A SKOROKHOD CRITERION FOR THE EXISTENCE OF SEMIMARTINGALES

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Abstract. We prove the existence of quasi-left continuous semimartingales with continuous local semimartingale characteristics which satisfy a Lyapunov-type or a linear growth condition, where latter takes the whole history of the paths into consideration. The proof is based on an approximation and a tightness argument and the martingale problem method.

1. Introduction

Existence theorems for solutions to stochastic equations are of fundamental interest in many areas of probability theory. In the context of weak solutions to stochastic differential equations (SDEs) important contributions were made by Skorokhod and by Stroock and Varadhan. Skorokhod (see [23]) showed that SDEs with continuous coefficients of linear growth have weak solutions. Stroock and Varadhan (see [24]) introduced the concept of the martingale problem, which is nowadays one of the most important tools for studying existence, uniqueness and limit theorems for stochastic processes. In many of the classical monographs on stochastic analysis (e.g., [17, 22]) Skorokhod’s existence result is proven by the martingale problem argument of Stroock and Varadhan. The main idea is to construct an approximation sequence of probability measures on a path space, to show its tightness and finally to use the martingale problem method to verify that any of its accumulation points is the law of a weak solution.

If one considers SDEs with Wiener noise and coefficients of linear growth, the tightness can be verified via Kolmogorov’s tightness criterion. Gatarek and Goldys [6] propose a more direct argument for tightness based on the compactness of a fractional operator and the factorization method of Da Prato, Kwapien and Zabczyk [4]. This method was used by Hofmanová and Seidler [8] to replace the linear growth assumption in Skorokhod’s criterion by a Lyapunov-type condition.

Skorokhod’s original theorem is not restricted to path continuous settings. For general semimartingales Jacod and Memin [11] proved conditions for tightness in terms of the so-called semimartingale characteristics. These criteria were used by Jacod and Memin [12] to prove continuity and uniform boundedness criteria for the existence of weak solutions to SDEs driven by general semimartingales.

Refinements of the tightness criteria from [11] are proved in the monograph [13] of Jacod and Shiryaev. The conditions are applied to prove a Skorokhod-type existence result for semimartingales. More precisely, Jacod and Shiryaev consider a candidate for semimartingale characteristics on the Skorokhod space and formulate continuity and uniform boundedness conditions which imply the existence of a probability measure for which the coordinate process is a semimartingale with the candidate as semimartingale characteristics.

In this article we generalize the existence result of Jacod and Shiryaev for the quasi-left continuous case by replacing the uniform boundedness assumptions by local boundedness assumptions together with a Lyapunov-type or a linear growth condition. The linear growth condition takes the whole history of the paths into consideration. We prove the result as follows: First, we construct an approximation sequence with the help of the existence result of Jacod and Shiryaev. Second, we show tightness by a localization of a criterion from [13] together...
with a Lyapunov-type or a Gronwall-type argument. In this step we also adapt arguments used by Liptser and Shiryaev \cite{22}. Finally, we use arguments based on the martingale problem for semimartingales to verify that any accumulation point of our approximation sequence is the law of a semimartingale with the correct semimartingale characteristics.

Let us shortly comment on continual problems. The weak convergence argument heavily relies on Lévy’s continuity theorem, which is applicable when the coefficients have a continuity property. It is only natural to ask what can be said for discontinuous coefficients. We do not touch this topic in this article and refer the curious reader to the recent articles \cite{8,18} where interesting progress in this direction is made.

The article is structured as follows. In Section 2.1 we explain the mathematical setting of the article. In Section 2.2 we state our main result. In particular, we discuss its assumptions. Finally, we comment on the method based on the extension of local solutions and on a possible expansion of our result via Girsanov-type arguments. In Section 2.3 we apply our result in a jump-diffusion setting. The proof of our main result is given in Section 3.

The topic of this article is of course very classical and the basic definitions can be found in many textbooks. Our main reference is the monograph of Jacod and Shiryaev \cite{13}. As far as possible we will refer to results in this monograph. Furthermore, all non-explained terminology can also be found there.

2. Formulation of the Main Results

2.1. The Mathematical Setting. Let \( \Omega \) be the Skorokhod space of càdlàg functions \( \mathbb{R}_+ \rightarrow \mathbb{R}^d \) equipped with the Skorokhod topology (see \cite{13} for details). We denote the coordinate process on \( \Omega \) by \( X \), i.e. \( X_t(\omega) = \omega(t) \) for \( t \in \mathbb{R}_+ \) and \( \omega \in \Omega \). Let \( F \triangleq \sigma(X_t, t \in \mathbb{R}_+) \) and \( F_t \triangleq \bigcap_{s \geq t} F_s \), where \( F_s \triangleq \sigma(X_t, t \in [0, s]) \). Except stated otherwise, when we use terms such as adapted, predictable, etc. we refer to the right-continuous filtration \((F_t)_{t \geq 0}\).

Throughout the article we fix a continuous truncation function \( h : \mathbb{R}^d \rightarrow \mathbb{R}^d \), i.e. a bounded continuous function which equals the identity around the origin.

A càdlàg \( \mathbb{R}^d \)-valued adapted process \( Y \) is a semimartingale if it admits a decomposition \( Y = Y_0 + M + V \), where \( M \) is a càdlàg local martingale and \( V \) is a càdlàg adapted process of finite variation. To a semimartingale \( Y \) we associate a quadruple \((b,c,K;A)\) consisting of a \( \mathbb{R}^d \)-valued predictable process \( b \), a predictable process \( c \) taking values in the set \( \mathbb{S}_d \) of symmetric non-negative definite \( d \times d \) matrices, a predictable kernel \( K \) from \( \Omega \times \mathbb{R}_+ \) into \( \mathbb{R}^d \) and a predictable increasing càdlàg process \( A \), see \cite{13} Definition II.2.6, Proposition II.2.9, II.2.12 – II.2.14) for precise definitions and properties. Providing an intuition, \( b \) represents the drift and depends on the truncation function \( h \), \( c \) encodes the continuous local martingale component and \( K \) reflects the jump structure. The quadruple \((b,c,K;A)\) is called the local characteristics of \( Y \). In addition, for \( i, j = 1, \ldots, d \) we define by

\[
\tilde{c}^{ij} \triangleq c^{ij} + \int h^i(x)h^j(x)K(dx) - \Delta A \int h^i(x)K(dx) \int h^j(x)K(dx)
\]

a modified second characteristic, see \cite{13} Proposition II.2.17].

Let us shortly comment on the role played by the initial law. For SDEs with Wiener noise it was proven by Kaltenberg \cite{14} that weak solutions exist for all initial laws if, and only if, weak solutions exist for all degenerated initial laws. Although the result is fairly old, it seems not to be commonly known. We now state a version for a general semimartingale setting. The proof is similar as in the diffusion case and can be found in Appendix A

**Proposition 1.** Assume that for all \( z \in \mathbb{R}^d \) there exists a probability measure \( P_z \) on \((\Omega,F)\) such that the coordinate process is a \( P_z \)-semimartingale with local characteristics \((b,c,K;A)\) and initial law \( \delta_z \). Then, for any Borel probability measure \( \eta \) on \( \mathbb{R}^d \) there exists a probability measure \( P_\eta \) on \((\Omega,F)\) such that the coordinate process is a \( P_\eta \)-semimartingale with local characteristics \((b,c,K;A)\) and initial law \( \eta \).

From now on we fix a (deterministic) continuous increasing function \( A : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( A_0 = 0 \) and a Borel probability measure \( \eta \) on \( \mathbb{R}^d \). Next, we define a so-called candidate triplet \((b,c,K)\) on \((\Omega,F)\). Let us shortly clarify some notations: For \( x, y \in \mathbb{R}^d \) we write \( \|x\| \) for the
Euclidean norm, $\langle x, y \rangle$ for the Euclidean scalar product, and for $M \in \mathbb{S}^d$ we write $\|M\| \triangleq \text{trace } M$.

**Definition 1.** We call $(b, c, K)$ a candidate triplet, if it consists of the following:

(i) A predictable $\mathbb{R}^d$-valued process $b$ such that $\int_0^t \|b_s(\omega)\|dA_s < \infty$ for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$.

(ii) A predictable $\mathbb{S}^d$-valued process $c$ such that $\int_0^t \|c_s(\omega)\|dA_s < \infty$ for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$.

(iii) A predictable kernel $(\omega, s) \mapsto K_s(\omega; dx)$ from $\Omega \times \mathbb{R}_+$ into $\mathbb{R}^d$ such that for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$ we have $K_t(\omega; \{0\}) = 0$ and $\int_0^t \|x\|^2 K_s(\omega; dx)dA_s < \infty$.

In the following we fix also a candidate triplet $(b, c, K)$. The goal is to find a probability measure $P$ on $(\Omega, F)$ such that the coordinate process $X$ is a $P$-semimartingale with local characteristics $(b, c, K)$ and initial law $\eta$.

### 2.2. Existence Conditions for Semimartingales

Let $C_2(\mathbb{R}^d)$ be the set of all continuous bounded function $\mathbb{R}^d \to \mathbb{R}$ which vanish around zero. Moreover, let $C_1(\mathbb{R}^d)$ be a subclass of the non-negative functions in $C_2(\mathbb{R}^d)$ which contains all functions $g(x) = (a\|x\| - 1)^+ \wedge 1$ for $a \in \mathbb{Q}$ and is convergence determining for the weak convergence induced by $C_2(\mathbb{R}^d)$ (see [13, p. 395] for more details).

For a twice continuously differentiable function $f: \mathbb{R}^d \to \mathbb{R}$ and $a \in \mathbb{R}_+$ we set

$$\tilde{c}^{ij,a}_t \triangleq c^{ij}_t + \int_{\|x\| \leq a} x^i x^j K(dx), \quad b^a_t \triangleq b_t - \int (h(x) - x1\{\|x\| \leq a\}) K(dx),$$

$$(\mathcal{L}_a f)(\omega; t) \triangleq \sum_{k=1}^d \partial_k f(\omega(t^-))b^{k,a}_t(\omega) + \frac{1}{2} \sum_{k,j=1}^d \partial^2_{kj} f(\omega(t^-))c^{kj}_t(\omega)$$

$$+ \int_{\|x\| \leq a} \left( f(\omega(t^-) + x) - f(\omega(t^-)) - \sum_{k=1}^d \partial_k f(\omega(t^-)) x^k \right) K_t(\omega; dx),$$

and

$$(\mathcal{L} f)(\omega; t) \triangleq \sum_{k=1}^d \partial_k f(\omega(t^-))b^k_t(\omega) + \frac{1}{2} \sum_{k,j=1}^d \partial^2_{kj} f(\omega(t^-))c^{kj}_t(\omega)$$

$$+ \int \left( f(\omega(t^-) + x) - f(\omega(t^-)) - \sum_{k=1}^d \partial_k f(\omega(t^-)) h^k(x) \right) K_t(\omega; dx).$$

For $m > 0$ we define

$$\Theta_m \triangleq \left\{ (t, \omega) \in [0, m] \times \Omega : \sup_{s \in [0,t]} \|\omega(s^-)\| \leq m \right\}.$$

The first main result of this article is the following:

**Theorem 1.** Suppose the following:

(i) Local majorization property of $(b, c, K)$: For all $m > 0$ it holds that

$$\sup_{(t, \omega) \in \Theta_m} \left( \|b_t(\omega)\| + \|c_t(\omega)\| + \int (1 \wedge \|x\|^2) K_t(\omega; dx) \right) < \infty.$$

(ii) Skorokhod continuity property of $(b, c, K)$: For all $\alpha \in \Omega$ each of the maps

$$\omega \mapsto b_t(\omega), \tilde{c}_t(\omega), \int g(x) K_t(\omega; dx), \quad g \in C_1(\mathbb{R}^d),$$

is continuous at $\alpha$ for $dA_t$-a.a. $t \in \mathbb{R}_+$.

(iii) Local uniform continuity property of $(b, c, K)$: For all $t \in \mathbb{R}_+, g \in C_1(\mathbb{R}^d), i, j = 1, \ldots, d$ and all compact sets $K \subset \Omega$ each $k \in \{ \omega \mapsto b^i_t(\omega), \tilde{c}^i_t(\omega), \int g(x) K_t(\omega; dx) \}$ has the following uniform continuity property: For all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $\omega, \alpha \in K$

$$\sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta \implies |k(\omega) - k(\alpha)| < \varepsilon.$$
Moreover, assume that one of the following conditions holds:

(vi) Big jump property of $K$: For all $m > 0$ we have
\[ \lim_{a \to \infty} \sup_{t \in [0, m]} \sup_{\omega \in \Omega} K_s(\omega; \{x \in \mathbb{R}^d : \|x\| > a\}) = 0. \]

Furthermore, assume that one of the following conditions holds:

(v) Lyapunov condition: There exists a $\theta \in \mathbb{R}_+$ such that for all $a \in (\theta, \infty)$ there exist Borel functions $V_a : \mathbb{R}^d \to (0, \infty)$, $\gamma_a : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta_a : \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties:

(a) $V_a \in C^2(\mathbb{R}^d)$.

(b) $\int_0^t \gamma_a(s)dA_s < \infty$ for all $t \in \mathbb{R}_+$.

(c) $\beta_a$ is increasing and $\lim_{a \to \infty} \beta_a(n) = \infty$.

(d) For all $\omega \in \Omega$ we have $V_a(\omega(t)) \geq \beta_a(\|\omega(t)\|)$ and
\[ \gamma_a(t)V_a(\omega(t)) - (L_aV)(\omega; t) \geq 0 \]
for $dA_t$-a.a. $t \in \mathbb{R}_+$.

(v)' Linear growth condition: There exists a $\theta \in \mathbb{R}_+$ such that for all $a \in (\theta, \infty)$ there exists a Borel function $\gamma_a : \mathbb{R}_+ \to \mathbb{R}_+$ such that we have $\int_0^t \gamma_a(s)dA_s < \infty$ for all $t \in \mathbb{R}_+$ and for all $\omega \in \Omega$ and for $dA_t$-a.a. $t \in \mathbb{R}_+$
\[ (2.1) \quad \|b^t_a(\cdot)\|^2 + \|\bar{c}^t_a(\cdot)\|^2 \leq \gamma_a(t) \left( 1 + \sup_{s \in [0, t]} \|\omega(s)\|^2 \right). \]

Then, there exists a probability measure $P$ on $(\Omega, \mathcal{F})$ such that $X$ is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$.

The theorem can be viewed as a generalization of [13, Theorem IX.2.31], which replaces the uniform boundedness assumptions by local boundedness assumptions and a Lyapunov-type condition or a linear growth condition. We stress that $A$ needs not to be continuous in [13, Theorem IX.2.31]. The continuity of $A$ implies that any semimartingale with local characteristics $(b, c, K; A)$ is quasi-left continuous, see [13, Proposition II.2.9]. Theorem 1 is proven in Section 3 below.

We need the big jump condition on $K$ to obtain the existence of our approximation sequence and to show its tightness. In fact, [13, Theorem VI.4.18] explains that a (weaker) condition of this type is necessary for tightness of our approximation sequence. The big jump condition on $K$ can be replaced by a local big jump condition when the big jumps are also taken into consideration in the Lyapunov and the linear growth condition. We state this modification as a second main result:

**Theorem 2.** Suppose that (i) – (iii) from Theorem 1 hold and that the following local big jump condition holds: For all $m > 0$
\[ \lim_{a \to \infty} \sup_{(s, \omega) \in \Theta_m} K_s(\omega; \{x \in \mathbb{R}^d : \|x\| > a\}) = 0. \]

Moreover, assume that one of the following conditions holds:

(vi) Lyapunov condition: There exist Borel functions $V : \mathbb{R}^d \to (0, \infty)$, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties:

(a) $V \in C^2(\mathbb{R}^d)$.

(b) $\int_0^t \gamma(s)dA_s < \infty$ for all $t \in \mathbb{R}_+$.

(c) $\beta$ is increasing and $\lim_{n \to \infty} \beta(n) = \infty$.

(d) For all $\omega \in \Omega$ we have $V(\omega(t)) \geq \beta(\|\omega(t)\|)$ and
\[ \gamma(t)V(\omega(t)) - (LV)(\omega; t) \geq 0 \]
for $dA_t$-a.a. $t \in \mathbb{R}_+$. 

\[ (2.1) \quad \|b^t_a(\cdot)\|^2 + \|\bar{c}^t_a(\cdot)\|^2 \leq \gamma_a(t) \left( 1 + \sup_{s \in [0, t]} \|\omega(s)\|^2 \right). \]
(vi)' Linear growth condition: There exists a Borel function \( \gamma: \mathbb{R}_+ \to \mathbb{R}_+ \) such that we have
\[
\int_0^t \gamma(s) dA_s < \infty \text{ for all } t \in \mathbb{R}_+ \text{ and } \Omega \text{ and } \text{for } dA_t \text{-a.a. } t \in \mathbb{R}_+. 
\]
\[
\|b_t(\omega)\|^2 + \|\tilde{c}_t(\omega)\| + \int_{0}^{t} \|h'(x)\|^2 K_t(dx) \leq \gamma(t) \left( 1 + \sup_{s \in [0,t]} \|\omega(s)\|^2 \right)^{\frac{1}{2}},
\]
(2.2)

where \( h'(x) \triangleq x - h(x) \).

Then, there exists a probability measure \( P \) on \( (\Omega, \mathcal{F}) \) such that \( X \) is a \( P \)-semimartingale with local characteristics \((b, c, K; \Lambda)\) and initial law \( \eta \).

Theorem 2 is also proven in Section 3 below.

**Remark 1.** In the statement of the Theorems 1 and 2 the set \( \Theta_m \) can be replaced by
\[
\Theta_m^* \triangleq \left\{ (t, \omega) \in [0, m] \times \Omega : \sup_{s \in [0,t]} \|\omega(s)\| \leq m \right\} \subset \Theta_m.
\]

Furthermore, in (2.1) and (2.2) one can replaced \( \sup_{s \in [0,t]} \|\omega(s)\| \) by \( \sup_{s \in [0,t]} \|\omega(s)\| \). This follows from part (d) of [13, Lemma III.2.43], which states that for a predictable process \( H \) and all \( t > 0 \) and \( \omega, \alpha \in \Omega \)
\[
\omega(s) = \alpha(s) \text{ for all } s < t \Rightarrow H_t(\omega) = H_t(\alpha).
\]

Due to this observation, we expect assumption (i) in Theorem 1 to be close to optimal for a local boundedness condition. We give some examples for functions satisfying the Skorokhod continuity property and the local uniform continuity property:

**Example 1.** Let \( g: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) be a Borel function such that \( x \mapsto g(t, x) \) is continuous for all \( t \in \mathbb{R}_+ \). Furthermore, fix \( t > 0 \).

(a) The map \( \omega \mapsto g(t, \omega(t-)) \) is continuous at each \( \alpha \in \Omega \) such that \( t \not\in J(\alpha) \triangleq \{ s > 0 : \alpha(s) \neq \alpha(s-), \} \), see [13, VI.2.3]. Since any càdlàg function has at most countably many discontinuities, the set \( J(\alpha) \) is a \( dA_t \)-null set and the Skorokhod continuity property holds. Furthermore, the local uniform continuity property holds. To see this, note that for each compact set \( K \subset \Omega \) there exists a compact set \( K_t \subset \mathbb{R}^d \) such that \( \omega(s) \in K_t \) for all \( \omega \in K \) and \( s \in [0, t] \), see [2, Problem 16, p. 152]. Using that continuous functions on compact sets are uniformly continuous, for each \( \epsilon > 0 \) there exists a \( \delta = \delta(\epsilon) > 0 \) such that
\[
x, y \in K_t : \|x - y\| < \delta \Rightarrow |g(t, x) - g(t, y)| < \epsilon.
\]

Now, if \( \omega, \alpha \in K \) are such that \( \sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta \) we have \( \omega(t) = \alpha(t) \in K_t \), because \( K_t \) is closed, and \( |\omega(t) - \alpha(t)| = \|\omega(s) - \alpha(s)\| < \delta \). Consequently, we have
\[
|g(t, \omega(t)) - g(t, \alpha(t))| < \epsilon.
\]

This shows that the local uniform continuity property holds.

(b) If \( g \) is continuous, the map \( \omega \mapsto \int_0^t g(s, \omega(s-)) dA_s \) is continuous. This follows from the fact that \( \omega \mapsto g(s, \omega(s-)) \) is continuous at each \( \alpha \in \Omega \) such that \( s \not\in J(\alpha) \), the dominated convergence theorem and the fact that \( J(\alpha) \) is a \( dA_t \)-null set. Furthermore, the map \( \omega \mapsto \int_0^t g(s, \omega(s-)) dA_s \) also has the local uniform continuity property. To see this, let \( K \subset \Omega \) and \( K_t \subset \mathbb{R}^d \) be as in part (a) and fix \( \epsilon > 0 \). Without loss of generality we assume that \( A_t > 0 \). Because \( g \) is uniformly continuous on \([0, t] \times K_t \) we find a \( \delta = \delta(\epsilon) > 0 \) such that
\[
x, y \in K_t : \|x - y\| < \delta \Rightarrow |g(s, x) - g(s, y)| < \frac{\epsilon}{2A_t}.
\]
for all $s \in [0, t]$. Now, for all $\omega, \alpha \in K$ such that $\sup_{s \in [0, t]} \|\omega(s) - \alpha(s)\| < \delta$ we have

$$
\left| \int_0^t g(s, \omega(s))dA_s - \int_0^t g(r, \alpha(r))dA_r \right|
\leq \int_0^t \left| g(s, \omega(s)) - g(s, \alpha(s)) \right| dA_s < \varepsilon,
$$

which gives the local uniform continuity property.

(c) If $g$ is continuous, the map $\omega \mapsto \sup_{s \in [0, t]} g(s, \omega(s))$ is continuous at each $\alpha \in \Omega$ such that $t \not\in J(\alpha)$. This can be seen with the arguments used in the proof of Lemma A below. Furthermore, the local uniform continuity property holds, which follows with the argument from part (b) and the inequality

$$
\left| \sup_{s \in [0, t]} g(s, \omega(s)) - \sup_{s \in [0, t]} g(s, \alpha(s)) \right| \leq \sup_{s \in [0, t]} \left| g(s, \omega(s)) - g(s, \alpha(s)) \right|.
$$

We now comment on the big jump property and the local big jump property.

**Example 2.**

(a) If $K_t(\omega; dx) = F(dx)$ for a Lévy measure $F$, then the big jump property of $K$ holds, because

$$
F(\{x \in \mathbb{R}^d : \|x\| > a\}) \to 0 \text{ with } a \not\to \infty.
$$

However, the linear growth condition (vi)' can fail, because $\|h'\|$ might not be $F$-integrable, i.e. $F$ corresponds to a Lévy process with infinite mean.

(b) When we consider a one-dimensional SDE of the type

$$
dX_t = g_t(X) dL_t,
$$

where $g$ is predictable and $L$ is a Lévy process, then $\Delta X_t = g_t(X)\Delta L_t$ and, consequently, we consider

$$
K_t(G) = \int 1_{G\setminus\{0\}}(g_t(X)y)F(dy), \quad G \in \mathcal{B}(\mathbb{R}),
$$

where $F$ is the Lévy measure corresponding to $L$. In this case, we obtain

$$
K_t(\{x \in \mathbb{R} : \|x\| > a\}) = \int 1\{y \in \mathbb{R} : \|g_t(X)\| > a\}F(dy).
$$

If $g$ is bounded, i.e. there is a constant $c > 0$ such that $|g_t(\omega)| \leq c$ for all $t \in \mathbb{R}^+, \omega \in \Omega$, then we have

$$
\int 1\{y \in \mathbb{R} : \|g_t(X)\| > a\}F(dy) \leq F(\{y \in \mathbb{R} : \|y\| > \frac{a}{c}\}) \to 0 \text{ with } a \not\to \infty.
$$

However, if $g$ is unbounded, the big jump condition might fail, while the local big jump conditions and the linear growth condition (iv)' might hold.

Next, we provide examples to understand the Lyapunov-type condition:

**Example 3.**

(a) For $V(x) \equiv 1 + \|x\|^2$ the Lyapunov-type conditions (v) and (vi) correspond to a linear growth condition. For example, if there exists a Borel function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}^+$ we have $\int_0^t \gamma(s)dA_s < \infty$ and

$$
\int_{\|x\| \leq a} \left( (\|X_{t-} + x\|^2 - \|X_{t-}\|^2 - 2\langle X_{t-}, x \rangle) K_t(dx) \right)
+ 2\langle X_{t-}, b_t \rangle + \|c_t\| \leq \gamma(t)(1 + \|X_{t-}\|^2),
$$

then the Lyapunov-type condition (v) is satisfied. This linear growth condition is different from (v)'. On one hand, the growth condition (2.3) allows an interplay of the coefficients. For example, if $d = 1$ and

$$
b_t \equiv -X_{t-}^3, \quad c_t \equiv 2X_{t-}^4, \quad K \equiv 0,
$$

then

$$
2\langle X_{t-}, b_t \rangle + \|c_t\| = -2X_{t-}^4 + 2X_{t-}^4 = 0 \leq 1 + X_{t-}^2,
$$
although $|b_t|$ and $|c_t|$ are not of linear growth. On the other hand, the linear growth condition (vi)' takes the whole history of the paths into consideration.

(b) Let us consider the case $d = 1$ where $b \equiv K \equiv 0$, i.e. we are looking for a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that the coordinate process is a one-dimensional continuous local $P$-martingale with quadratic variation process $\int_0^t c_t dA_s$. Suppose there exists an $a > 1$ and a constant $\zeta < \infty$ such that for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$: $\|\omega(t-\cdot)\| < a$ we have $c_t(\omega) \leq \zeta$. Then, the Lyapunov-type conditions (v) and (vi) with $\gamma(t) \triangleq \frac{a^2 \zeta}{\log(a^2)}$ and $V(x) \triangleq \log(a^2 + |x|^2)$. To see this, note that

$$
\gamma(t)V(X_t) - (\mathcal{L}V)(t) = \frac{a^2 \zeta}{\log(a^2)} V(X_t) + \left(\frac{|X_{t-}|^2 - a^2}{(a^2 + |X_{t-}|^2)^2}\right)c_t \\
\geq a^2(\zeta - c_t 1\{X_{t-} < a\}) \geq 0.
$$

In particular, the Lyapunov-type condition holds when $c_s(\omega) = \tau(\omega(s-))\iota_s(\omega)$ for a locally bounded function $\tau: \mathbb{R} \to \mathbb{R}_+$ and a bounded process $\iota$. This observation is in accordance with a theorem of Engelbert and Schmidt (see, e.g., [17 Theorem 5.5.4]) which implies that one-dimensional SDEs of the type

$$dX_t = \sqrt{\tau(X_t)} \, dW_t$$

have weak solutions whenever the coefficient $\tau: \mathbb{R} \to \mathbb{R}_+$ is continuous. In fact, the theorem of Engelbert and Schmidt provides one sufficient and necessary condition.

**Remark 2.** As already indicated in Example [2] if we have

$$K_t(\omega; G) = \int 1_{G \setminus \{0\}}(v(t, \omega, y)) F(dy), \quad G \in \mathcal{B}(\mathbb{R}^d),$$

where $v$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and $F$ is a Lévy measure on $\mathbb{R}^d$, then $(b, c, K)$ corresponds to an SDE driven by Lévy noise, see [13 Theorem III.2.26]. In this case, the linear growth condition (vi)' is in the spirit of the linear growth conditions from [10 Theorems 14.23, 14.95] and [13 Theorem III.2.32], which are stated together with local Lipschitz conditions. In particular, (vi)' holds under the following linear growth condition: *There exist two Borel functions $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ and $\theta: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ such that for all $(t, \omega, y) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$ we have $\int_0^t (\gamma(s) + \int |\theta(s, x)|^2 F(dx)) dA_s < \infty$ and

$$
\|b_t(\omega)\|^2 + \|c_t(\omega)\| \leq \gamma(t) \left(1 + \sup_{s \in [0, t]} \|\omega(s)\|^2\right),
$$

$$
\|h'(v(t, \omega, y))\| \leq \left(\theta(t, y) \wedge |\theta(t, y)|^2\right) \left(1 + \sup_{s \in [0, t]} \|\omega(s)\|^2\right)^{\frac{1}{2}}.
$$

Local Lipschitz conditions imply the existence of a local solution. We do not work with a local solution, but construct a solution by approximation. The local Lipschitz conditions also imply uniqueness, which is a property not provided by the approximation argument. Uniform boundedness and continuity conditions for the existence of weak solutions to SDEs driven by semimartingales were proven by Jacod and Mémant [12] and Lebedev [19]. Lebedev [20] also proved Lyapunov-type conditions.

As already indicated in the previous remark, Lyapunov-type and linear growth conditions for the existence of weak solutions to SDEs are sometimes combined with conditions implying the existence of a local solution. Next, we explain the method used by Stroock and Varadhan [24] to construct a global solution from a local solution and discuss some differences between the arguments based on extension and approximation.

The following proposition is a version of Tulcea’s extension theorem, which follows from [24 Theorem 1.1.9] in the same manner as its continuous analogous [24, Theorem 1.3.5] does.

---

1Here, $\mathcal{P}$ denotes the predictable $\sigma$-field.
Proposition 2. Let \((\tau_n)_{n \in \mathbb{N}}\) be an increasing sequence of \((F^i_t)_{t \geq 0}\)-stopping times and let \((P^n)_{n \in \mathbb{N}}\) be a sequence of probability measures on \((\Omega, \mathcal{F})\) such that \(P^n = P^{n+1}\) on \(F^\tau_{\tau_n}\) for all \(n \in \mathbb{N}\). If \(\lim_{n \to \infty} P^n(\tau_n \leq t) = 0\) for all \(t \in \mathbb{R}_+\), then there exists a unique probability measure \(P\) on \((\Omega, \mathcal{F})\) such that \(P = P^n\) on \(F^\tau_{\tau_n}\) for all \(n \in \mathbb{N}\).

Supposing that \((P^n)_{n \in \mathbb{N}}\) is a local solution, the consistency assumption shows that the extension, provided it exists, is a global solution.

Stroock and Varadhan \cite{24} construct a consistent sequence as in Proposition 2 under a uniqueness condition. In general semimartingale cases, the consistency holds when the sequence \((P^n)_{n \in \mathbb{N}}\) has a local uniqueness property as define in \cite{13} Definition III.2.37. Local uniqueness is a strong concept of uniqueness, which in particular implies (global) uniqueness. In Markovian settings, such as the diffusion setting of Stroock and Varadhan, local uniqueness is implied by the existence of (globally) unique solutions for all degenerated initial laws, see \cite{13} Theorem III.2.40.\footnote{The assumed kernel property in \cite{13} Theorem III.2.40] is implied by the uniqueness assumption. This follows from Lemma 2 in Appendix A and Kuratowski’s theorem.} In more general cases, however, local uniqueness is considered to be difficult to show, see the comment in the beginning of \cite{13} Section III.2.4.1. In our opinion, using local uniqueness is a natural approach to verify the consistency hypothesis. The approximation argument requires no uniqueness condition. However, it also provides no uniqueness statement.

A version of the convergence criterion \(\lim_{n \to \infty} P^n(\tau_n \leq t) = 0\) from Proposition 2 is also verified in the tightness argument as presented in Section 3.2 below. This is a similarity between the extension and the approximation argument and illustrates that both are soul mates in the point that they prevent a loss of mass.

In some cases it is possible to construct a consistent sequence as in Proposition 2 without a uniqueness assumption. An example for such a case arises from a local change of measure. A version of the convergence criterion \(\lim_{n \to \infty} P^n(\tau_n \leq t) = 0\) is also verified in the tightness argument as presented in Section 3.2 below. This is a similarity between the extension and the approximation argument and illustrates that both are soul mates in the point that they prevent a loss of mass.

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Then, if for all $\gamma$ the function $h^i(y)h^j(y)K_t(x, dy)$ are continuous, then both the Skorokhod continuity property and the local uniform continuity property hold, see part (a) of Example 1. Furthermore, if the map $s$ is left-continuous and adapted. Set $\phi \triangleq \sup_{t \in [0,m]} \|x\| > a$ such that for all $n, n' \in \mathbb{N}$ and $x \in \mathbb{R}^d$ we have $\int_0^t \gamma(s) dA_s < \infty$ and $\|\bar{b}(t, x)\|^2 + \|\bar{c}(t, x)\|^2 + \int (1 + \|y\|^2) K_t(x, dy) \leq \gamma(t) (1 + \|x\|^2)$.

Then, there exists a probability measure $P$ on $(\Omega, \mathcal{F})$ such that $X$ is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$.

This existence result generalizes Corollary IX.2.33 by replacing the global boundedness assumptions on the coefficients by local boundedness assumptions and a Lyapunov-type or a linear growth condition. The corollary is also a direct generalization of the main result in [8] to a setting including jumps.

3. Proof of Theorem 1 and Theorem 2

In view of Proposition 1 it suffices to show the claim for all degenerate initial laws, i.e. we assume that $\eta = \delta_z$, where $z \in \mathbb{R}^d$ is chosen arbitrary. Of course, here $\delta$ denotes the Dirac measure. The proof is split into three steps: First, we construct a sequence of probability measures. Second, we show that the sequence is tight and, third, we identify any accumulation point of the sequence as a probability measure under which the coordinate process is a semimartingale with local characteristics $(b, c, K; A)$ and initial law $\delta_z$.

In general, we assume that (i) – (iii) from Theorem 1 hold and that $K$ has the local big jump property as formulated in Theorem 2. Additional assumptions will be announced in the beginning of each section.

3.1. The Approximation Sequence $(P^n)_{n \in \mathbb{N}}$. Let $\phi^n : \mathbb{R} \to [0, 1]$ be a sequence of cutoff functions, i.e. $\phi^n \in C^\infty_c(\mathbb{R})$ with $\phi^n(x) = 1$ for $x \in [-n, n]$ and $\phi^n(x) = 0$ for $|x| \geq n + 1$. We define $X^n_t \triangleq \sup_{e \in [0,t]} \|X^n_e\|$ for $t \in \mathbb{R}_+$ and note that $X^n$ is a predictable process, because it is left-continuous and adapted. Set

$$b^n_t \triangleq \phi^n(X^n_t) 1\{t \leq n + 1\} b_t,$$
$$c^n_t \triangleq \phi^n(X^n_t) 1\{t \leq n + 1\} c_t,$$
$$K^n_t(dy) \triangleq \phi^n(X^n_t) 1\{t \leq n + 1\} K_t(dy).$$

It is clear that $(b^n, c^n, K^n)$ is a candidate triplet. Fix $n \in \mathbb{N}$. Our goal is to apply Theorem IX.2.31 to conclude that there exists a probability measure $P^n$ such that the coordinate process is a $P^n$-semimartingale with local characteristics $(b^n, c^n, K^n; A)$ and initial law $\delta_z$. We proceed by checking the prerequisites of Theorem IX.2.31.
By the local majorization property of the candidate triplet \((b, c, K)\) the modified triplet \((b^n, c^n, K^n)\) has the following global majorization property:

\[
\sup_{t \in \mathbb{R}_+} \sup_{\omega \in \Omega} \left( \left\| b^n_t(\omega) \right\| + \left\| c^n_t(\omega) \right\| + \int (1 + \|x\|^2) K^n_t(\omega; dx) \right)
\]

\[
= \sup_{t \in \mathbb{R}_+} \sup_{\omega \in \Omega} \phi^n(X^n_t(\omega)) \mathbf{1}\{t \leq n + 1\} \left( \left\| b_t(\omega) \right\| + \left\| c_t(\omega) \right\| + \int (1 + \|x\|^2) K_t(\omega; dx) \right)
\]

\[
\leq \sup_{t, \omega \in \Theta_{n+1}} \left( \left\| b_t(\omega) \right\| + \left\| c_t(\omega) \right\| + \int (1 + \|x\|^2) K_t(\omega; dx) \right) < \infty.
\]

Furthermore, the triplet \((b^n, c^n, K^n)\) has the following modified Skorokhod continuity property: For all \(t \in \mathbb{R}_+\) and \(g \in C_1(\mathbb{R}^d)\) the maps

\[
\omega \mapsto \int_0^t b^n_s(\omega) dA_s, \int_0^t c^n_s(\omega) dA_s, \int_0^t g(x) K_s(\omega; dx) dA_s
\]

are continuous for the Skorokhod topology. To see this, we first note the following:

**Lemma 1.** The map \(\omega \mapsto \phi^n(X^n_t(\omega))\) is continuous at \(\alpha \in \Omega\) for all \(t \not\in J(\alpha) = \{s > 0: \alpha(s) \neq \alpha(s-)\}\).

**Proof:** Let \((\alpha_n)_{n \in \mathbb{N}} \subset \Omega\) such that \(\alpha_n \to \alpha\) as \(n \to \infty\). By [13, Theorem VI.1.14] there exists a sequence \((\lambda_n)_{n \in \mathbb{N}}\) of strictly increasing continuous functions \(\mathbb{R}_+ \to \mathbb{R}_+\) such that \(\lambda_n(0) = 0, \lambda_n(t) \not\to \infty\) as \(t \to \infty\) and for all \(N \in \mathbb{N}\)

\[
\sup_{s \in [0,N]} |\lambda_n(s) - s| + \sup_{s \in [0,N]} \|\alpha_n(\lambda_n(s)) - \alpha(s)\| \to 0
\]

as \(n \to \infty\). Now, we have

\[
\left| X^n_t(\alpha_n) - X^n_{\lambda^{-1}_n(t)}(\alpha) \right| = \sup_{s \in [0,\lambda^{-1}_n(t)]} \|\alpha_n(\lambda_n(s)) - \alpha(s)\| \leq \sup_{s \in [0,\lambda^{-1}_n(t)]} \|\alpha_n(\lambda_n(s)) - \alpha(s)\| \to 0
\]

as \(n \to \infty\) by (3.1). Because \(t \not\in J(\alpha)\), (3.1) also yields that

\[
\left| X^n_{\lambda^{-1}_n(t)}(\alpha) - X^n_t(\alpha) \right| \to 0 \text{ as } n \to \infty.
\]

Using again [13, Theorem VI.1.14] yields that \(\omega \mapsto X^n_t(\omega)\) is continuous at \(\alpha\) for all \(t \not\in J(\alpha)\). Because \(\phi^n\) is continuous, this implies the claim. \(\square\)

Because càdlàg functions have at most countably many discontinuities, for each \(\alpha \in \Omega\) the set \(J(\alpha)\) is at most countable. Thus, because the function \(t \mapsto A_t\) is assumed to be continuous, the set \(J(\alpha)\) is a \(dA_t\)-null set. Now, the modified Skorokhod continuity property of \((b^n, c^n, K^n)\) follows from the Skorokhod continuity property of \((b, c, K)\) and the dominated convergence theorem.

Finally, we also note that the modified triplet \((b^n, c^n, K^n)\) has the following modified local uniform continuity property:

**Lemma 2.** For all \(t \in \mathbb{R}_+, g \in C_1(\mathbb{R}^d), i, j = 1, \ldots, d\) and all compact sets \(K \subset \Omega\) any \(k \in \{\omega \mapsto b_{ij}^n(\omega), c_{ij}^n(\omega), \int g(x) K^n_t(\omega; dx)\}\) has the uniform continuity property that for all \(\varepsilon > 0\) there exists a \(\delta = \delta(\varepsilon) > 0\) such that for all \(\omega, \alpha \in K\)

\[
\sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta \implies |k(\omega) - k(\alpha)| < \varepsilon.
\]
Proof: By the local uniform continuity property of \((b, c, K)\) it suffices to consider \(k(\omega) = \phi^n(X^*_i(\omega))g(\omega)\), where \(g\) already has the uniform continuity property and \(|g|\) is bounded by a constant \(\|g\|_{\infty} > 0\). We fix \(\varepsilon > 0\). There exists a \(\delta^* = \delta^*(\varepsilon) > 0\) such that for all \(\omega, \alpha \in K\)

\[
\sup_{s \in [0, t]} \|\omega(s) - \alpha(s)\| < \delta^* \Rightarrow |g(\omega) - g(\alpha)| < \frac{\varepsilon}{2}.
\]

Because smooth functions with compact support are Lipschitz continuous, there exists a constant \(L > 0\) such that

\[
|\phi^n(X^*_i(\omega)) - \phi^n(X^*_i(\alpha))| \leq L \sup_{s \in [0, t]} \|\omega(s) - \alpha(s)\|
\]

\[
\leq L \sup_{s \in [0, t]} \|\omega(s) - \alpha(s)\|.
\]

Now, choose \(\delta = \min(\delta^*, \varepsilon(2L\|g\|_{\infty})^{-1})\). Then, we obtain for all \(\omega, \alpha \in K\): \(\sup_{s \in [0, t]} \|\omega(s) - \alpha(s)\| < \delta\) that

\[
|k(\omega) - k(\alpha)| \leq \|g\|_{\infty} |\phi^n(X^*_i(\omega)) - \phi^n(X^*_i(\alpha))| + |g(\omega) - g(\alpha)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

We conclude that \(k\) has the uniform continuity property. \(\square\)

Finally, we note that for all \(t \in \mathbb{R}^+\)

\[
\lim_{a \to \infty} \sup_{\omega \in \Omega} K^n(\omega; \{x \in \mathbb{R}^d : \|x\| > a\}) = \lim_{a \to \infty} \sup_{\omega \in \Omega} \phi(X^*_i(\omega))1\{t \leq n + 1\} K(t; \{x \in \mathbb{R}^d : \|x\| > a\}) \leq \lim_{a \to \infty} \sup_{(s, \omega) \in \Theta_n} K(s; \{x \in \mathbb{R}^d : \|x\| > a\}) = 0,
\]

by the big jump property of \(K\). In summary, we conclude that the prerequisites of [13, Theorem IX.2.31] are fulfilled. Consequently, there exists a probability measure \(P^n\) such that the coordinate process \(X\) is a \(P^n\)-semimartingale with local characteristics \((b^n, c^n, K^n; A)\) and initial law \(\delta_z\).

3.2. Tightness of \((P^n)_{n \in \mathbb{N}}\). For \(m > 0\) we define the stopping time

\[
\rho_m \triangleq \inf\{t \in \mathbb{R}_+ : \|X^*_i\| > m\} \land m.
\]

For \(m > 0\) and \(n \in \mathbb{N}\) we define \(P^{n, m}\) to be the law of the stopped process \(X_{\cdot \wedge \rho_m}\) under \(P^n\). Our strategy is first to show tightness for \((P^{n, m})_{n \in \mathbb{N}}\) and then to deduce the tightness of \((P^n)_{n \in \mathbb{N}}\) with one of the assumptions (v), (v)', (vi) or (vi)'.

3.2.1. Tightness of \((P^{n, m})_{n \in \mathbb{N}}\). We check the assumptions from [13, Theorem VI.5.10]. Let \((b^{n, m}, c^{n, m}, K^{n, m}; A)\) be the local characteristics of \(X_{\cdot \wedge \rho_m}\) under \(P^n\). Due to [16, Lemma 2.3], we have

\[
b^{n, m} = 1_{[0, \rho_m]}b^n, \quad c^{n, m} = 1_{[0, \rho_m]}c^n, \quad K^{n, m}(dx) = 1_{[0, \rho_m]}K^n(dx),
\]

where \([0, \rho_m] \triangleq \{(t, \omega) \in \mathbb{R}_+ \times \Omega : 0 \leq t \leq \rho_m(\omega)\}\). The condition (i) of [13, Theorem VI.5.10] is trivially satisfied. We deduce from Chebychev’s inequality that for all \(t \in \mathbb{R}_+\) and \(\varepsilon > 0\)

\[
\limsup_{n \to \infty} P^{n, m} \left(\int_0^t K^{n, m}(\{x \in \mathbb{R}^d : \|x\| > a\})dA_s > \varepsilon\right) \leq \frac{A_t}{\varepsilon} \sup_{(s, \omega) \in \Theta_m} K(s; \{x \in \mathbb{R}^d : \|x\| > a\}) \to 0 \text{ with } \rho \nearrow \infty,
\]

by the local big jump property of \(K\). Thus, part (ii) of [13, Theorem VI.5.10] holds. Let

\[
\gamma^i \triangleq \sup_{(s, \omega) \in \Theta_m} |b^i_s(\omega)|, \quad i = 1, \ldots, d.
\]

Clearly, for all \(i = 1, \ldots, d\) we have

\[
\sup_{s \in \mathbb{R}_+} \sup_{\omega \in \Omega} |b^{n, m, i}_s(\omega)| \leq \sup_{s \in \mathbb{R}_+} \sup_{\omega \in \Omega} |b^i_s(\omega)| 1\{s \leq \rho_m(\omega)\} \leq \gamma^i.
\]
and $\gamma^i < \infty$, by the local majorization property. Denote by $\text{Var}(\cdot)$ the variation process. Because the process
\[
\sum_{i=1}^{d} \gamma^i A - \sum_{i=1}^{d} \text{Var} \left( \int_0^1 b^{n,m,i}_s dA_s \right) = \sum_{i=1}^{d} \int_0^1 (\gamma^i - |b^{n,m,i}_s|) dA_s
\]
is an increasing process, [13, Propositions VI.3.35, VI.3.36] yield that (iii) of [13, Theorem VI.5.10] holds. It follows in the same manner that (iv) of [13, Theorem VI.5.10] holds with (C1) as defined in [13, VI.5.1]. We conclude from [13, Theorem VI.5.10] that $(P^{n,m})_{n \in \mathbb{N}}$ is tight.

3.2.2. Non-Explosion implies Tightness. We recall [7, Theorem 15.47]: A sequence $(Q^n)_{n \in \mathbb{N}}$ of probability measures on $(\Omega, \mathcal{F})$ is tight if, and only if, for every $N \in \mathbb{N}$ and $\varepsilon, \delta > 0$ there exist $K, M > 0$ such that
\[
\limsup_{n \to \infty} Q^n \left( \sup_{t \in [0,N]} \|X_t\| \geq K \right) \leq \varepsilon,
\]
and
\[
\limsup_{n \to \infty} Q^n \left( w'(M, X, N) \geq \delta \right) \leq \varepsilon,
\]
where $w'$ is defined as on p. 438 in [7]. We only require the following property of $w'$: For every random time $\tau$ we have
\[
w'(M, X, N) = w'(M, X_{\wedge \tau}, N)
\]
on $\{N < \tau\}$. Fix $N \in \mathbb{N}$ and $\varepsilon, \delta > 0$. Because $(P^{n,m})_{n \in \mathbb{N}}$ is tight, there exist $K, M > 0$, which depend on $m$, such that
\[
\limsup_{n \to \infty} P^n \left( \sup_{t \in [0,T]} \|X_{t \wedge \rho_m}\| \geq K \right) \leq \frac{\varepsilon}{2},
\]
and
\[
\limsup_{n \to \infty} P^n \left( w'(M, X_{\wedge \rho_m}, N) \geq \delta \right) \leq \frac{\varepsilon}{2}.
\]
Now, we have
\[
P^n \left( \sup_{t \in [0,N]} \|X_t\| \geq K \right)
\]
\[
= P^n \left( \sup_{t \in [0,N]} \|X_t\| \geq K, N < \rho_m \right) + P^n \left( \sup_{t \in [0,N]} \|X_t\| \geq K, N \geq \rho_m \right)
\]
\[
\leq P^n \left( \sup_{t \in [0,N]} \|X_{t \wedge \rho_m}\| \geq K \right) + P^n \left( N \geq \rho_m \right),
\]
and
\[
P^n \left( w'(M, X, N) \geq \delta \right) \leq P^n \left( w'(M, X_{\wedge \rho_m}, N) \geq \delta \right) + P^n \left( N \geq \rho_m \right).
\]
Thus, using (3.2) and (3.3), $(P^n)_{n \in \mathbb{N}}$ is tight if we can chose $m > 0$ such that
\[
\limsup_{n \to \infty} P^n \left( N \geq \rho_m \right) \leq \frac{\varepsilon}{2}.
\]
Of course, we would first determine $m > 0$ and afterwards $K, M > 0$.

From this point on the strategies for the conditions from the Theorems 1 and 2 distinguish. To prove Theorem 1 we separate the big jumps, which is a step we do not require in the proof of Theorem 2.
3.2.3. Separation of the Big Jumps. In this section we use ideas from the proof of [21, Theorem 6.4.1]. We fix \( \alpha \in (0, \infty) \) and let \( m > \max(N, 2) \). Set
\[
Y^{\alpha} = \sum_{s \leq t} \Delta X_s \mathcal{1}_{\{\|\Delta X_s\| > \alpha\}}, \quad X^{\alpha} = X - Y^{\alpha}.
\]
Because \( X \) has càdlàg paths, \( \mathcal{1}_{\{\|\Delta X_s\| > \alpha\}} = 1 \) only for finitely many \( s \in [0, t] \). Thus, \( Y^{\alpha} \) is well-defined. Note that for two non-negative random variables \( U \) and \( V \) we have
\[
P(U + Y \geq 2\epsilon) = P(U + Y \geq 2\epsilon, X \geq \epsilon) + P(U + Y \geq 2\epsilon, X < \epsilon) \leq P(X \geq \epsilon) + P(Y \geq \epsilon).
\]
Hence, we obtain
\[
P^n(N \geq \rho_m) \leq P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|X_s\| \geq m \right) \leq P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|Y^\alpha_s\| \geq \frac{m}{T} \right) + P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|X^\alpha_s\| \geq \frac{m}{T} \right).
\]
Clearly, \( \sup_{s \in [0, N \wedge \rho_m]} \|Y^\alpha_s\| \geq 1 \) can only be true when at least one jump happens before time \( N \wedge \rho_m \), whose norm is larger than \( \alpha \), i.e.,
\[
\left\{ \sup_{s \in [0, N \wedge \rho_m]} \|Y^\alpha_s\| \geq 1 \right\} \subseteq \left\{ \sum_{s \in [0, N \wedge \rho_m]} \mathcal{1}_{\{\|\Delta X_s\| > \alpha\}} \geq 1 \right\}.
\]
Thus, we deduce from Lenglart’s domination property, see [13, Lemma 1.3.30], and Chebychev’s inequality that for all \( \epsilon > 0 \)
\[
P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|Y^\alpha_s\| \geq 1 \right) \leq P^n\left( \sum_{s \in [0, N \wedge \rho_m]} \mathcal{1}_{\{\|\Delta X_s\| > \alpha\}} \geq 1 \right) \leq \frac{\epsilon}{\bar{A}^\alpha} P^n\left( \int_0^{N \wedge \rho_m} K^\alpha_s(\{x \in \mathbb{R}^d : \|x\| > \alpha\})dA_s \geq \bar{A}^\alpha \right) \leq \frac{\epsilon}{\bar{A}^\alpha} \sup_{s \in [0, N]} \sup_{\omega \in \Omega} K^\alpha_s(\{x \in \mathbb{R}^d : \|x\| > \alpha\})
\]
Now, when the global big jump property from Theorem [2] holds we can choose a finite \( \alpha > \theta \) independent of \( m \) such that
\[
P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|Y^\alpha_s\| \geq \frac{m}{T} \right) \leq P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|Y^\alpha_s\| \geq 1 \right) \leq \frac{\epsilon}{\bar{A}^\alpha}.
\]
If \( K \) is not assumed to have the global big jump property (i.e. in the case of Theorem [2]) we choose \( \alpha \equiv \infty \). Because \( Y^\infty = 0 \), in this case we clearly have
\[
P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|Y^\infty_s\| \geq \frac{m}{T} \right) = 0.
\]
We stress again that \( \alpha \) is independent of \( m \). These choices for \( \alpha \) stay fix from now on. Set
\[
\zeta_m \defeq \inf\{t \in \mathbb{R}_+ : \|X^\alpha_{t \wedge \rho_m}\| > 1\}.
\]
Note that
\[
P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|X^\alpha_s\| \geq \frac{m}{T} \right) = P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|X^\alpha_s\| \geq \frac{m}{T}, N \leq \zeta_m \right) + P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|X^\alpha_s\| \geq \frac{m}{T}, \zeta_m < N \right) \leq P^n\left( \sup_{s \in [0, N \wedge \rho_m]} \|X^\alpha_s\| \geq \frac{m}{T} \right) + P^n\left( \zeta_m < N \right) \leq P^n\left( \sum_{s \in [0, N \wedge \rho_m \wedge \zeta_m]} \|X^\alpha_s\| \geq \frac{m}{T} \right) + \frac{\epsilon}{\bar{A}^\alpha}.
3.2.4. Non-Explosion under (v) or (vi). In this section we assume that the big jump property and condition (v) from Theorem 4 hold or that condition (vi) from Theorem 5 holds.

Due to [13, Theorem II.2.21, Proposition II.2.24], if \( a < \infty \) the process \( X^n \) is a \( P^n \)-semimartingale with local characteristics \((b^n, c^n, K^n, A)\) corresponding to the truncation function \( x1\{\|x\| \leq a\} \), where

\[
\begin{align*}
\hat{b}_n \triangleq \phi^n(X^n_t)1\{t \leq n+1\}b^n_t, \\
\hat{K}_n(q) \triangleq 1\{\|x\| \leq q\}K^n_q(dq), \\
t \in \mathbb{R}_+.
\end{align*}
\]

From this point on the proof is identical under both assumptions. We prove it under (vi) and note that when (v) holds one only has to replace \( \gamma, \nu, \beta, \mathcal{L} \) and \( X \) by \( \gamma_a, \nu_a, \beta_a, \mathcal{L}_a \) and \( X^a \). Set

\[
Z \triangleq e^{-\int_0^\infty \gamma(s)\,dA_s}V(X)
\]

and

\[
Y \triangleq Z + \int_0^e e^{-\int_0^\infty \gamma(s)\,dA_s} \left( (\mathcal{L}V)(s)\phi^n(X^n_s)1\{s \leq n+1\}\right)\,dA_s.
\]

Itô’s formula (see, e.g., [13, Theorem I.4.57]) yields that \( Y \) is a local \( P^n \)-martingale. For each \( \omega \in \Omega \) we have for \( dA_{t}\)-a.a. \( t \in \mathbb{R}_+ \)

\[
\gamma(t)V(\omega(t)) - (\mathcal{L}V)(t)\phi^n(X^n_t)1\{t \leq n+1\} \geq 0,
\]

by the Lyapunov condition. Thus, \( Y \geq Z \geq 0 \), which implies that \( Y \) is a non-negative local \( P^n \)-martingale and hence a \( P^n \)-supermartingale by Fatou’s lemma. Because \( \beta \) is increasing with \( \beta(m) \to \infty \) as \( m \to \infty \), we find an \( m > \max(N, 2) \) such that

\[
\beta(k) \geq e^{\frac{-\int_0^k \gamma(s)\,dA_s}{\varepsilon}} = \frac{\varepsilon V(s)}{\beta(\|X_t\|)}
\]

for all \( k \geq \frac{\varepsilon}{2} \). Using that for all \( t \in [0, N] \) we have

\[
Y_t \geq Z_t \geq e^{-\int_0^t \gamma(s)\,dA_s}V(X_t) \geq e^{-\int_0^N \gamma(s)\,dA_s} \beta(\|X_t\|),
\]

we conclude from the supermartingale inequality (see, e.g., [17, Theorem 1.3.8 (ii)]) that

\[
P^n \left( \sup_{s \in [0, N]} \|X_s\| \geq \frac{\varepsilon}{2} \right) \leq P^n \left( \sup_{s \in [0, N]} \beta(\|X_s\|) \geq e^{\int_0^N \gamma(s)\,dA_s} \frac{\varepsilon V(s)}{\beta(\|X_t\|)} \right)
\]

\[
\leq P^n \left( \sup_{s \in [0, N]} Y_s \geq \frac{\varepsilon V(s)}{\beta(\|X_t\|)} \right) \leq \frac{\varepsilon V(s)}{\beta(\|X_t\|)} = \frac{\varepsilon}{6}.
\]

We conclude that \( (P^n)_{n \in \mathbb{N}} \) is tight.

3.2.5. Non-Explosion under (v)’. In this section we assume that the big jump property and condition (v)’ from Theorem 4 hold. We use an argument based on Gronwall’s lemma.

Fix \( T > N \) and set

\[
M^n \triangleq X^n - \int_0^T b^n_t\,dA_s - X_0.
\]

Due to [13, Theorem II.2.21, Proposition II.2.24] the process \( M^n \) is a square-integrable local \( P^n \)-martingale with predictable quadratic variation process

\[
\langle M^n, M^n \rangle = \int_0^\infty \tilde{c}^{n,a}_s\,dA_s,
\]

where

\[
\tilde{c}^{n,a}_s \triangleq \phi^n(X^n_t)1\{t \leq n+1\}c^n_t, \\
t \in \mathbb{R}_+.
\]

Thus, using Doob’s inequality (see, e.g., [13, Theorem I.1.43]), we obtain

\[
E^{P^n} \left[ \sup_{s \in [0, N]} \|X^n_s\|^2 \right] \leq 4E^{P^n} \left[ \int_0^N \|\tilde{c}^{n,a}_s\|^2\,dA_s \right]
\]

\[
\leq 4 \int_0^N \gamma_a(s)\,dA_s + 4 \int_0^N \gamma_a(s)E^{P^n} \left[ \sup_{t \in [0, s]} \|X_{t-}\|^2 \right]\,dA_s.
\]
Hölder’s inequality yields that
\[
\sup_{t \in [0,N \wedge \rho_m \wedge \zeta_m]} \| \int_0^t b^n_s \, dA_s \|^2 \leq T \int_0^T \| b^n_s \|^2 \, dA_s
\]
\[
\leq T \int_0^T \gamma_\alpha(s) \, dA_s + T \int_0^N \gamma_\alpha(s) \, \sup_{t \in [0,s \wedge \rho_m \wedge \zeta_m]} \| X_t \|^2 \, dA_s.
\]
By the definition of \( \zeta_m \), we deduce from the inequality \( (a_1 + a_2)^2 \leq 2(a_1^2 + a_2^2) \) that
\[
\sup_{t \in [0,s \wedge \rho_m \wedge \zeta_m]} \| X_t \|^2 \leq 2 \left( \sup_{t \in [0,s \wedge \rho_m \wedge \zeta_m]} \| Y^n_t \|^2 + \sup_{t \in [0,s \wedge \rho_m \wedge \zeta_m]} \| X^n_t \|^2 \right)
\]
\[
\leq 2 \left( 1 + \sup_{t \in [0,s \wedge \rho_m \wedge \zeta_m]} \| X^n_t \|^2 \right).
\]
Now, using the inequality \( (a_1 + a_2 + a_3)^2 \leq 3(a_1^2 + a_2^2 + a_3^2) \), we conclude that there exist a constant \( c^* > 0 \) and a \( dA_t \)-integrable Borel function \( \iota : [0,T] \to \mathbb{R}_+ \), which only depend on \( z, T \) and \( \gamma_\alpha \), such that
\[
E^{P^n} \left[ \sup_{s \in [0,N \wedge \rho_m \wedge \zeta_m]} \| X^n_s \|^2 \right] \leq c^* + \int_0^N \iota(s) E^{P^n} \left[ \sup_{t \in [0,s \wedge \rho_m \wedge \zeta_m]} \| X^n_t \|^2 \right] \, dA_s.
\]
Applying the Gronwall-type lemma [21, Theorem 2.4.3] to the function
\[
[0,T] \ni t \mapsto E^{P^n} \left[ \sup_{s \in [0,T \wedge \rho_m \wedge \zeta_m]} \| X^n_s \|^2 \right]
\]
we obtain
\[
E^{P^n} \left[ \sup_{s \in [0,N \wedge \rho_m \wedge \zeta_m]} \| X^n_s \|^2 \right] \leq c^* \, e^{\int_0^T \iota(s) \, dA_s}.
\]
Chebychev’s inequality yields that
\[
\lim_{n \to \infty} P^n \left( \sup_{s \in [0,N \wedge \rho_m \wedge \zeta_m]} \| X^n_s \| \geq \frac{m}{\sqrt{2}} \right) \leq \limsup_{n \to \infty} \frac{4E^{P^n} \left[ \sup_{s \in [0,N \wedge \rho_m \wedge \zeta_m]} \| X^n_s \|^2 \right]}{m^2} \leq \frac{4c^* \, e^{\int_0^T \iota(s) \, dA_s}}{m^2}.
\]
Now, we find \( m > \max(N,2) \) such that
\[
\limsup_{n \to \infty} P^n \left( \sup_{s \in [0,N \wedge \rho_m \wedge \zeta_m]} \| X^n_s \| \geq \frac{m}{\sqrt{2}} \right) \leq \frac{m}{\sqrt{2}}
\]
and therefore we conclude that \((P^n)_{n \in \mathbb{N}}\) is tight.

3.2.6. Non-Explosion under (vi)’. In this section we assume that condition (vi)’ from Theorem 2 holds. The argument is almost identical to the one given in the previous section. The only difference is that we have an additional big jump term. Namely, we have
\[
X = X_0 + M(h) + N(h) + \int_0^t b^n_s \, dA_s + \int_0^t h'(x) K^n_s(\, dx) \, dA_s,
\]
where
\[
M(h) \triangleq X - \int_0^t b^n_s \, dA_s - \sum_{s \leq t} h'(\Delta X_s) - X_0,
\]
and
\[
N(h) \triangleq \sum_{s \leq t} h'(\Delta X_s) - \int_0^t h'(x) K^n_s(\, dx) \, dA_s.
\]
Here, \( N(h) \) is well-defined due to second part of the linear growth condition (vi)’. Moreover, [12, Proposition II.1.28, Theorem II.1.33] imply that \( N(h) \) is a square integrable local \( P^n \)-martingale with predictable quadratic variation process
\[
\langle N^i(h), N^j(h) \rangle = \int_0^t \langle h'(x) \rangle \, dK^n_{A_s}, \quad i = 1, \ldots, d.
\]
We deduce from Doob’s inequality that
\[
E^{P^n} \left[ \sup_{s \in [0,N \wedge \rho_m]} \| N(b)_s \|^2 \right] \leq 4 \int_0^T \gamma(s) dA_s + 4 \int_0^N \gamma(s) E^{P^n} \left[ \sup_{t \in [0,s \wedge \rho_m]} \| X_t - \| \right] dA_s.
\]
Furthermore, Hölder’s inequality yields that
\[
\sup_{t \in [0,N \wedge \rho_m]} \left\| \int_0^t h'(y) K^n_s(dy) dA_s \right\|^2 \leq \left( \int_0^N h'(y) \| K^n_s(dy) dA_s \right)^2 \leq \left( \int_0^T \gamma(s) \left( 1 + \sup_{t \in [0,s \wedge \rho_m]} \| X_t - \|^2 \right) dA_s \right)^2 \leq \left( \int_0^T \gamma(s) dA_s \right)^2 + \left( \int_0^T \gamma(s) dA_s \right) \int_0^N \gamma(s) \sup_{t \in [0,s \wedge \rho_m]} \| X_t - \|^2 dA_s.
\]
Now, identical to the previous section, the Gronwall-type lemma [21], Theorem 2.4.3] yields that
\[
E^{P^n} \left[ \sup_{s \in [0,N \wedge \rho_m]} \| X_s \|^2 \right] \leq c^* \epsilon f^n_0 \| b \| dA_s,
\]
for a constant \( c^* > 0 \) and a non-negative Borel function \( \epsilon \), which satisfies \( \int_0^N \| b \| dA_s < \infty \). Chebychev’s inequality finishes the proof of the tightness of \((P^n)_{n \in \mathbb{N}}\).

3.3. Martingale Problem Argument. In this section we show that for every accumulation point of \((P^n)_{n \in \mathbb{N}}\) the coordinate process is a semimartingale with local characteristics \((b,c,K;A)\) and initial law \(\delta_\omega\).

Let \( P \) be an accumulation point of \((P^n)_{n \in \mathbb{N}}\). Without loss of generality, we assume that \( P^n \to P \) weakly as \( n \to \infty \). Because \( \omega \mapsto \omega(0) \) is continuous, it follows from the continuous mapping theorem that \( P \circ X_0^{-1} = \delta_\omega \).

Set \( \tau_m \triangleq \inf \{ t \in \mathbb{R}_+: \| X_{t-} \| \geq m \text{ or } \| X_t \| \geq m \}, \ m > 0, \)
and for \( \alpha \in \Omega \) set
\[
V(\alpha) \triangleq \left\{ m > 0: \tau_m(\alpha) < \tau_m(\alpha) \right\},
V'(\alpha) \triangleq \left\{ m > 0: \Delta \alpha(\tau_m(\alpha)) \neq 0, \| \alpha(\tau_m(\alpha)) \| = m \right\}.
\]
Finally, we define
\[
U \triangleq \left\{ m > 0: P(\{ m \in V \cup V' = 0 \}) = 0 \right\}.
\]
Fix \( m \in U \) and denote by \( P_{n,m} \) the law of \( X_{\wedge \tau_m} \) under \( P^n \) and by \( P_m \) the law of \( X_{\wedge \tau_m} \) under \( P \). Due to [13, Proposition VI.2.12] and the definition of \( U \), the map \( \omega \mapsto X_{\wedge \tau_m(\omega)}(\omega) \) is \( P\)-a.s. continuous. Thus, due to the continuous mapping theorem, we have \( P_{n,m} \to P_m \) weakly as \( n \to \infty \).

Due to [16, Lemma 2.3], the stopped coordinate process \( X_{\wedge \tau_m} \) is a \( P^n\)-semimartingale with local characteristics \((1_{[0,\tau_m]} b^n, 1_{[0,\tau_m]} c^n, 1_{[0,\tau_m]} K^n; A)\).

Next, we use [13, Theorem IX.2.11] to conclude that the stopped coordinate process \( X_{\wedge \tau_m} \) is a \( P\)-semimartingale with local characteristics \((1_{[0,\tau_m]} b, 1_{[0,\tau_m]} c, 1_{[0,\tau_m]} K; A)\).

Due to the local majorization property, the Skorokhod continuity property and the fact that the map \( \omega \mapsto \tau_m(\omega) \) is \( P\)-a.s. continuous, because \( m \in U \) and [13, Proposition VI.2.11], we deduce from [13, IX.3.42] that for all \( t \in \mathbb{R}_+ \) and \( g \in C_1(\mathbb{R}^d) \) the maps
\[
\omega \mapsto \int_0^{t \wedge \tau_m(\omega)} b_s(\omega) dA_s, \int_0^{t \wedge \tau_m(\omega)} c_s(\omega) dA_s, \int_0^{t \wedge \tau_m(\omega)} g(x) K_s(\omega; dx) dA_s
\]
are $P$-a.s. continuous. This is assumption (iii) from [13, Theorem IX.2.11]. Moreover, assumption (ii) from [13, Theorem IX.2.11] follows from the local majorization property, because for each $g \in C_{1}(\mathbb{R})$ we find a constant $c^* > 0$ such that $g(x) \leq c^*(1 \wedge \|x\|^2)$. It remains to show part (i) from [13, Theorem IX.2.11]. Let $(k, k^n)$ be one of the following processes: $(b, b^n), (c, c^n)$ or $(f g(x) K(dx), f g(x) K^n(dx))$, where $g \in C_{1}(\mathbb{R})$. Chebychev’s inequality yields that for all $t \in \mathbb{R}_+$ and $\varepsilon > 0$

$$P^n\left(\left\| \int_0^{t \wedge \tau_m} (k_s - k^n_s) dA_s \right\| \geq \varepsilon \right) \leq \frac{1}{\varepsilon} E\left\| \int_0^{t \wedge \tau_m} (k_s - k^n_s) dA_s \right\|$$

$$\leq \frac{1}{\varepsilon} E\left( \int_0^{t \wedge \tau_m} \|k_s\| \left(1 - \phi^n(X_s^\Lambda)\right) dA_s \right) \leq A_s \sup_{s(\omega) \in \Theta_{m,n}} \|k_s(\omega)\| \sup_{|x| \leq m} (1 - \phi^n(x)) \to 0$$

with $n \to \infty$. In other words, assumption (i) from [13, Theorem IX.2.11] holds.

In summary, [13, Theorem IX.2.11] and [16, Lemma 2.3] yield the stopped coordinate process $X_{\wedge \tau_m}$ is a $P_m$-semimartingale with local characteristics $(1_{[0, \tau_m]}b, 1_{[0, \tau_m]}c, 1_{[0, \tau_m]}K; A)$.

Due to [13, Theorem II.2.42] the following are equivalent:

(i) The stopped coordinate process $X_{\wedge \tau_m}$ is a $P$-semimartingale with local characteristics $(1_{[0, \tau_m]}b, 1_{[0, \tau_m]}c, 1_{[0, \tau_m]}K; A)$.

(ii) For all bounded $f \in C^2(\mathbb{R}^d)$ the process

$$M^f \triangleq f(X_{\wedge \tau_m}) - f(X_0) - \int_0^{\tau_m} (\mathcal{L}f)(s) dA_s$$

is a local $P$-martingale.

Fix a bounded $f \in C^2(\mathbb{R}^d)$ and let $M^f$ be as in (3.4). The local majorization property yields that $M^f$ is bounded on finite time intervals and therefore a martingale whenever it is a local martingale. Because $X_{\wedge \tau_m}$ is a $P_m$-semimartingale with local characteristics $(1_{[0, \tau_m]}b, 1_{[0, \tau_m]}c, 1_{[0, \tau_m]}K; A)$, [13, Theorem II.2.42] also implies that the process $M^f$ is a $P_m$-martingale. Let $\rho$ be a bounded $(F_t)$$_{t \geq 0}$-stopping time. Due to [13, Lemma III.2.43] we have $M^f_\rho \circ X_{\wedge \tau_m} = M^f_\rho$. Thus, the optional stopping theorem yields that

$$E^P[M^f_\rho] = E^P[M^f_\rho] = 0.$$

Because predictable processes are $(\mathcal{F}_{t-})_{t \geq 0}$-adapted, see [13, Proposition I.2.4], and $\mathcal{F}_{t-} \subseteq \mathcal{F}_t$ for $t > 0$, see [13, p. 159], we conclude that $M^f$ is $(\mathcal{F}_t)$$_{t \geq 0}$-adapted. Hence (3.5) and [22, Proposition II.1.4] yield that $M^f$ is a $(\mathcal{F}_t)$$_{t \geq 0}$, $(P)$-martingale. Finally, the backward martingale convergence theorem yields that $M^f$ is a $P$-martingale, too.

Consequently, we conclude that the stopped coordinate process $X_{\wedge \tau_m}$ is a $P$-semimartingale with local characteristics $(1_{[0, \tau_m]}b, 1_{[0, \tau_m]}c, 1_{[0, \tau_m]}K; A)$.

Recall that $m \in U$ was arbitrary. As in the proof of [13, Proposition IX.1.17] we see that the complement of $U$ is at most countable. Consequently, we find a sequence $(m_k)_{k \in \mathbb{N}} \subset U$ such that $m_k \nearrow \infty$ as $k \to \infty$. In particular, we have $\tau_{m_k} \nearrow \infty$ as $k \to \infty$. It follows now from [13, Theorem II.2.42] that the coordinate process is a $P$-semimartingale with local characteristics $(b, c, K; A)$. The proof of the Theorems 1 and 2 is complete. \qed

Comment. The proof of [13, Theorem IX.2.11] completely relies on the martingale problem method, i.e. certain processes are identified to be local martingales, which implies the conclusion due to [13, Theorem II.2.21].

\footnote{or [13, Theorem II.2.21] or from the fact that the class of semimartingales is stable under localization and the definition of the local characteristics}
We first introduce the martingale problem for semimartingales. Let \( \mathcal{C}^+ (\mathbb{R}^d) \) be a countable sequence of test functions as defined in [12, II.2.20]. In particular, any function in \( \mathcal{C}^+ (\mathbb{R}^d) \) is bounded and vanishes around zero. We set
\[
X(h) \triangleq X - \sum_{s \leq h} (\Delta X_s - h(\Delta X_s))
\]
and
\[
M(h) \triangleq X(h) - \int_0^h b_s dA_s - X_0.
\]
Let \( \mathcal{X} \) be the set of the following processes:

(i) \( M^i(h) \) for \( i = 1, \ldots, d \).
(ii) \( M^i(h)M^j(h) - \int_0^h \tilde{\epsilon}_s^i dA_s \) for \( i, j = 1, \ldots, d \).
(iii) \( \sum_{s \leq} g(\Delta X_s) - \int_0^h g(x)K_s(dx)dA_s \) for \( g \in \mathcal{C}^+ (\mathbb{R}^d) \).

For \( n \in \mathbb{N} \) and a càdlàg process \( Y \) we set
\[
\tau_n^Y \triangleq \inf \{ t \in \mathbb{R}_+ : |Y_t| \geq n \text{ or } |Y_t| \geq n \}.
\]
Moreover, we define
\[
\tau_n^i \triangleq \tau_n^Y \text{ with } Y = M^i(h),
\]
\[
\tau_n^{ij} \triangleq \tau_n^Y \text{ with } Y = M^i(h)M^j(h) - \int_0^h \tilde{\epsilon}_s^i dA_s,
\]
\[
\tau_n^g \triangleq \tau_n^Y \text{ with } Y = \sum_{s \leq} g(\Delta X_s) - \int_0^h g(x)K_s(dx)dA_s.
\]

Let \( \mathcal{X}_{\text{loc}} \) be the set of the following processes:

(i) \( M^i(h)_{\wedge \tau_n^i} \) for \( i = 1, \ldots, d \) and \( n \in \mathbb{N} \).
(ii) \( (M^i(h)M^j(h) - \int_0^h \tilde{\epsilon}_s^i dA_s)_{\wedge \tau_n^{ij}} \) for \( i, j = 1, \ldots, d \) and \( n \in \mathbb{N} \).
(iii) \( (\sum_{s \leq} g(\Delta X_s) - \int_0^h g(x)K_s(dx)dA_s)_{\wedge \tau_n^g} \) for \( g \in \mathcal{C}^+ (\mathbb{R}^d) \) and \( n \in \mathbb{N} \).

We stress that the set \( \mathcal{X}_{\text{loc}} \) is countable.

Due to [13, Theorem II.2.21], \( X \) is a P-semimartingale with local characteristics \( (b, c, K; A) \) and initial law \( \eta \) if, and only if, \( P \circ X_0^{-1} = \eta \) and all processes in \( \mathcal{X} \) (or, equivalently, all processes in \( \mathcal{X}_{\text{loc}} \)) are local \( P \)-martingales.

For a bounded function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) we set \( \|f\|_\infty \triangleq \sup x \in \mathbb{R}^d |f(x)| \). We note that
\[
|\Delta M^i(h)| = \|h^i(\Delta X) - \Delta A \int h^i(x)K(dx)\| \leq 2\|h^i\|_\infty,
\]
\[
|\Delta \left( \sum_{s \leq} g(\Delta X_s) - \int_0^h g(x)K_s(dx)dA_s \right) | = |g(\Delta X) - \Delta A \int g(x)K_s(dx)| \leq 2\|g\|_\infty,
\]
and
\[
|\Delta \left( \int_0^h \tilde{\epsilon}_s^i dA_s \right) | = \left| \Delta A \int h^i(x)h^i(x)K(dx) - \Delta A \int h^i(x)K(dx) \int h^i(x)K(dx) \right| \leq \|h^i\|_\infty ||h^i||_\infty \|h^i\|_\infty,
\]
see [13, II.2.11, Proposition II.2.17]. Furthermore, we note that for all \( t \leq \tau_n^i \wedge \tau_n^{ij} \)
\[
|\Delta (M^i(h)M^j(h))| = \left| \Delta M^i(h)_{\wedge \tau_n^i} \Delta M^j(h)_{\wedge \tau_n^{ij}} + M^i(h)_{\wedge \tau_n^i} \Delta M^j(h)_{\wedge \tau_n^{ij}} + M^j(h)_{\wedge \tau_n^j} \Delta M^i(h)_{\wedge \tau_n^i} \right| \leq \|h^i\|_\infty \|h^j\|_\infty + \tau_n^i\|h^i\|_\infty + \tau_n^{ij}\|h^j\|_\infty.
\]
Hence, because for all \( t \in \mathbb{R}_+ \) we have
\[
|Y_{t \wedge \tau_n^Y} | \leq n + |\Delta Y_{t \wedge \tau_n^Y} |,
\]
\(^4\) because the class of local martingales is stable under localization and \( \tau_n^i, \tau_n^{ij}, \tau_n^g \to \infty \) as \( n \to \infty \).
we conclude that all processes in $X_{\text{loc}}$ are bounded and therefore martingales whenever they are local martingales. Furthermore, because predictable processes are $(\mathcal{F}_{t^-})_{t \geq 0}$-adapted, see \cite[Proposition 1.2.4]{13}, and $\mathcal{F}_{t^-} \subseteq \mathcal{F}^0_t$ for $t > 0$, see \cite[p.~159]{13}, all processes in $\mathcal{X}$ are $(\mathcal{F}^0_t)_{t \geq 0}$-adapted. Because, due to \cite[Proposition 2.1.5]{2}, the random time $\tau^Y$ is an $(\mathcal{F}^0_t)_{t \geq 0}$-stopping time whenever $Y$ is $(\mathcal{F}^0_t)_{t \geq 0}$-adapted, all processes in $X_{\text{loc}}$ are $(\mathcal{F}^0_t)_{t \geq 0}$-martingales if, and only if, they are $(\mathcal{F}^0_t)_{t \geq 0}$-martingales. Here, the implication $\Rightarrow$ follows from the tower rule and the implication $\Leftarrow$ follows from the backward martingale convergence theorem.

In summary, we proved the following:

**Lemma 3.** For a probability measure $P$ on $(\Omega, \mathcal{F})$ the coordinate process is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$ if, and only if, $P \circ X_0^{-1} = \eta$ and all processes in $X_{\text{loc}}$ are $(\mathcal{F}^0_t)_{t \geq 0}$-martingales.

With this observation at hand we are in the position to prove Proposition 1 along the lines of the proof of \cite[Proposition 2]{14}.

Let $\mathcal{P}$ be the set of all probability measures $P$ on $(\Omega, \mathcal{F})$ such that the coordinate process is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\delta_z$ for all $z \in \mathbb{R}^d$. We consider $\mathcal{P}$ as a subspace of the Polish space $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ equipped with the topology of convergence in distribution. We note that the space $\mathcal{P}$ is separable and metrizable.

**Lemma 4.** The set $\mathcal{P}$ is a Borel subset of $\mathcal{P}$.

**Proof:** Let $I \triangleq \{ P \in \mathcal{P} : P \circ X_0^{-1} \in \{ \delta_x \cap \mathbb{R}^d \} \}$ and let $J$ be the set of all $P \in \mathcal{P}$ such that

$$E_P[(Y_t - Y_s)1_{G}] = 0,$$

for all $Y \in X_{\text{loc}}, 0 \leq s < t < \infty$ and $G \in \mathcal{F}^0_t$. In \cite[(A.1)]{14} we can restrict ourselves to rational $0 \leq s < t < \infty$ because of the right-continuity of $Y$. Furthermore, the $\sigma$-field $\mathcal{F}^0_t = \sigma(X_r, r \in [0, s] \cap \mathbb{Q}, A)$ is countable generated, i.e. contains a countable determining class. Thus, in \cite[(A.1)]{14} it also suffices to take only countably many sets from $\mathcal{F}^0_t$ into consideration. We conclude that $J$ is Borel due to \cite[Theorem 15.13]{13}. Due to \cite[Theorem 8.3.7]{2} the set $\{ \delta_x, x \in S \}$ is Borel. Thus, since $P \mapsto P \circ X_0^{-1}$ is continuous by the continuous mapping theorem, we also conclude that $I$ is Borel. In view of Lemma 3 it follows that $\mathcal{P} = I \cap J$ is Borel. \hfill $\square$

In view of \cite[Theorem A.1.6]{13}, the previous lemma implies that $\mathcal{P}$ is a Borel space in the sense of \cite[p.~456]{13}. Let $\Phi: \mathcal{P} \to \mathbb{R}^d$ be the map such that $\Phi(P)$ is the starting point associated to $P \in \mathcal{P}$. We claim that $\Phi$ is continuous and therefore Borel. To see this let $(P^n)_{n \in \mathbb{N}}, P \subset \mathcal{P}$ such that $P^n \to P$ weakly as $n \to \infty$. Denote $\Phi(P^n) = x_n \in \mathbb{R}^d$ and $\Phi(P) = x \in \mathbb{R}^d$. We have to show that $x_n \to x$ as $n \to \infty$. For all $f \in C_b(\mathbb{R}^d)$ we have

$$f(x_n) = E^{P^n}[f(X_0)] \to E^P[f(X_0)] = f(x) \quad \text{as } n \to \infty. \quad (A.2)$$

This follows from the definition of convergence in distribution because the map $\omega \mapsto f(\omega(0))$ is continuous and bounded. Since \cite[(A.2)]{14} holds for all $f \in C_b(\mathbb{R}^d)$ the convergence $x_n \to x$ as $n \to \infty$ follows from \cite[Corollaries 2.57, 2.74]{1}. We conclude that $\Phi$ is continuous. Furthermore, its graph $G \triangleq \{ (P, \Phi(P)) : P \in \mathcal{P} \}$ is a Borel subset of $\mathcal{P} \times \mathbb{R}^d$ due to \cite[Proposition 8.1.8]{2}. We have $B(\mathcal{P} \times \mathbb{R}^d) = B(\mathcal{P}) \otimes B(\mathbb{R}^d)$, see \cite[Proposition 8.1.7]{2}, and

$$\bigcup_{P \in \mathcal{P}} \{ x \in \mathbb{R}^d : x = \Phi(P) \} = \mathbb{R}^d,$$

by the assumptions of Proposition 1. Thus, by the section theorem \cite[Theorem A.1.8]{15} there exists a Borel map $x \mapsto P_x$ and a $\eta$-null set $N \in B(\mathbb{R}^d)$ such that $(P_x, x) \in G$ for all $x \notin N$. By the definition of $G$, for all $x \notin N$ the coordinate process is a $P_x$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\delta_x$. Clearly, the probability measure $P_\eta \triangleq \int P_x \eta(dx)$ satisfies $P_\eta \circ X_0^{-1} = \eta$. Furthermore, for all $x \notin N$ we have

$$E^{P_\eta}[(Y_t - Y_s)1_{F}] = 0,$$
for all $0 \leq s < t < \infty$, $F \in \mathcal{F}_s^o$ and $Y \in \mathcal{X}_{loc}$. Consequently, integrating and using Lemma 8 yields that the coordinate process is a $P_\eta$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$. This completes the proof. □

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