A pedagogical presentation of a $C^*$-algebraic approach to quantum tomography

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Abstract

It is now well established that quantum tomography provides an alternative picture of quantum mechanics. It is common to introduce tomographic concepts starting with the Schrödinger–Dirac picture of quantum mechanics on Hilbert spaces. In this picture, states are a primary concept and observables are derived from them. On the other hand, the Heisenberg picture, which has evolved in the $C^*$-algebraic approach to quantum mechanics, starts with the algebra of observables and introduces states as a derived concept. The equivalence between these two pictures amounts, essentially, to the Gelfand–Naimark–Segal construction. In this construction, the abstract $C^*$-algebra is realized as an algebra of operators acting on a constructed Hilbert space. The representation that is defined may be reducible or irreducible, but in either case it allows us to identify a unitary group associated with the $C^*$-algebra by means of its invertible elements. In this picture both states and observables are appropriate functions on the group; it also follows that quantum tomograms are strictly related with appropriate functions (positive-type) on the group. In this paper we present, using very simple examples, a tomographic description emerging from the set of ideas connected with the $C^*$-algebra picture of quantum mechanics. In particular, we introduce the tomographic probability distributions for finite and compact groups, and formulate an autonomous criterion to recognize a given probability distribution as a tomogram of quantum state.

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1. Introduction

The problem of describing quantum states has been the subject of intensive investigation from the beginning of quantum mechanics [1–8]. The association of quantum states with quasi-distributions [4–7] allowed the description of states in quantum mechanics to be similar to that of particle states in classical statistical mechanics by means of probability distributions in phase space. However, the class of quasi-distributions introduced in quantum mechanics cannot contain all classical distribution functions because of the uncertainty relation [9–11].

In view of the uncertainty relations, there cannot exist a joint probability distribution function, say, of any random position and momentum pair since they cannot be measured simultaneously. It is admissible to have a probability distribution function of only one of the two conjugate variables, e.g. position.

The problem of discussing the position and the momentum probability distributions was discussed by Pauli [12]. Although this problem as formulated by Pauli gave a negative solution, it triggered investigation in this direction and it turned out that one can introduce a family of actual probability distributions of one random variable (position) called tomographic probability distributions or...
simply tomograms. These distributions provide a description of quantum states in complete analogy with that in classical statistical mechanics [13]; see also the recent review [14] and [15–17]. It is worth noting that the probability approach to describing the quantum states was discussed in [18–21] but the tomographic version of this description has appeared only as a result of experiments on homodyne detection of quantum photon states [22, 23]. These are based on optical tomograms whose relation with the Wigner functions was found in [24, 25]. In these papers the tomograms, being measurable probability distributions, were considered as a technical tool to measure the photon quantum states identified with the Wigner functions. So tomograms were not considered as primary objects providing an alternative picture of quantum mechanics.

In [13, 15–17] a new factor in the tomographic approach to quantum mechanics arose because the tomographic distribution itself is identified with the quantum state. In other words, by knowing a quantum tomogram, one can obtain all the quantum mechanical properties such as the energy spectrum, quantum transition probabilities, quantum state evolution in the form of an equation for the probability distribution, etc. Thus the tomogram can be used as an alternative to such primary concepts of state as the notions of wave function or density operator (we also call it a density state).

According to the tomographic approach, for any density state (or wave function) one can construct the tomogram and, vice versa, from any given tomogram one can reconstruct the quantum state density operator $\rho$, which has the properties: hermiticity, i.e. $\rho^\dagger = \rho$, trace normalization $\text{Tr}[\rho] = 1$ and non-negativity, i.e. $\rho \geq 0$. The tomographic probability distribution provides the density operator by using the inversion formulae that are available in explicit form for all kinds of tomograms such as optical [26], symplectic [27, 28], spin [29], photon number [30] and center of mass [31]. The problem of measuring continuous position and momentum in connection with the tomographic description of quantum states was discussed in [32], and the discrete-spin variables were considered in an analogous representation in [33].

If we consider tomograms as conceptual primary objects associated with quantum states, the problem arises from the outset of finding conditions to recognize whether a given probability distribution is a tomogram of a quantum state. The common solution is that one has to use the inversion formula to obtain an operator $\hat{\rho}$ and then to check if it has all the properties characterizing density operators. However, this solution is unsatisfactory because it requires us to switch from the tomographic description to other quantum mechanical pictures. An analogous problem was considered for Wigner functions and the criterion was formulated in [34] on the basis of the so-called Kastler–Loupis–Miracle Sole (KLM) conditions [35–37]. The connection of tomograms with Wigner functions could be used [38], but again it would be unsatisfactory. One needs autonomous criteria to solve the problem.

In this paper we provide self-contained conditions for a probability distribution to be a tomogram of a quantum state, i.e. a quantum tomogram. We will formulate such properties using the Naimark method [39] and the Gelfand–Naimark–Segal (GNS) construction [40] to describe quantum states in terms of vectors of a suitable Hilbert space.

It is worth noting that the tomographic approach can be formulated in the framework of a star-product scheme [41–43].

Our strategy is to find the connection between functions (which are diagonal matrix elements of a unitary representation of a group $G$) and quantum tomograms. Any such function in [39] was shown to have properties of positivity, recalled in the following. Then, based on this property and in view of the connection with tomograms, we can establish the properties characterizing quantum tomograms among other probability distributions.

The paper is organized as follows. Sections 2 and 3 contain introductory remarks on $C^*$-algebras, and a simple example is given. A concrete case of the $C^*$-algebra, the group algebra, is discussed in section 4 for a finite group. The fundamental notion of positive-type group functions is recalled in section 5. Section 6 is devoted to tomographic descriptions of quantum states based on irreducible representations of a finite group, via a positive group function. Section 6 is the core of this paper: its definitions and results, which are discussed in extreme detail in the case of the group of permutations of three points $S_3$ in section 7, are straightforwardly extended to compact groups like $U(n)$ in section 8, after a caveat on the necessity of using the Gelfand–Zetlin bases [44, 45]. Also, the tomographic reconstruction formula provided in section 8 is evaluated in detail for the case of $SU(2)$. The Gelfand–Zetlin bases are discussed in some detail in section 9. The paradigmatic case of $SU(3)$ in section 10 illustrates the previous results. In section 11 the important necessary and sufficient conditions for a given family of stochastic vectors to be a tomogram are formulated in terms of a suitable positive-type group function, both for finite and compact groups. An example based on $S_3$ illustrates the theory. Moreover, the possibility of checking the positivity of a compact group function via the restriction to a finite subgroup is analyzed. Finally, in section 12 some conclusions and perspectives are drawn.

2. Introductory remarks on $C^*$-algebras

It is known that the formulation of quantum mechanics stemming from the Heisenberg picture is given by using a $C^*$-algebra formalism [46]. In this formalism neither a Hilbert space nor operators is used. Instead, an associative algebra $A$ with identity $E$ and a $*$-involution, such that $(AB)^* = B^*A^*$ are used, as well as an appropriate norm $\| \cdot \|$ to introduce a topology. The norm $\| \cdot \|$ satisfies the continuity requirement for the product $\|AB\| \leq \|A\|\|B\|$ and the compatibility condition $\|A^*A\| = \|A\|^2$.

The observables of the theory are real (also called self-adjoint) elements, $A^* = A$. States are normalized positive continuous linear functionals $\rho$ on this algebra, that is, continuous linear maps such that $\rho(A^*A) \geq 0$, and $\rho(E) = 1$, (replacing the trace property for density states). The mean value of an observable $A$ in the state $\rho$, say $\langle A \rangle_\rho$, is just the real number $\rho(A)$, the evaluation of $\rho$ on $A$.
Some elements of the algebra $A$ have an inverse. The elements $U$ for which $U^* = U^{-1}$ are called unitary elements in the C*-algebra, and they form a group $\mathcal{U}$.

Starting from a C*-algebra $A$, the GNS construction, given a fiducial state $\rho$, provides a Hilbert space carrying a *-cyclic representation $\Pi$ of the algebra, $\Pi(A^*) = (\Pi(\lambda))^\dagger$. In this way one finds density operators for states and Hermitian operators for observables of the usual formulation of quantum mechanics.

One of the aims of our work is to introduce the tomographic approach at the level of the C*-algebra formulation of quantum mechanics and to relate it to the standard formulation by means of the GNS construction.

The idea of a tomographic picture in a C*-algebra is based on the possibility of representing an observable $A$ (at least for group algebras based on compact groups as will be discussed in the following) as a real linear combination of projectors; this is in the form

$$A = \sum_{a,k} \lambda^a_k P^a_k,$$

where $\lambda^a_k$ are real numbers, the observables $P^a_k = P^a_k P^{a\ast}_k \delta_{a\sigma} \delta_{jk}$, and satisfying $\sum_{a,k} P^a_k = E$. It follows that $A P^a_k = \lambda^a_k P^a_k$.

The same kind of decomposition (1) for $U(g)$ gives $\lambda^a_k = \exp(i\theta^a_k), \theta^a_k \in \mathbb{R}$. Now, for any state $\rho$, $\rho(P^a_k) = \rho(P^a_k P^{a\ast}_k) \geq 0$, so that we may interpret the formula $\rho(U) = \sum_{a,k} \exp(i\theta^a_k) \rho(P^a_k)$ as the evaluation of the state $\rho$ in $U$, providing the value of each random phase $\theta^a_k$ with the probability $W^a_k(\rho, U) := \rho(P^a_k)$. In other words, we have thus defined the tomographic probability $W^a_k(\rho, U)$ of random index $k$ for any given $\alpha$, and write $\langle U \rangle_\rho := \sum_{a,k} \exp(i\theta^a_k) W^a_k(\rho, U)$.

We complete the construction by introducing the notion of the Naimark matrix $\mathcal{N}_{ij} = \rho(U^{\dagger} U)_{ij}$, where $i, j$ vary over any finite set of natural numbers. If it is positive semi-definite, that is $\sum_{i,j} \mathcal{N}_{ij} \xi^i \xi^j \geq 0$ for all $\xi^i \in \mathbb{C}$, by definition $\rho(U)$ is a positive-type function on the group $\mathcal{U}$. Finally, particular realizations of C*-algebras as unitary irreducible representations of different groups provide corresponding standard definitions of tomography. The use of C*-algebras constructed from groups makes explicit state reconstruction from its tomogram possible.

### 3. An introductory example

In this section we illustrate the notion of C*-algebra by considering a simple finite dimensional example. Given three orthonormal vectors in a Hilbert space $|\alpha_1\rangle, |\alpha_2\rangle, |\alpha_3\rangle$, let us consider the linear space $\mathcal{A}_0$ with nine base vectors organized in a table.

$$
\begin{bmatrix}
|\alpha_1\rangle & |\alpha_2\rangle & |\alpha_3\rangle & \langle \alpha_1| & \langle \alpha_2| & \langle \alpha_3| & A_1 & A_2 & A_3 \\
|\alpha_2\rangle & |\alpha_2\rangle & |\alpha_2\rangle & \langle \alpha_2| & \langle \alpha_2| & \langle \alpha_2| & A_2 & A_2 & A_2 \\
|\alpha_3\rangle & |\alpha_3\rangle & |\alpha_3\rangle & \langle \alpha_3| & \langle \alpha_3| & \langle \alpha_3| & A_3 & A_3 & A_3 \\
\end{bmatrix}
= \begin{bmatrix}
A_1 & A_2 & A_3 \\
A_2 & A_2 & A_2 \\
A_3 & A_3 & A_3 \\
\end{bmatrix}.
$$

We define the table of products for the base vectors corresponding to the products of projectors

$$|\alpha_i\rangle \langle \alpha_m| |\alpha_n\rangle = \delta_{m,n} |\alpha_i\rangle \langle \alpha_i|.$$
To any state \( \rho \) we can associate a vector \( A_\rho \) in the C*-algebra \( A_0 \) by means of the formula
\[
A_\rho = \sum_{m=1}^{3} \rho (A_m^*) A_m = \sum_{j,k=1}^{3} \rho (|a_j\rangle \langle a_i|) |a_j\rangle \langle a_i|.
\]
(13)

The vector \( A_\rho \) is real, \( A_\rho^* = A_\rho \), because states must be Hermitian functionals \( \rho(A^*) = \langle \rho(A) \rangle^* \). Moreover, the Hermitian matrix \( (\rho_{jk}) = \rho(|a_j\rangle \langle a_i|) \) has trace one because \( \rho(E) = \rho(A_1 + A_3 + A_5) = \rho(\sum_{k=1}^{3} |a_k\rangle \langle a_k|) = \sum_{k=1}^{3} \rho a_k \).

The vector \( A_\rho \) is definite positive because \( \rho(A^*) = \sum_{j,m,k,n} \rho_{jm} \rho_{kn} |a_j\rangle \langle a_i| |a_m\rangle \langle a_n| \) for arbitrary \( e_{km}. \) The positive definite property of the matrix \( \rho_{kj} \) can also be seen by considering the orbits of the transitive action of unitary group \( U \) through the vectors.

Thus, if we consider a diagonal state \( \rho^0 \), i.e. a state such that \( \rho(|a_i\rangle \langle a_i|) = \rho_n^0 \geq 0, n = 1, 2, 3 \) and zero otherwise, we have
\[
A_{\rho^0} = \sum_{n=1}^{3} \rho_n^0 (|a_n\rangle \langle a_n|) |a_n\rangle \langle a_n| = \sum_{n=1}^{3} \rho_n^0 |a_n\rangle \langle a_n|.
\]
(14)

If \( U \) is a unitary element of \( A_0 \) with the representative matrix \( u \), then we will denote by \( \rho_U \) the state \( \rho_U(A) = \rho(U^*AU) \), which gives all the state vectors \( \rho \) as
\[
A_{\rho_U} = \sum_{j,k=1}^{3} u_{jk}|a_j\rangle \langle a_i| \sum_{n=1}^{3} \rho_n^0 |a_n\rangle \langle a_n| \sum_{k,l=1}^{3} \rho_{kl}^* |a_k\rangle \langle a_l| = \sum_{k,l=1}^{3} (u^0 u^*)_{jk}|a_j\rangle \langle a_k| = \sum_{j,k=1}^{3} \rho_{jk} |a_j\rangle \langle a_k|,
\]
(15)
where \( (u^0 u^*)_{mn} = \rho_0^0 \delta_{mn}. \)

Vice versa, in the dual space of the C*-algebra, any vector \( A \) has a dual partner: the base partners are
\[
|a_j\rangle \langle a_k| \leftrightarrow \delta_{jk} : \delta_{jk} |a_m\rangle \langle a_n| = \delta_{jm} \delta_{kn},
\]
so that from equation (7) we get
\[
\delta_{jk}(A) = c_{jk}
\]
(17)
and for the partner of \( A \)
\[
A \rightarrow \alpha_A = \sum_{j,k=1}^{3} c_{jk}^* \delta_{jk},
\]
(18)
giving
\[
\alpha_A(A) = \sum_{j,k=1}^{3} c_{jk}^* c_{kj} = \text{Tr}[c^* c] = \|A\|^2.
\]
(19)

Note that one has to rescale the above Hilbert–Schmidt norm giving \( \|E\| = 3 \), in order to have the property \( \|E\| = 1. \) Given any state, positive-type functions on the unitary group \( U \) are introduced as
\[
\psi(U) = \langle U \rangle_\rho = \rho(U).
\]
(20)

They satisfy the positive semidefinite \( n \times n \) matrix condition: for any \( n \)
\[
\psi(U_j^* U_k) \geq 0, \quad \forall U_j, U_k \in U, \quad j, k = 1, 2, \ldots, n, \forall n.
\]
(21)

The positive-type functions \( \psi(U) \) may be expanded in terms of tomograms \( W_k(\rho, U) \) by diagonalizing the unitary matrix representing \( U \):
\[
\psi(U) = \rho \left( \sum_{j,k} u_{jk} |a_j\rangle \langle a_k| \right)
\]
\[
= \sum_{j,k} v_{jk} \exp(i \theta_k) v_{kj}^* \rho |a_j\rangle \langle a_k|
\]
\[
= \sum_{j,k} \exp(i \theta_k) \left( \sum_{j,h} v_{hk}^* \rho_{jh} v_{jk} \right)
\]
\[
= \sum_{j,k} \exp(i \theta_k) W_k(\rho, U).
\]
(22)

Note that the tomogram component \( W_k(\rho, U) = (v^\dagger v)_{kk} \geq 0 \) is a component of a stochastic vector, \( \sum_k (v^\dagger v)_{kk} = 1 \).

**Remark.** We could have started with two vectors \(|a_1\rangle, |a_2\rangle\). Then the unitary group of the resulting C*-algebra is isomorphic to \( U(2) \). In that case one can embed the permutation group of three elements into \( U(2) \), via a unitary irreducible representation, so that the corresponding positive-type functions allow for a tomographic reconstruction of the state \( \rho \), as will be discussed in the following.

### 4. Group algebra

Another example of C*-algebra is the so called group algebra, which is an important tool in itself [39].

Following [39], we review below some properties of a group algebra, focusing first on finite groups. Given a finite group of order \( K : G = G_K = \{g_1, g_2, \ldots, g_k\} \), consider the complex valued functions on the group \( f : G \rightarrow \mathbb{C} \). The group algebra consists of formal linear combinations of group elements,
\[
A_f = \sum_{j=1}^{K} f(g_j) g_j,
\]
(23)
and will be denoted by \( \mathbb{C}[G] \) or \( A_G \). Each element \( A \in A_G \) is represented by the coefficients \( f(g_j) \) of the combination and vice versa. We have a one-to-one correspondence between elements of the group algebra and complex valued functions on the group.

If
\[
A_f = \sum_{j=1}^{K} f(g_j) g_j, \quad A_h = \sum_{j=1}^{K} h(g_j) g_j,
\]
(24)
we have \( A_f + A_h = A_{f+h} \). Components of a product are obtained from
\[
A_f \cdot A_h = \sum_{j,k} f(g_j) h(g_k) g_{jk} = \sum_{j,k} f(g_j) h(g_k^{-1}) g_k g_j
\]
\[
= \sum_{j,k} f(g_j) h(g_k^{-1}) g_{jk} = A_{f \cdot h}.
\]
(25)
where, on the algebra of group functions, the convolution product (star-product) is defined as
\[
(f \cdot h)(g) = \sum_{j=1}^{K} f(g_j)h(g_j^{-1}g_k).
\]  
(26)

The conjugate \(A^*\) of \(A\) is defined by setting \(g^* = g\) and
\[
A^*_f = \left( \sum_{j=1}^{K} f(g_j) g_j \right)^* = \sum_{j=1}^{K} f^*(g_j)g_j = A_{f^*},
\]  
(27)
i.e. \(f^*(g) = [f(g)]^*\). We also introduce the transpose \(A^T\) of \(A\) by \(g^T = g^{-1}\) and
\[
A^T_f = \left( \sum_{j=1}^{K} f(g_j)g_j \right)^T = \sum_{j=1}^{K} f(g_j)g_j^{-1}
\]  
(28)
thus \(f^T(g) = f(g^{-1})\). For a product \(A \cdot B\) we have \((A \cdot B)^T = B^T \cdot A^T\).

Hermitian conjugation is now defined as the composition of conjugation and transposition: \(A^* = (A^T)^*\) or \(g^* = g^{-1}\) and
\[
A^*_f = \sum_{j=1}^{K} f^*(g_j)g_j^{-1} = \sum_{j=1}^{K} f^*(g_j^{-1})g_j = A_{f^*},
\]  
(29)
i.e. \(f^*(g) = f^*(g^{-1})\). For a product \(A \cdot B\) we have \((A \cdot B)^* = B^* \cdot A^*\).

All above operations are involutions, that is,
\[
(A^*)^* = A, \quad (A^T)^T = A, \quad (A^*)^* = A.
\]  
(30)
We observe that only the \(\ast\)-involutions satisfies the condition \(\Pi(A^*) = \Pi^1(A)\) for any unitary representation \(\Pi\) of the algebra.

The trace of an element \(A \in \mathcal{A}_G\) is defined by \(\text{Tr}[g] = 1\) for \(g = e\), the group unity, and \(\text{Tr}[g] = 0\) otherwise. We have
\[
\text{Tr}[A_f] = f(e).
\]  
(31)
We now introduce the scalar product \(\langle A_f, A_h \rangle\) in the group algebra \(\mathcal{A}_G\) by
\[
\langle A_f, A_h \rangle = \text{Tr}[A^*_f \cdot A_h] = \sum_{j=1}^{K} f^*(g_j)h(g_j),
\]  
(32)
which agrees with the standard inner product on complex valued functions on \(G\) considered as vectors on \(\mathbb{C}^K\). It follows that \(\text{Tr}[A^*_f \cdot A_h] = (\text{Tr}[A^*_f \cdot A_f])^*\). It is worth noting that \(A^*\) is the Hermitian conjugate of \(A\) with the scalar product we have just defined.

The associativity of the group \(G\) implies the associativity of the group algebra. The scalar product is preserved by left and right action: \(g_k \mapsto L_g(g_k) = g_kg_k^{-1}g_k, g_l \mapsto R_g(g_l) = g_lg_k^{-1}g_l\), and similarly under conjugation \(g_k \mapsto C_g(g_k) = g_jg_k^{-1}g_k\). It is also invariant under transposition \(A \mapsto A^T\) and multiplication by a phase \(A \mapsto \exp(i\theta)A\). Under the transformations \(A \mapsto A^*\) and \(A \mapsto A^T\) the scalar product goes into its complex conjugate. Moreover the transformations \(A \mapsto gAg^{-1}\) and \(A \mapsto A^*\) are automorphisms of the group algebra. We should also mention that a pointwise product is available,
\[
A_f \circ A_h = \sum_{j} f(g_j)h(g_j)g_j = A_{f\cdot h},
\]  
(33)
which is called the Hadamard product.

4.1. Representations of group algebras
All irreducible representations of a finite group \(G\) of order \(K\) are finite dimensional and equivalent to unitary representations. If \(\{D^\alpha\}\) with \(\dim D^\alpha = n_\alpha\) is an irreducible unitary representation of \(G\), then, because of Schur’s lemma, we get the orthogonality conditions
\[
\sum_{j=1}^{K} (D^\alpha_{rs}(g_j))^* D^\beta_{pq}(g_j) = \frac{K}{n_\alpha} \delta_{r\alpha,\delta_{\beta,s}} \delta_{k,q},
\]  
(34)
that imply that the matrix elements \(\{D^\alpha_{rs}(g_j)\}\) of the set of irreducible unitary representations of \(G\) form an orthogonal set on the algebra \(\mathcal{A}_G\). Notice that the subspace of \(\mathcal{A}_G\) spanned by the elements \((r, s)\) of the irreducible representation \(D^\alpha\) is invariant under left (or right) translations, hence they define invariant subspaces of the regular representation, i.e., the canonical representation of the group \(G\) on its algebra \(\mathcal{A}_G\) by left translations. Hence, all irreducible representations are contained in the regular representation. Thus, there is a finite number of irreducible representations labelled by \(\alpha\), and the matrix elements \(\{D^\alpha_{rs}(g_j)\}\) form an orthogonal basis in the algebra of group functions:
\[
f(g_j) = \sum_{\alpha, r,s=1}^{n_\alpha} c^\alpha_{rs} D^\alpha_{rs}(g_j).
\]  
(35)
Moreover, the dimensions \(n_\alpha\) of the irreducible representations \(\{D^\alpha\}\) satisfy the equation
\[
\sum_{\alpha} n_\alpha^2 = K.
\]  
(36)
One can use a unitary (reducible or irreducible) representation \(U(g)\) of the group acting on an \(N\)-dimensional Hilbert space to introduce a representation of the group algebra by means of operators on the same Hilbert space. The operator \(\hat{A}_f\) corresponding to the group algebra element \(A_f\) will be
\[
\hat{A}_f = \sum_{j=1}^{K} f(g_j)U(g_j).
\]  
(37)
In view of
\[
U(g_j^{-1}g_l) = U(g_j^{-1})U(g_l),
\]  
(38)
one finds
\[
\hat{A}_f \hat{A}_h = \sum_{j=1}^{K} f(g_j)U(g_j) \sum_{l=1}^{K} h(g^{-1}_l g_l)U(g_j^{-1} g_l) = \sum_{j=1}^{K} f(g_j)h(g_j^{-1} g_l)U(g_l) = \sum_{j=1}^{K} (f \cdot h)(g_j)U(g_j) = \hat{A}_{f \cdot h}.
\]  
(39)
When \( U(g) \) is an irreducible representation \( D^\alpha(g) \) the orthogonality relations \((34)\) may be used to obtain the inversion formula
\[
f(g) = \frac{n_\alpha}{K} \text{Tr}[\hat{A}_f D^\alpha(g)].
\] (40)

This shows that we are in the framework of a star-product scheme, where the quantizer and dequantizer operators are \( D(g) \) and \( D^\dagger(g) \), respectively \([41–43]\).

**Remark.** When the group is finite, the group \( \mathcal{U} \) of unitary elements in the group algebra may be readily determined. \( \mathcal{U} \) consists, by definition, of the elements corresponding to group functions \( f \), satisfying the relation
\[
f^{-1} = f^*,
\] (41)
where we recall that \( f^* \) is defined as \( f^*(g) = f(g^{-1}) \).

Condition \((41)\) expresses unitarity at the abstract level of group algebra.

This implies that equation \((41)\) is equivalent to the condition of unitarity for the operator \( u_f = \sum_{j=1}^K f(g_j) D(g_j) \), for any unitary irreducible representation of the group.

For finite groups the set of such representations is finite and known, so condition \((41)\) gives explicitly \( \mathcal{U} \). We have
\[
f \in \mathcal{U} \iff u_f^\dagger = \sum_{j=1}^K f(g_j) D^\dagger(g_j) \in U(n_\alpha), \forall \alpha,
\] (42)
where \( D^\alpha \) is an irreducible \( n_\alpha \)-dimensional representation of the finite group, and \( U(n_\alpha) \) the corresponding unitary group. When \( D^\alpha \) varies in the set \( \{D^\alpha\} \) of all irreducible representations of the finite group, we get a set of \( \sum_\alpha n_\alpha^2 = K \) linear inhomogeneous equations in the \( K \) variables \( f(g_j) \) with the known terms \( u_f^\dagger \), \( \ldots \), \( u_f^\dagger \in U(n_1) \times \ldots \times U(n_\alpha) \).

The determinant of this system does not vanish because its rows are made by matrix elements \( D^\alpha(g_j) \), an orthonormal set of functions on the group. The linear system has a unique solution \( \{f(g_j)\} \) for any given \( g = u_1, \ldots, u_K \in U(n_1) \times \ldots \times U(n_\alpha) \) and determines an isomorphism between \( U(1) \times \ldots \times U(n_\alpha) \) and \( \mathcal{U} : f_g \cdot f_h = f_{gh} \forall g, h \in U(n_1) \times \ldots \times U(n_\alpha) \).

For instance, in the simplest case of the group \( S^3 \times S^1 \) there are only two representations \((1D)\), \( \mathcal{U} \) is isomorphic with \( S^1 \times S^1 \) and the isomorphism is given by
\[
(e^\alpha, e^\beta) \in S^1 \times S^1 \iff \alpha, \beta = \left( \frac{e^{2\alpha} + e^{-\beta}}{2}, \frac{e^{2\alpha} - e^{-\beta}}{2} \right).
\] (44)

This result can be easily obtained by solving equation \((41)\) directly, which yields
\[
\frac{1}{f^2(g_1) - f^2(g_2)} \begin{pmatrix} f(g_1) & f^*(g_1) \\ f(g_2) & f^*(g_2) \end{pmatrix} = \begin{pmatrix} f^* & f \\ f & f^* \end{pmatrix},
\] (45)
or the equivalent linear system \((42)\) which reads
\[
\begin{pmatrix} f(g_1) + f(g_2) \\ f(g_1) - f(g_2) \end{pmatrix} = \begin{pmatrix} e^{i\alpha} \\ e^{i\beta} \end{pmatrix}.
\] (46)

5. **Positive-type group functions**

To deal with states and tomograms we need to recall the definition of positive-type group functions. Given any group \( G \), a group function \( \varphi(g) \) is of positive-type if the corresponding matrix,
\[
N_{jk}(\varphi) := \varphi(g_j^{-1}g_k), \quad g_j, g_k \in \{g_1, g_2, \ldots, g_n\} \subseteq G,
\] (47)
is positive semidefinite, for any \( n \)-tuple \( \{g_1, g_2, \ldots, g_n\} \) of elements of \( G \), and for any \( n \in \mathbb{N} \). We may call \( N_{jk}(\varphi) \) the Naimark matrix of \( \varphi \).

For any unitary representation of the group \( U(g) \), it is possible to define a positive-type group function by means of a pure state corresponding to the vector \( \xi \):
\[
\varphi^U_{\xi}(g) := (\xi, U(g)\xi) = \text{Tr}[\rho_\xi U(g)],
\] (48)
Here, \( \rho_\xi \) is the density state corresponding to \( \xi \). In fact, the quadratic form
\[
\sum_{j,k=1}^n \lambda_j^* \lambda_k \text{Tr}[\rho_\xi U(g_j^{-1}g_k)]
\]
\[
= \text{Tr} \left[ \rho_\xi \sum_{j=1}^n \lambda_j^* U^\dagger(g_j) \sum_{k=1}^n \lambda_k U(g_k) \right]
\]
\[
= \text{Tr} \left[ \rho_\xi V^\dagger V \right] \geq 0,
\] (49)
where the \( \lambda \) are arbitrary complex numbers, is positive semidefinite.

The above form can be generalized by using any density state \( \rho \) instead of a pure state \( \rho_\xi \), and this will be very useful in the tomographic framework.

It should be stressed here that the form of equation \((48)\) is canonical. Because of Naimark’s representation theorem \([39]\), for any positive-type group function \( \varphi(g) \), there exist a Hilbert space, a unitary representation of the group \( U(g) \) and a cyclic vector \( \xi \) such that
\[
\varphi(g) = (\xi, U(g)\xi).
\] (50)

We recall that a vector \( \xi \) is called cyclic if the set \( \{U(g)\xi | g \in G\} \) spans the Hilbert space.

Notably, for a finite group the positivity of a group function may be checked by considering only one Naimark matrix, constructed with all the elements of the group. Thus we have the following:

**Proposition 1.** A group function \( \psi \), defined on a finite group \( G_K = \{g_1, g_2, \ldots, g_K\} \) of order \( K \), is of positive type iff the \( K \times K \)-matrix,
\[
N(\psi)_{ij} = \psi(g_i^{-1}g_j), \quad i, j = 1, \ldots, K,
\] (51)
is positive semidefinite.

**Proof.** Consider the Naimark matrix \( N(\psi)_{ij} \) of order \( K + 1 \) obtained by adding a repeated element \( g_{K+1} = g_h \) to \( \{g_1, g_2, \ldots, g_K\} \). Then, \( N(\psi)_{ij} \) has two equal rows. So, \( \det N(\psi)_{ij} = 0 \). By induction, \( \det N(\psi)_{ij} = 0 \) for \( g_1, g_2 \in \{g_1, \ldots, g_K, \ldots, g_{K+1}\} \), \( \forall p \), and the proposition is proven. □
6. Finite groups and tomography

We introduce a tomography for any density state \( \rho \) by means of a positive-type function on \( G \), defined as

\[
\psi^D_\rho (g) := \text{Tr}[\rho D(g)],
\]

(52)

where \( \{D(g)\} \) is a unitary irreducible representation of \( G \) on the Hilbert space where the density state \( \rho \) is defined.

In this section, we consider again a finite group of order \( K : G = G_K = \{g_1, g_2, \ldots, g_K\} \). We may suppose the representation \( D \) to be \( n \)-dimensional (\( n^2 < K \) if \( D \) is irreducible.) For any group element \( g \) corresponding matrix \( D(g) \) can be put in the form of a diagonal unitary matrix \( d_g \) by means of a unitary matrix \( V_g \):

\[
D(g) = V_g d_g V_g^\dagger, \quad d_g = \text{diag}[e^{i\theta_1}, \ldots, e^{i\theta_n}].
\]

(53)

We observe that, in general, neither \( d_g \) nor \( V_g \) separately is a group representation. Moreover, \( V_g \) is not uniquely determined: for if \( C_r, C_l \) are unitary matrices commuting with \( d_g \) and \( \rho \), respectively, we have

\[
\psi^D_\rho (g) := \text{Tr}[\rho V_g d_g V_g^\dagger] = \text{Tr}[C_r^\dagger \rho C_l V_g C_r d_g C_l^\dagger V_g] = \text{Tr}
\[
[(C_r V_g C_r)^\dagger \rho (C_l V_g C_r) d_g],
\]

(54)

so that this ambiguity does not affect the associate function, and we may write unambiguously

\[
\psi^D_\rho (g) = \text{Tr}[d_g (V_g^\dagger \rho V_g)] = \sum_{m=1}^n e^{i\theta_m} \rho_{mm} = \sum_{m=1}^n e^{i\theta_m} W_m (g, \rho).
\]

(55)

In the last equation, we introduced the components

\[
W_m (g, \rho) := (V_g^\dagger \rho V_g)_{mm} \quad (m = 1, \ldots, n)
\]

(56)

of the vector \( W(g, \rho) \) defining the tomogram of \( \rho \) in the chosen representation of \( G_K \). We note that, as \( V_g^\dagger \rho V_g \) is again a density state, the tomogram is by definition a stochastic vector, i.e.

\[
\sum_{m=1}^n W_m (g, \rho) = \sum_{m=1}^n (V_g^\dagger \rho V_g)_{mm} = \text{Tr}[\rho] = 1,
\]

(57)

\[
W_m (g, \rho) \geq 0 \quad (m = 1, \ldots, n), \forall g \in G_K.
\]

(58)

The knowledge of the tomograms \( \{W(g_j, \rho)\}_{j=1}^K \) allows for reconstructing the density state. In fact, as the diagonal matrices \( d_g \) depend only on the representation \( D \), and are supposed to be known, the function \( \psi^D_\rho \) is readily obtained as

\[
\psi^D_\rho (g_j) = \sum_{m=1}^n e^{i\theta_m} W_m (g_j, \rho).
\]

(59)

Then the state is given by the reconstruction formula

\[
\frac{n}{K} \sum_{j=1}^K \psi^D_\rho (g_j) D(g_j) = \rho,
\]

(60)

which is based on the orthogonality relations of the matrix elements of \( D(g) \):

\[
\frac{n}{K} \sum_{j=1}^K \psi^D_\rho (g_j)^* D_{rs}(g_j) = \frac{n}{K} \sum_{j=1}^K \text{Tr}[\rho^* D^s(g_j)] D_{rs}(g_j) = \sum_{q, m=1}^n \rho_{qm}^* \sum_{j=1}^K D_{mq}(g_j) D_{rs}(g_j)
\]

(61)

Now, suppose that \( \varphi \) is any positive-type function on \( G_K \). We recall that, according to Naimark’s theorem, there exist a Hilbert space acted upon by a unitary representation \( U \) of \( G_K \) and a cyclic vector \( \xi \) such that

\[
\varphi(g_j) = \langle \xi, U(g_j)\xi \rangle = \text{Tr}[\rho_U (g_j)].
\]

(62)

In general the above representation \( U \) becomes reducible as a direct sum of all the irreducible representations \( \{D^\alpha\}, \dim D^\alpha \geq 1 \) of the group, each block \( D^\alpha \) with the multiplicity \( m_{\alpha} \):

\[
U = \bigoplus_{\alpha} \bigoplus_{s=1}^{m_{\alpha}} D^\alpha_s.
\]

(63)

Out of the matrix representing \( \rho_U \), we can extract the same blocks of the reduction of \( U \) to construct a new matrix \( \tilde{\rho} \), with the remaining entries as zero. Moreover, \( \tilde{\rho} \) is still a state, as the determinants of its blocks are principal minors of \( \rho_U \). They are non-zero because \( \rho_U \) is cyclic. Then, by construction, the function \( \text{Tr}[\rho_U (g_j)] \) coincides with the above function \( \varphi(g_j) \), i.e.

\[
\varphi(g_j) = \text{Tr}[\rho_U (g_j)] = \text{Tr}[\tilde{\rho} U(g_j)].
\]

(64)

Now we sum together the blocks \( \rho^\alpha_s \) of \( \tilde{\rho} \) associated with the same \( D^\alpha \):

\[
\tilde{\rho}^\alpha = \sum_{s=1}^{m_{\alpha}} \rho^\alpha_s,
\]

(65)

and finally we can write

\[
\varphi(g_j) = \text{Tr}[\tilde{\rho} U(g_j)] = \sum_{\alpha} \text{Tr}[\tilde{\rho}^\alpha D^\alpha(g_j)].
\]

(66)

The function \( \tilde{\varphi} \) is normalized, i.e. on the identity element \( e \) of the group, \( \varphi(e) = 1 \). Then \( \tilde{\rho}^\alpha \) can be written as \( \tilde{\rho}^\alpha = \gamma^\alpha \rho^\alpha \), where \( 0 \leq \gamma^\alpha \leq 1 \), \( \sum_{\alpha} \gamma^\alpha = 1 \) and \( \rho^\alpha \) is a density state.

So, we have proven:
Proposition 2. Any positive-type function \( \varphi \) on \( G_K \) can be decomposed as a convex sum of the positive-type functions, \( \varphi^\alpha \), related tomographically to the irreducible representations, \( D^\alpha \), of the group:

\[
\varphi(g_j) = \sum_\alpha \gamma_\alpha^\varphi \varphi^\alpha(g_j), \quad \varphi^\alpha(g_j) = \text{Tr}[\rho^\alpha D^\alpha(g_j)].
\]  

(67)

We remark that any \( \varphi^\alpha \) can be written, again using the Naimark theorem, in terms of a pure cyclic state and a representation \( U^\alpha \) as

\[
\psi^\alpha(g_j) = (\zeta^\alpha, U^\alpha(g_j)\zeta^\alpha) = \text{Tr}[\rho_{\zeta^\alpha} U^\alpha(g_j)].
\]

(68)

So, the problem arises of relating \( U^\alpha \) to \( D^\alpha \) and \( \rho_{\zeta^\alpha} \) to \( \rho^\alpha \). Dropping the label \( \alpha \), we can state the following:

Proposition 3. If the density state \( \rho \) is of rank \( r \), the above representation \( U \) becomes reducible as a direct sum of \( r \) blocks, each one unitarily equivalent to the irreducible representation \( U^\alpha \) as

\[
\psi^\alpha(g_j) = (\zeta^\alpha, U^\alpha(g_j)\zeta^\alpha) = \text{Tr}[\rho_{\zeta^\alpha} U^\alpha(g_j)].
\]

(69)

Then, the convolution product on the algebra of group functions (26) for \( X, Y \) reads

\[
(X \cdot Y)(g_j) = \sum_{i=1}^K X(g_i)Y(g_i^{-1}g_j).
\]

(71)

and may be expanded as

\[
(X \cdot Y)(g_j) = \sum_{i=1}^K \sum_\alpha \sqrt{\frac{n_\alpha}{K}} \sum_{q,p=1}^{n_\alpha} X_{qp}^\alpha D_{qp}^\alpha(g_i) \times \sqrt{\frac{\beta}{K}} \sum_{m,s=1}^{n_\beta} Y_{ms}^\beta D_{ms}^\beta(g_i^{-1}g_j).
\]

(72)

From

\[
D_{ms}^\beta(g_i^{-1}g_j) = \sum_{r=1}^{n_\beta} D_{mr}^\beta(g_i^{-1}D_{rs}^\beta(g_j)
\]

\[
= \sum_{r=1}^{n_\beta} (D_{mr}^\beta(g_i))^* D_{rs}^\beta(g_j)
\]

(73)

and the orthogonality relations in equation (34), the convolution product may be written as

\[
(X \cdot Y)(g_j) = \sum_\alpha \sum_{q,p,s=1}^{n_\alpha} X_{qp}^\alpha Y_{ps}^\alpha D_{qs}^\alpha(g_j)
\]

\[
= \sum_\alpha \sum_{q,p,s=1}^{n_\alpha} (XY)_{qs}^\alpha D_{qs}^\alpha(g_j).
\]

(74)

By introducing the function

\[
X^\dagger(g) := X^\ast(g^{-1}) = \sum_\alpha \sqrt{\frac{n_\alpha}{K}} \sum_{p,q=1}^{n_\alpha} (X_{pq}^\alpha)^* D_{qp}^\alpha(g),
\]

(75)

we define a seminorm

\[
F(X^\ast \cdot X) = \sum_{j=1}^K (X^\ast \cdot X)(g_j)\varphi(g_j))^* = \sum_{j=1}^K \sum_\alpha \sum_{q,p,s=1}^{n_\alpha} (X_{pq}^\alpha)^* X_{ps}^\alpha D_{qs}^\alpha(g_j)
\]

\[
= \sum_{j=1}^K \sum_{p,q,s=1}^{n_\alpha} \sum_{m,s=1}^{n_\beta} \psi_{mr}^\ast D_{ms}^\ast(g_j))^* \psi_{qs}^\ast,
\]

(76)

Without any loss of generality, we may suppose that the density state is diagonal: \( \rho = \text{diag}(\rho_1, \rho_2, \ldots, \rho_n) \). Upon diagonalization

\[
\psi(g) := \text{Tr}[\rho D(g)] = \text{Tr}[\text{diag}(\rho_1, \rho_2, \ldots, \rho_n)V^\dagger D(g)V],
\]

(77)

and we could, in the previous discussion, have chosen \( V^\dagger D(g)V \) instead of \( D(g) \) from the outset. Then

\[
\psi_{qs} = \rho_{qs} \delta_{q,s}
\]

(78)

and the seminorm reads

\[
F(X^\ast \cdot X) = \frac{K}{n} \sum_{q=1}^{n} \left( \sum_{p=1}^{n} |X_{pq}|^2 \right) \rho_{qs} = \frac{K}{n} \sum_{q=1}^{n} \|X_q\|^2 \rho_{qs},
\]

(79)

where the vector \( X_q \) is the \( q \)th column of the matrix of coefficients of \( D(g) \) in the harmonic expansion of \( (X^\ast \cdot X) \).

Now, suppose the density state \( \rho \) has rank \( r \), with non-zero entries \( \{\rho_{11}, \rho_{22}, \ldots, \rho_{rr}\} \).

(80)

Then, in view of equation (79), the seminorm kernel \( F_0 = \{X: F(X^\ast \cdot X) = 0\} \) is given by the functions \( X \) such that the columns

\[
\{X_{11}, X_{22}, \ldots, X_{rr}\}
\]

of the representative matrix \( (X_{pq}) \) vanish. So, \( F \) is a norm on the quotient \( F/F_0 \) of the algebra of group functions
with respect to the kernel. Equivalence classes are labelled by the entries of the columns \{X_{ij}, X_{k}, \ldots, X_{n}\}. A class representative can be chosen with vanishing expansion coefficients, but those of the above columns of the matrix \((X_{pq})\) we denote as \(X_{(X_{ij}, X_{k}, \ldots, X_{n})}\). In other words, we have
\[
X_{(X_{ij}, X_{k}, \ldots, X_{n})}(g) = \sqrt{\frac{n}{K}} \sum_{p=1}^{n} \sum_{q=1}^{r} X_{pq} D_{pq}(g)\]
The \(r\) columns labeling the classes determine a Hilbert space of dimension \(rn\), and a corresponding group representation \(U^*\) may be defined as
\[
(U^*(h)X_{(x_i, x_2, \ldots, x_n)})(g) := X_{(x_i, x_2, \ldots, x_n)}(h^{-1}g)
\]
\[
= \sqrt{\frac{n}{K}} \sum_{m=1}^{n} \sum_{q=1}^{r} \left( \sum_{p=1}^{n} D_{mp}(h) X_{pq} \right) D_{mq}(g)
\]
\[
= X_{(D^*(h)x_i, D^*(h)x_2, \ldots, D^*(h)x_n)}(g).
\]
In other words, we have \(U^* = \oplus_{s=1}^r D^*_s\), or \(U = \oplus_{s=1}^r D_s\), and the sum has \(r\) terms.
We can use generalized orthogonality relations to get
\[
\frac{n}{K} \sum_{j=1}^{r} \left( \psi(g_j) U(g) \right) = \sum_{j=1}^{r} \rho_j, \quad \rho_j = \rho \quad \forall s
\]
where the sum, which has \(r\) terms equal to \(\rho\), is no longer a density state.

Now, we construct a \(rn\)-dimensional column vector state \(\xi = [\xi_m]_m\) using the non-zero rows of \(\rho\),
\[
\xi_m = \sum_{j=1}^{r} \sqrt{\rho_j} \delta_{m, \nu(j+1) \nu j}, \quad m = 1, 2, \ldots, rn
\]
which defines a pure state \(\rho_\xi\), cyclic for \(U\) and such that \((\xi, U(g)\xi) = Tr[\rho D(g)]\).
This completes the proof.

7. The example of \(S_3\): the permutation group of three elements

We now examine \(S_3\), the permutation group of three elements which is isomorphic to the group of symmetries of a triangle, to show how the considerations of the previous sections appear in a concrete example.

\(S_3\) has six elements \(\{g_k : k = 1, \ldots, 6\}\) with a law of multiplication encoded in the following table,
\[
R = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 1 & 5 & 6 & 4 \\
3 & 1 & 2 & 6 & 4 & 5 \\
4 & 6 & 5 & 1 & 3 & 2 \\
5 & 4 & 6 & 2 & 1 & 3 \\
6 & 5 & 4 & 3 & 2 & 1
\end{bmatrix}
\]
from which one can obtain the group law via
\[
g_1 g_2 = g_{g_{12}}\]
For example, the table gives \(2 \cdot 3 = 1\). The inverse elements are given by
\[
g_1^{-1} = g_1, g_2^{-1} = g_3, g_3^{-1} = g_2, g_4^{-1} = g_4,
\]
\[
g_5^{-1} = g_5, g_6^{-1} = g_6.
\]
For example, from this rule we get \(2^{-1} = 3\). From (84) and (86) we find the table for \(g_1^{-1} g_k\) which reads
\[
L = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 1 & 2 & 6 & 4 & 5 \\
2 & 3 & 1 & 5 & 6 & 4 \\
4 & 6 & 5 & 1 & 3 & 2 \\
5 & 4 & 6 & 2 & 1 & 3 \\
6 & 5 & 4 & 3 & 2 & 1
\end{bmatrix}
\]
The space of group functions on \(S_3\) is isomorphic to \(\mathbb{C}^6 : f_k := f(g_k) \in \mathbb{C}^6; \quad k = 1, \ldots, 6\). The Naimark matrix of \(f\) is the \(6 \times 6\) matrix with entries \(f(g_k^{-1} g_m) = f(g_{km})\) obtained by computing \(f\) in the points labelled by \(L\).

The left regular representation \(D^L\) of the group acting on functions \(f\) is defined as
\[
(D^L(g_k) f) (g_m) = f(g_k^{-1} g_m) = f(g_{km}),
\]
therefore, in the \(kth\) row of the matrix \(L f\), we find the six values of \(D^L(g_k) f\); the left regular representation is made by the following six \(6 \times 6\) matrices \(D^L(g_k)_{mn} = \delta_{m, L_{kn}}\). Analogously, by using the transpose of \(R\) instead of \(L\), we achieve the right action. The characters of the left regular representation are easily computed to be \(\chi^L = (6, 0, 0, 0, 0, 0)\).

\(S_3\) has three unitary irreducible representations: \(D^0 : \{1, 1, 1, 1, 1, 1\}\), with the character \(\chi^0 : \{1, 1, 1, 1, 1, 1\}\); \(D^1 : \{1, 1, 1, 1, 1, 1\}\), with the character \(\chi^1 = \{1, 1, 1, 1, 1, 1\}\); and \(D^2\):
\[
\left\{ \begin{array}{c}
(1 & 0)

\begin{pmatrix}
\sqrt{\frac{3}{2}} & -1
\end{pmatrix}

\begin{pmatrix}
\sqrt{\frac{3}{2}} & 1
\end{pmatrix}

\begin{pmatrix}
1 & 0
\end{pmatrix}

\begin{pmatrix}
\sqrt{\frac{3}{2}} & 1
\end{pmatrix}

\end{array} \right\},
\]
where
\[
\lambda = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\sqrt{\frac{3}{2}},
\]
with the character
\[
\chi^2 = [2, 2 \text{ Re } \lambda, 2 \text{ Re } \lambda^2, 0, 0, 0] = [2, -1, -1, 0, 0, 0].
\]
The character \(\chi^L = (6, 0, 0, 0, 0, 0)\) can be decomposed as
\[
\chi^L = \chi^0 + 2 \chi^2
\]
and therefore the left regular representation is unitarily equivalent to
\[
D^0 \oplus D^1 \oplus D^2 \oplus D^2.
Examples of positive-type functions are the diagonal elements of any unitary representation, and any linear combination of them with positive coefficients. Characters are therefore positive-type functions. For instance the Naimark matrix of $\chi^2$ is

$$N(\chi^2) = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix},$$

(89)

and has the eigenvalues: 0, 0, 3, 3, 3, 3.

One can write the most general positive-type function on the group. The most general $N(f)$, which only takes care of Hermiticity conditions, must be proportional to the matrix

$$\begin{bmatrix} 1 & a+ib & a-ib & r & s & t \\ a-ib & 1 & a+ib & t & r & s \\ a+ib & a-ib & 1 & s & t & r \\ r & s & t & 1 & a-ib & a+ib \\ s & r & t & a+ib & 1 & a-ib \\ t & s & r & a-ib & a+ib & 1 \end{bmatrix},$$

(90)

where $a, b, r, s, t$ are real. The different eigenvalues are

$$2a + 1 \pm (r+s+t),$$

$$1 - a \pm \sqrt{3b^2 - r - s + st - rs + r^2 + s^2 + t^2}.$$

The function

$$f = (1, a+ib, a-ib, r, s, t)$$

(92)

is of positive-type iff these eigenvalues are non-negative.

Let us consider the matrix $M$, constructed by taking as rows the matrix elements with the same row label in all the irreducible group representation matrices, normalized to be a unity norm vector. The matrix $M$ is unitary, due to the orthogonality relations (34). It reads

$${}^\dagger M = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \end{bmatrix}.$$  

The matrix $M$ diagonalizes the Naimark matrix of $\chi^2$:

$$M^* N(\chi^2) M = \text{diag} \{ 0, 0, 3, 3, 3, 3 \}. $$

(93)

Using the orthogonality relations, it can be shown that, for any finite or compact group, the Naimark matrix of characters is diagonalized by the corresponding $M$ matrix.

Finally, recalling the remark at the end of section 4.1, we note that the above explicit form of the unitary matrix $M$ solves the problem of determining the unitary elements $f$ in the group algebra of $S_3$ as

$$f = M'(u^0, u^1, u^2_{11}, u^2_{12}, u^2_{21}, u^2_{22})^T,$$

(94)

where $u^0, u^1 \in U(1)$ and the matrix $u^2$ belongs to $U(2)$.

### 7.1. The group algebra of a compact Lie group

The notion of group algebra can be extended to compact Lie groups. The essential aspect for the definition of a group algebra is the existence of an (bi-)invariant measure, the Haar measure $d\mu$. Thus any continuous function $f: G \to \mathbb{C}$ on a compact Lie group is integrable with respect to the Haar measure,

$$\int_G f(g) d\mu < \infty,$$

(95)

and the integral is invariant under left as well as right actions:

$$\int_G f(gh) d\mu = \int_G f(g) d\mu = \int_G f(g) d\mu.$$

(96)

The measure $d\mu$ is normalized in such a way that the volume of the group is one. We will consider the algebra $A_G$ as consisting of all integrable functions on the group $G$, i.e. $A_G = L^1(G, d\mu)$, together with the convolution product. Thus if $A$ is the element on $A_G$ represented by the function $f_A$, we will find that the element $A \cdot B$ is represented by the function

$$\int_G f_A(h) f_B(h^{-1}) d\mu = \int_G f_A(h^{-1}) f_B(h) d\mu = f_A B (g).$$

(97)

along with

$$\text{Tr}[A^* B] = \int_G f_A^*(g) f_B(g) d\mu.$$

(98)

Other properties of the finite group algebra are extended very easily in terms of representing functions.

For instance, consider the group $U(1)$, with the circle $0 \leq \theta < 2\pi$ as the group manifold. The Abelian group $U(1)$ has irreducible 1D representations labelled by integers

$$D^m: \theta \mapsto \exp(i m \theta), \quad m \in \mathbb{Z},$$

(99)

and their characters are: $\chi^m(\theta) = \exp(i m \theta)$.

The corresponding $M$ matrix has discrete row and continuous column labelling indices

$$(M_{m'}^m) = \frac{1}{\sqrt{2\pi}} \left( \exp(i m \theta) \right).$$

(100)

Of course, it is unitary, that is

$$\sum_{\theta} (M_{m'}^m) (M_{m''}^{m'}) = \frac{1}{2\pi} \int_0^{2\pi} \exp(i (m-m') \theta) d\theta = \delta_{m,m'},$$

(101)

and

$$\sum_m (M_{m'}^m)^* (M_{m''}^m) = \frac{1}{2\pi} \sum_m \exp(i (m-m') \theta) = \delta(m-m').$$

(102)
The Pauli matrices. The eigenvalues of $\chi^m$ has elements \(\exp[i(m\theta' - \theta)]/2\pi\) and is diagonalized by $M$:

$$
(M^tN(\chi^m)M)_{m,m_2} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \exp[i(m_1 - m)\theta + i(m - m_2)\theta'] d\theta d\theta' = \delta_{m,m_2}.
$$

(103)

7.2. States and tomograms in two dimensions

States in two dimensions are parametrized by points of the 3D solid sphere

$$
\rho = \frac{1}{2} \begin{bmatrix}
1 & 0 & x \sigma_x + y \sigma_y + z \sigma_z = \frac{1}{2} \begin{bmatrix}
1 + z & x - iy \\
x + iy & 1 - z
\end{bmatrix},
\end{bmatrix}
$$

(104)

where

$$
\sigma_x = \frac{1}{2} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad \sigma_y = \frac{1}{2} \begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix}, \quad \sigma_z = \frac{1}{2} \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
$$

are the Pauli matrices. The eigenvalues of $\rho$ are

$$
\rho_{\pm} = \frac{1}{2} (1 \pm r),
$$

(106)

whereby the positivity condition \(r^2 = x^2 + y^2 + z^2 \leq 1\), so \((x, y, z)\) is a point of a ball (Bloch sphere) of radius 1 centered at the origin, and the pure states are points on the surface \(x^2 + y^2 + z^2 = 1\).

The diagonal matrices $d_g$ for $D^2$ are

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix}, \quad \begin{bmatrix}
\lambda^2 & 0 \\
0 & \lambda^2
\end{bmatrix}, \quad \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix},
$$

(107)

while the diagonalizing $V_g$, such that $V_g^t D^2(g) V_g = d_g$, are, respectively,

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}_{j=1,2,3}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix},
$$

(108)

$$
\frac{1}{\sqrt{2}} \begin{bmatrix}
-e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\
e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}}
\end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix}
-e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\
e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}}
\end{bmatrix}.
$$

The $V_g$ are determined up to phases, one for each column; tomograms \((V_g^t \rho V_g)_{num}\) are invariant under the change of these phases. The first, $V_g = V_e$, is an arbitrary unitary matrix, here chosen as the identity. At the point $g = e$ the tomogram is an arbitrary stochastic vector: this is in agreement with the probabilistic interpretation of the tomogram as the probability of getting the eigenvalues of $D(e)$ in a measure.

The matrices \(V_g^t \rho V_g\) for a generic 2D state with respect to the representation $D^2$ are the stochastic vectors

$$
\begin{bmatrix}
\frac{1}{2} & 1 \\
1 & \frac{1}{2}
\end{bmatrix}_{j=1,2,3}, \quad \begin{bmatrix}
1 & \frac{1}{2} \\
\frac{1}{2} & 1
\end{bmatrix}_{j=4,5,6},
$$

(110)

7.3. Positive-type functions

In view of proposition 2, any positive group function has the form $\varphi_\rho = \text{Tr}(\rho D)$, with $D = D^0 \oplus D^1 \oplus D^2$ and $\rho$ decomposes accordingly. The $4 \times 4$ density state $\rho$ has the form

$$
\rho = \begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & \beta & 0 & 0 \\
0 & 0 & \frac{1}{2} \gamma (y + iy) & \frac{1}{2} \gamma (y - iy) \\
0 & 0 & \frac{1}{2} \gamma (y + iy) & \frac{1}{2} \gamma (y - iy)
\end{bmatrix},
$$

(111)

where $\alpha, \beta, \gamma \geq 0$, with $\alpha + \beta + \gamma = 1$, while the positive-type function $\varphi_\rho = \text{Tr}(\rho D)$ has the values

$$
\varphi_\rho = \begin{bmatrix}
\alpha + \beta - \frac{1}{2} \gamma (1 + i \sqrt{3}z) \\
\alpha + \beta - \frac{1}{2} \gamma (1 - i \sqrt{3}z) \\
\alpha - \beta + \gamma x \\
\alpha - \beta + \gamma x
\end{bmatrix}.
$$

(112)

This vector gives explicitly the form previously obtained in equation (92). For $\alpha = \beta = 0$ this gives positive functions when only $D^2$ is present.

8. Tomograms associated with $U(n)$ groups

In this section we introduce the tomograms of states, associating the tomographic probabilities with the group $U(n)$.
and other compact Lie groups \( G \). Since \( U(n) \) can be factorized as \( U(1) \times SU(n) \) up to a quotient by \( \mathbb{Z}_n \), we will mainly be concerned with \( SU(n) \).

As a general remark, we observe that all of the previous results can be extended to the present case in a straightforward manner. The diagonalization procedure leading to the tomographic scheme for finite groups is recovered by means of the theory of maximal tori for compact groups. In fact, the diagonalization procedure provides a set of eigen-projectors, containing a family of rank-one projectors which is tomographic, i.e. it is a resolution of the identity. In the compact group case, the tomographic family is obtained by group action on a fiducial set of rank-one projectors, obtained by the eigenvectors of a complete set of commuting observables.

We begin with a review of some results on compact Lie groups \( G \) that will be needed in what follows [47].

Any element \( g \) of \( G \) lies on a one-parameter subgroup \( L \), which does not need to be compact, and whose closure is a torus \( T \). Every such torus \( T \) is contained in a maximal torus \( T_{\text{max}} \) so that any element of the group belongs to a maximal torus at least. All maximal tori are conjugated: if \( T_{\text{max}} \) and \( T'_{\text{max}} \) are maximal tori, there exists an element \( g \) such that \( T_{\text{max}} = g T'_{\text{max}} g^{-1} \). So, maximal tori have the same dimension \( r \), the rank of the group \( G \). Moreover, \( G \) may be obtained by conjugating a fixed maximal torus \( T_{\text{max}} \) by all elements of \( G \) or, more simply, by representative elements \( g \) of cosets \( [g] \) of \( G/T_{\text{max}} \):

\[
G = \bigcup_{g \in G} g T_{\text{max}} g^{-1} = \bigcup_{[g] \in G/T_{\text{max}}} g T_{\text{max}} g^{-1}.
\] (113)

An element \( t \) of \( T_{\text{max}} \) is called regular if it does not belong to any other maximal torus, otherwise the element is singular. In other words, \( t \) is singular iff there exists \( g \notin T_{\text{max}} \) such that \( g t g^{-1} = t \), in particular the unity of \( G \) is singular.

From a tomographic point of view, it is necessary to describe the previous results by using the Lie algebra \( g \) of \( G \) and a Cartan subalgebra \( h \subset g \), which is mapped in a maximal torus \( T_{h} \) by the exponential map. To characterize the singular elements \( t \) of \( T_{h} \), we introduce a basis of \( d \) generators of \( g \): \( E_1, \ldots, E_{d-r}, H_1, \ldots, H_r \). Upon putting \( t = \exp \xi^b H_b \), \( g = \exp \eta^e E_e \), (hereafter we adopt the Einstein summation convention) we look for solutions of \( g t g^{-1} = t \), \( g \notin T_{h} \), at the level of Lie algebra, in the form

\[
[\eta^e E_e, \xi^b H_b] = 0.
\] (114)

This amounts to

\[
C_{a,b}^{a'} \xi^{a'} \eta^b = 0; \ a, a' = 1, \ldots, d - r,
\] (115)

where \( C_{a,b}^{a'} \) are structure constants of the algebra of the group \( G \). The above square linear system in the unknowns \( [\eta^e] \) yields the commutant, external to the Cartan subalgebra, of the given element \( \xi^b H_b \). Non-trivial solutions correspond to singular elements \( t = \exp (\xi^b H_b) \). If the compact Lie group \( G \) is semisimple we can identify its Lie algebra and its dual by means of the Killing–Cartan form. The dimension of the orbit of the \((co)-adjoint\) action of the group on its Lie algebra through a singular point, \( \xi^b H_b \), is smaller than that of the orbit through a regular point, which is \( d - r \). The same holds for the action of the group on itself by conjugation. We recall that all the co-adjoint orbits, both regular and singular, are symplectic manifolds, hence endowed with invariant measures. From a measure theoretical point of view, the set of all singular orbits has zero Haar measure in the group. As a consequence, integration of functions on the group may be performed via Fubini’s theorem, integrating over a maximal torus \( T_{h} \), the integral over a regular orbit through \( t \), times a Jacobian, taking into account the dependence on \( t \). This Jacobian can, in general, be evaluated for any compact Lie group [48].

Quantum tomography requires the use of an irreducible unitary group representation \( D(g) \). Assume \( D \) is the defining representation of \( G = SU(n) \). Then the Cartan subalgebra generators \( \{H_b\} \) become a complete set of commuting observables of a physical system. From the previous analysis, we know that the spectrum degeneracy of \( \xi^b H_b \) is even for singular points.

The adjoint action on the maximal torus gives rise to the family of unitary operators \( D(g) \exp (i \xi^b H_b) D(g)^* \) (\( g \)). By decomposing the vector space \( g = h \oplus e \) as a direct sum of orthogonal subspaces, and choosing accordingly the basis \( H_1, \ldots, H_r, E_1, \ldots, E_{d-r} \), we observe that the elements \( \exp \eta^e E_e \) parametrize \( G/T_{h} \) so that \( D(g) \exp (i \xi^b H_b) D(g)^* \) can be parametrized by \( (\xi^b, \eta^e) \), i.e.

\[
D(g) \exp (i \xi^b H_b) D(g)^* = D(\tilde{g}),
\] (116)

where \( \tilde{g} = (\xi^b, \eta^e) \) covers the whole group \( G \) almost everywhere. In other words, \( D(\tilde{g}) \) is diagonalized by \( D(g) \) and is iso-spectral with \( \exp (i \xi^b H_b) \). Both these matrices belong to the representation, in contrast to the finite group case, where the diagonalizing matrix \( V_{\xi^b} \) and the diagonal matrix \( d_{\xi^b} \) do not belong to the representation.

We note that, as \( D(g) \) diagonalizes \( D(\tilde{g}) \) for any \( g \in [g] \), one can choose \( g = (0, \eta^e) \) to avoid redundancies. The above invariant integration on the group may be performed according to that parametrization.

By using the projector valued measure (PVM) \( \Pi (\xi^b H_b) \) associated with the Hermitian operator \( \xi^b H_b \), the spectral decomposition of \( D(\tilde{g}) \) may be written as

\[
D(\tilde{g}) = \int_\mathbb{R} \exp (i \eta^e E_e) \Pi (\xi^b H_b)(d\lambda) \exp (-i \eta^e E_e).
\] (117)

By means of a density state of a physical system \( \rho \), we define a positive-type group function \( \psi (\tilde{g}) = \text{Tr} \left[ \rho D(\tilde{g}) \right] \) in terms of a probability measure \( \mathcal{M}_\rho \):

\[
\psi (\tilde{g}) = \int_\mathbb{R} \exp (i \eta^e E_e) \text{Tr} \left[ \rho (\xi^b H_b)(d\lambda) \exp (-i \eta^e E_e) \right] \] (118)

The probability measure \( \mathcal{M}_\rho \), which is labelled by \( \tilde{g} = (\xi^b, \eta^e) \), is related to the tomogram associated with the density state \( \rho \), in the tomographic scheme based on the group \( G \).

More precisely,

\[
\mathcal{M}_\rho (\xi^b, \eta^e)(d\lambda) (119)
\]
is the probability that a measure of the observable $\xi^b H_b$ in the rotated state $\exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a)$ belongs to the Borel set $B$ of the real line. As a consequence

$$
\int_B \mathcal{M}_\rho(\xi^b, \eta^a)(d\lambda) = \int_{kB} \mathcal{M}_\rho(k\xi^b, k\eta^a)(d\lambda')
$$

and we get the homogeneity property

$$
\mathcal{M}_\rho(k\xi^b, k\eta^a; d\lambda) = \frac{1}{|k|} \mathcal{M}_\rho(\xi^b, \eta^a; d\lambda). \tag{120}
$$

In view of the compactness of $G$, all the unitary irreducible representations (UIRs) are finite dimensional and the PVM of $\xi^b H_b$ is concentrated on a set of $n = \dim D$ points, $|\mu_m|_m$, $\mu_m = k^b m_b$, where $m_b$ is an eigenvalue of $H_b$, $b = 1, \ldots, r$, while $m = 1, \ldots, n$:

$$
\Pi(\xi^b)(d\lambda) = \sum_{m_b} P_{\mu_m}(\lambda - k^b m_b)d\lambda. \tag{122}
$$

Here a Gelfand–Zetlin basis has been chosen in such a way that the rank-one projector $P_{\mu_m}$ projects on the eigenspace of the eigenvalue $\xi^b m_b$, which is the same eigenspace of the eigenvalues $m_b$, for any $b$. Then we can define the tomogram of the state $\rho$, $W_\rho(\eta^a; m)$ with respect to the representation $D$ of the group $G$:

$$
\text{Tr}(\rho D(\tilde{g})) = \sum_{|m|} \exp(i\xi^b m_b) \text{Tr}\left[ \exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a) P_{|m|} \right] = \sum_{|m|} \exp(i\xi^b m_b) \text{Tr}\left[ \rho \exp(i\eta^a E_a) P_{|m|} \exp(-i\eta^a E_a) \right]
$$

$$
= \sum_{|m|} \exp(i\xi^b m_b) W_\rho(\eta^a; |m|). \tag{123}
$$

In other words, the tomogram $\{W_\rho(\eta^a; |m|)\}$ is a stochastic vector:

$$
\sum_{|m|} W_\rho(\eta^a; |m|) = 1. \tag{124}
$$

The component $W_\rho(\eta^a; |m|)$ is the joint probability that a measure of any $H_b$ in the rotated state $\tilde{\rho} = \exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a)$ is $m_b$:

$$
\text{Tr}[\tilde{\rho} H_b] = \text{Tr}\left[ \exp(-i\eta^a E_a) \rho \exp(i\eta^a E_a) \sum_{|m|} m_b P_{|m|} \right] = \sum_{|m|} m_b W_\rho(\eta^a; |m|). \tag{125}
$$

We observe explicitly that the tomogram can be viewed equivalently as a measure of the rotated observable $\exp(i\eta^a E_a) H_b \exp(-i\eta^a E_a)$ in the state $\rho$. In other words, from the fiducial set of rank one projectors $P_{|m|}$, we find a tomographic set of rotated rank-one projectors. Of course, the density state $\rho$ can be reconstructed from its tomogram $W_\rho$.

To this aim, we observe that from the orthogonality relations we get

$$
d^{(D)} \int_G \psi(\tilde{g})^* D(\tilde{g}) d\tilde{g} = d^{(D)} \int_G \text{Tr}[\tilde{\rho} D(\tilde{g})]^* D(\tilde{g}) d\tilde{g} = \rho, \tag{126}
$$

where $d^{(D)}$, the formal dimension, is the dimension of $D$ divided by the Haar volume of the group. That is, taking into account the reality of the tomograms,

$$
d^{(D)} \int_G \sum_{|m|} \exp(-i\xi^b m_b) W_\rho(\eta^a; |m|) D(\tilde{g}) d\tilde{g} = \rho. \tag{127}
$$

We observe that the above equation may be further detailed in particular cases, for example, in the $SU(2)$ case with $D = D^j$ of $2j + 1$ dimensions. We first note that, in general, as $\tilde{g} = gt g^{-1}$ with $t \in T$ and $g \in G$, for any summable group function $f$,

$$
\int_T \int_G f(\tilde{g}) \mu_G(d\tilde{g}) = \int_G \int_T f(g t g^{-1}) \mu_G(dg) \mu_T(dr), \tag{128}
$$

where $\mu_G$ and $\mu_T$ are normalized invariant measure on $G$ and $T$ respectively.

Then, in the canonical basis of the eigenvectors $\{|m|\}, m = -j, \ldots, j$, of $J$ we have

$$
\rho_{m_{m'j}} = d^{(D)} \int_G \sum_{m_{m'j}} \exp(-i\xi^b m_b) W_\rho(\eta^a; m) D_{m_{m'j}}(\tilde{g}) d\tilde{g}
$$

$$
= \frac{d^{(D)}}{2\pi} \int_0^{2\pi} d\xi \int_{G \times |m|} \sum_{m_{m'j}} \exp[i\xi(m' - m)]
$$

$$
	imes W_\rho(g; m) (D(g) |m'| |m| D^*(g))_{m_{m'j}} d^2 g
$$

$$
= d^{(D)} \int_{|m|} \int_G W_\rho(g; m) (D(g) |m| D^*(g))_{m_{m'j}} d^2 g. \tag{129}
$$

The expression $D(g) |m| D^*(g)$ is just the action of the group on $\mathcal{H} \otimes \mathcal{H}^*$, where $\mathcal{H}$ is the carrier space of $D$ and $\mathcal{H}^*$ its dual, the carrier space of the transpose representation $D^T(g^{-1}) : D^T(g^{-1}) (|m|) = (m, D(g^{-1}) \cdot |m|) = (m) D^T(g)$.

For $SU(2)$, the representations $D(g)$ and its complex conjugate $D^*(g) = D^T(g^{-1})$ are equivalent for any $j$, so we can use the contravariant basis ([49], section 41), $|m| \rightarrow (-1)^{j-m} |m| - m)$, in such a way that the group action on $\mathcal{H} \otimes \mathcal{H}^*$ is equivalent with the group action on $\mathcal{H} \otimes \mathcal{H}$. This allows us to use the group action $D \otimes D$ on $\mathcal{H} \otimes \mathcal{H}$ and the addition theorem to decompose the product representation:

$$
D_j \otimes D_j = \bigoplus_{j=0}^{2j} D_j. \tag{130}
$$
Finally, the reconstruction formula for the matrix element $\rho_{m_1 m_2}$ reads

$$
\rho_{m_1 m_2} = \sum_{m = -j}^{j} d^{(D)} \int_{G} W_{\rho}(g ; m) \times \langle D(g) | m | D^\dagger(g) \rangle_{m_1 m_2} \, dg
$$

$$
= \sum_{m = -j}^{j} \sum_{m' = 0}^{J} \rho^*_{m' m_1} \sum_{j = 0}^{j} \frac{(-1)^{j - m - m_2} d^{(D)}}{J!} \times \sum_{m = -m_2}^{m} \int_{G} W_{\rho}(g ; m) \langle m_1 | -m_2 J M \rangle \langle j j j | m \rangle \langle -m | m \rangle \, dg
$$

$$
= \sum_{m = -j}^{j} \sum_{m' = 0}^{J} \rho^*_{m' m_1} \sum_{j = 0}^{j} \frac{(-1)^{j - m - m_2} (2J + 1)}{J! \times \sum_{M = 0}^{J} \int_{G} W_{\rho}(g ; m) D_{M}^{J}(g) \, dg},
$$

(131)

where the Wigner $3j$-symbols are introduced. The above equation may be related to the reconstruction formulae contained in [50].

By observing that

$$
W_{\rho}(g ; m) = W^{\rho}_{\nu}(g ; m)
$$

$$
= \sum_{m_1, m_2 = -j}^{j} \rho^*_{m_2 m_1} \sum_{J = 0}^{j} \frac{(-1)^{J - m - m_1} d^{(D)}}{J! \times \sum_{M = 0}^{J} \int_{G} W_{\rho}(g ; m) D_{M}^{J}(g) \, dg},
$$

(132)

the integration over the group yields $\delta_{j, j'} \delta_{M, M'}$. By means of the well known identities

$$
(2J + 1) \sum_{m = -j}^{j} \frac{(-1)^{m - m_1} d^{(D)} \sum_{J = 0}^{j} \sum_{M = 0}^{J} \delta_{m, -m_1} \delta_{j, j'} \delta_{M, M'} \int_{G} W_{\rho}(g ; m) D_{M}^{J}(g) \, dg}{J! \times \sum_{m = -m_2}^{m} \int_{G} W_{\rho}(g ; m) \langle m_1 | -m_2 J M \rangle \langle j j j | m \rangle \langle -m | m \rangle \, dg} = 1,
$$

$$
\sum_{J = 0}^{j} \sum_{M = 0}^{J} \frac{(-1)^{J - m - m_1} d^{(D)} \sum_{m = -m_2}^{m} \int_{G} W_{\rho}(g ; m) \langle m_1 | -m_2 J M \rangle \langle j j j | m \rangle \langle -m | m \rangle \, dg}{J! \times \sum_{m = -m_2}^{m} \int_{G} W_{\rho}(g ; m) \langle m_1 | -m_2 J M \rangle \langle j j j | m \rangle \langle -m | m \rangle \, dg} = \delta_{m_1, m_2} \delta_{m_1, m_2},
$$

(133)

the lhs of equation (131) eventually gives $\rho^*_{m_2 m_1} = \rho_{m_1 m_2}$.

In the general $U(n)$ case, when the representation used and its conjugate are equivalent, one can try to follow the previous route to perform the reconstruction.

However, we note that $SU(2)$ can be embedded irreducibly in the defining representation of $U(n)$, for any $n$. So, the above reconstruction formula is general and can be used to reconstruct density states out of the restriction of the $U(n)$ tomograms to the subgroup $SU(2)$.

Retuning to the general analysis, we remark that as $\varphi(\tilde{g}) = \text{Tr} \left( \rho D(\tilde{g}) \right)$ is a function on the group $G$ of positive type, the theorem of Naimark [39] states that there exists a unitary representation $U$ on a Hilbert space determined by a GNS construction and a cyclic vector $\psi_0$ such that

$$
\varphi(g) = \langle \psi_0, U(g) \psi_0 \rangle.
$$

(134)

As a result, following a procedure similar to that discussed in section 6, if $\rho$ is a pure state $| \psi \rangle \langle \psi |$, then $U$ and $\psi_0$ are unitarily equivalent to $D$ and $\psi$, respectively. When $\rho$ is a mixed state of rank $r$, then $U$ is reducible, and can be put in a block form of $r$ blocks, $\mathcal{V}$, unitarily equivalent to $D : \mathcal{V}(g) = V D(g) V^\dagger$. Then $\rho$ can be reconstructed by

$$
\rho^{-1}(U) \int_{G} \varphi^*(g) \mathcal{V}(g) \, dg = V \rho V^\dagger.
$$

(135)

This extends proposition 3 of section 6 to the compact group case.

Proposition 2 of section 6 can also be extended to the present case. However, we note that when an arbitrary irreducible representation has been chosen instead of the defining one, the Cartan subalgebra operators are no longer a complete set, and a Gelfand–Zetlin operators are no longer a complete set, and a Gelfand–Zetlin basis construction in connection with tomographic representations. In fact, tomograms depend not only on the group parameters, playing the role of ‘positions’ in configuration space, but also on Gelfand–Zetlin basis labels, that play the role of ‘conjugate momenta’.

9. Gelfand–Zetlin bases

The tomograms constructed using a unitary representation of a group $G$ are connected not only with the group itself, but also with the choice of the chain of the subgroups of the group which is used to determine the basis in the Hilbert space on which the irreducible representation of the group is acting. In fact, the tomogram $W^{\rho}_{\nu}(g, m)$ is a function of the group element $g$, of the Casimir label of the representation $\alpha$ and of the collective label $m$ which determines the basis vector in the corresponding Hilbert space. We recall how this label $m$ is determined. For example, for the $SU(2)$-group, the natural choice of the parameter $m$ is the spin projection on the z-axis for a given value $\alpha = J$ of the Casimir operator $J^2$.

In a purely group-theoretical formalism that does not use any “physical” interpretation of the index $m$ (and index $\alpha$ as angular momentum $J$), the basis is determined by the Lie algebra generator $J_z$ of the subgroup $U(1)$ of the group $SU(2)$; this gives the chain $SU(2) \supset U(1)$. In the case of $SU(3)$, the Gelfand–Zetlin basis is determined by the chain $SU(3) \supset SU(2) \supset U(1)$ of subgroups embedded into $SU(3)$. In fact, we determine the basis using first the Casimir operators of $SU(3)$, then using the Casimir operator of $SU(2)$ (corresponding to the value of the isotopic spin $T^z$) and the generators of the Cartan subalgebra of $SU(2)$ providing the weights $m_1, m_2$. Due to multiplicity of the weights, in order to label Cartan generators as eigenvectors
one needs the Casimir operator $T^2$ of the subgroup $SU(2)$, to be embedded into the initial group $SU(3)$. For any higher group $SU(n)$, the Gelfand–Zetlin basis is constructed by using the chain $SU(n) \supset SU(n-1) \supset \cdots \supset U(1)$ of embedded subgroups.

Other possibilities to use different chains of subgroups embedded into the initial group $G$ exist. For example, one can construct the basis for the irreducible representations of the group $SU(6)$ by using the subgroup $SU(3) \otimes SU(2)$ embedded into $SU(6)$. The basis obtained in this manner provides the possibility to find ‘quantum numbers’ corresponding to standard spins (i.e. associated with the manner provides the possibility to find ‘quantum numbers’). We can construct the basis for the irreducible representations embedded into the initial group $G$ in terms of the corresponding rank-one projectors this reads:

$$
\rho_g = \sum_{m_1,m_2,m_3} e^{i\xi_1 m_1 + i\xi_2 m_2} |m_1, m_2, m_3; g\rangle \langle m_1, m_2, m_3|,
$$

where $P_{m_1,m_2,m_3}(g)$ is the rank-one projector corresponding to $|m_1, m_2, m_3; g\rangle$. In other words, $\{P_{m_1,m_2,m_3}(g)\}$ is the PVM of the observable $\xi^1 H_1 + \xi^2 H_2$, which is a concentrated measure on the points $\{\xi^1 m_1 + \xi^2 m_2\} \subset \mathbb{Z}$. We find an expression for the positive function in the form

$$
\varphi_\rho(\xi) = \sum_{m_1,m_2,m_3} e^{i\xi_1 m_1 + i\xi_2 m_2} |m_1, m_2, m_3; g\rangle \langle m_1, m_2, m_3| W_\rho (m_1, m_2, m_3; g).
$$

Here we have defined the tomogram of $\rho$ in the irreducible representation $D^{(\xi)}_{C_1,C_2}$ of $SU(3)$ as the function $W_\rho (m_1, m_2, m_3; g) = \mathrm{Tr}[\rho P_{m_1,m_2,m_3}(g)]$.

Let us rotate the basis $|m_1, m_2; m_3\rangle$ by applying the representation matrix $D(g)$ of $SU(3)$. We find a new basis

$$
|m_1, m_2; m_3; g\rangle := D(g) |m_1, m_2; m_3\rangle.
$$

Then, we consider for a group element, $g$, the mean value of $D(\xi)$ in the density state $\rho$ belonging to the Hilbert space of the irreducible representation; in other words we find the Naimark positive function

$$
\varphi^{(\xi),C_1,C_2}_{\rho}(\xi) = \mathrm{Tr}[\rho D^{(\xi)}_{C_1,C_2}(g)](\xi).
$$

Here, dropping the label $(C_1, C_2)$,

$$
D(g) = D(g) D(t) D^\dagger(g), \quad D(t) = \exp[\frac{\imath}{\hbar} (\xi^1 H_1 + \xi^2 H_2)].
$$

Consider the standard spectral decomposition of the unitary matrix $D(g)$:

$$
D(g) = \sum_{m_1,m_2,m_3} e^{i(m_1+m_2+3m_3)} |m_1, m_2, m_3; g\rangle \langle m_1, m_2, m_3|,
$$

where $P_{m_1,m_2,m_3}(g)$ is the rank-one projector corresponding to $|m_1, m_2, m_3; g\rangle$. In other words, $\{P_{m_1,m_2,m_3}(g)\}$ is the PVM of the observable $\xi_1 H_1 + \xi_2 H_2$, which is a concentrated measure on the points $\{\xi_1 m_1 + \xi_2 m_2\} \subset \mathbb{Z}$. We find an expression for the positive function in the form

$$
\varphi_\rho(g) = \sum_{m_1,m_2,m_3} e^{i(m_1+m_2+3m_3)} \mathrm{Tr}[\rho P_{m_1,m_2,m_3}(g)]
$$

$$
= \sum_{m_1,m_2,m_3} e^{i(m_1+m_2+3m_3)} W_\rho (m_1, m_2, m_3; g).
$$

10. The paradigmatic case of $SU(3)$

We illustrate the previous analysis by considering the paradigmatic example of the group $SU(3)$.

The basis vectors of irreducible representations of $SU(3)$ are labelled by the eigenvalues $C_1$ and $C_2$ of the Casimir operators $C_1$ and $C_2$. These, in the case of the SU(2) group reduce to the spin Casimir operator $J^2$ with eigenvalues $j(j+1)$, $j=0, 1/2, \ldots$. After fixing the representation $D^{(C_1,C_2)}$ by a pair $(C_1, C_2)$, there is a Gelfand–Zetlin basis $\{|m_1, m_2, m_3; g\rangle\}$ of the Hilbert space acted upon by $D^{(C_1,C_2)}$, labelled by three quantum numbers $m_1, m_2$, and $m_3$.

The quantum numbers $m_1, m_2$ are the spectra of the Cartan subalgebra $[H_a]$ operators, i.e.

$$
H_a |m_1, m_2, m_3\rangle = m_a |m_1, m_2, m_3\rangle, \quad a = 1, 2,
$$

and $m_3$ is the eigenvalue of the Casimir operator $T^2$ associated with the $SU(2)$ (isotopic spin) subgroup of $SU(3)$

$$
T^2 |m_1, m_2, m_3\rangle = m_3(m_3+1) |m_1, m_2, m_3\rangle.
$$

Let us rotate the basis $|m_1, m_2; m_3\rangle$ by applying the representation matrix $D(g)$ of $SU(3)$. We find a new basis

$$
|m_1, m_2; m_3; g\rangle := D(g) |m_1, m_2; m_3\rangle.
$$

Then, we consider for a group element, $g$, the mean value of $D(g)$ in the density state $\rho$ belonging to the Hilbert space of the irreducible representation; in other words we find the Naimark positive function

$$
\varphi^{(\xi),C_1,C_2}_{\rho}(\xi) = \mathrm{Tr}[\rho D^{(\xi)}_{C_1,C_2}(g)](\xi).
$$

11. An inverse tomographic problem

Consider an operator $A$, acting on the same Hilbert space of the irreducible representation $D^\mu$ of the finite group $G_K$. Using the tomographic symbols $\{V^\mu_{\alpha} A V^\mu_{\gamma}\}_{\alpha\gamma}$ of the operator $A$, the formula holds:

$$
\frac{n_\alpha}{K} \sum_{j=1}^{n_\alpha} \sum_{m_1,m_2,m_3} \exp(-i\xi^a m_1) \left( V^\mu_{\alpha} A V^\mu_{\gamma}\right)_{m_1,m_2,m_3} D^\mu_{\gamma} (g_j)
$$

$$
= \frac{n_\alpha}{K} \sum_{j=1}^{n_\alpha} \mathrm{Tr}[A D^\mu (g_j)] D^\mu_{\gamma} (g_j)
$$

$$
= \sum_{\alpha, \rho} A^\mu_{\alpha \rho} \frac{n_\alpha}{K} \sum_{j=1}^{n_\alpha} D^\mu (g_j)_{\gamma \rho} D^\rho_{\gamma \gamma} (g_j) = A^\nu_{\nu} = A^\dagger_{\alpha}.
$$

When the operator $A$ is an observable, i.e. $A^\dagger = A$, the equation above is a reconstruction formula for $A$. 


Let us consider the family of $n_{α}$-dimensional vectors
\[
\{v^α_m(g_j)\} = \left\{(V^α_{sl}AV^α_{sl})_{mm}\right\}, \quad j = 1, \ldots, K. \tag{145}
\]
In view of the above formula, as $(V^α_{sl}AV^α_{sl})_{mm} = (V^α_{sl}A^TAV^α_{sl})_{mm}$, they satisfy the self-consistency relation written in terms of a reproducing kernel $R^α_{pm}(g_j, g_h)$:
\[
\frac{n_u}{K} \sum_{j=1}^{K} \sum_{m=1}^{n_u} R^α_{pm}(g_j, g_h) v^α_m(g_j) = v^α_p(g_h), \tag{146}
\]
\[
R^α_{pm}(g_j, g_h) := e^{-\alpha g_j g_h} \sum_{r,s=1}^{n_u} (V^α_{sr})_p D^α_{rs}(g_j) (V^α_{rs})_p. \tag{147}
\]

The vectors $v^α(g_j)$ can be chosen as stochastic vectors only if $A$ is a positive semidefinite observable, i.e. a density state $ρ$, after normalization.

In fact, after diagonalization, $A = U \text{diag} \times [λ_1, λ_2, \ldots, λ_n] U^†$, we may choose the arbitrary diagonalizing matrix $V$ associated with the neutral element $e$ of the group to be $U, V_α = U$. If $A$ is diagonal we choose the identity matrix as $V_α$. In this way, we find as a corresponding column vector simply $(λ_1, λ_2, \ldots, λ_n)^T$, which is a (normalizable) stochastic vector only when all the eigenvalues are non-negative.

However, the above condition is by no means sufficient: a family of stochastic vectors can be associated with any observable $A$.

For instance, in the triangle group case, consider the tomographic symbols $\{v^α_m(g_j)\}$ of the observable $A$,
\[
A = U \begin{bmatrix} λ_1 & 0 \\ 0 & -λ_2 \end{bmatrix} U^†,
\]
\[
= \begin{bmatrix} λ_1 \cos^2 \frac{θ}{2} - λ_2 \sin^2 \frac{θ}{2} & -λ_1 \lambda_2 \frac{1}{2} \sin \theta e^{iφ} \\ -λ_1 \lambda_2 \frac{1}{2} \sin \theta e^{-iφ} & λ_1 \sin^2 \frac{θ}{2} - λ_2 \cos^2 \frac{θ}{2} \end{bmatrix}, \tag{148}
\]
where
\[
λ_1, λ_2 > 0, \quad U = \begin{bmatrix} \cos \frac{θ}{2} e^{iφ} & \sin \frac{θ}{2} e^{iφ} \\ -\sin \frac{θ}{2} e^{-iφ} & \cos \frac{θ}{2} e^{-iφ} \end{bmatrix}. \tag{149}
\]

The tomographic symbols $\{(V^α_{sl}AV^α_{sl})_{mm}\}$ of $A$ are in consequence
\[
\left\{\begin{array}{c}
\{λ_1 \cos^2 \frac{θ}{2} - λ_2 \sin^2 \frac{θ}{2}\}
\\
\{λ_1 \sin^2 \frac{θ}{2} - λ_2 \cos^2 \frac{θ}{2}\}
\\
\{λ_1 - λ_2 + (λ_1 + λ_2) \cos (φ + α) \sin θ\}
\\
\{λ_1 - λ_2 - (λ_1 + λ_2) \cos (φ + α) \sin θ\}
\end{array}\right\}_{j=1,2,3}, \tag{150}
\]
\[
Picking \ θ = π/2, \ λ_1 - λ_2 = 1, \ we \ have \ the \ vectors \ \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \frac{1}{2} \begin{bmatrix} 1 + (1 + 2λ_2) \cos (φ + α) \\ 1 - (1 + 2λ_2) \cos (φ + α) \end{bmatrix}, \tag{151}
\]
so that, putting $M(φ) = \max\{[\cos(φ + α)], α = 0, π/3, 4π/3\}$, we get the stochasticity condition
\[
(1 + 2λ_2) M(φ) \leq 1 \ Leftrightarrow 0 \leq λ_2 \leq \frac{1}{2M(φ)} - \frac{1}{2}, \tag{152}
\]
which gives a non-zero $λ_2$ for $M(φ) < 1$, that is, for $0 < φ < π/3$.

So, we have to address the problem of stating a sufficient, and necessary, condition for an assigned family of $n$-dimensional stochastic vectors, $\{τ(g_j)\}_{j=1}$, to be the tomogram of a state with respect to a given $n$-dimensional irreducible representation $D^α$ of the group $G_K$:
\[
\exists ρ : τ_m(g_j) = (V^α_{sl}ρV^α_{sl})_{mm}, \tag{153}
\]
\[
m = 1, \ldots, n, \quad j = 1, \ldots, K. \tag{154}
\]
A sufficient (and necessary) condition can be stated in terms of positivity of a suitable group function.

For the diagonal matrices, $D^α$, depend only on representation $D^α$ and are supposed to be known, we can define a normalized group function $ψ^α$ as
\[
ψ^α(g_j) = \sum_{m=1}^{n} e^{i\alpha g_j g_h} τ_m(g_j). \tag{155}
\]

Requiring that the tomographic symbols $\{v^α_m(g_j)\}$ of the operator
\[
\frac{n_u}{K} \sum_{j=1}^{K} \sum_{m=1}^{n} e^{-\alpha g_j g_h} τ_m(g_j) D^α(g_j), \tag{156}
\]
constructed using this group function, are just the assigned stochastic vectors, the self-consistency relation (146) yields
\[
\frac{n_u}{K} \sum_{j=1}^{K} \sum_{m=1}^{n} R^α_{pm}(g_j, g_h) τ_m(g_j) = v^α_p(g_h) = τ_p(g_h). \tag{157}
\]
This is a necessary condition that the stochastic vectors must satisfy in order to solve the posed problem, we may call it a condition of compatibility of $τ$ with the representation $D^α$.

Moreover, requiring that the operator (154) is self-adjoint gives
\[
\frac{n_u}{K} \sum_{j=1}^{K} \sum_{m=1}^{n} e^{-\alpha g_j g_h} τ_m(g_j) D^α(g_j) \tag{158}
\]
\[
= \frac{n_u}{K} \sum_{j=1}^{K} \sum_{m=1}^{n} e^{-\alpha g_j g_h} τ_m(g_j) D^α(g_j^{-1}). \tag{159}
\]

Finally, we check weather $ψ^α$ is a positive-type function. If the answer is affirmative, the observable (154) is just a density state $ρ_α^*$,
\[
ρ_α^* := \frac{n}{K} \sum_{j=1}^{K} (ψ^α(g_j))^* D^α(g_j). \tag{160}
\]
such that its tomogram with respect to $D^α$ is just the assigned family of stochastic vectors:
\[
\{W^α_j(g_j)\}_{j=1}^{K} = \{τ(g_j)\}_{j=1}^{K}. \tag{161}
\]
In this case, we call the tomogram the given family. Equivalently, we can write

$$\psi^a(g_j) = \text{Tr} \left[ \rho^a D^a(g_j) \right].$$

(159)

Thus, the positivity condition implies that the stochastic family is compatible with $D^a$. This, in turn, implies that, in the decomposition of a group function with respect to the matrix elements of all the irreducible representations, the normalized function $\psi^a$ has only components in the representation $D^a$. This completes the proof.

**Example.** We now illustrate the above analysis with the example of the $D^2$ representation of the triangle group.

The more general stochastic 2D distribution on the group reads

$$\tau(g_j) = \frac{1}{2} \left[ 1 + x_j \right], \quad -1 \leq x_j \leq 1, \quad j = 1, 2, \ldots, 6.$$ (160)

The compatibility condition with $D^2$ yields

$$x_4 + x_5 + x_6 = 0.$$ (161)

The Hermiticity condition gives

$$x_2 = x_3.$$ (162)

Construct the group function $\psi^2$ using equation (153) and the above self-consistency and Hermiticity relations. The Naimark matrix $\psi^2(g_1^{-1}g_j)$, $i, j = 1, \ldots, 6$, has the following distinct eigenvalues:

$$0, \frac{3}{2} \pm \frac{1}{2} \sqrt{3 \left(3 x_1^2 + 4 x_5 x_6 + 4 x_2^2 + 4 x_3^2\right)}.$$ (163)

Positivity requires that

$$3 x_1^2 + 4 x_5 x_6 + 4 x_2^2 + 4 x_3^2 \leq 3.$$ (164)

This constraint can be easily understood after diagonalization, putting

$$x_5 + x_6 = x, \quad x_5 - x_6 = \sqrt{3} y, \quad x_2 = z.$$ (165)

which yields

$$x^2 + y^2 + z^2 = r^2 \leq 1,$$ (166)

allowing the identification with the condition satisfied by density states in two dimensions, discussed in section 7.2. In other words, there exists a one-to-one correspondence between density states and stochastic distributions, satisfying the positivity condition, which simply become their tomograms.

In conclusion, in the space of parameters \{-1 \leq x_j \leq 1\}, $j = 1, \ldots, 6$, the relations \{x_2 = x_3, x_4 = -x_5 - x_6\} define the set of stochastic vectors in one-to-one correspondence with the tomographic symbols of observables in the representation $D^2$, which contains the unit ball of density states defined by the constraint

$$3 x_1^2 + 4 x_5 x_6 + 4 x_2^2 + 4 x_3^2 \leq 3.$$ (167)

To conclude this example, choose a tomogetic family of stochastic vectors by means of a suitable point $(x, y, z)$, corresponding to the density state

$$\rho = \frac{1}{2} \left[ x + iy \right] x - iy]$$ (168)

which is diagonalized, when $x + iy \neq 0$, by the unitary matrix $u$

$$u = \frac{1}{\sqrt{2}} \left[ (z - r) (r^2 - rz) - \frac{1}{2} \right] (z + r) (r^2 + rz)^{-\frac{1}{2}} (z + iy) (r^2 + rz)^{-\frac{1}{2}}.$$ (169)

The matrix $u$ corresponding to the diagonal case $x + iy = 0$ cannot be obtained by a limit procedure.

In view of the Naimark theorem and construction in section 6, it is possible to exhibit explicit formulae for a unitary representation and a pure cyclic vector state, $\xi$, to represent canonically the $\psi^2$ corresponding to the chosen point $(x, y, z)$.

One finds a 4D Hilbert space, acted upon by the following reducible representation of the group $S_3$,

$$\begin{pmatrix} D^2 & 0 \\ 0 & D^2 \end{pmatrix},$$ (170)

and the following density matrix for the pure cyclic state,

$$\rho_\xi = U \begin{pmatrix} \rho_- & 0 & 0 & \sqrt{\rho_- \rho_+} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\rho_- \rho_+} & 0 & 0 & \rho_+ \end{pmatrix} U^\dagger.$$ (171)

Here $\rho_\pm = \frac{1}{2} (1 \mp r)$ are the eigenvalues of $\rho$ and $U$ is a $4 \times 4$ matrix in block-form

$$U = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}.$$ (172)

We can check that the Hermitian matrix $\rho_\xi$ has trace one and $\text{Tr}[\rho_\xi^2] = 1$, so it is the density matrix of a pure state $\xi$. Since $U$ and $\rho$ are explicitly given in terms of the stochastic distribution $\tau(x, y, z)$, we find the relation between tomogetic probability distributions on the group and Naimark pure cyclic vector states $\xi$.

Returning to the general finite group case, suppose there are two (or more) irreducible different representations $D^a, D^b$ with the same dimensionality $n_a = n_b$, and that the stochastic vectors, $\{\tau^a(g_j)\}_{j=1}^n$, corresponding to a state $\rho^a$, make positive the function

$$\psi^a(g_j) = \sum_{m=1}^n e^{i \phi^a(g_j) \tau^a_m(g_j)}.$$ (173)

We can construct the set of stochastic vectors $\{\tau^b(g_j)\}_{j=1}^n$ corresponding to the same state $\rho^b$ and making positive the function

$$\psi^b(g_j) = \sum_{m=1}^n e^{i \phi^b(g_j) \tau^b_m(g_j)} = \text{Tr} \left[ \rho^b D^b(g_j) \right].$$ (174)
In view of the reconstruction formula (144), we have
\[
\frac{n_R}{\mathcal{K}} \sum_{j=1}^{n_R} \sum_{m=1}^{n_R} e^{-\psi_m(g_j)} e^{\psi_m(g_j)} = \sum_{r,s=1}^{n_R} (V_{rs}^g)_{pr} \rho_{rs} (V_{rs}^g)_{sp} = \frac{n_R}{\mathcal{K}} \sum_{r,s=1}^{n_R} (V_{rs}^g)_{pr} \rho_{rs} (V_{rs}^g)_{sp} = \tau_f^g (g_e).
\] (174)

It can happen that the same family of stochastic vectors satisfies the positivity condition of both the group functions \(\psi^+, \psi^\beta\). Then, in view of equation (158), the tomograms \(W^a, W^\beta\) are the same and two possibilities exist: either \(V^a = V^\beta\) or \(V^a \neq V^\beta\) for any group element \(g_e\).

In the first case, in view of equation (157), the reconstructed density states are the same: \(\rho_1^a = \rho_1^\beta\). For example, this is the case for the two inequivalent 2D representations of the tetrahedron group, related to the \(D^2\) representation of the triangular group \(S_3\) as \(D^2 = \{D^2, D^2\}^\ast\) and \(D^2 = \{D^2, -D^2\}\).

In the second case, the states are different: \(\rho_2^a \neq \rho_2^\beta\). This is the case, for example, for the 3D irreducible representations of \(SU(3)\) with \(D^3 = D^3\), where \(\rho_3^a = \rho_3^\beta\). This result is obtained by a straightforward and obvious generalization of all the above formulae and conditions to the case of compact groups.

Briefly, given on the group \(G\) an irreducible representation \(D\) and the stochastic vector function \(\tau_{[m_1, m_2]}(g)\), whose components are labelled by using a suitable Gelfand–Zetlin basis, one can construct the group function
\[
\psi (\tilde{g}) = \sum_{[m_1, m_2]} \exp(-i\delta_n m_2) \tau_{[m_1, m_2]}(g).
\] (175)

By using equation (126), a density state \(\rho\) can be recovered by \(\psi (\tilde{g})\) iff this function is of positive-type. Moreover, if the stochastic vector function is compatible with \(D\), it is the tomogram of \(\rho\),
\[
W_\rho (g ; [m_1; m_2]) = \tau_{[m_1, m_2]}(g),
\] (176)
and this completely solves the inverse tomographic problem.

The compatibility condition may be written as
\[
\tau_{[m_1', m_1]} (g') = d(D) \int_G \psi (\tilde{g})^* \times (D^t(g')D(g'))_{[m_1', m_1]} \tilde{g}
\]
\[
= d(D) \int_G \sum_{[m_1, m_2]} e^{-i\delta_n m_2} \tau_{[m_1, m_2]} (g)
\]
\[
\times (D^t(g')D(g'))_{[m_1', m_1]} \tilde{g}.
\] (177)

where \(\tilde{g} = g \exp(\tilde{g}^H H_b) g^{-1}\) and \(H_b\) are, as usual, the generators of the Cartan subalgebra.

We remark that checking the positivity of a compact group function, such as the above \(\psi (\tilde{g})\), amounts to an infinite number of operations.

However, if an irreducible representation \(D(G_K)\) of a finite group can be found in \(D(G)\), then one can limit checking the positivity condition on the finite group to only one \(K \times K\) matrix. Assume that this holds true. For example, this is the case of the defining representation of \(U(2)\), which contains the representation \(D^2\) of the group \(S_3\).

Moreover, suppose that \(\psi (\tilde{g})\) satisfies the compatibility condition with \(D\), so that it has no components with respect to other irreducible representations. In this situation the positivity of \(\psi\) on \(G\) can be checked on \(G_K\).

In fact, if \(\psi\) is positive on \(G_K\), we get a density state \(\rho\) on the \(n\)-dimensional Hilbert space on which \(D\) acts such that
\[
\psi (g_j) = \text{Tr}[\rho D(g_j)] = \sum_{r,s=1}^{n_R} \rho_{rs} D_r(s). \quad (178)
\]

By hypothesis, \(\psi\) can be expanded using only the matrix elements of \(D\),
\[
\psi (g) = \sum_{r,s=1}^{n_R} c_{rs} D_r(s). \quad (179)
\]

that are orthogonal on \(G\) as well on \(G_K\) :
\[
\delta_{r,s} \delta_{x,p} = \frac{n}{\mathcal{K}} \sum_{j=1}^{n_R} D_r^s(g_j)D_{sp}(g_j)
\]
\[
= d(D) \int_G D_r^s(g) D_{sp}(g) dg. \quad (180)
\]

It readily follows that
\[
c_{rs} = \rho_{rs} \Rightarrow \psi (g) = \text{Tr}[\rho D(g)] \quad (181)
\]
and \(\psi (g)\) is positive on \(G\).

12. Conclusion

For states of finite dimensional \(C^\ast\)-algebras, we have introduced the notion of tomographic probability distribution. This concept provides the possibility of clarifying new aspects of \(C^\ast\)-algebras related to information characteristics of probability distributions such as different kinds of entropies.

These tomograms were also introduced for finite and compact groups by using known unitary finite-dimensional irreducible representations of these groups. The tomographic probability vectors (tomograms) introduced for those groups were shown to contain complete information on the quantum states (Hermitian, trace-class, non-negative matrices) associated with the irreducible unitary representations of those groups.

The notion of the Naimark matrix and its properties were used to study necessary and sufficient conditions for the stochastic vectors defined on the finite or compact groups to be tomographic probability distributions. The Naimark theorem on positive-type group functions was shown to play a key role in the problem of connecting the tomographic probability vectors on the group with the density states on the Hilbert space of the irreducible representations of the group.

Paradigmatic examples of two groups, the group \(S_3\) of permutations of three points and \(SU(3)\), were discussed in detail. The general construction of the \(U(n)\) group (and other classical group) tomograms was presented by using the
Gelfand–Zetlin basis labels of the tomographic probability vectors.

The notion of tomographic probabilities introduced for finite C*-algebras was shown to coincide with that of tomographic probability vectors associated with finite unitary groups. The probability vectors defined on finite or compact groups establish a relation between the group structure and the structure of the simplexes containing those probability vectors. An analogous relation exists between finite C*-algebras and those simplexes, thanks to the existence of tomographic probability vectors defined on the C*-algebras.

Thus, for finite and compact groups, group algebras and abstract C*-algebras were considered in the unifying framework of the tomographic approach, where the tomograms provide the possibility of completely describing all the kinds of quantum states, both pure and mixed.

For example, the spin states (qu-dits) associated with SU(2)-group irreducible representations, can be alternatively described by the spin-tomographic probability distributions of measurable spin-projections on the quantization axes.

We will develop these aspects of the tomographic approach for systems with the discussed finite or compact symmetry groups in future papers.

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