Ricci Collineations in Friedmann-Robertson-Walker Spacetimes

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Abstract. Ricci collineations and Ricci inheritance collineations of Friedmann-Robertson-Walker spacetimes are considered. When the Ricci tensor is non-degenerate, it is shown that the spacetime always admits a fifteen parameter group of Ricci inheritance collineations; this is the maximal possible dimension for spacetime manifolds. The general form of the vector generating the symmetry is exhibited. It is also shown, in the generic case, that the group of Ricci collineations is six-dimensional and coincides with the isometry group. In special cases the spacetime may admit either one or four proper Ricci collineations in addition to the six isometries. These special cases are classified and the general form of the vector fields generating the Ricci collineations is exhibited. When the Ricci tensor is degenerate, the groups of Ricci inheritance collineations and Ricci collineations are infinite-dimensional. General forms for the generating vectors are obtained. Similar results are obtained for matter collineations and matter inheritance collineations.
1. Introduction

A vector field $\xi^i$ is said to generate a Ricci inheritance collineation (Duggal, 1993) if it satisfies the equation $\mathcal{L}_\xi R_{ij} = 2\phi R_{ij}$ or equivalently

$$R_{ij,k} \xi^k + R_{ik} \xi^k,_{j} + R_{jk} \xi^k,_{i} = 2\phi R_{ij} \quad (1)$$

If $\phi = 0$, then $\mathcal{L}_\xi R_{ij} = 0$ and $\xi^i$ is said to generate a Ricci collineation (Katzin et al., 1969). If $\xi^i$ is a Killing vector, then a fortiori $\mathcal{L}_\xi R_{ij} = 0$, thus any isometry is also a Ricci collineation. A similar result holds for homothetic Killing vectors $\xi^i$ for which $\mathcal{L}_\xi g_{ij} = 2\sigma g_{ij}$ with $\sigma,_{i} = 0$ as homotheties are also affine collineations and so $\mathcal{L}_\xi \Gamma_{jk}^i = 0$, $\mathcal{L}_\xi R_{ijkl} = 0$ and $\mathcal{L}_\xi R_{ij} = 0$. We will use the term proper Ricci collineation to denote a Ricci collineation which is not an isometry or a homothety.

We note immediately that equation (1), which involves partial rather than covariant derivatives, has the same form as the conformal Killing equations but with the metric tensor replaced by the Ricci tensor. Thus if the Ricci tensor is non-degenerate, that is has a non-zero determinant, we may apply standard results on conformal symmetries to deduce that the maximal dimension of the group of Ricci inheritance collineations (RICs) in a pseudo-Riemannian manifold of dimension $n$ is $(n + 1)(n + 2)/2$ and this occurs if and only if the Ricci tensor metric is conformally flat. By this we mean that the conformal curvature tensor $\text{Ric}^i_{\ jkl}$ calculated in the standard way but with $g_{ij}$ replaced everywhere by $R_{ij}$ (including in the definition of the connection) and with indices raised by the inverse of $R_{ij}$. Similarly when $\phi = 0$, equation (1) for Ricci collineations has the same form as Killing’s equations with $g_{ij}$ again replaced by the $R_{ij}$. If the Ricci tensor is non-degenerate, we may apply standard results on isometries to deduce that the maximal dimension of the group of Ricci collineations (RCs) in a pseudo-Riemannian manifold of dimension $n$ is $n(n+1)/2$ and this occurs if and only if the Ricci tensor metric has constant curvature. Thus, for spacetimes, the maximal dimensions of the groups of RICs and RCs are fifteen and ten respectively.

The Friedmann-Robertson-Walker (FRW) metric in stereographic coordinates is

$$ds^2 = dt^2 - S(t)^2(1 + K/4r^2)^{-2}(dx^2 + dy^2 + dz^2) \quad (2)$$

where $S$ is an arbitrary (non-zero) function of $t$, $K = 0, \pm 1$ and $r^2 = x^2 + y^2 + z^2$. The Ricci tensor is given by

$$R_{00} = -3\ddot{S}/S \quad R_{0\alpha} = 0 \quad R_{\alpha\beta} = -(2K + 2\dot{S}^2 + S\ddot{S})/S^2 g_{\alpha\beta} \quad (3)$$

where Greek indices take the values in the range $1 \ldots 3$. The Ricci tensor metric has the form

$$ds^2_{\text{Ric}} = R_{ij} dx^i dx^j = A(t) dt^2 + B(t)(1 + K/4r^2)^{-2}(dx^2 + dy^2 + dz^2) \quad (4)$$

where $A(t) = -3\ddot{S}/S$ and $B(t) = 2K + 2\dot{S}^2 + S\ddot{S}$. Thus the Ricci tensor metric is of FRW form with a generalised time coordinate. By rescaling the time coordinate...
\[ d\tilde{t} = |A|^{1/2}dt, \] we could set \( A = \pm 1, \) but the new coordinate would not then be proper time for the physical metric \( g_{ij}. \) In the rest of this section it will be assumed always that the Ricci tensor is non-degenerate, that is \( A \neq 0 \) and \( B \neq 0; \) the degenerate case will be considered in section 2. The signature of the Ricci tensor metric depends on the signs of \( A \) and \( B \) and is Lorentzian if they have opposite signs and is positive or negative definite if they have the same sign. Using Einstein’s field equations \( A = \kappa/2(\rho + 3p) \) and \( B = \kappa S^2(\rho - p)/2 \) where \( \rho \) and \( p \) are the fluid density and pressure respectively. Thus if \( \rho > 0 \) and \( p \geq 0 \) as would normally be assumed, the Ricci tensor is positive definite if and only if the energy condition \( \rho > p \) is satisfied.

For the FRW metric (2), equations (1) for RICs, or equivalently the conformal Killing equations for the metric (4), become

\[
B(\tilde{g}_{\alpha\beta\gamma}\xi^\gamma + \tilde{g}_{\alpha\gamma}\xi_{\beta}^\gamma + \tilde{g}_{\beta\gamma}\xi_{\alpha}^\gamma) = (2B\phi - \dot{B}\xi^0)\tilde{g}_{\alpha\beta} \quad (5a)
\]

\[
A\xi^0_{,\alpha} + B\tilde{g}_{\alpha\beta}\xi^\beta_{,t} = 0 \quad (5b)
\]

\[
2A\xi^0_{,t} + \dot{A}\xi^0 = 2\phi A \quad (5c)
\]

where \( \tilde{g}_{\alpha\beta} \) is the three-dimensional metric

\[
d\tilde{s}^2 = \tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = (1 + K/4r^2)^{-2}(dx^2 + dy^2 + dz^2) \quad (6)
\]

It is well-known (e.g. Stephani, 1967) that the FRW metrics are conformally flat (this conclusion is still true when the signature is definite rather than Lorentzian). Thus the *Ricci tensor metric* of a FRW spacetime is conformally flat and hence admits a fifteen parameter group of conformal symmetries or equivalently the physical metric admits a fifteen parameter group of Ricci inheritance collineations (RICs). It is also well-known (Maartens and Maharaj, 1986) that generic FRW metrics admit only a six-parameter isometry group, but that in special cases, they admit either a seven or ten-parameter complete isometry group. Furthermore for spatially flat FRW metrics the case of a complete seven-dimensional isometry group is excluded. Again these results hold when the signature is definite as well as for the Lorentzian case. When the complete group is seven-dimensional, the spacetime is a static Einstein universe (or its analogue with definite signature). When it is ten-dimensional the spacetime has constant curvature. Thus we may conclude immediately that group of the Ricci collineations (RCs) of a FRW spacetime with \( K \neq 0 \) is six, seven or ten-dimensional whereas for \( K = 0 \) the complete group of RCs is either six or ten-dimensional. Dimensions greater than six occur only when the Ricci tensor metric has constant curvature or is equivalent to an Einstein static universe.

A priori the possibility exists that a FRW spacetime always has a *Ricci tensor metric* that admits a seven or ten-dimensional group of isometries. If this were the case, then a generic FRW would admit a seven or ten-dimensional group of RCs including one or four proper RCs respectively. However, in actual fact for almost any form for \( S(t) \) other than a constant or a constant multiple of a power, an exponential function, hyperbolic sine and cosine or trigonometric sine and cosine, the complete
group of Ricci collineations is the six-dimensional isometry group. In special cases the isometry group of the physical metric can be six-dimensional whilst the Ricci tensor metric admits a seven or ten-dimensional isometry group and so proper Ricci collineations will exist in these cases. These are considered in greater detail in section 3.

If the isometry group of the physical metric is ten-dimensional, the spacetime has constant curvature. When the curvature is non-zero (i.e. for de Sitter and anti de Sitter spacetimes) the group of Ricci collineations and isometries coincide and there are no proper Ricci collineations, whereas for the case of zero curvature (flat space) any continuous transformation is trivially a Ricci collineation. When the physical metric admits a complete seven-dimensional isometry group (˙S = 0 and K ̸= 0, i.e. the Einstein static universes), the Ricci tensor is degenerate. Collineations when the Ricci tensor is degenerate are considered in the next section.

2. Collineations when the Ricci Tensor is Degenerate

From equation (3), the Ricci tensor of the FRW metric is degenerate if A(t) = −3S/S = 0 or B(t) = 2K + 2S2 + SS = 0 (or both). If A = B = 0 then either S(t) = S0 and K = 0 (Minkowski metric) or S(t) = t − t0 and K = −1 (Milne metric), where S0 and t0 are constants. The spacetime is flat and so this case is trivial as the Ricci tensor is zero and any vector field generates a Ricci collineation. For the case B = 0 and A ̸= 0, equation (5) reveals that the vector ξ generates a RIC if and only if ξ0 is a function of t only; the spatial components ξα are completely arbitrary. The scale function φ = ξ0 + ˚A/(2A)ξ0 is necessarily a function of t only, but is otherwise arbitrary. For RCs ξ0 satisfies the differential equation 2Aξ0 + ˚Aξ0 = 0. Hence ξ0 = d|A|−1/2 where d is a constant. The groups of RICs and RCs are both infinite-dimensional. A first integral of the equation B = 2K + 2S2 + SS = 0 is ˙S2 = c/S4 − K where c is a constant and hence ˙S = −2c/S5. For K = 0 the first integral may be integrated to give S(t) = S0(t − t0)1/3 where S0 and t0 are constants. The solution when K ̸= 0 involves elliptic integrals (see Appendix A).

If A = 0 but B ̸= 0, then S = αt + β and hence B = 2(K + α2) where α and β are constants where α ̸= 1 if K = −1 and α ̸= 0 if K = 0. Equation (5) shows that the spatial components ξα must be independent of t and must satisfy the conformal Killing equations of the three-dimensional metric (6) with conformal factor φ. It is well known (Robertson and Noonan, 1969) that there are ten independent conformal Killing vectors of this metric:

\[
\begin{align*}
X_1 &= \partial_x \\
X_2 &= \partial_y \\
X_3 &= \partial_z \\
X_4 &= y\partial_z - z\partial_y \\
X_5 &= z\partial_x - x\partial_z
\end{align*}
\]
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\[ X_6 = x\partial_y - y\partial_x \]  \hspace{1cm} (7f)
\[ X_7 = x\partial_x + y\partial_y + z\partial_z \]  \hspace{1cm} (7g)
\[ X_8 = (r^2 - 2x^2)\partial_x - 2xy\partial_y - 2xz\partial_z \]  \hspace{1cm} (7h)
\[ X_9 = -2xy\partial_x + (r^2 - 2y^2)\partial_y - 2yz\partial_z \]  \hspace{1cm} (7i)
\[ X_{10} = -2xz\partial_x - 2yz\partial_y + (r^2 - 2z^2)\partial_z \]  \hspace{1cm} (7j)

Thus the spatial components \( \xi^\alpha \) must have the form

\[ \xi^\alpha = \sum_{j=1}^{10} f_J X_j^\alpha \]  \hspace{1cm} (8)

where the \( f_J \)'s are constants. The associated conformal factor is given by

\[ \sigma = \frac{f_7(1 - K/4r^2) - K/2f\cdot r - 2g\cdot r}{1 + K/4r^2} \]  \hspace{1cm} (9)

where \( f = f_1i + f_2j + f_3k, \ g = f_8i + f_9j + f_{10}k \) and \( r = xi + yj + zk \); standard 3-dimensional vector notation has been used for conciseness. The time component \( \xi^0 \) is completely arbitrary and the scale factor of the RIC is given by \( \phi = \sigma \) where \( \sigma \) is given by equation (9). Thus the group of RICs is infinite dimensional.

For RCs the \( \xi^\alpha \)'s are independent of \( t \) and must satisfy the Killing equations of the metric (6). As is well-known there are six independent Killing vectors of this metric, namely:

\[ Y_1 = X_1 - K/4X_8 \]  \hspace{1cm} (10a)
\[ Y_2 = X_2 - K/4X_9 \]  \hspace{1cm} (10b)
\[ Y_3 = X_3 - K/4X_{10} \]  \hspace{1cm} (10c)
\[ Y_4 = X_4 \quad Y_5 = X_5 \quad Y_6 = X_6 \]  \hspace{1cm} (10d)

where the vectors \( X_j \) are as in equation (7). The most general form for \( \xi^\alpha \) is a general linear combination of these six vectors. The time component \( \xi^0 \) is completely arbitrary, and thus the group of RCs is infinite-dimensional.

If \( \alpha = 0 \) the spacetime is a static Einstein universe. If \( \alpha \neq 0 \), by a translation of the time coordinate we may set \( \beta = 0 \) and hence \( S = at \). We note that the physical metric admits a homothetic Killing vector given by \( Z = t\partial_t \) with associated conformal factor \( \sigma = 1 \) (Maartens and Maharaj, 1986). These results on Ricci collineations generalise those of Green et al. (1977) and Núñez et al. (1990) who investigated Ricci collineations in which the spatial components \( \xi^\alpha \) of the generating vector vanished or were purely radial respectively.

3. The Generating Vectors of RICs

In this section an expression is presented for the general form of a vector generating a Ricci inheritance collineation in a FRW spacetime when the Ricci tensor is non-degenerate. Since Ricci inheritance collineations are conformal symmetries of the
Ricci tensor metric which is of FRW form, essentially the results are already in
the literature. The isometries and conformal symmetries of FRW metrics have been
extensively studied by Maartens and Maharaj (1986) and by Keane and Barrett (2000)
and only need to be reinterpreted in the current context. However, in both these papers
the authors used curvature coordinates (rather than stereographic coordinates) for the
spatial metric and naturally proper time was used as a coordinate. Furthermore only
metrics with Lorentzian signature were considered. In the current context since proper
time of the physical metric will not usually coincide with ‘proper time’ of the Ricci
tensor metric, it is convenient to work with the metric in the form of equation (4)
without specialising the time coordinate and allowing both Lorentzian and definite
signatures. Stereographic coordinates are used rather than curvature coordinates as
this simplifies the form of the results slightly. Essentially the approach used is to
‘lift’ the ten conformal Killing vectors of the hypersurfaces of constant $t$ to become
conformal Killing vectors of the whole spacetime.

The spatial components $\xi^\alpha$ of any conformal Killing vector of the Ricci tensor
metric (4) satisfy the conformal Killing equations for the metric (6) and thus must
have the form

$$\xi^\alpha = \sum_{J=1}^{10} f_J(t) X_J^\alpha$$

where $X_J$ for $J = 1 \ldots 10$ are given by equation (6) and the $f_J$’s are functions of $t$.
The corresponding conformal factor is given by

$$\phi = \frac{f_7(1-K/4r^2) - K/2f_9r - 2g_7r}{1 + K/4r^2} + \frac{\dot{B}\xi^0}{2B}$$

where the three-dimensional vector notation of section 2 has again be used. Equation
(5b) leads to the following compatibility conditions

$$\dot{f}_8 = K/4\dot{f}_1 \quad \dot{f}_9 = K/4\dot{f}_2 \quad \dot{f}_{10} = K/4\dot{f}_3 \quad \dot{f}_4 = \dot{f}_5 = \dot{f}_6 = 0$$

For $K = 0$, integration of equations (5b) and (5c) gives

$$\xi^0 = -B/A(\ddot{f}.r + 1/2r^2\dot{f}_r) + h(t)$$

where the $f(t)$ and $h(t)$ satisfy the following compatibility conditions:

$$\ddot{f}_r = 1/2(\dot{A}/A - \dot{B}/B)\dot{f}_r + 2A/Bg$$

$$\ddot{f}_\tau = 1/2(\dot{A}/A - \dot{B}/B)\dot{f}_\tau$$

$$\dot{h} = -1/2(\dot{A}/A - \dot{B}/B)h + f_7$$

Equations (13) and (15)-(17) constitute a linear differential system for eleven
unknowns $f_J$ and $h$. Introducing new unknowns for the first derivatives of $f_1$, $f_2$, $f_3$ and
$f_7$ we may reduce it to a first order linear differential system for fifteen unknowns. The
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The general solution thus depends on 15 arbitrary constants and hence (as expected) there are 15 independent conformal Killing vectors of the Ricci tensor metric or equivalently 15 RICs of the physical metric. The general solution of the system is given by

\[ f_i(t) = a_i T + b_i + \epsilon c_i T^2 \] (18a)
\[ f_{3+i}(t) = d_i \] (18b)
\[ f_7(t) = k T + l \] (18c)
\[ f_{7+i}(t) = c_i \] (18d)
\[ h(t) = |B/A|^{1/2}(1/2kT^2 + lT + m) \] (18e)

where \( k, l, m, a_i, b_i, c_i \) and \( d_i \) \((i = 1 \ldots 3)\) are arbitrary constants; the constant \( \epsilon \) is +1 or −1 when the Ricci tensor metric has definite or Lorentzian signature respectively.

The quantity \( T \) is given by

\[ T = \int |A/B|^{1/2} dt \] (19)

and is a ‘conformal time’ coordinate for the Ricci tensor metric. For \( K = 0 \), the spatial components \( \xi^\alpha \) of any vector generating an RIC are given by equation (11) with the \( f_J(t) \)'s given by equation (18) and the time component \( \xi^0 \) is given by

\[ \xi^0 = |B/A|^{1/2}(1/2kT^2 + lT + m - \epsilon a.r - 2Tc.r - 1/2\epsilon kr^2) \] (20)

For \( K \neq 0 \) the corresponding results are

\[ \xi^0 = -B/A(\dot{\mathbf{f}}.\mathbf{r} - 2/K\dot{f}_7)/(1 + K/4r^2) + h(t) \] (21)

and

\[ \ddot{\mathbf{f}} = 1/2(\dot{A}/A - \dot{B}/B)\dot{\mathbf{f}} + \epsilon A/B(K/2\mathbf{f} + 2\mathbf{g}) \] (22)
\[ \ddot{f}_7 = 1/2(\dot{A}/A - \dot{B}/B)\dot{f}_7 + \epsilon KA/Bf_7 \] (23)
\[ \ddot{h} = -1/2(\dot{A}/A - \dot{B}/B)h - f_7 \] (24)

The linear system of differential equations (13) and (22)-(24) again has a fifteen parameter solution. When \( K \epsilon = -1 \),

\[ f_i(t) = a_i \cos(T) + b_i \sin(T) - 2Kc_i \] (25a)
\[ f_{3+i}(t) = d_i \] (25b)
\[ f_7(t) = k \cos(T) + l \sin(T) \] (25c)
\[ f_{7+i} = K/4(\mathbf{a} \cos(T) + \mathbf{b} \sin(T)) + 1/2c_i \] (25d)
\[ h(t) = |B/A|^{1/2}(-k \sin(T) + l \cos(T) + m) \] (25e)

where \( T, \epsilon \) etc. are as in the \( K = 0 \) case. The spatial components \( \xi^\alpha \) of any vector generating an RIC are given by equation (11) with the \( f_J(t) \)'s given by equation (25) and the time component \( \xi^0 \) is given by

\[ \xi^0 = |B/A|^{1/2} \left( \frac{\epsilon a.r \sin(T) - \epsilon b.r \cos(T) + 2k \sin(T) - 2l \cos(T)}{1 + K/4r^2} ight) \]
\[ - k \sin(T) + l \cos(T) + m \] for \( K \epsilon = -1 \) (26)
When $K\epsilon = +1$,

\begin{align}
    f_i(t) &= a_i \cosh(T) + b_i \sinh(T) - 2Kc_i \\
    f_{3+i}(t) &= d_i \\
    f_7(t) &= k \cosh(T) + l \sinh(T) \\
    f_{7+i} &= K/4(a_i \cosh(T) + b_i \sinh(T)) + 1/2c_i \\
    h(t) &= |B/A|^{1/2}(-k \sinh(T) - l \cosh(T) + m)
\end{align}

(27a) (27b) (27c) (27d) (27e)

The spatial components $\xi^0$ are given by equation (11) with the $f_i(t)$'s given by equation (27) and the time component $\xi^0$ is

$$
\xi^0 = |B/A|^{1/2}\left(\frac{-e.a.r \sinh(T) - e.b.r \cosh(T) + 2k \sinh(T) + 2l \cosh(T)}{1 + K/4r^2}
\right.
$$

$$
\left. - k \sinh(T) - l \cosh(T) + m\right) \quad \text{for } K \epsilon = +1
$$

(28)

4. Proper Ricci Collineations in the Non-degenerate Case

In this section we assume that the Ricci tensor of the FRW metric is not degenerate. We investigate the conditions on the scale factor $S(t)$ such that the spacetime admits proper RCs in addition to the six Killing vectors in equation (10). Thus by the discussion of section 1 we need to search for extra isometries of the Ricci tensor metric (4). The only possibilities are one extra Killing vector or four extra Killing vectors (Maartens and Maharaj, 1986).

When there is only one extra Killing vector, $K \neq 0$ and the Ricci tensor metric must be equivalent to the metric of an Einstein static universe (or its analogue with definite signature) and thus $\ddot{B} = 0$. The FRW scale factor $S$ and the vector $Z$ generating the proper RC satisfy

$$
\dot{S}^2 = d + c/S^4 \quad Z = S^3 \partial_t 
$$

(29)

where $c$ and $d$ are constants. The functions in the Ricci tensor metric satisfy $B = 2(K + d)$ and $A = 6c/S^6$ and hence for non-degeneracy: $d \neq -K$ and $c \neq 0$. When $d = 0$, the scale factor has the form $S(t) = S_0(t - t_0)^{1/3}$ where $S_0$ and $t_0$ are constants, but in general $S(t)$ will involve elliptic integrals (see Appendix A).

When there are four extra isometries the Ricci tensor metric has constant curvature; this condition is equivalent to

$$
\ddot{B} - \dot{B}(\ddot{B}/B + \dot{A}/(2A)) + 2KA = 0
$$

(30)

Eliminating $A$ and $B$ using equation (3), a complicated fourth order differential equation for $S(t)$ results. To date we have not succeeded in analysing this fully.

† see equation rc10 in the Reduce output file with URL: http://www.aston.ac.uk/~barnesa/cc.res
However when $K = 0$, one solution is $S(t) = c(t-t_0)^d$ where $c$, $d$ and $t_0$ are constants. When $d \neq 1$, this physical metric has only six Killing vectors plus a homothetic vector

$$Z = (t-t_0)/(1-d)\partial_t + x\partial_x + y\partial_y + z\partial_z \quad (31)$$

with corresponding conformal factor $\sigma = 1/(1-d)$. However it has three proper RCs plus seven RCs generated by the homothetic vector $Z$ and the six spatial Killing vectors. This example and the ones given in section 2 are the only FRW metrics admitting an homothetic vector (Maartens and Maharaj, 1986).

Other solutions of equation (30) are known:

$$S(t) = ce^{t/d} \quad \text{for } K = 0 \quad (32a)$$

$$S(t) = d \cosh((t-t_0)/d) \quad \text{for } K = 1 \quad (32b)$$

$$S(t) = d \sinh((t-t_0)/d) \quad \text{for } K = -1 \quad (32c)$$

$$S(t) = d \sin((t-t_0)/d) \quad \text{for } K = -1 \quad (32d)$$

However these all describe spacetimes of constant curvature and so the group of RCs coincides with the isometry group.

As we have seen above generic FRW metrics do not admit proper Ricci collineations; these exist only in special cases when the scale factor $S(t)$ has a special form such that the Ricci tensor metric is degenerate, or of constant curvature or isometric to the metric of a static Einstein universe. This result contradicts a claim by Carot et al. (1997) that all FRW metrics admit a RC with a generating vector $\xi^i$ which has only $t$ and $r$ components; we find that the vector given by their equation (75) only satisfies two of the four required relations given in their equation (74). According to our calculations the radial component can only be non-zero if $K = 0$ and if the scale factor $S(t)$ satisfies equation (30). For $K \neq 0$, the radial component must vanish and the scale factor must satisfy equation (29). Similar results have been derived under much more restrictive assumptions by Núñez et al. (1990).

The analysis above does not directly yield expressions for the additional Killing vectors of the Ricci tensor metric when these exist. However, in section 3 expressions for the most general vector $\xi^i$ generating a RIC were obtained. For RCs we must impose the condition $\phi = 0$ where $\phi$ is given by equation (12). Using equations (18) and (19) we see that $\phi = 0$ is a second order polynomial equation in $x$, $y$ and $z$ with coefficients which are functions of $t$ only. Equating these coefficients to zero leads to 4 conditions restricting $B$ and the constants $a_i$, $c_i$, $k$, $l$ and $m$.

For $K = 0$, equating the coefficient of $r^2$ to zero we find $kB = 0$. Now if $B = 0$, it follows that

$$f_i = a_iT + b_i \quad f_{i+3} = d_i \quad f_7 = f_{7+i} = 0 \quad h = |B/A|^{1/2}m \quad (33)$$

Thus the general vector generating an RC involves ten arbitrary constants; it has spatial components $\xi^\alpha$ given by equation (11) with the $f_j$’s given by equation (33) and time component given by $\xi^0 = |B/A|^{1/2}(m - \epsilon a r)$. There are 4 proper RCs plus the
usual six isometries. If $\dot{B} \neq 0$, then $k = 0$. The remaining equations are compatible only if $B = B_0/(T - T_0)^2$, where $B_0$ and $T_0$ are constants. However $T$ is only defined up to an additive constant and so, without loss of generality, we may set $T_0 = 0$. It follows that

$$f_i = b_i \quad f_{i+3} = d_i \quad f_7 = l \quad f_{7+i} = c_i \quad h = |B/A|^{1/2} l T$$  \hspace{1cm} (34)$$

Thus the general vector generating an RC involves ten arbitrary constants; its spatial components $\xi^\alpha$ are given by equation (8) with the $f_j$’s given by equation (34) and its time component is $\xi^0 = |B/A|^{1/2} T (l - 2c.r)$. Again there are 4 proper RCs plus the usual six isometries. The metric with scale factor $S(t) = ct^d$ discussed earlier in this section belongs to this class. Similar results (in curvature coordinates) have recently been obtained independently by Apostolopoulos and Tsamparlis (2001) for the spatially flat case ($K = 0$) only.

For $K \neq 0$, the condition $\phi = 0$ leads to the following relations

$$\dot{B}h - 2B f_7 = 0$$  \hspace{1cm} (35a)$$

$$2KA f_7 + \dot{B} f_7 = 0$$  \hspace{1cm} (35b)$$

$$A(K f + 4g) + \dot{B} f = 0$$  \hspace{1cm} (35c)$$

If $\dot{B} = 0$, it follows that $a_i = b_i = k = l = 0$ and $h(t) = m|B/A|^{1/2}$. Thus there is only one proper RC generated by the vector $Z = |B/A|^{1/2} \partial_t$. If $\dot{B} \neq 0$, the equations are only compatible for special forms of $B$. When $K \epsilon = -1$, essentially the only possibility is $B = B_0 / \sin^2(T)$. Then $b_i = l = m = 0$ and the general vector generating an RC involves ten arbitrary constants $a_i, c_i, d_i$ and $k$ and is given by equations (8) and (26) with the $f_j$’s given by equation (25). Again there are four proper RCs plus the usual six isometries. The case $K \epsilon = +1$ is similar except that now there are essentially three possibilities:

$$B = B_0 / \cosh^2(T) \quad B = B_0 / \sinh^2(T) \quad B = \exp(\pm 2T)$$  \hspace{1cm} (36)$$

In each case there is a ten-parameter group of RCs consisting of four proper RCs and the usual six isometries. For example when the first of equations (36) holds, $a_i = k = m = 0$ and the general vector generating an RC involves ten arbitrary constants $b_i, c_i, d_i$ and $l$ and is given by equation (8) and (28) with the $f_j$’s given by equation (27). The other two cases are similar.

The analysis of sections 3 and 4 involves some rather heavy algebraic manipulations and has been checked using the computer algebra systems CLASSI (Áman, 1987) and Reduce (Hearn, 1995).

5. Matter Collineations

In this section we consider briefly matter inheritance collineations and matter collineations admitted by FRW spacetimes. As we will assume the validity of Einstein’s
field equations $G_{ij} = \kappa T_{ij}$, a vector $\xi^i$ generates a matter inheritance collineation if $\mathcal{L}_\xi G_{ij} = 2\phi G_{ij}$. If $\phi = 0$, it is said to admit a matter collineation. If the Einstein tensor is non-degenerate, then following the same arguments as in the Introduction, we may conclude that the maximal groups of matter inheritance collineations and matter collineations are fifteen and ten respectively. Matter inheritance collineations and plain matter collineations are respectively conformal symmetries and isometries of the Einstein tensor metric.

For FRW spacetimes the Einstein tensor metric has FRW form:

$$ds_{\text{Ein}}^2 = G_{ij}dx^i dx^j = A(t)dt^2 + B(t)(1 + K/4r^2)^{-2}(dx^2 + dy^2 + dz^2)$$

(37)

where now

$$A = 3(K + \dot{S}^2)/S^2 = \kappa \rho \quad B = -(K + \dot{S}^2 + 2S\ddot{S}) = \kappa \rho S^2$$

(38)

In the rest of this section $A$ and $B$ will refer to those quantities defined in equation (38) rather than those defined immediately after equation (4).

The Einstein tensor metric is positive-definite when $\rho > 0$ and $p > 0$. Only two distinct degenerate cases occur: $A = B = 0$ and $B = 0$ with $A \neq 0$ as the condition $A = 0$ implies $B = 0$. In the first case the Einstein tensor vanishes and the spacetime is flat. Trivially any continuous transformation is a matter (inheritance) collineation. The second case corresponds to zero pressure (or matter-dominated) FRW models and in this case the groups of matter inheritance collineations and matter collineations are both infinite-dimensional. The generators have spatial components $\xi^\alpha$ which are completely arbitrary and the time component $\xi^0$ is a function of $t$ only; for inheritance collineations $\xi^0$ is otherwise arbitrary, whereas for plain matter collineations it is given by $\xi^0 = d|A|^{-1/2}$. The form of the scale factor $S(t)$ for pressure-free FRW models is well known and appears in many books on relativity and cosmology, for example Stephani (1982).

In the non-degenerate case the situation is again similar to, but slightly simpler than, the case of Ricci collineations discussed in sections 1, 3 and 4. There is always a fifteen parameter group of matter inheritance collineations which are the conformal symmetries of the Einstein tensor metric (which is conformally flat). The generating vectors take the same form as in section 3. For generic FRW metrics the physical metric and the Einstein tensor metric will admit the same isometry group; the group of matter collineations coincides with the usual six-dimensional group of motions generated by the Killing vectors given in equation (10). If the physical metric admits a (complete) isometry group of dimension seven, that is if the spacetime is an Einstein static universe ($k \neq 0$ and $\dot{S} = 0$), then the Einstein tensor metric is also isometric to that of an Einstein static universe. The functions $A$ and $B$ are both constant and the group of matter collineations coincides with the isometry group. If the physical metric has (non-zero) constant curvature, the group of matter collineations again coincides with the isometry group which is now ten-dimensional.
In all the non-degenerate cases discussed above there are no proper matter collineations, but the possibility remains that the physical metric admits only a six-dimensional isometry group whereas the Einstein tensor metric admits a higher dimensional group of motions (necessarily of dimension 7 or 10) and so admits matter collineations which are not isometries. As in section 4 two cases arise.

If \( K \neq 0 \), \( \dot{S} \neq 0 \) but \( \dot{B} = 0 \), the Einstein tensor metric is static whereas the physical metric is not. The group of matter collineations is seven-dimensional and is generated by the usual six Killing vectors plus the vector \( Z = |A|^{-1/2} \partial_t \). The scale factor \( S(t) \) satisfies \( 2\dot{S}\dot{S} + S^2 = -(K + B_0) \) where \( B_0 \) is a constant (\( = B \)). This is the same relation as that satisfied by the scale factor for a pressure-free FRW model, but with \( K \) replaced by \( K + B_0 \). Thus the integrated forms for \( S(t) \) can easily be obtained from the standard forms in the literature (Stephani, 1982).

If the Einstein tensor metric has constant curvature, but the physical metric does not, then the group of matter collineations will be ten-dimensional and hence there will be four independent proper matter collineations in addition to the usual six Killing vectors. The functions \( A \) and \( B \) appearing in equations (37) and (38) must satisfy equation (30) which again leads to a complicated fourth order non-linear differential expression for the scale factor \( S(t) \). Again we have not been able to completely analyse this expression; however when \( K = 0 \), \( S(t) = c(t-t_0)^d \) is again a solution. Thus the metric admits three proper matter collineations plus seven matter collineations generated by the homothetic vector \( Z \) in equation (31) and the six Killing vectors. The scale factors \( S(t) \) given by equation (32), all of which correspond to physical metrics of constant curvature again satisfy this fourth order differential expression.

The form of the generating vectors can be obtained using the methods of section 4. We conclude by briefly investigating the relation between Ricci and matter collineations. Suppose \( \xi^i \) satisfies \( \mathcal{L}_\xi R_{ij} = 2\phi R_{ij} \) and hence generates a Ricci inheritance collineation. If \( R = 0 \), a Ricci (inheritance) collineation is obviously a matter (inheritance) collineation and vice-versa. A simple calculation shows that

\[
\mathcal{L}_\xi G_{ij} = 2\phi G_{ij} + 1/2(h^{kl} R_{kl}g_{ij} - Rh_{ij})
\]

where \( h_{ij} = \mathcal{L}_\xi g_{ij} \). Hence if \( R \neq 0 \), a Ricci collineation is a matter collineation iff \( h_{ij} = \mathcal{L}_\xi g_{ij} \propto g_{ij} \); that is iff \( \xi^i \) is a conformal Killing vector. A similar results holds for inheritance collineations (with the same ‘inheriting’ factor \( \phi \)).

6. Summary

When the Ricci tensor is degenerate, the groups of Ricci collineations and Ricci inheritance collineations of a FRW spacetime are both infinite-dimensional. The FRW scale factors leading to these infinite-dimensional groups have been obtained in closed form. When the Ricci tensor is non-degenerate, Ricci collineations and Ricci

\[\text{‡ see equation mc10 in the Reduce output file with URL: http://www.aston.ac.uk/~barnesa/cc.res}\]
inheritance collineations are respectively isometries and conformal symmetries of the Ricci tensor metric. For FRW metrics the Ricci tensor metric also assumes FRW form and so it follows immediately that the group of Ricci inheritance collineations is of dimension fifteen and, for a generic FRW metric, the group of Ricci collineations is of dimension six and coincides with the isometry group. For special FRW metrics the group of Ricci collineations may be larger; it has dimension seven when the Ricci tensor metric is isometric to that of an Einstein static universe and dimension ten when the Ricci tensor metric has constant curvature. Conditions on the scale factor of the physical metric for these higher dimensional groups to occur have been obtained. Those scale factors leading to a seven-dimensional group are known explicitly whereas those leading to a ten-dimensional group are currently only known up to the solution of a complicated non-linear fourth order differential equation. A few special solutions of this equation have, however, been obtained.

The most general form of a vector generating a Ricci collineation or Ricci inheritance collineation has been obtained for both the degenerate and non-degenerate cases. A parallel set of results has been obtained for matter collineations and matter inheritance collineations of FRW metrics.

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Appendix A. Solution of the Equation $\dot{B}(t) = 0$

If the function $B(t)$ of section 4 is a constant, $B_0$ say, then the scale factor $S(t)$ satisfies $\ddot{S} + 2\dot{S}^2 = -2(K - B_0/2)$. Integration of this equation yields

$$t - t_0 = 3\int \frac{S^2dS}{(S_0 - 9(K - B_0/2)S^4)^{1/2}}$$

(A1)

where $t_0$ and $S_0$ are integration constants. It is possible to express the integral (A1) for all values of $K$ in terms of hypergeometric functions; it is given by the following unified form

$$t - t_0 = \frac{S^3}{S_0^{1/2}} 2F_1 \left(\frac{1}{2}, \frac{3}{4}; \frac{7}{4}; \frac{18K - B_0}{2S_0} S^4\right),$$

(A2)

where $2F_1$ is a hypergeometric function and $S_0 \neq 0$. The above unified form clearly depends on the values of $S_0$ and $K$.

In the special case $B(t) = 0$ given in section 2, the constant $B_0$ vanishes. Then $\rho = p$ and we have a fluid with the equation of state of ‘stiff’ matter. In this case, integrating the equation $B(t) = 0$, three kinds of solution for $S(t)$ occur: (i) $K = 0$, (ii) $K = +1$ and (iii) $K = -1$. In case (i) the integral is elementary, namely...
\[ S = S_0^{1/6}(t - t_0)^{1/3}. \] For the other two cases, in addition to the solution obtained by Núñez et al. (1990), the closed form of the solution of the integral (A1) may be found in terms of elliptic integrals of the second kind as follows:

\[ t - t_0 = \left( \frac{S_0}{729K^2} \right)^{1/4} \left[ E\left( \frac{\pi}{4}, 2 \right) - E(\gamma, 2) \right], \quad (A3) \]

where \( \gamma = (1/2) \cos^{-1}\left( \frac{3S^2}{\sqrt{S_0/K}} \right) \).

References

Apostolopoulos P S and Tsamparlis M 2001 Preprint: Comment on Ricci Collineations for spherically symmetric space-times arXiv:gr-qc/0108064

Carot J, Núñez L A and Percoco U 1997 Gen. Rel. Grav. 29 1223–37

Duggal K L 1993 Acta Appl. Math. 31 225

Green L H, Norris L K, Oliver D R and Davis W R 1977 Gen. Rel. Grav. 8 731

Hearn A C 1995 Reduce User’s Manual, Version 3.6 Rand, Santa Monica, CA

Katzin G H, Levine J and Davis H R 1969 J. Math. Phys. 10 617–9

Keane A J and Barrett R K 2000 Class. Quantum Grav. 17 201–18

Maartens R and Maharaj S D 1986 Class. Quantum Grav. 3 1005–11

Núñez L A, Percoco U and Villaiba V M 1990 J. Math. Phys. 31 137–9

Robertson H P and Noonan T W 1969 Relativity and Cosmology Sanders, Philadelphia PA

Stephani H 1967 Commun. Math. Phys. 4 137–42

Stephani H 1982 General Relativity Cambridge University Press, Cambridge UK

˚Aman J E 1987 Manual for CLASSI: classification program for geometries in general relativity University of Stockholm, Institute of Theoretical Physics technical report