Aspects of locally covariant quantum field theory

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Abstract

This thesis considers various aspects of locally covariant quantum field theory (see Brunetti et al., Commun. Math. Phys. 237 (2003), 31–68), a mathematical framework to describe axiomatic quantum field theories in curved spacetimes. Chapter 1 argues that the use of morphisms in this framework can be seen as a model for modal logic. To our knowledge this is the first interpretative description of this aspect of the framework. Chapter 2 gives an exposition of locally covariant quantum field theory which differs from the original in minor details, notably in the new notion of nowhere-classicality and the sharpened time-slice axiom, which puts a restriction on the state space as well as the algebras. Chapter 3 deals with the well-studied example of the free real scalar field and includes an elegant proof of the new general result that the commutation relations together with the Hadamard condition on the two-point distribution of a state completely fix the singularity structure of all $n$-point distributions. Chapter 4 describes the free Dirac field as a locally covariant quantum field, using a new representation independent approach, demonstrating that the physics is determined entirely by the relations between the adjoint map, charge conjugation and Dirac operator. It also proves the new result that the relative Cauchy evolution is related to the stress-energy-momentum tensor in the same way as for the free scalar field. Chapter 5 studies the Reeh-Schlieder property, both in the general setting and in specific examples. We obtain various interesting results concerning this property in curved spacetimes, most notably by using the idea of spacetime deformation, but some open questions and opportunities for further research remain. We will freely make use of smooth and analytic wave front sets throughout. These concepts are explained in appendix A, using a new and elegant way to generalise results for scalar distributions to Banach space-valued distributions, leading to some new but expected results.
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Foreword

... it is not his possession of knowledge, of irrefutable truth, that makes the man of science, but his persistent and recklessly critical quest for truth.

Karl Popper, [65] p.281

This thesis is based on research that was done at the University of York between October 2005 and June 2008. During those three years I learned a lot about science, about the world around us and also about myself. I feel that I have grown a lot as a mathematician, or perhaps as a mathematical physicist, although I don’t feel that my knowledge of physics has increased much. To a lesser extent I feel that I have grown as a philosopher of science, especially during the preparation of chapter 1 below, which is essentially the condensation of ideas that have been in my head since early 2004.

The most important thing that I learned about myself is exactly how ruthless I have to be to myself from time to time in order to get things done and to achieve the goals that I have set myself. My working attitude is perhaps best described in the words of my fellow PhD-student Paul Melvin, who told me time and again that I had been working like a machine. Maybe an insult to many, but to me these words were a compliment and they motivated me to go on and not to be tempted too much by York’s beautiful scenery, walks along the river Ouse and the taste of lukewarm, non-sparkling, English beer.
And so I went on in the quest for truth, as Popper describes it in the quote above. On some occasions I managed to prove a useful mathematical result that has consequences for the physical theories under investigation, as this thesis will indicate. On other occasions Mathematics denied me the proof that I was looking for and left me in the dark as to whether my gut feeling was right or wrong. Looking back on the results that were obtained I feel some gratification, of course, and pride for the knowledge I now possess due to all the hard work I have done. The dominant feeling, however, is curiosity. Curiosity aroused by the intriguing and tantalising new questions that emerged during the course of this research and that remain unanswered. For me too the quest for truth still goes on, and I am grateful for every opportunity I get to pursue it.

This thesis has been divided into six chapters and an appendix. Chapter 1 is of a more philosophical nature and the later chapters can be read independently of the first chapter. Conversely, chapter 1 only requires a superficial understanding of the framework of locally covariant quantum field theory, the main object of study in this thesis. The precise mathematical formulation of this framework is given in chapter 2. Chapters 3 and 4 describe in detail two examples of locally covariant quantum field theories, namely the real free scalar field and the free Dirac field. These chapters freely make use of the notion of wave front sets, which is explained in appendix A. It should be noted that the appendix provides an elegant and new approach to generalise results for scalar distributions to Banach space-valued ones and proves results that are more general than those existing in the literature. Chapter 5 studies the Reeh-Schlieder property in locally covariant quantum field theory,

1Probably more accurate than the word “truth”, at least for the physical aspect of mathematical physics, would be the word “verisimilitude” or the phrase “statements in which we have confidence”, see chapter I section 1.1. Of course these alternatives are far less aesthetic.
both in the general axiomatic setting and in the special examples of chapter 3. The final chapter 6 summarises the conclusions that can be drawn from the earlier chapters and discusses some opportunities for further research.

A remark about notations and conventions in this thesis is in order, although most notations are either standard or defined in the text when they are first introduced. The signatures of our spacetimes will be $(+ - - - )$, which agrees with most of the references, except e.g. [42, 88, 90, 6]. Lower case Greek letters are used to denote the components of vectors and covectors in a coordinate basis. Lower case Latin indices are used to indicate abstract indices of tensors (see [88] for a review of the abstract index notation), or to indicate the components of vectors and covectors in a vierbein in chapter 4. Capital Latin indices are used to indicate the components of spinors and cospinors in a spin frame, but for convenience these indices will often be dropped in favour of a matrix notation, as explained in chapter 4. Einstein’s summation convention is used throughout. Retarded fundamental solutions have their support to the future of the source function and are indicated by a superscript $^+$. Similarly, advanced fundamental solutions have their support in the past and are indicated by a $^-$. (A few of the references swap the names “retarded” and “advanced”, e.g. [85].) For quantisation we use the advanced-minus-retarded fundamental solution, as in [35, 66, 90].

Fourier transforms on $\mathbb{R}^n$ are defined by

$$\hat{f}(k) = \int e^{-ik \cdot x} f(x) \, dx,$$

where $\cdot$ denotes the pairing of $\mathbb{R}^n$ and its dual. This is unlike e.g. [26, 34, 35, 67] who omit the minus sign in the exponent. The Fourier inversion formula on $\mathbb{R}^n$ then reads:

$$f(x) = (2\pi)^{-n} \int e^{ik \cdot x} \hat{f}(k) \, dk.$$

For the real free scalar field in Minkowski spacetime the retarded (+) and advanced (−) fundamental solutions of the Klein-Gordon equation are given
by
\[
\hat{E}^\pm(k, l) = \lim_{\epsilon \to 0^+} \frac{-(2\pi)^4 \delta(k + l)}{(l_0 \pm i\epsilon)^2 - \|l\|^2 - m^2},
\]
where we have written \( l = (l_0, 1) \) and \( \delta \) is the four-dimensional Dirac distribution. The advanced-minus-retarded fundamental solution is given by (using e.g. [47] p.73)
\[
\hat{E}(k, l) = -(2\pi)^5 i\delta(k + l)\delta(l_2 - m^2) (\theta(l_0) - \theta(-l_0))
\]
where \( \theta \) is the Heaviside distribution. The two-point distribution of the Minkowski vacuum \( \omega_0 \) is given by
\[
\widehat{(\omega_0)_2}(k, l) = \widehat{(\omega_0)_{2+}}(k, l) + \frac{i}{2} \hat{E}(k, l) = (2\pi)^5 \delta(k + l)\theta(l_0)\delta(l_2 - m^2),
\]
where the symmetric part \( (\omega_0)_{2+} \) has been defined implicitly. Notice that
\[
(\omega_0)_2(\hat{f}, f) = (2\pi)^{-3} \int \theta(l_0)\delta(l_2 - m^2)|\hat{f}(-l)|^2 \, dl \geq 0
\]
and we have equality if \( \hat{f}(l) \) is supported in the half space \( l_0 \geq 0 \). In other words, positive frequency functions annihilate the vacuum, because then \( \|\Phi(f)\Omega_0\|^2 = (\omega_0)_2(\hat{f}, f) = 0 \). By analogy with Parseval’s formula,
\[
\int \Phi(x)f(x) \, dx = (2\pi)^{-4} \int \hat{\Phi}(k)\hat{f}(-k) \, dk
\]
(see [47] theorem 7.1.6), we then say that the quantum field \( \Phi(x) \) has positive energy in the vacuum state. We have \( WF(\omega_2) \subset \mathbb{R}^8 \times (N^- \times N^+) \), where \( N^+ \) denotes the future pointing null vectors and \( N^- \) the past pointing null vectors (both including 0) and \( WF \) denotes the wave front set (see appendix \[A\]).
Acknowledgements

I would like to thank the University of York for providing me with the opportunity to carry out the research on which this thesis is based and especially my supervisor, Dr. Chris Fewster, who has helped me with good advice on many occasions and has provided useful comments on drafts of this thesis at several stages of its development. I would also like to thank the University of Trento for its hospitality during my visit there in October 2007 and I am particularly grateful to Dr. Romeo Brunetti, who corrected a misconception of mine on the relative Cauchy evolution for the Dirac field at an early stage. Furthermore I would like to thank Dr. Alexander Strohmaier for a very helpful discussion on the Reeh-Schlieder property in curved spacetimes. Finally, while trying to help me understand its meaning, Esther Sanders (MA) has spotted a typographical error in the Greek quote on page 168, for which I am grateful.

Author’s declaration

Parts of chapters 2 and 5 are taken from a paper that has been made available online [74] and was submitted to Communications in Mathematical Physics for publication. The idea to apply a spacetime deformation argument to the Reeh-Schlieder property is due to Dr. Chris Fewster.
Introduction

Locally covariant quantum field theory was introduced in [16] as a mathematical framework to formulate axiomatic quantum field theories in curved spacetime and to give a precise meaning to Einstein’s general covariance principle for such theories. As such it provides an appropriate setting for the formulation of a semi-classical approximation to quantum gravitation. Moreover, as a matter of principle, quantum field theories are tested in the presence of gravity, so their formulation should not depend too much on the specific properties of Minkowski spacetime. In particular this means that the use of global symmetries and Fourier transformation should not be of crucial importance.

One major advance of recent years has been the realisation that the spectrum condition of Wightman field theories in Minkowski spacetime can be replaced by a microlocal spectrum condition in curved spacetimes [15, 66]. This has allowed the formulation of interacting quantum field theories in curved spacetime using perturbation theory, analogous to the Minkowski spacetime case [14, 45]. Another important idea has been the use of spacetime deformation arguments, which use the time-slice axiom to show that results on Minkowski spacetime can be carried over to (diffeomorphic) curved spacetimes [38]. One successful example of this is the spin-statistics theorem proved by [85].

Locally covariant quantum field theory can also serve as a reference struc-
ture for the philosophical discussion of quantum field theories in curved spacetime and possibly also quantum gravity. In this sense it would be analogous to algebraic quantum field theory, which serves the same purpose for quantum field theory in Minkowski spacetime \cite{10,22,11,21}, and indeed algebraic quantum field theory can be recovered from locally covariant quantum field theory \cite{16}. Crucial aspects for the theory in this context are its clear structure and the fact that the assumptions that are used are believed to be weak and general enough to encompass a sufficiently wide range of useful theories.

In this thesis we will use locally covariant quantum field theory as a reference structure as well as for the formulation of specific quantum field theories. In chapter 1 we will study the use of embeddings from a philosophical point of view. We believe this aspect deserves attention, because it is essentially new. (It differs from the setting of algebraic quantum field theory in Minkowski spacetime, because no fixed universe is present.) In chapters 2, 3 and 4 we will give a precise formulation of locally covariant quantum field theory and describe two examples of such theories, the real free scalar field and the free Dirac field, including some new results concerning the microlocal spectrum condition and Hadamard states. Chapter 5 then studies the Reeh-Schlieder property for quantum field theories in curved spacetimes. A state with this property has many non-local correlations, which makes this property of importance both for the physical and the philosophical aspects of quantum field theory in curved spacetime. On top of that it has useful and interesting mathematical implications. It is known that many physically interesting states in many spacetimes have this property \cite{69,80,79}, but whether this covers (almost) all physically interesting states in (almost) all interesting spacetimes is not at all clear. We will use the locally covariant framework to make several partial contributions towards answering this question.

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Chapter 1

Preliminary philosophical reflections

‘But how does it happen,’ I said with admiration, ‘that you were able to solve the mystery of the library looking at it from the outside, and were unable to solve it when you were inside?’

‘Thus God knows the world, because He conceived it in His mind, as if from the outside, before it was created, and we do not know its rule, because we live inside it, having found it already made.’

‘So one can know things by looking at them from the outside!’

Umberto Eco, The Name of the Rose, Third Day: Vespers

Looking at the history of science, especially the last few centuries, it is hard to imagine what mathematics or physics would have been like without each other. Nevertheless, the two disciplines are separate. Indeed, a purely mathematical argument is logically true, whether it accurately describes the world around us or not. If a physical theory allows a precise mathematical formulation, then the physics is in the formulation of the model, the assump-
tions that are made to arrive at it and the interpretation of the variables. Perhaps a mathematical physicist is the most prudent of physicists, because he checks the mathematical structure of physical theories for any shortcomings and tries to correct these. As a result, he can lay bare any assumptions of the theory that were previously hidden and these may provide new insight into the physical content of the theory.

However, there is more to physics than just the mathematical structure of its theories. There is also a philosophical side, which deals with questions like: what is physics? and how can we hope to learn something about the world around us in the first place? and what do our physical theories tell us about what the world is like? The philosophy of physics and mathematical physics are not independent of one another. Mathematical physics can provide a clear boundary between the logical (analytic) and the physical (synthetic) aspects of physical theories, thereby making the job of philosophers of physics easier. On the other hand, philosophical ideas can suggest alterations of physical theories, which then call for a sound mathematical formulation. (As an example one may think of Mach's principle, which influenced Einstein's thinking while he was formulating his general theory of relativity.)

In this light, some philosophical reflections are appropriate, even though this thesis is a work of mathematical physics. In fact, we feel there is an even more pressing reason for such reflections, because locally covariant quantum field theory, as described in chapter 2 is a relatively new and very general framework, whose mathematical structure introduces some interesting new ideas. In section 1.2 after an outline of some philosophical background material, we will argue that the novel use of morphisms lends itself excellently to make locally covariant quantum field theory a model for modal logic. To our knowledge the current chapter provides the first description of an interpretation of this important aspect of the theory. The author, trained as a mathematical physicist and not as a philosopher of physics, apologises in
advance for the relatively low standard of philosophical discussion.

1.1 Philosophical background information

Following Kant (see e.g. [70]) we may divide the reality of the world around us into two parts, namely those aspects of reality to which we, as observers, have direct epistemic access and those parts of reality for which this is not the case. By the phrase “direct epistemic access” we intend to describe all direct observations, experienced by whatever sense of a sentient being. Whatever we know, or believe to know, about the real world must be based on our observations. In these observations one may discern patterns and regularities, which lend themselves to abstraction and theoretical description. In particular, it often happens that different senses record certain patterns which tend to occur together. When we theorise about these observations, we tend to construct a single theoretical “object”, which is assumed to cause all the different perceptions. (According to [70] it is these theoretical objects which Kant calls “things in themselves” or “Dinge an sich”.)

Of course there is no way of knowing anything for certain about reality beyond the realm of our own observations, so whether our theoretical objects actually exist will always be unknown. In fact, one may take the philosophical position that reality consists of nothing else than ones own observations and theories (idealism). On the other hand there is the realist position, which postulates that there do exist things outside the realm of mere observations. In the realist’s words, the observations are appearances, and there must exist something that does the appearing. Note, however, that this does not mean that the theoretical objects of a specific theory must exist (see [70]).

Science is in the business of providing mathematical descriptions of observations. As such it makes no difference for science whether one takes an idealist or a realist position, although the meaning and importance that an
individual ascribes to science may depend on his philosophical position. Following Popper [65] we note that science adopts a particular way of theorising about observations. It deals with events that are reproducible and describes them by theories that are as universal in their range of application and precise in their description of observations as possible. Whenever a theory is falsified, i.e. whenever it has become clear that is not in agreement with observations, the theory is discarded and science will have to search for a better one. Another characteristic of science is, according to Popper, the persistence in attempting to falsify theories and lay bare the need for better ones.

The main difficulty in the characterisation of science seems to lie in the characterisation of the way that new theories are developed. Popper, following Hume, rejects the use of inductive logic as a characterisation of the scientific method, because it is not clear that inductive conclusions are justified ([65] section 1). Although induction can be used to formulate new theories (just like creativity or divine inspiration for that matter), Popper holds that it is the falsifiability and testing of such theories that is characteristic for science. The rejection of induction is a very cautious position, which seems to fit in well with the prudent nature of the mathematical physicist, but it does beg the question why unfalsified theories that have withstood serious testing are useful. Indeed, the obvious rational reason for their usefulness, namely that they will be successful in predicting the future, is based on an inductive argument (see [61]) and is therefore in contradiction with Popper’s position. Another less rational explanation for Popper’s choice of theory comes from the hypothesis that biological evolution has provided us with the inclination to choose such theories and to have confidence in them.

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1Popper remarks: “It follows that any controversy over the question whether events which are in principle unrepeatable and unique do occur cannot be decided by science: it would be a metaphysical controversy,” [65] section 8, p.24.
This is certainly consistent with the fact that these choices have served us well in the past. See [39, 77] for a further discussion along these lines.

As a final point we will comment on the social aspects of science, because contact and discussion between different scientists is often considered to be a crucial characteristic too. However, taking the words on direct epistemic access at the beginning of this section seriously, an observer should theorise on the basis of his own observations only and contact with other observers can only be included by treating it as a form of measurement or observation. This seems to be consistent both with relativity theory and quantum physics. Whether the opinions of other observers are accepted is then a question for the individual to decide and the fact that the opinions of others carry so much weight may perhaps be explained by another reference to evolution theory. For further comments on the social aspects of science and the characterisation of science as a social phenomenon we refer to [61].

1.2 Modal logic and locally covariant quantum field theory

Modal logic is the study of the truth values of statements and the validity of arguments that involve situations that are not actually the case. It deals with possibilities, with "it could have been that..." and "if only..." sentences. The analysis of such sentences and arguments is notoriously more difficult than that of proposition or predicate logic. Nevertheless, science uses such sentences in abundance when formulating hypothetical situations, e.g. in classical mechanics: "if a cylinder C would roll down a slope with angle $\alpha$...", a situation which need not actually be the case in order for us to analyse it. Indeed, if we want to make predictions it is necessary to think of situations that are not yet the case, but that may come about in the future.
Modal logic decides whether an argument is valid using the idea of models (see [36] for an introduction). In a specific model the validity of an argument can be evaluated explicitly. If a statement or argument is valid in every allowed model of a certain theory of modal logic, then the argument is said to be valid. The most common type of model consists of "possible worlds", a complete alternative for how things might have been. Objects may or may not exist at a certain possible world and propositions and predicate statements may or may not hold. The idea of possible worlds is well-known in quantum physics because of the "many worlds interpretation" of quantum physics, in which all possible measurement outcomes are considered to be real, but existing at different worlds.

It is sometimes said that locally covariant quantum field theory describes quantum fields on all possible spacetimes simultaneously. Here the word simultaneously clearly doesn’t mean "at the same instant of time", but it rather means the unified, systematic way in which the quantum field is described in all spacetimes. In fact, locally covariant quantum field theory deals with a category whose objects can be thought of as systems, indexed by the region of spacetime in which they live, and whose morphisms are embeddings, each of which can be thought of as a subsystem relation. (See [58] for more information on category theory.) Moreover, the framework assigns to each system a certain state space and it provides a map that restricts states to subsystems, this map being the dual to the embeddings of systems.

Taking things at face value it may be tempting to think of each spacetime as a possible world, in the sense of modal logic, and wonder whether there is an analogy to the many worlds interpretation of quantum mechanics. Ac-

\[\text{\[37\]}\] Van Fraassen’s modal interpretation of quantum physics also uses modal logic to describe possible measurement outcomes, although it only considers one of these as real, see [37].

\[\text{\[38\]}\] More precisely, one works with globally hyperbolic spacetimes, as will be explained in chapter [2].
tually, this idea fails at the first hurdle: a possible world in modal logic is supposed to be a complete set of circumstances, but when a spacetime can be embedded into a bigger one the description of the circumstances is clearly not complete. However, we can use a less well-known model theory for modal logic, which uses incomplete sets of circumstances called “possibilities” (see [30] pp.18-22). At a possibility, not all logical sentences need to be assigned a truth value, so they describe incomplete sets of circumstances. Moreover, a possibility can be refined by extending the set of circumstances, i.e. by extending the set of logical sentences which are assigned a truth value.

To see the correspondence with locally covariant quantum field theory we notice that we are using two types of modal operators. The first refers to possible systems, the systems that are the subject of the theory. The second refers to the set of possible states, for a given possible system. The first type of operator uses incomplete worlds and possibility semantics, whereas the second uses complete sets of circumstances because a state should provide a complete description of the circumstances in which a system finds itself. Putting everything together we could identify a possibility with a pair consisting of a system and the state it is in. A refinement necessarily corresponds to an embedding into a supersystem together with an extension of the state. Note that we will always identify a system with its image under a morphism, because this seems to correspond best to the idea of extension of circumstances and to the operational notion of subsystem.

In general refinements of a possibility are not unique: a possible system may have many supersystems and a state of a subsystem can be extended to a given supersystem in more than one way. Let us now turn to the interesting question whether a given state of a subsystem can be extended to a given supersystem at all. First suppose that a state cannot be extended to any supersystem. If a system is known to be in such a state, then it must clearly be the whole universe, for otherwise there would have to be some extension
to a supersystem. Now suppose that a state can only be extended to some
supersytems, but not to others. Such a state tells us not just something
about the system under consideration, but also about the nature of any
possible supersystems. This would be a strange situation, which would seem
to indicate that we have chosen the boundary between the system and the
rest of the universe poorly. The assumption that every state can be extended
to every supersystem is the principle of local physical equivalence introduced
in [33].

The possibility semantics seems to fit well with our generally prudent
approach and with experimental praxis: we would like to be able to make
predictions for a certain laboratory experiment, without prescribing a com-
plete set of circumstances for the entire world; making assumptions about
the system in question should be enough. For practical purposes, then, we
may stick with an instrumentalist interpretation of the framework, using a
Heisenberg cut between the system and the observer which may shift, ac-
cording to which system is under consideration. (Arguably this avoids the
measurement problem by denying that the theory deals with the universe as
a whole, excepting the special case of inextendible spacetimes.)

Let us emphasise the difference between possible worlds and possible sys-
tems by drawing some physically relevant conclusions. The difference be-
tween the two semantics is the idea of refinement, i.e. the embeddings of
locally covariant quantum field theory and the corresponding extension of
states. We will see in chapter 2 that these embeddings have a rich struc-
ture and they are a crucial part of the theory, because they express the idea
of local covariance. Now suppose that we may embed a subsystem $A$ into

\footnote{More precisely, [33] definition 4.1 requires that for every supersystem every state is
empirically equivalent with a state that can be extended. This prevents us from detect-
ing a state that cannot be extended, but it does allow the theory to make unphysical
idealisations.}
two distinct systems $B_1, B_2$, which cannot both be embedded into a single
supersystem, at least not when we identify the images of $A$ in both $B_i$ as we have chosen to do. (See [16] for an even more elaborate example in their discussion of states). In other words, both $B_1$ and $B_2$ are possible extensions of $A$, but it is not possible to have all the circumstances of both $B_1$ and $B_2$. This implies that not all embeddings can be actual at the same time. Furthermore, this shows that not all possible systems can be actual at the same time.

Similar problems occur when trying to obtain a theory of quantum gravity from locally covariant quantum field theory by allowing superpositions over different spacetimes, each with its own classical background gravitational field. If we would simply allow indiscriminate superpositions, we would disregard the subsystem relation altogether. Moreover, we would somehow jump from a theory of systems and possibilities to a quantum gravity theory of universes and possible worlds. This approach seems to be too naive and in fact it does not correspond to the many worlds interpretation of quantum mechanics. Indeed, in quantum mechanics one uses only a single system (the universe), so no superpositions of different systems appear. The modal aspect refers to measurement outcomes only, which can be formulated in terms of states. When trying to quantise gravity we believe it would be better to find a way that respects the notion of embeddings. This means we ought to allow only superpositions of different possibilities of the same system. This begs the question which spacetimes should be considered as the same system, but with a different background metric. This is obviously not the place to go into this difficult question.
Chapter 2

Locally covariant quantum field theory

As before, the Pequod steeply leaned over towards the Sperm Whale’s head, now, by the counterpoise of both heads, she regained her even keel; though sorely strained, you may well believe. So, when on one side you hoist in Locke’s head, you go over that way; but now, on the other side, hoist in Kant’s and you come back again; but in very poor plight. Thus, some minds for ever keep trimming boat. Oh, ye foolish! throw all these thunderheads overboard, and then you will float light and right.

Herman Melville, Moby Dick, Ch. 73

After the preliminary discussion in chapter [1] of the meaning of the categorical structure underlying locally covariant quantum field theory, we now come to the actual and detailed definition of this framework. Most of the following chapters is formulated in this framework, so this chapter serves to establish notations as well as to explain all the basic concepts. We will follow the original work [16] closely, but also refer to [33] for a slightly different
formulation. Furthermore our definition of the time slice axiom is slightly stronger than that of [16] and we introduce the new notion of nowhere-classicality. The basic facts from category theory that we will use can be found in [58] and we refer to [42, 88, 62] for background information on general relativity. Information on $C^\ast$-algebras, respectively $^\ast$-algebras, can be found in [49, 76], respectively. For the physical applications of these algebras we refer to [13, 2, 40].

2.1 Operational aspects

A quantum physical system will be described by a topological $^\ast$-algebra $\mathcal{A}$ with a unit $I$, whose self-adjoint elements are the observables of the system. For technical reasons it is often desirable to work with $C^\ast$-algebras, because they can be faithfully represented as algebras of bounded operators. However, both $C^\ast$-algebras and more general topological $^\ast$-algebras will appear in the following chapters, so for clarity we will develop both cases alongside each other. It will be advantageous to consider a whole class of possible systems rather than just one.

**Definition 2.1.1** The category $\mathcal{Alg}$ has as its objects topological $^\ast$-algebras $\mathcal{A}$ with unit $I$, and its morphisms are continuous, injective $^\ast$-homomorphisms $\alpha$ such that $\alpha(I) = I$. The product of morphisms is given by the composition of maps and the identity map $\text{id}_\mathcal{A}$ on a given object serves as an identity morphism. The category $\mathcal{CAlg}$ is the subcategory of $\mathcal{Alg}$ whose objects are unital $C^\ast$-algebras.

A morphism $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ in $\mathcal{Alg}$ expresses the fact that the system described by $\mathcal{A}_1$ is a sub-system of that described by $\mathcal{A}_2$, which is called a super-system.

$^1$We recall from [76] p.22 that a topological $^\ast$-algebra is a $^\ast$-algebra which is also a locally convex vector space such that the involution $^\ast$ is continuous and the product is separately continuous.
(see also the discussion in section \[1.2\]). The injectivity of the morphisms means that, as a matter of principle, any observable of a sub-system can always be measured, regardless of any practical restrictions that a super-system may impose.

A state of a system $\mathcal{A}$ is represented by a continuous linear functional $\omega$ on $\mathcal{A}$ which is positive, i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$, and normalised, $\omega(I) = 1$. The set of all states on $\mathcal{A}$ will be denoted by $\mathcal{A}_1^{+}$. Not all of these states are guaranteed to be of physical interest, so it will be convenient to have the following notion at our disposal:

**Definition 2.1.2** The category $\mathfrak{States}$ has as its objects all convex subsets $\mathcal{S} \subset \mathcal{A}_1^{+}$, for all objects $\mathcal{A}$ in $\mathfrak{TAlg}$, which are closed under operations from $\mathcal{A}$ (i.e. $\frac{\omega(A^*A)}{\omega(A^*A)} \in \mathcal{S}$ if $\omega \in \mathcal{S}$ and $A \in \mathcal{A}$ such that $\omega(A^*A) \neq 0$) and morphisms in $\mathfrak{States}$ are all affine maps $\sigma: \mathcal{S}_1 \to \mathcal{S}_2$, i.e. maps for which $\sigma(\lambda \omega_1 + (1 - \lambda)\omega_2) = \lambda \sigma(\omega_1) + (1 - \lambda)\sigma(\omega_2)$ for all $0 \leq \lambda \leq 1$ and $\omega_1, \omega_2 \in \mathcal{S}_1$. Again the product of morphisms is given by the composition of maps and the identity map $\text{id}_S$ on a given object serves as an identity morphism.

Each object $\mathcal{S}$ is a priori a suitable candidate for a state space of a system in $\mathfrak{TAlg}$. Using the category $\mathfrak{States}$ allows us to postpone a specific choice of state space until later.

If $\omega$ is a state on a (not necessarily topological) $\ast$-algebra with unit $\mathcal{A}$, then we can perform the GNS-construction. To explain this we need the following definitions (see \[7.6\]):

**Definition 2.1.3** A $\ast$-representation of $\mathcal{A}$ is called closed if and only if it represents $\mathcal{A}$ as an algebra of closable operators on a Hilbert space $\mathcal{H}$ which have as a common, dense and invariant domain $\mathcal{D}_\pi = \bigcap_{A \in \mathcal{A}} \text{dom}(\pi(A))$.

The graph topology of $\mathcal{D}_\pi$ is the locally convex topology determined by the family of semi-norms $\{ \phi \mapsto \|\pi(A)\phi\| \mid A \in \mathcal{A} \}$. 

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A closed $^*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is called cyclic if and only if there is a cyclic vector $\phi \in \mathcal{D}_\pi$, i.e. a vector such that $\pi(A)\phi \subset \mathcal{D}_\pi$ is dense in the graph topology.

If $\pi$ is a closed $^*$-representation with $\mathcal{D}_\pi = \mathcal{H}$, then $\pi$ represents $\mathcal{A}$ as an algebra of bounded operators (by the closed graph theorem, [19] theorem 1.8.6) and hence the graph topology coincides with the norm topology of $\mathcal{H}$.

In general, a cyclic vector for a cyclic representation $\pi$ is also weakly cyclic, i.e. $\pi(A)\phi \subset \mathcal{H}$ is dense in the norm topology of $\mathcal{H}$.

**Theorem 2.1.4 (GNS-representation)** Let $\omega$ be a state on $\mathcal{A}$. Then there exists a closed cyclic $^*$-representation $\pi_\omega$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ with a cyclic vector $\Omega_\omega$ in the dense domain $\mathcal{D}_\omega := \mathcal{D}_{\pi_\omega}$ such that $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$ for all $A \in \mathcal{A}$.

If $\pi$ is a closed cyclic $^*$-representation of $\mathcal{A}$ with a cyclic vector $\phi$ such that $\omega(A) = \langle \phi, \pi(A)\phi \rangle$ for all $A \in \mathcal{A}$, then there is a unique unitary equivalence $U$ between $\pi$ and $\pi_\omega$ such that $U(\phi) = \Omega_\omega$.

This follows from theorem 8.6.4 of [76]. The representation $\pi_\omega$ is called the **GNS-representation**. In the special case that $\mathcal{A}$ is a $C^*$-algebra one can show that $\mathcal{D}_\omega = \mathcal{H}_\omega$ and the triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is then called the **GNS-triple** (see e.g. the GNS-construction in [49] for the $C^*$-algebraic case). In general we will call $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \mathcal{D}_\omega)$ the **GNS-quadruple**.

If $\mathcal{B} \subset \mathcal{A}$ is a sub-$^*$-algebra and $\omega' := \omega|_\mathcal{B}$ then the GNS-quadruple (or GNS-triple) associated to $\omega'$ is related to that of $\omega$ by $\mathcal{H}_{\omega'} = \overline{\pi_\omega(\mathcal{B})\Omega_\omega}$, $\pi_{\omega'} := P\pi_\omega|_\mathcal{B}P^*$ where $P : \mathcal{H}_\omega \rightarrow \mathcal{H}_{\omega'}$ is the orthogonal projection and $\Omega_{\omega'} := \Omega_\omega$.

This follows from the uniqueness part of theorem 2.1.4 and we will often use this fact in the subsequent chapters.


2.2 Spacetimes

After these operational aspects we now turn to the physical ones. The systems we will consider are intended to model quantum fields living in a (region of a) spacetime which is endowed with a fixed Lorentzian metric (a background gravitational field). The relation between sub-systems will come about naturally by considering sub-regions of spacetime. More precisely we consider the following:

**Definition 2.2.1** By the term globally hyperbolic spacetime we will mean a connected, Hausdorff, paracompact, $C^\infty$ Lorentzian manifold $M = (\mathcal{M}, g)$ of dimension $d = 4$, which is oriented, time-oriented and admits a Cauchy surface (i.e. a continuous hypersurface which is intersected exactly once by every inextendible time-like curve, see e.g. [8]).

A subset $\mathcal{O} \subset \mathcal{M}$ of a globally hyperbolic spacetime $M$ is called causally convex iff for all $x, y \in \mathcal{O}$ all causal curves from $x$ to $y$ lie entirely in $\mathcal{O}$. A non-empty open set which is connected and causally convex is called a causally convex region or cc-region. A cc-region whose closure is compact is called a bounded cc-region.

The category $\text{Man}$ has as its objects globally hyperbolic spacetimes $M = (\mathcal{M}, g)$ and its morphisms $\Psi$ are given by all maps $\psi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ which are smooth isometric embeddings (i.e. $\psi : \mathcal{M}_1 \rightarrow \psi(\mathcal{M}_1)$ is a diffeomorphism and $\psi_*g_1 = g_2|_{\psi(\mathcal{M}_1)}$) such that the orientation and time-orientation are preserved and $\psi(\mathcal{M}_1)$ is causally convex. Again the product of morphisms is given by the composition of maps and the identity map $\text{id}_M$ on a given object serves as a unit.

A region $\mathcal{O}$ in a globally hyperbolic spacetime is causally convex if and only if $\mathcal{O}$ is a globally hyperbolic region in the sense of [12] section 6.6. It then follows that $\mathcal{O}$ is a globally hyperbolic spacetime in its own right. However,
the converse does not hold, i.e. if $\mathcal{O}$ is a globally hyperbolic spacetime in its own right it does not follow that it is causally convex (see e.g. the helical strip on p.177 of [53]).

The image of a morphism is by definition a cc-region. Notice that the converse also holds. If $\mathcal{O} \subset \mathcal{M}$ is a cc-region then $\mathcal{O} = (\mathcal{O},g|_{\mathcal{O}})$ defines a globally hyperbolic spacetime in its own right. In this case there is a canonical morphism $I_{\mathcal{M},\mathcal{O}}: \mathcal{O} \rightarrow \mathcal{M}$ given by the canonical embedding $\iota: \mathcal{O} \rightarrow \mathcal{M}$. We will often drop $I_{\mathcal{M},\mathcal{O}}$ and $\iota$ from the notation and simply write $\mathcal{O} \subset \mathcal{M}$.

The importance of causally convex sets is that for any morphism $\Psi$ the causality structure of $\mathcal{M}_1$ coincides with that of $\Psi(\mathcal{M}_1)$ in $\mathcal{M}_2$:

$$
\psi(J^\pm_{\mathcal{M}_1}(x)) = J^\pm_{\mathcal{M}_2}(\psi(x)) \cap \psi(\mathcal{M}_1), \quad x \in \mathcal{M}_1.
$$

(2.1)

If this were not the case then the behaviour of a physical system living in $\mathcal{M}_1$ could depend in an essential way on the super-system, which makes it practically impossible to study the smaller system as a sub-system in its own right. This possibility is therefore excluded from the mathematical framework.

Equation (2.1) allows us to drop the subscript in $J^\pm$ if we introduce the convention that $J^\pm$ is always taken in the largest spacetime under consideration. This simplifies the notation without causing any confusion, even when $O \subset \mathcal{M}_1 \subset \mathcal{M}_2$ with canonical embeddings, because then we just have $J^\pm(\mathcal{O}) := J^\pm_{\mathcal{M}_2}(\mathcal{O})$ and $J^\pm_{\mathcal{M}_1}(\mathcal{O}) = J^\pm(\mathcal{O}) \cap \mathcal{M}_1$. We adopt a similar convention for the domain of dependence and the causal complement,

$$
D(\mathcal{O}) := D_{\mathcal{M}_2}(\mathcal{O}),
$$

$$
\mathcal{O}^\perp := \mathcal{O}^\perp_{\mathcal{M}_2} := \mathcal{M}_2 \setminus J(\mathcal{O}),
$$

and we deduce from causal convexity that $D_{\mathcal{M}_1}(\mathcal{O}) = D(\mathcal{O}) \cap \mathcal{M}_1$ and $\mathcal{O}^\perp_{\mathcal{M}_1} = \mathcal{O}^\perp \cap \mathcal{M}_1$. The following lemma gives some ways of obtaining causally convex sets in a globally hyperbolic spacetime.
Lemma 2.2.2 Let \( M = (\mathcal{M}, g) \) be a globally hyperbolic spacetime and \( O \subset \mathcal{M} \) an open subset. Then:

1. the intersection of two causally convex sets is causally convex,
2. for any subset \( Q \subset \mathcal{M} \) the sets \( I^{\pm}(Q) \) are causally convex,
3. \( O^\perp \) is causally convex,
4. \( O \) is causally convex iff \( O = J^+(O) \cap J^-(O) \),
5. for any achronal set \( P \subset \mathcal{M} \) the sets \( \text{int}(D(P)) \) and \( \text{int}(D^\pm(P)) \) are causally convex,
6. if \( O \) is a cc-region, then \( D(O) \) is a cc-region,
7. if \( R \subset \mathcal{M} \) is an acausal continuous hypersurface then \( D(R) \), \( D(R) \cap I^+(R) \) and \( D(R) \cap I^-(R) \) are open and causally convex.

Proof. The first two items follow directly from the definitions and the fact that a piecewise smooth, causal curve which is time-like on some neighbourhood can be deformed to a smooth time-like curve (see e.g. [88] p.191 or [62]). The fourth follows from \( O \subset J^+(O) \cap J^-(O) = \bigcup_{p,q \in O} (J^+(p) \cap J^-(q)) \), which is contained in \( O \) if and only if \( O \) is causally convex. The fifth item follows from the first two and theorem 14.38 and lemma 14.6 in [62].

To prove the third item, assume that \( \gamma \) is a causal curve between points in \( O^\perp \) and \( p \in \overline{J(O)} \) lies on \( \gamma \). By perturbing one of the endpoints of \( \gamma \) in \( O^\perp \) we may ensure that the curve is time-like (see [88] [62] loc. cit.). Then we may perturb \( p \) on \( \gamma \) so that \( p \in \text{int}(J(O)) \) and \( \gamma \) is still causal. This gives a contradiction, because there then exists a causal curve from \( O \) through \( p \) to either \( x \) or \( y \).

For the sixth statement we note that \( O \) is globally hyperbolic (see [42] section 6.6), we let \( C \subset O \) be a smooth Cauchy surface for \( O \) (see [9]) and
note that $D(O)$ is non-empty, connected and $D(O) = D(C)$. The causal
convexity of $O$ implies that $C \subset \mathcal{M}$ is acausal, which reduces this case to
statement seven. The first part of statement seven is just lemma 14.43 and
theorem 14.38 in [62]. The rest of statement seven follows from statement
one and two together with the openness of $I^+(C)$.

As a matter of notation we define for any subset $S \subset T^*\mathcal{M}$ the set $-S$
by $-S := \{(x, \xi)| (x, -\xi) \in S\}$ and

\begin{align*}
\mathcal{N}^+ &:= \{(x, \xi) \in T^*\mathcal{M}| g^{\mu\nu}\xi_{\nu} \text{ is a future pointing light–like vector,} \\
onumber &\quad \text{or } \xi = 0\}, \\
\mathcal{N}^- &:= -\mathcal{N}^+ , \quad \mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^- , \\
\mathcal{V}^+ &:= \{(x, \xi) \in T^*\mathcal{M}| g^{\mu\nu}\xi_{\nu} \text{ is a future pointing causal vector,} \\
onumber &\quad \text{or } \xi = 0\}, \\
\mathcal{V}^- &:= -\mathcal{V}^+ , \quad \mathcal{V} := \mathcal{V}^+ \cup \mathcal{V}^- , \\
\mathcal{Z} &:= \{(x, 0) \in T^*\mathcal{M}\}.
\end{align*}

Strictly speaking we should index these sets with the spacetime or manifold
on which they are defined. However, we will avoid this cumbersome notation,
because it will always be clear from the context what spacetime or manifold
is meant. In particular, when $S \subset T^*\mathcal{M}$, the expressions $S \setminus \mathcal{Z}$ and $S \cup \mathcal{Z}$
are meant to imply that $\mathcal{Z}$ is the zero section of $T^*\mathcal{M}$.

### 2.3 Spacetimes with a spin structure

In order to describe the Dirac field we need more geometric structure than
for the scalar field. This section gives the relevant definitions to formulate
a locally covariant quantum field theory in this setting. More details on the
$Spin_{1,3}$ group can be found in section 4.1.2.
Given a globally hyperbolic spacetime $M$, the frame bundle $FM$, which consists of all oriented, time-oriented frames of the tangent bundle $TM$, is a principal $L_{+}^{\uparrow}$-bundle over $M$, where the proper orthochronous Lorentz group $L_{+}^{\uparrow}$ acts from the right. In other words, given $e = (x, e_0, \ldots, e_3) \in FM$, where $x \in M$ and $e_a \in T_xM$ such that $g_x(e_a, e_b) = \eta_{ab} = \text{diag}(1, -1, -1, -1)$ and $e_0$ future pointing, the action of $\Lambda$ is defined by $R_{\Lambda}e = e' = (x, e_0', \ldots, e_3')$ where $e'_a = e_b\Lambda^b_a$. The universal covering group of $L_{+}^{\uparrow}$ is a double covering, namely $\text{Spin}^{0}_{1,3}$, the identity connected component of the Spin group.

**Definition 2.3.1** A spin structure on $M$ is a pair $(SM, p)$, where $SM$ is a principal $\text{Spin}^{0}_{1,3}$-bundle over $M$, the spin frame bundle, which carries a right action $R_S$, $S \in \text{Spin}^{0}_{1,3}$, and $p: SM \to FM$ is a base-point preserving bundle homomorphism such that

$$p \circ R_S = R_{\Lambda(S)} \circ p,$$

where $S \mapsto \Lambda(S)$ is the canonical universal covering map of proposition 4.1.12.

A globally hyperbolic spin spacetime $\hat{M} = (M, g, SM, p)$ is a globally hyperbolic spacetime $M = (M, g)$ which is endowed with the spin structure $(SM, p)$.

The category $\mathcal{SMan}$ has as its objects globally hyperbolic spin spacetimes $\hat{M} = (M, g, SM, p)$ and its morphisms $\Psi: \hat{M}_1 \to \hat{M}_2$ are all pairs of maps $\Psi = (\psi, \chi)$ such that

1. $\psi: M_1 \to M_2$ is a morphism in $\text{Man}$ between $M_1 = (M_1, g_1)$ and $M_2 = (M_2, g_2)$,

2. $\chi: SM_1 \to SM_2$ is smooth and satisfies $\chi \circ (R_1)_S = (R_2)_S \circ \chi$ and $p_2 \circ \chi = \tilde{d}\psi \circ p_1$, where $\tilde{d}\psi: FM_1 \to FM_2$ denotes the canonical extension of $d\psi: TM_1 \to TM_2$. 
Again the product of morphisms is given by the composition of maps and the
identity map $\text{id}_M$ on a given object serves as a unit.

Every globally hyperbolic spacetime admits a spin structure, which need not
be unique [30]. Different spin structures on the same spacetime define distinct
spin spacetimes and are therefore to be regarded as distinct systems. We will
often drop the hat $\hat{}$ from our notation, when it is clear from the context that
we are dealing with a spin spacetime rather than a spacetime.

To keep the framework unified it will be useful to have at our disposal
a forgetful functor $F : \mathcal{SMan} \to \mathcal{Man}$, which maps the spin spacetime
$(\mathcal{M}, g, SM, p)$ to the spacetime $(\mathcal{M}, g)$. This functor is surjective, but not
necessarily injective. A functor $A_0 : \mathcal{Man} \to \mathcal{C}$ to some category $\mathcal{C}$ gives rise
to a functor $A : \mathcal{SMan} \to \mathcal{C}$ defined by $A := A_0 \circ F$. Whenever $A$ is of this
form we can recover $A_0$ using the surjectivity of $F$.

### 2.4 Locally covariant quantum field theory

We now come to the main set of definitions, which combine the notions
introduced above (cf. [16, 33]).

**Definition 2.4.1** A locally covariant quantum field theory is a covariant
functor $A : \mathcal{SMan} \to \mathcal{TAlg}$, written as $M \mapsto A_M, \Psi \mapsto \alpha_\Psi$.

A state space for a locally covariant quantum field theory $A$ is a con-
travariant functor $S : \mathcal{SMan} \to \mathcal{States}$, such that for all objects $M$ we
have $M \mapsto \mathcal{I}_M \subset (A_M)^{\ast +}$ and for all morphisms $\Psi : M_1 \to M_2$ we have
$\Psi \mapsto \alpha_\Psi^\ast \mid_{\mathcal{I}_M}$. The set $\mathcal{I}_M$ is called the state space for $M$.

When it is clear that $\Psi = (\iota, \kappa) = I_{M,O}$ is a canonical embedding $\iota : \mathcal{O} \to \mathcal{M}$, $\kappa : SM\mid_{\mathcal{O}} \to SM$, of a cc-region $\mathcal{O}$ in a globally hyperbolic spacetime $\mathcal{M}$,
i.e. when $O \subset M$ as spin spacetimes, we will often simply write $A_O \subset A_M$.
instead of using $\alpha_\Psi$. For a morphism $\Psi : M \to M'$ which restricts to a morphism $\Psi|_O : O \to O' \subset M$ we then have

$$\alpha_\Psi|_O = \alpha_\Psi|_A$$

(2.2)

rather than $\alpha_{I_{M',O'}} \circ \alpha_\Psi|_O = \alpha_\Psi \circ \alpha_{I_{M,O}}$, as one can see from a commutative diagram.

As a special case we may consider locally covariant quantum field theories $A : S\text{Man} \to \mathcal{CAlg}$, which use $C^*$-algebras only. This is a generalisation of algebraic quantum field theory (see [16, 40]). We will indicate it explicitly when we restrict attention to $C^*$-algebras only.

We now proceed to define and discuss several physically desirable properties that a locally covariant quantum field theory and its state space may have (cf. [16], but note that our time-slice axiom is stronger because it places a restriction on the state spaces as well as the algebras; see also [33]; the last property is original).

**Definition 2.4.2** A locally covariant quantum field theory $A$ is called causal iff for any two morphisms $\Psi_i : M_i \to M$, $i = 1, 2$, such that $\psi_1(M_1) \subset (\psi_2(M_2))^\perp$ in $M$ we have $[\alpha_{\Psi_1}(A_{M_1}), \alpha_{\Psi_2}(A_{M_2})] = \{0\}$ in $A_M$.

A locally covariant quantum field theory $A$ with state space $S$ satisfies the time-slice axiom iff for all morphisms $\Psi : M_1 \to M_2$ such that $\psi(M_1)$ contains a Cauchy surface for $M_2$ we have $\alpha_\Psi(A_{M_1}) = A_{M_2}$ and $\alpha_\Psi^*(\mathcal{I}_{M_2}) = \mathcal{I}_{M_1}$.

A locally covariant quantum field theory $A$ with state space $S$ respects local physical equivalence iff for every morphism $\Psi : M_1 \to M_2$ the state spaces $\mathcal{I}_{M_1}$ and $\alpha_\Psi^*(\mathcal{I}_{M_2})$ have the same weak$^*$ closures in $A_{M_1}$.

A locally covariant quantum field theory $A : S\text{Man} \to \mathcal{TAlg}$ is called additive iff $A_O = \vee_{i \in I} A_{O_i}$, where the $\{O_i\}_{i \in I}$ form a locally finite open covering of $O$ and the right-hand side denotes the smallest algebra generated by the algebras $A_{O_i}$. Similarly, $A : S\text{Man} \to \mathcal{CAlg}$ is called additive iff $A_O = \overline{\vee_{i \in I} A_{O_i}}$, where we take the completion on the right-hand side.
Given a locally covariant quantum field theory \( \mathbf{A} : \text{SM} \rightarrow \text{CAlg} \), a state space \( \mathbf{S} \) for \( \mathbf{A} \) is called locally quasi-equivalent iff for all \( M_2 \) every pair of states in \( \mathcal{I}_{M_2} \) is locally quasi-equivalent, i.e. iff for every morphism \( \Psi : M_1 \rightarrow M_2 \) such that \( \psi(M_1) \subset M_2 \) is bounded and for every pair of states \( \omega, \omega' \in \mathcal{I}_{M_2} \) the GNS-representations \( \pi_\omega, \pi_\omega' \) of \( \mathcal{A}_{M_2} \) restricted to \( \alpha_\Psi(\mathcal{A}_{M_1}) \) are quasi-equivalent (see the discussion and definition below). The local von Neumann algebras \( \mathcal{R}_M^\omega := \pi_\omega(\alpha_\Psi(\mathcal{A}_{M_1}))'' \) are then *-isomorphic for all \( \omega \in \mathcal{I}_{M_2} \).

A locally covariant quantum field theory \( \mathbf{A} : \text{SM} \rightarrow \text{CAlg} \) with a state space functor \( \mathbf{S} \) is called nowhere classical iff for every morphism \( \Psi : M_1 \rightarrow M_2 \) and for every state \( \omega \in \mathcal{I}_{M_2} \) the local von Neumann algebra \( \mathcal{R}_M^\omega \) is not commutative.

Note that the condition that \( \psi_1(M_1) \subset (\psi_2(M_2))^\perp \) in \( M \) is symmetric in \( i = 1, 2 \), because \( \psi_i(M_i) \) is open and hence:

\[
\psi_1(M_1) \subset (\psi_2(M_2))^\perp \iff \psi_1(M_1) \cap J(\psi_2(M_2)) = \emptyset \iff \psi_1(M_1) \cap J(\psi_2(M_2)) = \emptyset \iff J(\psi_1(M_1)) \cap \psi_2(M_2) = \emptyset.
\]

The causality condition formulates how the quantum physical system interplays with the classical gravitational background field, whereas the time-slice axiom expresses the existence of a causal dynamical law. Classical theories can be described by commutative algebras, which motivates the definition of nowhere-classicality (see also section 5.1 for comments on non-local correlations in nowhere-classical theories). The condition of a locally quasi-equivalent state space is more technical in nature and means that all states of a system can be described in the same Hilbert space representation, as long as we only consider operations in a small (i.e. bounded) cc-region of the spacetime. More precisely:

**Definition 2.4.3** The folium of a representation \( \pi \) of a \( \text{C}^* \)-algebra \( \mathcal{A}_M \) on
a Hilbert space $\mathcal{H}$ is the set of all states $\rho$ on $\mathcal{A}_M$ of the form $\omega_\rho(A) = \text{Tr}_{\mathcal{H}} \rho \pi(A)$ with some trace-class operator $\rho$.

Two representations are called quasi-equivalent iff their folia are equal.

The condition that $\psi(M_1)$ contains a Cauchy surface for $M_2$ is equivalent to $D(\psi(M_1)) = M_2$, because a Cauchy surface $S \subset M_1$ maps to a Cauchy surface $\psi(S)$ for $D(\psi(M_1))$. On the algebraic level this yields:

**Lemma 2.4.4** For a locally covariant quantum field theory $\mathcal{A}$ with a state space $\mathcal{S}$ satisfying the time-slice axiom, an object $M = (\mathcal{M}, g) \in \text{Man}$ and a cc-region $O \subset M$ we have $\mathcal{A}_O = \mathcal{A}_{D(O)}$ and $\mathcal{I}_O = \mathcal{I}_{D(O)}$. If $O$ contains a Cauchy surface of $M$ we have $\mathcal{A}_O = \mathcal{A}_M$ and $\mathcal{I}_O = \mathcal{I}_M$.

**Proof.** Note that both $(O, g|_O)$ and $(D(O), g|_{D(O)})$ are objects of $\text{Man}$ (by lemma 2.2.2) and that a Cauchy surface $S$ for $O$ is also a Cauchy surface for $D(O)$. (The causal convexity of $O$ in $M$ prevents multiple intersections of $S$ by inextendible causal curves in $D(O)$, cf. the comments below definition 2.2.1.) The first statement then reduces to the second. Leaving the canonical embedding implicit in the notation, the result follows immediately from the time-slice axiom. □

2.5 Quantum fields

The functorial dependence of an algebra $\mathcal{A}_M$ on a spacetime $M$ is not specific enough for many purposes. Instead, we would like to have certain elements in these algebras, (smeared) quantum fields, which depend in a functorial way on the spacetime. Our formulation of such quantum fields follows closely the treatment of [16, 33, 85]. For simplicity we will first describe the case of the scalar field. Here the sets of test-functions are simply $C^\infty_0(M)$ in the
test-function topology.

**Definition 2.5.1** The category \( \mathcal{Top} \) has as objects all topological spaces and as morphisms all continuous maps.

The functor \( D : \mathcal{Man} \to \mathcal{Top} \) maps each object \( M \) to the linear space \( C_0^\infty(M) \) in the test-function topology and each morphism \( \Psi = (\psi) \) to the push-forward \( \psi_* \), extending functions by 0 outside the image of \( \psi \).

A locally covariant scalar quantum field \( \Phi \) is a natural transformation between the functor \( D \) and a locally covariant quantum field theory \( A \), i.e. for each \( M \) in \( \mathcal{Man} \) we have a continuous map \( \Phi_M : C_0^\infty(M) \to A_M \) such that \( \alpha_\Psi \circ \Phi_{M_1} = \Phi_{M_2} \circ \psi_* \) for every morphism \( \Psi : M_1 \to M_2 \) in \( \mathcal{Man} \).

For Dirac fields we will need to use test-sections of a certain vector bundle instead, namely the Dirac double spinor bundle \( DM \oplus D^*M \), which will be introduced in chapter 4. All we need to know for now is that there is a functorial dependence of these vector bundles on the spin spacetime \( M \):

**Definition 2.5.2** The category \( \mathcal{VB} \) has as its objects the (finite dimensional) vector bundles \( X \) on every globally hyperbolic spin spacetime \( M \) and as its morphisms the vector bundle homomorphisms \( \lambda : X_1 \to X_2 \) such that for some morphism \( \Psi = (\psi, \chi) \) in \( \mathcal{SMan} \) we have \( \pi_2 \circ \lambda = \psi \circ \pi_1 \), where \( \pi_1, \pi_2 \) are the projections of \( X_1, X_2 \) on \( M \). As usual the products of morphisms are given by composition of maps and the identity maps serve as units.

Given a functor \( X : \mathcal{SMan} \to \mathcal{VB} \), written as \( M \mapsto X_M \) and \( \Psi \mapsto \lambda \), the functor \( D^X : \mathcal{SMan} \to \mathcal{Top} \) maps each object \( M \) to the linear space \( C_0^\infty(X_M) \) of compactly supported smooth sections of \( X_M \) in the test-section topology and

\[^2\text{As a matter of convention we will always identify a distribution density on a spacetime } M \text{ with a distribution, using the metric volume element } d\nu_g \text{ on } M \text{ (see [17] section 6.3). To remind the reader of this fact we will write } C_0^\infty(M) \text{ instead of } C_0^\infty(M).\]

\[^3\text{A natural transformation can only exist between two functors with the same target category, so strictly speaking } \Phi \text{ should be defined as a natural transformation between } D \text{ and } F \circ A, \text{ where } F : \mathcal{TAlg} \to \mathcal{Top} \text{ is the forgetful functor.}\]
each morphism $\Psi$ to the push-forward $\lambda$, extending sections by 0 outside the image of $\lambda$.

In the first part of the definition above we specifically use spin spacetimes rather than spacetimes, because the vector bundles we have in mind, the Dirac double spinor bundles, are constructed from the spin structure. Of course a similar definition can equally well be made on the category of spacetimes $\mathcal{M}an$. The definition of a locally covariant quantum field is now straightforward:

**Definition 2.5.3** A locally covariant quantum field $\Phi$ with test-section functor $X$ is a natural transformation between the functor $D^X$ and a locally covariant quantum field theory $A$, i.e. for each $M$ in $\mathcal{S}\mathcal{M}an$ we have a continuous map $\Phi_M : C^\infty_0(X_M) \to A_M$ such that $\alpha_\Psi \circ \Phi_{M_1} = \Phi_{M_2} \circ \lambda_\Psi$ for every morphism $\Psi : M_1 \to M_2$ in $\mathcal{M}an$, where $\Psi \mapsto \lambda$ under $X$.

Notice that we may think of $\Phi_M$ as a generalised distributional density, which is a section of $X_M^*$, the vector bundle dual to $X_M$, and which takes values in $A_M$. ($\Phi_M$ need not be a distribution in the usual sense of the word, because we do not require it to be linear.)
Chapter 3
The real free scalar field

If thou tellest thy tale in this manner, cried Don Quixote, repeating every circumstance twice over; it will not be finished these two days: proceed therefore, connectedly, and rehearse it, like a man of understanding: otherwise thou hadst better hold thy tongue.

Miguel de Cervantes, Don Quixote, Vol. 1 Book 3 Ch. 6

As a first example of a locally covariant quantum field theory we will now describe the real free scalar field in two different ways. First we give the distributional description using the Borchers-Uhlmann algebra in section 3.1, followed by the $C^*$-algebraic description using the CCR-algebra (or Weyl-algebra) in section 3.2. Because the free scalar field is a well-known test ground for quantum field theory in curved spacetime it is instructive to describe it in some detail before we treat the more complex case of the free Dirac field. We also give an elegant proof in proposition 3.1.13 of the fact that the commutation relations together with the Hadamard condition on the two-point distribution of a (not necessarily quasi-free) state completely fix the singularity structure of all $n$-point distributions. This result appears to be hitherto unknown in this generality.
3.1 Distributional approach to the free scalar field

In this section we will make use of a topological *-algebra that is not a $C^*$-algebra, namely the Borchers-Uhlmann algebra. This algebra naturally gives rise to unbounded field operators. After describing a general real scalar field and the microlocal spectrum condition we will specialise to the real free scalar field and introduce the important class of Hadamard states. We refer to appendix A for results on wave front sets. Our presentation in this section is largely based on [29, 16, 66, 84].

3.1.1 The real scalar field

On a spacetime $M$ in $\mathcal{M}$an we make the following definition:

**Definition 3.1.1** The Borchers-Uhlmann algebra is the direct sum

$$\mathcal{U}_M := \bigoplus_{n=0}^{\infty} C_0^\infty (M^{\times n})$$

(in the algebraic sense, i.e. only a finite number of terms in the sum are non-zero), equipped with:

1. the product $f(x_1, \ldots, x_n)g(x_{n+1}, \ldots, x_{n+m}) := (f \otimes g)(x_1, \ldots, x_{n+m})$, extended linearly,

2. the *-operation $f(x_1, \ldots, x_n)^* := \overline{f}(x_n, \ldots, x_1)$, extended anti-linearly,

3. a topology such that $f_j = \bigoplus_n f_j^{(n)}$ converges to $f = \bigoplus_n f^{(n)}$ if and only if for all $n$ we have $f_j^{(n)} \rightarrow f^{(n)}$ in $C_0^\infty (M^{\times n})$ and for some $N > 0$ we have $f_j^{(n)} = 0$ for all $j$ and $n \geq N$.

More precisely, as a topological space $\mathcal{U}_M$ is the strict inductive limit $\mathcal{U}_M = \bigcup_{N=0}^{\infty} \bigoplus_{n=0}^N C_0^\infty (K_N^{\times n})$, where $K_N$ is an exhausting (and increasing) sequence of
compact subsets of $\mathcal{M}$ and each $C^\infty_0 (K_N^x)$ is given the test-function topology, see [73] theorem 2.6.4. Following our convention for $C^\infty_0 (M)$ we will write $\mathcal{U}_M$ instead of $\mathcal{U}_M$ (see the footnote on page [36]). It should be noted that the algebra $\mathcal{U}_M$ restricts the field to be Hermitian by property 2, but it does not contain any dynamical information.

**Lemma 3.1.2** The Borchers-Uhlmann algebra is a topological $^*$-algebra with unit and a continuous linear functional $\omega$ consists of a sequence of distributions $\omega_n$ on $\times^n$, which are called the $n$-point distributions.

**Proof.** The given topology makes $\mathcal{U}_M$ a locally convex topological vector space, $^*$ is continuous and multiplication is separately continuous, i.e. $\mathcal{U}_M$ is a topological $^*$-algebra (see [76] p.22). The unit $I$ is $1 \in C^\infty_0 (M^x) := \mathbb{C}$, i.e. $I = 1 \oplus 0 \oplus \ldots$. A continuous linear functional on $\mathcal{U}_M$ gives rise to continuous linear functionals $\omega_n$ on all $C^\infty_0 (M^x)$ and vice versa and therefore corresponds to a sequence of distributions $\omega_n$. □

The Borchers-Uhlmann algebra is not a $C^*$-algebra and it cannot be represented faithfully as an algebra of bounded operators. Nevertheless, most of the ideas of locally covariant quantum field theory that apply to $C^*$-algebras also apply in the case of more general topological $^*$-algebras. The following proposition shows that the map $M \mapsto \mathcal{U}_M$ can be made into a covariant functor from $\text{Man}$ into $\mathcal{TAlg}$.

**Proposition 3.1.3** If $\Psi : M_1 \mapsto M_2$ is a morphism in $\text{Man}$ then there is a unique injective $^*$-algebra homomorphism $\nu_\Psi : \mathcal{U}_{M_1} \mapsto \mathcal{U}_{M_2}$ determined by $\nu_\Psi (f) := \psi^* f = f \circ \psi^{-1}$ on $C^\infty_0 (M_1)$, where we extend $\psi^* f$ by 0 outside $\psi (M_1)$.

$^1$Therefore, $\mathcal{U}_M$ is an LF-space, which is by definition the strict inductive limit of an increasing sequence of Fréchet spaces.
Proof. Using finite sums of finite tensor products of elements in $C_0^\infty(M_1)$ the given relation determines $\nu_\Psi$ uniquely on $\oplus_{n=0}^{\infty}(C_0^\infty(M_1))^{\otimes n} \subset \mathcal{U}_{M_1}$, where we take the algebraic direct sum and tensor product. The map so defined is an injective $^*$-algebra homomorphism of a dense subalgebra of $\mathcal{U}_{M_1}$ into $\mathcal{U}_{M_2}$ and extends by continuity in a unique way to a $^*$-algebra homomorphism $\nu_\Psi$ of $\mathcal{U}_{M_1}$ into $\mathcal{U}_{M_2}$. To prove that $\nu_\Psi$ is injective we note that $\nu_\Psi(f^{(n)}(x_1, \ldots, x_n)) = \psi_\ast f^{(n)}(x_1, \ldots, x_n) = f^{(n)}(\psi^{-1}(x_1), \ldots, \psi^{-1}(x_n))$. □

Definition 3.1.4 The Borchers-Uhlmann functor $\mathbf{U} : \text{Man} \to \mathfrak{T}\text{Alg}$ assigns to each globally hyperbolic spacetime $M$ the Borchers-Uhlmann algebra $\mathcal{U}_M$ and to each morphism $\Psi : M_1 \to M_2$ the morphism $\nu_\Psi$ of proposition 3.1.3.

Proposition 3.1.5 The Borchers-Uhlmann functor $\mathbf{U}$ defines an additive locally covariant quantum field theory.

Proof. If $O = \cup_i O_i$ and $\chi_i$ is a partition of unity on $O$ such that supp $\chi_i \subset O_i$, then every $f \in C_0^\infty(O)$ can be written as $f = \sum_i f_i$ with $f_i := f\chi_i$. The inclusion $\mathcal{U}_O \subset \vee_i \mathcal{U}_{O_i}$ now follows by decomposing every test-function in an element $A \in \mathcal{U}_O$ in this way and the converse inclusion is trivial. □

A locally covariant quantum field, in the sense of definition [2.5.1] is given in the current setting by

$$\Phi_M : C_0^\infty(M) \to \mathcal{U}_M : f \mapsto 0 \oplus f \oplus 0 \oplus 0 \ldots .$$

This takes care of the operators of the theory and the fields. Now let us turn our attention to the states. The following class of states is often of special importance. In analogy to theorem [A.1.3] in appendix [A] we can define the wave front set of the distribution $\Phi_M$ as $WF(\Phi_M) := \cup_l WF(l \circ \Phi_M) \setminus \mathbb{Z}$, where the union is taken over all continuous linear functionals $l$ on $\mathcal{U}_M$. This makes perfect sense, provided we can generalise lemma [A.1.1] to the case of $\mathcal{U}_M$-valued distributions. If $l = (l_n)_{n \in \mathbb{N}}$ is any continuous linear functional on $\mathcal{U}_M$, then $(l \circ \Phi_M)(f) = l_1(f)$ and hence $WF(\Phi_M) = T^*M \setminus \mathbb{Z}$ by theorem 8.1.4 in [47].
interest, because they arise from the canonical quantisation of a linear field equation.

Definition 3.1.6 A state $\omega$ on $U_M$ is called quasi-free iff $\omega_n = 0$ for $n$ odd and for $m \geq 1$:

$$\omega_{2m}(f_1, \ldots, f_{2m}) = \sum_{\pi \in \Pi_m} \omega_2(f_{\pi(1)}, f_{\pi(2)}) \cdots \omega_2(f_{\pi(2m-1)}, f_{\pi(2m)}),$$

where $\Pi_m$ is the set of permutations of $\{1, \ldots, 2m\}$ such that

1. $\pi(1) < \pi(3) < \ldots < \pi(2m - 1),$

2. $\pi(2i - 1) < \pi(2i), i = 1, \ldots, m.$

A quasi-free state is completely determined by its two-point distribution (note that $\omega_0 = 1$) and definition [3.1.6] tells us that the higher $n$-point distributions can be obtained using the combinatorics that is familiar from flat spacetime quantum field theory. Indeed, for a $2n$-point distribution we sum over all pairings of the indices, where we preserve the left-right ordering within each pair (we put the smaller index of each pair on the left by the second condition on $\pi$) and we only count every pairing once by the first condition on $\pi$.

If $U = U_M$ we may define smeared field operators by\footnote{For these representation specific entities we drop the subscript $M$ to ease the notation. This causes no confusion, because it is clear that $\omega$ itself is defined on a specific spacetime.}

$$\Phi_\omega^{(\omega)}(f) := \pi_\omega(\Phi_M(f)). \quad (3.1)$$

These are unbounded operators on $\mathcal{H}_\omega$ with a common dense and invariant domain $\mathcal{D}_\omega$ (see theorem [2.1.4]). We also define $\mathcal{H}_\omega$-valued $n$-point distributions by

$$\phi_n^{(\omega)}(f_n, \ldots, f_1) := \pi_\omega(f_n \otimes \ldots \otimes f_1)\Omega_\omega. \quad (3.2)$$

For all $n, m$ and all $f_i, g_j \in C_0^\infty(M)$ we have the identity

$$\langle \phi_n^{(\omega)}(f_n, \ldots, f_1), \phi_m^{(\omega)}(g_m \ldots g_1) \rangle = \omega_{n+m}(\bar{f}_1, \ldots, \bar{f}_n, g_m \ldots g_1). \quad (3.3)$$
As our state space for $M$ we can select the class of states that satisfy the microlocal spectrum condition of [13]. To formulate this condition we need to introduce some new terminology. Let $G_n$ be the set of directed graphs with $n$ vertices in which every edge that appears also appears in the opposite direction. An immersion of such a graph into $M$ assigns to every vertex $\nu_i$ a point $x_i$ and to every edge $e_r$ from $\nu_i$ to $\nu_j$ a piecewise smooth curve $\gamma_r$ from $x_i$ to $x_j$ and a causal covector field $k_r$ on $\gamma_r$ which is covariantly constant ($\nabla k_r = 0$) along the curve in such a way that

1. if $e_{-r}$ is the edge $e_r$ in the opposite direction, then $\gamma_{-r}$ is the curve $\gamma_r$ in the opposite direction and $k_{-r} = -k_r$,

2. if $e_r$ is a curve from $x_i$ to $x_j$ with $i < j$ then $k_r$ is future directed.

Intuitively one may think of the vectors $k_r$ as “singularities”, “propagating” along the curves $\gamma_r$ between points $x_i$ and $x_j$. We now define a set of allowed singularities as follows:

$$\Gamma_n := \left\{ (x_n, \xi_n; \ldots; x_1, \xi_1) \in T^* M^n \setminus \mathbb{Z} \mid \exists G \in G_n \text{ and an immersion of } G \text{ such that } \nu_i \mapsto x_i, \text{ and } \xi_i = \sum_{e_r, s(e_r) = x_i} k_r(x_i) \right\}, \quad (3.4)$$

where $s(e_r)$ denotes the source of the edge $e_r$.

**Definition 3.1.7** A state $\omega$ on $\mathcal{U}_M$ is said to satisfy the microlocal spectrum condition ($\mu SC$) if and only if for all $n \in \mathbb{N}$:

$$WF(\omega_n) \subset \Gamma_n.$$

4 A directed graph is a graph in which each edge $e$ is given a direction, so that it goes from a source vertex to a target vertex.

5 Note that we have ordered the indices of $(x_n, k_n; \ldots; x_1, k_1)$ in the opposite way to [13], because we want the singularities to originate on the right-hand side in the $n$-point distributions and to travel to the left as time progresses, cf. definition 3.1.7 below.
The microlocal spectrum condition restricts the set of singularities of the \( n \)-point distributions to the sets \( \Gamma_n \) and the usefulness of this condition follows from the special properties of the \( \Gamma_n \):

**Proposition 3.1.8** The sets \( \Gamma_n \subset T^* M^\times n \) have the following properties:

1. Each \( \Gamma_n \subset T^* M^\times n \backslash \mathcal{Z} \) is a convex cone,
2. \( \Gamma_n \cap -\Gamma_n = \emptyset \),
3. \( \pi((\Gamma_{n_1} \cup \mathcal{Z}) \times \cdots \times (\Gamma_{n_m} \cup \mathcal{Z})) \subset \Gamma_{n_1+\cdots+n_m} \cup \mathcal{Z} \), where \( \pi \) is a permutation acting on the indices such that \( \pi(1) < \pi(2) < \cdots < \pi(n_1) \); \( \pi(n_1+1) < \cdots < \pi(n_1+n_2) \); \( \cdots; \pi(n_1+\cdots+n_{m-1}+1) < \cdots < \pi(n_1+\cdots+n_m) \).

**Proof.** We refer to [15] lemma 4.2 for a proof of the first property. The second property follows from the first and the third property follows immediately from the definitions, using the unions of disjoint graphs (cf. [15] proposition 4.3).

It follows from the last two items that a quasi-free state satisfies \( \mu\text{SC} \) if and only if \( WF(\omega_2) \subset \Gamma_2 \). It also seems that the first two items are sufficient to guarantee that products of \( n \)-point distributions and Wick powers can be defined [15] [47], even without the commutator property that we will introduce in the next section.\(^6\) This forms the starting point of the perturbative treatment of interacting quantum field theories on curved spacetimes [14].

**Proposition 3.1.9** One can define a state space functor \( Q: \operatorname{Man} \rightarrow \operatorname{States} \) for the locally covariant quantum field theory \( \mathbf{U} \) that assigns to each globally hyperbolic spacetime \( M \) the set \( \mathcal{D}_M \) of states on \( \mathcal{U}_M \) that satisfy the \( \mu\text{SC} \).

\(^6\)Note that the difference of two two-point distributions with the \( \mu\text{SC} \) does not have to be smooth unless we also impose the commutator property [66]. [15] assumes the commutator property, but it does not appear to be necessary for their proofs.
Proof. We first note that the set of states is convex by theorem A.1.5. To show that it is closed under operations from $\mathcal{U}_M$ we note that for fixed $f \in C_0^\infty(M \times m)$ and $h \in C_0^\infty(M \times r)$ we have

$$WF(\omega_{m+n+r}(f, x_1, \ldots, x_n, h)) \subset \{(y_1, 0; \ldots; y_m, 0; x_1, k_1; \ldots; x_n, k_n; z_1, 0; \ldots z_r, 0) \in \Gamma_{m+n+r} \} \subset \Gamma_n,$$

using [47] theorem 8.2.12. The same holds for linear combinations of such terms, so if $\omega(A^*A) \neq 0$ then the state $B \mapsto \frac{\omega(A^*BA)}{\omega(A^*A)}$ satisfies the $\mu$SC if $\omega$ does.

The action of $Q$ on morphisms is defined implicitly by the statement that $Q$ is a state space for $U$. That this action is well-defined follows from the fact that wave front sets transform as a subset of the cotangent bundle (see appendix A) and the cones $\Gamma_n$ are subsets of the cotangent bundle that are constructed from the metric and hence covariant under isometric diffeomorphisms.

The locally covariant quantum field theory $U$ with state space $Q$ is not causal and does not satisfy the time-slice axiom. These shortcomings are due to the fact that we have not put any constraints on the dynamics or causality. This will be our next task.

3.1.2 The real free scalar field

In order to arrive at the usual description of the real free scalar field we will put in some physically motivated restrictions. These restrictions can be put either on the state or on the algebra and we will describe both approaches in that order.

Classically, two operations performed in space-like separated regions cannot influence each other. It seems reasonable to postulate that this must remain true for the expectation values of quantum physical operators. We
therefore say that a state \( \omega \) is causal iff (cf. definition 2.4.2)

\[
\omega_n(f_1, \ldots, f_i, f_{i+1}, \ldots, f_n) = 0,
\]

whenever \( \text{supp } f_i \subset (\text{supp } f_{i+1})^\perp \). Here \((,\) denotes anti-symmetrisation.

A state \( \omega \) on \( \mathcal{U}_M \) is a state of the free field iff the dynamics is described by the Klein-Gordon equation. The classical form of the Klein-Gordon equation is

\[
K \phi := (\Box + m^2 + \xi R)\phi = 0,
\]

where \( K \) is the Klein-Gordon operator, \( \Box = \nabla^a \nabla_a \) is the d’Alembertian, \( m \geq 0 \) is the mass of the field \( \phi \in C^\infty(M) \), \( R \) is the Ricci scalar of \( M \) and \( \xi \) is a coupling parameter. Here \( \xi \) and \( m \) are assumed to be independent of \( M \).

A state \( \omega \) on \( \mathcal{U}_M \) is a state for the free field iff for all \( n \geq 1 \) and all \( 1 \leq i \leq n \):

\[
\omega_n(f_1, \ldots, K f_i, \ldots, f_n) = 0.
\]

As the Klein-Gordon operator \( K \) is formally self-adjoint (or more precisely: the dual of \( K \) is an extension of \( K \)) these equations can also be written as \( K^{(i)} \omega_n = 0 \), where the upper index indicates that \( K \) acts on the \( i \)'th variable of the distribution \( \omega_n \).

Because we assume that the spacetime \( M \) is globally hyperbolic there are unique advanced (\( - \)) and retarded (\( + \)) fundamental solutions \( E^\pm : C^\infty_0(M) \to C^\infty(M) \) such that \( K E^\pm f = f, E^\pm K f = f \) and \( \text{supp}(E^\pm f) \subset J^\pm(\text{supp } f) \) for all \( f \in C^\infty_0(M) \) (see \[6\] theorem 3.3.1). Setting \( E := E^- - E^+ \) we see that \( Ef \) is a solution of the Klein-Gordon equation whose intersection with each Cauchy surface of \( M \) is compact. Conversely, every solution which has compact intersection with all Cauchy surfaces can be obtained in this way by \[29\] lemma A.3.

A stronger requirement than causality is the commutator property. A state \( \omega \) is said to have the commutator property if and only if

\[
\Phi^{(\omega)}(f)\Phi^{(\omega)}(h) - \Phi^{(\omega)}(h)\Phi^{(\omega)}(f) = iE(f, h),
\]

(3.6)
where we view $E$ as the bidistribution $E(f, h) := \int_M fE_h \, d\text{vol}_g$. This condition arises naturally from the canonical quantisation of the classical Klein-Gordon field.

For quasi-free states the causality condition, Klein-Gordon equation and commutator property reduce to the corresponding conditions on the two-point distribution:

$$\omega_2(f, h) = 0, \quad \text{supp } f \subset (\text{supp } h)^\perp,$$

$$K^{(1)}\omega_2 = K^{(2)}\omega_2 = 0,$$

$$\omega_2(f, h) := \omega_2(f, h) - \omega_2(h, f) = iE(f, h).$$

Instead of putting the causality, reality, dynamics and commutator property in the state we can incorporate this information directly in the algebra as follows. Let $J \subset U_M$ be the closed *-ideal generated by all elements of the form $Kf$ or $f \otimes h - h \otimes f - iE(f, h)I$. The quotient space $U^0_M := U_M / J$ is another locally convex topological vector space ([75] p.54) and the *-operation, respectively multiplication, on $U_M$ descends to a continuous, respectively separately continuous, map on $U^0_M$. In other words, $U^0_M$ is another topological *-algebra. The easiest way to show that the algebra $U^0_M$ is not trivial is to show that it has a non-trivial (faithful) representation.

**Proposition 3.1.10** If $\Psi : M_1 \to M_2$ is a morphism in $\text{Man}$ and $p_i : U_{M_i} \to U^0_{M_i}$, $i = 1, 2$, is the quotient map, then $\nu_\Psi$ descends to an injective *-algebra homomorphism $\nu^0_\Psi$ on $U^0_{M_1}$.

**Proof.** Let $J_i \subset U_{M_i}$ be the closed *-ideal generated by elements of the form $K_if$ or $f \otimes h - h \otimes f - iE_i(f, h)I$, where $K_i$ respectively $E_i$ are the Klein-Gordon operator and its advanced-minus-retarded fundamental solution on $M_i$. For $f, h \in C^\infty_0(M_1)$ set $f' := \nu_\Psi(f)$ and $h' := \nu_\Psi(h)$. Because of the covariance of the Klein-Gordon operator, $K_2 \circ \psi = \psi \circ K_1$, we see that
Similarly we can use the uniqueness of the advanced and retarded fundamental solutions and equation \(2.1\) to conclude that \(v_\Psi(E_1^+ f) = E_2^+(f')|_{\psi(M_1)}\), and therefore \(v_\Psi(E_1(f, h))I = E_2(f', h')I\), because on the right-hand side we integrate over a compact region in \(\psi(M_1)\). This then yields \(v_\Psi(f \otimes h - h \otimes f - iE_1(f, h)I) = f' \otimes h' - h' \otimes f' - iE_2(f', h')I\). By continuity we conclude that \(v_\Psi(J_1) = J_2 \cap v_\Psi(M_1)\), which means that \(v_\Psi\) descends to a well-defined *-algebra homomorphism \(v_\Psi^0\) on \(\mathcal{U}_M^0\) which is injective. \(\square\)

**Definition 3.1.11** The free field Borchers-Uhlmann functor \(U^0 : \mathfrak{Man} \to \mathfrak{TAlg}\) assigns to each globally hyperbolic spacetime \(M\) the algebra \(\mathcal{U}^0_M\) and to each morphism \(\Psi : M_1 \to M_2\) the morphism \(v_\Psi^0\) of proposition 3.1.10.

If \(p : \mathcal{U}_M \to \mathcal{U}^0_M\) denotes the quotient map, then \(I_0 := p(I)\) is the unit for \(\mathcal{U}^0_M\) and a state \(\omega'\) on \(\mathcal{U}^0_M\) gives rise to a state \(\omega := \omega' \circ p\) on \(\mathcal{U}_M\) because \(p\) is continuous. By construction, \(\omega\) is a causal state for the free field with the commutator property and the n-point distributions of \(\omega'\) and \(\omega\) are related by \(\omega_n = \omega'_n \circ p\). The GNS-quadruples of \(\omega\) and \(\omega'\) satisfy \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \mathcal{D}_\omega) = (\mathcal{H}_{\omega'}, \pi_{\omega'} \circ p, \Omega_{\omega'}, \mathcal{D}_{\omega'})\) by the uniqueness part of theorem 2.1.4.

A locally covariant quantum field \(\Phi^0_M\) can be defined as \(\Phi^0_M := p_M \circ \Phi_M\). It follows that \(K \Phi^0_M = 0\) in the weak sense, i.e. \((K \Phi^0_M)(f) = \Phi^0_M(Kf) = p_M(\Phi_M(Kf)) = 0\). Moreover,

\[
\Phi^0_M(f)\Phi^0_M(h) - \Phi^0_M(h)\Phi^0_M(f) = p_M(f \otimes h - h \otimes f) = iE(f, h)I_0
\]

in \(\mathcal{U}^0_M\), so the field \(\Phi^0_M\) is an \(\mathcal{U}^0_M\)-valued distribution that satisfies the Klein-Gordon equation and has the commutator property.\(^7\)

\(^7\)Strictly speaking \(\omega'_n\) is not a distribution, because it is not defined on the space of test-functions, but rather on a quotient of that space.

\(^8\)Again we can define the wave front set of \(\Phi^0_M\) in analogy to theorem A.1.3 as \(WF(\Phi^0_M) := \bigcup_l WF(l \circ \Phi^0_M) \setminus \mathcal{N}\), where the union is taken over all continuous linear functionals \(l\) on \(\mathcal{U}^0_M\). If \(l = (l_n)_{n \in \mathbb{N}}\) is any continuous linear functional on \(\mathcal{U}^0_M\), then \((l \circ \Phi^0_M)(f) = l_1(f)\) and \((Kl \circ \Phi^0_M)(f) = l \circ \Phi^0_M(Kf) = l(\Phi^0_M(Kf)) = 0\), so \(WF(\Phi^0_M) \subset \mathcal{N} \setminus \mathcal{Z}\) by theorem A.1.5 (recall the definition of \(\mathcal{N}\) on page 30).
A class of states that is of special importance for the free scalar field in curved spacetime is the class of Hadamard states. The original definition of the Hadamard condition in curved spacetimes of Kay and Wald [54] is equivalent to the following definition, due to a theorem of Radzikowski [66]. (Recall the definition of $N^\pm$ on page 30.)

**Definition 3.1.12** A (not necessarily quasi-free) state $\omega$ on $\mathcal{U}_M^0$ is called a Hadamard state iff

$$WF(\omega) = \{(x,\xi; y,\xi') \in N^- \times N^+ | (x, -\xi) \sim (y, \xi')\} \setminus \mathcal{Z},$$

where $(x, -\xi) \sim (y, \xi')$ if and only if $(x, -\xi) = (y, \xi')$ or there is an affinely parameterised light-like geodesic between $x$ and $y$ to which $-\xi, \xi'$ are cotangent (and hence $-\xi$ and $\xi'$ are parallel transports of each other along the geodesic).

Note that the principal symbol of $K$ is the metric $g_{\mu\nu}$, so by theorem A.1.5 the wave front set can only contain null-covectors. Moreover, the propagation of singularities theorem of Duistermaat and Hörmander ([32] theorem 6.1.1, also quoted in [66]) implies that these singularities propagate under the Hamiltonian flow on $T^*M$ determined by the principal symbol. It turns out that this means that null covectors propagate along the null geodesics to which they are cotangent, which gives rise to the equivalence relation $\sim$.

The two-point distributions of two Hadamard states on $\mathcal{U}_M^0$ differ by a smooth function, so the expectation value of the stress-energy-momentum tensor of the free scalar field can be renormalised (see [66], [90] section 4.6). A free field state satisfying the $\mu$SC is Hadamard [66]. Conversely, it is known (and easy to see) that a quasi-free Hadamard state satisfies the $\mu$SC [15]. In fact, we will now prove the new result that this is even true for general (not necessarily quasi-free) Hadamard states:

**Proposition 3.1.13** Let $\omega$ be a state on $\mathcal{U}_M^0$ which is Hadamard on a neighbourhood $\mathcal{W} \subset M$ of a Cauchy surface in $M$. Then $\omega$ satisfies the $\mu$SC on $M$. 

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Proof. Suppose that \((x_1, k_1; \ldots; x_n, k_n) \in WF(\omega_n)\) for \(n \geq 1\). (Note that \(\omega_0 = 1\) is always smooth.) For each index \(i\) we have \((x_i, k_i) \in \mathcal{N}\), because of the equation of motion (see theorem A.1.5). Moreover, if \(k_n \neq 0\) then we can apply theorem A.1.6 first to \(\omega_n = \langle \phi_{n-1}^{(\omega)}, \phi_1^{(\omega)} \rangle\) (see equation (3.3)) to find \((x_n, k_n) \in WF(\phi_1^{(\omega)})\) and then again to \(\omega_2 = \langle \phi_1^{(\omega)}, \phi_1^{(\omega)} \rangle\) to obtain \((x_n, -k_n; x_n, k_n) \in WF(\omega_2)\). We may then apply the propagation of singularities theorem ([32] theorem 6.1.1, [66]) to find \((y, l) \in \mathcal{N} \cap T^*\mathcal{W}\) such that \(y, l \sim (x_n, k_n)\) (see definition 3.1.12) and \((y, -l; y, l) \in WF(\omega_2)\). If \(\omega\) is Hadamard on \(\mathcal{W}\) we conclude that \((y, l) \in \mathcal{N}^+\) and hence \((x_n, k_n) \in \mathcal{N}^+\). Similarly, if \(k_1 \neq 0\) then \((x_1, k_1) \in \mathcal{N}^-.\) In particular, for \(n = 1\) we find that \((x_1, k_1) \in \mathcal{N}^+ \cap \mathcal{N}^- = \mathcal{Z}\), so \(WF(\omega_1) = \emptyset\) and \(\omega_1\) is smooth.

We now argue by contradiction. Let \(n \geq 2\) be the smallest number for which we can find a point \((x_1, k_1; \ldots; x_n, k_n) \in WF(\omega_n) \setminus \Gamma_n\). There must then be an index \(i\) such that \(k_i \neq 0\). Assume first that \((x_i, k_i) \in \mathcal{N}^-\). Now we interchange the points \(x_i\) and \(x_{i+1}\) in \(\omega_n\) to find:

\[
\omega_n(x_1, \ldots, x_n) = \omega_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) + i\omega_{n-2}(x_1, \ldots, \hat{x_i}, \hat{x_{i+1}}, \ldots, x_n) E(x_i, x_{i+1}),
\]

where the hats denote that these points are omitted. Using theorem A.1.5 we see that \((x_1, k_1; \ldots; x_n, k_n)\) must be in the wave front set of one of the terms on the right-hand side of equation (3.7). Suppose that it is in the wave front set of the second term. This wave front set can be estimated by ([47] theorem 8.2.9)

\[
WF(\omega_n \otimes E) \subset (WF(\omega_n \otimes E) \cup \mathcal{Z}) \times (WF(E) \cup \mathcal{Z}).
\]

If \((x_1, k_1; \ldots; x_n, k_n) \in WF(\omega_n \otimes E)\), then the assumption on \(k_i\) implies \((x_i, k_i; x_{i+1}, k_{i+1}) \in WF(E) \cap (\mathcal{N}^- \times T^*\mathcal{M}) \subset \Gamma_2\) by proposition A.1.7. By the minimality of \(n\) and proposition 3.1.8 we find \((x_1, k_1; \ldots; x_n, k_n) \in \Gamma_n\), which is a contradiction. Hence, \((x_1, k_1; \ldots; x_n, k_n)\) must be in the wave front
set of the first term of equation (3.7) and

\[(x_1, k_1; \ldots; x_{i+1}, k_{i+1}; x_i; k_i; \ldots; x_n, k_n) \in WF(\omega_n).\]

Proceeding in this way we can permute the point \((x_i, k_i)\) all the way to the right. Then we have \((x_i, k_i) \in \mathcal{N}^-\) by assumption and \((x_i, k_i) \in \mathcal{N}^+\) by the first paragraph of the proof. Similarly, if we had started with \((x_i, k_i) \in \mathcal{N}^+\) we could have permuted this point to the left to conclude that \((x_i, k_i) \in \mathcal{N}^-\). In both cases we get a contradiction, because \(k_i \neq 0\), but \(\mathcal{N}^+ \cap \mathcal{N}^- = \mathbb{Z}\).

This completes the proof. \(\Box\)

The argument in the proof of proposition 3.1.13 can also be used to show that the immersed graphs that occur in \(WF(\omega_n)\) are disjoint unions of pieces of light-like geodesics, to which the cotangent vectors are parallel or antiparallel.

For completeness we also prove a result concerning truncated \(n\)-point distributions, although we will not use it in this thesis. For \(n \geq 1\) we let \(\mathcal{P}_n\) denote the set of all partitions of the set \(\{1, \ldots, n\}\) into pairwise disjoint ordered sets and for each set \(r\) in the partition \(P \in \mathcal{P}_n\) we denote its elements by \(r(1), \ldots, r(|r|)\) where \(|r|\) is the number of elements of \(r\). We then define the truncated \(n\)-point distributions \(\omega_T^n\) implicitly through

\[\omega_T^n(x_1, \ldots, x_n) = \sum_{P \in \mathcal{P}_n} \prod_{r \in P} \omega_T^{|r|}(x_{r(1)}, \ldots, x_{r(|r|)}). \tag{3.8}\]

Note that this equation can be solved iteratively for the \(\omega_T^n\) order by order.

In their discussion of perturbative quantum field theory [45] impose the Hadamard condition together with the condition that \(\omega_T^n\) is smooth for all \(n \neq 2\) and \(WF(\omega_T^2) = WF(\omega_2)\). The same condition has also been considered by Kay in [52]. Our result states that the Hadamard condition already implies this condition on the truncated \(n\)-point distributions, so this extra condition is superfluous.
Proposition 3.1.14 If $\omega$ is a (not necessarily quasi-free) Hadamard state on $\mathcal{U}_M^0$, then $\omega_T^n$ is smooth for all $n \neq 2$ and $WF(\omega_T^n) = WF(\omega_2)$.

Proof. First note that $\omega_T^1(x_1) = \omega_1(x_1)$ and $\omega_T^2(x_1, x_2) = \omega_2(x_1, x_2) - \omega_1(x_1)\omega_1(x_2)$ by equation (3.8). Now, $\omega_T^1$ is smooth by the proof of proposition 3.1.13 and hence $WF(\omega_T^2) = WF(\omega_2)$. We prove the result for $n \geq 3$ by induction.

Suppose that $(x_1, k_1; \ldots; x_n, k_n) \in WF(\omega_T^n)$ and let $a$ be an index such that $k_a \neq 0$. Expanding equation (3.8) and using the induction hypothesis it follows that $(x_1, k_1; \ldots; x_n, k_n)$ is in the wave-front set of

$$
\omega_n(x_1, \ldots, x_n) = \sum_{i \leq a-1} \omega_{n-2}(x_1, \ldots, \hat{x}_i \ldots, \hat{x}_a \ldots, x_n)\omega_2(x_i, x_a)
- \sum_{i \geq a+1} \omega_{n-2}(x_1, \ldots, \hat{x}_a \ldots, \hat{x}_i \ldots, x_n)\omega_2(x_a, x_i),
(3.9)
$$

because it cannot be in the wave front set of any of the other terms. Notice that $(x_n, k_n) \in \mathcal{N}^+$ and $(x_1, k_1) \in \mathcal{N}^-$, because the $\omega_n$ satisfy the $\mu$SC by proposition 3.1.13.

Now we note what happens when we use the commutation relations for the indices $a$ and $a+1$ in expression (3.9). The only changes occur in the first term and in the term $i = a+1$ under the second summation symbol, namely:

$$
\omega_n(x_1, \ldots, x_n) - \omega_{n-2}(x_1, \ldots, \hat{x}_a \ldots, \hat{x}_{a+1} \ldots, x_n)\omega_2(x_a, x_{a+1}) = \\
\omega_n(x_1, \ldots, x_{a+1}, x_a, \ldots, x_n) - \omega_{n-2}(x_1, \ldots, \hat{x}_{a+1} \ldots, \hat{x}_a \ldots, x_n)\omega_2(x_{a+1}, x_a).
$$

Substituting this in expression (3.9) we see that $(x_1, k_1; \ldots; x_n, k_n)$ is in the wave front set of

$$
\omega_n(x_1, \ldots, x_{a+1}, x_a, \ldots, x_n)
- \sum_{i \leq a-1 \text{ or } i = a+1} \omega_{n-2}(x_1, \ldots, \hat{x}_i \ldots, \hat{x}_a \ldots, x_n)\omega_2(x_i, x_a)
- \sum_{i \geq a+2} \omega_{n-2}(x_1, \ldots, \hat{x}_a \ldots, \hat{x}_i \ldots, x_n)\omega_2(x_a, x_i).
$$

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It follows that \((x_1, k_1; \ldots; x_{a+1}, k_{a+1}; x_a, k_a; \ldots; x_n, k_n)\) is in the wave front set of expression (3.9) with \(a + 1\) substituted for \(a\). Hence it is also in the wave front set of \(\omega^T_n\). Hence, if \(k_a \neq 0\) we can swap the points \((x_a, k_a)\) and \((x_{a+1}, k_{a+1})\) in \((x_1, k_1; \ldots, x_n, k_n)\). Now move \(k_a\) to the \(n\)th position to see that \((x_a, k_a) \in \mathcal{N}^+\). Then move \((x_a, k_a)\) to the first position to find that \((x_a, k_a) \in \mathcal{N}^-\). This implies \(k_a = 0\), so there can be no non-zero vector \(k_a\). This proves that the wave front set of \(\omega^T_n\) is empty and hence \(\omega^T_n\) is smooth for \(n \geq 3\). \(\square\)

**Definition 3.1.15** The state space functor \(Q^0 : \text{Man} \to \text{States}\) for the locally covariant quantum field theory \(U^0\) assigns to every globally hyperbolic spacetime \(M\) the set of Hadamard states \(Q^0_M\) on \(U^0_M\).

For each globally hyperbolic spacetime \(M\) the set \(Q^0_M\) is the subset of states in \(\mathcal{D}_M\) characterised by the extra conditions that they solve the Klein-Gordon equation and have the commutator property. This class of states is convex and closed under operations from \(U^0_M\), because these extra conditions are invariant under convex linear combinations and under operations from \(U^0_M\). This last point uses proposition 3.1.9 and the fact that the Hadamard condition implies the \(\mu\text{SC}\), proposition 3.1.13. The action of \(Q^0\) on morphisms is implicitly defined by the statement that \(Q^0\) is a state space functor for \(U^0\) and this action is well-defined, because both wave front sets and the cones \(\Gamma_n\) behave covariantly under isometric diffeomorphisms of the spacetime (see the proof of proposition 3.1.9).

The following lemma contains the core of the proof of the time-slice axiom for the free scalar field and is adapted from [29].

**Lemma 3.1.16** Let \(M\) be a globally hyperbolic spacetime, \(W \subset M\) a neighbourhood of a Cauchy surface and \(\chi \in C^\infty(M)\) such that \(\chi \equiv 1\) on \(J^+(W) \setminus W\) and \(\chi \equiv 0\) on \(J^-(W) \setminus W\). For every \(f \in C^\infty_0(M)\) we have \(f = f' + Kh\), where \(f' := K(\chi Ef) \in C^\infty_0(W)\) and \(h := E^-(f - f') \in C^\infty_0(M)\).
Proof. Clearly \( \text{supp} f' \subset \text{supp}(Ef) \) and \( f' \equiv 0 \) on a neighbourhood of \( M \setminus W \), so that \( f' \in C_0^\infty(W) \). (This uses the results of [8] and corollary A.5.4 of [6].) We have \( h := E^-(f-f') = (1-\chi)E^-f + \chi E^+f \), which is compactly supported in \( M \) and \( Kh = f - f' \).

**Proposition 3.1.17** The locally covariant quantum field theory \( U^0 \) with the state space \( Q^0 \) is causal, additive and satisfies the time-slice axiom.

Proof. Because \( E(f,h) = 0 \) whenever \( \text{supp} f \subset (\text{supp} h)^\perp \) it is immediately verified that the free field Borchers-Uhlmann functor defines a causal locally covariant quantum field theory. Additivity follows from proposition 3.1.5 by choosing a representative in \( U_M \) for each element of \( U^0_M \). To prove the time-slice axiom we suppose that \( \Psi : M_1 \to M_2 \) is a morphism such that \( \psi(M_1) \subset M_2 \) contains a Cauchy surface \( C \). For any \( f \in C_0^\infty(M_2) \) we use lemma 3.1.16 to find \( f' \in C_0^\infty(\psi(M_1)) \) such that \( f = f' + Kh \) for some \( h \in C_0^\infty(M) \). Therefore, \( \Phi_{M_2}(f) = \Phi_{M_2}(f') = \psi_0(\Phi_{M_1}(f' \circ \psi)) \). Because the elements \( \Phi_{M_2}(f) \) generate \( U^0_{M_2} \) we conclude that \( \psi_0 \) is an isomorphism. We already noted that \( (\psi_0^*)^* \) maps a state satisfying the \( \mu \text{SC} \) on \( U_{M_2}^0 \) to a state satisfying the \( \mu \text{SC} \) on \( U_{M_1}^0 \) (see proposition 3.1.9). Conversely, every such state on \( U_{M_1}^0 \) can be obtained in this way as follows. First such a state gives rise to a state on \( U_{\psi(M_1)}^0 \) which satisfies the \( \mu \text{SC} \). This state in turn determines a state on \( U_{M_2}^0 \) with the \( \mu \text{SC} \) by proposition 3.1.13. □

### 3.2 A \( C^* \)-algebraic description of the real free scalar field

We now describe the real free scalar field as a locally covariant quantum field theory using \( C^* \)-algebras, which is often convenient because \( C^* \)-algebras can
be represented as algebras of bounded operators \cite{49} (in particular the GNS-representation yields an algebra of bounded operators). We will follow the usual practice and use the CCR-algebra or Weyl-algebra for this purpose, following \cite{29, 90, 54, 13, 16, 84}. An alternative would be to use the resolvent algebra instead \cite{18}.

Given a globally hyperbolic spacetime $M$ we choose a smooth Cauchy surface $C \subset M$ and consider the linear space $K_C(M) := C^\infty_0(C, \mathbb{R}) \oplus C^\infty_0(C, \mathbb{R})$, where $C^\infty_0(C, \mathbb{R})$ is the space of real-valued test-functions on $C$. An element $(f, \dot{f})$ in $K_C(M)$ specifies a unique solution $\phi$ to the Klein-Gordon equation on $M$ with initial data $\phi|_C = f$ and $n^a \nabla_a \phi|_C = \dot{f}$, where $n^a$ is the future pointing normal vector field on $C$. In this way $K_C(M)$ can be identified with a linear space of classical solutions to the Klein-Gordon equation. We endow $K_C(M)$ with the non-degenerate symplectic structure

$$\sigma_C((f, \dot{f}), (h, \dot{h})) := \int_C f \dot{h} - \dot{f} h,$$

where we integrate with respect to the volume element associated to the metric on $C$ that is induced by the metric $g$ of $M$. Before we quantise the classical system that is described by the symplectic space $(K_C(M), \sigma_C)$ we show that it is independent of the choice of Cauchy surface (see \cite{29, 84}).

**Proposition 3.2.1** Define the symplectic space $(\mathcal{K}(M), \sigma)$, where $\mathcal{K}(M) := C^\infty_0(M, \mathbb{R})/\ker E$ and

$$\sigma(f, h) := E(f, h) = \int_M f Eh \, d\text{vol}_g.$$

Then $(\mathcal{K}(M), \sigma)$ is isomorphic as a symplectic space to $(\mathcal{K}(M)_C, \sigma_C)$ for every smooth Cauchy surface $C$.

**Proof.** Note that each element $f \in \mathcal{K}(M)$ determines a unique solution $Ef$ of the Klein-Gordon equation which has compact intersection with each Cauchy surface of $M$, so we can define a linear map $k : \mathcal{K}(M) \to K_C(M)$ by
\[ k(f) := (Ef \mid C, (n^a \nabla_a E f) \mid C). \] This map is surjective, because every smooth solution of the Klein-Gordon equation which has a compact intersection with every Cauchy surface can be obtained in this way by \cite{29} lemma A.3. It remains to check that \( \sigma_C(k(f), k(h)) = \sigma(f, h) \). Leaving the metric volume elements on \( M \) and \( C \) implicit we have:

\[
\begin{align*}
\sigma(f, h) &= \int_M f E h = \int_{J^+(C)} (K E^{-} f)(E h) + \int_{J^-(C)} (K E^{+} f)(E h) \\
&= \int_{J^+(C)} \nabla_a((\nabla^a E^{-} f)(E h)) - \nabla_a((E^{-} f)(\nabla^a E h)) + 0 \\
&\quad + \int_{J^-(C)} \nabla_a((\nabla^a E^{+} f)(E h)) - \nabla_a((E^{+} f)(\nabla^a E h)) + 0 \\
&= \int_C -(n_a \nabla^a E^{-} f)(E h) + (E^{-} f)(n_a \nabla^a E h) \\
&\quad + \int_C (n_a \nabla^a E^{+} f)(E h) - (E^{+} f)(n_a \nabla^a E h) \\
&= \sigma_C(k(f), k(h)),
\end{align*}
\]

where we used \( K E h = 0, E = E^{-} - E^{+} \) and a partial integration (see e.g. \cite{88} (B.2.26), but note the different sign convention; in this case the sign can easily be checked by studying the example of Minkowski spacetime). \( \square \)

The symplectic space \( (\mathcal{K}(M), \sigma) \) gives a covariant and Cauchy-surface independent description of the classical Klein-Gordon field. Also note that \( \sigma \) is non-degenerate by proposition \[3.2.1\] because \( \sigma_C \) is non-degenerate. To the symplectic space \( (\mathcal{K}(M), \sigma) \) we may associate the CCR-algebra \( \mathcal{A}_0^0 \), i.e. the (simple) \( C^* \)-algebra \( \mathcal{A}_0^0 \) of canonical commutation relations \cite{59,13}. This algebra is generated by the set of Weyl-operators \( W(f), f \in \mathcal{K}(M) \) satisfying the Weyl-relations

\[
W(f)W(h) = e^{-\frac{i}{\hbar} \sigma(f, h)}W(f + h), \quad W(f)^* = W(-f). \quad (3.10)
\]

**Proposition 3.2.2** One can define a locally covariant quantum field theory \( \mathcal{A}^0 : \text{Man} \to \text{CAlg} \) which maps each \( M \) to \( \mathcal{A}_0^0 \) and each morphism \( \Psi : M \to \)
M' in Man to the morphism $\alpha \Psi : A^0_M \to A^0_M'$ determined by $\alpha \Psi (W(f)) = W' (\psi_* f)$, where $W'$ denotes the Weyl operators that generate $A^0_{M'}$. This locally covariant quantum field theory is causal and additive.

**Proof.** For the proof that $A^0$ is a causal locally covariant quantum field theory we refer to [16]. Additivity follows from [13] proposition 5.2.10. □

Let us now explain the relation between the $C^*$-algebra $A^0_M$ and the Borchers-Uhlmann algebra $U^0_M$. If $\omega$ is a quasi-free state on $U^0_M$ and $f \in C^\infty_0 (M, \mathbb{R})$, then $\Phi^{(\omega)}(f)$ is a self-adjoint (unbounded) operator and we can define the unitary operator $W(f) := e^{i\Phi^{(\omega)}(f)}$ (see [84] proposition 3.2 and [13] theorem 5.2.3 and 5.2.4). These unitary operators satisfy the Weyl-relations (3.10) and therefore generate a $C^*$-algebra that is isomorphic to $A^0_M$, [13].

In order to go in the opposite direction, i.e. to obtain the Borchers-Uhlmann algebra from the Weyl-algebra, we need to restrict our attention to a special class of states on $A^0_M$:

**Definition 3.2.3** We call a state $\omega$ on $A^0_M$ regular if and only if for every $f \in \mathcal{K}(M)$ the unitary group $t \mapsto \pi_{\omega}(W(tf))$ is strongly continuous with self-adjoint (unbounded) generator $\Phi^{(\omega)}(f)$.

A regular state $\omega$ on $A^0_M$ is called $C^\infty$-regular if and only if the maps

$$\omega_n (f_1, \ldots, f_n) := \partial_{t_1} \cdots \partial_{t_n} \omega(W(t_1 f_1) \cdots W(t_n f_n)) |_{t_1 = \ldots = t_n = 0}$$

are distributions, after extending them by linearity to $\mathbb{C}$-valued test-functions.

A $C^\infty$-regular state $\omega$ on $A^0_M$ also defines a continuous state on $U^0_M$ via the $n$-point distributions (see [13] or [35] section A.5). The notation $\Phi^{(\omega)}(f)$ coincides with that of equation (3.1). This is justified, because the operators $\Phi^{(\omega)}$ are linear in their argument and they generate an algebra that is isomorphic
to $\pi_\omega(\mathcal{U}_M^0)$ (see [13] lemma 5.2.12). We have for example
\[
\Phi^{(\omega)}(f)\Phi^{(\omega)}(h) - \Phi^{(\omega)}(h)\Phi^{(\omega)}(f) =
-\partial_s\partial_t\pi_\omega(W(tf)W(sh) - W(sh)W(tf))|_{s=t=0} =
-\partial_s\partial_t(e^{-ist\delta(f,h)} - 1)\pi_\omega(W(sh)W(tf))|_{s=t=0} =
i\sigma(f, h)I = iE(f, h)I
\]
on a dense domain of $\mathcal{H}_\omega$, i.e. we recover equation (3.6). To make the correspondence with section 3.1 precise we should extend the real scalar field $\Phi^{(\omega)}$ of this section by linearity to complex-valued test-functions.

**Definition 3.2.4** A (not necessarily quasi-free) state $\omega$ on $\mathcal{A}_M^0$ is called Hadamard iff $\omega$ is $C^\infty$-regular and defines a Hadamard state on $\mathcal{U}_M^0$.

The state space functor $\mathcal{S}^0: \text{Man} \to \text{States}$ for the locally covariant quantum field theory $\mathcal{A}_M^0$ assigns to each globally hyperbolic spacetime $M$ the set of states on $\mathcal{A}_M^0$ which are locally quasi-equivalent to a quasi-free Hadamard state.

To define the state space functor we used the fact that a quasi-free Hadamard state on $M$ restricts to a quasi-free Hadamard state on any given sub-spacetime and the same is then true for any state locally quasi-equivalent to a quasi-free Hadamard state. In our choice of state space functor we have followed [10], who also prove some of the following properties in their theorem 3.4:

**Proposition 3.2.5** The locally covariant quantum field theory $\mathcal{A}^0$ with state space $\mathcal{S}^0$ is causal, additive, satisfies the time-slice axiom, respects local physical equivalence, is locally quasi-equivalent and nowhere classical.

**Proof.** We already noted causality and additivity in proposition 3.2.2 The condition on $\mathcal{A}^0$ needed for the time-slice axiom follows from proposition 3.2.1 For the condition on $\mathcal{S}^0$ we first note that a state $\omega$ which is Hadamard
on a neighbourhood $N$ of a Cauchy surface $C$ is Hadamard everywhere by proposition 3.1.13 and if $\omega$ is quasi-free on $N$ it is quasi-free everywhere by lemma 3.1.16. Now let $O \subset M$ be any bounded cc-region and note that $J(\partial O)$ has a compact intersection with $C$. This means we can find a bounded cc-region $V \subset N$ such that $O \subset D(V)$. If the state $\omega'$ is locally quasi-equivalent to $\omega$ on $N$, then the map $\pi_{\omega'}(A) \mapsto \pi_\omega(A)$ for all $A \in \mathcal{A}_0^V$ is well-defined and can be extended to a $^*$-isomorphism $\alpha$ of the local von Neumann algebras $\mathcal{R}_V^{(\omega)}$ and $\mathcal{R}_V^{(\omega')}$ (see [3] pp.212-213). It follows that $\alpha$ restricts to a $^*$-isomorphism of the von Neumann algebras $\mathcal{R}_O^{(\omega)}$ and $\mathcal{R}_O^{(\omega')}$, which proves that the restrictions of $\pi_\omega$ and $\pi_{\omega'}$ to $\mathcal{A}_0^O$ are quasi-equivalent ([3] loc. cit.). We can therefore conclude that a state which is locally quasi-equivalent to a quasi-free Hadamard state on $N$ remains locally quasi-equivalent to a quasi-free Hadamard state. Local physical equivalence is proved in proposition 4.3 of [33]. Local quasi-equivalence follows from [81] and the theory is nowhere classical because of the Weyl-relations (3.10) and the fact that the symplectic structure $\sigma$ is not identically 0. \[ \square \]

If we take $\mathcal{A}_M^0$ in the norm topology, then $f \mapsto W(f)$ is not a locally covariant quantum field. Indeed, $\|W(f) - I\| = 2$ for all $f \neq 0$, [13] proposition 5.2.4. However, in the strong operator topology on $\mathcal{H}_\omega$ for any Hadamard state $\omega$, $W(f_n) \to W(f)$ as $f_n \to f$ in $C_0^\infty(M)$ ([13] loc. cit.). Because the locally covariant quantum field theory $\mathbf{A}^0$ is additive and locally quasi-equivalent we could define the strong topology unambiguously on a norm-dense subset of each $\mathcal{A}_M^0$, but we will not pursue this approach further. It is worth noting, however, that $W$, as a locally covariant quantum field, is non-linear and does not satisfy the same equation of motion as $\Phi^0$. 

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Chapter 4

The free Dirac field

One’s ideas must be as broad as Nature if they are to interpret Nature,

Arthur Conan Doyle, A study in scarlet, Ch. 5

After our treatment of the real free scalar field in chapter 3 we now broaden our perspective a little and describe the free Dirac field as a locally covariant quantum field along the same lines. In section 4.1 we present a construction of the classical Dirac field in a four-dimensional globally hyperbolic spacetime, describing the necessary algebraic, group theoretic and geometric aspects in sufficient detail in order to point out some pitfalls (such as the change of spacetime signature, $+---$ or $-+++$) and to correct a few typos that appear in parts of the literature. Our treatment differs from the existing literature by proving that the construction is essentially independent of the chosen representation of the Dirac algebra. More precisely, we will impose certain relations on Dirac spinors and cospinors, concerning their adjoints, charge conjugation and the Dirac operator. Given these relations, different choices of representation give rise to isomorphic Dirac spinor bundles. This shows that the physics is determined entirely by the relations we imposed
and can be described in a coherent and unified (representation-independent) way within the locally covariant framework. It should be noted that [30] discusses a similar idea, namely the independence of the algebras on the choice of representation of the canonical anti-commutation relations. However, it does not seem to consider different representations of the Dirac algebra or to determine the theory by imposing relations between the adjoint map, charge conjugation and the Dirac operator.

Next we will quantise the theory in section 4.2, noting that the distributional and $C^*$-algebraic description in this case coincide. In that section we also describe the class of Hadamard states and show that the Hadamard condition implies the $\mu SC$, exactly as for the real free scalar field. We discuss the causality and time-slice properties of the free Dirac field and we indicate how Majorana spinors can be quantised in the same, representation independent way.

In the final section of this chapter we consider the relative Cauchy evolution of the free Dirac field. For this we use the time-slice axiom to identify the algebra of a neighbourhood of a Cauchy surface $C_{\pm}$ in a spin spacetime $M$ with the algebra of a neighbourhood of a Cauchy surface $C_{\mp}$ in the future of $C_{\pm}$. This identification is a $^*$-isomorphism, which depends on the spin spacetime in between the two regions. A variation of the metric and/or the spin structure in the intermediate region can be encoded in such $^*$-isomorphisms, which is the idea behind the relative Cauchy evolution. We will then consider the functional derivative of the relative Cauchy evolution with respect to the metric and prove a relation between this quantity and the stress-energy-momentum tensor, where we describe the latter using a point-splitting procedure. The relation we obtain is the direct analogue of that which is already known to hold for the free scalar field [16].

For our presentation of the Dirac field in curved spacetime we largely follow [30, 26, 34]; for results on Clifford algebras we refer to [57] chapter 1.
4.1 The classical free Dirac field

The description of the classical Dirac field is much more involved than that of the scalar field. Whereas the classical scalar field is a section of a trivial vector bundle over $M$ (either $M \times \mathbb{C}$ or $M \times \mathbb{R}$), the Dirac field is a section of a four-dimensional complex vector bundle $DM$, the Dirac spinor bundle, that is intimately related to the spacetime geometry. Before we define the Dirac spinor bundle and the Dirac equation (subsection 4.1.3 and 4.1.4 respectively), we will give a review of the Dirac algebra (subsection 4.1.1), i.e. the algebra of gamma-matrices, and the Spin group (subsection 4.1.2). This is necessary in order to prove the representation independence of the Dirac spinor bundle in proposition 4.1.23 as well as to fix our notation and to point out some confusions and typos in the literature.

4.1.1 The Dirac algebra

To add clarity to our description of the Dirac algebra we will take the more general point of view of Clifford algebras at the beginning of this subsection. For a detailed treatment of Clifford algebras we refer to chapter 1 of [57] (but note the difference in sign convention in the Clifford multiplication).

Let $\mathbb{R}^{r,s}$ be the finite dimensional real vector space of dimension $n = r + s$, equipped with a non-degenerate bilinear form $\Omega_{ab}$ which has $r$ positive and $s$ negative eigenvalues. As a special case we note that $M_0 := \mathbb{R}^{1,3}$ is Minkowski spacetime, where the bilinear form is $\eta = \text{diag}(1, -1, -1, -1)$ when expressed in the orthonormal basis $g_a$, $a = 0, 1, 2, 3$, with $\|g_0\|^2 = 1$.

**Definition 4.1.1** The Clifford algebra $Cl_{r,s}$ of $\mathbb{R}^{r,s}$ is defined as the real-linear associative algebra generated by a unit element $I$ and an orthonormal basis $e_a$ of $\mathbb{R}^{r,s}$ subject to the Clifford relations

$$e_a e_b + e_b e_a = 2\Omega_{ab} I. \quad (4.1)$$
The even, respectively odd, subspace of \( \text{Cl}_{r,s} \) is the real-linear space spanned by monomials of even, respectively odd, degree in the basis vectors \( e_a \) and is denoted by \( \text{Cl}^{0}_{r,s} \), respectively \( \text{Cl}^{1}_{r,s} \).

The Dirac algebra \( D := \text{Cl}_{1,3} \) is the Clifford algebra of Minkowski space-time \( M_0 \) and is characterised by

\[
g_a g_b + g_b g_a = 2 \eta_{ab} I. \quad (4.2)
\]

The definition of Clifford algebra is independent of the choice of basis ([57] section 1.1). As a real-linear space \( \text{Cl}_{r,s} \) has a basis consisting of \( I \) and all elements \( e_{a_1} \cdots e_{a_m} \) with \( a_1 < \cdots < a_m, \ m \leq r + s \), which shows that the dimension of \( \text{Cl}_{r,s} \) is \( 2^{r+s} \). The even and odd subspaces are well-defined, because the Clifford relations are purely even. Note that the even subspace \( \text{Cl}^{0}_{r,s} \) is a subalgebra. We will identify \( \mathbb{R}^{r,s} \subset \text{Cl}_{r,s} \) as the subspace of monomials of degree 1 in the basis \( e_a \). In particular we will identify \( M_0 \subset D \).

For convenience we define the volume element \( g_5 \) of the Dirac algebra by \( g_5 := g_0 g_1 g_2 g_3 \). The following lemma lends a geometric interpretation to Clifford multiplication and will allow us to construct the Spin group as a subset of the Dirac algebra in subsection 4.1.2.

**Lemma 4.1.2** We have \( g_5^2 = -I \),

\[
g_5 v g_5^{-1} = -v g_5 g_5^{-1} = -v, \quad v \in M_0. \quad (4.3)
\]

Moreover, if \( u \in M_0 \) has \( u^2 = \|u\|^2 I \neq 0 \), with the norm taken in \( M_0 \), then \( u^{-1} = \frac{1}{\|u\|^2} u \) and \( v \mapsto -uvu^{-1} \) defines a reflection of \( M_0 \) in the hyperplane perpendicular to \( u \).

**Proof.** This follows directly from the Clifford relations (4.2). Indeed, we have \( g_5 e_a = -e_a g_5 \) for each \( a \), which implies equation (4.3) and \( g_5^2 = -I \). For the last claim we compute:

\[
-uvu^{-1} = v - (uv + vu)u^{-1} = v - \frac{2 \langle u, v \rangle}{\|u\|^2} u, \quad v \in M_0.
\]
Definition 4.1.3 A complex representation of the real algebra \( D \) is a real-linear representation \( \pi : D \to M(n, \mathbb{C}) \) for some \( n \in \mathbb{N} \).

In order to characterise the complex representations of the Dirac algebra we first note the following. Using standard arguments with Clifford algebras (theorem I.3.7, equation (I.1.7) and section I.4) we have:

\[
D = Cl_{1,3} \simeq Cl_{1,4}^0 \simeq Cl_{4,1}^0, \quad Cl_{4,1} \simeq M(4, \mathbb{C}).
\]

In fact, \( Cl_{4,1} \) is generated by the generators \( g_a \) of \( D \) together with a central element \( \omega \), which corresponds to the matrix \( iI \in M(4, \mathbb{C}) \), and hence:

\[
M(4, \mathbb{C}) \simeq \mathbb{C} \otimes_{\mathbb{R}} D. \tag{4.4}
\]

This implies that the center of \( D \) is spanned by \( I \) (over \( \mathbb{R} \)). Moreover, it brings the well-known representation theory of \( M(4, \mathbb{C}) \) into the study of complex representations of \( D \). The following fundamental theorem contains what is usually known as Pauli’s theorem [63]. Our method of proof is close to the approach of [57] and is shorter, but less elementary, than Pauli’s (loc. cit.)

Theorem 4.1.4 (Fundamental Theorem) The Dirac algebra \( D \) is simple and has a unique irreducible complex representation, up to equivalence. This is the representation \( \pi_0 : D \to M(4, \mathbb{C}) \) determined by \( \pi_0(g_a) = \gamma_a \) with the Dirac matrices \( \gamma_a \) given by\(^1\)

\[
\gamma_0 := \begin{pmatrix} O & I \\ I & 0 \end{pmatrix}, \quad \gamma_i := \begin{pmatrix} O & -\sigma_i \\ \sigma_i & 0 \end{pmatrix},
\]

\(^1\)This set of gamma-matrices is taken from [50] equation (I.3.45) and is the same as the Weyl or chiral representation of [54] equation (3.25) up to a sign in the \( \gamma_i \).
where $\sigma_i$ are the Pauli matrices:

$$
\sigma_1 := \begin{pmatrix} O & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} O & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The equivalence with another irreducible complex representation $\pi$ of $D$ is implemented by $\pi(S) = L\pi_0(S)L^{-1}$ for all $S \in D$, where $L \in GL(4, \mathbb{C})$ is unique up to a non-zero complex factor.

Consequently, for every set of matrices $\gamma'_a \in M(4, \mathbb{C})$ satisfying equation (4.2) there is an $L \in GL(4, \mathbb{C})$, unique up to a non-zero complex factor, such that

$$
\gamma'_a = L\gamma_a L^{-1}.
$$

Proof. One can show that $D \cong M(2, \mathbb{H})$ ([57] section I.4), which is simple, because it is a full matrix algebra. Indeed, suppose that $J \subset D$ is an ideal which contains a non-zero element $A$. Let $E_{ij}$ denote the matrix whose only non-zero entry is the $(ij)$-entry, which is 1. If the $(i_0j_0)$-entry of $A$ is $a \neq 0$, then $J$ contains $E_{1i0}AE_{j01} + E_{2i0}AE_{j02} = aI$, $I$ being the $2 \times 2$ identity matrix. As $a \in \mathbb{H}$ is invertible we have $I \in J$ and hence $J = D$.

It can be checked by direct computation that the given matrices $\gamma_a$ satisfy the Clifford relations (4.2) and therefore extend to a representation of $D$ in $M(4, \mathbb{C})$ (see [57], chapter I proposition 1.1). Any complex representation $\pi : D \to M(n, \mathbb{C})$ extends to a complex representation $\tilde{\pi}$ of $M(4, \mathbb{C})$ by (4.4), which is irreducible if $\pi$ is irreducible. As $M(4, \mathbb{C})$ has only one irreducible representation up to equivalence ([57] section 16, p.75), this determines $\pi$ up to equivalence, as stated. If $K, L \in GL(4, \mathbb{C})$ are two matrices which implement the same equivalence, then $KL^{-1}$ commutes with $D$ and hence with all of $M(4, \mathbb{C})$ by (4.4). The center of $M(4, \mathbb{C})$ is $\mathbb{C}I$, so we conclude $K = cL$ and $c \in \mathbb{C}$ is non-zero because $K$ is invertible.

Note that $\pi'(g_a) := \gamma'_a$ extends to a complex representation of $D$ in $M(4, \mathbb{C})$. The last statement therefore follows from the previous one. □
For notational consistency we define $\gamma_5 := \pi_0(g_5)$. As special cases of theorem 4.1.4 we now consider the adjoint and complex conjugate matrices, respectively, that will be used in subsection 4.1.3 to define the adjoint and charge conjugation maps on Dirac spinors, respectively.

**Definition 4.1.5** We say that $A, C \in GL(4, \mathbb{C})$ satisfy assumption (4.5) w.r.t. an irreducible complex representation $\pi$ if and only if

\[
A = A^*, \quad \pi(g_a)^* = A\pi(g_a)A^{-1}, \quad A\pi(n) > 0, \quad (4.5)
\]

\[
\overline{CC} = I, \quad -\pi(g_a) = C\pi(g_a)C^{-1}
\]

for all future pointing time-like vectors $n^P$.

Here the condition $A\pi(n) > 0$ means that $A\pi(n) = An^a\gamma_a$ is a positive matrix, i.e. $\langle z, A\pi(n)z \rangle > 0$ for all non-zero $z \in \mathbb{C}^4$. Note that the sets of matrices $\pi(g_a)^*$ and $-\pi(g_a)$, $a = 0, 1, 2, 3$, both satisfy the Clifford relations (4.2), so by theorem 4.1.4 the matrices $A$ and $C$ are uniquely determined up to non-zero complex factors.

**Theorem 4.1.6** For any irreducible complex representation $\pi$ of $D$ there are $A, C \in GL(4, \mathbb{C})$ which satisfy assumption (4.5) w.r.t. $\pi$. The matrix $A$ is uniquely determined up to a positive factor, $C$ up to a phase factor and we have $A = -C^*AC$. Moreover, if $A_i, C_i \in M(4, \mathbb{C})$, $i = 1, 2$, satisfy assumption (4.5) w.r.t. irreducible complex representations $\pi_i$ of $D$, then there is an $L \in GL(4, \mathbb{C})$, unique up to a sign, such that $L^*A_1L = A_2$, $L^{-1}C_1L = C_2$ and $\pi_2 = L^{-1}\pi_1L$ on $D$.

---

2On a general representation space of complex dimension four one can define many complex conjugations $z \mapsto \bar{z}$ and Hermitean inner products $\langle, \rangle$. We desire to obtain certain equalities involving adjoint and charge conjugate spinors in section 4.1.3 which requires the complex conjugation and Hermitean inner product to be compatible: $\langle \bar{w}, \bar{z} \rangle = \overline{\langle w, z \rangle}$. In this case we can use the standard complex conjugation and Hermitean inner product on $\mathbb{C}^4$ without loss of generality.
Proof. This result is essentially already contained in [63]. To prove existence in the representation $\pi_0$ we take $A = A_0 := \gamma_0$, $C = C_0 := \gamma_2$ and check assumption (4.5) by direct computation, using the Clifford relations (4.2). Note for example that
\[
\gamma_0 n^a \gamma_a = \left( \begin{array}{cc}
 n^0 I + n^i \sigma_i & 0 \\
 0 & n^0 I - n^i \sigma_i 
\end{array} \right) > 0,
\]
because $\det(n^0 I \pm n^i \sigma_i) = n^a n_a = 1$ and $Tr(n^0 I \pm n^i \sigma_i) = 2n^0 > 0$. Also, $-C_0^* A_0 C_0 = \gamma_2 \gamma_0 \gamma_2 = -\gamma_0 \gamma_2^2 = \gamma_0 = A_0$. To prove existence in a general irreducible complex representation $\pi$ we use theorem [4.1.4] to write $\gamma_a = K \pi(g_a) K^{-1}$ for some $K \in GL(4, \mathbb{C})$. One can then verify by direct computation that $A = K^* A_0 K$ and $C = K^{-1} C_0 K$ satisfy assumption (4.5) and $A = -C^* A C$. This proves the existence.

The matrices $A$ and $C$ are uniquely determined up to non-zero complex factors $a$ and $c$ by theorem [4.1.4]. Because $A = A^*$ and $C C = I$ we see that $a \in \mathbb{R}$ and $|c| = 1$. Moreover, as $A \pi(n) > 0$ for future pointing time-like vectors we must have $a > 0$. Now, the relation $A = -C^* A C$ is invariant under changes of $a$ and $c$ and we saw that for any $\pi$ there exist matrices $A, C$ satisfying assumption (4.5) and this equality. Therefore any $A, C$ satisfying assumption (4.5) w.r.t. $\pi$ necessarily satisfy this equality.

Given matrices $A_i, C_i \in GL(4, \mathbb{C})$ for the representations $\pi_i$, $i = 1, 2$, we can fix $K \in GL(4, \mathbb{C})$ such that $\pi_1 = K \pi_2 K^{-1}$ on $D$ by the fundamental theorem [4.1.4]. Setting $A'_2 := K^* A_1 K$ and $C'_2 := K^{-1} C_1 K$ we can verify by direct computation that $A'_2$ and $C'_2$ satisfy assumption (4.5) w.r.t. $\pi_2$, as in the first paragraph of this proof. By the uniqueness this means that $A'_2 = a A_2$ and $C'_2 = c C_2$ for some $a > 0$, $|c| = 1$. The desired matrix $L$ must be $L = z K$ for some $z \neq 0$, by the fundamental theorem. To get the right intertwining relations for $A_i$ and $C_i$ we need $|z|^2 = a$ and $z = c \bar{z}$, which fixes $z$ up to a sign.

□
As another application of the fundamental theorem we can introduce a determinant and trace on $D$:

**Definition 4.1.7** The determinant and trace functions on $D$ are defined by $\det S := \det \pi(S)$ and $\Tr(S) := \Tr(\pi(S))$ for all $S \in D$, where $\pi$ is any irreducible complex representation of $D$.

This is well-defined by the fundamental theorem. The following lemma will be useful in what follows:

**Lemma 4.1.8** $\Tr(g_ag_b) = 4\eta_{ab}$ and $\Tr([g_b, g_c] g_d g_a) = 8(\eta_{cd} \eta_{ba} - \eta_{bd} \eta_{ca})$.

**Proof.** Using the cyclicity of the trace and the Clifford relations [4.2] we find:

$$\Tr(g_ag_b) = \frac{1}{2} \Tr(g_ag_b + g_bg_a) = \Tr(\eta_{ab} I) = 4\eta_{ab}$$

and

$$\Tr([g_b, g_c] g_d g_a) = \Tr(g_b \{g_c, g_d\} g_a - g_b g_d \{g_c, g_a\})$$

$$= 2 \Tr(\eta_{cd} g_b g_a - g_b g_d \eta_{ca}) = 8(\eta_{cd} \eta_{ba} - \eta_{bd} \eta_{ca}).$$

□

### 4.1.2 The Spin$_{1,3}$ group

We now turn to the Spin group, which is the universal covering group of the proper Lorentz group and which can be constructed in an elegant way as a subset of the Dirac algebra.

**Definition 4.1.9** The Pin and Spin groups of $Cl_{r,s}$ are defined as

$$\text{Pin}_{r,s} := \left\{ S \in Cl_{r,s} \mid S = u_1 \cdots u_k, \ k \in \mathbb{N}, \ u_i \in \mathbb{R}^{r,s}, \ u_i^2 = \pm I \right\},$$

$$\text{Spin}_{r,s} := \text{Pin}_{r,s} \cap \text{Cl}^0_{r,s}.$$
We also define the Lorentz group $\mathcal{L} := O_{1,3}$, the proper Lorentz group $\mathcal{L}^+ := SO_{1,3}$ and the proper orthochronous Lorentz group $\mathcal{L}_+^1 := SO^0_{1,3}$, which is the connected component of $\mathcal{L}_+^1$ containing the identity.

The proper orthochronous Lorentz group preserves the time-orientation as well as the orientation. Note that $\text{Pin}$ is indeed a group and that $I \in \text{Pin}_{r,s}$ because of the following equivalent characterisation (cf. [20] p.66 and p.334):

**Proposition 4.1.10** $\text{Pin}_{1,3} = \{ S \in D | \det S = 1, \forall v \in M_0 \ S v S^{-1} \in M_0 \}$.

**Proof.** For $S \in \text{Pin}_{1,3}$ the map $v \mapsto S v S^{-1}$ on $M_0$ is a product of reflections (up to a sign), by lemma 4.1.2, so $S v S^{-1} \in M_0$ for all $v \in M_0$. Because $\det u = \| u \|^4 I$ for all $u \in M_0$, which can be verified by direct computation, we also have $\det S = 1$.

For the converse we suppose that $S \in D$ has $\det S = 1$ and $S v S^{-1} \in M_0$ for all $v \in M_0$. Notice that the adjoint action of $S$ is a linear map on $M_0$ which preserves the Lorentzian inner product, because it preserves the right-hand side of the equality $v w + w v = 2 v^a w^a I$. Hence, the adjoint action of $S$ determines a Lorentz transformation $\Lambda$, which can be written as a finite product of reflections in non-null hyperplanes ([4] theorem 3.20). Let $u_i$ be unit normal vectors to these hyperplanes, where $1 \leq i \leq k$ for some $k$. If $k$ is even we let $T := u_k \cdots u_1$ and otherwise we let $T := u_k \cdots u_1 g_5$. Notice that $T \in \text{Pin}_{1,3}$ and that in both cases we have $S v S^{-1} = \Lambda(v) = T v T^{-1}$.

---

3The definition of the Spin group in [20] corresponds to our group $\text{Pin}_{1,3}$. In [30] and [34] one uses the term Spin group for the group

$$S := \{ S \in M(4, \mathbb{C}) | \det S = 1, \ S v S^{-1} \in M_0 \text{ for all } v \in M_0 \}.$$ 

Note that this group cannot give a double covering of the Lorentz group, as claimed in [30] (but not in [34]), because for any $S \in S$ the matrices $i S, - S, - i S$ are in $S$ too. Its usefulness is based on its simple definition and the fact that $S^0 = Spin^0_{1,3}$.
by lemma 4.1.2 (Each reflection is given by $v \mapsto -u_i v u_i$ and we used $g_5$ to cancel the extra sign in case of an odd number of reflections.) In particular, $T^{-1} S g_a S^{-1} T = g_a$, so by theorem 4.1.4 we have $T^{-1} S = c I$ and hence $S = c S$ for some non-zero $c \in \mathbb{C}$. Because $S, T \in D$ we must have $c \in \mathbb{R}$ by equation (4.4). Moreover, we have $d S = 1$ as in the first paragraph and $d T = 1$ by assumption, so $S = \pm T$. Finally, $-T = (g_1)^2 T \in Pin_{1,3}$ too, so in any case $S \in Pin_{1,3}$.

It can be seen from proposition 4.1.10 that $Pin_{1,3}$ and $Spin_{1,3}$ are indeed Lie groups, using the embedding of equation (4.4). We let $Spin^0_{1,3}$ denote the connected component of $Spin_{1,3}$ which contains the identity. We now prove the following lemma concerning the Lie algebras of these Lie groups (cf. [57] proposition I.6.1):

**Lemma 4.1.11** The Lie algebras $spin^0_{1,3} = spin_{1,3} = pin_{1,3}$ are spanned by $g_a g_b, 0 \leq a < b \leq 3$.

**Proof.** We consider the curves $c_i : [0, 1] \to Spin^0_{1,3}$ for $i = 1, 2, 3$ and $d_{ij} : [0, 1] \to Spin^0_{1,3}$ for $1 \leq i < j \leq 3$ all starting at $I$ and defined by

\[
    c_i(t) := g_0 (\cosh(t) g_0 + \sinh(t) g_i) \\
    d_{ij}(t) := -g_i (\cos(t) g_i - \sin(t) g_j).
\]

The derivatives of these curves at $t = 0$ are the six linearly independent elements $g_a g_b$ with $0 \leq a < b \leq 3$. Conversely, for every curve $S(t) \subset Pin_{1,3}$ with derivative $s$ at $t = 0$ the condition $S(t) g_a S^{-1}(t) \in M_0$ for all $t$ (see proposition 4.1.10) implies $[s, g_a] \in M_0$. If we express $s$ as a real-linear combination of products of $g_a$’s then an elementary computation shows that this condition implies $s = \alpha_0 I + \alpha^{ab} g_a g_b$ for some $\alpha_0, \alpha^{ab} \in \mathbb{R}$. The condition $\det S(t) = 1$ implies $Tr(s) = 0$ and hence $\alpha_0 = 0$. We conclude that all three Lie algebras are equal and spanned by the given elements. \qed
After these preparations we can now turn to the relation between the Pin group and the Lorentz group. We define a mapping \( \Lambda : \text{Pin}_{1,3} \to \mathcal{L} \) by \( S \mapsto \Lambda^a_b(S) \) such that

\[
S g_b S^{-1} = g_a \Lambda^a_b(S).
\]

(4.6)

The matrix \( \Lambda^a_b(S) \) exists by proposition 4.1.10 is unique and determines a Lorentz transformation because the adjoint action of \( S \) leaves the right-hand side of \( vw + wv = 2v^a w_a I \) invariant.

**Proposition 4.1.12** The map \( \Lambda \) defined in equation (4.6) is a surjective double covering homomorphism of Lie groups, which restricts to a double covering homomorphism \( \text{Spin}^0_{1,3} \to \mathcal{L}^+_+ \). We have:

\[
\Lambda^a_b(S) = \frac{1}{4} \eta^{ac} Tr (g_c S g_b S^{-1}),
\]

\[
\Lambda^a_b(S^{-1}) = \eta^{ac} \eta_{bd} \Lambda^d_c(S),
\]

\[
(d\Lambda)^{-1}(\lambda^a_b) = \frac{1}{4} \lambda^a_b \eta^{ac} g_b g_c,
\]

where \( d\Lambda \) is the derivative \( d\Lambda : \text{spin}^0_{1,3} \to \mathcal{L}^+_+ \) at \( S = I \).

**Proof.** (Cf. [57] theorem I.2.10.) To check the homomorphism property we note that \( \Lambda^a_b(I) = \delta^a_b \) by (4.6) and for \( S, T \in \text{Pin}_{1,3} \):

\[
g_a \Lambda^a_c(ST) = ST g_c T^{-1} S^{-1} = g_a \Lambda^b_c(T) S^{-1}
\]

\[
= S g_b S^{-1} \Lambda^c_b(T) = g_a \Lambda^a_b(S) \Lambda^b_c(T)
\]

and hence \( \Lambda^a_c(ST) = \Lambda^a_b(S) \Lambda^b_c(T) \). Next we compute

\[
\Lambda^a_b(S) = \frac{1}{4} \eta^{ac} Tr (\eta_{cd} \Lambda^d_b(S) I) = \frac{1}{8} \eta^{ac} Tr ((g_c g_d + g_d g_c) \Lambda^d_b(S))
\]

\[
= \frac{1}{4} \eta^{ac} Tr (g_c g_d \Lambda^d_b(S)) = \frac{1}{4} \eta^{ac} Tr (g_c S g_b S^{-1}),
\]

and hence also

\[
\Lambda^a_b(S^{-1}) = \frac{1}{4} \eta^{ac} Tr (g_c S^{-1} g_b S) = \frac{1}{4} \eta^{ac} \eta_{bd} \eta^{de} Tr (g_e S g_c S^{-1}) = \eta^{ac} \eta_{bd} \Lambda^d_c(S).
\]

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(Of course this also follows from the fact that $\Lambda$ is a group homomorphism and that $(\Lambda^{-1})^a_b = \eta^{ac} \eta_{bd} \Lambda^d_c$ for $\Lambda \in \mathcal{L}$.) To find $d\Lambda$ we expand $\Lambda(S)$ for $S = I + \epsilon s + O(\epsilon^2)$ up to second order in $\epsilon$:

$$
\Lambda^a_b(S) = \frac{1}{4} \eta^{ac} Tr(g_c(I + \epsilon s)g_b(I - \epsilon s)) + O(\epsilon^2)
$$

$$
= \Lambda^a_b(I) + \frac{\epsilon}{4} \eta^{ac} Tr([g_b, g_c] s) + O(\epsilon^2),
$$

where we used the cyclicity of the trace. We can now immediately read off $d\Lambda^a_b(s) = \frac{1}{4} \eta^{ac} Tr([g_b, g_c] s)$. Notice that $\dim l^+_1 = 6 = \dim spin^0_{1,3}$ (real dimensions). We will show that the map $L : l^+_1 \to spin^0_{1,3}$ defined by

$$
L(\lambda^a_b) := \frac{1}{4} \lambda^a_b \eta^{bc} g_a g_c
$$

is an inverse of $d\Lambda^a_b$. First note that $\lambda^a_b \eta^{bc} + \lambda^c_b \eta^{ba} = 0$ for $\lambda^a_b \in l^+_1$, so $L$ is in the linear span of $g_a g_b$ with $a < b$ and hence $L$ takes values in $spin^0_{1,3}$ by lemma 4.1.11. Now we use lemma 4.1.8 to compute:

$$
d\Lambda^a_b(L(\lambda^d_e)) = \frac{1}{16} \eta^{ac} \lambda^d_e \eta^{ef} Tr([g_b, g_c] g_d g_f) = \frac{1}{2} \eta^{ac} \lambda^d_e \eta^{ef} (\eta_{cd}\eta_{bf} - \eta_{bd}\eta_{cf})
$$

$$
= \frac{1}{2} (\lambda^a_b - \eta^{ac} \eta_{bd} \lambda^d_e) = \lambda^a_b,
$$

where we used the symmetry properties of $\lambda^d_e$ again in the last line.

Because $d\Lambda$ is invertible $\Lambda$ is a local diffeomorphism (using the inverse function theorem). The surjectivity follows as in the proof of proposition 4.1.10 by expressing any Lorentz transformation $\Lambda \in \mathcal{L}$ as a finite product of reflections in non-null hyperplanes [4].

To find the kernel of $\Lambda$ we suppose that $S \in Pin_{1,3}$ has $\Lambda^a_b(S) = \delta^a_b$. Then, by definition, $Sg_a = g_a S$ and $S = cI$ by the fundamental theorem 4.1.4. As $S \in D$ we see that $c$ must be real by equation (4.4). By proposition 4.1.10 we have $1 = \det S = c^4$, so $c = \pm 1$ and $S = \pm I$. Note that $I = g_0^2$ and $-I = g_1^2$ are both in $Spin^0_{1,3} \subset Pin_{1,3}$, so the kernel of $\Lambda$ is $\{I, -I\}$. It now
follows that $\Lambda$ restricts to a local diffeomorphism of $\text{Spin}_{1,3}^0$ onto $\mathcal{L}_+^1$, which also is a double covering.

One can show that $\Lambda$ and its restrictions to $\text{Pin}_{1,3}$, $\text{Spin}_{1,3}$, $\text{Spin}_{0,3}$ are the universal coverings of $\mathcal{L}$, $\mathcal{L}_+$ and $\mathcal{L}_+^1$, respectively and all of these are double coverings (see [57] chapter I theorem 2.10 and the remarks below).

In the next subsection we will need the following lemma, which establishes a relationship between $\text{Spin}_{1,3}^0$ and matrices satisfying assumption (4.5):

**Lemma 4.1.13** Let $\pi$ be a complex irreducible representation of $D$ and let $A, C \in \text{GL}(4, \mathbb{C})$ satisfy assumption (4.5) w.r.t. $\pi$. Then for all $S \in \text{Spin}_{1,3}^0$:

$\pi(S)^* A \pi(S) = A, \quad \pi(S^{-1}) C^{-1} \overline{\pi(S)} = C^{-1}.$

**Proof.** For a unit vector $u = u^a g_a$ we have $u^2 = \|u\|^2 I = \pm I$ and hence

$\pi(u)^* A \pi(u) = u^a u^b \pi(g_a)^* A \pi(g_b) = u^a u^b A \pi(g_a g_b) = A \pi(u^2) = \pm A.$

By definition 4.1.9 we must therefore have $\pi(S)^* A \pi(S) = \pm A$ for $S \in \text{Pin}_{1,3}$.

If $S = I$ the sign is a plus, so by continuity we conclude that $\pi(S)^* A \pi(S) = A$ for all $S \in \text{Spin}_{1,3}^0$. For $C$ we use the fact that for $u \in M_0$

$\pi(u^{-1}) C^{-1} \overline{\pi(u)} = -\pi(u^{-1}) \pi(u) C^{-1} = -C^{-1}$

and hence $\pi(S^{-1}) C^{-1} \overline{\pi(S)} = C^{-1}$ for all $S \in \text{Spin}_{1,3}$, because $S$ is a product of an even number of $u$'s.

Note that $g_5 \in \text{Spin}_{1,3} \setminus \text{Spin}_{1,3}^0$. Indeed, using $\pi_0$ and $A = A_0 = \gamma_0$ in lemma 4.1.13 we see that $\gamma_5^* A_0 \gamma_5 = -A_0$, so $g_5$ is in $\text{Spin}_{1,3}$ by definition, but not in $\text{Spin}_{1,3}^0$ by the lemma.

### 4.1.3 The Dirac spinor and cospinor bundles

After presenting the algebraic and group theoretical background information in the previous subsections we will now start the formulation of the classical
Dirac field in curved spacetime. Our first task will be to construct the vector bundles in which the Dirac spinor and cospinor fields take values. For this purpose we choose an irreducible complex representation $\pi$ of $D$ and matrices $A, C \in GL(4, \mathbb{C})$ satisfying assumption [1.5]. Such matrices exist by theorem 4.1.6 and we will show afterwards, in proposition 4.1.23, that different choices give rise to equivalent constructions.

Let $M = (\mathcal{M}, g, SM, p)$ be a globally hyperbolic spin spacetime. We define the associated vector bundle

$$DM := SM \times_{Spin_{0,3}} \mathbb{C}^4,$$

where $Spin_{0,3}$ acts on $SM$ from the right as usual and on $\mathbb{C}^4$ from the left via the representation $\pi$. In other words, $DM$ is obtained from the product bundle $SM \times \mathbb{C}^4$ by identifying

$$[E, z] = [R_S E, \pi(S^{-1})z],$$

where we think of $z \in \mathbb{C}^4$ as a column vector. (Recall that $R_S$ denotes the right action of the group $Spin_{0,3}$ on the principal vector bundle $SM$, see definition 2.3.1.) We denote the dual vector bundle by $D^*M$ and note the equivalence relation $[E, w^*] = [R_S E, w^* \pi(S)]$, where we used the standard anti-isomorphism $w \mapsto w^* := \langle w, . \rangle$ between $\mathbb{C}^4$ and its dual $(\mathbb{C}^4)^*$ and we treat $w^*$ as a row vector. There is then a canonical pairing of the fibers of $DM^*$ and $DM$ over any point in $\mathcal{M}$, which is given by:

$$\langle [E, w^*], [E, z] \rangle := w^*(z) = \langle w, z \rangle.$$

Note that the element $E$ must be the same in both entries. This can always be accomplished by using the equivalence relation of $DM$ or $D^*M$, because the action of $Spin_{0,3}$ on each fiber of $SM$ is transitive.

---

4The claim of [34] that the map $L_S[E, z] := [E, \pi(S)z]$ defines a left action is to be understood as follows. If we fix the local section $E$ of $SM$, i.e. if we choose a local gauge, then the right-hand side is well-defined and defines a left action.
**Definition 4.1.14** The vector bundle $DM$ is the (Dirac) spinor bundle, its elements are (Dirac) spinors and a section of it is a (Dirac) spinor field. The space of all smooth spinor fields is denoted by $C^\infty(\mathcal{M})$, and the space of all compactly supported smooth spinor fields by $C^\infty_0(\mathcal{M})$.

The dual vector bundle $D^*M$ of $DM$ is called the Dirac cospinor bundle, its elements are (Dirac) cospinors and a section of it is a (Dirac) cospinor field. The space of all smooth cospinor fields is denoted by $C^\infty(D^*M)$ and that of the compactly supported smooth cospinor fields by $C^\infty_0(D^*M)$.

We indicate the canonical pairing of a spinor field $u$ and a cospinor field $v$ by writing them next to each other: $vu(x) := \langle v(x), u(x) \rangle$. This pairing therefore defines a sesquilinear map $C^\infty(D^*M) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$. As a matter of notation we will write $-\lbrack E, z \rbrack := \lbrack E, -z \rbrack$ and $-\lbrack E, z^* \rbrack := \lbrack E, -z^* \rbrack$, which is well-defined because $-I$ commutes with the action of $\pi(D)$ on $\mathbb{C}^4$.

We now turn to the adjoint and charge conjugation maps. We first define these maps for spinors and cospinors, then for spinor and cospinor fields.

**Lemma 4.1.15** We can define maps $^+ : DM \to D^*M$, $^+ : D^*M \to DM$, $^c : DM \to DM$ and $^c : D^*M \to D^*M$ by:

$$[E, z]^+ := \lbrack E, z^* A \rbrack \quad [E, z^*]^+ := \lbrack E, A^{-1} z \rbrack$$

$$[E, z]^c := \lbrack E, C^{-1} z \rbrack, \quad [E, z^*]^c := \lbrack E, z^* C \rbrack.$$

These maps are base-point preserving vector bundle anti-isomorphisms. For $q = [E, z] \in DM$ and $p = [E, w^*] \in D^*M$ we have:

$$q^{++} = q = q^{cc} \quad p^{++} = p = p^{cc}$$

$$q^{+c} = -q^{c^+} \quad p^{+c} = -p^{c^+}$$

$$\langle q^+, p^+ \rangle = \overline{\langle p, q \rangle} = \langle p^c, q^c \rangle.$$

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Proof. It follows from lemma 4.1.13 that the maps \( + \) and \( c \) are well-defined. As an example we compute for all \( S \in \text{Spin}_{1,3}^0 \):

\[
[R_S E, \pi(S^{-1})z]^+ = [R_S E, z^*\pi(S^{-1})^*A] = [R_S E, z^*A\pi(S)] = [E, z^*A] = [E, z]^+.
\]

The proof for the other maps is similar. By their definition the maps \( + \) and \( c \) are seen to be antilinear and to preserve the base-point. That they are isomorphisms follows from the relations (4.7), which we will prove next.

From assumption (4.5) we see that \( A = A^* \), \( C^{-1}\overline{C}^{-1} = (\overline{CC})^{-1} = I \) and \( \overline{CC} = I \), so for any \( q = [E, z] \) in \( DM \) and \( p = [E, w] \) in \( D^*M \) we find:

\[
q^{++} = [E, z^*A]^+ = [E, A^{-1}A^*z] = [E, z] = q
\]

\[
p^{++} = [E, A^{-1}w]^+ = [E, w^*(A^{-1})^*A] = [E, w^*] = p
\]

\[
q^{cc} = [E, C^{-1}z]^c = [E, C^{-1}\overline{C}^{-1}z] = [E, z] = q
\]

\[
p^{cc} = [E, \overline{w}^*C]^c = [E, w^*\overline{C}C] = [E, w^*] = p.
\]

From theorem 4.1.6 we find that \( A = -C^*\overline{A}C \), and hence

\[
q^{c+} = [E, z^*A]^c = [E, \overline{z}^*\overline{A}C] = -[E, \overline{z}^*(C^*)^{-1}A] = -[E, C^{-1}z]^+ = -[E, z]^{c+} = -q^{c+}.
\]

The result \( p^{c+} = -p^{c+} \) now follows, because \( p = q^+ \) for some \( q \) and hence \( p^{c+} = q^c \) whereas \( p^{c+} = q^{c+} = -q^{c+} = -q^c \). Finally, \( \langle p, q \rangle = w^*(z) \) and hence:

\[
\langle q^+, p^+ \rangle = \langle [E, z^*A], [E, A^{-1}w] \rangle = z^*(w) = \overline{w^*(z)} = \overline{\langle p, q \rangle}
\]

\[
\langle p^c, q^c \rangle = \langle [E, \overline{w}^*C], [E, C^{-1}z] \rangle = \overline{w^*(z)} = \overline{w^*(z)} = \overline{\langle p, q \rangle}.
\]

\( \square \)
Definition 4.1.16 The maps $+^c$ are called the (Dirac) adjoint maps and the maps $c$ are called the charge conjugation maps.

For spinor and cospinor fields we define the adjoint maps and the charge conjugation maps pointwise.

This means that for $u \in C^\infty(DM)$ we have e.g. $u^+(x) := u(x)^+$. Notice that the adjoint and charge conjugation maps preserve the support. The identities (4.7) of lemma 4.1.15 translate as:

\[
\begin{align*}
  u^{++} &= u = u^{cc} & v^{++} &= v = v^{cc} \\
  u^{+c} &= -u^{c+} & v^{+c} &= -v^{c+} \\
  u^+ v^+ &= \overline{vu} = v^c u^c.
\end{align*}
\] (4.8)

Taking tensor products of $DM, D^* M, TM, T^* M$ we can form a mixed spinor-tensor algebra in a natural way. In order to perform computations in this mixed spinor-tensor algebra it will be useful to work in suitable local frames, which we will now describe.

Given a local section $E$ of $SM$ and an orthonormal basis $b_A$ of $\mathbb{C}^4$ such that $\overline{b_A} = b_A$ we obtain local frames $E_A := [E, b_A]$ of $DM$ and $p \circ E = e = \{e_a\}_{a=0,...,3}$ of $TM$, where $p : SM \to FM$ is the projection of the spin structure. We denote the dual frames of $e_a$ and $E_A$ by $e^b$ and $E^B$, respectively, so that $e^b(e_a) = \delta^b_a$ and $E^B E_A = \delta^B_A$, where the Kronecker $\delta$'s are regarded as constant functions on $M$. Together these local frames give rise to local frames for the spinor-tensor algebra.

A different local section $E'$ of $SM$ over the same region $O \subset M$ can always be expressed as $E' = R_{S^{-1}} E$, where we allow $S$ to depend on $x \in O$, i.e. $S : O \to Spin^0_{1,3}$. To find the corresponding change of frames we compute:

\[
E'_A = [E', b_A] = [R_{S^{-1}} E, b_A] = [E, b_B \pi(S^{-1})^B_A] = E_B \pi(S^{-1})^B_A.
\]

It then follows that

\[
E'_A = E_B \pi(S^{-1})^B_A, \quad (E')^A = \pi(S)^A_B E^B
\]
\[ e'_a = e_b \Lambda(S^{-1})_a^b, \quad (e')^a = \Lambda(S)^a_b e^b. \]

The components of a spinor \( u = E_A u^A = E'_A (u')^A \) transform under a change of section as \( (u')^A = \pi(S)^A_B u^B \) and for general spinor-tensors we get similar expressions, e.g.

\[ (T')^A_a = \pi(S)_C^A \Lambda(S)^a_c \pi(S^{-1})^D_B \Lambda(S^{-1})^d_b T^C_d. \]

For the frame \( e_a \) of \( TM \) we can use \( g_{\mu \nu} e^\mu_a e^\nu_b = \eta_{ab} \) to derive

\[ e^\mu_a = g_{\mu \nu} \eta^{ab} e^\nu_b, \quad (4.9) \]

because both sides have the same action on basis vectors. It follows that \( \eta_{ab} e^\mu_a e^\nu_b = g_{\mu \nu}, \) \( g^{\mu \nu} e^\mu_a e^\nu_b = \eta^{ab} \) and \( \eta^{ab} e^\mu_a e^\nu_b = g^{\mu \nu}. \) A vector \( x \) can be expressed as \( x = x^a e_a \) where \( x^a := e^\mu_a x^\mu \) and similarly for covectors. We see that we can raise and lower indices in the frame \( e_a \) with \( \eta^{ab} \) and \( \eta_{ab}. \)

Using the matrix expressions of \( A, A^{-1}, C^{-1}, C \) in the bases \( b_A, b^A \) we can also express the adjoint and charge conjugation maps in components.

Because of \( E^+_A = \delta_{AB} A^B C^C E^C \) and \( E^+_A = E_B (C^{-1})^B_A \) we find:

\[ u^+_A = \overline{u^C} \delta_{CB} A^B_A, \quad (u^c)^A = (C^{-1})^A_B u^B, \]
\[ (v^+)^A = (A^{-1})^B_A \delta^{BC} \overline{v^C}, \quad v^c_A = \overline{v^B} C^B_A. \]

**Lemma 4.1.17** There is a smooth section \( \gamma \in C^\infty(T^*M \otimes DM \otimes D^*M) \) such that:

\[ \gamma = \gamma^B_A e^a \otimes E^A_B \otimes E^A, \]

for every local section \( E \) of \( SM \), where \( \gamma^B_A \) denotes the entries of \( \pi(g_a) \) in the basis \( b^A \) and its dual basis \( b_B \).

**Proof.** Using a different local section \( E' = R_{S^{-1}} E \) and, using the definitions of \( E^A, E_B, e^a \) and that of \( \Lambda \) in equation \( (4.6) \), we see that we can define \( \gamma \) locally by the given expression, independent of the choice of \( E \). Covering the
manifold by suitable regions (e.g. contractible ones) we can then extend $\gamma$ to a global section, which is given by the formula in any local frame constructed from a local section $E$ of $SM$.

When we perform computations in components we will ease the notation considerably by dropping the spinorial (capital) indices and using a matrix notation instead, whenever this is possible. In this notation we think of spinors as column vectors and cospinors as row vectors, so $\gamma^A_B u^B$ becomes $\gamma_a u$ and $v_A \gamma^A_B$ becomes $v \gamma_a$. This should cause no confusion, as long as we remember which objects carry spinor indices and we are careful with the non-commutative matrix products. This only works as long as no object carries more than one upper or lower spinorial index, but this will usually be the case in what follows. For vector fields $v$ and covector fields $k$ we also introduce the Feynman slash notation: $\not{v} := v^a \gamma_a$, $\not{k} := k_a \gamma^a$.

4.1.4 The spin connection, Dirac operator and Dirac equation

The dynamics of the free Dirac field is described by a partial differential equation which contains a covariant derivative for sections of the Dirac spinor bundle $DM$. There is a natural choice of a connection for this bundle, which is related to the Levi-Civita connection on the tangent bundle $TM$. The latter is therefore the starting point of this subsection.

Let $\nabla$ be the Levi-Civita connection, i.e. the unique connection on $TM$ which is torsion free and compatible with the metric. In local coordinates we can define the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ in terms of the coordinate derivatives through

$$\nabla v = (\nabla_\mu v^\rho) dx^\mu \otimes \frac{\partial}{\partial x^\rho} = (\partial_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu) dx^\mu \otimes \frac{\partial}{\partial x^\rho}.$$
These Christoffel symbols are given by the expression (3.1.30)

\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \]  

(4.10)

We may define connection coefficients \( \Gamma^a_{bc} \) for any local frame \( \{ e_a \}_{a=0,...,3} \) through

\[ \nabla v = (\nabla_b v^a) e^b \otimes e_a = (\partial_b v^a + \Gamma^a_{bc} v^c) e^b \otimes e_a, \]  

(4.11)

where \( \partial_b = e^\mu_b \partial_\mu \) denotes the action of the vector field \( e_b \) as a derivative. The connection coefficients \( \Gamma^a_{bc} \) may be compared to the Christoffel symbols in a coordinate basis\(^5\), which yields:

\[ \partial_b v^a + \Gamma^a_{bc} v^c = (\partial_\mu v^a + \Gamma^a_{\mu\nu} v^\nu) e^\mu_b = \partial_b (v^a e^\mu_b) - v^\nu \partial_b e^a_\nu + \Gamma^a_{\mu\nu} v^\nu e^\mu_b e^a_\nu \]

and hence, using \( \partial_b (e^a_\rho e^\rho_c) = \partial_b \delta^a_c = 0 \),

\[ \Gamma^a_{bc} = -e^c_\rho \partial_b e^a_\rho + e^a_\rho e^\mu_b e^\nu_c \Gamma^\rho_{\mu\nu} = e^a_\rho \partial_b e^\rho_c + e^a_\rho e^\mu_b e^\nu_c \Gamma^\rho_{\mu\nu}. \]  

(4.12)

Equivalently the Levi-Civita connection can be described by the connection one-forms \( \omega^a_c := \Gamma^a_{bc} e^b_c \). These one-forms can be regarded as a single one-form taking values in \( l^+ \), because equation (4.10) implies:

\[ g_{\tau\rho} \Gamma^\rho_{\mu\nu} + g_{\nu\rho} \Gamma^\rho_{\mu\tau} = \partial_\mu g_{\tau\nu} \Rightarrow \]

\[ \eta_{da} \Gamma^a_{bc} + \eta_{ca} \Gamma^a_{bd} = -e^\rho_c \partial_b (e^a_d g_{\rho\sigma}) + e^a_c g_{\sigma\rho} \partial_b e^\rho_d + e^\tau_d e^\mu_c e^\nu_c (g_{\tau\rho} \Gamma^\rho_{\mu\nu} + g_{\nu\rho} \Gamma^\rho_{\mu\tau}) \]

\[ = -e^\rho_c e^\sigma_d \partial_b g_{\rho\sigma} + e^\rho_c e^\sigma_d \partial_b g_{\sigma\rho} = 0. \]  

(4.13)

To find the spin connection on \( DM \) we use a third equivalent description of the Levi-Civita connection in terms of the principal \( l^+_1 \)-bundle \( FM \). Indeed, there is an \( l^+_1 \)-valued one-form \( \Omega^a_c \) on \( FM \) such that for every local section \( e \) of \( FM \) the pull-back satisfies \( e^* \Omega^a_c = \omega^a_c \) and which behaves in a particular way under the action of \( L^+_1 \). We refer to [55] chapter 2 proposition

\(^5\)It is important to note that our indices are not abstract indices, that only indicate the type of a tensor, but actually number the specific vector fields of a frame.
1.1 for the detailed description of this behaviour, because it is not essential for our discussion. The one-form $\Omega^{a}_{c}$ can be pulled back by $p: SM \rightarrow FM$ and lifted from $l^{1}_{+}$ to $spin_{1,3}^{0}$, which yields a $spin_{1,3}^{0}$-valued one-form $\Sigma$ on $SM$:

$$\Sigma := (d\Lambda)^{-1} p^{*}(\Omega^{a}_{c}) = \frac{1}{4} p^{*}(\Omega^{a}_{c}) \gamma^{a} \gamma^{c}$$

(see proposition 4.1.12), where we used $\gamma^{c} = \eta^{cb} \gamma^{b}$. Because $p$ intertwines the actions of the structure groups appropriately $\Sigma$ defines a connection on $DM$ (see [55] loc. cit.), which we call the spin connection. In a local section $E$ of $SM$ the spin connection one-forms $\sigma^{A}_{b} C$ are given by the pull-back of $\Sigma$ by $E$. Because of $E^{*}p^{*} = (p \circ E)^{*} = e^{*}$ we obtain:

$$\sigma^{A}_{b} = \frac{1}{4} \Gamma^{a}_{bc} \gamma^{a} \gamma^{c}. \quad (4.14)$$

We define the covariant derivative of spinor fields $u$ by:

$$\nabla u = (\nabla_{b} u^{A}) e^{b} \otimes E_{A} = (\partial_{b} u^{A} + \sigma^{A}_{b} C u^{C}) e^{b} \otimes E_{C},$$

and for cospinor fields $v$ via $\partial_{a} (vu) = \nabla_{a} vu + v \nabla_{a} u$, i.e.

$$\nabla v = (\nabla_{b} v^{C}) e^{b} \otimes E^{C} = (\partial_{b} v^{C} - v_{A} \sigma^{A}_{b} C) e^{b} \otimes E^{C},$$

In components, using the shorthand matrix notation, these definitions read:

$$\nabla_{b} u = \partial_{b} u + \sigma_{b} u, \quad \nabla_{b} v = \partial_{b} v - v \sigma_{b}. \quad (4.15)$$

We can now define a covariant derivative $\nabla$ for mixed spinor-tensors as follows. For spinorial indices we use the spin connection as in equation (4.15). For tensor indices we use the Levi-Civita connection as usual. The following lemma gives a typical illustration:

---

6Note the mistaken sign in the expression for the spin connection in [30] [34]. With the wrong sign we do not obtain a connection on $SM$, because one of the properties of [55] proposition 1.1 is not satisfied, and lemma 4.1.18 would no longer hold.

7Here again it is important that our indices are not abstract indices, but denote the components in specific frames.
Lemma 4.1.18 The section $\gamma$ is covariantly constant.

Proof. Because the coefficients $\gamma^A_B$ in a local frame are constant we have

$$\nabla_b \gamma^A_B = \sigma_b^D \gamma^A_B - \Gamma^c_{ba} \gamma^A_B - \sigma^D_b \gamma^A_D,$$

or, dropping the spinor indices and using equation (4.14):

$$\nabla_b \gamma^a = \sigma_b \gamma^a - \gamma_a \sigma_b - \Gamma^c_{ba} \gamma_c = \frac{1}{4} \Gamma^c_{bd} (\gamma^d \gamma_a - \gamma_a \gamma^d) - \Gamma^c_{ba} \gamma^a = - \frac{1}{2} \Gamma^c_{bd} (\delta^d_a \gamma_c + \eta_{ac} \gamma^d) = 0.$$

Here we used the Clifford relations (4.2), which also hold pointwise for the section $\gamma^a$, and the anti-symmetry of the $l^1_{\downarrow}$-valued connection one-form (4.13).

Remark 4.1.19 We warn the reader for the following. When applying $\nabla$ to a spinor-tensor $T$ we find the usual expression, involving the coordinate derivative $\partial T$ and a term for each index. Because we often leave the spinorial indices implicit to ease our notation, it is tempting to forget the corresponding terms in the expression for $\nabla T$.

We now define the following operators:

Definition 4.1.20 The Dirac operator $\nabla : C^\infty(DM) \to C^\infty(DM)$ is the first order partial differential operator defined by $\nabla := \gamma^a \nabla_a$, where we view $\gamma^a$ as a map from $DM$ to itself, acting on the left.

The Dirac operator $\nabla : C^\infty(D^*M) \to C^\infty(D^*M)$ is defined by the same expression, $\nabla := \gamma^a \nabla_a$, where we now view $\gamma^a$ as a map from $D^*M$ to itself, acting on the right.
When expressed in a local frame the Dirac operator on spinor and cospinor fields is given by:

\[
\nabla u = E_A(\nabla u)^A = E_A \gamma^{aA} B \nabla_a u^B = E_A \gamma^{aA} B (\partial_a u^B + \sigma_a B u^C),
\]

\[
\nabla v = (\nabla v)_A E^A = (\nabla_a v^B) \gamma^{aB} A E^A = (\partial_a v^B - v C \sigma_a B) \gamma^{aB} A E^A,
\]

or dropping the spinorial indices:

\[
\nabla u = \gamma^a \nabla_a u = \gamma^a (\partial_a u + \sigma_a u),
\]

\[
\nabla v = \nabla_a v^a = (\partial_a v - v \sigma_a)^a.
\]

**Definition 4.1.21** The Dirac equation for \( u \in C^\infty(DM) \), respectively for \( v \in C^\infty(D^*M) \), is

\[
(-i\nabla + m)u = 0, \quad (i\nabla + m)v = 0,
\]

for a constant mass \( m \geq 0 \).

Note that the spinor field \( u \) is a solution to the Dirac equation if and only if the cospinor field \( u^+ \) is a solution, because of the following lemma:

**Lemma 4.1.22** For all spinor fields \( u \) and cospinor fields \( v \) we have

\[
(\nabla u)^+ = \nabla u^+, \quad (\nabla v)^+ = \nabla v^+,
\]

\[
(\nabla u)^c = -\nabla u^c, \quad (\nabla v)^c = -\nabla v^c
\]

and \( u^+ \eta u \geq 0 \) everywhere on \( M \), for any future pointing time-like vector field \( n \).

**Proof.** Using assumption \( (4.5) \) and the fact that the entries of \( A \) and \( C \) are constant we can compute in a local frame \( E \):

\[
(\nabla v)^c = ((\partial_a v + v \sigma_a) \gamma^a)\gamma^c = (\partial_a v - v \sigma_a) \gamma^c C
\]

\[
= - (\partial(v C) - v C \sigma_a) \gamma^a = - \nabla(v C) = - \nabla v^c,
\]

\[
(\nabla u)^+ = (\gamma^a (\partial_a u + \sigma_a u))^+ = (\partial_a u^* + u^* \sigma_a^* (\gamma^a)^* A
\]

\[
= (\partial_a (u^* A) - u^* A \sigma_a) \gamma^a = \nabla(u^* A) = \nabla u^+,
\]
where the minus sign in the last line appears because the order of the two factors of $\gamma$ in the expression $(4.14)$ for $\sigma_a$ needs to be reversed. It follows that

$$(\nabla v)^+ = (\nabla v^{++})^+ = (\nabla v^+) = \nabla v^+$$

$$(\nabla u)^c = (\nabla u^+)^c = -(\nabla u^c)^+ = (\nabla u^c)^+ = -(\nabla u^c)^+ = -\nabla u^c.$$  

Finally, if $u(x) = [E, z]$ at $x \in M$, then $u^+(x)\phi(x)u(x) = (z, Au(x)z) \geq 0$, because $\phi(x)$ is just $n(x)$ considered as an element of the Dirac algebra. □

We have now used all parts of assumption $(4.5)$ to define the adjoint and charge conjugation maps and to establish their interrelations with each other (equation $(4.8)$), with the Dirac equation and with the time-orientation of the spacetime (lemma $4.1.22$). The next proposition shows that these relations completely characterise the Dirac field, independent of the choice of representation and of the matrices $A, C \in GL(4, \mathbb{C})$ satisfying assumption $(4.5)$.

**Proposition 4.1.23** Consider the Dirac spinor bundle $DM_0$ and cospinor bundle $D^*M_0$, defined analogously to $DM$ and $D^*M$ but using $\pi_0$ instead of $\pi$. Let $^\dagger$ and $^-$ be defined analogously to $^+$ and $^c$, using the matrices $A_0 = \gamma_0$ and $C_0 = \gamma_2$, and let $\nabla_0$ be the Dirac operator defined through the representation $\pi_0$. Then there exists a base-point preserving, vector bundle isomorphism $\lambda: DM \to DM_0$ with induced isomorphism $\lambda^*: D^*M \to D^*M_0$ such that $\lambda\circ^+ = ^\dagger \circ \lambda^*$, $\lambda\circ^c = ^- \circ \lambda$ and $\lambda \circ \nabla = \nabla_0 \circ \lambda$. This isomorphism is unique, up to an overall sign.

**Proof.** On each fiber the bundle isomorphism $\lambda$ must be given by the formula $\lambda: [E, z] \mapsto [E, Lz]_0$ for some $L \in GL(4, \mathbb{C})$ by the fundamental theorem $4.1.4$. The induced morphism $\lambda^*$ is then $\lambda^*: [E, z^*] \mapsto [E, z^*L^{-1}]_0$. To make $\lambda$ well-defined and to obtain the correct intertwining with adjoint
operation, charge conjugation and the Dirac operator we need:

\[ LA^{-1} = A_0(L^*)^{-1}, \quad LC^{-1} = C_0^{-1} \bar{L}, \quad L\pi = \pi_0 L. \]

Using the fact that \( A_0 = A_0^{-1} \) we can apply theorem 4.1.6 to conclude that \( L \) exists and is unique up to a sign. By continuity \( L \) must be locally constant on \( M \), but then the map \( \lambda : [E, z] \mapsto [E, Lz]_0 \) is globally well-defined and is unique up to a global sign (by connectedness of \( M \)). □

The bundle isomorphisms which intertwine the relations between the adjoint operation, charge conjugation and the Dirac operator form a group. Because this group preserves these relations it will also leave the theory’s predictions invariant, so we can think of this group as a gauge group for the Dirac field. Choosing a specific representation is then like fixing a gauge, leaving as a residual gauge freedom only the involutive bundle isomorphism \( u \mapsto -u \). We will divide out this residual gauge freedom after quantisation of the Dirac field.

A change in the signature convention, \( \tilde{\eta} := -\eta \), should not have any physical consequences either and indeed it can be seen that this is easily compensated for by changing the sign in equation (4.2). This does not change the Dirac algebra and any other constructions that follow from it, although we do get signs for all covectors when raising or lowering indices with \( \tilde{\eta}^\dagger \).

\[ \tilde{\gamma}_a := i\gamma_a, \text{ in which case the Dirac algebra would become} \]

\[ D \simeq Cl_{3,1}. \] Because \( Cl_{3,1}^0 = Cl_{1,3}^0 \) we have \( Spin_{1,3}^0 = Spin_{3,1}^0 \), so nothing changes in the representation of the Spin group. We can also keep the same matrices \( A, C \), which now must satisfy the relations \( -\tilde{\gamma}_a^* = A\tilde{\gamma}_a A^{-1} \) and \( \tilde{\gamma}_a = C\gamma_a C^{-1} \). The spinor and cospinor bundle and the adjoint and charge conjugation maps remain the same as before. All relations between these operations and the Dirac equation remain valid if we drop the factor \( i \) in front of the Dirac operator in the Dirac equation.

Notice that a complex irreducible representation of \( Cl_{3,1} \) extends to an irreducible representation of \( M(4, \mathbb{C}) \) and therefore also gives a complex irreducible representation of \( Cl_{3,1} \) and vice versa. The standard Clifford algebra isomorphism \( Cl_{3,1} \simeq M(4, \mathbb{R}) \) appears

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4.2 The free Dirac field as a LCQFT

Now that we have described spinor and cospinor fields and their equation of motion, the Dirac equation, it is time to quantise the theory. In subsection 4.2.1 we will describe a suitable space of classical solutions and quantise the theory on a single spin spacetime, largely following [30, 26, 34]. Then we consider Hadamard states and states with the microlocal spectrum condition in subsection 4.2.2, obtaining the new result that every Hadamard state satisfies the microlocal spectrum condition, just as in the scalar field case. Finally we show that that we indeed obtain a locally covariant quantum field theory in subsection 4.2.2 and we establish some of its properties. There we also note that different choices of representation give rise to equivalent theories.

4.2.1 Quantisation

It will be convenient to deal with spinor and cospinor fields simultaneously, for which purpose we introduce the following:

**Definition 4.2.1** The double spinor bundle is defined as the vector bundle $DM \oplus D^*M$. A double spinor (field) is a smooth section of this vector bundle. The space of double spinors will be denoted by $\mathcal{D}(M) := C^\infty(DM \oplus D^*M)$.

The space of double test-spinors is the space of compactly supported smooth double spinors $\mathcal{D}_0(M) := C^\infty_0(DM \oplus D^*M)$ in the topology of uniform convergence on a fixed compact set.

If and only if the representation of $Cl_{1,3}$ is a Majorana representation, i.e. iff $\bar{\gamma}_a = -\gamma_a$. In that case we also find

$$Pin_{3,1} \simeq \{ S \in M(4, \mathbb{R}) | \det S = 1, \forall v \in M_0 \Rightarrow S v S^{-1} \in M_0 \} \neq Pin_{1,3}.$$  

The group $Pin_{3,1}$ is also sometimes called Spin group (see e.g. [20] p.334).
We define the adjoint map and charge conjugation map as anti-linear isomorphisms from $DM \oplus D^* M$ to itself by $[E,z,w^*]^+ := [E,A^{-1}w,z^* A]$ and $[E,z,w^*]^c := [E,C^{-1}z,w^* C]$, respectively. For double spinors we define the adjoint and charge conjugation maps pointwise, i.e. $(u \oplus v)^+ := v^+ \oplus u^+$ and $(u \oplus v)^c := u^c \oplus v^c$. We also introduce the operators $D := (-i\nabla + m) \oplus (i\nabla + m)$ and $\tilde{D} = (i\nabla + m) \oplus (-i\nabla + m)$.

Note that the Dirac equation (4.16) translates as $Df = 0$ for $f \in D(M)$. Every $u_1 \oplus v_1 \in D(M)$ determines a distribution on the double test-spinors $D_0(M)$ by the sesquilinear pairing
\[
\langle u_1 \oplus v_1, u_2 \oplus v_2 \rangle := \int_M u_1^+ u_2 - v_2 v_1^+ d\text{vol}_g. \tag{4.17}
\]
This pairing is non-degenerate, but not positive (because $A$ itself is not a positive matrix). The following relations hold between the adjoint and charge conjugate maps, $D$ and the pairing of equation (4.17):

**Lemma 4.2.2** We have for all $f \in D_0(M)$ and $h \in D(M)$:

1. $Dh^+ = (Dh)^+$, $Dh^c = (Dh)^c$ and $h^+ = -h^c$,
2. $\langle f^+, h^+ \rangle = \langle f^c, h^c \rangle = -\overline{\langle f, h \rangle}$ = $-\langle h, f \rangle$,
3. $f^+, f^c \in D_0(M)$ and $\langle f, Dh \rangle = \langle Df, h \rangle$.

**Proof.** The first set of equations follows from lemma 4.1.22. E.g.
\[
((-i\nabla + m)u)^c = i(\nabla u)^c + (mu)^c = -i\nabla u^c + mu^c = (-i\nabla + m)u^c
\]
and similarly for cospinors $v$, which implies $Df^c = (Df)^c$. The second item follows directly from the equations (4.8) and (4.17). For the last item we note that $\text{supp } h^c = \text{supp } h^+ = \text{supp } h$ for every $h \in D(M)$ by definition 4.1.16 and that we can perform a partial integration as follows [30]: note
that for all \( u \oplus v \in \mathcal{D}(M) \) we have \( \nabla_u (v \gamma^a u) = (\nabla v) u + v \nabla u \), because \( \gamma \) is covariantly constant. If either \( u \) or \( v \) is compactly supported we can integrate this equation over \( M \) to get \( \int_M (\nabla v) u = - \int_M v \nabla u \). Together with equation (4.17) this implies the result.

The second order operator \( \tilde{\Box} D = D \tilde{\Box} \) has as its principal part the wave operator \( \Box = g^{\mu \nu} \nabla_\mu \nabla_\nu \), which is diagonal in the spinorial indices. Due to global hyperbolicity of \( M \) there exist\(^9\) unique advanced (-) and retarded (+) fundamental solutions \( E^\pm : \mathcal{D}_0(M) \to \mathcal{D}(M) \) for the operator \( D \tilde{\Box} \), i.e. for all \( f \in \mathcal{D}_0(M) \) we have \( D \tilde{\Box} E^\pm f = f = E^\pm D f \) and \( \text{supp}(E^\pm f) \subset J^\pm(\text{supp} f) \) (see [6] theorem 3.3.1). The fundamental solutions \( E^\pm \) help us to find fundamental solutions for the operator \( D \) as follows (see [30]):

**Proposition 4.2.3** The maps \( S^\pm : \mathcal{D}_0(M) \to \mathcal{D}(M) \) defined by \( S^\pm := \tilde{D} E^\pm \) are the unique advanced (-) and retarded (+) fundamental (left and right) solutions for \( D \) such that \( \text{supp} S^\pm f \subset J^\pm(\text{supp} f) \) for all \( f \in \mathcal{D}_0(M) \). Moreover, \( S^\pm f^+ = (S^\pm f)^+ \), \( S^\pm f^- = (S^\pm f)^- \) and \( \langle f, S^\pm h \rangle = \langle S^\pm f, h \rangle \) for all \( f, h \in \mathcal{D}_0(M) \).

**Proof.** For \( f \in \mathcal{D}_0(M) \) we see that \( S^\pm f \) has the correct support property and \( DS^\pm f = f \), so \( S^\pm \) is a right fundamental solution. For the fact they are left fundamental solutions and their uniqueness we refer to [30] theorem 2.1. Given \( f, h \in \mathcal{D}_0(M) \) we choose \( \chi \in C^\infty_0(M) \) with \( \chi \equiv 1 \) on the compact sets \( J^\pm(\text{supp} f) \cap J^\mp(\text{supp} h) \). Using lemma 4.2.2 and the support properties we then compute:

\[
\langle f, S^\pm h \rangle = \langle DS^\mp f, S^\pm h \rangle = \langle D(\chi S^\mp f), \chi S^\pm h \rangle = \langle \chi S^\mp f, D(\chi S^\pm h) \rangle = \langle S^\mp f, DS^\pm h \rangle = \langle S^\mp f, h \rangle.
\]

\(^9\)As a slight abuse of notation we use the same symbols \( E^\pm \) here as in chapter 3, although the operators are not the same. (They do not even have the same domain.)
Finally we compute for all $f, h \in \mathcal{D}_0(M)$:

\[
\langle S^{\pm} f^+, h \rangle = \langle f^+, S^{\mp} h \rangle = \langle (DS^{\pm} f)^+, S^{\mp} h \rangle = \langle D(S^{\pm} f)^+, S^{\mp} h \rangle = \langle (S^{\pm} f)^+, DS^{\mp} h \rangle = \langle (S^{\pm} f)^+, h \rangle,
\]

from which it follows that $S^{\pm} f^+ = (S^{\pm} f)^+$. The proof for charge conjugation is similar. □

Analogous to the scalar field case we define $S := S^- - S^+$. The proof of proposition 4.2.3 also works for the spinor and cospinor cases separately, yielding fundamental solutions $S^{\pm}_{sp}$, $S_{sp}$, $S^{\pm}_{cosp}$ and $S_{cosp}$ using the obvious notation. By the uniqueness part of the proposition we find $S = S_{sp} \oplus S_{cosp}$, $(S_{sp} u)^+ = S_{cosp} u^+$ and $\int_M v(S_{sp} u) = -\int_M (S_{cosp} v) u$ (see [30, 34]).

**Lemma 4.2.4** \( \ker S = D(\mathcal{D}_0(M)) \) and the bilinear map \( (f, h) := i\langle f, Sh \rangle \) defines an inner product on \( \mathcal{D}_0(M) / \ker S \). The adjoint and charge conjugation maps descend to this quotient space too and \( (f^+, h^+) = (f^c, h^c) = \overline{(f, h)} = (h, f) \).

**Proof.** If $f = Dh$ for $h \in \mathcal{D}_0(M)$ then $Sf = S^- Dh - S^+ Dh = 0$, so $D(\mathcal{D}_0(M)) \subset \ker S$. Conversely, if $Sf = 0$ with $f \in \mathcal{D}_0(M)$ then $h := S^- f = S^+ f$ has its support in the compact set $J^+(\text{supp } f) \cap J^-(\text{supp } f)$ and $f = Dh$, so $D(\mathcal{D}_0(M)) = \ker S$ Now \( (f, h) = -i\langle Sf, h \rangle = \overline{(h, f)} \) by proposition 4.2.3 and lemma 4.2.2 so \( (, \) \) is a well-defined sesquilinear map on \( \mathcal{D}_0(M) / \ker S \). By proposition 4.2.3 again the adjoint and charge conjugation maps descend to the quotient space \( \mathcal{D}_0(M) / \ker S \) and with $^*$ denoting either $^+$ or $^c$ we compute:

\[
(f^*, h^*) = i\langle f^*, Sh^* \rangle = i\overline{\langle f^*, (Sh)^* \rangle} = -i\overline{\langle f, Sh \rangle} = \overline{\langle f, h \rangle} = (h, f).
\]

It remains to show that $(f, f) \geq 0$ for all $f \in \mathcal{D}_0(M)$, with equality only if $Sf = 0$. Leaving the metric induced volume element implicit in the notation
we have,

\[(u \oplus v, u \oplus v) = i \int_M u^+ S_{sp} u - (S_{cosp} v)v^+ = i \int_M u^+ S_{sp} u + v S_{sp} v^+\]

and for any Cauchy surface \(C \subset M\) and \(D_{sp} := -i \nabla + m:\)

\[
i \int_M u^+ S_{sp} u = i \int_{J^+(C)} (D_{sp} S_{sp}^- u)^+ S_{sp} u + i \int_{J^-(C)} (D_{sp} S_{sp}^+ u)^+ S_{sp} u
\]

\[
= i \int_{J^+(C)} (S_{sp}^- u)^+ (D_{sp} S_{sp}^- u) + i \nabla_a((S_{sp}^- u)^+ \gamma^a S_{sp} u)
\]

\[
+ i \int_{J^-(C)} (S_{sp}^+ u)^+ (D_{sp} S_{sp}^- u) + i \nabla_a((S_{sp}^+ u)^+ \gamma^a S_{sp} u)
\]

\[
= - \int_{J^+(C)} \nabla_a((S_{sp}^- u)^+ \gamma^a S_{sp} u) - \int_{J^-(C)} \nabla_a((S_{sp}^+ u)^+ \gamma^a S_{sp} u)
\]

\[
= \int_C \eta_a(S_{sp}^- u - S_{sp}^+ u)^+ \gamma^a S_{sp} u = \int_C (S_{sp} u)^+ \eta^a S_{sp} u,
\]

where \(n^a\) is the future pointing normal vector field to \(C\) and we used equation (B.2.26) for the final partial integration. The integrand is smooth and pointwise positive by lemma [4.1.22] so the result follows. \(\square\)

We define \(\mathcal{L}(M) := \overline{D_0(M)/\ker S}\) to be the Hilbert space completion in the inner product \((,\)\). The continuous extensions of \(+\) and \(c\) to \(\mathcal{L}(M)\) will be

denoted by the same symbol.

In order to quantise the free Dirac field we first define the following.

**Definition 4.2.5** The exterior tensor product \(\mathcal{V}_1 \boxtimes \mathcal{V}_2\) of two vector bundles \(\mathcal{V}_i\) over \(\mathcal{M}_i\), with fiber \(V_i, i = 1, 2\), is the vector bundle over \(\mathcal{M}_1 \times \mathcal{M}_2\) whose fiber is \(V_1 \otimes V_2\) and whose local trivializations are determined by local trivialisations \(O_i \times V_i\) of \(\mathcal{V}_i\) as \((O_1 \times O_2) \times (V_1 \otimes V_2)\).

We can extend the adjoint and charge conjugation maps from \(DM \oplus D^*M\) to its exterior powers \((DM \oplus D^*M)^{\boxtimes m}\) by anti-linear extension of

\[(p_1 \otimes \ldots \otimes p_m)^+ := (p_1^+ \otimes \ldots \otimes p_m^+),\]

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\[(p_1 \otimes \ldots \otimes p_m)^\ast := (p_1^\ast \otimes \ldots \otimes p_m^\ast),\]

for all \(p_1, \ldots, p_m \in DM \oplus D^* M\).

**Definition 4.2.6** The Dirac Borchers-Uhlmann algebra \(\mathcal{F}_M\) is the direct sum

\[
\mathcal{F}_M := \bigoplus_{n=0}^{\infty} C^\infty_0((DM \oplus D^* M)^{\otimes n})
\]

(in the algebraic sense), equipped with:

1. the product \(f(x_1, \ldots, x_n)h(x_{n+1}, \ldots, x_{n+m}) := (f \otimes h)(x_1, \ldots, x_{n+m}),\) extended linearly,

2. the \(^*\)-operation \(f(x_1, \ldots, x_n)^* := f^+(x_n, \ldots, x_1) = (f(x_n, \ldots, x_1))^+,\) extended anti-linearly,

3. a topology such that \(f_j = (\bigoplus_{n=0}^{\infty} f_j^{(n)})\) converges to \(f = (f^{(n)})\) if and only if for all \(n\) we have \(f_j^{(n)} \to f^{(n)}\) in \(C^\infty_0((DM \oplus D^* M)^{\otimes n})\) and for some \(N > 0\) we have \(f_j^{(n)} = 0\) for all \(j\) and \(n \geq N\).

Much like the Borchers-Uhlmann algebra for the real scalar field, \(\mathcal{F}_M\) is the strict inductive limit \(\mathcal{F}_M = \bigcup_{N=0}^{\infty} \bigoplus_{n=0}^{N} C^\infty_0((DM \oplus D^* M)^{\otimes n})_{K_N}\), where \(K_N\) is an exhausting (and increasing) sequence of compact subsets of \(\mathcal{M}\), the vector bundle \((DM \oplus D^* M)^{\otimes n}_{K_N}\) is the restriction of \((DM \oplus D^* M)^{\otimes n}\) to \((K_N)^{\times n}\) and each \(C^\infty_0((DM \oplus D^* M)^{\otimes n}_{K_N})\) is given the test-function topology, see [75] theorem 2.6.4.

As in lemma 3.1.2 one can show that \(\mathcal{F}_M\) is a topological \(^*\)-algebra and that a continuous state \(\omega\) on \(\mathcal{F}_M\) consists of a sequence of \(n\)-point distributions \((\omega_n)\) acting on the smooth, compactly supported sections of \((DM \oplus D^* M)^{\otimes n}\). Another analogy with the scalar field case is that \(\mathcal{F}_M\) does not carry any dynamical information or anti-commutation relation. As in chapter 3 this can be remedied by dividing out a certain ideal:
Definition 4.2.7 We define the free Dirac Borchers-Uhlmann algebra as the topological $\ast$-algebra $\mathcal{F}_M^0 := \mathcal{F}_m/J$, where $J \subset \mathcal{F}_M$ is the closed $\ast$-ideal generated by all elements of the form $Df$ or $f \otimes h + h \otimes f - (f, h)I$, where $f, h \in \mathcal{D}_0(M)$.

We have presented the Dirac Borchers-Uhlmann algebras $\mathcal{F}_M$ and $\mathcal{F}_M^0$ to indicate the analogy with the Borchers-Uhlmann algebras $\mathcal{U}_M$ and $\mathcal{U}_M^0$ of the real scalar field. However, there is a more direct way to obtain $\mathcal{F}_M^0$, which is analogous to the algebra $\mathcal{A}_M^0$ for the free scalar field. This approach also shows that $\mathcal{F}_M^0$ can be completed to a $C^\ast$-algebra.

Proposition 4.2.8 The algebra $\mathcal{F}_M^0$ can be completed to a $C^\ast$-algebra $\mathcal{F}_M^0$, which is the unique $C^\ast$-algebra generated by elements $B_M(f), f \in \mathcal{L}(M)$, such that

1. $f \mapsto B_M(f)$ is $C$-linear,
2. $B_M(f^\ast) = B_M(f)^\ast$,
3. $\{B_M(f)^\ast, B_M(g)\} = (f, g)I$.

Recall that $f \in \mathcal{L}(M) = \overline{\mathcal{D}_0(M) / \ker S}$, so the equation of motion is implicit.

Proof. To prove that $\mathcal{F}_M^0$ has a $C^\ast$-norm we note that it is an infinite-dimensional Clifford algebra, by the anti-commutation relations. Each finite dimensional Clifford algebra has a $C^\ast$-norm, because it can be represented faithfully as an algebra of bounded operators [57]. The algebra $\overline{\mathcal{F}_M^0}$ is the $C^\ast$-algebraic inductive limit of its finite dimensional subalgebras. We refer to [13] theorem 5.2.5 (or [2] lemma 4.1 and 3.3) for a proof that $\mathcal{F}_M^0$ is the unique $C^\ast$-algebra generated by elements $B_M(f)$ with $f \in \mathcal{L}(M)$ and satisfying the stated properties. □

We formulate a charge conjugation map on the algebra $\overline{\mathcal{F}_M^0}$ as follows:
Proposition 4.2.9 The map \( f \mapsto f^c \) gives rise to a \(*\)-isomorphism \( \alpha_C \) of \( \mathcal{F}_M^0 \) determined by \( \alpha_C(B_M(f)) := B_M(f^c) \) and we have \( \alpha_C^2 B_M(f) = -B_M(f) \).

Proof. Note that \( f \mapsto f^c \) is a linear isomorphism of \( \mathcal{L}(M) \), because of \( (f^c, g^c) = (g^c, f^c) = (f, g) \) by lemma 4.2.4. The result then follows (see [13] section 5.2.2.1, p.18).

We have now quantised the Dirac field on a single spin spacetime. In subsection 4.2.3 we will investigate the properties of this construction as a locally covariant quantum field theory. The map \( B_M : \mathcal{D}_0(M) \rightarrow \mathcal{F}_M^0 \) of proposition 4.2.8 will then be a candidate for a locally covariant quantum field. For now we will only introduce the following notation.

Definition 4.2.10 Leaving the quotient map \( \mathcal{D}_0(M) \rightarrow \mathcal{L}(M) \) implicit, we define the maps \( \psi_M : C^\infty_0(D^*M) \rightarrow \mathcal{F}_M^0 \) and \( \psi_M^+ : C^\infty_0(DM) \rightarrow \mathcal{F}_M^0 \) by
\[
\psi_M(v) := B_M(0 \oplus v), \quad \psi_M^+(u) := B_M(u \oplus 0).
\]
We also define
\[
\psi_M^c(v^c) := \alpha_C(B_M(0 \oplus v^c)) = \psi_M(v)^*,
\]
\[
\psi_M^{c+}(u^c) := \alpha_C(B_M(u^c \oplus 0)) = \psi_M^+(u)^*.
\]

Proposition 4.2.11 The maps \( B_M, \psi_M \) and \( \psi_M^+ \) are distributions valued in the \( C^*\)-algebra \( \mathcal{F}_M^0 \) and:

1. \( \psi_M^+(u) = \psi_M(u^+)^* \),
2. \( \{ \psi_M^+(u), \psi_M(v) \} = (v^+ \oplus 0, u \oplus 0)I = i \int_M v(S_{ap}u)I \) and all the other anti-commutators vanish,
3. \( (-i\nabla + m)\psi_M = 0 \) and \( (i\nabla + m)\psi_M^+ = 0 \), where we have set \( (\gamma_a U)(v) := U(v\gamma_a) \) and \( (\nabla_a U)(v) := -U(\nabla_a v) \) for any distribution \( U \) on smooth sections of \( D^*M \).
Proof. The first two items follow straightforwardly from the definitions of \( \psi_M, \psi^+_M, B_M \) and the inner product \((,\)\). For the third we have:

\[
((-i\nabla + m)\psi_M)(v) = \psi_M((i\nabla + m)v) = B_M(D(0 \oplus v)) = 0,
\]

because \( SD(0 \oplus v) = 0 \), and similarly for \((i\nabla + m)\psi^+_M = 0\).

It remains to show that \( B_M, \psi_M \) and \( \psi^+_M \) are \( \mathbb{C}^* \)-algebra-valued distributions. The \( \mathbb{C}^* \)-sub-algebra of \( \mathcal{F}_M^0 \) generated by \( I, \psi_M(v), \psi_M(v)^* \) is a Clifford algebra which is isomorphic to \( M(2, \mathbb{C}) \) and an explicit isomorphism is given by \( \psi_M(v) \mapsto \begin{pmatrix} 0 & \sqrt{c} \\ 0 & 0 \end{pmatrix} \), where \( c = (0 \oplus v, 0 \oplus v) = i \int_M v(S_{sp}v^+) > 0 \). It follows that \( \|\psi_M(v)\| = \sqrt{c} \) is the operator norm of the corresponding matrix, i.e.\(^{10}\)

\[
\|\psi_M(v)\|^2 = i \int_M v(S_{sp}v^+)d\text{vol}_g.
\]

In the test-spinor topology we then have continuous maps \( v \mapsto v \oplus v^+ \mapsto i \int_M v(S_{sp}v^+) \), from which it follows that \( v \mapsto \psi_M(v) \) is norm continuous, i.e. it is a \( \mathbb{C}^* \)-algebra-valued distribution. The proof for \( \psi^+_M \) is analogous and the result for \( B_M \) then follows. \( \square \)

Remark 4.2.12 The quantisation of Majorana spinors proceeds in a largely analogous way. The space of test-spinors \( C^\infty_0(DM)/\ker S_{sp} \) can be given the inner product \((u_1, u_2)^\prime := (u_1 \oplus 0, u_2 \oplus 0) \) and then completed to a Hilbert space \( \mathcal{L}'(M) \). The map \( \mathcal{C} \) provides a conjugation map on this space, so we can quantise to obtain a \( \mathbb{C}^* \)-algebra \( \mathcal{F}_M' \) generated by elements \( B'_M(u), u \in \mathcal{L}'(M) \), satisfying

1. \( u \mapsto B'_M(u) \) is \( \mathbb{C} \)-linear,

\( \text{[10]} \)The factor 2 in [34] remark 2, p.340 seems to be erroneous. The sign is due to the fact that [34] uses for \( S \) the retarded-minus-advanced rather than the advanced-minus-retarded fundamental solution.
2. $B'_M(u^*) = B'_M(u)^*$ and 
3. $\{B'_M(u_1)^*, B'_M(u_2)^*\} = (u_1, u_2)'I$.

For cospinors $v$ we can then define $B'_M(v) := B'_M(v^+)^*$, which adds nothing new to the algebra. However, $\{B'_M(v)^*, B'_M(u)^*\} = (v^+, u)'I$, but on the other hand $\{\psi_M(v)^*, \psi_M^+(u)\} = \{B_M(0 \oplus v)^*, B_M(u \oplus 0)\} = 0$. Furthermore, the charge conjugation map $\alpha_C$ used in propositions 4.2.9 and 4.2.13 reduces to the identity map in the case of Majorana spinors.

To conclude this subsection we deal with the residual gauge freedom:

**Proposition 4.2.13** The bundle isomorphism $\lambda : DM \rightarrow DM$ defined by $\lambda(u) := -u$ gives rise to an involutive $^* -$ isomorphism $\tau$ of $\overline{F}^0_M$. The set $\mathcal{B}_M \subset \overline{F}^0_M$ of $\tau$-invariant elements is a $C^*$-algebra and is generated by elements of the form $B_M(f)B_M(g)$ (i.e. it is the even subalgebra of $\overline{F}^0_M$). The $^*$-isomorphism $\alpha_C$ restricts to an involutive $^*$-isomorphism of $\mathcal{B}_M$.

**Proof.** Note that $\lambda$ extends to $\mathcal{L}(M)$ as $\lambda(f) = -f$ and then descends to $\mathcal{L}(M)$, where it is the linear map $-I$. It then gives rise to the map $\tau$ on $\overline{F}^0_M$ defined by $\tau(B_M(f)) := B_M(\lambda(f)) = -B_M(f)$ extended as an algebra homomorphism. As $(-I)^2 = I$ we see that $\tau^2 = id$ and we can define the linear space $\mathcal{B}_M$ of $\tau$-invariant elements. As $\tau$ is a $^*$-isomorphism, $\mathcal{B}_M$ is a closed $^*$-subalgebra of $\overline{F}^0_M$ and hence a $C^*$-algebra in its own right. Clearly, $\mathcal{B}(M)$ contains all even powers of $B_M$, i.e. $B_M(f)B_M(g)$. Conversely, any $\tau$-invariant $A \in \overline{F}^0_M$ can be approximated by a sequence of polynomials $A_n$, which we can choose to be $\tau$-invariant. As the $\tau$-invariant polynomials $A_n$ only contain even powers of the $B_M(f)$'s we see that these even polynomials generate $\mathcal{B}_M$. Finally, $\alpha_C$ maps the even polynomials onto themselves, so $\alpha_C(B_M) = B_M$, and $\alpha_C^2B_M(f) = -B_M(f) = \tau(B_M(f))$ implies that $\alpha_C^2$ is the identity on $\mathcal{B}_M$. $\square$
Although the physical information should be contained entirely in the
gauge-invariant algebra $\mathcal{B}_M$, it will be convenient to have $\mathcal{F}_M^0$ at our disposal
too, because the putative locally covariant quantum fields $B_M$, $\psi_M$ and $\psi_M^+$
take values in $\mathcal{F}_M^0$ rather than $\mathcal{B}_M$.

4.2.2 States of the Dirac field

If $\omega$ is a state on $\mathcal{F}_M^0$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ its GNS-triple, then we may consider
for each $n \in \mathbb{N}$ the $\mathcal{H}_\omega$-valued distribution on $DM \oplus D^*M$ defined by:

$$\varphi_n(f_n, \ldots, f_1) := \pi_\omega(B_M(f_n) \cdots B_M(f_1)) \Omega_\omega$$

and $B(\omega)(f) := \pi_\omega(B_M(f))$.

**Definition 4.2.14** A state $\omega$ on $\mathcal{F}_M^0$ is called Hadamard if and only if

$$WF(\varphi_1) \subset \mathbb{N}^+.$$ 

A state $\omega$ on $\mathcal{F}_M^0$ satisfies the microlocal spectrum condition ($\mu SC$) if
and only if $WF(\omega_n) \subset \Gamma_n$ for all $n \in \mathbb{N}$.

A state $\omega$ on $\mathcal{B}_M$ or $\mathcal{F}_M^0$ is called Hadamard, respectively satisfies the
microlocal spectrum condition, if and only if it can be extended to a state on $\mathcal{F}_M^0$ which is Hadamard, respectively satisfies the $\mu SC$.

Note that every state on $\mathcal{B}_M$ can be extended to a state on $\mathcal{F}_M^0$ by the Hahn-
Banach theorem [49] and every state on $\mathcal{F}_M^0$ has an extension to $\mathcal{F}_M^0$ only
if it is continuous in the $C^\ast$-norm, in which case the extension is unique by
continuity. The Hadamard condition on $\mathcal{B}_M$ is independent of the choice of
extension, because it depends solely on the two-point distribution, as the
following proposition shows, and the same is true for the $\mu SC$ by proposition
4.2.17 below. The following proposition also shows that the definition of
Hadamard states on $\mathcal{F}_M^0$ is analogous to definition 3.1.12 for the free scalar
field.
Proposition 4.2.15 For a state $\omega$ on $\mathcal{F}_M^0$ the following three conditions are equivalent:

1. $\omega$ is Hadamard,

2. the two-point distribution $\omega_2(f_1, f_2) := \omega(B_M(f_1)B_M(f_2))$ has

   
   $$WF(\omega_2) \subset C := \{(x, \xi; y, \xi') \in \mathcal{N}^- \times \mathcal{N}^+ | (x, -\xi) \sim (y, \xi')\},$$

   where again $(x, -\xi) \sim (y, \xi')$ if and only if $(x, -\xi) = (y, \xi')$ or there is an affinely parameterised light-like geodesic between $x$ and $y$ to which $-\xi, \xi'$ are cotangent (and hence $-\xi$ and $\xi'$ are parallel transports of each other along the geodesic),

3. there is a two-point distribution $w$ such that $\omega_2(f_1, f_2) = iw(Df_1, f_2)$ and $WF(w) \subset C$.

Proof. First note that $\omega_2$ is a bidistribution on $DM \oplus D^*M$, because $B_M$ is an $\mathcal{F}_M^0$-valued distribution and multiplication in $\mathcal{F}_M^0$ and $\omega$ are continuous. For the equivalence of the first two statements we adapt the argument in [80], proposition 6.1. If $WF(\omega_2) \subset C$, then $WF(\varphi_1) \subset \mathcal{N}^+$ by theorem \[A.1.6\] so $\omega$ is Hadamard. For the converse we suppose that $\omega$ is Hadamard. Again by theorem \[A.1.6\] we see that $WF(\omega_2) \subset \mathcal{N}^- \times \mathcal{N}^+$. Defining $\tilde{\omega}_2(f_1, f_2) := \omega_2(f_2, f_1)$ we find $WF(\tilde{\omega}_2) \cap WF(\omega_2) = \emptyset$. Now, $(\omega_2 + \tilde{\omega}_2)(f_1, f_2) = 2i(f_1^+, Sf_2)$, so $WF(\omega_2) \subset WF(S) \cup WF(\tilde{\omega}_2)$ and hence $WF(\omega_2) \subset WF(S) \subset WF(E)$, because $S = \tilde{D}E$. By proposition \[A.1.7\] and $E = E^- - E^+$ we find $WF(E) \cap (\mathcal{N}^- \times \mathcal{N}^+) \subset C$ and therefore $WF(\omega_2) \subset C$.

To prove the equivalence of the second and third statement we now assume that $\omega_2(f_1, f_2) = iw(Df_1, f_2)$ and $WF(w) \subset C$. If $D^*$ is the formal adjoint of $D$, then $WF(\omega_2) = WF((D^* \otimes I)w) \subset WF(w) \subset C$. For the converse we suppose that $\omega_2$ satisfies condition 2 and we choose a smooth real-valued function $\chi^+$ on $M$ such that $\chi^+ \equiv 0$ to the past of some Cauchy surface $C_-$.
and such that \( \chi^- := 1 - \chi^+ \equiv 0 \) to the future of another Cauchy surface \( C_+ \). We then define \( w(f_1, f_2) := -i\omega_2(\chi^+ S^- f_1 + \chi^- S^+ f_1, f_2) \). Note that \( w \) is a bidistribution which is well-defined, because \( \chi^+ S^- f_1 \) and \( \chi^- S^+ f_1 \) are compactly supported. It is easy to verify that \( i w(D f_1, f_2) = \omega_2(f_1, f_2) \).

We now estimate the wave front set of \( w \) as follows. The wave front set of \( S^\pm = \tilde{D}E^\pm \) is contained in \( WF(E^\pm) \), which we collected in proposition A.1.7. Then we may apply theorem 8.2.9 and 8.2.13 in [47] to estimate the wave front sets of the tensor product \( \chi^\pm(y) S^\mp(y, x) \delta(y', x') \) respectively and, using \( WF(\omega_2) \subset C \), we find:

\[
WF(i w) \subset \bigcup_{\pm} \{ (x, k; x', k') | \exists (y, l; y', l') \in WF(\omega_2) \text{ such that } (y, -l; x, k; y', -l'; x', k') \in WF(S^\pm \otimes \delta) \} \
\subset WF(\omega_2).
\]

Using scaling limits one can even show that \( WF(\omega_2) = C \) if \( WF(\omega_2) \subset C \). In this form the equivalence of statements 2 and 3 was already known ([73] definitions 5.1 and 5.3 and theorem 5.8). The first characterisation appears to be new, but is analogous to the result for the free scalar field [80].

The following definition of quasi-free states is analogous to the free field case, definition 3.1.6.

**Definition 4.2.16** A state \( \omega \) on either of the algebras \( \mathcal{F}_M^0 \) or \( \overline{\mathcal{F}}_M^0 \) is called quasi-free if and only if \( \omega_n = 0 \) for \( n \) odd and for \( m \geq 1 \):

\[
\omega_{2m}(f_1, \ldots, f_{2m}) = \sum_{\pi \in \Pi_m} \omega_2(f_{\pi(1)}, f_{\pi(2)}) \cdots \omega_2(f_{\pi(2m-1)}, f_{\pi(2m)}),
\]

where \( \Pi_m \) is the set of permutations of \( \{1, \ldots, 2m\} \) such that

1. \( \pi(1) < \pi(3), \ldots < \pi(2m - 1) \),
2. \( \pi(2i-1) < \pi(2i), \ i = 1, \ldots, m. \)

The set of all states on \( \mathcal{F}_M^0 \) satisfying the \( \mu SC \) will be denoted by \( \mathcal{R}_M^0 \).

The set of all states on \( \mathcal{F}_M^0 \) which are locally quasi-equivalent to a quasi-free Hadamard state is denoted by \( \mathcal{T}_M^0 \).

The set of all states on \( \mathcal{B}_M \) which are locally quasi-equivalent to the restriction of a state on \( \mathcal{F}_M^0 \) which is in \( \mathcal{T}_M^0 \) is denoted by \( \mathcal{T}_M \).

Adapting the proof of proposition 3.1.13 we now show that every Hadamard state on \( \mathcal{F}_M^0 \) satisfies the \( \mu SC \). (The quasi-free Hadamard case was essentially known, because the proof is the same as for the scalar field algebra \( \mathcal{U}_M^0 \), see [15] proposition 4.3).

**Proposition 4.2.17** Let \( \omega \) be a state on \( \mathcal{F}_M^0 \) which is Hadamard on a neighbourhood \( \mathcal{W} \subset M \) of a Cauchy surface in \( M \). Then \( \omega \) satisfies the \( \mu SC \) on \( M \).

**Proof.** The proof is completely analogous to that of proposition 3.1.13 with the following modifications. We use the parts of theorems [A.1.5] and [A.1.6] that pertain to distributions on vector bundle-valued sections and the operator \( P = \tilde{D}D \). The propagation of the wave front set in this case follows from the work of [27] concerning polarisation sets. We use the fact that \( WF(S) = WF(\tilde{D}E) \subset WF(E) \) and finally we need to use the anticommutation relations instead of the commutation relations:

\[
\omega_n(x_1, \ldots, x_n) = -\omega_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \quad (4.18)
\]

\[
i \omega_{n-2}(x_1, \ldots, \hat{x}_i, \hat{x}_{i+1}, \ldots, x_n)S(x_i, x_{i+1}).
\]

Here we view \( S \) as the bidistribution \( S(f, g) := \langle f, Sg \rangle = -i(f, g), \) cf. lemma 4.2.4. Finally we need to replace \( \varphi_1 \) and \( \Phi^{(\omega)} \) by \( \varphi_1 \) and \( B^{(\omega)} \) respectively. \( \square \)

We now check that \( \mathcal{R}_M^0, \mathcal{T}_M^0 \) and \( \mathcal{T}_M \) are suitable candidates to construct a state space functor, in the sense of definition 2.1.2 (cf. proposition 3.1.9 and 3.1.17 for analogous results for the scalar field):
Proposition 4.2.18 The set $\mathcal{R}_M$, $\mathcal{T}_M$, respectively $\mathcal{F}_M$, is convex and closed under operations from $\mathcal{F}^0_M$, $\mathcal{F}^0_M$, respectively $\mathcal{B}_M$.

Proof. $\mathcal{R}_M$ is convex by theorem A.1.5. To show that it is closed under operations from $\mathcal{F}^0_M$ we note that for fixed $f \in C^\infty_0((DM \oplus D^*M)\otimes m)$ and $h \in C^\infty_0((DM \oplus D^*M)\otimes r)$ we have

$$WF(\omega_{m+n+r}(f, x_1, \ldots, x_n, h)) \subset \{(y_1, 0; \ldots; y_m, 0; x_1, k_1; \ldots; x_n, k_n; z_1, 0; \ldots; z_r, 0) \in \Gamma_{m+n+r}\} \subset \Gamma_n,$$

using [47] theorem 8.2.12, which can also be applied to vector bundle-valued sections by using local sections and expressing $\omega_n$ as a sum of components (cf. appendix A). The same inclusion holds for linear combinations of such terms, so if $\omega(A^*A) \neq 0$ then the state $B \mapsto \frac{\omega(A^*BA)}{\omega(A^*A)}$ satisfies the $\mu$SC if $\omega$ does.

$\mathcal{T}_M$ is convex, because all quasi-free Hadamard states of $\mathcal{T}^0_M$ are locally quasi-equivalent [24]. From the definition of $\mathcal{T}^0_M$ we see that it is closed under operations from $\mathcal{F}^0_M$.

To see that $\mathcal{F}_M$ is convex we note that the restrictions of any two states $\omega_1, \omega_2$ in $\mathcal{F}^0_M$ to $\mathcal{B}_M$ are locally quasi-equivalent. Indeed, for any bounded $cc$-region a $\ast$-isomorphism between the von Neumann algebras $\pi_{\omega_i}(\mathcal{F}^0_M)''$ restricts to a $\ast$-isomorphism of the von Neumann algebras $\pi_{\omega_i}(\mathcal{B}_M)''$ and the claim then follows from [3] pp.212-213. That $\mathcal{F}_M$ is closed under operations from $\mathcal{B}_M$ follows directly from definition 4.2.16 again.

The following lemma is analogous to the free field case, lemma 3.1.16

Lemma 4.2.19 Let $M$ be a globally hyperbolic spin spacetime, $\mathcal{W} \subset M$ a neighbourhood of a Cauchy surface and $\chi \in C^\infty_0(M)$ such that $\chi \equiv 1$ on $J^+(\mathcal{W}) \setminus \mathcal{W}$ and $\chi \equiv 0$ on $J^-(\mathcal{W}) \setminus \mathcal{W}$. For every $f \in \mathcal{D}_0(M)$ we have $f = f' + Dh$, where $f' := D(\chi S f) \in \mathcal{D}_0(\mathcal{W})$ and $h := S^{-}(f - f') \in \mathcal{D}_0(M)$.

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Proof. Notice that $\text{supp } f' \subset \text{supp}(Sf)$ and $f' \equiv 0$ on a neighbourhood of $M \setminus W$, so $f' \in \mathcal{D}_0(W)$. We have $h = S^-(f - f') = (1 - \chi)S^-f + \chi S^+f$ which is compactly supported in $M$ and $Dh = f - f'$.

If, in the situation of lemma 4.2.19, we set $\chi^+ := \chi$ and $\chi^- := 1 - \chi$ we note that $\chi^+Sf + S^+f$ and $\chi^-Sf - S^-f$ have compact support in $M$ and hence

$$SD(\chi^\pm Sf) = \mp Sf. \quad (4.19)$$

4.2.3 The free Dirac field as a LCQFT

In the previous subsections we have quantised the Dirac field and discussed interesting classes of states on a single spin spacetime. In this section we will show how the free Dirac field can be described as a locally covariant quantum field theory. For that purpose we will need to investigate how our quantisation and our classes of states behave under morphisms, paying special attention to the residual gauge freedom that arises from the choice of representation (see proposition 4.1.23).

Proposition 4.2.20 Given a morphism $\Psi : M_1 \to M_2$ in $\mathfrak{G}\mathfrak{M}an$ such that $\Psi = (\psi, \chi)$ (see definition 2.3.1), there exist exactly two bundle isomorphisms $\lambda_{\pm} : DM_1 \to DM_2_{|\psi(M)}$ which intertwine the adjoint, charge conjugation and Dirac operator and we have $\lambda_{+}(u) = -\lambda_{-}(u)$. Each of these bundle isomorphisms gives rise to a morphism $\beta_{\pm} : \mathcal{F}^0_{M_1} \to \mathcal{F}^0_{M_2}$ in $\mathfrak{C}\mathfrak{A}\mathfrak{l}\mathfrak{g}$ and we have $\beta_{+} = \tau \circ \beta_{-}$ where $\tau$ is the $^*$-isomorphism of $\mathcal{F}^0_{M_2}$ of proposition 4.2.13.

Proof. First we note that $\chi(SM_1) = SM_2_{|\psi(M)}$, because $\chi$ maps the fiber of $SM_1$ over $x \in M_1$ to the fiber of $SM_2$ over $\psi(x)$ and it intertwines the action of the structure group and the bundle projection appropriately. Note in particular that $\chi$ is a diffeomorphism of each fiber, because the action of the structure group is transitive. Because the Dirac spinor bundle is constructed from the spin frame bundle we can apply proposition 4.1.23 to
conclude that there are only two bundle isomorphisms $\lambda_\pm : DM_1 \to DM_2 |_{\psi(M)}$ which intertwine the adjoint, charge conjugation and Dirac operator in the required way and that $\lambda_+(u) = -\lambda_-(u)$.

The bundle isomorphisms $\lambda_\pm$ extend in the canonical way to the cospinor bundle and the double spinor bundle and we denote these extensions by the same symbol. Therefore the $\lambda_\pm$ give rise to two linear maps $(\lambda_\pm)_* : D(M_1) \to D(M_2)$. Now let $S^\pm_i$ be the fundamental advanced ($-$) and retarded ($+$) solution to the Dirac equation on the globally hyperbolic spin spacetimes $M_i$. Let $f \in D_0(\psi(M_1))$, let $D_i$ denote the operator $D$ on the spin spacetime $M_i$ and fix a sign $s = \pm$. Then we have (cf. the proof of proposition 3.1.10):

$$f = (\lambda_s)_* \lambda_s^* f = (\lambda_s)_* (D_1 S^\pm_1 \lambda_s^* f) = D_2 ((\lambda_s)_* S^\pm_1 \lambda_s^* f)$$

and

$$\text{supp}((\lambda_s)_* S^\pm_1 \lambda_s^* f) \subset \psi(J^\pm(\text{supp}(\lambda_s^* f))) = J^\pm(\text{supp} f) \cap \psi(M_1)$$

by causal convexity. The uniqueness part of proposition 4.2.3 now shows that $S^\pm_2 |_{\psi(M_1)} = (\lambda_s)_* S^\pm_1 \lambda_s^*$ and hence $S_2 |_{\psi(M_1)} = (\lambda_s)_* S_1 \lambda_s^*$.

If $f \in D_0(M_1)$ has $S_1 f = 0$, then $f = D_1 h$ for some $h \in D_0(M_1)$ by lemma 4.2.4 and hence $S_2 (\lambda_s)_* f = S_2 (\lambda_s)_* D_1 h = S_2 D_2 (\lambda_s)_* h = 0$. Therefore $(\lambda_s)_* : D_0(M_1) \to D_0(M_2)$ descends to a map $\kappa : L(M_1) \to L(M_2)$ (see lemma 4.2.4) which is injective and isometric. Indeed, leaving the metric induced volume elements implicit:

$$(f, h)_{L(M_1)} = i \int_{M_1} f S_1 h = i \int_{M_2} \kappa_s f S_2 \kappa_s h = (\kappa_s f, \kappa_s h)_{L(M_2)},$$

because $\kappa_s f = (\lambda_s)_* f \equiv 0$ outside $\psi(M_1)$. It now follows that there are morphisms $\beta_\pm : \mathcal{F}^0_{M_1} \to \mathcal{F}^0_{M_2}$ in $\text{Alg}$ defined by $\beta_\pm B_{M_1} (f) := B_{M_2} (\kappa_\pm f)$ (see [13]). Finally, as $\lambda_+(f) = -\lambda_-(f)$ we have $\beta_+ B_{M_1} (f) = -\beta_- B_{M_1} (f)$ and hence $\beta_+ = \tau \circ \beta_-$. □
To describe the $\lambda_\pm$ explicitly we fix a choice of complex irreducible representation $\pi$ of the Dirac algebra $D$ and of matrices $A, C \in GL(4, \mathbb{C})$ satisfying assumption (4.5). We use this choice to construct the Dirac spinor bundle on every spin spacetime in $\mathcal{S}\text{Man}$. For the morphism $\Psi : M_1 \rightarrow M_2$ with $\Psi = (\psi, \chi)$ we can then define

$$\lambda_\pm([E, z])_1 := [\chi(E), \pm z]_2 \quad (4.20)$$

and note that this is a linear, base-point preserving bundle homomorphism, which is well-defined because $\chi$ intertwines the right action of $Spin_{1,3}^0$ on both spin frame bundles. Moreover, $\lambda_\pm$ intertwines the adjoint and charge conjugation maps, because $\pm I$ commutes with $A$ and $C$. Hence, $\lambda_\pm$ are the maps of proposition 4.2.20.

Choosing the representation $\pi$ and the matrices $A, C$ does not fix the gauge completely. There are still bundle-automorphisms of the Dirac spinor bundle that leave all physical equations invariant. There are two ways to proceed in order to deal with this residual gauge freedom. The first is to fix it by hand in a locally covariant way. The second is to divide out the gauge freedom. We will present both approaches in that order.

To fix the residual gauge freedom we need to choose a sign for the Dirac spinor bundle on each spin spacetime in a locally covariant way. The following proposition shows that this can be done.

**Proposition 4.2.21** Fix a choice of $\pi$, $A$ and $C$. We can define a locally covariant quantum field theory $\mathcal{F}_0^0 : \mathcal{S}\text{Man} \rightarrow \mathcal{C}\text{Alg}$ which assigns to a globally hyperbolic spin spacetime $M$ the algebra $\mathcal{F}_M^0$ and to every morphism $\Psi : M_1 \rightarrow M_2$ the morphism $\phi_\Psi := \beta_+$ of proposition 4.2.20 associated to the bundle isomorphism $\lambda_+$ defined in equation (4.20).

**Proof.** The maps are well-defined, so we only need to check that they define a covariant functor. The identity morphism gets mapped to the identity
morphism and for a composition of morphisms $\Psi = \Psi_1 \circ \Psi_2$ we have in the obvious notation, $\lambda_+ = (\lambda_1)_+ \circ (\lambda_2)_+$ and hence $\beta_+ = (\beta_1)_+ \circ (\beta_2)_+$, which proves the proposition.

The functor $F^0$ of proposition 4.2.21 seems to depend on the choice of $\pi$, $A$ and $C$, however we will now prove that all choices give rise to equivalent functors. Recall that two functors $F_i : \mathfrak{Man} \to \mathfrak{Alg}$, $i = 1, 2$, are equivalent iff there is a natural transformation between $F_1$ and $F_2$, given by maps $T_M : F^1_M \to F^2_M$, such that each $T_M$ is an isomorphism in the category $\mathfrak{Alg}$ (see e.g. [58, 16]).

**Proposition 4.2.22** Let $\pi$ and $\pi'$ be two complex irreducible representations of the Dirac algebra $D$ and let $A, C$ and $A', C'$ be matrices in $\text{GL}(4, \mathbb{C})$ satisfying assumption (4.5) w.r.t. $\pi$ and $\pi'$, respectively. We let $F^0$ and $(F')^0$ be the corresponding functors of proposition 4.2.21. Then there are two equivalences between the functors $F^0$ and $(F')^0$.

**Proof.** Let $L_\pm$ be the matrices of theorem 4.1.6 which intertwine $\pi, A, C$ and $\pi', A', C'$. For each globally hyperbolic spin spacetime $M$ we then define bundle isomorphisms of the Dirac spinor bundle by

$$\eta^\pm_M([E, z]) := [E, L_\pm z]$$

and we let $\beta^\pm_M$ be the associated $^\ast$-isomorphisms of $F^0_M$ as in proposition 4.2.20.

Now let $\Psi : M_1 \to M_2$ be a morphism in $\mathfrak{Man}$ and let $\lambda_+$ and $\lambda'_+$ be the bundle homomorphisms of equation (4.20) using the representation $\pi$ and $\pi'$.

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11The fact underlying the proof of proposition 4.2.22 is the following. Given a choice of representation $\pi$ and matrices $A, C$ we can define a functor from the category $\mathfrak{Man}$ to the category $\mathfrak{VB}$ of vector bundles over spin manifolds, which maps each spin spacetime to the associated Dirac bundle and which uses $\lambda_+$ to describe embeddings. Different choices of representation then give rise to equivalent functors and there is an equivalence from such a functor to itself which is given by the bundle isomorphism $[E, z] \mapsto [E, -z]$.  

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respectively. Notice that $\eta_{M_2}^+ \circ \lambda^+ = \lambda^+_M \circ \eta_{M_1}^+$. The corresponding equation for the $\ast$-isomorphisms of $\mathcal{F}_M^0$ is then $\beta^\pm_{M_2} \circ \phi_{\Psi} = \phi'_{\Psi} \circ \beta^\pm_{M_1}$, which shows that $\beta^\pm_{M}$ defines an equivalence for each choice of the sign.

**Corollary 4.2.23** We can define a locally covariant quantum field theory $\mathbf{F}^0 : \mathcal{SM}\rightarrow \mathcal{T}\text{Alg}$ which assigns to every globally hyperbolic spin spacetime $M$ the algebra $\mathcal{F}_M^0$ and to every morphism $\Psi : M_1 \rightarrow M_2$ the restriction of the morphism $\phi_{\Psi}$ in $\mathcal{C}\text{Alg}$ to $\mathcal{F}_M^0$. If we define the functor $(\mathbf{F'})^0$ in the same way, but for a different choice of representation $\pi'$ and matrices $A'$, $C'$, then there are two equivalences between $\mathbf{F}^0$ and $(\mathbf{F'})^0$.

**Proof.** This follows from propositions 4.2.21 and 4.2.22. We only need to check that for each morphism the image $\phi_{\Psi}(\mathcal{F}_{M_1}^0)$ is contained in $\mathcal{F}_{M_2}^0$, which follows from the definition of $\phi_{\Psi}$ in the proof of proposition 4.2.20.

The following corollary describes the result of dividing out the residual gauge symmetry:

**Corollary 4.2.24** We can define a locally covariant quantum field theory $\mathbf{B} : \mathcal{SM}\rightarrow \mathcal{C}\text{Alg}$ which assigns to every globally hyperbolic spin spacetime $M$ the algebra $\mathcal{B}_M$ and to every morphism $\Psi : M_1 \rightarrow M_2$ the restriction of the morphism $\phi_{\Psi}$ in $\mathcal{C}\text{Alg}$ to $\mathcal{B}_M$. If we define the functor $\mathbf{B'}$ in the same way, but for a different choice of representation $\pi'$ and matrices $A'$, $C'$, then there is an equivalence between $\mathbf{B}$ and $\mathbf{B'}$.

**Proof.** Again this follows from propositions 4.2.21 and 4.2.22 if we check that for each morphism the image $\phi_{\Psi}(\mathcal{B}_{M_1})$ is contained in $\mathcal{B}_{M_2}$, which follows from the definition of $\phi_{\Psi}$ in the proof of proposition 4.2.20. Note that the two natural transformations of proposition 4.2.20 coincide on the even algebras $\mathcal{B}_M$, so now we only find one equivalence.

Finally we prove the properties of the locally covariant quantum field theories and their associated state spaces:
Proposition 4.2.25 We can define a state space functor $R^0$ for the locally covariant quantum field theory $F^0 : \mathcal{S}\text{Man} \to \mathcal{C}\text{Alg}$, which assigns to every globally hyperbolic spin spacetime $M$ the set $\mathcal{R}_M^0$. Together these functors satisfy the time-slice axiom, additivity and nowhere-classicality.

Proof. We know that each $\mathcal{R}_M^0$ is convex and closed under operations from $\mathcal{F}_M^0$ by proposition 4.2.18. To see that the functor $R^0$ is well-defined we need to show that $\phi_\Psi(\mathcal{R}_M^0) \subset \mathcal{R}_M^0$ for every morphism $\Psi : M_1 \to M_2$. This holds, because a state $\omega$ on $\mathcal{F}_M^0$ which satisfies the $\mu\text{SC}$ restricts to a state on $\mathcal{F}_{\Psi(M_1)}^0$ satisfying the $\mu\text{SC}$ and hence maps to a state on $\mathcal{F}_{M_1}^0$ satisfying the $\mu\text{SC}$, as in the proof of proposition 3.1.9.

Additivity follows as in proposition 3.1.17 by using a partition of unity after choosing representatives in $\mathcal{F}_M$. The time-slice axiom follows from lemma 4.2.19 and proposition 4.2.17 and covariance of the functors $F^0$ and $R^0$. To prove nowhere-classicality we use that fact that for each globally hyperbolic spin spacetime $M$ the $C^*$-algebra $\mathcal{F}_M^0$ is simple, theorem 5.2.5. It is also non-commutative, because any subspace of $\mathcal{L}(M)$ of dimension at least 2 generates a non-commutative Clifford-algebra, which is a sub-algebra of $\mathcal{F}_M^0$. By our definition of the $\mu\text{SC}$, definition 4.2.14, the state $\omega$ of $\mathcal{F}_M^0$ extends to a state on $\mathcal{F}_M^0$, which is necessarily faithful, because $\mathcal{F}_M^0$ is simple. It follows that $\pi_\omega(\mathcal{F}_M^0)$ is not commutative and hence the dense sub-algebra $\pi_\omega(\mathcal{F}_M)$ cannot be commutative.

Proposition 4.2.26 We can define a state space functor $T^0$ for the locally covariant quantum field theory $\overline{F}^0 : \mathcal{S}\text{Man} \to \mathcal{C}\text{Alg}$, which assigns to every globally hyperbolic spin spacetime $M$ the set $\mathcal{R}_M^0$. Together these functors satisfy the time-slice axiom, local physical equivalence, local quasi-equivalence, additivity and nowhere-classicality.

Proof. We know that each $\mathcal{R}_M^0$ is convex and closed under operations from $\overline{\mathcal{F}}_M^0$ by proposition 4.2.18. To see that the functor $T^0$ is well-defined we need
to show that $\phi^*_\Psi(\mathcal{T}^0_{M_2}) \subset \mathcal{T}^0_{M_1}$ for every morphism $\Psi : M_1 \to M_2$. For this we
first notice that a quasi-free Hadamard state $\omega_2$ on $\mathcal{F}^0_{M_2}$ restricts to a quasi-
free Hadamard state $\omega'_1$ on $\mathcal{F}^0_{\psi(M_1)}$ which maps to a quasi-free Hadamard
state $\omega_1$ on $\mathcal{F}^0_{M_1}$. Next we note that a state which is locally quasi-equivalent
to $\omega_2$ restricts to a state which is locally quasi-equivalent to $\omega'_1$ and then
maps to a state which is locally quasi-equivalent to $\omega_1$, by covariance.

The additivity and nowhere-classicality of $\mathcal{F}^0$ and $\mathcal{T}^0$ follow from proposition [4.2.25] by taking the norm closure. The same is true for the time-slice axiom, if we notice in addition that a state which is locally quasi-equivalent to a quasi-free Hadamard state on a neighbourhood of a Cauchy surface extends to a state with the same property, just like in proposition [3.2.5]. Because $\mathcal{F}^0_M$ is simple (theorem 5.2.5 in [13]) the local physical equivalence follows from proposition 4.3 in [33]. Local quasi-equivalence of $\mathcal{T}^0$ was already shown in the proof of proposition [4.2.18]. \[\square\]

**Proposition 4.2.27** We can define a state space functor $\mathbf{T}$ for the locally
covariant quantum field theory $\mathbf{B} : \mathcal{SMan} \to \mathcal{CAlg}$, which assigns to every
globally hyperbolic spin spacetime $M$ the set $\mathcal{T}_M$. Together these functors satisfy causality, the time-slice axiom, local quasi-equivalence and additivity.

**Proof.** We know that each $\mathcal{T}_M$ is convex and closed under operations from
$\mathcal{B}_M$ by proposition [4.2.18]. The fact that the functor $\mathbf{T}$ is well-defined follows from proposition [4.2.26].

To prove causality we choose $f_1, f_2, g_1, g_2 \in \mathcal{D}_0(M)$ such that $\text{supp } f_i \subset O$
and $\text{supp } g_i \subset O^\perp$ for some cc-region $O \subset M$. Then

\[
[B(f_1)B(f_2), B(g_1)B(g_2)] = \\
B(f_1) [B(f_2), B(g_1)B(g_2)] + [B(f_1), B(g_1)B(g_2)] B(f_2) = \\
B(f_1)(\{B(f_2), B(g_1)\} B(g_2) - B(g_1) \{B(f_2), B(g_2)\}) \\
+ (\{B(f_1), B(g_1)\} B(g_2) - B(g_1) \{B(f_1), B(g_2)\}) B(f_2) = 0,
\]
because all the anti-commutators vanish in $\mathcal{F}_M^0$ by the support properties of the $f_i$ and $g_i$.

To prove the time-slice axiom we let $\Psi : M_1 \rightarrow M_2$ be any morphism in $\mathcal{G}\mathfrak{Man}$ for which $\psi(M_1)$ contains a Cauchy surface of $M_2$. We then use the time-slice axiom of proposition 4.2.26 together with the fact that the isomorphism $\phi_\Psi$ preserves the even and odd subspaces of the $\mathcal{F}_M^0$.

That all states in $\mathcal{F}_M$ are locally quasi-equivalent follows from the fact that the restrictions of all states in $\mathcal{F}_M^0$ to $B_M$ are locally quasi-equivalent, which was shown in the proof of proposition 4.2.18.

Additivity follows as in proposition 3.1.17 by using a partition of unity after choosing representatives in $\mathcal{F}_M$. □

Note that $B_M$, $\psi_M$ and $\psi_M^+$ are linear locally covariant quantum fields with values in $\mathcal{F}_M$, but not in $B_M$. To find a locally covariant quantum field for the theory $B$ we could choose e.g. the non-linear field $\Theta_M(f) := B_M(f)B_M(f)$, but we have little need for this field in what follows, because the linear field $B_M$ is much easier to work with.

### 4.3 Relative Cauchy evolution and the stress-energy-momentum tensor for the free Dirac field

Using lemma 4.2.19 which is an explicit expression for the time-slice axiom, we can now consider the relative Cauchy evolution of the free Dirac field (cf. [10]). This means that we will consider the $^*$-isomorphism $\beta$ between the algebras of two cc-regions $N_\pm$ in a spin spacetime $M$, $N_+$ being to the future of $N_-$ and each containing a Cauchy surface for $M$. We study how $\beta$ varies when we vary the metric and/or the spin structure in a compact set in the region between $N_-$ and $N_+$. We will show that we obtain commutators with
the stress-energy-momentum tensor, in complete analogy with the case of
the free scalar field ([14] theorem 4.3).

As a preparation we will first discuss the stress-energy-momentum tensor
in subsection [4.3.1] where we use a point-splitting procedure to obtain an
expression for its commutator with a smeared field operator.

4.3.1 The stress-energy-momentum tensor

In a local frame $e_a$ the stress-energy-momentum tensor for the classical free
Dirac field $\psi$ on a spin spacetime $M$ has the form

$$T_{ab} = \frac{i}{2} \left( \psi^+ \gamma(a \nabla_b) \psi - \nabla(a \psi^+ \gamma_b) \psi \right), \quad (4.21)$$

where the brackets around indices denote symmetrisation as an idempotent
operation. (In the following, indices between $\parallel$ are not to be excluded
from the symmetrisation over.) Following [34] we want to find a point-split bidistri-
bution which acts on scalar test-functions and which is analogous to $T_{ab}$.

For this purpose we use the components $\gamma_a^A_B$ of $\gamma_a$ in a spin frame $E_A$. Recall
that these components are constant and note that

$$T_{ab}^{s}(x, y) := \frac{i}{2} \left( (\psi^+ E_A(x)) \gamma_a^A_B (E_B^B \gamma_b^B \nabla_\mu \psi)(y) 
- (e^\mu_a \nabla_\mu \gamma_b^B \nabla_\mu \psi)(x) \gamma_b^A_B (E_B^B \psi)(y) \right) \quad (4.22)$$

reduces to $T_{ab}$ in the limit $y \to x$. We write $T_{ab}^{s}$ as a bidistribution of scalar
test-functions $f, h$ after performing a partial integration, $\int \nabla_\mu (e_a^\mu \nu u) = 0$:

$$T_{ab}^{s}(f, h) = \frac{i}{2} \left( -\psi^+ (E_A f) \gamma_a^A_B \psi(E_B \gamma_b^B h)) 
+ \psi^+ (\nabla_\mu (e^\mu_a E_A f) \gamma_b^B \psi(E_B h)) \right). \quad (4.23)$$

Equation (4.23) can be promoted to the quantised case by replacing $\psi$ and
$\psi^+$ with the operator-valued distributions $\psi_M$ and $\psi^+_M$ of definition 4.2.10.

The expression (4.22) can be viewed as a formal expression for the same
bidistribution when we substitute the quantised field operators.
Proposition 4.3.1 Writing $\gamma_a(u \oplus v) := (\gamma_a u) \oplus (v \gamma_a)$ and $R(u \oplus v) := u \ominus v$ we have for all $f \in D_0$ and $h \in C^\infty_0(M)$:

$$\int_M [B_M(f), T^a_{ab}(x,x)]\, h(x) \, d\text{vol}_g(x) =$$

$$\frac{1}{2} \{ (\nabla_a B_M(h) S_R f)(x) - B_M(\gamma_b \nabla_a)(S_R f)(x) \}.$$ 

Proof. For $f = u \oplus v$ we use proposition 4.2.11 to obtain:

$$\{ B_M(f), (\psi^+ A_E)(h) \} = (v^+ \oplus 0, E_A h \oplus 0) I = i\langle (S_{\text{cosp}} v)(E_A), h \rangle I$$

$$\{ B_M(f), (E^B e^\mu_b \nabla_\mu \psi_M)(h) \} = -(0 \oplus u^+, 0 \oplus \nabla_\mu (e^\mu_b E^B h)) I = -i\langle E^B (\nabla_b S_{\text{sp}} u), h \rangle I$$

$$\{ B_M(f), (e^\mu_a \nabla_\mu \psi^+_M E_A)(h) \} = -(v^+ \oplus 0, \nabla_\mu (e^\mu_a E_A h) \oplus 0) I = i\langle (\nabla_a S_{\text{cosp}} v)(E_A), h \rangle I$$

$$\{ B_M(f), (E^B \psi_M)(h) \} = (0 \oplus u^+, 0 \oplus E^B h) I = i\langle E^B (S_{\text{sp}} u), h \rangle I$$

where the pairing $\langle \cdot, \cdot \rangle$ on the right-hand side denotes the action of a scalar distribution on $h$. Together with equation (4.22), the anti-commutation relations and $[A, BC] = \{A, B\} C - B \{A, C\}$ this implies:

$$[B_M(f), T^a_{ab}(x,y)] = \frac{1}{2} \left\{ \langle (S_{\text{cosp}} v)(E_A)(x)\gamma_a^A |_{B_l}(E^B e^\mu_b \nabla_\mu \psi_M)(y) - (\psi^+_M E_A)(x)\gamma_a^A |_{B_l}(E^B (\nabla_b S_{\text{sp}} u))(y)angle + \langle (\nabla_a S_{\text{cosp}} v)(E_A)(x)\gamma_b^A_B(E^B \psi_M)(y) - (e^\mu_a \nabla_\mu \psi^+_M E_A)(x)\gamma_b^A_B(E^B (S_{\text{sp}} u))(y) \rangle \right\}.$$ 

In this expression we may take the coincidence limit, which yields:

$$[B_M(f), T^a_{ab}(x,x)] = \frac{1}{2} \left\{ \nabla (\psi^+_M ((S_{\text{cosp}} v) \gamma_a))(x) - \psi^+_M (\gamma_a \nabla_b (S_{\text{sp}} u))(x) - \nabla (\psi^+_M \gamma_b) S_{\text{sp}} u) + \psi_M (\nabla_a (S_{\text{cosp}} v) \gamma_b)(x) \right\}$$

$$= \frac{1}{2} \{ \nabla B_M (\gamma_b) S_R (x) - B_M (\gamma_b) (S_R f)(x) \},$$

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from which the result follows.

Note that the point-split stress-energy-momentum tensor of equation (4.22) can also be used to renormalise the expectation value of the stress-energy-momentum tensor by subtracting a term that cancels out the divergence. In the proof of proposition 4.3.1 however, we used the tensor in a commutator, so any (divergent) multiple of the unit operator $I$ cancels out. For that reason we did not need to subtract any divergent part. For more details on the stress-energy-momentum tensor, its renormalisation and its conservedness we refer to [60, 46], which deal with the real scalar field.

The result of proposition 4.3.1 can be written for spinor and cospinor fields separately as:

$$
\int_M [\psi_M(v), T^s_{ab}(x,x)] h(x) d\text{vol}_g(x) = \frac{-1}{2} \{ \nabla_a \psi_M(\gamma_b S_{\text{cosp}} v h) - \psi_M(\gamma_b \nabla_a (S_{\text{cosp}} v h)) \} \tag{4.24}
$$

$$
\int_M [\psi^+_M(u), T^s_{ab}(x,x)] h(x) d\text{vol}_g(x) = \frac{1}{2} \{ \nabla_a \psi^+_M(\gamma_b S_{\text{sp}} u h) - \psi^+_M(\gamma_b \nabla_a (S_{\text{sp}} u h)) \}.
$$

### 4.3.2 Relative Cauchy evolution

In this subsection we will prove that we can obtain the expressions on the right-hand side of equation (4.24) also via a relative Cauchy evolution. We will first have to explain what such a relative Cauchy evolution means in the case of the free Dirac field theory $\tilde{\mathbf{F}}^0$ (cf. [16]).

Suppose that we have two objects in $\mathcal{GW}an$, $M = (\mathcal{M},g,SM,p)$ and $M' = (\mathcal{M},g',SM',p')$, where the manifold $\mathcal{M}$ is the same in both cases and such that both spin spacetimes are the same outside a compact subset $K \subset \mathcal{M}$, i.e. $g'|_{\mathcal{M}\setminus K} = g|_{\mathcal{M}\setminus K}$ and $(SM',p')|_{\mathcal{M}\setminus K} = (SM,p)|_{\mathcal{M}\setminus K}$. Now let $N^\pm \subset M$ be cc-regions, each containing a Cauchy surface for $M$ and such
that $K$ lies to the future of $N^-$ (i.e. $K \subset J^+(N^-) \setminus N^-$ in $M$ and hence also in $M'$) and to the past of $N^+$. We view $N^\pm$ as objects in $\mathcal{SM}an$ and consider the canonical embedding morphisms $\iota^\pm : N^\pm \to M$ and $(\iota')^\pm : N^\pm \to M'$. By the time-slice axiom, proposition [4.2.26], these give rise to $^*\text{-isomorphisms}$ $\beta^\pm : \mathcal{F}_M^0 \to \mathcal{F}_M^0$ and $(\beta')^\pm : \mathcal{F}_{M'}^0 \to \mathcal{F}_{M'}^0$. We then define the $^*$-automorphism $\beta_g'$ of $\mathcal{F}_M^0$ by

$$\beta_g' := \beta^+ \circ ((\beta')^+)\circ (\beta')^- \circ (\beta^-)^{-1}.$$  (4.25)

This $^*$-automorphism can easily be characterised in terms of its action on the generators $B_M(f)$ of $\mathcal{F}_M^0$ as follows:

**Proposition 4.3.2** If $f \in D_0(M)$ with supp $f \subset N^+$, then $\beta_g B_M(f) = B_M(T_g f)$, where

$$T_g f = D'\chi_+ S'D\chi_- S f.$$  

Here the superscripts on $D$ and $S$ indicate whether they are the objects defined on $M$ or $M'$ and the smooth functions $\chi_\pm$ are such that $\chi_\pm \equiv 1$ to the past of some Cauchy surface in $N^\pm$ and $\chi_\pm \equiv 0$ to the future of some other Cauchy surface in $N^\pm$.

**Proof.** Note that $(\beta')^- \circ (\beta^-)^{-1}B_M(\tilde{f}) = B_{M'}(\tilde{f})$ for any $\tilde{f} \in D_0(N^-)$. Similarly, for $f' \in D_0(N^+)$ we have $\beta^+ \circ ((\beta')^+)\circ (\beta')^- \circ (\beta^-)^{-1}B_{M'}(f') = B_M(f')$. The functions $\chi_\pm, 1 - \chi_\pm$ have been chosen appropriately in order to apply equation [4.19]. We then have $S\tilde{f} = S f$ and hence $B_M(\tilde{f}) = B_M(f)$, where $\tilde{f} := D\chi_- S f$. Notice that $\tilde{f}$ indeed has a compact support in $N^-$. Similarly we have $B_{M'}(\tilde{f}) = B_{M'}(f')$, where $f' := D'\chi_+ S'\tilde{f}$ has support in $N^+$. Putting everything together yields for $f' = T_g f$:

$$\beta_g' B_M(f) = \beta_g' B_M(\tilde{f}) = \beta^+ \circ ((\beta')^+)\circ (\beta')^- \circ (\beta^-)^{-1}B_{M'}(\tilde{f}) = \beta^+ \circ ((\beta')^+)\circ (\beta')^- \circ (\beta^-)^{-1}B_{M'}(f') = B_M(f').$$

$\square$
We will want to compute the variation of the \( * \)-isomorphism \( \beta_{g'} \) with respect to the metric \( g' \). For this purpose we suppose that the compact set \( K \subset \mathcal{M} \) has a contractible neighbourhood \( O \) which doesn’t intersect either \( N^\pm \). Now let \( \epsilon \mapsto g_\epsilon \) be a smooth curve from \([0, 1] \) into the space of Lorentzian metrics on \( \mathcal{M} \) starting at \( g \) and such that \( g_\epsilon = g \) outside \( K \) for every \( \epsilon \).

The spin bundle \( SM_\epsilon \) must be trivial over the contractible region \( O \). If we assume it to be diffeomorphic to \( SM \) outside \( K \) we can simply take \( SM_\epsilon := SM \) as a manifold and, choosing a fixed complex irreducible representation \( \pi \) and matrices \( A, C \) satisfying assumption \([4.5] \) to construct the Dirac spinor bundle, we obtain \( DM_\epsilon = DM \). The deformation of the spin structure is contained entirely in the \( \epsilon \)-dependence of the projection \( p_\epsilon : SM \to FM_\epsilon \). Now let \( E \) be a section of \( SM \) over \( O \) and set \( (e_\epsilon)_a := p_\epsilon(E) \). We require that \( e_\epsilon \) varies smoothly with \( \epsilon \) and that \( (e_\epsilon)_a = (e)_a = p_\epsilon(E) \) outside \( K \). To show that projections \( p_\epsilon \) with these properties exist we can apply the Gram-Schmidt orthonormalisation procedure for all \( \epsilon \) simultaneously, starting with the frame \( (e)_a \), which yields a smooth family of frames \( (e_\epsilon)_a \). The assignment \( p_\epsilon : E \mapsto e_\epsilon \) then determines \( p_\epsilon \) completely, because of the intertwining properties of \( p_\epsilon \) and the transitive action of \( Spin^0_{1,3} \) on the spin frame bundle. The family of frames \( e_\epsilon \) determines principal fiber bundle isomorphisms \( f_\epsilon : FM_\epsilon \to FM \) between the frame bundles by

\[
 f_\epsilon : \{(e_\epsilon)_a\} \mapsto \{(e)_a\}
\]
on \( K \) and extending it by the identity on the rest of \( \mathcal{M} \). By definition \( f_\epsilon \) intertwines the action of \( \mathcal{L}_+^1 \) on the frame bundles.

There may be many deformations of the spin structure, i.e. many families of projections \( p_\epsilon \) which satisfy our requirements. However, the variation of \( D_\epsilon f \) will not depend on this choice. Indeed, if \( p'_\epsilon \) is a different deformation of the spin structure, then \( e'_\epsilon := p'_\epsilon(E) = R_{\Lambda_\epsilon} e_\epsilon = p_\epsilon(R_{S_\epsilon} E) \) for some smooth curve \( S_\epsilon \) in \( Spin^0_{1,3} \). However, \( v \in DM_\epsilon = DM \) and \( D_\epsilon v \) are invariant under
the action of the gauge group $\text{Spin}^0_{1,3}$ and therefore the variation will be too.

On each spin spacetime $M_\epsilon = (\mathcal{M}, g_\epsilon, SM, p_\epsilon)$ we can now quantise the Dirac field and obtain relative Cauchy evolutions $\beta_\epsilon := \beta_{g_\epsilon}$ on $\mathcal{F}^0_M$ as in equation 4.25.

**Proposition 4.3.3** Writing $\delta := \partial_\epsilon|_{\epsilon=0}$ we have for all $f \in \mathcal{D}_0(M)$ with $\text{supp} \ f \subset N^+$:

$$\delta(\beta_\epsilon B_M(f)) = B_M((\delta D_\epsilon)Sf).$$

**Proof.** Using the fact that $B_M$ is a $C^*$-algebra-valued distribution and proposition 4.3.2 we find:

$$\delta(\beta_\epsilon B_M(f)) = \delta(B_M(D_\epsilon \chi_+ S_\epsilon D_\chi_- Sf)) = B_M(\delta(D_\epsilon \chi_+ S_\epsilon)D_\chi_- Sf)$$

$$= B_M(\delta(D_\epsilon)\chi_+ SD_\chi_- Sf) \quad + \quad B_M(D_\chi_+ \delta(S_\epsilon)D_\chi_- Sf).$$

Now, because $D_\chi_- Sf \in \mathcal{D}_0(N^-)$ and $N^-$ is to the past of $K$ we see that $\delta(S_\epsilon)D_\chi_- Sf$ vanishes on $J^-(N^-)$ and that $\chi_+ \delta(S_\epsilon)D_\chi_- Sf$ has compact support. Because $B_M$ solves the Dirac equation we conclude that the second term vanishes. The first term can be rewritten using equation (4.19), which yields:

$$\delta(\beta_\epsilon B_M(f)) = B_M(\delta(D_\epsilon)\chi_+ Sf) = B_M(\delta(D_\epsilon)Sf).$$

For the last equality we used the fact that $\delta(D_\epsilon)$ is supported in $K$, where $\chi_+ \equiv 1$. \hfill \Box

To compute the variation of $D_\epsilon$ we may work in a local frame on the contractible region $O$, because that is where $\delta(D_\epsilon)$ is supported. Recall that $D = (-i\nabla + m) \oplus (i\nabla + m)$, so essentially we just need to find the variation of $\nabla_\epsilon$ on spinor and cospinor fields. In fact, we will next show that it is sufficient to know the variation of this operator on cospinor fields, because we can then derive the case of spinor fields using the adjoint map. This uses the fact that the Dirac adjoint map is independent of $\epsilon$. Also note that the
components $\gamma^B_a$, in a local frame determined by the section $E$ of $SM$ over $O$, are constant and independent of $\epsilon$. This follows immediately from the definition of $\gamma$, in lemma \[4.1.17\]

**Lemma 4.3.4** For $v \in C^\infty_0(D^*M)$ we have $\delta(\nabla)v = (\delta(\nabla)v^+)^+$.

**Proof.** Because the adjoint operation between spinor and cospinor fields is continuous we have:

$$
\delta(\nabla)v = \partial_\epsilon \nabla_\epsilon v|_{\epsilon=0} = \partial_\epsilon (\nabla_\epsilon v^+)|_{\epsilon=0} = (\partial_\epsilon \nabla_\epsilon v^+|_{\epsilon=0})^+ = (\delta(\nabla)v^+)^+.
$$

□

We now start the computation of the variation of the Dirac operator on a cospinor field. For this purpose we will work in components and in local coordinates on the contractible neighbourhood $O$. To ease the notation we will drop the subscript $\epsilon$ on the local frame $e^\mu_a$. As $\gamma^a$ is independent of $\epsilon$ we may use equations \[4.14\] and \[4.12\] to vary the following equation for $v \in C^\infty_0(D^*M_0)$:

$$
\nabla v = \left( \partial_a v - \frac{1}{4} \Gamma^{c}_{\alpha\beta} v \gamma^c \gamma^\alpha \gamma^\beta \right) \gamma^a
$$

$$
= e^a_a \left( \partial_a v + \frac{1}{4} e^\beta_b \left( \partial_a e^c_b - e^c_c \Gamma^\alpha_{\alpha\beta} \right) v \gamma^c \gamma^\beta \right) \gamma^a, \tag{4.26}
$$

which yields:

$$
\delta \nabla v = \delta e^a_a \partial_a \nabla v \gamma^a - \frac{1}{4} \delta e^\beta_b e^\beta_{\gamma\delta} \Gamma^{c}_{\alpha\beta} v \gamma^c \gamma^\beta \gamma^\alpha a + \frac{1}{4} \delta \partial_a e^\gamma_c e^\gamma_b v \gamma^c \gamma^\beta \gamma^\alpha
$$

$$
- \frac{1}{4} \delta \gamma^\alpha e^a_b v \gamma^c \gamma^\beta \gamma^\alpha a - \frac{1}{4} \delta \Gamma^\gamma_{\alpha\beta} e^a_a e^\gamma_c e^\gamma_b v \gamma^c \gamma^\beta \gamma^\alpha, \tag{4.27}
$$

where we inserted a factor $\delta^\gamma = e^\gamma_\beta e^\gamma_a$ twice to simplify the first two terms.

We now define $D_c := i\nabla + m$ acting on cospinor fields and we try to get terms with this operator acting on $v$ or on the whole expression. These are harmless when we compute $B_M(\delta \nabla Sf)$, because $B_M$ and $v = Sf$ solve the
Dirac equation. We start by performing what is essentially an integration by parts as follows:

\[
\frac{1}{4} \partial_a \delta e^c_{\beta} e^\beta_b v_{\gamma c} \gamma^b = -\frac{i}{4} D_c (\delta e^c_{\beta} e^\beta_b v_{\gamma c} \gamma^b) + \frac{i}{4} \delta e^c_{\beta} e^\beta_b D_c (v_{\gamma c} \gamma^b) \\
-\frac{1}{4} \delta e^c_{\beta} \partial_a e^\beta_b v_{\gamma c} \gamma^b \gamma^a \\
+\frac{1}{4} \delta e^c_{\beta} e^\beta_b \Gamma^b_{ad} v_{\gamma c} \gamma^d \gamma^a \\
= -\frac{i}{4} D_c (\delta e^c_{\beta} e^\beta_b v_{\gamma c} \gamma^b) + \frac{i}{4} \delta e^c_{\beta} e^\beta_b (D_c v) \gamma_c \gamma^b \\
-\frac{1}{4} \delta e^c_{\beta} e^\beta_b \nabla_a v \left[ \gamma_c \gamma^b, \gamma^a \right] - \frac{1}{4} \delta e^c_{\beta} \partial_a e^\beta_b v_{\gamma c} \gamma^b \gamma^a \\
+\frac{1}{4} \delta e^c_{\beta} e^\beta_b \Gamma^c_{ad} v_{\gamma c} \gamma^b \gamma^a \\
\tag{4.28}
\]

Because \([\gamma_c \gamma^b, \gamma^a] = \gamma_c \left\{ \gamma^b, \gamma^a \right\} - \{ \gamma_c, \gamma^a \} \gamma^b = 2\eta^{ab} \gamma_c - 2\delta_c^a \gamma^b\) we can write:

\[
-\frac{1}{4} \delta e^c_{\beta} e^\beta_b \nabla_a v \left[ \gamma_c \gamma^b, \gamma^a \right] = -\frac{1}{2} \delta (g_{\mu \nu} \eta^{cd} e^\mu_d) e^\beta_b \eta^{ab} \nabla_a v_{\gamma c} + \frac{1}{2} \delta e^c_{\beta} e^\beta_b \nabla_c v \gamma^b \\
= -\frac{1}{2} \delta g_{\mu \nu} \eta^{cd} e^\mu_d e^\beta_b \eta^{ab} \nabla_a v_{\gamma c} - \delta e^\mu_c e^\mu_d \nabla_d v_{\gamma d} \\
= \frac{1}{2} \delta g^{\alpha \beta} e^\alpha_a e^\alpha_b \nabla_a v_{\gamma b} - \delta e^\alpha_c e^\alpha_d \nabla_d v_{\gamma a}. \tag{4.29}
\]

When substituting equations (4.28) and (4.29) in (4.27) we can recombine the terms

\[
-\frac{1}{4} \delta e^c_{\beta} \partial_a e^\beta_b v_{\gamma c} \gamma^b \gamma^a \\
-\frac{1}{4} \delta e^c_{\gamma} e^\gamma_a e^\gamma_b \Gamma^{\gamma}_{\alpha \gamma} v_{\gamma c} \gamma^b \gamma^a = -\frac{1}{4} \delta e^c_{\gamma} e^\gamma_d \Gamma^{d}_{\alpha \gamma} v_{\gamma c} \gamma^b \gamma^a
\]

to obtain

\[
\delta \nabla v = -\frac{i}{4} D_c (\delta e^c_{\beta} e^\beta_b v_{\gamma c} \gamma^b) + \frac{i}{4} \delta e^c_{\beta} e^\beta_b (D_c v) \gamma_c \gamma^b \\
+\frac{1}{2} \delta g^{\alpha \beta} e^\alpha_a e^\alpha_b \nabla_a v_{\gamma b} - \frac{1}{4} \delta \Gamma^{\gamma}_{\alpha \beta} e^\alpha_a e^\alpha_b e^\gamma_c v_{\gamma c} \gamma^b \gamma^a. \tag{4.30}
\]

Note that the variations of the frame \(\delta e^a_a\) cancel out, except in the terms with \(D_c\). Therefore, the final answer will not depend on variations of the frame, as desired.
In the last term of equation (4.30) we can use the symmetry of the Christoffel symbol in the lower indices:

\[
- \frac{1}{4} \delta \Gamma^\gamma_{\alpha \beta} e^\alpha_a e^\beta_b \epsilon^c_v \gamma^b \gamma^c = - \frac{1}{4} \delta \Gamma^\gamma_{\alpha \beta} e^\alpha_a e^\beta_b \epsilon^c_v \gamma^b \gamma^c = - \frac{1}{4} \delta g^{\gamma \mu} g_{\mu \nu} \Gamma^\nu_{\alpha \beta} g^{\alpha \beta} \epsilon^c_v \gamma^c - \frac{1}{4} \partial_\alpha \delta g_{\beta \mu} e^\mu_a g^{\alpha \beta} \epsilon^c_v \gamma^c
\]

\[
+ \frac{1}{8} \partial_\mu \delta g_{\alpha \beta} e^\mu_a g^{\alpha \beta} \epsilon^c_v \gamma^c = - \frac{1}{4} \frac{\partial_\alpha}{8} \delta g_{\alpha \beta} g^{\alpha \beta} \epsilon^c_v \gamma^c.
\]

The second term in equation (4.31) is:

\[
- \frac{1}{4} \partial_\alpha \delta g_{\alpha \beta} g^{\alpha \beta} \epsilon^c_v \gamma^c = \frac{1}{4} \partial_\alpha (\delta g_{\alpha \beta} g_{\mu \nu} \Gamma^\mu_{\alpha \beta} g_{\rho \sigma} \partial_\sigma (e^\rho_a e^\mu_b g^{\mu \nu} v^\gamma_a)
\]

\[
= \frac{1}{4} \partial_\alpha (\delta g_{\alpha \beta} e^\alpha_a e^\beta_b v^\gamma_a) - \frac{1}{4} \delta g^{\alpha \beta} g_{\alpha \mu} g_{\beta \nu} \partial_\sigma (e^\rho_a e^\mu_b g^{\mu \nu} v^\gamma_a)
\]

\[
= \frac{1}{4} \nabla_\alpha (\delta g_{\alpha \beta} e^\alpha_a e^\beta_b v^\gamma_a) - \frac{1}{4} \delta g_{\alpha \beta} \Gamma^a_{bc} e^b_{\alpha} e^c_{\beta} v^\gamma_a - \frac{1}{4} \delta g^{\alpha \beta} g_{\alpha \mu} g_{\beta \nu} \partial_\sigma (e^\rho_a e^\mu_b g^{\mu \nu} v^\gamma_a).
\]

The first term of equation (4.33) is

\[
\frac{1}{4} \nabla_\alpha (\delta g_{\alpha \beta} e^\alpha_a e^\beta_b v^\gamma_a) = \frac{1}{4} \nabla_\alpha (\delta g_{\alpha \beta} e^\alpha_a e^\beta_b v^\gamma_a) - \frac{1}{4} \delta g^{\alpha \beta} e^\alpha_a e^\beta_b \nabla_\alpha v^\gamma_a.
\]
The other terms can be simplified using equation (4.12) and some computation, which yields:

\[- \frac{1}{4} \delta g^{\alpha \beta} (\Gamma^a_{\beta c} e^c_\alpha e^b_\beta + \Gamma^b_{\alpha c} e^a_\alpha e^c_\beta + g_{\alpha \mu} g_{\beta \nu} \eta^{ac} \partial_b (e^\mu_c e^b_\rho g^{\rho \nu})) v_{\gamma a} = \]

\[- \frac{1}{4} \delta g^{\alpha \beta} (e^\alpha_\gamma \partial_\beta e^\gamma_\beta + e^\alpha_\gamma \Gamma^\gamma_{\beta a} + e^\alpha_\alpha \partial_\mu e^\mu_\epsilon e^\beta_\epsilon + e^\alpha_\alpha \Gamma^\mu_{\mu \beta} + e^\alpha_\alpha \partial_\rho g^{\rho \nu} + e^\alpha_\alpha \partial_\beta e^\epsilon_\beta + g_{\alpha \mu} \eta^{ac} \partial_\beta e^\mu_\epsilon) v_{\gamma a} = \]

\[- \frac{1}{4} \delta g^{\alpha \beta} (e^\alpha_\gamma \Gamma^\gamma_{\beta a} - e^\alpha_\alpha \partial_\epsilon e^\beta_\epsilon + e^\alpha_\alpha \Gamma^\mu_{\mu \beta} - e^\alpha_\alpha \partial_\rho g^{\rho \nu} + e^\alpha_\alpha \partial_\beta e^\epsilon_\beta + g_{\alpha \mu} \eta^{ac} \partial_\beta e^\mu_\epsilon) v_{\gamma a} = \]

\[- \frac{1}{4} \delta g^{\alpha \beta} (-\eta^{ac} e^\mu_\epsilon \partial_\beta g_{\alpha \mu} + e^\alpha_\alpha \Gamma^\gamma_{\beta a} + e^\alpha_\alpha \Gamma^\mu_{\mu \beta} - e^\alpha_\alpha g^{\rho \nu} \partial_\rho g_{\beta \nu}) v_{\gamma a} = \]

\[- \frac{1}{4} \delta g^{\alpha \beta} (-2 e^\alpha_\gamma g^{\mu \nu} \partial_\beta g_{\alpha \mu} + e^\alpha_\gamma g^{\gamma \mu} (2 \partial_\beta g_{\alpha \mu} - \partial_\mu g_{\alpha \beta}) + e^\alpha_\alpha g^{\rho \nu} \partial_\beta g_{\mu \nu} - 2 e^\alpha_\alpha g^{\rho \nu} \partial_\rho g_{\beta \nu}) v_{\gamma a} = \]

\[- \frac{1}{8} \delta g^{\alpha \beta} (e^\alpha_\gamma g^{\gamma \mu} \partial_\mu g_{\alpha \beta} + 2 e^\alpha_\alpha g_{\beta \nu} g^{\rho \nu} \Gamma^\mu_{\rho \nu}) v_{\gamma a}. \tag{4.35} \]

Substituting equations (4.31,4.35) into (4.30) yields:

\[ \delta \nabla u = -i D_c (\delta e^c_\beta e^\beta_\beta \nabla \gamma_c e_{\gamma b}) + \frac{i}{4} D_c (\delta e^c_\beta e^\beta_\beta (D_c v)) \gamma_c e_{\gamma b} - \frac{i}{8} D_c (\delta g_{\alpha \beta} g^{\alpha \beta} v) + \frac{i}{8} \delta g_{\alpha \beta} g^{\alpha \beta} D_c v + \frac{1}{4} \delta g^{\alpha \beta} e^a_\alpha e^b_\beta \nabla a v_{\gamma b} + \frac{1}{4} \nabla_b (\delta g^{\alpha \beta} e^a_\alpha e^b_\beta v_{\gamma a}). \tag{4.36} \]

Using lemma 4.3.4 and introducing \( D_s := -i \nabla + m \) we find for a spinor field \( u \in C^\infty (DM) \):

\[ \delta \nabla u = \frac{i}{4} D_s (\delta e^c_\beta e^\beta_\beta \gamma_c e_{\gamma b}) - \frac{i}{4} \delta e^c_\beta e^\beta_\beta \gamma_c (D_s u) + \frac{i}{8} D_s (\delta g_{\alpha \beta} g^{\alpha \beta} u) - \frac{i}{8} \delta g_{\alpha \beta} g^{\alpha \beta} D_s u + \frac{1}{4} \delta g^{\alpha \beta} e^a_\alpha e^b_\beta \nabla a u + \frac{1}{4} \nabla_b (\delta g^{\alpha \beta} e^a_\alpha e^b_\beta u). \tag{4.37} \]

Using the same notation as in proposition 4.3.1 we find the result:
Theorem 4.3.5 For a double test-spinor \( f \in \mathcal{D}_0(M) \) with \( \text{supp} \ f \subset N^+ \) and for \( x \in K \):

\[
\frac{\delta}{\delta g^{\alpha\beta}(x)} (\beta_g B_M(f)) = B_M \left( \frac{\delta}{\delta g^{\alpha\beta}(x)} D_g S f \right) = \frac{i}{2} e^a \epsilon_b [B_M(f), T^{s}_{ab}(x,x)] .
\] (4.38)

**Proof.** Using proposition 4.3.3 and equations (4.36, 4.37) we notice that the terms with \( D_c \) and \( D_s \) cancel out, because \( B_M \) and \( S f \) satisfy the (doubled) Dirac equation:

\[
\delta(g^{\alpha\beta}(x)) \beta_g B_M(f)) = B_M(\delta D_c S f) = \]

\[
\frac{-i}{4} B_M(\delta g^{\alpha\beta} e^a \epsilon_b \gamma_a \nabla_a SRf) - \frac{i}{4} B_M(\nabla_b(\delta g^{\alpha\beta} e^a \epsilon_b \gamma_a SRf)) = \frac{-i}{4} \delta g^{\alpha\beta} e^a \epsilon_b (B_M(\gamma(b \nabla_a) SRf) - \nabla(b B_M(\gamma(a) SRf)).
\] (4.39)

We now compare with proposition 4.3.1 to obtain the result. \( \square \)

This result compares well with the scalar field case, theorem 4.3 in [16]. As particular cases we obtain for \( \psi_M \) and \( \psi_M^* \):

\[
\frac{\delta}{\delta g^{\alpha\beta}(x)} (\beta_g \psi_M(v)) = \frac{i}{2} e^a \epsilon_b [\psi_M(v), T^{s}_{ab}(x,x)] ,
\]

\[
\frac{\delta}{\delta g^{\alpha\beta}(x)} (\beta_g \psi_M^*(u)) = \frac{i}{2} e^a \epsilon_b [\psi_M^*(u), T^{s}_{ab}(x,x)] .
\]

**Corollary 4.3.6** Let \( X \in \mathcal{F}_M^0 \) and \( x \in K \), then

\[
\frac{\delta}{\delta g^{\alpha\beta}(x)} (\beta_g X) = \frac{i}{2} e^a \epsilon_b [X, T^{s}_{ab}(x,x)] .
\]

**Proof.** Theorem 4.3.5 tells us that the equation is true if \( X = B_M(f) \) for any double test-spinor \( f \in \mathcal{D}_0(M) \) with \( \text{supp} \ f \subset N^+ \). The same is then true for any monomial of such terms, because for \( X_1, X_2 \in \mathcal{F}_M^0 \) we have \( \beta_g(X_1 X_2) = \beta_g(X_1) \beta_g(X_2) \) and hence

\[
\frac{\delta}{\delta g^{\alpha\beta}(x)} (\beta_g(X_1 X_2)) = \frac{\delta}{\delta g^{\alpha\beta}(x)} (\beta_g(X_1)) X_2 + X_1 \frac{\delta}{\delta g^{\alpha\beta}(x)} (\beta_g(X_2))
\]

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and $[X_1X_2, T] = [X_1, T]X_2 + X_1[X_2, T]$ for any operator $T$. Finally, because the equation is linear in $X$ it holds for any polynomial of terms $B_M(f)$ with $f$ supported in $N^+$. Any $X \in \mathcal{F}_M^0$ is of this form, by lemma 4.2.19. This completes the proof.
Chapter 5

The Reeh-Schlieder property in curved spacetime

Er stürzte sich allerdings in das Unermessliche, das die astro-physische Wissenschaft zu messen sucht, nur um dabei zu Maßen, Zahlen, Größenordnungen zu gelangen, zu denen der Menschengeist gar kein Verhältnis mehr hat, und die sich im Theoretischen und Abstrakten, im völlig Unsinnlichen, um nicht zu sagen: Un-sinnigen verlieren.

Thomas Mann, Doktor Faustus, Ch. 27

The Reeh-Schlieder theorem [69] is a result in axiomatic quantum field theory which states that for a scalar Wightman field in Minkowski spacetime any state in the Hilbert space can be approximated arbitrarily well by acting on the vacuum with operations performed in any prescribed open region. The physical meaning of this is that the vacuum state has very many non-local correlations and an experimenter in any given region can exploit the vacuum fluctuations by performing a suitable measurement in order to produce any desired state up to arbitrary accuracy.
The original proof uses analytic continuation arguments, an approach which was extended to analytic spacetimes in [80] by replacing the spectrum condition of the Wightman axioms in Minkowski spacetime by an analytic microlocal spectrum condition. For spacetimes which are not analytic a result by Strohmaier [79] (see also [83]) shows that in a stationary spacetime all ground and thermal (KMS-)states of several types of free fields (including the Klein-Gordon, Dirac and Proca field) also have the Reeh-Schlieder property. To prove the existence of such states directly one may need to make further assumptions, depending on the type of field (see [79]).

In this chapter we will investigate whether Reeh-Schlieder states exist in general globally hyperbolic spacetimes, which may be neither analytic nor stationary. First we will define and discuss the Reeh-Schlieder property in the context of locally covariant quantum field theory in section 5.1 and discuss the relevant fact that not all states on an algebra need to be in the physical state space. Next we will prove some general results in section 5.2, namely that the Reeh-Schlieder property is local and stable under purifications. We then proceed to discuss the possibility of deforming a Reeh-Schlieder state on one spacetime into a Reeh-Schlieder state on a diffeomorphic (but not isometric) spacetime in section 5.3. For this we use the time-slice axiom and the technique of spacetime deformation as pioneered in [38] and as applied successfully to prove a spin-statistics theorem in curved spacetime in [85]. We will prove that, given a Reeh-Schlieder state on the initial globally hyperbolic spacetime, we can find for every region in the deformed spacetime a state in the physical state space that has the Reeh-Schlieder property for that particular region (but maybe not for all regions). After these general results we specialise in section 5.4 to the Borchers-Uhlmann functor $U$ of chapter 3 and give a smoothly covariant condition on states that guarantees that a state has the Reeh-Schlieder property and satisfies the $\mu$SC. Next we specialise even further to the real free scalar field in Minkowski spacetime and we
prove the existence of many Hadamard Reeh-Schlieder states in section 5.5. Finally, we draw some conclusions concerning the Reeh-Schlieder property in locally covariant quantum field theory in section 5.6.

5.1 The Reeh-Schlieder property in a locally covariant quantum field theory

In locally covariant quantum field theory we define the Reeh-Schlieder property as follows:

**Definition 5.1.1** Consider a locally covariant quantum field theory \( \mathcal{A} \) with a state space \( S \). A state \( \omega \in S_M \) has the Reeh-Schlieder property for a cc-region \( O \subset M \) iff

\[
\pi_\omega(A_O)\Omega_\omega = \mathcal{H}_\omega.
\]

We then say that \( \omega \) is a Reeh-Schlieder state for \( O \). We say that \( \omega \) is a (full) Reeh-Schlieder state, or that \( \omega \) has the (full) Reeh-Schlieder property, iff it is a Reeh-Schlieder state for all cc-regions in \( M \).

The original result of [69] then states that the vacuum state \( \omega \) of a scalar Wightman field theory in Minkowski spacetime \( M_0 \) has the full Reeh-Schlieder property. It implies that the vacuum is an entangled state even over causally disjoint regions of spacetime [50, 21]. (Even if a state has the Reeh-Schlieder property only for a certain cc-region \( O \) and the theory is nowhere classical, there exist non-local correlations between \( O \) and any cc-region \( V \) space-like to it [68]). Furthermore, the entanglement can be improved using a distillation procedure (see [86]) to approximate a maximal violation of the Bell inequalities (for appropriate observables in the two disjoint regions). Moreover, it is argued in [22] that these non-local correlations cannot easily be avoided. Indeed, if there is one vector in a Hilbert space which defines a
state with the Reeh-Schlieder property for a theory defined by $C^*$-algebras, and this is the case for example for the Minkowski vacuum of the real free scalar field in terms of the local Weyl algebras (i.e. using the functor $A^0$ of chapter [3], then the same is true for a generic vector in that Hilbert space. (We will make this statement more precise below in definition [5.1.2].) Therefore, by Fell’s theorem (see [40] theorem 3.2.2.13), we cannot distinguish a Reeh-Schlieder vector-state from a vector-state that does not have the Reeh-Schlieder property.\footnote{[22] contrasts this with the entanglement that can occur between two systems in quantum mechanics and that can be undone by performing a measurement on one of the systems. However, [22] also argues that scientific methodology is not in danger due to another property commonly found in quantum field theories, namely the split property (see e.g. [40]). This allows one to isolate systems “for all practical purposes”.

Another consequence of the Reeh-Schlieder theorem is that every nontrivial positive local operator has a strictly positive vacuum expectation value (see e.g. proposition 5.2.2 ahead). The Reeh-Schlieder theorem therefore poses a problem for a notion of localised particles, because it is impossible to create, annihilate or even just count particles using local operations and again these problems cannot easily be avoided [41].

Apparently the conclusion must be that entanglement is the rule rather than the exception. On the other hand, it can be argued that the energy required to approximate a given state increases with the desired accuracy (see [40] p.254, [22]). In other words, it may take an increasingly large ensemble in order for a selective measurement to give a positive result within the desired range of accuracy. It is also known that the strength of any non-classical correlations decay exponentially with the separation between the regions of interest (see [81]). The Reeh-Schlieder theorem also has many theoretical implications, both of a mathematical and of a physical nature. On the mathematical side it can be used to determine the type of local von Neumann algebras [48, 1] and it allows the application of Tomita-Takesaki
modular theory, which has led to a field of research in its own right (see [12] for a review).

An extension of the Reeh-Schlieder theorem to curved spacetimes would have physical implications as well. One of its consequences would be the existence of correlations between observables localised in regions which are causally disjoint, or even in regions which have always been causally disjoint. This has led to speculations about its importance for the physical understanding of e.g. black holes ([5]) and the early universe. As Wald puts it in the beautiful little essay [89]:

The point I do wish to make is that – at the very least – it is far from obvious, a priori, that the relevant correlations beyond the horizon in a quantum field theory model will be small, and the neglect of such correlations in analyzing any phenomenon must be justified by quantitative estimates rather than by a simple appeal to a lack of causal communication. Indeed, because of the fundamental and universal nature of these correlations, it would be surprising if they did not play some important role in our understanding of the nature of the early universe.

Wald then goes on to state that

... the strength and generality of the Reeh-Schleider [sic] theorem in flat spacetime is such that it seems inconceivable that similar correlations could fail to be present for essentially all states and over essentially all regions in any curved spacetime, including cosmological spacetimes with horizons.

As we already mentioned, the Reeh-Schlieder property has indeed been shown to hold in curved spacetimes too, under certain conditions. However, there are also some arguments that suggest it may not be quite as general
as Wald suggests. Indeed, even in Minkowski spacetime it can be argued that the property may not be as omnipresent as the references above make us believe. To be specific, let us examine the case of the Minkowski vacuum state \( \omega_0 \) of the real free scalar field, described by a net of algebras \([40, 16]\). If this field is described by the functor \( A^0 \) (see chapter \([3]\)), then the results of \([22, 31]\) do show that a generic vector in \( \mathcal{H}_{\omega_0} \) defines a state in the state space \( \mathcal{S}_{M_0}^0 \) (for the state space defined in chapter \([3]\)) which has the Reeh-Schlieder property. But now suppose that we describe the real free scalar field by the free field Borchers-Uhlmann functor \( U^0 \). Although both functors are meant to describe the same field, and despite the correspondence between the descriptions (see the discussion above and below definition \([3.2.3]\)), the mathematical situation is very different. One can check in this concrete example that a Reeh-Schlieder state for \( U^0_{M_0} \) is also a Reeh-Schlieder state for \( A^0_{M_0} \) (see e.g. \([3]\) theorem 4.16, \([17]\)), however, how do we know that such a Reeh-Schlieder state, defined by a generic vector in the Hilbert space \( \mathcal{H}_{\omega_0} \), is actually in the state space \( \mathcal{D}_{M_0}^0 \)? In other words, how do we know if such a Reeh-Schlieder state is of any physical interest?

The next question is therefore: how big is the difference between the state spaces \( \mathcal{D}_{M_0}^0 \) for \( U^0_{M_0} \) and \( \mathcal{S}_{M_0}^0 \) for \( A^0_{M_0} \)? Note that both of these state spaces contain all quasi-free Hadamard states, which are certainly of physical interest. However, to make \( \mathcal{S}_{M_0}^0 \) closed under operations from \( A^0_{M_0} \), which is required by our definition of a state space (see definition \([2.1.2]\)), we included all states that are locally quasi-equivalent to a quasi-free Hadamard state. This makes the state space \( \mathcal{S}_{M_0}^0 \) much larger than \( \mathcal{D}_{M_0}^0 \), at least in the sense that a generic vector in \( \mathcal{H}_{\omega} \) does not define a Hadamard state on \( U^0_{M_0} \). To close this subsection we will make this statement precise, which requires some more terminology (see e.g. \([23]\)).

**Definition 5.1.2** A \( G_\delta \) set in a topological space \( T \) is a countable intersection of open sets. An \( F_\sigma \) set is the complement of a \( G_\delta \) set.
A Baire space is a topological space in which any countable intersection of dense open sets is a dense set.

We say that a property is generic in a Baire space $T$ iff there is a dense $G_δ$ in $T$ of elements which have this property.

It is known that every complete pseudo-metric space, and in particular every Hilbert space, is a Baire space by Baire’s theorem (see [23] section 9.2b). Note that for any countable sequence of generic properties $P_n$ the property $P$ of having all $P_n$ is still generic. It follows from [31] (see also [22]) that the property that a vector in $\mathcal{H}_{ω_0}$ defines a Reeh-Schlieder state is a generic property. We will now argue that the property of not defining a Hadamard state is also generic. Indeed, a vector $ψ$ which defines a Hadamard state must be in the domain of the unbounded field operator $Φ^{(ω₀)}(f)$ for every $f ∈ C^∞_0(M₀)$. Using $Φ^{(ω₀)}(f) = Φ^{(ω₀)}(f)^*|_{\mathcal{D}_{ω₀}}$, we see that $ψ$ must also be in the domain of $T := Φ^{(ω₀)}(f)^*Φ^{(ω₀)}(f)^*$, which is a self-adjoint operator (see [49] theorem 2.7.8v). The domain of $T$, although dense, is the complement of a dense $G_δ$:

**Lemma 5.1.3** The domain of a self-adjoint operator $T$ on a Hilbert space $\mathcal{H}$ is a meagre $F_σ$, i.e. it is the complement of a dense $G_δ$.

**Proof.** For every $n ∈ \mathbb{N}$ we define $V_n := \{ψ ∈ \text{dom}(T) | \|Tψ\| ≤ n\|ψ\|\}$, where $\text{dom}(T)$ denotes the domain of $T$. Note that $\text{dom}(T) = \bigcup_n V_n$. To show that $V_n$ is closed we choose a Cauchy sequence $ψ_i ∈ V_n$ such that $ψ_i → ψ ∈ \mathcal{H}$. We let $E_{[-r,r]}$ denote the spectral projection of $T$ on the interval $[-r,r]$ and compute: $\|TE_{[-r,r]}ψ\| ≤ \|TE_{[-r,r]}(ψ - ψ_i)\| + \|TE_{[-r,r]}ψ_i\| ≤ r\|ψ - ψ_i\| + n\|ψ_i\|$. Taking $i → ∞$ shows that $\|TE_{[-r,r]}ψ\| ≤ n\|ψ\|$ for all $r$ and hence $\|Tψ\| ≤ n\|ψ\|$, i.e. $ψ ∈ V_n$. Finally the sets $V_n$ are nowhere dense, because $T$ is unbounded. This completes the proof. \(\square\)
5.2 Some general results on the Reeh-Schlieder property

As a prelude to our study of the Reeh-Schlieder property in curved spacetimes we will now prove some relatively easy statements which hold under very general assumptions. In this subsection we will first consider a fixed globally hyperbolic spacetime $M$ and a locally covariant quantum field theory $A : \mathcal{G} \to \mathcal{C} \mathcal{A} \mathcal{G}$ with a state space $S$.

We first prove a well-known consequence of the Reeh-Schlieder theorem (see e.g. [40]).

**Definition 5.2.1** A vector $v$ in a Hilbert space $\mathcal{H}$ is a separating vector for a C*-algebra $A$ of operators on $\mathcal{H}$ iff $Av = 0$ with $A \in A$ implies $A = 0$.

**Proposition 5.2.2** Let $\omega \in \mathcal{S}_M$ be a state on the C*-algebra $A_M$ which has the Reeh-Schlieder property for a cc-region $O \subset M$. Assume that $A$ is causal and let $V \subset O^\perp$ be a cc-region, then $\Omega_\omega$ is a separating vector for $\mathcal{R}_V^{(\omega)}$.

**Proof.** Suppose that $A\Omega_\omega = 0$ for some $A \in \mathcal{R}_V^{(\omega)}$, then $A\pi_\omega(B)\Omega_\omega = \pi_\omega(B)A\Omega_\omega = 0$ for all $B \in A_O$. By the Reeh-Schlieder property the set $\pi_\omega(B)\Omega_\omega$ is dense, so by continuity of $A$ we find $Av = 0$ for all $v \in \mathcal{H}$ and hence $A = 0$. □

This result implies that every non-zero positive operator $A^*A$ in $\mathcal{R}_V^{(\omega)}$ has a strictly positive expectation value in the state $\omega$, because if we have $\omega(A^*A) = \|A\Omega_\omega\|^2 = 0$ then $A = 0$.

Next we prove that the Reeh-Schlieder property is stable under purification, which appears to be a hitherto unknown result:

**Proposition 5.2.3** Let $\omega \in \mathcal{S}_M$ be a state which has the Reeh-Schlieder property for a cc-region $O \subset M$ and suppose that $\omega$ is a mixture of $\omega_1, \omega_2 \in$
$\mathcal{S}_M$, i.e. $\omega = \lambda \omega_1 + (1 - \lambda)\omega_2$ with $0 < \lambda \leq 1$. Then $\omega_1$ also has the Reeh-Schlieder property for $O$.

**Proof.** We fix arbitrary $\psi \in \mathcal{H}_{\omega_1}$ and $\epsilon > 0$ and use theorem 2.1.4 to find an $A \in \mathcal{A}_M$ such that $\|\psi - \pi_{\omega_1}(A)\Omega_{\omega_1}\| < \frac{\epsilon}{2}$. We can then find $B \in \mathcal{A}_O$ such that $\omega((A - B)^*(A - B)) = \|\pi_\omega(A - B)\Omega_\omega\|^2 < \frac{\lambda\epsilon^2}{4}$, by the assumed Reeh-Schlieder property. Then

$$\|\pi_{\omega_1}(A - B)\Omega_{\omega_1}\|^2 = \omega_1((A - B)^*(A - B)) \leq \frac{1}{\lambda} \omega((A - B)^*(A - B)) < \frac{\epsilon^2}{4}$$

and hence $\|\psi - \pi_{\omega_1}(B)\Omega_{\omega_1}\| \leq \|\psi - \pi_{\omega_1}(A)\Omega_{\omega_1}\| + \|\pi_{\omega_1}(A - B)\Omega_{\omega_1}\| < \epsilon$. $\square$

It is interesting to note that the same purification argument works for the Hadamard condition on states of the real free scalar field:

**Proposition 5.2.4** If $\omega$ is a Hadamard state on $U^0_M$ and $\omega = \lambda \omega_1 + (1 - \lambda)\omega_2$ with $0 < \lambda \leq 1$ for any states $\omega_i$ on $U^0_M$, then $\omega_1$ is a Hadamard state.

**Proof.** By positivity of $\omega_1$ we have the Cauchy-Schwarz inequality for all $A, B \in U^0_M$: $|\omega_1(B^*A)|^2 \leq \omega_1(B^*B)\omega_1(A^*A)$. In particular, $|\omega_1(A)|^2 \leq \omega_1(A^*A) \leq \frac{1}{\lambda} \omega_1(A^*A)$. If $A_n \to A$ in $U^0_M$ then $(A - A_n)^*(A - A_n) \to 0$ by definition 3.1.1 and the continuity of the canonical projection $p: U_M \to U^0_M$. Hence $|\omega_1(A_n - A)| \to 0$, which proves that $\omega_1$ is a continuous state on $U^0_M$. For the two-point distribution we have $(\omega_1)_2(\mathcal{T}, f) \leq \frac{1}{\lambda} \omega_2(\mathcal{T}, f)$ which can be rewritten in terms of Hilbert-space-valued distributions as $\|\phi_1^{(\omega)}(f)\| \leq \|\phi_1^{(\omega)}(f)\|$. (Note that these distributions may take values in different Hilbert spaces.) This implies $WF(\phi_1^{(\omega)}) \subset WF(\phi_1^{(\omega)})$ and hence by theorem A.1.6 $WF((\omega_1)_2) \subset \mathcal{N}^- \times \mathcal{N}^+$. Using the commutation relations and propagation of singularities as in the proof of proposition 3.1.13 it now follows that $\omega_1$ is Hadamard. $\square$

To conclude this section we prove that for an additive $C^*$-algebraic theory the Reeh-Schlieder property is a local property. This is a new and interesting result, but we will not need it elsewhere.
Theorem 5.2.5 Consider an additive locally covariant quantum field theory \( A : \text{Man} \to \text{Alg} \), a globally hyperbolic spacetime \( M \) and a state \( \omega \) on \( A_M \). Assume that every point \( p \in M \) is contained in a cc-region \( O \) such that \( \omega|_{A_O} \) has the Reeh-Schlieder property, i.e. such that for every cc-region \( V \subset O \) we have \( \pi_{\omega}(A_V)\Omega_\omega = \pi_{\omega}(A_O)\Omega_\omega = H_{\omega|_{A_O}} \). Then \( \omega \) is a Reeh-Schlieder state.

As a matter of terminology we will say that \( \omega \) has the Reeh-Schlieder property on \( O \) if \( \omega|_{A_O} \) has the Reeh-Schlieder property. (Cf. definition 5.1.1).

Proof. Let \( U_i, i \in I \) be an open covering of \( M \) by cc-regions such that \( \omega|_{A_{U_i}} \) has the Reeh-Schlieder property on \( U_i \) and let \( O \subset M \) be an arbitrary cc-region. We wish to show that \( \omega \) has the Reeh-Schlieder property for \( O \). This is certainly the case if \( \omega \) has the Reeh-Schlieder property for a subset of \( O \), so without loss of generality we may shrink \( O \) and assume that \( O \subset U_a \) for some index \( a \in I \). Now suppose for the moment that \( \omega \) has the Reeh-Schlieder property for \( U_a \) (and not just on \( U_a \)). Given arbitrary \( \psi \in H_\omega \) and \( \epsilon > 0 \) we can then find \( A \in A_{U_a} \) such that \( \| \psi - \pi_{\omega}(A)\Omega_\omega \| < \frac{\epsilon}{2} \). Moreover, because the restriction \( \omega' \) of \( \omega \) to \( A_{U_a} \) is a Reeh-Schlieder state we can find a \( B \in A_O \) such that

\[
\| \pi_{\omega}(A - B)\Omega_\omega \|^2 = \omega((A - B)^*(A - B)) = \omega'((A - B)^*(A - B)) = \| \pi_{\omega'}(A - B)\Omega_{\omega'} \|^2 < \frac{\epsilon^2}{4}.
\]

Together this implies that \( \| \psi - \pi_{\omega}(B)\Omega_\omega \| < \epsilon \), so \( \omega \) then has the Reeh-Schlieder property for \( O \). It remains to prove that \( \omega \) has the Reeh-Schlieder property for \( U_a \). For this we prove the following lemma.

Lemma 5.2.6 For all \( \epsilon > 0 \), \( i, j \in I \) and \( A \in A_{U_i} \) there is a \( B \in A_{U_j} \) such that \( \| \pi_{\omega}(A - B)\Omega_\omega \| < \epsilon \).

Proof. We refer to figure 5.1 for a depiction of the geometry of this proof.
Let $\epsilon > 0$, $i,j \in I$ and $A \in \mathcal{A}_{U_i}$ be given. Because $M$ is connected we can find a continuous curve $\gamma : [0, 1] \rightarrow M$ between points $p := \gamma(0) \in U_j$ and $q := \gamma(1) \in U_i$. For each point $t \in [0, 1]$ we choose $k(t) \in I$ such that $U_{k(t)}$ contains the point $\gamma(t)$. The image of $\gamma$ is a compact subset of $M$, because it is the continuous image of a compact set. Therefore, we can find a finite number of points $t_1, \ldots, t_n \in [0, 1]$ such that the cc-regions $V_n := U_{k(t_n)}$ cover the image of $\gamma$. We now choose a subcover inductively as follows. First we take $p \in V_0 := U_j$. We let $t_1 := \min \{s \in [0, 1] | \gamma(s) \notin V_0\}$ and choose $V_1$ such that $t_1 \in V_1$. We then set $t_2 := \min \{s \in [t_1, 1] | \gamma(s) \notin V_1\}$ and choose $V_2$ such that $t_2 \in V_2$. We proceed in this way until we find an index $m$ with $q \in V_m$. Extending the sequence $V_0, \ldots, V_m$ by one set if necessary we may assume that $V_m := U_i$. Notice that by construction $V_{l-1} \cap V_l \neq \emptyset$ for $l = 1, \ldots, m$, so we can choose cc-regions $X_l \subset V_{l-1} \cap V_l$ for $l = 1, \ldots, m$. Using the same calculation as in equation (5.1) and the hypothesis that $\omega$ is a Reeh-Schlieder state on each $V_l$ we can find $B_m \in \mathcal{A}_{X_m}$ such that

$$\|\pi_{\omega}(A - B_m)\Omega_{\omega}\| < \frac{\epsilon}{m + 1},$$

and proceed inductively to find $B_l \in \mathcal{A}_{V_l}$ for $l = m - 1, \ldots, 0$ such that
\[ \|\pi_\omega(B_{l+1} - B_l)\Omega_\omega\| < \frac{\epsilon}{m+1}. \]

This provides us with a \( B := B_0 \in \mathcal{A}_{U_j} \) such that
\[ \|\pi_\omega(A - B)\Omega_\omega\| \leq \|\pi_\omega(A - B_m)\Omega_\omega\| + \sum_{l=1}^{m} \|\pi_\omega(B_m - B_{m-1})\Omega_\omega\| < \epsilon. \]

\[ \square \]

We resume the proof of theorem 5.2.3 and prove that \( \omega \) has the Reeh-Schlieder property for \( U_a \) for a fixed but arbitrary index \( a \in I \). We consider a monomial \( A_1 \cdots A_n \) where \( A_i \in \mathcal{A}_{U_{k(i)}} \) for some indices \( k(i) \in I \) and set \( k(0) = a \). We assume that all \( A_i \) are non-zero and we define \( r > 0 \) by \( r := 2 \max_{i=1,\ldots,n} \|\pi_\omega(A_i)\| \). Given \( \epsilon > 0 \) we can then find elements \( B_i \in \mathcal{A}_{U_{k(i)}} \) for \( i = 0, \ldots, n - 1 \) such that
\[ \|\pi_\omega(A_i+1 \cdots A_n - B_i)\Omega_\omega\| < \frac{\epsilon}{r^{i+1}}. \tag{5.2} \]

Indeed, for \( i = n - 1 \) this follows directly from the lemma. We can then proceed inductively to find \( B_n-2, \ldots, B_1, B_0 \) as follows. We notice that \( A_iB_i \in \mathcal{A}_{U_{k(i)}} \) for \( i \geq 1 \) and apply the lemma to choose \( B_{i-1} \in \mathcal{A}_{U_{k(i-1)}} \) such that \( \|\pi_\omega(A_iB_i - B_{i-1})\Omega_\omega\| < \frac{\epsilon}{2r^{i-1}} \). Then we use the estimate
\[ \|\pi_\omega(A_i \cdots A_n - B_{i-1})\Omega_\omega\| \]
\[ \leq \|\pi_\omega(A_i)\| \cdot \|\pi_\omega(A_{i+1} \cdots A_n - B_i)\Omega_\omega\| + \|\pi_\omega(A_iB_i - B_{i-1})\Omega_\omega\| \]
\[ < \frac{r}{2} \cdot \frac{\epsilon}{r^{i+1}} + \|\pi_\omega(A_iB_i - B_{i-1})\Omega_\omega\| < \frac{\epsilon}{r^{i+1}}, \]

which is \( (5.2) \) for the index \( i - 1 \). This provides us with \( B_0 \in \mathcal{A}_{U_a} \) such that \( \|\pi_\omega(A_1 \cdots A_n - B_0)\Omega_\omega\| < \epsilon \).

Now let \( \mathcal{P} \) be the *-algebra of all (finite) polynomials of elements in \( \bigcup_{i \in I} \mathcal{A}_{U_i} \). Given \( P \in \mathcal{P} \) and \( \epsilon > 0 \) we can apply the result of the previous paragraph to each monomial in \( P \) and find a \( B \in \mathcal{A}_{U_a} \) such that \( \|\pi_\omega(P - B)\Omega_\omega\| < \epsilon \). In other words, \( \pi_\omega(\mathcal{A}_{U_a})\Omega_\omega \) is dense in \( \pi_\omega(\mathcal{P})\Omega_\omega \). Notice that \( \mathcal{P} \) is the smallest *-algebra that contains all algebras \( \mathcal{A}_{U_i} \) and that we have \( \mathcal{A}_M = \overline{\mathcal{P}} \) by additivity, where we take the norm closure. It follows that \( \pi_\omega(\mathcal{P})\Omega_\omega \) is dense in \( \mathcal{H}_\omega \). This completes the proof. \( \square \)
5.3 The Reeh-Schlieder property under spacetime deformation

The existence of Hadamard states of the free scalar field in certain curved spacetimes was proved in [38] by deforming Minkowski spacetime into another globally hyperbolic spacetime. Using a similar but slightly more technical spacetime deformation argument [85] proved a spin-statistics theorem for locally covariant quantum field theories with a spin structure, given that such a theorem holds in Minkowski spacetime. In subsection 5.3.2 we will assume the existence of a Reeh-Schlieder state in one globally hyperbolic spacetime and try to deduce the existence of such states on a deformed spacetime along the same lines. As a geometric prerequisite we will state and prove in subsection 5.3.1 a spacetime deformation result employing similar methods to the references mentioned above.

5.3.1 Spacetime deformation

First we recall the spacetime deformation result due to [38]:

**Proposition 5.3.1** Consider two globally hyperbolic spacetimes $M_i$, $i = 1, 2$, with space-like Cauchy surfaces $C_i$ both diffeomorphic to $C$. Then there exists a globally hyperbolic spacetime $M' = (\mathbb{R} \times C, g')$ with space-like Cauchy surfaces $C'_i$, $i = 1, 2$, such that $C'_i$ is isometrically diffeomorphic to $C_i$ and an open neighbourhood of $C'_i$ is isometrically diffeomorphic to an open neighbourhood of $C_i$.

We omit the proof of this result, because we will prove the stronger proposition 5.3.3 later on. Note, however, the following interesting corollary (cf. [16] section 4):
**Corollary 5.3.2** Two globally hyperbolic spacetimes $M_i$ with diffeomorphic Cauchy surfaces are mapped to isomorphic $\ast$-algebras $\mathcal{A}_{M_i}$ by any locally covariant quantum field theory $\mathbf{A}$ satisfying the time-slice axiom (with some state space $\mathbf{S}$).

**Proof.** Consider two diffeomorphic globally hyperbolic spacetimes $M_i$ for $i = 1, 2$, let $M'$ be the deforming spacetime of proposition 5.3.1 and let $W_i \subset M_i$ be open neighbourhoods of the Cauchy surfaces $C_i \subset M_i$ which are isometrically diffeomorphic under $\psi_i$ to the open neighbourhoods $W_i' \subset M'$ of the Cauchy surfaces $C_i' \subset M'$. We may take the $W_i$ and $W_i'$ to be cc-regions (as will be shown in proposition 5.3.3), so that the maps $\psi_i : W_i \to W_i'$ determine isomorphisms $\Psi_i$ in $\mathfrak{Man}$. It then follows from lemma 2.4.4 that

$$\mathcal{A}_{M_i} \simeq \mathcal{A}_{W_i} \simeq \mathcal{A}_{\psi_i^{-1}(W_i')} \simeq \alpha_{\psi_i}^{-1}(\mathcal{A}_{W_i'}) \simeq \alpha_{\psi_i}^{-1}(\mathcal{A}_{M'})$$

$$\simeq \alpha_{\psi_i}^{-1} \circ \alpha_{\psi_2}(\mathcal{A}_{M_2}),$$

where the $\alpha_{\psi_i}$ are $\ast$-isomorphisms. This proves the assertion. \hfill \Box

At this point a warning seems in place. When $g_1, g_2$ are two Lorentzian metrics on a manifold $\mathcal{M}$ such that both $M_i := (\mathcal{M}, g_i)$ are objects in $\mathfrak{Man}$, corollary 5.3.2 gives a $\ast$-isomorphism $\alpha$ between the algebras $\mathcal{A}_{M_i}$. Hence, if $O \subset \mathcal{M}$ is a cc-region for $g_1$ then $\alpha$ is a $\ast$-isomorphism from $\mathcal{A}_{(O, g_1)}$ into $\mathcal{A}_{M_2}$. However, the image cannot always be identified with $\mathcal{A}_{(O, g_2)}$, because $O$ need not be causally convex for $g_2$, in which case the object is not defined.

We now formulate and prove our spacetime deformation result. The geometric situation is schematically depicted in figure 5.2.

**Proposition 5.3.3** Consider two globally hyperbolic spacetimes $M_i$, $i = 1, 2$, with diffeomorphic Cauchy surfaces and a bounded cc-region $O_2 \subset M_2$ with non-empty causal complement, $O_2^\perp \neq \emptyset$. Then there are a globally hyperbolic spacetime $M' = (\mathcal{M}', g')$, space-like Cauchy surfaces $C_i \subset M_i$ and $C_1', C_2' \subset M'$.
Figure 5.2: Sketch of the geometry of proposition 5.3.3.

\( M' \) and bounded cc-regions \( U_2, V_2 \subset M_2 \) and \( U_1, V_1 \subset M_1 \) such that the following hold:

1. There are isometric diffeomorphisms \( \psi_i : W_i \rightarrow W'_i \) where \( W_1 := I^-(C_1) \), \( W'_1 := I^-(C'_1) \), \( W_2 := I^+(C_2) \) and \( W'_2 := I^+(C'_2) \),

2. \( U_2, V_2 \subset W_2, U_2 \subset D(O_2), O_2 \subset D(V_2) \),

3. \( U_1, V_1 \subset W_1, U_1 \neq \emptyset, V_1^\perp \neq \emptyset, \psi_1(U_1) \subset D(\psi_2(U_2)) \) and \( \psi_2(V_2) \subset D(\psi_1(V_1)) \).

**Proof.** First we recall the result of [2] that for any globally hyperbolic spacetime \((\mathcal{M}, g)\) there is a diffeomorphism \( F : \mathcal{M} \rightarrow \mathbb{R} \times C \) for some smooth three dimensional manifold \( C \) in such a way that for each \( t \in \mathbb{R} \) the surface \( F^{-1}\{\{t\} \times C\} \) is a space-like Cauchy surface. The pushed-forward metric \( g' := F_* g \) makes \((\mathbb{R} \times C, g')\) a globally hyperbolic manifold, where \( g' \) is given by

\[
g'_{\mu\nu} = \beta dt_{\mu} dt_{\nu} - h_{\mu\nu}.
\]  

(5.3)

Here \( dt \) is the differential of the canonical projection on the first coordinate \( t : \mathbb{R} \times C \rightarrow \mathbb{R} \), which is a smooth time function, \( \beta \) is a strictly positive smooth function and \( h_{\mu\nu} \) is a (space and time dependent) Riemannian metric on
The orientation and time-orientation of \( M \) induce an orientation and time-orientation on \( \mathbb{R} \times C \) via \( F \). (If necessary we may compose \( F \) with the time-reversal diffeomorphism \((t, x) \mapsto (-t, x)\) of \( \mathbb{R} \times C \) to ensure that the function \( t \) increases in the positive time direction.) Applying the above to the \( M_i \) gives us two diffeomorphisms \( F_i : M_i \to M' \), where \( M' = \mathbb{R} \times C \) as a manifold. Note that we can take the same \( C \) for both \( i = 1, 2 \) by the assumption that the \( M_i \) have diffeomorphic Cauchy surfaces.

Define \( O'_2 := F_2(O_2) \) and let \( t_{\min} \) and \( t_{\max} \) be the minimum and maximum value that the function \( t \) attains on the compact set \( \overline{O_2} \). We now prove that \( F_2^{-1}((t_{\min}, t_{\max}) \times C) \cap O_2^\perp \neq \emptyset \). Indeed, if this were empty, then we see that \( J(O_2) \) contains \( F_2^{-1}([t_{\min}, t_{\max}] \times C) \) and hence also \( C_{\max} := F_2^{-1}([t_{\max}] \times C) \) and \( C_{\min} := F_2^{-1}([t_{\min}] \times C) \). In fact, we have \( C_{\min} \subset J^-(O_2) \). Indeed, if \( p := F_2^{-1}(t_{\min}, x) \) is in \( J^+(O_2) \) then we can consider a basis of neighbourhoods of \( p \) of the form \( I^-(F_2^{-1}(t_{\min} + 1/n, x)) \cap I^+(F_2^{-1}([t_{\min} - 1/n] \times C)) \). Now, if \( q_n \in J^+(O_2) \) is in such a basic neighbourhood, then the same neighbourhood also contains a point \( p_n \in O_2 \). Hence, given a sequence \( q_n \) in \( J^+(O_2) \) converging to \( p \) we find a sequence \( p_n \) in \( O_2 \) converging to \( p \) and we conclude that \( p \in \overline{O_2} \subset \overline{J^-(O_2)} \). Similarly we can show that \( C_{\max} \subset J^+(O_2) \). It then follows that \( I^+(C_{\max}) \subset J^+(O_2) \) and \( I^-(C_{\min}) \subset J^-(O_2) \), so together with \( F_2^{-1}([t_{\min}, t_{\max}] \times C) \subset J(O_2) \) we find that \( \overline{J(O_2)} = M \) and \( O_2^\perp = \emptyset \). This contradicts our assumption on \( O_2 \), so we must have \( F_2^{-1}((t_{\min}, t_{\max}) \times C) \cap O_2^\perp \neq \emptyset \). Then we may choose \( t_2 \in (t_{\min}, t_{\max}) \) such that \( C_2 := F_2^{-1}([t_2] \times C) \) intersects both \( O_2 \) and \( O_2^\perp \). We define \( C'_2 := F_2(C_2) \), \( W_2 := I^+(C_2) \) and \( W'_2 := (t_2, \infty) \times C \).

Note that \( C_2 \cap J(\overline{O_2}) \) is compact (see [6] corollary A.5.4). This means that we can find relatively compact open sets \( K, N \subset C \) such that \( K' := \{t_2\} \times K \), \( K_2 := F_2^{-1}(K') \), \( N' := \{t_2\} \times N \) and \( N_2 := F_2^{-1}(N') \) satisfy \( K \neq \emptyset \), \( N \neq C \), \( \overline{K_2} \subset O_2 \) and \( C_2 \cap J(\overline{O_2}) \subset N_2 \). We let \( C_{\max} := F_2^{-1}([t_{\max}] \times C) \) and define \( U_2 := D(K_2) \cap I^+(K_2) \cap I^-(C_{\max}) \) and \( V_2 := D(N_2) \cap I^+(N_2) \cap I^-(C_{\max}) \). It
follows from lemma 2.2.2 that $U_2, V_2$ are bounded cc-regions in $M_2$. Clearly $U_2, V_2 \subset W_2$, $U_2 \subset D(O_2)$, $O_2 \subset D(V_2)$ and $V_2^+ \neq \emptyset$.

Next we choose $t_1 \in (t_{\text{min}}, t_2)$ and define $C'_1 := \{t_1\} \times C$, $C_1 := F^{-1}_1(C'_1)$, $W_1 := I^-(C_1)$ and $W'_1 := (-\infty, t_1) \times C$. Let $N', K' \subset C$ be relatively compact connected open sets such that $K' \neq \emptyset$, $N' \neq C$, $\overline{K'} \subset K$ and $\overline{N} \subset N'$. We define $N'_1 := \{t_1\} \times N'$, $K'_1 := \{t_1\} \times K'$, $N_1 := F^{-1}_1(N'_1)$, $K_1 := F^{-1}_1(K'_1)$ and $C_{\text{min}} := F^{-1}_1(\{t_{\text{min}}\} \times C)$. Let $U_1 := D(K_1) \cap I^-(K_1) \cap I^+(C_{\text{min}})$ and $V_1 := D(N_1) \cap I^-(N_1) \cap I^+(C_{\text{min}})$. Again by lemma 2.2.2 these are bounded cc-regions in $M_1$. Note that $U_1, V_1 \subset W_1$ and $V_1^+ \neq \emptyset$.

The metric $g'$ of $\mathcal{M}'$ is now chosen to be of the form

$$g'_{\mu\nu} := \beta dt_\mu dt_\nu - f \cdot (h_1)_{\mu\nu} - (1 - f) \cdot (h_2)_{\mu\nu}$$

where we have written $((F_i)_*g_i)_{\mu\nu} = \beta_i dt_\mu dt_\nu - (h_i)_{\mu\nu}$, $f$ is a smooth function on $\mathcal{M}'$ which is identically 1 on $W'_1$, identically 0 on $W'_2$ and $0 < f < 1$ on the intermediate region $(t_1, t_2) \times C$ and $\beta$ is a strictly positive smooth function which is identically $\beta_i$ on $W'_i$. It is then clear that the maps $F_i$ restrict to isometric diffeomorphisms $\psi_i : W_i \to W'_i$.

The function $\beta$ may be chosen small enough on the region $(t_1, t_2) \times C$ to make $(\mathcal{M}, g')$ globally hyperbolic. (As pointed out in [38] in their proof of proposition 5.3.1 choosing $\beta$ small “closes up” the light cones and prevents causal curves from “running off to spatial infinity” in the intermediate region.) Furthermore, using the compactness of $(t_1, t_2) \times N'$ and the continuity of $(h_i)_{\mu\nu}$ we see that we may choose $\beta$ small enough on this set to ensure that any causal curve through $\overline{K'_1}$ must also intersect $K'_2$ and any causal curve through $\overline{N'_2}$ must also intersect $N'_1$. This means that $\overline{K'_1} \subset D(K'_2)$ and $\overline{N'_2} \subset D(N'_2)$ and hence $\psi_1(U_1) \subset D(\psi_2(U_2))$ and $\psi_2(V_2) \subset D(\psi_1(V_1))$. This completes the proof. \(\square\)

The analogue of corollary 5.3.2 for the situation of proposition 5.3.3 is:
Proposition 5.3.4 Consider a locally covariant quantum field theory $A$ with a state space $S$ satisfying the time-slice axiom and let $M_i, i=1,2,$ be two globally hyperbolic spacetimes with diffeomorphic Cauchy surfaces. For any bounded cc-region $O_2 \subset M_2$ with non-empty causal complement there are bounded cc-regions $U_1, V_1 \subset M_1$ and a $^*$-isomorphism $\alpha : A_{M_2} \to A_{M_1}$ such that $V_1^\perp \neq \emptyset$ and

$$A_{U_1} \subset \alpha(A_{O_2}) \subset A_{V_1}. \quad (5.4)$$

Moreover, if the space-like Cauchy surfaces of the $M_i$ are non-compact and $P_2 \subset M_2$ is any bounded cc-region, then there are bounded cc-regions $Q_2 \subset M_2$ and $P_1, Q_1 \subset M_1$ such that $Q_i \subset P_i^\perp$ for $i=1,2$ and

$$\alpha(A_{P_2}) \subset A_{P_1}, \quad \alpha(Q_1) \subset \alpha(A_{Q_2}), \quad (5.5)$$

where $\alpha$ is the same $^*$-isomorphism as in the first part of this proposition.

Proof. We apply proposition 5.3.3 to obtain sets $U_i, V_i$ with and isomorphisms $\Psi_i : W_i \to W'_i$ associated to the isometric diffeomorphisms $\psi_i$. As in the proof of corollary 5.3.2 the $\Psi_i = (\psi_i)$ give rise to $^*$-isomorphisms $\alpha_{\psi_i}$ and $\alpha := \alpha_{\psi_2}^{-1} \circ \alpha_{\psi_1}$ is a $^*$-isomorphism from $A_{M_2}$ to $A_{M_1}$.

Using the properties of $U_i, V_i$ stated in proposition 5.3.3 we deduce:

$$A_{U_1} = \alpha_{\psi_1}^{-1}(A_{U'_1}) \subset \alpha_{\psi_1}^{-1}(A_{D(U'_2)}) = \alpha_{\psi_1}^{-1}(A_{U'_2}) = \alpha(A_{U_2}) \subset \alpha(A_{O_2})$$

$$\subset \alpha(A_{V_2}) = \alpha_{\psi_1}(A_{V_2}) \subset \alpha_{\psi_1}(A_{D(V'_1)}) = \alpha_{\psi_1}^{-1}(A_{V'_1}) = A_{V_1}.$$ 

Here we repeatedly used equation (2.2) and lemma 2.4.4 (the time-slice axiom). This proves the first part of the proposition.

Now suppose that the Cauchy-surfaces are non-compact and let $P_2$ be any bounded cc-region. We refer to figure 5.3 for a depiction of this part of the proof.

First choose Cauchy surfaces $T_2, T_+ \subset W_2$ such that $T_+ \subset I^+(T_2)$. Note that $J(P_2) \cap T_2$ is compact, so it has a relatively compact connected open
neighbourhood $N_2 \subset T_2$. Choosing $T_+$ appropriately we see that $R := D(N_2) \cap I^+(N_2) \cap I^-(T_+)$ is a bounded cc-region in $M_2$ by lemma 2.2.2 and as usual we set $R' := \psi_2(R)$.

Now let $T'_-, T'_+ \subset W'_1$ be Cauchy surfaces such that $T'_- \subset I^-(T'_+)$ and note that $J(R') \cap T'_+$ is again compact, so we can find a relatively compact connected open neighbourhood $N'_1 \subset T'_+$ and use lemma 2.2.2 to define the bounded cc-region $P'_1 := D(N'_1) \cap I^-(N'_1) \cap I^+(T'_+) \cap I^-(T'_-) \cap I^+(T'_+) \cap I^-(T'_-) \cap I^+(T'_+)$ and $P_1 := \psi_1^{-1}(P'_1)$.

Next we let $L'_1 \subset T'_+$ be a connected relatively compact set such that $L'_1 \cap N'_1 = \emptyset$. Such an $L'_1$ exists because $T'_+$ is non-compact. We then define $Q'_1 := D(L'_1) \cap I^-(L'_1) \cap I^+(T'_+) \cap I^-(T'_-) \cap I^+(T'_+) \cap I^-(T'_-) \cap I^+(T'_+)$ and $Q_1 := \psi_1^{-1}(Q'_1)$. We see that $Q_1 \subset P'_1$ is a bounded cc-region and $Q'_1 \subset D(\psi_2(L_2))$ where $L_2 \subset T_2 \setminus N$ is a relatively compact open set. In fact, we can choose $L_2$ to be connected because $Q'_1$ lies in a connected component $C$ of $D(\psi_2(T_2 \setminus N))$. We now define the bounded cc-region $Q_2 := D(L_2) \cap I^+(L_2) \cap I^-(T_+) \cap I^+(T_+)$ and $Q'_2 := \psi_2(Q_2)$, so that $Q_1 \subset P'_1$ and $Q'_1 \subset D(Q'_2)$.

So far the geometry of the proof. Now note that $\mathcal{A}_{P_2} \subset \mathcal{A}_R$ by lemma 2.4.4 on $D(N_2) \cap I^+(N_2)$ and that $\mathcal{A}_{R'} = \alpha_{\psi_2}(\mathcal{A}_R)$. Applying lemma 2.4.4 in $D(N'_1) \cap I^-(N'_1)$ we see that $\mathcal{A}_{R'} \subset \mathcal{A}_{P'_1}$ and we have $\mathcal{A}_{P'_1} = \alpha_{\psi_1}^{-1}(\mathcal{A}_{P'_1})$. Putting this together yields the inclusion:

$$\alpha(\mathcal{A}_{P_2}) \subset \alpha(\mathcal{A}_R) = \alpha_{\psi_2}^{-1}(\mathcal{A}_{R'}) \subset \alpha_{\psi_1}^{-1}(\mathcal{A}_{P'_1}) = \mathcal{A}_{P_1}.$$ 

Similarly we have $\mathcal{A}_{Q_1} = \alpha_{\psi_1}^{-1}(\mathcal{A}_{Q'_1})$, $\mathcal{A}_{Q'_2} = \alpha_{\psi_2}(\mathcal{A}_{Q_2})$ and $\mathcal{A}_{Q'_1} \subset \mathcal{A}_{Q'_2}$ by lemma 2.4.4. This yields the inclusion:

$$\alpha(\mathcal{A}_{Q_2}) = \alpha_{\psi_1}^{-1}(\mathcal{A}_{Q'_2}) \supset \alpha_{\psi_1}^{-1}(\mathcal{A}_{Q'_1}) = \mathcal{A}_{Q_1}.$$ 

\[\square\]
5.3.2 Deformation of the Reeh-Schlieder property

We will now describe some of the consequences of the spacetime deformation argument of the previous subsection for the Reeh-Schlieder property. Unfortunately it is not clear that we can deform a Reeh-Schlieder state into another (full) Reeh-Schlieder state, but we do have the following more limited result:

**Theorem 5.3.5** Consider a locally covariant quantum field theory $\mathcal{A}$ with state space $\mathcal{S}$ which satisfies the time-slice axiom. Let $M_i$, $i = 1, 2$, be two globally hyperbolic spacetimes with diffeomorphic Cauchy surfaces and suppose that $\omega_1 \in \mathcal{S}_{M_1}$ is a Reeh-Schlieder state. Then given any bounded cc-region $O_2 \subset M_2$ with non-empty causal complement, $O^\perp_2 \neq \emptyset$, there is an *-isomorphism $\alpha : \mathcal{A}_{M_2} \to \mathcal{A}_{M_1}$ such that $\omega_2 := \alpha^*(\omega_1)$ has the Reeh-Schlieder property for $O_2$.

Moreover, if the Cauchy surfaces of the $M_i$ are non-compact and $P_2 \subset M_2$ is a bounded cc-region, then there is a bounded cc-region $Q_2 \subset P^{\perp}_2$ for which $\omega_2$ has the Reeh-Schlieder property.

**Proof.** For the first statement let $\alpha$ and $U_1$ be as in the first part of proposition [5.3.4] and note that $\alpha$ gives rise to a unitary map $U_\alpha : \mathcal{H}_{\omega_2} \to \mathcal{H}_{\omega_1}$. This
map is the expression of the essential uniqueness of the GNS-representation, so that $u_\alpha \Omega_2 = \Omega_1$ and $u_\alpha \pi_\omega u^*_\alpha = \pi_\omega \circ \alpha$. The Reeh-Schlieder property for $O_2$ then follows from the observation that $u_\alpha \pi_\omega (A_{O_2}) u^*_\alpha \supset \pi_\omega (A_{U_1})$:

$$
\pi_\omega (A_{O_2}) \Omega_2 \supset u^*_\alpha \pi_\omega (A_{U_1}) \Omega_1 = u^*_\alpha \mathcal{H}_{\omega_1} = \mathcal{H}_{\omega_2}.
$$

Similarly for the second statement, given a bounded cc-region $P_2$ and choosing $Q_1, Q_2$ as in the second statement of proposition 5.3.4 we see that $u_\alpha \pi_\omega (A_{Q_2}) u^*_\alpha \supset \pi_\omega (A_{Q_1})$. □

The second part of theorem 5.3.5 means that $\omega_2$ is a Reeh-Schlieder state for all cc-regions that are big enough. Indeed, if $V_2$ is a sufficiently small cc-region then $V_2^\perp$ is connected (recall that we work with four-dimensional spacetimes) and therefore $\omega_2$ has the Reeh-Schlieder property for some cc-region in $V_2^\perp$ and hence also for $V_2^\perp$ itself.

In the remainder of this subsection we consider only $C^*$-algebraic theories, because they allow us to draw stronger conclusions than for general topological $*$-algebras. We begin with the following consequence of theorem 5.3.5:

**Corollary 5.3.6** In the situation of theorem 5.3.5 if $A : \text{Man} \rightarrow \text{CAlg}$ is a causal locally covariant quantum field theory, then $\Omega_{\omega_2}$ is a cyclic and separating vector for the local von Neumann algebra $\mathcal{R}_{\omega_2}^{\omega_2}$. If the Cauchy surfaces are non-compact $\Omega_{\omega_2}$ is a separating vector for all $\mathcal{R}_{P_2}^{\omega_2}$ where $P_2$ is a bounded cc-region.

**Proof.** Recall that a vector is a separating vector for a von Neumann algebra $\mathcal{R}$ iff it is a cyclic vector for the commutant $\mathcal{R}'$ (49 proposition 5.5.11, see also our proof of proposition 5.2.2). Choosing $V_1$ as in the first part of proposition 5.3.4 we have $u_\alpha \pi_\omega (A_{O_2}) u^*_\alpha \subset \pi_\omega (A_{V_1})$ by the inclusion (5.4). Therefore the commutant of $u_\alpha \mathcal{R}_{O_2}^{\omega_2} u^*_\alpha$ contains $(\mathcal{R}_{V_1}^{\omega_1})'$. As $V_1^\perp \neq \emptyset$.
this commutant contains the local algebra of some cc-region for which $\Omega_{\omega_1}$ is cyclic. Hence $\Omega_{\omega_1}$ is a separating vector for $\mathcal{R}_{\psi_1}^{\omega_1}$ and $\Omega_{\omega_2}$ for $\mathcal{R}_{\psi_2}^{\omega_2}$.

If the Cauchy surfaces are non-compact, $P_2$ is a bounded region and $Q_2$ is as in theorem 5.3.5, then $(\mathcal{R}_{P_2}^{\omega_2})'$ contains $\pi_{\omega_2}(A_{Q_2})$, for which $\Omega_{\omega_2}$ is cyclic. It follows that $\Omega_2$ is separating for $\mathcal{R}_{P_2}^{\omega_2}$.

If the theory is nowhere classical then this corollary implies that there exist non-local correlations between $O_2$ and any cc-region $V_2$ space-like to it, just as in the Minkowski spacetime case (see e.g. [68]). Also, if the Cauchy surfaces are non-compact, any localised non-trivial positive observable has a strictly positive expectation value.

If the state space is locally quasi-equivalent and large enough it is possible to show the existence of full Reeh-Schlieder states. The proof uses abstract existence arguments, as opposed to the proof of theorem 5.3.5 which is constructive, at least in principle.

Theorem 5.3.7 Let $A : \text{Man} \rightarrow \mathcal{CAlg}$ be a locally covariant quantum field theory with a locally quasi-equivalent state space $\mathcal{S}$ which is causal and satisfies the time-slice axiom. Assume that $\mathcal{S}$ is maximal in the sense that for any state $\omega$ on some $A_M$ which is locally quasi-equivalent to a state in $\mathcal{S}_M$ we have $\omega \in \mathcal{S}_M$.

Let $M_i$, $i = 1, 2$, be two globally hyperbolic spacetimes with diffeomorphic non-compact Cauchy surfaces and assume that $\omega_1 \in \mathcal{S}_{M_1}$ is a Reeh-Schlieder state. Then $\mathcal{S}_{M_2}$ contains a (full) Reeh-Schlieder state.

Proof. Let $\{O_n\}_{n \in \mathbb{N}}$ be a countable basis for the topology of $M_2$ consisting of bounded cc-regions with non-empty causal complement. (That every open set contains a cc-region can be seen by using a convex normal neighbourhood and choosing a sufficiently small region of the form $I^+(p) \cap I^-(q)$, cf. [88] theorem 8.1.2 and our lemma 2.2.2). We then apply theorem 5.3.5 to each $O_n$ to obtain a sequence of states $\omega_2^n \in \mathcal{S}_{M_2}$ which have the Reeh-Schlieder
property for $O_n$. We write $\omega := \omega^1_2$ and let $(\mathcal{H}, \pi, \Omega)$ denote its GNS-triple.

For all $n \geq 2$ we now find a bounded cc-region $V_n \subset M_2$ such that $V_n \supset O_1 \cup O_n$. For this purpose we first choose a Cauchy surface $C \subset M_2$ and note that $K_n := C \cap J(O_n)$ is compact. Letting $L_n \subset C$ be a compact connected set containing $K_1 \cup K_n$ in its interior it suffices to choose $V_n := \text{int}(D(L_n)) \cap I^-(C_+) \cap I^+(C_-)$ for Cauchy surfaces $C_{\pm}$ to the future, respectively to the past, of $O_1$, $O_n$ and $C$. Note that $\Omega$ and $\Omega_{\omega^2_n}$ are cyclic and separating vectors for $\mathcal{R}_{V_n}$ and $\mathcal{R}_{V_n}^{\omega^2}$ respectively, by $O_1 \cup O_n \subset V_n$ and by corollary 5.3.6. Because $\omega$ and $\omega^2_n$ are locally quasi-equivalent there is a $^*$-isomorphism $\phi : \mathcal{R}_{V_n}^{\omega^2} \to \mathcal{R}_{V_n}^\omega$. In the presence of the cyclic and separating vectors $\Omega$ and $\Omega_{\omega^2_n}$ the $^*$-isomorphism $\phi$ is implemented by a unitary map $U_n : \mathcal{H}_{\omega^2_n} \to \mathcal{H}$ (see [49] theorem 7.2.9). We claim that $\psi_n := U_n \pi_{\omega^2_n} \Omega_{\omega^2_n}$ is cyclic for $\mathcal{R}_{V_n}^{\omega^2}$. Indeed, by the definition of quasi-equivalence we have $\phi \circ \pi_{\omega^2_n} = \pi_{\omega}$ on $A_{V_n}$, so

$$\pi_{\omega}(A_{O_n}) \psi_n = U_n \pi_{\omega^2_n}(A_{O_n}) \Omega_{\omega^2_n} = U_n \mathcal{H}_{\omega^2_n} = \mathcal{H}_{\omega}.$$  

We now apply the results of [31] to conclude that $\mathcal{H}$ contains a dense set of vectors $\psi$ which are cyclic and separating for all $\mathcal{R}_{O_n}^\omega$ simultaneously. Because each cc-region $O \subset M_2$ contains some $O_n$ we see that $\omega_{\psi} : A \mapsto \frac{\langle \psi, \pi_{\omega}(A) \psi \rangle}{\|\psi\|^2}$ defines a full Reeh-Schlieder state. Finally, because the GNS-triple of $\omega_{\psi}$ is just $(\mathcal{H}, \pi, \psi)$ we see that it is locally quasi-equivalent to $\omega$ and hence $\omega_{\psi} \in \mathscr{S}_{M_2}$.  

Although the state space may in general not be big enough to contain full Reeh-Schlieder states, theorem 5.3.5 is already enough for some useful applications. As an example we present the following conclusion concerning the type of local von Neumann algebras:

**Corollary 5.3.8** Consider a nowhere classical causal locally covariant quantum field theory $\mathbf{A} : \text{Man} \to \mathcal{CAlg}$ with a locally quasi-equivalent state space $\mathbf{S}$
which satisfies the time-slice axiom. Let $M_i$, $i = 1, 2$, be two globally hyperbolic spacetimes with diffeomorphic Cauchy surfaces and let $\omega_1 \in \mathcal{I}_{M_i}$ be a Reeh-Schlieder state. Then for any state $\omega \in \mathcal{I}_{M_i}$ and any cc-region $O \subset M_i$ the local von Neumann algebra $\mathcal{R}_{\omega}^O$ is not finite.

**Proof.** We will use proposition 5.5.3 in [7], which says that $\mathcal{R}_{\omega}^O$ is not finite if the GNS-vector $\Omega$ is a cyclic and separating vector for $\mathcal{R}_{\omega}^O$ and for a proper sub-algebra $\mathcal{R}_{\psi}^V$. Note that we can drop the superscript $\omega$ if $O$ and $V$ are bounded, by local quasi-equivalence.

First we consider $M_1$. For any bounded cc-region $O_1 \subset M_1$ such that $O_1^\perp \neq \emptyset$ we can find bounded cc-regions $O' \subset O_1^\perp$ and $U, V \subset O_1$ such that $U \subset V^\perp$. By the Reeh-Schlieder property the GNS-vector $\Omega_{\omega_1}$ is cyclic for $\mathcal{R}_V$ and hence also for $\mathcal{R}_{O_1}$. Moreover it is cyclic for $\mathcal{R}_{O_1} \supset \mathcal{R}_{O'}$ and therefore it is separating for $\mathcal{R}_{O_1}$ and $\mathcal{R}_V$. Now suppose that $\mathcal{R}_{O_1} = \mathcal{R}_V$. Then, by causality:

$$\pi_{\omega}(A_U) \subset \pi_{\omega}(A_V)' = \pi_{\omega}(A_{O_1})' \subset \pi_{\omega}(A_U)' .$$

It follows that $\mathcal{R}_U \subset \mathcal{R}_V'$, which contradicts the nowhere-classicality. Therefore, the inclusion $\mathcal{R}_V \subset \mathcal{R}_{O_1}$ must be proper and the cited theorem applies.

Of course, if $O \subset M_1$ is a cc-region that is not bounded, then it contains a bounded cc-region $O_1$ as above and $\mathcal{R}_{O_1} \supset \mathcal{R}_{O_1}$ isn’t finite either for any $\omega \in \mathcal{I}_{M_1}$. (If $V$ is a partial isometry in the smaller algebra such that $I = V^*V$ and $E := VV^* < I$ then the same $V$ shows that $I$ is not finite in the larger algebra.)

Next we consider $M_2$ and let $O \subset M_2$ be any cc-region. It contains a cc-region $O_2$ with $O_2^\perp \neq \emptyset$, so we can apply theorem 5.3.5. Using the unitary map $U_\alpha : \mathcal{H}_{\omega_2} \to \mathcal{H}_{\omega_1}$ we see that $\mathcal{R}_{O_2} \simeq \mathcal{R}_{O_1}^{\omega_2}$ contains $\alpha^{-1}(\mathcal{R}_{O_1}^{\omega_1})$, which is not finite by the first paragraph. Hence $\mathcal{R}_{O_2}$ is not finite and the statement for $O$ then follows again by inclusion. \qed
Instead of the nowhere-classicality we could have assumed that the local von Neumann algebras in $M_1$ are infinite, which allows us to derive the same conclusion for $M_2$. Unfortunately it is in general impossible to completely derive the type of the local algebras using this kind of argument. Even if we know the types of the algebras $A_{U_1}$ and $A_{V_1}$ in the inclusions \([5.4]\), we can't deduce the type of $A_{O_2}$.

Another important consequence of proposition \([5.3.5]\) in the $C^\ast$-algebraic case is that corollary \([5.3.6]\) enables us to apply the Tomita-Takesaki modular theory to $\mathcal{R}^\omega_{O_2}$ (or to the von Neumann algebra of any bounded cc-region $V_2$ which contains $O_2$, if the Cauchy surfaces are non-compact). More precisely, let $O_2 \subset M_2$ be given and let $U_1, V_1 \subset M_1$ be the bounded cc-regions and $\alpha : A_{M_1} \to A_{M_1}$ the $\ast$-isomorphism of proposition \([5.3.4]\) so that $A_{O_1} \subset \alpha(A_{O_2}) \subset A_{V_1}$. We can then define $\mathcal{R} := \bigcup_\alpha \mathcal{R}^\omega_{O_2} \mathcal{U}_0^\ast$ and obtain $\mathcal{R}^\omega_{U_1} \subset \mathcal{R} \subset \mathcal{R}^\omega_{V_1}$. It then follows that the respective Tomita-operators are extensions of each other, $S_{U_1} \subset S_R \subset S_{V_1}$ (see e.g. \([49]\)).

### 5.4 The quasi-analytic wave front set and the Reeh-Schlieder property for scalar fields

After the general results on the Reeh-Schlieder property presented in sections \([5.2]\) and \([5.3.1]\) we now specialise to the real scalar field, described by the Borchers-Uhlmann functor $U$. The main result of this section is a smoothly covariant condition on the continuous states of this algebra that guarantees the Reeh-Schlieder property as well as the fulfillment of the microlocal spectrum condition. This condition, which we call the quasi-analytic microlocal spectrum condition, is analogous to the analytic microlocal spectrum condition of \([80]\) and the microlocal spectrum condition. As a preparation to the formulation of our condition we need to study analytic wave front sets in
more detail in subsection 5.4.1 in particular their relation to the boundary of the support of a distribution. We refer to appendix A for the definition of analytic wave front sets and their properties.

5.4.1 Wave front sets and the support

We begin with a definition concerning the boundary of a closed set (see e.g. [47]):

**Definition 5.4.1** Let \( O \) be a closed subset of a smooth manifold \( M \). The exterior normal set \( N_e(O) \) of \( O \) consists of all \( (x, k) \in T^*M \) such that \( x \in O \) and there is a real-valued function \( f \in C^2(M) \) with \( f(y) \leq f(x) \) for all \( y \in O \) and \( df(x) = k \neq 0 \).

The normal set \( N(O) \) of \( O \) consists of \( (x, k) \in T^*M \) such that either \( (x, k) \in N_e(O) \) or \( (x, -k) \in N_e(O) \).

If \( (x, k) \in N_e(O) \) then \( x \) cannot be in the interior of \( O \), because an extremum of a \( C^2 \) function \( f \) can only be attained in the interior of \( O \) if \( df = 0 \), as is well-known. Conversely,

**Lemma 5.4.2** For a closed subset \( O \) of a smooth manifold \( M \) the projection of \( N_e(O) \) on \( M \) is dense in the boundary of \( O \).

**Proof.** See [47] proposition 8.5.8. \( \square \)

Thus the normal set characterises the boundary of the closed set \( O \) very well. Moreover, it is by definition a subset of \( T^*M \setminus Z \) and it is seen to be conic by multiplying the function \( f \) in the definition by a positive real number. This means that it can be compared with the wave front set:

**Proposition 5.4.3** Let \( u \) be a scalar distribution on an open set \( X \subset \mathbb{R}^n \), then \( N(\text{supp } u) \setminus Z \subset WF_A(u) \).
Remark 5.4.4 As an illustration we consider the case where the analytic wave front set of the distribution $u$ is empty, $WF_A(u) = \emptyset$. Proposition 5.4.3 then tells us that $N(supp u) = \emptyset$ and by lemma 5.4.2 the boundary of $supp u$ must be empty. Another way to reach the same conclusion is to notice that $u$ is an analytic function, so if there is an $x \in X$ which is not in the support of $u$, then $u \equiv 0$ on an open subset of $X$ and hence $u \equiv 0$ by analyticity. The support of $u$ is either all of $X$ or empty and in any case the boundary of the support is empty. It would be unreasonable to expect a similar result for the smooth wave front set, because one can easily construct smooth compactly supported functions.

Another instructive example, which shows that the support of a distribution cannot be characterised entirely in terms of the analytic wave front set, is the following. Let $\delta_0(x)$ be the Dirac measure on $\mathbb{R}$ at the point $x = 0$ and consider the distribution $u(x) := 1 + \delta_0(x)$ on $\mathbb{R}$. The support of $u$ is all of $\mathbb{R}$, so its normal set is empty. On the other hand, $WF_A(u) = \{0\} \times (\mathbb{R} \setminus \{0\})$. Indeed, $u$ fails to be analytic only at $x = 0$, so there must be some $k \neq 0$ with $(x,k) \in WF_A(u)$ and because $u$ is real-valued we then also have $(x,-k) \in WF_A(u)$. The conclusion then follows because $WF_A(u)$ is conic.

It follows from proposition 5.4.3 and lemma 5.4.2 that the analytic wave front set gives only an upper bound on the boundary of the support of a distribution. Moreover, because this result requires the use of analytic wave front sets it seems that it can only be formulated on an analytic manifold. Of course every $C^1$ manifold allows an analytic structure compatible with the $C^1$ structure ([4] section 2.5), so this is not really a restriction. However, this analytic structure is highly non-unique (unlike the $C^k$ and $C^\infty$ structures, which are unique, [4] theorem 2.2.9 and 2.3.4) and there does not seem to be
a natural choice. The following definition helps us to avoid making a choice and allows us to sharpen the result of proposition 5.4.3 considerably:

**Definition 5.4.5** Let $u$ be a distribution on a smooth manifold $\mathcal{M}$ with values in a Banach space $\mathcal{B}$. The quasi-analytic wave front set $WF_{qA}(u)$ of $u$ is defined to be the conic subset of $T^*\mathcal{M} \setminus Z$ consisting of all $(x, k) \in T^*\mathcal{M}$ such that for every smooth coordinate map $\kappa: O \subset \mathcal{M} \to \mathbb{R}^n$ near $x$ we have

$$(x, k) \in \kappa^*(WF_A(\kappa_*u)),$$

where $\kappa^*(y, l) := (\kappa^{-1}(y), d\kappa^{-1}l)$.

The quasi-analytic wave front set is a closed conic subset of $T^*\mathcal{M} \setminus Z$, because it is locally the intersection of the sets $\kappa^*(WF_A(\kappa_*u))$ which are closed in $T^*\mathcal{M} \setminus Z$. It is worth noting that

$$\bigcup_{\phi \in \mathcal{B}} WF_{qA}(\phi(u)) \setminus Z \subset WF_{qA}(u),$$

because of theorem A.2.2 and the closedness of $WF_{qA}(u)$, but it is not clear that equality holds in general, because the union over $\phi$ does not commute with the intersection over the choices of coordinates. Also some of the results in theorem A.2.4, such as the estimate for the wave front set of a sum, will fail in general for the quasi-analytic wave front set. Its usefulness is entirely based on the fact that it is by definition covariant under smooth diffeomorphisms, but at the same time contains some of the information of the analytic wave front set (in any given analytic structure on the manifold $\mathcal{M}$). More precisely:

**Lemma 5.4.6** Let $u$ be a distribution on an analytic manifold $\mathcal{M}$ with values in a Banach space $\mathcal{B}$. Then $WF(u) \subset WF_{qA}(u) \subset WF_A(u)$.

**Proof.** In any choice of local coordinates $\kappa$ we have $WF(\kappa_*u) \subset WF_A(\kappa_*u)$, from which the first inclusion follows. For the second we only need to choose $\kappa$ to be analytic and use the definition. \qed
In general neither of these inclusions is an equality. For the second inclusion we can see this by considering \( u = f \circ \phi \), where \( f \) is an analytic function on \( M \) and \( \phi \) is a smooth diffeomorphism of \( M \) which is not analytic. In this case \( u \) cannot be expected to be analytic. For the first inclusion we can choose \( u \) to be a compactly supported smooth function and use the following sharpening of proposition 5.4.3.

**Proposition 5.4.7** For a distribution \( u \) on a smooth manifold \( M \) with values in a Banach space \( B \) we have \( \overline{N(\text{supp} \, \phi(u))} \setminus Z \subset WF_{qA}(u) \) for all \( \phi \in B' \).

**Proof.** If \( (x, k) \in \overline{N(\text{supp} \, \phi(u))} \setminus Z \) and \( \kappa \) is a smooth choice of coordinates near \( x \) then \( d\kappa^T(x, k) \in WF_A(\kappa_*\phi(u)) \subset WF_A(\kappa_*u) \) by proposition 5.4.3 and theorem A.2.2. \( \square \)

**Remark 5.4.8** To see that proposition 5.4.7 is indeed a sharpening of proposition 5.4.3 one can consider the example of a smooth real-valued function \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \). We endow \( \mathbb{R} \) with the usual analytic structure. It is known that a generic real-valued function in \( C^\infty(\mathbb{R}) \) is nowhere analytic \([19, 25]\) in which case we have \( WF_A(f) = T^*\mathbb{R} \setminus Z \), i.e. the analytic wave front set could not be larger. The same is presumably true on an analytic manifold. On the other hand, a generic smooth function is a Morse function \(([44] \text{ theorem 6.1.2})\), which can be expressed locally as a polynomial of degree \( \leq 2 \) in suitable coordinates. In these coordinates the function is certainly analytic, so in the generic case we have \( WF_{qA}(f) = \emptyset \). In the two statements above the notion “generic” actually refers to two distinct topologies on the set of smooth functions, namely the weak and the strong topology respectively (see [44] section 2.1). However, for a compact manifold these topologies coincide, which would imply that at least on a compact manifold we generically have that \( WF_A(f) \) is maximal, whereas \( WF_{qA}(f) = \emptyset \). The conclusion is that
the many possible choices of local coordinates allow us to get a much stricter upper bound of the normal set \( N(\text{supp } f) \).

The following proposition is a microlocal analogue of (a corollary of) the edge-of-the-wedge theorem (see e.g. [78] theorem 2.16 and 2.17 or [47] theorem 9.3.5):

**Proposition 5.4.9** Let \( u \) be a distribution on a connected smooth manifold \( \mathcal{M} \) with values in a Banach space \( \mathcal{B} \) such that

\[
WF_{qA}(u) \cap -WF_{qA}(u) = \emptyset.
\]

If \( \mathcal{O} \subset \mathcal{M} \) is a non-empty open region, \( \phi \in \mathcal{B}' \) and \( \phi(u)|_{\mathcal{O}} = 0 \), then \( \phi(u) \equiv 0 \).

**Proof.** If \( (x,k) \in N(\text{supp } \phi(u)) \), then \( (x,-k) \in N(\text{supp } \phi(u)) \) so by lemma 5.4.7 both \( (x,k) \) and \( (x,-k) \) are in \( WF_{qA}(u) \), which contradicts the assumption. This means that \( N(\text{supp } \phi(u)) \) is empty and hence the boundary of \( \text{supp } \phi(u) \) is empty too by lemma 5.4.2. As \( \mathcal{M} \) is connected and \( \text{supp } \phi(u) \) is not all of \( \mathcal{M} \), we must have \( \text{supp } \phi(u) = \emptyset \), i.e. \( \phi(u) \equiv 0 \). \( \square \)

### 5.4.2 The quasi-analytic microlocal spectrum condition

We are now in a position to prove the Reeh-Schlieder property for states that satisfy an appropriate microlocal condition. We will first reproduce the result of [80], using an analytic microlocal spectrum condition. Then we will generalise this to all states that satisfy a certain quasi-analytic microlocal spectrum condition and we will discuss the implications and usefulness of this condition.

The analytic microlocal spectrum condition of [80] is a direct generalisation of the (smooth) microlocal spectrum condition:

**Definition 5.4.10** A spacetime \( \mathcal{M} \) is an analytic spacetime if it is endowed with an analytic structure in which the metric is analytic. (Equivalently, all
component functions $g_{\mu\nu}$ of the metric are analytic in any choice of coordinates on the analytic manifold $M$.)

A state $\omega$ on the Borchers-Uhlmann algebra $\mathcal{U}_M$ of an analytic spacetime $M$ satisfies the analytic microlocal spectrum condition $(A\mu SC)$ if and only if $WF_A(\omega_n) \subset \Gamma_n$ for all $n \in \mathbb{N}$.

Note that the $A\mu SC$ implies the $\mu SC$, because $WF_\omega(n) \subset WF_A(\omega_n)$. It also implies the Reeh-Schlieder property as follows:

**Theorem 5.4.11** Let $\omega$ be a state on the Borchers-Uhlmann algebra $\mathcal{U}_M$ of an analytic globally hyperbolic spacetime $M$ that satisfies the $A\mu SC$. Then $\omega$ has the Reeh-Schlieder property.

**Proof.** Our proof follows that of [80], which is a generalisation of the proof in [78] for the case of a Wightman field in Minkowski spacetime, using the spectrum condition of the Wightman axioms.

Let $O \subset M$ be any cc-region and set $D_O := \{ \pi_\omega(A)\Omega_\omega | A \in \mathcal{U}_O \}$. Notice that $D_O \subset \mathcal{H}_\omega$ is dense if and only if $D_O^\perp = \{0\}$. We now suppose that $\psi \in D_O^\perp$, which means that the distribution

$$w_n(x_n, \ldots, x_1) := \langle \psi, \phi_n^{(\omega)}(x_n, \ldots, x_1) \rangle$$

is identically zero on the open neighbourhood $O^{\times n}$ in $M^{\times n}$ for all $n \in \mathbb{N}$. Now suppose that $(x, \pm k) \in WF_A(\phi_n^{(\omega)})$ for both choices of the sign. By theorem A.2.4 we then have $(x, \pm k; x, \pm k) \in WF_A(\omega_{2n}) \subset \Gamma_{2n}$, or $(x, -k; x, k) \in \Gamma_{2n} \cap -\Gamma_{2n} = \emptyset$ by proposition 3.1.8. This is a contradiction, which proves that $WF_A(\phi_n^{(\omega)}) \cap -WF_A(\phi_n^{(\omega)}) = \emptyset$ and hence also $WF_A(w_n) \cap -WF_A(w_n) = \emptyset$. We may therefore apply proposition 5.4.9 to conclude that $w_n = 0$ on all of $M^{\times n}$. This in turn implies that $\langle \psi, \pi_\omega(A)\Omega_\omega \rangle = 0$ for all $A \in \mathcal{U}_M$. The vectors $\pi_\omega(A)\Omega_\omega$ with $A \in \mathcal{U}_M$ form a dense subspace of $\mathcal{H}_\omega$, so we conclude $\psi \in \mathcal{H}_\omega^\perp = \{0\}$, which completes the proof. $\square$
The proof of theorem 5.4.11 can be generalised considerably. In fact, the $A\mu SC$ allows us to derive $WF_A(\phi^{(n)}_n) \cap -WF_A(\phi^{(n)}_n) = \emptyset$, but we only need $WF_{qA}(\phi^{(n)}_n) \cap -WF_{qA}(\phi^{(n)}_n) = \emptyset$ to arrive at the conclusion of theorem 5.4.11. We will now show that the $A\mu SC$ can be weakened to a quasi-analytic microlocal spectrum condition which is still strong enough to make the proof above work. There is a subtlety involved, however, because theorem A.2.4 does not hold for quasi-analytic wave front sets without modification. We therefore define the following:

**Definition 5.4.12** Let $u_n$ be a distribution on the $n$-fold product $\mathcal{M}^\times_n$ of a smooth manifold $\mathcal{M}$ with values in a Banach space $\mathcal{B}$. We define the wave front set $WF_{qA}(u_n)$ to be the conic subset of $T^*\mathcal{M}^\times_n \backslash \mathcal{Z}$ consisting of all points $(x_1, k_1; \ldots; x_n, k_n)$ such that

$$(x_1, k_1; \ldots; x_n, k_n) \in (\kappa^{\times n})^*(WF_A((\kappa^{\times n})_*u)),$$

where $\kappa$ is a smooth coordinate map on a neighbourhood of all $x_i$, $1 \leq i \leq n$.

Given a finite number of points $x_1, \ldots, x_n$ we can always find a coordinate map which contains these points in its domain by theorem 16.26.9 of [28]. Moreover, when these points are distinct we can choose the analytic structure near every point arbitrarily by theorem 8.3.1 of [44].

Notice that $WF_{qA}(u_n) \subset WF_{qA}^{(n)}(u_n)$, but the converse is not true. The point of this definition is the following result, which presumably fails for $WF_{qA}$:

**Theorem 5.4.13** Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a smooth manifold and $u_i$, $i = 1, 2$, two $\mathcal{H}$-valued distributions on $\mathcal{M}^\times_{n_i}$ for some $n_i \in \mathbb{N}$. We define the distributions $w_{ij}$ on $\mathcal{M}^\times_{(n_i+n_j)}$ by $w_{ij}(f_1, f_2) := \langle u_i(\overline{f}_1), u_j(f_2) \rangle$. Then

$$(x, k) \in WF_{qA}^{(n_1)}(u_1) \iff (x, -k; x, k) \in WF_{qA}^{(2n_1)}(w_{11})$$
and
\[ \text{WF}_{qA}^{(n_i+n_j)}(w_{ij}) \subset \left(-\text{WF}_{qA}^{(n_i)}(u_i) \cup Z\right) \times \left(\text{WF}_{qA}^{(n_j)}(u_j) \cup Z\right). \]

**Proof.** If \((x, -k; x, k) \notin \text{WF}_{qA}^{(2n_1)}(w_{11})\) then there is a choice of coordinates \(\kappa\) near all \(x_i\) such that \((x, -k; x, k) \notin (\kappa^{2n_1})^*WF_A((\kappa^{2n_1})_wu_{11})\). Theorem \[\text{A.2.5}\] therefore implies \((x, k) \notin (\kappa^{n_1})^*WF_A((\kappa^{n_1})_wu_{1})\) and hence \((x, k) \notin \text{WF}_{qA}^{(n_1)}(u_1)\). This proves one direction of the first statement. On the other hand, if \((x, k) \notin \text{WF}_{qA}^{(n_i)}(u_i)\) with \(k \neq 0\) then there exists a choice of coordinates \(\kappa\) such that \((x, k) \notin (\kappa^{n_i})^*WF_A((\kappa^{n_i})_wu_i)\). Now consider any point \((x', k') \in T^*M^{x_{n_i}}\). We can find a choice of coordinates \(\lambda\) such that \((x, x')\) and \((x', x)\) are in the domain of \((\lambda)^{n_i+n_j}\) \([\text{loc. cit.}]\). By composing \(\lambda\) with a suitable diffeomorphism we can ensure that the analytic structure determined by \(\lambda\) near \(x\) coincides with that of \(\kappa\) \([\text{loc. cit.}]\). It then follows from theorem \[\text{A.2.5}\] that \((x', k'; x, k) \notin (\lambda^{n_i+n_j})^*WF_A((\lambda^{n_i+n_j})_w_{ij})\) and \((x, -k; x', k') \notin (\lambda^{n_i+n_j})^*WF_A((\lambda^{n_i+n_j})_w_{ji})\), which completes the proof. \(\Box\)

**Definition 5.4.14** We say that a state \(\omega\) on the Borchers-Uhlmann algebra \(\mathcal{U}_M\) of a globally hyperbolic spacetime \(M\) satisfies the quasi-analytic microlocal spectrum condition \((qA\mu\text{SC})\) iff \(WF_{qA}^{(n)}(\omega_n) \subset \Gamma_n\) for all \(n\).

Note that this condition is well-defined on a smooth spacetime \(M\) and that it is independent of a choice of coordinates. Moreover we have

**Corollary 5.4.15** Let \(\omega\) be a state on the Borchers-Uhlmann algebra \(\mathcal{U}_M\) of a globally hyperbolic spacetime \(M\) that satisfies the \(qA\mu\text{SC}\). Then \(\omega\) has the Reeh-Schlieder property.

**Proof.** The proof is essentially the same as that of theorem \[5.4.11\]. Note in particular that for any \(\psi \in \mathcal{H}_\omega\) we have
\[ WF_{qA}(\langle \psi, \phi_{n}^{(\omega)} \rangle) \subset WF_{qA}^{(n)}(\langle \psi, \phi_{n}^{(\omega)} \rangle) \subset WF_{qA}^{(n)}(\phi_{n}^{(\omega)}) \]
and \( WF_{qA}^{(n)}(\phi_n^{(\omega)}) \cap -WF_{qA}^{(n)}(\phi_n^{(\omega)}) = \emptyset \) by theorem 5.4.13 and the qA\(\mu\)SC. □

The qA\(\mu\)SC is a condition that implies the \(\mu\)SC as well as the Reeh-Schlieder property, but its practical use is limited unless we can find states that satisfy this condition. Indeed, on a generic smooth spacetime the class that is singled out by this condition could be empty. Unfortunately the condition is very hard to work with in this respect. It is clear that any state that satisfies the A\(\mu\)SC of [80] also satisfies the qA\(\mu\)SC, but in order to find more examples one would need to have a better understanding of how the analytic wave front set changes under a smooth diffeomorphism. Even for ground and KMS-states on a stationary spacetime, states that are known to have the Reeh-Schlieder property [79], it is not clear whether they satisfy the qA\(\mu\)SC. One approach would be to consider spacetimes whose metric is analytic in a time-coordinate. For such metrics some unique continuation results are known [71] and these can possibly be extended to our situation, although we were unable to obtain a result in this direction. Of course this idea may be criticised, because a generic metric is not analytic in a time coordinate. On the other hand it can be argued that we are not interested in generic metrics, but only in solutions to Einstein’s equation, which makes the situation less clear. In fact, it may well be easier to use another approach to find Reeh-Schlieder states with the \(\mu\)SC, for example using the spacetime deformation argument of section 5.3. This means that the importance of the qA\(\mu\)SC is largely academic, but it does prove the existence of a smoothly covariant condition that ensures the \(\mu\)SC and the full Reeh-Schlieder property.
5.5 The Reeh-Schlieder property for the real free scalar field on Minkowski spacetime

It is well-known that the Minkowski vacuum $\omega_0$ has the Reeh-Schlieder property (both as a state on $A^0_{M_0}$ and on $U^0_{M_0}$). It is also known [22, 31] that $H_{\omega_0}$ contains a dense $G_\delta$ of vectors which define Reeh-Schlieder states on $A^0_{M_0}$, and these include at least all states of bounded energy [40]. We now turn to the question whether we can find many vectors in $H_{\omega_0}$ that define Reeh-Schlieder states that are also Hadamard states.

For a first result we consider the space $S(\mathbb{R}^n)$ of Schwartz-functions, which is a Fréchet space ([47] definition 7.1.2), and we define the following algebra, in analogy with the Borchers-Uhlmann algebra $^2$

**Definition 5.5.1** We define the algebra $U'_{M_0} := \oplus_{n=0}^\infty S(M_0^{\times n})$, (in the algebraic sense), equipped with:

1. the product $f(x_1, \ldots, x_n)g(x_{n+1}, \ldots, x_{n+m}) := (f \otimes g)(x_1, \ldots, x_{n+m})$, extended linearly,

2. the $^*$-operation $f(x_1, \ldots, x_n)^* := \overline{f(x_n, \ldots, x_1)}$, extended anti-linearly,

3. a topology such that $f_j := \oplus_n f_j^{(n)}$ converges to $f = \oplus_n f^{(n)}$ if and only if for all $n$ we have $f_j^{(n)} \to f^{(n)}$ in $S(M_0^{\times n})$ and for some $N > 0$ we have $f_j^{(n)} = 0$ for all $j$ and $n \geq N$.

This is a topological $^*$-algebra in the same way as $U_{M_0}$. In fact, it contains $U_{M_0}$ as a dense linear subspace and the canonical embedding is a continuous linear map (see [47] lemma 7.1.8). Note that the multiplication in $U'_{M_0}$ is jointly continuous, because the map $(f^{(i)}, h^{(j)}) \mapsto f^{(i)} \otimes h^{(j)}$ is jointly continuous. A

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$^2$Actually, Borchers [11] works in Minkowski spacetime and defines the algebra $U'_{M_0}$, so by right this algebra can also be called “Borchers-Uhlmann algebra”. Note however that the notion of Schwartz-functions cannot be generalised to general manifolds.
state $\omega$ on $U_{M_0}'$ consists of a sequence of $n$-point distributions $\omega_n$, which are tempered distributions.

The state $\omega_0$ on $U_{M_0}^0$ has $n$-point distributions which are tempered, so if $p: U_{M_0} \rightarrow U_{M_0}^0$ is the canonical projection map, then the state $\omega_0 \circ p$ on $U_{M_0}$ can be extended in a unique way to $U_{M_0}'$. We will denote this extension by $\omega_0$ too.

**Theorem 5.5.2** Consider the Minkowski vacuum state $\omega_0$ on $U_{M_0}'$ for a positive mass $m > 0$ and an $A \in U_{M_0}'$ such that $v := \pi_\omega(A) \Omega_\omega \neq 0$. Then there exists a sequence of elements $A_n \in U_{M_0}'$ such that $A_n \rightarrow A$ in $U_{M_0}'$ as $n \rightarrow \infty$ and such that the vectors $v_n := \pi_\omega(A_n) \Omega_\omega$ define states on $U_{M_0}'$ which restrict to Hadamard Reeh-Schlieder states on $U_{M_0}$.

**Proof.** For all $n \in \mathbb{N}$ we set $h_n(x) := \frac{n^4}{\pi^2} e^{-n^2\|x\|^2}$, where $\|x\|^2$ denotes the Euclidean norm on $\mathbb{R}^4$. Notice that $h_n \in S(M_0)$ is analytic for all $n$ and that $h_n \rightarrow \delta_0$ in the space $S'(M_0)$ of tempered distributions.

Every element $A'$ in $U_{M_0}'$ can be approximated by an elements $A \in U_{M_0}'$ of the form $A = \oplus_{i=0}^N f^{(i)}_1 \otimes \ldots \otimes f^{(i)}_i$ with $f^{(i)}_j \in S(M_0)$ and the result for $A'$ follows from that for $A$. To prove it for an $A$ of this form we define the $A_n$ by $A_n := \oplus_{i=0}^N (h_n \ast f^{(i)}_1) \otimes \ldots \otimes (h_n \ast f^{(i)}_i)$, where $\ast$ denotes the convolution,

$$h_n \ast f^{(i)}_j(x) = \int h_n(x-y_j)f^{(i)}_j(y_j)dy_j.$$  

We then have $h_n \ast f^{(i)}_j \rightarrow f^{(i)}_j$ in $S(M_0)$ and hence $A_n \rightarrow A$ as $n \rightarrow \infty$.

Now we let $u_j \in C_0^\infty(M_0^{\times 2})$ be a sequence such that $u_j \rightarrow (\omega_0)_2$ as $j \rightarrow \infty$ (see [47] theorem 4.1.5 for the existence of such a sequence). For every pair of functions $\phi_1, \phi_2 \in C_0^\infty(M_0)$ we then have (cf. Parseval’s formula, [47] theorem 7.1.6)

$$\int u_j(x, y)\phi_1(x)\phi_2(y)dx \ dy = (2\pi)^{-4} \int e^{ix\cdot\xi}u_j(\xi, \eta)\phi_1(-\eta)dx \ d\xi \ d\eta.$$  

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Using the fact that $(\omega_0)_2(\xi, \eta) = (2\pi)^5 \delta(\xi + \eta) \delta(\eta^2 - m^2)\theta(\eta_0)$ (see the notations and conventions in the preface) and taking the limit $j \to \infty$ yields:

$$(\omega_0)_2(\phi_1, \phi_2) = (2\pi)^{-4} \int e^{ix\xi} (\omega_0)_2(\xi, \eta) \phi_1(x) \hat{\phi}_2(-\eta) dx \, d\xi \, d\eta$$

$$= 2\pi \int e^{-ix\eta} \delta(\eta^2 - m^2)\theta(\eta_0) \phi_1(x) \hat{\phi}_2(-\eta) dx \, d\eta.$$  (5.6)

This uses the fact that a distribution on $M_0^{x^2}$ is uniquely determined by its action on functions of the form $\phi_1 \otimes \phi_2$ (see [47] theorem 5.1.1). Notice that $\int e^{-ix\eta} \delta(\eta^2 - m^2)\theta(\eta_0) \phi_1(x) dx = \hat{\phi}_1(\eta) \delta(\eta^2 - m^2)\theta(\eta_0)$ is a tempered distribution in $\eta$, so using the fact that $C_0^\infty(M_0) \subset \mathcal{S}(M_0)$ is dense ([47] lemma 7.1.8) we may extend equation (5.6) to all $\phi_2 \in \mathcal{S}(M_0)$. We can then write for all $\phi_2 \in \mathcal{S}(M_0)$:

$$(\omega_0)_2(x, \phi_2) = 2\pi \int e^{-ix\eta} \delta(\eta^2 - m^2)\theta(\eta_0) \hat{\phi}_2(-\eta) d\eta,$$

where the expression on the right hand side is well-defined for each point $x$. We now substitute $\phi_2 = h_n \ast f$ with $f \in \mathcal{S}(M_0)$ so that

$$(\omega_0)_2(x, h_n \ast f) = 2\pi \int e^{-ix\eta} \delta(\eta^2 - m^2)\theta(\eta_0) e^{-\|\eta\|^2/(4n^2)} \hat{f}(-\eta) d\eta$$

$$= 2\pi \int e^{-ix\eta} \delta(\eta_0 - \omega_\eta^2) e^{-\|\eta\|^2/(4n^2)} \hat{f}(-\eta) \frac{d\eta}{2\eta_0}.$$  (5.7)

where $\omega_\eta := \sqrt{\|(\eta_1, \eta_2, \eta_3)\|^2 + m^2}$ and $\|\eta\|$ again denotes the Euclidean norm on $\mathbb{R}^4$. The Gaussian on the right-hand side ensures that this expression is well-defined for every $x \in \mathbb{C}$, so $(\omega_0)_2(x, h_n \ast f)$ can be extended to a function on $\mathbb{C}$. Moreover, for all $z \in \mathbb{C}^4$ we can substitute $e^{-i(x-z)\cdot \eta}$ for $e^{-ix\eta}$ in equation (5.7) and use $e^{-i(x-z)\cdot \eta} = e^{-ix\cdot \eta}(1 - iz \cdot \eta + R(z, \eta))$, where the remainder term $R(z, \eta) = -(z \cdot \eta)^2 \int_0^1 (1 - s) e^{-isz\cdot \eta} ds$ satisfies $|R(z, \eta)| \leq |z \cdot \eta|^2(1 + |e^{-i\eta}|)$. Using the Gaussian for convergence it follows that $(\omega_0)_2(x, h_n \ast f)$ is complex differentiable and hence analytic. The same is then true for $(\omega_0)_2(h_n \ast f, y) = (\omega_0)_2(y, h_n \ast f)$.
Notice that $v_n \to v$, by $\|v_n - v\|^2 = \omega_0((A_n - A)^*(A_n - A))$ and the joint continuity of the multiplication in $\mathcal{U}_M$. So if $v \neq 0$ then $v_n \neq 0$ for all sufficiently large $n$. By dropping a finite number of indices at the start of the sequence we may assume that this holds for all $n$. We now claim that each $v_n$ defines a state that satisfies the $A\mu SC$, from which the result follows. To prove this claim we note the fact that $\omega_0$ is quasi-free, that $(\omega_0)_2(h_n*f)$ and $(\omega_0)_2(h_n*f,..)$ are analytic for all $f = f^{(i)}_j$ and that $WF_{A}(\omega_0)_2 \subset \Gamma_2$ (see [80] theorem 6.3 and [2] section 4.2). The result then follows from definition 3.1.6 and proposition 3.1.8.

To conclude this section we prove that the Minkowski vacuum has a property which is stronger than the Reeh-Schlieder property.

**Theorem 5.5.3** Consider the Minkowski vacuum state $\omega_0$ on $\mathcal{U}_M$ with its GNS-quadruple $(\mathcal{H}_{\omega_0}, \pi_{\omega_0}, \Omega_{\omega_0}, \mathcal{D}_{\omega_0})$. If $O \subset M_0$ is any non-empty cc-region, then $\pi_{\omega_0}(\mathcal{U}_O)\Omega_{\omega_0}$ is dense in $\mathcal{D}_{\omega_0}$ in the graph topology (see definition 2.1.3).

**Proof.** Recall that we need to find for each $\phi \in \mathcal{D}_{\omega_0}$ a sequence of vectors $\phi_n \in \pi_{\omega_0}(\mathcal{U}_O)\Omega_{\omega_0}$ such that $\pi_{\omega_0}(A)\phi_n \to \pi_{\omega_0}(A)\phi$ for all $A \in \mathcal{U}_M$. We will use the fact that $\omega_0$ is a quasi-free state that satisfies the $A\mu SC$. First we consider the real Hilbert space $\mathcal{H}^1 := \{\phi^{(\omega_0)}_1(f) | f \in C^\infty(M_0, \mathbb{R})\}$ and the subspace $\mathcal{H}^1_2 := \{\phi^{(\omega_0)}_1(f) | f \in C^\infty_0(O, \mathbb{R})\}$ with $(\psi, \chi) := \text{Re}(\langle \psi, \chi \rangle)$ as inner product. (See [81] appendix A1 for a similar one-particle Reeh-Schlieder result.) If $\psi \in \mathcal{H}^1$ is in $(\mathcal{H}^1_0)^\perp$, where $\perp$ refers to the inner product $(,)$ on $\mathcal{H}^1_0$, then the $\mathcal{H}_{\omega_0}$-valued distribution $w(x) := \langle \psi, \phi^{(\omega_0)}_1(x) \rangle$ is identically 0 on $O$ by complex linearity. By the $A\mu SC$ we find that $WF_{A}(w) \subset \mathcal{N}^+$, so $WF_{A}(w) \cap -WF_{A}(w) = \emptyset$ and $w \equiv 0$ everywhere by proposition 5.4.9.

Hence, for every $f \in C^\infty_0(M_0, \mathbb{R})$ we can find a sequence $f_n$ of elements in $C^\infty_0(O, \mathbb{R})$ such that $\phi^{(\omega_0)}_1(f_n) \to \phi^{(\omega_0)}_1(f)$ in $\mathcal{H}^1$ as $n \to \infty$. This also means that:

$$\|\phi^{(\omega_0)}_1(f - f_n)\|^2_{\mathcal{H}_{\omega_0}} = \langle \phi^{(\omega_0)}_1(f - f_n), \phi^{(\omega_0)}_1(f - f_n) \rangle \to 0.$$
In other words, \( \phi_1^{(\omega)}(f_n) \to \phi_1^{(\omega)}(f) \) in \( \mathcal{H}_{\omega_0} \). If we decompose \( f \in C_0^\infty(M_0) \) as \( f = u + iv \) where \( u, v \in C_0^\infty(M_0, \mathbb{R}) \), then we may apply the previous reasoning to find sequences \( u_n \) and \( v_n \) in \( C_0^\infty(O, \mathbb{R}) \) such that \( \phi_1^{(\omega)}(u - u_n) \to 0 \) and \( \phi_1^{(\omega)}(v - v_n) \to 0 \) as \( n \to \infty \). This implies that for \( f_n := u_n + iv_n \) we have \( \phi_1^{(\omega)}(f - f_n) \to 0 \) and \( \phi_1^{(\omega)}(\overline{f} - \overline{f_n}) \to 0 \) as \( n \to \infty \).

Now consider the two homogeneous elements \( A := f^i \otimes \ldots \otimes f^1 \) and \( B := h^{(r)} \) of \( \mathcal{U}_{\mathcal{M}_0}^0 \) with \( f^j \in C_0^\infty(M_0) \) and \( h^{(r)} \in C_0^\infty(M_0^{*r}) \). By the previous paragraph we can find sequences \( f^j_n \) in \( C_0^\infty(O) \) such that \( \phi_1^{(\omega)}(f^j_n - f^j) \to 0 \) and \( \phi_1^{(\omega)}(\overline{f}^j_n - \overline{f}^j) \to 0 \) as \( n \to \infty \). We set \( A_n := f^j_n \otimes \ldots \otimes f^j_n \) and notice that

\[
B(A - A_n) = h^{(r)} \otimes f^i \otimes \ldots \otimes f^2 \otimes (f^1 - f^1_n) \\
+ h^{(r)} \otimes f^i \otimes \ldots \otimes f^3 \otimes (f^2 - f^2_n) \otimes f^1_n \\
+ \ldots + h^{(r)} \otimes (f^i - f^i_n) \otimes f^i_n \otimes \ldots \otimes f^1_n.
\] (5.8)

We wish to show that \( \pi_{\omega_0}(B(A - A_n))\Omega_{\omega_0} \to 0 \) as \( n \to \infty \). For this it is sufficient to show that each term in equation (5.8) converges to 0, so we consider a fixed term containing \( f^j_n - f^j_i \). Without loss of generality we may absorb the factor \( f^i \otimes \ldots \otimes f^{j+1} \) into \( h^{(r)} \), so that the term looks like

\[
h^{(r)} \otimes (f^j_n - f^j_i) \otimes f^i_n \otimes \ldots \otimes f^1_n.
\]

The norm squared of \( \pi_{\omega_0}(\cdot)\Omega_{\omega_0} \) of this term is of the form:

\[
\|\pi_{\omega_0}(h^{(r)} \otimes (f^j_n - f^j_i) \otimes f^i_n \otimes \ldots \otimes f^1_n)\Omega_{\omega_0}\|^2 = \\
\omega_0(\overline{f}^j_n \otimes \ldots \otimes f^i_n \otimes (\overline{f}^j_n - \overline{f}^{j_i}) \otimes (h^{(r)})^* \otimes h^{(r)} \otimes (f^j_n - f^j_i) \otimes f^i_n \otimes \ldots \otimes f^1_n).
\] (5.9)

Using the fact that \( \omega_0 \) is quasi-free we write this as a sum of terms and show that each term converges to 0. To see this we first note that the sequences \( \|\phi_1^{(\omega)}(f^j_n)\| \) and \( \|\phi_1^{(\omega)}(\overline{f}^j_n)\| \) remain bounded, so all factors of the
form \((\omega_0)_2(f_n, f'_n)\), where \(f_n\) and \(f'_n\) are either \(f^i_n\) or \(\overline{f}^j_n\), remain bounded by the Cauchy-Schwarz inequality. Next we notice that

\[
\left| \int (\omega_0)_2(x, y) \chi(x) h(y, y') dx \, dy \right|^2 \leq (\omega_0)_2(\chi, \chi) \cdot \int (\omega_0)_2(x, y) \overline{h}(x, y') h(y, y') dx \, dy
\]

is a compactly supported smooth function of \(y'\). If we substitute for \(\chi\) either \(f^i_n\) or \(\overline{f}^j_n\), the right-hand side is estimated by a bounded constant times a compactly supported smooth function of \(y'\). Using these facts we can integrate out all but a few variables in a summand of equation (5.9) to obtain:

\[
\left| \int \overline{f}^j_n(x_1) \cdots \overline{f}^j_{n-1}(x_{j-1}) (\overline{f}^j_n - \overline{f}^j_n')(x_j) \overline{h}(x_{j+r}, \ldots, x_{j+1}) h(x_{j+r+1}, \ldots, x_{j+2r}) (f^j_j - f^j_n)(x_{j+2r+1}) f^j_n(x_{j+2r+2}) \cdots f^j_{n-1}(x_{2r+2}) \right|
\]

\[
C \sum_{\pi} \left| \int (f^j_j - f^j_n)(x_1) H(x_2)(\omega_0)_2(x_{\pi(1)}, x_{\pi(2)}) dx_1 \, dx_2 \right| (5.10)
\]

\[
+ C \sum_{\pi} \left| \int (f^j_j - f^j_n)(x_1) \chi(x_2)(\omega_0)_2(x_{\pi(1)}, x_{\pi(2)}) dx_1 \, dx_2 \right|
\]

where \(C > 0\) is a constant, \(H \in C^\infty_0(M_0)\), \(\chi\) is either \(f^i_n\) or \(\overline{f}^j_n\) for some index \(i\) and we sum over both permutations \(\pi\) of the set \(\{1, 2\}\). The first term appears whenever the variable \(y\) occurs in \(\overline{f}^j \otimes h^* \otimes h\) rather than in some \(f^i_n\) or \(\overline{f}^j_n\). Both terms in equation (5.10) can be estimated using the Cauchy-Schwarz inequality and are then seen to converge to 0, because \(\|\phi_1^{(\omega_0)}(f^j_j - f^j_n)\|\) and \(\|\phi_1^{(\omega_0)}(\overline{f}^j_n - \overline{f}^j_n)\|\) converge to 0 as \(n \to \infty\). This proves that the norm-squared in equation (5.9) converges to 0. The same conclusion remains true when we replace \(B\) by a finite sum of homogenous terms and hence \(\pi_{\omega_0}(A - A_n)\Omega_{\omega_0}\) converges to 0 in the graph topology.

By definition of \(\pi_{\omega_0}\) the linear space \(\pi_{\omega_0}(U^0_{M_0})\Omega_{\omega_0}\) is dense in \(\mathcal{D}_{\omega_0}\) in the graph topology (see definition 2.1.3 and theorem 2.1.4). We can approximate
every \( A \in \mathcal{U}_{M_0} \) by a sequence \( A_n \) of elements of the form \( A_n = \oplus_{i=0}^N f_{1,n}^{(i)} \otimes \ldots \otimes f_{i,n}^{(i)} \) with \( f_{j,n}^{(i)} \in C_0^\infty(M_0) \). By joint continuity of the multiplication we find that \( (A - A_n)^* B^* B (A - A_n) \) converges to 0 for every \( B \in \mathcal{U}_{M_0} \) and hence that \( \pi_{\omega_0}(B(A - A_n))\Omega_{\omega_0} \) converges to 0 for all \( B \). Hence, the elements of the form \( \sum_{i=1}^n \phi_{i}^{(\omega_0)}(f_{1}^{(i)}, \ldots, f_{i}^{(i)}) \), where \( f_{j}^{(i)} \in C_0^\infty(M_0) \) and \( \phi_{i}^{(\omega_0)} \) are the Hilbert space-valued distributions, are dense in \( \mathcal{D}_{\omega_0} \) in the graph topology. By the previous paragraph every term in this sum can be approximated in the graph topology by a term of the same form but with \( f_{j}^{(i)} \in C_0^\infty(O) \). \( \square \)

5.6 The Reeh-Schlieder property for the real free scalar field

To conclude this chapter we apply the results of the previous sections to the free scalar and Dirac field as presented in chapters 3 and 4 and consider what conclusions we may draw. We first consider the free field Borchers-Uhlmann functor \( U^0 \) with the state space \( Q^0 \) of Hadamard states defined in chapter 3.

**Proposition 5.6.1** Let \( M \) be a globally hyperbolic spacetime, let \( O \subset M \) a bounded cc-region with non-empty causal complement and assume that the mass \( m > 0 \) is strictly positive. Then there is a state \( \omega \in \mathcal{D}^0_M \) on \( \mathcal{U}^0_M \) which has the Reeh-Schlieder property for \( O \).

**Proof.** We can find an ultrastatic (and hence stationary) spacetime \( M' \) diffeomorphic to \( M \). Because \( m > 0 \) we may apply the results of [50], which imply the existence of a regular quasi-free ground state \( \omega' \) on \( A^0_{M'} \). This state is a Reeh-Schlieder state (see [79]) and is Hadamard because it satisfies the microlocal spectrum condition (see [80, 66]). It follows that \( \omega' \) also defines a Hadamard state \( \bar{\omega} \) on \( \mathcal{U}^0_{M'} \). To see that \( \bar{\omega} \) is a Reeh-Schlieder state we choose
a non-empty cc-region $O \subset M'$ and compare the GNS-representations of $\rho' := \omega'|_{A_0^O}$ and $\tilde{\rho} := \tilde{\omega}|_{U_0^O}$. Notice that we may take $\mathcal{H}_\rho \subset \mathcal{H}_{\rho'}$ with $\Omega_\rho = \Omega_{\rho'}$, by the essential uniqueness of the GNS-representation (theorem 2.1.4). Because we can identify $\pi_{\omega'}(W(f)) = \exp(i\Phi^{(\omega)}(f))$ (see [13] proposition 5.2.4) we see that $\Omega_{\rho'}$ must be cyclic for $\pi_{\omega'}(U_0^O)$, otherwise it would not be cyclic for $\pi_{\omega'}(A_0^O)$. The fact that $\tilde{\omega}$ has the Reeh-Schlieder property therefore follows from the fact that $\omega'$ has it. Now recall that the locally covariant quantum field theory $U^0$ and state space $Q^0$ satisfy the time-slice axiom (see proposition 3.1.17). We can therefore apply theorem 5.3.5 with the state $\tilde{\omega}$, from which the result follows immediately. □

As we noticed in subsection 5.3.2 we can draw stronger conclusions when the theory is $C^*$-algebraic:

**Proposition 5.6.2** Let $M$ be a globally hyperbolic spacetime with a non-compact Cauchy surface and assume that the mass $m > 0$ is strictly positive. Then there is a state $\omega \in \mathcal{S}_M^0$ on $A^0_M$ such that $\mathcal{H}_\omega$ contains a dense $G_\delta \mathcal{G}$ of vectors which define (full) Reeh-Schlieder states. For all bounded cc-regions $V \subset M$ the local von Neumann algebra $\mathcal{R}_V$ is not finite and if $V$ has non-zero causal complement then each vector $\psi \in \mathcal{G}$ is cyclic and separating for $\mathcal{R}_V$.

**Proof.** The theory is causal, locally quasi-equivalent, satisfies the time-slice axiom and is nowhere classical (see proposition 3.2.5). Note that $\mathcal{R}_V$ is well-defined, independent of $\omega \in \mathcal{S}_M^0$ by local quasi-equivalence. As in the proof of proposition 5.6.1 we can find a Reeh-Schlieder state $\omega'$ on $A^0_{M'}$, where $M'$ is a spacetime diffeomorphic (but not isometric) to $M$. Now theorem 5.3.7 and the definition of $\mathcal{S}_M^0$ prove the existence of a full Reeh-Schlieder state $\omega \in \mathcal{S}_M^0$ and the results of [31] (see also the proof of theorem 5.3.7) provide the dense $G_\delta$ set $\mathcal{G}$ in $\mathcal{H}_\omega$. The other conclusions then follow from proposition 5.2.2 and corollary 5.3.8. □
Note that stronger results on the type of the local algebras are known [84], but we have used a different and interesting method of proof.

It seems likely that our deformation results can be extended from spacetimes to spin spacetimes, so that similar results can be obtained for the Dirac. In the case of the functors $F^0$ and $R^0$ we formulate:

**Conjecture 5.6.3** Let $M$ be a globally hyperbolic spin spacetime and let $O \subset M$ be a bounded cc-region with non-empty causal complement. Then there is a state $\omega \in \mathcal{R}^0$ on $\mathcal{F}^0_M$ which has the Reeh-Schlieder property for $O$.

**Sketch of proof.** We can find an ultrastatic (and hence stationary) spacetime $M'$ diffeomorphic to $M$. There then exists a quasi-free KMS state $\omega'$ on $\mathcal{F}^0_{M'}$, which has the Reeh-Schlieder property (see [79]). By [72] this state is Hadamard. Because $\mathcal{F}^0_{M'}$ is dense in $\mathcal{F}^0_M$, we see that $\omega'$ defines a Reeh-Schlieder state on $\mathcal{F}^0_M$, which is Hadamard by definition 4.2.14 and hence satisfies the $\mu$SC by proposition 4.2.17. The locally covariant quantum field theory $F^0$ and the state space $R^0$ satisfy the time-slice axiom (see proposition 4.2.25). The proof then comes down to a generalisation of theorem 5.3.5 with the state $\omega'$, from which the result would follow immediately.

To find full Reeh-Schlieder states for the free Dirac field we could again use the $C^*$-algebraic approach and a generalisation of theorem 5.3.7. However, theorem 5.3.7 requires the theory to be causal, which means that we would have to use $B$ and not $\mathcal{F}^0$.

**Conjecture 5.6.4** Let $M$ be a globally hyperbolic spin spacetime with a non-compact Cauchy surface. Then there is a state $\omega \in \mathcal{F}_M$ on $\mathcal{B}_M$ such that $\mathcal{H}_\omega$ contains a dense $G_\delta \mathcal{G}$ of vectors which define (full) Reeh-Schlieder states. For all bounded cc-regions $V \subset M$ with non-zero causal complement each vector $\psi \in \mathcal{G}$ is cyclic and separating for $\mathcal{R}_V$.

**Sketch of proof.** As in the proof of proposition 5.6.3 we can find an ultrastatic (and hence stationary) spacetime $M'$ diffeomorphic to $M$ and a quasi-free
Reeh-Schlieder state on $\mathcal{F}_{M'}^0$, which is Hadamard. The restriction of $\omega'$ to $\mathcal{F}_{M'}^0$ satisfies the $\mu$SC by proposition 4.2.17 and hence $\omega'$ as a state on $\mathcal{F}_{M'}^0$ satisfies the $\mu$SC, because this condition depends only on the $n$-point distributions (see definition 4.2.14).

In the Hilbert space $\mathcal{H}_{\omega'}$ we can define the closed subspaces $\mathcal{H}^0$ respectively $\mathcal{H}^1$, generated by the even respectively odd polynomials of elements $B_{M'}(f), f \in D_0(M')$. Because $\omega'$ is quasi-free we see that these spaces are orthogonal and hence $\mathcal{H}_{\omega'} = \mathcal{H}^0 \oplus \mathcal{H}^1$. The restriction of $\omega'$ to $B_{M'}$ has the GNS-triple $(\mathcal{H}^0, \pi_{\omega'}|_{B_{M'}}, \Omega_{\omega'})$, by the essential uniqueness of the GNS-representation (theorem 2.1.4). Because $\omega'$ has the Reeh-Schlieder property we see that for a non-empty cc-region $O$ the linear space $\pi_{\omega'}(B_{O})\Omega_{\omega'}$ is dense in $\mathcal{H}^0$ and the space spanned by the odd polynomials of $B_{M'}(f)$ with $f \in D_0(O)$ is dense in $\mathcal{H}^1$. The first of these two statements implies that the restriction of $\omega'$ to $B_{M'}$ is a Reeh-Schlieder state.

The locally covariant quantum field theory $\mathcal{B}$ with the state space functor $T$ is causal, locally quasi-equivalent and satisfies the time-slice axiom by proposition 4.2.27. A generalisation of theorem 5.3.7 and the definition of $\mathcal{T}_M$ (definition 4.2.16) would then prove the existence of a full Reeh-Schlieder state $\omega \in \mathcal{T}_M$ and the results of [31] (see also the proof of theorem 5.3.7) provide the dense $G_{\delta}$ set $\mathcal{G}$ in $\mathcal{H}_{\omega}$. The final conclusion follows from proposition 5.2.2.

To conclude this chapter we return to the question whether Hadamard states with the (full) Reeh-Schlieder property exist for the free scalar field in any globally hyperbolic spacetime. Whereas proposition 5.6.1 provides us with Hadamard states that have the Reeh-Schlieder property only for a fixed but arbitrary region, proposition 5.6.2 provides us with full Reeh-Schlieder states that are possibly not Hadamard (recall lemma 5.1.3). The main problem is that theorem 5.3.7 is formulated in the Hilbert space topology, a topology which is not suitable to obtain results on Hadamard states.
We believe that the invariant dense domain $\mathcal{D}_\omega$ in the graph topology, where $\omega$ is any Hadamard state, might be more suited for this purpose. A first question of interest is whether this space can be shown to be a Baire space. The result of theorem 5.5.3 may also be of interest for investigations along these lines.
Chapter 6

Conclusions

In this thesis we have presented and discussed results on several aspects of locally covariant quantum field theory[16].

First of all we have tried to put the theory in a philosophical context in chapter 1 and described how its morphisms can be interpreted as a subsystem relation, which makes the framework a model for modal logic.

In chapter 2 we gave a precise mathematical formulation of locally covariant quantum field theory, following closely the existing literature except for the sharpened definition of the time-slice axiom and the introduction of nowhere-classicality.

Chapter 3 and 4 describe two examples of locally covariant quantum fields, namely the real free scalar field and the free Dirac field. The scalar field is described in two well-known approaches in chapter 3, namely the distributional approach based on the Borchers-Uhlmann algebra and the $C^*$-algebraic approach which uses the CCR-algebra (or Weyl-algebra). This chapter also contains the elegant new results that the Hadamard condition on the two-point distribution of a state automatically implies the $\mu$SC and that all its truncated $n$-point distributions are smooth for $n \neq 2$, due to the commutation relations. Chapter 4 describes the free Dirac field as a locally covariant
quantum field and shows that this can be done in a representation independent way, so that the physics is determined entirely by the relations between the adjoint map, charge conjugation and the Dirac operation. This chapter also contains the proof of a relation between the stress-energy-momentum tensor and the relative Cauchy evolution, similar to a result that was already known for the scalar field [16].

In chapter 5 we considered the Reeh-Schlieder property in locally covariant quantum field theories. We discussed the meaning and importance of this property and proved several general results and their application to the real free scalar field. The main issue in finding full Reeh-Schlieder states was the size of the state space. If the state space is sufficiently large, we can find many such states. However, if we restrict our attention to Hadamard states, we have only proved the existence of Hadamard states with the Reeh-Schlieder property for an arbitrarily given region. The question whether Hadamard states with the full Reeh-Schlieder property exist in general curved space-times is still open, although we have given a smoothly covariant sufficient condition in terms of the new notion of quasi-analytic wave front sets. We also suggested that the use of the graph topology could be useful to answer it, if it can be shown that $\mathcal{D}_\omega$ is a Baire space. As a first result in this direction we proved that the Minkowski vacuum state has a strong form of the Reeh-Schlieder property.

Finally, the appendix explains the notion of smooth and analytic wave front set and gives a systematic and elegant treatment of these notions for distributions with values in a Banach space, including some new (but expected) results.
Appendix A

Some results on wave front sets

‘…[A]s Aristotle expressly declares on page 633 of the Louvre edition:

Εντελεχεία τις ἦστι καὶ λόγος τοῦ δυνάμιν ἐχόντος
tōu ὁδοῖ ἐπαί.

‘I am not very well versed in Greek,’ said the giant.
‘Nor I either,’ said the philosophical mite.
‘Why then do you quote that same Aristotle in Greek?’ resumed the Sirian.
‘Because,’ answered the other, ‘it is but reasonable we should quote what we do not comprehend in a language we do not understand.’

Voltaire, Micromegas: a philosophical tale, Ch. 7

In this appendix we will explain the language of wave front sets, which is used to formulate some of the results in this thesis. We will define smooth and analytic wave front sets for Banach space-valued distributions on complex vector bundles and derive a number of useful results in an elegant way that
directly generalises the scalar-valued cases. For a detailed introduction to scalar distributions we refer to [47]. More information on Hilbert and Banach space-valued distributions can be found in [80, 35] and for distributions on vector bundles we refer to [73] and also [27].

A.1 The smooth wave front set

Let $B$ be a Banach space with continuous dual space $B'$ and let $u$ be a $B$-valued distribution on an open set $X \subset \mathbb{R}^n$, i.e. $u : C_0^\infty(X) \to B$ is a continuous linear map, where $C_0^\infty(X)$ is the space of test-functions on $X$ in the test-function topology. This means that for every compact subset $K \subset X$ there are constants $C > 0$ and $m \in \mathbb{N}$ such that

$$\|u(f)\| \leq C \sum_{|\alpha| \leq m} \sup_{x \in K} |D^\alpha f(x)|$$

for all $f \in C_0^\infty(K)$. The following lemma will come in useful:

**Lemma A.1.1** Let $u$ be a $B$-valued distribution on an open set $X \subset \mathbb{R}^n$, where $B$ is a Banach space. If $O \subset X$ is an open subset, then $u$ is smooth on $O$ if and only if $\phi \circ u$ is smooth on $O$ for all $\phi \in B'$.

**Proof.** If $u$ is smooth on $O$ then $\phi \circ u$ is smooth on $O$ for each $\phi \in B'$, because $\phi : B \to \mathbb{C}$ is smooth. Notice that we can identify every continuous function $u : O \to B$, and hence also every smooth function, with a distribution using Bochner integrals (see e.g. [13] section 7.5 for the definition of Bochner integrals). This works as follows. For each $f \in C_0^\infty(O)$ the product $fu$ is Bochner-integrable and $u(f) \equiv \int_O fu$ is the unique element in $B$ such that $\phi(u(f)) = (\phi \circ u)(f) = \int_O (\phi \circ u)f$ for all $\phi \in B'$. Clearly $f \mapsto u(f)$ is linear and

$$\|u(f)\| \leq \int_O \|u(x)\| \cdot |f(x)| dx.$$
To prove that \( u \) is a distribution we note that \( \| u(x) \| \) attains a maximum \( C \geq 0 \) on any given compact set \( K \), so equation (A.2) implies for \( f \in C_0^\infty(K) \) that \( \| u(f) \| \leq C \int_O |f(x)| dx \), which implies equation (A.1).

For the converse we suppose that \( u \) is a distribution on \( X \) such that \( \phi \circ u \) is smooth on \( O \) for all \( \phi \in \mathcal{B}' \). For any compact subset \( K \subset O \) we consider the space \( C^0(K) \) of continuous functions on \( K \), which is a Banach space in the supremum norm \( \| f \|_{C^0} := \sup_K |f| \). The Banach space dual of \( C^0(K) \) is \( \mathcal{E}^0(K) \), the space of distributions of order 0 with support in \( K \), which has the norm \( \| v \|_{\mathcal{E}^0} := \sup_{f \neq 0} \frac{|v(f)|}{\| f \|_{C^0}} \). For each \( \phi \in \mathcal{B}' \) and \( f \in C_0^\infty(K) \subset \mathcal{E}^0(K) \) we then have

\[
|\phi \circ u(f)| = \left| \int_O (\phi \circ u) f \right| \leq C_\phi \| f \|_{\mathcal{E}^0}
\]

for some constant \( C_\phi \geq 0 \). For each \( f \neq 0 \) the map \( \phi \mapsto \frac{1}{\| f \|_{\mathcal{E}^0}} \phi \circ u(f) \) is a bounded linear map on \( \mathcal{B} \), so we can apply the uniform boundedness principle ([49] theorem 1.8.10) to find \( \| u(f) \| \leq C \| f \|_{\mathcal{E}^0} \) for all \( f \in C_0^\infty(K) \).

(Here we also use the fact that the canonical map \( \mathcal{B} \subset \mathcal{B}'' \) is isometric, [43] theorem 7.2.2, so the norm \( \| u(f) \| \) can be taken to be the norm in \( \mathcal{B} \).) Moreover, because \( C_0^\infty(K) \subset \mathcal{E}^0(K) \) is dense we can extend \( u \) to a bounded linear map from \( \mathcal{E}^0(K) \) to \( \mathcal{B} \). Because we can do this for all compact subsets \( K \subset O \) we can obtain a continuous linear map \( u : \mathcal{E}^0(O) \to \mathcal{B} \), where \( \mathcal{E}^0(O) \) is the space of compactly supported distributions on \( O \) of order 0. This space contains the Dirac delta distribution \( \delta_x \) at each point \( x \in O \), so we can define a function \( L : O \to \mathcal{B} \) by \( L(x) := u(\delta_x) \). We wish to show that \( L \) is smooth and gives rise to the original distribution \( u \).

For each convex compact subset \( K \subset O \) and each \( \phi \in \mathcal{B}' \) we can find a constant \( C_\phi \) such that \( |\phi \circ L(x) - \phi \circ L(y)| = |\phi \circ u(x) - \phi \circ u(y)| \leq C_\phi \| x - y \| \) for all \( x, y \in K \), because the first order derivatives of the smooth function \( \phi \circ u \) remain bounded on \( K \). Applying the uniform boundedness principle (and the isometry \( \mathcal{B} \subset \mathcal{B}'' \)) again we find a constant \( C \) such that
\[ \| L(x) - L(y) \| \leq C \| x - y \| \text{ for } x, y \in K, \] showing that \( L \) is continuous. For all \( f \in C^\infty_0(O) \) and \( \phi \in \mathcal{B}' \) the Bochner integral \( L(f) \) satisfies \( \phi \circ L(f) = \int (\phi \circ L)(x)f(x) \, dx = \int (\phi \circ u)(x)f(x) \, dx = \phi \circ u(f) \), i.e. \( L(f) = u(f) \) and we may identify \( u \) with the continuous function \( L \) on \( O \).

Applying the argument of the previous paragraphs to the distributions \( \partial^\alpha u \) for each multi-index \( \alpha \) gives rise to continuous functions \( L^\alpha : O \to \mathcal{B} \). To see that the \( L^\alpha \) really are the derivatives of \( L \) we argue as follows. For each \( x \in O, i \in \{1, \ldots, n\} \), multi-index \( \alpha \) and \( \phi \in \mathcal{B}' \) there is a constant \( C_{\alpha, \phi} \) such that for all sufficiently small \( h \in \mathbb{R}, h \neq 0 \) we have:

\[
\left| \frac{\phi \circ L^\alpha(x + he_i) - \phi \circ L^\alpha(x)}{h} - \phi \circ L^{\alpha'}(x) \right| \leq C_{\alpha, \phi} |h|
\]

by Taylor's theorem. Here \( e_i \) is a basis vector of \( \mathbb{R}^n \) and \( \alpha' \) is the multi-index obtained from \( \alpha \) by increasing \( \alpha_i \) by one. The maps \( \frac{1}{h}(L^\alpha(x + he_i) - L^\alpha(x)) - L^{\alpha'}(x) \) are continuous linear maps on \( \mathcal{B}' \), so by the uniform boundedness principle we obtain

\[
\left\| \frac{L^\alpha(x + he_i) - L^\alpha(x)}{h} - L^{\alpha'}(x) \right\| \leq C_\alpha |h|
\]

for some constant \( C_\alpha \). Hence, \( L^{\alpha'} \) is the derivative of \( L^\alpha \) in the direction \( e_i \). It follows that all derivatives of \( L \) exist and are continuous, so \( L \) is smooth.

\[ \square \]

**Definition A.1.2** A smooth regular direction for a Banach space-valued distribution \( u \) is a point \( (x, k) \in X \times (\mathbb{R}^n \setminus \{0\}) \) for which there exist an \( f \in C^\infty_0(X) \) with \( f(x) \neq 0 \), a conic open neighbourhood \( V \subset (\mathbb{R}^n \setminus \{0\}) \) of \( k \) (i.e. an open neighbourhood such that \( \xi \in V \) and \( r > 0 \) imply \( r\xi \in V \)) and a sequence of constants \( C_N, N \in \mathbb{N} \), such that \( \|u(e^{-i\xi} f)\| \leq \frac{C_N}{1+\|\xi\|^N} \) for all \( \xi \in V \), where \( \|\xi\| \) denotes the Euclidean norm.

The wave front set \( WF(u) \) of \( u \) is defined as

\[ WF(u) := \{ (x, k) \in X \times (\mathbb{R}^n \setminus \{0\}) | (x, k) \text{ is not a smooth regular direction for } u \} . \]
It is clear from the definition that the wave front set is a closed conic subsets of \( X \times (\mathbb{R}^n \setminus \{0\}) \). The case that \( B = \mathbb{C} \) and the general case are related by the following new theorem, which also gives an alternative way of defining the wave front set for Banach space-valued distributions.

**Theorem A.1.3** \( WF(u) = \bigcup_{\phi \in B'} WF(\phi \circ u) \setminus \mathcal{Z} \).

**Proof.** We let \( R_u \) and \( R_\phi \) denote the set of regular directions for \( u \) and \( \phi \circ u(.) \) respectively, where \( \phi \in B' \). If \((x,k) \in R_u\) then there are an open neighbourhood \( O \) of \( x \) and an open conic neighbourhood \( V \) of \( k \) such that \( O \times V \subset R_u \). For any point \((x',k') \in O \times V \) and any \( \phi \in B' \) we then have \((x',k') \in R_\phi \), because \( \| \phi \circ u(e^{-ik' \cdot f}) \| \leq \| \phi \| \cdot \| u(e^{-ik' \cdot f}) \| \) where \( \| \phi \| < \infty \).

Therefore,

\[
R_u \subset \text{int}(\cap_{\phi \in B'} R_\phi).
\]

To prove the converse of this inclusion we let \((x,k) \in \text{int}(\cap_{\phi \in B'} R_\phi)\). It follows that there are an open neighbourhood \( O \) of \( x \) and a conic open neighbourhood \( V \) of \( k \) such that \( O \times V \subset \text{int}(\cap_{\phi \in B'} R_\phi) \). Now choose a function \( f \in C_0^\infty(O) \) such that \( f(x) \neq 0 \) and a conic open neighbourhood \( V' \) of \( k \) such that \( V' \setminus \{0\} \subset V \). For each \( \phi \in B' \) we can then find constants \( C_{N,\phi} \) such that

\[
|\phi \circ u(e^{-ik' \cdot f})| \leq \frac{C_{N,\phi}}{1 + \|k'\|^N} \quad (A.3)
\]

for all \( k' \in V' \) and \( N \in \mathbb{N} \) by [17] lemma 8.2.1. We now consider the family \((1 + \|k'\|^N)u(e^{-ik' \cdot f})\) for all \( k' \in V' \) and for fixed (but arbitrary) \( N \in \mathbb{N} \) as a family of bounded linear operators on the Banach space \( B' \). By the estimate \( (A.3) \) these linear operators are bounded pointwise on each \( \phi \in B' \).

The uniform boundedness principle ([19] theorem 1.8.10) implies that we can choose constants \( C_N \) independently of \( \phi \) such that

\[
\| u(e^{-ik' \cdot f}) \| \leq \frac{C_N}{1 + \|k'\|^N}
\]

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for all \( k' \in V' \) and \( N \in \mathbb{N} \). Hence, \((x,k) \in R_u \) and \( R_u = \text{int}(\bigcap_{\phi \in \mathcal{V}} R_\phi) \) and therefore:

\[
WF(u) \cup \mathcal{Z} = R_u^c = (\text{int}(\bigcap_{\phi} R_\phi))^c = \bigcup_{\phi} R_\phi^c = \bigcup_{\phi} WF(\phi \circ u(.)) \cup \mathcal{Z} = \bigcup_{\phi} WF(\phi \circ u(.)) \cup \mathcal{Z}.
\]

\[
WF(u) = \bigcup_{\phi} WF(\phi \circ u(.)) \setminus \mathcal{Z}. \tag{A.4}
\]

\[\square\]

Theorem A.1.3 allows some standard results on scalar distributions (see [47]) to be generalised as follows:

**Theorem A.1.4** If \( u, v \) are \( \mathcal{B} \)-valued distributions on an open set \( X \subset \mathbb{R}^n \) and \( \mathcal{B} \) is a Banach space, then

1. \( \text{sing supp}(u) \) is the projection of \( WF(u) \) on the first variable,

2. \( u \in C^\infty(X, \mathcal{B}) \) if and only if \( WF(u) = \emptyset \),

3. \( WF(u + v) \subset WF(u) \cup WF(v) \),

4. if \( P \) is a linear partial differential operator on \( X \) with smooth coefficients and principal symbol\(^1\) \( p(x, \xi) \), then

\[
WF(Pu) \subset WF(u) \subset WF(Pu) \cup \text{Char}(P),
\]

where \( \text{Char}(P) := \{(x, \xi) \in X \times (\mathbb{R}^n \setminus \{0\}) | \xi \neq 0, p(x, \xi) = 0\} \),

5. if \( f : Y \to X \) is a diffeomorphism between open sets \( X, Y \subset \mathbb{R}^n \) and \( \text{supp} \ u \subset X \), then \( WF(f^*u) = f^*(WF(u)) \), where the wave front set is pulled back as a subset of the cotangent bundle \( T^*X \).

\(^1\)We refer to [3] definition A.4.2 for the definition of the principal symbol.
The last three statements follow directly from equation (A.4) and the statements for scalar distributions, which are proved in [47]. The second statement follows from the first, so it remains to prove the first statement.

The distribution $u$ is smooth on the open set $O$ if and only if $\phi \circ u$ is smooth on $O$ for all $\phi \in B'$ by lemma A.1.1. This is true if and only if $WF(\phi \circ u) \cap (O \times (\mathbb{R}^n \setminus \{0\}))$ for all $\phi$, by [47] section 8.1. In view of theorem A.1.3 this is true if and only if $WF(u) \cap (O \times (\mathbb{R}^n \setminus \{0\})) = \emptyset$. □

The last item of theorem A.1.4 allows us to define the wave front set of a distribution $u$ on a manifold $\mathcal{M}$ as a subset of the cotangent bundle which is closed in $T^*\mathcal{M} \setminus \mathcal{Z}$ and which coincides in each coordinate chart $\kappa$ with $\kappa^*WF(u \circ \kappa^{-1})$.

If $\mathcal{X}$ is an $m$-dimensional (complex) vector bundle on an $n$-dimensional manifold $\mathcal{M}$ then the space of compactly supported smooth sections of $\mathcal{X}$ can be given a test-function topology. We can define the wave front set of a $\mathcal{B}$-valued distribution on such test-functions in a local trivialisation. Let $\{e_i\}_{i=1,...,m}$ be a local frame for $\mathcal{X}$ and define the $\mathcal{B}$-valued distributions $u_i$ by $u_i(h) := u(he_i)$. Then $u$ is determined completely by $u(\sum_i f_i e_i) = \sum_i u_i(f_i)$.

We define

$$WF(u) := \cup_{i=1}^m WF(u_i).$$

If $e'_i$ is a different local frame, then $e'_i = e_j M^j_i$ for a local $\text{Aut}(\mathbb{C}^m)$-valued function $M$. Using theorem A.1.4 it follows that $WF(u)$ is independent of the choice of local frame and transforms as a subset of the cotangent bundle.

**Theorem A.1.5** If $u, v$ are $\mathcal{B}$-valued distributions on smooth sections of a complex vector bundle $\mathcal{X}$ over a smooth manifold $\mathcal{M}$ and $\mathcal{B}$ is a Banach

---

2Note that $u$ is locally equivalent to a distribution $\tilde{u}$ with values in the Banach space $\mathcal{B} \otimes (\mathbb{C}^m)^*$ and defined by: $\tilde{u}(h) := \sum_i u(fe_i) \otimes d^i$, where $d^i$ is a basis of $(\mathbb{C}^m)^*$. We can recover $u$ as $u(\sum_i f_i e_i) = \sum_i \langle \tilde{u}(f_i), d_i \rangle$, where $d_i$ is a basis of $\mathbb{C}^m$ dual to $d^i$ and the brackets denote the action of the second factor of $\mathcal{B} \otimes (\mathbb{C}^m)^*$ on $\mathbb{C}^m$. In this case we have $WF(u) = WF(\tilde{u})$ by theorem A.1.3
space, then

1. sing supp(u) is the projection of WF(u) on the first variable,
2. \( u \in C^\infty(\mathcal{X}^*, \mathcal{B}) \) if and only if \( WF(u) = \emptyset \),
3. \( WF(u + v) \subset WF(u) \cup WF(v) \),
4. if \( P \) is a linear partial differential operator on \( \mathcal{X} \) with smooth coefficients and (matrix-valued) principal symbol \( p(x, \xi) \), then \( WF(Pu) \subset WF(u) \subset WF(Pu) \cup \Omega_P \), where
   \[ \Omega_P := \{ (x, \xi) \in T^*M | \xi \neq 0, \det p(x, \xi) = 0 \} \].

**Proof.** These results follow directly from theorem A.1.4 and the definition of the wave front set for a distribution on vector-bundle-valued sections on a manifold, except the second inclusion of the last statement. For this result we refer to [27]. \( \Box \)

We now follow [80] and prove a useful result in the case where \( \mathcal{B} \) is a Hilbert space. We refer to definition 4.2.5 for the exterior tensor product \( \boxtimes \) of two vector bundles.

**Theorem A.1.6** Let \( \mathcal{H} \) be a Hilbert space and \( \mathcal{X}_i, i = 1, 2, \) two finite dimensional (complex) vector bundles over smooth \( n_i \)-dimensional manifolds \( M_i \) with complex conjugations \( J_i \), i.e. the \( J_i \) are anti-linear, base-point preserving bundle isomorphisms \( J_i : \mathcal{X}_i \to \mathcal{X}_i \) such that \( J_i^2 = \text{id} \). Let \( u_i, i = 1, 2, \) be two \( \mathcal{H} \)-valued distributions on the test-sections of \( \mathcal{X}_i \) and let \( w_{ij} \) be the distributions on sections of the vector bundle \( \mathcal{X}_i \boxtimes \mathcal{X}_j \) over \( M_i \times M_j \) determined by \( w_{ij}(f_1 \boxtimes f_2) := (u_i(Jf_1), u_j(f_2)) \). Then

\[ (x, k) \in WF(u_1) \iff (x, -k; x, k) \in WF(w_{11}) \]

and

\[ WF(w_{ij}) \subset (-WF(u_i) \cup \mathcal{Z}) \times (WF(u_j) \cup \mathcal{Z}) \].

---

See [8] definition A.4.2 for the definition of the principal symbol.
Note that $w_{ij}$ does indeed uniquely define a distribution on sections of $\mathcal{X}_i \otimes \mathcal{X}_j$, essentially by the Schwartz kernel theorem ([47] theorem 5.2.1).

**Proof.** The proof is a straightforward generalisation of the proof of proposition 3.2 part (iii) in [35], where we notice the following. We may work in local coordinates on $\mathcal{M}_i$ and choose a local frame $\{e_r^{(i)}\}$ that is real w.r.t $J_i$, i.e. such that $J_i e_r^{(i)} = e_r^{(i)}$. Notice that $e_r^{(i)} \times e_s^{(j)}$ is a local frame for $\mathcal{X}_i \otimes \mathcal{X}_j$ and for $w_{ij}$ there is no loss of generality in using the same frame in both entries, because any two points in $\mathcal{M}_i$ can be contained in a single local trivialisation (using [28] theorem 16.26.9). This, together with the complex conjugation and theorem A.1.4 part 3), essentially reduces the problem to distributions on test-functions rather than test-sections. In [35] one takes the inner product of two distributions on the same manifold, but the key ingredient of the proof, the Cauchy-Schwarz inequality, still works if we allow the manifolds to be different.

Finally we collect the wave front sets of some useful distributions, which may be found in [66]:

**Proposition A.1.7** Let $E^\pm$ be the advanced ($-$) and retarded (+) fundamental solutions of the Klein-Gordon operator $K$ or of the operator $\tilde{D}D$ of section 4.2.1 on a globally hyperbolic spin spacetime $M$, then

$$WF(E^\pm) = \{(x,\xi; y, \xi') \in T^*(M \times M)| (x, -\xi) \sim (y, \xi'), x \in J^\pm(y)\} \setminus Z,$$

where $(x, -\xi) \sim (y, \xi')$ if and only if $(x, -\xi) = (y, \xi')$ or there is an affinely parameterised light-like geodesic between $x$ and $y$ to which $-\xi, \xi'$ are cotangent (and hence $-\xi$ and $\xi'$ are parallel transports of each other along the geodesic).

Strictly speaking, [66] only states this proposition for advanced and retarded fundamental solutions of the scalar Klein Gordon operator, not for the Lichnerowicz wave operator $\tilde{D}D$. The latter acts on sections of a vector bundle, which complicates the situation somewhat. Nevertheless, the principal part
is diagonal and this is what determines the bicharacteristic strips and allows
the construction of the advanced and retarded fundamental solutions. Therefore we believe the result should still hold, although we could not produce a reference for this fact.

A.2 The analytic wave front set

Results for analytic wave front sets are mostly analogous to those for smooth
wave front sets, except that they are more involved to formulate. The difficulty is that we cannot localise singularities at a point $x$ by multiplying with
a compactly supported analytic function $f$ with $f(x) \neq 0$.

Consider again a $\mathcal{B}$-valued distribution $u$ on an open set $X \subset \mathbb{R}^n$.

**Definition A.2.1** An analytic regular direction for $u$ is a point $(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ for which there exist an open neighbourhood $O$ of $x$, a conic open neighbourhood $V$ of $k$, a bounded sequence of compactly supported dis-
tributions $u_N$, $N \in \mathbb{N}$, which equal $u$ on $O$ and a constant $C > 0$ such that
\[ \|u_N(e^{-i\xi})\| \leq C \left( \frac{C(N+1)}{\|\xi\|} \right)^N \text{ for all } \xi \in V \text{ and } N \in \mathbb{N}, \text{ where } \|\xi\| \text{ denotes the Euclidean norm.} \]

The analytic wave front set $WF_A(u)$ of $u$ is defined as
\[ WF_A(u) := \{(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})| \ (x,k) \text{ is not an analytic regular direction for } u\} . \]

Like the smooth wave front set the analytic wave front set is closed and
we have $WF(u) \subset WF_A(u)$. Analogous to theorem A.1.3 we have the fol-
lowing equivalent characterisation of the analytic wave front set of a Banach
space-valued distribution in terms of the analytic wave front sets of scalar
distributions:

**Theorem A.2.2** $WF_A(u) = \overline{\bigcup_{\phi \in \mathcal{B}} WF_A(\phi \circ u)} \setminus \mathcal{Z}$.
Proof. If \((x, k)\) is an analytic regular direction for \(u\) then there are an open neighbourhood \(O\) of \(x\) and an open conic neighbourhood \(V\) of \(k\) such that \(O \times V \cap WF_A(U) = \emptyset\). For any \(\phi \in \mathcal{B}'\) any point \((x', k') \in O \times V\) is then an analytic regular direction, because \(\|\phi \circ u_N(e^{-ik'})\| \leq \|\phi\| \cdot \|u_N(e^{-ik'})\|\) where \(\|\phi\| < \infty\).

For the converse we suppose that \((x, k) \notin \bigcup_{\phi \in \mathcal{B}'} WF_A(\phi \circ u)\) for \(k \neq 0\) and we choose an open neighbourhood \(O\) of \(x\) and a closed conic neighbourhood \(V\) of \(k\) such that \(O \times (V \setminus \{0\}) \cap \bigcup_{\phi \in \mathcal{B}'} WF_A(\phi \circ u) = \emptyset\). If \(K \subset O\) is a compact neighbourhood of \(x\) then we may find a sequence \(\chi_N \in C_0^\infty(O)\) such that \(\chi_N \equiv 1\) on \(K\) and \(\sup_K |D^{\alpha+\beta}\chi_N| \leq C_{\alpha}^{1+|\beta|}(N + 1)^{|\beta|}\) for \(|\beta| \leq N\) (see [47] theorem 1.4.2, cf. the proof of proposition 8.4.2 and lemma 8.4.4). By [47] lemma 8.4.4 we then have for some constants \(C_\phi > 0\) and all \(\xi \in V\):

\[
\left(\frac{\|\xi\|}{N + 1}\right)^N |\phi \circ u(\chi_N e^{-i\xi})| \leq C_\phi^N + 1. \tag{A.5}
\]

Now define for each \(p \in \mathbb{N}\) the Banach space

\[
l_p^\infty := \left\{ x = \{x_i\}_{i \in \mathbb{N}} \in \mathcal{B}^{\times \mathbb{N}} \mid \sup_{i \in \mathbb{N}} \|x_i\|p^{-i} < \infty \right\}
\]

and the inductive limit \(k^\infty := \bigcup_{p \in \mathbb{N}} l_p^\infty\), which is a locally convex space (cf. [10] section 3). The estimate (A.5) now means that for a fixed \(\phi \in \mathcal{B}'\) the set

\[
\left\{ \left(\frac{\|\xi\|}{N + 1}\right)^N |\phi \circ u(\chi_N e^{-i\xi})| \right\}_{N \in \mathbb{N}} \subset k^\infty
\]

is bounded. By the (generalised) uniform boundedness principle, theorem 3.4.2 in [75], the set \(X := \left\{ \left(\frac{\|\xi\|}{N + 1}\right)^N \|u(\chi_N e^{-i\xi})\| \right\}_{N \in \mathbb{N}} \subset k^\infty\) is bounded. This means that \(X \subset l_p^\infty\) is a bounded subset for some \(p \in \mathbb{N}\) (cf. [75] 2.6.5) and hence we have for some \(p \in \mathbb{N}\) and all \(\xi \in V\):

\[
\left(\frac{\|\xi\|}{N + 1}\right)^N \|u(\chi_N e^{-i\xi})\| \leq C p^N \tag{A.6}
\]
We conclude that $(x, k) \not\in WF_A(u)$. □

Analogously to theorem \ref{A.1.4} we can now generalise some results for scalar distributions:

**Theorem A.2.3** If $u, v$ are $\mathcal{B}$-valued distributions on an open set $X \subset \mathbb{R}^n$ and $\mathcal{B}$ is a Banach space, then

1. $\text{sing supp}_A(u)$ is the projection of $WF_A(u)$ on the first variable,
2. $u \in C^\omega(X, \mathcal{B})$ if and only if $WF_A(u) = \emptyset$,
3. $WF_A(u + v) \subset WF_A(u) \cup WF_A(v)$,
4. if $P$ is a linear partial differential operator on $X$ with real-analytic coefficients and principal symbol $p(x, \xi)$, then

$$WF_A(Pu) \subset WF_A(u) \subset WF_A(Pu) \cup \text{Char}(P),$$

where \( \text{Char}(P) := \{(x, \xi) \in X \times (\mathbb{R}^n \setminus \{0\}) \mid \xi \neq 0, p(x, \xi) = 0\} \),

5. if $f : Y \to X$ is an analytic diffeomorphism between open sets $X, Y \subset \mathbb{R}^n$ and $\text{supp } u \subset X$, then $WF_A(f^*u) = f^*(WF_A(u))$, where the wave front set is pulled back as a subset of the cotangent bundle $T^*X$.

**Proof.** The last three statements follow directly from theorem \ref{A.2.2} and the corresponding statements for scalar distributions, which are proved in \cite{47}. The second statement follows from the first, so it remains to prove the first statement.

If the distribution $u$ is analytic on the open set $O$ then $\phi \circ u$ is analytic on $O$ for all $\phi \in \mathcal{B}'$ and hence $WF_A(u) \cap (O \times (\mathbb{R}^n \setminus \{0\})) = \emptyset$ by theorem \ref{A.2.2}. Conversely, if $WF_A(u) \cap (O \times (\mathbb{R}^n \setminus \{0\})) = \emptyset$ then $u$ is a smooth function by theorem \ref{A.1.4} and the fact that $WF(u) \subset WF_A(u)$. It remains to prove that $u$ is analytic on $O$. In the case $O = \mathbb{R}$ this is \cite{10} proposition 9 (see also
the references there). For completeness we prove the required generalisation. Given $x \in O$ we can choose a compact neighbourhood $K \subset O$ of $x$ and functions $\chi_N$ as in the proof of theorem A.2.2. For each $\phi \in \mathcal{B}'$ and $x \in K$ we then have

$$
\phi(\partial^\alpha u)(x) = \partial^\alpha (\phi(\chi_N u))(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} (i\xi)^\alpha \hat{\phi}(\chi_N u)(\xi) d\xi. \quad (A.7)
$$

Because $u$ is a distribution we have $|\hat{\phi}(\chi_N u)(\xi)| \leq C \|\phi\|(1 + \|\xi\|)^M$ for some order $M$, which we use to estimate the integral over $\|\xi\| \leq 1$. For $\|\xi\| \geq 1$ we use the estimate (A.6) with $N = |\alpha| + n$ to find:

$$
|\phi(\partial^\alpha u)(x)| \leq C^{N+1} (N+1)^N \|\phi\|
$$

for some $C > 0$ and hence $\|\partial^\alpha u(x)\| \leq C^{N+1} (N+1)^N$, which implies

$$
\|\partial^\alpha u(x)\| \leq C^{1+|\alpha|} (|\alpha| + 1)^{|\alpha|}
$$

for some $C > 0$, using $(|\alpha| + 1)^n \leq c e^{|\alpha|}$ and $(|\alpha| + n + 1) \leq (|\alpha| + 1)(n + 1)$. Now let $r > 0$ be such that the disc around $x_0$ with radius $r$ is contained in $K$. A general term in the Taylor series of $u$ can then be estimated by

$$
\left\| \frac{(x-x_0)^\alpha}{\alpha!} \partial^\alpha u(x_0) \right\| \leq C(nrC)^{|\alpha|} (N+1)^N \frac{N!}{N!}.
$$

Here we used $n^N = (1 + \ldots + 1)^N = \sum_{|\alpha| \leq N} \frac{N!}{\alpha!}$ (by Newton’s binomial theorem) to obtain $\frac{1}{\alpha!} \leq \frac{n^N}{N!}$. The Taylor series contains no more than $n^N$ terms with $|\alpha| = N$, so

$$
\sum_{\alpha} \left\| \frac{(x-x_0)^\alpha}{\alpha!} \partial^\alpha u(x_0) \right\| \leq \sum_{N=0}^{\infty} C(n^2 rC)^{|\alpha|} (N+1)^N \frac{N!}{N!}.
$$

Because $\frac{(N+1)^N}{N!} \leq c^N$ for some constant $c > 0$ we can choose $r$ small enough to ensure that the series is absolutely convergent. For all $\phi \in \mathcal{B}'$ and $\|x-x_0\|
within the radius of convergence we then have
\[ φ \left( \sum_α \frac{(x - x_0)^α}{α!} \partial^α u(x_0) \right) = \sum_α \frac{(x - x_0)^α}{α!} \partial^α (φ ◦ u)(x_0) = φ ◦ u(x). \]
This shows that the limit of the series is the function \( u(x) \) itself. □

The last statement of theorem A.2.3 implies that we can define analytic wave front sets on analytic manifolds as a subset of the cotangent bundle in a similar way as for the smooth wave front set on smooth manifolds.

As in the smooth case we can consider an \( m \)-dimensional (complex) real-analytic vector bundle \( X \) on an \( n \)-dimensional analytic manifold \( M \) and endow the space of compactly supported smooth sections of \( X \) with a test-function topology. Given an analytic local frame \( \{e_i\}_{i=1,\ldots,m} \) for \( X \), the analytic wave front set of a \( B \)-valued distribution \( u \) on test-sections of \( X \) can be defined as
\[ WF_A(u) := \bigcup_{i=1}^m WF_A(u_i), \]
where \( u_i(h) := u(he_i) \) are \( B \)-valued distributions as before. If \( e'_i \) is a different analytic local frame, then \( e'_i = e_j M^j_i \) for an analytic local \( \text{Aut}(\mathbb{C}^m) \)-valued function \( M \). Using theorem A.2.3 it follows that \( WF_A(u) \) is independent of the choice of local frame and transforms as a subset of the cotangent bundle.

**Theorem A.2.4** If \( u, v \) are \( B \)-valued distributions on smooth sections of a complex, real-analytic vector bundle \( X \) over an analytic manifold \( M \) and \( B \) is a Banach space, then

1. \( \text{sing supp}_A(u) \) is the projection of \( WF_A(u) \) on the first variable,
2. \( u \in C^\infty(X^*, B) \) if and only if \( WF_A(u) = \emptyset \),
3. \( WF_A(u + v) \subset WF_A(u) \cup WF_A(v) \),
4. if \( P \) is a linear partial differential operator on \( X \) with real-analytic coefficients and (matrix-valued) principal symbol \( p(x, \xi) \), then \( WF_A(Pu) \subset WF_A(u) \).

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Proof. These results follow directly from theorem \( \text{A.2.3} \) and the definition of the wave front set for a distribution on vector-bundle-valued sections on a manifold. □

Again we can follow [80] and prove a useful result in the case where \( \mathcal{B} \) is a Hilbert space (and again we refer to definition [4.2.5] for the exterior tensor product \( \boxtimes \) of two vector bundles):

**Theorem A.2.5** Let \( \mathcal{H} \) be a Hilbert space and \( \mathcal{X}_i, i = 1, 2 \), two finite dimensional (complex) real-analytic vector bundles over analytic \( n_i \)-dimensional manifolds \( \mathcal{M}_i \) with real-analytic complex conjugations \( J_i : \mathcal{X}_i \rightarrow \mathcal{X}_i \). Let \( u_i, i = 1, 2 \), be two \( \mathcal{H} \)-valued distribution on the test-sections of \( \mathcal{X}_i \) and let \( w_{ij} \) be the distributions on sections of the vector bundle \( \mathcal{X}_i \boxtimes \mathcal{X}_j \) over \( \mathcal{M}_i \times \mathcal{M}_j \) determined by \( w_{ij}(f_1, f_2) := \langle u_i(Jf_1), u_j(f_2) \rangle \). Then

\[
(x, k) \in WF_A(u_1) \iff (x, -k; x, k) \in WF_A(u_{11})
\]

and

\[
WF_A(w_{ij}) \subset (-WF_A(u_i) \cup \mathcal{Z}) \times (WF_A(u_j) \cup \mathcal{Z})
\]

Proof. The proof is a straightforward generalisation of the proof of proposition 2.6 part 2) in [80], where we notice the following. We may work in local coordinates on \( \mathcal{M}_i \) and choose a local frame \( \{ e_r^{(i)} \} \) that is real w.r.t \( J_i \), i.e. such that \( J_i e_r^{(i)} = e_r^{(i)} \). Notice that \( e_r^{(i)} \times e_s^{(j)} \) is a local frame for \( \mathcal{X}_i \boxtimes \mathcal{X}_j \) and for \( w_{ii} \) there is no loss of generality in using the same frame in both entries, because any two points in \( \mathcal{M}_i \) can be contained in a single local trivialisation (using [28] theorem 16.26.9 and [44] theorem 8.3.1 to guarantee that we have the right analytic structure near the given points). In [80] one takes the inner product of a distribution with itself, but the key ingredient of the proof, the Cauchy-Schwarz inequality, still works if we take the inner product of two different distributions. □
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