Noncommutative SUSY Black Holes

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In this paper we propose a generalization to the Schwarzschild metric and define noncommutative SUSY black holes. We introduce the noncommutative deformation to the minisuperspace variables and derive the noncommutative supersymmetric (SUSY) Wheeler-DeWitt (WDW) equation for the Schwarzschild black hole. We calculate the metric and find that the singularities are not removed.
I. INTRODUCTION

Black Holes are one of the most enigmatic and fascinating objects in physics. It has been a very active area of research, and due to the spectacular observations from the LIGO \cite{1} and the Event Horizon collaborations \cite{2}, there has been a resurgence in the study of theoretical puzzles in black hole physics. Moreover, because there are black hole solutions in alternative theories of gravity, black holes are used as workhorses to probe fundamental aspects of these theories (i.e. supergravity, noncommutative gravity, Hordenski gravity, etc.).

The idea of a noncommutative space-time was revived at the beginning of the century. There have been several attempts to study the possible effects of noncommutativity in the cosmological scenario. In previous works \cite{3–5}, it has been argued that there is a possible relationship between the cosmological constant and the noncommutative parameters. Moreover, the effects of noncommutativity during inflation were explored \cite{6}, but noncommutativity was only incorporated in the matter Lagrangian, neglecting the gravitational sector. The difficulties of analyzing noncommutative gravitational models arise from the complicated structure of noncommutative gravity. Writing a noncommutative theory of gravity which only depends on the commutative fields and their derivatives has field equations that are highly nonlinear \cite{7, 8}. To avoid these difficulties, it has been proposed to introduce the effects of noncommutativity at the quantum level \cite{9}, namely quantum cosmology, by deforming the minisuperspace through a Moyal deformation of the Wheeler-DeWitt (WDW) equation. It is then possible to proceed in noncommutative quantum mechanics \cite{20}.

Using this approach for noncommutative cosmology and using the diffeomorphism to transform the Schwarzschild metric into the Kantowski-Sachs (KS) metric, one obtains the noncommutative Wheeler-DeWitt (NC-WDW) equation for the Schwarzschild black hole \cite{11}. From the NC-WDW equation and the Feynman-Hibbs method, they calculate the entropy of the noncommutative black hole. This method was originally used for the Schwarzschild black hole, to reproduce the known results \cite{12}.

We know that supergravity is the supersymmetric generalization of general relativity (GR). Supersymmetric black hole solutions of supergravity theories played a crucial role in important developments in string black hole physics. In supergravity, only black holes that satisfy the BPS constraints are well understood. Consequently non BPS black holes have not been extensively studied.

One approach that has been used to study SUSY black holes is to use the relationship between

\footnote{The simplest case for a spherical symmetric solution that meets this criteria is the Reissner-Nordström extremal black hole.}
the KS and the Schwarzschild metrics, to introduce supersymmetry. Along this line of reasoning, classical (and quantum) supersymmetric Schwarzschild and Schwarzschild-(anti) de Sitter black hole models were proposed [13, 14]. By supersymmetrizing the WDW equation associated with the standard Schwarzschild black hole. The authors derive a modified (SUSY quantum) Hamiltonian and its corresponding classical equations that in this sense define a supersymmetric generalization of the Schwarzschild and Schwarzschild-(anti) de Sitter space-times.

Exploiting the relationship between the KS and Schwarzschild metric we can introduce new physical ideas to black holes. It was effective to introduce noncommutative effects using the WDW equation allowing us to construct noncommutative Schwarzschild black hole. It is reasonable to assume that noncommutative constructions which deal with the supersymmetric versions of the WDW equation have a similar behaviour [13], this motivate us to study the effects of noncommutativity on SUSY black holes, using the NC-SUSY WDW equation, therefore the main objective of this paper is to explore black holes in the context of noncommutativity and supersymmetry.

The paper is organized as follows. In section II we review the proposal for SUSY black holes [13], we also discuss noncommutative black holes. In Section III we present our proposal for noncommutative SUSY black Holes. Section IV is devoted to discussion and final remarks.

II. MODIFYING THE WHEELER-DEWITT EQUATION

In this section we discuss how to introduce the new physics to the Schwarzschild black hole. We will briefly discuss SUSY black holes from the WDW equation. We also discuss the introduction of noncommutativity to black holes by deforming the WDW equation.

A. Wheeler-DeWitt equation and the SUSY black hole

Let us start by recalling the classical and quantum aspects of the supersymmetric cosmological KS model and the Schwarzschild black hole. We use the square root and operator method. Where the resulting Hamiltonian has terms that allow us to find a supersymmetric solution [13]. From the Schwarzschild metric

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \]  

we can find the relationship between the cosmological Kantowski-Sachs metric and the Schwarzschild metric, by doing the coordinate transformation \( t \leftrightarrow r \). Also, \( g_{tt} \) and \( g_{rr} \) change their sign and \( \partial_t \)
becomes a space-like vector. Finally, we compare the Kantowski-Sachs metric with the parameterization by Misner and identify

\[ N^2 = \left( \frac{2M}{t} - 1 \right)^{-1}, \quad e^{2\sqrt{3}\beta} = \frac{2M}{t} - 1, \quad e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} = t^2, \]  

(2.2)

where we know that

\[ ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dr^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \]  

(2.3)

We canonically quantize this model and get the Wheeler-DeWitt equation for the Kantowski-Sachs metric. Which, with some particular factor ordering, is given by

\[ \left[ -\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \beta^2} + 48e^{-2\sqrt{3}\Omega} \right] \psi(\Omega, \beta) = 0. \]  

(2.4)

To construct the supersymmetric generalization for the WDW equation Eq.(2.4), we follow the procedure for SUSY quantum cosmology [13]. First we need to find a diagonal Hamiltonian operator constructing the supercharges and get the supersymmetric KS WDW equation

\[ \left[ -\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \beta^2} + 12e^{-2\sqrt{3}\Omega} (4 \pm e^{\sqrt{3}\Omega}) \right] \psi_{\pm}(\Omega, \beta) = 0. \]  

(2.5)

For the metric, we apply the WKB method and from the semiclassical equivalent to Eq.(2.5), we find

\[ 4e^{-2\sqrt{3}\Omega} \left( 1 + 2t\sqrt{3}\Omega \right) - t^2 \left( 4 \pm e^{\sqrt{3}\Omega} \right) = 0, \]  

(2.6)

for our purposes, the only physically relevant asymptotic region of Eq.(2.6) is \( 4 \ll e^{\sqrt{3}\Omega} \). From the solutions for the asymptotic regions we get

\[ e^{-\sqrt{3}\Omega} = \left( \frac{3}{4} \right)^{1/3} r^{2/3} \left( \pm 1 + \frac{C}{\sqrt{r}} \right)^{1/3}, \]  

(2.7)

where \( C \) is a constant. Using these solution and Eq.(2.3) we construct the metric

\[ ds^2 = -\left( \frac{3}{4} \right)^{2/3} r^{-2/3} \left( \pm 1 + \frac{C}{\sqrt{r}} \right)^{2/3} dt^2 + \left( \frac{4}{3} \right)^{2/3} r^{2/3} \left( \pm 1 + \frac{C}{\sqrt{r}} \right)^{-2/3} dr^2 \]

\[ + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \]  

(2.8)

which is formally equivalent to Eq.(2.3). Note that there are two cases in Eq.(2.8), it was shown that the interesting and physical case is for \( C > 0 \). For this case, there are two singularities in \( r = 0 \) and \( r = C^2 \). Moreover, it was suggested that because the semiclassical limit of Eq.(2.5) contains the “fermionic” information, consequently, it is reasonable to expect the lack of a horizon. This was analyzed by solving the Dirac equation in the Schwarzschild and Kerr backgrounds and determining that the spinors destroy the horizon [17].
B. Noncommutative Black Hole

The starting point for the analysis is to consider the noncommutative proposal of quantum cosmology [9]. Where the Cartesian coordinates $\Omega$ and $\beta$ of the KS minisuperspace variables are introduced as a canonical deformation in the algebra of the minisuperspace operators,

$$[\hat{\Omega}, \hat{\beta}] = \theta, \quad [\hat{\Omega}, \hat{P}_\Omega] = [\hat{\beta}, \hat{P}_\beta] = 1, \quad [\hat{P}_\Omega, \hat{P}_\beta] = 0. \quad (2.9)$$

Given the Weyl quantization procedure in the context of the above description, the realization of the commutation relation Eq.(2.9) between the minisuperspace variables is made by a specific Moyal product

$$f(\Omega, \beta) \ast g(\Omega, \beta) = f(\Omega, \beta)e^{(i\theta/2)(\partial_\Omega \partial_\beta - \partial_\beta \partial_\Omega)} g(\Omega, \beta). \quad (2.10)$$

This leads to a shift in the variables

$$\hat{\Omega} = \Omega - \theta P_\beta, \quad \hat{\beta} = \beta + \frac{\theta}{2} P_\Omega. \quad (2.11)$$

The Moyal product functions will be applied to find the WDW equation, this gives a modified WDW equation for the noncommutative model

$$\left[-P_\Omega^2 + P_\beta^2 - 48e^{-2\sqrt{3}\Omega}\right] \ast \psi(\Omega, \beta) = 0. \quad (2.12)$$

As is known in noncommutative quantum mechanics, the original phase-space is modified. It is possible to reformulate in terms of the commutative variables and the ordinary product of functions, if the new variables satisfy Eq.(2.11). Consequently, the original WDW equation changes, with a modified potential $V(\Omega, \beta)$,

$$V(\Omega, \beta) \ast \psi(\Omega, \beta) = V \left(\Omega - \theta P_\beta, \beta + \frac{\theta}{2} P_\Omega\right) \psi(\Omega, \beta), \quad (2.13)$$

so the NC-WDW equation takes the form

$$\left[-\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \beta^2} + 48e^{(-2\sqrt{3}\Omega + \sqrt{3}\theta P_\beta)}\right] \psi(\Omega, \beta) = 0. \quad (2.14)$$

The consequences of the NC-WDW equation were originally analyzed in cosmology [9]. At the quantum level, it gives several new maxima on the probability density depending on the value of the noncommutative parameter $\theta$. The classical solutions have been obtained for the model [18]. Moreover, using the NC-WDW in Eq.(2.14) with the Feynman-Gibbs approach to statistical mechanics, the thermodynamics of noncommutative black holes were studied [11]. Moreover, using Eq.(2.14), the singularity of noncommutative quantum black holes was discussed [19].
As already stated in the previous sections, to construct the noncommutative SUSY black hole we will start with the SUSY WDW equation for the Kantowski-Sachs cosmological model Eq.(2.5). This gives the noncommutative SUSY WDW equation. After introducing the deformed algebra in Eq.(2.9) we get a SUSY generalization of the Schwarzschild black hole
\[
-\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \beta^2} + 12e^{2\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})}(4 \pm e^{\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})}) \psi(\Omega, \beta) = 0.
\] (3.1)

Then we apply the WKB method to the NC SUSY WDW equation. Assuming that the wave function has the form
\[
\psi(\Omega, \beta) = e^{i(S_1(\Omega) + S_2(\beta))},
\] (3.2)
we can construct the Einstein-Hamilton-Jacobi (EHJ) equation. Finally one can derive the equation of motion[13]. This approach is equivalent to using a modified Hamiltonian that includes the noncommutative deformation as well as the SUSY generalization.

From the EHJ equation one can identify \(dS_1(\Omega)/d\Omega \rightarrow P_\Omega \) and \(dS_2(\beta)/d\beta \rightarrow P_\beta \), where \(P_\Omega \) and \(P_\beta \) are the conjugate momentum to \(\Omega \) and \(\beta \), respectively.

We can also consider the noncommutative relations Eq.(2.9) and the shift in the noncommutative variable Eq.(2.11) in Eq.(2.5). From the noncommutative SUSY WDW equation we can write the Hamiltonian as
\[
H = P_\Omega^2 - P_\beta^2 + 12e^{2\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})}(4 \pm e^{\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})}).
\] (3.3)
This is the same Hamiltonian we obtain from Eq. (3.1). We are interested in asymptotic region where supersymmetry can dominate [13]. Therefore, we take the approximation \(e^{\sqrt{3}\Omega} \gg 4 \). This can be considered the “classical supersymmetric” limit. In this limit, the equations of motion are
\[
\dot{\Omega} = -\frac{e^{2\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})}P_\Omega}{12}, \quad \dot{P}_\Omega = +\frac{\sqrt{3}}{2}e^{\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})},
\]
\[
\dot{\beta} = +\frac{\sqrt{3}}{4}e^{\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})} + \frac{e^{2\sqrt{3}(\Omega - \frac{\vartheta}{2}P_{\beta})}P_{\beta}}{12}, \quad \dot{P}_\beta = 0.
\] (3.4)
As one expects that the effects of noncommutativity are very small, and to simplify the calculations, we take the approximation \(\vartheta \ll 1 \). Moreover, one can obtain definitions for the momentum from the Hamiltonian
\[
P_\Omega = -12e^{-2\sqrt{3}\Omega} - 144\sqrt{3}\vartheta e^{-4\sqrt{3}\Omega} \dot{\Omega},
\]
\[
P_\beta = \pm3\sqrt{3}\vartheta e^{-\sqrt{3}\Omega} + 12e^{-2\sqrt{3}\Omega} \dot{\beta} + 144\sqrt{3}\vartheta e^{-4\sqrt{3}\Omega} \dot{\beta}^2.
\] (3.5)
From the mentioned approximations we get

\[-3\Omega^2 - 72\sqrt{3} \Omega e^{-2\sqrt{3}\Omega \beta t^2} + \frac{3\sqrt{3}}{2} \vartheta e^{\sqrt{3}\Omega \beta t^2} + 3\beta^2 \]

\[+ 72\sqrt{3} \vartheta e^{-2\sqrt{3}\Omega \beta t^2} + \frac{1}{4} e^{3\sqrt{3}\Omega t} = 0. \tag{3.7} \]

Setting \[t^2 = e^{-2\sqrt{3}\Omega - 2\sqrt{3}\beta} \] and using the change of variable \[u = e^{-3\sqrt{3}\Omega} \], we get

\[\pm \frac{9}{4} \vartheta t^{-3/2} u^{2/3} + \frac{3}{4} \vartheta t^{-1/2} u^{-1/3} \pm \frac{3}{8} t^{-1/2} - \frac{3}{2} t^{-5/2} u + t^{-3/2} \vartheta \]

\[+ 36 \vartheta t^{-7/2} u^{5/3} - 36 \vartheta t^{-5/2} u^{2/3} \vartheta + 8 \vartheta t^{-3/2} u^{-1/3} u^2 = 0. \tag{3.8} \]

To solve this equation, we use a perturbative method. We start by proposing that \[u = \pm \frac{3}{4} + ct^{3/2} + \vartheta u_1 + O(\vartheta^2) \]. The first and second terms are the same as for the SUSY black hole\[13\]. Substituting, we obtain

\[t^{-3/2} \vartheta u_1 - \frac{3}{2} t^{-5/2} u_1 = \pm \frac{27 (2c \pm \sqrt{t})}{8 \sqrt{3} ct^{3/2} + 6t^2} \tag{3.9} \]

After solving for \(t\) and interchanging \(r \leftrightarrow t\), we arrive at

\[e^{-\sqrt{3}\Omega} = \left( \frac{3}{4} \right)^{1/3} r^{2/3} \left[ \pm 1 + Cr^{-1/2} + \frac{9}{40} \vartheta r^{-1/2} (9C \pm 4\sqrt{r}) (6C \pm 6\sqrt{r})^{2/3} \right]^{1/3} \tag{3.10} \]

\[e^{-\sqrt{3}\beta} = \left( \frac{4}{3} \right)^{1/3} r^{1/3} \left[ \pm 1 + Cr^{-1/2} + \frac{9}{40} \vartheta r^{-1/2} (9C \pm 4\sqrt{r}) (6C \pm 6\sqrt{r})^{2/3} \right]^{-1/3} \tag{3.11} \]

Finally, substituting the solutions in Eq. (3.7), we find the noncommutative SUSY generalization to the Schwarzschild metric

\[
ds^2 = - \left( \frac{3}{4} \right)^{2/3} r^{-2/3} \left[ \pm 1 + \frac{C}{\sqrt{r}} + \frac{9\vartheta (9C \pm 4\sqrt{r}) (6C \pm 6\sqrt{r})^{2/3}}{40\sqrt{r}} \right]^{2/3} dt^2 \\
+ \left( \frac{4}{3} \right)^{2/3} r^{2/3} \left[ \pm 1 + \frac{C}{\sqrt{r}} + \frac{9\vartheta (9C \pm 4\sqrt{r}) (6C \pm 6\sqrt{r})^{2/3}}{40\sqrt{r}} \right]^{-2/3} dr^2 \\
+ r^2 (d\vartheta^2 + \sin\theta d\varphi^2). \tag{3.12} \]

These solutions are valid only in the asymptotic region \[e^{\sqrt{3}\Omega} >> 4 \]. The different solutions to reconstruct the metric give rise to some branches that are not physical as well as some singularities, these are summarized in table I.

We can see that for \( \vartheta = 0 \) one recovers the results for the SUSY black Hole\[13\]. As in the commutative case, for the noncommutative SUSY case, we have a singularity in \( r = 0 \). This singularity is independent of the value of \( C \). We start with the solution with the positive sign,
TABLE I. Singular and regular points for the noncommutative SUSY metric.

| Type of region | Positive/Negative | $r$ | $C$ | $r = 0$ | Any value | Singular point |
|----------------|-------------------|-----|-----|---------|-----------|----------------|
| Positive       | $r > 0$           | $C > 0$ | Regular region |
| Positive       | $r = C^2$        | $C < 0$ | Singular point |
| Negative       | $r > 0$           | $C < 0$ | Invalid solution for the metric |
| Negative       | $r = C^2$        | $C > 0$ | Singular point |

there are two possibilities, $C > 0$ and $C < 0$. The first case is a regular for $r > 0$, for the second case we have a singularity in $r = C^2$. Taking the solution with the negative sign, again we have two cases. For positive $C$ the time component of the metric vanishes. Therefore, there is a singularity in $r = C^2$. The case for negative $C$ is negative and therefore this case is discarded.

In order to exhibit the singularities, the Kretschmann invariant is calculated and it is given by

\[
K = \frac{1}{864 \sqrt{6} \left( \pm 1 + \frac{C}{r} \right)^{8/3} r^{22/3}} \times \left[ \begin{array}{l}
3888 \cdot 6^{2/3} C^4 + 216 C^3 \sqrt{7} \left( \pm 63 \cdot 6^{2/3} - 48 \sqrt{7} \left( C + \sqrt{r} \right)^{1/3} \right) \\
+27 C^2 \left( 659 \cdot 6^{2/3} r + 1152 r^{3/2} \left( C + \sqrt{7} \right)^{1/3} + 128 \sqrt{6} r^{2} \left( C + \sqrt{7} \right)^{2/3} \right) \\
+24 C r^{3/2} \left( 431 \cdot 6^{2/3} + 1296 r^{-1/2} \left( C + \sqrt{7} \right)^{1/3} + 288 \sqrt{6} r \left( C + \sqrt{7} \right)^{2/3} \right) \\
+32 r^2 \left( 108 \sqrt{6} r \left( C + \sqrt{7} \right)^{2/3} + 71 \cdot 6^{2/3} + 324 r^{1/2} \left( C + \sqrt{7} \right)^{1/3} \right)
\end{array} \right] + \frac{\partial}{960 \left( \pm 1 + \frac{C}{r} \right)^{4} r^{15/2}} \times \left[ \begin{array}{l}
69984 C^5 + 972 C^4 \sqrt{7} \left( \pm 281 - 16 \sqrt{6} r^{1/2} \left( C + \sqrt{7} \right)^{1/3} \right) \\
+27 C^3 r \left( 15355 + 1984 \sqrt{6} r^{1/2} \left( C + \sqrt{7} \right)^{1/3} \right) + 13952 r^{5/2} \\
+9 C^2 r^{3/2} \left( 33767 - 7488 \sqrt{6} r^{1/2} \left( C + \sqrt{7} \right)^{1/3} \right) \\
+84 C r^2 \left( 1267 + 432 \sqrt{6} r^{1/2} \left( C + \sqrt{7} \right)^{1/3} \right) - 6912 \sqrt{6} r^3 \left( C + \sqrt{7} \right)^{1/3}
\end{array} \right].
\]

From this equation, we can see that there are singularities for $r = 0$ and $r = C^2$. Moreover, from the Kretschmann invariant we conclude (as in the commutative case) that the singularity in the SUSY region remains.
IV. CONCLUDING REMARKS

In this paper we have combined the ideas of noncommutative black holes \cite{13} and SUSY black holes \cite{11} to give a proposal for a noncommutative SUSY black hole. The approach is based on using the diffeomorphism between the KS and Schwarzschild metrics and introducing noncommutativity and supersymmetry to the KS-WDW equation. Furthermore, by solving the semiclassical approximation, we can reconstruct the metric for the noncommutative SUSY metric for the Schwarzschild black hole. Moreover, we calculate the Kretschmann scalar from which we conclude that for $\vartheta = 0$ we recover the result for the SUSY case \cite{13}. This gives us confidence that this is a noncommutative generalization of “SUSY Schwarzschild metric” \cite{13}. We also see that the singularities for $r = 0$ and $r = C^2$ are not removed by combining supersymmetry with noncommutativity \cite{2}. Therefore, the behavior of this noncommutative SUSY black hole is qualitatively the same as for the proposed SUSY Schwarzschild black hole \cite{13}.

Alternatively, we could have started with the NC-WDW and constructed the SUSY generalization. In particular, supersymmetric versions of noncommutative quantum mechanical models have been constructed \cite{20} with the noncommutative deformation introduced using the Bopp shift. One characteristic feature of these models, is that angular momentum type interaction are introduced. Therefore, it can be expected that in the SUSY deformation of the NC-WDW equation, new angular momentum type interaction might appear.

Finally, the procedure used in this paper can be applied to other black holes, in particular the (anti)de-Sitter black hole \cite{14}. These ideas are under research and will be reported elsewhere.

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\footnote{Even if in the metric it seems that for $r = C^2$ and taking the positive sign, the time component of the metric is non zero.}
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