Superfluidity and vortices: A Ginzburg-Landau model

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Abstract

The paper deals with the study of superfluidity by a Ginzburg-Landau model that investigates the material by a second order phase transition, in which any particle has simultaneously a normal and superfluid motion. This pattern is able to describe the classical effects of superfluidity as the phase diagram, the vortices, the second sound and the thermomechanical effect. Finally, the vorticities and turbulence are described by an extension of the model in which the material time derivative is used.

Keywords:
Superconductivity, Ginzburg-Landau equations, critical fields, existence and uniqueness.

1 Introduction

The behavior of superfluids is very different from the phenomenology of a perfect fluid, as well as superconductivity [14] is a phenomenon very different from perfect conductivity (see Bardeen [4], Chandrasekhar [6], Landau [17], [16], London [19], Mendelsshon [20], Fabrizio, Gentili and Lazzari [10]). Meanwhile, there is an evident similarity between the behavior of superfluids and superconductors. By the way, London claims in [19] § 22 "... in either case, superconductivity and superfluid helium, the basic equations can be written in the same form", and Mendelsshon asserts "However, frictionless flow and persistent currents are not the only features in which He II resembles a superconductor. As at the transition of a metal to the superconductive state, there occurs in helium also a rapid drop in the entropy. Moreover, superfluid flow, just as a persistent supercurrent, is distinguished by zero entropy". In view of the apparent analogies between

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superfluidity and superconductivity, in this paper we present a phase transition model, which use the two fluids Ginzburg-Landau viewpoint, in which the equation governing the motion of superfluid component $v_s$ assumes an analogous form as the equation for superconducting electrons. It turns out in our analysis that $\nabla \times v_s \neq 0$. Accordingly, the equations allow to predict existence of vortices, which have been widely studied by several authors (see: [21] cap. 6,[1], [2], [3], [5]). Moreover in this paper, we characterize superfluidity as a second order phase transition\(^1\). Thus, our model for superfluidity is able to explain the phase diagram represented in Fig.1, at least in the region close to the $\lambda$–line, which is the most significant area. At the same time, it allows to prove the existence of a critical velocity of superflow, beyond which the frictionless flow breaks down.

\[\nabla \times v_s \neq 0.\]

| Normal Liquid | Solid | Normal Liquid | Gas |
|---------------|------|---------------|-----|
| Pressure (MPa) | 1 2 3 4 5 | Pressure (MPa) | 1 2 3 4 5 |
| Temperature (K) | | Temperature (K) | |

Fig.1. Helium II. Phase diagram.

A remarkable feature of our model is the constraint on the rotational motion by means of the equation $\nabla \times v_s = v_n$ which relates $v_s$ with the normal component of the velocity $v_n$. So that the velocity is defined by

$$v = v_n + f^2 v_s. \quad (1)$$

The formula (1) does not allow to define the superfluids by means of two components, the treatment of the liquid as a mixture of normal and superfluid parts is simply a form of words ....Like any description of quantum effects in classical terms, it is not entirely adequate. It does not at all mean that the liquid can

\(^1\)There is a marked difference between first-order and second-order phase transitions. In first-order transitions, below the critical temperature the whole body occurs in a single phase. Instead, in second-order transitions, below the critical temperature the body consists of the simultaneous occurrence of both phases. That is why we can view the material, below the critical temperature, as consisting of both the normal fluid and the superfluid (two fluid model).
actually be separated into parts” [15]. Actually as in (1), any superfluid particle is capable of two types of motion simultaneously, corresponding to the energy spectrum of phonons and rotons. "One of these motions is normal, i.e. has the same properties as that of an ordinary viscous liquid; the other is superfluid" [15].

In Section 2 we will examine thermodynamical consistence of the model. Then, in view of the similarity between superfluidity and superconductivity, we will prove existence of vortices, whenever a superfluid, contained in a cylindrical bucket, rotates rapidly around its axis. In particular, in accordance with experimental results, we will show that, if the angular velocity is low, the superfluid stays at rest. Instead, when the velocity exceeds a threshold value, there occur vortices, similar to the vortices observed in superconductivity. Moreover, the proposed model allows to explain thermomechanical effect, in which we observe a motion of liquid produced by heat, but in the same direction to heat flux. Finally, we prove the existence of the phenomenon of second sound.

In section 3, we investigate the problem which arising when the flow is such that we need to consider the material time derivative of the velocities. This study introduces new non-linear terms that increase the complexity of the model and require a suitable change of the previous pattern.

In the last part of the paper we study the connection between this model and the turbulence. So that, as observed by Mendelsshon [20] "It is now clear that the dependence of the heat conduction on the heat current originally observed in Cambridge in 1937 ......is evidence of turbulence". A remarkable feature of our model is the constraint on the rotational motion \( \nabla \times \mathbf{v}_s = \mathbf{v}_n \), that relates \( \mathbf{v}_s \) with the normal component of the velocity \( \mathbf{v}_n \) and which provides the behavior of a disordered set of thin vortices.

2 A first model for superfluidity

In this section we shall provide a first approximate model for the study of superfluidity in a domain \( \Omega \subset \mathbb{R}^3 \) as a second order phase transition, by means of the Ginzburg-Landau equation

\[
\frac{\partial f}{\partial t} = \frac{1}{\kappa} \nabla^2 f - f(f^2 - 1 + u + \lambda p + v_s^2) \tag{2}
\]

where \( f \) denotes the order parameter (or phase field), while \( p \) is the pressure, \( u \) is the absolute temperature and \( \kappa, \lambda \) are positive constants. As studied in [8], the Ginzburg-Landau equation has to be considered as a new field equation, that we obtain by the balance law on the structure order (See Appendix). Indeed, following Landau, we suppose that the transition of the HeII from a normal to a superfluid state induces a change in the internal structure order. The phase of this transition is represented by the scalar parameter \( f \in [-1, 1] \) that is linked with the density \( \rho_s \) of the superfluid particles by the formula

\[
\rho_s = f^2.
\]
Hence the phase field $f^2 = 0$ denotes the normal state, while $f^2 \in (0, 1]$ describes a superfluid state. In this framework, in equation (2) if

$$R = u + \lambda p + v_s^2 - 1 > 0,$$

then the fluid is in the normal state, while when $R < 0$ the pattern describes the superfluid phase. In this work we do not employ the classical two fluids model which introduces two different velocities and densities, as for a mixture, but we suppose the velocity of any particle composed of two excitements, the normal component $v_n$ and the superfluid component $v_s$, such that

$$v = v_n + f^2 v_s. \quad (3)$$

Because superfluidity as well as superconductivity must be studied as a second order phase transition, then under the transition temperature, the phase of some particles $f$ can be still equal to zero. Nevertheless, according as the particles are in the normal phase ($f^2 = 0$) or in the superfluid one ($f^2 > 0$), we cannot interpret this different behavior as two different fluids.

Henceforth in this paper we choose $\lambda = 0$, and the density $\rho$ a positive constant. In addition, in this first approximate model we suppose the motion such that the acceleration $a_n = \frac{\partial v_n}{\partial t}$. Then, for the velocity $v_n$ we propose the modified Navier-Stokes equation

$$\frac{\partial v_n}{\partial t} = -\nabla p - \mu \nabla \times \nabla \times v_n - \mu \nabla \times f^2 v_s + g \quad (4)$$

where $g$ denotes the external force and $\mu$ is the viscosity coefficient. Since we have supposed the superfluid as an incompressible material, then the continuity equation provides

$$\nabla \cdot v_n = 0 \quad (5)$$

Furthermore, we suggest the following equations for the component $v_s$

$$\frac{\partial v_s}{\partial t} = -\nabla \phi - \mu \nabla \times \nabla \times v_s - \mu f^2 v_s + \nabla u + \mu h \quad (6)$$

$$\nabla \cdot f^2 v_s = -\tau f^2 \phi \quad (7)$$

where $\phi$ is a suitable scalar function and $\tau$ is a positive constant. While the vector $h$ is such that

$$\mu \nabla \times h = g - \nabla p, \quad \nabla \cdot h = 0, \quad h \times n|_{\partial \Omega} = 0 \quad (8)$$

Now, using (8), it is easy to prove that equation (4) can be obtained from equation (6) if

$$\nabla \times v_s = v_n \quad (9)$$

Therefore, under the latter condition it will be sufficient to work only with system (6)-(9), which contains equation (4) as a consequence.
3 Thermodynamics

In order to obtain the heat equation, let us consider the first law of thermodynamics in the form
\[ \dot{E} = \mathcal{P}^f + \mathcal{P}^{\nu s} + h, \]
(10)
where \( E \) is the total energy and \( h \) is the rate at which the heat is absorbed by the material. The internal powers \( \mathcal{P}^f, \mathcal{P}^{\nu s} \), related to the variables \( f, \nu_s \), are given\(^2\) by
\[
\mathcal{P}^f = f_t^2 + \frac{1}{2k} [ \nabla f )^2]_t + \frac{1}{4} [(1 - f^2)^2]_t + f f_t (u + v_s^2) \]
(11)
\[
\mathcal{P}^{\nu s} = \frac{1}{2} [v_{\nu s}^2]_t + \mu^{-1} (v_{\nu s} + \nabla \phi - \nabla u)^2 + f^2 v_{\nu s} \cdot \nu_{st} + u \nabla \cdot f^2 v_s + \tau f^2 \phi^2 \]
(12)
From first law (10) we have
\[
h = c(u)u_t - f_t^2 - \mu^{-1} (v_{\nu s} + \nabla \phi - \nabla u)^2 - u f f_t - u \nabla \cdot f^2 v_s - \tau f^2 \phi^2, \]
(13)
where \( c(u) \) is the specific heat, whose behaviour is represented in Fig.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{specific_heat.png}
\caption{Specific heat of Helium.}
\end{figure}

If \( T = \frac{1}{2} v_n^2 + \frac{1}{2} f^2 v_s^2 \) denotes the kinetic energy, and
\[
e = \int c(u) du + \frac{1}{4} (1 - f^2)^2 + \frac{1}{2k} (\nabla f)^2, \]
(14)
is the internal energy, then the total energy \( E = e + T \) is given by
\[
E = \frac{1}{2} v_n^2 + \frac{1}{2} f^2 v_s^2 + \int c(u) du + \frac{1}{4} (1 - f^2)^2 + \frac{1}{2k} (\nabla f)^2, \]
(15)
\(^2\)See Appendix.
In our model the Fourier theory of heat conduction will not be modified. Accordingly, the constitutive equation relating the heat flux \( q \) to the gradient of the temperature, assumes the classical form

\[
q = -k(u)\nabla u,
\tag{16}
\]

where the conductivity \( k \) depends on the absolute temperature. In this framework, since the conductivity is very small when the absolute temperature is close to zero, we suppose that

\[
k(u) = k_0 u, \quad k_0 > 0.
\tag{17}
\]

The heat balance is expressed by the equation

\[
h = -\nabla \cdot q + r,
\tag{18}
\]

from which we deduce the entropy equation (see [7])

\[
\frac{h}{u} - \frac{1}{u^2} q \cdot \nabla u = -\nabla \cdot \frac{q}{u} + \frac{r}{u}.
\tag{19}
\]

Therefore,

\[
K = \frac{h}{u} - \frac{1}{u^2} q \cdot \nabla u
\tag{20}
\]

can be identified with the rate (per unit volume) at which entropy is absorbed, while \( q/u \) and \( r/u \) represent the entropy flux and the entropy supply. Then equation (19) describes the entropy balance [7], [11]. Because of the non-local constitutive model given by system (2-7), we can state the following (see [7], [9])

**Second law of thermodynamics (non-local form).** There exists a function \( \eta(u, f) \), called entropy function, such that

\[
\frac{\partial \eta}{\partial t} \geq K + \nabla \cdot k = \frac{h}{u} - \frac{1}{u^2} q \cdot \nabla u + \nabla \cdot k.
\tag{21}
\]

where \( k \) is a suitable flux vector such that \( \mathbf{k} \cdot \nabla \phi \bigg|_{\partial \Omega} = 0 \).

It is easy to prove that system (2-7) with constitutive equations (13), (16) agrees with the inequality (21), if we regard as entropy function the functional

\[
\eta(u, f) = \int \frac{c(u)}{u} du - G(f).
\tag{22}
\]

where \( G(f) = \frac{1}{2} f^2 \) and as flux vector \( \mathbf{k} = f^2 \mathbf{v}_s \). Finally, in the sequel we shall consider an approximation of (19), namely

\[
\frac{c(u)}{u} u_t - \dot{G}(f) = \nabla \cdot (f^2 \mathbf{v}_s) + k_0 \nabla^2 u + \frac{r}{u},
\tag{23}
\]

where we have neglected the dissipative terms \( f_t \), \( (\nabla \phi - \nabla u)^2 \), \( (\nabla u)^2 \), \( \tau f^2 \phi_t^2 \).
Now, let us evaluate the free energy $\psi$ defined by

$$\psi = e - u\eta.$$  

From expressions (14) and (22) of the internal energy and the entropy, it follows that

$$\psi = \frac{1}{2\kappa}(\nabla f)^2 + \frac{1}{4}(1 - f^2)^2 + uG(f) + (1 - u) \int \frac{c(u)}{u} du.$$  

(24)

We will prove that system (2), (3), (4), (23) allows to describe the main aspects of superfluidity. First, we examine the phase diagram represented in fig.1. If we focus our attention on phenomena in which the pressure is below 25 atm, the transition occurs along the $\lambda-$line, which we can assume close to a vertical line. This phenomenon is explained by equation (2). Indeed, by determining the transition states $(u_T, v_{sT})$, we observe that the line which separates the normal and the superfluid phases is given by the curve

$$u + \lambda p + v_{s}^2 = 1$$  

(25)

If we suppose $\lambda \ll 1$ and consider the equilibrium states, i.e. $v_{s} = 0$, then from (25) we get the equation $u = 1$, which is a good approximation of the $\lambda-$line (see fig.1). Instead, when $v_{s} \neq 0$, the model is naturally able to account for the existence of a critical velocity, namely a threshold value above which the frictionless flow breaks down. This follows directly from equation (25).

The latent heat $l$ related to the transition is given by

$$l = u_T(\eta(u_T, f_1) - \eta(u_T, f_2))$$

where $f_1$ and $f_2$ are the local minima of the free energy at the transition points $u + v_{s}^2 = 1$. In this case $f_1 = f_2 = 0$. Hence, $l = 0$, which means that the transition is of second order.

In summary, our differential system assumes the form

$$\frac{\partial f}{\partial t} = \frac{1}{\kappa} \nabla^2 f - f(f^2 - 1 + u + v_{s}^2)$$  

(26)

$$\frac{\partial v_{s}}{\partial t} = -\nabla \phi - \mu \nabla \times \nabla \times v_{s} - \mu f^2 v_{s} + \nabla u + \mu h$$  

(27)

$$\nabla \cdot f^2 v_{s} = -\tau f^2 \phi$$  

(28)

$$\frac{c(u)}{u} \frac{\partial u}{\partial t} - f \frac{\partial f}{\partial t} = \nabla \cdot (f^2 v_{s}) + k_0 \nabla^2 u + \frac{r}{u}$$  

(29)

Moreover, we associate to system (26-29) the boundary conditions on the domain $\Omega$

$$\nabla f \cdot n|_{\partial \Omega} = 0, \quad (\nabla \times v_{s}) \times n|_{\partial \Omega} = \omega$$  

(30)

$$v_{s} \cdot n|_{\partial \Omega} = 0, \quad v_{n}|_{\partial \Omega} = 0, \quad u|_{\partial \Omega} = u_b$$  

(31)
It is worth noting that, when $f = 0$, in view of (3), $\mathbf{v} = \mathbf{v}_n$ and equation (6) is inessential, since the motion is described by means of the Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \mu \nabla \times \nabla \times \mathbf{v} + \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0,$$  \hspace{1cm} (32)

obtained from (4),(5). On the contrary, when the transition occurs, $f > 0$ and the consequent breaking of symmetry leads to system (26)-(29). Well-posedness of differential problem (26)-(29) in the case $f = 0$ has to be investigated carefully in order to prove uniqueness of solutions. Indeed, when $f = 0$ equation (27) reads

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \phi - \mu \nabla \times \nabla \times \mathbf{v}_s - \nabla u + \mu \mathbf{h},$$  \hspace{1cm} (33)

while (28) is satisfied identically. The temperature $u$ is univocally determined through (29), endowed with suitable initial and boundary conditions. However, since equation (33) is no longer coupled with (28), uniqueness of solutions cannot be proved. For this reason, it is necessary a choice of a gauge, which consists in decomposing the velocity $\mathbf{v}_s$ as

$$\mathbf{v}_s = \mathbf{A} + \nabla \phi, \quad \text{or} \quad \nabla \times \mathbf{A} = \mathbf{v}_n$$

The vector $\mathbf{A}$ is specified by fixing a value for its divergence and choosing suitable boundary conditions. By analogy with superconductivity, we will choose London’s gauge

$$\nabla \cdot \mathbf{A} = 0, \quad \mathbf{A} \cdot \mathbf{n}|_{\partial \Omega} = 0.$$  

Therefore, from (33) we deduce

$$\frac{\partial \mathbf{A}}{\partial t} = -\nabla (\phi + \dot{\phi}) - \mu \nabla \times \nabla \times \mathbf{A} - \nabla u + \mu \mathbf{h} \quad \hspace{1cm} (34)$$

Finally, from equation (34) we get equation (32). The main result which this model allows to prove is the presence of quantized vortices, when a cylindrical bucket containing Helium II rotates around its axis. Consider the stationary problem, in addition let us suppose the temperature gradient $\nabla \tilde{u} = 0$. Hence the differential system (26)-(28) with $\mathbf{h} = 0$ assumes the form

$$\frac{1}{\kappa} \nabla^2 f - f (f^2 - 1 + \tilde{u} + \gamma v_s^2) = 0 \quad \hspace{1cm} (35)$$

$$\mathbf{v}_s \cdot \nabla = 0, \quad (\nabla \times \mathbf{v}_s) \times \mathbf{n}|_{\partial \Omega} = \omega, \quad \mathbf{v}_s \cdot \mathbf{n}|_{\partial \Omega} = 0.$$  \hspace{1cm} (38)
a very small penetration depth close to the boundary\(^3\). This equilibrium state is called Landau’s state. The phenomenon is analogous to the diamagnetism shown in superconductivity by the Meissner effect, when for a second type superconductor the magnetic field is lower that a given field \(H_{c_1}\). When the angular velocity is greater that \(\omega_{c_1}\) we observe the creation of vortex lines in an exactly similar way as in superconductivity. Since, problem (35)-(38) is identical to the Ginzburg-Landau system in stationary case, studied by Abrikosov [1], for superconductors of type II. Therefore, when \(\omega > \omega_{c_1}\) with the same analysis, we may prove the existence of vortex lines in a superfluid. Moreover, as observed in [21], when the rotation velocity exceeds a second threshold value \(\omega_{c_2}\), owing to increase of vortices and to the overlapping of their cores, superfluidity is destroyed. Even this effect is analogous with the corresponding phenomena in superconductivity and it can be explained by (34). Indeed, for sufficiently large values of \(v_s\) we have

\[ \ddot{u} + \nu_s^2 > 1, \]

so that the fluid is in the normal phase. In every practical situations, angular velocities \(\omega\) satisfy the condition

\[ \omega < \omega_{c_2}. \]

Moreover, the first critical velocity is such that

\[ \omega_{c_1} \ll 1. \]

Another important phenomenon occurring in superfluidity, it is the thermomechanical effect according to which the particles of the superfluid flow in the same direction to heat flux. This phenomenon, which contradicts the classical behavior of thermo-fluid mechanics, can be explained by means of equation (6). Indeed in (6) the sign of \(\nabla u\) is the same as the acceleration \(v_{st}\) of the superfluid

\(^3\)Consider a fluid in a long straight wire of circular cross-section carrying a rotation around the axis. We suppose cylindrical symmetry and introduce the coordinates \(z, r, \theta \ (k, r, j)\). Then we have

\[
\begin{align*}
\nu_s(r) &= v \phi(r) j \\
\phi &= \phi(r) \\
\nu_n(r) &= \nabla \times \nu_s(r) = \frac{\partial v \phi(r)}{\partial r} k
\end{align*}
\]

Therefore, from (36) we obtain

\[
\begin{align*}
\nabla \phi(r) &= 0 \\
\frac{d^2 v \phi}{dr^2} &= -\frac{1}{r} \frac{dv \phi}{dr} + (f^2 + 1) v \phi
\end{align*}
\]

The solutions related with a given field \(v_0\) at the surface of the cylinder of radius \(R\) is

\[ v_0 = v_0 \frac{I_1(r)}{I_1(R)} \]

where \(I_1\) is the modified Bessel functions of imaginary argument. Because of the behavior of \(I_1(r)\) for \(r < R\), the velocity \(v_0(r)\) will be confined to a thin layer next to the boundary.
component. This means that the gradient of the temperature causes an increase of the velocity $v_s$ in the same direction. Moreover, to understand better the phenomenon, we have to observe that in the thermo-mechanical effect the tube may be straight and very narrow. As a consequence, the motion of the superfluid phase ($f \neq 0$) through the tube is required to satisfy $\nabla \times v_s = v_n = 0$ and hence the velocity is given by $v = f^2 v_s$. In other words, since the normal component is absent, the motion is necessarily that of the superfluid component. This in turn allows for the flow through very narrow tubes. It follows from (6) that, in stationary conditions, the motion is governed by the equation

$$-\nabla \phi - f^2 v_s + \nabla u + h = 0,$$

which, consistent by (39), shows that the term $\nabla u$ favors the particle displacement toward regions at a higher temperature. Moreover, because of the narrowness of the tube, only superfluid particles ($f = 0$) are allowed to flow in. Instead the normal component undergoes to a viscous resistance which forbids the flow to cross the tube.

Finally, we study the second sound effect. For this problem, let us suppose $f = \text{const.}$ and the supplies $h = 0$, $r = 0$. Then, from (27)-(29) it follows that

$$\nabla \cdot v_{st} = -\nabla^2 \phi - \mu \nabla \cdot f^2 v_s + \nabla^2 u$$

$$\alpha u_{tt} = \nabla \cdot (f^2 v_{st}) + k_0 \nabla^2 u_t$$

where $\alpha = \frac{4(u)}{u}$ will be supposed a positive constant. Moreover, in the superfluid framework we can suppose the thermal conductivity $k_0 \ll 1$, therefore we get from (37) and (41)

$$\alpha u_{tt} - f^2 \nabla^2 u = f^2 (\tau \mu f^2 \phi - \nabla^2 \phi)$$

In other words, the temperature satisfies a wave equation that is able to provide a new kind of wave.

### 4 General case

In the previous sections the material time derivative $\frac{dF}{dt} = \frac{\partial F}{\partial t} + v \cdot \nabla F$ was always completely ignored, because we have supposed slow motions. In this section we shall study the general case and then we use the material time derivative $\frac{dF}{dt}$ which we denote only with $\dot{F}$. In addition we suppose the fluid incompressible with a given constant density $\rho = 1$. Then, on the domain $\Omega$, the differential system assumes the new form

$$\dot{f} = \frac{1}{\kappa} \nabla^2 f - f(f^2 - 1 + u + v_s^2)$$

$$\dot{v}_n = -\nabla p - \mu \nabla \times \nabla \times v_n - \mu \nabla \times f^2 v_s + g$$

$$\nabla \cdot v_n = 0$$
\[ \dot{v}_s = -\nabla \phi - \mu \nabla \times v_n - \mu f^2 v_s + \nabla u + \mu h \quad (44) \]
\[ \nabla \cdot f^2 v_s = -\tau f^2 \phi \quad (45) \]

Because of the use of the material time derivative, even if we suppose \( \mu \nabla \times h = g - \nabla p \), it is not possible to prove that from equations (43) and (44) we obtain equation (4). Otherwise, if we consider the curl of the equation (44) and compare this new equation with (43), we have the restriction
\[ \nabla \times \dot{v}_s = \dot{v}_n \quad (46) \]

Moreover, the internal powers \( P^f \), \( P^{v_s} \), assume the analogous form
\[ P^f = f^2 + \frac{1}{2\kappa} \left[ (\nabla f)^2 \right] + \frac{1}{4} \left[ (1 - f^2)^2 \right] + f \dot{f} (u + v_n^2) \quad (47) \]
\[ P^{v_s} = \mu \nabla \times v_n \cdot \dot{v}_s + \mu^{-1} (\dot{v}_s + \nabla \phi - \nabla u)^2 + 2 f^2 v_s \cdot \dot{v}_s + u \nabla \cdot f^2 v_s + \tau f^2 \phi^2 \quad (48) \]

In particular, from (43-44) we have
\[ 0 = \int_{\Omega} \left( \frac{1}{2} [v_n^2] - \mu (\nabla \times v_n + f^2 v_s) \cdot \nabla \times v_n - (g - \nabla p) \cdot v_n \right) dx = \int_{\Omega} \left( \frac{1}{2} [v_n^2] - \nabla \times v_n \cdot \dot{v}_s \right) dx \quad (49) \]

Hence, if (9) holds the identity (49) is satisfied, but the vice versa is not true. However, from (49) and (31) we obtain the restriction
\[ \int_{\Omega} \left( \frac{1}{2} [v_n^2] - \nabla \times \dot{v}_s \cdot v_n \right) dx = 0 \quad (50) \]

which is satisfied by means of (46).

Then, from the first law (10), we get
\[ \dot{e} = h + f^2 + \frac{1}{2\kappa} \left[ (\nabla f)^2 \right] + \frac{1}{4} \left[ (1 - f^2)^2 \right] + f \dot{f} (u + v_n^2) + \nabla \cdot v_n \cdot \dot{v}_s \quad (51) \]
\[ \mu^{-1} (\dot{v}_s + \nabla \phi - \nabla u)^2 + 2 f^2 v_s \cdot \dot{v}_s + u \nabla \cdot f^2 v_s + \tau f^2 \phi^2 \]

where the internal energy is given by
\[ e = \int c(u) du + \frac{1}{4} (1 - f^2)^2 + \frac{1}{2\kappa} (\nabla f)^2, \quad (52) \]

which implies
\[ h = c(u) \dot{u} - \dot{f}^2 - (\dot{v}_s + \nabla \phi - \nabla u)^2 - u f \dot{f} - u \nabla \cdot f^2 v_s - \tau f^2 \phi^2, \quad (53) \]

where \( c(u) = c_u(u) \) is the specific heat.

With similar arguments considered in the previous section we obtain the same function for the entropy
\[ \eta(u, f) = \int \frac{c(u)}{u} du - G(f), \quad (54) \]
and under the same approximations, the following heat equation
\[ \frac{c(u)}{u} \dot{u} - \dot{G}(f) = \nabla \cdot (f^2 v_s) + k_0 \nabla^2 u + \frac{r}{u}, \tag{55} \]

From expressions (15) and (54) of the internal energy and the entropy, it follows that
\[ \psi = \frac{1}{2\kappa} (\nabla f)^2 + \frac{1}{4} (1 - f^2)^2 + uG(f) + (1 - u) \int \frac{c(u)}{u} du. \tag{56} \]

We will prove that system (42)-(45), (55) allows to describe the main aspects of superfluidity, but in more complex form, because of the presence of total time derivative and of equation (50).

## 5 Turbulence

It is well known that turbulence is a phenomenon which has been observed quite frequently in superfluidity. "Hydrodynamic flow in both classical and quantum fluids can be either laminar or turbulent. To describe the latter, vortices in turbulent flow are modelled with stable vortex filaments. While this is an idealization in classical fluids, vortices are real topologically stable quantized objects in superfluids. Thus superfluid turbulence is thought to hold the key to new understanding on turbulence in general." [5], (see also [12]).

This effect is connected with the definition (3) of velocity and with the relation (9) or (50) between normal and superfluid components. In particular, if we consider the model described by the system (2), (3), (4), (23), then we observe that, under the critical temperature \( T_c \), the path of the particles assumes an helicoidal motion, whose circulation, defined around the vortex core, is quantized. If the temperature goes below a second critical temperature \( T_{c2} \), the number of the superfluid particles increases. In such a case the approximation (2), (3), (4), (23) is not correct, because we need to consider the time material derivative. Hence the motion assume a chaotic form, typical of the turbulence flow. Otherwise, it is possible to approximate the system (42-45), (55) considering the material time derivative for \( v_n \) and \( v_s \), then we obtain the system

\[ \frac{\partial f}{\partial t} = \frac{1}{\kappa} \nabla^2 f - f(f^2 - 1 + u + v_s^2) \tag{57} \]
\[ \dot{v}_n = -\nabla p - \mu \nabla \times \nabla \times v_n - \mu \nabla \times f^2 v_s + g \tag{58} \]
\[ \nabla \cdot \dot{v}_n = \dot{v}_n , \quad \nabla \cdot v_n = 0 \tag{59} \]
\[ \dot{v}_s = -\nabla \phi - \mu \nabla \times v_n - \mu f^2 v_s + \nabla u + \mu h \tag{60} \]
\[ \nabla \cdot f^2 v_s = -\tau f^2 \phi \tag{61} \]
\[ c_0 \frac{\partial u}{\partial t} - \dot{G}(f) = \nabla \cdot (f^2 v_s) + k_0 \nabla^2 u + \frac{r}{u} \tag{62} \]

where in this framework we have supposed \( \frac{c(u)}{u} = c_0 \) positive constant. This system is completed by the boundary conditions (30-31).
Now, let us consider the phenomenon of turbulence in viscous fluids. As for a superfluid we suppose the velocity given by (3). While, we suggest for the differential problem a first order phase transition model by the following system

\[ \dot{f} = \frac{1}{\kappa} \nabla^2 f - \frac{1}{2} (N_R^2 F'(f) + \dot{v}_s^2 G'(f)) \]  

\[ \dot{v}_n = -\nabla p - \mu \nabla \times \nabla \times v_n - \mu \nabla \times G(f)v_s + g \]  

\[ \nabla \cdot \dot{v}_s = \dot{v}_n, \quad \nabla \cdot v_n = 0 \]  

\[ \dot{v}_s = -\nabla \phi - \mu \nabla \times v_n - \mu G(f)v_s + \mu h \]  

\[ \nabla \cdot G(f)v_s = -\nu G(f)\phi \]  

where \( g \) and \( h \) are related by (8), \( N_R \) is a constant related with the Reynolds number and \( \nu \) a positive constant. Moreover, because of the first order phase transition described by the system (63-67), the functions \( F(f) \) and \( G(f) \) are now given by the following fourth order polynomials

\[ F(f) = \frac{f^4}{4} - \frac{f^3}{3} \quad G(f) = \frac{f^4}{4} - \frac{2f^3}{3} + \frac{f^2}{2} \]  

Finally, in this framework the internal powers connected with the variables \( f, \) and \( v_s \) are given by

\[ \mathcal{P}^f = \dot{f}^2 + \frac{1}{2\kappa} \left[ (\nabla f)^2 \right] + N_R \dot{F}(f) + \dot{G}(f)v_s^2 \]  

\[ \mathcal{P}^{v_s} = \dot{v}_s^2 + \mu^{-1} (\dot{v}_s + \nabla \phi)^2 + \frac{1}{2} \left[ v_n^2 \right] \]  

\[ \mathcal{P}^{v_n} = \mu (\nabla \times v_n)^2 + G(f)v_s \cdot \nabla \times v_n + \frac{1}{2} \left[ v_n^2 \right] \]  

For this isothermal problem the laws of thermodynamics are given by the following

**Dissipation Principle.** There exists a functional \( \psi(f, v_s) \), called free energy, such that

\[ \psi \leq \dot{f}^2 + \frac{1}{2\kappa} \left[ (\nabla f)^2 \right] + N_R \dot{F}(f) + \mu^{-1} (\dot{v}_s + \nabla \phi)^2 \]

Then, the free energy is function only of \( f \) and is defined by

\[ \psi(f) = \frac{1}{2\kappa} (\nabla f)^2 + N_R F(f) \]
6 Appendix

In many natural phenomena, as phase transitions, chemical reactions, biological (processes) models, we can observe a natural evolution toward change of material structure order. We suggest that this behavior is related with a natural process which we may represent as a balance law on the structure order and it will be given as a function of order parameter $f$. Moreover, as for the other field equations, to this new law we may connect an internal power, which must be considered in the energy balance law. In this framework the structure order is a new form of energy, because during the transformation we observe a variation of the structural energy. Actually, if we consider a phase transition we notice a transformation from a less ordered material structure to a more ordered one or vice versa (see [8], [13]). Moreover, below a critical temperature, the structure order of many materials is greater then above. We meet, analogous behaviors, during biological processes and chemical reactions, always connected with structure order variations.

In order to obtain a balance law on the structure order. Consider a body $B$, for any sub-body $S \subset B$, we denote with $S^i(S)$ the rate at which structure order is absorbed by the material per unit time, given by

$$S^i(S) = \int_S \rho k dV$$

(72)

where $\rho$ is the density and $k$ the internal specific structure order. While, the external order structure $S^e(S)$ assumes the form

$$S^e(S) = \int_{\partial S} p \cdot n ds + \int_S \rho \delta dV$$

(73)

where the vector $p$ denotes the order structure flux and $\delta$ the structure order supply.

Hence, the order structure balance is given for any $S \subset B$ by the equality

$$\int_S \rho k dV = \int_{\partial S} p \cdot n ds + \int_S \rho \delta dV$$

(74)

In local form the integral equality (74) implies the identity

$$\rho k = \nabla \cdot p + \rho \delta$$

(75)

In any model of phase transitions the functions $k$ and $p$ are usually defined by

$$k = f_t + F'(f) + mG'(f)$$

(76)

$$p = \frac{1}{\kappa} \nabla f$$

(77)

where $m$ is a suitable coefficient depending of the field. While $F$ and $G$ are functions that characterize the order and the feature of the transition. For a second order phase transition as superfluidity $F$ and $G$ are defined by

$$F(f) = \frac{f^4}{4} - \frac{f^2}{2}, \quad G(f) = \frac{f^2}{2}$$

(78)
while \( m = u + \lambda p + v^2 \), from which we obtain by (75), (76) and (77) the equation

\[
\rho f_t = \frac{1}{\kappa} \nabla^2 f - \rho F'(f) - \rho (u + \lambda p + v^2 s) G'(f)
\]

(79)
i.e. the equation (2) with \( \rho = 1 \)

Because the equation (75) is joined with an energy, we have to introduce the power balance connected with this equation, namely

\[
\rho k f_t + \mathbf{p} \cdot \nabla f_t = \nabla \cdot (\mathbf{f}_t \mathbf{p}) + \rho f_t \delta
\]

(80)

Finally, we denote by

\[
\mathcal{P}^f = \rho k f_t + \mathbf{p} \cdot \nabla f_t, \quad \mathcal{P}^f = \nabla \cdot (\mathbf{f}_t \mathbf{p}) + \rho f_t \delta
\]

(81)

the internal and the external order structure power density respectively.

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