Spin-mixing-tunneling network model for Anderson transitions in two-dimensional disordered spinful electrons

Jie Lu¹, Bin Xi² and Mei Li³,⁴

¹ College of Physics and Information Engineering, Hebei Advanced Thin Films Laboratory, Hebei Normal University, Shijiazhuang 050024, People’s Republic of China
² College of Physics Science and Technology, Yangzhou University, Yangzhou 225002, People’s Republic of China
³ Physics Department, Shijiazhuang University, Shijiazhuang, Hebei 050035, People’s Republic of China
⁴ Author to whom any correspondence should be addressed.

E-mail: limeijim@163.com

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Abstract

We consider Anderson transitions in two-dimensional (2D) spinful electron gases subject to random scalar potentials with time-reversal-symmetric spin-mixing tunneling (SMT) and spin-preserving tunneling (SPT) at potential saddle points (PSPs). A symplectic quantum network model (QNM), named as SMT-QNM, is constructed in which SMT and SPT have the same status and contribute independent tunneling channels rather than sharing a total-probability-fixed one. 2D continuous Dirac Hamiltonian is then extracted out from this discrete network model as the generator of certain unitary transformation. With the help of high-accuracy numerics based on transfer matrix technique, finite-size analysis on two-terminal conductance and normalized localization length provides a phase diagram drawn in the SMT–SPT plane. As a manifestation of symplectic ensembles, a normal-metal (NM) phase emerges between the quantum spin Hall (QSH) and normal-insulator (NI) phases when SMT appears. We systematically analyze the quantum phases on the boundary and in the interior of the phase space. Particularly, the phase diagram is closely related to that of disordered three-dimensional weak topological insulators by appropriate parameter mapping. At last, if time-reversal symmetry in electron trajectories between PSPs is destroyed, the system falls into unitary class with no more NM phase. A direct SMT-driven transition from QSH to NI phases exists and can be explained by spin-flip backscattering between the degenerate doublets at the same sample edge.

1. Introduction

Anderson transitions (ATs), i.e., transitions between localized and delocalized quantum phases in disordered electronic systems, have attracted intense and continuous attention since its proposal [1] due to its fundamental significance in condensed matter physics [2–5]. In 1970s and 1980s, scaling-theory and field-theory approaches revealed the connections between ATs and conventional second-order phase transitions [2–4]. In 1990s, the symmetry classification of disordered systems was achieved based on its relation to the classical symmetric spaces [6–8]. Later, the completeness of this classification is proved in 2005 [9]. Now we know there are totally ten symmetry classes according to how many discrete symmetries are obeyed by the underlying physical system. When a system only has symmetries translationally invariant in energy, such as the time-reversal symmetry (TRS) and spin–rotation symmetry (SRS), it falls into one of the three traditional Wigner–Dyson classes (unitary, orthogonal and symplectic) [10, 11]. However, if we focus on some particular value of energy, extra discrete symmetries could arise and lead to novel symmetry classes. In condensed matter systems described by tight-binding models on a bipartite lattice with randomness only residing in hopping terms, three chiral classes are identified [6]. The remaining four were discovered in superconducting systems and known as the Bogoliubov-de Gennes classes [7]. In the past decades, ATs in these ten classes have been investigated intensively and considerable progress has been made in various directions, such as their scaling-theory and field-theory
descriptions [2–4], multifractality in critical wave functions [12–18] and level statistics at criticality [19–23], etc. Most theoretical works focus on discrete random equations (such as tight-binding models) and uncorrelated disorders. Their main results can be summarized as: ‘in one- and two-dimensions (2D) all bulk states are localized by disorder regardless of its strength; while in higher dimensions this only holds for sufficiently strong disorder’. Interestingly, when correlation appears in disorder, some localizing states can be delocalized [24, 25], even in one-dimension (1D). Further studies on 1D continuous wave equations show that for disorder with arbitrary strength, localization fades out when the roughness length vanishes; while at intermediate roughness, localization is maximized [26, 27].

Recently, the spin–orbit–induced topological materials, named as topological insulators (TIs), have received intensive attention [28–34]. In TIs, the interplay between topology and symmetry greatly enriches our knowledge of quantum states [34–38]. First, the TRS is crucial for their realization and stabilization. Second, the spin–orbit–coupling (SOC) destroys the SRS, thus makes TIs belong to the Wigner–Dyson symplectic class. In 2D, they are the well-known quantum spin Hall (QSH) ensembles. In disordered QSH systems, ATs can be extracted from traditional metal–insulator transitions to a broader sense which includes transition between topologically trivial and non-trivial phases [5]. In the past decade, great efforts have been devoted into this issue [38–47]. The widely-used framework is to construct a discrete quantum network model (QNM) which consists of two copies of Chalker–Coddington random network model (CC-RNM) [48] describing up and down spins, as well as certain coupling describing spin–flip process. If spin–flip occurs in electron trajectories between potential saddle points (PSPs), it is the well-known SOC. While if it takes place at the PSPs, it is the spin-mixing tunneling (SMT) which is the main focus in this work. Recently, Ryu et al proposed a $Z_2$ QNM ($Z_2$-QNM) which belongs to the Wigner–Dyson symplectic class and provides a systematic description of ATs in 2D disordered spinful electron gases (2D-DSEGs) [38–43]. In $Z_2$-QNM, at PSPs the total tunneling probability are fixed, which means SMT takes away part of the probability from spin–preserving tunneling (SPT) process. From the basic principles of quantum tunneling, SMT provides an additional channel and should not affect the existing SPT. In this work, we treat the SMT as an independent quantum tunneling channel and build a new discrete QNM, namely the ‘SMT–QNM’, to provide an alternative perspective to understand ATs in 2D-DSEGs. Throughout this work, the disorder is selected to be uncorrelated for simplicity.

This paper is organized as follows. In section 2 the SMT-QNM is systematically built up based on probability conservation and TRS at PSPs. Then the 2D continuous Dirac Hamiltonian with ‘valley’ degree of freedom is extracted out. In section 3 numerical algorithms using transfer matrix technique for finite-size analysis on two-terminal conductance and normalized localization length are reviewed. Based on them, in section 4 the quantum phases of SMT-QNM are investigated and a phase diagram is then obtained. We discuss its close connection with that of the disordered 3D weak TIs. In section 5, we consider the case when TRS in electron trajectories between PSPs is destroyed. The system then falls into unitary class. We briefly summarize the quantum phases and phase transitions therein. Finally, the concluding remarks are provided in the last section.

2. The SMT-QNM

2.1. Brief review of CC-RNM

Under a strong magnetic field $B = B\hat{e}_z$, the motion of an electron in a smooth enough 2D random scalar potential $V(r)$ can be decomposed into a rapid cyclotron gyration and a slow drift of the guiding center along an equipotential contour which is generally composed of numerous loops around potential valleys or peaks [48, 49]. The drifting direction of electrons in each loop is uni-directional (chiral): $\mathbf{v}(r) = \nabla V(r) \times \mathbf{B} / (\mathbf{eB}^2)$. At PSPs, electrons reflecting along equipotential lines and their mutual tunneling are the essential physical ingredients for constructing a network model describing quantum criticality in disordered 2D systems. For modelization, the PSPs are arranged to form a 2D square lattice with the interconnected links representing electron flows along equipotential lines. The potential peaks and valleys distribute alternatively in the square plaquettes enclosed by the links. This endues definite propagating direction of electron flows on the links and then divides the PSPs into two subgroups: the S- and S’-types (see figures 1 (a) and (b)). At each PSP, two incoming and two outgoing electron flows intersect hence lead to a $2 \times 2$ scattering matrix. Quantum tunneling only occurs at PSPs and in the simplest case can be assumed identical. At last, disorder is introduced by random phases along links. This is the basic framework of CC-RNM. In all illustration figures in this paper, we adopt the following sketch rules: if $r > t$, the reflecting (tunneling) routes are depicted by solid (dash) curves and vice versa.

For a S-type PSP at position $R$, its scattering matrix $S_{\text{CC}}$ satisfies,

$$
\begin{bmatrix}
Z^0_2 \\
Z^1_2
\end{bmatrix}
= s_{\text{CC}}
\begin{bmatrix}
Z^0_1 \\
Z^1_1
\end{bmatrix},
$$

where $s_{\text{CC}} = \psi_{\text{CC}}^\dagger S_{\text{CC}} \psi_{\text{CC}}$. 

(1)
where $Z_{ji}^{\text{out}}$ is the outgoing (incoming) electron flow amplitude at link $j$, $\Psi_{R}^{\text{in}} \equiv \text{diag}(e^{i\phi_{R}}, e^{i\phi_{L}})$ is a diagonal matrix with $\phi_{R/L}$ being the dynamical phase an electron acquires when propagating on link $j$ between the observation point and the PSP at $R$. The kernel matrix $S_{CC}$ has the general form,

$$S_{CC} = \begin{bmatrix} r & t \\ \eta_{+} t & \eta_{-} r \end{bmatrix}$$

where $r = \sqrt{p}$ ($t = \sqrt{1-p}$) measuring the reflecting (tunneling) amplitude at a PSP, with $p$ being related to the Fermi level of the system \cite{49}, and $\eta_{+/-}$ are undetermined coefficients. In steady states, probability conservation at any PSP requires $\eta_{+} = e^{i\phi_{R}}$ and $|\phi_{R} - \phi_{L}| = (2n + 1)\pi$. Clearly,

$$(i\eta_{-}) S_{CC}(-i\eta_{+}) = \eta_{-}^{-1} S_{CC} = S_{CC}$$

which means TRS is broken thus the CC-RNM belongs to the unitary class. Throughout this work, $\eta_{-} = -\eta_{+} = -1$ which is also the choice in most literatures.

### 2.2. Scatter matrices of SMT-QNM

To describe ATs in 2D-DSEGs, the CC-RNM should be generalized to include spins, providing the following hypotheses. First, the potential profile is identical for any spin orientation. Second, the absence of external magnetic fields makes TRS possible which turns the original uni-directed electron flow on each link to a Kramers doublet. Opposite spin components then 'feel' opposite effective magnetic fields, forming two copies of CC-RNM with opposite chirality. Third, appropriate coupling should be introduced between the two copies of CC-RNM to describe spin-flip process. Generally, spin-flip can occur anywhere. In real modelization, two strategies are most common: (a) it only occurs on the links between PSPs; (b) it only occurs at the PSPs. The first strategy reflects the SOC while the second one is the SMT.

The $Z_{2}$-QNM proposed by Ryu \textit{et al} \cite{38–42} follows the second strategy, in which SMT and SPT are considered as two competing processes sharing a fixed probability $'a'^{2}$. While in this work, the SPT channel remains unperturbed. Meantime we treat SMT as an independent quantum tunneling channel and construct the SMT-QNM to understand ATs in 2D-DSEGs. For S-type PSPs (see figure 1(c)), the scattering matrix at position $R$ reads,

$$S_{R}^{\text{SMT}} \equiv \begin{bmatrix} Z_{11}^{\text{SMT}} & Z_{12}^{\text{SMT}} \\ Z_{21}^{\text{SMT}} & Z_{22}^{\text{SMT}} \end{bmatrix}$$

$$S_{R}^{\text{S}} = \begin{bmatrix} \Psi_{R}^{1234} S_{R}^{\text{SMT}} \Psi_{R}^{1234}^{*} \\ \Psi_{R}^{2341} S_{R}^{\text{SMT}} \Psi_{R}^{2341}^{*} \end{bmatrix}$$

Figure 1. Schematics of CC-RNM and SMT-QNM. (a) and (b) show the S- and $S'$-type PSPs in CC-RNM. At each PSP, two incoming and two outgoing electron flows intersect with tunneling amplitude $\sqrt{1-p}$. Blue (green) circles with '+' '-' inside denote the potential peaks (valleys). (c) and (d) show the counterparts of (a) and (b) in SMT-QNM where the spin degree of freedom is included. The original chiral electron flow on each link is generalized to a Kramers doublet. Throughout this paper, black (red) means spin-up (down). In addition, at each PSP a SMT with amplitude $'r' \sin \theta'$ is introduced.
where $Z_{j0}^{(i)}$ is the outgoing (incoming) electron flow amplitude at link $j$ with spin $\sigma ( \uparrow \text{ or } \downarrow )$, 

\[ \Psi_R^{\text{link}} = \text{diag}(e^{i\varphi_i}, e^{i\varphi_2}, e^{i\varphi_4}, e^{i\varphi_6}) \] with $\varphi_i$ representing the phase an electron acquires when propagating on link $j$ between the observation point and the PSP at $R$. We have neglected the spin index since the Kramers pair of electron flows have the same accumulated phase on the same link. To mimic the randomness in PSP distribution for uncorrelated disorders, these phases are distributed uniformly and independently in the region $[0, 2\pi]$. If we focus on the very point where a PSP locates, $\Psi_R^{\text{link}}$ then becomes unity. The kernel matrix $S_{\text{SMT}}$ describes the reflecting and tunneling at a general S-type PSP and has the following structure

\[ S_{\text{SMT}} = \begin{bmatrix} n & 0 \\ 0 & n^* \end{bmatrix} \begin{bmatrix} Q \\ Q^t \end{bmatrix}, \]

(5)

where ‘$\dagger$’ means matrix complex conjugate. For this scattering matrix, several points need to be clarified. First, it is Hermitian due to TRS. Second, $|n| \leq r$ since SMT is an additional tunneling channel hence takes probability away from reflecting rather than SPT process. For simplicity, $r_1$ can be defined as $r_1 = r \cos \theta$ (thus is real), with $\theta \in [0, \pi/2]$ describing the strength of SMT. Third, probability conservation in steady states at any PSP requires the scattering matrix to be unitary

\[ S_{\text{SMT}}^\dagger S_{\text{SMT}} = S_{\text{SMT}} S_{\text{SMT}}^\dagger = \sigma_0 \otimes \sigma_0, \]

(6)

which gives

\[ QQ^t = (r^2 + r^2 \sin^2 \theta) \sigma_0, \]

(7)

where $\sigma_0$ is the $2 \times 2$ unit matrix. Fourth, TRS requires

\[ \begin{bmatrix} i\sigma_y & 0 \\ 0 & i\sigma_y \end{bmatrix} S_{\text{SMT}} \begin{bmatrix} -i\sigma_y & 0 \\ 0 & -i\sigma_y \end{bmatrix} = S_{\text{SMT}}^\dagger, \]

(8)

where $\sigma_{x,y,z}$ are the Pauli matrices. This gives,

\[ Q = \sigma_y Q^t \sigma_y. \]

(9)

By writing $Q$ as

\[ Q = a_0 \sigma_0 + \sum_k a_k \sigma_k, \quad a_0, a_k \in \mathbb{C}, \]

(10)

equation (7) turns to

\[ \sum_k |a_k|^2 = t^2 + r^2 \sin^2 \theta, \quad \sigma = 0, x, y, z, \]

\[ \text{Re}(a_0^* a_k) = \text{Im}(\epsilon_{klm} a_l^* a_m), \quad k, l, m = x, y, z, \]

(11)

in which $\epsilon_{klm}$ is the 3D Levi-Civita symbol. In addition equation (9) gives

\[ a_0 = a_0^*, \quad a_k = -a_k^*, \quad k = x, y, z. \]

(12)

Summarizing these two conditions, a reasonable solution to $a_k$ is

\[ a_0 = t \cos \phi_1, \quad a_x = \text{ir} \sin \theta \sin \phi_2, \]

\[ a_y = \text{ir} \sin \phi_1, \quad a_z = \text{ir} \cos \phi_2, \]

(13)

leading to a physical realization of $Q$ as

\[ Q = \begin{bmatrix} te^{i\phi_1} & re^{i\phi_2} \sin \theta \\ -re^{-i\phi_2} \sin \theta & te^{-i\phi_1} \end{bmatrix}. \]

(14)

Obviously $\phi_1$ and $\phi_2$ are the phase shifts associated with SPT and SMT processes, respectively. At last, by rotating S-type PSPs $90^\circ$ clockwise, we get S'-type PSPs and their scattering matrix can be easily obtained from equation (4).

Therefore, in our SMT-QNM at any PSP (S- and S'-type), for an incoming electron flow with some certain spin orientation and probability 1, it tunnels into an outgoing flow with the same spin orientation via SPT process with probability $r^2$ and also into an outgoing flow with opposite spin orientation via SMT process with probability $r^2 \sin^2 \theta$, leaving a probability $r^2 \cos^2 \theta$ residing in the original equipotential line.

In addition, our SMT-QNM is not equivalent to the existing $\mathbb{Z}_2$-QNM. To see this, we focus on S-type PSPs. First, by carefully checking the basis in figure 1(c) of our work and figure 1(b) in [38] (as well as figure 5 in [42]), it turns out that the bases as well as their order are the same. Second, if our scattering matrix $S_{\text{SMT}}$ could be converted to the counterpart in $\mathbb{Z}_2$-QNM (see equation (4) in [38] and equation (2.3) in [42]) by some basis...
change, then they were related by a unitary transformation, thus must have the same eigenvalues. However, direct calculations yield that the eigenvalue spectrum of $S_{\text{SMT}}$ is \{-1, -1, +1, +1\}, while that in $\mathbb{Z}_2$-QNM is $\{-1, +1\}$. This shows the inequivalence between these two QNMs.

2.3. 2D Dirac Hamiltonian from SMT-QNM

The mapping from CC-RNM to 2D Dirac Hamiltonian was accomplished in 1996 [50], and the connection between the $\mathbb{Z}_2$-QNM and 2D Dirac Hamiltonian was established in 2010 [38]. The main strategy of both works is to view the unitary (due to probability conservation) scattering matrices as a unitary time evolution operator whose infinitesimal generator is the required Hamiltonian, as we all know that a unitary matrix is the exponential of a Hermitian one. In this subsection, we follow this strategy and succeed in extracting the 2D Dirac Hamiltonian from our SMT-QNM and recognizing the roles of phase shifts in SMT and SPT at PSPs. Also, this part of work lays the foundation for understanding the close connection between the phase diagram of our SMT-QNM and that of disordered 3D weak TIs (see section 4.4).

2.3.1. Preparations

We arrange the S-type and S’-type PSPs alternatively in a 2D Cartesian plane to form a bipartite square lattice, as shown in figure 2. Then following the sketch rules in figures 1(c) and (d), a series of closed square plaquettes are obtained, with each edge bearing two opposite-directed links. For $r > t$, the centers of these closed plaquettes are the potential valleys, while the potential peaks reside in the blanks outside. For $r < t$, the situation is just reversed. Quantum tunnelings (SPT and SMT) occur at the plaquette corners, which are the PSPs. We take the $r > t$ case as the framework for our discussion, which does not affect the generality of our results. If one of these plaquettes is assigned with coordinate $(0, 0)$, then the position of anyone in this set is

$$R_{x,y} = xe_x + ye_y, \quad x, y \in \mathbb{Z}, \quad \mod(x + y, 2) = 0. \quad (15)$$

They form a square lattice and is our main concern. The eight directed links on the edges of a plaquette are labeled by $(n\sigma)$ with $n = 1, 2, 3, 4$ and $\sigma = \uparrow$ or $\downarrow$.

For the plaquette with coordinate $(x, y)$, the scattering event at the S-type PSP on its upper-right corner (urc) is as follows

$$\begin{pmatrix} Z_{21}(x, y) \\ Z_{11}(x, y) \\ Z_{41}(x + 1, y + 1) \\ Z_{31}(x + 1, y + 1) \end{pmatrix} = S_{\text{SMT}}^{\text{urc}} \begin{pmatrix} Z_{11}(x, y) \\ Z_{21}(x, y) \\ Z_{31}(x + 1, y + 1) \\ Z_{41}(x + 1, y + 1) \end{pmatrix}, \quad (16)$$

Figure 2. A bipartite square lattice composed of S-type and S’-type PSPs. Electron flows illustrated in figures 1(c) and (d) ($r > t$ case) generate a series of corner-shared closed square plaquettes around potential valleys with integral coordinates. The black (red) lines represent the links for electron flows with up (down) spin. The scattering-basis order on the edge of each plaquette is specified with Arabic numerals. Blue (green) circles indicate potential peaks (valleys).
with

\[ S^{\text{lrc}}_{x,y} \equiv U_{x,y}S_{\text{SMT}}V_{x,y}, \]
\[ U_{x,y} = \text{diag}[e^{i\psi(x,y)}, e^{i\psi(x,y)}], \]
\[ V_{x,y} = \text{diag}[e^{i\psi(x,y)}, e^{i\psi(x,y)}], \]

in which TRS has been invoked in writing \( U_{x,y} \) and \( V_{x,y} \). While that of the \( S' \)-type PSP on the lower-right corner (lrc) reads

\[
\begin{bmatrix}
Z_{21}(x, y) \\
Z_{21}(x, y) \\
Z_{41}(x + 1, y - 1) \\
Z_{41}(x + 1, y - 1)
\end{bmatrix} = S^{\text{lrc}}_{x,y}
\begin{bmatrix}
Z_{21}(x, y) \\
Z_{21}(x, y) \\
Z_{41}(x + 1, y - 1) \\
Z_{41}(x + 1, y - 1)
\end{bmatrix},
\]

with

\[ S^{\text{lrc}}_{x,y} \equiv U'_{x,y}S_{\text{SMT}}V'_{x,y}, \]
\[ U'_{x,y} = \text{diag}[e^{i\psi(x,y)}, e^{i\psi(x,y)}], \]
\[ V'_{x,y} = \text{diag}[e^{i\psi(x,y)}, e^{i\psi(x,y)}], \]

Next the displacement operators \( \tau^{\pm}_{z(x,y)} \) are introduced as

\[
\tau_{x,y}^{\pm} f_{x,y}(x, y) = f_{x,y}(x \pm 1, y),
\]

where \( f_{x,y}(x, y) \) is an arbitrary function defined at \( R_{x,y} \). By definition, they are commutative and

\[
[\tau_{x,y}^{\pm}]^{-1} = \tau_{x,y}^{\mp}\]

By rearranging the amplitudes in the order of ‘2, 4, 1, 3’, we rewrite equation (16) into the form

\[
\begin{bmatrix}
Z_{21}(x, y) \\
Z_{41}(x, y) \\
Z_{11}(x, y) \\
Z_{31}(x, y)
\end{bmatrix} = \Omega_S
\begin{bmatrix}
Z_{21}(x, y) \\
Z_{41}(x, y) \\
Z_{11}(x, y) \\
Z_{31}(x, y)
\end{bmatrix}, \quad \Omega_S \equiv O_{x,y} \cdot M_S \cdot O_{x,y},
\]

with

\[ O_{x,y} = \text{diag}[e^{i\psi(x,y)}, e^{i\psi(x,y)}, e^{i\psi(x,y)}, e^{i\psi(x,y)}] \]

and

\[
M_S = \begin{bmatrix}
\Phi_2 M_2 \Phi_2 & \Phi_1 M_{0d} \Phi_1 \\
\Phi_1 M_{0d} \Phi_1 & -\Phi_2^* M_2 \Phi_2^*
\end{bmatrix},
\]

\[ \Phi_1 = \text{diag}(e^{-i\frac{x}{2}}, e^{i\frac{y}{2}}), \]

\[ \Phi_2 = e^{i\frac{x}{2} \sigma_0}, \]

\[
M_d = \begin{bmatrix}
0 & r \sin \theta \tau^x_{x,y} \\
-r \sin \theta \tau^x_{x,y} & 0
\end{bmatrix},
\]

\[
M_{0d} = \begin{bmatrix}
0 & r \cos \theta \tau^y_{x,y} \\
-r \cos \theta \tau^y_{x,y} & 0
\end{bmatrix},
\]

in which ‘d (od)’ means diagonal (off-diagonal). Similarly, equation (18) is changed to

\[
\begin{bmatrix}
Z_{21}(x, y) \\
Z_{41}(x, y) \\
Z_{11}(x, y) \\
Z_{31}(x, y)
\end{bmatrix} = \Omega'_S
\begin{bmatrix}
Z_{21}(x, y) \\
Z_{41}(x, y) \\
Z_{11}(x, y) \\
Z_{31}(x, y)
\end{bmatrix}, \quad \Omega'_S = O_{x,y} \cdot M'_S \cdot O_{x,y},
\]
where

\[
M_{S'} = \begin{bmatrix}
\Phi_{1}^{2} M_{o}^{2} \Phi_{1}^{2} & \Phi_{1} M_{o}^{2} \Phi_{1} \\
\Phi_{1} \sigma_{x} M_{o}^{2} \sigma_{x} \Phi_{1} & -\Phi_{2} \sigma_{x} M_{o}^{2} \sigma_{x} \Phi_{2}
\end{bmatrix},
\]

\[
M_{d} = \begin{bmatrix}
0 & -r \sin \theta \tau_{x}^{+} \\
r \sin \theta \tau_{x}^{-} & 0
\end{bmatrix},
\]

\[
M_{d}' = \begin{bmatrix}
\tau_{x}^{+} \tau_{y} & r \cos \theta \\
-r \cos \theta & \tau_{x}^{-} \tau_{y}
\end{bmatrix},
\]

(26)

By defining the total amplitude vector \(Z_{x,y}\) composed of all eight links along the edges of plaquette at \(R_{x,y}\) as

\[
Z_{x,y} \equiv (Z_{2}, Z_{4}, Z_{1}, Z_{3}, Z_{2}, Z_{4}, Z_{1}, Z_{3})^{T},
\]

(27)

with the superscript ‘T’ indicating matrix transpose and introducing \(\mu (\in \mathbb{Z})\), the elementary imaginary discrete-time evolution of \(Z_{x,y}\) is,

\[
(Z_{x,y})_{t+1} = \begin{bmatrix}
0 & \Omega_{S'} \\
\Omega_{S'} & 0
\end{bmatrix} (Z_{x,y})_{t}.
\]

(28)

To acquire decoupled equations, the ‘two-step’ time evolution,

\[
(Z_{x,y})_{t+2} = \begin{bmatrix}
\Omega_{S} \Omega_{S'} & 0 \\
0 & \Omega_{S'} \Omega_{S}
\end{bmatrix} (Z_{x,y})_{t},
\]

(29)

is more convenient since it is diagonal. We will focus on \(\Omega_{S} \Omega_{S'} \equiv \Omega_{SS'}\) in the rest of this work. Also we make the transformation

\[
\psi_{4} \to \psi_{4} + \frac{\pi}{2}
\]

(30)
to raise the reference point of the total phase flux of each plaquette by \(\pi\) (see equations (22) and (25)), which is crucial for the extraction of 2D Dirac Hamiltonian. It can be easily checked that \(\Omega_{SS'}\) is unitary, thus provide a Hamiltonian as its infinitesimal generator,

\[
\mathcal{H}_{SMT} = i \ln \Omega_{SS'} \approx i(\Omega_{SS'} - 1).
\]

(31)

We then demonstrate that in the close vicinity of the CC-RNM critical point

\[
(R_{c}, \theta_{c}) = \left(\frac{1}{2}, 0\right),
\]

(32)

how \(\mathcal{H}_{SMT}\) is mapped to 2D Dirac Hamiltonian by expanding \(\Omega_{SS'}\) to the leading-order powers of

\[
\theta, \quad m \equiv 1 - \frac{p}{P_{c}}, \quad \partial_{x(y)} \equiv \ln \tau_{x(y)}^{\pm}, \quad \psi_{1,2,3,4}, \quad \text{and} \quad \phi_{1,2}.
\]

(33)

2.3.2. 2D Dirac Hamiltonian around \(\theta = 0\)

At \(\theta = 0\), \(M_{d} = M_{d}' = 0\). Equations (22) and (25) thus provide

\[
Q_{S}^{(0)} = \begin{bmatrix}
0 & A^{(0)} \\
B^{(0)} & 0
\end{bmatrix}, \quad Q_{S}^{(0)} = \begin{bmatrix}
0 & C^{(0)} \\
D^{(0)} & 0
\end{bmatrix},
\]

(34)

with

\[
A^{(0)} = \begin{bmatrix}
e^{i\phi_{2} + \phi_{1}} \tau_{x}^{+} & e^{i\phi_{2} + \phi_{1} + \phi_{3}} \tau_{x}^{-} \\
i e^{i\phi_{2} + \phi_{1} - \phi_{3}} \tau_{y}^{-} & -i e^{i\phi_{2} + \phi_{1}} \tau_{y}^{-}
\end{bmatrix},
\]

\[
B^{(0)} = \begin{bmatrix}
e^{i\phi_{2} + \phi_{1}} \tau_{x}^{+} & i e^{i\phi_{2} + \phi_{1} - \phi_{3}} \tau_{y}^{-} \\
e^{i\phi_{2} + \phi_{1} + \phi_{3}} \tau_{x}^{-} & -i e^{i\phi_{2} + \phi_{1}} \tau_{y}^{-}
\end{bmatrix},
\]

\[
C^{(0)} = \begin{bmatrix}
e^{i\phi_{2} + \phi_{1} - \phi_{3}} \tau_{x}^{+} & e^{i\phi_{2} + \phi_{1}} \tau_{x}^{-} \\
-i e^{i\phi_{2} + \phi_{1}} \tau_{y}^{-} & i e^{i\phi_{2} + \phi_{1} + \phi_{3}} \tau_{y}^{-}
\end{bmatrix},
\]

\[
D^{(0)} = \begin{bmatrix}
e^{i\phi_{2} + \phi_{1} + \phi_{3}} \tau_{x}^{+} & -i e^{i\phi_{2} + \phi_{1}} \tau_{x}^{-} \\
e^{i\phi_{2} + \phi_{1}} \tau_{y}^{-} & i e^{i\phi_{2} + \phi_{1} + \phi_{3}} \tau_{y}^{-}
\end{bmatrix},
\]

(35)
Then
\[
\mathcal{H}^{(0)}_{\text{SMT}} = i \begin{bmatrix} A^{(0)} D^{(0)} - \sigma_0 & 0 \\ 0 & B^{(0)} C^{(0)} - \sigma_0 \end{bmatrix} \equiv \begin{bmatrix} J_+ & 0 \\ 0 & J_- \end{bmatrix}
\] (36)

Under the following assumptions:

(a) displacement operators act on smooth enough functions thus
\[
\tau^{\pm}_{x(y)} \rightarrow 1 \pm \partial_{x(y)},
\] (37)

(b) the phases \(\psi_{n-1,2,3,4}\) and \(\phi_1\) are small enough hence
\[
e^{\pm i\psi_n} \rightarrow 1 \pm i\psi_n, \quad e^{\pm i\phi_1} \rightarrow 1 \pm i\phi_1,
\] (38)

(c) in the close vicinity of CC-RNM critical point one has
\[
J_+ = A_0 \sigma_0 + (-i \partial_x + A_x) \sigma_x - (-i \partial_y + A_y) \sigma_y - m \sigma_z,
\]
\[
J_- = A_0 \sigma_0 - (-i \partial_x - A_x) \sigma_x - (-i \partial_y - A_y) \sigma_y + m \sigma_z,
\] (40)

acting as a scalar/vector potential, respectively.

Then the system is driven away slightly from the critical point (32) along the \(\theta\)-line. Hence \(\psi_n = m = 0\) and \(\tau^{\pm}_{x(y)} = 1\), and to the leading-order of \(\theta\) one has
\[
\Omega_{SS'} = \Omega^{(0)}_{SS'} + \theta (\Omega^{(1)}_{S} \Omega^{(0)}_{S'} + \Omega^{(0)}_{S} \Omega^{(1)}_{S'} + \cdots),
\] (42)

with
\[
\Omega^{(1)}_{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} -e^{i\phi_2} & 0 \\ 0 & -e^{-i\phi_2} \end{bmatrix} \otimes \sigma_y,
\]
\[
\Omega^{(1)}_{S'} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\phi_2} & 0 \\ 0 & e^{i\phi_2} \end{bmatrix} \otimes \sigma_y.
\] (43)

Correspondingly, the SMT Hamiltonian turns to
\[
\mathcal{H}_{\text{SMT}} = \begin{bmatrix} J_+ & J_0 \\ J_0 & J_- \end{bmatrix}, \quad J_0 \equiv \theta e^{i\phi_0} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}
\] (44)

After performing a unitary transformation
\[
\mathcal{U} = \begin{bmatrix} e^{i\phi_{\alpha}} \cdot e^{i\phi_y} \cdot e^{-i\phi_1} & 0 \\ 0 & e^{-i\phi_{\alpha}} \cdot e^{-i\phi_y} \cdot e^{i\phi_1} \end{bmatrix} \sigma_0
\] (45)

we get the final Hamiltonian
\[
\mathcal{H}_f = \mathcal{U}^\dagger \mathcal{H}_{\text{SMT}} \mathcal{U} = \begin{bmatrix} \mathcal{H}_0^{\alpha} & \sqrt{2} \theta \sigma_0 \\ \sqrt{2} \theta \sigma_0 & \mathcal{H}_0^{\alpha} \end{bmatrix}
\] (46)

with
\[
\mathcal{H}_0^{\alpha} = A_0 \sigma_0 + (-i \partial_x \pm A_x) \sigma_x - (-i \partial_y \pm A_y) \sigma_y \pm m \sigma_z.
\] (47)

Obviously \(\mathcal{H}_f\) describes a pair of Dirac fermions (with mass \(\pm m\)) subject to the same random scalar potential \(A_0\) and respective random vector potential \(\pm (A_x, A_y)\), meantime bearing a mutual coupling \(\sqrt{2} \theta \sigma_0\). By introducing a ‘valley’ space distinguishing these two Dirac fermions (different locations of Dirac cones in Brillouin zone), the final Hamiltonian can be rewritten as
$\mathcal{H}_f = s_0 \otimes (-i\partial_x \sigma_x - i\partial_y \sigma_y) + s_0 \otimes A_0 \sigma_0$

$+ s_x \otimes (A_x \sigma_x + A_y \sigma_y + m\sigma_z) + s_x \otimes \sqrt{2} \theta \sigma_0$,  \hspace{1cm} (48)$

where $s_0$ and $s_{x,y,z}$ are the identity and Pauli matrices in valley space. Therefore our SMT-QNM belongs to the symplectic class and should be an effective model for ATs in QSH ensembles. Also, the above analytics shows that the phase shifts in SPT and SMT processes at PSPs have different roles during the extraction of 2D Dirac Hamiltonian. The former ($f_1$) enters the vector potentials thus could have impacts on geometric phase accumulated along the plaquette edges. While the latter ($f_2$) resides in the coupling matrix $J_\theta$ between $J_{\pm}$ and then manifests itself in the unitary transformation that changes $\mathcal{H}_{\text{SMT}}$ to $\mathcal{H}_f$, hence acts as a gauge field describing the spin-flip interaction.

3. Algorithms for finite-size analysis

3.1. Two-terminal conductance $G_{2T}$

For numerical convenience, after rotating figure 2 by $45^\circ$ clockwise, we obtain a 2D PSP lattice composed of $L$ principal layers (PLs), as shown in figure 3. Each PL consists of $W$ $S$-type and $W$ $S'$-type PSPs. $4W$ electron flows ($2W$ incoming and $2W$ outgoing) distribute regularly on each side of the network and are related by the total transfer matrix $T_L$.

\begin{equation}
\hat{Z}_{1\text{s}}^{2w} = \begin{bmatrix}
Z_{1\text{s}}^{2w} \\
Z_{2\text{s}}^{2w}
\end{bmatrix}, \quad T_L = U_1 \cdot T_0 \cdot U_2,
\end{equation}

in which

\begin{align*}
U_1 &= \text{diag}(e^{-i\phi_1}, e^{-i\phi_1}, e^{-i\phi_1}, e^{-i\phi_1}), \\
U_2 &= \text{diag}(e^{-i\phi_2}, e^{-i\phi_2}, e^{-i\phi_2}, e^{-i\phi_2}),
\end{align*}

and

\begin{align*}
T_0 &= (1 - r^2 \cos^2 \theta)^{-1} [\sigma_0 \otimes (T_0)^d + \sigma_z \otimes (T_0)^{ad}], \\
(T_0)^d &= \begin{bmatrix}
t & r^2 \sin \theta \cos \theta \\
-r^2 \sin \theta \cos \theta & t
\end{bmatrix}, \\
(T_0)^{ad} &= \begin{bmatrix}
-r t \cos \theta & -r \sin \theta \\
-r \sin \theta & -rt \cos \theta
\end{bmatrix}.
\end{align*}

Figure 3. An example of the layout of a SMT-QNM network with boundary nodes being $S'$-type PSPs. The light (dark) gray circles indicate $S$-type ($S'$-type) PSPs. This 2D PSP lattice is composed of $L$ PLs. Each PL (orange vertical strip) consists of $W$ $S$-type and $W$ $S'$-type PSPs. $4W$ electron flows ($2W$ incoming and $2W$ outgoing) distribute regularly on each side of the network and are related by the total transfer matrix $T_L$. 

\[ T_L = U_1 \cdot T_0 \cdot U_2 \]
While at a S’-type PSP, the counterpart is

\[
\begin{bmatrix}
Z_{41}^i \\
Z_{41}^o \\
Z_{31}^i \\
Z_{31}^o
\end{bmatrix}
= T_{0}^i \begin{bmatrix}
Z_{11}^i \\
Z_{11}^o \\
Z_{21}^i \\
Z_{21}^o
\end{bmatrix}, \quad T_{0}^r = U_{i}^r \cdot T_{0}^r \cdot (U_{i})^r,
\]

(52)

where

\[
U_{i}^r = \text{diag}(e^{i\phi_{1}}, e^{-i\phi_{2}}, e^{-i\phi_{3}}, e^{i\phi_{4}})
\]

and

\[
\begin{aligned}
T_{0}^d &= r^{-1}[\sigma_z \otimes (T_{0}^d)^{\text{d}} + i\sigma_y \otimes (T_{0}^d)^{\text{d}}], \\
(T_{0}^d)^{\text{d}} &= \begin{bmatrix}
\cos \theta & -t \sin \theta \\
t \sin \theta & \cos \theta
\end{bmatrix}, \\
(T_{0}^d)^{\text{d}} &= \begin{bmatrix}
-t \cos \theta & \sin \theta \\
-\sin \theta & -t \cos \theta
\end{bmatrix}
\end{aligned}
\]

(54)

Then the transfer matrix for the kth PL is

\[
T^{(k)} = V_{4}^{(k)} V_{3} V_{2}^{(k)} V_{1},
\]

(55)

where the boundary nodes are selected to be S’-type PSPs as an example (see figure 3). \(V_{1}\) is the transfer matrix of the sub-layer composed merely by S-type PSPs with the following form

\[
V_{1} = \text{diag}(T_{0}, \ldots, T_{0}),
\]

(56)

\(V_{3}\) is the transfer matrix of the S’-type sub-layer

\[
V_{3} = \begin{bmatrix}
B_{1} & 0 & \cdots & 0 & B_{2} \\
0 & T_{0} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & T_{0} \\
B_{3} & 0 & \cdots & 0 \\
\end{bmatrix},
\]

(57)

where \(B_{1,2,3,4}\) are 2 \(\times\) 2 matrices and determined by the choice of boundary condition in transverse direction. When we focus on edge modes, the reflecting boundary condition (RBC) is imposed. The Kramers pair is totally reflected without any spin-flip at boundary nodes, thus

\[
\begin{bmatrix}
B_{1} & B_{2} \\
B_{2} & B_{1}
\end{bmatrix} = \sigma_0 \otimes \sigma_0.
\]

(58)

If bulk behaviors are the main concern, the periodic boundary condition (PBC) is adopted, which means

\[
\begin{bmatrix}
B_{1} & B_{2} \\
B_{2} & B_{1}
\end{bmatrix} = T_{0}^r
\]

(59)

At last, \(V_{2}^{(2)}\) and \(V_{4}^{(2)}\) are 4W \(\times\) 4W diagonal matrices,

\[
[V_{\alpha}^{(2)}]_{\text{Im}} = \delta_{\alpha w} e^{i\phi_{\alpha w}}, \quad (\alpha = 2, 4),
\]

(60)

desccribing the left-to-right intra- and inter-PL random phases in 4W links connecting S-type and S’-type PSPs in adjacent sub-layers. Note that TRS ensures in any link, spin-up electron flowing in a certain direction acquires the same dynamical phase with that of a spin-down electron in the opposite direction. Thus one has the ’phase pairing rule’

\[
\psi_{\alpha w,1}^{(k)} + \psi_{\alpha w-1}^{(k)} = 0, \quad (w = 1, \ldots, 2W).
\]

(61)

In practice, for certain \(\alpha\) the 2W phases \(\phi_{\alpha w}\) are independently and uniformly distributed in \([0, 2\pi]\).

Multiplying \(T^{(k)}\) sequentially, the total transfer matrix \(T_{W}\), which relates the electron flows on the left of the network \((Z_{11}^L, Z_{11}^L, \ldots, Z_{2W}^{L}, Z_{2W}^{L})^T\) and those on the right \((Z_{11}^R, Z_{11}^R, \ldots, Z_{2W}^{R}, Z_{2W}^{R})^T\), is then obtained.
\[
\begin{bmatrix}
Z_{1}^{\phi_{0_i}} & Z_{1}^{\phi_{1i}} \\
Z_{2}^{\phi_{0_i}} & Z_{2}^{\phi_{1i}} \\
\vdots & \vdots \\
Z_{2W}^{\phi_{0_i}} & Z_{2W}^{\phi_{1i}}
\end{bmatrix}\] = \[T_{W}^{1} = T_{W}^{(L)} \cdots T_{W}^{(1)}.
\] (62)

By introducing a unitary matrix \(O\) with
\[
O_{mn} = \begin{cases} 
1, & (m, n) = (4w - 3, w) \text{ or } (4w - 2, 3W + w) \text{ or } (4w - 1, 2W + w) \text{ or } (4w, W + w) \text{, } w = 1, \ldots, W \\
0, & \text{otherwise}
\end{cases}
\] (63)

the electron flows on each side of the system are reordered into four subgroups marked by \((\alpha = i/o, \sigma = \uparrow \downarrow)\), i.e.,
\[
\begin{bmatrix}
Z_{o_i}^{\phi} \\
Z_{l_i}^{\phi} \\
Z_{o_i}^{\phi} \\
Z_{l_i}^{\phi}
\end{bmatrix} = \tilde{T}, \quad \tilde{T} = O'T_{W}^{l}O.
\] (64)

On the other hand, the entire network can be viewed as a whole hence its transport features are provided by a \(4W \times 4W\) scattering matrix \(S_{i}\),
\[
\begin{bmatrix}
Z_{o_i}^{\phi} \\
Z_{l_i}^{\phi} \\
Z_{o_i}^{\phi} \\
Z_{l_i}^{\phi}
\end{bmatrix} = S_{i}, \quad S_{i} = \begin{bmatrix} R & T' \\ T & R' \end{bmatrix}
\] (65)

where \(T\) and \(T'\) (\(R\) and \(R'\)) are \(2W \times 2W\) transmission (reflection) matrices. The Landauer formula tells us that the total two-terminal charge conductance, \(G_{2T}\), is
\[
G_{2T} = \frac{e^{2}}{h} \text{Tr}(T'^{t}T').
\] (66)

Finally, by comparing equations (64) and (65), one has
\[
\tilde{T} = \begin{bmatrix}
\tilde{T}_{11} & \tilde{T}_{12} \\
\tilde{T}_{21} & \tilde{T}_{22}
\end{bmatrix} = \begin{bmatrix} T - R'T'^{-1}R & R'T'^{-1} \\ -T'^{-1}R & T'^{-1} \end{bmatrix}
\] (67)

which leads to \(T' = (\tilde{T}_{22})^{-1}\).

This provides the main algorithm of calculating the two-terminal conductance. Before ending this subsection, a few points need to be addressed. First, the diagonal phases in \(U_{1,2}\) and \(U'_{1,2}\) are already grouped to pairs with opposite signs, thus can be absorbed into random phase matrices \(V_{1,2}\). This feature has two consequences: (i) we directly use \(T_{0} (T_{0}')\) rather than \(T_{k} (T_{k}')\) to build \(V_{1} (V_{1}')\), (ii) in real calculations, usually \(\phi_{1,2}\) are assumed to be distributed independently and uniformly in \([0, 2\pi]\) or even neglected. Second, during the calculation of \(\tilde{T}_{22}\), the numerical instability of multiplying iteratively \(T^{(k)}\), \(k = 1, \ldots, L\), can be fixed by performing QR decompositions where needed.

### 3.2. Lyapunov exponents (LEs) and normalized localization length

For a quasi 1D (Q1D) system \((W\text{ finite, } L \rightarrow \infty)\), generally the Anderson localization effect makes the two-terminal transmission decays exponentially. The corresponding decay length is called the Q1D localization length \(\xi_{W}\), which is the function of Fermi level \((p)\), SMT \((\theta)\) and transverse dimension \((W)\).

Now we define a real \(4W \times 4W\) symmetric matrix
\[
\Xi = \ln \left([T_{W}^{l}]^{T}T_{W}^{l}\right).
\] (68)

The TRS makes the \(4W\) eigenvalues of \(\Xi\) doubly degenerate into \(2W\) pairs, and further fall into \(W\) groups with opposite signs due to the current conservation request. In other words, the eigenvalues of \(\Xi\) can be written as \(\pm \omega_{i}, i = 1, \ldots, 2W\) meantime satisfying \(0 < \omega_{1} < \omega_{2} < \omega_{3} < \omega_{4} < \cdots < \omega_{2W-1} = \omega_{2W}\). The LEs associated with this Q1D network system with fixed width \(W\) are then defined by the following limit
\[ \Gamma_i = \lim_{L \to \infty} \omega_i / (2L), \tag{69} \]

and are self-averaging random variables.

The Q1D localization length of electrons is defined as the reciprocal of the \textit{smallest positive} LE,

\[ \xi_W = 1 / \Gamma_i, \tag{70} \]

since the decay of transmission should be controlled by the lowest decay rate in this system. Finally, the criticality of the 2D system is determined by the behavior of normalized localization length \( \Lambda \),

\[ \Lambda \equiv \xi_W / W, \tag{71} \]

as the transverse dimension \( W \) increases: the system falls into metallic (insulating) phases when \( \Lambda \) is an increasing (decreasing) function of \( W \) for sufficient large \( W \).

In practical numerical calculations, the LEs are not obtained by directly diagonalizing \( \Xi \), which comes from iterative multiplication of transfer matrices and turns out to be numerically unstable. Instead, we employ the following algorithm to achieve satisfactory estimations for both the LEs and their precision

\[ K^{(j)} M^{(j)} = [T^{(jm)} \ldots T^{(j-1)m+1})] K^{(j-1)}, \tag{72} \]

where \( j = 1, \ldots, sr \), and \( K^{(j)} \) are \( 4W \times 2W \) matrices with orthogonal columns and \( M^{(j)} \) are \( 2W \times 2W \) upper triangular matrices with positive diagonal elements.

The total length \( L \) is divided into \( s \) segments and each consists of \( r \cdot m \) PLs. In the \( k \)-th segment (\( 1 \leq k \leq s \)), we calculate,

\[ \gamma^{(k)}_{2W+1-w} = \frac{1}{rm} \sum_{j=(k-1)r+1}^{kr} \ln M^{(j)}_{w,w}, \quad 1 \leq w \leq 2W. \tag{73} \]

The \( 2W \) LEs are then evaluated as,

\[ \Gamma_i = \gamma_i = \frac{1}{s} \sum_{k=1}^{s} \gamma^{(k)}_i, \quad 1 \leq i \leq 2W, \tag{74} \]

If each segment \((rm)\) is long enough, it is reasonable to assume that \( \gamma^{(k)}_i (1 \leq k \leq s) \) are statistically independent. The standard error \( \sigma_i \) of \( \Gamma_i \) is given by,

\[ \sigma_i = \frac{1}{\sqrt{s-1}} (\overline{\gamma_i} - \overline{\gamma}^2)^{1/2}, \quad \overline{\gamma} = \frac{1}{s} \sum_{k=1}^{s} (\gamma^{(k)}_i)^2. \tag{75} \]

In most cases, \( \epsilon_1 = \sigma_1 / \Gamma_1 = 1\% \) is an acceptable criterion for a good estimation of \( \Gamma_1 \) and thus \( \Lambda \).

### 4. Quantum phases and phase transitions in SMT-QNM

The simplest S-type PSP is realized by 2D quadratic potential \( V_{S-PSP} = U_0 \cdot (y^2 - x^2) \) with \( U_0 > 0 \), which is identical for arbitrary spin orientation. In CC-RNM, the total Hamiltonian of an electron close to a S-type PSP

\[ \mathcal{H}_{S-PSP} = (-i \hbar \nabla + e\mathbf{A})^2 / (2m) + V_{S-PSP}, \tag{76} \]

is quadratic hence can be diagonalized. Under symmetric gauge of the vector potential \( \mathbf{A} = (B/2)(-y, x) \), the reflecting probability is [49, 51]

\[ p = r^2 = (1 + e^{\epsilon})^{-1}, \tag{77} \]

with

\[ \epsilon = [E_p - (2n + 1)E_r] / E_r, \]
\[ E_r = [\pm [\pm (\hbar \omega / A)^2 - \lambda^2])^{1/2}, \quad \lambda = hU_0 / (eB), \]
\[ \hbar \omega = [\lambda^2 + (\hbar \omega_B / 4)^2]^{1/2}, \quad \omega_B = eB / m, \tag{78} \]

in which \( E_r \) is the Fermi energy of the system. If \( E_r \) is well below (above) the saddle point energy, the quantum tunneling probability vanishes (approaches to 1). When \( E_r \) lies exactly at the PSP energy, \( \epsilon = 0 \) hence \( p_c = 1/2 \) being the CC-RNM critical point.

In SMT-QNM, there are no external magnetic fields. However we preserve the mathematical structure in equation (77) and in the simplest case assume \( \epsilon \equiv E_r \) without loss of generality. Then the mapping from \( E_r \in [-\infty, +\infty] \) to \( p \in [0, 1] \) is bijective with \( E_r = 0 \) corresponding to \( p = p_c \). Hence the ‘SMT–SPT’ phase space is isomorphic to \( \theta - p^2 \) parameter space.
When SMT is absent

Following the algorithms in section 3, numerical data, the complete phase diagram of SMT-QNM is obtained, as plotted in figure 1. Also, the re

in the absence of SMT. For any $p_i (p_i > p_c)$, by definition we have the following mappings under PBC,

\[ \Omega_1 \equiv \{ (p, \theta) | 0 \leq p \leq 1, 0 \leq \theta \leq \pi/2 \} \] (79)

are investigated in details.

### 4.1. Phase diagram of SMT-QNM in $\Omega_1$

Following the algorithms in section 3, $G_{\Sigma}$ and $\Lambda$ are calculated under PBC and/or RBC. Based on these numerical data, the complete phase diagram of SMT-QNM is obtained, as plotted in figure 4(a). Several important features are collected and explained as follows.

#### 4.1.1. Symmetry about $p = p_c$ when $\theta = 0$

When SMT is absent ($\theta = 0$), the SMT-QNM is nothing but two decoupled copies of CC-RNM with opposite chiralities meantime bearing opposite spin orientations. At all PSPs, When $p \rightarrow 0$ the quantum tunneling $t = \sqrt{1 - 2} \text{along equipotential lines. Hence all electron current loops around potential peaks become closed. On the contrary, when $p \rightarrow 1$ at PSPs the quantum tunneling gets weak and the reflecting along equipotential lines dominates. All electron current loops around potential valleys then become closed.}$

Under PBC, these two cases are equivalent and both lead to NI phase. Between these two phases, $p_c = 0.5 (P_{\text{CC-point in figure 4}})$ is the quantum critical point, which can be obtained from the infinitesimal Migdal–Kadanoff transformation for real-space renormalization of CC-RNM [52]. While under RBC, different choices of marginal PSP nodes result in different boundary modes on network edges. In figure 5 we illustrate the case in which $S'$-type PSPs reside in boundaries thus a quantum doublet emerges on each edge leading to the QSH state when $p < p_c$.

In addition, under PBC the quantum phases on $\theta = 0$ line are symmetric about $p = p_c$. There are two strategies to understand this symmetry. The first one comes from global considerations. To begin with, a given arbitrary random scalar potential profile (with statistical average being zero) is denoted as $\Sigma$. Then, following our sketch rules, we define $\mathcal{A}_{\Sigma}(p)$ as the network composed of all solid closed loops around potential valleys with up(down) spin. They are inter-connected by dashed SPTs for $p > p_c (E_F < 0)$, as shown in figure 1(c). Since $\theta = 0$, $\mathcal{A}_{B_{\Sigma}}(p)$ is decoupled from $\mathcal{A}_{B_{\Sigma}}(p)$, although they coincide with each other in real-space. Similarly the network including all SPT-interconnected closed loops around potential peaks with up(down) spin for $p' < p_c (E_F > 0)$ are defined as $\mathcal{B}_{B_{\Sigma}}(p')$ (not shown in figure 1). Also $\mathcal{B}_{B_{\Sigma}}(p')$ is unrelated to $\mathcal{B}_{B_{\Sigma}}(p')$ in the absence of SMT. For any $p_i (p_i > p_c)$, by definition we have the following mappings under PBC,

![Phase diagrams of the SMT-QNM in phase space $\Omega_1$ (a) and $\Omega_2$ (b). In each of them, NM phase is sandwiched by QSH and NI phases. The symmetries on the boundaries and in the interior of $\Omega_1$, are discussed in the main context in details. Through the mapping (94), the asymmetric phase diagram in $\Omega_1$ becomes the symmetric counterpart in $\Omega_2$. Lines with the same colors indicate the correspondence between the boundaries of the two phase spaces. The CC-RNM critical point $(p_0, \theta_0) = (X_0, Y_0) =$ (1/2, 0) is denoted by yellow solid squares in both phase spaces.](image)
Note that both \( \Sigma \) and \( \bar{\Sigma} \) are examples of ‘random scalar potential with zero statistical average’. Then naturally \( G^{\mathrm{PBC}}_T \) and \( A^{\mathrm{PBC}} \) are both statistically symmetric about \( p = \rho_c \).

The second strategy focuses locally on each PSP, in which \( T_0 \) and \( T'_0 \) (kernels of transfer matrices \( T_S \) and \( T'_S \)) are the main concern. At \( \theta = 0 \), for an arbitrary \( p \) (\( 0 < p < 1 \)), equations (51) and (54) provide

\[
T_0(p, 0) = \frac{1}{\sqrt{1 - p}} \sigma_0 \otimes \sigma_0 - \frac{p}{\sqrt{1 - p}} \sigma_0 \otimes \sigma_0,
\]

\[
T'_0(p, 0) = \frac{1}{\sqrt{p}} \sigma_0 \otimes \sigma_0 - \frac{1 - p}{p} \sigma_0 \otimes \sigma_0.
\]

Then the following connections

\[
T_0(1 - p, 0) = [\sigma_0 \otimes \sigma_0] \cdot T_0(p, 0)
\]

\[
T'_0(1 - p, 0) = [\sigma_0 \otimes \sigma_0] \cdot T'_0(p, 0)
\]

hold thus to lead the symmetry about \( p = \rho_c \).

It should be clarified that the \( p \rightarrow 1 - p \) mapping does not change the random scalar potential profile. S-type (\( S' \)-type) PSPs are always S-type (\( S' \)-type). What it really changes is the Fermi level of this system, i.e. from \( E_0(p) \) to \( -E_0(p) \) due to equation (77), since we have fixed the energy reference point to be zero. Under our sketch rules, at a S-type PSP, for \( p(>\rho_c) \), the electron flows are shown in figure 1(c). For \( 1 - p' \), the valley-peak-distribution is unchanged but the electron flows change to those depicted in figure 1(d). Now the PSP is still S-type and only its scattering matrix takes a similar mathematical format as a \( S' \)-type PSP. Bearing this in mind, the connection (84) actually means at a certain PSP, the transfer matrix at \( p' \) in an original closed equipotential loop surrounding a potential valley (peak) is mathematically related to the transfer matrix at \( 1 - p' \) in a mapped loop around a potential peak (valley). This is exactly what equation (80) tells us. Therefore mathematically S-type and \( S' \)-type PSPs exchange their roles in constructing the total transfer matrix. Hence under PBC, the total transfer matrix is unchanged, resulting in the symmetry about \( p = \rho_c \).
4.1.2. Asymmetry about \( p = p_c \) when \( \theta > 0 \)

When \( \theta > 0 \) (SMT appears), an intermediate NM phase emerges between the two NI phases (PBC) or ‘QSH+NI’ phases (RBC), as a manifestation of Wigner–Dyson symplectic ensembles. Numerical data in figure 4(a) clearly show that the phase diagram is asymmetric about \( p = p_c \). This can also be explained by the global and local strategies introduced in the above subsection.

From the global strategy, the definitions of \( A^{(1)}_{S\\square} (p) \) and \( B^{(1)}_{S\\square} (p') \) are unchanged. However, now \( A^{(1)}_{S\\square} (p) \) is coupled with \( A^{(1)}_{L\\square} (p) \) via SMT. The situation is similar for \( B^{(1)}_{S\\square} (p') \). The mappings in equation (80) still hold. But the symmetry in equation (81) fails due to the SMT terms. Hence the final mappings in equation (82) do not exist, leading to the asymmetry about \( p = p_c \) line when \( \theta > 0 \).

From the local strategy, the general form of \( T_0(p, \theta) \) and \( T'_0(p, \theta) \) are given in equations (51) and (54). For \( 0 < \theta < \pi/2 \), generally \( T_0(1 - p, \theta) \) and \( T'_0(1 - p, \theta) \) have no explicit connections with \( T_0(p, \theta) \) and \( T'_0(p, \theta) \) as in equation (84). This also explains the asymmetry about \( p = p_c \).

4.1.3. QSH phase on \( p = 0 \)

On the vertical \( p = 0 \) line in figure 4(a), equation (51) becomes

\[
T_0(0, \theta) = \sigma_0 \otimes \sigma_0,
\]

which is irrelevant to \( \theta \), meaning that the SMT has no effects on ‘left-to-right’ transfer of electron flows. However from equation (34), \( T'_0(0, \theta) \) provides singularity since \( \tau = \sqrt{p} = 0 \). This is due to the fact that when \( p = 0 \), at \( S' \)-type PSPs in the bulk, terminals on the left-hand side are decoupled from those on the right-hand side, thus have no contributions to left-to-right transfer. All electron current loops around potential peaks then become completely closed. Under RBC, at boundary \( S' \)-type PSPs, the completely reflecting of electron flows results in dissipationless edge modes thus make the system fall into QSH phase.

4.1.4. One-to-one mapping between \( p = 1 \) and \( \theta = \pi/2 \) lines

On \( p = 1 \) line, one has

\[
T_0(1, \theta) = (\cot \theta) \sigma_0 \otimes i\sigma_y - (\csc \theta) \sigma_y \otimes i\sigma_x,
\]

\[
T'_0(1, \theta) = (\cos \theta) \sigma_x \otimes \sigma_0 + (\sin \theta) i\sigma_y \otimes i\sigma_y.
\]

While on \( \theta = \pi/2 \) line, the counterparts are

\[
T_0(p, \pi/2) = \sqrt{1-p} \sigma_0 \otimes \sigma_0 - \sqrt{p} \sigma_x \otimes i\sigma_y,
\]

\[
T'_0(p, \pi/2) = -\sqrt{1-p} \sigma_z \otimes i\sigma_y + \frac{1}{\sqrt{p}} i\sigma_y \otimes i\sigma_y.
\]

If we perform the bijection

\[
\sqrt{p} \leftrightarrow \sin \theta, \quad \sqrt{1-p} \leftrightarrow \cos \theta
\]

between the two line segments \( \{ p = 1, \theta \in (0, \pi/2) \} \) and \( \{ \theta = \pi/2, p \in (0, 1) \} \), then the following connections

\[
T_0(p, \pi/2) = T'_0(1, \theta) \cdot [-\sigma_z \otimes \sigma_0]
\]

\[
T'_0(p, \pi/2) = [-\sigma_z \otimes \sigma_0] \cdot T_0(1, \theta)
\]

hold. Note although the unitary matrix \(-\sigma_z \otimes \sigma_0\) lies on different sides, its \( \pi \)-phases (originated from diagonal ‘–1’ elements) can be absorbed into phase matrices \( U_e \) and \( U'_e \), thus do not affect the mathematical role-reversal of \( S \)- and \( S' \)-type PSPs under bijection (88). Then \( p = 1 \) and \( \theta = \pi/2 \) lines are equivalent and both fall into NI phase under PBC. Under RBC, on \( \theta = \pi/2 \) line dissipationless edge modes appear at boundary \( S \)-type PSPs thus make the system fall into QSH phase. While for \( p = 1 \) line, similar to figure 5(b), closed electron flows loop around potential valleys turn the system to NI state.

At last, at the phase point \( (p, \theta) = (1, \pi/2) \), which is the cross point of the above two line segments, one has

\[
T_0(1, \pi/2) = -\sigma_x \otimes i\sigma_y.
\]

These completely skew diagonal transfer matrices fully mix the up and down spins and meantime greatly enhance the itinerant range of electrons. Thus at this very point, the system becomes metallic.

4.2. Mapping to phase diagram of \( Z_2 \)-QNM

In fact, we can map our phase diagram (figure 4(a)) to a more symmetric one. However, before do that, it is interesting to point out that our phase diagram has close connection with that from the existing \( Z_2 \)-QNM (see figures 8 and 11 in [39]): under PBC, they are symmetric about the vertical \( p = p_c \) line.
The reason is straightforward. By mapping the horizontal axis ‘x’ in [39] to the counterpart in this work ‘p’ through \( p = \tanh^2 x \), we rewrite their equation (2.3) in terms of ‘r’ and ‘y’ as

\[
T_0^{Z_2} = (1/t) [\sigma_0 \otimes (T_0^{Z_2})^d + \sigma_2 \otimes (T_0^{Z_2})^{yd}],
\]

\[
(T_0^{Z_2})^d = \begin{bmatrix}
\cos \theta & r \sin \theta \\
-r \sin \theta & \cos \theta
\end{bmatrix},
\]

\[
(T_0^{Z_2})^{yd} = \begin{bmatrix}
-r \cos \theta & -\sin \theta \\
\sin \theta & -r \cos \theta
\end{bmatrix}.
\]

By comparing them with the transfer matrix kernels \( T_0 \) and \( T'_0 \) of our SMT-QNM, we have the following connections

\[
T_0^{Z_2}(p, \theta) = [\sigma_2 \otimes \sigma_2] \cdot T'_0(1 - p, \theta) \cdot [\sigma_0 \otimes \sigma_2],
\]

\[
T_0^{Z_2}(p, \theta) = [\sigma_2 \otimes \sigma_2] \cdot T'_0(1 - p, \theta) \cdot [\sigma_0 \otimes \sigma_2],
\]

which are quite similar to equation (84). Therefore, similar discussions as in the end of section 4.1.1 can be performed.

In figure 6 we illustrate a typical mapping starting from the SMT-QNM with \( p > p_c \) (i.e. \( r > t \)). The main procedure is: (s1) \( p \rightarrow 1 - p \), or equivalently exchange \( t \) and \( r \); (s2) following our sketch rules, the electron flows are redrawn; (s3) by exchanging up and down spins, the S-type (S'-type) PSPs in SMT-QNM has the same electron flow structure as the S'-type (S-type) PSPs in \( Z_2 \)-QNM. Then it is understandable that under PBC by performing a mirror-symmetry operation on our phase diagram (figure 4(a)) about \( p = p_c \) line, one gets the phase diagram of the \( Z_2 \)-QNM. Note that in this mapping only the electron flows are converted. The potential valleys and peaks are unchanged. Based on this result, the critical exponent and normalized localization length at phase transitions should be the same as those from \( Z_2 \)-QNM. This is confirmed by numerical calculations within error permissibility. To save space, we do not show this part of our data here. However, this close connection should not downgrade the significance of SMT-QNM constructed in this work. First, in our SMT-QNM, SMT process is an additional tunneling channel and does not take probability away from the existing SPT channel, which is a more physical assumption. Second, the symmetry about \( p = p_c \) line between these two phase diagrams indicates a possible way to check which network model provides better description to real 2D-DSEGs. From equation (77), \( p \) is directly related to system Fermi level. By sweeping the Fermi level and check out the quantum phase a 2D-DSEG falls in, experimentally one can make reasonable judgment. Third, as will be shown next, the phase diagram of SMT-QNM.
can be topologically transformed to a symmetric one which is highly similar to the phase diagram of disordered 3D weak TIs. This enriches the possible applications of our 2D SMT-QNM.

4.3. Mapping to a symmetric phase diagram

The asymmetry of phase diagram in the original \((p, \theta)\) phase space is unfavorable for a deep understanding of ATs in 2D-DSEGs. Fortunately, its features summarized in section 4.1 provide us enough information to topologically transform it to a completely symmetric one. Mathematically, the following mapping

\[ X = \frac{p \cos^2 \theta}{1 - p \sin^2 \theta}, \quad Y = p \sin^2 \theta \]  

(94)

perfectly achieves this target.

(a) The original phase space \(\Omega_1\) (see equation (79)) is mapped to a new phase space

\[ \Omega_2 \equiv \{(X, Y)|0 \leq X \leq 1, 0 \leq Y \leq 1\}. \]  

(95)

(b) The original \(\theta = 0\) line is mapped to \(Y = 0\) line with the one-to-one correspondence \(p \leftrightarrow X\), hence \(Y = 0\) line is symmetric about the vertical line \(X = X_c(=p)\).

(c) The original \(p = 0\) line shrinks to a single point \((X, Y) = (0, 0)\).

(d) The metallic phase point \((p, \theta) = (1, \pi/2)\) stretches itself to \(Y = 1\) line, which is indeed a singularity of the mapping in equation (94).

(e) The \(\theta = \pi/2\) and \(p = 1\) lines are mapped to \(X = 0\) and \(X = 1\) lines, respectively. The combination of mappings in equations (88) and (94) generates a one-to-one correspondence of phase points on \(X = 0\) and \(X = 1\) lines with the same \(Y\). Therefore these two lines are completely symmetric about \(X = X_c\) line.

(f) For an arbitrary point \((X, Y)\) in the interior region of \(\Omega_2\), one has

\[ T_0(X, Y) = (1 - X + XY)^{-1}[\sigma_0 \otimes \Delta^0 + \sigma_1 \otimes \Delta^{\text{od}}], \]

\[ \Delta^0 = \sqrt{(1 - X)(1 - Y)} \sigma_0 + \sqrt{XY(1 - Y)} i \sigma_y, \]

\[ \Delta^{\text{od}} = -(1 - Y) \sqrt{(1 - X)} \sigma_0 - \sqrt{Y} i \sigma_y, \]  

and

\[ T_0'(X, Y) = (X + Y - XY)^{-1}[\sigma_0 \otimes \Delta^{\text{od}} + i \sigma_1 \otimes \Delta^0], \]

\[ \Delta^0 = \sqrt{(1 - Y)} \sigma_0 - \sqrt{(1 - X)Y(1 - Y)} i \sigma_y, \]

\[ \Delta^{\text{od}} = -(1 - Y) \sqrt{(1 - X)} \sigma_0 + \sqrt{Y} i \sigma_y. \]  

(96)

(97)

Then the following connections

\[ T_0(1 - X, Y) = [\sigma_2 \otimes \sigma_1] \cdot T_0'(X, Y) \cdot [\sigma_0 \otimes \sigma_1] \]

\[ T_0'(1 - X, Y) = [\sigma_2 \otimes \sigma_1] \cdot T_0(X, Y) \cdot [\sigma_0 \otimes \sigma_1] \]  

(98)

indicate the symmetry of mapped phase diagram about \(X = X_c\) line in \(\Omega_2\).

Following the mapping in equation (94), we transform the phase diagram in \(\Omega_1\) into the one in \(\Omega_2\) which is plotted in figure 4(b). Obviously, the new phase diagram looks better. However, it is not completely symmetric about \(X = X_c\) due to the finite-size effect during our calculation, since we only perform calculations on normalized localization length to \(W = 2^3\) limited by our existing computing capability. It is expected that when \(W\) is sufficient large, the symmetry in \(\Omega_2\) should be more apparent.

The results in this subsection have several potential applications. First, phase boundaries in \(\Omega_1\) can be double-checked through the mapping (94) and its inversion, since in \(\Omega_2\) phase boundaries should be symmetric about \(X = X_c\). Second, the narrow and long NI (or QSH) phase in the close vicinity of \((p, \theta) = (1, \pi/2)\), which is hard to precisely determined due to strong symmetry-crossover effects, is enlarged a bit in \(\Omega_2\). This should be helpful for better determination of NM-NI (QSH) boundaries.

4.4. Connection with disordered 3D weak TIs

In addition, the new phase diagram (figure 4(b)) shows apparent similarity with that of disordered 3D weak TIs (see figure 1 in [53]), indicating a close connection between 2D-DSEGs described by our SMT-QNM and the helical surface modes of 3D weak TIs under scalar disorder potentials respecting TRS. Comparing our 2D Dirac
Hamiltonian (see equation (48) in this work) and the effective Hamiltonian in [53] therein, the energy gap of the system is $|m|^2$ with $m = \pm m$ meaning on $Y = 0$ line in figure 4(b), the intermediate metallic region shrinks to a single critical point $X = X_c$. In the presence of uncorrelated disorder which couples the two Dirac fermions (with mass $\pm m$) with strength $V_0 = \sqrt{2} \theta$, direct transitions between the insulating phases (NI and QSH) are forbidden due to the stability of the symplectic metal, which results in the finite width of intermediate metallic phase. In addition, the disorder strength

$$g \sim |V_0|^2 \sim \theta^2 \sim \sin^2 \theta \propto Y$$

in the vicinity of the critical point $(X, Y) = (X_c, 0)$. All these correspondences confirm the close connection we mentioned at the beginning of this section. This implies the possible application of our SMT-QNM on investigations of disordered helical surface modes of 3D weak TIs. For our SMT-QNM, this is an interesting direction but out of the scope of this work.

5. Quantum phases and phase transitions in TRS-breaking SMT-QNM

The TRS–preserving SMT-QNM introduced above can be downgraded to the counterpart which still preserves TRS at PSPs but breaks it in the links between PSPs. Physically, this corresponds to 2D–DSEGs with TRS-breaking (usually called magnetic) isotropic impurities. These impurities inevitably affect the random potential profile, however will not create PSPs at their very locations due to the isotropic nature, thus can be described by the TRS-breaking SMT-QNM. In these systems, spin-flip backscattering on each link between PSPs emerges thus destroys the original Krammer’s doublet. For modelization, this can be simply realized by neglecting the ‘phase pairing rule’ in equation (61), meanwhile leaving the rest unchanged. Here we briefly summarize our data and provide reasonable explanations.

5.1. Phase diagram

Now the system falls into Wigner–Dyson unitary class (TRS fails, regardless of SRS) and generally no intermediate NM phase exists. This is confirmed by finite-size analysis on $G_{2T}$ and $\Lambda$. The resulting phase diagram is plotted in figure 7 and totally different from the TRS-preserving SMT-QNM. Our data show that QSH state only survives on the line segment $0 \leq p < 0.5, \theta = 0$. The point $p_{c1}$ is the CC-RNM critical point $(p, 0)$. The point $X$ is a typical QSH phase point with the coordinate: $X: (0.3, 0)$. For modelization, this can be simply realized by neglecting the ‘phase pairing rule’ in equation (61), meanwhile leaving the rest unchanged. Here we briefly summarize our data and provide reasonable explanations.

5.2. The NI phase

The TRS-breaking in links connecting PSPs will turns both NM and QSH phases (except for the segment on $\theta = 0$ line) into NI phase, which is the typical behavior of unitary ensembles. To check for this, we perform numerical calculations of $G_{2T}$ under RBC for enough dense grid of the phase space $\Omega_{1}$. For all phase points, the network size $W(=L)$ increases from $2^4$ to $2^6$. Further enlargement of $W$ is out of our computing capability. The sample number is always 128, which is enough to provide sufficient small error. To save space, we summarize the
main features and present typical data, if necessary. First, for all phase points in \( \Omega_1 \), \( \langle G_{2T}^{\text{RBC}} \rangle \) is smaller than 1 and decrease with \( W \) for sufficient large \( W \) without sign of convergence. Obviously this can not be QSH state. In addition, we have known that in NM phase (if exists) of systems with unitary symmetry, \( \langle G_{2T}^{\text{RBC}} \rangle \) should converge to the Boltzmann conductance \( G_0(\geq 1) \) \(^{[54]}\). Hence our data clearly show that the system falls into neither QSH nor unitary metallic phase. The only possibility is the NI phase.

Next, the finite-size calculations of the normalized localization length \( \Lambda \) for \( \theta = 0.1 \pi \) and \( \theta = 0.3 \pi \) are performed and the data are plotted in figure 8. In these calculates, RBC in transverse direction is imposed and the sample number is chosen to be 32. In addition, the relative standard error of the first LE is set to be 1% and the sample number is 32. All error bars are smaller than data symbols.

5.3. Direct transition from QSH to NI phases

The QSH phase on line segment \( \{ 0 \leq p < 0.5, \ \theta = 0 \} \) is absolutely unstable to SMT. This means no matter how small the SMT is, the QSH state will be destroyed completely. To see it, the point \( X(p = 0.3, \ \theta = 0) \) in figure 7 is selected as an example. We set \( W = L \) and vary \( W \) from \( 2^4 \) to \( 2^9 \). In the close vicinity of point \( X \), \( \langle G_{2T}^{\text{RBC}} \rangle \) and the corresponding error of 128 independent configurations are calculated and plotted in figure 9. For point \( X \), numerical data (hollow squares in figure 9) show that when the system size increases to \( W = 2^9 \), \( \langle G_{2T}^{\text{RBC}} \rangle \) approaches the quantized value 2, with the standard error as small as \( 5.6 \times 10^{-6} \). This validates that point \( X \) belongs to the QSH phase. Next we perform calculations for \( \theta = 0.01 \). The result is shown in figure 9 by solid magenta squares. As the system size gets larger, \( \langle G_{2T}^{\text{RBC}} \rangle \) falls to \( 10^{-3} \) or even smaller. We then gradually approach the point \( X \) by decreasing \( \theta \) by an order of magnitude and calculate the corresponding \( \langle G_{2T}^{\text{RBC}} \rangle \) until \( \theta \) reaches 1.0 \( \times \) 10\(^{-6}\). The results are plotted in figure 9, showing that as \( \theta \) decreases, the deviation of \( \langle G_{2T}^{\text{RBC}} \rangle \) from quantized value 2 gets weaker at \( W = L = 2^9 \). However, it always exists and has no sign of convergence. Even for \( q = 1.0 \times 10^{-6} \) (solid black squares), if the system size is further increased to \( 2^{10} \), \( \langle G_{2T}^{\text{RBC}} \rangle \) deviates from 2 evidently. Limited by computing capability, we can not perform calculations to the system size at which \( \langle G_{2T}^{\text{RBC}} \rangle \) falls to zero. However, the data in figure 9 clearly imply that QSH state can not survive when SMT emerges, no matter how small it is.

For cross-validation, we also perform finite-size calculations for the normalized localization length \( \Lambda \), in which RBC is adopted and the sample number is 16. In addition, the relative standard error of the first LE is 1%, leading to the network length \( L \sim 10^6 \). The network width \( W \) increases from \( 2^2 \) to \( 2^3 \) and the SMT strength \( \theta \) varies from 0 to 0.01 with the step \( d\theta = 0.001 \). The resulting data are plotted in figure 10. It is clear that \( \Lambda \) for \( \theta = 0 \) always increases with \( W \). This comes from the dissipationless edge modes and confirms the fact that point \( X \) belongs to QSH phase. On the other hand, when \( \theta \geq d\theta \), \( \Lambda \) eventually decreases as \( W \) increases to \( 2^3 \). This validates the conclusion based on data from \( \langle G_{2T}^{\text{RBC}} \rangle \) that \( (p = 0.3, \ \theta \geq d\theta) \) falls into NI phase. Further increase

---

**Figure 8.** Finite-size calculations of the reduced localization length \( \Lambda \) along (a) \( \theta = 0.1\pi \) and (b) \( \theta = 0.3\pi \) with RBC in transverse direction. The relative standard error of the first LE is set to be 1% and the sample number is 32. All error bars are smaller than data symbols.
of $W$ and decrease of $d\theta$ are beyond our present computing capability. However, the data in figure 10 already provide enough cross-validation evidences of the absolute instability of QSH state.

This result can be understood by the physical process sketched in figure 11. Initially the system is in QSH phase, i.e. the situation depicted in figure 5(a). For simplicity, we take the part close to the upper edge as an example and redraw it in figure 11. We focus on an electron with up spin propagating in the dissipationless left-to-right edge channel. Suppose at some moment, the electron is at point A which is set as the starting point. When SMT is absent, the electron at most tunnels into the closed loops with up spins via SPT and can not fall into trajectories associated with down spins. Hence the electron will never be backscattered into the right-to-left edge channel with down spin at the upper edge. On the other hand, the backscattering into the right-to-left edge channel with down spin at the upper edge. On the other hand, the backscattering into the right-to-left edge channel with down spin at the upper edge.
channel with up spin at the lower edge (not depicted in figure 11) by means of multi-SPTs through closed spin-up loops will be suppressed when the network is wide enough since this is a high-order process. When SMT emerges, the situation is completely different. When a spin-up electron propagates from point A and reaches point B, SMT at this S-type PSP allows it to tunnel into the closed loop associated with down spins (point C on the red loop). After circling this loop (C to D to E), the electron comes back to this PSP and tunnels into the right-to-left edge channel (point F) with down spin via SPT and then go to point G and even leftward. Now we realize a spin-flip backscattering event ($A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G$) which includes only one step of SMT. Therefore this process is not a high-order one and should take effect as long as SMT appears. Combing with the fact that a number of S-type PSPs distribute around the upper edge, it is understandable that the QSH state should be absolutely unstable with respect to SMT. Therefore, this process is not a high-order one and should take effect as long as SMT appears. Combing with the fact that a number of S-type PSPs distribute around the upper edge, it is understandable that the QSH state should be absolutely unstable with respect to SMT. Therefore, the QSH line segment acts as a critical line rather than a phase boundary. Hence the critical exponent for this direct transition can hardly be extracted out using the standard finite-size scaling procedure [55].

5.4. Revisit of the CC-RNM critical point ($p_c, 0$)

In the end of this section, we turn back to the CC critical point $P_{CC} \equiv (p_c, 0)$ in figure 7, where the network decouples into two copies of CC-RNM with opposite chiralities. As mentioned in section 4.1.1, the infinitesimal Migdal-Kadanoff transformation for real-space renormalization of CC-RNM provides that it is the quantum critical point that separates two insulating (NI and QSH) phases in the bulk. Consequently one may conjecture that in TRS-breaking SMT-QNM, this point should also be a quantum critical point along $\theta = 0$ line for bulk. To check this, we perform various finite-size calculations and the resulting data are plotted in figure 12.

First, we calculate $(G_{TT})$ under both RBC and PBC in the vicinity of $p = p_c$ along $\theta = 0$ line. The results are shown in figures 12(a) and (b), respectively. In these calculations, $W (= L)$ varies from $2^2$ to $W_{\text{max}} = 2^7$ and the step of $p$ is $dp = 0.01$. Meantime, $N = 10^5$ independent samples are generated for acceptable averages. Generally, $dp$ is limited by our computing capability ($W_{\text{max}}$ and $N$) hence can not be arbitrarily small. In this sense, we can only detect the criticality of $P_{CC}$ at the level of $dp$. For RBC, $(G_{TT}^{\text{RBC}})$ is an increasing function of $W$ when $p \leq p_c$. When $p \geq p_c + dp$, $(G_{TT}^{\text{RBC}})$ decreases for sufficient large $W$. For PBC, $(G_{TT}^{\text{PBC}})$ is an increasing function of $W$ when $|p - p_c| \leq dp$, at least for $W \leq W_{\text{max}}$. While if $|p - p_c| \geq 2dp$, $(G_{TT}^{\text{PBC}})$ eventually decreases at sufficient large $W$.

To further check, we calculate the normalized localization length $\Lambda$ and the data are plotted in figure 12(c). In these calculations, several points should be clarified. First, PBC is adopted to eliminate the effect of dissipationless edge states in QSH phase. Second, the relative standard error $\epsilon_1$ of the first LE is set to be 0.5%,
which results in the strip length $L \sim 10^7 \gg \xi_\omega$ for $W = 2\ell$. Third, the sample number is 16 and proved to be enough for sufficiently small error bars. The data in figure 12(c) clearly show that only at $p = p_\omega$, $\Lambda$ is an increasing function of the width $W$ (at least for $W \leq W_{\text{max}}$). For other $p$ satisfying $|p - p_\omega| \gg dp$, $\Lambda$ decreases for sufficient large strip width, which means in the bulk limit they fall into some insulating phase. Further verifications need smaller $dp$, greater network width $M_{\text{max}}$ and sample number $N$, which are out of our present computing capability. In fact, if $P_{\text{cc}}$ is indeed a critical point, we can hardly prove it numerically by finite-size calculations in foreseeable future since decrease (by magnitude) of $dp$ leads to geometric growth of calculation time.

6. Conclusions

In this work, we have constructed the symplectic SMT-QNM by recognizing the SMT as an independent tunneling channel. By leading-order expansion method, the 2D Dirac Hamiltonian is extracted out from SMT-QNM in the close vicinity of CC–RNM critical point, with the SMT strength associated with the spin–flip coupling. A sandwiched (QSH-NM-NI) phase diagram in original phase space $\Omega_1$ is then obtained by finite-size analysis of two-terminal conductance and normalized localization length. It is first mapped to the phase diagram of the existing $Z_2$-QNM, and then closely related to the counterpart of disordered 3D weak TIs. In the end, the TRS-breaking (in the links between PSPs) version of SMT-QNM is considered and turns out to fall into unitary class. Its phase diagram is filled by NI phase except for a marginal line segment still hosting QSH phase. A direct transition from QSH to NI phases exists and is explained by the SMT-induced spin–flip backscattering.

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