THE PRIMITIVE COHOMOLOGY OF THE THETA DIVISOR OF AN ABELIAN FIVEFOLD

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Dedicated to Herb Clemens

Abstract. The primitive cohomology of the theta divisor of a principally polarized abelian variety of dimension $g$ contains a Hodge structure of level $g - 3$ which we call the primal cohomology. The Hodge conjecture predicts that this is contained in the image, under the Abel-Jacobi map, of the cohomology of a family of curves in the theta divisor. In this paper we use the Prym map to show that this version of the Hodge conjecture is true for the theta divisor of a general abelian fivefold.

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Let $A$ be a principally polarized abelian variety \((ppav)\) of dimension $g \geq 4$, let $\Theta$ be a symmetric theta divisor in $A$, and assume that $\Theta$ is smooth. The cohomology group

$$H^{g-1}(\Theta, \mathbb{Z})$$

contains a natural sublattice of rank $g! - \frac{1}{g+1}(2^g_g)$ (see [IvS, p. 561])

$$\mathbb{K} := \text{Ker}(H^{g-1}(\Theta, \mathbb{Z}) \to H^{g+1}(A, \mathbb{Z})),$$

which we call the primal cohomology of $\Theta$. There is also a Hodge structure $\mathbb{H} \subset H^{g-1}(\Theta, \mathbb{Z})$ which fits in an exact sequence

$$0 \to \mathbb{K} \to \mathbb{H} \to H^{g-3}(A, \mathbb{Z}) \to 0.$$ 

By [IvS, p. 562], these Hodge structures are all of level $g - 3$. For a rational Hodge structure $V := (V_\mathbb{Q}, V_\mathbb{Q} \otimes \mathbb{C} = \oplus_{p+q=n} V^{p,q})$ of weight $n$, the level $l(V)$ of $V$ is defined as the positive integer

$$l(V) := \max \{|p - q| \mid V^{p,q} \neq 0\}.$$ 

Grothendieck’s version of the Hodge conjecture states that if $H^{g-1}(\Theta, \mathbb{Q})$ contains a Hodge substructure of level $g - 3$, then it is contained in the image, under Gysin push-forward, of the cohomology of a smooth (possibly reducible) variety of dimension $g - 2$. After tensoring with $\mathbb{Q}$ we have

$$\mathbb{H}_\mathbb{Q} := \mathbb{H} \otimes \mathbb{Q} = \mathbb{K}_\mathbb{Q} \oplus \theta \cdot H^{g-3}(A, \mathbb{Q})$$

where $\theta := [\Theta]$ is the cohomology class of $\Theta$ and $\theta \cdot H^{g-3}(A, \mathbb{Q})$ is the image of $H^{g-3}(A, \mathbb{Q}) \cong H^{g-3}(\Theta, \mathbb{Q})$ in $H^{g-1}(\Theta, \mathbb{Q})$. The subspace $\theta \cdot H^{g-3}(A, \mathbb{Q})$ is also a Hodge substructure of level $g - 3$ and satisfies the Hodge conjecture since it is in the image, for instance, of the cohomology of an intersection of a
translate of $\Theta$ with $\Theta$. Therefore the Hodge conjecture for $\mathbb{H}_Q$ is equivalent to the Hodge conjecture for $\mathbb{K}_Q$.

An equivalent formulation of the Hodge conjecture is that $\mathbb{H}_Q$ or $\mathbb{K}_Q$ is contained in the image, under the Abel-Jacobi map, of the cohomology of some family of curves in $\Theta$ (see e.g. [I, pp. 492-493] for a proof of this elementary fact). For $g = 4$, it was proved in [IvS] that the family of Prym-embedded curves in $\Theta$ is a solution to this problem for $\mathbb{H}_Q$.

For $g = 5$ a general ppav is again a Prym variety. However, in this case, every component of the family of Prym-embedded curves in $\Theta$ parametrizes curves that are translates of a single curve. Therefore the image of the cohomology of any of these components is contained in $\theta \cdot H^{g-3}(A, \mathbb{Q})$. Hence the family of Prym-embedded curves in $\Theta$ cannot be a solution to the Hodge conjecture for the primal cohomology $\mathbb{K}_Q$.

Denote by $A_g$ the coarse moduli space of principally polarized abelian varieties of dimension $g$. Representing $(A, \Theta)$ as a Prym variety and using some interesting geometric constructions, we construct a different family of curves in $\Theta$ which is a solution to the Hodge conjecture for $\mathbb{H}_Q$ for $(A, \Theta)$ general in $A_5$.

**Theorem 1.** For $(A, \Theta)$ in a non-empty Zariski open subset of $A_5$, the general Hodge conjecture holds for the Hodge structure $\mathbb{H}_Q \subset H^4(\Theta, \mathbb{Q})$ and hence $\mathbb{K}_Q \subset H^4(\Theta, \mathbb{Q})$.

As the rational cohomology of $\Theta$ is the sum of $\mathbb{K}_Q$ and the rational cohomology of $A$, our result, together with the main result of [H], implies

**Corollary 2.** For $(A, \Theta)$ in the complement of countably many proper Zariski closed subsets of $A_5$, the general Hodge conjecture holds for $\Theta$.

Note that there are relatively few examples of lower level Hodge substructures of the cohomology of algebraic varieties that are not already contained in the images of the cohomologies of subvarieties for trivial reasons. Some of the most interesting such examples are provided by abelian varieties, such as abelian varieties of Weil type (see [I]) and the primal cohomology of theta divisors. In fact we are not aware of any nontrivial examples that do not involve abelian varieties in some way. As far as we are aware, the primal cohomology of the theta divisor of an abelian fivefold is the first nontrivial case of a proof of the Hodge conjecture for a family of fourfolds of general type. The proof was considerably
more difficult than the case of of the theta divisor of the abelian fourfold worked out in [IvS] and required a difficult degeneration argument and nontrivial and interesting geometric constructions. As is often the case with deep conjectures such as the Hodge conjecture, the level of difficulty goes up rapidly with the dimension of the varieties concerned or, perhaps more accurately, with their Kodaira dimension.

It would be interesting to know whether the Hodge structure $\mathcal{K}$ is irreducible. This would considerably simplify our computation of the Abel-Jacobi map as in that case we would only have to prove that its image intersects $\mathcal{K}$ nontrivially.

Letting $\mathcal{R}_6$ denote the moduli space of étale double covers of curves of genus 6, further note that the monodromy group of the Prym map $\mathcal{R}_6 \to \mathcal{A}_5$ is the Weyl group $W(E_6)$ of the exceptional Lie algebra $E_6$ (see [Do, Theorem 4.2]). Also, the lattice $\mathcal{K}$ has rank 78 for $g = 5$ which is equal to the dimension of $E_6$. So one might wonder whether it is possible to define a natural isomorphism between $\mathcal{K}_C := \mathcal{K} \otimes \mathbb{C}$ and $E_6$.

We now explain the general outline of our proof.

A general ppav of dimension 5 is the Prym variety of an étale double cover of smooth curves $\tilde{X} \to X$ with $X$ general of genus 6.

Using the 5-gonal construction (see [ILS]), we construct a family of curves in $\Theta$ (see Section 1)

$$
\begin{array}{ccc}
F_r & \xrightarrow{\rho_2} & \Theta \\
\rho_1 & \downarrow & \\
\tilde{G}^1_{5} & \\
\end{array}
$$

dependent on the choice of a general point $r \in \tilde{X}$. Here $\tilde{G}^1_{5}$ is an étale double cover of the variety $G^1_{5}(X)$ parametrizing pencils of degree 5 on $X$ ($\cong W^1_{5}(X)$ if $X$ is not a plane quintic), which is a smooth irreducible surface for $X$ sufficiently general. The Abel-Jacobi map for this family of curves is, by definition,

$$
\rho_2^* : H^2(\tilde{G}^1_{5}) \to H^4(\Theta).
$$

The image of the Abel-Jacobi map defines a Hodge substructure of level $\leq 2$ of the cohomology of $\Theta$. Theorem 1 is a direct consequence of
Theorem 3. The Hodge structure $H^2_{\bar{Q}}$ is the sum of $\theta \cdot H^2(A, \mathbb{Q})$ and the image of $\rho_2 \circ \rho_1^*$. 

We prove Theorem 3 by specializing the étale double cover $\tilde{X} \to X$ to a Wirtinger cover. To define a Wirtinger cover, choose two general points $p$ and $q$ on a general curve $C$ of genus 5 and let $C_{pq}$ be the nodal curve of genus 6 obtained from $C$ by identifying $p$ and $q$. The Wirtinger cover $\tilde{C}_{pq}$ is obtained as the union of two copies $C_1$ and $C_2$ of $C$, with the copy of $p$ on each curve identified with the copy of $q$ on the other. The Prym variety of the Wirtinger cover $\tilde{C}_{pq} \to C_{pq}$ is naturally isomorphic to the polarized Jacobian $(J(C), \Theta_C)$ of the curve $C$ (see e.g. Section 2.4 below).

In most of the paper we work with a one-parameter family $X \to T$ of curves of genus 6 over an analytic disc $T$ with smooth total space, with general fiber $X_t$ a general curve of genus 6 and special fiber $X_0 = C_{pq}$ at $0 \in T$ a general one-nodal curve of genus 6. We also assume given an étale double cover $\tilde{X} \to X$ whose special fiber $(\tilde{X}_0 \to X_0) = (\tilde{C}_{pq} \to C_{pq})$ is the Wirtinger cover described above. To this family one associates the family of polarized Prym varieties $(A, \Theta) \to T$ with special fiber $(A_0, \Theta_0)$.

The plan of the paper is as follows.

In Section 1 we construct the family of curves $F_r$ in the general case. In Section 2 we describe the family of curves in the Wirtinger double cover case. We also explicitly describe the flat limit $G_0$ of the base $G_t := \tilde{G}_5^1(X_t)$ of the family. This is the transverse union of two smooth isomorphic surfaces. We prove that the total space $G \to T$ of the family of the $G_t$ is smooth.

In Section 3 we describe the total space of the family of theta divisors $\Theta \to T$. The singular locus of $\Theta_0$ is a translate of the smooth genus 11 curve $W_4^1 \subset \text{Pic}^4 C \cong JC$ parametrizing pencils of degree 4. We prove that the total space $\Theta$ has ten ordinary double points corresponding to the five $g_i \in W_4^1$, $i = 1, ..., 5$, such that $h^0(g_i - p - q) > 0$, and their residuals $h_i := |K_C - g_i|$.

In Section 4, we construct a semistable reduction $\tilde{\Theta}$ of the family $\{\Theta_t\}$. The central fiber $\tilde{\Theta}_0$ of the new family has two components $M_1$ and $M_2$, where $M_1$ is a resolution of $\Theta_0$ and $M_2$ is the exceptional divisor. During this process $T$ is replaced by a double cover ramified only at 0 and we also replace the family $G$ by $\tilde{G}$, which is a resolution of the base change of $G$ to this double cover.

In Section 5 we recall the necessary background material about the Clemens-Schmid exact sequence and limit mixed Hodge structures.
In Section 6 we compute the limit mixed Hodge structure induced by the family \( \tilde{\Theta} \) on the cohomology of \( \Theta_t \). The weight filtration is nonzero only in weights 3, 4, 5 with associated graded pieces as follows:

\[
\text{Gr}_3 H^4(\Theta_t) \cong \text{Gr}_5 H^4(\Theta_t) \cong \mathbb{Q}^{12}
\]

and

\[
\text{Gr}_4 H^4(\Theta_t) \cong \mathbb{Q}^{264}.
\]

To extend the family of curves to the central fiber we first assume given a section \( r: T \to \tilde{X}, t \mapsto r_t \) of the family of curves \( \tilde{X} \). Next we replace the families \( F_{r_t} \) by their images in the products \( G_t \times \Theta_t \). The Abel-Jacobi map on the fiber at \( t \) can then be described as the map induced by the cycle \( (\rho_1, \rho_2)_*[F_{r_t}] \in H^6(G_t \times \Theta_t) \):

\[
H^2(G_t) \xrightarrow{\rho_1^*} H^2(G_t \times \Theta_t) \xrightarrow{\cup (\rho_1, \rho_2)_*[F_{r_t}]} H^8(G_t \times \Theta_t) \xrightarrow{\rho_2^*} H^4(\Theta_t).
\]

To compute the limit of these maps at 0, we need a semistable reduction of the fiber product \( \tilde{G} \times_T \tilde{\Theta} \). This is constructed in Section 7. The resulting space \( \mathcal{P} \) is a small resolution of the fiber product \( \tilde{G} \times_T \tilde{\Theta} \).

In Section 8 we show how the computation of the Abel-Jacobi map on the general fiber can be reduced to computing it on (the strata of) the special fiber. We summarize the latter computations in Propositions 8.1-8.4 and show how Theorem 3 follows from them.

Sections 9 and 10 describe the limit families of curves at \( t = 0 \).

In Section 11 we prove Propositions 8.1-8.4. In other words, we compute the image of the Abel-Jacobi map AJ on the graded level with respect to the weight filtration:

\[
(0.1) \quad \text{Gr}_2 H^2(\tilde{G}) \to \text{Gr}_4 H^4(\tilde{\Theta})
\]

and

\[
(0.2) \quad \text{Gr}_1 H^2(\tilde{G}) \to \text{Gr}_3 H^4(\tilde{\Theta})
\]

Finally in the Appendix (Section 12) we gather some technical results needed in the rest of the paper.
Remark 4. (1) For $g \leq 2$, $g! - \frac{1}{g+1}(\binom{2g}{g}) = 0$ so $\mathbb{K} = 0$. For $g = 3$, the lattice $\mathbb{K}$ has rank 1 and level 0, i.e., it is generated by a Hodge class. The abelian variety $(A, \Theta) = (JC, \Theta_C)$ is the Jacobian of a curve of genus 3. The theta divisor is isomorphic to the second symmetric power $C^{(2)}$ of $C$ and $\mathbb{K}$ is generated by the class $\theta - 2\eta$ where $\eta$ is the cohomology class of the image of $C$ in $C^{(2)}$ via addition of a point $p$ of $C$:

\[ C \hookrightarrow C^{(2)} \]
\[ t \mapsto t + p. \]

(2) The primitive cohomology, in the sense of Lefschetz, is the subspace

\[ H^4_{pr}(\Theta, \mathbb{Q}) := \text{Ker} \left( H^4(\Theta, \mathbb{Q}) \xrightarrow{\cup \theta} H^6(\Theta, \mathbb{Q}) \right). \]

The relation between the primitive and the primal cohomology is

\[ H^4_{pr}(\Theta, \mathbb{Q}) = \mathbb{K}_\mathbb{Q} \oplus j^* H^4_{pr}(A, \mathbb{Q}), \]

where

\[ H^4_{pr}(A, \mathbb{Q}) := \text{Ker} \left( H^4(A, \mathbb{Q}) \xrightarrow{\cup \theta^2} H^8(A, \mathbb{Q}) \right). \]

Note that in the case of hypersurfaces in projective space the primal and primitive cohomology coincide.

**Notation and Conventions**

(1) Unless otherwise specified, all singular cohomology groups are with $\mathbb{Q}$-coefficients.

(2) For a smooth curve $C$ of genus $g$ and integer $k > 0$, choose a symplectic basis

\[ \xi_i \in H^1(C, \mathbb{Z}) \cong H^1(\text{Pic}^k C, \mathbb{Z}), \quad i = 1, ..., 2g. \]

Put $\xi'_i := \xi_{i+g}$, $\sigma_i = \xi_i \xi'_i$ for $i = 1, ..., g$ and denote $\theta = \sum_{i=1}^g \sigma_i$ the class of the theta divisor in $\text{Pic}^k C$. We also denote $\xi_i$, $\sigma_i$ and $\theta$ the pull backs to the $k$-th symmetric power $C^{(k)}$ under the natural map

\[ C^{(k)} \to \text{Pic}^k C. \]

Finally, denote $\eta \in H^2(C^{(k)}, \mathbb{Z})$ the class of the cycle $p + C^{(k-1)} \subset C^{(k)}$ for some $p \in C$.

(3) We will interchangeably refer to elements of $\text{Pic}^k C$ as invertible sheaves or complete linear systems. We use $\equiv$ to denote linear equivalence between divisors and $D_1 \leq D_2$ means $D_2 - D_1$.
is an effective divisor. As usual, we denote $W_r^d \subset \text{Pic}^d$ the scheme parametrizing complete linear systems of degree $d$ and dimension $r$. By a $g^r_d$ we will mean a linear system of degree $d$ and dimension $r$.

(4) For products of symmetric powers of $C$, we denote $\omega_k := pr_k^*\omega \in H^\bullet(C(n_1) \times ... \times C(n_k) \times ...)$, where $\omega \in H^\bullet(C(n_k))$ and $pr_k$ is the $k$-th projection.

(5) Via translation by an invertible sheaf of degree $g - 1$, we identify $JC = \text{Pic}^0 C$ with $\text{Pic}^{g-1} C$ so that $\Theta_C$ is identified with Riemann’s theta divisor $W_{g-1}^0 \subset \text{Pic}^{g-1} C$.

(6) As usual, $\omega_C$ will denote the dualizing sheaf of $C$ and $K_C$ an arbitrary canonical divisor on $C$.

1. The family of curves in $\Theta$: the general case

Let $X$ be a smooth curve of genus 6 with an étale double cover $\tilde{X}$ of genus 11. For a pencil $M$ of degree 5 on $X$ consider the curve $B_M$ defined by the pull-back diagram

$$
\begin{array}{ccc}
B_M & \subset & \tilde{X}^{(5)} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 = |M| & \subset & X^{(5)}.
\end{array}
$$

By [B2, p. 360] the curve $B_M$ has two isomorphic connected components, say $B_M^1$ and $B_M^2$. Put $M' = |K_X - M|$. Then for any $D \in B_M \subset \tilde{X}^{(5)}$ and any $D' \in B_{M'} \subset \tilde{X}^{(5)}$, the push-forward to $X$ of $D + D'$ is a canonical divisor on $X$. Hence the image of

$$
B_M \times B_{M'} \rightarrow \text{Pic}^{10} \tilde{X} \\
(D, D') \mapsto \mathcal{O}_{\tilde{X}}(D + D')
$$

is contained in the preimage of $\omega_X$ by the Norm map $N:\text{Pic}^{10} \tilde{X} \rightarrow \text{Pic}^{10} X$. This preimage has two connected components, say $A_1$ and $A_2$, each a principal homogeneous space under the Prym variety $(A, \Theta)$ of the cover $\tilde{X} \rightarrow X$ and parametrizing divisors whose spaces of global sections are even, respectively odd, dimensional. If we have labeled the connected components of $B_M$ and $B_{M'}$ in such a way that $B_M^1 \times B_{M'}^1$ maps into $A_1$, then $B_M^2 \times B_{M'}^2$ also maps into $A_1$ while $B_M^1 \times B_{M'}^2$ and $B_M^2 \times B_{M'}^1$ map into $A_2$.

In order to obtain a family of curves in the theta divisor $\Theta = \Theta_{\tilde{X} \rightarrow X} = \frac{1}{2} \Theta_{\tilde{X}}|_{A_1}$ of the Prym variety $A$, we globalize the above construction.
The scheme $W^1_5(X)$ parametrizing complete linear systems of degree 5 and dimension at least 1 on $X$ has a determinantal structure which is smooth for $X$ sufficiently general. Let $G^1_5(X)$ denote the scheme over $W^1_5(X)$ parametrizing pencils of degree 5. Note that $W^1_5(X)$ is a surface unless $X$ is hyperelliptic.

The universal family $P^1_5$ of divisors of the elements of $G^1_5$ is a $\mathbb{P}^1$ bundle over $G^1_5$ with natural maps

\[
P^1_5 \longrightarrow X^{(5)}
\]
\[
\downarrow
\]
\[
G^1_5
\]
\[
\downarrow
\]
\[
W^1_5
\]

whose pull-back via $\tilde{X} \to X$ gives us the family of the curves $B_M$ as $M$ varies:

\[
B \longrightarrow \tilde{X}^{(5)}
\]
\[
\downarrow \quad \downarrow
\]
\[
P^1_5 \longrightarrow X^{(5)}
\]
\[
\downarrow
\]
\[
G^1_5
\]

The parameter space of the connected components of the curves $B_M$ is an étale double cover $\tilde{G}^1_5$ of $G^1_5$.

The family of curves in the theta divisor of the Prym variety will be constructed as follows. Assuming that $X$ is not a plane quintic, the natural map $G^1_5 \to W^1_5$ is an isomorphism. We have the involution $\iota: M \mapsto M' := |K_X - M|$ on $W^1_5$ and hence also on $G^1_5$. First define a family of surfaces $'F$ over $G^1_5$ as the fiber product

\[
'F \longrightarrow B
\]
\[
\downarrow \quad \downarrow^{\iota}\rho
\]
\[
B \longrightarrow G^1_5.
\]

As noted above, the image of $'F$ in Pic$^{10} \tilde{X}$ maps into Nm$^{-1}(\omega_X) \subset$ Pic$^{10} \tilde{X}$ which also shows that $'F$ has two connected components. One component, denoted $'F_1$, maps into $A_1$ and the other, denoted $'F_2$, maps into $A_2$. The fiber of $'F_1$ over a point $|M| \in G^1_5$ has two connected components $B^1_M \times B^1_{M'}$ and $B^2_M \times B^2_{M'}$. 
Therefore, if we make the base change

\[
\begin{array}{c}
''F_1 \\ \downarrow \\
\tilde{G}_5^1 
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
G_5^1,
\end{array}
\]

''\(F_1\) splits into two connected components (both isomorphic to '\(F_1\) over \(\mathbb{C}\) but their maps to \(\tilde{G}_5^1\) differ by the involution of \(\tilde{G}_5^1\)). We denote \(F\) the component which has fiber \(B_M^1 \times B_M^1\) over the point parametrizing \(B_M^1\).

Finally, we think of \(F\) as a correspondence

\[
F \quad \longleftrightarrow \quad \tilde{G}_5^1 \times \tilde{X}(5) \times \tilde{X}(5)
\]

and define our family of curves \(F_r\) by intersecting \(F\) with the pull back of the divisor \(r + \tilde{X}(4)\) in the first factor \(\tilde{X}(5)\) for a general point \(r \in \tilde{X}\). The variety \(F_r'\) is the image of \(F_r\) in \(\tilde{G}_5^1 \times \Theta\):

\[
\begin{array}{c}
F_r \quad \stackrel{(\rho_1,\rho_2)}{\longrightarrow} \\
\downarrow \\
\tilde{G}_5^1
\end{array} \quad \begin{array}{c}
F_r' \subset \tilde{G}_5^1 \times \Theta \\
F_r' \subset \tilde{G}_5^1 \times \Theta \\
F_r' \subset \tilde{G}_5^1 \times \Theta \\
F_r' \subset \tilde{G}_5^1 \times \Theta \\
F_r' \subset \tilde{G}_5^1 \times \Theta
\end{array}
\]

**Remark 1.1.** It is easy to check that \(F_r\) maps generically injectively to \(\tilde{G}_5^1 \times \Theta\). So the push-forward of the cycle class \([F_r]\) is the cycle class \([F_r']\).

---

2. The family of curves in \(\Theta\): the degeneration to a Wirtinger cover

Let \(\tilde{X} \to X\) be the family of étale double covers over \(T\) specializing to the Wirtinger cover \(\tilde{C}_{pq} \to C_{pq}\) at \(0 \in T\) as explained in the introduction.

Also assume that \(X\) and \(\tilde{X}\) are smooth.

Consider the smooth one-parameter family

\[
\begin{array}{c}
J^5C_{pq} \\
\downarrow \\
0 \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array} \quad \begin{array}{c}
J^5 \\
\downarrow \\
\downarrow \\
\end{array} \quad \begin{array}{c}
0 \quad \in \\
T
\end{array}
\]
obtained as a compactification of the relative degree 5 Picard scheme of $X$. The fiber of $J^5 \to T$ is $\text{Pic}^5 X_t$ for $t \neq 0$ and the fiber at $t = 0$ is the usual compactification $J^5 C_{pq}$ of $\text{Pic}^5 C_{pq}$ obtained as follows.

2.1. The compactified Jacobian of $C_{pq}$. Let $\mathbb{P} \text{Pic}^5 C_{pq}$ be the unique projective line bundle over $\text{Pic}^5 C$ containing the $G_m$-bundle $\text{Pic}^5 C_{pq} \to \text{Pic}^5 C$. Then $\mathbb{P} \text{Pic}^5 C_{pq} \setminus \text{Pic}^5 C_{pq}$ is the union of the zero section $\text{Pic}_0^5 \cong \text{Pic}^5 C$ and the infinity section $\text{Pic}^5_{\infty} \cong \text{Pic}^5 C$ of $\mathbb{P} \text{Pic}^5 C_{pq} \to \text{Pic}^5 C$. The compactification $J^5 C_{pq}$ is obtained from $\mathbb{P} \text{Pic}^5 C_{pq}$ by identifying $x \in \text{Pic}^5 C = \text{Pic}_0^5$ with $x \otimes \mathcal{O}_C(p - q) \in \text{Pic}^5 C = \text{Pic}^5_{\infty}$. The points of $J^5 C_{pq} \setminus \text{Pic}^5 C_{pq}$ are the push-forwards $\nu_* N$ where $\nu : C \to C_{pq}$ is the normalization map and $N \in \text{Pic}^4 C$.

2.2. The support of $W^1_5(C_{pq})$ and of its compactification $\overline{W}^1_5(C_{pq})$. Let $\overline{W}^1_5(C_{pq})$ be the subvariety of $J^5 C_{pq}$ parametrizing torsion-free rank 1 sheaves $M$ of degree 5 such that $h^0(M) \geq 2$. Let $W_{pq} \subset W^1_5(C) \subset \text{Pic}^5 C$ be the surface consisting of those $L$ such that $h^0(L - p - q) > 0$, and let $X_p$ and $X_q$ be the two curves $p + W^1_4(C)$ and $q + W^1_4(C)$ in $W_{pq}$. Pull-back via the normalization map gives a morphism

$$ \nu^* : W^1_5(C_{pq}) \longrightarrow W_{pq} $$

whose image is $W_{pq} \setminus X_p \cup X_q$. We have

**Lemma 2.1.** The morphism $\nu^* : W^1_5(C_{pq}) \to W_{pq}$ is injective. Its inverse extends to a birational morphism

$$(\nu^*)^{-1} : W_{pq} \longrightarrow \overline{W}^1_5(C_{pq})$$

that is bijective on $W_{pq} \setminus X_p \cup X_q$ and sends $p + g^1_4$ and $q + g^1_4$ to $\nu_* g^1_4$.

The involution $\iota$ extends to $W_{pq}$ and sends $L$ to $|K_C + p + q - L|$. It also descends to $\overline{W}^1_5(C_{pq})$ and sends $\nu_* g^1_4$ to $\nu_* (|K_C - g^1_4|)$.

**Proof.** If $M \in \overline{W}^1_5(C_{pq})$ is invertible, then the pull-back $\nu^* M$ is an invertible sheaf of degree 5 on $C$ and we have the usual exact sequence

$$ 0 \longrightarrow M \longrightarrow \nu_* \nu^* M \longrightarrow sk \longrightarrow 0 $$

where $sk$ is a skyscraper sheaf of length 1 supported at the singular point of $C_{pq}$. It follows that if $h^0(M) \geq 2$, then $h^0(\nu^* M) \geq 2$ also. Since $C$ is a general curve of genus 5 and $\nu^* M$ has degree 5, we
have $h^0(\nu^*M) \leq 2$. So the map $H^0(\nu^*M) = H^0(\nu_*\nu^*M) \to H^0(sk)$ obtained from the above sequence is zero. Since this map factors through the evaluation map $H^0(\nu^*M) \to (\nu^*M)_p \oplus (\nu^*M)_q$, a moment of reflection will show that a map $\nu_*\nu^*M \to sk$ that is zero on global sections and has locally free kernel exists if and only if neither $p$ nor $q$ are base points of $|\nu^*M|$ and the unique nonzero section of $\nu^*M$ that vanishes at $p$ also vanishes at $q$.

Conversely, given an invertible sheaf $L$ of degree 5 on $C$ such that neither $p$ nor $q$ are base points of $|L|$ and the unique nonzero section of $L$ vanishing at $p$ also vanishes at $q$, one sees immediately that there is a unique quotient map

\[ \nu_*L \to sk \]

onto a skyscraper sheaf of rank 1 supported at the singular point of $C_{pq}$ such that the resulting map on global sections

\[ H^0(\nu_*L) \to H^0(sk) \]

is zero. The kernel of such a map is also immediately seen to be an invertible sheaf of degree 5 on $C_{pq}$. Thus $W^1_5(C_{pq})$ maps injectively into $W_{pq}$ under $\nu^*$.

If $M$ is not locally free, then it is the direct image of a $g_1^4 \in W^1_4(C)$. We have two exact sequences

\[ 0 \to \nu_*g_4^1 \to \nu_*(p + g_4^1) \to sk \to 0, \]
\[ 0 \to \nu_*g_4^1 \to \nu_*(q + g_4^1) \to sk \to 0, \]

that give us two representations of $M$ as the kernel of a surjective map from the pushforward of an invertible sheaf to $sk$. Thus $\nu^*$ maps $p + g_4^1$ and $q + g_4^1$ to $\nu_*g_4^1$. The statements about $\iota$ are immediate.

Note that $W_{pq} \subset \text{Pic}^5 C$ naturally embeds in $C^{(3)}$ via two different maps: $q_1 : L \mapsto \Gamma_3 := |K_C - L|$ and $q_2 : L \mapsto \Gamma'_3 := |L - p - q|$. We have

**Proposition 2.2.** The surface $W_{pq}$ is smooth for $C$, $p$ and $q$ general.

**Proof.** For $L \in W_{pq}$, via the two embeddings of $W_{pq}$ in $C^{(3)}$, the tangent space to $W_{pq}$ at $L$ is contained in the tangent spaces to $C^{(3)}$ at $\Gamma_3$ and $\Gamma'_3$. Embedding $C^{(3)}$ in $\text{Pic}^0 C$ via subtraction of a fixed divisor of degree 3, the projectivizations of these two tangent spaces can be identified (after a translation) with the respective spans $\langle \Gamma_3 \rangle$ and $\langle \Gamma'_3 \rangle$ of $\Gamma_3$ and $\Gamma'_3$ in the canonical space $|K_C|^* \cong \mathbb{P}T_0 \text{Pic}^0 C$. To
prove $W_{pq}$ is smooth at $L$, i.e. $T_L W_{pq}$ has dimension 2, it suffices to show that $\langle \Gamma_3 \rangle \neq \langle \Gamma'_3 \rangle$, since the intersection $\langle \Gamma_3 \rangle \cap \langle \Gamma'_3 \rangle$ is then a projective line which contains $\mathbb{P} T_L W_{pq}$.

Using Riemann Roch and Serre Duality it is immediately seen that a divisor of degree $\geq 5$ on $C$ cannot span a space of dimension $\leq 2$ in $|K_C|^*$. So, if $\langle \Gamma_3 \rangle = \langle \Gamma'_3 \rangle$, then $\Gamma_3$ and $\Gamma'_3$ have a divisor of degree at least 2 in common: $\Gamma_3 = \Gamma_2 + t$ and $\Gamma'_3 = \Gamma_2 + t'$ for some $\Gamma_2 \in C^{(2)}$ and $t, t' \in C$. Note that by our assumptions $\Gamma_3 + \Gamma'_3 + p + q \in |K_C|$ is a canonical divisor.

If $t = t'$, then the span $\langle \Gamma_3 + \Gamma'_3 + p + q \rangle$ is a hyperplane in $|K_C|^*$ which is tangent to the canonical image of $C$ at three distinct points or has even higher tangency to the canonical curve. Such hyperplanes form a family of dimension 1 for $C$ general, hence choosing $p$ and $q$ sufficiently general, this can be avoided.

If $t \neq t'$, then the span $\langle \Gamma_2 + t + t' \rangle$ is a plane, and, by Riemann Roch and Serre Duality, $|\Gamma_2 + t + t'| \in W'_4(C)$, hence $|K_C - \Gamma_2 - t - t'| \in W'_4(C)$. However $|K_C - \Gamma_2 - t - t'| = |p + q + \Gamma_2|$. The divisors of $g^1_2 := |\Gamma_2 + t + t'|$ and $h^1_2 := |K_C - \Gamma_2 - t - t'| = |p + q + \Gamma_2|$ are cut on $C$ by the two rulings of a quadric of rank 4 (since $C$ is general, it is not contained in any quadrics of rank 3). Since $\Gamma_2$ appears in both $g^1_2$ and $h^1_2$, the line $\langle \Gamma_2 \rangle$ contains the singular point of this quadric of rank 4. There is a one-parameter family of such secants to $C$ and for each such secant $\langle \Gamma_2 \rangle$, there are exactly 5 (counted with multiplicities) divisors $p + q$ such that $h^0(\Gamma_2 + p + q) \geq 2$. Therefore there is a one-parameter family of divisors $p + q$ such that $h^0(\Gamma_2 + p + q) \geq 2$ for some $\Gamma_2$ such that $\langle \Gamma_2 \rangle$ contains the singular point of some quadric of rank 4 containing $C$. Taking $p + q$ outside this one-parameter family this case is also eliminated.

The computation of the Hilbert polynomial of $\overline{W}^1_5(C_{pq})$ in Lemma 12.5 shows that $\overline{W}^1_5(C_{pq})$, with its reduced scheme structure, is the flat limit of $W^1_5(X_t)$. Therefore, if $W^1_5 \subset J^5$ is the family of sheaves with at least two independent global sections on each fiber $X_t$, then $W^1_5 \to T$ is flat. Moreover, we have

**Proposition 2.3.** The total space $W^1_5$ is smooth.

**Proof.** This is clear at any invertible sheaf $M \in W^1_5(X_t)$ or $M \in W^1_5(C_{pq})$. Suppose therefore that $M \in \overline{W}^1_5(C_{pq}) \setminus W^1_5(C_{pq})$. We need to prove that the Zariski tangent space to $W^1_5$ at $M$ is 3-dimensional, i.e., it is equal to the Zariski tangent space to $\overline{W}^1_5(C_{pq})$ at $M$. By Lemma 12.5, the morphism $W^1_5 \to T$ is flat and its scheme-theoretical fiber at 0 is $\overline{W}^1_5(C_{pq})$ with its reduced structure. So the tangent space
to $\overline{W}_5^1(C_{pq})$ at $M$ is the kernel of the differential of the map $W_5^1 \to T$ at $M$ and it is equal to the tangent space to $W_5^1$ if and only this differential is zero. Now the fact that this differential is zero follows from the fact that the differential of the map $\mathcal{J}^5 \to T$ is zero at $M$ because $\mathcal{J}^5$ is smooth and $\mathcal{J}^5 \to T$ is flat. \hfill $\Box$

2.3. The surface $\tilde{G}_5^1$ in the Wirtinger double cover case. Let $P_5^1$ be the universal $\mathbb{P}^1$ bundle over $\overline{W}_5^1(C_{pq})$ whose fiber over $M \in \overline{W}_5^1(C_{pq})$ is $\mathbb{P} H^0(C_{pq}, M)$. The following fibered diagram is the limit of the analogous diagram in the smooth case:

\begin{equation}
\begin{array}{ccc}
B & \longrightarrow & \tilde{C}_{pq}^{(5)} \\
\downarrow & & \downarrow \\
P_5^1 & \longrightarrow & C_{pq}^{(5)} \\
\downarrow & & \downarrow \\
\overline{W}_5^1(C_{pq}). \\
\end{array}
\end{equation}

The horizontal map in the second row is as follows. If $M \in W_5^1(C_{pq})$ and $0 \neq s \in H^0(C_{pq}, M)$, then the image of $s$ in $C_{pq}^{(5)}$ is $\text{div}(s) = \nu_*(\text{div}(\nu^*s))$. If $M \in \overline{W}_5^1(C_{pq}) \setminus W_5^1(C_{pq})$, then $M = \nu_*g_4^1$ and the image of $s \in H^0(C_{pq}, M) = H^0(C, g_4^1)$ is $\nu_*(\text{div}(s) + p) = \nu_*(\text{div}(s) + q) \in C_{pq}^{(5)}$.

**Lemma 2.4.** The surface $\tilde{G}_5^1(C_{pq})$ is the union of two copies of $W_{pq}$, denoted $W_1$ and $W_2$, where $X_{kp} = W_4^1(C) + p \subset W_k$ is identified with $X_{3-k,q} = W_4^1(C) + q \subset W_{3-k}$ for $k = 1, 2$.

**Proof.** First note that $\tilde{C}_{pq}^{(5)}$ has the following irreducible components:

$$
C_1^{(5)} \cup (C_1^{(4)} \times C_2) \cup (C_1^{(3)} \times C_2^{(2)}) \cup (C_1^{(2)} \times C_2^{(3)}) \cup (C_1 \times C_2^{(4)}) \cup C_2^{(5)}.
$$

Accordingly, for a given $M \in W_5^1(C_{pq})$, the two connected components $B_M^1$ and $B_M^2$ of the curve $B_M$ embed, respectively, into

$$
(C_1^{(4)} \times C_2) \cup (C_1^{(2)} \times C_2^{(3)}) \cup C_2^{(5)}
$$

and

$$
C_1^{(5)} \cup (C_1^{(3)} \times C_2^{(2)}) \cup (C_1 \times C_2^{(4)}).
$$
This first shows that \( \tilde{G}_5^1(C_{pq}) \) has two irreducible components and that the double cover \( \tilde{G}_5^1(C_{pq}) \to G_5^1(C_{pq}) \) is split away from \( \nu_*W_4^1(C) \). The claim of the lemma over \( \nu_*W_4^1(C) \) follows from the fact that, for a fixed \( M \), the components \( B^1_M \) and \( B^2_M \) are exchanged by the involution induced by that exchanging \( C_1 \) and \( C_2 \) in \( \tilde{C}_{pq} \). □

An immediate consequence of Lemma 2.4 is

**Corollary 2.5.** For a general double cover \( \tilde{X} \to X \), the double cover \( \tilde{G}_5^1 \to G_5^1 \) is nontrivial.

2.4. **The pair** \((A_0, \Theta_0)\). Denote \((A, \Theta) \to T\) the family of principally polarized Prym varieties of the above family of double covers of curves.

Following Beauville [B1, pp. 175-176], the Prym variety \( A_0 \) associated to the Wirtinger cover is given by the following diagram

\[
\begin{array}{ccc}
A_0 & \cong & J(C) \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\mathbb{C}^*} & J(\tilde{C}_{pq}) \\
\downarrow & \cong & \downarrow \nu^* \leftarrow \downarrow Nm \\
1 & \xrightarrow{\mathbb{C}^*} & J(C_{pq}) \xrightarrow{\mathbb{C}^*} J(C) \to 0 \\
\end{array}
\]

where \( J(\tilde{C}_{pq}) \) and \( J(C_{pq}) \) are the generalized (noncompact) Jacobians, \( Nm \) is the Norm map, and \( \nu: \tilde{C} = C_1 \amalg C_2 \to \tilde{C}_{pq} \) is the normalization map.

To obtain a canonical theta divisor in \( A_0 \), we fix a bidegree \((d_1, d_2)\) such that \( d_1 + d_2 = 10 \) and the following holds.

◊ There exists a line bundle \( N \) on \( \tilde{C}_{pq} \) of multidegree \((d_1, d_2)\), so that \( h^0(N) = 0 \).

By [B1, p. 153], the only bidegrees satisfying ◊ are \((6, 4)\), \((4, 6)\) and \((5, 5)\). If there is a one-parameter family of line bundles \( N \) on \( \tilde{X} \) with \( N_0 = N|_{\tilde{C}_{pq}} \) of bidegree \((d_1, d_2)\), we can modify \( N \) by twisting with a component of \( \tilde{C}_{pq} \) so that \( N_0 \) has either bidegree \((6, 4)\) or \((5, 5)\).

**Proposition 2.6.** We have a canonical identification \((A_0, \Theta_0) \cong (\text{Pic}^4 C, \Theta_C)\).

**Proof.** Denote \( \text{Pic}^{6,4} (\tilde{C}_{pq}) \) the principal homogeneous space over \( J(\tilde{C}_{pq}) \) parametrizing line bundles of bidegree \((6, 4)\) on \( \tilde{C}_{pq} \). We identify \( A_0 \) with the subvariety of \( \text{Pic}^{6,4} (\tilde{C}_{pq}) \) consisting of line bundles \( N \)
such that $\text{Nm}(N) = \omega_{C_{pq}}$. Note that $\text{Nm}^{-1}(\omega_{C_{pq}})$ only has one connected component by the diagram above ($\text{Pic}^{6,4}(C_{pq})$ is not compact). We then define the theta divisor $\Theta_0$ of $A_0$ as the locus of line bundles $N \in A_0 \subset \text{Pic}^{6,4}(C_{pq})$ such that $h^0(N) > 0$.

We have an isomorphism $A_0 \cong \text{Pic}^4$ which sends $N$ to $N|_{C_2}$. This is an isomorphism because $N$ is determined by $\nu^*N$ by the diagram above and $N|_{C_2}$ determines $\nu^*N$ since $\text{Nm}(\nu^*N) = N|_{C_1} \otimes N|_{C_2} \cong \omega_{C}(p + q)$. We claim that if $h^0(\tilde{C}_{pq}, N) \neq 0$, then $h^0(C_2, N|_{C_2}) \neq 0$. If not, let $0 \neq s \in H^0(\tilde{C}_{pq}, N)$ be such that $s|_{C_2} = 0$: then $s|_{C_1}$ vanishes at $p$ and $q$. Thus $0 \neq s|_{C_1} \in H^0(C_1, N|_{C_1}(-p - q))$. However, since $N|_{C_1}(-p - q) \otimes N|_{C_2} \cong \omega_{C}$, we have $h^0(C_1, N|_{C_1}(-p - q)) = h^1(C_2, N|_{C_2}) = h^0(C_2, N|_{C_2}) = 0$, a contradiction. Thus the canonical identification sends $\Theta_0$ isomorphically to $\Theta_C$. 

\[ \square \]

2.5. The family of curves in the limit. Denote the total space of the family of the double covers $G_t := \tilde{G}^4_5(X_t)$ by $G$. The space $G$ is an étale double cover of $\mathcal{W}^4_5$ and therefore smooth by Proposition 2.3. The central fiber $G_0$ of $G$ is described in Lemma 2.4.

Assume we are also given a section $r: t \mapsto r_t$ of $\tilde{\mathcal{X}} \to T$. Let $F$ (resp. $F_r$) be the closure in $\mathcal{G} \times_T \tilde{\mathcal{X}}^{(5)} \times_T \tilde{\mathcal{X}}^{(5)}$ of the family of fourfolds (resp. threefolds) $F_t$ (resp. $F_{r_t}$) constructed in Section 1 for the fibers over $t \neq 0$.

By construction, the central fiber $F_0$ of $F$ fibers over $G_0 = W_1 \cup W_2$. The fiber over $M \in W_k$ is the surface $B^k_M \times B^k_{M'}$, where $B^1_M$ and $B^1_{M'}$ live in

$$ (C_1^{(4)} \times C_2) \cup (C_1^{(2)} \times C_2^{(3)}) \cup C_2^{(5)} $$

and $B^2_M$ and $B^2_{M'}$ live in

$$ C_1^{(5)} \cup (C_1^{(3)} \times C_2^{(2)}) \cup (C_1 \times C_2^{(4)}) $$

3. The degeneration of theta divisors

Let $(\mathcal{A}, \Theta) = \{(A_t, \Theta_t)\}_{t \in T}$ be a 1-parameter family of principally polarized Abelian varieties of dimension 5 with smooth total space $\mathcal{A}$. Assume that for $t \neq 0$, the fiber $\Theta_t$ of $\Theta$ is smooth and that the fiber of $(\mathcal{A}, \Theta)$ at 0 is the polarized Jacobian $(A_0 = JC, \Theta_0 = \Theta_C)$ of a smooth curve $C$ of genus 5.

We will obtain information about the cohomology of $\Theta_t$ from the cohomology of $\Theta_0$ using limit mixed Hodge structures. We shall see below that the total space $\Theta$ is singular. We first need to modify the
family \((A, \Theta)\) using base change and blow-ups to obtain a family of theta divisors with smooth total space whose central fiber is a divisor with simple normal crossings.

3.1. The singularities of \(\Theta\). Denote by \(H\) the Siegel upper half space and consider the Riemann theta function \(\theta(z, \tau)\) on \(\mathbb{C}^5 \times H\). After possibly replacing \(T\) with a finite cover we can assume that there is a map \(\tau: T \to H\) such that the family \((A, \Theta)\) is the inverse image, via \(\tau\), of the universal family of polarized abelian varieties over \(H\). In particular, we can assume that the family \(\Theta\) is defined by
\[
\{(z, t) \in \mathbb{C}^5 \times T: \theta(z, \tau(t)) = 0\}
\]
(modulo the action of the lattice of \(A_t\)). Denote \(F(z, t) := \theta(z, \tau(t))\).

Since \(\Theta_t\) is smooth for \(t \neq 0\), the total space \(\Theta\) is smooth away from the special fiber \(\Theta_0\). In the case we are interested in, the singularities of the special fiber \(\Theta_0\) are all double points and hence the singularities of the total family \(\Theta\) are at worst double points.

We compute the singularities of \(\Theta\) locally, using the heat equation: \(\partial \theta/\partial \tau_{ij} = \partial^2 \theta/\partial z_i \partial z_j\) modulo multiplication by a constant. Here the \(\tau_{ij}\) denote coordinates on \(H\) and the \(z_i\) coordinates on an abelian variety \(A\). We write \(\tau(t) = (\tau_{ij}(t))_{1 \leq i,j \leq 5}\) and let \(\dot{\tau}_{ij}(t)\) denote the derivative of \(\tau_{ij}(t)\) with respect to \(t\). We also put \(\dot{\tau} := \dot{\tau}(0)\) and \(\dot{\tau}_{ij} := \dot{\tau}_{ij}(0)\).

Throughout the rest of this section, we use the notation \(\partial_i := \partial/\partial z_i\), \(\partial_{ij} := \partial^2/\partial z_i \partial z_j\) and \(\partial_t := \partial^2/\partial z_i \partial t\), etc. We also use the summation convention: when an index appears twice in a single term, it implies summation of that term as the index goes from 1 to 5.

**Proposition 3.1.** A point \((z, 0)\) is a singular point of \(\Theta\) exactly when \((z, 0)\) is a singular point of \(\Theta_0\) such that the equation \(q_z \in S^2 H^1(O_{A_0})^*\) of the quadric tangent cone to \(\Theta_0\) at \(z\) vanishes on the infinitesimal deformation direction \(\dot{\tau} = (\dot{\tau}_{ij})_{1 \leq i,j \leq 5} \in S^2 H^1(O_{A_0})\) under duality.

**Proof.** By the heat equation, at a point \((z, t) \in \Theta\) the equation of the tangent hyperplane to \(\Theta\) in \(A\) is the pullback from the Siegel space of the equation

\[
Z_i \partial_i \theta + T_{ij} \partial_{ij} \theta = 0,
\]

where the \(Z_i\) are the coordinates on the tangent space to a fiber \(A_t\) and the \(T_{ij}\) coordinates on the tangent space to \(H\) at \(\tau(t)\).

This gives the equation

\[
\partial_t F(z, t) Z_i + (\dot{\tau}_{ij}(t) \partial_{ij} F(z, t)) \Omega = 0,
\]
where $\Omega$ is the coordinate on the tangent space to $T$ at $t$.

So the point $(z,0)$ is singular on $\Theta$ if and only if it is singular on $\Theta_0$ and

$$\hat{\tau}_{ij}(t) \partial_{ij} F(z,0) = 0.$$ 

Since

$$q_z = Z_i Z_j \partial_{ij} F(z,0),$$

the proposition follows. \hfill \Box

The partial derivatives of $F$ are:

$$\partial_t F(z,t) = \partial_{ij} \theta(z,\tau(t)) \hat{\tau}_{ij}(t) = \partial_{ij} F(z,t) \hat{\tau}_{ij}(t),$$

$$\partial_{it} F(z,t) = \partial_i (\partial_t F(z,t)) = \partial_{ijk} F(z,t) \hat{\tau}_{jk}(t),$$

$$\partial_{tt} F(z,t) = \partial_{ijkl} F(z,t) \hat{\tau}_{ij}(t) \hat{\tau}_{kl}(t) + \partial_{ij} \partial_{kl} F(z,t) \hat{\tau}_{ij}(t).$$

3.2. The case $A_0 \simeq J(C) \simeq \text{Pic}^4 C$. In this case the theta divisor $\Theta_0 = W_4^0(C)$ of the special fiber is smooth outside the curve $W_4^1 := W_4^1(C)$ and $W_4^1$ is an ordinary double curve on it. Therefore we have

$$F(p,0) = 0, \quad \forall p \in W_4^1,$$

$$\partial_i F(p,0) = 0, \quad \forall i, \forall p \in W_4^1,$$

$$\text{rank} \left( \partial_{ij} F(p,0) \right)_{1 \leq i,j \leq 5} = 4, \quad \forall p \in W_4^1.$$

**Theorem 3.2.** For $\tau$ sufficiently general, the singularities of $\Theta$ consist of ten ordinary double points.

In the case where $(A,\Theta)$ is the family of Prym varieties of a family of double covers $(\tilde{X},X)$ as in 2.4, the ten distinct singular points $g_1,\ldots,g_5,h_1,\ldots,h_5$ of $\Theta$ are the $g_i$’s cut on $C$ by quadrics of rank 4 containing $C$ and its secant $\langle p+q \rangle$. In other words, $h^0(g_i-p-q) > 0$ and $h_i = |K-g_i|$ up to relabeling.

**Proof.** We use the calculations in Section 3.1. The annihilator of the deformation direction

$$\hat{\tau} = (\hat{\tau}_{ij})_{1 \leq i,j \leq 5} \in S^2 H^1(O_{A_0})$$

is a hyperplane in $S^2 H^0(\omega_C)$ which, for $\tau$ sufficiently general, gives a hyperplane in the space $I_2(C)$ of quadrics containing the canonical image of $C$ and hence a line $l$ in $\mathbb{P}I_2(C) \cong \mathbb{P}^2$. The quadrics of rank 4 containing the canonical model of $C$ are the elements of $Q$, a plane quintic in $\mathbb{P}I_2(C)$. Those
whose equations vanish on $\tau$ are the elements of the intersection $l \cap Q$ which, for $\tau$ sufficiently general, consists of 5 distinct points, say $q_1, \ldots, q_5$. There are ten distinct points in the singular locus $W_1'$ of $\Theta_0$ above these five points: the $g_1^1$'s cut on $C$ by the rulings of $q_1, \ldots, q_5$. Hence we see that $\Theta$ has exactly ten distinct singular points.

In the case where our family of abelian varieties is a family of Prym varieties of double covers with central fiber a Wirtinger cover, the deformation direction $\dot{\tau}$ is the image, via the differential of the Prym map, of the infinitesimal deformation direction, say $\eta$, of double covers induced by the family $(\tilde{X}, \mathcal{X})$. As the Prym map sends the locus $\mathcal{W}_6$ of Wirtinger covers in $\mathcal{R}_6$ into the Jacobian locus $J_5$, its differential induces a linear map from the 1-dimensional normal space $N_{C_{pq}}$ to $\mathcal{W}_6$ to the 3-dimensional normal space $N_{JC}$ to $J_5$. It is well known, see e.g. [DS, p. 45], that the normal space to $J_5$ at $JC$ can be canonically identified with the dual $I_2(C)\ast$ to $I_2(C)$. By [DS, p. 86], the image of $\mathbb{P}N_{C_{pq}}$ in $\mathbb{P}I_2(C)\ast = \mathbb{P}N_{JC}$ is the pencil of quadrics containing the canonical image of $C$ together with its secant $\langle p + q \rangle$. This is also the line that we denoted $l$ above. Therefore the points $q_1, \ldots, q_5$ are the quadrics of rank 4 containing $C$ and $\langle p + q \rangle$. The line $\langle p + q \rangle$ is contained in exactly one ruling of $q_i$ and we denote $g_i$ the $g_1^1$ cut on $C$ by that ruling. We then have $h^0(g_i - p - q) > 0$. The second ruling of $q_i$ cuts $h_i := |K_C - g_i|$ on $C$.

It remains to prove that the ten singular points are ordinary double points. The degree 2 term of the Taylor expansion of $F$ near a singular point $(z, t)$ is (using the heat equation up to a scalar):

$$Z_i Z_j \partial_{ij} F + (Z_i \dot{\tau}_{jk} \partial_{ijk} F) \Omega + (\dot{\tau}_{ij} \dot{\tau}_{kl} \partial_{ijkl} F + \ddot{\tau}_{ij} \partial_{ij} F) \Omega^2.$$ 

The first part of the above is the equation of the quadric $q_z$ which has rank 4. In a basis adapted to $q_z$ we have the matrix of second partials of $F$:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \dot{\tau}_{jk} \partial_{1jk} F \\
1 & 0 & 0 & 0 & 0 & \dot{\tau}_{jk} \partial_{2jk} F \\
0 & 0 & 0 & 1 & 0 & \dot{\tau}_{jk} \partial_{3jk} F \\
0 & 0 & 1 & 0 & 0 & \dot{\tau}_{jk} \partial_{4jk} F \\
0 & 0 & 0 & 0 & 0 & \dot{\tau}_{jk} \partial_{5jk} F \\
\dot{\tau}_{jk} \partial_{ijkl} F & \dot{\tau}_{jk} \partial_{2jk} F & \dot{\tau}_{jk} \partial_{3jk} F & \dot{\tau}_{jk} \partial_{4jk} F & \dot{\tau}_{jk} \partial_{5jk} F & \ddot{\tau}_{ij} \partial_{ijkl} F + \dddot{\tau}_{ij} \partial_{ij} F
\end{pmatrix}
$$
So we need to see that this matrix has rank 6 at the points of $W_1^1$. In other words, for $\tau$ sufficiently general the coefficient $\dot{\tau}_{ij} \partial_{\dot{\tau}_{ij}} F(z, 0)$ is not zero. Taking $\dot{\tau}_{ij} = \lambda_i \lambda_j$ such that the point $(\lambda_i) \in \mathbb{C}^g = H^1(O_{A_0}) = T_0 A_0$ is on the osculating cone to $\Theta_0$ and is otherwise general, this means that the vertex of the quadric $q_z$ is not contained in the tangent space to the osculating cone to $\Theta_0$ at the point $(\lambda_i)$. For a general choice of the $\lambda_i$ this is a consequence of [KS] page 353. \[\Box\]

4. The semistable reduction of the family of theta divisors

As before denote $(\mathcal{A}, \Theta) \to T$ the family of principally polarized Prym varieties associated to the étale cover $\tilde{\mathcal{X}} \to \mathcal{X}$. The central fiber is the Jacobian $A_0 \cong \text{Pic} C$ of a general curve of genus 5. By Theorem 3.2, the total space $\Theta$ has ten ordinary double points on $W_1^1$: $g_1, \ldots, g_5$, which satisfy $h^0(g_i - p - q) > 0$, and $h_i := |K_C - g_i|$. We will construct a semistable reduction of $\Theta$ and, in Section 6, use the Clemens-Schmid exact sequence to compute the cohomology of $\Theta_t$.

4.1. The base change and first blow-ups. To construct our semistable reduction, we first make a base change of degree 2, then resolve singularities. Let $T^b \to T$ be a degree 2 cover. After possibly shrinking $T$, we assume that the cover $T^b \to T$ has a unique branch point at $0 \in T$. Pulling back, we obtain the family $\Theta^b \subset \mathcal{A}^b \to T^b$ and $\Theta^b$ is singular along $W_1^1 \subset \Theta_0$. We define $\tilde{\Theta}$ as the blow-up of $\Theta^b$ along its singular locus $W_1^1$. We will see that $\tilde{\Theta}$ is a resolution of $\Theta^b$ whose special fiber $\tilde{\Theta}_0$ is a simple normal crossings divisor ($\mathcal{A}^b \to T^b$ is still a smooth family):

$$
\begin{array}{ccc}
\tilde{\Theta} & \to & \Theta^b \\
\downarrow & & \downarrow \\
T^b & \to & T.
\end{array}
$$

To make our family of curves compatible with the base change, we also need to take the base change of $\mathcal{G}$ to $T^b$ and then blow up along the singular locus of $\mathcal{G}^b$ to obtain a semistable family. The resulting space is $\tilde{\mathcal{G}}$:

$$
\begin{array}{ccc}
\tilde{\mathcal{G}} & \to & \mathcal{G}^b \\
\downarrow & & \downarrow \\
T^b & \to & T.
\end{array}
$$

Recall that, by Lemma 2.4, the fiber of $\mathcal{G}$ at $t = 0$ is the union of two copies of $W_{pq}$, denoted $W_1$ and $W_2$, where $X_{kp} = W_1^1(C) + p \subset W_k$ is identified with $X_{3-k,q} = W_1^1(C) + q \subset W_{3-k}$. After blowing up
along the singular locus $X_{1p} \sqcup X_{1q} \subset G^b$, the central fiber $\tilde{G}_0$ of $\tilde{G}$ has four components: $W_1, W_2, P_1$ and $P_2$, where $P_1$, resp. $P_2$, is a $\mathbb{P}^1$ bundle over $X_{1p} = X_{2q}$, resp. $X_{1q} = X_{2p}$. The four components meet as below:

\[
\begin{array}{ccc}
X_{1p} & X_{2q} & P_i \\
X_{1q} & X_{2p} & P_2 \\
W_1 & W_2 & \\
\end{array}
\]

**Notation 4.1.** From now on we will replace $T$ by $T^b$ and the families $\Theta \subset \mathcal{A} \to T$, $\mathcal{G} \to T$, $\mathcal{F} \to T$ and $\tilde{\mathcal{X}} \to \mathcal{X} \to T$ by their base changes to $T^b$.

### 4.2. The central fiber $\tilde{\Theta}_0$.

**Proposition 4.2.** The total space $\tilde{\Theta}$ is smooth. Its special fiber $\tilde{\Theta}_0$ is a divisor with simple normal crossings with the following two irreducible components.

1. The component $M_1$ which is the blow-up of $\Theta_0$ along $W_4^1$.
2. The component $M_2$ which is the exceptional divisor, i.e. the projectivized normal cone to $\Theta$ along $W_4^1$. Therefore $M_2$ is a fibration over $W_4^1$. At the points $g_i$ and $h_i$ the fibers $Q_3^{\text{sing}}$ of $M_2$ are isomorphic to the singular quadric $Q_3^{\text{sing}}$ of rank 4 in $\mathbb{P}^4$. At all the other points of $W_4^1$, the fibers of $M_2$ are isomorphic to the smooth quadric hypersurface $Q_3 \subset \mathbb{P}^4$.

The intersection $M_{12} = M_1 \cap M_2$ is a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over $W_4^1$. In particular, it is smooth.

**Proof.** It immediately follows from the definition of $\tilde{\Theta}$ that $\tilde{\Theta}_0$ has two components, one of which is the blow-up $M_1$ of $\Theta_0$ and the other the projectivized normal cone $M_2$ of $W_4^1$ in $\Theta$. To prove the assertions about the smoothness of $M_1, M_2$ and $M_{12}$ and the fibers of $M_2$ over $W_4^1$ we work in local coordinates near each of the ten points $g_i$ and $h_i$ of Theorem 3.2.

By Theorem 3.2, before our base change of degree 2 the local equation of $\Theta$ near one of the points $g_i$ or $h_i$ can be written as $xy + zw + st = 0$ in $\mathbb{A}^6$ where $t$ is a local analytic coordinate on $T$ centered at 0. Hence, after base change, the local equation is

\[xy + zw + st^2 = 0.\]
In the above coordinates, the equations of $W_4^1$ are

$$x = y = z = w = t = 0.$$ 

Hence, locally, $\tilde{\Theta}$ is obtained from $\Theta$ by blowing up the ideal $\mathcal{I} = (x, y, z, w, t)$. Furthermore, $s$ gives a local coordinate on $W_4^1$. For any given nonzero value of $s$, the local equation of $\Theta$ defines a quadric of rank 5 in $\mathbb{P}^4$ and, for $s = 0$, the equation defines a quadric of rank 4 in $\mathbb{P}^4$. This proves the assertions about the fibers of $M_2 \to W_4^1$.

Now let $X, Y, Z, W, T$ be the homogeneous coordinates on the blow-up. We have the relations

$$\text{rank} \begin{pmatrix} x & y & z & w & t \\ X & Y & Z & W & T \end{pmatrix} \leq 1.$$

By symmetry, we only need to check the following cases.

(1) $\{ X \neq 0 \}$. $\tilde{\Theta}$ is locally isomorphic to

$$\text{Spec} \frac{\mathbb{C}[x, Y, Z, W, T, s]}{(Y + ZW + sT^2)},$$

which is clearly smooth. $\tilde{\Theta}_0$ is given by the equation $t = 0$, i.e., $xT = 0$, hence has two smooth components meeting transversely, defined locally by the equations $T = 0$ and $x = 0$. The equation $T = 0$ locally defines the component $M_1$ while $x = 0$ locally defines $M_2$.

(2) $\{ T \neq 0 \}$. $\tilde{\Theta}$ is locally isomorphic to

$$\text{Spec} \frac{\mathbb{C}[X, Y, Z, W, t, s]}{(XY + ZW + s)}.$$

In this open subset, the total space and the central fiber are both smooth and $t = 0$ locally defines the component $M_1$.

$\square$

**Proposition 4.3.**

(1) The divisor $M_1$ can be identified with the correspondence

$$M_1 = \{ (D_4, B_4) \in C^{(4)} \times C^{(4)} | D_4 + B_4 \in |K_C| \}$$
with the two projections \( p_1 \) and \( p_2 \) to \( C^{(4)} \). We have a fibered diagram

\[
\begin{array}{c}
\phi \\
\downarrow \\
\Theta_0 \\
\end{array} 
\begin{array}{c}
\otimes \\
\downarrow \\
\otimes \\
\end{array} 
\begin{array}{c}
M_1 \\
\downarrow \\
C^{(4)} \\
\downarrow \\
C^{(4)} \\
\end{array} 
\begin{array}{c}
\otimes \\
\downarrow \\
\otimes \\
\end{array} 
\begin{array}{c}
\psi \\
\downarrow \\
\Theta_0 \\
\end{array}
\]

where \( \phi \) is the natural map sending \( D_4 \) to \( \mathcal{O}_C(D_4) \) and \( \psi \) sends \( B_4 \) to \( \omega_C(-B_4) \).

(2) Both \( p_1 \) and \( p_2 \) are birational morphisms and can be realized as the blow-up of \( C^{(4)} \) along the smooth surface

\[
C^1_4 = \{ D \in C^{(4)} \mid h^0(\mathcal{O}_C(D)) \geq 2 \}.
\]

(3) The double locus \( M_{12} \) is the fiber product

\[
\begin{array}{c}
\phi \\
\downarrow \\
W^1_4 \\
\end{array} 
\begin{array}{c}
\otimes \\
\downarrow \\
\otimes \\
\end{array} 
\begin{array}{c}
M_{12} \\
\downarrow \\
C_4^1 \\
\downarrow \\
C_4^1 \\
\end{array} 
\begin{array}{c}
\otimes \\
\downarrow \\
\otimes \\
\end{array} 
\begin{array}{c}
\psi \\
\downarrow \\
\Theta_0 \\
\end{array}
\]

Proof. Immediate. \( \square \)

### 5. General facts about the Clemens-Schmid exact sequence

We briefly review some general facts about the Clemens-Schmid exact sequence in this section. In Section 6, we will apply this general theory to compute the cohomology of \( \tilde{\Theta}_0 \) and \( \tilde{\Theta}_t \).

#### 5.1. The Clemens-Schmid exact sequence

Given any semistable degeneration

\[
Y_0 \longrightarrow Y \\
\downarrow \\
\{ 0 \} \longrightarrow T
\]
of relative dimension \( n \), where \( \mathcal{Y} \) deformation retracts to \( Y_0 \), denote

\[
H^m_t := H^m(Y_t, \mathbb{Q}),
\]

\[
H^m := H^m(\mathcal{Y}, \mathbb{Q}) \cong H^m(Y_0, \mathbb{Q}),
\]

\[
H_m := H_m(\mathcal{Y}, \mathbb{Q}) \cong H_m(Y_0, \mathbb{Q}).
\]

It follows from the work of Clemens-Schmid [C], [Sc], and Steenbrink [St], that one can define mixed Hodge structures on \( H^* \) and \( H_* \) such that we have an exact sequence of mixed Hodge structures

\[
\begin{array}{cccccccc}
H_{2n+2-m} & \xrightarrow{\alpha} & H^m & \xrightarrow{i^*_t} & H^m_t & \xrightarrow{N} & H^m_t & \xrightarrow{\beta} & H_{2n-m} & \xrightarrow{\alpha} & H^{m+2} \\
& & & & & & & & & & \\
\end{array}
\]

(5.1)

where \( N \) is the logarithm of the monodromy operator, \( i_t : Y_t \hookrightarrow \mathcal{Y} \) is the inclusion of the general fiber into the total space, \( \alpha \) is the composition

\[
H_{2n+2-m}(\mathcal{Y}) \xrightarrow{PD} H^m(\mathcal{Y}, \partial \mathcal{Y}) \xrightarrow{\alpha} H^m(\mathcal{Y}),
\]

where PD denotes Poincaré Duality, and \( \beta \) is the composition

\[
H^m(Y_t) \xrightarrow{PD} H_{2n-m}(Y_t) \xrightarrow{i^*_t} H_{2n-m}(\mathcal{Y}).
\]

5.2. **The weight filtrations on \( H^m \) and \( H_m \).** Recall from [Mo, p. 103] that there is a Mayer-Vietoris type spectral sequence abutting to \( H^\bullet(Y_0) \) with \( E_1 \) term

\[
E_1^{p,q} = H^q(Y_0^{[p]}).
\]

Here \( Y_0^{[p]} \) is the disjoint union of the codimension \( p \) strata of \( Y_0 \), i.e.,

\[
Y_0^{[p]} := \bigsqcup_{i_0, \ldots, i_p} Z_{i_0} \cap \ldots \cap Z_{i_p}
\]

where the \( Z_{i_j} \) are distinct irreducible components of \( Y_0 \).

The differential \( d_1 \)

\[
\begin{array}{cccccc}
E_1^{p,q} & \xrightarrow{d_1} & E_1^{p+1,q} \\
\cong & & & \cong \\
H^q(Y_0^{[p]}) & \xrightarrow{d_1} & H^q(Y_0^{[p+1]})
\end{array}
\]
is the alternating sum of the restriction maps on all the irreducible components. By [Mo, p. 103] this sequence degenerates at $E_2$.

The weight filtration is given by

$$W_k H^m := \bigoplus_{p+q=m, \; q \leq k} E^{p,q}_\infty = \bigoplus_{p+q=m, \; q \leq k} E^{p,q}_2.$$ 

Therefore the weights on $H^m$ go from 0 to $m$ and

$$\text{Gr}_k H^m \cong E^{m-k,k}_2 = \frac{\text{Ker}(d_1: H^k(Y_0^{[m-k]}) \to H^k(Y_0^{[m-k+1]}))}{\text{Im}(d_1: H^k(Y_0^{[m-k-1]}) \to H^k(Y_0^{[m-k]}))}.$$ 

We also put a weight filtration on $H_m$:

$$W_{-k} H_m := (W_{k-1} H^m)^\perp$$ 

under the perfect pairing between $H^m$ and $H_m$. With this definition,

$$\text{Gr}_{-k} H_m \cong (\text{Gr}_k H^m)^\vee.$$ 

5.3. The monodromy weight filtration on $H^m_t$. Associated to the nilpotent operator $N$ is an increasing filtration of $\mathbb{Q}$-vector spaces

$$0 \subset W_0 \subset W_1 \subset \ldots \subset W_{2m} = H^m_t.$$ 

Let $K^m_t := \text{Ker} N \subset H^m_t$ be the monodromy invariant subspace. It inherits an induced weight filtration from $H^m_t$. We refer to [Mo, pp. 106-109] for the precise definition of the monodromy weight filtration and the fact that this filtration on $H^m_t$ can be computed via its induced filtration on $K^m_t$:

$$\text{Gr}_k H^m_t \cong \text{Gr}_k K^m_t \oplus \text{Gr}_{k-2} K^m_t \oplus \ldots \oplus \text{Gr}_{k-2 \lfloor \frac{m}{2} \rfloor} K^m_t$$ (5.2) 

for $k \leq m$, and

$$\text{Gr}_k H^m_t \cong \text{Gr}_{2m-k} H^m_t$$ (5.3) 

for $k > m$.

The weight filtrations on $H^m$ and $K^m_t$ are related by the Clemens-Schmid exact sequence. Below are the basic facts we will use (see [Mo, pp. 107-109]).
(1) The pull-back map $i^*_t$ induces an isomorphism

$$\text{Gr}_k H^m \xrightarrow{\cong} \text{Gr}_k K^m_t \quad \text{for } k \leq m - 1. \quad (5.4)$$

(2) There is an exact sequence

$$0 \longrightarrow \text{Gr}_{m-2} K^{m-2}_t \longrightarrow \text{Gr}_{m-2n-2} H^{2n+2-m} \longrightarrow \alpha \longrightarrow \text{Gr}_m H^m \longrightarrow \text{Gr}_m K^m_t \longrightarrow 0. \quad (5.5)$$

5.4. Mixed Hodge structures on $H_c^\bullet(Y)$. Now suppose furthermore that $Y$ is an analytic open subset of a smooth projective variety $\overline{Y}$ of dimension $n + 1$. We have a sequence of isomorphisms

$$H^{2n+2-m}_c(Y) \cong H^{2n+2-m}(\overline{Y}, \overline{Y} \setminus Y) \cong H^{2n+2-m}(\overline{Y}, \overline{Y} \setminus Y_0),$$

where the last isomorphism follows from the fact that $\overline{Y} \setminus Y_0$ deformation retracts to $\overline{Y} \setminus Y$.

Both $H^\bullet(\overline{Y})$ and $H^\bullet(\overline{Y} \setminus Y_0)$ admit canonical mixed Hodge structures ([De], [Du, 1022-1024]). The relative singular cochain complex $S^\bullet(\overline{Y}, \overline{Y} \setminus Y_0)$ is quasi isomorphic to the mapping cone of the chain map

$$S^\bullet(\overline{Y}) \rightarrow S^\bullet(\overline{Y} \setminus Y_0).$$

Using a standard mapping cone construction (see, for instance, [Du, pp. 1205-1207]), we can put a canonical mixed Hodge structure on $H^\bullet(\overline{Y}, \overline{Y} \setminus Y_0)$, and therefore on $H_c^\bullet(Y)$, such that the maps in the long exact sequence

$$\cdots \longrightarrow H^{m-1}(\overline{Y} \setminus Y_0) \longrightarrow H^m(\overline{Y}, \overline{Y} \setminus Y_0) \longrightarrow H^m(\overline{Y}) \longrightarrow H^m(\overline{Y} \setminus Y_0) \longrightarrow \cdots \quad (5.6)$$

are morphisms of mixed Hodge structures.

There is also a spectral sequence [Du, pp. 1025-1027] for the mapping cone, dual to the Mayer-Vietoris type spectral sequence in Section 5.2, abutting to $H_c^\bullet(Y)$. This spectral sequence is in the second quadrant, degenerates at $E_2$, and has $E_1$ terms

$$E_{1,c}^{p,q} = H^{q+2p-2}(Y_0^{[\cdot]})^c,$$
for \( p \leq 0 \). The differential

\[
\begin{array}{ccc}
E^{p,q}_{1,c} & \longrightarrow & E^{p+1,q}_{1,c} \\
\| & & \| \\
H^{q+2p-2}(Y_0^{-[p]}) & \overset{d_{1,c}}{\longrightarrow} & H^{q+2p}(Y_0^{-[p-1]})
\end{array}
\]

is the alternating sum of Gysin morphisms. We have the duality

\[
(E^{p,q}_1)^\vee \cong E_{1,c}^{-p,2n+2-q}
\]

The increasing weight filtration is given by

\[
W_k H^m_c(Y) = \bigoplus_{p+q=m,q \leq k} E^{p,q}_{2,c}.
\]

The weights on \( H^m_c(Y) \) go from \( m \) to \( 2m-2 \) and, for \( m \leq k \leq 2m-2 \),

\[
\text{Gr}_k H^m_c(Y) \cong E^{m-k,k}_{2,c} = \frac{\text{Ker}(H^{2m-k-2}(Y_0^{[k-m]}) \to H^{2m-k}(Y_0^{[k-m-1]}))}{\text{Im}(H^{2m-k-4}(Y_0^{[k-m+1]}) \to H^{2m-k-2}(Y_0^{[k-m]}))}
\]

with the convention that \( Y_0^{-[1]} = \emptyset \).

The mixed Hodge structures on \( H^m_c(Y) \) and \( H^2_{c}^{2n+2-m}(Y) \cong H^{2n+2-m}(Y, Y \setminus Y_0) \) are dual to each other. We have

\[
\text{Gr}_k H^m(Y)^\vee \cong \text{Gr}_{2n+2-k} H^2_c^{2n+2-m}(Y).
\]

6. The monodromy weight filtration on the cohomology of \( \Theta_t \)

We apply the general theory in Section 5 to the case \( Y = \tilde{\Theta} \) to compute the cohomology of \( \Theta_t \) in this section. By the Hard Lefschetz Theorem

\[
H^m(\Theta_t) \cong H^{8-m}(\Theta_t)
\]

and, by the Lefschetz Hyperplane Theorem,

\[
H^m(\Theta_t) \cong H^m(A_t) \cong \mathbb{Q}^{(10)}_{(m)}
\]

for \( m \leq 3 \). The only remaining case is the middle cohomology \( H^4(\Theta_t) \). We will describe the monodromy weight filtration on it.
According to the general theory explained in Section 5, in order to compute the monodromy weight filtration on the cohomology of $\Theta_t$, we first need to compute the cohomology of the central fiber $\tilde{\Theta}_0 = M_1 \cup M_2$.

6.1. **The cohomology of the strata of $\tilde{\Theta}_0$.** In this subsection we compute the cohomology of $M_1$, $M_2$ and $M_{12}$ and describe their generators. The various spaces fit into the commutative diagram with Cartesian squares

\[
\begin{array}{ccccccccc}
M_2 & \leftarrow & M_{12} & \leftarrow & M_1 \\
\pi_2 & | & \pi_{12} & | & \pi_1 \\
W_4 & \leftarrow & C_4^1 & \leftarrow & C^{(4)} \\
\phi' & \downarrow & l & \downarrow & \phi \\
\downarrow & & \downarrow & & \downarrow \\
\phi & \leftarrow & \pi_{41} & \leftarrow & \pi_{41} \\
\end{array}
\]

(6.3)

where we denote $p'_1$ (resp. $\phi'$) the restriction of $p_1$ (resp. $\phi$) to $M_{12}$ (resp. $C_4^1$) and $j_k : M_{12} \to M_k$ the inclusion map.

**Lemma 6.1.** We have the following table of Betti numbers.

|       | $h^0$ | $h^1$ | $h^2$ | $h^3$ | $h^4$ | $h^5$ | $h^6$ | $h^7$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $C_4^1$ | 1     | 22    | 2     | 22    | 1     | 0     | 0     | 0     |
| $C^{(4)}$ | 1    | 10    | 46    | 130   | 256   | 130   | 46    | 10    |
| $M_1$  | 1     | 10    | 47    | 152   | 258   | 152   | 47    | 10    |
| $Q_3$  | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 0     |
| $Q_{3,\text{sing}}$ | 1   | 0     | 1     | 0     | 2     | 0     | 1     | 0     |
| $M_{12}$ | 1    | 22    | 3     | 44    | 3     | 22    | 1     | 0     |
| $M_2$  | 1     | 22    | 2     | 22    | 12    | 22    | 2     | 22    |

**Proof.** This is a straightforward computation so we only sketch the idea.

(1) By Proposition 4.3, $M_1$ is the blow-up of $C^{(4)}$ along $C_4^1$. So we have $H^\bullet(M_1) = p_1^*H^\bullet(C^{(4)}) \oplus j_{1*}p_1^*H^{\bullet-2}(C_4^1)$. The cohomology of $C^{(4)}$ was computed by Macdonald [Ma]:

\[
H^k(C^{(4)}) = \bigoplus_{\beta=0}^{[k/2]} \eta^\beta \cdot H^{k-2\beta}(\text{Pic}^4 C).
\]
(2) Since $M_{12}$ (resp. $C^1_4$) is a smooth fibration over $W^1_4$ with fibers $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{P}^1$), we can apply the Leray spectral sequence to $\pi_{12}: M_{12} \to W^1_4$ (resp. $\phi'$) to compute the cohomology of $M_{12}$ and $C^1_4$.

(3) The variety $M_2$ is a fibration over $W^1_4$ with general fiber isomorphic to the smooth quadric threefold $Q_3$ and ten special fibers isomorphic to the singular quadric $Q^*_3$ of rank 4. Since the base is a curve, the Leray spectral sequence for $\pi_2$ degenerates at $E_2$.

We present the Leray spectral sequence computation for $H^4(M_2, \mathbb{Q})$, the other cohomology groups are similar and somewhat easier to compute. The $E_2$ terms are

$$E_2^{p,q} = H^p(W^1_4, R^q \pi_2^* \mathbb{Q}).$$

Let $U \subset W^1_4$ be a small analytic disc, open neighborhood of a critical value of $\pi_2$. Then $\pi_2^{-1}(U)$ is homotopic to a smooth fiber $\pi_2^{-1}(t) = Q_3$ with a real 4-cell $B^4$ attached to $\pi_2^{-1}(t)$ along a vanishing sphere $S^3$. Since $h^3(Q_3) = 0$, this amounts to increasing $h^4$ by 1. Thus

$$R^q \pi_2^* \mathbb{Q} \cong \begin{cases} \mathbb{Q} \oplus (\oplus_{i=1}^{10} \mathbb{Q}_i) & q = 4, \\ \mathbb{Q} & q = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{Q}_i$ is the skyscraper sheaf with stalk $\mathbb{Q}$ supported at the $i$-th critical point. Therefore

$$\dim \mathbb{Q} E_2^{p,4-p} = \dim \mathbb{Q} E_\infty^{p,4-p} = h^p(W^1_4, R^{4-p} \pi_2^* \mathbb{Q}) = \begin{cases} 11 & p = 0, \\ 1 & p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This gives $h^4(M_2) = 12$. \hfill \Box

**Notation 6.2.** Denote $e_k \in H^2(M_k)$ the class of $M_{12}$ in $M_k$, $f \in H^2(M_{12})$ the class of a fiber $\pi_{12}^{-1}(t)$ and $\tau_1 := p_1'^{*}l^* \eta \in H^2(M_{12})$ (see Diagram (6.3)). Here $l^* \eta$ is represented by the curve $C^1_4 \cap (x + C^{(3)}) \subset C^1_4$ for $x \in C$ general, and $\tau_1$ is represented by a $\mathbb{P}^1$ bundle over this curve. The product $\tau_1 \cdot f \in H^4(M_{12})$ is the class of the ruling of $\pi_{12}^{-1}(t) \cong \mathbb{P}^1 \times \mathbb{P}^1$ which projects to a point under $p_1$, and the product $j_2^* e_2 \cdot f$ is the hyperplane class in $\pi_{12}^{-1}(t)$. We also have the relation

$$-j_1^* e_1 = j_2^* e_2.$$
Furthermore, denote \([\mathbb{P}^2_i], i = 1, \ldots, 5\) (resp. \(i = 6, \ldots, 10\)) the class of the projective plane spanned by a line of the ruling corresponding to \(\tau_1 \cdot f\) and the vertex of the singular quadric \(Q_{3i}^{\text{sing}} = \pi_2^{-1}(g_i)\) (resp. \(Q_{3i}^{\text{sing}} = \pi_2^{-1}(h_{i-5})\)).

**Lemma 6.3.** We have the following generators for the cohomology:

\[
\begin{align*}
H^2(M_{12}) &= \langle f, \tau_1, j_2^*e_2 \rangle \cong \mathbb{Q}^3, \\
H^3(M_{12}) &= \tau_1 \cdot \pi_{12}^*H^1(W_4^1) \oplus j_2^*e_2 \cdot \pi_{12}^*H^1(W_4^1) \cong \mathbb{Q}^{14}, \\
H^4(M_{12}) &= \langle f \cdot \tau_1, f \cdot j_2^*e_2, \tau_1 \cdot j_2^*e_2 \rangle \cong \mathbb{Q}^3, \\
H^3(M_1) &= p_1^*H^3(C^{(4)}) \oplus j_1\pi_{12}^*H^1(W_4^1) \cong \mathbb{Q}^{152}, \\
H^4(M_1) &= p_1^*H^4(C^{(4)}) \oplus \langle j_1f, j_1\tau_1 \rangle \cong \mathbb{Q}^{258}, \\
H^3(M_2) &= e_2 \cdot \pi_2^*H^1(W_4^1) \cong \mathbb{Q}^{22}, \\
H^4(M_2) &= \langle [\mathbb{P}^2], j_2^*f, j_2^*\tau_1 | i = 1, \ldots, 10 \rangle \cong \mathbb{Q}^{12}.
\end{align*}
\]

**Proof.** The statements about \(M_1\) follow from the formula for the cohomology of a blow-up and the statements about \(M_2\) and \(M_{12}\) follow from the Leray spectral sequence. \(\square\)

### 6.2. The cohomology of \(\widetilde{\Theta}_0\).

Recall that \(Q \subset \mathbb{P}I_2(C)\) is the plane quintic parametrizing quadrics of rank 4 and also the quotient of \(W_4^1\) by the involution exchanging \(g_4^1\) with \(|K_C - g_4^1|\) (see the proof of Theorem 3.2). We have

**Proposition 6.4.** The weight filtration on \(H^4 := H^4(\widetilde{\Theta}_0) = H^4(M_1 \cup M_2)\) is as follows:

\[
\begin{align*}
\text{Gr}_k H^4 &= 0 \quad \text{for } k \leq 2; \\
\text{Gr}_3 H^4 &\cong \frac{H^1(W_4^1)}{h^*H^1(\text{Pic}^3 C)} \cong H^1(Q) \cong \mathbb{Q}^{12}; \\
\text{Gr}_4 H^4 &= \text{Ker}(H^4(M_1) \oplus H^4(M_2) \xrightarrow{j_1^*-j_2^*} H^4(M_{12})) \cong \mathbb{Q}^{267}.
\end{align*}
\]

**Proof.** We apply the spectral sequence of Section 5.2, which degenerates at \(E_2\), to the case \(\mathcal{Y} = \widetilde{\Theta}\). Since \(\widetilde{\Theta}_0 = M_1 \cup M_2\), the \(E_1\) term of the spectral sequence has only two nonzero columns corresponding...
to \( p = 0 \) and \( p = 1 \). Thus, from the definition of the weight filtration, we obtain

\[
\text{Gr}_k H^4 \cong E_{2}^{4-k,k} = 0 \quad \text{for } k \leq 2,
\]

\[
\text{Gr}_3 H^4 \cong E_{2}^{1,3} = \text{Coker}(H^3(M_1) \oplus H^3(M_2) \xrightarrow{j_1^* - j_2^*} H^3(M_{12})�)
\]

\[
\text{Gr}_4 H^4 \cong E_{2}^{0,4} = \text{Ker}(H^4(M_1) \oplus H^4(M_2) \xrightarrow{j_1^* - j_2^*} H^4(M_{12})�).
\]

We compute \( E_{2}^{1,3} \) in Lemma 6.5. By Lemma 6.6, the image of \( j_1^* - j_2^* \) is equal to \( H^4(M_{12}) \), therefore \( E_{2}^{0,4} \cong \mathbb{Q}^{267} \) by a dimension count. \( \square \)

**Lemma 6.5.** We have isomorphisms \( \text{Coker}(H^3(M_1) \oplus H^3(M_2) \xrightarrow{j_1^* - j_2^*} H^3(M_{12}) \cong \frac{H^1(W_1)}{h^* H^1(\text{Pic}^4 C)} \cong H^1(Q) \cong \mathbb{Q}^{12}. \)

**Proof.** By Lemma 6.3,

\[
H^3(M_1) = p_1^* H^3(C^{(4)}) \oplus j_1* \pi_{12}^* H^1(W_4)
\]

and, by [Ma, p. 325],

\[
H^3(C^{(4)}) = H^3(\text{Pic}^4 C) \oplus \eta \cdot H^1(\text{Pic}^4 C).
\]

Note that

\[
H^3(\text{Pic}^4 C) \xrightarrow{p_1^* \circ \phi^*} H^3(M_1) \xrightarrow{j_1^*} H^3(M_{12})
\]

is zero since \( \phi \circ p_1 \circ j_1 = h \circ \phi' \circ p'_1 \) (see Diagram 6.3) and \( H^3(W_4^1) = 0 \).

Furthermore, we see from Lemma 6.3 that the image of

\[
H^3(M_2) = e_2 \cdot \pi_1^* H^1(W_4^1) \xrightarrow{j_2^*} H^3(M_{12})
\]

is equal to \( j_1^*(j_1* \pi_{12}^* H^1(W_4^1)) = j_1^* e_1 \cdot \pi_{12}^* H^1(W_4^1) \). This is because

\[
j_1^* \circ j_1* = j_1^* e_1 \cup \bullet = -j_2^* e_2 \cup \bullet.
\]

Therefore we have

\[
\text{Coker}(j_1^* - j_2^*) \cong \frac{\tau_1 \cdot \pi_{12}^* H^1(W_4^1)}{j_1^* p_1^* (\eta \cdot H^1(\text{Pic}^4 C))} \cong \frac{H^1(W_4^1)}{h^* H^1(\text{Pic}^4 C)} \cong H^1(Q) \cong \mathbb{Q}^{12}.
\]

\( \square \)
Lemma 6.6. The map $j_2^*: H^4(M_2) \to H^4(M_{12})$ acts as follows

$$j_2^*: j_2^* f \mapsto f \cdot j_2^* e_2,$$
$$j_2^* \tau_1 \mapsto \tau_1 \cdot j_2^* e_2,$$
$$[\mathbb{P}^2_1] \mapsto f \cdot \tau_1 \text{ for } i = 1, \ldots, 10.$$

The map $j_1^*: H^4(M_1) = p_1^* H^4(C^{(4)}) \oplus \langle j_1^* f, j_1^* \tau_1 \rangle \to H^4(M_{12})$ acts as follows

$$j_1^*: - j_1^* f \mapsto f \cdot j_2^* e_2,$$
$$- j_1^* \tau_1 \mapsto \tau_1 \cdot j_2^* e_2,$$

$$p_1^* \omega \mapsto p_1^* i_1^* \omega \in Qf \cdot \tau_1, \forall \omega \in H^4(C^{(4)}). \tag{6.4}$$

As a consequence, $j_1^*: H^4(M_k) \to H^4(M_{12})$ is surjective for $k = 1, 2$.

Proof. The Lemma follows from the formula

$$j_k^* \circ j_k^* = - \cup j_k^* e_k$$

for $k = 1, 2$ and the definition of $[\mathbb{P}^2_1]$ (see Notation 6.2). \hfill \Box

6.3. The monodromy weight filtration on $H^4_t$.

Proposition 6.7. The weight filtration on $H^4(\Theta_t)$ is as follows:

1. $\text{Gr}_k H^4_t = 0$, for $k \leq 3$, or $k \geq 6$.
2. $\text{Gr}_5 H^4_t \cong \text{Gr}_3 H^4_t = i_t^* \text{Gr}_3 H^4 \cong \frac{H^1(W^4)}{h^1(W^4)\text{Pic}^3 C} \cong H^1(Q) \cong Q^{12}$.
3. There is an exact sequence

$$0 \longrightarrow H^2(M_{12}) \xrightarrow{(-j_1^* + j_2^*)} \text{Gr}_4 H^4 \xrightarrow{i_t^*} \text{Gr}_4 H^4_t \longrightarrow 0.$$ 

Consequently, $\text{Gr}_4 H^4_t \cong Q^{264}$ and $H^4(\Theta_t) \cong Q^{288}$.

Proof. If $k \leq 3$, by (5.2) and (5.4),

$$\text{Gr}_k H^4_t \cong \text{Gr}_k K^4_t \oplus \cdots \oplus \text{Gr}_{k-\lfloor \frac{k}{2} \rfloor} K^4_t \cong \text{Gr}_k H^4 \oplus \cdots \oplus \text{Gr}_{k-\lfloor \frac{k}{2} \rfloor} H^4.$$
Therefore, the statements about $\text{Gr}_k H^4_t$ for $k \leq 3$ follow immediately from the computation of the weight filtration on $H^4$ in Proposition 6.4.

For $k = 4$,

$$\text{Gr}_4 H^4_t \cong \text{Gr}_4 K^4_t \oplus \text{Gr}_2 K^4_t \oplus \text{Gr}_0 K^4_t \cong \text{Gr}_4 K^4_t.$$  

The exact sequence (5.5) becomes,

$$0 \longrightarrow \text{Gr}_2 K^2_t \longrightarrow \text{Gr}_{-6} H_6 \alpha \longrightarrow \text{Gr}_4 H^4 \longrightarrow \text{Gr}_4 K^4_t \longrightarrow 0.$$  

By Lemma 6.9 below, the image of $\alpha$ is equal to $(-j_1^*, j_2^*)H^2(Pic^4 C)$ and $(-j_1^*, j_2^*)$ is clearly injective. Therefore (3) holds. The statements for $k \geq 5$ follow by symmetry (see Section 5.3). □

Proposition 6.8. The induced monodromy filtration on the primal cohomology $K^0_t \subset H^4_t$ and $H^4_t = K^0_t \oplus \theta H^2(A_t)$ satisfies the following:

1. $\text{Gr}_k K^4_t \cong \text{Gr}_k H^4_t$ for $k = 3, 5$.
2. We have an exact sequence

$$(I \oplus H^4(M_2)) \cap \text{Gr}_4 H^4 \xrightarrow{i^*_t} \text{Gr}_4 \mathbb{H}_t \xrightarrow{} 0,$$

where $I := p_t^*(\theta H^2(Pic^4 C) \oplus \eta H^2(Pic^4 C) \oplus \eta^2) \oplus \langle j_1^* f, j_1^* \tau^*_1 \rangle \subset H^4(M_1)$.

Proof. Since the family of Prym varieties $A_t$ does not degenerate, we have $\text{Gr}_6 H^6(A_t) \cong H^6(A_t)$ and $\text{Gr}_5 H^6(A_t) = \text{Gr}_7 H^6(A_t) = 0$. Therefore $\text{Gr}_3 H^4(\Theta_4)$ and $\text{Gr}_5 H^4(\Theta_t)$ map to zero under Gysin push-forward. This implies the first statement of the proposition. Now consider the commutative diagram

$$
\begin{array}{ccc}
H^4(M_1 \cup M_2) & \xrightarrow{\cong} & H^4(\Theta) \\
\downarrow j_{0*} & & \downarrow j^* \\
H^6(Pic^4 C) & \xrightarrow{\cong} & H^6(A)
\end{array}
\xrightarrow{i_t^*} \begin{array}{ccc}
\text{Gr}_4 H^4(\Theta) & \xrightarrow{i_t^*} & \text{Gr}_4 \mathbb{H}_t \\
\downarrow j_{1*} & & \downarrow j_{1*} \\
\text{Gr}_4 H^4(A_t) & \xrightarrow{\cong} & \text{Gr}_4 H^4(A_t).
\end{array}
$$

Since the induced Gysin map on the graded piece

$$\text{Gr}_4 H^4(M_1 \cup M_2) \xrightarrow{i_t} \text{Gr}_6 H^6(Pic^4 C) \cong H^6(Pic^4 C)$$

sends $(I \oplus H^4(M_2)) \cap \text{Gr}_4 H^4$ to the subspace $\theta^2 H^2(Pic^4 C)$ and the bottom horizontal maps are isomorphisms, we see that $i_t^*$ sends $(I \oplus H^4(M_2)) \cap \text{Gr}_4 H^4$ into $\text{Gr}_4 \mathbb{H}_t$. By Proposition 6.7 (3), the
kernel of $i_l^*$ is 3-dimensional, therefore $i_l^*$ sends $(I \oplus H^4(M_2)) \cap \text{Gr}_4 H^4$ onto $\text{Gr}_4 \mathbb{H}_l$ by a simple dimension count. □

It remains to describe the 3-dimensional image of $\alpha$ in (5.5). Recall from the definition of the spectral sequence in Section 5.2 that $\text{Gr}_4 H^4$ fits in the exact sequence

$$0 \longrightarrow \text{Gr}_4 H^4 \longrightarrow H^4(M_1) \oplus H^4(M_2) \overset{d_1^{4,4} = j_1^* - j_2^*}{\longrightarrow} H^4(M_{12}) \longrightarrow 0.$$ 

The composition of the natural map

$$H^2(M_{12}) \overset{(-j_1^*, j_2^*)}{\longrightarrow} H^4(M_1) \oplus H^4(M_2)$$

with $d_1^{4,4}$ is zero, therefore $(-j_1^*, j_2^*)$ factors through $\text{Gr}_4 H^4$:

$$0 \longrightarrow \text{Gr}_4 H^4 \longrightarrow H^4(M_1) \oplus H^4(M_2) \overset{d_1^{4,4}}{\longrightarrow} H^4(M_{12}) \longrightarrow 0.$$

**Lemma 6.9.** The image of $\alpha$: $\text{Gr}_6 H^6 \longrightarrow \text{Gr}_4 H^4$ is equal to the image of

$$H^2(M_{12}) \overset{(-j_1^*, j_2^*)}{\longrightarrow} \text{Gr}_4 H^4 \subset H^4(M_1) \oplus H^4(M_2).$$

**Proof.** We have the isomorphism $\text{Gr}_6 H^6 \cong \text{Gr}_6 H^6$ and the latter fits into the exact sequence

$$0 \longrightarrow \text{Gr}_6 H^6 \longrightarrow H^6(M_1) \oplus H^6(M_2) \overset{j_1^* - j_2^*}{\longrightarrow} H^6(M_{12}) \longrightarrow 0$$

whose Poincaré dual is

$$0 \longrightarrow H^0(M_{12}) \overset{(j_1^*, - j_2^*)}{\longrightarrow} H^2(M_1) \oplus H^2(M_2) \longrightarrow (\text{Gr}_6 H^6)\overset{\vee}{\longrightarrow} 0.$$ 

The map $\alpha$ is induced by

$$H_6(\bar{\Theta}) \overset{\text{PD}}{\longrightarrow} H^4(\bar{\Theta}, \partial \bar{\Theta}) \longrightarrow H^4(\bar{\Theta}) \cong H^4(M_1 \cup M_2).$$
On the graded level,
\[ \alpha: \text{Gr}_6 H_6 = (\text{Gr}_5 H^6)^* = \frac{H^2(M_1) \oplus H^2(M_2)}{H^0(M_{12})} \rightarrow \text{Gr}_4 H^4 \]
is induced by the map
\[ (\gamma_1, \gamma_2) \mapsto (-j_1^* (j_1^* \gamma_1 - j_2^* \gamma_2), j_2^* (j_1^* \gamma_1 - j_2^* \gamma_2)). \]
Since the map
\[ j_1^* - j_2^*: H^2(M_1) \oplus H^2(M_2) \rightarrow H^2(M_{12}) \]
is surjective, the image of \( \alpha \) is equal to the 3-dimensional image of \( H^2(M_{12}) \) via \((-j_1^*, j_2^*)\). \( \square \)

7. The semistable reduction of the fiber product

7.1. We need to construct a semistable reduction for the fiber product \( \tilde{G} \times_T \tilde{\Theta} \). The central fibers of \( \tilde{G} \) and \( \tilde{\Theta} \) are described in Section 4.1 and Proposition 4.2 respectively. We follow the notation there. The total space of \( \tilde{G} \times_T \tilde{\Theta} \) is singular along \( X_{kp} \times M_{12} \) and \( X_{kq} \times M_{12} \) for \( k = 1, 2 \). The semistable reduction is simply the blow-up \( \mathcal{P} \) of \( \tilde{G} \times_T \tilde{\Theta} \) along the union of \( W_1 \times M_1 \) and \( W_2 \times M_1 \) and it sits in the commutative diagram with Cartesian squares

![Diagram](image)

**Proposition 7.1.** The blow-up \( \mathcal{P} \) of \( \tilde{G} \times_T \tilde{\Theta} \) along the union of \( W_1 \times M_1 \) and \( W_2 \times M_1 \) is a semistable family whose central fiber \( \mathcal{P}_0 \) has eight components:
(1) For $k = 1, 2$, the total transform $\widetilde{W_k \times M_1}$ of $W_k \times M_1$, which is isomorphic to the blow-up of $W_k \times M_1$ along $X_{kp} \times M_{12} \cup X_{kq} \times M_{12}$.

(2) The proper transforms $\widetilde{P_1 \times M_2}$ and $\widetilde{P_2 \times M_2}$ of $P_1 \times M_2$ and $P_2 \times M_2$ respectively, which are isomorphic to the blow-ups of $P_1 \times M_2$ and $P_2 \times M_2$ along $X_{1p} \times M_{12} \cup X_{2q} \times M_{12}$ and $X_{1q} \times M_{12} \cup X_{2p} \times M_{12}$ respectively.

(3) The proper transforms of $P_1 \times M_1, P_2 \times M_1, W_1 \times M_2$ and $W_2 \times M_2$, which are unchanged under the blow-up.

Proof. We check locally that this is indeed a semistable reduction. Locally, the total space of the fiber product near, say, $X_{1p} \times M_{12}$, is isomorphic to the product of an affine space and

\[
\text{Spec } \frac{\mathbb{C}[x, y, z, w, t]}{(xy - t, zw - t)} \cong \text{Spec } \frac{\mathbb{C}[x, y, z, w]}{xy - zw}.
\]

In the above local coordinates, $X_{1p} \times M_{12}$ is defined by the ideal $(x, y, z, w)$ and blowing up $\tilde{G} \times_T \tilde{\Theta}$ along $W_1 \times M_1$ amounts to blowing up (7.1) along the ideal $(x, z)$. Let $X, Z$ be the corresponding homogeneous coordinates in the blow-up. By symmetry, it is sufficient to check the result on the chart $\{ X \neq 0 \}$. Here $\mathcal{P}$ is isomorphic to the product of an affine space and

\[
\text{Spec } \frac{\mathbb{C}[x, y, Z, w]}{y - Zw} \cong \text{Spec } \mathbb{C}[x, Z, w]
\]

which is smooth. The central fiber in this chart is given by $t = xy = xZw$ which is a simple normal crossing divisor.

The other assertions about the components of the central fiber are immediate. $\square$
7.2. The eight components of the central fiber $\mathcal{P}_0$ meet as follows

The lines with the same label indicate the subvarieties that are glued together to form the double loci of the central fiber. The horizontal lines represent the loci that project onto $M_{12}$ via $\rho_2$ and the vertical lines the loci that project onto either $X_{kp}$ or $X_{kq}$ by $\rho_1$. The slanted lines represent exceptional loci: these are $\mathbb{P}^1$-bundles over the products $X_{kp} \times M_{12}$ and $X_{kp} \times M_{12}$, hence are contracted by $\rho_1$ and $\rho_2$.

The dual graph of the central fiber is

The four vertices of the inside square correspond to the four components in the top row of the previous picture and the four vertices of the outside square to the bottom row. The shaded triangles correspond to triple intersections in the central fiber.

7.3. Let $\text{Pic}^{(10)}(\mathcal{X}/T)$ be the (noncompact) relative Picard scheme whose central fiber is $\text{Pic}^{6,4}(\tilde{C}_{pq})$. There is a rational map $\psi: \tilde{\mathcal{X}}^{(5)} \times_T \tilde{\mathcal{X}}^{(5)} \dashrightarrow \text{Pic}^{(10)}(\mathcal{X}/T)$ which is regular on the fibers over $t \neq 0$. We will show in Proposition 9.1 that the rational map $id \times \psi: \mathcal{G} \times_T \tilde{\mathcal{X}}^{(5)} \times_T \tilde{\mathcal{X}}^{(5)} \dashrightarrow \mathcal{G} \times_T \text{Pic}^{(10)}(\mathcal{X}/T)$
restricted to $\mathcal{F} \subset \mathcal{G} \times T \tilde{\mathcal{X}}(5) \times T \tilde{\mathcal{X}}(5)$ is regular. In other words, we have the following commutative diagram

\[
\begin{array}{c}
\mathcal{F} \hookrightarrow \mathcal{G} \times T \tilde{\mathcal{X}}(5) \times T \tilde{\mathcal{X}}(5) \\
\downarrow \\
\mathcal{G} \times T \Theta \hookrightarrow \mathcal{G} \times T \mathcal{A} \hookrightarrow \mathcal{G} \times T \text{Pic}^{(10)}(\mathcal{X}/T).
\end{array}
\]

**Notation 7.2.** Denote $\mathcal{F}', \mathcal{F}_r'$ the images of $\mathcal{F}, \mathcal{F}_r$ in $\mathcal{G} \times T \Theta$, and $\mathcal{F}'', \mathcal{F}_r''$ and $\mathcal{F}''', \mathcal{F}_r'''$ the proper transforms of $\mathcal{F}'$ and $\mathcal{F}_r'$ in $\tilde{\mathcal{G}} \times T \Theta, \mathcal{P}$ respectively. We summarize the relations between the various spaces in the diagram below:

\[
\begin{array}{c}
\mathcal{F}_r''' \subset \mathcal{F}''', \hookrightarrow \mathcal{P} \\
\downarrow \\
\tilde{\mathcal{G}} \times T \tilde{\Theta} \\
\downarrow \\
\mathcal{F}_r'' \subset \mathcal{F}''', \hookrightarrow \tilde{\mathcal{G}} \times T \Theta \hookrightarrow \tilde{\mathcal{G}} \times T \mathcal{A} \\
\downarrow \\
\mathcal{F}_r \subset \mathcal{F}, \hookrightarrow \mathcal{F}_r', \hookrightarrow \mathcal{G} \times T \Theta \hookrightarrow \mathcal{G} \times T \mathcal{A}.
\end{array}
\]

### 8. Abel-Jacobi maps on the generic and special fibers: outline of the proof of Theorem 3

The Abel-Jacobi map $\text{AJ}$ on the total space is the composition

\[
H^2(\tilde{\mathcal{G}}) \xrightarrow{\rho_1^*} H^2(\mathcal{P}) \xrightarrow{\cup [\mathcal{F}''']} H^8(\mathcal{P}) \xrightarrow{\rho_{2*}} H^4(\tilde{\Theta}),
\]

where the Gysin map $\rho_{2*}$ is defined as

\[
H^8(\mathcal{P}) \xrightarrow{\text{PD}} H^6_c(\mathcal{P})^\vee \xrightarrow{(\rho_2^*)^\vee} H^6_c(\tilde{\Theta})^\vee \xrightarrow{\text{PD}} H^4(\tilde{\Theta}),
\]

where PD denotes Poincaré duality. As explained in Section 5.4, there exist canonical mixed Hodge structures on $H^6_c(\mathcal{P})$ and $H^6_c(\tilde{\Theta})$, such that $\rho_2^*$ (and therefore $(\rho_2^*)^\vee$) is a morphism of mixed Hodge structures. Thus the Abel-Jacobi map $\text{AJ}$, as a composition of such, is also a morphism of mixed Hodge structures.
By functoriality of the morphisms involved, we have a commutative diagram

\[
\begin{array}{ccc}
H^2(\tilde{G}) & \longrightarrow & H^2(G_t) \\
\downarrow \Lambda J & & \downarrow \Lambda J_t \\
H^4(\tilde{\Theta}) & \longrightarrow & H^4(\Theta_t),
\end{array}
\]

where the images of the horizontal maps are the monodromy invariant parts of the cohomology groups of \( G_t \) and \( \Theta_t \).

8.1. The map \( \Lambda J \) on the \( E_1 \) terms. The maps \( \rho_1^* \), \( \cup [F''']_r \) and \( \rho_2^* \) are defined on the \( E_1 \) terms of the spectral sequences in Section 5 and commute with the differentials \( d_1 \).

For \( k = 0, 1 \), the map \( \rho_1^* \) on the \( E_1 \) terms is

\[
\begin{array}{ccc}
\tilde{g}E_1^{k,2-k} & \longrightarrow & pE_1^{k,2-k} \\
\downarrow & & \downarrow \\
H^2-k(\tilde{G}_0[k]) & \longrightarrow & H^2-k(P_0[k]).
\end{array}
\]

If, for a stratum \( S \) in \( P_0^k \), \( \rho_1(S) \) is not contained in \( \tilde{G}_0[k] \), then the projection of \( \rho_1^* \) onto the summand \( H^2-k(S) \subset H^2-k(P_0[k]) \) is zero (some components of \( P_0^1 \) map onto components in \( \tilde{G}_0^0 \), c.f. Section 7).

Cup-product with \( [F''']_r \) induces the horizontal maps

\[
\begin{array}{ccc}
pE_1^{k,2-k} & \longrightarrow & pE_1^{k,8-k} \\
\downarrow & & \downarrow \\
H^2-k(P_0^k) & \longrightarrow & H^8-k(P_0^k),
\end{array}
\]

where the lower horizontal map is cup-product with the cycle class of the scheme-theoretic intersection of \( \mathcal{F}_r''' \) with each component in \( P_0^k \).

The map \( \rho_2^* \) on cohomology with compact supports is

\[
\begin{array}{ccc}
\tilde{g}E_1^{-k,k+6} & \longrightarrow & pE_1^{-k,k+6} \\
\downarrow & & \downarrow \\
H^4-k(\tilde{\Theta}_0[k]) & \longrightarrow & H^4-k(P_0[k]).
\end{array}
\]
Similarly to the case of \( \rho_1^* \) above, we only pull back to the strata of \( \mathcal{P}_0^k \) which map to \( \tilde{\Theta}_0^k \). Thus the dual map \( \rho_2^* = (\rho_2^*)^\vee \) is induced by the usual Gysin maps between the relevant strata:

\[
\begin{array}{c}
p E_1^{k,8-k} \xrightarrow{\rho_2^*} \tilde{\Theta}_0^{k,4-k} \\
H^{8-k}(\mathcal{P}_0^k) \xrightarrow{\rho_2^*} H^{4-k}(\tilde{\Theta}_0^k). 
\end{array}
\]

To compute the graded parts of the Abel-Jacobi maps (see (0.1) and (0.2)), we first compute the Abel-Jacobi map \( AJ_k \) on the \( E_1 \) terms for \( k = 0, 1 \):

\[
AJ_k: H^{2-k}(\tilde{g}_0^k) \xrightarrow{\rho_1^*} H^{2-k}(\mathcal{P}_0^k) \xrightarrow{\cup \{F''\}} H^{8-k}(\mathcal{P}_0^k) \xrightarrow{\rho_2^*} H^{4-k}(\tilde{\Theta}_0^k),
\]

then pass to the \( E_2 \) terms of the corresponding spectral sequences.

### 8.2. Proof of the main theorem.

Notation as in Section 6.1. We divide the proof of Theorem 3 into four propositions.

For the Abel-Jacobi map on the \( E_1 \) terms, we write

\[
AJ^0 =: (AJ_1^0, AJ_2^0): H^2(\tilde{G}_0^0) \to H^4(\tilde{\Theta}_0^0) = H^4(M_1) \oplus H^4(M_2).
\]

We have

**Proposition 8.1.** The image of the map \( AJ_1^0: H^2(\tilde{G}_0^0) \to H^4(M_1) \) contains the subspace

\[
I := p_1^*(\theta H^2(\text{Pic}^4 C) \oplus \eta H^2(\text{Pic}^4 C) \oplus \eta^2) \oplus \langle j_1, f, j_1, \tau_1 \rangle
\]

modulo \( \langle j_1, f, j_1, \tau_1 \rangle \oplus p_1^*(\theta H^2(\text{Pic}^4 C)) \).

**Proposition 8.2.** The map \( AJ_2^0: H^2(\tilde{G}_0^0) \to H^4(M_2) \) is surjective modulo \( \langle j_2, f, j_2, \tau_1 \rangle \).

For \( AJ^1 \), we have

**Proposition 8.3.** The image of \( AJ^1: H^1(\tilde{G}_0^1) \to H^3(\tilde{\Theta}_0^1) = H^3(M_{12}) \) contains \( \tau_1 \cdot \pi_{12}^* H^1(W_4^1) \).

Next we pass to the Abel-Jacobi map on the \( E_2 \) terms.
**Proposition 8.4.** The image of the restriction of $AJ^0 = (AJ^0_1, AJ^0_2)$ to

$$\text{Gr}_2 H^2(\tilde{G}) = \text{Ker}(H^2(\tilde{\Theta}) \to H^2(\tilde{\Theta}^1))$$

contains $(I \oplus H^4(M_2)) \cap \text{Gr}_4 H^4(\tilde{\Theta})$ modulo $(-j_1, j_2^*)H^2(M_{12}) + (\theta H^2(\text{Pic}^4 C), 0)$.

Assuming the above four propositions, we can prove our main theorem.

**Proof.** of **Theorem 3.** Identifying $H^4(A_t)$ with a subspace of $H^4(\Theta_t)$ via pull-back, we have $H^4(\Theta_t) = (\mathbb{K}_t \otimes \mathbb{Q}) \oplus H^4(A_t)$, and, since $A_t$ does not degenerate,

$$\text{Gr}_3 H^4(\Theta_t) = \text{Gr}_3(\mathbb{K}_t \otimes \mathbb{Q}),$$

and

$$\text{Gr}_4 H^4(\Theta_t) = \text{Gr}_4(\mathbb{K}_t \otimes \mathbb{Q}) \oplus H^4(A_t).$$

Consider the commutative diagram

$$\begin{array}{ccc}
H^2(\tilde{G}) & \xrightarrow[i^*_t]{\text{AJ}_t} & H^2(G_t) \\
\downarrow & & \downarrow \text{AJ}_t \\
H^4(\tilde{\Theta}) & \xrightarrow[i^*_t]{\text{AJ}_t} & H^4(\Theta_t).
\end{array}$$

Proposition 8.3 implies that the image of $\text{AJ}_t$ sends $\text{Gr}_1 H^2(G_t) = i^*_t \text{Gr}_1 H^2(\tilde{G})$ surjectively to $\text{Gr}_3 H^4(\Theta_t) = i^*_t \text{Gr}_3 H^4(\tilde{\Theta}) \cong \frac{H^1(W_1)}{H^1(\text{Pic}^4 C)} = H^1(Q)$. Since the logarithm of the monodromy operator $N$ induces an isomorphism from $\text{Gr}_5 H^4(\Theta_t)$ to $\text{Gr}_3 H^4(\Theta_t)$ and from $\text{Gr}_3 H^2(G_t)$ to $\text{Gr}_1 H^2(G_t)$, we conclude that $\text{AJ}_t$ sends $\text{Gr}_3 H^2(G_t)$ surjectively to $\text{Gr}_5 H^4(\Theta_t)$.

Next, by Lemma 6.9, the ambiguity $(-j_1, j_2^*)H^2(M_{12})$ restricts to zero under $i^*_t$. Therefore, by Propositions 6.8 and 8.4, the image of $\text{Gr}_2 H^2(G_t)$ by $\text{AJ}_t$ contains $\text{Gr}_4(\mathbb{H}_t \otimes \mathbb{Q})$ modulo $\theta_t H^2(A_t)$.

Combining the above, we see that the image of $\text{AJ}_t$ contains $\mathbb{H}_t \otimes \mathbb{Q}$ modulo $\theta_t H^2(A_t)$. Since, as we observed earlier, $\theta_t H^2(A_t)$ is always contained in the image of $H^2(\Theta_t \cap \Theta_{ta})$ for $a \in A_t$ general, the theorem follows (here $\Theta_{ta}$ is the translate of $\Theta_t$ by $a$).
9. The cycles at time zero: before resolving the family of theta divisors

9.1. The central fiber $F_0$. We list the intersections of the central fiber $F_0$ of $\mathcal{F}$ with each component $W_k \times (C_1^{(d_1)} \times C_2^{(d_2)}) \times (C_1^{(e_1)} \times C_2^{(e_2)})$ in Tables 2 and 3. The left column lists the ambient spaces of all possible bidegrees. The middle column gives the conditions defining the cycles $F_0$ in each ambient space.

For each pair of bidegrees $(d_1, d_2)$ and $(e_1, e_2)$, we define a morphism

$$\psi_{(d_1,d_2)(e_1,e_2)} : F_0 \cap \left( W_k \times (C_1^{(d_1)} \times C_2^{(d_2)}) \times (C_1^{(e_1)} \times C_2^{(e_2)}) \right) \longrightarrow \Theta_0 \subset \text{Pic}^4 C$$

$$(L, D_{d_1}, D_{d_2}, D'_{e_1}, D'_{e_2}) \mapsto \mathcal{O}_C(D_{d_2} + D'_{e_2} - m(p + q)),$$

where $m$ is the integer such that $d_2 + e_2 = 4 + 2m$. These morphisms are listed case by case in the rightmost column of Tables 2 and 3.

9.2. The morphism to $\Theta_0$.

**Proposition 9.1.** The rational map

$$\text{id} \times \psi : \mathcal{G} \times_T \tilde{\mathcal{X}}(5) \times_T \tilde{\mathcal{Y}}(5) \longrightarrow \mathcal{G} \times_T \text{Pic}^{10}(\mathcal{X}/T)$$

extends to a morphism when restricted to $\mathcal{F} \subset \mathcal{G} \times_T \tilde{\mathcal{X}}(5) \times_T \tilde{\mathcal{Y}}(5)$ (see Section 7.3 for the notation).

**Proof.** We need to extend the rational map $\psi$ to the central fiber $F_0$ of $\mathcal{F}$. As explained in Section 2.4, the natural extension of the map $\psi$ to a general point of $W_k \times (C_1^{(d_1)} \times C_2^{(d_2)}) \times (C_1^{(e_1)} \times C_2^{(e_2)})$ is given by $\psi_{(d_1,d_2)(e_1,e_2)}$. Therefore we need to show that the morphisms $\psi_{(d_1,d_2)(e_1,e_2)}$ coincide on the intersection of $F_0$ with the overlaps of the different components of $G_0 \times \tilde{\mathcal{C}}(5) \times \tilde{\mathcal{C}}(5)$. For instance, a point $(p + g_1^1, D_2, D_3 = B_2 + p, D'_4 = B'_3 + q, a') \in F_0 \cap W_1 \times (C_1^{(2)} \times C_2^{(2)}) \times (C_1^{(4)} \times C_2)$ is identified with $(q + g_1^1, D_2 + q, B_2, B'_3, a' + p) \in F_0 \cap W_2 \times (C_1^{(3)} \times C_2^{(2)}) \times (C_1^{(3)} \times C_2^{(2)})$. The images under $\psi_{(2,3)(4,1)}$ and $\psi_{(3,2)(3,2)}$ are both equal to $\mathcal{O}_C(B_2 + a' + p)$. Therefore all the $\psi_{(d_1,d_2)(e_1,e_2)}|_{F_0}$ glue together and we obtain a morphism from $F_0$ to $\Theta_0$. \qed
Recall that we have a tower of blow-ups and algebraic cycles in each blow-up.

![Diagram of blow-ups and cycles](image)

**Table 2. Cycles in** $W_1 \times C_{pq}^{(5)} \times C_{pq}^{(5)}$

| Ambient Spaces | $F_0$ | Image under $\psi$ |
|----------------|-------|---------------------|
| $(L, D_4, a, D'_4, a') \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(4)} \times C_2)$ | $\begin{cases} D_4 + a \in |L| \\ D'_4 + a' \in |L'| \end{cases}$ | $\mathcal{O}_C(a + a' + p + q)$ |
| $(L, D_4, a, D'_2, D'_3) \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(2)} \times C_2^{(3)})$ | $\begin{cases} D_4 + a \in |L| \\ D'_2 + D'_3 \in |L'| \end{cases}$ | $\mathcal{O}_C(a + D'_3)$ |
| $(L, D_4, a, D'_3) \in W_1 \times (C_1^{(4)} \times C_2) \times C_2^{(5)}$ | $\begin{cases} D_4 + a \in |L| \\ D_5 \in |L'| \end{cases}$ | $K_C(-D_4)$ |
| $(L, D_2, D_3, D'_4, a') \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times (C_1^{(4)} \times C_2)$ | $\begin{cases} D_2 + D_3 \in |L| \\ D'_3 + a' \in |L'| \end{cases}$ | $\mathcal{O}_C(D_3 + a')$ |
| $(L, D_2, D_3, D'_2, D'_3) \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times (C_1^{(2)} \times C_2^{(3)})$ | $\begin{cases} D_2 + D_3 \in |L| \\ D'_2 + D'_3 \in |L'| \end{cases}$ | $K_C(-D_2 - D'_2)$ |
| $(L, D_2, D_3, D'_5) \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times C_2^{(5)}$ | $\begin{cases} D_2 + D_3 \in |L| \\ D_5 \in |L'| \end{cases}$ | $K_C(-D_2 - p - q)$ |
| $(L, D_5, D'_4, a') \in W_1 \times C_2^{(5)} \times (C_1^{(4)} \times C_2)$ | $\begin{cases} D_5 \in |L| \\ D'_4 + a' \in |L'| \end{cases}$ | $K_C(-D'_4)$ |
| $(L, D_5, D'_2, D'_3) \in W_1 \times C_2^{(5)} \times (C_1^{(2)} \times C_2^{(3)})$ | $\begin{cases} D_5 \in |L| \\ D'_2 + D'_3 \in |L'| \end{cases}$ | $K_C(-D'_2 - p - q)$ |
| $(L, D_5, D'_5) \in W_1 \times C_2^{(5)} \times C_2^{(5)}$ | $\begin{cases} D_5 \in |L| \\ D'_5 \in |L'| \end{cases}$ | $K_C(-2p - 2q)$ |
Proposition 9.2. (1) The cycle $F_{r_0}$ is the push-forward under $\lambda$ of

$$F_{r_0} := F_{(1,1)(4,1)} \amalg F_{(4,1)(2,3)} \amalg F_{(2,3)(4,1)} \amalg F_{(2,3)(2,3)}.$$
(2) The cycle \( F''_{r_0} |_{W_2 \times \Theta_0} \) is the push-forward of
\[
F_{r_02} := F_{(5,0),(3,2)} \amalg F_{(5,0),(1,4)} \amalg F_{(3,2),(3,2)} \amalg F_{(3,2),(1,4)} \amalg F_{(1,4),(3,2)} \amalg F_{(1,4),(1,4)}.
\]

(3) The cycle \( F''_{r_0} |_{P_k \times \Theta_0} \) is the image of the fiber product
\[
\begin{array}{c}
F_{r_0} |_{P_k} \\
\downarrow \\
P_k
\end{array} \longrightarrow \begin{array}{c}
F_{r_0} |_{X_{kp}} \\
\downarrow \\
X_{kp},
\end{array}
\]

where \( F_{r_0} |_{P_k} \) maps to \( P_k \) via projection and to \( \Theta_0 \) via \( \lambda_2 \).

**Proof.** Since any component of \( F_0 \) with bidegree \((0,5) + (e_1, e_2)\) does not intersect \( F_{r_0} \), we see that \( F_{(0,5)(e_1,e_2)} \) is empty. From Tables 2 and 3, we see that \( F_{(4,1),(0,5)}, F_{(2,3),(0,5)} \) and \( F_{(d_1,d_2),(5,0)} \) are contracted by \( \lambda \) and their image by \( \lambda \) is contained in the closure of the image of cycles of other bidegrees. For other bidegrees, \( \lambda \) is generically injective on any irreducible component. This proves the first two statements. The third statement follows immediately from the construction of \( F'' \).

10. The cycles at time zero: after resolving the family of theta divisors

10.1. The cycle \( F'''_{r_0} \) is the proper transform of \( F''_{r_0} \) under
\[
\mathcal{P} \longrightarrow \tilde{G} \times_T \tilde{\Theta} \longrightarrow \tilde{G} \times_T \Theta
\]

where the arrow on the right is the blow-up of \( \tilde{G} \times_T \Theta \) along \( \tilde{G}_0 \times W_4^1 \) and the arrow on the left, which is a small resolution, is the blow-up of \( \tilde{G} \times_T \tilde{\Theta} \) along \( \Pi_k(W_k \times M_1) \). The central fiber of \( \mathcal{P} \) has 8 components (see Section 7), where \( \tilde{W}_k \times M_1 \) and \( P_k \times M_1 \) are the main components and \( W_k \times M_2, P_k \times M_2 \) are the exceptional components.

**Proposition 10.1.** \( F'''_{r_0} |_{\tilde{W}_k \times M_1} \) is the proper transform of \( F''_{r_0} |_{W_k \times \Theta_0} \) under the birational morphism
\[
\tilde{W}_k \times M_1 \longrightarrow W_k \times M_1 \stackrel{(id,p_1)}{\longrightarrow} W_k \times C^{(4)} \stackrel{(id,\phi)}{\longrightarrow} W_k \times \Theta_0.
\]

**Proof.** The inverse of the birational morphism
\[
\tilde{W}_k \times M_1 \longrightarrow W_k \times M_1 \longrightarrow W_k \times \Theta_0
\]
is defined on the open subset \((W_k \setminus (X_p \cup X_q)) \times (\Theta_0 \setminus W_4^1)\). This open subset contains an open dense subset of \(F''_{r_k}|_{W_k \times \Theta_0}\).

\[\square\]

10.2. The center of the blow-up. Next we study \(F''_{r_0}|_{W_k \times M_2}\), which is the scheme-theoretic intersection of \(F''_r\) with the exceptional divisor \(W_k \times M_2\). So \(F''_{r_0}|_{W_k \times M_2}\) is the projectivized normal cone to the scheme-theoretic intersection \(F''_{r_0} \cap (W_k \times W_4^1) \subset F''_{r_0}|_{W_k \times \Theta_0}\) in \(F''_r\). We first study the center of the blow-up.

By Proposition 9.2, \(F''_{r_0}|_{W_k \times \Theta_0}\) is the image of

\[
F_{r_0,k} \xrightarrow{\lambda_1, \lambda_2} F''_{r_0}|_{W_k \times \Theta_0} \subset W_k \times \Theta_0.
\]

Denote \(Z_k \subset F_{r_0,k}\) the inverse image scheme of \(F''_{r_0} \cap (W_k \times W_4^1) \subset F''_{r_0}|_{W_k \times \Theta_0}\) and put \(Z := Z_1 \cup Z_2\) and \(Z_{(d_1,d_2)(e_1,e_2)} = Z \cap F_{(d_1,d_2)(e_1,e_2)}\). Then \(Z_k\) maps onto \(W_4^1 \subset \Theta_0\) by \(\lambda_2\) and we have the Cartesian diagram

\[
\begin{array}{ccc}
Z_k & \xrightarrow{\lambda_2} & W_4^1 \\
\downarrow & & \downarrow \lambda_2 \\
F_{r_0,k} & \xrightarrow{\lambda_1} & \Theta_0.
\end{array}
\]

Proposition 10.2. The cycle \(Z_1\) is 1-dimensional and, for \(s_4^1 \neq g_i\) or \(h_i\), the fiber \(\lambda_2^{-1}(s_4^1) \cap F_{r_01}\) is finite. For \(i = 1, \ldots, 5\), the fiber \(\lambda_2^{-1}(g_i) \cap F_{r_01}\) is 1-dimensional (modulo finitely many points) and its support is listed in Table 4.

The fiber \(\lambda_2^{-1}(h_i) \cap F_{r_01}\) is also 1-dimensional with support described in Table 5.

Proof. We study \(Z_1\) case by case according to the bidegree. The proof is divided into three Lemmas: 10.4, 10.5 and 10.7. \(\square\)

Proposition 10.3. \(Z_2\) is 1-dimensional and, for \(s_4^1 \neq g_i\) or \(h_i\), the fiber \(\lambda_2^{-1}(s_4^1) \cap F_{r_02}\) is finite. For \(i = 1, \ldots, 5\), the fiber \(\lambda_2^{-1}(g_i) \cap F_{r_02}\) is 1-dimensional with support described in Table 6.

The fiber \(\lambda_2^{-1}(h_i) \cap F_{r_02}\) is 1-dimensional with support described in Table 7.

Proof. The proof is entirely analogous to that of Proposition 10.2, we omit the details. \(\square\)
Lemma 10.4. For any $s_4^1 \in W_4^1$, the intersection $\lambda_2^{-1}(s_4^1) \cap F_{(4,1)(4,1)}$ is empty except when $s_4^1 = g_i$. The support of the intersection $\lambda_2^{-1}(g_i) \cap F_{(4,1)(4,1)}$ is of pure dimension 1 and equal to

$$\{ (L, D_4, a, D'_4, a') \mid a + a' + p + q \equiv g_i, \ h^0(L - r_0 - a) > 0, D_4 \equiv L - a, D'_4 \equiv L' - a' \}.$$
Proof. The map $\lambda: F_{\langle 4,1 \rangle(4,1)} \to W_1 \times \Theta_0$ factors through the projection of $F_{\langle 4,1 \rangle(4,1)} \subset W_1 \times (C^{(4)}_1 \times C_2) \times (C^{(4)}_1 \times C_2)$ to $W_1 \times C_2 \times C_2$, which is generically injective. The image of $F_{\langle 4,1 \rangle(4,1)}$ in $W_1 \times C_2 \times C_2$ consists of $(L, a, a')$ such that

$$h^0(L - r_0 - a) > 0.$$ 

The image of $Z_{\langle 4,1 \rangle(4,1)} = Z \cap F_{\langle 4,1 \rangle(4,1)}$ under the projection is defined scheme-theoretically by imposing an extra condition

$$h^0(a + a' + p + q) > 1.$$ 

If $a + a' + p + q \equiv s^1_4 \in W_1^1$, then $s^1_4$ is equal to one of the $g_i$. This first shows that $\lambda_{\langle 4,1 \rangle}^{-1}(s^1_4)$ is empty unless $s^1_4 = g_i$ for some $i$. Then it shows that there are only finitely many choices for $a$, hence $\lambda_{\langle 4,1 \rangle}^{-1}(g_i) \cap F_{\langle 4,1 \rangle(4,1)}$ of pure dimension 1 and is as described .
**Lemma 10.5.** For $s_4^1 \neq g_i$ or $h_i$, the intersection $\lambda_2^{-1}(s_4^1) \cap F_{(4,1)(2,3)}$ is finite. The intersection $\lambda_2^{-1}(h_i) \cap F_{(4,1)(2,3)}$ (up to finitely many points) has support

$$\{(L, D_4, a, D'_2, D'_3) \mid L = a + g_i, a \in C, r_0 \leq D_4 \equiv g_i, D'_3 \equiv h_i - a, D'_2 = p + q\},$$

and the intersection $\lambda_2^{-1}(g_i) \cap F_{(4,1)(2,3)}$ (again, up to finitely many points) has support

$$\{(L, D_4, a, D'_2, D'_3) \mid L = a + g_j, j \neq i, a \in C, r_0 \leq D_4 \equiv g_j, D'_3 \equiv g_i - a, D'_2 \equiv h_i - (g_i - p - q)\}$$

and

$$\{(L, D_4, a, D'_2, D'_3) \mid L = h_i + p + q - c, c \in C, h^0(L - r_0 - a) > 0, D'_3 \equiv g_i - a, D'_2 = a + c\}.$$

**Proof.** Consider the projection of $Z_{(4,1)(2,3)}$ to $W_1 \times C_2 \times C_2^{(3)}$ consisting of $(L, a, D'_3)$ satisfying equations

\begin{align*}
(10.2) & \quad h^0(L' - D'_3) > 0 \\
(10.3) & \quad h^0(L - r_0 - a) > 0 \\
(10.4) & \quad h^0(a + D'_3) > 1.
\end{align*}

Fix any $s_4^1 \in W_4^1(C)$. Suppose $a + D'_3 \equiv s_4^1$. In the canonical space $|K_C|^*$, the span $\langle D'_3 \rangle$ is a plane (C is not trigonal). By Riemann-Roch, $a \in \langle D'_3 \rangle$. We have two cases:

1. $a \not\in \Gamma^\prime_3 := K_C - L'$. In this case, $h^0(L' - D'_3) = h^0(K_C - \Gamma^\prime_3 - D'_3) > 0$ implies $h^0(K_C - \Gamma^\prime_3 - s_4^1) > 0$, i.e. $h^0(L' - s_4^1) > 0$. If $s_4^1 \neq g_i$, then $L' = p + s_4^1$ or $L' = q + s_4^1$ because $h^0(L' - p - q) > 0$. In either case, there are finitely many choices of $a$ satisfying condition (10.3) and therefore there are finitely many points in $Z_{(4,1)(2,3)}$ that map to $s_4^1$. If $s_4^1 = g_i$, then there exists $c \in C$ such that $L' = c + g_i$ and (10.3) becomes

$$h^0(K + p + q - (c + g_i) - r_0 - a) = h^0(h_i + p + q - c - r_0 - a) > 0.$$

For each $c$, there are 4 choices of $a$ satisfying the above condition, so $(L, a, D'_3)$ belongs to

$$\{(L = h_i + p + q - c, a, D'_3 = g_i - a) \mid c \in C, h^0(h_i + p + q - c - r_0 - a) > 0\}.$$

Finally, there is a unique lifting of such $(L, a, D'_3)$ to a point $(L, D_4, a, D'_2, D'_3)$ in $Z_{(4,1)(2,3)}$ as described in the statement.
(2) $a \leq \Gamma_3$. Write $\Gamma'_3 = a + \Gamma'_2$. The conditions defining the fiber of $Z_{(4,1)(2,3)}$ over $s_4^1$ are

$$
\begin{align*}
&h^0(K_C - a - \Gamma'_2 - D'_3) = h^0(K_C - s_4^1 - \Gamma'_2) > 0 \\
&h^0(K_C - \Gamma'_3 - a - r_0) = h^0(\Gamma'_3 + p + q - a - r_0) = h^0(\Gamma'_2 + p + q - r_0) > 0 \\
&a + D'_3 \equiv s_4^1
\end{align*}
$$

Put $h_4^1 := |K_C - s_4^1|$ so that, by the above, $h^0(h_4^1 - \Gamma'_2) > 0$. There are two subcases:

(a) $h^0(\Gamma'_2 + p + q) = 2$. So the second condition above is automatically satisfied.

Here $\Gamma'_2 + p + q \in g_i$ for some $i$.

**Claim 10.6.** The five $g_i$'s containing $\Gamma'_2$ are $g_i$ and $h_j$ for $j \neq i$.

To prove this, denote $l_{pq}$ the line in $\mathbb{P}^2 = \mathbb{P}(I_2(C))$ consisting of quadrics vanishing on the secant line $\langle p + q \rangle$ in $|K_C|^*$. There are five rank 4 quadrics $Q_j, j = 1, \ldots, 5$ in $l_{pq}$, corresponding to the intersection of $l_{pq}$ with the quintic curve parametrizing rank 4 quadrics in $\mathbb{P}(I_2(C))$. For each $j$, $g_j$ is cut on $C$ by one ruling of $Q_j$. Let $S$ be the base locus of the pencil $l_{pq}$. Then $S$ is a Del Pezzo surface of degree 4. By construction $\langle p + q \rangle$ is contained in $S$. Since the span $\langle p + q + \Gamma'_2 \rangle$ is a plane in $|K_C|^*$, $S \cap \langle p + q + \Gamma'_2 \rangle$ is a conic containing $\langle p + q \rangle$, thus $S \cap \langle p + q + \Gamma'_2 \rangle = \langle p + q \rangle \cup \langle \Gamma'_2 \rangle$. Therefore the pencil of quadrics containing $\langle \Gamma'_2 \rangle$ is also $l_{pq}$. We know that $\langle p + q + \Gamma'_2 \rangle \subset Q_i$. For all $j \neq i$, $Q_j \cap \langle p + q + \Gamma'_2 \rangle = S \cap \langle p + q + \Gamma'_2 \rangle = \langle p + q \rangle \cup \langle \Gamma'_2 \rangle$. So $\Gamma'_2$ and $\langle p + q \rangle$ belong to different rulings of $Q_j$, i.e., $\Gamma'_2$ is contained in the ruling of $Q_j$ corresponding to $h_j$ for $j \neq i$. The claim is proved.

Thus $h_4^1 = g_i$ or $h_4^1 = h_j$ for some $j \neq i$.

So those $(L, a, D'_3)$ which map to $s_4^1 = h_i$ are

$$
\{ (L = a + g_i, a, D'_3 \equiv h_i - a) \mid a \in O \}.
$$

Similarly, the $(L, a, D'_3)$ which map to $s_4^1 = g_j$ for $j \neq i$ are

$$
\{ (L = a + g_i, a, D'_3 \equiv g_j - a) \mid a \in O, \ j \neq i \}.
$$

There are unique liftings to points in $Z_{(4,1)(2,3)}$ as described in the statement of the proposition.
(b) $h^0(\Gamma_2' + p + q) = 1$. Then the second condition implies $h^0(\Gamma_2' - r_0) > 0$. For each $s_1^i$, there are finitely many choices of $\Gamma_2' = r_0 + b$ satisfying the first condition and for each choice of $\Gamma_2'$, there are finitely many choices of $a$ such that $a + \Gamma_2' \in W_{pq}$ (because this means $h^0(K_C - a - \Gamma_2' - p - q) > 0$, and, since $h^0(\Gamma_2' + p + q) = 1$, we have $h^0(K_C - \Gamma_2' - p - q) = 1$ as well). Therefore, there are no positive dimensional fibers.

□

Lemma 10.7. (1) The only positive dimensional fibers in $Z_{(2,3)(4,1)}$ are $\lambda_2^{-1}(g_i) \cap Z_{(2,3)(4,1)}$. For each $i$, the 1-dimensional components of $\lambda_2^{-1}(g_i) \cap Z_{(2,3)(4,1)}$ are supported on the curve

$$\{ (L, D_2, D_3, D_4', r_0) \mid L = c + g_i, c \in C, D_2 = r_0 + c, D_3 \equiv g_i - r_0, D_4' \equiv h_i + p + q - c - r_0 \}$$

and

$$\{ (L, D_2, D_3, D_4', a') \mid L = r_0 + g_i, D_2 = a' + r_0, D_3 \equiv g_i - a', D_4' \equiv h_i + p + q - r_0 - a', a' \in C \}.$$

(The second curve is contracted by $\lambda = (\lambda_1, \lambda_2)$, and therefore does not contribute to the Abel-Jacobi map in Section 11.)

(2) All fibers in $Z_{(2,3)(2,3)}$ are finite.

Proof. (1) The projection of $Z_{(2,3)(4,1)}$ to $W_{pq} \times C_2^{(3)} \times C_2$ is the locus of $(L, D_3, a')$ satisfying

$$\begin{cases} h^0(L - D_3 - r_0) > 0 \\ h^0(a' + D_3) > 1. \end{cases}$$

As in the previous lemma, only the inverse image of $g_i$ is positive dimensional. It is equal to

$$\{ (L = c + g_i, a' = r_0, D_3 \equiv g_i - r_0) \mid c \in C \} \cup \{ (L = r_0 + g_i, D_3 \equiv g_i - a', a') \mid a' \in C \}.$$

As before, we can uniquely lift these curves to $Z_{(2,3)(4,1)}$. 
(2) The projection of $Z_{(2,3)(2,3)}$ to $W_{pq} \times C^{(2)}_1 \times C^{(2)}_1$ is the locus of $(L, D_2, D'_2)$ satisfying
\[
\begin{cases}
  h^0(L - D_2) > 0 \\
  h^0(L' - D'_2) > 0 \\
  r_0 \leq D_2 \\
  h^0(K_C - D_2 - D'_2) > 1.
\end{cases}
\]

These cycles are also 1-dimensional but there are only finitely many points mapping to a fixed $s^1_4$ (we choose $r_0$ general such that $r_0 + p + q$ is not in any $g^1_4$).

□

By the previous three Lemmas, Proposition 10.2 is proved.

We also need to describe the components of $Z_1$ which lie over $X_{1p}$ under $\lambda_1$. This will be needed in the computation of the Abel-Jacobi map in Section 11.3.

**Lemma 10.8.** The scheme $Z_1$ has the following components which map onto $X_{1p}$ by $\lambda_1$. Each component maps onto $W^1_4$ by $\lambda_2$. They are supported on
\[
\begin{align*}
\{ (L, D_2, D_3, D'_4, r_0) & \mid L = p + g^1_4, D_3 \equiv g^1_4 - r_0, D_2 = p + r_0, D'_4 \equiv p + g^1_4 - r_0 \} \subset Z_{(2,3)(4,1)}, \\
\{ (L, D_4, p, D'_2, D'_3) & \mid L = p + g^1_4, r_0 \leq D_4 \equiv g^1_4, D'_3 \equiv K_C - g^1_4 - a \} \subset Z_{(4,1)(2,3)},
\end{align*}
\]

and
\[
\{ (L, D_4, a, D'_2, D'_3) \mid L = p + g^1_4, h^0(g^1_4 - r_0 - a) > 0, D_4 \equiv p + g^1_4 - a, D'_3 \equiv K_C - g^1_4 - a, D'_2 = a + q \}
\]
\[\subset Z_{(4,1)(2,3)}.
\]

**Proof.** Fix a general $L = p + g^1_4 \in X_{1p}$. One easily sees from Table 2 that only $Z_{(2,3)(4,1)}$ and $Z_{(4,1)(2,3)}$ have a point over $L$.

In $Z_{(2,3)(4,1)}$, the condition $h^0(p + g^1_4 - D_3 - r_0) > 0$ implies that either $D_3 \equiv g^1_4 - r_0$ or $D_3 = p + B_2$ with $h^0(g^1_4 - r_0 - B_2) > 0$. In the first case, $h^0(a' + D_3) > 1$ implies $a' = r_0$. This is because $D'_3$ is contained in at most one pencil of degree 4. Thus we obtain the first curve in the statement of the lemma. The second case does not happen because $|a' + p + B_2|$ cannot be a pencil for $p$ and $g^1_4$ general.
In $Z_{(4,1)(2,3)}$, there are four choices of $a$ such that $h^0(p + g_4^1 - r_0 - a) > 0$. The condition $h^0(L' - D_3') = h^0(q + K_C - g_4^1 - D_3') > 0$ implies that either $h^0(K_C - g_4^1 - D_3') > 0$ or $D_3' = p + B_2'$ with $h^0(K_C - g_4^1 - B_2') > 0$. In the first case, $h^0(a + D_3') > 1$ implies that $D_3' \equiv K_C - g_4^1 - a$. This is because $D_3'$ is contained in at most one pencil of degree 4. Thus we obtain the curves in the statement of the lemma. In the second case, $|a + q + D_2'|$ is not a pencil for $q$ and $g_4^1$ general. Note that the last component is a degree 3 cover of $X_{1p}$ under $\lambda_1$. □

10.3. **Infinitesimal study of $F_{r_0}$ and $Z$.** In this subsection, we prove that each irreducible component of the center of the blow-up $F''_{r_0} \cap (W_k \times W_4^1)$ is generically smooth, or equivalently, generically reduced. We also prove that $F''_{r_0}$ is smooth at a general point in each irreducible component of $F''_{r_0} \cap (W_k \times W_4^1)$.

The infinitesimal study is similar for all components. So let us take one component, say the image in $W_1 \times \Theta_0$ of the curve in $Z_{(4,1)(2,3)}$

$$\{ (L, D_4, a, D_2', D_3') \mid L = a + g_i, a \in C, r_0 \leq D_4 \equiv g_i, D_3' \equiv h_i - a, D_2' = p + q \}.$$

This curve projects isomorphically to (with identification $C_1 = C_2 = C$)

$$Z'_{(4,1)(2,3)} = \{ (L, a, D_3') \mid L = a + g_i, a, D_3' \equiv h_i - a, a \in C \} \subset W_1 \times C \times C(3).$$

(10.5) It suffices to show that the curve $Z'_{(4,1)(2,3)}$ is generically reduced. To this end, recall that by [ACGH, p 189], for any line bundle $M$ of degree $d$ on $C$, and $v \in H^1(\mathcal{O}_C) = T_M \text{Pic}^d C$ a tangent vector, all sections in $H^0(C, M)$ extend to first order along $v$ if and only if

$$(v, \text{Im } \mu_M)_S = 0$$

where $(,)_S$ is the pairing for Serre duality and

$$\mu_M : H^0(M) \otimes H^0(K_C - M) \rightarrow H^0(K_C)$$

is the multiplication map.

Note that Im $\mu_{g_i}$ is of codimension 1 in $H^0(K_C)$ by the base point free pencil trick.
If we embed $W_1 \times C \times C^{(3)}$ in $\text{Pic}^5 C \times \text{Pic}^1 C \times \text{Pic}^3 C$, by the previous paragraph, the tangent space to $W_1 \times C \times C^{(3)}$ at the point $(L, a, D'_3)$ consists of $(v_1, v_2, v_3) \in H^1(O_C)^{\oplus 3}$ such that

(10.6) \hspace{1cm} v_1 \in \text{Im} \mu^\perp_L \cap \text{Im} \mu^\perp_{L'},

(10.7) \hspace{1cm} v_2 \in H^0(K_C - a)^\perp,

(10.8) \hspace{1cm} v_3 \in H^0(K_C - D'_3)^\perp.

**Lemma 10.9.** A tangent vector $(v_1, v_2, v_3) \in H^1(O_C)^{\oplus 3}$ of $W_{pq} \times C \times C^{(3)}$ is tangent to $Z'_{(4,1)(2,3)} \subset W_{pq} \times C \times C^{(3)}$ at $(L = a + g_i, a, D'_3 = h_i - a)$ if, in addition, the following holds

(10.9) \hspace{1cm} v_1 + v_3 \in H^0(K_C - p - q)^\perp,

(10.10) \hspace{1cm} v_1 - v_2 \in H^0(K_C - (g_i - r_0))^\perp,

(10.11) \hspace{1cm} v_2 + v_3 \in \text{Im} \mu^\perp_{g_i}.

**Proof.** The cycle $Z'_{(4,1)(2,3)}$ is defined scheme-theoretically by (10.2), (10.3), and (10.4). These translate into the above conditions for infinitesimal deformations. \hfill \square

**Proposition 10.10.** Each irreducible component of $F''_{r_0} \cap (W_k \times W'_1)$ is generically smooth.

**Proof.** We only prove the proposition for the component which is the image in $W_1 \times \Theta_0$ of $Z'_{(4,1)(2,3)}$.

Fix a general point $(L = a + g_i, a, D'_3 \equiv h_i - a)$. Consider the linear map from the tangent space of $Z_{(4,1)(2,3)}$ to $H^1(O_C)$ which sends $(v_1, v_2, v_3)$ to $v_2 + v_3$. Its image is 1-dimensional by (10.11). To show the tangent space of $Z_{(4,1)(2,3)}$ is 1-dimensional, it suffices to show that the kernel of this linear map is trivial, i.e. if $v_2 + v_3 = 0$, then $v_1 = v_3 = 0$.

So assume $v_2 + v_3 = 0$. Then

$$v_1 + v_3 = v_1 - v_2 \in H^0(K_C - p - q)^\perp \cap H^0(K_C - (g_i - r_0))^\perp.$$

Since the pencil $K_C - (g_i - r_0) = h_i + r_0$ does not have base points at $p$ or $q$ and can separate $p$ and $q$, we conclude

$$H^0(K_C - p - q)^\perp \cap H^0(K_C - (g_i - r_0))^\perp = (H^0(K_C - p - q) + H^0(K_C - (g_i - r_0)))^\perp = 0.$$
Therefore $v_1 = -v_3$. Now by (10.6) and (10.7), $v_1 = v_2 = -v_3 \in \text{Im } \mu L^2 \cap \text{Im } \mu L^3 \cap H^0(K_C - a) = 0$ for $a \in C$ general, this implies $v_1 = v_2 = v_3 = 0$. \hfill \square

**Proposition 10.11.** The scheme $F_{r_0 k}$ is smooth at a general point of each component of $Z_k$.

**Proof.** Again we only check the proposition for a general point of the image in $W_1 \times C \times C(3)$ of (10.5). The defining conditions for $F(41)(23) \subset W_1 \times C \times C(3)$ are (10.2) and (10.3). The tangent space of $F_{r_0}$ at $(L = a + g, a, D_3' = h_i - a)$ consists of $(v_1, v_2, v_3)$ satisfying the conditions from (10.6) to (10.10). Projection to the $v_1$ summand of $(v_1, v_2, v_3)$ is surjective and the kernel of this projection is 1-dimensional. The proposition follows. \hfill \square

### 10.4. The structure of the projectivized normal cone.

Note that $F''_{r_0}|W_k \times M_2$ is the projectivized normal cone of $F_{r_0} \cap (W_k \times \Theta_0)$ in $F''_r$.

We have the commutative diagram

\begin{equation}
\begin{array}{ccc}
C_k & \xrightarrow{\lambda_1, \lambda_2} & F''_{r_0}|W_k \times M_2 \\
\downarrow & & \downarrow \rho_2 \\
Z_k & \xrightarrow{(\lambda_1, \lambda_2)} & M_2 \\
\downarrow \pi_2 & & \downarrow \\
W_k & \xrightarrow{Pr_1} & W_1 \\
\end{array}
\end{equation}

where $C_k$ is defined by the left square being a fiber product.

**Proposition 10.12.** $C_k$ is generically a $\mathbb{P}^2$ bundle over the curve $\lambda_2^{-1}(\cup_i \{g_i, h_i\}) \cap Z_k$.

**Proof.** Since $W_k \times M_2$ is a divisor in the total space $P$, $F''_{r_0}|W_k \times M_2 = F''_r \cap (W_k \times M_2)$ is purely 3-dimensional. Furthermore, by Propositions 10.10 and 10.11, at a generic point of any component of $\lambda_2^{-1}(\cup_i \{g_i, h_i\}) \cap Z_k$, both $Z_k$ and $F_{r_0 k}$ are smooth. Thus there is an open dense subset of $\lambda_2^{-1}(\cup_i \{g_i, h_i\}) \cap Z_k$ where the dominant map $C_k \to Z_k$ is a $\mathbb{P}^2$ bundle. So the general fiber of $C_k$ is a 2-dimensional linear subspace of the singular quadric threefold $Q_3^{\text{sing}}$ which is the fiber of $M_2$ over one of the $g_i$ or $h_i$. Therefore the general fiber is a $\mathbb{P}^2$ passing through the vertex of $Q_3^{\text{sing}}$. \hfill \square

### 11. The Abel-Jacobi map

We are now ready to prove Propositions 8.1 to 8.4.
11.1. **Proof of Proposition 8.1:** The map $\mathcal{A}J_0^0 : H^2(\tilde{G}_0^{[0]}) \to H^4(M_1)$.

We will show that it is enough to compute the restriction of $\mathcal{A}J_0^0$ to the direct summand $H^2(W_1)$ of $H^2(\tilde{G}_0^{[0]})$. This map is the correspondence induced by the cycle $[F'''_{r_0}]_{W_1 \times M_1} \in H^6(W_1 \times M_1)$. We use the notation introduced in Section 6.1.

There are two reduction steps:

1. First, since we are computing $\mathcal{A}J_0^0$ modulo $\langle j_1^* f, j_1^* \tau_1 \rangle$ in Proposition 8.1 (recall that $H^4(M_1) \cong p_1^* H^4(C^{(4)}) \oplus \langle j_1^* f, j_1^* \tau_1 \rangle$), it suffices to check that the image of the composition

$$H^2(W_1) \xrightarrow{\mathcal{A}J_0^0} H^4(M_1) \xrightarrow{p_1^*} H^4(C^{(4)})$$

contains $\eta H^2(\text{Pic}^4 C) \oplus \eta^2$ modulo $\theta H^2(\text{Pic}^4 C)$. Recall (see Proposition 10.1) that $F'''_{r_0}|_{W_1 \times M_1}$ is the proper transform of $F''_{r_0}|_{W_1 \times \Theta_0}$ under

$$\tilde{W}_1 \times \tilde{M}_1 \to W_1 \times M_1 \to W_1 \times C^{(4)} \to W_1 \times \Theta_0.$$ 

By the projection formula, $p_1^* \circ \mathcal{A}J_0^0$ is induced as a correspondence map by the proper transform $F'''_{r_0}|_{W_1 \times C^{(4)}}$ of $F'''_{r_0}|_{W_1 \times \Theta_0}$ in the intermediate space $W_1 \times C^{(4)}$:

$$H^2(W_1) \xrightarrow{\cup [F'''_{r_0}|_{W_1 \times C^{(4)}}]} H^8(W_1 \times C^{(4)}) \xrightarrow{H^4(C^{(4)})}.$$ 

2. Second, we will prove that in fact the image by $p_1^* \circ \mathcal{A}J_0^0$ of the subspace $(q_1, q_2)^* H^2(C^{(3)} \times C^{(3)})$ of $H^2(W_1)$ contains $\eta H^2(\text{Pic}^4 C) \oplus \eta^2$ modulo $\theta H^2(\text{Pic}^4 C)$. We therefore compute the composition $\overline{\mathcal{A}J_0^0}$

$$\overline{\mathcal{A}J_0^0} : H^2(C^{(3)} \times C^{(3)}) \xrightarrow{(q_1, q_2)^*} H^2(W_1) \xrightarrow{p_1^* \circ \mathcal{A}J_0^0} H^4(C^{(4)}) \xrightarrow{\theta H^2(\text{Pic}^4 C)} H^4(C^{(4)}).$$

where

$$(q_1, q_2) : W_{pq} \to C^{(3)} \times C^{(3)}$$

$L \mapsto (\Gamma_3, \Gamma'_3)$

is the embedding used in Section 2.2.
**Lemma 11.1.** The Kunneth component of $[F''_{r_0}|_{W_1 \times C^{(4)}}] \in H^6(W_1 \times C^{(4)})$ in $H^2(W_1) \otimes H^4(C^{(4)})$ is the restriction to $W_1 \times C^{(4)} \subset C^{(3)} \times C^{(3)} \times C^{(4)}$ of

\[(11.1)\]  
\[
(-2\theta_1 + 4\eta_1 + 4\eta_2)\eta_3^2 + 4\delta_{13}^2 \eta_3 + (\theta_1 - \eta_1)\theta_3 \eta_3.
\]

in $H^6(C^{(3)} \times C^{(3)} \times C^{(4)})$ modulo $\theta_3 H^2(\text{Pic}^4 C)$, where $\delta_{kl} = \sum_{i=1}^5 (\xi_{ki}\xi'_{li} + \xi_{li}\xi'_{ki})$.

**Proof.** The Kunneth component of $[F''_{r_0}|_{W_1 \times C^{(4)}}]$ in $H^2(W_1) \otimes H^4(C^{(4)})$ is computed case by case for each bidegree in Appendix 12.4. It is the sum of the classes in (12.3), (12.4), (12.5), (12.6), (12.7), which is equal to the restriction to $W_1 \times C^{(4)} \subset C^{(3)} \times C^{(3)} \times C^{(4)}$ of

\[(11.2)\]  
\[
[-2\theta_1 + 4\eta_1 + 4\eta_2] \eta_3^2 + \left[2\delta_{23}^2 + \delta_{13}^2 - \delta_{13} \delta_{23} + (\theta_1 - \eta_1)\theta_3 \right] \eta_3.
\]

Consider the commutative diagram

\[
\begin{array}{ccc}
W_{pq} & \xrightarrow{q_1} & C^{(3)} \\
\downarrow q_2 & & \downarrow \ \\
C^{(3)} & \xrightarrow{\tau} & \text{Pic}^3(C)
\end{array}
\]

where $\tau$ is the involution sending $M$ to $K_C - p - q - M$. Since $\tau^*(\xi_i) = -\xi_i$, we see immediately that

\[
q_1^*(\xi_i) = -q_2^*(\xi_i),
\]

\[
\delta_{13}|_{W_1 \times C^{(4)}} = -\delta_{23}|_{W_1 \times C^{(4)}}.
\]

Therefore (11.2) simplifies to (11.1). \qed

For any $\omega \in H^2(C^{(3)})$, denote $\omega_1$ its pull back to $C^{(3)} \times C^{(3)}$ under the first projection (see Notation (4)). Now, using the class (11.1), we obtain

\[
\overline{AJ}_1''(\omega_1) = p_{r(C^{(4)})*} \left\{ \omega_1 \left[ (-2\theta_1 + 4\eta_1 + 4\eta_2)\eta_3^2 + 4\delta_{13}^2 + (\theta_1 - \eta_1)\theta_3 \eta_3 \right] \right\}|_{W_1 \times C^{(4)}}.
\]
Expanding \( \delta_{13}^2 = \sum_{i,j=1}^{5} \left[ 2\xi_i\xi_j\xi_i^j\xi_j - \xi_i\xi_j\xi_i\xi_j + \xi_i\xi_j\xi_i\xi_j \right] \), we obtain

\[
\overline{AJ}_1^0(\omega_1) = \left[ \int_{C^{(3)}} \omega(-2\theta + 4\eta)(\theta - \eta) + 4 \int_{C^{(3)}} \omega(\frac{1}{2}\theta^2 - \theta\eta + \eta^2) \right] \eta^2 + 8 \sum_{i,j=1}^{5} \left[ \int_{C^{(3)}} \omega\xi_i\xi_j(\theta - \eta) \right] \xi_i\xi_j - 4 \sum_{i,j=1}^{5} \left[ \int_{C^{(3)}} \omega\xi_i\xi_j(\theta - \eta) \right] \xi_i\xi_j - 4 \sum_{i,j=1}^{5} \left[ \int_{C^{(3)}} \omega(\theta - \eta)^2 \right] \theta\eta.
\]

Noting that the class of the image of \( W_1 \) in \( C^{(3)} \) by \( q_1 \) or \( q_2 \) is \( \theta - \eta \), and \( q_1q_2\eta = \frac{1}{2}\theta^2 - \theta\eta + \eta^2 \in H^2(C^{(3)}) \), the above formula becomes

\[
(11.3) \quad \overline{AJ}_1^0(\omega_1) = \left[ \int_{C^{(3)}} \omega(-2\theta + 4\eta)(\theta - \eta) + 4 \int_{C^{(3)}} \omega(\frac{1}{2}\theta^2 - \theta\eta + \eta^2) \right] \eta^2 + 8 \sum_{i,j=1}^{5} \left[ \int_{C^{(3)}} \omega\xi_i\xi_j(\theta - \eta) \right] \xi_i\xi_j - 4 \sum_{i,j=1}^{5} \left[ \int_{C^{(3)}} \omega\xi_i\xi_j(\theta - \eta) \right] \xi_i\xi_j - 4 \sum_{i,j=1}^{5} \left[ \int_{C^{(3)}} \omega(\theta - \eta)^2 \right] \theta\eta.
\]

Now a simple computation using the ring structure of \( H^\bullet(C^{(3)}) \) described in Macdonald [Ma] gives

\[
\overline{AJ}_1^0(\eta_1) = 10\eta^2 - 11\theta\eta,
\]

\[
\overline{AJ}_1^0(\xi_i\xi_j) = c_i j \xi_i\xi_j\eta \text{ for } 0 \neq i,j \in \mathbb{Z}, j \neq i \pm 5,
\]

\[
\overline{AJ}_1^0(\sigma_k) = 8\eta^2 - 11\theta\eta + 16\sigma_k\eta,
\]

Thus the image of \( \overline{AJ}_1^0 \) contains \( \eta H^2(\text{Pic}^4 C) \oplus \eta^2 \) modulo \( \theta H^2(\text{Pic}^4 C) \).

\[ \square \]

11.2. **Proof of Proposition 8.2:** The map \( AJ_2^0 : H^2(\overline{C}^{[0]}_0) \rightarrow H^4(M_2) \).

We will work with the restriction of \( AJ_2^0 \) to the direct summand \( H^2(W_1) \oplus H^2(W_2) \) of \( H^2(\overline{C}^{[0]}_0) \):

\[
H^2(W_k) \xrightarrow{\rho_1^*} H^2(W_k \times M_2) \xrightarrow{\cup[F_{\rho_1}^{[m]}|_{W_k \times M_2}]} H^8(W_k \times M_2) \xrightarrow{\rho_2^*} H^4(M_2).
\]

The relations between the various spaces involved are summarized in diagram (10.12). The projection of \( F_{\rho_1}^{[m]}|_{W_k \times M_2} \) to \( W_k \) is supported on curves. By Section 10.2, the image curve contains the following
special curves in $W_k$

$$C_i := \{ c + g_i \mid c \in C \}, C_i' := \iota(C_i), i = 1, \ldots, 5,$$

$$X_{1p} = \{ p + g_4^1 \mid g_4^1 \in W^1_4(C) \},$$

$$X_{1q} = \{ q + g_4^1 \mid g_4^1 \in W^1_4(C) \},$$

where $\iota(L) = |K_C + p + q - L|$. By Lemma 6.3, $H^4(M_2)$ is generated by $j_{2*}, j_{2*}\tau_1$, $[\mathbb{P}^2_i]$ and $[\mathbb{P}^2_{i+5}]$ (recall that $f$ is the class of the fiber of $\pi_{12}: M_{12} \to W^1_4$ and see Lemma 6.3 for the definition of $\mathbb{P}^2_i$).

**Lemma 11.2.** Put $[C]_{tot} := [C_1] + \ldots + [C_5]$. For any $(\alpha, \beta) \in H^2(W_1) \oplus H^2(W_2)$,

$$AJ^0_2(\alpha) = \sum_{i=1}^{5} \left( \int_{W_1} \alpha \cdot [C_i] \right) [\mathbb{P}^2_{i+5}] + \sum_{i=1}^{5} \left( \int_{W_1} \alpha \cdot ([C]_{tot} + 4[C_i'] + q_1^*(\theta - \eta)) \right) [\mathbb{P}^2_i]$$

modulo $\langle j_{2*}, f \rangle$, and,

$$AJ^0_2(\beta) = -\sum_{i=1}^{5} \left( \int_{W_2} \beta \cdot (3[C_i] + [C_i']) \right) [\mathbb{P}^2_{i+5}] + \sum_{i=1}^{5} \left( \int_{W_2} \beta \cdot ([C_i'] + q_1^*(\theta - \eta)) \right) [\mathbb{P}^2_i]$$

modulo $\langle j_{2*}, f \rangle$.

**Proof.** By Sections 10.2 and 10.3, the scheme $F_{r_0}^\prime \cap (W_k \times W^1_4)$ is of pure dimension 1 and generically reduced on each of its components.

Represent $\alpha$ as the cohomology class of a real 2-chain in general position. By definition, $AJ^0_2(\alpha)$ is the push-forward to $M_2$ of the pull-back of $\lambda_1^*\alpha \cup [Z_1]$ to $C_k$. By Proposition 10.12, the fibers of $C_k$ over $\lambda_2^{-1}(\cup_i \{ g_i, h_i \}) \cap Z_k$ are isomorphic to $\mathbb{P}^2$.

Since we are computing $AJ^0_2$ modulo $\langle j_{2*}, f \rangle \in H^4(M_2)$, we only need to compute the intersection of $\lambda_1^*\alpha$ with $\lambda_2^{-1}(\cup_i \{ g_i, h_i \}) \cap Z_1$. The components of $\lambda_2^{-1}(\cup_i \{ g_i, h_i \})) \cap Z_1$ are described in Proposition 10.2. For instance, the curve supported on

$$\{ (L, D_4, a, D'_4, a') \mid a + a' + p + q \equiv g_i, \ h^0(L - r_0 - a) > 0, D_4 \equiv L - a, D'_4 \equiv L' - a' \}$$

has two components since we can switch $a$ and $a'$. Each component projects to a curve in $W_1$ whose class is $(\theta_1 - \eta_1)|_{W_1}$ by the secant plane formula (Section 12.2). Thus the contribution of this curve is $\int_{W_1} \alpha \cdot 2(\theta_1 - \eta_1)[\mathbb{P}^2_i]$. The formula for $AJ^0_2(\alpha)$ now easily follows.
The computation of $AJ'_2(\beta)$ is analogous. The minus sign in the formula for $AJ'_2(\beta)$ comes from the fact that the maps to $\Theta_0$ on the curves

$$\{(L, D_3, D_2, a', D'_4) \mid L = c + g_i, c \in C, r_0 \leq D_3, a' + D_3 \equiv g_i, D_2 = a' + c, D'_4 \equiv h_i + p + q - c - a' \}$$

$$\subset Z_{(3,2)(1,4)}$$

are given by $O_C(K_C - D_3 - a')$ and $O_C(K_C - D'_3 - a)$ respectively (instead of $O_C(D_3 + a')$ and $O_C(D'_3 + a)$). Thus the $\mathbb{P}^2$ fibers over these curves are in the rulings opposite to those of $\mathbb{P}^2_{i+5}$. Since we work modulo $j_{2*}f$, the two rulings differ by a minus sign. 

We need the following Lemma to study the rank of $AJ'_2$.

**Lemma 11.3.** We have the following intersection numbers in the smooth surface $W_{pq}$

$$C_i^2 = C'_i = -2, \quad C_iC'_j = C'_iC_j = C'_iC'_j = 0, \quad C_iC'_j = 2, \text{ for } i \neq j.$$  

**Proof.** Clearly $C_iC'_j = C'_iC_j = 0$ for $i \neq j$. To compute $C_i^2$, consider the exact sequence

$$0 \longrightarrow N_{C_i|W_{pq}} \longrightarrow N_{C_i|C^{(3)}} \longrightarrow N_{q_2(W_{pq})|C^{(3)}|C_i} \longrightarrow 0.$$ 

Under the embedding $q_2 : W_{pq} \rightarrow C^{(3)}$ sending $L$ to $|L - p - q|$, $C_i$ is a complete intersection with cohomology class $\eta^2 \in H^4(C^{(3)})$. Therefore, $c_1(N_{C_i|C^{(3)}}) = 2$. We also have $c_1(N_{W_{pq}|C^{(3)}|C_i}) = \int_{C^{(3)}|W_{pq}} \cdot [C_i] = \int_{C^{(3)}}(\theta - \eta)\eta^2 = 4$. We conclude that $C_i^2 = -2$.

Now we compute $C_iC'_j$. Suppose $x + g_i \sim p + q + h_j - y$ for some $x, y \in C$. Then

$$D_{2i} := g_i - p - q = h_j - x - y.$$ 

By Claim 10.6, for a fixed $i$, the $g_i$'s containing $D_{2i}$ are $g_i$ and $h_l$ for $l \neq i$. This implies $C_iC'_i = 0$ and $C_iC'_j = 2$ (embedding $C^{(3)}$ in Pic$^3C$, one easily sees that the intersection of $C_i$ and $C'_j$ is transverse for a general choice of $p + q$). 

\qed
Using the formula in Lemma 11.2 and the intersection numbers in Lemma 11.3 we compute

$$\text{AJ}_2^0 : H^2(W_1) \oplus H^2(W_2) \rightarrow H^4(M_2)/\langle j_2 \ast f \rangle$$

(11.4) \( (12[C_i] - 3[C]_{\text{tot}} + 2q^*_2(\eta - \sigma_i), 3[C_i]) \rightarrow -58[p^2_i] + 44 \sum_{j \neq i, j=1}^5 [p^2_j] \mod \langle j_2 \ast f \rangle. \)

It immediately follows that the image of \( \text{AJ}_2^0 \) contains \( \langle [p^2_i] \mid i = 1, \ldots, 5 \rangle \) modulo \( j_2 \ast f \). We then compute that

(11.5) \( \text{AJ}_2^0([C_i], [C'_i]) = -6 \sum_{j \neq i, j=1}^5 [p^2_{j+5}] \mod \langle [p^2_i], j_2 \ast f \mid i = 1, \ldots, 5 \rangle. \)

Proposition 8.2 follows immediately. \( \square \)
11.3. **Proof of Proposition 8.3:** The map $AJ^1: H^1(\widetilde{G_0}^{[1]}) \to H^3(M_{12})$.

It follows from Sections 7.2 and 8.1 that the only double loci of the central fiber $P_0$ inducing non-trivial Abel-Jacobi maps are those which map to $X_{kp}$ or $X_{kq}$ under $\rho_1$ and map to $M_{12}$ under $\rho_2$. These are the slanted lines in the picture in Section 7.2. Recall (see Section 4.1) that $H^1(\widetilde{G_0}^{[1]}) = H^1(X_{1p}) \oplus H^1(X_{1q}) \oplus H^1(X_{2p}) \oplus H^1(X_{2q})$ and $H^3(M_{12}) = \tau_1 \cdot \pi_{12}^* H^1(W_1^1) \oplus j_2^* \cdot \pi_{12}^* H^1(W_4^1)$ (see Lemma 6.3). To prove Proposition 8.3, it is sufficient to prove that the image of the summand $H^1(X_{1q})$ by $AJ^1$ contains $\tau_1 \cdot \pi_{12}^* H^1(W_4^1)$. The map $AJ^1$ on this summand is given by

\[
H^1(X_{1p}) \xrightarrow{\rho_1^*} H^1(E_{1p}) \xrightarrow{\cup [F''_{r_0}|E_{1p}]} H^7(E_{1p}) \xrightarrow{\rho_{2*}} H^3(M_{12}),
\]

where $E_{1p}$ corresponds to the slanted line labeled $b$ in the picture in Section 7.2. Therefore $E_{1p}$ is a $\mathbb{P}^1$-bundle over $X_{1p} \times M_{12}$ and fits into the diagram

\[
\begin{array}{ccc}
E_{1p} & \xrightarrow{\rho_1^*} & H^1(E_{1p}) \xrightarrow{\cup [F''_{r_0}|E_{1p}]} H^7(E_{1p}) \xrightarrow{\rho_{2*}} H^3(M_{12}) \\
\downarrow & & \downarrow \\
X_{1p} \times M_{12} & \xrightarrow{\tau_1} & M_{12} \\
\downarrow & & \downarrow \\
X_{1p},
\end{array}
\]

By the projection formula, to compute (11.6), it suffices to compute the correspondence induced by the push-forward cycle of $[F''_{r_0}|E_{1p}]$ to $X_{1p} \times M_{12}$. Denote $Y'$ the projectivized normal cone of $F''_{r_0} \cap (W_1 \times W_4^1)$ in $F''_{r_0}|W_1 \times \Theta_0$. By construction, $Y'$ has dimension 2 and $Y' = (W_1 \times M_{12}) \cap F''_{r_0}|W_1 \times M_2$.

The components of $Z_1$ which dominate $X_{1p}$ are described in Lemma 10.8. Let $Z_{1p}$ denote the union of these components and let $Y$ be the fiber product $Z_{1p} \times F''_{r_0} \cap (W_k \times W_4^1) Y'$, which is generically a $\mathbb{P}^1$-bundle over $Z_{1p}$ (the $\mathbb{P}^1$ in the ruling corresponds to $\tau_1$ because the map $\lambda_2$ from $F_{(4,1)(2,3)}$ and $F_{(2,3)(4,1)}$
factors through $C^{(4)} \xrightarrow{\phi} \Theta_0$:

$$
\begin{array}{cccc}
Y & \rightarrow & Y' & \rightarrow & M_{12} \\
\downarrow & & \downarrow & & \downarrow_{\pi_{12}} \\
Z_{1p} & \rightarrow & F''_{r_0} \cap (W_k \times W^1_4) & \rightarrow & W^1_4 \\
\downarrow_{\lambda_1} & & \downarrow_{Pr_2} & & \downarrow_{Pr_1} \\
X_{1p} & \rightarrow & W_1. \\
\end{array}
$$

For a real 1-cycle $\alpha$ in general position in $X_{1p}$, the inverse image of $\alpha$ in $Y$ is a $\mathbb{P}^1$-bundle over $\alpha$. The push-forward of the class of this $\mathbb{P}^1$-bundle to $M_{12}$ is a class in $H^3(M_{12})$. As the class of $\alpha$ varies in $H^1(X_{1p}) \cong H^1(W^1_4)$, the class in $H^3(M_{12})$ spans $\tau_1 \cdot \pi_{12}^* H^1(W_4^1)$ because $X_{1p}$ and $W^1_4$ are isomorphic to each other. \qed
11.4. **Proof of Proposition 8.4**: Passage to the $E_2$ terms.

Recall that $\tilde{G}_0$ has four components and $E_2^{0,2} = \text{Gr}_2 H^2(\tilde{G}_0)$ is the kernel of

\[
\begin{array}{ccc}
H^2(\tilde{G}_0^{[0]}) & \xrightarrow{d_1} & H^2(\tilde{G}_0^{[1]}) \\
\cong & & \cong \\
\bigoplus_{k=1}^2 H^2(W_k) \oplus H^2(P_k) & \xrightarrow{\bigoplus_{k=1}^2 H^2(X_{kp}) \oplus H^2(X_{kq})}
\end{array}
\]

Consider the subspace of $\text{Gr}_2 H^2(\tilde{G}_0)$ consisting of $(x_1, x_2, \beta_1, \beta_2)$ with $x_k \in H^2(W_k)$ and $\beta_k \in H^2(P_k)$ such that $\beta_k$ is a multiple of the class of fiber of the $\mathbb{P}^1$-bundle $P_k$. Since we always have

\[
\int_{X_{kp}} \beta_k = \int_{X_{3-kq}} \beta_k,
\]

the compatibility condition defining $\text{Ker}(d_1)$ becomes

\[(11.7)\]

\[
\int_{X_{kp}} x_k = \int_{X_{3-kq}} x_{3-k}.
\]

Because the cycles $F'''_{r_0}|_{P_k \times M_1}$ and $F'''_{r_0}|_{P_k \times M_2}$ come from a base change (Proposition 9.2), the maps $\text{AJ}_1^0$ and $\text{AJ}_2^0$ are trivial on $\beta_k \in H^2(P_k)$. We will therefore write $\text{AJ}_1^0(x_1, x_2) := \text{AJ}_1^0(x_1, x_2, \beta_1, \beta_2)$.

Now start with $(\gamma_1, \gamma_2) \in (I \oplus H^4(M_2)) \cap \text{Gr}_4 H^4(\tilde{\Theta}_0)$. The condition $(\gamma_1, \gamma_2) \in \text{Gr}_4 H^4(\tilde{\Theta}_0)$ means $j_1^* \gamma_1 = j_2^* \gamma_2 \in H^4(M_{12})$ by Proposition 6.4. By Proposition 8.2, we can choose $(x_1, x_2) \in H^2(W_1) \oplus H^2(W_2)$ such that

\[
\gamma_2 - \text{AJ}_2^0(x_1, x_2) \in \langle j_2^* f, j_2^* \tau_1 \rangle.
\]

Furthermore, note that in formula $(11.4)$ and $(11.5)$, we have chosen $x_1$ and $x_2$ so that

\[
\int_{X_{1p}} x_1 = \int_{X_{2q}} x_2, \quad \int_{X_{1q}} x_1 = \int_{X_{2p}} x_2.
\]

Subtracting $(\text{AJ}_1^0(x_1, x_2), \text{AJ}_2^0(x_1, x_2))$ from $(\gamma_1, \gamma_2)$, we may assume $\gamma_2 \in \langle j_2^* f, j_2^* \tau_1 \rangle \subset H^4(M_2)$. Now choose $\omega \in H^2(C^{(3)})$ such that for $i = 1, \ldots, 5$ (see Section 11.2 for the notation),

\[(11.8)\]

\[
\int_{X_{1p}} q_1^* \omega = \int_{X_{1q}} q_1^* \omega = 0
\]
and

\[(11.9) \quad \int_{C_i} q_i^* \omega = \int_{W_1} q_i^* \omega \cdot ([C]_\text{tot} + 4[C'_i] + 2q_i^*(\theta - \eta)) = 0.\]

The equations (11.8) imply \((q_i^* \omega, 0) \in \text{Gr}_2 H^2(\tilde{G}_0)\). The equations (11.9) imply

\[AJ_2^0(q_i^* \omega, 0) \in \langle j_2, f, j_2^* \tau_1 \rangle\]

by the formula for \(AJ_2^0\) in Lemma 11.2.

By the secant plane formula,

\begin{align*}
q_1^*[C_i] &= \frac{1}{2} \theta^2 - \theta \eta + \eta^2, \\
q_1^*[C_i'] &= \eta^2, \\
q_1^*[X_{1p}] &= q_1^*[X_{1q}] = \frac{1}{2} \theta^2 - \theta \eta.
\end{align*}

Therefore the equations (11.8) and (11.9) together impose two conditions on \(\omega\) since

\[\langle q_1^*[C_i], q_1^*(2[C'_i] + q_i^*(\theta - \eta)), q_1^*[X_{1p}], q_1^*[X_{1q}] \rangle = \langle \theta^2, \theta \eta, \eta^2 \rangle = \langle \theta \eta, \eta^2 \rangle = H^4(C^{(3)}).\]

So, if we choose

\[\omega \in \langle \xi_i \xi_j, \sigma_k - \sigma_1 \mid i \neq j \pm 5, k = 2, \ldots, 5 \rangle = \langle \theta \eta, \eta^2 \rangle,\]

by the formula for \(AJ_2^0\) in Lemma 11.2, \(AJ_2^0(q_i^* \omega, 0) \in \langle j_2, f, j_2^* \tau_1 \rangle\). Similarly, we can choose \(\omega' \in H^2(C^{(3)})\) such that \(q_i^*(\omega')\) satisfies (11.8) and

\[AJ_2^0(0, q_i^* \omega') \in \langle j_2, f, j_2^* \tau_1 \rangle.\]

By formula (11.3), if we modify \((\gamma_1, \gamma_2)\) by a linear combination of \((AJ_1^0(q_i^* \omega, 0), AJ_2^0(q_i^* \omega, 0)), (AJ_1^0(0, q_i^* \omega'), AJ_2^0(0, \theta H^2(\text{Pic}^4 C), 0))\), we have \(\gamma_1 = -j_1^* y_1\) and \(\gamma_2 = j_2^* y_2\) for \(y_1, y_2 \in H^2(M_{12})\). But since

\[j_1^* \gamma_1 = j_2^* \gamma_2 \in H^4(M_{12}),\]

we conclude immediately that \(y_1 = y_2\), thus \((\gamma_1, \gamma_2) \in \text{Im}(j_1^*, j_2^*)\).

12. Appendix

12.1. The cohomology of \(C^{(k)}\). For a smooth curve \(C\) of genus \(g\), let \(m\) be the natural map from the Cartesian power \(C^k\) to \(C^{(k)}\). We identify the cohomology \(H^\bullet(C^{(k)})\) with its image under \(m^*\), which is the invariant subring of \(H^\bullet(C^k)\) under the action of the symmetric group \(\mathfrak{S}_k\).
Macdonald [Ma] proved that the cohomology ring $H^\bullet(C^{(k)},\mathbb{Z})$ is generated by (see Notation and Conventions (2))

$$\xi_i \in H^1(C^{(k)},\mathbb{Z}) \cong H^1(\text{Pic}^k(C),\mathbb{Z}), \ i = 1,...,2g$$

and the class $\eta \in H^2(C^{(k)},\mathbb{Z})$ subject to the following relations:

(12.1) \hspace{1cm} \xi_i \xi'_j (\sigma_K - \eta) \eta^d = 0

where $I,J,K$ are mutually disjoint subsets of $\{1,...,g\}$ and $|I| + |J| + 2|K| + d = k + 1$, $\xi_I = \Pi_{i \in I} \xi_i$, $(\sigma_K - \eta) = \Pi_{i \in K} (\sigma_i - \eta)$, etc.

12.2. The secant plane formula [ACGH, p. 342]. Let $|V| \subset |L|$ be a $g_d^r$. Fix $d \geq k \geq r$ and consider the following cycle

$$\{ D \in C^{(k)} | E - D \geq 0 \text{ for some } E \in |V| \} \subset C^{(k)}.$$  

The cohomology class of the above cycle is given by

(12.2) \hspace{1cm} \sum_{i=0}^{k-r} \binom{d - g - r}{l} \frac{\eta^l g^{k-r-l}}{(k-r-l)!}

12.3. The Gysin maps. If $\omega \in H^\bullet(C^k,\mathbb{Z})$, the Gysin push-forward for the sum map

$$m_s : H^\bullet(C^k,\mathbb{Z}) \to H^\bullet(C^{(k)},\mathbb{Z})$$

is given by

$$m_s(\omega) = \sum_{\sigma \in \mathfrak{S}_k} \sigma^*(\omega).$$

If $\omega$ is $\mathfrak{S}_k$-invariant, then

$$m_s(\omega) = k! \ \omega$$

reflecting the fact that $m$ is generically $k!$ to 1.

Fix $k_1 + k_2 = k$, and let $m_1$ and $m_2$ be the symmetrization maps

$C^k \xrightarrow{m_1} C^{(k_1)} \times C^{(k_2)} \xrightarrow{m_2} C^{(k)}.$
For a cohomology class $\omega' \in H^\bullet(C^{(k_1)} \times C^{(k_2)})$ we have

$$m_{2*}(\omega') = \frac{1}{\deg(m_1)} m_*(m_1^*\omega') = \frac{1}{\deg(m_1)} \sum_{\sigma \in S_k} \sigma^*(m_1^*\omega').$$

In our case $g_C = 5$ and we have the following lemmas (whose proofs are straightforward computations).

**Lemma 12.1.** The Gysin map $m_* : H^2(C^{(2)} \times C^{(2)}) \longrightarrow H^2(C^{(4)})$ acts as follows:

- $1 \otimes \theta \mapsto \theta + 10\eta,$
- $1 \otimes \eta \mapsto 3\eta,$
- $\xi_i \otimes \xi_j \mapsto 2\xi_i \xi_j$ for $j \neq i \pm 5,$
- $\xi_i \otimes \xi_{i\pm 5} \mapsto 2\xi_i \xi_{i\pm 5} \mp 2\eta,$
- $\xi_i \xi_j \otimes 1 \mapsto \xi_i \xi_j$ for $j \neq i \pm 5.$

**Lemma 12.2.** The Gysin map $m_* : H^4(C^{(2)} \times C^{(2)}) \longrightarrow H^4(C^{(4)})$ acts as follows:

- $\eta \otimes (\xi_i \cdot \xi_{i+5}) \mapsto \eta \xi_i \xi_{i+5} + \eta^2,$
- $\eta \otimes \xi_i \xi_j \mapsto \eta \xi_i \xi_j$ for $j \neq i \pm 5,$
- $\eta \otimes \eta \mapsto 2\eta^2,$
- $\eta \xi_i \otimes \xi_j \mapsto \eta \xi_i \xi_j$ for $j \neq i \pm 5,$
- $\eta \xi_i \otimes \xi_{i\pm 5} \mapsto \eta \xi_i \xi_{i\pm 5} \mp \eta^2,$
- $\sigma_k \otimes \sigma_k \mapsto 2\sigma_k \eta$ for $k = 1, .., 5,$
- $\sigma_k \otimes \sigma_l \mapsto \sigma_k \sigma_l + \eta^2 \ k \neq l,$
- $\sigma_k \otimes \xi_k \xi_j \mapsto \xi_k \xi_j \eta$ for $j \neq k + 5,$
- $\sigma_k \otimes \xi_i \xi_j \mapsto \sigma_k \xi_i \xi_j$ for $i, j \notin \{k, k + 5\},$
- $\eta^2 \otimes 1 \mapsto \eta^2.$
Lemma 12.3. The Gysin map $m^*: H^4(C \times C^{(3)}) \rightarrow H^4(C^{(4)})$ acts as follows:

\[
\begin{align*}
\eta \otimes \xi_i \xi_j & \mapsto \eta \cdot \xi_i \xi_j \quad \text{for } 1 \leq i, j \leq 10, \\
1 \otimes \eta \xi_i \xi_j & \mapsto \eta \cdot \xi_i \xi_j \quad \text{for } j \neq i \pm 5, \\
1 \otimes \eta \sigma_i & \mapsto \eta \cdot \sigma_i + \eta^2, \\
\xi_i \otimes \xi_j \xi_k \xi_l & \mapsto \xi_i \xi_j \xi_k \xi_l \quad \text{for } j, k, l \neq i \pm 5, \\
\xi_i \otimes \xi_{i \pm 5} \xi_k \xi_l & \mapsto \xi_i \xi_{i \pm 5} \xi_k \xi_l + \eta \cdot \xi_k \xi_l \quad \text{for } k, l \neq i \pm 5, \\
\xi_i \otimes \eta \xi_j & \mapsto \eta \xi_i \xi_j \quad \text{for } j \neq i \pm 5, \\
\xi_i \otimes \eta \cdot \xi_{i \pm 5} & \mapsto \eta \cdot \xi_i \xi_{i \pm 5} + \eta^2, \\
\eta \otimes \eta & \mapsto \eta^2, \\
1 \otimes \eta^2 & \mapsto 2\eta^2.
\end{align*}
\]

12.4. The cycle class of $F''_{r_0}|_{W_1 \times C^{(4)}}$. We use the secant plane formula (Section 12.2) to compute the cycle class of $F''_{r_0}|_{W_1 \times C^{(4)}}$ in each bidegree. For each bidegree $(d_1, d_2) + (e_1, e_2)$, the corresponding cycle $F(d_1, d_2)(e_1, e_2) \subset W_k \times C_1^{(d_1)} \times C_1^{(e_1)} \times C_2^{(d_2)} \times C_2^{(e_2)}$ projects generically injectively to a product of some of the factors. Since the map $\lambda: F(d_1, d_2)(e_1, e_2) \rightarrow W_k \times \Theta_0$ factors through these projections, we only need the cycle class of the projection of $F(d_1, d_2)(e_1, e_2)$.

(1) $(4,1)+(2,3)$ We first compute the class of the projection of $F_{(4,1)(2,3)}$ to $C^{(3)} \times C^{(3)} \times C \times C^{(3)}$ (with the identification $C_1 = C_2 = C$ and the embedding of $W_1$ into $C^{(3)} \times C^{(3)}$ via $(q_1, q_2)$).

The cycles are given by the following conditions

\[
(\Gamma_3, \Gamma'_3, a, D'_3) \in C^{(3)} \times C^{(3)} \times C \times C^{(3)}
\]

\[
\begin{align*}
h_0^0(K_C - p - q - \Gamma_3 - \Gamma'_3) & > 0 \\
h_0^0(K_C - \Gamma'_3 - D'_3) & > 0 \\
h_0^0(K_C - \Gamma_3 - r_0 - a) & > 0
\end{align*}
\]

The map $\lambda_2|_{F_{(4,1)(2,3)}}$ factors through $m$ which sends $(\Gamma_3, \Gamma'_3, a, D'_3)$ to $(\Gamma_3, \Gamma'_3, a + D'_3) \in C^{(3)} \times C^{(3)} \times C^{(4)}$. 

By the secant plane formula (12.2), the cycle class is given by the pull-back under the sum map from $C^{(3)} \times C^{(3)}$ (the first and fourth factor) to $C^{(6)}$ of the class

$$\frac{1}{2} \theta^2 - \eta \theta + \eta^2 \in H^4(C^{(6)})$$

cupped with the pull-back to $C^{(3)} \times C$ (second and third factor) of

$$\theta - \eta \in H^2(C^{(4)}),$$

then restriction to $W_1 \times C \times C^{(3)}$. Thus we obtain (c.f. Notation (4))

$$\left[\frac{1}{2} (\theta + \eta + \delta_{24})^2 - (\eta + \eta_4)(\theta + \eta + \delta_{24}) + (\eta + \eta_4)^2\right] \cdot [(\theta_1 + \theta_3 + \delta_{13}) - (\eta_1 + \eta_3)].$$

We only need the Kunneth component of this cycle class in $H^2(C^{(3)} \times C^{(3)}) \otimes H^4(C \times C^{(3)})$. We organize the terms according to the types in the Kunneth decomposition.

(a) Type $(2, 0, 0, 4)$.

$$\left(\frac{1}{2} \theta_4^2 - \theta_3 \eta_4 + \eta_4^2\right) (\theta_1 - \eta_1) = \left(\sum_{i<j} [\sigma_{4i} \sigma_{4j}] - \theta_4 \eta_4 + \eta_4^2\right) (\theta_1 - \eta_1)$$

$$= \left(\sum_{i<j} [(\sigma_{4i} + \sigma_{4j}) \cdot \eta_4 - \eta_4^2] - \theta_4 \eta_4 + \eta_4^2\right) (\theta_1 - \eta_1)$$

$$= 3 (\theta_4 \eta_4 - 3 \eta_4^2) (\theta_1 - \eta_1)$$

(b) Type $(0, 2, 2, 2)$

$$\left(\frac{1}{2} \delta_{24}^2 + \theta_2 \theta_4 - \theta_2 \eta_4 - \theta_4 \eta_2 + 2 \eta_2 \eta_4\right) (\theta_3 - \eta_3) = \left(\frac{1}{2} \delta_{24}^2 + \theta_2 \theta_4 - \theta_2 \eta_4 - \theta_4 \eta_2 + 2 \eta_2 \eta_4\right) 4 \eta_3$$

(c) Type $(1, 1, 1, 3)$

$$(\theta_4 \delta_{24} - \eta_4 \delta_{24}) \delta_{13}$$

By Lemma 12.3, the push-forwards of these classes to $C^{(3)} \times C^{(3)} \times C^{(4)}$ are

(a)

$$m_* \left(3 \theta_4 \eta_4 - 9 \eta_4^2\right) (\theta_1 - \eta_1) = 3 (\theta_3 \eta_3 - \eta_3^2) (\theta_1 - \eta_1)$$
(b) 

\[ m_* \left( \frac{1}{2} \delta_{24}^2 + \theta_2 \theta_4 - \theta_2 \eta_4 - \theta_4 \eta_2 + 2 \eta_2 \eta_4 \right) 4 \eta_3 \]

\[ = m_* \left( 2 \eta_3 \delta_{24}^2 \right) + 4 m_* \left[ (\theta_2 \theta_4 - \theta_2 \eta_4 - \theta_4 \eta_2 + 2 \eta_2 \eta_4) \eta_3 \right] \]

\[ = m_* 2 \eta_3 \sum_{i,j=1}^{5} \left[ -\xi_{2i} \xi_{2j} \xi_{3i} \xi_{3j} + 2 \xi_{2i} \xi_{2j} \xi_{3i} \xi_{3j} - \xi_{2i} \xi_{2j} \xi_{3i} \xi_{3j} \right] \]

\[ + 4 m_* \left[ (\theta_2 \theta_4 - \theta_2 \eta_4 - \theta_4 \eta_2 + 2 \eta_2 \eta_4) \eta_3 \right] \]

\[ = 2 \eta_3 \sum_{i,j=1}^{5} \left[ -\xi_{2i} \xi_{2j} \xi_{3i} \xi_{3j} + 2 \xi_{2i} \xi_{2j} \xi_{3i} \xi_{3j} - \xi_{2i} \xi_{2j} \xi_{3i} \xi_{3j} \right] \]

\[ + 4 \left[ \theta_2 \eta_3 \theta_3 - \theta_2 \eta_3^2 - \eta_2 \eta_3 \theta_3 + 2 \eta_2 \eta_3^2 \right] \]

\[ = 2 \eta_3 \delta_{23}^2 + 4 \left[ \theta_2 \eta_3 \theta_3 - \theta_2 \eta_3^2 - \eta_2 \eta_3 \theta_3 + 2 \eta_2 \eta_3^2 \right] \]

(c) For \( i \neq j \), using the formula

\[ m_* \xi_{3i} \xi_{4j} \sigma_{4k} = \begin{cases} 0, & k = j \\ \xi_{3i} \xi_{4j} \eta_3, & k = i \\ \xi_{3i} \xi_{4j} \sigma_{3k}, & k \neq i, j \end{cases} \]

and

\[ m_* \xi_{3i} \xi_{4j} \eta_4 = \xi_{3i} \xi_{3j} \eta_3, \]
we compute that

\[(12.5) \quad m_*(\theta_4 - \eta_4) \delta_{13} \delta_{24} \]

\[= m_*(\theta_4 - \eta_4) \sum_{i,j=1}^{5} \left[ -\xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} - \xi_{1i} \xi'_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} \right] \]

\[= m_*(\theta_4 - \eta_4) \sum_{i=1}^{5} \left[ -\xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} - \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} \right] \]

\[+ m_*(\theta_4 - \eta_4) \sum_{i,j=1, i\neq j}^{5} \left[ -\xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} - \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} \right] \theta_3 \]

\[= 0 + \sum_{i=1}^{5} \left( -\sigma_3 \theta_3 + \eta_3 \theta_3 - \eta_3^2 \right) + \xi_{1i} \xi_{2j} (\sigma_3 \theta_3 - \eta_3 \theta_3 + \eta_3^2) \]

\[+ \sum_{i,j=1, i\neq j}^{5} \left[ -\xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} - \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} \right] \theta_3 \]

\[= \delta_{12}(\eta_3 \theta_3 - \eta_3^2) + \delta_{13} \delta_{23} \theta_3 \]

(2) (3.2)+(4.1)

The cycle is

\[
\{ (\Gamma_3, \Gamma_3', a', D_3) \in C^{(3)} \times C^{(3)} \times C \times C^{(3)} \mid h^0(\mathcal{O}_C(K_C - r_0 - \Gamma_3 - D_3)) > 0 \}.
\]

The map \(m\) sends \((\Gamma_3, \Gamma_3', a', D_3)\) to \((\Gamma_3, \Gamma_3', a' + D_3)\) \(\in C^{(3)} \times C^{(3)} \times C^{(4)}\).

Its class is the pull-back under the sum map to \(H^6(C^{(3)} \times C^{(3)})\) of

\[
\frac{\theta^3}{6} - \frac{\eta \theta^2}{2} + \eta^2 \theta - \eta^3 \in H^6(C^{(6)}).
\]

which is equal to

\[
\frac{1}{6} (\theta_1 + \theta_4 + \delta_{14})^3 - \frac{1}{2} (\eta_1 + \eta_4) (\theta_1 + \theta_4 + \delta_{14})^2 + (\eta_1 + \eta_4)^2 (\theta_1 + \theta_4 + \delta_{14}) - (\eta_1 + \eta_4)^3.
\]
The contributing terms in the Kunneth decomposition have type (2, 0, 0, 4):

\[
\frac{1}{2} (\theta_1 \theta_4^2 + \theta_4 \delta_{14}^2) - \frac{1}{2} (\eta_1 \theta_4^2 + \eta_4 \delta_{14}^2) - \theta_1 \eta_4 \theta_4 + 2 \eta_1 \eta_4 \theta_4 + \theta_1 \eta_4^2 - 3 \eta_1 \eta_4^2
\]

\[
= \frac{1}{2} (\theta_1 - \eta_1) (8 \theta_4 \eta_4 - 20 \eta_1^2) + \frac{1}{2} (\theta_4 - \eta_4) \delta_{14}^2 + (2 \eta_1 - \theta_1) \eta_4 \theta_4 + (\theta_1 - 3 \eta_1) \eta_4^2
\]

\[
= 2 (\theta_1 - \eta_1) (2 \theta_4 \eta_4 - 5 \eta_1^2) + (\eta_4 \delta_{14}^2 + 4 \theta_1 \eta_4^2 - \theta_1 \eta_4 \theta_4) + (2 \eta_1 - \theta_1) \eta_4 \theta_4 + (\theta_1 - 3 \eta_1) \eta_4^2
\]

\[
= \eta_4 \delta_{14}^2 + 2 (\theta_1 - \eta_1) \eta_4 \theta_4 + (-5 \theta_1 + 7 \eta_1) \eta_4^2
\]

Pushing forward to \( C^{(3)} \times C^{(3)} \times C^{(4)} \):

\[
m_+ [\eta_4 \delta_{14}^2 + 2 (\theta_1 - \eta_1) \eta_4 \theta_4 + (-5 \theta_1 + 7 \eta_1) \eta_4^2]
\]

\[
= (\eta_3 \delta_{13}^2 - 2 \theta_1 \eta_3^2) + 2 (\theta_1 - \eta_1) (\eta_3 \theta_3 + 5 \eta_3^2) + 2 (-5 \theta_1 + 7 \eta_1) \eta_3^2
\]

\[
= \eta_3 \delta_{13}^2 + 2 (\theta_1 - \eta_1) \eta_3 \theta_3 + 2 (-\theta_1 + 2 \eta_1) \eta_3^2
\]

(3) (2,3)+(2,3)

The cycle consists of \((\Gamma_3, \Gamma_3', D_2, D_2') \in C^{(3)} \times C^{(3)} \times C^{(2)} \times C^{(2)}\) given by the conditions

\[
\begin{cases}
    h^0(K_C - \Gamma_3 - D_2) > 0, \\
    h^0(K_C - \Gamma_3' - D_2') > 0, \\
    r_0 \in D_2.
\end{cases}
\]

The map \(m\) sends \((\Gamma_3, \Gamma_3', D_2, D_2')\) to \((\Gamma_3, \Gamma_3', D_2 + D_2') \in C^{(3)} \times C^{(3)} \times C^{(4)}\). Note that in this bidegree, \(m\) is not a lifting of \(\lambda_2|_{F_{(2,3)(2,3)}}\). As in the previous case, the cycle class is the restriction to \(W_1 \times C^{(2)} \times C^{(2)}\) of

\[
[(\theta_1 + \theta_3 + \delta_{13}) - (\eta_1 + \eta_3)] \cdot [(\theta_2 + \theta_4 + \delta_{24}) - (\eta_2 + \eta_4)] \cdot \eta_3.
\]

The contributing terms in the Kunneth decomposition are

(a) Type (2, 0, 2, 2)

\[
[\theta_1 \theta_4 + \eta_1 \eta_4 - \eta_1 \theta_4 - \theta_1 \eta_4] \cdot \eta_3.
\]

(b) Type (2, 0, 4, 0)

\[
(\theta_3 - \eta_3)(\theta_2 - \eta_2)\eta_3 = 4(\theta_2 - \eta_2) \cdot \eta_3^2.
\]
(c) Type (1, 1, 3, 1)

\[ \delta_{13} \delta_{24} \cdot \eta_3. \]

Pushing these classes forward to \( H^4(C^{(3)} \times C^{(3)} \times C^{(4)}) \) by \( m_* \), we obtain

(a)

\[ m_* \left[ (\theta_1 - \eta_1)\eta_3 \theta_4 + (\eta_1 - \theta_1)\eta_3 \eta_4 \right] = (\theta_1 - \eta_1) \left( \eta_3 \theta_3 + 5\eta_3^2 \right) + (\eta_1 - \theta_1) \left( 2\eta_3^2 \right) \]

\[ = (\theta_1 - \eta_1) \eta_3 \theta_3 + 3(\theta_1 - \eta_1)\eta_3^2 \]

(b)

\[ m_* \left[ 4(\theta_2 - \eta_2) \cdot \eta_3^2 \right] = 4(\theta_2 - \eta_2) \cdot \eta_3^2 \]

(c)

\[ m_* \left[ (\delta_{13} \delta_{24}) \eta_3 \right] \]

\[ = m_* \sum_{i,j=1}^{5} \left[ -\xi_{1i} \xi_{2j} \xi_{3i}' \xi_{4j}' + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{4j} + \xi_{1i}' \xi_{2j} \xi_{3i} \xi_{4j}' - \xi_{1i}' \xi_{2j}' \xi_{3i}' \xi_{4j} \right] \eta_3 \]

\[ = \sum_{i,j=1}^{5} \left[ -\xi_{1i} \xi_{2j} \xi_{3i}' \xi_{3j}' + \xi_{1i} \xi_{2j} \xi_{3i} \xi_{3j} + \xi_{1i}' \xi_{2j} \xi_{3i} \xi_{3j}' - \xi_{1i}' \xi_{2j}' \xi_{3i}' \xi_{3j} \right] \eta_3 + \eta_3^2 \sum_{i=1}^{5} \left[ \xi_{1i} \xi_{2i}' - \xi_{1i}' \xi_{2i} \right] \]

\[ = \delta_{13} \delta_{23} \eta_3 + \delta_{12} \eta_3^2 \]

Finally, since \( \lambda_2 \) sends \( (\Gamma_3, \Gamma_3', D_2, D_2') \) to \( K_C(-D_2 - D_2') \) (instead of \( O_C(D_2 + D_2') \)), we apply the involution \( p_2^* p_1^* \) to the sum of the classes in (a), (b), (c) as in Lemma 12.4 to obtain the cycle class

\[ (\theta_1 - \eta_1) \theta_3 (\theta_3 - \eta_3) + 3(\theta_1 - \eta_1) \left( \frac{1}{2} \theta_3^2 - \theta_3 \eta_3 + \eta_3^2 \right) \]

\[ + 4(\theta_2 - \eta_2) \left( \frac{1}{2} \eta_3^2 - \theta_3 \eta_3 + \eta_3^2 \right) + \delta_{13} \delta_{23} (\theta_3 - \eta_3) + \delta_{12} \left( \frac{1}{2} \theta_3^2 - \theta_3 \eta_3 + \eta_3^2 \right). \]

Lemma 12.4. The correspondence

\[ M_1 = \{ (D_4, B_4) \in C^{(4)} \times C^{(4)} \mid D_4 + B_4 \equiv K_C \} \]
induces an involution $p_2^*p_1^*: H^4(C^{(4)}) \to H^4(C^{(4)})$ where $p_1$ and $p_2$ are the two birational projections to $C^{(4)}$. Under the decomposition

$$H^4(C^{(4)}) \cong H^4(\text{Pic}^4 C) \oplus \eta H^2(\text{Pic}^4 C) \oplus C \cdot \eta^2,$$

$p_2^*p_1^*$ acts as identity on $H^4(\text{Pic}^4 C)$, sends $\eta \cdot \omega$ to $(\theta - \eta) \cdot \omega$ for any $\omega \in H^2(\text{Pic}^4 C)$, and $\eta^2$ to $\frac{\theta^2}{2} - \eta \theta + \eta^2$.

**Proof.** First note that the proper transform of the algebraic cycle $r_0 + C^{(3)}$ under the birational map $p_2p_1^{-1}$ is the cycle

$$\{ B_4 \in C^{(4)}| \ h^0(K_C - r_0 - B_4) > 0 \}$$

whose cohomology class is $\theta - \eta$ by the secant plane formula (12.2). Therefore $p_2^*p_1^*$ sends $\eta$ to $\theta - \eta$. Similarly, the proper transform of $2r_0 + C^{(2)}$ is

$$\{ B_4 \in C^{(4)}| \ h^0(K_C - 2r_0 - B_4) > 0 \}$$

whose cohomology class is $\frac{\theta^2}{2} - \eta \theta + \eta^2$, i.e.

$$p_2^*p_1^*\eta^2 = \frac{\theta^2}{2} - \eta \theta + \eta^2.$$

Now let us prove the statement on the summand $\eta H^2(\text{Pic}^4 C)$. Consider the commutative diagram

$$\begin{array}{ccc}
M_1 & \xrightarrow{p_1} & C^{(4)} \\
\downarrow & & \downarrow \phi \\
\text{Pic}^4 C & \xleftarrow{\tau} & \text{Pic}^4 C \\
\downarrow & & \downarrow \phi \\
& & C^{(4)} \\
\end{array}$$

where $\tau$ sends $L$ to $K_C - L$. For any $\omega \in H^2(C^{(4)})$,

$$p_1^*(\eta \cdot \phi^* \omega) = p_1^*\eta \cdot p_1^*\phi^* \omega = p_1^*\eta \cdot (p_2^*\phi^* \tau^* \omega)$$

By the projection formula,

$$p_2^*p_1^*(\eta \cdot \phi^* \omega) = (p_2^*p_1^*\eta) \cdot (\phi^* \tau^* \omega) = (\theta - \eta) \cdot \phi^* \omega$$
Similarly, the statement about the $H^4(\text{Pic}^4 C)$ summand is a consequence of the projection formula. □

12.5. The reducedness of $W^1_5(C_{pq})$ and of its compactification $\overline{W}^1_5(C_{pq})$.

Lemma 12.5. The surface $\overline{W}^1_5(C_{pq})$ is reduced and is the flat limit of the family of $W^1_5(C_t)$ as $t$ goes to 0.

Proof. We will prove that $\overline{W}^1_5(C_{pq})$ with its reduced scheme structure is the flat limit of the family of $W^1_5(C_t)$ as $t$ goes to 0. By [So] the family of theta divisors specializes to the ample Cartier divisor 

$$\Theta_{pq} := \{ M \in J^5 C_{pq} \mid h^0(M) > 0 \}$$

on $J^5 C_{pq}$. We will prove that the Hilbert polynomial of $\overline{W}^1_5(C_{pq})$ with its reduced scheme structure and with respect to $\Theta_{pq}$ is equal to the Hilbert polynomial of $W^1_5(C_t)$ with respect to $\Theta_t$ for $t \neq 0$.

To compute the Hilbert polynomial of $\overline{W}^1_5(C_{pq})$, we use the normalization map (see Lemma 2.1)

$$\mu = (\nu^*)^{-1} : W_{pq} \longrightarrow \overline{W}^1_5(C_{pq}).$$

From this we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{W}^1_5(C_{pq})} \longrightarrow \mu_* \mathcal{O}_{W_{pq}} \longrightarrow \mathcal{M} \longrightarrow 0$$

where $\mathcal{M}$ is a sheaf supported on the image of $W^1_4(C)$ in $\overline{W}^1_5(C_{pq})$. It is immediately seen, by restricting the above sequence to $W^1_4(C)$, that

$$\mathcal{M} \cong \mathcal{O}_{W^1_4(C)}$$

so that we have the exact sequence

$$(12.8) \quad 0 \longrightarrow \mathcal{O}_{\overline{W}^1_5(C_{pq})} \longrightarrow \mu_* \mathcal{O}_{W_{pq}} \longrightarrow \mathcal{O}_{W^1_4(C)} \longrightarrow 0.$$

To compute $\chi(\mathcal{O}_{\overline{W}^1_5(C_{pq})}(n\Theta_{pq}))$, we therefore compute $\chi(\mathcal{O}_{W_{pq}}(n\Theta_{pq}))$ and $\chi(\mathcal{O}_{W^1_4(C)}(n\Theta_{pq}))$.

By [BC, p. 57], the inverse image of the divisor $\Theta_{pq}$ in $\mathbb{P} \text{Pic}^5 C_{pq}$ is numerically equivalent to the sum of reduced divisors

$$\overline{(\nu^*)^{-1}\Theta_{C,x} + \text{Pic}^5_0}$$

where we use the notation of 2.1, $\Theta_{C,x}$ is the image of $\Theta_C \subset \text{Pic}^4 C$ in $\text{Pic}^5 C$ by the addition of the general point $x \in C$ and $\overline{(\nu^*)^{-1}\Theta_{C,x}}$ is the closure of $\overline{(\nu^*)^{-1}\Theta_{C,x}} \subset \text{Pic}^5 C_{pq}$ in $\mathbb{P} \text{Pic}^5 C_{pq}$. 
Now, we have
\[(\nu^*)^{-1}\Theta_{C,x} = \{ M \in \text{Pic}^5 C_{pq} \mid h^0(\nu^*M(-x)) > 0 \}.\]
The trace of \((\nu^*)^{-1}\Theta_{C,x}\) on the image of \(W_{pq}\) in \(\mathbb{P}\text{Pic}^5 C_{pq}\) is reduced for a general choice of \(x\) and is equal to
\[\Theta_{C,x}|_{W_{pq}} = \{ L \in W_{pq} \mid h^0(L(-x)) > 0 \}.\]
Furthermore, it is immediate that \(\text{Pic}^5_0|_{W_{pq}} = X_q\).
To compute the degree of \(\Theta_{pq}\) on \(W_1^4(C)\), we use the isomorphism \(W_1^4(C) \cong X_p\). In this way we immediately see that the restriction of \(\text{Pic}^5_0\) to \(W_1^4(C)\) is zero while \((\nu^*)^{-1}\Theta_{C,x}\) pulls back to \(\Theta_{C}|_{W_1^4(C)}\) via the natural embedding \(W_1^4(C) \subset \text{Pic}^4 C\). Therefore, summarizing the above, we have
\[\chi(n\Theta_{pq}|_{W_{pq}}) = \chi(n\Theta_{C,x}|_{W_{pq}} + X_q)\]
and
\[\chi(n\Theta_{pq}|_{W_1^4(C)}) = \chi(n\Theta_{C}|_{W_1^4(C)}).\]
To compute \(\chi(n\Theta_{C}|_{W_{pq}})\), we use the embedding \(q_1\) of \(W_{pq}\) in \(C^{(3)}\) given by \(g_5^1 \mapsto |K - g_5^1|\). Via this embedding \(W_{pq}\) is identified with the reduced surface in \(C^{(3)}\)
\[\{ \Gamma_3 \mid h^0(K_C - p - q - \Gamma_3) > 0 \}\]
whose cohomology class by the secant plane formula (Section 12.2) is \(\theta - \eta\). By Hirzebruch-Riemann-Roch
\[\chi(n(\Theta_C|_{W_{pq}} + X_q)) = \frac{1}{2} n(\Theta_C|_{W_{pq}} + X_q)(c_1(T_{W_{pq}}) + n(\Theta_C|_{W_{pq}} + X_q)) + \frac{1}{12} (c_1^2(T_{W_{pq}}) + c_2(T_{W_{pq}})).\]
By [Ma, p. 332 (14.5)], the total Chern class of \(C^{(3)}\) is
\[(1 + \eta)^6 \prod_{i=1}^{5}(1 + \eta + \sigma_i) = 1 - \eta - \theta - 9\eta^2 + 6\eta\theta - 56\eta^3.\]
So, using the tangent bundle sequence
\[0 \rightarrow T_{W_{pq}} \rightarrow T_{C^{(3)}|_{W_{pq}}} \rightarrow \mathcal{O}_{W_{pq}}(W_{pq}) \rightarrow 0,\]
we compute
\[ c_1(T_{W_{pq}}) = (1 - 2\theta - 9\eta^2 + 4\eta \cdot \theta + 2\theta^2) |_{W_{pq}}. \]

Now, since \( X_q \) is the restriction of the zero section of a \( \mathbb{P}^1 \)-bundle to \( W_{pq} \), we have
\[ X_q^2 = 0. \]

Furthermore, the degree of \( \Theta_C \) on \( X_q \) is 10 since this is a Prym-embedded curve in \( \text{Pic}^4 C \). By the above,
\[ c_1(T_{W_{pq}}) = 2\theta|_{W_{pq}}, \]

hence the degree of \( c_1(T_{W_{pq}}) \) on \( X_q \) is 20. Putting all this together with the relations in [Ma, p. 325 (6.3)], we obtain
\[ \chi(n\Theta_{pq}|_{W_{pq}}) = 30n^2 - 50n + 22. \]

To compute \( \chi(n\Theta_C|_{W_4^1(C)}) \), note that \( W_4^1(C) \) has genus 11 and its cohomology class in \( \text{Pic}^4 C \) is twice the minimal class, i.e.,
\[ [W_4^1(C)] = 2[\Theta_C]^4/4!. \]

Therefore, by Riemann-Roch for curves,
\[ \chi(n\Theta_C|_{W_4^1(C)}) = 1 - 11 + \deg(n\Theta_C|_{W_4^1(C)}) = 10n - 10. \]

Finally, by (12.8),
\[ \chi(O_{W_5^1(C_{pq})}(n\Theta_{pq})) = \chi(n\Theta_{pq}|_{W_{pq}}) - \chi(n\Theta_{pq}|_{W_4^1(C)}) = 30n^2 - 60n + 32. \]

To compute the Hilbert polynomial of \( W_5^1(C_t) \) for \( t \neq 0 \), we only need to do so for one smooth curve \( X \) of genus 6 such that \( \dim_C W_5^1(X) = 2 \). If \( X \) is trigonal, \( W_5^1(X) \) is the reduced union of two copies of \( X^{(2)} \) (see [T]):
\[ W_5^1(X) = X^{(2)} + g_3^1 \cup K_X - (X^{(2)} + g_3^1). \]

The intersection of these two components is the reduced curve
\[ X_2(g_4^1) = \{ D_2 \mid h^0(g_4^1 - D_2) > 0 \} \subset X^{(2)} \]
where $g_4^1 = |K_X - 2g_3^1|$. As in the previous case, we have the normalization exact sequence

$$0 \to \mathcal{O}_{W_3^1(X)} \to \mu_* \mathcal{O}_{X^{(2)}} |_{X^{(2)}} \to \mathcal{O}_{X_2(g_4^1)} \to 0.$$ 

So

$$\chi\left( n\Theta_X|_{W_3^1(X)} \right) = 2\chi\left( n\Theta_X|_{X^{(2)}} \right) - \chi\left( n\Theta_X|_{X_2(g_4^1)} \right).$$

This time, using similar methods, we compute

$$\chi\left( n\Theta_X|_{X^{(2)}} \right) = 15n^2 - 24n + 10,$$

$$\chi\left( n\Theta_X|_{X_2(g_4^1)} \right) = 12n - 12$$

and

$$\chi\left( n\Theta_X|_{W_3^1(X)} \right) = 30n^2 - 60n + 32.$$

\[\square\]

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