LARGE DEVIATION PRINCIPLE FOR THE MICROPOLAR, MAGNETO-MICROPOLAR FLUID SYSTEMS

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Abstract. Micropolar fluid and magneto-micropolar fluid systems are systems of equations with distinctive feature in its applicability and also mathematical difficulty. The purpose of this work is to follow the approach of [8] and show that another general class of systems of equations, that includes the two-dimensional micropolar and magneto-micropolar fluid systems, is well-posed and satisfies the Laplace principle, and consequently the large deviation principle, with the same rate function.

1. Introduction. The theory of large deviations is an important direction of research and has been studied by many (e.g. [13, 20], Chapter 12 [12], [7]). In particular, the authors in [16] developed an approach to this theory through proving the convergence of solutions to variational problems, based on the fact that the large deviation principle (LDP) in a Polish space is equivalent to Laplace principle (see Theorem 1.2.3 [16]). Subsequently, the authors in [4, 5] proved a type of extended contraction principle that consists of a weak convergence and compactness conditions, (see Assumption 4.3 [4], and also Assumption 1 on pg. 1401 [5]) that guarantees the Laplace principle with a rate function and hence the LDP with the same rate function.

Various authors have studied the LDP results for many models (e.g. [6, 8, 9, 28, 35]). In particular, the work in [8] covered many models that include the Navier-Stokes equations (NSE), magnetohydrodynamics (MHD) system, Bénard problem, magnetic Bénard problem, Leray α-model and shell models of turbulence; we also refer to its accompanying paper [9] for Wong-Zakai approximation results. However, the micropolar and magneto-micropolar fluid systems, which have also attracted much attention from many researchers for reasons to be described in a subsequent section, do not seem to be covered by the work of [8], precisely due to the singular terms that do not exist in most other models of fluid mechanics. The purpose of this work is to establish the well-posedness and LDP of a new class of systems of equations that includes the two-dimensional (2-d) micropolar and magneto-micropolar fluid systems. The well-posedness result on the micropolar and magneto-micropolar fluid system extends the work of [37] in the three-dimensional (3-d) case, the claim

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2. Preliminaries and statement of results. Let us consider $D$ to be a bounded open and simply-connected domain, while the mappings $u(x, t)$, $w(x, t)$, $b(x, t)$, the velocity, the micro-rotational velocity and the magnetic vector fields respectively, and $\pi(x, t)$ the hydrostatic pressure scalar field. We denote the physical quantities: $\chi, \mu, \gamma, \nu$ the vortex viscosity, the kinematic viscosity, the spin viscosity and the magnetic diffusivity respectively. Finally, we let $f_u, f_w, f_b$ be the external forces to be elaborated subsequently, $\partial_t = \frac{\partial}{\partial t}$, and state the boundary-value problem of the magneto-micropolar fluid (MMPF) system:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla\pi &= \chi \nabla \times w + (\mu + \chi)\Delta u + f_u, \\
\partial_t w + (u \cdot \nabla)w &= -2\chi w + \chi \nabla \times u + \gamma \Delta w + f_w, \\
\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u &= -\nu \nabla \times \nabla \times b + f_b,
\end{align*}
\]

\[
\left\{ \begin{array}{ll}
\nabla \cdot u = \nabla \cdot b = 0, & \forall t \in [0, T], \\
u|_{\partial D} = w|_{\partial D} = 0, & b \cdot n|_{\partial D} = \nabla \times b|_{\partial D} = 0, & \forall t \in (0, T).
\end{array} \right.
\]

Individual particles of complex fluids may consist of different shape, shrink, expand or even rotate independently of the rotation and movement of the fluid and the NSE cannot take into account of such micro-structural aspect. In order to emphasize the micro-structure of fluids, the theory of microfluids and thereafter micropolar fluids were introduced by Eringen in [17, 18]; the micropolar fluid (MPF) system is the MMPF system (1a)-(1c) with $b \equiv 0$. The authors in [1] proposed coupling it further with a magnetic field to study the motion of incompressible electrically conducting micropolar fluid. We note that from the original MMPF system introduced in [1], we made the appropriate modification in (1a)-(1c) by letting $u = (u_1, u_2, 0)$, $w = (0, 0, w_3)$, $b = (b_1, b_2, 0)$ (see pg. 185 [26]). As the MPF system models some fluid better than the NSE, e.g. fluid consisting of bar-like elements such as liquid crystals, dumbbell molecules and animal blood, this system has caught much attention from researchers such as physicists, engineers and mathematicians (e.g. [21, 30, 32]). We also wish to emphasize that the MPF system has some similarity with the Boussinesq equations, equivalently the Bénard problem via an introduction of a new function (see pg. 133 [36]):

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u + \nabla\pi &= \chi w e_2 + (\mu + \chi)\Delta u + f_u, \\
\partial_t w + (u \cdot \nabla)w &= \chi u_2 + \gamma \Delta w + f_w,
\end{align*}
\]

where $(e_1, e_2)$ is an orthonormal basis of $\mathbb{R}^2$. We note that the linear terms $\chi \nabla \times w$, $\chi \nabla \times u$ in (1a), (1b) respectively are one derivative more singular than $\chi we_2$, $\chi u_2$ of (3a), (3b) respectively. This difference is so immense that despite the fact that in [22], a global well-posedness result for the Boussinesq equation was obtained for the case of $\gamma = 0$ and the dissipativity strength being only half of $(\mu + \chi)\Delta u$ (see equation (1.1) [22] for details in terms of a fractional Laplacian), such a result is absent and seems very difficult in the case of the MPF system (see [14, 38]). Due to this difference, we are interested in the stochastic analysis of MPF and MMPF systems; moreover, this is precisely the reason why this model was excluded from the general case that was covered by the comprehensive result in [8].

In order to present our results, we now recall the standard notations for fluid mechanics mathematical literature (e.g. [10, 36, 37]). Firstly, to emphasize the
significance of a constant on certain parameters, we write $A \lesssim_{a,b} B$ to imply the existence of a constant $c = c(a,b)$ such that $A \leq c(a,b)B$; similarly we write $A \asymp_{a,b} B$ if $A = c(a,b)B$.

For the MPF and the MMPF systems, we may denote

- $V_1 \triangleq \{ \phi \in (C^\infty_c(D))^3 : \nabla \cdot \phi = 0 \}$, $V_2 \triangleq \{ \phi \in \mathbb{H}^1_0 : \nabla \cdot \phi = 0 \}$,
- $V_3 \triangleq \{ \phi \in (C^\infty_c(D))^3 : \nabla \cdot \phi = 0, \phi \cdot n|_{\partial D} = 0 \}$,
- $H_1 = H_2 \triangleq \{ \phi \in L^2 : \nabla \cdot \phi = 0, \phi \cdot n|_{\partial D} = 0 \}$, $H_2 \triangleq L^2$.

and the inner products of $H_i, i = 1, 2, 3$ by $(\phi, \psi) \triangleq \sum_{i=1}^3 \int_D \phi_i(x)\psi_i(x)dx$. We may set $H \triangleq \prod_{i=1}^3 H_i$ and define for $\Phi_i = (X^i, Y^i, Z^i), i = 1, 2$,

- $(\Phi^1, \Phi^3) \triangleq (X^1, X^2) + (Y^1, Y^2) + (Z^1, Z^2), (\Phi^i, \Phi^i) \triangleq |\Phi^i|^2$.

Furthermore, for the MPF and the MMPF systems we may define

- $\langle A_1 X^1, X^2 \rangle \triangleq - (\mu + \chi) \langle \Delta X^1, X^2 \rangle$,
- $\langle A_2 Y^1, Y^2 \rangle \triangleq - (\gamma \Delta Y^1 + \chi Y^1, Y^2)$,
- $\langle A_3 Z^1, Z^2 \rangle \triangleq \nu (\nabla \times \nabla \times X^1, Z^2), D(A_3) \triangleq H_1 \cap \{ b \in \mathbb{H}^2 : \nabla \times b|_{\partial D} = 0 \}$,
- $\langle A_1 X^1, X^2 \rangle + \langle A_2 Y^1, Y^2 \rangle + \langle A_3 Z^1, Z^2 \rangle$,

and set $V \triangleq \prod_{i=1}^3 V_i$ with $\|\Phi^i\|^2_V \triangleq \langle A\Phi^i, \Phi^i \rangle = ((\Phi^i, \Phi^i))$: specifically, this implies

- $\|\Phi^i\|^2_V = (\mu + \chi) \|\nabla X^i\|^2 + \gamma \|\nabla Y^i\|^2 + \chi \|Y^i\|^2 + \nu \|\nabla \times X^i\|^2$.

Moreover, we may denote

- $B_1(X^1, X^2) \triangleq (X^1, \nabla)X^2$,
- $B_2(Z^1, Z^2) \triangleq (Z^1, \nabla)Z^2$,
- $B_3(X^1, Y^2) \triangleq (X^1, \nabla)Y^2$,
- $B_4(X^1, Z^2) \triangleq (X^1, \nabla)Z^2$,

- $(B(\Phi^1, \Phi^2), \Phi^3) \triangleq (B_1(X^1, X^2), X^3) - (B_2(Z^1, Z^2), X^3)
  + (B_3(X^1, Y^2), Y^3) + (B_4(X^1, Z^2), Z^3) - (B_5(Z^1, X^2), Z^3)$;

where e.g. $B_1(u, u) = B_1(u)$ and analogously $B(y, y) = B(y)$.

In this paper, we study a general class of equations (see (10)) that includes the MPF and the MMPF systems. We now state the general conditions on an operator $A$ and $B : V \times V \mapsto V'$, $V = D(A^\frac{1}{2})$ where $V'$ is the dual of $V$.

**Condition (C1)**

1. With $(H, \|\cdot\|)$ a separable Hilbert space, $A$ is an unbounded self-adjoint positive linear operator on $H$ such that $V = D(A^\frac{1}{2})$ with $\|\cdot\|_V = \|A^\frac{1}{2}\|.$
2. The operator $B$ is a bilinear continuous mapping such that $\langle B(\Phi^1, \Phi^2), \Phi^3 \rangle = - \langle B(\Phi^1, \Phi^3), \Phi^2 \rangle$, $\forall \Phi^i = (X^i, Y^i, Z^i) \in V, i = 1, 2, 3$.
3. There exists a Banach space $\mathcal{H}$ such that the following properties hold:
   (a) $V \subset \mathcal{H} \subset H$.
   (b) There exists a constant $c_0 > 0$ such that $\|\Phi^i\|^2_\mathcal{H} \leq c_0 \|\Phi^i\| \|\Phi^i\|_V$.
   (c) For all $\delta > 0$, there exists a constant $c_\delta = c(\delta) > 0$ such that

   - $\|B(\Phi^1, \Phi^2), \Phi^3\| \leq \delta \|\Phi^3\|^2_\mathcal{H} + c_\delta \|\Phi^1\|^2_\mathcal{H} \|\Phi^2\|^2_\mathcal{H}$,
   - equivalently (see Remark 2.1 (1) pg. 383 [8])

   - $\|B(\Phi^1, \Phi^2), \Phi^3\| \leq c ||\Phi^1||_\mathcal{H} ||\Phi^2||_V ||\Phi^3||_\mathcal{H}$.


Finally, for the MPF and the MMPF systems, we may denote
\[ R(u, w, b) \triangleq (-\chi \nabla \times w, \chi w - \chi \nabla \times u, 0). \]  
(4)

We emphasize already that although the authors in \[8\] considered a wide class of different systems, they required that \( R \) is a linear bounded operator in \( H \) and hence does not apply for the stochastic MPF and the MMPF systems of our consideration.

Concerning the external forces \( f_u, f_w, f_b \) in (1a), (1b), (1c) respectively, we let \( Q \) be a linear positive operator in \( H \) such that it is in the trace class and hence compact (see \[12\]). Let \( H_0 \triangleq Q^{1/2} H \) so that \( H_0 \) is still a Hilbert space with a scalar product \( \langle \phi, \psi \rangle_0 \triangleq \langle Q^{-1/2} \phi, Q^{-1/2} \psi \rangle \), \( |\phi|_0 \triangleq \sqrt{\langle \phi, \phi \rangle_0} \) for all \( \phi, \psi \in H_0 \). The embedding \( i: H_0 \to H \) is Hilbert-Schmidt and hence compact; moreover, \( i \cdot i^* = Q \) with \( i^* \) being the adjoint operator of \( i \). We denote by \( L_Q \triangleq L_Q(H_0, H) \) the space of all linear operators \( S : H_0 \mapsto H \) such that \( SQ^2 : H \mapsto H \) is also Hilbert-Schmidt, endowed with a norm
\[ |S|^2_{L_Q} \triangleq \text{tr}(SQS^*) = \text{tr}([SQ^2][SQ^2]^*) = \sum_{i=1}^{\infty} |SQ^2_i e_i|^2 = \sum_{i=1}^{\infty} |[SQ^2_i]^* e_i|^2 \]  
(5)

with \( S^* \) being the adjoint operator of \( S \), and \( \{e_i\}_i \) an orthonormal basis of \( H \). We let \( W \) be the \( H \)-valued Wiener process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with a covariance operator \( Q \) so that \( W \) is Gaussian, has time independent increments, \( \mathbb{E}[(W(s), f)] = 0 \), \( \mathbb{E}[(W(s), f)(W(t), g)] = (s \wedge t)\langle f, g \rangle \), and
\[ W(t) = \lim_{m \to \infty} W_m(t) \triangleq \lim_{m \to \infty} \sum_{i=1}^{m} q_i \beta_i(t)e_i, \quad \forall s, t \geq 0, f, g \in H \]  
(6)

where \( \{\beta_i\}_i \) are mutually independent standard Wiener processes, and \( \{q_i\}_i \) satisfy \( q_i e_i \triangleq Q e_i \) for all \( i \). Moreover, \( (\mathcal{F}_t)_{t \geq 0} \) is the Brownian filtration, the smallest right-continuous complete filtration with respect to which \( \{W(t)\}_{t \geq 0} \) is adapted. With these notations we may reconsider from (1a), (1b), (1c), the MPF system with \( y = (u, w, 0) \) and the MMPF system with \( y = (u, w, b) \) as
\[ \partial_t y + Ay + B(y) + R y = \sigma(t, y) \partial W \]  
(7)

where \( \sigma : [0, T] \times V \mapsto L_Q(H_0, H) \) is the noise intensity. For the general class of equations which we will study, let us state the condition on \( \sigma \):

**Condition (C2)**

\( \sigma \in C([0, T] \times V; L_Q(H_0, H)) \) and there exist constants \( K_i, L_i \geq 0, i = 0, 1, 2, \) such that for all \( t \in [0, T], y_i \in V, i = 1, 2, \)
\begin{enumerate}
  \item \( |\sigma(t, y_i)|_{L_Q}^2 \leq K_0 + K_1 |y_i|^2 + K_2 \|y_i\|_V^2, \)
  \item \( |\sigma(t, y_1) - \sigma(t, y_2)|_{L_Q}^2 \leq L_1 |y_1 - y_2|^2 + L_2 \|y_1 - y_2\|_V^2. \)
\end{enumerate}

Now we let \( \mathcal{A} \) be the class of \( H_0 \)-valued \( \mathcal{F}_t \)-predictable processes \( \phi \) such that \( \mathbb{P}(\{\omega \in \Omega : \int_0^T |\phi(s)|_V^2 ds < \infty\}) = 1 \) so that for \( M \in \mathbb{N}, \)
\[ S_M \triangleq \left\{ h \in L^2([0, T]; H_0) : \int_0^T \|h(s)\|^2_0 ds \leq M \right\}, \]  
(8)

endowed with the weak topology that is defined by the metric of \( d_1(y_1, y_2) \triangleq \sum_{i=1}^{\infty} \frac{1}{i!} \left| \int_0^T (y_1(s) - y_2(s), \xi_i(s))_0 ds \right| \), where \( \{\xi_i\}_{i=1}^{\infty} \) is an orthonormal basis for the space \( L^2([0, T]; H_0) \), becomes a Polish space. We furthermore define
\[ \mathcal{A}_M \triangleq \{ h \in \mathcal{A} : h(w) \in S_M \mathbb{P}\text{-almost surely} \}. \]  
(9)
Finally, we let $M > 0$, $h \in \mathcal{A}_M$, $\zeta \in H$ and write

\begin{equation}
\begin{aligned}
dy(t) + [Ay(t) + B(y(t)) + \tilde{R}(t, y(t))]dt \\
= \sigma(t, y(t))dW(t) + \tilde{\sigma}(t, y(t))h(t)dt, \quad y(0) = \zeta,
\end{aligned}
\end{equation}

where $\tilde{\sigma} : [0, T] \times V \mapsto L(H_0, H)$ and $\tilde{R} : [0, T] \times V \mapsto H$ are assumed to satisfy the following condition.

**Condition (C3)**

1. $\tilde{\sigma} \in C([0, T] \times V; L(H_0, H))$ and there exist constants $\tilde{K}_i, \tilde{L}_j$, $i = 0, 1, 2$, $j = 1, 2$ such that for all $y_k \in H \cap V, k = 1, 2$,

\begin{equation}
|\tilde{\sigma}(t, y_k)|^2_{L(H_0, H)} \leq \tilde{K}_0 + \tilde{K}_1 |y_k|^2 + \tilde{K}_2 |y_k|^2,
\end{equation}

2. $\tilde{R} \in C([0, T] \times V; H)$ and there exist constants $R_i, i = 0, 1$ such that for all $y_j \in V, j = 1, 2$,

\begin{equation}
|\tilde{R}(t, 0)| \leq R_0, \quad |\tilde{R}(t, y_1) - \tilde{R}(t, y_2)| \leq R_1 |y_1 - y_2|_V.
\end{equation}

We define the Polish space $X \triangleq C([0, T]; H) \cap L^2([0, T]; V)$ endowed with the norm $\|y\|_X^2 \triangleq \sup_{t \in [0, T]} |y(t)|^2 + \int_0^T |y|^2_V dt$ and now define a weak solution to (10) as follows.

**Definition 2.1.** $y_h(t, \omega)$ is a weak solution to (10) on $[0, T]$ if $y_h(0) = \zeta, y_h \in X, y_h$ is $\mathcal{F}_t$-predictable, and for all $v \in D(A), t \in [0, T]$,

\begin{equation}
(y_h(t), v) - (\zeta, v) + \int_0^t (y_h(s), Av) + (B(y_h(s)), v) + (\tilde{R}(s, y_h(s)), v)ds
\end{equation}

\begin{equation}
\quad \quad = \int_0^t (\sigma(s, y_h(s))dW(s), v) + \int_0^t (\tilde{\sigma}(s, y_h(s))h(s), v)ds, \quad \mathbb{P}\text{-almost surely.}
\end{equation}

We may now state our first result on the well-posedness of a class of systems of equations, that includes the MPF and the MMPF systems (1a), (1b), (1c).

**Theorem 2.2.** For the system (10), suppose

1. the Conditions (C1), (C2) hold,
2. and either
   (a) $\sigma = \tilde{\sigma}$ and $\tilde{R}$ satisfies the Condition (C3)(2)
   (b) or (C3).

It holds that for all $M > 0$ and $T > 0$, there exists $K_2 = K_2(T, M) > 0$ such that if $K_2$ of Condition (C2) satisfies $K_2 \in [0, K_2]$, the $L_2$ of the Condition (C2) satisfies $L_2 < 2$, $\mathbb{E}[|\zeta|^4] < \infty$ and $h \in \mathcal{A}_M$, then (10) admits a weak solution $y_h$ such that

\begin{equation}
\mathbb{E}[\sup_{t \in [0, T]} |y_h(t)|^4 + \int_0^T \|y_h\|^2_V + \|y_h(t)\|^4_{V} dt] \lesssim_{T, M, K_i, L_i, \tilde{K}_i, \tilde{L}_i} 1 + \mathbb{E}[|\zeta|^4],
\end{equation}

$i = 0, 1, 2$. Moreover, if $L_2$ of the Condition (C2) is sufficiently small, then the solution is unique. Even if $L_2 < 2$ is not sufficiently small, the uniqueness still holds if either $\tilde{\sigma} = \sigma$ or $h$ possesses a deterministic bound, i.e. there exists a deterministic $\psi(t) \in L^4([0, T])$ such that $|h(t)|_0 \leq \psi(t)$ $\mathbb{P}$-almost surely.
Remark 1. 1. The operators $A$ and $B$ that we defined for the MPF and the MMMPF systems (7) satisfy the Condition (C1) with $\mathcal{H} = L^4(D)$ (see Lemma 6.2 pg. 70 [25] and also [27]), and also the Condition (C3) because $R$ defined in (4) satisfies the role of $\tilde{R}$ in Condition (C3)(2).

2. The well-posedness result of stochastic equations in fluid mechanics has been investigated by many mathematicians; for brevity we only mention the most relevant work on the stochastic NSE here with no intention to be complete (e.g. [2, 19, 29]). We also note that for the LDP result, we only need the well-posedness result in the case $0 < L_2 < 2, \tilde{\sigma} = \sigma$. As in the case of [8], we chose to state Theorem 2.2 in all possible cases for completeness.

Next, we prepare to state our result on the LDP for a class of systems of equations, that includes the systems (7). We let $\epsilon > 0$ and $y^\epsilon$ solve

$$dy^\epsilon(t) + [Ay^\epsilon(t) + B(y^\epsilon(t)) + \tilde{R}(t, y^\epsilon(t))]dt = \sqrt{\epsilon}\sigma(t, y^\epsilon(t))dW(t), \quad y^\epsilon(0) = \zeta. \quad (14)$$

By Theorem 2.2, we see that for any $K_2, L_2 > 0$, if $\epsilon > 0$ is sufficiently small, there exists a unique solution $y^\epsilon = G^\epsilon(\sqrt{\epsilon}W)$ to (14), where $G^\epsilon : C([0, T]; H) \rightarrow X$ is a Borel measurable function (see pg. 310 [23] an also [31] for details). Now we let $B(X)$ denote the Borel $\sigma$-field generated by $X$ and recall some definitions relevant to LDP theory. Firstly, with the standard convention that an infimum over an empty set is $+\infty$, we recall the definition of LDP (cf. [13]).

Definition 2.3. The family of random variables $\{y^\epsilon\}_\epsilon$ satisfy the Large Deviation Principle (LDP) on $X$ with the good rate function $I$ if the following conditions hold, with $I(A) \triangleq \inf_{\phi \in A} I(\phi)$ for all $A \in B(X)$:

1. $I$ is a good rate function; i.e. $I : X \rightarrow [0, \infty]$ and for all $M \in [0, \infty)$, the level set $\{\phi \in X : I(\phi) \leq M\}$ is a compact subset of $X$.
2. For any closed subset $F \subset X$, $\limsup_{\epsilon \rightarrow 0} \epsilon \log P(\{y^\epsilon \in F\}) \leq -I(F)$.
3. For any open subset $G \subset X$, $\liminf_{\epsilon \rightarrow 0} \epsilon \log P(\{y^\epsilon \in G\}) \geq -I(G)$.

Now for any $h \in L^2([0, T]; H_0)$, we let $y_h$ be the solution to a corresponding control equation

$$dy_h(t) + [Ay_h(t) + B(y_h(t)) + \tilde{R}(t, y_h(t))]dt = \sigma(t, y_h(t))h(t)dt, \quad y_h(0) = \zeta. \quad (15)$$

We denote a space $C_0 \triangleq \{\int_0^t h(s)ds : h \in L^2([0, T]; H_0)\} \subset C([0, T]; H_0)$ as well as an operator $\mathcal{G}^0 : C([0, T]; H_0) \rightarrow X$, defined by

$$\mathcal{G}^0(g) \triangleq \begin{cases} y_h & \text{if } g = \int_0^t h(s)ds \in C_0, \\ 0 & \text{otherwise.} \end{cases}$$

We need the following additional assumption of the Hölder regularity on the noise intensity $\sigma(\cdot, y)$ in time $t$:

Condition (C4)

There exist $\gamma > 0, c \geq 0$ such that for any $t_1, t_2 \in [0, T], y \in V,$

$$|\sigma(t_1, y) - \sigma(t_2, y)|_{L_0} \leq c(1 + \|y\|_V)|t_1 - t_2|^\gamma.$$

We now state our LDP result.

Theorem 2.4. For the system (14), suppose the Conditions (C1), (C2) with $K_2 = L_2 = 0$, (C3)(2) and (C4) hold. Then the family of solutions $\{y^\epsilon\}_\epsilon$ to (14) satisfies
the LDP on $X = C([0, T]; H) \cap L^2([0, T]; V)$ with the good rate function

$$I_\zeta(\psi) \triangleq \inf \left\{ \frac{1}{2} \int_0^T |h(s)|_H^2 ds : h \in L^2([0, T]; H_0), \psi = G_0(\int_0^\cdot h(s)ds) \right\}.$$ 

**Remark 2.** Most importantly, the difference between Theorem 2.4 and Theorem 3.2 of [8] is the Condition (C3)(2). By Poincare’s inequality we see that any $\tilde{R}$ that satisfies

$$|\tilde{R}(t, y_1) - \tilde{R}(t, y_2)| \leq R_1|y_1 - y_2|$$

also satisfies the Condition (C3)(2) with a modified constant; thus, our work recovers the result of [8] as well.

**Remark 3.** We remark that the work within this manuscript focuses on the 2-d case. In the 3-d case, the LDP result will face difficulty because the uniqueness of the global weak solution is unknown; this problem is actually analogous to the deterministic case. While LDP result on the local unique strong solution may be pursued, it seems to require a different approach from what is considered in this manuscript.

Moreover, even in the 2-d case, it is not clear to the author if the well-posedness and LDP results may be extended to the stochastic density-dependent or nonhomogeneous fluid equations. In particular, we mention the work of [3] which proved the local existence of a unique strong solution to the 3-d deterministic nonhomogeneous MPF system, and [11, 33] which established the global existence of a weak solution to the 3-d stochastic nonhomogeneous NSE. The work of [40] proved the global existence of a weak solution to the stochastic 3-d nonhomogeneous MHD system and following the proof therein immediately deduces an analogous results for the 3-d stochastic nonhomogeneous MPF and MMPF systems. These works in [11, 33, 40] all may be extended to the 2-d case with no difficulty; however, the uniqueness of the global weak solution remains unknown due to technical difficulty and therefore, whether or not the LDP result may be obtained for such 2-d stochastic density-dependent or nonhomogeneous fluid equations remain unknown.

In subsequent sections we prove Theorem 2.2 and Theorem 2.4. We elaborate in proof where the difference with the work of [8] is significant, in particular when estimates involve $\tilde{R}$ or Condition (C3)(2). Moreover, we intentionally be brief when our proof can be similarly done as in the work of [8] while give details when it was missing in [8]; some parts of our proofs distinctively differ, e.g. choice of $r(t)$ in (47).

3. **Proof of Theorem 2.2.** We define a $V'$-valued mapping from $[0, T] \times V$ by $F(t, y) \triangleq -Ay - B(y) - \tilde{R}(t, y)$ and note its properties: for any $\phi, \psi \in V, \eta > 0$, there exist constants $R_1, c_\eta > 0$ such that

$$\langle F(\phi) - F(\psi), \phi - \psi \rangle \leq -(1 - \eta)\|\phi - \psi\|_{V'}^2 + c_\eta(R_1^2 + \|\phi\|_{H_0}^4)\|\phi - \psi\|^2.$$  

(16)

This is same as the equation (A.1) in the Appendix of [8] except the term $\tilde{R}$ in our case, which we compute carefully. We may write

$$\langle F(\phi) - F(\psi), \phi - \psi \rangle = \langle -A(\phi - \psi) - [B(\phi) - B(\psi)] - [\tilde{R}(t, \phi) - \tilde{R}(t, \psi)], \phi - \psi \rangle.$$  

(17)

Firstly,

$$\langle -A(\phi - \psi), \phi - \psi \rangle = -\|\phi - \psi\|_{V'}^2.$$  

(18)
by Condition (C1)(1). Secondly
\[-[B(\phi) - B(\psi)], \phi - \psi \leq \frac{\eta}{2} \|\phi - \psi\|^2 + c_\eta |\phi - \psi|^2 \|\phi\|_{H^4}^2 \] (19)
by that \([B(\phi, \phi - \psi), \phi - \psi] = 0\) due to Condition (C1)(2), (C1)(3c), (C1)(3b), and Young’s inequality. Finally,
\[-[\bar{R}(t, \phi) - \bar{R}(t, \psi)], \phi - \psi \leq R_1 \|\phi - \psi\|_V \|\phi - \psi\| \leq \frac{\eta}{2} \|\phi - \psi\|^2 + c_\eta R_1^2 \|\phi - \psi\|^2 \] (20)
by Hölder’s inequality, Condition (C3)(2) and Young’s inequality. Thus, in sum of (18), (19), (20) applied to (17), we obtain (16).

Now we let \(\{\phi_j\}\) be an orthonormal basis of \(H\) such that by denseness we assume \(\phi_j \in D(A)\) for any \(j \geq 1\), define \(H_n \triangleq \text{span}\{\phi_1, \ldots, \phi_n\} \subset D(A)\) and denote \(P_n : H \to H_n\) as the orthogonal projection from \(H\) onto \(H_n\), and \(\sigma_n \triangleq P_n \sigma, \tilde{\sigma}_n \triangleq P_n \tilde{\sigma}\). Thus, \(|\sigma_n(y)|_{L^2}^2 \leq |\sigma(y)|_{L^2}^2\) by (5) and because \(P_n\) is a contraction on \(H\). For \(h \in A_M\) and \(v \in H_n\), we consider the following approximating system of (10):
\[
\begin{aligned}
d(y_{n, h}(t), v) &= [\langle F(y_{n, h}(t)), v \rangle + \langle \tilde{\sigma}(y_{n, h}(t))h(t), v \rangle]dt \\
&\quad + \langle \sigma(y_{n, h}(t))dW_n(t), v \rangle, \quad y_{n, h}(0) = P_n \zeta, \\
\end{aligned}
\] (21)
where \(W_n(t) = \sum_{i=1}^n q_i^2 \beta_i(t) e_i\) (cf. (6)). It follows that for \(v \in H_n\), the map \(y \mapsto \langle Ay + \tilde{R}y, v \rangle \in H_n\) is globally Lipschitz uniformly in \(t\) because
\[
\langle Ay_1(t) + \tilde{R}(y_1(t)), v \rangle - \langle Ay_2(t) + \tilde{R}(y_2(t)), v \rangle \\
\leq \|y_1(t) - y_2(t)\|_V \|v\|_V + |\tilde{R}(y_1(t)) - \tilde{R}(y_2(t))| \leq c(1 + R_1) \|y_1(t) - y_2(t)\|_V
\]
by Hölder’s inequality, the Condition (C3)(2) and because \(v \in H_n \subset D(A)\). By analogous computations using Conditions (C1)(2), (C1)(3c) and (C1)(3a), the map \(y \mapsto \langle B(y), v \rangle, v \in H_n\) may be shown to be locally Lipschitz. Moreover, similarly to (19), we can also compute for any \(\eta > 0\),
\[
\langle B(y_1), y_2 \rangle \leq \eta \|y_1\|_V^2 + c_\eta \|y_1\|^2 \|y_2\|_H^4
\] (22)
by Conditions (C1)(2), (C1)(3c), (C1)(3b) and Young’s inequalities. Furthermore, there exists a constant \(c = c(n)\) such that \(\|v\|_V \leq c(n) \|v\|\) for all \(v \in H_n\).

Now since \(v \in H_n = \text{span}\{\phi_1, \ldots, \phi_n\}\), we may substitute \(\phi_j, j \in \{1, \ldots, n\}\) for \(v\) in (21). By hypothesis of Theorem 2.2, the Condition (C2) holds, and either \(\sigma = \tilde{\sigma}\) and \(\tilde{R}\) satisfies Condition (C3)(2) or (C3) holds. It follows that \(y \in H_n \mapsto (\sigma_n(y)h(t), \phi_j)_{1 \leq j \leq n}\) is globally Lipschitz from \(H_n\) to \(n \times n\) matrices and \(y \in H_n \mapsto (\tilde{\sigma}_n(y)h(t), \phi_j)_{1 \leq j \leq n}\) is globally Lipschitz from \(H_n\) to \(\mathbb{R}^n\) uniformly in \(t\); this is because for all \(y_i \in H_n, i = 1, 2\), if the Condition (C2)(2) holds, then
\[
|\langle \sigma_n(y_1), h(t), \phi_j \rangle - \langle \sigma_n(y_2), h(t), \phi_j \rangle| \leq \|\sigma_n(y_1) - \sigma_n(y_2)\|_{L^2} \|h\|_0 \lesssim |y_1 - y_2|_{H_n}. \]
(23)

If \(\sigma = \tilde{\sigma}\), then \(\sigma_n = P_n \sigma = P_n \tilde{\sigma} = \tilde{\sigma}_n\) so that the computation in (23) shows that \(y \in H_n \mapsto (\tilde{\sigma}_n(y)h(t), \phi_j)_{1 \leq j \leq n}\) is also globally Lipschitz from \(H_n\) to \(\mathbb{R}^n\). On the other hand, if \(\sigma \neq \tilde{\sigma}\), then using the hypothesis that the Condition (C3) holds and in particular (12), it follows that \(y \in H_n \mapsto (\tilde{\sigma}_n(y)h(t), \phi_j)_{1 \leq j \leq n}\) is also globally Lipschitz from \(H_n\) to \(\mathbb{R}^n\). Thus, by existence and uniqueness theorem for stochastic ordinary differential equations (e.g. [24]), there exists a unique solution \(y_{n, h} = \sum_{j=1}^n(y_{n, h}, \phi_j)\phi_j\) and a stopping time \(\tau_{n, h} \leq T\) such that (21) holds for all \(t < \tau_{n, h}\). To prove the next proposition that deduces that \(T_{n, h} = T\), we rely on the following Gronwall's inequality type result from [8]:
Lemma 3.1. (cf. Lemma A.1 [8]) Let $X(t), Y(t), I(t)$ and $\phi(t)$ be non-negative processes and $Z(t)$ be a non-negative integrable random variable. Suppose that $I(t)$ is non-decreasing in $t$ and there exist non-negative constants $C, \alpha, \beta, \gamma, \delta$ such that
\[
\int_0^T \phi(s)ds \leq C \quad \mathbb{P}\text{-a.s.,} \quad 2\beta e^C \leq 1, \quad 2\delta e^C \leq \alpha, \tag{24}
\]
\[
X(t) + \alpha Y(t) \leq Z(t) + \int_0^t \phi(r)X(r)dr + I(t), \quad \mathbb{P}\text{-a.s., for } 0 \leq t \leq T,
\]
\[
\mathbb{E}[I(t)] \leq \beta \mathbb{E}[X(t)] + \gamma \int_0^t \mathbb{E}[X(r)]dr + \delta \mathbb{E}[Y(t)] + \tilde{C}, \quad \text{for } 0 \leq t \leq T
\]
where $\tilde{C} > 0$ is a constant. If $X \in L^\infty([0, T] \times \Omega)$, then
\[
\mathbb{E}[X(t) + \alpha Y(t)] \leq 2e^{C + 2\gamma e^C} (\mathbb{E}[Z(t)] + \tilde{C}), \quad t \in [0, T].
\]

Proposition 1. Let $M, T > 0, h \in \mathcal{A}_M$. Suppose that the Conditions (C1), (C2), (C3) hold with
\[
|\check{\sigma}(t, v)|^2_{L(H_0, H)} \leq \tilde{K}_0 + \tilde{K}_1|v|^2 + \tilde{K}_2||v||_V^2 \quad \forall \ t \in [0, T], v \in V, \tag{25}
\]
instead of (11). Then for any $p \geq 1$, there exists a constant $K_2 = K_2(p, T, M)$ such that $0 \leq K_2 \leq \tilde{K}_2$ and for $\zeta \in L^{2p}(\Omega; H)$, there exist a unique solution $y_{n, h} \in C([0, T]; H_0)$ to (21) on $[0, T]$ with a modification such that
\[
\sup_n \sup_{t \in [0, T]} |y_{n, h}(t)|^{2p} + \int_0^T ||y_{n, h}(s)||_V^2 |y_{n, h}(s)|^{2(p-1)} ds \leq c(\mathbb{E}[|\zeta|^{2p}] + 1). \tag{26}
\]

Proof. We let $y_{n, h}(t)$ be the approximate solution to (21) and set the stopping time for any $N \in \mathbb{R}^+$, $\tau_N \triangleq \inf\{t \in [0, T] : |y_{n, h}(t)| \geq N\} \wedge T$. Now we know $y_{n, h}$ exists at least locally on $[0, t]$, $t < \tau_{n, h}$ where $\tau_{n, h}$ is a stopping time such that $\tau_{n, h} \leq T$ and $\lim_n \tau_{n, h} [y_{n, h}(t)] = \infty$. We also set $\pi_n : H_0 \rightarrow H_0$ the projection operator such that $\pi_n y \triangleq \sum_{i=1}^n (y, e_i)e_i$ for $\{e_i\}_i$, an orthonormal basis of $H$. Applying Itô’s formula on (21) with $f(t, x) = x^p$ and again with $f(t, x) = x^{p-1}$ gives
\[
d|y_{n, h}|^{2p} + 2p|y_{n, h}|^{2(p-1)}|y_{n, h}|_V^2 dt = -2p\langle \tilde{R}(y_{n, h}(t)), y_{n, h} \rangle |y_{n, h}|^{2(p-1)} dt + 2p\langle \sigma_n(y_{n, h}(t), y_{n, h}(t))|y_{n, h}|^{2(p-1)} dt
\]
\[
+ 2p\langle \check{\sigma}(y_{n, h}(t), y_{n, h}(t))h, y_{n, h} \rangle |y_{n, h}|^{2(p-1)} dt + p|\sigma_n(y_{n, h}(t))\pi_n Y_{n, h}|_L \|y_{n, h}|^{2(p-1)} dt
\]
\[
+ 2p(p-1)|\pi_n \sigma_n(y_{n, h}(t))y_{n, h}(t)|_V^2 |y_{n, h}|^{2(p-2)} dt. \tag{27}
\]

Now we estimate
\[
-2p\langle \tilde{R}(y_{n, h}(t)), y_{n, h} \rangle |y_{n, h}|^{2(p-1)} dt \leq 2p|\tilde{R}_0 + R_1|y_{n, h}|_V|y_{n, h}|^{2(p-1)} dt
\]
\[
\leq p|y_{n, h}|^{2(p-1)}\|y_{n, h}\|_V^2 dt + p(2R_0|y_{n, h}|^{2p-1} + R_1^2|y_{n, h}|^{2p}) dt
\]
by Hölder’s inequality, Condition (C3)(2) and Young’s inequality. We apply this estimate to (27) and subtract $p|y_{n, h}|^{2(p-1)}\|y_{n, h}\|_V^2 dt$ from both sides, integrate over $[0, t \wedge \tau_N]$ to obtain
\[
|y_{n, h}(t \wedge \tau_N)|^{2p} + p \int_0^{t \wedge \tau_N} |y_{n, h}(r)|^{2(p-1)}|y_{n, h}(r)|_V^2 dr
\]
\[ \leq |P_n \zeta|^{2p} + p \int_0^{t \wedge T_N} (2R_0 + R_1^2 |y_{n,h}|)|y_{n,h}|^{2p-1} \, dr \\
+ 2p \int_0^{t \wedge T_N} (\sigma_n(y_{n,h}(r))) dW_n(r), y_{n,h}(r))|y_{n,h}|^{2(p-1)} \, dr \\
+ 2p \int_0^{t \wedge T_N} (\bar{\sigma}_n(y_{n,h}(r))h(r), y_{n,h}(r))|y_{n,h}|^{2(p-1)} \, dr \\
+ p \int_0^{t \wedge T_N} |\sigma_n(y_{n,h}(r))\pi_n|^2_T |y_{n,h}(r)|^{2(p-1)} \, dr \\
+ 2p(p-1) \int_0^{t \wedge T_N} |\pi_n \sigma_n^*(y_{n,h}(r))y_{n,h}(r)|^2 |y_{n,h}|^{2(p-2)} \, dr \\
\triangleq |P_n \zeta|^{2p} + \sum_{j=1}^{5} T_j(t) \]

where we used that by (21), \( y_{n,h}(0) = P_n \zeta \). Now the following estimates may be proven (see pg. 408 [8]):

\[ T_3(t) \leq \frac{p}{8} \int_0^{t \wedge T_N} \|y_{n,h}(r)\|^2 \|y_{n,h}(r)\|^{2(p-1)} \, dr + c_p \sqrt{K_0} \int_0^{t \wedge T_N} |h(r)|_0 \, dr \]
\[ + c_p \int_0^{t \wedge T_N} ((\sqrt{K_0} + \sqrt{K_1})|h(r)|_0 + \tilde{K}_2|h(r)|_0^2)|y_{n,h}(r)|^{2p} \, dr, \]  

\[ T_4(t) + T_5(t) \leq p(2p-1)K_2 \int_0^{t \wedge T_N} \|y_{n,h}\|^2 \|y_{n,h}\|^{2(p-1)} \, dr \\
+ c_p \int_0^{t \wedge T_N} K_0 + (K_0 + K_1)|y_{n,h}|^{2p} \, dr, \]  

(30)

by Conditions (C3)(1), (C2)(1) that \( \pi_n \) is a contraction on \( |\cdot| \) and

\[ |\sigma(u)|_{L(H_0, H)} = |\sigma^*(u)|_{L(H, H_0)} \leq |\sigma(u)|_{L_Q}. \]  

(31)

Therefore, if \( K_2 \leq \frac{1}{2(2p-1)} \), then taking this bound on \( T_4 + T_5 \) in (30) and that on \( T_3 \) in (29) into (28), and subtracting \( \frac{3p}{8} \int_0^{t \wedge T_N} \|y_{n,h}\|^{2p} \|y_{n,h}\|^{2(p-1)} \, dr \) from both sides, it follows that with

\[ \phi(r) \triangleq c_p(2R_0 + R_1^2 + (\sqrt{K_0} + \sqrt{K_1})|h(r)|_0 + \tilde{K}_2|h(r)|_0^2 + K_0 + K_1) \]  

(32)

and \( I(t) \triangleq \sup_{r \in [0,t]} T_2(r) \), we may deduce

\[ \sup_{r \in [0,t \wedge T_N]} \|y_{n,h}(r)\|^{2p} + \frac{3p}{8} \int_0^{t \wedge T_N} \|y_{n,h}(r)\|^{2(p-1)} \|y_{n,h}(r)\|^{2} \, dr \]
\[ \leq Z(t) + \int_0^{t \wedge T_N} \phi(r) \sup_{s \in [0,r \wedge T_N]} \|y_{n,h}(s)\|^{2p} \, dr + I(t) \]  

(33)

where we furthermore denoted

\[ Z(t) \triangleq |P_n \zeta|^{2p} + c_p|2R_0T + K_0T + \sqrt{K_0} \int_0^{t \wedge T_N} |h(r)|_0 \, dr|. \]  

(34)

Now for \( C \triangleq \int_0^T \phi(s) \, ds \) from (32), we can fix \( \beta \) such that \( 2\beta e^C \leq 1 \) so that it follows that (see (A.10), pg. 409 [8] for detail)
\[
\mathbb{E}[I(t)] \leq E\left[ \sup_{r \in [0,t \wedge \tau_N]} |y_{n,h}(r)|^{2p} + K_2 \frac{\epsilon^2}{4\alpha} \int_0^{t \wedge \tau_N} \|y_{n,h}(r)\|^2 \, dr \right]
\]
\[
+ \frac{c_2^2}{4\alpha} \left[ K_0 \left( \frac{p-1}{p} \right) + K_1 \right] E\left[ \int_0^{t \wedge \tau_N} \sup_{s \in [0,r \wedge \tau_N]} |y_{n,h}(s)|^{2p} \, dr \right] + \frac{c_2^2 K_0}{43p} T
\]
by (28), Burkholder-Davis-Gundy inequality (e.g. pg. 166 [23]), that \( \pi_n \) is a contraction on \(|-|\) norm, Condition (C2)(1) and Young’s inequalities. Thus, with \( \alpha \geq 2\delta e^C \) for \( \delta \) sufficiently small and \( K_2 \leq \frac{\epsilon^2}{4\alpha} \left( \frac{8\alpha}{3p} \right) \leq \delta \) by taking \( K_2 \in [0, K_2] \) sufficiently small and finally \( \gamma \geq \frac{\epsilon^2}{4\alpha} \left( \frac{p-1}{p} \right) + K_1 \), from (33) with \( X(t) \triangleq \sup_{r \in [0,t \wedge \tau_N]} |y_{n,h}(r)|^{2p} \), \( \alpha Y(t) \triangleq \frac{3p}{4} \int_0^{t \wedge \tau_N} \|y_{n,h}(r)\|^2 \, dr \), \( \beta \triangleq \frac{c_2^2 K_0}{43p} T \), we have proven the hypothesis of Lemma 3.1 so that
\[
\sup_n \mathbb{E}\left[ \sup_{r \in [0,t \wedge \tau_N]} |y_{n,h}(r)|^{2p} + \int_0^{t \wedge \tau_N} \|y_{n,h}(r)\|^2 \, dr \right] \leq c(p) \tag{35}
\]
with \( c(p) \) independent of \( N \). We obtain \( \tau_N = \inf\{ t : |y_{n,h}(t)| \geq N \} \wedge T \nrightarrow \tau_n \) by taking \( N \rightarrow \infty \). Now \( |y_{n,h}(t)| \nrightarrow \) as \( t \uparrow \tau_n \) so that on \( \{ \tau_n, \omega < T \} \), \( \sup_{0 \leq s \leq \tau_N} |y_{n,h}(s)| \rightarrow \infty \). However, we showed \( \sup_n \mathbb{E}\left[ \sup_{r \in [0,t \wedge \tau_N]} |y_{n,h}(r)|^{2p} \right] \leq c(p) \). Thus, \( \mathbb{P}\{ \tau_n < T \} = 0 \) for a.a. \( \omega \in \Omega \), and thus \( \tau_N(\omega) = T \) for \( N(\omega) \) large enough.\( \square \)

**Proof of Theorem 2.2.** In the hypothesis of Theorem 2.2 (2), let us first assume (b). We let \( \Omega_T \triangleq [0, T] \times \Omega \) be endowed with the product measure \( ds \otimes d\mathbb{P} \) on \( \mathcal{B}([0, T]) \otimes \mathcal{F} \). We take \( \overline{K}_{2,1} \) for \( p = 1 \) in Proposition 1 so that for \( K_2 \in [0, \overline{K}_{2,1}] \), we have
\[
\sup_n \mathbb{E}\left[ \int_0^T \|y_{n,h}\|^2 \, ds \right] \leq c\mathbb{E}[\|\zeta\|^2 + 1]. \tag{36}
\]
Now we take \( \overline{K}_{2,2} \) for \( p = 2 \) in Proposition 1 so that for \( K_2 \in [0, \min\{ \overline{K}_{1,2}, \overline{K}_{2,2} \}] \),
\[
\sup_n \mathbb{E}\left[ \int_0^T \|y_{n,h}(s)\|_H^2 \, ds \right] \lesssim \sup_n \mathbb{E}\left[ \int_0^T \|y_{n,h}(s)\|^2 \|y_{n,h}(s)\|_V^2 \, ds \right] \lesssim \mathbb{E}[\|\zeta\|^4 + 1] \tag{37}
\]
by Condition (C1)(3b) and (26).

**Step 1.** We claim that weak compactness and Banach-Alaoglu theorem imply that there exists a subsequence of \( \{ y_{n,h} \}_{n \geq 0} \), which we relabel, and
\[
y_h \in X \triangleq L^4(\Omega; L^\infty([0,T]; H)) \cap L^2(\Omega_T; V) \cap L^4(\Omega_T; \mathcal{H}),
\]
\[
F_h \in L^2(\Omega_T; V'), \quad S_h, S_h \in L^2(\Omega_T; L_Q), \quad \tilde{y}_h(T) \in L^2(\Omega; H), \tag{38}
\]
such that as \( n \rightarrow \infty \), the following convergence results hold:
\[
y_{n,h} \rightarrow y_h \text{ weakly in } L^2(\Omega_T; V), \tag{39a}
y_{n,h} \rightarrow y_h \text{ weakly in } L^4(\Omega_T; \mathcal{H}), \tag{39b}
y_{n,h} \rightarrow y_h \text{ weak* in } L^4(\Omega, L^\infty([0,T]; H)), \tag{39c}
y_{n,h}(T) \rightarrow \tilde{y}_h(T) \text{ weakly in } L^2(\Omega; H), \tag{39d}
F(y_{n,h}) \rightarrow F_h \text{ weakly in } L^2(\Omega_T; V'), \tag{39e}
\]
\[ \sigma_n(y_{n,h}) \pi_n \to S_h \text{ weakly in } L^2(\Omega_T;L_Q), \]  
\[ \bar{\sigma}_n(y_{n,h}) h \to \bar{S}_h \text{ weakly in } L^\frac{1}{2}(\Omega_T;H). \] 

The convergence in (39a), (39b), (39c), (39d) follow from (36), (37), (21) and (26) with \( p = 2 \). For (39e), due to definition of \( F \), Conditions (C1)(2), (C1)(3c), (C1)(3b), (37) and (36), we see that 

\[
\|F(y_{n,h})\|_{L^2(\Omega_T')}^2 \lesssim \mathbb{E}[\int_0^T \|y_{n,h}\|_{H^t}^2 dt] + \mathbb{E}[\int_0^T |\bar{R}(t,0) + \bar{R}(t,y_{n,h}) - \bar{R}(t,0)|^2 dt] 
\lesssim 1 + \mathbb{E}[\int_0^T R_0^2 + R_{1h}^2 \|y_{n,h}\|_{V_t}^2 dt] \lesssim 1.
\]

Thus, by weak compactness we obtain (39c). For (40a), (40b), one may show 

\[
\sup_n \mathbb{E}[\int_0^T |\sigma_n(y_{n,h}(t))\pi_n|_{L_Q}^2 dt] \lesssim 1, \quad \mathbb{E}[\int_0^T |\bar{\sigma}_n(y_{n,h}(s))h(s)|_{V_t}^\frac{1}{2} ds] \lesssim 1
\]

using that \( \pi_n \) is a contraction on \(|-|\)-norm, Conditions (C2)(1), (C3)(1) and Proposition 1 (see pg. 411 [8]). Therefore, by weak compactness (40a), (40b) follow.

**Step 2.** For a fixed \( \delta > 0 \), we let \( f \in H^1((-\delta,T+\delta)) \) such that \( \|f\|_{L^\infty} = 1, f(0) = 1 \) and \( g_j(t) \triangleq \phi_j f(t) \). From (21) with \( v \) replaced by \( \phi_j \),

\[ d(y_{n,h}(t),\phi_j) = [(F(y_{n,h}(t)),\phi_j) + (\bar{\sigma}(y_{n,h}(t))h(t),\phi_j)]dt + (\sigma(y_{n,h}(t))dW_n,\phi_j). \]

We let \( G(t,x) = xf(t) \) so that by Ito’s formula, we obtain 

\[
(y_{n,h}(T),g_j(T)) = (y_{n,h}(0),g_j(0)) + \sum_{i=1}^4 I^4_{n,j}
\]

where 

\[
I^1_{n,j} \triangleq \int_0^T (y_{n,h}(s),\phi_j)f'(s)ds, \quad I^2_{n,j} \triangleq \int_0^T (\sigma_n(y_{n,h}(s))dW_n(s),g_j(s)),
\]

\[
I^3_{n,j} \triangleq \int_0^T \langle F(y_{n,h}(s)),g_j(s) \rangle ds, \quad I^4_{n,j} \triangleq \int_0^T (\bar{\sigma}(y_{n,h}(s))h(s),g_j(s))ds.
\]

Using (39d) for \( (y_{n,h}(T),g_j(T)), (21) \) for \( y_{n,h}(0), (39a) \) for \( I^1_{n,j}, (40a) \) for \( I^2_{n,j}, (39c) \) for \( I^3_{n,j} \) and (40b) for \( I^4_{n,j} \), it may be shown from (41) that (see pg. 412 [8])

\[
(\bar{y}_h(T),\phi_j)f(T) = (\zeta,\phi_j) + \int_0^T (y_{h}(s),\phi_j)f'(s)ds
\]

\[
+ \int_0^T (S_h(s)dW(s),\phi_jf) + \int_0^T (F_h(s),\phi_jf)ds + \int_0^T (\bar{S}_h(s),\phi_jf)ds
\]

as \( g_j(t) = \phi_j f(t) \) and \( f(0) = 1 \).

Now for the already fixed \( \delta > 0 \), we let \( k > \frac{1}{3}, t \in [0,T], f_k \in H^1((-\delta,T+\delta)) \) be such that \( \|f_k\|_{L^\infty} = 1 \), and

\[
f_k(s) \triangleq \begin{cases} 
1 & \text{for } s \in (-\delta,t-\frac{1}{k}), \\
0 & \text{for } s \in (t,T+\delta)
\end{cases}
\]
so that \(f_k \to 1_{(-\delta,\delta)}\) in \(L^2\), \(f_k \to -\delta\) in distribution as \(k \to \infty\) where \(\delta\) is the Delta function at \(t\). Rewriting (42) with \(f\) replaced by \(f_k\) so that as \(g_j = \phi_j f_k\), using that \(f_k(0) = 1\) and convergence of \(f_k\) as \(k \to \infty\), we obtain for a.a. \(t \in [0,T]\),

\[
(y_h(t), \phi_j) = (\zeta, \phi_j) + \int_0^t (S_h(s)dW, \phi_j) + \int_0^t (F_h(s), \phi_j) + (\tilde{S}_h(s), \phi_j)ds. \tag{43}
\]

Now \(j\) is arbitrary and \(S_h\) is the weak limit of \(\sigma_n(y_{n,h})\pi_n\) in \(L^2(\Omega_T; L_Q)\) and \(\tilde{S}_h \in L^2(\Omega_T; L_Q)\) by (40a). Hence, for any \(t \in [0,T]\), it follows that

\[
y_h(t) = \zeta + \int_0^t S_h(s)dW(s) + \int_0^t F_h(s)ds + \int_0^t \tilde{S}_h(s)ds. \tag{44}
\]

Letting \(f = 1_{(-\delta,T+\delta)}\) so that \(f'(s) = 0\) for all \(s \in [0,t]\) in (42) allows us to deduce

\[
\tilde{y}_h(T) = \zeta + \int_0^T S_h(s)dW(s) + \int_0^T F_h(s)ds + \int_0^T \tilde{S}_h(s)ds. \tag{45}
\]

Comparing (44) and (45) shows that \(\tilde{y}_h(T) = y_h(T)\) \(P\)-almost surely.

**Step 3.** The purpose of this Step is to show that \(ds \otimes dP\) a.e. on \(\Omega_T\), \(S_h(s) = \sigma(y_h(s)), F_h(s) = F(y_h(s))\), \(\tilde{S}_h(s) = \tilde{\sigma}(y_h(s))h(s)\). We let \(v \in \mathcal{X}\) where \(\mathcal{X}\) is defined in (38). By hypothesis of Theorem 2.2, the constant \(L_2\) of the Condition (C2) satisfies \(L_2 < 2\) so that we can find \(\epsilon \in (0, 2 - L_2)\) and then choose

\[
\eta \in \left(0, \frac{2 - (L_2 + \epsilon)}{2}\right). \tag{46}
\]

Then for any \(t \in [0,T]\), we set

\[
r(t) \triangleq \int_0^t 2c_n(R_1^2 + \|v(t)\|^2_4) + L_1 + 2^3 \sqrt{L_1|h|_0} + \frac{\epsilon^2}{2} L_2 |h|_0^2 ds \tag{47}
\]

where \(c_n\) is the constant from (16); we emphasize that our choice of \(r(t)\) here differs from that of equation (A.17) on pg. 413 [8]. It is clear that \(0 \leq r(t) < \infty\) \(P\)-a.s. because \(v \in \mathcal{X}\) implies by (38) that \(v \in L^4(\Omega_T; \mathcal{H})\) and by hypothesis of Theorem 2.2, \(h \in \mathcal{A}_M\) so that by (8), (9), \(P\)-a.s., \(h \in L^2([0,T]; H_0)\). Hence \(r \in L^1(\Omega_T; L^\infty(0,T)), e^{-r} \in L^\infty(\Omega_T)\) and

\[
r'(t) = 2c_n R_1^2 + 2c_n \|v(t)\|^2_4 + L_1 + 2^3 \sqrt{L_1|h(t)|_0} + \frac{2L_2}{\epsilon} |h(t)|_0^2 \in L^1(\Omega_T). \tag{48}
\]

Therefore,

\[
\|r'e^{-r}\|_{L^1(\Omega_T)} \leq \|r'\|_{L^1(\Omega_T)} \|e^{-r}\|_{L^\infty(\Omega_T)} \lesssim 1. \tag{49}
\]

Now using the fact that \(P_n \zeta \to \zeta\) in \(H\) as \(n \to \infty\),

\[
\lim_{n \to \infty} \mathbb{E} \|y_h(T) e^{-r(T)} - |y_{n,h}(T)|^2 e^{-r(T)}\|^2 = 0
\]

and \(\|(y_h(T) + y_{n,h}(T))e^{-r(T)}\|^2_{L^2(\Omega; H)} < \infty\), it follows that

\[
\mathbb{E} \|y_h(T) e^{-r(T)}\|^2 - \mathbb{E} \|\zeta\|^2 \leq \liminf_{n \to \infty} \mathbb{E} \|y_{n,h}(T)|^2 e^{-r(T)}\| - \mathbb{E} \|P_n \zeta\|^2
\]

by Fatou’s lemma. With \(f(t, x) = xe^{-r(t)}\), we obtain by Ito’s formula, in general for any \(v \in \mathcal{X}\),

\[
\mathbb{E} \|v(T)^2 e^{-r(T)}\|^2 - \mathbb{E} \|v(0)\|^2 = \mathbb{E} \left[ \int_0^T e^{-r(s)} |v|^2 ds \right] - \mathbb{E} \left[ \int_0^T r'(s)e^{-r(s)} |v|^2 ds \right]. \tag{50}
\]
Thus, by (50) with \( v = y_h \), using Ito’s formula on (44) with \( f(t, x) = x^2 \) gives

\[
\mathbb{E}[|y_h(T)|^2 e^{-r(T)}] - \mathbb{E}[|y_h(0)|^2] = \mathbb{E}[\int_0^T e^{-r(s)} 2\langle y_h, S_h(s) dW(s) \rangle] + \mathbb{E}[\int_0^T e^{-r(s)} 2\langle y_h, F_h(s) + \tilde{S}_h(s) \rangle + |S_h|^2_{L_2} ds] - \mathbb{E}[\int_0^T r'(s) e^{-r(s)}|y_h(s)|^2 ds]
\]

while by (50) with \( v = y_{n,h} \), using Ito’s formula on (21) with \( f(t, x) = x^2 \) gives

\[
\mathbb{E}[|y_{n,h}(T)|^2 e^{-r(T)}] - \mathbb{E}[|y_{n,h}(0)|^2] = \mathbb{E}[\int_0^T e^{-r(s)} 2\langle y_{n,h}, F(y_{n,h}(s)) + \tilde{\sigma}(y_{n,h}(s)) h(s) \rangle ds] + \mathbb{E}[\int_0^T e^{-r(s)} \sigma_n(y_{n,h}(s)) \pi_n |\tilde{\sigma}(y_{n,h}(s)) h(s), y_{n,h} \rangle ds] - \mathbb{E}[\int_0^T r'(s) e^{-r(s)}|y_{n,h}(s)|^2 ds].
\]

Now using \(|y_n - v|^2 + 2(y_n - v, v) = |y_n|^2 - |v|^2\) and defining,

\[
X_n \triangleq \mathbb{E}[\int_0^T e^{-r(s)}|\tilde{r}'(v)(|y_{n,h}(s) - v(s)|^2 + 2(y_{n,h}(s) - v(s), v(s)) + 2\langle F(y_{n,h}(s)), y_{n,h} \rangle + |\sigma_n(y_{n,h}(s)) \pi_n |_{L_Q}^2 + 2(\tilde{\sigma}(y_{n,h}(s)) h(s), y_{n,h}) ds, (51)
\]

it follows that due to Fatou’s lemma,

\[
\mathbb{E}[\int_0^T e^{-r(s)}|\tilde{r}'(v)(|y_{n,h}(s) - v(s)|^2 + 2(y_{n,h}(s) - v(s), v(s))) + 2\langle F_h(s), y_h(s) \rangle + |S_h(s)|^2_{L_Q} + 2(\tilde{S}_h(s), y_h(s)) ds \leq \liminf_{n \to +\infty} X_n. (52)
\]

Thus, for \( \epsilon > 0 \) chosen in (46),

\[
V_n \triangleq \mathbb{E}[\int_0^T e^{-r(s)}|\tilde{r}'(v)(|y_{n,h} - v|^2 + 2(F(y_{n,h}) - F(v), y_{n,h} - v) + |\sigma_n(y_{n,h}) \pi_n - \sigma_n(v) \pi_n |^2_{L_Q} + 2(\{\tilde{\sigma}(y_{n,h}) - \tilde{\sigma}_n(v)\} h(s), y_{n,h} - v) ds \leq 0. (53)
\]

by (16), the Conditions (C2)(2), (C3)(1), (48) and (46). Hence, \( X_n - V_n = Z_n^1 + Z_n^2 \) where

\[
Z_n^1 \triangleq \mathbb{E}[\int_0^T e^{-r(s)}|2r'(s)(y_{n,h}(s) - v(s), v(s)) + 2\langle F(y_{n,h}(s)), v(s) \rangle + 2\langle F(v(s), y_{n,h}(s)) - 2(F(v(s)), v(s)) + 2\sigma_n(y_{n,h}(s)) \pi_n, \sigma(v(s)) |_{L_Q} + 2(\tilde{\sigma}_n(y_{n,h}(h(s), v(s)) + 2(\tilde{\sigma}(v) h(s), y_{n,h}(s)) - 2(P_n \tilde{\sigma}(v(s)) h(s), v(s))) ds \]
\]

and

\[
Z_n^2 \triangleq \mathbb{E}[\int_0^T e^{-r(s)}|2(\sigma_n(y_{n,h}(s)) \pi_n, [\sigma(v(s)) \pi_n - \sigma(v(s))] |_{L_Q} - P_n \sigma(v(s)) \pi_n |^2_{L_Q} | ds.
\]
By (39a)-(40b), we see that $Z_n^1 \to Z^1$ as $n \to \infty$ where

\begin{equation}
Z^1 = \mathbb{E}\left[\int_0^T e^{-r(s)}\left[-2r'(s)(y_h(s) - v(s), v(s)) + 2\langle F_h(s), v(s) \rangle + 2\langle F(v(s)), v(s) \rangle + 2\langle S_h(s), \sigma(v(s)) \rangle_{L_Q} + 2\langle \tilde{S}_h(s), v(s) \rangle + 2\langle \tilde{\sigma}(v(s))h(s), y_h(s) \rangle - 2\langle \tilde{\sigma}(v(s))h(s), v(s) \rangle \right] ds\right].
\end{equation}

Moreover, as $\mathbb{P}$-a.s., $\int_0^T e^{-r(s)}\sigma(v(s))(\pi_n - Id_{H_0})^2_{L_Q} ds \leq 1$ by the fact that $e^{-r(s)} \in L^\infty(\Omega_T)$ and Condition (C2)(1), the dominated convergence theorem implies that

\[
\lim_{n \to \infty} \mathbb{E}\left[\int_0^T e^{-r(s)}\sigma(v(s))(\pi_n - Id_{H_0})^2_{L_Q} ds\right] = 0.
\]

Similarly by the dominated convergence theorem,

\[
Z_n^2 \to -\mathbb{E}\left[\int_0^T e^{-r(s)}\sigma(v(s)) (\pi_n - Id_{H_0})^2_{L_Q} ds\right]
\]

as $n \to \infty$ so that $\pi_n \to Id_{H_0}$. Thus, by , we deduce

\[
\mathbb{E}\left[\int_0^T e^{-r(s)}\left(-r'(s)y_h(s) - v(s)\right)^2 + 2\langle F_h(s) - F(v(s)), y_h(s) - v(s) \rangle + |S_h(s) - \sigma(v(s))|^2_{L_Q} + 2\langle \tilde{S}_h(s) - \tilde{\sigma}(v(s))h(s), y_h(s) - v(s) \rangle \right] ds
\]

\[
= \lim_{n \to \infty} X_n - Z^1 + \mathbb{E}\left[\int_0^T e^{-r(s)}\left|\sigma(v(s))\right|^2_{L_Q} ds\right] \leq 0
\]
due to (51), (54), (39a), (39b), (39c), (39d), (39e), (40a), (40b), (53), (54), (55).

Now we let $v = y_h$ in (56) and hence see that $S_h(t) = \sigma(y_h(t)), ds \otimes d\mathbb{P}$ almost everywhere. Next, for $\lambda \in \mathbb{R}$, let $v_\lambda = y_h - \lambda \tilde{v}$ for some $\tilde{v} \in L^\infty(\Omega_T; X)$ and hence $\|v_\lambda\|_{X} \leq 1$ by (38) and because $y_h \in X$ by Proposition 1 and $V \subset H$ by Condition (C1)(3a). Furthermore, we may let $v = v_\lambda = y_h - \lambda \tilde{v}$ in (56) so that with

\[
r_\lambda(t) \triangleq \int_0^t 2c_\eta(R_1^2 + \|v_\lambda\|^2_{H}) + L_1 + 2 \sqrt{L_1} |h|_0 + \left(\frac{\varepsilon}{2}\right) L_2 |h|^2 ds
\]

instead of $r(t)$ in (47), we obtain

\[
\mathbb{E}\left[\int_0^T e^{-r_\lambda(s)}\left[-\lambda^2 r_\lambda(s)|\tilde{v}(s)|^2 + 2\lambda\langle F_h(s) - F(v_\lambda(s)), \tilde{v}(s) \rangle + \|S_h(s) - \sigma(v_\lambda(s))\|_{L_Q}^2 + 2\lambda\langle \tilde{S}_h(s) - \tilde{\sigma}(v_\lambda(s))h(s), \tilde{v}(s) \rangle \right] ds \leq 0.
\]

Now since we already showed that $S_h(s) = \sigma(y_h(s)) \ ds \otimes d\mathbb{P}$ almost everywhere, we know $|S_h(s) - \sigma(v_\lambda(s))|^2_{L_Q} \to 0$ as $\lambda \to 0$ so that $v_\lambda = y_h - \lambda \tilde{v} \to y_h$. Moreover, due to Condition (C3)(1), by the dominated convergence theorem, we see that

\[
\lim_{\lambda \to 0} \mathbb{E}\left[\int_0^T e^{-r_{\lambda}(s)} (\tilde{S}_h(s) - \tilde{\sigma}(v_\lambda(s))h(s), \tilde{v}(s)) ds\right] = \mathbb{E}\left[\int_0^T e^{-r_0(s)} (\tilde{S}_h(s) - \tilde{\sigma}(y_h(s))h(s), \tilde{v}(s)) ds\right]
\]

where $r_0(s)$ is $r_\lambda(s)$ with $\lambda = 0$ so that $v_0 = y_h$. Furthermore, using that $v_\lambda = y_h - \lambda \tilde{v}$ and (16) deduces by dominated convergence theorem,

\[
\mathbb{E}\left[\int_0^T e^{-r_\lambda(s)} \langle F_h - F(v_\lambda), \tilde{v}(s) \rangle ds\right] \to \mathbb{E}\left[\int_0^T e^{-r_0(s)} \langle F_h - F(y_h), \tilde{v}(s) \rangle ds\right]
\]
as $\lambda \to 0$. Thus, dividing (57) by $\lambda > 0$, letting $\lambda \to 0$ and applying (58) and (59) in (57), and repeating same procedure with $\lambda < 0$, we obtain

$$
\mathbb{E}\left[ \int_0^T e^{-\tau_\lambda(s)}[\langle F_h(s) - F(y_h(s)), \bar{v}(s) \rangle + (\bar{S}_h(s) - \bar{s}(y_h(s))h(s), \bar{v}(s))ds \right] = 0 \tag{60}
$$

because clearly $\mathbb{E}\left[ \int_0^T e^{-\tau_\lambda(s)}\lambda r_h'(s)|\bar{v}(s)|^2ds \right] \to 0$ as $\lambda \to 0$ by dominated convergence theorem. From (60), we see that $F_h(s) = F(y_h(s))$, $\bar{S}_h(s) = \bar{s}(y_h(s))h(s)$ so that along with the fact that $S_h(s) = \sigma(y_h(s))$, ds $\otimes d\mathbb{P}$ a.e., we obtain from (44),

$$
y_n(t) = \zeta + \int_0^t \sigma(y_n(s))dW(s) + \int_0^t [F(y_n(s)) + \bar{s}(y_h(s))h(s)]ds. \tag{61}
$$

We have thus proven the existence of the solution to (10). The bound of (13) follows from (39a), (39b), (39c), (26) at $p = 1, 2$ and applications of (C1)(3b).

**Step 4.** Here we show that $y_n \in C([0,T]; H) \mathbb{P}$-almost surely. Firstly, for all $\delta > 0$, $e^{-\delta A}t$ maps $H$ to $V, V'$ to $H$. Thus, $e^{-\delta A} \int_0^T F(y_h(s))ds \in C([0,T]; H)$ because $F(t, y) = -Ay - B(y) - \tilde{R}(t, y)$ and $\mathbb{P}$-a.s., in particular

$$
\|e^{-\delta A} \int_0^t \tilde{R}(y_h)ds\|_{C([0,T]; H)} \lesssim \int_0^T \|\tilde{R}(y_h)\|_{V'} ds
$$

by Conditions (C3)(2), (C1)(3a) and (13). Next, by Condition (C3)(1) and (13), $\int_0^T \bar{s}(y_h(s))h(s)ds \in C([0,T]; V')$ and hence $e^{-\delta A} \int_0^T \bar{s}(y_h(s))h(s)ds \in C([0,T]; H)$. Finally, using Burkholder-Davis-Gundy inequality, (5), Condition (C2)(1) and (13) shows that $\mathbb{E}\left[ \|\int_0^T e^{-\delta A}\sigma(y_h(s))dw(s)\|_{C([0,T]; H)} \right] \lesssim 1$. Thus, using (61) we have shown that $e^{-\delta A}y_n \in C([0,T]; H) \mathbb{P}$-almost surely. Hence, in order to show $y_h \in C([0,T]; H) \mathbb{P}$-a.s., it suffices to show

$$
\lim_{\delta \to 0} \mathbb{E}\left[ \sup_{t \in [0,T]} |y_h(t) - e^{-\delta A}y_h(t)|^2 \right] = 0. \tag{62}
$$

We let $G_\delta \triangleq I - e^{-\delta A}$, apply $G_\delta$ on (61), and apply Ito’s formula on the resulting equation with $f(t, x) = x^2$ to obtain

$$
|G_\delta y_h(t)|^2 = |G_\delta \zeta|^2 - 2 \int_0^t \|G_\delta y_h\|^2 ds + 2I(t) + \int_0^t |G_\delta \sigma(y_h(s))|^2_{L_Q} ds \tag{63}
$$

where $I(t) \triangleq \int_0^t \langle B(y_h(s)) + \tilde{R}(y_h(s)) + \bar{s}(y_h(s))h(s), G_\delta^2 y_h(s) \rangle ds$.

Firstly, we estimate

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} I(t) \right] \leq \frac{1}{4} \mathbb{E}\left[ \sup_{t \in [0,T]} |G_\delta y_h(t)|^2 \right] + c \mathbb{E}\left[ \int_0^T |G_\delta \sigma(y_h(s))|^2_{L_Q} ds \right] \tag{64}
$$

by Burkholder-Davis-Gundy inequality and Young’s inequality. We take supremum over $t \in [0,T]$ on the right and then left hand sides of (63), take expected value and then apply (64) to obtain

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} |G_\delta y_h(t)|^2 \right] \lesssim |G_\delta \zeta|^2 + \mathbb{E}\left[ \int_0^T |G_\delta \sigma(y_h(s))|^2_{L_Q} ds \right]
$$
where we took by Condition (C3)(2) and Young’s inequality to deduce from (66)
Now we let (C2) and (13) so that
\[ E\{ \text{any orthonormal basis } v \} \]
Clearly for all \( \delta \)
Finally, \( \sup_{\delta > 0} |G_3 y(t)|^2 \in L^1(\Omega \times [0, T]) \) by (5), Condition (C2) and (13) so that \( E[0^T |G_3 y(t)|^2 \Omega] ds \rightarrow 0 \) as \( \delta \rightarrow 0 \) by (5), and dominated convergence theorem. Similarly for all \( v \in V \), \( \|G_3^2 v\|_V \rightarrow 0 \) as \( \delta \rightarrow 0 \) and \( \sup_{\delta > 0} |G_3|_{L(V, V)} \leq 2 \). This leads to
\[
E[0^T (B(y(t)) + \tilde{R}(y(t)) + \tilde{\sigma}(y(t)) h(s), G_3^2 y(t)) ds] \rightarrow 0
\]
as \( \delta \rightarrow 0 \) by Hölder’s inequalities, Conditions (C1)(3c), (C3)(2), (C3)(1) and (13). Finally, \( |G_3| \rightarrow 0 \) as \( \delta \rightarrow 0 \). Hence, taking \( \delta \rightarrow 0 \) in (65) shows (62).

**Step 5.** We now show the uniqueness of \( y_\in X = C([0, T]; H) \cap L^2([0, T]; V) \) We let \( v \in X \) be another solution to (61) and set
\[
\tau_N \triangleq \inf\{ t \geq 0 : |y_\in(t)| \geq N \} \wedge \inf\{ t \geq 0 : |v(t)| \geq N \} \wedge T.
\]
As \( \delta \rightarrow 0 \) we obtain by Ito’s formula with \( f(t, x) = x^2 \) and subsequently with \( f(t, x) = e^{-a t} f_0 \|y\|_H^4 dr \), for \( a > 0 \) to be determined subsequently,
\[
e^{-a f_0^{\tau^\ast \tau_N} \|y\|_H^4 ds} \Psi(s)^2 = \int_0^{\tau^\ast \tau_N} \Psi(s) ds + \Phi(t \wedge \tau_N)
\]
where
\[
\Psi(s) \triangleq e^{-a f_0 \|y\|_H^4 dr} [-a \|y\|_H^4 Y^2 - 2 \|y\|_V^2 - 2 (B(y) - B(v), Y) + |\sigma(y) - \sigma(v)|^2 + 2 (\tilde{\sigma}(y) - \tilde{\sigma}(v)) h, Y] - 2 (\tilde{R}(y) - \tilde{R}(v), Y)
\]
and
\[
\Phi(t \wedge \tau_N) \triangleq 2 \int_0^{\tau^\ast \tau_N} e^{-a f_0 \|y\|_H^4 dr} (Y(s), \sigma(y) - \sigma(v)) dW.
\]
Now for any \( \eta > 0 \) we estimate in particular
\[
-2 (\tilde{R}(y) - \tilde{R}(v), Y) \leq 2 |\tilde{R}(y) - \tilde{R}(v)| ||Y| \leq 2 R_1 ||Y||_V ||V| \leq \eta ||Y||_V^2 + c(\frac{R_1}{\eta}) ||Y||^2
\]
by Condition (C3)(2) and Young’s inequality to deduce from (66)
\[
e^{-a f_0^{\tau^\ast \tau_N} \|y\|_H^4 ds} ||Y(t \wedge \tau_N)||^2 \leq \int_0^{\tau^\ast \tau_N} e^{-a f_0 \|y\|_H^4 dr} [-2 - 2 \eta \|Y\|_V^2 + c(\eta)(1 + ||h||_0^2) ||Y||^2] ds + \Phi(t \wedge \tau_N)
\]
where we took \( a = 2c_2^4 \) and in particular computed that
\[
\Psi(s) \leq e^{-a f_0 \|y(r)\|_H^4 dr} [-2 - 2 \eta \|Y(s)||_V^2 + c(\eta)(1 + ||h||_0^2) ||Y(s)||^2].
\]
Now we let
\[
X(t) \triangleq \sup_{s \in [0, t]} e^{-a f_0^{s \wedge \tau_N} \|y\|_H^4 dr} ||Y(s \wedge \tau_N)||^2,
\]
\[
Y(t) \triangleq \int_0^{t \wedge \tau_N} e^{-a f_0 \|y\|_H^4 dr} ||Y(s)||_V^2 ds.
\]
Thus,

\[ X(t) + (2 - 3\eta - L_2)Y(t) \leq c(\eta) \int_0^t (1 + |h(s)|^2)X(s)ds + I(t) \]  \hspace{1cm} (69)

where \( I(t) \triangleq \sup_{s \in [0,t]} |\Phi(s \land \tau_N)| \) and one may further estimate (see pg. 418 [8])

\[ \mathbb{E}[I(t)] \leq \beta \mathbb{E}[X(t)] + \frac{9L_1}{\beta} \int_0^t \mathbb{E}[X(s)]ds + \frac{9L_2}{\beta} \mathbb{E}[Y(t)] \]

by Burkholder-Davis-Gundy inequality, Condition (C2)(2) and Young’s inequalities.

In order to apply Lemma 3.1, we denote by hypothesis of Theorem 2.2, and 2\( \beta e^C \leq \alpha \) if \( L_2 > 0 \) is sufficiently small. Thus, under the hypothesis that \( L_2 > 0 \) is sufficiently small, we can apply Lemma 3.1 to obtain \( \mathbb{E}[\sup_{s \in [0,T]} e^{-a\int_0^s \|y_k(s)\|_H^4 dr} |Y(s \land \tau_N)|^2] = 0 \). Taking \( N \to \infty \) so that \( \tau_N \to T \), by (13) we obtain \( |Y(s, \omega)| = 0 \) \( \mathbb{P} \)-a.s. on \( \Omega_T \).

Now suppose the other hypothesis of Theorem 2.2, namely only that 0 < \( L_2 < 2 \), not necessarily small, and there exists a scalar function \( \psi(t) \in L^2([0, T]) \) such that \( |h(t)|_0 \leq \psi(t) \) \( \mathbb{P} \)-almost surely. Since \( L_2 \in (0, 2) \), similarly to (46), we may still take \( \eta \in (0, \frac{2 - L_2}{3}) \) so that \( 2 - 3\eta - L_2 > 0 \). Hence,

\[ \mathbb{E}[e^{-a\int_0^t \|y_k\|_H^4 \text{dr}} |Y(t \land \tau_N)|^2] \leq c(\eta) \int_0^t e^{-a\int_0^s \|y_k(r)^2\|_H^4 \text{dr}} |Y(s \land \tau_N)|^2 ds \]

by (66), (67), (68), and that by hypothesis of Theorem 2.2, we have \( |h(s)|_0 \leq \psi(s) \).

By Gronwall’s inequality, again, we obtain \( |Y(s, \omega)| = 0 \) \( \mathbb{P} \)-a.s. on \( \Omega_T \).

The last case of hypothesis, namely when \( L_2 \in (0, 2) \) and \( \tilde{\sigma} = \sigma \) may be attained by the standard method of defining for \( h \neq 0 \), \( \tilde{W}^h(t) = W(t) + \int_0^t h(s)ds \), \( \tilde{\mathbb{P}} \) by \( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\int_0^t h(s)dW(s) - \frac{1}{2}\int_0^t |h(s)|^2 ds} \) and relying on Girsanov theorem (e.g. [12]); we refer to [8] for detail.

4. Proof of Theorem 2.4. We let \( \sigma, \tilde{\sigma}, \tilde{R} \) satisfy the Conditions (C2) and (C3), let \( h \in \mathcal{A}_M \) and consider

\[
\begin{aligned}
\frac{dy_k^h(t)}{dt} &= \tilde{A}y_k^h(t) + B(y_k^h(t)) + \tilde{R}(t, y_k^h(t))dt \\
&= \tilde{\sigma}(t, y_k^h(t))dW(t) + \tilde{\sigma}(t, y_k^h(t))h(t)dt,
\end{aligned}
\]

\hspace{1cm} (70)

Now for all \( n \in \mathbb{N} \), we let \( \psi_n : [0, T] \mapsto [0, T] \) be a measurable mapping such that for any \( s \in [0, T] \),

\[ s \leq \psi_n(s) \leq (s + c2^{-n}) \land T \]

(71)

for some \( c > 0 \). For some \( N > 0, h \in \mathcal{A}_M, t \in [0, T] \), we define

\[ G_N(t) \triangleq \{ \omega \in \Omega : \sup_{s \in [0,t]} |y_k^h(s)|^2 + (\int_0^t |y_k^h(s)|^2 ds) \leq N \} \]

Lemma 4.1. Let \( \epsilon_0, M, N > 0, \sigma \) satisfy the Condition (C2), and \( \tilde{\sigma} \) satisfy (12) and (25) instead of (11). Suppose that the Condition (C1) holds, \( \zeta \in L^4(\Omega; H) \) and there exists a solution \( y_k \) of (70) with the properties of the solution in Theorem 2.2. Then
there exists a constant $c = c(K_i, \tilde{K}_j, L_j, \tilde{L}_j, R_1, T, M, \epsilon_0) > 0, i = 0, 1, 2, j = 1, 2$

such that for all $h \in \mathcal{A}_h, \epsilon \in [0, \epsilon_0]$, \[I_n(h, \epsilon) \triangleq \mathbb{E}[G_N(T) \int_0^T |y_h^n(\psi_n(s)) - y_h^n(s)|^2 ds] \leq c2^{-\frac{n}{2}}. \tag{72}\]

**Proof.** We fix $h \in \mathcal{A}_h, \epsilon \in [0, \epsilon_0]$. We use Ito’s formula on (70) to obtain \[|y_h^n(\psi_n(s)) - y_h^n(s)|^2 = 2 \int_s^{\psi_n(s)} (y_h^n(r) - y_h^n(s), dy_h^n(r)) + \epsilon \int_s^{\psi_n(s)} |\sigma(y_h^n)|^2_L dr. \tag{73}\]

Integrating over $[0, T]$, multiplying by $1_{G_N(T)}$, and taking expected value, we obtain \[I_n(h, \epsilon) = \sum_{i=1}^6 I_{n,i}, \tag{74}\]

de due to (70), (72) and (73). It is clear by definition that $G_N(T) \subset G_N(r)$ for all $r \in [0, T]$ and hence on $G_N(r), 0 \leq s \leq r \leq T$, \[|y_h^n(r)| + |y_h^n(s)| \leq \sqrt{N} + \sqrt{N} = 2\sqrt{N}. \tag{75}\]

Let us elaborate on the estimate of $I_{n,6}$ while brief on others, referring to [8] for details. We can estimate as follows: $|I_{n,1}| \leq c_12^{-\frac{2n}{2}}, |I_{n,2}| \leq c_22^{-n}, |I_{n,3}| \leq c_32^{-n}, |I_{n,4}| \leq cN2^{-(n+1)}, |I_{n,5}| \leq cN2^{-(n+1)} + 2c^2\epsilon N^22^{-n}$ (see pg. 395–397 [8]) and \[|I_{n,6}| \leq 2[\mathbb{E}[G_N(T) \int_0^T |\mathcal{R}(y_h^n(s))||y_h^n(s)|ds] \tag{76}\]

where we used (75), Condition (C3)(2) and Fubini’s theorem. Collecting these estimates in (74), the proof of Lemma 4.1 is complete. \[\square\]

Now for all $n \in \mathbb{Z}$, we define a step function $\psi_n(s) \triangleq \pi_n$ on $[0, T]$ where \[s \mapsto \pi_n = t_{k+1} \triangleq (k + 1)T2^{-n} \quad \forall s \in [kT2^{-n}, (k + 1)T2^{-n}) \quad \tag{77}\]
so that by (71), \( s \leq \psi_n(s) \leq (s + c2^{-n}) \wedge T \) with \( c = T \). Now for a fixed \( \epsilon_0 > 0 \), let \( \{h_{\epsilon} \}_{\epsilon \in (0, \epsilon_0)} \) be a family of random elements taking values in \( A_M \) in (9). We let \( y_{\epsilon} \) be the solution of

\[
\begin{align*}
dy_{\epsilon} &= [Ay_{\epsilon} + B(y_{\epsilon}) + \tilde{R}(t, y_{\epsilon})]dt \\
&= \sigma(t, y_{\epsilon})\tilde{h}_\epsilon(t)dt + \sqrt{\epsilon}\sigma(t, y_{\epsilon})dW(t), \quad y_{\epsilon}(0) = \zeta.
\end{align*}
\]

As discussed in Section 2, for \( W^\epsilon \triangleq W + \frac{1}{\sqrt{\epsilon}} \int_0^\infty h_\epsilon(s)ds \), we have \( y_{\epsilon} = G^\epsilon(\sqrt{\epsilon}W^\epsilon) \) for (78) whereas for (15), \( y_\epsilon = G^0(\int_0^\infty h(s)ds) \). Next, we prove the following proposition:

**Proposition 2.** Suppose that the Conditions (C1), (C2) are satisfied with \( K_2 = L_2 = 0 \) and that \( \tilde{R} \) and \( \sigma \) satisfy the Conditions (C3)(2) and (C4). Let \( \zeta \) be \( F_0 \)-measurable and satisfy \( E[|\zeta|^2] \leq \infty \), \( h_\epsilon \to h \) as \( \epsilon \to 0 \) in distribution (as random elements that are \( A_M \)-valued), endowed with the weak topology of \( L^2([0,T];H_0) \).

Then, the solution \( y_{\epsilon} \) of (78) converges to the solution \( y_\epsilon \) of (15) in distribution as \( \epsilon \to 0 \) in \( X = C([0,T];H) \cap L^2([0,T];V) \); i.e. \( G^\epsilon(\sqrt{\epsilon}W^\epsilon) \) converges in distribution to \( G^0(\int_0^\infty h(s)ds) \) in \( X \).

**Proof.** We first remark that due to the hypothesis of Proposition 2, by Theorem 2.2 we know there exists a unique solution to (78) that satisfies (13).

By hypothesis of Proposition 2, \( h_\epsilon \to h \) as \( \epsilon \to 0 \) in distribution (as \( A_M \)-valued random elements). As \( A_M \) is a Polish space, by Skorokhod’s Representation theorem (see [34], also [12]), there exists \((\tilde{h}_\epsilon, \tilde{h}, \tilde{W}^\epsilon)\) such that

1. the joint distribution of \((\tilde{h}_\epsilon, \tilde{W}^\epsilon)\) is the same as that of \((h_\epsilon, W^\epsilon)\),
2. the distribution of \( \tilde{h} \) coincides with that of \( h \),
3. \( \tilde{h}_\epsilon \to \tilde{h} \) \( P \)-a.s. in the weak topology of \( S_M \) so that \( P \)-a.s. for all \( t \in [0,T] \),

\[
\int_0^t \tilde{h}_\epsilon(s)ds - \int_0^t \tilde{h}(s)ds \to 0 \text{ weakly in } H_0.
\]

Let us write \((h_\epsilon, h, W)\) instead of \((\tilde{h}_\epsilon, \tilde{h}, \tilde{W}^\epsilon)\) for simplicity of notation. Now we let \( Y^\epsilon \triangleq y_{\epsilon} - y_\epsilon \) so that by (15) and (78), Ito’s formula with \( f(t, x) = x^2 \) gives

\[
\begin{align*}
|Y^\epsilon(t)|^2 + 2\int_0^t &\|Y^\epsilon\|^2_t ds \\
&= -2\int_0^t (Y^\epsilon, B(y_{\epsilon}) - B(y_\epsilon))ds - 2\int_0^t (Y^\epsilon, \tilde{R}(s, y_{\epsilon}) - \tilde{R}(s, y_\epsilon))ds \\
&\quad + 2\int_0^t (Y^\epsilon, \sigma(y_{\epsilon})\tilde{h}_\epsilon - \sigma(s, y_\epsilon)h)ds \\
&\quad + 2\sqrt{\epsilon}\int_0^t (Y^\epsilon, \sigma(y_{\epsilon})dW(s)) + \epsilon\int_0^t |\sigma(y_{\epsilon})|^2 ds
\end{align*}
\]

where in particular we estimate

\[
-2\int_0^t (Y^\epsilon, \tilde{R}(s, y_{\epsilon}) - \tilde{R}(s, y_\epsilon))ds \leq 2\int_0^t |Y^\epsilon||\tilde{R}(s, y_{\epsilon}) - \tilde{R}(s, y_\epsilon)|ds
\]

\[
\leq 2R_1\int_0^t |Y^\epsilon||Y^\epsilon|_V ds \leq \frac{1}{2}\int_0^t |Y^\epsilon|^2 ds + 2R_1^2\int_0^t |Y^\epsilon|^2 ds
\]

by Hölder’s inequality, Condition (C3)(2) and Young’s inequality. Besides, using Conditions (C1)(2), (C1)(3c), (C1)(3b), (C2)(2) with \( L_2 = 0 \) leads to

\[
|Y^\epsilon(t)|^2 + \int_0^t \|Y^\epsilon\|^2 ds
\]
Thus, as by (C1)(3b), H"older's inequality, (84), that
where
\[ G_{N}(t) \triangleq \{ \omega : \sup_{s \in [0,t]} |y_{h}(s)|^{2} \leq N \} \cap \{ \omega : \int_{0}^{t} \| y_{h}(s) \|_{V}^{2} ds \leq N \}, \]
\[ G_{N,\epsilon}(t) \triangleq G_{N}(t) \cap \{ \omega : \sup_{s \in [0,t]} |y_{h}(s)|^{2} \leq N \} \cap \{ \omega : \int_{0}^{t} \| y_{h} \|_{V}^{2} ds \leq N \}. \]

**Step 1.** We compute
\[
\mathbb{P}(G_{N,\epsilon}(T)^{c}) \leq \frac{1}{N} \sup_{h,h_{\epsilon} \in \mathcal{A}_{M}} \mathbb{E} \left[ \sup_{s \in [0,t]} |y_{h}(s)|^{2} + \sup_{s \in [0,t]} |y_{h_{\epsilon}}(s)|^{2} + \int_{0}^{T} \| y_{h} \|_{V}^{2} + \| y_{h_{\epsilon}} \|_{V}^{2} ds \right] \to 0 \] (85)
as \( N \to \infty \) by (83), (84), Chebyshev inequality and (13). Therefore, for all \( \epsilon_{0} \in (0,1] \),
\[
\lim_{N \to \infty} \sup_{\epsilon \in (0,\epsilon_{0})} \sup_{h,h_{\epsilon} \in \mathcal{A}_{M}} \mathbb{P}(G_{N,\epsilon}(T)^{c}) = 0. \] (86)

**Step 2.** We fix \( N > 0, h, h_{\epsilon} \in \mathcal{A}_{M} \) such that \( h_{\epsilon} \to h \) \( \mathbb{P} \)-a.s. in the weak topology of \( L^{2}([0,T];H_{0}) \) as \( \epsilon \to 0 \). Now on \( G_{N,\epsilon}(T) \),
\[
T_{2}(t,\epsilon) = \epsilon \int_{0}^{t} K_{0} + K_{1}|y_{h_{\epsilon}}(s)|^{2} ds \leq \epsilon T(K_{0} + K_{1}N) \] (87)
by (82) and (84). Moreover, on \( G_{N,\epsilon}(T) \),
\[
2 \int_{0}^{T} c_{2} \| y_{h} \|_{H}^{4} + R_{1}^{2} + \sqrt{L_{1}}|h_{\epsilon}(s)|_{0} ds \leq 2a_{0}c_{2}N^{2} + 2R_{1}^{2}T + 2\sqrt{L_{1}TM} \] (88)
by (C1)(3b), Hölder’s inequality, (84), that \( h_{\epsilon} \in \mathcal{A}_{M} \) and (9). Thus, applying (88), Gronwall’s inequality on (81) gives
\[
\sup_{t \in [0,T]} |Y^{\epsilon}(t)|^{2} \lesssim \sup_{t \in [0,T]} \sum_{i=1}^{3} T_{i}(s,\epsilon) e^{2a_{0}c_{2}N^{2}+2R_{1}T+2\sqrt{L_{1}TM}}. \] (89)
Thus,
\[
\mathbb{E}[1_{G_{N,\epsilon}(T)}|Y^{\epsilon}|^{2}_{X}] \leq \mathbb{E}[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} \sum_{i=1}^{3} T_{i}(t,\epsilon) + c \sup_{t \in [0,T]} |Y^{\epsilon}(t)|^{2}] \leq \tilde{c}(\epsilon + \mathbb{E}[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} (T_{1}(t,\epsilon) + T_{3}(t,\epsilon))]) \] (90)
by definition of the X-norm, (81), (88), (89), as $1_{G_{N,\epsilon}(T)}T_2(t,\epsilon) \leq cT(K_0 + K_1N)$ by (87). Now by (82) and the fact that $G_{N,\epsilon}(T) \subset G_{N,\epsilon}(s)$ for all $s \leq T$,

$$
E[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} |T_1(t,\epsilon)|] \leq 2\sqrt{\epsilon} E[\sup_{t \in [0,T]} \int_0^t 1_{G_{N,\epsilon}(s)}(Y^\epsilon(s), \sigma(s, y_h(s))dW(s))]
$$

where the right hand side vanishes in $L^1(\Omega)$ as $\epsilon \to 0$ because

$$
2\sqrt{\epsilon} \left( \sup_{t \in [0,T]} \int_0^t 1_{G_{N,\epsilon}(s)}(Y^\epsilon(s), \sigma(s, y_h(s))dW(s)) \right) \leq 6\sqrt{\epsilon} E\left( \int_0^T 1_{G_{N,\epsilon}(s)}|Y^\epsilon(s)|^2 d|\sigma(s, y_h(s))|_{L^2}^2 \right) \frac{1}{\epsilon} \leq c(T, N)\sqrt{\epsilon}
$$

by Burkholder-Davis-Gundy inequality, Condition (C2)(1) with $K_2 = 0$, and that $\sup_{s \in [0,T]}|Y^\epsilon(s)| \leq 2N$ by (84) and (83). Next, we set $t_k = kT2^{-n}$ for $0 \leq k \leq 2^n$ as we did in (77) so that from (82) we may write

$$
E[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} |T_3(t,\epsilon)|] 
$$

$$
\leq 2(E[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} \int_0^t (\sigma(s, y_h(s))(h_s(s) - h(s)), |Y^\epsilon(s) - Y^\epsilon(\tilde{s}_n)|)ds)]
$$

$$
+ E[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} \int_0^t (\sigma(s, y_h(s)) - \sigma(\tilde{s}_n, y_h(s))(h_s(s) - h(s)), Y^\epsilon(\tilde{s}_n))ds]
$$

$$
+ E[1_{G_{N,\epsilon}(T)} \sup_{1 \leq k \leq 2^n} \sup_{t \in [t_{k-1}, t_k]} |(\sigma(t_k, y_h(t_k)) \int_{t_{k-1}}^{t_k} (h_s(s) - h(s))ds, Y^\epsilon(t_k))|]
$$

$$
+ E[1_{G_{N,\epsilon}(T)} \sum_{k=1}^{2^n} |(\sigma(t_k, y_h(t_k)) \int_{t_{k-1}}^{t_k} (h_s(s) - h(s))ds, Y^\epsilon(t_k))|]
$$

$$
\triangleq 2T_1(N, n, \epsilon) + 2E[T_5(N, n, \epsilon)]
$$

where $\tilde{s}_n$ is defined in (77). The estimates of $\sum_{i=1}^{4} \tilde{T}_i(N, n, \epsilon) + 2E[T_5(N, n, \epsilon)]$ does not require the Condition (C3)(2) and hence we just note that it is shown on pg. 400–402 [8] that $\tilde{T}_1(N, n, \epsilon) \leq c2^{-\frac{n}{2}}, \tilde{T}_2(N, n, \epsilon) \leq c2^{-n-1}, \tilde{T}_3(N, n, \epsilon) \leq c2^{-\frac{n}{2}}, \tilde{T}_4(N, n, \epsilon) \leq c2^{-\frac{n}{2}}$ and $\lim_{\epsilon \to 0} E[T_5(N, n, \epsilon)] = 0$ so that

$$
\lim_{\epsilon \to 0} E[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} |T_3(t,\epsilon)|] = 0.
$$

Thus,\n
$$
E[1_{G_{N,\epsilon}(T)} \|Y^\epsilon\|_X^\epsilon] \leq c(\epsilon + c(T, N)\sqrt{\epsilon} + E[1_{G_{N,\epsilon}(T)} \sup_{t \in [0,T]} |T_3(t,\epsilon)|]) \to 0
$$

as $\epsilon \to 0$ by (90), (91) and (93). Hence, we have shown for all $\delta > 0$,

$$\mathbb{P}(\|Y^\epsilon\|_X > \delta) \leq \mathbb{P}(G_{N,\epsilon}(T)) + \frac{1}{\delta^2} E[1_{G_{N,\epsilon}(T)} \|Y^\epsilon\|_X^\epsilon] \to 0$$

as $\epsilon \to 0, N \to \infty$ due to (85) and (94). Thus, $\lim_{\epsilon \to 0} \mathbb{P}(\|Y^\epsilon\|_X > \delta) = 0$ for all $\delta > 0$. This completes the proof of Proposition 2. \Box
Proposition 3. Let \( K_M \stackrel{\triangle}{=} \{ y_h \in X : y_h \text{ solves } (15) \text{ uniquely with } h \in S_M \} \) where \( M > 0, \xi \in H \) and \( X = C([0,T];H) \cap L^2([0,T];V) \). Suppose that the Conditions (C1), (C2) with \( K_2 = L_2 = 0 \), (C3)(2) and (C4) hold. Then \( K_M \) is a compact subset of \( X \).

Proof. Let \( \{ y_n \} \) be a family of solutions in \( K_M \), corresponding to (15) with \( \{ h_n \} \) in \( S_M \):

\[
dy_n + [Ay_n(t) + B(y_n(t)) + \tilde{R}(t, y_n(t))]| dt = \sigma(t, y_n(t))h_n(t)| dt, \quad y_n(0) = \xi. \tag{95}
\]

Now \( S_M \) from (9) is a bounded closed subset in the Hilbert space \( L^2([0,T];H_0) \); thus, it is weakly compact and hence there exists a subsequence of \( \{ h_n \} \), which we relabel by \( \{ h_n \} \), so that \( h_n \to h \) weakly in \( L^2([0,T];H_0) \). Then \( h \in S_M \) as \( S_M \) is closed. We show that the subsequence of solution \( \{ y_n \} \) corresponding to \( \{ h_n \} \), still denoted by \( \{ y_n \} \), converges to \( y \) in \( X \) and \( y \) solves

\[
dy(t) + [Ay(t) + B(y(t)) + \tilde{R}(t, y(t))]| dt = \sigma(t, y(t))h(t)| dt, \quad y(0) = \xi. \tag{96}
\]

This will imply by arbitrariness of \( \{ y_n \} \) in \( K_M \), that every sequence in \( K_M \) has a convergent subsequence and hence \( K_M \) is compact. We let \( Y_n \stackrel{\triangle}{=} y_n - y \). From (95), (96), we obtain

\[
|Y_n(t)|^2 + 2 \int_0^t \| Y_n \|^2_V ds = -2 \int_0^t \langle B(y_n) - B(y), Y_n \rangle ds - 2 \int_0^t \langle \tilde{R}(s, y_n) - \tilde{R}(s, y), Y_n \rangle ds + 2 \int_0^t \langle [\sigma(s, y_n) - \sigma(s, y)]h_n, Y_n(s) \rangle + \langle [\sigma(s, y)h_n(s) - h(s)], Y_n(s) \rangle ds \tag{97}
\]

where in particular we may compute

\[
-2|\tilde{R}(s, y_n(s)) - \tilde{R}(s, y(s))| |Y_n(s)| \leq 2|\tilde{R}(s, y_n(s)) - \tilde{R}(s, y(s))| |Y_n(s)| \leq 2R_1\|y_n(s) - y(s)\|_V |Y_n(s)| \leq \frac{1}{2}||Y_n(s)||^2_V + \frac{1}{2}R_1||Y_n(s)||^2 \tag{98}
\]

by Hölder’s inequality, Condition (C3)(2) and Young’s inequality. Using Conditions (C1)(2), (C1)(3c), (C1)(3b) and (C2)(2) with \( L_2 = 0 \) on other terms leads to

\[
|Y_n(t)|^2 + \int_0^t \| Y_n \|^2_V ds \leq \int_0^t 2c_1^2 |Y_n|^2 |y||^4_V + 2R_1^2 |Y_n|^2 + 2\sqrt{L_1} |Y_n|^2 |h_n|ds + 2 \int_0^t \langle [\sigma(s, y(s))h_n(s) - h(s)], Y_n(s) \rangle ds \tag{99}
\]

(see pg. 403 [8]). By (13), we know that there exists a constant \( c_0 > 0 \) such that

\[
\sup_{t \in [0,T]} \left[ \sup_n |y(t)|^2 + |y_n(t)|^2 + \int_0^T \| (y(t)) \|^2_V + \| y(s) \|^4_V + \| y_n(s) \|^2_V ds \right] \leq c_0 \tag{100}
\]

P-a.s. and hence Gronwall’s and Hölder’s inequalities give

\[
\sup_{t \in [0,T]} |Y_n(t)|^2 + \int_0^T \| Y_n(t) \|^2_V dt \leq e^{c(2c_1^2c_0 + R_1^2 + \sqrt{L_1}MT)} \sum_{i=1}^5 I_{n,N} \tag{101}
\]
where

\[ I_{n,N}^1 \triangleq \int_0^T |[(\sigma(s, y(s)) [h_n(s) - h(s)], Y_n(s) - Y_n(\tau_N))]| ds, \]

\[ I_{n,N}^2 \triangleq \int_0^T \left| \frac{1}{2 \sqrt{2\pi T}} \int_{\mathbb{R}^2} \right| (x - y) e^{-\frac{(x-y)^2}{2T}} ds, \]

\[ I_{n,N}^3 \triangleq \int_0^T \left| \frac{1}{2 \sqrt{2\pi T}} \int_{\mathbb{R}^2} \right| (x - y) e^{-\frac{(x-y)^2}{2T}} ds, \]

\[ I_{n,N}^4 \triangleq \sup_{2 \leq k \leq 2^N} \sup_{t \in [t_{k-1}, t_k]} |(\sigma(t_k, y(t_k)) \int_{t_{k-1}}^t (h_n(s) - h(s)) ds, Y_n(t_k))|, \]

\[ I_{n,N}^5 \triangleq \left| \sum_{k=1}^{2^N} (\sigma(t_k, y(t_k)) \int_{t_{k-1}}^{t_k} [h_n(s) - h(s)] ds, Y_n(t_k)) \right|, \]

with \( \tau_N = t_{k+1} = (k + 1) T 2^{-N}, \psi_N(s) = \tau_N \) as defined in (77). In the estimates of \( I_{n,N}^i, i = 1, 2, 3, 4, 5 \) in (102), the Condition (C3)(2) is not needed and we hence just note that it is shown on pg. 404–405 that P-a.s. \( I_{n,N}^1 \leq c 2^{-\frac{N}{2}}, \)

\( I_{n,N}^2 \leq 2^{-N/2}, I_{n,N}^3 \leq c 2^{-\frac{N}{2}}, I_{n,N}^4 \leq c 2^{-\frac{N}{2}}, \) \( \lim_{n \to \infty} I_{n,N}^5 = 0. \) Considering these estimates in (101) leads to \( \lim_{n \to \infty} \sup_{s \in [0, T]} |Y_n|_{2} \leq 2^{-N/2} \) by the definition of \( X \)-norm. As \( N \) \( \in \mathbb{Z}^+ \) is arbitrary and \( \gamma > 0 \) by Condition (C4), we conclude that \( |Y_n|_{\infty} \to 0 \) as \( n \to \infty \). Thus, \( K_M \) is sequentially compact in \( X \). Let \( \{y_n\} \) be a sequence of elements of \( K_M \) such that \( y_n \to v \) in \( X \) as \( n \to \infty \). Then there exists \( \{y_{n_k}\}_{k=1}^{\infty} \) such that \( y_n \to y_{n_k} \in K_M \) in \( X \) as \( k \to \infty \). Thus, \( v = y_h \). This completes the proof of Proposition 3.

**Proof of Theorem 2.4.** Due to Proposition 2 and Proposition 3, we may apply Theorem 4.4 of [4] (also Theorem 5 [5]) so that the proof of Theorem 2.4 is complete by Theorem 1.2.3 [16].

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