Fermion Condensation and Non Fermi Liquid Behavior in a Model with Long Range Forces.

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Abstract. The phenomenon of the so called Fermion condensation, a phase transition analogous to Bose condensation but for Fermions, postulated in the past to occur in systems with strong momentum dependent forces, is reanalysed in a model with infinite range interactions. The strongly non Fermi Liquid behavior of this system is demonstrated analytically at $T = 0$ and at $T \neq 0$ in the superconducting and normal phases. The validity of the quasiparticle picture is investigated and seems to hold true for temperatures less than the characteristic temperature $T_f$ of the Fermion condensation.

1 Introduction

Recently, proceeding from the Landau approach [1] which treats the ground state energy $E$ of a Fermi system as a functional of its quasiparticle distribution $n_{p,\sigma}$, Khodel and Shaginyan [2] proposed the idea of the so called Fermion Condensation (FC). This phenomenon can occur in systems with a sufficiently strong momentum dependence of the effective interaction between particles giving rise to a downswing of the quasiparticle dispersion in the vicinity of the chemical potential $\mu$ (Fig. 1).

A distinct feature of this phenomenon is the emergence of a flat (dispersionless) portion of extension $\Omega$ in the quasiparticle spectrum $\varepsilon_{p,\sigma} = \frac{\delta E}{\delta n_{p,\sigma}}$, located at the chemical potential $\mu$: $\varepsilon_{p,\sigma} = \mu$, for $p \in \Omega$. This plateau is obtained from the Maxwell construction as shown in Fig.1 and as explained in [2]. As a consequence of the high degeneracy of this spectrum the distribution $n_{p,\sigma}$ varies with $p$ continuously with a finite derivative $\partial n_{p,\sigma}/\partial p$ in contrast to $n_F(p) = \theta(p - p_F)$ of ordinary Fermi liquids. A macroscopic part of quasiparticles can then occupy the region $\Omega$ (the Fermion condensate), a feature very similar to Bose-Einstein condensation of bosons. The condensate wave function turns out to be degenerate. This degeneracy may be removed by the interactions excluded from the initial functional $E[n_{p,\sigma}]$, e.g. pairing correlations [2]. As a result, the plateau in the spectrum of single-particle excitations is distorted, but, as a rule, the spectrum remains quite flat since the strength of pairing force is rather small.

Some difficulties with the concept of FC have later been discussed by P. Nozieres within the Hartree-Fock approach by using a schematic model of infinite range forces [3] providing an extremely strong momentum dependence of the effective interaction entering the Fock term. In fact, the radius $r_s$ of the force is supposed to greatly exceed the interparticle distance $r_0$ and then the model [3] provides us with the leading terms of the $r_0/r_s$ expansion. This model proves to be very useful, since it
enables us to analytically evaluate most of the basic quantities. We will adopt it to a large extent in this paper.

The purpose of the present work is to further elucidate the rather unusual properties of Fermi systems which undergo FC. The outline of the paper is as follows. In Sect. 2 we will address the question of the interplay between superconductivity and FC at $T = 0$. Sect. 3 is devoted to finite temperatures, and in Sect. 4 we discuss the validity of the quasiparticle picture. Finally in Sect. 5 we conclude our paper.

## 2 Interplay of superconductivity and Fermion condensation.

In order to formulate our studies in a convenient way we take up the model of infinite range forces [3] in a slightly more general way. We start out from the following Hamiltonian

$$H = H_0 + H_{\text{int}},$$

where

$$H_0 = \sum_{p,\sigma} \epsilon_0^p a^+_{p,\sigma} a_{p,\sigma}, \quad \epsilon_0^p = \varepsilon_0^p - \mu,$$

and

$$H_{\text{int}} = \frac{1}{2} \sum_{p_1, p_2, q, \alpha_1, \ldots, \alpha_4} [U_0(q) \delta_{\alpha_1, \alpha_3} \delta_{\alpha_2, \alpha_4} + U_s(q) \sigma^i_{\alpha_1, \alpha_3} \sigma^i_{\alpha_2, \alpha_4}] a^+_{p_1, \alpha_1} a^+_{p_2, \alpha_2} a_{p_1+q, \alpha_3} a_{p_2-q, \alpha_4}$$

with $\varepsilon_0^p = \frac{p^2}{2m}$, $U_0(q) = U_0 \delta(q)$ and $U_s(q) = U_s \delta(q)$. The constants $U_0$ and $U_s$ are assumed to be small compared to the Fermi energy $\varepsilon_F^0 = \frac{p_F^2}{2m}$. The only difference with the model [3] is that the Hamiltonian (1-3) contains an additional spin exchange mixture.

In order to study the superconducting properties of our model let us calculate the ground state energy from the usual singlet BCS wave function

$$|BCS> = \Pi \left( u_p + v_p a^+_{p,\sigma} a_{-p,-\sigma} \right) |\text{vac}>.$$  

This ground state is known to be the vacuum to the quasiparticle operators $\alpha_{p,\sigma}$ (i.e. $\alpha_{p,\sigma} |BCS> = 0$) which are related to the original operators through the Bogoliubov transformation

$$a_{p,+} = u_p \alpha_{p,+} + v_p \alpha_{p,-}, \quad a_{p,-} = u_p \alpha_{p,-} - v_p \alpha_{p,+},$$
with the coefficients $u_p$ and $v_p$ obeying the normalization condition $u_p^2 + v_p^2 = 1$. The expectation value of $H$ is easily calculated and we obtain

$$F = \langle BCS|H|BCS \rangle \equiv E - \mu N = \sum_p [2\xi_p^0n_p + V_1n_p^2 - V_2\kappa_p^*\kappa_p],$$  \hspace{1cm} (6)$$

where

$$n_p = \frac{1}{2}\sum_\sigma <a^+_{p,\sigma}a_{p,\sigma}> = v_p^2, \quad \kappa_p = <a_{p,+}a_{-p,-}> = u_pv_p,$$  \hspace{1cm} (7)$$

are the occupation numbers and the pair distribution function, respectively. A straightforward calculation yields $V_1 = -(U_0 + 3U_s)$, $V_2 = 3U_s - U_0$. For the model [3] with $U_s = 0$, the condition $V_1 > 0$, necessary for the existence of the Fermion condensate, is fulfilled only for attractive forces $U_0 < 0$ that also leads inevitably to Cooper pairing, since in this case $V_2 > 0$. However, by adding a spin-dependent term $U_s < 0$ we can, for example, take repulsive $U_0 > 0$ and choose the spin-dependent constant $U_s < 0$ such that FC exists while Cooper pairing not. Note, that the Hartree term reducing to a renormalization of the chemical potential $\mu$ is omitted in (6). We also want to indicate that the infinite range of $V_1$ and $V_2$ only is a convenient tool to trigger FC and to qualitatively investigate the interplay between pairing and FC.

At $T = 0$, the BCS extremum equation $\frac{\delta F}{\delta v_p} = 0$, corresponding to the minimum of $E$ provided $V_1 + V_2 > 0$, has the well known form (see, e.g. [4]):

$$\xi_p - \Delta_p(u_p^2 - v_p^2)/2u_pv_p = 0,$$  \hspace{1cm} (8)$$

where $\xi_p = \varepsilon_p - \mu$ and the Landau quasiparticle energy $\varepsilon_p = \frac{\delta E}{\delta n_p}$ and the gap $\Delta_p = -\frac{\delta F}{\delta \xi_p}$ are given by

$$\varepsilon_p = \varepsilon_p^0 + V_1n_p,$$  \hspace{1cm} (9)$$

and

$$\Delta_p = V_2u_pv_p.$$  \hspace{1cm} (10)$$

It is worth noting that the substitution of the Fermi step $n_p = \theta(p - p_F)$ into eq.(9) yields a downswing in $\varepsilon_p$ with a vertical slope lying exactly at the Fermi surface.

Upon substitution of (10) into (8) with the replacement of $u_p^2 - v_p^2$ by $1 - 2n_p$ we obtain

$$\xi_p - V_2(1 - 2n_p)/2 = 0.$$  \hspace{1cm} (11)$$

Together with (9) this yields

$$n_p = \frac{\mu + V_2/2 - p^2/2m}{V_1 + V_2}, \quad \xi_p = \frac{p^2/2m - \mu + V_1/2}{V_1 + V_2}V_2.$$  \hspace{1cm} (12)
From (12) we see, as discussed in the introduction, that $\varepsilon_p - \mu$ is dispersionless if $V_2$ is zero, i.e. for vanishing pairing interaction.

The occupation numbers $n_p$ must lie between 0 and 1. We therefore have $n_p = 1$ for $p \leq p_i$ and $n_p = 0$ for $p \geq p_f$. Upon inserting these conditions into (12) one finds

$$p_f^2 = 2m(\mu + V_2/2), \quad p_i^2 = 2m(\mu - V_2/2 - V_1).$$

(13)

With the help of (13) $n_p$ and $\kappa_p$ can be rewritten in a very compact form

$$n_p = \frac{p_f^2 - p_i^2}{p_f^2 - p_i^2}, \quad \kappa_p = \frac{\sqrt{(p_f^2 - p_i^2)(p_f^2 - p_i^2)}}{p_f^2 - p_i^2}, \quad p_i < p < p_f,$$

(14)

while outside the above domain $n_p$ coincides with the usual Fermi gas distribution (see Fig.2) where $\kappa_p$ is zero. We infer that the minimum of the free energy $F$ is attained inside the functional space $[n_p]$ only if the momentum $p$ belongs to the condensate region, otherwise this minimum gets into a boundary point of the space $[n_p]$ at which the gap $\Delta_p$ vanishes.

An additional relation, linking $p_i$ to $p_f$ and appropriate for the evaluation of the chemical potential $\mu$, follows from the equality between the quasiparticle and particle numbers. We have

$$\frac{p_f^3}{3\pi^2} = \frac{p_i^3}{3\pi^2} + \int_{p_i}^{p_f} \frac{dp^2 p_i^2}{\pi^2 p_f^2 - p_i^2}.$$ 

The result of the integration is

$$p_f^3 = \frac{2(p_f^5 - p_i^5)}{5(p_f^2 - p_i^2)}.$$ 

(15)

In this article we shall concentrate on the case $V_2 << V_1$ in which the pairing forces are weaker than the particle-hole interaction $V_1$ since the case $V_1 \simeq V_2$ has already been analyzed in [3] while the choice $V_1 << V_2$ leads to ordinary BCS theory. The quasiparticle spectrum $E_p$ can be evaluated by rewriting (11) as $2n_p = 1 - 2\xi_p/V_2$ and comparing to the standard BCS form

$$2n_p = 1 - \xi_p/E_p$$ 

(12'),

where $E_p$ is the quasiparticle energy. Then we find that the spectrum $E_p$ in the FC region is dispersionless as in [3]:

$$E_p = \sqrt{\xi_p^2 + \Delta_p^2} = \frac{V_2}{2}, \quad p_i < p < p_f.$$ 

(16)
Relation (16) can also be verified directly using (7,10,12,14). With (16) it is straightforward to show that the usual expression for the pair distribution

\[ \kappa_p = \frac{\Delta_p}{2E_p} \]  

(14')
is compatible with (14).

In the case \( V_2 = 0 \), the quasiparticle distribution \( n_p \) (12) transforms to

\[ n_p = \frac{\mu - p^2/2m}{V_1}, \quad p^0_i < p < p^0_f, \]  

(17)

where \( p^0_i = \sqrt{2m\mu} \) and \( p^0_i = \sqrt{2m(\mu - V_1)} \). In this case the expression (17) coincides with that determined by the minimum equation [2]: \( \frac{\delta E}{\delta n_p} = \mu \) and the spectrum \( \epsilon_p \) from (12) does have the plateau \( \epsilon_p = \mu \) inside the interval \( p^0_i < p < p^0_f \).

Using (13), (15) and keeping only leading order terms in \( V_i/\epsilon^0_F \) we get for the correction to the chemical potential:

\[ \mu - \mu_F \simeq -(V_1 + V_2)^2/48\mu_F. \]  

(18)

Here we have used the formula \( \mu_F = p_F^2/2m + V_1/2 \) for the HF chemical potential. With the chemical potential (18) one can evaluate the spectrum of the collective excitations that are sound vibrations. The sound velocity \( c_s \) is found from the relation \( mc_s^2 = \rho \frac{\partial \mu}{\partial \rho} \) and is given by

\[ c_s^2 = \frac{p_F^2}{3m^2} \left( 1 + \frac{(V_1 + V_2)^2}{48(\mu_F)^2} \right). \]  

(19)

We see that the rearrangement of the single particle degrees of freedom influences little the collective ones since the second term is a small correction to the first one.

From the above considerations we can draw our first conclusions. For example, let us consider the limit \( V_2 \to 0 \). From (12, 12', 16) we see that the distribution \( n_p \) is an absolutely smooth and continuous function of \( V_2 \) down to \( V_2 = 0 \). The fall off width of \( n_p \) is practically independent of \( V_2 \) and is dominated by the particle-hole component \( V_1 \). Furthermore, the superconductivity order parameter (14,14') is also a continuous function of \( V_2 \) down to the limit \( V_2 = 0 \) which thus differs from zero even if \( \Delta_p = 0 \). In other words, we can say that even in the absence of pairing forces, if the situation for the Fermion condensation prevails, the system
spontaneously undergoes a transition to the state with broken gauge invariance that is traditionally associated with superconductivity. This is also clear from the fact that, according to what we just have said, the BCS state (4) still holds true at $V_2 = 0$. (However, for $V_2 = 0$, the critical temperature $T_c = 0$, see Sect.3). Another way to see this stems from the usual Thouless criterium for the onset of superconductivity which we here consider using the Bethe Salpeter equation for the 2-particle Green function (in a symbolic writing) [4]

$$G_2 = G^0 + G^0 V_2 G_2,$$

(20)

where $G^0 = (1 - 2n_p)/(\omega - 2(\varepsilon_p - \mu))$. For attractive $V_2 \neq 0$ eq.(20) yields the usual Cooper pole singularity. However, in systems with FC, eq.(20) shows a pole at $\omega = 0$ for the condensate state with $\varepsilon_p = \mu$ even if $V_2 \equiv 0$.

A consequence of these considerations concerns the entropy $S$ of systems with FC. Since the ground state wave function (4) does not change its BCS structure as a function of $V_2$ down to $V_2 = 0$ and since $|BCS>$ is a pure state, we conclude that the entropy $S$ of the system is zero even at $V_2 \equiv 0$. Thus, the entropy paradox inferred from a nonzero value of $S(T = 0)$, when calculated on the basis of the ordinary Fermi gas formula

$$S = -2 \int \frac{d^3 p}{(2\pi)^3} [n_p \ln n_p + (1 - n_p) \ln(1 - n_p)],$$

(21)

with the distribution (12), is removed. Instead we should use the BCS expression for $S$ (see below) which again shows that $S = 0$ at $T = 0$.

The superfluid properties following from the formulas obtained, turn out to be quite unusual. First, in the general case of finite range forces when the strength of the pairing term $V_2 << V_1$, we seemingly deal with the weak coupling limit of BCS theory. However, in this case, as first shown in [2] and also seen from (16), the gap in the spectrum of the single particle excitations has no BCS exponential smallness: $\Delta_{BCS} \sim \exp(-\varepsilon_F/V_2)$, since on the contrary, it turns out to be linear in the particle-particle (p-p) effective coupling constant $V_2$. Furthermore, outside the interval $p_i < p < p_f$, the gap $\Delta_p$ vanishes, the boundaries $p_i$ and $p_f$ practically depending only on the value of $V_1$. Thus, the gap $\Delta_p$ exists only in the region occupied by the Fermion condensate. These properties and the fact that $\kappa_p \neq 0$ even for $V_2 = 0$ (and thus $\Delta_p = 0$) show that the strong BCS pairing is triggered by $V_1$ (in conjunction with the quasiparticle plateau). Pairing with such unusual properties is a “shadow” of the Fermion condensation.
A quite unexpected point is that a nontrivial solution $\Delta_p \neq 0$ exists in our frame of singlet pairing even if $V_2$ becomes negative, i.e. repulsive, contradicting completely the Cooper pairing pattern. This can be seen in the following way. First of all, nothing dramatic happens if the sign of $V_2$ in $\Delta_p (10)$ and $\kappa_p (14, 14')$ is turned around. Of course, the form $V_2 \kappa_p^2$ in the energy (6) indicates a loss of energy for finite $\kappa_p$ rather than a gain as this was the case for $V_2 > 0$, i.e. attraction. However, the other option, which is no pairing, means unambiguously that $n_p$ is back to the Fermi step $n_F(p) = \theta(p-p_F)$. Because of the square of the momentum distribution $n_p$ this in turn entails an energy loss in the term $V_1 n_p^2$ even greater than if we had kept $\kappa_p$ finite ($V_2$ repulsive!). We convinced ourselves from numerical studies that the inclusion of the kinetic energy does not turn over the situation of the energy balance, at least for the case $|V_2| << V_1$ considered here. That a gain in energy prevails even if $V_2$ is repulsive can also be seen from eq.(18): the sign of the correction to the chemical potential $-(V_1 + V_2)^2/48\varepsilon_F^0$ is independent of the sign of $V_2$, as long as $|V_2| << V_1$ (otherwise higher order terms must be included). Another way to characterize the situation is that FC is analogous to Bogoliubov pairing in a nonideal Bose gas. Indeed, according to (10), $V_2$ is nothing but the factor of proportionality in the relation between the gap $\Delta_p$ and the anomal density $\kappa_p$. In contrast to BCS theory, $\kappa_p$ is practically independent of the p-p interaction $V_2 << V_1$ (see eq.(14),(14')). It is “prepared” by the interaction $V_1$ entailing a non-Fermi-step distribution $n_p$ (see Fig.2).

3 Finite temperature

The extension of our model to finite temperature is, in principle, straightforward. It follows along the lines of ordinary BCS theory [4]. First, the quasiparticle distribution $n_p$ and the anomal density $\kappa_p$ are modified as follows

$$n_p = v_p^2(1 - f_p) + u_p^2 f_p, \quad \kappa_p = u_p v_p (1 - 2f_p), \quad \quad (22)$$

where $f_p = \langle \alpha_p^+ \alpha_p \rangle$ is the distribution function for the quasiparticle excitations (see below). They are described as a gas and the entropy $S$ is given by the integral

$$S = -2 \int \frac{d^3 p}{(2\pi)^3} [f_p \ln f_p + (1 - f_p) \ln(1 - f_p)]. \quad \quad (23)$$
We have now two different equations of the minimum: $\frac{\delta \Omega}{\delta v_p} = 0$ and $\frac{\delta \Omega}{\delta f_p} = 0$ with $\Omega = E - \mu N - TS$. The last one is written as

$$\xi_p (1 - 2v_p^2) + 2u_p v_p \Delta_p - T \ln \frac{1 - f_p}{f_p} = 0.$$  \hspace{1cm} (24)

where $(\varepsilon_p - \mu) = \xi_p$ as before. When deriving this equation we again have used the definitions $\varepsilon_p = \frac{\delta E}{\delta n_p}$, and $\Delta_p = -\frac{\delta F}{\delta \kappa_p}$. The first equation of the minimum keeps the same form as at $T = 0$ (eq.(8)). Therefore introducing the quasiparticle energy $E_p$ by a finite temperature generalization of (16)

$$E_p(T) = \sqrt{\xi_p^2(T) + \Delta_p^2(T)},$$  \hspace{1cm} (25)

we can, as before, solve eq.(24) yielding the ordinary relations for the $u$- and $v$-factors:

$$v_p^2 = \frac{1}{2} \left( 1 - \frac{\xi_p}{E_p} \right), \quad u_p^2 = \frac{1}{2} \left( 1 + \frac{\xi_p}{E_p} \right).$$  \hspace{1cm} (26)

Then inserting these relations together with (25) into (24) we are led to a seemingly ordinary Fermi-Dirac distribution

$$f_p = \frac{1}{e^{E_p/T} + 1}.$$  \hspace{1cm} (27)

As a matter of fact, this relation represents an equation for $f_p$ since $E_p$ implicitly depends on $f_p$.

a. The case $T < T_c$.

Let us turn to the solution of the equation for the gap $\Delta_p(T)$ following from its definition and the relation (22) for $\kappa_p(T)$

$$\Delta_p(T) = V_2 u_p(T) v_p(T)(1 - 2f_p(T))$$  \hspace{1cm} (28)

Upon substituting into this equation relations (25-27) we get

$$\Delta_p = V_2 \frac{\Delta_p}{2E_p(T)} \tanh \frac{E_p(T)}{2T}$$  \hspace{1cm} (29)

We shall see that a nontrivial solution of this equation written as

$$2E_p(T) = V_2 \tanh \frac{E_p(T)}{2T}, \quad p_1(T) < p < p_2(T)$$  \hspace{1cm} (30)
only exists in an interval \([p_1, p_2]\), narrower than the interval \([p_i, p_f]\) (13) of FC. Being independent of \(p\), \(E_p(T)\) drops with \(T\) and vanishes at \(T = T_c = V_2/4\) [3]. This is also seen in Fig.3 where a graphical solution of (30) is given.

From (30) one finds \(\tanh \frac{E_p}{2T} = 2E_p/V_2\) which allows us to rewrite (27) as

\[
f_p(T) = \frac{1}{2} - \frac{E_p(T)}{V_2}, \quad p_1 < p < p_2.
\]  
(31)

With this relation, eq.(22) for \(n_p\) takes the same form as the first of eqs.(12) with the only replacement of \(\mu(T = 0)\) by \(\mu(T)\). Taking into account that the second relation between these quantities following from the definition \(\varepsilon_p = \frac{\Delta E}{\delta n_p}\) stays unchanged (see (9)) we have for \(n_p\) and \(\varepsilon_p\) practically the same form as at \(T = 0:\)

\[
n_p(T) = \frac{\mu(T) + V_2/2 - p^2/2m}{V_1 + V_2},
\]

\[
\varepsilon_p(T) = \mu(T) = \frac{p^2/2m - \mu(T) + V_1/2}{V_1 + V_2}, \quad p_1 < p < p_2.
\]  
(32)

Outside this interval the gap \(\Delta_p = 0\) and the quasiparticle distribution becomes

\[
n_p = \frac{1}{e^{\frac{\varepsilon_p(T) - \mu(T)}{T}} + 1}, \quad p < p_1 \text{ or } p > p_2.
\]  
(33)

as can be seen from the first of eqs.(22). Since relation (9), linking the quasiparticle energy \(\varepsilon_p\) to the distribution \(n_p\), also holds at \(T \neq 0\), this means that eq. (33) is a quite intricate equation rather than an explicit formula for the calculation of \(n_p\).

We can rewrite relation (33) between these quantities in the form

\[
\varepsilon_p - \mu = T \ln \frac{1 - n_p}{n_p}
\]  
(34)

and substitute it into (9). Then one obtains

\[
p^2/2m - \mu + V_1 n_p = \varepsilon_p - \mu = T \ln \frac{1 - n_p}{n_p}, \quad p < p_1, \text{ or } p > p_2.
\]  
(35)

This is a closed equation for the evaluation of \(n_p\). Finding \(n_p\) we can insert it into eq.(34) to calculate \(\varepsilon_p\). This procedure will be considered in more detail for the case \(T > T_c\) later on.
Now we turn to the calculation of the boundaries $p_1(T)$, $p_2(T)$, determining the volume occupied by the superfluid phase. They are found from matching the distribution $n_p$ given by the formulas (32) and (33). At these boundaries, eqs. (32) and (33), we can rewrite the matching equation in the form

$$-\frac{\xi(p_k)}{V_2} + \frac{1}{2} = \frac{1}{e^{\frac{\xi(p_k)}{T}} + 1} \quad k = 1, 2$$

or equivalently

$$2\xi(p_k) = V_2 \tanh \frac{\xi(p_k)}{2T}$$

where $p_k = p_1$ or $p_2$. It is the same equation as (30) but for $\Delta p = 0$, as it must be. It also has a nontrivial solution only if $T < T_c = V_2/4$. The distance between the points $p_1(T)$ and $p_2(T)$ determining the volume occupied by the superfluid phase diminishes with $T$ and they go to meet each other at $T = T_c$ at the midpoint $p_M$ of the interval $[p_1, p_f]$ where $n_{pM} = 1/2$.

Upon expanding the r.h.s. of (37) which vanishes at $T = T_c$ one finds for the support of the superfluid phase close to $T_c$:

$$p_2 - p_1 = 4m \frac{(V_1 + V_2)T_c}{p_M V_2} \sqrt{3 \left(1 - \frac{T}{T_c}\right)}.$$  

The same procedure applied to eq. (30) yields

$$E_p = 2T_c \sqrt{3 \left(1 - \frac{T}{T_c}\right)}, \quad p_1 < p < p_2.$$  

Thus, the gap in the spectrum of the single particle excitations has a quite unusual behavior. It is limited by the boundaries $p_1$ and $p_2$ and the support shrinks as $T$ approaches $T_c$. Since the absolute value of the gap also diminishes similar to the usual BCS gap, it is like “an ice cube which is melting”. This behavior is shown in Fig.4. It can easily be obtained from $\Delta_p = \sqrt{E_p^2(T) - \xi_p^2(T)}$ using the solution of eq. (30) shown in Fig.3. It is in strong contrast with the $T$-dependence of the usual BCS gap which “melts” only in “height” but whose $p$-extension, given by the Debye temperature, is independent of $T$. The shrinkage of the volume of the superfluid phase near $T_c$ significantly influences the density of states $\rho(\varepsilon)$ and may exhibit itself in tunneling phenomena.
b. The case \( T > T_c \). Let us now turn to the normal state of the system, i.e. to \( T > T_c \). Since then \( \Delta_p = \kappa_p = 0 \), eqs. (33,34) and (35) hold in the whole domain. It is straightforward to solve eq.(35) with (33) for \( \varepsilon_p \). The result is shown in Fig.5 for \( T << \varepsilon_0^p \) (\( T_c \) can always be chosen arbitrarily small in the case \( V_2 << V_1 << \varepsilon_0^p \) considered here). We see that in comparison with the zero temperature case of Fig.1 the plateau is just a little tilted and rounded off at the end points. Inserting \( \varepsilon_p(T) \) of Fig.5 (see also eq. (41) below) into (33) we realise that even for \( T > T_c \) the distribution function has qualitatively the same behavior as the one seen in Fig.2 with only corners rounded by the temperature effects. The spread of \( n_p \) for \( T \geq T_c \) is therefore still governed by features of the Fermion condensation and not by temperature. Thus at \( T_c < T < V_1 \) we are faced with a quite interesting situation. Superfluid current is absent while phenomena which occur due to the spread of the momentum distribution \( n_p \), usually associated with superconductivity (e.g. deviations from the Korringa law in the spin-lattice relaxation rate [5]), persist.

To have more quantitative and analytical insight into what is going on let us first find the low temperature expansion for \( n_p \) in the limit \( V_2 \rightarrow 0 \). For small \( T \) the r.h.s. of eq.(35) is small. The lowest order of \( n_p \) is obtained in omitting the r.h.s. completely. The result, of course, coincides with (17). The next correction \( n_p^{(1)} \) is found by the substitution of (17) into the r.h.s. of (35) that yields

\[
n_p^{(1)} = \frac{\mu - p^2/2m}{V_1} + \frac{T}{V_1} \ln \frac{V_1 - \mu + p^2/2m}{\mu - p^2/2m}.
\]

This correction contains a small parameter \( T/V_1 \). The iterations can be continued giving rise to a low temperature expansion of the quasiparticle distribution \( n_p \) in the case \( \Delta_p = 0 \). There is a one to one correspondence between iterating the solutions for the distribution \( n_p(T) \) and the spectrum \( \xi_p(T) \). To linear order the latter is given by (see eq.(34))

\[
\xi_p = T \ln \frac{1 - n_p^{(0)}}{n_p^{(0)}} = T \ln \frac{V_1 - \mu + p^2/2m}{\mu - p^2/2m} + O(T^2).
\]

In the limit \( \varepsilon_p \rightarrow \mu \), at \( T \) finite, we can further simplify (41), since then the argument of the logarithm must be close to one. This leads to

\[
\xi_p(T) \approx 2T \left[ \frac{2\varepsilon_0^p}{V_1} + 1 \right] + O(T^2).
\]

This result can also be obtained more directly upon inserting (33) into (35) and taking the limit \( \xi_p \rightarrow 0 \) at finite \( T << V_1 \). However (42) is valid only in the
immediate vicinity of \( \varepsilon_p \) to \( \mu \) whereas (41) holds for a longer range of \( p \)-values. From (41) and (42) we see that the quasiparticle spectrum \( \varepsilon_p(T) \) in the region of the Fermion condensate depends linearly on \( T \). This feature is very different from the ordinary Fermi liquid with its \( T^2 \) dependence of \( \varepsilon_p(T) \) stemming from the Landau formula for the variation of the quasiparticle energy

\[
\delta \varepsilon_p(T) = 2 \int \frac{d^3p_1}{(2\pi)^3} f(p, p_1) \delta n_{p_1},
\]

where \( f \) is the Landau scattering amplitude. A linear temperature law may seem quite surprising, since in the case considered the specific heat \( C_V = \partial E/\partial T = T \delta S/\partial T = -T \Sigma_p(\varepsilon_p - \mu) \partial n_p/\partial T \) is a linear function of \( T \) [3] and hence \( E(T) - E(0) \sim T^2 \) as in ordinary Fermi liquids (this is easily understandable, since for FC we have \( \xi_p/T = \text{const} \) (see eq.(42)) and thus \( \partial n_p/\partial T = 0 \). For this reason the condensate does not contribute to \( C_V \).

Let us therefore pin down the point where the conventional argument leading to the \( T^2 \)-dependence of \( \varepsilon_p(T) \) at low \( T \) fails. To this purpose, we differentiate both sides of the previous equality with respect to \( T \) and obtain

\[
\frac{\partial \varepsilon_p(T)}{\partial T} = -2 \int \frac{d^3p_1}{(2\pi)^3} f_0(p, p_1) \frac{\xi_{p_1}}{T^2} n_{p_1}(1 - n_{p_1}). \tag{43}
\]

Here \( f_0 \) is the s-wave part of \( f \) and use has been made of (33) for \( \partial n_p/\partial T = (\varepsilon_p - \mu)n_p(1 - n_p)/T^2 \) neglecting, as usual for normal Fermal liquids, the insignificant \( T \)-dependence of the quasiparticle energies on \( T \). One can then introduce \( x_p = \xi_p/T \) as a new integration variable and extend the integration limit at low \( T \) with good accuracy to \( \pm \infty \). Since \( xn(x)(1 - n(x)) \) is an odd function of \( \xi_p \) we have to expand the remainder of the integrand, i.e. \( \sqrt{\varepsilon_p/(\partial \varepsilon_p/\partial p)} \) to linear order in \( \xi_p \) and hence: \( \partial \varepsilon_p(T)/\partial T \sim T \) or \( \varepsilon_p \sim T^2 \). In a system with FC this demonstration, however, does not hold, since \((d\varepsilon_p/ dp)^{-1}\) diverges exactly at the Fermi surface. Therefore all arguments in favor of the \( T^2 \) behaviour of the spectrum \( \varepsilon_p(T) \) fail and formula (41) survives.

A straightforward comparison demonstrates a remarkable correspondence between the present results and those of the Landau theory: most relations of ordinary Fermi liquid theory also hold for systems with FC, we simply have to remember that the effective mass diverges as \( T^{-1} \):

\[
m^* = \frac{1}{p} \left( \frac{\partial \varepsilon_p}{\partial p} \right)^{-1}_{p_F} = \frac{(p_F^2 - p_i^2)(p_F^2 - p_i^2)}{2T(p_i^2 - p_F^2)} \approx m V_i \frac{1}{4T}. \tag{44}
\]
We will use this relation in Sect. 4. (For quantities which contain the derivative \( \partial n_p / \partial T \) like the specific heat \( C_V \sim p_F m^* T \), the insertion of (44) is not valid, since otherwise \( C_V \) would be a constant in \( T \) in contrast to what we have seen above).

The memory of FC for temperatures beyond \( T_c \) can be seen in a particularly transparent way from the density of states

\[
\rho(\varepsilon) = 2 \int \frac{d^3 p}{(2\pi)^3} \delta(\varepsilon - \varepsilon_p(T)).
\]  

(45)

This integral can be calculated analytically since \( \partial \varepsilon_p / \partial p \) can be obtained by differentiating eq.(35). The result is

\[
\rho(\varepsilon) = \frac{V_1}{2\pi^2} \left( 2m \right)^{3/2} [\varepsilon - V_1 n(\varepsilon - \mu)]^{1/2} [1 + \frac{V_1}{T} n(\varepsilon - \mu) n(\mu - \varepsilon)]
\]

\[
\simeq N_F^0 \frac{p_f^2 - p_i^2}{2mT} n(\varepsilon - \mu) n(\mu - \varepsilon), \quad \mu \simeq \varepsilon,
\]  

(46)

where \( n(\xi) = (e^{\xi/T} + 1)^{-1} \) and \( N_F^0 = p_F m / \pi^2 \). For typical situations \( T_c < T < V_1 \) we show \( \rho(\varepsilon) \) in Fig. 5.

We see that the condensate contribution is spread near the Fermi surface over the interval \( \delta \varepsilon \sim T \) and it is enhanced by the factor \( (p_f^2 - p_i^2) / 2mT \sim V_1 / T > 1 \) if \( T \sim T_c \). Its magnitude drops as \( T^{-1} \). It becomes of the order of the regular contribution \( N_F^0 \) to the density of states at a characteristic temperature, denoted further by \( T_f \) which is

\[
T_f \sim \frac{p_f^2 - p_i^2}{2m} \sim \varepsilon_0^c \frac{\Omega_c}{\Omega_F};
\]  

(47)

where \( \Omega_c \) stands for the condensate volume and \( \Omega_F \), for the volume of the Fermi sphere. Above \( T_f \) the effects of FC become insignificant. With (47) we can write the enhancement factor \( \rho(\varepsilon = \mu) / N_F^0 \) of the density of states near the Fermi surface in terms of \( T_f \) as

\[
\frac{\rho(\varepsilon = \mu)}{N_F^0} \simeq \frac{T_f}{T}.
\]  

(48)

It is worth noting that in such a form the estimates (47,48) are still valid for an arbitrary form of the effective particle-hole interaction as long as it gives rise to FC. We also realise that there is not a definite \( T \) value at which the effect of FC stops, its influence diminishes gradually.

As a further peculiar point characterizing FC we want to consider the entropy at \( T > T_c \). Using (23) in the low temperature limit we obtain
\[ S(T) \simeq -2 \int \frac{d^3p}{(2\pi)^3} n_p(T=0) \ln n_p(T=0) \sim V_1 \ln \left( \frac{\varepsilon_F}{V_1} \right). \] (49)

Thus, the entropy of the system with the Fermion condensate at \( T \geq T_c \) drastically exceeds ordinary superconductor values \( S_{BCS}(T_c) \sim T_c \sim V_2 \). This is a specially peculiar “visit card” of FC in homogeneous systems.

Summing up this section we may say that systems with the Fermion condensate exhibit a quite unusual non Fermi liquid temperature behavior. The main characteristic features are that the gap as a function of \( T \) not only shrinks in magnitude but also in phase space volume and that the level density enhancement and flattening of the quasiparticle spectra, i.e. the typical Fermion condensate features, persist up to temperatures \( T \simeq T_f \) far greater than \( T_c \). Furthermore, the entropy for \( T > T_c \) is drastically enhanced with respect to ordinary Fermi liquid values.

Let us just add a word of the behavior of the various quantities as a function of \( T \) for \( T < T_c \). It is readily realized that the presence of a finite gap barely alters the characteristic features found for \( T > T_c \) indicating once again that in this kind of physics FC is the driving mechanism and not pairing. The latter is a host, itself strongly influenced by FC.

### 4 Validity of the quasiparticle picture

All the results of this article have been obtained on the assumption that the approach based on the quasiparticle picture is valid implying that the width \( \gamma \) of the relevant single particle states (in our case, the condensate) does not exceed the quasiparticle energy \( \varepsilon_p \). This assumption is fulfilled for superfluid systems with the Fermion condensate because of the presence of the gap in the spectrum of the single particle excitations. Thus, at \( T < T_c \) the quasiparticle picture is valid.

We shall see that the width \( \gamma \) remains small even if \( T \) exceeds \( T_c \). This implies that the quasiparticle picture survives turning to the normal state of these systems. Considering this problem we can treat it, as it was said before, within the ordinary Fermi liquid approach by simply taking into account the \( T \) dependence of the effective mass \( m^*(T) \sim T^{-1} \) (see eq.(44)). Much work has been done within the ordinary Fermi liquid approach to simplify the calculations and we shall make use of well known results. First of all, the width \( \gamma \) is inversely proportional to the
collision time for which the following formula can be derived [6-8]

\[ \frac{1}{\tau_0} = \frac{(m^*)^3 T^2}{8\pi^4} \frac{W(\theta, \phi)}{\cos^2 \theta}, \quad (50) \]

where \( W(\theta, \phi) \) is the transition probability depending on the angle \( \theta \) between the vectors \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) of the incoming particles and the angle \( \phi \) between the planes defined by the vectors \( \mathbf{p}_1, \mathbf{p}_2 \) and \( \mathbf{p}'_1, \mathbf{p}'_2 \). The brackets \( < ... > \) denote angular averaging.

It is useful to introduce a dimensionless quantity \( w \) defined as follows: \( < W/ \cos \theta > = 2\pi w / N_F^2 \) where \( N_F = p_F m^* / \pi^2 \) is the density of states at the Fermi surface. With this result the formula for the collision time \( \tau_0 \) becomes

\[ \frac{1}{\tau_0} = \frac{\pi m^* T^2}{4 p_F^2} w, \quad (51) \]

where [6-9]

\[ w = \frac{1}{2\pi} \int_0^\pi d\phi \int_{-1}^1 d\cos \theta \left[ \frac{1}{4} |A^s(\theta, \phi) + A^a(\theta, \phi)|^2 + \frac{1}{2} |A^s(\theta, \phi) - A^a(\theta, \phi)|^2 \right], \quad (52) \]

and \( A^s \) and \( A^a \) are components of the scattering amplitude. The triplet \( A_T \) and singlet \( A_S \) scattering amplitudes are related to \( A^s \) and \( A^a \) as follows: \( A_T = A^s + A^a \) and \( A_S = A^s - 3A^a \). Let us note that in a Fermi liquid, without spontaneous breaking of symmetry, the knowledge of the component \( A^s(\theta, \phi) \) is enough to reconstruct on the basis of symmetry relations the other component \( A^a(\theta, \phi) \). These relations are written for the triplet and singlet scattering amplitudes separately

\[ A_T(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = -A_T(\mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4), \]
\[ A_S(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = A_S(\mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4). \quad (53) \]

Substituting here \( A^s \) and \( A^a \) leads to

\[ A^a(\cos \theta, q^2) = -\frac{1}{3} A^s(\cos \theta, q^2) - \frac{2}{3} A^s(\cos \theta, q'^2). \quad (54) \]

We have used that \( \cos \theta \) remains unaltered under the interchange of the initial momenta while the transferred momentum \( q \) is replaced by \( q' = \mathbf{p}_3 - \mathbf{p}_2 \). One has [8]: \( q^2 = (\mathbf{p}_3 - \mathbf{p}_1)^2 = 4p_F^2 \sin^2 \frac{\theta}{2} \sin^2 \frac{\phi}{2} \) and \( q'^2 = 2p_F^2 (1 - \cos \theta) - q^2 \).
The variables \( \cos \theta \) and \( q \) are particularly convenient if \( A^s \) is linked to the Landau scattering amplitude \( f \) at an arbitrary value of the momentum transferred \([6,9]\). The corresponding equation has the form

\[
A^s(p_1, p_2, q) = f(p_1, p_2, q) + 2 \int \frac{d^3 p}{(2\pi)^3} f(p_1, p, q) \frac{n_{p+q/2} - n_{p-q/2}}{\varepsilon_{p+q/2} - \varepsilon_{p-q/2} - \omega} A^s(p, p_2, q).
\]

(55)

This equation is, as usual, solved by expanding both sides in Legendre polynomials. The familiar result for each component is:

\[
A^s_l(q) = f_l(q) \frac{N_F}{1 + N_F f_l(q) L(q)/(2l + 1)}.
\]

(56)

Here the dimensionless Lindhardt function \( L(q) = -\chi_0/N_F \) is introduced where \( \chi_0(q) \) is the linear response function of noninteracting quasiparticles

\[
\chi_0(q) = 2 \int \frac{d^3 p}{(2\pi)^3} \frac{n_{p+q/2} - n_{p-q/2}}{\varepsilon_{p+q/2} - \varepsilon_{p-q/2} - \omega}.
\]

(57)

where the energy transfer \( \omega = \varepsilon_{p_1'} - \varepsilon_{p_1} \). This function exists in closed form \([6]\) and varies smoothly with the parameters \( q/p_F \) and \( \omega/qv_F \).

Since nothing is known about the Landau parameters \( f_l \) with \( l \geq 2 \) we keep only the \( l=0 \) and \( l=1 \) harmonics of \( f(\theta) \) neglecting all others with \( l \geq 2 \) as is usual in Fermi liquid theory \([9]\). In addition, we shall consider \( f_0(q) \) and \( f_1(q) \) as smooth functions of \( q \). Hence, for \( T \to 0 \) we shall neglect the "1" in the denominator of (56), since the density of states diverges for \( T \to 0 \). Then the scattering amplitude \( A \) can fully be constructed. As a result, one obtains:

\[
A^s(\theta, \phi) = \frac{1 + 3 \cos \theta}{L(q)},
\]

\[
A^a(\theta, \phi) = -\frac{1 + 3 \cos \theta}{3L(q)} - 2\frac{1 + 3 \cos \theta}{3L(q')}.
\]

(58)

Thus, we infer that in the strong coupling limit the scattering amplitude \( A(\theta, \phi) \) becomes a universal function independent of any parameter characterizing the interaction between particles.

At this point one may wonder whether screening corrections to the Landau scattering amplitude \( f \) in (55) do not play a very important role as well (this point had also been stressed in \([3]\)). In fact, this is not the case. A convenient frame to
discuss this point are the selfconsistent equations for $f$ derived by Babu and Brown [10]. In that work $f$ is split into a direct and an induced part. The direct part is a Bruckner G-matrix and the induced part $f^{\text{ind}}$ corresponds to an exchange of an (RPA) phonon between the particle and the hole (schematically: $f = G + f^{\text{ind}}$). For our purpose it is sufficient to restrict to the s-wave part $f_0$ of $f$. Then the induced part is given by (eq.(4.23) and eq.(5.20) of [10]):

$$- f^{\text{ind}} = 2 f_0^2 \frac{N_F L(q)}{1 + f_0 N_F L(q)}. \quad (59)$$

As before we can neglect the "1" in the denominator of (59) and obtain $f^{\text{ind}} = -2 f_0$ which is a finite result. Therefore screening corrections do not invalidate our above conclusion.

As mentioned already, the $q$-dependence of the Lindhardt function is insignificant and, in first approximation, it can be ignored by taking $L(q) = L(0) = 1$. Then $w$ is evaluated analytically and after a simple integration one obtains

$$w = \frac{64}{5}. \quad (60)$$

With this result the width $\gamma$ is calculated in closed form:

$$\gamma(T) = \frac{1}{\tau_0} = \frac{16\pi}{5} \frac{m^* T^2}{p_F^2}. \quad (61)$$

Inserting here $m^*/m = T_f/4T$ from (44) we finally find

$$\gamma(T) = \frac{2\pi T_f}{5 \epsilon_F} T. \quad (62)$$

Thus, $\gamma(T)$ is a smooth function of $T$, the width of the condensate states turns out to be rather small, and no inconsistency emerges in our calculations. We therefore can infer that the quasiparticle picture turns out to be valid in all the temperature interval $0 < T < T_f$ of our interest.

It should be mentioned that, had we treated $\gamma$ in the Born approximation, the quasiparticle lifetime would go to zero with $T$ as in [3]. The reason for this difference stems from our interaction which takes into account screening effects. In different words, within perturbation theory the product of the effective interaction and the density of states goes to infinity. However, if screening effects are properly taken into account this product stays of the order of one in these systems.
It is useful to discuss also the $\varepsilon$-dependence of the imaginary part $\text{Im}\Sigma(p,\varepsilon)$ of the mass operator $\Sigma(p,\varepsilon)$ in systems with FC. Usually, to obtain the $\varepsilon$-dependence of $\text{Im}\Sigma$, if its $T$-dependence is already known, it is enough to replace $T$ by $\xi = \varepsilon - \mu$. This rule implies that in the system with FC, the imaginary part of the mass operator is proportional to $\xi$ in contrast to the $\xi^2$-dependence inherent in normal Fermi systems. A straightforward calculation does confirm the linearity of $\text{Im}\Sigma$ as a function of $\xi$ [2]. The difference stems from the fact that, in systems with FC one of the three quasiparticles of the decay channel can belong to the condensate. Since the condensate energy is fixed, there is no additional integration related to this particle and the phase volume has a value as if only two-quasiparticles were present in the decay. This leads to the linear behavior of $\text{Im}\Sigma(p,\varepsilon)$ in $\varepsilon - \mu$.

5 Discussion and Conclusion

In this article, we have again carefully analysed the implications and the physics of the Fermion condensation phenomenon. This phase transition was postulated to occur in strongly correlated Fermi systems several years back [2]. Our investigations were carried out in a slightly generalized version of the model of infinite range forces [3] which enabled us to analytically demonstrate the non Fermi liquid behavior of systems with FC. In our case we decoupled the link $V_1 = V_2$ considered in [3] between the parameters of the particle-hole and particle-particle channels and analysed the case $V_2 << V_1 << \varepsilon_F^0$. The first thing which our study revealed is that the BCS ground state persists even if pairing forces ($V_2$) are absent. The smearing of the quasiparticle momentum distribution $n_p$ is therefore due to the particle-hole ($V_1$) rather than the particle-particle ($V_2$) interaction. This implies that the system with FC is simultaneously in the paired state with phase locking independent of whether there is a pairing force or not. Since the BCS ground state is a pure state there is no residual entropy problem at $T = 0$ as this was evoked earlier [3]. Even more dramatic, we found that, due to an unusual balance, the BCS solution persists even for repulsive $V_2$ as long as $|V_2| << V_1$. In these limits the BCS state therefore exists independent of the sign of $V_2$ and the entropy always is zero as $T \to 0$. At finite $T$ there exists a critical temperature $T_c = V_2/4$ where the gap disappears. This disappearance occurs, however, in a non-Fermi-liquid fashion: $\Delta_p$ not only shrinks in magnitude but also in momentum extension (phase space) which may be a measurable effect. Once the gap has disappeared the system still shows Fermion condensation behavior up to temperatures $T_f \sim V_1 > T_c$: the typical flat plateau in
the single particle spectrum $\varepsilon_p$ is only slightly tilted ($\sim T$) and rounded off at the edges and the fall off width of $n_p$ is of the order of $V_1$ and not of the order of $T < V_1$. This is another example of non-Fermi-liquid behavior. The effect nicely shows up in a strong enhancement of the level density at the Fermi level for $T \geq T_c < T_f$. One can also say that the standard Fermi liquid expressions remain valid with one replacement: for $T > T_c$ the effective mass behaves itself as $m^* \sim T^{-1}$.

An important issue of FC is the validity of the quasiparticle picture. This was put into question in [3]. However, there particle collisions have been treated in the Born approximation whereas the diverging density of states demands a nonperturbative treatment. Including polarization effects yields a quasiparticle width $\gamma$ of the order of $T$ and the quasiparticle concept is preserved.

Of course, the phenomena analyzed here can occur not only in systems with infinite range forces. Solvable Hartree-Fock-like models suggested in [2,3] do confirm the conclusion that the case of finite range forces merely changes quantitative details without changing qualitative features of the Fermion condensation such as the plateau in $\varepsilon_p$ etc. In particular, the shrinkage of the volume occupied by the superfluid phase also survives with only one modification: the gap $\Delta_p$ acquires an exponentially small tail instead of a sharp cut off. No change in the conclusions also occurs if we go beyond the Hartree-Fock approximation. For example, we can add to the functional (7) a term of non HF nature of the form $\delta F \sim \Sigma_{p,p_1} n_p^2 n_{p_1}^2$. With this term the problem again can be solved nearly analytically and we can verify that all qualitative results, obtained above, hold in this model, too. The same is true for the relation (44) between the characteristic temperature $T_f$ and the size of the region occupied by the Fermion condensate. Further, when calculating the width $\gamma$, we did not use the HF approach at all so that the quasiparticle picture persists beyond the HF approximation.

A separate question concerns the possibility of the existence of Fermion condensation in real physical systems. First of all, it is related to the evaluation of the single particle spectra in such systems. A theory of these spectra in normal Fermi liquid based on a functional approach has been constructed in [2]. It is in agreement with the RPA results obtained by Gell-Mann-Brueckner-Galitskii [11,12]. Results of calculation of the single particle excitation spectra carried out for electron systems within the jellium model demonstrate that the point in the dispersion $\varepsilon_p$, where the derivative $d\varepsilon_p/dp = 0$, a precursor of the Fermion condensation, does emerge in these systems at the densities at which the effective interaction between electrons becomes strong enough [13]. In this case one can also calculate the width $\gamma$ for the condensate particles and again find $\gamma(T) \sim T$ validating the quasiparticle picture.
An issue of considerable interest is the finding [14] within the local density approximation (LDA) of bifurcated saddle points in double-plane materials like YBa$_2$Cu$_3$O$_7$, YBa$_2$Cu$_4$O$_8$, and Bi$_2$Sr$_2$CaCu$_2$O$_{8+δ}$. It seems that, for a certain range of doping, these saddle points are pinned to the Fermi level and that they lie exactly where experimentally an anomalous flattening of the single particle dispersion in two-dimensional strongly correlated electron systems is observed [15]. With the Fermi level in the region of the bifurcated saddle point again the Maxwell construction seems applicable [16], leading to straight segments of Fermi lines instead of the two-dimensional plateaus discussed in this work. It should, however, be noted that these flat Fermi lines also lead to a diverging density of states $\rho(\varepsilon = \mu)$ [17]. We intend to investigate this problem in a future publication.

To summarize, we have analysed the rather unusual properties of Fermi systems which undergo a phase transition to Fermion condensation. Strong non Fermi liquid behavior has been demonstrated for those systems whereas the the quasiparticle picture seems to survive: for $T < T_c$ due to the gap in the single particle spectrum, for $T > T_c$, due to a screening suppression of the matrix elements responsible for the decay process.

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**Figure Captions**

Fig.1. Schematic view of a quasiparticle dispersion with a downswing at the Fermi energy and the corresponding Maxwell construction.

Fig.2. Normal ($n_p$) and anomalous ($\kappa_p$) occupation numbers in the model with the Fermion condensate.

Fig.3. The quasiparticle energy $E_p$ (eq.(30)) as a function of temperature (in units of $V_1$).

Fig.4. The gap function $\Delta_p = \sqrt{E_p^2 - \xi_p^2}$ as a function of $\xi_p = \varepsilon_p - \mu$ for various temperatures.

Fig.5. The quasiparticle dispersion around the chemical potential for various temperatures $T \geq T_c$.

Fig.6. The level density $\rho(\varepsilon)$ corresponding to the same situation as in Fig.5.