Particle-antiparticle pair production from vacuum and the imaginary proper time method

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Abstract. Using the imaginary proper time method, based on the Fock method of solving the relativistic wave equations, the process of fermionic and scalar pairs production from a vacuum by a constant uniform electric field is considered. The relative (conditional) probability of the one pair production, which does not depend on the possible production of other pairs, was obtained with exponential accuracy, and the absolute probability of pair production, which takes into account the particle statistics, was obtained with pre-exponential accuracy.

1. Introduction
The semiclassical imaginary time method in the theory of ionization of atoms and ions by an external electromagnetic field was proposed in [1, 2], and its relativistic version — in [3, 4]. The process of particles-antiparticles pair production was considered on its basis as a special case of “ionization” of vacuum [5, 6], see also the review [7] and further literature references therein.

A natural generalization of this method in the relativistic case is the imaginary proper time method based on the Fock solution [8, 9] of relativistic wave equations. Here we consider on its basis the process of production from a vacuum by a constant uniform electric field of both fermionic and scalar pairs, for the latter of which the concept of the Dirac sea is not applicable.

The relative (conditional) probability of the production of one pair, which does not depend on the possible production of other pairs, will be obtained with exponential accuracy. At the same time, one can obtain the absolute probability of pair production, which takes into account the particle statistics, with pre-exponential accuracy. This is due to the fact that in a constant uniform field the semiclassical approximation is valid almost everywhere. In particular, when the electric field $\mathcal{E}$ is close to its critical value $\mathcal{E}_{cr}$ [10, 11], it is applicable everywhere, except for a narrow vicinity of turning points in the effective potential of the order of the Compton wave of particles. Therefore, the expressions for the absolute probability of the production of one fermionic or scalar pair from vacuum can be compared with the exact results of Schwinger [11], which are based on the Fock proper time method.

2. Proper time and the solution of relativistic wave equations
1. By introducing the independent variable $\tau$, as Fock did [8, 9], one can consider the space-time variables as generalized coordinates and use all the analytical tools of classical mechanics [12].
The Fock action integral of a particle in an electromagnetic field then reads

\[ S = \int_{\tau_0}^{\tau} L \, d\tau, \quad L = -\frac{1}{2} m (\dot{x}_\mu \dot{x}^\mu + c^2) + \frac{e}{c} A_\mu(x) \dot{x}^\mu, \quad e > 0 \]  \hspace{1cm} (1)

where \( x = \{x^\mu(\tau)\} = \{ct(\tau), r(\tau)\}, \) \( x_\mu = g_{\mu\nu} x^\nu, \) \( \{g_{\mu\nu}\} = \text{diag}(1, -1, -1, -1), \) \( \mu = 0, 1, 2, 3, \) and the dot denotes the derivative \( \dot{x} \equiv dx/d\tau \) with respect to the independent variable \( \tau. \)

Since the Lagrange function in (1) does not explicitly depend on \( \tau, \) the energy is an integral of motion. Assuming the value of this integral to be equal to zero, we obtain the condition

\[ \mathcal{H} = \dot{x}_\mu \frac{\partial L}{\partial \dot{x}^\mu} - L = \frac{1}{2} m (c^2 - \dot{x}_\mu \dot{x}^\mu) = 0, \]  \hspace{1cm} (2)

due to which the variable \( \tau \) becomes the proper time, and the Lagrange equations coincide with the standard [9, 14] equations of motion of an electron in proper time. Expressing "velocities" \( \dot{x}_\mu \) in terms of generalized momenta, we obtain the Hamiltonian of the system \( \mathcal{H}, \) and with it the Hamilton–Jacobi equation

\[ \frac{\partial S}{\partial \tau} + \frac{1}{2m} \left[ m^2 c^2 - \left( \frac{\partial S}{\partial x^\mu} - \frac{e}{c} A_\mu \right) \left( \frac{\partial S}{\partial \dot{x}_\mu} - \frac{e}{c} A^\mu \right) \right] = 0, \]  \hspace{1cm} (3)

where the action \( S(x; x_0; \tau - \tau_0) \) is considered as a function of the final \( x = x(\tau) \) and initial \( x_0 = x(\tau_0) \) coordinates and "time" \( \tau. \) And due to the equations of motion

\[ \frac{\partial}{\partial \tau} S(x; x_0; \tau - \tau_0) = -\mathcal{H} = 0, \]  \hspace{1cm} (4)

so that the independent variable \( \tau \) is the proper time.

2. As is known [9, 13], the solution of the Dirac equation [16] \( \mathcal{D} \psi = 0 \) for an electron in an external electromagnetic field can be represented as

\[ \psi = \mathcal{D}_+ \Psi, \quad \mathcal{D}_\pm = \gamma^\mu \left( i \hbar \frac{\partial}{\partial x^\mu} + \frac{e}{c} A_\mu \right) \pm mc. \]  \hspace{1cm} (5)

Here \( \Psi \) is the solution to the quadrated equation

\[ \mathcal{D} \Psi = 0, \quad \mathcal{D} = \mathcal{K} + i \frac{e}{\hbar c} (\alpha \mathcal{E} + i \Sigma \mathcal{B}), \]  \hspace{1cm} (6)

where \( \mathcal{K} \) is the operator of the Klein–Fock–Gordon (KFG) equation [17–19]

\[ \mathcal{K} \varphi(x) = 0, \quad \mathcal{K} = \left( i \frac{\partial}{\partial x^\mu} + \frac{e}{\hbar c} A_\mu \right) \left( i \frac{\partial}{\partial x^\mu} + \frac{e}{\hbar c} A^\mu \right) - \frac{m^2 c^2}{\hbar^2}, \]  \hspace{1cm} (7)

and four-row matrices \( \alpha, \Sigma \) were introduced by Dirac [16].

According to Fock [8, 9], the solution to equation (6) can be represented as a contour integral with respect to the variable \( \tau \)

\[ \Psi(x; x_0) = \int_C \mathcal{R} e^{i S(x; x_0; \tau - \tau_0)} f \, d\tau, \]  \hspace{1cm} (8)

where the values of the integrand must coincide at the ends of the integration contour \( C. \) The action integral \( S(x; x_0; \tau - \tau_0) \) in (8) satisfies the Hamilton–Jacobi equation (3), the function \( \mathcal{R} \) is expressed through a determinant composed of the second derivatives of the action, and the bispinor \( f \) satisfies the first order differential equation [9].
3. Semiclassical approximation and the imaginary proper time method

1. The integral in (8) in the semiclassical approximation can be calculated using the stationary phase method, and the stationary (saddle) point is determined by the equation (4). Therefore, the transition amplitude with exponential accuracy is

\[ T(x; x_0) \simeq \exp \left\{ \frac{i}{\hbar} S(x; x_0; \tau - \tau_0) \right\}, \tag{9} \]

where according to (4) the variable \( \tau \) is the proper time.

According to the Feynman formulation of quantum mechanics, the calculation of the probability amplitude of a certain particle motion is reduced to a sum (continual integral) over all possible alternatives (paths or trajectories) \[20\]. In the semiclassical approximation, the main contribution to the transition probability is determined only by the extreme trajectory \[1, 2\], which minimizes the imaginary part of the action integral,

\[ w = |T(x; x_0)|^2 = \exp \left\{ -\frac{2}{\hbar} \text{Im} S(x; x_0; \tau - \tau_0) \right\}. \tag{10} \]

This relation is a natural generalization of the imaginary time method to the relativistic case.

2. The Lagrange equations, conservation laws and the condition (2), which determines the proper time, solve the problem of calculating the particle trajectories in parametric form. As an example, let us consider the motion of an electron in a constant uniform electric field \( \mathcal{E} \), when the solution is obtained in closed form. In this case, calculating the action integral along the trajectory of motion and expressing the integrals of motion in terms of the initial values of the generalized coordinates, \( x(0) = x_0 \), we obtain

\[
S(x; x_0; \tau) = \frac{1}{2} m c_0 \cot \left( \frac{\omega_0 \tau}{2} \right) \left[ (x - x_0)^2 - c^2 (t - t_0)^2 \right] - \frac{1}{2} m \omega_0 c (t - t_0) (x + x_0) + \frac{m}{2\tau} \left( (y - y_0)^2 + (z - z_0)^2 \right) - \frac{1}{2} m c^2 \tau, \tag{11}\]

which, up to notation, coincides with Fock’s result \[9\] for \( \mathcal{B} = 0 \). Here \( \omega_0 \equiv e \mathcal{E}/mc \) and we set \( \tau_0 = 0 \). The function \( \mathcal{R} \) and bispinor \( f \) in (8) in this case are

\[
\mathcal{R} = \frac{\omega_0}{2\tau \sinh(\omega_0 \tau/2)}, \quad f = f_0 \left[ \cosh \left( \frac{\omega_0 \tau}{2} \right) - \alpha_x \sinh \left( \frac{\omega_0 \tau}{2} \right) \right]. \tag{12}\]

3. Let us find using the imaginary proper time method the relative (conditional), i.e. disregarding the Pauli exclusion principle, probability of the production of an \( e^+e^- \)-pair from a vacuum by a constant uniform electric field. Let \( \tau_0 \) be the proper time moment of the beginning of the electron creation process, when its total “kinetic” energy is equal to zero, \( mc^2 \varepsilon(\tau_0) = 0 \). Since the energy of a real electron is greater than or equal to \( mc^2 \), the creation process is of a tunnelling nature. It is convenient to consider that the electron leaves the effective barrier at \( \tau = 0 \), after which it moves in the classically allowed region and its action is defined in (11).

The condition \( \dot{t}(\tau_0) = \varepsilon(\tau_0) = 0 \) for the classical “trajectory of motion” leads to the purely imaginary initial proper time moment \( \tau_0 = i \eta_0/\omega_0, \eta_0 = \pi/2 \), so that all the sub-barrier motion of an electron occurs in imaginary proper time \( \tau = i \eta/\omega_0 \), which becomes real only after leaving the barrier, i.e. at \( \tau = 0 \). Calculating the action (11) along the extreme trajectory, we obtain

\[
S = i \frac{\pi mc^2}{4 \omega_0} \left( 1 + \frac{p^2}{m^2 c^2} \right), \quad \mathbf{p}_\perp \equiv (p_y, p_z). \tag{13}\]

\[1\] We use the gauge \( A_0 = -\mathcal{E} x, A_0 = 0 \), i.e. we assume that the field is directed along the \( x \) axis.
Then for the relative probability of the creation of an electron with momentum \(-\mathbf{p} = (-p_\parallel, -p_\perp)\) and a fixed spin from vacuum according to [10] we get

\[
\begin{align*}
\wp^{(e^-)}_{-\mathbf{p}} &= \exp \left[ -\frac{\pi \mathcal{E}_{cr}}{\mathcal{E}} \left( 1 + \frac{p_\parallel^2}{m^2c^2} \right) \right], \quad \mathcal{E}_{cr} = \frac{m^2c^3}{e\hbar},
\end{align*}
\]

where \(\mathcal{E}_{cr}\) is the critical field of quantum electrodynamics [10, 11].

By virtue of the electric charge conservation, the probability of the creation of a positron from vacuum is equal to the probability of creation of an electron, \(w^{(e^+)}_{\mathbf{p}} = w^{(e^-)}_{-\mathbf{p}}\), so for the conditional probability \(w^{(1/2)}_{\mathbf{p}}\), which we will call the probability of production of an \(e^+e^-\) pair with momentum \(\mathbf{p}\) from vacuum, we have \(w^{(1/2)}_{\mathbf{p}} = w^{(e^+)_{\mathbf{p}}}_{\mathbf{p}}w^{(e^-)}_{-\mathbf{p}}\). Since the concept of spin cannot be introduced at \(\hbar \to 0\), the result (14) does not depend on particle statistics, i.e. it is also valid for the production of a scalar particle, so for the production of a scalar pair with momentum \(\mathbf{p}\) we have \(w^{(0)}_{\mathbf{p}} = w^{(1/2)}_{\mathbf{p}}\). So the relative probability of the production of a particle-antiparticle pair with momentum \(\mathbf{p}\) from the vacuum in the semiclassical approximation reads

\[
\begin{align*}
\wp = \wp^{(1/2)} = w^{(0)}_{\mathbf{p}} &= \exp \left[ -\frac{\pi \mathcal{E}_{cr}}{\mathcal{E}} \left( 1 + \frac{p_\parallel^2}{m^2c^2} \right) \right].
\end{align*}
\]

### 4. Exact results
1. To obtain the absolute probability \(P^{(1/2)}_{\mathbf{p}}\) of an \(e^+e^-\) pair production with momentum \(\mathbf{p}\) from vacuum, according to Feynman [21], it is necessary to take into account the Pauli exclusion principle, according to which the lower continuum of solutions can be considered in the one-particle approach as the Dirac sea filled with vacuum electrons. Then the process of an \(e^+e^-\) pair production by the electric field can be considered as a result of electron tunneling from the lower continuum to the upper one, modified by the external field, see, for example, Fig. 1 in [22].

   Since the motion of an electron with negative energy can be considered as a backward motion in time of a positron [22] then for the absolute probability of \(e^+e^-\) pair production we have

\[
\begin{align*}
P^{(1/2)}_{\mathbf{p}} &= \exp \left[ -\frac{\pi \mathcal{E}_{cr}}{\mathcal{E}} \left( 1 + \frac{p_\parallel^2}{m^2c^2} \right) \right].
\end{align*}
\]

It turns out that this result, unlike (15), is accurate. As noted above, this is related to the fact that in a constant uniform electric field the motion of a charge is semiclassical almost everywhere, except for turning points. To connect the absolute \(P^{(1/2)}_{\mathbf{p}}\) and relative \(w^{(1/2)}_{\mathbf{p}}\) probabilities, we will introduce the notation \(C^{(1/2)}_{\mathbf{p},\nu}\) for the probability that no pair was born in a state with a momentum \(\mathbf{p}\) and a given spin, i.e. for the probability of the vacuum-vacuum transition. Then the absolute probability of one pair production is \(P^{(1/2)}_{\mathbf{p}} = C^{(1/2)}_{\mathbf{p},\nu}w^{(1/2)}_{\mathbf{p}}\) and, by virtue of the exclusion principle, two pairs cannot be born. Due to unitarity

\[
\begin{align*}
C^{(1/2)}_{\mathbf{p},\nu} + C^{(1/2)}_{\mathbf{p},\nu}w^{(1/2)}_{\mathbf{p}} = 1, \quad C^{(1/2)}_{\mathbf{p},\nu} = \left( 1 + w^{(1/2)}_{\mathbf{p}} \right)^{-1},
\end{align*}
\]

and for the average number of particles we have

\[
\begin{align*}
\mathcal{N}^{(1/2)}_{\mathbf{p}} = \frac{C^{(1/2)}_{\mathbf{p},\nu}}{C^{(1/2)}_{\mathbf{p},\nu}w^{(1/2)}_{\mathbf{p}}} = P^{(1/2)}_{\mathbf{p}}.
\end{align*}
\]

\(^2\) This circumstance is associated with the technical advantage of the imaginary proper time method, since a total change in imaginary time when moving along a sub-barrier trajectory occurs upon going around the branch point \(t_0 = imc/e\mathcal{E}\), see, for example, Fig. 2 in [22].
Hence, for the conditional probability \( w_\text{p}^{(1/2)} \), taking into account (17) and (16), we obtain
\[
 w_\text{p}^{(1/2)} = \frac{\mathcal{N}_\text{p}^{(1/2)}}{1 - \mathcal{N}_\text{p}^{(1/2)}} = \left\{ \exp \left[ \pi \frac{\mathcal{E}_\text{cr}}{\mathcal{E}} \left( 1 + \frac{p^2}{m^2c^2} \right) \right] - 1 \right\}^{-1},
\]
(19)
which agrees with exponential accuracy with the Schwinger’s result [11].

The total number of e\(^+\)e\(^-\)-pairs produced from vacuum in volume \( V \) during the observation time \( T \) is
\[
\mathcal{N}^{(1/2)} = 2 \int \frac{V d^3p}{(2\pi\hbar)^3} e^{-\pi\mathcal{E}_\text{cr} (1 + p^2/m^2c^2)/\mathcal{E}} = \Omega_4 \frac{e^2\mathcal{E}^2}{4\pi^3\hbar^2c} e^{-\pi\mathcal{E}_\text{cr}/\mathcal{E}}.
\]
(20)
Here the invariant 4-volume \( \Omega_4 = cTV \) and it is taken into account that for a pulse parallel to the field we have \( dp = e\mathcal{E} dt \), and the factor 2 is due to two possible directions of the electron spin. Equation (20) is the exact result [23], which confirms the validity of equation (16).

The total probability of the vacuum-vacuum transition, \( C_{\text{v}}^{(1/2)} \equiv \exp \{ -2\text{Im} W^{(1/2)}/\hbar \} \), by definition is
\[
C_{\text{v}}^{(1/2)} = \prod_{\text{p, s}} C_{\text{p, s}}^{(1/2)} = \exp \left\{ -2 \int \frac{V d^3p}{(2\pi\hbar)^3} \ln \left[ 1 + w_\text{p}^{(1/2)} \right] \right\}.
\]
Hence, for the imaginary part of the effective Lagrangian \( \mathcal{L}_{\text{eff}}^{(1/2)} = W^{(1/2)}/VT \), we obtain
\[
\text{Im} \mathcal{L}_{\text{eff}}^{(1/2)} = \frac{e^2\mathcal{E}^2}{8\pi^3\hbar c} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi\mathcal{E}_\text{cr}/\mathcal{E}},
\]
(21)
which completely coincides with the Schwinger’s result [11].

The total probability of pair production is
\[
P^{(1/2)} = 1 - C_{\text{v}}^{(1/2)} = 1 - \exp \left\{ -2\Omega_4 \mathcal{L}_{\text{eff}}^{(1/2)}/\hbar c \right\},
\]
(22)
According to Schwinger, see the text after equation (6.41) in [11],
\[
P^{(1/2)} \approx 2\Omega_4 \mathcal{L}_{\text{eff}}^{(1/2)}/\hbar c, \quad \mathcal{E} \ll \mathcal{E}_\text{cr},
\]
(23)
so that the probability of one pair production in a unit 4-volume is
\[
w_1^{(1/2)} \approx \frac{c}{4\pi^3\Omega_c} \left( \frac{\mathcal{E}}{\mathcal{E}_\text{cr}} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi\mathcal{E}_\text{cr}/\mathcal{E}}, \quad \Omega_c \equiv \left( \frac{\hbar}{mc} \right)^4,
\]
(24)
where \( \Omega_c \) is the Compton’s 4-volume. However, the approximate equation (23) follows from the exact one (22) when
\[
K e^{-\pi\mathcal{E}_\text{cr}/\mathcal{E}} \ll 1, \quad K \equiv \frac{1}{4\pi^3} \frac{\Omega_4}{\Omega_c} \left( \frac{\mathcal{E}}{\mathcal{E}_\text{cr}} \right)^2.
\]
(25)
Since the pre-exponential factor \( K \) in (25) can be very large even for \( \mathcal{E} \ll \mathcal{E}_\text{cr} \), in the expansion
\[
P^{(1/2)} \approx K \left\{ e^{-\pi\mathcal{E}_\text{cr}/\mathcal{E}} + \frac{1}{4} e^{-2\pi\mathcal{E}_\text{cr}/\mathcal{E}} \right\} + \frac{1}{4} K^2 e^{-2\pi\mathcal{E}_\text{cr}/\mathcal{E}} + \ldots
\]
you can’t keep even the second term in curly braces. Therefore, the consistent formula for the probability of an e\(^+\)e\(^-\)-pair production per unit volume per unit time has the form
\[
w_1^{(1/2)} = \frac{c}{4\pi^3\Omega_c} \left( \frac{\mathcal{E}}{\mathcal{E}_\text{cr}} \right)^2 e^{-\pi\mathcal{E}_\text{cr}/\mathcal{E}}
\]
(26)
i.e. the presence of the sum in the Schwinger’s formula (24) is excessive accuracy.

2. In the semiclassical approximation the dynamics does not depend on particle statistics. Therefore, similarly to the fermionic case, see also [23], for the probability of one scalar pair production per unit volume per unit time we obtain the expression

$$w_1^{(0)} = \frac{e}{(2\pi)^3\Omega_c} \left( \frac{\mathcal{E}}{\mathcal{E}_{cr}} \right)^2 e^{-\pi\mathcal{E}_{cr}/\mathcal{E}},$$

(27)

which differs only by the factor $1/2$ from the probability (26) of an $e^+e^-$-pair production.

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