Short generators without quantum computers: the case of multiquadratics

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Joint work with Jens Bauch & Henry de Valence & Tanja Lange
Part I: Introduction

INTRODUCTIONS
Should always start with a handshake

Daily HAHA
How secure is SVP?

“How lattice-based crypto is secure because lattice problems are hard.”

Really? How hard are they? Which cryptosystems are secure?
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- $c \approx 0.415$: 2008 Nguyen–Vidick.
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- $c \approx 0.378$: 2011 Wang–Liu–Tian–Bi.
- $c \approx 0.337$: 2014 Laarhoven.
- $c \approx 0.298$: 2015 Laarhoven–de Weger.
- $c \approx 0.292$: 2015 Becker–Ducas–Gama–Laarhoven.
- Quantum algorithm: $c \approx 0.268$: 2014 Laarhoven–Mosca–van de Pol.

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How secure is approx SVP?

2002 Micciancio–Goldwasser (emphasis added): “To date, the best known polynomial time (possibly randomized) approximation algorithms for SVP and CVP achieve worst-case (over the choice of the input) approximation factors $\gamma(n)$ that are essentially exponential in the rank $n$.”

2007 Regev:

2013 Micciancio: “Smooth trade-off between running time and approximation: $\gamma \approx 2^{O(n \log \log T / \log T)}$”
Quantum attacks against cyclotomic lattice problems

STOC 2014 Eisenträger–Hallgren–Kitaev–Song: poly-time quantum algorithm for $K \mapsto \mathcal{O}_K^\times$.

$K$: number field.
$\mathcal{O}_K$: ring of algebraic integers in $K$.
$\mathcal{O}_K^\times$: group of units in $\mathcal{O}_K$. 

This recovers secret keys in, e.g.,
2000 Buchmann–Maurer–Möller cryptosystem using cyclotomics,
STOC 2009 Gentry FHE system using cyclotomics,
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2015 (and SODA 2016) Biasse–Song, also using an idea from 2014 Campbell–Groves–Shepherd: poly-time quantum algorithm for $K, g\mathcal{O}_K \mapsto \zeta_m^j g$ for some $j$, assuming cyclotomic $K = \mathbb{Q}(\zeta_m)$, small $h_m^+$, very short $g$. This recovers secret keys in, e.g., 2000 Buchmann–Maurer–Möller cryptosystem using cyclotomics, STOC 2009 Gentry FHE system using cyclotomics, Eurocrypt 2013 Garg–Gentry–Halevi multilinear-map system, etc.
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More lattice problems of interest:

- $I \mapsto$ shortest nonzero vector in $I$. ("Exact Ideal-SVP".)
- $I \mapsto$ close to shortest nonzero vector in $I$. ("Approximate Ideal-SVP".)

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Counterargument: attack is poly time against arbitrary principal ideals for approx factor $2^{N^{1/2+o(1)}}$ in degree-$N$ cyclotomics, assuming small $h^+$. See, e.g., 2016 Cramer–Ducas–Peikert–Regev.
Is the attack idea limited to principal ideals?

2015 Peikert:
“Although cyclotomics have a lot of structure, nobody has yet found a way to exploit it in attacking Ideal-SVP/BDD . . . For commonly used rings, principal ideals are an extremely small fraction of all ideals. . . . The weakness here is not so much due to the structure of cyclotomics, but rather to the extra structure of principal ideals that have short generators.”
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Counterargument, 2016 Cramer–Ducas–Wesolowski:
Ideal-SVP attack for approx factor $2^{N^{1/2+o(1)}}$ in degree-$N$ cyclotomics, under plausible assumptions about class-group generators etc. Starts from Biasse–Song, uses more features of cyclotomic fields.

This shreds the standard approx-Ideal-SVP tradeoff picture.
Non-cyclotomic lattice-based cryptography

Cyclotomics are scary. Let’s explore alternatives:

- Eliminate the ideal structure.
  e.g., use LWE instead of Ring-LWE.
  But this limits the security achievable for key size $K$. 
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- 2016 Bernstein–Chuengsatiansup–Lange–van Vredendaal “NTRU Prime” (preliminary announcement 2014.02, before these attacks): as in discrete-log crypto, eliminate unnecessary ring morphisms.
  Use prime degree, large Galois group: e.g., $x^p - x - 1$. 

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- This talk: Switch from cyclotomics to other Galois number fields.
  Another popular example in algebraic-number-theory textbooks:
  multiquadratics; e.g., $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \sqrt{17}, \sqrt{19}, \sqrt{23})$. 
A reasonable multiquadratic cryptosystem

Case study of a lattice-based cryptosystem that was already defined in detail for arbitrary number fields: 2010 Smart–Vercauteren, optimized version of 2009 Gentry.

Parameter: \( R = \mathbb{Z}[\alpha] \) for an algebraic integer \( \alpha \).
Secret key: very short \( g \in R \).
Public key: \( gR \).
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To handle multiquadratics better, we generalized beyond $\mathbb{Z}[\alpha]$; fixed a keygen speed problem; used twisted Hadamard transforms as replacement for FFTs; adapted 2011 Gentry–Halevi cyclotomic speedups to multiquadratics.

Like Smart–Vercauteren, took $N \in \lambda^{2+o(1)}$ for target security $2^\lambda$.
Checked security against standard lattice attacks: nothing better than exponential time.
Part II: Some preliminaries
Definition

A number field is a field $L$ containing $\mathbb{Q}$ with finite dimension as a $\mathbb{Q}$-vector space. Its degree is this dimension.

Definition

The ring of integers $\mathcal{O}_L$ of a number field $L$ is the set of algebraic integers in $L$. The invertible elements of this ring form the unit group $\mathcal{O}_L^\times$.

Problem

Recover a “small” $g \in \mathcal{O}_L$ (modulo roots of unity) given $g\mathcal{O}_L$.

Definition (for this talk)

A multiquadratic field is a number field that can be written in the form $L = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n})$, where $(d_1, \ldots, d_n)$ are distinct primes.

The degree of the multiquadratic field is $N = 2^n$. 
General strategy to recover \( g \)

1. Compute the unit group \( \mathcal{O}_L^\times \)
General strategy to recover $g$

0. Compute the unit group $\mathcal{O}_L^\times$

1. Find some generator $ug$ of principal ideal $g\mathcal{O}_L$
   - subexponential time algorithm [1990 Buchmann, 2014 Biasse–Fieker, 2014 Biasse]
   - quantum poly-time algorithm [2016 Biasse–Song]

    (BDD: bounded-distance decoding; i.e., finding a lattice vector close to an input point.)
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2. Solve BDD for $\log ug$ in the log-unit lattice to find $\log u$
   - 2014 Campbell–Groves–Shepherd pointed out this was easy for cyclotomic fields with $h^+$ small
   - 2015 Schanck confirmed experimentally
   - 2015 Cramer–Ducas–Peikert–Regev proved pre-quantum polynomial time for these fields

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Definition

Fix a number field $L$ of degree $N$ and fix distinct complex embeddings $\sigma_1, \ldots, \sigma_N$ of $L$. The **Dirichlet logarithm map** is defined as

\[
\text{Log} : L^\times \mapsto \mathbb{R}^N \\
\text{Log} : x \mapsto (\log |\sigma_1(x)|, \ldots, \log |\sigma_N(x)|)
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Theorem (Dirichlet Unit Theorem)

The kernel of $\text{Log}_{\mathcal{O}_L - \{0\}}$ is the cyclic group of roots of unity in $\mathcal{O}_L$. Let $\Lambda = \text{Log} \mathcal{O}_L^\times \subset \mathbb{R}^N$. $\Lambda$ is a lattice of rank $r + c - 1$, where $r$ is the number of real embeddings and $c$ is the number of complex-conjugate pairs of non-real embeddings of $L$. 
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Fact

If $h\mathcal{O}_L = g\mathcal{O}_L$ and $g \neq 0$ then $h = ug$ for some $u \in \mathcal{O}_L^\times$, and

$$\text{Log } g \in \text{Log } h + \Lambda.$$
Part III: The algorithm

algorithm

Word, used by programmers
When they do not want to
Explain what they did.

https://starecat.com/algorithm-word-used-by-programmers-when-they-do-not-want-to-explain-what-they-did/
Algorithm idea 1: subfields

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Algorithm idea 2: the subfield relation

Let $\sigma$ be the automorphism of $L$ that negates $\sqrt{d_n}$ and fixes other $\sqrt{d_j}$. Define $K_\sigma = \{x \in L : \sigma(x) = x\}$ as the field fixed by $\sigma$. The norm $N_\sigma(x)$ of $x \in L$ is defined as $x\sigma(x)$. Then $N_\sigma(x) \in K_\sigma$. 
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Let $\tau$ be the automorphism of $L$ that negates $\sqrt{d_{n-1}}$ and fixes other $\sqrt{d_j}$.

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N_\sigma(x) = x\sigma(x) \\
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Algorithm idea 3: computing units via subfields

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If $U_L = \mathcal{O}_{K\sigma}^\times \cdot \mathcal{O}_{K\tau}^\times \cdot \sigma(\mathcal{O}_{K_{\sigma\tau}}^\times)$, then

$$(\mathcal{O}_L^\times)^2 \subseteq U_L \subseteq \mathcal{O}_L^\times$$

So if we can find a basis for $(\mathcal{O}_L^\times)^2$, taking square roots gives $\mathcal{O}_L^\times$. 
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1966 Wada: We can do this—in exponential time!
Check which products of subsets of basis vectors for $U_L$ are squares.
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Better: polynomial time, adapting 1991 Adleman idea from NFS.
Define many quadratic characters $\chi_i : \mathcal{O}_L^\times \rightarrow \mathbb{Z}/2\mathbb{Z}$.
Almost certainly $(\mathcal{O}_L^\times)^2 = U_L \cap (\bigcap_i \text{Ker } \chi_i)$. Compute by linear algebra.
Algorithm idea 4: recovering generators via subfields

**Fact**

*Can compute* $N_\sigma(g)O_{K_\sigma}$ *quickly from* $gO_L$.

Apply algorithm recursively to find generator $h_\sigma$ of $N_\sigma(g)O_{K_\sigma}$.

i.e. $h_\sigma = u_\sigma N_\sigma(g)$ for some unit $u_\sigma$. 
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Similarly $h_{\tau}$, $h_{\sigma\tau}$. Compute

$$h = \frac{h_{\sigma}h_{\tau}}{\sigma(h_{\sigma\tau})} = \frac{u_{\sigma}N_{\sigma}(g)u_{\tau}N_{\tau}(g)}{\sigma(u_{\sigma\tau})\sigma(N_{\sigma\tau}(g))}.$$ 

Subfield relation: $h = ug^2$ for some $u \in \mathcal{O}_L^\times$. 
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Last step is to shorten the generator $u'g = \sqrt{vh}$ by solving the BDD problem in the log-unit lattice.
Algorithm 1: MQPIP(\(L, I\))

**Input:** Real multiquadratic field \(L\) and a basis matrix for a principal ideal \(I\) of \(O_L\)

**Result:** A short generator \(g\) for \(I\)

1. **if** \([L : \mathbb{Q}] = 2\) **then**
2. **return** MQPIP\((L, I)\)

3. \(\sigma, \tau \leftarrow \text{Gal}(L/\mathbb{Q})\)

4. **for** \(\ell \in \{\sigma, \tau, \sigma \tau\}\) **do**
5. **Set** \(K_\ell\) so that \(\text{Gal}(L/K_\ell) = \langle \ell \rangle\)
6. \(I_\ell \leftarrow (I \cdot \sigma_\ell(I)) \cap K_\ell = N_\ell(I)\)
7. \(g_\ell, U_\ell \leftarrow \text{MQPIP}(K_\ell, I_\ell)\)

8. \(O_L^\times, X \leftarrow \text{UnitGroupFromSubgroup}(U_\ell)\)
9. \(h \leftarrow g_\sigma g_\tau \sigma (g_{\sigma \tau}^{-1})\)
10. \(g' \leftarrow \text{IdealSqrt}(h, O_L^\times, X)\)
11. \(g \leftarrow \text{ShortenGen}(g', O_L^\times)\)
12. **return** \(g, O_L^\times\)
Algorithm 1: MQPIP($L, \mathcal{I}$)

**Input:** Real multiquadratic field $L$ and a basis matrix for a principal ideal $\mathcal{I}$ of $\mathcal{O}_L$

**Result:** A short generator $g$ for $\mathcal{I}$

1. **if** $[L : \mathbb{Q}] = 2$ **then**
   
   2. **return** MQPIP($L, \mathcal{I}$)  

   \[ N^2 \cdot \exp((\ln |D|)^{1/2} + o(1)) \]

3. $\sigma, \tau \leftarrow \text{Gal}(L/\mathbb{Q})$

4. **for** $\ell \in \{\sigma, \tau, \sigma\tau\}$ **do**

5. Set $K_\ell$ so that $\text{Gal}(L/K_\ell) = \langle \ell \rangle$

6. $\mathcal{I}_\ell \leftarrow (\mathcal{I} \cdot \sigma_\ell(\mathcal{I})) \cap K_\ell = N_\ell(\mathcal{I})$

7. $g_\ell, U_\ell \leftarrow \text{MQPIP}(K_\ell, \mathcal{I}_\ell)$

8. $\mathcal{O}_L^\times, X \leftarrow \text{UnitGroupFromSubgroup}(U_\ell)$ \[ O(N^5B) \]

9. $h \leftarrow g_\sigma g_\tau \sigma(g_{\sigma\tau}^{-1})$ \[ O(NB) \]

10. $g' \leftarrow \text{IdealSqrt}(h, \mathcal{O}_L^\times, X)$ \[ O(N^4B) \]

11. $g \leftarrow \text{ShortenGen}(g', \mathcal{O}_L^\times)$ \[ O(N^5) \]

12. **return** $g, \mathcal{O}_L^\times$
Part IV: Results
### Speed Results (in seconds)

| $n$ | $2^n$ | units               | keygen | attack |
|-----|-------|---------------------|--------|--------|
| 3   | 8     | 0.05 old tower      | 0.03   | 0.006  | 0.24   |
|     |       | 0.03 old absolute   | 0.83   |        |        |
| 4   | 16    | 0.51 old tower      | 0.24   | 0.007  | 1.10   |
|     |       | 0.24 old absolute   | 2.23   |        |        |
| 5   | 32    | 7.24 new            | 5.98   | 0.025  | 4.24   |
|     |       | >700000.00 keygen   | 31.12  |        |        |
|     |       | >700000.00 attack   |        |        |        |
| 6   | 64    | >700000.00          |        | 0.050  | 18.78  |
|     |       | 176.98              | 0.171  |        |        |
| 7   | 128   | 1855.05             | 0.834  |        |        |
|     |       |                      |        |        |        |
| 8   | 256   |                      |        |        |        |
Coefficient Results

Vertical axis: Average absolute coefficients of $\log g$ on MQ basis.
Horizontal axis: $1.11/(2^{n/2} \log(u_D))$. 
Failure Results

Vertical axis: Failure probability of simple rounding (without enumeration). Horizontal axis: $d_1$, using $n$ consecutive primes for $(d_1, \ldots, d_n)$. 

![Graph showing failure results with vertical axis representing failure probability and horizontal axis representing $d_1$ values for consecutive primes.](image)
Figure: A multitude of quads.

Questions?