ON INTEGRATION IN BANACH SPACES AND TOTAL SETS

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Dedicated to the memory of Joe Diestel

Abstract. Let $X$ be a Banach space and $\Gamma \subseteq X^*$ a total linear subspace. We study the concept of $\Gamma$-integrability for $X$-valued functions $f$ defined on a complete probability space, i.e. an analogue of Pettis integrability by dealing only with the compositions $\langle x^*, f \rangle$ for $x^* \in \Gamma$. We show that $\Gamma$-integrability and Pettis integrability are equivalent whenever $X$ has Plichko’s property $(D')$ (meaning that every $w^*$-sequentially closed subspace of $X^*$ is $w^*$-closed). This property is enjoyed by many Banach spaces including all spaces with $w^*$-angelic dual as well as all spaces which are $w^*$-sequentially dense in their bidual. A particular case of special interest arises when considering $\Gamma = T^*(Y^*)$ for some injective operator $T : X \to Y$. Within this framework, we show that if $T : X \to Y$ is a semi-embedding, $X$ has property $(D')$ and $Y$ has the Radon-Nikodým property, then $X$ has the weak Radon-Nikodým property. This extends earlier results by Delbaen (for separable $X$) and Diestel and Uhl (for weakly $K$-analytic $X$).

1. Introduction

A result attributed to Delbaen, which first appeared in a paper by Bourgain and Rosenthal (see [6, Theorem 1]), states that if $T : X \to Y$ is a semi-embedding between Banach spaces (i.e. an injective operator such that $T(B_X)$ is closed), $X$ is separable and $Y$ has the Radon-Nikodým property (RNP), then $X$ has the RNP as well (cf. [7, Theorem 4.1.13]). That result was also known to be true if $X$ is weakly $K$-analytic, see [14, footnote on p. 160]. No proof nor authorship info of that generalization was given in [14]. In our last email exchange I asked Prof. Joe Diestel about that and he told me:

"The result was realized as so, around the time that semi-embeddings were being properly appreciated, as almost immediate consequences of the more general notions of $K$-analyticity. Jerry and I understood what we understood thru Talagrand’s papers. So if attribution is the issue, it’s Talagrand’s fault!"

Loosely speaking, a key point to get such kind of results is to deduce the integrability of an $X$-valued function $f$ from the integrability of the $Y$-valued composition $T \circ f$, 

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which quite often reflects on the family of real-valued functions
\[ \{y^*, T \circ f : y^* \in Y^*\} = \{x^*, f : x^* \in T^*(Y^*)\}, \]
i.e. the compositions of \( f \) with elements of the total linear subspace \( T^*(Y^*) \subseteq X^* \).

In this paper we study Pettis-type integration of Banach space valued functions with respect to a total linear subspace of the dual. Throughout \((\Omega, \Sigma, \nu)\) is a complete probability space and \( X \) is a Banach space. Let \( \Gamma \subseteq X^* \) be a total linear subspace and let \( f : \Omega \to X \) be a function. Following [28], \( f \) is said to be:

(i) \( \Gamma \)-scalarly integrable if \( \langle x^*, f \rangle \) is integrable for all \( x^* \in \Gamma \); (ii) \( \Gamma \)-integrable if it is \( \Gamma \)-scalarly integrable and for every \( A \in \Sigma \) there is an element \( \int_A f \, d\nu \in X \)
such that \( x^*(\int_A f \, d\nu) = \int_A \langle x^*, f \rangle \, d\nu \) for all \( x^* \in \Gamma \). This generalizes the classical Pettis and Gelfand integrals, which are obtained respectively when \( \Gamma = X^* \) or \( X \) is a dual space and \( \Gamma \) is the predual of \( X \). When does \( \Gamma \)-integrability imply Pettis integrability? Several authors addressed this question in the particular case of Gelfand integrability. For instance, Diestel and Faires (see [12, Corollary 1.3]) proved that Gelfand and Pettis integrability coincide for any strongly measurable function \( f : \Omega \to X^* \) whenever \( X^* \) contains no subspace isomorphic to \( \ell_\infty \), while Musiał (see [28, Theorem 4]) showed that one can give up strong measurability if the assumption on \( X \) is strengthened to being \( w^* \)-sequentially dense in \( X^{**} \). More recently, \( \Gamma \)-integrability has been studied in [2, 25] in connection with semigroups of operators.

This paper is organized as follows.

Section 2 contains some preliminaries on scalar measurability of Banach space valued functions and total sets. Special attention is paid to Banach spaces having Gulisashvili's property \((D)\) [22], i.e. those for which scalar measurability can be tested by using any total subset of the dual.

In Section 3 we analyze \( \Gamma \)-integrability and discuss its coincidence with Pettis integrability. We show (see Theorem 3.2) that this is the case whenever \( X \) has Plichko's property \((D')\) [32], which means that every \( w^* \)-sequentially closed subspace of \( X^* \) is \( w^* \)-closed. By the Banach-Dieudonné theorem, property \((D')\) is formally weaker than property \((E')\) of [27], which means that every \( w^* \)-sequentially closed convex bounded subset of \( X^* \) is \( w^* \)-closed. The class of Banach spaces having property \((E')\) includes those with \( w^* \)-angelic dual as well as all spaces which are \( w^* \)-sequentially dense in their bidual (see [1 Theorem 5.3]). On the other hand, we also study conditions ensuring the \( \Gamma \)-integrability of a \( \Gamma \)-scalarly integrable function. This is connected in a natural way to the Mazur property of \((\Gamma, w^*)\) and the completeness of the Mackey topology \( \mu(X, \Gamma) \) associated to the dual pair \( (X, \Gamma) \), which have been topics of recent research in [3, 20, 21]. If \( X \) has property \((E')\) and \( \Gamma \) is norming, we prove that a function \( f : \Omega \to X \) is Pettis integrable whenever the family \( \{\langle x^*, f \rangle : x^* \in \Gamma \cap B_{X^*}\} \) is uniformly integrable (Theorem 3.7).

Finally, in Section 4 we focus on the interplay between the integrability of a function \( f : \Omega \to X \) and that of the composition \( T \circ f : \Omega \to Y \), where \( T \) is an injective operator from \( X \) to the Banach space \( Y \). We extend the aforementioned results of Delbaen, Diestel and Uhl by proving that if \( T \) is a semi-embedding, \( Y \) has the RNP.
and $X$ has property ($D'$), then $X$ has the weak Radon-Nikodým property (WRNP), see Theorem 4.8. In this statement, the RNP of $X$ is guaranteed if the assumption on $X$ is strengthened to being weakly Lindelöf determined (Corollary 4.12).

**Notation and terminology.** We refer to [7, 13] (resp. [29, 35]) for detailed information on vector measures and the RNP (resp. Pettis integrability and the WRNP). All our linear spaces are real. An operator between Banach spaces is a continuous linear map. By a subspace of a Banach space we mean a closed linear subspace. No closedness is assumed when we just talk about “linear subspaces”. We write $B_X = \{ x \in X : \|x\| \leq 1 \}$ (the closed unit ball of $X$). The topological dual of $X$ is denoted by $X^*$. The evaluation of $x^* \in X^*$ at $x \in X$ is denoted by either $x^*(x)$ or $\langle x^*, x \rangle$. A set $\Gamma \subseteq X^*$ is total (over $X$) if it separates the points of $X$, i.e. for every $x \in X \setminus \{0\}$ there is $x^* \in \Gamma$ such that $x^*(x) \neq 0$. A linear subspace $\Gamma \subseteq X^*$ is said to be norming if the formula

$$|||x||| = \sup \{ x^*(x) : x^* \in \Gamma \cap B_{X^*} \}, \quad x \in X,$$

defines an equivalent norm on $X$. The weak topology on $X$ and the weak $^*$ topology on $X^*$ are denoted by $w$ and $w^*$, respectively. Given another Banach space $Z$, we write $X \not\supseteq Z$ if $X$ contains no subspace isomorphic to $Z$. The convex hull and the linear span of a set $D \subseteq X$ are denoted by $co(D)$ and $span(D)$, respectively. The Banach space $X$ is said to be weakly Lindelöf determined (WLD) if $(B_{X^*}, w^*)$ is a Corson compact, i.e. it is homeomorphic to a set $K \subset [-1,1]^I$ (for some non-empty set $I$), equipped with the product topology, in such a way that $\{ i \in I : k(i) \neq 0 \}$ is countable for every $k \in K$. Every weakly $K$-analytic (e.g. weakly compactly generated) Banach space is WLD, but the converse does not hold in general, see e.g. [23] for more information on WLD spaces. Finally, we recall that a locally convex Hausdorff space $E$ is said to have the Mazur property if every sequentially continuous linear functional $\varphi : E \to \mathbb{R}$ is continuous.

## 2. Preliminaries on scalar measurability and total sets

Given any set $\Gamma \subseteq X^*$, we denote by $\sigma(\Gamma)$ the smallest $\sigma$-algebra on $X$ for which each $x^* \in \Gamma$ is measurable (as a real-valued function on $X$). According to a result of Edgar (see [15, Theorem 2.3]), $\sigma(X^*)$ coincides with the Baire $\sigma$-algebra of $(X,w)$, that is, $\sigma(X^*) = \text{Baire}(X,w)$ (cf. [29, Theorem 2.1]). The following Banach space property was introduced by Gulisashvili [22].

**Definition 2.1.** The Banach space $X$ is said to have property ($D$) if the equality $\text{Baire}(X,w) = \sigma(\Gamma)$ holds for every total set $\Gamma \subseteq X^*$.

Equivalently, $X$ has property ($D$) if and only if for every total set $\Gamma \subseteq X^*$ and every measurable space $(\tilde{\Omega}, \tilde{\Sigma})$ we have:

- a function $f : \tilde{\Omega} \to X$ is $\Gamma$-scalarly measurable (i.e. $\langle x^*, f \rangle$ is measurable for every $x^* \in \Gamma$) if and only if it is scalarly measurable (i.e. $\langle x^*, f \rangle$ is measurable for every $x^* \in X^*$).
Any Banach space with $w^*$-angelic dual has property (D), see [22, Theorem 1]. The converse fails in general: this is witnessed by the Johnson-Lindenstrauss space $JL_2(\mathcal{F})$ associated to any maximal almost disjoint family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ (see the introduction of [11] and the references therein). More generally, property (D') implies property (D), see [32, Proposition 12].

Lemma 2.2. If $X$ has property (D), then any subspace of $X$ has property (D).

Proof. Let $Y \subseteq X$ be a subspace. Given any set $\Gamma \subseteq Y^*$, for each $\gamma \in \Gamma$ we take $\tilde{\gamma} \in X^*$ such that $\tilde{\gamma}|_Y = \gamma$. Define $\tilde{\Gamma} := \{ \tilde{\gamma} : \gamma \in \Gamma \} \subseteq X^*$. The following three statements hold true without the additional assumption on $X$:

(i) $\text{Baire}(Y, w) = \{ Y \cap A : A \in \text{Baire}(X, w) \}$, as an application of the aforementioned result of Edgar (see e.g. [29, Corollary 2.2]).

(ii) $\sigma(\Gamma) = \{ Y \cap A : A \in \sigma(\tilde{\Gamma} \cup Y^\perp) \}$. Indeed, since every element of $\tilde{\Gamma} \cup Y^\perp$ is $\sigma(\Gamma)$-measurable when restricted to $Y$, the inclusion operator from $Y$ into $X$ is $\sigma(\Gamma)$-$\sigma(\tilde{\Gamma} \cup Y^\perp)$-measurable, that is, $A := \{ Y \cap A : A \in \sigma(\tilde{\Gamma} \cup Y^\perp) \} \subseteq \sigma(\Gamma)$.

On the other hand, $A$ is a $\sigma$-algebra on $Y$ for which every element of $\Gamma$ is $A$-measurable, hence $\sigma(\Gamma) = A$.

(iii) If $\Gamma$ is total (over $Y$), then $\tilde{\Gamma} \cup Y^\perp$ is total (over $X$). Indeed, just bear in mind that $Y^\perp \cap X = Y$ (by the Hahn-Banach separation theorem).

So, if $X$ has property (D) and $\Gamma$ is total, then (i), (ii) and (iii) ensure that $\text{Baire}(Y, w) = \sigma(\Gamma)$. This shows that $Y$ has property (D). □

It is easy to check that any WLD Banach space has $w^*$-angelic dual (so it has property (D)). On the other hand, a non-trivial fact is that any WLD Banach space has a Markushevich basis (see e.g. [23, Theorem 5.37]). The following result might be compared with that of Vanderwerff, Whitfield and Zizler that a Banach space having a Markushevich basis and Corson’s property (C) is WLD (see [36, Theorem 3.3], cf. [23, Theorem 5.37]).

Proposition 2.3. If $X$ has a Markushevich basis and property (D), then $X$ is WLD.

Proof. Let $\{(x_i, x_i^*) : i \in I\} \subseteq X \times X^*$ be a Markushevich basis of $X$. Then $\{x_i^* : i \in I\}$ is total and property (D) ensures that $\text{Baire}(X, w) = \sigma(\{x_i^* : i \in I\})$. Take any $x^* \in X^*$. By [34, Lemma 3.5], there is a countable set $I_{x^*} \subseteq I$ such that $x^* \in \overline{\text{span}\{x_i^* : i \in I_{x^*}\}}^{w^*}$, hence $x^*(x_i) = 0$ for every $i \in I \setminus I_{x^*}$. This implies that $X$ is WLD (see e.g. [23, Theorem 5.37]). □

Bearing in mind that $\ell_1(\omega_1)$ is not WLD, Lemma 2.2 and Proposition 2.3 yield the following known result (cf. [32, Lemma 11]).

Corollary 2.4. A Banach space having property (D) contains no subspace isomorphic to $\ell_1(\omega_1)$.

To the best of our knowledge, the next question remains open:
Question 2.5 (Gulisashvili, [22]). Is \((X^{**}, w^*)\) angelic whenever \(X^*\) has property \((D)\)?

This question has affirmative answer if \(X\) is separable (see [22, Theorem 2]). Indeed, the Odell-Rosenthal and Bourgain-Fremlin-Talagrand theorems ensure that \((X^{**}, w^*)\) is angelic whenever \(X\) is a separable Banach space such that \(X \not\supseteq \ell^1\) (see e.g. [35, Theorem 4.1]). On the other hand, property \((D)\) of \(X^*\) implies that \(X \not\supseteq \ell^1\), even for a non-separable \(X\). Indeed, this follows from the following result (see [34, Proposition 3.9]):

Fact 2.6. Let us consider the following statements:

(i) \(X^*\) has property \((D)\).

(ii) Baire\((X^*, w) = \sigma(X)\).

(iii) \(X\) is \(w^*\)-sequentially dense in \(X^{**}\).

(iv) \(X \not\supseteq \ell^1\).

Then (i) \(\Rightarrow\) (ii) \(\Leftrightarrow\) (iii) \(\Rightarrow\) (iv).

It is known that \(X^*\) is WLD if and only if \(X\) is Asplund and \(X\) is \(w^*\)-sequentially dense in \(X^{**}\), see [10, Theorem III-4, Remarks III-6] and [31, Corollary 8]. Recall that \(X\) is said to be Asplund if every subspace of \(X\) has separable dual or, equivalently, \(X^*\) has the RNP (see e.g. [13, p. 198]). Therefore, Question 2.5 has affirmative answer for Asplund spaces:

Corollary 2.7. If \(X\) is Asplund and \(X^*\) has property \((D)\), then \(X^*\) is WLD.

In particular, since a Banach lattice is Asplund if (and only if) it contains no subspace isomorphic to \(\ell_1\) (see [13, p. 95] and [19, Theorem 7]), we have:

Corollary 2.8. If \(X\) is a Banach lattice and \(X^*\) has property \((D)\), then \(X^*\) is WLD.

3. Integration and total sets

Throughout this section \(\Gamma \subseteq X^*\) is a total linear subspace. Given a function \(f : \Omega \to X\), we write

\[ Z_{f,D} := \{ \langle x^*, f \rangle : x^* \in D \} \]

for any set \(D \subseteq X^*\). Note that if \(f\) is \(\Gamma\)-integrable, then the map

\[ I_f : \Sigma \to X, \quad I_f(A) := \int_A f \, d\nu, \]

is a finitely additive vector measure vanishing on \(\nu\)-null sets. It is countably additive (and it is called the indefinite Pettis integral of \(f\)) in the particular case \(\Gamma = X^*\).

Statement (i) of our next lemma is an application of a classical result of Diestel and Faires [12], while part (iii) is similar to [9, Lemma 3.1]. Recall first that a set \(H \subseteq L_1(\nu)\) is called uniformly integrable if it is bounded and for every \(\varepsilon > 0\) there is \(\delta > 0\) such that \(\sup_{h \in H} \int_A |h| \, d\nu \leq \varepsilon\) for every \(A \in \Sigma\) with \(\nu(A) \leq \delta\).

Lemma 3.1. Let \(f : \Omega \to X\) be a \(\Gamma\)-integrable function.

(i) If \(X \not\supseteq \ell_\infty\), then \(I_f\) is countably additive.
(ii) If $I_f$ is countably additive, then $Z_{f,\Gamma\cap B_X^*}$ is uniformly integrable.
(iii) If $I_f$ is countably additive and there is a partition $\Omega = \bigcup_{n\in\mathbb{N}} A_n$ into countably many measurable sets such that each restriction $f|_{A_n}$ is Pettis integrable, then $f$ is Pettis integrable.

Proof. (i) follows from [12, Theorem 1.1] (cf. [13, p. 23, Corollary 7]).
(ii) Since $I_f$ is countably additive, it is bounded and $\lim_{\nu(A)\to 0} \|I_f\|(A) = 0$, where $\|I_f\|(\cdot)$ denotes the semivariation of $I_f$ (see e.g. [13, p. 10, Theorem 1]).

Now, the uniform integrability of $Z_{f,\Gamma\cap B_X^*}$ follows from the inequality

$$\sup_{x^*\in \Gamma\cap B_X^*} \int_A |\langle x^*, f \rangle|\,d\nu \leq \|I_f\|(A) \quad \text{for all } A \in \Sigma.$$ 

(iii) We write $\Sigma_A := \{B \cap A : B \in \Sigma\}$ for every $A \in \Sigma$. Clearly, $f$ is scalarly measurable. We first prove that $f$ is scalarly integrable. Fix $x^* \in X^*$. Take any $N \in \mathbb{N}$ and set $B_N := \bigcup_{n=1}^N A_n$. Note that $f|_{B_N}$ is Pettis integrable and its indefinite Pettis integral coincides with $I_f$ on $\Sigma_{B_N}$. Hence

$$\int_{B_N} |\langle x^*, f \rangle|\,d\nu \leq \|I_f\|(B_N) \leq \|I_f\|{\Omega} < \infty.$$ 

As $N \in \mathbb{N}$ is arbitrary, $\langle x^*, f \rangle$ is integrable. This shows that $f$ is scalarly integrable.

Fix $E \in \Sigma$. Since $I_f$ is countably additive, the series $\sum_{n\in\mathbb{N}} I_f(E \cap A_n)$ is unconditionally convergent in $X$ and for each $x^* \in X^*$ we have

$$x^* \left( \sum_{n\in\mathbb{N}} I_f(E \cap A_n) \right) = \sum_{n\in\mathbb{N}} x^* \left( I_f(E \cap A_n) \right) \overset{(\ast)}{=}= \sum_{n\in\mathbb{N}} \int_{E \cap A_n} \langle x^*, f \rangle\,d\nu = \int_E \langle x^*, f \rangle\,d\nu,$$

where equality $(\ast)$ holds because $I_f|_{\Sigma_{A_n}}$ is the indefinite Pettis integral of $f|_{A_n}$ for every $n \in \mathbb{N}$. This proves that $f$ is Pettis integrable. \hfill $\square$

The Banach space $X$ is said to have the $\nu$-Pettis Integral Property ($\nu$-PIP) if every scalarly measurable and scalarly bounded function $f : \Omega \to X$ is Pettis integrable. Recall that a function $f : \Omega \to X$ is called scalarly bounded if there is a constant $c > 0$ such that for each $x^* \in X^*$ we have $|\langle x^*, f \rangle| \leq c\|x^*\| \nu$-a.e. (the exceptional set depending on $x^*$). A Banach space is said to have the PIP if it has the $\nu$-PIP with respect to any complete probability measure $\nu$. In general:

$X$ has property $(D') \Rightarrow (X^*, w^*)$ has the Mazur property \Rightarrow $X$ has the PIP.

Indeed, the first implication is easy to check, while the second one goes back to [10] (cf. Theorem 3.7(i) below).

**Theorem 3.2.** Suppose $X$ has property $(D')$. Then every $\Gamma$-integrable function $f : \Omega \to X$ is Pettis integrable.

Proof. Since $X$ has property $(D)$, $f$ is scalarly measurable. Then there is a partition $\Omega = \bigcup_{n\in\mathbb{N}} A_n$ into countably many measurable sets such that $f|_{A_n}$ is scalarly bounded for every $n \in \mathbb{N}$ (see e.g. [29, Proposition 3.1]). On the other hand, $X$ has the PIP according to the comments preceding the theorem. Therefore, each $f|_{A_n}$ is Pettis integrable. Bearing in mind that a Banach space having property $(D)$ cannot contain subspaces isomorphic to $\ell_\infty$ (by Corollary 2.3 and the fact that $\ell_1(c)$ embeds
isomorphically into $\ell_\infty$), an appeal to Lemma 3.1 allows us to conclude that $f$ is Pettis integrable. \hfill $\square$

**Remark 3.3.** From the proof of Theorem 3.2 it follows that the result still holds true if $X$ has property $(D)$ and the $\nu$-PIP. There exist Banach spaces having the PIP but failing property $(D)$, like $\ell_1(\omega_1)$ (see [10] Theorem 5.10, cf. [29] Proposition 7.2). However, we do not know an example of a Banach space having property $(D)$ but failing the PIP.

**Question 3.4.** Does property $(D)$ imply the PIP?

The previous question has affirmative answer for dual spaces. Indeed, if $X^*$ has property $(D)$, then $X$ is $w^*$-sequentially dense in $X^{**}$ (Fact 2.6) and so $(X^{**}, w^*)$ has the Mazur property, as it can be easily checked. As a consequence:

**Corollary 3.5.** Suppose $X^*$ has property $(D)$ and let $\tilde{\Gamma} \subseteq X^{**}$ be a total linear subspace. Then every $\tilde{\Gamma}$-integrable function $f : \Omega \to X^*$ is Pettis integrable.

Given any set $D \subseteq X^*$, we denote by $S_1(D) \subseteq X^*$ the set of all limits of $w^*$-convergent sequences contained in $D$. More generally, for any ordinal $\alpha \leq \omega_1$, the $\alpha$-th $w^*$-sequential closure $S_\alpha(D)$ is defined by transfinite induction as follows: $S_0(D) := D$, $S_\alpha(D) := S_1(S_\beta(D))$ if $\alpha = \beta + 1$ and $S_\alpha(D) := \bigcup_{\beta<\alpha} S_\beta(D)$ if $\alpha$ is a limit ordinal. Then $S_\alpha(D)$ is the smallest $w^*$-sequentially closed subset of $X^*$ containing $D$. Clearly, the Banach space $X$ has property $(D')$ (resp. $(\mathcal{E}')$) if and only if $S_\alpha(D) = \overline{D}^{w^*}$ for every subspace (resp. convex bounded set) $D \subseteq X^*$.

**Lemma 3.6.** Let $f : \Omega \to X$ be a $\Gamma$-scalarly integrable function. For each $A \in \Sigma$, let $\varphi_{f, A} : \Gamma \to \mathbb{R}$ be the linear functional defined by

$$\varphi_{f, A}(x^*) := \int_A \langle x^*, f \rangle \, d\nu \quad \text{for all } x^* \in \Gamma.$$  

The following statements hold:

(i) $f$ is $\Gamma$-integrable if and only if $\varphi_{f, A}$ is $w^*$-continuous for every $A \in \Sigma$.

(ii) Let $D \subseteq \Gamma$. If $Z_{f, D}$ is uniformly integrable, then:

(ii.1) $\varphi_{f, A}$ is $w^*$-sequentially continuous on $D$ for every $A \in \Sigma$;

(ii.2) $Z_{f, S_\omega(D)}$ is a uniformly integrable subset of $L_1(\nu)$.

**Proof.** (i) The “only if” part is obvious, while the “if” part follows from the fact that any $w^*$-continuous linear functional $\varphi : \Gamma \to \mathbb{R}$ is induced by some $x \in X$ via the formula $\varphi(x^*) = \langle x, x^* \rangle$ for all $x^* \in \Gamma$ (see e.g. [24] §10.4).

(ii.1) Let $(x_n^*)_{n \in \mathbb{N}}$ be a sequence in $D$ which $w^*$-converges to some $x^* \in D$. Then $(\langle x_n^*, f \rangle)_{n \in \mathbb{N}}$ is uniformly integrable and $\lim_{n \to \infty} \langle x_n^*, f \rangle = \langle x^*, f \rangle$ pointwise on $\Omega$. By Vitali’s convergence theorem, we have $\lim_{n \to \infty} \|\langle x_n^*, f \rangle - \langle x^*, f \rangle\|_{L_1(\nu)} = 0$. Hence, $\lim_{n \to \infty} \varphi_{f, A}(x_n^*) = \varphi_{f, A}(x^*)$ for every $A \in \Sigma$.

(ii.2) Fix $c > 0$ such that $Z_{f, D} \subseteq cB_{L_1(\nu)}$ and a function $\delta : (0, \infty) \to (0, \infty)$ such that

$$\sup_{A \in \Sigma} \sup_{\nu(A) \leq \delta(c)} \sup_{x^* \in D} \int_A \|\langle x^*, f \rangle\| \, d\nu \leq \varepsilon \quad \text{for every } \varepsilon > 0.$$
We will prove, by transfinite induction, that for each $\alpha \leq \omega_1$ we have

\[(p_\alpha) \quad Z_{f,S_\alpha(D)} \subseteq cB_{L_1(\nu)} \quad \text{and} \quad (q_\alpha) \quad \sup_{A \in \Sigma} \sup_{x^* \in S_\alpha(D)} \int_A |\langle x^*, f \rangle| \, d\nu \leq \varepsilon \quad \text{for every } \varepsilon > 0.\]

The case $\alpha = 0$ is obvious. Suppose now that $0 < \alpha \leq \omega_1$ and that $(p_\beta)$ and $(q_\beta)$ hold true for every $\beta < \alpha$. If $\alpha$ is a limit, then $S_\alpha(D) = \bigcup_{\beta<\alpha} S_\beta(D)$ and so $(p_\alpha)$ and $(q_\alpha)$ also hold. Suppose on the contrary that $\alpha = \beta + 1$. Fix an arbitrary $x^* \in S_\alpha(D) = S_1(S_\beta(D))$. Then there is a sequence $(x^*_n)_{n \in \mathbb{N}}$ in $S_\beta(D)$ which $w^*$-converges to $x^*$, hence $\lim_{n \to \infty} \langle x^*_n, f \rangle = \langle x^*, f \rangle$ pointwise on $\Omega$. Since $Z_{f,S_\beta(D)}$ is uniformly integrable (by $(p_\beta)$ and $(q_\beta)$), we can apply Vitali’s convergence theorem to conclude that $\langle x^*, f \rangle \in L_1(\nu)$ and $\lim_{n \to \infty} \|\langle x^*_n, f \rangle - \langle x^*, f \rangle\|_{L_1(\nu)} = 0$. Clearly, this shows that $(p_\alpha)$ and $(q_\alpha)$ hold.

**Theorem 3.7.** Let $f : \Omega \to X$ be a $\Gamma$-scalarly integrable function such that $Z_{f,\Gamma \cap B_X^*}$ is uniformly integrable.

(i) If $(\Gamma, w^*)$ has the Mazur property, then $f$ is $\Gamma$-integrable.

(ii) If $X$ has property $(\mathcal{E}')$ and $\Gamma$ is norming, then $f$ is Pettis integrable.

**Proof.** (i) follows at once from Lemma 3.6.

(ii) The fact that $\Gamma$ is norming is equivalent to saying that $\overline{\Gamma \cap B_X^{w^*}} \supseteq cB_{X^*}$ for some $c > 0$ (by the Hahn-Banach separation theorem). Since $X$ has property $(\mathcal{E}')$, we have $S_{\omega_1}(\Gamma \cap B_X^{w^*}) = \overline{\Gamma \cap B_X^{w^*}} \supseteq cB_{X^*}$. An appeal to Lemma 3.6(ii.2) ensures that $f$ is scalarly integrable and $Z_{f,B_X^*}$ is uniformly integrable. Since $(X^*, w^*)$ has the Mazur property (by property $(\mathcal{D}')$ of $X$), statement (i) (applied to $\Gamma = X^*$) implies that $f$ is Pettis integrable.

**Remark 3.8.** Theorem 3.7(ii) generalizes an earlier analogous result for spaces with $w^*$-anglic dual, see [9] pp. 551–552.

The Mackey topology $\mu(X, \Gamma)$ is defined as the (locally convex Hausdorff) topology on $X$ of uniform convergence on absolutely convex $w^*$-compact subsets of $\Gamma$. When $\Gamma$ is norm-closed, $(X, \mu(X, \Gamma))$ is complete if (and only if) it is quasi-complete (see e.g. [3]). This completeness assumption was used by Kunze [25] to find conditions ensuring the $\Gamma$-integrability of a $\Gamma$-scalarly integrable function provided that $\Gamma$ is norming and norm-closed. After Kunze’s work, the completeness of $(X, \mu(X, \Gamma))$ has been discussed in [3] [5] [20] [21]. For instance, $(X, \mu(X, \Gamma))$ is complete whenever $(X^*, w^*)$ is angelic and $\Gamma$ is norming and norm-closed (see [21] Theorem 4)). There is also a connection between the completeness of $(X, \mu(X, \Gamma))$ and the Mazur property of $(\Gamma, w^*)$, see [21].

The following result is a refinement of [25] Theorem 4.4. We denote by $\sigma(X, \Gamma)$ the (locally convex Hausdorff) topology on $X$ of pointwise convergence on $\Gamma$.

**Theorem 3.9.** Suppose $(X, \mu(X, \Gamma))$ is complete. Let $f : \Omega \to X$ be a $\Gamma$-scalarly integrable function such that $Z_{f,\Gamma \cap B_X^*}$ is uniformly integrable and there is a $\sigma(X, \Gamma)$-separable linear subspace $X_0 \subseteq X$ such that $f(\omega) \in X_0$ for $\mu$-a.e. $\omega \in \Omega$. Then $f$ is $\Gamma$-integrable.
To deal with Theorem 3.9 we need the following folk lemma:

**Lemma 3.10.** If $X$ is $\sigma(X,\Gamma)$-separable, then every $w^*$-compact subset of $\Gamma$ is $w^*$-metrizable.

**Proof.** Let $(x_n)_{n \in \mathbb{N}}$ be a $\sigma(X,\Gamma)$-dense sequence in $X$. The map

$$\Gamma \ni x^* \mapsto (x^*(x_n))_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$$

is ($w^*$-pointwise)-continuous and injective. Hence any $w^*$-compact subset of $\Gamma$ is homeomorphic to a (compact) subset of the metrizable topological space $\mathbb{R}^\mathbb{N}$. □

**Proof of Theorem 3.9.** We will prove that $f$ is $\Gamma$-integrable by checking that $\varphi_{f,A}$ is $w^*$-continuous for every $A \in \Sigma$ (see Lemma 3.6(i)).

Note that $X_1 := X_0 \sigma(X,\Gamma)$ is a subspace of $X$. We denote by $r : X^* \to X_1^*$ the restriction operator, i.e. $r(x^*) := x^*|_{X_1}$ for all $x^* \in X^*$. Then

$$\Gamma_1 := r(\Gamma) = \{x^*|_{X_1} : x^* \in \Gamma\} \subseteq X_1^*$$

is a total linear subspace (over $X_1$). Of course, we can assume without loss of generality that $f(\Omega) \subseteq X_1$, so $f$ can be seen as an $X_1$-valued $\Gamma_1$-scalarly integrable function.

Fix $A \in \Sigma$. Since $(X,\mu(X,\Gamma))$ is complete, in order to check that $\varphi := \varphi_{f,A}$ is $w^*$-continuous it suffices to show that the restriction $\varphi|_K$ is $w^*$-continuous for each absolutely convex $w^*$-compact set $K \subseteq \Gamma$ (see e.g. [21, §21.9]). Since $r$ is ($w^*$-$w^*$)-continuous, $r(K)$ is a $w^*$-compact subset of $\Gamma_1$. Note that $\varphi|_K$ factors as

$$\begin{array}{ccc}
K & \xrightarrow{\varphi|_K} & \mathbb{R} \\
\downarrow{r|_K} & & \downarrow{\psi} \\
r(K) & & \\
\end{array}$$

where $\psi(y^*) := \int_A \langle y^*, f \rangle \, d\nu$ for all $y^* \in r(K)$. Observe that $Z_{f,K}$ is uniformly integrable (because $K$ is a bounded subset of $\Gamma$ and $Z_{f,\Gamma \cap B_X}$ is uniformly integrable). From Lemma 3.6(ii.1) (applied to $f$ as an $X_1$-valued $\Gamma_1$-scalarly integrable function and $D = r(K)$) it follows that $\psi$ is $w^*$-sequentially continuous on $r(K)$. Since $r(K)$ is $w^*$-metrizable (bear in mind the $\sigma(X_1,\Gamma_1)$-separability of $X_1$ and Lemma 3.10), we conclude that $\psi$ is $w^*$-continuous, and so is $\varphi|_K$. The proof is finished. □

### 4. Integration and operators

Throughout this section $Y$ is a Banach space. Any operator $T : X \to Y$ is $(\text{Baire}(X,w)\text{-Baire}(Y,w))$-measurable, in the sense that $T^{-1}(B) \in \text{Baire}(X,w)$ for every $B \in \text{Baire}(Y,w)$. In fact, we have the following lemma, whose proof is included for the sake of completeness.

**Lemma 4.1.** Let $T : X \to Y$ be an operator. Then

$$\sigma(T^*(Y^*)) = \{T^{-1}(B) : B \in \text{Baire}(Y,w)\}.$$
Proof. Note that $A := \{ T^{-1}(B) : B \in \text{Baire}(Y,w) \}$ is a $\sigma$-algebra on $X$. Since $T^*(y^*) = y^* \circ T : X \to \mathbb{R}$ is $A$-measurable for every $y^* \in Y^*$, the inclusion $\sigma(T^*(Y^*)) \subseteq A$ holds. On the other hand, define
$$B := \{ B \in \text{Baire}(Y,w) : T^{-1}(B) \in \sigma(T^*(Y^*)) \},$$
which is clearly a $\sigma$-algebra on $Y$. Since each $y^* \in Y^*$ is $B$-measurable, we have $\text{Baire}(Y,w) = B$ and so $\sigma(T^*(Y^*)) = A$. □

Corollary 4.2. Let $T : X \to Y$ be an operator. The following statements are equivalent:

(i) $\text{Baire}(X,w) = \{ T^{-1}(B) : B \in \text{Baire}(Y,w) \}$.

(ii) For every measurable space $(\tilde{\Omega}, \tilde{\Sigma})$ and every function $f : \tilde{\Omega} \to X$ we have: $f$ is scalarly measurable if (and only if) $T \circ f$ is scalarly measurable.

An injective operator satisfying the statements of Corollary 4.2 is said to be a weak Baire embedding. This concept was considered by Andrews [11, p. 152] in the particular case of adjoint operators. Plainly, if $T : X \to Y$ is an injective operator, then $T^*(Y^*)$ is a total linear subspace of $X^*$. Therefore:

Corollary 4.3. If $X$ has property $(D)$, then every injective operator $T : X \to Y$ is a weak Baire embedding.

As to strong measurability, in [11, Proposition 4.3(ii)] it was shown that if $X$ is WLD and $T : X \to Y$ is an injective operator, then a function $f : \Omega \to X$ is strongly measurable if (and only if) $T \circ f$ is strongly measurable. The following proposition extends that result:

Proposition 4.4. Suppose $X$ is weakly measure-compact and has property $(D)$. Let $T : X \to Y$ be an injective operator and let $f : \Omega \to X$ be a function. If $T \circ f$ is strongly measurable, then so is $f$.

Recall that the Banach space $X$ is said to be weakly measure-compact if every probability measure $P$ on $\text{Baire}(X,w)$ is $\tau$-smooth, i.e. for any net $h_\alpha : X \to \mathbb{R}$ of bounded weakly continuous functions which pointwise decreases to 0, we have $\lim \alpha \int_X h_\alpha \, dP = 0$. Every weakly Lindelöf (e.g. WLD) Banach space is weakly measure-compact, see [15, Section 4].

Proof of Proposition 4.4. By Corollary 4.2 $f$ is scalarly measurable. Since $X$ is weakly measure-compact, a result of Edgar (see [13, Proposition 5.4], cf. [29, Theorem 3.3]) ensures that $f$ is scalarly equivalent to a strongly measurable function $f_0 : \Omega \to X$. Scalar equivalence means that for each $x^* \in X^*$ we have $\langle x^*, f \rangle = \langle x^*, f_0 \rangle$ $\nu$-a.e. (the exceptional set depending on $x^*$). Note that $T \circ f$ and $T \circ f_0$ are scalarly equivalent and strongly measurable, hence $T \circ f = T \circ f_0$ $\nu$-a.e. The injectivity of $T$ implies that $f = f_0$ $\nu$-a.e., so that $f$ is strongly measurable. □

Remark 4.5. The class of weakly measure-compact Banach spaces having property $(D)$ is strictly larger than that of WLD spaces. Indeed, there are compact Hausdorff topological spaces $K$ with the following properties:
(i) $K$ is scattered of height 3,
(ii) $C(K)$ is weakly Lindelöf,
(iii) $K$ is not Corson,
see [33] (cf. [18]). Condition (i) implies that $(B_{C(K)′}, w^*)$ is sequential (see [27]
Theorem 3.2]), thus $C(K)$ has property $(E′)$ and so property $(D)$. (ii) implies that
$C(K)$ is weakly measure-compact. On the other hand, $C(K)$ is not WLD by (iii).

Obviously, if $T : X \to Y$ is an injective operator and the function $f : \Omega \to X$
is $T^*(Y^*)$-integrable, then $T \circ f$ is Pettis integrable. In the opposite direction, we
have the following result:

**Proposition 4.6.** Let $T : X \to Y$ be a semi-embedding and let $f : \Omega \to X$ be a
function for which there is a constant $c > 0$ such that, for each $y^* \in Y^*$, we have

$$|(T^*(y^*), f)| \leq c\|T^*(y^*)\| \nu\text{-a.e.}$$

(4.1)

If $T \circ f$ is Pettis integrable, then $f$ is $T^*(Y^*)$-integrable.

**Proof.** Since $T$ is a semi-embedding, $T(cB_X)$ is closed. We claim that

$$\frac{1}{\nu(A)} \int_A T \circ f \, d\nu \in T(cB_X)$$

for every $A \in \Sigma$ with $\nu(A) > 0$. Indeed, suppose this is not the case. Then the
Hahn-Banach separation theorem ensures the existence of some $y^* \in Y^*$ such that

$$\frac{1}{\nu(A)} \int_A \langle y^*, T \circ f \rangle \, d\nu = y^*\left(\frac{1}{\nu(A)} \int_A T \circ f \, d\nu\right) > \sup_{x \in B_X} y^*(T(cx)) = c\|T^*(y^*)\|.$$  

But, on the other hand, we have

$$\frac{1}{\nu(A)} \int_A \langle y^*, T \circ f \rangle \, d\nu = \frac{1}{\nu(A)} \int_A \langle T^*(y^*), f \rangle \, d\nu \overset{(4.1)}{\leq} c\|T^*(y^*)\|,$$

which is a contradiction. This proves (4.2). Therefore, $\int_A T \circ f \, d\nu \in T(X)$ for every
$A \in \Sigma$, which clearly implies that $f$ is $T^*(Y^*)$-integrable.

**Remark 4.7.** Let $T : X \to Y$ be a tauberian operator (i.e. $(T^{**})^{-1}(Y) = X$).
In this case, the argument of [17] Proposition 8] shows that a scalarly measurable and
scalarly bounded function $f : \Omega \to X$ is Pettis integrable whenever $T \circ f$ is Pettis
integrable. We stress that an operator $T : X \to Y$ is injective and tauberian if and
only if the restriction $T|_Z$ is a semi-embedding for every subspace $Z \subseteq X$, see [30].

We arrive at the main result of this section:

**Theorem 4.8.** Let $T : X \to Y$ be a semi-embedding. If $X$ has property $(D′)$ and
$Y$ has the $\nu$-RNP, then $X$ has the $\nu$-WRNP.

**Proof.** Let $I : \Sigma \to X$ be a countably additive vector measure of $\sigma$-finite variation
such that $I(A) = 0$ for every $A \in \Sigma$ with $\nu(A) = 0$. We will prove the existence of a
Pettis integrable function $f : \Omega \to X$ such that $I(A) = \int_A f \, d\nu$ for all $A \in \Sigma$.
We can assume without loss of generality that there is a constant $c > 0$ such that
$\|I(A)\| \leq c\nu(A)$ for every $A \in \Sigma$ (see e.g. [35] proof of Lemma 5.9)).
Define
\[ \tilde{I} : \Sigma \to Y, \quad \tilde{I}(A) := T(I(A)) \]
so that \( \tilde{I} \) is a countably additive vector measure satisfying \( \|\tilde{I}(A)\| \leq \|T\|c\nu(A) \) for every \( A \in \Sigma \). Since \( Y \) has the \( \nu \)-RNP, there is a Bochner integrable function \( g : \Omega \to Y \) such that
\[ (4.3) \quad T(I(A)) = \tilde{I}(A) = \int_A g \, d\nu \quad \text{for all} \quad A \in \Sigma. \]
The Bochner integrability of \( g \) implies that
\[ g(\omega) \in H := \left\{ \frac{1}{\nu(A)} \int_A g \, d\nu : A \in \Sigma, \nu(A) > 0 \right\}^{\|\cdot\|} \quad \text{for} \quad \nu \text{-a.e.} \ \omega \in \Omega, \]
see e.g. [26, Lemma 4.3] (cf. [8, Lemma 3.7]), so we can assume without loss of generality that \( g(\Omega) \subseteq H \). On the other hand, (4.3) yields
\[ H \subseteq T(cB_X)^{\|\cdot\|} = T(cB_X), \]
where the equality holds because \( T \) is a semi-embedding. Hence \( g(\Omega) \subseteq T(cB_X) \), so there is a function \( f : \Omega \to cB_X \) such that \( T \circ f = g \).

We claim that \( f \) is Pettis integrable. Indeed, \( f \) is \( T^*(Y^*) \)-integrable by Proposition 4.6. Bearing in mind that \( X \) has property \( (D') \), an appeal to Theorem 3.2 ensures that \( f \) is Pettis integrable. Since \( T \) is injective and
\[ T\left( \int_A f \, d\nu \right) = \int_A T \circ f \, d\nu = \int_A g \, d\nu \overset{\text{4.3}}{=} T(I(A)) \quad \text{for all} \quad A \in \Sigma, \]
we have \( \int_A f \, d\nu = I(A) \) for every \( A \in \Sigma \). This proves that \( X \) has the \( \nu \)-WRNP. \( \square \)

**Remark 4.9.** In the previous proof, an alternative way to check the Pettis integrability of \( f \) is as follows. Since \( f \) is \( T^*(Y^*) \)-scalarly measurable and \( X \) has property \( (D') \), \( f \) is scalarly measurable. On the other hand, \( f \) is bounded, hence the \( \nu \)-PIP of \( X \) ensures that \( f \) is Pettis integrable.

At this point it is convenient to recall that the RNP (resp. WRNP) is equivalent to the \( \lambda \)-RNP (resp. \( \lambda \)-WRNP), where \( \lambda \) is the Lebesgue measure on \( [0,1] \), see e.g. [13, p. 138, Corollary 8] (resp. [29, Theorem 11.3]).

**Remark 4.10.** The conclusion of Theorem 4.8 might fail if \( Y \) is only assumed to have the WRNP. Indeed, let \( Y \) be any Banach space having the WRNP but failing the RNP (e.g. \( Y = JT^* \), where \( JT \) is the James tree space). Since the RNP is a separably determined property (see e.g. [13, p. 81, Theorem 2]), there is a separable subspace \( X \subseteq Y \) without the RNP. Then \( X \) has property \( (D') \) and the inclusion operator from \( X \) into \( Y \) is a semi-embedding, but \( X \) fails the WRNP (which is equivalent to the RNP for separable Banach spaces).

**Remark 4.11.** Theorem 4.8 cannot be improved by replacing the WRNP of \( X \) by the RNP, even if \( X \) has \( w^* \)-angelic dual. Indeed, let \( Z \) be a separable Banach space not containing subspaces isomorphic to \( \ell_1 \) such that \( X := Z^* \) is not separable (e.g. \( Z = JT \)). Then \( (X^*, w^*) \) is angelic (as we already pointed out in Section 2),
X fails the RNP and there is a semi-embedding $T : X \to \ell_2$ (because $X$ is the dual of a separable Banach space, see e.g. [7 Lemma 4.1.12]).

The $\nu$-WRNP and the $\nu$-RNP are equivalent for weakly measure-compact Banach spaces (thanks to [15 Proposition 5.4]). Since every WLD Banach space is weakly measure-compact and has property $(D')$, from Theorem 4.8 we get:

Corollary 4.12. Let $T : X \to Y$ be a semi-embedding. If $X$ is WLD and $Y$ has the $\nu$-RNP, then $X$ has the $\nu$-RNP.

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