LEAST ENERGY RADIAL SIGN-CHANGING SOLUTION FOR THE SCHRÖDINGER-POISSON SYSTEM IN $\mathbb{R}^3$
UNDER AN ASYMPTOTICALLY CUBIC NONLINEARITY

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Abstract. In this paper we consider the following Schrödinger-Poisson system in the whole $\mathbb{R}^3$,
\[
\begin{aligned}
-\Delta u + u + \lambda \phi u &= f(u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 
\end{aligned}
\]
where $\lambda > 0$ and the nonlinearity $f$ is “asymptotically cubic” at infinity. This implies that
the nonlocal term $\phi u$ and the nonlinear term $f(u)$ are, in some sense, in a strict competition.
We show that the system admits a least energy sign-changing and radial solution obtained by
minimizing the energy functional on the so-called nodal Nehari set.

1. INTRODUCTION

A great attention has been given in the last decades to the so called Schrödinger-Poisson
system, namely
\[
\begin{aligned}
-\Delta u + u + \lambda \phi u &= f(u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 
\end{aligned}
\]
due especially to its importance in many physical applications but also since it presents
difficulties and challenges from a mathematical point of view.

It is known that the system can be reduced to the equation
\[
-\Delta u + u + \lambda \phi u = f(u) \quad \text{in } \mathbb{R}^3,
\]
and that its solutions can be found as critical points in $H^1(\mathbb{R}^3)$ of the energy functional
\[
I(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx,
\]
where
\[
F(t) := \int_0^t f(\tau) \, d\tau, \quad \phi u = \frac{1}{4\pi |\cdot|} \ast u^2.
\]
Before anything else, we observe that $\phi u$ is automatically positive and univocally defined by $u$;
hence words like “solution”, “positive”, “sign-changing” always refer to the unknown $u$ of the
system.

Observe that since $\phi u$ is $3-$homogeneous, in the sense that
\[
\phi_{tu}(tu) = t^3 \phi u, \quad t \in \mathbb{R},
\]
2010 Mathematics Subject Classification. 35J50, 35Q60, 58E30.
Key words and phrases. Schrödinger-Poisson system, variational methods, standing waves solutions, nodal
Nehari set.

Edwin Murcia was supported by Department of Mathematics, Pontificia Universidad Javeriana. Gaetano
Siciliano was partially supported by Fapesp, CNPq and Capes, Brazil.
there is a further difficulty in the problem exactly when the nonlinearity $f$ behaves “cubically” at infinity, we say it is asymptotically cubic, being in this case in competition with the nonlocal term $\phi_u u$.

The number of papers which have studied the Schrödinger-Poisson system in the mathematical literature is so huge that it is almost impossible to give a satisfactory list. Indeed many papers deal with the problem in bounded domain or in the whole space (see e.g. [4, 5, 8, 15, 17, 18] and the references therein) and some other papers deal with the fractional counterpart (see e.g [9, 14] and its references). In all the cited papers various type of solutions have been found under different assumptions on the nonlinearity. However the solutions found are positive or with undefined sign and the nonlinearity $f$ is “supercubic” at infinity (in a sense that will be specified below) an this fact helps in many computations since it gains on the nonlocal term $\phi_u u$.

Nevertheless some results have been obtained also in the asymptotically cubic case: for example, in the remarkable paper [3] the authors consider the existence of solutions under a very general nonlinearity $f$ of Berestycki-Lions type. However they found a positive solution, for small values of the parameter $\lambda > 0$ (and as we will see the smallness of $\lambda$ is necessary).

However beside the existence of positive solutions it is also interesting to find sign-changing solutions and indeed many authors began recently to address this issue. We cite the interesting paper [19] which deals with the case $f(u) = |u|^{p-1}u$ and $p \in (3, 5)$ and where the authors search for least energy sign-changing solutions, that is the sign-changing solution whose functional has minimal energy among all the others sign-changing solutions. Their idea is to study the energy functional on a new constraint, a subset of the Nehari manifold which contains all the sign-changing solutions.

Another interesting paper is [2] which deals with a more general nonlinearity, not necessarily of power type, where the authors assume that

\[ \lim_{t \to \infty} \frac{F(t)}{t^4} = +\infty. \]

In this sense [2] and [19] deal with a supercubic nonlinearity $f$.

The above condition is also required in [1], for the case of the bounded domain, and in [11], for the case of the whole space, where a least energy sign-changing solution is obtained.

In all these papers concerning sign-changing solutions, one of the main task is to prove that the new constraint on which minimize the functional is not empty. To show this, the fact that the nonlinearity is supercubic is strongly used.

Motivated by the previous discussion, a natural question which arises concerns the case when the nonlinearity is “cubic” at infinity. More specifically in this paper we address the problem (1.1) under the following conditions. Let $\lambda > 0$ and assume

\begin{enumerate}
  \item[(f1)] $f \in C(\mathbb{R}, \mathbb{R})$;
  \item[(f2)] $f(t) = -f(-t)$ for $t \in \mathbb{R}$;
  \item[(f3)] $\lim_{t \to 0} f(t)/t = 0$;
  \item[(f4)] $\lim_{t \to \infty} f(t)/t^3 = 1$ and $f(t)/t^3 < 1$ for all $t \in \mathbb{R}$;
  \item[(f5)] the function $t \mapsto f(t)/t^3$ is strictly increasing on $(0, \infty)$;
  \item[(f6)] recalling that $F(t) = \int_0^t f(\tau)d\tau$, \[ \lim_{t \to \infty} [f(t)t - 4F(t)] = +\infty. \]
\end{enumerate}
Assumption (f4) is what we called *asymptotically cubic* behaviour for the nonlinearity and (f6) is the analogous of the usual *non-quadraticity condition*.

Our result is the following

**Theorem 1.** If \( \lambda > 0 \) is sufficiently small, under the conditions (f1)-(f6), problem (1.1) has a radial sign-changing ground state solution. Moreover it changes sign exactly once in \( \mathbb{R}^3 \).

A function \( f \) satisfying our assumptions is

\[
f(t) = \frac{t^5}{1+t^2}, \quad t \in \mathbb{R},
\]

which has as primitive

\[
F(t) = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} \ln(1+t^2).
\]

Clearly this function does not satisfy the assumption

\[
\lim_{t \to \infty} \frac{F(t)}{t^4} = +\infty
\]

required in [2]. Moreover, the case \( f(t) = |t|^{p-1} t, p \in (3, 5) \) studied in [19] does not satisfies (f4). So the present paper gives a new contribution in studying sign-changing solutions for the Schrödinger-Poisson problem in the asymptotically cubic nonlinearity and can be seen as a counterpart of the papers [2, 3, 19].

As we said before, we use variational methods: the solution will be found as the minimum of \( I \), in the context of radial functions, on the constraint already introduced in [19]. Nevertheless, the main difficulty is to show that the constraint on which minimize the functional is nonempty under our assumptions on \( f \); indeed all the techniques of the above cited papers concerning the supercubic case (see also [13] for the single equation) do not work and some new ideas have been necessary.

The organisation of the paper is the following.

In Section 2 we recall and give some preliminary facts.

In Section 3 the set on which minimize the functional is introduced, and some of its properties proved. Indeed this Section collects all the ingredients we need in order to prove the result. In particular it is stated Proposition 4 which says that the set on which we minimize is nonempty.

In Section 4 the main result is proved.

Finally in the Appendix we prove the technical Proposition 4.

**Notations.** We conclude this Introduction by introducing few basic notations. In all the paper, \( H^1(\mathbb{R}^3) \) is the usual Sobolev space with norm

\[
\|u\| = (\|\nabla u\|_2^2 + |u|_2^2)^{1/2},
\]

where \( | \cdot |_p \) is the usual \( L^p \)-norm in \( \mathbb{R}^3 \). Moreover \( H^1_{rad}(\mathbb{R}^3) \) denotes the subspace of radial functions. We need also the space \( D^{1,2}(\mathbb{R}^3) \), which is defined as the completion of the test functions with respect to the norm \( \| \cdot \|_D := |\nabla \cdot |_2 \). As usual, \( D^{1,2}_{rad}(\mathbb{R}^3) \) is the subspace of radial functions. We denote with \( \| \cdot \|_* \) the norm for the space of continuous linear functionals defined on \( H^1_{rad}(\mathbb{R}^3) \). As customary, \( \text{meas}(\cdot) \) takes the Lebesgue measure of a set. Furthermore, we use the letter \( C \) to denote a positive constant whose value may change from line to line. Other notations will be introduced as soon as we need.
2. Preliminaries and known results

We begin by recalling that in particular from (f1), (f3) and (f4), given $\varepsilon > 0$ and $q \in (4, 6)$, there exists $C_\varepsilon > 0$ constant such that

$$
|f(t)| \leq \varepsilon |t| + C_\varepsilon |t|^{q-1} \quad \text{and} \quad |F(t)| \leq \frac{\varepsilon}{2} t^2 + \frac{C_\varepsilon}{q} |t|^q, \quad \forall t \in \mathbb{R}.
$$

Moreover from (f5) it follows that

$$
f(t) - 4F(t) \geq 0, \quad \forall t \in \mathbb{R}. \tag{2.2}
$$

(see e.g. [10, Lemma 2.3]) and then by (f2) and (2.2) we infer

$$
f(t) - 4F(t) \geq 0, \quad \forall t \in \mathbb{R}. \tag{2.3}
$$

Let us recall also a result on the whole $\mathbb{R}^3$. This will have a major role in all our analysis. In the paper [3] it was studied the problem

$$
\begin{align*}
\left\{ 
-\Delta u + \lambda \phi u &= g(u) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= \lambda u^2 \quad \text{in } \mathbb{R}^3,
\end{align*} \tag{2.4}
$$

under the following assumptions on $g$:

1. $g \in C(\mathbb{R}, \mathbb{R})$,
2. $-\infty < \lim \inf_{t \to 0^+} g(t)/t \leq \lim \sup_{t \to 0^+} g(t)/t = -m < 0$,
3. $-\infty \leq \lim \sup_{t \to \infty} g(t)/t^5 \leq 0$,
4. there exists $\zeta > 0$ such that $G(\zeta) := \int_0^{\zeta} g(t) dt > 0$.

In other words $g$ is a general nonlinearity satisfying the Berestycki-Lions assumptions. The authors prove that there is a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ the problem (2.4) has a nontrivial positive and radial solution.

Our problem (1.1) can be written as (2.4) just renaming the nonlinearity and the potential $\phi$; then in virtue of [3] we have the following

**Lemma 1.** Under conditions (f1), (f3) and (f4) there is a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ problem (1.1) has a positive and radial solution $u$.

Recall also that if $\phi_u$ is the unique solution of the Poisson equation

$$
-\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3
$$

for a given $u \in H^1_{rad}(\mathbb{R}^3)$, then

$$
\exists C > 0 : \|\phi_u\|^2_{D} = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq C \|u\|^4. \tag{2.5}
$$

For this see e.g. [16].

The smallness of $\lambda$ stated in Lemma 1 is a necessary condition in order to have a positive solution for problem (1.1). In our case, that is in presence of a nonlinearity which is not of power type, this can be seen in the following way. Let $u_\lambda \in H^1(\mathbb{R}^3)$ be a positive solution of (1.1) for a certain value of $\lambda > 0$ and let $\phi_\lambda := \phi_{u_\lambda} \in D^{1,2}(\mathbb{R}^3)$. Multiplying the first equation of (1.1) by $u_\lambda$ and integrating on $\mathbb{R}^3$ we obtain

$$
0 < \|u_\lambda\|^2 = \int_{\mathbb{R}^3} \left( \frac{f(u_\lambda)}{u_\lambda^2} - \lambda \frac{\phi_\lambda}{u_\lambda^2} \right) u_\lambda^4 dx < \int_{\mathbb{R}^3} \left( 1 - \lambda \frac{\phi_\lambda}{u_\lambda^2} \right) u_\lambda^4 dx = \int_{\mathbb{R}^3} (u_\lambda^4 - \lambda \phi_\lambda u_\lambda^2) dx \tag{2.6}
$$
so that

\[ (2.7) \quad \lambda < \frac{\int_{\mathbb{R}^3} u_\lambda^4 \, dx}{\int_{\mathbb{R}^3} \phi u_\lambda^2 \, dx}. \]

We define the set made by scalar multiple of solutions, precisely

\[ S_\lambda := \{ \| \phi \|_D^{-1/2} u_\lambda \in H^1(\mathbb{R}^3) \setminus \{0\} : u_\lambda \text{ solves } (1.1) \} \neq \emptyset. \]

Now if \( v \in S_\lambda \) then \( v = \| \phi \|_D^{-1/2} u_\lambda \) for some solution \( u_\lambda \) and \( \| \phi \|_D = 1 \). Then by (2.5),

\[ 1 = \| \phi \|_D^2 = \int_{\mathbb{R}^3} |\nabla \phi_v|^2 \, dx = \int_{\mathbb{R}^3} \phi v^2 \, dx \leq C\|v\|^4 \]

showing that \( K := \inf_{v \in S_\lambda} \|v\|^4 > 0 \). Moreover recalling (2.7),

\[ \lambda < \frac{\| \phi \|_D^{-2} \int_{\mathbb{R}^3} u_\lambda^4 \, dx}{\| \phi \|_D^{-2} \int_{\mathbb{R}^3} \phi u_\lambda^2 \, dx} = \frac{\int_{\mathbb{R}^3} v^4 \, dx}{\int_{\mathbb{R}^3} \phi v^2 \, dx} \leq \int_{\mathbb{R}^3} v^4 \, dx \leq C\|v\|^4 \]

and then \( \lambda/C \leq K \) i.e. \( \lambda \leq KC \).

Recall also that if \( u \) is radial, the unique solution \( \phi_u \) of the Poisson equation is also radial, and by the Newton’s Theorem it can be written as (we omit from now on the factor \( 1/4\pi \))

\[ \phi_u(r) = \frac{1}{r} \int_0^\infty u_\lambda(s) \min\{s, r\} \, ds. \]

From this representation it is easy to see that \( \phi_u \) is decreasing in the radial coordinate. Indeed let \( 0 < r_1 < r_2 \); we have that

\[ \phi(r_1) = \frac{1}{r_1} \int_0^{r_1} u^2(s) s \min\{s, r_1\} \, ds \]
\[ = \frac{1}{r_1} \int_0^{r_1} u^2(s) s^2 \, ds + \int_{r_1}^{r_2} u^2(s) s \, ds \]
\[ = \frac{1}{r_1} \int_0^{r_1} u^2(s) s^2 \, ds + \int_{r_1}^{r_2} u^2(s) s \, ds + \int_{r_2}^{\infty} u^2(s) \, ds \]
\[ > \frac{1}{r_2} \int_0^{r_2} u^2(s) s^2 \, ds + \frac{1}{r_2} \int_{r_1}^{r_2} u^2(s) s^2 \, ds + \int_{r_2}^{\infty} u^2(s) \, ds \]
\[ = \frac{1}{r_2} \int_0^{r_2} u^2(s) s^2 \, ds + \int_{r_2}^{\infty} u^2(s) \, ds \]
\[ = \frac{1}{r_2} \int_0^{r_2} u^2(s) s \min\{s, r_2\} \, ds \]
\[ = \phi(r_2). \]

The radial solutions of (1.1) are critical points of the \( C^1 \) functional \( I : H_{rad}^1(\mathbb{R}^3) \to \mathbb{R} \) defined in (1.2); indeed the derivative at \( u \) has the expression

\[ I'(u)[v] = \int_{\mathbb{R}^3} (\nabla u \nabla v + u v) \, dx + \lambda \int_{\mathbb{R}^3} \phi_u u v \, dx - \int_{\mathbb{R}^3} f(u) v \, dx, \quad \text{for } v \in H_{rad}^1(\mathbb{R}^3) \]

and clearly all the nontrivial critical points of \( I \) belong to the so called Nehari set

\[ \mathcal{N} := \{ u \in H_{rad}^1(\mathbb{R}^3) \setminus \{0\} : I'(u)[u] = 0 \}. \]
Let us define the map
\[ \gamma(u) := I'(u)[u] = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \lambda \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \int_{\mathbb{R}^3} f(u)u \, dx, \]
evidently continuous. Two basic properties of \( \mathcal{N} \) are the following. Others will be shown in the next section.

First, \( \mathcal{N} \) is bounded away from zero in \( H^1_{\text{rad}}(\mathbb{R}^3) \). Indeed from \( u \neq 0 \) and \( \gamma(u) = 0 \) by (2.1) with \( \varepsilon = 1/2 \) and \( q \in (4,6) \) we have
\[
\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx < \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \lambda \int_{\mathbb{R}^3} \phi_u u^2 \, dx
= \int_{\mathbb{R}^3} f(u)u \, dx
\leq \frac{1}{2} \int_{\mathbb{R}^3} u^2 \, dx + C_{1/2} \int_{\mathbb{R}^3} |u|^q \, dx
\]
so that
\[
(2.8) \quad \frac{1}{2} \|u\|^2 < \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \frac{1}{2} u^2\right) \, dx < C_{1/2} \int_{\mathbb{R}^3} |u|^q \, dx < C\|u\|^q.
\]
Remark 2. Of course the above computation also holds if \( \gamma(u) < 0 \). Moreover we can deduce from (2.8) and the Sobolev embeddings, that there exists \( L > 0 \) such that
\[
u \in \mathcal{N}, \quad \text{or,} \quad u \in H^1_{\text{rad}}(\mathbb{R}^3) \text{ with } \gamma(u) < 0 \implies 0 < L = |u|_q, \|u\|.
\]
Second, the functional \( I \) is bounded from below on \( \mathcal{N} \); indeed, for \( u \in \mathcal{N} \),
\[
I(u) = I(u) - \frac{1}{4} I'(u)[u]
= \frac{1}{4} \|u\|^2 - \int_{\mathbb{R}^3} F(u) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} f(u)u \, dx
\leq \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u)u - F(u) \right] \, dx
\geq 0.
\]
in virtue of (2.3). Through the paper we will use repeatedly inequalities like
\[
I(u) \geq \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u)u - F(u) \right] \, dx, \quad \text{whenever} \quad u \in \mathcal{N}.
\]
Remark 3. Recall also the following. Given a sequence \( \{u_n\} \subset H^1_{\text{rad}}(\mathbb{R}^3) \) with \( u_n \rightharpoonup u \) in \( H^1_{\text{rad}}(\mathbb{R}^3) \), since \( H^1_{\text{rad}}(\mathbb{R}^3) \hookrightarrow \hookrightarrow L^p(\mathbb{R}^3) \) for \( 2 < p < 6 \), we have
\[
\int_{\mathbb{R}^3} f(u_n)u_n \, dx \to \int_{\mathbb{R}^3} f(u)u \, dx \quad \text{and} \quad \int_{\mathbb{R}^3} F(u_n) \, dx \to \int_{\mathbb{R}^3} F(u) \, dx,
\]
and (see [16, Lemma 2.1])
\[
\phi_{u_n} \to \phi_u \quad \text{in } D^{1,2}_{\text{rad}}(\mathbb{R}^3).
\]
Even more, if \( \{v_n\} \) is another sequence in \( H^1_{\text{rad}}(\mathbb{R}^3) \) with \( v_n \rightharpoonup v \) in \( H^1_{\text{rad}}(\mathbb{R}^3) \), then
\[
\int_{\mathbb{R}^3} \phi_{u_n} v_n^2 \, dx \to \int_{\mathbb{R}^3} \phi_u v^2 \, dx,
\]
because \( \phi_{u_n} \to \phi_u \) in \( L^6(\mathbb{R}^3) \) and \( v_n^2 \to v^2 \) in \( L^{6/5}(\mathbb{R}^3) \).
3. Some useful results

In this Section, by following [19] we introduce the set on which minimize the functional $I$ and give all the properties we need to work with.

Let us denote hereafter, $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \min\{u(x), 0\}$. In this way we have the decomposition $u = u^+ + u^-$. 

Now, although $\phi_u = \phi_{u^+} + \phi_{u^-}$ (see [19]) it results

$$I(u) = I(u^+) + I(u^-) + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 dx$$

then an extra term, with respect to the case $\lambda = 0$, appears in the decomposition of $I$. We use here, and throughout the paper, that

$$\forall u, v \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \phi_u v^2 dx = \int_{\mathbb{R}^3} \phi_v u^2 dx.$$ 

Moreover, if $u^\pm \neq 0$,

$$I'(u)[u^+] = I'(u^+)[u^+] + \lambda \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 > I'(u^+)[u^+],$$

$$I'(u)[u^-] = I'(u^-)[u^-] + \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 > I'(u^-)[u^-],$$

and this implies that, if $u$ is a sign changing solution, hence in particular $u \in \mathcal{M}$, then $I'(u^+)[u^+], I'(u^-)[u^-] < 0$ and so $u^\pm \notin \mathcal{N}$.

For this reason $\mathcal{N}$ is not a good set on which find the critical points of $I$; a natural choice of the set on which study the functional $I$ is

$$\mathcal{M} := \{ u \in \mathcal{N} : I'(u)[u^+] = 0; u^\pm \neq 0 \}.$$ 

Observe that if $u \in \mathcal{M}$ then $I'(u)[u^-] = 0$ and that any sign-changing solution is on $\mathcal{M}$. Moreover $\mathcal{M}$ is not a smooth manifold.

By the above decomposition we have, for $\alpha, \beta > 0$ and $u \in H^1(\mathbb{R}^3)$:

$$(3.1) \quad I'(\alpha u^+ + \beta u^-)[\alpha u^+] = I'(\alpha u^+)[\alpha u^+] + \alpha^2 \beta^2 \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx$$

that will be useful later on.

The main results of this Section are the following:

- $\mathcal{M}$ is not empty (see Proposition 4),
- if $\inf \mathcal{M} I$ is achieved, it is a critical level (see Proposition 11).

Note that the second item is not immediate since $\mathcal{M}$ is not a smooth manifold, and indeed a preliminary work is necessary.

**Proposition 4.** For $\lambda$ sufficiently small, the set $\mathcal{M}$ is not empty.

The proof of this fact is postponed to the Appendix being somehow technical. It strongly uses the solution $u$ found in Lemma 1 for $\lambda$ small. In virtue of this, from now on the parameter $\lambda$ has to be considered fixed in $(0, \lambda_0)$.

If we define the maps

$$\gamma_{\pm} : u \in H^1_{rad}(\mathbb{R}^3) \mapsto I'(u)[u^\pm] \in \mathbb{R}$$

we can write

$$\mathcal{M} = \{ u \in \mathcal{N} : \gamma_+(u) = 0, u^\pm \neq 0 \}.$$
By the continuity of \( u \in H^1_{rad}(\mathbb{R}^3) \mapsto u^+ \in H^1_{rad}(\mathbb{R}^3) \), see [7, Proposition 7.2], we deduce that \( \gamma_\pm \) are continuous. By the definitions it also holds

\[
\gamma_+(u) = I'(u^+) [u^+] + \lambda \int_{\mathbb{R}^3} \phi_{u^+} (u^-)^2 \, dx = \gamma(u^+) + \lambda \int_{\mathbb{R}^3} \phi_{u^+} (u^-)^2 \, dx
\]

and

\[
\gamma_-(u) = I'(u^-) [u^-] + \lambda \int_{\mathbb{R}^3} \phi_{u^+} (u^-)^2 \, dx = \gamma(u^-) + \lambda \int_{\mathbb{R}^3} \phi_{u^+} (u^-)^2 \, dx.
\]

Due to the continuity of the maps defined above, we have the following useful result.

**Proposition 5.** The set \( \mathcal{M} \) is closed.

**Proof.** We know that

\[
\mathcal{M} = \gamma^{-1}([0]) \cap \gamma_+^{-1}([0]) \cap \{ v \in H^1_{rad}(\mathbb{R}^3) : v^\pm \neq 0 \},
\]

and

\[
\gamma^{-1}([0]), \gamma_+^{-1}([0]) \text{ are closed sets in } H^1_{rad}(\mathbb{R}^3).
\]

Let us consider \( \{ u_n \} \subset \mathcal{M} \) such that \( u_n \to u \) in \( H^1_{rad}(\mathbb{R}^3) \). In particular

\[
\gamma_+(u_n) = 0 \text{ and } \gamma(u_n^+) = -\lambda \int_{\mathbb{R}^3} \phi_{u_n^+} (u_n^-)^2 < 0
\]

and then by Remark 2 it is \( |u_n^+| > L > 0 \) with \( q \in (4,6) \). The sequence \( \{ u_n^+ \} \) is bounded in \( H^1_{rad}(\mathbb{R}^3) \), being \( \| u_n^+ \| \leq \| u_n \| \), and then by the compact embedding \( H^1_{rad}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3) \) we get the existence of some function \( v \) such that

\[
u_n^+ \to v \geq 0 \text{ in } L^q(\mathbb{R}^3) \text{ and } |v|_q \geq L > 0.
\]

Since it is also

\[
\mathcal{M} = \gamma^{-1}([0]) \cap \gamma_+^{-1}([0]) \cap \{ v \in H^1_{rad}(\mathbb{R}^3) : v^\pm \neq 0 \},
\]

we can argue as before by using \( \gamma_-, u_n^- \) and (3.3). In such a way we arrive at

\[
u_n^- \to w \leq 0 \text{ in } L^q(\mathbb{R}^3) \text{ and } |w|_q \geq L > 0.
\]

But since \( u_n = u_n^+ + u_n^- \to u^+ + u^- \) in \( L^q(\mathbb{R}^3) \) it has to be

\[
u^+ = v, \quad u^- = w
\]

and by (3.5) and (3.6) we infer

\[
u \in \{ v \in H^1_{rad}(\mathbb{R}^3) : v^\pm \neq 0 \}.
\]

Then (3.4) and (3.7) give the conclusion. \qed

Also in the asymptotically cubic case there is the unicity of the projection on the Nehari set, and on the fibers the functional \( I \) achieves the maximum on \( \mathcal{N} \). More precisely we have the following:

**Lemma 6.** If \( u \in \mathcal{N} \), then

(i) \( I(tu) < I(u) \) for every \( t > 0, \ t \neq 1 \);

(ii) \( tu \notin \mathcal{N} \) for every \( t > 0, \ t \neq 1 \). More specifically,

\[
\gamma(tu) > 0 \text{ for } t \in (0,1) \quad \text{and} \quad \gamma(tu) < 0 \text{ for } t > 1.
\]
Lemma 7. If $u \in \mathcal{N}$ and define the function
\[
\xi(t) := \left( t^2 - \frac{t^4}{4} \right) (|\nabla u|^2 + u^2) + \frac{t^4}{4} f(u)u - F(tu), \quad t \geq 0
\]
on the set $\{x \in \mathbb{R}^3 \mid u(x) \neq 0\}$ which has positive measure. For any $t > 0$
\[
\xi'(t) = t(1-t^2) (|\nabla u|^2 + u^2) + t^3 u^4 \left[ \frac{f(u)}{u^3} - \frac{f(tu)}{t^3 u^3} \right],
\]
and thus, by (55), $\xi(t) < \xi(1)$, for every $t > 0$, $t \neq 1$. Then, after integration on $\mathbb{R}^3$ and using that $\gamma(u) = I'(u)[u] = 0$, we have
\[
I(tu) = \left( \frac{t^2}{2} - \frac{t^4}{4} \right) \|u\|^2 + \int_{\mathbb{R}^3} \left[ \frac{t^4}{4} f(u)u - F(tu) \right] dx
\]
\[
\leq \frac{1}{4} \|u\|^2 + \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u)u - F(u) \right] dx
\]
\[
= I(u).
\]
To prove (ii), simply observe that by (55), in case $0 < t < 1$ we have
\[
\gamma(tu) = I'(tu)[tu] > t^4 \left[ \|u\|^2 + \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} \frac{\phi(tu)}{t^3} u dx \right] > t^4 \gamma(u) = 0,
\]
while in case $t > 1$ we have
\[
\gamma(tu) = I'(tu)[tu] < t^4 \left[ \|u\|^2 + \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} \frac{\phi(tu)}{t^3} u dx \right] < t^4 \gamma(u) = 0
\]
and the proof is completed. \qed

A similar assertion can be done for the elements of $\mathcal{M}$.

Lemma 7. If $u \in \mathcal{M}$, then
(i) $I(su^+ + tu^-) < I(u)$ for $s,t \geq 0$, $s \neq 1$ or $t \neq 1$;
(ii) $tu \notin \mathcal{M}$ for every $t > 0$, $t \neq 1$. More specifically,
\[
\gamma_{\pm}(tu) > 0 \quad \text{for } t \in (0,1) \quad \text{and} \quad \gamma_{\pm}(tu) < 0 \quad \text{for } t > 1.
\]
Proof. If $u \in \mathcal{M}$, then $\gamma(u) = I'(u)[u] = \gamma_{\pm}(u) = I'(u)[u^\pm] = 0$. By making use of (3.1)-(3.3) we get
\[
I(su^+ + tu^-) = I(su^+ + tu^-) - \frac{s^4}{4} I'(u)[u^+] - \frac{t^4}{4} I'(u)[u^-]
\]
\[
= \left( I(su^+) - \frac{s^4}{4} I'(u)[u^+] \right) + \left( I(tu^-) - \frac{t^4}{4} I'(u^-)[u^-] \right)
\]
\[
- \frac{\lambda}{4} (s^2 - t^2)^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx
\]
\[
< A + B
\]
where
\[
A := I(su^+) - \frac{s^4}{4} I'(u^+)[u^+], \quad B := I(tu^-) - \frac{t^4}{4} I'(u^-)[u^-].
\]
Explicitly
\[
A = \left( \frac{s^2}{2} - \frac{s^4}{4} \right) \|u^+\|^2 + \int_{\mathbb{R}^3} \left[ \frac{s^4}{4} f(u^+)[u^+] - F(su^+) \right] dx
\]
so that, arguing as in Lemma 6 by using the function
\[ \xi_+(s) := \left( \frac{s^2}{2} - \frac{s^4}{4} \right) \left[ |\nabla u^+|^2 + (u^+)^2 \right] + \frac{s^4}{4} f(u^+) u^+ - F(su^+), \]
we obtain \( I(su^+) < I(u^+) \) and then
\[ A = I(su^+) - \frac{s^4}{4} I'(u^+)[u^+] < I(u^+) - \frac{1}{4} I'(u^+)[u^+], \quad \text{for } s > 0, \ s \neq 1. \]
Similarly we have
\[ B = I(tu^-) - \frac{t^4}{4} I'(u^-)[u^-] < I(u^-) - \frac{1}{4} I'(u^-)[u^-], \quad \text{for } t > 0, \ t \neq 1, \]
and consequently by (3.8) we infer
\[
I(su^+ + tu^-) < \left( I(u^+) - \frac{1}{4} I'(u^+)[u^+] \right) + \left( I(u^-) - \frac{1}{4} I'(u^-)[u^-] \right)
= \left( I(u^+) - \frac{1}{4} I'(u)[u^+] \right) + \left( I(u^-) - \frac{1}{4} I'(u)[u^-] \right) + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi'_{tu^+}(u^-)^2 dx
= I(u^+) + I(u^-) + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi'_{tu^+}(u^-)^2 dx
= I(u),
\]
and (i) follows.

For (ii), first note that
\[
\gamma_+(tu) = I'(tu)[tu^+]
= I'(tu^+)[tu^+] + \lambda \int_{\mathbb{R}^3} \phi'_{tu^+}(tu^-)^2 dx
= t^2 \|u^+\|^2 + \lambda t^4 \int_{\mathbb{R}^3} \phi'_{u^+}u^2 dx - \int_{\mathbb{R}^3} f(tu^+) tu^+ dx.
\]
Since \( \gamma_+(u) = 0 \), by (15), in case \( 0 < t < 1 \)
\[
\gamma_+(tu) > t^4 \left[ \|u^+\|^2 + \lambda \int_{\mathbb{R}^3} \phi'_{u^+}u^2 dx - \int_{\mathbb{R}^3} \frac{f(tu^+)}{t^3} u^+ dx \right] > t^4 \gamma_+(u) = 0,
\]
while in case \( t > 1 \)
\[
\gamma_+(tu) < t^4 \left[ \|u^+\|^2 + \lambda \int_{\mathbb{R}^3} \phi'_{u^+}u^2 dx - \int_{\mathbb{R}^3} \frac{f(tu^+)}{t^3} u^+ dx \right] < t^4 \gamma_+(u) = 0.
\]
Furthermore from \( \gamma(u) = I'(u)[u] = 0 \) and \( \gamma_+(u) = 0 \), we have \( \gamma_-(u) = I'(u)[u^-] = 0 \), and similar computations, together with (2), give us
\[
\gamma_-(tu) > 0, \quad \text{when } 0 < t < 1, \quad \text{and } \gamma_-(tu) < 0, \quad \text{when } t > 1
\]
concluding the proof. \( \square \)

Remark 8. By using the same ideas of Lemma 7 we can establish the following generalization.
Assume that \( u \in \mathcal{N} \) is such that
1. \( u = u_1 + u_2 + u_3, \) with \( \operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset, \) for \( i < j \) where \( i, j = 1, 2, 3; \)
2. \( u_i \neq 0, \) for every \( i = 1, 2, 3; \)
3. \( I'(u)[u_i] = 0, \) for every \( i = 1, 2, 3. \)
Then we have

\[ I(t_1 u_1 + t_2 u_2 + t_3 u_3) < I(u), \quad \text{for } t_i \geq 0, \ i = 1, 2, 3, \ \text{with at least one } t_i \neq 1. \]

Indeed, denoting with \( \phi_i := \phi_{u_i} \) for \( i = 1, 2, 3 \), under our assumptions

\[
I(t_1 u_1 + t_2 u_2 + t_3 u_3) = \sum_{i=1}^{3} I(t_i u_i) + \frac{\lambda}{2} \sum_{i < j} t_i^2 t_j^2 \int_{\mathbb{R}^3} \phi_i \phi_j^2 dx \\
I'(u)[u_i] = I'(u_i)[u_i] + \lambda \sum_{j \neq i} \int_{\mathbb{R}^3} \phi_j u_i^2 dx,
\]

so that

\[
I(t_1 u_1 + t_2 u_2 + t_3 u_3) = I(t_1 u_1 + t_2 u_2 + t_3 u_3) - \sum_{i=1}^{3} \frac{t_i^4}{4} I'(u)[u_i] \\
= \sum_{i=1}^{3} \left( I(t_i u_i) - \frac{t_i^4}{4} I'(u_i)[u_i] \right) - \frac{\lambda}{4} \sum_{i < j} (t_i^2 - t_j^2)^2 \int_{\mathbb{R}^3} \phi_i \phi_j^2 dx.
\]

By defining the functions

\[
\xi_i(t_i) := \left( \frac{t_i^2}{2} - \frac{t_i^4}{4} \right) [\nabla u_i^2 + (u_i)^2] + \frac{t_i^4}{4} f(u_i) - F(t_i u_i), \quad i = 1, 2, 3,
\]

and arguing as in Lemma 6 and Lemma 7 we have that

\[ I(t_i u_i) - \frac{t_i^4}{4} I'(u_i)[u_i] < I(u_i) - \frac{1}{4} I'(u_i)[u_i], \quad \text{for every } t_i > 0, \ t_i \neq 1. \]

Thus

\[
I(t_1 u_1 + t_2 u_2 + t_3 u_3) < \sum_{i=1}^{3} \left( I(u_i) - \frac{1}{4} I'(u_i)[u_i] \right) \\
= \sum_{i=1}^{3} \left( I(u_i) - \frac{1}{4} I'(u_i)[u_i] \right) + \frac{\lambda}{2} \sum_{i < j} \int_{\mathbb{R}^3} \phi_j u_i^2 dx \\
= \sum_{i=1}^{3} I(u_i) + \frac{\lambda}{2} \sum_{i < j} \int_{\mathbb{R}^3} \phi_j u_i^2 dx \\
= I(u_1 + u_2 + u_3) \\
= I(u),
\]

and we are done.

The next result will be useful for dealing with minimising sequences.

**Proposition 9.** Let \( \{u_n\} \subset \mathcal{M} \) be a sequence such that \( \{I(u_n)\} \) is bounded. Then \( \{u_n\} \) is bounded.

**Proof.** Assume that the statement does not hold, i.e., there exists a subsequence again denoted by \( \{u_n\} \) such that \( \{I(u_n)\} \) is bounded but \( \|u_n\| \to \infty \) when \( n \to \infty \). Then, up to a subsequence, \( I(u_n) \to l \geq 0 \) by (2.9).

If \( l > 0 \), we define \( v_n := 2\sqrt{l} u_n/\|u_n\| \), so that \( \|v_n\| = 2\sqrt{l} \). If \( l = 0 \), we first define \( t_n := 1/\|u_n\| \) and then \( v_n := t_n u_n \), so that \( \|v_n\| = 1 \).
Claim: there exist numbers \( r, d > 0 \) and a sequence \( \{y_n\} \subset \mathbb{R}^3 \) such that

\[
\liminf_{n \to \infty} \int_{B_r(y_n)} v_n^2 \, dx \geq d > 0.
\]

If the claim does not hold, then by Lions’ lemma, \( v_n \to 0 \) in \( L^p(\mathbb{R}^3) \), for \( 2 < p < 6 \). Hence, in particular for \( q \in (4, 6) \), using (2.1) we obtain

\[
\left| \int_{\mathbb{R}^3} F(v_n) \, dx \right| \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^3} v_n^2 \, dx + \frac{C\varepsilon}{q} \int_{\mathbb{R}^3} |v_n|^q \, dx,
\]

and thus we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} F(v_n) \, dx = 0.
\]

Therefore, when \( l > 0 \)

\[
I(v_n) = \frac{1}{2} \|v_n\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \, dx - \int_{\mathbb{R}^3} F(v_n) \, dx = 2l + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \, dx + o_n(1),
\]

while when \( l = 0 \)

\[
\liminf_{n \to \infty} I(v_n) \geq \liminf_{n \to \infty} \left[ \frac{1}{2} \|v_n\|^2 - \int_{\mathbb{R}^3} F(v_n) \, dx \right] = \frac{1}{2}.
\]

On the other hand, since \( I'(u_n)(u_n) = 0 \), using Lemma 6 we get

\[
I(tu_n) \leq I(u_n), \quad \forall t > 0.
\]

Taking \( t = t_n := 2\sqrt{l/\|u_n\|} \) when \( l > 0 \), and \( t = t_n := 1/\|u_n\| \) when \( l = 0 \), we have

\[
I(v_n) \leq I(u_n) = l + o_n(1).
\]

This fact implies, when \( l = 0 \), that \( I(v_n) \leq I(u_n) = o_n(1) \), contrary to (3.10), while when \( l > 0 \),

\[
2l + o_n(1) < 2l + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \, dx + o_n(1) = I(v_n) \leq I(u_n) = l + o_n(1),
\]

or

\[
0 < l < o_n(1),
\]

an absurd. Therefore the Claim holds.

However by the Claim we infer a contradiction in both cases: when \( \{y_n\} \) is bounded or unbounded. This will complete the proof.

Hereafter the arguments can be used indistinctly for \( l \geq 0 \) real number.

Case 1: \( \{y_n\} \) is bounded

Then there is \( \tilde{r} > 0 \) with \( \{y_n\} \subset B_{\tilde{r}} \). By (3.9), eventually

\[
\int_{B_r(y_n)} v_n^2 \, dx > \frac{d}{2}.
\]

Thus we can choose \( \tilde{r} > r + \tilde{r} \); for this radius it holds \( B_{\tilde{r}}(y_n) \subset B_{\tilde{r}} \) and hence

\[
\int_{B_{\tilde{r}}} v_n^2 \, dx > \frac{d}{2}.
\]

Since \( \|v_n\| \) is constant, there exists a subsequence still denoted by \( \{v_n\} \) such that \( v_n \to v \) in \( H^1_{rad}(\mathbb{R}^3) \), from which \( v_n \to v \) in \( L^p_{loc}(\mathbb{R}^3) \) for \( 1 \leq p < 6 \), and \( v_n(x) \to v(x) \) a.e. \( x \in \mathbb{R}^3 \). In particular

\[
\int_{B_{\tilde{r}}} v_n^2 \, dx \to \int_{B_{\tilde{r}}} v^2 \, dx, \quad \text{and thus} \quad \int_{B_{\tilde{r}}} v^2 \, dx \geq \frac{d}{2} > 0,
\]
implying that \( v \neq 0 \). But then there is \( \Lambda \subset B_r \) with \( \text{meas}(\Lambda) > 0 \) such that \( v(x) \neq 0 \), for every \( x \in \Lambda \). Hence for \( x \in \Lambda \) fixed and a constant \( k > 0 \), \( v_n(x) = ku_n(x)/\|u_n\| \neq 0 \) for every \( n \) large enough, so we can claim that \( u_n(x) \neq 0 \) eventually. As a consequence of \( \|u_n\| \to \infty \), \( |u_n(x)| \to \infty \). Hence \( |u_n(x)| \to \infty \) for every \( x \in \Lambda \). Since

\[
I(u_n) \geq \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u_n)u_n - F(u_n) \right] dx \geq \int_{\Lambda} \left[ \frac{1}{4} f(u_n)u_n - F(u_n) \right] dx,
\]

by (6) and Fatou’s lemma

\[
\liminf_{n \to \infty} I(u_n) \geq \int_{\Lambda} \liminf_{n \to \infty} \left[ \frac{1}{4} f(u_n)u_n - F(u_n) \right] dx = \infty,
\]

from which \( I(u_n) \to \infty \), contradicting that \( I(u_n) \to l \in \mathbb{R} \).

**Case 2: \( \{y_n\} \) is unbounded**

In this case we define a new sequence \( \hat{v}_n := v_n(\cdot + y_n) \); note that \( \|\hat{v}_n\| = \|v_n\| \) is constant. Thus, up to a subsequence \( \hat{v}_n \to \hat{v} \) in \( H^1_{rad}(\mathbb{R}^3) \), so that \( \hat{v}_n \to \hat{v} \) in \( L^p_{lo} \mathbb{R}^3 \) for \( 1 \leq p < 6 \), and \( \hat{v}_n(x) \to \hat{v}(x) \) a.e. \( x \in \mathbb{R}^3 \). From (3.9)

\[
\liminf_{n \to \infty} \int_{B_r(y_n)} v_n^2 dx = \liminf_{n \to \infty} \int_{B_r} \hat{v}_n^2 dx \geq d
\]

and hence

\[
\int_{B_r} \hat{v}^2 dx \geq d > 0,
\]

implying that \( \hat{v} \neq 0 \). But then there is \( \Lambda \subset B_r \) with \( \text{meas}(\Lambda) > 0 \) such that \( \hat{v}(x) \neq 0 \), for every \( x \in \Lambda \). Hence for \( x \in \Lambda \) fixed and a constant \( k > 0 \), \( \hat{v}_n(x) = ku_n(x+y_n)/\|u_n\| \neq 0 \) for every \( n \) large enough, so that we can claim that \( u_n(x+y_n) \neq 0 \) eventually. As a consequence of \( \|u_n\| \to \infty \), \( |u_n(x+y_n)| \to \infty \). Hence, as before, \( |u_n(x+y_n)| \to \infty \) for every \( x \in \Lambda \). Therefore we have

\[
I(u_n) \geq \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u_n)u_n - F(u_n) \right] dx
\]

\[
\geq \int_{B_r(y_n)} \left[ \frac{1}{4} f(u_n)u_n - F(u_n) \right] dx
\]

\[
= \int_{B_r} \left[ \frac{1}{4} f(u_n(x+y_n))u_n(x+y_n) - F(u_n(x+y_n)) \right] dx
\]

\[
\geq \int_{\Lambda} \left[ \frac{1}{4} f(u_n(x+y_n))u_n(x+y_n) - F(u_n(x+y_n)) \right] dx,
\]

and by (6) and Fatou’s lemma

\[
\liminf_{n \to \infty} I(u_n) \geq \int_{\Lambda} \liminf_{n \to \infty} \left[ \frac{1}{4} f(u_n(x+y_n))u_n(x+y_n) - F(u_n(x+y_n)) \right] dx = \infty,
\]

from which \( I(u_n) \to \infty \), which is again a contradiction. \( \Box \)

The above proposition allows us to get the next result.

**Corollary 10.** We have \( c := \inf_\mathcal{M} I > 0 \).

**Proof.** By (2.9) we know that \( c \geq 0 \). The proof is by contradiction. Assume that there is a sequence \( \{u_n\} \subset \mathcal{M} \) with \( I(u_n) \to 0 \). By Proposition 9, \( \{u_n\} \) is bounded; let \( \bar{u} \in H^1_{rad}(\mathbb{R}^3) \)
such that, up to a subsequence, $u_n \rightharpoonup \overline{u}$ in $H^1_{rad}(\mathbb{R}^3)$ and $u_n(x) \to \overline{u}(x)$ a.e. in $\mathbb{R}^3$. Using Fatou’s lemma and (2.3)

$$0 = \liminf_{n \to \infty} I(u_n)$$

$$= \liminf_{n \to \infty} \left( I(u_n) - \frac{1}{4} I'(u_n)[u_n] \right)$$

$$\geq \liminf_{n \to \infty} \frac{1}{4} \|u_n\|^2 + \liminf_{n \to \infty} \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u_n)u_n - F(u_n) \right] \, dx$$

$$\geq \frac{1}{4} \|\overline{u}\|^2 + \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(\overline{u})\overline{u} - F(\overline{u}) \right] \, dx$$

$$\geq \frac{1}{4} \|\overline{u}\|^2,$$

from which $\|\overline{u}\| = 0$, which is a contradiction since $\mathcal{N}$ (and hence $\mathcal{M}$) is bounded away from zero by (2.8).

Recall that $\mathcal{M}$ is not a smooth manifold, hence the next result will be fundamental.

**Proposition 11.** If the infimum $c$ of $I$ on $\mathcal{M}$ is attained, $c$ is a critical value of $I$.

**Proof.** Let be $u \in \mathcal{M}$ such that $I(u) = c = \inf_{\mathcal{M}} I$. We need to prove that $I'(u) = 0$. By contradiction assume that $I'(u) \neq 0$. Since $I$ is a $C^1$ functional on $H^1_{rad}(\mathbb{R}^3)$, there are $\delta > 0$ and $\varepsilon_0 > 0$ such that

$$\forall v \in H^1_{rad}(\mathbb{R}^3) \text{ with } \|v - u\| \leq 2\delta, \text{ we have } \|I'(v)\|_* \geq \varepsilon_0.$$

Since $u \in \mathcal{M}$, we know by Remark 2 there is $L > 0$ such that $\|u^+\|, \|u^-\| > L > 0$ and we can assume that $6\delta < L$.

On the set $Q := \left[ \frac{1}{2}, \frac{3}{2} \right] \times \left[ \frac{1}{2}, \frac{3}{2} \right]$ we define the function

$$h : (\alpha, \beta) \in Q \mapsto \alpha u^+ + \beta u^- \in H^1_{rad}(\mathbb{R}^3).$$

Since $I'(u)[u^+] = 0$, by Lemma 7, $I(su^+ + tu^-) < I(u)$, for any $s, t > 0$, with $s \neq 1$ or $t \neq 1$, and then

$$I(h(\alpha, \beta)) = I(\alpha u^+ + \beta u^-) < I(u) = c,$$

for every pair $(\alpha, \beta) \in Q$ with $\alpha \neq 1$ or $\beta \neq 1$. Therefore

$$c_0 := \max_{\partial Q} I \circ h < c.$$

Defining $\varepsilon := \min\{(c - c_0)/2, 5\varepsilon_0/8\}$, by (3.11) we have

$$v \in I^{-1}(\{c - 2\varepsilon, c + 2\varepsilon\}) \cap B_{2\delta}(u) \implies \|I'(v)\|_* \geq \varepsilon_0.$$

But then, by the deformation lemma (see [20, Lemma 2.3]) there exists a deformation $\eta \in C([0,1] \times H^1_{rad}(\mathbb{R}^3), H^1_{rad}(\mathbb{R}^3))$ such that

(i) $\eta(t, v) = v$, if $t = 0$ or $v \notin I^{-1}(\{c - 2\varepsilon, c + 2\varepsilon\}) \cap B_{2\delta}(u)$.

(ii) Denoting $I^a := \{u \in H^1_{rad}(\mathbb{R}^3) : I(u) \leq a\}$ for $a \in \mathbb{R}$, then $\eta(1, I^{c+\varepsilon} \cap B_\delta(u_c)) \subset I^{c-\varepsilon}$.

(iii) For every $v \in H^1_{rad}(\mathbb{R}^3)$, $I(\eta(., v))$ is non increasing.

In particular, from (i) and (iii) it holds

$$I(\eta(1, v)) \leq I(\eta(0, v)) = I(v), \text{ for every } v \in H^1_{rad}(\mathbb{R}^3).$$
For \((\alpha, \beta) \in Q\) we have two possibilities; if \(\alpha \neq 1\) or \(\beta \neq 1\), from (3.12) and (3.13),
\[
I(\eta(1, h(\alpha, \beta))) \leq I(h(\alpha, \beta)) < c.
\]
If \((\alpha, \beta) = (1, 1)\), so that \(h(1, 1) = u\), it holds \(h(1, 1) \in I^{\varepsilon, \varepsilon} \cap B_\delta(u)\), and by (ii)
\[
I(\eta(1, h(1, 1))) \leq c - \varepsilon < c.
\]

Then
\[
(3.14) \quad \max_{(\alpha, \beta) \in Q} I(\eta(1, h(\alpha, \beta))) < c.
\]

**Claim:** we have
\[
\eta(1, h(Q)) \cap \mathcal{M} \neq \emptyset.
\]
Indeed, for \((\alpha, \beta) \in Q\), we define the functions given by
\[
\varphi(\alpha, \beta) := \eta(1, h(\alpha, \beta)) \in H^1_{rad}(\mathbb{R}^3),
\]
and
\[
\Psi(\alpha, \beta) := (\psi_1(\alpha, \beta), \psi_2(\alpha, \beta)) := (I'(\varphi(\alpha, \beta))[\varphi(\alpha, \beta)^+], I'(\varphi(\alpha, \beta))[\varphi(\alpha, \beta)^-]).
\]
The claim holds if there exists \((\alpha_0, \beta_0) \in Q\) such that \(\Psi(\alpha_0, \beta_0) = (0, 0)\). Since
\[
\|u - h(\alpha, \beta)\| = \|(u^+ + u^-) - (\alpha u^+ + \beta u^-)\|
\]
\[
= |1 - \alpha||u^+||1 - \beta||u^-||
\]
\[
\geq |1 - \alpha||u^+||
\]
\[
\geq |1 - \alpha|L
\]
\[
> |1 - \alpha|6\delta
\]
\[
(3.15) \quad > 2\delta \iff \alpha < \frac{2}{3} \text{ or } \alpha > \frac{4}{3},
\]
using (i) and (3.15), for \(\alpha = \frac{1}{2}\) and for every \(\beta \in [\frac{1}{2}, \frac{3}{2}]\) we have \(\varphi(\frac{1}{2}, \beta) = h(\frac{1}{2}, \beta)\), so that
\[
\Psi(\frac{1}{2}, \beta) = (I'(h(\frac{1}{2}, \beta))[h(\frac{1}{2}, \beta)^+], I'(h(\frac{1}{2}, \beta))[h(\frac{1}{2}, \beta)^-])
\]
\[
= (I'(\frac{1}{2}u^+ + \beta u^-)[\frac{1}{2}u^+], I'(\frac{1}{2}u^+ + \beta u^-)[\beta u^-]).
\]

By (3.1), (3.2) and (ii) of Lemma 7 we infer
\[
I'(\frac{1}{2}u^+ + \beta u^-)[\frac{1}{2}u^+] = I'(\frac{1}{2}u^+)[\frac{1}{2}u^+] + \frac{\beta^2}{4}\lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx
\]
\[
\geq I'(\frac{1}{2}u^+)[\frac{1}{2}u^+] + (\frac{1}{2})^4 \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx
\]
\[
= \gamma_+(\frac{1}{2}u) > 0,
\]
from which we obtain
\[
(3.16) \quad \psi_1(\frac{1}{2}, \beta) = I'(\frac{1}{2}u^+ + \beta u^-)[\frac{1}{2}u^+] > 0, \quad \text{for every } \beta \in [\frac{1}{2}, \frac{3}{2}].
\]
If $\alpha = \frac{2}{3}$, for every $\beta \in \left[\frac{1}{2}, \frac{3}{2}\right]$, using (i) above and (3.15), we have $\varphi(\frac{2}{3}, \beta) = h(\frac{2}{3}, \beta)$, and arguing as before we get
\[
I'(\frac{2}{3}u^+ + \beta u^-)[\frac{3}{2}u^+] = I'(\frac{3}{2}u^+)[\frac{3}{2}u^+] + (\frac{3}{2})^2 \beta^2 \lambda \int_{\mathbb{R}^3} \phi_{u^+} - (u^+)^2 dx \\
\leq I'(\frac{3}{2}u^+)[\frac{3}{2}u^+] + (\frac{3}{2})^4 \lambda \int_{\mathbb{R}^3} \phi_{u^+} - (u^+)^2 dx \\
= \gamma_+(\frac{3}{2}u) < 0,
\]
so that
\[
(3.17) \quad \psi_1(\frac{2}{3}, \beta) = I'(\frac{3}{2}u^+ + \beta u^-)[\frac{3}{2}u^+] < 0, \quad \text{for every } \beta \in \left[\frac{1}{2}, \frac{3}{2}\right].
\]

Analogously we have, by using (3.1), (3.3) and (ii) of Lemma 7,
\[
(3.18) \quad \psi_2(\alpha, \frac{2}{3}) = I'(\alpha u^+ + \frac{2}{3}u^-)[\frac{3}{2}u^+] > 0, \quad \text{for every } \alpha \in \left[\frac{1}{2}, \frac{3}{2}\right],
\]
\[
(3.19) \quad \psi_2(\alpha, \frac{2}{3}) = I'(\alpha u^+ + \frac{3}{2}u^-)[\frac{3}{2}u^+] < 0, \quad \text{for every } \alpha \in \left[\frac{1}{2}, \frac{3}{2}\right].
\]
Since $\Psi$ is continuous on $Q$, because $\eta$, $h$ are continuous, and (3.16)-(3.19) hold, by Miranda’s theorem (see [12]), there is $(\alpha_0, \beta_0) \in Q$ such that $\Psi(\alpha_0, \beta_0) = (0, 0)$. Therefore, the Claim holds.

But then $I(\varphi(\alpha_0, \beta_0)) \geq \min_{\mathcal{M}} I = c$, in contradiction with (3.14). Therefore $I'(u) = 0$ and the proof is completed. \hfill \square

4. Proof of the main result

Before proving our result let us observe the following:

\[
(4.1) \quad \exists \delta > 0 \text{ such that } \forall w \in H^1_{rad}(\mathbb{R}^3) \text{ with } ||w|| \leq \delta : \gamma_{\pm}(w) = I'(w)[w^\pm] \geq \frac{1}{4} ||w^\pm||^2.
\]

In fact, fixing $0 < \varepsilon < 1/2$, by (2.1) we have
\[
\gamma_{\pm}(w) \geq I'(w^\pm)[w^\pm] \geq ||w^\pm||^2 - \int_{\mathbb{R}^3} f(w^\pm) w^\pm dx \\
\geq ||w^\pm||^2 - \varepsilon \int_{\mathbb{R}^3} (w^\pm)^2 dx - C_\varepsilon \int_{\mathbb{R}^3} |w^\pm|^q dx \\
= \frac{1}{2} ||w^\pm||^2 - C_\varepsilon \int_{\mathbb{R}^3} |w^\pm|^q dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w^\pm|^2 dx + \left(\frac{1}{2} - \varepsilon\right) \int_{\mathbb{R}^3} (w^\pm)^2 dx \\
\geq \frac{1}{4} ||w^\pm||^2 + \left(\frac{1}{4} ||w^\pm||^2 - C_\varepsilon ||w^\pm||^q\right).
\]

Choosing $\delta \leq 1/(4C_\varepsilon)^{1/(q-2)}$ we conclude.

4.1. Proof of Theorem 1. Denote $c = \inf_{\mathcal{M}} I > 0$ and let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence, i.e., $I(u_n) \to c$. By Proposition 9 $\{u_n\}$ is bounded and we can assume that $u_n \rightharpoonup u$ in $H^1_{rad}(\mathbb{R}^3)$. Since $u_n \in \mathcal{M}$, $\gamma_+(u_n) = I'(u_n)[u_n^+] = 0$ and $\gamma_-(u_n) = I'(u_n)[u_n^-] = 0$. Then

\[
0 = \gamma_+(u_n) = \gamma(u_n^+) + \lambda \int_{\mathbb{R}^3} \phi_{u_n^-} (u_n^+)^2 dx
\]

implying that $\gamma(u_n^+) < 0$. Similarly, by using (3.3), we get $\gamma(u_n^-) < 0$; but then by Remark 2, $|u_n^+|_q, |u_n^-|_q \geq L > 0$, and using the compact embedding $H^1_{rad}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, we have $|u^+|_q, |u^-|_q \geq L > 0$. Hence $u^+, u^- \neq 0$, and $u = u^+ + u^-$ is a sign-changing function.
The function \( u \) is the candidate to be the element where the minimum of \( I \) on \( \mathcal{M} \) is attained; however for this we need to show that \( u \in \mathcal{M} \).

Note that

\[
\begin{align*}
  u^+_n \rightarrow u^+ & \text{ and } u^-_n \rightarrow u^- \text{ in } H^1_{rad}(\mathbb{R}^3).
\end{align*}
\]

Indeed, \(|u^+_n| \leq |u_n|\) so that the sequences \( \{u^+_n\} \), \( \{u^-_n\} \) are also bounded. Assume that 
\( u^+_n \rightarrow v \geq 0 \) and \( u^-_n \rightarrow w \leq 0 \) in \( H^1_{rad}(\mathbb{R}^3) \). Thus \( u_n = u^+_n + u^-_n \rightarrow v + w \) in \( H^1_{rad}(\mathbb{R}^3) \), and then 
\( u = v + w \), from which \( u^+ = v, u^- = w \).

**Claim:** it holds

(4.2)

\[
\begin{align*}
  u^+_n \rightarrow u^+ \text{ in } H^1_{rad}(\mathbb{R}^3).
\end{align*}
\]

We just prove the claim concerning the positive parts, since similarly it can be shown that 
\( u^-_n \rightarrow u^- \) in \( H^1_{rad}(\mathbb{R}^3) \).

Suppose that this is not true, i.e., \(|u^+| < \inf_{n \rightarrow \infty} |u^+_n|\). Since \( u^+_n \rightarrow u^+ \) in \( H^1_{rad}(\mathbb{R}^3) \), by Remark 3

\[
\begin{align*}
  \int_{\mathbb{R}^3} f(u^+_n)u^+_n dx & \rightarrow \int_{\mathbb{R}^3} f(u^+)u^+ dx, \\
  \int_{\mathbb{R}^3} F(u^+_n) dx & \rightarrow \int_{\mathbb{R}^3} F(u^+) dx,
\end{align*}
\]

besides of 

\[
\begin{align*}
  \int_{\mathbb{R}^3} \phi_{u^+_n}(u^+_n)^2 dx & \rightarrow \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 dx \quad \text{and} \\
  \int_{\mathbb{R}^3} \phi_{u^-_n}(u^+_n)^2 dx & \rightarrow \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 dx.
\end{align*}
\]

Therefore

(4.3)

\[
\begin{align*}
  \gamma_+(u) < \liminf_{n \rightarrow \infty} \left[ |u^+_n|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u^+_n}u^+_n dx - \int_{\mathbb{R}^3} f(u^+_n)u^+_n dx \right] = \liminf_{n \rightarrow \infty} \gamma_+(u_n) = 0.
\end{align*}
\]

Also recall that \( \gamma_-(u_n) = 0 \), and since

\[
\begin{align*}
  |u^-|^2 \leq \liminf_{n \rightarrow \infty} |u^-_n|^2,
\end{align*}
\]

we have

(4.4)

\[
\begin{align*}
  \gamma_-(u) \leq \liminf_{n \rightarrow \infty} \left[ |u^-_n|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u^-_n}u^-_n dx - \int_{\mathbb{R}^3} f(u^-_n)u^-_n dx \right] = \liminf_{n \rightarrow \infty} \gamma_-(u_n) = 0.
\end{align*}
\]

Now we take \( \delta > 0 \) satisfying (4.1) and we choose \( \zeta > 0 \) satisfying \( \zeta ||u|| \leq \delta \). Thence

\[
\begin{align*}
  \gamma_+(\zeta u^+ + tu^-) & \geq \frac{\zeta^2}{4} |u^+|^2 > 0, \quad \text{for every } t \in [\zeta, 1],
\end{align*}
\]

and

\[
\begin{align*}
  \gamma_-(su^+ + \zeta u^-) & \geq \frac{\zeta^2}{4} |u^-|^2 > 0, \quad \text{for every } s \in [\zeta, 1].
\end{align*}
\]

On the other hand, by (4.3), for every \( t \in [\zeta, 1] \) we have

\[
\begin{align*}
  \gamma_+(u^+ + tu^-) & = I'(u^+)[u^+] + t^2 \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx \\
  & \leq I'(u^+)[u^+] + \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx = \gamma_+(u) < 0,
\end{align*}
\]

and by (4.4), for every \( s \in [\zeta, 1] \)

\[
\begin{align*}
  \gamma_-(su^+ + u^-) & = I'(u^-)[u^-] + s^2 \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx \\
  & \leq I'(u^-)[u^-] + \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx = \gamma_-(u) \leq 0.
\end{align*}
\]
Hence, by Miranda’s theorem, there is a point \((\alpha, \beta) \in [\zeta, 1] \times [\zeta, 1]\) for which the function given by
\[
\Phi(s, t) := (\gamma_+(su^+ + tu^-), \gamma_-(su^+ + tu^-)), \quad (s, t) \in [\zeta, 1] \times [\zeta, 1],
\]
satisfies \(\Phi(\alpha, \beta) = (0, 0)\), i.e., \(\alpha u^+ + \beta u^- \in \mathcal{M}\). Therefore using (2.2)
\[
I(\alpha u^+ + \beta u^-) = I(\alpha u^+ + \beta u^-) - \frac{1}{4} I'(\alpha u^+ + \beta u^-)[\alpha u^+ + \beta u^-]
\]
\[
= \frac{\alpha^2}{4} ||u^+||^2 + \frac{\beta^2}{4} ||u^-||^2 + \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(\alpha u^+)\alpha u^+ - F(\alpha u^+) \right] dx
\]
\[
+ \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(-\beta u^-)(-\beta u^-) - F(-\beta u^-) \right] dx,
\]
and since \(0 < \alpha < 1, 0 < \beta \leq 1\) and (2.2) holds, we have (again by (2.2))
\[
I(\alpha u^+ + \beta u^-) < \frac{1}{4} ||u||^2 + \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u)u - F(u) \right] dx
\]
\[
< \liminf_{n \to \infty} \frac{1}{4} ||u_n||^2 + \liminf_{n \to \infty} \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(u_n)u_n - F(u_n) \right] dx,
\]
where the last inequality follows from our assumption and Fatou’s lemma. Thus
\[
I(\alpha u^+ + \beta u^-) < \liminf_{n \to \infty} \left( I(u_n) - \frac{1}{4} I'(u_n)[u_n] \right)
\]
\[
= \liminf_{n \to \infty} I(u_n)
\]
an absurd. Then (4.2) follows.

In virtue of the Claim, \(u_n \to u\) in \(H^1_{rad}(\mathbb{R}^3)\). Since \(\mathcal{M}\) is closed in \(H^1_{rad}(\mathbb{R}^3)\), \(u \in \mathcal{M}\), so that
\(I(u) = c = \inf_{\mathcal{M}} I\). By Proposition 11, \(u\) is a critical point of \(I\), and thus, a radial sign-changing solution of (1.1) in \(H^1_{rad}(\mathbb{R}^3)\) which has minimal energy among all the radial and sign-changing solutions.

Let us see that \(u\) changes the sign exactly once. We adapt the proof from the arguments given in [2] and [6]. In fact by the regularity of the solution \(u\) (see e.g. [16]), the set \(E := \{x \in \mathbb{R}^3 : u(x) \neq 0\}\) is open. If \(E\) has more than two components, since \(u\) changes of sign, without lost of generality we can assume the existence of connected components \(E_1, E_2\) and \(E_3\) such that we have the decomposition \(u = u_1 + u_2 + u_3\) with
\[
u_1 > 0\) on \(E_1\), and \(u_1 = 0\) on \(E_2 \cup E_3\),
\[
u_2 < 0\) on \(E_2\), and \(u_2 = 0\) on \(E_1 \cup E_3\),
\[
u_3 \neq 0\) on \(E_3\), and \(u_3 = 0\) on \(E_1 \cup E_2\).
\]
Also we define \(v := u_1 + u_2\), so that \(v^+ = u_1\) and \(v^- = u_2\). Since \(u\) is critical point, \(I'(u) = 0\), from which (here we use the notation \(\phi_3 := \phi_{u_3}\))
\[
I'(u)[u_1] = I'(u)[v^+] = I'(v)[v^+] + \lambda \int_{\mathbb{R}^3} \phi_3(v^+)^2 dx = 0,
\]
\[
I'(u)[u_2] = I'(u)[v^-] = I'(v)[v^-] + \lambda \int_{\mathbb{R}^3} \phi_3(v^-)^2 dx = 0,
\]
\[
I'(u)[u_3] = 0,
\]
and hence, in particular,

\[(4.5) \quad \gamma_+(v) = -\int_{\mathbb{R}^3} \phi_3 u_1^2 dx < 0 \quad \text{and} \quad \gamma_-(v) = -\int_{\mathbb{R}^3} \phi_3 u_2^2 dx < 0.\]

Let \( \delta > 0 \) as in (4.1) and let \( \zeta > 0 \) be such that \( \zeta \|v\| \leq \delta \). Then for every \( t \in [\zeta, 1] \)

\[\gamma_+ (\zeta v^+ + tv^-) \geq \frac{\zeta^2}{4} \|v^+\|^2 > 0,\]

and, by (4.5),

\[\gamma_+ (v^+ + tv^-) = I'(v^+) [v^+] + t^2 \lambda \int_{\mathbb{R}^3} \phi_{v^-} (v^+)^2 dx \leq I'(v^+) [v^+] + \lambda \int_{\mathbb{R}^3} \phi_{v^-} (v^+)^2 dx = \gamma_+ (v) < 0.\]

Similarly, for every \( s \in [\zeta, 1] \)

\[\gamma_- (sv^+ + \zeta v^-) \geq \frac{\zeta^2}{4} \|v^-\|^2 > 0,\]

and using again (4.5)

\[\gamma_- (sv^+ + v^-) = I'(v^-) [v^-] + s^2 \lambda \int_{\mathbb{R}^3} \phi_{v^-} (v^+)^2 dx \leq I'(v^-) [v^-] + \lambda \int_{\mathbb{R}^3} \phi_{v^-} (v^+)^2 dx = \gamma_- (v) < 0.\]

Summarizing, Miranda’s Theorem can be applied to the function \( \Psi \) defined on \([\zeta, 1] \times [\zeta, 1] \) by

\[\Psi (s, t) := (\gamma_+ (sv^+ + tv^-), \gamma_- (sv^+ + tv^-)),\]

so there is a point \((\alpha, \beta) \in [\zeta, 1] \times [\zeta, 1] \) such that

\[\gamma_+ (\alpha v^+ + \beta v^-) = \gamma_- (\alpha v^+ + \beta v^-) = 0,\]

i.e., \( \alpha v^+ + \beta v^- \in \mathcal{M} \). On the other hand, the assumptions in Remark 8 are satisfied, so that

\[I(\alpha v^+ + \beta v^-) = I(\alpha u_1 + \beta u_2) < I(u_1 + u_2 + u_3) = I(u) = c = \inf_{\mathcal{M}} I,\]

a contradiction. Therefore \( E \) has exactly two connected components, and this completely proves Theorem 1.

**APPENDIX**

In this appendix we prove Proposition 4, however some preliminaries are in order. In all that follows, \( u \) denotes the solution obtained in Lemma 1 in correspondence of a small fixed value of \( \lambda \).

Let us start by observing that (2.6) reads as

\[\|u\|^2 = \int_{\mathbb{R}^3} (f(u)u - \lambda \phi_4 u^2) dx < \int_{\mathbb{R}^3} (u^4 - \lambda \phi_4 u^2) dx,\]

so that

\[(4.6) \quad 0 < \int_{\mathbb{R}^3} (u^2 - \lambda \phi_4) u^2 dx.\]

In the following we will always adopt the convention that, for a radial function \( z : \mathbb{R}^3 \rightarrow \mathbb{R} \), we use the same notation to denote the function \( r \in [0, +\infty) \mapsto z(r) \in \mathbb{R} \) where \( r = |x| \). It will
be clear from the context if we mean \( z(x) \) or \( z(r) \). As usual we will denote always by \( \phi_z \) the solution of \(-\Delta \phi = z^2\) in \( \mathbb{R}^3 \). In particular the above convention applies to \( u \) and \( \phi_u \).

By the regularity of \( u \) and \( \phi_u \), there are \( r_1, r_4 \) real numbers with \( 0 < r_1 < r_4 \) such that
\[
|u|^2 - \lambda \phi_u > 0 \quad \text{on} \quad [r_1, r_4].
\]

Given \( 0 < s < p \), we use the notation \( A_{s,p} \) for the annulus \( B_p \setminus \overline{B}_s \). Here \( B_a \) denotes the ball centred in zero with radius \( a > 0 \) and \( \overline{B}_a \) its closure.

Let \( \delta > 0 \) be such that
\[
\int_{A_{r_1, r_4}} (u^2 - \lambda \phi_u)u^2 dx > \frac{3}{2} \delta,
\]
which is possible by (4.6). We can find numbers \( r_2, r_3 > 0 \) with \( r_1 < r_2 < r_3 < r_4 \) such that
\[
\int_{A_{r_2, r_3}} (u^2 - \lambda \phi_u)u^2 dx > \delta \quad \text{and} \quad \int_{A_{r_i, r_{i+1}}} (u^2 + \lambda \phi_u)u^2 dx < \frac{\delta}{4}, \quad \text{for} \quad i = 1, 3.
\]

Let \( \nu, \eta \in C^\infty(\mathbb{R}^3; [0, 1]) \) be radial cut-off functions satisfying:
- \( \nu = 0 \) in \( B_{r_1} \), \( \nu \) strictly increasing (in the radial coordinate) in \( A_{r_1, r_2} \), with \( \nu = 1 \) outside \( B_{r_2} \).
- \( \eta = 1 \) in \( B_{r_3} \), \( \eta \) strictly decreasing (in the radial coordinate) in \( A_{r_3, r_4} \), with \( \eta = 0 \) outside \( B_{r_4} \).

For brevity we define the functions
\[
\nu := \nu \eta \quad \text{and} \quad e_t := t \nu, \quad \text{for} \quad t > 1.
\]

We also define the functional \( G : H^1_{\text{rad}}(\mathbb{R}^3) \to \mathbb{R} \) by
\[
G(u) := \int_{\mathbb{R}^3} (|\nabla u|^2 + 2u^2)dx + \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} f(u) u dx.
\]

Lemma 12. With the above notations, there exists \( T_1 > 1 \) such that
\[
\forall t \geq T_1 : \quad G(e_t) < 0.
\]

Proof. Computing we have, for \( t > 1 \),
\[
G(e_t) = t^2 \left( \|\nu\|^2 + \int_{\mathbb{R}^3} \nu^2 dx \right) + \lambda t^4 \int_{\mathbb{R}^3} \phi_v \nu^2 dx - \int_{\mathbb{R}^3} \frac{f(t \nu) t \nu dx}{t^3},
\]
so that
\[
\frac{G(e_t)}{t^4} < \frac{1}{t^2} \left( \|\nu\|^2 + \int_{\mathbb{R}^3} \nu^2 dx \right) + \lambda \int_{\mathbb{R}^3} \phi_v \nu^2 dx - \frac{f(t \nu)}{t^3} \nu dx,
\]
from which
\[
\limsup_{t \to \infty} \frac{G(e_t)}{t^4} \leq \lambda \int_{\mathbb{R}^3} \phi_v \nu^2 dx + \limsup_{t \to \infty} \left[ - \int_{\mathbb{R}^3} \frac{f(t \nu)}{t^3} \nu dx \right]
\]
\[
= \lambda \int_{\mathbb{R}^3} \phi_v \nu^2 dx - \liminf_{t \to \infty} \int_{\mathbb{R}^3} \frac{f(t \nu)}{t^3} \nu dx
\]
\[
\leq \lambda \int_{\mathbb{R}^3} \phi_v \nu^2 dx - \int_{A_{r_1, r_4}} \left[ \liminf_{t \to \infty} \frac{f(t \nu)}{(t \nu)^3} \right] \nu dx
\]
\[
= \int_{A_{r_1, r_4}} (\lambda \phi_v - \nu^2) \nu^2 dx.
\]
The last inequality above follows from Fatou’s lemma. Therefore
\[
\limsup_{t \to \infty} \frac{G(e_t)}{t^4} \leq \int_{A_{r_1,r_2}} (\lambda \phi_{ru} - \nu^2 u^2) \nu^2 u^2 dx + \int_{A_{r_2,r_3}} (\lambda \phi_u - u^2) u^2 dx
\]
\[
+ \int_{A_{r_3,r_4}} (\lambda \phi_{u\eta} - u^2 \eta^2) u^2 \eta^2 dx
\]
\[
=: I_1 + I_2 + I_3,
\]
Let us estimate every integral. By (4.7) we have
\[
I_1 \leq \int_{A_{r_1,r_2}} |\lambda \phi_{ru} - \nu^2 u^2| \nu^2 u^2 dx \leq \int_{A_{r_1,r_2}} (u^2 + \lambda \phi_u) u^2 dx < \frac{\delta}{4}.
\]
Similarly
\[
I_3 < \frac{\delta}{4},
\]
and, again by (4.7),
\[
I_2 = \int_{A_{r_2,r_3}} (\lambda \phi_u - u^2) u^2 dx < -\delta.
\]
Then
\[
\limsup_{t \to \infty} \frac{G(e_t)}{t^4} < \frac{\delta}{4} - \frac{\delta}{2} = -\frac{\delta}{2},
\]
from which the conclusion follows. \(\square\)

For \(t > 0\), define the functional \(H_t : H^1_{rad}(\mathbb{R}^3) \to \mathbb{R}\) by
\[
H_t(u) := t \int_{\mathbb{R}^3} |\nabla u|^2 dx + \left(\frac{1}{t} + t^2\right) \int_{\mathbb{R}^3} \lambda \phi_u^2 dx + \frac{\lambda}{t} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1}{t^2} \int_{\mathbb{R}^3} f(tu) u dx.
\]

**Lemma 13.** With the notations in (4.8), there exists \(T_2 > 1\) such that
\[
\forall t \geq T_2 : H_t(e_t) < 0.
\]

**Proof.** Note that
\[
H_t(e_t) = t \int_{\mathbb{R}^3} |\nabla e_t|^2 dx + \left(\frac{1}{t} + t^2\right) \int_{\mathbb{R}^3} e_t^2 dx + \frac{\lambda}{t} \int_{\mathbb{R}^3} \phi_{e_t} e_t^2 dx - \frac{1}{t^2} \int_{\mathbb{R}^3} f(t e_t) e_t dx
\]
\[
= t^3 \int_{\mathbb{R}^3} |\nabla v|^2 dx + (t^4 + 1) t \int_{\mathbb{R}^3} v^2 dx + t^3 \lambda \int_{\mathbb{R}^3} \phi_{v^2} dx - \frac{1}{t} \int_{\mathbb{R}^3} f(t^2 v) v dx,
\]
thus
\[
\frac{H_t(e_t)}{t^5} = \frac{1}{t^2} \int_{\mathbb{R}^3} |\nabla v|^2 dx + \left(\frac{1}{t^4} + \frac{1}{t}\right) \int_{\mathbb{R}^3} v^2 dx + \lambda \frac{1}{t^2} \int_{\mathbb{R}^3} \phi_{v^2} dx - \int_{\mathbb{R}^3} f(t^2 v) \frac{t^2 v}{t^6} dx,
\]
and then
\[
\limsup_{t \to \infty} \frac{H_t(e_t)}{t^5} = \limsup_{t \to \infty} \left[ - \int_{\mathbb{R}^3} \frac{f(t^2 v)}{t^6} v dx \right]
\]
\[
= - \liminf_{t \to \infty} \int_{\mathbb{R}^3} \frac{f(t^2 v)}{t^6} v dx
\]
\[
\leq - \int_{A_{r_1,r_4}} \liminf_{t \to \infty} \left[ \frac{f(t^2 v)}{(t^2 v)^3} \right] v^4 dx
\]
\[
= - \int_{A_{r_1,r_4}} v^4 dx,
\]
where the last inequality follows from Fatou’s lemma. Therefore we conclude.

We can prove now that $\mathcal{M}$ is nonempty.

### 4.2. Proof of Proposition 4

Under the above notations, consider the element

$$u := T_0 v \nu \eta \in H^1_{rad}(\mathbb{R}^3),$$

where $T_0 > 1$ is chosen such that

$$T_0 \geq \max\{T_1, T_2\}, \quad \left[\frac{r_1}{T_0}, \frac{r_4}{T_0}\right] \cap [r_1, r_4] = \emptyset, \quad \lambda \phi_u(T_0 r_1) < 1,$$

with $T_1, T_2$ given in Lemmas 12 and 13. Note that $\text{supp}(u) = A_{r_1, r_4}$. It will be useful the rescaled function

$$w(x) := u(T_0 x).$$

Before to proceed with the proof, let us show other preliminary facts.

First note that,

$$\int_{\mathbb{R}^3} |\nabla w|^2 dx = \frac{1}{T_0^2} \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad \int_{\mathbb{R}^3} w^2 dx = \frac{1}{T_0^3} \int_{\mathbb{R}^3} u^2 dx,$$

that the nonlinear terms are

$$\int_{\mathbb{R}^3} F(w) dx = \frac{1}{T_0^3} \int_{\mathbb{R}^3} F(u) dx, \quad \int_{\mathbb{R}^3} f(w) w dx = \frac{1}{T_0^3} \int_{\mathbb{R}^3} f(u) u dx,$$

and that the nonlocal term can be written as

$$\int_{\mathbb{R}^3} \phi_w w^2 dx = \frac{1}{T_0^3} \int_{\mathbb{R}^3} \phi_u u^2 dx \quad \text{ since } \phi_w(x) = \frac{1}{T_0^3} \phi_u(T_0 x).$$

Moreover, recalling that $\phi_u$ is decreasing and (4.9), we get

$$\lambda \int_{\mathbb{R}^3} \phi_w u^2 dx = 4\pi \lambda \int_{\mathbb{R}^3}^\infty \phi_w(r) u^2(r) r^2 dr = \frac{4\pi}{T_0^2} \int_{r_1}^{r_4} \lambda \phi_u(T_0 r) u^2(r) r^2 dr$$

$$< \frac{4\pi}{T_0^2} \int_{r_1}^{r_4} u^2(r) r^2 dr = \frac{1}{T_0^3} \int_{\mathbb{R}^3} u^2 dx.$$

Defining $v := u - w$, we have $\text{supp}(u) \cap \text{supp}(w) = \emptyset$ by (4.9), so that $v^+ = u$ and $v^- = -w$.

Note that for every $\tau \in (0, T_0]$,

$$I'(T_0 v^+ + \tau v^-)[T_0 v^+] = T_0^2 \int_{\mathbb{R}^3} |\nabla v^+|^2 + (v^+)^2 dx + T_0^4 \lambda \int_{\mathbb{R}^3} \phi_{v^+}(v^+)^2 dx$$

$$+ T_0^4 \lambda \int_{\mathbb{R}^3} \phi_{v^+}(v^-)^2 dx - \int_{\mathbb{R}^3} f(T_0 v^+) T_0 v^+ dx$$

$$\leq T_0^2 \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx + T_0^4 \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx$$

$$+ T_0^4 \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} f(T_0 u) T_0 u dx,$$

from which

$$I'(T_0 v^+ + \tau v^-)[T_0 v^+] \leq \|T_0 u\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{T_0 u}(T_0 u)^2 dx - \int_{\mathbb{R}^3} f(T_0 u) T_0 u dx + T_0^4 \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx.$$
Then we have by (4.10) and by recalling the notation in (4.8) (concretely that $e_{T_0^2} = T_0u$)

$$I'(T_0v^+ + \tau v^-)[T_0v^+] \leq ||e_{T_0^2}||^2 + \lambda \int_{\mathbb{R}^3} \phi_{e_{T_0^2}^2}^2 dx - \int_{\mathbb{R}^3} f(e_{T_0^2})e_{T_0^2} dx + \int_{\mathbb{R}^3} e_{T_0^2}^2 dx$$

(4.11)

$$= \ G(e_{T_0^2}) < 0,$$

by Lemma 12. Similarly, for every $\theta \in (0, T_0]$, we get:

$$I'(\theta v^+ + T_0v^-)[T_0v^-] = T_0^2 \int_{\mathbb{R}^3} ||\nabla v^-||^2 + (v^-)^2 \ dx + T_0^4 \lambda \int_{\mathbb{R}^3} \phi_{v^-}(v^-)^2 dx$$

$$+ T_0^4 \theta^2 \lambda \int_{\mathbb{R}^3} \phi_{v^+}(v^+)^2 dx - \int_{\mathbb{R}^3} f(T_0v^-)T_0v^- dx$$

$$\leq T_0^2 \int_{\mathbb{R}^3} ||\nabla w||^2 + w^2 \ dx + T_0^4 \lambda \int_{\mathbb{R}^3} \phi_{w}w^2 dx$$

$$+ T_0^4 \lambda \int_{\mathbb{R}^3} \phi_{w}w^2 dx - \int_{\mathbb{R}^3} f(T_0w)T_0w \ dx$$

$$= T_0 \int_{\mathbb{R}^3} |\nabla u|^2 \ dx + \frac{1}{T_0} \int_{\mathbb{R}^3} u^2 \ dx + \frac{\lambda}{T_0} \int_{\mathbb{R}^3} \phi_{u}u^2 \ dx$$

$$+ T_0^4 \lambda \int_{\mathbb{R}^3} \phi_{u}u^2 \ dx - \frac{1}{T_0} \int_{\mathbb{R}^3} f(T_0u)T_0u \ dx$$

$$\leq T_0 \int_{\mathbb{R}^3} |\nabla u|^2 \ dx + \frac{1}{T_0} \int_{\mathbb{R}^3} u^2 \ dx + \frac{\lambda}{T_0} \int_{\mathbb{R}^3} \phi_{u}u^2 \ dx$$

$$+ T_0^2 \int_{\mathbb{R}^3} u^2 \ dx - \frac{1}{T_0^3} \int_{\mathbb{R}^3} f(T_0u)T_0u \ dx$$

$$= H_{T_0}(u),$$

where the last inequality holds true because of (4.10). Then, by Lemma 13, for every $\theta \in (0, T_0]$ we get:

(4.12) $$I'(\theta v^+ + T_0v^-)[T_0v^-] \leq H_{T_0}(u) = H_{T_0}(e_{T_0^2}) < 0.$$ 

On the other hand, it is clear that

$$\int_{\mathbb{R}^3} f(tv^+)tv^+ dx = t^2 \int_{\text{supp}(u)} \frac{f(tu)}{tu} u^2 dx.$$ 

By (f3) and (f4) it holds

$$\lim_{t \to 0} \frac{f(tu)}{tu} = 0 \quad \text{and} \quad \frac{f(tu)}{tu} < (tu)^2 \quad \text{a.e.} \ x \in \text{supp}(u),$$

and since

$$\int_{\mathbb{R}^3} (tu)^2 u^2 dx = t^2 \int_{\mathbb{R}^3} u^4 dx \to 0 \quad \text{when} \ t \to 0,$$

we deduce

$$\int_{\mathbb{R}^3} \frac{f(tu)}{t} u dx = \int_{\mathbb{R}^3} \frac{f(tu)}{tu} u^2 dx < \int_{\mathbb{R}^3} (tu)^2 u^2 dx \to 0 \quad \text{when} \ t \to 0.$$

We infer that

(4.13) $$\exists t_0 \in (0, 1) : ||u||^2 > \int_{\mathbb{R}^3} \frac{f(t_0u)}{t_0} u dx.$$
Furthermore, for every $\tau \in [t_0, T_0]$, we have

$$I'(t_0v^+ + \tau v^-)[t_0v^+] = t_0^3 \int_{\mathbb{R}^3} \left[ |\nabla v^+|^2 + (v^+)^2 \right] dx + t_0^4 \lambda \int_{\mathbb{R}^3} \phi_{v^+}(v^+)^2 dx + \frac{t_0^2}{2} \int_{\mathbb{R}^3} \phi_{v^-}(v^-)^2 dx - \int_{\mathbb{R}^3} f(t_0v^+)t_0v^+ dx$$

$$> t_0^2 \left[ \|u\|^2 - \int_{\mathbb{R}^3} \frac{f(t_0u)}{t_0} u dx \right]$$

(4.14)

Finally, for every $\theta \in [t_0, T_0]$, we have

$$I'(\theta v^+ + t_0v^-)[t_0v^-] = t_0^3 \int_{\mathbb{R}^3} \left[ |\nabla v^-|^2 + (v^-)^2 \right] dx + t_0^4 \lambda \int_{\mathbb{R}^3} \phi_{v^-}(v^-)^2 dx + \frac{t_0^2}{2} \int_{\mathbb{R}^3} \phi_{v^+}(v^+)^2 dx - \int_{\mathbb{R}^3} f(t_0v^-)t_0v^- dx$$

$$> t_0^2 \left[ \|u\|^2 - \int_{\mathbb{R}^3} \frac{f(t_0u)}{t_0} u dx \right]$$

(4.15)

Summarizing, there exists $t_0 > 0$, the one given in (4.13), and $T_0 > t_0$, the one given in (4.9), such that, by (4.14) and (4.15)

$$I'(t_0v^+ + \tau v^-)[t_0v^+] \geq I'(\theta v^+ + t_0v^-)[t_0v^-] > 0, \quad \forall \theta, \tau \in [t_0, T_0],$$

and, by (4.11) and (4.12),

$$I'(T_0v^+ + \tau v^-)[T_0v^+] \geq I'(\theta v^+ + T_0v^-)[T_0v^-] < 0, \quad \forall \theta, \tau \in [t_0, T_0].$$

This means that, if we define $\Phi : [t_0, T_0] \times [t_0, T_0] \rightarrow \mathbb{R}^2$ by

$$\Phi(\theta, \tau) := (I'(\theta v^+ + \tau v^-)[\theta v^+], I'(\theta v^+ + \tau v^-)[\tau v^-]),$$

as a consequence of Miranda’s Theorem, there is $(\alpha_0, \beta_0) \in [t_0, T_0] \times [t_0, T_0]$ that satisfies $\Phi(\alpha_0, \beta_0) = (0, 0)$, i.e.

$$I'(\alpha_0v^+ + \beta_0v^-)[\alpha_0v^+] = 0, \quad I'(\alpha_0v^+ + \beta_0v^-)[\beta_0v^-] = 0.$$

Therefore $\alpha_0v^+ + \beta_0v^- \in \mathcal{M}$ and we conclude the proof.

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