Explicit Construction of MBR Codes for Clustered Distributed Storage

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Abstract—This paper considers capacity-achieving coding for the clustered form of distributed storage that reflects practical storage networks. The suggested coding scheme is shown to exactly regenerate the arbitrary failed node with minimum required bandwidth, i.e., the proposed scheme is a minimum-bandwidth-regenerating (MBR) code of clustered DSSs with general parameter setting. The proposed code is a generalization of the existing MBR code designed for non-clustered DSSs. This code can be implemented in data centers with multiple racks (clusters), depending on the desired system parameters.

I. INTRODUCTION

Motivated by the need to handle the data deluge in modern networks, large-scale distributed storage systems (DSSs) are now widely deployed. With the aid of network coding, DSSs are highly tolerant to failure events, allowing users reliably access to the stored data. The early work of [1] obtained a closed-form expression for capacity $C(\alpha, \gamma)$, the maximum reliably storable file size, as a function of two important system parameters: the node capacity $\alpha$ and the bandwidth $\gamma$ for regenerating a failed node. The authors of [1] also found that there exists a trade-off relationship between $\alpha$ and $\gamma$, to satisfy $C(\alpha, \gamma) \geq M$, i.e., to reliably store a given file with size $M$. Based on the information-theoretic analysis of DSS in [1], several researchers [2]–[5] have developed practical solutions for implementing DSSs. The authors of [2]–[4] have suggested explicit network coding schemes which achieve capacity of DSSs, while [5] considered locally repairable codes which can repair any failed node by contacting a limited number of survived nodes. All these works considered homogeneous setting where each node has identical storage capacity and communication bandwidth.

However, in the real world, data centers arrange their storage devices into multiple racks forming clusters, where the available cross-rack communication bandwidth is far less than the available intra-rack bandwidth. In an effort to reflect this practical feature of data centers, several researchers recently considered the concept of clustered topologies in DSSs [6]–[8]. Especially in [6], the authors of the present paper considered clustered DSSs with $n$ storage nodes dispersed in $L$ clusters, where each node has storage capacity of $\alpha$. To reflect the difference between intra-rack and cross-rack bandwidth, [6] adopts two parameters for indicating repair bandwidths: $\beta_{1}$ for the repair bandwidth among nodes in the same cluster and $\beta_{c}$ for the repair bandwidth among nodes in different clusters. Under this setting, the storage capacity $C(\alpha, \beta_{1}, \beta_{c})$ of clustered DSSs — the maximum reliably retrievable file size via a contact of arbitrary $k < n$ nodes — has been obtained. Moreover, the existence of capacity-achieving network coding has been proved in [6], while explicit coding schemes which achieve capacity of the clustered distributed storage were not available.

This paper designs an explicit coding scheme which achieves capacity of clustered DSSs as computed in [6]. The proposed scheme is shown to have the minimum required repair bandwidth, i.e., the proposed scheme is a minimum-bandwidth-regenerating (MBR) code of clustered DSSs, for arbitrary system parameter values of $n, k, L, \beta_{1}, \beta_{c}$. Naturally, the proposed coding scheme can be implemented in real data centers with multiple racks or clusters, depending on the desired system parameters. The work of [2] provided an insightful design of MBR codes achieving capacity of DSSs. Using the properties of the incidence matrix of a complete graph, the authors of [2] showed that their coding scheme enables the exact repair of arbitrary failed node, while allowing reliable data retrieval through a contact of arbitrary $k$ nodes out of $n$. There have been some research [8]–[9] on designing regenerating codes for DSSs with multi-rack structure, but to a limited extent. The coding scheme suggested in [9], for example, is well suited for multi-rack systems, but not proven to be an MBR code. The code designed in [8] cannot be applied in general $n, k, L$ setting, while the present paper suggests an MBR code for general $n, k, L, \beta_{1}, \beta_{c}$ setting. The code proposed in the present paper can be viewed as a generalization of the MBR coding scheme of [2] for application to clustered DSSs.

II. BACKGROUND & NOTATIONS

Here, we briefly summarize the clustered distributed storage systems, which is originally suggested in [6]. Consider a file with $M$ symbols to be stored. The file is encoded and distributed into $n$ storage nodes, which are uniformly dispersed into $L$ clusters. We use the notation $n_{f} = n/L$ to indicate the number of nodes in each cluster. Fig. 1 provides the system model for $n = 12, L = 3$ case. When a node fails, the node is regenerated by contacting $n_{f} - 1$ survived nodes in the same cluster and $n - n_{f}$ nodes in other clusters. In the regeneration
where contacting arbitrary maximum amount of data that can be reliably retrievable by (In the clustered DSS with the given parameters of \[6\]): The clustered DSS is obtained as Theorem 1 of \[6\]. The capacity is expressed as \(\gamma\) stored in each storage node and 2) the amount \(\alpha\) of bandwidth required for regenerating a failed node. According to the \(\gamma\) figure, the point with minimum required repair bandwidth \(\gamma\) is called the minimum-bandwidth-regenerating (MBR) point. The explicit regenerating coding schemes which achieve the \(\gamma\) figure, the point with minimum required repair bandwidth \(\gamma\) is called the MBR codes. According to Corollary 3 of \[10\], the node capacity is equal to the repair bandwidth, i.e., \(\alpha = \gamma\) for the MBR codes. Throughout the paper, we use some useful additional notations. For a positive integer \(n\), we denote \(\{1, 2, \cdots, n\}\) as \([n]\). For given parameters of \(k\) and \(n_t\), we use the notations of \(q := \left\lfloor \frac{k}{n_t} \right\rfloor\), \(r := k \mod n_t = k - q n_t\). Note that clustered DSSs can be expressed as a two-dimensional representation, as in Fig. 3. In this structure, we denote the \(j^{th}\) storage node in \(l^{th}\) cluster as \(N(l, j)\). A vector is denoted as \(v\) using a bold-faced lower case letter. For given integers \(k\) and \(n\), the binomial coefficient \(\binom{n}{k}\) is written as \(\binom{n}{k}\).

\[C(\alpha, \beta_I, \beta_c) = \sum_{i=1}^{n_t} \sum_{j=1}^{g_i} \min\{\alpha, \rho_i \beta_I + \phi_i^{(j)} \beta_c\},\]  \hspace{1cm} (2)

where

\[\rho_i = n_t - i\]  \hspace{1cm} (3)

\[\phi_i^{(j)} = n - l_i^{(j)} - \rho_i\]  \hspace{1cm} (4)

\[l_i^{(j)} = (\sum_{m=1}^{i-1} g_m) + j\]  \hspace{1cm} (5)

\[g_m = \begin{cases} \left\lfloor \frac{k}{n_t} \right\rfloor + 1, & m \leq \text{mod}(k, n_t) \\ \left\lfloor \frac{k}{n_t} \right\rfloor, & \text{otherwise}. \end{cases}\]  \hspace{1cm} (6)

Fig. 1: Clustered DSS for \(n = 12, L = 3, n_t = n/L = 4\)

Recall that clustered DSSs have two important system parameters: node capacity \(\alpha\) and repair bandwidth \(\gamma\). The feasible \(\alpha, \gamma\) points which satisfy \(C(\alpha, \gamma) \geq M\) are obtained in a closed-form solution in Corollaries 1 and 2 of \[10\]. The set of feasible \(\alpha, \gamma\) points are illustrated in Fig. 2. In this figure, the point with minimum required repair bandwidth \(\gamma\) is called the MBR point. The explicit regenerating coding schemes which achieve the MBR point is called the MBR codes. According to Corollary 3 of \[10\], the node capacity is equal to the repair bandwidth, i.e.,

\[\alpha = \gamma\]  \hspace{1cm} (7)

III. MBR Code Design for \(\beta_c = 0\) Case

We construct an MBR code and confirm that the proposed code achieves capacity of clustered distributed storage. This section considers scenarios with zero cross-cluster repair bandwidth, i.e., \(\beta_c = 0\).
A. Parameter Setting for MBR Point

We consider the MBR point \((\alpha, \gamma) = (\alpha_{mbr}, \gamma_{mbr})\) which can reliably store file \(M\). Without a loss of generality, we set \(\beta_1 = 1\). Then, the node storage capacity is
\[
\alpha_{mbr} = \gamma_{mbr} = (n_1 - 1)\beta_1 = n_1 - 1.
\]
From Corollary 3 of \([10]\), the minimum repair bandwidth for \(\beta_3 = 0\) is
\[
\gamma_{mbr} = M/\lambda_0
\]
where \(\lambda_0\) is defined as
\[
\lambda_0 = q \sum_{i=0}^{n_1-1} \left(1 - \frac{i}{n_1-1}\right) + \sum_{i=0}^{r-1} \left(1 - \frac{i}{n_I-1}\right)
\]
from (36) of \([10]\). Combining (10) and (11), the maximum reliably storable file size \(M\) is expressed as
\[
M = \lambda_0 \gamma_{mbr} = \lambda_0 (n_1 - 1).
\]

B. Code Construction

We now describe an explicit coding scheme with parameters of \(\alpha = n_1 - 1, \gamma = n_1 - 1, \beta_1 = 1\), which satisfies the following:

- A failed node can be repaired within each cluster.
- Contacting any \(k\) out of \(n\) nodes can recover file \(M\).

The suggested coding scheme is a modified version of the existing coding scheme of \([2]\), with the modification required to reflect the clustered structure. Let \(f = [f_1, f_2, \cdots, f_M]^T\) be a vector of \(M\) source symbols. Define
\[
\theta = \left(\frac{n_1}{2}\right) L,
\]
which represents the length of a codeword. Consider a set of \(\theta\) vectors \(\{v_1, \cdots, v_\theta\}\) which constructs a \([\theta, M]\) maximum-distance-separable (MDS) code. In other words, \(v_i\) is a vector of length \(M\) for all \(i \in [\theta]\), where codewords \(c = [c_1, c_2, \cdots, c_\theta]^T\) are constructed by the following rule
\[
c^T = [c_1, c_2, \cdots, c_\theta] = [f_1, f_2, \cdots, f_M][v_1, \cdots, v_\theta]
\]
satisfy the MDS property. Namely, obtaining any \(M\) out of \(\theta\) codewords \(\{c_i\}_{i=1}^\theta\) is sufficient to recover \(M\) source symbols \(f = [f_1, f_2, \cdots, f_M]^T\).

Now, consider the incidence matrix \(V_i\) of a fully connected graph \(G_i\) with \(t\) vertices. Fig. 4 gives an example for \(t = 4\). The incidence matrix is a \(\theta \times \binom{n_I}{2}\) matrix given by:
\[
V_i(j, i) = \begin{cases} 
1, & \text{if } j^{th} \text{ edge is connected to } j^{th} \text{ node} \\
0, & \text{otherwise}
\end{cases}
\]

To construct an MBR code, here we consider an incidence matrix \(V_{i, MBR}\). For a given codeword \(c = [c_1, \cdots, c_\theta]\) with \(\theta\) coded symbols, we have the following assignment rule for coded symbols:

- Node \(N(l, j)\) stores the symbol \(c_{(l-1)\binom{\gamma}{2}} + i = f^T v_{(l-1)\binom{\gamma}{2}} + i\) if and only if \(V_{i, MBR}(j, i) = 1\) holds.

Fig. 4: Incidence matrix \(V_i\) of a fully connected graph \(G_i\), for \(t = 4\) case

Here, the ranges of parameters are
\[
l \in [L], \quad j \in [n_I], \quad i \in \left[\binom{n_I}{2}\right].
\]

The suggested coding scheme has the following properties.

**Proposition 1.** The proposed MBR code for \(\beta_3 = 0\) satisfies

(a) Each coded symbol is stored in exactly two different storage nodes.

(b) Nodes in different clusters do not share any coded symbols.

(c) Nodes in the same cluster share exactly one coded symbol.

(d) Each node contains \(\alpha = n_1 - 1\) coded symbols.

**Proof.** See Appendix A

Now we prove that the suggested coding scheme satisfies the exact regeneration property and the data reconstruction property.

**Exact Regeneration:** Suppose that \(N(l_0, j_0)\), the \(j_0^{th}\) storage node in \(l_0^{th}\) cluster, is out of order. From Proposition 1(a) and Proposition 1(b), any coded symbol in \(N(l_0, j_0)\) is also stored in another node in the \(l_0^{th}\) cluster. In other words, the set \(\cup_{j \neq j_0} N(l_0, j)\) contains all symbols stored in \(N(l_0, j_0)\). Thus, \(N(l_0, j_0)\) can be regenerated by contacting other nodes in the \(l_0^{th}\) cluster only. From Proposition 1(c) we can confirm that the node \(N(l_0, j_0)\) can be exactly regenerated by downloading \(\beta_1 = 1\) symbol from each of the \(n_I - 1\) nodes in \(\cup_{j \neq j_0} N(l_0, j)\). Moreover, the regeneration process does not incur cross-cluster repair traffic, i.e., \(\beta_3 = 0\).

**Data Reconstruction:** The data collector (DC) connects to arbitrary \(k\) out of \(n\) nodes to recover the original source symbol vector \(f = [f_1, f_2, \cdots, f_M]^T\). Here, we prove that contacting arbitrary \(k\) nodes can obtain \(M\) coded symbols \(\{c_i\}_{i=1}^\theta\), which is sufficient to recover the vector \(f\) of \(M\) source symbols. Consider \(k\) nodes contacted by DC. Define the corresponding contact vector \(a = [a_1, \cdots, a_L]\) where \(a_l\) represents the number of contacted nodes in the \(l^{th}\) cluster. Then, the set of possible contact vectors is expressed as
\[
A = \left\{ a = [a_1, \cdots, a_L] : \sum_{l=1}^L a_l = k, a_l \in \{0, 1, \ldots, n_I\} \right\}.
\]

Thus, for an arbitrary contact of \(k\) nodes, there exists a corresponding contact vector \(a \in A\). Let \(n(a)\) be the number
of distinct coded symbols obtained by contacting $k$ nodes with the corresponding contact vector being $a$. Then, we establish the following bound on $n(a)$.

**Proposition 2.** Consider the proposed MBR code for $\beta_e = 0$. Let DC contact arbitrary $k$ nodes with the contact vector $a$. Then, the number of distinct coded symbols $\{c_i\}$ retrieved by DC is lower bounded by $M$ in (17). In other words,

$$n(a) \geq M \quad \forall a \in A.$$  

(18)

**Proof.** See Appendix [5] □

Therefore, for arbitrary contacting $k$ nodes, the suggested coding scheme guarantees at least $M$ distinct coded symbols to be retrieved. Using the MDS property of the $[\theta, M]$ code in [15], we can confirm that the original source symbol $f$ can be obtained from the retrieved $M$ distinct coded symbols. This concludes the proof for the data reconstruction property.

**C. Example**

Consider the scenario with $n = 12, L = 3, n_I = n/L = 4$. According to the suggested coding scheme, node $N(l, j)$ stores $c_{6(l-1)+j}$ if and only if $V_4(i, l) = 1$. Here, the ranges of parameters are $l \in [3], j \in [4]$ and $i \in [6]$. Using the $V_4$ matrix in Fig. 4, the coded symbols are distributed as in Fig. 5. Each circle represents a storage node, which stores $3$ coded symbols. For example, in the case of $(l, j) = (2, 3)$, the node $N(2, 3)$ stores three coded symbols: $c_8, c_{10}$ and $c_9$.

**Exact Regeneration:** Suppose node $N(2, 3)$ fails. Then, node $N(2, 1)$ gives $c_8$, node $N(2, 2)$ gives $c_{10}$, and node $N(2, 4)$ gives $c_8$. These three symbols are stored in the new storage node for replacing $N(2, 3)$. This regeneration process satisfies $\beta_I = 1, \beta_e = 0$. The same argument holds for arbitrary single-node failure events.

**Data Reconstruction:** Suppose that the data collector connects to $k = 6$ nodes, say, $N(1, 1), N(1, 2), N(1, 3), N(1, 4), N(2, 1), \text{and} N(2, 2)$. It is easy to see that this process allows the retrieval of the eleven coded symbols $\{c_i\}_{i=1}^{11}$. From (12) and (13), we have $M = 11$, while $\theta = 18$ holds from (14). By using the MDS property of the $[\theta, M] = [18, 11]$ code in (15), we can easily obtain $\{f_i\}_{i=1}^{11}$ from the retrieved coded symbols $\{c_i\}_{i=1}^{11}$. The same can be shown for any arbitrary choice of $k = 6$ nodes.

**IV. MBR Code Design for $\beta_e \neq 0$ Case**

We now construct an MBR code for scenarios with nonzero cross-cluster repair bandwidth, i.e., $\beta_e \neq 0$.

**A. Parameter Setting for MBR Point**

Let us assume that $\chi := \beta_I / \beta_e$ is a natural number. Without losing generality, we set $\beta_I = \chi$ and $\beta_e = 1$. Then, the node storage capacity is expressed as

$$\alpha_{nbr} = \gamma_{nbr} = (n_I - 1)\chi + (n - n_I)$$

from (1) and (7). Under this setting, the following proposition provides a simplified form of the capacity expression.

**Proposition 3.** For the case of $\beta_e = 1$ and $\beta_I = \chi$, the capacity of clustered distributed storage can be expressed as

$$M = k\alpha - \frac{1}{2}(\chi - 1)(qn^2 + r^2 - k) - \frac{k(k - 1)}{2}$$

(19)

where $q$ and $r$ are defined in (8) and (9), respectively.

**Proof.** See Appendix C □

**B. Code Construction**

We use the same notations defined in the $\beta_e = 0$ case with a slight modification. First, let $f = [f_1, f_2, \cdots, f_M]^T$ be a vector of $M$ source symbols. Define

$$\theta = (\chi - 1)\left(\frac{n_I}{2}\right) + \left(\frac{n}{2}\right),$$

(20)

which represents the length of a codeword. Consider the $[\theta, M]$ MDS code

$$c^T = [c_1, c_2, \cdots, c_M] = [f_1, f_2, \cdots, f_M][v_1, \cdots, v_\theta]$$

$$= [f^Tv_1, \cdots, f^Tv_\theta]$$

(21)

where $\theta$ is in (20) and $M$ is in (19). Now, define two incidence matrices $V_n$ and $V_n$. Note that $V_n$ is a $n \times \binom{n}{2}$ matrix, while $V_n$ is a $n \times \binom{n}{2}$ matrix. An example for the incidence matrix is shown in Fig. 4. For a given codeword $c = [c_1, c_2, \cdots, c_M]^T$ with $\theta$ coded symbols, we have the following rules for storing coded symbols into each node:

- Node $N(l, j)$ stores $c_i$ if and only if $V_n(n_I(l-1)+i)$.
- For $t \in [x-1]$, node $N(l, j)$ stores $c_{(2)+t}$ if and only if $V_n(j, i)$.

Here, the ranges of the parameters are

$$l \in [L], \quad j \in [n_I], \quad i_1 \in \left[\frac{n(n-1)}{2}\right], \quad i_2 \in \left[\frac{n_I(n_I-1)}{2}\right].$$

The suggested coding scheme has the following properties.

**Proposition 4.** The proposed MBR code for $\beta_e \neq 0$ satisfies

(a) Each coded symbol is stored in exactly two different storage nodes.
(b) Nodes in different clusters share one coded symbol.
(c) Nodes in the same cluster share $\chi$ coded symbols.
(d) Each node contains $\alpha = (n_1 - 1)\chi + (n - n_1)\chi$ coded symbols.

Proof. See Appendix D

Now we prove that the suggested coding scheme satisfies the exact regeneration property and the data reconstruction property.

Exact Regeneration: Consider $N(l_0, j_0)$, the $j_0^{th}$ storage node in the $l_0^{th}$ cluster, is broken. From Proposition 3(a) and Proposition 4(c), each survived node in the $l_0^{th}$ cluster contains $\chi$ distinct coded symbols which are stored in $N(l_0, j_0)$. Therefore, the set $\cup_{j \neq j_0} \{N(l_0, j)\}$ contains $(n_1 - 1)\chi$ symbols stored in $N(l_0, j_0)$. Similarly, from Proposition 3(b) and Proposition 4(b) each survived node not in the $l_0^{th}$ cluster contains one distinct coded symbol which is stored in $N(l_0, j_0)$. Therefore, the set $\cup_{j \neq j_0} \{N(l, j)\}$ contains $(n - n_1)$ symbols stored in $N(l_0, j_0)$. In summary, $\alpha = (n_1 - 1)\chi + (n - n_1)$ coded symbols stored in the failed node $N(l_0, j_0)$ can be recovered by contacting $n_1 - 1$ nodes in the $l_0^{th}$ cluster and $n - n_1$ nodes in other $l \neq l_0$ clusters; $n_1 - 1$ nodes within the same cluster transmit $\chi$ coded symbols each, while $n - n_1$ nodes in other clusters contribute one coded symbol each. Therefore, this process satisfies $\beta_1 = \chi, \beta_2 = 1$, as described in Section IV-A.

Data Reconstruction: Similar to the $\beta_2 = 0$ case, we here prove that contacting arbitrary $k$ out of $n$ nodes can recover the original source symbol $f = [f_1, f_2, \ldots, f_M]^T$. We use the notations $n(a)$ and $A$, which are defined in Section III-B.

Similar to Proposition 2 we have the following proposition for $\beta_0 \neq 0$. This proposition completes the proof for the data reconstruction property, in a similar way that Proposition 2 completes the proof for $\beta_0 = 0$.

Proposition 5. Consider the proposed MBR code for $\beta_0 \neq 0$. Let DC contact arbitrary $k$ nodes with a being the corresponding contact vector. Then, the number of distinct coded symbols $\{c_i\}$ retrieved by DC is lower bounded by $M$ in $\mathcal{M}[19]$. In other words,

\[ n(a) \geq \mathcal{M} \quad \forall a \in A. \]  

(22)

Proof. See Appendix E

C. Example

Consider the case $n = 6, L = 2, n_1 = n/L = 3, \chi = 3$. According to the suggested coding scheme, $N(l, j)$ stores

- $c_{i_1}$ if and only if $V_0(3(l - 1) + j, i_1) = 1$.
- $c_{15+3(2l+1)-3}+i_2$ if and only if $V_0(j, i_2) = 1$, for $t = 1, 2$

Here, $l \in [2], j \in [3], i_1 \in [15], i_2 \in [3]$. Using the following $V_0$ and $V_3$ matrices,

\[
V_0 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
V_3 = \begin{bmatrix}
1 & 1 & 0
1 & 0 & 1
0 & 1 & 0
0 & 0 & 1
0 & 0 & 0
\end{bmatrix}
\]

The coded symbols are distributed as in Fig. 6. Each rectangle represents a storage node, which stores $\alpha = (n_1 - 1)\chi + (n - n_1) = 9$ coded symbols. The numbers in a rectangle represent the indices of the coded symbols contained in the node. For example, when $(l, j) = (1, 2)$, the node $N(l, j) = N(1, 2)$ stores $\{c_i\}$ for $i = 1, 6, 7, 8, 9, 16, 18, 19, 21$.

Exact Regeneration: Suppose the node $N(1, 2)$ is broken. Then, $N(1, 1)$ provides $c_1, c_{16}, c_{19}$, while $N(1, 3)$ gives $c_6, c_{18}, c_{21}$. The nodes $N(2, 1), N(2, 2), N(2, 3)$ transmit $c_7, c_8, c_9$, respectively. This completely regenerates the failed node $N(1, 2)$. Any single node failure can be repaired in a similar way.

Data Reconstruction: For $\alpha = 9, n = 6, L = 2, k = 3, \beta_1 = 3, \beta_2 = 1$, the capacity value in $[19]$ is $\mathcal{M} = 18$. Consider an arbitrary contact of $k = 3$ nodes, say, three nodes in the $1^{st}$ cluster in Fig. 6. Note that this contact can retrieve 18 coded symbols $c_i$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 18, 19, 20, 21\}$. By using the $[\theta, \mathcal{M}] = [27, 18]$ MDS code in $[21]$, we obtain $\{f_i\}_{i=1}^{18}$ from the retrieved 18 coded symbols. The same can be shown for any arbitrary contact of $k = 3$ nodes.

V. CONCLUSION

Focusing on clustered distributed storage systems which reflect the structure of real data centers, an MBR coding scheme which achieves capacity has been presented. For arbitrary parameter values of $n, k, L, \beta_1, \beta_2$, the proposed coding scheme is shown to satisfy two key requirements: 1) exact regeneration of an arbitrary failed node with minimum repair bandwidth and 2) data reconstruction by contacting arbitrary $k$ nodes. The coding scheme is a generalization of the MBR coding scheme of $[2]$ developed for homogeneous (non-clustered) DSSs.

APPENDIX A

PROOF OF PROPOSITION 4

We begin with reviewing four properties of incidence matrix $V_t$.

Proposition 6. The incidence matrix $V_t$ of a fully connected graph $G_t$ with $t$ vertices has the following four properties as summarized in $[2]$:

(a) Each element is either 0 or 1.
(b) Each row has exactly $(t - 1)$ 1’s.

| $j = 1$ | $j = 2$ | $j = 3$ |
|--------|--------|--------|
| $[1, 2, 3, 4, 5]$ | $[1, 6, 7, 8, 9]$ | $[2, 6, 10, 11, 12]$ |
| $[16, 17, 19, 20]$ | $[16, 18, 19, 21]$ | $[17, 18, 20, 21]$ |

1st cluster ($l = 1$)

| $[3, 7, 10, 13, 14]$ | $[4, 8, 11, 13, 15]$ | $[5, 9, 12, 14, 15]$ |
| $[22, 23, 25, 26]$ | $[22, 24, 25, 27]$ | $[23, 24, 26, 27]$ |

2nd cluster ($l = 2$)
(c) Each column has exactly two 1’s.
(d) Any two rows have exactly one section of 1’s.

Recall that for a given codeword \( c = [c_1, \cdots, c_\theta] \) with \( \theta = \binom{n_f}{2} L \) coded symbols, the node \( N(l, j) \) stores the symbol \( c_{(l-1)(n_f^2) + i} \) if and only if \( V_{n_1}(j, i) = 1 \). Note that any natural number \( s \in (\theta) \) can be uniquely expressed as an \((l_0, i_0)\) pair where

\[
s = (l_0 - 1) \binom{n_f}{2} + i_0,
\]

\( l_0 \in \{1, 2, \cdots, L\} \),

\( i_0 \in \{1, 2, \cdots, \binom{n_f}{2}\} \)

holds.

Therefore, a coded symbol \( c_s = c_{(l_0-1)(n_f^2)+i_0} \) is stored at the node \( N(l_0, j) \) if and only if \( V_{n_1}(j, i_0) = 1 \). From the Proposition 6, each column of \( V_{n_1} \) has exactly two 1’s. In other words,

\[
V_{n_1}(j_1, i_0) = V_{n_1}(j_2, i_0) = 1
\]

holds for some \( j_1, j_2 \in \{1, 2, \cdots, n_1\} \). Therefore, nodes \( N(l_0, j_1) \) and \( N(l_0, j_2) \) store the coded symbol \( c_s \). Note that no other nodes can store \( c_s \) since \((A.1)\) is the unique expression of \( s \) into \((l_0, i_0)\) pair. This proves Proposition \([1]\) and Proposition \([2]\).

Finally, according to the second property of Proposition \([2]\) for \( t = n_1 \), each row of \( V_{n_1} \) has \((n_1-1)\) number of 1’s. Thus, \( V_{n_1}(j, i_0) = 1 \) holds for some \( i_0 \in \{1, 2, \cdots, n_1\} \).

Therefore, node \( N(l, j) \) contains \((n_1-1)\) coded symbols of \( \{c_{(l-1)(n_f^2) + i_0} \}_{i_0=1}^{n_f} \). This proves Proposition \([1]\) and Proposition \([4]\).

APPENDIX B

PROOF OF PROPOSITION \([4]\)

For a given contacting vector \( a = [a_1, a_2, \cdots, a_L] \), we know that \( a_1 \) nodes are contacted in 1st cluster, \( a_2 \) nodes are contacted in 2nd cluster, and so on. Moreover, the total number of contacted nodes is \( \sum_{l=1}^{L} a_l = k \) from the definition of contacting vector. From Proposition \([1]\) and Proposition \([4]\) we have

\[
n(a) = k \alpha - \sum_{l=1}^{L} \left( \frac{a_l}{2} \right)
\]

for a given \( a \in \mathcal{A} \). Here, the first term represents the total number of coded symbols retrieved from \( k \) nodes, each containing \( \alpha \) symbols. The second term represents the number of symbols which are retrieved twice.

Consider \( a^*_1 = [a_1^*, \cdots, a_L^*] \) where

\[
a^*_i = \begin{cases} 
  n_1, & i \leq \lfloor k/n_1 \rfloor \\
  k - \lfloor \frac{k}{n_1} \rfloor n_1, & i = \lfloor k/n_1 \rfloor + 1 \\
  0, & \text{otherwise}
\end{cases}
\]

Given the sequence \( a_1^*, \cdots, a_L^* \), let the notation \( a_{i}^* \) refers to the permuted sequence such that

\[
a_{i}^* \geq a_{i}^* \geq a_{i}^* \cdots
\]

holds. Moreover, for a given arbitrary \( a = [a_1, \cdots, a_L] \in \mathcal{A} \), define \( a_{i}^* \) as the permuted sequence such that

\[
a_{i}^* \geq a_{i}^* \geq a_{i}^* \cdots
\]

holds. Then, we have

\[
\sum_{i=1}^{L} a_{i}^* = \sum_{i=1}^{L} a_{i} = k,
\]

\[
\sum_{i=1}^{L} a_{i}^* \geq \sum_{i=1}^{L} a_{i} \text{ for } t = 1, 2, \cdots, L.
\]

In other words, for arbitrary \( a \in \mathcal{A} \), we can conclude that \( a^* \) majorizes \( a \) (the definition of majorization is in \([11]\) ), which is denoted as

\[
a^* \succ a.
\]

Note that \( g(x) \triangleq x^2 \) is convex for real number \( x \). Then, from Thm 21.3 of \([11]\),

\[
f(a) \triangleq \sum_{i=1}^{L} a_i = \sum_{i=1}^{L} a_i^2
\]

is a schur-convex function on \( \mathbb{R}^L \), where \( \mathbb{R} \) is the set of real number. From the definition of Schur-convexity (definition 21.4 of \([11]\) ), \( x \succ y \) implies that \( f(x) \geq f(y) \). Thus, from \([B.4]\),

\[
f(a^*) = \sum_{i=1}^{L} (a_{i}^*)^2 \geq \sum_{i=1}^{L} (a_{i})^2 = f(a)
\]

holds for arbitrary \( a \in \mathcal{A} \). Therefore,

\[
a^* = \arg\max_{a \in \mathcal{A}} \sum_{i=1}^{L} (a_{i})^2 = \arg\max_{a \in \mathcal{A}} \sum_{i=1}^{L} \frac{a_{i}(a_{i} - 1)}{2}
\]

where the last equality is from \([B.3]\). From \([B.1]\), we have

\[
n(a) = k\alpha - \sum_{i=1}^{L} \frac{a_{i}(a_{i} - 1)}{2} \geq k\alpha - \sum_{i=1}^{L} \frac{a_{i}^*(a_{i}^* - 1)}{2}
\]

\[
= k\alpha - \frac{1}{2} (a^* \cdot a^* - k)\alpha.
\]

Using \([8]\), \([9]\) and \([B.2]\), this can be reduced as

\[
n(a) \geq k\alpha - \frac{1}{2} (qa_1^2 + r^2 - k) = k\alpha - \frac{1}{2} (k(1 - r)q_1 + r^2 - k)
\]

\[
= k\alpha - \frac{1}{2} ((n_1-1)k - r(n_1 - r))
\]

Using \( \alpha = (n_1 - 1) \) from \([10]\), we have

\[
n(a) \geq \frac{k\alpha}{2} + \frac{1}{2} r(n_1 - r).
\]

According to the Proposition 5 of \([10]\), the capacity in \([13]\) is
expressed as
\[ M = \lambda_0(n_I - 1) = \frac{1}{2} [k(n_I - 1) + r(n_I - r)] \]
\[ = \frac{k\alpha}{2} + \frac{1}{2} r(n_I - r), \]
where the last equality is from (9). Therefore, we have \( n(a) \geq M \) for all \( a \in A \), which completes the proof.

**APPENDIX C**

**PROOF OF PROPOSITION 3**

We begin with three properties, which help to prove Proposition 3. Here, we use several definitions: \( g_i, q \) and \( r \) are defined in (6), (8), and (9), respectively.

**Property 1:**
\[ \sum_{i=1}^{n_I} g_i = k \quad \text{(C.1)} \]
holds.

**Proof.** Note that
\[ g_i = \begin{cases} q + 1, & i \leq r \\ q, & \text{otherwise} \end{cases} \]
Therefore,
\[ \sum_{i=1}^{n_I} g_i = (q + 1)r + q(n_I - r) = r + qn_I = k, \]
where the last equality is from (9).

**Property 2:**
\[ \sum_{i=1}^{n_I} ig_i = \frac{1}{2}(qn_I^2 + r^2 + k) \]
holds.

**Proof.** From (C.2),
\[ \sum_{i=1}^{n_I} ig_i = \sum_{i=1}^{r} (q + 1)i + \sum_{i=r+1}^{n_I} q_i 
= q \sum_{i=1}^{n_I} i + \sum_{i=1}^{r} i 
= q \frac{n_I(n_I + 1)}{2} + r(r + 1) 
= \frac{1}{2}(qn_I^2 + r^2 + qn_I + r) 
= \frac{1}{2}(qn_I^2 + r^2 + k) \]
where the last equality is from (9).

**Property 3:**
\[ \sum_{i=1}^{n_I} \sum_{j=1}^{i-1} \sum_{m=1}^{j} g_m + j = \frac{k}{2} + \frac{k^2}{2} \]
holds.

**Proof.**
\[ \text{(LHS)} = \sum_{i=1}^{n_I} \left\{ g_i \sum_{m=1}^{i-1} g_m + \frac{g_i(g_i + 1)}{2} \right\} \]
\[ = \frac{1}{2} \left\{ \sum_{i=1}^{n_I} \left[ n_I g_i g_m + \sum_{i=1}^{g_i} g_i^2 \right] + \frac{1}{2} \sum_{i=1}^{n_I} g_i \right\} \]
\[ = \frac{1}{2} \left\{ \sum_{i=1}^{n_I} \left[ n_I g_i g_m + \sum_{i=1}^{g_i} g_i^2 \right] + \frac{k}{2} \right\} \]
\[ = \frac{k^2}{2} + \frac{k}{2}, \]
where the second-last equality is from (C.1), and the last equality is from (C.3).

Note that we have
\[ \rho_i \beta_I + \phi_i^{(j)} \beta_c \leq \gamma, \quad \forall i \in \{n_I\}, \forall j \in \{g_i\} \quad \text{(C.5)} \]
according to the proposition 2 of [10]. Combined with (7), the capacity expression in (2) is reduced as
\[ M = \sum_{i=1}^{n_I} \sum_{j=1}^{g_i} (\rho_i \beta_I + \phi_i^{(j)} \beta_c) \quad \text{(C.6)} \]
Combining (1) and (C.6), we have
\[ M = \sum_{i=1}^{n_I} \sum_{j=1}^{g_i} \{ g_i (\gamma - (i - 1) \beta_I) - (j - i + 1 - \sum_{m=1}^{g_i-1} g_m) \beta_c \} \quad \text{(C.3)} \]
Since \( \alpha = \gamma, \beta_I = \chi, \beta_c = 1 \) holds, the capacity expression reduces to
\[ M = \sum_{i=1}^{n_I} \sum_{j=1}^{g_i} \{ \alpha - \chi(i - 1) + j - \sum_{m=1}^{g_i-1} g_m \} \]
\[ = \sum_{i=1}^{n_I} (\alpha + \chi) g_i - \sum_{i=1}^{n_I} (\chi - 1) g_i \]
\[ - \sum_{i=1}^{n_I} \sum_{j=1}^{g_i} \sum_{m=1}^{g_i-1} g_m \]
Using (C.1), (C.3), and (C.4), this reduces to
\[ M = k(\alpha + \chi) - (\chi - 1) \frac{1}{2} (qn_I^2 + r^2 + k) - \left( \frac{k}{2} + \frac{k^2}{2} \right) \]
\[ = k\alpha + (\chi - 1)k + k \]
\[ - (\chi - 1) \frac{1}{2} (qn_I^2 + r^2 + k) - \left( \frac{k}{2} + \frac{k^2}{2} \right) \]
\[ = k\alpha - (\chi - 1) \frac{1}{2} (qn_I^2 + r^2 - k) + k - \left( \frac{k}{2} + \frac{k^2}{2} \right) \quad \text{(C.4)} \]
which completes the proof.
APPENDIX D
PROOF OF PROPOSITION 4

Recall that the suggested coding scheme has the following rule: for \( l \in \langle L \rangle \) and \( j \in \langle n_I \rangle \), the node \( N(l, j) \) stores

- \( c_{ij} \) if and only if \( V_{ni}(l-1) + j, i \rangle = 1 \) for \( i \in \langle n_I \rangle \)
- \( c_{ij} \) if and only if \( V_{ni}(l, i_2) = 1 \) for \( i \in \langle n_I \rangle \) and \( i_2 \in \langle n_2 \rangle \)

Note that the first rule deals with storing \( c_{ij} \) with \( s \in S_1 \), and the second rule stores \( c_{ij} \) with \( s \in S_2 \), where

\[
S_1 \triangleq \{ 1, 2, \cdots, \left( \frac{n}{2} \right) \} \tag{D.1} \\
S_2 \triangleq \left\{ \left( \frac{n}{2} \right) + 1, \left( \frac{n}{2} \right) + 2, \cdots, \theta \right\} \tag{D.2}
\]

Here,

\[
\theta = \left( \frac{n}{2} \right) + (\chi - 1) \left( \frac{n_I}{2} \right) L. \tag{D.3}
\]

as in (20).

We first focus on the coded symbols \( c_{ij} \) for \( s \in S_1 \). The mathematical results are summarized in following remark, with proofs at below.

Remark 1. Consider coded symbols \( c_{ij} \) for \( s \in S_1 \) only. Then,

- Each coded symbol is stored in exactly two different storage nodes.
- Nodes in different clusters share one coded symbol.
- Nodes in the same cluster share one coded symbol.
- Each node contains \( n - 1 \) coded symbols.

Proof. The first statement is directly obtained from the Proposition 3. The second and third statement is obtained from the Proposition 3. Finally, the last statement is from Proposition 3.

Now we focus on the coded symbols \( c_{ij} \) for \( s \in S_2 \). First, note that we can represent \( s = s' + \left( \frac{n}{2} \right) \) for \( s' \in S'_2 \) where

\[
S'_2 \triangleq \{ 1, 2, \cdots, \left( \chi - 1 \right) \left( \frac{n_I}{2} \right) L \}. \tag{D.4}
\]

Moreover, \( s' \in S'_2 \) can be uniquely represented as \((l, t, i_2)\) tuple, through the following steps. For ease of understanding, please refer to Fig. 7

1) Divide \( S'_2 \) into \( L \) partitions \( P_1, P_2, \cdots, P_L \), where each partition \( P_1 \) has

\[
\Delta \triangleq (\chi - 1) \left( \frac{n_I}{2} \right) \tag{D.5}
\]

elements. To be specific, the \( L \) partitions are

\[
P_1 = \{ 1, 2, \cdots, \Delta \},
\]

\[
P_2 = \{ \Delta + 1, \Delta + 2, \cdots, 2\Delta \},
\]

\[
\vdots
\]

\[
P_L = \{ (L - 1)\Delta + 1, (L - 1)\Delta + 2, \cdots, L\Delta \}.
\]

For each \( s' \in S'_2 \), we can uniquely assign \( l \), the index of partition which includes \( s' \). For example, since \( 2\Delta \in P_2 \), we assign \( l = 2 \) to \( s' = 2\Delta \).

2) Consider a specific \( P_l \) of size \( \Delta \). Divide it into \( (\chi - 1) \) partitions \( P_{l,1}, P_{l,2}, \cdots, P_{l,\chi-1} \), where each partition \( P_{l,t} \) has

\[
\delta \triangleq \left( \frac{n_I}{2} \right) \tag{D.6}
\]

elements. To be specific, for \( l \in \langle L \rangle \), we have

\[
P_{l,1} = \{ \left( l - \frac{1}{2} \right) \Delta + 1, \cdots, \left( l - \frac{1}{2} \right) \Delta + \delta \},
\]

\[
P_{l,2} = \{ \left( l - \frac{1}{2} \right) \Delta + \delta + 1, \cdots, \left( l - \frac{1}{2} \right) \Delta + 2\delta \},
\]

\[
\vdots
\]

\[
P_{l,\chi-1} = \{ \left( l - \frac{1}{2} \right) \Delta + (\chi - 2)\delta + 1, \cdots, \left( l - \frac{1}{2} \right) \Delta + (\chi - 1)\delta \}
\]

For each \( s' \in S'_2 \), we can uniquely assign \( (l, t) \), the index pair of partition which includes \( s' \). For example, since \( 2\Delta \in P_{2,(\chi-1)} \), we assign \( (l, t) = (2, \chi - 1) \) to \( s' = 2\Delta \).

3) Note that each \( s' \in S'_2 \) belongs to a specific \( P_{l,t} \). Let \( i_2 \) be the position of \( s' \) within the set \( P_{l,t} \). For example, since \( 2\Delta \) is located at the last (ie., \( \delta^{th} \)) element of \( P_{2,(\chi-1)} \), we assign \( i_2 = \delta \) to \( s' = 2\Delta \). Therefore, \( s' = 2\Delta \) can be uniquely expressed as \((l, t, i_2) = (2, \chi - 1, \delta)\) tuple.

Using the \((l, t, i_2)\) representation of \( s' \in S'_2 \), we can express

\[
s = \left( \frac{n}{2} \right) + s'
\]

\[
= \left( \frac{n}{2} \right) + (l - 1)(\chi - 1) \left( \frac{n_I}{2} \right) + t \left( \frac{n_I}{2} \right) + i_2
\]

\[
= \left( \frac{n}{2} \right) + (\chi l - \chi - l + t) \left( \frac{n_I}{2} \right) + i_2 \tag{D.7}
\]

for \( s \in S_2 \). Note that there exists a one-to-one mapping between \( s \) and \((l, t, i_2)\) tuple. Obtaining \( s \) from \((l, t, i_2)\) is expressed in (D.7), while the other direction is obtained as follows, which is from Fig. 7.

\[
s' = s - \left( \frac{n}{2} \right), \quad l = \left[ \frac{s'}{\Delta} \right], \quad t = \left[ \frac{s' - (l - 1)\Delta}{\delta} \right]
\]
\[ i_2 = s' - (l - 1)\Delta - (t - 1)\delta. \]

Now we move onto our second remark, with is proved below.

**Remark 2.** Consider coded symbols \( c_s \) for \( s \in S_2 \) only. Then,
- Each coded symbol is stored in exactly two different storage nodes.
- Nodes in different clusters do not share any coded symbol.
- Nodes in the same cluster share \((\chi - 1)\) coded symbols.
- Each node contains \((\chi - 1)(n_t - 1)\) coded symbols.

**Proof.** Consider arbitrary coded symbol \( c_s \) for \( s \in S_2 \). Then, there exists an unique corresponding \((l, t, i_2)\) tuple. From Proposition 6–(c), there exists \( j \) such that \( V_{n_1}(j, i_2) = V_{n_2}(j_1, i_2) = 1 \) holds. Therefore, nodes \( N(l, j_1) \) and \( N(l, j_2) \) store \( c_s \). Since \((l, t, i_2)\) notation is unique, no other node can store \( c_s \). This proves the first statement. Moreover, since both \( N(l, j_1) \) and \( N(l, j_2) \) nodes are in the \( t^{th} \) cluster, each coded symbol \( c_s \) is stored in exactly two different nodes in the same cluster. This proves the second statement.

Consider arbitrary two nodes in the same cluster, denoted as \( N(l, j_1) \) and \( N(l, j_2) \). From the Proposition 6–(d), there exists unique \( i_2 \in \{i_2^{(j)}\}_{j=1}^{n_t-1} \) such that \( V_{n_1}(j_1, i_2) = V_{n_2}(j_1, i_2) = 1 \). Therefore, both \( N(l, j_1) \) and \( N(l, j_2) \) nodes store
\[
C(i_2^{(j)} + (\chi - l + t + 1)(i_2^{(j)}) + i_2)
\]
for \( t = 1, 2, \ldots, \chi - 1 \). In other words, two nodes in the same cluster share \((\chi - 1)\) coded symbols. This proves the third statement.

Consider arbitrary node \( N(l, j) \). From Proposition 6–(b), there exist \( \{i_2^{(j)}\}_{j=1}^{n_t-1} \) such that \( V_{n_1}(j, i_2^{(j)}) = 1 \) holds for \( v = 1, 2, \ldots, n_t - 1 \). Thus, node \( N(l, j) \) stores
\[
C(i_2^{(j)} + (\chi - l + t + 1)(i_2^{(j)}) + i_2)
\]
for every \( t \in \{\chi - 1\} \) and \( v \in \{n_t - 1\} \). Therefore, each node store \((\chi - 1)(n_t - 1)\) coded symbols, which completes the proof of fourth statement.

From the remarks 1 and 2, we obtain the Proposition 4.

**APPENDIX E**

**Proof of Proposition 5**

For a given contacting vector \( \mathbf{a} = [a_1, a_2, \ldots, a_L] \), we know that \( a_1 \) nodes are contacted in 1st cluster, \( a_2 \) nodes are contacted in 2nd cluster, and so on. Moreover, the total number of contacted nodes is \( \sum_{l=1}^{L} a_l = k \) from the definition of contacting vector. From Proposition 6–(b) and Proposition 6–(c), we obtain
\[
n(\mathbf{a}) = k\alpha - \binom{k}{2} - (\chi - 1)\sum_{l=1}^{L} \binom{a_l}{2}, \tag{E.1}
\]
for a given \( \mathbf{a} \in \mathcal{A} \). Here, the first term represents the total number of coded symbols retrieved from \( k \) nodes, each containing \( \alpha \) symbols. Since any two distinct nodes share one coded symbol, we substract the second term. Moreover, since nodes in the same cluster share \((\chi - 1)\) extra symbols, we subtract the third term.

In Appendix [3] it is shown that the vector \( \mathbf{a}^{*} \) defined in (B.2) satisfies the equation (B.3), which says
\[
\mathbf{a}^{*} = \arg\max_{\mathbf{a} \in \mathcal{A}} \sum_{l=1}^{L} \binom{a_l}{2}.
\]
Therefore, combining with (E.1), we have
\[
n(\mathbf{a}) = k\alpha - \binom{k}{2} - (\chi - 1)\sum_{l=1}^{L} \binom{a_l}{2} \\
\geq k\alpha - \binom{k}{2} - (\chi - 1)\sum_{l=1}^{L} \binom{a_l^{*}}{2} \\
= k\alpha - \binom{k}{2} - \frac{1}{2}(\chi - 1)(q\gamma_t^2 + r^2 - k) \\
= M
\]
for all \( \mathbf{a} \in \mathcal{A} \), where the second last equality is from (B.2), and the last equality is from (19). This completes the proof of Proposition 5.

**REFERENCES**

[1] A. G. Dimakis, P. B. Godfrey, Y. Wu, M. J. Wainwright, and K. Ramchandran, “Network coding for distributed storage systems,” IEEE Transactions on Information Theory, vol. 56, no. 9, pp. 4539–4551, 2010.

[2] K. Rashmi, N. B. Shah, P. V. Kumar, and K. Ramchandran, “Explicit construction of optimal exact regeneration codes for distributed storage,” in Communication, Control, and Computing, 2009. Allerton 2009. 47th Annual Allerton Conference on. IEEE, 2009, pp. 1243–1249.

[3] V. R. Cadambe, S. A. Jafar, H. Maleki, K. Ramchandran, and C. Suh, “Asymptotic interference alignment for optimal repair of mds codes in distributed storage,” IEEE Transactions on Information Theory, vol. 59, no. 5, pp. 2974–2987, 2013.

[4] T. Ernvall, “Codes between mbr and mrs point with exact repair property,” IEEE Transactions on Information Theory, vol. 60, no. 11, pp. 6993–7005, 2014.

[5] D. S. Papailiopoulos and A. G. Dimakis, “Locally repairable codes,” IEEE Transactions on Information Theory, vol. 60, no. 10, pp. 5843–5855, 2014.

[6] J. Y. Sohn, B. Choi, S. W. Yoon, and J. Moon, “Capacity of clustered distributed storage,” in 2017 IEEE International Conference on Communications (ICC), May 2017.

[7] N. Prakash, V. Atrashkin, and M. Médard, “The storage vs repair-bandwidth trade-off for clustered storage systems,” arXiv preprint arXiv:1701.04909, 2017.

[8] Y. Hu, X. Li, M. Zhang, P. P. Lee, X. Zhang, P. Zhou, and D. Feng, “Optimal repair layering for erasure-coded data centers: From theory to practice,” arXiv preprint arXiv:1704.03696, 2017.

[9] M. A. Tebibi, T. H. Chan, and C. W. Sung, “A code design framework for multi-rack distributed storage,” in Information Theory Workshop (ITW), 2014 IEEE. IEEE, 2014, pp. 55–59.

[10] J. Sohn, B. Choi, S. W. Yoon, and J. Moon, “Capacity of clustered distributed storage.” CoRR, vol. abs/1710.02821, 2017. [Online]. Available: http://arxiv.org/abs/1710.02821

[11] P. Vaidyanathan, S.-M. Phoong, and Y.-P. Lin, *Signal processing and optimization for transceiver systems*. Cambridge University Press, 2010.