THE ASYMPTOTICS OF GROUP RUSSIAN ROULETTE

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Abstract. We study the group Russian roulette problem, also known as the shooting problem, defined as follows. We have \( n \) armed people in a room. At each chime of a clock, everyone shoots a random other person. The persons shot fall dead and the survivors shoot again at the next chime. Eventually, either everyone is dead or there is a single survivor. We prove that the probability \( p_n \) of having no survivors does not converge as \( n \to \infty \), and becomes asymptotically periodic and continuous on the log \( n \) scale, with period 1.

1. Introduction and main result

In [14], Peter Winkler describes the following probability puzzle, called group Russian roulette, and also known as the shooting problem. We start at time \( t = 0 \) with \( n \) people in a room, all carrying a gun. At time \( t = 1 \), all people in the room shoot a randomly chosen person in the room; it is possible that two people shoot each other, but no one can shoot him- or herself. We assume that every shot instantly kills the person shot at. After this first shooting round, a random number of people have survived, and at time \( t = 2 \) we repeat the procedure with all survivors. Continuing like this, eventually we will reach a state with either no survivors, or exactly one survivor. Denote by \( p_n \) the probability that eventually there are no survivors. We are interested in the behavior of \( p_n \) as \( n \to \infty \).

Observe that the probability that a given person survives the first shooting round is \( (1 - (n-1)^{-1})^{n-1} \approx 1/e \), so that the expected number of survivors of the first round is approximately \( n/e \). This fact motivates us to plot \( p_n \) against \( \log n \), see Figure 1 below. Figure 1 suggests that \( p_n \) does not converge as \( n \to \infty \), and becomes asymptotically periodic on the \( \log n \) scale, with period 1. This turns out to be correct, and is perhaps surprising. One may have anticipated that, as \( n \) gets very large, the fluctuations at every round will somehow make the process forget its starting point, but this is not the case. Indeed, here we prove the following:

Theorem 1.1. There exists a continuous, periodic function \( f : \mathbb{R} \to [0, 1] \) of period 1, satisfying \( \sup f \geq 0.515428 \) and \( \inf f \leq 0.477449 \), such that

\[
\sup_{x \geq x_0} \left| p_{[\exp x]} - f(x) \right| \to 0 \quad \text{as } x_0 \to \infty.
\]

The solution to the group Russian roulette problem as it is stated in Theorem 1.1 was already stated in [14], without the explicit bounds on the
limit function. However, [14] does not provide a proof, and as far as we know, there is no proof in the literature.

A number of papers [2–4, 6, 8–11] study the following related problem and generalizations thereof. Suppose we have \( n \) coins, each of which lands heads up with probability \( p \). Flip all the coins independently and throw out the coins that show heads. Repeat the procedure with the remaining coins until 0 or 1 coins are left. The probability of ending with 0 coins does not converge as \( n \to \infty \) and becomes asymptotically periodic and continuous on the \( \log n \) scale [6, 11]. For \( p = 1 - 1/e \), the limit function takes values between 0.365879 and 0.369880, see [6, Corollary 2].

The coin tossing problem for \( p = 1 - 1/e \) has some similarities with group Russian roulette. In view of Theorem 1.1 and the results in [6, 11], the asymptotic behavior of these two models is qualitatively similar but their limit functions have different average values and amplitudes. In the above-mentioned papers, explicit expressions for the probability of ending with no coins could be obtained because of the independence between coin tosses. Analytic methods were subsequently employed to evaluate the limit. This strategy does not seem applicable to the group Russian roulette problem for the simple reason that no closed-form expressions can be obtained for the relevant probabilities. Our approach is, therefore, very different, and we end this introduction with an overview of our strategy.

We recursively compute rigorous upper and lower bounds on \( p_n \) for \( n = 1, \ldots, 6000 \), using *Mathematica*. Based on these computations, we identify values of \( n \) where \( p_n \) is high (the “hills”) and values of \( n \) where \( p_n \) is low (the “valleys”). To prove the non-convergence of the \( p_n \), we explicitly construct intervals \( H_k \) and \( V_k \) \((k = 0, 1, \ldots)\) in such a way that, if \( n \in H_k \) for some \( k \), then with high probability uniformly in \( k \), the number of survivors in the

\[
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{\( p_n \) as a function of \( \log n \) up to \( n = 6000 \).}
\end{figure}
The asymptotics of group Russian roulette shooting process starting with $n$ people will, during the first $k$ shooting rounds, visit each of the intervals $H_{k-1}, H_{k-2}, \ldots, H_0$ (in that order), and similarly for the $V_k$. By our rigorous bounds on $p_n$ we know that $H_0$ is a hill and $V_0$ a valley. This implies that the values of $p_n$ on the respective intervals $H_k$ and $V_k$ are separated from each other, uniformly in $k$.

We stress that, although we make use of Mathematica, our proof of Theorem 1.1 is completely rigorous. There are no computer simulation methods involved, and we use only integer calculations to avoid rounding errors. To make this point clear, we isolated the part of the proof where we use Mathematica as a separate lemma, Lemma 3.2. In the proof of this lemma we explain how we compute the rigorous bounds we need. Our Mathematica notebook and bounds on the $p_n$ are available online at http://arxiv.org/format/1507.03805.

A generic bound on the probability that the number of survivors after each round successively visits the intervals in a carefully constructed sequence, appears in Section 2.3 below. To obtain a good bound on this probability, we make crucial use of a coupling, introduced in Section 2.1, which allows us to compare the random number of survivors of a single shooting round with the number of empty boxes remaining after randomly throwing balls into boxes. For this latter random variable reasonably good tail bounds are readily available, and we provide such a bound in Section 2.2.

The coupling is also crucial in proving the asymptotic continuity and periodicity of the $p_n$ on the log $n$ scale. To prove continuity, we consider what happens if we start the shooting process from two different points in the same interval, using for every round an independent copy of the coupled numbers of survivors for each point. By carefully analyzing the properties of our coupling, we will show that we can make the two coupled processes collide with arbitrarily high probability before reaching 0 or 1, by making the intervals sufficiently narrow on the log $n$ scale, and taking the interval we start from far enough to the right. This shows that for our two starting points, the probabilities of eventually having no survivors must be very close to each other. Periodicity follows because our argument also applies when we start from two points that lie in different intervals, and the distance between the intervals in our construction is 1 on the log $n$ scale.

The proof of non-convergence of the $p_n$, based on the coupling and tail bounds from Section 2, is in Section 3. The proof of asymptotic periodicity and continuity follows in Section 4. Together, these results give Theorem 1.1.

2. Coupling and tail bounds

2.1. Coupling and comparison. Let $S_n$ be the number of survivors after one round of the shooting process starting with $n$ people. Using inclusion-exclusion, the distribution of $S_n$ can be written down explicitly:

\[
P(S_n = k) = \binom{n}{k} (n-1)^{-n} \times \\
\sum_{r=0}^{n-k-2} \binom{n-k}{r} (-1)^r (n-k-r)^{k+r} (n-k-r-1)^{n-k-r}.
\]
We use this formula in Section 3 but not in the rest of our analysis. Instead, let \( Y_n \) be a random variable that counts the number of boxes that remain empty after randomly throwing \( n-1 \) balls into \( n-1 \) (initially empty) boxes. Similarly, let \( Z_n \) be the result of adding 1 to the number of boxes that remain empty after randomly throwing \( n \) balls into \( n-1 \) boxes. It turns out that these random variables \( Y_n \) and \( Z_n \) are very close in distribution to \( S_n \), and are more convenient to work with.

In this section we describe a coupling between the \( S_n \), \( Y_n \) and \( Z_n \), for all \( n \geq 2 \) simultaneously, in which (almost surely) \( S_n, Y_n \) and \( Z_n \) are within distance 1 from each other for all \( n \), and the \( Y_n \) and \( Z_n \) are ordered in \( n \) (see Lemma 2.7 below). This last fact has the useful implication that in the shooting problem, if the number \( n \) of people alive in the room is known to be in an interval \([a, b]\), then the probability that the number of survivors of the next shooting round will lie in some other interval \([\alpha, \beta]\) can be estimated by considering only the two extreme cases \( n = a \) and \( n = b \) (see Corollary 2.2 below). At the end of the section, we extend our coupling to a coupling we can use to study shooting processes with multiple shooting rounds.

To describe our coupling, we construct a Markov chain as follows. Number the people \( 1, 2, \ldots, n \) and define \( A^i_n \subset \{1, \ldots, n\} \) as the set of people who are not shot by any of the persons \( 1 \) up to \( i \) (inclusive). In this formulation, \( S_i^n := |A^i_n| \) represents the number of survivors if only persons \( 1 \) up to \( i \) shoot, and we can write

\[
S_n = S_n^n = |A^n_n|.
\]

The sets \( A^i_n \) (\( i = 1, \ldots, n \)) form a Markov chain inducing the process \((S^i_n)_i\) with transition probabilities given by

\[
P(S^i_{n+1} = S^i_n - 1 | A^i_n) = 1 - P(S^i_{n+1} = S^i_n | A^i_n) = \frac{S^i_n - \mathbb{1}(i + 1 \in A^i_n)}{n - 1}.
\]

Indeed, when person \( i + 1 \) selects his target, the number of persons who will survive the shooting round decreases by 1 precisely when person \( i + 1 \) aims at someone who has not already been targeted by any of the persons \( 1 \) up to \( i \), where we must take into account that person \( i + 1 \) cannot shoot himself (hence the subtraction of \( \mathbb{1}(i + 1 \in A^i_n) \) in the numerator).

An explicit construction of the process described above can be given as follows. Suppose that on some probability space, we have random variables \( U_1, U_2, \ldots \) uniformly distributed on \((0, 1]\), and, for all finite subsets \( A \) of \( \mathbb{N} \) and all \( i \in \mathbb{N} \), random variables \( V_{A,i} \) uniformly distributed on the set \( A \setminus \{i\} \), all independent of each other. Now fix \( n \geq 2 \). Set \( S^0_n := n \) and \( A^0_n := \{1, \ldots, n\} \), and for \( i = 0, 1, \ldots, n - 1 \), recursively define

\[
A_{i+1}^n := \begin{cases} 
A^n_i \setminus \{V_{A^n_i,i+1}\} & \text{if } U_{n-i} \leq \frac{|A^n_i| - \mathbb{1}(i + 1 \in A^n_i)}{n - 1}, \\
A^n_i & \text{otherwise};
\end{cases}
\]

and set \( S_{i+1}^n := |A_{i+1}^n| \). In this construction, the variable \( U_{n-i} \) is used first to decide whether person \( i + 1 \) aims at someone who will not be shot by any of the persons \( 1 \) up to \( i \), and then we use \( V_{A^n_i,i+1} \) to determine his victim. Clearly, this yields a process with the desired distribution, and provides a coupling of the processes \((S^i_n)_i\) for all \( n \geq 2 \) simultaneously.
We now extend this coupling to include new processes \((Y^n_i)\), and \((Z^n_i)\), as follows. For fixed \(n \geq 2\), we first set \(Y^n_0 := n\) and \(Z^n_0 := n\), and then for \(i = 0, 1, \ldots, n - 1\) we recursively define

\[
Y^n_{i+1} := \begin{cases} 
Y^n_i - 1 & \text{if } U_{n-i} \leq \frac{Y^n_i}{n-1}; \\
Y^n_i & \text{otherwise};
\end{cases}
\]

and

\[
Z^n_{i+1} := \begin{cases} 
Z^n_i - 1 & \text{if } U_{n-i} \leq \frac{Z^n_i - 1}{n-1}; \\
Z^n_i & \text{otherwise}.
\end{cases}
\]

Then, by construction, \((Y^n_i)\) and \((Z^n_i)\) are Markov chains with the respective transition probabilities

\[
P(Y^n_{i+1} = Y^n_i - 1 \mid Y^n_i) = 1 - P(Y^n_{i+1} = Y^n_i \mid Y^n_i) = \frac{Y^n_i}{n-1};
\]

\[
P(Z^n_{i+1} = Z^n_i - 1 \mid Z^n_i) = 1 - P(Z^n_{i+1} = Z^n_i \mid Z^n_i) = \frac{Z^n_i - 1}{n-1}.
\]

The similarity with \(2.2\) is clear, and we see that we can interpret \(Y^n_i\) as the number of empty boxes after throwing \(i\) balls into \(n\) boxes, where the first ball is thrown into the \(n\)th box and the remaining balls are thrown randomly into the first \(n-1\) boxes only. Likewise, \(Z^n_i\) is the number of empty boxes after throwing \(i\) balls into the first \(n-1\) of a total of \(n\) boxes (so that the \(n\)th box remains empty throughout the process). If we now set

\[S_n := S^n_n, \quad Y_n := Y^n_n\]  

and \(Z_n := Z^n_n\),

then \(S_n, Y_n\) and \(Z_n\) have the interpretations described at the beginning of this section. The next lemma shows they have the properties we mentioned:

**Lemma 2.1.** The coupling of the \(S_n, Y_n\) and \(Z_n\) described above satisfies

1. \(Y_n \leq Y_{n+1} \leq Y_n + 1\) and \(Z_n \leq Z_{n+1} \leq Z_n + 1\) for all \(n \geq 2\);
2. \(Y_n \leq S_n \leq Z_n \leq Y_n + 1\) for all \(n \geq 2\).

**Proof.** As for (1), we claim that the \(Y^n_i\) satisfy the stronger statement that

\[
Y^n_i \leq Y^n_{i+1} \leq Y^n_i + 1 \quad \text{for all } n \geq 2 \text{ and } i = 0, 1, \ldots, n.
\]

To see this, first note that necessarily, \(Y^n_{n+1} = n = Y^n_0\). Now suppose that \(Y^n_i = Y^n_{i+1} + 1\) for some index \(i\). Then \(2.3\) implies that if \(Y^n_{i+2} = Y^n_{i+1} - 1\), we also have \(Y^n_{i+1} = Y^n_i - 1\). Hence the ordering is preserved, proving that \(Y^n_i \leq Y^n_{i+1}\) for all \(i \leq n\). Likewise, if \(Y^n_{i+1} = Y^n_i + 1\) and \(Y^n_{i+1} = Y^n_i - 1\) for some \(i\), then \(2.3\) implies that \(Y^n_{i+1} = Y^n_i + 1\). This proves \(2.7\) and hence (1) for the \(Y^n_i\). The proof for the random variables \(Z^n_i\) is similar.

As for property (2), observe that if \(Y^n_i = Z^n_i\) for some index \(i\) and \(Z^n_{i+1} = Z^n_i - 1\), then also \(Y^n_{i+1} = Y^n_i - 1\). On the other hand, if \(Y^n_i = Z^n_i - 1\) for some index \(i\), then it follows from the construction that \(Y^n_j = Z^n_j - 1\) for all \(j = i, i+1, \ldots, n\). Since \(Y^n_0 = Z^n_0 = n\), we conclude that

\[
i \leq Z^n_i \leq Y^n_i + 1 \quad \text{for all } n \geq 2 \text{ and } i = 0, 1, \ldots, n.
\]
Furthermore, if \( Y_i^n = S_i^n \) and \( S_{i+1}^n = S_i^n - 1 \), then our construction implies that \( Y_{i+1}^n = Y_i^n - 1 \). Similarly, if \( S_i^n = Z_i^n \) and \( Z_{i+1}^n = Z_i^n - 1 \), then in our coupling we also have that \( S_{i+1}^n = S_i^n - 1 \). It follows that
\[
Y_i^n \leq S_i^n \leq Z_i^n
\]
for all \( n \geq 2 \) and \( i = 0, 1, \ldots, n \), and this together with (2.8) establish property (2). \( \square \)

**Corollary 2.2.** Suppose we have coupled the \( S_n \) as described above. Then, for any intervals \([a, b]\) and \([\alpha, \beta]\), with \( a, b, \alpha, \beta \) integers,
\[
P(\exists n \in [a, b] : S_n \notin [\alpha, \beta]) \leq P(Y_a \leq \alpha - 1) + P(Y_b \geq \beta).
\]

*Proof.* Let the \( S_n \) and \( Y_n \) be coupled as described above. By Lemma 2.1
\[
P(\forall n \in [a, b] : S_n \in [\alpha, \beta]) \geq P(\forall n \in [a, b] : Y_n \in [\alpha, \beta - 1])
\]
\[
= P(Y_a \geq \alpha, Y_b \leq \beta - 1).
\]
By taking complements the desired result follows. \( \square \)

**Remark 2.3.** The distribution of \( Y_i^n \) is related to Stirling numbers of the second kind, as follows. Recall that \( Y_{i+1}^n \) can be interpreted as the number of empty boxes after throwing \( i \) balls randomly into \( n \) boxes. We claim that
\[
P(Y_{i+1}^n = n-k) = P(n-k \text{ boxes empty, } k \text{ boxes non-empty})
\]
\[
= \frac{n!}{(n-k)!} \frac{1}{n^i} S(i, k),
\]
with \( S(i, k) \) a Stirling number of the second kind. Indeed, \( S(i, k) \) is by definition the number of ways of partitioning the set of \( i \) balls into \( k \) non-empty subsets. Balls in the same subset are thrown into the same box. The number of ways to assign these subsets to \( k \) distinct boxes equals \( n!/(n-k)! \). Finally, \( n^i \) is the number of ways of distributing \( i \) balls over \( n \) boxes.

We now extend our coupling to a coupling we can use for an arbitrary number of shooting rounds, and for shooting processes starting from different values of \( n \). Since the shooting rounds must be independent, we take an infinite number of independent copies of the coupling described above, one for each element of \( \mathbb{Z} \) (so including the negative integers). The idea is to use a different copy for each round of a shooting process. For reasons that will become clear, we want to allow the copy that is used for the first round to vary with the starting point \( n \).

To be precise, let \( X^n_i \) represent the number of survivors after round \( i \) of a shooting process started with \( n \) people in the room. Let \( k_n \) be the number of the copy of our coupling that is to be used for the first round of this process, and denote the \( i \)-th copy of \( S_n \) by \( S_{n}^{(i)} \). We recursively define
\[
X_0^n := n, \quad X_{i+1}^n := S_{X_i^n}^{(k_n - i)} \quad \text{for } i \geq 0.
\]
In this way, the \((k_n - i)\)-th copy of the \( S_n \) is used to determine what happens in round \( i + 1 \) of the process. Note that the index \( k_n - i \) becomes negative when \( i > k_n \). Our setup is such that if \( k_m = k_n \), then the shooting processes started from \( m \) and \( n \) are coupled from the first shooting round onward, but if \( k_m = k_n + l \) with \( l > 0 \), then the shooting process started from \( m \) will first undergo \( l \) independent shooting rounds before it becomes coupled.
with the shooting process started from \( n \). Thus, by varying the \( k_n \), we can
choose after how many rounds shooting processes with different starting
points become coupled.

2.2. Tail bounds. In Corollary 2.2 we have given a bound on the probabil-
ity that the shooting process, starting at any point in some interval, visits
another interval after one shooting round. This bound is in terms of the
tails of the distribution of the random variables \( Y_n \). In this section, we show
that \( Y_n \) in fact has the same distribution as a sum of independent Bernoulli
random variables, and we use this result to obtain tail bounds for \( Y_n \).

**Lemma 2.4.** For every \( n \geq 3 \), there exist \( n-2 \) independent Bernoulli
random variables \( W_1, \ldots, W_{n-2} \) such that \( Y_n \) has the same distribution as
\( W_1 + W_2 + \cdots + W_{n-2} \).

**Proof.** The proof is based on the following beautiful idea due to Vatutin and
Mikhailov \cite{13}: we will show that the generating function of \( Y_n \) has only real
roots, and then show that this implies the statement of the lemma. For the
first step we observe that 
\[
(Y_n)^k = \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \mathbb{1}(\text{boxes } i_1, \ldots, i_k \text{ empty})
\]
\[
= \binom{n-1}{k} \mathbb{P}(\text{boxes } 1, \ldots, k \text{ empty}) = \binom{n-1}{k} \left(\frac{n-k-1}{n-1}\right)^{n-1}.
\]

Hence, if we define
\[
R(z) = \sum_{k=0}^{n-1} \binom{n-1}{k} (n-k-1)^{n-1} z^k,
\]
then we see that
\[
E(z^{Y_n}) = E\left(\sum_{k=0}^{n-1} \binom{n-1}{k} (z-1)^k\right) = (n-1)^{-(n-1)} R(z-1).
\]

We want to show that \( R \) has only real roots, for which it is enough to
show that \( z^{n-1} R(1/z) \) has only real roots. To show this, we now write
\[
z^{n-1} R(1/z) = \sum_{k=0}^{n-1} \binom{n-1}{k} (n-k-1)^{n-1} z^{n-k-1}
\]
\[
= \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{d}{dz}\right)^{n-1} z^{n-k-1},
\]
from which it follows that
\[
z^{n-1} R(1/z) = \left(\frac{d}{dz}\right)^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} z^{n-k-1} = \left(\frac{d}{dz}\right)^{n-1} (z+1)^{n-1}.
\]

Now observe that if a polynomial \( f(z) \) has only real roots, then so do the
polynomials \( zf(z) \) and \( f'(z) \) (one way to see the latter is to observe that
between any two consecutive zeroes of \( f \), there must be a local maximum
or minimum). Therefore, our last expression for \( z^{n-1} R(1/z) \) above has only real roots and hence so does \( R \).

It follows that the generating function \( E(z^{Y_n}) \) of \( Y_n \) has only real roots. Now note that \( E(z^{Y_n}) \) is a polynomial of degree \( n-2 \) which cannot have positive roots. Let its roots be \( -d_1, -d_2, \ldots, -d_{n-2} \), with all the \( d_i \geq 0 \), and let \( W_1, \ldots, W_{n-2} \) be independent Bernoulli random variables such that

\[
P(W_i = 1) = 1 - P(W_i = 0) = \frac{1}{1 + d_i}, \quad i = 1, \ldots, n-2.
\]

Note that these are properly defined random variables because of the fact that \( d_i \geq 0 \) for all \( i \). Writing \( W = W_1 + \cdots + W_{n-2} \), we now have

\[
E(z^W) = \prod_{i=1}^{n-2} E(z^{W_i}) = \prod_{i=1}^{n-2} \frac{z + d_i}{1 + d_i} = E(z^{Y_n}),
\]

so \( Y_n \) and \( W \) have the same distribution.

Tail bounds for sums of independent Bernoulli random variables are generally derived from a fundamental bound due to Chernoff [5] by means of calculus, see e.g. [1, Appendix A]. Here we use the following result:

**Theorem 2.5.** Let \( W \) be the sum of \( n \) independent Bernoulli random variables, and let \( p = EW/n \). Then for all \( u \geq 0 \) we have

\[
P(W \leq EW - u) \leq \exp \left( -\frac{1}{2 np(1-p)} \frac{u^2}{1-u(1-2p)/3} \right), \tag{2.10}
\]

and

\[
P(W \geq EW + u) \leq \exp \left( -\frac{1}{2 np(1-p)} \frac{u^2}{1+u(1-2p)/3} \right). \tag{2.11}
\]

**Proof.** This theorem has been proved by Janson, see [7, Theorems 1 and 2]. For the convenience of the reader we outline the main steps of the proof of inequality (2.11) here. Inequality (2.10) follows by symmetry.

By [1, Theorem A.1.9] we have for all \( \lambda > 0 \),

\[
P(W \geq EW + u) < e^{-\lambda p n (pe^\lambda + (1-p))^n e^{-\lambda u}}. \tag{2.12}
\]

By the remark following [1, Theorem A.1.9], for given \( p, n, u \), the value of \( \lambda \) that minimizes the right hand side of inequality (2.12) is

\[
\lambda = \log \left( \frac{1-p}{p} \left( \frac{u + np}{n - (u + np)} \right) \right). \tag{2.13}
\]

In [1] suboptimal values of \( \lambda \) are substituted into (2.12) to obtain bounds. Substituting the optimal value (2.13) into (2.12), and letting \( q = 1-p \) and \( x = u/n \in [0, q] \), yields [7] Inequality (2.1):

\[
P(W \geq EW + u) \leq \exp \left( -n(p+x) \log \frac{p+x}{p} - n(q-x) \log \frac{q-x}{q} \right).
\]

Following [7] we define, for \( 0 \leq x \leq q \),

\[
f(x) = (p+x) \log \frac{p+x}{p} + (q-x) \log \frac{q-x}{q} - \frac{x^2}{2(pq + x(q-p)/3)}.
\]
Then $f(0) = f'(0) = 0$ and
\[
    f''(x) = \frac{3pq(q-p)^2x^2 + \frac{1}{3}q^2x^3 + p^2q^2x^2}{(x+p)(q-x)(pq + x(q-p)/3)^3} \geq 0,
\]
for $0 \leq x \leq q$. Hence $f(x) \geq 0$ for $0 \leq x \leq q$, which proves (2.11). \qed

**Corollary 2.6.** Let $n \geq 4$. Then for all $u \geq 0$,
\[
    P\left(Y_n \leq \frac{n - 5/3}{e} - u\right) \leq \exp\left(-\frac{1}{2} \frac{e^2u^2}{(n-1)(e-1)^2}\right)
\]
and
\[
    P\left(Y_n \geq \frac{n - 3/2}{e} + u\right) \leq \exp\left(-\frac{1}{2} \frac{e^2u^2}{(n-1)(e-1) + ue(e-2)/3}\right).
\]

**Proof.** Recall that $Y_n$ can be interpreted as the number of empty boxes after randomly throwing $n-1$ balls into $n-1$ boxes. Thus we have
\[
    EY_n = (n-1) \left(1 - \frac{1}{n-1}\right)^{n-1}.
\]
We will bound this expectation using the following two inequalities, which hold for all $u \in (0, 1)$:
\[
    (1 - u)^{1/u} \leq (1 - \frac{1}{2}u)/e, \quad (1 - u)^{1/u} \geq (1 - \frac{1}{2}u - \frac{1}{2}u^2)/e.
\]
To prove these inequalities, we define
\[
    h_1(u) = u + \log(1-u) - u \log(1 - \frac{1}{2}u), \quad h_2(u) = u + \log(1-u) - u \log(1 - \frac{1}{2}u - \frac{1}{2}u^2).
\]
Then $h_1(0) = h_2(0) = h_1'(0) = h_2'(0) = 0$ and moreover
\[
    h_1''(u) = -\frac{u(5 - 5u + u^2)}{(1-u)^2(2-u)^2}, \quad h_2''(u) = \frac{u(7 + 2u)}{(1-u)(2-u)^2}.
\]
Hence $h_1''(u) < 0$ and $h_2''(u) > 0$ for $u \in (0, 1)$. Therefore, $h_1(u) < 0$ and $h_2(u) > 0$ for all $u \in (0, 1)$, which implies (2.15) and (2.16).

By (2.14) and (2.15), we have that
\[
    EY_n \leq \frac{n - 3/2}{e}.
\]

Similarly, using (2.14) and (2.16), we obtain that
\[
    EY_n \geq \frac{n - 3/2}{e} - \frac{1}{2e(n-1)} \geq \frac{n - 5/3}{e} \quad \text{for } n \geq 4.
\]
Since, by Lemma 2.4, $Y_n$ has the same distribution as a sum of $n - 2$ independent Bernoulli random variables, Theorem 2.5 applies to the $Y_n$. It follows from (2.17) and (2.18) that in applying this theorem to $Y_n$ for $n \geq 4$, we can use that
\[
    (n-2)p \leq \frac{n-1}{e}, \quad 1-p \leq \frac{e-1}{e}, \quad 0 \leq 1 - 2p \leq \frac{e-2}{e},
\]
where $p = EY_n/(n-2)$. This yields the desired result. \qed
2.3. Visiting consecutive intervals. Corollaries 2.2 and 2.6 together give an explicit upper bound on the probability that the shooting process, starting anywhere in some interval, visits a given other interval after the next shooting round. In this section, we extend this result to more than one round. We give an explicit construction of a sequence of intervals $I_0, I_1, \ldots$ and, using Corollaries 2.2 and 2.6, we estimate the probability that the shooting process successively visits each interval in this specific sequence.

To start our construction, suppose that the (real) numbers $I_0^+, I_0^- \geq 2$, with $I_0^- < eI_0^+$, and a parameter $\gamma \in (0, 1]$ are given. Set

\begin{equation}
(2.20) \quad c_0 := \left( \sqrt{I_0^+} - \sqrt{I_0^-} \right) \frac{\gamma}{s_0 \sqrt{e}}.
\end{equation}

For all $k \geq 1$, we now define the real numbers $I_k^-$ and $I_k^+$ by

\begin{align*}
(2.21) \quad I_k^- & := I_0^- e^{c_k \left( 1 + c_0 \sqrt{\frac{e}{I_0^+}} \sum_{i=1}^{k} \sqrt{i} e^{-i/2} \right)}, \\
(2.22) \quad I_k^+ & := I_0^+ e^{c_k \left( 1 - c_0 \sqrt{\frac{e}{I_0^-}} \sum_{i=1}^{k} \sqrt{i} e^{-i/2} \right)},
\end{align*}

and we set $I_k := [I_k^-, I_k^+]$ for all $k \geq 0$. These specific choices for $I_k^+$ and $I_k^-$ may look peculiar, but the reader will see in our calculations below why they are convenient. At this point, let us just note that our intervals are disjoint (since $I_{k+1}^- > I_0^- e^{c_{k+1}} > I_0^- e^{c_k} > I_k^+$) and their lengths are (roughly) given by the relatively simple expression

$$I_k^+ - I_k^- = (I_0^+ - I_0^-) e^{c_k \left( 1 - \frac{2}{s_0} \sum_{i=1}^{k} \sqrt{i} e^{-i/2} \right)}.$$ 

For $\gamma = 1$, this reduces to

$$I_k^+ - I_k^- = (I_0^+ - I_0^-) e^{c_k \left( 1 - \frac{1}{s_0} \sum_{j=1}^{k} \sqrt{j + k} e^{-(j+k)/2} \right) \geq (I_0^+ - I_0^-) e^{k/2},}$$

which shows that the lengths of our intervals $I_k$ grow to infinity with $k$.

We want to consider shooting processes starting from any $n \in \bigcup_{k=1}^{\infty} I_k$, and we couple these processes as in (2.9), where we take $k_n$ equal to the index of the interval containing $n$. In this way, all shooting processes starting from the same interval are coupled from the first round onward, while a shooting process starting from a point in $I_{k+1}$ first undergoes $l$ independent shooting rounds (and, with high probability, reaches $I_k$), before it becomes coupled to a shooting process starting from a point in $I_k$. The following lemma gives an estimate of the probability that a shooting process starting from any $n \in \bigcup_{k=1}^{\infty} I_k$ visits each of the intervals $I_{k-1}, I_{k-2}, \ldots, I_0$, in that order.

**Lemma 2.7.** Let the numbers $I_0^+, I_0^-$ and the parameter $\gamma \in (0, 1]$ be given, and define the intervals $I_k$ $(k \geq 0)$ by (2.21) and (2.22), as explained above.
Likewise, define together. This proves (2.23). □

For each $n \in \bigcup_{k=1}^{\infty} I_k$, let $k_n$ be the index of the interval containing $n$, and define $X_n^i (i \geq 0)$ by (2.9). Then

(2.23) \quad P\left(\text{for some } n \in \bigcup_{k=1}^{\infty} I_k \text{ and } i \leq k_n, X_n^i \notin I_{k_n-i}\right)
\leq \frac{1}{e c_1 - 1} + \frac{1}{e c_2 - 1}

where

$$c_1 = \frac{e c_0^2 / 2}{(e - 1)(1 + c_0 s_0 \sqrt{e / I_0^+})} \quad \text{and} \quad c_2 = \frac{e c_0^2 / 2}{e - 1 - c_0 (2e - 1) / 3 \sqrt{I_0^+}},$$

with $s_0$ and $c_0$ defined as in (2.19) and (2.20). Note that the right hand side of (2.23) depends, via $I_0^+$, $I_0^-$ and $\gamma$.

Proof. Let the $S_n^{(k)}$, for $n \geq 2$ and $k \geq 1$, be coupled as in Section 2.1. Suppose we are on the event that for all $k \geq 1$ and $n \in I_k$ it holds that $S_n^{(k)} \notin I_{k-1}$. Then, by our coupling, it follows that for all $k \geq 1$, $n \in I_k$ and $i \leq k$, $X_i^c \notin I_{k-i}$. The latter statement is equivalent to saying that for all $n \in \bigcup_{k=1}^{\infty} I_k$ and $i \leq k_n$, $X_i^c \notin I_{k_n-i}$. Therefore, the left hand side of (2.23) is bounded above by

(2.24) \quad P(\exists k \geq 1, \exists n \in I_k : S_n^{(k)} \notin I_{k-1}) \leq \sum_{k=1}^{\infty} P(\exists n \in I_k : S_n \notin I_{k-1}).

By Corollary 2.2 we have for $k \geq 1$,

(2.25) \quad P(\exists n \in I_k : S_n \notin I_{k-1})
\leq P(Y_{I_k^-} \leq \lfloor I_{k-1}^- \rfloor - 1) + P(Y_{r^+} \geq \lfloor I_{k-1}^+ \rfloor).

To bound the right hand side of (2.25), we use Corollary 2.6, which applies for all $k \geq 1$ since $I_1 \geq e I_0 > 4$. We first note that since $\frac{e c_0^2}{e - 1} - 1 < 0$,\n
(2.26) \quad \lfloor I_{k-1}^- \rfloor - 1 - \frac{\lfloor I_k^- \rfloor - 5/3}{e} \leq \frac{I_{k-1}^- - I_k^-}{e} = -c_0 \sqrt{I_0^-} \frac{e (k-1)^{1/2}}{\sqrt{k}}.

By Corollary 2.6 with $n = \lfloor I_k^- \rfloor$ and $u = e$ equal to the right hand side of (2.26), using $\lfloor I_k^- \rfloor - 1 \leq I_0^- e \left(1 + c_0 s_0 \sqrt{e / I_0^-}\right)$, we see that

(2.27) \quad P(Y_{I_k^-} \leq \lfloor I_{k-1}^- \rfloor - 1) \leq \exp \left( -\frac{1}{2} \frac{e c_0^2 k}{(e - 1)(1 + c_0 s_0 \sqrt{e / I_0^-})} \right).

Likewise,

$$\lfloor I_{k-1}^+ \rfloor - \frac{\lfloor I_k^+ \rfloor - 3/2}{e} \geq \frac{I_{k-1}^+ - I_k^+}{e} = c_0 \sqrt{I_0^-} \frac{e (k-1)^{1/2}}{\sqrt{k}}.$$

By Corollary 2.6, using $\lfloor I_k^+ \rfloor - 1 \leq I_0^+ e k - c_0 e \sqrt{I_0^+}$ and $e (k-1)^{1/2} \sqrt{k} \leq e^{k-1}$,

(2.28) \quad P(Y_{r^+} \geq \lfloor I_{k-1}^+ \rfloor) \leq \exp \left( -\frac{1}{2} \frac{e c_0^2 k}{e - 1 - c_0 (2e - 1) / 3 \sqrt{I_0^+}} \right).

By (2.25), (2.27) and (2.28), the right hand side of (2.24) is bounded above by the sums over all $k \geq 1$ of the right hand sides of (2.27) and (2.28), added together. This proves (2.23). ∎
3. Non-convergence

In this section we prove non-convergence of the $p_n$:

**Theorem 3.1** (Non-convergence). *It is the case that*

$$\limsup_{n \to \infty} p_n \geq 0.515428 \quad \text{and} \quad \liminf_{n \to \infty} p_n \leq 0.477449.$$  

The idea of the proof of Theorem 3.1 is as follows. We will take intervals $H_0$ and $V_0$ around $n = 2795$ and $n = 4608$, the last peak and valley in Figure 1 respectively, so that $p_n$ is high on $H_0$ and low on $V_0$. Then we will construct sequences of intervals $H_1, H_2, \ldots$ and $V_1, V_2, \ldots$ in such a way that, if $n \in H_k$ for some $k$, then with high probability uniformly in $k$, the number of survivors in the shooting process starting with $n$ persons will, during the first $k$ shooting rounds, visit each of the intervals $H_{k-1}, H_{k-2}, \ldots, H_0$ (in that order), and similar for $V_k$. As a consequence, $p_n$ must be high on all intervals $H_k$, and low on all intervals $V_k$.

To make this work, the intervals $H_0$ and $V_0$ should be big enough to make the probability high that the number of survivors after $k$ rounds will lie in them when we start from $H_k$ or $V_k$, but small enough so that the values taken by the $p_n$ on the respective intervals $H_0$ and $V_0$ are sufficiently separated from each other. It turns out that $H_0 = [2479, 3151]$ and $V_0 = [4129, 5143]$ work, and these intervals form our starting point.

The next three intervals $H_1$, $H_2$ and $H_3$ are constructed as follows. We choose the right boundary $H_1^+$ of $H_1$ such that $E S_{H_1}$ lies roughly $3.56$ standard deviations away from the right boundary of $H_0$, and we choose the left boundary $H_1^-$ of $H_1$ similarly. In this way, we expect that after one shooting round we will end up in $H_0$ with high probability, when we start in $H_1$. The intervals $H_2$ and $H_3$ are constructed similarly, and so are the intervals $V_1$ and $V_2$. We need this special treatment only for two (instead of three) intervals $V_1$ and $V_2$, because $V_0$ lies to the right of $H_0$. We end up with

$$H_0 = [2479, 3151], \quad V_0 = [4129, 5143],$$

$$H_1 = [6991, 8290], \quad V_1 = [11553, 13623],$$

$$H_2 = [19425, 22086], \quad V_2 = [31952, 36447],$$

$$H_3 = [53501, 59301].$$

The remaining intervals are now constructed as explained in Section 2.3, taking $H_3$ and $V_2$ as the respective starting intervals. To be more precise, we first set $I_0 := H_3$, take $\gamma = 1$, and then for $k \geq 4$ define the intervals $H_k = [H_k^-, H_k^+] := [[I_{k-3}^-], [I_{k-3}^+]]$ using equations (2.21) and (2.22) for the endpoints. In the same way we define the intervals $V_k$ for $k \geq 3$, taking $I_0 := V_2$ as the initial interval in the construction from Section 2.3.

The following lemma tells us that the values of the $p_n$ on the intervals $H_0$ and $V_0$ are sufficiently separated from each other and that, when we start in $H_3$, the number of survivors in the shooting process will visit each of the intervals $H_2, H_1, H_0$ with high probability, and similarly for $V_2, V_1, V_0$. We obtain the desired bounds using computations in *Mathematica*. We explain how we can perform the computations in such a way that we avoid introducing rounding errors, and thus obtain rigorous results.
Lemma 3.2. We have
\[ \min\{ p_n : n \in H_0 \} \geq 0.5163652651, \]
\[ \max\{ p_n : n \in V_0 \} \leq 0.4767018688, \]
and moreover
\[ \sum_{k=1}^{3} \left[ P(Y_{H_k}^- \leq H_{k-1}^- - 1) + P(Y_{H_k}^+ \geq H_{k-1}^+) \right] \leq 0.0010954222, \]
\[ \sum_{k=1}^{2} \left[ P(Y_{V_k}^- \leq V_{k-1}^- - 1) + P(Y_{V_k}^+ \geq V_{k-1}^+) \right] \leq 0.0006060062. \]

Proof. The explicit bounds in the first part of Lemma 3.2 are based on exact calculations in Mathematica of bounds on the numbers \( p_n \) up to \( n = 6000 \) using the recursion
\[ p_n = \sum_{k=0}^{n-2} P(S_n = k) p_k \quad (n \geq 2), \]
with \( p_0 = 1 \) and \( p_1 = 0 \). To obtain lower bounds on \( p_n \) from this recursion, we need lower bounds on the \( P(S_n = k) \). To this end, write
\[ t_{k,r} = \binom{n}{k} \binom{n-k}{r} (n-k-r)^k \binom{n-k-r-1}{n-k-r}, \]
for the terms that appear in the inclusion-exclusion formula (2.1). Observe that these are integer numbers. Now, for fixed \( n \) and \( k \), define \( r_{\text{max}} \) by
\[ r_{\text{max}} := \min\{ r \geq 0 : 10^{10} t_{k,2r} < (n-1)^n \}. \]
Since truncating the sum in the inclusion-exclusion formula after an even number of terms yields a lower bound on \( P(S_n = k) \), we have that
\[ P(S_n = k) \geq (n-1)^{-n} \sum_{r=0}^{2r_{\text{max}}-1} (-1)^r t_{k,r} n^r. \]
By our choice of \( r_{\text{max}} \), we know that the difference between the left and right hand sides of this inequality is smaller than \( 10^{-10} \).

However, this rational lower bound on \( P(S_n = k) \) is numerically awkward to work with, because the numerator and denominator become huge for large \( n \). We therefore bound \( P(S_n = k) \) further by the largest smaller rational number of the form \( m/10^{10} \) with \( m \in \mathbb{N} \). Stated in a different way, we bound the quantity \( 10^{10} P(S_n = k) \) from below by the integer
\[ P_{n,k} := 0 \vee \left[ 10^{10} \sum_{r=0}^{2r_{\text{max}}-1} (-1)^r t_{k,r} n^r / (n-1)^n \right], \]
where we remark that for integers \( a \) and \( b \), \( \lfloor a/b \rfloor \) is just the quotient of the integer division \( a/b \).

We now return to (3.1). Suppose that we are given nonnegative integers \( \hat{p}_0, \hat{p}_1, \ldots, \hat{p}_{n-1} \) that satisfy \( 10^{10} \hat{p}_k \geq \hat{p}_k \) for \( k = 0, 1, \ldots, n-1 \). Let
\[ \hat{p}_n := \left[ \sum_{k=k_1}^{k_2} P_{n,k} \hat{p}_k / 10^{10} \right], \]
where
\[ k_1 = 0 \lor \left\lceil \frac{n/e}{\sqrt{5n}} \right\rceil, \quad k_2 = (n-2) \land \left\lceil \frac{n/e}{\sqrt{5n}} \right\rceil. \]

Then it follows from (3.1) and the fact that \(10^{10} \Pr(S_n = k) \geq P_{n,k}\), that \(10^{10}p_n \geq \hat{p}_n\). In this way, starting from the values \(\hat{p}_0 = 10^{10}\) and \(\hat{p}_1 = 0\), we recursively compute integer lower bounds on the numbers \(10^{10}p_n\), or equivalently, rational lower bounds on \(p_n\), up to \(n = 6000\). We emphasize that this procedure involves only integer calculations, that could in principle be done by hand. For practical reasons, we invoke the aid of Mathematica to perform these calculations for us, using exact integer arithmetic.

In the same way (now starting the recursion from \(\hat{p}_0 = 0\) and \(\hat{p}_1 = 10^{10}\)), we compute exact bounds on the probabilities \(1 - p_n\) of ending up with a single survivor. Taking complements, this gives us rational upper bounds on the \(p_n\) up to \(n = 6000\). The first part of Lemma 3.2 follows from these exact bounds, and Figure 1 shows the lower bounds as a function of \(\log n\).

As it turns out, the largest difference between our upper and lower bounds is 527, up to \(n = 527\). In this way, starting from the values \(\hat{p}_0 = 0\) and \(\hat{p}_1 = 10^{10}\), we recursively compute integer lower bounds on the numbers \(10^{10}p_n\) or equivalently, rational lower bounds on \(p_n\) up to \(n = 6000\). We emphasize that this procedure involves only integer calculations, that could in principle be done by hand. For practical reasons, we invoke the aid of Mathematica to perform these calculations for us, using exact integer arithmetic.

The second part of Lemma 3.2 again follows from exact integer calculations with the aid of Mathematica. Inclusion-exclusion tells us that

\[ P(Y_{n+1} = k + i) = \frac{1}{n^n} \sum_{r=0}^{n-k} (-1)^{r-i} \binom{n}{k+i} \binom{n-k-i}{r-i} (n-k-r)^n. \]

Summing over \(i = 0, \ldots, n-k\), interchanging the order of summation, and reorganising the binomial coefficients yields

\[ P(Y_{n+1} \geq k) = \frac{1}{n^n} \sum_{r=0}^{n-k} (-1)^r \sum_{i=0}^{r} (-1)^i \binom{k+r}{k+i} \binom{n}{k+r} (n-k-r)^n. \]

Using the binomial identity
\[ \sum_{i=0}^{r} (-1)^i \binom{k+r}{k+i} = \binom{k+r-1}{r} \quad (k \geq 1, r \geq 0), \]
which is easily proved by induction in \(r\), we conclude that for \(k \geq 1\),

\[ P(Y_{n+1} \geq k) = \frac{1}{n^n} \sum_{r=0}^{n-k} (-1)^r \frac{k^n}{k+r} \binom{n}{k} \binom{n-k}{r} (n-k-r)^n. \]

Since \(P(Y_{n+1} \leq k) = 1 - P(Y_{n+1} \geq k) + P(Y_{n+1} = k)\), the previous equation together with (3.2) for \(i = 0\) gives

\[ P(Y_{n+1} \leq k) = 1 + \frac{1}{n^n} \sum_{r=0}^{n-k} (-1)^r \frac{r^n}{k+r} \binom{n}{k} \binom{n-k}{r} (n-k-r)^n. \]

We note that from our derivation it follows that, as before, the terms that appear in the sums above are integers. This allows us to compute the rational numbers \(P(Y_{n+1} \leq k)\) and \(P(Y_{n+1} \geq k)\), and hence the sums in the second part of Lemma 3.2 using only exact integer arithmetic. Bounding these sums above by rational numbers of the form \(m/10^{10}\) (which again involves only integer arithmetic) yields the second part of Lemma 3.2. \(\square\)
Proof of Theorem 3.1. Let the intervals \( H_k \), for \( k \geq 0 \), be constructed as explained below the statement of Theorem 3.1. Similarly as in Section 2.3, for \( n \in \bigcup_{k=1}^{\infty} H_k \) we now define \( X^n_i \) by (2.9), with \( k_n \) equal to the value of \( k \) such that \( n \in H_k \). Recall that \( X^n_i \) represents the number of survivors after round \( i \) of the shooting process started from \( n \). Fix a \( k \geq 1 \) and \( n \in H_k \).

We are interested in the event \( G_n = \{ X^n_i \in H_{k-i} \text{ for all } i = 1, \ldots, k \} \).

It follows from

\[
P(G_n^c) = P(\exists i \leq k - 3: X^n_i \notin H_{k-i}) + \sum_{k=1}^{3} P(\exists m \in H_k: S_m \notin H_{k-1})
\]

and Corollary 2.2, that \( P(G_n^c) \) is bounded from above by

\[
(3.3) \quad P(\exists i \leq k - 3: X^n_i \notin H_{k-i}) + \sum_{k=1}^{3} \left[ P(Y_{H_k^-} \leq H_{k-1}^- - 1) + P(Y_{H_k^+} \geq H_{k-1}^+) \right].
\]

We use Lemma 2.7 to compute an upper bound on the first term in (3.3), and Lemma 3.2 to bound the sum in the second term. This gives

\[
P(G_n^c) \leq 0.0007188677 + 0.0010954222 = 0.0018142899,
\]

uniformly for all \( k \geq 1 \) and \( n \in H_k \). Using the first part of Lemma 3.2, this gives

\[
p_n \geq P(G_n) \min_{m \in H_0} p_m \geq 0.9981857101 \times 0.5163652651 \geq 0.515428,
\]

for all \( n \in \bigcup_{k=1}^{\infty} H_k \). In a similar way, we bound the values \( 1 - p_n \) from below, and hence the \( p_n \) from above, on the intervals \( V_k \). \( \square \)

### 4. Periodicity and Continuity

#### 4.1. Main theorem

In this section we prove the convergence of the \( p_n \) on the log \( n \) scale to a periodic and continuous function \( f \). Together with Theorem 3.1 (non-convergence), this gives Theorem 1.1.

**Theorem 4.1** (Asymptotic periodicity and continuity). There exists a periodic and continuous function \( f: \mathbb{R} \to [0, 1] \) of period 1 such that

\[
\sup_{x \geq x_0} |p_{\exp x} - f(x)| \to 0 \quad \text{as } x_0 \to \infty.
\]

To prove Theorem 4.1, we consider coupled shooting processes started from different points that lie in one of the intervals

\[
J_0 = \left[ e^{k_0 + w - 3\delta}, e^{k_0 + w - 3\delta} + (e^{k_0 + w - 3\delta})^{2/3} \right],
\]

\[
J_k = \left[ e^{k_0 + w + k - \delta}, e^{k_0 + w + k + \delta} \right], \quad k \geq 1,
\]

for some \( k_0 \), \( w \) and \( \delta \) specified in Proposition 4.2 below. Observe that the intervals \( J_k \) for \( k \geq 1 \) have length \( 2\delta \) on the log \( n \) scale. We will show in three steps that with high probability, the distance between the numbers of survivors in these shooting processes decreases, and the coupled processes
collide before the number of survivors has reached 0 or 1. The three steps are respectively described by Propositions 4.2, 4.3 and 4.4 below.

**Proposition 4.2.** For all \( \varepsilon > 0 \) and \( a_2 \) there exist \( \delta \in (0, \frac{1}{2}) \) and \( k_0 > 1 + \log a_2 \) such that, with the intervals \( J_k \) as in (4.1) and (4.2),

\[
\inf_{w \in [0,1]} P\left( \text{for all } n \in \bigcup_{k=1}^{\infty} J_k, \ X_{kn}^n \in J_0 \right) \geq 1 - \varepsilon,
\]

where for all \( n \in \bigcup_{k=1}^{\infty} J_k \) and \( i \geq 0 \), \( X_i^n \) is defined by (2.9), with \( k_n \) equal to the value of \( k \) such that \( n \in J_k \).

Note that on the event considered in Proposition 4.2, the number of survivors \( X_{kn}^n \) after \( k_n \) shooting rounds for different starting points \( n \in \bigcup_{k=1}^{\infty} J_k \) are all in the same interval \( J \). By (2.9), from this moment onward the processes \( X_{kn+i}^n \) for different \( n \) will be coupled together. Our next two propositions explore what will happen when we are in a situation like this.

**Proposition 4.3.** For all \( n \geq 2 \) and \( i \geq 0 \), let the \( X_i^n \) be coupled as in (2.9), with \( k_n = 0 \) for all \( n \). Then for all \( \varepsilon > 0 \) there exist \( a_0 \) and \( d \) such that, for all \( a, b \) with \( a_0 \leq a < b \leq a + a^{2/3} \),

\[
P\left( -1 \leq X_i^b - X_i^a \leq d \right. \left. \text{ and } X_i^a \geq a^{0.01} \right) \geq 1 - \varepsilon.
\]

**Proposition 4.4.** For all \( n \geq 2 \) and \( i \geq 0 \), let the \( X_i^n \) be coupled as in (2.9), with \( k_n = 0 \) for all \( n \). Then for all \( \varepsilon > 0 \) and \( d \) there exists \( a_1 \) such that, for all \( a, b \) with \( a_1 \leq a < b \leq a + d \),

\[
P\left( X_i^a = X_i^b \right. \left. \text{ for some } i \right) \geq 1 - \varepsilon.
\]

We will now prove Theorem 4.1 using these three propositions, and defer the proofs of Propositions 4.2, 4.3 and 4.4 to Sections 4.2 and 4.3.

**Proof of Theorem 4.1.** We define, for all \( x \geq 0 \) and integer \( k \),

\[
f_k(x) := p_{\exp(k+x)}.
\]

First we will use Propositions 4.2, 4.3 and 4.4 to prove that for all \( \varepsilon > 0 \), there exist \( \delta > 0 \) and \( k_0 \) such that, for all \( u, v \in [0,1] \) with \( |u - v| \leq \delta \),

\[
|f_k(u) - f_l(v)| \leq \varepsilon \quad \text{for all } k, l \geq k_0.
\]

(4.3)

Let \( \varepsilon > 0 \). Choose \( a_0 \) and \( d \) according to Proposition 4.3 such that, for all \( a, b \) with \( a_0 \leq a < b \leq a + a^{2/3} \),

\[
P\left( |X_i^b - X_i^a| \leq d \right. \left. \text{ and } X_i^a \geq a^{0.01} - 1 \right. \left. \text{ for some } i \right) \geq 1 - \frac{\varepsilon}{3}.
\]

Next, choose \( a_1 \) according to Proposition 4.4 such that, for all \( a, b \) with \( a_1 \leq a < b \leq a + d \),

\[
P\left( X_i^a = X_i^b \right. \left. \text{ for some } i \right) \geq 1 - \frac{\varepsilon}{3}.
\]

(4.5)

Recall that in both (4.4) and (4.5), the shooting processes \( X_i^n \) are coupled from the first shooting round onward.

Finally, we define \( a_2 := \max\{a_0, (a_1 + 1)^{1000}\} \), and choose \( \delta \in (0, \frac{1}{2}) \) and \( k_0 > 1 + \log a_2 \) according to Proposition 4.2 such that

\[
\inf_{w \in [0,1]} P\left( \text{for all } n \in \bigcup_{k=k_0}^{\infty} J_k, \ X_{kn}^n \in J_0 \right) \geq 1 - \frac{\varepsilon}{3},
\]

(4.6)
where the $X_i^\alpha$ are coupled as in (2.9), with $k_n$ equal to the index of the interval $J_k$ containing $n$. We claim that (4.3) holds for these $\delta$ and $k_0$.

In order to prove this, let $u, v \in [0,1]$ be such that $|u - v| \leq \delta$ and let $k, l \geq k_0$. Write $w = (u + v)/2$ and set

$$\alpha := \lfloor \exp(k + u) \rfloor, \quad \beta := \lfloor \exp(l + v) \rfloor.$$ 

Note from (4.1) and (4.2) that $\alpha$ and $\beta$ are defined and coupled as described above. We need to show that $|p_\alpha - p_\beta| \leq \varepsilon$, but we will actually prove the stronger statement that

$$P(X_{i+1}^\alpha = X_{i+1}^\beta \text{ for some } i) \geq 1 - \varepsilon. \tag{4.7}$$

To prove (4.7), first note that by (4.6) and (4.1),

$$P(X_{i+1}^\alpha, X_{i+1}^\beta \in [e^{k_0+w-3\delta}, e^{k_0+w-3\delta} + (e^{k_0+w-3\delta})^{2/3}]) \geq 1 - \varepsilon/3. \tag{4.8}$$

Since $k_0 > 1 + \log a_2$ and $\delta < 1/3$, we have that $e^{k_0+w-3\delta} \geq a_2 \geq a_0$. Using the fact that $X_{i+1}^\alpha$ and $X_{i+1}^\beta$ are coupled together for all $i \geq 0$, and since $a_2^{0.01} \geq a_1 + 1$, it now follows from (4.8) and (4.4) that

$$P(\lfloor X_{i+1}^\alpha - X_{i+1}^\beta \rfloor \leq d \text{ and } X_{i+1}^\alpha, X_{i+1}^\beta \geq a_1 \text{ for some } i) \geq 1 - \frac{2\varepsilon}{3}. \tag{4.9}$$

By (4.9) and (4.5), we have that (4.7) holds. This proves (4.3).

Next we prove that (4.3) implies the theorem. Let $\varepsilon > 0$, and let $\delta > 0$ and $k_0$ be such that (4.3) holds for this $\varepsilon$. Fix $x \geq 0$. Taking $u = v = x - \lfloor x \rfloor$ in (4.3) and using $f_k(x) = f_k(\lfloor x \rfloor + u) = f_{k+\lfloor x \rfloor}(u)$, we get

$$|f_k(x) - f_l(x)| = |f_{k+\lfloor x \rfloor}(u) - f_{l+\lfloor x \rfloor}(u)| \leq \varepsilon, \quad \text{for all } k, l \geq k_0. \tag{4.10}$$

In particular, $\sup_{k \geq k_0} f_k(x) \leq \varepsilon + \inf_{k \geq k_0} f_k(x)$ and hence $\lim_{k \to \infty} f_k(x)$ exists. We define

$$f(x) := \lim_{k \to \infty} f_k(x), \quad x \geq 0.$$ 

Since $f_k(l+x) = f_{k+l}(x)$ for integer $k, l$, the limit function $f$ is periodic with period 1. Furthermore, since (4.10) holds uniformly for all $x \geq 0$, by taking $k = k_0$ and letting $l \to \infty$ we obtain

$$\varepsilon \geq \sup_{x \geq 0} |f_{k_0}(x) - f(x)| = \sup_{x \geq 0} |p_{\exp(k_0+x)} - f(k_0 + x)|,$$

which proves the desired uniform convergence to the limit function $f$. Finally, by (4.3) we obtain that, for all $u, v \in [0,1]$ with $|u - v| \leq \delta$,

$$|f(u) - f(v)| = \lim_{k \to \infty} |f_k(u) - f_k(v)| \leq \varepsilon,$$

which shows that $f$ is continuous. This completes the proof. \qed
4.2. One shooting round. In this section we give two key ingredients for the proof of Propositions 4.2, 4.3 and 4.4. These two ingredients give information about one shooting round of coupled shooting processes starting at two different points a and b. The first ingredient is the following lemma:

Lemma 4.5. Let \( S_n, n \geq 2 \), be coupled as in Section 2.1. For all \( a, b \) such that \( a < b \leq \frac{5}{4}a \),

\[
P(S_a = S_b) \geq e^{-(b-a)}.\]

Proof. Let \( Y_i^n, S_i^n \) and \( Z_i^n \) be coupled as in Section 2.1. By Lemma 2.1, we have that \( Y_a^a \leq S_a \leq Z_b^b \) and \( Y_a^a \leq S_b \leq Z_b^b \). Hence it suffices to show that

\[
P(Y_a^a = Z_b^b) \geq e^{-(b-a)}.\]

Note that the Markov chain \((Z_b^b)\), first takes \( b - a \) steps independently, before its steps are coupled to the Markov chain \((Y_a^a)\). To prove (4.11), we will first estimate the probability that the \( Z_b^b \) process decreases to the height \( a \) in these first \( b - a \) steps, and then estimate the probability that in the remaining \( a \) steps, the distance between \( Z_{b-a+i}^b \) and \( Y_i^a \) never increases.

For the first part, note that by Robbins’ version of Stirling’s formula [12],

\[
P(Z_{b-a}^b = a) = \frac{b - 1}{b - 1} - \frac{a}{b - 1} = \frac{(b - 1)!}{(a - 1)!} \frac{e^{1/(12b-12)}}{b - 1},
\]

which implies

\[
P(Z_{b-a}^b = a) \geq e^{-(b-a)}.\]

Next we consider the probability that in the remaining steps, the processes \( Y_i^a \) and \( Z_{b-a+i}^b \) stay at the same height. By the coupled transition probabilities (2.3) and (2.4), for all \( i = 0, \ldots, a - 1 \) and \( k < a \) we have

\[
P(Y_{i+1}^a = Z_{b-a+i+1}^b \mid Y_i^a = Z_{b-a+i}^b = k) = 1 - \frac{k}{a - 1} + \frac{k - 1}{b - 1} \geq 1 - \frac{2(b-a)}{a} - \frac{b-a}{a}.
\]

By our assumption that \( b \leq \frac{5}{4}a \) and the inequality \( 1 - u \geq e^{-2u} \), which holds for \( 0 \leq u \leq \frac{3}{4} \), this gives

\[
P(Y_{i+1}^a = Z_{b-a+i+1}^b \mid Y_i^a = Z_{b-a+i}^b = k) \geq e^{-6(b-a)/a}.
\]

A separate computation shows that this bound also holds for \( k = a \). Since this bound holds for each of the remaining \( a \) steps, we conclude that

\[
P(Y_a^a = Z_b^b \mid Z_{b-a}^b = a) \geq e^{-6(b-a)}.
\]

Together with (4.12), this gives (4.11), which completes the proof.

The second key ingredient is Lemma 4.7 below. For the proof, we need the following preliminary result:
Lemma 4.6. Let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1/\lambda_2$ is an integer. Let $T_x$, $x = 0, 1, \ldots, \lambda_1/\lambda_2$, be independent random variables such that $T_x$ has the exponential distribution with parameter $\lambda_1 - x\lambda_2$ (where $T_{\lambda_1/\lambda_2} = \infty$ with probability 1). Define

$$X_t = \min\{x : T_0 + T_1 + \cdots + T_x > t\}.$$

Then $X_t$ has the binomial distribution with parameters $n = \lambda_1/\lambda_2$ and $p = 1 - e^{-\lambda_2 t}$.

Proof. Let $n = \lambda_1/\lambda_2$. Consider $n$ independent Poisson processes, each with rate $\lambda_2$. Let $X'_i$ be the number of Poisson processes that have at least 1 jump before time $t$. Clearly, $X'_i$ has the binomial distribution with parameters $n = \lambda_1/\lambda_2$ and $p = 1 - e^{-\lambda_2 t}$. We will prove that $X'_i$ has the same law as $X_t$, which implies the statement of the lemma.

Let $T'_0$ be the waiting time until one of the $n$ Poisson processes has a jump. Then $T'_0$ has the exponential distribution with parameter $n\lambda_2 = \lambda_1$, hence $T'_0$ has the same law as $T_0$. Without loss of generality, suppose this jump occurs in Poisson process 1. Let $T'_1$ be the waiting time from time $T'_0$ until one of the Poisson processes 2 through $n$ has a jump. Then $T'_1$ has the exponential distribution with parameter $(n - 1)\lambda_2 = \lambda_1 - \lambda_2$, hence $T'_1$ has the same law as $T_1$. Moreover, $T'_0$ and $T'_1$ are independent. Continuing in this way, we construct independent random variables $T'_0, T'_1, \ldots, T'_{n-1}$ that have the same laws as $T_0, T_1, \ldots, T_{n-1}$. Finally, we define $T_n = \infty$. We then have that $X'_i = \min\{x : T'_0 + T'_1 + \cdots + T'_x > t\}$. It follows that $X'_i$ has the same law as $X_t$. \hfill \Box

Lemma 4.7. Let $S_n$, $n \geq 2$, be coupled as in Section 2.1. There exist $a_0, c_1, c_2 > 0$ such that, for all $a, b$ with $a_0 \leq a < b \leq a + a^{2/3}$,

$$P(S_b - S_a < \frac{1}{2}(b - a)) \geq 1 - c_1 e^{-c_2(b-a)}.$$

Proof. Let $Y^n_i, S^n_i$ and $Z^n_i$ be coupled as in Section 2.1. First we will show that there exists $c_3 > 0$ such that, for all $a, b$ sufficiently large and satisfying $a < b \leq a + a^{2/3},$

$$(4.14) \quad P(Z^n_{b-a} - a > 0.01(b-a)) \leq e^{-c_3(b-a)}.$$

Note that $Z^n_{b-a} - a$ is the number of times the $Z^n_b$ process does not decrease in the first $b-a$ steps. By (2.6), for $0 \leq i < b-a$, the conditional probability that $Z^n_b$ does not decrease in the $(i+1)$-th step satisfies

$$P(Z^n_{i+1} = Z^n_i \mid Z^n_i) \leq 1 - \frac{a + 1 - \frac{1}{b-1}}{a + 1 - \frac{1}{b-1}} \leq 1 - \frac{a}{b}.$$

Therefore, $Z^n_{b-a} - a$ is stochastically smaller than a random variable $W$ having the binomial distribution with parameters $n = b-a$ and $p = 1 - \frac{a}{b}$. If $a$ and $b$ are such that $1 - \frac{a}{b} < 0.01$, then we can use Hoeffding’s inequality to bound the left hand side of (1.14) by

$$P(W > 0.01(b-a)) \leq \exp[-2(b-a)(0.01-p)^2].$$

It follows that (4.14) holds.

Next we will show that there exist $c_4, c_5 > 0$ such that, for all $a, b$ sufficiently large and satisfying $a < b \leq a + a^{2/3},$

$$(4.15) \quad P(Z^n_b - Y^n_a > \frac{1}{2}(b-a) \mid Z^n_{b-a} - a \leq 0.01(b-a)) \leq c_4 e^{-c_5(b-a)}.$$
To prove (4.15), we consider the process of differences \(Z_{b-a+i}^b - Y_i^a\) at the steps \(i\) at which the \(Y\) process decreases, and bound the probability that at such steps the \(Z\) process does not decrease. Using the coupled transition probabilities (2.3) and (2.4), for \(k < a\) and \(l > 0\) we have

\[
P(Z_{b-a+i+1}^b = k + l \mid Y_{i+1}^a = k - 1, Y_i^a = k, Z_{b-a+i}^b = k + l) = \left[\left(\frac{k}{a-1} - \frac{k+l-1}{b-1}\right) a - 1\right]^+ \leq \left[1 - \frac{a - 1 + l - 1}{b - 1}\right]^+ \leq \left[\frac{b-a-(l-2)}{b}\right]^+.
\]

(4.16)

The upper bound (4.16) also holds for \(k = a\).

Next we define a pure birth process \(X_t, t = 0, 1, \ldots\) with the properties that (i) \(X_0 \geq \lfloor 0.01(b-a) \rfloor\), and (ii) when their heights are the same, the birth process \(X_t\) increases with a higher probability than the process of differences \(Z_{b-a+i}^b - Y_i^a\). To define the process \(X_t\), let

\[X_0 = x_0 := \max\{2, \lfloor 0.01(b-a) \rfloor\},\]

and let the dynamics of \(X_t\) be given by

\[P(X_{t+1} = X_t + 1 \mid X_t = x_t + x) = 1 - P(X_{t+1} = X_t \mid X_t = x_t + x) = \frac{b-a-x}{b}\]

for \(x = 0, 1, \ldots, b-a\). Since the two processes can be coupled in such a way that on the event \{\(Z_{b-a}^b - a \leq 0.01(b-a)\}\}, the birth process \(X_t\) dominates the process of differences, we have that

\[
P(Z_b^b - Y_a^a > \frac{1}{2}(b-a) \mid Z_{b-a}^b - a \leq 0.01(b-a)) \leq P(X_{t_0} > \frac{1}{2}(b-a)) + P(Y_a^a < \frac{a}{e} - (\frac{a}{e})^{5/6}),
\]

(4.17)

where

\[t_0 = \left[a - \left(\frac{a}{e} - \left(\frac{a}{e}\right)^{5/6}\right)\right].\]

To bound the first term on the right in (4.17), we use a continuous-time version of the process \(X_t\). Let \(T_x, x = 0, 1, \ldots, b-a\), be independent such that \(T_x\) has the geometric distribution with parameter \(p = (b-a-x)/b\) (where \(T_{b-a} = \infty\) with probability 1). We can then write

\[X_t = x_0 + \min\{x : T_0 + \cdots + T_x > t\},\]

Now let \(T'_x, x = 0, 1, \ldots, b-a\), be independent such that \(T'_x\) has the exponential distribution with parameter

\[\lambda = \log \frac{b-a-x}{b-a} \frac{\log b - \log a}{b-a}.
\]

We define the continuous-time process \(X'_t, t \geq 0\), by

\[X'_t = x_0 + \min\{x : T'_0 + \cdots + T'_x > t\}.
\]

Since

\[
P(T_x > t) = \left(1 - \frac{b-a-x}{b}\right)^{[t]} \geq e^{-t\left[-\log\left(1 - \frac{b-a-x}{b}\right)\right]} \geq e^{-t\left[\sum_{k=1}^{\infty} \frac{1}{b}(b-a)^k\right]} = e^{-t\left[\frac{b-a-x}{b-a} \log \frac{b}{a}\right]} = P(T'_x > t),
\]

we have for \(t \geq 0\),

\[P(T_x > t) \leq \frac{b-a-x}{b} \frac{\log b - \log a}{b-a} \frac{b}{a} = \frac{b-a-x}{b-a} \log \frac{b}{a}.
\]
we have that $T'_{x}$ is stochastically less than $T_{x}$. It follows that $X_{t}$ is stochastically dominated by $X'_{t}$, hence

$$P(X_{t_0} > \frac{1}{2}(b - a)) \leq P(X'_{t_0} > \frac{1}{2}(b - a)).$$

By Lemma 4.6, $X'_{t_0} - x_{0}$ has the same law as a random variable $W'$ having the binomial distribution with parameters

$$n' = b - a, \quad p' = 1 - \exp\left(-\frac{\log b - \log a}{b - a} t_0\right).$$

We have, as $a \to \infty$,

$$p' \leq 1 - \exp\left(-\frac{1}{a}\left[a - \frac{a}{e} + \frac{a^1}{e^5/6}\right]\right) \to 1 - e^{-(1-\varepsilon^{-1})} \approx 0.4685.$$

Therefore, if $0.01(b-a) \geq 2$ and $a$ is sufficiently large, then using Hoeffding’s inequality we can bound $P(X'_{t_0} > \frac{1}{2}(b - a))$ above by

$$P(W' > 0.49(b - a)) \leq \exp[-2(b-a)(0.49 - p')^2] \leq e^{-c_6(b - a)}$$

for some constant $c_6 > 0$ that does not depend on $a, b$. Hence

$$P(X'_{t_0} > \frac{1}{2}(b - a)) \leq e^{200c_6 e^{-c_6(b-a)}}$$

for all values of $b - a$ and sufficiently large $a$.

By Corollary 2.6 and the assumption that $b-a \leq a^{2/3}$, the second term on the right in (4.17) satisfies

$$P(Y_{a} \leq \frac{4}{e} - (\frac{a}{e})^{5/6}) \leq e^{-c_7 a^{2/3}} \leq e^{-c_7(b - a)},$$

for some constant $c_7 > 0$ that does not depend on $a, b$. Combining (4.17), (4.18), (4.19) and (4.20) gives (4.15). Since $Y_{a} \leq S_{a}$ and $S_{b} \leq Z_{b}^{0}$, (4.14) and (4.15) imply the statement of the lemma.

4.3. Proof of Propositions 4.2, 4.3 and 4.4

Proof of Proposition 4.2. The proposition is a corollary of Lemma 2.7 applied for a specific sequence of intervals, as we explain below. We define, for every $x \geq 0$, a sequence of intervals $I_{k}(x)$, $k \geq 0$, as follows. Let

$$\delta_{x} = \frac{1}{12} e^{-\frac{1}{2}x},$$

and let $I_{0}(x) = \left[[I_{0}^{-}(x)], [I_{0}^{+}(x)]\right]$ with

$$I_{0}^{-}(x) = e^{x-2\delta_{x}}, \quad I_{0}^{+}(x) = e^{x+2\delta_{x}}.$$

Let $s_{0}$ be defined by (2.19), let $\gamma = 1/4$ and let $c_{0} = c_{0}(x)$ be given by (2.20), i.e.,

$$c_{0}(x) = \frac{1}{4s_{0}} e^{\frac{1}{2}\gamma - \frac{1}{2}(e^{\delta_{x}} - e^{-\delta_{x}})}.$$

For $k \geq 1$, we define the interval $I_{k}(x)$ by (2.21) and (2.22), with $I_{0}(x)$ and $c_{0} = c_{0}(x)$ as above, i.e., $I_{k}(x) = \left[[I_{k}^{-}(x)], [I_{k}^{+}(x)]\right]$ with

$$I_{k}^{-}(x) = e^{x+k-2\delta_{x}} \left(1 + c_{0}(x) e^{\frac{1}{2}\gamma - \frac{1}{2}x + \delta_{x}} \sum_{i=1}^{k} i^{\frac{1}{2}} e^{-i/2}\right),$$

$$I_{k}^{+}(x) = e^{x+k+2\delta_{x}} \left(1 - c_{0}(x) e^{\frac{1}{2}\gamma - \frac{1}{2}x - \delta_{x}} \sum_{i=1}^{k} i^{\frac{1}{2}} e^{-i/2}\right).$$
Proof of Proposition 4.3. The idea is to repeatedly apply Lemma 4.7 to the coupled processes \( X^a \) and \( X^b \), with \( a < b \leq a + 2/3 \), until the distance \( X^b - X^a \) has decreased to a constant. To this end, however, \( X^a \) and \( X^b \) should satisfy the conditions of Lemma 4.7 at each round, and this requires that we first strengthen the statement of the lemma somewhat.

Let \( S_a, n \geq 2, \) be coupled as in Section 2.1. By Lemma 2.1 and Corollary 2.6 and since \( \frac{1}{11} < e^{-1} \), there exists \( q_0 > 0 \) such that for all \( a, \)

\[
P(S_a \geq 4a) \geq 1 - e^{-q_0 a}.
\]
Next, note that it is a deterministic fact that if \(S_b - S_a \leq \frac{1}{2}(b-a), \ b-a \leq a^{2/3}\) and \(S_a \geq \frac{4}{31}a\), then

\[
(4.25) \quad S_b - S_a \leq \frac{1}{2}a^{2/3} \leq \frac{1}{2}(\frac{11}{14}S_a)^{2/3} \leq S_a^{2/3}.
\]

By (4.24), (4.25) and Lemma 4.7, there exist \(a_0, c_1, c_2 > 0\) such that, for all \(a, b\) with \(a_0 \leq a < b \leq a + a^{2/3},\)

\[
(4.26) \quad P(S_b - S_a \leq \min\{\frac{1}{2}(b-a), S_a^{2/3}\}, S_a \geq \frac{4}{11}a) \geq 1 - c_1 e^{-c_2(b-a)}.
\]

The additional statements that \(S_b - S_a \leq S_a^{2/3}\) and \(S_a \geq \frac{4}{11}a\) in (4.26) make this version of the statement of Lemma 4.7 suitable for repeated application to the coupled processes \(X_i^a\) and \(X_i^b\).

Let \(\varepsilon > 0\). Define \(a_0 := (a_0^{(1)})_{100}\) and \(d\) such that \(\sum_{k=0}^{\infty} c_1 \exp(-c_2 k) \leq \varepsilon,\) let \(a, b\) be such that \(a_0 \leq a < b \leq a + a^{2/3}\), and let \(T_0 = \inf\{i: X_i^b - X_i^a \leq d\}.\)

We claim that

\[
(4.27) \quad P(-1 \leq X_i^b - X_i^a \leq d \text{ and } X_i^a \geq a^{0.01} \text{ for some } i) \\
\geq P(X_{i+1}^b - X_{i+1}^a \leq \frac{1}{2}(X_i^b - X_i^a), X_{i+1}^a \geq \frac{4}{31}X_i^a \text{ for all } i < T_0).
\]

Indeed, the following three facts together imply (4.27):

1. If \(X_{i+1}^b - X_{i+1}^a \leq \frac{1}{2}(X_i^b - X_i^a)\) for every \(i < 0.97 \log a\), then \(X_i^b - X_i^a \leq d\) for some \(i \leq 0.97 \log a\) (and hence \(T_0 \leq 0.97 \log a\)), since

\[
(b-a)\left(\frac{1}{2}\right)^{0.97 \log a} \leq 2a^{2/3} \left(\frac{1}{2}\right)^{0.97 \log a} \leq 2 \leq d.
\]

2. If \(X_{i+1}^a \geq \frac{4}{31}X_i^a\) in every round \(i < 0.97 \log a\), then \(X_i^a \geq a^{0.01}\) for all \(i \leq 0.97 \log a\), since

\[
a\left(\frac{4}{31}\right)^{0.97 \log a} \geq a^{0.01}.
\]

3. If \(X_i^b - X_i^a > 0\), then \(X_{i+1}^b - X_{i+1}^a \geq -1\) a.s. by (2.9) and Lemma 2.1.

The right hand side of (4.27) is at least

\[
(4.28) \quad P(X_{i+1}^b - X_{i+1}^a \leq \min\{\frac{1}{2}(X_{i+1}^b - X_{i+1}^a), X_{i+1}^a\}^{2/3}) \\
\geq 1 - \sum_{k=d}^{\infty} c_1 e^{-c_2 k} \geq 1 - \varepsilon,
\]

where the first bound on the probability follows from repeated application of (4.26). Note that on the event considered in (4.28), we have that \(X_i^a \geq a^{0.01} \geq a_0^{0.01} \geq a_0^*\) for all \(i < T_0\), so that we can indeed apply (4.26). The sum in (4.28) is over all possible values that the distance \(X_i^b - X_i^a\) can assume, and all larger values. The second inequality in (4.28) follows from the definition of \(d\). Combining (4.27) and (4.28) yields the desired result. \(\square\)

**Proof of Proposition 4.4.** Let the \(S_n, n \geq 2,\) be coupled as in Section 2.1.

By Lemma 2.1 and Corollary 2.6, and since \(\frac{1}{11} < e^{-1}\), there exists \(c_0 > 0\) such that for all \(a,\)

\[
(4.29) \quad P(S_a \geq \frac{4}{11}a) \geq 1 - e^{-c_0 a}.
\]

By Lemma 4.7, there exist \(a_0, c_1, c_2 > 0\) such that for all \(a, b\) satisfying \(a_0 \leq a < b \leq a + a^{2/3},\)

\[
(4.30) \quad P(S_b - S_a \leq \frac{1}{2}(b-a)) \geq 1 - c_1 e^{-c_2(b-a)}.
\]
Let $\varepsilon > 0$ and $d$ be given. Choose $d_0 \geq d$ such that
\begin{equation}
 e^{-d_0} \leq \frac{\varepsilon}{3} \quad \text{and} \quad d_0 e^{d_0} e^{14d_0 - c_2d_0^2} \leq \frac{\varepsilon}{3},
\end{equation}
and define $T := d_0 \lceil \exp(14d_0) \rceil$. Now choose $a_1^*$ such that
\begin{equation}
 2Te^{-c_0a_1^*} \leq \frac{\varepsilon}{3} \quad \text{and} \quad a_1^* \geq \max\{a_0, (2d_0)^{3/2}, 8d_0\},
\end{equation}
and set $a_1 := a_1^*(11/4)^T$. Let $a, b$ be such that $a_1 \leq a < b \leq a + d$, and consider the coupled processes $X_i^a$ and $X_i^b$. Define the events
\begin{align*}
 A_k &:= \{X_i^a, X_i^b \geq a_1^* \text{ for all } i \leq k\}, \\
 B_k &:= \{|X_i^b - X_i^a| \leq 2d_0 \text{ for all } i \leq k\}, \\
 C_k &:= \{|X_i^b - X_i^a| \neq 0 \text{ for all } i \leq k\}.
\end{align*}

Our goal is to show that $C_T$ has small probability, which implies that with high probability, $|X_i^b - X_i^a| = 0$ for some $i$. Since
\begin{equation}
 P(C_T) \leq P(A_T^c) + P(A_T \cap B_T^c) + P(A_T \cap B_T \cap C_T),
\end{equation}
it suffices to prove that $P(A_T^c)$, $P(A_T \cap B_T^c)$ and $P(A_T \cap B_T \cap C_T)$ are small.

We start with $P(A_T^c)$. Observe that it is a deterministic fact that if $a \geq a_1$ and $X_i^a \geq \frac{a}{T} X_i^{a-1}$ for all $i \leq T$, then by definition of $a_1$, $X_i^a \geq a_1^*$ for all $i \leq T$. Hence, by (4.29) and (4.32),
\begin{equation}
 P(A_T^c) \leq 2Te^{-c_0a_1^*} \leq \frac{\varepsilon}{3}.
\end{equation}

Next, we turn to $P(A_T \cap B_T^c)$. Observe that we can consider the absolute differences $|X_i^b - X_i^a|$, $i = 0, 1, \ldots$, as a random walk starting at $b - a \leq d_0$. By the definition (2.9) of the $X_i^a$ and Lemma 2.1,
\begin{equation*}
 |X_{i+1}^b - X_{i+1}^a| \leq |X_i^b - X_i^a| + 1 \text{ a.s. for all } i \geq 0.
\end{equation*}
This implies that it can only be the case that $|X_i^b - X_i^a| > 2d_0$ for some $i \leq T$, if there exists a $k < T - d_0$ such that $d_0 \leq |X_{k+j}^b - X_{k+j}^a| \leq d_0 + j$ holds for $j = 0, 1, 2, \ldots, d_0$. Hence, if we introduce the notation
\begin{equation}
 D_{k,i} := A_{k+i} \cap \{d_0 \leq |X_{k+j}^b - X_{k+j}^a| \leq d_0 + j \text{ for } j = 0, 1, \ldots, i\},
\end{equation}
then we have that $P(A_T \cap B_T^c) \leq \sum_{k<d_0} P(D_{k,d_0})$, which is the same as
\begin{equation}
 P(A_T \cap B_T) \leq \sum_{k=0}^{T-d_0-1} \prod_{i=1}^{d_0} P(D_{k,i} \mid D_{k,i-1}) \cdot P(D_{k,0}).
\end{equation}

Now, by (4.32), if $X_i^a, X_i^b \geq a_1^*$ and $|X_i^b - X_i^a| \leq 2d_0$, then it also holds that $|X_i^b - X_i^a| \leq \min\{X_i^a, X_i^b\}/2$. Moreover, if the absolute difference $|X_i^b - X_i^a|$ is strictly less than $2d_0$, it will drop below $d_0$ if it decreases by at least $\frac{1}{2} |X_i^b - X_i^a|$ at the next step. Therefore, it follows from (4.30) that
\begin{equation}
 P(D_{k,i} \mid D_{k,i-1}) \leq P(|X_{k+i}^b - X_{k+i}^a| \geq d_0 \mid D_{k,i-1}) \leq c_1 e^{-c_2d_0}
\end{equation}
for $i \leq d_0$. Together, (4.35), (4.36) and (4.31) give
\begin{equation}
 P(A_T \cap B_T^c) \leq (T - d_0)(c_1 e^{-c_2d_0})^{d_0} \leq d_0 c_1 d_0 e^{14d_0 - c_2d_0^2} \leq \frac{\varepsilon}{3}.
\end{equation}
Finally, we consider \( P(\{X_b^T - X_a^T\mid E_i\} = 0) \). To simplify the notation, write \( E_i = A_i \cap B_i \cap C_i \) for \( i \geq 0 \). Then we have

\[
P(E_T) = \prod_{i=1}^{T} P(E_i \mid E_{i-1}) P(E_0) \leq \prod_{i=1}^{T} P(C_i \mid E_{i-1}).
\]

Since on the event \( E_i \), \( |X_b^i - X_a^i| \leq 2d_0 \) and \( 2d_0 \leq \frac{1}{2} d_i^* \leq \frac{1}{2} \min\{X_b^i, X_a^i\} \), by Lemma 4.5 each factor in this product is bounded above by \( 1 - e^{-u} \) and (4.31).

(4.38) \( P(\{X_b^T - X_a^T\mid E_i\} = 0 \text{ for some } i \leq T) = 1 - P(C_T) \geq 1 - \varepsilon. \)

Combining (4.33), (4.34), (4.37), and (4.38) gives

\[
P(\{X_b^T - X_a^T\mid E_i\} = 0 \text{ for some } i \leq T) = 1 - P(C_T) \geq 1 - \varepsilon. \]

\[\square\]

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References

[1] N. Alon and J. H. Spencer, The probabilistic method, Third, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2008. With an appendix on the life and work of Paul Erdős. MR2437651 (2009j:60004)

[2] J. S. Athreya and L. M. Fidkowski, Number theory, balls in boxes, and the asymptotic uniqueness of maximal discrete order statistics, Integers (2000), A3, 5. MR1759421 (2002f:60104)

[3] J. J. A. M. Brands, F. W. Steutel, and R. J. G. Wilms, On the number of maxima in a discrete sample, Statist. Probab. Lett. 20 (1994), no. 3, 209–217. MR1294106 (95e:60010)

[4] F. T. Bruss and C. A. O’Cinneide, On the maximum and its uniqueness for geometric random samples, J. Appl. Probab. 27 (1990), no. 3, 598–610. MR1067025 (92a:60096)

[5] H. Chernoff, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, Ann. Math. Statistics 23 (1952), 493–507. MR0057518 (15,241c)

[6] B. Eisenberg, G. Stengle, and G. Strang, The asymptotic probability of a tie for first place, Ann. Appl. Probab. 3 (1993), no. 3, 731–745. MR1233622 (95d:60044)

[7] S. Janson, Large deviation inequalities for sums of indicator variables, arXiv:1609.00533 [math.PR] (2016).

[8] J. Kinney, Tossing Coins Until All Are Heads, Math. Mag. 51 (1978), no. 3, 184–186. MR1572269

[9] S. Li and C. Pomerance, Primitive roots: a survey, Sūrikaisekikenkyūsho Kōkyūroku 1274 (2002), 77–87. New aspects of analytic number theory (Japanese) (Kyoto, 2001). MR1948333

[10] H. Prodinger, How to select a loser, Discrete Math. 120 (1993), no. 1-3, 149–159. MR1235902 (94g:05010)

[11] L. Rade, P. Griffin, and O. P. Lossers, Problems and Solutions: Solutions: E3436, Amer. Math. Monthly 101 (1994), no. 1, 78–80. MR1542469

[12] H. Robbins, A remark on Stirling’s formula, Amer. Math. Monthly 62 (1955), 26–29. MR0069528 (16,102a)

[13] V. A. Vatutin and V. G. Mikhaylov, Limit theorems for the number of empty cells in an equiprobable scheme for the distribution of particles by groups, Teor. Veroyatnost. i Primenen. 27 (1982), no. 4, 684–692. MR681461 (84c:60020)

[14] P. Winkler, Mathematical puzzles: a connoisseur’s collection, A K Peters, Ltd., Natick, MA, 2004. MR2034896 (2006c:00002)
