Simple unified derivation and solution of Coulomb, Eckart and Rosen-Morse potentials in prepotential approach

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Abstract

The four exactly-solvable models related to non-sinusoidal coordinates, namely, the Coulomb, Eckart, Rosen-Morse type I and II models are normally being treated separately, despite the similarity of the functional forms of the potentials, their eigenvalues and eigenfunctions. Based on an extension of the prepotential approach to exactly and quasi-exactly solvable models proposed previously, we show how these models can be derived and solved in a simple and unified way.

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I. INTRODUCTION

Exactly-solvable models are important in any branch of physics. They allow a complete understanding of the dynamics of the corresponding systems, and thus serve as paradigmatic examples for the field of studies concerned. However, exactly-solvable systems are rather scanty, and hence any means to find them are always welcome. It is thus very interesting that most exactly-solvable one-dimensional quantum systems can be obtained in the framework of supersymmetric quantum mechanics, based only on the requirement of shape invariance [1]. Recently, the number of physical systems which we can study analytically has been greatly enlarged by the discovery of the so-called quasi-exactly solvable models [2–7]. These are spectral problems for which it is possible to determine analytically a part of the spectrum but not the whole spectrum.

In [8] a unified approach to both the exactly and quasi-exactly solvable systems is presented. This is a simple constructive approach, based on the so-called prepotential [9, 10], which gives the potential as well as the eigenfunctions and eigenvalues simultaneously. The novel feature of the approach is the realization that both exact and quasi-exact solvabilities can be solely classified by two integers, the degrees of two polynomials which determine the change of variable and the zero-th order prepotential. Most of the well-known exactly and quasi-exactly solvable models, and many new quasi-exactly solvable ones, can be generated by appropriately choosing the two polynomials. This approach can be easily extended to the constructions of exactly and quasi-exactly solvable Dirac, Pauli, and Fokker-Planck equations.

The exactly solvable models that are generated by the prepotential approach in [8] are related to the so-called sinusoidal coordinates [11]. These are coordinates \( z(x) \) whose derivative-squared \( z'^2(x) \) (henceforth the prime denotes derivative w.r.t. the variable \( x \)) is at most a quadratic polynomial of \( z \) (or equivalently, \( z'' \) is at most linear in \( z \)). Such coordinates include quadratic polynomial, exponential, trigonometric, and hyperbolic types. The corresponding exactly solvable models are the six systems listed in [1]: the shifted oscillator, three-dimensional oscillator, the Morse, Scarf type I and II, and generalized Pöschl-Teller potentials. However, the other four models in [1], namely, the Coulomb, Eckart, Rosen-Morse type I and II models cannot be generated by the approach as presented in [8]. These
four models are based on a change of coordinates which are non-sinusoidal,\(^1\) which according to the discussions in [8] can only be quasi-exactly solvable. This is consistent with the Lie-algebraic approach to quasi-exactly models, as these four models, unlike the other six systems, cannot be obtained as the exactly solvable limits of some quasi-exactly solvable systems [3, 4].

The potentials of these later four systems are usually given as follows [1]:

\[
\begin{align*}
\text{Coulomb} & : A (A - 1) \frac{1}{x^2} - 2B \frac{1}{x}; \\
\text{Eckart} & : A (A - \alpha) \cosec^2 \alpha x - 2B \coth \alpha x; \\
\text{Rosen-Morse II} & : -A (A + \alpha) \sech^2 \alpha x + 2B \tanh \alpha x; \\
\text{Rosen-Morse I} & : A (A - \alpha) \cosec^2 \alpha x + 2B \cot \alpha x,
\end{align*}
\]

where \(A, B\) and \(\alpha\) are real constants. Despite the similarity of the functional forms of the potentials, their eigenvalues and eigenfunctions, these models are normally being treated separately in the literature.

The purpose of this paper is to show how these four systems can be generated by a simple extension of the prepotential presented in [8]. What is more, they can be treated in a unified manner. Put simply, the required extension is simply to allow the coefficients in the prepotential, which are assumed to be constants in [8], to be dependent on the number of nodes of the wave functions in such a way that the coefficients of all terms involving \(z\) in the potential are real constants.

To bring out the close similarity of these four potentials, we find it convenient to express the functions cosech, sech and cosec in terms of coth, tanh and cot, respectively, in (1). The resulted forms of the potentials, the relevant change of coordinates, and their eigenvalues are listed in Table 1. For clarity of presentation, we adopt the unit system in which \(\hbar\) and the mass \(m\) of the particle are such that \(\hbar = 2m = 1\). Also, without loss of generality, we have absorbed the scale factor \(\alpha\) into \(x\), or equivalently, we set \(\alpha = 1\).

From the table, we see that these four models involve a change of variable \(z(x)\) whose derivative is of the form

\[
z' = \lambda - z^2,
\]

\(^1\) We note here that the Coulomb potential can in fact be treated using sinusoidal coordinate, which we shall present in the Appendix.
TABLE I: The four exactly solvable models based on non-sinusoidal coordinates. The potential $V(x)$, its relevant non-sinusoidal coordinate $z(x)$, the derivative $z'(x)$, and the eigenvalues $E_N$ ($N = 0, 1, \ldots$) are listed. Without loss of generality, we absorb $\alpha$ into $x$, or equivalently, we set $\alpha = 1$. The range of $x$ is: $x \in [0, \infty)$ for the Coulomb and Eckart potentials, $x \in (-\infty, \infty)$ for the Rosen-Morse II potential, and $x \in [0, \pi]$ for the Rosen-Morse I potential.

| $V(x)$      | $z(x)$ | $z'(x)$ | $E_N$                        |
|-------------|--------|---------|------------------------------|
| Coulomb     | $A(\Lambda - 1)\frac{1}{x^2} - 2B\frac{1}{x}$ | $\frac{1}{x}$ | $-z^2$ | $-\frac{B^2}{(A+N)^2}$ |
| Eckart      | $A(\Lambda - 1)\coth^2 x - 2B\coth x \coth x$ | $1 - z^2$ | $-\frac{B^2}{(A-N)^2} - A(2N+1) - N^2$ |
| Rosen-Morse II | $A(\Lambda + 1)\tanh^2 x + 2B\tanh x \tanh x$ | $1 - z^2$ | $-\frac{B^2}{(A-N)^2} + A(2N+1) - N^2$ |
| Rosen-Morse I  | $A(\Lambda - 1)\cot^2 x + 2B\cot x \cot x$ | $-1 - z^2$ | $-\frac{B^2}{(A+N)^2} + A(2N+1) + N^2$ |

and that the potentials can be cast in the same form

$$V(x) = A(\Lambda - 1)z^2(x) - 2Bz(x).$$

In this form, the four potentials are regular in the variable $z$, with singularity only at $z = \infty$ and/or $z = -\infty$. As the function $z'^2$ is a forth-degree polynomial in $z$, the coordinate $z(x)$ is non-sinusoidal. Naively any system based on such coordinate will be quasi-exactly solvable according to the discussion in [8]. In the rest of the paper, we shall show how the exactly solvable models in (4) can be obtained from (2).
The organization of the paper is as follows. In Sect. II we review the main points of the prepotential approach of [8] relevant to our present discussions. Sect. III shows how the exactly solvable models (4) can be derived from the non-sinusoidal coordinates (2) by a simple extension of the approach in [8]. The four specific models are then discussed in detail in Sect. IV to VII. Sect. VIII concludes the paper. In the Appendix, we show how the Coulomb system is treated based on sinusoidal coordinate.

II. PREPOTENTIAL APPROACH

The essence of the prepotential approach is as follows. Consider a wave function \( \phi_N(x) \) \((N: \text{non-negative integer})\) which is defined as

\[
\phi_N(x) \equiv e^{-W_0(x)} p_N(z),
\]

with

\[
p_N(z) \equiv \begin{cases} 
1, & N = 0; \\
\prod_{k=1}^{N}(z - z_k), & N > 0.
\end{cases}
\]

Here \( z = z(x) \) is some real function of the basic variable \( x \), \( W_0(x) \) is a regular function of \( z(x) \), and \( z_k \)'s are the roots of \( p_N(z) \). The variable \( x \) is defined on the full line, half-line, or finite interval, as dictated by the choice of \( z(x) \). We have assumed that the only singularities of the system are \( z = \infty \) and/or \( z = -\infty \), as is the case for the four models of concerned here. The function \( p_N(z) \) is a polynomial in an \((N + 1)\)-dimensional Hilbert space with the basis \( \langle 1, z, z^2, \ldots, z^N \rangle \). \( W_0(x) \) defines the ground state wave function.

We rewrite \( \phi_N \) as

\[
\phi_N = \exp \left( -W_N(x, \{z_k\}) \right),
\]

with \( W_N \) given by

\[
W_N(x, \{z_k\}) = W_0(x) - \sum_{k=1}^{N} \ln |z(x) - z_k|.
\]

Operating on \( \phi_N \) by the operator \(-d^2/dx^2\) results in a Schrödinger equation \( H_N \phi_N = 0 \), where

\[
H_N = -\frac{d^2}{dx^2} + V_N,
\]

\[
V_N \equiv W_N^2 - W_N''.
\]
Hence the potential $V_N$ is defined by $W_N$, and we shall call $W_N$ the $N$th order prepotential. From Eq. (8), one finds that $V_N$ has the form $V_N = V_0 + \Delta V_N$:

$$
V_0 = W'_0^2 - W''_0, \\
\Delta V_N = -2 \left( W'_0 z' - \frac{z''}{2} \right) \sum_{k=1}^{N} \frac{1}{z - z_k} + \sum_{k,l} \frac{z'^2}{(z - z_k)(z - z_l)}. 
$$

(11)

Thus the form of $V_N$, and consequently its solvability, are determined by the choice of $W_0(x)$ and $z'^2$ (or equivalently by $z'' = (dz'^2/dz)/2$). Let $W'_0 z'$ and $z'^2$ be taken as polynomials in $z$. In [8], it was shown that if the degree of $W'_0 z'$ is no higher than one, and the degree of $z'^2$ no higher than two, then in $V_N(x)$ the parameter $N$ and the roots $z_k$’s, which satisfy the so-called Bethe ansatz equations (BAE) to make the potential analytic, will only appear in an additive constant and not in any term involving powers of $z$. Such system is then exactly solvable. If the degree of one of the two polynomials exceeds the corresponding upper limit, the resulted system is quasi-exactly solvable.

Now we are interested in constructing a system based on a transformed variable $z(x)$ which is a solution of $z' = \lambda - z^2$. Since in this case the degree of $z'^2$ is four, a direct application of the arguments in [8] would mean that no exactly-solvable system can be generated with whatever choice of $W_0$.

But it turns out that with a slight extension of the methods in [8], one can generate the four potentials in the prepotential approach. The main observation is this. The classification of solvability given in [8] is valid as long as the coefficients of the powers of $z$ in $W'_0 z'$ are constants which are independent of $N$. If one allows these coefficients to be $N$-dependent, then it may be possible that in the $V_N$ obtained, all the coefficients of terms involving powers of $z$ are $N$-independent constants, thus giving rise to an exactly-solvable system. This is in fact the case for the four systems mentioned above. We shall demonstrate this in what follows.

### III. EXACTLY-SOLVABLE MODELS WITH NON-SINUSOIDAL COORDINATES

Let us choose $z$ from a solution of $z' = \lambda - z^2$, and take

$$
W'_0 = A_1 z + A_0, 
$$

(12)
where $A_1$ and $A_0$ are real parameters. With this choice of $W_0'$ and $z^2$, we obtain from (11)

$$V_0 = A_1(A_1 + 1)z^2 + 2A_1A_0z + A_0^2 - \lambda A_1$$

(13)

and

$$\Delta V_N = - (\lambda - z^2) \left\{ 2 [(A_1 + 1)z + A_0] \sum_{k=1}^{N} \frac{1}{z - z_k} - (\lambda - z^2) \sum_{k,l}^{N} \frac{1}{(z - z_k)(z - z_l)} \right\}.$$  

(14)

Using the identities

$$\sum_{k=1}^{N} \frac{z}{z - z_k} = \sum_{k=1}^{N} \frac{z_k}{z - z_k} + N,$$

(15)

$$\sum_{k,l=1}^{N} \frac{1}{(z - z_k)(z - z_l)} = 2 \sum_{k,l=1}^{N} \frac{1}{z - z_k} \left( \frac{1}{z_k - z_l} \right),$$

(16)

$$\sum_{k,l=1}^{N} \frac{z^2}{(z - z_k)(z - z_l)} = 2 \sum_{k,l=1}^{N} \frac{1}{z - z_k} \left( \frac{z_k^2}{z_k - z_l} \right) + N(N - 1),$$

(17)

we rewrite $\Delta V_N$ as

$$\Delta V_N = - (\lambda - z^2) \left\{ N (2A_1 + N + 1) + 2 \sum_{k=1}^{N} \frac{1}{z - z_k} \left[ (A_1 + 1) z_k + A_0 + \sum_{l \neq k} \frac{z_k^2 - \lambda}{z_k - z_l} \right] \right\}.$$  

(18)

To remove the poles in $\Delta V_N$, we must demand that $z_k$’s satisfy the BAE

$$\sum_{l \neq k} \frac{z_k^2 - \lambda}{z_k - z_l} + (A_1 + 1) z_k + A_0 = 0, \quad k = 1, 2, \ldots, N.$$  

(19)

With these $z_k$’s, only the first term in $\Delta V_N$ remains, and the potential $V_N$ becomes

$$V_N = (A_1 + N) (A_1 + N + 1) z^2 + 2A_1A_0z + A_0^2 - \lambda [(2N + 1) A_1 + N (N + 1)].$$  

(20)

Now it is seen that $N$ appears in the coefficient of $z^2$ term, and thus $V_N$ represents a quasi-exactly solvable system, if $A_0$ and $A_1$ are some fixed constants.

But in this case it is easy to obtain an exactly solvable potential. Let us choose $A_1$ and $A_0$ such that the combinations $A \equiv -(A_1 + N)$ and $B \equiv -A_1A_0$ are $N$-independent real constants. The choice of the signs in the definitions of $A$ and $B$ are for convenience and are not essential for the moment (they will have to be determined by the normalizability of the wave functions for physical systems). Consequently, $A_1$ and $A_0$ depend on $N$:

$$A_1 = - (A + N), \quad A_0 = \frac{B}{A + N}.$$  

(21)
Then the potential $V_N$ becomes $V_N(x) = V(x) - E_N$, where

$$V(x) = A(A-1)z^2(x) - 2Bz(x), \quad (22)$$

as advertised in Sect. I, and

$$E_N = -\frac{B^2}{(A+N)^2} - \lambda \left[ A(2N+1) + N^2 \right]. \quad (23)$$

Now $V(x)$ is independent of $N$, and can be taken to be the potential of an exactly-solvable system, with eigenvalues $E_N$ ($N = 0, 1, 2, \ldots$). The corresponding wave functions $\phi_N$ are given by (5) together with (12) and (21):

$$\phi_N \sim e^{(A+N)\int^x dxz(x) - \frac{B}{A+N}p_N(x)}, \quad N = 0, 1, \ldots \quad (24)$$

From (19) and (21), the BAE satisfied by the roots $z_k$'s are

$$\sum_{l \neq k} \frac{z_k^2 - \lambda}{z_k - z_l} - (A + N - 1)z_k + \frac{B}{A + N} = 0, \quad k = 1, 2, \ldots, N. \quad (25)$$

In the sections below, we will show how the four potentials mentioned emerge from (22) with different $\lambda$. But first a comment on the choice of the form of $V(x)$ is in order. One notes that there is an ambiguity in the definitions of $V(x)$ and $E_N$ in $V_N = V - E_N$: $V_N$ is invariant under $V \rightarrow V - \alpha$ and $E_N \rightarrow E_N - \alpha$ for any real $\alpha$. This amounts to the choice of the zero point of $V(x)$. The form $V^{\text{SUSY}}$ adopted in supersymmetric quantum mechanics (e.g., in [1]), corresponds to the choice $\alpha = E_0$ so that the ground state energy is zero, i.e. $E_0 = 0$. In our case, $V^{\text{SUSY}}$ is obtained from the zero-th order prepotential $W_0(N = 0)$ with $N = 0$ (remember now that $A_0$ and $A_1$ in $W_0$ depend on $N$):

$$W''_0(N = 0) = -Az + \frac{B}{A},$$

$$V^{\text{SUSY}} = W''_0(N = 0) - W'''_0(N = 0) = A(A-1)z^2 - 2Bz + \frac{B^2}{A^2} + \lambda A. \quad (26)$$

The energies are

$$E^{\text{SUSY}}_N = \lambda \left[ A^2 - (A + N)^2 \right] + \frac{B^2}{A^2} - \frac{B^2}{(A + N)^2}. \quad (27)$$

In the literature on supersymmetric quantum mechanics, $W'_0(N = 0)$ is generally called the superpotential.
IV. \( \lambda = 0: \) THE COULOMB CASE

In this case, \( \lambda = 0 \) and \( z' = -z^2 \). A solution is \( z = 1/x \). The domain of \( x \) is \( x \in [0, \infty) \).

The potential is
\[
V(x) = \frac{A(A - 1)}{x^2} - 2B/x. \tag{28}
\]

From (24) the wave function is
\[
\phi_N(x) \sim x^{A+N} e^{-\frac{B}{x}x} p_N(z). \tag{29}
\]

For \( \phi_N \to 0 \) at \( x = 0 \) and \( \infty \), one must have \( A > 0 \) and \( B > 0 \). The eigen-energies are given by (23)
\[
E_N = -\frac{B^2}{(A+N)^2}, \quad N = 0, 1, \ldots \tag{30}
\]

The roots \( z_k \)'s satisfy the BAE (25) with \( \lambda = 0 \). It should be emphasized that the above expressions are valid for any real \( A > 0 \) (not necessary integer) and \( B > 0 \).

The ordinary Coulomb potential is obtained by setting \( A = l + 1 \) \((l = 0, 1, \ldots)\) and \( B = e^2/2 \), where \( l \) is the orbital angular quantum number and \( e \) the electric charge. This gives the well known form of the Coulomb energies \( E_N = -e^2/4(N + l + 1)^2 \). To write the wave functions (29) in the familiar form, we express \( z \) and \( z_k \) in \( p_N(z) \) in terms of \( x = 1/z \) and \( x_k = 1/z_k \). This gives (by factoring out all \( z \)'s and \( z_k \)'s)
\[
\phi_N(x) \sim x^{l+1} e^{-\frac{e^2}{2(N + l + 1)} x} \prod_{k=1}^{N} (x - x_k), \quad N = 1, 2, \ldots \tag{31}
\]

Recall that \( p_N(x) = 1 \) for \( N = 0 \). The BAE satisfied by \( x_k \)'s are obtained from (25) by setting \( \lambda = 0 \) and changing \( z_k, z_l \) to \( 1/x_k, 1/x_l \). The final result can be written as
\[
\sum_{l \neq k} \frac{1}{x_k - x_l} + \frac{l + 1}{x_k} = \frac{e^2}{2(N + l + 1)}, \quad k = 1, 2, \ldots, N. \tag{32}
\]

Letting \( y \equiv e^2x/(N + l + 1) \), we can simplify the BAE to
\[
\sum_{l \neq k} \frac{1}{y_k - y_l} + \frac{\gamma/2}{y_k} = \frac{1}{2}, \quad k = 1, 2, \ldots, N, \tag{33}
\]

where \( \gamma \equiv 2(l+1) \). Eq. (33) is just the set of equations satisfied by the roots of the Laguerre polynomials \( L_N^{\gamma-1}(y) \) [9, 12]. Hence in terms of the variable \( y \), the wave functions are
\[
\phi_N(y) \sim y^{l+1} e^{-\frac{e^2}{2}y} L_N^{l+1}(y), \tag{34}
\]

which are exactly the form given in Table 4.1 of [1].
V. \( \lambda = 1 \): THE ECKART CASE

We take \( z = \coth x \). The range of \( x \) is again the half-line \( x \in [0, \infty) \). The potential, energies and wave functions are

\[
V(x) = A(A - 1) \coth^2 x - 2B \coth x,
\]
\[
E_N = -\frac{B^2}{(A + N)^2} - A(2N + 1) - N^2,
\]
\[
\phi_N \sim (\sinh x)^{A+N} e^{-\frac{B}{A+N}x} P_N(z).
\]

Now the boundary conditions \( \phi_N \to 0 \) as \( x \to 0 \) and \( x \to \infty \) require \( A > 0 \) and \( B > (A+N)^2 \), respectively. Hence \( B \) must be greater than \( A^2 \), i.e. \( B > A^2 \), in order to admit at least one bound state (corresponding to \( N = 0 \)). For fixed \( A \) and \( B > A^2 \), the maximal value of \( N \) is such that \( B > (A+N)^2 \) remains valid. Thus the number of bound states is \( N + 1 \).

Since \( z_k^2 \neq 1 \) one can divide (25) (with \( \lambda = 1 \)) by \( z_k^2 - 1 \). Writing the result in partial fractions, we obtain

\[
\sum_{l \neq k} \frac{1}{z_k - z_l} + \frac{1}{2} (\alpha + 1) \frac{1}{z_k - 1} + \frac{1}{2} (\beta + 1) \frac{1}{z_k + 1} = 0, \quad k = 1, 2, \ldots, N, \quad (36)
\]

with

\[
\alpha = -A - N + \frac{B}{A + N}, \quad \beta = -A - N - \frac{B}{A + N}.
\]

One recognizes that (36) are the equations satisfied by the roots of the Jacobi polynomial \( P_N^{(\alpha,\beta)}(z) \) [9, 12]. Hence the wave functions for the Eckart potential are

\[
\phi_N \sim (\sinh x)^{A+N} e^{-\frac{B}{A+N}x} P_N^{(\alpha,\beta)}(z).
\]

It is easy to check that \( \phi_N \) can be expressed as

\[
\phi_N(x) \sim (z - 1)^{\frac{\alpha}{2}} (z + 1)^{\frac{\beta}{2}} P_N^{(\alpha,\beta)}(z).
\]

This is the form presented in [1].

VI. \( \lambda = 1 \): THE ROSEN-MORSE II CASE

Here \( \lambda = 1 \) is the same as for the Eckart potential, but now we take a different solution \( z(x) = \tanh x \). This choice implies the variable \( x \) is defined on the whole line, \(-\infty < x < \infty \). The wave functions are

\[
\phi_N \sim (\cosh x)^{A+N} e^{-\frac{B}{A+N}x} P_N(z).
\]
The boundary conditions $\phi_N \to 0$ as $|x| \to \infty$ lead to

$$A + N < 0, \quad |B| < (A + N)^2. \quad (41)$$

So in this case $A$ must be negative, and $B$ can have any sign as long as the inequality in (41) for $B$ is satisfied. For easy comparison with the corresponding expressions in [1], we change $A \to -A$ and $B \to -B$. With these new definitions of $A$ and $B$, the potential (22) and energies (23) are written as

$$V(x) = A(A+1) \tanh^2 x + 2B \tanh x$$
$$E_N = -\frac{B^2}{(A-N)^2} + A(2N+1) - N^2,$$
$$A > 0, \quad |B| < (A - N)^2. \quad (42)$$

For fixed $A$ and $B$ the maximal value of $N$ is such that $|B| < (A - N)^2$ remains valid. The roots $z_k$’s now satisfy a BAE which is the same as that in the Eckart case, but with the changes $A \to -A$ and $B \to -B$. Hence the wave functions are

$$\phi_N \sim (\cosh x)^{-(A-N)} e^{-\frac{B}{A-N} x} P_N^{(\alpha,\beta)}(z), \quad (43)$$

where

$$\alpha = A - N + \frac{B}{A-N}, \quad \beta = A - N - \frac{B}{A-N}. \quad (44)$$

Again, one can rewrite $\phi_N$ in the form given in [1]

$$\phi_N(x) \sim (1-z)^{\frac{1}{2}} (1+z)^{\frac{1}{2}} P_N^{(\alpha,\beta)}(z). \quad (45)$$

We note there that in this case the two models defined by only a difference in the sign of $B$ are simply mirror images of each other: they are related by the parity transformation $x \to -x$.

VII. $\lambda = -1$: THE ROSEN-MORSE I CASE

Finally we come to the Rosen-Morse potential defined with the solution $z = \cot x$ for $\lambda = -1$. The system is defined on the finite interval $x \in [0, \pi]$. From the wave function

$$\phi_N \sim (\sin x)^{A+N} e^{-\frac{B}{A+N} x} p_N(z), \quad (46)$$
one infers that $A > 0$ while $B$ is arbitrary. Again in order to facilitate comparison with the expressions in [1], we change $B \rightarrow -B$. The two systems differing only in the signs of $B$ are equivalent by reflection and periodicity.

The potential and energies read

$$V(x) = A (A - 1) \cot^2 x + 2B \cot x$$

$$E_N = -\frac{B^2}{(A + N)^2} + A (2N + 1) + N^2. \quad (47)$$

The BAE (25) is

$$\sum_{l \neq k} \frac{z_k^2 + 1}{z_k - z_l} - (A + N - 1) z_k - \frac{B}{A + N} = 0, \quad k = 1, 2, \ldots, N. \quad (48)$$

This set of BAE is related to the Jacobi polynomials, but with imaginary argument. To see this, let us rewrite (48) in terms of $y \equiv iz$. One gets

$$\sum_{l \neq k} \frac{1}{y_k - y_l} + \frac{1}{2} (\alpha + 1) \frac{y_k + 1}{y_k - 1} + \frac{1}{2} (\beta + 1) y_k = 0, \quad k = 1, 2, \ldots, N, \quad (49)$$

where

$$\alpha = -A - N - i \frac{B}{A + N}, \quad \beta = -A - N + i \frac{B}{A + N}. \quad (50)$$

Hence the wave functions are

$$\phi_N \sim (\sin x)^{A+N} e^{\frac{B}{A+N} x} P_N^{(\alpha,\beta)} (iz). \quad (51)$$

This is consistent with the expression in [1] in the form

$$\phi_N \sim \left(z^2 + 1\right)^{-\frac{A+N}{2}} e^{\frac{B}{A+N} \cot^{-1} z} P_N^{(\alpha,\beta)} (iz). \quad (52)$$

**VIII. SUMMARY**

In [8] a new approach to both exact and quasi-exact solvabilities was proposed. In this approach the solvability of a one-dimensional quantum system can be solely classified by two integers, the degrees of two polynomials which determine the change of variable and the zero-th order prepotential. It was shown that exactly solvable models can only involve a change of variable which is sinusoidal, otherwise the system is quasi-exactly solvable. As such, four out of the ten exactly solvable models classified by shape invariance in supersymmetric quantum mechanics [1], namely, the Coulomb, Eckart, Rosen-Morse type I and II models,
are not covered by the approach as presented in [8]. In this paper, we have shown how these four models could be easily generated in a unified way in the prepotential approach by allowing the coefficients in the prepotential, which are assumed to be constants in [8], to be dependent on the number of nodes of the wave functions.

Thus with the results presented in this paper and those in [8], all the well known one-dimensional exactly solvable Schrödinger models have been generated by the prepotential approach in a way which, we believe, is much simpler and direct than other approaches.

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APPENDIX: COULOMB POTENTIAL IN SINUSOIDAL COORDINATES

Of the four potentials discussed in the main text, Coulomb potential can be treated in terms of sinusoidal coordinate, namely, \( z(x) = x \) (as in the main text, for simplicity, we absorb any scale factor into \( x \)). Unlike those discussed in the main text, in the present case what we want is to generate a potential that is singular at \( x = 0 \). According to [8], the prepotential \( W_0(x) \) should be modified to \( \tilde{W}_0(x) = W_0(x) - A \ln |x| \), where \( A \) is some real parameter, to ensure proper behavior of the wave function at the singular point \( x = 0 \). Naively, from the discussions in [8], the degree of \( W_0'z' \) should not be more than one in order to get exactly solvable model. So let us try \( W_0'(x) = b \) (\( z' = 1 \)). Thus the ground state wave function is \( \phi_0 \sim \exp(-\tilde{W}_0) \sim x^A \exp(-bx) \). Normalizability of \( \phi_0 \) requires \( A > 0 \) and \( b > 0 \).

Replacing \( W_0 \) in (11) by \( \tilde{W}_0 \), we obtain

\[
V_0 = \frac{A(A - 1)}{x^2} - \frac{2Ab}{x} + b^2
\]  

(A.1)
and
\[ \Delta V_N = -\frac{2A}{x} \sum_{k=1}^{N} \frac{1}{x_k} + 2 \sum_{k=1}^{N} \frac{1}{x - x_k} \left[ \sum_{l \neq k} \frac{1}{x_k - x_l} + \frac{A}{x_k} - b \right]. \] (A.2)

Simple poles are removed if \( x_k \)'s satisfied the BAE
\[ \sum_{l \neq k} \frac{1}{x_k - x_l} + \frac{A}{x_k} = b = 0, \quad k = 1, 2 \ldots, N. \] (A.3)

Summing over \( k \) in (A.3) gives
\[ A \sum_{k=1}^{N} \frac{1}{x_k} - bN = 0, \] (A.4)
and hence \( \Delta V_N = -2bN/x \). Finally, we have
\[ V_N = \frac{A(A - 1)}{x^2} - \frac{2b(A + N)}{x} + b^2. \] (A.5)

Since \( N \) appears in the \( 1/x \) term in \( V_N \), the system is quasi-exactly solvable in this form. But as discussed in the main text, one can make this system exactly solvable by allowing \( b \) to depend on \( N \) in such a way that the coefficient of \( 1/x \) term is \( N \)-independent, i.e.
\[ b = \frac{B}{A + N}, \] (A.6)
where \( B \) is a real constant. So the Coulomb potential is given by
\[ V_N = \frac{A(A - 1)}{x^2} - \frac{2B}{x} + \frac{B^2}{(A + N)^2}. \] (A.7)

This is consistent with the results given in (28) and (30). The BAE (A.3) becomes
\[ \sum_{l \neq k} \frac{1}{x_k - x_l} + \frac{A}{x_k} = \frac{B}{A + N}, \quad k = 1, 2 \ldots, N. \] (A.8)

Letting \( A = l + 1 \) and \( B = e^2/2 \), we recover the BAE in (32), and subsequently the wave function \( \phi_N \) in (34).

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