The information capacity of entanglement-assisted continuous variable quantum measurement

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Abstract

The present paper is devoted to the investigation of the entropy reduction and entanglement-assisted classical capacity (information gain) of continuous variable measurements. These quantities are computed explicitly for multimode Gaussian measurement channels. For this we establish a fundamental property of the entropy reduction of a measurement: under a restriction on the second moments of the input state it is maximized by a Gaussian state (providing an analytical expression for the maximum). In the case of one mode, the gain of entanglement assistance is investigated in detail.

Keywords: quantum measurement, entropy reduction, entanglement-assisted capacity, continuous variable system, multimode Gaussian observable, gain of entanglement assistance

(Some figures may appear in colour only in the online journal)

1. Introduction

Continuous variable (CV) systems constitute one of the prospective platforms for implementation of quantum communication and computation protocols [1–3]. During the past few years, an important chapter of quantum information science—quantum Shannon theory—is being developed for CV systems, which requires mathematical tools of infinite-dimensional Hilbert space. The theory of various channel capacities and related entropic quantities was elaborated, in particular, for bosonic Gaussian channels (see e.g. [4] and references therein).

The notion of quantum channel presupposed quantumness of both the input and output systems, making necessary a separate treatment of quantum observables, which do not allow a simple reduction to quantum channels in the CV case (contrary to the discrete
finite-dimensional case). In particular, this fully applies to quantum bosonic Gaussian observables. Thus we are led to the study of quantum measurement channels, which map CV quantum input into CV classical output, and to computation of their information-processing and entropic characteristics.

An important quantity characterizing information-processing performance of the quantum measurement channel is its classical capacity [5–7]. The computation of the classical capacity for multi-mode quantum Gaussian measurement channels, based on the progress in the solution of the quantum Gaussian optimizer conjecture [8, 9], was recently developed in [10] under the assumption of global gauge symmetry (‘phase insensitivity’), and in [11] under certain more general ‘threshold condition’.

In the present paper we study another important characteristic of a CV measurement channel—the entropy reduction [12, 13] (see equation (3) below), which is strongly related to its quantum mutual information [14, 15] and to the entanglement-assisted classical capacity [7, 16, 17]. In finite dimensions related notions were studied by a number of authors under the names purification capacity, measurement strength, information gain of the measurement (see [18] where one can find also a detailed survey of the subject and further references). Thus the entropy reduction and the entanglement-assisted classical capacity of a quantum measurement are of considerable interest from various points of view in quantum Shannon theory.

By using results previously obtained in [16, 17], we derive here computable expressions for the entropy reduction and entanglement-assisted capacity of multimode Gaussian measurement channels. We prove that under a restriction on second moments, the entropy reduction of Gaussian observable is maximized by a Gaussian state, explicitly giving the value of the maximum. This fundamental property of the entropy reduction is parallel to a similar property of quantum mutual information for quantum Gaussian channels, however the proof is somewhat more intricate due to absence of the Schmidt decomposition and symmetry between parts of a composite hybrid (classical–quantum) system. As an application we consider in detail the case of one mode and study the gain of the entanglement-assisted vs unassisted classical capacities of the measurement channel. Our findings give another evidence of the remarkable fact that measurement channels—while being entanglement-breaking—can show unlimited gain of entanglement assistance in the classical capacity [7]. At this point we would like to add that recently a quantum communication scheme was proposed that utilizes pre-shared CV entanglement, and in principle can demonstrate theoretically predicted capacity enhancement for noisy quantum attenuator channel [19]. It would be worthwhile to investigate designs which can achieve a similar goal for entanglement-assisted quantum CV measurements.

The plan of the paper is as follows: in section 2 we recall the notion of measurement channel and its entropy reduction. In section 3 we briefly describe the protocol of entanglement-assisted measurement and summarize in theorem 1 relevant results from our papers [16, 17] concerning entanglement-assisted capacity. A detailed proof of the main result concerning the extremal property of the entropy reduction of Gaussian measurement channel is given in section 4 in the case of the global gauge symmetry; then the gain of entanglement assistance is demonstrated on the example of one mode in section 5. We have chosen to consider first the phase insensitive case because it is of special importance in applications while admitting relatively direct treatment and transparent description. Finally, the extension of the basic extremal property to the case of general Gaussian observables is outlined in section 6.
2. Entropy reduction of a measurement channel

Let \( \mathcal{H} \) be a separable Hilbert space of a quantum system, \( \mathcal{S}(\mathcal{H}) \) the set of all density operators (quantum states), and let \( (\Omega, \mathcal{F}, \mu) \) be a standard measurable space, where \( \mu \) is a \( \sigma \)-finite measure on the \( \sigma \)-algebra \( \mathcal{F} \).

A quantum observable with values in \( \Omega \) is a probability operator-valued measure (POVM) \( M = \{ M(A), A \in \mathcal{F} \} \) on \( (\Omega, \mathcal{F}) \). The probability distribution of observable \( M \) on the state \( \rho \in \mathcal{S}(\mathcal{H}) \) is given by the formula

\[
P_\rho(A) = \text{Tr} \rho M(A), \quad A \in \mathcal{F}.
\]

A measurement channel \( \mathcal{M} \) is an affine map \( \rho \rightarrow P_\rho(d\omega) \) of the convex set of quantum states \( \mathcal{S}(\mathcal{H}) \) into the set of probability distributions on \( \Omega \).

We will deal with the special class of observables which have bounded operator valued density

\[
m_\omega = \int_{\Omega} m(\omega) \mu(d\omega), \quad A \in \mathcal{F},
\]

where \( m(\omega) = V(\omega)^* V(\omega) \) and \( V(\omega) \) is a weakly measurable function with values in the algebra of bounded operators in \( \mathcal{H} \) such that

\[
\int_{\Omega} V(\omega)^* V(\omega) \mu(d\omega) = I,
\]

and the integral converges weakly. With any measurable factorization of \( m(\omega) \) one can associate an efficient instrument (see [13, 15]) defined by the probability distribution \( P_\rho(d\omega) \) with density

\[
p_\rho(\omega) = \text{Tr} \rho V(\omega)^* V(\omega)
\]

with respect to the measure \( \mu \), and by the family of posterior states

\[
\hat{\rho}(\omega) = \begin{cases}
p_\rho(\omega)^{-1} V(\omega) \rho V(\omega)^*, & \text{if } p_\rho(\omega) \neq 0; \\
\hat{\rho}_0, & \text{otherwise}
\end{cases}
\]

for any state \( \rho \in \mathcal{S}(\mathcal{H}) \) (\( \hat{\rho}_0 \) is a fixed state).

Following [13, 15], one defines the entropy reduction of the efficient instrument by the formula

\[
ER(\rho, \mathcal{M}) = H(\rho) - \int_{\Omega} p_\rho(\omega) H(\hat{\rho}(\omega)) \mu(d\omega),
\]

where \( H(\rho) = -\text{Tr} \rho \log \rho \), provided \( H(\rho) < \infty \). In [13] it was shown that the entropy reduction of an efficient instrument is nonnegative. In [15] the entropy reduction was related to quantum mutual information of the instrument and hence shown that it is a concave, subadditive, lower semicontinuous function of \( \rho \).

Let us make an important observation: the entropies of posterior states depend only on \( m(\omega) \) and not on the chosen measurable factorization \( m(\omega) = V(\omega)^* V(\omega) \) because the trace-class operator \( V(\omega) \rho V(\omega)^* = (V(\omega) \sqrt{\rho}) (V(\omega) \sqrt{\rho})^* \) has the same eigenvalues as \( (V(\omega) \sqrt{\rho})^* (V(\omega) \sqrt{\rho}) = \sqrt{\rho} m(\omega) \sqrt{\rho} \). Thus the entropy reduction (3) is uniquely defined for any observable \( M \) of the form (1), which justifies the notation \( ER(\rho, \mathcal{M}) \).
Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators and $\mathcal{S}(\mathcal{H})$ the Banach space of trace-class operators in $\mathcal{H}$. A hybrid classical-quantum (cq) system [20] is described by the von Neumann algebra $\mathcal{L} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mu, \mathcal{B}(\mathcal{H}))$, consisting of weakly measurable, essentially bounded functions $\chi(\omega), \omega \in \Omega$ with values in $\mathcal{B}(\mathcal{H})$. The elements of the predual space $\mathcal{L}_* = \mathcal{L}^1(\Omega, \mathcal{F}, \mu, \mathcal{S}(\mathcal{H}))$ are measurable functions $\varrho = \{\rho(\omega)\}$ with values in $\mathcal{S}(\mathcal{H})$, integrable with respect to the measure $\mu$. An element $\varrho = \{\rho(\omega)\} \in \mathcal{L}_*$, such that
\[
\rho(\omega) \geq 0 \; \text{(mod } \mu), \quad \int_\Omega \text{Tr} \rho(\omega) \mu(d\omega) = 1,
\]
is called cq-state. The partial c-state is the probability measure on $(\Omega, \mathcal{F})$, determined by the density $\rho(\omega) = \text{Tr} \rho(\omega)$ with respect to the measure $\mu(d\omega)$; the partial q-state is the density operator $\rho = \int_\Omega \rho(\omega) \mu(d\omega)$.

Let $\mathcal{M}$ be a measurement channel introduced above, then the relation $\varrho = \{V(\omega)\rho V(\omega)^*\}$ defines a cq-state. The map $\rho \rightarrow \mathcal{E}[\rho] = \{V(\omega)\rho V(\omega)^*\}$ is a channel with quantum input and cq-output.

3. Entanglement-assisted capacity of a measurement channel

In the ordinary (unassisted) measurement scenario there are two parties—the quantum system $A$ (the measured system) and the classical system $\Omega$ (the meter), and the measurement channel $\mathcal{M} : A \rightarrow \Omega$. In the case of infinite-dimensional $\mathcal{H}$ one usually introduces an energy constraint $\text{Tr} \rho H \leq E$ on the input states $\rho$ of the channel. Here $H$ is positive selfadjoint (in general unbounded) ‘energy operator’ (Hamiltonian) on the space $\mathcal{H}$ of the system, $E$ is a positive constant, and the trace is understood e.g. as in section 11.1 of [4]. A natural (but not the unique) measure of information-processing performance of the measurement is the energy-constrained classical capacity $C(\mathcal{M}, H, E)$ of the channel $\mathcal{M}$, see [17].

The protocol of entanglement-assisted classical communication via finite-dimensional quantum channel was introduced in [21, 22]. A modification of this protocol for quantum measurement channels, which requires the notion of hybrid cq system, was studied in [16, 17].

We give here a brief description of the entanglement-assisted measurement protocol including the resulting capacity formula (6) which is sufficient for our purposes. In this scenario the meter is a classical-quantum system $\Omega B$ where $B$ is its quantum part. The composite quantum system $AB$ is initially in a pure entangled state $\rho_{AB}$. The party $A$ performs encoding $x \rightarrow \mathcal{E}_x$ of the classical signal $x$, where $\mathcal{E}_x$ are operations on the measured system. Thereafter the measurement channel $\mathcal{M} : A \rightarrow \Omega$ is applied so that the meter $\Omega B$ is transformed into one of the cq-states $\mathcal{M} \circ \mathcal{E}_x \otimes \text{id}_B[\rho_{AB}]$. The goal is to extract the maximum information about $x$ basing on measurements in the hybrid system $\Omega B$. With the block coding, this procedure should be applied to the channel $\mathcal{M}^{(n)}$ whose input states satisfy the corresponding energy constraint. The asymptotic (as $n \rightarrow \infty$) capacity of this protocol is called the energy-constrained classical entanglement-assisted capacity $C_{ea}(\mathcal{M}, H, E)$ of the measurement channel $\mathcal{M}$. We refer to [16, 17] for explanation of the relevant details.

In the following theorem we summarize the relevant results from [4, 17] giving a convenient expression for $C_{ea}(\mathcal{M}, H, E)$ in terms of the entropy reduction.

**Theorem 1.** Let $\mathcal{M}$ be a measurement channel with observable of the form (1) such that
\[
\sup_{\rho, \text{Tr} \rho H \leq E} H_{\mathcal{A}}(\mathcal{M}(\rho)) < \infty,
\]
where \( H_{c}(\mathcal{M}(\rho)) \) is the classical differential entropy of the output probability density (2) of the channel.

Assume that the energy operator \( H \) satisfies the Gibbs condition

\[
\text{Tr} \exp(-\beta H) < \infty \quad \text{for all} \quad \beta > 0.
\]

Then the energy-constrained entanglement-assisted capacity is finite and is given by the formula

\[
C_{ea}(\mathcal{M}, H, E) = \max_{\rho : \text{Tr} \rho H \leq E} \text{ER}(\rho, \mathcal{M}).
\]

Moreover, if the channel \( \mathcal{M} \) is such that

\[
\sup_{\rho} \text{ER}(\rho, \mathcal{M}) = +\infty,
\]

then the maximum in (6) is achieved on a density operator \( \rho \), such that \( \text{Tr} \rho H = E \).

**Proof (Sketch).** By theorem 3 of [16], the conditions (4) and (5) imply

\[
C_{ea}(\mathcal{M}, H, E) = \sup_{\rho : \text{Tr} \rho H \leq E} \text{ER}(\rho, \mathcal{M}).
\]

Next notice that the quantum entropy \( H(\rho) \) is bounded and continuous on the set \( \{ \rho : \text{Tr} \rho H \leq E \} \) by lemma 11.8 of [4], provided the operator \( H \) satisfies the condition (5). This implies that \( \text{ER}(\rho, \mathcal{M}) \) is well-defined and also continuous by theorem 2 of [15]. The set \( \{ \rho : \text{Tr} \rho H \leq E \} \) is compact by lemma 11.5 of [4], hence the supremum of the entropy reduction is achieved on this set, and the formula (6) holds.

By using the concavity of the entropy reduction, the second statement can be proved similarly to the corresponding statement of proposition 11.26 of [4]. □

In the recent paper [11] it was shown that the condition (4) is fulfilled in the case of Gaussian measurement channel with the constraint given by an oscillator-system Hamiltonian (which fulfills (5)). Thus the relation (6) holds in this case to which we pass in the next section.

### 4. Gauge-covariant Gaussian measurements

In what follows \( \mathcal{H} \) will be the space of a strongly continuous irreducible representation of bosonic canonical commutation relations (CCR) (see e.g. [4, 23] for a detailed account) describing quantization of a linear classical system with \( s \) degrees of freedom such as finite number of physically relevant electromagnetic modes in a receiver’s cavity (see e.g. [3]). Let \( a_j, a_j^\dagger; j = 1, \ldots, s \) be the annihilation/creation operators of the modes, let \( z \in \mathbb{C}^s \) be a column vector with complex coordinates \( z_j, j = 1, \ldots, s \), and \( z^* \) denote Hermitian conjugate row vector. Then the CCR are conveniently written in terms of the unitary displacement operators \( D(z) = \exp \sum_{j=1}^s (a_j^\dagger z_j - z_j^* a_j) \), namely

\[
D(z)D(w) = \exp (-i \text{Im} z^* w) D(z + w), \quad z, w \in \mathbb{C}^s.
\]

The (global) gauge group acts as \( z \to e^{i\varphi} z, (\varphi \text{ is real phase}) \) in the space \( \mathbb{C}^s \), and via the unitary group \( \varphi \to U_\varphi = \exp (-i\varphi \mathcal{N}) \) in \( \mathcal{H} \) (here \( \mathcal{N} = \sum_{j=1}^s a_j^\dagger a_j \) is the total number operator), so that

\[
U_\varphi^* D(z) U_\varphi = D(e^{i\varphi} z).
\]
An operator $A$ is gauge-invariant if $U_\varphi A U_\varphi^* = A$ for all $\varphi$. A gauge-invariant Gaussian state has the quantum characteristic function [23]

$$\text{Tr} \rho_A D(w) = \exp \left[ -w^* \left( \Lambda + \frac{I_s}{2} \right) w \right],$$

(9)

where $\Lambda = \text{Tr} a \rho_A a^\dagger$ is the complex correlation matrix, satisfying $\Lambda \geq 0$. (We denote by $I_s$ the unit $s \times s$-matrix, as distinct from the unit operator $I$ in a Hilbert space). The case $\Lambda = 0$ in (9) corresponds to the vacuum state $\rho_0 = |0\rangle \langle 0|$. The coherent state vectors are $|z\rangle = D(z)|0\rangle$.

The displaced state $\rho_{\Lambda,z} = D(z)\rho_{\Lambda} D(z)^*$ has the quantum characteristic function

$$\text{Tr} \rho_{\Lambda,z} D(w) = \exp \left[ 2i \text{Im } z^* w - w^* \left( \Lambda + \frac{I_s}{2} \right) w \right].$$

(10)

We will use the $P$-representation in the case of nondegenerate $\Lambda$:

$$\rho_{\Lambda,z} \equiv D(z)\rho_{\Lambda} D(z)^*$$

$$= \int |w\rangle \langle w| \exp \left[ -(w - z)^* \Lambda^{-1} (w - z) \right] \frac{d^2 w}{\pi^s \det \Lambda}.$$  

(11)

The formula (10) remains valid for arbitrary correlation matrix $\Lambda \geq 0$, while (11) needs modification by introducing Gaussian measure on $C^s$ with zero mean and complex correlation matrix $\Lambda$. For the sake of clarity, we will deal with the case of nondegenerate $\Lambda$, while the resulting formulas remain valid for arbitrary $\Lambda \geq 0$.

In this section we will consider the gauge-covariant Gaussian observable (POVM) with values in $\Omega = C^s$ defined by

$$M(d^2z) = D(z)\rho_N D(z)^* \frac{d^2z}{\pi^s},$$

(12)

where $N \geq 0$ is the correlation matrix of the measurement noise (see [10] for more detail). The case $N = 0$ corresponds to the multimode heterodyne measurement. Put $\mu(dz) = \frac{d^2z}{2\pi}$, then observable (12) has the form (1) with $m(z) = D(z)\rho_N D(z)^*$ taking values in the space of trace-class operators. For any input state $\rho$ the output probability density (2) of the corresponding measurement channel $M$ is

$$p_\rho(z) = \text{Tr} \rho D(z)\rho_N D(z)^*.$$  

(13)

Choosing $V(z) = \sqrt{\rho_N} D(z)^*$, (which is a weakly continuous factorization of the density $m(z)$), we get the posterior states

$$\hat{\rho}(z) = p_\rho(z)^{-1} V(z) \rho V(z)^* = p_\rho(z)^{-1} \sqrt{\rho_N} D(z)^* \rho D(z) \sqrt{\rho_N},$$

(14)

and the entropy reduction is given by (3).

Assume that

$$H = \sum_{j,k=1}^{s} \epsilon_{jk} a_j^\dagger a_k$$

(15)

is a quadratic gauge-invariant oscillator-type Hamiltonian, where $\epsilon = \epsilon_{jk}$ is positive definite Hermitian matrix. In [11] we have shown that in the case of observable (12) and the
Hamiltonian (15) the conditions (4) and (5) are fulfilled, making the formula (6) for \( C_{ea}(\mathcal{M}; H, E) \) applicable.

Throughout this paper we use the fact that for any state \( \rho \) with finite second moments \( H(\rho) \) is finite (it is upperbounded by the entropy of the Gaussian state with the same second moments), hence the entropy reduction is well-defined.

**Proposition 1.** Let \( \mathcal{M} \) be the measurement channel corresponding to the observable (12). Then for any state \( \rho \) with finite second moments there is a gauge-invariant state \( \rho_{gi} \) such that

\[
ER(\rho, \mathcal{M}) \leq ER(\rho_{gi}, \mathcal{M}); \quad \text{Tr} \, \rho H = \text{Tr} \, \rho_{gi} H. \tag{16}
\]

The proof is similar to that of corollary 12.39 in [4]. Define the gauge-invariant state

\[
\rho_{gi} = \int_0^{2\pi} U^*_\varphi \rho U_\varphi \frac{d\varphi}{2\pi},
\]

then the second relation in (16) follows from the fact that \( U_\varphi H U^*_\varphi = H \), and the first one—from Jensen’s inequality relying upon nonnegativity, concavity and lower semicontinuity of \( ER(\rho, \mathcal{M}) \) [24].

By \( \mathcal{S}(\Lambda) \) we denote the set of all states which have finite second moments with the complex correlation matrix

\[
\left[ \text{Tr} \, a_j \rho a_k^\dagger \right]_{j, k = 1, \ldots, s} = \Lambda.
\]

If \( \rho \in \mathcal{S}(\Lambda) \), then \( \rho_{gi} \in \mathcal{S}(\Lambda) \) and it has zero first moments, and second moments such as \( \text{Tr} \, a_j \rho a_k^\dagger \) vanishing. This follows from the identities \( U^*_\varphi a_j U_\varphi = a_j e^{-i\varphi} \), \( U^*_\varphi a_j^\dagger U_\varphi = a_j^\dagger e^{i\varphi} \). Moreover the normal second moments such as \( \text{Tr} \, a_j \rho a_k^\dagger \) coincide for \( \rho \) and \( \rho_{gi} \). There is a unique gauge-invariant Gaussian state \( \rho_\Lambda \) in \( \mathcal{S}(\Lambda) \) and the normal second moments coincide for \( \rho \in \mathcal{S}(\Lambda) \) and \( \rho_\Lambda \).

We will study the following quantity

\[
ER(\mathcal{M}; \Lambda) = \sup_{\rho \in \mathcal{S}(\Lambda)} ER(\rho, \mathcal{M}). \tag{17}
\]

This quantity, which is interesting on its own, is of the main importance in computing the energy-constrained classical entanglement-assisted capacity of the measurement channel. Indeed, assume that the Hamiltonian is given by (15), so that the mean energy of the input state \( \rho \) is equal to

\[
\text{Tr} \, \rho H = \sum_{j, k = 1}^s \epsilon_{jk} \Lambda_{kj} = \text{Sp} \, \epsilon \Lambda,
\]

where \( \text{Sp} \) denotes trace of \( s \times s \)–matrices as distinct from the trace of operators. Then the energy constraint has the form \( \text{Sp} \, \epsilon \Lambda \leq E \), and according to (6) the energy-constrained entanglement-assisted classical capacity of the channel \( \mathcal{M} \) is

\[
C_{ea}(\mathcal{M}; H, E) = \max_{\Lambda: \text{Sp} \, \epsilon \Lambda \leq E} ER(\mathcal{M}; \Lambda). \tag{18}
\]

Given an explicit expression for \( ER(\mathcal{M}; \Lambda) \) such as (19) below, computation of the last supremum is a separate optimization problem. Moreover, from (19) it follows that
sup_\Lambda ER(\rho_\Lambda, \mathcal{M}) = +\infty$, which implies the condition (7) in theorem 1. Hence the maximum in (18) is attained on a $\Lambda$ satisfying $\mathcal{S}\rho = E$.

**Theorem 2.** Let $\mathcal{M}$ be the measurement channel corresponding to the observable (12). Then the maximum of entropy reduction $ER(\rho, \mathcal{M})$ on $\mathcal{S}(\Lambda)$ is attained on the gauge-invariant Gaussian state $\rho_\Lambda$. Moreover,

$$ER(\mathcal{M}; \Lambda) = Sp \ g(\Lambda) - Sp \ g(\tilde{N}),$$

where $g(x) = (x + 1) \log(x + 1) - x \log x$, and

$$\tilde{N} = \sqrt{N(N + L_z)^{-1}} \Lambda(\Lambda + N + L_z)^{-1} \sqrt{N(N + L_z)}.$$  

**Proof.** Due to proposition 1, in consideration of the maximum of $ER(\rho, \mathcal{M})$ we can restrict to gauge-invariant states $\rho \in \mathcal{S}(\Lambda)$. Denote $V(z) = \sqrt{p_N}(D(z))^*$ and consider the cq-states

$$\varrho = \{\rho(z)\}, \quad \rho(z) = V(z)\rho V(z)^*, \quad \varrho_\Lambda = \{\rho_\Lambda(z)\}, \quad \rho_\Lambda(z) = V(z)\rho_\Lambda V(z)^*,$$

then the c-states $P$ and $P_\Lambda$ are defined by densities $p(z) = \text{Tr} \rho(z)$ and $p_\Lambda(z) = \text{Tr} \rho_\Lambda(z)$. We have

$$ER(\rho_\Lambda, \mathcal{M}) - ER(\rho, \mathcal{M}) = H(\rho\parallel \rho_\Lambda) - H_{cq}(\varrho || \varrho_\Lambda) + H_c(P || P_\Lambda) + \text{Tr}(\rho - \rho_\Lambda) \log \rho_\Lambda$$

$$+ \int \text{Tr}(\rho_\Lambda(z) - \rho(z)) \log \rho_\Lambda(z) \frac{d^2z}{\pi^t}, \quad (21)$$

where $\rho_\Lambda(z) = \rho_\Lambda(z)/p_\Lambda(z)$ are the posterior states corresponding to the input state $\rho_\Lambda$, 

$$H_c(P || P_\Lambda) = \int p(z) \log \left( \frac{p(z)}{p_\Lambda(z)} \right) \frac{d^2z}{\pi^t},$$

is the classical relative entropy of $P, P_\Lambda$ and

$$H_{cq}(\varrho || \varrho_\Lambda) = \int \text{Tr} \rho(z) (\log \rho(z) - \log \rho_\Lambda(z)) \frac{d^2z}{\pi^t} = \mathcal{H}(P || \rho_\Lambda),$$

is the relative entropy of cq-states (see equation (3) in [20]). Here we use the channel $\mathcal{E}[\rho] = \{V(z)\rho V(z)^*\}$ with quantum input and hybrid cq output.

Monotonicity of the relative entropy for cq-states ([20], theorem 1) then implies

$$H_{cq}(\varrho || \varrho_\Lambda) \leq H(\rho\parallel \rho_\Lambda),$$

hence we have for the first three terms in the right-hand side of (21)

$$H(\rho\parallel \rho_\Lambda) - H_{cq}(\varrho || \varrho_\Lambda) + H_c(P || P_\Lambda) \geq 0. \quad (22)$$
As we have assumed, $\Lambda$ is non-degenerate hence $\log \rho_\Lambda$ exists and is a linear combination of the operators $I, a_j a_k$. This follows from the exponential form of the density operator (theorem 12.23 in [4]). Since $\rho, \rho_\Lambda \in \mathcal{S}(\Lambda)$, then the normal second moments of the states $\rho$ and $\rho_\Lambda$ coincide and hence

$$\text{Tr}(\rho - \rho_\Lambda) \log \rho_\Lambda = 0. \quad (23)$$

It remains to show that also

$$\int \text{Tr}(\rho_\Lambda(z) - \rho(z)) \log \hat{\rho}_\Lambda(z) \frac{d^2z}{\pi^2} = 0. \quad (24)$$

**Lemma 1.** For $\rho = \rho_\Lambda$, the posterior state $\rho_\Lambda(z)$ is the Gaussian state

$$\hat{\rho}_\Lambda(z) = D(Kz)^\dagger \rho_\Lambda D(Kz), \quad (25)$$

where

$$K = \sqrt{N(N + I_\ell)(\Lambda + N + I_\ell)^{-1}},$$

$$\hat{N} = \sqrt{N(N + I_\ell)^{-1} \Lambda(\Lambda + N + I_\ell)^{-1} \sqrt{N(N + I_\ell)}}.$$

**Proof.** By using the quantum Parceval relation

$$\text{Tr} \rho \sigma^\dagger = \int \text{Tr} \rho D(w) \text{Tr} \sigma D(w) \frac{d^2w}{\pi^2}, \quad (26)$$

the relation (13) for $\rho = \rho_\Lambda$ and the characteristic functions of the Gaussian states, we can show as in [10] that

$$p_\Lambda(z) = \frac{1}{\det(\Lambda + N + I_\ell)} \exp \left(-z^\dagger(\Lambda + N + I_\ell)^{-1}z\right).$$

It is known (see [25]) that the square root of a Gaussian density operator is proportional to another Gaussian density operator. By using the Fock basis in $\mathcal{H}$ associated with the eigenvectors of the matrix $N$ (see e.g. appendix in [10]), we obtain

$$\sqrt{\rho_N} = c \rho_L, \quad L = N + \sqrt{N(N + I_\ell)},$$

$$c^2 = \det(2L + I_\ell) = \det \left(\sqrt{N} + \sqrt{N + I_\ell}\right)^2. \quad (27)$$

A calculation in this Fock basis shows also that, similarly to equation (33) of [10],

$$\sqrt{\rho_N} |w\rangle = \frac{1}{\det \sqrt{N + I_\ell}} \exp \left(-w^\dagger(N + I_\ell)^{-1}w\right) \left|\sqrt{N(N + I_\ell)^{-1}w}\right\rangle. \quad (28)$$
whence, by using the P-representation (11) for $\rho_{\Lambda,-z} = D(z)^* \rho_{\Lambda} D(z)$,

$$\hat{\rho}_{\Lambda}(z) = p_{\Lambda}(z)^{-1} \sqrt{\rho_{N}} p_{\Lambda,-z} \sqrt{\rho_{N}}$$

$$= \frac{p_{\Lambda}(z)^{-1}}{\det \Lambda(N + I_s)} \int \left| \sqrt{N(N + I_s)^{-1} w} \right| \left| \sqrt{N(N + I_s)^{-1} w} \right| \times \exp \left( -w^* (N + I_s)^{-1} w - (w + z)^* \Lambda^{-1} (w + z) \right) \frac{d^2 w}{\pi s}$$

$$= \frac{1}{\det N} \int |w| |u| \exp \left( -(u + Kz)^* \tilde{N}^{-1} (u + Kz) \right) \frac{d^2 u}{\pi s}$$

$$= D(Kz)^* \rho_{\tilde{N}} D(Kz), \quad (29)$$

where we made the change of variable $u = \sqrt{N(N + I_s)^{-1} w}$. □

Substituting the posterior state (25) into the right-hand side of (24), we obtain

$$\int \text{Tr}(\rho_{\Lambda}(z) - \rho(z)) \log \hat{\rho}_{\Lambda}(z) \frac{d^2 z}{\pi s} = \text{Tr} \left[ \Phi_M(\rho_{\Lambda}) - \Phi_M(\rho) \right] \log \rho_{\tilde{N}}, \quad (30)$$

where we have introduced the channel

$$\Phi_M(\sigma) = \int D(Kz) \sqrt{\rho_{\tilde{N}}} D(z)^* \sigma D(z) \sqrt{\rho_{\tilde{N}}} D(Kz) \frac{d^2 z}{\pi s}. \quad (31)$$

**Lemma 2.** $\Phi_M$ is a gauge-covariant Gaussian channel.

We will give the proof in a moment, but first let us explain how this lemma implies the required identity (24). The state $\rho \in \mathcal{S}(\Lambda)$ and $\rho_{\Lambda}$ have the same normal second moments, which are transformed similarly under the action of a gauge-covariant Gaussian channel. Hence $\Phi_M(\rho)$ and $\Phi_M(\rho_{\Lambda})$ also have the same normal second moments. Without loss of generality, we can assume that $N$ hence $\tilde{N}$ is nondegenerate. Then $\log \rho_{\tilde{N}}$ exists and is a linear combination of the operators $I, a_j^\dagger a_k$, hence,

$$\text{Tr} \left[ \Phi_M(\rho_{\Lambda}) - \text{Tr} \Phi_M(\rho) \right] \log \rho_{\tilde{N}} = 0.$$

Taking into account (30) this implies (24). Together with (22) and (23) this implies that the difference (21) is nonnegative, i.e. the first statement of the theorem. The formula (19) follows from

$$ER(\rho_{\Lambda}, \mathcal{M}) = H(\rho_{\Lambda}) - \int p_{\Lambda}(z) H(\hat{\rho}_{\Lambda}(z)) \frac{d^2 z}{\pi s},$$

and the fact that $H(\hat{\rho}_{\Lambda}(z)) = H(\rho_{\tilde{N}})$ because of the unitary equivalence (25).

It remains to prove lemma 2. We will do this by checking the definition of the dual Gaussian channel (see [4], section 12.4.2). We have

$$\Phi^*_M(D(w)) = \int \exp (-2i \text{Im}(w^* Kz)) D(z)^* D(z) \frac{d^2 z}{\pi s} \equiv X,$
where
\[ \gamma = \sqrt{\rho} D(w) \sqrt{\rho}. \]

One has
\[ X = \left[ \text{Tr} \gamma D(K^* w) \right] D(K^* w), \]
indeed by using composition of displacements (8)
\[ XD(K^* w)^* = \int D(z) \gamma D(K^* w)^* D(z)^* \frac{d^2 z}{\pi^2} = \text{Tr} \left( \gamma D(K^* w)^* \right) I, \]
where the second equality follows from the orthogonality relations for the irreducible representation \( z \rightarrow D(z) \) (see e.g. section I.3.5 of [23]). Thus
\[ \Phi^*_M(D(w)) = \varphi(w) D(K^* w), \tag{32} \]
where \( \varphi(w) = \text{Tr} \sqrt{\rho} D(w) \sqrt{\rho} D(K^* w)^*. \)

By using (27) and the quantum Parceval relation (26) we have
\[ \varphi(w) = c^2 \int \text{Tr} \rho_1 D(w) D(z) \frac{d^2 z}{\pi^2} \]
\[ \times \exp \left( i \text{Im} \left( I_z (I_z + K^* w) - z^* R z - \text{Re} z^* R w \right) \right) \frac{d^2 z}{\pi^2}, \tag{33} \]
which is Gaussian characteristic function with
\[ B = \frac{1}{4} \left( \left( I_z + K \right) R^{-1} \left( I_z + K^* \right) + \left( I_z - K \right) R \left( I_z - K^* \right) \right). \tag{35} \]

Then (32) with (34) mean that \( \Phi_M \) is a gauge-covariant Gaussian channel. \( \square \)

The matrix (35) of the quadratic form in the exponent (34) must satisfy the general necessary and sufficient condition for quantum channels (see equation (12.170) in [4])
\[ B \geq \pm \frac{1}{2} \left( I_z - K K^* \right). \tag{36} \]

Although this should follow automatically from the complete positivity of the map (31), let us give an independent check. Denote by \( r_j \) the eigenvalues, \( v_j \) the eigenvectors of the positive
definite Hermitian matrix \( R = 2L + I \). Let \( z \) be an arbitrary vector from \( C^s \) and \( w = K^* z \). Then (36) amounts to

\[
(z + w)^* R^{-1} (z + w) + (z - w)^* R (z - w) \geq \pm 2 \left( z^* z - w^* w \right),
\]

or

\[
\sum_{j=1}^{s} \left[ r_j^{-1} |a_j + b_j|^2 + r_j |a_j - b_j|^2 \right] \geq \pm 2 \sum_{j=1}^{s} \left( |a_j|^2 - |b_j|^2 \right),
\]

where \( a_j = v_j^* z, \ b_j = v_j^* w \). But for arbitrary \( r > 0 \) and complex \( a, b \)

\[
r^{-1} |a + b|^2 + r |a - b|^2 \geq 2 \left| |a|^2 - |b|^2 \right|,
\]

implying the required inequality.

5. One mode

In this section we consider the channel \( \tilde{\mathcal{M}} \) defined by the POVM (12) with \( s = 1 \). Take the energy operator \( H = a^\dagger a \), then theorems 1 and 2 imply that the maximum in (6) is attained on the gauge-invariant Gaussian state \( \rho_\Lambda \) with \( \Lambda = E \) (see remark just before theorem 2), and the energy-constrained entanglement-assisted capacity is given by the formula

\[
C_{ea} \equiv C_{ea}(\tilde{\mathcal{M}}, H, E) = g(E) - g \left( \frac{NE}{N + E + 1} \right).
\]

For \( N = 0 \) we recover the formula \( C_{ea} = g(E) \) obtained in [7].

Let us compare (37) with the unassisted capacity, which was calculated in [10], i.e.

\[
C \equiv C(\tilde{\mathcal{M}}, H, E) = \log (N + E + 1) - \log (N + 1).
\]

We will use the asymptotic

\[
g(E) = (E + 1) \log(E + 1) - E \log E \sim \begin{cases} -E \log E, & E \to 0 \\ \log E, & E \to \infty \end{cases}.
\]

Then it is easy to see that in the limit \( E \to 0 \) (weak signal, noise \( N \) fixed)

\[
C \sim \frac{E}{N + 1} \log e, \quad C_{ea} \sim \frac{E}{N + 1} \log E,
\]

so that the gain of entanglement assistance

\[
G = C_{ea} / C \sim - \ln E, \quad E \to 0.
\]

When \( E \to \infty \) (strong signal) we have

\[
C_{ea} \sim C \sim \log E,
\]
Figure 1. Comparison of the assisted capacity $C_{ea}$ (solid line) and unassisted capacity $C$ (dotted line).

so that $G \sim 1$, while

$$C_{ea} - C = \log \left(1 + \frac{1}{E}\right)^E - \log \left(1 + \frac{N + E + 1}{NE}\right)^{\frac{NE}{N+E+1}},$$

and

$$\lim_{E \to \infty} (C_{ea} - C) = \log e - \log \left(1 + \frac{1}{N}\right)^N.$$

Another interesting limit is $N \to \infty$ (strong noise, $E$ fixed). Using the relation $g'(E) = \log \left(\frac{E+1}{E}\right)$, we obtain

$$C_{ea} \sim g'(E) \left(E - \frac{NE}{N+E+1}\right) = \log \left(\frac{E+1}{E}\right) \frac{E(E+1)}{N+E+1},$$

while $C \sim \log e \frac{E}{N+1}$ whence

$$G \sim (E + 1) \log \left(\frac{E+1}{E}\right),$$

which varies from $\infty$ for $E \to 0$ to 1 for $E \to \infty$.

The plots of the two capacities and of the gain for the values of measurement noise $N = 0, 1, 10$ are shown in figures 1 and 2.

6. Arbitrary Gaussian measurements

Consider the 2s-dimensional symplectic space $(\mathbb{R}^{2s}, \Delta)$ with

$$\Delta = \text{diag} \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]_{j=1,\ldots,s}. \quad (39)$$
In order to spare the symbols, we will preserve the notations for the real vector \( z = [x, y] \) and for the corresponding element of the volume. Let \( \mathcal{H} \) be the space of an irreducible representation \( z \rightarrow W(z); z \in \mathbb{R}^2 \), of the Weyl–Segal canonical commutation relations
\[
W(z)W(z') = \exp \left[ -\frac{i}{2} \Delta z' \right] W(z + z').
\] (40)

Here \( W(z) = \exp \left( \sum_{j=1}^s (x_j q_j + y_j p_j) \right) \) are the unitary Weyl operators, where \( q_j, p_j \) are the canonical observables of the quantum system. We denote by \( \rho_\alpha, \rho_\beta \) centered Gaussian states with correlation matrices \( \alpha, \beta \) (see e.g. chapter 12 of [4] for a detailed description).

We will consider the Gaussian observable given by the POVM
\[
M(d^2z) = W(z)\rho_\beta W(z)^* \frac{d^2z}{(2\pi)^s}.
\] (41)

and the corresponding measurement channel \( \mathcal{M} \) (see e.g. [11]). Let \( \mathcal{G}(\alpha) \) be the set of all of centered states \( \rho \) with correlation matrix \( \alpha \). We will study the following entropic characteristic of the Gaussian measurement channel \( \mathcal{M} \) underlying its entanglement-assisted capacity
\[
ER(\mathcal{M}; \alpha) = \sup_{\rho(\mathcal{G}(\alpha))} ER(\rho, \mathcal{M}).
\] (42)

**Theorem 3.** The supremum in (42) is attained on the Gaussian state \( \rho_\alpha \) and is equal to
\[
ER(\mathcal{M}; \alpha) = \frac{1}{2} \left[ \text{Sp} \ g \left( \text{abs}(\Delta^{-1} \alpha) - \frac{I_2}{2} \right) - \text{Sp} \ g \left( \text{abs}(\Delta^{-1} \tilde{\alpha}) - \frac{I_2}{2} \right) \right]
\] (43)

where
\[
\tilde{\alpha} = \beta - \sqrt{I_2 + (2\beta \Delta^{-1})^{-2} \beta (\alpha + \beta)^{-1} \beta \sqrt{I_2 + (2\Delta^{-1} \beta)^{-2}}}
\] (44)
and \( \text{abs}(\Delta^{-1} \alpha) \) is the matrix with eigenvalues equal to modulus of eigenvalues of \( \Delta^{-1} \alpha \) and with the same eigenvectors.

In this theorem we do not assume the gauge symmetry: \( \alpha \) and \( \beta \) need not share the common complex structure. Notice that in the gauge-invariant case we have the correspondence \( \alpha \rightarrow \Lambda + I_c/2, \beta \rightarrow N + I_c/2, \Delta^{-1} \beta \rightarrow i(N + I_c/2) \) [4], and (44) turns into (20).

**Proof (Sketch).** For the difference \( ER(\rho_{\alpha}, M) - ER(\rho, M) \) we have a representation similar to (21). It follows that to prove \( ER(\rho_{\alpha}, M) - ER(\rho, M) \geq 0, \rho \in \mathcal{S}(\alpha) \), it is sufficient to establish the analog of (24), i.e.,

\[
\int \text{Tr}(\rho_{\alpha}(z) - \rho(z)) \log \hat{\rho}_{\alpha}(z) \frac{d^2z}{(2\pi)^2} = 0, \quad (45)
\]

where \( \rho(z) = \sqrt{\rho_\beta} W(z)^* \rho W(z) \sqrt{\rho_\beta}, \rho_\alpha(z) = \sqrt{\rho_\beta} W(z)^* \rho_\alpha W(z) \sqrt{\rho_\beta}, \hat{\rho}_{\alpha}(z) = \rho_\alpha(z)/p_{\alpha}(z) \) and \( p_{\alpha}(z) = \text{Tr} \rho_{\alpha}(z) = \text{Tr} \rho_\alpha W(z) W(z)^* \).

To establish (45), we first prove generalization of lemma 1: for the input state \( \rho = \rho_{\alpha} \), the posterior states are Gaussian, namely

\[
\hat{\rho}_{\alpha}(z) = W(Kz)^* \rho_{\tilde{\alpha}} W(Kz),
\]

where \( K \) is a real square matrix and \( \tilde{\alpha} \) is real correlation matrix (44) of the centered Gaussian state \( \rho_{\tilde{\alpha}} \). This is established with the help of the formula for the characteristic function of product of Gaussian states established in the appendix of [26]. More specifically, the correlation matrix of the operator \( \sqrt{\rho_1} \rho_2 \sqrt{\rho_1} \) where \( \rho_1, \rho_2 \) are Gaussian, was computed in [27] (see also [28]) and the formula (44) was given in [29], equation (3.27).

Then similarly to (30) we have

\[
\int \text{Tr}(\rho_{\alpha}(z) - \rho(z)) \log \hat{\rho}_{\alpha}(z) \frac{d^2z}{(2\pi)^2} = \text{Tr} \Phi_M(\rho_{\alpha}) - \Phi_M(\rho) \log \rho_{\tilde{\alpha}}, \quad (46)
\]

where

\[
\Phi_M(\sigma) = \int W(Kz)^* \sigma W(z) \sqrt{\rho_\beta} W(Kz)^* \frac{d^2z}{(2\pi)^2}.
\]

The proof that \( \Phi_M \) is Gaussian channel and hence the right-hand side of (46) is equal to zero follows the same lines as in lemma 2. This proves

\[
ER(M; \alpha) = ER(\rho_{\alpha}, M)
\]

\[
= H(\rho_{\alpha}) - \int p_{\alpha}(z)H(W(Kz)^* \rho_{\tilde{\alpha}} W(Kz)) \frac{d^2z}{(2\pi)^2}
\]

\[
= H(\rho_{\alpha}) - H(\rho_{\tilde{\alpha}}).
\]

The formula (43) now follows from the expression for the entropy of an arbitrary Gaussian state given by equation (12.110) in [4].
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