CLASSIFICATION OF NATURALLY GRADED ZINBIEL ALGEBRAS WITH CHARACTERISTIC SEQUENCE EQUAL TO \((n - p, p)\)

J. K. Adashev,¹ M. Ladra,²,³ and B. A. Omirov⁴

UDC 512.5

The present work continues our description of some classes of nilpotent Zinbiel algebras. We focus our attention on the investigation of Zinbiel algebras with restrictions imposed on the gradation and characteristic sequence. Namely, we propose a classification of naturally graded Zinbiel algebras with characteristic sequence equal to \((n - p, p)\).

1. Introduction

The present paper is devoted to investigation of algebras that are Koszul dual to the Leibniz algebras. These algebras were introduced in the middle of the 1990 by the French mathematician J.-L. Loday [15] and called Zinbiel algebras (Leibniz written in the reverse order).

A crucial fact from the theory of finite-dimensional Zinbiel algebras is the nilpotency of these algebras over a field with characteristic zero [12]. Since the description of finite-dimensional complex Zinbiel algebras is a boundless problem (even if they are nilpotent), their investigation should be carried out by introducing some additional restrictions (for the index of nilpotency, gradation, characteristic sequence, etc.).

In general, the investigation of Zinbiel algebras goes parallel with the investigation of nilpotent Leibniz algebras. Thus, the \(n\)-dimensional Leibniz algebras with nilindices \(n + 1\) and \(n\) (which is equivalent to admitting the characteristic sequences equal to \((n)\) and \((n - 1, 1)\), respectively) were described in [5] and [14]. A similar description for the Zinbiel algebras was obtained in [3].

In the study of \(n\)-dimensional Leibniz algebras with nilindex \(n - 1\) (see [10]), it was indicated that the characteristic sequences of these algebras are equal either to \((n - 2, 1, 1)\) or to \((n - 2, 2)\). The descriptions of Leibniz (Zinbiel) algebras with these characteristic sequence were obtained in [9] and [10] (respectively, [4]).

Later, the naturally graded Leibniz algebras with nilindex \(n - 2\) admitting the following characteristic sequences:

\[(n - 3, 3), \ (n - 3, 2, 1), \ (n - 3, 1, 1, 1)\]

were investigated in the papers [6, 7, 11], respectively. The description of naturally graded Zinbiel algebras with these properties was given in [1] and [2].

Finally, the latest progress in the description of the structure of nilpotent Leibniz algebras was achieved in [11] and [16]. Thus, in particular, the authors described naturally graded nilpotent \(n\)-dimensional Leibniz algebras with characteristic sequences equal to \((n - p, p)\) and \((n - p, 1, \ldots, 1)\). Since the description of \(p\)-filiform Zinbiel algebras (i.e., Zinbiel algebras with characteristic sequence equal to \((n - p, 1, \ldots, 1)\)) was obtained in [8], in the present paper, in order to complete the description similar to that obtained in [11], we propose a description (to within an isomorphism) of naturally graded Zinbiel algebras with characteristic sequence equal to \((n - p, p)\).

¹ Institute of Mathematics, Uzbekistan National University, Tashkent, Uzbekistan; e-mail: adashevjq@mail.ru.
² University of Santiago de Compostela, Santiago de Compostela, Spain; e-mail: manuel.ladra@gmail.com.
³ Corresponding author.
⁴ Institute of Mathematics, Uzbekistan National University, Tashkent, Uzbekistan; e-mail: omirovb@mail.ru.

Published in Ukrain’s’kyi Matematychnyi Zhurnal, Vol. 71, No. 7, pp. 867–883, July, 2019. Original article submitted July 15, 2016.
All algebras and vector spaces considered in the present work are assumed to be finite-dimensional and complex. In order to make the tables of multiplication of algebras shorter, we omit zero products.

2. Preliminaries

In this section, we present definitions and known results required to proceed further to the main part of the work.

Definition 2.1. An algebra $A$ over a field $F$ is called a Zinbiel algebra if, for any $x, y, z \in A$, the following identity holds:

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y),$$

where $\circ$ is the multiplication of the algebra $A$.

For an arbitrary Zinbiel algebra we define the lower series as follows:

$$A_1 = A, A_{k+1} = A \circ A_k, k \geq 1.$$

Definition 2.2. A Zinbiel algebra $A$ is called nilpotent if there exists $s \in \mathbb{N}$ such that $A^s = 0$. The minimal number of this kind is called the nilindex of $A$.

Definition 2.3. An $n$-dimensional Zinbiel algebra $A$ is called null-filiform if $\dim A^i = (n + 1) - i$ for $1 \leq i \leq n + 1$.

By definition, it is clear that the null-filiformity of an algebra $A$ is equivalent to admitting the maximal possible nilindex.

Let $x$ be an element of the set $A \setminus A^2$. For an operator of left multiplication $L_x$ (defined as $L_x(y) = x \circ y$) we consider a descending sequence $C(x) = (n_1, n_2, \ldots, n_k)$, where $n = n_1 + n_2 + \ldots + n_k$, formed by the sizes of Jordan blocks of the operator $L_x$. On the set of sequences of this kind, we introduce the lexicographical order, i.e.,

$$(n_1, n_2, \ldots, n_k) \leq (m_1, m_2, \ldots, m_s)$$

if there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for all $j < i$ and $n_i < m_i$.

Definition 2.4. A sequence

$$C(A) = \max_{x \in A \setminus A^2} C(x)$$

is called the characteristic sequence of the algebra $A$.

Example 2.1. Let $C(A) = (1, 1, \ldots, 1)$. Then the algebra $A$ is Abelian.

Example 2.2. An $n$-dimensional Zinbiel algebra $A$ is null-filiform if and only if $C(A) = (n)$.

Let $A$ be a finite-dimensional Zinbiel algebra with nilindex $s$. We set

$$A_i := A^i/A^{i+1}, \quad 1 \leq i \leq s - 1,$$

$$\text{gr } A := A_1 \oplus A_2 \oplus \cdots \oplus A_{s-1}.$$
From the condition \( A_i \circ A_j \subseteq A_{i+j} \), we derive a graded algebra \( \text{gr} A \). The graduation constructed in this way is called natural graduation. If a Zinbiel algebra \( A \) is isomorphic to the algebra \( \text{gr} A \), then the algebra \( A \) is called a naturally graded Zinbiel algebra.

Further, we need the following lemmas.

**Lemma 2.1** [13]. For any \( n, a \in \mathbb{N} \), the following equality holds:

\[
\sum_{k=0}^{n} (-1)^k C^n_k C^{-k}_{a+n-k} = 0.
\]

**Lemma 2.2** [12]. Let \( A \) be a Zinbiel algebra with the following known products:

\[
e_1 \circ e_i = e_{i+1}, \quad 1 \leq i \leq k - 1.
\]

Then

\[
e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i + j \leq k,
\]

where \( C^b_a = \binom{a}{b} \) denotes the binomial coefficient.

3. Main Results

Let \( A \) be a Zinbiel algebra and let \( C(A) = (n_1, n_2, \ldots, n_k) \) be its characteristic sequence. Then there exists a basis \( \{e_1, e_2, \ldots, e_n\} \) such that the matrix of the operator of left multiplication by an element \( e_1 \) has the form

\[
L_{e_1, \sigma} = \begin{pmatrix}
J_{n_{\sigma(1)}} & 0 & \cdots & 0 \\
0 & J_{n_{\sigma(2)}} & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \cdots & J_{n_{\sigma(s)}}
\end{pmatrix},
\]

where \( \sigma(i) \) belongs to \( \{1, 2, \ldots, s\} \).

By a suitable permutation of basis elements, we can guarantee that

\[
n_{\sigma(2)} \geq n_{\sigma(3)} \geq \cdots \geq n_{\sigma(s)}.
\]

Let \( A \) be a naturally graded Zinbiel algebra with characteristic sequence equal to \( (n_1, n_2, \ldots, n_k) \).

**Proposition 3.1.** There are no naturally graded Zinbiel algebras with \( n_{\sigma(1)} = 1 \) and \( n_{\sigma(2)} \geq 4 \).

**Proof.** By the condition of proposition, we get the following products:

\[
e_1 \circ e_1 = 0, \quad e_1 \circ e_i = e_{i+1}, \quad 2 \leq i \leq 4,
\]

\[
e_1 \circ e_i = e_{i+1}, \quad \sum_{k=1}^{t} n_{\sigma(k)} + 1 \leq i \leq \sum_{k=1}^{t+1} n_{\sigma(k)} - 1, \quad 3 \leq t \leq s - 1,
\]

\[
e_1 \circ e_i = 0, \quad i = \sum_{k=1}^{t} n_{\sigma(k)}, \quad 3 \leq t \leq s.
\]
By the property of the Zinbiel algebras

\[(a \circ b) \circ c = (a \circ c) \circ b,\]

we obtain

\[e_3 \circ e_1 = (e_1 \circ e_2) \circ e_1 = (e_1 \circ e_1) \circ e_2 = 0 \Rightarrow e_3 \circ e_1 = 0.\]

The chain of equalities

\[0 = (e_1 \circ e_1) \circ e_3 = e_1 \circ (e_1 \circ e_3) + e_1 \circ (e_3 \circ e_1) = e_1 \circ e_4 = e_5\]

implies that \(e_5 = 0\), i.e., we arrive at a contradiction with the condition \(n_{\sigma(2)} \geq 4\), which completes the proof of the proposition.

The next example shows that the condition \(n_{\sigma(2)} \geq 4\) of Proposition 3.1 is essential.

**Example 3.1.** Let \(A\) be a four-dimensional Zinbiel algebra with the following multiplication table:

\[e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_3.\]

Then \(C(A) = (3, 1)\) and the matrix of the operator of left multiplication on \(e_1\) takes the form

\[
\begin{pmatrix}
    J_1 & 0 \\
    0 & J_3
\end{pmatrix}.
\]

Let \(A\) be an arbitrary Zinbiel algebra with characteristic sequence equal to \((n - p, p)\). Then the matrix of the operator of left multiplication by \(e_1\) admits one of the following forms:

I. \[
\begin{pmatrix}
    J_{n-p} & 0 \\
    0 & J_p
\end{pmatrix};
\]

II. \[
\begin{pmatrix}
    J_p & 0 \\
    0 & J_{n-p}
\end{pmatrix}, \quad n \geq 2p.
\]

**Definition 3.1.** A Zinbiel algebra is called an algebra of the first type (of type I) if the operator \(L_{e_1}\) has the form \(\begin{pmatrix} J_{n-p} & 0 \\ 0 & J_p \end{pmatrix}\); otherwise it is called an algebra of the second type (of type II).

In view of the results obtained in [4] and [3], we consider only \(n\)-dimensional naturally graded Zinbiel algebras with \(C(A) = (n - p, p), \quad p \geq 3\).

**3.1. Classification of Zinbiel Algebras of Type I.** Let \(A\) be a Zinbiel algebra of type I. Then we conclude that there exists a basis \(\{e_1, e_2, \ldots, e_{n-p}, f_1, f_2, \ldots, f_p\}\) such that the products containing the element \(e_1\) on the left are as follows:

\[e_1 \circ e_i = e_{i+1}, \quad 1 \leq i \leq n - p - 1.\]

It follows from Lemma 2.2 that

\[e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i + j \leq n - p, \quad e_1 \circ e_p = 0,\]

\[e_1 \circ f_i = f_{i+1}, \quad 1 \leq i \leq p - 1, \quad e_1 \circ f_p = 0.\]
It is easy to see that

\[ A_1 = \langle e_1, f_1 \rangle, \quad A_2 = \langle e_2, f_2 \rangle, \ldots, \quad A_p = \langle e_p, f_p \rangle, \quad A_{p+1} = \langle e_{p+1}, f_{p+1} \rangle, \ldots, \quad A_{n-p} = \langle e_{n-p} \rangle. \]

Let

\[ f_1 \circ e_i = \alpha_i e_{i+1} + \beta_i f_{i+1}, \quad 1 \leq i \leq p - 1, \]

\[ f_1 \circ e_i = \alpha_i e_{i+1}, \quad p \leq i \leq n - p - 1, \]

\[ f_1 \circ f_i = \gamma_i e_{i+1} + \delta_i f_{i+1}, \quad 1 \leq i \leq p - 1, \]

\[ f_1 \circ f_p = \gamma_p e_{p+1}. \tag{2} \]

**Proposition 3.2.** Let \( A \) be a Zinbiel algebra of type I. Then, for the structural constants \( \alpha_i, \beta_i, \gamma_i, \) and \( \delta_i \), the following restrictions are true:

\[ \alpha_{i+1} = 0, \quad 1 \leq i \leq n - p - 2, \]

\[ \beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1}, \quad 1 \leq i \leq p - 2, \]

\[ (i + 1)\gamma_i = \beta_1 \left( 2\gamma_1 + \sum_{k=2}^{i} \gamma_k \right), \quad 1 \leq i \leq n - p - 2, \]

\[ (i + \beta_1)\delta_i = \beta_1 \left( 2\delta_1 + \sum_{k=2}^{i} \delta_k \right), \quad 1 \leq i \leq p - 2. \]

**Proof.** First, we compute the products \( f_i \circ e_1 \) and \( f_2 \circ e_i \).

Consider

\[ f_2 \circ e_1 = e_1 \circ (f_1 \circ e_1) + e_1 \circ (e_1 \circ f_1) = \alpha_1 e_3 + (1 + \beta_1) f_3. \]

By using the chain of equalities

\[ f_i \circ e_1 = (e_1 \circ f_{i-1}) \circ e_1 = e_1 \circ (f_{i-1} \circ e_1) + e_1 \circ (e_1 \circ f_{i-1}), \]

we conclude that

\[ f_i \circ e_1 = \alpha_1 e_{i+1} + (i - 1 + \beta_1) f_{i+1} \]

for \( 1 \leq i \leq p - 1 \) and \( f_i \circ e_1 = \alpha_1 e_{i+1} \) for \( p \leq i \leq n - p - 1. \)

By using the equality

\[ e_i \circ f_1 = (e_1 \circ e_{i-1}) \circ f_1 = e_1 \circ (e_{i-1} \circ f_1) + e_1 \circ (f_1 \circ e_{i-1}), \]
we obtain

\[ e_i \circ f_1 = \sum_{k=1}^{i-1} \alpha_k e_{i+1} + \sum_{k=0}^{i-1} \beta_k f_{i+1} \]

for \( 1 \leq i \leq p - 1 \) and

\[ e_i \circ f_1 = \sum_{k=1}^{i-1} \alpha_k e_{i+1} \]

for \( p \leq i \leq n - p - 1 \). Moreover,

\[ f_2 \circ e_2 = e_1 \circ (f_1 \circ e_2) + e_1 \circ (e_2 \circ f_1) = (\alpha_1 + \alpha_2) e_4 + (1 + \beta_1 + \beta_2) f_4. \]

From

\[ f_2 \circ e_i = (e_1 \circ f_1) \circ e_i = e_1 \circ (f_1 \circ e_i) + e_1 \circ (e_i \circ f_1), \]

we get

\[ f_2 \circ e_i = \sum_{k=1}^{i} \alpha_k e_{i+2} + \sum_{k=0}^{i} \beta_k f_{i+2}, \quad 1 \leq i \leq p - 2, \]

\[ f_2 \circ e_i = \sum_{k=1}^{i} \alpha_k e_{i+2}, \quad p - 1 \leq i \leq n - p - 2. \]

We now compute the products \( f_i \circ f_1 \) and \( f_2 \circ f_i \). We have

\[ f_2 \circ f_1 = (e_1 \circ f_1) \circ f_1 = 2 \gamma_1 e_3 + 2 \delta_1 f_3, \]

\[ f_3 \circ f_1 = (e_1 \circ f_2) \circ f_1 = (2 \gamma_1 + \gamma_2) e_4 + (2 \delta_1 + \delta_2) f_4. \]

By induction, we obtain

\[ f_i \circ f_1 = \left( 2 \gamma_1 + \sum_{k=2}^{i-1} \gamma_k \right) e_{i+1} + \left( 2 \delta_1 + \sum_{k=2}^{i-1} \delta_k \right) f_{i+1}, \quad 2 \leq i \leq p - 1, \]

\[ f_p \circ f_1 = \left( 2 \gamma_1 + \sum_{k=2}^{p-1} \gamma_k \right) e_{p+1}. \]

Similarly, it follows from

\[ f_2 \circ f_i = (e_1 \circ f_1) \circ f_i = e_1 \circ (f_1 \circ f_i) + e_1 \circ (f_i \circ f_1) \]

that

\[ f_2 \circ f_i = \left( 2 \gamma_1 + \sum_{k=2}^{i} \gamma_k \right) e_{i+2} + \left( 2 \delta_1 + \sum_{k=2}^{i} \delta_k \right) f_{i+2}, \quad 2 \leq i \leq p - 2, \]
\[ f_2 \circ f_i = \left( 2\gamma_1 + \sum_{k=2}^{i} \gamma_k \right) e_{i+2}, \quad p - 1 \leq i \leq p. \]

If \( \beta_1 = 1 \), then, by using the equality \((f_1 \circ f_1) \circ e_1 = (f_1 \circ e_1) \circ f_1 \) we get

\[ 2\gamma_1(1 - \beta_1) = \alpha_1^2 - \delta_1 \alpha_1, \quad (1 - \beta_1)\delta_1 = \alpha_1(1 + \beta_1). \]

Consequently, \( \alpha_1 = 0 \).

Let \( \beta_1 \neq 1 \). Then, by the following change of variables:

\[ e'_1 = e_1, \quad f'_1 = \frac{\alpha_1}{\beta_1 - 1} e_1 + f_1, \]

we obtain \( \alpha'_1 = 0 \).

By virtue of the equalities

\[ f_1 \circ e_{i+1} = f_1 \circ (e_1 \circ e_i) = (f_1 \circ e_1) \circ e_i - if_1 \circ e_{i+1} = \beta_1 f_2 \circ e_i - if_1 \circ e_{i+1}, \]

we find

\[ (i + 1)f_1 \circ e_{i+1} = \beta_1 f_2 \circ e_i. \]

Therefore,

\[ (i + 1)\alpha_{i+1} e_{i+2} + (i + 1)\beta_{i+1} f_{i+2} = \beta_1 \left( \sum_{k=1}^{i} \alpha_k e_{i+2} + \sum_{k=0}^{i} \beta_k f_{i+2} \right). \]

We now compare the coefficients of the basis elements. As a result, by induction, we conclude that

\[ \alpha_{i+1} = 0, \quad 1 \leq i \leq n - p - 2, \]

\[ \beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1}, \quad 1 \leq i \leq p - 2. \]

By analyzing the equality \((f_1 \circ f_1) \circ e_1 = (f_1 \circ e_1) \circ f_1\), we get the remaining restrictions of the proposition. In the next proposition, we compute the products \( e_i \circ f_j \) and \( f_j \circ e_i \).

**Proposition 3.3.** Let \( A \) be a Zinbiel algebra of type I. Then the following expressions are true:

\[ e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j} \quad \text{for} \quad 2 \leq i + j \leq p, \quad (3) \]

\[ f_i \circ e_j = \sum_{k=0}^{j-2} C_{i+j-2-k}^{i-2} \beta_k f_{i+j} \quad \text{for} \quad 2 \leq i + j \leq p, \quad (4) \]

where \( \beta_0 = 1 \).
Proof. We prove (3), (4) by induction. In view of (1) and (3.1), we conclude that relations (3) and (4) are true for $i = 1$.

Consider (4) for $j = 1$. Since

$$f_1 \circ e_1 = \beta_1 f_2 \quad \text{and} \quad f_2 \circ e_1 = e_1 \circ (f_1 \circ e_1) + e_1 \circ (e_1 \circ f_1) = (1 + \beta_1)f_3,$$

by using the equalities

$$f_i \circ e_1 = (e_1 \circ f_{i-1}) \circ e_1 = e_1 \circ (f_{i-1} \circ e_1) + e_1 \circ (e_1 \circ f_{i-1})$$

and induction, we conclude that $f_i \circ e_1 = (i - 1 + \beta_1)f_{i+1}$ for $1 \leq i \leq p - 1$. In view of

$$e_i \circ f_1 = (e_1 \circ e_{i-1}) \circ f_1 = e_1 \circ (e_{i-1} \circ f_1) + e_1 \circ (f_1 \circ e_{i-1}),$$

we obtain

$$e_i \circ f_1 = \sum_{k=0}^{i-1} \beta_k f_{i+1} \quad \text{for} \quad 1 \leq i \leq p - 1.$$

Therefore, equalities (3) are true for $j = 1$ and arbitrary $i$.

Suppose that expressions (3), (4) are true for $i$ and any value of $j$. The proof of the expressions for $i + 1$ is obtained by using the following chain of equalities:

$$e_{i+1} \circ f_j = e_1 \circ (e_i \circ f_j) + e_1 \circ (f_j \circ e_i)$$

$$= e_1 \circ \left( \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j} + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k f_{i+j} \right)$$

$$= \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j+1} + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k f_{i+j+1}$$

$$= \left( \sum_{k=0}^{i-1} C_{i+j-1-k}^{j-1} \beta_k + \sum_{k=0}^{i} C_{i+j-1-k}^{j-2} \beta_k \right) f_{i+j+1}$$

$$= \sum_{k=0}^{i} C_{i+j-1-k}^{j-1} \beta_k f_{i+j+1}.$$

Here, we have used the well-known formula

$$C_n^{m-1} + C_n^m = C_{n+1}^m.$$  

The proof of expressions (4) is similar.

In what follows, we clarify the restrictions imposed on the structural constants of the algebra and related to the dimension and to the parameter $\beta_1$. 
Proposition 3.4. Let $A$ be a Zinbiel algebra of type I. Then the following restrictions are true:

(1) Case $\dim A \geq 2p + 1$. If $\beta_1 \neq 1$, then

$\gamma_i = 0, \quad 1 \leq i \leq p - 1,$

$\delta_i = 0, \quad 1 \leq i \leq p - 2,$

$(p - 1 + \beta_1)\gamma_p = 0,$

$(p - 2 + \beta_1)\delta_{p-1} = 0.$

If $\beta_1 = 1$, then

$\beta_i = 1, \quad 1 \leq i \leq p - 1,$

$\gamma_i = \gamma_1, \quad 1 \leq i \leq p,$

$\delta_i = \delta_1, \quad 1 \leq i \leq p - 1.$

(2) Case $\dim A = 2p$. If $\beta_1 \neq 1$, then

$\gamma_i = 0, \quad 1 \leq i \leq p - 2,$

$\delta_i = 0, \quad 1 \leq i \leq p - 2,$

$(p - 2 + \beta_1)\gamma_{p-1} = 0,$

$(p - 2 + \beta_1)\delta_{p-1} = 0.$

If $\beta_1 = 1$, then

$\beta_i = 1, \quad 1 \leq i \leq p - 1,$

$\gamma_i = \gamma_1, \quad 1 \leq i \leq p - 1,$

$\delta_i = \delta_1, \quad 1 \leq i \leq p - 1.$

Proof. Let $\dim A \geq 2p + 1$. Then it follows from Proposition 3.2 that

$$(i + 1)\gamma_i = \beta_1 \left(2\gamma_1 + \sum_{k=2}^{K} \gamma_k\right), \quad 1 \leq i \leq p - 1,$$

$$\tag{5} (i + \beta_1)\delta_i = \beta_1 \left(2\delta_1 + \sum_{k=2}^{K} \delta_k\right), \quad 1 \leq i \leq p - 2.$$
Consider
\[(f_1 \circ f_i) \circ e_1 = f_1 \circ (f_i \circ e_1) + f_1 \circ (e_1 \circ f_i)\]
\[= (i + \beta_1) (\gamma_i e_i + \delta_i f_i).\]

On the other hand,
\[(f_1 \circ f_i) \circ e_1 = (\gamma_i e_i + \delta_i f_i) \circ e_1\]
\[= (i + 1) \gamma_i e_i + (i + \beta_1) \delta_i f_i.\]

Hence,
\[(i + 1) \gamma_i = (i + \beta_1) \gamma_{i+1}, \quad 1 \leq i \leq p - 1,\]
\[(i + \beta_1) \delta_i = (i + \beta_1) \delta_{i+1}, \quad 1 \leq i \leq p - 2.\] (6)

If we consider the cases \(\beta_1 \neq 1\) and \(\beta_1 = 1\) together with expressions (5) and (6), then we get the restrictions for the case
\[\dim A \geq 2p + 1.\]

The proof of the remaining case is carried out in a similar way.

Consider a general change of basis for the algebra \(A\). It is known that, for naturally graded Zinbiel algebras, it is sufficient to change the basis as follows:
\[e'_1 = Ae_1 + Bf_1, \quad f'_1 = Ce_1 + Df_1,\]
where \(AD - BC \neq 0\).

**Proposition 3.5.** Let \(A\) be a Zinbiel algebra of type I and let \(\beta_1 \neq 1\). Then
\[e'_{i+1} = e'_{i} \circ e'_{i}, \quad 1 \leq i \leq n - p - 1,\]
\[f'_{i+1} = e'_{i} \circ f'_{i}, \quad 1 \leq i \leq p - 1,\]
\[e'_i = A^i e_i + A^{i-1} B \sum_{k=0}^{i-1} \beta_k f_i, \quad 1 \leq i \leq p - 1,\]
\[f'_i = A^{i-1} Df_i, \quad 1 \leq i \leq p - 1,\]
\[C = 0.\]

**Proof.** In view of \(f'_1 \circ f'_1 = 0\), we get \(C = 0\). The proof of proposition is completed by analyzing the products
\[e'_{i} \circ e'_{i} = e'_{i+1} \quad \text{and} \quad e'_1 \circ f'_i = f'_{i+1}.\]
Theorem 3.1. Let $A$ be an $n$-dimensional ($n \geq 2p + 2$) Zinbiel algebra of type I and with characteristic sequence equal to $(n - p, p)$. Then it is isomorphic to one of the following nonisomorphic algebras:

$$A_1:\left\{\begin{array}{ll}
e_i \circ e_j = C^j_{i+j-1}e_{i+j}, & 2 \leq i + j \leq n - p, \\
e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p, \\
f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p,
\end{array}\right.$$ 

where $\beta_{i+1} = \frac{i}{k+1} \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1}$ for $1 \leq i \leq p - 2$ and $\beta_1 \in \mathbb{C}$,

$$A_2:\left\{\begin{array}{ll}
e_i \circ e_j = C^j_{i+j-1}e_{i+j}, & 2 \leq i + j \leq n - p, \\
e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p, \\
f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p,
\end{array}\right.$$ 

where $\beta_i = (-1)^i C^i_{p-2}$ for $1 \leq i \leq p - 2$, and

$$A_3:\left\{\begin{array}{ll}
e_i \circ e_j = C^j_{i+j-1}e_{i+j}, & 2 \leq i + j \leq n - p, \\
e_i \circ f_j = f_i \circ e_j = f_i \circ f_j = C^j_{i+j-1}f_{i+j}, & 2 \leq i + j \leq p.
\end{array}\right.$$ 

Proof. By virtue of Proposition 3.4 with $\beta_1 \neq 1$, we get the following multiplication table for the algebra:

$$e_i \circ e_j = C^j_{i+j-1}e_{i+j}, \quad 2 \leq i + j \leq n - p,$$

$$f_1 \circ f_i = \delta_i f_{i+1}, \quad 1 \leq i \leq p - 1, \quad f_i \circ f_j = \varphi(\delta_1, \delta_2, \ldots, \delta_s)f_{i+j}, \quad 2 \leq i + j \leq p,$$

$$e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, \quad 2 \leq i + j \leq p,$$

where $(p - 1 + \beta_1)\gamma_p = 0$ and $(p - 2 + \beta_1)\delta_{p-1} = 0$.

Consider

$$(f_1 \circ f_p) \circ e_1 = f_1 \circ (f_p \circ e_1) + f_1 \circ (e_1 \circ f_p) = 0.$$ 

On the other hand,

$$(f_1 \circ f_p) \circ e_1 = \gamma_p e_{p+1} \circ e_1 = (p + 1)\gamma_p e_{p+2}.$$ 

Therefore, we conclude that $\gamma_p = 0$. 
Applying Proposition 3.5 to the general change of the basis, by virtue of the equalities

\[ f'_1 \circ f'_{p-1} = (Df_1) \circ (A^{p-2}Df_{p-1}) = A^{p-2}D^2 \delta_{p-1}f_p \]

\[ = \delta'_{p-1}f'_p = \delta'_{p-1}(A^{p-1}D + A^{p-2}BD\delta_{p-1})f_p, \]

we get

\[ \delta'_{p-1} = \frac{D\delta_{p-1}}{A + B\delta_{p-1}}. \]

If \( \delta_{p-1} = 0 \), then \( \delta'_{p-1} = 0 \), and we obtain the algebra \( A_1 \).

Further, if \( \delta_{p-1} \neq 0 \), then, by choosing \( D = \frac{A + B\delta_{p-1}}{\delta_{p-1}} \) and using \( (p - 2 + \beta_1)\delta_{p-1} = 0 \), we obtain

\[ \delta'_{p-1} = 1 \quad \text{and} \quad \beta_1 = 2 - p, \]

i.e., we get the algebra \( A_2 \).

In case \( \beta_1 = 1 \), we have \( \delta_i = \delta_1, \ 1 \leq i \leq p - 1 \). Setting \( D = \frac{A + B\delta_1}{\delta_1} \), we find

\[ \delta'_1 = 1. \]

Consequently, \( f_1 \circ f_i = f_{i+1} \) for \( 1 \leq i \leq p - 1 \). From Lemma 2.2, we deduce

\[ f_i \circ f_j = C_{i+j-1}^{j}f_{i+j}, \quad 2 \leq i + j \leq p. \]

Thus, we get the algebra \( A_3 \).

In the following theorem, we present the classification for \( n = 2p + 1 \).

**Theorem 3.2.** Let \( A \) be a Zinbiel algebra of type I and with characteristic sequence equal to \( (p+1, p) \). Then it is isomorphic to one of the following nonisomorphic algebras:

\[ A_4: \begin{cases}
  e_i \circ e_j = C_{i+j-1}^{j}e_{i+j}, & 2 \leq i + j \leq p + 1, \\
  e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1}\beta_k f_{i+j}, & 2 \leq i + j \leq p, \\
  f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2}\beta_k f_{i+j}, \\
  f_1 \circ f_p = e_{p+1},
\end{cases} \]

where \( \beta_0 = 1 \) and \( \beta_i = (-1)^iC_{p-1}^{i} \) for \( 1 \leq i \leq p - 1 \);

\[ A_5: \begin{cases}
  e_i \circ e_j = C_{i+j-1}^{j}e_{i+j}, & 2 \leq i + j \leq p + 1, \\
  e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1}\beta_k f_{i+j}, & 2 \leq i + j \leq p, \\
  f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2}\beta_k f_{i+j}, \\
  \text{where } \beta_0 = 1, \ \beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1} \text{ for } 1 \leq i \leq p - 2, \text{ and } \beta_1 \in \mathbb{C};
\end{cases} \]
where $\beta_0 = 1$, $\beta_i = (-1)^i C_{p-2}^i$ for $1 \leq i \leq p-2$, and $\beta_{p-1} = 0$; and

\[
A_7:\begin{cases}
e i \circ e_j = C_{i+j-1}^i e_{i+j}, & 2 \leq i + j \leq p + 1, \\
e i \circ f_j = f_i \circ e_j = C_{i+j-1}^i f_{i+j}, & 2 \leq i + j \leq p, \\
f_i \circ f_j = \gamma_1 C_{i+j-1}^i e_{i+j} + \delta_1 C_{i+j-1}^i f_{i+j}, & 2 \leq i + j \leq p, \\
f_i \circ f_j = \gamma_1 C_{i+j-1}^i e_{p+1}, & i + j = p + 1,
\end{cases}
\]

where $\gamma_1, \delta_1 \in \mathbb{C}$.

**Proof.** By using Proposition 3.4 with $\beta_1 \neq 1$, we obtain the multiplication table for $A$:

\[
e i \circ e_j = C_{i+j-1}^i e_{i+j}, & 2 \leq i + j \leq p + 1, \\
e_i \circ f_j = f_i \circ e_j = C_{i+j-1}^i f_{i+j}, & 2 \leq i + j \leq p, \\
f_i \circ f_j = \delta_{p-1} f_p, & f_1 \circ f_p = \gamma_p e_{p+1}, \\
e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{i-1} \beta_k f_{i+j}, & \quad f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i+j-2} \beta_k f_{i+j}, & 2 \leq i + j \leq p,
\]

where

\[
\beta_0 = 1, \quad (p - 1 + \beta_1) \gamma_p = 0 \quad \text{and} \quad (p - 2 + \beta_1) \delta_{p-1} = 0.
\]

As above, we consider the general change of basis. Thus, it follows from Proposition 3.5 that

\[
e_1' = A^p e_p + \left( A^{p-1} B \sum_{k=0}^{p-1} \beta_k + A^{p-2} B^2 \sum_{k=0}^{p-2} \beta_k \delta_{p-1} \right) f_p,
\]

\[
e_{p+1}' = \left( A^{p+1} + \left( A^{p-1} B^2 \sum_{k=0}^{p-1} \beta_k + A^{p-2} B^3 \sum_{k=0}^{p-2} \beta_k \delta_{p-1} \right) \gamma_p \right) e_{p+1},
\]

\[
f_p' = \left( A^{p-1} D + A^{p-2} B D \delta_{p-1} \right) f_p.
\]

In the new basis, the equality $e_1' \circ f_p' = 0$ implies that $B \gamma_p = 0$. 
Case 1. Let $\gamma_p \neq 0$. Then $B = 0$ and $\beta_1 = 1 - p$, $\delta_{p-1} = 0$.
From the equality $f'_p \circ f'_p = \gamma'_p e'_{p+1}$, we derive
$$A^2 \gamma'_p = D^2 \gamma'_p.$$ Setting $D = \frac{A}{\sqrt{\gamma'_p}}$, we obtain $\gamma'_p = 1$. Thus, we get the algebra $A_4$.

Case 2. Let $\gamma_p = 0$. Then, in view of the equality $f'_p \circ f'_{p-1} = \delta'_{p-1} f'_p$, we deduce
$$\delta'_{p-1} = \frac{D \delta_{p-1}}{A + B \delta_{p-1}}.$$ If $\delta_{p-1} = 0$, then $\delta'_{p-1} = 0$, i.e., we arrive at the algebra $A_5$.
If $\delta_{p-1} \neq 0$, then $\beta_1 = 2 - p$. Thus, setting $D = \frac{A + B \delta_{p-1}}{\delta_{p-1}}$, we get $\delta'_{p-1} = 1$ and the algebra $A_6$.

We now consider the case $\beta_1 = 1$. By using Proposition 3.4, we obtain the algebra $A_7$.

We now present the classification of Zinbiel algebras with characteristic sequence equal to $C(A) = (p, p)$.

**Theorem 3.3.** Let $A$ be a Zinbiel algebra with characteristic sequence $(p, p)$. Then it is isomorphic to one of the following nonisomorphic algebras:

- **$A_8$:**
  
  \[
  \begin{cases}
  e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \\
  e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p, \\
  f_i \circ e_j = \sum_{k=0}^{j-1} C^i_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p,
  \end{cases}
  \]

  where $\beta_0 = 1$, $\beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1}$ for $1 \leq i \leq p - 2$, and $\beta_1 \in \mathbb{C}$;

- **$A_9$:**
  
  \[
  \begin{cases}
  e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \\
  e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p, \\
  f_i \circ e_j = \sum_{k=0}^{j-1} C^i_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p,
  \end{cases}
  \]

  where $\beta_0 = 1$, $\beta_i = (-1)^i \sigma_{p-2}$ for $1 \leq i \leq p - 2$, and $\beta_{p-1} = 0$;

- **$A_{10}$:**
  
  \[
  \begin{cases}
  e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \\
  e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p, \\
  f_i \circ e_j = \sum_{k=0}^{j-1} C^i_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i + j \leq p,
  \end{cases}
  \]

  where $\beta_0 = 1$, $\beta_i = (-1)^i \sigma_{p-2}$ for $1 \leq i \leq p - 2$, $\beta_{p-1} = 0$, and $\delta_{p-1} \in \mathbb{C}$.
CLASSIFICATION OF NATURALLY GRADED ZINBIEL ALGEBRAS WITH CHARACTERISTIC SEQUENCE

A_{11}: e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad e_i \circ f_j = f_i \circ e_j = C_{i+j-1}^j f_{i+j}, \quad 2 \leq i + j \leq p;

A_{12}:
\begin{align*}
e_i \circ e_j &= C_{i+j-1}^j e_{i+j}, & \quad 2 \leq i + j \leq p, \\
e_i \circ f_j &= f_i \circ e_j = f_i \circ f_j = C_{i+j-1}^j f_{i+j}, & \quad 2 \leq i + j \leq p.
\end{align*}

Proof. The proof of this theorem is carried out by applying the same methods and arguments as in the proof of Theorems 3.1 and 3.2.

3.2. Classification of the Zinbiel algebras of type II. Consider a Zinbiel algebra of type II. From the condition imposed on the operator \( L_{e_1} \), we get the existence of a basis \( \{ e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_{n-p} \} \) such that the products containing \( e_1 \) on the left-hand side have the form

\[ e_1 \circ e_i = e_{i+1}, \quad 1 \leq i \leq p - 1. \]

Applying Lemma 2.2, we find

\[ e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i + j \leq p, \quad e_1 \circ e_{p} = 0, \]
\[ e_1 \circ f_i = f_{i+1}, \quad 1 \leq i \leq n - p - 1, \quad e_1 \circ f_{n-p} = 0. \]

It is easy to see that

\[ A_1 = \langle e_1, f_1 \rangle, \quad A_2 = \langle e_2, f_2 \rangle, \ldots, \]
\[ A_p = \langle e_p, f_p \rangle, \quad A_{p+1} = \langle f_{p+1} \rangle, \ldots, \quad A_{n-p} = \langle f_{n-p} \rangle. \]

We now introduce the notation:

\[ f_1 \circ e_i = \alpha_i e_{i+1} + \beta_i f_{i+1}, \quad 1 \leq i \leq p - 1, \quad f_1 \circ e_p = \beta_p f_{p+1}, \]
\[ f_1 \circ f_i = \gamma_i e_{i+1} + \delta_i f_{i+1}, \quad 1 \leq i \leq p - 1, \]
\[ f_1 \circ f_i = \delta_i f_{i+1}, \quad p \leq i \leq n - p - 1, \quad f_1 \circ f_{n-p} = 0. \]

The following proposition can be proved by analogy with Proposition 3.2.

Proposition 3.6. Let \( A \) be a Zinbiel algebra of type II. Then the following restrictions are true for the structural constants \( \alpha_i, \beta_i, \gamma_i \) and \( \delta_i \):

\[ \alpha_{i+1} = 0, \quad 1 \leq i \leq p - 2, \]
\[ \beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1}, \quad 1 \leq i \leq p - 1, \]
\[(i + 1)\gamma_i = \beta_1 \left( 2\gamma_1 + \sum_{k=2}^{i} \gamma_k \right), \quad 1 \leq i \leq p - 2,\]

\[(i + \beta_1)\delta_i = \beta_1 \left( 2\delta_1 + \sum_{k=2}^{i} \delta_k \right), \quad 1 \leq i \leq n - p - 2.\]

**Proposition 3.7.** Let \( A \) be a Zinbiel algebra of type II. Then the following relations hold:

\[e_i \circ f_j = \sum_{k=0}^{i-1} C^i_{i+j-2-k}^j \beta_k f_{i+j}, \quad 1 \leq i \leq p, \quad p + 1 \leq i + j \leq n - p,\]  

\[f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k}^i \beta_k f_{i+j}, \quad 1 \leq j \leq p, \quad p + 1 \leq i + j \leq n - p,\]

where \( \beta_0 = 1. \)

**Proof.** We prove the assertion of the proposition by induction. Clearly, the relation (7) is true for \( i = 1. \)

We have

\[f_2 \circ e_1 = e_1 \circ (f_1 \circ e_1) + e_1 \circ (e_1 \circ f_1) = (1 + \beta_1) f_3.\]

By using the chain of equalities

\[f_i \circ e_1 = (e_1 \circ f_{i-1}) \circ e_1 = e_1 \circ (f_{i-1} \circ e_1) + e_1 \circ (e_1 \circ f_{i-1}),\]

and induction, we conclude that

\[f_i \circ e_1 = (i - 1 + \beta_1) f_{i+1} \quad \text{for} \quad 1 \leq i \leq p - 1\]

and

\[f_i \circ e_1 = (i - 1 + \beta_1) f_{i+1} \quad \text{for} \quad p \leq i \leq n - p - 1.\]

Therefore, relation (8) is true for \( j = 1. \)

Assume that the relations (7), (8) are true for \( i \) and any value of \( j. \) The proof of these relations for \( i + 1 \) follows from the chain of equalities:

\[e_{i+1} \circ f_j = e_1 \circ (e_i \circ f_j) + e_1 \circ (f_j \circ e_i)\]

\[= e_1 \circ \left( \sum_{k=0}^{i-1} C^i_{i+j-2-k}^j \beta_k f_{i+j} + \sum_{k=0}^{i} C^i_{i+j-2-k}^j \beta_k f_{i+j} \right)\]

\[= \sum_{k=0}^{i-1} C^i_{i+j-2-k}^j \beta_k f_{i+j+1} + \sum_{k=0}^{i} C^i_{i+j-2-k}^j \beta_k f_{i+j+1}\]
The remaining relations of the proposition can be checked similarly.

As in the case of Zinbiel algebra of type I, for algebras of type II, we obtain the restrictions for the structure constants related to the parameter $\beta_1$.

**Proposition 3.8.** Let $A$ be a Zinbiel algebra of type II.

If $\beta_1 \neq 1$, then

\[
\gamma_i = 0, \quad 1 \leq i \leq p - 2, \\
\delta_i = 0, \quad 1 \leq i \leq n - p - 2, \\
(p - 2 + \beta_1)\gamma_{p-1} = 0, \\
(n - p - 2 + \beta_1)\delta_{n-p-1} = 0; \\
\]

if $\beta_1 = 1$, then

\[
\beta_i = 1, \quad 1 \leq i \leq p, \\
\gamma_i = \gamma_1, \quad 1 \leq i \leq p - 1, \\
\delta_i = \delta_1, \quad 1 \leq i \leq n - p - 1. \\
\]

In the next theorem, we prove that there are no $n$-dimensional Zinbiel algebras of type II with $n \geq 3p + 2$.

**Theorem 3.4.** There are no Zinbiel algebras of type II with characteristic sequence equal to $(n - p, p)$ for $n \geq 3p + 2$.

**Proof.** For $1 \leq i \leq p + 1$, we consider the equalities

\[
0 = (e_1 \circ e_p) \circ f_i = e_1 \circ (e_p \circ f_i) + e_1 \circ (f_i \circ e_p). \\
\]

Applying relations (7), (8) and the arguments similar to the arguments used in the proof of Proposition 3.1, we obtain the relation

\[
\sum_{k=0}^{p} C_{p+i-1-k}^{i-1} \beta_k = 0, \\
\]

where $\beta_0 = 1$ and $1 \leq i \leq n - p - 1$. 

We now consider the determinant of the matrix of order $p + 1$:

\[
M = \begin{vmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
C_{p+1}^1 & C_p^1 & C_{p-1}^1 & \ldots & C_2^1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
C_{2p-1}^{p-1} & C_{2p-2}^{p-1} & C_{2p-3}^{p-1} & \ldots & C_p^{p-1} & 1 \\
C_{2p}^p & C_{2p-1}^p & C_{2p-2}^p & \ldots & C_{p+1}^p & 1
\end{vmatrix}.
\]

By using the identity $C_m^{n-1} + C_m^n = C_{n+1}^m$ and subtracting the previous row from each row of this determinant, we obtain

\[
M = \begin{vmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
C_p^1 & C_{p-1}^1 & C_{p-2}^1 & \ldots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
C_p^{p-1} & 1 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0
\end{vmatrix} = -1.
\]

Since $M = -1$, the system of equations (9) for $i = p + 1$ possesses solely the trivial solution with respect to unknown variables $\beta_i$. In particular, $\beta_0 = 0$. However, $\beta_0 = 1$, i.e., we arrive at a contradiction with the condition

\[
i = p + 1 \leq n - p - 1,
\]

which means that the algebras with $n \geq 3p + 2$ do not exist.

Let $A$ be an $n$-dimensional algebra with a basis $\{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_{n-p}\}$ and a multiplication table

\[
e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i + j \leq p, \quad e_1 \circ e_p = 0,
\]

\[
e_1 \circ f_{n-p} = 0, \quad f_i \circ f_j = 0, \quad 1 \leq i, \quad j \leq n - p,
\]

\[
e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{j-2} \beta_k e_{i+j}, \quad 2 \leq i + j \leq n - p,
\]

where

\[
\beta_i = (-1)^i C_p^i, \quad 0 \leq i \leq p.
\]

It is easy to see that this algebra is a Zinbiel algebra.

The validity of relation (9) for the parameters $\beta_i$ and $n = 3p + 1$ follows from Lemma 2.1. Thus, the condition $n \geq 3p + 2$ is necessary.
We now present three theorems on the description of Zinbiel algebras of type II without proofs. These theorems can be proved by using the arguments similar to the reasoning used above.

**Theorem 3.5.** A Zinbiel algebra of type II with characteristic sequence equal to \((p + 1, p)\) is isomorphic to one of the following nonisomorphic algebras:

\[
\widetilde{A}_1: \begin{cases} 
    e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \leq i + j \leq p, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{i-1} \beta_k f_{i+j}, & 2 \leq i + j \leq p + 1, \\
    f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, & \end{cases}
\]

where \(\beta_0 = 1, \beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1}\) for \(1 \leq i \leq p - 1,\) and \(\beta_1 \in \{-p, -(p - 1), \ldots, -2, -1\}\);

\[
\widetilde{A}_2: \begin{cases} 
    e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \leq i + j \leq p, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{i-1} \beta_k f_{i+j}, & 2 \leq i + j \leq p + 1, \\
    f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, & \end{cases}
\]

where \(\beta_i = (-1)^i C_{p-2}^i\) for \(0 \leq i \leq p - 2\) and \(\beta_{p-1} = \beta_p = 0;\)

\[
\widetilde{A}_3: \begin{cases} 
    e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \leq i + j \leq p, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{i-1} \beta_k f_{i+j}, & 2 \leq i + j \leq p + 1, \\
    f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, & \end{cases}
\]

where \(\beta_i = (-1)^i C_{p-1}^i\) for \(0 \leq i \leq p - 1\) and \(\beta_p = 0;\)

\[
\widetilde{A}_4: \begin{cases} 
    e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \leq i + j \leq p, \\
    e_i \circ f_j = f_i \circ e_j = C_{i+j-1}^j f_{i+j}, & 2 \leq i + j \leq p + 1; \\
\end{cases}
\]

\[
\widetilde{A}_5: \begin{cases} 
    e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \leq i + j \leq p, \\
    e_i \circ f_j = f_i \circ e_j = C_{i+j-1}^j f_{i+j}, & 2 \leq i + j \leq p + 1, \\
    f_i \circ f_j = C_{i+j-1}^j f_{i+j}, & \end{cases}
\]
Theorem 3.6. A Zinbiel algebra of type II with characteristic sequence equal to \((p + 2, p)\) is isomorphic to one of the following nonisomorphic algebras:

\[
\tilde{A}_6: \begin{cases} 
    e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq i \leq p, \quad 2 \leq i + j \leq p + 2, \\
    f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq j \leq p, \quad 2 \leq i + j \leq p + 2,
\end{cases}
\]

where \(\beta_0 = 1, \beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1} \) for \(1 \leq i \leq p - 1,\) and \(\beta_1 \in \{-p, -(p - 1), \ldots, -2, -1\}\);

\[
\tilde{A}_7: \begin{cases} 
    e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \quad f_1 \circ f_{p-1} = e_p, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq i \leq p, \quad 2 \leq i + j \leq p + 2, \\
    f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq j \leq p, \quad 2 \leq i + j \leq p + 2,
\end{cases}
\]

where \(\beta_i = (-1)^i C^i_{p-2} \) for \(0 \leq i \leq p - 2\) and \(\beta_{p-1} = \beta_p = 0;\)

\[
\tilde{A}_8: \begin{cases} 
    e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \quad f_1 \circ f_{p+1} = f_{p+2}, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq i \leq p, \quad 2 \leq i + j \leq p + 2, \\
    f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq j \leq p, \quad 2 \leq i + j \leq p + 2,
\end{cases}
\]

where \(\beta_i = (-1)^i C^i_{p} \) for \(0 \leq i \leq p.\)

Theorem 3.7. A Zinbiel algebra of type II with characteristic sequence equal to \((p + t, p), 3 \leq t \leq p + 1,\) is isomorphic to one of the following nonisomorphic algebras:

\[
\tilde{A}_9: \begin{cases} 
    e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq i \leq p, \quad 2 \leq i + j \leq p + t, \\
    f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq j \leq p, \quad 2 \leq i + j \leq p + t,
\end{cases}
\]

where \(\beta_0 = 1, \beta_{i+1} = \prod_{k=0}^{i} \frac{k + \beta_1}{k + 1} \) for \(1 \leq i \leq p - 1,\) and \(\beta_1 \in \{-p, -(p - 1), \ldots, -(t - 1)\}\);

\[
\tilde{A}_{10}: \begin{cases} 
    e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i + j \leq p, \quad f_1 \circ f_{p-1} = e_p, \\
    e_i \circ f_j = \sum_{k=0}^{i-1} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq i \leq p, \quad 2 \leq i + j \leq p + t, \\
    f_i \circ e_j = \sum_{k=0}^{j} C^j_{i+j-2-k} \beta_k f_{i+j}, & 1 \leq j \leq p, \quad 2 \leq i + j \leq p + t,
\end{cases}
\]

where \(\beta_i = (-1)^i C^i_{p-2} \) for \(0 \leq i \leq p - 2\) and \(\beta_{p-1} = \beta_p = 0.\)
The present work was partially supported by the Agencia Estatal de Investigación (Spain), Grant MTM2016-79661-P (European FEDER support included, UE) and by the Xunta de Galicia, Grant ED431C2019/10 (European FEDER support included).

REFERENCES

1. J. Q. Adashev, “Description of $n$-dimensional Zinbiel algebras of nilindex $k$ with $n - 2 \leq k \leq n + 1$, Ph. D. Thesis, Uzbekistan (2011).
2. J. Q. Adashev, L. M. Camacho, S. Gómez-Vidal, and I. A. Karimjanov, “Naturally graded Zinbiel algebras with nilindex $n - 3$,” *Linear Algebra Appl.*, **443**, 86–104 (2014).
3. J. Q. Adashev, A. K. Khudoyberdiyev, and B. A. Omirov, “Classifications of some classes of Zinbiel algebras,” *J. Gen. Lie Theory Appl.*, **4**, Art. ID S090601, 10 p. (2010)
4. J. Q. Adashev, A. K. Khudoyberdiyev, and B. A. Omirov, “Classification of complex naturally graded quasifiliform Zinbiel algebras,” *Contemp. Math.*, **483**, 1–11 (2009).
5. Sh. A. Ayupov and B. A. Omirov, “On some classes of nilpotent Leibniz algebras,” *Sib. Math. J.*, **42**, No. 1, 18–29 (2001); *English translation*: *Sib. Math. J.*, **42**, No. 1, 15–24 (2001).
6. J. M. Cabezas, L. M. Camacho, J. R. Gómez, and B. A. Omirov, “On the description of Leibniz algebras with nilindex $n - 3$,” *Acta Math. Hungar.*, **133**, No. 3, 203–220 (2011).
7. L. M. Camacho, E. M. Cañete, J. R. Gómez, and S. B. Redjepov, “Leibniz algebras of nilindex $n - 3$ with characteristic sequence $(n - 3, 2, 1)$,” *Linear Algebra Appl.*, **438**, No. 4, 1832–1851 (2013).
8. L. M. Camacho, E. M. Cañete, S. Gómez-Vidal, and B. A. Omirov, “$p$-Filiform Zinbiel algebras,” *Linear Algebra Appl.*, **438**, No. 7, 2958–2972 (2013).
9. L. M. Camacho, J. R. Gómez, A. J. González, and B. A. Omirov, “Naturally graded quasifiliform Leibniz algebras,” *J. Symbolic Comput.*, **44**, No. 5, 527–539 (2009).
10. L. M. Camacho, J. R. Gómez, A. J. González, and B. A. Omirov, “Naturally graded 2-filiform Leibniz algebras,” *Comm. Algebra*, **38**, No. 10, 3671–3685 (2010).
11. L. M. Camacho, J. R. Gómez, A. J. González, and B. A. Omirov, “The classification of naturally graded $p$-filiform Leibniz algebras,” *Comm. Algebra*, **39**, No. 1, 153–168 (2011).
12. A. S. Dzhumadil’daev and K. M. Tulenbaev, “Nilpotency of Zinbiel algebras,” *J. Dyn. Control Syst.*, **11**, No. 2, 195–213 (2005).
13. A. O. Gel’fond, *Calculus of Finite Differences*, Hindustan Publ. Corp., Delhi (1971).
14. J. R. Gómez and B. A. Omirov, “On classification of filiform Leibniz algebras,” *Algebra Colloq.*, **22**, Spec. Issue 1, 757–774 (2015).
15. J.-L. Loday, “Cup-product for Leibniz cohomology and dual Leibniz algebras,” *Math. Scand.*, **77**, No. 2, 189–196 (1995).
16. K. K. Masutova, B. A. Omirov, and A. K. Khudoyberdiyev, “Naturally graded Leibniz algebras with characteristic sequence $(n - m, m)$,” *Math. Notes*, **93**, No. 5–6, 740–755 (2013).