CR-RELATIVES KÄHLER MANIFOLDS

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Abstract. In this paper we show that two Kähler manifolds which do not share a Kähler submanifold, do not share either a Levi degenerate CR–submanifold with constant dimension Levi kernel. In particular, they do not share a CR–product. Further, we obtain that a Levi degenerate CR-submanifold of \( \mathbb{C}^n \) cannot be isometrically immersed into a flag manifold.

1. Introduction and statement of the main result

Two Kähler manifolds \( (\tilde{M}_1, \omega_1), (\tilde{M}_2, \omega_2) \), are called relatives when they share a common Kähler submanifold, that is when there exists a Kähler manifold \((M, \omega)\) and two holomorphic maps \( f_1 : M \to \tilde{M}_1 \), \( f_2 : M \to \tilde{M}_2 \), such that \( f_1^* \omega_1 = f_2^* \omega_2 \). The problem of understanding when two Kähler manifolds are relatives appears in literature in Umehara’s work [20], where it is proven that two complex space form with holomorphic sectional curvature of different signs can not be relative. Umehara’s proof relies on Calabi’s criterion for the existence of a holomorphic and isometric immersion of a Kähler manifold into complex space forms [5] (see also [15]). In [10] it is shown that Hermitian symmetric spaces with different compact types are not relatives. Further, in [13] it has been proven that Euclidean spaces and Hermitian symmetric spaces of noncompact type are not relatives, and the same result for Hermitian symmetric spaces of compact type follows by Umehara’s result and Nakagawa-Takagi embedding of Hermitian symmetric spaces of compact type into the complex projective space. The relativity problem has been investigated also in [9], where it has been reformulated in terms of the Umehara’s algebra, and in [18, 19, 21].

In this paper we are interested in studying when two Kähler manifolds are CR–relatives, i.e. when they share a CR–submanifold. The notion of CR–submanifold of a Kähler manifold was introduced and firstly studied by A. Bejancu in [2, 3]. In [7] B. Y. Chen studied a special family of CR–submanifolds into complex space forms called CR–products. In particular, he proved that a CR–product isometrically immersed in \( \mathbb{C}^m \) is globally a Riemannian product
of a holomorphic submanifold and a totally real one, and he gave a characterization of CR–products of maximal dimension isometrically immersed in a complex projective space. In \[8\] the existence of a foliation with totally real leaves is investigated, and further results on CR–submanifolds of complex space forms can be found in \[4\]. For a more comprehensive exposition on this topic, we refer the reader to [16]. Our main result is the following:

**Theorem 1.** Two Kähler manifolds which are not relatives do not share a Levi degenerate CR–submanifold with constant dimension Levi kernel.

Notice that the Levi degenerate condition cannot be dropped, as shown for example by the canonical CR-structure on the 3-sphere, which is induced from both the flat Kähler structure on $\mathbb{C}^2$ and the Fubini–Study one on $\mathbb{C}P^2$, proving that $\mathbb{C}^2$ and $\mathbb{C}P^2$ are an example of CR–relatives Kähler manifolds which, by Umehara’s result Th. \[3\] below, are not relatives. It is worth pointing out that if two Kähler manifolds are relative then they are also CR–relatives in a trivial way, since a Kähler submanifold can be viewed as a (holomorphic) CR–submanifold. Finally, the condition on the Levi kernel is a natural assumption which is fundamental to obtain the existence of a foliation with complex leaves.

As direct consequence of Theorem \[1\] we get the following:

**Corollary 1.** Kähler manifolds which are not relatives do not share a CR–product.

Further, combining Theorem \[1\] with a result of A. Loi and R. Mossa (see \[14\] or Th. \[4\] below) we get the following:

**Corollary 2.** A Levi degenerate with constant dimension Levi kernel CR-submanifold of $\mathbb{C}^m$ cannot be isometrically immersed into a flag manifold.

In particular the CR–submanifolds of hypersurface type of $\mathbb{C}^m$ given in Example \[1\] cannot be isometrically immersed as CR–submanifolds into a flag manifolds.

The paper is organized as follows. In Section \[2\] we collects some results about Kähler relatives and Kähler immersions into the complex flat and complex projective spaces, while in Section \[3\] we summarize what we need on CR–manifolds and CR–submanifolds of a Kähler manifold. Finally in Section \[4\] we prove Theorem \[1\] and its corollaries.

2. Preliminaries on Kähler manifolds

Throughout this section, let $M$ be a Kähler manifold endowed with a Kähler metric $g$, and let us denote by $\omega$ the closed $(1,1)$-form associated to $g$, i.e. $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$. Recall that a Kähler manifold is characterized by the existence in a neighborhood $U$ of each point $x \in M$ of a Kähler potential $\varphi: U \to \mathbb{R}$ such that $\omega|_U := \frac{i}{2} \partial \bar{\partial} \varphi$.

Denote by $g_0$ the flat metric on the complex euclidean space $\mathbb{C}^m$, and by $\omega_0$ the associate Kähler form, i.e. $\omega_0 := \frac{i}{2} \partial \bar{\partial} ||z||^2$ (where $z = (z_1, \ldots, z_m)$ are holomorphic coordinates on
C^m). Further denote by CP^m_b = (CP^m, g_b) the complex projective space endowed with the Fubini–Study metric g_b of holomorphic sectional curvature 4b > 0. More precisely, given homogeneous coordinates [Z_0, . . . , Z_m] on CP^m and setting affine coordinates (z_1, . . . , z_m), with z_j := Z_j/Z_0, j = 1, . . . , m, on the open set U_0 = {Z_0 ≠ 0}, a Kähler potential for the metric g_b is given by:

\( \Phi_b(z) := \frac{1}{b} \log \left( 1 + b \sum_{j=1}^{m} |z_j|^2 \right) \).

We denote by \( \omega_b \) the Kähler form associated to \( g_b \), i.e. \( \omega_b|_{U_0} = \frac{i}{2} \partial \bar{\partial} \Phi_b \). When \( b = 1 \) we usually use the notation \( g_{FS} \) and \( \omega_{FS} \) for \( g_1 \) and \( \omega_1 \) respectively.

We say that a Kähler immersion is full if the submanifold is not contained in any totally geodesic submanifold of the ambient space, that is in our context when the manifold M Kähler immersed in C^m (or CP^m_b) is not contained into C^{m'} (or CP^{m'}_b) for any \( m' < m \).

When the ambient space is a complex space form, E. Calabi [5] proved that a full Kähler immersion determines uniquely the dimension of the ambient space. In particular for C^m and CP^m Calabi’s result reads as follows.

**Theorem 2 (Calabi Rigidity Theorem [5]).** If a neighborhood \( V \) of a point \( p \) admits a full Kähler immersion into C^m or CP^m, then \( m \) is uniquely determined by the metric and the immersion is unique up to rigid motions of C^m (respectively CP^m).

Calabi Rigidity Theorem is a very useful tool to study relatives Kähler manifolds. The following result of M. Umehara [20] together with Calabi’s Rigidity proves that C^m and CP^{m'} are not relatives for any \( m, m' < \infty \).

**Theorem 3 (M. Umehara [20]).** Any Kähler submanifold of C^m, \( m \leq \infty \), admits a full Kähler immersion into CP^\infty_b, for any value of \( b > 0 \).

Finally, we recall that up to rescale the metric by a positive factor, a homogeneous compact Kähler manifold admits a local Kähler immersion into CP^m_b, \( m < \infty \).

**Theorem 4 (A. Loi, R. Mossa [14]).** Let (M, g) be a simply-connected homogeneous Kähler manifold with associated Kähler form \( \omega \) integral. Then there exists a positive real number \( \lambda \) such that (M, \lambda g) admits a Kähler immersion in (CP^m, g_{FS}).

Observe that if we drop the hypothesis for M to be simply connected, Theorem 4 still applies locally. For details and further results about Kähler immersions of Kähler manifolds into complex space forms we refer the reader to [15] and references therein.

### 3. Preliminaries on Cauchy–Riemann manifolds

For the arguments in this section we refer the reader to [11]. Let M be a smooth manifold of real dimension \( 2n + k \). A CR–structure (M, D, J) of type \( (n, k) \) on M consists of the data
of a smooth subbundle $D \subseteq TM$ of fiber dimension $2n$ and a fiber preserving smooth vector bundle isomorphism $J : D \to D$, satisfying $J^2 = -\text{Id}$ and the integrability condition on the Nijenhuis tensor:

$$N(X, Y) = [X, Y] - [JX, JY] - J([JX, Y] + [X, JY]) = 0, \quad \forall X, Y \in \Gamma(M, D).$$

The numbers $n$ and $k$ are called respectively the CR–dimension and CR–codimension of $M$. When $k = 0$ $M$ is holomorphic, while if $n = 0$ it is called totally real. Prime examples of CR–structures of CR–dimension $n$ are provided by the real submanifolds $M$ of a complex manifold $M$ for which $\dim_C(TM_x \otimes \mathbb{C} \cap T^{1,0}_x M) = n \ \forall x \in M$. In this case $D := \text{Re}(TM \otimes \mathbb{C} \cap T^{0,1} \tilde{M}) = JT M \cap TM$ is a smooth real subbundle (of real fiber dimension $2n$) of $TM$.

The real vector valued Levi form is the real bilinear form:

$$(3.1) \quad L^R : D \times D \to TM/\Sigma D, \quad L^R(X, Y) := \pi^R([JX, Y] - [X, JY])$$

where $\pi^R$ is the canonical projection $TM \to TM/\Sigma D$. Observe that (3.1) measures whether the subbundles $D$ of $TM$ is integrable and how the complex structure $J$ interplays with the integrability on the respective fibers. Define the Levi Kernel at $x \in M$ as the null space of the Levi form:

$$(3.2) \quad \text{Null}(L^R) = \{X \in \Gamma(M, D) | L^R(X, Y) = 0, \forall Y \in \Gamma(M, D)\}.$$

We recall here the definition of Levi degenerate CR–manifold, recent results of a more general notion can be found e.g. in [17].

**Definition 1.** A CR–manifold $(M, D, J)$ of type $(n, k)$ is Levi degenerate in $x \in M$ if $\dim \text{Null}_x(L^R) \neq 0$. Levi degenerate CR–manifolds with $\text{Null}(L^R) = D$ are called Levi flat.

On any Levi flat CR manifold $M$ there is a unique foliation $F$ by complex manifolds such that $TF$ is the Levi distribution $\text{Null}_x(L^R)$ of $M$, see [1]. This can be extended to the case when $(M, J, D)$ is Levi degenerate with $\dim \text{Null}_x(L^R) = 2n' = \text{cost} (2n' \leq 2n)$. Since $\text{Null}_x(L^R)$ is involutive and $J$ invariant, one can applies both Frobenius and Newlander-Niremberg to obtain a foliation $F$ of $M$ by complex $n'$–dimensional manifolds, i.e. $M$ can be realized as a disjoint union $M = \bigcup_\alpha F_\alpha$ by complex $n'$–dimensional manifolds $F_\alpha$ with $TF_\alpha \simeq \text{Null}_x(L^R)$. This is essentially the following:

**Theorem 5** (M. Freeman [12]). Let $D$ and $\text{Null}(L^R)$ have constant dimension on $M$. Then for each $x \in M$ there exist a neighborhood $U$ of $x$ and a unique smooth foliation of $M \cap U$ by complex submanifolds, such that the tangent space to the leaf through $x$ is $\text{Null}_x(L^R)$.

Such foliation is called the Levi foliation of $M$. The following example shows a Levi flat CR–manifold admitting a Levi foliation.

**Example 1.** (E. Cartan, [6]) Let $M$ be a Levi-flat real-analytic smooth hypersurface in $\mathbb{C}^m$. It was shown by E. Cartan that near each point $x \in M$, there exist local holomorphic
coordinates \((z, w) \in \mathbb{C}^{m-1} \times \mathbb{C}\) vanishing at \(x\), such that \(M\) near \(x\) is described by \(\text{Im} w = 0\).

The distribution \(\mathcal{D}\) can be given as the tangent space to the leaves of the Levi foliation \(\{(z, w) : w = t | t \in \mathbb{R}\}\).

We are interested in CR–structures induced by a Riemannian immersion into a Kähler manifold. Let \(\tilde{M}\) be a \(m\)-dimensional Kähler manifold with complex structure \(J\) and Kähler metric \(g\). Following \([7]\), an isometric Riemannian submanifold \(M\) of \(\tilde{M}\) is a CR–submanifold if there exists a differentiable distribution \(\mathcal{D} : M \rightarrow T\tilde{M}\) of constant dimension such that:

1. \(\mathcal{D}\) is holomorphic, i.e. \(JD_x = \mathcal{D}_x\) for each \(x \in M\);

2. the complementary orthogonal distribution \(\mathcal{D}^\perp : x \mapsto \mathcal{D}^\perp_x \subset T_x M\) is totally real, i.e. \(JD_x^\perp \subset T_x^\perp M\), where \(T_x^\perp M\) is the normal space of \(M\) in \(\tilde{M}\) at \(x\).

Observe that the triple \((M, \mathcal{D}, J)\) is a CR–manifold in the sense above.

Any isometric Riemannian submanifold of a Kähler manifold carries a natural structure of CR–submanifold as follows. Define \(\mathcal{D} = JTM \cap TM\) and let \(\mathcal{D}^\perp\) be the orthogonal complement of \(\mathcal{D}\) in TM with respect to \(g\), i.e. \(T_x M = \mathcal{D}_x \oplus \mathcal{D}^\perp_x\). Then, \(\mathcal{D}\) is holomorphic by definition, and \(JD^\perp_x \subset T^\perp_x M\) for \(g(J\mathcal{D}^\perp_x, T_x M) = g(J(JT_x M) \cap T_x M, T_x M) = 0\).

Finally, we recall the definition of CR–product (see \([7, 8]\) for details and further results).

**Definition 2.** A CR–submanifold of a Kähler manifold is called a CR–product if it is locally a Riemann product of a holomorphic submanifold and a totally real one.

**Remark 1.** Observe that our choice of \(\mathcal{D} = \langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m} \rangle \subset\) in Example \([7]\) gives \(M\) a structure of CR–product with the complex structure and the flat metric induced by the canonical ones in \(\mathbb{C}^m\), where \(\mathcal{D}^\perp = \langle \frac{\partial}{\partial \bar{z}} \rangle \subset\).

To conclude this section we present an example of two complex manifolds which are CR–relatives but not relatives.

**Example 2.** (The 3-sphere in \(\mathbb{C}^2\) and in \(\mathbb{CP}^2\)) We describe here how the natural inclusions of the 3-sphere \(S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}\) in \(\mathbb{C}^2\) and \(\mathbb{CP}^2\) induce the same structure of CR–submanifold. Let us start with \(\mathbb{C}^2\). Consider the usual identification \(\mathbb{C}^2 \simeq \mathbb{R}^4\), \((z_1, z_2) \mapsto (x_1, y_1, x_2, y_2)\), where \(z_1 = x_1 + iy_1\), \(z_2 = x_2 + iy_2\). Let \(p = (z_1, z_2) = (x_1, y_1, x_2, y_2) \in S^3\). Denote by \(X^\perp\) the normal vector, i.e.:

\[
X^\perp := x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}.
\]

Then a basis for \(T_p S^3\) is given by \(\{V_1, V_2, V_3\}\) with:

\[
V_1 = y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_2},
V_2 = x_2 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2},
V_3 = -y_2 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2}.
\]
The complex structure \( J \) acts on the \( V_j \)'s by \( J V_1 = X^\perp, J V_2 = -V_3, J V_3 = V_2 \), thus \( \mathcal{D} := TS^3 \cap J(TS^3) \) is spanned by \( V_2 \) and \( V_3 \). Then \( (S^3, \mathcal{D}, J) \) has a natural structure of CR–submanifold of \( \mathbb{C}^2 \).

Consider now homogeneous coordinates \([Z_0 : Z_1 : Z_2] \) on \( \mathbb{C}P^2 \) and define affine coordinates \( z_1 = \frac{Z_1}{Z_0}, z_2 = \frac{Z_2}{Z_0} \) on the open set \( U_0 = \{ Z_0 \neq 0 \} \). The 3-sphere \( S^3 \subseteq U_0 \) is realized as CR–submanifold of \( \mathbb{C}P^2 \) in the following way. The Fubini–Study metric \( g_{FS} \) is described on \( U_0 \) by the Kähler potential \( \Phi(x) = \log \left(1 + |z_1|^2 + |z_2|^2 \right) \) (see (2.1) above). Let \( X^\perp \) be as in (3.3), then it is not hard to see that \( g_{FS}(X^\perp, \cdot) = g_0(X^\perp, \cdot) \), where \( g_0 \) is the euclidean metric on \( U_0 \). Thus, a basis for \( T_pS^3 \) is given again by \( \{ V_1, V_2, V_3 \} \) as in the \( \mathbb{C}^2 \) case, and \( \mathcal{D} \) is spanned by \( V_2 \) and \( V_3 \). We conclude that the induced structure of CR–submanifold is the same on \( S^3 \) and it can be seen that it is Levi non degenerate.

4. Proof of Theorem 1 and Corollaries

**Proof of Theorem 1** Let \((\tilde{M}_1, J_1, g_1), (\tilde{M}_2, J_2, g_2)\) be two Kähler manifolds which are not relatives and let \( M \) be a common CR–submanifold. Denote by \( f_1 : M \to \tilde{M}_1, f_2 : M \to \tilde{M}_2 \) the isometric immersions and by \((\mathcal{D}, J, g)\) the holomorphic distribution, the almost complex structure and the Riemannian metric on \( M \) induced by both \( \tilde{M}_1 \) and \( \tilde{M}_2 \), i.e. \( J_j := f_j^*J \) and \( f_j^*g_j = g \), for \( j = 1, 2 \). Observe that \( \dim \mathcal{D}^\perp_x \neq 0 \), otherwise \( M \) would be a common Kähler submanifold of \( \tilde{M}_1 \) and \( \tilde{M}_2 \). By Th. 5 if \( M \) is degenerate with constant Levi kernel, then it carries a holomorphic foliation \( \mathcal{F} \) with leaves \( M_c \) such that \((M_c, J|_{M_c})\) is a complex submanifold of \( M \). It remains to shows that the metric induced by the inclusions of \( M_c \subseteq M \) in \( \tilde{M}_1 \) and \( \tilde{M}_2 \) induce the same Kähler structure on \( M_c \). The map \( i_j = f_j \circ i_c : M_c \to \tilde{M}_j \), for \( j = 1, 2 \), where \( i_c \) is the inclusion \( i_c : M_c \to M \), is holomorphic, for

\[
i_j^*J|_{M_c}(X) = f_j^*i_c^*J|_{M_c}(X) = f_j^*J(X) = J_j(X) = J_j(i_j^*X)
\]

for any \( X \in \mathcal{N}ull_x(\mathcal{L}^R) \). Since \( i_1^*g_1 = i_2^*g_2 = i_2^*g_2 \) on \( M_c \), we need only to show that \((M_c, J|_{M_c}, i_c^*g)\) is Kähler. Let us first show that it is hermitian. Since \( i_c^*g(J|_{M_c}X, J|_{M_c}Y) = g(i_c^*J|_{M_c}X, i_c^*J|_{M_c}Y) = g(JX, JY) = g(X, Y) \) for any \( X, Y \in \mathcal{N}ull_x(\mathcal{L}^R) \), where we used that \( \mathcal{N}ull_x(\mathcal{L}^R) = J|_{M_c} \) invariant. Conclusion follows by finally observing that \((1,1)\)-form \( \omega_c(X, Y) := i_j^*g(J|_{M_c}X, Y) \) is the pull-back through the holomorphic map \( i_j \) of the Kähler form \( \omega_j \) on \( \tilde{M}_j \) and thus it is closed.

**Proof of Corollary 1** A CR–product is Levi flat, thus the integrable manifold of its distribution \( \mathcal{D} \) is a holomorphic submanifold. Further, \( \mathcal{D} = \mathcal{N}ull(\mathcal{L}^R) \) has constant dimension. Conclusion follows by Theorem 1.

**Proof of Corollary 2** Let \((M, g)\) be a flag manifold and let \( \omega \) be its Kähler form. Since a flag manifold is a homogeneous variety, \( \omega \) is integral up to rescaling. If \( M \) is not simply connected, Theorem 1 applies locally, proving that there exists \( \lambda > 0 \) such that \((M, \lambda g)\) admits a local
Kähler immersion in $\mathbb{CP}^m$. We claim that $\frac{1}{\sqrt{\lambda}} f$ is a local Kähler immersion of $(M, g)$ in $\mathbb{CP}^m$. If the claim holds true, since by Theorem 3 combined with Calabi Rigidity Theorem 2 $\mathbb{CP}^m$ and $\mathbb{CP}^m'$ are not relatives for any $m, m', \lambda > 0$, conclusion follows by Theorem 1. In order to prove the claim observe that locally if we denote by $\varphi$ a Kähler potential for $g$, since $f^* \omega_{FS} = \omega$, by (2.1) we have:

$$f^* \Phi_1 = \log \left(1 + \sum_{j=1}^{m} |f_j|^2\right) = \lambda (\varphi + h + \bar{h}),$$

for some holomorphic function $h$ on $U$, and conclusion follows by:

$$\left(\frac{1}{\sqrt{\lambda}} f\right)^* \Phi_\lambda = \frac{1}{\lambda} \log \left(1 + \sum_{j=1}^{m} |f_j|^2\right) = \varphi + h + \bar{h}.\]$$

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