Instantons on Noncommutative $\mathbb{R}^4$ and Projection Operators

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Abstract

I carefully study noncommutative version of ADHM construction of instantons, which was proposed by Nekrasov and Schwarz. Noncommutative $\mathbb{R}^4$ is described as algebra of operators acting in Fock space. In ADHM construction of instantons, one looks for zero-modes of Dirac-like operator. The feature peculiar to noncommutative case is that these zero-modes project out some states in Fock space. The mechanism of these projections is clarified when the gauge group is $U(1)$. I also construct some zero-modes when the gauge group is $U(N)$ and demonstrate that the projections also occur, and the mechanism is similar to the $U(1)$ case. A physical interpretation of the projections in IIB matrix model is briefly discussed.

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1 Introduction

One of the most important reasons to consider physics on noncommutative spacetime is that the behavior of the theory at short distance is expected to become manageable due to the noncommutativity of the spacetime coordinates. It was shown [1] that noncommutative geometry appears in a definite limit of string theory, BFSS matrix theory [2] and IIB matrix theory [3]. In these cases noncommutativity should be relevant to the short scale physics of D-branes.

Among D-brane systems, Dp-brane-D(p+4)-brane bound states are of interest because this system has two different descriptions: the one in terms of the worldvolume theory of Dp-branes and another in terms of the worldvolume theory of D(p+4)-branes. The D-flat condition of the worldvolume theory of Dp-brane coincides with ADHM equations [5][6], and Dp-branes are described as instantons in D(p+4)-brane worldvolume theory. These descriptions should be equivalent because both describe the same system, and indeed the moduli space of the worldvolume theory of Dp-branes is identical to the instanton moduli space. Turning constant NS-NS B-field in the worldvolume of D(p+4)-branes causes noncommutativity in the worldvolume theory of D(p+4)-branes, and adds Fayet-Iliopoulos D-term in the worldvolume theory of Dp-branes [16][11]. The equivalence of two descriptions follows from the pioneering work of Nekrasov and Schwarz [12]. In order to construct instantons on noncommutative $\mathbb{R}^4$, one adds a constant (corresponding to the Fayet-Iliopoulos term) to the ADHM equations[1]. The modified ADHM equations describe the resolutions of singularities in the moduli space of instantons on $\mathbb{R}^4$. This moduli space has been an important clue to the nonperturbative aspects of string theory [17][18][19][20][21][22] and matrix theory [11][23][24][25][26]. Further studies from the viewpoints of both string theory and noncommutative geometry were recently given by [15]. More recently Braden and Nekrasov constructed instantons on blowups of $\mathbb{C}^2$, which are conjectured to be related to instantons on noncommutative $\mathbb{R}^4$ [13][14].

In [12] Nekrasov and Schwarz explicitly constructed some instanton solutions and showed that they are non-singular. An interesting point is that they are non-singular even if their commutative counterparts in original ADHM construction are singular, so-called small instantons. In these cases noncommutativity of the coordinates actually eliminates the singular behavior of the field configurations. What is special to the noncommutative case is the appearance of projection operators which project out potentially dangerous states in Fock space, where Fock space is introduced to describe the noncommutative $\mathbb{R}^4$ [12]. The purpose of this paper is to investigate this mechanism. It is shown that this mechanism has rich structures, and gives insight in the short scale structures near the core of instantons on noncommutative space. An important point is that the existence

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1 The case of equivariant instanton is studied in [14].
2 In most part of this paper I use the term “commutative” with usual commutative $\mathbb{R}^4$ in mind, and explicitly refer to [13] when I compare our noncommutative descriptions to their commutative descriptions.
of the projection forces us to express gauge fields in reduced Fock space where some of the states have been projected out. It is shown that this modification of the Fock space corresponds to the modification of the spacetime topology.

An outline of this paper is as follows. In section 2, gauge theory on noncommutative space and the ADHM construction on commutative \( \mathbf{R}^4 \) are briefly reviewed. In section 3, the ADHM construction on noncommutative \( \mathbf{R}^4 \) is studied. The reason why we must consider the projections is explained. In section 4, the mechanism of the projections is clarified when the gauge group is \( U(1) \), utilizing Nakajima’s beautiful results \([8][10]\). In section 5, it is demonstrated that the similar projections also occur in \( U(N) \) case. In section 6, embedding of the \( U(1) \) instanton solution to IIB matrix model is considered. The solution is understood as D-instantons within D3-brane in IIB matrix model. It is shown that the role of the projection is to remove anti-D-instantons and make holes on D3-brane worldvolume.

When the previous version of this paper was at the final stage of preparation, the paper \([13]\) appeared. Some issues discussed in this paper have commutative counterparts in \([13]\). Explanations on the modification of spacetime topology have been added after taking into due consideration of the relation to their work.\(^3\)

2 Preliminaries

In this section we briefly review the theory of gauge fields on noncommutative \( \mathbf{R}^4 \) and ADHM construction on commutative \( \mathbf{R}^4 \), as preliminaries to ADHM construction on noncommutative \( \mathbf{R}^4 \).

2.1 Gauge Fields on Noncommutative \( \mathbf{R}^4 \)

Noncommutative \( \mathbf{R}^4 \) is described by an algebra generated by \( x^\mu (\mu = 1, \ldots, 4) \) obeying the commutation relations:

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu},
\]

where \( \theta^{\mu\nu} \) is real and constant. In this paper we restrict ourselves to the case where \( \theta^{\mu\nu} \) is self-dual and set\(^4\)

\[
\theta^{12} = \theta^{34} = \frac{\zeta}{4}.
\]

\(^3\)I would like to thank N. Nekrasov for explaining their work to me, and pointing out my misleading statement in the concluding section in the earlier version of this paper.

\(^4\)See \([13]\) for the meaning of this choice of parameters in string theory.
Then the algebra depends only one constant parameter $\zeta$.

Introduce the generators of noncommutative $\mathbb{C}^2 \approx \mathbb{R}^4$ by

$$z_1 = x_2 + ix_1, \quad z_2 = x_4 + ix_3.$$  \hfill (2.3)

Their commutation relations are:

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\frac{\zeta}{2}, \quad \text{(others: zero).} \hfill (2.4)$$

We choose $\zeta > 0$. The commutation relations (2.1) have a group of automorphisms of the form $x^\mu \mapsto x^\mu + c^\mu$, where $c^\mu$ is a commuting real number. We denote the Lie algebra of this group by $\mathfrak{g}$. Following [12], we start with the algebra $\text{End } \mathcal{H}$ of operators acting in the Fock space $\mathcal{H} = \sum_{(n_1, n_2) \in \mathbb{Z}^2 \geq 0} \mathbb{C} |n_1, n_2 \rangle$, where $z, \bar{z}$ are represented as creation and annihilation operators:

$$\sqrt{2\zeta} z_1 |n_1, n_2 \rangle = \sqrt{n_1 + 1} |n_1 + 1, n_2 \rangle, \quad \sqrt{2\zeta} \bar{z}_1 |n_1, n_2 \rangle = \sqrt{n_1} |n_1 - 1, n_2 \rangle,$$

$$\sqrt{2\zeta} z_2 |n_1, n_2 \rangle = \sqrt{n_2 + 1} |n_1, n_2 + 1 \rangle, \quad \sqrt{2\zeta} \bar{z}_2 |n_1, n_2 \rangle = \sqrt{n_2} |n_1, n_2 - 1 \rangle. \hfill (2.5)$$

The algebra $\text{End } \mathcal{H}$ has a subalgebra of operators which have finite norm; we define the norm of operators by $||a|| := \sup ||a\phi||/||\phi||; \ a \in \text{End} \mathcal{H}, \ |\phi\rangle \neq 0, \ |\phi\rangle \in \text{Dom}(a) \subset \mathcal{H}$. $\text{Dom}(a)$ is a domain of operator $a$ and $||\phi|| := (\langle \phi | \phi \rangle)^{1/2}$. We denote this algebra by $\mathcal{A}_\zeta$. Whenever we consider the derivative of an operator $a \in \mathcal{A}_\zeta$, we assume that it is also contained in $\mathcal{A}_\zeta$, i.e. $\partial_\mu a \in \mathcal{A}_\zeta$ ($\partial_\mu$ is understood as the action of $\mathfrak{g} = \mathbb{R}^4$ on $\mathcal{A}_\zeta$ by translation). The $U(N)$ gauge field on noncommutative $\mathbb{R}^4$ is defined as follows. First we consider $N$-dimensional vector space $\mathcal{E} := (\mathcal{A}_\zeta)^{\otimes N}$ which carries a right representation of $\mathcal{A}_\zeta$:

$$\mathcal{E} \times \mathcal{A}_\zeta \ni (e, a) \mapsto ea \in \mathcal{E}, \quad e(ab) = (ea)b,$$

$$e(a + b) = ea + eb,$$

$$(e + e')a = ea + e'a, \hfill (2.6)$$

for any $e, e' \in \mathcal{E}$ and $a, b \in \mathcal{A}_\zeta$. The elements of $\mathcal{E}$ can be thought of as an $N$-dimensional vector with entries in $\mathcal{A}_\zeta$. Let us consider unitary action of $U$ on the element of $\mathcal{E}$:

$$e \rightarrow Ue, \hfill (2.7)$$

where $U$ is an $N \times N$ matrix with its components in $\mathcal{A}_\zeta$, satisfying $UU^\dagger = U^\dagger U = \text{Id}_\mathcal{H} \otimes \text{Id}_N$. $\text{Id}_\mathcal{H}$ is an identity operator in $\mathcal{A}_\zeta$ and $\text{Id}_N$ is an $N \times N$ identity matrix. Under this unitary transformation, $De$, the covariant derivative of $e \in \mathcal{E}$, is required to transform covariantly:

$$De \rightarrow UD_e. \hfill (2.8)$$

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$^5$\text{$\mathcal{E}$ is a right module over $\mathcal{A}_\zeta$. See, for example, [31] [12].}$
The covariant derivative $D$ is written as

$$D = d + A.$$ (2.9)

Here the $U(N)$ gauge field $A$ is introduced to ensure the covariance, as explained below. $A$ is a matrix valued one-form: $A = A_\mu dx^\mu$ with $A_\mu$ being an anti-hermitian $N \times N$ matrix.

The action of exterior derivative $d$ is defined as:

$$da := (\partial_\mu a) \, dx^\mu, \quad a \in \mathcal{A}_\zeta. \quad (2.10)$$

$dx^\mu$'s commute with $x^\mu$ and anti-commute among themselves, and hence $d^2 a = 0$ for $a \in \mathcal{A}_\zeta$. From (2.7) and (2.8), the covariant derivative transforms as

$$D \rightarrow UDU^\dagger.$$ (2.11)

Hence the gauge field $A$ transforms as

$$A \rightarrow UdU^\dagger + UAU^\dagger.$$ (2.12)

The field strength is defined by

$$F := D^2 = dA + A^2.$$ (2.13)

We can construct a gauge invariant action $S$ by

$$S = -\frac{1}{4g^2} \text{Tr}_{H_\zeta} F_{\mu\nu} F^{\mu\nu}. \quad (2.14)$$

For later purpose, let us consider a projection operator $P \in M_N(\mathcal{A}_\zeta)$, $P^\dagger = P$, $P^2 = P$, where $M_N(\mathcal{A}_\zeta)$ denotes the algebra of $N \times N$ matrices with their entries in $\mathcal{A}_\zeta$. For every projection operator $P$, we can consider vector space $PE$:

$$e \in PE \iff e \in \mathcal{E}, \; e = Pe.$$ (2.15)

We can consider unitary action on $PE$:

$$e \rightarrow U_\rho e, \quad U_\rho = PU_\rho = U_\rho P, \quad U_\rho^\dagger U_\rho = U_\rho U_\rho^\dagger = P.$$ (2.16)

We can construct covariant derivative $D_\rho$ for $PE$ by

$$D_\rho = Pd + A, \quad A = PA = AP.$$ (2.17)

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6In this paper we only consider the case where the metric on $\mathbb{R}^4$ is flat: $g_{\mu\nu} = \delta_{\mu\nu}$.

7$PE$ is a right projective module over $\mathcal{A}_\zeta$. 

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Notice that $D_P = PD_P$. We require $D_P e$ to transform as

$$D_P e \rightarrow U_P D_P e. \quad (2.18)$$

Then the covariant derivative $D_P$ must transform as

$$D_P \rightarrow U_P D_P U_P^\dagger. \quad (2.19)$$

For any $e \in P\mathcal{E}$, one can show

$$U_P D_P U_P^\dagger e = U_P (Pd + A) U_P^\dagger e = U_P d(P(U_P^\dagger e)) + U_P A U_P^\dagger e$$

$$= U_P Pd U_P^\dagger e + PU_P (U_P^\dagger de) + U_P A U_P^\dagger e \quad (U_P P = PU_P)$$

$$= Pde + (U_P P d U_P^\dagger + U_P A U_P^\dagger) e. \quad (2.20)$$

Hence the gauge transformation rule of the gauge field $A$ is given by

$$A \rightarrow U_P d U_P^\dagger + U_P A U_P^\dagger. \quad (2.21)$$

The field strength becomes

$$F := D_P^2$$

$$= PdA + A^2 + PdPd. \quad (2.22)$$

Indeed, for $e \in P\mathcal{E}$, one can show

$$Fe = (Pd + A)(Pde + Ae)$$

$$= Pd(Pde) + Pd(Ae) + APde + A^2 e$$

$$= Pd(Pde) + PdAe + A^2 e, \quad (2.23)$$

and since $e = Pe$ and $P^2 = P$, we can rewrite (2.23) using following equations:

$$Pd(Pde) = Pd(Pd(Pe))$$

$$= Pd(PdPe + Pde)$$

$$= PdPdPe - PdPde + PdPde$$

$$= PdPdPe. \quad (2.24)$$

Hence we obtain (2.22). We can construct a gauge invariant action $S_P$ by

$$S_P = -\frac{1}{4g^2} \text{Tr}_{U(N)} PF_{\mu\nu} F^{\mu\nu} P. \quad (2.25)$$

Gauge field $A$ is called anti-self-dual, or instanton, if its field strength satisfies the conditions:

$$F^+ := \frac{1}{2} (F + *F) = 0, \quad (2.26)$$

where $\ast$ is the Hodge star.
2.2 Review of ADHM Construction on Commutative $\mathbb{R}^4$ 

ADHM construction \[7\] is the way to obtain anti-self-dual gauge field on $\mathbb{R}^4$ from solutions of some quadratic matrix equations. More specifically, in order to construct anti-self-dual $U(N)$ gauge field with instanton number $k$, one starts from the following data (ADHM data):

1. A pair of complex hermitian vector spaces $V = \mathbb{C}^k$ and $W = \mathbb{C}^N$.

2. The operators $B_1, B_2 \in \text{Hom}(V, V), I \in \text{Hom}(W, V), J = \text{Hom}(V, W)$ satisfying the equations

   $$\mu_R = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0,$$

   $$\mu_C = [B_1, B_2] + IJ = 0.$$  \hspace{1cm}(2.27)

Next define Dirac-like operator $D_z : V \oplus V \oplus W \rightarrow V \oplus V$ by

$$D_z = \begin{pmatrix} \tau_z & \sigma_z^\dagger \\ \sigma_z & \tau_z^\dagger \end{pmatrix},$$

$$\tau_z = (B_2 - z_2, B_1 - z_1, I),$$

$$\sigma_z^\dagger = (-B_1^\dagger - \bar{z}_1), B_2^\dagger - \bar{z}_2, J^\dagger).$$ \hspace{1cm}(2.28)

(2.27) is equivalent to the set of equations

$$\tau_z\tau_z^\dagger = \sigma_z^\dagger \sigma_z, \hspace{0.5cm} \tau_z \sigma_z = 0,$$ \hspace{1cm}(2.29)

which are important conditions in ADHM construction. There are $N$ zero-modes of $D_z$:

$$D_z \psi^{(a)} = 0, \hspace{0.5cm} a = 1, \ldots, N.$$ \hspace{1cm}(2.30)

We can choose orthonormal basis of the space of zero-modes:

$$\psi^{(a)^\dagger} \psi^{(b)} = \delta^{ab}.$$ \hspace{1cm}(2.31)

The change of basis in the space of orthonormalized zero-modes $\psi^{(a)}$ becomes $U(N)$ gauge symmetry. Anti-self-dual $U(N)$ gauge field is constructed by the formula

$$A^{ab} = \psi^{(a)^\dagger} d\psi^{(b)}.$$ \hspace{1cm}(2.32)

There is an action of $U(k)$ that does not change (2.32):

$$(B_1, B_2, I, J) \mapsto (gB_1 g^{-1}, gB_2 g^{-1}, gI, gJ g^{-1}), \hspace{0.5cm} g \in U(k).$$ \hspace{1cm}(2.33)

The moduli space of anti-self-dual $U(N)$ gauge field with instanton number $k$ is given by

$$\mathcal{M}(k, N) = \mu_R^{-1}(0) \cap \mu_C^{-1}(0)/U(k),$$ \hspace{1cm}(2.34)

where the action of $U(k)$ is the one given in (2.33). When $(B_1, B_2, I, J)$ is a fixed point of $U(k)$ action, $\mathcal{M}(k, N)$ is singular. Such a singularity corresponds to an instanton shrinking to zero size.
3 ADHM Construction on Noncommutative R\(^4\) and the Appearance of Projection Operator

The singularities in (2.34) has a natural resolution \([8]\). Modify (2.27) to

\[
\begin{align*}
\mu_R &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \text{Id}_V, \\
\mu_C &= [B_1, B_2] + IJ = 0,
\end{align*}
\]

and consider the space

\[
\mathcal{M}_\zeta(k, N) = \mu_R^{-1}(\zeta \text{Id}_V) \cap \mu_C^{-1}(0)/U(k).
\]

Then \(\mathcal{M}_\zeta(k, N)\) is a smooth \(4kN\) dimensional hyper-Kähler manifold. Although the absence of singularities is interesting from the physical point of view, construction of instantons from (3.1) does not work straightforwardly. The main obstruction is that the key equations in (2.29) are not satisfied on the usual commutative \(R^4\). However, Nekrasov and Schwarz noticed that \(\tau_z\) and \(\sigma_z\) do satisfy (2.29) if the coordinates are noncommutative as in (2.4) \([12]\). Once (2.29) is satisfied, we can expect that the construction of instantons is similar to the usual commutative case. But there are some features peculiar to the noncommutative case. Especially, since the ADHM construction on noncommutative \(R^4\) starts from (3.2) where small instanton singularities have been resolved, one expects that crucial difference will appear when the size of the instanton is small. It is interesting to study such situations and see how the effects of noncommutativity appear.

The ADHM construction on noncommutative \(R^4\) is as follows \([12]\). We define operator

\[
\mathcal{D}_z : (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta
\]

by the same formula (2.28):

\[
\begin{align*}
\mathcal{D}_z &= \begin{pmatrix}
\tau_z \\
\sigma_z^\dagger
\end{pmatrix}, \\
\tau_z &= (B_2 - z_2, B_1 - z_1, I), \\
\sigma_z^\dagger &= (- (B_1^\dagger - \bar{z}_1), B_2^\dagger - \bar{z}_2, J^\dagger).
\end{align*}
\]

The operator \(\mathcal{D}_z \mathcal{D}_z^\dagger : (V \oplus V) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta\) has a block diagonal form

\[
\mathcal{D}_z \mathcal{D}_z^\dagger = \begin{pmatrix}
\Box_z & 0 \\
0 & \Box_z
\end{pmatrix}, \quad \Box_z \equiv \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z
\]

which is a consequence of (2.29) and important for ADHM construction. Next we look for solutions to the equation

\[
\mathcal{D}_z \Psi^{(a)} = 0 \quad (a = 1, \ldots, N),
\]
where the components of \( \Psi^{(a)} \) are operators: \( \Psi^{(a)} : \mathcal{A}_\zeta \to (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta \). If we can normalize \( \Psi^{(a)} \)'s as

\[
\Psi^{(a)*} \Psi^{(b)} \stackrel{?}{=} \delta^{ab} \text{Id}_\mathcal{H},
\]

we can construct anti-self-dual \( U(N) \) gauge field by the same formula \( \text{(2.32)} \):

\[
A^{ab} = \Psi^{(a)*} d \Psi^{(b)},
\]

where \( a \) and \( b \) are \( U(N) \) indices. Then the field strength becomes

\[
F = F^{\text{ADHM}} = \Psi^{(a)*} \left( \frac{1}{\partial_z} d\tau_z + \frac{1}{\partial_{\bar{z}}} d\sigma_{\bar{z}} \right) \Psi
\]

where we have written

\[
\Psi \equiv \left( \begin{array}{c} \psi_1 \\ \psi_2 \\ \xi \end{array} \right),
\]

\[
\begin{array}{c}
\psi_1 : C^N \otimes \mathcal{A}_\zeta \to V \otimes \mathcal{A}_\zeta, \\
\psi_2 : C^N \otimes \mathcal{A}_\zeta \to V \otimes \mathcal{A}_\zeta, \\
\xi : C^N \otimes \mathcal{A}_\zeta \to W \otimes \mathcal{A}_\zeta.
\end{array}
\]

The derivation is similar to the commutative case. The field strength in \( \text{(3.8)} \) is anti-self-dual.

However, as we will see shortly, there are some states in \( \mathcal{H} \) which are annihilated by \( \Psi^{(a)} \) for some \( a \). More precisely, all the components of \( \Psi^{(a)} \) annihilate those states. This is not a special phenomenon, and the study of this phenomenon is the purpose of this paper. Let us consider the case where there is one such zero-mode \( \Psi^{(1)} \). In that case we cannot normalize \( \Psi^{(1)} \) as in \( \text{(3.6)} \). We may normalize \( \Psi^{(1)} \) as

\[
\Psi^{(1)*} \Psi^{(1)} = P,
\]

where \( P \in \mathcal{A}_\zeta \) is a projection operator that projects out the states annihilated by \( \Psi^{(1)} \). However, the projection operator gives additional contribution to the field strength, because the projection operator depends on \( z \) and \( \bar{z} \). The derivative of the projection operator gives additional contribution to the field strength, which is not anti-self-dual.

The appearance of the projection operator \( P \) indicates that we should consider restricted vector space \( P \mathcal{E} \) rather than \( \mathcal{E} \). Indeed, as we will see shortly, ADHM construction perfectly works in this setting.

\[\text{For example, } |0,0\rangle \langle 0,0| = e^{-\frac{i}{2}(z_1\bar{z}_1 + z_2\bar{z}_2)}:, \text{ where : : means normal ordering.}\]
Let us concentrate on the simplest $U(1)$ case. The covariant derivative is given by the formula (2.17):

$$D_P = Pd + A,$$  (3.10)

with $A = PAP$. The field strength is given by (2.22):

$$F = PdA + A^2 + PdPdP.$$  (3.11)

We can construct anti-self-dual gauge field by putting

$$A = \Psi^\dagger d\Psi P,$$  (3.12)

where $\Psi$ is a zero-mode of $D_z$ and normalized as $\Psi^\dagger \Psi = P$. Note that $\Psi^\dagger = P\Psi^\dagger$. Let us check that (3.12) is really anti-self-dual. The first term in (3.11) becomes

$$PdA = Pd\Psi^\dagger d\Psi P - P\Psi^\dagger d\Psi dP.$$  (3.13)

The last term above can be rewritten as

$$P\Psi^\dagger d\Psi dP = P(d(\Psi^\dagger \Psi) - d\Psi^\dagger \Psi)dP = PdPdP - Pd\Psi^\dagger \Psi dP.$$  (3.14)

The first term in (3.14) cancels $PdPdP$ in (3.11). The last term in (3.14) vanishes when acting on $e = Pe \in P\mathcal{E}$, since $\Psi dPP = -\Psi Pd(1 - P)P = 0$. The second term in (3.11) becomes

$$A^2 = P\Psi^\dagger d\Psi^\dagger d\Psi P = P(d(\Psi^\dagger \Psi)) - d\Psi^\dagger \Psi\Psi^\dagger d\Psi P = PdP\Psi^\dagger d\Psi - Pd\Psi^\dagger \Psi \Psi^\dagger d\Psi P.$$  (3.15)

The first term in (3.15) vanishes because $PdP\Psi^\dagger = -Pd(1 - P)P\Psi^\dagger = 0$. Then the field strength becomes

$$F = Pd\Psi^\dagger (1 - \Psi\Psi^\dagger) d\Psi P = PF_{\text{ADHM}}^- P = F_{\text{ADHM}}^-,$$  (3.16)

where $F_{\text{ADHM}}^-$ is defined in (3.8) and is anti-self-dual. Generalization to $U(N)$ case is straightforward.

The absence of the singular behavior in the field configuration follows rather straightforwardly from the explicit formula (3.8). Since we have normalized the zero-modes in the subspace where zero-modes do not vanish, these normalized zero-modes are well defined. Moreover, as shown in appendix A, the operator $\Box_z$ has no zero-mode and hence its inverse does not cause divergences. Therefore from the explicit formula (3.8), we cannot see any source of divergences in (3.8) or (3.16), either.
4 U(1) Instantons and Projection Operators

4.1 Projection Operators in U(1) Instanton Solutions and Relation to the Ideal

In the previous section it is shown how to construct anti-self-dual gauge field when the zero-mode annihilates some states. Then the natural question is: “How the states annihilated by the zero-modes should be determined?” In this section the answer to this question is given when the gauge group is U(1).

Let us consider the solution to the equation
\[ D_z |U\rangle = 0, \] (4.1)
where \(|U\rangle \in \mathcal{H}^\oplus \oplus \mathcal{H}^\oplus \oplus \mathcal{H}, \) i.e. the components of \(|U\rangle\) are vectors in the Fock space \(\mathcal{H}.\) We call \(|U\rangle\) “vector zero-mode” and call \(\Psi\) in (3.3) “operator zero-mode”. We can construct operator zero-mode if we know all the vector zero-modes. The advantage of considering vector zero-modes is that we can relate them to the ideal discussed in \([9][10].\) The point is that we can regard vector zero-modes as holomorphic vector bundle described in purely commutative terms. Noncommutativity appears when we construct operator zero-mode treating all the vector zero-modes as a whole.

Let us write
\[ |U\rangle = \begin{bmatrix} |u_1\rangle \\ |u_2\rangle \\ |f\rangle \end{bmatrix}, \quad |u_1\rangle \equiv u_1(z_1, z_2) |0, 0\rangle, \quad |u_2\rangle \equiv u_2(z_1, z_2) |0, 0\rangle, \quad |f\rangle \equiv f(z_1, z_2) |0, 0\rangle \] (4.2)
where \(|u_1\rangle, |u_2\rangle \in \mathcal{H}^\oplus\) i.e. they are vectors in \(V = \mathbb{C}^k\) and vectors in \(\mathcal{H},\) and \(|f\rangle \in \mathcal{H}.\)

The space of the solutions of (4.1), i.e. \(\ker D_z = \ker \tau_z \cap \ker \sigma_z^+ \simeq \ker \tau_z/\text{Im} \sigma_z\) is isomorphic to the ideal \(\mathcal{I}\) defined by
\[ \mathcal{I} = \left\{ f(z_1, z_2) \mid f(B_1, B_2) = 0 \right\}, \] (4.3)
where \(B_1\) and \(B_2\) together with \(I\) and \(J\) give a solution to (3.3). In U(1) case, one can show \(J = 0,\) and the isomorphism is given by the inclusion of the third factor in (4.2) \([9][10].\)

\[ \ker \tau_z/\text{Im} \sigma_z \hookrightarrow \mathcal{O}_{C^2} : \quad |U\rangle = \begin{bmatrix} |u_1\rangle \\ |u_2\rangle \\ |f\rangle \end{bmatrix} \hookrightarrow f(z_1, z_2). \] (4.4)

9Notice that since \(\tau_z \sigma_z = 0 (2.29),\) \(\ker \tau_z/\text{Im} \sigma_z\) is well defined. \(\ker \tau_z \cap \ker \sigma_z^+ \simeq \ker \tau_z/\text{Im} \sigma_z\) is understood as follows: the condition \(\ker \sigma_z^+ |U\rangle = 0\) fixes the “gauge freedom” mod \(\text{Im} \sigma_z\) in \(\ker \tau_z/\text{Im} \sigma_z.\)
Let us define “ideal state” by

\[ |\varphi\rangle \in \text{ideal states of } \mathcal{I} \iff \exists f(z_1, z_2) \in \mathcal{I}, \quad |\varphi\rangle = f(z_1, z_2) |0, 0\rangle, \] (4.5)

and denote the space of all the ideal states by \( \mathcal{H}_\mathcal{I} \). We define \( \mathcal{H}/\mathcal{I} \) as a subspace in \( \mathcal{H} \) orthogonal to \( \mathcal{H}_\mathcal{I} \):

\[ |g\rangle \in \mathcal{H}/\mathcal{I} \iff \forall f(z_1, z_2) \in \mathcal{I}, \quad \langle 0 | f^\dagger(\bar{z}_1, \bar{z}_2) |g\rangle = 0. \] (4.6)

\( \mathcal{H}/\mathcal{I} \) is a \( k \) dimensional space \(^{10}\). Let us denote the complete basis of \( \mathcal{H}/\mathcal{I} \) by \( |g_\alpha\rangle, \alpha = 1, 2, \ldots, k \), and the orthonormalized complete basis of \( \mathcal{H}_\mathcal{I} \) by \( |f_i\rangle, i = k + 1, k + 2, \ldots \). They altogether span the complete basis of \( \mathcal{H} \). We can label them by positive integer \( n \):

\[ \{ |h_n\rangle, \ n \in \mathbb{Z}_+ \} = \{ |g_\alpha\rangle, |f_i\rangle, \alpha = 1, 2, \ldots, k, \ i = k + 1, k + 2, \ldots \}. \] (4.7)

As we can see from (4.4), zero-modes (4.1) are completely determined by the ideal \( f_i(z_1, z_2) \) \(^{10}\):

\[ |U(f_i)\rangle = \begin{pmatrix} |u_1(f_i)\rangle \\ |u_2(f_i)\rangle \\ |f_i\rangle \end{pmatrix}. \] (4.8)

We can construct operator zero-mode (3.5) by the following formula:

\[ \Psi = \sum_i \sum_n (\Psi)_{in} |U(f_i)\rangle \langle h_n|, \] (4.9)

where \((\Psi)_{in}\) is a commuting number. From (4.3), one can see that there are infinitely many operator zero-modes. Since the Fock space \( \mathcal{H} \) is divided into two orthogonal subspaces \( \mathcal{H}_\mathcal{I} \) and \( \mathcal{H}/\mathcal{I} \) through the isomorphism (4.3), it is natural to restrict the action of operators to \( \mathcal{H}_\mathcal{I} \). We call \( \Psi_0 \) “minimal operator zero-mode” if it has the form:

\[ \Psi_0 = \sum_{i,j} (\Psi_0)_{ij} |U(f_i)\rangle \langle f_j|, \quad |f_j\rangle \in \mathcal{H}_\mathcal{I}, \] (4.10)

i.e. \((\Psi_0)_{in} = 0\) for \( n = \alpha = 1, 2, \cdots, k \). We call it “normalized” minimal operator zero-mode if it is normalized in \( \mathcal{H}_\mathcal{I} \):

\[ \Psi_0^\dagger \Psi_0 = P_\mathcal{I}, \] (4.11)

where \( P_\mathcal{I} \) is a projection operator which represents the projection to \( \mathcal{H}_\mathcal{I} \), the space of ideal states. The uniqueness of the normalized minimal operator zero-mode (up to gauge

\(^{10}\)The meaning of this notation is as follows: \( \mathcal{H}/\mathcal{I} \) corresponds to \( \mathbb{C}[z_1, z_2]/\mathcal{I} \), where \( \mathbb{C}[z_1, z_2] \) is the ring of polynomials of \( z_1, z_2 \).
transformation) is shown in appendix [3]. This means that the normalized minimal operator zero-mode contains minimal information of the ideal (4.3). From above definition, minimal operator zero-mode annihilates states in $\mathcal{H}/\mathcal{I}$, i.e. $\Psi_0 |\varphi\rangle = 0$ for $|\varphi\rangle \in \mathcal{H}/\mathcal{I}$. Note that if we write

$$\Psi_0 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix},$$

(4.12)

then $\xi |\varphi\rangle = 0 \Rightarrow \psi_1 |\varphi\rangle = \psi_2 |\varphi\rangle = 0$. Hence the states annihilated by minimal operator zero-mode $\Psi_0$ are completely determined by the third factor $\xi$ in (4.12).

An interesting point is that the noncommutative operators appear from the ideal described in purely commutative terms, by treating infinite number of the elements of ideal simultaneously.

As an illustration, let us construct $U(1)$ one-instanton solution from the ideal. First, let us recall $U(1)$ one-instanton solution constructed in [12]. The solution to the modified ADHM equations (3.1) is given by

$$B_1 = B_2 = 0, \quad I = \sqrt{\zeta}, \quad J = 0.$$  

(4.13)

There is a solution $\tilde{\Psi}_0$ to the equation $D_2 \tilde{\Psi}_0 = 0$:

$$\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} \sqrt{\zeta} \bar{z}_2 \\ \sqrt{\zeta} \bar{z}_1 \\ \bar{z}_1 \bar{z}_1 + \bar{z}_2 \bar{z}_2 \end{pmatrix}.$$  

(4.14)

Notice that all the components of $\tilde{\Psi}_0$ annihilate $|0,0\rangle$. As a consequence $\tilde{\Psi}_0^\dagger \tilde{\Psi}_0 = (z_1 \bar{z}_1 + z_2 \bar{z}_2)(z_1 \bar{z}_1 + z_2 \bar{z}_2 + \zeta)$ annihilates $|0,0\rangle$. Therefore the inverse of $(z_1 \bar{z}_1 + z_2 \bar{z}_2)$ is only defined in the subspace of Fock space where $|0,0\rangle$ is projected out:

$$(z_1 \bar{z}_1 + z_2 \bar{z}_2)^{-1} := P(z_1 \bar{z}_1 + z_2 \bar{z}_2)^{-1} P,$$  

(4.15)

where $P$ is a projection operator that project out $|0,0\rangle$:

$$P = \text{Id}_{\mathcal{H}} - |0,0\rangle \langle 0,0|.$$  

(4.16)

Therefore

$$\Psi_0 = \tilde{\Psi}_0 (\tilde{\Psi}_0^\dagger \tilde{\Psi}_0)^{-1/2}$$  

(4.17)

is normalized as $\Psi_0^\dagger \Psi_0 = P$.

Let us reconstruct this zero-mode from ideal. The ideal which corresponds to (4.13) is $\mathcal{I} = (z_1, z_2)$. The basis vector of the $\mathcal{H}/\mathcal{I}$ is $|0,0\rangle$ which is orthogonal to all the ideal
states. We can use $|n_1, n_2\rangle, (n_1, n_2) \neq (0, 0)$ as basis vectors of $\mathcal{H}_I$, the space of ideal states. The solutions of $D_z |U\rangle = 0$ are given by

$$|U_{n_1 n_2}\rangle = \begin{pmatrix} |u_{1,n_1 n_2}\rangle \\ |u_{2,n_1 n_2}\rangle \\ |f_{n_1 n_2}\rangle \end{pmatrix}, \quad (n_1, n_2) \neq (0, 0). \quad (4.18)$$

From (4.18), we obtain operator zero-mode $D_z \Psi = 0$:

$$\Psi = \sum_{(m_1, m_2) \neq (0, 0)} \sum_{(n_1, n_2)} (\Psi)_{(m_1, m_2)(n_1, n_2)} |U_{m_1 m_2}\rangle \langle n_1, n_2|; \quad (4.19)$$

The normalized minimal operator zero-mode $\Psi_0$ is required to satisfy

$$\Psi_0^{\dagger} \Psi_0 = \text{Id}_{\mathcal{H}} - |0, 0\rangle \langle 0, 0| . \quad (4.20)$$

From the normalization condition in (4.20), we obtain

$$\sum_{(m_1, m_2) \neq (0, 0)} \frac{1}{2} (m_1 + m_2)(m_1 + m_2 + 2) \langle \Psi^{\dagger}\rangle_{(l_1 l_2)(m_1, m_2)} (\Psi)_{(m_1, m_2)(n_1, n_2)} = \delta_{(l_1 l_2)(n_1, n_2)}. \quad (4.21)$$

The solution of (4.21) is

$$(\Psi_0)_{(m_1, m_2)(n_1, n_2)} = \sqrt{\frac{2}{(n_1 + n_2)(n_1 + n_2 + 2)}} \delta_{(m_1, m_2)(n_1, n_2)} . \quad (4.22)$$

(4.20) and (4.22) are equivalent to (4.14) and (4.17).

### 4.2 Some $U(1)$ Instanton Solutions

The construction of operator zero-mode from vector zero-modes is useful for understanding the notion of minimal operator zero-mode. But in some simple cases it is easier to directly look for the operator zero-modes. It is interesting to observe that the obtained operator zero-modes which are most naturally obtained really annihilate states in $\mathcal{H}_I$.

**$U(1)$ two-instanton solution**

Let us study the two-instanton solutions degenerating at the origin. The corresponding solution to the matrix equations (3.1) is given by

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta} \lambda_1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \sqrt{\zeta} \lambda_2 \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & \sqrt{2\zeta} \\ 0 & 0 \end{pmatrix}, \quad J = 0. \quad (4.23)$$

---

11 Although we only consider solutions of matrix equation (3.1) and do not construct gauge field explicitly, we call the solutions “instanton solutions” because in principle we can construct instantons from the matrix data. We regard $k$ as a number of instantons.
where \( \lambda_1 \) and \( \lambda_2 \) are complex numbers satisfying \(|\lambda_1|^2 + |\lambda_2|^2 = 1\). Notice that \( B_1 \) and \( B_2 \) are upper half triangle matrices. \( \lambda_1, \lambda_2 \) (partially) remember the direction before two instantons collide \([10]\). The corresponding ideal is \( \mathcal{I} = (z_1^2, -\lambda_2 z_1 + \lambda_1 z_2) \). Hence the states orthogonal to all the ideal states are annihilated by \( \bar{B} \).

Therefore the basis vectors of the states orthogonal to all the ideal states are \( \bar{z} \) and \( \bar{z}' \):

\[
z'_1 \equiv \lambda_1^* z_1 + \lambda_2^* z_2, \quad z'_2 \equiv -\lambda_2 z_1 + \lambda_1 z_2, \quad |0, 0\rangle = |0, 0\rangle_\lambda, \]

\[
\sqrt{\frac{2}{\zeta}} z'_1 |n'_1, n'_2\rangle_\lambda = \sqrt{n'_1 + 1} |n'_1 + 1, n'_2\rangle_\lambda, \quad \sqrt{\frac{2}{\zeta}} \bar{z}'_1 |n'_1, n'_2\rangle_\lambda = \sqrt{n'_1} |n'_1 - 1, n'_2\rangle_\lambda,
\]

\[
\sqrt{\frac{2}{\zeta}} z'_2 |n'_1, n'_2\rangle_\lambda = \sqrt{n'_2 + 1} |n'_2 + 1, n'_1\rangle_\lambda, \quad \sqrt{\frac{2}{\zeta}} \bar{z}'_2 |n'_1, n'_2\rangle_\lambda = \sqrt{n'_2} |n'_2, n'_1 - 1\rangle_\lambda.
\]

(4.24)

Then the states annihilated by \( \bar{z}'_2 = -\lambda_2^* \bar{z}_1 + \lambda_1^* \bar{z}_2 \) are \( |n'_1, 0\rangle_\lambda \) for all non-negative \( n'_1 \). Therefore the basis vectors of the states orthogonal to all the ideal states are \( |0, 0\rangle, |1, 0\rangle_\lambda \)

Now let us study operator zero-mode. The (unnormalized) minimal operator zero-mode can be directly obtained from (4.23):

\[
\tilde{\Psi}_0 = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \bar{\xi} \end{pmatrix}, \quad \tilde{\psi}_1 = \begin{pmatrix} \sqrt{\zeta} \bar{z}'_1 \\ \bar{z}_2 \frac{\zeta}{2} (\hat{N} - 1) + \zeta \lambda_1 \bar{z}'_2 \end{pmatrix}, \quad \tilde{\psi}_2 = \begin{pmatrix} \bar{z}_1 \frac{\zeta}{2} (\hat{N} - 1) - \zeta \lambda_2 \bar{z}'_2 \end{pmatrix}
\]

\[
\bar{\xi} = \frac{1}{\sqrt{2\zeta}} \left( \frac{\zeta}{2} \right)^2 (\hat{N} - 1 + 2n_2), \quad (4.25)
\]

where \( \frac{\zeta}{2} \hat{N} \equiv z_1 \bar{z}_1 + z_2 \bar{z}_2 \), \( \frac{\zeta}{2} n_2' \equiv z'_2 \bar{z}'_2 \). (4.23) is really minimal: \( |0, 0\rangle \) and \( |1, 0\rangle_\lambda \) are annihilated by all the components of \( \tilde{\Psi}_0 \).

\textbf{U(1) three-instanton solutions}

Let us consider the \( k = 3 \) solution corresponding to the following simple ideal: \([\hat{1}^3]\)

\[
\mathcal{I} = \left\{ f(z_1, z_2) = \sum_{n_1, n_2} a_{n_1 n_2} z_1^{n_1} z_2^{n_2} \quad | a_{n_1 n_2} = 0 \text{ when } (n_1, n_2) \text{ belongs } \text{to the Young tableau (Y1)} \right\} \quad (4.26)
\]

\footnote{This kind of ideal corresponds to a fixed points of \( T^2 \) action in \([8, 10].\)}
The solution to (3.1) is given by
\[
B_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \sqrt{\zeta} \\
0 & 0 & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad I = \begin{pmatrix}
0 \\
0 \\
\sqrt{3\zeta}
\end{pmatrix}, \quad J = 0. \tag{4.28}
\]

We can find the (unnormalized) minimal operator zero-mode:
\[
\tilde{\Psi}_0 = \begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2 \\
\tilde{\xi}
\end{pmatrix}, \quad \tilde{\psi}_1 = \begin{pmatrix}
\sqrt{\zeta^2} \\
\sqrt{\bar{\zeta} \bar{z}_2} \\
\frac{\sqrt{3}}{2} \bar{N} \bar{z}_2
\end{pmatrix}, \quad \tilde{\psi}_2 = \begin{pmatrix}
\sqrt{\zeta^2} \\
\sqrt{\bar{\zeta} \bar{z}_2} \\
\frac{\sqrt{3}}{2} \bar{N} \bar{z}_1
\end{pmatrix},
\]
\[
\tilde{\xi} = \frac{1}{\sqrt{3\zeta}} \left( \frac{\zeta}{2} \right)^2 \bar{N} (\bar{N} - 1). \tag{4.29}
\]

(4.29) really annihilates \(|0,0\rangle, |1,0\rangle, |0,1\rangle\) and hence minimal.

Next consider the ideal corresponding to the following Young tableau (Y2):
\[
\begin{array}{c}
(2,0) \\
(1,0) \\
(0,0)
\end{array} \quad \text{(Y2)}
\]

The solution to (3.1) is given by
\[
B_1 = \begin{pmatrix}
0 & \sqrt{\zeta} & 0 \\
0 & 0 & \sqrt{2\zeta} \\
0 & 0 & 0
\end{pmatrix}, \quad B_2 = 0, \quad I = \begin{pmatrix}
0 \\
0 \\
\sqrt{3\zeta}
\end{pmatrix}, \quad J = 0. \tag{4.31}
\]

The (unnormalized) minimal operator zero-mode is given as
\[
\tilde{\Psi}_0 = \begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2 \\
\tilde{\xi}
\end{pmatrix}, \quad \tilde{\psi}_1 = \begin{pmatrix}
2\zeta \bar{z}_1^2 \bar{z}_2 \\
\sqrt{2\zeta^2} \bar{N} \bar{z}_1 \bar{z}_2 \\
\left( \frac{\zeta}{2} \right)^2 \{ (\bar{N} + 1)(\bar{N} + 4) - 2(\tilde{n}_1 - 1) \} \bar{z}_2
\end{pmatrix},
\]
\[
\tilde{\psi}_2 = \begin{pmatrix}
2\zeta \bar{z}_1^3 \\
\sqrt{2\zeta^2} \bar{N} \bar{z}_1^2 \\
\left( \frac{\zeta}{2} \right)^2 \{ (\bar{N} + 1)(\bar{N} + 4) - 2(\tilde{n}_1 - 1) \} \bar{z}_1
\end{pmatrix},
\]
\[
\tilde{\xi} = \frac{1}{\sqrt{3\zeta}} \left( \frac{\zeta}{2} \right)^3 \bar{N} \{ \bar{N}(\bar{N} + 3) - 2(3\tilde{n}_1 - 1) \}. \tag{4.32}
\]

We can check (4.32) annihilates \(|0,0\rangle, |1,0\rangle, |2,0\rangle\).
5 \( U(N) \) Instantons and Projection Operators

In the previous section we have clarified the notion of the minimal operator zero-mode for \( U(1) \) case. In this section we will study the \( U(2) \) instanton solutions and observe that the projection of states by zero-modes also occurs. Since the \( U(N) \) instanton solutions are essentially embeddings of \( U(2) \) instanton solutions to \( U(N) \), this means that the projection of states is a general phenomenon in the ADHM construction of instantons on noncommutative \( \mathbb{R}^4 \). In the following, we will make two observations:

1. The minimal operator zero-mode appears in the \( U(1) \) subgroup of \( U(2) \) gauge group. It annihilates some states even when the size of instanton is not small.

2. When the size of instanton becomes small, only the contribution from \( U(1) \) subgroup described by the minimal operator zero-mode remains.

Although we have not defined minimal operator zero-mode for \( U(N) \) case, zero-modes similar to the minimal operator zero-mode in \( U(1) \) case appear in explicit solutions. Hence in the above we have also called them minimal operator zero-modes. The second observation can be understood as follows. We may define “small instanton” on noncommutative \( \mathbb{R}^4 \) as \( J = 0 \) solution of the modified ADHM equations \( (3.1) \). Then the solution is essentially the embedding of \( U(1) \) instanton to \( U(N) \).

\( U(2) \) one-instanton solution

The solution to the modified ADHM equations \( (3.1) \) is given by\(^\text{13}\)

\[
B_1 = B_2 = 0, \quad I = \left( \begin{array}{cc} \sqrt{\rho^2 + \zeta} & 0 \\ 0 & \rho \end{array} \right), \quad J^\dagger = \left( \begin{array}{c} 0 \\ \rho \end{array} \right),
\]

where \( \rho \) is a real non-negative number and parameterizes the size of the instanton. The two “orthonormalized” operator zero-modes of \( D_z \) are given by

\[
\Psi^{(1)} = \left( \begin{array}{c} \psi_1^{(1)} \\ \psi_2^{(1)} \\ \xi^{(1)} \end{array} \right) = \left( \begin{array}{c} \sqrt{\rho^2 + \zeta} z_2 \\ \sqrt{\rho^2 + \zeta} z_1 \\ (z_1 z_1 + z_2 z_2) \end{array} \right) \left( (z_1 z_1 + z_2 z_2)(z_1 z_1 + z_2 z_2 + \zeta + \rho^2) \right)^{-1/2}, \quad (5.2)
\]

\[
\Psi^{(2)} = \left( \begin{array}{c} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \xi^{(2)} \end{array} \right) = \left( \begin{array}{c} -\rho z_1 \\ \rho z_2 \\ 0 \end{array} \right) \left( (z_1 z_1 + z_2 z_2 + \zeta)(z_1 z_1 + z_2 z_2 + \zeta + \rho^2) \right)^{-1/2}.
\]

\( \text{13} \)There are of course family of solutions with different orientation in gauge group \( U(2) \). The resulting conclusions are the same.

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The zero-mode $\Psi^{(1)}$ is a straightforward modification of (4.17). $\Psi^{(1)}$ annihilates $|0,0\rangle$ for any values of $\rho$, and normalized in the subspace where $|0,0\rangle$ is projected out. The zero-mode $\Psi^{(2)}$ annihilates no state in $\mathcal{H}$ and manifestly non-singular even if $\rho = 0$. When $\rho = 0$, $\psi^{(2)}_1 = \psi^{(2)}_2 = 0$, and from (3.3) $\Psi^{(2)}$ does not contribute to the field strength. Therefore the structure of the instanton at $\rho = 0$ is completely determined by the $U(1)$ subgroup described by minimal operator zero-mode $\Psi^{(1)}$.

**U(2) two-instanton solution**

We can also construct a two-instanton solution and check the statements in the beginning of this section. Here we only construct one simple solution. The solution of modified ADHM equations (3.1) is given by

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta} \\ 0 & 0 \end{pmatrix}, \quad B_2 = 0, \quad I = \begin{pmatrix} 0 & 0 \\ \sqrt{2(\rho^2 + \zeta)} & 0 \end{pmatrix}, \quad J^t = \begin{pmatrix} 0 & 0 & \rho \end{pmatrix}. \quad (5.4)$$

We can obtain two (unnormalized) zero-modes orthogonal to each other:

$$\Psi^{(1)} = \begin{pmatrix} \psi^{(1)}_1 \\ \psi^{(1)}_2 \\ \xi^{(1)} \end{pmatrix}, \quad \Psi^{(2)} = \begin{pmatrix} \psi^{(2)}_1 \\ \psi^{(2)}_2 \\ \xi^{(2)} \end{pmatrix}, \quad \psi^{(1)}_1 = \frac{\sqrt{\rho^2 + \zeta}}{\sqrt{\rho^2 + \zeta\bar{z}_1z_2}} \left( \frac{1}{\sqrt{2}} \left( \sqrt{\rho^2 + \zeta\bar{z}_1z_2} \left( z_1\bar{z}_1 + z_2\bar{z}_2 + \frac{\zeta}{2} \right) \right) \right),$$

$$\psi^{(1)}_2 = \frac{\sqrt{\rho^2 + \zeta}}{\sqrt{\rho^2 + \zeta\bar{z}_1z_2}} \left( \frac{1}{\sqrt{2}} \left( \sqrt{\rho^2 + \zeta\bar{z}_1z_2} \left( z_1\bar{z}_1 + z_2\bar{z}_2 - \frac{\zeta}{2} \right) \right) \right),$$

$$\xi^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \left( z_1\bar{z}_1 + z_2\bar{z}_2 \right) \left( z_1\bar{z}_1 + z_2\bar{z}_2 + \frac{\zeta}{2} \right) + \zeta z_2z_2 \\ 0 \end{pmatrix}, \quad (5.5)$$

and

$$\psi^{(2)}_1 = \frac{\rho}{\sqrt{\rho^2 + \zeta\bar{z}_1z_2}} \left( \frac{1}{\sqrt{2}} \left( \sqrt{\rho^2 + \zeta\bar{z}_1z_2} \left( z_1\bar{z}_1 + z_2\bar{z}_2 + \frac{\zeta}{2} \right) \right) \right),$$

$$\psi^{(2)}_2 = \frac{\rho}{\sqrt{\rho^2 + \zeta\bar{z}_1z_2}} \left( \frac{1}{\sqrt{2}} \left( \sqrt{\rho^2 + \zeta\bar{z}_1z_2} \left( z_1\bar{z}_1 + z_2\bar{z}_2 - \frac{\zeta}{2} \right) \right) \right),$$

$$\xi^{(2)} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \left( z_1\bar{z}_1 + z_2\bar{z}_2 \right) \left( z_1\bar{z}_1 + z_2\bar{z}_2 + \frac{\zeta}{2} \right) + \zeta(z_2z_2 + \frac{\zeta}{2}) \end{pmatrix}. \quad (5.6)$$

$\Psi^{(1)}$ is a slight modification of (4.25) with $(\lambda_1, \lambda_2) = (1, 0)$. It annihilates $|0,0\rangle, |1,0\rangle$. $\Psi^{(2)}$ is apparently non-singular and $\psi^{(2)}_1 = \psi^{(2)}_2 = 0$ when $\rho = 0$. Hence when the size of the instanton is small, only the $U(1)$ subgroup described by $\Psi^{(1)}$ contributes to the field strength.
6 D-Instanton Makes a Hole on D3-Brane

The existence of the projection operator forces us to consider the reduced Fock space. In this section it is shown that the projection can be interpreted as modification of spacetime topology. Usual Yang-Mills theory cannot describe such spacetime topology change. However, as we will see shortly, IIB matrix model \[3\] gives an appropriate framework. The action of the IIB matrix model is obtained by dimensionally reducing ten-dimensional $U(N)$ super Yang-Mills theory down to zero dimension:

$$\begin{align}
S &= -\frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [X_\mu, X_\nu] [X^\mu, X^\nu] + \frac{1}{2} \Theta \Gamma_\mu [X^\mu, \Theta] \right), \quad (6.1)
\end{align}$$

where $X_\mu$ and $\Theta$ are $N \times N$ hermitian matrices and each component of $\Theta$ is a Majorana-Weyl spinor. The action (6.1) has the following $\mathcal{N} = 2$ supersymmetry:

$$\begin{align}
\delta^{(1)} \Theta &= \frac{i}{2} [X_\mu, X_\nu] \Gamma^{\mu \nu} \epsilon^{(1)}, \\
\delta^{(1)} X_\mu &= i \epsilon^{(1)} \Gamma_\mu \Theta, \\
\delta^{(2)} \Theta &= \epsilon^{(2)}, \\
\delta^{(2)} X_\mu &= 0. \quad (6.2)
\end{align}$$

The classical equation of motion is given by

$$[X_\mu, [X_\mu, X_\nu]] = 0. \quad (6.3)$$

IIB matrix model has classical D-brane solutions:

$$\begin{align}
X_\mu &= i \hat{\partial}_\mu, \\
[i \hat{\partial}_\mu, i \hat{\partial}_\nu] &= -i B_{\mu \nu}, \quad (6.4)
\end{align}$$

where $B_{\mu \nu}$’s are real constants. Hereafter we will consider (Euclidean) D3-brane solution, i.e. the rank of $B_{\mu \nu}$ is four and $B_{\mu \nu} = 0$ when $\mu, \nu \neq 1, 2, 3, 4$. We define “coordinate matrices” $\hat{x}_\mu$ by

$$\hat{x}_\mu = -i \theta^{\mu \nu} \hat{\partial}_\nu, \quad (6.5)$$

where $\theta^{\mu \nu}$ is an inverse matrix of $B_{\mu \nu}$. Then their commutation relations are the same as those in (2.1):

$$[\hat{x}_\mu, \hat{x}_\nu] = i \theta^{\mu \nu}. \quad (6.6)$$

\[14\] We have slightly changed the notations from those in the previous sections: in this section $N$ denotes the rank of the gauge group of IIB matrix model. We will only consider $U(1)$ instantons in the following.
Hence by setting $\theta^{\mu\nu}$ (or equivalently $B_{\mu\nu}$) self-dual as in (2.2), i.e. $\theta^{12} = \theta^{34} = \zeta/4$, and replacing operators to infinite rank matrices, we can embed the instanton solution (3.12) to IIB matrix model:

$$X_\mu = P(i\hat{\partial}_\mu + iA_\mu)P$$ (6.7)

where $A_\mu$ is the $U(1)$ instanton solution obtained by ADHM construction:

$$A_\mu = \Psi^\dagger[\hat{\partial}_\mu, \Psi]P, \quad (6.8)$$

where $\Psi$ is a zero-mode (3.3). $P$ is the projection operator determined by the zero-mode, as described in section 4. From (6.7) the solution can be represented within reduced Fock space $\mathcal{PH} := \sum_{(n_1, n_2) \in \mathbb{Z}_{\geq 0}^2} C(P|n_1, n_2)$. Therefore the solution is realized by $N \times N$ matrices with $N = (\dim \mathcal{H} - k)$, where $k$ is an instanton number. Notice that in (6.7) instanton and geometry(D3-brane) are combined into single solution. Indeed, we can rewrite (6.7) into simpler form:

$$X_\mu = P(i\hat{\partial}_\mu + iA_\mu)P = P(i\hat{\partial}_\mu)P + P(i\Psi^\dagger\hat{\partial}_\mu\Psi)P - P(i\Psi^\dagger\psi P\hat{\partial}_\mu)P = iP\Psi^\dagger\hat{\partial}_\mu\psi P = i\Psi^\dagger\hat{\partial}_\mu\psi \quad (6.9)$$

From (6.7) we obtain

$$[X_\mu, X_\nu] = P(-iB_{\mu\nu} - F^-_{\mu\nu\text{ADHM}})P \quad (6.10)$$

The derivation is similar to (3.12) $\sim$ (3.16) and $F^-_{\mu\nu\text{ADHM}}$ is anti-self-dual. From (6.10) it is easy to check that $X_\mu$ in (6.9) solves the equation of motion (6.3).

Let us consider the supersymmetry transformation in this background:

$$\delta^{(1)}\Theta = \frac{i}{2}[X_\mu, X_\nu]\Gamma^{\mu\nu}\epsilon^{(1)}$$

$$= \frac{i}{2}P(-iB_{\mu\nu} - F^-_{\mu\nu\text{ADHM}}\frac{1+\Gamma_5}{2})P\Gamma^{\mu\nu}\epsilon^{(1)},$$

$$\delta^{(2)}\Theta = \epsilon^{(2)}. \quad (6.11)$$

From (6.11) we can see that the solution (6.7) preserves one fourth of supersymmetry [4]:

$$\Gamma_5 \epsilon^{(1)} = -\epsilon^{(1)}, \quad \epsilon^{(2)} = -\frac{1}{2}PB_{\mu\nu}P\Gamma^{\mu\nu}\epsilon^{(1)} \quad (6.12)$$

Notice that the projection operator is an identity operator in the reduced Fock space $\mathcal{PH}$. Hence the second supersymmetry transformation is proportional to the identity matrix in $U(N)$ IIB matrix model, with $N = \dim \mathcal{H} - k$.

\footnote{[6,4] is not satisfied in $U(N)$ IIB matrix model with finite $N$.}
The physical interpretation of the projection in this setting is as follows. The $B_{\mu\nu}$ in (6.4) is interpreted as NS-NS B-field on D3-brane worldvolume \cite{4}. We have set $B_{\mu\nu}$ self-dual. Since the self-dual $B$-field on D3-brane induces negative D-instanton charge,\footnote{Our convention is: D-instanton $\sim$ instanton $\sim$ anti-self-dual.} we can regard that the D3-brane is made of infinitely many constituent anti-D-instantons. Now let us consider D-instantons within these infinite number of anti-D-instantons. In order for this configuration to become BPS, it is necessary to change the configurations of constituent anti-D-instantons. The projection removes anti-D-instantons at the place of D-instantons and makes holes on D3-brane worldvolume.

We can express the holes made by the projections by rewriting above operator formulas using ordinary functions and star-product. More precisely, we map operators to normal symbols (see appendix C).\footnote{Here we use normal symbols only to give concrete expressions of holes on $\mathbb{R}^4 \approx \mathbb{C}^2$. It may be interesting to formulate field theory on noncommutative $\mathbb{R}^4$ using normal symbols. It may be also interesting to investigate the relation to superstring theory. These are, however, beyond the scope of this paper.} For example, consider the projection corresponding to the ideal $\mathcal{I}$ generated by $(z_1 - w_i^1, z_2 - w_i^2) (i = 1, \cdots, k)$. Then, all normal symbols corresponding to operators acting in the reduced Fock space $\text{End} \mathcal{P} \mathcal{H}$ vanish at $(z_1, z_2) = (w_i^1, w_i^2) (i = 1, \cdots, k)$. This can be equivalently stated as the points $(z_1, z_2) = (w_i^1, w_i^2) (i = 1, \cdots, k)$ do not exist, or appear as holes.

Using operator symbols, one can show that the projection removes $k$ units of anti-D-instanton charge. Let us calculate (anti-)instanton number when there are no D-instantons. Using (C.4),

$$\frac{1}{16\pi^2} \int d^4x B_{\mu\nu} \tilde{B}^{\mu\nu} = \frac{1}{16\pi^2} \left( \frac{2\pi \zeta}{4} \right)^2 \text{Tr} \mathcal{H} \left( \frac{4}{\zeta} \right)^2 = \text{Tr} \mathcal{H},$$
$$\tilde{B}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}^{\rho\sigma} B_{\rho\sigma}. \quad (6.13)$$

Since the projection reduces dimension of Fock space by $k$, it reduces $k$ units of anti-D-instanton charge (Of course there are also contributions from anti-self-dual part. Here we only mention the role of the projection). This fact also supports the idea that the projection removes anti-D-instantons.

**Conclusions and Speculations**

**Conclusions**

In this paper we have learned that the appearance of projection operators is a general phenomenon in the ADHM construction on noncommutative $\mathbb{R}^4$. It has been shown how to treat these projections. The existence of the projection operator forces us to...
consider gauge fields on reduced Fock space. Since noncommutative $\mathbb{R}^4$ is defined by algebra over whole Fock space, the projection means the change of the spacetime topology from (noncommutative) $\mathbb{R}^4$. In order to describe such change of spacetime topology, it seems appropriate to consider theory which can describe both gauge theory and geometry. Therefore we have embedded the instanton solution to IIB matrix model. In IIB matrix model, instanton and geometry are combined into single classical solution.

Speculations

In [13] it was conjectured that the $U(1)$ instanton on noncommutative $\mathbb{R}^4 \approx \mathbb{C}^2$ can be transformed to $U(1)$ instanton on commutative Kähler manifold which is a blowup of $\mathbb{C}^2$, via field redefinition described in [13]. The ideal used to describe projection in this paper is essentially the same as the one used to describe blowup in [13]. Since both instantons are constructed from the same ADHM data, the correspondence is of course one-to-one. It is interesting if this correspondence is understood as field redefinition along the lines of [13].

In section 6 we have embeded instanton solutions to IIB matrix model. Instantons on noncommutative $\mathbb{R}^4$ represent D-instantons within D3-brane worldvolume. We interpret D3-brane as bound states of infinitely many anti-D-instantons. Then the bound states of D-instantons and D3-brane are interpreted as bound states of D-instantons and anti-D-instantons. As shown in (6.11) and (6.12), this coexistence of positive and negative D-instanton charges still preserves one fourth of supersymmetry. However, anti-D-instantons are removed at the place of D-instantons. This fact strongly suggests the relation to brane-anti-brane pair annihilation [27]. IIB matrix model describe above D-instanton-D3-brane bound states simply as its classical solution. This fact indicates the power of IIB matrix model in the description of the fate of brane-anti-brane unstable systems. It is also straightforward to embed the noncommutative instanton solution to BFSS matrix model. It is interesting to study the instanton solution in IIB matrix model or BFSS matrix model from the point of view of brane-anti-brane pair annihilation [28]. In order to classify the topological charges which will be preserved during pair annihilations, investigations from K-theoretical viewpoints may be important [29][30].

From above considerations, D3-brane may be regarded as a kind of “Dirac sea” for D-instantons. This gives new viewpoints to the second quantization of branes [20][21].

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A  The Absence of Zero-mode of $\Box_z$

In this section we show that $\Box_z$ in (3.4) has no zero-mode. Suppose

$$\Box_z |v\rangle = 0$$  \hspace{1cm} (A.1)

for some $|v\rangle$, where $|v\rangle \in H^\oplus k$, i.e. $|v\rangle$ is vector in $V = C^k$ and vector in $H$. Then,

$$\langle v| \Box_z |v\rangle = 0 \Rightarrow \langle v| \tau_z \tau_z^\dagger |v\rangle = 0, \langle v| \sigma_z^\dagger \sigma_z |v\rangle = 0, \langle v| II^\dagger |v\rangle = 0, \langle v| J^\dagger J |v\rangle = 0.$$  \hspace{1cm} (A.2)

Since the norm of vectors in $V$ are non-negative,

$$(B_1^\dagger - \bar{z}_1) |v\rangle = 0, \quad (B_2^\dagger - \bar{z}_2) |v\rangle = 0, \quad I^\dagger |v\rangle = 0,$$

$$(B_1 - z_1) |v\rangle = 0, \quad (B_2 - z_2) |v\rangle = 0, \quad J |v\rangle = 0.$$  \hspace{1cm} (A.3)

From (A.3), we obtain

$$\langle v| \zeta |v\rangle = \langle v|[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J |v\rangle$$

$$= \langle v|[z_1, \bar{z}_1] + [z_2, \bar{z}_2] |v\rangle$$

$$= - \langle v| \zeta |v\rangle.$$  \hspace{1cm} (A.4)

This means $|v\rangle = 0$.

B  The Uniqueness of the Normalized Minimal Operator Zero-mode

In this appendix we show the uniqueness of the normalized minimal operator zero-mode (up to gauge transformation) when the gauge group is $U(1)$. Let us consider operator zero-mode which has the following form:

$$\Psi_0 = \sum_{i,j} (\Psi_0^*)_{ij} |U(f_i)\rangle \langle f_j|.$$  \hspace{1cm} (B.1)

Then its norm is:

$$\Psi_0^\dagger \Psi_0 = \sum (\Psi_0^*)_{ik} (\Psi_0)_{kj} |U(f_k)| U(f_i) \langle f_j|.$$  \hspace{1cm} (B.2)
where

\[ \langle U(f_k)|U(f_l) \rangle = \langle u_1(f_k)|u_1(f_l) \rangle + \langle u_2(f_k)|u_2(f_l) \rangle + \langle f_k|f_l \rangle. \quad (B.3) \]

Let us rewrite the equation \( D_z |U(f_i)\rangle = 0 \) as

\[ D u(f_i) = -f_i, \quad (B.4) \]

where

\[ D = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 \\ -(B_1^\dagger - \bar{z}_1) & B_2 - \bar{z}_2 \end{pmatrix}, \quad u(f_i) = \begin{pmatrix} |u_1(f_i)\rangle \\ |u_2(f_i)\rangle \end{pmatrix}, \quad f_i = \begin{pmatrix} |f_i\rangle I_0 \\ 0 \end{pmatrix}. \quad (B.5) \]

Since the correspondence between the elements of ideal and vector zero-modes is one-to-one, we can consider the inverse operator of \( D \):

\[ u(f_i) = -\frac{1}{D} f_i. \quad (B.6) \]

Then, (B.3) can be written as

\[ \langle U(f_k)|U(f_l) \rangle = u^\dagger(f_k) u(f_l) + f_k^\dagger f_l = f_k^\dagger \left( \frac{1}{DD^\dagger} + 1 \right) f_l \]

\[ = \langle f_k| I^\dagger \left( \frac{1}{DD^\dagger} \right)_{ii} + 1 \rangle I |f_l\rangle, \quad (B.7) \]

where we denote the components of \((DD^\dagger)^{-1}\) as

\[ (DD^\dagger)^{-1} = \begin{pmatrix} (DD^\dagger)_{11}^{-1} & (DD^\dagger)_{12}^{-1} \\ (DD^\dagger)_{21}^{-1} & (DD^\dagger)_{22}^{-1} \end{pmatrix}. \quad (B.8) \]

From (B.7), the matrix \( C_{kl} = \langle U(f_k)|U(f_l) \rangle \) has no zero-eigenvalue-vector and we can consider \((C^{-1})_{kl}\). The normalized minimal operator zero-mode is uniquely determined (up to phase factor):

\[ (\Psi_0)_{ij} = (C^{-1/2})_{ij}. \quad (B.9) \]

### C Calculations by the Method of Operator Symbols

One can represent the equations over the algebra \( \mathcal{A}_c \) by mapping operators to ordinary c-number functions (operator symbols) and using star product. Some calculations become simpler by the use of operator symbols. The map from operators to ordinary functions depends on operator ordering procedures. In order to express holes on D3-brane (see section 3), we utilize normal symbol which corresponds to the normal ordering. Here we
review this normal symbol. For more detailed arguments on the operator symbols, see for example [33] and references therein. In this appendix, we use \( \hat{\cdot} \) to denote the operators: \( \hat{x}^\mu \)'s are noncommutative operators and \( x^\mu \)'s are c-number coordinates of \( \mathbb{R}^4 \).

Let us consider normal ordered operator of the form

\[
\hat{f}(\hat{x}) = \int \frac{dk}{(2\pi)^4} \tilde{f}(k) \ e^{ik\hat{x}} : ,
\]

where \( k\hat{x} := k_\mu \hat{x}^\mu \). : : denotes the normal ordering. For the operator valued function (C.1), the corresponding normal symbol is defined by

\[
f_N(x) = \int \frac{dk}{(2\pi)^4} \tilde{f}(k) \ e^{ikx} ,
\]

where \( x^\mu \)'s are commuting coordinates of \( \mathbb{R}^4 \). We define \( \Omega_N \) as a map from operators to the normal symbols:

\[
\Omega_N(\hat{f}(\hat{x})) = f_N(x) := \int \frac{dk}{(2\pi)^4} \left( \left( \frac{2\pi}{4} \right)^2 \text{Tr}_\mathcal{H} \{ \hat{f}(\hat{x}) : e^{-ik\hat{x}} : \} \right) e^{ikx} \quad (C.3)
\]

Notice that from the relation \( \text{Tr}_\mathcal{H} \{ : \exp (ik\hat{x}) : \} = \left( \frac{2\pi}{4} \right)^2 \delta^{(4)}(k) \), it follows

\[
\left( \frac{2\pi}{4} \right)^2 \text{Tr}_\mathcal{H} \hat{f}(\hat{x}) = \int d^4x f_N(x) . \quad (C.4)
\]

The inverse map of \( \Omega_N \) is given by

\[
\Omega_N^{-1}(f(x)) = \hat{f}^N(\hat{x}) := \int \frac{dk}{(2\pi)^4} \left( \int d^4x f(x) \ e^{-ikx} \right) : e^{ik\hat{x}} : . \quad (C.5)
\]

The star product of functions is defined by:

\[
f(x) \star_{\Omega_N} g(x) := \Omega_N(\Omega_N^{-1}(f(x))\Omega_N^{-1}(g(x))) . \quad (C.6)
\]

Since

\[
e^{ik\hat{x}} : e^{ik\hat{x}} := e^{\bar{w}\hat{z}} e^{\bar{w}'\hat{z}'} e^{w\hat{z}'} e^{w'\hat{z}} = e^{\bar{\hat{z}} w \hat{z}'} e^{(\bar{\hat{z}} + \bar{w}')\hat{z}'} e^{(w + w')\hat{z}} , \quad (C.7)
\]

where

\[
w_1 = -\frac{i}{2}(k_2 + ik_1), \quad w_2 = -\frac{i}{2}(k_4 + ik_3), \quad \bar{w}\hat{z} = \bar{w}_1\hat{z}_1 + \bar{w}_2\hat{z}_2, \quad \text{etc.} , \quad (C.8)
\]

the explicit form of the star product is given by

\[
f(z, \bar{z}) \star_{\Omega_N} g(z', \bar{z}') = \left. \frac{\partial}{\partial z'} \frac{\partial}{\partial \bar{z}'} \tilde{f}(z, \bar{z})g(z', \bar{z}') \right|_{z' = z, \bar{z}' = \bar{z}} . \quad (C.9)
\]
From the definition (C.6), the star product is associative:

\[(f(x) \ast_{\Omega_N} g(x)) \ast_{\Omega_N} h(x) = f(x) \ast_{\Omega_N} (g(x) \ast_{\Omega_N} h(x)).\]  
(C.10)

If we use coherent states, the expression of the normal symbol becomes simpler. The coherent states \(|\bar{z}_1, \bar{z}_2\rangle\) are eigen states of annihilation operators \(\hat{\bar{z}}_1, \hat{\bar{z}}_2\):

\[
\begin{align*}
\hat{\bar{z}}_1 |\bar{z}_1, \bar{z}_2\rangle &= \bar{z}_1 |\bar{z}_1, \bar{z}_2\rangle, \\
\hat{\bar{z}}_2 |\bar{z}_1, \bar{z}_2\rangle &= \bar{z}_2 |\bar{z}_1, \bar{z}_2\rangle.
\end{align*}
\]  
(C.11)

Then the normal symbol of operator \(\hat{f}\) is given by

\[f_N(z, \bar{z}) = \langle \bar{z}_1, \bar{z}_2| \hat{f} |\bar{z}_1, \bar{z}_2\rangle.\]  
(C.12)

(C.12) follows from (C.1),(C.2) and

\[
\langle \bar{z}_1, \bar{z}_2| : e^{ik\hat{\bar{z}}}: |\bar{z}_1, \bar{z}_2\rangle = \langle \bar{z}_1, \bar{z}_2| e^{i\bar{w}\bar{z}} e^{w\bar{z}} |\bar{z}_1, \bar{z}_2\rangle = e^{i\bar{w}z},
\]  
(C.13)

(we have normalized the coherent states as \(\langle \bar{z}_1, \bar{z}_2| \bar{z}_1, \bar{z}_2\rangle = 1\)). From (C.12) it is easy to see that the normal symbol \(f_N(z, \bar{z})\) vanishes at \((z_1, z_2)\) when the corresponding operator \(\hat{f}\) annihilates \(|\bar{z}_1, \bar{z}_2\rangle\) or \(\langle \bar{z}_1, \bar{z}_2|\), i.e.

\[\hat{f} |\bar{z}_1, \bar{z}_2\rangle = 0 \text{ or } \langle \bar{z}_1, \bar{z}_2| \hat{f} = 0 \quad \Rightarrow \quad f_N(z, \bar{z}) = 0.\]  
(C.14)
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