Some integrals involving the Stieltjes constants

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Abstract

Some new integrals involving the Stieltjes constants are developed in this paper.

1. Introduction

Using the binomial theorem we obtain

\[
\int_0^\infty e^{-ax} (1-e^{-x})^n x^{-p-1} \, dx = \int_0^\infty e^{-ax} \sum_{j=0}^n \binom{n}{j} (-1)^j e^{-jx} \, dx
\]

\[
= \sum_{j=0}^n \binom{n}{j} (-1)^j \int_0^\infty e^{-(u+j)x} x^{-p-1} \, dx
\]

and employing the definition of the gamma function for Re\( (p) > 0 \)

\[
\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} \, dx
\]

we see for Re\( (u) > 0 \) that

\[
\frac{\Gamma(p)}{u^p} = \int_0^\infty e^{-ux} x^{p-1} \, dx
\]

Hence for Re\( (u) > 0 \) and Re\( (p) > 0 \) we have

\[
(1.1) \quad \int_0^\infty e^{-ax} (1-e^{-x})^n x^{-p-1} \, dx = \Gamma(p) \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{(u+j)^p}
\]

With \( n = 0 \) in (1.1) we have \( \int_0^\infty e^{-ux} x^{p-1} \, dx = \frac{\Gamma(p)}{u^p} \) and therefore (1.1) is also valid for \( n = 0 \). The integral (1.1) appears in Gradshteyn and Ryzhik [16, p.357, Eq. 3.432.1] where it is attributed to Bierens de Haan.
As \( p \to 0 \) the right-hand side of (1.1) is indeterminate but we may use L’Hôpital’s rule when considering the ratio

\[
\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{(u + j)^p} \frac{1}{\Gamma(p)}
\]

(since \( \sum_{j=0}^{n} \binom{n}{j} (-1)^j = \delta_{n,0} \), we require that \( n \neq 0 \) in order to ensure that the numerator is equal to zero as \( p \to 0 \)).

As regards the numerator, it is easily seen that

\[
\frac{d}{dp} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{(u + j)^p} = -\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j \log(u + j)}{(u + j)^p}
\]

We have using Euler’s reflection formula for the gamma function

\[
\frac{d}{dp} \left[ \frac{1}{\Gamma(p)} \right] = \frac{1}{\pi} \frac{d}{dp} \left[ \Gamma(1 - p) \sin \pi p \right]
\]

\[
= \Gamma(1 - p) \cos \pi p - \frac{1}{\pi} \Gamma'(1 - p) \sin \pi p
\]

and therefore we see that where \( r \) is a positive integer

\[
\lim_{p \to 1-r} \frac{d}{dp} \left[ \frac{1}{\Gamma(p)} \right] = (-1)^{r+1} \Gamma(r)
\]

This gives us

\[
(1.2) \quad \int_{0}^{\infty} \frac{e^{-ux}(1-e^{-x})^n}{x^r} \, dx = \frac{(-1)^r}{\Gamma(r)} \sum_{j=0}^{n} \binom{n}{j} (-1)^j (u + j)^{r-1} \log(u + j)
\]

The above integral was originally given by Anglesio [3] in 1997. [Could one adopt a Eulerian approach at this stage and assume that \( r \) was a continuous variable with the representation \((-1)^r = \cos \pi r ?\)]

We now let \( u = 1 \) and \( r = 1 \) to obtain
(1.3) \[
\int_0^\infty \frac{e^{-x} (1-e^{-x})^n}{x} \, dx = -\sum_{j=0}^n \binom{n}{j} (-1)^j \log(1+j)
\]

Making the substitution \( y = e^{-x} \), equation (1.3) becomes for \( n \geq 1 \)

(1.4) \[
\int_0^1 \frac{(1-y)^n}{\log y} \, dy = \sum_{j=0}^n \binom{n}{j} (-1)^j \log(1+j)
\]

and this is in agreement with G&R [16, p.541, Eqn. 4.267.1] as corrected by Boros and Moll (see the Errata to G&R [16] dated 26 April 2005).

Making the same substitution \( y = e^{-x} \) in (1.2) we see that for \( n \geq 1 \)

(1.5) \[
\int_0^1 \frac{y^{n-1} (1-y)^n}{\log y} \, dy = \frac{1}{\Gamma(r)} \sum_{j=0}^n \binom{n}{j} (-1)^j (u+j)^{-r} \log(u+j)
\]

and we have the summation (where we are required to start at \( n = 1 \))

\[
\sum_{n=1}^\infty \frac{1}{n+1} \int_0^1 \frac{y^{n-1} (1-y)^n}{\log y} \, dy = \frac{1}{\Gamma(r)} \sum_{n=1}^\infty \frac{1}{n+1} \sum_{j=0}^n \binom{n}{j} (-1)^j (u+j)^{-r} \log(u+j)
\]

Using the logarithmic expansion

\[
\sum_{n=1}^\infty \frac{(1-y)^n}{n+1} = -\frac{1}{1-y} [\log y + 1 - y]
\]

we therefore obtain

(1.6) \[
\int_0^1 \left[ \frac{1}{\log y} + \frac{1}{1-y} \right] \frac{y^{n-1}}{\log y} \, dy = -\frac{1}{\Gamma(r)} \sum_{n=1}^\infty \frac{1}{n+1} \sum_{j=0}^n \binom{n}{j} (-1)^j (u+j)^{-r} \log(u+j)
\]

Starting the summation at \( n = 0 \) gives us

(1.7) \[
\int_0^1 \left[ \frac{1}{\log y} + \frac{1}{1-y} \right] \frac{y^{n-1}}{\log y} \, dy = \frac{1}{\Gamma(r)} u^{-r} \log u - \frac{1}{\Gamma(r)} \sum_{n=0}^\infty \frac{1}{n+1} \sum_{j=0}^n \binom{n}{j} (-1)^j (u+j)^{-r} \log(u+j)
\]

and reference to the Hasse identity [18]
\[(s - 1)\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right) \frac{(-1)^j}{(u+j)^{s-1}}\]

shows that the right-hand side of (1.7) is related to \(\zeta'(2-r,u)\).

With \(r = 1\) we get

\[(1.8) \int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{u-1} \, dy = \log u - \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right) (-1)^j \log(u+j)\]

and it is known from [17] and [12] that the digamma function \(\psi(u)\) may be represented by

\[(1.9) \psi(u) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right) (-1)^j \log(u+j)\]

Equation (1.8) then becomes

\[(1.10) \int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{u-1} \, dy = \log u - \psi(u)\]

This integral is due to Binet [22, p.175] and with \(u = 1\) in (1.10) we obtain the well-known integral for Euler’s constant [6, p.178]

\[(1.11) \gamma = \int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \, dy\]

The identity (1.9) may also be obtained in the following way by employing a method due to Dirichlet [25]. We see that

\[
\frac{\Gamma(u+h) - \Gamma(u)}{h} = \frac{\Gamma(u+h)}{\Gamma(1+h)} \Gamma(h) - \frac{\Gamma(u+h)}{\Gamma(1+h)} \Gamma(u) \Gamma(h) - \frac{\Gamma(u+h)}{\Gamma(1+h)} [\Gamma(h) - B(u,h)]
\]

\[
= \frac{\Gamma(u+h)}{\Gamma(1+h)} \int_0^1 \left[ \frac{1}{\log^{1-h}(1/y)} - \frac{y^{u-1}}{(1-y)^{-h}} \right] \, dy
\]

and in the limit as \(h \to 0\) we have
\[ \Gamma'(u) = -\Gamma(u) \int_{0}^{1} \left( \frac{1}{\log y} + \frac{u^{-1}}{1 - y} \right) \, dy \]

Therefore we have

\[ \psi(u) = -\int_{0}^{1} \left( \frac{1}{\log y} + \frac{u^{-1}}{1 - y} \right) \, dy \] (1.12)

We see that

\[ \frac{1}{u} = \int_{0}^{1} y^{u-1} \, dy \]

and integration of this results in the well-known Frullani integral [8, p.472]

\[ \log u = \int_{0}^{1} \left[ \frac{y^{u-1}}{\log y} - \frac{1}{\log y} \right] \, dy \] (1.13)

We then have

\[ \log u - \psi(u) = \int_{0}^{1} \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] y^{u-1} \, dy \]

and, comparing this with (1.8), results in (1.9).

Making the substitution \( y = e^{-x} \) in (1.10) we see that

\[ I = \int_{0}^{1} \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] y^{u-1} \, dy = \int_{0}^{\infty} \left[ \frac{1}{1 - e^{-x}} - \frac{1}{x} \right] e^{-ux} \, dx \]

and noting that

\[ \frac{1}{1 - e^{-x}} = \frac{e^x}{e^x - 1} = 1 + \frac{1}{e^x - 1} \]

we obtain

\[ I = \frac{1}{u} + \int_{0}^{\infty} \left[ \frac{1}{e^x - 1} - \frac{1}{x} \right] e^{-ux} \, dx \]

Therefore we have from (1.10) [26, p.15]
Integrating (1.14) with respect to $u$ over the interval $[0, t]$ gives us a Binet-type integral representation for $\log \Gamma(u)$

$$
(1.15) \quad \int_0^\infty \left[ \frac{1}{e^x - 1} - \frac{1}{x} \right] e^{-tx} \, dx = \frac{1}{t} \log t - t \log(1 + t)
$$

Alternatively, integrating (1.14) with respect to $u$ over the interval $[1, t]$ gives us

$$
(1.16) \quad \int_0^\infty \left[ \frac{1}{e^x - 1} - \frac{1}{x} \right] e^{tx} e^{-tx} \, dx = \frac{1}{t} \log t + 1 - \log(1 + t)
$$

which recently appeared in [15].

Further integrations would in turn develop integrals for the Barnes multiple gamma functions defined in [26, p.24]. This is considered further in [12].

Further consideration of integrals such as (1.10) is contained in Appendix B to this paper.

2. Some integrals involving the Stieltjes constants

We now boldly assume that (1.6) is also valid in the case where $r$ is a continuous variable (perhaps Carlson’s Theorem [27, p.185] could be utilised to provide a rigorous proof: further advice on this aspect would be appreciated). In this regard, it may be noted that Adamchik [1] made a similar assumption in making the transition from equation (23) to equation (24) in his 1997 paper, “A class of logarithmic integrals”.

Differentiation of (1.6) with respect to $r$ results in

$$
(2.1) \quad \int_0^1 \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] y^{n-1} \log \left| \frac{\log y}{\log (r^{-1} y)} \right| \, dy
$$

$$
= \frac{1}{\Gamma(r)} \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \left( \frac{n}{j} \right) (-1)^j (u + j)^{r-1} \log^2(u + j) - \frac{\psi(r)}{\Gamma(r)} \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \left( \frac{n}{j} \right) (-1)^j (u + j)^{r-1} \log(u + j)
$$

Letting $r = 1$ results in

$$
\int_0^1 \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] y^{n-1} \log |\log y| \, dy
$$
\[
\begin{align*}
&= \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \log^2(u+j) + \gamma \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \log(u+j)
\end{align*}
\]

and starting the summation at \( n = 0 \) gives us

\[
\int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{n-1} \log |\log y| dy = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \log^2(u+j)
\]

\[
+ \gamma \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \log(u+j)) - \log^2 u - \gamma \log u
\]

We have previously shown in [13] that for integer \( p \geq 0 \)

\[
(2.2) \quad \gamma_p(u) = -\frac{1}{p+1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \log^{p+1}(u+j)
\]

where the Stieltjes constants \( \gamma_n(u) \) are the coefficients in the Laurent expansion of the Hurwitz zeta function \( \zeta(s,u) \) about \( s = 1 \)

\[
(2.3) \quad \zeta(s,u) = \sum_{n=0}^{\infty} \frac{1}{(n+u)^s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(u)(s-1)^n
\]

and \( \gamma_0(u) = -\psi(u) \), where \( \psi(u) \) is the digamma function which is the logarithmic derivative of the gamma function \( \psi(u) = \frac{d}{du} \log \Gamma(u) \). It is easily seen from the definition of the Hurwitz zeta function that \( \zeta(s,1) = \zeta(s) \) and accordingly that \( \gamma_n(1) = \gamma_n \).

Hence we obtain

\[
(2.4) \quad \int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{n-1} \log |\log y| dy = -2\gamma_1(u) - \gamma_0(u) - \log^2 u - \gamma \log u
\]

Letting \( u = 1 \) results in (however see (B.30) in Appendix B)

\[
(2.5) \quad \int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log |\log y| dy = -(2\gamma_1 + \gamma^2)
\]

We define \( f(y) \) as
Using L'Hôpital's rule we can easily deduce that \( \lim_{y \to 0} f(y) = 1 \) and \( \lim_{y \to 1} f(y) = -1/2 \) and, since \( f'(y) < 0 \), we note that \( f(y) \) is monotonically decreasing on the interval \([0,1]\). However, the range of \( \log|\log y| \) is \((-\infty, \infty)\) and hence it is not straightforward to determine the sign of the integral in (2.5). Another approach is therefore necessary.

With the substitution \( y = 1/t \) in (2.5) we get

\[
\int_0^1 \left[ \frac{1}{1-t} + \frac{1}{\log y} \right] \log|\log y| dy = \int_1^\infty \left[ \frac{1}{t-1} - \frac{1}{t \log t} \right] \frac{\log|\log(1/t)|}{t} dt
\]

We denote \( g(t) \) by

\[
g(t) = \frac{1}{t-1} - \frac{1}{t \log t} = \frac{t \log t - (t-1)}{t(t-1) \log t} = \frac{\log t - (1-1/t)}{(t-1) \log t}
\]

and using L'Hôpital’s rule we obtain

\[
\lim_{t \to 1} g(t) = 1/2 \quad \lim_{t \to \infty} g(t) = 0
\]

Since \( \log t > 1-1/t \) we see that \( g(t) > 0 \) for \( t \in [1, \infty) \)

We see that \( h(t) = \frac{\log|\log(1/t)|}{t} \to -\infty \) as \( t \to 1 \) and using L'Hôpital’s rule we have

\[
\frac{\log|\log(1/t)|}{t} \to 0 \quad \text{as} \quad t \to \infty
\]

We have the derivative

\[
h'(t) = \frac{1}{\log(1/t) \left| \log(1/t) \right|} - \log|\log(1/t)|
\]

Since \( \log t > 1-1/t \) for \( t > 0 \) we also have
\[ \log|\log(1/t)| > 1 - \frac{1}{\log(1/t)} \quad \text{for } |\log(1/t)| > 0 \]

and therefore \( h'(t) \) is negative with the result that \( h(t) \) is monotonic decreasing and negative for \( t \in [1, \infty) \). We accordingly deduce that the integrand in (2.5) is negative and that the integral is also negative. This then enables us to conclude that

\[ 2\gamma_1 + \gamma^2 > 0 \]

The constants \( \eta_k \) are defined by the following Maclaurin expansion

\[
\log[(s-1)\zeta(s)] = -\sum_{k=0}^{\infty} \frac{\eta_k}{k+1} (s-1)^{k+1} = -\sum_{k=1}^{\infty} \frac{\eta_{k-1}}{k} (s-1)^k
\]

where

\[ \eta_0 = -\gamma \]

\[ \eta_1 = 2\gamma_1 + \gamma^2 \]

We note the approximate numerical values of the first three Stieltjes constants from [2]

\[ \gamma = 0.5772\cdots \quad \gamma_1 = -0.0728\cdots \quad \gamma_1 = -0.0096\cdots \]

Coffey [9a] has shown that the sequence \( (\eta_n) \) has strict sign alteration, i.e.

\[ \eta_n = (-1)^{n+1} \varepsilon_n \]

where \( \varepsilon_n \) are positive constants. It may be noted that the above analysis has confirmed, without performing any numerical calculations, that \( \eta_1 \) is indeed positive.

We see that \( \frac{\log^{2n}|\log(1/t)|}{t} > 0 \) and with \( j(t) = \frac{\log^{2n+1}|\log(1/t)|}{t} \) we see that

\[
j'(t) = -\frac{(2n+1)\log^{2n}|\log(1/t)| \frac{1}{|\log(1/t)|} + \log^{2n+1}|\log(1/t)|}{t^2}
\]

\[
= -\frac{\log^{2n}|\log(1/t)|}{t^2} \left( \frac{(2n+1) + \log|\log(1/t)|}{|\log(1/t)|} \right)
\]
Since \( \log t > 1 - 1/t \) for \( t > 0 \) we also have

\[
\log^{2n+1} t > (1 - 1/t)^{2n+1} > 1 - \frac{2n+1}{t}
\]

where, in the last step, we have employed Bernoulli’s inequality. We then have

\[
\log^{2n+1} \left| \log(1/t) \right| > 1 - \frac{2n+1}{\log(1/t)} \quad \text{for} \quad \left| \log(1/t) \right| > 0
\]

and conclude that \( j'(t) \) is negative.

We have therefore determined that

\[
\left(2.6\right) \quad \int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log y \, dy = \int_1^\infty \left[ \frac{1}{t-1} - \frac{1}{t \log t} \right] \frac{\log^2 \left| \log(1/t) \right|}{t} \, dt = (-1)^n d_n
\]

where \( d_n \) are positive constants. This relationship is used in (3.15) below.

For convenience, we now write (2.1) as

\[
\left(2.7\right) \quad \int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log^{\nu-1} \left| \log y \right| \log^r y \, dy = \frac{1}{\Gamma(r)} S(r,u,2) - \frac{\psi(r)}{\Gamma(r)} \frac{S(r,u,1)}{r}
\]

where \( S(r,u,p) \) is defined as

\[
S(r,u,p) = \sum_{n=0}^\infty \frac{1}{n+1} \sum_{j=0}^n \left( \begin{array}{c} n \\ j \end{array} \right) (-1)^j (u+j)^{-r-1} \log^p (u+j)
\]

and note that \( \frac{d}{dr} S(r,u,p) = S(r,u,p+1) \).

Differentiation of (2.7) with respect to \( r \) gives us

\[
\left(2.8\right) \quad -\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log^{\nu-1} \left| \log y \right| \log^r y \, dy = \frac{1}{\Gamma(r)} S(r,u,3) - \frac{\psi(r)}{\Gamma(r)} S(r,u,2)
\]

\[
-\frac{\psi'(r) - \left[ \psi(r) \right]^2}{\Gamma(r)} S(r,u,1)
\]
We additionally define \( S_0(r, u, p) \) by

\[
S_0(r, u, p) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j (u + j)^{r-1} \log^p (u + j)
\]

where the summation starts at \( n = 0 \) and we note that

\[
S_0(r, u, p) = S(r, u, p) + u^{r-1} \log^p u
\]

and

\[
S_0(r,1, p) = S(r,1, p)
\]

We also see from (2.2) with \( r = 1 \) that

\[
S(1, u, p+1) = -(p+1)\gamma_p(u) - \log^{p+1} u
\]

and

\[
S(1,1, p+1) = -(p+1)\gamma_p
\]

We then have with \( r = 1 \) in (2.8)

\[
-\int_0^1 \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] y^{u-1} \log^2 |\log y| dy = S(1,u,3) - 2\psi(1)S(1,u,2) - \left( \psi'(1) - \left[ \psi(1) \right]^2 \right) S(1,u,1)
\]

which is equivalent to

\[
\int_0^1 \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] y^{u-1} \log^2 |\log y| dy = 3\gamma_2(u) + \log^3 u + 2\gamma \left[ 2\gamma_1(u) + \log^2 u \right] - [\zeta(2) - \gamma^2] [\gamma(u) + \log u]
\]

Therefore we get with \( u = 1 \)

\[
\int_0^1 \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] \log^2 |\log y| dy = 3 \gamma_2 + 4 \gamma_1 - [\zeta(2) - \gamma^2] \gamma
\]

Therefore we see that

\[
3 \gamma_2 + 4 \gamma_1 - [\zeta(2) - \gamma^2] \gamma > 0
\]
Coffey [9b] has given the following recurrence relation for the \( \eta_n \) constants
\[
\eta_n = (-1)^{n+1} \left[ \frac{n+1}{n!} \gamma_n' + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(n-k-1)!} \gamma_{n-k-1}' \eta_k \right]
\]
from which we obtain
\[
(2.12) \quad \eta_2 = -\left[ \frac{3}{2} \gamma_2 - \eta_0 \gamma_1 + \eta_1 \gamma \right] = -\left[ \frac{3}{2} \gamma_2 + 3 \gamma_1 + \gamma^3 \right]
\]
We then see that
\[
\left[ \frac{3}{2} \gamma_2 + 3 \gamma_1 + \gamma^3 \right] - \gamma \left[ \gamma_1 \gamma + \gamma \frac{1}{2} \gamma + \gamma \frac{1}{6} \right] > 0
\]
but it is not clear if such inequalities are germane in connection with the Riemann Hypothesis via the Li/Keiper criterion.

Differentiation of (2.8) with respect to \( r \) gives us
\[
\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{n-1} \log^3 \log y \frac{dy}{\log^{-1} y} = \frac{1}{\Gamma(r)} S(r, u, 4) - 3 \frac{\psi(r)}{\Gamma(r)} S(r, u, 3)
\]
(2.13)
\[
-3 \frac{\psi'(r) - [\psi(r)]^2}{\Gamma(r)} S(r, u, 2) + \psi''(r) - 2 \psi(r) \psi'(r) + [\psi(r)]^3 S(r, u, 1)
\]
Further differentiations will result in integrals
\[
\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{n-1} \log^n \log y dy
\]
involve the Stieltjes constants \( \gamma_n(u) \) (together with the resulting inequalities).

Differentiation of (2.4) with respect to \( u \) results in
\[
\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{n-1} \log y \cdot \log \log y dy = -2 \gamma'_1(u) - \gamma'_{0}(u) - \frac{\gamma + 2 \log u}{u}
\]
and with \( u = 1 \) we get using \( \gamma'_0(u) = \psi'(u) \)
\[
-\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log y \cdot \log \log y dy = 2 \gamma'_1(1) - \gamma(2) + \gamma
\]
Using equation (4.3.244) in [13]

\[ \gamma'_1(1) = 2\pi^2 \zeta'(-1) + \zeta(2)(\gamma + \log 2\pi) \]

we may write this as

\[ (2.14) \quad -\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log y \cdot \log \left| \log y \right| dy = 4\pi^2 \zeta'(-1) + \zeta(2)(\gamma + 2\log 2\pi) + \gamma \]

Using (C.1) this may be written as

\[ (2.15) \quad -\int_0^1 \frac{\log y \cdot \log \left| \log y \right|}{1-y} dy = 4\pi^2 \zeta'(-1) + \zeta(2)(\gamma + 2\log 2\pi) \]

3. An application of the (exponential) complete Bell polynomials

Alternatively, we may write (1.6) as

\[ (3.1) \quad \Gamma(r) \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \frac{y^{n-1}}{\log^{r-1} y} dy = -\sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j (u + j)^{r-1} \log(u + j) \]

Letting \( h(r) \) be defined as

\[ h(r) = \frac{\Gamma(r)}{\log^{r-1} y} \]

we have

\[ \log h(r) = \log \Gamma(r) - (r-1) \log \left| \log y \right| \]

and differentiation results in

\[ (3.2) \quad h'(r) = h(r)[\psi(r) - \log \left| \log y \right|] \]

Using (3.2) we now differentiate (3.1) with respect to \( r \) to obtain

\[ \Gamma(r) \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \frac{y^{n-1} \left[ \psi(r) - \log \left| \log y \right| \right]}{\log^{r-1} y} dy = -\sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j (u + j)^{r-1} \log^2(u + j) \]

which may be expressed as
Letting \( r = 1 \) this becomes

\[
\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{r-1} [\gamma + \log \| \log y \|] \, dy = -2\gamma_1(u) - \log^2 u
\]

and with \( u = 1 \) we have

\[
\int_0^1 \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] [\gamma + \log \| \log y \|] \, dy = -2\gamma_1
\]

which of course is consistent with our previous results.

We now wish to obtain a formula for the \( m \)th derivative of (3.1).

It is shown in Appendix A that

\[
d^m \frac{d}{dx^m} e^{f(x)} = e^{f(x)} Y_m \left( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m)}(x) \right)
\]

where the (exponential) complete Bell polynomials \( Y_n(x_1, \ldots, x_n) \) are defined by \( Y_0 = 1 \) and for \( n \geq 1 \)

\[
Y_n(x_1, \ldots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! k_2! \ldots k_n!} \left( \frac{x_1}{1!} \right)^{k_1} \left( \frac{x_2}{2!} \right)^{k_2} \ldots \left( \frac{x_n}{n!} \right)^{k_n}
\]

where the sum is taken over all partitions \( \pi(n) \) of \( n \), i.e. over all sets of integers \( k_j \) such that

\[ k_1 + 2k_2 + 3k_3 + \ldots + nk_n = n \]

Suppose that \( h'(x) = h(x)g(x) \) and let \( f(x) = \log h(x) \). We see that

\[
f'(x) = \frac{h'(x)}{h(x)} = g(x)
\]

and then using (3.3) above we have
(3.5) \[ \frac{d^m}{dx^m} h(x) = \frac{d^m}{dx^m} e^{\log h(x)} = h(x) Y_m \left( g(x), g^{(1)}(x), \ldots, g^{(m-1)}(x) \right) \]

As an example, letting \( h(x) = \Gamma(x) \) in (3.5) we obtain

(3.6) \[ \frac{d^m}{dx^m} e^{\log \Gamma(x)} = \Gamma^{(m)}(x) = \Gamma(x) Y_m \left( \psi(x), \psi^{(1)}(x), \ldots, \psi^{(m-1)}(x) \right) \]

\[ = \int_0^\infty t^{x-1} e^{-t} \log^m t dt \]

and since [26, p.22]

(3.7) \[ \psi^{(p)}(x) = (-1)^{p+1} p! \zeta(p+1, x) \]

we may express \( \Gamma^{(m)}(x) \) in terms of \( \psi(x) \) and the Hurwitz zeta functions. In particular, Köhlig [20] noted that

(3.8) \[ \Gamma^{(m)}(1) = Y_m(-\gamma, x_1, \ldots, x_{m-1}) \]

where \( x_p = (-1)^{p+1} p! \zeta(p+1) \). Values of \( \Gamma^{(m)}(1) \) for \( m \leq 10 \) are reported in [26, p.265] and the first three are

\[ \Gamma^{(1)}(1) = -\gamma \]
\[ \Gamma^{(2)}(1) = \zeta(2) + \gamma^2 \]
\[ \Gamma^{(3)}(1) = -[2\zeta(3) + 3\gamma \zeta(2) + \gamma^3] \]

As shown in Appendix C, we note that \( \Gamma^{(n)}(1) \) has the same sign as \((-1)^n\). This was also reported as an exercise in Apostol’s book [4, p.303].

As a variation of (3.5) above, suppose that \( j'(x) = j(x)[g(x) + \alpha] \) where \( \alpha \) is independent of \( x \) and let \( f(x) = \log j(x) \). We see that

\[ f'(x) = \frac{j'(x)}{j(x)} = g(x) + \alpha \quad \text{and} \quad f^{(k+1)}(x) = g^{(k+1)}(x) \quad \text{for} \quad k \geq 1 \]

and therefore we obtain
(3.9) \[
\frac{d^n}{dx^n} j(x) = \frac{d^n}{dx^n} e^{\log j(x)} = j(x)Y_n \left( g(x) + \alpha, g^{(1)}(x), \ldots, g^{(m-1)}(x) \right)
\]

It is also shown in Appendix A that

\[
Y_n(x_1 + \alpha, \ldots, x_n) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} Y_k(x_1, \ldots, x_k)
\]

and we then determine that

(3.10) \[
\frac{d^n}{dx^n} j(x) = j(x) \sum_{k=0}^{m} \binom{m}{k} \alpha^{m-k} Y_k \left( g(x), g^{(1)}(x), \ldots, g^{(k-1)}(x) \right)
\]

Now, referring back to (3.2), we see that with \( g(r) = \psi(r) - \log|\log y| \)

\[
\frac{d^n}{dr^n} \frac{\Gamma(r)}{\log^{-1} y} = \frac{\Gamma(r)}{\log^{-1} y} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \log^{-m-k} \log y \cdot Y_k \left( \psi(r), \psi^{(1)}(r), \ldots, \psi^{(k-1)}(r) \right)
\]

and since we have from (3.5)

\[
\Gamma^{(k)}(r) = \Gamma(r)Y_k \left( \psi(r), \psi^{(1)}(r), \ldots, \psi^{(k-1)}(r) \right)
\]

we obtain

\[
\frac{d^n}{dr^n} \frac{\Gamma(r)}{\log^{-1} y} = \frac{\Gamma(r)}{\log^{-1} y} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}(r) \log^{-m-k} \log y
\]

Hence, differentiating (3.1) \( m \) times with respect to \( r \) results in

(3.11) \[
\Gamma(r) \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}(r) \int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{-1} \log^{-m-k} \log y \, dy
\]

\[
= - \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^{j} (u+j)^{-1} \log^{m+1} (u+j)
\]

Letting \( r = 1 \) this becomes

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}(1) \int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] y^{-1} \log^{-m-k} \log y \, dy
\]
\[= -\sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n} \binom{n}{j} (-1)^{j} \log^{n+1}(u + f)\]

\[= (m+1)\gamma_m(u) + \log^{m+1} u\]

With \( u = 1 \) we obtain

\[(3.12) \quad (m+1)\gamma_m = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Gamma^{(k)}(1) \int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log^{m-k} |\log y| dy\]

For example, with \( m = 1 \) we get

\[2\gamma_1 = -\int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log|\log y| dy - \gamma_1 \int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] dy\]

which is in agreement with our previous results.

Equation (3.12) sheds a little light upon the complexity involved in the alteration in the signs of \( \gamma_m \) because

\[(3.13) \quad \Gamma^{(k)}(1) = (-1)^k c_k\]

\[(3.14) \quad \int_{0}^{1} \left[ \frac{1}{1-y} + \frac{1}{\log y} \right] \log^{m-k} |\log y| dy = (-1)^{m-k} d_{m-k}\]

where \( c_k \) and \( d_k \) are positive constants. The relationship (3.13) is derived in Appendix C and (3.14) was shown in (2.6) above. We then have

\[(3.15) \quad (m+1)\gamma_m = \sum_{k=0}^{m} \binom{m}{k} (-1)^{k} c_k d_{m-k}\]

Referring back to (1.1) we make the summation

\[\sum_{n=0}^{\infty} \frac{e^{-\alpha}}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(a+k)^p} \]

and we have
\[ \sum_{n=0}^{\infty} \frac{1}{n+1} e^{-ax} (1-e^{-x})^n x^{n+1}dx = \int_{0}^{\infty} e^{-ax} x^{n+1} \sum_{n=0}^{\infty} \frac{(1-e^{-x})^n}{n+1}dx \]

\[ = \int_{0}^{\infty} e^{-ax} x^{n+1} \sum_{n=0}^{\infty} \frac{(1-e^{-x})^{n+1}}{n+1}dx \]

\[ = \int_{0}^{\infty} e^{-ax} x^{n+1} \log(1-e^{-x}) \frac{1}{1-e^{-x}} \]

Therefore we obtain

\[ \int_{0}^{\infty} e^{-ax} x^{n+1} \log(1-e^{-x}) \frac{1}{1-e^{-x}} \] \[ = \Gamma(p) \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \]

and with \( a = 1 \) we have

\[ \int_{0}^{\infty} e^{-x} x^{n+1} \log(1-e^{-x}) \frac{1}{1-e^{-x}} \] \[ = \Gamma(p) \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \]

We note that the Bernoulli polynomials may be represented by [17]

\[ B_k(a) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (a+k)^k \]

and hence for \( m \geq 1 \)

\[ B_{2m+1}(1) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1+k)^{2m+1} = 0 \]

We now consider the limit as \( p \rightarrow -(2m+1) \) where \( m \) is a positive integer and \( m \geq 1 \);
the right-hand side of (\ref{eq:limit}) is indeterminate but we may use L'Hôpital’s rule when considering the ratio

\[ \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(a+k)^p} \]

\[ \frac{1}{1/\Gamma(p)} \]

As shown previously, using Euler’s reflection formula we have

\[ \frac{d}{dp} [1/\Gamma(p)] = \Gamma(1-p) \cos \pi p - \frac{1}{\pi} \Gamma'(1-p) \sin \pi p \]
and therefore we see that where \( p \to -(2m+1) \)

\[
\lim_{p \to -(2m+1)} \frac{d}{dp} \left[ 1/ \Gamma(p) \right] = (-1)^{2m+1} \Gamma(2m)
\]

We thereby obtain

\[
\int_0^\infty \frac{e^{-x} \log(1-e^{-x})}{(1-e^{-x})x^m} \, dx = (-1)^{2m+1} \Gamma(2m) \sum_{n=0}^\infty \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} (-1)^k (1+k)^{2m+1} \log(1+k)
\]

**Appendix A**

A brief survey of the (exponential) complete Bell polynomials

The (exponential) complete Bell polynomials are defined by \( Y_0 = 1 \) and for \( n \geq 1 \)

\[
(A1) \quad Y_n(x_1, \ldots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! \, k_2! \ldots k_n!} \left( \frac{x_1}{1!} \right)^{k_1} \left( \frac{x_2}{2!} \right)^{k_2} \ldots \left( \frac{x_n}{n!} \right)^{k_n}
\]

where the sum is taken over all partitions \( \pi(n) \) of \( n \), i.e. over all sets of integers \( k_j \) such that

\[ k_1 + 2k_2 + 3k_3 + \ldots + nk_n = n \]

The complete Bell polynomials have integer coefficients and the first six are set out below [11, p.307]

\[
(A2) \quad Y_1(x_1) = x_1 \\
Y_2(x_1, x_2) = x_1^2 + x_2 \\
Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3 \\
Y_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4 \\
Y_5(x_1, x_2, x_3, x_4, x_5) = x_1^5 + 10x_1^3x_2 + 10x_1^2x_3 + 15x_1x_4 + 10x_2^2 + 5x_3x_4 + 10x_3x_5 + x_5 \\
Y_6(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^6 + 6x_1^4x_2 + 15x_1^3x_3 + 15x_1^2x_4 + 15x_1x_5 + 60x_2x_3 + x_6
\]
\[ +20x_1^3x_3 + 45x_1^2x_2^2 + 15x_1^4x_1 + x_6 \]

The total number of terms \( \pi(n) \) increases rapidly; for example, as reported by Bell [5] in 1934, we have \( \pi(22) = 1002 \) terms.

The modus operandi of the summation in (A.1) is easily illustrated by the following example: if \( n = 4 \), then \( k_1+2k_2+3k_3+4k_4 = 4 \) is satisfied by the integers in the following array

\[
\begin{bmatrix}
k_1 & k_2 & k_3 & k_4 \\
4 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The complete Bell polynomials are also given by the exponential generating function (Comtet [11, p.134])

(A.3) \[
\exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = 1 + \sum_{n=1}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!}
\]

Let us now consider a function \( f(x) \) which has a Taylor series expansion around \( x \): we have

\[
e^{f^{(x+r)}} = \exp \left( \sum_{j=0}^{\infty} f^{(j)}(x) \frac{t^j}{j!} \right) = e^{f(x)} \exp \left( \sum_{j=1}^{\infty} f^{(j)}(x) \frac{t^j}{j!} \right)
\]

\[
= e^{f(x)} \left\{ 1 + \sum_{n=1}^{\infty} Y_n \left( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x) \right) \frac{t^n}{n!} \right\}
\]

We see that

\[
\frac{d^m}{dx^m} e^{f(x)} = \frac{\partial^m}{\partial x^m} e^{f^{(x+r)}} \bigg|_{r=0} = \frac{\partial^m}{\partial t^m} e^{f^{(x+r)}} \bigg|_{t=0}
\]

and we therefore obtain (as noted by Köblig [20])
Differentiating (A.4) we see that

\[ Y_{m+1}(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m+1)}(x)) = \left( f^{(1)}(x) + \frac{d}{dx} \right) Y_{m}(f^{(1)}(x), f^{(2)}(x), \ldots, f^{(m)}(x)) \]

Suppose that \( h'(x) = h(x)g(x) \) and let \( f(x) = \log h(x) \). We see that

\[ f'(x) = \frac{h'(x)}{h(x)} = g(x) \]

and then using (A.4) above we have

\[ \frac{d^m}{dx^m} \log h(x) = \frac{d^m}{dx^m} e^{\log h(x)} = h(x)Y_{m}\left(g(x), g^{(1)}(x), \ldots, g^{(m-1)}(x)\right) \]

As an example, letting \( f(x) = \log \Gamma(x) \) in (A.4) we obtain

\[ \frac{d^m}{dx^m} e^{\log \Gamma(x)} = \Gamma^{(m)}(x) = \Gamma(x)Y_{m}\left(\psi(x), \psi^{(1)}(x), \ldots, \psi^{(m-1)}(x)\right) \]

\[ = \int_0^\infty t^{x-1} e^{-t} \log^m t \, dt \]

and since [26, p.22]

\[ \psi^{(p)}(x) = (-1)^{p+1} p! \zeta(p + 1, x) \]

we may express \( \Gamma^{(m)}(x) \) in terms of \( \psi(x) \) and the Hurwitz zeta functions. In particular, Kölblig [20] notes that

\[ \Gamma^{(m)}(1) = Y_{m}\left(-\gamma, x_1, \ldots, x_{m-1}\right) \]

where \( x_p = (-1)^{p+1} p! \zeta(p + 1) \). Values of \( \Gamma^{(m)}(1) \) are reported in [26, p.265] for \( m \leq 10 \) and the first three are

\[ \Gamma^{(1)}(1) = -\gamma \]

\[ \Gamma^{(2)}(1) = \zeta(2) + \gamma^2 \]
\[ \Gamma^{(3)}(1) = -[2\zeta(3) + 3\gamma \zeta(2) + \gamma^3] \]

The general form is

\[ \Gamma^{(m)}(1) = (-1)^m \sum_{j=1}^{m} \varepsilon_{mj} \]

where \( \varepsilon_{mj} \) are positive constants.

We could also, for example, let \( f(x) = \log \sin(\pi x) \) in (A.4) to obtain complete Bell polynomials involving the derivatives of \( \cot(\pi x) \).

We now consider a minor modification of the arguments of the exponential generating function

\[ \exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} \]

where we let \( x_i \to x_i + \alpha \) and the other arguments remain unchanged. We then see that

\[ \exp \left( \alpha t + \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = \sum_{n=0}^{\infty} Y_n(x_1 + \alpha, \ldots, x_n) \frac{t^n}{n!} \]

Since

\[ \exp \left( \alpha t + \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) = \exp(\alpha t) \exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) \]

we easily determine that

\[ (A.9) \quad \exp(\alpha t) \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_n(x_1 + \alpha, \ldots, x_n) \frac{t^n}{n!} \]

Using the Cauchy series product formula we find that

\[ \sum_{n=0}^{\infty} \alpha^n \frac{t^n}{n!} \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\alpha^{n-k}}{(n-k)!} \frac{Y_k(x_1, \ldots, x_k)}{k!} t^n \]

and equating coefficients we see that

\[ (A.10) \quad Y_n(x_1 + \alpha, \ldots, x_n) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} Y_k(x_1, \ldots, x_k) \]
As a variation of (A.5) above, suppose that \( j'(x) = j(x)[g(x) + \alpha] \) where \( \alpha \) is independent of \( x \) and let \( f(x) = \log j(x) \). We see that

\[
f'(x) = \frac{j'(x)}{j(x)} = g(x) + \alpha \quad \text{and} \quad f^{(k+1)}(x) = g^{(k)}(x) \quad \text{for} \quad k \geq 1
\]

and therefore we obtain

\[
(A.11) \quad \frac{d^m}{dx^m} j(x) = \frac{d^m}{dx^m} e^{\log j(x)} = j(x) Y_m \left( g(x) + \alpha, g^{(1)}(x), \ldots, g^{(m-1)}(x) \right)
\]

Using (A.10) we then determine that

\[
(A.12) \quad \frac{d^m}{dx^m} j(x) = j(x) \sum_{k=0}^{n} \binom{m}{k} \alpha^{m-k} Y_k \left( g(x), g^{(1)}(x), \ldots, g^{(k-1)}(x) \right)
\]

The relation (A.10) may be generalised as follows to

\[
(A.13) \quad Y_n(x_1 + y_1, \ldots, x_n + y_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(x_1, \ldots, x_{n-k}) Y_k(y_1, \ldots, y_k)
\]

and we note that

\[Y_k(\alpha, 0, \ldots, 0) = \alpha^k\]

Equation (A.13) follows by noting that

\[
Y_n(x_1 + y_1, \ldots, x_n + y_n) = \exp \left( \sum_{j=1}^{\infty} \binom{n}{j} x_j \frac{t^j}{j!} \right) = \exp \left( \sum_{j=1}^{\infty} x_j \frac{t^j}{j!} \right) \exp \left( \sum_{j=1}^{\infty} y_j \frac{t^j}{j!} \right)
\]

\[
= \sum_{n=0}^{\infty} Y_n(x_1, \ldots, x_n) \frac{t^n}{n!} \sum_{n=0}^{\infty} Y_n(y_1, \ldots, y_n) \frac{t^n}{n!}
\]

and, as before, we apply the Cauchy series product formula.
Appendix B

Miscellaneous integral identities

In [24] Rivoal recently gave an elementary proof of the following lemma:

\[(B.1) \quad \int_0^1 x^n \Omega(x) \, dx = \gamma - \left[ H_n - \log(n+1) \right] \]

where, for convenience \( \Omega(x) \), is defined as \( \Omega(x) = \frac{1}{1-x} + \frac{1}{\log x} \) and \( H_n \) is the harmonic number defined by

\[ H_n = \sum_{k=1}^{n} \frac{1}{k} \]

Using L’Hôpital’s rule it is easily seen that \( \Omega(x) \) is continuous on \([0,1]\). For \( x \in (0,1) \) we also have by a straightforward integration

\[ \Omega(x) = \int_0^1 \frac{1-x'}{1-x} \, dt \]

We see that

\[ \int_0^1 \Omega(x) \, dx = \int_0^1 \left( \frac{1}{1-x} + \frac{1}{\log x} \right) \, dx = \int_0^\infty \left[ \frac{1}{1-e^{-u}} - \frac{1}{u} \right] e^{-u} \, du = \gamma \]

Using the well-known integral representation of the Riemann zeta function

\[ \zeta(s)\Gamma(s) = \int_0^\infty \frac{u^{s-1}}{e^u - 1} \, du \]

we also see that \( \Gamma(s) = (s-1)\Gamma(s-1) = (s-1)\int_0^\infty e^{-u}s^{-2} \, du \). Therefore we get

\[ \left[ \zeta(s) - \frac{1}{s-1} \right] \Gamma(s) = \int_0^\infty u^{s-1} \left[ \frac{1}{e^u - 1} - \frac{1}{ue^u} \right] \, du \]

Hence, in the limit as \( s \to 1 \), we obtain the familiar limit

\[ \lim_{s \to 1} \left[ \zeta(s) - \frac{1}{s-1} \right] \Gamma(s) = \lim_{s \to 1} \left[ \zeta(s) - \frac{1}{s-1} \right] = \int_0^\infty \left[ \frac{1}{1-e^{-u}} - \frac{1}{u} \right] e^{-u} \, du = \gamma \]
We now continue with Rivoal’s lemma.

We have by simple algebra

\[
\int_0^1 x^n \Omega(x) \, dx = \int_0^1 \Omega(x) \, dx - \int_0^1 \frac{x^n - 1}{x - 1} \, dx + \int_0^1 \frac{x^n - 1}{\log x} \, dx
\]

and this gives us

(B.2) \[
\int_0^1 x^n \left( \frac{1}{1-x} + \frac{1}{\log x} \right) \, dx = \gamma - H_n + \frac{1}{\log x} \int_0^1 \frac{x^n - 1}{\log x} \, dx = \gamma - \left[ H_n - \log(n+1) \right]
\]

It may be noted that this is equivalent to letting \( u = n + 1 \) in (1.10) because

\[ \psi(n+1) = -\gamma + H_n \]

Hence by summation of (B.2) we obtain for \( \text{Re}(s) > 1 \)

(B.3) \[
\gamma_\infty(s) - \sum_{n=1}^\infty \frac{H_n}{n^s} + \sum_{n=1}^\infty \frac{\log(n+1)}{n^s} = \int_0^1 Li_s(x) \left( \frac{1}{1-x} + \frac{1}{\log x} \right) \, dx
\]

where \( Li_s(x) \) is the polylogarithm function defined by

\[ Li_s(x) = \sum_{n=1}^\infty \frac{x^n}{n^s} \]

In 1997 Candelpergher et al. [9] produced a somewhat similar result in the case where \( s = 2 \) by reference to “Ramanujan summation” but unfortunately I am not au fait with the underlying analysis contained in that paper.

We have from [23]

(B.4) \[
H_n - \log n - \gamma = \int_0^\infty e^{-nx} \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx
\]

and on summation we obtain

\[
\sum_{n=1}^\infty \frac{H_n}{n^s} + \psi'(s) - \gamma_\infty(s) = \sum_{n=1}^\infty \frac{1}{n^s} \int_0^\infty e^{-nx} \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx
\]

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In particular we have

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{0}^{\infty} e^{-x} \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx = \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-nu} \int_{0}^{\infty} e^{-nv} \int_{0}^{\infty} e^{-nx} \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx \]

\[ = \sum_{n=1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-n(x+u+v)} \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{e^{x+u+v} - 1} \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx \]

We have

\[ \int_{0}^{\infty} \frac{1}{e^{x+u+v} - 1} \, du = \int_{0}^{\infty} e^{-(x+u+v)} \, du = \log[1 - e^{-(x+u+v)}] \Big|_0^\infty = -\log[1 - e^{-(x+v)}] \]

and the Wolfram Integrator tells us that

\[ \int \log[1 - e^{-(x+v)}] \, dv = \frac{1}{2} v^2 + v \log \left[ \frac{1 - e^{-(x+v)}}{1 - e^{-(x+v)}} \right] = \text{Li}_2[e^{(x+v)}] \]

With the substitution \( v = e^{-v} \) we can easily find that

\[ \int_{0}^{\infty} \log[1 - e^{-(x+v)}] \, dv = -\text{Li}_2[e^{-x}] \]

and hence we obtain

\[ \sum_{n=1}^{\infty} \frac{H_n}{n^2} + \zeta'(2) - \gamma \zeta(2) = \int_{0}^{\infty} \text{Li}_2[e^{-x}] \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx \]

or equivalently

\[ (B.5) \quad 2\zeta(3) + \zeta'(2) - \gamma \zeta(2) = \int_{0}^{\infty} \text{Li}_2[e^{-x}] \left[ \frac{1}{x} - \frac{1}{e^x - 1} \right] \, dx \]

where we have employed the well-known Euler sum [26, p.103] \( \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3). \)

With the substitution \( t = e^{-x} \) this may be written as
We see from (1.13) that

\[
\sum_{n=1}^{\infty} \frac{\log n}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \int_{0}^{\infty} \frac{e^{-x} - e^{-nx}}{x} \, dx
\]

which may be written as

\[
-\zeta'(s) = \int_{0}^{\infty} \frac{\zeta(s)e^{-x} - Li[e^{-x}]}{x} \, dx
\]

With \( t = e^{-x} \) this becomes

\[
\zeta'(s) = \int_{0}^{1} \frac{\zeta(s)t - Li[t]}{t \log t} \, dt
\]

and with \( s = 2 \) we get

\[
\zeta'(2) = \int_{0}^{1} \frac{\zeta(2)t - Li[t]}{t \log t} \, dt
\]

We note from (B.6) that

\[
2\zeta(3) + \zeta'(2) - \gamma \zeta(2) = \int_{0}^{1} Li_2(t) \left[ \frac{1}{t-1} - \frac{1}{t \log t} \right] \, dt
\]

\[
= \int_{0}^{1} \left[ \frac{Li_2(t)}{t-1} - \frac{Li_2(t)}{t \log t} \right] \, dt
\]

\[
= \int_{0}^{1} \left[ \frac{Li_2(t)}{t-1} - \frac{\zeta(2)}{\log t} + \frac{\zeta(2)}{\log t} - \frac{Li_2(t)}{t \log t} \right] \, dt
\]

\[
= \int_{0}^{1} \left[ \frac{Li_2(t)}{t-1} - \frac{\zeta(2)}{\log t} \right] \, dt + \int_{0}^{1} \left[ \frac{\zeta(2)}{\log t} - \frac{Li_2(t)}{t \log t} \right] \, dt
\]

and using (B.8) this becomes
We therefore see that
\[(B.10) \quad 2\zeta(3) - \gamma\zeta(2) = \int_0^1 \left[ \frac{Li_2(t)}{t-1} - \frac{\zeta(2)}{\log t} \right] dt + \zeta'(2) \]

We note from (B.3) with \( s = 2 \) that
\[2\zeta(3) - \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} - \gamma\zeta(2) = \int_0^1 Li_2(t) \left( \frac{1}{t-1} - \frac{1}{\log t} \right) dt \]

Subtracting (B.6) results in
\[(B.11) \quad \zeta'(2) + \sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} = \int_0^1 \frac{(t-1)Li_2(t)}{t \log t} dt \]
or equivalently
\[\sum_{n=1}^{\infty} \frac{\log(n+1)}{n^2} - \sum_{n=1}^{\infty} \frac{\log n}{n^2} = \int_0^1 \frac{(t-1)Li_2(t)}{t \log t} dt \]

We may write (B.10) as the limit
\[2\zeta(3) - \gamma\zeta(2) = \lim_{x \to 1} \int_0^x \left[ \frac{Li_2(t)}{t-1} - \frac{\zeta(2)}{\log t} \right] dt \]

Using integration by parts we can easily determine that
\[\int \frac{Li_2(t)}{t-1} dt = Li_2(t) \log(1-t) + \log t \log^2(1-t) + 2Li_2(1-t) \log(1-t) - 2Li_3(1-t) \]

and we therefore have
\[\int_0^x \frac{Li_2(t)}{t-1} dt = Li_2(x) \log(1-x) + \log x \log^2(1-x) + 2Li_2(1-x) \log(1-x) - 2Li_3(1-x) + 2\zeta(3) \]
Using L’Hôpital’s rule we find that
\[
\lim_{x \to 1} [(1-x) \log(1-x)] = 0
\]
and hence we have
\[
\lim_{x \to 1} [\text{Li}_2(1-x) \log(1-x)] = 0
\]
We therefore have the limit
\[
\lim_{x \to 1} \int_0^x \frac{\text{Li}_2(t)}{t^2-1} dt = \lim_{x \to 1} [\text{Li}_2(x) \log(1-x) + \log x \log^2(1-x)] + 2\zeta(3)
\]
which then implies that
\[
-\gamma \zeta(2) = \lim_{x \to 1} \left[ \text{Li}_2(x) \log(1-x) + \log x \log^2(1-x) - \int_0^x \frac{\zeta(2)}{\log t} dt \right]
\]
We therefore have
(B.12) \[\gamma \zeta(2) = \lim_{x \to 1} \left[ -\text{Li}_2(x) \log(1-x) - \log x \log^2(1-x) + \zeta(2) \text{li}(x) \right] \]
where \(\text{li}(x)\) is the logarithmic integral. Nielsen [21, p.3] has shown that for \(0 < x < 1\)
(B.13) \[\text{li}(x) = \gamma + \log(-\log x) + \sum_{n=1}^\infty \frac{\log^n x}{n!n} \]
and hence we have
\[
\gamma \zeta(2) = \lim_{x \to 1} \left[ -\text{Li}_2(x) \log(1-x) - \log x \log^2(1-x) + \zeta(2)[\gamma + \log(-\log x)] \right]
\]
which gives us
\[
\lim_{x \to 1} \left[ -\text{Li}_2(x) \log(1-x) - \log x \log^2(1-x) + \zeta(2) \log(-\log x) \right] = 0
\]
Using Euler’s identity
\[
\zeta(2) = \log x \log(1-x) + \text{Li}_2(x) + \text{Li}_2(1-x)
\]
we may write this as
\[
\lim_{x \to 1} \left[ Li_x(1-x) \log(1-x) + \zeta(2) \log(1-x) + \zeta(2) \log(-\log x) \right] = 0
\]

and hence we have

(B.14) \[ \lim_{x \to 1} \left[ \log(1-x) + \log(-\log x) \right] = 0 \]

Since

\[
\frac{d}{dt} \log[\log(1/t)] = -\frac{1}{t \log(1/t)} = \frac{1}{t \log t}
\]

integration by parts gives us

\[
\int_0^x \log[\log(1/t)]dt = t \log[\log(1/t)]\big|_0^x - \int_0^x \frac{dt}{\log t}
\]

Using L'Hôpital’s rule we see that

\[
\lim_{t \to 0} t \log[\log(1/t)] = \lim_{t \to 0} \frac{d}{dt} \log[\log(1/t)] = 0
\]

and therefore we obtain

(B.15) \[ \int_0^x \log[\log(1/t)]dt = x \log[\log(1/x)] - li(x) \]

Since \[ \int_0^1 \log[\log(1/t)]dt = -\gamma \] we find that

(B.16) \[ \lim_{x \to 1} [x \log[\log(1/x)] - li(x)] = -\gamma \]

Integration gives us

\[
\int_0^x \left[ \frac{1}{1-t} + \frac{1}{\log t} \right]dt = -\log(1-t) + li(t)\big|_0^x
\]

\[
= -\log(1-x) + li(x)
\]

We note from (B.2) that
\[ \gamma = \int_0^1 \left[ \frac{1}{1 - y} + \frac{1}{\log y} \right] dy \]

and therefore we have

\[ \lim_{x \to 1^+} [-\log(1 - x) + li(x)] = \gamma \]

Therefore we see that

\[ \lim_{x \to 1^+} [x \log(\log(1/x)) - \log(1 - x)] = 0 \]

Multiplying (B.12) by \( x \), upon taking the limit, we see that

\[ \lim_{x \to 1^+} [x li(x) - x \log(-\log x)] = \gamma \]

Combining this with (B.18) we obtain

\[ \lim_{x \to 1^+} [x li(x) - li(x) + \log(1 - x) - x \log(-\log x)] = 0 \]

and hence

\[ \lim_{x \to 1^+} [(x - 1) li(x)] = 0 \]

This may also be verified using L’Hôpital’s rule

\[ \lim_{x \to 1^+} [(x - 1) li(x)] = \lim_{x \to 1^+} \frac{li(x)}{1/(x - 1)} = -\lim_{x \to 1^+} \frac{(x - 1)^2}{\log x} = -\lim_{x \to 1^+} 2x(x - 1) = 0 \]

The limit (B.19) in conjunction with (B.12) implies that

\[ \lim_{x \to 1^+} [(x - 1) \log(-\log x)] = 0 \]

We refer back to the Frullani integral (1.13)

\[ \log u = \int_0^1 \left[ \frac{y^{u-1}}{\log y} - \frac{1}{\log y} \right] dy \]

where the third derivative results in
\[
\frac{2}{u^3} = \int_0^1 y^{n-1} \log y^2 \, dy
\]

We now let \( u = n \) and make a summation to obtain the well-known integral

\[
2\zeta(3) = 2 \sum_{n=1}^{\infty} \frac{1}{n^3} = \int_0^1 \frac{\log y^2}{1 - y} \, dy
\]

The following is extracted from Brede’s dissertation [7]. In 1999 Coppo showed that there exists a polynomial \( p_n(z) \) such that

\[
x^n = \int_0^\infty p_n(x - \log z) e^{-z} \, dz
\]

With \( p_n(z) = \sum_{k=0}^{n} a_k z^k \) we have

\[
\int_0^\infty p_n(x - \log z) e^{-z} \, dz = \int_0^\infty \sum_{k=0}^{n} a_k (x - \log z)^k e^{-z} \, dz
\]

\[
= \sum_{k=0}^{n} a_k \sum_{l=0}^{k} \binom{k}{l} x^l (-\log z)^{k-l} e^{-z} \, dz
\]

\[
= \sum_{k=0}^{n} a_k \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \Gamma(k-l+1) x^l
\]

Hence we have

\[
\int_0^\infty p_n(x - \log z) e^{-z} \, dz = \sum_{k=0}^{n} \left[ \sum_{l=0}^{n-k} (-1)^l \binom{k+l}{l} \Gamma(l+1) a_{k+l} \right] x^k
\]

and we then select the coefficients \( a_k \) such that

\[
\sum_{l=0}^{n-k} (-1)^l \binom{k+l}{l} \Gamma(l+1) a_{k+l} = \begin{cases} 
1 & \text{for } k = n \\
0 & \text{for } k \text{ less than } n
\end{cases}
\]

resulting in
\[ x^n = \int_0^\infty p_n(x - \log z)e^{-z}dz \]

For example we have

\[ p_0(z) = 1 \]

\[ p_1(z) = z - \gamma \]

\[ p_2(z) = z^2 - 2\gamma z + \gamma^2 - \zeta(2) \]

Letting \( x \to \log x \) we obtain

\[ \log^n x = \int_0^\infty p_n(\log x - \log z)e^{-z}dz \]

and, with the substitution \( z/x = -\log t \), we have

\[ \frac{\log^n x}{x} = \int_0^\infty p_n[-\log(1/t)]t^{x-1}dt \]

Referring to the well-known expression for the Stieltjes constants [19, p.4]

\[ \gamma_n = \sum_{k=1}^{\infty} \left[ \log^n \frac{k}{k} - \log^n \frac{k+1}{k} - \log^n \frac{k}{k} \right] = \sum_{k=1}^{\infty} \left[ \log^n \frac{k}{k} - \int_k^{k+1} \log^n \frac{t}{t} dt \right] \]

\[ = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \log^n \frac{k}{k} - \int_1^{N+1} \log^n \frac{t}{t} dt \right] \]

we then see that

\[ \gamma_n = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \int_0^1 p_n[-\log(1/t)]t^{x-1}dt - \int_0^1 \int_{1}^{N+1} p_n[-\log(1/t)]t^{x-1}dt dx \right] \]

\[ = \lim_{N \to \infty} \int_0^1 p_n[-\log(1/t)] \left[ \sum_{k=0}^{N-1} t^k - \int_1^{N+1} t^x dx \right] dt \]

\[ = \lim_{N \to \infty} \int_0^1 p_n[-\log(1/t)] \left[ \frac{t^N - 1}{t - 1} - \frac{t^{N-1}}{\log t} \right] dt \]
Hence, as shown more completely in Brede’s dissertation, we obtain as $N \to \infty$

$$(B.21) \quad \gamma_n = \int_0^1 p_n[-\log \log(1/t)] \left[ \frac{1}{\log t} - \frac{1}{t-1} \right] dt$$

Since $\frac{1}{\log t} - \frac{1}{t-1} = \int_0^1 \frac{1-u'}{1-u} du$ we may write this as a double integral

$$\gamma_n = \int_0^1 \int_0^1 p_n[-\log \log(1/t)] \frac{1-u'}{1-u} du dt$$

□

The following is extracted from Coppo’s paper [14]. With the substitution $u = xt$ we see that

$$\int_0^\infty e^{-xt} \log^n t dt = \frac{1}{x} \int_0^\infty e^{-u} (\log u - \log x)^n du$$

$$= \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{\log^k x}{x} \int_0^\infty e^{-u} \log^{n-k} u du$$

$$= \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{\log^k x}{x} \Gamma^{(n-k)}(1)$$

Let us assume that

$$(B.22) \quad \sum_{k=0}^n \binom{n}{k} (-1)^k \Gamma^{(n-k)}(1) \hat{a}_k(t) = \log^n t$$

where $\hat{a}_k(t)$ are to be determined. With $n = 0$ we have

$$\hat{a}_0(t) = 1$$

and $n = 1$ gives us

$$\Gamma^{(1)}(1) - \hat{a}_1(t) = \log t$$

We then multiply (B.22) by $e^{-xt}$ and integrate to obtain
We then have

\[ (B.23) \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma^{(n-k)}(1) \left[ \log^k x - \int_0^\infty e^{-xt} \hat{a}_k(t) dt \right] = 0 \]

and Coppo deduces that

\[ \frac{\log^k x}{x} = \int_0^\infty e^{-xt} \hat{a}_k(t) dt \]

It seems to me that this solution is only unique provided the coefficients appearing in (B.23) are linearly independent.

The associated Stieltjes functions \( \hat{\gamma}_k(x) \) are defined by

\[ \hat{\gamma}_k(x) = \int_0^\infty e^{-xt} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) \hat{a}_k(t) dt \]

and we have in particular

\[ \hat{\gamma}_0(x) = \int_0^\infty e^{-xt} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) dt = \log x - \psi(x) \]

and with \( x = 1 \) we have

\[ \hat{\gamma}_0(1) = \int_0^\infty e^{-t} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) dt = \gamma \]

It is easily seen that

\[ (B.24) \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma^{(n-k)}(1) \hat{\gamma}_k(x) = \int_0^\infty e^{-xt} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) \log^n t dt \]

We have the well-known integral representation for the Hurwitz zeta function [26, p.92]
and from the definition of the gamma function we see that

\[
\frac{x^{1-s}}{s-1} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} e^{-ut} dt
\]

Hence we have for \( \text{Re}(s) > 1 \)

\[
\zeta(s, x) - \frac{x^{1-s}}{s-1} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-ut} \left( \frac{1}{1-e^{-t}} - \frac{1}{t} \right) t^{s-1} dt
\]

which we write as

(B.25) \( \left( \zeta(s, x) - \frac{x^{1-s}}{s-1} \right) \Gamma(s) = \int_0^\infty e^{-ut} \left( \frac{1}{1-e^{-t}} - \frac{1}{t} \right) t^{s-1} dt \)

We have

\[
\zeta(s, x) - \frac{x^{1-s}}{s-1} = \zeta(s, x) - \frac{1}{s-1} + \frac{1}{s-1} - \frac{x^{1-s}}{s-1}
\]

and we denote \( f(x) \) as

\[
f(x) = \frac{1-x^{1-s}}{s-1}
\]

We can represent \( f(x) \) by the following integral

\[
f(x) = \frac{1-x^{1-s}}{s-1} = \int_1^x t^{s-1} dt
\]

so that

\[
f^{(r)}(x) = (-1)^r \int_1^x t^{s-r-1} \log t dt
\]

and thus

\[
f^{(r)}(1) = (-1)^r \int_1^1 \log t \frac{dt}{t} = (-1)^r \frac{\log^{r+1} x}{r+1}
\]

Differentiating (B.25) \( n \) times gives us

\[
\frac{d^n}{ds^n} \left( \zeta(s, x) - \frac{x^{1-s}}{s-1} \right) \Gamma(s) = \int_0^\infty e^{-ut} \left( \frac{1}{1-e^{-t}} - \frac{1}{t} \right) t^{s-1} \log^n t dt
\]
The Stieltjes constants \( \gamma_n(u) \) are the coefficients in the Laurent expansion of the Hurwitz zeta function \( \zeta(s,u) \) about \( s = 1 \)

\[
\zeta(s,x) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k(x)(s-1)^k
\]

and we therefore have

\[
\frac{d^n}{ds^n} \left[ \zeta(s,x) - \frac{1}{s-1} \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k(x)k(k-1)\cdots(k-r)(s-1)^{k-r}
\]

and thus

\[
\lim_{s \to 1} \frac{d^n}{ds^n} \left[ \zeta(s,x) - \frac{1}{s-1} \right] = (-1)^r \gamma_r(x)
\]

\[
\lim_{s \to 1} \frac{d^n}{ds^n} \left[ \zeta(s,x) - \frac{x^{1-s}}{s-1} \right] = (-1)^r \gamma_r(x) + (-1)^r \frac{\log^{r+1} x}{r+1}
\]

Applying the Leibniz rule we have

\[
\lim_{s \to 1} \frac{d^n}{ds^n} \left( \zeta(s,x) - \frac{x^{1-s}}{s-1} \right) \Gamma(s) = \sum_{k=0}^{\infty} \binom{n}{k} (-1)^{k} \Gamma^{(n-k)}(1) \left[ \gamma_k(x) + \frac{\log^{k+1} x}{k+1} \right]
\]

and hence using (B.24) we see that

\[
(B.26) \quad \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma^{(n-k)}(1) \left[ \gamma_k(x) + \frac{\log^{k+1} x}{k+1} \right] = \int_0^\infty \frac{1 - e^{-t} - t^{-1}}{1 - e^{-t}} \log^n t \, dt
\]

We then see from (B.24) that (subject to the previous comment regarding the coefficients being linearly independent)

\[
\hat{\gamma}_k(x) = \gamma_k(x) + \frac{\log^{k+1} x}{k+1}
\]

and accordingly that

\[
\hat{\gamma}_k(1) = \gamma_k(1) = \gamma_k
\]

With \( x = 1 \) we have
We also have

\[ \zeta(s, x) = \frac{x^{1-s}}{s-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \hat{\gamma}_k(x)(s-1)^k \]

With the substitution \( t = \log(1/u) \) we obtain

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma^{(n-k)}(1) \hat{\gamma}_k(x) = \int_0^1 u^{x-1} \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \log^n \log \left( \frac{1}{u} \right) du
\]

and

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma^{(n-k)}(1) \gamma_k = \int_0^1 \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \log^n \log \left( \frac{1}{u} \right) du
\]

and in particular we have

\[ A_0(x) = \hat{\gamma}_0(x) = \log x - \psi(x) \]

\[ A_1(x) = \Gamma^{(1)}(x) - \hat{\gamma}_1(x) \]

This gives us

\[ \hat{\gamma}_1 = -\gamma \left[ \log x - \psi(x) \right] - \int_0^1 u^{x-1} \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \log \log \left( \frac{1}{u} \right) du \]

\[ \gamma_1 = -\gamma^2 - \int_0^1 \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \log \log \left( \frac{1}{u} \right) du \]

For \( u \in [0,1] \) we have

\[ \log \left( \frac{1}{u} \right) = \left| \log \left( \frac{1}{u} \right) \right| = -\log |u| = \log |u| \]

and hence we obtain
(B.30) \[ \gamma_1 = -\gamma^2 - \int_0^1 \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \log|\log u| \, du \]

which unfortunately differs from (2.5).

Appendix C

Derivatives of the gamma function

The gamma function is defined as

\[ \Gamma(x) = \int_0^\infty y^{x-1} e^{-y} \, dy \]

and, with the substitution \( y = -\log t \), this becomes

\[ \Gamma(x) = \int_0^1 \left( \log(1/t) \right)^{x-1} \, dt \]

We therefore have

\[ \Gamma'(x) = \int_0^1 \left( \log(1/t) \right)^{x-1} \log(1/t) \, dt \]

and thus

\[ (C.1) \quad \Gamma'(1) = \int_0^1 \log(1/t) \, dt = -\gamma \]

More generally we see that

\[ \Gamma^{(n)}(x) = \int_0^1 \left( \log \frac{1}{t} \right)^{x-n} \left( \log \log \frac{1}{t} \right)^n \, dt \]

and as we shall see below

\[ \Gamma^{(n)}(1) = \int_0^1 \left( \log \log \frac{1}{t} \right)^n \, dt = (-1)^n c_n \]

where \( c_n \) are positive constants.

We have
\[ \Gamma^{(n)}(x) = \int_0^\infty e^{-t} t^{x-1} \log^n t \, dt \]

and thus

\[ \Gamma^{(n)}(1) = \int_0^\infty e^{-t} \log^n t \, dt \]

We see that

\[ \int_0^\infty e^{-t} \log^n t \, dt = \int_0^1 e^{-t} \log^n t \, dt + \int_1^\infty e^{-t} \log^n t \, dt \]

and with the substitution \( t = 1/y \) we have

\[ \int_1^\infty e^{-t} \log^n t \, dt = (-1)^n \int_0^1 e^{-1/y} y^{-2} \log^n y \, dy \]

We therefore obtain

\[ \Gamma^{(n)}(1) = \int_0^1 \left[ t^2 + (-1)^n e^{-t} \right] e^{-t} t^2 \log^n t \, dt \]

and it is clear that the integrand is positive in the interval \([0,1]\) in the case where \( n \) is an even integer. We now consider the case where \( n \) is an odd integer and we want to prove that for \( t \in [0,1] \)

\[ t^2 \geq e^{-1/t} \]

and this is easily seen by considering the logarithmic equivalent

\[ 2 \log t \geq t - 1/t \]

This inequality is easily proved by considering

\[ f(t) = 2t \log t - t^2 + 1 \]

where \( f(0) = 1 \) and \( f'(t) = 2[\log t + 1 - t] \) and it is known that \( \log t > 1 - t \) for \( t > 0 \).

Therefore we see that \( \Gamma^{(n)}(1) \) has the same sign as \((-1)^n\). This result was reported as an exercise in Apostol’s book [4, p.303] and we used it in (3.13) above.
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