An application of a matrix inequality in quantum information theory

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April 1, 2022

Abstract

Quantum information theory has generated several interesting conjectures involving products of completely positive maps on matrix algebras, also known as quantum channels. In particular it is conjectured that the output state with maximal $p$-norm from a product channel is always a product state. It is shown here that the Lieb-Thirring inequality can be used to prove this conjecture for one special case, namely when one of the components of the product channel is of the type known as a diagonal channel.
1 Introduction

The minimal output entropy of a quantum channel $\Phi$ is defined by

$$S_{\text{min}}(\Phi) = \inf_{\rho} S(\Phi(\rho))$$

(1)

where $S$ is the von Neumann entropy, and the inf runs over states in the domain of $\Phi$. The following additivity property is conjectured.

Conjecture 1 Let $\Phi$ and $\Psi$ be any quantum channels, that is completely positive, trace-preserving maps on finite-dimensional matrix algebras. Then

$$S_{\text{min}}(\Phi \otimes \Psi) = S_{\text{min}}(\Phi) + S_{\text{min}}(\Psi)$$

(2)

Within the last year, Shor [11] proved that Conjecture 1 is equivalent to several other outstanding conjectures in quantum information theory, among them additivity of Holevo capacity of a quantum channel, and additivity of the entanglement of formation. Thus a proof of Conjecture 1 would settle quite a few outstanding problems in the field.

The additivity conjecture has been proved for several special classes of channels [10], [6], [7]. The Lieb-Thirring inequality [9] was a key ingredient in several of those proofs. The purpose of this paper is to show how the Lieb-Thirring inequality can be used to demonstrate (4) for another class known as the ‘diagonal’ channels.

The problem is attacked by making use of the maximal output $p$-norm for $p \geq 1$, also called the maximal output purity of a channel [1], which is defined by

$$\nu_p(\Phi) = \sup_{\rho} ||\Phi(\rho)||_p = \sup_{\rho} \left( \text{Tr} \left( \Phi(\rho) \right)^p \right)^{1/p}$$

(3)

The derivative of $\nu_p(\Phi)$ at $p = 1$ is the negative minimal output entropy, so Conjecture 1 is a consequence of the following stronger conjecture.

Conjecture 2 There is some $p_0 > 1$, such that for all quantum channels $\Phi$ and $\Psi$, and all $1 \leq p \leq p_0$,

$$\nu_p(\Phi \otimes \Psi) = \nu_p(\Phi) \nu_p(\Psi)$$

(4)
In this paper we consider a class of channels known as the ‘diagonal’ channels, and show how the Lieb-Thirring inequality can be used to derive (4) under the assumption that at least one of the channels Φ or Ψ is in this class. It turns out that the multiplicativity result for diagonal channels holds for all \( p \geq 1 \), so this might lead one to hope that \( p_0 = \infty \) in Conjecture [2]. However it is known that multiplicativity fails in general for \( p \geq 5 \) [12], and indeed this probably provides evidence that the strategy used in this paper cannot be directly extended to prove Conjecture [2] in the general case. Nevertheless it seems worthwhile to explain the approach used, as the techniques may be useful for other reasons, and the results may have other applications in quantum information theory. The method of proof is quite similar to the approach used by the author to prove Conjecture [2] for the class of entanglement-breaking channels [5].

2 Statement of results

Since we will be concerned with \( p \)-norms from now on, the trace-preserving condition for quantum channels is unimportant, and so we deal instead with completely positive maps. The diagonal class of channels was described by Landau and Streater [8]. Recall that the Hadamard product of two \( n \times n \) matrices \( A \) and \( B \) is defined by

\[
(A \ast B)_{ij} = A_{ij}B_{ij}
\]  

(5)

**Definition 3** The CP map \( \Phi \) is called diagonal if there is a positive semidefinite matrix \( C \) such that

\[ \Phi(\rho) = C \ast \rho \]  

(6)

If \( C = |\psi\rangle\langle\psi| \) is rank one, then (6) can be written

\[ C \ast \rho = \text{Diag}(\psi) \rho \text{Diag}(\psi)^* \]  

(7)

where \( \text{Diag}(\psi) \) is the diagonal \( n \times n \) matrix with the components of \( |\psi\rangle \) along the diagonal. Using the spectral representation it follows that a map is diagonal if and only if it has a Kraus representation with all diagonal matrices.

Our main result is stated below in Theorem [4]
Theorem 4 Let $\Phi$ be a diagonal map, and let $\Psi$ be any other CP map. Then for all $p \geq 1$,
\[ \nu_p(\Phi \otimes \Psi) = \nu_p(\Phi) \nu_p(\Psi) \]  
(8)

The main tool used in the proof is the Lieb-Thirring inequality \[9\], which we now state. Let $K \geq 0$ be a positive semidefinite $n \times n$ matrix, and let $V$ be any $k \times n$ matrix. Then for all $p \geq 1$,
\[ \text{Tr} \left( VKV^* \right)^p \leq \text{Tr}(V^*V)^{p/2} K^p (V^*V)^{p/2} = \text{Tr}(V^*V)^p K^p \]  
(9)

There are several proofs of this inequality \[9\], \[2\]. The original proof of Lieb and Thirring employs Epstein’s concavity theorem \[3\], which is based on a combination of spectral theory and analytic continuation methods.

3 The factorization

The goal of this section is to rewrite the output of the product channel $\Phi \otimes \Psi$ in the factorized form $VKV^*$ so that (9) can be applied. We assume that $\Phi$ is a diagonal channel which acts by Hadamard product with the $n \times n$ matrix $C$. Let $\rho$ be a state on $\mathbb{C}^{kn}$, that is a positive semidefinite $kn \times kn$ matrix with trace 1, for some $k \geq 1$. Then $\rho$ can be written as a $n \times n$ block matrix where the blocks $(\rho)_{ij}$ are $k \times k$ matrices. The diagonal blocks $(\rho)_{ii}$ are positive semidefinite, and we define
\[ \alpha_i = \text{Tr}(\rho)_{ii} \]  
(10)

Define a new $kn \times kn$ matrix $\tau$ with blocks
\[ (\tau)_{ij} = (\alpha_i \alpha_j)^{-1/2} (\rho)_{ij} \]  
(11)

and let $A$ denote the $n \times n$ matrix with entries $A_{ij} = (\alpha_i \alpha_j)^{1/2}$. Then $\rho$ can be written as a Hadamard product of $A$ with $\tau$, that is
\[ \rho = (A \otimes J_k) * \tau \]  
(12)

where $J_k$ is the $k \times k$ matrix with all entries equal to 1:
\[ J_k = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \]  
(13)
Notice that $J_k$ acts as the identity for the Hadamard product. Furthermore this decomposition commutes with the action of $\Psi$ on the second factor, that is

$$(I \otimes \Psi)(\rho) = (A \otimes J_k) \ast \left( (I \otimes \Psi)(\tau) \right) \tag{14}$$

The map $\Phi \otimes I$ acts on (14) by a Hadamard product with the matrix $C$ on the first factor. This Hadamard product acts just on the matrix $A$, and the result is

$$(\Phi \otimes \Psi)(\rho) = \left( \Phi(A) \otimes J_k \right) \ast \left( (I \otimes \Psi)(\tau) \right) \tag{15}$$

The next step is to factorize the matrix $(I \otimes \Psi)(\tau)$. To do this, let $V_1, \ldots, V_n$ be the $k \times kn$ matrices which are the block-rows of its square root, that is

$$\left( (I \otimes \Psi)(\tau) \right)^{1/2} = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \tag{16}$$

Then it follows that

$$(I \otimes \Psi)(\tau) = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \begin{pmatrix} V_1^* & \cdots & V_n^* \end{pmatrix} = \begin{pmatrix} V_1V_1^* & \cdots & V_1V_n^* \\ \vdots & \ddots & \vdots \\ V_nV_1^* & \cdots & V_nV_n^* \end{pmatrix} \tag{17}$$

Notice that the diagonal terms are $\Psi((\tau)_{ii}) = V_iV_i^*$, and since $(\tau)_{ii}$ is positive semidefinite with $\text{Tr}(\tau)_{ii} = 1$ it follows that

$$\|V_iV_i^*\|_p \leq \nu_p(\Psi) \tag{18}$$

Applying the factorization (17) to (15) gives

$$(\Phi \otimes \Psi)(\rho) = \begin{pmatrix} \Phi(A)_{11}V_1V_1^* & \cdots & \Phi(A)_{1n}V_1V_n^* \\ \vdots & \ddots & \vdots \\ \Phi(A)_{n1}V_nV_1^* & \cdots & \Phi(A)_{nn}V_nV_n^* \end{pmatrix} \tag{19}$$

Now the right side of (19) can be rewritten as a product of three matrices:

$$\begin{pmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & V_n \end{pmatrix} \begin{pmatrix} \Phi(A)_{11}I' & \cdots & \Phi(A)_{1n}I' \\ \vdots & \ddots & \vdots \\ \Phi(A)_{n1}I' & \cdots & \Phi(A)_{nn}I' \end{pmatrix} \begin{pmatrix} V_1^* & 0 & \cdots & 0 \\ 0 & V_2^* & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & V_n^* \end{pmatrix} \tag{20}$$
where the $I'$ in the middle term is the $kn \times kn$ identity matrix. This is the same as

$$\begin{pmatrix} V_1 & 0 & \ldots & 0 \\ 0 & V_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & V_n \end{pmatrix} (\Phi(A) \otimes I') \begin{pmatrix} V_1^* & 0 & \ldots & 0 \\ 0 & V_2^* & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & V_n^* \end{pmatrix}$$  \hspace{1cm} (21)

Therefore (19) has been written in the factorized form

$$(\Phi \otimes \Psi)(\rho) = V K V^*$$  \hspace{1cm} (22)

where $V$ is the $kn \times kn^2$ matrix

$$V = \begin{pmatrix} V_1 & 0 & \ldots & 0 \\ 0 & V_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & V_n \end{pmatrix}$$  \hspace{1cm} (23)

and

$$K = (\Phi(A) \otimes I')$$  \hspace{1cm} (24)

4 Applying the inequality

The last step is to apply the Lieb-Thirring inequality (9) to (22). It follows from (23) that $(V^*V)^p$ is block diagonal, that is

$$\left( \begin{array}{cccc} (V_1^*V_1)^p & 0 & \ldots & 0 \\ 0 & (V_2^*V_2)^p & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & (V_n^*V_n)^p \end{array} \right)$$  \hspace{1cm} (25)

Also $K^p = (\Phi(A))^p \otimes I'$, so the diagonal blocks of $K^p$ are just the diagonal entries of $(\Phi(A))^p$ multiplied by the identity matrix $I'$. Hence

$$\text{Tr} \left( V^*V \right)^p K^p = \sum_{i=1}^{n} \text{Tr}(V_i^*V_i)^p \left( (\Phi(A))^p \right)_{ii}$$  \hspace{1cm} (26)
The matrices $V_i^*V_i$ and $V_i^*V_i$ share the same nonzero spectrum, and so (18) can be used to bound the terms $\text{Tr}(V_i^*V_i)^p$ on the right side of (26). This gives

$$\text{Tr}(V^*V)^p K^p \leq \sum_{i=1}^{n} \left( \nu_p(\Psi) \right)^p \left( (\Phi(A))^p \right)_{ii}$$  \hspace{1cm} (27)

$$= \left( \nu_p(\Psi) \right)^p \text{Tr}(\Phi(A))^p$$  \hspace{1cm} (28)

Furthermore since $\rho$ is a state, it follows that $\text{Tr}A = \text{Tr}\rho = 1$, and hence $\text{Tr}(\Phi(A))^p$ can be bounded using the definition (3). Putting it all together we deduce

$$\text{Tr}(\Phi \otimes \Psi)(\rho)^p \leq \left( \nu_p(\Psi) \right)^p \left( \nu_p(\Phi) \right)^p$$  \hspace{1cm} (29)

From this it follows that

$$\nu_p(\Phi \otimes \Psi) \leq \nu_p(\Phi) \nu_p(\Psi)$$  \hspace{1cm} (30)

The inequality in the other direction follows easily by restricting to product states, hence Theorem 2 is proved.

**Acknowledgements** This work was supported in part by National Science Foundation Grant DMS–0101205. The author is grateful to E. Lieb and M. B. Ruskai for first demonstrating that the Lieb-Thirring inequality could be used to address the additivity problem, and for allowing their work to be included in the Appendix of the paper [4].

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