Small-d MSR Codes with Optimal Access, Optimal Sub-Packetization and Linear Field Size

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Abstract

This paper presents an explicit construction of a class of optimal-access, minimum storage regenerating (MSR) codes, for small values of the number $d$ of helper nodes. The construction is valid for any parameter set $(n, k, d)$ with $d \in \{k + 1, k + 2, k + 3\}$ and employs a finite field $\mathbb{F}_q$ of size $q = O(n)$. We will refer to the constructed codes as Small-d MSR codes. The sub-packetization level $\alpha$ is given by $\alpha = s^{\lceil \frac{d}{2} \rceil}$, where $s = d - k + 1$. By an earlier result on the sub-packetization level for optimal-access MSR codes, this is the smallest value possible.

Keywords: coding theory, distributed storage, regenerating codes, minimum storage regenerating (MSR) codes, optimal access repair, Small-d codes, optimal sub-packetization level codes.

I. INTRODUCTION

Erasure codes are of strong interest in distributed storage systems as they offer reliability at lower values of storage overhead in comparison with replication. In the setting of distributed storage, the $B$ symbols of a given data file $\mathcal{F}$ are stored in redundant fashion, across $n$ storage units (nodes), such as a hard disk or a flash memory unit. Among the class of erasure codes, Maximum Distance Separable (MDS) codes are of particular interest as they offer reliability at lowest possible value of storage overhead. Apart from reliability and storage overhead, an additional important concern in a distributed storage system is that of efficient single node repair [2]. Efficient repair could either call for the amount of data download needed to repair a failed node to be kept to a low level or else, the number of helper nodes contacted for repair to be kept small. The focus in the present paper, is on the first criterion, i.e., on lowering the amount of data download needed for node repair, also termed as the repair bandwidth.

A. Minimum Storage Regenerating (MSR) Codes

Regenerating codes [3] are codes that protect against data loss as well as single node failure with less repair bandwidth. These codes have a vector symbol alphabet, given by $\mathbb{F}_q^\alpha$ where $\alpha$ is termed the level of sub-packetization of the regenerating code. Thus each storage unit stores $\alpha$ symbols from $\mathbb{F}_q$ associated to the file $\mathcal{F}$. Protection against data loss is ensured by requiring that the stored data file be retrievable even in the face of the loss of up to $r, 1 \leq r \leq n$ storage units. Thus the minimum Hamming distance $d_{\text{min}}$ of the regenerating code must satisfy $d_{\text{min}} \geq r + 1$. We define the parameter $k = n - r$. Node repair is ensured by requiring that a failed node be repaired by downloading $\beta$ symbols over $\mathbb{F}_q$ from each node within a set of $d$ nodes, where the $d$ nodes are arbitrarily selected from the surviving $(n - 1)$ nodes. Within the class of regenerating codes, the subclass of Minimum Storage Regenerating (MSR) codes are of particular interest, as this subclass falls within the class of MDS codes, and hence incur least-possible storage overhead when required to recover from the failure of $r$ storage units. To qualify for being called an MDS code, the regenerating code must satisfy the Singleton bound

$$q^B = \text{size of the code} = q^{\alpha(n - d_{\text{min}} + 1)} = q^{\alpha k},$$

i.e., it must be that

$$B = k\alpha.$$
It turns out that the minimal number symbols downloaded $\beta$, from each helper node in an MSR code is necessarily given by

$$\beta = \frac{\alpha}{d - k + 1} \text{ (see [3]).}$$

This is obtained by quantifying the condition under which the repair bandwidth $\beta$ for the repair of a failed node, in an MDS code over $\mathbb{F}_q$ for a fixed value of $\{n, k, d, \alpha\}$, is the least possible.

Thus, MSR codes are characterized by the parameter set

$$\{(n, k, d), (\alpha, \beta), B, \mathbb{F}_q\},$$

where

- $\mathbb{F}_q$ is the underlying finite field,
- $n$ is the number of code symbols $\{c_i\}_{i=1}^n$, each of which is stored on a distinct node and
- each code symbol $c_i$ is an element of $\mathbb{F}_q^\alpha$.

Since each code symbol $c_i$ is stored on a distinct node, it follows that the index $i$ of a code symbol is synonymous with the index of the node upon which that code symbol is stored. Throughout this paper, we will focus on a linear MSR code i.e., on MSR codes where, the mapping from symbols comprising the data file and the symbols stored in the storage network takes on the linear form

$$m^T G = [c_1^T, \ldots, c_n^T],$$

where $G$ is an $(k\alpha \times n\alpha)$ generator matrix over $\mathbb{F}_q$ and where $m$ is a $(k\alpha \times 1)$ message vector over $\mathbb{F}_q$ corresponding to the $B = k\alpha$ message symbols of the data file, encoded by the MSR code.

### B. Desirable Properties of an MSR Code

While MSR codes are MDS codes that need smallest repair bandwidth possible for single node repair, there is still scope for optimization with the class of MSR codes. The additional features of interest are listed below.

- **Optimal-Access:** Optimal-access MSR codes [4] are a subclass of MSR codes having the property that during repair, the $\beta$ symbols that are transmitted by a helper node during repair, are simply a subset of the $\alpha$ symbols contained in the node. This has two important and desirable, practical consequences. Firstly, the number of symbols accessed in the node is as small as possible and secondly, no computations are required to generate the transmitted repair symbols.

- **Low Values of Sub-Packetization:** Low values of sub-packetization level are desirable both to reduce complexity as well as to permit smaller file sizes $B = k\alpha$ to be encoded.

- **Low Field Size:** The need for a low field size is clear since the smaller the size of the finite field, the lesser is the implementation complexity.

### C. Prior Work on MSR Codes

Several constructions of an MSR code can be found in the literature. In addition, there are constructions of systematic MDS codes in the literature where it is only the systematic nodes that can be recovered with minimal repair bandwidth, i.e., repaired by downloading $\frac{\alpha d}{\alpha + \beta - 1}$ symbols. We will refer to this latter class of codes as systematic MSR codes. A detailed survey on MSR code constructions and sub-packetization level bounds can be found in [5]. The product matrix construction in [6] for any $2k - 2 \leq d \leq n - 1$ is one of the earliest constructions of an MSR code. These codes have smallest possible sub-packetization level possible of an MSR code, since $\beta = 1$ in the product matrix construction of an MSR code. However, and as a consequence of this, the rate of a product-matrix MSR code is bounded above by the quantity $\frac{1}{2} + \frac{1}{2n}$.

In [7], the authors provide a construction for a high-rate MSR code that makes use of Hadamard designs for any $(n, k, d)$ parameter set of the form $(n, k = n - 2, d = n - 1)$ with sub-packetization level $\alpha = 2k + 1$. In [8], high-rate systematic MSR codes termed as Zigzag codes were constructed for $d = n - 1$. These codes however have large field size and sub-packetization level that is exponential in the parameter $k$. This construction was subsequently extended in [9] to enable the repair of parity nodes. The existence of MSR codes for any value of $(n, k, d)$ as $\alpha$ tends to infinity is shown in [10].
1) Sub-Packetization Level: An open problem in the literature on regenerating codes is that of determining the smallest possible sub-packetization level $\alpha$ of an MSR code, for given parameters $\{n, k, d = (n-1)\}$. This question is addressed in [11], where a lower bound on $\alpha$ for MSR codes is presented by showing that $k \leq \alpha(\alpha/(n-k))$. In [12] it is established that:

$$k \leq 2\log_2(\alpha)([\log_{r^{-1}}(\alpha)] + 1),$$

while in [13] the authors prove that:

$$k \leq 2\log_r(\alpha)([\log_{r^{-1}}(\alpha)] + 1).$$

Most recently, in [14] the authors prove that $\alpha \geq e^{O(\frac{1}{s})}$ for any general MSR code.

2) Optimal-Access MSR Codes: For the special case of an optimal-access MSR code, it was shown in [11] that:

$$\alpha \geq \frac{k-1}{r}. $$

The constructions presented in [8], [9] satisfy the optimal access property. However, they have sub-packetization level exponential in $k$ for a fixed rate. In [15], an optimal-access systematic MSR code is constructed for the case $d = n-1$ with $\alpha = r^k$. This was followed by in [16], by the construction of an optimal-access MSR for the case $d = n-1$ with $\alpha = r^{\frac{k}{2}}$. The construction in [16] was extended to any $d \leq n-1$ in [17] with $\alpha = s^{\frac{d}{n}}$ where $s = d-k+1$. The constructions in [15]–[17] are not explicit and need large field size. In [18] explicit MSR constructions for any $(n, k, d)$ with field size $O(n)$ and $\alpha = s^n$ are provided. In [19], the authors improve upon the lower bound for optimal-access MSR case to $\alpha \geq s^2$ where $s = d - k + 1$. This turns out to settle the problem of determining the smallest sub-packetization level of an optimal-access MSR code as the optimal-access MSR code constructions in [16], [17], [20–22] achieve this lower bound on $\alpha$ with equality.

3) The Coupled-Layer MSR Code: In [21], Ye-Barg presented an explicit construction of optimal-access MSR codes having parameters $(n, k, d = n-1)$ with sub-packetization level $\alpha = r^{\frac{k}{2}}$ and field size $q \geq n - 1$. In independent work, that followed shortly after, the authors of [20], came up with essentially the same construction, but one that was presented from a coupled-layer perspective that involved the application of a pairwise coupling transform applied to a data cube wherein each horizontal layer was an MDS code. Flexibility in selecting this MDS code meant that it was possible to construct a binary coupled layer MSR code by starting from a MDS code built over a binary vector alphabet. Unknown to the authors of [20], in [22], the authors had employed the same coupling transform to transform an MDS code to one in which certain symbols could be optimally repaired. This was later extended by the authors of [22], after the appearance of [21] and [20] to show how an MDS code could be transformed to yield an MSR code through iterated application of the pairwise coupling transform.

We will refer to the MSR code resulting form the constructions presented in [20–22] as the Clay code (Clay for Coupled LAYER), following the nomenclature introduced in [23], where a detailed performance evaluation of the Clay code was conducted.

D. Our Contributions

As discussed above, the Clay code is an optimal-access MSR code that is optimal with respect to sub-packetization level and has linear field size. However, the Clay code construction applies only to the case when the number of helper nodes $d$ contacted equals $(n-1)$ which is the largest possible. As pointed out in the literature on locally recoverable codes, there is practical interest in minimizing the number of helper nodes that are contacted. While the prior literature contains optimal-access MSR constructions for $d < n-1$, the resultant codes are either non-explicit with large field size [17], [24] or else have large sub-packetization level [18].

In the present paper, we present explicit construction of MSR codes that are also optimal access, have optimal sub-packetization level, linear field size, but where this time, $d$ is as small as possible. For this reason, we term these codes as Small-$d$ MSR codes. Specifically, we provide constructions for the cases $d \in \{k + 1, k + 2, k + 3\}$. The case $d = k$ is uninteresting since setting $d = k$ results in $\beta = \alpha$ from equation (1) and thus there is no saving in repair bandwidth to be had in this case. The parameters of the Small-$d$ MSR codes constructed over a finite field $\mathbb{F}_q$ are given by:

$$(n, k, d \in \{k + 1, k + 2, k + 3\}), \quad (\alpha = s^t, \beta = s^{t-1}), \quad B = ka,$$
where
\[ r = n - k, \ s = d - k + 1, \ t = \left\lceil \frac{n}{s} \right\rceil, \ q = O(n). \]

It follows that the union of the Small-d and Clay code MSR constructions provide optimal-access, optimal sub-packetization level and linear-size code constructions for all \((n, k, d)\) parameter sets with \(n - k \leq 5\). Given the emphasis within industry on small block lengths and low values of storage overhead, this range is of practical interest. The parameters of some example Small-d MSR codes is presented in Table I.

| \(n\) | \(k\) | \(d\) | \(r\) | \(s\) | \(t\) | \(\alpha\) | \(\beta\) | \(B = k\alpha\) | \(d\beta\) |
|------|------|------|------|------|------|--------|--------|--------------|--------|
| 10   | 8    | 9*   | 2    | 2    | 5    | 32     | 16     | 256          | 144    |
| 9    | 6    | 7    | 3    | 2    | 5    | 32     | 16     | 192          | 112    |
| 14   | 10   | 11   | 4    | 2    | 7    | 128    | 64     | 1280         | 704    |
| 14   | 10   | 13*  | 4    | 4    | 4    | 256    | 64     | 2560         | 832    |

**TABLE I**

***PARAMETERS OF SOME EXAMPLE SMALL-d MSR CODES.*** Here, \(r = n - k, \ s = d - k + 1, \ t = \left\lceil \frac{n}{s} \right\rceil, \ \alpha = s^t\) and \(\beta = s^{t-1}\). An * attached to the parameter \(d\) identifies instances where the SMALL-d and CLAY-code constructions yield codes with identical parameters.

**E. Outline**

We start by presenting the description of Small-d MSR code in Section II and then introduce the notation and terminology in Section III that will be used to prove the MDS and optimal-access repair properties of the Small-d MSR code. In Section IV we show that for an example case of \(s = (d - k + 1) = 2\) and \(r = n - k = 3\), the Small-d MSR code is an optimal-access MSR code. In Section V we show that the MDS property of Small-d MSR code can be reduced to proving invertibility of a reduced matrix that is a sub-matrix of parity check matrix. Similar to Section VI in Section VII we show that the optimal-access property of Small-d MSR code can also be reduced to proving invertibility of a reduced matrix. Finally, in Section VII we show that the reduced matrix is invertible thereby proving that Small-d MSR code is an optimal-access MSR code for any \(s = d - k + 1 \in \{2, 3, 4\}\) and \(r \geq s\).

We will adopt the following notation throughout the paper.

1) \([a : b] = \{a, a + 1, \ldots, b\}, [a] = [1 : a] \) and \(\mathbb{Z}_s = [0 : s - 1]\).

2) Let \(\overline{z} = (z_0, z_1, \ldots, z_{t-1}) \in \mathbb{Z}_s^t \) and \(x \in [0 : s - 1]\). We define \(\overline{z}(x \rightarrow z_y)\) to be the vector obtained by replacing the \(y\)th component of \(\overline{z}\) by \(x\):

\[ \overline{z}(x \rightarrow z_y) = (z_0, \ldots, z_{y-1}, x, z_{y+1}, \ldots, z_{t-1}) \]

**II. SMALL-d CONSTRUCTION**

A description of the Small-d MSR code construction is provided in this section. This description includes associating a datacube structure with a codeword in a Small-d MSR code and identifying parity-checks that are imposed on this data structure. This is the same datacube structure that appears in the description of the Coupled-Layer MSR code in [20]. However, the parity-check equations take on a different form and this difference is explained in Section II-E. Proof of the data collection and node repair properties of the Small-d MSR codes is deferred to Sections V and VI.

Small-d MSR codes are constructed over a finite field \(\mathbb{F}_q\) of size \(q\) and have parameters given by

\[ (n = st, k = n - r, d = k + s - 1), \ (\alpha = s^t, \beta = s^{t-1}), \ s \in \{2, 3, 4\}, \]
where $r, t$ are integers such that $r \geq s \geq 2$, $t > 1$. The field size is linear in the length $n$ of the code, i.e., $q = O(n)$, with the precise relationship (see Theorem VII.5) dependent on the value of $d$ within the set $\{k + 1, k + 2, k + 3\}$.

A. Extension to General Parameter Sets

We note that through shortening, we can obtain codes for any $(n, k, d \in \{k + 1, k + 2, k + 3\})$. In particular if $s \nmid n$ where $s = d - k + 1$, then we can first set $t = \lceil \frac{n}{s} \rceil$ and $\delta = n - st$ and proceed to construct a Small-$d$ MSR code $C'$ having parameters $(n + \delta, k + \delta, d + \delta)$. We can shorten $C'$ thereafter, to obtain the MSR code $C$ having the desired parameters $(n, k, d)$.

B. Data Cube Representation of the Codeword

As in the case of the Coupled-Layer MSR code, each codeword in a Small-$d$ MSR code is associated to a datacube of dimension $(s \times t \times s^t)$ (see Figure 1(a)). It will be found convenient to view the datacube as a collection of $st$ planes, each of size $n = (s \times t)$. Thus the data cube contains in all $n\alpha = ts^{t+1}$ points. Each point in the data cube is indexed by the three tuple $(x, y, z)$ where $x \in [0 : s - 1], y \in [0 : t - 1]$ and $z \in \mathbb{Z}_t^{st}$ and is associated to a unique code symbol $C(x, y; z)$. The collection of $n\alpha$ code symbols are given by:

$$\{C(x, y; z) | x \in [0 : s - 1], y \in [0 : t - 1], z \in \mathbb{Z}_t^t\}.$$

Each $(x, y)$ 2-tuple is associated to a node or storage unit in the distributed data storage network comprising of $n = st$ nodes. The vector $z$ is used to index the $\alpha = st$ planes and also serves as an index for the $\alpha$ code symbols contained within a node.

C. Pictorial Identification of the Planes in the Datacube

We associate a plane-dot-representation to each $(s \times t)$ plane indexed by $z$ where a (red) dot is inserted in a location $(x,y)$ iff $z_y = x$. See Figure 1(b) for an example where for a plane $z = (1, 0, 1)$, the dots are inserted at locations $(1, 0), (0, 1), (1, 2)$. Thus the location of the dot within the plane serves to uniquely identify the plane.

D. Parity Check Matrix

The Small-$d$ MSR code will be identified via an $(r\alpha \times n\alpha)$ parity check (p-c) matrix $H$ that imposes $r\alpha$ p-c equations on the code symbols associated with the datacube. We associate $r$ parity checks to each plane $z$ and thus index a parity check equation or equivalently, a row of p-c matrix, using the pair $[\ell, z]$ where $\ell \in [r]$ and
\( z \in \mathbb{Z}_s^t \). The \( n \alpha = s \times t \times s^t \) columns of the parity check matrix \( H \) are indexed using the three tuple \( [x, y, z] \) with \( x \in [0 : s - 1] \), \( y \in [0 : t - 1] \) and \( z \in \mathbb{Z}_s^t \). The entries in the p-c matrix of the Small-d code are given by:

\[
H([\ell, z], [x, y, u]) = \begin{cases} \gamma_{u,x} \theta_{x,y,u}^{\ell-1}, & z = u, \\ 0, & \text{else,} \end{cases}
\]

where \( H([\ell, z], [x, y, u]) \) is the element in the \([\ell, z]\)-th row and \([x, y, u]\)-th column of the parity check matrix. Also,

\[
\gamma_{x,x'} = \begin{cases} \gamma & x < x' \\ 1 & x > x' \\ 0 & \text{Otherwise} \end{cases}
\]

such that \( \gamma \in \mathbb{F}_q \setminus \{0, 1\} \). Additionally, the element \( \theta_{x,y,x'} \) is the entry in the \( x \)-th row and \( x' \)-th column of the matrix \( \Lambda_{s,y} \), where

\[
\Lambda_{2,y} = \begin{bmatrix} \lambda_{0,y} & \lambda_{1,y} \\ \gamma \lambda_{1,y} & \lambda_{0,y} \end{bmatrix}, \quad \Lambda_{3,y} = \begin{bmatrix} \lambda_{0,y} & \lambda_{1,y} & \lambda_{2,y} \\ \gamma \lambda_{1,y} & \lambda_{0,y} & \lambda_{3,y} \\ \gamma \lambda_{2,y} & \gamma \lambda_{3,y} & \lambda_{0,y} \end{bmatrix}, \quad \Lambda_{4,y} = \begin{bmatrix} \lambda_{0,y} & \lambda_{1,y} & \lambda_{2,y} & \lambda_{3,y} \\ \gamma \lambda_{1,y} & \lambda_{0,y} & \lambda_{3,y} & \lambda_{2,y} \\ \gamma \lambda_{2,y} & \gamma \lambda_{3,y} & \lambda_{0,y} & \lambda_{1,y} \\ \gamma \lambda_{3,y} & \gamma \lambda_{2,y} & \gamma \lambda_{1,y} & \lambda_{0,y} \end{bmatrix}.
\]

Further, the entries of the matrices \( \Lambda_{s,y} \) are selected in such a way that for \( s = 2 \), all the elements in the set \( \{\lambda_{0,y}, \lambda_{1,y} | y \in [0 : t - 1]\} \) form a set of 3t distinct nonzero elements of \( \mathbb{F}_q \setminus \{0\} \). For \( s \in \{3, 4\} \), the analogous requirement is that all the elements in the set \( \{\lambda_{0,y}, \lambda_{i,y} | i \in [3], y \in [0 : t - 1]\} \) form a set of 7t distinct nonzero elements of \( \mathbb{F}_q \setminus \{0\} \). Finally, \( \mathbb{F}_q \) is a field of characteristic 2. In Theorem VII.5 we show how to pick the elements \( \{\lambda_{i,y}\} \) given that \( q \geq 6t + 2 \) for \( s = 2 \) and \( q \geq 18t + 2 \) for \( s = \{3, 4\} \).

The \([\ell, z]\)-th parity check equation is given by:

\[
\sum_{y=0}^{t-1} \sum_{x=0}^{s-1} \sum_{u \in \mathbb{Z}_s^t} H([\ell, z], [x, y, u])C(x, y; u) = 0.
\]

By applying the equation (2) we get:

\[
\sum_{y=0}^{t-1} \sum_{x=0}^{s-1} \left( \theta_{x,y,z,y}^{\ell-1} C(x, y; z) + 1_{x \neq z_y} \gamma_{x,y,z} \theta_{z,y,x}^{\ell-1} C(z, y; z(x \rightarrow z_y)) \right) = 0.
\]

By parity check equations associated to the plane \( z \), we will mean the p-c equations resulting from fixing \( z \in \mathbb{Z}_s^t \) and varying \( \ell \in [r] \). The symbols participating within a p-c equation can be differentiated as in-plane symbols and out-of-plane symbols as indicated in equation (5) and as illustrated using circles, in Fig 2.

Notice that there are \( n = st \) in-plane symbols and \( t(s - 1) \) out-of-plane symbols participating in parity check equation shown in (5). We will now show in Lemma II.1 that these \( (2s - 1)t \) symbols together are elements of an \((2s - 1)t, (2s - 1)t - r \) Generalized Reed Solomon (GRS) code by showing that the \( (2s - 1)t \) p-c variables \( \cup_{y=0}^{t-1} \{\theta_{x,y,z,y}, \theta_{z,y,x} | x \in [0 : s - 1]\} \) that appear in equation (5), are all distinct.

**Lemma II.1.** The collection of \( \theta \)'s shown below correspond to \((2s - 1)t\) distinct symbols in \( \mathbb{F}_q \) for any \( z \in \mathbb{Z}_s^t \).

\[
\cup_{y=0}^{t-1} \{\theta_{x,y,z,y}, \theta_{z,y,x} | x \in [0 : s - 1]\}.
\]

**Proof:** We will first show that there are \((2s - 1)\) distinct elements in \( A_y \). \( \{\theta_{x,y,z,y}, \theta_{z,y,x} | x \in [0 : s - 1]\} \) are \((2s - 1)\) elements in \((s \times s)\) matrix \( \Lambda_{s,y} \) where \( s \) elements are from column \( z_y \) of \( \Lambda_{s,y} \) and remaining \( s - 1 \) elements are non-diagonal elements from row indexed by \( z_y \) of matrix . From equation (4) it can be verified that these elements are distinct for any \( y \). It is clear to see that \( A_y \cap A_y' = \phi \) for \( y \neq y' \) as the elements in \( A_y \) and \( A_y' \) are picked from matrices \( \Lambda_{s,y} \), \( \Lambda_{s,y'} \) respectively and by definition these matrices have distinct symbols from \( \mathbb{F}_q \). □
E. Making a Connection with the Clay Code

An \((n = st, k = n - r, d = k + s - 1)\) Clay code can be defined using the \((rα \times nα)\) p-c matrix \(H_{\text{Clay}}\) shown below:

\[
H_{\text{Clay}}(\ell, z, [x, y, u]) = \begin{cases} 
\theta_{(x,y)}^{\ell-1}(z) & z = u \\
\gamma \theta_{(u,y)}^{\ell-1}(z) & z = u(x \rightarrow u_y), x \neq u_y \\
0 & \text{otherwise,}
\end{cases}
\]  \quad (6)

where the pair \([\ell, z]\) indexes the rows with \(\ell \in [r], z \in \mathbb{Z}_t^k\) and the triple \([x, y, u] \in [0 : s - 1] \times [0 : t - 1] \times \mathbb{Z}_s^t\) indexes the columns. Here, we impose the condition that \(γ^2 \neq 1\) and \(\{θ_{x,y} | x \in [0 : s - 1], y \in [0 : t - 1]\}\) are a collection of \(n = st\) distinct elements in \(\mathbb{F}_q\) where \(q \geq n\). If the symbols \(\{C(x, y, z) | x \in [0 : s - 1], y \in [0 : t - 1], z \in \mathbb{Z}_s^t\}\) are the \(nα\) code symbols of the Clay code, the parity check equations are then given by:

\[
\sum_{y=0}^{t-1} \sum_{x=0}^{s-1} \begin{pmatrix} \theta_{(x,y)}^{\ell-1}(z)C(x, y; z)_{\text{in-plane}} + 1_{x \neq z_0} \gamma \theta_{(x,y)}^{\ell-1}(z)C(z, y; x \rightarrow z_y)_{\text{out-of-plane}} 
\end{pmatrix} = 0 \quad \text{for any } \ell \in [r], z \in \mathbb{Z}_s^t.
\]  \quad (7)

Notice that \((2s - 1)t\) code symbols appear in the p-c equations associated to a plane \(z\). However, they do not form a GRS code unlike in the case of the Small-d code. As a result of this Clay code structure, when one attempts to carry out single-node repair using a collection of \(d < n - 1\) helper nodes, during repair of failed node \((x_0, y_0)\) the \((s - 1)\) nodes \(\{(x, y_0) | x \in [0 : s - 1] \setminus \{x_0\}\}\) must necessarily be part of the \(d\) helper nodes. The remaining \(d - s + 1 = k\) helper nodes can be chosen arbitrarily. Thus, one cannot choose any \(d\) nodes to aid in node repair as is required of a regenerating code. This problem is circumvented in the case of the Small-d MSR code construction by ensuring that all \((2s - 1)t\) p-c variables appearing in the p-c equation (5), are distinct.

III. Partitioning of Erasure Patterns and the Equivalence Classes of Planes

We will now introduce the notation and terminology that will be used to show that the Small-d MSR code is indeed an optimal-access MSR code. For this, we have to show that the Small-d MSR code is an MDS code and that it possesses the optimal-access repair property.
A. Steps Involved in Establishing the MDS and Optimal-Access Repair Properties

In order to prove the MDS property it is enough to show that the code is able to recover from the erasure of the code symbols associated to any \((n - k) = r\) nodes. This implies recovering \(r\) symbols from each of the \(\alpha = s^t\) planes. We provide a sequential decoding algorithm where the planes are first associated with an intersection score (see Definition 1) and are ordered by that score. The erased symbols corresponding to planes with lower intersection score are decoded first followed by planes having larger intersection score. The planes having the same intersection score, are partitioned into equivalence classes and all the planes within the same equivalence class are decoded together. The partitioning into equivalence classes is introduced in Definition 2. It will be shown in the subsequent section, Section V, how recovery of erased symbols reduces to the problem of proving the invertibility of certain sub-matrices of the p-c matrix that are introduced within the present section, in Definitions 4, 5. Specifying these sub-matrices calls for a partitioning of the erasure pattern into three distinct subsets (see Definition 3) of erasures, a partitioning that is dependent on the plane index \(z\).

To establish the optimal-access repair property, we provide a sequential repair algorithm in which the \((n-1-d) = r-s\) nodes that do not participate in the repair process are regarded as nodes that have been erased. We use the term aloof nodes to refer to these nodes. Only a subset \(\beta\) of the \(\alpha\) planes within the datacube participate in the repair process and are hereby referred to as repair planes. We associate an intersection score with each of the repair planes and as was the case with the sequential decoding algorithm described above, we partition the repair planes into equivalence classes and simultaneously repair planes lying within the same equivalence class. In Section VI we will show how recovery of failed node can be reduced to showing the invertibility of certain sub-matrices of the p-c matrix.

Remark 1. In this section, we will use the symbol \(E\) in two different ways. During the proof of the MDS property of the Small-\(d\) MSR code, \(E\) will denote the set of \(r\) erased nodes. During the proof of the optimal-access repair property of the Small-\(d\) MSR code, \(E\) will refer to the \((r-s)\) aloof nodes that do not participate in the repair process and thus may be regarded as being erased.

Definition 1. Intersection Score corresponding a plane \(z\) and an erasure pattern (aloof node) set \(E\) is given by:

\[
\sigma(E, z) = |\{(z_y, y) \in E | y \in [0 : t-1]\}| = |E_{0,z}|.
\]  

(8)

This can also be seen as the hole-dot count in the plane-dot representation, where holes (dotted-circles) correspond to the erasure (aloof-node) pattern and dots indicate the plane index. See Fig.3 for an illustration.

\[\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(a) \; z = (1,1,1), \; \sigma(E, z) = 0
\end{array}
\end{array}
\begin{array}{c}
(0,1,1), \; \sigma(E, z) = 1
\end{array}
\begin{array}{c}
(0,0,1), \; \sigma(E, z) = 2
\end{array}
\end{array}
\end{array}
\end{array}\]

Fig. 3. Illustration of the Intersection Scores of erasure (aloof-node) pattern \(E = \{(0, 0), (0, 1)\}\) over various planes and erasures (aloof-nodes) are indicated as holes (the dotted circles).

In the sequential decoding (repair) algorithm, the planes within an equivalence class are decoded (repaired) together. Given an erasure (aloof node) pattern \(E\) and a plane \(z\), we define the equivalence class of planes \(Q(E, z)\) below.

Definition 2 (Equivalence Class of \(z\)). Given a plane \(z\), we use \(Q(E, z)\) to denote the collection of planes

\[
Q(E, z) = S_0 \times S_1 \cdots S_{t-1},
\]
where the $S_y, y \in [0 : t - 1]$ are given by:

$$
S_y = \begin{cases} 
\{x|(x,y) \in E\} & (z_y,y) \in E \\
\{z_y\} & \text{otherwise}.
\end{cases}
$$

(9)

The collection $Q(E, z)$ contains $z$ and will turn out to represent the set of planes that are decoded together during the process of recovering the erased symbols $E$ contained within the plane $z$. The collection $Q(E, z)$ can be verified to satisfy the following closure property:

$$
 w \in Q(E, z) \iff Q(E, z) = Q(E, w).
$$

Fig. 4. Illustrating the definition of the equivalence class of planes that are decoded together for erasure pattern $E = \{(0,0), (1,0), (1,1)\}$: $Q(E, z_1) = Q(E, z_2) = \{z_1, z_2\}$, $Q(E, z_3) = Q(E, z_4) = \{z_3, z_4\}$ where $z_5 = (0,1,1)$ and $Q(E, z_6) = \{z_6\}$.

Remark 2. It is clear to see that $Q(E, z)$ is an equivalence class of $z$ as for any $w \in Q(E, z)$, $Q(E, w) = Q(E, z)$ and for any $w \notin Q(E, z)$, $Q(E, z) \cap Q(E, w) = \emptyset$.

We define below a partitioning of erasure pattern set $E$ into three subsets given a plane $z$. This will be used in defining the p-c sub-matrices whose invertibility will imply the MDS and optimal access properties. In Fig. 5, erasures are indicated as holes (the dotted circles).

Definition 3 (Erasure Patterns). Given an erasure pattern $E$ and a plane $z \in Z^t$ we define a partitioning of the erasure patterns as following:

$$
E_{0,z} = \{(z_y,y) \in E\} \text{ (hole-dot pairs)}
$$

$$
E_{1,z} = \{(x,y) \in E \mid (z_y,y) \notin E\} \text{ (holes without hole-dot pair in their column)}
$$

$$
E_{2,z} = \{(x,y) \in E \mid x \neq z_y, (z_y,y) \in E\} \text{ (holes with hole-dot pair in their column)}
$$

Fig. 5. Illustration of the erasure pattern partitioning for $E = \{(0,0), (1,0), (1,1)\}$ over various planes.

We will refer to the subsets in the partition $E_{0,z}, E_{1,z}$ and $E_{2,z}$ as mild, moderate and serious erasures respectively.

Lemma III.1. For any plane $w$ in the equivalence class of $z$, $w \in Q(E, z)$ it follows that:

1) The subset of moderate erasures remains the same i.e., $E_{1,w} = E_{1,z}$. 

2) The cardinality of the mild and serious erasures remains the same i.e., $|E_0,w| = |E_0,z|$, $|E_2,w| = |E_2,z|$.
3) The intersection score is the same i.e., $\sigma(E,w) = \sigma(E,z)$.
4) $|Q(E,z)| = \prod_{y=0}^{t-1} (e_y + 1)$, $|E_2,z| = \sum_{y=0}^{t-1} e_y$, $S_y = \{ z_y \} \cup E_2,z(y)$ where:

$$E_2,z(y) = \{ x' \mid (x', y') \in E_2,z, y' = y \} \text{ and } e_y = |E_2,z(y)|.$$
5) If $|E_2,z| = 0$ then $Q(E,z) = \{ z \}$.

Proof: The proof follows directly from Definitions [2], [3] and [1]. \square

B. MDS Sub-Matrix and The Reduced Matrix

Let $H$ denote the overall $p$-c matrix of the MSR code appearing in equation [2]. We will now define sub-matrices of the $p$-c matrix, $H_{E,z}$ for any $z \in \mathbb{Z}_n^t$, $E \subseteq [0 : s - 1] \times [0 : t - 1]$ such that $|E| = r$, whose invertibility would imply the MDS property. The proof of this appears in Theorem V.1.

Definition 4 (MDS Sub-Matrix). Given an erasure pattern $E$ such that $|E| = r$ and plane $z \in \mathbb{Z}_n^t$, we set $p = |Q(E,z)|$. We use $H_{E,z}$ to denote the $(rp \times rp)$ sub-matrix of $H$ obtained by restricting attention to $p$-c equations indexed by planes in $Q(E,z)$ and erased symbols $E$ within planes in $Q(E,z)$ i.e.,

$$H_{E,z}(\ell, v, [x, y, u]) = H(\ell, v, [x, y, u]), \quad u, v \in Q(E,z), \quad (x, y) \in E, \quad \ell \in [r].$$

We now define a further small sub-matrix of $p$-c matrix whose invertibility implies invertibility of $H_{E,z}$. This will be shown in Theorem V.2.

Definition 5 (The Reduced Matrix). Given an erasure or aloof-node pattern $E$ and plane $z \in \mathbb{Z}_n^t$, we set $p = |Q(E,z)|$ and $\mu = |E_2,z|$. We use $H_{E,z}^{rd}$ to denote the $(\mu p \times \mu p)$ sub-matrix of $H$ obtained by restricting attention to $\mu$ $p$-c equations indexed by planes in $Q(E,z)$ and erased symbols from set $E_2,u$ for planes $u \in Q(E,z)$ i.e.,

$$H_{E,z}^{rd}(\ell, v, [x, y, u]) = H(\ell, v, [x, y, u]), \quad u, v \in Q(E,z), \quad (x, y) \in E_2,u, \quad \ell \in [\mu].$$

IV. AN EXAMPLE CODE $s = 2$, $r = 3$

Before proving the MDS and optimal-access repair properties for the general case of any $s \in \{2, 3, 4\}$ and $r \geq s$, we present the proof for example case of $s = 2$ and $r = 3$. The ideas used to prove the lemmas in this section will help understand the reduction proofs for MDS property in Theorems V.1 V.2 and the reduction proofs for optimal-access repair property presented in Theorems VI.2 VI.3. The parameters of Small-d code for this example are as follows:

$$(n = 2t, k = n - 3, d = n - 2), \quad (\alpha = 2^t, \beta = 2^{t-1}).$$

Note that by using the idea of shortening described in Section III, one can construct optimal-access MSR code for any $(n, k = n - 3, d = n - 2)$. We will start by showing the MDS property for the example in Lemma IV.1.

Lemma IV.1 (MDS property for $s = 2$, $r = 3$). Small-d construction for $s = 2, r = 3$ is an MDS code.

Proof: MDS property can be shown by proving that we can recover from any $r = 3$ erasures. The type of erasure patterns $E$ can be classified in to two cases, (1) where all the three erasures have different $y$ and (2) where two erasures have same $y$. In both the cases erased symbols are recovered by arranging the planes sequentially in increasing order of intersection score and decoding erased symbols plane by plane.

1) Case 1: Three erasures with different $y$: Let $E = \{ (x_1, y_1), (x_2, y_2), (x_3, y_3) \}$ be the set of erasures where $y_1, y_2, y_3$ are distinct. In this case, for any $z \in \mathbb{Z}_2^t$, the equivalence class of $z$ contains $z$ alone i.e., $Q(E,z) = \{ z \}$. 0-th Step: Consider planes $z \in \mathbb{Z}_2^t$ with intersection score $\sigma(E,z) = 0$. In this case for all $y \in [0 : t - 1]$, $(z, y) \notin E$. Therefore all the out-of-plane symbols participating in the $[\ell, z]$-th $p$-c equation are known. Hence the $[\ell, z]$-th $p$-c equation (5) reduces to:

$$\sum_{(x,y) \in E} \theta_{x,y,z}^{t-1} C(x, y; z) = \kappa^*.$$
where $\kappa^*$ can be computed from the unerased symbols. Therefore by varying $\ell \in [3]$, erased symbols corresponding to this plane $\{C(x, y; \bar{z}) \mid (x, y) \in E\}$ can be recovered as $\theta_{x,y,z}$’s are distinct for $(x, y) \in E$. The $j$-th step: consider planes $\bar{z} \in \mathbb{Z}_2^t$ with intersection score $\sigma(E, \bar{z}) = j$. Then the $[\ell, \bar{z}]$-th p-c equation can be written as:

$$\sum_{(x,y)\in E} \theta_{x,y,z}^{\ell-1} C(x, y; \bar{z}) + \sum_{y(z, y, z)\in E, x\neq z} \gamma_{x,y,z}^{\ell-1} C(z, y; \bar{z}(x \rightarrow z)) = \kappa^*, \quad (\ell, \bar{z})$$

where $\kappa^*$ can be computed from the unerased symbols. We will now make an observation that the out-of-plane symbols appearing in the above equation are known. For $y$ such that $(z, y) \in E$, by the choice of erasure pattern, there are no more erasures in with same $y$ i.e., for any $x \neq z$, $(x, y) \notin E$ and therefore $\sigma(\bar{z}(x \rightarrow z)) = j - 1$. Hence the out-of-plane symbol $C(z, y; \bar{z}(x \rightarrow z))$ is recovered in $(j - 1)$-th step and is available during the $j$-th step. Therefore the $[\ell, \bar{z}]$-th p-c equation reduces to equation (12) and the erased symbols in this plane can be recovered due to distinctness of $\theta_{x,y,z}$’s for $(x, y) \in E$.

By end of all steps we have recovered all the erased symbols $\{C(x, y; \bar{z}) \mid (x, y) \in E, \bar{z} \in \mathbb{Z}_2^t\}.$

2) Case 2: Two erasures with same $y$: Let $E = \{(x, y_1), (y_1, y_2), (x_2, y_2)\}$ be the set of erasures where $y_1 \neq y_2$. The intersection scores that are possible in this case are 1, 2 with plane $\bar{z}$ having intersection score 1 when $z_{y_2} \neq x_2$ and intersection score 2 when $z_{y_2} = x_2$.

1 - st step: Consider planes $\bar{z} \in \mathbb{Z}_2^t$ such that $y_{z_1} = 0, z_{y_2} \neq x_2$. These planes have intersection score $\sigma(E, \bar{z}) = 1$. The $[\ell, \bar{z}]$-th p-c equation (5) reduces to:

$$\sum_{(x,y)\in E} \theta_{x,y,z}^{\ell-1} C(x, y; \bar{z}) + \sum_{y(z, y, z)\in E, x\neq z} \gamma_{x,y,z}^{\ell-1} C(z, y; \bar{z}(x \rightarrow z)) = \kappa^*.$$  

Here the out-of-plane symbol $C(0, y_1; \bar{z}(1 \rightarrow y_1))$ is unknown as the intersection score of the plane $w = \bar{z}(1 \rightarrow y_1)$ is $\sigma(E, w) = 1$. Therefore there are 4 unknowns and 3 p-c equations by varying $\ell \in [3]$. The equivalence class of $\bar{z}$ in this case is given by $Q(E, \bar{z}) = \{z, w\}$. We will therefore also consider the $[\ell, w]$-th p-c equations for $\ell \in [3]$:

$$\sum_{(x,y)\in E} \theta_{x,y,w}^{\ell-1} C(x, y; w) + \gamma_{0,1}^{\ell-1} C(1, y_1; w(0 \rightarrow w_{y_1})) = \kappa^*.$$  

Together the 6 equations in (13) and (14) have 6 unknowns and the equations are as shown below.

$$\begin{bmatrix}
\theta_{0,y_1,0} & \theta_{1,y_1,0} & \gamma \\
\theta_{0,y_1,0} & \theta_{1,y_1,0} & \gamma \\
\gamma & \gamma & \gamma
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
C(0, y_1; \bar{z}) \\
C(1, y_1; \bar{z}) \\
C(x_2; y_2; \bar{z})
\end{bmatrix}$$

Note that $\gamma_{1,0} = 1$ and $\gamma_{0,1} = \gamma$ from equation (5). Therefore the erased symbols corresponding to the planes $\bar{z}, w$ can be recovered given the following matrix is invertible.

$$H_{E, \bar{z}} = \begin{bmatrix}
\theta_{0,y_1,0} & \theta_{1,y_1,0} & \gamma \\
\theta_{0,y_1,0} & \theta_{1,y_1,0} & \gamma \\
\gamma & \gamma & \gamma
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}$$

Let the vector $f = (f_0, 0, f_0, 1, f_0, 2, f_1, 0, f_1, 1, f_1, 2)^T$ be in the left null space of $H_{E, \bar{z}}$ i.e., $f^T H_{E, \bar{z}} = 0$ and let
\[ f_1(x) = \sum_{\ell=1}^{3} f_{1,\ell} \theta^{\ell-1} \] for \( i = 0, 1 \). It is clear to see that:

\[
\begin{align*}
    f_0(\theta_{0,y_i,0}) &= f_0(\theta_{x_2,y_2,z_{y_2}}) = 0 \\
    f_1(\theta_{1,y_i,1}) &= f_1(\theta_{x_2,y_2,z_{y_2}}) = 0 \\
    f_0(\theta_{0,y_i,1}) + f_1(\theta_{0,y_i,1}) &= 0 \\
    f_0(\theta_{1,y_i,0}) + \gamma f_1(\theta_{1,y_i,0}) &= 0.
\end{align*}
\]

By the assignment of coefficients shown in equation (4), \( \theta_{0,y_i,0} = \theta_{1,y_i,1} = \lambda_{0,y_i} \). Therefore, both the polynomials \( f_0, f_1 \) have \( \lambda_{0,y_i} \) and \( \theta_{x_2,y_2,z_{y_2}} \) as roots and hence can be expressed as \( f_i(\theta) = f^R_{i}(\theta - \lambda_{0,y_i})(\theta - \theta_{x_2,y_2,z_{y_2}}) \) for \( i = 0, 1 \) where \( f^R_0, f^R_1 \) are constants. Substituting this expression in equations (16) and (17) we get:

\[
\begin{bmatrix}
1 \\
1 \\
\gamma
\end{bmatrix}
\begin{bmatrix}
f^R_0 \\
f^R_1
\end{bmatrix}
= 0.
\]

Therefore, \( f^R_0 = f^R_1 = 0 \) as \( \gamma \neq 1 \) implying that the polynomials \( f_0, f_1 \) are zeroes and that \( f = 0 \). Therefore the erased symbols corresponding to the planes \( z, z(1 \to z_{y_1}) \) given by \( \{C(x,y,z), C(x,y,z(1 \to z_{y_1})) \mid (x,y) \in E\} \) can be recovered.

2-nd Step: Consider planes \( z \in \mathbb{Z}_2^t \) such that \( z_{y_1} = 0 \) and \( z_{y_2} = x_2 \). These planes have intersection score \( \sigma(E,z) = 2 \) and the \([\ell,z]-th \) p-c equation can be written as:

\[
\sum_{(x,y) \in E} \theta^{\ell-1}_{x,y,z_0} C(x,y;z) + \sum_{y:z \neq z_y} \gamma_{x,y} \theta^{\ell-1}_{\gamma_{x,y}} C(x,y;z) = \kappa_{s}
\]

\[
\sum_{(x,y) \in E} \theta^{\ell-1}_{x,y,z_0} C(x,y;z) + \gamma_{x,y} \theta^{\ell-1}_{\gamma_{x,y}} C(0,z_0;1 \to z_{y_1}) + \gamma_{x_2,y_2,z_2} \theta^{\ell-1}_{\gamma_{x_2,y_2,z_2}} C(x,y,z_0;1 \to z_{y_2}) = \kappa_{s}.
\]

The plane \( z((x_2 \oplus 1) \to z_{y_2}) \) has intersection score \( \sigma(E,z((x_2 \oplus 1) \to z_{y_2})) = 1 \), therefore the symbol \( C(x_2,y_2,z((x_2 \oplus 1) \to z_{y_2})) \) is already recovered in the first step. Hence the \([\ell,z]-th \) p-c equation reduces to equation (13). The equivalence class of \( z, Q(E,z) = \{z, z(1 \to z_{y_1})\} \). Therefore, we look at p-c equations corresponding to plane \( w = z(1 \to z_{y_1}) \). The \([\ell,w]-th \) p-c equation reduces to equation (14). Therefore erased symbols corresponding to planes \( z, z(1 \to z_{y_1}), \{C(x,y,z), C(x,y,z(1 \to z_{y_1})) \mid (x,y) \in E\} \) can be recovered due to invertibility of \( H_{E,z} \) shown in equation (15).

By the end of the two steps all the erased symbols are recovered.

We will now prove the optimal-access property which along with previous lemma proves that the Small-d code is an MSR code for \( s = 2, r = 3 \).

**Lemma IV.2 (Optimal Access Property for \( s = 2, r = 3 \)).** \( \text{Small-d code for } s = 2, r = 3 \text{ satisfies the optimal-access repair property.} \)

**Proof:** Let \( (x_0,y_0) \) be the failed node and \( E = \{(x_1,y_1)\} \) be the set of aloof nodes. The number of aloof nodes in this case is \( n - 1 - d = (r-s) = 1 \). We consider two cases for aloof nodes (1) when aloof node has same \( y \) as the failed node (2) when aloof node has different \( y \).

A helper node \( (x,y) \) sends symbols \( \{C(x,y;z) \mid z \in R\} \) from the repair planes \( R = \{z \in \mathbb{Z}_2^t \mid z_{y_0} = x_0\} \). Therefore, the number of symbols downloaded from each helper node is \( \beta = 2^{t-1} \).

1) Case 1: Aloof node is \((x_1,y_1)\) where \( y_0 = y_1 \) and \( x_1 = x_0 \oplus 1 \). The \([\ell,z]-th \) p-c equation corresponding to a repair plane \( z \in R \) reduces to:

\[
\theta^{\ell-1}_{x_0,y_0,x_0} C(x_0,y_0;z) + \theta^{\ell-1}_{x_1,y_1,z_0} C(x_1,y_1;z) + \gamma_{x_0,1,x_0} \theta^{\ell-1}_{x_0,y_0,x_0} C(x_0,y_0;z((x_0 \oplus 1) \to z_{y_0})) = \kappa_{s}.
\]

This is because all the in-plane symbols of \( z \) other than the failed node and aloof node symbol given by \( \{C(x_0,y_0;z), C(x_1,y_1;z)\} \) are known. Among the out-of-plane symbols given by \( \{C(z_{y_0},y_0,z((x_0 \oplus 1) \to z_{y_0})) \mid y \in [0 : t-1], x \neq z_y \} \) the symbol \( C(z_{y_0},y_0,z((x_0 \oplus 1) \to z_{y_0})) = C(x_0,y_0,z((x_0 \oplus 1) \to z_{y_0})) \) is a failed node symbol and is hence unknown. For \( y \neq y_0, (z,y) \notin E \) and therefore it is a helper node and the
out-of-plane symbol \( C(z_y, y; \bar{z}(x \rightarrow z_y)) \) belongs to plane \( \bar{z}(x \rightarrow z_y) \) which belong to the repair planes set \( R \). Therefore the symbol \( C(z_y, y; \bar{z}(x \rightarrow z_y)) \) is available as helper information. Now by varying \( \ell \in [3] \) in equation \( \ref{eq:5} \) there are 3 equations and 3 unknowns and by Lemma [I.1] the symbols corresponding to failed node \( \{C(x_0, y_0, \bar{z}), C(x_0, y_0, \bar{z}(x_0 + 1) \rightarrow z_{y_0})\} \) can be recovered along with one aloof node symbol \( C(x_1, y_1; \bar{z}) \). By varying \( z \in R \) we can recover

\[
\{C(x_0, y_0; \bar{z}(x \rightarrow z_{y_0})) \mid z \in R, x \in \mathbb{Z}_2\} = \{C(x_0, y_0; \bar{z}) \mid z \in \mathbb{Z}_2^k\}.
\]

2) Case 2: Aloof node set is \( E = \{(x_1, y_1)\} \) where \( y_1 \neq y_0 \). The repair in this case is sequential. First the repair planes \( z \in R \) such that \( z_{y_1} \neq x_1 \) are repaired as the intersection score for such planes is \( \sigma(E, z) = 0 \) and then the repair planes with \( z_{y_1} = x_1 \) are looked at as they have intersection score 1.

0 -th Step: Let \( z \in R \) such that \( z_{y_1} = x_1 + 1 \). Then the \( [\ell, z] \)-th p-c equation reduces to equation \( \ref{eq:5} \) as in this case too the only unknown symbols are the in-plane symbols \( \{C(x_0, y_0, z), C(x_1, y_1, z)\} \) and the out-of-plane symbol \( C(x_0, y_0; \bar{z}(x_0 + 1) \rightarrow z_{y_0}) \). Therefore the failed node symbols \( C(x_0, y_0; \bar{z}) \) and the aloof node symbol \( C(x_1, y_1; \bar{z}) \) can be recovered. By the end of Step 0, we would have recovered all the aloof node symbols in plane \( z \in R \) such that \( z_{y_1} = x_1 + 1 \).

1 -th Step: Let \( z \in R \) such that \( z_{y_1} = x_1 \), the \( [\ell, z] \)-th p-c equation in this case is given by:

\[
\theta_{y_0, x_0; x_0}^{\ell-1} C(x_0, y_0, \bar{z}) + \gamma_{\ell, x_0; x_0}^{\ell-1} C(x_0, y_0, \bar{z}(x_0 + 1) \rightarrow z_{y_0}) + \theta_{y_1, x_1; y_1}^{\ell-1} C(x_1, y_1; z) + \gamma_{\ell, x_1; x_1}^{\ell-1} C(x_1, y_1; \bar{z}(x_0 + 1) \rightarrow z_{y_1}) = \kappa^*.
\]

This is because the only unknown in-plane symbols are \( \{C(x_0, y_0, z), C(x_1, y_1, z)\} \) and out-of-plane symbols are \( \{C(x_0, y_0; \bar{z}(x_0 + 1) \rightarrow z_{y_0}), C(x_1, y_1; \bar{z}(x_0 + 1) \rightarrow z_{y_1})\} \). However the aloof-node symbol \( C(x_1, y_1; \bar{z}(x_0 + 1) \rightarrow z_{y_1}) \) is already recovered in the first step. Therefore \( [\ell, z] \)-th p-c equation reduces to equation \( \ref{eq:5} \) and hence the failed node symbols \( C(x_0, y_0; \bar{z}), C(x_0, y_0, \bar{z}(x_0 + 1) \rightarrow z_{y_0}) \) and aloof node symbol \( C(x_1, y_1; \bar{z}) \) can be recovered. Therefore by the end of the algorithm all the \( \alpha \) symbols corresponding to the failed node \( \{C(x_0, y_0, z) \mid z \in \mathbb{Z}_2\} \) are recovered.

\[\square\]

V. MDS PROPERTY: THE REDUCTIONS

In this section we first start by showing in Theorem [V.1] that invertibility of the MDS Sub-Matrix \( H_{E, z} \) (see Definition [I.4]) for any erasure pattern \( E \subseteq [0 : s - 1] \times [0 : t - 1] \) such that \( |E| = r \) and any plane \( z \in \mathbb{Z}_2^k \), implies that the Small-d code satisfies the MDS property. We follow this up with Theorem [V.2] where we show that invertibility of further reduced matrix \( H_{E, z}^{\text{Red}} \) implies invertibility of \( H_{E, z} \).

Theorem V.1 (The Reduction I: MDS property). To show that Small-d construction yields an MDS code, it suffices to show that for any erasure pattern \( E \) such that \( |E| = r \), and for any plane \( z \in \mathbb{Z}_2^k \), the matrix \( H_{E, z} \) is invertible, where \( H_{E, z} \) is as shown in Definition [I.4]

Proof: To show that Small-d code is an MDS code, it is enough to show that the code can recover from any erasure pattern \( E \) such that \( |E| = r \).

Given an erasure pattern we recover the erased symbols sequentially by ordering the planes in increasing order of their intersection scores, starting from 0 and recovering erased symbols lying in planes \( z \) having intersection score \( \sigma(E, z) = 0 \), then \( \sigma(E, z) = 1 \) and so on. Among the planes that have same intersection score, say for plane \( z \) such that \( \sigma(E, z) = j \), we look at the planes in equivalence class of \( z \), \( Q(E, z) \) and decode them together. This can be done because all the planes in \( Q(E, z) \) have same intersection score by Lemma [III.1].

Step 0: Let \( z \in \mathbb{Z}_2^k \) with intersection score \( \sigma(E, z) = 0 \). Then in the \( [\ell, z] \)-th p-c equation shown in \( \ref{eq:5} \), all the out-of-plane symbols \( \{C(z_y, y; \bar{z}(x \rightarrow z_y)) \mid y \in [0 : t - 1], x \in [0 : s - 1] \setminus \{z_y\}\} \) are known as for any \( y \in [0 : t - 1], (z_y, y) \notin E \). Therefore the \( [\ell, z] \)-th p-c equation \( \ref{eq:5} \) reduces to:

\[\sum_{(x, y) \in E} \theta_{x, y; z_y}^{\ell-1} C(x, y; z) = \kappa^* \text{ for all } \ell \in [r],\]
where $\kappa^*$ indicates the quantity that can be computed from known symbols. The equivalence class of $\bar{z}$, $Q(E, \bar{z})$ consists of just the single plane $\{\bar{z}\}$, therefore $H_{E, \bar{z}}$ is an $(r \times r)$ matrix and $H_{E, \bar{z}}([\ell, \bar{z}], [x, y, \bar{z}]) = \theta_{x,y,\bar{z}}^{\ell-1}$ for any $\ell \in [r]$ and $(x, y) \in E$. It follows that if $H_{E, \bar{z}}$ is invertible, the erased symbols corresponding to this plane $\bar{z}$ can be recovered.

Step j: Let $\bar{z} \in \mathbb{Z}_q^r$ be such that $\sigma(E, \bar{z}) = j$ and let us assume that the erased symbols corresponding to planes $\psi$ having intersection score $\sigma(E, \psi) \leq (j - 1)$ have already been recovered, then the $[\ell, \bar{z}]$-th p-c equation further simplifies to:

$$\sum_{(x,y) \in E} \theta^{\ell-1}_{x,y,\bar{z}} C(x, y; z) + \sum_{\gamma_{x,y,z} \in E : x = 0} \sum_{s=0}^{s-1} \gamma_{x,y,z} \theta_{x,y,z}^{\ell-1} C(z, y, \bar{z}; x \to z) = \kappa^*$$

for all $\ell \in [r]$.

Let $(z, y) \in E$, then $\sigma(E, \bar{z}; x \to y) \leq j$ with $\sigma(E, \bar{z}; x \to y) = j$ iff $(x, y) \in E$. This implies that the symbols $C(z, y, \bar{z}; x \to y)$ are recovered in the $(j - 1)$ step if $(x, y) \not\in E$, whereas they are unknown if $(x, y) \in E$. Therefore the $[\ell, \bar{z}]$-th p-c equation further simplifies to:

$$\sum_{(x,y) \in E} \theta^{\ell-1}_{x,y,\bar{z}} C(x, y; \bar{z}) + \sum_{(x,y) \in E, \bar{z}} \gamma_{x,y,z} \theta_{x,y,z}^{\ell-1} C(z, y, \bar{z}; x \to y) = \kappa^*$$

for all $\ell \in [r]$. (19)

This follows as $\gamma_{x,y,z} = 0$ if $x = z$ and by the Definition of the set $E_2, \bar{z}$. Notice that for the case when $|E_2, \bar{z}| = 0$ there are $r$ equations and $r$ unknowns in the above equation. From Lemma [III.1] $Q(E, \bar{z}) = \{\bar{z}\}$ and $H_{E, \bar{z}}$ is an $r \times r$ matrix with $H_{E, \bar{z}}([\ell, \bar{z}], [x, y, \bar{z}]) = \theta_{x,y,\bar{z}}^{\ell-1}$ for all $\ell \in [r]$ and $(x, y) \in E$. Invertibility of $H_{E, \bar{z}}$ implies that we can recover the erased symbols $\{C(x, y; \bar{z}) \mid (x, y) \in E\}$.

In the case when $|E_2, \bar{z}| > 0$, the number of erased symbols appearing in the p-c equation (19) associated to plane $\bar{z}$ is greater than the number of equations $r$. It turns out that in this case, if we consider the p-c equations corresponding to all the planes in the equivalence class of $\bar{z}$ then we have a situation where the number of unknowns $rp$ equals the number of equations $rp$ where $p = |Q(E, \bar{z})|$. The p-c matrix associated to this set of equations is precisely $H_{E, \bar{z}}$ and hence if this p-c matrix is invertible, then all such erasures can be recovered. We will now go ahead and show that the unknowns in the p-c equations defined by indices $\{[\ell, \psi] \mid \psi \in Q(E, \bar{z}), \ell \in [r]\}$ correspond to the erased symbols in planes $Q(E, \bar{z})$. This will imply that the number of unknowns and number of equations is $rp$.

For any $\psi \in Q(E, \bar{z})$, the p-c equations are given by:

$$\sum_{(x,y) \in E} \theta^{\ell-1}_{x,y,\psi} C(x, y; \psi) + \sum_{(x,y) \in E, \bar{z}} \gamma_{x,y,z} \theta_{x,y,z}^{\ell-1} C(z, y, \bar{z}; x \to y) = \kappa^*$$

for all $\ell \in [r]$, and the symbol $C(w, y, w(x \to w))$ corresponds to the plane $w(x \to w)$ where $(x, y) \in E_2, w$. Therefore from Lemma [III.1] it follows that $w(x \to w) \in Q(E, \psi)$ and from the definition of equivalence class of planes in Definition [3] it is clear that $Q(E, \psi) = Q(E, \bar{z})$ implying $w(x \to w) \in Q(E, \bar{z})$. This would mean that the parity checks corresponding to planes in $Q(E, \bar{z})$ also involve erased symbols corresponding to planes in $Q(E, \bar{z})$ alone and therefore invertibility of sub matrix $H_{E, \bar{z}}$ would imply recoverability of erased symbols in planes $Q(E, \bar{z})$.

\[ \square \]

**Theorem V.2 (The Reduction II: MDS property).** Let $E$ be an erasure pattern of size $r$ and let $z$ be a plane. For the case when $|E_2, \bar{z}| = 0$, $H_{E, \bar{z}}$ is invertible. Otherwise, $H_{E, \bar{z}}$ is invertible if $H_{E, \bar{z}}^{\text{red}}$ is invertible.

**Proof:** Let $p = |Q(E, \bar{z})|$ and $f$ be a vector in $\mathbb{F}_q^p$ such that $f^T H_{E, \bar{z}} = 0$ and $f^T = (f_{\ell, \psi} \mid \ell \in [r], \psi \in Q(E, \bar{z}))$. Let $f_{\psi}$ be a polynomial defined as:

$$f_{\psi}(x) = \sum_{\ell=1}^{r} f_{\ell, \psi} x^{\ell-1}$$

Given $f^T H_{E, \bar{z}} = 0$ we want to show that $f = 0$ to prove the invertibility of $H_{E, \bar{z}}$. $f^T H_{E, \bar{z}} = 0$ implies that:

$$\sum_{\psi \in Q(E, \bar{z})} r \sum_{\ell=1}^{r} f_{\ell, \psi} H([\ell, \psi], [x, y, u]) = 0$$

for any $(x, y) \in E$ and $u \in Q(E, \bar{z})$. (20)
By definition of Small-d construction, the assignment of $H([\ell, \nu], [x, y, u])$ is given by:

$$H([\ell, \nu], [x, y, u]) = \begin{cases} \theta_{x,y,u}^{\ell-1} & \nu = u \\ \gamma_{u,y} x \theta_{x,y,u}^{\ell-1} & \nu = u(x \rightarrow u_y), x \neq u_y \\ 0 & \text{otherwise.} \end{cases}$$

$H([\ell, \nu], [x, y, u])$ is non-zero only when $\nu = u$ and $\nu = u(x \rightarrow u_y)$. For any $y$ such that $(z_y, y) \in E_{0, \hat{z}}$, and for any $u \in Q(E, \hat{z})$ it is implied that $(u_y, y) \in E$ by the definition of equivalence class of planes in Definition 2. By considering $[x, y, u] = [u_y, y, u]$, equation (20) reduces to:

$$\sum_{\ell=1}^{r} f_{\ell,u} \theta_{x,y,u}^{\ell-1} = 0 \implies f_{\ell,u}(\theta_{x,y,u}) = 0 \text{ from equation (21) } f_{\ell,u}(\lambda_{0,y}) = 0 \text{ for all } u \in Q(E, \hat{z}), (z_y, y) \in E_{0, \hat{z}}. \quad (21)$$

For $(x, y) \in E_{1, \hat{z}}$ it implies that $(z_y, y) \notin E$ and therefore $S_y = \{z_y\}$ (in definition shown in equation (9)) and for any $u \in Q(E, \hat{z})$, $u_y = z_y$ and $u(x \rightarrow u_y) \notin Q(E, \hat{z})$. Equation (20) in this case reduces to:

$$\sum_{\ell=1}^{r} f_{\ell,u} \theta_{x,y,u}^{\ell-1} = 0 \implies f_{\ell,u}(\theta_{x,y,u}) = f_{\ell,u}(\theta_{x,y,z_y}) = 0 \text{ for all } u \in Q(E, \hat{z}), (x, y) \in E_{1, \hat{z}}. \quad (22)$$

For any $u \in Q(E, \hat{z})$ and $(x, y) \in E_{2, \hat{u}}$, it is implied that $(u_y, y) \in E$, $x \neq u_y$ and $u(x \rightarrow u_y) \in Q(E, \hat{z})$, therefore equation (20) in this case reduces to:

$$\sum_{\ell=1}^{r} \left( f_{\ell,u} \theta_{x,y,u}^{\ell-1} + \gamma_{u,y} x f_{\ell,u}(x \rightarrow u_y) \theta_{x,y,u}^{\ell-1} \right) = 0 \implies f_{\ell,u}(\theta_{x,y,u}) + \gamma_{u,y} x f_{\ell,u}(x \rightarrow u_y)(\theta_{x,y,u}) = 0, \quad (23)$$

for all $u \in Q(E, \hat{z})$ and $(x, y) \in E_{2, \hat{u}}$. For the case when $|E_{2, \hat{z}}| = 0$, equations (21) and (22) imply that there are $|E_{0, \hat{z}}| + |E_{1, \hat{z}}| = r$ roots for $f_{\ell,u}(x)$ for any $u \in Q(E, \hat{z})$ given by:

$$\{\theta_{x,y,u} \mid (x, y) \in E\}.$$

By Lemma II.1 all these $r$ roots are distinct. But $f_{\ell,u}(x)$ is an $(r - 1)$ degree polynomial implying that $f_{\ell,u}(x) = 0$ for all $u \in Q(E, \hat{z})$. This also implies that $\ell = 0$ and hence $H_{E, \hat{z}}$ is invertible.

For the case when $\mu = |E_{2, \hat{z}}| > 0$, from equations (21) and (22) it is implied that:

$$f_{\ell,u}(x) = \left( \prod_{(z_y, y) \in E_{0, \hat{z}}} (x - \lambda_{0,y}) \right) \left( \prod_{(\hat{x}, y) \in E_{1, \hat{z}}} (x - \theta_{x,y,z_y}) \right) f_{\ell,u}^{\text{Red}}(x), \quad (24)$$

where $f_{\ell,u}^{\text{Red}}(x)$ is a polynomial of degree $\mu - 1$.

By substituting equation (24) in (23) we get that for any $u \in Q(E, \hat{z}), (x, y) \in E_{2, \hat{u}}$:

$$\left( \prod_{(z_y, y) \in E_{0, \hat{z}}} (\theta_{x,y,u_y} - \lambda_{0,y}) \right) \left( \prod_{(\hat{x}, y) \in E_{1, \hat{z}}} (\theta_{x,y,u_y} - \theta_{x,y,z_y}) \right) f_{\ell,u}^{\text{Red}}(\theta_{x,y,u_y}) + \gamma_{u_y,y} x f_{\ell,u}^{\text{Red}}(x \rightarrow u_y)(\theta_{x,y,u_y}) = 0. \quad (25)$$

The term $P_1$ is clearly non-zero as $(x, y) \in E_{2, \hat{u}}$ it is implied that $x \neq u_y$ therefore by the assignment in equation (4), $\theta_{x,y,u_y} \neq \lambda_{0,y}$ for any $y \in [0 : t - 1]$ as $\theta_{x,y,u_y}$ corresponds to a non-diagonal element of $\Lambda_{x,y}$. We will now look at term $P_2$. By the definition of erasure partitions if $(\hat{x}, \hat{y}) \in E_{1, \hat{z}}$ it is implied that $(z_{\hat{y}}, \hat{y}) \notin E$. It follows from equation (9) that $S_\hat{y} = \{z_{\hat{y}}\}$ and therefore for any $u \in Q(E, \hat{z})$, $u_{\hat{y}} = z_{\hat{y}}$. From Lemma III.1 $E_{1, \hat{z}} = E_{1, \hat{u}}$, hence the term $P_2$ can be written as:

$$P_2 = \left( \prod_{(\hat{x}, \hat{y}) \in E_{1, \hat{z}}} (\theta_{x,y,u_y} - \theta_{x,y,u_y}) \right).$$

From Lemma II.1 it is clear that $P_2 \neq 0$. Therefore it follows from equation (25) that for any $u \in Q(E, \hat{z})$,
\[(x, y) \in E_{2,\mathcal{U}}:\]
\[f^{\text{Red}}_{\mathcal{U}}(\theta_{x,y,u_y}) + \gamma_{u_y,x}f^{\text{Red}}_{\mathcal{U}(x \rightarrow u_y)}(\theta_{x,y,u_y}) = 0.\]  \(26\)

Let \(f^{\text{Red}}_{\mathcal{U}}(x) = \sum_{\ell=1}^{\mu} f^{\text{Red}}_{\mathcal{U}(x \rightarrow u_y)} \ell - 1\) and let \(\underline{f}^{\text{Red}}_{\mathcal{U}}\) be a vector in \(\mathbb{P}^q\) and \(f^{\text{Red}}_{\mathcal{U}} = (f^{\text{Red}}_{\mathcal{U}} | \ell \in [\mu], \nu \in Q(E, z))^T\). Equation \((26)\) can be rewritten as:
\[
\sum_{\ell=1}^{\mu} f^{\text{Red}}_{\mathcal{U}}(H([\ell, u], [x, y, u])) + f^{\text{Red}}_{\mathcal{U}(x \rightarrow u_y)} H([\ell, u(x \rightarrow u_y)], [x, y, u]) = 0 \text{ for all } u \in Q(E, z), (x, y) \in E_{2,\mathcal{U}}
\]
\[
\sum_{\ell=1}^{\mu} \sum_{\nu \in Q(E, z)} f^{\text{Red}}_{\mathcal{U}} H([\ell, \nu], [x, y, u]) = 0 \text{ for all } u \in Q(E, z), (x, y) \in E_{2,\mathcal{U}}
\]
\[
\underline{f}^{\text{Red}}_{\mathcal{U}}(\underline{H}^{\text{Red}_{E,z}}) = 0 \text{ from the definition in equation } (11).
\]

If \(H^{\text{Red}_{E,z}}\) is invertible, this would imply that \(\underline{f}^{\text{Red}}_{\mathcal{U}} = 0\). From equation \((24)\), if \(\underline{f}^{\text{Red}}_{\mathcal{U}} = 0\), it follows that \(\underline{f} = 0\) implying \(H^{\text{Red}_{E,z}}\) is invertible.

The Theorems \(\text{VI.1}\) and \(\text{VI.2}\) together imply that it is enough to show invertibility of reduced matrix \(H^{\text{Red}_{E,z}}\) to prove the MDS property. We prove that reduced matrix is invertible in Section \(\text{VII}\).

VI. OPTIMAL ACCESS REPAIR PROPERTY: THE REDUCTIONS

Recall that during a single node repair, \(d\) helper nodes are contacted among the remaining \(n - 1\) nodes. Therefore \((n - 1 - d) = r - s\) nodes remain aloof in the repair process. To prove the optimal-access property we will first introduce a sub-matrix of the \(p\)-\(c\) matrix \(H\) called Repair Sub-Matrix. For any failed node \((x_0, y_0)\), aloof node set \(E \subseteq ([0 : s - 1] \times [0 : t - 1]) \setminus \{(x_0, y_0)\}\) such that \(|E| = r - s\) and repair plane \(z \in R\), where \(R = \{z \in \mathbb{Z}_t^s | z_{y_0} = x_0\}\) we define the sub-matrix \(H^{\text{Red}_{E,(x_0,y_0),z}}\) in Definition \(6\). We will later show in Theorem \(\text{VI.2}\) that the invertibility of this sub-matrix for any aloof node set \(E\), repair plane \(z\) would imply the optimal-access property.

We first show in the following lemma that for any repair plane \(z \in R\), the planes in equivalence class of \(z\) are in indeed repair planes.

Lemma VI.1. Let \((x_0, y_0)\) be a failed node and let \(E\) be the set of \((r - s)\) aloof nodes such that \((x_0, y_0) \notin E\). Let \(z \in R\) where \(R = \{z \in \mathbb{Z}_t^s | z_{y_0} = x_0\}\), then \(Q(E, z) \subseteq R\).

Proof: By the Definition \(2\) the equivalence class of \(z\) \(Q(E, z) = S_0 \times S_1 \times \cdots \times S_{t-1}\) where \(S_q\) is defined as shown in equation \((9)\). It is clear to see that \(S_{y_0} = \{x_0\}\) as \((z_{y_0}, y_0) = (x_0, y_0) \notin E\). Therefore for any \(u \in Q(E, z)\), \(u_{y_0} = x_0\) i.e., \(u \in R\).

We now define the repair sub-matrix \(H^{\text{Red}_{E,(x_0,y_0),z}}\) by looking at \(p\)-\(c\) equations corresponding to the planes in equivalence class of \(z\), \(Q(E, z)\) and the failed node, aloof node symbols that participate in those equations.

Definition 6 (Repair Sub-Matrix). Given a node \((x_0, y_0)\), an aloof node pattern \(E\), such that \((x_0, y_0) \notin E\), \(|E| = r - s\), and plane \(z \in R\), where \(R = \{z \in \mathbb{Z}_t^s | z_{y_0} = x_0\}\), \(H^{\text{Red}_{E,(x_0,y_0),z}}\) is defined as an \((rp \times rp)\) sub-matrix of the parity check matrix \(H\) where \(p = |Q(E, z)|\).

We can index the rows of the matrix by \([\ell, \nu]\) where \(\ell \in [r]\) and \(\nu \in Q(E, z)\) and columns by \([x, y, u]\) where,
\[
[x, y, u] \in (E \times Q(E, z)) \cup \{(x_0, y_0, w) | \hat{x} \in [0 : s - 1], w \in Q(E, z)\}
\]
\[
H^{\text{Red}_{E,(x_0,y_0),z}}([\ell, \nu], [x, y, u]) = H([\ell, \nu], [x, y, u]).
\]  \(27\)

Columns of this matrix correspond to the \((r - s)p\) aloof node symbols within the planes \(Q(E, z)\) and \(sp\) failed node symbols that are not limited to planes in \(Q(E, z)\).

Using the repair sub-matrix definition we will show that its invertibility implies the optimal-access repair property in Theorem \(\text{VI.2}\).
Theorem VI.2 (The Reduction I: Optimal-Access Property). Small-d construction satisfies the optimal-access repair property, if for any node \((x_0, y_0)\), aloof node pattern \(E \subseteq \{(0 : s - 1) \times [0 : t - 1]\} \setminus \{(x_0, y_0)\}\) such that \(|E| = (r - s)\), and for any plane \(z \in R\), where \(R = \{z \in \mathbb{Z}_s^t | z_{y_0} = x_0\}\) the repair sub-matrix \(H_{E,(x_0,y_0),z}\) is invertible, where \(H_{E,(x_0,y_0),z}\) is defined as shown in Definition 6.

Proof: To show that Small-d construction satisfies the optimal-access property, we will show that it can recover any node \((x_0, y_0)\) with the help of contact with any \(d\) helpers. Let \(E\) denote the set of aloof nodes that do not participate in repair. Therefore, \(|E| = (n - 1 - d) = r - s\). The helper information sent by a node \((x, y) \notin E \cup \{(x_0, y_0)\}\) is given by:

\[
\{C(x, y; z) | z \in R, R = \{z \in \mathbb{Z}_s^t | z_{y_0} = x_0 \} \}
\]

Given an aloof node pattern \(E\) we recover the failed node symbols sequentially by first ordering the repair planes, \(R\) by the intersection scores and then recovering failed node symbols and aloof node symbols within the repair plane. In this method, the repair planes belong to the same equivalence class are repaired together.

Step 0: \(z \in R\) such that \(\sigma(E, z) = 0\), then \(Q(E, z) = \{z\}\) and the \((\ell, z)\)-th p-c equation reduces to:

\[
\theta_{x, y_0, x_0}^\ell C(x_0, y_0; z) + \sum_{(x, y) \in E} \theta_{x, y, z_0}^\ell C(x, y; z) + \sum_{x \neq x_0} \gamma_{x, x_0} \theta_{x, y_0, x_0}^\ell C(x_0, y_0; z(x \to y_0)) = \kappa^* \text{ for all } \ell \in [r].
\]

This is because the only unknown in-plane symbols are \(\{C(x, y; z) | (x, y) \in E\} \cup \{C(x_0, y_0, z)\}\) and the unknown out-of-plane symbols are \((s - 1)\)-symbols \(\{C(x_0, y_0; z(x \to y_0)) | x \in [0 : s - 1] \setminus \{x_0\}\}\). The remaining \((s - 1)(t - 1)\) out-of-plane symbols \(\{C(z, y; z(x \to y_0)) | y \neq [0 : t - 1] \setminus \{y_0\}, x \in [0 : s - 1] \setminus \{z_0\}\}\) belong to a helper node as \((z, y) \notin E\) and are part of repair planes as \(z(x \to y_0) \in R\).

Therefore, there are \(|E| + s = r\) unknowns in the above equations corresponding to failed node symbols and \((r - s)\) aloof node symbols. Clearly, for this plane \(z\), \(H_{E,(x_0,y_0),z}\) is an \(r \times r\) matrix and if it is invertible then we can recover the failed node symbols \(\{C(x_0, y_0; z(x \to y_0)) | x \in [0 : s - 1]\}\) and \((r - s)\) aloof node symbols \(\{C(x, y; z) | (x, y) \in E\}\).

Step j: Let \(z \in R\) such that \(\sigma(E, z) = j\) and let us assume by induction that the aloof node symbols corresponding to repair planes with intersection score \(< j\) are already recovered, then the \((\ell, z)\)-th p-c equation reduces to:

\[
\theta_{x, y_0, x_0}^\ell C(x_0, y_0; z) + \sum_{x \neq x_0} \gamma_{x, x_0} \theta_{x, y_0, x_0}^\ell C(x_0, y_0; z(x \to y_0)) + \sum_{(x, y) \in E} \theta_{x, y, z_0}^\ell C(x, y; z) + \sum_{y(z, y) \in E} \sum_{x=0}^{s-1} \gamma_{x, z_0} \theta_{x_0, y, x}^\ell C(z, y; z(x \to y_0)) = \kappa^* \text{ for all } \ell \in [r].
\]

Let \((z, y) \in E\), then \(\sigma(E, z(x \to y_0)) = j\) iff \((x, y) \in E\). This implies that the symbols \(C(z, y; z(x \to y_0))\) are recovered in the \((j - 1)\) step if \((x, y) \notin E\), whereas they are unknown if \((x, y) \in E\). Therefore the \((\ell, z)\)-th p-c equation further simplifies to:

\[
\theta_{x, y_0, x_0}^\ell C(x_0, y_0; z) + \sum_{x \neq x_0} \gamma_{x, x_0} \theta_{x, y_0, x_0}^\ell C(x_0, y_0; z(x \to y_0)) + \sum_{(x, y) \in E} \theta_{x, y, z_0}^\ell C(x, y; z) + \sum_{(x, y) \in E_{x, y}} \gamma_{x, z_0} \theta_{x_0, y, x}^\ell C(z, y; z(x \to y_0)) = \kappa^* \text{ for all } \ell \in [r],
\]

by the Definition 3 of \(E_{x, y, z}\). Clearly, when \(|E_{x, y, z}| = 0\) there are \(r\) equations and \(r\) unknowns in the above equation. From Lemma III.1, \(|Q(E_{x, y, z})| = \{z\}\) when \(|E_{x, y, z}| = 0\) and \(H_{E,(x_0,y_0),z}\) is the \(r \times r\) matrix corresponding to the \(r\) p-c equations indexed by \(\{[\ell, z] | \ell \in [r]\}\) and the \(r\) unknowns participating. Therefore its invertibility implies recoverability of \(s\) failed node symbols \(\{C(x_0, y_0; z(x \to y_0)) | x \in [0 : s - 1]\}\) and \((r - s)\) aloof node symbols \(\{C(x, y; z) | (x, y) \in E\}\).

When \(|E_{x, y, z}| > 0\), the number of unknowns \(r + |E_{x, y, z}|\) is greater than the number of equations \(r\), therefore we need to use more parity checks in order to recover aloof node symbols corresponding to the plane \(z\). It turns out that in this case, if we consider the p-c equations corresponding to planes in equivalence class of \(z\) then we have a situation where the number of equations \(rp\) equals the number of equations \(r\) where \(p = |Q(E_{x, y, z})|\). The
be a vector in $\mathbb{C}^n$.

We will now go ahead and show that the unknowns in the p-c equations indexed by $\{[\ell, w] \mid w \in Q(E, z), \ell \in [r]\}$ correspond to the $(r-s)p$ aloof symbols in planes $Q(E, z)$ and $sp$ failed node symbols given by $\{C(x_0, y_0; w(x \to z_{y_0})) \mid x \in [0 : s-1], w \in Q(E, z)\}$. This will imply that the number of unknowns and number of equations is $rp$.

We therefore consider all the equations corresponding to the planes in $Q(E, z)$. From Lemma VI.1 $Q(E, z) \subseteq R$.

For any $w \in Q(E, z)$, the $[\ell, w]$-th p-c equation is given by:

$$\theta_{x_0, y_0, x_0}^\ell C(x_0, y_0; w) + \sum_{x \neq x_0} \gamma_{x, x_0} \theta_{x, y_0}^{\ell - 1} C(x, y_0; w(x \to w_{y_0})) + \sum_{(y, y_0) \in E} \theta_{x, y_0}^{\ell - 1} C(x, y; w) + \sum_{(y, y_0) \in E} \gamma_{x, w_0} \theta_{w_0, y_0, x}^{\ell - 1} C(w_{y_0}, y; w(x \to w_y)) = \kappa^\ell$$

and the symbol $C(w_{y_0}, y; w(x \to w_y))$ corresponds to the plane $w(x \to w_y)$. For $(x, y) \in E_{2, w}$ it is clear from the Definition 2 of the equivalence class of planes that $w(x \to w_y) \in Q(E, w)$. It is also known that $Q(E, w) = Q(E, z)$. Therefore $w(x \to w_y) \in Q(E, z)$. This would mean that the parity checks corresponding to planes in $Q(E, z)$ involve $(r-s)p$ aloof node symbols corresponding to planes within $Q(E, z)$ and $sp$ failed node symbols in planes $\{w(x \to w_{y_0}) \mid x \in [0 : s-1], w \in Q(E, z)\}$, therefore invertibility of sub matrix $H_{E,(x_0,y_0),z}$ would imply recoverability of aloof node symbols in planes $Q(E, z)$ and failed node $(x_0, y_0)$’s symbols in planes $\{w(x \to w_{y_0}) \mid x \in [0 : s-1], w \in Q(E, z)\}$.

Therefore, by the end of all the steps invertibility of $H_{E,(x_0,y_0),z}$ for all $z \in R$ implies recovery of all the $\alpha$ failed node symbols:

$$\{C(x_0, y_0, z_0(x \to z_{y_0}) ; z \in R\} = \{C(x_0, y_0, z_0) ; z \in Z^\ell\}$$

and recovery of $\beta(r-s)$ aloof node symbols $\{C(x, y ; z) ; (x, y) \in E, z \in R\}$.

Now, in Theorem VI.3 we show that invertibility of reduced matrix introduced in Definition 5 implies invertibility of the repair sub-matrix.

**Lemma VI.3 (The Reduction II: Optimal Access Property).** Let $(x_0, y_0)$ be the failed node and $E$ be an aloof node pattern of size $(r-s)$ such that $(x_0, y_0) \notin E$ and let $z \in R$ where $R = \{w \in Z^\ell \mid u_{y_0} = x_0\}$. For the case when $|E_{2, z}| = 0$, $H_{E,(x_0,y_0),z}$ is invertible. Otherwise, $H_{E,(x_0,y_0),z}$ is invertible if $H_{E,z}^{\text{Red}}$ (see Definition 3) is invertible.

**Proof:** Let $p = |Q(E, z)|$ and $f$ be a vector in $\mathbb{F}_q^p$ such that $f^T H_{E,(x_0,y_0),z} = 0$ and $f^T = (f_{\ell, v} \mid \ell \in [r], v \in Q(E, z))$. Let $f_{\ell, v}$ be a polynomial defined as:

$$f_{\ell, v} (x) = \sum_{\ell = 1}^r f_{\ell, v} x^{\ell - 1}.$$ 

Given $f^T H_{E,(x_0,y_0),z} = 0$ we want to show that $f = 0$. $f^T H_{E,(x_0,y_0),z} = 0$ implies that:

$$\sum_{\ell = 1}^r \sum_{v \in Q(E, z)} f_{\ell, v} H([\ell, v], [x, y, u]) = 0, \quad \text{(28)}$$

for any $[x, y, u] \in (E \times Q(E, z)) \cup \{(x_0, y, w(x \to w_{y_0})) \mid x \in [0 : s-1], w \in Q(E, z)\}$.

By definition of Small-d construction, $H([\ell, v], [x, y, u])$ is non-zero only for $v = u, u(x \to y)$. For any $y$ such that $(z, y) \in E_{0, z}$, and for any $u \in Q(E, z)$ it is implied that $(u_y, y) \in E$, by considering $[x, y, u] = [u_y, y, u]$, equation (28) reduces to:

$$\sum_{\ell = 1}^r f_{\ell, u} \theta_{u_y, y, u_y}^{\ell - 1} = 0 \implies f_{\ell, u} \theta_{u_y, y, u_y} = 0 \text{ from equation (28)} \implies f_{\ell, u} (\lambda_{0, y}) = 0 \text{ for all } u \in Q(E, z), (z, y) \in E_{0, z} \quad \text{(29)}$$

For $(x, y) \in E_{1, z}$ it implies that $(z, y) \notin E$ and that $x \neq z_y$, therefore $S_y = \{z_y\}$ (see definition in equation (9)).
and given \( u \in Q(E, \bar{z}) \), \( u(x \to u_y) \not\in Q(E, \bar{z}) \) and \( u_y = z_y \). Equation (28) in this case reduces to:

\[
\sum_{\ell=1}^{r} f_{\ell, u} \theta_{x,y,u}^{-1} = 0 \implies f_{\ell, u}(\theta_{x,y,u}) = 0 \text{ for all } u \in Q(E, \bar{z}), (x, y) \in E_{1,\bar{z}} = E_{1,u}. \tag{30}
\]

For any \( u \in Q(E, \bar{z}) \) and \( (x, y) \in E_{2,u} \), it is implied that \( (u_y, y) \in E, x \neq u_y \) and \( u(x \to u_y) \in Q(E, \bar{z}) \), therefore:

\[
\sum_{\ell=1}^{r} \left( f_{\ell, u} \theta_{x,y,u}^{-1} + \gamma_{u_y} x f_{\ell, u}(x \to u_y) \theta_{x,y,u}^{-1} \right) = 0 \implies f_{\ell, u}(\theta_{x,y,u}) + \gamma_{u_y} x f_{\ell, u}(x \to u_y)(\theta_{x,y,u}) = 0. \tag{31}
\]

For \( (x, y, u) = (x_0, y_0, w(\hat{x} \to w_{y_0})) \) and \( w \in Q(E, \bar{z}) \), \( \hat{x} \in [0 : s - 1] \), the \( \bar{w} \) where \( H([\ell, \bar{v}], [x, y, u]) \) is non-zero is \( \bar{v} = \bar{w}, w(\hat{x} \to w_{y_0}) \). However for \( \hat{x} \neq w_{y_0} = x_0 \), \( w(\hat{x} \to w_{y_0}) \notin R \). From Lemma [\ref{L1}] \( Q(E, \bar{z}) \subseteq R \). Therefore \( w(\hat{x} \to w_{y_0}) \notin Q(E, \bar{z}) \) and the only \( \bar{v} \in Q(E, \bar{z}) \) for which \( H([\ell, \bar{v}], [x, y, u]) \) is non-zero is \( \bar{v} = \bar{w} \). Equation (28) reduces to:

\[
\sum_{\ell=1}^{r} f_{\ell, u} \theta_{x,y,u}^{-1} = 0 \implies f_{\ell, u}(\theta_{x,y,u}) = 0 \text{ for all } w \in Q(E, \bar{z}), \hat{x} \in [0 : s - 1]. \tag{32}
\]

For the case when \( \mu = |E_{1,\bar{z}}| = 0 \), (29), (30) and (32) imply that there are \( |E_{0,\bar{z}}| + |E_{1,\bar{z}}| + s = |E| + s = r \) roots for \( f_{\ell, u}(x) \) for any \( u \in Q(E, \bar{z}) \) given by:

\[
\{\theta_{x,y,z_y} \mid (x, y) \in E\} \cup \{\theta_{x_0,y_0,x} \mid x \in [0 : s - 1]\}.
\]

From Lemma [\ref{L1}] that there are \( r \) distinct roots for \( f_{\ell, u}(x) \). But \( f_{\ell, u}(x) \) is an \( r - 1 \) degree polynomial implying that \( f_{\ell, u}(x) = 0 \) for all \( u \in Q(E, \bar{z}) \). This also implies that \( f = 0 \) and hence \( H_{E,(x_0,y_0)\bar{z}} \) is invertible.

For the case when \( \mu = |E_{1,\bar{z}}| > 0 \), from (29), (30) and (32) it is implied that:

\[
f_{\ell, u}(x) = \left( \prod_{\hat{x} = 0}^{s-1} (x - \theta_{x_0,y_0,\hat{x}}) \right) \left( \prod_{(z_0, y) \in E_{0,\bar{z}}} (x - \lambda_{0,y}) \right) \left( \prod_{(\hat{x}, y) \in E_{1,\bar{z}}} (x - \theta_{\hat{x},y,u_y}) \right) f_{\ell, u}^{\text{red}}(x), \tag{33}
\]

where \( f_{\ell, u}^{\text{red}}(x) \) is a polynomial of degree \( \mu - 1 \).

By substituting (33) in (31) we get that for any \( u \in Q(E, \bar{z}), (x, y) \in E_{2,u} \):

\[
P_1 P_2 P_3 \left( f_{\ell, u}^{\text{red}}(\theta_{x,y,u_y}) + \gamma_{u_y} x f_{\ell, u}^{\text{red}}(x \to u_y)(\theta_{x,y,u_y}) \right) = 0,
\]

where \( P_1 = \left( \prod_{\hat{x} = 0}^{s-1} (\theta_{x,y,u_y} - \theta_{x_0,y_0,\hat{x}}) \right), P_2 = \left( \prod_{(z_0, y) \in E_{0,\bar{z}}} (\theta_{x,y,u_y} - \lambda_{0,y}) \right), P_3 = \left( \prod_{(\hat{x}, y) \in E_{1,\bar{z}}} (\theta_{x,y,u_y} - \theta_{\hat{x},y,u_y}) \right) \).

It follows from Lemma [\ref{L1}] that \( P_1, P_2, P_3 \) are non-zero and hence equation (33) for any \( u \in Q(E, \bar{z}) \) and \( (x, y) \in E_{2,u} \), reduces to:

\[
f_{\ell, u}^{\text{red}}(\theta_{x,y,u_y}) + \gamma_{u_y} x f_{\ell, u}^{\text{red}}(x \to u_y)(\theta_{x,y,u_y}) = 0. \tag{34}
\]

Let \( f_{\ell, u}^{\text{red}}(x) = \sum_{\ell=1}^{\mu} f_{\ell, u}^{\text{red}}x^{-\ell} \) and let \( \ell^{\text{red}} \) be a vector in \( \mathbb{F}_q^{\mu} \) such that \( \ell^{\text{red}} = (f_{\ell, u}^{\text{red}} \mid \ell \in [\mu], u \in Q(E, \bar{z}) \text{\}T} \). (34) can be rewritten as:

\[
\sum_{\ell=1}^{\mu} f_{\ell, u}^{\text{red}} H([\ell, \bar{u}], [x, y, u]) + f_{\ell, u}^{\text{red}}(x \to u_y) H([\ell, \bar{u}(x \to u_y)], [x, y, u]) = 0 \text{ for all } u \in Q(E, \bar{z}), (x, y) \in E_{2,u}
\]

\[
\sum_{\ell=1}^{\mu} \sum_{u \in Q(E, \bar{z})} f_{\ell, u}^{\text{red}} H([\ell, \bar{u}], [x, y, u]) = 0 \text{ for all } u \in Q(E, \bar{z}), (x, y) \in E_{2,u}
\]

\[
f_{\ell, u}^{\text{red}} H_{E,z_y}^{\text{red}} = 0 \text{ from the definition in (11).}
\]
If \( H_{E,\bar{z}}^{\text{Red}} \) is invertible, this would imply that \( f^\text{Red} = 0 \). From (33), it is clear that \( f = 0 \) implying that \( H_{E,(x_0,y_0),\bar{z}} \) is invertible.

The Theorems VII.2 and VII.3 together imply that it is enough to show invertibility of reduced matrix \( H_{E,\bar{z}}^{\text{Red}} \) to prove the optimal-access repair property.

VII. INVERTIBILITY OF REDUCED MATRIX

We will now prove that the matrix \( H_{E,\bar{z}}^{\text{Red}} \) is invertible. To do so we will first introduce some notation to prove this inductively. The induction will be over the number of \( y \)-columns over which there are non-zero erasures appearing in \( E_{2,\bar{z}} \). Throughout the reminder of the paper we continue to work with set \( E \) indicating erasure (aloof node) pattern and plane \( \bar{z} \in \mathbb{Z}_2 \) and ignore indicating them in the notations.

Definition 7. Given an erasure (aloof node) pattern \( E \), plane \( \bar{z} \in \mathbb{Z}_2 \) we define the erasure vector that indicates the number of erasures in \( E_{2,\bar{z}} \) with same \( y \)-value

\[
e = (e_y \mid y \in [0 : t - 1]) \text{ where } e_y = |E_{2,\bar{z}}(y)|, \ E_{2,\bar{z}}(y) = \{x' \mid (x', y') \in E_{2,\bar{z}}, y' = y\}
\]

Fig. 6. Illustration of an example erasure pattern for \( E = \{(0, 0), (1, 0), (0, 2), (1, 2), (0, 3), (1, 3)\} \) indicated by dotted circles (holes) in plane \( \bar{z} = (0, 1, 0, 1, 0, 0) \). \( E_{2,\bar{z}} = \{(0, 1), (1, 2), (0, 3)\} \).

Definition 8. Let \( Y = \text{Supp}(e) \) be the support set of erasure weight vector \( e \), \( |Y| = m, Y = \{y_1, \ldots, y_m\} \). We define a subset of planes \( Q^j(E, \bar{z}) \subseteq Q(E, \bar{z}) \) for any \( j \in [m] \) as

\[
Q^j(E, \bar{z}) = S_0 \times S_1 \times \cdots \times S_y \times \{z_{y+j+1}\} \times \cdots \times \{z_{t-1}\}
\]

(35)

where \( S_y \) for \( y \in [0 : t - 1] \) is defined as shown in equation (9).

Lemma VII.1. The number of planes in \( Q^j(E, \bar{z}) \) is given by \( p_j = \prod_{i=1}^j (e_{y_i} + 1) \) for all \( j \in [m] \).

Proof: From the Lemma III.1 \( S_y = E_{2,\bar{z}}(y) \cup \{(z_y, y)\} \). Therefore \( |S_y| = (e_y + 1) \) and

\[
p_j = |Q^j(E, \bar{z})| = \prod_{y=0}^{y_j} |S_y| = \prod_{i=1}^j (e_{y_i} + 1).
\]

For the example erasure pattern and plane shown in the Fig 6 \( e = (1, 0, 1, 1, 0, 0), m = |\text{Supp}(e)| = 3 \) and \( S_0 = S_2 = S_3 = \{0, 1\}, S_1 = \{1\}, S_4 = S_5 = \{0\} \). The number of planes in the sets \( Q^1(E, \bar{z}), Q^2(E, \bar{z}) \) and \( Q^3(E, \bar{z}) \) are 2, 4 and 8 respectively. We can now define the sub-matrix of \( H_{E,\bar{z}}^{\text{Red}} \) that will be used to prove its invertibility by induction.

Definition 9 (Induction Matrix). Let \( j \in [m], d \leq \mu \) we define a matrix \( M_j,d \) of size \( (dp_j \times \mu_j p_j) \) where \( \mu_j = \sum_{i=1}^j e_{y_i} \), \( p_j = \prod_{i=1}^j (e_{y_i} + 1) \) as below:

\[
M_j,d([\ell, v], [x, y, u]) = H([\ell, v], [x, y, u])
\]
where $\ell \in [d]$ and $u, v \in Q^j(E, z)$ and $(x, y) \in E_{2, u}$ such that $y \in \{y_1, y_2, \ldots, y_j\}$.

**Remark 3.** It follows from the Definition that $H^d_{E, z} = M_{m, \mu m}$ and $p_m = p = |Q(E, z)|$, $\mu_m = \mu = |E_{2, z}|$.

**Remark 4.** $e_y \leq s - 1$ for all $i \in [m]$ and therefore in the proofs that follow, we consider cases of $e_y \in \{1, 2, 3\}$ as $s \in \{2, 3, 4\}$.

We will show in Lemma VII.2 that $M_{1, \mu i}$ is invertible and then shown in Lemma VII.3 that $M_{i+1, \mu i+1}$ is invertible given $M_{i, \mu i}$ is invertible for any $i \in [m - 1]$. This will imply that $H^d_{E, z} = M_{m, \mu m}$ is invertible.

**Lemma VII.2.** For any plane $z \in Z^d_s$ and any erasure pattern $E$ such that $|E| \leq r$, the matrix $M_{1, \mu i}$ is invertible.

**Proof:** It is clear that $z \in Q^j(E, z)$ and the planes in $Q^j(E, z)$ are given by $\{z(x \rightarrow y)\}$ $x \in S_{y_i}$. As $s \in \{2, 3, 4\}$ there are three possibilities for the $|S_{y_i}| = e_y + 1 \in \{2, 3, 4\}$. We will look at the invertibility of $M_{1, \mu i}$ for each of these cases separately. Let

$$V_{i,j} = \begin{bmatrix} \gamma \lambda_{i,y} d & \lambda_{i,y} d \end{bmatrix}$$

be a $(d \times 2)$ matrix where $\theta^d = [1 \theta \cdots \theta^{d-1}]^T$ and $\Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}$.

**Case 1:** $e_y = 1$, $\mu = 1$, $p = 2$: Let $S_{y_1} = \{x_0, x_1\}$ and $x < x_1$, then $M_{1,d}$ is an $(2d \times 2)$ matrix given by:

$$M_{1,d} = \begin{bmatrix} \theta_{x_0,y_0,x_0} d & \theta_{x_0,y_1,x_1} d \\ \gamma \theta_{x_0,y_0,x_0} d & \theta_{x_0,y_1,x_1} d \\ u_{y_1} = x_0 & u_{y_1} = x_1 \end{bmatrix} \begin{bmatrix} v_{y_1} = x_0 \\ v_{y_1} = x_1 \end{bmatrix}$$

where the first $d$ p-c equations correspond to plane $v = \tilde{x}(z) \rightarrow z_{y_1}$, the last $d$ p-c equations correspond to plane $v = \tilde{x}(x \rightarrow y)$, and the first, second columns correspond to the symbols $(x, y, u) = (x_1, y, \tilde{x}(z) \rightarrow z_{y_1})$ and $(x, y, u) = (x_0, y, \tilde{x}(z) \rightarrow z_{y_1})$ respectively.

**Case 2:** $e_y = 2$, $\mu = 2$, $p = 3$: Let $S_{y_1} = \{x_0, x_1, x_2\}$ and $x < x_1 < x_2$, then:

$$M_{1,d} = \begin{bmatrix} \theta_{x_0,y_0,x_0} d & \theta_{x_0,y_1,x_1} d & \theta_{x_0,y_2,x_2} d \\ \gamma \theta_{x_0,y_0,x_0} d & \theta_{x_0,y_1,x_1} d & \theta_{x_0,y_2,x_2} d \\ \gamma \theta_{x_2,y_0,x_0} d & \theta_{x_2,y_1,x_1} d & \theta_{x_2,y_2,x_2} d \\ u_{y_1} = x_0 & u_{y_1} = x_1 & u_{y_1} = x_2 \end{bmatrix} \begin{bmatrix} v_{y_1} = x_0 \\ v_{y_1} = x_1 \\ v_{y_1} = x_2 \end{bmatrix}$$

is a $(3d \times 6)$ matrix. In $M_{1,d}$, first $d$ p-c equations correspond to plane $v = \tilde{x}(z) \rightarrow z_{y_1}$, the next $d$ p-c equations correspond to plane $v = \tilde{x}(z) \rightarrow z_{y_1}$ and the last $d$ p-c equations correspond to plane $v = \tilde{x}(z) \rightarrow z_{y_1}$ whereas the first 2 columns correspond to erasures in plane $u = \tilde{x}(z) \rightarrow z_{y_1}$, next 2 columns correspond to erasures in plane $u = \tilde{x}(z) \rightarrow z_{y_1}$ and the last 2 columns correspond to erasures in plane $u = \tilde{x}(z) \rightarrow z_{y_1}$. After permuting
the columns, $M_{1,d}$ can be written as:

$$M_{1,d} = \begin{bmatrix}
\theta_{x_1,y_1,x_0} & \theta_{x_0,y_1,x_1} & \theta_{x_2,y_1,x_0} & \theta_{x_0,y_1,x_2} \\
\gamma \theta_{x_1,y_1,x_0} & \theta_{x_0,y_1,x_1} & \gamma \theta_{x_2,y_1,x_0} & \theta_{x_0,y_1,x_2} \\
V_{i_1 \Gamma} & V_{i_2 \Gamma} & V_{i_3 \Gamma} & V_{i_4 \Gamma}
\end{bmatrix}$$

where $(i_1, i_2, i_3) = \{(1, 2, 3), (1, 3, 2), (2, 3, 1), (3, 2, 1)\}$.

The assignment of $(i_1, i_2, i_3)$ follows from the $\theta$ to $\lambda$ assignment defined in [4].

$$|M_{1,2}| = \begin{vmatrix}
\gamma \lambda_{i_1,y_1} & \lambda_{i_1,y_1} & 1 \\
\gamma \lambda_{i_2,y_1} & \lambda_{i_2,y_1} & 1 \\
\gamma \lambda_{i_3,y_1} & \lambda_{i_3,y_1} & 1
\end{vmatrix}
$$

The determinant of matrix $M_{1,2}$ can be computed to be equal to $\gamma (1 - \gamma)^4 \lambda_{i_1,y_1}(\lambda_{i_2,y_1} - \lambda_{i_1,y_1})(\lambda_{i_2,y_1} - \lambda_{i_3,y_1})$. This is non-zero by the distinctness of $\lambda$'s as defined in Section [4]. Note that the field $\mathbb{F}_q$ being of characteristic 2 is used in the determinant computation.

**Case 3:** $e_{y_1} = 3$, $\mu_1 = 3$, $p_1 = 4$: $S_{y_1} = \{0, 1, 2, 3\}$ as $s \leq 4$ and the $(4d \times 12)$ matrix $M_{1,d}$ is given by:

$$M_{1,d} = \begin{bmatrix}
\theta_{y_0,0} & \theta_{y_0,1} & \theta_{y_0,2} & \theta_{y_0,3} \\
\gamma \theta_{y_0,0} & \theta_{y_0,1} & \gamma \theta_{y_0,2} & \theta_{y_0,3} \\
u_0 = 0 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3} \\
u_1 = 1 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3} \\
u_2 = 2 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3} \\
u_3 = 3 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3}
\end{bmatrix}
$$

where

$$\begin{align*}
v_0 &= 0 \\
v_1 &= 1 \\
v_2 &= 2 \\
v_3 &= 3
\end{align*}$$

In $M_{1,d}$, first $d$ p-c equations correspond to plane $\nu = \overline{z}(x_0 \rightarrow z_{y_1})$, the next $d$ p-c equations correspond to plane $\nu = \overline{z}(x_1 \rightarrow z_{y_1})$ followed by $d$ p-c equations corresponding to plane $\nu = \overline{z}(x_2 \rightarrow z_{y_1})$ and the last $d$ p-c equations corresponding to plane $\nu = \overline{z}(x_3 \rightarrow z_{y_1})$. Similarly, the first 3 columns correspond to erasures in plane $u = \overline{z}(x_0 \rightarrow z_{y_1})$, next 3 columns correspond to erasures in plane $u = \overline{z}(x_1 \rightarrow z_{y_1})$ followed by 3 erased symbols from plane $u = \overline{z}(x_2 \rightarrow z_{y_1})$ and the last 3 erasures in plane $u = \overline{z}(x_3 \rightarrow z_{y_1})$. After permuting columns $M_{1,d}$ can be written as:

$$M_{1,d} = \begin{bmatrix}
\theta_{y_0,0} & \theta_{y_0,1} & \theta_{y_0,2} & \theta_{y_0,3} \\
\gamma \theta_{y_0,0} & \theta_{y_0,1} & \gamma \theta_{y_0,2} & \theta_{y_0,3} \\
u_0 = 0 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3} \\
u_1 = 1 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3} \\
u_2 = 2 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3} \\
u_3 = 3 & \gamma \theta_{y_0,0} & \gamma \theta_{y_0,1} & \gamma \theta_{y_0,2} & \gamma \theta_{y_0,3}
\end{bmatrix}
$$

where

$$\begin{align*}
v_0 &= 0 \\
v_1 &= 1 \\
v_2 &= 2 \\
v_3 &= 3
\end{align*}$$
The determinant of matrix $M_{1:3}$ from Lemma A.3 is $\gamma^4(1-\gamma)^6(\lambda_1, y_1 - \lambda_2, y_1)^2(\lambda_1, y_1 - \lambda_3, y_1)^2(\lambda_2, y_1 - \lambda_3, y_1)^4(\lambda_1, y_1 - \gamma \lambda_3, y_1)(\gamma \lambda_1, y_1 - \lambda_3, y_1)(\lambda_1, y_1 - \gamma \lambda_2, y_1)(\gamma \lambda_1, y_1 - \lambda_2, y_1)$. This determinant is non-zero due to the $\lambda$ assignment conditions presented in Section II-D. Hence we proved that $M_{1, m_1}$ is invertible.

**Lemma VII.3.** For any $i \in [2 : m - 1]$, $M_{i, m_i}$ is invertible given $M_{i-1, m_{i-1}}$ is invertible.

**Proof:** Proof is provided in Appendix.

**Corollary VII.4.** For any $E$ such that $|E| \leq r$ and any plane $z \in \mathbb{Z}_q^t$, $H_{E_i, z}^{rd}$ is invertible.

**Proof:** $H_{E_i, z}^{rd} = M_{m, m_n}$. Therefore the proof follows from Lemma VII.2 and Lemma VII.3.

We will now show that Small-$d$ code is an MSR code with field size $q = O(n)$.

**Theorem VII.5.** Small-$d$ code is an optimal-access MSR code with parameters of the form $(n = st, d = k + s - 1)$ and $\alpha = s^t$ where $s \in \{2, 3, 4\}$ over field $\mathbb{F}_q$ that is an extension of binary field $q = 2^w$, such that $q = O(n)$.

**Proof:** The MDS property of Small-$d$ code follows from Theorems VII.1 and Corollary VII.2. The optimal-access property follows from Theorems VII.2, VII.3 and Corollary VII.4.

We show here a way to do the $\lambda$ assignment that satisfies the conditions presented in Section II-D with a field $\mathbb{F}_q$ of size $q = O(n)$. Consider a multiplicative sub-group $G$ of $\mathbb{F}_q \setminus \{0\}$ and cosets $\gamma G, \gamma^2 G$. Let $m_0$ be the number of distinct $\lambda$’s in $\Lambda_{s, y}$ matrix defined in (4) excluding $\lambda_{0, y}$. It is clear to see that $m_0 = 1$ for $s = 2$ and $m_0 = 3$ for $s = 3, 4$. We pick coefficients for $\{\lambda_{i, y} \mid y \in [0: t - 1], i \in [m_0]\}$ from $G$ and the corresponding $\gamma$ multiples can be picked from $\gamma G$ and the remaining $\lambda$’s corresponding to $\{\lambda_{0, y} \mid y \in [0: t - 1]\}$ are picked from $\gamma^2 G$. When $w$ is even $3t^w - 1$ and define $G = \{\psi^i : 0 \leq i \leq \frac{2^w - 1}{3} - 1\}$, where $\psi$ is primitive element of $\mathbb{F}_2$, and set $\gamma = \psi$. Therefore, by choosing field size such that $|G| > m_0 t$, we get $q \geq 3m_0 t + 1$. If the smallest possible field size $q$ that satisfies $2^w = q \geq 3m_0 t + 1$ results in a odd $w$, we just take double the field size. Therefore field size $q \geq 6t + 2$ for $s = 2$ and $q \geq 18t + 2$ for $s = 3, 4$.

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We will now introduce notation and prove few lemmas on factors of polynomials that will be used to prove Lemma VII.3

**Lemma A.1.** Let \( g(x) \) be a polynomial in \( \mathbb{F}_q[x] \) and \( f(x_1, \ldots, x_m) \) be a multivariate polynomial in \( m \) variables i.e., \( f \in \mathbb{F}_q[x_1, \ldots, x_m] \) such that:

\[
g(x) = f(x_1 = x, x_2 = x, \ldots, x_m = x)
\]

\[
f(x_1, \ldots, x_{i-1}, x_i = a, x_{i+1}, \ldots, x_m) = 0 \text{ for any } i \in [n] \text{ where } n \leq m.
\]

where \( a \) is an element in \( \mathbb{F}_q \). Then \( (x-a)^n \) divides \( g(x) \).

**Proof:** It is clear to see that \( \prod_{i=1}^{n} (x_i - a) | f(x_1, x_2, \ldots, x_m) \). Therefore \( (x-a)^n \) divides \( g(x) \).

**Lemma A.2.** Let \( g(x) \) be a polynomial in \( \mathbb{F}_q[x] \) and let \( f(x_1, \ldots, x_{2n}) \) be a multivariate polynomial in \( 2n \) variables i.e, \( f \in \mathbb{F}_q[x_1, \ldots, x_{2n}] \) such that:

\[
g(x) = f(x_1 = x, x_2 = x, \ldots, x_{2n} = x)
\]

\[
f(x_1, \ldots, x_{2n}) \mid \text{on setting any } n+1 \text{ variables to } a = 0
\]

where \( a \in \mathbb{F}_q \). Then \( (x-a)^n \) divides \( g(x) \).

**Proof:** We divide \( f(x_1, \ldots, x_{2n}) \) by \( (x_1 - a) \) to get:

\[
f(x_1, \ldots, x_{2n}) = (x_1 - a)f_1(x_1, \ldots, x_{2n}) + R_1(x_2, \ldots, x_{2n}).
\]
We now recursively define \( R_i(x_{i+1}, \ldots, x_{2n}) \) to be the remainder obtained on dividing \( R_{i-1}(x_1, \ldots, x_{2n}) \) by \( (x_i - a) \) for \( 2 \leq i \leq (n + 1) \). Let:

\[
R_{i-1}(x_1, \ldots, x_{2n}) = (x_i - a) f_i(x_1, \ldots, x_{2n}) + R_i(x_{i+1}, \ldots, x_{2n}). \tag{38}
\]

From equations (37) and (38) we get:

\[
f(x_1, \ldots, x_{2n}) = (x_1 - a) f_1(x_1, \ldots, x_{2n}) + \cdots + (x_{n+1} - a) f_{n+1}(x_1, \ldots, x_{2n}) + R_{n+1}(x_{n+2}, \ldots, x_{2n}). \tag{39}
\]

By setting \( x_1 = x_2 = \cdots = x_{n+1} = 0 \) in equation (39) we get:

\[
R_{n+1}(x_{n+2}, \ldots, x_{2n}) = 0.
\]

Therefore:

\[
f(x_1, \ldots, x_{2n}) = (x_1 - a) f_1(x_1, \ldots, x_{2n}) + \cdots + (x_{n+1} - a) f_{n+1}(x_1, \ldots, x_{2n}). \tag{40}
\]

On similarly expanding \( f_i(x_1, \ldots, x_{2n}) \) for \( i \in [n+1] \) by recursively dividing it by \( (x_1 - a), (x_2 - a), \ldots, (x_{i-1} - a), (x_{i+1} - a), \ldots, (x_{n+2} - a) \) we get:

\[
f_i(x_1, \ldots, x_{2n}) = \sum_{i_2=1 \atop i_2 \neq i_1}^{n+2} (x_{i_2} - a) f_{i_1,i_2}(x_1, \ldots, x_{2n}) + R(x_i, x_{n+3}, \ldots, x_{2n}). \tag{41}
\]

On setting \( x_1 = x_2 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_{n+1} = x_{n+2} = a \) in equation (40) we get:

\[
0 = (x_i - a) f_i(x_1, \ldots, x_{2n}) \bigg|_{x_1 = x_2 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_{n+2} = a}
\]

\[
0 = f_i(x_1, \ldots, x_{2n}) \bigg|_{x_1 = x_2 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_{n+2} = a} \quad \text{as it is true for } x_i \neq a
\]

\[
= R(x_i, x_{n+3}, \ldots, x_{2n}) \quad \text{follows from equation (41)}
\]

From equations (40), (41), (42) we get:

\[
f(x_1, \ldots, x_{2n}) = \sum_{i_1=1}^{n+1} \sum_{i_2=1 \atop i_2 \neq i_1}^{n+2} \sum_{i_3=1 \atop i_3 > i_1}^{n+2} \sum_{i_4=1 \atop i_4 > i_3}^{n+2} \cdots \sum_{i_k=1 \atop i_k > \cdots > i_1}^{n+k} \prod_{j=1}^{k} (x_{i_j} - a) \quad \text{for } 2 \leq k \leq n - 1
\]

\[
= \sum_{i_1=1 \atop i_1 \neq \cdots \neq i_k}^{n+k+1} \prod_{j=1}^{k} (x_{i_j} - a) \quad \text{for } 2 \leq k \leq n - 1.
\]

for \( 2 \leq k \leq n - 1 \). Expanding \( g_{i_1,i_2,\cdots,i_k}(x_1, \ldots, x_{2n}) \) similar using factors \( (x_{i_{k+1}} - a) \) for \( i_{k+1} \in [1, n + k + 1] \setminus \{i_1, \cdots, i_k\} \) we get:

\[
g_{i_1,i_2,\cdots,i_k}(x_1, \ldots, x_{2n}) = \sum_{i_{k+1}=1 \atop i_{k+1} \neq \{i_1, \cdots, i_k\}}^{n+k+1} (x_{i_{k+1}} - a) f_{i_1,\cdots,i_{k+1}}(x_1, \ldots, x_{2n}) + R(x_{i_1}, \ldots, x_{i_k}, x_{n+k+2}, \ldots, x_{2n}). \tag{44}
\]

On setting \( x_j = a \) for all \( j \in [1, n + k + 1] \setminus \{i_1, \cdots, i_k\} \) in equation (43) we get:

\[
0 = \left( \prod_{j=1}^{k} (x_{i_j} - a) \right) g_{i_1,i_2,\cdots,i_k}(x_1, \ldots, x_{2n}) |_{x_j = a \text{ for all } j \in [1, n + k + 1] \setminus \{i_1, \cdots, i_k\}}
\]

\[
0 = g_{i_1,i_2,\cdots,i_k}(x_1, \ldots, x_{2n}) |_{x_j = a \text{ for all } j \in [1, n + k + 1] \setminus \{i_1, \cdots, i_k\}}
\]

\[
= R(x_{i_1}, \ldots, x_{i_k}, x_{n+k+2}, x_{2n}) \quad \text{from equation (44)}
\]

(45)
Equations (43, 44, 45) imply that:

\[
f(x_1, \cdots x_{2n}) = \sum_{i_1 = 1 \atop i_2 > i_1}^{n+1} \cdots \sum_{i_k > i_{k-1}}^{n+k} \prod_{j=1}^{k+1} (x_{i_j} - a) f_{i_1, i_2, \cdots, i_{k+1}}(x_1, \cdots, x_{2n})
\]

where \( f_{i_1, i_2, \cdots, i_{k+1}} = f_{i_{k+1}, i_1, i_2, \cdots, i_k} + f_{i_1, i_{k+1}, i_2, \cdots, i_k} + \cdots + f_{i_1, i_2, \cdots, i, i_{k+1}} \). Setting \( k = n - 1 \) in equation (46) we get:

\[
f(x_1, \cdots x_{2n}) = \sum_{i_1 = 1 \atop i_2 > i_1}^{n+1} \cdots \sum_{i_n > i_{n-1}}^{2n} \prod_{j=1}^{n} (x_{i_j} - a) g_{i_1, i_2, \cdots, i_n}(x_1, \cdots, x_{2n}).
\]

It now clearly follows that \((x - a)^{n+1} \mid g(x)\) where \( g(x) = f(x_1 = x, x_2 = x, \cdots, x_{2n} = x) \). \( \square \)

**Lemma A.3.** The determinant of matrix:

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\gamma \lambda_1 & \lambda_1 & \gamma \lambda_2 & \lambda_2 & \gamma \lambda_3 & \lambda_3 & \gamma \lambda_4 & \lambda_4 \\
\gamma^2 \lambda_1 & \lambda_1 & \gamma^2 \lambda_2 & \lambda_2 & \gamma^2 \lambda_3 & \lambda_3 & \gamma^2 \lambda_4 & \lambda_4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\gamma^3 \lambda_1 & \lambda_1 & \gamma^3 \lambda_2 & \lambda_2 & \gamma^3 \lambda_3 & \lambda_3 & \gamma^3 \lambda_4 & \lambda_4 \\
\gamma^4 \lambda_1 & \lambda_1 & \gamma^4 \lambda_2 & \lambda_2 & \gamma^4 \lambda_3 & \lambda_3 & \gamma^4 \lambda_4 & \lambda_4 \\
\gamma^5 \lambda_1 & \lambda_1 & \gamma^5 \lambda_2 & \lambda_2 & \gamma^5 \lambda_3 & \lambda_3 & \gamma^5 \lambda_4 & \lambda_4 \\
\gamma^6 \lambda_1 & \lambda_1 & \gamma^6 \lambda_2 & \lambda_2 & \gamma^6 \lambda_3 & \lambda_3 & \gamma^6 \lambda_4 & \lambda_4 \\
\end{bmatrix}
\]

is \( \gamma^4(1 - \gamma)^6(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^4(\gamma \lambda_1 - \lambda_3)(\gamma \lambda_1 - \lambda_2)(\gamma \lambda_1 - \lambda_2) \).

**Proof:** Let the determinant of matrix \( M \) be a polynomial \( f(\lambda_1) \). It can be observed the degree of this polynomial is almost 8. Let

\[
M(\nu_1, \nu_2) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\nu_1 & \nu_2 & \gamma \lambda_2 & \lambda_2 & \gamma \lambda_3 & \lambda_3 & \gamma \lambda_4 & \lambda_4 \\
\gamma \nu_1 & \nu_2 & \gamma^2 \lambda_2 & \lambda_2 & \gamma^2 \lambda_3 & \lambda_3 & \gamma^2 \lambda_4 & \lambda_4 \\
\gamma \nu_1^2 & \nu_2^2 & \gamma^3 \lambda_2 & \lambda_2 & \gamma^3 \lambda_3 & \lambda_3 & \gamma^3 \lambda_4 & \lambda_4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\gamma^4 \lambda_2 & \lambda_2 & \gamma^4 \lambda_3 & \lambda_3 & \gamma^4 \lambda_4 & \lambda_4 \\
\gamma^5 \lambda_2 & \lambda_2 & \gamma^5 \lambda_3 & \lambda_3 & \gamma^5 \lambda_4 & \lambda_4 \\
\gamma^6 \lambda_2 & \lambda_2 & \gamma^6 \lambda_3 & \lambda_3 & \gamma^6 \lambda_4 & \lambda_4 \\
\end{bmatrix}
\]


and $g(\nu_1, \nu_2) = \det(M(\nu_1, \nu_2))$. Then $f(\lambda_1) = g(\gamma \lambda_1, \lambda_1)$. We will now show that $g(\nu_1, \nu_2)$ is divisible by $(v_1 - \lambda_2)(v_1 - \lambda_3)(v_1 - \gamma \lambda_2)(v_1 - \gamma \lambda_3)(v_2 - \lambda_2)(v_2 - \lambda_3)(v_2 - \gamma \lambda_2)(v_2 - \gamma \lambda_3)$. It can be seen that:

\[
\begin{align*}
g(\gamma \lambda_2, \nu_2) &= 0 \text{ as } \tilde{c}_1 + \tilde{c}_3 + \gamma c_7 + \gamma c_9 = 0 \\
g(\lambda_2, \nu_2) &= 0 \text{ as } \tilde{c}_1 + \tilde{c}_4 + c_7 + \gamma c_{10} = 0 \\
g(\nu_1, \lambda_2) &= 0 \text{ as } \tilde{c}_2 + \tilde{c}_4 + \gamma c_8 + \gamma c_{10} = 0 \\
g(\nu_1, \gamma \lambda_2) &= 0 \text{ as } \tilde{c}_2 + \tilde{c}_5 + \gamma c_8 + c_9 = 0 \\
g(\nu_1, \lambda_3) &= 0 \text{ as } \tilde{c}_2 + \tilde{c}_6 + \gamma c_8 + \gamma c_{11} = 0 \\
g(\nu_1, \gamma \lambda_3) &= 0 \text{ as } \tilde{c}_2 + \tilde{c}_5 + \gamma c_8 + \gamma c_{11} = 0
\end{align*}
\]

where $\tilde{c}_i$ is the $i$-th column of the matrix $M(\nu_1, \nu_2)$. We will now show that the matrix $M(\lambda_3, \nu_2)$ has determinant 0. This is implied if there exists a non-zero vector $\mu$ such that $\mu^T M(\lambda_3, \nu_2) = 0$. Let $\mu^T = [\gamma \mu_1^T \gamma \mu_2^T \mu_1^T \mu_2^T]$ and $\mu^T = [m_0, m_1, m_2]$ such that the polynomial $m(x) = m_0 + m_1 x + m_2 x^2 = (x - \lambda_3)(x - \lambda_2)$. It can be seen that $\mu^T M(\lambda_3, \nu_2) = 0^T$.

We will now similarly show $M(\gamma \lambda_3, \nu_2)$ has determinant zero. Let $\mu^T = [\mu_1^T \mu_2^T \mu_1^T \mu_2^T]$ and $\mu^T = [m_0, m_1, m_2]$ such that the polynomial $m(x) = m_0 + m_1 x + m_2 x^2 = (x - \gamma \lambda_3)(x - \gamma \lambda_2)$. It can be seen that $\mu^T M(\gamma \lambda_3, \nu_2) = 0^T$.

This implies that:

\[
g(\nu_1, \nu_2) = g' \prod_{\nu=1}^{2} (\nu_1 - \lambda_2)(\nu_1 - \lambda_3)(\nu_2 - \gamma \lambda_2)(\nu_2 - \gamma \lambda_3),
\]

where $g'$ is a constant dependant only on $\lambda_2, \lambda_3$ as $g$ is of degree 4 in variables $\nu_1, \nu_2$. Therefore:

\[
f(\lambda_1) = g(\gamma \lambda_1, \lambda_1) = g' \gamma^2 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_1 - \gamma \lambda_2)(\lambda_1 - \gamma \lambda_3)(\gamma \lambda_1 - \lambda_2)(\gamma \lambda_1 - \lambda_3)
\]

We will now determine $g'$ by computing $f(0) = \det(M(0, 0))$.

\[
M(0, 0) = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^2 \lambda_2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^2 \lambda_2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^2 \lambda_3 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^2 \lambda_3 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^2 \lambda_3 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^2 \lambda_3 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_2^2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_2^2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_2^2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_2^2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_3^2 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_3^2 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_3^2 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^3 \lambda_3^2 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\det(M(0, 0)) = (1 - \gamma)^2 \gamma^4 \lambda_2^2 \lambda_3^2
\]

\[
= (1 - \gamma)^6 \gamma^6 \lambda_2^2 \lambda_3^2 (\lambda_2 - \lambda_3)^4
\]

by computing the determinant of the $8 \times 8$ matrix above

\[
f(0) = g' \gamma^4 \lambda_2^4 \lambda_3^4
\]

\[
g' = (1 - \gamma)^6 \gamma^2 (\lambda_2 - \lambda_3)^4
\]
\[ f(\lambda_1) = (1 - \gamma)^6 \gamma^4 (\lambda_2 - \lambda_3)^4 \prod_{i=2}^{3} (\lambda_1 - \lambda_i)^2 (\lambda_1 - \gamma \lambda_i)(\gamma \lambda_1 - \lambda_i) \]

\[ \Box \]

A. Recursive definition of \( M_{i,d} \)

The notation introduced in Definitions 7, 8 and 9 within Section VII will be used throughout this appendix. Note that the induction matrix \( M_{i,y} \) introduced in Definition 9 is a sub matrix of \( H_{E}^{\text{rd}_{-z}} \) where \( E \) is an erasure (aloof node) pattern and \( z \) is plane in \( \mathbb{Z}_q \). Since we restrict our attention to a particular \( E, \tilde{z} \) we do not mention it explicitly in the notation and this holds for any \( |E| \leq r \) and \( \tilde{z} \in \mathbb{Z}_q \).

We note here the recursive definition of \( M_{i,d} \) in terms of \( M_{i-1,d} \). The recursive description is dependent on the value of \( e_y \in \{1, 2, 3\} \).

Recall that from the Definition 9 the inductive matrix \( M_{i,d} \) is a \((dp_i \times \mu_i p_i)\) sub-matrix of the parity-check matrix \( H \) where \( p_i = |Q^i(E, \tilde{z})| \). The rows are indexed by \([\ell, u]\) and columns by \([x, y, u]\) where \( \ell \in [d], u, v \in Q^i(E, \tilde{z}), (x, y) \in E_{2,u} \) such that \( y \in \{y_1, y_2, \ldots, y_i\} \). Let,

\[
D_{d,j}^i = \begin{bmatrix}
V_{d,j}^i & \cdots & V_{d,j}^i \\
(d_{j,p_i} \times 2p_i - 1)
\end{bmatrix} \quad \Psi = \begin{bmatrix}
\Gamma & \cdots & \Gamma \\
(2p_i \times 2p_i - 1)
\end{bmatrix} \quad \text{where} \quad V_{d,j}^i = \begin{bmatrix}
1 & 1 \\
\gamma \lambda_{j,y}, & \lambda_{j,y} \\
\vdots & \vdots \\
(\gamma \lambda_y)_{d-1}, & \lambda_y^d_{-1}
\end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix}
\gamma & \Gamma
\end{bmatrix}.
\]

1) \( e_y = 1 \): For this case \( |S_{y_i}| = (e_y + 1) = 2 \). Let \( S_{y_i} = \{x_0, x_1\} \) where \( x_0 < x_1 \) then it can be seen that:

\[ M_{i,d} = \begin{bmatrix}
M_{i-1,d} & D_{1,i}^d & D_{1,i}^d \Psi \\
M_{i-1,d} & D_{1,i}^d & D_{1,i}^d \Psi \\
y \neq y_i & y \neq y_i & y \neq y_i \\
y_i = y_i & y_i = y_i & x, u_{y_i} \in \{x_0, x_1\}
\end{bmatrix} \quad \Psi = \begin{bmatrix}
v_{y_i} = x_0 \\
v_{y_i} = x_1
\end{bmatrix} \quad \text{where} \quad i_1 = \begin{cases}
1 & S_{y_i} = \{0, 1\} \text{ or } \{2, 3\} \\
2 & S_{y_i} = \{0, 2\} \text{ or } \{1, 3\} \\
3 & S_{y_i} = \{0, 3\} \text{ or } \{1, 2\}
\end{cases}
\]

due to the \( \theta \) to \( \lambda \) assignment shown in equation (4). It can be noted that number of rows in \( M_{i,d} \) is two times the rows of \( M_{i-1,d} = 2dp_i - 1 = dp_i \). Number of columns of \( M_{i,d} \) is equal to \( 2\mu_{i-1} p_i - 1 + 2p_i - 1 = (\mu_{i-1} + 1)p_i = \mu_i p_i \).

2) \( e_y = 2 \): For this case \( |S_{y_i}| = (e_y + 1) = 3 \). Let \( S_{y_i} = \{x_0, x_1, x_2\} \) where \( x_0 < x_1 < x_2 \) then it can be seen that:

\[ M_{i,d} = \begin{bmatrix}
M_{i-1,d} & D_{1,i}^d & D_{1,i}^d \Psi \\
M_{i-1,d} & D_{1,i}^d & D_{1,i}^d \Psi \\
y \neq y_i & y \neq y_i & y \neq y_i \\
y_i = y_i & y_i = y_i & x, u_{y_i} \in \{x_0, x_1\}
\end{bmatrix} \quad \Psi = \begin{bmatrix}
v_{y_i} = x_0 \\
v_{y_i} = x_1 \\
v_{y_i} = x_2
\end{bmatrix} \quad \text{where} \quad i_1 = \begin{cases}
1, 2, 3 & S_{y_i} = \{0, 1, 2\} \\
1, 3, 2 & S_{y_i} = \{0, 1, 3\} \\
2, 3, 1 & S_{y_i} = \{0, 2, 3\} \\
3, 2, 1 & S_{y_i} = \{1, 2, 3\}
\end{cases}
\]

due to (4).

Number of rows in \( M_{i,d} \) is equal to \( 3dp_i - 1 = dp_i \) and number of columns in \( M_{i,d} \) is equal to \( 3\mu_{i-1}p_i - 1 + 6p_i - 1 = 3p_i - 1(\mu_{i-1} + 2) = p_i \mu_i \).

3) \( e_y = 3 \): For this case \( |S_{y_i}| = (e_y + 1) = 4 \). \( S_{y_i} = \{0, 1, 2, 3\} \) and the recursion for \( M_{i,d} \) is given by:

\[
\begin{bmatrix}
M_{i-1,d} & D_{1,i}^d & D_{1,i}^d \Psi \\
M_{i-1,d} & D_{1,i}^d & D_{1,i}^d \Psi \\
y \neq y_i & y \neq y_i & y \neq y_i \\
y_i = y_i & y_i = y_i & x, u_{y_i} \in \{x_0, x_1\}
\end{bmatrix} \quad \Psi = \begin{bmatrix}
v_{y_i} = 0 \\
v_{y_i} = 1 \\
v_{y_i} = 2 \\
v_{y_i} = 3
\end{bmatrix}
\]
Number of rows in $M_{i,d}$ is equal to $4dp_{i-1} = dp_i$ and number of columns in $M_{i,d}$ is equal to $4\mu_{i-1}p_{i-1} + 12p_{i-1} = 4p_{i-1}(\mu_{i-1} + 3) = p_{i\mu_i}$.

We prove the Lemma VII.3 case by case depending on the value of $e_{y_i} \in \{1, 2, 3\}$.

**B. Proof of Lemma VII.3 for $e_{y_i} = 1$**

We will first prove that the determinant of $M_{i,\mu_i}$ has certain factors in Lemma A.4. This particular lemma holds for $e_{y_i} \in \{1, 2, 3\}$. We use this to prove that $M_{i,\mu_i}$ is invertible given $M_{i-1,\mu_i-1}$ is invertible when $e_{y_i} = 1$. Proving this implies that $H_{E,\mathbb{L}}^{\text{Red}}$ is invertible for $s = 2$ case completing the MSR property proof for the case when $s = 2$ i.e., $d = k + 1$.

**Lemma A.4.** $f_{\text{coup}}(\theta)$ divides $|M_{i,\mu_i}|$ for all $\theta \in \{\theta_{x,y,x',y'}^i \mid x_i, x'_i \in S_{y_i}, x_i \neq x'_i\}$ ($S_{y_i}$ is defined in equation (9) and $M_{i,\mu_i}$ defined in Definition 8) where

$$f_{\text{coup}}(\theta) = \prod_{j=1}^{p_{i-1}} \prod_{x_j \neq x'_j} \left( \theta - \theta_{x,y,x',y'}^j \right)^{p_{i-1} - 1}. $$

**Proof:** We use the Lemma A.1 to prove the current lemma. Determinant of matrix $M_{i,\mu_i}$ can be expressed as a polynomial in $\theta_{x,y,x',y'}$ where $x_i, x'_i \in S_{y_i}, \theta_{x,y,x',y'}$ appears in columns of $M_{i,\mu_i}$ indexed by:

$$\{ [x_i, y_i, u] \mid u \in Q^i(E, z), u_{y_i} = x'_i \}.$$  

The number of columns where $\theta_{x,y,x',y'}$ appears is equal to $\frac{p_{i-1}}{e_{y_i} + 1}$. Let $j \in [i - 1]$, $x_j, x'_j \in S_{y_i}$ with $x_j \neq x'_j$. To show that $(\theta_{x,y,x',y'})$ is a factor with multiplicity $\frac{p_{i-1}}{e_{y_i} + 1}$, we consider $\theta_{x,y,x',y'}$ in columns $[x_i, y_i, u], [x_i, y_i, u(x_j \rightarrow u_{y_i})]$ to be equal to $\phi_{u}$ for all $u \in C$ where $C = \{u \in Q^i(E, z) \mid u_{y_i} = x'_i, u_{y_j} = x'_j\}$. We will now show that on substituting $\phi_{u} = \theta_{x,y,x',y'}$ in columns $[x_i, y_i, u], [x_i, y_j, u(x_j \rightarrow u_{y_i})]$ for any $u \in C$, we get the determinant of the matrix to be 0. From Lemma A.1 it follows that $(\theta_{x,y,x',y'})$ is a factor with multiplicity $|C| = \frac{p_i}{|S_{y_i}|} = \frac{p_{i-1}}{e_{y_i} + 1}$ implying the lemma.

This columns indexed by $[x_i, y_i, u], [x_i, y_j, u(x_j \rightarrow u_{y_i})], [x_j, y_j, u], [x_j, y_i, u(x_j \rightarrow u_{y_i})]$ have support only in rows indexed by $[\ell, v]$ where $v \in \{u, u(x_i \rightarrow u_{y_i}), u(x_j \rightarrow u_{y_i}), u(x_i \rightarrow u_{y_i})\}$ indexed by $[\ell, v]$ as shown below:

$$\begin{bmatrix}
\phi_{u}\mu_i \\
\gamma_{x_{i-1},x_{i},u} \\
\gamma_{x_{i},y_{i},x_{i-1},u} \\
x_{i}, y_{i}, u
\end{bmatrix}
\begin{bmatrix}
\theta_{x,y,x',y'}^i \\
\gamma_{x_{i-1},x_{i},y_{i},x_{i-1},y_{i}}^i \\
\gamma_{x_{i},y_{i},y_{i},x_{i-1},y_{i}}^i \\
x_{i}, y_{i}, y_{i}, u(x_j \rightarrow u_{y_i})
\end{bmatrix}
\begin{bmatrix}
\theta_{x,y,x',y'}^j \\
\gamma_{x_{i-1},x_{i},y_{i},x_{i-1},y_{j}}^j \\
\gamma_{x_{i},y_{i},y_{j},x_{i-1},y_{j}}^j \\
x_{j}, y_{j}, y_{j}, u(x_j \rightarrow u_{y_i})
\end{bmatrix}
\begin{bmatrix}
v = u \\
v = u(x_i \rightarrow u_{y_i}) \\
v = u(x_j \rightarrow u_{y_i}) \\
v = u(x_i \rightarrow u_{y_i})(x_j \rightarrow u_{y_i})
\end{bmatrix}$$

where $\theta^{d} = \begin{bmatrix}
1 & \theta & \cdots & \theta^{d-1}
\end{bmatrix}$. It can be observed that $c_1 + c_2 + \gamma_{x_{i-1},x_{i},c_3} + \gamma_{x_{i},x_{i},c_4} = 0$ on substituting $\phi_{u} = \theta_{x,y,x',y'}$ where $c_l$ is $\ell$-th column in the matrix shown above. Therefore the determinant is zero on substitution $\phi_{u} = \theta_{x,y,x',y'}$ for any $u \in C$. 

**Proof of Lemma VII.3 for $e_{y_i} = 1$:** Let $S_{y_i} = \{x_0, x_1\}$. From the recursive definition shown in Section A.A we know that:

$$M_{i,\mu_i} = \begin{bmatrix}
M_{i-1,\mu_i} & D_{1,i}^{\mu_i} \\
D_{i-1}^{\mu_i} & D_{1,i}^{\mu_i}
\end{bmatrix}
\begin{bmatrix}
v_{y_i} = x_0 \\
v_{y_i} = x_1
\end{bmatrix}
\begin{bmatrix}
1 & S_{y_i} = \{0, 1\} \text{ or } \{2, 3\} \\
2 & S_{y_i} = \{0, 2\} \text{ or } \{1, 3\} \text{ (due to (4)} \\
3 & S_{y_i} = \{0, 3\} \text{ or } \{1, 2\}
\end{bmatrix}$$
We will prove factors of determinant of $M_{i,\mu}$ (considering this determinant as polynomial in $\lambda$’s) corresponding to $\lambda_{i_1,y_i}$, that can match up to maximum degree $2(\mu_i - 1)p_i - 1$ it can take. Hence determinant of $M_{i,\mu}$, can be written as product of these factors involving $\lambda_{i_1,y_i}$ and polynomial in rest of the $\lambda$’s. Now we substitute $\lambda_{i_1,y_i} = 0$ in $M_{i,\mu}$, and show that resulting determinant is non-zero given $|M_{i-1,\mu_{i-1}}| \neq 0$ which implies that the factor of determinant of $M_{i,\mu}$, corresponding to polynomial in rest of the $\lambda$’s is non zero. First, we show factors that can account to a degree of $2p_i - 1(\mu_i - 1)$.

From Lemma [A.4] for $\theta \in \{x_0,y_i,x_1,\theta_{x_1,y_i},x_0\} = \{\lambda_{i_1,y_i}, \gamma \lambda_{i_1,y_i}\}$, $f_{coup}(\theta)$ divides $|M_{i,\mu_i}|$. Therefore,

$$f_{coup}(\lambda_{i_1,y_i}) f_{coup}(\gamma \lambda_{i_1,y_i}) \text{ divides } |M_{i,\mu_i}|,$$

where $f_{coup}(\theta) = \prod_{j=1}^{i-1} \prod_{x_j, x'_j \in S_{y_j}} (\theta - \theta_{x_j,y_j,x'_j})^{\frac{p_{i-1}}{v_{y_j}} + 1}$.

This amounts to degree equal to $2 \sum_{j=1}^{i-1} e_{y_j}(e_{y_j} + 1) \frac{p_{i-1}}{v_{y_j}} = 2p_i - 1\mu_i - 1 = 2p_i - 1(\mu_i - e_{y_i}) = 2p_i - 1(\mu_i - 1)$.

We have all the factors of $\lambda_{i_1,y_i}$ that accounts to the degree $2p_i - 1(\mu_i - 1)$. Therefore,

$$|M_{i,\mu_i}| = f_{coup}(\lambda_{i_1,y_i}) f_{coup}(\gamma \lambda_{i_1,y_i}) c,$$

c is not a function of $\lambda_{i_1,y_i}$. We will now show that $|M_{i,\mu_i}|$ when evaluated at $\lambda_{i_1,y_i} = 0$ is non-zero. This will imply that:

$$|M_{i,\mu_i}|_{\{\lambda_{i_1,y_i} = 0\}} = cf_{coup}^2(0) \neq 0,$$

indicating that $c \neq 0$ as $f_{coup}(0) \neq 0$ by choice of $\gamma$’s and therefore $|M_{i,\mu_i}| \neq 0$. Recall that:

$$M_{i,\mu_i} = \begin{bmatrix}
M_{i-1,\mu_i} & D_{i_1,i}^\mu \\
D_{i_1,i}^\mu & D_{i_1,i}^\psi
\end{bmatrix}.$$

On substitution of $\lambda_{i_1,y_i} = 0$:

$$D_{i_1,i}^\mu = \begin{bmatrix}
v_1 & v_1 \\
\vdots & \vdots \\
v_1 & v_1
\end{bmatrix} \quad \text{where} \quad v_1 = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \quad \text{at} \quad (\mu, p_{i-1} \times 2p_{i-1}).$$

By doing column operations on the matrix $M_{i,\mu_i}$, we can remove the rows corresponding to non-zero entries of $\begin{bmatrix}
D_{i_1,i}^\mu \\
D_{i_1,i}^\psi
\end{bmatrix}$ while calculating the determinant and the $2p_i - 1$ columns of $\begin{bmatrix}
D_{i_1,i}^\mu \\
D_{i_1,i}^\psi
\end{bmatrix}$ with an effect of the factor $(1 - \gamma)^{p_{i-1}}$. In the resultant matrix, any column $[x,y,u]$ has $\theta_{x,y,u}$ as a factor as we removed the first row corresponding to $\theta_{x,y,u}$. We therefore get:

$$|M_{i,\mu_i}|_{\{\lambda_{i_1,y_i} = 0\}} = (1 - \gamma)^{p_{i-1}} \left( \prod_{u \in Q_{i-1} (E_{ij}), \quad (x,y) \in E_{ij}, y \leq y_{i-1}} \theta_{x,y,u} \right)^2 \left| M_{i-1,\mu_{i-1}} \right| M_{i-1,\mu_{i-1}} \left| M_{i-1,\mu_{i-1}} \right|^2$$

$$= (1 - \gamma)^{p_{i-1}} \left( \prod_{j=1}^{i-1} \prod_{x_j, x'_j \in S_{y_j}, x_j \neq x'_j} \left( \theta_{x_j,y_j,x_j'} \right)^{2p_{i-1}} \right) \left| M_{i-1,\mu_{i-1}} \right|^2$$

$$= (1 - \gamma)^{p_{i-1}} f_{coup}^2(0) \left| M_{i-1,\mu_{i-1}} \right|^2 = c f_{coup}^2(0).$$

Therefore, $|M_{i,\mu_i}| = |M_{i-1,\mu_{i-1}}|^2 (1 - \gamma)^{p_{i-1}} f_{coup}(\lambda_{i_1,y_i}) f_{coup}(\gamma \lambda_{i_1,y_i}) \neq 0$ and hence invertible.
C. Proof of Lemma VII.3 for $e_{y_i} = 2$

Before proving that $M_{i,\mu_i}$ is invertible given $M_{i-1,\mu_{i-1}}$ is invertible, we first show certain factors of $|M_{i,\mu_i}|$ in the following lemma for the case when $e_{y_i} = 2$. We will use this along with Lemma [A.4] to prove the invertibility of $M_{i,\mu_i}$ for the case when $e_{y_i} = 2$. Note that this along with the proof for the case when $e_{y_i} = 1$ imply that $H_{E,\mathbb{Z}}^{\text{Red}}$ is invertible for $s = 3$ proving the MSR property for $s = 3$ i.e., $d = k + 2$.

**Lemma A.5.** For the case when $e_{y_i} = 2$, $f_{\text{base}}(\lambda_{i_2, y_i})$ divides $\det(M_{i,\mu_i}) = |M_{i,\mu_i}|$ where

$$f_{\text{base}}(\theta) = ((\theta - \lambda_{1, y_i})(\theta - \lambda_{i_2, y_i}))^{p_i-1} \text{ and } (i_1, i_2, i_3) = \begin{cases} (1, 2, 3) & S_{y_i} = \{0, 1, 2\} \\ (1, 3, 2) & S_{y_i} = \{0, 1, 3\} \\ (2, 3, 1) & S_{y_i} = \{0, 2, 3\} \\ (3, 2, 1) & S_{y_i} = \{1, 2, 3\} \end{cases}$$

**Proof:** The proof shown here in similar to Lemma [A.4]. Let $S_{y_i} = \{x_0, x_1, x_2\}$ and $x_0 < x_1 < x_2$. We will first show $(\lambda_{i_2, y_i} - \lambda_{i_1, y_i})^{p_i-1}$ divides $M_{i,\mu_i}$. To do this we substitute $\lambda_{i_2, y_i} = \lambda_{i_1, y_i}$ in any $p_i-1$ columns out of the $2p_i-1$ columns where $\lambda_{i_2, y_i}$ appears. The set of columns where $\lambda_{i_2, y_i}$ appears is given by $C_2 = \{(x_0, y_i; u(x_2 \rightarrow u_{y_i})), (x_2, y_i; u(x_0 \rightarrow u_{y_i})) \mid u \in \Psi_{i_2}^{-1}(E, \mathbb{Z})\}$ by Section A-A. Let the set of the selected $p_i-1$ columns be $\mathcal{P} \subset C_2$. We will show that upon substitution the determinant $|M_{i,\mu_i}|$ is zero by showing that a non-zero vector exists in the null space of $M_{i,\mu_i}, M_{i,\mu_i}$ can be recursively expressed in the following form when $e_{y_i} = 2$.

$$M_{i,\mu_i} = \begin{bmatrix} M_{i-1,\mu_i} & D_{i_1,i}^\mu \hline M_{i-1,\mu_i} & D_{i_2,i}^\mu \hline M_{i-1,\mu_i} & D_{i_3,i}^\mu \hline (\mu, p_i-1 \times 2p_i-1) & (\mu, p_i-1 \times 2p_i-1) \hline (\mu, p_i-1 \times 2p_i-1) & (\mu, p_i-1 \times 2p_i-1) \hline \end{bmatrix}$$

Now we add these ($\mathcal{P}$) columns to corresponding columns in $C_1$ resulting in an equivalent matrix $M'$ (same determinant). Let $M(; [x, y, u])$ be the column at index $[x, y, u]$ in matrix $M_{i,\mu_i}$, and let $M'(; [x, y, u])$ be a column in matrix $M'$. The column operations to result in $M'$ are given by:

$$M'(; [x, y, u]) = \begin{cases} M(;[x, y, u]) + M(;[x_0, y, u(x_1 \rightarrow u_{y_i})]) & [x, y, u] \in \mathcal{P}, u_{y_i} = x_2, x = x_0, y = y_i \\ M(;[x, y, u]) + M(;[x_1, y, u(x_0 \rightarrow u_{y_i})]) & [x, y, u] \in \mathcal{P}, u_{y_i} = x_0, x = x_2, y = y_i \\ M(;[x, y, u]) & \text{otherwise} \end{cases}$$

This implies that after setting $\lambda_{i_2, y_i} = \lambda_{i_1, y_i}$ in columns given by $\mathcal{P}$ we get:

$$|M'|_{\lambda_{i_2, y_i} = \lambda_{i_1, y_i}, \text{columns given by } \mathcal{P}} = \begin{bmatrix} M_{i-1,\mu_i} & D_{i_1,i}^\mu \hline M_{i-1,\mu_i} & D_{i_2,i}^\mu \hline M_{i-1,\mu_i} & D_{i_3,i}^\mu \hline (\mu, p_i-1 \times 2p_i-1) & (\mu, p_i-1 \times 2p_i-1) \hline (\mu, p_i-1 \times 2p_i-1) & (\mu, p_i-1 \times 2p_i-1) \hline \end{bmatrix}$$

where $\begin{bmatrix} D_2' \Psi'' \hline D_2' \Psi'' \hline \end{bmatrix}$ is a submatrix of $\begin{bmatrix} D_{i_1,i}^\mu \hline D_{i_2,i}^\mu \hline D_{i_3,i}^\mu \hline \end{bmatrix}$ that contains columns from $C_2 \setminus \mathcal{P}$ and the submatrix $\begin{bmatrix} D_1' \Psi' \hline D_1' \Psi' \hline \end{bmatrix}$ is the set of $p_i-1 + 1$ columns $\mathcal{P}$ after the column operations.

Consider a vector $(f_1', f', f)$ where, $f_1', f \in \mathbb{F}_q^{p_i-1}$ are any vectors in the left null space of $M_{i-1,\mu_i}$ such that $f$...
vector satisfies the conditions:
\[
\ell = (f_{\ell,u} \mid \ell \in [\mu_i], u \in Q^{i-1}(E,z)),\quad f_{\ell}(\gamma \lambda_{i,y_i}) = 0
\]
for all \( u \in Q^{i-1}(E,z) \), where \( f_{\ell}(x) = \sum_{\ell=1}^{\mu_i} f_{\ell,u} x^{\ell-1} \)

With these conditions \((f_1, f, f)\) is a vector in null space of:

\[
\begin{bmatrix}
M_{i-1,\mu_i} & M_{i-1,\mu_i} & D_1' \Psi' & D_{i+3,\mu_i}'
\end{bmatrix}
\]

The dimension of matrix \(M_{i-1,\mu_i}\) is \((\mu_i \times p_{i-1})\). Therefore the dimension of its left null space is at least \((\mu_i - \mu_i) p_{i-1} - e_i p_{i-1} = 2 p_{i-1}\). But, by additional conditions that need to be satisfied by \(f\) shown in equation (47), the dimension of vector space in which \(f\) can take values reduces to \(p_{i-1}\). However, \(f_1\) can take values in a vector space of dimension \(2 p_{i-1}\). Now we will show that there should be at least one non-zero vector \((f_1, f, f)\) that is also in the left null space of matrix:

\[
D = \begin{bmatrix}
D_{i+3,\mu_i}^\prime & D_2^\prime \\
D_{i+3,\mu_i}^\prime & D_2^\prime \\
(3 \mu_i \times p_{i-1}, p_{i-1})
\end{bmatrix}
\]

The dimension of left null space of \(D\), \(\text{NS}(D)\) is at least \(3 \mu_i - 1 p_{i-1} + 1\). Let the space of vectors \((f_1, f, f)\) be \(\mathcal{F}\). \(\dim \mathcal{F} \geq 3 p_{i-1}\) and \(\dim(\mathcal{F} + \text{NS}(D)) \leq 3 \mu_i p_{i-1}\) as the vector \((f_1, f, f)\) is \(\mathbb{H}_q^{3 \mu_i, p_{i-1}}\). Therefore,

\[
\dim(\mathcal{F} \cap \text{NS}(D)) = \dim \mathcal{F} + \dim(\text{NS}(D)) - \dim(\mathcal{F} + \text{NS}(D)) \geq 1
\]

implying that \(\mathcal{F} \cap \text{NS}(D) \neq \phi\) and there is non zero vector in the null space of \(M_{i,\mu_i}\).

This shows that on substituting \(\lambda_{i,y_i} = \lambda_{i,y_i}\) in any \(p_{i-1} + 1\) columns out of \(2 p_{i-1}\) columns, the determinant is zero. This results in \((\lambda_{i,y_i} - \lambda_{i,y_i}) p_{i-1}\) being a factor of \(|M_{i,\mu_i}|\) by Lemma A.2. The proof for \((\lambda_{i,y_i} - \lambda_{i,y_i}) p_{i-1}\) being a factor of \(|M_{i,\mu_i}|\) follows in the same lines, in that case, we set \(\lambda_{i,y_i} = \lambda_{i,y_i}\) in \(p_{i-1} + 1\) columns. This proves that \(f_{\text{base}}(\lambda_{i,y_i})\) divides \(|M_{i,\mu_i}|\). □

**Proof of Lemma VII.3** for \(e_{y_i} = 2\): Let \(S_{y_i} = \{x_0, x_1, x_2\}\). From the recursion defined in Section A-A:

\[
M_{i,\mu_i} = \begin{bmatrix}
M_{i-1,\mu_i} & M_{i-1,\mu_i} & D_{i+3,\mu_i}' & D_{i+3,\mu_i}' \\
\{y \neq y_i, u_{y_i} = x_0, u_{y_i} = x_1, u_{y_i} = x_2\} & \{y = y_i, x, u_{y_i} \in \{x_0, x_1\}\} & \{y = y_i, x, u_{y_i} \in \{x_0, x_2\}\} & \{y = y_i, x, u_{y_i} \in \{x_1, x_2\}\}
\end{bmatrix}
\]

where \((i_1, i_2, i_3) = (1, 2, 3), (1, 3, 2), (2, 3, 1), (3, 2, 1)\) due to (4).

We will prove factors of determinant of \(M_{i,\mu_i}\), corresponding to \(\lambda_{i,y_i}\), that can match up to maximum degree \(2 p_{i-1}(\mu_i - 1)\) it can take. Hence determinant of \(M_{i,\mu_i}\) can be written as product of these factors involving \(\gamma \lambda_{i,y_i}\) and polynomial in rest of the \(\lambda\)’s. Now we substitute \(\lambda_{i,y_i} = 0\) in \(M_{i,\mu_i}\) and show that resulting determinant is non-zero given \(|M_{i-1,\mu_i,1}| \neq 0\) which implies that the factor of determinant of \(M_{i,\mu_i}\), corresponding to polynomial in rest of the \(\lambda\)’s is non zero.

From Lemma A.4, Lemma A.5 and the fact that the factors are coprime, \(f_{\text{coup}}(\lambda_{i,y_i}) f_{\text{coup}}(\gamma \lambda_{i,y_i}) f_{\text{base}}(\lambda_{i,y_i})\)
divides $|M_{i,\mu_i}|$. This amounts to degree equal to $2 \sum_{j=1}^{i-1} e_{y_j} (e_{y_j} + 1) \frac{p_{r-1}}{e_{y_j} + 1} + 2p_{r-1} = 2p_{r-1}(\mu_{r-1} + 1) = 2p_{r-1}(\mu_i - e_{y_i} + 1) = 2p_{r-1}(\mu_i - 1)$.

Hence we have all the factors involving $\lambda_{i_2,y_i}$ in the polynomial $|M_{i,\mu_i}|$. Therefore, $|M_{i,\mu_i}|$ can be written as:

$$|M_{i,\mu_i}| = f_{coup}(\lambda_{i_2,y_i}) f_{coup}(\gamma \lambda_{i_2,y_i}) f_{base}(\lambda_{i_2,y_i}),$$

which $c$ is a polynomial not involving $\lambda_{i_2,y_i}$. Now its enough to prove $c \neq 0$ for the chosen $\lambda$’s. To show that we set $\lambda_{i_2,y_i} = 0$ in (48) and prove that the polynomial $|M_{i,\mu_i}|$ is not equal to zero for the chosen $\lambda$’s when $\lambda_{i_2,y_i} = 0$. We will therefore, prove that $|M_{i,\mu_i}|(\lambda_{i_2,y_i}=0) \neq 0$. Recall that

$$M_{i,\mu_i} = \begin{bmatrix} M_{i-1,\mu_i} & D_{i_2,i_3}^{\mu_i} & D_{i_2,i_3}^{\mu_i} \\ M_{i-1,\mu_i} & D_{i_2,i_3}^{\mu_i} & D_{i_2,i_3}^{\mu_i} \\ M_{i-1,\mu_i} & D_{i_2,i_3}^{\mu_i} & D_{i_2,i_3}^{\mu_i} \end{bmatrix}.$$

On substituting, $\lambda_{i_2,y_i} = 0$ to prove that $M_{i,\mu_i}$ is invertible, it is enough to show that the only vector in the left null space of $M_{i,\mu_i}$ is zero. On substituting $\lambda_{i_2,y_i} = 0$ we have:

$$D_{i_2,i_3}^{\mu_i} = \begin{bmatrix} v_1 & v_1 \\ \vdots & \vdots \\ v_1 & v_1 \end{bmatrix} \text{ where } v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence by doing columnn operations, we can remove all rows corresponding to non-zero entries in $D_{i_2,i_3}^{\mu_i}$, $D_{i_2,i_3}^{\mu_i} \Psi$ from $M_{i,\mu_i}$ and all columns corresponding to $D_{i_2,i_3}^{\mu_i}$, $D_{i_2,i_3}^{\mu_i} \Psi$ from $M_{i,\mu_i}$ without affecting the invertibility of determinant of $M_{i,\mu_i}$ as $\gamma \neq 0,1$. This can be seen as follows:

$$|M_{i,\mu_i}|(\lambda_{i_2,y_i}=0) = (1 - \gamma)^{p_{i-1}} \times \left( \prod_{u \in Q^{i-1}(E \not\rightarrow z)} \theta_{u,x,y,u} \right)^2 \times |M'|$$

where $M' = \begin{bmatrix} M_{i-1,\mu_i} & \lambda_{i_2,y_i} D_{i_2,i_3}^{\mu_i-1} \\ M_{i-1,\mu_i} & \lambda_{i_2,y_i} D_{i_2,i_3}^{\mu_i-1} \Psi \\ M_{i-1,\mu_i} & \lambda_{i_2,y_i} D_{i_2,i_3}^{\mu_i-1} \Psi^2 \end{bmatrix}$.

$$|M_{i,\mu_i}|(\lambda_{i_2,y_i}=0) = (1 - \gamma)^{p_{i-1}} \times \left( \prod_{j=1}^{i-1} \prod_{x' \in S_{y_j}} (\theta_{x,y,x'})^{2p_{i-1}} \right) \times |M'|$$

$$= (1 - \gamma)^{p_{i-1}} \times f_{coup}(0)^2 |M'|$$

$$= f_{coup}(0) f_{base}(0) c \text{ from equation (48)}$$

$$c = \frac{1}{f_{base}(0)} (1 - \gamma)^{p_{i-1}} |M'|.$$

Hence its enough to show that the left null space of matrix $M'$ is zero to show invertibility of $M'$ and hence invertibility of $M_{i,\mu_i}$. Let the vector in left null space of matrix $M'$ be of the form $F = [F_1, F_2, F_3]$ where $F_1 = (f_{1,\ell,x} \mid \ell \in [\mu_i - 1], x \in Q^{i-1}(E \not\rightarrow z))$, $F_2 = (f_{2,\ell,y} \mid \ell \in [\mu_i], y \in Q^{i-1}(E \not\rightarrow z))$, $F_3 = (f_{3,\ell,z} \mid \ell \in [\mu_i - 1], z \in Q^{i-1}(E \not\rightarrow z))$ and $f_{1,\ell,x}(x) = \sum_{\ell=1}^{\mu_i-1} f_{1,\ell,x} x^{\ell-1}$, $f_{2,\ell,y}(x) = \sum_{\ell=1}^{\mu_i-1} f_{2,\ell,y} x^{\ell-1}$ and $f_{3,\ell,z}(x) = \sum_{\ell=1}^{\mu_i-1} f_{3,\ell,z} x^{\ell-1}$.
Any vector \(F = [F_1, F_2, F_3]\) in the left null space of \(M'\) must be such that \(F' = [F_1, F_2]\) is in left null space of:

\[
M'' = M' \left[ \begin{array}{c|c|c}
\lambda_{i_1, y_1} D_{i_1, i}^{\mu_1-1} \\
\hline
0 & 0 & 0
\end{array} \right] \Psi.
\]

The above matrix is a \((2\mu_1 - 1)p_{i-1} \times 2(\mu_1 - 1)p_{i-1}\) matrix. \(M''\) can be shown to be of rank \(2(\mu_1 - 1)p_{i-1}\) by showing that the matrix \(M''\) created by appending \(p_{i-1}\) columns is of full rank. Let:

\[
M'' = \left[ \begin{array}{c|c|c|c|c|c}
M_{i-1, \mu_1-1} & 0 & \lambda_{i, y} D_{i, i}^{\mu_1-1} & 0 & 0 & 0 \\
0 & M_{i-1, \mu_1} & D_{i, i}^{\mu_1-1} & v_1 & \cdots & v_{p_{i-1}}
\end{array} \right],
\]

where \(v_j\) is a \(\mu_1p_{i-1} \times 1\) vector with 1 at \((j - 1)\mu_i + 1\)th component and with 0 at other components. As \(v_j\)'s are columns with single non-zero element, we can remove the rows corresponding to these non-zero elements and bring out the factors that are common to the columns without affecting the determinant.

\[
|M''| = (\gamma \lambda_{i, y})^{p_{i-1}} \prod_{j=1}^{i-1} \prod_{x \neq x'} (\theta_{x, y_1, x'})^{p_{i-1}} |M_{i-1, \mu_1-1} - 0| M_{i-1, \mu_1-1}
\]

The last equality follows from the derivation for the case when \(e_{y_1} = 1\) presented in Section A-B. Hence if we produce a set of vectors \(F' = [F_1, F_2]\) forming \(p_{i-1}\) dimensional left null space for \(M''\) then it is the exact full left null space of \(M''\) as rank of \(M''\) is exactly \(2(\mu_1 - 1)p_{i-1}\). Let \(F_1\) be vector in left null space of \(M_{i-1, \mu_1-1}\). Define \(F_2\), such that \(f_{2, \ell, \nu} = \lambda_{i, y} f_{1, \ell, \nu}\) for all \(\ell \in [\mu_i - 1]\), \(v \in Q^{i-1}(E, z)\) and \(f_{2, \mu, \nu} = 0\) for all \(v \in Q^{i-1}(E, z)\). Now it can be seen that \([F_1, F_2]\) is in the left null space of \(M''\) and we can produce such a vector \(F' = [F_1, F_2]\) in left null space of \(M''\) for every vector \(F_1\) in left null space of \(M_{i-1, \mu_1-1}\). By invertibility of \(M_{i-1, \mu_1-1}\), the left null space of \(M_{i-1, \mu_1-1}\) has dimension \((\mu_1 - 1)p_{i-1} - \mu_1 - p_{i-1} = (\mu_1 - \mu_1 - 1)p_{i-1} = p_{i-1}\) and we have produced a \(p_{i-1}\) dimensional left null space for \(M''\) which is the exactly full left null space of \(M''\). Now any vector in left null space of \(M'\) must be of the form \([F_1, F_2, F_3]\) where \(F_1, F_2\) is as described just now in the left null space of \(M''\). As \([F_1, F_2, F_3]\) \(M' = 0\). By looking at the last \(2p_{i-1}\) columns of \(M'\) we get:

\[
f_{2, \nu}(\gamma \lambda_{i, y_1}) = \gamma^2 \lambda_{i, y} f_{3, \nu}(\gamma \lambda_{i, y_1}) \text{ for all } \nu \in Q^{i-1}(E, z) \quad (49)
\]

We now define for any \(\nu \in Q^{i-1}(E, z)\) \(g_{\nu}(x) = f_{2, \nu}(x) - \lambda_{i, y} f_{3, \nu}(x) = \sum_{\ell=1}^{\mu_1-1} g_{\ell, \nu} x^{\ell-1}\). From equation (50), \(g_{\nu}(x)\) has root at \(\lambda_{i, y_1}\).

From the definition of \(F_2, F_3\), it is implied that \(G = (g_{\nu} | \ell \in [\mu_i - 1], \nu \in Q^{i-1}(E, z))\) is in left null space of \(M_{i-1, \mu_1-1}\). We will show that the condition that \(G M_{i-1, \mu_1-1}\) is 0 together with constraints that \(g_{\nu}(\lambda_{i, y_1}) = 0\) will force \(G\) to be a null vector. Let \(g_{\nu}(x) = (x - \lambda_{i, y_1}) g_{\nu}'(x)\) where \(g_{\nu}'(x)\) is of degree \(\mu_i - 3\) and let \(g_{\nu}'(x) = \sum_{\ell=1}^{\mu_2-2} g_{\ell, \nu} x^{\ell-1}\). \(G M_{i-1, \mu_1-1}\) implies that for any \(\nu \in Q^{i-1}(E, z)\) and \((x, y) \in E_{2, y}, y \leq y_{i-1}:

\[
g_{\nu}(\theta_{x, y, u}) + \gamma_{u, x} g_{\nu}(x \rightarrow u_y) (\theta_{x, y, u}) = 0
\]

\[
\Rightarrow (\theta_{x, y, u} - \lambda_{i, y}) g_{\nu}'(\theta_{x, y, u}) + \gamma_{u, x} g_{\nu}'(x \rightarrow u_y) (\theta_{x, y, u}) = 0
\]

By setting \(G' = (g_{\ell, \nu}) | \ell \in [\mu_i - 2], \nu \in Q^{i-1}(E, z)\) it is implied that:

\[
G' M_{i-1, \mu_1-2} = 0
\]

\[
G' = 0 \text{ as } M_{i-1, \mu_1-1} \text{ is invertible and } \mu_i = \mu_i + 2.
\]

Therefore \(G = 0\) and \(f_{2, \nu}(x) = \lambda_{i, y} f_{3, \nu}(x)\) for all \(\nu \in Q^{i-1}(E, z)\). Substituting this in equation (49) we get:

\[
\lambda_{i, y} f_{3, \nu}(\gamma \lambda_{i, y_1}) = \gamma^2 \lambda_{i, y} f_{3, \nu}(\gamma \lambda_{i, y_1})
\]

\[
(1 - \gamma^2) f_{3, \nu}(\gamma \lambda_{i, y_1}) = 0
\]

(52)
Let \( f_{3,y}(x) = (x - \gamma \lambda_{1,y}) f_{3,y}(x) \) for any \( y \in Q^{i-1}(E, z) \) where \( f_{3,y}(x) \) is of degree \( \mu_i - 3 \). \( F_3 \) is null space of \( M_{1-\mu_i-1} \). Therefore for any \( \nu \in Q^{i-1}(E, z) \), \((x, y) \) \( \in \mathcal{E}_2, y \leq y_{i-1} \):

\[
\begin{align*}
\frac{f_{3,y}}{f_{3,y}(\theta_{x,y,\nu}) + \gamma_{u,x} f_{3,y}(x \rightarrow u_x) (\theta_{x,y,\nu})} &= 0 \\
(\theta_{x,y,\nu} - \gamma \lambda_{1,y}) \left( \frac{f_{3,y}}{f_{3,y}(\theta_{x,y,\nu}) + \gamma_{u,x} f_{3,y}(x \rightarrow u_x) (\theta_{x,y,\nu})} \right) &= 0
\end{align*}
\]

By setting \( F_3' = \{ f_{3,y}(\ell, \nu) \mid \ell \in [\mu_i - 2], y \in Q^{i-1}(E, z) \} \) it is implied that:

\[
\begin{align*}
F_3'M_{1-\mu_i-2} &= 0 \\
F_3' &= 0 \text{ as } M_{1-\mu_i-1} \text{ is invertible.}
\end{align*}
\]

This implies that \( F_3 = 0 \) and therefore \( F_2 = 0 \) and \( F_1 = 0 \). Hence the left null space of \( M' \) has only zero vector and hence is invertible. It follows that \( M_{i,\mu_i} \) is invertible.

\[\square\]

**D. Proof of Lemma VII.3 for \( e_{y_i} = 3 \)**

Similar to the proof for the case where \( e_{y_i} = 2 \), before proving that \( M_{i,\mu_i} \) is invertible given \( M_{1-\mu_i-1} \), we first show certain factors of \( |M_{i,\mu_i}| \) in the following lemma for the case when \( e_{y_i} = 3 \). We will use this along with Lemma VIII.4 to prove the invertibility of \( M_{i,\mu_i} \) for the case when \( e_{y_i} = 3 \). Note that this along with the proof for the case when \( e_{y_i} = 1, 2 \) imply that \( H^E_{E, 3} \) is invertible for \( s \leq 4 \).

**Lemma A.6.** For the case when \( e_{y_i} = 3 \), \( f_{\text{base,2}}(\lambda_{1,y_i}) f_{\text{base,3}}(\lambda_{1,y_i}) \) divides \( |M_{i,\mu_i}| \), where

\[
f_{\text{base,2}}(\theta) = ((\theta - \lambda_{j,y_i})^2(\theta - \gamma \lambda_{j,y_i})((\gamma \theta - \lambda_{j,y_i}))^\mu_i.
\]

**Proof:**

**Case 1:** \( f_{\text{base,2}}(\lambda_{1,y_i}) \) divides \( |M_{i,\mu_i}| \): Set \( \lambda_{1,y_i} = \lambda_{2,y_i} \) in columns given by \( \{ (0, y_i, \nu(1 \rightarrow u_{y_i})), (2, y_i, \nu(3 \rightarrow u_{y_i})) \} \) for any \( \nu \in Q^{i-1}(E, z) \). This is because \( \theta_{0,y_i,1} = \theta_{2,y_i,3} = \lambda_{1,y_i} \) from the \( \theta \) to \( \lambda \) assignment shown in equation (4). Then we will show that the columns:

\[
\{ (0, y_i, \nu(1 \rightarrow u_{y_i})), (2, y_i, \nu(3 \rightarrow u_{y_i})), (0, y_i, \nu(2 \rightarrow u_{y_i})), (1, y_i, \nu(3 \rightarrow u_{y_i})) \}
\]

are linearly dependent. It can be seen that the rows where they have non zero elements is indexed by \( [\ell, \nu], \ell \in [\mu_i], \nu \in \{ \nu(0 \rightarrow u_{y_i}), \nu(1 \rightarrow u_{y_i}), \nu(2 \rightarrow u_{y_i}), \nu(3 \rightarrow u_{y_i}) \} \) and restricted to these rows the columns are as shown below:

\[
\begin{bmatrix}
\gamma_{1,0} \theta_{0,y_i,1}^\mu_i & \gamma_{3,2} \theta_{2,y_i,3}^\mu_i & \gamma_{3,1} \theta_{1,y_i,3}^\mu_i \\
\theta_{0,y_i,1}^\mu_i & \theta_{2,y_i,3}^\mu_i & \theta_{1,y_i,3}^\mu_i \\
(0, y_i, \nu(1 \rightarrow u_{y_i})) & (2, y_i, \nu(3 \rightarrow u_{y_i})) & (0, y_i, \nu(2 \rightarrow u_{y_i})) & (1, y_i, \nu(3 \rightarrow u_{y_i}))
\end{bmatrix}
\]

It is clear to see that on setting \( \lambda_{1,y_i} = \lambda_{2,y_i} \) the columns in matrix shown above sum to zero. This is true for any \( \nu \in Q^{i-1}(E, z) \) resulting in \( (\lambda_{1,y_i} - \lambda_{2,y_i})^\mu_i \) being a factor of \( |M_{i,\mu_i}| \).

Now set \( \lambda_{1,y_i} = \lambda_{2,y_i} \) in the columns given by \( \{ (1, y_i, \nu(0 \rightarrow u_{y_i})), (3, y_i, \nu(2 \rightarrow u_{y_i})) \} \) for a fixed \( \nu \in Q^{i-1}(E, z) \). Then we show that the columns:

\[
\{ (1, y_i, \nu(0 \rightarrow u_{y_i})), (3, y_i, \nu(2 \rightarrow u_{y_i})), (2, y_i, \nu(0 \rightarrow u_{y_i})), (3, y_i, \nu(1 \rightarrow u_{y_i})) \}
\]

are linearly dependent. It can be seen that the rows where they have non zero elements is indexed by \([\ell, \nu], \ell \in [\mu_i], \nu \in \{ \nu(0 \rightarrow u_{y_i}), \nu(1 \rightarrow u_{y_i}), \nu(2 \rightarrow u_{y_i}), \nu(3 \rightarrow u_{y_i}) \} \) and restricted to these rows the columns are as shown.
below:

\[
\begin{bmatrix}
\gamma \lambda_{1,y_i}^{\mu_i} & \gamma \lambda_{2,y_i}^{\mu_i} \\
\gamma \lambda_{1,y_i}^{\mu_i} & \gamma \lambda_{2,y_i}^{\mu_i} \\
\gamma \lambda_{1,y_i}^{\mu_i} & \gamma \lambda_{2,y_i}^{\mu_i} \\
\gamma \lambda_{1,y_i}^{\mu_i} & \gamma \lambda_{2,y_i}^{\mu_i}
\end{bmatrix}
\]

It is clear to see that on setting \(\lambda_{1,y_i} = \lambda_{2,y_i}\), \(c_1 + \gamma c_2 + c_3 + c_4 = 0\) where \(c_i\) is \(i\)-th column of the matrix shown above. This is true for any \(u \in Q^{i-1}(E, z)\) resulting in a factor of \((\lambda_{1,y_i} - \lambda_{2,y_i})^{p_{i-1}}\). Therefore \((\lambda_{1,y_i} - \lambda_{2,y_i})^{2^{p_{i-1}}}\) divides \(|M_{i,\mu_i}|\).

We now set \(\lambda_{1,y_i} = \gamma \lambda_{2,y_i}\) in columns indexed by \((0,y_i, u(1 \rightarrow u_{y_i})), (2,y_i, u(3 \rightarrow u_{y_i}))\) for some \(u \in Q^{i-1}(E, z)\). Then we show that the columns \(\{ (0, y_i, u(1 \rightarrow u_{y_i})), (2, y_i, u(3 \rightarrow u_{y_i})) \} \) are linearly dependent. It can be seen that the rows where they have non zero elements is indexed by \([\ell, u], \ell \in [\mu_i], u \in \{u(x \rightarrow u_{y_i}) \mid x \in [0, 3]\}\) and restricted to these rows the columns are as shown below:

\[
\begin{bmatrix}
\lambda_{1,y_i}^{\mu_i} & \lambda_{1,y_i}^{\mu_i} & \lambda_{1,y_i}^{\mu_i} & \lambda_{1,y_i}^{\mu_i} \\
\lambda_{2,y_i}^{\mu_i} & \lambda_{2,y_i}^{\mu_i} & \lambda_{2,y_i}^{\mu_i} & \lambda_{2,y_i}^{\mu_i}
\end{bmatrix}
\]

It is clear to see that \(c_1 + \gamma c_2 + c_3 + c_4 = 0\) where \(c_i\) is \(i\)-th column of the matrix shown above. This is true for any \(u \in Q^{i-1}(E, z)\) resulting in a factor of \((\lambda_{2,y_i} - \gamma \lambda_{1,y_i})^{p_{i-1}}\). Therefore \(f_{base,2}(\lambda_{1,y_i})\) divides \(|M_{i,\mu_i}|\).

**Case 2:** \(f_{base,3}(\lambda_{1,y_i})\) divides \(|M_{i,\mu_i}|\). Set \(\lambda_{1,y_i} = \lambda_{3,y_i}\) in columns given by \(\{ (0, y_i, u(1 \rightarrow u_{y_i})), (2, y_i, u(3 \rightarrow u_{y_i})) \}\) for any \(u \in Q^{i-1}(E, z)\). Then we will show that the columns

\[
\{ (0, y_i, u(1 \rightarrow u_{y_i})), (2, y_i, u(3 \rightarrow u_{y_i})) \}
\]

are linearly dependent. It can be seen that the rows where they have non zero elements is indexed by \([\ell, u], \ell \in [\mu_i], u \in \{u(x \rightarrow u_{y_i}) \mid x \in [0, 3]\}\) and restricted to these rows the columns are as shown below:

\[
\begin{bmatrix}
\lambda_{1,y_i}^{\mu_i} & \lambda_{1,y_i}^{\mu_i} & \lambda_{1,y_i}^{\mu_i} & \lambda_{1,y_i}^{\mu_i} \\
\lambda_{3,y_i}^{\mu_i} & \lambda_{3,y_i}^{\mu_i} & \lambda_{3,y_i}^{\mu_i} & \lambda_{3,y_i}^{\mu_i}
\end{bmatrix}
\]

It is clear to see that \(c_1 + c_2 + c_3 + c_4 = 0\) where \(c_i\) is \(i\)-th column of the matrix shown above. This is true for any \(u \in Q^{i-1}(E, z)\) resulting in a factor of \((\lambda_{1,y_i} - \gamma \lambda_{3,y_i})^{p_{i-1}}\). Set \(\lambda_{1,y_i} = \lambda_{3,y_i}\) in columns given by \(\{ (1, y_i, u(0 \rightarrow u_{y_i})), (3, y_i, u(2 \rightarrow u_{y_i})) \}\) for some fixed \(u \in Q^{i-1}(E, z)\). We will show that there exists a non zero vector in the null space of the matrix. From the recursion in Section A-A the matrix \(M_{i,\mu_i}\) can be expressed as following:

\[
\begin{bmatrix}
M_{i-1,\mu_i} & D_{1,i}^{\mu_i} & D_{2,i}^{\mu_i} & D_{3,i}^{\mu_i} & D_{4,i}^{\mu_i} \\
M_{i-1,\mu_i} & D_{1,i}^{\mu_i} & D_{2,i}^{\mu_i} & D_{3,i}^{\mu_i} & D_{4,i}^{\mu_i}
\end{bmatrix}
\]
The matrix $M_{i-1,\mu_i}$ is of dimension $(\mu_i p_{i-1} \times \mu_i p_{i-1}-1)$ and therefore has a left null space of dimension at least $(\mu_i - \mu_i-1)p_{i-1} = 3p_{i-1}$. Let $f = (f_{\langle \ell,\mu \rangle} | \ell \in [\mu_i], \mu \in Q^{-1}(E, z))$ be a vector in null space of $M_{i-1,\mu_i}$. We use notation $f_{\langle \ell,\mu \rangle}(x) = \sum_{j=1}^{\mu_i} f_{\langle \ell,\mu \rangle} x^{\ell-1}$ and introduce some more conditions on the vector $f$ such that the vector $[ f_1 f_2 f_3 f_4 ]$ is in null space of $M_{i,\mu_i}$. Let $f_{\langle \gamma,3,y,\mu \rangle} = f_{\langle \gamma,2,y,\mu \rangle} = 0$ for all $y \in Q^{-1}(E, z)$ and $f_{\langle \gamma,1,y,\mu \rangle} = 0$ for all $y \in Q^{-1}(E, z) \setminus \{ \gamma \}$. These conditions will ensure that $[ f_1 f_2 f_3 f_4 ]$ is in null space of $M_{i,\mu_i}$, as we are setting $\lambda_{1,y_i} = \lambda_{3,y_i}$ in columns given by $\{(1,y_i,u(0 \to u_{y_i})),(3,y_i,u(2 \to u_{y_i}))\}$. This introduces $3p_{i-1} - 1$ extra conditions on $f$ apart from the constraint that $f$ is in null space of $M_{i-1,\mu_i}$. Therefore, the null space is of dimension at least 1 implying that determinant $|M_{i,\mu_i}|$ is 0 on setting $\lambda_{1,y_i} = \lambda_{3,y_i}$. This is true for any $y \in Q^{-1}(E, z)$. Therefore, it follows from Lemma A.1 that $(\lambda_{1,y_i} - \lambda_{3,y_i})^{p_{i-1}}$ divides $|M_{i,\mu_i}|$. In total we have $(\lambda_{1,y_i} - \lambda_{3,y_i})^{2p_{i-1}}$ divides $M_{i,\mu_i}$. This is because we used $4p_{i-1}$ distinct columns containing $\lambda_{1,y_i}$ while substituting it in two columns to be equal to $\lambda_{3,y_i}$ at a time.

We now set $\lambda_{3,y_i} = \gamma_1\lambda_{3,y_i}$ in columns given by $\{(0,y_i,u(3 \to u_{y_i})),(1,y_i,u(2 \to u_{y_i}))\}$ for some fixed $y \in Q^{-1}(E, z)$. We will show that there exists a non zero vector in the null space of the matrix. We use similar ideas as before. The matrix $M_{i-1,\mu_i}$ has a left null space of dimension $(\mu_i - \mu_i-1)p_{i-1} = 3p_{i-1}$. Let $f = (f_{\langle \ell,\mu \rangle} | \ell \in [\mu_i], \mu \in Q^{-1}(E, z))$ be a vector in null space of $M_{i-1,\mu_i}$. We introduce some more conditions on the vector $f$ such that the vector $[ f_1 f_2 f_3 f_4 ]$ is in null space of $M_{i,\mu_i}$. Let $f_{\langle \gamma,1,y,\mu \rangle} = f_{\langle \gamma,2,y,\mu \rangle} = 0$ for all $\gamma \in Q^{-1}(E, z)$ and $f_{\langle \gamma,3,y,\mu \rangle} = 0$ for all $\gamma \in Q^{-1}(E, z) \setminus \{ \mu \}$. This introduces $3p_{i-1} - 1$ extra conditions on $f$ and with these conditions there exists a non zero vector of the form $[ f_1 f_2 f_3 f_4 ]$ is in null space of $M_{i,\mu_i}$ obtained after substitution. This implies that $|M_{i,y_i} - \gamma_1\lambda_{3,y_i})|$ divides $|M_{i,\mu_i}|$. Hence, $(\lambda_{1,y_i} - \lambda_{3,y_i})^{p_{i-1}}$ is a factor of $|M_{i,\mu_i}|$.

Set $\lambda_{1,y_i} = \gamma_3\lambda_{3,y_i}$ in columns given by $\{(0,y_i,u(1 \to u_{y_i})),(2,y_i,u(3 \to u_{y_i}))\}$ for some fixed $y \in Q^{-1}(E, z)$. Then we will show that the columns $(0,y_i,u(1 \to u_{y_i})),(2,y_i,u(3 \to u_{y_i})),(3,y_i,u(0 \to u_{y_i})),(2,y_i,u(1 \to u_{y_i}))$ are linearly dependent. It can be seen that the rows where they have non zero elements is indexed by $[\ell,\mu], \mu \in [\mu_i], \mu \in Q(x \to u_{y_i}) | x \in [0,3])$ and restricted to these rows the columns are as shown below:

\[
\begin{bmatrix}
\lambda_{1,y_i}^{\mu_i} \\
\lambda_{3,y_i}^{\mu_i} \\
\lambda_{3,y_i}^{\mu_i} \\
\lambda_{3,y_i}^{\mu_i}
\end{bmatrix}
\begin{bmatrix}
\gamma_{1,y_i}^{\mu_i} \\
\gamma_{3,y_i}^{\mu_i} \\
\gamma_{3,y_i}^{\mu_i} \\
\gamma_{3,y_i}^{\mu_i}
\end{bmatrix}
\]

It is clear to see that $c_1 + \gamma c_2 + c_3 + c_4 = 0$ where $c_i$ is $i$-th column of the matrix shown above. This is true for any $y \in Q^{-1}(E, z)$ resulting in a factor of $(\lambda_{1,y_i} - \gamma_3\lambda_{3,y_i})^{p_{i-1}}$.

Hence $f_{base,3}(\lambda_{1,y_i})$ divides $|M_{i,\mu_i}|$. \qed

**Proof of Lemma VII.3** for $e_{y_i} = 3$: From Lemma A.4 and Lemma A.6

\[ f_{coup}(\lambda_{1,y_i})^2 f_{coup}(\gamma\lambda_{1,y_i})^2 f_{base,2}(\lambda_{1,y_i}) f_{base,3}(\lambda_{1,y_i}) \text{ divides } |M_{i,\mu_i}|, \]

as $\theta_{0,y_i,1} = \theta_{2,y_i,3} = \lambda_{1,y_i}$ and $\theta_{1,y_i,0} = \theta_{3,y_i,2} = \gamma\lambda_{1,y_i}$. This accounts to degree $4 \sum_{j=1}^{i-1} e_{y_j}(e_{y_j} + 1) \frac{p_{i-1}}{e_{y_j+1}} + 8p_{i-1} = 4(\mu_i - 1)2p_{i-1} = 4(\mu_i - 1)p_{i-1}$. Hence we have all the factors involving $\lambda_{1,y_i}$ in the polynomial $|M_{i,\mu_i}|$. Therefore, $|M_{i,\mu_i}|$ can be written as:

\[ |M_{i,\mu_i}| = f^2_{coup}(\lambda_{1,y_i}) f^2_{coup}(\gamma\lambda_{1,y_i}) f_{base,2}(\lambda_{1,y_i}) f_{base,3}(\lambda_{1,y_i}) c \]

where $c$ is a polynomial not involving $\lambda_{1,y_i}$. We will now show that $|M_{i,\mu_i}|$ when evaluated at $\lambda_{1,y_i} = 0$ is invertible for the chosen $\lambda$. Given that it is true:

\[ |M_{i,\mu_i}|_{\lambda_{1,y_i},=0} = f^4_{coup}(0) f_{base,2}(0) f_{base,3}(0) c \neq 0 \]

We know that $f^4_{coup}(0) f_{base,2}(0) f_{base,3}(0) \neq 0$

Hence: $c \neq 0$

From equation (53) $|M_{i,\mu_i}| \neq 0$. 

Therefore, we prove that $|M_{i,\mu_i}\{\lambda_{1,y_i}=0\} \neq 0$. Recall,

$$M_{i,\mu_i} = \begin{bmatrix}
M_{i-1,\mu_i,1} & D_{1,i}^{\mu_i} & D_{2,i}^{\mu_i} & D_{3,i}^{\mu_i} \\
M_{i-1,\mu_i,1} & M_{i-1,\mu_i,1} & D_{2,i}^{\mu_i} & D_{3,i}^{\mu_i} \\
M_{i-1,\mu_i,1} & M_{i-1,\mu_i,1} & M_{i-1,\mu_i,1} & D_{3,i}^{\mu_i} \\
M_{i-1,\mu_i,1} & M_{i-1,\mu_i,1} & M_{i-1,\mu_i,1} & D_{3,i}^{\mu_i}
\end{bmatrix}$$

It is enough to prove that the only vector in the left null space of the above matrix is zero vector on substituting $\lambda_{1,y_i} = 0$. We have that on substituting $\lambda_{1,y_i} = 0$:

$$D_{1,i}^{\mu_i} = \begin{bmatrix} v_1 & v_1 \\ \\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
\end{bmatrix} \text{ where } v_1 = \begin{bmatrix} 1 \\ 0 \\ \\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
\end{bmatrix}$$

where $v_1$ is a vector with $\mu_i$ components with 1 at the first component and with 0 at rest of the components. Hence by doing column operations, we can remove all rows corresponding to non-zero entries in $D_{1,i}^{\mu_i}$, $D_{1,i}^{\mu_i}$ from $M_{i,\mu_i}$ and all columns corresponding to $D_{1,i}^{\mu_i}$, $D_{1,i}^{\mu_i}$ from $M_{i,\mu_i}$ without affecting the invertibility of determinant of $M_{i,\mu_i}$ as $\gamma \neq 0,1$. This can be seen as follows:

$$|M_{i,\mu_i}\{\lambda_{1,y_i}=0\} = (1 - \gamma)^{2p_{h-1}} \times \prod_{j=1}^{i-1} \prod_{x,x' \in S_{y_j} \atop x \neq x'} (\theta_{x,y_j,x'})^{4p_{h-1}} \times (\lambda_{2,y_j,3,y_j})^{4p_{h-1}} \times \gamma^{4p_{h-1}} \times$$

$$\begin{bmatrix}
M_{i-1,\mu_i,1} & D_{2,i}^{\mu_i-1} & D_{3,i}^{\mu_i-1} \\
M_{i-1,\mu_i,1} & D_{2,i}^{\mu_i-1} & D_{3,i}^{\mu_i-1} \\
M_{i-1,\mu_i,1} & D_{2,i}^{\mu_i-1} & D_{3,i}^{\mu_i-1} \\
\end{bmatrix}$$

Hence its enough to show that the left null space of following matrix is zero in order to prove $|M_{i,\mu_i}\{\lambda_{1,y_i}=0\}$.

$$M' = \begin{bmatrix}
M_{i-1,\mu_i,1} & D_{2,i}^{\mu_i-1} & D_{3,i}^{\mu_i-1} \\
M_{i-1,\mu_i,1} & D_{2,i}^{\mu_i-1} & D_{3,i}^{\mu_i-1} \\
M_{i-1,\mu_i,1} & D_{2,i}^{\mu_i-1} & D_{3,i}^{\mu_i-1} \\
\end{bmatrix}$$

Let the vector in left null space of above matrix $M'$ be of the form $F = [F_1, F_2, F_3, F_4]$ where for $1 \leq j \leq 4, F_j = (f_j,\ell,U | \ell \in [\mu_i - 1], U \in Q^{i-1}(E,\mathbb{Z}))$, and $f_j,\ell,U$ a polynomial of degree $\mu_i - 2$ given by $f_j,\ell,U(x) = \sum_{\ell=1}^{\mu_i-1} \int_{\ell,\ell,U} x^{\ell-1}$.

1) Since $FM' = 0$, we write the null space equations corresponding to $2p_{h-1}$ columns $C_1$.

$$f_1,\ell,U(\lambda_{2,y_i}) = f_3,\ell,U(\lambda_{2,y_i}) \text{ for all } U \in Q^{i-1}(E,\mathbb{Z})$$

$$f_1,\ell,U(\gamma \lambda_{2,y_i}) = \gamma f_3,\ell,U(\lambda_{2,y_i}) \text{ for all } U \in Q^{i-1}(E,\mathbb{Z}).$$

Equation (54) implies that $f_1,\ell,U(x) - f_3,\ell,U(x)$ have $\lambda_{2,y_i}$ as root. Let:

$$g_1,\ell,U(x) = f_3,\ell,U(x) - f_1,\ell,U(x) \text{ for all } U \in Q^{i-1}(E,\mathbb{Z})$$

$$= \sum_{\ell=1}^{\mu_i-1} g_1,\ell,U x^{\ell-1}$$

where $g_1,\ell,U(x)$ has root at $\lambda_{2,y_i}$. Let $G_1 = (g_1,\ell,U | \ell \in [\mu_i - 1], U \in Q^{i-1}(E,\mathbb{Z}))$. Hence $F_3 = F_1 + G_1$. By
equation (55) and (56):
\[
\begin{align*}
 f_{1,\mathbf{v}}(\gamma \lambda_{2,y}) &= \gamma f_{1,\mathbf{v}}(\gamma \lambda_{2,y}) + \gamma g_{1,\mathbf{v}}(\gamma \lambda_{2,y}) \\
 g_{1,\mathbf{v}}(\gamma \lambda_{2,y}) &= \gamma^{-1} (1 - \gamma) f_{1,\mathbf{v}}(\gamma \lambda_{2,y}) 
\end{align*}
\]
(57)

Since \( F_3 = F_1 + G_1 \) and \( F_3, F_1 \) are in left null space of \( M_{i-1,\mu_i-1} \), we have that \( G_1 \) is also in left null space of \( M_{i-1,\mu_i-1} \).

2) Since \( FM' = 0 \), we write the \( 2p_{i-1} \) null space equations corresponding to \( C_2 \). For any \( \mathbf{v} \in Q^{i-1}(E, \mathbb{Z}) \):
\[
\begin{align*}
 f_{1,\mathbf{v}}(\lambda_{3,y_i}) &= f_{3,\mathbf{v}}(\lambda_{3,y_i}) \\
 f_{1,\mathbf{v}}(\gamma \lambda_{3,y_i}) &= \gamma f_{3,\mathbf{v}}(\gamma \lambda_{3,y_i}) 
\end{align*}
\]
Equation (58) implies that \( f_{1,\mathbf{v}}(x) - f_{3,\mathbf{v}}(x) \) has \( \lambda_{3,y_i} \) as root. Let:
\[
g_{2,\mathbf{v}}(x) = f_{3,\mathbf{v}}(x) - f_{1,\mathbf{v}}(x) = \sum_{\ell=1}^{\mu_i-1} g_{2,\ell,\mathbf{v}} x^{\ell-1}
\]
where \( g_{2,\mathbf{v}} \) has root at \( \lambda_{3,y_i} \). Let \( G_2 = (g_{2,\ell,\mathbf{v}} | \ell \in [\mu_i - 1], \mathbf{v} \in Q^{i-1}(E, \mathbb{Z}) \)\). Hence \( F_4 = F_1 + G_2 \). By same argument that led to equation (57), we have that:
\[
g_{2,\mathbf{v}}(\gamma \lambda_{3,y_i}) &= \gamma^{-1} (1 - \gamma) f_{1,\mathbf{v}}(\gamma \lambda_{3,y_i}) 
\]
(61)

Since \( F_4 = F_1 + G_2 \) and \( F_4, F_1 \) are in left null space of \( M_{i-1,\mu_i-1} \), we have that \( G_2 \) is also in left null space of \( M_{i-1,\mu_i-1} \).

3) Since \( FM' = 0 \), we write the null space equations corresponding to \( 2p_{i-1} \) columns \( C_3 \). For any \( \mathbf{v} \in Q^{i-1}(E, \mathbb{Z}) \):
\[
\begin{align*}
 f_{2,\mathbf{v}}(\lambda_{2,y_i}) &= f_{4,\mathbf{v}}(\lambda_{2,y_i}) \\
 f_{2,\mathbf{v}}(\gamma \lambda_{2,y_i}) &= \gamma f_{4,\mathbf{v}}(\gamma \lambda_{2,y_i}) 
\end{align*}
\]
Equation (62) implies that \( f_{2,\mathbf{v}}(x) - f_{4,\mathbf{v}}(x) \) has \( \lambda_{2,y_i} \) as root. Let:
\[
g_{3,\mathbf{v}}(x) = f_{4,\mathbf{v}}(x) - f_{2,\mathbf{v}}(x) = \sum_{\ell=1}^{\mu_i-1} g_{3,\ell,\mathbf{v}} x^{\ell-1}
\]

By above equation and equation (60):
\[
f_{2,\mathbf{v}}(x) = f_{1,\mathbf{v}}(x) + g_{2,\mathbf{v}}(x) + g_{3,\mathbf{v}}(x)
\]
(64)

where \( g_{3,\mathbf{v}}(x) \) has root at \( \lambda_{2,y_i} \). Let \( G_3 = (g_{3,\ell,\mathbf{v}} | \ell \in [\mu_i - 1], \mathbf{v} \in Q^{i-1}(E, \mathbb{Z}) \). Hence \( F_2 = F_1 + G_2 + G_3 \) and \( G_3 \) is in left null space of \( M_{i-1,\mu_i-1} \). 4) Since \( FM' = 0 \), we write the null space equations corresponding to \( 2p_{i-1} \) columns \( C_4 \). For any \( \mathbf{v} \in Q^{i-1}(E, \mathbb{Z}) \):
\[
\begin{align*}
 f_{2,\mathbf{v}}(\lambda_{3,y_i}) &= f_{3,\mathbf{v}}(\lambda_{3,y_i}) \\
 f_{2,\mathbf{v}}(\gamma \lambda_{3,y_i}) &= \gamma f_{3,\mathbf{v}}(\gamma \lambda_{3,y_i}) 
\end{align*}
\]
Equation (65) implies that \( f_{2,\mathbf{v}}(x) - f_{3,\mathbf{v}}(x) \) has \( \lambda_{3,y_i} \) as root. Let:
\[
g_{4,\mathbf{v}}(x) = f_{3,\mathbf{v}}(x) - f_{2,\mathbf{v}}(x)
\]

By above equation and equation (56):
\[
f_{2,\mathbf{v}}(x) = f_{1,\mathbf{v}}(x) + g_{1,\mathbf{v}}(x) + g_{4,\mathbf{v}}(x)
\]
(67)

where \( g_{4,\mathbf{v}}(x) \) has root at \( \lambda_{3,y_i} \). Let \( G_4 = (g_{4,\ell,\mathbf{v}} | \ell \in [\mu_i - 1], \mathbf{v} \in Q^{i-1}(E, \mathbb{Z}) \). Hence \( F_2 = F_1 + G_1 + G_4 \) and \( G_4 \) is in left null space of \( M_{i-1,\mu_i-1} \).
5) From equations (64) and (67), we have that:
\[
f_1,\mathbf{v}(x) + g_1,\mathbf{v}(x) + g_4,\mathbf{v}(x) = f_1,\mathbf{v}(x) + g_2,\mathbf{v}(x) + g_3,\mathbf{v}(x)
\]
\[
g_1,\mathbf{v}(x) + g_4,\mathbf{v}(x) = g_2,\mathbf{v}(x) + g_3,\mathbf{v}(x)
\]
\[
g_1,\mathbf{v}(x) - g_3,\mathbf{v}(x) = g_2,\mathbf{v}(x) - g_4,\mathbf{v}(x)
\]
\[(68)\]

We know that \(g_1,\mathbf{v}\) and \(g_3,\mathbf{v}\) have \(\lambda_2,\gamma\) as root and \(g_2,\mathbf{v}\) and \(g_4,\mathbf{v}\) have \(\lambda_3,\gamma\) as root. By equation (68), it follows that \(g_1,\mathbf{v} - g_3,\mathbf{v}\) has root at both \(\lambda_2,\gamma\), \(\lambda_3,\gamma\), this constraint when combined with the condition that \((G_1 - G_3)M_{i-1,\mu_i-1} = 0\) as \(G_1, G_3\) are in left null space of \(M_{i-1,\mu_i-1}\). This implies that for any \(\mathbf{v} \in Q^{i-1}(E, \mathbf{z})\), \((x, y) \in E_{2,\mathbf{u}}\) such that \(y \leq y_i-1\):
\[(g_1,\mathbf{v} - g_3,\mathbf{v})(\theta_{x,y,u}) - (g_1,\mathbf{v}(x \rightarrow u_y) + g_3,\mathbf{v}(x \rightarrow u_y))((\theta_{x,y,u})) = 0.\]
\[(69)\]

As \(g_1,\mathbf{v} - g_3,\mathbf{v}\) has roots at \(\lambda_2,\gamma\), \(\lambda_3,\gamma\) for every \(\mathbf{v} \in Q^{i-1}(E, \mathbf{z})\), we can write \((g_1,\mathbf{v} - g_3,\mathbf{v})(x) = (x - \lambda_2,\gamma)(x - \lambda_3,\gamma)g'_{13,\mathbf{v}}(x)\) where \(g'_{13,\mathbf{v}}(x) = \sum_{\ell=1}^{\mu_i-3} g_{13,\mathbf{v}} \ell^\ell-1\) is a polynomial of degree \(\mu_i - 4\). Substituting this in equation (69) we get that for any \(\mathbf{v} \in Q^{i-1}(E, \mathbf{z})\), \((x, y) \in E_{2,\mathbf{u}}\) such that \(y \leq y_i-1\):
\[g'_{13,\mathbf{v}}(\theta_{x,y,u}) + g'_{13,\mathbf{v}(x \rightarrow u_y)}((\theta_{x,y,u})) = 0.\]
\[(70)\]

By setting \(G'_{13} = (g'_{13,\mathbf{v}} \mid \ell \in [\mu_i - 3], \mathbf{v} \in Q^{i-1}(E, \mathbf{z}))\) equation (70) implies that \((G'_{13})M_{i-1,\mu_i-3} = 0\). But since \(\mu_i - 3 = \mu_i-1\) and \(M_{i-1,\mu_i-1}\) is invertible it follows that \(G'_{13} = 0\). By the definition of \(G'_{13}\) it follows that \(G_1 - G_3 = 0\), \(G_2 - G_4 = 0\). Hence
\[f_2,\mathbf{v}(x) = f_1,\mathbf{v}(x) + g_1,\mathbf{v}(x) + g_2,\mathbf{v}(x)\]
\[(71)\]

6) By equation (63), equation (71), equation (60):
\[f_2,\mathbf{v}(\gamma\lambda_2,\gamma) = \gamma f_4,\mathbf{v}(\gamma\lambda_2,\gamma)\]
\[(72)\]

From (72) and equation (57) we have:
\[\gamma^{-1}(1 - \gamma)f_1,\mathbf{v}(\gamma\lambda_2,\gamma) = -(1 - \gamma)(f_1,\mathbf{v} + g_2,\mathbf{v})(\gamma\lambda_2,\gamma)\]
\[g_2,\mathbf{v}(\gamma\lambda_2,\gamma) = \gamma^{-1}(1 - \gamma)f_1,\mathbf{v}(\gamma\lambda_2,\gamma)\]
\[(73)\]

The constraint that \(G_2M_{i-1,\mu_i-1} = 0\) and the constraints given in equation (73) and equation (61) form a set of linear constraints as shown below:
\[G_2 \begin{bmatrix}
\frac{M_{i-1,\mu_i-1}}{(\mu_i-1)p_i \times p_i-1)}V_2 \\
\frac{V_3}{(\mu_i-1)p_i \times p_i-1)}
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{F_{1,2}}{1 \times p_i-1} \\
\frac{F_{1,3}}{1 \times p_i-1}
\end{bmatrix}\]
\[(74)\]

where \(V_j = \begin{bmatrix}
v_j \\
\vdots \\
v_j
\end{bmatrix}\), \(V_j = \begin{bmatrix}
1 \\
\gamma \lambda_{j,y_i} \\
\vdots \\
(\gamma \lambda_{j,y_i})^{\mu_i-2}
\end{bmatrix}\), and \(F_{1,j} = (\gamma^{-1}(1 - \gamma)f_1,\mathbf{v}(\gamma\lambda_{j,y_i}) \mid \mathbf{v} \in Q^{i-1}(E, \mathbf{z}))^T\) for \(j \in \{2, 3\}\).

We will now show that the appended matrix is invertible implying that there is a unique solution to \(G_2\). We prove the invertibility by showing that only in its null space is the zero vector. Let \(G'_2\) be in the null space of
the extended matrix. Then:

$$G'_2 \begin{bmatrix} M_{i-1,\mu_i-1} & V_2 & V_3 \end{bmatrix} = 0 \text{ where } G'_2 = (g'_{2,\ell,\nu} \mid \ell \in [\mu_i - 1], \nu \in Q^{i-1}(E, z)), $$

$$g'_{2,\nu}(\gamma \lambda_{2,\nu}) = g'_{2,\nu}(\gamma \lambda_{3,\nu}) = 0 \text{ where } g'_{2,\nu} = \sum_{\ell=1}^{\mu_i-1} g'_{2,\ell,\nu} x^{\ell-1}. $$

Let us now define $G''_2 = (g''_{2,\ell,\nu} \mid \ell \in [\mu_i - 3], \nu \in Q^{i-1}(E, z))$ where $g''_{2,\nu}(x) = (x - \gamma \lambda_{2,\nu})(x - \lambda_{3,\nu})g''_{2,\nu}(x)$, $g''_{2,\nu} = \sum_{\ell=1}^{\mu_i-3} g''_{2,\ell,\nu} x^{\ell-1}$. It can be shown that $G'_2 M_{i-1,\mu_i-1} = 0$ implies that $G''_2 M_{i-1,\mu_i-3} = 0$. Since $M_{i-1,\mu_i-3} = M_{i-1,\mu_i-1}$ is invertible it follows that $G''_2 = 0$ and therefore $G'_2 = 0$. Hence, the extended matrix is invertible and $G_2$ in equation (74) has unique solution. But is clear to see that $\gamma^{-1}(1 - \gamma)F_1$ is a solution. Hence $G_2 = \gamma^{-1}(1 - \gamma)F_1,$

$$g_{2,\nu}(x) = \gamma^{-1}(1 - \gamma) f_{1,\nu}(x). \quad (75)$$

By similar argument it can be proven that $G_1 = \gamma^{-1}(1 - \gamma)F_1$ and therefore:

$$g_{1,\nu}(x) = \gamma^{-1}(1 - \gamma) f_{1,\nu}(x) \quad (76)$$

7) $F_1 M_{i-1,\mu_i-1} = 0$ and by equation (75), (76), $f_{1,\nu}$ has roots at $\lambda_{2,\nu}, \lambda_{3,\nu}$. From an argument similar to the one shown in step 5 to prove $G_1 - G_3 = 0$ can be followed to show that $F_1 = 0$ and hence $G_1 = G_2 = 0$ and hence $F_3 = F_4 = F_2 = 0$. Therefore, the only vector in left null space of $M'$ is zero vector implying $M'$ is invertible and hence $M_{i,\mu_i}$ to be invertible.