A Shimorin-type analytic model for left-invertible operators on an annulus and applications

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Abstract. A new analytic model for left-invertible operators is introduced and investigated. We show that left-invertible operator \( T \), which satisfies certain conditions can be modelled as a multiplication operator \( M_z \) on a reproducing kernel Hilbert space of vector-valued analytic functions on an annulus or a disc. Similar result for composition operators in \( \ell^2 \)-spaces is established.

1. Introduction

The classical Wold decomposition theorem (see \cite{42}) states that if \( U \) is isometry on Hilbert space \( \mathcal{H} \) then \( \mathcal{H} \) is the direct sum of two subspaces reducing \( U \), \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_p \) such that \( U|_{\mathcal{H}_u} \in \mathcal{B}(\mathcal{H}_u) \) is unitary and \( U|_{\mathcal{H}_p} \in \mathcal{B}(\mathcal{H}_p) \) is unitarily equivalent to a unilateral shift. This decomposition is unique and the canonical subspaces are defined by

\[
\mathcal{H}_u = \bigcap_{n=1}^{\infty} U^n \mathcal{H} \quad \text{and} \quad \mathcal{H}_p = \bigoplus_{n=1}^{\infty} U^n E,
\]

where \( E = \mathcal{N}(U^*) = \mathcal{H} \ominus U \mathcal{H} \). The Wold decomposition theorem and results analogous to this theorem plays an important role in many areas of operator theory, including the invariant subspace problem for Hilbert spaces of holomorphic functions. The interested reader is referred to \cite{35, 30, 40, 23, 24, 26}

In \cite{40} S. Shimorin obtain a weak analog of the Wold decomposition theorem, representing operator close to isometry in some sense as a direct sum of a unitary operator and a shift operator acting in some reproducing kernel Hilbert space of vector-valued holomorphic functions defined on a disc. The construction of the Shimorin’s model for a left-invertible analytic operator \( T \in \mathcal{B}(\mathcal{H}) \) is as follows. Let \( E := \mathcal{N}(T^*) \) and define a vector-valued holomorphic functions \( U_x \) as

\[
U_x(z) = \sum_{n=0}^{\infty} (P_E T^* T^* x) z^n, \quad z \in \mathbb{D}(T'^{-1}),
\]

where \( T' \) is the Cauchy dual of \( T \). Then we equip the obtained space of analytic functions \( \mathcal{H} := \{ U_x : x \in \mathcal{H} \} \) with the norm induced by \( \mathcal{H} \). The operator \( U : \mathcal{H} \ni x \to U_x \in \mathcal{H} \) becomes a unitary operator. Moreover, Shimorin proved that \( \mathcal{H} \)

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is a reproducing kernel Hilbert space and the operator $T$ is unitary equivalent to the operator $M_z$ of multiplication by $z$ on $H$ and $T^*$ is unitary equivalent to the operator $L$ given by the
\[(L f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in H.
\]
Following [40], the reproducing kernel for $H$ is an $B(E)$-valued function of two variables $\kappa_H : \Omega \times \Omega \to B(E)$ that
\[
\text{(i) for any } e \in E \text{ and } \lambda \in \Omega \\
\kappa_H(\cdot, \lambda)e \in H
\]
\[
\text{(ii) for any } e \in E, f \in H \text{ and } \lambda \in \Omega \\
(f(\lambda), e)_E = (f, \kappa_H(\cdot, \lambda)e)_H
\]

The class of weighted shifts on a directed tree was introduced in [17] and intensively studied since then [18, 16, 15, 8, 10, 12, 27]. The class is a source of interesting examples (see e.g., [9, 18, 31, 41, 13, 19, 22, 20]). In [15] S. Chavan and S. Trivedi showed that a weighted shift $S_\lambda$ on a rooted directed tree with finite branching index is analytic therefore can be modelled as a multiplication operator $M_z$ on a reproducing kernel Hilbert space $H$ of $E$-valued holomorphic functions on a disc centered at the origin, where $E := N(S_\lambda^*)$. Moreover, they proved that the reproducing kernel associated with $H$ is multi-diagonal.

2. Preliminaries

In this paper, we use the following notation. The fields of rational, real and complex numbers are denoted by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbols $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$ and $\mathbb{R}_+$ stand for the sets of integers, positive integers, nonnegative integers, and nonnegative real numbers, respectively. Set $D(r) = \{z \in \mathbb{C} : |z| \leq r\}$ and $\mathcal{A}(r^-, r^+) = \{z \in \mathbb{C} : r^- \leq |z| \leq r^+\}$ for $r, r^-, r^+ \in \mathbb{R}_+$. The expression "a countable set" means a finite set or a countably infinite set.

All Hilbert spaces considered in this paper are assumed to be complex. Let $T$ be a linear operator in a complex Hilbert space $H$. Denote by $T^*$ the adjoint of $T$. We write $B(H)$ for the $C^*$-algebra of all bounded operators and the cone of all positive operators in $H$, respectively. The spectrum and spectral radius of $T \in B(H)$ is denoted by $\sigma(T)$ and $r(T)$ respectively. Let $T \in B(H)$. We say that $T$ is left-invertible if there exists $S \in B(H)$ such that $ST = I$. The Cauchy dual operator $T'$ of a left-invertible operator $T \in B(H)$ is defined by
\[T' = T(T^*T)^{-1}\]

The notion of the Cauchy dual operator has been introduced and studied by Shimorin in the context of the wandering subspace problem for Bergman-type operators [40]. We call $T$ analytic if $H_\infty := \bigcap_{n=1}^{\infty} T^n = \{0\}$.

Let $X$ be a set and $\phi : X \to X$. If $n \in \mathbb{N}$ then the $n$-th iterate of $\phi$ is given by $\phi^{(n)} = \phi \circ \phi \circ \cdots \circ \phi$, $\phi$ composed with itself $n$-times. For $x \in X$ the set
\[[x]_\phi = \{y \in X : \text{there exist } i, j \in \mathbb{N} \text{ such that } \phi^{(i)}(x) = \phi^{(j)}(y)\}\]
is called the orbit of $f$ containing $x$. If $x \in X$ and $\phi^{(i)}(x) = x$ for some $i \in \mathbb{Z}_+$ then the cycle of $\phi$ containing $x$ is the set
\[\mathcal{C}_\phi = \{\phi^{(i)}(x) : i \in \mathbb{N}\}\]
Define the function \( \phi : X \to \mathbb{N} \) by

\begin{enumerate}
\item \( \phi(x) = 0 \) if \( x \) is in the cycle of \( \phi \)
\item \( \phi(x^*) = 0 \), where \( x^* \) is a fixed element of \( X \) not containing a cycle,
\item \( \phi(\phi(x)) = [\phi](x) + 1 \) if \( x \) is not in a cycle of \( \phi \).
\end{enumerate}

We set

\[ \text{Gen}_\phi (m, n) := \{ x \in X : m \leq [\phi](x) \leq n \} \]

for \( m, n \in \mathbb{N} \).

Let \( (X, \mathcal{A}, \mu) \) be a \( \mu \)-finite measure space, \( \phi : X \to X \) and \( w : X \to \mathbb{C} \) be measurable transformations. By a weighted composition operator \( C_{\phi,w} \) in \( L^2(\mu) \) we mean a mapping

\begin{equation}
(2.1) \quad D(C_{\phi,w}) = \{ f \in L^2(\mu) : w(f \circ \phi) \in L^2(\mu) \}, \quad C_{\phi,w} f = w(f \circ \phi), \quad f \in D(C_{\phi,w}).
\end{equation}

Let us recall some useful properties of composition operator we need in this paper:

**Lemma 2.1.** Let \( X \) be a countable set, \( \phi : X \to X \) and \( w : X \to \mathbb{C} \) be measurable transformations. If \( C_{\phi,w} \in \mathbf{B}(l^2(X)) \) then for any \( x \in X \) and \( n \in \mathbb{N} \)

\begin{enumerate}
\item \( C_{\phi,w}^* e_x = \frac{w(x)}{w(\phi(x))} e_{\phi(x)} \)
\item \( C_{\phi,w}^n e_x = \frac{w(x) \cdots w(\phi^{n-1}(x))}{w(\phi^n(x))} e_{\phi^n(x)} \)
\item \( C_{\phi,w}^* e_x = \sum_{y \in \phi^{-1}(x)} \frac{w(y) \cdots w(\phi^{n-1}(y))}{w(\phi^n(y))} e_y \)
\item \( C_{\phi,w}^* C_{\phi,w} e_x = \left( \sum_{y \in \phi^{-1}(x)} |w(y)|^2 \right) e_x \)
\end{enumerate}

We now describe Cauchy dual of weighted composition operator

**Lemma 2.2.** Let \( X \) be a countable set, \( \phi : X \to X \) and \( w : X \to \mathbb{C} \) be measurable transformations. If \( C_{\phi,w} \in \mathbf{B}(l^2(X)) \) is left-invertible operator then the Cauchy dual \( C'_{\phi,w} \) of \( C_{\phi,w} \) is also a weighted composition operator and is given by:

\[ C'_{\phi,w} e_x = \sum_{y \in \phi^{-1}(x)} \frac{w(y)}{\sum_{z \in \phi^{-1}(y)} |w(z)|^2} e_y \]

Let \( \mathcal{T} = (V; E) \) be a directed tree \((V, E)\) are the sets of vertices and edges of \( \mathcal{T} \), respectively. For any vertex \( u \in V \) we put \( \text{Chi}(u) = \{ v \in V : (u, v) \in E \} \). Denote by \( \text{par} \) the partial function from \( V \) to \( V \) which assigns to a vertex \( u \) a unique \( v \in V \) such that \((v,u) \in E\). A vertex \( u \in V \) is called a root of \( \mathcal{T} \) if \( u \) has no parent. If \( \mathcal{T} \) has a root, we denote it by root. Put \( V^\circ = V \setminus \{ \text{root} \} \) if \( \mathcal{T} \) has a root and \( V^\circ = V \) otherwise. The Hilbert space of square summable complex functions on \( V \) equipped with the standard inner product is denoted by \( l^2(V) \). For \( u \in V \), we define \( c_u \in l^2(V) \) to be the characteristic function of the set \( \{ u \} \). It turns out that the set \( \{ c_v \}_{v \in V} \) is an orthonormal basis of \( l^2(V) \). We put \( V_\varsigma := \{ v \in V : \text{card}(\text{Chi}(v)) \geq 2 \} \) and call the a member of this set a branching vertex of \( \mathcal{T} \).

Given a system \( \lambda = \{ \lambda_v \}_{v \in V_\varsigma} \) of complex numbers, we define the operator \( S_\lambda \) in \( l^2(V) \), which is called a weighted shift on \( \mathcal{T} \) with weights \( \lambda \), as follows

\[ D(S_\lambda) = \{ f \in l^2(V) : A_\mathcal{T} f \in l^2(V) \} \quad \text{and} \quad S_\lambda f = A_\mathcal{T} f \quad \text{for} \quad f \in D(S_\lambda), \]

where

\[ (A_\mathcal{T} f)(v) = \begin{cases} \lambda_v f(\text{par}(v)) & \text{if} \ v \in V^\circ, \\ 0 & \text{otherwise}. \end{cases} \]
Lemma 2.3. If $S_\lambda$ is a densely defined weighted shift on a directed tree $T$ with weights $\lambda = \{\lambda_v\}_{v \in V^0}$, then

$$\mathcal{N}(S_\lambda^*) = \begin{cases} \langle e_{\text{root}} \rangle \oplus \bigoplus_{u \in V_\lambda} (\ell^2(\text{Chi}(u)) \oplus \langle \lambda^u \rangle) & \text{if } T \text{ has a root,} \\ \bigoplus_{u \in V_\lambda} (\ell^2(\text{Chi}(u)) \oplus \langle \lambda^u \rangle) & \text{otherwise,} \end{cases}$$

where $\lambda^u \in \ell^2(\text{Chi}(u))$ is given by $\lambda^u : \ell^2(\text{Chi}(u)) \ni v \mapsto \lambda_v \in \mathbb{C}$.

A subgraph of a directed tree $T$ which itself is a directed tree will be called a subtree of $T$. We refer the reader to [17] for more details on weighted shifts on directed trees.

3. Analytic model

Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator and $E$ be a subspace of $\mathcal{H}$ denote by $[E]_{T, T^*}$ the direct sum of the smallest $T^*$-invariant subspace containing $E$ and the smallest $T^*$-invariant subspace containing $E$:

$$[E]_{T, T^*} := \bigvee \{T^n x : x \in E, n \in \mathbb{N}\} \oplus \bigvee \{T'^n x : x \in E, n \in \mathbb{N}\},$$

where $T'$ is the Cauchy dual of $T$.

To avoid the repetition, we state the following assumption which will be used frequently in this section.

(LI) The operator $T \in \mathcal{B}(\mathcal{H})$ is left-invertible and $E$ is a subspace of $\mathcal{H}$ such that $[E]_{T, T^*} = \mathcal{H}$.

Suppose (LI) holds. In this case we may construct a Hilbert $\mathcal{K}$ associated with $T$, of formal Laurent series with vector coefficients. We proceed as follows. For each $x \in \mathcal{H}$, define a formal Laurent series $U_x$ with vector coefficients as

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^n x) z^n.$$

Let $\mathcal{H}$ denote the vector space of formal Laurent series with vector coefficients of the form $U_x$, $x \in \mathcal{H}$. Consider the map $U : \mathcal{H} \to \mathcal{K}$ defined by $U_x = U_x$. As shown in Lemma 3.1 below, by the assumption $U$ is injective. In particular, we may equip the space $\mathcal{H}$ with the norm induced from $\mathcal{H}$, so that $U$ is unitary.

Lemma 3.1. Suppose $\mathcal{H}$, $\mathcal{K}$, $E$ and $U$ are as above. Then $\mathcal{N}(U) = \{0\}$.

Proof. Suppose that $x \in \mathcal{H}$ is such that

$$P_E T^n x = 0 \quad \text{and} \quad P_E T'^n x = 0, \quad n \in \mathbb{N}.$$ 

Then for every $y \in E$

$$\langle T^n x, y \rangle = 0 \quad \text{and} \quad \langle T'^n x, y \rangle = 0, \quad n \in \mathbb{N}.$$ 

$$\langle x, T^* y \rangle = 0 \quad \text{and} \quad \langle x, T'^* y \rangle = 0, \quad n \in \mathbb{N}.$$ 

We see that the above condition is equivalent to the following one

$$x \perp [E]_{T, T^*} (= \mathcal{H}).$$

This completes the proof. \qed
We now prove that the operator \( T \) is unitary equivalent to the operator \( \mathcal{M}_z : \mathcal{H} \to \mathcal{H} \) of multiplication by \( z \) on \( \mathcal{H} \) given by
\[
(\mathcal{M}_z f)(z) = zf(z), \quad f \in \mathcal{H}
\]
and operator \( T^* \) is unitary equivalent to the operator \( \mathcal{L} : \mathcal{H} \to \mathcal{H} \) given by
\[
(\mathcal{L} f)(z) = f(z) - (P_N(\mathcal{M}_z^*) f)(z), \quad f \in \mathcal{H}
\]

**Theorem 3.2.** Suppose (LI) holds. Then the following assertions are valid:

(i) \( UT = \mathcal{M}_z U \),

(ii) \( UT^* = \mathcal{L} U \).

**Proof.** Since
\[
(UT^* x)(z) = \sum_{n=1}^{\infty} (P_E T^n T^* x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T^{n+1} x) z^n
\]

\[
= \sum_{n=1}^{\infty} (P_E T^{n-1} (I - P_{N(T^*)}) x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T^{n+1} x) z^n
\]

\[
= \frac{(Ux)(z) - (UP_{N(T^*)}x)(z)}{z}
\]

the proof is complete. \( \square \)

We now show that in the case of a left-invertible and analytic operators our analytic model with \( E = N(T^*) \) coincides with the Shimorin analytic model.

**Theorem 3.3.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be left-invertible and analytic, \( \mathcal{H}_1, U_1 \) be the Hilbert space and the unitary map constructed in (3.1) with \( E := N(T^*) \) and \( \mathcal{H}_2, U_2 \) be the Hilbert space and the unitary map obtained in Shimorin construction. Then \( \mathcal{H}_1 = \mathcal{H}_2 \) and \( U_1 = U_2 \).

**Proof.** By [40, Proposition 2.7], \( \mathcal{H}_1 = [E]_{T^*} \). Since \( T \) is analytic \( \mathcal{H}_\infty = \{0\} \), we see that \( [E]_{T^*} = \mathcal{H} \). By kernel-range decomposition, \( P_{N(T^*)} T^n = 0 \) for \( n \in \mathbb{N} \). Hence, the first sum in (3.1) vanishes.

**Lemma 3.4.** Suppose (LI) holds. Then \( [T^n E]_{T^* T} = \mathcal{H} \) for every \( n \in \mathbb{N} \).

For left-invertible operator \( T \in \mathcal{B}(\mathcal{H}) \), among all subspaces satisfying condition (LI) we will distinguish those subspaces \( E \) which satisfy the following condition
\[
E \perp T^n E \quad \text{and} \quad E \perp T^n E, \quad n \in \mathbb{Z}_+
\]

**Theorem 3.5.** Suppose (LI) holds. Then the following assertions hold:

(i) If additionally (3.3) holds then \( \mathcal{E} = U(E) \) is a copy of \( E \) in \( \mathcal{H} \), the subspace consisting of constant functions;

(ii) \( (\mathcal{M}_z^* \mathcal{M}_z)^{-1} \mathcal{M}_z^* = \mathcal{L} \)

**Proof.** (i) This is obvious.

(ii) Fix any \( x \in \mathcal{H} \). Combining Theorem 3.2 and the kernel-range decomposition, we deduce that
\[
(\mathcal{L} \mathcal{M}_z Ux)(z) = \frac{z(Ux)(z) - (UP_{N(T^*)} T x)(z)}{z} = (Ux)(z),
\]
which means that \( L' \) is a left-inverse of \( M_z \). Since
\[
L' M_z = I \quad \text{and} \quad (M_z^* M_z)^{-1} M_z^* M_z = I,
\]
we see that \( L'|_{R(M_z)} = (M_z^* M_z)^{-1} M_z^* |_{R(M_z)} \). One can verify that \( L'|_{N(M_z)} = (M_z^* M_z)^{-1} M_z^* |_{N(M_z)} \), which completes the proof.

Now we shall discuss the extent to which our formal Laurent series actually represent analytic functions.

**Theorem 3.6.** Suppose (LI) holds. Let
\[
r^+ := \liminf_{n \to \infty} \| P_E T^{n+} \|^{-\frac{1}{n}} \\
r^- := \limsup_{n \to \infty} \| P_E T^n \|^{\frac{1}{n}}
\]
If \( r^+ > r^- \), then formal Laurent series (3.1) represent analytic functions on annulus \( A(r^-, r^+) \).

**Theorem 3.7.** Suppose (LI) holds and the series (3.1) is convergent absolutely in \( E \) on \( \Omega \subset \mathbb{C} \). For any \( z, \lambda \in \Omega \) and \( e_0, e_1 \in E \), we have
\[
\langle \kappa, \varphi(z, \lambda) e_0, e_1 \rangle_E = \langle \kappa, \varphi(\cdot, \lambda) e_0, \kappa, \varphi(\cdot, z) e_1 \rangle_E
\]
\[
= \left\langle \left[ \frac{T^*}{\lambda} \left( I - \frac{T^*}{\lambda} \right)^{1} + (I - \lambda T')^{-1} \right] e_0, \left[ \frac{T^*}{z} \left( I - \frac{T^*}{z} \right)^{-1} + (I - z T')^{-1} \right] e_1 \right\rangle.
\]

**Proof.** Let \( \lambda \in \Omega \) and \( e_0 \in E \). Then
\[
(f, e_0)_E = \sum_{n=1}^{\infty} \langle P_E T^n x, \frac{1}{\lambda^n} \rangle + \sum_{n=0}^{\infty} \langle P_E T^n x, \lambda^n e_0 \rangle
\]
\[
= \langle U^{-1} f, \sum_{n=1}^{\infty} P_E T^n x, \frac{1}{\lambda^n} \rangle + \sum_{n=0}^{\infty} \langle P_E T^n x, \lambda^n e_0 \rangle
\]
\[
= \langle U^{-1} f, \left[ \frac{T^*}{\lambda} \left( I - \frac{T^*}{\lambda} \right)^{-1} + (I - \lambda T')^{-1} \right] e_0 \rangle
\]
for any \( f \in \mathcal{H} \). As a consequence, we obtain
\[
\kappa(\cdot, \lambda) = U \left[ \frac{T^*}{\lambda} \left( I - \frac{T^*}{\lambda} \right)^{-1} + (I - \lambda T')^{-1} \right].
\]
This implies that
\[
\langle \kappa, \varphi(z, \lambda) e_0, e_1 \rangle_E = \langle \kappa, \varphi(\cdot, \lambda) e_0, \kappa, \varphi(\cdot, z) e_1 \rangle_E
\]
\[
= \left\langle \left[ \frac{T^*}{\lambda} \left( I - \frac{T^*}{\lambda} \right)^{1} + (I - \lambda T')^{-1} \right] e_0, \left[ \frac{T^*}{z} \left( I - \frac{T^*}{z} \right)^{-1} + (I - z T')^{-1} \right] e_1 \right\rangle,
\]
which completes the proof.

It follows from Theorem 3.7 that the reproducing kernel \( \kappa, \varphi \) has form
\[
\kappa(\varphi(z, \lambda) = P_E \left[ \frac{T^*}{z} \left( I - \frac{T^*}{z} \right)^{-1} + (I - z T')^{-1} \right] \left[ \frac{T^*}{\lambda} \left( I - \frac{T^*}{\lambda} \right)^{-1} + (I - \lambda T')^{-1} \right]|_E.
\]
This can be rewritten in the following form
\[
\kappa_{\mathcal{H}}(z, \lambda) = \sum_{i,j \geq 1} P_E T^{i} T^{*j} |E \frac{1}{z^i \lambda^j} + \sum_{i \geq 1, j \geq 0} P_E T^{i} T^{*j} |E \frac{1}{z^i \lambda^j} + \sum_{i,j \geq 1} P_E T^{*i} T^{j} |E z^i \lambda^j.
\]

Now, we turn to the properties of the Cauchy dual operator \(T'\). The Cauchy dual operator \(T'\) of left-invertible operator \(T\) is itself left-invertible. Assume now that there exist a subspace \(E \subset \mathcal{H}\) such that \([E]_{T,T'} = \mathcal{H}\) and \([E]_{T',T} = \mathcal{H}\) hold. Then for both operators \(T\) and \(T'\) we can construct a Hilbert spaces \(\mathcal{H}\) and \(\mathcal{H}'\) of \(E\)-valued Laurent series. Then
\[
U'_a(z) = \sum_{n=1}^{\infty} (P_E T^{in} x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E (T^{*n} x) z^n).
\]

**Theorem 3.8.** Let \(E \subset \mathcal{H}\) be such that \([E]_{T,T'} = \mathcal{H}\), \([E]_{T',T} = \mathcal{H}\) and (3.3) holds. Let \(f\) and \(g\) be \(E\)-valued series
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n.
\]
Then
\[
\langle U^{-1} f, U'^{-1} g \rangle = \sum_{n=0}^{\infty} \langle a_n, b_n \rangle.
\]

**Proof.** It suffices to consider the case \(f(z) = e_0 z^n\) and \(g(z) = e_1 z^m, e_0, e_1 \in E\). Observe that
\[
\langle U^{-1} f, U'^{-1} g \rangle = \begin{cases} \langle T^{n-m} e_0, e_1 \rangle & \text{if } n \geq m, \\ \langle T^{m-n} e_0, e_1 \rangle & \text{otherwise.} \end{cases}
\]
Since (3.3), we deduce that \(\langle U^{-1} f, U'^{-1} g \rangle = \delta_{m-n} \langle e_0, e_1 \rangle\). This finishes the proof. □

**Theorem 3.9.** Suppose (II) holds and the series (3.1) is convergent absolutely in \(E\) on \(\Omega \subset \mathbb{C}\), such that int \(\Omega \neq \emptyset\).

(i) the point spectrum of \(T\) is empty \(\sigma_p(T) = \emptyset\),

(ii) \(\Omega \subset \sigma_p(T')\)

**Proof.** Let \(f(z) = \sum_{n=\infty}^{\infty} a_n z^n \in \mathcal{H}\). Suppose, to derive a contradiction, that \(\lambda \in \sigma_p(T)\). Then by Theorem 3.2
\[
(z - \lambda) \sum_{n=\infty}^{\infty} \langle a_n, e \rangle z^n = 0, \quad z \in \Omega, e \in E,
\]
and thus
\[
\langle a_n, e \rangle = \lambda^{-n} \langle a_0, e \rangle, \quad n \in \mathbb{Z}.
\]
Clearly, \(\sum_{n=\infty}^{\infty} (\frac{1}{\lambda})^n\) is divergent, which contradicts our assumption.

By Theorem 3.7, we have
\[
\langle U_z, \mathcal{M}_z \kappa_{\mathcal{H}}(\cdot, \lambda) g \rangle = \langle \mathcal{M}_z U_z, \kappa_{\mathcal{H}}(\cdot, \lambda) g \rangle = \langle \lambda U_z(\lambda), g \rangle = (U_z(\lambda), \tilde{g}) = \langle U_z, \tilde{\lambda} \kappa_{\mathcal{H}}(\cdot, \lambda) g \rangle,
\]
for $x \in \mathcal{H}$, $\lambda \in \Omega$ and $g \in E$. This gives the equality

$$\mathcal{M}_z^* \kappa_{\scriptscriptstyle \mathcal{H}}(\cdot, \lambda)g = \bar{\lambda} \kappa_{\scriptscriptstyle \mathcal{H}}(\cdot, \lambda)g,$$

which means that $\bar{\lambda} \in \sigma_p(T^*)$. \hfill $\Box$

4. Weighted composition operators.

Let $\mathcal{T} = (V, E)$ be a rootless directed tree. Following [15], we say that $\mathcal{T}$ has finite branching index if there exist $m \in \mathbb{N}$ such that

$$\text{Chi}^k(V_\omega) \cap V_\omega = \emptyset, \quad k \geq m.$$

Lemma 4.1. ([15]) Let $\mathcal{T} = (V, E)$ be a rootless directed tree with finite branching index $m$. Then there exist a vertex $\omega \in V_\omega$ such that

$$\text{card}(\text{Chi}(\text{par}^{(k)}(\omega))) = 1, \quad n \in \mathbb{Z}_+.$$

Moreover, if $V_\omega$ is non-empty then there exists a unique $\omega \in V_\omega$ satisfying (4.1).

We extend the notion of finite branching index to transformations $\phi : X \to X$. We say that $\phi$ has finite branching index if

$$\sup \{ |\phi|(x) : \text{card}(\phi^{-1}(x)) \geq 2, \ x \in X \} < \infty.$$

Let $X$ be a countable set, $w : X \to \mathbb{C}$ be a complex function on $X$, $\phi : X \to X$ be a transformation of $X$ and $C_{\phi, w}$ be a weighted composition operator in $\ell^2(X)$. We will need only consider composition functions with one orbit, since an orbit induces a reducing subspace to which the restriction of the weighted composition operator is again a weighted composition operator.

Lemma 4.2. Let $X$ be a countable set, $w : X \to \mathbb{C}$ be a complex function on $X$ and $\phi : X \to X$ be a transformation of $X$, which has finite branching index. Let $C_{\phi, w}$ be a weighted composition operator in $\ell^2(X)$ and

$$E := \left\{ \begin{array}{ll} \bigoplus_{x \in \text{Gen}(1, 1)} \mathcal{N}(\{(C_{\phi, w}|_{\ell^2(\text{Des}(x))})^* \} & \text{when } \phi \text{ has a cycle,} \\
\langle \omega \rangle \oplus \mathcal{N}(S^*_\lambda) & \text{otherwise,} \end{array} \right.$$  

where $\text{Des}(x) := \bigcup_{n=0}^{\infty} \phi^{-1}(\omega)$. Then the subspace $E$ is such that $[E]_{C_{\phi, w}, C_{\phi', w}} = \mathcal{H}$ and (3.3) holds.

Proof. First, we consider the case when $\phi$ does not have a cycle. Then weighted composition operator on $L^2$ space with respect to the counting measure is a weighted shift on a rootless directed tree. We show that the subspace $E := \langle \omega \rangle \oplus \mathcal{N}(S^*_\lambda)$ satisfies (3.3) and $[E]_{S_\lambda, S_{\lambda'}} = \ell^2(X)$. The subtree $\mathcal{T}_{\text{Des}(\omega)}$ of $\mathcal{T}$ is a directed tree with root $\omega$ and by [15, Lemma 3.4] the Cauchy dual of the weighted shift $S_\lambda|_{\ell^2(\text{Des}(\omega))} \in \mathcal{B}(\mathcal{H})$ is analytic and $[E]_{S_\lambda|_{\ell^2(\text{Des}(\omega))}} = \ell^2(\text{Des}(\omega))$. Note that the space $\ell^2(\text{Des}(\omega))$ is invariant for $S_\lambda$ and

$$\left(S_\lambda|_{\ell^2(\text{Des}(\omega))}\right)' = S_\lambda|_{\ell^2(\text{Des}(\omega))}(S_\lambda|_{\ell^2(\text{Des}(\omega))})^*S_\lambda|_{\ell^2(\text{Des}(\omega))}^{-1} = S_\lambda^*(S_\lambda)^{-1}|_{\ell^2(\text{Des}(\omega))} = S_\lambda|_{\ell^2(\text{Des}(\omega))}.$$  

This implies that

$$\forall \{S^*_\lambda x : x \in E, n \in \mathbb{N}\} = [E]_{S_\lambda|_{\ell^2(\text{Des}(\omega))}} = \ell^2(\text{Des}(\omega)).$$

The assertion (i) of Lemma 2.1, shows that

$$\forall \{S^*_\lambda x : x \in E, n \in \mathbb{N}\} = \ell^2(X \setminus \text{Des}(\omega)).$$
which together with (4.3) give $[E]S_{\phi,w}(e) = f^2(X)$. According to Lemma 2.3, $E = \langle \omega \rangle \oplus \mathcal{N}(S^*_{\phi})$. If $e, f \in E$, then
\[ \langle C_{\phi,w}^{*n}e, f \rangle = \langle C_{\phi,w}^{*n}P_{(\omega)}e, f \rangle = 0, \]
where in the last step we used assertion (ii) of Lemma 2.1. Using
\[ \langle C_{\phi,w}^{*n}e, f \rangle = \langle C_{\phi,w}^{*n}P_{(\omega)}e, f \rangle, \]
and following steps analogous to those above, we obtain $\langle C_{\phi,w}^{*n}e, f \rangle = 0$. This immediately yields that condition (3.3) holds, which completes the proof of the case when $\phi$ does not have a cycle.

If $\phi$ has a cycle, then the operator
\[ C_{\phi,w}|_{\ell^2(\text{Des}(x))} \quad \text{for} \quad x \in \text{Gen}(1,1) \]
is a weighted shift on directed tree with root $x$. Using [15, Lemma 3.4] again and arguing as in the previous case we obtain
\[ (4.4) \quad \bigcap \{S^*_n x: x \in \mathcal{N}((C_{\phi,w}|_{\ell^2(\text{Des}(x))}^*)^n, n \in \mathbb{N} \} = \ell^2(\text{Des}(x)), \quad x \in \text{Gen}(1,1). \]
Applying the assertion (i) of Lemma 2.1 again, we see that
\[ \bigcap \{S^*_n x: x \in E, n \in \mathbb{N} \} = \ell^2(X \setminus \bigcup_{x \in \text{Gen}(1,1)} \text{Des}(x)). \]
This and (4.4), implies that $[E]_{C_{\phi,w},C_{\phi,w}^*} = \mathcal{H}$. The same reasoning as in the previous case shows that (3.3) holds. This completes the proof.

Suppose (LI) holds with $C_{\phi,w}$ in place of $T$. Define
\[ k_\phi(E) := \min\{n \in \mathbb{N}: E \subset \text{Gen}(1,n)(\phi)\} \]
and subsets of $X$ by
\[ W^E_0 := \begin{cases} \text{Gen}(1,k_\phi(E))(\phi) & \text{when} \ \phi \ \text{has a cycle} \\ \text{Gen}(1,k_\phi(E))(\phi) \cap \text{Des}(\omega) & \text{otherwise} \end{cases} \]
\[ W^E_n := \begin{cases} \phi(-n)(W^E_0) & n \in \mathbb{N} \ \text{when} \ \phi \ \text{has a cycle} \\ \phi(-n)(W^E_0) & n \in \mathbb{Z} \ \text{otherwise} \end{cases} \]
\[ r^+_{w,\phi} := \liminf_{n \to \infty} \left( \sum_{x \in W^E_n} |w'(x)w'(\phi(x)) \cdots w'(\phi^{(n-1)}(x))|^2 \right)^{\frac{1}{2}} \]
\[ r^-_{w,\phi} := \begin{cases} \prod_{x \in e_\phi} |w(x)| & \text{if} \ \phi \ \text{has a cycle} \\ \limsup_{n \to \infty} \sqrt[n]{|w(\phi^2(\omega))w(\phi^3(\omega)) \cdots w(\phi^n(\omega))|} & \text{otherwise} \end{cases} \]

**Theorem 4.3.** Let $X$ be a countable set, $w : X \to \mathbb{C}$ be a complex function on $X$ and $\phi : X \to X$ be a transformation of $X$, which has finite branching index. Let $C_{\phi,w}$ be a weighted composition operator in $\ell^2(X)$, which satisfy condition (LI) with $C_{\phi,w}$ in place of $T$, where $E$ is as in (4.2) and $C_{\phi,w}^*$ be the Cauchy dual of $C_{\phi,w}$. If $r^+_{w,\phi} > r^-_{w,\phi}$, then there exist a $z$-invariant reproducing kernel Hilbert space $\mathcal{H}$ of $E$-valued holomorphic functions defined on the annulus $h(r_{w,\phi},r^+_{w,\phi})$ and a unitary mapping $U : \ell^2(X) \to \mathcal{H}$ such that $\mathcal{M}_U = UC_{\phi,w}$, where $\mathcal{M}_z$ denotes the operator of multiplication by $z$ on $\mathcal{H}$. Moreover,
(i) The reproducing kernel $\kappa_{\mathcal{H}} : \mathcal{A}(r_{w,\phi}^+, r_{w,\phi}^-) \times \mathcal{A}(r_{w,\phi}^+, r_{w,\phi}^-) \to \mathbf{B}(E)$ associated with $\mathcal{H}$ satisfies $\kappa_{\mathcal{H}}(\cdot, w)g \in \mathcal{H}$ and $(Uf, \kappa_{\mathcal{H}}(\cdot, w)g) = \langle (Uf)(w), g \rangle$ for $f, g \in \ell^2(X)$.

(ii) $\kappa_{\mathcal{H}}(z, \lambda) = \sum_{i,j \geq 1} A_{i,j} \frac{1}{z^i \lambda^j} + \sum_{i \geq 1, j \geq 0} B_{i,j} \frac{1}{z^i \lambda^j} + \sum_{i \geq 0, j \geq 1} C_{i,j} z^i \frac{1}{\lambda^j} + \sum_{i,j \geq 0} D_{i,j} z^i \lambda^j,$

where $A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j} \in \mathbf{B}(E)$ and when $\phi$ has no cycle then

$A_{i,j} = 0$ if $|i - j| > k_{\phi(E)}$

$B_{i,j} = 0$ if $i + j > k_{\phi(E)}$

$C_{i,j} = 0$ if $i + j > k_{\phi(E)}$

$D_{i,j} = 0$ if $|i - j| > k_{\phi(E)}$

Proof. We begin by showing that the $E$-valued series

$$\sum_{n=0}^{\infty} P_E C_{\phi, w}^n f z^n$$

converges absolutely in $E$ on the disc $\mathbb{D}(r_{w,\phi}^+)$. Let $f = \sum_{x \in X} f(x) e_x$. Applying Lemma 2.1, we obtain

$$P_E C_{\phi, w}^n f = \sum_{x \in X} f(x) w'(x) w'(\phi(x)) \cdots w'(\phi^{(n-1)}(x)) P_E e_{\phi^{(n)}(x)}$$

By the Cauchy-Schwarz inequality, we have

$$\| \sum_{n=0}^{k} P_E C_{\phi, w}^n f z^n \| \leq \sum_{x \in W_{n,\phi}^E} \| f(x) w'(x) w'(\phi(x)) \cdots w'(\phi^{(n-1)}(x)) z^n \|$$

$$\leq \left( \sum_{n \geq 0} \| f(x) \|^2 \right)^{\frac{1}{2}} \left( \sum_{n \geq 0} \| w'(x) w'(\phi(x)) \cdots w'(\phi^{(n-1)}(x)) z^n \|^2 \right)^{\frac{1}{2}}$$

$$\leq k_{\phi(E)} + 1 \| f \| \left( \sum_{n \geq 0} \| w'(x) w'(\phi(x)) \cdots w'(\phi^{(n-1)}(x)) z^n \|^2 \right)^{\frac{1}{2}}$$

Now we show that the $E$-valued series

$$\sum_{n=0}^{\infty} P_E C_{\phi, w}^n f \frac{1}{z^n}$$
converges absolutely in $E$ on $\mathbb{C} \setminus \mathcal{D}(r_{w, \phi}^-)$. First, we consider the case when $\phi$ does not have a cycle. For this, note that using Lemma 2.1 again,

$$P_E C_{\phi, w}^n f = \sum_{x \in X} f(x) \sum_{y \in \varphi^{-n}(x)} w(y) w(\varphi(y)) \cdots w(\varphi^{(n-1)}(y)) P_E e_y$$

$$= \sum_{x \in W_{n, \phi}} f(x) \sum_{y \in \varphi^{-n}(x)} w(y) w(\varphi(y)) \cdots w(\varphi^{(n-1)}(y)) P_E e_y$$

$$= \sum_{x \in W_{n, \phi}} f(x) w(\varphi^{-1}(x)) w(\varphi^{-2}(x)) \cdots w(\varphi^{k-n}(x)) P_E C_{\phi, w}^n e_{\varphi^{k-n}(x)}$$

for $n \geq k_{\phi(E)}$. Put $M := \max \{ \|C_{\phi, w}\|, \|C_{\phi, w}\|^{k_{\phi(E)}} \}$. Hence, by the Cauchy-Schwarz inequality again, we have

$$\left\| \sum_{n=0}^{k} P_E C_{\phi, w}^n f \right\| \leq M \sum_{n=0}^{k} \left\| f(x) w(\varphi^{-1}(x)) w(\varphi^{-2}(x)) \cdots w(\varphi^{k-n}(x)) \right\| \frac{1}{2^n}$$

$$\leq M \left( \sum_{n=0}^{k} \left\| f(x) \right\| \right)^{2/3} \left( \sum_{n=0}^{k} \left\| w(\varphi^{-1}(x)) w(\varphi^{-2}(x)) \cdots w(\varphi^{k-n}(x)) \right\| \frac{1}{2^n} \right)^{2/3}$$

$$\leq M \left[ k_{\phi(E)} + 1 \right] \left\| f \right\| \left( \sum_{n=0}^{k} \left\| w(\varphi^{-1}(x)) w(\varphi^{-2}(x)) \cdots w(\varphi^{k-n}(x)) \right\| \frac{1}{2^n} \right)^{2/3}$$

We now consider the other case when $\phi$ has a cycle. It is easily seen that

$$\sum_{n=0}^{\infty} P_E C_{\phi, w}^n f z^n = \sum_{n=0}^{\infty} P_E C_{\phi, w}^n P_\mathcal{H}_\phi f z^n + \sum_{n=0}^{\infty} P_E C_{\phi, w}^n P_\mathcal{H}_\phi f z^n,$$

where $\mathcal{H}_\phi := \operatorname{lin} \{ e_x : x \in \mathcal{E}_\phi \}$. If $h \in \mathcal{H}_\phi^+ \cap \operatorname{lin} \{ e_x : x \in \text{Gen}^\phi (m, n) \}$ for $m, n \in \mathbb{N}$, then $C_{\phi, w} h \in \operatorname{lin} \{ e_x : x \in \text{Gen}^\phi (m+1, n+1) \}$, which yields

$$\left\| \sum_{n=0}^{\infty} P_E C_{\phi, w}^n P_\mathcal{H}_\phi f z^n \right\| = \left\| \sum_{n=0}^{\infty} P_E C_{\phi, w}^n P_\mathcal{H}_\phi f z^n \right\| \leq \sum_{n=0}^{k_{\phi(E)}} \| C_{\phi, w} \| z^n.$$

Let us now observe that

$$P_{W_i} C_{\phi, w}^n e_x = c P_{W_i} C_{\phi, w}^n e_x, \quad x \in \mathcal{E}_\phi,$$
where \( c = \prod_{x \in \mathcal{C}} w(x) \), \( \tau = \text{card} \mathcal{C} \) and \( W_i = \text{lin} \{ e_x : x \in \text{Gen}^\phi(i, i + 1) \} \). We now apply this observation to estimate the following sum.

\[
\sum_{n=0}^{\infty} P_E C_{\phi,w}^n P_{H_\phi} f \frac{1}{z^n} = \sum_{n=0}^{\infty} P_E C_{\phi,w}^n \left( \sum_{x \in \mathcal{C}} f(x) e_x \right) \frac{1}{z^n} = \sum_{x \in \mathcal{C}} f(x) \sum_{n=0}^{\infty} P_E C_{\phi,w}^n e_x \frac{1}{z^n}
\]

\[
= \sum_{x \in \mathcal{C}} \sum_{i=0}^{k_{\phi(x)}} f(x) \sum_{n=0}^{\infty} P_E P_{W_i} C_{\phi,w}^n e_x \frac{1}{z^n}
\]

\[
= \sum_{x \in \mathcal{C}} \sum_{i=0}^{k_{\phi(x)}} \sum_{j=0}^{\tau} f(x) \sum_{n=0}^{\infty} P_E P_{W_i} C_{\phi,w}^{n+j} e_x \frac{1}{z^{n+j}}
\]

Now we turn to the proof of the “moreover” part.

It suffices to notice that

\[
T^{m+n} E \subset W^{E,\phi}_{m+n}, \quad T^{m} T^{n} E \subset W^{E,\phi}_{m+n},
\]

\[
T^{m+n} T^{n} E \subset W^{E,\phi}_{m+n}, \quad T^{m} T^{n} E \subset W^{E,\phi}_{m+n}.
\]

\[ \square \]

5. Examples

Example 5.1. Fix \( n \in \mathbb{N} \). Set \( X = \{1, 2, \ldots, n\} \). Define a mapping \( \phi : X \to X \) by \( \phi(i) = i+1 \) for \( i < n \), \( \phi(n) = 1 \) and \( w : X \to \mathbb{C} \) by \( w(i) = \lambda_i \) for some \( \lambda_i \in \mathbb{C} \setminus \{0\} \) for \( i \in X \). Set \( \Lambda = \lambda_1 \lambda_2 \ldots \lambda_n \). Let \( C_{\phi,w} \) be the composition operator in \( \mathbb{C}^n \). The matrix of this operator is

\[
C_{\phi,w} = \begin{bmatrix}
0 & \lambda_1 & 0 & \cdots & 0 \\
0 & 0 & \lambda_2 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_n & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

Let \( E := \text{lin} \{ e_1 \} \). It is easy to see that \( [E]_{S_\Lambda, S_\Lambda^*} = \mathcal{H} \). Using Lemma 2.1 and Lemma 2.2, one can verify that

\[
P_E C_{\phi,w}^{nk+r} x = \lambda^k (\lambda_1 \lambda_2 \ldots \lambda_r) x_{r+1} e_1, \quad r < n, \ r, k \in \mathbb{Z},
\]

\[
P_E C_{\phi,w}^{nk+r} x = \frac{1}{\lambda^k} \frac{1}{\lambda_1 \cdots \lambda_n} x_{r+1} e_1, \quad r < n, \ r, k \in \mathbb{Z},
\]

take the following form:

\[
U_x(z) = \sum_{k=0}^{\infty} \sum_{r=0}^{n-1} \left( \lambda^k (\lambda_1 \lambda_2 \ldots \lambda_r) x_{r+1} e_1 \right) \frac{1}{z^{nk+r}}
\]

\[
+ \sum_{k=0}^{\infty} \left( \sum_{r=0}^{n-1} \frac{1}{\lambda^k} \frac{1}{\lambda_1 \cdots \lambda_n} x_{r+1} e_1 \right) z^{nk+r}.
\]
Example 5.2. (Bilateral weighted shift)
Consider the bilateral weighted shift

\[ S_\lambda e_n = \lambda_{n+1} e_n, \quad n \in \mathbb{Z} \]

Let \( E := \text{lin} \{ e_0 \} \). It is easy to see that \( [E, S_\lambda, S_\lambda^*] = \mathcal{H} \). It is a matter of routine to verify that the Cauchy dual \( S_\lambda^* \) of \( S_\lambda \) has the following form

\[ S_\lambda^* e_n = \frac{1}{\lambda_n} e_{n-1}, \quad n \in \mathbb{Z}. \]

It is now easily seen that

\[ P_E(S_\lambda^*)^n x = \left( \prod_{i=1}^{n} \lambda_i \right)^{-1} x_n e_0, \quad n \in \mathbb{Z}_+\]

and

\[ P_E S_\lambda^n x = \left( \prod_{i=-n+1}^{0} \lambda_i \right) x_{-n} e_0, \quad n \in \mathbb{Z}_+. \]

\[ U_x(z) = \sum_{n=1}^{\infty} \left( \prod_{i=-n+1}^{0} \lambda_i \right) x_{-n} e_0 + \sum_{n=0}^{\infty} \left( \prod_{i=1}^{n} \lambda_i \right)^{-1} x_n. \]

Example 5.3. Fix \( k \in \mathbb{N} \). Set \( X = \{(1, i) : i \in \mathbb{N}, i \leq k\} \sqcup \mathbb{N} \). Define a mapping \( \phi : X \to X \) by

\[ \phi(x) = \begin{cases} (1, i - 1) & \text{for } x = (1, i), i \in \mathbb{N} \setminus \{0\} \\ k & \text{for } x = (1, 0) \\ i - 1 & \text{for } x = i \text{ and } i \in I_k \setminus \{0\} \\ k & \text{for } x = 0 \end{cases} \]

and \( w : X \to \mathbb{C} \) by \( w(x) = \lambda_x \) for some \( \lambda_x \in \mathbb{C} \setminus \{0\} \) and \( x \in X \).

\[ C_{\phi, w} e_x = \begin{cases} \lambda(1, i+1) e_{(1, i+1)} & \text{for } x = (1, i), i \in \mathbb{N} \setminus \{0\} \\ \lambda_{i+1} e_{i+1} & \text{for } x = i \text{ and } i \in I_{k-1} \\ \lambda_0 e_0 + \lambda_{(1,0)} e_{(1,0)} & \text{for } x = k \end{cases} \]

\[ P_E(C_{\phi', w})^n x = \left( \prod_{i=1}^{n} \lambda_{(1, i)} \right)^{-1} x_n e_{(1,0)}, \quad n \in \mathbb{Z}_+, \]

\[ P_E C_{\phi, w} e_{nk+r} x = \Lambda^k \lambda(1,0)(\lambda_{k} \lambda_{k-1} \ldots \lambda_{k+2-r}) e_{(1,0)} x_{k+1-r}, \quad r < k, \quad r, k \in \mathbb{Z} \]

\[ U_x(z) = \sum_{n=1}^{k} \sum_{r=0}^{k} \left( \Lambda^k \lambda(1,0)(\lambda_{k} \lambda_{k-1} \ldots \lambda_{k+2-r}) e_{(1,0)} x_{k+1-r} \right) \frac{1}{z^{nk+r}} + \sum_{n=0}^{\infty} \left( \prod_{i=1}^{n} \lambda_{(1, i)} \right)^{-1} x_n z^{nk+r}. \]
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