Eventually and asymptotically positive semigroups on Banach lattices

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November 18, 2015

Abstract

We develop a theory of eventually positive $C_0$-semigroups on Banach lattices, that is, of semigroups for which, for every positive initial value, the solution of the corresponding Cauchy problem becomes positive for large times. We give characterisations of such semigroups by means of spectral and resolvent properties of the corresponding generators, complementing existing results on spaces of continuous functions. This enables us to treat a range of new examples including the square of the Laplacian with Dirichlet boundary conditions, the bi-Laplacian on $L^p$-spaces, the Dirichlet-to-Neumann operator on $L^2$ and the Laplacian with non-local boundary conditions on $L^2$ within the one unified theory. We also introduce and analyse a weaker notion of eventual positivity which we call “asymptotic positivity”, where trajectories associated with positive initial data converge to the positive cone in the Banach lattice as $t \to \infty$. This allows us to discuss further examples which do not fall within the above-mentioned framework, among them a network flow with non-positive mass transition and a certain delay differential equation.

Mathematics Subject Classification (2010): 47D06, 47B65, 34G10, 35B09, 47A10

Keywords: One-parameter semigroups of linear operators; semigroups on Banach lattices; eventually positive semigroup; asymptotically positive semigroup; positive spectral projection; eventually positive resolvent; asymptotically positive resolvent; Perron-Frobenius theory

1 Introduction

While the study of positive operator semigroups is by now a classical topic in the theory of $C_0$-semigroups (see e.g. \textsuperscript{3} for a survey), the analysis of \textit{eventually} positive semigroups,

\textsuperscript{*}Supported by a scholarship within the scope of the LGFG Baden-Württemberg, Germany.
\textsuperscript{†}Partly supported by a fellowship of the Alexander von Humboldt Foundation, Germany.
i.e. semigroups which only become positive for positive, possibly large, times, in various contexts seems to have emerged only within the last decade. Probably the first example, the idea to consider matrices whose powers are eventually positive, is somewhat older and was in large part motivated for example by the consideration of inverse eigenvalues problems (see e.g. [11, 26, 54] or [18, p 48–54]) and by an attempt to generalise the classical Perron–Frobenius type spectral results to a wider class of matrices (see e.g. [21, 36]).

An analysis of continuous-time eventually positive matrix semigroups can for example be found in [42]. The phenomenon of eventually positive solutions of Cauchy problems was also observed at around the same time in an infinite-dimensional setting in the context of biharmonic equations; see [25] and [27]. Another infinite-dimensional occurrence of eventual positivity was analysed in [17], where it was proven that the semigroup generated by the Dirichlet-to-Neumann operator on a disk is eventually positive but not positive in some cases. A first attempt to develop a unified theory of eventually positive $C_0$-semigroups was subsequently made by the current authors in [18], providing some spectral results on Banach lattices and a characterisation of eventually strongly positive semigroups on $C(K)$-spaces with $K$ compact.

The current paper has two principal aims. The first aim is to characterise eventual strong positivity of resolvents and $C_0$-semigroups on general Banach lattices, not just in $C(K)$; see Sections 3–5. Unlike in $C(K)$-spaces, the positive cone in general Banach lattices may have empty interior, a fact which poses considerable challenges, but allows us to consider a wide variety of new examples: on Hilbert spaces, on $L^p$-spaces and on spaces of continuous functions vanishing at the boundary of a sufficiently smooth bounded domain; see Section 6.

The second aim is to cover situations where the $C_0$-semigroup does not satisfy the assumptions made in Sections 3–5, but where there is nevertheless some weaker form of “eventual positivity”. This is done in Sections 7 and 8, where we introduce and characterise a notion which we call asymptotic positivity, where, roughly speaking, denoting our semigroup by $(e^{tA})_{t \geq 0}$, the distance of $e^{tA}f$ to the positive cone of the Banach lattice converges to zero as $t \to \infty$, whenever $f$ itself is in the cone. In this framework we are able to drop the distinction between individual and uniform eventual behaviour, which was necessary in our theory on eventual positivity so far.

We note that there are other notions of eventual or asymptotic positivity, for example in [13], where a positive forcing term is introduced to obtain a form of asymptotic positivity. We shall investigate eventual positivity as an inherent property of the semigroup, in particular without any such forcing term.

Let us now formulate two theorems giving a somewhat incomplete – summary of our main results, the first on eventual and the second on asymptotic positivity. In what follows, we will denote by $E_+$ the positive cone of a Banach lattice $E$; if $u \in E_+$, then $E_u$ is the principal ideal generated by $u$, and we will write $v \gg u$ 0 and say $v$ is strongly positive with respect to $u$ if there is a $c > 0$ such that $v \geq cu$. We refer to Section 2 for a complete description of our notation, and to Definitions 4.1 and 5.1 for the relevant terminology.

**Theorem 1.1.** Let $(e^{tA})_{t \geq 0}$ be a real and eventually differentiable $C_0$-semigroup with $\sigma(A) \neq \emptyset$ on a complex Banach lattice $E$. Suppose that the peripheral spectrum $\sigma_{\text{per}}(A)$ is finite and consists of poles of the resolvent. If $u \in E_+$ such that $D(A) \subseteq E_u$, then the following assertions are equivalent.

(i) The semigroup $(e^{tA})_{t \geq 0}$ is individually eventually strongly positive with respect to $u$. 

The spectral bound \( s(A) \) of \( A \) is a dominant spectral value and the resolvent \( R(\cdot, A) \) is individually eventually strongly positive with respect to \( u \) at \( s(A) \).

(iii) \( s(A) \) is a dominant spectral value and the spectral projection \( P \) associated with \( s(A) \) fulfils \( Pf \gg u \) for every \( f > 0 \).

(iv) \( s(A) \) is a dominant spectral value. Moreover, \( \ker(s(A)I - A) \) is spanned by a vector \( v \gg u \) and \( \ker(s(A)I - A') \) contains a strictly positive functional.

The assertions of the above theorem are shown in Corollary 3.3, Theorem 4.4 and Corollary 5.3.

The eventual differentiability of the semigroup and the domination condition \( D(A) \subseteq E_u \) can be partially weakened at the expense of losing the equivalent assertion (ii) and of needing an additional boundedness condition in assertions (iii) and (iv); see Theorem 5.2 and Corollary 3.3.

For the precise definition of asymptotic positivity which appears in the next theorem we refer the reader to Definitions 7.2 and 8.1.

**Theorem 1.2.** Let \( (e^{tA})_{t \geq 0} \) be an eventually norm continuous \( C_0 \)-semigroup with \( \sigma(A) \neq \emptyset \) on a complex Banach lattice \( E \). Suppose that the peripheral spectrum \( \sigma_{\per}(A) \) is finite and consists of simple poles of the resolvent. Then the following assertions are equivalent:

(i) The semigroup \( (e^{tA})_{t \geq 0} \) is individually asymptotically positive.

(i') The semigroup \( (e^{tA})_{t \geq 0} \) is uniformly asymptotically positive.

(ii) \( s(A) \) is a dominant spectral value of \( A \) and \( R(\cdot, A) \) is individually asymptotically positive.

(ii') \( s(A) \) is a dominant spectral value of \( A \) and \( R(\cdot, A) \) is uniformly asymptotically positive.

(iii) \( s(A) \) is a dominant spectral value of \( A \) and the associated spectral projection \( P \) is positive.

This theorem follows from Theorems 7.6 and 8.3 and from the comments subsequent to Theorem 8.3.

2 Notation and preliminaries

Throughout this article, we will use the following notation. We assume that the reader is familiar with the theory of real and complex Banach lattices (see for instance \([46, 41]\)), and with the theory of \( C_0 \)-semigroups (see for instance \([22, 23]\)). Throughout, we suppose \( E \) and \( F \) are Banach spaces and denote by \( \mathcal{L}(E, F) \) the space of bounded linear operators from \( E \) to \( F \) (or by \( \mathcal{L}(E) \) if \( F = E \)); in the special case where \( E \subseteq F \) and the natural embedding is continuous we write \( E \hookrightarrow F \).
Positivity and related notions. Suppose \( E \) and \( F \) are Banach lattices. We denote by
\[
E_+ = \{ u \in E : u \geq 0 \}
\]
the positive cone in \( E \). An element \( u \in E_+ \) is called positive. We write \( u > 0 \) to say that \( u \geq 0 \) and \( u \neq 0 \). For \( f \in E \) we denote by
\[
d_+(f) := \text{dist}(f, E_+)
\]
the distance of \( f \) to the positive cone \( E_+ \). As usual, the principal ideal generated by \( u \in E_+ \) is given by
\[
E_u := \{ f \in E : \exists c \geq 0, |f| \leq cu \}.
\]
If \( E \) is a real Banach lattice, we define the gauge norm of \( f \in E_u \) with respect to \( u \) by
\[
\| f \|_u := \inf\{ \lambda \geq 0 : f \leq \lambda u \}.
\]
If \( E \) is a complex Banach lattice, and thus the complexification of a real Banach lattice \( E_R \), then we define the gauge norm \( \| \cdot \|_u \) on \( E_u \) to be the lattice norm complexification (see \([46, \text{Section II.11}]\)) of the gauge norm \( \| \cdot \|_u \) on \( (E_R)_u \). When endowed with the gauge norm, \( E_u \) embeds continuously into \( E \). What is important for our purposes is that \( E_u \) with the gauge norm is a Banach lattice and we have an isometric lattice isomorphism
\[
E_u \cong C(K)
\]
for some compact Hausdorff space \( K \); see \([46, \text{Corollary to Prop II.7.2 and Theorem II.7.4}]\).

We call \( u \in E_+ \) a quasi-interior point of \( E_+ \) and write \( u \gg 0 \) if \( E_u \) is dense in \( E \). If \( u \) is a quasi-interior point of \( E_+ \), then we say that \( f \in E \) is strongly positive with respect to \( u \) and write \( f \gg_u 0 \) if there is a \( c > 0 \) such that \( f \geq cu \). We say \( f \) is strongly negative with respect to \( u \) and write \( f \ll_u 0 \) if \( -f \gg_u 0 \).

An operator \( T \in \mathcal{L}(E, F) \) is called positive if \( TE_+ \subseteq F_+ \); we say \( T \) is strongly positive and write \( T \gg 0 \) if \( Tf \gg 0 \) whenever \( f > 0 \). Given a quasi-interior point \( u \in E_+ \), we say that \( T \) is strongly positive with respect to \( u \) and write \( T \gg_u 0 \) if \( Tf \gg_u 0 \) whenever \( f > 0 \). We call \( T \) strongly negative with respect to \( u \) and write \( T \ll_u 0 \) if \( -T \gg_u 0 \).

The dual space of \( E \) is denoted by \( E' \) and is again a Banach lattice. A linear functional \( \varphi \in E' \) is called strictly positive if \( \langle \varphi, f \rangle > 0 \) for every \( f > 0 \); note that \( \varphi \) is automatically strictly positive if it is a quasi-interior point of \( (E')_+ \), but the converse is not true.

Linear operators, resolvent and spectrum. The domain of an operator \( A \) on a Banach space \( E \) will always be denoted by \( D(A) \), and if not stated otherwise, \( D(A) \) will be assumed to be endowed with the graph norm. If \( A \) is densely defined, then its adjoint is well defined and we denote it by \( A' \). Let \( E \) and \( F \) be complex Banach lattices, i.e. let them be complexifications of real Banach lattices \( E_R \) and \( F_R \). We call an operator \( A : D(A) \subseteq E \to F \) real if \( D(A) = D(A) \cap E_R + iD(A) \cap E_R \) and if \( T(D(A) \cap E_R) \subseteq F_R \). Positive and, in particular, strongly positive operators are automatically real.

Let \( A \) be a closed linear operator on a complex Banach space \( E \); we denote its spectrum by \( \sigma(A) \), its resolvent set by \( \rho(A) := \mathbb{C} \setminus \sigma(A) \), and for each \( \lambda \in \rho(A) \) the operator \( R(\lambda, A) := (\lambda I - A)^{-1} \) denotes the resolvent of \( A \) at \( \lambda \). The spectral bound of \( A \) is given by
\[
s(A) := \sup\{ \text{Re} \lambda : \lambda \in \sigma(A) \} \in [-\infty, \infty].\]
If $s(A) \in \mathbb{R}$, the set
\[ \sigma_{\text{per}}(A) := \sigma(A) \cap (s(A) + i\mathbb{R}) \]
is called the \textit{peripheral spectrum} of $A$. We call $s(A)$ a \textit{dominant spectral value} of $A$ if $\sigma_{\text{per}}(A) = \{s(A)\}$. In particular, this includes the assertion that $s(A) \in \sigma(A)$.

A spectral value $\mu \in \sigma(A)$ is called a \textit{pole of the resolvent} of $A$ if the analytic mapping $\rho(A) \to \mathcal{L}(E)$, $\lambda \mapsto R(\lambda, A)$ has a pole at $\mu$. We will make extensive use of the Laurent series expansion of $R(\cdot, A)$ about its poles and of the spectral projection $P$ associated with a pole of $R(\cdot, A)$. Details on this can be found in \cite[Section 2]{18}, \cite[Section-III.6.5]{37}, \cite[Section VIII.8]{53}, \cite[Section IV.1]{22} or \cite{12}.

\textbf{Semigroups.} Suppose the operator $A$ on a Banach space $E$ generates a $C_0$-semigroup, which will be denoted by $(e^{tA})_{t \geq 0}$. This semigroup is called \textit{eventually differentiable} if there is a $t_0 > 0$ such that $e^{t_0 A} E \subseteq D(A)$, \textit{eventually norm continuous} if there is $t_0 \geq 0$ such that the mapping $[t_0, \infty) \to \mathcal{L}(E)$, $t \mapsto e^{tA}$ is continuous with respect to the operator norm on $\mathcal{L}(E)$, and \textit{uniformly exponentially stable} if $\|e^{tA}\|_{\mathcal{L}(E)} \to 0$ as $t \to \infty$. A $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on a complex Banach lattice $E$ is called \textit{real} if each operator $e^{tA}$ is real. It is easy to check that a $C_0$-semigroup on $E$ is real if and only if its generator is real.

\section{3 Strongly positive projections}

In this section, we consider eigenvalues of linear operators on complex Banach lattices and characterise when the corresponding spectral projection is strongly positive. Our first result is the following analogue of \cite[Proposition 3.1]{18} for arbitrary Banach lattices.

\textbf{Proposition 3.1.} Let $A$ be a closed, densely defined and real operator on a complex Banach lattice $E$. Let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of $A$ and a pole of $R(\cdot, A)$. Let $P$ be the corresponding spectral projection. Then the following assertions are equivalent.

(i) $P$ is positive and irreducible.

(ii) $P \gg 0$.

(iii) The eigenvalue $\lambda_0$ of $A$ is geometrically simple. Moreover, $\ker(\lambda_0 I - A)$ contains a quasi-interior point of $E_+$ and $\ker(\lambda_0 I - A')$ contains a strictly positive vector.

(iv) The eigenvalue $\lambda_0$ of $A$ is algebraically simple, $\ker(\lambda_0 I - A)$ contains a quasi-interior point of $E_+$ and $\mathrm{im}(\lambda_0 I - A) \cap E_+ = \{0\}$.

If assertions (i)--(iv) are fulfilled, then $\lambda_0$ is a simple pole of the resolvents $R(\cdot, A)$ and $R(\cdot, A')$ and $\lambda_0$ is the only eigenvalue of $A$ having a positive eigenvector.

The proof requires some properties of positive projections which are given in the next lemma and which are based on standard techniques from the Perron–Frobenius theory of positive operators; compare \cite[Sections V.4 and V.5]{46}.

\textbf{Lemma 3.2.} Let $E$ be a complex Banach lattice and let $P \in \mathcal{L}(E)$ be a positive, non-zero and irreducible projection. Then every non-zero element of $E_+ \cap \mathrm{im} P$ is a quasi-interior point of $E_+$ and $\mathrm{im} P'$ contains a strictly positive functional. Moreover, $\dim(\mathrm{im} P') = \dim(\mathrm{im} P'') = 1$. 

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Proof. Since \( P \) is positive and non-zero, \( \text{im} \, P \) contains a vector \( u > 0 \). As \( E_u \) is \( P \)-invariant and \( P \) is irreducible we conclude that \( E_u \) is dense in \( E \) and hence \( u \gg 0 \). In fact, this argument shows that every non-zero element of \( \text{im}(P) \cap E_+ \) is a quasi-interior point of \( E_+ \). Since the adjoint projection \( P' \) is also positive and non-zero, \( \text{im} \, P' \) contains a functional \( \varphi > 0 \). As \( P \) is irreducible, we easily conclude that \( \varphi \gg 0 \).

Note that \( \dim(\text{im} \, P) = \dim(\text{im} \, P') \) (see [37, Section III.6.6]), so it remains to show that \( \dim(\text{im} \, P) = 1 \). To this end, let us show first that \( |v| \in \text{im} \, P \) whenever \( v \in \text{im} \, P \). If \( v \in \text{im} \, P \) and \( \varphi \in \text{im} \, P' \) with \( \varphi \gg 0 \), then \( P|v| \geq |Pv| = |v| \) and hence

\[
0 \leq \langle \varphi, P|v| - |v| \rangle = \langle P|v|, |v| \rangle - \langle \varphi, |v| \rangle = 0.
\]

As \( \varphi \gg 0 \) we conclude that \( P|v| = |v| \), so indeed \( |v| \in \text{im} \, P \).

By definition, the complex Banach lattice \( E \) is the complexification of a real Banach lattice \( E_\mathbb{R} \). If we define \( F_\mathbb{R} := E_\mathbb{R} \cap \text{im} \, P \), then \( \text{im} \, P = F_\mathbb{R} + iF_\mathbb{R} \) since \( PE_\mathbb{R} \subseteq E_\mathbb{R} \). Hence it is sufficient to show that \( F_\mathbb{R} \) is one-dimensional over \( \mathbb{R} \). We have shown that \( |v| \in F_\mathbb{R} \) for each \( v \in F_\mathbb{R} \), and so \( F_\mathbb{R} \) is a sublattice of \( E_\mathbb{R} \). Clearly \( F_\mathbb{R} \) is a normed vector lattice with respect to the norm induced by \( E_\mathbb{R} \) and therefore it is Archimedean (see [46, Proposition II.5.2(ii)]). Hence, to show that \( F_\mathbb{R} \) is one-dimensional we only need to show that \( F_\mathbb{R} \) is totally ordered, see [46, Proposition II.3.4]. To do so, let \( v_1, v_2 \in F_\mathbb{R} \) and set \( v := v_1 - v_2 \). Then the positive part \( v^+ \) lies in \( \text{im} \, P \) and by what we have shown above either \( v^+ = 0 \) or \( v^+ \gg 0 \). Hence, either \( v_1 - v_2 \leq 0 \) or \( v_1 - v_2 \gg 0 \), showing that \( F_\mathbb{R} \) is totally ordered and thus one-dimensional over \( \mathbb{R} \).

Regarding the assumptions of Lemma 3.2, we point out that a positive irreducible projection on a complex Banach lattice is automatically non-zero whenever \( \dim \, E \geq 2 \).

Proof of Proposition 3.1. We may assume without loss of generality that \( \lambda_0 = 0 \), since otherwise we may replace \( A \) with \( A - \lambda_0 I \). We prove \( (i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (i) \).

“(i) \Rightarrow (iii)” If (i) holds, then Lemma 3.2 implies (iii), the lemma in particular asserts that \( \dim(\text{im} \, P) = 1 \), that is, \( \lambda_0 = 0 \) is algebraically and hence geometrically simple.

“(iii) \Rightarrow (iv)” By (iii), \( 0 \) is a geometrically simple eigenvalue. To show that it is algebraically simple we have to prove that \( \ker A^2 = \ker A \). Let \( u \in \ker A^2 \). Then \( Au \in \ker A \) and by (iii) there exists \( \alpha \in \mathbb{C} \setminus \{0\} \) such that \( \alpha Au \geq 0 \). By assumption there is a strictly positive functional \( \varphi \in \ker A' \). Thus \( \langle \varphi, \alpha Au \rangle = \langle A' \varphi, \alpha u \rangle = 0 \). As \( \varphi \) is strictly positive, we conclude that \( \alpha Au = 0 \), that is, \( u \in \ker A \) as claimed. Finally, let \( v = Au \in E_+ \cap \ker A \). Then \( \langle \varphi, v \rangle = \langle A' \varphi, u \rangle = 0 \) and thus \( v = 0 \).

“(iv) \Rightarrow (ii)” Since \( 0 \) is algebraically simple we have \( \ker P = \text{im} \, A \), \( \text{im} \, P = \ker A \) and \( E = \text{im} \, P \cap \ker P \). Hence, we can decompose each \( f \in E_+ \setminus \{0\} \) uniquely in the form \( f = Pf + g \), where \( Pf \in \ker A \) and \( g \in \text{im} \, A \). From (iv) we have that \( Pf = \alpha u \) for a quasi-interior point \( u \in E_+ \) and for some scalar \( \alpha \in \mathbb{C} \). Since \( A \) is real, so is \( P \), and hence we have \( \alpha \in \mathbb{R} \). Assume for a contradiction that \( \alpha \leq 0 \). Then \( 0 < f \leq f - \alpha u = g \in \text{im} \, A \) which contradicts (iv). Hence we must have \( \alpha > 0 \) and thus \( Pf = \alpha u \gg 0 \).

The implication “(ii) \Rightarrow (i)” is obvious.

Now assume that the equivalent assertions (i)–(iv) hold. By Lemma 3.2 we have \( \dim \, P = \dim \, P' = 1 \). Hence \( \lambda_0 \) is an algebraically simple eigenvalue of \( A \) and \( A' \) and thus a simple pole of \( R(\cdot, A) \) and \( R(\cdot, A') \). Finally, let \( \lambda \in \mathbb{R} \setminus \{0\} \) be an eigenvalue of \( A \) and \( u \) a corresponding eigenvector. Then \( 0 \neq u = \lambda^{-1} Au \in \text{im} \, A \). As \( E_+ \cap \text{im} \, A = \{0\} \) by (iv), \( u \) cannot be positive. 

\[ \Box \]
As pointed out above, Proposition 3.1 is an analogous result to [18, Proposition 3.1] where the situation on $C(K)$-spaces was considered. However, $u \gg 0$ in $C(K)$ means that $u$ is an interior point of the positive cone, whereas in a general Banach lattice the interior of the positive cone is empty. This is the main obstacle when seeking to generalise the results from [18]. For this reason we will not focus on the relation $\gg$, but on the stronger property of being strongly positive with respect to a given quasi-interior point (see Section 2 for details). The following corollary translates Proposition 3.1 into this setting.

**Corollary 3.3.** Let $A$ be a closed, densely defined and real operator on a complex Banach lattice $E$. Let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of $A$ and a pole of the resolvent and denote by $P$ the corresponding spectral projection. If $u \in E_+$ is a quasi-interior point, then the following assertions are equivalent:

(i) $P \gg_u 0$.

(ii) The eigenvalue $\lambda_0$ of $A$ is geometrically simple. Moreover, $\ker(\lambda_0 I - A)$ contains a vector $x \gg_u 0$ and $\ker(\lambda_0 I - A^\dagger)$ contains a strictly positive vector.

(iii) The eigenvalue $\lambda_0$ of $A$ is algebraically simple, $\ker(\lambda_0 I - A)$ contains a vector $x \gg_u 0$ and $\text{im}(\lambda_0 I - A) \cap E_+ = \{0\}$.

If assertions (i)–(iii) are fulfilled, then $\lambda_0$ is a simple pole of the resolvents $R(\cdot, A)$ and $R(\cdot, A^\dagger)$ and $\lambda_0$ is the only eigenvalue of $A$ having a positive eigenvector.

**Proof.** We assume throughout the proof that $\lambda_0 = 0$.

“(i) $\Rightarrow$ (ii)” Assertion (i) clearly implies that $P \gg 0$. Hence by Proposition 3.1 we only have to show that $\ker A$ contains a vector $v \gg_u 0$. We already know that $\ker A$ contains a quasi-interior point $v \gg 0$. As $v \in \text{im} P$ we indeed have $v = P v \gg_u 0$.

“(ii) $\Rightarrow$ (iii)” By (ii) we already know that $\ker A$ contains a vector $v \gg_u 0$. The remaining assertions follow from Proposition 3.1.

“(iii) $\Rightarrow$ (i)” Let $v \in \ker A$ with $v \gg_u 0$. If $f > 0$, then $P f \gg 0$ by Proposition 3.1. As 0 is algebraically simple we have $\text{im} P = \ker A$ and $\dim \text{im} P = 1$. Hence, $P f = \alpha v$ for some $\alpha \in \mathbb{C}$. As $P f \gg 0$ we see that $\alpha > 0$. Thus $P f \gg_u 0$.

Suppose now that (i)–(iii) are fulfilled. Then we clearly have $P \gg 0$ due to (i) and hence the remaining assertions follow from Proposition 3.1. \qed

### 4 Eventually strongly positive resolvents

To prepare for our analysis of eventually positive semigroups, we first consider what we shall call eventually positive resolvents. Here we will generalise certain results on $C(K)$-spaces from [18, Section 4] to the technically more demanding case of general Banach lattices. As pointed out before Corollary 3.3 it seems appropriate in this setting not to consider merely strong positivity, but strong positivity with respect to a fixed quasi-interior point.

**Definition 4.1.** Let $A$ be a closed, real operator on a complex Banach lattice $E$, let $u \in E_+$ be a quasi-interior point and let $\lambda_0$ be either $-\infty$ or a spectral value of $A$ in $\mathbb{R}$.

(a) The resolvent $R(\cdot, A)$ is called individually eventually strongly positive with respect to $u$ at $\lambda_0$ if there exists $\lambda_2 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_2) \subset \rho(A)$ and for each $f \in E \setminus \{0\}$ there is a $\lambda_1 \in (\lambda_0, \lambda_2]$ such that $R(\lambda, A) f \gg_u 0$ for all $\lambda \in (\lambda_0, \lambda_1]$. 

(b) The resolvent \(R(\cdot, A)\) is called \textit{uniformly eventually strongly positive with respect to} \(u\) at \(\lambda_0\) if there exists \(\lambda_1 > \lambda_0\) with the following properties: \((\lambda_0, \lambda_1] \subset \rho(A)\) and \(R(\lambda, A) \gg_u 0\) for all \(\lambda \in (\lambda_0, \lambda_1]\).

While eventual positivity focuses on what happens to the resolvent in a right neighbourhood of a spectral value, we might also ask what happens in a left neighbourhood. As we will see, \textit{eventual negativity} is the appropriate notion to describe this behaviour in our setting.

**Definition 4.2.** Let \(A\) be a closed, real operator on a complex Banach lattice \(E\), let \(u \in E_+\) be a quasi-interior point and let \(\lambda_0\) be either \(\infty\) or a spectral value of \(A\) in \(\mathbb{R}\).

(a) The resolvent \(R(\cdot, A)\) is called \textit{individually eventually strongly negative with respect to} \(u\) at \(\lambda_0\) if there exists \(\lambda_2 < \lambda_0\) with the following properties: \([\lambda_2, \lambda_0) \subset \rho(A)\) and for each \(f \in E \setminus \{0\}\) there is a \(\lambda_1 \in [\lambda_2, \lambda_0)\) such that \(R(\lambda, A)f \ll_u 0\) for all \(\lambda \in [\lambda_2, \lambda_0)\).

(b) The resolvent \(R(\cdot, A)\) is called \textit{uniformly eventually strongly negative with respect to} \(u\) at \(\lambda_0\) if there exists \(\lambda_1 < \lambda_0\) with the following properties: \([\lambda_1, \lambda_0) \subset \rho(A)\) and \(R(\lambda, A) \ll_u 0\) for all \(\lambda \in [\lambda_1, \lambda_0)\).

Concerning eventual strong positivity of resolvents (with respect to a quasi-interior point) we can make similar observations on arbitrary Banach lattices as were made for \(C(K)\)-spaces in [18, Propositions 4.2 and 4.3]. However, we do not pursue this in detail here. Instead, we concentrate on proving a characterisation of individually eventually strongly positive resolvents. To state this characterisation, the following notion concerning powers of a given operator will be useful.

**Definition 4.3.** Let \(T\) be a bounded linear operator on a complex Banach lattice \(E\) and let \(u \in E_+\) be a quasi-interior point.

(a) The operator \(T\) is called \textit{individually eventually strongly positive with respect to} \(u\) if for every \(f \in E_+ \setminus \{0\}\) there is an \(n_0 \in \mathbb{N}\) such that \(T^n f \gg_u 0\) for all \(n \geq n_0\).

(b) The operator \(T\) is called \textit{uniformly eventually strongly positive with respect to} \(u\) if there is an \(n_0 \in \mathbb{N}\) such that \(T^n \gg_u 0\) for all \(n \geq n_0\).

We can now formulate the following theorem, which was previously known only on \(C(K)\)-spaces, cf. [18, Theorem 4.5]. Beside its wider applicability, the main difference in the case of general Banach lattices is that the cone might not contain interior points; but if instead we take a quasi-interior point \(u\), then we cannot control other vectors by \(u\) unless they are contained in the principal ideal \(E_u\). This makes it necessary to impose an additional domination hypothesis, which in turn requires more technical proofs; nevertheless, in many practical examples, this condition seems to be satisfied, suggesting it is in a sense quite natural. See also the discussion below.

**Theorem 4.4.** Let \(A\) be a closed, densely defined and real operator on a complex Banach lattice \(E\). Suppose that \(\lambda_0 \in \mathbb{R}\) is an eigenvalue of \(A\) and a pole of the resolvent. Denote by \(P\) the corresponding spectral projection. Moreover, let \(0 \leq u \in E\) and assume that \(D(A) \subseteq E_u\). Then \(u\) is a quasi-interior point and the following assertions are equivalent:

(i) \(P \gg_u 0\).
(ii) The resolvent $R(\cdot, A)$ is individually eventually strongly positive with respect to $u$ at $\lambda_0$.

(iii) The resolvent $R(\cdot, A)$ is individually eventually strongly negative with respect to $u$ at $\lambda_0$.

If $\lambda_0 = s(A)$, then (i)–(iii) are also equivalent to the following assertions.

(iv) There exists $\lambda > s(A)$ such that the operator $R(\lambda, A)$ is individually eventually positive with respect to $u$.

(v) For every $\lambda > s(A)$ the operator $R(\lambda, A)$ is individually eventually positive with respect to $u$.

Before we prove the above theorem, a few remarks on the condition $D(A) \subseteq E_u$ are in order. First, if we endow $E_u$ with the (complexification of the) gauge norm, then the embedding $D(A) \hookrightarrow E_u$ is automatically continuous due to the closed graph theorem, a fact of which we will make repeated use. Second, it is natural to ask whether the condition $D(A) \subseteq E_u$ in the above theorem can be omitted, but Example 5.4 below shows that it is required. Finally, one might wonder how to check this condition in applications. In a typical situation, the Banach lattice $E$ is an $L^p(\Omega)$-space on a finite measure space $\Omega$, $u$ is the constant function 1 and the principal ideal $E_u$ is thus given by $L^\infty(\Omega)$. If $\Omega$ is an open set in $\mathbb{R}^n$ and $A$ is a differential operator which is defined on some Sobolev space, then the condition $D(A) \subseteq E_u$ is fulfilled if an appropriate Sobolev embedding theorem holds. Some concrete examples of this type can be found in Section 6, but see also the example and remark below.

Example 4.5. The above theorem contains what is often referred to as an anti-maximum principle. Let $A$ be a closed densely defined real operator on the Banach Lattice $E$. Suppose that $\lambda_0 \in \mathbb{R}$ a pole of the resolvent $R(\cdot, A)$ with spectral projection $P$. We consider the equation

$$\lambda f - Af = g$$

in $E$ (4.1)

with $\lambda < \lambda_0$ close to $\lambda_0$ and $g > 0$. If we assume that there exists $u \gg 0$ such that $D(A) \subseteq E_u$ and $P \gg_u 0$, then Theorem 4.4(iii) implies that for every $g > 0$ there exists $\lambda_g < \lambda_0$ such that the solution $f$ of (4.1) satisfies $f \ll_u 0$ whenever $\lambda \in (\lambda_g, \lambda_0)$. This is known as a (non-uniform) anti-maximum principle and has been the focus of many papers such as [14, 15, 32, 50], looking at standard second order elliptic equations, but also higher order elliptic equations of order $2m$. The key assumptions in our language are that $P \gg_u 0$ and that $D(A) \subseteq E_u$. The first one is most conveniently obtained by checking Proposition 3.1(iii). In many applications the dual problem has the same structure as the original problem, and therefore guarantees the existence of a positive eigenfunction for both. The condition that $D(A) \subseteq E_u$ follows from elliptic regularity theory as well as boundary maximum principles. If these regularity conditions are violated, an anti-maximum principle may fail as shown in [49].

Remark 4.6. The known anti-maximum principles also show that uniform strong eventual positivity is not in general equivalent to uniform strong eventual negativity of the resolvent. As an example, for second order elliptic boundary value problems, the strong maximum principle implies $R(\lambda, A) \gg 0$ for all $\lambda > s(A)$. However, as shown in [32], the anti-maximum principle is not necessarily uniform. At an abstract level, it does not seem to be obvious what guarantees uniform strong eventual negativity (or positivity); in [32] it is a certain kernel estimate that does this.
For the proof of Theorem 4.4 we need a few lemmata which are generalisations and extensions of similar auxiliary results in [18, Section 4]. In particular, in the next lemma we obtain convergence in a stronger norm than in [18, Lemma 4.7(ii)].

**Lemma 4.7.** Let $A$ be a closed linear operator on a complex Banach space $E$. Suppose that $0$ is an eigenvalue of $A$ and a simple pole of $R(\cdot, A)$. Let $P$ be the corresponding spectral projection.

(i) We have $\lambda R(\lambda, A) \to P$ in $\mathcal{L}(E, D(A))$ as $\lambda \to 0$.

(ii) If in addition $0 = s(A)$, then for every $\lambda > 0$ we have $[\lambda R(\lambda, A)]^n \to P$ in $\mathcal{L}(E, D(A))$ as $n \to \infty$.

**Proof.** (i) As $0$ is a simple pole of $R(\lambda, A)$ and $P$ is the corresponding residue, $\lambda R(\lambda, A) \to P$ in $\mathcal{L}(E)$ as $\lambda \to 0$ and $\text{im} \, P = \ker A$. Therefore

$$A\lambda R(\lambda, A) = \lambda(\lambda R(\lambda, A) - I) \to 0 = AP$$

in $\mathcal{L}(E)$ as $\lambda \downarrow 0$ and so the required limit holds in $\mathcal{L}(E, D(A))$.

(ii) By [18, Lemma 4.7(ii)] and its proof we have that the limit holds in $\mathcal{L}(E)$, and that $\lambda R(\lambda, A)P = P$. Since $R(\lambda, A) \in \mathcal{L}(E, D(A))$ due to the closed graph theorem, it follows that

$$[\lambda R(\lambda, A)]^n = \lambda R(\lambda, A)[\lambda R(\lambda, A)]^{n-1} \to \lambda R(\lambda, A)P = P$$

as $n \to \infty$ in $\mathcal{L}(E, D(A))$. \qed

**Lemma 4.8.** Let $E$ be a complex Banach lattice and let $u \in E_+$ be a quasi-interior point. Let $(J, \preceq)$ be a non-empty, totally ordered set and let $T = (T_j)_{j \in J}$ be a family in $\mathcal{L}(E)$ whose fixed space is denoted by

$$F := \{v \in E : T_jv = v \text{ for all } j \in J\}$$

Assume that for every $f \in E_+ \setminus \{0\}$ there exists $j_f \in J$ such that $T_jf \gg_0 0$ for all $j \succeq j_f$.

(i) Suppose that for every $j_0 \in J$ the family $(T_j|_{E_{u}})_{j \leq j_0}$ is bounded in $\mathcal{L}(E_u, E)$ and that $F$ contains an element $v_0 > 0$. Then the entire family $(T_j|_{E_u})_{j \in J}$ is bounded in $\mathcal{L}(E_u, E)$.

(ii) Let $P > 0$ be a projection in $\mathcal{L}(E)$ with $\text{im} \, P \subseteq F$ and suppose that each operator $T_j$, $j \in J$, leaves $\ker P$ invariant. Then $P \gg_0 0$.

**Proof.** (i) By the uniform boundedness principle we only have to show that $(T_jf)_{j \in J}$ is bounded in $E$ for every $0 < f \in E_u$. By assumption there is a vector $0 < v_0 \in F$ and $j_0 \in J$ such that $v_0 = T_{j_0}v_0 \gg_0 0$. For $0 < f \in E_u$ we can thus find a constant $c \geq 0$ such that $cv_0 \pm f \geq 0$. Hence we have $T_j(cv_0 \pm f) \geq 0$ and thus $|T_jf| \leq cv_0$ for all sufficiently large $j$. This yields the assertion.

(ii) If $f > 0$ then $Pf \in F$ and $Pf \geq 0$. In the case that $Pf \neq 0$ we have $Pf = T_jPf \gg_0 0$ for some $j \in J$. To show that $Pf \neq 0$ for every $f > 0$ fix $f > 0$. As $P \neq 0$ and $E_u$ is dense in $E$ there is an element $0 < g \in E_u$ such that $Pg > 0$. By assumption $T_jf \gg_0 0$ for some $j \in J$, and thus $T_jf - cg \geq 0$ for some $c > 0$. Hence, $PT_jf \geq cPg > 0$ and in particular $PT_jf \neq 0$. Since $T_j$ leaves $\ker P$ invariant, this implies that $Pf \neq 0$. \qed
We are now able to prove Theorem 4.4.

Proof of Theorem 4.4. We may assume throughout the proof that \( \lambda_0 = 0 \). First, observe that the domination condition \( D(A) \subseteq E_u \) implies that \( u \) is a quasi-interior point of \( E_+ \). Since \( D(A) \) is dense in \( E \). We shall prove \( (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \) and \( (i) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i) \).

"(i) \Rightarrow (ii)" If (i) holds, then Proposition 3.1 implies that 0 is a simple pole of \( R(\cdot, A) \). Let \( f > 0 \). Lemma 4.7(i) now yields that \( \lambda R(\lambda, A)f \to Pf \gg_u 0 \) in \( D(A) \) as \( \lambda \downarrow 0 \). By the closed graph theorem, \( D(A) \to E_u \) if \( E_u \) is endowed with the gauge norm with respect to \( u \). Hence, \( \lambda R(\lambda, A)f \to Pf \gg_u 0 \) in \( E_u \). Since \( \lambda R(\lambda, A)f \in \mathcal{L} \) for every \( \lambda \in (0, \infty) \cap \rho(A) \) and \( E_u = C(K) \) for some compact Hausdorff space \( K \), this implies that \( \lambda R(\lambda, A)f \gg_u 0 \) for sufficiently small \( \lambda > 0 \).

"(ii) \Rightarrow (i)" First we show that the eigenvalue 0 of \( A \) admits a positive eigenvector. Let \( m \geq 1 \) be the order of 0 as a pole of \( R(\cdot, A) \) and \( U \in \mathcal{L}(E) \) be the coefficient of \( \lambda^{-m} \) in the Laurent expansion of \( R(\cdot, A) \) about 0. Then \( U \neq 0 \) and \( \text{im} U \) consists of eigenvectors of \( A \) (see for instance [18, Remark 2.1]). Moreover, \( \lambda^m R(\lambda, A) \to U \) in \( \mathcal{L}(E) \). As the resolvent is individually eventually positive, \( U \gg 0 \) and so \( A \) has an eigenvector \( v > 0 \) corresponding to the eigenvalue 0.

We now apply Lemma 4.8 to the operator family \( \{\lambda R(\lambda, A)\}_{\lambda \in (0, \varepsilon)} \) and let \( m \) be the order of \( 0 \) as a pole of \( R(\cdot, A) \) and \( \varepsilon > 0 \). Then \( \mathcal{L}(E) \) is bounded in \( \mathcal{L}(E,E_u) \) for every \( \lambda \in (0, \varepsilon) \cap \rho(A) \) and \( E_u = C(K) \) for some compact Hausdorff space \( K \), this implies that \( \lambda R(\lambda, A)f \gg_u 0 \) for sufficiently small \( \lambda > 0 \).

"(i) \Leftrightarrow (ii)" Note that the fixed space of this operator family coincides with \( \ker A \). Therefore, all assumptions of part (i) of the Lemma are fulfilled, and we conclude that the operator family \( \{\lambda R(\lambda, A)|_{E_u}\}_{\lambda \in (0, \varepsilon)} \) is bounded in \( \mathcal{L}(E_u, E) \).

If we fix some \( \mu \in \rho(A) \), then by the resolvent identity

\[
\lambda R(\lambda, A) = (\mu - \lambda)\lambda R(\lambda, A)R(\mu, A) + \lambda R(\mu, A)
\]

for all \( \lambda \in (0, \varepsilon) \). We have \( R(\mu, A) \in \mathcal{L}(E, D(A)) \). As \( D(A) \to E_u \), we conclude that \( R(\mu, A) \in \mathcal{L}(E, E_u) \). Hence we have

\[
\|\lambda R(\lambda, A)\|_{\mathcal{L}(E)} \leq |\mu - \lambda| \|\lambda R(\lambda, A)\|_{\mathcal{L}(E)} \|R(\mu, A)\|_{\mathcal{L}(E, E_u)} + |\lambda| \|R(\mu, A)\|_{\mathcal{L}(E)}
\]

for every \( \lambda \in (0, \varepsilon) \). The operator family \( \{\lambda R(\lambda, A)\}_{\lambda \in (0, \varepsilon)} \) is therefore bounded in \( \mathcal{L}(E) \), showing that 0 is a simple pole of \( R(\cdot, A) \).

Hence, we have \( P = U > 0 \). Moreover, \( \text{im} P = \ker A \) and therefore the fixed space of \( \{\lambda R(\lambda, A)\}_{\lambda \in (0, \varepsilon)} \) is im \( P \). Applying Lemma 4.8 we conclude that \( P \gg_u 0 \).

"(i) \Leftrightarrow (iii)" Note that 0 is also an eigenvalue of \( -A \) and that the corresponding spectral projection is also \( P \). Thus, (i) holds true if and only if \( R(\cdot, -A) \) is individually eventually strongly negative with respect to \( u \) at 0. This however is true if and only if \( R(\cdot, A) \) is individually eventually strongly negative with respect to \( u \) at 0.

From now on we assume that \( \lambda_0 = s(A) = 0 \).

"(i) \Rightarrow (v)" We argue similarly as in the implication "(i) \Rightarrow (ii)", but use part (ii) of Lemma 4.7 instead of part (i) to conclude that for every \( f > 0 \) we have \([\lambda R(\lambda, A)]^n f \gg_u 0 \) for all \( n \) sufficiently large.

"(v) \Rightarrow (iv)" This implication is obvious.

"(iv) \Rightarrow (i)" We proceed similarly as in the proof of "(ii) \Rightarrow (i)", so we only provide an outline. Let \( \lambda > 0 \) such that \( R(\lambda, A) \) is individually strongly positive with respect to \( u \). We apply [18, Lemma 4.8] to \( T := \lambda R(\lambda, A) \) to conclude that \( \lambda R(\lambda, A) \) admits an eigenvector \( v > 0 \) for the eigenvalue 1. Then we apply Lemma 4.8(i) to the operator family \( \{[\lambda R(\lambda, A)]^n\}_{n \in \mathbb{N}_0} \) and conclude that its restriction to \( E_u \) is bounded in \( \mathcal{L}(E_u, E) \).
Since \( \lambda R(\lambda, A) \in \mathcal{L}(E, E_u) \), the family is also bounded in \( \mathcal{L}(E) \) and hence 0 is a simple pole of \( R(\cdot, A) \). It follows from Lemma 4.7(ii) that \( P \) is positive and since 0 is a spectral value of \( A \), \( P \) is non-zero. As above, we can now apply Lemma 4.8(ii) to conclude that \( P \gg u_0 \).

In the proof of the implication “(i) \( \Rightarrow \) (ii)” we used Lemma 4.7(ii) asserting that \( \lambda R(\lambda, A) \to P \) with respect to the operator norm in \( \mathcal{L}(E, D(A)) \) as \( \lambda \downarrow 0 \). One might thus be tempted to conjecture that \( R(\cdot, A) \) is uniformly eventually strongly positive with respect to \( u \). However, a counterexample from [18, Example 5.7] with \( E = C(K) \) and \( u = 1 \) shows that this is not the case. Although for each \( f \in E \) we have \( f \geq cu \) for some \( c > 0 \), the problem is that, as \( f \) varies, the constant \( c \) can be become arbitrarily small, even if we require \( \|f\| = 1 \).

5 Eventually strongly positive semigroups

We are finally ready turn to one of the main topics of our article and consider eventually strongly positive semigroups. We start with the precise definitions that we will use.

**Definition 5.1.** Let \((e^{tA})_{t \geq 0}\) be a real \( C_0 \)-semigroup on a complex Banach lattice \( E \). Let \( u \in E_+ \) be a quasi-interior point.

(a) The semigroup \((e^{tA})_{t \geq 0}\) is called individually eventually strongly positive with respect to \( u \) if for each \( f \in E_+ \setminus \{0\} \) there is a \( t_0 > 0 \) such that \( e^{tA}f \gg u_0 \) for each \( t \geq t_0 \).

(b) The semigroup \((e^{tA})_{t \geq 0}\) is called uniformly eventually strongly positive with respect to \( u \) if there is a \( t_0 > 0 \) such that \( e^{tA} \gg u_0 \) for each \( t \geq t_0 \).

Our main characterisation of individually eventually strongly positive semigroups is the following theorem. While we needed the domination \( D(A) \subseteq E_u \) in Theorem 4.4 we now require the “smoothing” assumption \( e^{tA}E \subset E_u \), which in practice is usually weaker. See Corollary 5.3 below for a connection between the two conditions.

**Theorem 5.2.** Let \((e^{tA})_{t \geq 0}\) be a real \( C_0 \)-semigroup with \( \sigma(A) \neq \emptyset \) on a complex Banach lattice \( E \). Suppose that the peripheral spectrum \( \sigma_{\text{per}}(A) \) is finite and consists of poles of the resolvent. If \( u \in E_+ \) is a quasi-interior point and if there exists \( t_0 \geq 0 \) such that \( e^{tA}E \subseteq E_u \), then the following assertions are equivalent:

(i) The semigroup \((e^{tA})_{t \geq 0}\) is individually eventually strongly positive with respect to \( u \).

(ii) The semigroup \((e^{(A-s(A)I)}t)_{t \geq 0}\) is bounded, \( s(A) \) is a dominant spectral value, and its associated spectral projection \( P \) fulfils \( P \gg u_0 \).

(iii) The semigroup \( e^{(A-s(A)I)} \) converges to some operator \( Q \gg u_0 \) with respect strong operator topology as \( t \to \infty \).

If assertions (i) – (iii) hold, then \( P = Q \).

Again, this theorem was previously only known in the case \( E = C(K) \) and \( u = 1 \) [18, Theorem 5.4]. In this case we have \( E_u = E \), hence the condition \( e^{tA} \subset E_u \) is automatically satisfied; this shows that the known result on \( C(K) \)-spaces is indeed a special case of our general Theorem 5.2 above.
**Proof of Theorem 5.2.** We may assume throughout that $s(A) = 0$.

“(i) $\Rightarrow$ (ii)” By [15, Theorem 7.6] $s(A) = 0$ is a spectral value of $A$ and by [15, Theorem 7.7(i)] it is even an eigenvalue and admits a positive eigenvector. Applying Lemma 5.3(i) to the operator family $(e^{tA})_{t \in [0, \infty)}$ we conclude that the family $(e^{tA}|_{E_u})_{t \in [0, \infty)}$ is bounded in $L(E_u, E)$. By assumption there exists $t_0 > 0$ such that $e^{t_0 A}E \subseteq E_u$ and thus, due to the closed graph theorem, $e^{t_0 A} \in L(E, E_u)$. Hence, the operator family $(e^{tA})_{t \in [t_0, \infty)}$ is bounded in $L(E)$ and therefore also $(e^{tA})_{t \geq 0}$ is bounded in $L(E)$.

As $(e^{tA})_{t \geq 0}$ is the spectral bound $s(A) = 0$ is a simple pole of $R(\cdot, A)$ and so $P = \ker A$. By [15, Theorem 7.7(ii)], $\sigma_{per}(A)$ consists of simple poles of $R(\cdot, A)$. Theorem 8.3 below then implies (under even weaker positivity assumptions) that $s(A)$ is a dominant spectral value of $A$ and that $P$ is positive. As $s(A)$ is an eigenvalue of $A$, $P$ is non-zero and thus Lemma 4.8(ii) applied to the operator family $(e^{tA})_{t \in [0, \infty)}$ implies that $P \gg u$.

“(ii) $\Rightarrow$ (iii)” Since all spectral values of $A|_{\ker P}$ have strictly negative real part and since the semigroup $(e^{tA})_{t \geq 0}$ is bounded, it follows from [2, Theorem 2.4] or [51, Corollary 5.2.6] that $e^{tA}$ converges strongly to 0 on $\ker P$ as $t \to \infty$. As $P \gg u$, 0 is a simple pole of $R(\cdot, A)$ according to Proposition 3.1 and hence we have $im P = \ker A$. Thus, $e^{tA}f \to Pf$ as $t \to \infty$ for each $f \in E$. In particular, (iii) holds with $Q = P$.

“(iii) $\Rightarrow$ (i)” Let $f > 0$. By assumption $\lim_{t \to \infty} e^{tA}f = Qf$ in $E$ and clearly, $Qf$ is a fixed point of the operator $e^{tA}$. As $e^{t_0 A} \in L(E, E_u)$ for some $t_0 > 0$ we conclude for $t \geq t_0$ that

$$e^{tA}f = e^{t_0 A}e^{(t-t_0)A}f \to e^{t_0 A}Qf = Qf \quad \text{in } E_u \quad \text{as } t \to \infty.$$  

Since $e^{tA}f$ is real and $Qf \gg u$, 0, this implies that $e^{tA}f \gg u$ for all sufficiently large $t$. □

As in [15, Corollary 5.6], the boundedness condition in Theorem 5.2(ii) is redundant if the semigroup $(e^{tA})_{t \geq 0}$ is eventually norm-continuous. If we assume that $(e^{tA})_{t \geq 0}$ is a little more regular, then we can also give a criterion to check the condition $e^{t_0 A} \subset E_u$:

Recall that a $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on a Banach space $E$ is called eventually differentiable if there exists $t_0 \geq 0$ such that $e^{t_0 A}E \subset D(A)$. In that case $e^{tA}E \subset D(A)$ for all $t \geq t_0$. Note that each analytic semigroup is eventually (in fact immediately) differentiable.

**Corollary 5.3.** Let $(e^{tA})_{t \geq 0}$ be a real, eventually differentiable $C_0$-semigroup with $\sigma(A) \neq \emptyset$ on a complex Banach lattice $E$. Suppose that the peripheral spectrum $\sigma_{per}(A)$ is finite and consists of poles of the resolvent. If $u \in E_+$ is a quasi-interior point and if there exists $n \in \mathbb{N}$ such that $D(A^n) \subset E_u$, then the following assertions are equivalent:

(i) The semigroup $(e^{tA})_{t \geq 0}$ is individually eventually strongly positive with respect to $u$.

(ii) $s(A)$ is a dominant spectral value, and its associated spectral projection $P$ fulfills $P \gg u$.

**Proof.** Let $t_0 \geq 0$ such that $e^{t_0 A}E \subset D(A)$. Then $e^{nt_0 A}E \subset D(A^n) \subset E_u$ and hence the assumptions of Theorem 5.2 are fulfilled. The implication “(i) $\Rightarrow$ (ii)” therefore follows.

Now assume that (ii) is true. To conclude from Theorem 5.2(i) we only have to show that $(e^{(A-s(A))}t \geq 0)$ is bounded. Without loss of generality we assume that $s(A) = 0$. Since $P \gg u$, 0, Proposition 3.1 tells us that $s(A) = 0$ is a simple pole of the resolvent; hence, $(e^{tA})_{t \geq 0}$ is bounded on $im P$. Since the semigroup is eventually differentiable, it is in particular eventually norm-continuous, and hence $\sigma(A) \cap \{z \in \mathbb{C} : \Re z \geq \alpha \}$ is bounded for every $\alpha \in \mathbb{R}$, see [22, Theorem II.4.18]. Since $s(A) = 0$ is a dominant spectral value of $A$ this implies that $s(A|_{\ker P}) < 0$. Using again the eventual
norm-continuity of the semigroup we conclude that the growth bound of \((e^{tA}|_{\ker P})_{t \geq 0}\) is strictly negative; in particular, \((e^{tA}|_{\ker P})_{t \geq 0}\) is bounded. \(\square\)

It is interesting to note that we needed the condition \(D(A) \subset E_u\) to prove Theorem 4.4 about resolvents, while we only need the weaker assumption \(D(A^0) \subset E_u\) for some power \(n \in \mathbb{N}\) in Corollary 5.3 about (eventually differentiable) semigroups. When we consider the bi-Laplacian with Dirichlet boundary conditions in Section 6, this will allow us to prove stronger results on the semigroup than on the resolvent (compare Propositions 6.5 and 6.6).

We now adapt [18, Example 5.7] to show that we cannot in general drop the domination conditions \(D(A) \subset E_u\) and \(e^{t_0A}E \subset E_u\) in Theorems 4.4 and 5.2. Interestingly, the example was used in [18] as a counterexample to a rather different question.

**Example 5.4.** Let \(p \in [1, \infty)\) and \(E = L^p((-1,1))\). We denote by \(1\) the constant function with value one and by \(\varphi: E \to \mathbb{C}\) the continuous linear functional given by \(\varphi(f) = \int_{-1}^{1} f(\omega) \, d\omega\). Consider the decomposition
\[
E = (1) \oplus F \quad \text{with} \quad F := \ker \varphi.
\]
Let \(S: F \to F\) denote the reflection operator given by \(S f(\omega) = f(-\omega)\) for all \(\omega \in (-1,1)\). As \(S^2 = I_F\) we have \(\sigma(S) = \{-1,1\}\). We define \(A \in \mathcal{L}(E)\) by
\[
A := 0_{(1)} \oplus (-2 I_F - S).
\]
Using \(S^2 = I_F\), we can immediately check that
\[
e^{tA} = I_{(1)} \oplus e^{-2t} \left( \cosh(t) I_F - \sinh(t) S \right) \quad \text{and} \quad \tag{5.1}
R(\lambda, A) = \frac{1}{\lambda} I_{(1)} \oplus \frac{1}{(\lambda + 2)^2 - 1} \left( (\lambda + 2) I_F - S \right), \quad \tag{5.2}
\]
for all \(t \geq 0\) and all \(\lambda \in \rho(A) = \mathbb{C} \setminus \{0,-1,-3\}\). In particular \(\sigma(A) = \{0,-1,-3\}\).

Now let \(P\) be the spectral projection associated with \(s(A) = 0\). We clearly have \(P f = \frac{1}{\lambda} \varphi(f) \cdot 1\) for all \(f \in E\). Thus, \(P\) is strongly positive with respect to \(u = 1\). Moreover, all assumptions of Theorems 4.4 and 5.2 are fulfilled, except that \(D(A) = E \not\subset L^\infty((-1,1)) = E_u\) and \(e^{t_0A}E = E \not\subset E_u\) for each \(t_0 \geq 0\).

Now, consider \(f \in E\) given by \(f(\omega) = (1 - \omega)^{-\frac{1}{p-1}}\) for all \(\omega \in (-1,1)\). Note that \(f\) is bounded on \([-1,0]\), but unbounded for \(\omega\) close to 1. By splitting \(f\) into \(P f \in \langle 1 \rangle\) and \((1 - P)f \in F\) and applying the formulae (5.1) and (5.2), we see that \(e^{tA}f \nRightarrow 0\) for all \(t > 0\) and \(R(\lambda, A)f \nRightarrow 0\) for all \(\lambda > 0\), that is, both the semigroup and the resolvent are not individually eventually positive. In particular, they are not individually eventually strongly positive with respect to \(u\).

### 6 Applications of eventual strong positivity

We shall now give some applications of the theory developed so far. Several applications were already given in [18, Section 6], but now we have much more freedom since we are not confined to \(C(K)\)-spaces. Our first two examples are concerned with bi-harmonic operators with different boundary conditions and on different spaces. Then we show how our results can be reformulated in the setting of a self-adjoint operator on a Hilbert lattice, which we apply to the Dirichlet-to-Neumann operator in two dimensions realised on \(L^2\)-spaces. Our final example is a class of Laplacians with non-local boundary conditions.
The square of the Dirichlet Laplacian  In [IS Section 6.4] it was shown that, under sufficiently strong regularity conditions, the negative square of the Robin Laplacian on a bounded domain $\Omega$ of class $C^2$ generates an eventually strongly positive semigroup on $C(\overline{\Omega})$. However, the negative square of the Dirichlet Laplacian $\Delta_D$ does not fit into that framework, since it generates a $C_0$-semigroup on $C_0(\Omega)$. Here we want to show that our theory on general Banach lattices naturally allows us to deal with such an operator. The Dirichlet Laplacian is given by

$$D(\Delta_D) := \{ f \in C_0(\Omega) : \Delta f \in C_0(\Omega) \}, \quad \Delta_D f := \Delta f,$$

where $\Delta f$ is understood in the sense of distributions.

**Theorem 6.1.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain of class $C^2$. On the Banach lattice $E = C_0(\Omega)$, consider the operator

$$D(A) = \{ f \in D(\Delta_D) : \Delta_D f \in D(\Delta_D) \}, \quad Af = -\Delta_D^2 f.$$

Let $u \in C_0(\Omega)$ be given by $u(x) := \text{dist}(x, \partial\Omega)$. Then $A$ generates a holomorphic $C_0$-semigroup on $C_0(\Omega)$ of angle $\pi/2$ which is not positive, but individually eventually strongly positive with respect to $u$.

**Proof.** It is known that $\Delta_D$ generates a holomorphic $C_0$-semigroup of angle $\pi/2$ on $C_0(\Omega)$; see [IS Theorem 2.3]. We have $\sigma(\Delta_D) \subseteq (-\infty, 0)$. Therefore, $A = -\Delta_D^2$ also generates a holomorphic $C_0$-semigroup of angle $\pi/2$ on $C_0(\Omega)$ as shown in the first part of [IS Proposition 6.5].

To show that $R(0, \Delta_D) \gg_u 0$ assume that $f \in D(\Delta_D)$ with $-\Delta_D f = g > 0$. As $\Omega$ is of class $C^2$ and $C_0(\Omega) \subseteq L^p(\Omega)$ for all $p \in (1, \infty)$ classical regularity theory implies that $f \in W^{2,p}(\Omega) \cap C_0(\Omega)$ for all $p > n$. In particular, by standard Sobolev embedding theorems, $f \in C^1(\overline{\Omega})$. Applying a Sobolev space version of the maximum principle and the strong boundary maximum principle we see that $\partial f/\partial \nu < 0$ on the compact manifold $\partial\Omega$, where $\nu$ is the outer unit normal, see [10] or [1 Theorem 6.1]. It follows that $R(0, \Delta_D) \gg_u 0$ and thus $R(0, A) = R(0, \Delta_D)^2 \gg_u 0$. For $\lambda \in (s(A), 0)$ we obtain

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-\lambda)^n R(0, A)^{n+1} \gg_u 0,$$

so the resolvent $R(\cdot, A)$ is uniformly eventually strongly positive with respect to $u$.

By the Sobolev embedding theorem, for $p > n$, $D(A) \hookrightarrow W^{2,p}(\Omega) \cap C_0(\Omega) \hookrightarrow C_0(\Omega) = E$ is compact, so $A$ has compact resolvent, and hence, $s(A)$ is a pole of the resolvent. By [10], we have $\partial u/\partial \nu < 0$ on $\partial\Omega$ and hence $D(A) \subseteq C^1(\overline{\Omega}) \cap C_0(\Omega) \subseteq C_0(\Omega)_u$. Theorem [4] now yields that the spectral projection $P$ associated with $s(A)$ fulfills $P \gg_u 0$. As $s(A)$ is dominant, Theorem [5] finally implies that the semigroup $(e^{tA})_{t \geq 0}$ is individually eventually strongly positive with respect to $u$. That the semigroup is not positive follows from [5] Proposition 2.2].

**The bi-Laplace operator with Dirichlet boundary conditions** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain of class $C^{\infty}$. Consider the bi-Laplace operator $A_p$ with Dirichlet boundary conditions on $L^p(\Omega)$ ($1 < p < \infty$), given by

$$A_p : D(A_p) := W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \to L^p(\Omega), \quad f \mapsto Af := -\Delta^2 f.$$

This operator has the following properties:
Proposition 6.2. For $p \in (1, \infty)$ the operator $A_p$ is a closed, densely defined operator on $L^p(\Omega)$ having compact resolvent, and $\sigma(A_p)$ is independent of $p \in (1, \infty)$. Moreover, the resolvent operators are consistent on the $L^p$-scale for $p \in (1, \infty)$.

Proof. Clearly, $A_p$ densely defined. If $p_1 < p_2$ and $\lambda \in \mathbb{C}$ is an eigenvalue of $A_{p_2}$, then $\ker(\lambda I - A_{p_2}) \subseteq \ker(\lambda I - A_{p_1})$ since $D(A_{p_2}) \subseteq D(A_{p_1})$. On the other hand, [28 Corollary 2.21] together with a simple bootstrapping argument shows that each function in $\ker(\lambda I - A_{p_1})$ is continuous up to the boundary, hence in $L^{p_2}(\Omega)$, and therefore in $\ker(\lambda I - A_{p_1})$. Hence, the point spectrum of $A_p$ and the corresponding eigenspaces do not depend on $p$.

Since under our assumptions on $\Omega$ the problem $\Delta^2 f = 0$ only admits the trivial solution in $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$, it follows from [28 Corollary 2.21] that $0 \in \rho(A_p)$. In particular, $A_p$ is closed. Since $D(A_p)$ is compactly embedded in $L^p(\Omega)$ and since $\rho(A_p) \neq \emptyset$, we conclude that $A_p$ has compact resolvent. Therefore, $\sigma(A_p)$ consists of eigenvalues only and is independent of $p$ by what we have shown above.

To see that the resolvent operators are consistent on the $L^p$-scale, let $p_1 < p_2$ and suppose $\lambda \notin \sigma(A_{p_1}) = \sigma(A_{p_2})$. If $f \in L^{p_2}$, then $w = R(\lambda, A_{p_2})f$ is a function in $D(A_{p_2}) \subseteq D(A_{p_1})$ and $(\lambda - A_{p_2})w = f$. Hence $(\lambda - A_{p_1})w = f$ and thus $R(\lambda, A_{p_1})$ and $R(\lambda, A_{p_2})$ agree on $D(A_{p_2})$.

We shall consider the function $u: \Omega \to \mathbb{C}$, $u(x) = \text{dist}(x, \Omega)^2$; $u$ is a quasi-interior point of $L^p(\Omega)_+$ for every $p \in [1, \infty)$. The following result was proved by Grunau and Sweers in [31 Theorem 5.2].

Theorem 6.3. Let $1 < p < \infty$. Suppose that $\Omega$ is sufficiently close to the unit ball in $\mathbb{R}^n$ in the sense of [31 Theorem 5.2] (where we have $m = 2$). Then the eigenvalue of the operator $A_p$ for the largest real eigenvalue is spanned by a function $v \gg 0$.

In [31 Proposition 5.3] Grunau and Sweers used this result to prove that for sufficiently large $p$ the resolvent $R(\cdot, A_p)$ is individually eventually strongly positive with respect to $u$ (though they did not use this terminology). We now demonstrate that this result fits into our general theory; we also do not require their assumption $p \geq 2$. In fact, for the semigroup, we do not even need to assume that $p > n/2$.

Lemma 6.4. Let $p \in (1, \infty)$ and let $\Omega \in C^\infty$ be such that the conclusion of Theorem 6.3 holds. Then $\lambda_0 := s(A_p)$ is a dominant spectral value of $A_p$ and a simple pole of $R(\cdot, A_p)$; the corresponding spectral projection $P$ satisfies $P \gg u 0$.

Proof. Since $A_2$ is self-adjoint, all of its spectral values are real. Proposition 6.2 thus implies that $s(A_p)$ is the largest real eigenvalue and a dominant spectral value of $A_p$; moreover, it is a pole of the resolvent $R(\cdot, A_p)$ since the resolvent is compact. According to Theorem 6.3 there is an eigenfunction $v$ for the eigenvalue $s(A_p)$ such that $v \gg u 0$. Again, as $A_2$ is self-adjoint, $v$ is also an eigenfunction of the adjoint operator $A_2' = A_2$ corresponding to $\lambda_0$. As $\Omega$ is of class $C^\infty$, we have $v \in C^\infty(\Omega)$ by standard regularity theory, and so $v$ is an eigenfunction of $(A_p)'$ as well. Corollary 3.3 now yields that the spectral projection $P$ associated with the eigenvalue $s(A_p)$ of $A_p$ is strictly positive with respect to $u$ and that $s(A_p)$ is an algebraically simple eigenvalue; in particular, it is a simple pole of the resolvent.

Proposition 6.5. Let $p \in (n/2, \infty)$ and let $\Omega \in C^\infty$ be as in Theorem 6.3. Then the resolvent $R(\cdot, A_p)$ is individually eventually strongly positive with respect to $v$ at the largest real eigenvalue $\lambda_0 = s(A_p)$ of $A_p$.  

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Proof. By Theorem [1.4] using Lemma [6.4] it remains to show that \( D(A_p) \subseteq L^p(\Omega)_u \). As \( p > n/2 \), we know that \( D(A_p) \subseteq W^{4,p}(\Omega) \hookrightarrow C^2(\Omega) \). For every \( f \in D(A_p) \), the trace of \( f \) and of its weak gradient \( \nabla w \) on \( \partial\Omega \) are 0. Hence, \( f = 0 \) and \( \nabla f = 0 \) on \( \partial\Omega \) in the classical sense. Hence \( D(A_p) \subseteq L^p(\Omega)_u \). \[\square\]

**Proposition 6.6.** Let \( p \in (1, \infty) \) and let \( \Omega \in C^\infty \) be as in Theorem [6.3]. Then the semigroup \( (e^{tA_p})_{t \geq 0} \) is individually eventually strongly positive with respect to \( u \).

Proof. The semigroup \( (e^{tA_p})_{t \geq 0} \) is analytic and, using Lemma [6.4] by Corollary [5.3] it only remains to show that \( D(A_p^t) \subseteq L^p(\Omega)_u \) for some \( n \in \mathbb{N} \). However, we see from [28] Corollary 2.21] that \( D(A_p^t) \subseteq W^{4n,p}(\Omega) \) for all \( n \in \mathbb{N} \). Hence, the Sobolev embedding theorem yields \( D(A_p^t) \subseteq C^2(\Omega) \) for all sufficiently large \( n \). Since we also have \( D(A_p^t) \subseteq W_0^{2,p}(\Omega) \) for all \( n \), we can now conclude as in the proof of Proposition [6.5] that \( D(A_p^t) \subseteq L^p(\Omega)_u \). \[\square\]

It seems quite interesting that we need the assumption \( p \in (n/2, \infty) \) only for the individual eventual strong positivity of the resolvent \( R(\cdot, A_p) \), but not for the same property of the semigroup \( (e^{tA_p})_{t \geq 0} \). This is of course due to the fact that Theorem [1.4] requires the condition \( D(A) \subseteq E_u \), while Corollary [5.3] only requires the weaker assumption \( D(A^t) \subseteq E_u \) for some \( n \in \mathbb{N} \); compare also the related discussion after Corollary [5.3].

**Eventual strong positivity for self-adjoint operators on Hilbert lattices**

In this paragraph we reformulate our results for the special case of self-adjoint operators on Hilbert lattices. Recall that a *Hilbert lattice* is a Banach lattice \( H \) whose norm is induced by an inner product. For every measure space \( \Omega \) the space \( L^2(\Omega) \) is a Hilbert lattice and conversely, every Hilbert lattice \( H \) is isometrically lattice isomorphic to \( L^2(\Omega) \) for some measure space \( \Omega \) (see [10] Theorem 6.7] for a slightly stronger result).

For self-adjoint operators on Hilbert lattices, our main result can be summarised as follows.

**Theorem 6.7.** Let \( H \) be a complex Hilbert lattice and let \( u \in H_+ \) be a quasi-interior point. Let \( A \) be a densely defined, self-adjoint operator on \( H \) and assume that \( s(A) \in \mathbb{R} \) is an isolated point of \( \sigma(A) \). Moreover, suppose that \( D(A) \subseteq H_u \). Then the following assertions are equivalent:

(i) The eigenvalue \( s(A) \) is geometrically simple and has an eigenvector \( v \gg u \).

(ii) The spectral projection \( P \) associated with \( s(A) \) satisfies \( P \gg u \).

(iii) The resolvent \( R(\cdot, A) \) is individually eventually strongly positive with respect to \( u \) at \( s(A) \).

(iv) The semigroup \( (e^{tA})_{t \geq 0} \) is individually eventually strongly positive with respect to \( u \).

Proof. As mentioned above, \( H \) can be identified with \( L^2(\Omega) \) for some measure space \( \Omega \). This shows that, when restricted to the real part of \( H \), the canonical identification \( H \simeq H' \) is a lattice isomorphism. Under this identification, the Hilbert space adjoint of \( A \) coincides with the Banach space dual of \( A \) on the real part of \( H \), so the equivalence “(i) \iff (ii)” follows from Proposition [3.1]. The equivalence “(ii) \iff (iii)” follows from Theorem [1.4]. Moreover, \( s(A) \) is a dominant eigenvalue and since the semigroup \( (e^{t(A-s(A))})_{t \geq 0} \) is analytic, the equivalence “(ii) \iff (iv)” follows from Corollary [5.3]. \[\square\]
The Dirichlet-to-Neumann operator in two dimensions In [18, Section 6.2] the Dirichlet-to-Neumann operator on $C(\Gamma)$ was analysed, where $\Gamma \subseteq \mathbb{R}^2$ is the unit circle. Using our theory for general Banach lattices we can now consider the more natural setting of $L^2$-spaces on more general domains.

We assume for simplicity that $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with $C^\infty$ boundary, although much of what follows still holds under weaker assumptions. Let $\lambda \in \mathbb{R}$ be contained in the resolvent set of the Dirichlet Laplacian $\Delta_D$ on $L^2(\Omega)$. For $g \in L^2(\partial\Omega)$ we solve, whenever possible, the Dirichlet problem

$$\Delta f = \lambda f \quad \text{in } \Omega, \quad f = g \quad \text{on } \partial\Omega.$$ 

Afterwards, we map the solution $f$ to its (distributional) normal derivative $\partial f/\partial \nu$ on the boundary $\partial\Omega$, if this is in $L^2(\partial\Omega)$. The operator $D_\lambda: g \mapsto \partial f/\partial \nu$ thus defined is called the Dirichlet-to-Neumann operator for the domain $\Omega$ and for the parameter $\lambda$. For a precise definition of the Dirichlet-to-Neumann operator $D_\lambda$ we refer the reader to [4] or [7].

It can be shown that $-D_\lambda$ is a densely defined self-adjoint operator on $L^2(\partial\Omega)$ with spectral bound $s(-D_\lambda) \in \mathbb{R}$ and compact resolvent on $L^2(\partial\Omega)$; see [7, Proposition 2]. In [17] it was shown that the semigroup $(e^{-tD_\lambda})_{t \geq 0}$ is uniformly eventually positive, but not positive for certain $\lambda$ if $\Omega$ is the disk in $\mathbb{R}^2$. The abstract theory developed in this paper allows us to give a characterisation of the semigroups $(e^{-tD_\lambda})_{t \geq 0}$ that are individually eventually strongly positive with respect to 1.

**Proposition 6.8.** Let $\Omega \subseteq \mathbb{R}^2$ be a domain with $C^\infty$ boundary and let $\lambda \in \mathbb{R}$ be contained in the resolvent set of the Dirichlet Laplacian on $L^2(\Omega)$. Then the following assertions are equivalent.

(i) The semigroup $(e^{-tD_\lambda})_{t \geq 0}$ is individually eventually strongly positive with respect to 1.

(ii) The largest real eigenvalue of $-D_\lambda$ is geometrically simple and admits an eigenfunction which is strongly positive with respect to 1.

**Proof.** It follows from [3, Theorem 5.2] that $D(D_\lambda) = H^1(\partial\Omega)$. Since $\partial\Omega$ is a smooth one-dimensional manifold, standard embedding theorems imply $H^1(\partial\Omega) \subseteq C(\partial\Omega)$, the latter clearly being contained in $L^\infty(\partial\Omega) = L^2(\partial\Omega)_1$. Hence the proposition follows from Theorem 6.7. \qed

The Laplace operator with non-local Robin boundary conditions It is well known that the Laplace operator with Dirichlet or Neumann boundary conditions (or more generally with Robin boundary conditions) generates a positive $C_0$-semigroup on $L^2(\Omega)$ whenever $\Omega \subseteq \mathbb{R}^n$ is a sufficiently regular bounded domain. We consider the non-local Robin problem

$$-\Delta u = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} + B\gamma(u) = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain, $B \in \mathcal{L}(L^2(\partial\Omega))$ a bounded linear operator and $\gamma \in \mathcal{L}(H^1(\Omega), L^2(\partial\Omega))$ the trace operator. The usual local Robin boundary condition can be recovered as a special case, by choosing $B$ to be a multiplication operator of the form $Bu = \beta u$ for $\beta \in L^\infty(\partial\Omega)$. 

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There has been considerable interest in non-local Robin boundary conditions in recent times, possibly beginning with [29]. In [30], conditions for positivity of the semigroup are given. Positivity and loss of positivity in a simple model of a thermostat appear in [33], and [47] deals with applications to Bose condensates. We discuss three examples where eventual positivity occurs, but before we do so we look at some general properties of (6.1).

First note that the sesquilinear form corresponding to (6.1) is given by

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \langle B \gamma(u), \gamma(u) \rangle$$

(6.2)

for all $u, v \in H^1(\Omega)$. Since $B$ and $\gamma$ are bounded operators, the form (6.2) is bounded from below. Therefore the induced operator $-A$ generates an analytic semigroup on $L^2(\Omega)$. As $D(A) \subseteq H^1(\Omega)$, $A$ has compact resolvent. We can say a bit more if $B$ is self-adjoint.

**Lemma 6.9.** Assume that $B \in \mathcal{L}(L^2(\partial\Omega))$ is self-adjoint and positive semi-definite, and that $\langle B \mathbf{1}, \mathbf{1} \rangle > 0$. Then the operator $A$ induced by (6.2) on $L^2(\Omega)$ is self-adjoint and $[0, \infty) \subseteq \rho(-A)$.

**Proof.** By assumption $B$ is positive semi-definite. Hence $a(u, u) \geq 0$ for all $u \in H^1(\Omega)$ and so $(0, \infty) \subseteq \rho(-A)$. We now show that $A$ is injective and therefore $0 \in \rho(-A)$.

Assume that $u \in D(A)$ such that $Au = 0$, that is, $0 = a(u, u) = \|\nabla u\|^2 + \langle B \gamma(u), \gamma(u) \rangle$. As $\langle B \gamma(u), \gamma(u) \rangle \geq 0$ we conclude that $\|\nabla u\|^2 = \langle B \gamma(u), \gamma(u) \rangle = 0$. In particular, $\nabla u = 0$ on $\Omega$ and therefore $u = c \mathbf{1}$ for some constant $c \in \mathbb{C}$. Hence $\langle B \gamma(c \mathbf{1}), \gamma(c \mathbf{1}) \rangle = |c|^2 \langle B \mathbf{1}, \mathbf{1} \rangle = 0$. By assumption $\langle B \mathbf{1}, \mathbf{1} \rangle > 0$, so $c = 0$. Therefore $u = 0$, showing that $a$ is coercive and that $A$ is injective. \hfill \Box

We now proceed to discuss the specific examples. The first is a simple model of a thermostat of the form (6.1) with $\Omega = (0, \pi) \subseteq \mathbb{R}$,

$$\gamma(u) = \begin{bmatrix} u(0) \\ u(\pi) \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix},$$

(6.3)

where $\beta \in \mathbb{R}$; note that $L^2(\partial\Omega) \simeq \mathbb{C}^2$ here. An explicit calculation in [33, Theorem 6.1] or [34, Section 3] shows that $s(A)$ is a positive, dominant and geometrically simple eigenvalue with an eigenfunction $v \geq 1 \ 0$ if and only if $\beta < 1/2$. It is also shown there that the corresponding semigroup is positive if and only if $\beta \leq 0$. The dual interchanges the roles of the boundary points 0 and $\pi$ and therefore it has the same spectrum, with correspondingly reflected eigenfunctions. By Corollary 5, the spectral projection associated with $s(A)$ is strongly positive with respect to $\mathbf{1}$. Because $s(A)$ is a simple dominant eigenvalue, the semigroup $(e^{t(A-I_A)})_{t \geq 0}$ is bounded. Regarding the domination condition, note that $D(A) \subseteq H^1((0, 1)) \subseteq C([0, 1])$, so that $D(A) \subseteq E_1$. Hence, applying Theorem 5.2 we have proved the following theorem.

**Theorem 6.10.** Let $\Omega = (0, \pi)$ and let $B = B(\beta)$ be given as in (6.3). Denote by $A$ the operator associated with the form (6.2). Then the semigroup $(e^{-tA})_{t \geq 0}$ is individually eventually strongly positive with respect to $\mathbf{1}$ but not positive if and only if $\beta \in (0, 1/2)$.

Let us now give a second example where we have eventual positivity without positivity; as we are not interested in a general theoretical development, we will merely consider one
special case, more precisely taking $\Omega := (0, 1)$ and

$$B := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

Denote by $A$ the operator induced by (6.2) on $L^2((0, 1))$. Then we have the following theorem.

**Theorem 6.11.** The operator $A$ defined above has the following properties:

(i) $\sigma(-A) \subseteq (-\infty, 0)$ and $R(0, -A) \gg 1$. 0.

(ii) The semigroup $(e^{-tA})_{t \geq 0}$ is individually eventually strongly positive with respect to 1.

(iii) The semigroup $(e^{-tA})_{t \geq 0}$ is not positive.

**Remark 6.12.** The above example was considered in [8, Example 4.5], where it was claimed that the associated semigroup dominates the semigroup associated with the Dirichlet Laplacian on $L^2((0, 1))$. Khalid Akhlil observed that the semigroup is not in fact positive (private communication), meaning the claimed domination cannot hold, but we see that the semigroup is at least “almost”, that is, eventually, positive.

**Proof of Theorem 6.11** (i) Lemma 6.9 implies that $\sigma(-A) \subseteq (-\infty, 0)$. Now let $f \in L^2$. One easily verifies that the resolvent at 0 is given by

$$(R(0, -A)f)(x) = \frac{1}{2} \int_0^x \int_0^1 f(z) \, dz \, dy + \frac{1}{2} \int_x^1 \int_y^1 f(z) \, dz \, dy \quad \text{for all } x \in [0, 1]$$

for all $f \in L^2((0, 1))$. If $f > 0$, then we have $(R(0, -A)f)(x) > 0$ for every $x \in [0, 1]$. By continuity we conclude that even $R(0, -A)f = u \gg 1$. 0.

(ii) For $\lambda \in (s(A), 0)$ we have the power series expansion

$$R(\lambda, -A) = \sum_{n=0}^{\infty} (-\lambda)^n R(0, -A)^{n+1}$$

and hence $R(\lambda, A) \gg 1$. 0. As $\sigma(A) \subseteq \mathbb{R}$, $R(0, A) \gg 1$ 0 and $A$ has compact resolvent, we conclude that $s(A)$ is a spectral value and a pole of the resolvent. Since $D(A) \subseteq H^1((0, 1)) \subseteq C([0, 1]) \subseteq (L^2)_1$, we conclude from Theorem 6.7 that $(e^{tA})_{t \geq 0}$ is individually eventually strongly positive with respect to 1.

(iii) By the Beurling-Deny criterion [44, Theorem 2.6], the semigroup $(e^{-tA})_{t \geq 0}$ is positive if and only if the form $a$ satisfies the estimate $a(u^+, u^-) \leq 0$ for each $u \in H^1((0, 1); \mathbb{R})$. However, if we choose $u \in H^1((0, 1); \mathbb{R})$ such that $u(0) = -1$ and $u(1) = 1$, this condition is not fulfilled. Hence $(e^{tA})_{t \geq 0}$ is not positive.

Our third and final example of non-local boundary conditions of the form (6.1) comes from Bose condensation as studied in [47, 24]. As in [47] we will consider the example of the unit disc $\Omega$ in $\mathbb{R}^2$ and a convolution operator $B$. We express functions on $\Omega$ in terms of polar coordinates $r \in [0, 1]$ and $\theta \in (-\pi, \pi]$ and let $B$ be defined by

$$(Bf)(\theta) = (q * f)(\theta) := \int_{-\pi}^{\pi} q(\theta - \varphi)f(\varphi) \, d\varphi,$$ 6.4
where \( q \in L^1((-\pi, \pi)) \) and \( f \in L^2((-\pi, \pi)) \). We identify \( q \) and \( f \) with \( 2\pi \)-periodic functions on \( \mathbb{R} \) so that the integral in (6.4) makes sense. We consider conditions under which \((e^{-\lambda A})_{\lambda \geq 0}\), called the Schrödinger semigroup, is individually eventually strongly positive but not positive.

By Young’s inequality for convolutions we have \( \|Bf\|_2 = \|q * f\|_2 \leq \|q\|_1 \|f\|_2 \) for all \( f \in L^2(\partial \Omega) \) and therefore \( B \in \mathcal{L}(L^2(\partial \Omega)) \). To ensure that \( B \) is self-adjoint and positive semi-definite we assume that \( q \) and its Fourier coefficients \( q_k \) satisfy

\[
q(\theta) = \overline{q(-\theta)} \quad \text{and} \quad q_k := \int_{-\pi}^{\pi} q(\theta) e^{-ik\theta} \, d\theta \geq 0 \quad (6.5)
\]

for all \( \theta \in \mathbb{R} \) and \( k \in \mathbb{Z} \), respectively.

**Theorem 6.13.** Let \( \Omega \) be the unit disc in \( \mathbb{R}^2 \). Let \( B \) be the convolution operator (6.4) with \( q \in L^1(\partial \Omega) \) so that (6.5) is satisfied with \( q_0 > 0 \). Then \( B \) is positive definite and the operator \( A \) associated with (6.2) on \( L^2(\Omega) \) has the following properties.

(i) \( A \) has compact resolvent and \( s(-A) < 0 \) is an algebraically simple eigenvalue.

(ii) The spectral projection \( P \) associated with \( s(-A) \) is strongly positive with respect to \( 1 \).

(iii) \((e^{-\lambda A})_{\lambda \geq 0}\) is individually eventually strongly positive with respect to \( 1 \), but not positive.

**Proof.** We start by showing that \( B \) given by (6.4) is positive semi-definite. Let \( f \in L^2(\partial \Omega) \) with Fourier coefficients \( f_k = \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \, d\theta \in \mathbb{C} \). The convolution theorem for Fourier series asserts that

\[
(Bf)(\theta) = (q * f)(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} q_k f_k e^{ik\theta}
\]

in \( L^2(\partial \Omega) \); see [38, Section 1.7]. Hence, by the orthogonality of \((e^{ik\theta})_{k \in \mathbb{Z}}\) in \( L^2(\partial \Omega) \) and (6.5)

\[
\langle Bf, f \rangle = \frac{1}{(2\pi)^2} \left( \sum_{k=-\infty}^{\infty} q_k f_k e^{ik\theta} \right) \left( \sum_{k=-\infty}^{\infty} f_k e^{ik\theta} \right) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} q_k |f_k|^2 \geq 0. \quad (6.6)
\]

Hence \( B \) is positive semi-definite. If we choose \( f = 1 \), then \( f_0 = 1 \) and \( f_k = 0 \) otherwise, so \( \langle B1, 1 \rangle = q_0 \). Hence the condition \( \langle B1, 1 \rangle > 0 \) is equivalent to \( q_0 > 0 \).

(i) Having shown that \( B \) is positive definite on \( L^2(\partial \Omega) \) we deduce from Lemma 6.9 and the discussion preceding it that \( A \) has compact resolvent and \( s(-A) < 0 \).

(ii) To show that the spectral projection associated with \( s(A) > 0 \) is strongly positive we compute the eigenvalues and eigenfunctions. Let \( J_k \) be the Bessel functions of the first kind whose properties we freely use, see for instance [16, Chapter VII]. The function \( u(r, \theta) = J_k(\sqrt{\lambda} r) e^{ik\theta} \) is a solution of \( \Delta u + \lambda u = 0 \) for every \( \lambda > 0 \); see [16, Section 5.5]. The values of \( \lambda > 0 \) such that \( u \) satisfies the boundary conditions in (6.1) are eigenvalues. We require that

\[
\frac{\partial}{\partial v} u + Bu = \frac{\partial}{\partial r} J_k(\sqrt{\lambda} r) e^{ik\theta} \bigg|_{r=1} + J_k(\sqrt{\lambda}) \int_{-\pi}^{\pi} q(\theta - \varphi) e^{ik\varphi} \, d\varphi
\]

\[
= \sqrt{\lambda} J_k'(\sqrt{\lambda} r) e^{ik\theta} + q_k J_k(\sqrt{\lambda}) e^{ik\theta} = 0. \quad (6.7)
\]
Due to (6.5) we have \( q_{-k} = q_k \). As \( J_{-k} = (-1)^k J_k \) we seek \( \lambda > 0 \) such that
\[
\sqrt{\lambda} J_k' (\sqrt{\lambda}) + q_k J_k (\sqrt{\lambda}) = 0 \tag{6.8}
\]
for some \( k \in \mathbb{N}_0 \). Denote by \( j_{k,l} \) (\( l = 1, 2, \ldots \)) the positive zeros of \( J_k \). Note that \( J_k'(s) > 0 \) and \( J_k(s) > 0 \) for all \( s \in (0, j_{0,1}) \) and all \( k \geq 1 \). Moreover, \( J_0'(s) < 0 \) and \( J_0(s) > 0 \) for all \( (0, j_{0,1}) \). Hence, as \( q_0 > 0 \), the smallest possible value of \( \lambda \) satisfying (6.8) occurs for \( k = 0 \) and \( \sqrt{\lambda} \in (0, j_{0,1}) \). Here we use that \( J_0'(0) = 0 \) and \( J_0(j_{0,1}) = 0 \) so that (6.8) with \( k = 0 \) has a unique solution \( \lambda_1 \in (0, j_{0,1}^2) \). Then \( u_1(r, \theta) := J_0(\sqrt{\lambda_1} r) > 0 \) is the only eigenfunction corresponding to \( \lambda_1 \). In particular, as \( A \) is self-adjoint, \( \lambda_1 \) is algebraically simple and \( u_1 \gg 1 \). Hence the corresponding spectral projection \( P \) satisfies \( P \gg 1 \).

To be sure that \( \lambda_1 \) is the dominant eigenvalue we need to know that the system of eigenfunctions we have constructed is complete; we sketch a proof of this fact. For any given \( k \in \mathbb{Z} \) let \( \lambda_{kj} \) (\( j \in \mathbb{N} \)) be the positive zeros of (6.8). Then, the functions \( v_{kj}(r) = J_k(\sqrt{\lambda_{kj}} r) \) (\( j \in \mathbb{N} \)) form a complete orthonormal system in the weighted space \( L^2((0, 1); r) \). A proof of this fact for Dirichlet boundary conditions, but easily modified for our conditions, appears in [52, Example 7.12 and Theorem 14.10]. As \( e^{ik\theta} \) (\( k \in \mathbb{Z} \)) is a complete system on the circle it follows that \( u_{kj}(r, \theta) = v_{kj}(r) e^{ik\theta} \) (\( k \in \mathbb{Z}, j \in \mathbb{N} \)) is a complete system in \( L^2(\Omega) \).

(iii) It follows from (ii) and Theorem [6.7] that \( (e^{-tA})_{t \geq 0} \) is individually eventually strongly positive with respect to 1, if in addition \( D(A) \subseteq E_1 \). To see this note that every solution \( u \in H^1(\Omega) \) of (6.1) can be written as \( u = w + v \), where
\[
- \Delta w = f \quad \text{in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega. \tag{6.9}
\]
and
\[
- \Delta v = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} = -Bu \quad \text{on } \partial \Omega. \tag{6.10}
\]
By standard regularity theory, \( w \in H^2(\Omega) \). As \( Bu \in L^2(\Omega) \) it follows from [33] that \( v \in H^{3/2}(\Omega) \). Hence \( u \in H^{3/2}(\Omega) \) and by the usual Sobolev embedding theorems we conclude \( D(A) \subseteq H^{3/2}(\Omega) \hookrightarrow C(\Omega) \subseteq E_1 \), as required.

To show that \( (e^{-tA})_{t \geq 0} \) is not positive we use the Beurling-Deny criterion which states that \( (e^{-tA})_{t \geq 0} \) is positive if and only if \( a(u^+, u^-) \leq 0 \) for all \( u \in H^1(\Omega) \), see [17, Theorem 2.6]. We will show that the criterion is violated for the harmonic function \( u_m(r, \theta) := r^m \sin m \theta \) for some choice of \( m \in \mathbb{N} \). Similarly as in [17, Proposition 4.8], an explicit calculation shows that
\[
u^+_m(1, \theta) = \frac{1}{4t} (e^{i \theta} + e^{-i \theta}) + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{(2k)^2 - 1} e^{2ik \theta}
\]
in \( L^2(\partial \Omega) \). As \( u^-_1 = u^+_1 - u_1 \) and \( u^+_m(1, \theta) = u^+_m(1, m \theta) \) we see that
\[
u^+_m(1, \theta) = \frac{1}{4t} (e^{im \theta} - e^{-im \theta}) + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{(2k)^2 - 1} e^{2ikm \theta} \tag{6.11}
\]
Applying the convolution theorem for Fourier series as before we see that
\[
(B_\gamma(u^+_m^1))(\theta) = (q \gamma (u^+_m^1))(\theta) = \frac{1}{4t} (q_m e^{im \theta} - q_{-m} e^{-im \theta}) + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{q_{2km}}{(2k)^2 - 1} e^{2ikm \theta}. \tag{6.12}
\]
Using (6.11), (6.12), the orthogonality of \((e^{ik\theta})_{k\in\mathbb{Z}}\) and the fact that \(q_k = q_{-k}\) we deduce that

\[
a(u_m^+, u_m^-) = \int_\Omega \nabla u_m^+ \nabla u_m^- \, dx + \langle B \gamma(u_m^+), \gamma(u_m^-) \rangle
\]

\[
= 0 + 2\pi \left( -\frac{q_m + q_{-m}}{16} + \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{q_{2km}}{(2k^2 - 1)^2} \right)
\]

\[
= \frac{2}{\pi} q_0 - \frac{\pi}{4} q_m + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{q_{2km}}{(2k^2 - 1)^2}
\]

(6.13)

for all \(m \geq 1\). As \(q_k \to 0\) by as \(k \to \infty\) by the Riemann-Lebesgue lemma and \(q_0 > 0\) we can choose \(m \geq 1\) such that \(2q_0/\pi - q_m\pi/4 > 0\). Since \(q_{2km} \geq 0\) for all \(k \geq 1\) by assumption, we conclude that \(a(u_m^+, u_m^-) > 0\) for this \(m\). This violates the Beurling-Deny criterion for the positivity of the semigroup generated by \(-A\) and therefore \((e^{-tA})_{t\geq0}\) is not positive.

### 7 Asymptotically positive resolvents

In [18] and the preceding sections we considered semigroups and resolvents which were, in some appropriate sense, eventually strongly positive. Nevertheless, the results presented have some limitations.

First, for our characterisations of eventual positivity, we have always required a domination or smoothing condition such as \(D(A) \subseteq E_u\), cf. Theorems 4.4 and 5.2 and Corollary 5.3. As Example 5.4 illustrates, such conditions cannot in general be dropped.

Second, the relationship between individual and uniform eventual positivity properties is not clear. We showed in [18, Examples 5.7 and 5.8] that it is essential to distinguish between individual and uniform eventual positivity, even under rather strong regularity and compactness assumptions.

Third, one might suspect that in certain applications a form of eventual positivity could occur which cannot be described in terms of strong positivity. At first glance the following notions seem to be appropriate to describe such a more general behaviour: we recall from [18, Section 7] that a \(C_0\)-semigroup \((e^{tA})_{t\geq0}\) on a complex Banach lattice \(E\) is called \(\textit{individually eventually positive}\) if for every \(f \in E_+\) there is a \(t_0 \geq 0\) such that \(e^{tA}f \in E_+\) whenever \(t \geq t_0\). Similarly, if \(A\) is a closed operator on \(E\) and \(\lambda_0\) is either \(-\infty\) or a spectral value of \(A\) in \(\mathbb{R}\), then we call the resolvent \(R(\cdot, A)\) on \(E\) \(\textit{individually eventually positive at} \ \lambda_0\) if there is a \(\lambda_2 > \lambda_0\) with the following properties: \((\lambda_0, \lambda_2) \subseteq \rho(A)\) and for every \(f \in E_+\) there is a \(\lambda_1 \in (\lambda_0, \lambda_2]\) such that \(R(\lambda, A)f \in E_+\) for all \(\lambda \in (\lambda_0, \lambda_1]\). Unfortunately, it turns out that these eventual positivity properties are difficult to characterise. This is demonstrated by the following two examples.

**Examples 7.1.** (a) If \(s(A)\) is a dominant spectral value and the associated spectral projection \(P\) is positive, then \((e^{tA})_{t\geq0}\) is not necessarily individually eventually positive. Indeed, consider the linear operator on \(\mathbb{C}^3\) given by

\[
A := \begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & -1 \\
0 & 1 & -1
\end{bmatrix}.
\]
with respect to the canonical basis on \( \mathbb{C}^3 \). Then \( \sigma(A) = \{0, i-1, -i-1\} \) and the spectral projection \( P \) associated with \( s(A) = 0 \) is the projection onto the first component. In particular \( s(A) \) is dominant and \( P \) is positive. However, the semigroup

\[
e^{tA} = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^{-t}\cos t & -e^{-t}\sin t \\
0 & e^{-t}\sin t & e^{-t}\cos t
\end{bmatrix}.
\]

is not eventually positive. Yet, we observe that the orbits spiral towards the \( x \)-axis, so the distance to the positive cone approaches zero for every positive initial condition.

(b) Even if \( s(A) \) is a dominant spectral value, individual eventual positivity of the resolvent at \( s(A) \) is in general not equivalent to individual eventual positivity of the semigroup. Indeed, let \( E = \mathbb{C}^n \) and let \( A \in \mathcal{L}(E) \) with \( s(A) = 0 \) such that \( (e^{tA})_{t \geq 0} \) is bounded and eventually positive, but not positive (such semigroups exist in all dimensions \( n \geq 3 \), cf. [18, Remark 5.3(a)]). Then \( \{ \lambda > 0 : R(\lambda, A) \not\geq 0 \} \) is non-empty and open in \((0, \infty)\). We choose an arbitrary element \( \lambda_0 \) from this set.

Now, consider the operator \( B := (A - \lambda_0 I) \oplus 0 \) on \( E \oplus \mathbb{C} \). Then \( 0 \) is a dominant spectral value of \( B \) and a simple pole of its resolvent. Clearly, \( B \) generates an eventually positive semigroup on \( E \times \mathbb{C} \) with \( s(B) = 0 \). However, the resolvent of \( B \) is not eventually positive at \( 0 \), since \( R(\lambda, B)|_E = R(\lambda_0 + \lambda, A) \not\geq 0 \) for small \( \lambda > 0 \).

In [18, Example 8.2] the reader can find another example of a \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) (on an infinite dimensional Banach lattice) which is uniformly eventually positive but whose resolvent is not individually eventually positive at \( s(A) \). However, in that example the semigroup is nilpotent and therefore \( s(A) = -\infty \).

For the reasons described above it seems appropriate to introduce yet another concept of eventual positivity which does not exhibit the above mentioned disadvantages. An indication of what concept this should be can be found in Examples 7.1 Observe that in Example (a) the semigroup is not eventually positive, but its “negative part” tends to \( 0 \) as time evolves. Similarly, in Example (b) the resolvent is not eventually positive but, despite having a pole in \( 0 \), its “negative part” remains bounded as \( \lambda \downarrow 0 \). These observations motivate Definitions 7.2, 7.3 and 8.1 below.

Recall that for every element \( f \) of a Banach lattice \( E \), we denote by \( d_+(f) \) the distance of \( f \) to the positive cone \( E_+ \) as defined in (2.1).

**Definition 7.2.** Let \( A \) be a closed linear operator on a complex Banach lattice \( E \). Suppose that \( \lambda_0 \in \mathbb{R} \) is a spectral value of \( A \) such that \( (\lambda_0, \lambda_0 + \varepsilon] \subset \rho(A) \) for some \( \varepsilon > 0 \) and such that \( R(\cdot, A) \) satisfies the Abel-type growth condition

\[
\limsup_{\lambda \uparrow \lambda_0} \| (\lambda - \lambda_0) R(\lambda, A) \| < \infty. \tag{7.1}
\]

(a) The resolvent \( R(\cdot, A) \) is called *individually asymptotically positive* at \( \lambda_0 \) if for each \( f \geq 0 \) we have \( (\lambda - \lambda_0) d_+(R(\lambda, A)f) \to 0 \) as \( \lambda \downarrow \lambda_0 \).

(b) The resolvent \( R(\cdot, A) \) is called *uniformly asymptotically positive* at \( \lambda_0 \) if for each \( \varepsilon \geq 0 \) there is a \( \lambda_1 > \lambda_0 \) with the following properties: \( (\lambda_0, \lambda_1] \subset \rho(A) \) and \( (\lambda - \lambda_0) d_+(R(\lambda, A)f) \leq \varepsilon \| f \| \) for all \( f \in E_+ \) and all \( \lambda \in (\lambda_0, \lambda_1] \).

Note that, in contrast to Definition 4.1, we do not allow for the case \( \lambda_0 = -\infty \) here, since in this case the growth condition (7.1) does not make sense. Let us also introduce a somewhat stronger refinement of the above definitions.
Definition 7.3. Let $A$ be a closed linear operator on a complex Banach lattice $E$. Suppose that $\lambda_0 \in \mathbb{R}$ is a spectral value of $A$ such that $(\lambda_0, \lambda_0 + \varepsilon] \subset \rho(A)$ for some $\varepsilon > 0$ and such that $R(\cdot, A)$ satisfies (7.1).

(a) The resolvent $R(\cdot, A)$ is called **individually asymptotically positive of bounded type at $\lambda_0$** if for every $f \in E_+$ there exists $\lambda_1 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_1] \subset \rho(A)$ and the set $\{d_+(R(\lambda, A)f) : \lambda \in (\lambda_0, \lambda_1]\}$ is bounded.

(b) The resolvent $R(\cdot, A)$ is called **uniformly asymptotically positive of bounded type at $\lambda_0$** if there exist $\lambda_1 > \lambda_0$ and $K \geq 0$ with the following properties: $(\lambda_0, \lambda_1] \subset \rho(A)$ and $d_+ ((R(\lambda, A)f)) \leq K\|f\|$ for all $f \in E_+$ and all $\lambda \in (\lambda_0, \lambda_1]$.

Clearly, if the resolvent of $R(\cdot, A)$ is individually asymptotically positive of bounded type, then it is also individually asymptotically positive and the same observation also holds for the uniform properties.

We could also define asymptotic negativity of the resolvent from the left just as we defined eventual negativity of the resolvent in Definition 4.2. However, this definition would not lead to any fundamentally new concepts, and it does not seem to have applications similar to the anti-maximum principle that we considered in Example 4.5. We shall therefore not discuss it in detail.

Note that in Definition 7.3(a) $\lambda_1 > \lambda_0$ can always be chosen independently of $f$, but with respect to $f$ in the unit ball, no uniform upper bound for
$$\{d_+ (R(\lambda, A)f) : \lambda \in (\lambda_0, \lambda_1]\}$$
is guaranteed. In contrast, (b) requires the existence of such a uniform bound.

Remark 7.4. Note that if $\lambda_0 \in \sigma(A)$ is a pole of the resolvent, then the growth condition (7.1) is fulfilled if and only if $\lambda_0$ is a simple pole.

To state our main theorem of this section it will be useful to introduce the notion of asymptotic positivity not only for resolvents of unbounded operators, but also for powers of bounded operators. We recall that a bounded operator $T$ is called power bounded if the family $(T^n)_{n \in \mathbb{N}_0}$ is bounded.

Definition 7.5. Let $T$ be a bounded linear operator on a complex Banach lattice $E$ with $r(T) > 0$ such that $\frac{T^n}{r(T)^n}$ is power bounded.

(a) We call $T$ **individually asymptotically positive** if $d_+ \left( \frac{T^n}{r(T)^n} f \right) \to 0$ as $n \to \infty$ for every $f \in E_+$.

(b) We call $T$ **uniformly asymptotically positive** if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}_0$ such that such that $d_+ \left( \frac{T^n}{r(T)^n} f \right) \leq \varepsilon\|f\|$ for every $n \geq n_0$ and every $f \in E_+$.

We can now state our main theorem on asymptotically positive resolvents.

Theorem 7.6. Let $A$ be a closed linear operator on a complex Banach lattice $E$ and suppose that $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ is a simple pole of $R(\cdot, A)$. Then the following assertions are equivalent.

(i) The spectral projection $P$ associated with $\lambda_0$ is positive, that is, $P \geq 0$.

(ii) The resolvent $R(\cdot, A)$ is individually asymptotically positive at $\lambda_0$. 

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(iii) The resolvent $R(\cdot, A)$ is uniformly asymptotically positive of bounded type at $\lambda_0$.

If $\lambda_0 = s(A)$, then the above assertions are also equivalent to:

(iv) There is a $\lambda > s(A)$ such that the operator $R(\lambda, A)$ is individually asymptotically positive.

(v) For each $\lambda > s(A)$ the operator $R(\lambda, A)$ is uniformly asymptotically positive.

A few remarks are in order. First, note that in contrast to Theorem 4.4 we now assume $\lambda_0$ to be a simple pole. Indeed, we cannot expect an asymptotically positive resolvent to have a simple pole in $\lambda_0$ automatically. To see this simply consider a two-dimensional Jordan block with eigenvalue 0. Second, note that we did not need any domination assumption such as $D(A) \subseteq E_u$. Hence the theorem is applicable in a wider range of situations. Third, the above theorem yields the desired equivalence between individual and uniform eventual behaviour which is not true for eventual (strong) positivity.

Proof of Theorem 7.6 We may assume that $\lambda_0 = 0$. We shall prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (i).

“(i) $\Rightarrow$ (iii)” Choose $\varepsilon > 0$ sufficiently small that the closed punctured disk of radius $\varepsilon$ around 0 is contained in $\rho(A)$. As 0 is a simple pole we have that

$$K := \sup_{\lambda \in (0, \varepsilon)} \|R(\lambda, A)\|_{\ker P} < \infty.$$ 

Moreover, $\lambda R(\lambda, A)P = P$ for all $\lambda \in (0, \varepsilon)$. Thus, using that $P \geq 0$, for every $f \geq 0$ and every $\lambda \in (0, \varepsilon)$

$$d_+(R(\lambda, A)f) \leq d_+(\lambda^{-1} Pf) + \|R(\lambda, A)(f - Pf)\| \leq 0 + K\|f - Pf\| \leq K\|I - P\||f|.$$

Clearly (iii) implies (ii). To see the implication “(ii) $\Rightarrow$ (i)” recall that because $\lambda_0$ is a simple pole of the resolvent we have $\lambda R(\lambda, A) \rightarrow P$ in $\mathcal{L}(E)$ as $\lambda \downarrow \lambda_0$. Hence for $f \geq 0$ we have $d_+(Pf) = \lim_{\lambda \downarrow \lambda_0} d_+(\lambda R(\lambda, A)f) = 0$ and so $P \geq 0$ as claimed.

From now on, assume that $s(A) = \lambda_0 = 0$.

“(i) $\Rightarrow$ (v)” Let $\lambda > 0$. It is sufficient to show that $\lambda R(\lambda, A)$ is uniformly asymptotically positive. Note that $r(\lambda R(\lambda, A)) = 1$. Moreover, as $s(A) = 0$ is a simple pole, $(\lambda R(\lambda, A))^n \rightarrow P$ in $\mathcal{L}(E)$ as $n \rightarrow \infty$, see Lemma 4.7 ii). Thus, given $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\|(\lambda R(\lambda, A))^n - P\| \leq \varepsilon$ for all $n \geq n_0$. Using that $P \geq 0$ we see that

$$d_+((\lambda R(\lambda, A))^n f) \leq \|(\lambda R(\lambda, A))^n - P\||f| + d_+(Pf) \leq \varepsilon\|f\|.$$ 

for all $f \in E_+$ and all $n \geq n_0$. Hence $\lambda R(\lambda, A)$ is uniformly asymptotically positive.

Clearly, (v) implies (iv). To show that (iv) implies (i), let $f \geq 0$ and observe that

$$d_+(Pf) = \lim_{n \rightarrow \infty} d_+((\lambda R(\lambda, A))^n f) = 0.$$ 

Hence, $P \geq 0$ as claimed. \hfill $\square$

In Proposition 3.1 results about strongly positive spectral projections are given. In the setting of asymptotic positivity, we are instead interested in projections which are merely positive. Hence, the following proposition and its corollary can sometimes be useful:
Proposition 7.7. Let $E$ be a (real or complex) Banach lattice and let $P$ be a projection. If $\text{im} \, P$ is one-dimensional and if $\text{im} \, P$ and $\text{im} \, P'$ contain positive non-zero vectors, then $P$ is positive.

Proof. Since $\text{im} \, P$ is one-dimensional, so is $\text{im} \, P'$; c.f. [37] Section III.6.6]. Let $0 < u \in \text{im} \, P$ and $0 < \varphi \in \text{im} \, P'$. We can find a vector $\psi \in E'$ such that $\langle \psi, u \rangle = 1$. Hence, we also have $\langle P'\psi, u \rangle = 1$. Since $\text{im} \, P'$ is one-dimensional, the vector $\varphi$ is a non-zero scalar multiple of $P'\psi$. Thus we have $\langle \varphi, u \rangle \neq 0$ and hence $\langle \varphi, u \rangle > 0$. After an appropriate rescaling of $u$ we may assume that $\langle \varphi, u \rangle = 1$. Since $\text{im} \, P$ is spanned by $u$, one now immediately computes that $Pf = \langle \varphi, f \rangle u$ for every $f \in E$. Hence, $P$ is positive.

Corollary 7.8. Let $E$ be a complex Banach lattice, let $A$ be a closed operator on $E$ and let $\lambda_0 \in \sigma(A)$ be a simple pole of the resolvent. Assume that $\ker(\lambda_0 I - A)$ is one-dimensional and contains a non-zero, positive vector and assume that $\ker(\lambda_0 I - A')$ contains a non-zero, positive functional. Then the spectral projection $P$ associated with $\lambda_0$ is positive.

Proof. Since $\lambda_0$ is a simple pole of the resolvent, $\text{im} \, P$ coincides with $\ker(\lambda_0 I - A)$ and is thus one-dimensional. The assertion now follows from Proposition 7.7.

Proposition 3.11 gives conditions under which $\lambda_0$ is a first-order pole. However, these assumptions are very strong if we are only interested in positive projections, and therefore the following proposition should be useful in situations where we do not know a priori whether or not $\lambda_0$ is a first-order pole.

Proposition 7.9. Let $E$ be a complex Banach lattice, let $A$ be a closed operator on $E$ and let $\lambda_0 \in \sigma(A)$ be a pole of the resolvent. Assume that $\ker(\lambda_0 I - A)$ is one-dimensional and that both $\ker(\lambda_0 I - A)$ and $\ker(\lambda_0 I - A')$ contain positive, non-zero vectors. Furthermore assume that at least one of the following two assumptions is fulfilled:

(a) $\ker(\lambda_0 I - A)$ contains a quasi-interior point of $E_+$.

(b) $\ker(\lambda_0 I - A')$ contains a strictly positive functional.

Then $\lambda_0$ is an algebraically simple eigenvalue of $A$ (in particular, a first-order pole of the resolvent $R(\cdot, A)$) and the corresponding spectral projection is positive.

Proof. We may assume that $\lambda_0 = 0$. Since 0 is a geometrically simple eigenvalue of $A$ by assumption, we only have to prove that it is a first-order pole of the resolvent in order to obtain that it is algebraically simple. We assume for a contradiction that $\lambda_0$ is not a first-order pole, i.e. there is an element $f \in \ker(A^2) \setminus \ker A$.

Let $v \in \ker A$ and $\varphi \in \ker A'$. Since $Af \in \ker A \setminus \{0\}$ and $\ker A$ is one-dimensional, we have $\alpha Af = v$ for some $\alpha \in \mathbb{C}$. We thus have

$$\langle \varphi, v \rangle = \alpha \langle \varphi, Af \rangle = 0.$$ 

for all $v \in \ker A$ and all $\varphi \in \ker A'$.

(a) Now assume that (a) is fulfilled. Then there exists a functional $0 < \varphi \in \ker A'$ and a quasi-interior point $v \in E_+$ which is contained in $\ker A$. For such elements we cannot have $\langle \varphi, v \rangle = 0$, so we have arrived at a contradiction.

(b) If (b) is true, then there is a vector $0 < v \in \ker A$ and a strictly positive functional $\varphi \in \ker A'$. Again we cannot have $\langle \varphi, v \rangle = 0$, and thus we obtain a contradiction.

We have proved that $\lambda_0$ is an algebraically simple eigenvalue of $A$. Corollary 7.8 now implies that the corresponding spectral projection is positive.
8 Asymptotically positive semigroups

In this section we characterise asymptotically positive semigroups. We begin with the major definitions.

Definition 8.1. Let \((e^{tA})_{t \geq 0}\) be a \(C_0\)-semigroup on a complex Banach lattice \(E\) with \(s(A) > -\infty\) and assume that \((e^{t(A-s(A))})_{t \geq 0}\) is bounded.

(a) The semigroup \((e^{tA})_{t \geq 0}\) is called individually asymptotically positive if for every \(f \geq 0\) we have \(d_+ \left( e^{t(A-s(A))} f \right) \to 0\) as \(t \to \infty\).

(b) The semigroup \((e^{tA})_{t \geq 0}\) is called uniformly asymptotically positive if for every \(\varepsilon > 0\) there is a \(t_0 \geq 0\) such that \(d_+ \left( e^{t(A-s(A))} f \right) \leq \varepsilon \|f\|\) for all \(t \geq t_0\) and all \(f \in E_+\).

Before proceeding, let us first note the following simple density condition for individual asymptotic positivity. Its proof is a simple \(2\varepsilon\)-argument.

Proposition 8.2. Let \((e^{tA})_{t \geq 0}\) be a \(C_0\)-semigroup on a complex Banach lattice \(E\) and suppose that \((e^{t(A-s(A))})_{t \geq 0}\) is bounded. Suppose that \(D \subseteq E_+\) is dense in \(E_+\) and that \(d_+ \left( e^{t(A-s(A))} g \right) \to 0\) as \(t \to \infty\) for all \(g \in D\). Then \((e^{tA})_{t \geq 0}\) is individually asymptotically positive.

We now state our main theorem which characterises asymptotic positivity. In contrast to Theorem 7.6 on resolvents we have to be a bit more careful here concerning the equivalence between the statements on individual and uniform asymptotic positivity.

Theorem 8.3. Let \((e^{tA})_{t \geq 0}\) be a \(C_0\)-semigroup on a complex Banach lattice \(E\), \(s(A) > -\infty\) and suppose that \((e^{t(A-s(A))})_{t \geq 0}\) is bounded. Assume furthermore that \(\sigma_{\text{per}}(A)\) is non-empty and finite and consists of poles of the resolvent. Then the following assertions are equivalent:

(i) \(s(A)\) is a dominant spectral value of \(A\) and the associated spectral projection \(P\) is positive.

(ii) The semigroup \((e^{tA})_{t \geq 0}\) is individually asymptotically positive.

(iii) The operators \(e^{t(A-s(A))}\) converge strongly to a positive operator \(Q\) as \(t \to \infty\).

If assertions (i)-(iii) are fulfilled, then \(P = Q\). If \((e^{t(A-s(A))})_{t \geq 0}\) is uniformly exponentially stable on the spectral space associated with \(\sigma(A) \setminus \sigma_{\text{per}}(A)\), then (i)-(iii) are equivalent to

(iv) The semigroup \((e^{tA})_{t \geq 0}\) is uniformly asymptotically positive.

Note that the additional assumption that \((e^{t(A-s(A))})_{t \geq 0}\) be stable on the spectral space associated with \(\sigma(A) \setminus \sigma_{\text{per}}(A)\) is for example fulfilled if the semigroup \((e^{tA})_{t \geq 0}\) is eventually norm continuous. Also note that if \((e^{tA})_{t \geq 0}\) is eventually norm continuous, then \(\sigma_{\text{per}}(A)\) is automatically non-empty. Moreover, if we suppose that \((e^{tA})_{t \geq 0}\) is eventually norm continuous and that \(\sigma_{\text{per}}(A)\) is finite and consists of poles of the resolvent, then the boundedness of \((e^{t(A-s(A))})_{t \geq 0}\) is equivalent to the assertion that \(\sigma_{\text{per}}(A)\) consists of simple poles of the resolvent.

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Proof of Theorem 8.3. We may assume that \( s(A) = 0 \).

“(i) \(\Rightarrow\) (ii)” Since the semigroup is bounded and since \( s(A) \) is a dominant spectral value, it follows from [2, Theorem 2.4] that the semigroup converges strongly to 0 on \( \ker P \). Moreover, 0 must be a simple pole of \( R(\cdot, A) \) due to the boundedness of the semigroup. Hence, \( e^{tA}|_{\text{im } P} = I_{\text{im } P} \) for all \( t \geq 0 \). We thus conclude that for every \( f \geq 0 \)

\[
d_+(e^{tA}f) \leq d_+(Pf) + \|e^{tA}(f - Pf)\| = \|e^{tA}(f - Pf)\| \to 0
\]
as \( t \to \infty \). Hence, the semigroup is individually asymptotically positive.

“(ii) \(\Rightarrow\) (i)” Let \( P_{\text{per}} \) be the spectral projection associated with \( \sigma_{\text{per}}(A) \). Note that \( \sigma_{\text{per}}(A) \) consists of simple poles of \( R(\cdot, A) \) since the semigroup is bounded. Hence, by virtue of [18, Proposition 2.3], we can find a sequence \( 0 \leq t_n \to \infty \) such that \( e^{t_nA}P_{\text{per}}f \to P_{\text{per}}f \) for all \( f \in E \). For every \( f \in E_+ \) and every \( t \geq 0 \) we have

\[
d_+(e^{tA}P_{\text{per}}f) = \lim_{n \to \infty} d_+ (e^{(t+t_n)A}P_{\text{per}}f) \leq \lim_{n \to \infty} d_+ (e^{(t+t_n)A}f) + \lim_{n \to \infty} \|e^{(t+t_n)A}(P_{\text{per}}f - f)\| = 0,
\]

where the last limit is 0 because the semigroup converges strongly to 0 on \( \ker P_{\text{per}} \) (this follows from [2, Theorem 2.4]). Hence \( e^{tA}P_{\text{per}} \geq 0 \) for all \( t \geq 0 \). In particular \( P_{\text{per}} = e^{0A}P_{\text{per}} \geq 0 \). Thus, \( \text{im } P_{\text{per}} \) is a complex Banach lattice with respect to an appropriate equivalent norm, see [16, Proposition III.11.5]. Moreover, we have shown that the \( C_0 \)-semigroup \( (e^{tA}|_{\ker P_{\text{per}}})_{t \geq 0} \) is positive. As \( \sigma_{\text{per}}(A|_{\ker P_{\text{per}}}) \neq \emptyset \) we conclude that \( s(A) = s(A|_{\ker P_{\text{per}}}) \subset \sigma(A|_{\ker P_{\text{per}}}) \subset \sigma(A) \). Moreover, \( \sigma_{\text{per}}(A|_{\ker P_{\text{per}}}) = \sigma_{\text{per}}(A) \) is imaginary additively cyclic; see [3, Definition B-III.2.5, Proposition C-III.2.9 and Theorem C-III.2.10]. Since \( \sigma_{\text{per}}(A) \) is finite, it follows that \( \sigma_{\text{per}}(A) = \{0\} \). This in turn implies that \( P = P_{\text{per}} \geq 0 \).

“(i) \(\Rightarrow\) (iii)” If (i) is true, then according to [2, Theorem 2.4], \( e^{tA} \to 0 \) strongly on \( \ker P \) as \( t \to \infty \). Since \( e^{tA}|_{\text{im } P} = I_{\text{im } P} \) we have \( e^{tA} \to P \geq 0 \) strongly. In particular (iii) holds with \( Q = P \).

“(iii) \(\Rightarrow\) (i)” If \( s(A) \) is not a spectral value or not dominant, then there is an eigenvalue \( \lambda \in i\mathbb{R} \setminus \{0\} \). Hence, \( e^{tA} \) does not converge strongly as \( t \to \infty \). Thus (iii) implies that \( s(A) \) must be a dominant spectral value. Then, however, \( e^{tA} \) converges strongly to \( P \) as \( t \to \infty \), which in turn implies \( P = Q \geq 0 \).

Clearly, (iv) implies (ii). Now, assume that \( (e^{tA})_{t \geq 0} \) is uniformly exponentially stable on \( \ker P \) and (i) is true. If \( \varepsilon > 0 \), then we can find a \( t_0 \geq 0 \) such that \( \|e^{tA}|_{\ker P} - P\| \leq \varepsilon \) for all \( t \geq t_0 \). This implies that

\[
d_+(e^{tA}f) \leq d_+(Pf) + \|e^{tA}(f - Pf)\| \leq \varepsilon \|f - Pf\| \leq \varepsilon \|I - P\| \|f\|
\]

for all \( f \in E_+ \) and all \( t \geq t_0 \) and so \( (e^{tA})_{t \geq 0} \) is uniformly asymptotically positive. \( \square \)

We shall now give several (counter-) examples regarding the assumptions and conditions in Theorem 8.3.

Examples 8.4. (a) The assumption that \( \sigma_{\text{per}}(A) \) be non-empty is essential in Theorem 8.3. Indeed, let \( B \in \mathbb{R}^{2 \times 2} \) be a matrix with \( \sigma(B) = \{-i, i\} \) and define \( A_n = nB - \frac{1}{n}I \) for every \( n \in \mathbb{N} \). If we endow \( \mathbb{C}^2 \) with the Euclidean norm and let \( E = l^2(\mathbb{N}; \mathbb{C}^2) \), then \( E \) is a complex Banach lattice. Let \( A \) be the matrix multiplication operator on \( E \) with symbol \( (A_n)_{n \in \mathbb{N}} \), that is,

\[
D(A) := \{(x_n) \in E : (A_n x_n) \in E\}, \quad A(x_n) := (A_n x_n).
\]

(8.1)
Then $A$ generates a bounded $C_0$-semigroup on $E$ and $\sigma(A) = \{\pm ni - \frac{1}{n} : n \in \mathbb{N}\}$. In particular, $\sigma_{\text{per}}(A) = \emptyset$, which implies that $e^{tA} \to 0$ strongly as $t \to \infty$ (see [24, Theorem 2.4]). Since $s(A) = 0$, this implies that $(e^{tA})_{t \geq 0}$ is in particular trivially individually asymptotically positive, even though $s(A)$ is not a spectral value of $A$.

(b) If the semigroup $(e^{tA})_{t \geq 0}$ is not uniformly exponentially stable on the spectral space associated with $\sigma(A) \setminus \sigma_{\text{per}}(A)$ then assertions (i)–(iii) of Theorem [8.3] do not imply (iv) in general. To see this, let $B \in \mathbb{R}^{3 \times 3}$ be such that $e^{tB}$ is the rotation of angle $t$ about the line in the direction of the vector $(1, 1, 1)$. Let $Q$ be the projection along $(1, 1, 1)$ onto its orthogonal complement and define $A_n = nB - \frac{1}{n}Q$ for every $n \in \mathbb{N}$. Endow $\mathbb{C}^3$ with the Euclidean norm and consider the complex Banach lattice $E = l^2(\mathbb{N}; \mathbb{C}^3)$. If $A$ is the matrix multiplication operator on $E$ with symbol $(A_n)$ analogous to (8.1), then $A$ generates a bounded $C_0$-semigroup on $E$ which is individually but not uniformly asymptotically positive. Moreover, 

$$\sigma(A) = \{0\} \cup \left\{ \pm ni - \frac{1}{n} : n \in \mathbb{N} \right\},$$

so that $\sigma_{\text{per}}(A) = \{0\}$, where $0$ is a simple pole of the resolvent. Thus, all assumptions of Theorem [8.3] are fulfilled, but assertions (i) and (iv) are not equivalent.

(c) The assumption that the peripheral spectrum be isolated cannot be omitted in Theorem [8.3]. Indeed, let $B$ and $Q$ be as in (b), but this time, define $A_n = B - \frac{1}{n}Q$. As above, denote by $A$ the matrix multiplication operator with symbol $(A_n)$ on $E = l^2(\mathbb{N}; \mathbb{C}^3)$. Then $A$ generates a bounded $C_0$-semigroup which is easily seen to be asymptotically eventually positive (though not uniformly asymptotically positive). However, $\sigma(A) = \{0, \pm i\} \cup \{\pm i - 1/n : n \in \mathbb{N}\}$, so $s(A) = 0$ is not a dominant spectral value.

Remark 8.5. In Example [8.4(b)] the semigroup $(e^{tA})_{t \geq 0}$ is not uniformly asymptotically positive. However, according to Theorem [8.3] the spectral projection $P$ associated with $0$ is positive. Because the assumptions of Theorem [7.6] are fulfilled the resolvent $R(\cdot, A)$ is uniformly asymptotically positive at $s(A)$. In particular, the resolvent of a generator can be uniformly asymptotically positive at $s(A)$ even if the semigroup is only individually asymptotically positive.

9 Applications of asymptotic positivity

In this penultimate section we shall give some applications of our results on asymptotic positivity. We begin with an analysis of the finite-dimensional case. Then we revisit the bi-Laplacian with Dirichlet boundary conditions and formulate a result on asymptotic positivity of the resolvent which in some manner complements Proposition [6.5]. We again consider the special case of self-adjoint operators on Hilbert spaces, with an application to the Dirichlet-to-Neumann operator on $L^2(\partial \Omega)$, as well as a transport process on a metric graph and a one-dimensional delay differential equation.

The finite-dimensional case We consider the special case of matrices $A \in \mathbb{C}^{n \times n}$ and characterise when the matrix exponential $(e^{tA})_{t \geq 0}$ is asymptotically positive.

A characterisation of eventual strong positivity of matrix semigroups was first given in [22, Theorem 3.3], and later in [18, Theorem 6.1] as an application of the general $C(K)$-theory. By characterising asymptotically positive matrix semigroups, Theorem [4.1] below adds new aspects to the finite-dimensional theory. Moreover, since the matrix $A$
in Theorem 9.1 is not required to be real, the theorem also contributes to the Perron–Frobenius theory of matrices with entries in \( \mathbb{C} \), a topic which was the focus of \[43\]. We also refer to \[45\], where generalisations of Perron–Frobenius theory to complex matrices are approached from a rather different perspective.

It is evident that in finite dimensions individual and uniform asymptotic positivity are equivalent. Hence we merely speak of “asymptotic positivity”.

**Theorem 9.1.** Let \( A \in \mathbb{C}^{n \times n} \) and assume that \((e^{t(A-s(A))})_{t \geq 0}\) is bounded or equivalently that all \( \lambda \in \sigma_{\text{per}}(A) \) are simple poles of \( R(\cdot, A) \). Then the following assertions are equivalent.

(i) \((e^{tA})_{t \geq 0}\) is asymptotically positive.

(ii) There is a number \( c \in \mathbb{R} \) such that \( A + c I \) has positive spectral radius, \( \frac{A+cI}{r(A+cI)} \) is power bounded and \( A + c I \) is asymptotically positive.

**Proof.** “(i) \( \Rightarrow \) (ii)” It follows from Theorem \[23\] that \( s(A) \) is a dominant spectral value and that the associated spectral projection \( P \) is positive. Now, choose \( c > 0 \) sufficiently large such that \( s(A) + c I > 0 \) is larger than the modulus of any other spectral value of \( A + c I \). Then in particular \( r := r(A + c I) = s(A) + c \). The spectral projection of \( \frac{A+cI}{r} \) associated with the spectral value 1 is \( P \), and \( \text{im} P \) coincides with the fixed space of \( \frac{A+cI}{r} \).

Thus, we clearly have

\[
\left( A + c I \right)^n \to P \geq 0 \quad \text{as } n \to \infty,
\]

which implies that \( \frac{A+cI}{r} \) is power bounded and that \( A + c I \) is asymptotically positive.

“(ii) \( \Rightarrow \) (i)” First, let \( T \in \mathbb{C}^{n \times n} \), \( r(T) > 0 \), such that \( \frac{T}{r(T)} \) is power bounded and such that \( T \) is asymptotically positive. Let \( Q \) be the spectral projection associated with \( \sigma(T) \cap r(T) \mathbb{T} \), where \( \mathbb{T} \) denotes the unit circle in \( \mathbb{C} \). Since \( \frac{T}{r(T)} \) is power bounded, all eigenvalues on the circle \( r(T) \mathbb{T} \) are simple poles of the resolvent, so the image of \( Q \) is spanned by eigenvectors of \( T \) belonging to eigenvalues of modulus \( r(T) \).

We can find a sequence \( n_k \to \infty \) of positive integers such that \( \left( \lambda r(T)^{-1} \right)^{n_k} \to 1 \) as \( k \to \infty \) for each \( \lambda \in r(T) \mathbb{T} \); this follows from the same argument that was used in the proof of \[18\, \text{Proposition 2.3}\]. This implies that

\[
\left( \frac{T}{r(T)} \right)^{n_k} \to Q \quad \text{as } k \to \infty,
\]

which in turn shows that \( Q \geq 0 \). Hence, \( \text{im} Q \) is a (finite-dimensional) complex Banach lattice when equipped with an appropriate norm, see \[46\, \text{Proposition III.11.5}\]. Moreover, for each \( 0 \leq f \in \text{im} Q \), we have

\[
\frac{T}{r(T)} f = \lim_{k \to \infty} \left( \frac{T}{r(T)} \right)^{n_k+1} f \geq 0,
\]

so \( T|_{\text{im} Q} \geq 0 \). This implies that the spectral radius \( r(T|_{\text{im} Q}) = r(T) \) is contained in \( \sigma(T|_{\text{im} Q}) \) and hence in \( \sigma(T) \).

Next we show that the spectral projection \( P \) associated with \( r(T) \) is positive. Using
the Neumann series expansion \( R(\lambda, T) = \sum_{k=0}^{\infty} \lambda^{-k+1} T^k \) valid for \( |\lambda| > r(T) \) we have

\[
d_+(Pf) = d_+ \left( \lim_{\lambda \downarrow r(T)} (\lambda - r(T)) R(\lambda, T)f \right) \leq \limsup_{\lambda \downarrow r(T)} \left( (\lambda - r(T)) \sum_{n=0}^{\infty} \frac{d_+(T^nf)}{\lambda^{n+1}} \right)
\]

\[
= \limsup_{\lambda \downarrow r(T)} \left( \frac{\lambda}{r(T)} - 1 \right) \sum_{n=0}^{\infty} \frac{d_+ \left( \left( \frac{T}{r(T)} \right)^n f \right)}{\left( \frac{\lambda}{r(T)} \right)^{n+1}} = 0
\]

for all \( f \geq 0 \), where we have used that \( d_+ \left( \left( \frac{T}{r(T)} \right)^n f \right) \to 0 \) as \( n \to \infty \). Hence, \( P \geq 0 \).

Finally assume that \( A + cI \) fulfills condition (ii). Then by what we have just shown \( r(A + cI) \in \sigma(A + cI) \), and hence \( r(A + cI) = s(A + cI) \) is a dominant spectral value of \( A + cI \). Moreover, the associated spectral projection \( P \) is positive. Hence, \( s(A) \) is a dominant spectral value of \( A \) and since the associated spectral projection is still \( P \), we conclude from Theorem \([8,3]\) that \( (e^{tA})_{t \geq 0} \) is asymptotically positive.

In \([18, Proposition 6.2]\) we proved that real, eventually positive semigroups in two dimensions are automatically positive. This is also true for asymptotically positive semigroups.

**Proposition 9.2.** Let \( A \in \mathbb{R}^{2 \times 2} \) such that \( (e^{t(A-s(A))})_{t \geq 0} \) is bounded. If the semigroup \( (e^{tA})_{t \geq 0} \) is asymptotically positive, then it is positive.

**Proof.** We may assume \( s(A) = 0 \). From Theorem \([8,3]\) we know that \( s(A) = 0 \) is a dominant spectral value of \( A \). The boundedness of \( (e^{t(A-s(A))})_{t \geq 0} \) implies that \( s(A) \) is a simple pole of the resolvent and hence the algebraic and geometric multiplicities coincide. If the multiplicity of 0 is two, then \( A = 0 \) and the semigroup is positive.

Now assume that \( A \) has two distinct simple eigenvalues. Then \( A \) has a real eigenvalue \( \lambda < 0 = s(A) \). If \( P \) is the spectral projection associated with 0, then \( P \geq 0 \) by Theorem \([8,3]\) and hence for every \( t \geq 0 \)

\[
e^{tA} = P + e^M(I-P) = e^M I + (1 - e^M) P \geq 0
\]

as \( e^\lambda \leq 1 \).

The reader should note that the assertion of the above proposition fails if \( A \) is allowed to be a complex matrix; for example the semigroup generated by the matrix

\[
A = \begin{bmatrix}
0 & 0 \\
0 & -1 + i
\end{bmatrix}
\]

is asymptotically positive, but not positive.

**The resolvent of the bi-Laplace operator with Dirichlet boundary conditions**

Here we consider the same operator \( A_p \ (1 < p < \infty) \) as in the second paragraph of Section \([6]\) and we use the properties of \( A_p \) given in Proposition \([6,2]\). The eventual positivity of the resolvent \( R(\cdot, A_p) \) was analysed in Proposition \([6,5]\). The disadvantage there is that, in contrast to the semigroup, our results on eventual strong positivity only apply to large \( p \) and/or small dimensions \( n \). We now show that the resolvent is at least asymptotically positive for all \( p \in (1, \infty) \), independent of the dimension.
Proposition 9.3. Let $p \in (1, \infty)$ and let $\Omega \in C^\infty$ be such that the conclusion of Theorem 6.3 holds. Then the resolvent $R(\cdot, A_p)$ is uniformly asymptotically positive at $s(A)$.

Proof. According to Lemma 6.4, $s(A_p)$ is a simple pole of the resolvent $R(\cdot, A_p)$ and a dominant spectral value of $A_p$; moreover, the corresponding spectral projection is positive. The assertion therefore follows from Theorem 7.6. \hfill \Box

Asymptotic positivity for self-adjoint operators on Hilbert lattices and the Dirichlet-to-Neumann operator In this section we again consider self-adjoint operators $A$ on a Hilbert lattice, c.f. the corresponding paragraph in Section 6. In Theorem 6.7 we provided a characterisation of eventual strong positivity under the assumption that $D(A) \subseteq E_u$ for some $u \gg 0$. If we do not assume this domination property, we are still able give a sufficient condition for the asymptotic positivity of the resolvent and the semigroup.

Theorem 9.4. Let $H$ be a complex Hilbert lattice and let $A$ be a densely defined, self-adjoint operator on $H$ such that $s(A) \in \mathbb{R}$ is an isolated point of the spectrum of $A$. Moreover, assume that the eigenspace $\ker(s(A)I - A)$ is one-dimensional and contains a non-zero positive vector. Then the resolvent $R(\cdot, A)$ is uniformly asymptotically positive at $s(A)$ and the semigroup $(e^{tA})_{t \geq 0}$ is uniformly asymptotically positive.

Proof. Using the same argument as at the beginning of the proof of Theorem 6.7, we deduce that the Banach space adjoint of $A$ has a positive functional as an eigenvector for the eigenvalue $s(A)$. Since $A$ is self-adjoint, $s(A)$ is a simple pole of the resolvent and therefore Corollary 7.8 implies that the spectral projection $P$ associated with $s(A)$ is positive. Hence, $R(\cdot, A)$ is uniformly asymptotically positive at $s(A)$ by Theorem 7.6. Because $s(A)$ is a dominant spectral value and since $(e^{tA})_{t \geq 0}$ is analytic, the semigroup $(e^{t(A-s(A)I)})_{t \geq 0}$ is bounded; since by assumption $s(A)$ is isolated (and $\sigma(A) \subseteq \mathbb{R}$), $(e^{t(A-s(A)I)})_{t \geq 0}$ is even uniformly exponentially stable on $\ker P$. Hence, $(e^{tA})_{t \geq 0}$ is uniformly asymptotically positive by Theorem 8.3. \hfill \Box

Of course, similar assertions as in the above theorem also hold for the eventual positivity of the resolvent at other spectral values than $s(A)$.

Remarks 9.5. (a) Suppose $A$ is a densely defined, self-adjoint operator on a Hilbert lattice such that $s(A)$ is isolated in $\sigma(A)$. Then Theorem 8.3 implies that a necessary condition for the asymptotic positivity (uniform or individual) of $(e^{tA})_{t \geq 0}$ is that the spectral projection $P$ associated with $s(A)$ is positive. Simple examples show that we cannot in general say more about $P$ without further assumptions on $A$; if $A$ for example has compact resolvent, we at least obtain the existence of a positive eigenvector associated with $s(A)$ as a necessary condition.

(b) One might wonder whether under the assumptions of the above theorem, the semigroup $(e^{tA})_{t \geq 0}$ is individually eventually positive. If we set $p = 2$ in Example 5.4 we can see that this is not in general the case without a domination or smoothing assumption.

Example 9.6. We recall the Dirichlet-to-Neumann operator $D_\lambda$ from Section 6. Here we may assume that $\Omega \subset \mathbb{R}^n$ is a general bounded domain with sufficiently smooth boundary; as mentioned earlier, $D_\lambda$ is a densely defined, self-adjoint operator on $L^2(\partial \Omega)$ with compact resolvent. It follows directly from the definition that the eigenspace associated with $s(-D_\lambda)$ is given by the finite-dimensional span in $L^2(\partial \Omega)$ of the traces of all.
eigenfunctions of the Laplacian associated with the Robin problem
\[ \Delta f = \lambda f \quad \text{in } \Omega, \quad \frac{\partial}{\partial \nu} f = s(-D\lambda)f \quad \text{on } \partial \Omega. \]  
(9.1)

By Theorem 9.4 we have that \((e^{-tD\lambda})_{t \geq 0}\) is uniformly asymptotically positive if there is a solution of (9.1) which is unique up to scalar multiples and which has non-zero positive trace on \(\partial \Omega\). Conversely, a necessary condition for the asymptotic positivity of \((e^{-tD\lambda})_{t \geq 0}\) is the existence of (at least one) solution of (9.1) with positive trace.

A network flow with non-positive mass diversion  
Consider a directed graph with \(n\) edges \(e_k\) of length \(l_k\), \(k = 1, \ldots, n\), and suppose that we are given a mass distribution on every edge. Further, assume that a transport process shifts the mass along the edges with a given velocity. Whenever some mass arrives at a vertex, it is diverted to the outgoing edges of this vertex according to some pre-defined weights. Such a transport process is often called a network flow and it can be described by means of a \(C_0\)-semigroup on the space \(\bigoplus_{k=1}^n L^1([0,l_k])\).

During the last decade a deep and extensive theory of network flow semigroups has been developed which deals, among other topics, with the long time behaviour of the flow and relates it to properties of the underlying graph; see e.g. [33, 26, 19]. However, it seems that so far only positive weights for the mass diversion in the vertices have been considered. In this section we want to demonstrate by means of an example that it is possible to consider non-positive mass diversion and that in such a situation asymptotic positivity can occur. It is however not our intention to develop a general theory here.

We consider a directed graph as shown in (9.2). It consists of two vertices \(v_{-1}, v_0\), an edge \(e_1\) of length 1 directed from \(v_{-1}\) to \(v_0\), a “looping” edge \(e_2\) of length 1 going from \(v_0\) to itself and another “looping” edge \(e_3\) of length \(l\), again going from \(v_0\) to itself. We assume \(l > 0\) to be an irrational number.

\[ (9.2) \]

We assume that the mass is shifted along the edges with constant velocity 1, and that the mass diversion in the vertices is as follows: Since \(v_{-1}\) has no incoming edges, no mass arrives at \(v_{-1}\) and hence no mass is inserted to \(e_1\) from \(v_{-1}\). Two thirds of the mass arriving at \(v_0\) from \(e_1\) is diverted to \(e_3\); the other third is diverted to \(e_2\), but with a flipped sign. One half of the mass arriving at \(v_0\) from \(e_2\) is diverted to \(e_2\) itself, the other half is diverted to \(e_3\). Similarly, one half of the mass arriving at \(v_0\) from \(e_3\) is diverted to \(e_2\) and the other half is diverted to \(e_3\) itself.

Note that the mass diversion in \(v_0\) contains a somewhat finer structure than is usually considered in the literature: in most other models, all the incoming mass at a vertex is summed up, and then the entire mass is distributed to the outgoing edges according to certain weights. In our model, however, the diversion of mass in \(v_0\) depends on the edge it arrives from.

We model the mass distribution by a function \(f = (f_1, f_2, f_3) \in L^1([0,1]) \oplus L^1([0,1]) \oplus L^1([0,l]) =: E\), where \(f_k\) describes the mass distribution on the edge \(e_k\); for each \(k = 1, 2, 3\) the number 0 in the interval \([0,1]\) (or \([0,l]\), respectively) shall denote the starting point
of the edge $e_k$. The time evolution of our network flow can be described by the abstract Cauchy problem $df/dt = Af$ where the operator $A$ on $E$ is given by

$$D(A) = \left\{ (f_1, f_2, f_3) \in W^{1,1}([0,1]) \oplus W^{1,1}([0,1]) \oplus W^{1,1}([0,l]) : \right. \right.$$

$$f_2(0) = \frac{1}{2} f_2(1) + \frac{1}{2} f_3(1) - \frac{1}{3} f_1(1),$$

$$f_3(0) = \frac{1}{2} f_2(1) + \frac{1}{2} f_3(1) + \frac{2}{3} f_1(1), \quad f_1(0) = 0 \left. \right\},$$

$$A(f_1, f_2, f_3) = -(f_1', f_2', f_3').$$

Now we can prove the following properties of the abstract Cauchy problem associated with $A$:

**Theorem 9.7.** The operator $A$ defined above is closed, densely defined and has the following properties:

(i) Any complex number $\lambda$ is an eigenvalue of $A$ if and only if it is a spectral value of $A$ if and only if the matrix

$$S(\lambda) := \begin{bmatrix} e^{-\lambda} - 2 & e^{-\lambda} \\ e^{-\lambda} & e^{-\lambda} - 2 \end{bmatrix}$$

is singular. Moreover, for each eigenvalue $\lambda$ of $A$ the corresponding eigenspace is one-dimensional.

(ii) $A$ is dissipative and generates a contractive $C_0$-semigroup on $E$.

(iii) $A$ has compact resolvent.

(iv) $(0, 1_{[0,1]}, 1_{[0,l]}) \in \ker(A)$ and, moreover, $(\frac{1}{2} 1_{[0,1]}, 1_{[0,1]}, 1_{[0,l]}) \in \ker(A')$.

(v) $s(A)$ equals 0 and is a dominant spectral value of $A$.

(vi) The semigroup $(e^{tA})_{t \geq 0}$ is individually asymptotically positive, but not positive.

**Proof.** Obviously, $A$ is closed and densely defined.

(i) Let $\lambda$ be a complex number. A straightforward computation shows that there exists a non-trivial function $f \in \ker(\lambda I - A)$ if and only if $S(\lambda)$ is singular. Moreover, if such a function $f$ exists, then the same computation shows that $f$ is unique up to scalar multiples, so $\ker(\lambda I - A)$ is one-dimensional. Finally, another simple (but somewhat lengthy) computation shows that $\lambda I - A$ is surjective if $S(\lambda)$ is not singular. This proves (i).

(ii) Using the boundary condition satisfied by functions in $D(A)$ it is easy to check that $A$ is indeed dissipative. Since $\det S(\lambda)$ is an entire function which is not identically 0, $S(\lambda)$ must be regular for some $\lambda > 0$; hence $\lambda \in \rho(A)$ for some $\lambda > 0$ and we conclude that $A$ generates a contractive $C_0$-semigroup on $E$.

(iii) We have seen in (ii) (or we can conclude immediately from (i)) that $\rho(A) \neq \emptyset$. Since $D(A)$ compactly embeds into $E$, it follows that the resolvent is compact.

(iv) The first assertion is obvious and the second assertion can easily be checked by using the definition of the adjoint.

(v) Since $A$ is dissipative and has non-empty resolvent set, no spectral value of $A$ can have strictly positive real part (alternatively, we could also conclude this from (i)). Since
0 ∈ σ(A) according to (iv), we have indeed s(A) = 0. Now assume for a contradiction that \( iβ \) (\( 0 \neq β \in \mathbb{R} \)) is another spectral value of \( A \) on the imaginary axis. Then it follows from (ii) that

\[
0 = \det S(iβ) = -2(e^{-iβ} + e^{-iβl}) + 4.
\]

Taking the real part of the above equation we obtain \( \cos(β) + \cos(βl) = 0 \) and hence \( \beta \in \pi/2 + \mathbb{Z} \) and \( βl \in \pi(1/2 + \mathbb{Z}) \) which is a contradiction since \( l \) is irrational.

(vii) Due to (iii) \( s(A) = 0 \) is a pole of the resolvent and since the semigroup \( (e^{tA})_{t \geq 0} \) is bounded, 0 is even a first order pole of the resolvent. Hence, Corollary 7.8 together with assertion (iv) implies that the associated spectral projection \( P \) is positive. As \( s(A) \) is a dominant spectral value according to (v), individual asymptotic positivity of the semigroup follows from Theorem 8.3.

That the semigroup is not positive is obvious if we consider a positive initial mass distribution \( f \) which lives only on the first edge: after some time some of the mass of \( f \) is diverted with a negative sign to the second edge, and when this first happens, there is no mass close the end of \( e_2 \) and \( e_3 \) which could compensate those negative values.

A delay differential equation In [18, Section 6.5] a delay differential equation was considered as an example for eventual strong positivity on a \( C(K) \)-space. Here, we want to consider another delay differential equation whose solution semigroup is only asymptotically positive.

Consider a time-dependent complex value \( y(t) \) whose time evolution is governed by the equation

\[
\dot{y}(t) = y(t - 2) - y(t - 1).
\]

As shown in [22, Section VI.6], this equation can be rewritten as an abstract Cauchy problem \( \dot{u} = Au \) in the space \( E = C([-2, 0]) \), where the operator \( A \) is given by

\[
D(A) = \{ f \in C^1([-2, 0]) : f'(0) = f(-2) - f(-1) \}
\]

\[
Af = f'
\]

(one has to set \( r = 2, Y = \mathbb{C}, B = 0 \) and \( \Phi(f) = f(-2) - f(-1) \) in [22, Section VI.6] to obtain our example). There it is also shown that the operator \( A \) generates a \( C_0 \)-semigroup on \( E \). We are now going to prove the following theorem on this semigroup.

**Theorem 9.8.** Let the operator \( A \) on \( E = C([-2, 0]) \) be given by (9.4). Then the operator \( A \) and the \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) on \( E \) have the following properties:

(i) \( A \) has compact resolvent and the spectral bound \( s(A) \) equals 0 and is a dominant spectral value.

(ii) \( s(A) \) is an algebraically simple eigenvalue of \( A \) and the associated spectral projection \( P \) is positive.

(iii) \( (e^{tA})_{t \geq 0} \) is uniformly asymptotically positive, but neither positive nor individually eventually strongly positive with respect to any quasi-interior point of \( E_+ \).
Proof. (i) By the Arzelà–Ascoli Theorem the embedding $D(A) \hookrightarrow E$ is compact, and since $A$ has non-empty resolvent set its resolvent is compact. In particular, $\lambda \in \sigma(A)$ if and only if $\lambda$ is an eigenvalue of $A$. A short computation shows that this is the case if and only if
\[ \lambda = e^{-2\lambda} - e^{-\lambda}. \tag{9.5} \]
(alternatively, this follows from [22, Proposition VI.6.7]). Obviously, $\lambda = 0$ is a solution of (9.3), so we have to show that (9.5) has no other solution with non-negative real part. If $\lambda \neq 0$ and if we set $z = \lambda/2$, then (9.5) is equivalent to
\[ e^{3z} = -\frac{\sinh z}{z}. \tag{9.6} \]
It is easy to see that (9.6) does not have a solution on $i\mathbb{R} \setminus \{0\}$. We now show that (9.6) does not have a solution $z = \alpha + i\beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$ either. A short calculation using that $\sin^2 \beta \leq \beta^2$ and $\sinh^2 \alpha \geq \alpha^2$ shows that
\[ \frac{\sinh z}{z} = \frac{\sinh^2 \alpha + \sin^2 \beta}{\alpha^2 + \beta^2} \leq \frac{\sinh^2 \alpha + \alpha^2}{\alpha^2 + \beta^2} \leq \frac{\sinh^2 \alpha - \alpha^2}{\alpha^2} + 1 = \frac{\sinh^2 \alpha}{\alpha^2}. \]
Using the Taylor expansions for $\exp$ and $\sinh$ about $z = 0$ we therefore have
\[ |e^{3z}| = e^{3\alpha} > e^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} > \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(2k)!} > \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(2k+1)!} = \frac{\sinh \alpha}{\alpha} \geq \frac{|\sinh z|}{|z|} \]
for all $\alpha > 0$. Hence (9.6) cannot have a solution with non-negative real part except for $z = 0$.

(ii) Since the resolvent of $A$ is compact, $s(A) = 0$ is a pole of the resolvent. To show that 0 is an algebraically simple eigenvalue of $A$ we verify the assumptions of Proposition 7.9. To see that they are fulfilled, note that $\ker A$ is one-dimensional and spanned by the quasi-interior point $1_{[-2,0]}$ of $E_+$. Moreover, one can easily check that the positive functional $\varphi \in E'$, given by $\varphi(f) = f(0) + \int_{-2}^{-1} f(x) \, dx$, is contained in $\ker A'$.

(iii) Since the semigroup $(e^{tA})_{t \geq 0}$ is eventually norm-continuous [22, Theorem VI.6.6] it follows from Theorem 8.34 that it is uniformly asymptotically positive (see also the explanation below Theorem 8.34). By [3] Example B-II.1.22 the semigroup is not positive.

Finally, assume for a contradiction that the semigroup is individually eventually strongly positive with respect to a quasi-interior point $u$ of $E_+$. Since $u \gg 1_{[-2,0]}$, the semigroup is then individually eventually strongly positive with respect to $1_{[-2,0]}$ and Theorem 5.2 implies that the spectral projection $P$ corresponding to $s(A) = 0$ fulfills $P \gg 1_{[-2,0]}$. However, Proposition 3.1 then yields that $\ker(A')$ contains a strictly positive functional $\hat{\varphi}$. Since 0 is an algebraically simple eigenvalue of $A$, it is also an algebraically simple eigenvalue of $A'$ and hence $\ker A'$ is one-dimensional; see [37, Section III.6.6]. Thus, $\hat{\varphi}$ has to be a scalar multiple of the functional $\varphi$ from (b), which is clearly a contradiction, since $\varphi$ is not strictly positive.

It is currently unclear whether the semigroup $(e^{tA})_{t \geq 0}$ is individually eventually positive in the sense that for each $f \in E_+$ there exists $t_0 \geq 0$ such that $e^{tA} f \geq 0$ for all $t \geq t_0$. 

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10 Open problems

We have seen in several examples that the notion of “eventual positivity” on a general Banach lattice is difficult from a structural point of view, and therefore additional assumptions on the spectrum seem to be necessary to obtain good descriptions. It is therefore natural to ask if these assumptions can be changed or even weakened, and if there are possible alternative definitions. Let us explicitly formulate the following open problems:

(a) In our characterisations of strong eventual and asymptotic positivity we always assumed the peripheral spectrum to be finite. However, in some important examples, as e.g. in some transport equations, this assumption is not fulfilled. We therefore ask:

How can asymptotic positivity of a semigroup be characterised if the peripheral spectrum $\sigma_{\text{per}}(A)$ is allowed to be infinite and even unbounded?

(b) Example 5.4 shows that strong eventual positivity of the resolvent or the semigroup cannot be characterised by strong positivity of the spectral projection if the assumption $D(A) \subset E_u$ is dropped. One could ask the following question:

Suppose that all assumptions of Theorem 5.2 are fulfilled except for the condition $D(A) \subseteq E_u$. Can individual eventual strong positivity of the semigroup still be characterised by individual eventual strong positivity of the resolvent at $s(A)$ plus a spectral condition?

(c) We only defined the notion of asymptotic positivity of a semigroup $(e^{tA})_{t \geq 0}$ under the assumption that the rescaled semigroup $(e^{t(A-s(A))})_{t \geq 0}$ be bounded. If this assumption is not fulfilled, it is not clear to the authors if the condition $d_+(e^{t(A-s(A))}f) \to 0$ for each $f \geq 0$ should still be used to define individual asymptotic positivity, or if e.g. the condition $d_+(\frac{e^{t(A-s(A))}f}{\|e^{t(A-s(A))}\|}) \to 0$ for each $f \geq 0$ would be more appropriate. The same question arises for asymptotic positivity of the resolvent if the Abel-boundedness condition in Definition 7.2 is dropped:

How should asymptotic positivity of semigroups and resolvents be defined without additional boundedness assumptions?

(d) The following problem is concerned with eventual positivity rather than eventual strong positivity: In Example 7.1(b) and in [18, Example 8.2] we showed that individual eventual positivity of a semigroup does not imply individual eventual positivity of the resolvent at $s(A)$, even in finite dimensions. However, if the spectral bound $s(A)$ is a dominant spectral value, one might ask whether at least the converse implication is true:

Let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup with dominant spectral value $s(A)$ of $A$ and suppose that the resolvent is individually eventually positive at $s(A)$. Does it follow (maybe under some additional regularity assumptions) that $(e^{tA})_{t \geq 0}$ is individually eventually positive?

Acknowledgements. The authors would like to express their warmest thanks to Wolfgang Arendt for many stimulating and helpful discussions, Anna Dall’Acqua for her invaluable assistance concerning the bi-Laplace operator, and Khalid Akhlil for suggesting to consider the Laplacian with non-local boundary conditions. The first author wants to express his gratitude for a pleasant stay at Ulm University, where part of the work was done.
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