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Electrodynamic theory of ferromagnetic resonance and its applications in precise measurements of ferromagnetic linewidth, permeability tensor and saturation magnetization

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ABSTRACT
This review paper describes the state of the art in measurements of ferromagnetic linewidth, permeability tensor and saturation magnetization employing electrodynamic theory of ferromagnetic resonance. It is shown that the electrodynamic theory allows significant improvements of measurement accuracy of these parameters with respect to the commonly used perturbation and magnetostatics theories. Contrary to the perturbation method the electrodynamic theory is not limited to small samples. It allows determination of the resonance frequencies and $Q$-factors for arbitrary size spherical and cylindrical gyromagnetic samples in suitably chosen metal enclosures. Results obtained with the electrodynamic theory for very small samples are identical to those obtained with perturbation and magnetostatics theories. Results of measurements of the ferromagnetic linewidth, permeability tensor and saturation magnetization at microwave frequencies are presented.

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I. INTRODUCTION

The ferromagnetic resonance (FMR) phenomenon has been known for more than 80 years and a lot of papers have been already published on this topic.¹⁻¹⁸ For a uniformly magnetized ferromagnetic material its permeability becomes a tensor quantity (1). When the static magnetic field is sufficiently strong, to saturate ferromagnetic medium i.e. $H_0 > M_S$, and it is aligned along z-axis of Cartesian or cylindrical coordinate system then $\mu_\parallel = 1$ and the permeability tensor is known as Polder’s tensor.² Then:

$$\bar{\mu} = \mu_0 \begin{bmatrix} \mu & j\kappa & 0 \\ -j\kappa & \mu & 0 \\ 0 & 0 & \mu_\parallel \end{bmatrix},$$

where:

$$\mu = 1 + \frac{H_0 + j\omega}{H_0^2 - \omega^2 + 2j\alpha H_0 \omega},$$

$$\kappa = -\frac{\omega}{H_0^2 - \omega^2 + 2j\alpha H_0 \omega},$$

where: $H_0 = H_0/M_S$, $\omega = \tilde{f} f_m$, $f_m = \gamma M_S$, $H_0$ is the static magnetic field inside the sample (the internal static magnetic field), $M_S$ is the saturation magnetization, $\alpha$ is Gilbert damping factor, $\gamma$ is the effective gyromagnetic ratio $\gamma = g_e = 35.21719$ MHz/(kA/m) and $\tilde{f}$ is the complex frequency. Alternatively to the Gilbert damping factor, the relaxation time, $\tau = 1/(\alpha \gamma H_0)$, and the ferromagnetic resonance linewidth, $\Delta H = 2a H_0 = 2/(\gamma \tau)$, are used to describe the losses in the ferromagnetic material.
FMR frequency is defined as the frequency for which the imaginary parts of $\mu$ and $\kappa$ approach maxima.

For a circularly polarized electromagnetic (EM) field the “effective” permeability of ferromagnetic material biased with a static magnetic field, orthogonally polarized with respect to the applied EM field, is defined as

$$\mu_{eff} = \mu \pm \kappa. \quad (4)$$

A positive (negative) sign in Equation (4) corresponds to the clockwise (counterclockwise) polarization of the magnetic field.

For a lossless ferromagnetic medium $\mu_{l,s} = 1 + 1/(H_{0l} \pm w)$.

From the two quantities only

$$\mu_t = 1 + 1/(H_{0t} - w) \quad (5)$$

exhibits resonance character. It takes place when $H_{0t} = w$.

For most known ferromagnetic materials FMR resonance appears at microwave frequency range. Only two physical quantities namely the saturation magnetization and the Gilbert damping factor are necessary to describe the permeability tensor of a saturated ferromagnetic medium. For partially magnetized medium theoretical models describing dependence of permeability tensor components on the static magnetic field bias are approximate and measurements of all permeability tensor components are desirable. The saturation magnetization of ferromagnetic materials is routinely measured at microwave frequencies at uniform static magnetic field and resonance frequencies in spheroidal ferromagnetic samples (Fig. 1) Walker employed quasi-magnetostatics theory. He found the relationship between the normalized resonance frequencies and the normalized internal static magnetic field for many modes (eigenmodes) that can be excited in such samples. This theory is so popular that the resonance modes in ferromagnetic samples are often called magnetostatic modes.\(^{19}\) The dominant magnetostatics mode a spherical sample is called as the mode of uniform precession when $w - H_{0t} = 1/3$. One can find from Equation (5) that for this mode $\mu_t = -2$. The magnetostatic model does not allow to consider magnetic losses to determine the $Q$-factors of a specific mode. So far perturbation theory has been commonly used for this purpose.

II. THEORETICAL MODELS DESCRIBING FMR

Experimental studies of FMR phenomena are usually performed on small samples, having spherical, cylindrical or planar (thin films) shape. For spherical samples the internal static magnetic field is uniform when they are placed in a uniform static magnetic field. For cylindrical and planar samples, the internal field is only approximately uniform when the aspect ratio (length to diameter or lateral dimension to thickness) is sufficiently large.

Therefore, spherical samples have been preferably used in the ferromagnetic resonance linewidth measurements. Narrow ferromagnetic linewidth samples behave as tunable microwave resonators. Resonances in them can be excited either in free space or in close metal cavities. Resonance structures having spherical symmetry for which exact solutions of Maxwell’s equations are available\(^7\) are shown in Fig. 1.

![Fig. 1. Resonance structures having spherical symmetry: a) Sample in spherical cavity, b) sample in free space.](image)

A. Magnetostatics theory

To find the relationship between the internal static magnetic field and resonance frequencies in spheroidal ferromagnetic samples (Fig. 1) Walker\(^8\) employed quasi-magnetostatics theory. He found the relationship between the normalized resonance frequencies and the normalized internal static magnetic field for many modes (eigenmodes) that can be excited in such samples. This theory is so popular that the resonance modes in ferromagnetic samples are often called magnetostatic modes.\(^{19}\) The dominant magnetostatics mode a spherical sample is called as the mode of uniform precession when $w - H_{0t} = 1/3$. One can find from Equation (5) that for this mode $\mu_t = -2$. The magnetostatic model does not allow to consider magnetic losses to determine the $Q$-factors of a specific mode. So far perturbation theory has been commonly used for this purpose.

B. Perturbation theory and the external susceptibilities

Perturbation theory is one of the most the most frequently used methods to determine the resonance frequency and $Q$-factor changes of resonance cavities containing small specimens. In perturbation theory the relationship between the complex frequency of an empty and the complex frequency of the cavity containing ferromagnetic sample is expressed as follows.

$$\frac{\tilde{f} + \tilde{f}_0}{f} = -C_1F(\mu, \kappa, \mu_0) - C_2G(\varepsilon f), \quad (6)$$

where: $\tilde{f}_0 = f'_0 + j\frac{\varepsilon f_0}{2}, \tilde{f} = f' + j\frac{\varepsilon f}{2}$ denote complex frequencies of the empty and perturbed cavity, respectively, $Q_0$ and $Q$ represent the corresponding unloaded $Q$-factors, $C_1$ and $C_2$ are perturbation constants, that are the order of $V_0$/$V$, where $V_0$ and $V$ denote the volume of the sample and the cavity, respectively.

Functions $F(\mu, \kappa, \mu_0)$ and $G(\varepsilon f)$ depend on the shape of the sample under test and its position in the cavity. Perturbation formulas have been derived for spherical, thin cylindrical rod, and thin disc samples being the degenerate forms of a spheroid. The most frequently used magnetic perturbation functions are: $F = 3(\mu + \kappa - 1)/(\mu + \kappa + 2)$, for spherical sample situated in circularly polarized magnetic field, $F = 2(\mu + \kappa - 1)/(\mu + \kappa + 1)$, for thin, cylindrical rod situated in circularly polarized magnetic field. Functions $F(\mu, \kappa, \mu_0)$ are often referred to as the external susceptibilities of the samples.\(^{11}\) For spherical sample situated in a clockwise circularly polarized magnetic field $F = 3(\mu_t - 1)/(\mu_t + 2)$. This function has a singularity (resonance) when $\mu_t = -2$ which corresponds to the resonance condition for the magnetostatic mode of uniform precession.

In experiments with spherical samples the dominant resonance appears not at FMR frequency where $\mu_t$ becomes singular but at frequency where $\mu_t = -2$ or in other words at frequency where the external susceptibility of a spherical sample has a singularity i.e. when $f = yH_{ext}$ where $H_{ext}$ is the external static magnetic field. It is often said that for such case demagnetization factors for the DC and the microwave fields cancel out.

C. Electrodynamic theory

The authors of this paper employed rigorous electrodynamic theory (ED) to study resonances in dispersive gyromagnetic sphere.\(^{12}\)
FIG. 2. a) Normalized resonance frequencies \( w - H_{0r} \) of the TE\(_{101} \) mode versus \( d/\lambda \). b) Normalized resonance frequencies \( w - H_{0r} \) of the first TE\(_{n01} \) modes versus \( H_{0r} \). Computations were performed for spherical samples in free space (Fig. 1b) with \( d = 0.5 \) mm. 

and cylinder.\(^{18}\) They proved that dominant modes in such samples have physical properties of magnetic plasmon resonances (MPR). Using ED theory, it is possible to determine the resonance frequencies and their dependence on the sizes of the sample and the metal enclosure. Q-factors can be rigorously analyzed considering magnetic, dielectric and conductor losses in the whole resonance structure. Effect of size of a spherical ferromagnetic sample on its dominant resonance frequency was measured\(^7\) and it was theoretically analyzed.\(^{11,12}\) In all these papers it is shown that the fundamental mode resonance frequency for larger samples is shifted down with respect to the value predicted by the magnetostatics theory. The frequency shift depends on the saturation magnetization of the sample, its permittivity and \( \frac{d}{\lambda} \), where \( d \) denotes the diameter of the sample and \( \lambda = c/f \). In Fig. 2 results of computations of the normalized resonance frequencies \( w - H_{0r} \) are shown for spherical samples depicted in Fig. 1b having \( d = 0.5 \) mm. Results in Fig. 2a

FIG. 3. Comparison between rigorous electrodynamic modelling and perturbation theory for spherical cavity operating at 8576 MHz on the TE\(_{101} \) mode (\( R_2 = 25 \) mm) containing spherical ferromagnetic samples with \( R_1 = 0.5 \) mm, and \( M_S = 1750 \) Gs. a) Resonance frequency shifts for \( \Delta H = 5 \) Oe. b) Q-factors (total) for \( \Delta H = 5 \) Oe. c) Resonance frequency shifts for broad linewidth samples. d) Q-factors due to magnetic losses for broad linewidth samples, e) Schematic of the empty cavity and the cavity containing spherical sample for which computations were performed.
were obtained employing ED model (solid lines) and approximate Hurd and Marcereau theories (broken lines). Results in Fig. 2b were obtained employing ED model. It is seen that for small samples when \( \frac{d}{\lambda} < 1 \), the \( w - H_{0r} \) values converge to fixed numbers predicted by the magnetostatics model. Computations presented in Fig. 2 were obtained for small samples \((d = 0.5 \text{ mm})\). For larger samples discrepancies between rigorous ED theory and magnetostatics model increase. It should be noted that \( w - H_{0r} \) values also depend on the diameter of metal enclosure and permittivity of medium outside the sample as it has been shown in Ref. 21.

ED computations of the resonance frequencies can be used to improve accuracy of measurements of the effective gyromagnetic ratio and the saturation magnetization. The effective gyromagnetic ratio \( g_{\text{eff}} \) is usually determined from the measured resonance frequency of the mode of uniform precession (TE\(_{101}\) mode) for a given external static magnetic field \( H_{\text{ext}} \) using the following formula:

\[
g_{\text{eff}} = \frac{f}{H_{\text{ext}}}
\]

This formula is valid if \( w - H_{0r} = 1/3 \) which takes place for infinitesimally small samples. As it is seen in Fig. 2a for larger samples \( w - H_{0r} < 1/3 \) and appropriate corrections should be introduced to evaluate \( g_{\text{eff}} \) accurately. The saturation magnetization can be determined from measurements of the frequency spacing \((f_n - f_m)\) between two well identified modes\(^{15-22}\) from the formula:

\[
M_S = (f_n - f_m)(w_n - w_m)/\gamma,
\]

where \( w_n \) denotes computed normalized resonance frequency of the mode \( n \). As it is seen in Fig. 2b, \( w_n - w_m \) differences between consecutive TE\(_{\text{101}}\) modes are not constants, as predicted by magnetostatics model, so such approach would lead to significant errors in the determination of saturation magnetization. As it has been described in Ref. 23 the ED model allows for accurate determination of this parameter.

Measurements of the ferromagnetic linewidth (or Gilbert damping factor) can be performed either in closed metal cavities where one of the cavity resonances interacts with resonance in ferromagnetic sample or by direct observation of the resonance in the sample which is connected do VNA (vector network analyzer) using appropriate coupling system. The first method has been described as a standard\(^{24,25}\) for measurements of the ferromagnetic linewidth on small spherical samples but it limited to specimens having \( \Delta H > 10 \text{ Oe} \). As it has been already proved in Ref. 26 the 10 Oe limit
is related to limits of perturbation theory. For a narrow linewidth sample perturbation of the cavity is large and fails even if the sample is very small because perturbation function for spherical sample has singularity at $\mu_r = -2$. It is demonstrated in Fig. 3 showing theoretical computations of the resonance frequency and Q-factor of the $\text{TE}_{101}$ mode of spherical cavity (as shown in Fig. 3e) employing ED model and perturbation theory. It is seen in Fig. 3c and Fig. 3d that for medium and broad linewidth samples ($\Delta H = 100 \text{ Oe}$ and $200 \text{ Oe}$) results of ED and perturbation theory well agree while for the sample having $\Delta H = 5 \text{ Oe}$ and $\Delta H = 10 \text{ Oe}$ (Fig. 3a and Fig. 3b) perturbation theory fails. Results of computations shown in Fig. 3 were obtained for the sample having diameter of 1 mm. For smaller samples the minimum $\Delta H$ value for which perturbation theory is acceptable decreases but even for samples having diameter of 0.2 mm perturbation theory errors become unacceptable large when $\Delta H < 10 \text{ Oe}$. Detailed procedure of $\Delta H$ determination from the measured values of the resonance frequencies and Q-factors, when perturbation theory fails, (such as shown in Fig. 3a and 3b) employing ED model has been presented in Ref. 26. The linewidth of the sample is determined from the measured Q-factor value for such DC magnetic bias when the Q-factors for the lower and for the upper branch are equal to each other (Fig. 3b). The saturation magnetization, the diameters of the sample and the cavity and the Q-factor of the empty cavity have to be known, because they are involved in the ED computations. Being rigorous the ED theory allows to overcome the limits of perturbation theory and use cavities described in the standard methods for the determination of $\Delta H$ for samples having arbitrary size and arbitrary $\Delta H$ values. The advantage of the standard single frequency measurement method supplemented with ED theory is that it allows measurements of samples having arbitrary $\Delta H$ values.

For narrow linewidth materials such as monocrystalline garnets the resonance frequency and the Q-factor of the MPR mode (the mode of uniform precession) can be directly measured in a subwavelength cavities (shown in Fig. 4c). Such measurements are usually performed as a function of the static magnetic field, and therefore frequency, below the first frequency of the empty cavity. Observed resonance corresponds to the lower branch of MPR mode as it is seen in Fig. 3a and Fig. 3b. ED theory allows to consider conductor losses in the metal shield and dielectric losses in the sample. The linewidth from such broad frequency band measurements is determined as $\Delta H = H_{\text{int}}/Q_m$, where $Q_m$ is the Q-factor related to the magnetic losses in the sample. Conductor losses in the metal enclosure are taken into account in determination of $Q_m$ in the manner described in Ref. 26. Results of the ferromagnetic linewidth and Gilbert damping factor determination for pure and Ga-doped single crystal YIG samples are presented in Fig. 4. Details of experiment have been presented in Ref. 27. Fig. 4 shows that both the ferromagnetic linewidth and the Gilbert damping factor are not constant versus frequency.

Early researchers measured permeability tensor in metal cavities employing perturbation theory. ED model was used for the first time in 1957 but at that time permeability tensor components were considered as numbers that are calculated from the resonance equation. Exact transcendental equations are currently available for gyromagnetic samples situated in resonance structure having cylindrical symmetry and they were used in measurements...
setup shown in Fig. 5a for measurements of all permeability tensor components.\textsuperscript{33} The measurements setup employs three modes in two dielectric resonators having the same height but different external diameters. The larger resonator operates on the $\text{TE}_{011}$ mode while the smaller on the $\text{HE}_{111}$ mode. The external diameter of the smaller resonator is chosen such that its $\text{HE}_{111}$ mode resonant frequency is close to the $\text{TE}_{011}$ mode resonant frequency of the larger resonator in order to reduce the influence of frequency on measurements results. Results of measurements of the resonance frequencies versus the static field bias for polycrystalline YIG samples having diameter of 2 mm and 0.46 mm respectively are shown in Fig. 5b (after Ref. 32). Results of permeability tensor components determination (scaled to 9.82 GHz) based on these measurements are shown in Fig. 3c as solid circles (for 2 mm sample) and as empty squares (for 0.46 mm sample). When the static magnetic field bias is sufficiently large to saturate the sample i.e. when $H_{0r} > 1$ measured permeability tensor components precisely follow theoretical expressions (2) and (3). As it is seen in Fig. 3b) for the clockwise circularly polarized magnetic fields the resonance frequencies of the $\text{HE}_{111}$ mode converge to the magnetic plasmon asymptotic line, corresponding to $\mu_r = -1$ for cylindrical rod shape sample, when perturbation function has singularity. If measurements are performed in resonators operating on different modes e.g. in frequently used cylindrical cavities employing $TM_{110}^r$ modes, the asymptotic line for the $TM_{110}^r$ mode corresponds to $\mu_r = 0$.\textsuperscript{11} For practical applications, in design of microwave circulators and phase shifters, measurements results. More details about permeability tensor measurements can be found in the overview paper.\textsuperscript{33}

### III. CONCLUSIONS

Rigorous electrodynamic theory improves modes computation in ferromagnetic spheres and cylinders and allows to take into account the influence of various factors, such as permittivity, conductivity of the sample and conductivity of metal shield on the resonance frequencies and $Q$-factors. Employing ED theory accurate microwave measurements of the ferromagnetic linewidth, the saturation magnetization and permeability tensor components can be performed, regardless of the size of samples. Limits of magnetostatics and perturbation theories for ferromagnetic linewidth and saturation magnetization measurements have been established.

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