MEAN-SQUARE DELAY-DISTRIBUTION-DEPENDENT
EXPONENTIAL SYNCHRONIZATION OF CHAOTIC NEURAL
NETWORKS WITH MIXED RANDOM TIME-VARYING DELAYS
AND RESTRICTED DISTURBANCES

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Abstract. This paper investigates the delay-distribution-dependent exponential synchronization problem for a class of chaotic neural networks with mixed random time-varying delays as well as restricted disturbances. Given the probability distribution of the time-varying delay, stochastic variable that satisfying Bernoulli distribution is formulated to produce a new system which includes the information of the probability distribution. Based on the Lyapunov-Krasovskii functional method, the Jensen’s integral inequality theory and linear matrix inequality (LMI) technique, several delay-distribution-dependent sufficient conditions are developed to guarantee that the chaotic neural networks with mixed random time-varying delays are exponentially synchronized in mean square. Furthermore, the derived results are given in terms of simplified LMI, which can be straightforwardly solved by Matlab. Finally, two numerical examples are proposed to demonstrate the feasibility and the effectiveness of the presented synchronization scheme.

1. Introduction. During the recent two decades, the study of neural networks has attracted a great deal of interest and has been successfully applied in various areas such as associative memory, image and signal processing, optimization [9, 4]. It has been shown that time delays may lead to some complex dynamical behaviours such as instability, bifurcation and chaos in neural networks model [6, 27, 13]. Therefore, the synchronization problems for neural networks with time delay gradually began to get the attention of researchers and many sufficient conditions are proposed to
guarantee the synchronization of neural networks with different types of time delays such as finite and unbounded distributed delay \([18, 23, 10, 25, 19]\), neutral type delay \([12, 26]\) and leakage delay \([5, 14, 22, 8, 17]\). In \([18]\), by utilizing Lyapunov-Krasovskii functional, linear matrix inequalities technique and free-weighting matrix method, a delay-dependent feedback controllers is schemed to guarantee the mean square exponential synchronization for Markovian jump stochastic chaotic neural networks with mixed delays and sector-bounded non-linearities. In \([26]\), the mean-square robust adaptive synchronization conditions are established for Markovian jumping stochastic neural networks of neutral-type with mixed delays and robust adaptive feedback controller, which are also based on Wirtinger-based integral inequality and a reciprocally convex combination technique. In \([8]\), the researchers discussed the problem synchronization for impulsive complex-valued neural networks with leakage delay and mixed time-varying delays, where several sufficient criteria are established to ensure the global \(\mu\)-synchronization of the considered neural networks.

In practice, the time-varying delay in some neural networks often exists in a random form \([15, 1]\), and its probabilities can be measured by the statistical methods such as Bernoulli distribution, normal distribution, uniform distribution and Poisson distribution. It is uncomplicated to see that disturbances invariably exist, which can lead to instability and poor performances always real physical systems \([11, 3]\). Consequently, how to cut down the effect of disturbances in the synchronization process for chaotic systems has become a significant issue. It’s worth noting that the important problem of the mean square delay-distribution-dependent exponential synchronization of chaotic neural networks with mixed random time-varying delays as well as restricted disturbances has not been completely considered, so this situation motivates to our present study.

Motivated by the above analysis, our research have mainly focused on developing a new approach to analyzing the delay-distribution-dependent exponential synchronization problem for a class of chaotic neural networks with mixed random time-varying delays as well as restricted disturbances. The main contributions of this paper as follows: (i) By introducing a stochastic variable which satisfies Bernoulli distribution, the information of probabilistic time-varying delay is equivalently transformed into the deterministic time-varying delay with stochastic parameters. (ii) A suitable Lyapunov-Krasovskii functional is constructed with the full information of probabilistic time-varying delay. (iii) By employing the Jensen’s integral inequality theory, several delay-distribution-dependent sufficient conditions have been derived in terms of simplified LMI. (iv) Two numerical examples are given to illustrate the feasibility and the effectiveness of the theoretical results.

The remainder of this paper is organized as follows. In Section \(2\), the master system and the slave system are introduced, some necessary assumptions, definitions and lemmas are given. Our main results and their rigorous proofs are described in Section \(3\). In Section \(4\), two numerical examples and their simulations are given to illustrate the effectiveness of our results. In Section \(5\), conclusions are given.

2. Preliminaries. Consider the following n-neuron chaotic neural networks with mixed delays \([23, 10]\):

\[
\begin{aligned}
\frac{dx(t)}{dt} &= -Cx(t) + A\tilde{f}(x(t)) + B\tilde{g}(x(t - \tau(t))) + D\int_{t-\sigma(t)}^{t} \tilde{h}(x(s)) ds + J, \\
x(t) &= \varphi_1(t), \quad t \in \lbrack -d, 0\rbrack, \quad d = \max\{\tau, \sigma\},
\end{aligned}
\]  
(1)
where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) is the state vector associated with the \( n \) neurons at time \( t \); \( C = \text{diag}(c_1, c_2, \ldots, c_n) \in \mathbb{R}^{n \times n} \) is the state feedback coefficient matrix with \( c_i > 0 \) \( (i = 1, 2, \ldots, n) \), \( A \in \mathbb{R}^{n \times n} \) is the connection weight matrix, \( B \in \mathbb{R}^{n \times n} \) is the time-varying delay connection weight matrix and \( D \in \mathbb{R}^{n \times n} \) is the distributed delay connection weight matrix; \( J = [J_1, J_2, \ldots, J_n]^T \in \mathbb{R}^n \) is a constant external input vector; \( \tau(t) \) and \( \sigma(t) \) are the time-varying transmission delay and distributed time-varying delay satisfying the following conditions:

\[
0 \leq \underline{\tau} \leq \tau(t) \leq \bar{\tau}, \quad 0 \leq \sigma(t) \leq \bar{\sigma},
\]

(2)

where \( \underline{\tau}, \bar{\tau}, \sigma \) are constants; Here, \( \varphi(t) \in \mathbb{R}^n \) denotes the initial condition function on \([-d, 0] \), and

\[
\begin{align*}
\dot{f}(x(t)) &= \left[ f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)) \right]^T \in \mathbb{R}^n, \\
\dot{g}(x(t)) &= \left[ g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)) \right]^T \in \mathbb{R}^n, \\
\dot{h}(x(t)) &= \left[ h_1(x_1(t)), h_2(x_2(t)), \ldots, h_n(x_n(t)) \right]^T \in \mathbb{R}^n,
\end{align*}
\]

represent the neuron activation functions satisfying the following assumption.

**Assumption 1.** (see [21]) There exist constants \( F_j^-, F_j^+, G_j^-, G_j^+, H_j^-, H_j^+ \) such that the neuron activation functions \( \bar{f}_j \), \( \bar{g}_j \), \( \bar{h}_j \) satisfies the following conditions:

\[
F_j^- \leq \frac{\bar{f}_j(u) - \bar{f}_j(v)}{u - v} \leq F_j^+ \quad G_j^- \leq \frac{\bar{g}_j(u) - \bar{g}_j(v)}{u - v} \leq G_j^+, \\
H_j^- \leq \frac{\bar{h}_j(u) - \bar{h}_j(v)}{u - v} \leq H_j^+ \quad j = 1, 2, \ldots, n
\]

for any \( u, v \in \mathbb{R}, u \neq v \).

**Assumption 2.** (see [2]) In order to describe the probability distribution of the time-varying delay, we assume that there exists a constant \( \tau_0 \) \( (\underline{\tau} \leq \tau_0 \leq \bar{\tau}) \) such that time-varying delay \( \tau(t) \) takes values in two subintervals \( [\underline{\tau}, \tau_0] \) and \( (\tau_0, \bar{\tau}] \) with certain probability. Therefore, \( \tau(t) \) is a random variable that takes values in two subintervals \( [\underline{\tau}, \tau_0] \) and \( (\tau_0, \bar{\tau}] \) and its probability distribution is assumed to be

\[
\text{Prob}\{\underline{\tau} \leq \tau(t) \leq \tau_0\} = \gamma_0, \quad \text{Prob}\{\tau_0 < \tau(t) \leq \bar{\tau}\} = 1 - \gamma_0,
\]

where \( \gamma_0 \in [0, 1] \) is known constant. Then, a stochastic variable \( \gamma(t) \) is defined as

\[
\gamma(t) = \begin{cases} 
1, & \underline{\tau} \leq \tau(t) \leq \tau_0, \\
0, & \tau_0 < \tau(t) \leq \bar{\tau}.
\end{cases}
\]

**Assumption 3.** (see [2]) \( \gamma(t) \) is a Bernoulli distributed while sequence with

\[
\text{Prob}\{\gamma(t) = 1\} = \text{Prob}\{\underline{\tau} \leq \tau(t) \leq \tau_0\} = \text{E}\{\gamma(t)\} = \gamma_0,
\]

\[
\text{Prob}\{\gamma(t) = 0\} = \text{Prob}\{\tau_0 < \tau(t) \leq \bar{\tau}\} = \text{E}\{1 - \gamma(t)\} = 1 - \gamma_0.
\]

Furthermore, we get

\[
\text{E}\{\gamma(t) - \gamma_0\} = 0, \quad \text{E}\{(\gamma(t) - \gamma_0)^2\} = \gamma_0(1 - \gamma_0).
\]

Define two time-varying delays

\[
\tau_1(t) = \begin{cases} 
\tau(t), & \tau(t) \in [\underline{\tau}, \tau_0] \\
\tau_0, & \tau(t) \in (\tau_0, \bar{\tau}]
\end{cases}, \quad \tau_2(t) = \begin{cases} 
\tau_0, & \tau(t) \in [\underline{\tau}, \tau_0] \\
\tau(t), & \tau(t) \in (\tau_0, \bar{\tau}]
\end{cases}
\]
and satisfy
\[ \hat{\tau}_1(t) \leq \rho_1 < \infty, \quad \hat{\tau}_2(t) \leq \rho_2 < \infty, \]
where \( \rho_1, \rho_2 \) are constants. Therefore, the system (1) is rewritten as follows:
\[ \frac{dx(t)}{dt} = -Cx(t) + A\tilde{f}(x(t)) + \gamma(t)B\tilde{g}(x(t - \tau(t))) + (1 - \gamma(t))B\tilde{g}(x(t - \tau_2(t))) + D\int_{t-\tau(t)}^{t} \tilde{h}(x(s))ds + J, \]
\[ x(t) = \varphi_1(t), \quad t \in [-d, 0], \quad d = \max\{\tau, \sigma\}. \]  

In this paper, neural networks system (3) is regarded as the master system, the corresponding slave system will be described by:
\[ \frac{dy(t)}{dt} = -Cy(t) + A\tilde{f}(y(t)) + \gamma(t)B\tilde{g}(y(t - \tau_1(t))) + (1 - \gamma(t))B\tilde{g}(y(t - \tau_2(t))) + D\int_{t-\tau(t)}^{t} \tilde{h}(y(s))ds + J + u(t) + \delta(e(t)), \]
\[ y(t) = \varphi_2(t), \quad t \in [-d, 0], \quad d = \max\{\tau, \sigma\}, \]
where \( y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n \) is the state vector; \( \varphi_2(t) \in \mathbb{R}^n \) denotes the initial conditions; \( u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T \in \mathbb{R}^n \) is the control input; \( e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T \in \mathbb{R}^n \) is the synchronization error vector; \( \delta(e(t)) \in \mathbb{R}^n \) represents the system disturbance satisfying the following assumption.

**Assumption 4.** \( \delta(e(t)) : \mathbb{R}^n \to \mathbb{R}^n \) is locally Lipschitz and there exist symmetric matrix \( \Gamma \in \mathbb{R}^{n \times n} \) and positive constant \( \varepsilon \) such that
\[ \delta^T(z)[\delta(z) - \Gamma z] \leq 0, \]
\[ [\delta^T(z) - \delta^T(\hat{z})][\varepsilon \delta(z) - \varepsilon \delta(\hat{z}) - (z - \hat{z})] \leq 0, \]
for any \( z, \hat{z} \in \mathbb{R}^n \).

**Notations.** Throughout this paper, let \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space and \( \mathbb{R}^{n \times m} \) is the set of \( n \times m \) real matrices. Let the superscript “\( \top \)” denote the transpose of matrix or vector, the symbol “\( \ast \)” denotes the symmetric term in a symmetric, \( I \) denotes the identity matrix with compatible dimensions, \( \dot{x}(t) \) denotes the derivative of \( x(t) \). \( \mathbb{E}(\cdot) \) stands for the mathematical expectation of a stochastic process. \( \mathcal{L} \) represents the diffusion operator. If \( A \) is a symmetric matrix, \( \lambda_{\max}(A) \) or \( \lambda_{\min}(A) \) denotes the maximum eigenvalue of matrix \( A \) or the minimum eigenvalue of matrix \( A \). \( \| \cdot \| \) stands for the Euclidean norm in \( \mathbb{R}^n \). \( \text{diag}(\ldots) \) stands for the block diagonal matrix.

In order to investigate the synchronization problem for (3) and (4), we define the synchronization error state as \( e(t) = y(t) - x(t) \) and the control input \( u(t) \) in the slave system is given by
\[ u(t) = K_1e(t) + K_2 \int_{t-\sigma(t)}^{t} e(s)ds, \]  

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where $K_1, K_2 \in \mathbb{R}^{n \times n}$ are controller gain matrices to be determined. Then, the synchronization error dynamical system for (3) and (4) can be obtained as follows

$$\frac{de(t)}{dt} = (K_1 - C)e(t) + Af(e(t)) + \gamma(t)Bg(e(t - \tau_1(t))) +$$

$$\left((1 - \gamma(t))Bg(e(t - \tau_2(t))) + D\int_{t - \sigma(t)}^{t} h(e(s))ds +
\right)$$

$$K_2\int_{t - \sigma(t)}^{t} e(s)ds + \delta(e(t)),$$

$$e(t) = \varphi(t) = \varphi_2(t) - \varphi_1(t), \ t \in [-d, 0], \ d = \max\{\tau, \sigma\},$$

which can further be expressed as

$$\frac{de(t)}{dt} = (K_1 - C)e(t) + Af(e(t)) + \gamma_0 Bg(e(t - \tau_1(t))) +$$

$$\left((1 - \gamma_0)Bg(e(t - \tau_2(t))) +
\right)$$

$$\left((\gamma(t) - \gamma_0)B[g(e(t - \tau_1(t))) - g(e(t - \tau_2(t)))]ight) +$$

$$D\int_{t - \sigma(t)}^{t} h(e(s))ds + K_2\int_{t - \sigma(t)}^{t} e(s)ds + \delta(e(t)),$$

$$e(t) = \varphi(t) = \varphi_2(t) - \varphi_1(t), \ t \in [-d, 0], \ d = \max\{\tau, \sigma\},$$

where $f(e(t)) = \hat{f}(x(t) + e(t)) - \hat{f}(x(t)), \ g(e(t)) = \hat{g}(x(t) + e(t)) - \hat{g}(x(t)), \ h(e(t)) = \hat{h}(x(t) + e(t)) - \hat{h}(x(t)).$

To prove the main results, we introduce the following definitions and lemmas.

**Definition 2.1.** (see [24]) For a given functional $V : C([-d, 0]; \mathbb{R}^n) \times \mathbb{R}^+ \rightarrow \mathbb{R}$, its infinitesimal operator $\mathcal{L}$ is defined as

$$\mathcal{L}V(t, x(t)) = \lim_{\Delta \rightarrow 0^+} \frac{1}{\Delta} \left[ E\left( V(t + \Delta, x(t + \Delta)|x(t)) - V(t, x(t)) \right) \right].$$

**Definition 2.2.** (see [10]) The master system (3) and the slave system (4) under the controller (5) are said to be exponentially synchronized in mean square if the error dynamical system (7) is exponentially stable in mean square, i.e., there exist positive constants $\alpha$ and $\beta$ such that

$$E\left\{ \|e(t)\|^2 \right\} \leq \beta \sup_{d \in \mathbb{R}^+} E\left\{ \|\varphi(\theta)\|^2 \right\} e^{-\alpha t}, \ t \geq 0,$$

where $\varphi \in C([-d, 0], \mathbb{R}^n)$. Moreover, the constant $\alpha$ is defined as the exponential synchronization rate.

**Lemma 2.3.** (Jensen’s inequality, see [7]): For any constant matrix $W \in \mathbb{R}^{n \times n}, W = W^T > 0$, scalar $\tau > 0$, vector function $\dot{x} : [\tau, 0] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$-\tau \int_{\tau}^{0} \dot{x}^T(s + t)W\dot{x}(s + t)ds \leq -\left( \int_{\tau}^{0} \dot{x}(s + t)ds \right)^T W \left( \int_{\tau}^{0} \dot{x}(s + t)ds \right).$$

Rearranging the term $\int_{\tau}^{0} \dot{x}(s + t)ds$ with $x(t) - x(t - \tau)$, one can yield the following inequality:

$$-\tau \int_{\tau}^{0} \dot{x}^T(s + t)W\dot{x}(s + t)ds \leq \left[ \begin{array}{c} x(t) \\ x(t - \tau) \end{array} \right]^T \left[ \begin{array}{cc} -W & W \\ * & -W \end{array} \right] \left[ \begin{array}{c} x(t) \\ x(t - \tau) \end{array} \right].$$
Lemma 2.4. (see [16]) For any constant matrix $W \in \mathbb{R}^{n \times n}$, $W = W^T > 0$, scalars $\underline{z} \leq \tau(t) \leq \overline{z}$ and vector function $\dot{z} : [\underline{z}, \overline{z}] \rightarrow \mathbb{R}^n$ such that the integration concerned is well defined, then it holds that
\[
-(\overline{z} - \underline{z}) \int_{t-\overline{z}}^{t-\underline{z}} \dot{x}^T(s)W \dot{x}(s)ds \leq \begin{bmatrix}
  x(t-\overline{z}) & x(t-\tau(t)) & x(t-\underline{z})
\end{bmatrix}^T \begin{bmatrix}
  -W & W & 0 \\
  * & -2W & W \\
  * & * & -W
\end{bmatrix} \begin{bmatrix}
  x(t-\overline{z}) \\
  x(t-\tau(t)) \\
  x(t-\underline{z})
\end{bmatrix}.
\]

3. Main results. In this section, we will firstly present a delay-distribution-dependent global exponential stability criteria for the synchronization error dynamical system (6). By designing the feedback controller with distributed delay (5), we will derive a LMIs based criteria such that the master system (3) and the slave system (4) are delay-distribution-dependent globally exponentially synchronized in the mean square.

Theorem 3.1. Under Assumptions 1–4, the synchronization error dynamical system (6) is globally exponentially stable in the mean square if there exist positive-definite matrices $P$, $Q_s$, $R_s$, $S_s$ ($s = 1, 2, 3$), $T$, $Z \in \mathbb{R}^{n \times n}$ and diagonal positive-definite matrices $U$, $\Lambda_k$ ($k = 1, 2, 3$), $M \in \mathbb{R}^{n \times n}$ and controller gain $K_1$, $K_2 \in \mathbb{R}^{n \times n}$ such that the following matrix inequality hold:
\[
\Xi = \begin{bmatrix}
  \Xi_{11} & \Xi_{12} & \Xi_{13} \\
  * & \Xi_{22} & 0 \\
  * & * & \Xi_{33}
\end{bmatrix} < 0,
\]
where $\Xi_{11} = (\Omega_{i,j})_{15 \times 15}$ and
\[
\begin{align*}
\Omega_{1,1} &= \alpha P + P(K_1 - C) + (K_1 - C)^T P + Q_1 + Q_2 + Q_3 + R_1 + S_1 + \sigma^2 F_2 - \Lambda_1 G_1 - M \Lambda_1, \\
\Omega_{1,2} &= \Omega_{1,3} = \Omega_{1,4} = \Omega_{1,5} = \Omega_{1,6} = 0, \\
\Omega_{1,7} &= PA + UF_2, \quad \Omega_{1,8} = \Lambda_1 G_2, \quad \Omega_{1,9} = MH_2, \quad \Omega_{1,10} = \gamma_0 PB, \\
\Omega_{1,11} &= (1 - \gamma_0)PB, \quad \Omega_{1,12} = PD, \quad \Omega_{1,13} = PK_2, \quad \Omega_{1,14} = P + I + \Gamma, \\
\Omega_{1,15} &= -I, \quad \Omega_{2,2} = -e^{-\alpha} Q_1 - R_3, \quad \Omega_{2,3} = R_3, \quad \Omega_{2,4} = \Omega_{2,5} = \Omega_{2,6} = 0, \\
\Omega_{2,7} &= \Omega_{2,8} = \Omega_{2,9} = \Omega_{2,10} = \Omega_{2,11} = \Omega_{2,12} = \Omega_{2,13} = \Omega_{2,14} = \Omega_{2,15} = 0, \\
\Omega_{3,3} &= -e^{-\alpha_j} R_1 + \rho_1 e^{-\alpha_j} R_2 - 2R_3 - \Lambda_2 G_1, \quad \Omega_{3,4} = R_3, \quad \Omega_{3,5} = \Omega_{3,6} = 0, \\
\Omega_{3,7} &= \Omega_{3,8} = \Omega_{3,9} = 0, \quad \Omega_{3,10} = \Lambda_2 G_2, \quad \Omega_{3,11} = \Omega_{3,12} = \Omega_{3,13} = \Omega_{3,14} = 0, \\
\Omega_{3,15} &= 0, \quad \Omega_{4,4} = -e^{-\alpha_j} Q_2 - R_1 - S_3, \quad \Omega_{4,5} = S_3, \quad \Omega_{4,6} = \Omega_{4,7} = 0, \\
\Omega_{4,8} &= \Omega_{4,9} = \Omega_{4,10} = \Omega_{4,11} = \Omega_{4,12} = \Omega_{4,13} = \Omega_{4,14} = \Omega_{4,15} = 0, \quad \Omega_{5,5} = 0, \\
\Omega_{5,10} &= 0, \quad \Omega_{5,11} = \Lambda_3 G_2, \quad \Omega_{5,12} = \Omega_{5,13} = \Omega_{5,14} = \Omega_{5,15} = 0, \quad \Omega_{6,6} = -e^{-\alpha_j} Q_3 - S_3, \quad \Omega_{6,7} = \Omega_{6,8} = \Omega_{6,9} = \Omega_{6,10} = \Omega_{6,11} = \Omega_{6,12} = \Omega_{6,13} = 0, \\
\Omega_{6,14} &= \Omega_{6,15} = 0, \quad \Omega_{7,7} = -U, \quad \Omega_{7,8} = \Omega_{7,9} = \Omega_{7,10} = \Omega_{7,11} = \Omega_{7,12} = 0, \\
\Omega_{7,13} &= \Omega_{7,14} = \Omega_{7,15} = 0, \quad \Omega_{8,8} = R_2 + S_2 - \Lambda_1, \quad \Omega_{8,9} = \Omega_{8,10} = \Omega_{8,11} = 0, \\
\Omega_{8,12} &= \Omega_{8,13} = \Omega_{8,14} = \Omega_{8,15} = 0, \quad \Omega_{9,9} = \frac{1}{\alpha}(e^{\alpha_j} - 1)Z - M, \quad \Omega_{9,10} = 0,
\end{align*}
\]
where

\[ \Omega_{9,11} = \Omega_{9,12} = \Omega_{9,13} = \Omega_{9,14} = \Omega_{9,15} = 0, \quad \Omega_{10,10} = -e^{-\alpha \tau_0} R_2 + \rho_1 e^{-\alpha \tau_2} R_2 - A_2, \quad \Omega_{10,11} = \Omega_{10,12} = \Omega_{10,13} = \Omega_{10,14} = \Omega_{10,15} = 0, \quad \Omega_{11,11} = -e^{-\alpha \tau_3} S_2 + \rho_2 e^{-\alpha \tau_0} S_2 - A_3, \quad \Omega_{11,12} = \Omega_{11,13} = \Omega_{11,14} = \Omega_{11,15} = 0, \]
\[ \Omega_{12,12} = -Z, \quad \Omega_{12,13} = \Omega_{12,14} = \Omega_{12,15} = 0, \quad \Omega_{13,13} = -e^{-\alpha \tau T}, \quad \Omega_{13,14} = -I, \quad \Omega_{13,15} = I, \quad \Omega_{14,14} = -2(\varepsilon + 1) I, \quad \Omega_{14,15} = 2\varepsilon I, \quad \Omega_{15,15} = -2\varepsilon I, \]

\[ \Xi_{12} = \begin{bmatrix} \sqrt{l_1}(K_1 - C) & 0 & 0 & 0 & 0 & 0 & \sqrt{l_1} A & 0 & 0 & \sqrt{l_1 \tau_0} B \\ \sqrt{l_2}(K_1 - C) & 0 & 0 & 0 & 0 & 0 & \sqrt{l_2} A & 0 & 0 & \sqrt{l_2 \tau_0} B \\ \sqrt{l_1}(1 - \tau_0) B & \sqrt{l_1} D & \sqrt{l_1} K_2 & \sqrt{l_1} I & 0 & 0 & T \\ \sqrt{l_2}(1 - \tau_0) B & \sqrt{l_2} D & \sqrt{l_2} K_2 & \sqrt{l_2} I & 0 & 0 & T \end{bmatrix}, \]
\[ \Xi_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{l_1 \tau_0} B & -\sqrt{l_1 \tau_0} B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{l_2 \tau_0} B & -\sqrt{l_2 \tau_0} B & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \]
\[ \Xi_{22} = \Xi_{33} = \text{diag} \left[ -R_3^{-1}, -S_3^{-1} \right], \quad l_1 = \frac{1}{\alpha} (\tau_0 - T) (e^{\alpha \tau_0} - e^{\alpha T}), \]
\[ l_2 = \frac{1}{\alpha} (T - \tau_0) (e^{\alpha \tau} - e^{\alpha \tau_0}), \quad \tau_0 = \gamma_0 (1 - \gamma_0). \]

**Proof.** To proceed with the synchronization problem for the neural networks (3) and (4), we construct the following Lyapunov-Krasovskii functional candidate:

\[ V(t, e(t)) = \sum_{i=1}^{6} V_i(t, e(t)), \]

where

\[ V_1(t, e(t)) = e^{\alpha t} e^T(t) P e(t), \]
\[ V_2(t, e(t)) = \int_{t-T}^{t} e^{\alpha s} e^T(s) Q_1 e(s) ds + \int_{t-\tau_0}^{t} e^{\alpha s} e^T(s) Q_2 e(s) ds + \int_{t-\tau}^{t} e^{\alpha s} e^T(s) Q_3 e(s) ds, \]
\[ V_3(t, e(t)) = \int_{t-\tau_1(t)}^{t} e^{\alpha s} e^T(s) R_1 e(s) ds + \int_{t-\tau_1(t)}^{t} e^{\alpha s} e^T(s) R_2 g(e(s)) ds + (\tau_0 - T) \int_{t-\tau_0}^{t} \int_{t-\theta}^{t} e^{\alpha(s-\theta)} \dot{e}^T(s) R_3 \dot{e}(s) ds d\theta, \]
\[ V_4(t, e(t)) = \int_{t-\tau_2(t)}^{t} e^{\alpha s} e^T(s) S_1 e(s) ds + \int_{t-\tau_2(t)}^{t} e^{\alpha s} e^T(s) S_2 g(e(s)) ds + (T - \tau_0) \int_{t-\tau_0}^{t} \int_{t-\theta}^{t} e^{\alpha(s-\theta)} \dot{e}^T(s) S_3 \dot{e}(s) ds d\theta, \]
\[ V_5(t, e(t)) = \sigma \int_{t-\sigma}^{t} (s - t + \sigma) e^{\alpha s} e^T(s) T e(s) ds, \]
\[ V_6(t, e(t)) = \sigma \int_{t-\sigma}^{t} \int_{t+\theta}^{t} e^{\alpha(s-\theta)} h^T(e(s)) Z h(e(s)) ds d\theta. \]
Using the infinitesimal operator (8), we have
\[
\mathcal{L}V(t, e(t)) = \sum_{i=1}^{6} \mathcal{L}V_{i}(t, e(t)),
\]
\[\text{(11)}\]
where
\[
\mathcal{L}V_{1}(t, e(t)) = e^{at} \left[ e^{T}(t) \alpha P e(t) + 2e^{T}(t) \dot{P}e(t) \right] = e^{at} \left\{ e^{T}(t) \left[ \alpha P + P(K_{1} - C) + (K_{1} - C)^{T} P \right] e(t) + 2e^{T}(t) PAf(e(t)) + 2\gamma_{0}e^{T}(t) PBg(e(t - \tau_{1}(t))) + 2(1 - \gamma_{0})e^{T}(t) PBg(e(t - \tau_{2}(t))) + 2e^{T}(t) PD \int_{t-\sigma(t)}^{t} h(e(s)) ds + 2e^{T}(t) PK_{2} \int_{t-\sigma(t)}^{t} e(s) ds + 2e^{T}(t) P\delta(e(t)) \right\},
\]
\[\text{(12)}\]
\[
\mathcal{L}V_{2}(t, e(t)) = e^{at} \left[ e^{T}(t)(Q_{1} + Q_{2} + Q_{3}) e(t) - e^{-\alpha t} e^{T}(t - \tau) Q_{1} e(t - \tau) - e^{-\alpha t} e^{T}(t - \tau_{0}) Q_{2} e(t - \tau_{0}) - e^{-\alpha t} e^{T}(t - \tau_{1}) Q_{3} e(t - \tau_{1}) \right],
\]
\[\text{(13)}\]
\[
\mathcal{L}V_{3}(t, e(t)) = e^{at} \left[ e^{T}(t) R_{1} e(t) + g^{T}(e(t)) R_{2} g(e(t)) - e^{-\alpha \tau_{1}(t)} e^{T}(t - \tau_{1}(t)) R_{1} e(t - \tau_{1}(t)) + \tau_{1}(t) e^{-\alpha \tau_{1}(t)} e^{T}(t - \tau_{1}(t)) R_{1} e(t - \tau_{1}(t)) - e^{-\alpha \tau_{1}(t)} g^{T}(e(t - \tau_{1}(t))) R_{2} g(e(t - \tau_{1}(t))) + \tau_{1}(t) e^{-\alpha \tau_{1}(t)} g^{T}(e(t - \tau_{1}(t))) R_{2} g(e(t - \tau_{1}(t))) + \frac{1}{\alpha} (\tau_{0} - \tau) (e^{\alpha \tau_{0} - \alpha \tau} e^{T}(t) R_{3} \dot{e}(t) - (\tau_{0} - \tau) \int_{t-\tau_{0}}^{t-\tau} e^{T}(s) R_{3} \dot{e}(s) ds \right]\leq e^{at} \left[ e^{T}(t) R_{1} e(t) - e^{-\alpha \tau_{0}} e^{T}(t - \tau_{1}(t)) R_{1} e(t - \tau_{1}(t)) + \rho_{1} e^{-\alpha \tau_{0}} e^{T}(t - \tau_{1}(t)) R_{1} e(t - \tau_{1}(t)) + g^{T}(e(t)) R_{2} g(e(t)) - e^{-\alpha \tau_{0}} g^{T}(e(t - \tau_{1}(t))) R_{2} g(e(t - \tau_{1}(t))) + \rho_{1} e^{-\alpha \tau_{0}} g^{T}(e(t - \tau_{1}(t))) R_{2} g(e(t - \tau_{1}(t))) + \frac{1}{\alpha} (\tau_{0} - \tau) (e^{\alpha \tau_{0} - \alpha \tau} e^{T}(t) R_{3} \dot{e}(t) - (\tau_{0} - \tau) \int_{t-\tau_{0}}^{t-\tau} e^{T}(s) R_{3} \dot{e}(s) ds \right],
\]
\[\text{(14)}\]
\[
\mathcal{L}V_{4}(t, e(t)) \leq e^{at} \left[ e^{T}(t) S_{1} e(t) - e^{-\alpha \tau} e^{T}(t - \tau_{2}(t)) S_{1} e(t - \tau_{2}(t)) + \rho_{2} e^{-\alpha \tau_{0}} e^{T}(t - \tau_{2}(t)) S_{1} e(t - \tau_{2}(t)) + g^{T}(e(t)) S_{2} g(e(t)) - \rho_{2} e^{-\alpha \tau_{0}} g^{T}(e(t - \tau_{2}(t))) S_{2} g(e(t - \tau_{2}(t))) \right] + \rho_{2} e^{-\alpha \tau_{0}} e^{T}(t - \tau_{2}(t)) S_{1} e(t - \tau_{2}(t)) + g^{T}(e(t)) S_{2} g(e(t)) - \rho_{2} e^{-\alpha \tau_{0}} g^{T}(e(t - \tau_{2}(t))) S_{2} g(e(t - \tau_{2}(t))) \right],
\]
\[
e^{-\alpha \tau} g^T (e(t - \tau_2(t))) S_2 g(e(t - \tau_2(t))) + \\
p_2 e^{-\alpha \tau_0} g^T (e(t - \tau_2(t))) S_2 g(e(t - \tau_2(t))) + \\
\frac{1}{\alpha} (\tau - \tau_0) (e^{\alpha \tau} - e^{\alpha \tau_0}) e^T (t) S_3 \dot{e}(t) - \\
(\tau - \tau_0) \int_{t-\tau}^{t} e^T (s) S_3 \dot{e}(s) ds, \tag{15}
\]

\[
\mathcal{L}V_5(t, e(t)) = e^{\alpha t} \sigma^2 e^T (t) T e(t) - \sigma \int_{t-\tau}^{t} e^{\alpha s} e^T (s) T e(s) ds \\
\leq e^{\alpha t} \sigma^2 e^T (t) T e(t) - e^{\alpha (t-\tau)} \sigma \int_{t-\tau}^{t} e^T (s) T e(s) ds \\
= e^{\alpha t} \left[ \sigma^2 e^T (t) T e(t) - e^{-\alpha \tau} \sigma \int_{t-\tau}^{t} e^T (s) T e(s) ds \right], \tag{16}
\]

\[
\mathcal{L}V_6(t, e(t)) = e^{\alpha t} \left[ \frac{1}{\alpha} \sigma (e^{\alpha \tau} - 1) h^T (e(t)) Z h(e(t)) - \right. \\
\left. \sigma \int_{t-\tau}^{t} h^T (e(s)) Z h(e(s)) ds \right]. \tag{17}
\]

Based on Assumption 1, we can acquire the following inequality:

\[
(f_j(e_j(t)) - F_j^+ e_j(t)) (f_j(e_j(t)) - F_j^- e_j(t)) \leq 0,
\]

where \(j = 1, 2, \ldots, n\). Then, there exist matrix \(U = \text{diag}(u_1, u_2, \ldots, u_n) > 0\), such that

\[
e^{\alpha t} \sum_{j=1}^{n} u_j \left[ \begin{array}{c} e(t) \\ f(e(t)) \end{array} \right]^T \left[ \begin{array}{cc} F_j^+ & F_j^- v_j e_j^T \\ F_j^+ & F_j^- v_j e_j^T \end{array} \right] \left[ \begin{array}{c} e(t) \\ f(e(t)) \end{array} \right] \leq 0, \tag{18}
\]

where \(v_j\) denotes the unit column vector having “1” element on its \(j\)-th row and zeros elsewhere,

\[
F_1 = \text{diag} \left[ \frac{F_1^+}{2}, \frac{F_2^+}{2}, \ldots, \frac{F_n^+}{2} \right], \\
F_2 = \text{diag} \left[ \frac{F_1^-}{2}, \frac{F_2^-}{2}, \ldots, \frac{F_n^-}{2} \right].
\]

In a Similar way, we have

\[
e^{\alpha t} \left[ \begin{array}{c} e(t) \\ g(e(t)) \end{array} \right]^T \left[ \begin{array}{cc} \Lambda_1 G_1 & -\Lambda_1 G_2 \\ -\Lambda_1 G_2 & \Lambda_1 \end{array} \right] \left[ \begin{array}{c} e(t) \\ g(e(t)) \end{array} \right] \leq 0, \tag{19}
\]

\[
e^{\alpha t} \left[ \begin{array}{c} e(t - \tau_1(t)) \\ g(e(t - \tau_1(t))) \end{array} \right]^T \left[ \begin{array}{cc} \Lambda_2 G_1 & -\Lambda_2 G_2 \\ -\Lambda_2 G_2 & \Lambda_2 \end{array} \right] \left[ \begin{array}{c} e(t - \tau_1(t)) \\ g(e(t - \tau_1(t))) \end{array} \right] \leq 0, \tag{20}
\]

\[
e^{\alpha t} \left[ \begin{array}{c} e(t - \tau_2(t)) \\ g(e(t - \tau_2(t))) \end{array} \right]^T \left[ \begin{array}{cc} \Lambda_3 G_1 & -\Lambda_3 G_2 \\ -\Lambda_3 G_2 & \Lambda_3 \end{array} \right] \left[ \begin{array}{c} e(t - \tau_2(t)) \\ g(e(t - \tau_2(t))) \end{array} \right] \leq 0, \tag{21}
\]

\[
e^{\alpha t} \left[ \begin{array}{c} e(t) \\ h(e(t)) \end{array} \right]^T \left[ \begin{array}{cc} M H_1 & -M H_2 \\ -M H_2 & M \end{array} \right] \left[ \begin{array}{c} e(t) \\ h(e(t)) \end{array} \right] \leq 0, \tag{22}
\]
where \( \Lambda_k = \text{diag}(\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn}) > 0 \), \((k = 1, 2, 3)\) and \( M = \text{diag}(m_1, m_2, \ldots, m_n) > 0 \) are diagonal matrices,

\[
G_1 = \text{diag}\left[ G^+_1 G^-_1, G^+_2 G^-_2, \ldots, G^+_n G^-_n \right],
\]

\[
G_2 = \text{diag}\left[ \frac{G^+_1 + G^-_1}{2}, \frac{G^+_2 + G^-_2}{2}, \ldots, \frac{G^+_n + G^-_n}{2} \right],
\]

\[
H_1 = \text{diag}\left[ H^+_1 H^-_1, H^+_2 H^-_2, \ldots, H^+_n H^-_n \right],
\]

\[
H_2 = \text{diag}\left[ \frac{H^+_1 + H^-_1}{2}, \frac{H^+_2 + H^-_2}{2}, \ldots, \frac{H^+_n + H^-_n}{2} \right].
\]

According to Assumption 4, we can obtain the following inequalities:

\[
e^{\alpha t} \begin{bmatrix} e(t) \\ \delta(e(t)) \end{bmatrix}^T \begin{bmatrix} 0 & -I \\ \ast & 2I \end{bmatrix} \begin{bmatrix} e(t) \\ \delta(e(t)) \end{bmatrix} \leq 0,
\]

and

\[
e^{\alpha t} \begin{bmatrix} \int_{t-\sigma(t)}^t e(s) ds \\ \delta \left( \int_{t-\sigma(t)}^t e(s) ds \right) \end{bmatrix}^T \begin{bmatrix} 0 & 0 & -I & I \\ \ast & \ast & 2\varepsilon I & -2\varepsilon I \\ \ast & \ast & \ast & 2\varepsilon I \end{bmatrix} \begin{bmatrix} \int_{t-\sigma(t)}^t e(s) ds \\ \delta \left( \int_{t-\sigma(t)}^t e(s) ds \right) \end{bmatrix} \leq 0.
\]

On the other hand, we can derive from Lemma 2.3 and Lemma 2.4 that

\[
-(\tau_0 - \tau) \int_{t-\tau_0}^{t-\tau} e^{T}(s) R_3 \dot{e}(s) ds 
\]

\[
\leq \begin{bmatrix} e(t - \tau) \\ e(t - \tau_1(t)) \\ e(t - \tau_0) \end{bmatrix}^T \begin{bmatrix} -R_3 & R_3 & 0 \\ \ast & -2R_3 & R_3 \\ \ast & \ast & -R_3 \end{bmatrix} \begin{bmatrix} e(t - \tau) \\ e(t - \tau_1(t)) \\ e(t - \tau_0) \end{bmatrix},
\]

and

\[
-(\tau - \tau_0) \int_{t-\tau}^{t-\tau_0} e^{T}(s) S_3 \dot{e}(s) ds 
\]

\[
\leq \begin{bmatrix} e(t - \tau_0) \\ e(t - \tau_2(t)) \\ e(t - \tau) \end{bmatrix}^T \begin{bmatrix} -S_3 & S_3 & 0 \\ \ast & -2S_3 & S_3 \\ \ast & \ast & -S_3 \end{bmatrix} \begin{bmatrix} e(t - \tau_0) \\ e(t - \tau_2(t)) \\ e(t - \tau) \end{bmatrix},
\]

and

\[
-\sigma \int_{t-\sigma}^t e^{T}(s) Te(s) ds \leq -\sigma(t) \int_{t-\sigma(t)}^{t-\sigma} e^{T}(s) Te(s) ds 
\]

\[
\leq - \left( \int_{t-\sigma(t)}^t e(s) ds \right)^T T \left( \int_{t-\sigma(t)}^t e(s) ds \right),
\]

\[
-\sigma \int_{t-\sigma}^t h^{T}(e(s)) Z h(e(s)) ds 
\]

\[
\leq - \left( \int_{t-\sigma(t)}^t h(e(s)) ds \right)^T Z \left( \int_{t-\sigma(t)}^t h(e(s)) ds \right).
\]
It can be obtained from (7) that
\[
e^{T}(t)R_{3}\dot{e}(t) = \xi^{T}(t)A_{1}^{T}R_{3}A_{1}\xi(t) + (\gamma(t) - \gamma_{0})^{2}\xi^{T}(t)A_{2}^{T}R_{3}A_{2}\xi(t) + 2(\gamma(t) - \gamma_{0})\xi^{T}(t)A_{1}^{T}R_{3}A_{2}\xi(t),
\]
and
\[
e^{T}(t)S_{3}\dot{e}(t) = \xi^{T}(t)A_{1}^{T}S_{3}A_{1}\xi(t) + (\gamma(t) - \gamma_{0})^{2}\xi^{T}(t)A_{2}^{T}S_{3}A_{2}\xi(t) + 2(\gamma(t) - \gamma_{0})\xi^{T}(t)A_{1}^{T}S_{3}A_{2}\xi(t),
\]
where
\[
\xi(t) = \left[ e^{T}(t), e^{T}(t - \tau), e^{T}(t - \tau_{1}(t)), e^{T}(t - \tau_{2}(t)), e^{T}(t - \tau_{0}(t)), e^{T}(t - \tau), f^{T}(e(t)), g^{T}(e(t)), h^{T}(e(t)), g^{T}(e(t - \tau_{1}(t))), g^{T}(e(t - \tau_{2}(t))), \int_{t-\tau(t)}^{t} h(e(s))ds \right]^{T},
\]
\[
\dot{\xi}(t) = \left[ \left( \int_{t-\tau(t)}^{t} e(s)ds \right)^{T}, \left( \int_{t-\tau(t)}^{t} e(s)ds \right)^{T} \right]^{T} \delta^{T}(t),
\]
\[
A_{1} = \begin{bmatrix} K_{1} - C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ \end{bmatrix},
\]
\[
A_{2} = \begin{bmatrix} \gamma_{0}B & (1 - \gamma_{0})B & D & K_{2} & I & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & B & -B & 0 & 0 \ \end{bmatrix}.
\]

Combining (11)–(30) and taking the mathematical expectation on both sides of (11), we get
\[
\mathbb{E}\{\mathcal{L}V(s, e(s))\} \leq e^{\alpha t}\mathbb{E}\{\xi^{T}(t)\Xi\xi(t)\},
\]
where
\[
\Xi = \Xi_{11} + \frac{1}{\alpha}(\tau_{0} - \bar{\tau})(e^{\alpha\tau_{0}} - e^{\alpha\bar{\tau}})A_{1}^{T}R_{3}A_{1} + \frac{1}{\alpha}(\tau - \tau_{0})(e^{\alpha\tau} - e^{\alpha\tau_{0}})A_{1}^{T}S_{3}A_{1} + \gamma_{0}(1 - \gamma_{0})\left( \frac{1}{\alpha}(\tau_{0} - \bar{\tau})(e^{\alpha\tau_{0}} - e^{\alpha\bar{\tau}})A_{2}^{T}R_{3}A_{2} + \frac{1}{\alpha}(\tau - \tau_{0})(e^{\alpha\tau} - e^{\alpha\tau_{0}})A_{2}^{T}S_{3}A_{2} \right).
\]

Applying the Schur complement formula to (9) yields following results:
\[
\mathbb{E}\{\mathcal{L}V(s, e(s))\} \leq 0.
\]

In addition, it follows from (32) and the generalized Itô’s formula that
\[
\mathbb{E}\{V(t, e(t))\} - \mathbb{E}\{V(0, e(0))\} = \int_{0}^{t} \mathbb{E}\{\mathcal{L}V(s, e(s))\}ds \leq 0.
\]
Hence, for any \( t > 0 \), we obtain
\[
\mathbb{E}\{V(t, e(t))\} \leq \mathbb{E}\{V(0, e(0))\} < \infty.
\]
From the definition of function \( V(t, e(t)) \), it can be deduced
\[
e^{\alpha t}\mathbb{E}\{\|e(t)\|^{2}\} \leq \mathbb{E}\{V(t, e(t))\} \leq \mathbb{E}\{V(0, e(0))\}.
\]
Let \( F_m = \max_{1 \leq j \leq n} \{ |F_j^-|, |F_j^+| \}, \) \( G_m = \max_{1 \leq j \leq n} \{ |G_j^-|, |G_j^+| \}, \) \( H_m = \max_{1 \leq j \leq n} \{ |H_j^-|, |H_j^+| \}, \) together with Assumption 4 and a scalar \( L > 0, \) we have that

\[
\mathbb{E} \{ V(0, e(0)) \} = E \left\{ e^T(0) P e(0) + \int_{-T}^0 e^{\alpha s} e^T(s) Q_1 e(s) ds + \right. \\
\left. \int_{-\tau_0}^0 e^{\alpha s} e^T(s) Q_2 e(s) ds + \int_{-\tau}^0 e^{\alpha s} e^T(s) Q_3 e(s) ds + \int_{-\tau_1(0)}^0 e^{\alpha s} e^T(s) R_1 e(s) ds + \int_{-\tau_1(0)}^0 e^{\alpha s} g^T(\theta(s)) R_2 g(s) ds + \right. \\
\left. \int_{-\tau_2(0)}^0 e^{\alpha s} e^T(s) S_1 e(s) ds + \int_{-\tau_2(0)}^0 e^{\alpha s} g^T(\theta(s)) S_2 g(s) ds + \right. \\
\left. (\tau_0 - T) \int_{-\tau_0}^{t-\theta} \int_{t+\theta}^T e^{\alpha(s-\theta)} e^T(s) R_3 e(s) ds d\theta + \int_{-\tau_0}^0 e^{\alpha s} e^T(s) T e(s) ds + \right. \\
\left. \sigma \int_{-\sigma}^0 \int_{t+\theta}^T e^{\alpha(s-\theta)} h^T(\theta(s)) Z_h(\theta(s)) ds d\theta \right\} \\
\leq \left\{ \lambda_{\max}(P) + \frac{1}{\alpha} \left( \lambda_{\max}(Q_1) + \lambda_{\max}(Q_2) + \lambda_{\max}(Q_3) + \lambda_{\max}(R_1) + \lambda_{\max}(R_2) G_m^2 + \lambda_{\max}(S_1) + \lambda_{\max}(S_2) G_m^2 + \lambda_{\max}(T) \sigma^2 \right) + \lambda_{\max}(R_3) \mathbb{E} \left\{ \Pi^T \Pi \right\} \left( \frac{\tau_0 - \tau}{\sigma^2} \right) [e^{\alpha \tau_0} - e^{\alpha \tau} - \alpha (\tau_0 - \tau)] + \lambda_{\max}(R_3) \mathbb{E} \left\{ \Pi^T \Pi \right\} \left( \frac{\tau - \tau_0}{\sigma^2} \right) [e^{\alpha \tau} - e^{\alpha \tau_0} - \alpha (\tau - \tau_0)] + \lambda_{\max}(Z) \sigma H_m^2 \left( \frac{\tau - \tau_0}{\sigma^2} \right) [e^{\alpha \tau} - 1 - \alpha \sigma] \right\} \sup_{-d \leq \theta \leq 0} \mathbb{E} \left\{ ||e(\theta)||^2 \right\},
\]

where

\[
||\Pi|| = ||K_1 - C|| + F_m ||A|| + G_m ||B|| + \sigma H_m ||D|| + \sigma ||K_2|| + L.
\]

By Definition 2.2, it has

\[
\mathbb{E} \left\{ ||e(t)||^2 \right\} \leq \frac{\beta}{\lambda_{\min}(P)} \sup_{-d \leq \theta \leq 0} \mathbb{E} \left\{ ||e(\theta)||^2 \right\} e^{-\alpha t},
\]

where

\[
\beta = \lambda_{\max}(P) + \frac{1}{\alpha} \left( \lambda_{\max}(Q_1) + \lambda_{\max}(Q_2) + \lambda_{\max}(Q_3) + \lambda_{\max}(R_1) + \lambda_{\max}(R_2) G_m^2 + \lambda_{\max}(S_1) + \lambda_{\max}(S_2) G_m^2 + \lambda_{\max}(T) \sigma^2 \right) + \lambda_{\max}(R_3) \mathbb{E} \left\{ \Pi^T \Pi \right\} \left( \frac{\tau_0 - \tau}{\sigma^2} \right) [e^{\alpha \tau_0} - e^{\alpha \tau} - \alpha (\tau_0 - \tau)] + \lambda_{\max}(R_3) \mathbb{E} \left\{ \Pi^T \Pi \right\} \left( \frac{\tau - \tau_0}{\sigma^2} \right) [e^{\alpha \tau} - e^{\alpha \tau_0} - \alpha (\tau - \tau_0)] + \lambda_{\max}(Z) \sigma H_m^2 \left( \frac{\tau - \tau_0}{\sigma^2} \right) [e^{\alpha \tau} - 1 - \alpha \sigma].
\]
\[ \lambda_{\text{max}}(R_3) [ \Pi^T \Pi ] \left( \frac{(\tau - \tau_0)}{\alpha^2} \right) [ e^{\alpha \tau} - e^{\alpha \tau_0} - \alpha (\tau - \tau_0) ] + \lambda_{\text{max}}(Z) \sigma H_m^2 \left( \frac{(\tau - \tau_0)}{\alpha^2} \right) [ e^{\alpha \sigma} - 1 - \alpha \sigma ] . \]

Therefore, it can be concluded that the synchronization error dynamical system \((6)\) is globally exponentially stable in the mean square. This completes the proof. \( \square \)

On the basis of Theorem 3.1, we can derive a LMI based criteria to design the feedback controller with distributed delay \((5)\) such that the master system \((3)\) and the slave system \((4)\) are delay-distribution-dependent globally exponentially synchronized in the mean square.

**Theorem 3.2.** Under Assumptions 1–4, the master system \((3)\) and the slave system \((4)\) are globally exponentially synchronized in the mean square if there exist positive-definite matrices \(\hat{P}, \hat{Q}_s, \hat{R}_s, \hat{T}, \hat{Z}, \hat{X}_s (s = 1, 2, 3) \in \mathbb{R}^{n \times n}\), diagonal positive-definite matrices \(\hat{U}, \hat{\Lambda}_k (k = 1, 2, 3)\), \(\hat{M} \in \mathbb{R}^{n \times n}\) and matrices \(Y_1 \in \mathbb{R}^{n \times n}\), \(Y_2 \in \mathbb{R}^{n \times n}\) such that the following LMI hold:

\[
\hat{\Delta} = \begin{bmatrix} \hat{\Delta}_{11} & \hat{\Delta}_{12} & \hat{\Delta}_{13} \\ \hat{\Delta}_{21} & \hat{\Delta}_{22} & \hat{\Delta}_{23} \\ \hat{\Delta}_{31} & \hat{\Delta}_{32} & \hat{\Delta}_{33} \end{bmatrix} < 0, \quad (34)
\]

where \(\hat{\Delta}_{11} = \left( \hat{\Omega}_{\cdot, \cdot} \right)_{15 \times 15}\) and

\[
\begin{align*}
\hat{\Omega}_{1,1} &= \alpha P + Y_1 - CX_1 + Y_1^T - X_1^T C^T + \hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 + \hat{R}_1 + \hat{S}_1 + \\
\hat{\Omega}_{1,1} &= \sigma^2 \hat{T} - \hat{U} F_1 - \hat{\Lambda}_1 G_1 - \hat{M} H_1, \quad \hat{\Omega}_{1,2} = \hat{\Omega}_{1,3} = \hat{\Omega}_{1,4} = \hat{\Omega}_{1,5} = \hat{\Omega}_{1,6} = 0, \\
\hat{\Omega}_{1,7} &= AX_1 + \hat{U} F_2, \quad \hat{\Omega}_{1,8} = \hat{\Lambda}_1 G_2, \quad \hat{\Omega}_{1,9} = \hat{M} H_2, \quad \hat{\Omega}_{1,10} = \gamma_0 B X_1, \\
\hat{\Omega}_{1,11} &= (1 - \gamma_0) B X_1, \quad \hat{\Omega}_{1,12} = DX_1, \quad \hat{\Omega}_{1,13} = Y_2, \quad \hat{\Omega}_{1,14} = X_1^T + I + \\
\hat{\Omega}_{1,15} &= -X_1^T, \quad \hat{\Omega}_{2,2} = -e^{-\alpha \sigma} \hat{Q}_1 - \hat{R}_3, \quad \hat{\Omega}_{2,3} = \hat{R}_3, \quad \hat{\Omega}_{2,4} = \\
\hat{\Omega}_{2,5} &= \hat{\Omega}_{2,6} = \hat{\Omega}_{2,7} = \hat{\Omega}_{2,8} = \hat{\Omega}_{2,9} = \hat{\Omega}_{2,10} = \hat{\Omega}_{2,11} = \hat{\Omega}_{2,12} = \hat{\Omega}_{2,13} = \\
\hat{\Omega}_{2,14} &= \hat{\Omega}_{2,15} = 0, \quad \hat{\Omega}_{3,3} = -e^{-\alpha \tau_0} \hat{R}_1 + \rho_1 e^{-\alpha \tau} \hat{R}_1 - 2 \hat{R}_3 - \hat{\Lambda}_2 G_1, \\
\hat{\Omega}_{3,4} &= \hat{\Omega}_{3,5} = \hat{\Omega}_{3,6} = \hat{\Omega}_{3,7} = \hat{\Omega}_{3,8} = \hat{\Omega}_{3,9} = 0, \quad \hat{\Omega}_{3,10} = \hat{\Lambda}_2 G_2, \\
\hat{\Omega}_{3,11} &= \hat{\Omega}_{3,12} = \hat{\Omega}_{3,13} = \hat{\Omega}_{4,14} = \hat{\Omega}_{3,15} = 0, \quad \hat{\Omega}_{4,4} = -e^{-\alpha \tau_0} \hat{Q}_2 - \\
\hat{\Omega}_{4,12} &= \hat{\Omega}_{4,13} = \hat{\Omega}_{4,14} = \hat{\Omega}_{4,15} = 0, \quad \hat{\Omega}_{5,5} = -e^{-\alpha \tau} \hat{S}_1 + \rho_2 e^{-\alpha \tau} \hat{S}_1 - \\
2 \hat{S}_3 - \hat{\Lambda}_3 G_1, \quad \hat{\Omega}_{5,6} = \hat{\Omega}_{5,7} = \hat{\Omega}_{5,8} = \hat{\Omega}_{5,9} = \hat{\Omega}_{5,10} = 0, \quad \hat{\Omega}_{5,11} = \\
\hat{\Omega}_{5,12} = \hat{\Omega}_{5,13} = \hat{\Omega}_{5,14} = \hat{\Omega}_{5,15} = 0, \quad \hat{\Omega}_{6,6} = -e^{-\alpha \tau} \hat{Q}_3 - \hat{S}_5, \\
\hat{\Omega}_{6,7} &= \hat{\Omega}_{6,8} = \hat{\Omega}_{6,9} = \hat{\Omega}_{6,10} = \hat{\Omega}_{6,11} = \hat{\Omega}_{6,12} = \hat{\Omega}_{6,13} = \hat{\Omega}_{6,14} = \\
\hat{\Omega}_{6,15} &= 0, \quad \hat{\Omega}_{7,7} = -\hat{U}, \quad \hat{\Omega}_{7,8} = \hat{\Omega}_{7,9} = \hat{\Omega}_{7,10} = \hat{\Omega}_{7,11} = \hat{\Omega}_{7,12} = \\
\hat{\Omega}_{7,13} &= \hat{\Omega}_{7,14} = \hat{\Omega}_{7,15} = 0, \quad \hat{\Omega}_{8,8} = \hat{R}_2 + \hat{S}_2 - \hat{\Lambda}_1, \quad \hat{\Omega}_{8,9} = \hat{\Omega}_{8,10} = \\
\hat{\Omega}_{8,11} &= \hat{\Omega}_{8,12} = \hat{\Omega}_{8,13} = \hat{\Omega}_{8,14} = \hat{\Omega}_{8,15} = 0, \quad \hat{\Omega}_{9,9} = \frac{1}{\alpha} \sigma (e^{\alpha \sigma} - 1) \hat{Z} - \\
\hat{\Omega}_{9,10} &= \hat{\Omega}_{9,11} = \hat{\Omega}_{9,12} = \hat{\Omega}_{9,13} = \hat{\Omega}_{9,14} = \hat{\Omega}_{9,15} = 0, \quad \hat{\Omega}_{10,10} = \\
\end{align*}
\]
Theorem 3.2. This completes the proof. □

Remark 1. It is noted that Theorem 3.2 depends on all the delay constants. But in many practical cases, the time-varying delay is often continuous but not derivable, so the information of the time-varying delay derivatives is difficult to obtain. At present, the time-varying delays \( \tau(t) \) and \( \sigma(t) \) satisfy \( 0 \leq \tau \leq \tau(t) \leq \mathcal{T} \) and \( 0 \leq \sigma(t) \leq \sigma \), respectively. By using the similar proof method of Theorem 3.1 and 3.2, we obtain the corresponding delay-distribution-dependent, but the delay derivative independent synchronization criteria.

Theorem 3.3. Under Assumptions 1–4, the master system (3) and the slave system (4) are globally exponentially synchronized in the mean square if there exist positive-definite matrices \( P, Q_s \) (s = 1, 2, 3), \( R, \dot{S}, \dot{T}, \dot{Z}, X_1, X_4, X_5 \in \mathbb{R}^{n \times n}, \)
diagonal positive-definite matrices $\hat{U}$, $\hat{A}_k$ ($k = 1, 2, 3$), $\hat{M} \in \mathbb{R}^{n \times n}$ and matrices $Y_1 \in \mathbb{R}^{n \times n}$, $Y_2 \in \mathbb{R}^{n \times n}$ such that the following LMI hold:

$$
\Phi = \begin{bmatrix}
\hat{\xi}_{12} & \hat{\xi}_{13} \\
* & \hat{\xi}_{22} \\
* & * & \hat{\xi}_{33}
\end{bmatrix} < 0,
$$

(35)

where $\hat{\xi}_{12}$, $\hat{\xi}_{13}$ follows the same definitions as those in Theorem 3.2, $\hat{\Phi}_{11} = \left(\hat{\Phi}_{1,..}\right)_{15 \times 15}$ and

\begin{align*}
\hat{\Phi}_{1,1} &= \alpha \hat{P} + Y_1 - CX_1 + Y_1^T X_1^T C^T + \hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 + \alpha^2 \hat{T} - \\
&\hat{U} F_1 - \hat{A}_1 G_1 - \hat{M} H_1, \\
\hat{\Phi}_{1,2} &= \hat{\Phi}_{1,3}, \\
\hat{\Phi}_{1,4} &= \hat{\Phi}_{1,5}, \\
\hat{\Phi}_{1,6} &= 0, \\
\hat{\Phi}_{1,7} &= A X_1 + \hat{U} F_2, \\
\hat{\Phi}_{1,8} &= \hat{A}_1 G_2, \\
\hat{\Phi}_{1,9} &= \hat{M} H_2, \\
\hat{\Phi}_{1,10} &= \gamma_0 B X_1, \\
\hat{\Phi}_{1,11} &= (1 - \gamma_0) B X_1, \\
\hat{\Phi}_{1,12} &= D X_1, \\
\hat{\Phi}_{1,13} &= Y_2, \\
\hat{\Phi}_{1,14} &= X_1^T + I + X_1^T G, \\
\hat{\Phi}_{1,15} &= -X_1^T, \\
\hat{\Phi}_{2,2} &= -e^{-\alpha \tau} \hat{Q}_1 - \hat{R}, \\
\hat{\Phi}_{2,3} &= \hat{R}, \\
\hat{\Phi}_{2,4} &= \hat{\Phi}_{2,5}, \\
\hat{\Phi}_{2,6} &= \hat{\Phi}_{2,7}, \\
\hat{\Phi}_{2,8} &= \hat{\Phi}_{2,9}, \\
\hat{\Phi}_{2,10} &= \hat{\Phi}_{2,11}, \\
\hat{\Phi}_{2,12} &= \hat{\Phi}_{2,13}, \\
\hat{\Phi}_{2,14} &= \hat{\Phi}_{2,15} = 0, \\
\hat{\Phi}_{3,3} &= -2\hat{R} - \hat{A}_2 G_1, \\
\hat{\Phi}_{3,4} &= \hat{R}, \\
\hat{\Phi}_{3,5} &= \hat{\Phi}_{3,6} = \\
\hat{\Phi}_{3,7} &= \hat{\Phi}_{3,8}, \\
\hat{\Phi}_{3,9} &= 0, \\
\hat{\Phi}_{3,10} &= \hat{A}_2 G_2, \\
\hat{\Phi}_{3,11} &= \hat{\Phi}_{3,12} = \hat{\Phi}_{3,13} = \\
\hat{\Phi}_{3,14} &= \hat{\Phi}_{3,15} = 0, \\
\hat{\Phi}_{4,4} &= -e^{-\alpha \tau} \hat{Q}_2 - \hat{R} - \hat{S}, \\
\hat{\Phi}_{4,5} &= \hat{S}, \\
\hat{\Phi}_{4,6} &= \\
\hat{\Phi}_{4,7} &= \hat{\Phi}_{4,8}, \\
\hat{\Phi}_{4,9} &= \hat{\Phi}_{4,10}, \\
\hat{\Phi}_{4,11} &= \hat{\Phi}_{4,12} = \hat{\Phi}_{4,13} = \hat{\Phi}_{4,14} = \\
\hat{\Phi}_{4,15} &= 0, \\
\hat{\Phi}_{5,5} &= -2\hat{S} - \hat{A}_3 G_1, \\
\hat{\Phi}_{5,6} &= \hat{S}, \\
\hat{\Phi}_{5,7} &= \hat{\Phi}_{5,8} = \hat{\Phi}_{5,9} = \\
\hat{\Phi}_{5,10} &= 0, \\
\hat{\Phi}_{5,11} &= \hat{A}_3 G_2, \\
\hat{\Phi}_{5,12} &= \hat{\Phi}_{5,13} = \hat{\Phi}_{5,14} = \hat{\Phi}_{5,15} = 0, \\
\hat{\Phi}_{6,6} &= -e^{-\alpha \tau} \hat{Q}_3 - \hat{S}, \\
\hat{\Phi}_{6,7} &= \hat{\Phi}_{6,8} = \hat{\Phi}_{6,9} = \hat{\Phi}_{6,10} = \hat{\Phi}_{6,11} = \\
\hat{\Phi}_{6,12} &= \hat{\Phi}_{6,13} = \hat{\Phi}_{6,14} = \hat{\Phi}_{6,15} = 0, \\
\hat{\Phi}_{7,7} &= -\hat{U}, \\
\hat{\Phi}_{7,8} &= \hat{\Phi}_{7,9} = \\
\hat{\Phi}_{7,10} &= \hat{\Phi}_{7,11} = \hat{\Phi}_{7,12} = \hat{\Phi}_{7,13} = \hat{\Phi}_{7,14} = \hat{\Phi}_{7,15} = 0, \\
\hat{\Phi}_{8,8} &= -\hat{A}_1, \\
\hat{\Phi}_{8,9} &= \hat{\Phi}_{8,10} = \hat{\Phi}_{8,11} = \hat{\Phi}_{8,12} = \hat{\Phi}_{8,13} = \hat{\Phi}_{8,14} = \hat{\Phi}_{8,15} = 0, \\
\hat{\Phi}_{9,9} &= \frac{1}{\alpha} \sigma (e^{\alpha \tau} - 1) \hat{Z} - \hat{M}, \\
\hat{\Phi}_{9,10} &= \hat{\Phi}_{9,11} = \hat{\Phi}_{9,12} = \hat{\Phi}_{9,13} = \hat{\Phi}_{9,14} = \\
\hat{\Phi}_{9,15} &= 0, \\
\hat{\Phi}_{10,10} &= -\hat{A}_2, \\
\hat{\Phi}_{10,11} &= \hat{\Phi}_{10,12} = \hat{\Phi}_{10,13} = \hat{\Phi}_{10,14} = \\
\hat{\Phi}_{10,15} &= 0, \\
\hat{\Phi}_{11,11} &= -\hat{A}_3, \\
\hat{\Phi}_{11,12} &= \hat{\Phi}_{11,13} = \hat{\Phi}_{11,14} = \hat{\Phi}_{11,15} = 0, \\
\hat{\Phi}_{12,12} &= -\hat{Z}, \\
\hat{\Phi}_{12,13} &= \hat{\Phi}_{12,14} = \hat{\Phi}_{12,15} = 0, \\
\hat{\Phi}_{13,13} &= -X_1^T, \\
\hat{\Phi}_{13,14} &= -X_1^T, \\
\hat{\Phi}_{14,14} &= -2(\varepsilon + 1) I, \\
\hat{\Phi}_{14,15} &= 2\varepsilon I, \\
\hat{\Phi}_{15,15} &= -2\varepsilon I, \\
\Phi_{22} &= \Phi_{33} = \text{diag} \begin{bmatrix} -X_1 & -X_2 \end{bmatrix}.
\end{align*}

Moreover, the feedback controller gain matrix can be designed by $K_1 = Y_1 X_1^{-1}$ and $K_2 = Y_2 X_2^{-1}$.

Proof. We choose the following Lyapunov-Krasovskii functional:

$$
V(t, e(t)) = \sum_{i=1}^{6} V_i(t, e(t)),
$$

where $V_i(t, e(t))$ are chosen to satisfy

$$
\dot{V}_i(t, e(t)) < 0 \quad \forall t \in [0, +\infty), \quad e(t) \in \mathbb{R}^n.
$$

By applying the aforementioned LMIs and using the Lyapunov-Krasovskii functional, we can show that the error system is globally exponentially stable.
where

\[ V_1(t, e(t)) = e^{\alpha t} e^T(t) P e(t), \]
\[ V_2(t, e(t)) = \int_{t-\tau}^t e^{\alpha s} e^T(s) Q_1 e(s) ds + \int_{t-\tau}^t e^{\alpha s} e^T(s) Q_2 e(s) ds + \int_{t-\tau}^t e^{\alpha s} e^T(s) Q_3 e(s) ds, \]
\[ V_3(t, e(t)) = (\tau_0 - \tau) \int_{-\tau_0}^{-t} \int_{t+\theta}^t e^{\alpha (s-\theta)} e^T(s) R e(s) ds d\theta, \]
\[ V_4(t, e(t)) = (\tau - \tau_0) \int_{-\tau_0}^{-t} \int_{t+\theta}^t e^{\alpha (s-\theta)} e^T(s) S e(s) ds d\theta, \]
\[ V_5(t, e(t)) = \sigma \int_{t-\sigma}^t (s - t + \sigma) e^{\alpha s} e^T(s) T e(s) ds, \]
\[ V_6(t, e(t)) = \sigma \int_{-\sigma}^0 \int_{t+\theta}^t e^{\alpha (s-\theta)} h^T(e(s)) Z h(e(s)) ds d\theta. \]

Then the proof is similar to Theorem 3.1 and 3.2, and it is omitted here. \(\square\)

In this case, take \(D = 0\), then we can obtain from Theorem 3.2 and 3.3 that the two following corollaries.

**Corollary 1.** Under Assumptions 1–4, the master system (3) and the slave system (4) are globally exponentially synchronized in the mean square if there exist positive-definite matrices \(\hat{P}, \hat{Q}_s, \hat{R}_s, \hat{S}_s, X_s (s = 1, 2, 3) \in \mathbb{R}^{n \times n}\), diagonal positive-definite matrices \(\hat{U}, \hat{\lambda}_k (k = 1, 2, 3) \in \mathbb{R}^{n \times n}\) and matrix \(Y_1 \in \mathbb{R}^{n \times n}\) such that the following LMI hold:

\[
\Theta = \begin{bmatrix}
    \Theta_{11} & \Theta_{12} & \Theta_{13} \\
    * & \hat{Z}_{22} & 0 \\
    * & * & \hat{Z}_{33}
\end{bmatrix} \leq 0,
\]

where \(\hat{Z}_{22}, \hat{Z}_{33}\) follows the same definitions as those in Theorem 3.2, \(\Theta_{11} = (\hat{\Theta}_{11})_{11 \times 11}\) and

\[
\hat{\Theta}_{11} = \alpha \hat{P} + Y - CX_1 + X_1^T C^T + \hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 + \hat{R}_1 + \hat{S}_1 - \hat{U} F_1 - \hat{\lambda}_1 G_1 - M H_1, \quad \hat{\Theta}_{1,2} = \hat{\Theta}_{1,3} = \hat{\Theta}_{1,4} = \hat{\Theta}_{1,5} = \hat{\Theta}_{1,6} = 0, \quad \hat{\Theta}_{1,7} = AX_1 + \hat{U} F_2, \quad \hat{\Theta}_{1,8} = \hat{\lambda}_1 G_2, \quad \hat{\Theta}_{1,9} = \gamma_0 BX_1, \quad \hat{\Theta}_{1,10} = (1 - \gamma_0) BX_1, \hat{\Theta}_{1,11} = I + X_1^T \Gamma, \quad \hat{\Theta}_{2,2} = -e^{-\alpha T} \hat{Q}_1 - \hat{R}_3, \quad \hat{\Theta}_{2,3} = \hat{R}_3, \quad \hat{\Theta}_{2,4} = \hat{\Theta}_{2,5} = \hat{\Theta}_{2,6} = \hat{\Theta}_{2,7} = \hat{\Theta}_{2,8} = \hat{\Theta}_{2,9} = \hat{\Theta}_{2,10} = \hat{\Theta}_{2,11} = 0, \quad \hat{\Theta}_{3,3} = -e^{-\alpha T} \hat{R}_1 + \rho_1 e^{-\alpha T} \hat{R}_1 - 2 \hat{R}_3 - \hat{\lambda}_2 G_2, \quad \hat{\Theta}_{3,4} = \hat{R}_3, \quad \hat{\Theta}_{3,5} = \hat{\Theta}_{3,6} = \hat{\Theta}_{3,7} = \hat{\Theta}_{3,8} = 0, \quad \hat{\Theta}_{3,9} = \hat{\lambda}_2 G_2, \quad \hat{\Theta}_{3,10} = \hat{\Theta}_{3,11} = 0, \quad \hat{\Theta}_{4,4} = -\hat{R}_3 - e^{-\alpha T} \hat{Q}_2 - \hat{S}_3, \quad \hat{\Theta}_{4,5} = \hat{\Theta}_{4,6} = \hat{\Theta}_{4,7} = \hat{\Theta}_{4,8} = \hat{\Theta}_{4,9} = \hat{\Theta}_{4,10} = \hat{\Theta}_{4,11} = 0, \quad \hat{\Theta}_{5,5} = -e^{-\alpha T} \hat{S}_1 + \rho_2 e^{-\alpha T} \hat{S}_1 - 2 \hat{S}_3 - \hat{\lambda}_3 G_1, \quad \hat{\Theta}_{5,6} = \hat{\Theta}_{5,7} = \hat{\Theta}_{5,8} = \hat{\Theta}_{5,9} = 0, \quad \hat{\Theta}_{5,10} = \rho_2 e^{-\alpha T} \hat{S}_1 - 2 \hat{S}_3 - \hat{\lambda}_3 G_1, \quad \hat{\Theta}_{5,6} = \hat{\Theta}_{5,7} = \hat{\Theta}_{5,8} = \hat{\Theta}_{5,9} = 0, \quad \hat{\Theta}_{5,10} = \rho_2 e^{-\alpha T} \hat{S}_1 - 2 \hat{S}_3 - \hat{\lambda}_3 G_1. \]
Corollary 2. Under Assumptions 1–4, the master system (3) and the slave system (4) are globally exponentially synchronized in the mean square if there exist positive-definite matrices $\hat{P}_1, \hat{Q}_s (s = 1, 2, 3)$, $\hat{R}$, $\hat{S}$, $X_1$, $X_4$, $X_5 \in \mathbb{R}^{n \times n}$, diagonal positive-definite matrices $\hat{U}$, $\hat{\Lambda}_k (k = 1, 2, 3) \in \mathbb{R}^{n \times n}$ and matrix $Y_1 \in \mathbb{R}^{n \times n}$ such that the following LMI hold:

\[
\Theta_{12} = \begin{bmatrix}
\sqrt{\tau_1} (Y_1 - CX_1) & 0 & 0 & 0 & 0 & \sqrt{\tau_1} AX_1 & 0 & \sqrt{\tau_1 \gamma_0} BX_1 \\
\sqrt{\tau_2} (Y_1 - CX_1) & 0 & 0 & 0 & 0 & \sqrt{\tau_2} AX_1 & 0 & \sqrt{\tau_2 \gamma_0} BX_1 \\
\sqrt{\tau_1} (1 - \gamma_0) BX_1 & \sqrt{\tau_1 I} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{\tau_2} (1 - \gamma_0) BX_1 & \sqrt{\tau_2 I} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T,
\]

\[
\Theta_{13} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T.
\]

Moreover, the feedback controller gain matrix can be designed by $K_1 = Y_1 X_1^{-1}$.

4. Numerical examples. To illustrate the effectiveness of the obtained results in this paper, consider the two following numerical examples.
Example 1. Consider the neural networks (3) and (4) with the following parameters:

\[
A = \begin{bmatrix} 1 + \frac{\pi}{4} & 20 \\ 0.1 & 1 + \frac{\pi}{4} \end{bmatrix}, \quad B = \begin{bmatrix} -1.3\sqrt{\pi} / 4 & 0.1 \\ 0.1 & -1.3\sqrt{\pi} / 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
D = \begin{bmatrix} 1 - \frac{\sqrt{\pi}}{4} \\ 0 & 1 - \frac{\sqrt{\pi}}{4} \end{bmatrix}, \quad J = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.
\]

The neuron activation functions

\[
\tilde{f}(x(t)) = \tilde{g}(x(t)) = \tilde{h}(x(t)) = \begin{bmatrix} 0.5 \left( |x_1(t) + 1| - |x_1(t) - 1| \right) \\ 0.5 \left( |x_2(t) + 1| - |x_2(t) - 1| \right) \end{bmatrix},
\]

the time-varying transmission delay \( \tau(t) = 0.5 (1 + |\cos(3t)|) \), distributed time-varying delay \( \sigma(t) = 0.2 |\sin(t)| \) and disturbance \( \delta(e(t)) = 0.1 \tanh(e(t)) \). For the given value \( \gamma_0 = 0.25 \), and time delay bounds \( \tau = 0.5, \quad \tau_0 = 0.85, \quad \tau = 1, \quad \sigma = 0.2 \). Fig.1 (a) shows the chaotic behavior of neural networks (3) with the initial condition \((x_1(s), x_2(s))^T = (0.4, 0.4)^T\) for \(-1 \leq s \leq 0\). In the absence of control input \( u(t) \), Fig.1 (b) depicts the chaotic behavior of neural networks (4) with the initial condition \((y_1(s), y_2(s))^T = (0.6, 0.6)^T\) for \(-1 \leq s \leq 0\).

![Fig. 1. (a) Chaotic behavior of the neural networks (3). (b) Chaotic behavior of the neural networks (4) without control input u(t).](image)

Clearly, the neural networks (3) and (4) satisfy Assumption 1 with

\[
F_1 = G_1 = H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = G_2 = H_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.
\]

Let \( \alpha = 0.5, \quad \varepsilon = 100, \quad \rho_1 = 1.5, \quad \rho_2 = 1.5. \) By the LMI toolbox in Matlab, we obtain the feasible solutions of the inequalities (34) as follows:

\[
\hat{P} = \begin{bmatrix} 4.2967 & -0.0763 \\ -0.0763 & 4.2025 \end{bmatrix}, \quad \hat{Q}_1 = \begin{bmatrix} 7.7416 & 0.0894 \\ 0.0894 & 7.9103 \end{bmatrix}, \\
\hat{Q}_2 = \begin{bmatrix} 4.0048 & -0.0840 \\ -0.0840 & 4.1832 \end{bmatrix}, \quad \hat{Q}_3 = \begin{bmatrix} 6.9709 & 0.0573 \\ 0.0573 & 7.1936 \end{bmatrix},
\]
\[
\hat{R}_1 = \begin{bmatrix} 1.7330 & 0.0007 \\ 0.0007 & 1.7025 \end{bmatrix}, \quad \hat{R}_2 = \begin{bmatrix} 1.6066 & -0.0141 \\ -0.0141 & 1.5934 \end{bmatrix}, \quad \hat{R}_3 = \begin{bmatrix} 2.1910 & -0.0247 \\ -0.0247 & 2.1811 \end{bmatrix}, \quad \hat{S}_1 = \begin{bmatrix} 1.6410 & 0.0011 \\ 0.0011 & 1.6124 \end{bmatrix}, \quad \hat{S}_2 = \begin{bmatrix} 1.4814 & -0.0024 \\ -0.0024 & 1.4735 \end{bmatrix}, \quad \hat{S}_3 = \begin{bmatrix} 2.2193 & -0.0350 \\ -0.0350 & 2.2020 \end{bmatrix}, \\
\hat{T} = \begin{bmatrix} 1.7335 & -0.0005 \\ -0.0005 & 1.7324 \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} 1.6918 & -0.0008 \\ -0.0008 & 1.6866 \end{bmatrix}, \\
X_1 = \begin{bmatrix} 0.2121 & -0.0363 \\ -0.0363 & 0.1943 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2.1502 & 0.0117 \\ 0.0117 & 2.0969 \end{bmatrix}, \\
X_3 = \begin{bmatrix} 2.1289 & 0.0083 \\ 0.0083 & 2.0574 \end{bmatrix}, \quad \hat{\Upsilon} = \begin{bmatrix} 1.5972 & 0 \\ 0 & 1.6262 \end{bmatrix}, \\
\hat{A}_1 = \begin{bmatrix} 6.8790 & 0 \\ 0 & 6.7710 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 1.1479 & 0 \\ 0 & 1.1478 \end{bmatrix}, \\
\hat{A}_3 = \begin{bmatrix} 1.2110 & 0 \\ 0 & 1.2270 \end{bmatrix}, \quad \hat{M} = \begin{bmatrix} 1.6509 & 0 \\ 0 & 1.6572 \end{bmatrix}, \\
Y_1 = \begin{bmatrix} -1.4581 & -0.0125 \\ -0.0132 & -1.4489 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -1.7320 & -4.0951 \\ -2.8721 & 2.6137 \end{bmatrix}.
\]

Then the controller gain matrices \( K_1 \) and \( K_2 \) are designed as follows:
\[
K_1 = \begin{bmatrix} -7.0903 & -1.2603 \\ -1.3827 & -7.7153 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -12.1619 & -23.3483 \\ -11.6103 & 11.2828 \end{bmatrix}.
\]

In order to show the effectiveness of the proposed LMI based criteria for the mean square exponential synchronization of the neural networks (3) and (4), the following simulation results are presented in Fig.2. Fig.2(a) depicts the chaotic behavior of the slave system (4) with the initial condition \((y_1(s), y_2(s))^T = (0.6, 0.6)^T\) for \(-1 \leq s \leq 0\). Fig.2(b) gives the temporal evolution of \(x_1(t), y_1(t)\). Fig.2(c) presents the temporal evolution of \(x_2(t), y_2(t)\). Fig.2(d) provides the synchronization error of the neural networks (3) and (4). It is obviously seem from Fig.2 that the master system (3) and the slave system (4) are globally exponentially synchronized in the mean square.

**Example 2.** Consider the neural networks (3) and (4) with the following parameters:
\[
A = \begin{bmatrix} 2 & -0.08 \\ 5 & 2.8 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 \\ -0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
D = \begin{bmatrix} -0.3 & 0.03 \\ 0.1 & -0.2 \end{bmatrix}, \quad J = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}.
\]
The neuron activation functions \( \tilde{f}(x(t)) = \tilde{g}(x(t)) = \tilde{h}(x(t)) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix} \), the time-varying transmission delay \( \tau(t) = \frac{1}{t^2} \), distributed time-varying delay \( \sigma(t) = 0.5|\cos(t)| \), and disturbance \( \delta(e(t)) = 0.05(|e(t) + 1| - |e(t) - 1|) \). For the given value \( \gamma_0 = 0.2 \), and time delay bounds \( \underline{\tau} = 0.5, \gamma_0 = 0.8, \varpi = 1, \sigma = 0.5 \). Fig.3(a) shows the chaotic behavior of neural networks (3) with the initial condition \((x_1(s), x_2(s))^T = (0.4, 0.6)^T\) for \(-1 \leq s \leq 0\). In the absence of control input \( u(t) \),
Fig. 2. (a) Chaotic behavior of the neural networks (4). (b)-(c) State trajectories of the neural networks (3) and (4). (d) Synchronization error trajectories of the state variables between the neural networks (3) and (4).

Fig.3(b) depicts the chaotic behavior of neural networks (4) with the initial condition \((y_1(s), y_2(s))^T = (0.5, 0.7)^T\) for \(-1 \leq s \leq 0\).

Fig. 3. (a) Chaotic behavior of the neural networks (3). (b) Chaotic behavior of the neural networks (4) without control input \(u(t)\).
Clearly, the neural networks (3) and (4) satisfy Assumption 1 with
\[ F_1 = G_1 = H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = G_2 = H_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}. \]

Let \( \alpha = 0.5, \varepsilon = 200. \) By the LMI toolbox in Matlab, we obtain the feasible solutions of the inequalities (35) as follows:
\[
\begin{align*}
\hat{P} &= \begin{bmatrix} 9.4652 & 0.0068 \\ 0.0068 & 9.4402 \end{bmatrix}, \quad \hat{Q}_1 = \begin{bmatrix} 8.7910 & -0.0054 \\ -0.0054 & 8.7698 \end{bmatrix}, \\
\hat{Q}_2 &= \begin{bmatrix} 6.8848 & -0.0002 \\ -0.0002 & 6.8707 \end{bmatrix}, \quad \hat{Q}_3 = \begin{bmatrix} 7.5149 & -0.0076 \\ -0.0076 & 7.4956 \end{bmatrix}, \\
\hat{R} &= \begin{bmatrix} 2.4186 & -0.0002 \\ -0.0002 & 2.4161 \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} 2.4315 & 0.0009 \\ 0.0009 & 2.4296 \end{bmatrix}, \\
\hat{T} &= \begin{bmatrix} 1.4677 & -0.0004 \\ -0.0004 & 1.4666 \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} 1.3295 & -0.0007 \\ -0.0007 & 1.3273 \end{bmatrix}, \\
X_1 &= \begin{bmatrix} 0.1065 & 0.1073 \\ 0.1073 & 0.0765 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 2.3186 & 0.0009 \\ 0.0009 & 2.3116 \end{bmatrix}, \\
X_5 &= \begin{bmatrix} 2.2059 & -0.0015 \\ -0.0015 & 2.2007 \end{bmatrix}, \quad \hat{U} = \begin{bmatrix} 1.4840 & 0 \\ 0 & 1.4844 \end{bmatrix}, \\
\hat{\Lambda}_1 &= \begin{bmatrix} 1.4838 & 0 \\ 0 & 1.4833 \end{bmatrix}, \quad \hat{\Lambda}_2 = \begin{bmatrix} 1.2509 & 0 \\ 0 & 1.2499 \end{bmatrix}, \\
\hat{\Lambda}_3 &= \begin{bmatrix} 1.2145 & 0 \\ 0 & 1.2133 \end{bmatrix}, \quad \hat{M} = \begin{bmatrix} 1.7165 & 0 \\ 0 & 1.7161 \end{bmatrix}, \\
Y_1 &= \begin{bmatrix} 0.2855 & -0.0001 \\ -0.0002 & 0.2853 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.0234 & 0.0796 \\ 0.0440 & 0.0716 \end{bmatrix}.
\end{align*}
\]

Then the controller gain matrices \( K_1 \) and \( K_2 \) are designed as follows:
\[
K_1 = \begin{bmatrix} -6.4926 & 9.1054 \\ 9.1011 & -9.0350 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 2.0077 & -1.7751 \\ 1.2851 & -0.8659 \end{bmatrix}.
\]

To demonstrate the effectiveness of the proposed synchronization criteria of the neural networks (3) and (4), the simulation results are presented in Figs.4. Fig.4(a) depicts the chaotic behavior of neural networks (4) with the initial condition \((y_1(s), y_2(s))^T = (0.5, 0.7)^T\) for \(-1 \leq s \leq 0\). Fig. 4(b) gives the temporal evolution of \(x_1(t), y_1(t)\). Fig. 4(c) presents the temporal evolution of \(x_2(t), y_2(t)\). Fig.4(d) provides the synchronization error of the neural networks (3) and (4). It is obviously seen from Fig.4 that the master system (3) and the slave system (4) are globally exponentially synchronized in the mean square.

5. Conclusions. In this paper, we have dealt with the mean-square delay-distribution-dependent exponential synchronization for chaotic neural networks with mixed random time-varying delays. By constructing an appropriate Lyapunov-Krasovskii functional and utilizing the Jensen’s integral inequality, several delay-distribution-dependent sufficient conditions have been derived in the form of LMIs. Furthermore, the design of the control gain matrices have been successfully transformed into solving LMI. Two numerical examples have illustrated the feasibility and effectiveness of the theoretical results.
Fig. 4. (a) Chaotic behavior of the neural networks (4). (b)-(c) State trajectories of the neural networks (3) and (4). (d) Synchronization error trajectories of the state variables between the neural networks (3) and (4).

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