TRACES FOR FUNCTIONS OF BOUNDED VARIATION
ON MANIFOLDS WITH APPLICATIONS TO
CONSERVATION LAWS ON MANIFOLDS WITH
BOUNDARY

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Abstract. In this paper we show existence of a trace for functions of bounded variation on
Riemannian manifolds with boundary. The trace, which is bounded in $L^\infty$, is reached via $L^1$-convergence and allows an
integration by parts formula. We apply these results in order to show well-posedness and total variation estimates for the initial boundary
value problem for a scalar conservation law on compact Riemannian
manifolds with boundary in the context of functions of bounded vari-
ation via the vanishing viscosity method. The flux function is assumed
to be time-dependent and divergence-free.

1. Introduction

Numerous applications in continuum dynamics are modeled by hyperbolic
conservation laws, often posed on surfaces or manifolds. Examples are the
shallow water equations on a sphere, relativistic flows, transport processes
on interfaces or cell surfaces, just to mention a few. If the physical domain
contains a boundary or only a part of a larger closed manifold is of inter-
est, one has to consider initial boundary value problems on manifolds with
boundary.

In this work we will show existence of a trace for BV functions on Rie-
mannian manifolds with boundary and conclude some properties in order
to prove existence and uniqueness in the space of BV functions, and to-
tal variation estimates for a solution $u : \bar{M} \times (0,T) \to \mathbb{R}$ of the following
problem:

\begin{align*}
\partial_t u + \text{div}_g(f(u,x,t)) &= 0 \quad \text{in } M \times (0,T), \\
u(\cdot,0) &= u_0 \quad \text{in } M, \\
u &= 0 \quad \text{on the outflow part (cf. (4.17)) of } \partial M,
\end{align*}

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where $\bar{M}$ is a smooth, compact manifold with boundary and smooth Riemannian metric $g$, $M = \bar{M} \backslash \partial M$, $f(u, \cdot, t)$ a family of smooth vector fields on $\bar{M}$ parametrized smoothly by $u \in \mathbb{R}$ and $t \in [0, T]$ with $|\partial_u f|_g \in L^\infty(\mathbb{R} \times M \times (0, T))$, $(\text{div}_g f)(u, x, t) = 0$ and $u_0 \in BV(M)$.

Let us briefly summarize related results. The theory on BV functions and their traces is well established for Euclidean domains. For a comprehensive introduction to this subject we refer to [1, 12, 15]. Studying heat semigroups on Riemannian manifolds, parts of the theory for BV functions have been generalized to Riemannian manifolds without boundary in [22]. In particular, the heat semigroup allows an appropriate mollification of BV-functions in this setting. BV functions in a more general understanding are studied by Vittone [26]. He shows existence and some properties of a trace operator for BV functions defined on a special domain with compact boundary in Carnot-Carathéodory spaces and where the bounded variation is defined with respect to a family of vector fields.

The theory on BV functions plays a central role in the study of conservation laws in Euclidean space. Indeed, they form a natural solution space as the property of a bounded variation is conserved for scalar equations with respect to time evolution, and numerical analysis for these PDEs heavily relies on this regularity. For material on conservation laws in the Euclidean case we refer to [10] and the references therein. The initial boundary value problem for the Euclidean case was first solved by Bardos et al. [6] exploiting the fact that functions of bounded variation admit traces on the boundary. It was Otto [23], who established $L^\infty$-theory for the initial boundary value problem. To our knowledge, conservation laws on manifolds were studied for the first time by [24] showing existence and uniqueness for a geometry-independent variant of (1.1). For the case of compact manifolds without boundary the theoretical and numerical study of conservation laws on manifolds has reached significant progress [8, 2, 7, 3, 19, 11, 20, 13, 14] during the last decade. The Dirichlet problem for a geometry-independent formulation of (1.1) was addressed by Panov using a kinetic formulation.

Our results on traces as well as on well-posedness for the initial boundary value problem seem to be new in the context of Riemannian manifolds with boundary. Notation and preliminaries are presented in Section 2. In Section 3 we show existence of a trace via a partition of unity. This definition, based on local terms, turns out to be well-posed and thus, independent of the choice of coordinates. A key result for the application to conservation laws is the partial integration formula (3.1), which also guarantees uniqueness of the trace. Section 4 is devoted to the study of the initial boundary value problem. To this end a parabolic regularization of (1.1) is considered, adding small viscosity. Estimates for the solution of the regularized problem, uniform in the viscosity parameter, guarantee convergence of a subsequence to the entropy solution. Uniqueness is proved by transferring Kružkov’s doubling of variables to our setting.
2. Notation and preliminaries

In this section we give a short overview on Riemannian geometry, for a comprehensive introduction see e.g. [9, 18].

Throughout the whole paper, let $M$ be an $n$-dimensional, compact, oriented, smooth manifold with boundary $\partial M$. The inner of $M$, which is a manifold without boundary, we denote with $M := M \setminus \partial M$. Let $g := \langle \cdot, \cdot \rangle_g$ be a smooth Riemannian metric defined on $M$ and $\nabla^g = \nabla$ the associated Levi-Civita connection on the tangential bundle $TM$. With $\Gamma(TM)$ we denote the set of differentiable vector fields on $M$ and with $\Gamma_0(TM)$ the set of differentiable fields on $M$ of compact support contained in $M$. The space of smooth $(r,s)$-tensor fields is denoted by $\Gamma(T^r_s M)$. We call $(M, g)$ a Riemannian manifold. The pair $(\partial M, \bar{g})$ where $\bar{g}$ is the $g$-induced metric on $\partial M$, is an $(n-1)$-dimensional Riemannian manifold. The Riemannian distance of two points $x, y \in \bar{M}$ will be denoted by $d_g(x, y)$ and the geodesic ball around $x$ with radius $\rho$ by $B^g_\rho(x)$. Using Einstein’s summation convention we write for the scalar product of two tangential vectors $X, Y$ locally $\langle X, Y \rangle_g = g_{ij}X^iY^j$. The summation convention will be used throughout the whole paper. For $(r,s)$-tensors $F, G$ we define

$$\langle F, G \rangle_g := g^{i_1k_1} \ldots g^{i_\ell k_\ell} g_{j_1l_1} \ldots g_{j_\ell l_\ell} F_{i_1 \ldots i_\ell}^{j_1 \ldots j_\ell} G_{k_1 \ldots k_\ell}^{l_1 \ldots l_\ell}.$$ 

For the $g$-induced norm of a tensor $F$ we write $|F|_g := \langle F, F \rangle_g^{\frac{1}{2}}$. In local coordinates we write $X^i_j := \partial_j X^i + X^k \Gamma^i_{jk}$ for the covariant derivative of a vector field $X$ with the Christoffel symbols $\Gamma^i_{jk}$. By $\nabla^k$ we denote the $k$-fold application of $\nabla$. We can associate each covector field with a vector field by lowering the index. E.g., the covariant vector field $\nabla u$ can be associated with the vector field $\text{grad}_g u$ by $(\nabla u)_i = u_{,i} = \partial_i u = g_{ij} g^{jk} \partial_k u = g_{ij} (\text{grad}_g u)^j = g_{ij} u^j$. A generalization of the connection $\nabla$ for $(r,s)$-tensors $A \in \Gamma(T^r_s M)$ is given by

$$\nabla : \Gamma(TM) \times \Gamma(T^r_s M) \to \Gamma(T^r_s M)$$

$$(X, A) \mapsto \nabla_X A$$

with

$$(\nabla_X A)(\omega^1, \ldots, \omega^r, Y_1, \ldots Y_s) := X(A(\omega^1, \ldots, \omega^r, Y_1, \ldots Y_s)) - \sum_{j=1}^r A(\omega^1, \ldots, \nabla_X \omega^j, \ldots, \omega^r, Y_1, \ldots Y_s) - \sum_{i=1}^s A(\omega^1, \ldots, \omega^r, Y_1, \ldots, \nabla_X Y_i, \ldots Y_s).$$

For a smooth vector field $X$, we define $\text{div}_g X$ by

$$\int_M u \, \text{div}_g X \, dv_g = - \int_M \langle \text{grad}_g u, X \rangle_g \, dv_g \quad \forall u \in C_0^\infty(M)$$

where locally $dv_g = \sqrt{|g|} \, dz$ denotes the Riemannian volume element for positively oriented coordinates $z \in \mathbb{R}^n$ and $|g| := \det(g_{i,j})$. In local coordinates this yields $\text{div}_g X = X^i_{,i} = \frac{1}{\sqrt{|g|}} \partial_i (X^i \sqrt{|g|})$. For an arbitrary
(r, 0)-tensor $\alpha$ we define
\[
\text{div}_g(\alpha)_{i_1...i_{r-1}} := g^{jl} \nabla_j \alpha_{i_1...i_{r-1}} = \nabla^j \alpha_{ji_1...i_{r-1}}
\]
with $\nabla_j = \nabla_{\partial_j}$. We define the Laplace-Beltrami $\Delta_g u$ for $u \in C^\infty(M)$ as
\[
\Delta_g u := \text{div}_g(\text{grad}_g u)
\]
and in general
\[
\Delta_g = \text{div}_g \nabla = \text{trace} \nabla^2.
\]
For $u \in C^\infty(M)$ we define the commutator
\[
[\Delta_g, \nabla]u := \Delta_g \nabla u - \nabla \Delta_g u.
\]
Let $\mathcal{R}$ denote the Riemannian curvature tensor, i.e.
\[
\mathcal{R}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
\]
for $X, Y, Z \in \Gamma(TM)$ and $\text{Ric}$ the Ricci tensor defined in local coordinates as
\[
\text{Ric}_{jk} := \mathcal{R}_{ijk}^i.
\]
One can prove (cf. [20]) that in local coordinates
\[
\text{Ric}_{jk} u^k = \mathcal{R}_{ijk}^i u^k = u_{;ji} - u_{;ij}
\]
and thus
\[
(2.1) \quad [\Delta_g, \nabla]u = \Delta_g \nabla u - \nabla \Delta_g u = \text{Ric}(\nabla u, \cdot).
\]
By $\text{vol}_g(U) := \int_U 1 \, dv_g$ for a subset $U \subset M$ we define the volume measure on $M$ and, analogously, by $\text{vol}_{\tilde{g}}$ the volume measure on $\partial M$.

At several points we will work with local coordinates. Note that a chart $\varphi : U \to V$ maps a portion $U \subset M$ into the half space $\mathbb{H}^n := \{(z^1, \ldots, z^n) \in \mathbb{R}^n | z^n \geq 0\}$. Of special use are Riemannian normal coordinates centered at some point $x \in U \subset M$ which are given by a chart $\varphi : U \mapsto \varphi(U) \subset \mathbb{R}^n$ defined as the concatenation of the inverse of the exponential map $\exp_x$ with the isomorphism from the tangent space $T_x M$ to $\mathbb{R}^n$ induced by choosing a $g(x)$-orthonormal basis of $T_x M$. In these coordinates we obtain in $\varphi(x) = (0, \ldots, 0)$
\[
g_{ij} = g^{ij} = \delta_{ij}, \quad \partial_k g_{ij} = 0 \quad \text{and} \quad \Gamma_{ij}^k = 0.
\]
Furthermore, for every point $x \in \partial M$ there is a neighborhood $U \subset M$ on which we can define geodesic boundary coordinates centered in $x \in \partial M$ (see [16]). On $\partial M \cap U$ we define Riemannian normal coordinates $(x^1, \ldots, x^{n-1})$ related to $\tilde{g}$ and extend them by $x^n$ such that the $x^n$-curve is a geodesic on $M$ which is orthogonal to the $x^i$-curves. We obtain for these coordinates
\[
g_{mn} = g^{mn} = 1 \quad \text{and} \quad g_{ni} = g^{ni} = 0 \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{in} \quad U
\]
and as immediate implications for $\tilde{g}$ and the unit outer normal $N$ on $\partial M \cap U$
\[
N^n = -1,
\]
\[
N^i = 0 \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{and}
\]
\[
\sqrt{|g|} = \sqrt{|\tilde{g}|}.
\]
Lemma 2.1. Let \((\bar{M}, g)\) be a compact, oriented, smooth Riemannian manifold with boundary. Then for \(u \in C^\infty(\bar{M})\) we have
\[
\lim_{\eta \to 0} \int_{\{x \in \bar{M} \mid |u(x)| < \eta\}} |\text{grad}_g u(x)|_g \, dv_g = 0.
\]

Proof. The claim follows from \[6, p. 1020, Lemma 2\] via a partition of unity. \(\square\)

Lemma 2.2 (Lebesgue’s Theorem on manifolds). Let \(u \in L^1(M)\). Then we have for almost every \(x \in M\)
\[
\lim_{\rho \to 0} \int_{\rho \setminus \partial B_\rho(x)} |u(x) - u(y)| \, dv_g(y) = 0 \quad \text{and} \quad \lim_{\rho \to 0} \int_{\rho \setminus \partial B_\rho(x)} u(y) \, dv_g(y) = u(x).
\]

Proof. The proof for the Euclidean case can be repeated after a partition of unity. \(\square\)

At several places we will make use of functions \(R_\delta : \bar{M} \to \mathbb{C}^\infty \to [0, 1]\) for \(\delta > 0\) that are only supported in a small neighborhood of \(\partial M\), precisely we require
\[
R_\delta \equiv 0 \quad \text{in} \quad M_\delta := \{x \in \bar{M} | d_g(x, \partial M) > \delta\},
\]
\[
R_\delta \equiv 1 \quad \text{in} \quad \{x \in \bar{M} | d_g(x, \partial M) \leq \frac{\delta}{2}\}.
\]

Lemma 2.3. For a positive finite measure \(\mu\) on \(M\) we have
\[
\lim_{\delta \to 0} \int_M R_\delta \, d\mu = 0.
\]

Proof. Without loss of generality we can consider a monotone, positive sequence \((\delta_n)_{n \in \mathbb{N}}\) with \(\delta_n \to 0\) for \(n \to \infty\). For \(A_n := M \setminus M_{\delta_n}\) and the disjoint sets \(B_n := A_n \setminus A_{n+1}\) we obtain from the \(\sigma\)-additivity and finiteness of \(\mu\)
\[
\sum_{l=1}^k \mu(B_l) = \mu(A_1) - \mu(A_k) \leq \mu(A_1) < \infty.
\]

The positivity of \(\mu\) yields convergence of the sequence \(\sum_{l=1}^k \mu(B_l)\) towards \(\mu(A_1)\) for \(k \to \infty\) and hence
\[
\lim_{k \to \infty} \mu(A_k) = \mu(A_1) - \lim_{k \to \infty} \sum_{l=1}^k \mu(B_l) = 0
\]
which completes the proof. \(\square\)

3. Traces for functions of bounded variation on manifolds

In this section we will show the existence and fundamental properties of traces for functions of bounded variation on manifolds. The key result of this section is Theorem 3.4. Analogous results for the Euclidean case are given in \[12\] pp. 176-183].
Definition 3.1 (BV functions on manifolds). The total variation of a function $u \in L^1(M)$ on $M$ is defined as

$$TV(u, M) := \sup \left\{ \int_M u \, \text{div}_g X \, dv_g \mid X \in \Gamma_0(TM), |X|_g \leq 1 \right\}.$$ 

For smooth functions $u : M \to \mathbb{R}$ we have

$$TV(u, M) := \int_M |\text{grad}_g u|_g \, dv_g.$$ 

We define the set of functions of bounded variation on $M$ as

$$BV(M) := \{ u \in L^1(M) \mid TV(u, M) < \infty \}.$$ 

For the proof of Theorem 3.4 we will use the notation of the following Lemma from [12, p. 167, Theorem 1].

Lemma 3.2. Let $V \subset \mathbb{R}^n$ be open, $h \in BV_{\text{loc}}(V)$. Then there exist a Radon measure $|Dh|$ on $V$ and a $|Dh|$-measurable function $\sigma_h : V \to \mathbb{R}^n$, such that

$$|\sigma_h| = 1 \text{ } |Dh|\text{-almost everywhere and}$$

$$\int_V h \, \text{div} \phi \, dz = -\int_V \langle \phi, \sigma_h \rangle \, d|Dh| \text{ for all } \phi \in C_0^\infty(V, \mathbb{R}^n),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^n$.

An analogous result for manifolds is given in [22, p. 104]:

Lemma 3.3. For a function $u \in BV(M)$ there exist a finite measure $|Du|$ on $M$ and a $|Du|$-measurable function $\sigma_u : M \to TM$ such that

$$|\sigma_u|_g = 1 \text{ } |Du|\text{-almost everywhere and}$$

$$\int_M u \, \text{div}_g X \, dv_g = -\int_M \langle \sigma_u, X \rangle_g \, d|Du| \text{ for all } X \in \Gamma_0(TM).$$

Theorem 3.4 (Traces for functions of bounded variation on manifolds). There exists a unique linear operator

$$T : BV(M) \to L^1(\partial M; dv_g)$$

where $dv_g$ denotes the $(n - 1)$-dimensional Riemannian volume element on $\partial M$ such that

$$\int_M u \, \text{div}_g X \, dv_g = -\int_M \langle X, \sigma_u \rangle_g \, d|Du| + \int_{\partial M} \langle X, N \rangle_g \, Tu \, dv_g$$

for all $u \in BV(M) \cap L^\infty(M)$ and all $X \in \Gamma(TM)$, where $|Du|$ and $\sigma_u$ are defined as in Lemma 3.3 and $N$ denotes the unit outer normal.

Proof. Let $u \in BV(M)$ and $X \in \Gamma(TM)$. We can write

$$\int_M u \, \text{div}_g X \, dv_g = \int_M u \, \text{div}_g (XR_\delta) \, dv_g + \int_M u \, \text{div}_g (X(1 - R_\delta)) \, dv_g$$

for all $u \in BV(M) \cap L^\infty(M)$ and all $X \in \Gamma(TM)$, where $|Du|$ and $\sigma_u$ are defined as in Lemma 3.3 and $N$ denotes the unit outer normal.
with $R_\delta$ as in (2.4). The application of Lemma 3.3 yields existence of a finite measure $|Du|$ and a $|Du|$-measurable function $\sigma_u$ such that
\[
\int_M u \operatorname{div}_g(X(1 - R_\delta)) \, dv_g = - \int_M \langle \sigma_u, X(1 - R_\delta) \rangle_g \, d|Du| \\delta \wedge 0 - \int_M \langle \sigma_u, X \rangle_g \, d|Du|.
\]
since $X(1 - R_\delta) \in \Gamma_0(TM)$ and by the use of Lemma 2.3.

Considering the first term on the right hand side of (3.2) we introduce a finite collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$ in geodesic boundary coordinates which covers $M \setminus M_\delta$ and a subordinate partition of unity $\{\psi_i\}_{i \in I}$ such that
\[
\int_M u \operatorname{div}_g(XR_\delta) \, dv_g = \sum_{i \in I} \int_{\varphi_i(U_i)} (\psi_i u \operatorname{div}_g(XR_\delta)) \circ \varphi_i^{-1} \sqrt{|g_i|} \, dz.
\]

From [12, p. 177, Theorem 1] we know that for $V \subset \mathbb{R}^n$ open and bounded with $\partial V$ Lipschitz there exists a linear trace operator
\[
\Theta :BV(V) \to L^1(\partial V; H^{n-1}),
\]
where $H^{n-1}$ denotes the Hausdorff-measure restricted to $\partial V$, such that
\[
(3.4) \quad \int_V h \, \delta \circ \varphi^{-1} \in BV(V)
\]
for all $h \in BV(V)$ and $\delta \in C^\infty(V, \mathbb{R}^n)$, where $|Dh|$ and $\sigma_h$ are defined as in Lemma 3.3 and $\nu$ denotes the unit outer normal to $\partial V$ with respect to the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$.

We want to apply (3.4) to an arbitrary summand of the right-hand side of (3.3) and suppress the index $i$ in the following, i.e. $\varphi_i = \varphi : U \to \varphi(U) = V$ etc. It is easy to see that
\[
(3.5) \quad \bar{u} := (u \psi) \circ \varphi^{-1} \in BV(V)
\]
as on a compact set, i.e. particularly on the set $\text{supp}(\psi \circ \varphi^{-1}) \subset \mathbb{R}^n$, there exist constants $c, c' \in \mathbb{R}^+$ such that
\[
\frac{1}{c} \leq \sqrt{|g|} \leq c \quad \text{and} \quad \frac{1}{c'} |z| \leq |\varphi(z)| := \sqrt{g_{ij} z^i z^j} \leq c' |z|
\]
uniformly for $z \in \mathbb{R}^n$. After the introduction of $\phi_\delta := (\phi^1_\delta, \ldots, \phi^n_\delta)$ with $\phi^i_\delta := \hat{X}^i(R_\delta \circ \varphi^{-1}) \sqrt{|g|}$ and $\hat{X}^i$ being the local components of $X$ the application of (3.4) yields
\[
(3.6) \quad \int_V (\psi u \operatorname{div}_g(XR_\delta)) \circ \varphi^{-1} \sqrt{|g|} \, dz = \int_V \bar{u} \operatorname{div} \phi_\delta \, dz = S^\delta_1 + S^\delta_2 + S^\delta_3
\]
with
\[
S^\delta_1 := - \int_V \langle \phi_\delta, \sigma_u \rangle \, d|Du|,
\]
\[
S^\delta_2 := \int_{\partial V \setminus \partial \mathbb{R}^n} \langle \phi_\delta, \nu \rangle \Theta \bar{u} \, dH^{n-1},
\]
\[
S^\delta_3 := \int_{\partial V \setminus \partial \mathbb{H}^n} \langle \phi_\delta, \nu \rangle \Theta \bar{u} \, dH^{n-1}.
\]
The terms $S_1^\delta$ and $S_3^\delta$ converge to zero for $\delta \searrow 0$ which can be seen by Lemma 2.3 and the fact that $\Theta \bar{u} \in L^\infty(\partial V \setminus \partial H^n)$ (see [12, p. 181, Theorem 2]).

Regarding $S_2^\delta$ recall that we chose geodesic boundary coordinates and hence $\langle \phi_\delta, \nu \rangle = -X^n \sqrt{|g|} = \langle X, N \rangle_g \circ \varphi^{-1} \sqrt{|\tilde{g}|}$ on $\partial V \cap \partial H^n$. Consequently, $S_2^\delta = \hat{\partial V} \cap \partial H^n \langle \phi_\delta, \nu \rangle \Theta \bar{u} dH^n - 1 = \hat{\partial U} \cap \partial M \langle X, N \rangle_g T(u) d\tilde{g}$

with $T(u) := \Theta(\bar{u}) \circ \varphi : \partial U \cap \partial M \to \mathbb{R}$. Analogously, we define the trace for each $u \psi_i$ on $\partial M$ as

$$T^i u := \begin{cases} \Theta((u \psi_i) \circ \varphi^{-1}) \circ \varphi_i & \text{on } \partial M \cap \partial U_i, \\ 0 & \text{on } \partial M \setminus \partial U_i \end{cases}$$

and

$$Tu := \sum_{i \in I} T^i u.$$

For $\delta \searrow 0$ in (3.3) we finally obtain

$$\lim_{\delta \to 0} \int_M u \text{ div}_g(X \delta_R) d\tilde{g} = \int_{\partial M} \langle X, N \rangle_g Tu d\tilde{g}$$

which proves existence.

To prove uniqueness we assume that (3.1) holds for $Tu$ and for $v \in L^\infty(\partial M)$. Subtraction of the corresponding equations yields

$$\int_{\partial M} \langle X, N \rangle_g (Tu - v) d\tilde{g} = 0 \text{ for all } X \in \Gamma(TM).$$

This is true particularly in the limit $\delta \searrow 0$ for $X = X_\delta := R_\delta \phi N_\delta$ with an arbitrary $\phi \in C^\infty(M)$ and $N_\delta$ being an extension of $N$ to $M \setminus M_\delta$ for small $\delta$. The fundamental lemma of calculus of variations proves uniqueness up to a set of vol$\tilde{g}$-measure zero. Linearity can be proved by considering (3.1) for functions $u, v \in BV(M)$ and their sum $u + v$ for $X = X_\delta$ in the limit $\delta \searrow 0$.

From the proof of Theorem 3.4 we obtain the following corollary.

**Corollary 3.5.** For $u \in BV(M) \cap L^\infty(M)$ and $X \in \Gamma(TM)$ we have

$$\lim_{\delta \to 0} \int_M u(\text{grad}_g R_\delta, X)_g d\tilde{g} = \int_{\partial M} T U \langle X, N \rangle_g d\tilde{g}.$$
Corollary 3.6 (Properties of the trace $Tu$). The trace $Tu : \partial M \to \mathbb{R}$ satisfies

1. For $\text{vol}_{\tilde{g}}$-almost every $x_0 \in \partial M$ we have
   \[
   \lim_{\rho \to 0} \int_{B^g_{\rho}(x_0)} |u - Tu(x_0)| \, dv_g = 0
   \]
   and
   \[
   Tu(x_0) = \lim_{\rho \to 0} \int_{B^g_{\rho}(x_0)} u \, dv_g
   \]
2. $Tu \in L^\infty(\partial M)$.
3. For $h \in C^1([-z, z])$ with $[-z, z] \subset \mathbb{R}$ and $z > \|Tu\|_{L^\infty(\partial M)}$ it is
   \[
   T[h(u)] = h(Tu)
   \]
   almost everywhere on $\partial M$.

Proof. To prove claim (1), we choose normal coordinates on $B^g_{\rho}(x_0)$ and refer to the Euclidean case ([12, p. 181, Theorem 2]). Claim (2) follows immediately from claim (1). The mean value theorem together with claim (1) yield claim (3). The argumentation is analogous to the one in the Euclidean case.

Considering $M \times (0, T)$ as an $(n + 1)$-dimensional manifold endowed with the Riemannian metric $g_T$, locally defined by $g_T = dt^2 + g_{ij} dx^i dx^j$, we will need the following Lemma.

Lemma 3.7. For $u \in BV(M \times (0,T))$ we have

1. $u(\cdot, t) \in BV(M)$ for almost every $t \in (0, T)$ and
2. $u(x, \cdot) \in BV((0,T))$ for almost every $x \in M$.

Proof. This can be proved via a partition of unity and by then applying Lemma 1 from [6, p. 1019]. □

Lemma 3.8. Let $u \in BV(M) \cap L^\infty(M)$ and $F \in C^1(\mathbb{R} \times \bar{M})$. Then the function

\[x \mapsto F(u(x), x)\]

is in $BV(M)$.

Proof. For the proof we use Proposition 1.4, Theorem 2.1 and Theorem 3.3 from [22, p. 105, p. 109 and p. 117], where the existence of a sequence $(u_j)_{j \in \mathbb{N}} \subset C_0^\infty(M)$ with the properties

\[u_j \to u \text{ in } L^1(M),\]

\[\text{TV}(u, M) = \lim_{j \to \infty} \int_M |\text{grad}_g u_j| \, dv_g \text{ and}\]

\[\|u_j\|_{L^\infty(M)} \leq \|u\|_{L^\infty(M)} \text{ for all } j \in \mathbb{N}\]

is shown. Using the fact that

\[F(u_j, \cdot) \to F(u, \cdot) \text{ in } L^1(M)\]

and the boundedness of $\|\partial_u F(u_j(\cdot), \cdot)\|_{L^\infty(M)}$ and $\|(\text{grad}_g F)(u_j, \cdot)\|_{L^\infty(M)}$ uniformly in $j$ we obtain
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\[ \sup \left\{ \int_M F(u, \cdot) \, \text{div}_g X \, dv_g \mid X \in \Gamma_0(TM), |X|_g \leq 1 \right\} < \infty, \]

which proves the claim.

\[ \square \]

4. Application to scalar conservation laws

We will now use the previous results to show existence, uniqueness and total variation estimates for an entropy solution of problem (1.1), (1.2) with admissible boundary conditions. We will proceed as in [6] and emphasize only the boundary terms. For the rest we refer to [20] in which a generalization of problem (1.1), (1.2) to Riemannian manifolds (without boundary) is treated.

Let \( \bar{M}, u \) and \( f \) be defined as in the introduction. We will write \( u \) instead of \( u(x,t) \) and \( f(u) \) instead of \( f(u(x,t),x,t) \) whenever this should not lead to confusion.

Considering the characteristics of (1.1), (1.2) one can see that, in general, it is not possible to require \( u = 0 \) on the whole boundary.

To define admissible boundary conditions and to prove existence and uniqueness of a solution of (1.1), (1.2) with appropriate boundary conditions we will use the vanishing viscosity method which consists in passing to the limit, as \( \epsilon > 0 \) tends to zero, in the solution \( u^\epsilon \) of the parabolic regularization

\begin{align*}
\partial_t u^\epsilon + \text{div}_g f(u^\epsilon, x, t) - \epsilon \Delta_g u^\epsilon &= 0 \quad \text{in } \bar{M} \times (0, T), \\
\epsilon^\epsilon(\cdot, 0) &= u_0^\epsilon \quad \text{in } \bar{M}, \\
\epsilon^\epsilon &= 0 \quad \text{on } \partial M \times (0, T),
\end{align*}

where \( u_0^\epsilon : M \to \mathbb{R} \) denotes a sequence of \( u_0 \) mollifying functions satisfying

\begin{align*}
\|u_0^\epsilon\|_{L^\infty(M)} &\leq \|u_0\|_{L^\infty(M)}, \\
\|u_0^\epsilon - u_0\|_{L^1(M)} &\xrightarrow{\epsilon \to 0} 0, \\
\text{TV}(u_0^\epsilon, M) &\xrightarrow{\epsilon \to 0} \text{TV}(u_0, M), \\
\epsilon \|u_0^\epsilon\|_{H^{2,1}(M)} &\leq c_0 \text{TV}(u_0, M),
\end{align*}

for a constant \( c_0 > 0 \). Existence and uniqueness of a solution \( u^\epsilon \in C^\infty(\bar{M} \times (0, T)) \) of (4.1)-(4.3) is shown in [25].

4.1. Convergence of a parabolic regularization. In this section we will show convergence for a subsequence of the solutions \( \{u^\epsilon\}_{\epsilon > 0} \) of the regularized problem (4.1)-(4.3). To this end we prove boundedness of \( \{u^\epsilon\}_{\epsilon > 0} \) in \( L^\infty(M \times (0, T)) \cap H^{1,1}(M \times (0, T)) \) uniform in \( \epsilon \), where the product manifold \( M \times (0, T) \) is endowed with the Riemannian metric \( g_T \) and apply the following theorem from [1].

Theorem 4.1. Let \( (\bar{M}, g) \) be a compact Riemannian manifold with Lipschitz boundary \( \partial M \). Then for \( p, q \geq 1 \) with \( 1 \geq 1/q > 1/p - 1/n > 0 \) the embedding

\[ H^{1,p}(M) \hookrightarrow L^q(M) \]
is compact.

Proof. See [4] pp. 166-169, Theorem 11]. □

Theorem 4.2. The solutions \( \{u^\epsilon\}_{\epsilon > 0} \) of (4.1)-(4.3) are bounded in \( L^\infty(M \times (0,T)) \cap H^{1,1}(M \times (0,T)) \) uniformly in \( \epsilon \), precisely

(4.8) \( \|u^\epsilon\|_{L^\infty(M \times (0,T))} \leq \|u_0^\epsilon\|_{L^\infty(M)} \leq \|u_0\|_{L^\infty(M)} \),

(4.9) \( \|\partial_t u^\epsilon(\cdot, t)\|_{L^1(M)} \leq c_1 \text{TV}(u_0, M) \),

(4.10) \( \|\nabla u^\epsilon(\cdot, t)\|_{L^1(M)} \leq ((1 + c_2 t) \text{TV}(u_0, M) + o(1)) (1 + c_3 t e^{c_4 t}) \)

for every \( t \in (0, T) \) and consequently

(4.11) \( \|u^\epsilon\|_{H^{1,1}(M \times (0,T))} < c_4 \),

where the constants \( c_1, c_2, c_3, c_4 > 0 \) do only depend on the data \( M, g, T, f \), \( \|u_0\|_{L^\infty(M)} \), but not on \( \epsilon \).

Proof. Since there exist analogous proofs for problems without boundary conditions on manifolds, we will emphasize the argumentation for the boundary terms and refer to literature for the rest.

Concerning the \( L^\infty \)-estimate (4.8) note that the proof from [21, pp. 131-139, Theorem 8.4] can be transferred easily to our case, i.e. showing that \( \sup_{M \times (0,T)} u_\epsilon \leq \max\{\text{ess sup}_M u_0, 0\} \) we multiply (4.1) by \( \Phi_\zeta(u^\epsilon) \) where

\[
\Phi_\zeta(z) := \begin{cases} 
((z - m)^2 + \zeta^2)^{1/2} - \zeta & \text{if } z \geq m, \\
0 & \text{if } z < m,
\end{cases}
\]

\( m := \max\{\text{ess sup}_M u_0, 0\} \) and \( \zeta > 0 \), integrate over \( M \times (0, t) \) with \( t \in (0, T) \) and let \( \zeta \) tend to zero. Similarly we get \( \inf_{M \times (0,T)} u_\epsilon \geq \min\{\text{ess inf}_M u_0, 0\} \).

For the estimates of the time derivative and the total variation we proceed as in [20] and define a function \( S_\eta : \mathbb{R} \to \mathbb{R}_{\geq 0} \) for \( \eta > 0 \) by

\[
S_\eta(z) := \begin{cases} 
-z & \text{if } z < -\eta, \\
\frac{z^2}{2\eta} + \eta & \text{if } |z| \leq \eta, \\
z & \text{if } z > \eta.
\end{cases}
\]

The proof of (4.9) can be transferred from [8, 1022-1023], i.e. taking the time derivative of (4.1), multiplying with \( S_\eta^\prime(\partial_t u^\epsilon) \), integrating over \( M \times (0,t) \) for \( t \in (0, T) \) and letting \( \eta \) tend to zero. Note, that all boundary terms vanish due to our homogeneous boundary condition (1.3).

As the proof of (4.10) is a little more involved due to the boundary treatment, it will be presented in detail here. From now on we will write \( u \) instead of \( u^\epsilon \) for better readability. Taking the total covariant derivative of equation (4.1) and using (2.1) we obtain

(4.12) \( \partial_t \nabla u + \nabla \text{div}_g f(u) = \epsilon (\Delta_g \nabla u - \text{Ric}(\nabla u, \cdot)) \).

Multiplying (4.12) by \( \frac{S_\eta^\prime(|\nabla u_g|)}{|\nabla u_g|} \) and integration over \( M \) leads to

\[
\frac{d}{dt} \int_M S_\eta(|\nabla u_g|) \, dv_g + \int_M \langle \nabla \text{div}_g f(u), \nabla u_g \rangle \frac{S_\eta^\prime(|\nabla u_g|)}{|\nabla u_g|} \, dv_g
\]

(4.13) \( \epsilon \int_M \frac{S_\eta^\prime(|\nabla u_g|)}{|\nabla u_g|} (\langle \Delta_g \nabla u, \nabla u \rangle_g - \text{Ric}(\nabla u, \nabla u)) \, dv_g. \)
From the lines of \[20\] pp. 1721-1723, Proposition 5.3 we know that
\[
\langle \nabla_\theta f(u), \nabla u \rangle_g \frac{S'_\eta(|\nabla u|_g)}{|\nabla u|_g} = \left\langle \left( \frac{\nabla_\theta f}{|\nabla u|_g} \right)(u), \nabla_g u \right\rangle_g S'_\eta(|\nabla u|_g)
\]
+ \( \langle \nabla_\theta f(u) S_\eta(|\nabla u|_g) + \nabla_g(\partial_u f(u)) (|\nabla u|_g S'_\eta(|\nabla u|_g) - S_\eta(|\nabla u|_g)) \). Concerning the first term on the right-hand side of (4.13) integration by parts yields
\[
(4.14)
\int_M \frac{S'_\eta(|\nabla u|_g)}{|\nabla u|_g} (\Delta_g \nabla u, \nabla u)_g \, dv_g = - \int_M \left\langle \nabla^2 u, \nabla \left( \frac{S'_\eta(|\nabla u|_g)}{|\nabla u|_g} \nabla u \right) \right\rangle_g \, dv_g
\]
+ \( \int_{\partial M} \langle \nabla u \otimes N, \nabla^2 u \rangle_g \frac{S'_\eta(|\nabla u|_g)}{|\nabla u|_g} \, dv_g \),
where in local coordinates \( \langle \nabla u \otimes N, \nabla^2 u \rangle_g = u^i \, u_{ij} \, N^j \). From \[20\] pp. 1721-1723, Proposition 5.3 we obtain positivity for the integrand of the first term of the right-hand side of (4.13). Applying the divergence theorem on \( \int_M \langle \nabla \eta f(u) S_\eta(|\nabla u|_g) \rangle \, dv_g \) and letting \( \eta \) tend to zero in (4.13) we obtain
\[
(4.15)
\frac{d}{dt} \int_M |\nabla u|_g \, dv_g + \int_M \left\langle \left( \frac{\nabla_\theta f}{|\nabla u|_g} \right)(u), \nabla_g u \right\rangle_g \, dv_g
\leq -\epsilon \int_M \text{Ric}(\nabla u, \nabla u) \, dv_g + \liminf_{\eta \searrow 0} I^n.
\]
with
\[
I^n := \int_M \epsilon \left( \langle \nabla u \otimes N, \nabla^2 u \rangle_g \frac{S'_\eta(|\nabla u|_g)}{|\nabla u|_g} - \langle \partial_u f(0), N \rangle_g S_\eta(|\nabla u|_g) \right) \, dv_g.
\]
In order to study the limit \( \eta \searrow 0 \) for \( I^n \) we do some transformations first. By the fact that \( \nabla_g u = N(u) N \) on \( \partial M \) and consequently \( |\nabla u|_g = |N(u)| \), as \( u \equiv 0 \) on \( \partial M \), we have on \( \partial M \)
\[
\langle \nabla u \otimes N, \nabla^2 u \rangle_g = N(u) \langle N, \nabla_N \nabla_g u \rangle_g.
\]
Considering the regularized conservation law (1.1) on \( \partial M \), we obtain
\[
\langle \partial_u f(0), N \rangle_g N(u) = \epsilon \Delta_g u.
\]
Thus,
\[
I^n = \epsilon \int_{\partial M} S'_\eta(|N(u)|_g) \frac{N(u)}{|N(u)|_g} \langle N, \nabla_N \nabla_g u \rangle_g - \Delta_g u \frac{S_\eta(|N(u)|_g)}{N(u)} \, dv_g.
\]
For \( \eta \searrow 0 \) we obtain
\[
(4.16) \liminf_{\eta \searrow 0} |I^n| \leq \epsilon \int_{\partial M} \langle N, \nabla_N \nabla_g u \rangle_g - \Delta_g u \, dv_g.
\]
In geodesic boundary coordinates centered in \( x \in \partial M \) we have
\[
\Delta_g u(x) = \sum_{i=1}^n \partial_{g}^2 u(x) = \partial_{n}^2 u(x)
\]
regarding the fact that \( u \equiv 0 \) on \( \partial M \). Extending \( N \) onto a small neighborhood of \( \partial M \) by \( N = -\partial_n \) we obtain at \( x \in \partial M \)
\[
\Delta_g u = N(N(u)) = N(\langle N, \nabla_g u \rangle_g) = \langle \nabla_N N, \nabla_g u \rangle_g + \langle N, \nabla_N \nabla_g u \rangle_g.
\]
Note that the above expression is independent of the choice of coordinates. Thus,

\[
\liminf_{\eta \to 0} |I^\eta| \leq c \epsilon \int_{\partial \mathcal{M}} |\nabla u|_g \, dv_g
\]

with \( c := \|\nabla_N N\|_{L^\infty(\partial \mathcal{M})} < \infty \) since \( N \) is smooth and \( \partial \mathcal{M} \) compact. A repetition of the proof for the Euclidean case [5, p. 92, Lemma A.3] yields the analogous result,

\[
\int_{\partial \mathcal{M}} |\nabla u|_g \, dv_g \leq \int_M |\Delta_g u| \, dv_g,
\]
on manifolds. Using again equation (4.1), we obtain

\[
c_\epsilon \epsilon \int_{\partial \mathcal{M}} |\nabla u|_g \, dv_g \leq c' (\|\partial_t u\|_{L^1(M)} + \|\nabla u\|_{L^1(M)})
\]
for a constant \( c' = c'(\mathcal{M}, g, T, f, \|u_0\|_{L^\infty(M)}) > 0 \). Since \( \text{Ric}(\nabla u, \nabla u) \leq c'' |\nabla u|_g'^2 \) with a constant \( c'' = c''(\mathcal{M}, g, T) > 0 \), we obtain from (4.10)

\[
\frac{d}{dt} \int_M |\nabla u|_g \, dv_g \leq c_3 \|\nabla u\|_{L^1(M)} + c' \|\partial_t u\|_{L^1(M)}
\]

with

\[
c_3 := \sup_{u, X} \|\langle \nabla_X \partial_t f(\bar{u}), X \rangle_g \|_{L^\infty(M \times (0,T))} + \|\text{Ric} \|_{L^\infty(M)} + c',
\]

where the supremum is taken over all real numbers \( |\bar{u}| \leq \|u_0\|_{L^\infty(M)} \) and all smooth vector fields \( X \) with \( |X|_g \leq 1 \). Integration over \((0, t)\) together with (4.9) and (4.6) yields for almost every \( t \in (0, T) \)

\[
\int_M |\nabla u(\cdot, t)|_g \, dv_g \leq \|\nabla u_0\|_{L^1(M)} + c_3 \int_0^t \|\nabla u(\cdot, \tau)\|_{L^1(M)} \, d\tau + tc_1 \text{TV}(u_0, M)
\]

\[
\leq (1 + c_2 t) \text{TV}(u_0, M) + o(1) + c_3 \int_0^t \|\nabla u(\cdot, \tau)\|_{L^1(M)} \, d\tau
\]

with \( c_2 := c'c_1 \) and \( o(1) \to 0 \) for \( \epsilon \searrow 0 \). Finally, Gronwall’s Inequality completes the proof of (4.10) and consequently of (4.11).

\[ \Box \]

An application of Theorem 4.1 leads to the following corollary of Theorem 4.2.

**Corollary 4.3 (Viscosity limit).** There is a subsequence \((u^\epsilon_j)_{j \in \mathbb{N}}\) of \( \{u^\epsilon\}_{\epsilon > 0} \) and a function \( u \in L^1(M \times (0,T)) \) such that \( \|u^\epsilon_j - u\|_{L^1(M \times (0,T))} \to 0 \) for \( j \to \infty \). Such a limit function \( u \) is called viscosity limit of \((4.7) - (4.3)\).

### 4.2. Existence of an entropy solution

First, we motivate a formulation of boundary conditions analogously to [6] 126-127. For a moment, assume the flux function \( f = f(u, x, t) \) to be monotone in \( u \). In this case, outflow boundary points \( x \in \partial \mathcal{M} \) at time \( t \in (0, T) \) are characterized by the property

\[
\left\langle \frac{f(Tu) - f(k)}{Tu - k}, N \right\rangle_g > 0 \quad \forall k \in \mathbb{R}
\]

(4.17)
\[ \text{Tu denotes the trace of } u \text{ on } \partial M \text{ and inflow boundaries are characterized conversely. We want to have a boundary condition which} \]

(1) \text{ requires } u = 0 \text{ on } \partial M \text{ if the data are entering } \bar{M}, \text{ which means that } u \text{ is not determined by the initial data or some other boundary data and} \]

(2) \text{ is a trivial condition if the data are leaving } \bar{M}. \]

This is ensured by the following boundary condition:

\[ \min_{k \in I(Tu,0)} \{ \text{sgn}(Tu) \langle f(Tu) - f(k), N_g \rangle \} = 0 \quad (4.18) \]

almost everywhere on \( \partial M \times (0, T) \) with \( I(Tu,0) := [\min\{Tu,0\}, \max\{Tu,0\}] \).

Although these explanations only work for monotone functions \( f \), we will see that (4.18) is a valid boundary condition for problem (1.1), (1.2), even if \( f \) is not monotone.

We now give the definition for an entropy solution of problem (1.1), (1.2), (4.18).

**Definition 4.4 (Entropy solution).** We call a function \( u \in BV(M \times (0,T)) \cap L^\infty(M \times (0,T)) \) \textit{entropy solution} of problem (1.1), (1.2), (4.18) if for every \( k \in \mathbb{R} \) and every \( \phi \in C^\infty_0(\bar{M} \times (0,T)) \), \( \phi(x,t) \geq 0 \) the inequality

\[ \int_M \int_0^T |u - k| \partial_t \phi + \text{sgn}(u - k) \langle f(u) - f(k), \text{grad}_g \phi \rangle_g \ dt \ dv_g \\
+ \int_{\partial M} \int_0^T \text{sgn}(k) \langle f(Tu) - f(k), N_g \phi \rangle \ dt \ dv_g \geq 0 \quad (4.19) \]

holds and the initial condition (1.2) is fulfilled by the trace \( Tu|_{t=0} \) almost everywhere in \( M \).

**Remark 4.5.** Setting \( k = \sup_{M \times (0,T)} |u(x,t)| \) and \( k = \inf_{M \times (0,T)} |u(x,t)| \) in (4.19) it is easy to prove that an entropy solution is a weak solution of (1.1). The fact that (4.19) implies the boundary condition (4.18) is proved by setting \( \phi = R_5 \phi \) in (4.19) with \( \phi \in C^\infty_0(M \times (0,T)) \), \( \phi \geq 0 \) and \( R_5 \) as in (2.4) and letting \( \delta \) tend to zero.

**Theorem 4.6 (Existence of an entropy solution).** Any viscosity limit \( u \) of (4.1)-(4.3) is an entropy solution of problem (1.1), (1.2), (4.18).

To prove Theorem 4.6, we will use the following Corollary 4.7 and Lemma 4.8.

**Corollary 4.7.** Any viscosity limit \( u \) of (4.1)-(4.3) belongs to \( BV(M \times (0,T)) \cap L^\infty(M \times (0,T)) \) and has a trace for \( t = 0 \) satisfying the initial condition (1.2) almost everywhere on \( M \).

**Proof.** Since (4.8), (4.9) and (4.10) hold and since TV is lower semicontinuous w.r.t \( L^1 \)-convergence we obtain

\[ u \in BV(M \times (0,T)) \cap L^\infty(M \times (0,T)). \]
Let \( Tu|_{t=0} \) be the trace of \( u \) for \( t = 0 \) whose existence is ensured by Theorem 3.4. Then we have for \((u^\delta)_{\delta \in \mathbb{N}}\), a converging subsequence of \( \{u^\epsilon\}_{\epsilon > 0} \).

\[
\|Tu|_{t=0} - u_0\|_{L^1(M)} \leq \|Tu|_{t=0} - u(\cdot, t)\|_{L^1(M)} + \|u(\cdot, t) - u(\cdot, t)^\delta(\cdot, t)\|_{L^1(M)} + \|u(\cdot, t)^\delta(\cdot, t) - u_0(\cdot, t)^\delta(\cdot, t)\|_{L^1(M)}.
\]

Since (4.9) holds independently of \( \epsilon_j \), the third term of the right hand side is bounded by \( tc \) with a constant \( c > 0 \). Thus, letting first \( j \) tend to \( \infty \) and then \( t \) to zero, we obtain

\[
\lim_{\delta \searrow 0} \|Tu|_{t=0} - u_0\|_{L^1(M)} = 0,
\]

analogously, to the Euclidean case in [6, pp. 125-126]. □

**Lemma 4.8.** Let \( u \in BV(M) \cap L^\infty(M) \), \( R_\delta \) as in (2.4) and \( \phi \in C_0^\infty(M) \). Then we have

\[
(4.20) \quad \lim_{\delta \searrow 0} \int_M \langle f(u), \text{grad}_g R_\delta \rangle_g \phi \, dv_g = \int_{\partial M} \langle f(Tu), N \rangle_g \phi \, dv_{\tilde{g}}.
\]

**Proof.** Without loss of generality we neglect the \( t \)-dependence of \( f \) in the proof. For \( X \in \Gamma(TM) \) and \( v \in BV(M) \) we obtain by Corollary 3.3

\[
(4.21) \quad \lim_{\delta \searrow 0} \int_M v(\text{grad}_g R_\delta, X)_g \, dv_g = \int_{\partial M} T v(X, N)_g \, dv_{\tilde{g}}.
\]

Let \( \{(U_i, \varphi_i)\}_{i \in I} \) be a finite collection of charts which covers \( M \) and \( \{\psi_i\}_{i \in I} \) a partition of unity subordinate to this cover. For \( i \in I \) and \( 1 \leq l \leq n \) we define

\[
\hat{f}_i^l(x) := \begin{cases} f_i^l(u(x), x) & \text{if } x \in U_i, \\ 0 & \text{otherwise} \end{cases}
\]

where \( f_i^l(u(x), x) \) denotes the \( l \)-th component of \( f(u(x), x) \) on \( U_i \) relating to \( \varphi_i \). Note that \( f_i^l(u, \cdot) \in BV(U_i) \) due to Lemma 3.3 and hence an application of Theorem 3.4 yields

\[
TV(\hat{f}_i^l, M) = \sup \left\{ - \int_{U_i} \langle X, \sigma f_i^l(u, \cdot) \rangle_g \, dv_{\tilde{g}}, \quad T f_i^l(u, \cdot) \mid_{\partial U_i} \right\}
\]

\[
\leq TV(f_i^l(u, \cdot), U_i) + \|T f_i^l(u, \cdot)\|_{L^\infty(\partial U_i)} \text{vol}_{\tilde{g}}(\partial U_i)
\]

\[
< \infty
\]

and thus, (4.21) is true for \( v = \hat{f}_i^l \). Setting \( X = \phi \psi_i \partial_i \), where \( \partial_i \) is defined by \( \varphi_i \), we obtain

\[
\int_{U_i} f_i^l(u, \cdot) R_\delta^j \phi \psi_i g_{ji} \, dv_{\tilde{g}} \overset{\delta \searrow 0}{\longrightarrow} \int_{U_i \cap \partial M} T f_i^l(u, \cdot) \phi \psi_i N^j g_{ji} \, dv_{\tilde{g}}.
\]

After summation over \( i \in I \) it remains to show that we can replace \( T f_i^l(u, \cdot) \) by \( f_i^l(Tu, \cdot) \). To this end, let \( x_0 \in U_i \cap \partial M \) be fixed, meaning \( x_0 \) is not
the variable of integration, then due to Lemma 3.6 we have for \( \tilde{g} \)-almost every such \( x_0 \)

\[
\left| T[f_l^i(u(x_0), x_0)] - f_l^i(Tu(x_0), x_0) \right|
\]

\[
= \left| \lim_{\rho \to 0} \int_{B^\rho(x_0)} f_l^1(u(\cdot), \cdot) \, dv_g - \int_{B^\rho(x_0)} f_l^1(Tu(x_0), x_0) \, dv_g \right|
\]

\[
\leq \liminf_{\rho \to 0} \int_{B^\rho(x_0)} \left| f_l^1(u(\cdot), \cdot) - f_l^1(u(\cdot), x_0) \right| \, dv_g
\]

\[
+ \liminf_{\rho \to 0} \int_{B^\rho(x_0)} \left| f_l^1(u(\cdot), x_0) - f_l^i(Tu(x_0), x_0) \right| \, dv_g = 0
\]

because of Corollary 3.6 (3) and the regularity of \( u \) and \( f \).

\( \square \)

**Proof of Theorem 4.6** We define an approximation \( s_\eta : \mathbb{R} \to [-1, 1] \) of the sign function with

\[
s_\eta(z) := S'_\eta(z) = \begin{cases} 
  -1 & \text{if } z < -\eta \\
  \frac{z}{\eta} & \text{if } |z| \leq \eta, \\
  1 & \text{if } z > \eta.
\end{cases}
\]

Multiplying (4.1) by \( s_\eta(u_\epsilon^t - k) \phi \), where \( k \in \mathbb{R}, \phi \in C_0^\infty(\bar{M} \times (0, T)) \), \( \phi \geq 0 \) and integration over \( M \times (0, T) \) yields

\[
\int_M \int_0^T \left[ \int_k^{u_\epsilon^t} s_\eta(z - k) \, dz \right] \partial_t \phi \, dt \, dv_g
\]

\[
+ \int_M \int_0^T \langle f(u_\epsilon^t) - f(k), \text{grad}_g u_\epsilon^t \rangle_g s_\eta'(u_\epsilon^t - k) \phi \, dt \, dv_g
\]

\[
+ \int_M \int_0^T \langle f(u_\epsilon^t) - f(k), \text{grad}_g \phi \rangle_g s_\eta(u_\epsilon^t - k) \, dt \, dv_g
\]

\[
(4.22)
\]

\[
= \epsilon \int_M \int_0^T |\text{grad}_g u_\epsilon^t|^2_g s_\eta'(u_\epsilon^t - k) \phi \, dt \, dv_g
\]

\[
+ \epsilon \int_M \int_0^T \langle \text{grad}_g u_\epsilon^t, \text{grad}_g \phi \rangle_g s_\eta(u_\epsilon^t - k) \, dt \, dv_g
\]

\[
+ \epsilon \int_{\partial M} \int_0^T N(u_\epsilon^t) s_\eta(k) \phi \, dt \, dv_g
\]

\[
- \int_{\partial M} \int_0^T \langle f(0) - f(k), N \rangle_g s_\eta(k) \phi \, dt \, dv_g
\]

where we used integration by parts, \( \text{div}_g f(k) = 0 \) and the definition of \( \phi \).

Since the forth line of (4.22) is nonnegative and Lemma 2.1 yields that the
second line of (4.22) tends to zero for \( \eta \searrow 0 \), we obtain in the limit \( \eta \searrow 0 \):

\[
\int_M \int_0^T |u^\varepsilon - k| \partial_t \phi \, dt \, dv_g + \int_M \int_0^T \langle f(u^\varepsilon) - f(k), \text{grad}_g \phi \rangle_g \text{sgn}(u^\varepsilon - k) \, dt \, dv_g
\]

\[
\geq \epsilon \int_M \int_0^T (\text{grad}_g u^\varepsilon, \text{grad}_g \phi)_{\tilde{g}} \text{sgn}(u^\varepsilon - k) \, dt \, dv_{\tilde{g}}
\]

\[
+ \epsilon \int_{\partial M} \int_0^T N(u^\varepsilon) \text{sgn}(k) \phi \, dt \, dv_{\tilde{g}} - \int_{\partial M} \int_0^T \langle f(0) - f(k), N \rangle_g \text{sgn}(k) \phi \, dt \, dv_{\tilde{g}}.
\]

Next, we consider \( \epsilon \searrow 0 \). Since the total variation of \( u^\varepsilon \) on \( M \) is bounded uniformly in \( \epsilon \) (cf. (4.10)) the third line of (4.23) tends to zero for \( \epsilon \searrow 0 \).

With regard to the forth line of (4.23) we insert \( R_\delta \), defined in (2.4), apply the divergence theorem and use (4.1) and (4.10) in order to conclude

\[
\lim_{\epsilon \searrow 0} \epsilon \int_{\partial M} \int_0^T N(u^\varepsilon) \phi \, dt \, dv_{\tilde{g}} = \lim_{\epsilon \searrow 0} \epsilon \int_M \int_0^T \Delta_g u^\varepsilon \phi R_\delta + \langle \text{grad}_g u^\varepsilon, \text{grad}_g (\phi R_\delta) \rangle_g dt \, dv_g
\]

\[
= -\int_M \int_0^T (u \partial_t \phi + \langle f(u), \text{grad}_g \phi \rangle_g) R_\delta dt \, dv_g
\]

\[
- \int_M \int_0^T \langle f(u), \text{grad}_g R_\delta \rangle_g \phi dt \, dv_g + \int_{\partial M} \int_0^T \langle f(0), N \rangle_g \phi \, dt \, dv_{\tilde{g}}.
\]

With the fact that the first term on the right-hand side tends to zero for \( \delta \searrow 0 \) and with Lemma 4.8 applied to the second term on the right-hand side we conclude that (4.23) in the limit \( \epsilon \searrow 0 \) implies that any viscosity limit \( u \) of (4.1)-(4.3) fulfills the entropy inequalities (4.19). \( \square \)

4.3. **Uniqueness of the entropy solution.** To prove uniqueness we will use Kruzkov’s [17] technique of doubling the variables which was generalized by Lengeler and Müller [20] to the case of closed Riemannian manifolds. In this section we adapt their work to the case of compact Riemannian manifolds with boundary.

We need the following Lemma from Kruzkov [17].

**Lemma 4.9.** If a function \( h \in C(\mathbb{R}) \) satisfies a Lipschitz condition on an interval \( [-z, z] \subset \mathbb{R} \) with constant \( L > 0 \), then the function \( q(z_1, z_2) := \text{sgn}(z_1 - z_2)(h(z_1) - h(z_2)) \) satisfies the Lipschitz condition in \( z_1 \) and \( z_2 \) with the same constant \( L \).
Theorem 4.10 (Uniqueness of the entropy solution). The entropy solution of problem \([1.7], [1.2], [4.18]\) is unique.

Proof. We assume that there exist two entropy solutions \(u\) and \(v\). Using the doubling of variables technique of Kruzkov (cf. [17]) we consider \([1.19]\) with 
\[
\phi = \phi(x, t, y, s) \in C_0^\infty(M \times (0, T) \times M \times (0, T))
\]
first for \(u\) with \(k = v(y, s)\) and integrate over \(M \times (0, T)\) w.r.t. \((y, s)\), and then for \(v\) with \(k = u(x, t)\) and integrate over \(M \times (0, T)\) w.r.t. \((x, t)\). Summation of the two inequalities yields

\[
\int_0^T \int_M \int_0^T \int_M |u(x, t) - v(y, s)| (\partial_t \phi + \partial_x \phi) dt dy dx ds + \langle q(u(x, t), v(y, s), x, t), \text{grad}_x^\| \phi \rangle_g + \langle q(v(y, s), u(x, t), y, s), \text{grad}_y^\| \phi \rangle_g \geq 0
\]

with

\[
q(u, k, x, t) := \text{sgn}(u - k) (f(u, x, t) - f(k, x, t)).
\]

We set

\[
\phi(x, t, y, s) := \psi(t)\bar{\psi}(x)\omega_\epsilon(t - s)\kappa_\epsilon(x, y),
\]

where \(\psi \in C^\infty_C((0, T))\) with \(\psi \geq 0\) and \(\bar{\psi} \in C_0^\infty(M)\) with \(\bar{\psi} \geq 0\), \(\omega_\epsilon(s) := \frac{1}{\epsilon} \omega(\frac{s}{\epsilon})\) with \(\omega \in C^\infty(\mathbb{R})\), \(\text{supp}(\omega) \subset (-1, 1)\), \(\omega \geq 0\) and \(\int_{\mathbb{R}} \omega(s) ds = 1\) and \(\kappa_\epsilon(x, y) := \frac{1}{\epsilon^2} \kappa(\frac{d(x, y)}{\epsilon})\) with \(\kappa \in C^\infty(\mathbb{R})\), \(\text{supp}(\kappa) \subset (-1, 1)\), \(\kappa \geq 0\) and \(\int_{\mathbb{R}} \kappa(|z|) dz = 1\). Thus, (4.24) yields

\[
\int_0^T \int_M \psi'(t)\bar{\psi}(x) \int_0^T \int_M \omega_\epsilon(t - s)\kappa_\epsilon(x, y) |u(x, t) - v(y, s)| \text{dv}_y \text{dv}_x dt + \int_0^T \int_M \psi(t)\bar{\psi}(x) \int_0^T \int_M \omega_\epsilon(t - s) \left[ \langle q(u(x, t), v(y, s), x, t), \text{grad}_x^\| \kappa_\epsilon(x, y) \rangle_g \bar{\psi}(x) + \langle q(u(x, t), v(y, s), x, t), \text{grad}_y^\| \bar{\psi}(x) \rangle_g \kappa_\epsilon(x, y) + \langle q(v(y, s), u(x, t), y, s), \text{grad}_y^\| \kappa_\epsilon(x, y) \rangle_g \bar{\psi}(x) \right] \text{dv}_y \text{dv}_x dt \geq 0.
\]

With the same argumentation as in [20, 1714-1718] we obtain by subsequently letting \(\epsilon\) and \(\epsilon\) tend to zero

\[
\int_0^T \int_M \psi'\bar{\psi} |u - v| + \psi \text{sgn}(u - v) \langle \text{grad}_x^\| \bar{\psi}, f(u) - f(v) \rangle_g \text{dv}_x dt \geq 0.
\]

Setting \(\bar{\psi} := 1 - R_\delta\) we get with Lemma [18] for \(\delta \searrow 0\)

\[
(4.25) \int_M \int_0^T \psi|u - v| dt dv_g \geq \int_M \int_0^T \psi \text{sgn}(Tu - Tv) \langle f(Tu) - f(Tv), N \rangle_g dt dv_g.
\]

Here, we used that \(\text{sgn}(u - v)(f(u) - f(v)) = f(\max\{u, v\}) - f(\min\{u, v\})\) and the fact that \(\max\{u, v\}\) and \(\min\{u, v\}\) inherit a bounded variation from \(u\) and \(v\). In the next lines we will show that the right-hand side of (4.25)
nonnegative. With
\[
\tilde{k} := \begin{cases} 
Tu & \text{if } Tu \in I(0, Tv) \\
0 & \text{if } 0 \in I(Tu, Tv) \\
Tv & \text{if } Tv \in I(0, Tu),
\end{cases}
\]
where \(I(a, b) := [\min\{a, b\}, \max\{a, b\}]\) we obtain
\[
\int_{\partial M} \int_0^T \psi(t) \, \text{sgn}(Tu - Tv) \, \langle f(Tu) - f(Tv), N \rangle_g \, dt \, dv_g 
\]
\[
= \int_{\partial M} \int_0^T \psi(t) \, \text{sgn}(Tu - \tilde{k}) \, \langle f(Tu) - f(\tilde{k}), N \rangle_g \, dt \, dv_g 
\]
\[
+ \int_{\partial M} \int_0^T \psi(t) \, \text{sgn}(Tv - \tilde{k}) \, \langle f(Tv) - f(\tilde{k}), N \rangle_g \, dt \, dv_g.
\]

In order to show that each summand is nonnegative we exploit inequality \((4.19)\) with \(\phi = \tilde{\phi} R_{\delta} \) for \(\tilde{\phi} \in C_0^\infty(M \times (0, T))\), \(\tilde{\phi}(x, t) \geq 0\) and obtain
\[
\int_M \int_0^T |u - k| \, \partial_t \tilde{\phi} R_{\delta} + \text{sgn}(u - k) \, \langle f(u) - f(k), \text{grad}_g R_{\delta} \rangle_g \, \tilde{\phi} \, dt \, dv_g 
\]
\[
+ \text{sgn}(u - k) \, \langle f(u) - f(k), \text{grad}_g \tilde{\phi} \rangle_g \, R_{\delta} \, dt \, dv_g 
\]
\[
+ \int_{\partial M} \int_0^T \text{sgn}(k) \, \langle f(Tu) - f(k), N \rangle_g \, \tilde{\phi} \, dt \, dv_g \geq 0
\]
for all \(k \in \mathbb{R}\). By Lemma \((4.13)\) we have for \(\delta \searrow 0\)
\[
\int_{\partial M} \int_0^T (\text{sgn}(Tu - k) + \text{sgn}(k)) \, \langle f(Tu) - f(k), N \rangle_g \, \tilde{\phi} \, dt \, dv_g \geq 0
\]
for all \(k \in \mathbb{R}\). Since \(\tilde{\phi}\) was arbitrary, obviously
\[
(\text{sgn}(Tu - k) + \text{sgn}(k)) \, \langle f(Tu) - f(k), N \rangle_g \geq 0
\]
almost everywhere on \(\partial M \times (0, T)\). Using the fact that
\[
\text{sgn}(Tu - \tilde{k}) = \begin{cases} 
0 & \text{if } \tilde{k} = Tu, \\
\text{sgn}(Tu - \tilde{k}) + \text{sgn}(\tilde{k}) & \text{if } \tilde{k} = 0, \\
\frac{1}{2} (\text{sgn}(Tu - \tilde{k}) + \text{sgn}(\tilde{k})) & \text{if } \tilde{k} = Tv,
\end{cases}
\]
we see, after a repetition of the argumentation for \(v\), that the right-hand side of \((4.25)\) is nonnegative and consequently
\[
(4.26) \quad \int_M \int_0^T \psi(t) \, |u - v| \, dt \, dv_g \geq 0.
\]

Let \(\Psi\) denote the characteristic function of an arbitrary time interval \([t_0, t_1]\) \(\subset (0, T)\) and \(\psi_\epsilon = \Psi * \omega_\epsilon\) its mollification. For \(\psi = \psi_\epsilon\) in \((4.20)\) we obtain as \(\epsilon\) tends to zero
\[
(4.27) \quad \|u(\cdot, t_1) - v(\cdot, t_1)\|_{L^1(M)} \leq \|u(\cdot, t_0) - v(\cdot, t_0)\|_{L^1(M)}.
\]
Letting \(t_0\) tend to zero we obtain uniqueness.

**Corollary 4.11** \((L^1\text{-contraction property})\). Let \(u, v\) be two entropy solutions of problem \((1.7), (1.2), (4.18)\), then \((4.27)\) holds for \(0 \leq t_0 \leq t_1\).
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