Experimental proposal for measuring the Gouy phase of matter waves

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Abstract. The Schrödinger equation for an atomic beam predicts that it must have a phase anomaly near the beam waist analogous to the Gouy phase of an electromagnetic beam. We propose here a feasible experiment that allows for direct determination of this anomalous phase using Ramsey interferometry with Rydberg atoms. Possible experimental limitations are discussed, and shown to be completely under control within present-day technology. We also discuss how this finding can open the possibility of using the spatial mode wavefunctions of atoms as q-dits, since the Gouy phase is an essential ingredient for making rotations in the quantum states.

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1. Introduction

As early as 1890, Gouy [1, 2] demonstrated the existence of an anomalous phase, which carries his name, the Gouy phase, using classical electromagnetic fields in the paraxial wave regime. This anomaly exists for any wave, including acoustic waves, and results in a phase advance of $\pi$ for the amplitude of a Gaussian beam after passing through the focus, propagating from $-\infty$ to $+\infty$, in relation to a plane wave. This phase anomaly has important consequences for optics and has been highly exploited in recent experiments [3–9].

It was soon realized that the paraxial wave equation for light is formally equivalent to the two-dimensional (2D) Schrödinger equation for a free particle [10–13]. This implies that the Gouy phase, which is present in the solutions of the paraxial equation, must also be present in the solutions of the Schrödinger equation for a free particle. Recently, we showed [12] that the Gouy phase for matter waves is compatible with diffraction experiments performed with fullerene molecules [14] by noting that the matter counterpart of the Gouy phase is directly related to the covariance $\Delta x p = \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle / 2$, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators for the molecules’ center of mass in the $x$-direction. However, this is indirect evidence, and a direct measurement of the Gouy phase for matter waves might prove to be an interesting research tool. With the experimental advance that is actually in progress, the Gouy phase may soon acquire the same importance for matter waves as it already has for electromagnetic waves.

The paraxial Helmholtz equation describes the behavior of the amplitude $A(x, y, z)$ of an electromagnetic wave written as $E(r, t) = A(x, y, z) \exp(ikz - i\omega t)$ when $A(x, y, z)$ varies slowly with $z$ such that $\frac{\partial^2 A}{\partial z^2}$ may be disregarded in relation to $k \frac{\partial A}{\partial z}$. With this consideration, the wave equation gives [15]

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik \frac{\partial}{\partial z} \right] A(x, y, z) = 0. \quad (1)$$

The above equation is completely equivalent to the 2D Schrödinger equation for a free particle if we make the substitutions $k \rightarrow m/\hbar$ and $z \rightarrow \tau$, where $m$ is the mass of the particle, with the particle wavefunction $\psi(x, y, t)$ in the place of $A(x, y, z)$. One particular solution of the above equation is the Gaussian beam in the $x$-direction, where we disregard the behavior of the beam in the $y$-direction [15]

$$A_G(x, z) = A_0 \frac{w_0}{w(z)} \exp\left[ -\frac{x^2}{w(z)^2} + i \frac{kx^2}{2R(z)} - i\xi(z) \right]. \quad (2)$$
where the beam width \( w(z) \), the curvature radius \( R(z) \), the Gouy phase \( \xi(z) \) and the Rayleigh range \( z_0 \) are given by

\[
\begin{align*}
  w(z) &= w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}, & R(z) &= z \left[1 + \left(\frac{z_0}{z}\right)^2\right], \\
  \xi(z) &= \frac{1}{2} \arctan\left(\frac{z}{z_0}\right) \quad \text{and} \quad z_0 = \frac{k w_0^2}{2}.
\end{align*}
\]

The total variation of the Gouy phase when we go from \( z = -\infty \) to \( z = +\infty \) with this beam is \( \pi/2 \). If the \( y \) dependence of the amplitude is the same as the \( x \) dependence, the Gouy phase for the \( y \) confinement is summed to the phase due to the \( x \) confinement and a total phase of \( \pi \) is obtained. However, in our proposal the atomic beam, which will have behavior analogous to an electromagnetic beam, will be confined only in the \( x \)-direction. We will also consider that the energy associated with the momentum of the atoms in the \( z \)-direction is very high, such that we can consider a classical movement of atoms in this direction, with the time component given by \( t = z/v_z \). So the \( x \)-component of the atomic wave function will have the same form as (2) with the substitutions \( k \rightarrow m/\hbar \) and \( z \rightarrow z/v_z \).

2. Experimental proposal

Atoms interacting with electromagnetic fields can suffer mechanical effects, such as deflections or deviations of the motion of their center of mass [16, 17]. This property is useful in atomic optics, since it allows for the construction of devices that focus matter beams [13, 18–20], in an analogous way to the focalization of light beams by ordinary lenses. In wave optics, it is well known that the fields suffer the Gouy phase shift in the focus region, like the Gaussian beam described above around \( z = 0 \). In a previous work we explicitly showed that the same anomaly should occur around the focus of an atomic beam [13]. In order to experimentally observe this effect, we propose an experiment with a focused Gaussian atomic beam. We will use a cylindrical focusing in the \( x \)-direction, without changing the beam wavefunction in the \( y \)-direction, which makes the total Gouy phase \( \pi/2 \).

The experimental setup we propose for measuring the Gouy phase shift of matter waves is depicted in figure 1. This proposal is based on the system of [21]. Rubidium atoms are excited by laser to a circular Rydberg state with the principal quantum number 49 [22, 23], which will be called state \( |i⟩ \), and their velocity in the \( z \)-direction is chosen to have a fixed value \( v_z \). As was stated before, we will consider a classical movement of atoms in this direction, with the time component given by \( t = z/v_z \). A slit is used to prepare a beam with small width in the \( x \)-direction, but still without a significant divergence, such that the consideration that the atomic beam has plane-wave behavior is a good approximation. The relevant atomic states for the experiment, \( |g⟩, |e⟩ \) and \( |i⟩ \), along with the transition frequencies among them, are illustrated in figure 2. If we disregard the cavities \( C_1 \) and \( C_2 \), the setup is that of an atomic Ramsey interferometer [24]. The cavity \( R_1 \) has a field resonant or quasi-resonant with the transition \( |i⟩ \leftrightarrow |g⟩ \) and results in a \( \pi/2 \) pulse on the atoms, which exit the cavity in the state \( (|i⟩ + |g⟩) / \sqrt{2} \) [21, 24, 25]. After passing through the cavity \( R_1 \), the atoms propagate freely for time \( t \) until the cavity \( R_2 \), which also makes a \( \pi/2 \) pulse on the atoms. Calling \( \hbar \omega_g \) and \( \hbar \omega_i \) the energy of the internal states \( |g⟩ \) and \( |i⟩ \), respectively, \( \omega_r \) the frequency of the field in the cavities \( R_1 \) and \( R_2 \) and defining \( \omega_{gi} = \omega_g - \omega_i \), the probability that detector \( D \) measures each atom in

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Figure 1. Sketch of the experimental setup for measuring the Gouy phase for matter waves. Rydberg atoms are sent one by one with well-defined velocity along the z-axis. A slit is used to collimate the atomic beam in the x-direction. The Ramsey zones $R_1$ and $R_2$ are two microwave cavities fed by a common source $S$, whereas $C_1$ and $C_2$ are two high-$Q$ microwave cavities devised to work as thin lenses for the atomic beam. The field inside these cavities is supplied by a common source $S'$. The state of each atom is detected by the detector $D$.

Figure 2. Atomic energy levels compared with the frequency of the field inside the cavities $C_1$ and $C_2$.

the $|g\rangle$ state is $P_0 = \cos^2[(\omega_r - \omega_{gi})t]$ [21, 24, 26]. On slightly varying the frequency $\omega_r$ of the fields in cavities $R_1$ and $R_2$, the interference fringes can be seen [21, 24, 26].

The cavities $C_1$ and $C_2$ act as selective atomic lenses. Our description of atomic lenses is based on [19]. The frequency of the fields in these cavities $\omega$ is supposed to be strongly detuned from the atomic transition frequencies, not changing the internal states of the atoms with their passage. In this case, we can describe the interaction by an effective Hamiltonian [19] $H_{\text{eff}}^{(\omega)} = -\alpha_n|E(r)|^2$, where $E(r)$ represents the electric field in the cavity and $\alpha_n$ the atomic linear
susceptibility, which depends on the internal state \(|n\rangle\) of the atom

\[
\alpha_n = \sum_m \frac{|\langle m | \hat{P} | n \rangle|^2}{\hbar} \left[ \frac{1}{(\omega_m - \omega_n) + \omega} + \frac{1}{(\omega_m - \omega_n) - \omega} \right],
\]

where \(\hbar \omega_n\) is the energy of the level \(|n\rangle\) and \(\hat{P}\) an operator that corresponds to the atomic electric dipole moment. When the field frequency \(\omega\) is much closer to one particular difference \(\omega_m - \omega_n\) (but not sufficiently close to induce a transition), we can consider only the last term in square brackets in the sum above. So we can write the atomic linear susceptibilities for the states \(|i\rangle\) and \(|g\rangle\) as

\[
\alpha_i = \frac{|\langle g | \hat{P} | i \rangle|^2}{\hbar \Delta_i}, \quad \alpha_g = \frac{|\langle e | \hat{P} | g \rangle|^2}{\hbar \Delta_g},
\]

with \(\Delta_g \equiv (\omega_g - \omega_g) - \omega\) and \(\Delta_i \equiv (\omega_g - \omega_g) - \omega\). If we choose a frequency \(\omega\) such that \(\Delta_g \ll |\Delta_i|\), it is possible that the component with internal state \(|g\rangle\) of the atomic wavefunction suffers the influence of the cavity field, while the component with internal state \(|i\rangle\) does not.

If the field in the cavities \(C_1\) and \(C_2\) has an electric field node in the axis of the atomic beam and the width of the atomic beam is much smaller than the wavelength of the cavity field, we can consider the following Hamiltonian for the \(|g\rangle\) part of the atomic wavefunction in the cavity, expanded up to the second order in \(x\) [13, 19, 20]:

\[
H_{\text{eff}}^{(g)}(x) = -\alpha_g |E_0|^2 \sin^2(2\pi x/\lambda) \simeq -\alpha_g |E_0|^2 \left(\frac{2\pi}{\lambda}\right)^2 x^2
\]

\[
\simeq -\hbar N\Omega^2 \Delta_g \left(\frac{2\pi}{\lambda}\right)^2 x^2,
\]

where \(\lambda\) is the wavelength of the cavity field, \(|E_0|^2\) corresponds to an effective square of the electric field on an antinode of the cavity, which is an average of \(|E(r)|^2\) on the \(z\) position inside the cavity in an antinode, \(\Omega = |\langle g | \hat{P} | i \rangle \cdot E_0 / \hbar\) is the Rabi frequency per photon of the system multiplied by \(2\pi\) and \(N\) is the number of photons in each cavity. Let us disregard the kinetic energy term of the Hamiltonian inside the cavity [19]. If the atoms enter the cavity with a wavefunction \(\psi_0(x)|i\rangle + |g\rangle/\sqrt{2}\) and interact with the cavity field during an effective time \(t_i\), the \(|g\rangle\) component of the atomic wavefunction will exit the cavity as

\[
\psi'(x)|g\rangle = e^{-imx^2/(2\hbar)} \psi_0(x)|g\rangle e^{-i\omega_gt_i},
\]

with

\[
1/t_F = -\frac{2\hbar t_i N\Omega^2}{m \Delta_g} \left(\frac{2\pi}{\lambda}\right)^2.
\]

An optical converging cylindrical lens with focal distance \(f\) puts a quadratic phase \(-kx^2/(2f)\) on the electromagnetic beam [15]. By the analogy of the Schrödinger equation with the paraxial Helmholtz equation [10–12], we see that the cavities act on the \(|g\rangle\) component of the atomic beam as cylindrical lenses with ‘focal time’ \(t_F\) and focal distance \(z_F = v_z t_F\). If we have \(z_F = d/2\), where \(d\) is the distance between the cavities \(C_1\) and \(C_2\), the system will behave like the illustration in figure 3. The cavity \(C_1\) will transform the \(|g\rangle\) component of the wavefunction in a converging beam with the waist at a distance \(d/2\) (represented by solid lines). After its waist, the beam will diverge until the cavity \(C_2\). The \(|g\rangle\) component of the wavefunction at the
Figure 3. Illustration of the operation of the cavities $C_1$ and $C_2$ as thin lenses for the atomic beam. The dashed lines represent the width of the atomic beam if the cavities are empty. If a field is present, the solid lines represent the width of a beam composed of atoms in the state $|g\rangle$. $F$ denotes the focus region. On the other hand, if the beam is composed of atoms in the state $|i\rangle$, the width does not change significantly.

position of the cavity $C_2$ will have the same width and opposite quadratic phase of the state $\psi'(x)$ above, so the cavity $C_2$ will transform the divergent beam into a plane-wave beam again. The $|i\rangle$ component of the wavefunction, on the other hand, propagates as a plane-wave beam all the time (represented by dashed lines), as its interaction with the field of the cavities $C_1$ and $C_2$ is considered to be very small.

If we now repeat the Ramsey interference experiment, we will observe a change of the positions of the fringes, because now the $|g\rangle$ component acquires a $\pi/2$ Gouy phase due to the cylindrical focusing that is not shared by the $|i\rangle$ component. So, the interference pattern will be $P' = \cos^2((\omega_r - \omega_{gi})t - \pi/2)$. The difference of the positions of the minimums and maximums of the patterns, one constructed when the field that forms the atomic lenses is present on the cavities $C_1$ and $C_2$ and other constructed when the field is removed, should attest to the existence of the Gouy phase for matter waves.

3. Experimental parameters and discussion

As experimental parameters, we propose the velocity of the atoms $v_z = 50 \text{ m s}^{-1}$ and a slit that generates an approximately Gaussian wavefunction for the atoms $\psi_0(x) \propto e^{-x^2/w_0^2}$ with $w_0 = 10 \text{ m}\mu$. The mass of rubidium is $m = 1.44 \times 10^{-25} \text{ kg}$. With these parameters, the Rayleigh range of the atomic beam will be $z_r = k_z w_0^2/2 \approx 3.5 \text{ m}$ (where $k_z = m v_z / \hbar$), much larger than the length of the experimental apparatus, which justifies the plane-wave approximation. On the cavities $C_1$ and $C_2$, we consider an interaction time between the atoms and atomic lenses $t_i = 0.2 \text{ ms}$, which corresponds to a width $v_z t_i = 1 \text{ cm}$ for the field on the cavities. The wavelength of the field of the cavities $C_1$ and $C_2$ must be $\lambda \approx 5.8 \text{ mm}$ [21], with frequency near but strongly detuned from the resonance of the transition $|g\rangle \leftrightarrow |e\rangle$. The Rabi frequency is about $\Omega/(2\pi) \approx 47 \text{ kHz}$ [21] and the detuning chosen is $\Delta_e/(2\pi) = -30 \text{ MHz}$, which makes $\Delta_e/(2\pi) = +3.2 \text{ GHz}$, such that with $N = 3 \times 10^6$ photons, an effective classical field, the focal distance for the atomic lenses is $10.5 \text{ cm}$ for the $|g\rangle$ component and $-11 \text{ m}$ for the $|i\rangle$ component of the wavefunction. These parameters are consistent with a separation of $d = 21 \text{ cm}$ between $C_1$ and $C_2$. All the proposed parameters can be experimentally achieved [21, 26, 27].
The Rayleigh range $z'_r$ and the beam waist $w'_0$ of the focused atomic beam can be calculated using the analogy with the action of lenses in electromagnetic beams considering that the incident beam has plane wavefronts [15]: $z'_r = z_r/[1 + (z_r/z_f)^2]$, $w'_0 = w_0/\sqrt{1 + (z_r/z_f)^2}$, where $z_r$ and $w_0$ are the Rayleigh range and the beam waist of the incident beam and $z_f$ is the focal distance of the atomic lens. Using the proposed parameters, we have $z'_r \simeq 3$ mm and $w'_0 \simeq 0.3 \mu$m. The fact that $d > z'_r$ justifies our consideration that the $|g\rangle$ component of the beam acquires a $\pi/2$ Gouy phase.

The interaction between the atomic beam and the field in the cavities $C_1$ and $C_2$ depends on the position $x$, according to (6). If we do not want that photons be absorbed by the atoms, it is important that $4\pi^2 N\Omega^2 x^2/(\Delta^2 \lambda^2) \ll 1$ for the entire beam [28]. We have $4\pi^2 N\Omega^2 w'_0/(\Delta^2 \lambda^2) \simeq 8 \times 10^{-4}$ for the proposed parameters, where $w_0$ is the beam width, showing that the absorption of photons can be disregarded. It is also important that the cavities $C_1$ and $C_2$ have a large quality factor $Q$. This occurs becase in (6) it was considered that the intensity of the electric field is exactly 0 in $x = 0$, which is impossible for real cavities. In fact, the ratio between the maximum and the minimum of intensity in a cavity should roughly be equal to the quality factor $Q$. So, the $|g\rangle$ component of the beam also acquires a phase $N\Omega^2 t_i/(\Delta \sqrt{Q})$ on the passage in each cavity, and this phase will be added to the accumulated Gouy phase. If we want that this undesired phase be smaller than $\pi/20$, we need $Q > 4 \times 10^6$ for our proposed parameters. This can also be experimentally achieved [21, 26, 27].

In our treatment of the atomic lenses, we disregarded the kinetic energy term of the atoms in the Hamiltonian. To verify that this assumption is in fact reasonable, we performed the exact calculations for the problem, considering the kinetic energy term in the Hamiltonian following [20], section 20.4. The basic difference in the results for the proposed parameters was a difference inferior to 2% for the focal distance of the atomic lenses. So, disregarding the kinetic energy term does not represent a problem.

In our proposal, we used one particular combination of parameters for the experiment within today’s experimental capabilities. It is important to stress that many of the proposed parameters can be varied in a wide range, making it possible to choose the most appropriate ones in an experiment.

The Gouy phase for matter waves could have important applications in the field of quantum information. The transversal wavefunction of an atom in a beam state can be treated not only as a continuous variable system, but also as an infinite-dimensional discrete system. The atomic wavefunction can be decomposed into Hermite–Gaussian or Laguerre–Gaussian modes in the same way as an optical beam [15], which form an infinite discrete basis. This basis was used, for instance, to demonstrate entanglement in a two-photon system [29]. However, it is essential, for realizing quantum information tasks, that we have the ability to transform the states from one mode to another, making rotations in the quantum state. This can be done using the Gouy phase, constructing mode converters in the same way as for light beams [30, 31]. The mode converters can transform any mode (Hermite–Gaussian or Laguerre–Gaussian) into another mode of the same order. A mode converter is composed simply of two cylindrical lenses that can focus and collimate the beam in one direction. As different modes in general acquire different Gouy phases through focalization in one direction [15], for a beam in a superposition of modes, phase differences between the components can be included, and a combination of these converters is sufficient for making transformations between any two modes of the same order [31]. So, the Gouy phase may make it possible to use atomic beams of a determinate order in quantum information schemes as q-dits. Recently, electron beams in the Laguerre–Gaussian modes were
constructed [32, 33], and the same technique as that of [33], which uses diffraction gratings to generate the beams, can be used to generate atomic Laguerre–Gaussian beams to be used in such schemes. It should also be possible to implement a similar scheme with trapped atoms, since the Hermite–Gaussian modes are the eigenstates of harmonic oscillators. In this case, focalization of the wavefunction could be done turning electromagnetic fields on and off.

4. Conclusion

We have proposed a feasible experiment to directly measure the Gouy phase for matter waves using atomic Ramsey interferometry. The experimental parameters necessary for the implementation were shown to be accessible under current technology. The verification of the Gouy phase in matter waves has the possibility of generating a great deal of development in atomic optics, in the same way as the electromagnetic counterpart Gouy phase had contributed to electromagnetic optics. For instance, it can be used to construct mode converters for atomic beams and trapped atoms, with potential applications in quantum information.

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References

[1] Gouy L G 1890 C. R. Acad. Sci. Paris 110 1251
[2] Gouy L G 1891 Ann. Chim. Phys. Ser. 6 24 145
[3] Kandpal H C, Raman S and Methrotra R 2007 Opt. Lasers Eng. 45 249
[4] Ruffin A B, Rudd J V, Whitaker J F, Feng S and Winful H G 1999 Phys. Rev. Lett. 83 341
[5] Feurer T, Stoyanov N S, Ward D W and Nelson K A 2002 Phys. Rev. Lett. 88 257402
[6] Lindner F et al 2004 Phys. Rev. Lett. 92 113001
[7] Klaassen T, Hoogeboom A, van Exter M P and Woerdman J P 2004 J. Opt. Soc. Am. A 21 1689
[8] Zhu W, Agrawal A and Nahata A 2007 Opt. Express 15 9995
[9] Kawase D, Miyamoto Y, Takeda M, Sasaki K and Takeuchi S 2008 Phys. Rev. Lett. 101 050501
[10] Berman P R 1997 Atom Interferometry (San Diego, CA: Academic)
[11] Marte M A M and Stenholm S 1997 Phys. Rev. A 56 2940
[12] da Paz I G, Nemes M C, Pádua S, Monken C H and Peixoto de Faria J G 2010 Phys. Lett. A 374 1660
[13] da Paz I G, Nemes M C and Peixoto de Faria J G 2007 J. Phys.: Conf. Ser. 84 012016
[14] Nairz O, Arndt M and Zeilinger A 2002 Phys. Rev. A 65 032109
[15] Saleh B E A and Teich M C 1991 Fundamentals of Photonics (New York: Wiley)
[16] Ashkin A 1970 Phys. Rev. Lett. 24 156
[17] Gordon J P and Ashkin A 1980 Phys. Rev. A 21 1606
[18] Bjorkholm J E, Freeman R R, Ashkin A and Pearson D B 1978 Phys. Rev. Lett. 41 1361
[19] Averbukh I S, Akulin V M and Schleich W P 1994 Phys. Rev. Lett. 72 437
[20] Schleich W P 2001 Quantum Optics in Phase Space (Berlin: Wiley-VCH)
[21] Raimond J M, Brune M and Haroche S 2001 Rev. Mod. Phys. 73 565
[22] Nussenzveig P et al 1993 Phys. Rev. A 48 3991
[23] Gallagher T F 1994 Rydberg Atoms (Cambridge: Cambridge University Press)
[24] Ramsey N F 1985 *Molecular Beams* (New York: Oxford University Press)
[25] Kim J I et al 1999 *Phys. Rev. Lett.* 82 4737
[26] Nogues G et al 1999 *Nature* 400 239
[27] Gleyzes S et al 2007 *Nature* 446 297
[28] Scully M O and Zubary M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press)
[29] Mair A, Vaziri A, Weihs G and Zeilinger A 2001 *Nature* 412 313
[30] Allen L, Beijersbergen M W, Spreeuw R J C and Woerdman J P 1992 *Phys. Rev. A* 45 8185
[31] Beijersbergen M W, Allen L, van der Veen H E L O and Woerdman J P 1993 *Opt. Commun.* 96 123
[32] Uchida M and Tonomura A 2010 *Nature* 464 737
[33] McMorran B J et al 2011 *Science* 331 192