New 2D dilaton gravity for nonsingular black holes

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Received 10 December 2015, revised 9 February 2016
Accepted for publication 29 February 2016
Published 19 April 2016

Abstract
We construct a two-dimensional action that is an extension of spherically symmetric Einstein–Lanczos–Lovelock (ELL) gravity. The action contains arbitrary functions of the areal radius and the norm squared of its gradient, but the field equations are second order and obey Birkhoff’s theorem. In complete analogy with spherically symmetric ELL gravity, the field equations admit the generalized Misner–Sharp mass as the first integral that determines the form of the vacuum solution. The arbitrary functions in the action allow for vacuum solutions that describe a larger class of interesting nonsingular black hole spacetimes than previously available.

Keywords: black holes, quantum gravity, Birkhoff’s theorem, nonsingular black holes

1. Introduction

General relativity has to date passed all experimental and observational tests. However, the singularity theorems in classical general relativity imply that the appearance of singularities is generically an inevitable result of gravitational collapse of massive stars. This powerful result leads to the conclusion that there must be spacetime regions in our Universe where the curvature is so large that general relativity is no longer reliable. In such highly curved spacetime regions near the Big Bang or deep inside black holes, quantum gravitational effects must play a large role in the description of spacetime. While it is strongly believed that the
singularities of classical general relativity are ultimately cured by quantum gravity, a complete quantum theory of gravity is not yet at hand.

This situation is somewhat analogous to that of the early 20th century when Rutherford’s classical model of atoms faced the problem of the electromagnetic radiation instability. We now know that this serious problem in classical physics can only be solved by invoking quantum mechanics. It is nonetheless true that in the semi-classical equations of motion, quantum effects can be incorporated as an effective repulsive force that balances the attractive electromagnetic force between proton and electron. A detailed analysis of effective, semi-classical equations for the hydrogen atom based on the method of moments [1] can be found in [2]. Another suggestive result [3] shows that quantum corrections to the Raychaudhuri equation prevent focusing of geodesics and the formation of conjugate points. Based on this potential resolution of the instability problem in terms of modified classical equations of motion, it is reasonable to expect that the singularity problem in general relativity can also be addressed by considering suitable modified, semi-classical theories of gravity.

The spherically symmetric Schwarzschild black hole is a typical vacuum solution containing a spacelike curvature singularity. Generally, there are two possible global structures for nonsingular black holes that are obtained by modifying the Schwarzschild black hole. In the first class, the spacelike singularity is replaced by a regular Big-Bounce so that the spacetime in the interior is extended to a cosmological spacetime beyond the bounce. This class of nonsingular black holes has been obtained as exact solutions to a modified theory based on the polymer quantization of gravity [4, 5].

By contrast, nonsingular black holes in the second class contain a regular de Sitter core and have global structures similar to that of the Reissner–Nordström black hole, but with a regular center. Sakharov [6] was first to suggest that black holes might be nonsingular with a de Sitter core, while Bardeen explicitly constructed such a spacetime. It obeyed the weak energy condition, but as noted by Bardeen, contained spacelike surfaces that evolved from noncompact to compact. In fact, it was proven more recently by Borde [7] that for a large class of spacetimes that obey the weak energy condition, topology change is required in order for regular black holes to exist. Israel and Poisson [8] derived a similar nonsingular metric from considerations of semi-classical quantum gravity, while the stability of black holes with de Sitter cores has been extensively analyzed by Dymnikova and Galaktionov [9]. Such nonsingular black holes have been realized as exact solutions to a modified theory based on the polymer quantization of gravity [4, 5].

At present, it is not exactly known what types of modified gravity can be realized as effective theories of quantum gravity. Under these circumstances having at one’s disposal a large class of modified gravity theories admitting nonsingular black holes provides a firm ground for future studies of the singularity problem and its related puzzles such as the information loss problem.

In the present paper, we focus on two-dimensional (2D) dilaton gravity because it provides an effective theory for spherically symmetric spacetimes in higher dimensions. Indeed, it has been shown that particular types of 2D dilaton gravity [15] admit nonsingular black holes as exact solutions [16–20]. The purpose of this paper is to present a new and considerably larger class of 2D dilaton gravity theories that satisfy Birkhoff’s theorem. We show that there exist members of this class of theories that admit nonsingular black holes with a de Sitter core as unique vacuum solutions and with maximal curvature bounded above for arbitrarily large mass5. This latter feature was, to the best of our knowledge, not possible in the context of ordinary 2D dilaton gravity without the addition of matter.

5 GK is grateful to Valeri Frolov for impressing on him the importance of this latter criterion.
The paper is organized as follows: in the next section we review the forms of spherically symmetric Einstein gravity and Einstein–Lanczos–Lovelock (ELL) gravity that we wish to consider. Section 3 presents our generalization of spherically symmetric ELL gravity, derives the mass functions and proves Birkhoff’s theorem. The Hamiltonian analysis in our new theories is performed in section 4. Section 5 shows how to derive specific nonsingular black holes of physical interest, and also defines a subclass of theories, dubbed designer Lovelock (dL) gravity that are more closely connected to the original ELL gravity. Finally we close with a summary and prospects for future work. Details of lengthy calculations in the Hamiltonian analysis are presented in the appendix. We adopt units such that \( c = \hbar = 1 \).

2. 2D effective actions for spherically symmetric systems

In this section, we review the effective 2D actions for spherically symmetric general relativity and ELL gravity. As shown in the following section, our new 2D gravity generalizes the latter just as 2D dilaton gravity is a generalization of the former.

2.1. Einstein gravity

The Einstein–Hilbert action for general relativity in arbitrary \( n \)-dimensions is given by

\[
I_{EH} = \frac{1}{16\pi G_{(n)}} \int d^n x \sqrt{-g} \mathcal{R}[g],
\]

where \( G_{(n)} \) is the higher dimensional gravitational constant and \( \mathcal{R}[g] \) is the Ricci scalar calculated using the \( n \)-dimensional metric, \( g \). The most general metric for \( n \)-dimensional spherically symmetric spacetimes is given by

\[
d_{(n)}^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \bar{g}_{AB}(y) dy^A dy^B + R(y)^2 d\Omega_{(n-2)}^2,
\]

where \( g_{\mu\nu}(y) \) \((A, B = 0, 1)\) is the general 2D Lorentzian metric, \( d\Omega_{(n-2)}^2 \) is the line-element on the unit \((n - 2)\)-sphere, and \( R \) is the areal radius. After imposing spherical symmetry and integrating out the angular variables the action \((2.1)\) takes the form \([22]\)

\[
I_{EH(2)} = \frac{1}{l_{n-2}^2} \int d^2 y \sqrt{-\bar{g}} \left( R^{n-2} \mathcal{R}[\bar{g}] + (n - 2)(n - 3)R^{n-4}(D\bar{R})^2 + (n - 2)(n - 3)R^{n-4} \right),
\]

where \((D\bar{R})^2 := (D_A R)(D^A R)\) in which \( D_A \) is the 2D covariant derivative, and \( \mathcal{R}[\bar{g}] \) is the Ricci scalar of \( \bar{g} \), the 2D Lorentzian part of the higher dimensional metric. We have defined a length parameter \( l \) proportional to the Planck length:

\[
l_{n-2} := \frac{16\pi G_{(n)}}{\mathcal{A}_{(n-2)}},
\]

where \( \mathcal{A}_{(n-2)} \) is the invariant volume of a unit \((n - 2)\)-sphere. The variation of the action \((2.3)\) will give the same equations of motion as varying \((2.1)\) and then imposing spherical symmetry. This is the case whenever the symmetry group is a compact lie group, as is the case in this paper \([23–25]\).

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6 Note that there is no relationship between the class of theories we consider and designer gravity, as first introduced in \([21]\). In the latter work, one is concerned with the defining gravity theories that admit arbitrary boundary conditions generally in the context of AdS/CFT duality.
This system admits the Misner–Sharp quasi-local mass [26]:
\[ \mathcal{M} = \frac{(n - 2)R^{n-3}}{l^{n-2}} \left[ 1 - (DR)^2 \right], \] (2.5)
which satisfies \( D_a \mathcal{M} = 0 \) and hence \( \mathcal{M} \) is constant in vacuum. By Birkhoff’s theorem, the unique vacuum solution is the well-known Schwarzschild–Tangherlini solution:
\[ ds^2_{(n)} = - \left( 1 - \frac{l^{n-2}M}{(n - 2)R^{n-3}} \right) dt^2 + \left( 1 - \frac{l^{n-2}M}{(n - 2)R^{n-3}} \right)^{-1} dR^2 + R^2 d\Omega_{(n-2)}^2, \] (2.6)
where \( M = \mathcal{M} \) is the Arnowitt–Deser–Misner (ADM) mass.

### 2.2. ELL gravity

ELL gravity is a natural generalization of general relativity in arbitrary dimensions as a second-order quasilinear theory of gravity [27, 28]. The second-order field equations ensure the ghost-free nature of the theory and ELL gravity reduces to general relativity with a cosmological constant in four-dimensions. (see [29, 30] for a review of Lovelock black holes.)

The ELL action [27, 28] in vacuum is a sum of dimensionally extended Euler densities given by
\[ I_{EL} = \frac{1}{16\pi G_{(n)}} \int d^n x \sqrt{-g} \sum_{p=0}^{[n/2]} \alpha_{(p)} \mathcal{L}_{(p)}, \] (2.7)
where
\[ \mathcal{L}_{(p)} := \frac{1}{2^p} \delta_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} \mathcal{R}_{\mu_1 \nu_1} \nu_1 \cdots \nu_p \mathcal{R}_{\nu_p \nu_p} \mathcal{R}_{\mu_p \nu_p} \mathcal{R}_{\mu_1 \nu_1} \nu_1 \cdots \nu_p, \] (2.8)
\[ \delta_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} := p! \delta_{\nu_1 \cdots \nu_p}^{\mu_1 \cdots \mu_p}. \] (2.9)

The gravitational equation following from this action is given by
\[ \mathcal{G}_{\mu\nu} = 0, \] (2.10)
where
\[ \mathcal{G}_{\mu\nu} = \sum_{p=0}^{[n/2]} \alpha_{(p)} G_{\mu\nu}^{(p)}, \] (2.11)
\[ G_{\mu\nu}^{(p)} := -\frac{1}{2^{p+1}} \delta_{\nu_1 \cdots \nu_p}^{\mu_1 \cdots \mu_p} \mathcal{R}_{\eta_{\mu_1} \eta_{\nu_1}} \nu_1 \cdots \nu_p \mathcal{R}_{\mu_1 \nu_1} \nu_1 \cdots \nu_p, \] (2.12)
with \( G_{\mu\nu}^{(p)} \equiv 0 \) for \( p \geq [(n + 1)/2] \).

As a concrete example, when \( n = 4 \), the only nonzero contributions come from \( p = 0, 1, 2 \) and the action is
\[ I_{EL} = \frac{1}{16\pi G_{(4)}} \int d^4 x \sqrt{-g} \left( \alpha_{(0)} + \frac{\alpha_{(2)}}{2} (\mathcal{R}^2 - 4 \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma}) \right), \] (2.13)
where we have set \( \alpha_{(1)} = 1 \) without loss of generality. In four-dimensions, the quadratic term is topological and does not contribute to the gravitational equations, namely \( G_{\mu\nu}^{(2)} \equiv 0 \).

Consider an \( n \)-dimensional spherically symmetric spacetime (2.2). In [31–33] it was shown that the spherically symmetric ELL action takes the form
where
\[
\mathcal{L}_{(p)} = \frac{(n-2)!}{(n-2p)!} \left[ p \mathcal{R} [\tilde{g}] R^{2-2p} + (n-2p)(n-2p-1) \{ (1-Z)^p + 2pZ \} R^{-2p} \right. \\
\left. + p(n-2p) R^{1-2p} \{ (1-Z)^p+1 \} (D_A R) \left( \frac{D_A Z}{Z} \right) \right]
\]  
(2.15)

and we have defined
\[
Z := (DR)^2.
\]  
(2.16)

The generalized Misner–Sharp mass in ELL gravity was defined \([34]\) as
\[
\mathcal{M} := \frac{n-2}{p-2} \sum_{p=0}^{[n/2]} \tilde{\alpha}_{(p)} R^{n-1-2p} [1 - (DR)^2]^p,
\]  
(2.17)

where
\[
\tilde{\alpha}_{(p)} := \frac{(n-3)! \alpha_{(p+1)}}{(n-1-2p)!}.
\]  
(2.18)

We note that \(\tilde{\alpha}_{(p)} \equiv 0 \quad (p \geq 2)\) for \(n = 3, 4, 5, \ldots, 2p-1, 2p\) by definition.

The generalized Misner–Sharp mass \((2.17)\) satisfies \(D_A \mathcal{M} = 0\) and hence \(\mathcal{M}\) is constant in vacuum. The resulting Birkhoff’s theorem in Lovelock gravity \([35]\) shows that, under several technical assumptions, the unique vacuum solution is given by the following Schwarzschild–Tangherlini-type solution:
\[
d\chi^2 = -f(R) dt^2 + f(R)^{-1} dr^2 + r^2 d\Omega^2_{(n-2)},
\]  
(2.19)

where the function \(f(R)\) is determined by the following algebraic equation \([36, 37]\):
\[
\mathcal{M} = \frac{n-2}{p-2} \sum_{p=0}^{[n/2]} \tilde{\alpha}_{(p)} R^{n-1-2p} (1-f)^p,
\]  
(2.20)

where \(\mathcal{M} = \mathcal{M}\).

3. New 2D dilaton gravity

A natural way to generalize the spherically symmetric action \((2.3)\) in Einstein gravity is 2D dilaton gravity:
\[
I_{(2)} = \frac{1}{p-2} \int d^2 \sqrt{-\tilde{g}} \left\{ \phi(R) \mathcal{R} [\tilde{g}] + h(R) (DR)^2 + V(R) \right\},
\]  
(3.1)

where \(\phi(R), h(R),\) and \(V(R)\) are arbitrary functions of a scalar field \(R\). This class of theories was extensively studied in the 1990s in the hopes that they would shed light on the conundra associated with black hole thermodynamics. (see \([15]\) for an excellent review.)

The action \((3.1)\) is, up to reparametrizations, the most general action that contains at most two derivatives of the metric and areal radius. The resulting field equations are therefore trivially second-order. However, as shown in the action \((2.14)\), the field equations can be second order even if the action contains the higher derivative term with \(D_A Z\). This motivates us to generalize the spherically symmetric action \((2.14)\) of ELL gravity, as follows.
3.1. Action and field equations

In analogy with the action (3.1), we now consider the following natural extension of the spherically symmetric ELL action (2.14):

\[
I_{XL} = \frac{1}{\mu - 2} \int d^2y \sqrt{-\bar{g}} \left\{ \phi(R)\mathcal{R}[\bar{g}] + \eta(R, Z) + \chi(R, Z)(D^2R) \frac{(D^2Z)}{Z} \right\},
\]

(3.2)

\( \eta(R, Z) \) and \( \chi(R, Z) \) are as yet arbitrary functions of a scalar field \( R \) and \( Z \) defined by equation (2.16).

Equation (3.2) is the starting point for our analysis. Note that the XL action \( I_{XL} \) contains higher powers (potentially an infinite number of them) of \( Z \) and hence of the ‘velocity’ \( R_t \) of the areal radius. We conjecture that this is the most general 2D action involving only the metric and a scalar that yields second-order equations for both. Moreover, we will now show that for any given \( \phi(R) \) and \( \chi(R, Z) \), one can choose the function \( \eta(R, Z) \) so that the field equations obey Birkhoff’s theorem, i.e., there is a unique one parameter family of solutions that admit at least one Killing vector.

Here we note that our approach is to think of 2D dilaton gravity as deformed spherically symmetric gravity/Lovelock gravity and identify from the beginning the geometrical quantities \( \bar{g}_{AB} \) and \( R \) that correspond to the metric and areal radius in the higher dimensional theory. This leads to the general form of the 2D dilaton action (3.2). We could for example identify some function \( F(R) \) with the higher dimensional areal radius, but this would just change the definitions of the arbitrary functions in the action.

We now present the field equations. Varying the action (3.2) for \( g_{AB} \) and \( R \) gives

\[
0 = (\chi - \phi_{,R})(D_A D_B R - g_{AB} D^2 R) + g_{AB} \left( \phi_{,RR} Z - \frac{1}{2} \eta \right)
+ (\phi_{,RR} + \eta_{,Z} - \chi_{,R})(D_A R)(D_B R)
\]

(3.3)

and

\[
0 = - (\chi - \phi_{,R}) \mathcal{R} + \eta_{,R} + 2(\chi_{,RR} - \eta_{,ZZ}) Z + (4 \chi_{,R} - 2 \eta_{,Z}) D^2 R
+ 2 \chi_{,Z} [D^2 R]^2 - (D_A D_B R)(D^2 D^2 R) + 2(\chi_{,ZZ} - \eta_{,ZZ})(D_A Z)(D_B R),
\]

(3.4)

respectively, where a comma denotes partial differentiation. These equations are clearly second order. To obtain equation (3.3), it is useful to use the identity (A.6) in [22] but with \( r \rightarrow R^2 \). The derivation of equation (3.4) requires the identities (A.5) and (A.8) in [33].

3.2. Mass function

We have shown that our new theory (3.2) gives rise to the second-order field equations (3.3) and (3.4). In order to make the theory resemble ELL gravity as closely as possible, we require that it admit a generalized Misner–Sharp mass \( M \) as the first integral of the field equations. In ELL gravity, the mass function satisfies

\[
D_A M = \mathcal{G}^A_B(D^B R) - \mathcal{G}^B_A(D_A R),
\]

(3.5)

where the gravitational equations take the form \( \mathcal{G}^{AB} = 0 \), guaranteeing that the mass function is constant on shell.

7 Recently Tibrewala [38] used Hamiltonian techniques involving loop quantum gravity motivated variables to construct a set of new second derivative spherically symmetric gravity theories. Since the action was not written in covariant form it is difficult to say for sure whether or not it falls into the class described by (3.2).
The mass function $M$ satisfying equation (3.5) has a physical interpretation as quasi-local mass. In ELL gravity with a minimally coupled matter field, the gravitation equations in $n$-dimensions are given by $\mathcal{G}_{\mu\nu} = 8\pi G_m T_{\mu\nu}$, where the gravitational tensor $\mathcal{G}_{\mu\nu}$ is defined by equation (2.11) and $T_{\mu\nu}$ is the energy–momentum tensor for the matter field. Now we consider the following energy–momentum:

$$T^\mu_\nu = \text{diag}(\rho, P_r, P_r, \ldots, P_t),$$

where $\rho$, $P_r$, and $P_t$ are energy density, radial pressure, and tangential pressure, respectively. Then, in the following coordinates;

$$ds^2 = g_{tt}(t, r)dt^2 + g_{rr}(t, r)dr^2 + R(t, r)^2d\Omega^2_{(n-2)},$$

the component $A = r$ of equation (3.5) gives

$$M_r = A_{(n-2)} R^{n-2} R_t.$$

The above equation gives $M$ its physical interpretation as the quasi-local mass inside the areal radius $R$ on a spacelike hypersurface with constant $t$:

$$M = \int A_{(n-2)} R^{n-2} \frac{\partial R}{\partial r} dr.$$  

For this reason, we expect equation (3.5) to provide a suitable definition for the mass function $M$.

In the present case, the two-tensor $\mathcal{G}_{AB}$ is obtained directly from (3.3):

$$\mathcal{G}_{AB} = 2(\chi - \phi_R)(D_\alpha D_\beta R - g_{AB} D^2 R) + 2g_{AB}(\phi_{RR} Z - \frac{1}{2} \eta)$$

$$+ 2(\phi_{RR} + \eta_Z - \chi_{,R}) D_\alpha RD_\beta R$$

so that

$$\mathcal{G}_{AB}(D^2 R) - \mathcal{G}_{AB}(D_\alpha R) = (\chi - \phi_R) D_\alpha Z - 2(\phi_{RR} Z - \frac{1}{2} \eta) D_\alpha R.$$  

In order for the theory to have a mass function that obeys (3.5), equation (3.11) shows that $M = M(R, Z)$ must satisfy the following:

$$\frac{\partial M}{\partial Z} = \chi - \phi_R,$$

$$\frac{\partial M}{\partial R} = -2\phi_{RR} Z + \eta.$$  

The necessary and sufficient condition for the existence of such a mass function is therefore the integrability condition $\partial^2 M/\partial R\partial Z = \partial^2 M/\partial Z\partial R$, which gives the following constraint on the Lagrangian functions:

$$\phi_{RR} = \eta_Z - \chi_{,R}.$$  

Straightforward integration of (3.11) yields the following two integral forms for the mass function:

$$M = -\phi_R Z + \int Z \chi(R, Z)d\bar{Z} + M_0(R).$$
and

\[ \mathcal{M} = -2\phi_R Z + \int \eta(\tilde{R}, Z) d\tilde{R} + \mathcal{M}_4(Z). \]  

(3.16)

It is emphasized that this quantity may be interpreted as quasi-local mass because it satisfies equation (3.5) and hence equation (3.9) for the matter field (3.6). In order to uniquely determine the correct expression for the mass function \( \mathcal{M}(R, Z) \), one must in general evaluate both integrals and fix the two arbitrary functions appropriately. The integrability conditions guarantee that there will always be a choice of \( \mathcal{M}_0(R) \) and \( \mathcal{M}_4(Z) \) to make the expressions for the two expressions (3.15) and (3.16) consistent. For example, in Einstein–Gauss–Bonnet gravity, which is second-order ELL gravity, we have

\[ \phi(R) = R^{n-2} + 2(n - 2)(n - 3)\alpha(2)R^{n-4}, \]  

(3.17)

\[ \chi(R, Z) = 2(n - 2)(n - 3)(n - 4)\alpha(2)R^{n-5}Z, \]  

(3.18)

\[ \eta(R, Z) = (n - 2)(n - 3)R^{n-4}(1 + Z) \]

\[ + (n - 2)(n - 3)(n - 4)(n - 5)\alpha(2)R^{n-6}(1 + Z)^2. \]  

(3.19)

Integration of equation (3.16) gives the correct mass function [22] with \( \mathcal{M}_4 = 0 \):

\[ \mathcal{M} = (n - 2)R^{n-3}(1 - Z) + (n - 2)(n - 3)(n - 4)\alpha(2)R^{n-5}(1 - Z)^2. \]  

(3.20)

In contrast, since \( \eta(R, Z) \) contains terms depending only on \( R \), equation (3.15) gives the following:

\[ \mathcal{M} = (n - 2)R^{n-3}(1 - Z) + (n - 2)(n - 3)(n - 4)\alpha(2)R^{n-5} + \mathcal{M}_0(R). \]  

(3.21)

Consistency between (3.20) and (3.21) requires the choice:

\[ \mathcal{M}_0(R) = (n - 2)R^{n-3} + (n - 2)(n - 3)(n - 4)\alpha(2)R^{n-5}. \]  

(3.22)

### 3.3. Birkhoff’s theorem

We now show that the theory (3.2) with the condition (3.14) obeys Birkhoff’s theorem. Here we assume \( Z = (RdR)^2 \neq 0 \) for simplicity. Then we can choose without loss of generality \( R \) as a coordinate such that

\[ ds^2 = -f(t, R)e^{2\phi(t, R)}dt^2 + f(t, R)^{-1}dR^2. \]  

(3.23)

In this case, the components of the gravitational tensor \( G^R_i \) are

\[ G^R_i = G^R_i \]

\[ = \frac{f}{2}e^{2\phi(t, R)}(\chi - \phi(t, R)) - f(-\phi(t, R) + \chi(t, R)), \]  

(3.24)

\[ 2G^R_i = 2\phi(t, R) - \eta - f(t, R)(\chi - \phi(t, R)), \]  

(3.25)

\[ G^i_R = -\frac{1}{2}f^{-2}f_{,t}e^{2\phi(t, R)}, \]  

(3.26)

\[ G^i_R = \frac{1}{2}f_{,t}(\chi - \phi(t, R)), \]  

(3.27)

where \( \eta \) and \( \chi \) and their derivatives have been evaluated at \( Z = f(t, R) \). Equation (3.4) is an auxiliary equation in general. We will also assume that \( \chi - \phi(t, R) = 0 \) so that vanishing of the components (3.26) or (3.27) imply that \( f = f(R) \).
We also see that the integrability condition (3.14) makes the last term in equation (3.24) vanish, which implies $\delta = \delta(t)$. Since we can set $\delta(t) = 0$ without loss of generality by redefinition of $t$, the metric reduces to the usual Schwarzschild form:

$$ds^2 = -f(R)dr^2 + f(R)^{-1}d\rho^2.$$  (3.28)

The metric function $f(R)$ can now be determined algebraically from either equation (3.15) or (3.16), namely

$$\mathcal{M} = -\phi_Rf + \int f(R, f)df + \mathcal{M}_0(R)$$  (3.29)

or

$$\mathcal{M} = -2\phi_Rf + \int Rf(R, f)dR + \mathcal{M}_0(f),$$  (3.30)

where we used $Z = f$.

### 3.4. Birkhoff’s theorem with integrating factor

In the previous two subsections, we have seen that given the integrability condition (3.14), we can define the mass function (3.15) or (3.16) which in turn yields the unique, static vacuum solution given by equation (3.28). We now show that even if the Lagrangian functions do not satisfy the integrability condition (3.14), we can define a quantity analogous to the mass function that is constant on shell and leads to a unique and static vacuum solution. The form of the solution is, however, different from the standard Schwarzschild form, in that $g_00, g_11 \neq -1$. Moreover, this first integral of the field equations does not have the physical interpretation as the quasi-local mass, discussed after equation (3.5) above.

We define the quantity $\mathcal{M}$ by introducing an integrating factor $\Omega(R, Z)$ such that

$$D_\alpha \mathcal{M} = \Omega(R, Z)(G_{\alpha\beta}D_{\beta}R - G_{\beta\gamma}(D_\gamma R))$$

$$= \Omega(R, Z)\left\{ \chi - \phi_R D_\alpha Z - 2\left( \phi_{RR}Z - \frac{1}{2} \eta \right) D_\alpha R \right\}. $$  (3.31)

Assuming that the integrating factor does not vanish, $\mathcal{M}$ is constant on shell.

In this case, equation (3.31) requires $\mathcal{M} = \mathcal{M}(R, Z)$ to satisfy

$$\frac{\partial \mathcal{M}}{\partial Z} = \Omega(\chi - \phi_R),$$  (3.32)

$$\frac{\partial \mathcal{M}}{\partial R} = \Omega(-2\phi_{RR}Z + \eta)$$  (3.33)

and the integrability condition is now

$$\phi_{RR} + \chi_R - \eta Z = \frac{\Omega}{\Omega Z}(-2\phi_{RR}Z + \eta) - \frac{\Omega_R}{\Omega}(\chi - \phi_R).$$  (3.34)

This is is a first-order quasilinear partial differential equation for $\ln(\Omega)$, so that the integrating factor $\Omega(R, Z)$ always exists for given $\phi(R)$, $\chi(R, Z)$, and $\eta(R, Z)$ with sufficient differentiability. It is not, however, easy to find analytically in general.

Finally, given the integrability condition (3.34), $\mathcal{M}$ can be written in either of the two following integral forms:
\[ \mathcal{M} = \int^Z \Omega (\chi - \phi_R) dZ + \mathcal{M}_0(R) \]  
(3.35)

or

\[ \mathcal{M} = \int^R \Omega (\vec{R}, Z) (-2\phi_{,RR}Z + \eta(\vec{R}, Z)) d\vec{R} + \mathcal{M}_1(Z). \]  
(3.36)

Similar computations to those for the mass function without the integration factor (3.15) or (3.16) show that one can always find integration ‘constants’ \( \mathcal{M}_0(R) \) and \( \mathcal{M}_1(Z) \) that insure consistency of (3.35) and (3.36).

We now derive the vacuum solution. Starting from the metric (3.23), either equation (3.26) or (3.27) again implies that \( f = f(R) \). The metric function \( f(R) \) is again determined algebraically by equation (3.35) or (3.36), namely

\[ \mathcal{M} = \int^f \Omega (R, \bar{f}) (\chi (R, \bar{f}) - \phi_R) d\bar{f} + \mathcal{M}_0(R) \]  
(3.37)

or

\[ \mathcal{M} = \int^R \Omega (\vec{R}, f) (-2\phi_{,RR}f + \eta(\vec{R}, f)) d\vec{R} + \mathcal{M}_1(Z). \]  
(3.38)

The remaining metric function \( \delta(t, R) \) is determined by equation (3.24):

\[ \delta_{,R} = \frac{\phi_{,RR} - \eta_{,Z} + \chi_{,R}}{\phi_{,R} - \chi} \bigg|_{Z=f(R)}. \]  
(3.39)

Since the right-hand side depends only on \( R \), the solution is

\[ \delta(t, R) = \int^R \frac{\phi_{,RR} - \eta_{,Z} + \chi_{,R}}{\phi_{,R} - \chi} \bigg|_{Z=f(R)} d\vec{R} + \delta_z(t). \]  
(3.40)

Since \( \delta_z(t) \) can be set to zero by redefinition of \( t \), the unique vacuum solution is again static.

In summary, under the sole assumption \( \chi - \phi_R = 0 \), for a given solution \( \Omega(R, Z) \) of the partial differential equation (3.34), the metric function \( f(R) \) and \( \delta(R) \) are given by equations (3.35) (or (3.36)) and (3.40) with \( \delta_z(t) = 0 \), respectively.

### 3.5. Maximally symmetric vacua

We now derive the general conditions on the functions \( \phi(R), \chi(R, Z), \) and \( \eta(R, Z) \) in the action in order that the theory admit maximally symmetric vacuum solutions. We do this by simply inserting the vacuum solution into the field equations (3.24)–(3.27). Adopting coordinates such that

\[ ds^2 = -(1 - \lambda R^2) dt^2 + (1 - \lambda R^2)^{-1} dR^2, \]  
(3.41)

where \( \lambda \) is the effective cosmological constant, the field equations give

\[ 0 = -2\lambda R (\chi - \phi_R) + \eta + 2(1 - \lambda R^2)(\chi_{,R} - \eta_{,Z}), \]  
(3.42)

\[ 0 = -2\lambda (\chi - \phi_R) + \eta_{,R} + 2(1 - \lambda R^2)(\chi_{,RR} - \eta_{,RZ}) - 2\lambda R (4\chi_{,R} - 2\eta_{,Z}) \]  
\[ + 4\lambda^2 R^2 \chi_{,Z} - 4\lambda R (1 - \lambda R^2)(\chi_{,RZ} - \eta_{,ZZ}). \]  
(3.43)

where \( \chi, \eta \), and their derivatives are evaluated at \( Z = 1 - \lambda R^2 \).

For the Minkowski vacuum (\( \lambda = 0 \)), the equations above reduce to the following single condition:
0 = \eta + 2(\chi, R - \eta_Z), \quad (3.44)

where \( \chi, \eta \) and their derivatives are evaluated at \( Z = 1 \). If the integrability condition is satisfied, equation (3.44) implies that:

\[ \eta(R, 1) = 2\phi_{RR}. \quad (3.45) \]

We now consider the (A)dS vacuum \((\lambda = 0)\). In this case, differentiating equation (3.42) with respect to \( R \) and using equation (3.43), we obtain a necessary condition for existence of the (A)dS vacuum:

\[ \phi_{RR} = \eta_Z - \chi, R. \quad (3.46) \]

This is the same as the integrability condition (3.14) for the mass function but evaluated at \( Z = 1 - \lambda R^2 \). If this condition is satisfied, the (A)dS vacua can be obtained for lagrangian functions \( \chi \) and \( \phi \) that satisfy equation (3.42), namely

\[ 0 = -2\lambda R(\chi - \phi_R) + \eta - 2(1 - \lambda R^2)\phi_{RR}, \quad (3.47) \]

where we used equation (3.46) and \( \chi, \eta \) are evaluated at \( Z = 1 - \lambda R^2 \). Notice that when \( \lambda = 0 \) (3.47) yields (3.45).

4. Hamiltonian formalism

In this section, we perform the Hamiltonian analysis of our new 2D dilaton gravity (3.2), leaving the computational details to the appendix. The Hamiltonian analysis of spherically symmetric Einstein–Gauss–Bonnet gravity was done in [39] and again using a different formalism in [40]. The Hamiltonian analysis of spherically symmetric ELL gravity was discussed in [41, 42]. The following is based in large part on the methodology and results of [31, 32] and uses the notation and conventions of [18].

4.1. ADM decomposition

For convenience we separate the XL action (3.2) into two parts

\[ I_{XL} = I_G + I_L, \quad (4.1) \]

where

\[ I_G := \frac{1}{p^{n-2}} \int d^2y \sqrt{-\bar{g}} \left\{ \phi(R) R \bar{g} + \eta(R, Z) \right\}, \quad (4.2) \]

\[ I_L := \frac{1}{p^{n-2}} \int d^2y \sqrt{-\bar{g}} \left\{ \chi(R, Z)(D_R \bar{g}) \frac{(D^4Z)}{Z} \right\}, \quad (4.3) \]

\( \eta(R, Z) \) and \( \chi(R, Z) \) are as yet arbitrary functions of a scalar field \( R \) and \( Z \) defined by equation (2.16).

We henceforth assume that \( \chi(R, Z) \) has an expansion of the form

\[ \chi(R, Z) := \sum_l [\beta^{(l)}(R) W_{(l)}(Z) Z], \quad (4.4) \]

where \( \beta^{(l)}(R) \) and \( W_{(l)}(Z) \) are arbitrary functions of \( R \) and \( Z \), respectively. We note that this may be an infinite expansion. This assumption gives
To derive the Hamiltonian, we start with the general ADM metric in two spacetime dimensions:

\[ ds^2 = -N(t,x)^2 dt^2 + \Lambda(t,x)^2 (N_t(t,x) dt + dx)^2. \]  

The unit normal to the spacelike hypersurface with constant \( t \) is given by

\[ u^4 \frac{\partial}{\partial y^4} = N^{-1} \left( \frac{\partial}{\partial t} - N_t \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial u}. \]  

In this parametrization we have\(^8\)

\[ Z = -R_u^2 + \Lambda^{-2} \pi_x^2, \]  

where the operator \( \partial/\partial u \) is defined by equation (4.8). Using this notation, the action (4.2) becomes

\[ I_G = \frac{2}{p-2} \int \sqrt{-g} \{ \beta^{(1)}(R) W_{(1)}(Z)(D_t R)(D^3 Z) \}. \]  

The contributions from \( I_G \) to the momenta conjugate to \( \Lambda \) and \( R \) are given, respectively, by

\[ P^G_{\Lambda} = -2l^{-(n-2)} \phi_R^2 R_{,u}, \]  

\[ P^G_R = 2l^{-(n-2)} \left( -\Lambda, u \phi_R + \frac{N_{,xx} \Lambda}{N} \phi_R - R_{,u} \Lambda \frac{\partial \eta(R,Z)}{\partial Z} \right). \]  

We note for future reference that

\[ \delta Z = -2 R_{,u} \delta R_{,u} + \delta b, \]  

where we have defined for convenience

\[ b := \frac{R_x^2}{N^2}. \]  

In order to write \( I^{(1)}_u \) defined by equation (4.6) in terms of phase space variables, we assume that \( W_{(1)}(Z) := W_{(1)}(-R_{,u}^2 + b) \) has a Taylor expansion in \( Z \) and hence in \( R_{,u}^2 \), so that

\[ W_{(1)}(Z) = \sum_m W^{(m)}_{(1)}(b) R_{,u}^{2m}. \]  

where

\[ W^{(m)}_{(1)}(b) := \left. \frac{(-1)^m}{m!} \frac{d^m W_{(1)}}{d Z^m} \right|_{Z=b}. \]  

\( ^8 \) The following is a slight departure from the notation in [18], where \( y \) was used rather than \( u \) for \( R_u \).
This gives us the relation:

\[ W^{(f)}(Z)R_\mu \delta Z = W^{(f)}(Z)R_\mu (-2R_\mu \delta R_\mu + \delta b) = -2 \sum_{m=0}^\infty w^{(m)}(b) R_\mu 2m+2 \delta R_\mu + W^{(f)}(Z)R_\mu \delta b \]

\[ = -2 \sum_{m=0}^\infty w^{(m)}(b) \frac{\delta (R_\mu 2m+3)}{2m+3} + W^{(f)}(Z)R_\mu \delta b. \]  

(4.17)

Using the above, after another lengthy derivation (see appendix A.1), we obtain the Lagrangian density \( \mathcal{L}^{(f)} \) for each of the terms satisfying \( \mathcal{L}^{(f)}_i = \int d^4y \sqrt{-g} \mathcal{L}^{(f)}_i \):

\( I^{n-2} \mathcal{L}^{(f)}_i = I^{n-2}(P^{(f)}_\Lambda \Lambda + \Lambda N_i P^{(f)}_\Lambda, x) - (N_i P^{(f)}_\Lambda, x) \)

\[ + 2NR_\mu \left( \beta^{(f)}(R) W^{(f)}(Z)R_\mu \frac{R_x}{\Lambda} + 2\beta^{(f)}(R) R \sum_m \frac{dw^{(m)}(b)}{db} \right) \]

\[ - 2\beta^{(f)}(R) W^{(f)}(Z)R_\mu \frac{R_x 2m+3}{2m+3} + \frac{N}{\Lambda} \beta^{(f)}(R) R \frac{W^{(f)}(Z)Z_x}{x}, \]  

(4.18)

where we have dropped total divergences and defined

\[ p^{(f)}_\Lambda = \frac{1}{I^{n-2}} \left\{ -2 \sum_m \beta^{(f)}(R) \left( w^{(m)}(b) - 2 \frac{dw^{(m)}(b)}{db} \right) \frac{R_x 2m+3}{2m+3} \right\}. \]

(4.19)

A key feature of this class of theories is that the total action is linear in \( \Lambda \). Thus from the above it is easy to extract the expression for the total \( P_\Lambda \):

\[ P_\Lambda = -\frac{2}{I^{n-2}} \beta^{(f)} R_\mu + \sum_i p^{(f)}_\Lambda. \]  

(4.20)

Equation (4.20) provides implicitly an expression for \( R_\mu \) as a function of the other phase space variables. From (4.20) we see that \( P_R \), the momentum conjugate of \( R \), is conspicuously absent in the final expression for \( R_\mu \), which therefore depends only on \( \Lambda, P_\Lambda \), and \( R \). The expression for \( P_R \) analogous to (4.20) is significantly more complicated but another important aspect of spherically symmetric ELL gravity that is shared by our XL gravity is that one does not need to make use of this expression when deriving the Hamiltonian equations of motion.

The total Hamiltonian density is

\[ \mathcal{H}_{XL} = P_\Lambda \Lambda + P_R R + \mathcal{L}_{XL} \]

\[ = N\mathcal{H} + N_i \mathcal{H}_i, \]  

(4.21)

where the total Lagrangian density is

\[ \mathcal{L}_{XL} := \frac{1}{I^{n-2}} \left\{ \phi(R) \mathcal{R} + \eta(R, Z) + \chi(R, Z)(D_\mu R)(D^\mu Z) \right\}. \]

(4.22)
The terms linear in $\Lambda$, again cancel, leaving:

$$
\mathcal{H} = p_R r_x + \frac{1}{l^{n-2}} \left[ \frac{2}{\Lambda} \left( \frac{\phi_x}{\Lambda} \right)_x \right] = \Lambda \eta (R, Z)
$$

$$
- 2R_x \sum_i \left( \beta^{(i)}(R) W^{(i)}(Z) R_x \frac{R_x}{\Lambda} + 2 \beta^{(i)}(R) \frac{R_x}{\Lambda} \sum_m \frac{dW^{(m)}(b)}{db} \frac{R_u}{2m+3} \right)_{,x}
$$

$$
+ 2 \Lambda R_x \sum_R \beta^{(i)}(R) \sum_m W^{(m)}(b) \frac{R_u}{2m+3} - \sum_i \frac{R_x}{\Lambda} \beta^{(i)}(R) W^{(i)}(Z)_{,x}.
$$

(4.23)

Finally, the ADM form of the XL action (3.2) is given by

$$
I_{XL} = \int dx (\Lambda, p_{\Lambda} + R_x, p_R - N \mathcal{H} - N_t \mathcal{H}_t).
$$

(4.25)

Variation of the Lagrange multipliers $N$ and $N_t$ give the Hamiltonian constraint $\mathcal{H} = 0$ and the diffeomorphism constraint $\mathcal{H}_t = 0$, respectively. Note that since $R_u$ is an implicit function of $L, \Lambda,$ and $R$, the momentum conjugate to $R$, namely $P_R$, appears only in the first terms of both the Hamiltonian and diffeomorphism constraints.

### 4.2. Mass function

The general Hamiltonian procedure for obtaining the mass function is to take the linear combination of $\mathcal{H}$ and $\mathcal{H}_t$ that eliminates $P_R$:

$$
\mathcal{H} = \frac{R_x}{\Lambda} \mathcal{H}_t = \frac{R_u}{\Lambda} \mathcal{H}_t
$$

$$
= p_R r_x + \frac{1}{l^{n-2}} \left[ \frac{2}{\Lambda} \left( \frac{\phi_x}{\Lambda} \right)_x \right] - \eta (R, Z) R_x
$$

$$
- 2R_x \sum_i \left( \beta^{(i)}(R) W^{(i)}(Z) R_x \frac{R_x}{\Lambda} + 2 \beta^{(i)}(R) \frac{R_x}{\Lambda} \sum_m \frac{dW^{(m)}(b)}{db} \frac{R_u}{2m+3} \right)_{,x}
$$

$$
+ 2R_x R_u \sum_i \left( \beta^{(i)}(R) \sum_m W^{(m)}(b) \frac{R_u}{2m+3} - \left( \frac{R_x}{\Lambda} \right)^2 \beta^{(i)}(R) W^{(i)}(Z)_{,x} \right).
$$

(4.26)

By expanding the explicit expression for $P_{\Lambda}$ (see appendix A.2), we obtain

$$
\tilde{\mathcal{H}} = (2 \phi_R R_x - \eta (R, Z)) R_x + (\phi_R - \chi (R, Z)) R_x.
$$

(4.27)

The total Hamiltonian is now

$$
H = \int dx (\tilde{\mathcal{H}} + N_t \mathcal{H}_t) + H_B.
$$

(4.28)

where $H_B$ is the boundary term required to make the variational principle well defined and will be made explicit below. New Lagrange multipliers $\tilde{N}$ and $\tilde{N}_t$ are defined by

$$
\tilde{N} := \frac{N \Lambda}{R_x},
$$

(4.29)
\[ \mathcal{N}_t := N_t + N R_{ uu} / R_{,x}, \]  

(4.30)

whose variations give the constraint equations \( \mathcal{H} = 0 \) and \( \mathcal{H}_t = 0 \), respectively.

Given the form of the Hamiltonian density in (4.27), the procedure for finding a mass function \( \mathcal{M}(R, Z) \) follows as in section 3.2. In the Hamiltonian context, we have

\[
\begin{align*}
\mathcal{H} &= (2\phi_{,RZ}Z - \eta (R, Z)) R_{,x} + (\phi_{R} - \chi (R, Z))Z_{,x} \\
&= -\mathcal{M}_{,x}, 
\end{align*}
\]

(4.31)

so that \( \mathcal{M} = \mathcal{M}(t) \) is satisfied on the constraint surface. From the expression \( \mathcal{M}_{,x} = \mathcal{M}_{,R} R_{,x} + \mathcal{M}_{,Z} Z_{,x} \), we obtain the same integrability condition for the existence of such a mass function as before, namely (3.14). Finally, we obtain the mass function defined by equations (3.15) or (3.16). It is also straightforward to verify that the mass function commutes with the total Hamiltonian, and therefore is also independent of time.

Lastly, we derive the boundary term \( H_B \) in the asymptotically flat case. The variation of the total Hamiltonian \( H \) takes the form

\[
\delta H = \int \delta x (-\mathcal{N} \delta \mathcal{M}_{,x} + \mathcal{N}_t \delta \mathcal{H}_t) + \delta H_B = \int \delta x \{ \mathcal{N}_{,x} \delta \mathcal{M} + \mathcal{N}_t \delta \mathcal{H}_t - (\mathcal{N} \delta \mathcal{M})_{,x} \} + \delta H_B. 
\]

(4.32)

We have neglected the variation of the Lagrange multipliers, which merely enforces the constraints. In the asymptotically flat case, we assume \( \mathcal{N} \to 1 \) at infinity, while \( \mathcal{M} = M \) is constant for vacuum solutions. Thus, as anticipated, the required boundary term is

\[
H_B = \int \delta x (\mathcal{N}_t \mathcal{M})_{,x} \bigg|_{x=x_B} = M, 
\]

(4.33)

which is the ADM mass, where \( x = x_B \) corresponds to the asymptotically flat region. Moreover, \( \mathcal{N} \to 0 \) holds in the asymptotically flat case so that this is in fact the only boundary term required.

5. Designing nonsingular black holes

In this section, we show how to construct specific nonsingular black holes as exact solution by making appropriate choices for the functions in the XL action (3.2). We are interested in constructing ‘physical’ nonsingular black holes whose curvature is everywhere bounded for arbitrarily large \( M \).

5.1. Criterion for physical nonsingular black holes

The standard 2D dilaton gravity theory (3.1) is solvable and obeys Birkhoff’s theorem. By making suitable choices of the functions of \( R \) in the action, the solutions can describe nonsingular spacetimes that have either two horizons or, at least in one special case, one horizon. If one chooses, for example, \( h(R) = V(R) = \phi_{,Rk}(R) \), the most general solution is [18]
\begin{align}
\text{d}s^2 &= g_{AB}\text{d}y^A\text{d}y^B \\
&= -\left(1 - \frac{l^{n-2}M}{j(R)}\right)\text{d}t^2 + \left(1 - \frac{l^{n-2}M}{j(R)}\right)^{-1}\text{d}R^2,
\end{align}

where

\begin{equation}
 j(R) := \int V(R)\text{d}R.
\end{equation}

As expected this solution contains a single parameter, \(M\), and has at least one Killing vector \(\partial/\partial t\). There are Killing horizons whenever \(j = l^{n-2}M\), and the Killing vector is timelike in the asymptotic region, \(j > l^{n-2}M\).

A special case of particular interest when \(n = 4\) is \(\phi_R = j(R) = (R^2 + l^2)^{3/2}/R^2\), which produces the well-known Bardeen metric [43]:

\begin{equation}
\text{d}s_{(4)}^2 = -\left(1 - \frac{l^2MR^2}{(R^2 + l^2)^{3/2}}\right)\text{d}t^2 + \left(1 - \frac{l^2MR^2}{(R^2 + l^2)^{3/2}}\right)^{-1}\text{d}R^2 + R^2\text{d}\Omega_{(2)}^2.
\end{equation}

Near \(R = 0\) this metric approaches de Sitter spacetime with curvature of order \(M/l\). It asymptotes to the Schwarzschild solution at spatial infinity as required. Although the Bardeen spacetime (5.3) is everywhere nonsingular, the maximum value of the curvature clearly grows without bound as the mass of the black hole is increased. For this reason, we consider this class of nonsingular black holes to be unphysical.

Loosely speaking, the problem with the Bardeen spacetime (5.3) stems from the fact that the mass \(M\) appears only in the numerator of the second term in \(g_{00}\). It turns out to be more difficult to find theories in which the mass \(M\) appears in the denominator of the metric functions. In fact, to the best of our knowledge it cannot be done within the framework of the action for pure 2D dilaton gravity (3.1).

5.2. Exact solutions

5.2.1. Hayward black hole. The following Hayward nonsingular black hole [44, 45] is different from the Bardeen black hole (5.3) in that it is physical in the sense defined above:

\begin{equation}
 f(R) = 1 - \frac{l^2MR^2}{R^3 + l^2M}.
\end{equation}

This metric also approaches the de Sitter form near \(R = 0\), but this time the curvature goes as \(1/l^2\). The curvature is therefore bounded for arbitrarily large \(M\).

The Hayward black hole can be easily generalized in \(n\)-dimensions as

\begin{equation}
 f(R) = 1 - \frac{l^{n-2}MR^2}{R^{n-1} + l^nM}.
\end{equation}

The mass-horizon relation is given by

\begin{equation}
 M = \frac{R_h^{n-1}}{l^{n-2}(R_h^2 - l^2)},
\end{equation}

where \(R = R_h\) is the radius of the Killing horizon, defined by \(f(R_h) = 0\). This shows that, for \(n \geq 4\), a black hole configuration with outer and inner horizons is realized for \(M > M_{\text{ex}}\), where
The lower bound \( M = M_{ex} \) gives an extremal black hole. Unlike the Reissner–Nordström black hole, the radius of the inner horizon converges to \( l \) in the limit of \( M \to \infty \).

Let us identify the conditions on the function in the XL action \((3.2)\) to admit this nonsingular black hole. Replacing \( f \) by \( Z \) in equation \((5.5)\), we obtain

\[
\mathcal{M} = \frac{R^{n-1}(1 - Z)}{l^{n-2}R^2 - l^n(1 - Z)}
\]

and hence

\[
\frac{\partial \mathcal{M}}{\partial R} = \frac{(n - 3)l^{n-2}R^n(1 - Z) - (n - 1)l^nR^{n-2}(1 - Z)^2}{(l^{n-2}R^2 - l^n(1 - Z))^2},
\]

\[
\frac{\partial \mathcal{M}}{\partial Z} = -\frac{l^{n-2}R^{n+1}}{(l^{n-2}R^2 - l^n(1 - Z))^2}.
\]

Comparing this to equations \((3.12)\) and \((3.13)\), we find the conditions:

\[
\eta(R, Z) = 2\phi_{RR}Z + \frac{(n - 3)l^{n-2}R^n(1 - Z) - (n - 1)l^nR^{n-2}(1 - Z)^2}{(l^{n-2}R^2 - l^n(1 - Z))^2},
\]

\[
\chi(R, Z) = \phi_R = -\frac{l^{n-2}R^{n+1}}{(l^{n-2}R^2 - l^n(1 - Z))^2}.
\]

Note that since \( Z \) takes values in the range \((0, 1)\), the denominators in \( \chi \) and \( \eta \) above are nowhere vanishing. As well one can verify that \( \phi_R - \chi \) vanishes only at the point \( R = 0 \), just as in the general relativistic limit \( l = 0 \), so that the gauge condition is valid everywhere except at the trivial coordinate singularity \( R = 0 \).

### 5.2.2. Bardeen-type black hole.

It is possible to construct physical nonsingular black holes similar to the Bardeen black hole \((5.3)\) using the action \( I_{XL} \). The \( n \)-dimensional version of this Bardeen-type black hole is

\[
f(R) = 1 - \frac{l^{n-2}MR^2}{(R^2 + M^{2/n-1})^{2/n-1}l^{2(n-1)/n-1}R^n},
\]

where we have assumed \( M \geq 0 \). This metric also reduces to de Sitter with bounded curvature for large \( M \). The mass-horizon relation for this Bardeen-type black hole is

\[
M = \frac{R_h^{n-1}}{l^{n-2}(R_h^{4/n-1} + M^{2/n-1}l^{2(n-1)/n-1})^{2/n-1} l^{4/n-1}l^{2(n-1)/n-1}/2}
\]

and the parameter dependence for a black hole configuration is similar to the Hayward black hole. For this Bardeen-type black hole, the mass parameter for an extremal black hole is

\[
M_{ex} = \frac{1}{l} \left( \frac{n - 1}{n - 3} \right)^{(n-1)/4} \left( \frac{n - 3}{2} \right)^{(n-1)/2}.
\]
In this case, the mass function is given by
\[ \mathcal{M} = \frac{R^{n-1}(1 - Z)}{\{l^{2(n-2)/(n-1)}R^4/(n-1) - l^{2n/(n-1)}(1 - Z)2/(n-1)\}^{(n-1)/2}} \] (5.15)
and hence
\[ \frac{\partial \mathcal{M}}{\partial R} = \frac{R^{n-2}(1 - Z)\{(n - 3)\}^{2(n-2)/(n-1)}R^4/(n-1) - \{l^{2n/(n-1)}(1 - Z)2/(n-1)\}}{\{l^{2(n-2)/(n-1)}R^4/(n-1) - l^{2n/(n-1)}(1 - Z)2/(n-1)\}^{(n-1)/2}}, \] (5.16)
\[ \frac{\partial \mathcal{M}}{\partial Z} = -\frac{l^{2(n-2)/(n-1)}R^{n^2 - 2n + 5}/(n-1)}{\{l^{2(n-2)/(n-1)}R^4/(n-1) - l^{2n/(n-1)}(1 - Z)2/(n-1)\}^{(n-1)/2}}. \] (5.17)
Comparing this to equations (3.12) and (3.13), we find that
\[ \eta(R, Z) = 2\phi_{\eta R}Z + \frac{R^{n-2}(1 - Z)\{(n - 3)\}^{2(n-2)/(n-1)}R^4/(n-1) - \{l^{2n/(n-1)}(1 - Z)2/(n-1)\}}{\{l^{2(n-2)/(n-1)}R^4/(n-1) - l^{2n/(n-1)}(1 - Z)2/(n-1)\}^{(n-1)/2}}, \] \[ \chi(R, Z) = \phi_{\eta R} - \frac{l^{2(n-2)/(n-1)}R^{n^2 - 2n + 5}/(n-1)}{\{l^{2(n-2)/(n-1)}R^4/(n-1) - l^{2n/(n-1)}(1 - Z)2/(n-1)\}^{(n-1)/2}}. \] (5.18)

### 5.2.3. New nonsingular black hole.

Another physical nonsingular black hole is
\[ f(R) = 1 + \frac{R^{n+1}}{2n^2 + 2M} \left( 1 - \sqrt{1 + \frac{4l^{2nM^2}}{R^{2(n-1)}}} \right). \] (5.19)
This metric resembles the vacuum solution in Einstein–Gauss–Bonnet gravity \([36, 46]\). The metric reduces to de Sitter and therefore the curvature is bounded for large \(M\). The mass-horizon relation is
\[ M = \frac{R_{n+1}}{l^{n-2}(R_h^4 - l^4)}. \] (5.20)
and the global structure is similar to that of the Hayward black hole. For this new black hole, the mass parameter for an extremal black hole is given by
\[ M_{ex} = \frac{n - 3}{2l} \left( \frac{n + 1}{n - 3} \right)^{(n+1)/4}. \] (5.21)
In this case, the mass function is given by
\[ \mathcal{M} = \frac{R^{n+1}(1 - Z)}{l^{n-2}R^4 - l^{n+2}(1 - Z)^2} \] (5.22)
and hence
\[ \frac{\partial \mathcal{M}}{\partial R} = \frac{(n - 3)l^{n-2}R^{n+1}(1 - Z) - (n + 1)l^{n+2}R^n(1 - Z)^3}{\{l^{n+2}(1 - Z)^2 - l^{n-2}R^4\}^2}. \] (5.23)
Comparing this to equations (3.12) and (3.13), we see that

$$
\eta(R, Z) = 2\phi_{RR}Z + \frac{(n-3)R_{,R} - (n+1)R_{,I}J^I}{\{R^2 - \frac{1}{2} R^4\}^2}
$$

$$
\chi(R, Z) = \phi_{,R} + \frac{R_{,R} - (n+1)R_{,I}J^I}{\{R^2 - \frac{1}{2} R^4\}^2}.
$$

5.3. Designer Lovelock gravity

We define dL gravity by the dimensionally reduced action (2.14) for spherically symmetric ELL gravity, but assume that all the \(\phi_{(p)}\) are potentially nonzero for any value of \(n\). In this case the action can no longer be lifted to a higher-dimensional ELL gravity since the corresponding Lovelock terms vanish identically for \(p > n/2\). It does nonetheless provide us with an interesting 2D generalization of the spherical theory that can be interpreted in one of two ways:

(i) the large coupling limit \(\phi_{(p)} = \infty\) for \(p \geq [(n-1)/2]\),

(ii) the large \(n\) limit \((n \rightarrow \infty)\).

Under the above assumption, the metric function \(f(R)\) is determined just as in ELL gravity by

$$
\frac{n-2M}{(n-2)R^{n-1}} = \sum_{p=0}^{\infty} \phi_{(p)} \left(1 - \frac{f(R)}{R^2}\right)^p.
$$

Since the right-hand side is an infinite series, it may be written as an analytic function by choosing \(\phi_{(p)}\) appropriately:

$$
\frac{n-2M}{(n-2)R^{n-1}} = s(\nu),
$$

where

$$
\nu = \frac{1 - \frac{f(R)}{R^2}}{R^2}.
$$

A given dL gravity action is therefore determined by a free analytic function \(s(\nu)\) that in turn determines the vacuum solution. The Hayward black hole (5.5) is realized in dL gravity by choosing the coupling constants to give

$$
s(\nu) = \frac{\nu}{(n-2)(1-\nu)}
$$

$$
= \frac{\nu}{n-2} \{1 + \nu^2 + (\nu^2)^2 + \cdots\}.
$$
Furthermore, with the following choice;

\[ s(\nu) = \frac{\nu}{(n-2)\{1 - (l^2\nu)^2\}} \]
\[ = \frac{\nu}{n-2}\{1 + (l^2\nu)^2 + (l^2\nu)^4 + \cdots\}, \quad (5.30) \]

the new nonsingular black hole (5.19) is realized. Although the metric resembles the vacuum solution in Einstein–Gauss–Bonnet gravity [36, 46], the solution is realized with only odd-order Lovelock terms in the action.

On the other hand, the Bardeen-type black hole (5.12) is realized for

\[ s(\nu) = \frac{\nu}{(n-2)(1 - l^4/(n-1)\nu^2/(n-1)^{(n-1)/2})}. \quad (5.31) \]

This cannot be realized as \( s(\nu) = \sum_{p=0}^{\infty} \delta_{(p)}\nu^p \) for any choice of \( \delta_{(p)} \). Hence, this nonsingular black hole is not realized in \( dL \) gravity and one needs to consider the full XL action (3.2).

6. Summary and future prospects

We have presented a new class of gravity theories in two space–time dimensions. The action contains three arbitrary functions and may provide a reasonable 2D effective theory for the spherically symmetric sector of a large class of higher-dimensional gravity theories. Our actions are readily understood as extensions of spherically symmetric ELL gravity. They share many, if not all, of the latter’s desirable properties.

As shown in section 3, the field equations are second order, which implies that the theories are ghost-free. We have also identified the integrability condition for the theories to admit as first integral a mass function that coincides with the generalized Misner–Sharp quasi-local mass in spherically symmetric ELL gravity [34]. The Hamiltonian analysis performed in section 4 showed that the super-Hamiltonian of the system is proportional to minus the spatial derivative of the mass function, and that, as a consequence, the on-shell mass function is a constant, both spatially and with respect to time.

As a consequence of the existence of the mass function, the system obeys Birkhoff’s theorem. In contrast to ELL gravity, the extensions admit a large class of static black holes as unique vacuum solutions that are nonsingular and have bounded curvature for arbitrarily large mass. In section 5, we presented examples of some physical nonsingular black hole solutions. A subset of these nonsingular black holes are realized in what we call \( dL \) gravity, which is understood as spherically symmetric ELL gravity with infinitely large coupling constants or infinite number of spacetime dimensions.

One natural and important question concerns which of the extended 2D theories that we have constructed can be obtained by imposing spherical symmetry in a fully covariant higher dimensional, higher curvature theory of gravity. In this context, a class of higher dimensional, higher curvature theories called ‘quasi-topological gravity’ may play an important role [47, 48]. Unlike ELL gravity, these theories are actually higher derivative theories whose field equations nonetheless become second order in spherically symmetric spacetimes. They therefore admit a mass function, and obey Birkhoff’s theorem. Although these theories do not admit nonsingular black holes, it is of great interest to understand the connection between our class of 2D dilaton gravity theories and dimensionally reduced quasi-topological gravity. Indeed, it would be very valuable to be able to identify the most general class of higher-dimensional theories giving second-order field equations in spherically symmetric spacetimes.
An important physical application of our new theories is the formation of a nonsingular black hole via gravitational collapse. Quantum effects are expected to resolve the classical singularity in general relativity. While the notion of space and time itself may very well break down in the vicinity of the classical singularity, it is possible, and perhaps even likely, that the quantum effects could alter the conformal structure of the spacetime commonly thought to represent the formation and evaporation via Hawking radiation of black holes. This conformal structure lies at the heart of the information loss conundrum. In order to determine whether or not the absence of a singularity can potentially solve the information loss problem, it is necessary to have models that allow a quantitative study of the formation and evaporation of nonsingular black holes. In this context, our new 2D dilaton gravity may play an important role since the dynamics of quantum corrected nonsingular spacetimes can in principle be modeled by an effective theory of the form that we have presented.

Acknowledgments

GK is grateful Jack Gegenberg, Viqar Husain, Jorma Louko, and Jon Ziprick for helpful conversations and comments on the manuscript. He also thanks Valeri Frolov for providing motivation to look for an action that could produce nice, nonsingular, spherically symmetric black hole solutions. HM thanks the Theoretical Physics group in University of Winnipeg for hospitality and support, where this work was started. This work was funded in part by the Natural Sciences and Engineering Research Council of Canada. Support was also provided by the Perimeter Institute for Theoretical Physics (funded by Industry Canada and the Province of Ontario Ministry of Research and Innovation). This work has also been funded by the Fondecyt Grant No. 3140123. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

Appendix. Details of derivations

A.1. Lagrangian density

Here we present how to derive the Lagrangian density (4.18). The Lagrangian density for the action (4.6) in terms of ADM variables is given by

\[
\mathcal{L}_L^{(I)} = \frac{1}{l^{n-2}} \beta^{(I)}(R) W_{(I)}(Z) \Delta N \left\{ R_{(I)Z_{(I)}} g^{00} + (R_{(I)Z_{(I)}} + R_{(I)Z_{(I)}}) g^{01} + R_{(I)Z_{(I)}} g^{11} \right\}
\]

\[
= \frac{1}{l^{n-2}} \beta^{(I)}(R) W_{(I)}(Z) \Delta N \left\{ \frac{R_{(I)Z_{(I)}}}{N^2} + \frac{N_I}{N^2} (R_{(I)Z_{(I)}} + R_{(I)Z_{(I)}}) + R_{(I)Z_{(I)}} \left( \frac{1}{N^2} - \frac{N^2}{N^2} \right) \right\}
\]

\[
= - \frac{1}{l^{n-2}} \beta^{(I)}(R) W_{(I)}(Z) \Delta N \left( R_{(I)Z_{(I)}} - \frac{1}{N^2} R_{(I)Z_{(I)}} \right).
\]

Note that in this appendix we drop the sum over \( I \) for simplicity. The first term in the above is the interesting one since we need to eliminate the second time derivatives in \( Z_{(I)} \) in order to put the action into the Hamiltonian form. Inspired by the appendix in [33], we write
\[
\mathcal{L}_{L}^{(t)} = \frac{1}{p^{n-2}} \beta^{(i)}(R)\mathcal{W}_{(i)}(Z) \Lambda N \left\{ R_{,a} \left( \frac{1}{N} Z_{,a} - \frac{N_{,a}}{N} Z_{,a} \right) - \frac{R_{,a} Z_{,a}}{N} \right\}
\]
\[
= \mathcal{L}_{L1}^{(t)} + \mathcal{L}_{L2}^{(t)} + \mathcal{L}_{L3}^{(t)}, \quad (A.2)
\]

where
\[
\mathcal{L}_{L1}^{(t)} := \frac{1}{p^{n-2}} \beta^{(i)}(R) \Lambda \mathcal{W}_{(i)}(Z) R_{,a} Z_{,a},
\]
\[
\mathcal{L}_{L2}^{(t)} := \frac{1}{p^{n-2}} \beta^{(i)}(R) N_{,a} \Lambda \mathcal{W}_{(i)}(Z) R_{,a} Z_{,a},
\]
\[
\mathcal{L}_{L3}^{(t)} := \frac{1}{p^{n-2}} \frac{N_{,a} \beta^{(i)}(R) R_{,a} \Lambda \mathcal{W}_{(i)}(Z) Z_{,a}}{N} \quad (A.3)
\]

We note that
\[
\delta Z = -2 R_{,a} \delta R_{,a} + \delta b, \quad (A.4)
\]
where we have defined for convenience
\[
b := \frac{R_{,a}^{2}}{N}. \quad (A.5)
\]

We now assume that \( \mathcal{W}_{(i)}(Z) := \mathcal{W}_{(i)}(-R_{,a}^{2} + b) \) has a Taylor expansion in \( Z \) and hence in \( R_{,a}^{2} \), so that
\[
\mathcal{W}_{(i)}(Z) = \sum_{m} \mathcal{W}_{(i)}^{(m)}(b) R_{,a}^{2m}, \quad (A.6)
\]

where
\[
\mathcal{W}_{(i)}^{(m)}(b) := \left( -1 \right)^{m} \frac{d^{m} \mathcal{W}_{(i)}(b)}{m!} \bigg|_{Z=b}. \quad (A.7)
\]

Then, we have
\[
\mathcal{W}_{(i)}(Z) R_{,a} \delta Z = \mathcal{W}_{(i)}(Z) R_{,a} (-2 R_{,a} \delta R_{,a} + \delta b)
\]
\[
= -2 \sum_{m} \mathcal{W}_{(i)}^{(m)}(b) R_{,a}^{2m+2} \delta R_{,a} + \mathcal{W}_{(i)} R_{,a} \delta b
\]
\[
= -2 \sum_{m} \mathcal{W}_{(i)}^{(m)}(b) \frac{\delta (R_{,a}^{2m+3})}{2m + 3} + \mathcal{W}_{(i)} R_{,a} \delta b. \quad (A.8)
\]

The first term can now be integrated by parts term by term. This gives, up to total derivatives
\[
\mathcal{L}_{L1}^{(t)} := \frac{1}{p^{n-2}} \beta^{(i)}(R) \mathcal{W}_{(i)}(Z) \Lambda R_{,a} Z_{,a}
\]
\[
= - \frac{1}{p^{n-2}} \beta^{(i)}(R) \Lambda \left\{ -2 \sum_{m} \mathcal{W}_{(i)}^{(m)}(b) \frac{(R_{,a}^{2m+3})_{,a}}{2m + 3} + \mathcal{W}_{(i)}(Z) R_{,a} b_{,a} \right\}
\]
\[
= \frac{1}{p^{n-2}} \left\{ -2 \sum_{m} \beta^{(i)}(R) \Lambda \mathcal{W}_{(i)}^{(m)}(b) R_{,a}^{2m+3} - \beta^{(i)}(R) \Lambda \mathcal{W}_{(i)}(Z) R_{,a} b_{,a} \right\}. \quad (A.9)
\]
Similarly we obtain
\[
\mathcal{L}_{L2}^{(i)} = \frac{1}{p-2} \left\{ 2 \sum_m (N_i \beta^{(i)}(R) \Lambda w_{(l)}^{(m)}(b))_x \frac{R_{,u}^{2m+3}}{2m+3} + N_i \beta^{(i)}(R) \Lambda W_{(l)}(Z) R_{,u} b_x \right\}. \quad \text{(A.10)}
\]

Finally, \( \mathcal{L}_{L3}^{(i)} \) is straightforward because of
\[
W_{(l)}(Z) Z_x = (X_{(l)}(Z))_x,
\]
where
\[
W_{(l)}(Z) = X_{(l)}(Z) Z_x.
\]
Thus \( \mathcal{L}_{L3}^{(i)} \) is computed as
\[
\begin{align*}
\mathcal{L}_{L3}^{(i)} &= \frac{1}{p-2} \left( N \beta^{(i)}(R) W_{(l)}(Z) R_x Z_x ight) \\
&= \frac{1}{p-2} \left( N \beta^{(i)}(R) R_x X_{(l)}(Z) \right)_x \\
&= - \left( \frac{1}{p-2} N \beta^{(i)}(R) R_x \right)_x X_{(l)}(Z) \\
&= - \left( \beta^{(i)}(R) R_x \right)_x X_{(l)}(Z) \quad \text{(A.13)}
\end{align*}
\]
up to boundary terms.

Putting it all together, we have
\[
\mathcal{L}_L^{(i)} = \frac{1}{p-2} \left\{ -2 \sum_m (\beta^{(i)}(R) \Lambda w_{(l)}^{(m)}(b))_x \frac{R_{,u}^{2m+3}}{2m+3} - \beta^{(i)}(R) \Lambda W_{(l)}(Z) R_{,u} b_x \right\} + \frac{1}{p-2} \left\{ 2 \sum_m (N_i \beta^{(i)}(R) \Lambda w_{(l)}^{(m)}(b))_x \frac{R_{,u}^{2m+3}}{2m+3} + N_i \beta^{(i)}(R) \Lambda W_{(l)}(Z) R_{,u} b_x \right\} \\
- \left( \frac{1}{p-2} N \beta^{(i)}(R) R_x \right)_x X_{(l)}(Z). \quad \text{(A.14)}
\]
We will concentrate on the first line of the above, since it has the time derivatives of \( \Lambda \).
We will need
\[
b_{,t} = 2 \frac{R_x^2}{N^3} R_{,xt} - \frac{2 R_{,xt}^2}{N^3} \Lambda_{,t}.
\]
Thus we have
\[
\begin{align*}
p_{,l} \mathcal{L}_L^{(i)} &= -2 \sum_m (\beta^{(i)}(R) \Lambda w_{(l)}^{(m)}(b))_x \frac{R_{,u}^{2m+3}}{2m+3} - \beta^{(i)}(R) \Lambda W_{(l)}(Z) R_{,u} b_x \\
&= -2 \sum_m (\beta^{(i)}(R) \Lambda w_{(l)}^{(m)}(b))_x + \beta^{(i)}(R) \Lambda w_{(l)}^{(m)}(b) \\
&\quad + \beta^{(i)}(R) \Lambda d w_{(l)}^{(m)}(b) db \left( 2 \frac{R_x^2}{N^3} R_{,xt} - \frac{2 R_{,xt}^2}{N^3} \Lambda_{,t} \right) \frac{R_{,u}^{2m+3}}{2m+3} \\
&\quad - \beta^{(i)}(R) \Lambda W_{(l)}(Z) R_{,u} \left( 2 \frac{R_x^2}{N^3} R_{,xt} - \frac{2 R_{,xt}^2}{N^3} \Lambda_{,t} \right). \quad \text{(A.16)}
\end{align*}
\]
Collecting time derivatives of $\Lambda$, we get

$$I^{n-2} L_{\Lambda_1}^{(1)} = -2 \sum_m \left\{ \beta^{(j)}(R) w^{(m)}_{(j)}(b) \Lambda_x - \beta^{(j)}(R) \frac{d}{db} \left( -\frac{2 R_x^2}{N^3} \Lambda_x \right) \right\} \left( \frac{R_{aR}^{2m+3}}{2m+3} \right)$$

$$- \beta^{(j)}(R) \Lambda W_{(j)}(Z) R_{aR} \left( -\frac{2 R_x^2}{N^3} \Lambda_x \right)$$

$$- 2 \sum_m \left\{ \beta^{(j)}(R) \frac{\Lambda W^{(m)}_{(j)}(b)}{d} + \beta^{(j)}(R) \frac{d w^{(m)}_{(j)}(b)}{d} \left( \frac{2 R_x^2}{N^3} \Lambda_x \right) \right\} \left( \frac{R_{aR}^{2m+3}}{2m+3} \right)$$

$$- \beta^{(j)}(R) \Lambda W_{(j)}(Z) R_{aR} \left( -\frac{2 R_x^2}{N^3} \Lambda_x \right)$$

$$- 2 \sum_m \left\{ \beta^{(j)}(R) \frac{\Lambda W^{(m)}_{(j)}(b)}{d} + \beta^{(j)}(R) \frac{d w^{(m)}_{(j)}(b)}{d} \left( \frac{2 R_x^2}{N^3} \Lambda_x \right) \right\} \left( \frac{R_{aR}^{2m+3}}{2m+3} \right)$$

$$- \beta^{(j)}(R) \Lambda W_{(j)}(Z) R_{aR} \left( -\frac{2 R_x^2}{N^3} \Lambda_x \right)$$

$$= I^{n-2} P^{(1)}_{\Lambda} - \beta^{(j)}(R) \Lambda W_{(j)}(Z) R_{aR} \left( \frac{R_x^2}{N^3} \Lambda_x \right)$$

$$- 2 \sum_m \left\{ \beta^{(j)}(R) \frac{\Lambda W^{(m)}_{(j)}(b)}{d} + \beta^{(j)}(R) \frac{d w^{(m)}_{(j)}(b)}{d} \left( \frac{2 R_x^2}{N^3} \Lambda_x \right) \right\} \left( \frac{R_{aR}^{2m+3}}{2m+3} \right). \quad (A.17)$$

where

$$P^{(1)}_{\Lambda} = \frac{1}{I^{n-2}} \left\{ -2 \sum_m \beta^{(j)}(R) \left\{ w^{(m)}_{(j)}(b) - \frac{d}{db} \left( \frac{R_x^2}{N^3} \Lambda_x \right) \right\} \left( \frac{R_{aR}^{2m+3}}{2m+3} \right) \right\}$$

$$+ 2 \beta^{(j)}(R) \Lambda W_{(j)}(Z) R_{aR} \left( \frac{R_x^2}{N^3} \Lambda_x \right) \quad (A.18)$$

is the contribution to the conjugate momentum of $\Lambda$ from $L_{\Lambda_1}^{(1)}$.

We now repeat this for the second line of (A.17). Noting that it is exactly the same form as the first line, we obtain

$$I^{n-2} L_{\Lambda_2}^{(1)} = 2 \sum_{m=0} \left\{ \frac{N_a}{N} \beta^{(j)}(R) \Lambda w^{(m)}_{(j)}(b) \right\} \left( \frac{R_{aR}^{2m+3}}{2m+3} \right) + N_a \beta^{(j)}(R) \Lambda W_{(j)}(Z) R_{aR} b_x$$

$$= 2 N_a \sum_{m=0} \left\{ \frac{N_a}{N} \beta^{(j)}(R) \Lambda w^{(m)}_{(j)}(b) + \beta^{(j)}(R) \Lambda w^{(m)}_{(j)}(b) + \beta^{(j)}(R) \Lambda \Lambda w^{(m)}_{(j)}(b) \right\}$$

$$+ \beta^{(j)}(R) \Lambda \Lambda w^{(m)}_{(j)}(b) \frac{d}{db} \left( \frac{R_{aR}^{2m+3}}{2m+3} \right)$$

$$+ N_a \beta^{(j)}(R) \Lambda W_{(j)}(Z) R_{aR} \left( \frac{R_x^2}{N^3} \Lambda_x - \frac{2 R_x^2}{N^3} \Lambda_x \right). \quad (A.19)$$
Collecting terms in $\Lambda_s$, we get

$$\mathcal{L}^{(i)}_{L3} = \mathcal{L}^{(i)}_{L2} + \sum_n \left\{ \beta^{(i)}(R) \omega_{(m)}^{(n)}(b) + \beta^{(i)}(R) \frac{d \omega_{(m)}^{(n)}(b)}{db} \right\} R_s^{2m+3} + \beta^{(i)}(R) \Lambda W_{(i)}(Z) R_{u \Lambda} \left( \frac{2 R_s^2}{\Lambda^3} \right) R_s^{2m+3}$$

The third line of (A.14) can be integrated by parts to give

$$\mathcal{L}^{(i)}_{L3} = \frac{1}{\mu^2} \sum_n \left\{ \beta^{(i)}(R) \Lambda_s W_{(i)}(Z) \right\} R_{s \Lambda} \left( \frac{2 R_s^2}{\Lambda^3} \right) R_s^{2m+3}$$

Putting together (A.17), (A.20), and (A.21) and replacing $R_s$ by $N R_{u \Lambda} + N R_s$, we get

$$\mathcal{L}^{(i)}_{L3} = \mathcal{L}^{(i)}_{L2} - \sum_n \left\{ \beta^{(i)}(R) \omega_{(m)}^{(n)}(b) + \beta^{(i)}(R) \frac{d \omega_{(m)}^{(n)}(b)}{db} \right\} R_s^{2m+3} + \sum_n \beta^{(i)}(R) \Lambda W_{(i)}(Z) R_{u \Lambda} \left( \frac{2 R_s^2}{\Lambda^3} \right) R_s^{2m+3}$$

The first term will cancel with the corresponding Liouville term when constructing the Hamiltonian density. The second term gives the expected contribution to the diffeomorphism constraint. The next two terms give nontrivial contributions to the Hamiltonian constraint.

### A.2. Hamiltonian density

Here we present how to derive the Hamiltonian density (4.27).
The momentum conjugate to $\Lambda$ is

$$P_\Lambda = -\frac{2}{P_{\phi_K R,u}} \phi_K R,u + \sum_{i} P_{\phi_i}^{(i)},$$

(A.23)

where the second term is given in (A.18). The total Hamiltonian density is then

$$\mathcal{H}_{XL} = P_\Lambda \Lambda + P_K R,t - \mathcal{L}_{XL} = N\mathcal{H} + N_t \mathcal{H}_t,$$

(A.24)

where now

$$\mathcal{H} = P_K R,u + \frac{1}{P_{\phi_K R,u}} \sum_{i} \left\{ 2 \left( \frac{\phi_i}{\Lambda} \right)_{,i} - \Lambda \eta(R, Z) \right\}$$

and

$$2 R,u \left( \beta^{(i)}(R) W(Z) R,u \frac{R}{\Lambda} + 2 \beta^{(i)}(R) \frac{R}{\Lambda} \sum_m \frac{dw^{(m)}(b)}{db} R,u^{2m+3}_a \right)_{,i}$$

$$+ 2 \beta^{(i)}_{,K} \Lambda R,u \sum_m w^{(m)}(b) \frac{R,u^{2m+3}_a}{2m+3} - \frac{R,u}{\Lambda} \beta^{(i)}(R) W(Z) Z,s.$$  

(A.25)

$$\mathcal{H}_t = P_K R, s - P_\Lambda \Lambda.$$  

(A.26)

In the above $R,u$ is an implicit function of $\Lambda$, $P_\Lambda$, and $R$ given by (4.20).

As before we want to eliminate $P_K$ completely from the Hamiltonian constraint in the hopes of finding a suitable mass function, so we define

$$\tilde{\mathcal{H}} = \frac{R}{\Lambda} \mathcal{H} - \frac{R}{\Lambda} \mathcal{H}_t$$

$$= R,u P_{\Lambda,s} + \frac{1}{P_{\phi_K R,u}} \sum_{i} \left\{ 2 \left( \frac{\phi_i}{\Lambda} \right)_{,i} - \eta(R, Z) R,s \right\}$$

$$- 2 R,u \left( \beta^{(i)}(R) W_r(Z) R,u \frac{R}{\Lambda} + 2 \beta^{(i)}(R) \frac{R}{\Lambda} \sum_m \frac{dw^{(m)}(b)}{db} R,u^{2m+3}_a \right)_{,i}$$

$$+ 2 \beta^{(i)}_{,K} R,u \sum_m w^{(m)}(b) \frac{R,u^{2m+3}_a}{2m+3} - \left( \frac{R,u}{\Lambda} \right)^2 \beta^{(i)}(R) W(Z) Z,s.$$  

(A.27)

We need to express the first term in terms of $R,u$:

$$R,u P_{\Lambda,s} = -\frac{2}{P_{\phi_K R,u}} \sum_{i} \left\{ \phi_{i, K} R,u \right\}$$

$$+ \sum_m \beta^{(i)}(R) \left( w^{(m)}(b) - \frac{2 dw^{(m)}(b) R,u^2}{N^2} \right) \frac{R,u^{2m+3}_a}{2m+3} - \beta^{(i)}(R) W_r(Z) \frac{R,u}{N} \right\}$$

$$= \frac{1}{P_{\phi_K R,u}} \sum_{i} \left\{ -\frac{1}{\phi_{i, K}} (\phi_{i, K} R,u)^2 \right\}_{,i}$$

$$- 2 R,u \left\{ \sum_m \beta^{(i)}(R) \left( w^{(m)}(b) - \frac{2 dw^{(m)}(b) R,u^2}{N^2} \right) \frac{R,u^{2m+3}_a}{2m+3} \right\}$$

$$- \beta^{(i)}(R) W_r(Z) \frac{R,u^2}{N} \right\}_{,i}.$$

(A.28)
Putting (A.28) into (A.27) yields

\[
\hat{\mathcal{H}}^{n-2} = \frac{1}{\phi_R} \left\{ -\left( (\phi_R R)_{\alpha} \right)^2_x + \frac{2 \phi_R R_x}{\Lambda} \left( \frac{\phi_R}{\Lambda} \right)_x - \eta(R, Z) R_x \right\}
- \left( \frac{R_x}{\Lambda} \right)^2 \sum_i \beta^{(i)}(R) W_{(i)}(Z) Z_{ix} + \sum_i \beta^{(i)}(R) R_{ix} \left( \frac{R_x^2}{\Lambda^2} \right)_x
\]

\[
+ \sum_i 2 \beta^{(i)} R_{ix} \left( \frac{R_x^2}{\Lambda^2} \right)_x \sum_m \frac{dW_m^{(i)}(b)}{db} R_{ix}^{2m+3} - \sum_i 2 \beta^{(i)} R_{ix} \left( \frac{R_x^2}{\Lambda^2} \right)_x \sum_m \frac{dW_m^{(i)}(b)}{db} R_{ix}^{2m+3}
\]

\[
\frac{1}{\phi_R} \left( \phi_R Z \right)_x - \eta(R, Z) R_x - \left( \frac{R_x}{\Lambda} \right)^2 \sum_i \beta^{(i)}(R) W_{(i)}(Z) Z_{ix}
\]

\[
+ \sum_i \beta^{(i)} W R_{ix} \left( \frac{R_x^2}{\Lambda^2} \right)_x + \sum_i 2 \beta^{(i)} R_{ix} \left( \frac{R_x^2}{\Lambda^2} \right)_x \sum_m \frac{dW_m^{(i)}(b)}{db} R_{ix}^{2m+3}
\]

\[
- \sum_i 2 \beta^{(i)} R_{ix} \left( \frac{W_m^{(i)}(b)}{R_{ix}^{2m+3}} \right)_x.
\]  

(A.29)

Expanding the derivative in the last term gives

\[
-2 \beta^{(i)} R_{ix} \left( \sum_m W_m^{(i)}(b) \frac{R_{ix}^{2m+3}}{2m+3} \right)_x
\]

\[
= -2 \beta^{(i)} \left( \frac{R_x^2}{\Lambda^2} \right)_x \sum_m \frac{dW_m^{(i)}(b)}{db} R_{ix}^{2m+3} - \frac{2 \beta^{(i)} R_x^2}{\Lambda^2} \sum_m W_m^{(i)}(b) R_{ix}^{2m}
\]

\[
- \beta^{(i)} W R_{ix} \left( \frac{R_x^2}{\Lambda^2} \right)_x + \frac{2 \beta^{(i)} R_x^2}{\Lambda^2} \sum_m \frac{dW_m^{(i)}(b)}{db} R_{ix}^{2m+3} - \beta^{(i)} W R_{ix} \left( \frac{R_x^2}{\Lambda^2} \right)_x.
\]  

(A.30)

Putting the above into (A.29) gives, quite miraculously

\[
\hat{\mathcal{H}} = \frac{1}{\phi_R} \left( \phi_R^2 Z \right)_x - \eta(R, Z) R_x
\]

\[
+ \sum_i \left\{ \beta^{(i)} W_{(i)}(Z) R_{ix}^2 Z_{ix} - \frac{R_x^2}{\Lambda} \beta^{(i)}(R) W_{(i)}(Z) Z_{ix} \right\}
\]

\[
= (2 \phi_R^2 - \eta(R, Z)) R_x + (\phi_R - \chi(R, Z)) Z_{ix},
\]  

(A.31)

where $\chi(R, Z)$ is defined in (4.4).

References

[1] Bojowald M and Skirzewski A 2006 Effective equations of motion for quantum systems Rev. Math. Phys. 18 713

[2] Chacón-Acosta G and Hernández H H 2011 Effective quantum equations for the semiclassical description of the hydrogen atom arXiv:1110.3337

[3] Das S 2013 Quantum Raychaudhuri equation Phys. Rev. D 89 084068
[4] Peltola A and Kunstatter G 2009 A complete single-horizon quantum corrected black hole space–time Phys. Rev. D 79 061501
[5] Peltola A and Kunstatter G 2009 Effective polymer dynamics of D-dimensional black hole interiors Phys. Rev. D 80 044031
[6] Sakharov Ad 1966 The initial stage of an expanding universe and the appearanc of a nonuniform distribution of matter Sov. Phys. JETP 22 241
[7] Borde A 1997 Regular black holes and topology change Phys. Rev. D 55 7615–7
[8] Poisson E and Israel W 1988 Structure of the black hole nucleus Class. Quantum Grav. 5 L201
[9] Dymnikova I and Galaktionov E 2005 Stability of a vacuum nonsingular black hole Class. Quantum Grav. 22 2331–57
[10] Ayón-Beato E and Garcia A 1998 Regular black hole in general relativity coupled to nonlinear electrodynamics Phys. Rev. Lett. 80 5056–9
[11] Ayón-Beato E and Garcia A 1999 New regular black hole solution from nonlinear electrodynamics Phys. Lett. B 464 25
[12] Ayón-Beato E 1999 Non-singular charged black hole solution for nonlinear source Gen. Relativ. Gravit. 31 629–33
[13] Ayón-Beato E and Garcia A 2000 The Bardeen model as a nonlinear magnetic monopole Phys. Lett. B 493 149–52
[14] Ayón-Beato E and Garcia A 2005 Four parametric regular black hole solution Gen. Rel. Grav. 37 635
[15] Grumiller D, Kummer W and Vassilevich D V 2002 Dilaton gravity in two-dimensions Phys. Rep. 369 327–430
[16] Ziprick J and Kunstatter G 2010 Quantum corrected spherical collapse: a phenomenological framework Phys. Rev. D 82 044031
[17] Ziprick J 2009 Singularity resolution and dynamical black holes MSc Thesis University of Manitoba
[18] Taves T and Kunstatter G 2014 Modelling the evaporation of nonsingular black holes Phys. Rev. D 90 124062
[19] Grumiller D 2003 Deformations of the Schwarzschild black hole arXiv:gr-qc/0311011
[20] Grumiller D 2004 Long time black hole evaporation with bounded Hawking flux J. Cosmol. Astropart. Phys. JCAP5(2004)004
[21] Hertog T and Horowitz G T 2005 Designer gravity and field theory effective potentials Phys. Rev. Lett. 94 221301
[22] Maeda H and Nozawa M 2008 Generalized Misner–Sharp quasi-local mass in Einstein–Gauss–Bonnet gravity Phys. Rev. D 77 064031
[23] Palais R S 1979 The principle of symmetric criticality Commun. Math. Phys. 69 19–30
[24] Fels M E and Torre C G 2002 The principle of symmetric criticality in general relativity Class. Quantum Grav. 19 641–75
[25] Deser S and Tekin B 2003 Shortcuts to high symmetry solutions in gravitational theories Class. Quantum Grav. 20 4877–83
[26] Misner C W and Sharp D H 1964 Relativistic equations for adiabatic, spherically symmetric gravitational collapse Phys. Rev. 136 B571–6
[27] Lanczos C 1938 A remarkable property of the Riemann–Christoffel tensor in four-dimensions Ann. Math. 39 842.
[28] Lovelock D 1971 The Einstein tensor and its generalizations J. Math. Phys. 12 498–501
[29] Garraffo C and Giribet G 2008 The Lovelock black holes Mod. Phys. Lett. A 23 1801–18
[30] Charmousis C 2009 Higher order gravity theories and their black hole solutions Phys. Lecture Notes
[31] Kunstatter G, Taves T and Maeda H 2012 Geometrodynamics of spherically symmetric Lovelock gravity Class. Quantum Grav. 29 092001
[32] Kunstatter G, Maeda H and Taves T 2013 Hamiltonian dynamics of Lovelock black holes with spherical symmetry Class. Quantum Grav. 30 065002
[33] Taves T 2013 Black hole formation in Lovelock gravity PhD Thesis University of Manitoba
[34] Maeda H, Willison S and Ray S 2011 Lovelock black holes with maximally symmetric horizons Class. Quantum Grav. 28 165005
[35] Zegers R 2005 Birkhoffs theorem in Lovelock gravity J. Math. Phys. 46 072502
[36] Wheeler J T 1986 Symmetric solutions to the Gauss–Bonnet extended Einstein equations Nucl. Phys. B 268 737–46
[37] Whitt B 1988 Spherically symmetric solutions of general second order gravity Phys. Rev. D 38 3000

[38] Tibrewala R 2015 New second derivative theories of gravity for spherically symmetric spacetimes Class. Quantum Grav. 32 115007

[39] Louko J, Simon J and Winters-Hilt S 1997 Hamiltonian thermodynamics of a Lovelock black hole Phys. Rev. D 55 3525–35

[40] Taves T, Leonard C D, Kunstatter G and Mann R B 2012 Hamiltonian formulation of scalar field collapse in Einstein–Gauss–Bonnet gravity Class. Quantum Grav. 29 015012

[41] Deser S and Franklin J 2005 Birkhoff for Lovelock redux Class. Quantum Grav. 22 L103–6

[42] Teitelboim C and Zanelli J 1987 Dimensionally continued topological gravitation theory Class. Quantum Grav. 4 125–9

[43] Bardeen J 1968 Non-singular general-relativistic gravitational collapse Conf. Proc. GR5 (Tiblisi, U.S.S.R)

[44] Hayward S A 2006 Formation and evaporation of nonsingular black holes Phys. Rev. Lett. 96 031103

[45] Frolov V P 2014 Information loss problem and a black hole model with a closed apparent horizon J. High Energy Phys. JHEP14(2014)49

[46] Boulware D G and Deser S 1985 String-generated gravity models Phys. Rev. Lett. 55 2656–60

[47] Oliva J and Sourya R 2011 Birkhoff’s theorem in higher derivative theories of gravity Class. Quantum Grav. 28 175007

[48] Myers R and Robinson B 2010 Black holes in quasi-topological gravity J. High Energy Phys. JHEP08(2010)067