Supplement to Neuschel’s paper “Asymptotics for Ménage polynomials and certain hypergeometric polynomials of type $3F_1$”

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Abstract

Neuschel investigated the asymptotic expansion of certain hypergeometric polynomials of type $3F_1$ inside and outside a closed curve. We supplement this result by studying a subfamily of those polynomials on a part of the closed curve.

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1 Introduction

In [5], Neuschel studied the asymptotic expansion of the polynomials $F_n(z)$ defined by

$$F_n(z) = {}_3F_1\left(\begin{array}{cc}-n & n & \alpha \\ 1/2 & & \end{array} \bigg| \frac{z}{2n}\right),$$

where $\alpha$ is a positive integer and

$${}_3F_1\left(\begin{array}{cc}-n & n & \alpha \\ 1/2 & & \end{array} \bigg| \frac{z}{2n}\right) = \sum_{k=0}^{n} \frac{(-n)_k(n)_k(\alpha)_k}{(1/2)_k k!} z^k.$$

The polynomials $F_n(z)$ are concerned with what are called Ménage polynomials that appear in combinatorics ([5]).

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Calculating the behavior of $F_n(z)$ as $n \to \infty$ is a part of the vast field of asymptotic analysis of hypergeometric quantities. One can find many formulas and references in [2]. Relatively little is known about $\,\_3F_1$ and [5] is one of rare major results.

Set

$$[-i, i] = \{ i s \mid -1 \leq t \leq 1 \}, \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}, \mathbb{D}^c = \{ z \in \mathbb{C} \mid |z| > 1 \}.$$

Let the mapping

$$\mathbb{C} \setminus [-i, i] \to \mathbb{D}^c, z \mapsto z + \sqrt{z^2 + 1}$$

be defined as the inverse mapping of the conformal mapping (a variant of the Joukowsky transformation)

$$\mathbb{D}^c \to \mathbb{C} \setminus [-i, i], w \mapsto \frac{1}{2} \left( w - \frac{1}{w} \right).$$

The mapping

$$\varphi(z) = \left( z + \sqrt{z^2 + 1} \right) \exp \left( \frac{-2}{z + \sqrt{z^2 + 1} - 1} - 1 \right) = \left( z + \sqrt{z^2 + 1} \right) \exp \left( -\frac{1}{z} - \frac{\sqrt{z^2 + 1}}{z} \right)$$

is defined accordingly. We stipulate that the value of the mapping \([1]\) for $z = iy \in [-i, i]$ ($-1 \leq y \leq 1$) is $iy + \sqrt{1 - y^2}$, which is on the right half of the unit circle. The curve $\mathcal{C}$ is defined by $|\varphi(z)| = 1$ and it contains the line segment $[-i, i]$. Although $\mathcal{C}$ is continuous, it is not differentiable at $\pm i$.

Let $\mathcal{E}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ be the exterior and the interior respectively. Then the main result of [5]
is the following. In $\mathcal{E}(\mathbb{C})$, one has

$$F_n(z) = 3F_1\left(\begin{array}{c}
-n \\
1/2 \\
\end{array} \begin{array}{c}
\alpha \\
\frac{z}{2n} \\
\end{array} \right) \sim \frac{(-1)^n}{\Gamma(\alpha)} n^{-\frac{1}{2}} \sqrt{\frac{\pi}{2}} \left(\frac{1}{z} + \frac{\sqrt{z^2 + 1}}{z}\right)^{\alpha-1} \left(\frac{\sqrt{z^2 + 1}}{z}\right)^{-\frac{1}{2}} \varphi(z)^n$$

as $n \to \infty$. Notice that $|\varphi(z)| > 1$ in $\mathcal{E}(\mathbb{C})$. On the other hand, in $\mathcal{I}(\mathbb{C})$ one has

$$F_n(z) = 3F_1\left(\begin{array}{c}
-n \\
1/2 \\
\alpha \\
\end{array} \begin{array}{c}
\frac{z}{2n} \\
\end{array} \right) \sim \left(\frac{2}{n}\right)^\alpha \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}} \left(\frac{-1}{z}\right)^\alpha.$$

The behavior on $\mathbb{C}$ remained an open problem. In the present paper, we give some information about the case of $\alpha = 1$, $z \in [-i, i] \subset \mathbb{C}$. Notice that the value at $z = 0$ is trivial.

## 2 Finite Fourier transform of the Chebyshev polynomials

The Jacobi polynomials are defined by

$$P_n^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta (-1)^n \frac{d^n}{2^n n!} \left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{\alpha+n}{k} \binom{\beta+n}{n-k} (x-1)^{n-k} (x+1)^k \ (\alpha, \beta > -1).$$

The Chebyshev polynomials are

$$T_n(x) = \frac{(-1)^n}{(2n-1)!} (1-x^2)^{1/2} \frac{d^n}{dx^n} (1-x^2)^{n-1/2} = \frac{n!}{(\frac{1}{2})_n} P_n^{(-1/2,-1/2)}(x),$$

and we have

$$T_n(\cos \theta) = \cos n\theta.$$ 

According to [4], the finite Fourier transform of the Jacobi polynomials is given by

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(t) e^{i\lambda t} dt$$

$$= \frac{(\beta+1)_n}{i\lambda n!} (-1)^{n+1} e^{-i\lambda} 3F_1\left(\begin{array}{c}
n + \alpha + \beta + 1 \\
\beta + 1 \\
\end{array} \begin{array}{c}
\frac{1}{2i\lambda} \\
1 \\
\end{array} \right) + \frac{(\alpha+1)_n}{i\lambda n!} e^{i\lambda} 3F_1\left(\begin{array}{c}
n + \alpha + \beta + 1 \\
\alpha + 1 \\
\end{array} \begin{array}{c}
\frac{1}{2i\lambda} \\
1 \\
\end{array} \right).$$
Therefore by setting $\alpha = \beta = -1/2, \lambda = -n/y$, we get

$$S = \frac{in}{y} \int_{-1}^{1} T_n(t) e^{-int/y} dt,$$  

(2)

where

$$S := (-1)^{n+1} e^{in/y} \text{}_3F_1\left( \begin{array}{ccc} n & -n & 1 \\ 1/2 & | \frac{-iy}{2n} \end{array} \right) + e^{-in/y} \text{}_3F_1\left( \begin{array}{ccc} n & -n & 1 \\ 1/2 & | \frac{iy}{2n} \end{array} \right).$$

If $y$ is real, we have

$$S = \begin{cases} 
2i \text{Im} \left\{ e^{-in/y} \text{}_3F_1\left( \begin{array}{ccc} n & -n & 1 \\ 1/2 & | \frac{iy}{2n} \end{array} \right) \right\} & (n: \text{even}), \\
2 \text{Re} \left\{ e^{-in/y} \text{}_3F_1\left( \begin{array}{ccc} n & -n & 1 \\ 1/2 & | \frac{iy}{2n} \end{array} \right) \right\} & (n: \text{odd}).
\end{cases}$$

(3)

Our aim is to calculate the asymptotic behavior of $S$ by using (2). If $0 < |y| \leq 1$, the values of $\text{}_3F_1$ corresponds to the polynomial studied in [5] with $z = iy \in [-i, i] \setminus \{0\}$. Notice that the case of $z = iy = 0$ is trivial.

### 3 Asymptotic expansion on $[-i, i]$

In view of (2), it is enough to calculate

$$I_n = \int_{-1}^{1} T_n(t) \exp \left( -\frac{int}{y} \right) dt,$$

when $-1 \leq y \leq 1, y \neq 0$. Set $t = \cos \theta \ (0 \leq \theta \leq \pi)$. Then we have

$$I_n = \frac{1}{2} (I_n^+ + I_n^-), \quad I_n^\pm = \int_{0}^{\pi} \exp \left( in \left[ \frac{-\cos \theta}{y} \pm \theta \right] \right) \sin \theta \, d\theta.$$  

(4)

Now we apply the classical method of stationary phase ([1, 3]) as opposed to the saddle point method employed in [5]. Set

$$\varphi_{\pm}(\theta) = -\frac{\cos \theta}{y} \pm \theta,$$

then $\varphi'_{\pm}(\theta) = y^{-1}(\sin \theta \pm y), \varphi''_{\pm}(\theta) = y^{-1} \cos \theta$. If $0 < y \leq 1$, $\varphi_{+}(\theta)$ never vanishes.
3.1 Behavior at the interior of the line segment

We consider the case \( z = iy \) \((0 < y < 1)\). We have \( I^+_n = O(1/n) \) because \( \varphi'_+ \) never vanishes. This implies, by (2) and (4),

\[
S = -\frac{in}{y} I_n = -\frac{in}{2y} I_n^- + o(1). \tag{5}
\]

On the other hand, the asymptotic behavior of \( I^- \) can be calculated by using the method of stationary phase. The phase function \( \varphi_- \) has two stationary points \( \sin^{-1} y, \pi - \sin^{-1} y \) on \( 0 \leq \theta \leq \pi \). We have

\[
\begin{align*}
\varphi_-(\sin^{-1} y) &= -\sqrt{1 - y^2}/y - \sin^{-1} y, \\
\varphi_-(\pi - \sin^{-1} y) &= \sqrt{1 - y^2}/y + \sin^{-1} y - \pi, \\
\varphi''_-(\sin^{-1} y) &= \sqrt{1 - y^2}/y, \quad \varphi''_-(\pi - \sin^{-1} y) = -\sqrt{1 - y^2}/y,
\end{align*}
\]

Summing up the contribution from the two stationary points ([1, Lemma 6.3.3], [3, p.51]), we obtain

\[
I^-_n \sim \begin{cases} 
2y \left( \frac{2\pi y}{n\sqrt{1 - y^2}} \right)^{1/2} \cos \left[ n \left( \frac{\sqrt{1-y^2}}{y} + \sin^{-1} y \right) - \frac{\pi}{4} \right] & (n \text{ : even}), \\
-2iy \left( \frac{2\pi y}{n\sqrt{1 - y^2}} \right)^{1/2} \sin \left[ n \left( \frac{\sqrt{1-y^2}}{y} + \sin^{-1} y \right) - \frac{\pi}{4} \right] & (n \text{ : odd}).
\end{cases}
\]

Therefore we obtain, by (3) and (5), the following result.

**Theorem 1.** If \( 0 < y < 1 \), we have

\[
\begin{align*}
\text{Im} & \left\{ e^{-in/y} \text{}_3F_1 \left( \begin{array}{c} n \ -n \ 1 \\ 1/2 \end{array} \right| \frac{i y}{2n} \right\} \\
& \sim \begin{cases} 
\left( \frac{n\pi y}{2\sqrt{1 - y^2}} \right)^{1/2} \cos \left[ n \left( \frac{\sqrt{1-y^2}}{y} + \sin^{-1} y \right) - \frac{\pi}{4} \right] & (n \text{ : even}), \\
\left( \frac{n\pi y}{2\sqrt{1 - y^2}} \right)^{1/2} \sin \left[ n \left( \frac{\sqrt{1-y^2}}{y} + \sin^{-1} y \right) - \frac{\pi}{4} \right] & (n \text{ : odd})
\end{cases}
\end{align*}
\]

as \( n \to \infty \). The behavior on \(-1 < y < 0\) can be obtained by complex conjugation.

**Remark 2.** Only either the real or the imaginary part has been calculated here. This is less than satisfactory but at least it has been proved that the asymptotic behavior is different from that in \( \mathcal{E}(C) \) (power-times-exponential growth with oscillation) and \( \mathcal{I}(C) \) (decay of order \(-\alpha = -1\) with no oscillation). In the next section, we will see still another type of behavior at \( z = \pm i \). Nothing is known about the remaining part of \( C \).
3.2 Behavior at the end points

Assume \( z = i (y = 1) \). Since

\[
\varphi_-(\pi/2) = -\pi/2, \quad \varphi'_-(\pi/2) = \varphi''_-(\pi/2) = 0, \quad \varphi'''_-(\pi/2) = -1,
\]

we get

\[
I_n^- \sim \frac{\Gamma(1/3)(-i)^n}{\sqrt{3}} \left( \frac{6}{n} \right)^{1/3}.
\]

Therefore, we have the following.

**Theorem 3.** The asymptotic behavior at \( z = i \) as \( n \to \infty \) is

\[
\text{Im} \left\{ e^{-in/y} \, {}_3F_1 \left( \begin{array}{c} n \ -n \ 1 \\ 1/2 \ 2n \end{array} \right) \right\} \sim \frac{6^{1/3} \Gamma(1/3)(-1)^{n/2}}{4\sqrt{3}y} n^{2/3} (n : \text{even}),
\]

\[
\text{Re} \left\{ e^{-in/y} \, {}_3F_1 \left( \begin{array}{c} n \ -n \ 1 \\ 1/2 \ 2n \end{array} \right) \right\} \sim \frac{6^{1/3} \Gamma(1/3)(-1)^{(n+1)/2}}{4\sqrt{3}y} n^{2/3} (n : \text{odd}).
\]

The behavior at \( z = -i \) can be obtained by complex conjugation.

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