INVARIANT MEASURES FOR INTEGRABLE SPIN CHAINS
AND INTEGRABLE DISCRETE NLS

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ABSTRACT. We consider discrete analogues of two well-known open problems regarding invariant measures for dispersive PDE, namely, the invariance of the Gibbs measure for the continuum (classical) Heisenberg model and the invariance of white noise under focusing cubic NLS. These continuum models are completely integrable and connected by the Hasimoto transform; correspondingly, we focus our attention on discretizations that are also completely integrable and also connected by a discrete Hasimoto transform. We consider these models on the infinite lattice \( \mathbb{Z} \).

Concretely, for a completely integrable variant of the classical Heisenberg spin chain model (introduced independently by Haldane, Ishimori, and Sklyanin) we prove the existence and uniqueness of solutions for initial data following a Gibbs law (which we show is unique) and show that the Gibbs measure is preserved under these dynamics. In the setting of the focusing Ablowitz–Ladik system, we prove invariance of a measure that we will show is the appropriate discrete analogue of white noise.

We also include a thorough discussion of the Poisson geometry associated to the discrete Hasimoto transform introduced by Ishimori that connects the two models studied in this article.

1. Introduction

The research detailed in this paper began with the consideration of the following problem: Can one prove invariance of the Gibbs measure for the one-dimensional continuum (classical) Heisenberg model:

\[
\frac{d}{dt} \vec{S} = -\vec{S} \times \Delta \vec{S}
\]

where \( \vec{S} : \mathbb{R}_t \times \mathbb{R}_x \rightarrow S^2 \) describes the configuration of spins, \( \times \) denotes the cross-product, and \( \Delta = \partial^2_x \) is the spatial Laplacian.

This model is a special case of the Schrödinger maps equation (where general Kähler targets are allowed). It is also associated with the names of Landau–Lifshitz (see [29] or [32, §69]), who also introduced a damping term into these dynamics, and of Gilbert (see [16]), who further refined their theory at high damping. It is natural to also include an external magnetic field in (1), however, this would only complicate a problem that we already do not know how to solve.

Gibbs measure provides a statistical description of a physical system at thermal equilibrium and is dictated by the inverse temperature \( \beta > 0 \), the Hamiltonian (or energy functional), and the underlying symplectic volume.

From a physical point of view, (1) arises as the continuum limit of the classical Heisenberg spin-chain model

\[
\frac{d}{dt} \vec{S}_n = -\vec{S}_n \times (\vec{S}_{n+1} + \vec{S}_{n-1}),
\]

where \( \vec{S}_n : \mathbb{R}_t \times \mathbb{R}_x \rightarrow S^2 \) describes the configuration of spins, \( \times \) denotes the cross-product, and \( \Delta = \partial^2_x \) is the spatial Laplacian.
describing the dynamics of a chain of spins \( \vec{S} : \mathbb{R}^t \times \mathbb{Z} \to \mathbb{S}^2 \). This dynamics is Hamiltonian, being induced by the energy functional
\[
H_{\text{Heis}} := \sum_{n \in \mathbb{Z}} \frac{1}{2} |\vec{S}_n - \vec{S}_{n+1}|^2
\]  
with respect to the Poisson structure (4) below, which is merely the vestige (in classical mechanics) of the standard (quantum mechanical) commutation relations for spins. It is shown in [15] that the quantum mechanical spin chain reduces to this classical model in the limit of large spin per site.

**Definition 1.1 (Poisson bracket).** For fields \( \vec{S} : \mathbb{Z} \to \mathbb{S}^2 \subset \mathbb{R}^3 \), we define the Poisson bracket via
\[
\{ \vec{a} \cdot \vec{S}_n, \vec{b} \cdot \vec{S}_m \} = \delta_{nm} \vec{a} \cdot (\vec{S}_n \times \vec{b}).
\]  
The symplectic form associated to this Poisson bracket is the sum of the standard surface area on each copy of \( \mathbb{S}^2 \). As it comes from a (closed) symplectic structure, this Poisson bracket is immediately guaranteed to obey the Jacobi identity, although this can also be checked directly via Lagrange’s identity for the cross product.

Analogously, the continuum model (1) is naturally associated to the Hamiltonian
\[
\int_\mathbb{R} |\nabla \vec{S}(x)|^2 \, dx,
\]  
which (formally at least) tells us that the associated Gibbs measure simply corresponds to Brownian paths on the sphere. The key difficulty associated with the problem posed in the first paragraph of this paper is not to make sense of the Gibbs measure, but rather, to be able to make sense of the dynamics (1) for such irregular data.

The study of Hamiltonian PDE at low regularity has been a topic of intensive study for many years now and has made it possible to prove the existence of dynamics for initial data sampled from Gibbs measures and hence the invariance (under the flow) of these Gibbs measures for a variety of Hamiltonian PDE. We note, in particular, the pioneering work (on both fronts) of Bourgain, surveyed in [4].

At this moment, the most powerful method for studying the Schrödinger maps equation at low regularity is via the Hasimoto transform. Discovered in the study of vortex tubes in [20] and first applied to (1) in [28], this mapping transforms solutions to (1) into solutions to the focusing cubic NLS:
\[
i\psi_t = -\partial_x^2 \psi - \frac{1}{2} |\psi|^2 \psi.
\]  
Concretely, viewing \( x \mapsto \vec{S}(t, x) \) as the field of tangents to an arc-length parameterized curve in \( \mathbb{R}^3 \), one defines
\[
\psi(t, x) = \kappa(t, x) \exp \left\{ -i \int_{-\infty}^x \tau(t, x') \, dx' \right\}
\]  
where \( \kappa \) denotes the curvature of the curve and \( \tau \) its torsion. Note that the energy of the spin wave is carried over to the mass of the solution to NLS,
\[
\int_\mathbb{R} |\nabla \vec{S}(x)|^2 \, dx = \int_\mathbb{R} |\psi(x)|^2 \, dx,
\]  
rather than to the traditional Hamiltonian for (5). Evidently, the Hasimoto map is not a Poisson map with respect to the *standard* Poisson structure associated to NLS.
The presence of a second (compatible) Poisson structure for (5) is indicative of the well-known complete integrability of NLS (cf. [34]). The equation (1) has also been shown to be completely integrable, both directly [41] and via Hasimoto-type transformations [28, 33]. While the problem of constructing dynamics for (1) with initial data sampled from the Gibbs measure seems out of reach at the current moment, the complete integrability of this equation is, at least, propitious.

The original calculations used in deriving the Hasimoto transformation involve use of the Frenet–Serret formulae for curves. As is well-known, this approach to the differential geometry of curves is poorly adapted to vanishing curvature. These difficulties can be averted by adopting a parallel frame (cf. [3]) along the curve. Indeed, this approach has lead to the development of Hasimoto-like transformations in the context of general Kähler targets, as well as for higher dimensional arrays of spins; see [7, 10, 35, 38].

Regarded as a mapping of individual states (rather than trajectories), it is not difficult to see that the Hasimoto transform maps Brownian paths on the sphere to white noise on the line. Setting aside whether this can be extended to trajectories (in any sense), this raises the question of studying NLS with white noise initial data. This problem is well-known and currently open, for focusing and defocusing nonlinearities, both on the line and on the circle. In fact, one would formally expect white noise measure to be invariant under the NLS flow. For the state of the art in the low-regularity problem for NLS, we refer the reader to [0, 8, 9, 17, 18, 25, 26, 27], as well as [2, 24] which study low-regularity problems originating directly from (1). We include here several references considering problems on the circle or, what is equivalent, for periodic initial data. As white noise constitutes non-decaying (indeed ergodic) data on the line, there is a strong analogy with the circle case.

One thing that is clearly understood in the circle setting is that one must renormalize (5) to have any hope of treating data at regularities below $L^2$; see [18]. At the very least, one must employ Wick ordering, which amounts to removing an infinite phase shift from solutions to the equation.

Once one accepts that renormalization may be necessary to make sense of the model (1) for Gibbs distributed initial data, then one is compelled to return to the basic physics. Not only should one endeavor to renormalize in a physical way, but the break-down of the effective model should also be regarded as casting doubt on its derivation from more elementary principles. Concretely, one is lead to ask if (1) is the proper continuum limit of (2) in the setting of thermal equilibrium.

For smooth initial data, the convergence of (2) to (1) is shown rigourously in [40]. Our hesitation in assuming that this result extends to low regularity data is most easily explained through consideration of the continuum limit of the discrete linear Schrödinger equation

$$i\partial_t \psi_n = -\left(\psi_{n+1} + \psi_{n-1}\right),$$

(8)

with initial data constructed by choosing each $\psi_n$ independently and identically distributed according to a complex Gaussian law. It is easily shown (by Fourier transformation) that this measure is invariant under the flow. Now, this measure and indeed these dynamics are left invariant by the transformation

$$\psi_n \mapsto (-1)^n \tilde{\psi}_n$$

which shows that low-frequencies (slowly varying sequences) and very high frequencies (slowly varying modulus with alternating signs) contribute equally to the
problem in question. However, it is only for the low frequencies that one would traditionally conflate the Laplacian with its finite difference approximation. For the model $\mathbf{3}$ with white-noise initial data, one is lead to posit that the continuum limit should be described (at the very least) by a pair of linear Schrödinger equations: one for the low frequencies and one for the high frequencies.

While it is fair to say that the process of inverting the Hasimoto transform is one of integration, which would suppress the high frequencies, our immediate discussion has centered around the linear model $\mathbf{5}$. Nonlinearities would couple the low- and high-frequency portions of the solution and thus we cannot discount the possibility that the high-frequency components impact the low-frequency dynamics in a non-trivial way.

We should caution the reader that the preceding discussion is heuristic and that we are not asserting the existence of a Hasimoto-like transform attendant to $\mathbf{2}$. Nonetheless, we shall soon discuss a discrete spin chain model and a discrete nonlinear Schrödinger equation that are connected by such a Hasimoto-like transformation; moreover, both are completely integrable. On the other hand, numerical evidence $\mathbf{37}$ suggests that the model $\mathbf{2}$ is not completely integrable.

Low regularity problems in dispersive PDE are inherently difficult, notwithstanding the additional difficulties stemming from passing to the continuum limit of a discrete model. Past experience suggests the greatest chance of success if one works with a completely integrable model, which led us to seek out discrete analogues of $\mathbf{11}$ and $\mathbf{14}$ that retain complete integrability and which are connected by a Hasimoto-like transformation. This pursuit does not represent a disparagement of $\mathbf{2}$, but rather, the belief that it may be more fruitfully treated as a perturbation of such a completely integrable analogue, rather than attacked directly.

Our search for an integrable discrete analogue of $\mathbf{2}$ was a very short one. It is clearly documented in $\mathbf{14}$:

$$\frac{d}{dt} \mathbf{S}_n = -\mathbf{S}_n \times \left( \frac{2\mathbf{S}_{n+1}}{1 + \mathbf{S}_n \cdot \mathbf{S}_{n+1}} + \frac{2\mathbf{S}_{n-1}}{1 + \mathbf{S}_n \cdot \mathbf{S}_{n-1}} \right),$$  

which has Hamiltonian

$$H_{\text{LHM}} := \sum_n -2 \log \left( 1 - \frac{1}{4} |\mathbf{S}_n - \mathbf{S}_{n+1}|^2 \right)$$  

with respect to the standard Poisson structure $\mathbf{4}$. Following this reference, we will refer to this model as the Lattice Heisenberg Model (LHM), which appeared independently in three papers $\mathbf{19, 22, 39}$ in the same year.

The book $\mathbf{11}$ also describes (following $\mathbf{23}$) a transformation of the LHM to a completely integrable form of discrete NLS. However, this mapping is essentially a stereographic projection at each position along the lattice and so is unlike the Hasimoto transform, which acts like a derivative. It is not difficult to obtain a discrete analogue of the Hasimoto transform, starting from $\mathbf{11}$ and mimicking the arguments in $\mathbf{20}$; see the next section. However, the answer (found by a different method) appears already in $\mathbf{22}$, which shows that the LHM can be transformed to the (focusing) Ablowitz–Ladik system,

$$i \frac{d}{dt} \alpha_n = -\left( 1 + |\alpha_n|^2 \right) \left[ \alpha_{n+1} + \alpha_{n-1} \right] + 2\alpha_n.$$  

This model was introduced in $\mathbf{1}$ as an integrable discretization of $\mathbf{4}$.
Informed by the preceding discussion, our immediate goals with regard to the models (9) and (11) are now clear:

(i) Construct (unique) Gibbs measures for (9).
(ii) Prove the existence and uniqueness of the dynamics (9) with initial data sampled from this measure.
(iii) Show that these dynamics leave said Gibbs measures invariant.
(iv) Determine a suitable discrete analogue of white-noise that is connected to the Gibbs measure for (9) via a discrete Hasimoto transformation.
(v) Show that (11) is well-posed for initial data sampled from this ‘white noise’ measure and that the dynamics (11) leaves this measure invariant.

This is what will be achieved in this paper. The rather more challenging problem of taking a continuum limit in these results remains our ambition for the future.

We note that the approach to constructing invariant measures for NLS by taking a continuum limit of the Ablowitz–Ladik system has already been shown to be successful in [42]. In that paper, Vaninsky considers the defocusing problem on the circle and constructs an invariant measure associated to the conservation law at one degree of regularity higher than the Hamiltonian. For convergence in the deterministic setting, see [21], which works in the energy space, and references therein.

The existence and uniqueness of Gibbs measures for (9) will be proved in Proposition 5.1. While the prevailing method for proving dynamical invariance of Gibbs measures is based on finite-dimensional approximation, we eschew this methodology for the construction of the measure. Instead, we adopt the intrinsic definition of Gibbs measures introduced by Dobrushin, Lanford, and Ruelle; see [11, 30]. We prove uniqueness of such Gibbs measures by using the Perron–Frobenius Theorem to show that the underlying Markov chain is mixing; see (56).

In order to prove invariance of the Gibbs measure, we need a more direct construction than the abstract existence and uniqueness given by Proposition 5.1. This is effected by using the discrete Hasimoto transformation in reverse to construct initial data for (9) from initial data for (11). In fact, we will also construct solutions to (9) by this method, namely, by first constructing solutions to (11) and then transferring them to (9). The virtues of employing the discrete Hasimoto transform here are the same as in the continuum case — it transforms a quasilinear problem into a semilinear one, which makes it much easier to control both individual solutions and differences between pairs of solutions.

Up to now, we have avoided addressing one of the main deficiencies of the Hasimoto transform, namely, its failure to admit an invariant definition, both in the sense of dynamically invariant and in the sense of being independent of arbitrary choices. This problem stems from the incompatible gauge invariances of the two equations involved: The spin models (both continuum and discrete) have a global $SO(3)$ gauge invariance corresponding to a collective rigid rotation of all the spins, while (6) and (11) have global $U(1) \cong SO(2)$ phase invariance. In the study of individual solutions, this nuisance is usually handled by fixing a gauge for the initial data and propagating the resulting frame through time, as necessary. For statistical ensembles of solutions (as considered here) this is unsatisfactory — it leads to measurability issues and non-invariant measures (due to dynamical modifications of the gauge). The remedy we adopt here is to randomize the gauge and show that this randomization is dynamically invariant.
Our discussion of the discrete Hasimoto transform is divided into two parts: In Section 2 we present its construction by paralleling the classical approach of [20]. This will allow us to elucidate the Poisson structure of the discrete Hasimoto transformation more fully than appears to have been done before. On the other hand, this approach breaks down whenever consecutive spins are parallel — this is the discrete analogue of the problem of vanishing curvature in the Frenet–Serret description of curves.

In Section 3, we revisit the discrete Hasimoto transform in a manner parallel to modern treatments of the continuum version, which are based on parallel frames. This approach does not suffer from problems with vanishing curvature; moreover, it is well-suited to randomization of the gauge. Neither this approach nor that presented in Section 2 is very close to that adopted in [22], where the discrete Hasimoto transform was first discovered.

Already in Section 2 it is possible to deduce what distribution should be assigned to initial data for the Ablowitz–Ladik system so that it corresponds to the Gibbs measure for (9) via the discrete Hasimoto transform. The answer is given in (45). The values at each site are statistically independent, as one might well imagine for a measure mimicking white noise. However, their distribution is not Gaussian — it has very long tails. In fact, at inverse temperature \( \beta > 0 \), we have \( \alpha_n \in L^p(d\mathbb{P}) \) if and only if \( p < 2 + \beta \).

In Section 4, we first prove almost sure existence and uniqueness of solutions to (11) for initial data sampled from the measure (45). This is Theorem 4.3. We then show that this flow preserves the measure (45); this is Theorem 4.4. The key idea is to take a limit (uniform on bounded sets in space-time) of solutions to spatial truncations of the equation. For such finite systems, global well-posedness follows from standard ODE techniques; see Proposition 2.9. Note that these methods cannot be applied in infinite volume. First, as RHS (11) is not globally Lipschitz, one can only hope to apply contraction mapping on a small time interval whose length is dictated by the size of the data. But as our initial data is ergodic under translation, every possible local configuration will occur with positive density somewhere; thus no time interval is short enough to apply contraction mapping if one works globally in space. Secondly, to pass from local to global well-posedness, one would like to apply conservation laws; however, all conserved quantities are infinite in this case.

The method we employ is a close analogue of that used by Bourgain [5] to construct solutions to defocusing NLS on the line with initial data sampled from Gibbs measure. The principal novelty in this paper is in the implementation, where subtleties arise from the long tails in the distribution of the initial data.

The climax of the paper is Section 5 where we prove existence and uniqueness of the Gibbs measure for (9), construct unique solutions associated to such initial data, and prove the resulting dynamics leaves the Gibbs measure invariant. In summary, we prove

**Theorem 1.2** (Invariance of the Gibbs measure for LHM). Fix \( \beta > 0 \). For almost every initial data distributed according to the Gibbs measure \( d\mu^\beta_{\text{Gibbs}} \), there exists a unique global good solution to the spin chain model (9). Moreover, the Gibbs measure \( d\mu^\beta_{\text{Gibbs}} \) is left invariant by the flow of (9).
2. The discrete Hasimoto transform

Our goal in this section is to develop the discrete Hasimoto transform following closely the methodology expounded in the original work of Hasimoto [20].

**Definition 2.1.** For a field \( \vec{S} : \mathbb{Z} \rightarrow S^2 \), with no two consecutive spins parallel or antiparallel, we define coordinates \( \theta_n \in (0, \pi) \) and \( \gamma_n \in (-\pi, \pi] \) via

\[
\cos(\theta_n) = \vec{S}_n \cdot \vec{S}_{n+1},
\sin(\theta_n) e^{i\gamma_n} = (\vec{S}_{n-1} \times \vec{S}_n) \cdot (\vec{S}_n \times \vec{S}_{n+1}) + i \vec{S}_{n-1} \cdot (\vec{S}_n \times \vec{S}_{n+1}).
\]

Note that \( \theta_n \) measures the angle between consecutive spins and hence may be considered as a substitute for the curvature appearing in the original Hasimoto transformation. However, this is not quite the correct choice, as we will see below. The quantity \( \gamma_n \) measures the (signed) angle between the planes spanned by \( \{ \vec{S}_{n-1}, \vec{S}_n \} \) and \( \{ \vec{S}_n, \vec{S}_{n+1} \} \). As such, it is a natural analogue of the torsion of the curve appearing in the original Hasimoto transform. We note that while \( \gamma_n \) can be regarded as the torsion at site \( n \), one should really regard \( \theta_n \) as the curvature between sites \( n \) and \( n + 1 \). In this sense the coordinates are better seen as being indexed by interlacing lattices, which explains some asymmetry in the formulae that follow.

The functions \( (\theta_n, \gamma_n)_{n \in \mathbb{Z}} \) do not form a complete set of coordinates. Indeed, they are invariant under global rotations:

\[
\vec{S}_n \mapsto \mathcal{O} \vec{S}_n \quad \text{for all } n \in \mathbb{Z} \text{ and fixed } \mathcal{O} \in \text{SO}(3). \tag{12}
\]

This is the only obstruction to inverting this change of coordinates, as is evident from our next lemma.

**Lemma 2.2.** Given \( \vec{S}_0, \vec{S}_1 \in S^2 \), and \( (\theta_n, \gamma_n)_{n \in \mathbb{Z}} \), one can reconstruct the full spin field. Indeed,

\[
\vec{S}_{n+1} = \cos(\theta_n) \vec{S}_n + \frac{\sin(\theta_n)}{\sin(\theta_{n-1})} \left[ \sin(\gamma_n) \vec{S}_{n-1} \times \vec{S}_n + \cos(\gamma_n) (\vec{S}_{n-1} \times \vec{S}_n) \times \vec{S}_n \right],
\]

\[
\vec{S}_{n-1} = \cos(\theta_{n-1}) \vec{S}_n + \frac{\sin(\theta_{n-1})}{\sin(\theta_n)} \left[ \sin(\gamma_n) \vec{S}_n \times \vec{S}_{n+1} - \cos(\gamma_n) (\vec{S}_n \times \vec{S}_{n+1}) \times \vec{S}_n \right].
\]

These relations (and Definition 2.1) also show that

\[
\vec{S}_n \cdot \vec{S}_{n+1} = \cos(\theta_n),
\]

\[
\vec{S}_n \cdot \vec{S}_{n+2} = \cos(\theta_n) \cos(\theta_{n+1}) - \sin(\theta_{n+1}) \sin(\theta_n) \cos(\gamma_{n+1}),
\]

\[
\vec{S}_n \cdot \vec{S}_{n+3} = \cos(\theta_n) \left[ \cos(\theta_{n+1}) \cos(\theta_{n+2}) - \cos(\gamma_{n+2}) \sin(\theta_{n+1}) \sin(\gamma_{n+2}) \right]
+ \left[ - \left[ \sin(\theta_{n+1}) \cos(\theta_{n+2}) + \cos(\theta_{n+1}) \sin(\gamma_{n+2}) \right] \cos(\gamma_{n+1})
+ \sin(\theta_{n+2}) \sin(\gamma_{n+1}) \sin(\gamma_{n+2}) \right] \sin(\theta_n).
\]

**Proof.** Note that

\[
\frac{1}{\sin(\theta_{n-1})} (\vec{S}_{n-1} \times \vec{S}_n) \times \vec{S}_n, \quad \frac{1}{\sin(\theta_{n-1})} \vec{S}_{n-1} \times \vec{S}_n, \quad \text{and } \vec{S}_n \quad \tag{13}
\]

and

\[
\frac{1}{\sin(\theta_n)} (\vec{S}_n \times \vec{S}_{n+1}) \times \vec{S}_n, \quad \frac{1}{\sin(\theta_n)} \vec{S}_n \times \vec{S}_{n+1}, \quad \text{and } \vec{S}_n \quad \tag{14}
\]

form positively oriented orthonormal bases for \( \mathbb{R}^3 \). The first two identities follow by expressing \( \vec{S}_{n+1} \) using (13) and \( \vec{S}_{n-1} \) using (14). In particular, the first identity shows that \( \theta_n \) and \( \gamma_n \) are the traditional spherical polar coordinates for \( \vec{S}_{n+1} \) in this
frame. More precisely, $\theta_n$ represents the colatitude of $\vec{S}_{n+1}$ relative to a north pole $\vec{S}_n$. Analogously, $\gamma_n$ denotes the longitude of $\vec{S}_{n+1}$ with prime meridian passing through $-\vec{S}_{n-1}$; this is the sensible choice, since for a slowly varying curve $n \mapsto \vec{S}_n$, the points $\vec{S}_{n+1}$ and $\vec{S}_{n-1}$ will tend to be on opposite sides of $\vec{S}_n$.

To elucidate the Poisson structure introduced in Definition 1.1 at the level of $(\theta_n, \gamma_n)_{n \in \mathbb{Z}}$, we record the following proposition.

**Proposition 2.3.** Among the functions $\{\theta_n, \gamma_n : n \in \mathbb{Z}\}$, all non-zero Poisson brackets are as follows:

\[
\begin{array}{|c|c|}
\hline
f & \{f, \theta_n\} \\
\hline
\gamma_{n-1} & - \csc(\theta_{n-1}) \cos(\gamma_n) \\
\theta_{n-1} & \sin(\gamma_n) \\
\gamma_n & \cot(\theta_n/2) + \cot(\theta_{n-1}) \cos(\gamma_n) \\
\theta_n & 0 \\
\gamma_{n+1} & - \cot(\theta_{n-1}) - \cot(\theta_{n+1}) \cos(\gamma_{n+1}) \\
\theta_{n+1} & - \sin(\gamma_{n+1}) \\
\gamma_{n+2} & \csc(\theta_{n+1}) \cos(\gamma_{n+1}) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
f & \{f, \gamma_n\} \\
\hline
\gamma_{n-2} & - \sin(\gamma_{n-2}) \cot(\theta_{n-2}) \csc(\theta_{n-1}) \\
\gamma_{n-1} & \left[ \cot(\theta_{n-2}) \sin(\gamma_{n-1}) + \cot(\theta_{n-1}) \sin(\gamma_n) \right] \csc(\theta_{n-1}) \\
\gamma_n & 0 \\
\gamma_{n+1} & - \left[ \cot(\theta_{n-1}) \sin(\gamma_n) + \cot(\theta_{n+1}) \sin(\gamma_{n+1}) \right] \csc(\theta_{n+1}) \\
\gamma_{n+2} & \sin(\gamma_{n+1}) \cot(\theta_n) \csc(\theta_{n+1}) \\
\hline
\end{array}
\]

together with those determined by the above via anti-symmetry.

**Proof.** The exact calculations are lengthy; we summarize the method, rather than give all details.

Using Definitions 1.1 and 2.1 it is easy to compute

\[
\{ \vec{S}_m \cdot \vec{S}_{m+1}, \vec{S}_n \cdot \vec{S}_{n+1} \} = \delta_{m,n+1} \vec{S}_{m+2} \cdot (\vec{S}_{n+1} \times \vec{S}_n) - \delta_{m,n-1} \vec{S}_{m+1} \cdot (\vec{S}_{n+1} \times \vec{S}_n) = - \delta_{m,n+1} \sin(\theta_n) \sin(\gamma_{n+1}) + \delta_{m,n-1} \sin(\theta_n) \sin(\gamma_{m+1}) \sin(\gamma_{n+1}).
\]

On the other hand,

\[
\{ \vec{S}_m \cdot \vec{S}_{m+1}, \vec{S}_n \cdot \vec{S}_{n+1} \} = \{ \cos(\theta_m), \cos(\theta_n) \} = \sin(\theta_n) \sin(\gamma_{n+1}) \{ \theta_m, \theta_n \}.
\]

This yields all Poisson brackets of the form $\{\theta_m, \theta_n\}$.

By the Jacobi identity and the previous result,

\[
\cos(\gamma_m) \{ \gamma_m, \theta_n \} = \{ \sin(\gamma_m), \theta_n \} = \{ \{ \theta_{n-1}, \theta_m \}, \theta_n \}
\]
which shows (using the previous result again) that this quantity is zero unless \( m \in \{n - 1, n, n + 1, n + 2\} \). To actually determine the values in these four cases, we compute

\[
\{ \vec{S}_{m-1} \cdot (\vec{S}_m \times \vec{S}_{m+1}), \vec{S}_n \cdot \vec{S}_{n+1} \} = \{ \sin(\theta_{m-1}) \sin(\theta_m) \sin(\gamma_n), \cos(\theta_n) \}
\]

directly from Definition 1.1. As the example

\[
\{ \vec{S}_{n-2} \cdot (\vec{S}_{n-1} \times \vec{S}_n), \vec{S}_n \cdot \vec{S}_{n+1} \} = (\vec{S}_{n-2} \cdot \vec{S}_n) (\vec{S}_{n-1} \cdot \vec{S}_{n+1}) - (\vec{S}_{n-1} \cdot \vec{S}_n) (\vec{S}_{n-2} \cdot \vec{S}_{n+1})
\]

shows, this requires expressing various dot products in terms of \( \theta \) and \( \gamma \). This is possible through applications of Lemma 2.2. Performing these computations yields all the information presented in the first table.

Arguing as previously, we have

\[
\{ \sin(\gamma_n), \sin(\gamma_n) \} = \{ \{ \theta_{m-1}, \theta_m \}, \sin(\gamma_n) \}
\]

\[
= \{ \{ \sin(\gamma_n), \theta_m \}, \theta_{m-1} \} - \{ \{ \sin(\gamma_n), \theta_{m-1} \}, \theta_m \}.
\]

Thus the values shown in the second table can be deduced from those in the first, with only the expenditure of sufficient labour. □

**Definition 2.4** (Discrete Hasimoto transform). For a field \( \vec{S} : \mathbb{Z} \to \mathbb{S}^2 \), we define complex coordinates \( \alpha_n \in \mathbb{C} \) via

\[
\alpha_n = \tan(\theta_n/2)e^{-i\Gamma(n)} \quad \text{where} \quad \Gamma(n) := \sum_{\ell \leq n} \gamma_{\ell}
\]

and \( \theta_n \in (0, \pi) \) and \( \gamma_n \in (-\pi, \pi] \) are as in Definition 2.1.

Included in this definition is the assertion that \( \tan(\theta_n/2) \) is the proper discrete analogue of the curvature in (10). Unaware that it appears already in [22, equation (14a)], we originally intuited this relation by comparing conserved quantities for (9) and (11): see (16) below.

The domain of the functions \( \alpha_n \) is a rather thin set within all possible spin configurations. Not only must we avoid consecutive spins being parallel or antiparallel, but we must now also constrain the torsion \( \gamma_n \) to be summable. Below we will determine the Poisson brackets of these functions of the spins and find that the results are polynomials in these same functions. This induces a Poisson structure on the algebra of finitely supported smooth functions of the variables \( \alpha_n \), which may now be regarded as an independent object, free from the constraints just mentioned. From this perspective, one may simply take the results of Proposition 2.5 as the definition of a Poisson structure on such an algebra, which happens to be inspired by the spin model. However, before one simply accepts the formulae below as the definition of a Possion structure, one must verify the Jacobi identity.

While it is indeed elementary (though tedious) to verify the Jacobi identity directly — indeed, we did this as a check on our computations — this is unnecessary since the domain of the functions \( \alpha_n \) is nonetheless rich enough to guarantee that this identity is inherited from the corresponding relation for (4).
Proposition 2.5. Poisson brackets among the functions \( \{ \Re \alpha_n, \Im \alpha_n : n \in \mathbb{Z} \} \) are as follows
\[
\{ \Re \alpha_n, \Im \alpha_m \} = \begin{cases} 
- \frac{1 + |\alpha_m|^2}{2} \Im \alpha_n \Im(\alpha_{m-1} - \alpha_{m+1}), & n \geq m + 2, \\
- \frac{1 + |\alpha_m|^2}{2} \Im \alpha_n \Im(\alpha_{m-1} - \alpha_{m+1}) + \frac{1 + |\alpha_m|^2}{2}, & n = m + 1, \\
\frac{1 + |\alpha_m|^2}{2} - \frac{1 + |\alpha_n|^2}{2} \Re(\alpha_n \alpha_{n-1}), & n = m, \\
- \frac{1 + |\alpha_n|^2}{2} \Re \alpha_m \Re(\alpha_{n-1} - \alpha_{n+1}) + \frac{1 + |\alpha_m|^2}{2}, & n = m - 1, \\
- \frac{1 + |\alpha_n|^2}{2} \Re \alpha_m \Re(\alpha_{n-1} - \alpha_{n+1}), & n \leq m - 2,
\end{cases}
\]
and
\[
\{ \Re \alpha_n, \Re \alpha_m \} = - \frac{1 + |\alpha_m|^2}{2} \Im \alpha_n \Re(\alpha_{m-1} - \alpha_{m+1}), \quad \text{for} \quad n \geq m + 1,
\]
\[
\{ \Im \alpha_n, \Im \alpha_m \} = \frac{1 + |\alpha_m|^2}{2} \Re \alpha_n \Im(\alpha_{m-1} - \alpha_{m+1}), \quad \text{for} \quad n \geq m + 1.
\]
These determine all remaining cases through anti-symmetry.

Proof. Using Proposition 2.3 it is elementary to verify that
\[
\{ \Gamma(n), \theta_k \} = \begin{cases} 
- \tan(\theta_{k-1}/2) \cos(\gamma_k) + \tan(\theta_{k+1}/2) \cos(\gamma_{k+1}), & n \geq k + 2, \\
- \tan(\theta_{k-1}/2) \cos(\gamma_k) - \cot(\theta_{k+1}) \cos(\gamma_{k+1}), & n = k + 1, \\
- \tan(\theta_{k-1}/2) \cos(\gamma_k) + \cot(\theta_k/2), & n = k, \\
- \cosec(\theta_{k-1}) \cos(\gamma_k), & n = k - 1, \\
0, & n \leq k - 2.
\end{cases}
\]
To complete the calculations, we also need to know \( \{ \Gamma(n), \Gamma(m) \} \) for all \( n \) and \( m \). Due to the finite-range nature of the Poisson bracket detailed in Proposition 2.3, these are easily determined. Indeed,
\[
\{ \Gamma(m + 1), \Gamma(m) \} = \{ \Gamma(m + 1) - \Gamma(m), \Gamma(m) \} = \{ \gamma_{m+1}, \gamma_m + \gamma_{m-1} \}
= \left[ \tan(\theta_{m-1}/2) \sin(\gamma_m) - \cot(\theta_{m+1}) \sin(\gamma_{m+1}) \right] \cosec(\theta_m).
\]
Similarly, for \( n \geq m + 2 \), we have
\[
\{ \Gamma(n), \Gamma(m) \} = \left[ \tan(\theta_{m-1}/2) \sin(\gamma_m) + \tan(\theta_{m+1}/2) \sin(\gamma_{m+1}) \right] \cosec(\theta_m).
\]
These determine all other cases via antisymmetry.

Using the new coordinates, we can rewrite the Hamiltonian (10) as
\[
H_{LHM} = \sum_n 4 \log[\sec(\frac{\theta_n}{2})] = \sum_n 2 \log(1 + |\alpha_n|^2). \tag{16}
\]
This is the discrete analogue of (7). The right-hand side here is a well-known conservation law in the context of the Ablowitz–Ladik system, where it plays the role analogous to that played by the mass for the NLS equation. Concretely, for solutions to (11), we have
\[
\partial_t \log(1 + |\alpha_n|^2) = -2 \text{Im}(\bar{\alpha}_n \alpha_{n+1}) + 2 \text{Im}(\bar{\alpha}_{n-1} \alpha_n).
\]
As mentioned before, we initially derived (15) by finding what relation between \( \theta_n \) and \( |\alpha_n| \) was necessary to arrive at the identity (16).

For comparison, the Hamiltonian corresponding to the Heisenberg spin chain model (2) becomes
\[
H_{Heis} = \sum_n 2 \sin^2(\frac{\theta_n}{2}) = \sum_n \frac{2|\alpha_n|^2}{1 + |\alpha_n|^2}.
\]
Lemma 2.6. Consider the phase space \( \ell^2(\mathbb{Z}) \) endowed with the Poisson bracket laid out in Proposition 2.5. The Hamiltonian (16) induces the focusing Ablowitz–Ladik flow (11), which is globally wellposed.

Proof. It is evident that the infinite sum (16) converges for \( \alpha \in \ell^2(\mathbb{Z}) \). Moreover, from Proposition 2.5, we have

\[
i\{\alpha_n, 2 \log(1 + |\alpha_k|^2)\} = \begin{cases} 
-2 \text{Re}[\bar{\alpha}_k(\alpha_{k-1} - \alpha_{k+1})] \alpha_n, & n \geq k + 2 \\
-2 \text{Re}[\bar{\alpha}_k(\alpha_{k-1} - \alpha_{k+1})] \alpha_n - (1 + |\alpha_n|^2) \alpha_k, & n = k + 1 \\
-2 \text{Re}[\bar{\alpha}_k \alpha_{k-1}] \alpha_n + 2 \alpha_n, & n = k \\
-(1 + |\alpha_n|^2) \alpha_k, & n = k - 1 \\
0, & n \leq k - 2
\end{cases}
\]

which shows that the induced vector fields are also summable, yielding

\[
i \partial_t \alpha_n = \sum_k i\{\alpha_n, 2 \log(1 + |\alpha_k|^2)\} = -(1 + |\alpha_n|^2) [\alpha_{n+1} + \alpha_{n-1}] + 2 \alpha_n \quad (17)
\]

which is the Ablowitz–Ladik flow (11).

The local well-posedness of (17) is trivial, since RHS(17) defines a locally Lipschitz vector field on \( \ell^2(\mathbb{Z}) \). This extends to global well-posedness due to conservation of the Hamiltonian (16), which controls the \( \ell^2 \) norm. \( \square \)

While the context in which we derived Lemma 2.6 explains the connection of the Ablowitz–Ladik equation to (9), it does little to help us understand invariant measures. We would like to truncate in space, obtain invariant measures in that setting, and then pass to the infinite volume limit. Such spatial truncations are rather violently at odds with the infinite-range character of the Poisson structure given in Proposition 2.5.

Secondly, the traditional construction of invariant measures in Hamiltonian mechanics rests on the invariance of phase volume (Liouville’s Theorem). It is far from clear what phase volume we should associate with the Poisson structure we have studied thus far.

The remedy to both our troubles lies in the fact that the Ablowitz–Ladik equation is bi-Hamiltonian (in the sense of [34]), as we will explain. Let us begin by recalling the standard Hamiltonian formulation of the Ablowitz–Ladik equation, as laid out in [14], for example.

Definition 2.7. We define a second Poisson structure on the algebra generated by \( \{\text{Re} \alpha_n, \text{Im} \alpha_n : n \in \mathbb{Z}\} \) as follows:

\[
\{\text{Re} \alpha_n, \text{Im} \alpha_m\}_0 = -\{\text{Im} \alpha_n, \text{Re} \alpha_m\}_0 = (1 + |\alpha_n|^2) \delta_{nm}
\]

and all other brackets are zero.

We note that this corresponds the symplectic structure

\[
\omega_0 = \sum_{n \in \mathbb{Z}} (1 + |\alpha_n|^2)^{-1} d \text{Re}(\alpha_n) \wedge d \text{Im}(\alpha_n) \quad (18)
\]

and that the flow (11) is generated by

\[
H_{AL} := \sum_{n \in \mathbb{Z}} - \text{Re}(\bar{\alpha}_n \alpha_{n+1}) + \log(1 + |\alpha_n|^2), \quad (19)
\]

which Poisson commutes with \( H_{LHM} \).
While this shows that the Ablowitz–Ladik equation admits a second Hamiltonian interpretation, this is slightly less than being bi-Hamiltonian. One needs to show that the two Poisson structures are compatible, namely, that any linear combination of the two Poisson brackets remains a Poisson bracket. The only obstruction to compatibility is the Jacobi identity.

**Theorem 2.8.** The Poisson brackets of Proposition 2.5 and Definition 2.7 are compatible.

**Proof.** As we already know that each of the Poisson brackets obeys the Jacobi identity individually, it suffices to show that

$$\sum \{F, \{G, H\}\} + \{F, \{G, H\}\} = 0,$$

where the sum is taken over the three cyclic permutations of the functions \(F, G,\) and \(H\). Moreover, it suffices to select each of these three functions from the collection \(\{\text{Re} \alpha_n, \text{Im} \alpha_n : n \in \mathbb{Z}\}\). Due to the zero-range structure of the \(\{,\}\) bracket, these observations reduce matters to a finite collection of computations that one simply has to grind through. As a finite system of polynomial identities, this is also amenable to checking via computer algebra systems. □

While the existence of multiple Hamiltonian interpretations of the Ablowitz–Ladik system has been known for some time (see [33] and references therein), to the best of our knowledge no previous authors have verified compatibility; see, for example, [13, §5].

As described earlier, our interest in this alternate Poisson structure stems from the problem of constructing invariant measures for truncations of the system.

We obtain our finite-volume model by truncating the Hamiltonian (19):

\[ H_{KLH}^K := K^{-1} \sum_{n=-K}^{K} -\text{Re}(\bar{\alpha}_n \alpha_{n+1}) + \sum_{n=-K}^{K} \log(1 + |\alpha_n|^2) \] (20)

generates the following dynamics

\[ i\partial_t \alpha_n = \{\alpha_n, H_{KLH}^K\} = \begin{cases} - (1 + |\alpha_{-K}|^2)\alpha_{-K+1} + 2\alpha_{-K}, & n = -K, \\ - (1 + |\alpha_n|^2)\left(\alpha_{n+1} + \alpha_{n-1}\right) + 2\alpha_n, & |n| \leq K - 1, \\ - (1 + |\alpha_K|^2)\alpha_{K-1} + 2\alpha_K, & n = K, \end{cases} \] (21)

which is easily seen to conserve

\[ H_{LHM}^K := \sum_{|n| \leq K} 4 \log[\text{sec}(\frac{n\pi}{2})] = \sum_{|n| \leq K} 2 \log(1 + |\alpha_n|^2). \] (22)

At the level of the spins, \(H_{LHM}^K\) is the energy functional corresponding to free boundary conditions — the spins at the ends of the chain only couple to their one neighbour. One could also consider other boundary conditions. However, we will prove uniqueness of both the Gibbs measure and the dynamics in infinite volume; thus, the choice of boundary condition has no effect.

**Proposition 2.9.** The truncated Ablowitz–Ladik system (21) is globally wellposed and conserves the following ‘white-noise’ probability measure

\[ d\mu_{\beta,K}^{\text{wn}} = \prod_{-K \leq n \leq K} \frac{1 + 2\beta}{\pi} \frac{d\text{Area} (\alpha_n)}{(1 + |\alpha_n|^2)^{2+2\beta}} \] (23)
for any $\beta > 0$.

Proof. As RHS $21$ is a Lipschitz function on $C^{2K+1}$, local well-posedness follows immediately. This can be made global in time due to conservation of the coercive quantity $22$. By writing

$$d\mu_{\beta,K}^{\alpha} = \left(\frac{1+2\beta}{\pi}\right)^{2K+1} e^{-(\beta+\frac{1}{2})H_{LHM}^K} \prod_n \frac{d\text{Re}(\alpha_n) \wedge d\text{Im}(\alpha_n)}{(1+|\alpha_n|^2)},$$

we see that the preservation of this measure under the flow stems from conservation of $H_{LHM}^K$ and Liouville’s Theorem on the preservation of phase volume (cf. $18$).

We note that $24$ deviates rather sharply from the Gibbs measure one would naturally associate with the system $21$: the inverse temperature is shifted and multiplies the analogue of mass, rather than the Hamiltonian. These anomalies will disappear when we pass back through the discrete Hasimoto transform — we will see that under this correspondence, this measure does indeed map to the true Gibbs measure for the spin system. These anomalies also serve to remind us of the subtle interrelation between the two Hamiltonian structures.

### 3. The Discrete Hasimoto Transform via Parallel Frames

In this section we revisit the discrete Hasimoto transform from the modern perspective of parallel frames. In order to complete the program laid out in the introduction, we will need to show how to transfer solutions from the Ablowitz–Ladik system to the spin chain model. This is the major impetus of this section; see Theorem $3.4$. We start by introducing some notation. For $z \in \mathbb{C}$ we define the orthogonal matrix

$$Q(z) = \frac{1}{1 + |z|^2} \begin{bmatrix} 1 - \text{Re}(z^2) & \text{Im}(z^2) & 2\text{Re}(z) \\ \text{Im}(z^2) & 1 + \text{Re}(z^2) & -2\text{Im}(z) \\ -2\text{Re}(z) & 2\text{Im}(z) & 1 - |z|^2 \end{bmatrix}. \quad (25)$$

Note that $Q(z)$ is the exponential of the antisymmetric matrix

$$q(z) = \begin{bmatrix} 0 & 0 & 2\arctan(|z|)\frac{\text{Re}(z)}{|z|} \\ 0 & 0 & -2\arctan(|z|)\frac{\text{Im}(z)}{|z|} \\ -2\arctan(|z|)\frac{\text{Re}(z)}{|z|} & 2\arctan(|z|)\frac{\text{Im}(z)}{|z|} & 0 \end{bmatrix}. \quad (26)$$

Proposition 3.1. Let $\{\vec{S}_n\}_{n \in \mathbb{Z}}$ be a sequence of spins such that no two consecutive spins are antiparallel. Let $P_0 \in \text{SO}(3)$ be such that

$$\vec{S}_0 = P_0 e_3.$$ 

Then there exists a unique sequence $\{\alpha_n\}_{n \in \mathbb{Z}}$ of complex numbers such that with

$$Q_n = Q(\alpha_n) \quad \text{and} \quad P_{n+1} = P_n Q_n,$$

we have

$$\vec{S}_n = P_n e_3.$$
Moreover, for all $n \in \mathbb{Z}$ we have
\begin{equation}
\vec{S}_n \cdot \vec{S}_{n+1} = \frac{1 - |\alpha_n|^2}{1 + |\alpha_n|^2},
\end{equation}

\begin{equation}
(\vec{S}_{n-1} \times \vec{S}_n) \cdot (\vec{S}_n \times \vec{S}_{n+1}) + i \vec{S}_{n-1} \cdot (\vec{S}_n \times \vec{S}_{n+1}) = \frac{4\alpha_n \alpha_{n-1}}{(1 + |\alpha_n|^2)(1 + |\alpha_{n-1}|^2)},
\end{equation}

from which we see that the map $\{\vec{S}_n\}_{n \in \mathbb{Z}} \mapsto \{\alpha_n\}_{n \in \mathbb{Z}}$ agrees with the one constructed in Section 2 modulo $U(1)$ gauge invariance.

Before turning to the proof of this proposition, let us first explain the sense in which it encapsulates the modern approach to the Hasimoto transform via parallel frames. As $P_n \in SO(3)$, its columns form a positively oriented orthonormal basis for $\mathbb{R}^3$. By (28), the third column coincides with $\vec{S}_n$, which in the context of the original Hasimoto transform means that it is tangent to the vortex curve. The remaining two columns form an orthonormal basis normal to the curve.

In the continuum setting, one asks that the derivatives of these normal vectors along the curve be parallel to the tangent to the curve, that is, they are given by parallel transport. Equivalently, the frame $P : \mathbb{R} \to SO(3)$ obeys
\begin{equation}
\partial_x P = AP \quad \text{where} \quad A = \begin{bmatrix} 0 & 0 & \kappa_1(x) \\ 0 & 0 & \kappa_2(x) \\ -\kappa_1(x) & -\kappa_2(x) & 0 \end{bmatrix}
\end{equation}

and $\kappa_1, \kappa_2$ are functions (dictated by the geometry of the curve) that ensure $P(x)\vec{e}_3$ remains tangent to the curve. It is not difficult to verify that the modulus of $\kappa_1 + i\kappa_2$ coincides with the curvature of the curve, while the derivative of its argument is the torsion of the curve; see [3, 36] for details. Comparing with (15), we see that $\kappa_1(x) + i\kappa_2(x) = \psi(x)$ modulo a global phase rotation.

Let us now compare the continuum setup with that of Proposition 3.1. First we see that the distribution of non-zero entries in $A$ matches that in $q(z)$ given above; moreover, matching the non-zero entries in $A$ to those in $q(\alpha_n)$ leads via (15) to the relation $\kappa_1 + i\kappa_2 = \theta_n e^{i\Gamma(n)}$, which matches the continuum analogue. This further explains the appearance of the tangent function in (15).

Proof of Proposition 3.1. The key observation is that
\begin{equation}
\text{maps } \mathbb{C} \text{ bijectively onto } S^2 \setminus \{-\vec{e}_3\}; \text{ indeed it is essentially the inverse of the stereographic projection. As } \vec{S}_0 \cdot \vec{S}_1 \neq -1, \text{ it follows that there exists a unique } \alpha_0 \in \mathbb{C} \text{ such that }
\end{equation}

\begin{equation}
P_0^T \vec{S}_1 = Q(\alpha_0)\vec{e}_3 \quad \text{or equivalently, } \quad \vec{S}_1 = P_0 Q(\alpha_0)\vec{e}_3.
\end{equation}

Using this observation and arguing inductively, one easily constructs uniquely the remaining $\alpha_n$ such that (28) holds. It remains to verify (29) and (30).

Using that $P_n$ is an orthogonal matrix, we get
\begin{equation}
\vec{S}_n \cdot \vec{S}_{n+1} = P_n \vec{e}_3 \cdot P_n Q_n \vec{e}_3 = \vec{e}_3 \cdot Q_n \vec{e}_3 = \frac{1 - |\alpha_n|^2}{1 + |\alpha_n|^2},
\end{equation}

which is (29).
To continue, we use the fact that for any matrix $O \in SO(3)$ and any vector $\vec{v}$,

$$\langle O\vec{e}_3 \times \vec{v} \rangle = O \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} O^T \vec{v}. \quad (32)$$

This allows us to compute

$$\vec{S}_n \times \vec{S}_{n+1} = P_n \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P_n^T P_n e_3 = \frac{1}{1 + |\alpha_n|^2} P_n \begin{bmatrix} 2 \text{Im}(\alpha_n) \\ 2 \text{Re}(\alpha_n) \\ 0 \end{bmatrix}. \quad (32)$$

Thus,

$$\vec{S}_{n-1} \cdot (\vec{S}_n \times \vec{S}_{n+1}) = P_n Q_{n-1}^T e_3 \cdot \frac{1}{1 + |\alpha_n|^2} P_n \begin{bmatrix} 2 \text{Im}(\alpha_n) \\ 2 \text{Re}(\alpha_n) \\ 0 \end{bmatrix} = \frac{4 \text{Im}[\bar{\alpha}_n \alpha_{n-1}]}{(1 + |\alpha_n|^2)(1 + |\alpha_{n-1}|^2)}. \quad (33)$$

Using also (29), we get

$$(\vec{S}_{n-1} \times \vec{S}_n) \cdot (\vec{S}_n \times \vec{S}_{n+1}) = (\vec{S}_{n-1} \cdot \vec{S}_n)(\vec{S}_n \cdot \vec{S}_{n+1}) - \vec{S}_{n-1} \cdot \vec{S}_{n+1}$$

$$= (\vec{S}_{n-1} \cdot \vec{S}_n)(\vec{S}_n \cdot \vec{S}_{n+1}) - P_{n-1} e_3 \cdot P_{n-1} Q_{n-1} Q_{n} e_3$$

$$= \frac{(1 - |\alpha_{n-1}|^2)(1 - |\alpha_n|^2)}{(1 + |\alpha_{n-1}|^2)(1 + |\alpha_n|^2)} - Q_{n-1}^T e_3 \cdot Q_{n} e_3$$

$$= \frac{4 \text{Re}[\bar{\alpha}_n \alpha_{n-1}]}{(1 + |\alpha_n|^2)(1 + |\alpha_{n-1}|^2)}. \quad (34)$$

Collecting (33) and (34), we obtain (30). □

As announced earlier, the main goal of this section is to ‘invert’ the discrete Hasimoto transform. To this end, let $\alpha : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ be a solution to the Ablowitz–Ladik system (11). For $n \in \mathbb{Z}$, we define

$$Q_n(t) = Q(\alpha_n(t)) \quad (35)$$

and

$$A_n(t) = \begin{bmatrix} 0 & -2 \text{Re}(\bar{\alpha}_n \alpha_{n-1}) & -2 \text{Im}(\alpha_n - \alpha_{n-1}) \\ 2 \text{Re}(\alpha_n \alpha_{n-1}) & 0 & -2 \text{Re}(\alpha_n - \alpha_{n-1}) \\ 2 \text{Im}(\alpha_n - \alpha_{n-1}) & 2 \text{Re}(\alpha_n - \alpha_{n-1}) & 0 \end{bmatrix}. \quad (36)$$

Fix $O \in SO(3)$ and let $P_0(t)$ be the solution to the initial-value problem

$$\frac{d}{dt} P_0 = P_0 A_0 \quad \text{with} \quad P_0(t = 0) = O. \quad (37)$$

For all other $n \in \mathbb{Z} \setminus \{0\}$, we define $P_n(t)$ via the recurrence relation

$$P_{n+1}(t) = P_n(t) Q_n(t). \quad (38)$$
Lemma 3.2. Assume $\alpha : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ is a solution to the Ablowitz–Ladik system \((11)\). Let \(\{Q_n\}_{n \in \mathbb{Z}}\) and \(\{P_n\}_{n \in \mathbb{Z}}\) be as defined by \((35)\) through \((38)\). Then for all \(n \in \mathbb{Z}\), we have
\[
d\frac{dt}{dt} Q_n = Q_n A_{n+1} - A_n Q_n \tag{39}
\]
\[
d\frac{dt}{dt} P_n = P_n A_n. \tag{40}
\]

Remark 3.3. The identity \((39)\) can be interpreted as an \(SO(3)\)-valued zero-curvature representation of the Ablowitz–Ladik model. The usual \(2 \times 2\) representation (cf. \([1]\)) is inferior for our purposes since it leads to a less transparent action of the \(SO(3)\) gauge group of the spin chain model.

Proof. The claim \((39)\) follows from a lengthy computation, using \((11)\) to compute the time derivative of \(Q_n\). We omit the details.

To prove \((40)\), we argue by induction. For \(n = 0\), \((40)\) is precisely the definition of \(P_0\). Assuming \((40)\) holds for some \(n \geq 0\), and using \((38)\) and \((39)\), we compute
\[
P^T_{n+1} \frac{d}{dt} P_{n+1} = Q^T_n \frac{d}{dt} P_n + P_n \frac{d}{dt} Q_n
\]
\[
= Q^T_n \frac{d}{dt} P_n [P_n A_n Q_n + P_n (Q_n A_{n+1} - A_n Q_n)]
\]
\[
= A_{n+1}.
\]

Similarly, assuming that \((40)\) holds for some \(n + 1 \leq 0\), and using \((38)\), \((39)\), and the fact that the matrices \(A_n\) are antisymmetric, we compute
\[
P^T_n \frac{d}{dt} P_n = (P_{n+1} Q^T_n)^T \frac{d}{dt} (P_{n+1} Q^T_n)
\]
\[
= Q_n P^T_{n+1} \left[ \frac{d}{dt} P_{n+1} Q^T_n + P_{n+1} \frac{d}{dt} Q_n \right]
\]
\[
= Q_n P^T_{n+1} [P_{n+1} A_{n+1} Q^T_n + P_{n+1} (Q^T_{n+1} Q_n - Q^T_n A_n)]
\]
\[
= Q_n (A_{n+1} + A^T_{n+1}) Q^T_n - A^T_n
\]
\[
= A_n.
\]

This completes the proof of the lemma.

Theorem 3.4. Let \(\mathcal{O} \in SO(3)\) and let $\alpha : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ be a solution to the Ablowitz–Ladik system \((11)\). Let \(\{Q_n\}_{n \in \mathbb{Z}}\) and \(\{P_n\}_{n \in \mathbb{Z}}\) be as defined by \((35)\) through \((38)\). Then \(\tilde{S} : \mathbb{Z} \times \mathbb{R} \to \mathbb{S}^2\) given by
\[
\tilde{S}_n(t) = P_n(t) e_3 \tag{41}
\]
is a solution to the system \((39)\).

Proof. On one hand, using Lemma \(3.2\) we get
\[
\frac{d}{dt} \tilde{S}_n = \frac{d}{dt} P_n e_3 = P_n A_n e_3 = P_n \begin{bmatrix} -2 \text{Im}(\alpha_n - \alpha_{n-1}) \\ -2 \text{Re}(\alpha_n - \alpha_{n-1}) \\ 0 \end{bmatrix}. \tag{42}
\]

On the other hand, using \((35)\) we compute
\[
1 + \tilde{S}_n \cdot \tilde{S}_{n+1} = 1 + P_n e_3 \cdot P_n Q_n e_3 = 1 + \frac{1 - |\alpha_n|^2}{1 + |\alpha_n|^2} = \frac{2}{1 + |\alpha_n|^2}
\]
Using also (32), we find
\[
\vec{S}_n \times \vec{S}_{n+1} = P_n \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P_n^T P_n Q_n e_3 = \frac{1}{1 + |\alpha_n|^2} P_n \begin{bmatrix} 2 \text{Im}(\alpha_n) \\ 2 \text{Re}(\alpha_n) \\ 0 \end{bmatrix}.
\]

Thus,
\[
-\frac{2 \vec{S}_n \times \vec{S}_{n+1}}{1 + \vec{S}_n \cdot \vec{S}_{n+1}} - \frac{2 \vec{S}_n \times \vec{S}_{n-1}}{1 + \vec{S}_n \cdot \vec{S}_{n-1}} = P_n \begin{bmatrix} -2 \text{Im}(\alpha_n) \\ -2 \text{Re}(\alpha_n) \\ 0 \end{bmatrix} + P_n Q_n^T \begin{bmatrix} 2 \text{Im}(\alpha_{n-1}) \\ 2 \text{Re}(\alpha_{n-1}) \\ 0 \end{bmatrix}.
\]

It is easy to verify that for each \(n \in \mathbb{Z}\), the vector \(\begin{bmatrix} 2 \text{Im}(\alpha_n) & 2 \text{Re}(\alpha_n) & 0 \end{bmatrix}^T\) is an eigenvector for \(Q_n\) with eigenvalue 1. Indeed, this vector belongs to the kernel of \(q(\alpha_n)\), where \(q\) is the antisymmetric matrix defined in (20). Thus,
\[
-\frac{2 \vec{S}_n \times \vec{S}_{n+1}}{1 + \vec{S}_n \cdot \vec{S}_{n+1}} - \frac{2 \vec{S}_n \times \vec{S}_{n-1}}{1 + \vec{S}_n \cdot \vec{S}_{n-1}} = P_n \begin{bmatrix} -2 \text{Im}(\alpha_n - \alpha_{n-1}) \\ -2 \text{Re}(\alpha_n - \alpha_{n-1}) \\ 0 \end{bmatrix},
\]

which combined with (12) yields the claim. \(\square\)

4. INVARIANCE OF WHITE NOISE FOR ABLOWITZ–LADIK

**Definition 4.1.** We say that a global solution \(\alpha : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}\) to the Ablowitz–Ladik system (11) is a good solution if it satisfies the following two conditions:
\[
\int_{-T}^{T} \sum_{n \in \mathbb{Z}} (n)^{-q} |\alpha_n(t)|^{2p} dt < \infty \quad \text{for some } p > q > 1 \text{ and all } T > 0, \quad (43)
\]
\[
\sup_{|t| \leq T} \sum_{n \in \mathbb{Z}} e^{-c|n|} |\alpha_n(t)|^2 < \infty \quad \text{for some } c > 0 \text{ and all } T > 0. \quad (44)
\]

**Remark 4.2.** If \(\alpha(t) = \{\alpha_n(t)\}_{n \in \mathbb{Z}}\) is a good solution to (11), then so is
\[
\{e^{i\phi} \alpha_{n+m}(t + t_0)\}_{n \in \mathbb{Z}}
\]
for any \(m \in \mathbb{Z}\), \(\phi \in [0, 2\pi]\), and \(t_0 \in \mathbb{R}\). Indeed, one may use the same parameters \(p, q, c\) appearing in (43) and (44), respectively.

**Theorem 4.3** (Almost sure global existence and uniqueness for Ablowitz–Ladik). Fix \(\beta > 0\). Then for almost every initial data \(\alpha(0) = \{\alpha_n(0)\}_{n \in \mathbb{Z}}\) chosen according to the white noise measure
\[
d\mu_{wn}^\beta = \prod_{n \in \mathbb{Z}} \frac{1 + 2\beta}{\pi} \frac{d\text{Area}(\alpha_n)}{(1 + |\alpha_n|^2)^{2+2\beta}}
\]
there exists a unique global good solution \(\alpha : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}\) to the Ablowitz–Ladik system (11).

**Proof.** We begin by constructing global solutions to (11) for almost every initial data chosen according to the measure \(d\mu_{wn}^\beta\). We will do so by proving that increasingly large finite-volume solutions to the Ablowitz–Ladik system (21) converge to a solution to (11), uniformly on compact regions of spacetime.
Let $\alpha(0) = \{\alpha_n(0)\}_{n \in \mathbb{Z}}$ be chosen according to the measure $d\mu_{\alpha}^3$. For $4 \leq K \in 2\mathbb{Z}$, let $\alpha^K : \{-K, \ldots, K\} \times \mathbb{R} \rightarrow \mathbb{C}$ denote the unique global solution to (21) with initial data $\alpha^K(0) = \{\alpha_n(0)\}_{|n| \leq K}$ constructed in Proposition 2.9.

We will show that almost surely, the global solutions $\alpha^K$ converge uniformly on compact regions of spacetime as $K \rightarrow \infty$. To this end, we fix $T > 0$ and for each $|t| \leq T$ and $4 \leq K \in 2\mathbb{Z}$, we define

$$M_K(t) = \sum_{n \in \mathbb{Z}} e^{-4(n)}|\alpha_n^{2K}(t) - \alpha_n^K(t)|^2,$$

with the convention that $\alpha_n^L \equiv 0$ for $|n| > L$. Straightforward computations give

$$\frac{d}{dt} M_K(t)$$

$$= -2 \text{Im} \sum_{n \in \mathbb{Z}} e^{-4(n)}\left(\overline{\alpha_n^{2K}} - \overline{\alpha_n^K}\right)\left((1 + |\alpha_n^{2K}|^2)\left[\left(\alpha_n^{2K} - \alpha_n^K\right) + \left(\alpha_n^{2K} - \alpha_n^{2K}\right)\right] + \left(\alpha_n^{2K} + \alpha_n^{2K}\right)\left[\left(\alpha_n^{2K} - \alpha_n^K\right) + \alpha_n^{2K} - \alpha_n^{2K}\right]\right).$$

Using Cauchy–Schwarz and the fact that $2 + e^{4(n)-4(n-1)} + e^{4(n)-4(n+1)} \leq 100$ uniformly for $n \in \mathbb{Z}$, we get

$$\left|\frac{d}{dt} M_K(t)\right| \leq 100 \left[1 + \sup_n |\alpha_n^{2K}(t)|^2\right] M_K(t) + \left[6 \sup_n |\alpha_n^K(t)|^2 + 2 \sup_n |\alpha_n^{2K}(t)|^2\right] M_K(t) \leq A(t) M_K(t),$$

where

$$A(t) = 100 + 102 \sup_n |\alpha_n^{2K}(t)|^2 + 6 \sup_n |\alpha_n^K(t)|^2.$$

Therefore, by Gronwall,

$$\sup_{|t| \leq T} M_K(t) \leq M_K(0) \exp\left(\int_{-T}^{T} A(t) \, dt\right). \quad (46)$$

To continue, we compute

$$\mathbb{E} M_K(0) = \mathbb{E} \sum_{n \in \mathbb{Z}} e^{-4(n)}|\alpha_n^{2K}(0) - \alpha_n^K(0)|^2 = \mathbb{E} \sum_{K < |n| \leq 2K} e^{-4(n)}|\alpha_n(0)|^2 \lesssim_{\beta} e^{-4K}$$

and so

$$\mathbb{P}(M_K(0) \geq e^{-2K}) \lesssim_{\beta} e^{-2K}. \quad (47)$$

Using invariance of the measure for the finite-dimensional system (21), we find

$$\mathbb{E}(\sup_n |\alpha_n^L(t)|^2) = \mathbb{E}(\sup_n |\alpha_n^L(0)|^2) \leq \lambda + \lambda^{-\varepsilon} \mathbb{E}(\sup_n |\alpha_n^L(0)|^{2+2\varepsilon})$$

$$\leq \lambda + \lambda^{-\varepsilon} \sum_{n \in \mathbb{Z}} |\alpha_n(0)|^{2+2\varepsilon} \lesssim_{\beta} \lambda + \lambda^{-\varepsilon} L,$$

provided $\varepsilon < 2\beta$. Optimizing in $\lambda$, we get

$$\mathbb{E}(\sup_n |\alpha_n^L(t)|^2) \lesssim_{\beta} L^{1+\varepsilon}.$$

Thus,

$$\mathbb{P}\left(\int_{-T}^{T} A(t) \, dt \geq K\right) \leq K^{-1} \mathbb{E} \int_{-T}^{T} A(t) \, dt \lesssim_{\beta} K^{-1} T + K^{-1} TK^{1+\varepsilon} \lesssim_{\beta} TK^{-\frac{1}{1+\varepsilon}}. \quad (48)$$
Combining (46) through (48), we obtain
\[ \sup_{|t| \leq T} M_K(t) \lesssim e^{-K} \]
on a set \( \Omega_{T,K} \) satisfying
\[ \mathbb{P}(\Omega_{T,K}^c) \lesssim \beta \langle T \rangle (K^{-\frac{1}{1+\varepsilon}} + e^{-2K}), \]
whenever \( \varepsilon < 2\beta \).

Now let \( \Omega_T \) be the set of initial data defined via
\[ \Omega_T = \{ \alpha(0) : \sum_{4 \leq K \in 2^Z} \sup_{|t| \leq T} \sqrt{M_K(t)} < \infty \}. \]
By conservation of the Hamiltonian (22), for any \( K \geq 4 \) we have \( \sup_{t \in \mathbb{R}} M_K(t) < \infty \).

Thus,
\[ \Omega_T = \bigcup_{K_0 \geq 4} \left\{ \alpha(0) : \sum_{K_0 \leq K \in 2^Z} \sup_{|t| \leq T} \sqrt{M_K(t)} < \infty \right\} \supset \bigcup_{K_0 \geq 4} \bigcap_{K \geq K_0} \Omega_{T,K}. \]
In particular, for any \( \varepsilon < 2\beta \),
\[ \mathbb{P}(\Omega_T^c) \leq \sum_{K \geq K_0} \mathbb{P}(\Omega_{T,K}^c) \lesssim \langle T \rangle (K_0^{-\frac{1}{1+\varepsilon}} + e^{-2K_0}) \rightarrow 0 \quad \text{as} \quad K_0 \rightarrow \infty. \]

Finally, let \( T_n \) be a sequence of times diverging to infinity. Then \( \Omega = \bigcap \Omega_{T_n} \) is a set of full measure. Moreover, for an initial data \( \alpha(0) = \{ \alpha_n(0) \}_{n \in \mathbb{Z}} \in \Omega \), the unique global solutions \( \alpha^K : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{C} \) to (21) with truncated initial data \( \alpha^K(0) = \{ \alpha_n(0) \}_{|n| \leq K} \) satisfy
\[ \sum_{4 \leq K \in 2^Z} \sup_{|t| \leq T,n \in \mathbb{Z}} e^{-2|n|} |\alpha^K_n(t) - \alpha^K_n(t)| \lesssim \langle T \rangle \langle K \rangle^{-\frac{1}{1+\varepsilon}} + e^{-2K_0} \lesssim \beta T, \]
which shows that \( \alpha^K \) converge uniformly on compact regions of spacetime.

It follows from this that the pointwise limit \( \alpha : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{C} \) is a global solution to (11) with initial data \( \alpha(0) \). Furthermore, for any \( T > 0 \) this solution satisfies
\[ \sup_{|t| \leq T,n \in \mathbb{Z}} e^{-4|n|} |\alpha_n(t)|^2 < \infty, \]
which yields (44) in the definition of a good solution (with \( c > 4 \)).

Our next goal is to prove that the statistical ensemble of global solutions \( \alpha \) to (11) with initial data \( \alpha(0) \) that we constructed above satisfies
\[ \mathbb{E} \int_{-T}^{T} \sum_{n \in \mathbb{Z}} \langle n \rangle^{-q} |\alpha_n(t)|^{2p} dt < \infty \]
for any \( 1 < q < p < 1 + 2\beta \) and any \( T > 0 \). In this way, we see that (11) admits a global good solution for a full measure set of initial data.

Fix \( T > 0 \) and \( 4 \leq K \in 2^Z \). By invariance of the measure for the finite-dimensional system (21), we obtain
\[ \mathbb{E} \int_{-T}^{T} \sum_{n \in \mathbb{Z}} \langle n \rangle^{-q} |\alpha^K_n(t)|^{2p} dt = \int_{-T}^{T} \mathbb{E} \sum_{n \in \mathbb{Z}} \langle n \rangle^{-q} |\alpha^K_n(0)|^{2p} dt \lesssim \beta T, \]
provided merely \( q > 1 \) and \( p < 1 + 2\beta \). As \( \alpha^K \) converge uniformly on compact regions of spacetime to \( \alpha \), Fatou’s Lemma implies (49).
Finally, it remains to prove uniqueness in the class of good solutions. Let \( \alpha(t) \) and \( \beta(t) \) be two good solutions to (11) with initial data \( \alpha(0) = \beta(0) \). Assume, towards a contradiction, that the two solutions \( \alpha \) and \( \beta \) are not equal. Then, translating in space (cf. Remark 4.2) and reversing time if necessary, we may find \( T > 0 \) so that

\[
\alpha_0(T) \neq \beta_0(T).
\]

As \( \alpha \) and \( \beta \) verify (43) and (44), there exist \( \sigma \in (0, 1) \) and positive constants \( A_T, c, B_T \) such that

\[
\int_{-T}^{T} \sup_{|n| \leq 2N} \left[ 1 + |\alpha_n(t)|^2 + |\beta_n(t)|^2 \right] dt \leq A_T N^\sigma \quad \text{uniformly for } N \geq 1,
\]

\[
\sup_{|t| \leq T} \sum_{n \in \mathbb{Z}} e^{-c|n|} \left[ 1 + |\alpha_n(t)|^2 + |\beta_n(t)|^2 \right] \leq B_T.
\]

Indeed, in terms of the parameters appearing in (43), we may take

\[
\sigma = \max \left\{ \frac{3}{p_\alpha}, \frac{3}{p_\beta} \right\} \quad \text{and} \quad c = \max \{ c_\alpha, c_\beta \}.
\]

To continue, for \( t \in [-T, T] \) we define

\[
M(t) = \sum_{n \in \mathbb{Z}} e^{-3c|n|} |\alpha_n(t) - \beta_n(t)|^2.
\]

A straightforward computation yields

\[
\left| \frac{dM}{dt} \right| \leq \sum_{n \in \mathbb{Z}} e^{-3c|n|} \left[ (1 + |\alpha_n|^2) \left[ 2|\alpha_n - \beta_n|^2 + |\alpha_{n+1} - \beta_{n+1}|^2 + |\alpha_{n-1} - \beta_{n-1}|^2 \right] \right.
\]

\[
\left. + \sum_{n \in \mathbb{Z}} e^{-3c|n|} \left( 1 + |\alpha_n(t)|^2 + |\beta_n(t)|^2 \right) \right] \sup_{|n| \leq 2N} (1 + |\alpha_n(t)|^2 + |\beta_n(t)|^2) M(t) \]

\[
+ C e^{3c} \sum_{|n| \geq N} e^{-3c|n|} \left( 1 + |\alpha_n(t)|^2 + |\beta_n(t)|^2 \right)^2,
\]

for some absolute constant \( C \) and \( N \geq 2 \). Now employing (52) we obtain

\[
\left| \frac{dM}{dt} \right| \leq C e^{3c} \left\{ \sup_{|n| \leq 2N} (1 + |\alpha_n(t)|^2 + |\beta_n(t)|^2) M(t) + e^{-cN} B_T^2 \right\}
\]

uniformly for \( t \in [-T, T] \) and \( N \geq 2 \). By Gronwall and (51), this implies

\[
M(T) \leq C e^{3c} T B_T^2 \exp \left\{ -cN + C e^{3c} A_T N^\sigma \right\}.
\]

This contradicts (50), since the right-hand side above converges to zero as \( N \to \infty \), thereby completing the proof of uniqueness.

**Theorem 4.4** (Invariance of white noise for Ablowitz–Ladik). Fix \( \beta > 0 \). Then the white noise measure \( d\mu^2_{\mathbb{Z}} \) is left invariant by the flow of the Ablowitz–Ladik system (11).

**Proof.** Let \( \alpha(0) = \{ \alpha_n(0) \}_{n \in \mathbb{Z}} \) belong to the full-measure set of initial data for which Theorem 4.3 guarantees the existence of a unique global good solution to
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and let \( \alpha : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{C} \) denote this solution. To prove invariance of the white noise measure, it suffices to show that

\[
\int F(\alpha(t)) \, d\mu_{\text{wn}}^\beta(\{\alpha_n(0)\}) = \int F(\alpha(0)) \, d\mu_{\text{wn}}^\beta(\{\alpha_n(0)\})
\]

for all \( t \in \mathbb{R} \) and all bounded continuous functions \( F \) depending on only finitely many coordinates.

To proceed, we fix such an \( F \) and choose \( N \) large enough so that \( F \) is determined by \( \alpha_{-N}, \ldots, \alpha_N \). For \( K \geq N \), let \( \alpha^K \) denote the unique global solution to (21) with data \( \alpha^K(0) = \{\alpha_n(0)\}_{|n| \leq K} \); see Proposition 2.9. This proposition also shows that the measure

\[
d\mu_{\text{wn}}^{\beta,K}(\{\alpha_n(0)\}) = \prod_{-K \leq n \leq K} \frac{1 + 2\beta}{\pi} \frac{d\text{Area}(\alpha_n)}{(1 + |\alpha_n|^2)^{1+2\beta}}
\]

is left invariant by this flow. Thus for any \( t \in \mathbb{R} \),

\[
\int F(\alpha(0)) \, d\mu_{\text{wn}}^{\beta,K}(\{\alpha_n(0)\}) = \int F(\alpha_{-N}(0), \ldots, \alpha_N(0)) \, d\mu_{\text{wn}}^{\beta,K}(\{\alpha_n(0)\})
\]

\[
= \int F(\alpha_{-N}(0), \ldots, \alpha_N(0)) \, d\mu_{\text{wn}}^{\beta,K}(\{\alpha_n(0)\})
\]

\[
= \int F(\alpha^K_{-N}(t), \ldots, \alpha^K_N(t)) \, d\mu_{\text{wn}}^{\beta,K}(\{\alpha_n(0)\})
\]

\[
= \int F(\alpha^K_{-N}(t), \ldots, \alpha^K_N(t)) \, d\mu_{\text{wn}}^{\beta}(\{\alpha_n(0)\}).
\]

As \( \alpha^K \) converges to \( \alpha \) uniformly on compact regions of spacetime as \( K \to \infty \), so

\[
\int F(\alpha^K_{-N}(t), \ldots, \alpha^K_N(t)) \, d\mu_{\text{wn}}^{\beta}(\{\alpha_n(0)\}) \to \int F(\alpha_{-N}(t), \ldots, \alpha_N(t)) \, d\mu_{\text{wn}}^{\beta}(\{\alpha_n(0)\})
\]

as \( K \to \infty \). This completes the proof of the theorem. \( \square \)

5. Invariance of the Gibbs Measure for the Spin Model

In this section we prove almost sure global existence and uniqueness for the spin chain model (9) with initial data distributed according to the Gibbs measure. Moreover, we show that the flow of (9) leaves the Gibbs measure invariant.

Our first task is to make sense of the Gibbs measure for (9) if it satisfies the DLR condition. This condition takes its name from the work of Dobrushin, [11], and Lanford–Ruelle, [30]. In the setting of our model, it says the following: for any bounded and continuous function \( f \) and any integers \( a \leq b \),

\[
E_{\beta}\{f(\vec{S}_a, \ldots, \vec{S}_b) \mid \vec{S}_{a-1}, \vec{S}_{b+1}\} = \frac{1}{Z_{ab}} \int_{\mathbb{R}_{2a}} \cdots \int_{\mathbb{R}_{2b}} f(s_a, \ldots, s_b) p(\vec{S}_{a-1}, s_a) p(s_b, \vec{S}_{b+1}) \prod_{k=a}^{b-1} p(s_k, s_{k+1}) \, ds_a \cdots ds_b,
\]

where

\[
p(s, \sigma) = \frac{1 + 2\beta}{4\pi} \exp\{2\beta \log(1 - \frac{1}{4} |s - \sigma|^2)\} = \frac{1 + 2\beta}{4\pi} (\frac{1 + s \cdot \sigma}{2})^{2\beta},
\]

as dictated by (11). The numerical factor \( \frac{1 + 2\beta}{4\pi} \) is included here for later convenience. It is inconsequential in (53), because it is canceled by the normalization
constant
\[ Z_{a,b} = \int_{S^2} \cdots \int_{S^2} p(\vec{S}_{a-1}, s_a) p(s_b, \vec{S}_{b+1}) \prod_{k=a}^{b-1} p(s_k, s_{k+1}) ds_a \cdots ds_b. \]

Here and below, integration over the sphere is performed with respect to area measure; hence \( \int_{S^2} ds = 4\pi \). This is dictated by the symplectic structure underlying Definition 1.1.

**Proposition 5.1 (Existence and uniqueness of the Gibbs measure).** The spin chain model (9) admits a unique Gibbs measure at inverse-temperature \( \beta > 0 \). Moreover, for any integers \( n \leq m \),
\[
E_{\beta}\left\{ f(\vec{S}_n, \ldots, \vec{S}_m) \right\} = \int_{S^2} \cdots \int_{S^2} f(s_n, \ldots, s_m) \prod_{k=n}^{m-1} p(s_k, s_{k+1}) ds_n \cdots ds_m, \tag{55}
\]
using the notation (54). We denote this Gibbs measure by \( d\mu_{Gibbs}^\beta \).

**Remark 5.2.** The law (55) shows that the random variables \( \{\vec{S}_n\} \) can also be interpreted as the stationary Markov chain associated to the transition probabilities
\[
E_{\beta}\left\{ f(\vec{S}_{n+1}) | \vec{S}_n \right\} = \int_{S^2} f(s) p(s, \vec{S}_n) ds.
\]

**Proof.** The formula (55) gives a consistent family of marginals. Thus, by Kolmogorov’s extension theorem there exists a unique probability measure with these marginals. It is easy to verify directly from (55) that this probability measure satisfies the DLR condition (53). It thus remains to verify that any law \( E_{\beta} \) satisfying the DLR condition (53) has marginals given by (55).

To continue, we define inductively the kernels \( p_k : S^2 \times S^2 \to \mathbb{R} \) via
\[
p_1(s, \sigma) = p(s, \sigma) \quad \text{and} \quad p_{k+1}(s, \sigma) = \int_{S^2} p_k(s, v) p(v, \sigma) dv.
\]
With this notation, (53) implies that for any integers \( a < n \leq m < b \),
\[
E_{\beta}\left\{ f(\vec{S}_n, \ldots, \vec{S}_m) \right\} = E_{\beta}\left\{ E_{\beta}\left\{ f(\vec{S}_n, \ldots, \vec{S}_m) | \vec{S}_a, \vec{S}_b \right\} \right\}
= E_{\beta}\left\{ \int_{S^2} \cdots \int_{S^2} \frac{p_{n-a}(\vec{S}_a, s_n) p_{m-b}(s_m, \vec{S}_b)}{p_{n-a}(\vec{S}_a, \vec{S}_b)} f(s_n, \ldots, s_m) \right. \\
\left. \quad \times \prod_{k=n}^{m-1} p_1(s_k, s_{k+1}) ds_n \cdots ds_m \right\}.
\]
To obtain (55), it thus suffices to show that
\[
p_k(s, \sigma) \to \frac{1}{4\pi} \quad \text{uniformly as } k \to \infty. \tag{56}
\]

Let \( P \) denote the operator with kernel \( p_1 \); this operator is compact, self-adjoint, and positivity-improving; moreover, the constant functions are eigenvectors with eigenvalue 1. Therefore, by the Perron–Frobenius theorem, \( P^k \) converges in operator norm to projection onto constant functions as \( k \to \infty \). Writing
\[
p_{k+2}(s, \sigma) = \langle p(s, \cdot), P^k p(\cdot, \sigma) \rangle_{L^2(S^2)},
\]
this immediately implies (56) and so completes the proof of the proposition. \qed
Now that we have established existence and uniqueness of the Gibbs measure for the spin chain model \( \mathcal{M} \), at inverse temperature \( \beta > 0 \), we wish to prove almost sure global existence and uniqueness of solutions to \( \mathcal{M} \) for data distributed according to this measure. We will work with the following notion of solution:

**Definition 5.3.** We say that a global solution \( \tilde{S} : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{S}^2 \) to the spin chain model \( \mathcal{M} \) is a good solution if it satisfies the following:

\[
\int_{-T}^{T} \sum_{n \in \mathbb{Z}} \frac{\langle n \rangle^{-q}}{[1 + \tilde{S}_n(t) \cdot \tilde{S}_{n+1}(t)]^p} \, dt < \infty \quad \text{for some } p > q > 1 \text{ and all } T > 0, \tag{57}
\]

\[
\sup_{|t| \leq T} \sum_{n \in \mathbb{Z}} e^{-c(n)} < \infty \quad \text{for some } c > 0 \text{ and all } T > 0. \tag{58}
\]

Note that the property of being a good solution is invariant under rigid rotations (the natural gauge transformations), as well as space and time translations. In view of the denominators in \( \mathcal{M} \), it is necessary to avoid consecutive spins being antiparallel. The above restriction is a more quantitative version of this that allows us to prove uniqueness and is connected to our notion of good solution to \( \mathcal{M} \) via the discrete Hasimoto transform. We do not know if uniqueness holds for completely general classical solutions to \( \mathcal{M} \).

**Proposition 5.4** (Uniqueness of good solutions). Let \( \tilde{S}(t) \) and \( \tilde{U}(t) \) be global good solutions to \( \mathcal{M} \) with initial data \( \tilde{S}(0) = \tilde{U}(0) \). Then \( \tilde{S}(t) = \tilde{U}(t) \) for all \( t \in \mathbb{R} \).

**Proof.** Fix \( T > 0 \). As \( \tilde{S} \) and \( \tilde{U} \) verify \( \mathcal{M} \), there exist \( \sigma \in (0, 1), c > 0 \), and positive constants \( A_T \) and \( B_T \) such that

\[
\int_{-T}^{T} \sup_{|n| \leq 2N} \left[ 1 + \frac{1}{1 + \tilde{S}_n(t) \cdot \tilde{S}_{n+1}(t)} + \frac{1}{1 + \tilde{U}_n(t) \cdot \tilde{U}_{n+1}(t)} \right] \, dt \leq A_T N^p, \tag{59}
\]

uniformly for \( N \geq 2 \) and

\[
\sup_{|t| \leq T} \sum_{n \in \mathbb{Z}} e^{-c|n|} \left[ 1 + \frac{1}{1 + \tilde{S}_n(t) \cdot \tilde{S}_{n+1}(t)} + \frac{1}{1 + \tilde{U}_n(t) \cdot \tilde{U}_{n+1}(t)} \right] \leq B_T. \tag{60}
\]

To continue, for \( t \in [-T, T] \) we define

\[ M(t) = \sum_{n \in \mathbb{Z}} e^{-2c|n|} |\tilde{S}_n(t) - \tilde{U}_n(t)|^2, \]

where \( C > 0 \) denotes a large constant to be chosen later. A straightforward computation yields

\[
\frac{dM}{dt} = -4 \sum_{n \in \mathbb{Z}} e^{-2c|n|} (\tilde{S}_n - \tilde{U}_n) \left\{ \frac{\tilde{S}_n + \tilde{S}_{n+1}}{|\tilde{S}_n + \tilde{S}_{n+1}|^2} \times \tilde{S}_{n+1} - \frac{\tilde{U}_n + \tilde{U}_{n+1}}{|\tilde{U}_n + \tilde{U}_{n+1}|^2} \times \tilde{U}_{n+1} \right. \\
+ \left. \frac{\tilde{S}_n + \tilde{S}_{n-1}}{|\tilde{S}_n + \tilde{S}_{n-1}|^2} \times \tilde{S}_{n-1} - \frac{\tilde{U}_n + \tilde{U}_{n-1}}{|\tilde{U}_n + \tilde{U}_{n-1}|^2} \times \tilde{U}_{n-1} \right\}. 
\]

Using \( |\tilde{a} \times \tilde{b} - \tilde{c} \times \tilde{d}| \leq |\tilde{a} - \tilde{c}| |\tilde{b}| + |\tilde{c}| |\tilde{b} - \tilde{d}| \) followed by the arithmetic–geometric mean inequality, we get

\[
\left| \frac{dM}{dt} \right| \leq 4 \sum_{n \in \mathbb{Z}} e^{-2c|n|} |\tilde{S}_n - \tilde{U}_n| \left\{ \frac{|	ilde{S}_n - \tilde{U}_n| + |	ilde{S}_{n+1} - \tilde{U}_{n+1}|}{|\tilde{S}_n + \tilde{S}_{n+1}| |\tilde{U}_n + \tilde{U}_{n+1}|} + \frac{|	ilde{S}_{n+1} - \tilde{U}_{n+1}|}{|\tilde{U}_n + \tilde{U}_{n+1}|} \right. \\
+ \left. \frac{|	ilde{S}_n + \tilde{S}_{n-1}|}{|\tilde{S}_n + \tilde{S}_{n-1}|^2} \times |\tilde{S}_{n-1} - \tilde{U}_{n-1}| + \frac{|	ilde{U}_n + \tilde{U}_{n-1}|}{|\tilde{U}_n + \tilde{U}_{n-1}|^2} \times |\tilde{U}_{n-1} - \tilde{U}_{n-1}| \right\}.
\]
\[ + |S_n - U_n| + |S_{n-1} - U_{n-1}| + |S_{n-1} - U_{n-1}| \right) \]
\[ \leq C e^{2c} \sup_{|n| \leq 2N} \left\{ 1 + \frac{1}{1 + S_n \cdot S_{n+1}} + \frac{1}{1 + U_n \cdot U_{n+1}} \right\} M(t) \]
\[ + C e^{2c} \sum_{|n| \geq N} e^{-2c|n|} \left\{ 1 + \frac{1}{1 + S_n \cdot S_{n+1}} + \frac{1}{1 + U_n \cdot U_{n+1}} \right\} \]

for some absolute constant \( C \) and any \( N \geq 2 \). As \( M(0) = 0 \) by assumption, combining Gronwall with (59) and (60) yields
\[ \sup_{|t| \leq T} |M(t)| \leq C e^{2c} T B_T \exp \{ -cN + C e^{2c} A_T N^\sigma \} \rightarrow 0 \text{ as } N \rightarrow \infty. \]

Therefore, \( M(t) = 0 \) for all \( |t| \leq T \). As \( T > 0 \) was arbitrary, this shows that \( S(t) = U(t) \) for all \( t \in \mathbb{R} \).

We are now ready to tackle Theorem 1.2 whose proof will occupy the remainder of this section.

**Proof of Theorem 1.2.** We first address the existence of global good solutions to (9), for which we will rely on the results of Sections 3 and 4. Specifically, Theorem 4.3 guarantees the existence of a full measure set of initial data distributed according to the white noise measure \( d\mu_{\text{wn}} \) for which there exist unique global good solutions to (11). Let \( \alpha(0) \) belong to this full measure set of initial data and let \( \alpha : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{C} \) denote the unique global good solution to (11) with initial data \( \alpha(0) \). Let \( O \in \text{SO}(3) \) be an independent random variable distributed according to Haar measure. (This plays the role of a random choice of gauge.) For \( n \in \mathbb{Z} \), we define \( Q_n(t) \) and \( P_n(t) \) as in (35) through (38). By Theorem 3.4, \( \tilde{S}(t) = \{ \tilde{S}_n(t) \}_{n \in \mathbb{Z}} \) defined as in (11) is a global solution to (9). Moreover, since \( \alpha \) verifies (43) and (44), it is easy to check that \( \tilde{S} \) verifies (57) and (58), and so it is a global good solution to (9). Proposition 5.4 shows that this solution is uniquely determined by the initial data. This is important since (due to gauge invariance) each initial configuration \( \tilde{S}(0) \) results from continuum many choices of \( \alpha(0) \) and \( O \).

Next, we have to verify that the initial data \( \tilde{S}(0) \) for the solution to (9) constructed above is indeed distributed according to the Gibbs measure \( d\mu_{\text{Gibbs}} \). This is the scope of the next proposition. In fact, together with Proposition 5.4, our next result also proves that the Gibbs measure \( d\mu_{\text{Gibbs}} \) is left invariant by the flow of (9), thus completing the proof of Theorem 1.2.

**Proposition 5.5.** For any \( t \in \mathbb{R} \), the sequence \( \tilde{S}(t) = \{ \tilde{S}_n(t) \}_{n \in \mathbb{Z}} \) is distributed according to the Gibbs measure \( d\mu_{\text{Gibbs}} \).

**Proof.** The proof proceeds in two steps: First we verify the invariance of the joint law \( d\text{Haar} \mu_{\text{wn}}^\beta \) under the flow given by (11) and (37). Then we prove that the measure on the spins induced by \( d\text{Haar} \mu_{\text{wn}}^\beta \) agrees with \( d\mu_{\text{Gibbs}}^\beta \).

**Step 1.** To verify invariance of the joint law \( d\text{Haar} \mu_{\text{wn}}^\beta \) under the flow given by (11) and (37), it suffices to show that for any \( N \geq 0 \) and any bounded continuous
function $F : \text{SO}(3) \times \mathbb{R}^{2N+1} \to \mathbb{R}$ we have

$$
\int\int F(P_0(t), \alpha_{-N}(t), \ldots, \alpha_N(t)) \, d\text{Haar}(P_0(0)) \, d\mu_{\text{wn}}^\beta(\{\alpha(0)\})
= \int\int F(P_0(0), \alpha_{-N}(0), \ldots, \alpha_N(0)) \, d\text{Haar}(P_0(0)) \, d\mu_{\text{wn}}^\beta(\{\alpha(0)\})
$$

(61)

for all $t \in \mathbb{R}$.

To this end, let $\mathcal{A}$ denote the $\sigma$-algebra generated by the random variables $\alpha_n(0)$. For a full measure set of initial data, there exists a unique global good solution $\alpha(t)$ to (11). This shows that $\alpha(t)$ is $\mathcal{A}$-measurable for all $t \in \mathbb{R}$. Moreover, defining $A_0(t)$ via (60) and then $\Phi(t)$ by

$$
\frac{d}{dt}\Phi(t) = \Phi(t)A_0(t) \quad \text{with} \quad \Phi(0) = \text{Id},
$$

we see that $\Phi(t)$ is also $\mathcal{A}$-measurable. Note that $P_0(0)$ is independent of $\mathcal{A}$.

Thus, by right-invariance of the Haar measure followed by invariance of the white noise measure under the flow of (11), we obtain

$$
\text{LHS}(61) = E_\beta\left\{ E_\beta\left\{ F(P_0(t)\Phi(t), \alpha_{-N}(t), \ldots, \alpha_N(t)) | \mathcal{A} \right\} \right\}
= \int\int F(\mathcal{O}, \alpha_{-N}(t), \ldots, \alpha_N(t)) \, d\text{Haar}(\mathcal{O}) \, d\mu_{\text{wn}}^\beta(\{\alpha(0)\})
= \int\int F(\mathcal{O}, \alpha_{-N}(0), \ldots, \alpha_N(0)) \, d\text{Haar}(\mathcal{O}) \, d\mu_{\text{wn}}^\beta(\{\alpha(0)\}) = \text{RHS}(61).
$$

This proves invariance of the joint law $d\text{Haar} \, d\mu_{\text{wn}}^\beta$.

These arguments also yield the law of a single spin: In view of (38), for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, there is an $\mathcal{A}$-measurable matrix $\Phi_n(t) \in \text{SO}(3)$ so that

$$
\tilde{S}_n(t) = P_0(0)\Phi_n(t)e_3; \quad \text{indeed,} \quad \Phi_n(t) = \begin{cases} \Phi(t)Q_0(t) \cdots Q_{n-1}(t) & : n \geq 0 \\ \Phi(t)Q_{-1}(t)^T \cdots Q_n(t)^T & : n \leq 0. \end{cases}
$$

As $P_0(0)$ is Haar distributed and independent of $\mathcal{A}$,

$$
E_\beta\{g(\tilde{S}_n(t))\} = E_\beta\left\{ E_\beta\{g(P_0(0)\Phi_n(t)e_3) | \mathcal{A} \} \right\} = \frac{1}{\gamma} \int_{\mathbb{S}^2} g(s) \, ds.
$$

(62)

**Step 2.** To verify that the measure induced by the joint law $d\text{Haar} \, d\mu_{\text{wn}}^\beta$ on the spins $\{\tilde{S}_n(t)\}_{n \in \mathbb{Z}}$ agrees with the Gibbs measure $d\mu_{\text{Gibbs}}^\beta$, it suffices to verify that the induced measure gives the same marginals as (55).

To this end, fix $t \in \mathbb{R}$. For $k \in \mathbb{Z}$, we let $\mathcal{A}_k$ denote the $\sigma$-algebra generated by the random variables $\{P_n(t)\}_{n \leq k}$, or equivalently, by $\{P_k(t), \{\alpha_n(t)\}_{n \leq k-1}\}$. Note that $\tilde{S}_l(t)$ is $\mathcal{A}_k$ measurable if and only if $l \leq k$.

The key observation is the following:

**Lemma 5.6.** For any bounded and continuous function $f$ and any integers $n \leq m$,

$$
E_\beta\left\{ f(\tilde{S}_n(t), \ldots, \tilde{S}_m(t)) \bigg| \mathcal{A}_{m-1} \right\}
= \int_{\mathbb{S}^2} f(\tilde{S}_n(t), \ldots, \tilde{S}_{m-1}(t), s_m) \, p(\tilde{S}_{m-1}(t), s_m) \, ds_m.
$$

(63)
Proof. We use the notation of Section 3. As $\alpha_{m-1}(t)$ is independent of $A_{m-1}$,

$$E_\beta\{g(Q_{m-1}(t)\hat{e}_3)|A_{m-1}\} = \int_0^{2\pi} \int_0^\infty g\left(\frac{1}{1+r^2} \left[\begin{array}{c} 2r \cos(\theta) \\ 2r \sin(\theta) \end{array}\right]\right) \frac{(1+2\beta)r}{\pi(1+r^2)^{2+2\beta}} dr \, d\theta$$

for any bounded and continuous function $g$. Here we used polar coordinates to obtain the last equality. Consequently, \hfill $\square$

of initial data. Finally, Proposition 5.5 proves that the Gibbs measure $d\mu_{\beta}$

initial data for which one can construct global good solutions to (9). Proposition 5.4

left invariant by the flow of (9), thus completing the proof of Theorem 1.2.

\hfill $\square$

This completes the proof of the lemma.

Applying Lemma [5.6] inductively and then [62], we obtain

$$E_\beta\{f(S_n(t), \ldots, S_{m-1}(t))|A_{m-1}\}$$

for any bounded and continuous function $g$. Here we used polar coordinates in the form $\alpha_{m-1}(t) = re^{-i\theta}$. Changing variables via $\cos(\phi) = \frac{r^2}{1+r^2}$, with $\phi \in [0, \pi)$ yields

$$E_\beta\{g(Q_{m-1}(t)\hat{e}_3)|A_{m-1}\} = \int_0^{2\pi} \int_0^\pi g\left(\frac{\sin(\phi) \cos(\phi)}{\cos(\phi)}\right) \frac{1+2\beta}{4\pi} \left[\frac{1+\cos(\phi)}{2}\right]^{2\beta} \sin(\phi) \, d\phi \, d\theta$$

where we used spherical coordinates to obtain the last equality. Consequently,

$$\text{LHS}[63] = E_\beta\left\{f(S_n(t), \ldots, S_{m-1}(t), P_{m-1}(t)Q_{m-1}(t)\hat{e}_3)|A_{m-1}\right\}$$

$$= \int_{\mathbb{R}^2} f(S_n(t), \ldots, S_{m-1}(t), s) \frac{1+2\beta}{4\pi} \left[\frac{1+s \cdot \hat{e}_3}{2}\right]^{2\beta} \, ds = \text{RHS}[63].$$

This completes the proof of the lemma. \hfill $\square$

To recapitulate, Proposition 5.3 shows that there exists a full measure set of

initial data for which one can construct global good solutions to (9). Proposition 5.4

then guarantees the uniqueness of these global good solutions for a full measure set of

initial data. Finally, Proposition 5.5 proves that the Gibbs measure $d\mu_{\beta}$ is

left invariant by the flow of (9), thus completing the proof of Theorem 1.2. \hfill $\square$
[8] M. Christ, Power series solution of a nonlinear Schrödinger equation. Mathematical aspects of nonlinear dispersive equations, 131–155, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.
[9] J. Colliander and T. Oh, Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(T)$. Duke Math. J. 161 (2012), no. 3, 367–414.
[10] W. Ding, On the Schrödinger flows. Proceedings of the International Congress of Mathematicians (Beijing, 2002), Vol. II, 283–291, Higher Ed. Press, Beijing, 2002.
[11] P. L. Dobruschin, The description of a random field by means of conditional probabilities and conditions of its regularity. Theory Probab. Appl. 13 (1968), no. 2, 197–224.
[12] A. Doliwa and P. M. Santini, Integrable dynamics of a discrete curve and the Ablowitz-Ladik hierarchy. J. Math. Phys. 36 (1995), no. 3, 1259–1273.
[13] N. M. Ercolani and G. Lozano, A bi-Hamiltonian structure for the integrable, discrete nonlinear Schrödinger system. Phys. D 218 (2006), no. 2, 105–121.
[14] L. D. Faddeev and L. Takhtajan, Hamiltonian methods in the theory of solitons. Classics in Mathematics. Springer, Berlin, 2007.
[15] J. Fröhlich, A. Knowles, and E. Lenzmann, Semi-classical dynamics in quantum spin systems. Lett Math Phys 82 (2007), no. 2–3, 275–296.
[16] T. L. Gilbert, A phenomenological theory of damping in ferromagnetic materials. IEEE Transactions on Magnetics 40 (2004), no. 6, 3443–3449.
[17] A. Grünrock and S. Herr, Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data. SIAM J. Math. Anal. 39 (2008), no. 6, 1890–1920.
[18] Z. Guo and T. Oh, Non-Existence of Solutions for the Periodic Cubic NLS below $L^2$. Int. Math. Res. Not. IMRN 2018, no. 6, 1656–1729.
[19] F. D. M. Haldane, Excitation spectrum of a generalised Heisenberg ferromagnetic spin chain with arbitrary spin. J. Phys. C: Solid State Physics 15 (1982), no. 36, L1309–L1312.
[20] H. Hasimoto, A soliton on a vortex filament. J. Fluid Mech. 51 (1972), no. 3, 477–485.
[21] Y. Hong and C. Yang, Strong convergence for discrete nonlinear Schrödinger equations in the continuum limit. Preprint arXiv:1806.07542.
[22] Y. Ishimori, An integrable classical spin chain. J. Phys. Soc. Jpn. 51 (1982), no. 11, 3417–3418.
[23] A. G. Izergin and V. E. Korepin, A lattice model connected with a nonlinear Schrödinger equation. Dokl. Akad. Nauk SSSR 259 (1981), no. 1, 76–79.
[24] R. L. Jerrard and D. Smets, On the motion of a curve by its binormal curvature. J. Eur. Math. Soc. (JEMS) 17 (2015), no. 6, 1487–1515.
[25] R. Killip, M. Visan, and X. Zhang, Low regularity conservation laws for integrable PDE. To appear in Geom. Funct. Anal.
[26] N. Kishimoto, A remark on norm inflation for nonlinear Schrödinger equations. Preprint arXiv:1806.10066.
[27] H. Koch and T. Tataru, Conserved energies for the cubic NLS in 1-d. Preprint arXiv:1607.02534.
[28] M. Lakshmanan, Continuum spin system as an exactly solvable dynamical system. Phys. Lett. A 61 (1977), no. 1, 53–54.
[29] L. Landau and E. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. Phys. Z. Sowjet. 8 (1935), 153–169.
[30] O. E. Lanford and D. Ruelle, Observables at infinity and states with short range correlations in statistical mechanics. Comm. Math. Phys. 13 (1969), no. 3, 194–215.
[31] J. L. Lebowitz, H. A. Rose, and E. R. Speer, Statistical mechanics of the nonlinear Schrödinger equation. J. Stat. Phys. 50 (1988), no. 3–4, 657–687.
[32] E. M. Lifshitz and L. P. Pitaevskii, Statistical physics. Part 2. Theory of the condensed state. Course of theoretical physics Vol. 9. Translated from the Russian by J. B. Sykes and M. J. Reesl. Pergamon Press, Oxford-Elmsford, N.Y., 1980.
[33] G. I. Lozano, Poisson geometry of the Ablowitz-Ladik equations. Ph.D. thesis, The University of Arizona, 2004.
[34] F. Magri, A simple model of the integrable Hamiltonian equation. J. Math. Phys. 19 (1978), no. 5, 1156–1162.
[35] N. Koiso, The vortex filament equation and a semilinear Schrödinger equation in a Hermitian symmetric space. Osaka J. Math. 34 (1997), no. 1, 199–214.
P. Petersen, Classical Differential Geometry. Lecture notes, available from the authors webpage: http://www.math.ucla.edu/~petersen/DGnotes.pdf

J. A. G. Roberts and C. J. Thompson, Dynamics of the classical Heisenberg spin chain. J. Phys. A 21 (1988), no. 8, 1769–1780.

I. Rodnianski, Y. A. Rubinstein, and G. Staffilani, On the global well-posedness of the one-dimensional Schrödinger map flow. Anal. PDE 2 (2009), no. 2, 187–209.

E. K. Sklyanin, Some algebraic structures connected with the Yang-Baxter equation. Functional Anal. Appl. 16 (1982), no. 4, 263–270.

P.-L. Sulem, C. Sulem, and C. Bardos, On the continuous limit for a system of classical spins. Comm. Math. Phys. 107 (1986), no. 3, 431–454.

L. A. Takhtajan, Integration of the continuous Heisenberg spin chain through the inverse scattering method. Phys. Lett. A 64 (1977), no. 2, 235–237.

K. L. Vaninsky, An additional Gibbs’ state for the cubic Schrödinger equation on the circle. Comm. Pure Appl. Math. 54 (2001), no. 5, 537–582.

V. E. Zakharov and L. A. Takhtadzhyan, Equivalence of the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet. Theor. Math. Phys. 38 (1979), no. 1, 17–23.

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