Non-isospectral extension of the Volterra lattice hierarchy, and Hankel determinants

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Received 15 December 2017, revised 4 June 2018
Accepted for publication 19 June 2018
Published 7 August 2018

Abstract

For the first two equations of the Volterra lattice hierarchy and the first two equations of its non-autonomous (non-isospectral) extension, we present Riccati systems for functions $c_j(t)$, $j = 0, 1, \ldots$, such that an expression in terms of Hankel determinants built from them solves these equations on the right half of the lattice. This actually achieves a complete linearization of these equations of the extended Volterra lattice hierarchy.

Keywords: Volterra lattice hierarchy, Hankel determinant, orthogonal polynomials, Hirota bilinearization, non-isospectral

Mathematics Subject Classification numbers: 37K10, 37K15, 42C05

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1. Introduction

The Volterra lattice equation\(^7\)
\[
\frac{du_n}{dt} = u_n (u_{n+1} - u_{n-1}) =: V^{(1)}, \quad n \in \mathbb{Z},
\] (1.1)
is one of the most important integrable partial differential-difference equations \([6]\). In particular, it is a semi-discretization of the inviscid Burgers (also called Riemann or Hopf) equation, but also of the famous Korteweg–deVries (KdV) equation (see \([22, 35]\)). Because of the latter, it is sometimes referred to as ‘discrete KdV equation’ (which should not be confused with a full discretization). Another name used for it (mostly if it is presented in a certain equivalent form) is ‘Kac–van-Moerbeke equation’. The Volterra lattice is well-known for its use to model population dynamics in biological systems \([21, 50]\). It also models, for example, the propagation of electron density waves (‘Langmuir oscillations’) that originate from an instability when a periodic electric field is applied to a homogeneous and isotropic plasma. More precisely, the Volterra lattice describes the discrete chain of the peaks \([31, 53]\). Moreover, the Volterra lattice also models certain electric network (ladders) built with inductors and capacitors \([19, 20]\).

An integrable partial differential-difference equation typically belongs to a hierarchy, which is an infinite sequence of (somewhat similar) integrable equations, with increasing complexity and such that the flows mutually commute. The Volterra lattice equation (1.1) extends to the hierarchy (see, e.g. \([30, 37, 54]\)), given by
\[
\frac{du_n}{dt_k} = V^{(k)} = \mathcal{R}^{k-1} V^{(1)}, \quad k = 1, 2, \ldots,
\] (1.2)
with a ‘recursion operator’ \(\mathcal{R}\) specified in section 2. Requiring \(u\) to be a common solution, we have
\[
\frac{\partial^2 u_n}{\partial t_k \partial t_l} = \frac{\partial^2 u_n}{\partial t_l \partial t_k} \quad \forall k, l = 1, 2, \ldots, k \neq l,
\]
which imposes conditions on the right hand sides of the equation (1.2). Commutativity of flows means that these conditions are fulfilled as a consequence of the equations. This ‘symmetry condition’ has its roots in Lie theory. Each hierarchy equation is a symmetry of any other, see e.g. \([38]\). If \(V'\) denotes the Fréchet derivative (see e.g. \([38]\)) of \(V\) with respect to \(u_n\), and \([V, W] := V'[W] - W'[V]\), the commutativity of flows can be expressed as
\[
[V^{(k)}, V^{(j)}] = 0 \quad j, k = 1, 2, \ldots.
\]
The second equation of the Volterra lattice hierarchy is
\[
\frac{du_n}{dt_2} = u_n \left( u_{n+1}(u_n + u_{n+1} + u_{n+2}) - u_{n-1}(u_{n-2} + u_{n-1} + u_n) \right).
\] (1.3)
Explicit expressions of the higher flows of the Volterra lattice hierarchy can also be found in \([46, 47]\).

Moreover, in this work we address an extension of (1.2) by additional non-autonomous equations that possess a non-isospectral Lax pair. This means that such an equation arises as the compatibility condition of a system of two linear equations (Lax pair), depending on a (‘spectral’) parameter that is a function of the respective evolution variable\(^8\). Because of this

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7 The subscript of \(u_n\) corresponds to a point on a one-dimensional lattice. We hide away a corresponding subscript of \(V^{(1)}\), for simplicity.

8 Also see remark 2.1 below.
property, such equations are often referred to as ‘non-isospectral flows’. The non-autonomous extension of the Volterra lattice hierarchy,

\[ \frac{du_n}{d\tau_k} = V^{(k)} = R^k u_n, \quad k = 1, 2, \ldots, \]  

first appeared in a different form in [4] (also see appendix B), where more generally a non-autonomous extension of the Volterra lattice hierarchy has been treated (also see [3, 27] and [2, 33, 34]). It was later rediscovered in [30] and generalized to \(2 + 1\) dimensions in [16], a further extension, which will not be considered in this work. How (1.4) indeed leads to non-autonomous flows is explained in section 2. The flows (1.4) do not commute with each other and also not with the flows of the Volterra lattice hierarchy. We recall from [30] the commutation relations

\[ [V^{(k)}, V^{(j)}] = k V^{(k+j)}, \quad [V^{(j)}, V^{(k)}] = (k - j) V^{(j+k)}, \]

where \( j, k = 1, 2, \ldots \). The same structure is also known from non-isospectral extensions of other hierarchies [28, 29]. Despite of this non-commutativity of flows, the equation

\[ \frac{du_n}{dt} = \sum_{k=1}^{\infty} \left( \alpha_k(t) V^{(k)} + \beta_k(t) V^{(k)} \right) + \beta_0 u_n \]  

is integrable, in the sense of possessing a Lax pair, for any choice of functions \( \alpha_k, \beta_k \) of \( t \) [4, 30], also see remark 2.1 below.

The first non-autonomous equation of the extended hierarchy,

\[ \frac{du_n}{d\tau_1} = u_n \left[ (n+1) u_{n+1} + u_n - (n-2) u_{n-1} \right], \]  

already appeared in [11, 26, 41]. Quite surprisingly, this equation is deeply related to the problem of computing the coefficients in a continued fraction expansion of a given function, Thiele’s expansion formula. Underlying is a method to construct interpolation rational functions (Thiele’s interpolation formula), see e.g. [7, 15]. This interesting observation is elaborated in appendix D.

The second non-autonomous equation of the extended hierarchy is already quite involved. In section 2 we show that it can be chosen as

\[ \frac{du_n}{d\tau_2} = u_n \left( (3-n)u_{n-2}u_{n-1} + (3-n)u_{n-1}^2 + (2-n)u_{n-1}u_n + u_n^2 + (n+3)u_nu_{n+1} 
\]
\[ + (n+2)u_{n+1}^2 + (n+2)u_{n+1}u_{n+2} + 2(u_{n+1} - u_{n-1}) \sum_{i=1}^{n-1} u_i \right). \]  

Some integrable lattice equations, notably the famous Toda lattice, have been shown to be linearizable, via Hankel determinants, on the level of time evolution for the entries of the corresponding Hankel matrices [4, 12, 23, 40], also see appendix A. A Hankel matrix is a matrix that has the same entries in each ascending skew diagonal. There is a deep relation with the theory of orthogonal polynomials (see for example, [13] and [4, 22, 44]), of which not much is needed, however, to understand the present work. Here we just mention that the entries of the Hankel matrices have an interpretation as ‘moments’. Also see the discussion in section 5, and appendix A.

For \( m = 0, 1, \) and \( n \in \mathbb{Z}_+ \), let

\[ H_n^m = \det(c_{i+j+m})_{i,j=0}^{n-1} \]
be the determinant of the Hankel matrix with entries $c_{i+j+n}$, $i, j = 0, 1, \ldots, n - 1$. We further set $H_0^n = 1$. It can be inferred from [4] (which actually contains a lot more results) that, for any equation of the extended Volterra lattice hierarchy, there is a linear system of ordinary differential equations for $c_j$, $j = 0, 1, \ldots$, such that setting

$$
 u_{2n-1} = \frac{H_{n-1}^0 H_n^1}{H_n^0 H_{n-1}^1}, \quad u_{2n} = \frac{H_{n+1}^0 H_{n-1}^1}{H_n^0 H_n^1}, \quad n = 1, 2, \ldots, (1.8)
$$

yields a solution (of the respective equation of the extended Volterra lattice hierarchy), on the right half lattice and with boundary condition $u_0 = 0$. For (1.1), this is achieved by the simple linear evolution equations

$$
 \frac{dc_j}{dt_1} = c_{j+1}, \quad j = 0, 1, \ldots.
$$

For the non-autonomous equation (1.6), the corresponding system is

$$
 \frac{dc_j}{d\tau_1} = (j + 1) c_{j+1}, \quad j = 0, 1, \ldots.
$$

This extends to higher members of the Volterra lattice hierarchy via

$$
 \frac{dc_j}{d\tau_k} = c_{j+k}, \quad j = 0, 1, \ldots,
$$

and to members of the non-autonomous extension via

$$
 \frac{dc_j}{d\tau_k} = (j + k) c_{j+k}, \quad j = 0, 1, \ldots,
$$

where $k = 1, 2, \ldots$.

The main goal of the present work is to extend the aforementioned results to $u_0 \neq 0$, allowing an arbitrary function $u_0(t)$. In this case, the evolution equations for the $c_j$ acquire quadratic terms (see [23, 40] for the case of the Toda equation). This means they have to be extended to a Riccati system. For the Volterra lattice equation (1.1), this has already been done in [40]. In this way, we reach a description of the complete set of solutions of the first and second autonomous, as well as non-autonomous, flows, of the extended Volterra lattice hierarchy, on the right half lattice (with arbitrary boundary data).

The extended Volterra lattice hierarchy is described in more detail in section 2. According to our knowledge, this contains the first elaboration of the second non-autonomous Volterra lattice equation, moreover with a simplification leading to (1.7). In section 3 we collect our main results. Corresponding proofs are then presented in section 4. Section 5 contains some additional remarks. Appendix A explains the origin of the expressions in (1.8). Appendix B recalls a bit of material from [4], and in particular an expression for the extended Volterra lattice hierarchy, different from that given in [30]. These appendices shall help the reader to quickly understand the relation between (part of) [4] and [30]. Appendix C addresses a freedom in the (Hirota) bilinearization of (1.1), which is usually disregarded, but important in the present work. This is discussed in section 5. As already mentioned, appendix D establishes a relation between the first non-autonomous Volterra flow (1.6) and Thiele expansion.

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9 In terms of suitable dependent variables, integrable nonlinear equations can typically be cast into a certain bilinear form, see [18], which makes it easy to access some classes of exact solutions.
2. Volterra lattice hierarchy and a non-autonomous integrable extension

The recursion operator of the Volterra lattice hierarchy (1.2) is (see, e.g. [16, 30, 37, 42, 54])

\[ \mathcal{R} = u_n(S + 1)(u_nS - S^{-1}u_n)(S - 1)^{-1}u_n^{-1} = H_2H_1^{-1}, \]

with the shift operator \( S \) and

\[ H_1 = u_n(S - S^{-1})u_n, \]
\[ H_2 = u_n[(1 + S)u_n(1 + S) - (1 + S^{-1})u_n(1 + S^{-1})]u_n. \]

The latter are Hamiltonian operators [37, 42]. The operator \( \mathcal{R} \) is well-defined on the image of \( u_n(S - 1) \). Since functions independent of \( n \) are in the kernel of the latter, the result of an application of \( \mathcal{R} \) is only determined up to addition of an arbitrary such function times \( V^{(1)} \) (also see [25]). In the following computations we will mostly pick a representative and not display this freedom.

\( V^{(1)} \) can be written as

\[ V^{(1)} = u_n(S - 1)(u_n + u_{n-1}), \]

so that \(^{10}\)

\[ V^{(2)} = \mathcal{R}V^{(1)} = u_n(1 + S^{-1})(S u_n S - u_n)(u_n + u_{n-1}) \]
\[ = u_n\left( u_{n+1}(u_n + u_{n+1} + u_{n+2}) - u_{n-1}(u_{n-2} + u_{n-1} + u_n) \right), \]

which is the right hand side of (1.3). Since

\[ V^{(2)} = u_n(S - 1)\left( u_n(u_{n-1} + u_n + u_{n+1}) + u_{n-1}(u_{n-2} + u_{n-1} + u_n) \right), \]

we find

\[ V^{(3)} = \mathcal{R}V^{(2)} \]
\[ = u_n(1 + S^{-1})(u_{n+1}S^2 - u_n)\left( u_n(u_{n-1} + u_n + u_{n+1}) + u_{n-1}(u_{n-2} + u_{n-1} + u_n) \right) \]
\[ = u_n\left( u_{n+1}(u_n^2 + 2u_n u_{n+1} + u_{n+1}^2 + u_n u_{n+2} + 2u_{n+1}u_{n+2} + u_{n+1}^2 + u_{n+2}^2 + u_n + 2u_n + 3u_n + 3u_{n+2}) \right) \]
\[ - u_{n-1}(u_{n-2}^2 + 2u_{n-2}u_{n-1} + u_{n-1}^2 + u_n - 2u_n + 2u_{n-1}u_n + u_{n-1}^2 + u_{n-1} - 3u_{n-2}) \), \]

and so forth. In these expressions we disregarded possible additional terms arising from the aforementioned indeterminacy in the action of \( \mathcal{R} \), since they simply lead to linear combinations of hierarchy equations.

In [30] a non-autonomous extension of the (autonomous) Volterra lattice hierarchy has been constructed by equations that possess a non-isospectral Lax pair. A crucial observation is that \( u_n = u_n(n - (n - 1)) = u_n(S - 1)(n - 1) \) lies in the image of \( u_n(S - 1) \), and thus

\[ \mathcal{R}u_n = u_n\left( (n + 1)u_{n+1} + u_n - (n - 2)u_{n-1} \right) = V^{(1)} \]

is a local expression. It defines the non-autonomous flow (1.6), which is integrable and possesses a non-isospectral Lax pair. This equation extends to the sequence of non-autonomous

\(^{10}\) Besides the Volterra lattice equation, also a combination of the first two flows of the Volterra lattice hierarchy, \( d\xi/dt = V^{(2)} - 6V^{(1)} \), is an integrable semi-discretization of the KdV equation, see section 4.9 in [45].
flows in (1.4). However, for \( k > 1 \), these are non-local expressions. We will discuss this further below. In (1.4) we can also allow \( k = 0 \), which is the autonomous linear equation \( du_0/dt_0 = u_0 \).

**Remark 2.1.** Although the non-autonomous Volterra flows do not commute with each other and also not with the autonomous flows, the combination (1.5) of all flows of the extended Volterra lattice hierarchy is integrable. The reason for this lies in the fact that all the flows have one equation of their Lax pair in common [30], namely

\[
q_{n+1} = \lambda q_n - u_n q_{n-1}, \quad n \in \mathbb{Z},
\]

with a (‘spectral’) parameter \( \lambda \). This equation, restricted to the positive integers \( \mathbb{Z}_+ \), and with \( q_{-1} = 0, q_0 = 1 \), is known to be a recurrence relation for monic symmetric orthogonal polynomials (see, e.g. [13]). For the Volterra lattice equation (1.1), we have \( d\lambda/dt_1 = 0 \), and the second part of the Lax pair is

\[
\frac{dq_n}{dt_1} = -u_{n-1}u_n q_{n-2}.
\]

For the first non-autonomous flow (1.6), we have \( d\lambda/d\tau_1 = \frac{1}{2} \lambda^3 \) and

\[
\frac{dq_n}{d\tau_1} = \frac{n}{2} q_{n+2} + \left( \sum_{j=1}^{n-1} u_j + \frac{n}{2}(u_n + u_{n+1}) \right) q_n - \frac{n}{2} u_{n-1} u_n q_{n-2}.
\]

Let us return to the feature of non-locality of the flows (1.4) for \( k > 1 \). We can write

\[
\mathcal{V}^{(1)} = u_n (S - 1) \left( n u_n + (n - 2) u_{n-1} \right) + 2 u_n^2.
\]

In order to get a local expression for \( \mathcal{V}^{(2)} \), we would need \( \mathcal{V}^{(1)} \) to be in the image of \( u_n (S - 1) \), acting on local expressions. But this would mean that \( u_n \) can be expressed as \( S - 1 \) applied to a local term, which is impossible. Hence, \( \mathcal{V}^{(2)} \) contains non-local terms. We obtain

\[
\begin{align*}
\mathcal{V}^{(2)} &= u_n (S + 1)(u_n S - S^{-1} u_n) \left( n u_n + (n - 2) u_{n-1} + 2(S - 1)^{-1} (u_n) \right) \\
&= u_n \left( (3 - n) u_{n-2} u_{n-1} + (1 - n) u_{n-1}^2 + (2 - n) u_{n-1} u_n + u_n^2 \right) \\
&\quad + (n + 1) u_n u_{n+1} + n u_{n+1}^2 + (n + 2) u_n u_{n+2} \\
&\quad - 2u_{n-1} (S - 1)^{-1} (u_{n-1}) + 2u_n (S - 1)^{-1} (u_{n+2}) \\
&= u_n \left( (3 - n) u_{n-2} u_{n-1} + (3 - n) u_{n-1}^2 + (2 - n) u_{n-1} u_n + u_n^2 \right) \\
&\quad + (n + 2) u_{n+1}^2 + (n + 2) u_n u_{n+2} - 2u_{n+1} - u_{n-1} \sum_{k=n}^{\infty} u_k \right) + f \mathcal{V}^{(1)}.
\end{align*}
\]

Now we can use

\[
(S - 1)^{-1} = -\sum_{k=0}^{\infty} S^k
\]

to express the nonlocal parts in (2.1) in terms of values of \( u \) at positive lattice sites only.
Because of the indeterminacy in the action of the recursion operator, mentioned above, here we added an arbitrary \( n \)-independent function \( f(\tau_2) \) times the right hand side of the Volterra lattice equation (1.1). The reason is that, with the choice 

\[
f(\tau_2) = 2 \sum_{k=1}^{\infty} u_k,
\]

the second non-autonomous flow of the extended Volterra lattice hierarchy can be chosen as (1.7), where the non-locality has been considerably reduced.

**Remark 2.2.** Alternatively we can use

\[
 u_n = (S - 1) \left( \sum_{i=-\infty}^{n-1} u_i \right),
\]

to turn (2.1) into

\[
 V^{(2)} = u_0 \left( (3 - n) u_{n-2}u_{n-1} + (1 - n) u_{n-1}^2 + (2 - n) u_{n-1}u_n + 2u_{n-2}u_{n+1} + u_n^2 \right) 
\]

\[
 + (n + 3) u_nu_{n+1} + (n + 2) u_{n+1}^2 + (n + 2) u_{n+1}u_{n+2} + 2(u_{n+1} - u_{n-1}) \sum_{i=-\infty}^{n-2} u_i.
\]

In [4], the boundary condition \( u_0 = 0 \) has been chosen. Since

\[
 \sum_{i=-\infty}^{n-1} u_i = (S - 1)^{-1}(u_0),
\]

in this case the infinite sum in the above expression for \( V^{(2)} \) reduces to the finite sum \( \sum_{i=1}^{n-2} u_i \). The resulting equation is equivalent to (1.7). □

### 3. Main results

In this section, we state our main results. Proofs are provided in section 4. In an equivalent form, our first theorem has already been stated in [40].

**Theorem 3.1.** Let \( u_0 \) be a smooth function of \( t_1 \). Let \( c_j, j = 0, 1, \ldots, \) satisfy

\[
 \frac{dc_j}{dt_1} = c_{j+1} - \frac{u_0}{c_0} \sum_{i=0}^{j} c_ic_{j-i}.
\]

Then (1.8) determines a solution of the Volterra lattice equation (1.1) on the right half lattice.

**Remark 3.2.** Regarding the system (3.1) as a recurrence relation

\[
 c_{j+1} = \frac{dc_j}{dt_1} + \frac{u_0}{c_0} \sum_{i=0}^{j} c_ic_{j-i},
\]

it determines \( c_j(t_1), j > 0 \), recursively, starting from given functions \( u_0(t_1) \) and \( c_0(t_1) \neq 0 \).
the theorem yields a solution of (1.1) on the right half lattice, with arbitrary boundary data $u_0(t_1)$ and $u_1(t_1)$. Writing the Volterra lattice equation (1.1) in the form

$$u_{n+1} = \frac{1}{n+1} \left( \frac{du}{dt_1} + u_n \right),$$

makes evident that the solutions determined by the above system for the $c_j$ then exhaust the set of its solutions.

**Theorem 3.3.** Let $u_0$ be a smooth function of $\tau_1$. Let $c_j$, $j = 0, 1, \ldots$, satisfy

$$\frac{dc_j}{d\tau_1} = (j + 1) c_{j+1} + \frac{u_0}{c_0} \sum_{i=0}^{j} c_i c_{j-i}.$$  \hfill (3.2)

Then (1.8) determines a solution of the first non-autonomous Volterra lattice equation (1.6) on the right half lattice.

**Remark 3.4.** Writing the system (3.2) as a recurrence relation,

$$c_{j+1} = \frac{1}{j+1} \left( \frac{dc_j}{d\tau_1} - \frac{u_0}{c_0} \sum_{i=0}^{j} c_i c_{j-i} \right),$$

it determines $c_j(\tau_1)$, $j > 0$, recursively, starting from given functions $u_0(\tau_1)$ and $c_0(\tau_1)$. In the same way as in the case of the Volterra lattice equation (1.1), see remark 3.2, theorem 3.3 determines a solution of (1.6) on the right half lattice, with arbitrary boundary data $u_0(\tau_1)$ and $u_1(\tau_1)$. Writing (1.6) in the form

$$u_{n+1} = \frac{1}{n+1} \left( \frac{du}{d\tau_1} - u_n + (n-2) u_{n-1} \right),$$

assures that the solutions determined via (3.2) comprise all of its solutions.

**Theorem 3.5.** Let $u_1$ and $u_0$ be smooth functions of $t_2$. Let $c_j$, $j = 0, 1, \ldots$, satisfy

$$\frac{dc_j}{dt_2} = c_{j+2} - \frac{u_0}{c_0} \left( (u_{-1} + u_0) \sum_{i=0}^{j-1} c_i c_{j-i} + \sum_{i=1}^{j} c_i c_{j+1-i} \right).$$  \hfill (3.3)

Then (1.8) determines a solution of the second Volterra lattice equation (1.3) on the right half lattice.

**Remark 3.6.** The system (3.3), written in the form

$$c_{j+2} = \frac{dc_j}{dt_2} + \frac{u_0}{c_0} \left( (u_{-1} + u_0) \sum_{i=0}^{j-1} c_i c_{j-i} + \sum_{i=1}^{j} c_i c_{j+1-i} \right),$$
determines $c_j(t_2)$, $j > 1$, recursively, starting from given functions $u_{-1}(t_2)$, $u_0(t_2)$, $c_0(t_2)$ and $c_1(t_2)$. Since $u_1 = c_1/c_0$ and

$$u_2 = \frac{c_2}{c_1} - \frac{c_1}{c_0} = \frac{1}{c_1} \frac{d c_0}{d t_2} - u_1.$$  

Theorem 3.5 yields a solution of the second Volterra lattice equation on the right half lattice, with arbitrary boundary data $u_{-1}(t_2)$, $u_0(t_2)$, $u_1(t_2)$ and $u_2(t_2)$. Writing (1.3) in the form

$$u_{n+2} = \frac{1}{u_{n+1}} \left( \frac{d u_n}{d t_2} + u_{n-1}(u_{n-2} + u_{n-1} + u_0) \right) - u_n - u_{n+1},$$

shows that (3.3) reaches all of its solutions.

\[ \square \]

**Theorem 3.7.** Let $u_{-1}$ and $u_0$ be smooth functions of $\tau_2$. Let $c_j$, $j = 0, 1, \ldots$, satisfy

$$\frac{d c_j}{d \tau_2} = (j + 2) c_j + \frac{d u_0}{c_0} \left( 2(u_0 + u_{-1}) \sum_{i=0}^{j-1} c_i c_{j-i} + \sum_{i=1}^{j} c_i c_{j+1-i} \right).$$

(3.4) determines a solution of the second non-autonomous Volterra lattice equation (1.7) on the right half lattice.

**Remark 3.8.** Written in the form

$$c_{j+2} = \frac{1}{j + 2} \left[ \frac{d c_j}{d \tau_2} - \frac{d u_0}{c_0} \left( 2(u_0 + u_{-1}) \sum_{i=0}^{j-1} c_i c_{j-i} + \sum_{i=1}^{j} c_i c_{j+1-i} \right) \right],$$

(3.4) determines $c_j(\tau_2)$, $j > 1$, recursively, starting from given functions $u_{-1}(\tau_2)$, $u_0(\tau_2)$, $c_0(\tau_2)$ and $c_1(\tau_2)$. In the same way as described in remark 3.6, we can argue that theorem 3.7 yields a solution of (1.7) on the right half lattice, with arbitrary boundary data $u_{-1}(\tau_2)$, $u_0(\tau_2)$, $u_1(\tau_2)$ and $u_2(\tau_2)$. Writing (1.7) in the form

$$u_{n+2} = \frac{1}{(n + 2) u_{n+1}} \left( \frac{d u_n}{d \tau_2} - (3 - n) u_{n-2} u_{n-1} - (3 - n) u_{n-1}^2 - (2 - n) u_{n-1} u_n - u_n^2 \right. \left. - (n + 3) u_n u_{n+1} - (n + 2) u_{n+1}^2 - 2(u_{n+1} - u_{n-1}) \sum_{i=1}^{n-1} u_i \right),$$

shows that the solutions determined by (3.4) exhaust the set of its solutions.

\[ \square \]

The above remarks show that the Riccati systems in the theorems are actually equivalent to the respective equations of the extended Volterra lattice hierarchy.

In the subsequent subsection, following [40] (also see [39]) we present a reformulation of the Riccati systems that appear in the above theorems. According to our knowledge, a Riccati system (beyond the linear case) for entries of a Hankel determinant first appeared in [23], in the context of the Toda lattice equation.

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3.1. Riccati systems in terms of a Stieltjes function

Let us introduce the formal series

\[ \mathcal{F}(\lambda) = \sum_{j=0}^{\infty} \frac{c_j}{\lambda^{2j+1}}, \]

where \( \lambda \) is an indeterminate. This is the ‘Stieltjes function’ for symmetric orthogonal polynomials, a generating function for the moments \( s_i \), see (A.1). Since odd moments vanish in this case, only \( c_j := s_{2j} \) shows up. It is easily verified that (3.1) can be expressed as the Riccati equation

\[ \frac{d\mathcal{F}}{dt_1} = -c_0 \lambda + \lambda^2 \mathcal{F} - \frac{u_0}{c_0} \lambda \mathcal{F}^2. \]

The latter equation already appeared in [40], but with the restriction \( c_0 = 1 \).

In the same way, (3.3) can be expressed as the Riccati equation

\[ \frac{d\mathcal{F}}{dt_2} = -c_0 [\lambda^3 + (u_0 + u_1) \lambda] + [\lambda^4 + 2u_0 \lambda^2 + (u_{-1} + u_0)u_0] \mathcal{F} - \frac{u_0}{c_0} [\lambda^3 + (u_{-1} + u_0) \lambda] \mathcal{F}^2. \]

Allowing \( \lambda \) to be a function of \( \tau_1 \), respectively \( \tau_2 \), we find that (3.2) and (3.4) are equivalent, respectively, to the Riccati equations

\[ \frac{d\mathcal{F}}{d\tau_1} = -\frac{1}{2} \lambda^2 \mathcal{F} + \frac{u_0}{c_0} \lambda \mathcal{F}^2, \quad \text{where} \quad \frac{d\lambda}{d\tau_1} = \frac{1}{2} \lambda^3, \]

and

\[ \frac{d\mathcal{F}}{d\tau_2} = c_0 (u_0 - u_1) \lambda - \frac{1}{2} \lambda^4 + 2u_0 \lambda^2 + 2(u_{-1} + u_0)u_0] \mathcal{F} + \frac{u_0}{c_0} [\lambda^3 + 2(u_{-1} + u_0) \lambda] \mathcal{F}^2, \]

where

\[ \frac{d\lambda}{d\tau_2} = \frac{1}{2} \lambda^5. \]

It is well-known that Riccati equations can be linearized. In this way, the above theorems present linearizations of the respective equations of the extended Volterra lattice hierarchy. Further analysis can be carried out in analogy to the treatment of a Riccati equation for a Stieltjes function in section 6 of [40]. This also concerns the construction of special exact solutions.

4. Proofs of the theorems

The proofs presented in this section crucially use determinant identities. In particular, for any determinant \( D \), the Jacobi determinant identity [1, 10] reads

\[ D \cdot D \begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} = D \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} \cdot D \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} - D \begin{bmatrix} i_1 \\ j_2 \end{bmatrix} \cdot D \begin{bmatrix} i_2 \\ j_1 \end{bmatrix}, \quad (4.1) \]

where

\[ D \begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{bmatrix}, \quad i_1 < i_2 < \cdots < i_k, \quad j_1 < j_2 < \cdots < j_k, \]
denotes the determinant obtained from $D$ by removing the rows at positions $i_1, i_2, \ldots, i_k$, and the columns at positions $j_1, j_2, \ldots, j_k$, in the respective matrix.

In order to demonstrate that a certain Riccati system for $c_j$, $j = 0, 1, \ldots$, implies that (1.8) determines solutions of some equation from the extended Volterra lattice hierarchy, the starting point are the identities

\[
\dot{z}_n = \left( (H^0_{n+1}H^1_n - H^0_nH^1_{n+1})H^0_nH^1_n - (H^0_{n+1}H^1_n - H^0_nH^1_{n+1})H^0_nH^1_n \right)^{-2},
\]

\[
\dot{z}_{n-1} = \left( (H^0_{n+1}H^1_n - H^0_nH^1_{n+1})H^0_nH^1_n - (H^0_{n+1}H^1_n - H^0_nH^1_{n+1})H^0_nH^1_n \right)^{-2},
\]

which result from (1.8). Here the ‘overdot’ stands for any derivation.

4.1. Some notation and determinant identities

For $m = 0, 1$, we define

\[
G^m_n = \begin{pmatrix}
    c_m & c_{m+1} & \cdots & c_{m+n-1} \\
    c_{m+1} & c_{m+2} & \cdots & c_{m+n} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{m+n-2} & c_{m+n-1} & \cdots & c_{m+2n-3} \\
    c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n-1}
\end{pmatrix}, \quad n = 2, 3, \ldots,
\]

and $G^0_0 = -u_0$, $G^1_0 = 0$, $G^m_1 = c_{m+1}$. In sections 4.4 and 4.5, we need as well the determinants

\[
E^m_n = \begin{pmatrix}
    c_m & \cdots & c_{m+n-2} & c_{m+n+1} \\
    c_{m+1} & \cdots & c_{m+n-1} & c_{m+n+2} \\
    \vdots & \ddots & \vdots & \vdots \\
    c_{m+n-1} & \cdots & c_{m+2n-3} & c_{m+2n}
\end{pmatrix}, \quad n = 2, 3, \ldots,
\]

\[
F^m_n = \begin{pmatrix}
    c_m & \cdots & c_{m+n-3} & c_{m+n-1} & c_{m+n} \\
    c_{m+1} & \cdots & c_{m+n-2} & c_{m+n} & c_{m+n+1} \\
    \vdots & \ddots & \vdots & \vdots & \vdots \\
    c_{m+n-1} & \cdots & c_{m+2n} & c_{m+2n-1} & c_{m+2n+1}
\end{pmatrix}, \quad n = 3, 4, \ldots,
\]

in intermediate steps. We set $E^0_0 = -u_0u_1$, $E^1_0 = 0$, $E^m_1 = c_{m+2}$, $F^0_0 = u_0(u_1 + u_0 + u_1)$, $F^1_0 = u_0u_1$, $F^m_0 = 0$ and $F^{m+1}_2 = H^{m+1}_2$.

In the proofs, presented in the following subsections, we will also use the vectors

\[
A_j = \begin{pmatrix}
    c_j \\
    c_{j+1} \\
    \vdots \\
    c_{j+n-1}
\end{pmatrix}, \quad A_j^\dagger = \begin{pmatrix}
    (n-1)c_j \\
    (n-2)c_{j+1} \\
    \vdots \\
    c_{j+n-2}
\end{pmatrix},
\]

\[
B_j = \begin{pmatrix}
    \sum_{i=0}^j c_{i}c_{j-i} \\
    \sum_{i=0}^{j+1} c_{i}c_{j+1-i} \\
    \vdots \\
    \sum_{i=0}^{j+n-1} c_{i}c_{j+n-1-i}
\end{pmatrix}, \quad B_j^\dagger = \begin{pmatrix}
    \sum_{i=1}^j c_{i}c_{j+1-i} \\
    \sum_{i=1}^{j+1} c_{i}c_{j+2-i} \\
    \vdots \\
    \sum_{i=1}^{j+n-1} c_{i}c_{j+n-1-i}
\end{pmatrix}.
\]

where, for simplicity, the notation disregards the dependence on a fixed $n$. 

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Note that $|A_m, A_{m+1}, \ldots, A_{n-1+m}| = H_n^m$. The next lemma is only needed for the proof of lemma 4.2.

**Lemma 4.1.** For $m = 0, 1$, and $n = 1, 2, \ldots$, the following identities hold\(^{11}\),

\[
\sum_{j=m}^{n-1+m} |A_m, A_{j-1}, B_j^*, A_{j+1}, \ldots, A_{n-1+m}| = (n - 1)c_0 H_n^m, \tag{4.4}
\]

\[
\sum_{j=m}^{n-1+m} |A_m, A_{j-1}, \hat{B}_j^*, A_{j+1}, \ldots, A_{n-1+m}| = (n - 1)c_1 H_n^m, \tag{4.5}
\]

where

\[
B_j^* = \begin{pmatrix}
0 \\
\sum_{i=0}^{j+2} c_i c_{j+2-i} \\
\vdots \\
\sum_{i=0}^{n-2} c_i c_{j+n-1-i}
\end{pmatrix}, \quad \hat{B}_j^* = \begin{pmatrix}
0 \\
\sum_{i=1}^{j+3} c_i c_{j+3-i} \\
\vdots \\
\sum_{i=1}^{n-1} c_i c_{j+n-1-i}
\end{pmatrix}.
\]

**Proof.** Let $m = 0$. For $n = 1$, (4.4) is obviously satisfied. For $n > 1$, Laplacian determinant expansion of the left hand side of (4.4), with respect to the $(j + 1)$th column, yields

\[
\sum_{j=0}^{n-1} |A_0, \ldots, A_{j-1}, B_j^*, A_{j+1}, \ldots, A_{n-1}| = \sum_{k=2}^{n-1} c_k \sum_{j=0}^{n-k} (-1)^{j+k+1} \sum_{i=0}^{k-2} c_i c_{j+k-1-i} H_n^0 \left[ \begin{array}{c} k \\ j+1 \end{array} \right].
\]

This can be expressed as

\[
\sum_{j=0}^{n-1} |A_0, \ldots, A_{j-1}, B_j^*, A_{j+1}, \ldots, A_{n-1}| = \sum_{k=2}^{n-1} c_k \sum_{j=0}^{n-k} \left| \begin{array}{cccc}
c_0 & c_1 & \cdots & c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{k-2} & c_{k-1} & \cdots & c_{k+n-3} \\
c_{k-1-i} & c_{k-i} & \cdots & c_{k+n-2-i} \\
c_k & c_{k+1} & \cdots & c_{k+n-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n-1} & c_n & \cdots & c_{2n-2}
\end{array} \right|.
\]

In a similar way one proves the other identities. \(\square\)

\(^{11}\) The summand in (4.4) with $j = m$ is $|B_m^*, A_{m+1}, \ldots, A_{n-1+m}|$. 
Lemma 4.2. For $m = 0, 1$, and $n = 1, 2, \ldots$ we have
\[
\sum_{j=m}^{n-1+m} |A_j, A_{j+1}, \ldots, A_{n-1+m}| = (2n - 1 + m) c_0 H_n^m, \tag{4.6}
\]
\[
\sum_{j=m}^{n-1+m} |A_m, A_{j+1}, A_{j+2}, \ldots, A_{n-1+m}| = (2n - 2 + m) c_1 H_n^m, \tag{4.7}
\]
\[
\sum_{j=m}^{n-1+m} |A_m, A_{j+1}, A_{j+1}, \ldots, A_{n-1+m}| = 0, \tag{4.8}
\]
\[
\sum_{j=m}^{n-1+m} |A_m, A_{j+1}, A_{j+2}, A_{j+2}, \ldots, A_{n-1+m}| = -F_n^m. \tag{4.9}
\]

Proof.

(1) $m = 0$.

For $n = 1$, (4.6) is a trivial identity. Let $n > 1$. In the determinant on the left hand side of (4.6), to the $(j+1)$th column, $j \geq 1$, we add, for $k = 1, 2, \ldots, j$, the $k$th column, multiplied by $-c_{j+1-k}$. This yields
\[
\sum_{j=0}^{n-1} |A_0, A_{j+1}, A_{j+2}, \ldots, A_{n-1}| = \sum_{j=0}^{n-1} |A_0, A_{j+1}, A_{j+2}, \ldots, A_{n-1}| + nc_0 H_n^0.
\]

Now (4.6) follows by use of (4.4).

For $n = 1, 2$, (4.7) is easily verified. Let $n > 2$. In the determinant on the left hand side of (4.7), for $j \geq 2$ we add to the $(j+1)$th column the $k$th column, multiplied by $-c_{j+2-k}$, for $k = 2, 3, \ldots, j$, to obtain
\[
\sum_{j=0}^{n-1} |A_0, A_{j+1}, A_{j+2}, \ldots, A_{n-1}| = \sum_{j=0}^{n-1} |A_0, A_{j+1}, A_{j+2}, \ldots, A_{n-1}| + (n - 1)c_1 H_n^0.
\]

Use of (4.5) leads to the stated equation.

(4.8) and (4.9) are easily verified for $n = 1$. Let now $n > 1$. (4.8) is verified by expanding the determinant on the left hand side with respect to the $(j+1)$th column,
\[
\sum_{j=0}^{n-1} |A_0, A_{j+1}, A_{j+2}, A_{j+2}, \ldots, A_{n-1}| = \sum_{j=0}^{n-1} \sum_{k=1}^{n-1} (-1)^{k+j+1} (n-k) c_{j+k} H_n^0 \left[ \begin{array}{c} j+k \end{array} \right]^{k+j+1},
\]
which vanishes, since the second sum in the last expression is the Laplace expansion of a determinant having identical $k$th and $(k+1)$th rows, for $k = 1, \ldots, n-1$. Addressing
we expand the determinant on the left hand side with respect to the \((j+1)\)th column to obtain
\[
\sum_{j=0}^{n-1} |A_0, \ldots, A_{j-1}, A_j^+, A_{j+1}, \ldots, A_{n-1}| = \sum_{j=0}^{n-1} \sum_{k=1}^{n-1} (-1)^{k+j+1} (n-k)c_{k+j+1}H_n^0 \left[ \begin{array}{c} k \\ j+1 \end{array} \right]
\]
\[
= \sum_{k=1}^{n-1} (n-k) \sum_{j=0}^{n-1} (-1)^{k+j+1} c_{k+j+1}H_n^0 \left[ \begin{array}{c} k \\ j+1 \end{array} \right].
\]
Since
\[
\sum_{j=0}^{n-1} (-1)^{k+j+1} c_{k+j+1}H_n^0 \left[ \begin{array}{c} k \\ j+1 \end{array} \right] = 0, \quad k = 1, 2, \ldots, n - 2,
\]
which results from a determinant with two equal rows, we obtain
\[
\sum_{j=0}^{n-1} |A_0, \ldots, A_{j-1}, A_j^+, A_{j+1}, \ldots, A_{n-1}| = \sum_{j=0}^{n-1} (-1)^{n+j+1} c_{n+j}H_n^0 \left[ \begin{array}{c} n-1 \\ j+1 \end{array} \right] = -F_n^0.
\]
(2) \(m = 1\).

For \(n = 1\), (4.6) is a trivial identity. Let now \(n > 1\). In the determinant on the left hand side of (4.6), to the \(j\)th column, \(j > 1\), we add, for \(k = 1, 2, \ldots, j-1\), the \(k\)th column, multiplied by \(-c_{j-k}\), to obtain
\[
\sum_{j=1}^{n} |A_1, \ldots, A_{j-1}, B_j, A_{j+1}, \ldots, A_n|
\]
\[
= \sum_{j=1}^{n} \left( c_0H_n^0 + |A_1, \ldots, A_{j-1}, B_j^*, A_{j+1}, \ldots, A_n| + c_j |A_1, \ldots, A_{j-1}, A_0, A_{j+1}, \ldots, A_n| \right).
\]
Then (4.6) follows with the help of (4.4), and by Laplace expansion of \(|A_1, \ldots, A_{j-1}, A_0, A_{j+1}, \ldots, A_n|\) with respect to the \(j\)th column,
\[
\sum_{j=1}^{n} c_j |A_1, \ldots, A_{j-1}, A_0, A_{j+1}, \ldots, A_n| = \sum_{j=1}^{n} c_j \sum_{k=1}^{n} (-1)^{k+j} c_{k-1}H_n^1 \left[ \begin{array}{c} k \\ j \end{array} \right]
\]
\[
= \sum_{k=1}^{n} c_{k-1} \sum_{j=1}^{n} (-1)^{k+j} c_j H_n^1 \left[ \begin{array}{c} k \\ j \end{array} \right] = c_0 \sum_{j=1}^{n} (-1)^{n+j} c_j H_n^1 \left[ \begin{array}{c} 1 \\ j \end{array} \right] = c_0 H_n^1,
\]
where we used
\[
\sum_{j=1}^{n} (-1)^{k+j} c_j H_n^1 \left[ \begin{array}{c} k \\ j \end{array} \right] = 0, \quad k = 2, 3, \ldots, n,
\]
which results from a determinant with two equal rows.
For \(n = 1\), (4.7) is easily checked. Let now \(n > 1\). In the determinant on the left hand side of (4.7), to the \(j\)th column, \(j > 1\), we add, for \(k = 1, 2, \ldots, j-1\), the \(k\)th column, multiplied by \(-c_{j+1-k}\), to obtain
\[
\sum_{j=1}^{n} |A_1, \ldots, A_{j-1}, B_j, A_{j+1}, \ldots, A_n| = \sum_{j=1}^{n} \left( c_1 H_n^2 + |A_1, \ldots, A_{j-1}, B_j^*, A_{j+1}, \ldots, A_n| \right).
\]
Now (4.7) follows by use of (4.5).

(4.8) and (4.9), with \( m = 1 \), are proved analogously to the case \( m = 0 \).

\[ ]

**Lemma 4.3.** For \( n = 1, 2, \ldots \), the following identities hold,

\[
G_0^0 H_0^0 - G_0^1 H_0^1 - H_{n+1}^0 H_{n-1}^0 = 0, \tag{4.10}
\]

\[
G_0^1 H_{n-1}^0 - G_0^1 H_n^0 - H_{n-1}^1 H_n^1 = 0, \tag{4.11}
\]

\[
E_0^m H_n^m + E_n^m H_n^m - G_n^m G_n^{m-1} = 0, \quad m = 0, 1, \tag{4.12}
\]

\[
E_0^1 H_0^0 - E_0^0 H_1^1 = G_{n+1}^0 H_{n+1}^0, \tag{4.13}
\]

\[
F_0^0 H_n^0 - F_0^1 H_n^1 = G_{n+1}^0 H_{n+1}^0. \tag{4.14}
\]

**Proof.** Equation (4.10) is true for \( n = 1 \). For \( n \geq 2 \), it is the Jacobi identity, see (4.1), with \( D = H_{n+1}^0 \) and

\[
D \begin{bmatrix} 1 & n+1 \\ n & n+1 \end{bmatrix} = H_{n-1}^1, \quad D \begin{bmatrix} 1 & n \\ n & n+1 \end{bmatrix} = G_n^1.
\]

For \( n = 1 \), (4.11) is obvious. For \( n \geq 2 \) it is obtained as the Jacobi identity with

\[
D = \begin{vmatrix}
    c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} & c_n & 0 \\
    c_1 & c_2 & \cdots & c_{n-3} & c_{n-2} & c_{n-1} & c_n+1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    c_{n-1} & c_n & \cdots & c_{2n-3} & c_{2n-2} & c_{2n-1} & 0 \\
    0 & 0 & \cdots & 0 & 1 & 0 & 0
\end{vmatrix} = -G_n^0
\]

and

\[
D \begin{bmatrix} n & n+1 \\ 1 & n+1 \end{bmatrix} = H_{n-1}^1, \quad D \begin{bmatrix} n & n \\ 1 & n+1 \end{bmatrix} = -G_{n-1}^1.
\]

(4.12) is easily checked for \( n = 1, 2 \), using the definitions of the determinants. For \( n \geq 3 \), (4.12) is obtained as the Jacobi identity (4.1) with

\[
D = \begin{vmatrix}
    c_m & c_{m+1} & \cdots & c_{m+n-1} & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_{m+n-3} & c_{m+n-2} & \cdots & c_{m+2n-4} & 0 \\
    c_{m+n-2} & c_{m+n-1} & \cdots & c_{m+2n-3} & 1 \\
    c_{m+n-1} & c_{m+n} & \cdots & c_{m+2n-2} & 0 \\
    c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n-1} & 0
\end{vmatrix} = F_n^m
\]
and
\[
D \left[ \begin{array}{c} n \\ n+1 \end{array} \right] = H^m_n, \quad D \left[ \begin{array}{c} n \\ n+1 \end{array} \right] = -E^m_n.
\]

For \( n = 1 \), (4.13) is easily verified. For \( n \geq 2 \), (4.13) is the Jacobi identity (4.1) with
\[
D = G^m_{n+1}, \quad D = H^m_n - \frac{1}{2} E^m_n.
\]

(4.14) is quickly verified for \( n = 1, 2 \). For \( n \geq 3 \), it is the Jacobi identity (4.1) with
\[
D = H^m_{n+1}, \quad D = F^m_n.
\]

\begin{lemma}
4.4.
\[
H^m_n (F^m_n + E^m_n) = (G^m_n)^2 + H^m_{n+1} H^m_{n-1}, \quad n = 1, 2, \ldots, \quad m = 0, 1.
\]
\end{lemma}

\begin{proof}
For \( n = 2, 3, \ldots \), let us introduce
\[
S^m_n = \begin{vmatrix}
    c_m & \cdots & c_{m+n} \\
    \vdots & \ddots & \vdots \\
    c_{m+n-2} & \cdots & c_{m+2n-4} \\
    c_{m+n} & \cdots & c_{m+2n-2} \\
\end{vmatrix}.
\]

Moreover, we set \( S^0_n = u_0 (u_{-1} + u_0), \ S^1_n = u_0 u_1 \) and \( S^m_n = c_m + 2 \). The stated equation is then obtained by eliminating \( S^m_n \) from the following two equations,
\[
S^m_n H^m_n - (G^m_n)^2 - H^m_{n+1} H^m_{n-1} = 0, \quad n = 1, 2, \ldots,
\]
\[
F^m_n + E^m_n = S^m_n, \quad \text{for } n = 0, 1, 2, \ldots,
\]

which we shall prove now. For \( n = 1 \), the first equation is easily verified. For \( n \geq 2 \), it is the Jacobi identity (4.1) with \( D = H^m_{n+1} \) and
\[
D \left[ \begin{array}{c} n \\ n+1 \end{array} \right] = H^m_n, \quad D \left[ \begin{array}{c} n \\ n+1 \end{array} \right] = S^m_n.
\]

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For $n = 0, 1, 2$, the second equation can be checked directly. For $n \geq 3$, let
\[
f(x) = \sum_{k=0}^{+\infty} \frac{c_k}{k!} x^k, \quad \sigma^m_n = \det \left( f^{(i+j+m)} \right)_{0 \leq i,j \leq n-1},
\]
where $f^{(k)} = d^k f / dx^k$. Then we have $c_k = f^{(k)}(0)$ and
\[
d^2 \sigma^m_n / dx^2 = \begin{vmatrix}
    f^{(m)} & \cdots & f^{(m+n-2)} & f^{(m+n)} \\
    \vdots & \ddots & \vdots & \vdots \\
    f^{(m+n-2)} & \cdots & f^{(m+2n-4)} & f^{(m+2n-2)} \\
    f^{(m+n)} & \cdots & f^{(m+2n-2)} & f^{(m+2n)}
\end{vmatrix} + \begin{vmatrix}
    f^{(m)} & \cdots & f^{(m+n-3)} & f^{(m+n-1)} & f^{(m+n)} \\
    \vdots & \ddots & \vdots & \vdots & \vdots \\
    \vdots & \ddots & \vdots & \vdots & \vdots \\
    f^{(m+n-1)} & \cdots & f^{(m+2n-4)} & f^{(m+2n-2)} & f^{(m+2n)}
\end{vmatrix}.
\]
Setting $x = 0$ in the above expression, we find that $S^m_n = F^m_n + E^m_n$ is also true for $n \geq 3$. Also see [12] for a similar argument.

**Lemma 4.5.** For $m = 0, 1$, and $n = 1, 2, \ldots$,
\[
    \frac{G^m_n}{H^m_n} - \frac{G^m_{n-1}}{H^m_{n-1}} = u_{2n-2+m} + u_{2n-1+m},
\]
(4.16)
\[
    \frac{G^m_n}{H^m_n} = \sum_{i=1}^{2n-1+m} u_i,
\]
(4.17)
\[
    \frac{E^m_n}{H^m_n} - \frac{E^m_{n-1}}{H^m_{n-1}} = \frac{G^m_n}{H^m_n} (u_{2n-2+m} + u_{2n-1+m}) + u_{2n-1+m} u_{2n+m},
\]
(4.18)
\[
    \frac{E^m_n}{H^m_n} - \frac{E^m_{n-1}}{H^m_{n-1}} = \frac{G^m_n}{H^m_n} (u_{2n-2+m} + u_{2n-1+m}) - u_{2n-3+m} u_{2n-2+m}.
\]
(4.19)

**Proof.** First we note that (4.10), (4.11) and (1.8) imply (4.16), using $G^0_0 = -u_0$, $G^1_0 = 0$, $H^0_0 = 1$, $m = 0, 1$. Summing (4.16) at lattice sites 1 to $n$, leads to (4.17).

From the identity (4.12), we obtain
\[
    \frac{F^m_n}{H^m_n} + \frac{F^m_{n-1}}{H^m_{n-1}} = \frac{G^m_n}{H^m_n} \frac{G^m_{n-1}}{H^m_{n-1}}, \quad n = 1, 2, \ldots.
\]
Using (4.15) to eliminate either $F^m_n$ or $F^m_{n-1}$ in this equation, leads to the two equations
\[
    \frac{E^m_n}{H^m_n} - \frac{E^m_{n-1}}{H^m_{n-1}} = \frac{G^m_n}{H^m_n} \frac{G^m_{n-1}}{H^m_{n-1}} + \frac{H^m_{n+1} H^m_{n-1}}{(H^m_n)^2}, \quad n = 1, 2, \ldots,
\]
\[
    \frac{F^m_n}{H^m_n} - \frac{F^m_{n-1}}{H^m_{n-1}} = \frac{G^m_{n-1}}{H^m_{n-1}} \left( \frac{G^m_n}{H^m_n} - \frac{H^m_{n+2}}{H^m_{n-1}} \right) - \frac{H^m_n H^m_{n-2}}{(H^m_{n-1})^2}, \quad n = 2, 3, \ldots.
\]

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Now we use (4.16) and (1.8) to conclude that (4.18) holds for \( n = 1, 2, \ldots \), and (4.19) holds for \( n = 2, 3, \ldots \). Using the definitions of \( H^m_n, G^m_n, F^m_n, F^1_n \), \( m = 0, 1 \), one easily verifies that (4.19) also holds for \( n = 1 \). □

**Lemma 4.6.** For \( m = 0, 1 \), and \( n = 1, 2, \ldots \), we have

\[
\frac{E^m_n - F^m_n}{H^m_n} = \frac{E^m_{n-1} - F^m_{n-1}}{H^m_{n-1}} = \left(u_{2n-2+m} + u_{2n-1+m}\right)^2 + u_{2n-3+m}u_{2n-2+m} + u_{2n-1+m}u_{2n+m},
\]

(4.20)

\[
\frac{E^0_n - F^0_n}{H^0_n} = \frac{E^1_{n-1} - F^1_{n-1}}{H^1_{n-1}} = u_{2n-1}(u_{2n-2} + u_{2n-1} + u_{2n}),
\]

(4.21)

\[
\frac{E^1_n - F^1_n}{H^1_n} - \frac{E^0_n - F^0_n}{H^0_n} = u_{2n}(u_{2n-1} + u_{2n} + u_{2n+1}).
\]

(4.22)

**Proof.** Taking the difference between (4.18) and (4.19), and using (4.16), leads to (4.20). (4.21) is quickly verified for \( n = 1 \). For \( n \geq 2 \), we start from the trivial identity

\[
\frac{E^0_n - F^0_n}{H^0_n} = \frac{E^1_{n-1} - F^1_{n-1}}{H^1_{n-1}} = \frac{E^0_n}{H^0_n} - \frac{E^0_{n-1}}{H^0_{n-1}} + \frac{E^0_{n-1}H^1_{n-1} - H^0_{n-1}E^1_{n-1}}{H^0_{n-1}H^1_{n-1}} - \left(\frac{F^0_n}{H^0_n} - \frac{F^0_{n-1}}{H^0_{n-1}}\right) - \frac{F^0_{n-1}H^1_{n-1} - F^1_{n-1}H^0_{n-1}}{H^0_{n-1}H^1_{n-1}}.
\]

Using (4.13), (4.14), (4.18) and (4.19) with \( m = 0 \), and (1.8), we obtain

\[
\frac{E^0_n - F^0_n}{H^0_n} = \frac{E^1_{n-1} - F^1_{n-1}}{H^1_{n-1}} = \left\{G^0_n - G^0_{n-1}\right\}(u_{2n-2} + u_{2n-1}) + u_{2n-3}u_{2n-2} + u_{2n-1}u_{2n}
\]

\[- u_{2n-2}\left\{G^0_n - G^1_{n-1}\right\} + \left\{G^1_n - G^1_{n-1}\right\} - \left\{G^1_{n-2}\right\}, \quad n = 2, 3, \ldots .
\]

Applying (4.11), (4.16) and (1.8) again, shows that (4.21) holds for \( n = 2, 3, \ldots .\)

With the help of (4.13), (4.14) and (1.8), for \( n = 1, 2, \ldots \), we have

\[
\frac{E^1_n - F^1_n}{H^1_n} - \frac{E^0_n - F^0_n}{H^0_n} = \frac{E^1_n}{H^1_n} - \frac{E^0_n}{H^0_n} - \left(\frac{F^1_n}{H^1_n} - \frac{F^0_n}{H^0_n}\right) = \left\{G^0_{n+1} - G^0_n\right\} + \left\{G^0_n - G^1_n\right\} - \left\{G^1_{n-1}\right\}u_{2n}.
\]

Then, employing (4.11), (4.16) with \( m = 0 \), and (1.8), we confirm (4.22). □
4.2. Proof of theorem 3.1

In this subsection, a dot stands for a derivative with respect to $t_1$.

**Lemma 4.7.** For $m = 0, 1$, and $n = 1, 2, \ldots$ (3.1) implies

$$
\dot{H}_n^m = G_n^m - (2n - 1 + m) u_0 H_n^m.
$$

(4.23)

**Proof.** As a consequence of (3.1), we have

$$
\dot{A}_j = A_{j+1} - \frac{u_0}{c_0} B_j, \quad j = 0, 1, \ldots,
$$

where $A_j$ and $B_j$ have been defined in (4.3). This in turn implies

$$
\dot{H}_n^m = \sum_{j=m}^{n-1+m} |A_m, \ldots, A_{j-1}, \dot{A}_j, A_{j+1}, \ldots, A_{n-1+m}|
$$

(4.23) now follows by use of (4.6).

**Corollary 4.8.** As a consequence of (3.1), the following identities hold for $n = 1, 2, \ldots$,

$$
\begin{align*}
\dot{H}_n^0 + \dot{H}_n^1 & = H_n^0 H_n^1 + u_0 H_n^0 H_n^1, \\
\dot{H}_n^0 H_n^1 & = \dot{H}_n^1 H_n^1 - u_0 H_n^1 H_n^1.
\end{align*}
$$

**Proof.** This follows immediately from the identities (4.10) and (4.11), together with lemma 4.7.

This corollary expresses a bilinearization of the Volterra lattice equation (1.1), see appendix C.

For $n > 2$ in (1.1), theorem 3.1 is now a consequence of (4.2) and the preceding corollary. For $n = 1, 2$, (1.1) is easily verified directly.

4.3. Proof of theorem 3.3

**Lemma 4.9.** For $m = 0, 1$, and $n = 1, 2, \ldots$, (3.2) implies

$$
\dot{H}_n^m = (2n - 1 + m) G_n^m + (2n - 1 + m) u_0 H_n^m.
$$

(4.24)

**Proof.** As a consequence of (3.2), we have

$$
\dot{A}_j = (j + n) A_{j+1} - A_{j+1}^* + \frac{u_0}{c_0} B_j, \quad j = 0, 1, \ldots
$$

12 In the following subsections, a dot means a derivative with respect to the ‘time’ variable of the equation under consideration.
This implies

\[
\dot{H}_n^m = \sum_{j=m}^{n-1+m} |A_m, \ldots, A_{j-1}, \dot{A}_j, A_{j+1}, \ldots, A_{n-1+m}|
\]

\[
= (2n - 1 + m)G_n^m - \sum_{j=m}^{n-1+m} |A_m, \ldots, A_{j-1}, A_{j+1}', A_{j+1}, \ldots, A_{n-1+m}|
\]

\[
+ \frac{u_0}{c_0} \sum_{j=m}^{n-1+m} |A_m, \ldots, A_{j-1}, B_j, A_{j+1}, \ldots, A_{n-1+m}|.
\]

Now we use (4.8) and (4.6) to obtain (4.24).

\[\square\]

**Corollary 4.10.** For \( n = 1, 2, \ldots, (3.2) \) implies

\[
\dot{H}_n^0 + H_1^1 - H_n^0 + H_1^1 = 2n H_n^0 + G_n^1 + u_0 H_n^0 + H_1^0.
\]

\[
H_n^0 H_1^1 - H_n^0 H_1^1 = (2n - 1)H_n^0 + H_1^1 + H_n^0 H_1^1.
\]

**Proof.** This follows from the preceding lemma, together with (4.10) and (4.11).

For \( n > 2 \) in (1.6), theorem 3.3 follows from (4.2), (4.16), and the preceding corollary. For \( n = 1, 2 \), (1.6) is easily verified directly.

4.4. Proof of theorem 3.5

**Lemma 4.11.** For \( m = 0, 1, \) and \( n = 1, 2, \ldots, (3.3) \) implies

\[
\dot{H}_n^m = E_n^m - F_n^m - (n - 1 + m)(u_0 + u_{-1})u_0 H_n^m - (2n - 2 + m)u_0 u_1 H_n^m.
\]

(4.25)

**Proof.** Equation (3.3) implies

\[
\dot{A}_j = A_{j+2} - \frac{(u_0 + u_{-1})u_0}{c_0} (B_j - c_0 A_j) - \frac{u_0}{c_0} \dot{B}_j, \quad j = 0, 1, \ldots.
\]

This in turn leads to

\[
\dot{H}_n^m = \sum_{j=m}^{n-1+m} |A_m, \ldots, A_{j-1}, \dot{A}_j, A_{j+1}, \ldots, A_{n-1+m}|
\]

\[
= E_n^m - F_n^m - \frac{(u_0 + u_{-1})u_0}{c_0} \sum_{j=m}^{n-1+m} |A_m, \ldots, A_{j-1}, B_j, A_{j+1}, \ldots, A_{n-1+m}|
\]

\[
+ n(u_0 + u_{-1})u_0 H_n^m - \frac{u_0}{c_0} \sum_{j=m}^{n-1+m} |A_m, \ldots, A_{j-1}, B_j, A_{j+1}, \ldots, A_{n-1+m}|.
\]

Now (4.25) is obtained by use of (4.6), (4.7) and \( u_1 = c_1 / c_0 \).
Corollary 4.12. For $n = 1, 2, \ldots$ (3.3) implies
\[
\dot{H}^0_{n+1}H^1_n - \dot{H}^1_nH^0_{n+1} = (E^0_n - F^0_n)H^1_n - (E^1_n - F^1_n)H^0_{n+1} - u_0u_1H^0_{n+1}H^1_n,
\]
\[
\dot{H}^1_nH^0_{n-1} - \dot{H}^0_nH^1_{n-1} = (E^1_n - F^1_n)H^0_n - (E^0_n - F^0_n)H^1_{n-1} - (u_{-1} + u_0 + u_1)u_0H^0_nH^1_{n-1}.
\]

Proof. These are simple consequences of the preceding lemma.

Proof of theorem 3.5. Equation (1.3) is easily verified for $n = 1, 2$. We use (4.2) and the preceding corollary to find
\[
\dot{u}_{2n} = u_{2n} \left[ \frac{E^0_{n+1} - F^0_{n+1}}{H^0_{n+1}} - \frac{E^0_n - F^0_n}{H^0_n} - \left( \frac{E^1_n - F^1_n}{H^1_n} - \frac{E^1_{n-1} - F^1_{n-1}}{H^1_{n-1}} \right) \right]
\]
and
\[
\dot{u}_{2n-1} = u_{2n-1} \left[ \frac{E^1_n - F^1_n}{H^1_n} - \frac{E^0_{n-1} - F^0_{n-1}}{H^0_{n-1}} - \left( \frac{E^0_n - F^0_n}{H^0_n} - \frac{E^0_{n-1} - F^0_{n-1}}{H^0_{n-1}} \right) \right],
\]
for $n = 2, 3, \ldots$ Now (4.20) shows that the second Volterra lattice hierarchy equation is also satisfied for $n = 3, 4, \ldots$.

4.5. Proof of theorem 3.7

Lemma 4.13. For $m = 0, 1$, and $n = 1, 2, \ldots$ (3.4) implies
\[
\dot{H}^m_n = -(2n - 2 + m)F^m_n + (2n + m)E^m_n + 2(n - 1 + m)(u_0 + u_{-1})u_0H^m_n
\]
\[+ (2n - 2 + m)u_0u_1H^m_n.
\]

Proof. As a consequence of (3.4), we have
\[
\dot{A}_j = (j + n + 1)A_{j+1} - A^*_j + \frac{2(u_0 + u_{-1})u_0}{c_0} (B_j - c_0A_j) + \frac{u_0}{c_0} B_j, \quad j = 0, 1, \ldots,
\]
which implies
\[
\dot{H}^m_n = \sum_{j=0}^{n-1+m} \left| A_m, \ldots, A_{j-1}, \dot{A}_j, A_{j+1}, \ldots, A_{n-1+m} \right|
\]
\[= (2n + m)E^m_n - (2n - 1 + m)F^m_n - \sum_{j=0}^{n-1+m} \left| A_m, \ldots, A_{j-1}, A^*_j, A_{j+1}, \ldots, A_{n-1+m} \right|
\]
\[+ \frac{2(u_0 + u_{-1})u_0}{c_0} \sum_{j=0}^{n-1+m} \left| A_m, \ldots, A_{j-1}, B_j, A_{j+1}, \ldots, A_{n-1+m} \right|
\]
\[- 2n(u_0 + u_{-1})u_0H^m_n + \frac{u_0}{c_0} \sum_{j=0}^{n-1+m} \left| A_m, \ldots, A_{j-1}, \dot{B}_j, A_{j+1}, \ldots, A_{n-1+m} \right|.
\]
Now we use (4.9), (4.6) and (4.7) to obtain (4.26).

**Corollary 4.14.** For \( n = 1, 2, \ldots, (3.4) \) implies
\[
\begin{align*}
\dot{H}^0_{n+1} H^1_n - \dot{H}^1_n H^0_{n+1} &= (2n + 2)(E^0_{n+1} - F^0_{n+1}) H^1_n - 2n(E^1_n - F^1_n) H^0_{n+1} \\
&\quad + 2E^0_n H^1_n - (E^0_n + F^0_n) H^0_{n+1} + u_0 u_1 H^0_n H^1_n, \\
H^1_n H^0_{n-1} - \dot{H}^0_{n-1} H^1_n &= 2n(E^1_n - F^1_n) H^0_n - 2n(E^0_n - F^0_n) H^1_n - 2F^0_n H^1_n \\
&\quad + (E^1_n + F^1_n) H^0_n + (2u_0 + 2u_{-1} + u_1) u_0 H^1_n H^0_n.
\end{align*}
\]

**Proof.** These are simple consequences of the preceding lemma.

**Proof of theorem 3.7.** For \( n = 1, 2, (1.7) \) is quickly verified. Using (4.2) and the preceding corollary, we find
\[
\begin{align*}
\dot{u}_{2n} &= u_{2n} \left[ (2n + 2) \left( \frac{E^0_{n+1}}{H^0_{n+1}} - \frac{F^0_{n+1}}{H^0_{n+1}} - \frac{E^0_n}{H^0_n} + \frac{F^0_n}{H^0_n} \right) - 2n \left( \frac{E^1_n}{H^1_n} - \frac{F^1_n}{H^1_n} - \frac{E^1_{n-1}}{H^1_{n-1}} + \frac{F^1_{n-1}}{H^1_{n-1}} \right) \right] \\
&\quad + 2 \left( \frac{E^0_n}{H^0_n} - \frac{E^1_n}{H^1_n} - \frac{E^1_{n-1}}{H^1_{n-1}} + \frac{F^1_{n-1}}{H^1_{n-1}} \right) + 2 \left( \frac{F^0_{n+1}}{H^0_{n+1}} - \frac{F^0_n}{H^0_n} - \frac{E^0_{n-1}}{H^0_{n-1}} + \frac{E^0_{n-1}}{H^0_{n-1}} \right) - \left( \frac{E^1_n}{H^1_n} - \frac{F^1_n}{H^1_n} - \frac{E^1_{n-1}}{H^1_{n-1}} + \frac{F^1_{n-1}}{H^1_{n-1}} \right)
\end{align*}
\]
and
\[
\begin{align*}
\dot{u}_{2n-1} &= u_{2n-1} \left[ 2n \left( \frac{E^1_n}{H^1_n} - \frac{E^1_{n-1}}{H^1_{n-1}} - \frac{F^1_n}{H^1_n} + \frac{F^1_{n-1}}{H^1_{n-1}} \right) - 2n \left( \frac{E^0_n}{H^0_n} + \frac{F^0_n}{H^0_n} - \frac{E^0_{n-1}}{H^0_{n-1}} + \frac{F^0_{n-1}}{H^0_{n-1}} \right) \right] \\
&\quad + 2 \left( \frac{E^1_{n-1}}{H^1_{n-1}} - \frac{E^1_n}{H^1_n} - \frac{F^1_{n-1}}{H^1_{n-1}} + \frac{F^1_n}{H^1_n} \right) - 2 \left( \frac{F^0_n}{H^0_n} + \frac{F^0_{n-1}}{H^0_{n-1}} - \frac{E^0_n}{H^0_n} - \frac{E^0_{n-1}}{H^0_{n-1}} + \frac{F^1_n}{H^1_n} + \frac{F^1_{n-1}}{H^1_{n-1}} \right),
\end{align*}
\]
for \( n = 2, 3, \ldots \). Using (4.17)–(4.20) in both expressions, (4.21) in the first and (4.22) in the second, we see that (1.7) also holds for \( n = 3, 4, 5, \ldots \).

5. Conclusions and final remarks

Via expressing solutions in terms of Hankel determinants, we achieved a transformation of the first two autonomous and also the non-autonomous flows of the extended Volterra lattice hierarchy to Riccati systems. Since the latter are known to be linearizable, we thus achieved a linearization of these nonlinear partial differential-difference equations. This resolves corresponding results in [4] from the restriction to the boundary condition \( u_0 = 0 \).

Originally we had some hope to be able to extend the results of this work to the whole extended Volterra lattice hierarchy. However, a corresponding treatment of higher than second equations of the extended Volterra lattice hierarchy meets with rapidly increasing complexity, and the underlying structure is not yet visible.

Let \( u_{-2}, u_{-1}, u_0, u_1 \) be smooth functions of \( t \). Let \( c_j, j = 0, 1, \ldots, \) satisfy
\[
\frac{dc_j}{dt^3} = c_{j+3} - \frac{u_0}{c_0} \left( \sum_{i=2}^{j} c_i c_{j+2-i} + (u_{-1} + u_0) \sum_{i=1}^{j} c_i c_{j+1-i} \right) + \left( u_{-2} u_{-1} + u_{-1}^2 + 2 u_{-1} u_0 + u_0^2 + u_0 u_1 \right) \sum_{i=0}^{j-1} c_i c_{j-i} \), \quad j > 0,
\]
\[
\frac{dc_0}{dt^3} = c_3 + u_0 u_1 c_1.
\]

Then computer algebra computations suggest that (1.8) determines a solution of the third Volterra lattice equation, which is (1.2) with \( k = 3 \), on the right half lattice. For the 4th flow we thus expect \( k \) sums of quadratic terms in the evolution equations for the functions \( c_j \). In the step from \( k \) to \( k + 1 \), this means inclusion of products of two variables at a more remote distance on the lattice.

The linearizations of equations of the extended Volterra lattice hierarchy are actually obtained via a bilinearization, as an intermediate step. For the most prominent member (1.1), this is expressed in corollary 4.8 and further explained in appendix C. For the first non-autonomous flow, a bilinearization is given by corollary 4.10, together with (4.10) and (4.11). Corresponding bilinearizations of the second autonomous and non-autonomous flows have been obtained in corollaries 4.12 and 4.14, together with the respective equations stated in lemma 4.3 and (4.15).

Can we say something about regularity of solutions? This is subtle. According to a theorem of Hamburger [17], a sequence \( s_j, j = 0, 1, \ldots \) can be represented as a sequence of moments (see (A.1)), with a positive measure \( \mu \) on the real line, if and only if the Hankel matrices \( \hat{H}_n = (s_{i+j}) \), where \( i, j = 0, \ldots, n - 1, n = 1, 2, \ldots \), are positive semi-definite [17]. Moreover, if the support of \( \mu \) contains infinitely many points, then \( \det \hat{H}_n > 0 \), for \( n = 1, 2, \ldots [5, 43] \). If this is so, then it implies, via (A.2), regularity of a corresponding solution of an equation from the extended Volterra lattice hierarchy.

If the functions \( a_j, j = 0, 1, \ldots \), in (B.2) are positive and bounded, the spectral measure of the (then selfadjoint) operator (Jacobi matrix) \( L \) has the required properties [4], so that a solution \( u_0 \) of the form (1.8) is regular. This can be achieved at least at some value of time, say \( t = 0 \). It is then shown in [4] that the time evolution of the measure, induced by (B.3), leads to a solution of the corresponding equation of the extended Volterra lattice hierarchy on a finite time interval and on the right half lattice, with boundary data \( u_0 = 0 \).

In our work, we characterized solutions on the right half lattice in terms of boundary data, a point of view also taken in [40]. To determine those boundary data that correspond to regular solutions is an open problem, which might be solvable using the results and tools of [4], or other methods from the theory of orthogonal polynomials.

Besides the structural insights, in particular those expressed in our main results, the Riccati systems, shown to be equivalent to equations of the extended Volterra lattice hierarchy, provide us with an unfamiliar approach to exact solutions of the latter. Note that here we describe solutions in terms of determinants of (Hankel) matrices the size of which grows with the lattice site number. Since we reach all solutions, this raises the question how e.g. soliton solutions can be characterized in this way. The (integrable) equations of the extended Volterra lattice hierarchy possess solutions outside the familiar families of solitons and algebro-geometric (periodic) solutions, and we expect our results to be of help to reveal them. This still needs

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13 This is done in [4] moreover for an extension of the Toda lattice hierarchy.
elaboration, partly following [40]. All this should then also have some impact on scientific
problems modeled by the Volterra lattice equation or its companions.

Our results should also be useful to explore the structure of reductions of the extended
Volterra lattice hierarchy. In this context we should mention the application of the Hankel
determinant formula for the Toda lattice in order to prove the generic polynomiality of \( \tau \) func-
tions of the Painlevé equations [23, 24].

Acknowledgment

X-M Chen has been supported by the Sino-German (CSC-DAAD) Postdoc Scholarship
Program 2016 (57251553) and the German Academic Exchange Service (DAAD): Research
Grants—Short-Term Grants 2018 (57378443). She would like to thank A S Zhedanov for a
very enlightening correspondence concerning his work [40], and X K Chang for many very
helpful discussions. X-B Hu has also been partially supported by the National Natural Science
Foundation of China (Grant no. 11331008, 11571358). Last but not least we have to thank an
Editor-in-Chief for several suggestions that improved this work.

Appendix A. From Toda to Volterra

The Toda lattice equation in Flaschka variables reads

\[
\dot{a}_n = \frac{1}{2} a_n (b_{n+1} - b_n), \quad \dot{b}_n = a_n^2 - a_{n-1}^2 .
\]

Its inverse spectral problem (see [4] and references cited there) can be solved via

\[
a_n = \frac{\sqrt{\hat{H}_n \hat{H}_{n+2}}}{\hat{H}_{n+1}}
\]

and a corresponding expression for \( b_n \), see [4]. Here \( \hat{H}_n \) is the determinant of the Hankel
matrix \( \hat{H}_n = (s_{i+j}) \), where \( i, j = 0, \ldots, n-1 \), built with the moments

\[
s_j = \int_{\mathbb{R}} \lambda^j \, d\mu(\lambda) , \quad j = 0, 1, \ldots ,
\]

(A.1)

where \( \mu \) is an infinite positive measure on the real line. On the level of moments, the time
dependence corresponds to a simple deformation of the measure. This translates the Toda lat-
tice equation into a Riccati system for the moments [4, 40].

The reduction to the Volterra lattice involves choosing an odd measure, so that all odd
moments are zero. Setting

\[
c_j := s_{2j} ,
\]

by exchanges of rows and columns we find that

\[
\hat{H}_{2n} = H_{2n}^0 H_n^1 , \quad \hat{H}_{2n+1} = H_{n+1}^0 H_n^1 .
\]

(A.2)

In terms of \( u_n = a_{n-1}^2 \), we thus obtain

\[
u_{2n-1} = \frac{\hat{H}_{2(n-1)} \hat{H}_{2n}}{(\hat{H}_{2n-1})^2} = \frac{H_{2n-1}^0 H_n^1}{H_n^0 H_{n-1}^1} , \quad u_{2n} = \frac{\hat{H}_{2n-1} \hat{H}_{2n+1}}{(\hat{H}_{2n})^2} = \frac{H_{2n-1}^0 H_{n+1}^1}{H_n^0 H_{n-1}^0} .
\]

These are the expressions in (1.8). Also see [14, 48].
Appendix B. Another expression for the extended Volterra lattice hierarchy

In [4] (equation (7.14) therein), the following generalization of the Volterra lattice equation appeared,

\[
\dot{a}_n(t) = \{\Phi(L,t)\}_{n+1,n} + \frac{1}{2} a_n \{\{\Psi(L,t)\}_{n+1,n+1} - \{\{\Psi(L,t)\}_{n+1,n} \}
\]

(B.1)

where \( a_{-1} = 0 \) and \( n = 0, 1, \ldots \). Here \( \Phi \) and \( \Psi \) are polynomials, in a parameter \( \lambda \), of the form

\[
\Phi(\lambda,t) = \sum_{i=0}^{m} \varphi_{2i+1}(t) \lambda^{2i+1}, \quad \Psi(\lambda,t) = \sum_{i=0}^{m} \varphi_{2i}(t) \lambda^{2i},
\]

with coefficients that are functions of \( t \). Furthermore, \( L \) is the Jacobi matrix with the only non-zero entries given by

\[
L_{j+1,j} = a_j = L_{j,j+1}, \quad j = 0, 1, \ldots .
\]

Let \( P_k(\lambda,t) \), \( k = 0, 1, \ldots \), be a sequence of functions (actually symmetric orthogonal polynomials), satisfying the recurrence relation

\[
a_{k-1}P_{k-1} + a_k P_{k+1} = \lambda P_k, \quad k = 0, 1, \ldots ,
\]

(B.2)

where \( P_{-1} = 0 \). We define \( d_{jk} \), \( k = 1, 2, \ldots , j = 0, 1, \ldots , k - 1 \), by

\[
\frac{\partial P_k}{\partial \lambda} = \sum_{j=0}^{k-1} d_{jk} P_j, \quad k = 1, 2, \ldots .
\]

Then \( D_L \) is the strictly upper triangular matrix with entries \( d_{jk} \). These can be computed recursively using the following equation, which is obtained by differentiation of (B.2) with respect to \( \lambda \), and using the preceding relation,

\[
(a_{k-1}d_{0,k-1} + a_k d_{0,k+1} - a_0 d_{0,k}) P_0 + \sum_{j=1}^{k-2} (a_{k-1}d_{j,k-1} + a_k d_{j,k+1} - a_j d_{j+1,k} - a_{j-1} d_{j-1,k}) P_j
\]

\[
+ (a_{k-2}d_{k-1,k+1} - a_k d_{k-2,k}) P_{k-1} + (a_k d_{k,k+1} - a_{k-1} d_{k-1,k-1}) P_k = 0 .
\]

In particular, we find

\[
d_{k,k+1} = \frac{k+1}{a_k}, \quad d_{k,k+2} = 0, \quad d_{k,k+3} = \frac{2 \sum_{i=0}^{k} a_i^2 - (k+1)a_{k+1}^2}{a_k a_{k+1} a_{k+2}}, \quad k = 0, 1, \ldots .
\]

As a consequence of the recurrence relation (B.2), the Jacobi matrix \( L \) represents the operator of multiplication by \( \lambda \) in the Hilbert space spanned by \( \{P_k | k = 0, 1, \ldots \} \), whereas \( D_L \) represents the operator \( \partial / \partial \lambda \). \( L \) is a Lax operator for (B.1), and we have non-isospectrality if \( \Phi \neq 0 \), since

\[
\frac{d\lambda}{dt} = \Phi(\lambda,t).
\]

(B.3)

We refer to [4] for details.

The Volterra lattice equation (1.1) and the second hierarchy flow (1.3) are obtained from (B.1) by setting \( \Phi = 0 \) and \( \Psi(\lambda) = \lambda^3 \), respectively \( \Psi(\lambda) = \lambda^4 \), and using the transformation

\[
u_n = a_{n-1}^2, \quad n = 0, 1, \ldots .
\]

(B.4)
Note that the condition $a_{-1} = 0$ enforces $u_0 = 0$. The first non-autonomous flow (1.6) is obtained in the same way by choosing $\Phi(\lambda) = \frac{1}{2} \lambda^3$ and $\Psi = \lambda^2$.

**Example.** Let $\Phi(\lambda) = \frac{1}{2} \lambda^3$ and $\Psi = 2 \lambda^4$. Then (B.1) leads to
\[
\dot{a}_n = \frac{1}{2} a_n \left( (2 - n)a_n^2 a_{n-2}^2 - n a_{n-1}^4 + (1 - n)a_{n-2}^2 a_{n-1}^2 + a_n^4 + (4 + n) a_n^2 a_{n+1}^2 + 2(a_{n+1}^2 - a_{n-1}^2) \sum_{i=0}^{n-2} a_i^2 \right).
\]
In terms of (B.4), we obtain (1.7). □

It is quite evident now that the autonomous Volterra lattice hierarchy is contained in (B.1) with $\Phi = 0$, choosing for $\Psi$ the members of the sequence of even powers of $\lambda$. The non-autonomous equations are recovered, up to addition of multiples of the right hand side of autonomous flows, if we choose $\Psi = 0$ and for $\Phi(\lambda)$ the members of the sequence of odd powers of $\lambda$.

**Appendix C. Hirota bilinearization of the Volterra lattice equation**

From [36] (equation (22) therein, with $k = 2$, $m = 1$ and $t \mapsto -t$), we recall that, via
\[
u_n = \frac{\tau_{n-2}}{\tau_{n-2} \frac{1}{2} \tau_n ^{\frac{1}{2}}}.
\]
the following (Hirota) bilinearization of the Volterra lattice equation (1.1) is obtained,
\[
2 \sinh(D_n) \left[ \sinh \left( \frac{1}{2} D_n \right) D_n + 2 \sinh(D_n) \sinh \left( - \frac{1}{2} D_n \right) \right] \tau_n \cdot \tau_n = \chi \cosh \left( \frac{1}{2} D_n \right) \tau_n \cdot \tau_n,
\]
Here $D_n$ and $D_t$ are Hirota bilinear operators [18]. The latter equation is equivalent to
\[
\left[ \sinh \left( \frac{1}{2} D_n \right) D_t + 2 \sinh(D_n) \sinh \left( - \frac{1}{2} D_n \right) \right] \tau_n \cdot \tau_n = \chi \cosh \left( \frac{1}{2} D_n \right) \tau_n \cdot \tau_n,
\]
where $\chi$ is an arbitrary function of $t$, independent of $n$. Shifting by half a lattice spacing, and writing $\chi = 1 - u_0$, with a function $u_0(t)$, this can be expressed as
\[
\dot{\tau}_{n+1} \tau_n - \tau_{n+1} \dot{\tau}_n = \tau_{n-2} - u_0 \tau_{n+1} \tau_n.
\]
Via
\[
\tau_{2n} = H_n^0, \quad \tau_{2n-1} = H_{n-1}^1,
\]
is this turned into the equations in corollary 4.8. This is related to the two-field form of the Volterra lattice equation, see [20, 32, 45], for example.

In [36], the freedom expressed by the function $\chi$ is not taken into consideration. Equation (23) in [36] corresponds to the case where $\chi = 0$ (also see e.g. [49]). In the present context, where we express $r$-functions in terms of Hankel determinants, we see that the additional freedom is necessary. In [32] (see (1.5) therein), for example, a bilinearization of the two-field form of (1.1) is given, which is of the form of the equations in corollary 4.8, but with constant coefficient of the last term.
Appendix D. Thiele expansion and the first non-autonomous Volterra flow

Let us consider the algorithm [9]

\[ \gamma_{n+1}(t) = \gamma_n(t) + \frac{g_n(t)}{\gamma_n(t)} \quad n = 0, 1, \ldots, \]

where \( f(t) \) is a smooth function and \( g_n(t), n = 0, 1, \ldots, \) are given functions, assumed to be nowhere vanishing. Via the Miura transformation (see [8, 9] in the present context)

\[ u_n(t) = -M_{n-1}(t) M_n(t), \quad M_n(t) = \frac{\dot{\gamma}_n(t)}{\gamma_n(t)}, \quad n = 1, 2, \ldots, \]

does not appear in the algorithm, \( u_0 = 0 \) is a consequence.

If \( g_n(t) = 1 \) for all \( n \), this is the Volterra lattice equation (1.1), in which case the above algorithm is known as ‘confluent \( \varepsilon \)-algorithm’ [51, 52]. If \( g_n(t) = n \), (D.1) is the first non-autonomous Volterra lattice equation (1.6). If \( g_n(t) = n + 1 \), (D.1) is a combination of the Volterra lattice and the first non-autonomous Volterra lattice equation. More precisely, the right hand side is then \( V^{(1)} + \gamma^{(1)} \). In this case we are dealing with the ‘confluent form of the \( \rho \)-algorithm’ [51],

\[ \rho_{n+1}(t) = \rho_n(t) + n + \frac{1}{\rho_n(t)} \quad n = 0, 1, \ldots, \]

In particular, this computes the functions in Thiele’s expansion formula, a continued fraction expansion (see e.g. [7, 15], also for the notation) of \( f(t+h) \) around \( t \),

\[ f(t+h) = f(t) + \frac{h}{\rho_1(t) - \rho(t)} + \frac{h \rho_1(t) - \rho_0(t)}{\rho_2(t) - \rho_1(t)} + \frac{h \rho_2(t) - \rho_1(t)}{\rho_3(t) - \rho_2(t)} + \ldots. \]

Moreover, for \( g_n = \alpha n + \beta \), where \( \alpha \) and \( \beta \) are functions of \( t \), the \( \gamma_n \), defined via the above recurrence relation, have explicit expressions in terms of Hankel determinants,

\[ \gamma_{2k}(t) = \frac{\tilde{H}_{k+1} \gamma_{2n+1}(t)}{\tilde{H}_k(t)} \quad \gamma_{2k+1}(t) = \frac{\tilde{H}_{k+1}}{\tilde{H}_k(t)} \quad (D.2) \]

where

\[ \tilde{H}_0 = 1, \quad \tilde{H}_n = \det(\xi_{i+j+m})_{j=0}^{m-1}, \quad m = 0, 1, 2, \ldots, \]

\[ \xi_0 = f(t), \quad \xi_j(t) = g_j(t) \xi_{j+1}(t), \quad j = 0, 1, 2, \ldots. \]

Note that \( \xi_{n+1} = c_j, j = 0, 1, \ldots, \) with the \( c_j \) used elsewhere in this work. By using the Miura transformation, one can also obtain the determinant expressions (1.8) for \( u_n \) from (D.2).
Remark D.1. Our results motivate the following generalization of the $\gamma$-algorithm,

$$
\gamma_{-1}(t) = 0, \quad \gamma_0(t) = f(t), \\
\gamma_{n+1}(t) = \gamma_{n-1}(t) + \frac{g_n(t)}{\gamma_n(t) - \psi(t)}, \quad n = 0, 1, \ldots.
$$

(D.3)

For $g_n = \alpha n + \beta$, with functions $\alpha(t)$ and $\beta(t)$, the $\gamma_n$ are still given by the expressions in (D.2), but now with

$$
\xi_0 = f(t), \quad \dot{\xi}_0(t) = g_0(t) \xi_1(t) + \psi(t), \\
\dot{\xi}_j(t) = g_j(t) \xi_{j+1}(t) - \psi(t) \sum_{i=1}^{j} \xi_i \xi_{j+1-i}, \quad j = 1, 2, 3, \ldots.
$$

(D.4)

Using the Miura transformation, but now with $M_n(t) = (\dot{\gamma}_n(t) - \psi(t)) / g_n(t)$, we are again led to the second equation of (D.1), which is accompanied by

$$
\ddot{u}_1 = u_1 \left( g_2 u_2 + (g_1 - g_0) u_1 - \frac{\psi}{\gamma_1} \right).
$$

Let us set $\psi(t) = g_{-1}(t) \gamma_1(t) u_0(t)$, with arbitrary $u_0(t)$. Recall that $\xi_{j+1} = c_j$ and note that $\gamma_1 = 1/\xi_1 = 1/c_0$ by use of (D.2). If $g_0(t) = 1$, then $\psi(t) = u_0(t)/c_0(t)$ and the last equation of (D.4), expressed in terms of $c_j$, coincides with (3.1). This makes contact with theorem 3.1. If $g_0(t) = n$, then $\psi(t) = -u_0(t)/c_0(t)$ and the last equation of (D.4) coincides with (3.2). Here we meet the case described in theorem 3.3.

If $g_n(t) = n + 1$, we have $g_{-1} = 0$, which is indeed necessary in order to cast the above equation for $u_1$ and the second of (D.1) into the combination of autonomous and non-autonomous Volterra flows. However, this enforces $\psi = 0$, so that we are back to the old algorithm in this case.

The generalized algorithm (D.3) still needs further exploration. It is of interest since its solutions still admit Hankel determinant representations.

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