Thermalization of an Interacting Quantum Field in the CTP-2PI Next-to-leading-order Large N Scheme

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Abstract

In this paper we use an $O(N)$-invariant scalar field of unbroken symmetry to investigate whether an interacting quantum field at the next-to-leading order Large $N$ approximation may show signs of thermalization. We develop the closed time-path (CTP) two-particle irreducible (2PI) effective action in powers of $1/N$, retaining up to next to leading order ($O(1)$) terms, and write down the corresponding (truncated) Schwinger-Dyson equations for its two point function. We show that in this approximation, the only translation invariant solutions to the Schwinger - Dyson equations are thermal. This provides a useful temperature concept without invoking a heat bath. Coupled with the familiar Kadanoff-Baym approach to quantum kinetic theory our result shows that at this order of approximation thermalization can occur, at least if initial conditions are smooth enough that a derivative expansion is valid. Our analytic result provides support for similar claims in recent literature based on numerical evidence.

1 Introduction and Summary

The problem of thermalization in relativistic quantum fields has drawn much attention over time, both in its own right in our attempt to understand the origin of macroscopic irreversible

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behavior from microscopic theories, and in the context of nonequilibrium quantum field processes in the early Universe and in relativistic heavy ion collisions.

The goal of this paper is to investigate whether at the next-to-leading order (NLO) Large $N$ approximation an interacting quantum field may show signs of thermalization. We consider a $O(N)$ invariant scalar field of unbroken symmetry, develop the closed time-path (CTP) 2-particle irreducible (2PI) effective action in powers of $1/N$, retaining up to the next-to-leading order ($O(1)$) terms, and write down the corresponding (truncated) Schwinger-Dyson equations for its two point function. We show that in this approximation, the only translation invariant solutions to the Schwin ger- Dyson equations are thermal. Thus, without having it coupled to a heat bath this provides a useful temperature concept. Together with the familiar Kadanoff-Baym approach to quantum kinetic theory, this result shows that at this order of approximation thermalization can occur, at least if initial conditions are smooth enough that a derivative expansion is valid. Our analytic result provides support for similar claims in recent literature based on numerical evidence. We will not address the bigger question of whether the thermal solutions are in any sense an attractor with a nontrivial basin of attraction, as it is considerably more difficult. (We may look at the theory of glasses to see just how involved the long time behavior of a field theory can be.)

However, to appreciate this result, we must distinguish between a ‘true’ NLO approximation (truthful to the scheme) to the full quantum theory, and the theory which results from solving the NLO Schwinger-Dyson equations as if they were exact. By a ‘true’ NLO approximation we mean that, after deriving the Schwinger-Dyson equations up to NLO corrections, the solution to these equations is expanded as well, and terms of higher order which then result are discarded. In the second – call it the ‘exact’– procedure, once the equations are written down, a solution is sought, which will involve terms at all orders in $1/N$.

To give an example, the situation is similar to the usual textbook derivation of the running of coupling constants in an interacting field theory. A renormalization group equation, let us say for a $\lambda \phi^4$ scalar field theory, is derived within perturbation theory as

$$\frac{d \lambda}{d \mu} = \epsilon \lambda + a \lambda^2$$

where $\mu$ is a renormalization parameter. We then solve this equation as if it were exact and get

$$\lambda(\mu) = \left(\frac{\mu}{\mu_0}\right)^\epsilon \frac{\lambda(\mu_0)}{1 - \frac{a\lambda(\mu_0)}{\epsilon} \left(\frac{\mu}{\mu_0}\right)^\epsilon - 1}$$

Call this the ‘exact’ way. Of course, a Taylor expansion of $\lambda(\mu)$ should include all powers of $\lambda(\mu_0)$, but only up to two powers were considered in Eq. (1). A true second order approximation would be restricted to

$$\lambda(\mu) = \left(\frac{\mu}{\mu_0}\right)^\epsilon \lambda(\mu_0) \left\{1 + \frac{a\lambda(\mu_0)}{\epsilon} \left[\left(\frac{\mu}{\mu_0}\right)^\epsilon - 1\right]\right\}$$
Call this the ‘true’ way. We would obtain this same solution if $\lambda^2$ in the second term in Eq. (4) is approximated by

$$\lambda^2 \sim \left(\frac{\mu}{\mu_0}\right)^\epsilon \lambda(\mu_0)^2$$

However, we usually believe that Eq. (2) is meaningful, although it oversteps the bounds within which Eq. (1) was derived, because it can be shown that it captures the leading terms in the perturbative expansion (the so-called leading logs), and because in any case we only use it in the regime in which the coupling constant is small.

In the thermalization problem, this procedure breaks down. If we adhere strictly to the ‘true’ NLO approximation, then the only thermalization mechanism left in the theory is binary scattering of on-shell particles. Since this process conserves particle number, the truncated theory allows thermalization with non vanishing chemical potential. The ‘exact’ theory, represented by the full hierarchy, on the other hand, does not conserve particle number [24], and so the chemical potential must vanish in true equilibrium states. We must conclude that a ”true” NLO approximation, in the above sense, fails to describe thermalization.

This problem actually disappears if we solve the NLO Schwinger-Dyson equations ‘exactly’ to all orders in $1/N$ (like going from Eq. (1) to (2)). In this procedure the density of states takes on a Breit-Wigner form, and we have states with all masses. In particular, now a state with squared momentum $-p^2 \geq 9M^2$ [we use signature (-,+,+,+) for the Minkowski metric and $M^2$ is the leading order physical mass] may decay into three on-shell particles whereby particle number is no longer conserved.

In this way, we see that the theory built on the NLO Schwinger-Dyson equations is able to describe thermalization, including relaxation of the chemical potential. This relaxation is a higher than NLO effect, and we must raise the issue of whether our analysis is still a meaningful approximation to the full theory. The problem is that in this case there is no analog of the ”leading log” concept that validates the running coupling constant Eq. (2) over Eq. (3). On the contrary, the NLO approximation discards $2 \to 4$ scattering of on-shell particles, which also violates particle number conservation. The contribution to the relaxation rate from these higher order effects is comparable to the decay of off-shell excitations [20].

We therefore conclude that, while the next to leading order approximation describes thermalization, it overestimates the relaxation time, and may not be a realistic picture of thermalization in actual physical systems.

1.1 The meaning of thermalization

Let us begin with a discussion of the exact meaning of thermalization. In the strictest sense an isolated system depicted by quantum field theory undergoes unitary evolution and does not thermalize. However, one can still ask meaningful questions such as whether certain correlation functions may converge to their thermal forms in some well defined physical limit (weak thermalization).
As a matter of fact even asking this type of questions is too ambitious. The Schwinger -Dyson equations for the correlation functions form an infinite hierarchy. Physical limitations on the precision of our measurements amounts to truncating the hierarchy. When certain causal boundary condition such as Boltzmann’s molecular chaos hypothesis is imposed, the truncated subsystem will show signs of irreversibility and a tendency to equilibrate. In practical terms we have to deal with a truncated hierarchy to make the analysis possible. Moreover, nontrivial (point) field theories are plagued by divergences which can only be controlled by regularization and renormalization within some perturbative scheme. Therefore the question of weak thermalization can only make sense within a chosen approximation scheme, whether this is a 1/N expansion, loop expansion, expansion in powers of a coupling constant, etc., to allow us to organize the Schwinger - Dyson equation and evaluate the relative weight of different processes.

One must distinguish between two different viewpoints. If one accepts thermalization as an empirical fact, then there is only the question of coming up with a formulation which describes this process. For example, one may assume from the beginning that the relevant dynamics involves only the longest scales in space and time [25, 26, 27, 28], introduce a Wigner function as a partial Fourier transform of the propagators (as in the Kadanoff - Baym approach [17, 18, 19, 20]) and deduce a Boltzmann equation for the dynamics of this Wigner function [29, 30, 31]. Of course the resulting model describes thermalization.

However, this is not a proof that the original field theory thermalizes, even weakly, because there is no clear cut description of the set of initial conditions for which the Kadanoff-Baym approach, which depends upon a derivative expansion of the propagators, is valid. Mrowczynsky has shown that, for free theories, the only correlators that satisfy a reasonable almost-invariant condition are those which are exactly translation invariant [32]. In other words, the set of initial conditions which permits thermalization as described by the Kadanoff-Baym equations may be empty, but for the equilibrium solution itself. We may say that the Kadanoff - Baym formalism is useful for studying certain important questions, such as the determination of the transport coefficients [20], assuming one already knows on independent grounds that the system thermalizes. It describes certain important aspects of the thermalization process, but it does not address the conditions conducive to it.

On the other hand, if one does not assume thermalization, then one needs an approximation scheme which avoids the imposition of an arrow of time on the system by hand.

### 1.2 The Large $N$ Approximation Beyond Leading Order

The number $N$ of replicas of essentially identical fields (like the $N$ scalar fields in an $O(N)$ invariant theory, or the $N^2 - 1$ gauge fields in a $SU(N)$ invariant non-abelian gauge theory) suggests using 1/N as a natural small parameter, with a well defined physical meaning and that, unlike coupling constants, is not subjected to renormalization or radiative corrections. By ordering the perturbative expansion in powers of this small parameter, several nonperturbative effects (in terms of coupling constants) may be systematically investigated.
In the case of the $O(N)$ invariant theory, in the presence of a nonzero background field (or an external gravitational or electromagnetic field interacting with the scalar field) we may distinguish the longitudinal quantum fluctuations in the direction of the background field, in field space, from the $N-1$ transverse (Goldstone or pion) fluctuations perpendicular to it. To first order in $1/N$, the longitudinal fluctuations drop out of the formalism, so we effectively are treating the background field as classical. Likewise, quantum fluctuations of the external field are overpowered by the fluctuations of the $N$ scalar field. In this way, the $1/N$ framework provides a quantitative measure and concrete meaning to the semiclassical approximation \[33\].

To leading order (LO), the theory reduces to $N-1$ linear fields with a time dependent mass, which depends on the background field and on the linear fields themselves through a gap equation local in time. This depiction of the dynamics agrees both with the Gaussian approximation for the density matrix \[34, 35\] and with the Hartree approximation \[36\].

The ability of the $1/N$ framework to address the nonperturbative aspects of quantum field dynamics has motivated a detailed study of the properties of these systems. In non-equilibrium situations, this formalism has been applied to the dynamics of symmetry breaking \[11, 37, 38\] and self-consistent semiclassical cosmological models \[39, 40, 41, 42, 43\].

The LO $1/N$ theory is Hamiltonian \[36\] and time-reversal invariant. However, it does not thermalize. For example, if we set up conditions where both the background field and the self-consistent mass are space-time independent, then the particle numbers for each fluctuation mode will be conserved. The existence of these conservation laws precludes thermalization \[44\].

We note that the failure of the LO approximation to describe thermalization is indicative of a more general breakdown of the approximation at later times, where effects of particle interaction dominate. Both the distribution of energy among the field modes and the phase relationships (or lack thereof) among them affect the way quantum fluctuations react on the background or external fields. Therefore, from physical considerations, one can say that a theory which does not describe thermalization becomes unreliable for most other purposes as well \[45\].

This is where the next to leading order (NLO) approximation enters. It has been applied to quantum mechanics \[47, 48\], classical field theory \[49, 50, 51, 52\] and quantum field theory in 1 space dimension \[53\], being contrasted both to exact numerical simulations of these systems, as well as against other approximations purporting to go beyond LO. The NLO has been shown to be an accurate approximation, even at moderate values of $N$.

The $2PI$ formalism is also suitable for this question because, provided an auxiliary field is cleverly introduced, the $2PI$ CTP effective action can be found in closed form at each order in $1/N$ \[9, 46\].

1.3 This paper

This paper is organized as follows: In Section II we present our model, calculate the $2PI$ CTP EA, and discuss several properties of the propagators which hold to all orders in $1/N$. In Section III we discuss translation invariant solutions to the Schwinger-Dyson equations, still without
any explicit approximations. In Section IV we implement the large $N$ approximation, showing that it is possible to write down a closed expression for the 2PI CTP EA to next to leading order. In Section V we show that the only translation invariant solutions to the NLO Schwinger-Dyson equations are thermal. In Section VI we discuss the relaxation of the chemical potential. In the final Section, we ask a central question at the foundations of statistical mechanics: where does macroscopic irreversibility arise from microscopic reversible dynamics? We point out the exact spot where coarse-graining was introduced which leads to the appearance of thermalization.

2 The model

Let us consider a $O(N)$-invariant quantum scalar field $\Phi$, in the limit $N \to \infty$. The classical action

$$S = \int d^d x \left( -\frac{1}{2} \right) \left\{ \partial_\mu \phi^\alpha_\beta \partial^\mu \phi^\alpha_\beta + M^2_B \phi^\alpha_\beta \phi^\alpha_\beta + \frac{\lambda_B}{4N} \left( \phi^\alpha_\beta \phi^\alpha_\beta \right)^2 \right\}$$

(5)

where $\phi^\alpha_\beta$, $M^2_B$ and $\lambda_B$ are the bare wave function, mass parameter and coupling constant, soon to be renormalized, and the dimension $d = 4 - \varepsilon$. We introduce the bare wave function renormalization $Z_B$ by rescaling $\phi^\alpha_\beta = \sqrt{(N/Z_B)} \phi^\alpha$

$$S = \frac{N}{Z_B} \int d^d x \left( -\frac{1}{2} \right) \left\{ \partial_\mu \phi^\alpha \partial^\mu \phi^\alpha + M^2_B \phi^\alpha \phi^\alpha + \frac{\lambda_B}{4Z_B} \left( \phi^\alpha \phi^\alpha \right)^2 \right\}$$

(6)

Since later on we shall discuss in detail the conservation laws of the exact and approximated dynamics, let us observe that this theory conserves, besides energy-momentum, a number of Noether charges associated with the global $O(N)$ symmetry. Concretely, if the infinitesimal $O(N)$ transformation reads $\phi^\alpha \to \phi^\alpha + \varepsilon_A T^A_{\alpha\beta} \phi^\beta$ then the Noether charges are

$$Q^A = \frac{N}{Z_B} \int d^{d-1} x T^A_{\alpha\beta} \phi^\beta \dot{\phi}^\alpha$$

(7)

To investigate thermalization in total generality, we should allow for a Lagrange multiplier for each of these charges. Here we assume that all of these vanish, as well as the mean value of the charges themselves. Also, since there is no particle current the field is its own antiparticle, and the chemical potential must be zero in equilibrium. Discarding a constant term, we may rewrite the classical action as

$$S = \frac{N}{Z_B} \int d^d x \left( -\frac{1}{2} \right) \left\{ \partial_\mu \phi^\alpha \partial^\mu \phi^\alpha + \left[ \sqrt{\frac{Z_B}{\lambda_B}} M^2_B + \sqrt{\frac{\lambda_B}{Z_B}} \frac{\phi^\alpha \phi^\alpha}{2} \right]^2 \right\}$$

(8)

To set up the $1/N$ resummation scheme, it is customary to introduce the auxiliary field $\chi$ writing
\[
S = \frac{N}{Z_B} \int d^d x \left\{ -\frac{1}{2} \partial_\mu \phi^\alpha \partial^\mu \phi^\alpha + \left[ \sqrt{\frac{Z_B}{\lambda_B}} M_B^2 + \sqrt{\frac{\lambda_B}{Z_B}} \frac{\phi^\alpha \phi^\alpha}{2} \right]^2 - \frac{1}{2} \left[ \sqrt{\frac{Z_B}{\lambda_B}} \left( M_B^2 - \chi \right) + \sqrt{\frac{\lambda_B}{Z_B}} \frac{\phi^\alpha \phi^\alpha}{2} \right]^2 \right\}
\]
whence
\[
S = \frac{N}{Z_B} \int d^d x \left\{ -\frac{1}{2} \partial_\mu \phi^\alpha \partial^\mu \phi^\alpha - \chi \left[ \frac{Z_B}{\lambda_B} M_B^2 + \frac{\phi^\alpha \phi^\alpha}{2} \right] + \frac{Z_B}{2\lambda_B} \chi^2 \right\}.
\]
From now on, we consider \( \chi \) and \( \phi^\alpha \) as fundamental fields on equal footing. We assume a background field decomposition \( \phi \equiv \bar{\phi} + \varphi \) and that the background field \( \bar{\phi} \) is identically zero (because of the \( O(N) \) symmetry, the symmetric point must be a solution of the equations of motion) so we can focus on the dynamics of the fluctuation fields \( \varphi \). We also split the auxiliary field \( \chi \) field into a background \( \bar{\chi} \) and a fluctuation \( \tilde{\chi} \), \( \chi = \bar{\chi} + \tilde{\chi} \). The action becomes
\[
S = S_{\text{back}} + S_{\text{lin}} + S_{\text{quad}} + S_{\text{cub}}
\]
\( S_{\text{back}} \) is just the classical action evaluated at \( \varphi^\alpha = 0, \chi = \bar{\chi} \)
\[
S_{\text{back}} = \frac{N}{\lambda_B} \int d^d x \left\{ \frac{1}{2} \bar{\chi}^2 - M_B^2 \bar{\chi} \right\}
\]
\( S_{\text{lin}} \) contains terms linear on \( \tilde{\chi} \) and can be set to zero by a choice of the background field \( \bar{\chi} \).
\[
S_{\text{lin}} = \frac{N}{\lambda_B} \int d^d x \left\{ \bar{\chi} - M_B^2 \right\} \tilde{\chi}
\]
\( S_{\text{quad}} \) contains the quadratic terms and yields the tree - level inverse propagators
\[
S_{\text{quad}} = \frac{N}{Z_B} \int d^d x \left\{ \frac{-1}{2} \partial_\mu \varphi^\alpha \partial^\mu \varphi^\alpha - \frac{\bar{\chi}}{2} \varphi^\alpha \varphi^\alpha + \frac{Z_B}{2\lambda_B} \chi^2 \right\}
\]
Finally \( S_{\text{cub}} \) contains the bare vertex
\[
S_{\text{cub}} = \left( -\frac{N}{2Z_B} \right) \int d^d x \left\{ \bar{\chi} \varphi^\alpha \varphi^\alpha \right\}
\]
To write the \( 2PI \) CTP EA we double the degrees of freedom, incorporating a branch label \( a = 1, 2 \). We also introduce propagators \( G^{\alpha a, \beta b} \) for the path ordered expectation values
\[
G^{\alpha a, \beta b} (x, y) = \langle \varphi^{\alpha a} (x) \varphi^{\beta b} (y) \rangle
\]
and \( F^{ab} \) for
\[
F^{ab} (x, y) = \langle \bar{\varphi}^a (x) \bar{\varphi}^b (y) \rangle
\]
Because of symmetry, it is not necessary to introduce a mixed propagator, \( \langle \bar{\chi}^a(x) \varphi^{b\beta}(y) \rangle \equiv 0 \). The 2PI CTP EA reads

\[
\Gamma = S_{\text{back}}[\bar{\chi}^1] - S_{\text{back}}[\bar{\chi}^2] + \frac{1}{2} \int d^d x d^d y \left\{ D_{\alpha a,\beta b}(x,y) G^{\alpha a,\beta b}(x,y) + \frac{N}{\lambda_B} c_{ab} \delta(x,y) F^{ab}(x,y) \right\} - \frac{i \hbar}{2} [\text{Tr} \ln G + \text{Tr} \ln F] + \Gamma_Q
\]

(18)

where \( c_{11} = -c_{22} = 1, \ c_{12} = c_{21} = 0, \)

\[
D_{\alpha a,\beta b}(x,y) = \frac{N}{Z_B} \delta_{\alpha \beta} \left[ c_{ab} \partial_x^2 - c_{abc} \bar{\chi}^c \right] \delta(x,y)
\]

(19)

c_{abc} = 1 when all entries are 1, \( c_{abc} = -1 \) when all entries are 2, and \( c_{abc} = 0 \) otherwise. \( \Gamma_Q \) is the sum of all 2PI vacuum bubbles with cubic vertices from \( S_{\text{cub}} \) and propagators \( G^{\alpha a,\beta b} \) and \( F^{ab}(x,y) \). Observe that \( \Gamma_Q \) is independent of \( \bar{\chi}^c \).

Taking variations of the 2PI CTP EA and identifying \( \bar{\chi}^1 = \bar{\chi}^2 = \bar{\chi} \), we find the equations of motion

\[
\frac{N}{2Z_B} \delta_{\alpha \beta} c_{ab} D(x,y) - \frac{i \hbar}{2} \left[ G^{-1} \right]_{\alpha a,\beta b}(x,y) + \frac{1}{2} \Pi_{\alpha a,\beta b}(x,y) = 0
\]

(20)

\[
\frac{N}{2\lambda_B} c_{ab} \delta(x,y) - \frac{i \hbar}{2} \left[ F^{-1} \right]_{ab}(x,y) + \frac{1}{2} \Pi_{ab}(x,y) = 0
\]

(21)

\[
\frac{N}{\lambda_B} \left\{ \bar{\chi}(x) - M_B^2 \right\} - \frac{N}{2Z_B} \delta_{\alpha \beta} G^{\alpha 1,\beta 1}(x,x) = 0
\]

(22)

where \( D(x,y) = [\partial_x^2 - \bar{\chi}(x)] \delta(x,y), \)

\[
\Pi_{\alpha a,\beta b}(x,y) = 2 \frac{\delta \Gamma_Q}{\delta G^{\alpha a,\beta b}(x,y)}; \quad \Pi_{ab}(x,y) = 2 \frac{\delta \Gamma_Q}{\delta F^{ab}(x,y)}
\]

(23)

We shall seek a solution with the structure

\[
G^{\alpha a,\beta b}(x,y) = \frac{\hbar}{N} \delta_{\alpha \beta} G^{ab}(x,y)
\]

(24)

which is consistent with vanishing Noether charges. Then it is convenient to write

\[
F^{ab}(x,y) = \frac{\hbar}{N} H^{ab}(x,y); \quad \Pi_{\alpha a,\beta b}(x,y) = \delta_{\alpha \beta} P_{ab}(x,y); \quad \Pi_{ab}(x,y) = N Q_{ab}(x,y)
\]

(25)

The equations become
\[
\frac{1}{Z_B} c_{ab} D(x, y) - i \left[ G^{-1} \right]_{ab}(x, y) + \frac{1}{N} P_{ab}(x, y) = 0 \quad (26)
\]
\[
\frac{1}{\lambda_B} c_{ab} \delta(x, y) - i \left[ H^{-1} \right]_{ab}(x, y) + Q_{ab}(x, y) = 0 \quad (27)
\]
\[
\frac{1}{\lambda_B} \left\{ \bar{\chi}(x) - M_B^2 \right\} - \frac{\hbar}{2Z_B} G^{11}(x, x) = 0 \quad (28)
\]
Observe that
\[
P_{ab}(x, y) = \frac{2}{\hbar} \frac{\delta \Gamma_Q}{\delta \Gamma_{ab}(x, y)}; \quad Q_{ab}(x, y) = \frac{2}{\hbar} \frac{\delta \Gamma_Q}{\delta \Gamma_{ab}(x, y)} \quad (29)
\]
These are the exact equations we must solve. The successive 1/N approximations amount to different constitutive relations expressing \( P_{ab} \) and \( Q_{ab} \) in terms of the propagators.

### 2.1 The retarded propagator

For later use, we want an equation for the retarded propagator \( G_{ret} = i (G^{11} - G^{12}) = i (G^{21} - G^{22}) \).

Rewrite the Schwinger - Dyson equation above as
\[
\frac{1}{Z_B} D G^{ac}(x, y) + \frac{1}{N} \int d^d z \, P^a_b(x, z) G^{bc}(z, y) = i c^{ac} \delta(x, y) \quad (30)
\]
Subtracting the (11) from the (12) components in the above equation, we obtain
\[
\frac{1}{Z_B} D G_{ret}(x, y) + \frac{1}{N} \int d^d z \, P_{ret}(x, z) G_{ret}(z, y) = (-1) \delta(x, y) \quad (31)
\]
where \( P_{ret} = P_{11} + P_{12} \).

### 2.2 Some nonperturbative identities

We note some non-perturbative properties of the self-energy \( P_{ab} \), as follows. From the identity
\[
\frac{\delta^2 \Gamma_{1PI}}{\delta \varphi^{a a} \delta \varphi^{b b}} = i \hbar \left( G^{-1} \right)_{aa,bb} \quad (32)
\]
relating the inverse propagator to the CTP 1PI EA \( \Gamma_{1PI} \), the inverse propagators may be read off the Schwinger - Dyson equations, and \( \Gamma_{1PI} \) may be written as
\[
\Gamma_{1PI} = \frac{N}{2} \int d^d x d^d y \left\{ \left[ \varphi^{a1} - \varphi^{a2} \right](x) \left[ \frac{1}{Z_B} D(x, y) + P(x, y) \right] \left[ \varphi^{a1} + \varphi^{a2} \right](y) + i \left[ \varphi^{a1} - \varphi^{a2} \right](x) N(x, y) \left[ \varphi^{a1} - \varphi^{a2} \right](y) \right\} \quad (33)
\]
\[
+ i \left[ \varphi^{a1} - \varphi^{a2} \right](x) N(x, y) \left[ \varphi^{a1} - \varphi^{a2} \right](y) \quad (34)
\]
where $P$ is causal and $N$ is even, and both are real. Computing the derivatives we get

$$\frac{1}{N}P_{11} = P_{\text{even}} + iN$$  \hspace{1cm} (35)$$

$$\frac{1}{N}P_{12} = P_{\text{odd}} - iN$$  \hspace{1cm} (36)$$

$$\frac{1}{N}P_{21} = -P_{\text{odd}} - iN$$  \hspace{1cm} (37)$$

$$\frac{1}{N}P_{22} = -P_{\text{even}} + iN$$  \hspace{1cm} (38)$$

where

$$P_{\text{even}} (x, y) = \frac{1}{2} [P (x, y) + P (y, x)] ; \quad P_{\text{odd}} (x, y) = \frac{1}{2} [P (x, y) - P (y, x)]$$  \hspace{1cm} (39)$$

Observe that $P_{\text{ret}} = NP (x, y)$.

3 Translation-invariant solutions

Translation invariant solutions are functions only of the relative variable $x - y$ and may be Fourier transformed

$$G^{ab} (x - y) = \int \frac{d^d p}{(2\pi)^d} e^{ip(x - y)} G^{ab} (p)$$  \hspace{1cm} (40)$$

The Fourier transform of an even (odd) kernel is an even (odd) function of $p$. If a kernel is real, the real (imaginary) part of its Fourier transform is even (odd). Vice versa, if a kernel is imaginary, then the real (imaginary) part of its transform is odd (even).

It follows that, since $P_{\text{even}} (x - y)$ is real and even, $P_{\text{even}} (p)$ is also real and even, while since $P_{\text{odd}} (x - y)$ is real and odd, $P_{\text{odd}} (p)$ is odd and imaginary. We may write

$$P_{\text{odd}} (p) = i\pi \gamma (p) \text{ sign} (p^0)$$  \hspace{1cm} (41)$$

therefore $\gamma (p)$ is real and even.

3.1 The density of states

Let us introduce the density of states $\Delta (p)$ out of the Fourier transform of the Jordan propagator $G = G^{21} - G^{12}$

$$G (p) = 2\pi \Delta (p) \text{ sign} (p^0)$$  \hspace{1cm} (42)$$
The Jordan and retarded propagators are related through \( G(p) = 2 \text{Im} \ G_{\text{ret}}(p) \). From Eq. (31)

\[
G_{\text{ret}} = (-1) \left[ \frac{1}{Z_B} D(p) + \frac{1}{N} P_{\text{ret}}(p) \right]^{-1} \quad (43)
\]

Because of the retarded boundary conditions on \( G_{\text{ret}} \), it is understood that \( p^0 \) is replaced by \( p^0 + i\epsilon, \epsilon \rightarrow 0 \). We must distinguish two cases. As we shall see below, in the LO approximation, \( Z_B = 1 \) and \( P_{\text{ret}}(p) = 0 \). In this case we have the explicit expression for \( G_{\text{ret}} \)

\[
G_{\text{ret}}(p) = \frac{1}{-(p^0 + i\epsilon)^2 + \vec{p}^2 + \bar{\chi}} \quad (LO) \quad (44)
\]

Therefore

\[
G(p) = 2 \text{Im} G_{\text{ret}}(p) = 2\pi \text{sign} \left( p^0 \right) \delta \left( p^2 + \bar{\chi} \right) \quad (LO) \quad (45)
\]

and

\[
\Delta(p) = \delta \left( p^2 + \bar{\chi} \right) \quad (LO) \quad (46)
\]

In all higher approximations, \( P_{\text{ret}}(p) \neq 0 \). We get

\[
G(p) = 2 |G_{\text{ret}}(p)|^2 \text{Im} P_{\text{odd}}(p) \quad (NLO \text{ and higher}) \quad (47)
\]

that is

\[
\Delta(p) = |G_{\text{ret}}(p)|^2 \gamma(p) \quad (NLO \text{ and higher}) \quad (48)
\]

### 3.2 The distribution function and the fluctuation-dissipation relation

Now consider the (12) component of Eq. (30)

\[
\frac{1}{Z_B} D(p) G^{12}(p) + \frac{1}{N} \left[ P_{11}(p) G^{12}(p) + P_{12}(p) G^{22}(p) \right] = 0 \quad (49)
\]

Introducing the advanced propagator \( G_{\text{adv}} = (-i) (G^{22} - G^{12}) \) (\( G_{\text{adv}}(p) = [G_{\text{ret}}(p)]^* \)) we may rewrite this as

\[
[G_{\text{ret}}(p)]^{-1} = \frac{i}{N} P_{12}(p) G_{\text{adv}}(p) \Rightarrow G^{12}(p) = \frac{i}{N} P_{12}(p) |G_{\text{ret}}(p)|^2 \quad (50)
\]

and transform this into

\[
G^{12}(p) = \left[ -\pi \gamma(p) \text{sign}(p^0) + \mathbf{N}(p) \right] \frac{\Delta(p)}{\gamma(p)} \quad (51)
\]
In other words, for a translation invariant solution we must have

\[ G^{12} (p) = 2\pi F^{12} (p) \Delta (p) \]  

(52)

where (recall that \( \text{sign} (p^0) = 1 - 2\theta (-p^0) \))

\[ F^{12} (p) = \theta (-p^0) + f (p) \]  

(53)

Comparing both expressions for \( G^{12} \) we get

\[ f (p) = \frac{1}{2} \left[ \frac{N (p)}{\pi \gamma (p)} - 1 \right] \]  

(54)

It is more common to write this as

\[ N (p) = \pi \gamma (p) \left[ 1 + 2 f (p) \right] \]  

(55)

whereby we recognize the fluctuation - dissipation relation.

Given the Jordan and negative frequency propagators, it is easy to find all the others. In particular, the positive frequency propagator \( G^{21} (p) = G + G^{12} = 2\pi F^{21} (p) \Delta (p) \), with \( F^{21} (p) = \theta (p^0) + f (p) \).

### 3.3 A necessary condition for translation invariant solutions

The expression for \( f (p) \) above (eq. (54)) is equivalent to the identity

\[ P_{12} (p) G^{21} (p) - P_{21} (p) G^{12} (p) = 0 \]  

(56)

Indeed, from eqs. (56), (57), (11) and (54) we get

\[ P_{12} (p) G^{21} (p) - P_{21} (p) G^{12} (p) = P_{12} (p) \theta (p^0) - P_{21} (p) \theta (-p^0) + [P_{12} (p) - P_{21} (p)] f (p) \]  

(57)

but also

\[ P_{12} (p) - P_{21} (p) = 2NP_{\text{odd}} (p) = 2iN\pi \gamma (p) \text{ sign} (p^0) \]  

(58)

\[ P_{12} (p) \theta (p^0) - P_{21} (p) \theta (-p^0) = N \left[ P_{\text{odd}} (p) - iN (p) \text{ sign} (p^0) \right] \]

\[ = iN\pi \gamma (p) \left[ 1 - \frac{N (p)}{\pi \gamma (p)} \right] \text{ sign} (p^0) \]

\[ = -2iN\pi \gamma (p) f (p) \text{ sign} (p^0) \]  

(59)

Substituting these identities in Eq. (57) we get (56), which is therefore a necessary condition for translation invariant propagators.
4 The large $N$ approximation

So far we have shown that translation invariant solutions are defined by the density of states $\Delta (p)$ and the distribution function $f (p)$. To show that they correspond to thermal propagators, we must show that $f (p)$ is necessarily of the form of a Bose - Einstein distribution

$$f (p) = \left[ e^{\beta p} - 1 \right]^{-1}$$

To do this, we need explicit expressions for the $P$ kernels, which we can only find perturbatively. We shall adopt the large $N$ approximation, which consists of taking the limit of $N \to \infty$ in $\Gamma_Q$ and retaining only terms scaling like $N$ (LO), $1$ (NLO), $N^{-1}$ (NNLO), etc. The key observation is that in any given Feynman graph each vertex contributes a power of $N$, each internal line a power of $N^{-1}$, and each trace over group indices another power of $N$. We have both $G$ and $H$ internal lines, but the $G$ lines only appear in closed loops. On each loop, the number of vertices equals the number of $G$ lines, so there only remains one power of $N$ from the single trace over group labels. Therefore the overall power of the graph is the number of $G$ loops minus the number of $H$ lines. Now, since we only consider $2PI$ graphs, there is a minimum number of $H$ lines for a given number of $G$ loops. For example, if there are two $G$ loops, they must be connected by no less than 3 $H$ lines, and so this graph cannot be higher than NNLO. A graph with 3 $G$ loops can not have less than 5 $H$ lines, and so on.

4.1 The leading order approximation

We conclude that $\Gamma_Q$ vanishes at LO, and therefore $P_{ab} = Q_{ab} = 0$. Under the ansatz $Z_B = 1$, the equations we need to solve become

$$c_{ab} D (p) - i \left[ G^{-1} \right]_{ab} (p) = 0$$

$$\frac{1}{\lambda_B} \{ \bar{\chi} - M_B^2 \} - \frac{\hbar}{2} \int \frac{d^d p}{(2\pi)^d} G^{11} (p) = 0$$

We disregard the auxiliary field propagator $H^{ab}$, since to this order it is decoupled from the background auxiliary field and the other propagators.

For the retarded propagator, we have the expression Eq. (44), leading to Eq. (45) for the Jordan propagator and (46) for the density of states. Write Eq. (61) as

$$D (p) G^{ab} (p) = i c^{ab}$$

Setting $a = 2, b = 1$, we see that $G^{21}$ can only be nonzero at the zeroes of $D (p) = -p^2 - \bar{\chi}$, so we still can write $G^{21} (p) = 2\pi F^{21} (p) \Delta (p)$ . Also $G^{12} (p) = G^{21} (-p)$, and $G^{21} (p) - G^{12} (p) = G (p)$, so from Eq. (45) we conclude that $F^{21} (p) - F^{21} (-p) = \text{sign} (p^0)$ . Therefore we may write $F^{21} (p) = \theta (p^0) + f (p)$, with $f (p)$ a real and even function. From these results, we may
write all propagators in terms of the distribution function \( f(p) \). In particular, the Feynman propagator becomes

\[
G^{11} = (-i) G_{\text{ret}} + G^{12} \\
= \left\{ \frac{(-i)}{-(p^0 + i\varepsilon)^2 + \vec{p}^2 + \chi} \right\} + 2\pi \left[ \theta(-p^0) + f(p) \right] \delta(p^2 + \chi) \\
= \left\{ \frac{(-i)}{-p^0 + i\varepsilon + \vec{p}^2 + \chi} \right\} + 2\pi f(p) \delta(p^2 + \chi)
\]

(64)

So, assuming \( d = 4 - \varepsilon \) dimensions, we may evaluate

\[
\int \frac{d^d p}{(2\pi)^d} G^{11}(p) = \mu^{-\varepsilon} \left[ M_V^2 + M_f^2 \right]
\]

(65)

where \( \mu \) is some (so far) arbitrary renormalization scale,

\[
M_V^2 = \left\{ \frac{-\mu^2}{2\pi\varepsilon} \Gamma \left[ 1 + \frac{\varepsilon}{2} \right] \left( \frac{\chi}{4\pi\mu^2} \right)^{1-\frac{\varepsilon}{2}} \right\}
\]

(66)

\[
M_f^2 = \mu^\varepsilon \int \frac{d^d p}{(2\pi)^d} 2\pi f(p) \delta(p^2 + \chi)
\]

(67)

The gap equation becomes

\[
\frac{1}{\lambda_B} \left\{ \bar{\chi} - M_B^2 \right\} - \frac{\hbar}{2\mu^{-\varepsilon}} \left[ M_V^2 + M_f^2 \right] = 0
\]

(68)

We are now confronted with the formal need to show that Eq. (68) admits finite solutions for \( \bar{\chi} \) when \( \varepsilon \to 0 \) \[54\], as well as the physical need to show that the theory is reasonably stable against changes in the distribution functions \( f(p) \) \[54, 56, 57, 58\]. Let us interpret this equation as defining \( \bar{\chi} \) as a function of \( M_f^2 \). Taking one derivative, we obtain

\[
\left[ \frac{1}{\lambda_B} + \mu^{-\varepsilon} \frac{\hbar \Gamma \left[ 1 + \frac{\varepsilon}{2} \right]}{16\pi^2\varepsilon} \left( \frac{\bar{\chi}}{4\pi\mu^2} \right)^{-\frac{\varepsilon}{2}} \right] \frac{d\bar{\chi}}{dM_f^2} = \frac{\hbar}{2\mu^{-\varepsilon}} = 0
\]

(69)

This suggests defining a background field and renormalization scale dependent effective coupling constant \( \lambda \) from

\[
\frac{d\bar{\chi}}{dM_f^2} \left( \bar{\chi}, \mu^2 \right) = \frac{\hbar}{2} \lambda \left( \bar{\chi}, \mu^2 \right) \mu^{-\varepsilon}
\]

(70)

In other words
Now the gap equation

\[ \frac{-1}{\lambda_B} M_f^2 + \frac{1}{1 - \frac{\epsilon}{2}} \left[ \frac{1}{\lambda} + \frac{\epsilon}{2\lambda_B} \right] \bar{\chi} - \frac{\hbar}{2} \mu - \epsilon \bar{h} \left[ M_f^2 - M_{\text{crit}}^2 \right] = 0 \]  

(72)

shows that \( \bar{\chi} = 0 \) when \( M_f^2 = M_{\text{crit}}^2 \), given by

\[ \mu - \epsilon \bar{h} \frac{1}{2} \left[ M_f^2 - M_{\text{crit}}^2 \right] = 0 \]

(73)

So assuming \( M_{\text{crit}}^2 \) to be finite, we may rewrite the gap equation as

\[ \frac{1}{1 - \frac{\epsilon}{2}} \left[ \frac{1}{\lambda} + \frac{\epsilon}{2\lambda_B} \right] \bar{\chi} - \mu - \epsilon \bar{h} \frac{1}{2} \left[ M_f^2 - M_{\text{crit}}^2 \right] = 0 \]

(74)

We see that it is possible to find a solution with finite propagators and \( \bar{\chi} \) for any distribution function \( f \), provided \( M_f^2 \) is finite. On the other hand, (weak) thermalization would require that any solution eventually converges to the thermal form Eq. (60). Therefore, we conclude that the \( \text{LO} \) system does not thermalize.

4.2 The \( \text{NLO} \) approximation

Since the \( \text{LO} \) approximation admits a plurality of translation invariant solutions, it is necessary to go at least to \( \text{NLO} \) to study the issue of thermalization. There is only one \( \text{NLO} \) graph, consisting of a single \( G \) loop and a single \( H \) line (see Fig. 1). This graph leads to

\[ \Gamma_{Q}^{\text{NLO}} = (-i\hbar) \left( -\frac{1}{2} \right) \left( -N \right)^2 2N \left( \frac{\hbar}{N} \right)^3 c_{\alpha e c d f} c_{\alpha d e f} \int d^4xd^4y H^{\alpha e} (x, y) G^{\alpha e}(x, y) G^{\alpha f}(x, y) \]

(75)

Therefore, from

\[ P_{ab} (x, y) = \frac{2}{\hbar} \frac{\delta \Gamma_{Q}}{\delta G^{ab}(x, y)}; \quad Q_{ab} (x, y) = \frac{2}{\hbar} \frac{\delta \Gamma_{Q}}{\delta H^{ab}(x, y)} \]

(76)

we get

\[ P_{ab} (x, y) = \frac{i\hbar}{Z_B^2} c_{\alpha c d e f} c_{\alpha d e f} H^{\alpha e} (x, y) G^{\alpha f} (x, y) \]

(77)

\[ Q_{ab} (x, y) = \frac{i\hbar}{Z_B^2} c_{\alpha c d e f} G^{\alpha e} (x, y) G^{\alpha f} (x, y) \]

(78)
Since $Q_{ab}$ does not depend on $H^{ab}$, we may solve the corresponding equation

$$\frac{1}{\lambda_B} c_{ab} \delta(x, y) - i \left[ H^{-1} \right]_{ab} (x, y) + Q_{ab} (x, y) = 0$$  \hspace{1cm} (79)$$

in closed form. First, let us Fourier transform

$$\left[ H^{-1} \right]_{ab} (p) = (-i) \left( \frac{1}{\lambda_B} + \frac{Q_{11}}{Q_{21}} \frac{Q_{12}}{\lambda_B + Q_{22}} \right)$$  \hspace{1cm} (80)$$

To lowest order in $1/N$, we may use the LO propagators to compute the $Q'$s. In particular, we get

$$Q_{11} = \mu^{-\epsilon} \left[ Q_{V11} + Q^{(1)}_{f11} + Q^{(2)}_{f11} \right]$$  \hspace{1cm} (81)$$

where

$$Q_{V11} (p) = \left( -\frac{ih}{2} \right) \mu^\epsilon \int \frac{d^dq}{(2\pi)^d} \frac{1}{(p-q)^2 + \bar{\chi} - i\epsilon} \frac{1}{q^2 + \bar{\chi} - i\epsilon}$$  \hspace{1cm} (82)$$

$$Q^{(1)}_{f11} = \hbar \mu^\epsilon \int \frac{d^dq}{(2\pi)^d} 2\pi f(q) \delta (q^2 + \bar{\chi})$$  \hspace{1cm} (83)$$

$$Q^{(2)}_{f11} = \frac{i\hbar}{2} \mu^\epsilon \int \frac{d^dq}{(2\pi)^d} 4\pi^2 f(q) \delta (q^2 + \bar{\chi}) f(p-q) \delta ((p-q)^2 + \bar{\chi})$$  \hspace{1cm} (84)$$

We assume that $Q_{f11}^{(1)}$ and $Q_{f11}^{(2)}$ are well defined, and compute

$$Q_{V11} (p) = \frac{\hbar}{16\pi^2} \frac{\Gamma \left[ 1 + \frac{\epsilon}{2} \right]}{\epsilon} \int_0^1 dx \left( \frac{x(1-x)p^2 + \bar{\chi}}{4\pi\mu^2} \right)^{-\epsilon/2}$$  \hspace{1cm} (85)$$

Recalling the renormalization condition Eq. (71), we find

$$\frac{1}{\lambda_B} + \mu^{-\epsilon} Q_{V11} = \frac{1}{\lambda} + \frac{\hbar \Gamma \left[ 1 + \frac{\epsilon}{2} \right]}{16\pi^2\epsilon} \left( \frac{\bar{\chi}}{4\pi} \right)^{-\frac{\epsilon}{2}} \int_0^1 dx \left[ \left( 1 + x(1-x) \left( \frac{p^2}{\bar{\chi}} \right) \right)^{-\epsilon/2} - 1 \right] \equiv \frac{1}{\lambda} + Q_{V_{ren11}}$$  \hspace{1cm} (86)$$

$Q_{V_{ren11}}$ is explicitly finite. Let us call $Q_{ren11} = Q_{V_{ren11}} + \mu^{-\epsilon} \left[ Q_{f11}^{(1)} + Q_{f11}^{(2)} \right]$. Since $Q_{22} = -Q_{11}^{*}$, we may also write

$$\frac{-1}{\lambda_B} + Q_{22} = \frac{-1}{\lambda} + Q_{ren22}$$  \hspace{1cm} (87)$$
where \( Q_{\text{ren}22} = -Q_{\text{ren}11} + \mu^{-1} \left[ -Q_{f11}^{(1)} + Q_{f11}^{(2)} \right] \). Introduce \( Q_{\text{ret}} = Q_{\text{ren}11} + Q_{12} \), \( Q_{\text{adv}} = Q_{\text{ren}11} + Q_{21} = Q_{\text{ret}} \) and write this as (recall that \( Q_{\text{ren}22} + Q_{\text{ren}11} = Q_{22} + Q_{11} = -(Q_{12} + Q_{21}) \))

\[
\left[ H^{-1} \right]_{ab}(p) = (-i) \begin{pmatrix} \frac{1}{\lambda} + Q_{\text{ret}} - Q_{12} & Q_{12} \\ Q_{21} & \frac{1}{\lambda} + Q_{\text{adv}} + Q_{12} \end{pmatrix}
\]

Call

\[
\Sigma^{-1}(p) = \det \left[ H^{-1} \right]_{ab}(p) = \left[ \frac{1}{\lambda} + Q_{\text{ret}} - Q_{12} \right] \left[ \frac{1}{\lambda} + Q_{\text{adv}} + Q_{12} \right] + Q_{12}Q_{21}
\]

\[
= \left[ \frac{1}{\lambda} + Q_{\text{ret}} \right] \left[ \frac{1}{\lambda} + Q_{\text{adv}} \right] + Q_{12} [Q_{\text{ret}} - Q_{\text{adv}}] + Q_{12} [Q_{21} - Q_{12}]
\]

\[
= \left[ \frac{1}{\lambda} + Q_{\text{ret}} \right] \left[ \frac{1}{\lambda} + Q_{\text{adv}} \right]
\]

Observe that \( \Sigma(p) \) is real, finite and positive definite. Then

\[
H^{ab}(p) = i \Sigma(p) \begin{pmatrix} \frac{1}{\lambda} + Q_{\text{adv}} + Q_{12} & Q_{12} \\ Q_{21} & \frac{1}{\lambda} + Q_{\text{ret}} - Q_{12} \end{pmatrix}
\]

### 4.3 The NLO density of states

Beyond this point, our analysis will not depend upon the details of the NLO approximation, but only on a few structural features. One of these features is the fact that the NLO density of states is nonvanishing for \( -p^2 > 9 \bar{\chi} \). To establish this fact, it is enough to look at \( P_{21}(p) \). Recall that, from Eq. (37), we know that \( P_{21}(p) \) is imaginary. Its odd part determines the kernel \( P_{\text{odd}} \), and its even part the noise kernel \( N \). \( P_{\text{odd}} \) determines \( \gamma(p) \) through Eq. (41). For an actual translation invariant solution, \( N \) and \( \gamma \) are related by the fluctuation dissipation relation. The density of states \( \Delta \) and \( \gamma \) are related through Eq. (48). It is clear from this equation that their zeroes are exactly the same, and so we only need to show that \( \gamma \) is nonvanishing.

At NLO

\[
P_{21}(p) = \frac{-i\hbar}{Z_B^2} \int \frac{d^d q}{(2\pi)^d} H^{21}(q) G^{21}(p - q)
\]

\[
= \frac{\hbar}{Z_B^2} \int \frac{d^d q}{(2\pi)^d} \Sigma(q) Q_{21}(q) G^{21}(p - q)
\]

\[
= \frac{-i\hbar^2}{2Z_B^4} \int \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \Sigma(q) G^{21}(r) G^{21}(q - r) G^{21}(p - q)
\]

To simplify the analysis, we can make the rather drastic approximation \( \Sigma(q) \sim \lambda^2 \), and use the LO propagators.
\[ P_{21}(p) = \frac{-i\hbar^2\lambda^2}{2Z_B^4} \int \frac{d^dq \Delta(q)}{(2\pi)^{d-1}} \frac{d^dr \Delta(r)}{(2\pi)^{d-1}} \frac{d^ds \Delta(s)}{(2\pi)^{d-1}} \delta(q + r + s - p) \]
\[ \left[ \theta(q^0) + f(q) \right] \left[ \theta(r^0) + f(r) \right] \left[ \theta(s^0) + f(s) \right] \]

It is convenient to write the integrals in terms of future oriented momenta only. We get

\[ P_{21}(p) = \frac{-i\hbar^2\lambda^2}{2Z_B^4} \left[ I_3 + 3I_2 + 3I_1 + I_0 \right] \]  

where

\[ I_3 = \int DqDrDs \delta(q + r + s - p) \left[ 1 + f(q) \right] \left[ 1 + f(r) \right] \left[ 1 + f(s) \right] \]
\[ I_2 = \int DqDrDs \delta(q + r - s - p) \left[ 1 + f(q) \right] \left[ 1 + f(r) \right] f(s) \]
\[ I_1 = \int DqDrDs \delta(s - q - r - p) \left[ 1 + f(s) \right] f(r) f(q) \]
\[ I_0 = \int DqDrDs \delta(q + r + s + p) f(q) f(r) f(s) \]

and we have defined

\[ Dq = \frac{d^dq}{(2\pi)^{d-1}} \theta(q^0) \Delta(q) \]

It is clear that each of these integrals is nonnegative, so we only must show that for arbitrary \( p \) at least one is nonzero.

Let us assume the \( LO \) density of states within the integrand, so that momenta \( q, r \) and \( s \) are on-shell. Assume \( -p^2 > 9\bar{\chi} \). If \( p^0 > 0 \), \( I_3 \) and \( I_2 \) are nonvanishing, while \( I_1 \) and \( I_0 \) are zero. If \( p^0 < 0 \), it is the other way round. If \( p^0 > 0 \), moreover, \( I_3(p) > I_0(-p) \) and \( I_2(p) > I_1(-p) \), and so both the odd and even parts of \( P_{12} \) are nonvanishing, as we wanted to show. The fact that they are not only nonzero but actually proportional to each other only obtains for a special form of \( f(p) \), indeed, a thermal form. We shall show this in next Section.

It must be observed that in going from \( P_{12} \) to \( P_{\text{odd}} \) there is an extra factor of \( 1/N \) involved (cfr. Eq. (36)), and so the off-shell density of states, while non zero, is of higher order in \( 1/N \).

5 The only translation invariant solutions to \( NLO \) are thermal

We may now show that the only translation invariant solutions of the \( NLO \) equations are thermal. The solutions must satisfy the identity Eq. (56), which becomes
\[ \int \frac{d^{d}q}{(2\pi)^{d}} \left[ H^{12}(q) G^{12}(p-q) G^{21}(p) - H^{21}(q) G^{21}(p-q) G^{12}(p) \right] = 0 \] (99)

From the explicit solution for \( H \)

\[ \int \frac{d^{d}q}{(2\pi)^{d}} \Sigma(q) \left[ Q_{12}(q) G^{12}(p-q) G^{21}(p) - Q_{21}(q) G^{21}(p-q) G^{12}(p) \right] = 0 \] (100)

Finally, use the NLO approximation to \( Q_{ab} \)

\[ \int \frac{d^{d}q}{(2\pi)^{d}} \frac{d^{d}r}{(2\pi)^{d}} \Sigma(q) \left[ G^{12}(r) G^{12}(q-r) G^{12}(p-q) G^{21}(p) - G^{21}(r) G^{21}(q-r)(q) G^{21}(p-q) G^{12}(p) \right] = 0 \] (101)

It is more usual to write this as

\[ 0 = (2\pi)^{d} \int \frac{d^{d}q}{(2\pi)^{d}} \frac{d^{d}r}{(2\pi)^{d}} \frac{d^{d}s}{(2\pi)^{d}} \Sigma(p-q) \delta(q+r+s-p) \left\{ G^{12}(q) G^{12}(r) G^{12}(s) G^{21}(p) - G^{21}(q) G^{21}(r)(q) G^{21}(s) G^{12}(p) \right\} \] (102)

We recognize the usual Boltzmann collision term, with \( \Sigma(p-q) \) playing the role of cross section. The only solutions must be thermal.

There is one important observation to be made. Since this term is of order \( 1/N \), in a strict power expansion we would use the LO density of states Eq. (46) in the propagators. In the Introduction, we described this as making a ”true” approximation. In practice, it means that we would put all momenta on mass-shell. Then only binary collisions would be possible, and the Boltzmann term would admit solutions with nonvanishing chemical potential. These do not exist in the exact theory, and so the ”true” approximation does not describe thermalization.

We see, however, that if we keep the NLO density of states the problem disappears. In the Introduction, we called this procedure the ”exact” way. Because the density of states is nonzero everywhere, it is possible for one on-shell particle to decay into three off-shell ones, or vice versa, for one off-shell particle with momentum \(-p^2 > 9\chi\) to decay onto three on-shell particles (to show that this possibility is indeed open we have shown explicitly in the last Section that the NLO density of states is nonvanishing in this region). Particle number is no longer conserved, and only zero chemical potential is allowed.

Thermalization is described, but only as a higher than NLO phenomenon, since it depends on NLO corrections to the density of states within an expression which is itself a NLO construct. We must consider if some of the NNLO terms we have left out may not bear on this process at a similar level. We shall see below that this is indeed the case.
6 Relaxation of the chemical potential

We have seen so far that if nonperturbative corrections to the density of states are allowed, then the only translation invariant solutions to the Schwinger-Dyson equations are thermal propagators with vanishing chemical potentials. These are also the only true equilibrium solutions of the full theory. However, since we have stepped beyond the strict NLO approximation, we must ask ourselves if terms we have discarded would not affect thermalization at a similar level. To investigate this question, we shall consider the relaxation of the chemical potential.

To place the issue in the simplest possible context, we shall assume that only long wavelengths are involved, and adopt the Kadanoff-Baym approach. That is, we shall assume that all two-point functions $G(x, y)$ may be written as functions of a relative variable $u = x - y$ and a center of mass variable $X = (x + y)/2$, and that the $X$ dependence is weak, loosely meaning that $\partial_X G \sim (1/N) \partial_u G$.

We may introduce a mixed representation $G(X, p)$ by performing a Fourier transform on the $u$ variable (cfr. Eq. (40)). For fixed $X$, the manipulations in Section III are still valid, only now both the density of states and the distribution function display an extra $X$ dependence (we refer the reader to ref. [20] and references therein for a detailed discussion).

To obtain the dynamics of the distribution function $f(X, p)$, write the Schwinger-Dyson Eq. (26) for $G^{21}(x, y)$ as

$$\frac{1}{Z_B} D(x, y) G^{21}(x, y) - \frac{1}{N} \int d^d z \left[ P_{21}(x, z) G^{11}(z, y) + P_{22}(x, z) G^{21}(z, y) \right] = 0 \quad (103)$$

Recall that $D(x, y) = [\partial_x^2 - \bar{\chi}(x)] \delta(x, y)$ and Fourier transform with respect to the relative variable in each case to get, up to $1/N$ terms

$$[-\Omega + iL] G^{21}(X, p) = \frac{Z_B}{N} \left[ P_{21}(X, p) G^{11}(X, p) + P_{22}(X, p) G^{21}(X, p) \right] \quad (104)$$

where $\Omega = p^2 + \bar{\chi}(X)$, and $L$ is the Vlasov operator

$$L = p \frac{\partial}{\partial X} - \frac{1}{2} \frac{\partial \bar{\chi}}{\partial X} \frac{\partial}{\partial p} \quad (105)$$

Separating the imaginary part, we get the Boltzmann equation

$$L f = I_{col} \quad (106)$$

where (recall eqs. (55) to (58))

$$I_{col} = -\frac{iZ_B}{2N} \left[ P_{21}(X, p) G^{12}(X, p) - P_{12}(X, p) G^{21}(X, p) \right] \quad (107)$$

Observe that different approximations yield different collision integrals. Of course, the condition $I_{col} = 0$ for a translation invariant solution is just the necessary condition Eq. (56).
Manipulating the collision term as in the last Section, we get, to \textit{NLO}

\[
I_{\text{col}} = \frac{(2\pi)^d \lambda^2 \hbar^2}{4N \ Z_B^3} \int \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \delta (q + r + s - p) \\
\left\{ G^{12} (q) G^{12} (r) G^{12} (s) G^{21} (p) - G^{21} (q) G^{21} (r) (q) G^{21} (s) G^{12} (p) \right\}
\]

(108)

where we leave the \(X\) dependence implicit, and have approximated \(\Sigma \sim \lambda^2\) for simplicity.

Since we are only interested in the relaxation of the chemical potential, we may linearize the Boltzmann equation around equilibrium. Write the distribution function as

\[
f (X, p) = f_0 (p) \left[1 + (1 + f_0) \delta f (X, p)\right]
\]

(109)

where \(f_0\) is a thermal distribution with vanishing chemical potential. Neglecting the variation in the density of states, which gives a higher order contribution, we get

\[
G^{21,12} = 2\pi \Delta (p) \left[\theta (\pm p^0) + f_0 + f_0 (1 + f_0) \delta f (X, p)\right]
\]

(110)

A nonzero chemical potential corresponds to a \(p\)-independent perturbation \(\delta f (X, p) = \delta \mu (X)\). To isolate the dynamics of the chemical potential, we may integrate both sides of the Vlasov-Boltzmann equation with respect to \(p\). Since the distribution function is even, there is no loss in restricting the integration region to \(p^0 > 0\). We also write the collision integral in terms of integrals over positive energy momenta, to get

\[
B \frac{\partial}{\partial t} \delta \mu (X) = \frac{(2\pi)^d \ h^2}{2N \ Z_B^3} J
\]

(111)

where (recall the momentum space measure Eq. (98))

\[
B = \int Dp \ f_0 (1 + f_0) p^0
\]

(112)

\[
J = \int DpDqDrDs \ \delta (q - r - s - p) \left\{ f (q) [1 + f (r)] [1 + f (s)] [1 + f (p)] - [1 + f (q)] f (r) f (s) f (t)\right\}
\]

(113)

Linearizing the \(J\) integral we get

\[
J = -2K \delta \mu (X)
\]

(114)

\[
K = \int DpDqDrDs \ \delta (q - r - s - p) [1 + f (q)] [1 + f (r)] [1 + f (s)] [1 + f (p)] e^{-\beta q}
\]

(115)
The integral is nonzero, provided we use the NLO density of states. If we used the LO density of states, then the argument of the delta function would never vanish. This means we predict chemical potential relaxation on a time scale which grows at least as $N^2$, since to the $1/N$ factor in Eq. (111) we must add at least one more $1/N$ factor coming from the off-shell density of states.

The problem is that there are terms, coming from the NNLO approximation to the 2PI CTP EA, that modify the collision term in such a way to contribute to chemical potential relaxation at the same accuracy. Therefore the NLO prediction for the relaxation time is not accurate. We shall conclude this Section by showing this explicitly.

6.1 The NNLO approximation and particle number violation

As we already remarked, the power of $N$ associated to a given vacuum bubble is the number of $G$ loops minus the number of $H$ lines. Therefore, in the NNLO approximation we must look for graphs with one more $H$ line than $G$ loops. However, a 2PI graph with 3 $G$ loops must have no less than 5 $H$ lines, so the allowed number of $G$ loops is only 1 or 2. There are only two graphs with 1 $G$ loop and 2 $H$ lines, but only one of them is 2PI. (see Fig. 2) Similarly, there is only one 2PI graph with 2 $G$ loops and 3 $H$ lines (see Fig. 3). We therefore have two new graphs contributing to the 2PI CTP EA.

Variation of these graphs with respect to $G$ yields two new contributions to the $P_{ab}$ (Figs. 4 and 5)

Let us consider Fig. 4, and replace the $H$ lines by their expansion in powers of $\lambda$ (Fig. 6) The lowest order contribution yields the setting sun diagram (Fig. 7).

Introducing the first correction to one of the $H$ lines gives the graph in Fig. 8. Correcting both $H$ lines gives the graph in Fig. 9.

The second new graph in the 2PI CTP EA (Fig. 3) yields, upon variation, the graph in Fig. 5.

Replacing the $H$ lines by their lowest order expression, we find another contribution of the form of Fig. 8. The next order yields contributions proportional to Figs. 9 and 10.

It has been shown in ref. ([20]) that the graphs in Fig. 9 and 10, when translated in terms of the collision integral, describe scattering of 2 into 4 particles and vice versa. These scattering processes do not conserve particle number and therefore contribute to the relaxation of the chemical potential (we have shown explicitly the linearized collision operator in ref. ([20])). The resulting contribution is at least of the same order of magnitude as that found by allowing a nonperturbative density of states in the NLO collision term.

Therefore we conclude that the NLO prediction for the relaxation time of the chemical potential is not accurate. It is nevertheless remarkable that the NLO succeeds in predicting relaxation, in agreement with the claims of refs. [21] and [22].
7 How does macroscopic irreversibility appear?

As we mentioned at the beginning of this paper, the issue of thermalization in relativistic quantum fields, besides its concrete application to high energy plasmas, is relevant to the larger question of the origin of the thermodynamic arrow of time in physics. The LO Large $N$ approximation does not break time reversal invariance (it does not lead to thermalization either) and one could jump to the conclusion that this is a general feature of the Large $N$ perturbative scheme. However, as we have seen, the NLO or, at worst, the NNLO approximations show thermalization. It behooves us to identify at which point time-reversal invariance has been broken, and how.

To identify the crucial assumptions, let us return to the form Eq. (11) of the action for the theory. A variation with respect to the field yields the Heisenberg equations of motion for the field operator

$$\partial^2 \varphi^a (x) - \chi (x) \varphi^a (x) = 0$$  (116)

Multiplying from the left (say) by $\varphi^a(y)$, taking expectation values and summing over $a$, we obtain (recall eqs. (16) and (24))

$$\hbar \partial^2 x G^{21} (x, y) - \langle \chi (x) \varphi^a (x) \varphi^a (y) \rangle = 0$$  (117)

This Schwinger-Dyson equation is, of course, not a closed equation for the propagator; it only relates the propagator to a higher correlation function. If we wish to say something about this new correlation, one possibility is to repeat the argument. We may begin from the Heisenberg equation for the auxiliary field

$$\chi (x) = M^2_B + \frac{\lambda_B}{2} \varphi^a (x) \varphi^a (x)$$  (118)

and multiply by $\varphi^a (x) \varphi^a (y)$ to get

$$\langle \chi (x) \varphi^a (x) \varphi^a (y) \rangle = M^2_B \hbar G^{21} (x, y) + \frac{\lambda_B}{2} \langle \varphi^b (x) \varphi^b (x) \varphi^a (x) \varphi^a (y) \rangle$$  (119)

or else we go back to Eq. (116) to get

$$\partial^2 y \langle \chi (x) \varphi^a (x) \varphi^a (y) \rangle - \langle \chi (x) \varphi^a (x) \chi (y) \varphi^a (y) \rangle = 0$$  (120)

In either case, yet another higher correlation is involved.

Comparing Eq. (117) to Eq. (23), we see that

$$\frac{1}{\hbar} \langle \chi (x) \varphi^a (x) \varphi^a (y) \rangle = \bar{\chi} (x) G^{21} (x, y) - \frac{1}{N} \int d^4 z \left[ P^1_1 (x, z) G^{11} (z, y) + P^2_2 (x, z) G^{21} (z, y) \right]$$  (121)

Using Eq. (28) for $\bar{\chi}$, and comparing to Eq. (113), we get

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To be more concrete, observe that if Wick’s theorem held, then

$$\langle \varphi^b(x) \varphi^b(x) \varphi^a(x) \varphi^a(y) \rangle \sim \bar{\hbar}^2 \left( 1 + \frac{2}{N} \right) G^{21}(x, x) G^{21}(x, y)$$

(123)

This suggests writing

$$\langle \varphi^b(x) \varphi^b(x) \varphi^a(x) \varphi^a(y) \rangle \equiv \hbar^2 \left[ G^{21}(x, x) G^{21}(x, y) + \frac{1}{N} C(x, y) \right]$$

(124)

The NLO approximation consists in closing the Schwinger-Dyson hierarchy by writing the $P_{ab}$’s as in Eq. (77), whereby

$$\lambda_B \frac{2 \hbar}{2 \hbar} \langle \varphi^b(x) \varphi^b(x) \varphi^a(x) \varphi^a(y) \rangle = \frac{\lambda_B \hbar}{2} G^{21}(x, x) G^{21}(x, y) - \frac{1}{N} \int d^4 z \left[ P_1^2(x, z) G^{11}(z, y) + P_2^2(x, z) G^{21}(z, y) \right]$$

(122)

Observe that the integrand vanishes when $z^0 \gtrless x^0, y^0$. At this point, time reversal symmetry has been broken.

Let us investigate further the mechanism for breaking time symmetry. First decompose the $H$ propagators in singular and regular parts

$$H^{ab}(x, y) = H(x) c^{ab}(x - y) + H^{ab}_{\text{reg}}(x, y)$$

(126)

Then $C(x, y)$ is split into a reducible and an irreducible term

$$C(x, y) = -\frac{2i}{\lambda_B} H(x) G^{21}(x, x) G^{21}(x, y) + C_{\text{irr}}(x, y)$$

(127)

Time symmetry is broken because the irreducible term $C_{\text{irr}}(x, y)$ vanishes in the distant past while remains non-zero in the far future.

For completeness, let us observe that Eq. (27) shows that $H(x) = i\lambda_B + O(\lambda_B^2)$, and therefore Eq. (127) leads to Eq. (123) to leading order.

In summary, the scheme works because at a crucial point some higher correlation ($C$ in eq. (124)) is replaced by a perturbative expansion in terms of propagators. The replacement assumes that the irreducible part of the higher correlation ($C_{\text{irr}}$ in eq. (127)) vanishes in the distant past, in effect enforcing a variant of Boltzmann’s molecular chaos condition.

In order to restore time reversal symmetry, we ought to treat $C(x, y)$ as a dynamical variable in its own right, for example, by defining a higher generating functional with a new non local source coupled to four fields. We refer the reader to ref. [30] for a fuller discussion of this...
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Figure 1: Only \textit{NLO} contribution to the $2PI$ effective action. The full line denotes a $G$ propagator, while the dotted line stands for $H$.

Figure 2: Only \textit{NNLO} contribution to the $2PI$ effective action with a single $G$ loop.

Figure 3: Only \textit{NNLO} contribution to the $2PI$ effective action with 2 $G$ loops.
Figure 4: Contribution to $P_{ab}$ from the variation of Fig. 2.

Figure 5: Contribution to $P_{ab}$ from the variation of Fig. 3.

Figure 6: Expansion of $H$ in powers of the coupling constant.

Figure 7: Setting sun graph
Figure 8: First correction to Fig. 4

Figure 9: Second correction to Fig. 4

Figure 10: Higher order correction to Fig. 5