Open Mathematics

Research Article

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A new fourth power mean of two-term exponential sums

https://doi.org/10.1515/math-2019-0034
Received September 18, 2018; accepted January 9, 2019

Abstract: The main purpose of this paper is to use analytic methods and properties of quartic Gauss sums to study a special fourth power mean of a two-term exponential sums mod\(p\), with \(p\) an odd prime, and prove interesting new identities. As an application of our results, we also obtain a sharp asymptotic formula for the fourth power mean.

Keywords: Two-term exponential sums; fourth power mean; elementary method; identity; asymptotic formula

MSC: 11L05, 11L07

Dedicated to our supervisor Professor Zhang Wenpeng for his 60th birthday.

1 Introduction

Let \(q \geq 3\) be an integer. For any integer \(m\) and \(n\), the two-term exponential sum \(G(k, h, m, n; q)\) is defined as

\[
G(k, h, m, n; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + na^h}{q}\right),
\]

where as usual, \(e(y) = e^{2\pi iy}\), \(k\) and \(h\) are positive integers with \(k \neq h\).

Many scholars have studied various elementary properties of \(G(k, h, m, n; q)\) and obtained a series of results. For example, from the A. Weil’s important work [2], one can get the general upper bound estimate

\[
\left|\sum_{a=1}^{p-1} \chi(a)e\left(\frac{ma^k + na}{p}\right)\right| \ll \sqrt{p},
\]

where \(p\) is an odd prime, \(\chi\) is any Dirichlet character mod \(p\) and \((m, n, p) = 1\).

Zhang Han and Zhang Wenpeng [3] proved the identity

\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right)^4 = \begin{cases}
2p^2 & \text{if } 3 \mid p - 1, \\
2p^2 - 7p^2 & \text{if } 3 \nmid p - 1,
\end{cases}
\]

where \(p\) be an odd prime.

Zhang Han and Zhang Wenpeng [4] also obtained

\[
\sum_{m=0}^{p-1} \sum_{a=1}^{p-1} e\left(\frac{ma^5 + na}{p}\right)^4 = \begin{cases}
3p^3 - p^2 \left(8 + 2 \left(\frac{-1}{p}\right) + 4 \left(\frac{-3}{p}\right)\right) + 3p & \text{if } 5 \mid p - 1, \\
3p^3 + O\left(p^2\right) & \text{if } 5 \mid p - 1,
\end{cases}
\]

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where \( \left( \frac{r}{p} \right) = \chi^2 \) denotes the Legendre symbol mod \( p \).

Some other related mean value papers can also be found in [5] - [13]. If someone is interested in this field, please refer to these references. However, regarding the fourth power mean

\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e^{\left( \frac{ma^4 + a}{p} \right)} = \begin{cases} 2p^2 (p - 2) & \text{if } p = 12h + 7, \\ 2p^3 & \text{if } p = 12h + 11. \end{cases}
\]

\[\text{Theorem 1.} \quad \text{Let } p > 3 \text{ be a prime with } p \equiv 3 \mod 4, \text{ then we have}\]

\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e^{\left( \frac{ma^4 + a}{p} \right)} = \begin{cases} 2p^2 (p - 10p - 2\alpha^2) & \text{if } p = 24h + 1, \\ 2p^2 (p - 6p - 2\alpha^2) & \text{if } p = 24h + 5, \\ 2p^2 (p - 6p - 2\alpha^2) & \text{if } p = 24h + 13, \\ 2p^2 (p - 8p - 2\alpha^2) & \text{if } p = 24h + 17, \end{cases}
\]

where \( \alpha = a(p) = \sum_{a=1}^{p-1} \left( \frac{a + \overline{a}}{p} \right) \) is an integer satisfying the identity (state displayed identity ), where \( r \) is any quadratic non-residue mod \( p \), see Theorem 4-11 in [16].

\[
p = \alpha^2 + \beta^2 \equiv \left( \frac{1}{\alpha=1} \left( \frac{a + \overline{a}}{p} \right) \right)^2 + \left( \frac{1}{\alpha=1} \left( \frac{a + r\overline{a}}{p} \right) \right)^2,
\]

and \( r \) is any quadratic non-residue mod \( p \).

From these two theorems we may immediately deduce the following:

**Corollary.** For any odd prime \( p \), we have the asymptotic formula

\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e^{\left( \frac{ma^4 + a}{p} \right)} = 2p^3 + O \left( p^3 \right).
\]

### 2 Several Lemmas

To prove our theorems, we first need to give several necessary lemmas. Hereafter, we will use many properties of the classical Gauss sums, the fourth-order character mod \( p \) and the quartic Gauss sums. All of these contents can be found in any Elementary Number Theory or Analytic Number Theory book, such as references [1], [14] or [16]. These contents will not be repeated here. First we have the followings:

**Lemma 1.** If \( p \) is a prime with \( p \equiv 1 \mod 4 \), and \( \lambda \) is any fourth-order character mod \( p \), then we have

\[
r^2(\lambda) + r^2 \left( \frac{\lambda}{p} \right) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left( \frac{a + \overline{a}}{p} \right) = 2\sqrt{p} \cdot \alpha.
\]
Proof. In fact this is Lemma 2 of [15], so its proof is omitted.

Lemma 2. If \( p \) is a prime with \( p \equiv 1 \mod 4 \), then we have the identity

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^2 \sum_{c=0}^{p-1} e \left( \frac{-mc^4 - c}{p} \right) = \begin{cases} (2 + \chi_2(7)) p^2 + 2p \alpha & \text{if } p \equiv 5 \mod 8, \\ (2 + \chi_2(7)) p^2 - 6p \alpha & \text{if } p \equiv 1 \mod 8, \end{cases}
\]

where \( \chi_2 = \left( \frac{\cdot}{p} \right) \) denotes the Legendre’s symbol mod \( p \).

Proof. First applying trigonometric identity

\[
\sum_{m=1}^{q} e \left( \frac{nm}{q} \right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n, \end{cases}
\]

we have

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^2 \sum_{c=0}^{p-1} e \left( \frac{-mc^4 - c}{p} \right) = \sum_{m=0}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^2 \sum_{c=0}^{p-1} e \left( \frac{-mc^4 - c}{p} \right) = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e \left( \frac{m(a^4 + b^4 - c^4) + a + b - c}{p} \right) = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e \left( \frac{a + b - c}{p} \right) + \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e \left( \frac{c(a + b - 1)}{p} \right) = p - p^{p-1} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e \left( \frac{a + b - c}{p} \right) + 1 + p^2 - p^{p-1} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 - p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1. \tag{3}
\]

Let \( \lambda \) be a fourth-order character mod \( p \), if \( p \equiv 5 \mod 8 \), then note that \( \lambda(-1) = -1 \) we have

\[
-p \sum_{a=0}^{p-1} 1 = 0. \tag{4}
\]

Noting the identity \( \lambda \chi_2 = \bar{\lambda} \) and

\[
B(m) = \sum_{a=0}^{p-1} e \left( \frac{ma^4}{p} \right) = \chi_2(m) \sqrt{p} + \bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau \left( \bar{\lambda} \right). \tag{5}
\]

Applying (5) and Lemma 1 we have

\[
p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} e \left( \frac{ma^4 + b^4 - 1}{p} \right) = p^2 + \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4}{p} \right) \right)^2 e \left( \frac{-m}{p} \right).
\]
This proves Lemma 2.

Applying (5), Lemma 1 and note that

\[ p^{410} \equiv \sqrt[p]{p}^{-2} \equiv \sqrt[p]{p} \bmod p \]

It is clear that the congruences \( a^4 + b^4 \equiv 1 \bmod p \) and \( a + b \equiv 1 \bmod p \) imply that \( ab \left( 2a^2 + 3ab + 2b^2 \right) \equiv 0 \bmod p \) and \( a + b \equiv 1 \bmod p \). So we have

\[
p^2 + p^2 + 2p + 2\sqrt[p]{p} \equiv 2p^2 + p - 2pa. \quad (6)
\]

If \( p \equiv 1 \bmod 8 \), then noting that \( \lambda(-1) = 1 \) we have

\[
-p \sum_{a=0}^{p-1} \frac{1}{a^4 + 1 \equiv 0 \bmod p} = 4p. \quad (8)
\]

Applying (5), Lemma 1 and note that \( \tau(\lambda) \mid \sqrt[p]{p} \bmod p \) we have

\[
p^2 + \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \frac{1}{\lambda(a^4 + b^4 + 1 \equiv 0 \bmod p)} = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} \left( \frac{m(a^4 + b^4 - 1)}{p} \right) \left( \frac{m}{p} \right) \left( \frac{m}{p} \right)
\]

\[
= p^2 + \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} \left( \frac{ma^4}{p} \right) \right)^2 \left( \sum_{c=0}^{p-1} \left( \frac{-mc^4 - c}{p} \right) \right) = \left\{ \begin{array}{ll} (2 + \chi_2(7))^2 p^2 + 2pa & \text{if } p \equiv 5 \bmod 8, \\ (2 + \chi_2(7))^2 p^2 - 6pa & \text{if } p \equiv 1 \bmod 8. \end{array} \right. \quad (9)
\]

Combining (3), (4), (6), (9) we have the identity

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} \left( \frac{ma^4}{p} + a \right) \right)^2 \left( \sum_{c=0}^{p-1} \left( \frac{-mc^4 - c}{p} \right) \right) = \left\{ \begin{array}{ll} (2 + \chi_2(7))^2 p^2 + 2pa & \text{if } p \equiv 5 \bmod 8, \\ (2 + \chi_2(7))^2 p^2 - 6pa & \text{if } p \equiv 1 \bmod 8. \end{array} \right.
\]

This proves Lemma 2.

**Lemma 3.** If \( p \) is a prime with \( p \equiv 3 \bmod 4 \), then we have the identity

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} \left( \frac{ma^4}{p} + a \right) \right)^2 \left( \sum_{c=0}^{p-1} \left( \frac{-mc^4 - c}{p} \right) \right) = (2 - \chi_2(7))^2 p^2.
\]

**Proof.** If \( p = 4h + 3 \), then \( \chi_2(-1) = -1 \) and \( \tau(\chi_2) = i, i^2 = -1 \). For any integer \( m \) with \( (m, p) = 1 \), we have

\[
\sum_{a=0}^{p-1} \left( \frac{ma^4}{p} \right) = 1 + \sum_{a=1}^{p-1} \left( 1 + \chi_2(a) \right) \left( \frac{ma^2}{p} \right) = \sum_{a=0}^{p-1} \left( \frac{ma^2}{p} \right) = \chi_2(m) \tau(\chi_2)
\]

(10)
From identity (2) we have

\[-p \sum_{a=0}^{p-1} \sum_{a=0}^{p-1} \frac{ma^4 + a}{p} = 0.\]

(11)

From (10), (11) and the method of proving Lemma 2 we have

\[\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^2 \left( \sum_{c=0}^{p-1} e \left( -\frac{mc^4 - c}{p} \right) \right) = \left( 2 - \chi_2(7) \right) p.\]

This proves Lemma 3.

**Lemma 4.** If \( p \) is a prime with \( p \equiv 1 \mod 4 \), then we have the identity

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^2 \left( \sum_{c=0}^{p-1} e \left( -\frac{mc^4 - c}{p} \right) \right) \left( \sum_{d=1}^{p-1} e \left( -\frac{md^4 - d}{p} \right) \right) =
\begin{cases}
2p^2 - 22p - \chi_2(7)p - 4a^2 + 6a & \text{if } p = 24h + 1, \\
2p^2 - 10p - \chi_2(7)p - 4a^2 - 2a & \text{if } p = 24h + 5, \\
2p^2 - 14p - \chi_2(7)p - 4a^2 - 2a & \text{if } p = 24h + 13, \\
2p^2 - 18p - \chi_2(7)p - 4a^2 + 6a & \text{if } p = 24h + 17.
\end{cases}
\]

**Proof.** From identity (2) we have

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^2 \left( \sum_{c=0}^{p-1} e \left( -\frac{mc^4 - c}{p} \right) \right) \left( \sum_{d=1}^{p-1} e \left( -\frac{md^4 - d}{p} \right) \right) =
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=1}^{p-1} e \left( \frac{m(a^4 + b^4 - c^4 - d^4) + a + b - c - d}{p} \right)
\]

\[
= p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=1}^{p-1} e \left( \frac{d(a + b - c - 1)}{p} \right)
\]

\[
= p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{a+b\equiv c+1 \mod p} 1 - p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{a+b\equiv c+1 \mod p} 1.
\]

(12)

It is clear that the congruences \( a^4 + b^4 \equiv c^4 + 1 \mod p \) and \( a + b \equiv c + 1 \mod p \) imply that \((a - 1)(b - 1)(2a^2 + 3ab + 2b^2 - a - b + 1) \equiv 0 \mod p \). So we have

\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{a+b\equiv c+1 \mod p} 1 = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{b=0}^{p-1} 1
\]

\( (a-1)(b-1)(2a^2+2b^2+3ab-a-b+1)\equiv0 \mod p \)

\[
= 2p - 1 + \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1
\]

\( 2a^2+2b^2+3ab-a-b+1\equiv0 \mod p \)
\[
\begin{align*}
\lambda &= 2p - 1 + \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \frac{1 - 2 \sum_{a=0}^{p-1} \frac{1}{a^2 + 2b^2 + \frac{3ab}{a + b} - 1 \mod p}}{2a^2 + 2b^2 + 3ab - a - b + 1 \equiv 0 \mod p} \\
&= 2p - 1 + \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \frac{1 - 2 \sum_{a=2}^{p-1} \frac{1}{a^3 \equiv 1 \mod p}}{(4a + 3b - 1)^{2} - 7b^2 + 2b - 7 \mod p} \\
&= 2p + 1 + \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \frac{1}{a^2 - 7b^2 + 2b - 7 \mod p} \\
&= 2p + 1 + \sum_{b=0}^{p-1} \left(1 + \left(\frac{-7b^2 + 2b - 7}{p}\right)\right) - 2 \sum_{a=1}^{p-1} \frac{1}{a^3 \equiv 1 \mod p} \\
&= 3p + 1 + \left(\frac{7}{p}\right) \sum_{b=0}^{p-1} \left(\frac{(7b - 1)^2 + 48}{p}\right) - 2 \sum_{a=1}^{p-1} \frac{1}{a^3 \equiv 1 \mod p} \\
&= 3p + 1 + \left(\frac{7}{p}\right) \sum_{b=0}^{p-1} \left(\frac{b^2 + 48}{p}\right) - 2 \sum_{a=1}^{p-1} \frac{1}{a^3 \equiv 1 \mod p} \\
&= 3p + 1 - \left(\frac{7}{p}\right) - 2 \sum_{a=1}^{p-1} \frac{1}{a^3 \equiv 1 \mod p} \\
&= 3p + 1 - \left(\frac{7}{p}\right),
\end{align*}
\]

where we have used the identity \(\sum_{b=0}^{p-1} \left(\frac{b^2 + 48}{p}\right) = -1\).

If \(p = 24h + 13\) or \(p = 24h + 1\), then \(a^3 \equiv 1 \mod p\) has three solutions. So from (13) we have

\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 3p - 5 - \left(\frac{7}{p}\right).
\]  \hspace{1cm} (14)

If \(p = 24h + 5\) or \(p = 24h + 17\), then \(a^3 \equiv 1 \mod p\) has one solution. So from (13) we have

\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 3p - 1 - \left(\frac{7}{p}\right).
\]  \hspace{1cm} (15)

If \(p = 8h + 5\), then applying (5), Lemma 1 and noting that \(r(\lambda) \tau\left(\overline{\lambda}\right) = -p\) we have

\[
\begin{align*}
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 &= \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{m=0}^{p-1} e\left(\frac{m(a^6 + b^4 - c^6 - 1)}{p}\right) \\
&= \sum_{m=0}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^6}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^4}{p}\right)\right) \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^6}{p}\right)\right) \\
&= p^3 + \sum_{m=1}^{p-1} \left(3p - \chi_2(m) r^2(\lambda) - \chi_2(m) r^2\left(\overline{\lambda}\right)\right) \\
&\quad \times \left(\chi_2(m) \sqrt{p} + \chi(m) r(\lambda) + \lambda(m) r\left(\overline{\lambda}\right)\right) \left(\sum_{\lambda} \frac{-m}{p}\right)
\end{align*}
\]
Combining (12), (14) - (17) we have the identity

$$p^3 + 9p^2 + 4pa^2 + 2pa. \quad (16)$$

If $p = 8h + 1$, then applying (5), Lemma 1 and noting that $r(\lambda) \tau(\bar{\lambda}) = p$ we have

$$p = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{m(a^4 + b^4 - c^4 - d^4)}{p}\right)$$

$$= \sum_{m=0}^{p-1} \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) + \sum_{b=0}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{-mb^4}{p}\right) e\left(\frac{-m}{p}\right)$$

$$= p^3 + \sum_{m=1}^{p-1} \left(3p + 2\chi(2(m)\sqrt{p}) + 2\lambda(m)\sqrt{\lambda}\tau(\lambda) + 2\lambda(m)\sqrt{\lambda}\tau(\bar{\lambda})\right) e\left(\frac{-m}{p}\right)$$

$$\times \left(\chi(2(m)\sqrt{p} + \bar{\lambda}(m)r(\lambda) + \lambda(m)\tau(\bar{\lambda})\right) e\left(\frac{-m}{p}\right)$$

$$= p^3 + 17p^2 + 4pa^2 - 6pa. \quad (17)$$

Combining (12), (14) - (17) we have the identity

$$\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right) \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4 + c}{p}\right)\right) \left(\sum_{d=1}^{p-1} e\left(\frac{-md^4 + d}{p}\right)\right)$$

$$= p \begin{cases} 2p^2 - 14p - \chi(2(7))p - 4a^2 - 2a & \text{if } p = 24h + 13, \\
2p^2 - 22p - \chi(2(7))p - 4a^2 + 6a & \text{if } p = 24h + 1, \\
2p^2 - 10p - \chi(2(7))p - 4a^2 - 2a & \text{if } p = 24h + 5, \\
2p^2 - 18p - \chi(2(7))p - 4a^2 + 6a & \text{if } p = 24h + 17. \\
\end{cases}$$

This proves Lemma 4.

**Lemma 5.** If $p$ is a prime with $p \equiv 3 \mod 4$, then we have the identity

$$\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right) \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4 + c}{p}\right)\right) \left(\sum_{d=1}^{p-1} e\left(\frac{-md^4 + d}{p}\right)\right)$$

$$= \begin{cases} p^2 \left(2p - 6 + \chi(2(7))\right) & \text{if } p = 12h + 7, \\
p^2 \left(2p - 2 + \chi(2(7))\right) & \text{if } p = 12h + 11. \\
\end{cases}$$

**Proof.** Noting (11) and $\chi(-1) = -1$, from the methods of proving Lemma 4 we can easily deduce Lemma 5.

### 3 Proofs of the theorems

Now we prove our main results. First we prove Theorem 2. If $p = 24h + 1$, then from Lemma 2 and Lemma 4 we have

$$\sum_{m=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right|^4 = \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-ma^4 + c}{p}\right)\right) \left(\sum_{d=1}^{p-1} e\left(\frac{-md^4 + d}{p}\right)\right)$$

$$+ \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-ma^4 + c}{p}\right)\right)$$

$$= p \left(2p^2 - 22p - \chi(2(7))p - 4a^2 + 6a\right) + 2p^2 + \chi(2(7))p^2 - 6pa.$$
\[ = 2p \left( p^2 - 10p - 2a^2 \right). \tag{18} \]

Similarly, if \( p = 24h + 5 \), then from Lemma 2 and Lemma 4 we have
\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e \left( \frac{ma^h + a}{p} \right) \right| = 2p \left( p^2 - 4p - 2a^2 \right). \tag{19} \]

If \( p = 24h + 13 \), then we have
\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e \left( \frac{ma^h + a}{p} \right) \right| = 2p \left( p^2 - 6p - 2a^2 \right). \tag{20} \]

If \( p = 24h + 17 \), then we have
\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e \left( \frac{ma^h + a}{p} \right) \right| = 2p \left( p^2 - 8p - 2a^2 \right). \tag{21} \]

Now Theorem 2 follows from (18) - (21).

Similarly, from Lemma 3, Lemma 5 and the methods of proving Theorem 2 we can also deduce Theorem 1. This completes the proofs of all of our results.

Acknowledgement: This work was supported by the N. S. F. (Grant No. 11771351) of P. R. China.

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