Steps towards NNLO QCD calculations: collinear factorization at $\mathcal{O}(\alpha_S^2)^*$

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I consider the singular behaviour of tree-level QCD amplitudes when the momenta of three partons become simultaneously parallel and I discuss the universal factorization formula that controls the singularities of the multiparton matrix elements in this collinear limit.

1. Introduction

The properties of QCD in the soft and collinear limits play an important role in doing higher order calculations. The singularities arising in these limits prevents a straightforward integration of matrix elements to obtain physical cross sections. However, as far as NLO calculations are concerned, the universality property of these singularities provided a method to solve these problems. The singular behaviour at this order is well known and is given in terms of $\mathcal{O}(\alpha_S^2)$ factorization formulae for tree level [1] and one loop [2] amplitudes to obtain physical cross sections. The singularities in two loop corrections with real emission and double real emission have been discussed in Ref. [3]. The properties of QCD in the soft and collinear limits have been recently obtained at all order in $\epsilon$ in Ref. [4]. The singular behaviour of tree-level amplitudes has been studied in Ref. [4, 5]. Here I will concentrate on the collinear behaviour.

2. Notation and kinematics

We consider a generic scattering process involving final-state QCD partons with momenta $p_1, p_2, \ldots$. Non-QCD partons ($\gamma^*, Z^0, W^\pm, \ldots$), carrying a total momentum $Q$, are always understood. The corresponding tree-level matrix element is denoted by

$$\mathcal{M}_{a_1, a_2, \ldots}^{s_1, s_2, \ldots}(p_1, p_2, \ldots) ,$$

where $\{c_i\}$, $\{s_i\}$ and $\{a_i\}$ are respectively colour, spin and flavour indices. The matrix element squared summed over final-state colours and spins will be denoted by $|\mathcal{M}_{a_1, a_2, \ldots}(p_1, p_2, \ldots)|^2$. If the sum over the spin polarizations of the parton $a_1$ is not carried out, we define the following 'spin-polarization tensor'

$$\mathcal{T}_{a_1, \ldots}^{s_1, s'_1}(p_1, \ldots) \equiv \sum_{\text{spins } s_1, s'_1} \sum_{\text{colours}} \mathcal{M}_{a_1, a_2, \ldots}^{c_1, c_2, \ldots; s_1, s_2, \ldots} \times \left[ \mathcal{M}_{a_1, a_2, \ldots}^{c_1, c_2, \ldots; s_1', s_2, \ldots} \right]^\dagger .$$

In the evaluation of the matrix element, we use conventional dimensional regularization ($d = 4 - 2\epsilon$ space-time dimensions, two helicity states for massless quarks and $d - 2$ helicity states for gluons).

The relevant collinear limit at $\mathcal{O}(\alpha_S)$ is approached when the momenta of two partons, say $p_1$ and $p_2$, become parallel. This limit is defined by setting:

$$p_1^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2 p \cdot n} ,$$

$$p_2^\mu = (1 - z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1 - z} \frac{n^\mu}{2 p \cdot n} ,$$

and sending $k_\perp \to 0$. In Eq. [3] the light-like ($p^2 = 0$) vector $p^\mu$ denotes the collinear direction,
while \( n^\mu \) is an auxiliary light-like vector, which is necessary to specify the transverse component \( k_\perp \) \((k_\perp^2 < 0)\) \((k_\perp \cdot p = k_\perp \cdot n = 0)\) or, equivalently, how the collinear direction is approached. In the small-\( k_\perp \) limit (i.e. neglecting terms that are less singular than \( 1/k_\perp^2 \)), the square of the matrix element in Eq. (4) fulfils the following factorization formula

\[
|\mathcal{M}_{a_1, a_2, \ldots, p_1, p_2, \ldots, p_3, \ldots}|^2 \simeq \frac{4}{s_{12}^2} (4\pi \mu^{2x} \alpha_s)^2 \times \mathcal{T}_{a_1 a_2}(x, k_\perp; \epsilon),
\]

where \( \mu \) is the dimensional-regularization scale. The spin-polarization tensor \( \mathcal{T}_{a_1 a_2}(x, \ldots) \) is obtained by replacing the partons \( a_1 \) and \( a_2 \) on the right-hand side of Eq. (3) with a single parton denoted by \( a \). This parton carries the quantum numbers of the pair \( a_1 + a_2 \) in the collinear limit.

The kernel \( \tilde{P}_{a_1 a_2} \) in Eq. (4) is the 4-dimensional Altarelli–Parisi (AP) splitting function \([7]\). It depends on the momentum fraction \( x \) but also on the transverse momentum \( k_\perp \) and on the helicity of the parton \( a \) in the matrix element \( \mathcal{M}_{a_1, a_2, \ldots, p_1, p_2, \ldots, p_3, \ldots}^{s, s'}(p, \ldots) \). More precisely, \( \tilde{P}_{a_1 a_2} \) is in general a matrix acting on the spin indices \( s, s' \) of the parton \( a \) in the spin-polarization tensor \( \mathcal{T}_{a_1 a_2}(p, \ldots) \). Because of these spin correlations, the spin-average square of the matrix element \( \mathcal{M}_{a_1, a_2, \ldots, p_1, p_2, \ldots, p_3, \ldots}^{s, s'}(p, \ldots) \) cannot be simply factorized on the right-hand side of Eq. (4).

3. Collinear factorization at \( O(\alpha_s^3) \)

In the following we are interested in the collinear limit at \( O(\alpha_s^3) \). At this order there are two different collinear limits to be considered \([8]\). The first is when two pairs of parton momenta become independently parallel. In this case the factorization formula is obtained by a simple iteration of Eq. (4). In the second case three parton momenta can simultaneously become parallel. Denoting these momenta by \( p_1, p_2 \) and \( p_3 \), their most general parametrization is

\[
p_i^\mu = x_i p^\mu + k_i^\mu - \frac{k_i^2}{2x_i} n^\mu, \quad i = 1, 2, 3,
\]

where the notations are the same as in Eq. (3). Note that no constraint is imposed on the longitudinal and transverse variables \( x_i \) and \( k_{\perp i} \).

It can be shown \([9]\) that in the triple-collinear limit \((k_{\perp i} \rightarrow 0)\) the matrix element squared still fulfils a factorization formula analogous to Eq. (4), namely

\[
|\mathcal{M}_{a_1, a_2, a_3, \ldots, p_1, p_2, p_3, \ldots}|^2 \simeq \frac{4}{s_{123}^2} (4\pi \mu^{2x} \alpha_s)^2 \times \mathcal{T}_{a_1 a_2 a_3}(x, p, \ldots) \tilde{P}_{a_1 a_2 a_3}(x, k_{\perp 1}; \epsilon),
\]

As in Eq. (4), the spin-polarization tensor \( \mathcal{T}_{a_1 a_2 a_3}(x, \ldots) \) is obtained by replacing the partons \( a_1, a_2 \) and \( a_3 \) with a single parent parton, whose flavour \( a \) is determined by flavour conservation in the splitting process \( a \rightarrow a_1 + a_2 + a_3 \).

The three-parton splitting functions \( \tilde{P}_{a_1 a_2 a_3} \) generalize the AP splitting functions in Eq. (4). The spin correlations produced by the collinear splitting are taken into account in a universal way, i.e. independently of the specific matrix element on the right-hand side of Eq. (4). Besides depending on the spin of the parent parton, the functions \( \tilde{P}_{a_1 a_2 a_3} \) depend on the momenta \( p_1, p_2, p_3 \). However, due to their invariance under longitudinal boosts along the collinear direction, the splitting functions can depend in a non-trivial way only on the sub-energy ratios \( s_{ij}/s_{123} \) and on the following longitudinal and transverse variables:

\[
z_i = \frac{x_i}{x}, \quad k_i^\mu = k_{\perp i}^\mu - \frac{x_i}{x} \sum_{j=1}^{3} k_j^\mu,
\]

where \( x = \sum_{i=1}^{3} x_i \).

The method used to derive these results exploits the fact that interfering Feynman diagrams obtained by squaring the amplitude \( \mathcal{M}(p_1, \ldots, p_m, \ldots) \) are collinearly suppressed when computed in a physical gauge. Thus, in the evaluation of the triple collinear limit we can limit ourselves to consider the diagrams in Fig. 1. Details on the method and on our calculation are given in Ref. \([9]\). The basic observation is that if we rescale the transverse momenta as \( k_{\perp i} \rightarrow \lambda k_{\perp i} \), the matrix element squared has the singular behaviour

\[
|\mathcal{M}_{a_1, a_2, a_3, \ldots, p_1, p_2, p_3, \ldots}|^2 \sim 1/\lambda^4 + \ldots
\]

where dots stand for less singular contributions when \( \lambda \rightarrow 0 \). To extract the singular behaviour
Figure 1. Triple collinear limit: dominant diagrams in a physical gauge

one has to put on shell the parent parton leg in Fig. 1. Since $p_1 + p_2 + p_3 = xp + O(k_\perp)$, in order to do this we have to neglect $O(k_\perp)$ terms from the amplitude which may affect in a non-universal way the singular behaviour. But since we are interested in the most singular behaviour we can safely put $p_1 + p_2 + p_3 \rightarrow xp$ and reconstruct the gauge invariant spin polarization tensor in (6). Thus $\hat{P}_{a_1a_2a_3}$ can be computed by evaluating the process independent diagrams in Fig.1 in the collinear limit. Our explicit results are presented in [6]. We find that when the parent parton is a fermion, spin-correlations are absent. This is analogous to what happens at $O(\alpha_s)$ and it is a consequence of helicity conservation in the quark vector coupling. On the contrary, in the case in which the parent parton is a gluon spin correlations are highly non trivial.

A check of the calculation is provided by the strong-ordered limit. In this limit the three partons become collinear sequentially and $P_{a_1a_2a_3}^{ss'}$ factorize in the product of two AP splitting functions. A further non-trivial check of the calculation is provided by supersymmetry. As for AP splitting functions, we find that $\hat{P}_{a_1a_2a_3}$ obey a $N = 1$ SUSY identity in the limit $\epsilon \rightarrow 0$ and when $C_F = C_A = 2T_R$:

$$\left[\hat{P}_{q_1g_2g_3} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)\right] + \left[\hat{P}_{g_3g_1g_2} + (1 \leftrightarrow 3)\right] + \left[\hat{P}_{g_3q_1q_2} + 5 \text{ perm.}\right] = P_{g_1g_2g_3} + \left[\hat{P}_{g_3q_1q_2} + 5 \text{ perm.}\right]$$

which express the fact that the total quark (actually, gluino) and gluon decay probability are the same.

The $O(\alpha_s^2)$-collinear limit of tree-level QCD amplitudes has been independently considered by Campbell and Glover [5]. Taking for granted Eq. (6) they neglected spin correlations and computed only the spin-averaged splitting functions by performing the limit of known amplitudes. We have compared our results with those of Ref. [5] and found complete agreement.

4. Summary

I have discussed the three-parton collinear limit of tree-level QCD amplitudes. In this limit the singular behaviour of the matrix element squared is controlled by process-independent splitting functions, which are analogous to the Altarelli–Parisi splitting functions.

These splitting functions are one of the necessary ingredients to extend QCD predictions at higher perturbative orders. In particular, they will be relevant to set up general methods to compute jet cross sections at NNLO. The knowledge of the collinear splitting functions, when combined with a consistent analysis of soft-gluon coherence properties, could also be used to improve the logarithmic accuracy of parton showers available at present for Monte Carlo event generators.

The results presented in this talk have been obtained in collaboration with S. Catani.

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