Response to the review on the manuscript AA12260 titled “Superiority in dense coding through non-Markovian stochasticity”

Dear Dr. Bill Munro,

Thank you very much for sending us the report on the manuscript AA12260 titled “Superiority in dense coding through non-Markovian stochasticity”. We would first like to thank all the Reviewers for their comments and criticisms that helped us to further improve our work. As suggested by the Referees, we have now provided additional explanations and interpretations of our results to further enrich our work. We have answered all the comments by all the Referees, and have made the corresponding changes (marked in red) in the manuscript to improve it.

Herewith we resubmit the manuscript for your consideration as a regular article in Physical Review A. We hope that after all these changes, the current form of the manuscript is now suitable for publication. In the following, we attach a detailed response to the Referees’ comments.

Yours sincerely,

Authors
Reply to the Report of the First Referee

Referee’s Comment: The authors presented the impact of non-Markovianity on dephasing and depolarizing channels. Their most significant discovery is that the channel’s randomness can counteract the negative effects of non-Markovian depolarizing channels in the dense coding protocol. The authors’ content is made more convincing by including both theoretical and numerical results. It is a significant finding that they have analytically proven that the dense coding capacity increases as the level of randomness increases in quantum channels with random noise, supported by numerical results shown in several graphs and tables. The study has carried out thorough analysis for scenarios with three or fewer senders and receivers, which is crucial for evaluating the scalability for N networks. I fully support accepting this manuscript for publication in Physical Review A if the necessary revisions to the minor requirements outlined below are made.

Author’s Response 1: We thank the Referee for finding our results to be significant and thorough. We now proceed to address his/her comments in order.

Referee’s Comment: In the proof for Theorem 1, the purpose of the function f in the proof is not immediately obvious. Can you explain the physical significance of the function and whether it only requires four parameters x, a, p and θ_j without any additional conditions?

Author’s Response 2: We thank the Referee for this important question. In the following discussion, we explain the function \( f(x, \alpha, p, \theta_j) \) appearing in the proof. In particular, this function \( f \) does not have any physical significance, it only helps us to write the eigenvalues of the resulting state in a compact form.

The \((N + 1)\)-qubit gGHZ state is given by

\[
|gGHZ\rangle^{N+1} = x|0\rangle^{\otimes N+1} + \sqrt{1-x^2}|1\rangle^{\otimes N+1}.
\]

The senders perform unitaries \( U_{s_j}^{\min}(\omega_j, \theta_j, \delta_j) \), minimizing \( S(\tilde{\rho}) \) in the dense coding capacity \( C_{\max}^1(\rho_{s_1...s_NR}) \), in Eq. (2) of the revised manuscript, before being affected by the noise \( \Lambda \). For each sender, \( S_j \), the two-dimensional unitary \( U_{s_j}^{\min} \) can be parameterized (upto an overall phase) by three parameters \( \omega_j, \theta_j \) and \( \delta_j \) as discussed in Eq. (9) of the revised manuscript. The noisy state \( \Lambda(\tilde{\rho}_{gGHZ}) \) has \( 2^{N+1} \) eigenvalues, all of which are functions of \( \{x, \alpha, p, \theta_j, \omega_j, \delta_j\} \) with \( \alpha \) and \( p \) being the strength of non-Markovianity and noise respectively. The analytical form of the eigenvalues is too complicated to present in the manuscript,
but close observation reveals that they can be written as

\[ \mu_k = \exp\left[ i \sum_{j=0}^{N-1} (\omega_j + \delta_j) \right] \sqrt{\exp\left[ -2i \sum_{j=0}^{N-1} (\omega_j + \delta_j) \right]} f_k(x, \alpha, p, \theta_j) = \sqrt{f_k(x, \alpha, p, \theta_j)}, \quad (1) \]

where \( k = 1, 2, \ldots, 2^{N+1} \) identifies the eigenvalues. Note that the functional form of \( f_k \) may be different for different eigenvalues and the only condition that \( f_k \) must satisfy is that \( \sum_k \sqrt{f_k(x, \alpha, p, \theta_j)} = 1 \), to ensure normalization. Even though the form of \( f_k \) is complicated, Eq. (1) indicates that the minimization of \( S(\Lambda(\tilde{\rho}_{gGHZ})) \) needs to be performed only over the variables \( \theta_j \) of each \( U_{S_j}^{\min} \), and not over \( \omega_j \) and \( \delta_j \). Numerical minimization of \( S(\Lambda(\tilde{\rho}_{gGHZ})) \) over \( \theta_j \) for \( N \) up to 10 reveals that the minimum occurs at \( \theta_{j_{\text{opt}}} \approx n\pi \forall j \). Thus, \( \cos \theta_{j_{\text{opt}}} \approx \pm 1 \) and \( \sin \theta_{j_{\text{opt}}} \approx 0 \) implying that the minimizing unitaries are proportional to the identity operator, i.e., \( U_{S_j}^{\min} \propto I \) (where \( I \) is the \( 2 \times 2 \) identity operator). Hence the proof.

We have now modified the proof of Proposition 1 accordingly, to make the analysis more coherent.

**Referee’s Comment:** The formulas used in the text are only explained in the appendix, which makes it difficult to comprehend the flow of the equations throughout the text. It is crucial that important notations that are not solely defined in the appendix be clearly stated in the main text. For instance, the notations for Equation (4) can only be found in Appendix B. As this is a crucial aspect of the equation’s usage, it is suggested that the information from the appendix be relocated to the main text.

**Author’s Response 3:** We thank the Referee for this valuable suggestion. We have now defined all the important quantities required in the main text (see Sec. IIA.)

**Referee’s Comment:** The notations used in equations (B3), (B8), and (C1) may cause confusion. If it represents the dagger of \( U^{\min} \), it is necessary to have a standardized notation. In particular, the parentheses used in equation (C1) have a typo and need to be corrected.

**Author’s Response 4:** We thank the Referee for pointing out this confusion. We have now updated the manuscript with a uniform notation for \( U^{\min \dagger} \). We have also corrected the typo present in Eq. (C1) (now referred to as Eq. (B1) in the revised manuscript).

**Referee’s Comment:** On page 5, it only briefly states that the optimal unit \( U^{\min} \) can be calculated numerically using the NLOPT algorithm for the dephasing channel, but this is not clear. An explanation of the algorithm is required, including why it converges while
obtaining $U^\text{min}$.

**Author’s Response 5:** We thank the Referee for the comment. We briefly describe the NLOPT algorithm used for obtaining $U^\text{min}$ in the discussion given below.

We use the Improved Stochastic Ranking Evolution Strategy (ISRES) algorithm based on the method described in Ref. [?]. The quantity being minimized is $S(\Lambda((U^\text{min}_{S_1} \otimes \cdots \otimes U^\text{min}_{S_N} \otimes I_R) \rho_{S_1 \cdots S_N R}(U^\text{min}_{S_1} \otimes \cdots \otimes U^\text{min}_{S_N} \otimes I_R)))$. Since each minimizing two-dimensional unitary $U^\text{min}_{S_i}$ can be parameterized by three variables $\omega_i, \theta_i$, and $\delta_i$, the entropy function involves optimization over the $3N$ quantities $\{\omega_i, \theta_i, \delta_i\}_{i=1}^N$ corresponding to the $N$ senders. The ranges of the variables are set as $0 \leq \omega_i, \delta_i \leq 2\pi$ and $0 \leq \theta \leq \pi$. Non-linear optimization using the NLOPT library [?] in C++ indeed implements ISRES algorithm. The evolution strategy involves two steps - mutation rule (with a log-normal step-size update and exponential smoothing) and differential variation. Since our optimization problem does not involve any non-linear constraint, the objective function itself determines the fitness ranking. The optimization algorithm is executed $10^5$ times in order to locate the global minimum. We use the aforementioned algorithm since it has the heuristics to escape local extrema present within the variable range. The convergence is set to $10^{-5}$ in the NLOPT routine, which guarantees that the minimum of the entropy function is correct up to the fifth decimal place. We first use the algorithm to find $U^\text{min}_{S_i}$ for the depolarizing channel, for which the minimizing unitaries should reduce to the identity for each sender (since the noise is covariant). We verify that this is indeed the case, i.e., $U^\text{min}_{S_i} = I$ to ensure that the ISRES algorithm is suited for our purpose before applying it for other types of noise.

We have now updated the manuscript incorporating the discussion about the used algorithm in Sec. II D.

**Referee’s Comment:** Please carefully check all references.

**Author’s Response 6:** We thank the Referee for this important suggestion. We have now diligently checked all the references and updated them.
Reply to the Report of the Second Referee

Referee’s Comment: The manuscript “Superiority in dense coding through non-Markovian stochasticity”, investigated the dense coding protocol under the non-Markovian noise. The authors investigated the impact of non-Markovian noise on dense coding protocols for multiple senders and one or two receivers. Their results show that non-Markovian dephasing noise can increase the dense coding capacity for three-party gGHZ shared states, but the effect is less pronounced for W-states or as the number of parties increases. The authors also found that high non-Markovian strength can mitigate the negative impact of dephasing channels on certain classes of states, but not for depolarizing noise. Additionally, replacing Pauli matrices with random unitaries improves the quenched averaged dense coding capacity and eliminates the negative impact of non-Markovian depolarizing channels on the dense coding protocol.

Author’s Response 1: We thank the Referee for reviewing our manuscript and summarising our results. We will now respond to his/her queries in detail.

Referee’s Comment: The results of the manuscript are not sound. The manuscript is a sort of exercise without solid motivations (foundational) and the results are largely predictable and not much interesting. Overall, the manuscript confirms that the authors know well the formalism of dense coding and noisy model, however, it fails to report novelty and interesting results. I regret I’m unable to recommend this manuscript for publication.

Author’s Response 2: We thank the Referee for this comment. We would like to draw the attention of the Referee to certain points which are essentially the motivation of our paper.

- Quantum dense coding is one of the first information theoretic protocols to be proposed and then has been experimentally realized in several physical systems [? ? ? ?]. During its realization, it is imperative that environmental impacts would induce unwanted noise in the system. Several recent works have been dedicated to study the protocol under the influence of Markovian noise, typically characterized by paradigmatic noise models such as the dephasing and depolarizing channels. It is unexpected that the actual disorder affecting the experimental setup would always cater to such mathematical formulations. Therefore, the aim of our work is to analyze how exactly
the performance of the dense coding scheme would be altered, when the noise itself behaves in a random fashion or the noise is non-Markovian. This formalism introduces unknown stochastic elements into the implementation of the protocol, which are completely beyond the users’ control.

• We factor stochasticity in the noise models by replacing the involved Pauli matrices, which define a specific noise model, with unitaries comprising randomly chosen parameters. Such a noise model, if present, would have a profound impact on the realization of the dense coding scheme, for example, it would destroy the covariant nature of the depolarising noisy channel, thereby rendering the experiment invalid if the assumption of that particular channel is taken into account in the experiment. Furthermore, inexact knowledge of the noise model would also prevent the application of the entropy-minimizing unitaries for encoding which are crucial for obtaining quantum advantage in the protocol. Therefore, we feel that such a direction is interesting and important to explore, especially when building quantum networks is one of the primary foci in quantum communication.

• Our results reveal that as we deviate from the Pauli nature of the noise, the protocol furnishes better results in the sense that one can overcome the detrimental effects of noise on the protocol. In other words, stochasticity has constructive impacts on the efficiency of the protocol. Note that, such a study requires several parameters to vary which is, in general, not easy to handle. On increasing the standard deviation of the Gaussian probability distribution, from which we sample the noise parameters, the noise would revert to its Pauli characteristics at a certain point. Therefore, we investigate the interplay between the randomness present in the noise and the enhanced quantum advantage due to noise. In particular, quantum advantage actually persists, if and only if the stochastic nature of the noise is within close range of the Pauli matrices. Therefore, we limit our analysis to standard deviations up to unity, which also hints at how one could construct the minimizing unitaries (by guessing the standard deviation, it is possible to incorporate its effects on minimizing unitaries through simple numerical calculations) in the experiment. However, if the randomness in the noise is unbounded, not only would it impart more destructive effects on the protocol, but would also create hindrances towards the successful implementation through
the inability to gauge the minimizing unitaries. Furthermore, we also report that the inherent randomness present in the noise would help to counteract the destructive effects of non-Markovianity, thereby highlighting a positive impact of an otherwise undesirable element.

We feel that our work adds to the growing literature on networks, that deal with classical information transmission. Moreover, as emphasized before, our work can be interesting from a more realistic point of view and provide a way out to overcome the detrimental effects of noise in quantum communication protocols. We have now added this discussion to the conclusion of our manuscript.
Reply to the Report of the Third Referee

Referee’s Comment: Referee report for “AA12260 is being considered for publication in Physical Review A as a regular article. Superiority in dense coding through non-Markovian stochasticity by Abhishek Muhuri, Rivu Gupta, Srijon Ghosh, et al.”

In this paper the authors have investigated the distributed dense coding (DC) protocol, involving multiple senders and a single or two receivers under the influence of non-Markovian noise, acting on the encoded qubits, which contains the information needed to be sent from one place to another. It is interesting to investigate how much information the senders can send when the encoded part is affected by noise. In general the noise is modeled by a Kraus-operator acting on the encoded part, and the parameters in the Kraus-operator formalism quantify the effect of noise in the system. In some sense one can consider it as a static noise.

In this manuscript the Authors make one step forward and investigate the effect of non-Markovian noise, i.e., when the noise retains memories of earlier stages of the evolution and influences later noise processes in the system. In general the quantum advantage is very fragile under the influence of noise, whereas the Authors have found that the effect of non-Markovian noise sometimes eradicate the negative influence of noisy channels which is not observed for depolarizing noise (Theorem 2), which no doubts a good direction to explore.

Author’s Response 1: We thank the Referee for reviewing our manuscript and summarising our results.

Referee’s Comment: However, the paper is well written and well organized, and I think the authors deal with an interesting problem of dense coding for open quantum dynamics. In my opinion I will recommend the paper for publication, once the authors provide a details response of my following queries/doubts.

Author’s Response 2: We thank the Referee for recommending the paper for publication. We will now respond to the Referee’s comments in detail.

Referee’s Comment: First of all ˜ρ mentioned in theorem 1, has not defined earlier in the main text, I have no clue what it is, unless you go to the Appendix.

Author’s Response 3: We thank the Referee for this important suggestion. We have now defined the relevant quantities in the main text in Sec. II A.

Referee’s Comment: In Eq. 2 the authors mentioned the Krauss operators for Marko-
vian and non-Markovian noise model, but in my opinion they should also explain for the reader how the parameters like $\alpha$ and $p$ are related for Markovianity and non-Markovianity.

**Author’s Response 4:** We thank the Referee for this significant comment. We have now added the following discussions after Eq. (2) in Sec. II B. to make the relation between the non-Markovianity parameter $\alpha$ and the noise strength $p$.

The degree of non-Markovianity is denoted by $\alpha$, where a higher value indicates more backflow of information from the environment into the system. The quantity $p$, on the other hand, represents the strength of noise acting on the system, i.e., an increase in $p$ implies that a greater amount of noise is affecting the system. In the Markovian limit (i.e., with $\alpha = 0$), the noise parameter $p$ varies as $0 \leq p \leq 0.5$ for the dephasing channel and as $0 \leq p \leq 1$ in the case of the depolarising channel. The allowed range of $p$ in the non-Markovian regime is the same as the Markovian one for the dephasing channel, whereas for finite non-Markovian behavior ($\alpha > 0$) in the depolarising channel, the noise strength is constrained to remain within $p \in (0, \frac{1}{3\alpha})$. This is to ensure that the Kraus operators for the depolarising channel remain positive and the channel still remains a completely positive trace-preserving (CPTP) map.

**Referee’s Comment:** It is important to give the derivation of Eq. (3), somewhere in the text, as the given form is little bit unclear to me. How many non-zero eigenvalues are there? Does all of them has the same functional dependence “$f(x, \alpha, p, \theta_j)$”?

**Author’s Response 5:** We thank the Referee for this important question. We give a revised version of the proof of Theorem 1., where we explain the function $f(x, \alpha, \theta_j)$ in a more comprehensive manner. See Author’s response 2 of Referee 1.

**Referee’s Comment:** Without giving the explicit form of “$f$” on $\theta_j$, how can one analytically optimize it, and yields the optimal value $\theta_{j_{opt}} \approx n\pi$.

**Author’s Response 6:** We again thank the Referee for this important comment. We have now clearly stated in the revised manuscript that the analytical form of the function $f$ is too complicated to present in the manuscript and that the optimization has been performed numerically for any arbitrary number of senders. We have modified the proof of Theorem 1 (which we now call Proposition 1) accordingly and have explained the function $f$ more elaborately, to make our discussion clearer.

**Referee’s Comment:** $C_{\text{noise}}^1$ and $B_{\text{noise}}^2$ mentioned in Eq. (4) without proper references.

**Author’s Response 7:** We thank the Referee for this comment. We have now added
the relevant references pertaining to $C_{\text{noise}}^1$ and $B_{\text{noise}}^2$ in the revised manuscript, as Ref. [62] and Ref. [63] respectively in Sec. II A.

**Referee’s Comment:** In Fig. 1, the authors claim that “$C_{\text{noise}}^1$ against the non-Markovian dephasing noise parameter, $p$ (abscissa) for different non-Markovianity, $\alpha$, when the shared resource states are the three-qubit GHZ and W states”. My question is which curve is for GHZ and which one is for W state. Does the $C_{\text{noise}}^1$ overlap for both the states?

**Author’s Response 8:** We thank the Referee for this question. We have now modified the caption of Fig. 1 to remove the confusion regarding the curves corresponding to the GHZ and W states. In particular, Fig. 1(a) represents the DC for $|\text{GHZ}\rangle^3$ state while Fig. 1(b) is for $|\text{W}\rangle^3$ state. We now present the new caption of Fig. 1. below.

(Color online.) **Two senders and a single receiver, $2S - 1R$ case under non-Markovian dephasing noise.** (a) $C_{\text{noise}}^1$ (ordinate) against the non-Markovian dephasing noise parameter, $p$ (abscissa) for different non-Markovianity, $\alpha$, when the shared resource state is the three-qubit GHZ which is affected by dephasing noise after encoding. The inset in (a) highlights the constructive effect of non-Markovianity at high noise strength on the GHZ state; (b) A similar study is performed for the shared W state. (c) and (d) illustrate DCC (ordinate) of three-qubit $|g\text{GHZ}\rangle^3$ and $|W_{1/2}\rangle^3$ states respectively against the state parameters $x$ and $b$ (abscissa) respectively for $p = 0.4$. The different non-Markovianity parameters are represented from dark (blue) to light (green) as $\alpha = 0.0$ (squares), $\alpha = 0.8$ (circles), $\alpha = 0.9$ (triangles) and $\alpha = 0.99$ (diamonds) respectively. The $x$-axis is dimensionless whereas the $y$-axis is in bits.

**Referee’s Comment:** For random choice of noise model how the Krauss operators are affected due to the random choice of the parameters of unitaries, is not clear to me.

**Author’s Response 9:** We thank the Referee for this interesting question. For the dephasing and the depolarising channels, the Kraus operators are defined by the Pauli matrices $\sigma_x, \sigma_y, \text{ and } \sigma_z$. When we characterize a random noise model, we incorporate the fluctuation in the standard Pauli matrices, keeping the framework of the dephasing and depolarising noise models intact. Precisely, we use the same Kraus operators as standard noise models but with the Pauli matrices replaced by random unitaries $U$ each of which is a function of three parameters $\omega, \theta, \text{ and } \delta$ chosen from a Gaussian distribution of the fixed mean (which are the desired values for recovering the Pauli matrices with vanishing standard deviation representing Pauli noise) and a given standard deviation. This construction ensures that
the random noise still remains a completely positive trace preserving (CPTP) map and is physically realizable. Thus, the only modification in the Kraus operators is that they are defined by random unitaries instead of Pauli matrices (the coefficients remain the same as in the original definition of the dephasing and depolarising noise).

In the updated manuscript, we have added this discussion as in Sec II. B. before Eq. (10).

**Referee’s Comment:** The authors also clarify that if they change the Pauli matrices of the depolarizing channel, to some randomly chosen unitaries, then is it legitimate to call them noisy depolarizing channel. Is this also true for dephasing channel.

**Author’s Response 10:** We thank the Referee for this important comment. By replacing the Pauli matrices with random unitaries, the noise which affects the protocol becomes random, but since the Kraus operators retain their original form, the nature of the noise also remains the same. For example, the action of the dephasing channel is to leave the state (on which it acts) unaffected with some probability, $p$ or change its phase by acting $\sigma_z$ with probability $(1 - p)$. Similarly, for the random “noisy” dephasing channel, the action of the noise is to either keep the state unaffected with a certain probability ($p$) or to change it by acting on it with a unitary close to the $\sigma_z$ operator (with probability $1 - p$). A similar argument holds for the noisy depolarising channel, where noise acts on the state from all directions $x, y, z$ but the action is quantified by random unitaries instead of the original Pauli matrices. Therefore, even in the presence of random unitaries, it is possibly legitimate to refer to the noise models as “noisy dephasing” and “noisy depolarising” channels, since the characters of the noise remain unchanged through the coefficients of the Kraus operators, i.e., “how” the noise disturbed on the system does not change, but its eventual effect does not correspond to that of the well-known Pauli noise models.

We have now added the above discussion as a remark in Sec II B. before Theorem 1.