) \right] \sin \theta + \left[ \left( \tilde{\phi}_p^m, \tilde{p}_p^{\text{SB}} \right) \right] \sin \theta + \left[ \left( \tilde{\phi}_p^{\text{SB}}, \tilde{p}_p^m \right) \right] \sin \theta
\]

\[\text{(A1)}\]

Here the subscripts “$p$” and “$SB$” denote the pumps and sidebands in \(k_t\) space. The four terms at RHS correspond to \(\Gamma_1 \propto (\tilde{\phi}_p^m)^* \tilde{p}_p^m\), \(\Gamma_2 \propto (\tilde{\phi}_p^{\text{SB}})^* \tilde{p}_p^{\text{SB}}\), \(\Gamma_3 \propto (\tilde{\phi}_p^m)^* \tilde{p}_p^{\text{SB}}\) and \(\Gamma_4 \propto (\tilde{\phi}_p^{\text{SB}})^* \tilde{p}_p^m\), respectively.

Note that the pumps \(\tilde{\phi}_p\) and \(\tilde{p}_p\) are primary toroidal ITG modes, which satisfy the linear ITG dispersion relation. Based on the nonlinear coupling, the sidebands can be expressed as

\[
\tilde{\phi}_p^m \propto \tilde{\phi}_p^m \tilde{\phi}_q^m,
\]
\[\text{(A2)}\]
\[
\tilde{\phi}_p^{\text{SB}} \propto \tilde{\phi}_p^{\text{SB}} \tilde{\phi}_q^m,
\]
\[\text{(A3)}\]
\[
\tilde{p}_p^m = \left[ \tilde{\phi}_p^m \tilde{p}_p^m \right] + \left[ \tilde{\phi}_p^{\text{SB}} \tilde{p}_p^{\text{SB}} \right] \propto c_2 \tilde{\phi}_p^m \tilde{p}_p^m + c_2 \tilde{\phi}_p^{\text{SB}} \tilde{p}_p^{\text{SB}} \tilde{\phi}_q^m,
\]
\[\text{(A4)}\]
\[
\tilde{p}_p^{\text{SB}} = \left[ \tilde{\phi}_p^m \tilde{p}_p^{\text{SB}} \right] + \left[ \tilde{\phi}_p^{\text{SB}} \tilde{p}_p^m \right] \propto c_2 \tilde{\phi}_p^m \tilde{p}_p^{\text{SB}} + c_2 \tilde{\phi}_p^{\text{SB}} \tilde{p}_p^m \tilde{\phi}_q^m.
\]
\[\text{(A5)}\]

So substituting these relations into equation (A1), we have

\[
\left( \tilde{\phi}_p, \tilde{p}_p \right) \sin \theta = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \propto (\tilde{\phi}_p^m)^2 \tilde{p}_p^{m1} + \left[ (\tilde{\phi}_p^m)^* \tilde{p}_p^{m1} + (\tilde{\phi}_p^m)^* \tilde{p}_p^m \right] \tilde{\phi}_q.
\]
\[\text{(A6)}\]
These two terms correspond to the diagonal (~NG) and off-diagonal (~NU) parts. Considering that the GAM is an oscillatory eigenmode, the corresponding electrostatic potential, i.e., the zonal flow $\hat{\phi}_q$, includes two components with an oscillatory and a zero-frequency. In the case without GAMs, the nonlinear driving of the zonal flows due to the gradient of Reynolds stress is balanced by the collisional damping in the quasi-steady. When it couples with the GAM, it becomes to oscillate approximately with GAM frequency. Hence, in equation (12) for the zonal flow, although the nonlinear driving may have some influence, the coupling with GAMs is dominated by the linear GAM oscillator with spectral and phase’s matching. Based on this idea, we represent the relation between the zonal flows and the GAM dynamics as $\hat{\phi}_q \approx \hat{\phi}_{s1}$ within an approximation of the lowest order in an analytical minimal model. Under this assumption, we effectively absorb the off-diagonal term into the diagonal part with a modified coefficient $\alpha_i$.

Interestingly, some efforts have been also made in a recent work [19] to estimate the contribution of the off-diagonal terms to the GAM dynamics based on the ballooning character of ITG modes in order to resolve the complexity due to the off-diagonal effects in a simplified modeling. Further improvement of the approximation in an advanced modelling is in progress.

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