Abstract

We study the asymptotic behavior, when $\varepsilon \to 0$, of the minimizers $\{u_\varepsilon\}_{\varepsilon > 0}$ for the energy

$$E_\varepsilon(u) = \int_\Omega \left( |\nabla u|^2 + \frac{1}{\varepsilon^2} - 1 \right) |\nabla|u||^2,$$

over the class of maps $u \in H^1(\Omega, \mathbb{R}^2)$ satisfying the boundary condition $u = g$ on $\partial \Omega$, where $\Omega$ is a smooth, bounded and simply connected domain in $\mathbb{R}^2$ and $g : \partial \Omega \to S^1$ is a smooth boundary data of degree $D \geq 1$. The motivation comes from a simplified version of the Ericksen model for nematic liquid crystals with variable degree of orientation. We prove convergence (up to a subsequence) of $\{u_\varepsilon\}$ towards a singular $S^1$–valued harmonic map $u_*$, a result that resembles the one obtained in Bethuel et al. (Ginzburg–Landau Vortices, Birkhäuser, 1994) for an analogous problem for the Ginzburg–Landau energy. There are however two striking differences between our result and the one involving the Ginzburg–Landau energy. First, in our problem, the singular limit $u_*$ may have singularities of, degree strictly larger than one. Second, we find that the principle of “equipartition” holds for the energy of the minimizers, i.e., the contributions of the two terms in $E_\varepsilon(u_\varepsilon)$ are essentially equal.

Contents

1. Introduction .................................... 1010
2. Boundary Condition of Degree Zero ........................ 1018
   2.1. Convergence of the Minimizers ........................ 1019
   2.2. Uniqueness of the Minimizers for Small $\varepsilon$ .................. 1024
3. Boundary Condition of Degree $D \geq 1$: Preliminary Estimates ................ 1025
   3.1. Minimization Within the Radial Class ........................ 1025
   3.2. Asymptotic Behavior of the Energy ........................ 1026
Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply connected domain and $g : \partial \Omega \to S^1$ a smooth boundary condition. For each $\varepsilon > 0$ consider the energy
\[ E_\varepsilon(u) = \int_\Omega \left( |\nabla u|^2 + \frac{1}{\varepsilon^2} - 1 \right)|\nabla u|^2, \quad (1) \]
and let $u_\varepsilon$ denote a minimizer for $E_\varepsilon$ over
\[ H_{g}^1(\Omega) = H_{g}^1(\Omega; \mathbb{R}^2) := \{ u \in H^1(\Omega; \mathbb{R}^2) \text{ s.t. } u = g \text{ on } \partial \Omega \}. \]
We are interested in the limit of $u_\varepsilon$ when $\varepsilon$ goes to zero.

This problem can be viewed as a relaxation of the problem
\[ \min \{ \int_\Omega |\nabla u|^2 : u \in H_g^1(\Omega; S^1) \}. \quad (2) \]
In fact, when the degree of $g$—to be denoted hereafter by $D$—is zero, no relaxation is needed since the problem (2) has a solution. In this case there exists a (smooth) scalar function $\varphi_0$ such that $g = e^{i\varphi_0}$ and the (unique) minimizer in (2) is given by $u_0 = e^{i\tilde{\varphi}_0}$, where $\tilde{\varphi}_0$ is the harmonic extension of $\varphi_0$ to $\Omega$. When $D = 0$, we prove in Theorem 3 that $u_\varepsilon \to u_0$ in $C^m(\overline{\Omega})$, $\forall m$. This result is analogous to the one treating the zero degree case of the Ginzburg–Landau energy in [7].

The more interesting situation arises when $D = \deg g \neq 0$ because for such $g$ the set of competitors $H_g^1(\Omega; S^1)$ is empty (see e.g., [8, Introduction]) and the problem (2) has no solution. Even though the minimization problem (2) is by itself meaningless, one may still consider the limit of $u_\varepsilon$ when $\varepsilon$ goes to zero, as a “generalized minimizer”.

1. Introduction
A Variational Singular Perturbation Problem Motivated

This type of relaxation was carried out in the past for different energies. In their famous work, Bethuel, Brezis and Hélein [8] (see also [27]) studied the limit of the minimizers \{v_\varepsilon\} for the energy

$$F_\varepsilon(u) = \int_\Omega \left( |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right),$$

over \(H_0^1(\Omega)\). In the case \(\deg g = D \geq 1\) they showed for a subsequence that

$$v_\varepsilon \rightarrow u_* = e^{i\phi} \prod_{j=1}^D \frac{z - a_j}{|z - a_j|} \quad \text{in } C_{\text{loc}}^1(\Omega \setminus \{a_1, \ldots, a_D\}),$$

where \(\phi\) is a harmonic function determined by the constraint \(u_* = g\) on \(\partial \Omega\).

Moreover,

$$\lim_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) - 2\pi D |\ln \varepsilon| = \min_{b \in \Omega^\partial} W(b) + D\gamma,$$

where \(\gamma\) is a universal constant and \(W\) is the renormalized energy that was introduced in [8], see (17) and (19) below. In summary, the limit of a sequence of minimizers has \(D\) singularities of degree one, with their locations determined by minimization of \(W\) over all configurations of \(D\) distinct points in \(\Omega\). Interestingly, the same type of limit as in (4) is also obtained for a different relaxation, studied by Hardt and Lin [14]. In contrast with the case \(p = 2\), the set \(W_\varepsilon^1,p(\Omega; S^1) \neq \emptyset\) for \(p \in [1, 2)\). Denoting by \(w_p\) a minimizer for \(\int_\Omega |\nabla u|^p\) over \(W_\varepsilon^1,p(\Omega; S^1)\) for each \(p \in [1, 2)\), they showed for a subsequence \(p_n \nearrow 2\) that an analogous result to (4) holds, namely,

$$w_{p_n} \rightarrow u_* = e^{i\phi} \prod_{j=1}^D \frac{z - a_j}{|z - a_j|} \quad \text{in } C_{\text{loc}}^1(\Omega \setminus \{a_1, \ldots, a_D\}).$$

Moreover, an analogous formula to (5) holds in this case as well and the locations of the singularities \(a_1, \ldots, a_D\) are still determined by minimizing the same renormalized energy as above.

In view of these two examples, one may suspect that any “reasonable” relaxation would lead to the same limit. Somewhat surprisingly, we find that this isn’t the case for the limit of the minimizers \(u_\varepsilon\) of \(E_\varepsilon\) over \(H_0^1(\Omega)\). We will show that, for a subsequence, we have

$$u_{\varepsilon_n} \rightarrow u_* = e^{i\phi} \prod_{j=1}^N \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} \quad \text{in } C_{\text{loc}}^m(\Omega \setminus \{a_1, \ldots, a_N\}),$$

with degrees \(d_j \geq 1, \forall j\), i.e., the limit is the canonical harmonic map associated with \(g\), the singularities and their degrees (see [8]). However, in contrast to (4) and (6), we might have \(d_j \geq 2\) for some values of \(j\), so that a strict inequality \(N < D\) may occur (see Corollary 2 and Proposition 5 below). Moreover, the location of the
singularities and their degrees are determined by minimizing a different function than $W$.

An important property of the energy (1) is its conformal invariance, that is, we have $E_\varepsilon(u) = E_\varepsilon(u \circ F)$ for every conformal map $F$. We shall often use this property in the sequel. For example, it allows us to assume that the simply connected domain $\Omega$ is the unit disc (thanks to the Riemann mapping theorem). Our first result for the case $D \geq 1$ provides a convergence result and a partial description of the limit.

**Theorem 1.** Let $\Omega$ be a smooth, bounded, simply connected domain in $\mathbb{R}^2$. Let $g : \partial \Omega \to S^1$ be a smooth boundary condition of degree $D \geq 1$. Then,

$$\frac{2\pi D}{\varepsilon} \leq E_\varepsilon(u_\varepsilon) \leq \frac{2\pi D}{\varepsilon} + C. \tag{8}$$

Moreover, up to a subsequence we have

$$u_{\varepsilon_n} \to u_* \text{ in } C^m_{\text{loc}}(\overline{\Omega \setminus \{a_1, \ldots, a_N\}}), \quad \forall m,$$

where $u_*$ is a smooth $S^1$–valued harmonic map in $\overline{\Omega \setminus \{a_1, \ldots, a_N\}}$. The singularities $a_1, \ldots, a_N$ are distinct points in $\Omega$, the degree of $u_*$ around each $a_j$ is an integer $d_j > 0$, and the compatibility condition $\sum_{j=1}^{N} d_j = D$ holds. Moreover, $u_*$ is the canonical harmonic map associated with $g$, the points $a_1, \ldots, a_N$ and the degrees $d_1, \ldots, d_N$.

Our second result establishes a precise asymptotic expansion of the energy $E_\varepsilon(u_\varepsilon)$ by computing

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) - \frac{2\pi D}{\varepsilon}.$$  

This allows us to obtain a criterion for the choice of the points $a_1, \ldots, a_N$ and their associated degrees $d_1, \ldots, d_N$. In order to state the next theorem, we will need the following definitions:

For each integer $D \geq 1$ we set

$$\mathcal{H}_D(\partial \Omega) = \{g \in C^1(\partial \Omega; S^1) : \deg g = D \text{ and } g = G|_{\partial \Omega} \text{ for some holomorphic } G \in C^1(\overline{\Omega}; \mathbb{C}) \text{ s.t. } G(\partial \Omega) = \partial B_1\}.$$  

An explicit description of $\mathcal{H}_D(\partial \Omega)$ is available using the concept of Blaschke products. Indeed, when $\Omega = B_1$, to any configuration of $D \geq 1$ points $a \in B_1^D$ we associate a Blaschke product

$$B_a(z) := \prod_{j=1}^{D} \frac{z - a_j}{1 - \overline{a}_j z}.$$

Then we have

$$\mathcal{H}_D(\partial B_1) = \{e^{i\alpha} B_a(z)|_{\partial B_1} : \alpha \in \mathbb{R}, \ a \in B_1^D\}. \tag{11}$$
For an arbitrary smooth and simply connected $\Omega$ we may fix a Riemann mapping $F : \Omega \to B_1$ (with smooth extension to the boundary) and then clearly,

$$\mathcal{H}_D(\partial \Omega) = \{ g \circ F : g \in \mathcal{H}_D(\partial B_1) \},$$

so any function in $\mathcal{H}_D(\partial \Omega)$ has the form $e^{i\alpha} \prod_{j=1}^D \frac{F(z)-a_j}{\overline{F(z)}-\overline{a}_j}$, for some $\alpha \in \mathbb{R}$ and $a \in B_1^D$.

Let $g_1, g_2 : \partial \Omega \to S^1$ be two smooth maps, or more generally, maps in $H^{1/2}(\partial \Omega; S^1)$ with the same degree. We define a distance between the maps as follows:

$$d_{H^{1/2}}(g_1, g_2) = \inf \{ \| \nabla w \|_{L^2(\Omega)} : w \in H^1(\Omega; S^1), w = g_1 \overline{g_2} \text{ on } \partial \Omega \}.$$  \hfill (13)

Note that the assumption $\deg g_1 = \deg g_2$ implies that $\deg g_1 \overline{g_2} = 0$, whence we may write on $\partial \Omega$,

$$g_1 \overline{g_2} = e^{i\psi}$$

for some scalar function $\psi$ on $\partial \Omega$ (with $\psi$ smooth, or more generally in $H^{1/2}(\partial \Omega)$). It is then clear that

$$d_{H^{1/2}}(g_1, g_2) = \| \nabla \tilde{\psi} \|_{L^2(\Omega)},$$

where $\tilde{\psi}$ denotes the harmonic extension of $\psi$. Naturally we denote for $g \in C^1(\partial \Omega; S^1)$ of degree $D$,

$$d_{H^{1/2}}(g, \mathcal{H}_D(\partial \Omega)) = \inf_{f \in \mathcal{H}_D(\partial \Omega)} d_{H^{1/2}}(g, f).$$ \hfill (15)

It is easy to see that the infimum in (15) is actually attained. Note that when $\Omega = B_1$ we have

$$d_{H^{1/2}}^2(g, \mathcal{H}_D(\partial B_1)) = \min_{b \in B_1^D} \left\{ \int_{B_1} |\nabla \tilde{\psi}|^2 : \varphi \in H^{1/2}(\partial B_1) \text{ s.t. } e^{i\varphi} = g \overline{B}_b \right\},$$

where as usual $\tilde{\varphi}$ denotes the harmonic extension of $\varphi$. A similar expression can be written for a general $\Omega$, using the Riemann mapping $F : \Omega \to B_1$.

The “excess energy” $d_{H^{1/2}}^2(g, \mathcal{H}_D(\partial \Omega))$ is related to the notion of renormalized energy $W$ from [8], but there are important differences between the two, see Remark 2 below. Next, we present an explicit expression for $d_{H^{1/2}}^2(g, \mathcal{H}_D(\partial \Omega))$ using quantities that also appear in $W$. We begin by recalling one of the equivalent definitions of $W$ from [8]. It is convenient to denote by $(\Omega_N)^*$ the subset of $\Omega_N$ consisting only of configurations of distinct points. Given a boundary condition $g : \partial \Omega \to S^1$ of degree $D > 0$, the points $a \in (\Omega_N)^*$, and the degrees $d \in \mathbb{Z}_N$ satisfying $\sum_{j=1}^N d_j = D = \deg g$, we first consider the associated canonical harmonic map

$$u_0 = e^{i\tilde{\varphi}} \prod_{j=1}^N \left( \frac{z-a_j}{|z-a_j|} \right)^{d_j}.$$
where $\tilde{\varphi}$ is the harmonic extension of $\varphi$, which in turn is determined (up to an additive constant in $2\pi \mathbb{Z}$) by the requirement that $u_0 = g$ on $\partial \Omega$. Thm I.8 in [8] asserts that

$$\int_{\Omega \setminus \bigcup_{j=1}^{N} B_{\lambda}(a_j)} |\nabla u_0|^2 = 2\pi \left( \sum_{j=1}^{N} d_j^2 \right) \ln(1/\lambda) + W + O(\lambda^2), \text{ as } \lambda \to 0^+. \quad (17)$$

An explicit expression for $W = W(a, d, g)$ is given in [8, Thm I.7] (note that there is a factor of 2 difference between our definition and the one in [8]). This expression involves the solution $\tilde{\Phi}_0$ of

$$\begin{cases}
\Delta \tilde{\Phi}_0 = 2\pi \sum_{j=1}^{N} d_j \delta_{a_j} \text{ in } \Omega, \\
\frac{\partial \tilde{\Phi}_0}{\partial \nu} = g \times g_\tau \text{ on } \partial \Omega,
\end{cases} \quad (18)$$

with the normalization condition $\int_{\partial \Omega} \tilde{\Phi}_0 = 0$. Setting $R_0(x) = \tilde{\Phi}_0(x) - \sum_{j=1}^{N} d_j \ln |x - a_j|$, we have according to [8],

$$W(a, d, g) = \int_{\partial \Omega} \tilde{\Phi}_0(g \times g_\tau) \, d\tau - 2\pi \sum_{j=1}^{N} d_j R_0(a_j) - 2\pi \sum_{i \neq j} d_i d_j \ln |a_i - a_j|. \quad (19)$$

The relation between $d_{H^{1/2}}^2(g, \mathcal{H}_D(\partial \Omega))$ and $W$ is clarified in the next proposition. To state it, we define, as in [17],

$$\tilde{W}(a, d) = \inf \{ W(a, d, f) : f \in C^1(\partial \Omega; S^1), \deg f = D = \sum_{j=1}^{N} d_j \}. \quad (20)$$

**Proposition 1.** We have

$$d_{H^{1/2}}^2(g, \mathcal{H}_D(\partial \Omega)) = \inf \{ W(a, d, g) - \tilde{W}(a, d) : a \in (\Omega^*)^N, d \in \mathbb{Z}_+^N, \sum_{j=1}^{N} d_j = D, N \geq 1 \}. \quad (21)$$

Moreover, when $\Omega = B_1$,

$$d_{H^{1/2}}^2(g, \mathcal{H}_D(B_1)) = \min_{\sum_{j=1}^{N} d_j = D, a \in (B_1^*)^N} \int_{\partial B_1} \tilde{\Phi}_0(g \times g_\tau) \, d\tau - 2\pi \sum_{j=1}^{N} d_j R_0(a_j) - 2\pi \sum_{i,j=1}^{N} d_i d_j \ln |1 - a_i a_j|. \quad (22)$$
Comparing (22) to (19) we notice the absence from (22) of the last term in (19),

\[-2\pi \sum_{i \neq j} d_i d_j \ln |a_i - a_j|,

responsible for repulsion between vortices. This might explain the fact that vortices of degree \(d_j \geq 2\) are allowed for minimizers of \(E_\varepsilon\). In the context of Ginzburg–Landau-type problems, we are not aware of another situation where energy minimizers are characterized by point singularities that have unbounded energy in the limit \(\varepsilon \to 0\) and have degrees different from \(\pm 1\).

**Remark 1.** There is an alternative simple expression to the one in (22) in which the minimization is over all the configurations of \(D\) points in \(B_1\) (not necessarily distinct):

\[
d_H^2(g, H_D(B_1)) = \min_{a \in B_1^D} \int_{\partial B_1} \tilde{\Phi}_0(g \times g_\tau) d\tau - 2\pi \sum_{j=1}^D R_0(a_j)
- 2\pi \sum_{i,j=1}^D \ln |1 - a_i \bar{a}_j|,
\]

where \(\tilde{\Phi}_0\) is like in (18), but with \(N = D\) and \(d_j = 1\) for all \(j\) (and accordingly \(R_0(x) = \tilde{\Phi}_0(x) - \sum_{j=1}^D \ln |x - a_j|\)). The verification of (23) from (22) is straightforward.

**Remark 2.** We should emphasize that, although there are some common expressions in the explicit formulas for the renormalized energy \(W(a, d, g)\) and the “excess energy” \(d_{H^{1/2}}^2(g, H_D(\partial \Omega))\), there is a basic difference between the two. The renormalized energy has an intrinsic meaning. To cite from the Introduction in [8]: it is what remains in the energy after the singular “core energy” \(2\pi d |\log \lambda|\) has been removed in the problem of shrinking holes of radii \(\lambda\), which is closely related to (17). This feature of \(W\), namely, that it represents the “regular part” of the energy of singular \(S^1\)-valued harmonic maps, is the reason behind its appearance in many different variational problems. For example, in addition to the case involving the Ginzburg–Landau energy [8], one can find analogous \(W\) in the problems considered in [4,5,12,14]. The excess energy \(d_{H^{1/2}}^2(g, H_D(\partial \Omega))\) is quite different in that it is specific to the particular choice of the energy \(E_\varepsilon\) we consider. It does not represent a contribution from the phase alone (as does \(W\)) but rather a contribution from **both** the phase and the modulus of the maps.

We are now ready to state our second main theorem that provides a more precise information about the asymptotic behavior of the energy and the location of the singularities of the limit \(u_\varepsilon\). Note that we denote \(\rho_\varepsilon = |u_\varepsilon|\) throughout the manuscript.

**Theorem 2.** Let \(\Omega, g\) and \(u_\varepsilon\) together with the singular points \(a_1, \ldots, a_N\) and the degrees \(d_1, \ldots, d_N\) be as in Theorem 1. Then, up to a subsequence we have that,
\( \lim_{\varepsilon \to 0} \frac{\ln \rho_\varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\rho_\varepsilon - 1}{\varepsilon} = \Phi_0 \) in \( C^m_{loc}(\Omega \setminus \{a_1, \ldots, a_N\}) \), \( \forall m \geq 1 \), where \( \Phi_0 \) is the solution of

\[
\begin{cases}
\Delta \Phi_0 = 2\pi \sum_{j=1}^N d_j \delta_{a_j} \text{ in } \Omega, \\
\Phi_0 = 0 \text{ on } \partial \Omega.
\end{cases}
\]  

(iii) The configurations of points \( a = (a_1, \ldots, a_N) \) and degrees \( d = (d_1, \ldots, d_N) \) realize the minimum in (21).

The main feature of Theorem 2 is that it provides a simple criterion for the computation of the location of the singularities of the limit \( u_* \) (whence of \( u_* \) itself) for a given boundary condition \( g \). For simplicity we describe it for the case \( \Omega = B_1 \). What one has to do is to find the nearest point projection of \( g \) on the set \( \mathcal{H}_D(\partial B_1) \) (i.e., the set of all the finite Blaschke products with \( D \) factors) with respect to the metric \( d_{H^{1/2}} \). The singularities of \( u_* \) are precisely the zeros of the (extension to \( B_1 \) of the) Blaschke product which is the nearest point projection. To be exact, since we do not know whether the nearest point projection of \( g \) is unique, what we can only say is that \( u_* \) (any possible limit of a subsequence of \( \{u_\varepsilon\} \)) is one of the nearest point projections of \( g \). In the special case where \( g \in \mathcal{H}_D(\partial B_1) \) we can immediately say that the singularities of \( u_* \) coincide the zeros of \( g \). Actually, much more can be said in this case: for each \( \varepsilon > 0 \) we have an explicit formula for the (unique) minimizer \( u_\varepsilon \), whose zeros are the zeros of \( g \), see Proposition 5 in Subsection 3.3 for details.

Our original motivation to study the energy \( E_\varepsilon \) came from Ericksen’s model for nematic liquid crystals with variable degree of orientation [10]. In this model the nematic, confined to a domain \( \Omega \subset \mathbb{R}^3 \), is described by a pair \((s, n)\) with \( s : \Omega \rightarrow (-\frac{1}{2}, 1) \) and \( n : \Omega \rightarrow S^2 \). In its simplest form the energy of the nematic is given by

\[
\mathcal{F}_E(s, n) = \int_\Omega \left\{ k |\nabla s|^2 + s^2 |\nabla n|^2 + f(s) \right\},
\]  

for some smooth potential function \( f : (-\frac{1}{2}, 1) \rightarrow \mathbb{R}_+ \) that vanishes at a single point \( s_0 \in (-\frac{1}{2}, 1) \) and diverges at the endpoints of its interval of definition. A further simplification of the model can be achieved once we realize that the field \( s \) can be forced to deviate not too much from \( s_0 \) in \( \Omega \) by setting \( s|_{\partial \Omega} = s_0 \) and taking advantage of the fact that variations of \( s \) are penalized by the corresponding gradient term in (26). Here, larger values of the parameter \( k \) would result in smaller values of \( s - s_0 \) in \( \Omega \). Hence, we drop the potential \( f \) in (26), similarly to what Ambrosio and Virga did in [3] for different reasons (see also [21,28]). A possible physical justification for dropping \( f \) for polymeric liquid crystals was given in [28].
More recently, the same simplification was used in a numerical work [24] when simulating nematic configurations arising within the Ericksen model.

To see the connection between the energy (1) to the one in (26) we follow Lin [18] by representing the pair \((s, n)\) (in the case \(s \geq 0\)) by a single vector–valued function \(u = sn\), where \(u : \Omega \to \mathbb{R}^3\), so that \(s = |u|\) while \(n = u/|u|\) on the set \(\{s > 0\}\). This allows us to rewrite the energy in (26), in the case \(f = 0\), as:

\[
G_k(u) = \int_\Omega \left( (k - 1)|\nabla u|^2 + |\nabla u|^2 \right).
\]

Replacing the parameter \(k\) with \(\varepsilon = (1/k)^{1/2}\) we get that \(G_k(u) = E_\varepsilon(u)\) with \(E_\varepsilon\) given by (1). Note however that in (1) we consider \(\mathbb{R}^2\)–valued maps, while here the physical model leads us to consider \(\mathbb{R}^3\)–valued maps; we will return to this point below.

Associating with \(u\) the map \(w : \Omega \to C^3_k\) given by \(w(x) = (u(x), \sqrt{k - 1}|u|(x))\) (assuming \(k > 1\)) we notice that \(w\) takes its values in a circular cone \(C^3_k\) given by

\[
C^3_k = \{(y, t) \in \mathbb{R}^2 \times \mathbb{R} : t = \sqrt{k - 1}|y|\}.
\]

Moreover, \((k - 1)|\nabla u|^2 + |\nabla u|^2 = |\nabla w|^2\), whence

\[
G_k(u) = \int_\Omega |\nabla w|^2.
\]

Hence the energy (26) without the potential term has another interpretation, leading to the study of minimizing harmonic maps taking values in the circular cone \(C^3_k\). Properties of these maps, in particular their regularity, were studied extensively by Lin [18,19] and Hardt and Lin [13]. Delicate regularity results for the analogous problem when the cone \(C^3_k\) is replaced by a cone over the real projective plane were obtained recently by Alper, Hardt and Lin [2] and Alper [1].

Replacing the parameter \(k\) with \(\varepsilon = (1/k)^{1/2}\) we get that \(G_k(u) = E_\varepsilon(u)\) with \(E_\varepsilon\) given by (1). There are however two special features in the problem that are not present in the standard physical model. These are the assumptions that both the domain and the target are two-dimensional, i.e., the extension \(w\) of the original \(\mathbb{R}^2\)–valued map \(u\) takes values in

\[
C^2_k = \{(y, t) \in \mathbb{R}^2 \times \mathbb{R} : t = \sqrt{k - 1}|y|\}.
\]

These assumptions need some justification. In the Appendix, we present a possible physical motivation that led us to consider the present model, by showing that it can be derived as a thin film limit of a problem set in three dimensions.

The fact that our model constrains the minimizers to take values in \(\mathbb{R}^2\) and, hence, prevents them from “escaping to the third dimension” (see [6]) is crucial to our results. This is the reason why the energy of the minimizers blows up as \(\varepsilon \to 0\) (equiv. \(k \to \infty\)) whenever the boundary condition has a nonzero degree, leading to emergence of singularities in the limit \(\varepsilon \to 0\). Indeed, the energy of minimizers, for the same boundary condition but for \(C^3_k\)–valued maps, remains bounded uniformly in \(k\). The extra dimension of the target also influences the uniqueness issue. In fact, for minimizing harmonic maps taking values in the upper hemisphere \(S^2_+\),
uniqueness holds “almost always” (and in particular whenever the domain \( \Omega \) has a connected boundary). This is a special case of a result by Sandier and Shafrir [25] that takes advantage of a certain convexity property of the energy that holds thanks to the extra dimension. This result was strengthened and generalized—using a different elegant technique—in a recent paper by Ignat, Nguyen, Slastikov and Zarnescu [15]. We expect a similar phenomenon to hold in our setting, that is, for a boundary condition taking values in \( S^1 \) and each \( k > 1 \) there should be a unique \((C^3_k)^+\)-valued minimizing harmonic map. Here \((C^3_k)^+\) is defined by

\[
(C^3_k)^+ = \{(y,t) \in \mathbb{R}^3 \times \mathbb{R} : t = \sqrt{k-1}|y|, \ y_3 \geq 0\},
\]

and the \((C^3_k)^+\)-valued minimizer is one of two minimizers for the problem among \(C^3_k\)-valued maps—the other one is \((C^3_k)^-\)-valued, and is obtained from the first one via reflection w.r.t. the plane \( \{y_3 = 0\} \). For the problem we are concerned with in this manuscript, that is for \(C^2_k\)-valued maps, the question of uniqueness is widely open. The only two modest results we have in that respect are Proposition 5 that establishes uniqueness, for every \( \varepsilon > 0 \), in the special case of a boundary condition which is a Blaschke product and Theorem 4 that establishes uniqueness when the boundary condition has degree zero and \( \varepsilon \) is sufficiently small.

In a related work, Ignat, Nguyen, Slastikov and Zarnescu [16] considered a two-dimensional problem on a disk, involving the Landau–de Gennes model for nematic liquid crystals. They proved that, for sufficiently large radius and a symmetric boundary condition carrying a topological defect of degree \( D/2 \) (for \( D \) even), there exist exactly two minimizers—both retaining the symmetry of the boundary data—as well as non–minimizing critical points with \( D \)-fold symmetry. An interesting open problem for us is whether for \( \varepsilon \ll 1 \) and the boundary data \( g(e^{i\theta}) = e^{iD\theta} \) with \( D \geq 2 \) there exist local minimizers of \( E_\varepsilon \) having \( D \) vortices, each of degree one, arranged in a symmetric pattern. If true, the techniques required to prove this fact in our case, will likely have to be significantly different from [16] because harmonic maps for the limiting problem with an even \( D \) in [16] have bounded energy due to escape into the third dimension.

The paper is organized as follows. In Section 2 we examine the case deg \( g = 0 \). The rest of the paper is devoted to the case deg \( g \geq 1 \). Section 3 contains some preliminary results needed for the proof of the main theorems. Section 4 is devoted to the Proof of Theorem 1 while Section 5 is devoted to the Proof of Theorem 2. The Proof of Proposition 1 is given in Section 6. Finally, in the Appendix, we outline the dimension reduction argument that motivates our model.

### 2. Boundary Condition of Degree Zero

Throughout this section we suppose that \( g : \partial \Omega \to S^1 \) is a smooth boundary condition of degree zero and let \( g = e^{i\phi} \). Denote by \( \bar{\phi}_0 \) the harmonic extension of \( \phi_0 \) and let \( u_0 = e^{i\bar{\phi}_0} \). We mention in passing that the inequality \( |u_\varepsilon(x)| \leq 1 \) always holds in \( \Omega \) (regardless of the value of deg \( g \)). Indeed, otherwise we could reduce the energy by replacing \( u_\varepsilon(x) \) by \( u_\varepsilon(x)/|u_\varepsilon(x)| \) on the set \( \{x \in \Omega : |u_\varepsilon(x)| > 1\} \).
An alternative argument that yields the same inequality uses the sub-harmonicity of the function $|u_\varepsilon|^2$, see (88) below.

### 2.1. Convergence of the Minimizers

**Proposition 2.** We have $u_\varepsilon \to u_0$ strongly in $H^1(\Omega)$ as $\varepsilon \to 0$.

**Proof.** Since
\[
E_\varepsilon(u_\varepsilon) = \int_{\Omega} \left( |\nabla u_\varepsilon|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla|u_\varepsilon|\right)^2 \leq E_\varepsilon(u_0) = \int_{\Omega} |\nabla u_0|^2,
\]
there is a subsequence satisfying $u_{\varepsilon_n} \rightharpoonup u$ weakly in $H^1(\Omega)$. Therefore,
\[
\int_{\Omega} |\nabla u|^2 \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla u_{\varepsilon_n}|^2 \leq \int_{\Omega} |\nabla u_0|^2.
\]
Denoting $\rho_\varepsilon = |u_\varepsilon|$ and $\bar{\rho}_\varepsilon = \frac{1}{|\Omega|} \int_{\Omega} \rho_\varepsilon$ we have by Poincaré inequality:
\[
\int_{\Omega} |\rho_\varepsilon - \bar{\rho}_\varepsilon|^2 \leq C \int_{\Omega} |\nabla \rho_\varepsilon|^2 \to 0.
\]
Passing to a further subsequence we may assume that $\bar{\rho}_{\varepsilon_n} \to R$ for some constant $R \in [0, 1]$ and then by (31), $\rho_{\varepsilon_n} \to R$ strongly in $H^1$. It follows that $1 = \text{Tr}(\rho_{\varepsilon_n}) \to \text{Tr}(R) = R$ in $L^2(\partial \Omega)$, whence $R = 1$. It follows that $u \in H^1_g(\Omega; S^1)$ and the inequality $\int_{\Omega} |\nabla u|^2 \leq \int_{\Omega} |\nabla u_0|^2$ implies that $u = u_0$. From (30) we conclude that $u_{\varepsilon_n} \to u_0$ strongly in $H^1$, and the full convergence $u_\varepsilon \rightharpoonup u_0$ follows from the uniqueness of $u_0$. Going back to (29) we deduce that also
\[
\lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon^2} - 1 \right) \int_{\Omega} |\nabla \rho_\varepsilon||^2 = 0.
\]
\[\square\]

**Proposition 3.** Under the same assumptions as in Proposition 2 we have: $\rho_\varepsilon \to 1$ uniformly on $\Omega$. More precisely, we have
\[
1 - \rho_\varepsilon(x) \leq C\varepsilon, \quad \forall x \in \Omega.
\]

**Remark 3.** In Theorem 3 below we will improve the estimate in (33) to $1 - \rho_\varepsilon(x) \leq C\varepsilon^2$.

**Proof of Proposition 3.** Throughout the proof we will denote by $C$ different generic constants whose value is independent of $\varepsilon$. Thanks to the conformal invariance, we may assume that $\Omega = B_1$. By Proposition 2 and, in particular, (32) we have
\[
(\nabla u_\varepsilon, \left( \frac{1}{\varepsilon^2} - 1 \right)^{1/2} \nabla \rho_\varepsilon) \overset{L^2}{\to} (\nabla u_0, 0).
\]
Therefore, for any $\delta_0 \in (0, 1)$ we can find $r_0 > 0$ such that
\[
\int_{B_{r_0}(x_0) \cap \Omega} |\nabla u_\varepsilon|^2 + ((1/\varepsilon^2) - 1)|\nabla \rho_\varepsilon|^2 \leq \delta_0, \quad \forall x_0 \in \Omega. \tag{35}
\]
For reasons to become clear later we fix a value of $\delta_0 > 0$ satisfying
\[
\delta_0 < \frac{1}{4\pi}. \tag{36}
\]
In the sequel, we shall suppress for simplicity the subscript $\varepsilon$ and write for short, $u = u_\varepsilon$, $\rho = \rho_\varepsilon$, etc.. Recall that we also have
\[
\int_{\Omega} |\nabla u|^2 + ((1/\varepsilon^2) - 1)|\nabla \rho|^2 \leq C_0 := \int_{\Omega} |\nabla u_0|^2. \tag{37}
\]
We first consider the case $x_0 = 0$. By (35) we may choose $r'_0 \in (r_0/2, r_0)$ such that
\[
\int_{\partial B_{r'_0}} |\nabla u|^2 + ((1/\varepsilon^2) - 1)|\nabla \rho|^2 \leq \frac{2}{r'_0} \int_{B_{r'_0} \setminus B_{r_0/2}} |\nabla u|^2 + ((1/\varepsilon^2) - 1)|\nabla \rho|^2
\leq \frac{2\delta_0}{r'_0}. \tag{38}
\]
In particular, we deduce from (38) that
\[
|u(x_1) - u(x_2)| \leq \int_{\partial B_{r'_0}} |\nabla u| \leq (2\pi r'_0)^{1/2} \left(\int_{\partial B_{r'_0}} |\nabla u|^2\right)^{1/2}
\leq \left(4\pi \delta_0\right)^{1/2}, \quad \forall x_1, x_2 \in \partial B_{r'_0}. \tag{39}
\]
Similarly,
\[
|\rho(x_1) - \rho(x_2)| \leq \left(4\pi \delta_0\right)^{1/2} \varepsilon, \quad \forall x_1, x_2 \in \partial B_{r'_0}. \tag{40}
\]
We next define the radial function
\[
\bar{\rho}(r) = \frac{1}{2\pi r} \int_{\partial B_r} \rho \, d\tau, \quad r \in (0, 1]. \tag{41}
\]
By (37) we have
\[
C_0 \varepsilon^2 \geq \int_{B_1 \setminus B_{r'_0}} |\nabla \rho|^2 \geq \int_{B_1 \setminus B_{r'_0}} |\nabla \bar{\rho}|^2 = 2\pi \int_{r'_0}^1 \left|\frac{d\bar{\rho}}{dr}\right|^2 r \, dr \geq \frac{2\pi(1 - \bar{\rho}(r'_0))^2}{\ln(1/r'_0)}, \tag{42}
\]
whence
\[
1 - \bar{\rho}(r'_0) \leq \left\{\left(\frac{C_0 \varepsilon^2}{2\pi}\right) \ln(1/r'_0)\right\}^{1/2}. \tag{43}
\]
By (40) and (43) we get that
\[ \rho = 1 + O(\varepsilon) \text{ on } \partial B_{r_0}', \]  
(44)
while (39) and (36) imply that
\[ |u(x_1) - u(x_2)| < 1 \text{ on } \partial B_{r_0}'. \]  
(45)

In particular, it follows from (44)–(45) that the image of \( u/|u| \) at \( \partial B_{r_0}' \) is contained strictly in \( S^1 \) (for sufficiently small \( \varepsilon \)), whence \( \deg(u/|u|, \partial B_{r_0}') = 0 \). Therefore we may write \( u = \rho e^{i\phi} \) on \( \partial B_{r_0}' \).

Denote by \( \tilde{\rho} \) and \( \tilde{\phi} \) the harmonic extensions of \( \rho \) and \( \phi \), respectively, from \( \partial B_{r_0} \) to \( B_{r_0}' \). Recall that in dimension two any harmonic function \( h \) satisfies:
\[ \int_{B_{r_0}} |\nabla h|^2 \leq R \int_{\partial B_{r_0}} \left| \frac{\partial h}{\partial \tau} \right|^2. \]  
(46)

Using (46) and the fact that \( \rho^2 \geq 1/2 \) on \( \partial B_{r_0}' \) (for small \( \varepsilon \)) thanks to (44), we obtain:
\[ \int_{B_{r_0/2}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \leq \int_{B_{r_0}' \setminus B_{r_0}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \]
\[ \leq 2r' \int_{\partial B_{r_0}'} \rho^2 \left| \frac{\partial \phi}{\partial \tau} \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{\partial \rho}{\partial \tau} \right|^2 \leq 2r' \int_{\partial B_{r_0}'} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \]
\[ \leq 4 \int_{B_{r_0/2}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2, \]  
(47)

where in the last inequality we used (38). An immediate consequence of (47) is
\[ \int_{B_{r_0/2}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \leq \frac{4}{5} \int_{B_{r_0}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \leq \frac{4\delta_0}{5}. \]  
(48)

Next, we set \( r_1 = r_0/2 \) and choose, as in (38), \( r_1' \in (r_1/2, r_1) \) such that
\[ \int_{\partial B_{r_1}'} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \leq \frac{2}{r_1'} \int_{B_{r_1} \setminus B_{r_1/2}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \]
\[ \leq \frac{4}{5} \cdot \frac{2\delta_0}{r_1}. \]  
(49)

Similarly to (39)–(40) we get
\[ |u(x_1) - u(x_2)| \leq \left( 4\pi \delta_0 \cdot (4/5) \right)^{1/2}, \]  
\[ \forall x_1, x_2 \in \partial B_{r_1}'. \]  
(50)

\[ |\rho(x_1) - \rho(x_2)| \leq \left( 4\pi \delta_0 \cdot (4/5) \right)^{1/2} \varepsilon, \]  
\[ \forall x_1, x_2 \in \partial B_{r_1}'. \]  
(50)
By a similar argument to the one used in (42) we get
\[
\delta_0 \varepsilon^2 \geq \int_{B_{r_0}' \setminus B_1'} |\nabla \rho|^2 \geq \int_{B_{r_0}' \setminus B_1'} |\nabla \bar{\rho}|^2 \geq \frac{2\pi}{\ln(r_0'/r_1')} |\bar{\rho}(r_0') - \bar{\rho}(r_1')|^2
\]
\[
\geq \frac{2\pi}{\ln 4} |\bar{\rho}(r_0') - \bar{\rho}(r_1')|^2,
\]
whence
\[
|\bar{\rho}(r_0') - \bar{\rho}(r_1')| \leq \left( \frac{\delta_0 \ln 4}{2\pi} \right)^{1/2} \varepsilon.
\]  
(51)

Using the harmonic extensions of \( \rho \) and \( \varphi \) from \( \partial B_{r_1}' \) to \( B_{r_1}' \), as in (47), we obtain, analogously to (48):
\[
\int_{B_{r_1/2}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \leq \frac{4}{5} \int_{B_{r_1}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \leq \left( \frac{4}{5} \right)^2 \delta_0.
\]  
(53)

We continue by defining recursively \( r_j = r_{j-1}/2 = r_0/2^j \) and then choose \( r_j' \in (r_{j+1}, r_j) \) satisfying
\[
\int_{\partial B_{r_j'}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2 \leq \frac{2}{r_j'} \int_{B_{r_j'} \setminus B_{r_j/2}} |\nabla u|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla \rho|^2
\]
\[
\leq \left( \frac{2}{r_j'} \right) \left( \frac{4}{5} \right)^j \delta_0.
\]  
(54)

Analogously to (50) we get
\[
|u(x_1) - u(x_2)| \leq \left( 4\pi \delta_0 \right)^{1/2} \cdot \left( \frac{4}{5} \right)^{j/2} \varepsilon,
\]
\[
|\rho(x_1) - \rho(x_2)| \leq \left( 4\pi \delta_0 \right)^{1/2} \cdot \left( \frac{4}{5} \right)^{j/2} \varepsilon, \quad \forall x_1, x_2 \in \partial B_{r_j}'.
\]  
(55)

The argument used to obtain (52) yields
\[
|\bar{\rho}(r_{j-1}') - \bar{\rho}(r_j')| \leq \left( \frac{\delta_0 \ln 4}{2\pi} \right)^{1/2} \cdot \left( \frac{4}{5} \right)^{(j-1)/2} \varepsilon.
\]  
(56)

Combining (43) with (56) gives
\[
1 - \bar{\rho}(r_j') \leq 1 - \bar{\rho}(r_0') + \sum_{i=1}^{j} |\bar{\rho}(r_{i-1}') - \bar{\rho}(r_i')|
\]
\[
\leq \left\{ \left( \frac{C_0}{2\pi} \right) \ln(2/r_0) \right\}^{1/2} \varepsilon + \left\{ \sum_{i=1}^{j} \left( \frac{4}{5} \right)^{(i-1)/2} \right\} \left( \frac{\delta_0 \ln 4}{2\pi} \right)^{1/2} \varepsilon \leq C \varepsilon.
\]  
(57)

Letting \( j \) go to infinity in (57) yields \( 1 - \rho(0) \leq C \varepsilon \), which is (33) for \( x = 0 \).
Finally we consider the case \( x \in B_1 \setminus \{0\} \). First, denote by \( d_h \) the hyperbolic metric in \( B_1 \) with the convention that \( d_h(0, x) = \tanh^{-1}|x| \) (it is half of the hyperbolic distance commonly used in Geometry). In particular, Let \( D_r(x) \) denote hyperbolic disk of radius \( r \), centered at \( x \), that is
\[
D_r(x) = \{ y \in B_1 : d_h(x, y) < r \}.
\]

For a given \( x \neq 0 \) and \( r_0 \) as in (35) we let \( \tilde{r}_0 = \tanh^{-1} r_0 \), so that \( D_{\tilde{r}_0}(x) = M_x(D_{r_0}(0)) = M_x(B_{r_0}) \), where \( M_x \) denotes the Möbius transformation sending 0 to \( x \). It is easy to see that \( D_{\tilde{r}_0}(x) = B_s(y) \), for some \( y \in B_1 \) and \( s < r_0 \). By (35) and the conformal invariance of the energy we obtain that \( v := u \circ M_x \) satisfies
\[
\int_{B_{\tilde{r}_0}(x)} |\nabla v|^2 + \left( \frac{1}{\epsilon^2} - 1 \right)|\nabla v|^2 = \int_{B_s(y)} |\nabla u|^2 + \left( \frac{1}{\epsilon^2} - 1 \right)|\nabla \rho|^2 \leq \delta_0.
\]

By the first part of the proof, \( 1 - |u(x)| = 1 - |v(0)| \leq C \epsilon \) and (33) follows. \( \square \)

In the next theorem we improve the estimate (33).

**Theorem 3.** For a smooth boundary condition \( g = e^{i\varphi_0} \) of degree zero we have
\[
\|u_\epsilon - u_0\|_{C^m(\Omega)} \leq C_m \epsilon^2, \quad \forall m \geq 1.
\]

**Proof.** Note that for \( v \) with no zeros, i.e., of the form \( v = \rho e^{i\varphi} \), the energy in (1) takes the form
\[
E_\epsilon(v) = \int_{\Omega} \rho^2 |\nabla \varphi|^2 + \frac{1}{\epsilon^2} |\nabla \rho|^2.
\]

By Proposition 3, for \( \epsilon \) small enough, any minimizer \( u = u_\epsilon \) can be written as \( u = \rho_\epsilon e^{i\varphi_\epsilon} = \rho e^{i\varphi} \). It follows from (59) that the Euler-Lagrange system for \( \rho \) and \( \varphi \) reads as
\[
\begin{cases}
\text{div}(\rho^2 \nabla \varphi) = 0, \\
- \Delta \rho + \epsilon^2 \rho |\nabla \varphi|^2 = 0.
\end{cases}
\]

We write \( \varphi = \tilde{\varphi}_0 + \psi \) which allows us to write the equation satisfied by \( \psi \) as
\[
\begin{cases}
\Delta \psi = \text{div}((1 - \rho^2) \nabla \varphi) \text{ in } \Omega, \\
\psi = 0 \text{ on } \partial \Omega.
\end{cases}
\]

For any \( p > 2 \) we have by standard elliptic estimates and (33),
\[
\|\nabla \psi\|_p \leq C \| (1 - \rho^2) \nabla \varphi \|_p \leq C \| \nabla \varphi \|_p \leq C \epsilon (1 + \| \nabla \psi \|_p).
\]

It follows that \( \| \nabla \psi\|_p \leq C \epsilon \), whence
\[
\|\nabla \varphi\|_p \leq C_p, \quad \forall p > 2.
\]

Plugging (62) in the second equation in (60), yields \( \| \Delta \rho\|_p \leq C_p \epsilon^2 \), \( \forall p > 1 \), whence, since \( 1 - \rho = 0 \) on \( \partial \Omega \),
\[
\|1 - \rho\|_{W^{2,p}} \leq C_p \epsilon^2, \quad \forall p > 1.
\]
Using the first equation in (60) we obtain that
\[- \Delta \psi = - \Delta \varphi \equiv \frac{2}{\rho} (\nabla \rho \cdot \nabla \varphi), \tag{64}\]
so we can now conclude from (62) and (63) that \(\| \Delta \varphi \|_p \leq C_p \varepsilon^2, \forall p > 1\). Hence by elliptic estimates we get that also
\[\| \psi \|_{W^2,p} \leq C_p \varepsilon^2, \forall p > 1. \tag{65}\]
Next we claim that
\[\| \psi \|_{W_j,p} + \| 1 - \rho \|_{W_j,p} \leq C_j \varepsilon^2, \forall p > 1, \forall j \geq 2. \tag{66}\]
We prove (66) by induction on \(j\). For \(j = 2\) the result holds by (63) and (65). Assuming the result holds for \(j\), we see from (64) that \(\| \Delta \psi \|_{W_{j-1,p}} \leq C \varepsilon^2\), implying that \(\| \psi \|_{W_{j+1,p}} \leq C_j \varepsilon^2\). Similarly, the estimate for \(\| 1 - \rho \|_{W_{j+1,p}}\) follows from the second equation in (60). Finally, (58) follows from (66) and Sobolev embeddings. \(\square\)

2.2. Uniqueness of the Minimizers for Small \(\varepsilon\)

**Theorem 4.** If \(g\) is a smooth boundary condition of degree zero then there exists \(\varepsilon_0 > 0\) such that for all \(\varepsilon \leq \varepsilon_0\) the minimizer \(u_\varepsilon\) for \(E_\varepsilon\) over \(H^1(\Omega)\) is unique.

**Proof.** We follow an argument from [9]. By Theorem 3 there exists \(\varepsilon_1\) such that for \(\varepsilon \leq \varepsilon_1\) any minimizer \(u = u_\varepsilon\) satisfies \(1/2 \leq |u| \leq 1\). Let \(v = v_\varepsilon\) be any minimizer for \(\varepsilon \leq \varepsilon_1\), whence also \(1/2 \leq |v| \leq 1\). We may then write \(u = \rho e^{i\psi}\) and also \(w := v/u = \eta e^{i\psi}\) with \(1/2 \leq \eta \leq 2\) in \(\Omega\), \(\eta = 1\) on \(\partial \Omega\) and \(\psi = 0\) on \(\partial \Omega\). A direct computation yields
\[
E_\varepsilon(v) - E_\varepsilon(u) = \int_{\Omega} \rho^2 (\eta^2 - 1) |\nabla \varphi|^2 + \rho^2 \eta^2 (2 \nabla \varphi \cdot \nabla \psi + |\nabla \psi|^2)
+ \frac{1}{\varepsilon^2} \int_{\Omega} (\eta^2 - 1) |\nabla \rho|^2 + (\rho^2 |\nabla \eta|^2 + 2 \rho \eta \nabla \rho \cdot \nabla \eta). \tag{67}\]
Next we multiply the second equation in (60) by \(\rho(\eta^2 - 1)\) and integrate over \(\Omega\) to find
\[
\left(\int_{\Omega} (\eta^2 - 1) |\nabla \rho|^2 + 2 \rho \eta \nabla \rho \cdot \nabla \eta \right) + \varepsilon^2 \int_{\Omega} \rho^2 (\eta^2 - 1) |\nabla \varphi|^2 = 0. \tag{68}\]
Substituting (68) in (67) gives
\[
E_\varepsilon(v) - E_\varepsilon(u) = \int_{\Omega} \varepsilon^{-2} \rho^2 |\nabla \eta|^2 + 2 \rho^2 \eta^2 \nabla \varphi \cdot \nabla \psi + \rho^2 \eta^2 |\nabla \psi|^2. \tag{69}\]
On the other hand, multiplying the first equation in (60) by \(\psi\) and integrating, we conclude that
\[
\int_{\Omega} \rho^2 \nabla \varphi \cdot \nabla \psi = 0. \]
Plugging it in (69) yields that

\[ E_\varepsilon(v) - E_\varepsilon(u) = \int_{\Omega} \varepsilon^2 \rho^2 |\nabla \eta|^2 + 2 \rho^2 (\eta^2 - 1) \nabla \varphi \cdot \nabla \psi + \rho^2 \eta^2 |\nabla \psi|^2. \]  

(70)

By Theorem 3 we have \( \|\nabla \psi\|_\infty \leq c_0 \) for some constant \( c_0 > 0 \). Hence, by the Cauchy-Schwarz inequality we have

\[
\left| \int_{\Omega} 2 \rho^2 (\eta^2 - 1) \nabla \varphi \cdot \nabla \psi \right| \leq 4c_0^2 \int_{\Omega} (\eta^2 - 1)^2 + \frac{1}{4} \int_{\Omega} |\nabla \psi|^2 
\]

(71)

Applying Poincaré inequality to the function \( \eta^2 - 1 \in H^1_0(\Omega) \) yields

\[
\int_{\Omega} (\eta^2 - 1)^2 \leq C_P \int_{\Omega} |\nabla \eta|^2 \leq 16C_P \int_{\Omega} |\nabla \eta|^2. 
\]

(72)

Combining (71)–(72) with (70) yields

\[
E_\varepsilon(v) - E_\varepsilon(u) \geq \int_{\Omega} (\rho^2/\varepsilon^2 - 64C_P c_0^2) |\nabla \eta|^2 \geq \frac{1 - 256\varepsilon^2 c_0^2 C_P}{4\varepsilon^2} \int_{\Omega} |\nabla \eta|^2.
\]

It follows from the above and our assumption \( E_\varepsilon(v) = E_\varepsilon(u) \), that for \( \varepsilon < \frac{1}{16c_0\sqrt{C_P}} \) we must have \( |\nabla \eta| = 0 \) in \( \Omega \), whence \( \eta \equiv 1 \). Plugging it in (70) we finally get that \( \psi \equiv 0 \) and the equality \( v = u \) follows. \( \square \)

**Remark 4.** We do not know whether the uniqueness result of Theorem 4 holds without the assumption that \( \varepsilon \) is sufficiently small.

### 3. Boundary Condition of Degree \( D \geq 1 \): Preliminary Estimates

In this section we consider the case of boundary condition of nonzero degree. Without loss of generality we assume that \( \deg g = D \geq 1 \). We continue to assume that \( \Omega \) is a smooth, bounded and simply connected domain in \( \mathbb{R}^2 \); whenever convenient, we will suppose that \( \Omega \) is the unit disc \( B_1 = B_1(0) \).

**3.1. Minimization Within the Radial Class**

Consider the case \( \Omega = B_R = B_R(0) \) and \( g(Re^{i\theta}) = e^{iD\theta} \) with \( D \geq 1 \). Define

\[
V := \{ f \in H^1_{loc}(0, R) : \sqrt{r} f' , \frac{f}{\sqrt{r}} \in L^2(0, R), f(R) = 1 \}.
\]

For \( f \in V \) we have \( f e^{iD\theta} \in H^1_0(\Omega) \) and

\[
E_\varepsilon(f e^{iD\theta}) = 2\pi \int_0^R \left( \frac{f'^2}{\varepsilon^2} + \frac{D^2}{r^2} f^2 \right) r \, dr.
\]

We first solve the minimization problem under the restriction that the maps satisfy the above “\( D \)-radial symmetry” ansatz.
Lemma 1. For every $D \geq 1$ and $\varepsilon > 0$ we have
\[ \min_{f \in V} E_{\varepsilon}(f e^{iD\theta}) = \frac{2\pi D}{\varepsilon} \]
and the unique minimizer is
\[ \tilde{f}_{\varepsilon, D}(r) = \left( \frac{r}{R} \right)^{D\varepsilon}. \] (73)

Proof. First we note that for every $f \in V$ the following pointwise inequality holds on $(0, R)$:
\[ \frac{rf'^2}{\varepsilon^2} + \frac{D^2f^2}{r} = \left( \frac{\sqrt{r}f'}{\varepsilon} \right)^2 + \left( \frac{Df}{\sqrt{r}} \right)^2 \geq \frac{2}{\varepsilon} f f'. \] (74)
Integration of (74) over the interval $(0, R)$ yields $E_{\varepsilon}(f e^{iD\theta}) \geq \frac{2\pi D}{\varepsilon}$.
Equality holds in (74) if $\sqrt{r}f' = Df/\sqrt{r}$ a.e. on $(0, R)$. (75)
A simple integration of (75) yields $f = \tilde{f}_{\varepsilon, D}$ as given in (73).

We remark that the special solutions given by (73) are well-known in the literature. They appeared for example in [21] as part of the study of axially symmetric minimizers. In the next subsection, see Corollary 2 below, we will prove that $\tilde{f}_{\varepsilon, D}(e^{iD\theta})$ is the minimizer for $E_{\varepsilon}$ over the whole class $H^1_g(B_R)$ (for $g(e^{i\theta}) = e^{iD\theta}$), i.e., without assuming the $D$-radial symmetry ansatz.

3.2. Asymptotic Behavior of the Energy

In this subsection we will prove the following asymptotic formula for the energy: $E_{\varepsilon}(u) = \frac{2\pi D}{\varepsilon} + O(1)$. We start with the lower bound.

Proposition 4. Assume $g : \Omega \to S^1$ has degree $D > 0$. Then we have
\[ E_{\varepsilon}(u) \geq \frac{2\pi D}{\varepsilon}, \forall u \in H^1_g(\Omega). \] (76)

Proof. By density of smooth maps in $H^1_g(\Omega)$ it suffices to prove (76) for smooth $u$. Applying the Cauchy-Schwarz inequality gives
\[ E_{\varepsilon}(u) = \int_{\Omega \cap \{|u| \neq 0\}} \left( \varepsilon^{-2} |\nabla |u||^2 + |u|^2 |\nabla (u/|u|)||^2 \right) \]
\[ \geq \frac{2}{\varepsilon} \int_{\Omega \cap \{|u| \neq 0\}} |\nabla |u|||u||\nabla (u/|u|)||. \] (77)
For each $t \in (0, 1)$ set $\gamma_t = \{ x \in \Omega : |u| = t \}$. For almost every $t \in (0, 1)$, $\gamma_t$ is a union of a finite number closed smooth curves, $C_1, C_2, \ldots, C_m$. For such $t$, the boundary of the set $\Omega_t := \{ x \in \Omega : |u(x)| > t \}$ consists of $\partial \Omega \cup \bigcup_{k=1}^m C_k$,.
whence, the total winding number of the map \( u / |u| : \gamma_t \to S^1 \) around the origin equals \( D \). Hence,

\[
\int_{\gamma_t} |\nabla (u / |u|)| \geq \left| \int_{\gamma_t} (u / |u|) \wedge (u / |u|) \tau \, d\tau \right| = 2\pi D. \tag{78}
\]

The direction chosen for the unit vector \( \tau \) was dictated by the requirement that \((\nu, \tau)\) will be a direct frame, where \( \nu \) denotes the inward unit normal to \( \Omega_t \). Applying the coarea formula to the R.H.S. of (77), using (78), yields

\[
E_\epsilon(u) \geq \frac{2}{\epsilon} \int_0^1 \int_{\gamma_t} t |\nabla (u / |u|)| \, d\tau \, dt \geq \frac{4\pi D}{\epsilon} \int_0^1 t \, dt = \frac{2\pi D}{\epsilon}, \tag{79}
\]

and (76) follows.

**Corollary 1.** We have

\[
\frac{2\pi D}{\epsilon} \leq E_\epsilon(u_\epsilon) \leq \frac{2\pi D}{\epsilon} + C. \tag{80}
\]

**Proof.** The lower bound follows from Proposition 4. W.l.o.g. we may assume that \( 0 \in \Omega \). Fix any \( R > 0 \) such that \( B_R \subset \Omega \). For each \( \epsilon \) let \( U_\epsilon \) be equal to \( \tilde{f}_{\epsilon,D}(r)e^{iD\theta} \) in \( B_R \) (see (73)) and complete it in \( \Omega \setminus B_R \) by any \( S^1 \)-valued smooth map which equals \( e^{iD\theta} \) on \( \partial B_R \) and \( g \) on \( \partial \Omega \). By Lemma 1 we have \( E_\epsilon(U_\epsilon) \leq \frac{2\pi D}{\epsilon} + C \). □

**Corollary 2.** For \( \Omega = B_R \) and \( g(\Re i \theta) = e^{iD\theta} \), the map \( \tilde{f}_D(r)e^{iD\theta} \), with \( f_D \) as in (73), is a minimizer for \( E_\epsilon \) over \( H^1_g(\Omega) \).

**Proof.** This is an immediate consequence of Lemma 1 and Proposition 4. □

**Remark 5.** From Proposition 5 below it follows that \( \tilde{f}_D(r)e^{iD\theta} \) is the unique minimizer over \( H^1_g(\Omega) \) for the boundary condition \( g(\Re i \theta) = e^{iD\theta} \).

**Remark 6.** The combination of the Proof of Proposition 4 with the result of Corollary 1 demonstrates that the principle of “equipartition of the energy” holds for our problem, i.e., the contributions of the two terms in \( E_\epsilon(u_\epsilon) \) are essentially equal. It is well-known that this principle holds for *scalar* problems, like \( \Gamma \)-convergence of the Modica-Mortola functional, see [22,23,26] or its vector-valued analogues, see e.g. [26, Section 2] and [11]. In these works the equipartition is associated with phase-separation and, more specifically, with the profile of a minimizer being asymptotically one-dimensional. The equipartition of energy in our problem is of a quite different nature. Roughly speaking, it results from the approximate pointwise equality

\[
\epsilon |\nabla (\ln \rho_\epsilon)| \sim |\nabla \varphi_\epsilon|
\]

holding for a minimizer which can be written locally as \( u_\epsilon = \rho_\epsilon e^{i\varphi_\epsilon} \). Thus, in our case, equipartition reflects a strong coupling between the phase and the modulus of a minimizer.
3.3. When the Boundary Condition is a Blaschke Product

In this subsection we will show that the case considered above in Corollary 2, where we were able to give a simple explicit formula for the minimizers for each fixed $\varepsilon$, is a special case of a more general family of boundary data. In fact, let $\Omega = B_1$ and $g = F|_{\partial B_1}$ where $F \in C(\overline{B_1})$ is analytic function on $B_1$ that sends $\partial B_1$ to itself. It is well-known that such $F$ must be a finite Blaschke product, i.e., of the form

$$F(z) = e^{i\alpha} \prod_{j=1}^{D} \left( \frac{z - a_j}{\bar{a}_j z - 1} \right),$$  \hspace{1cm} (81)

for some $\alpha \in \mathbb{R}$ and $D$ points $a_1, a_2, \ldots, a_D$ (not necessarily distinct) in $B_1$. Note that the choice $a_j = 0, \forall j$ (and $\alpha = 0$) corresponds to the $D$-symmetric boundary data considered above.

**Proposition 5.** When $\Omega = B_1$ and $g = F|_{\partial B_1}$ with $F$ as in (81) we have for each $\varepsilon$: the map $U(\varepsilon) := U_{\varepsilon}(z) = |F(z)|^\varepsilon \left( \frac{F(z)}{|F(z)|} \right)$ is the unique minimizer of $E_\varepsilon$ over $H^1_g(B_1)$.

**Proof.** (i) We first prove that $U$ is a minimizer. Let us denote $\tilde{\rho} = |F|$ in $B_1$ and $h = \ln |\tilde{\rho}|$ in $E := B_1 \setminus \{a_1, \ldots, a_D\}$. Locally in $E$ we may write $F = \tilde{\rho} e^{i\varphi} = e^{h+i\varphi}$. The function $\varphi$ is then a harmonic conjugate of the harmonic function $h$, locally in $E$. Note that although $\varphi$ is defined only locally in $E$, its gradient is globally defined there since $\nabla \varphi = (F/|F|) \wedge \nabla (F/|F|)$. In particular, the equality

$$\left| \nabla \left( \frac{F(z)}{|F(z)|} \right) \right| = |\nabla h|,$$  \hspace{1cm} (82)

holds globally in $E$. Consider $U$, defined in $B_1$ as in the statement of the proposition, i.e., $U = \rho \left( \frac{F(z)}{|F(z)|} \right)$ with $\rho = (\tilde{\rho})^\varepsilon$, so locally in $E$ we have $U = \rho e^{i\varphi}$.

Next we notice that for $u = U$, the Cauchy-Schwarz inequality used in (77) reduces to an equality. Indeed, we need the pointwise equality $|\nabla \rho|/\varepsilon = |\nabla \varphi|$, which is equivalent to

$$|\nabla (\ln \rho)|/\varepsilon = |\nabla \varphi|.$$  \hspace{1cm} (83)

Since $\ln \rho = \varepsilon \ln h$ we finally deduce (83) from (82). To sum-up, so far we proved that

$$E_\varepsilon(U) = \frac{2}{\varepsilon} \int_E |\nabla \rho| |\nabla (F/|F|)| = \frac{2}{\varepsilon} \int_E |\nabla \rho| |\nabla (U/|U|)|.$$  \hspace{1cm} (84)

Next we continue to follow the Proof of Proposition 4 for the case $u = U$. We denote

$$\Gamma = \{ t \in (0, 1) : t is a regular value of \rho \}$$
$$= \{ t \in (0, 1) : t is a regular value of \tilde{\rho} \}.$$  \hspace{1cm} (85)
Clearly $\Gamma$ has full measure in $(0, 1)$. For each $t \in \Gamma$ the set $\gamma_t := \{\rho^{-1}(t)\}$ consists of a finite union of smooth closed curves, each encircles some of the points $\{a_1, \ldots, a_D\}$ (and the union of them encircle all the points). At each point of $\gamma_t$, with $t \in \Gamma$, we have $\frac{\partial \phi}{\partial \tau} = \frac{\partial (\ln \rho)}{\partial \nu} > 0$, since $|\nabla \rho| > 0$ on $\gamma_t$. Moreover, $\frac{\partial \rho}{\partial \nu} = -\frac{\partial (\ln \rho)}{\partial \tau} = 0$. Whence, for each $t \in \Gamma$, it holds that
\[
\int_{\gamma_t} |\nabla (U/|U|)| \, d\tau = \int_{\gamma_t} |\partial_{\tau} \phi| \, d\tau = \int_{\gamma_t} \partial_{\tau} \phi \, d\tau = \int_{\gamma_t} (U/|U|) \wedge (U/|U|)_{\tau} = \\
= \int_{\partial \Omega} (U/|U|) \wedge (U/|U|)_{\tau} = 2\pi D. \tag{86}
\]
Using (86) in conjunction with the coarea formula as in (79) gives
\[
\int_{E} |\nabla \rho||\nabla (U/|U|)| = \int_{\Gamma} \int t|\nabla (U/|U|)| \, d\tau \, dt = (2\pi D) \int_{0}^{1} t \, dt = \pi D. \tag{87}
\]
Combining (87) with (84) yields $E_\varepsilon(U) = 2\pi D/\varepsilon$. Applying Proposition 4 we finally conclude that indeed $U$ is a minimizer.

(ii) Next we will prove the uniqueness assertion – the main idea is to analyse the equality cases in the inequalities we used in part (i). We shall need a result of F.H. Lin [18] who derived (in a more general setting) the Euler-Lagrange equation satisfied by $\rho_\varepsilon^2 = |u_\varepsilon|^2$,
\[
\Delta(\rho_\varepsilon^2) = 2\varepsilon^2 \left(|\nabla u_\varepsilon|^2 + \left(\frac{1}{\varepsilon^2} - 1\right)|\nabla \rho_\varepsilon|^2\right), \tag{88}
\]
by using variations of the form $u^{(i)} = (1 + t\phi(x))u_\varepsilon$ (the same equation can be deduced from the second equation in (60) on the set $\{\rho_\varepsilon > 0\}$). In particular, the function $\rho_\varepsilon^2$ is subharmonic in $\Omega$. We next recall known regularity properties of any minimizer $u_\varepsilon$. By [13,19], the function $u_\varepsilon$ is Hölder continuous in $B_1$ and is real analytic in $B_1 \setminus S$, where $S$ is a finite set consisting of the zeros of $u_\varepsilon$. By (88) and the strong maximum principle $\rho_\varepsilon = |u_\varepsilon|$ satisfies $0 < \rho_\varepsilon < 1$ in $B_1$. Moreover, by Hopf lemma, $\frac{\partial (\rho_\varepsilon^2)}{\partial n} > 0$ on $\partial B_1$. Next we fix $r_0$ satisfying $\max_{1 \leq j \leq D} |a_j| < r_0 < 1$ and
\[
\frac{\partial \rho_\varepsilon}{\partial r} > 0 \text{ in } \{r_0 \leq |x| \leq 1\}. \tag{89}
\]
and let
\[
T := \max_{|x| = r_0} \rho_\varepsilon(x).
\]
Since by assumption $E_\varepsilon(u_\varepsilon) = E_\varepsilon(U) = 2\pi D$, equality must hold when we plug $u_\varepsilon$ for $u$ in all the inequalities in (77), (78), (79). Since by the maximum principle $\rho_\varepsilon < T$ in $B_{r_0}$, for each $t \in (T, 1)$ we have $\tilde{\gamma}_t := \{\rho_\varepsilon^{-1}(t)\} \subset \{r_0 < |x| < 1\}$. Moreover, thanks to (89) every $t \in (T, 1)$ is a regular value of $\rho_\varepsilon$, whence the set $\tilde{\gamma}_t$ consists of a single closed smooth curve (since the topology of $\tilde{\gamma}_t$ should be the same as that of $\tilde{\gamma}_1 = S^1$ in the absence of a critical point). For each such $t$ we write
locally $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ on the curve $\bar{\gamma}_t$. From the pointwise equality, $\nabla(u_\varepsilon/|u_\varepsilon|) = (u_\varepsilon/|u_\varepsilon|) \wedge (\rho_\varepsilon/|\rho_\varepsilon|)_\tau$ that holds in (78), we obtain that $\nabla\varphi_\varepsilon = \tau \perp \nabla\rho_\varepsilon$, i.e., taking into account orientation,

$$
((\varphi_\varepsilon)_x, (\varphi_\varepsilon)_y) = \lambda(x, y)(-(\rho_\varepsilon)_y, (\rho_\varepsilon)_x)
$$

locally on $\bar{\gamma}_t$. From the pointwise equality,

$$
\left|\nabla(\ln \frac{\rho_\varepsilon}{\varepsilon \rho_\varepsilon})\right| = \left|\nabla \varphi_\varepsilon\right|
$$

locally on $\bar{\gamma}_t$, $t \in (T, 1)$. (90)

for some $\lambda(x, y) > 0$. On the other hand, the pointwise equality

$$
\left|\nabla\rho_\varepsilon\right| = \left|\nabla \varphi_\varepsilon\right|
$$

locally on $\bar{\gamma}_t$, $t \in (T, 1)$. (91)

Combining (90) with (91) we deduce that locally on the set $\bigcup_{T < t < 1} \bar{\gamma}_t$, the pair of functions $\ln \rho_\varepsilon^{1/\varepsilon}$ and $\varphi_\varepsilon$ satisfy the Cauchy-Riemann equations ($\varphi_\varepsilon$ being a complex conjugate of $\ln \rho_\varepsilon^{1/\varepsilon}$). In particular, this holds locally on some annulus $A_{r_1} = \{r_1 < |x| < 1\} \subset \bigcup_{T < t < 1} \bar{\gamma}_t$ for some $r_1 \in (r_0, 1)$ and globally on $A_{r_1} \setminus \mathcal{L}$ where $\mathcal{L} = \{ (s, 0) : r_0 < s < 1 \}$. Since $U = u_\varepsilon$ on $\partial B_1$ we have $\varphi = \varphi_\varepsilon + 2\pi J$ on $\partial B_1 \setminus \mathcal{L}$ for some $J \in \mathbb{Z}$, and we may assume w.l.o.g. that $J = 0$. Since we also have $\frac{\partial \varphi}{\partial \nu} = \frac{\partial \varphi_\varepsilon}{\partial \nu} = 0$ on $\partial B_1 \setminus \mathcal{L}$ (using $\rho = \rho_\varepsilon = 1$ on $\partial B_1$ and the Euler-Lagrange equations) and since a harmonic function is uniquely determined by its values and the values of its normal derivative on the boundary, we deduce that $\varphi = \varphi_\varepsilon$ in $A_{r_1} \setminus \mathcal{L}$. By the Euler-Lagrange equations we obtain that also $\rho = \rho_\varepsilon$ in $A_{r_1} \setminus \mathcal{L}$, whence $u_\varepsilon = U$ in $A_{r_1}$. Finally, as both $u_\varepsilon$ and $U$ are real analytic in $B_1 \setminus (S \cup \{a_j\}_{j=1}^D)$ that coincide on the open subset $A_{r_1}$, they must coincide also on $B_1 \setminus (S \cup \{a_j\}_{j=1}^D)$, and then also on $B_1$.  

\[\square\]

4. Proof of Theorem 1

This section is devoted to the Proof of Theorem 1. Note that estimate (8) was already established in Corollary 1. Throughout most of this section we assume, as we may w.l.o.g., that $\Omega = B_1$.

4.1. Construction of the Bad Discs

The objective of this subsection is to show that the set where $|u_\varepsilon|$ is close to zero is “small”. This is established in Proposition 6 below, where we show that for some $\beta < 1$ the set $\{|u_\varepsilon| < \beta\}$ can be covered by a finite collection of discs of small radii whose number is bounded uniformly in $\varepsilon$. This is the same approach as that used in [8] for studying minimizers of the Ginzburg–Landau energy, but the technique we use here is different.

Recall that by Corollary 1 we have for some $c_1 > 0$,

$$
E_\varepsilon(u_\varepsilon) \leq \frac{c_1}{\varepsilon}, \forall \varepsilon \in (0, 1).
$$

(92)
In the sequel we fix a $\beta \in (1/\sqrt{2}, 1)$ that for reasons to become clear later we assume to satisfy
\[ \beta^2 > \frac{D}{D + 1}. \]  
(93)

**Lemma 2.** Let $u_\varepsilon$ be a minimizer satisfying
\[ r_0 \int_{\partial B_{r_0}} |\nabla u_\varepsilon|^2 + \left(\frac{1}{\varepsilon^2} - 1\right)|\nabla \rho_\varepsilon|^2 \leq \delta_0. \]  
(94)

with $\delta_0$ as in (36) and $r_0 > 0$ satisfying
\[ \frac{c_1 \varepsilon}{2\pi} \ln(1/r_0) < \frac{(1 - \beta)^2}{4}. \]  
(95)

Then, for $\varepsilon < \varepsilon_0$, we have
\[ |u_\varepsilon(0)| \geq \beta. \]  
(96)

**Proof.** For simplicity we shall drop the subscript $\varepsilon$. Analogously to (39) and (40) we have
\[ |u(x_1) - u(x_2)| \leq \int_{\partial B_{r_0}} |\nabla u| \leq (2\pi r_0)^{1/2} \left( \int_{\partial B_{r_0}} |\nabla u|^2 \right)^{1/2} \]
\[ \leq \sqrt{2\pi \delta_0}, \ \forall x_1, x_2 \in \partial B_{r_0}, \]  
(97)

\[ |\rho(x_1) - \rho(x_2)| \leq \sqrt{2\pi \delta_0} \varepsilon, \ \forall x_1, x_2 \in \partial B_{r_0}. \]  
(98)

Defining $\bar{\rho}$ as in (41), we find by (92), analogously to (42):
\[ c_1 \varepsilon \geq \int_{B_1 \setminus B_{r_0}} |\nabla \rho|^2 \geq \int_{B_1 \setminus B_{r_0}} |\nabla \bar{\rho}|^2 = 2\pi \int_{r_0}^{1} \left| \frac{d\bar{\rho}}{dr} \right|^2 r \, dr \geq \frac{2\pi (1 - \bar{\rho}(r_0))^2}{\ln(1/r_0)}. \]  
(99)

whence, by (95)
\[ 1 - \bar{\rho}(r_0) \leq \left\{ \frac{c_1 \varepsilon}{2\pi} \ln(1/r_0) \right\}^{1/2} < \frac{1 - \beta}{2}. \]  
(100)

By (100) and (98) we get that
\[ 1 - \rho(x) \leq \frac{1 - \beta}{2} + O(\varepsilon) \text{ on } \partial B_{r_0}, \]

so in particular,
\[ \rho^2 \geq (3/4)^2 > 1/2 \text{ on } \partial B_{r_0}. \]  
(101)

From (101) and (97) we conclude that $\deg(u/|u|, \partial B_{r_0}) = 0$. Therefore we may write on $\partial B_{r_0}$, $u = \rho e^{i\varphi}$. Using the harmonic extensions of $\rho$ and $\varphi$, as in the Proof of Proposition 3, we obtain (using (46) and (94)) that
\[ \int_{B_{r_0}} |\nabla u|^2 + \left(\frac{1}{\varepsilon^2} - 1\right)|\nabla \rho|^2 \leq \int_{B_{r_0}} |\nabla \varphi|^2 + \frac{1}{\varepsilon^2} |\nabla \bar{\rho}|^2 \]
\[
\leq r_0 \int_{\partial B_{r_0}} \left| \frac{\partial \varphi}{\partial \tau} \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{\partial \rho}{\partial \tau} \right|^2 \leq 2r_0 \int_{\partial B_{r_0}} \| \nabla u \|^2 + \left( \frac{1}{\varepsilon^2} - 1 \right) \| \nabla \rho \|^2 \leq 2\delta_0. \tag{102}
\]

Next we continue as in the Proof of Proposition 3, defining \( r_j = r_0/2^j \) for \( j \geq 1 \) and choosing successively, for \( j \geq 0, r_j' \in (r_{j+1}, r_j) \) satisfying

\[
\int_{\partial B_{r_j'}} |\nabla u|^2 + (1/\varepsilon^2 - 1)|\nabla \rho|^2 \leq \frac{2}{r_j'} \int_{B_{r_j} \setminus B_{r_j'/2}} |\nabla u|^2 + (1/\varepsilon^2 - 1)|\nabla \rho|^2. \tag{103}
\]

This allows us to conclude, arguing as in (53) and (54), that

\[
\int_{B_{r_{j+1}}} |\nabla u|^2 + (1/\varepsilon^2 - 1)|\nabla \rho|^2 \leq \frac{4}{5} \int_{B_{r_j}} |\nabla u|^2 + (1/\varepsilon^2 - 1)|\nabla \rho|^2 \leq 2\delta_0 \left( \frac{4}{5} \right)^{j+1}. \tag{104}
\]

Combining (104) with (103) yields

\[
\int_{\partial B_{r_j'}} |\nabla u|^2 + (1/\varepsilon^2 - 1)|\nabla \rho|^2 \leq \frac{4\delta_0}{r_j'} \left( \frac{4}{5} \right)^j,
\]

implying, in particular, that

\[
|\rho(x) - \rho(y)| \leq (8\pi \delta_0)^{1/2} \varepsilon \left( \frac{4}{5} \right)^j, \quad \forall x, y \in \partial B_{r_j'}. \tag{105}
\]

As in (52) and (56) we get that

\[
|\bar{\rho}(r_{j-1}') - \bar{\rho}(r_j')| \leq \left( \frac{\delta_0 \ln 4}{\pi} \right)^{1/2} \left( \frac{4}{5} \right)^{(j-1)/2} \varepsilon. \tag{106}
\]

Therefore, analogously to (57) we obtain

\[
1 - \bar{\rho}(r_j') \leq 1 - \bar{\rho}(r_0') + \sum_{i=1}^{j} |\bar{\rho}(r_{i-1}') - \bar{\rho}(r_i')| \leq \frac{1 - \beta}{2} + \left\{ \sum_{i=1}^{j} \left( \frac{4}{5} \right)^{(i-1)/2} \right\} \left( \frac{\ln 4}{\pi \delta_0} \right)^{1/2} \varepsilon. \tag{107}
\]

Thanks to (105)--(107), we have for each \( j \) that \( 1 - \rho(x) \leq \frac{1 - \beta}{2} + O(\varepsilon) \) on \( \partial B_{r_j'} \), which allows us to continue with the construction. Finally, letting \( j \) go to \( \infty \) in (107) we get that \( 1 - \rho(0) \leq (1 - \beta)/2 + O(\varepsilon) \) so, in particular, (96) holds for \( \varepsilon < \varepsilon_0. \)
Definition 1. (i) We shall say that 0 is a good point for $u_\varepsilon$ if there exists $r_0$ satisfying (95) and (94).
(ii) We shall say that $a \in B_1$ is a good point of $u_\varepsilon$ if 0 is a good point for $v_\varepsilon = u_\varepsilon \circ M_a$.

Corollary 3. (i) If $a$ is a good point for $u_\varepsilon$ then $|u_\varepsilon(a)| \geq \beta$.
(ii) If $|u_\varepsilon(0)| < \beta$ then

$$r \int_{\partial B_r} |\nabla u_\varepsilon|^2 + ((1/\varepsilon)^2 - 1) |\nabla \rho_\varepsilon|^2 > \delta_0 \text{ when } r > \rho_0(\varepsilon),$$

(108)

$$\int_{B_{r_2} \setminus B_{r_1}} |\nabla u_\varepsilon|^2 + ((1/\varepsilon)^2 - 1) |\nabla \rho_\varepsilon|^2 > \delta_0 \ln(r_2/r_1) \text{ when } r_1 > r_2 > r_1 \geq \rho_0(\varepsilon).$$

(109)

Here $\rho_0(\varepsilon) := \exp(-\pi(1-\beta)^2/2c_1)$, with $c_2 = \pi(1-\beta)^2/2c_1$.

(iii) There exists a constant $c_3 > 0$ such that, if $|u_\varepsilon(a)| < \beta$, then for $ho_1 = \rho_1(\varepsilon) := \rho_0(\varepsilon)^{1/2}$ we have

$$\int_{D_{\tanh^{-1} \rho_1(a)}} |\nabla u_\varepsilon|^2 + ((1/\varepsilon)^2 - 1) |\nabla \rho_\varepsilon|^2 \geq c_3/\varepsilon.$$  

(110)

Proof. Assertion (i) and (108) are immediate consequences of Lemma 2. The inequality (109) follows by integration of (108). The case $a = 0$ in (iii) follows from (ii) applied with $r_1 = \rho_0(\varepsilon)$ and $r_2 = \rho_1(\varepsilon)$. For general $a$ we use conformal invariance. $\square$

Definition 2. We denote the set of bad points of $u_\varepsilon$ by

$$S = S_\varepsilon = \{x \in B_1 : |u_\varepsilon(x)| < \beta\}.$$  

(111)

Proposition 6. For each $\varepsilon > 0$, there is a set of $m = m_\varepsilon$ points

$$\{x_j\}_{j=1}^m = \{x_j(\varepsilon)\}_{j=1}^m \subset S,$$

and a number $R = R(\varepsilon)$ satisfying

$$-\frac{A}{\varepsilon} < \ln R < -\frac{B}{\varepsilon}, \text{ where } A, B > 0,$$

(112)

such that the (hyperbolic) discs $\{D_{\tanh^{-1} R(x_j)}\}_{j=1}^m$ are mutually disjoint and the following properties hold:

(a) $S \subset \bigcup_{j=1}^m D_{\tanh^{-1}(R)(x_j)}$,

(113)

(b) $m \leq N$, for some $N$ independently of $\varepsilon$,

(c) $\kappa_j = \kappa_j^\varepsilon := \deg(u_\varepsilon/|u_\varepsilon|, \partial D_{\tanh^{-1} R(x_j)}) \in [1, C], \forall j,

\text{for some constant } C \geq 1,

(114)

(d) $R \int_{\partial B_R} \left( |\nabla (u_\varepsilon \circ M_{x_j})|^2 + ((1/\varepsilon)^2 - 1)|\nabla (\rho_\varepsilon \circ M_{x_j})|^2 \right) \leq c_4,

\forall j, \text{ for some constant } c_4 > 0.$$

(115)
Proof. The proof is divided to several steps. 
Step 1: Select an initial collection of bad discs. Let \( \rho_1 = \rho_1(\varepsilon) \) be as defined in Corollary 3. Applying Vitali covering lemma for the collection of discs \( \{D_{\tanh^{-1}\rho_1(x)}\}_{x \in S} \) yields a finite collection of mutually disjoint hyperbolic discs \( \{D_{\tanh^{-1}\rho_1(x_j)}\}_{j=1}^{m_\varepsilon} \), with \( \{x_j\}_{j=1}^{m_\varepsilon} = \{x_j^{(\varepsilon)}\}_{j=1}^{m_\varepsilon} \subset S \), such that

\[
S \subset \bigcup_{j=1}^{m_\varepsilon} D_{\tanh^{-1}\rho_1(x_j)}. \tag{117}
\]

Moreover, thanks to (110) and (92) we have \( m_\varepsilon \leq N \) for some \( N \), for all \( \varepsilon \).
Step 2: Construct the final collection of bad discs. First, we extend \( \rho_0 \) and \( \rho_1 \) from Corollary 3 to an infinite sequence by setting

\[
\rho_j = \rho_{j-1}^{1/2}, \quad j = 2, 3, \ldots, \text{i.e., } \rho_j = \exp(-\frac{c_2}{2j\varepsilon}). \tag{118}
\]

Consider the collection \( \{D_{\tanh^{-1}\rho_2(x_j)}\}_{j=1}^{m_\varepsilon} \) that clearly covers \( S \) by (117), since \( \tanh^{-1}\rho_2 \gg 5\tanh^{-1}(\rho_1) \). If the discs are mutually disjoint we are done. Otherwise, if for example \( D_{\tanh^{-1}\rho_2(x_1)} \cap D_{\tanh^{-1}\rho_2(x_2)} \neq \emptyset \), we keep \( D_{\tanh^{-1}\rho_2(x_1)} \) and drop \( D_{\tanh^{-1}\rho_2(x_2)} \). We relabel the new centers and keep the same notation for \( m = m_\varepsilon \) (which is strictly smaller than the original one) and consider the new collection \( \{D_{\tanh^{-1}\rho_3(x_j)}\}_{j=1}^{m_\varepsilon} \) that also covers \( S \). If these discs are all mutually disjoint we are done, otherwise we eliminate some discs, taking into account the intersections. We continue in this way until we reach \( l \) for which

\[
S \subset \bigcup_{j=1}^{m} D_{\tanh^{-1}\rho_{l-1}(x_j)} \text{ and } \{D_{\tanh^{-1}\rho_{l-1}(x_j)}\}_{j=1}^{m} \text{ are mutually disjoint.} \tag{119}
\]

This process must stop after at most \( N \) steps [\( N \) is given below (117)].

Let us assume for a moment that \( x_j = 0 \), and then \( D_{\tanh^{-1}\rho_{l-1}(0)} = B_{\rho_{l-1}} \). By the upper bound (92) we have

\[
\int_{B_{\rho_l} \setminus B_{\rho_{l-1}}} |\nabla u|^2 + ((1/\varepsilon^2) - 1)|\nabla \rho|^2 \leq \int_{\partial B_r} (|\nabla u|^2 + ((1/\varepsilon^2) - 1)|\nabla \rho|^2) \, dr \leq \frac{c_1}{\varepsilon}. \tag{120}
\]

Therefore, we can find \( R = R^{(\varepsilon)} \in (\exp(-\frac{c_3}{2^{l-1}\varepsilon}), \exp(-\frac{c_2}{2^{l}\varepsilon})) \) such that

\[
R \int_{\partial B_R} \left(|\nabla u|^2 + ((1/\varepsilon^2) - 1)|\nabla \rho|^2\right) \leq \frac{2^l c_1}{c_2} := c_4. \tag{121}
\]

For a general \( x_j \) (not necessarily \( x_j = 0 \)) we apply the above to \( \tilde{u} := u \circ M_{x_j} \). This gives first \( R_j \) such that (121) holds for \( \tilde{u} \). Actually we can apply the argument in a way that insures that the same \( R^{(\varepsilon)} := R = R_j \) works for all \( j \), so that (116) holds true. This completes the construction of the bad discs \( \{D_{\tanh^{-1}R(x_j)}\}_{j=1}^{m} \).
Step 3: Verify (115). Since $\rho \geq \beta$ on $\partial D_{\tanh^{-1}(R)}(x_j)$ for all $j$, the degree

$$\kappa_j = \kappa_j^\varepsilon = \deg(u/|u|, \partial D_{\tanh^{-1}(R)}(x_j)) \in \mathbb{Z}$$

is well defined and we may write

$$u = \rho e^{i(\kappa_j\beta + \eta)}$$
on $\partial D_{\tanh^{-1}(R)}(x_j)$, $j = 1, \ldots, m$

for some scalar function $\eta$. We first claim that

$$|\kappa_j| \leq C, \forall j$$

(122)
for some $C$, independently of $\varepsilon$. Again, it suffices to consider the case $x_j = 0$. The only interesting case is when $\kappa_j \neq 0$. Applying the argument used in the Proof of Proposition 4 yields, denoting this time $\gamma_t = \{x \in B_R : |u| = t\}$,

$$E_\varepsilon(u; B_R) \geq \frac{2}{\varepsilon} \int_0^\beta \int_{\gamma_t} t|\nabla(u/|u|)| \, dx \, dt \geq \frac{2}{\varepsilon} (2\pi |\kappa_j|) \int_0^\beta \varepsilon \, dt$$

(123)

since $\beta^2 > 1/2$. Our claim (122) clearly follows from (123) and the upper bound (92).

To conclude, we show that

$$\kappa_j > 0, \forall j.$$  

(124)

We first show the weaker inequality

$$\kappa_j \geq 0, \forall j.$$  

(125)

Indeed, combining (80) with (123) yields

$$\frac{2\pi D}{\varepsilon} + C \geq E_\varepsilon(u_\varepsilon) \geq \sum_{j=1}^m E_\varepsilon(u_\varepsilon; D_{\tanh^{-1}R}(x_j)) \geq \frac{2\pi \beta^2}{\varepsilon} \sum_{j=1}^m |\kappa_j|. \quad (126)$$

Sending $\varepsilon$ to 0 in (126) gives

$$D \geq \beta^2 \sum_{j=1}^m |\kappa_j|. \quad (127)$$

Combining (127) with (93) yields

$$D = |\sum_{j=1}^m \kappa_j| \leq \sum_{j=1}^m |\kappa_j| < D + 1.$$  

Therefore, necessarily $\sum_{j=1}^m |\kappa_j| = D$, implying that $\kappa_j = |\kappa_j|$ for all $j$, and (125) follows.

To prove that the inequality in (125) is strict, i.e., (124) holds, we fix one $j$ for which we may assume w.l.o.g. that $x_j = 0$. Looking for contradiction, suppose
that $\kappa_j = 0$. Then we may write $u = \rho e^{i\varphi}$ on $\partial BR$ and let again $\tilde{\rho}$ and $\tilde{\varphi}$ denote, respectively, the harmonic extensions of $\rho$ and $\varphi$ to $BR$. Analogously to (102) we obtain, using (116), that

$$
\int_{BR} |\nabla u|^2 + \left(\frac{1}{\epsilon^2} - 1\right) |\nabla \rho|^2 \leq \int_{BR} |\nabla \tilde{\varphi}|^2 + \frac{1}{\epsilon^2} |\nabla \tilde{\rho}|^2 \leq 2c_4.
$$

(128)

However the assumption $|u(0)| < \beta$ implies, by (110), that

$$
E_\varepsilon(u_\varepsilon; BR) \geq E_\varepsilon(u_\varepsilon; B_{\rho_1}) \geq c_3 \varepsilon,
$$

which clearly contradicts (128), for sufficiently small $\varepsilon$. \hspace{1cm} \Box

### 4.2. Control of the Phase Oscillations Away from the Bad Discs

To prove convergence of $u_\varepsilon$ away from the bad discs the main difficulty is to prove a bound on the oscillations of the phase. For that matter we shall use an appropriate modification of the strategy employed in [20] for a different problem. We denote

$$
\Omega_\varepsilon = B_1 \setminus \bigcup_{j=1}^{m} D_{\tanh^{-1} R(x_j)}.
$$

(129)

Whenever there is no confusion we shall drop the subscript $\varepsilon$. On $\Omega_\varepsilon$ we may write

$$
u(z) = \rho e^{i\eta(z)} \prod_{j=1}^{m} \left(\frac{M-x_j(z)}{|M-x_j(z)|}\right)^{\kappa_j}
$$

(130)

for some scalar function $\eta = \eta_\varepsilon$, which is unique up to addition of an integer multiple of $2\pi$. By adding an appropriate multiple of $2\pi$ we may assume then that

$$
\min_{\partial B_1} \eta \in [0, 2\pi).
$$

(131)

Since $g$ is smooth, we deduce from (131) that

$$
\|\eta\|_{L^\infty(\partial B_1)} \leq C(g).
$$

(132)

By (115) and (116) it follows that

$$
|\eta(x) - \eta(y)| \leq C, \quad \text{for all } x, y \in \partial D_{\tanh^{-1} R(x_j)}, \ j = 1, \ldots, m.
$$

(133)

We shall use the following estimate for $\int_{\Omega_\varepsilon} |\nabla \eta|^2$:

**Lemma 3.** We have

$$
\int_{\Omega_\varepsilon} |\nabla \eta|^2 \leq \frac{C}{\varepsilon}.
$$

(134)
Proof. By the upper bound (92), the representation (130) and (115) it suffices to show that
\[
\int_{\Omega_{\varepsilon}} \left| \nabla \left( \frac{M - x_j(z)}{|M - x_j(z)|} \right) \right|^2 \leq \frac{C}{\varepsilon}, \quad j = 1, \ldots, m.
\] (135)

In fact, (135) follows easily by using conformal invariance:
\[
\int_{\Omega_{\varepsilon}} \left| \nabla \left( \frac{M - x_j(z)}{|M - x_j(z)|} \right) \right|^2 \leq \int_{B_1 \setminus D_{\tanh^{-1} R(x_j)}} \left| \nabla \left( \frac{M - x_j(z)}{|M - x_j(z)|} \right) \right|^2
\]
\[
= \int_{B_1 \setminus B_R} \left| \nabla \left( \frac{z}{|z|} \right) \right|^2 = 2\pi \ln \frac{1}{R} \leq \frac{C}{\varepsilon}.
\] (136)

\[\square\]

Our first step consists of proving an \(L^\infty\) bound for \(\eta\). We will use the method of selection of “good rays”, that was introduced in [20]. This will be done by removing from \(\Omega_{\varepsilon}\) a collection of “rays”, that in our settings will be usually arcs of circles orthogonal to \(\partial B_1\), connecting the boundaries of the holes \(\partial D_{\tanh^{-1} R(x_j)}, j = 1, \ldots, m\), to the boundary of \(B_1\). The choice of these “good rays” will depend on energy considerations. Consider first the case where \(x_j = 0\). For any \(\alpha \in [0, 2\pi)\), let
\[
C_0(\alpha) := \{re^{i\alpha} : r \in [R, 1)\}.
\] (137)

In the general case, when \(x_j\) is any point in \(B_1\), we set
\[
C_{x_j}(\alpha) := \{M_{x_j}(re^{i\alpha}) : r \in [R, 1)\}.
\] (138)

Note that for \(x_j \neq 0\) the set \(C_{x_j}(\alpha)\) is an arc of a circle joining \(x_j\) to \(\partial B_1\) which is orthogonal to \(\partial B_1\) (a geodesic for the hyperbolic metric).

Lemma 4. There exists \(C > 0\) such that for each \(j = 1, \ldots, m\) and \(\varepsilon \in (0, 1/2)\) there exists \(\alpha_j = \alpha_j(\varepsilon) \in [0, 2\pi)\), such that the following holds:
\[
|\eta(x) - \eta(y)| \leq \frac{C}{\varepsilon}, \quad \text{for all} \ x, y \in C_{x_j}(\alpha_j) \cap \Omega_{\varepsilon}.
\] (139)

Proof. By (134) there exists \(\alpha_j \in [0, 2\pi)\) such that
\[
\int_{C_0(\alpha_j) \cap M_{-x_j}(\Omega_\varepsilon)} |\nabla (\eta \circ M_{x_j})|^2 r \, dr \leq \frac{1}{2\pi} \int_{M_{-x_j}(\Omega_\varepsilon)} |\nabla (\eta \circ M_{x_j})|^2
\]
\[
= \frac{1}{2\pi} \int_{\Omega_\varepsilon} |\nabla \eta|^2 \leq \frac{C}{\varepsilon}.
\] (140)

Therefore,
\[
\int_{C_0(\alpha_j) \cap M_{-x_j}(\Omega_\varepsilon)} \left| \frac{\partial (\eta \circ M_{x_j})}{\partial r} \right| \leq \left( \int_{R}^{1} \frac{dr}{r} \right)^{1/2}
\]
\[
\left( \int_{\gamma} \frac{1}{\sqrt{2 \pi r}} \, dr \right)^{1/2} \leq \left( \ln \left( \frac{1}{R} \right) \right)^{1/2} \left( \frac{C}{\varepsilon} \right)^{1/2} \leq \frac{C}{\varepsilon}.
\]

Here, \( \partial / \partial r \) stands for the tangential derivative along \( \gamma \). It follows that

\[ |(\eta \circ Mx_j)(x) - (\eta \circ Mx_j)(y)| \leq \frac{C}{\varepsilon}, \quad \text{for all } x, y \in \gamma \cap M - x_j \Omega \varepsilon, \]

which is clearly equivalent to (139). \( \square \)

Next, we denote \( \omega \varepsilon := \Omega \varepsilon \setminus \bigcup_{j=1}^m C_{x_j}(\alpha_j) \). For each \( j \), let \( \theta_j \) denote a polar coordinate around the point \( x_j \), taking values in \([\alpha_j, \alpha_j + 2\pi)\) associated with the factor \( M - x_j \varepsilon \), i.e.,

\[ \frac{M - x_j z}{|M - x_j z|} = e^{i \theta_j(z)}. \quad (141) \]

Then the function

\[ \Theta = \sum_{j=1}^m \kappa_j \theta_j, \quad (142) \]

is smooth in \( \omega \varepsilon \) and satisfies

\[ \| \Theta \|_{L^\infty(\omega \varepsilon)} \leq 4\pi \sum_{j=1}^m |\kappa_j|. \quad (143) \]

We define \( \varphi = \varphi \varepsilon := \eta + \Theta \) in \( \omega \varepsilon \), so that

\[ u = \rho e^{i \theta} \prod_{j=1}^m \left( \frac{M - x_j \varepsilon z}{|M - x_j \varepsilon z|} \right)^{\kappa_j} = \rho e^{i (\Theta + \eta)} = \rho e^{i \varphi} \text{ in } \omega \varepsilon. \]

Hence \( \varphi \) is a well-defined phase of \( u \) in \( \omega \varepsilon \).

**Lemma 5.** We have, for all \( 0 < \varepsilon < 1/2 \) that,

\[ \| \eta \varepsilon \|_{L^\infty(\omega \varepsilon)} \leq \frac{C}{\varepsilon}. \]

**Proof.** First we notice, combining (132)–(133) with (139), that

\[ \| \eta \|_{L^\infty(\partial \omega \varepsilon)} \leq \frac{C}{\varepsilon}. \]

(145)
Therefore, by the definition of \( \varphi \) we have
\[
\limsup_{\delta \to 0} \sup \{ |\varphi(x)| : x \in \omega_\delta, \text{dist}(x, \partial \omega_\delta) \leq \delta \} \leq \frac{C}{\varepsilon}.
\] (146)

We apply the maximum principle to \( \varphi \) on each component of the open set \( \{ x \in \omega_\varepsilon : \text{dist}(x, \partial \omega_\varepsilon) > \delta \} \), on which \( \varphi \) satisfies
\[
\text{div}(\rho^2 \nabla \varphi) = 0.
\]

Then we let \( \delta \to 0 \) and use (146) to obtain that
\[
\|\varphi\|_{L^\infty(\omega_\varepsilon)} \leq \frac{C}{\varepsilon}.
\] (147)

Finally, (144) follows from (147), (143), (115) and the definition of \( \varphi \).

4.3. An \( L^p \)-Bound for the Gradient, \( p \in [1, 2) \)

The main result of this subsection is

**Proposition 7.** We have \( \|\nabla u_\varepsilon\|_{L^p(B_1)} \leq C_p, 1 \leq p < 2 \).

The following simple lemma will be needed in the Proof of Proposition 7:

**Lemma 6.** For every \( a \in B_1 \), it holds that
\[
\left| \nabla \left( \frac{M_{-a}(z)}{|M_{-a}(z)|} \right) \right| \leq \frac{C}{|z - a|}, \quad \forall z \in B_1.
\] (148)

**Proof.** Using \( (M_{-a})'(z) = \frac{1 - |a|^2}{(1 - \bar{a}z)^2} \), we get that
\[
\left| \nabla \left( \frac{M_{-a}(z)}{|M_{-a}(z)|} \right) \right| \leq \frac{C}{|M_{-a}(z)|} \cdot \frac{1 - |a|^2}{|1 - \bar{a}z|^2} = \frac{C(1 - |a|^2)}{|z - a||1 - \bar{a}z|} \leq \frac{C}{|z - a|}.
\]

**Proof of Proposition 7.** Fix any \( p \in (1, 2) \). By standard elliptic estimates, there exists a constant \( A_p = A_p(\Omega) \) such that the solution \( w \) of the problem
\[
\begin{aligned}
-\Delta w &= \text{div} \, g \quad \text{in } \Omega \\
w &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\] (149)

with \( g \in (L^p(\Omega))^2 \) satisfies
\[
\|\nabla w\|_{L^p(\Omega)} \leq A_p \|g\|_{L^p(\Omega)}.
\] (150)

We now apply the bad discs construction of Proposition 6, but this time covering the bad set
\[
S = S_\varepsilon = \{ x \in B_1 : |u(x)| < \tilde{\beta} \}
\]
with $\tilde{\beta} \in [\beta, 1)$ that satisfies
\[ 0 < 1 - \tilde{\beta} < \frac{1}{4A_p}. \quad (151) \]

In the sequel, $\Omega_\varepsilon$ denotes the set given in (129) for the resulting bad discs from this choice of $\tilde{\beta}$. Note that the number of discs and the value of $l$ may change as well, but we shall use the same notation as before.

Let $H$ denote the harmonic function in $B_1$ satisfying $H = \eta$ on $\partial B_1$. By (132) and the maximum principle,
\[ \|H\|_{L^\infty(B_1)} = \|\eta\|_{L^\infty(\partial B_1)} \leq C(g). \quad (152) \]

Note that Lemma 6 implies that
\[ \left\| \prod_{j=1}^m \left( \frac{M_{-x_j}(z)}{|M_{-x_j}(z)|} \right)^{\kappa_j} \right\|_{W^{1,p}(B_1)} \leq C. \quad (153) \]

Therefore
\[ \|\eta\|_{W^{1-1/p, p}(\partial B_1)} \leq C, \]
and also
\[ \|H\|_{W^{1,p}(B_1)} \leq C. \quad (154) \]

Next we define the function $\xi_0$ in $B_{R^{1/2}}$ by
\[ \xi_0(z) = \begin{cases} 0 & |z| \leq R \\ -\frac{\ln(|z|/R)}{\ln \sqrt{R}} & R < |z| < \sqrt{R}. \end{cases} \quad (155) \]

Note that
\[ \int_{B_{\sqrt{R}}} |\nabla \xi_0|^2 = \frac{2\pi}{(\ln \sqrt{R})^2} \int_{\sqrt{R}} \frac{\sqrt{R}}{r} \frac{dr}{\ln \sqrt{R}} = -\frac{2\pi}{\ln \sqrt{R}} \leq C\varepsilon. \quad (156) \]

For $j = 1, \ldots, m$ we set in $D_j := D_{\tanh^{-1}\sqrt{R}(x_j)}$: $\xi_j(z) = \xi_0(M_{-x_j}(z))$. From (156) we deduce that
\[ \int_{D_j} |\nabla \xi_j|^2 = \int_{B_{\sqrt{R}}} |\nabla \xi_0|^2 \leq C\varepsilon. \quad (157) \]

We finally define a function $\xi$ in $B_1$ by
\[ \xi(z) = \begin{cases} \xi_j(z) & \text{if } z \in D_j \text{ for some } j, \\ 1 & \text{on } B_1 \setminus \bigcup_{j=1}^m D_j. \end{cases} \quad (158) \]
Note that for any $p \in [1, 2)$ we have by (157) and (112),

$$\int_{B_1} |\nabla \xi|^p = \sum_{j=1}^{m} \int_{D_j} |\nabla \xi|^2 \frac{(\int_{D_j} |\nabla \xi|^2)^{p/2}}{|D_j|^{1-p/2}} \leq C \varepsilon^{p/2} R^{1-p/2} \leq C \varepsilon^{p/2} \exp \left(-\frac{(2-p)B^2}{2\varepsilon}\right).$$

(159)

In $B_1$ we set $\tilde{\eta} := \xi^2 \eta$ and $\tilde{H} := \xi^2 H$. From (154) and (159) we conclude that

$$\|\tilde{H}\|_{W^{1,p}(B_1)} \leq C.$$  

(160)

The function $\tilde{\eta}$ satisfies

$$- \text{div}(\rho^2 \nabla \tilde{\eta}) = - \text{div}(\rho^2 \xi^2 \nabla \eta) - \text{div}(\rho^2 \eta \nabla (\xi^2))$$

$$= -\xi^2 \text{div}(\rho^2 \nabla \varphi) \rho^2 \xi^2 \cdot \nabla \varphi + \text{div}(\rho^2 \xi^2 \nabla \Theta) + \text{div}(-2 \rho^2 \eta \xi \nabla \xi)$$

$$:= F_1 + F_2 + \text{div} G_1 + \text{div} G_2.$$  

First we note that $F_1 = 0$ by (60). Therefore,

$$\begin{cases}
-\Delta (\tilde{\eta} - \tilde{H}) = F_2 + \text{div}(G_1 + G_2) + \text{div}(\rho^2 \nabla \Theta) + \text{div}((\rho^2 - 1) \nabla (\tilde{\eta} - \tilde{H})) & \text{in } B_1, \\
\tilde{\eta} - \tilde{H} = 0 & \text{on } \partial B_1.
\end{cases}$$  

(161)

By elliptic estimates, for any $p \in [1, 2]$ there exists $B_p = B_p(\Omega) > 0$ such that the solution $w$ of the problem

$$\begin{cases}
-\Delta w = v & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{cases}$$

(162)

with $v \in L^1(\Omega)$, satisfies

$$\|\nabla w\|_p \leq B_p \|v\|_1.$$  

(163)

We bound $F_2$ in $L^1$ by

$$\int_{B_1} |F_2| = \int_{B_1} |\rho^2 \nabla (\xi^2) \cdot \nabla \varphi| \leq 2 \sum_{j=1}^{m} \int_{D_j} |\nabla \xi| \|\nabla \varphi|$$

$$\leq C \left(\sum_{j=1}^{m} \left(\int_{D_j} |\nabla \xi|^2\right)^{1/2}\right) \left(\int_{B_1} |\nabla u|^2\right)^{1/2} \leq (C \varepsilon)^{1/2} \cdot \left(\frac{C}{\varepsilon}\right)^{1/2} \leq C,$$  

(164)

where we used (92) and (157).

Clearly (153) implies a bound

$$\|G_1\|_{L^p(B_1)} \leq C.$$  

(165)

To bound $G_2$ in $L^p$ we use (159) and (144) to get

$$\int_{B_1} |G_2|^p \leq C \|\eta\|_{L^p} \|\nabla \xi\|_{L^p(B_1)}^p \leq \left(\frac{C}{\varepsilon^p}\right)^{p/2}$$
\[ \exp\left(-\frac{(2 - p)B}{2\varepsilon}\right) \leq \exp\left(-\frac{c}{\varepsilon}\right), \]  

(166)

for some positive constant \( c \). A bound in \( L^p(B_1) \) for \( \rho^2 \nabla \tilde{H} \) follows from (160).

We also note that

\[ 1 - \rho^2 \leq 2(1 - \tilde{\beta}) \quad \text{on} \quad \text{supp}(\nabla(\tilde{\eta} - \tilde{H})) \subset \Omega_\varepsilon. \]

Using the above in (161) we get by (150) and (163) that

\[
\| \nabla(\tilde{\eta} - \tilde{H}) \|_{L^p} \leq A_p \left( \| (\rho^2 - 1) \nabla(\tilde{\eta} - \tilde{H}) \|_{L^p} + \| G_1 \|_{L^p} + \| G_2 \|_{L^p} + \| \rho^2 \nabla \tilde{H} \|_{L^p} \right) \\
+ B_p \| F_2 \|_{L^1} \leq 2A_p(1 - \tilde{\beta}) \| \nabla(\tilde{\eta} - \tilde{H}) \|_{L^p} + C. 
\]

(167)

Combining (151) and (167), we find that \( \| \nabla(\tilde{\eta} - \tilde{H}) \|_{L^p} \leq C \), which in conjunction with (160) implies that \( \| \nabla \tilde{\eta} \|_{L^p} \leq C \). Since \( \| \nabla \tilde{\Theta} \|_{L^p(\Omega_\varepsilon)} \leq C \), we obtain that

\[
\| \nabla u \|_{L^p(B_1 \setminus \bigcup_{j=1}^m D_j)} \leq C. 
\]

(168)

Finally we note that for each \( j = 1, \ldots, m \) we have

\[
\int_{D_j} |\nabla u|^p \leq \left( \int_{D_j} |\nabla u|^2 \right)^{p/2} |D_j|^{1-p/2} \leq C \varepsilon^{-p/2} R^{1-p/2} \\
\leq C \varepsilon^{-p/2} \exp\left(-\frac{(2 - p)B}{2\varepsilon}\right) = o_{\varepsilon}(1). 
\]

(169)

The conclusion of Proposition 7 follows from (168) and (169).

\[ \square \]

4.4. Some Identities Satisfied by \( u_\varepsilon \)

In this subsection we list some (essentially known) identities satisfied by the minimizers that will be useful in the proofs of both Theorems 1 and 2. An important property of the minimizers is that the associated Hopf differential is a holomorphic function (see [13, Lemma 3.1]). Note that in dimension two this property is equivalent to the “divergence free” property of the stress-energy tensor, that holds in higher dimensions (of the domain and the target), see e.g., [2] and the references therein. In this subsection we represent a point in \( \Omega \) as \( z = x_1 + i x_2 \) and we continue to drop the subscript \( \varepsilon \).

**Proposition 8.** For any \( \varepsilon > 0 \) the function

\[
\chi = \chi_\varepsilon = |u_{x_1}|^2 - |u_{x_2}|^2 - 2i u_{x_1} \cdot u_{x_2} + \left( \frac{1}{\varepsilon^2} - 1 \right) \\
\left( |u|_{x_1}^2 - |u|_{x_2}^2 - 2i |u|_{x_1} |u|_{x_2} \right) 
\]

(170)

is holomorphic in \( \Omega \) and the Cauchy-Riemann equations hold in the classical sense in a neighborhood of the boundary.
We emphasize that in (170) the dot product refers to scalar product of vectors in $\mathbb{R}^2$.

**Proof.** To see that the Cauchy-Riemann equations are satisfied in the sense of distributions, we consider the effect of a family of diffeomorphisms generated by an arbitrary vector field $X$ on the energy $E_\varepsilon$ (see [2]). Since $u$ is Hölder continuous on $\overline{\Omega}$, in a small enough neighborhood of the boundary it satisfies $|u| > 0$. Therefore $u$ is smooth in that neighborhood. We can then verify by a direct computation that the Cauchy-Riemann equations hold for $\chi$ in this neighborhood using (60). $\square$

From Proposition 8 we deduce the following Pohozaev identity:

**Corollary 4.** Every minimizer $u = u_\varepsilon$ satisfies

$$\int_{\partial B_1} \left( |\partial_\tau u|^2 - |\partial_r g|^2 \right) + \left( \frac{1}{\varepsilon^2} - 1 \right) |\partial_r \rho|^2 = 0. \quad (171)$$

**Proof.** We denote

$$U = \left( u, \left( \frac{1}{\varepsilon^2} - 1 \right) \rho \right). \quad (172)$$

Therefore

$$\chi = |U_{x_1}|^2 - |U_{x_2}|^2 - 2i U_{x_1} \cdot U_{x_2}. \quad (173)$$

Since $\chi$ is holomorphic in $B_1$ and continuous on $\overline{B}_1$ we have, in particular that

$$0 = \int_{\partial B_1} \chi z \, dz = i \int_0^{2\pi} \chi(e^{i\theta}) e^{2i\theta} \, d\theta. \quad (174)$$

A direct computation shows that

$$|U_\nu|^2 - |U_\tau|^2 = |x_1 U_{x_1} + x_2 U_{x_2}|^2 - | - x_2 U_{x_1} + x_1 U_{x_2}|^2 = \Re \left( \chi(z) z^2 \right) \text{ on } \partial B_1. \quad (175)$$

Combining (174) with (175) gives that

$$\int_{\partial B_1} |U_\nu|^2 - |U_\tau|^2 = 0,$$

which is equivalent to (171). $\square$

Next we present a weak formulation of the equation satisfied by the phase of $u$.

**Proposition 9.** We have

$$\frac{\partial}{\partial x_1} \left( u_\varepsilon \wedge (u_\varepsilon)_{x_1} \right) + \frac{\partial}{\partial x_2} \left( u_\varepsilon \wedge (u_\varepsilon)_{x_2} \right) = 0 \quad (176)$$

in the sense of distributions.

**Proof.** Fix $\phi \in C_c^\infty(\Omega)$ and for $t \in \mathbb{R}$ let $u_\varepsilon = (u_1, u_2)$ and $u_\varepsilon^{(t)} := e^{i\phi} u_\varepsilon$. From the minimality of $u_\varepsilon$ we derive by a simple computation that

$$0 = \frac{d}{dt} \big|_{t=0} E_\varepsilon(u_\varepsilon^{(t)}) = 2 \int_\Omega \sum_{j=1}^2 \left( (u_2)_x u_1 - (u_1)_x u_2 \right) \phi x_j. \quad (177)$$

Since $\phi$ is arbitrary we immediately deduce (176). $\square$
4.5. An $L^2$-Bound for $|\nabla u_\varepsilon|$ Away from the Singularities

We denote by $a_1, \ldots, a_N \in \overline{B}_1$ the different limits of the families $\{x_j^{(\varepsilon)}\}, j = 1, \ldots, m$ (possibly along a subsequence). Since two different families may converge to the same limit, we have $N \leq m$. At this point we do not exclude the possibility that some of the $a_i$’s belong to $\partial B_1$. Consider any $r > 0$ satisfying

$$r < \min\{|a_i - a_j| : i \neq j\} \text{ and } r < \text{dist}(a_j, \partial B_1), \ \forall j \text{ such that } a_j \in B_1,$$

(178)

We denote

$$\tilde{\Omega}_r := B_1 \setminus \bigcup_{j=1}^{N} B_r(a_j),$$

and by $d_j$ the degree of $u_\varepsilon$ on $\partial (B_s(a_j) \cap B_1)$ for a small $\varepsilon$ and (a small but fixed) $s$. The following equality is clear: if $J_j := \{\ell : x^{(\varepsilon)}_\ell \to a_j\}$, then

$$d_j = \sum_{\ell \in J_j} \kappa_\ell.$$

Theorem 5. For each $r$ as in (178) we have

$$\int_{\tilde{\Omega}_r} |\nabla u_\varepsilon|^2 \leq C(r). \quad (179)$$

Proof. Note that, dropping the subscript $\varepsilon$,

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2|\nabla \phi|^2 = |\nabla \rho|^2 + \rho^2|\nabla (\Theta + \eta)|^2.$$

(180)

Since $\int_{B_1} |\nabla \rho|^2 \leq C \varepsilon$ by (92), and $\int_{\tilde{\Omega}_r} |\nabla \Theta|^2 \leq C(r)$ thanks to Lemma 6 and (115), we only need to find a bound for $\int_{\tilde{\Omega}_r} |\nabla \phi|^2$. By the boundedness of $\{\nabla \eta\}$ in $L^1(\Omega_\varepsilon)$ (see Proposition 7), it follows that there exists $\tilde{r} = \tilde{r}(\varepsilon) \in (r/2, r)$ such that

$$\sum_{j=1}^{N} \int_{\partial B_r(a_j) \cap \Omega} |\nabla \eta| \, d\sigma \leq C_1(r). \quad (181)$$

Similarly, we can find for each

$$j \in I := \{k \in 1, \ldots, N \text{ such that } a_k \in B_1\}$$

a number $\beta_j \in [0, 2\pi)$ such that the set

$$\tilde{L}_j = \tilde{L}_j(\beta_j) := \{a_j + se^{i\beta_j} : s \geq \tilde{r}\} \cap \tilde{\Omega}_{\tilde{r}}$$

satisfies

$$\int_{\tilde{L}_j} \left| \frac{\partial \eta}{\partial s} \right| \, ds \leq C_2(r). \quad (182)$$

By the argument of the Proof of Lemma 5 and using (181) and (182), we find that

$$\|\eta\|_{L^\infty(\tilde{\Omega}_{\tilde{r}})} \leq C_3(r). \quad (183)$$
For $\varepsilon$ sufficiently small we have

$$|x^{(\varepsilon)}_\ell - a_j| < \tilde{r}/2, \forall \ell \in J_j, j = 1, \ldots, N.$$  \hspace{1cm} (184)

Next, we multiply the equation

$$- \text{div}(\rho^2 \nabla \eta) = \text{div}(\rho^2 \nabla \Theta)$$

by $\eta$, and integrate over $\tilde{\Omega}_{\tilde{r}}$. This yields

$$\int_{\tilde{\Omega}_{\tilde{r}}} \rho^2 |\nabla \eta|^2 = - \int_{\tilde{\Omega}_{\tilde{r}}} \rho^2 \nabla \Theta \cdot \nabla \eta + \int_{\partial \tilde{\Omega}_{\tilde{r}}} \rho^2 \frac{\partial \varphi}{\partial n} \eta := I_1 + I_2. \hspace{1cm} (185)$$

We first claim that

$$|I_2| \leq C_4(r). \hspace{1cm} (186)$$

Indeed, we use (171) and (183) for the integral on $\partial \tilde{\Omega}_{\tilde{r}} \cap \partial B_1$ and for the integral on $\partial B_\tilde{r}(a_j) \cap B_1$ we use (181) and the fact that thanks to (184) we have

$$\left| \frac{\partial \Theta}{\partial n} \right| \leq \frac{C}{r} \text{ on } \partial B_\tilde{r}(a_j).$$

Applying the Cauchy-Schwarz inequality to $I_1$ in conjunction with (186) in (185) yields

$$\int_{\tilde{\Omega}_{\tilde{r}}} \rho^2 |\nabla \eta|^2 \leq C_4(r) + \int_{\tilde{\Omega}_{\tilde{r}}} \frac{\rho^2}{2} |\nabla \eta|^2 + \int_{\tilde{\Omega}_{\tilde{r}}} \frac{\rho^2}{2} |\nabla \Theta|^2. \hspace{1cm} (187)$$

Since $\int_{\tilde{\Omega}_{\tilde{r}}} (\rho^2/2) |\nabla \Theta|^2 \leq C_5(r)(|\log r| + 1)$, we deduce from (187) that $\int_{\tilde{\Omega}_{\tilde{r}}} |\nabla \eta|^2 \leq C_6(r)$. It follows that also $\int_{\tilde{\Omega}_{\tilde{r}}} |\nabla \varphi|^2 \leq C_7(r)$, which in view of (180) clearly implies (179).

4.6. Convergence of $u_{\varepsilon_n}$

Next, we will prove convergence of $u_{\varepsilon_n}$ on $\bar{B}_1 \setminus \{a_1, \ldots, a_N\}$.

**Proposition 10.** Let $b \in B_1$ and $r_1 > 0$ be such that $B_{r_1}(b) \subset B_1 \setminus \{a_1, \ldots, a_N\}$. Then $u_{\varepsilon_n} \rightharpoonup u_0$ in $C^k(B_{r_1/2}(b))$ for all $k \geq 0$, where $u_0$ is a smooth $S^1$-valued harmonic map.

**Proof.** Since $|u_\varepsilon| \geq \tilde{\rho}$ in $B_{\tilde{r}}(b)$ for small $\varepsilon$, we may write $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$. By Theorem 5, $\int_{B_{\tilde{r}}(b)} |\nabla u_\varepsilon|^2 \leq C$. Also, $\int_{B_{\tilde{r}}(b)} |\nabla \rho_\varepsilon|^2 \leq C \varepsilon$ by (92). Hence by Fubini we can find $\tilde{r} \in ((3/4)r_1, r_1)$ such that

$$\int_{\partial B_{\tilde{r}}(b)} |\nabla \varphi_\varepsilon|^2 + \varepsilon^{-1} |\nabla \rho_\varepsilon|^2 \leq C. \hspace{1cm} (188)$$

Since $|\varphi_\varepsilon|_{\partial B_{\tilde{r}}(b)}$ is bounded in $H^1(\partial B_{\tilde{r}}(b))$, by passing to a subsequence we may assume that

$$\varphi_\varepsilon|_{\partial B_{\tilde{r}}(b)} \rightharpoonup \varphi_0 \text{ in } H^{1/2}(\partial B_{\tilde{r}}(b)) \text{ and uniformly on } \partial B_{\tilde{r}}(b). \hspace{1cm} (189)$$
As for \( \rho_\varepsilon \), from (188) we infer that
\[
\rho_\varepsilon|_{\partial B_r(b)} \to c_0 \text{ in } H^1(\partial B_r(b)) \text{ and uniformly on } \partial B_r(b),
\] (190)
for some constant \( c_0 \geq 0 \). We denote by \( \tilde{\varphi}_0 \) the harmonic extension of \( \varphi_0 \) to \( B_r(b) \),
and set \( u_0 = e^{i\tilde{\varphi}_0} \). We are going to prove that \( u_\varepsilon \to u_0 \) on \( B_r(b) \) in different norms,
starting with the \( H^1 \)-norm.

We denote as usual by \( \tilde{\varphi}_\varepsilon \) and \( \tilde{\rho}_\varepsilon \), respectively, the harmonic extensions of \( \rho_\varepsilon \)
and \( \varphi_\varepsilon \). First, by (189) we have
\[
\lim_{\varepsilon \to 0} \int_{B_r(b)} |\nabla \tilde{\varphi}_\varepsilon|^2 = \int_{B_r(b)} |\nabla \tilde{\varphi}_0|^2.
\] (191)
Next we claim that
\[
1 - C\varepsilon^{1/2} \leq \rho_\varepsilon \leq 1 \text{ on } \partial B_r(b).
\] (192)
Indeed, assuming first that \( b = 0 \), we have as in (42)–(43) that
\[
1 - \frac{1}{2\pi r} \int_{\partial B_r} \rho_\varepsilon \leq C\varepsilon^{1/2}.
\] (193)
Note the difference with respect to the situation in Subsection 2.1: here we have at
our disposal only the weaker upper bound \( \int_{B_r} |\nabla \rho_\varepsilon|^2 \leq C\varepsilon \). Since (188) implies that
\[
|\rho_\varepsilon(x) - \rho_\varepsilon(y)| \leq C\varepsilon^{1/2}, \quad \forall x, y \in \partial B_r(y),
\] (194)
we deduce (192) from (193)–(194) in the case \( b = 0 \). The general case follows
again by applying a Möbius transformation.

An immediate consequence of (192) is that \( c_0 = 1 \). Therefore, the bound
\( \int_{B_r(b)} |\nabla \rho_\varepsilon|^2 \leq C\varepsilon \) implies that
\[
\rho_\varepsilon \to 1 \text{ in } H^1(B_r(b)).
\] (195)
Next we use the harmonic extensions of \( \rho_\varepsilon \) and \( \varphi_\varepsilon \) to construct the comparison
map \( u_\varepsilon = \tilde{\rho}_\varepsilon e^{i\tilde{\varphi}_\varepsilon} \) on \( B_r(b) \). Clearly,
\[
E_\varepsilon(u_\varepsilon; B_r(b)) \leq E_\varepsilon(v_\varepsilon; B_r(b)) \leq \int_{B_r(b)} |\nabla \tilde{\varphi}_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla \tilde{\rho}_\varepsilon|^2.
\] (196)
Since \( \int_{B_r(b)} |\nabla \tilde{\rho}_\varepsilon|^2 \leq \int_{B_r(b)} |\nabla \rho_\varepsilon|^2 \), we deduce from (196) that
\[
E_\varepsilon(u_\varepsilon; B_r(b)) = \int_{B_r(b)} \rho_\varepsilon^2 |\nabla \varphi_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla \rho_\varepsilon|^2 \leq \int_{B_r(b)} |\nabla \tilde{\varphi}_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla \tilde{\rho}_\varepsilon|^2
\leq \int_{B_r(b)} |\nabla \tilde{\varphi}_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla \rho_\varepsilon|^2.
\] (197)
Therefore, \( \int_{B_r(b)} \rho_\varepsilon^2 |\nabla \varphi_\varepsilon|^2 \leq \int_{B_r(b)} |\nabla \tilde{\varphi}_\varepsilon|^2 \), and we obtain that
\[
\int_{B_r(b)} |\nabla u_\varepsilon|^2 = \int_{B_r(b)} \rho_\varepsilon^2 |\nabla \varphi_\varepsilon|^2 + |\nabla \rho_\varepsilon|^2 \leq \int_{B_r(b)} |\nabla \tilde{\varphi}_\varepsilon|^2 + C\varepsilon.
\] (198)
Next, consider a subsequence such that $u_{\varepsilon_n} \rightharpoonup u$ weakly in $H^1(B_r(b))$. By (195), $u = e^{i\varphi_0} = u_0$ on $\partial B_r(b)$, whence

$$\int_{B_r(b)} |\nabla u_0|^2 \leq \int_{B_r(b)} |\nabla u|^2. \quad (199)$$

Finally, by (198) and (191) we have

$$\int_{B_r(b)} |\nabla u|^2 \leq \limsup \int_{B_r(b)} |\nabla u_{\varepsilon_n}|^2 = \int_{B_r(b)} |\nabla u_0|^2. \quad (200)$$

Combining (199) with (200) we get that $u = u_0$ and then deduce the strong convergence (up to passing to a subsequence), $u_{\varepsilon_n} \to u_0$ in $H^1(B_r(b))$.

Next we write in $B_r$, $\varphi_{\varepsilon} = \tilde{\varphi}_{\varepsilon} + \psi_{\varepsilon}$, analogously to the notation we used in the proof of Theorem 3 (i.e., $\psi_{\varepsilon} = 0$ on $\partial B_r$). Note that $\rho_{\varepsilon}, \varphi_{\varepsilon}$ and $\psi_{\varepsilon}$ satisfy the equations (60)–(61). Since $\varphi_{\varepsilon} |_{\partial B_r(b)}$ is bounded in $H^1(\partial B_r(b))$, it follows that

$$\|\tilde{\varphi}_{\varepsilon}\|_{H^{3/2}(B_r(b))} \leq C. \quad (201)$$

Then from Sobolev embeddings it follows that

$$\|\tilde{\varphi}_{\varepsilon}\|_{W^{1,4}(B_r(b))} \leq C. \quad (202)$$

From the invariance of the equation

$$\Delta \psi_{\varepsilon} = \div((1 - \rho_{\varepsilon}^2) \nabla \varphi_{\varepsilon}) \quad (203)$$

with respect to scalings it follows that the constant $A_4$ in the inequality

$$\|\nabla \psi_{\varepsilon}\|_{L^4(B_r(b))} \leq A_4 \|1 - \rho_{\varepsilon}^2\| \nabla \varphi_{\varepsilon}\|_{L^4(B_r(b))} \quad (204)$$

can be chosen independently of the radius $\tilde{r}$. We may assume that $\tilde{\beta}$ that was used to construct the bad discs satisfies in addition

$$1 - \tilde{\beta} < \frac{1}{4A_4}. \quad (205)$$

By (202)–(205) we get that

$$\|\nabla \psi_{\varepsilon}\|_{L^4(B_r(b))} \leq 2(1 - \tilde{\beta})A_4\left(C + \|\psi_{\varepsilon}\|_{L^4(B_r(b))}\right),$$

implying that

$$\|\nabla \psi_{\varepsilon}\|_{L^4(B_r(b))} \leq C \text{ and } \|\nabla \varphi_{\varepsilon}\|_{L^4(B_r(b))} \leq C. \quad (206)$$

Next we deduce from the equation satisfied by $\rho_{\varepsilon}$ in (60) and elliptic estimates that

$$\|\nabla (\rho_{\varepsilon} - \tilde{\rho}_{\varepsilon})\|_{L^p(B_r(b))} \leq C_p \|\Delta \rho_{\varepsilon}\|_{L^2(B_r(b))} \leq C_p \varepsilon^2 \|\nabla \varphi_{\varepsilon}\|_{L^4(B_r(b))} \leq C\varepsilon^2, \forall p < \infty. \quad (207)$$
In particular, we deduce from \((207)\) that \( \| \rho_\varepsilon - \bar{\rho}_\varepsilon \|_{L^\infty(B_r(b))} \leq C \varepsilon^2 \). Since \( \| 1 - \bar{\rho}_\varepsilon \|_{L^\infty(B_r(b))} \leq C \varepsilon^{1/2} \) by \((192)\) and the maximum principle, it follows that
\[
\| \rho_\varepsilon - 1 \|_{L^\infty(B_r(b))} \leq C \varepsilon^{1/2} . \tag{208}
\]
We clearly have:
\[
\bar{\rho}_\varepsilon \text{ and } \bar{\phi}_\varepsilon \text{ are bounded in } W^{j,p}_{\text{loc}}(B_{\bar{r}}(b)), \forall j, \forall p. \tag{209}
\]
Using \((208)\) in \((203)\), taking into account \((209)\), we can deduce, as in the Proof of Theorem 3 that \(\{ \nabla \phi_\varepsilon \}_{\varepsilon > 0}\) are uniformly bounded in \(W^{1,p}_{\text{loc}}(B_{\bar{r}/2}(b) \cap B_1)\), for all \(p > 1\). We can now conclude the proof of the \(C^k\)-convergence by induction as in the Proof of Theorem 3. \(\square\)

We will also need a version of Proposition 10 in a neighborhood of the boundary.

**Proposition 11.** Let \(b \in \partial B_1\) and \(r_1 > 0\) be such that \(B_{r_1}(b) \subset \bar{B}_1 \setminus \{a_1, \ldots, a_N\}\). Then, \(u_{\varepsilon_n} \to u_0 \in C^k(B_{r_1/2}(b) \cap B_1)\) for all \(k \geq 0\), where \(u_0\) is a smooth \(S^1\)-valued harmonic map satisfying \(u_0 = g\) on \(B_{r_1}(b) \cap \partial B_1\).

**Proof.** As in the Proof of Proposition 10 we may use Fubini to find \(\tilde{r} \in ((3/4)r_1, r_1)\) such that
\[
\int_{\partial B_{\tilde{r}/2}(b) \cap B_1} |\nabla \varphi_\varepsilon|^2 + \frac{1}{\varepsilon} |\nabla \rho_\varepsilon|^2 \leq C. \tag{210}
\]
Denoting by \(q\) any of the two points in \(\partial B_{\tilde{r}}(b) \cap \partial B_1\), we obtain by the Cauchy-Schwarz inequality that
\[
|\rho_\varepsilon(x) - 1| = |\rho_\varepsilon(x) - \rho_\varepsilon(q)| \leq C \varepsilon^{1/2}, \quad \forall x \in \partial B_{\tilde{r}}(b) \cap B_1, \tag{211}
\]
which is the analogue of \((192)\) in our setting. The rest of the proof follows by the same arguments as in the Proof of Proposition 10. \(\square\)

### 4.7. Conclusion of the Proof of Theorem 1

As explained in the Introduction, we may assume that \(\Omega = B_1\).

**Proof of Theorem 1.** The inequality \((8)\) is the result of Corollary 1. The convergence result \((9)\) follows from Proposition 10 and Proposition 11. The fact that \(d_j > 0\) for all \(j\) follows from \((115)\).

Next we prove that \(a_j \in B_1\) for all \(j\), that is, singularities cannot occur on the boundary. The proof is the same as that of \([8, \text{Theorem X.4}]\), so we just describe the main idea. By Pohozaev identity \((171)\) and Proposition 11 it follows that
\[
\int_{\partial B_1} \left| \frac{\partial u_*}{\partial r} \right|^2 < \infty. \tag{212}
\]
Since by Proposition 7 we also have \(u_{\varepsilon_n} \rightharpoonup u_*\) weakly in \(W^{1,p}_1\), for all \(p \in (1, 2)\) it follows that \(u_* \in W^{1,p}_1(B_1; S^1)\) for all \(p \in [1, 2)\). Therefore, all the hypotheses
A Variational Singular Perturbation Problem Motivated

of [8, Lemma X.14] are satisfied, and we can conclude that \( u_* \) is smooth in a neighborhood of \( \partial B_1 \).

Finally we show that \( u_* \) is the canonical harmonic map associated with \( g \), the singularities and their degrees. By Proposition 7 we can pass to the limit \( \varepsilon \to 0^+ \) in (176) and deduce that
\[
\frac{\partial}{\partial x_1}(u_* \wedge (u_*)_{x_1}) + \frac{\partial}{\partial x_2}(u_* \wedge (u_*)_{x_2}) = 0 ,
\]
but by [8, Remark I.1], the only \( S^1 \)-valued harmonic map in \( W^{1,1}(\Omega) \) satisfying (173) is the canonical one.

\[\square\]

5. Proof of Theorem 2

5.1. An Improved Upper Bound for \( E_\varepsilon(u_\varepsilon) \)

We begin with the easy part, the upper bound, in the estimate (25).

**Proposition 12.** Under the assumptions of Theorem 1 we have
\[
\limsup_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon) - \frac{2\pi D}{\varepsilon} \leq \frac{d^2}{2} H_{1/2}(g, H_D(\partial \Omega)).
\]

(214)

In fact, for each fixed \( \varepsilon > 0 \), we have
\[
E_\varepsilon(u_\varepsilon) - \frac{2\pi D}{\varepsilon} \leq d^2_{H_{1/2}}(g, H_D(\partial \Omega)).
\]

(215)

**Proof.** As before we assume w.l.o.g. that \( \Omega = B_1 \). Fix any \( b \in B_1^D \). We know from Subsection 3.3 that
\[
U_{b,\varepsilon}(z) = |B_b(z)|^\varepsilon \left( \frac{B_b(z)}{|B_b(z)|} \right)
\]
is a minimizer for \( E_\varepsilon \) for its own boundary data, with \( E_\varepsilon(U_{b,\varepsilon}) = \frac{2\pi D}{\varepsilon} \). Set \( \tilde{\rho}_\varepsilon := |U_{b,\varepsilon}| = |B_b(z)|^\varepsilon \) and write
\[
U_{b,\varepsilon}(z) = \tilde{\rho}_\varepsilon(z) \exp(i\Theta(z)) .
\]

(216)

Note that although \( \Theta \) is well-defined only locally in \( \overline{B}_1 \setminus \{b_1, \ldots, b_D\} \), its gradient \( \nabla \Theta \) is globally defined. Let \( \psi \) be a smooth lifting of \( g/B_b|_{\partial B_1} \), that is, \( g = e^{i\psi} B_b \) on \( \partial B_1 \), and let \( \tilde{\psi} \) denote the harmonic extension of \( \psi \) to \( B_1 \). We set \( v_\varepsilon = e^{i\tilde{\psi}} U_{b,\varepsilon} \) and note that \( v_\varepsilon = g \) on \( \partial B_1 \). Using \( |v_\varepsilon| = |U_{b,\varepsilon}| = \tilde{\rho}_\varepsilon \) we get
\[
E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(v_\varepsilon) = \int_{B_1} \varepsilon^{-2} |\nabla \tilde{\rho}_\varepsilon|^2 + \tilde{\rho}_\varepsilon^2 (|\nabla \Theta|^2 + 2 \nabla \Theta \cdot \nabla \tilde{\psi} + |\nabla \tilde{\psi}|^2)
\]
\[
= E_\varepsilon(U_{b,\varepsilon}) + 2 \int_{B_1} \tilde{\rho}_\varepsilon^2 \nabla \Theta \cdot \nabla \tilde{\psi} + \int_{B_1} \tilde{\rho}_\varepsilon^2 |\nabla \tilde{\psi}|^2
\]
\[
= \frac{2\pi D}{\varepsilon} + 2 \int_{B_1} \tilde{\rho}_\varepsilon^2 \nabla \Theta \cdot \nabla \tilde{\psi} + \int_{B_1} \tilde{\rho}_\varepsilon^2 |\nabla \tilde{\psi}|^2 .
\]

(217)
Next we recall that $\Theta$ is a harmonic conjugate of $h := (1/\epsilon) \ln \bar{\rho}_\epsilon = \ln |B_b|$. The function $h$ is defined globally in $B_1$, having singularities at the points $b_1, \ldots, b_D$. Moreover,

$$h = 0 \quad \text{and} \quad \frac{\partial \Theta}{\partial \nu} = -\frac{\partial h}{\partial \tau} = 0 \quad \text{on} \quad \partial B_1. \quad (218)$$

Therefore,

$$\int_{B_1} \bar{\rho}_\epsilon^2 \nabla \Theta \cdot \nabla \tilde{\psi} = -\int_{B_1} \text{div}(\bar{\rho}_\epsilon^2 \nabla \Theta) \tilde{\psi} + \int_{\partial B_1} \bar{\rho}_\epsilon^2 \left( \frac{\partial \Theta}{\partial \nu} \right) \tilde{\psi} = 0, \quad (219)$$

where we used $\text{div}(\bar{\rho}_\epsilon^2 \nabla \Theta) = 0$ in $B_1$ and (218) on $\partial B_1$. Plugging (219) into (217) yields

$$E_\epsilon(u_\epsilon) \leq E_\epsilon(v_\epsilon) \leq \frac{2\pi D}{\epsilon} + \int_{B_1} \bar{\rho}_\epsilon^2 |\nabla \tilde{\psi}|^2. \quad (220)$$

Since the configuration $b \in B_1^D$ is arbitrary we deduce from the definition (16) of $d_{H/2}^2(g, \mathcal{H}_D(\partial B_1))$ and (220) that (215) holds, whence also (214).

5.2. The Limit of $\ln \rho_\epsilon/\epsilon$ and $(\rho_\epsilon - 1)/\epsilon$

We begin with a local $L^\infty$-bound for $|\nabla \rho_\epsilon|/\epsilon$, away from $\partial B_1$ and the points $a_1, \ldots, a_N$.

Lemma 7. For every small $\eta > 0$ we have

$$|\nabla \rho_\epsilon|/\epsilon \leq C_\eta \quad \text{on} \quad B_1-\eta \setminus \bigcup_{j=1}^N B_\eta(a_j), \quad \forall \epsilon \in (0, 1). \quad (221)$$

Proof. For simplicity we now drop the subscript $\epsilon$. From Corollary 4 we get that

$$\int_{\partial B_1} |u_\nu|^2 + \frac{\rho_\nu^2}{\epsilon^2} = \int_{\partial B_1} |u_\tau|^2 \leq C. \quad (222)$$

Therefore,

$$\int_{\partial B_1} (|u_x|^2 + |u_y|^2) + \frac{1}{\epsilon^2}(\rho_x^2 + \rho_y^2) = \int_{\partial B_1} (|u_\nu|^2 + |u_\tau|^2) + \frac{1}{\epsilon^2}(\rho_\nu^2 + \rho_\tau^2) \leq C. \quad (222)$$

Let us denote, as in (172), $U = (u, \left(\frac{1}{\epsilon^2} - 1\right)^{1/2}\rho)$ and consider the two harmonic functions $h_1 = |U_x|^2 - |U_y|^2$ and $h_2 = 2U_x \cdot U_y$. From (222) we deduce that

$$\int_{\partial B_1} |h_1| = \int_{\partial B_1} \left| U_x^2 - U_y^2 \right| \leq \int_{\partial B_1} U_x^2 + U_y^2 \leq C. \quad (223)$$
Similarly,

\[
\int_{\partial B_1} |h_2| \leq \int_{\partial B_1} 2|U_x||U_y| \leq \int_{\partial B_1} |U_x|^2 + |U_y|^2 \leq C.
\]  

(224)

From (223)–(224) and the Poisson formula it follows that

\[
\|h_1\|_{L^\infty(B_1-\eta)}, \; \|h_2\|_{L^\infty(B_1-\eta)} \leq C_\eta.
\]  

(225)

Thanks to Theorem 1 we also have,

\[
|\nabla u| \leq C_\eta \text{ on } B_1 \setminus \bigcup_{j=1}^N B_\eta(a_j).
\]  

(226)

Combining (225) with (226) yields

\[
\left| \left( \frac{\rho_x}{\varepsilon} \right)^2 - \left( \frac{\rho_y}{\varepsilon} \right)^2 \right| \leq C_\eta \text{ and } \left| \left( \frac{\rho_x}{\varepsilon} \right) \left( \frac{\rho_y}{\varepsilon} \right) \right| \leq C_\eta \text{ on } B_{1-\eta} \setminus \bigcup_{j=1}^N B_\eta(a_j).
\]  

(227)

Since \((\rho_x^2 - \rho_y^2)^2 + (2\rho_x\rho_y)^2 = (\rho_x^2 + \rho_y^2)^2\), (221) follows from (227).

The next result provides a crucial bound for the energy away from the singularities of \(u_*\).

**Proposition 13.** Let \(\eta > 0\) satisfy

\[
\eta < \frac{1}{2} \min_{i \neq j} |a_i - a_j| \text{ and } \eta < \min_j (1 - |a_j|).
\]  

(228)

Then,

\[
E_\varepsilon(u_\varepsilon; B_1 \setminus \bigcup_{j=1}^N B_\eta(a_j)) \leq C_\eta.
\]  

(229)

**Proof.** For \(j = 1, \ldots, N\) we denote

\[
m_j = m_j(\varepsilon, \eta) = \min_{\partial B_\eta(a_j)} \rho_\varepsilon \text{ and } M_j = M_j(\varepsilon, \eta) = \max_{\partial B_\eta(a_j)} \rho_\varepsilon.
\]  

(230)

Thanks to (221) we have

\[
M_j - m_j \leq C_\eta \varepsilon, \; j = 1, \ldots, N.
\]

Actually, connecting pairs of circles from \(\{\partial B_\eta(a_j)\}_{j=1}^N\) to each other by segments allows us to deduce from (221) that

\[
|M_j - m_i| \leq C_\eta \varepsilon, \; i, j = 1, \ldots, N.
\]  

(231)
Let us denote $\bar{m} = \min_j m_j$. By (231) and (221) we have
\[
|\rho_e - \bar{m}| \leq C_{\eta} \varepsilon \text{ on } \bigcup_{j=1}^{N} (B_{\eta}(a_j) \setminus B_{\eta/2}(a_j)).
\] (232)

Next we define a function $S \in H^1_0(B_1)$ by
\[
S(x) = \begin{cases} 
1 - \rho_e(x) & x \in B_1 \setminus \bigcup_{j=1}^{N} B_{\eta}(a_j), \\
1 - \frac{1}{\bar{m}}((|x - a_j| - \frac{\eta}{2})\rho_e(x) + (\eta - |x - a_j|)\bar{m}) & x \in B_{\eta}(a_j) \setminus B_{\eta/2}(a_j), 1 \leq j \leq N, \\
1 - \frac{1}{\bar{m}} & x \in B_{\eta/2}(a_j), 1 \leq j \leq N.
\end{cases}
\] (233)

Thanks to (221) and (232) we have
\[
\int_{B_1} |\nabla S|^2 \leq \int_{B_1 \setminus \bigcup_{j=1}^{N} B_{\eta}(a_j)} |\nabla \rho_e|^2 + C_{\eta} \varepsilon^2
\]
\[
\leq \varepsilon^2 E_{\varepsilon}(u_\varepsilon; B_1 \setminus \bigcup_{j=1}^{N} B_{\eta}(a_j)) + C_{\eta} \varepsilon^2.
\] (234)

Next we apply Trudinger’s inequality to $S$, similarly to the way it was used in the Proof of [8, Lemma X.5]. It yields, for some universal constants $\sigma_1, \sigma_2$,
\[
\int_{B_1} \exp \left( \frac{|S|}{\sigma_1 \|
abla S\|_2} \right) \leq \sigma_2 |B_1|.
\] (235)

In particular, we obtain from (235) that
\[
|B_{\eta/2}(a_1)| \exp \left( \frac{1 - \bar{m}}{\sigma_1 \|
abla S\|_2} \right) \leq \sigma_2 |B_1|,
\]
which after some manipulations and application of (234) leads to
\[
1 - \bar{m} \leq C_{\eta} \varepsilon \left( E_{\varepsilon}(u_\varepsilon; B_1 \setminus \bigcup_{j=1}^{N} B_{\eta}(a_j)) + 1 \right)^{1/2}.
\] (236)

Next, the same argument that was used in the Proof of Proposition 4 gives
\[
E_{\varepsilon}(u_\varepsilon; \bigcup_{j=1}^{N} B_{\eta}(a_j)) \geq \frac{2}{\varepsilon} (2\pi D) \int_{0}^{\bar{m}} t \, dt = \frac{2\pi D}{\varepsilon} \bar{m}^2.
\] (237)

Combining (236)–(237) with the upper bound from (80) yields
\[
\frac{2\pi D}{\varepsilon} \bar{m}^2 + C_{\eta} \frac{(1 - \bar{m})^2}{\varepsilon^2} \leq \frac{2\pi D}{\varepsilon} - C_{\eta},
\]

implying that
\[
1 - \bar{m} \leq C_{\eta} \varepsilon.
\] (238)

Finally, plugging (238) in (237) yields $E_{\varepsilon}(u_\varepsilon; \bigcup_{j=1}^{N} B_{\eta}(a_j)) \geq \frac{2\pi D}{\varepsilon} - C_{\eta}$, which together with (80) leads to (229). \qed
In the course of the Proof of Proposition 13 we also obtained the necessary information needed to prove that $1 - \rho_\varepsilon = O(\varepsilon)$ locally in $B_1 \setminus \{a_1, \ldots, a_N\}$. More precisely, we have

**Proposition 14.** For every small $\eta > 0$ we have

$$1 - \rho_\varepsilon \leq C_\eta \varepsilon \text{ in } B_{1-\eta} \setminus \bigcup_{j=1}^N B_\eta(a_j). \tag{239}$$

**Proof.** First, combining (238) with (231) yields

$$1 - \rho_\varepsilon \leq C_\eta \varepsilon \text{ on } \bigcup_{j=1}^N \partial B_\eta(a_j). \tag{240}$$

Any point $x \in B_{1-\eta} \setminus \bigcup_{j=1}^N B_\eta(a_j)$ can be connected to the closest circle, say $\partial B_\eta(a_{j_0})$. Using (240) in conjunction with (221) we conclude that $(1 - \rho_\varepsilon)(x) \leq C_\eta \varepsilon$. \hfill \Box

Next we strengthen further our estimate for $1 - \rho_\varepsilon$.

**Proposition 15.**

$$\lim_{\varepsilon \to 0} \frac{\rho_\varepsilon - 1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\ln \rho_\varepsilon}{\varepsilon} = \Phi_0 \text{ in } C^{k}_{\text{loc}}(\overline{B}_1 \setminus \{a_1, \ldots, a_N\}), \text{ for all } k \geq 1, \tag{241}$$

where $\Phi_0$ is the solution of (24).

**Proof.** The proof is divided to several steps.

Step 1: Convergence of $\frac{\rho_\varepsilon - 1}{\varepsilon}$ in $C^{k}_{\text{loc}}(B_1 \setminus \{a_1, \ldots, a_N\})$.

Let $x_0 \in B_1 \setminus \{a_1, \ldots, a_N\}$ be given. Choose $\eta > 0$ such that $B_\eta(x_0) \subset B_1 \setminus \{a_1, \ldots, a_N\}$. By (60) we have

$$\Delta \left( \frac{\rho_\varepsilon - 1}{\varepsilon} \right) = \varepsilon \rho_\varepsilon |\nabla \varphi_\varepsilon|^2. \tag{242}$$

Denoting as usual the harmonic extension of $\rho_\varepsilon$ by $\tilde{\rho_\varepsilon}$, we set $w_\varepsilon := \frac{\tilde{\rho_\varepsilon} - 1}{\varepsilon}$. It is a harmonic function that thanks to Proposition 14 satisfies

$$\|w_\varepsilon\|_{L^\infty(\partial B_\eta(x_0))} \leq C. \tag{243}$$

It follows that

$$\|w_\varepsilon\|_{C^k(B_{3\eta/4}(x_0))} \leq C, \forall k \geq 1. \tag{244}$$

In particular,

$$w_\varepsilon \to \Phi \text{ in } C^k(B_{\eta/2}(x_0)) \text{ for all } k, \tag{245}$$
and the limit \( \Phi \) is a harmonic function. Now, by (242) the function \( f_\varepsilon := \left( \frac{\rho_\varepsilon - 1}{\varepsilon} \right) - w_\varepsilon \) satisfies
\[
\begin{cases}
\Delta f_\varepsilon = \varepsilon \rho_\varepsilon |\nabla \varphi_\varepsilon|^2 & \text{in } B_\eta(x_0) \\
f_\varepsilon = 0 & \text{on } \partial B_\eta(x_0).
\end{cases}
\tag{246}
\]
It follows from (246) and Theorem 1 that \( \|f_\varepsilon\|_{C^k(B_\eta(x_0))} = O(\varepsilon) \), for all \( k \geq 1 \), which in conjunction with (245) yields that
\[
\frac{\rho_\varepsilon - 1}{\varepsilon} \to \Phi \text{ in } C^k(B_{\eta/2}(x_0)) \text{ for all } k.
\tag{247}
\]
Since \( x_0 \) is arbitrary, we deduce the convergence
\[
\frac{\rho_\varepsilon - 1}{\varepsilon} \to \Phi \text{ in } C^k_{\text{loc}}(B_1 \setminus \{a_1, \ldots, a_N\}).
\tag{248}
\]
Step 2: Convergence of \( \frac{\ln \rho_\varepsilon}{\varepsilon} \) in \( C^k_{\text{loc}}(B_1 \setminus \{a_1, \ldots, a_N\}) \).
To deduce the same convergence for \( \ln \rho_\varepsilon/\varepsilon \), we note first that this function satisfies in \( B_1 \setminus \{a_1, \ldots, a_N\} \) the equation
\[
\Delta(\ln \frac{\rho_\varepsilon}{\varepsilon}) = \varepsilon \left( |\nabla \varphi_\varepsilon|^2 - \left( \frac{|\nabla \rho_\varepsilon|}{\varepsilon \rho_\varepsilon} \right)^2 \right).
\tag{249}
\]
By Theorem 1 and (248) we obtain that locally in \( B_1 \setminus \{a_1, \ldots, a_N\} \), the R.H.S. of (249) is \( O(\varepsilon) \). Therefore, by the same argument as in the first part of the proof we can deduce that also
\[
\frac{\ln \rho_\varepsilon}{\varepsilon} \to \Phi \text{ in } C^k_{\text{loc}}(B_1 \setminus \{a_1, \ldots, a_N\}),
\tag{250}
\]
noting that the limit must be the same \( \Phi \) since locally in \( B_1 \setminus \{a_1, \ldots, a_N\} \) we have
\[
\frac{\ln \rho_\varepsilon}{\varepsilon} - \frac{\rho_\varepsilon - 1}{\varepsilon} = O\left( \frac{(1 - \rho_\varepsilon)^2}{\varepsilon} \right) = O(\varepsilon).
\]
Step 3: Convergence of \( \frac{\ln \rho_\varepsilon}{\varepsilon} \) and \( \frac{\rho_\varepsilon - 1}{\varepsilon} \) up to the boundary.
We recall that so far we haven’t shown even that \( |\nabla \rho_\varepsilon|/\varepsilon \) is bounded up to the boundary. Let \( \eta \) satisfy
\[
0 < \eta < \min\{1 - |a_j|\}_{j=1}^N.
\tag{251}
\]
Fix any point \( b \in \partial B_1 \). By Proposition 13 we have
\[
E_\varepsilon(u_\varepsilon; B_\eta(b) \cap B_1) \leq C.
\]
Therefore, by Fubini we can choose \( \tilde{\eta} \in (\eta/2, \eta) \) such that
\[
\int_{\partial B_{\tilde{\eta}}(b) \cap B_1} \frac{|\nabla \rho_\varepsilon|^2}{\varepsilon^2} \leq C;
\tag{252}
\]
A Variational Singular Perturbation Problem Motivated
note the improvement over (210). Denoting by \( q \) any of the two points in \( \partial B_\tilde{\eta}(b) \cap \partial B_1 \), we obtain by the Cauchy-Schwarz inequality that

\[
|\rho_\varepsilon(x) - 1| = |\rho_\varepsilon(x) - \rho_\varepsilon(q)| \leq C \varepsilon, \quad \forall x \in \partial B_\tilde{\eta}(b) \cap B_1, \tag{253}
\]

which is stronger than (211). We can now proceed as in the proof of the estimate around an interior point. In fact, setting \( w_\varepsilon := \tilde{\rho}_\varepsilon - \frac{1}{\varepsilon} \), where, as usual, \( \tilde{\rho}_\varepsilon \) denotes the harmonic extension of \( \rho_\varepsilon \) from \( \partial (B_\tilde{\eta}(b) \cap B_1) \) to \( B_\tilde{\eta}(b) \cap B_1 \), we have thanks to (253) that

\[
\|w_\varepsilon\|_{L^\infty(\partial(B_\tilde{\eta}(b) \cap B_1))} \leq C. \tag{254}
\]

Therefore, analogously to (244) we have

\[
\|w_\varepsilon\|_{C^k_{\text{loc}}(B_\tilde{\eta}(b) \cap B_1)} \leq C, \quad \forall k \geq 1. \tag{255}
\]

This allows us to repeat the argument of Step 2, using again the equation (242), to deduce that

\[
\frac{\rho_\varepsilon - 1}{\varepsilon} \to \Phi \text{ in } C^k(B_{\tilde{\eta}/2}(b) \cap B_1) \text{ for all } k. \tag{256}
\]

We can then argue as above to also obtain that

\[
\frac{\ln \rho_\varepsilon}{\varepsilon} \to \Phi \text{ in } C^k(B_{\tilde{\eta}/2}(b) \cap B_1) \text{ for all } k. \tag{257}
\]

Since the point \( b \in \partial B_1 \) is arbitrary, we deduce that both convergences, \( \frac{\rho_\varepsilon - 1}{\varepsilon} \to \Phi \) and \( \frac{\ln \rho_\varepsilon}{\varepsilon} \to \Phi \), hold in \( C^k \)-norm in a neighborhood of the boundary.

Step 4: Identification of the limit \( \Phi \) as \( \Phi_0 \)

We already know that \( \Phi \) is a harmonic function in \( B_1 \setminus \{a_1, \ldots, a_N\} \), which is continuous in \( B_1 \setminus \{a_1, \ldots, a_N\} \) and satisfies \( \Phi = 0 \) on \( \partial B_1 \). Recall the Hopf differentials \( \{\chi_\varepsilon\} \) introduced in Subsection 4.4. In the Proof of Lemma 7 we showed that \( \{\chi_\varepsilon\} \) are bounded in \( L^\infty(\text{loc}) (B_1) \) (see (225)). Therefore, we have \( \chi_\varepsilon \to \chi_* \) in \( C^k_{\text{loc}}(B_1) \) where \( \chi_* \) is holomorphic in \( B_1 \) and locally bounded. In fact, thanks to Step 3 and Theorem 1 we can assert that the convergence actually holds in \( C^k(B_1) \).

On the other hand, because of the convergences

\[
\nabla u_\varepsilon \to \nabla (e^{i\phi_*}) \quad \text{and} \quad \frac{\nabla \rho_\varepsilon}{\varepsilon} \to \nabla \Phi \text{ in } C^k(B_1 \setminus \{a_j\}_j^{N=1}) \tag{258}
\]

established in Theorem 1, and the previous steps, we have in \( B_1 \setminus \{a_j\}_j^{N=1} \) that

\[
\chi_* = \left( \frac{\partial \phi_*}{\partial x} \right)^2 - \left( \frac{\partial \phi_*}{\partial y} \right)^2 - 2i \frac{\partial \phi_*}{\partial x} \cdot \frac{\partial \phi_*}{\partial y} + \left( \frac{\partial \Phi}{\partial x} \right)^2 - \left( \frac{\partial \Phi}{\partial y} \right)^2
\]

\begin{equation}
-2i \left( \frac{\partial \Phi}{\partial x} \right) \cdot \left( \frac{\partial \Phi}{\partial y} \right). \tag{259}
\end{equation}

Here and in the sequel we use \( \phi_* \) to denote the phase of \( u_* \), but we keep in mind that this function is defined only \( \text{locally} \) in \( B_1 \setminus \{a_j\}_j^{N=1} \), and even there it is determined uniquely only up to an additive constant in \( 2\pi \mathbb{Z} \). Yet, the gradient \( \nabla \phi_* \) is globally
defined in $B_1 \setminus \{a_j\}_{j=1}^N$. Since $\chi_*$ belongs to $L^\infty(B_1)$, we may take the modulus in both sides of (259) and deduce that

$$|\nabla \Phi|^2 = |\nabla \varphi_*|^2 + O(1) \text{ in } B_1 \setminus \{a_j\}_{j=1}^N.$$  \hfill (260)

Since $|\nabla \varphi_*| \in L^p(B_1)$ for all $p \in [1, 2)$ it follows from (260) that also $|\nabla \Phi| \in L^p(B_1)$ for all $p \in [1, 2)$.

Since $\Phi$ is harmonic in $B_1 \setminus \{a_1, \ldots, a_N\}$ and $|\nabla \Phi| \in L^1_{\text{loc}}(B_1)$, we must have

$$\Delta \Phi = \sum_{j=1}^N (2\pi c_j) \delta_{a_j} \text{ in the distributions sense,}$$  \hfill (261)

for some constants $\{c_j\}_{j=1}^N$. Therefore we have

$$\Phi(z) = \sum_{j=1}^N c_j \ln |z - a_j| + H \text{ in } B_1$$  \hfill (262)

for some smooth harmonic function $H$.

We still need to determine the values of $\{c_j\}_{j=1}^N$. Fix any $j$ and assume for simplicity of notation that $a_j = 0$. In a punctured neighborhood of 0, $B_\eta^* = B_\eta \setminus \{0\}$, we have

$$e^{i\varphi_*} = e^{id_j \theta + f_j},$$  \hfill (263)

where $f_j$ is a smooth harmonic function in a neighborhood of 0 (including 0). Similarly, in $B_\eta^*$ we have also

$$\Phi(z) = c_j \ln |z| + h_j,$$  \hfill (264)

with $h_j$ having the same properties as $f_j$. Rewriting (259) as

$$\left(\frac{\partial \Phi}{\partial z}\right)^2 = -\left(\frac{\partial \varphi_*}{\partial z}\right)^2 + \chi_*/4,$$  \hfill (265)

and plugging (263)–(264), yields

$$\left(c_j z + 2 \frac{\partial h_j}{\partial z}\right)^2 = -\left(-i \frac{d_j}{z} + 2 \frac{\partial f_j}{\partial z}\right)^2 + \chi_* \text{ in } B_\eta^*.$$  \hfill (265)

Multiplying (265) by $z^2$ and sending $z$ to zero gives $c_j^2 = d_j^2$, so that $c_j = \pm d_j$. Since $\Phi \leq 0$ (as the limit of $\ln \rho_\varepsilon/\varepsilon$) we conclude that $c_j = d_j$. Using this for all $j$’s in (261) clearly implies that $\Phi = \Phi_0$, the function given in (24). \hspace{1em} □
5.3. A Precise Asymptotic Estimate for the Energy

Our next objective is to prove the lower bound in (25). Recall that for the points \(a_1, \ldots, a_N\) and degrees \(d_1, \ldots, d_N\) given by Theorem 1 we associate the function \(\Phi_0\) satisfying (24) and its conjugate harmonic function \(\Theta_0\) (which is well-defined only locally in \(B_1 \setminus \{a_1, \ldots, a_N\}\); \(\Theta_0\) is unique up to an additive constant in \(2\pi \mathbb{Z}\) that we can fix arbitrarily. Once a representative of \(\Theta_0\) is fixed, the function \(\phi = \varphi_* - \Theta_0\) is well defined on \(\partial B_1\) and we denote by \(\tilde{\varphi}\) its harmonic extension to \(B_1\). We keep in mind that \(\tilde{\varphi}\) is determined uniquely up to an additive constant which is an integer multiple of \(2\pi\).

**Lemma 8.** For each small \(\lambda > 0\) we have

\[
E_\varepsilon(u_\varepsilon) \geq \frac{2\pi D}{\varepsilon} + \int_{B_1} |\nabla \tilde{\varphi}|^2 + o_\varepsilon(1) + o_\varepsilon^{(\lambda)}(1),
\]

where \(o_\varepsilon^{(\lambda)}(1)\) denotes a quantity that tends to 0 with \(\varepsilon\), for each fixed \(\lambda\), while \(o_\varepsilon(1)\) denotes a quantity that tends to 0 with \(\lambda\) (independently of \(\varepsilon\)).

**Proof.** Fix a small \(\lambda > 0\) and denote \(\Omega_\lambda = B_1 \setminus \bigcup_{j=1}^N B_\lambda(a_j)\). By Proposition 15 and Theorem 1 we have

\[
E_\varepsilon(u_\varepsilon; \Omega_\lambda) = \frac{1}{\varepsilon^2} \int_{\Omega_\lambda} |\nabla \rho_\varepsilon|^2 + \int_{\Omega_\lambda} \rho_\varepsilon^2 |\nabla \varphi_\varepsilon|^2
= \int_{\Omega_\lambda} |\nabla \Phi_0|^2 + \int_{\Omega_\lambda} |\nabla \varphi_*|^2 + o_\varepsilon^{(\lambda)}(1).
\]

Since \(\varphi_* = \Theta_0 + \tilde{\varphi}\), we have

\[
\int_{\Omega_\lambda} |\nabla \varphi_*|^2 = \int_{\Omega_\lambda} \left( |\nabla \Theta_0|^2 + 2 \nabla \Theta_0 \nabla \tilde{\varphi} + |\nabla \tilde{\varphi}|^2 \right)
= \int_{\Omega_\lambda} |\nabla \Theta_0|^2 + \int_{\Omega_\lambda} |\nabla \tilde{\varphi}|^2 + 2 \sum_{j=1}^N \int_{\partial B_\lambda(a_j)} \frac{\partial \Theta_0}{\partial v} \tilde{\varphi} + 2 \int_{\partial B_1} \frac{\partial \Theta_0}{\partial v} \tilde{\varphi}.
\]

Here \(v\) stands for the outward normal w.r.t. the domain \(\Omega_\lambda = B_1 \setminus \bigcup_{j=1}^N B_\lambda(a_j)\) on each component of its boundary. Next we use the fact that \(\frac{\partial \Theta_0}{\partial v} = -\frac{\partial \Phi_0}{\partial \tau}\) which implies in particular that \(\frac{\partial \Phi_0}{\partial \tau} = 0\) on \(\partial B_1\). Therefore

\[
\int_{\Omega_\lambda} |\nabla \varphi_*|^2 = \int_{\Omega_\lambda} |\nabla \Theta_0|^2 + \int_{\Omega_\lambda} |\nabla \tilde{\varphi}|^2 - 2 \sum_{j=1}^N \int_{\partial B_\lambda(a_j)} \frac{\partial \Phi_0}{\partial \tau} \tilde{\varphi}.
\]

Since \(\left|\frac{\partial \Phi_0}{\partial \tau}\right| \leq C\) on each \(\partial B_\lambda(a_j)\) we have

\[
\left|\int_{\partial B_\lambda(a_j)} \frac{\partial \Phi_0}{\partial \tau} \tilde{\varphi}\right| \leq C \|\tilde{\varphi}\|_\infty (2\pi \lambda) = o_\lambda(1).
\]
Note that above we could have replaced \( \| \tilde{\varphi} \|_\infty \) by \( \min_{m \in \mathbb{N}} \| \tilde{\varphi} - 2\pi m \|_\infty \). By (269)–(270) we get

\[
\int_{\Omega_\lambda} |\nabla \varphi_*|^2 = \int_{\Omega_\lambda} |\nabla \Theta_0|^2 + \int_{\Omega_\lambda} |\nabla \tilde{\varphi}|^2 + o_\lambda(1). \tag{271}
\]

By (267), (271) and the relation \( |\nabla \Theta_0| = |\nabla \Phi_0| \) we finally obtain that

\[
E_\varepsilon(u_\varepsilon; \Omega_\lambda) = 2 \int_{\Omega_\lambda} |\nabla \Phi_0|^2 + \int_{\Omega_\lambda} |\nabla \tilde{\varphi}|^2 + o_\lambda(1) + o_\varepsilon^{(\lambda)}(1). \tag{272}
\]

We continue to estimate the first integral on the R.H.S. of (272). First we define for each \( j \),

\[
m_j = m_j(\lambda, \varepsilon) = \min_{x \in \partial B_\lambda(a_j)} \rho_\varepsilon(x).
\]

By Proposition 15 we have

\[
2 \int_{\Omega_\lambda} |\nabla \Phi_0|^2 = 2 \sum_{j=1}^N \int_{\partial B_\lambda(a_j)} \Phi_0 \frac{\partial \Phi_0}{\partial \nu} = 2 \sum_{j=1}^N \int_{\partial B_\lambda(a_j)} \left( \frac{\rho_\varepsilon - 1}{\varepsilon} + o_\varepsilon^{(\lambda)}(1) \right) \frac{\partial \Phi_0}{\partial \nu}.
\tag{273}
\]

Note that thanks again to Proposition 15 we have

\[
\left\| \frac{\rho_\varepsilon - 1}{\varepsilon} \right\|_{L_\infty(\partial B_\lambda(a_j))} \leq \max_{x, y \in \partial B_\lambda(a_j)} |\Phi_0(x) - \Phi_0(y)| + o_\varepsilon^{(\lambda)}(1) \leq o_\lambda(1) + o_\varepsilon^{(\lambda)}(1).
\tag{274}
\]

Therefore, for each \( j \) we have

\[
\int_{\partial B_\lambda(a_j)} \left( \frac{\rho_\varepsilon - 1}{\varepsilon} + o_\varepsilon^{(\lambda)}(1) \right) \frac{\partial \Phi_0}{\partial \nu} = \int_{\partial B_\lambda(a_j)} \left( \frac{m_j - 1}{\varepsilon} + o_\lambda(1) + o_\varepsilon^{(\lambda)}(1) \right) \frac{\partial \Phi_0}{\partial \nu} = -2\pi d_j \left( \frac{m_j - 1}{\varepsilon} \right) + o_\lambda(1) + o_\varepsilon^{(\lambda)}(1),
\tag{275}
\]

where we used the fact that \( \int_{\partial B_\lambda(a_j)} \frac{\partial \Phi_0}{\partial \nu} = -2\pi d_j \) thanks to (24). Plugging (275) in (273) yields

\[
2 \int_{\Omega_\lambda} |\nabla \Phi_0|^2 = -4\pi \sum_{j=1}^N d_j \left( \frac{m_j - 1}{\varepsilon} \right) + o_\lambda(1) + o_\varepsilon^{(\lambda)}(1).
\tag{276}
\]

On the other hand, the argument based on the coarea formula, used in the Proof of Proposition 4 (and again in (123)), gives that

\[
E_\varepsilon(u_\varepsilon; B_\lambda(a_j)) \geq \frac{2}{\varepsilon} \int_{B_\lambda(a_j)} \rho_\varepsilon |\nabla \rho_\varepsilon||\nabla \varphi_\varepsilon| \geq \frac{2\pi d_j m_j^2}{\varepsilon}, \forall j.
\tag{277}
\]
Combining (272), (276) and (277) we obtain
\[
E_\varepsilon(u_\varepsilon) \geq \frac{4\pi}{\varepsilon} \sum_{j=1}^{N} d_j \left( \frac{m_j^2}{2} - (m_j - 1) \right) + \int_{\Omega_\lambda} |\nabla \tilde{\varphi}|^2 + o_\varepsilon(1) + o_\varepsilon(\lambda)(1)
\]
\[
= \frac{2\pi D}{\varepsilon} + \frac{2\pi}{\varepsilon} \sum_{j=1}^{N} d_j (m_j - 1)^2 + \int_{\Omega_\lambda} |\nabla \tilde{\varphi}|^2 + o_\varepsilon(1) + o_\varepsilon(\lambda)(1)
\]
\[
= \frac{2\pi D}{\varepsilon} + \int_{\Omega_\lambda} |\nabla \tilde{\varphi}|^2 + o_\varepsilon(1) + o_\varepsilon(\lambda)(1),
\]
where in the last estimate we used the fact that \(1 - m_j \leq C_\lambda \varepsilon\), implying that \((m_j - 1)^2/\varepsilon \leq C_\lambda \varepsilon = o_\varepsilon(\lambda)(1)\). The desired conclusion (266) follows from (278) since
\[
\int_{\bigcup_{j=1}^{N} B_\lambda(a_j)} |\nabla \tilde{\varphi}|^2 = o_\lambda(1).
\]

5.4. Conclusion of the Proof of Theorem 2

**Proof of Theorem 2.** Assertion (i) follows from Proposition 15. The inequality “\(\leq\)” in (25) was proved in Proposition 12. To prove the inequality “\(\geq\)” we use Lemma 8. We first fix \(\lambda\) and send \(\varepsilon\) to 0 to get
\[
\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) - \frac{2\pi D}{\varepsilon} \geq \int_{B_1} |\nabla \tilde{\varphi}|^2 + o_\lambda(1).
\]
(279)

Then, sending \(\lambda\) to 0 in (279) yields
\[
\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) - \frac{2\pi D}{\varepsilon} \geq \int_{B_1} |\nabla \tilde{\varphi}|^2,
\]
and the conclusion follows. Finally, assertion (iii) is a direct consequence of assertion (ii).

\hspace{1cm} \square

6. Proof of Proposition 1.

This short section is devoted to the proof Proposition 1 that provides an explicit expression for the “excess energy” \(d^2_{H_{1/2}}(g, H_D(\partial \Omega))\) and clarifies its relation with the renormalized energy \(W\).

**Proof of Proposition 1.** Let \(\Theta_0\) be a conjugate harmonic function of \(\Phi_0\) as described in the beginning of Subsection 5.3, but this time for a general simply connected domain \(\Omega\). We have
\[
d^2_{H_{1/2}}(g, H_D(\partial \Omega)) = \int_{\Omega} |\nabla \tilde{\varphi}|^2.
\]
(280)
where \( \tilde{\varphi} \) is the harmonic extension of the function \( \psi \) given on \( \partial \Omega \) as \( \psi = \varphi - \Theta_0 \), i.e., \( e^{i\psi} = g / f_0 \), with \( f_0 = U_0 \mid_{\partial \Omega} \) where \( U_0 = e^{i\Theta_0} \). Therefore,

\[
u_* = U_0 e^{i\tilde{\varphi}} \text{ in } \Omega.
\] (281)

Next we apply (17) twice, first for \( u_* \),

\[
\int_{\Omega_1} |\nabla u_*|^2 = 2\pi \left( \sum_{j=1}^{N} d_j^2 \right) \ln(1/\lambda) + W(a, d, g) + O(\lambda^2), \text{ as } \lambda \to 0^+,(282)
\]

and then for \( U_0 \),

\[
\int_{\Omega_1} |\nabla U_0|^2 = 2\pi \left( \sum_{j=1}^{N} d_j^2 \right) \ln(1/\lambda) + W(a, d, f_0) + O(\lambda^2), \text{ as } \lambda \to 0^+. (283)
\]

Since \( |\nabla U_0| = |\nabla \Theta_0| \) and \( |\nabla u_*| = |\nabla \varphi_*| \), we infer from (282)–(283) and (271) that

\[
W(a, d, g) = W(a, d, f_0) + \int_{\Omega} |\nabla \tilde{\varphi}|^2. \quad (284)
\]

An immediate consequence of (284) is that the minimum of \( \tilde{W}(a, d) \) (see (20)) is attained by \( f_0 \). Clearly (21) follows from (284) and (280).

Finally we turn to the proof of (22). Here we need an explicit expression for \( \tilde{W}(a, d) \) in the case \( \Omega = B_1 \). Since now we know that the minimum defining \( \tilde{W}(a, d) \) in (20) is attained by \( f_0 \), we can rely on the formula (283) and compute an asymptotic expansion for \( \int_{\Omega_1} |\nabla \Phi_0|^2 \) as \( \lambda \to 0 \). This can be done rather easily but a similar computation was already done in [17, Prop. 1]:

\[
\tilde{W}(a, d) = -2\pi \sum_{j \neq k} d_j d_k \ln |a_j - a_k| + 2\pi \sum_{j, k} d_j d_k \ln |1 - \bar{a}_j a_k|.
\] (285)

Finally, by (19) and (285) we obtain that

\[
W(a, d, g) - \tilde{W}(a, d) = \int_{\partial B_1} \tilde{\Phi}_0(g \times g_\tau) - 2\pi \sum_{j=1}^{N} d_j R_0(a_j)
\]

\[
-2\pi \sum_{j, k=1}^{N} d_j d_k \ln |1 - a_j \bar{a}_k|, \quad (286)
\]

and the result follows from (21).
Thus Problem (287) is equivalent to the following problem:

\[
\text{Problem of the energy limit of a problem defined on a thin film, } \Omega_h:=\Omega \times (0, h) \subset \mathbb{R}^3 \text{ when the thickness } h \text{ goes to zero. We fix } \varepsilon \text{ and for each } h>0 \text{ let } w_h = w_{h, \varepsilon} \text{ denote a minimizer for the problem}
\]

\[
\min \left\{ F_h(u):= \int_{\Omega \times (0, h)} \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla u|^2 + |\nabla u|^2 : u \in \mathcal{V}_h \right\},
\]

where

\[
\mathcal{V}_h = \left\{ u \in H^1(\Omega_h; \mathbb{R}^3) : u(x, y, z) = g(x, y) \text{ for } (x, y, z) \in \partial \Omega \times (0, h), u \perp e_3 \text{ on } \Omega \times \{0, h\} \right\},
\]

with \( e_3 \) denoting a unit vector in the direction of the z-axis.

Next, for any \( u \in \mathcal{V}_h \) we use rescaling to define \( \tilde{u} \in H^1(\Omega \times (0, 1); \mathbb{R}^3) \) by setting

\[
\tilde{u}(x, y, z) = u(x, y, hz).
\]

A simple computation yields that

\[
\tilde{F}_h(\tilde{u}) := h^{-1} F_h(u) = \int_{\Omega \times (0, 1)} \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla_{x,y,\tilde{u}}|^2 + |\nabla_{x,y,\tilde{u}}|^2 + \frac{1}{h^2} \int_{\Omega \times (0, 1)} \left( \frac{1}{\varepsilon^2} - 1 \right) \frac{\partial |\tilde{u}|^2}{\partial z} + \left| \frac{\partial \tilde{u}}{\partial z} \right|^2.
\]

Thus Problem (287) is equivalent to the following problem:

\[
\min \left\{ \tilde{F}_h(\tilde{u}) : \tilde{u} \in H^1(\Omega \times (0, 1); \mathbb{R}^3), \tilde{u} = g \text{ on } \partial \Omega \times (0, 1), \tilde{u} \perp e_3 \right\}
\]

on \( \Omega \times \{0, 1\} \),

for which the minimizer is given by \( \tilde{w}_h(x, y, z) = w_h(x, y, hz) \).

**Proposition 16.** For a subsequence we have

\[
\lim_{h \to 0} \tilde{w}_{h,n} = U_{\varepsilon},
\]

where \( U_{\varepsilon}(x, y, z) = u_{\varepsilon}(x, y) \), with \( u_{\varepsilon} \) being a minimizer for \( E_{\varepsilon} \) over \( H^1_g(\Omega) \).

**Proof.** Let \( u_{\varepsilon} \) be any minimizer for \( E_{\varepsilon} \) over \( H^1_g(\Omega) \). Clearly \( U_{\varepsilon} \) is an admissible map for (291), whence

\[
\tilde{F}_h(\tilde{w}_h) \leq \tilde{F}_h(U_{\varepsilon}) = E_{\varepsilon}(u_{\varepsilon}).
\]

It follows from (293) and (290) that

\[
\lim_{h \to 0} \int_{\Omega \times (0, 1)} \left| \frac{\partial |\tilde{w}_h|}{\partial z} \right|^2 + \left| \frac{\partial \tilde{w}_h}{\partial z} \right|^2 = 0.
\]
Let \( \tilde{w}_{hn} \rightarrow V_\varepsilon \) weakly in \( H^1(\Omega \times (0, 1); \mathbb{R}^3) \). In particular, for the trace we have, \( \tilde{w}_{hn} \rightarrow V_\varepsilon \) strongly in \( L^2(\Omega \times [0, 1]; \mathbb{R}^3) \) and a.e., so that

\[
V_\varepsilon \perp e_3 \text{ on } \Omega \times \{0, 1\}. \tag{295}
\]

It follows from (294) that \( V_\varepsilon \) is independent of the \( z \)-variable, i.e., \( V_\varepsilon(x, y, z) = V_\varepsilon(x, y) \), while by (295) \( V_\varepsilon \) is \( \mathbb{R}^2 \)-valued. Passing to the limit in (293), using weak lower semicontinuity, we get

\[
\int_{\Omega \times (0, 1)} \left( \frac{1}{\varepsilon^2} - 1 \right) |\nabla_{x,y} V_\varepsilon||V_\varepsilon|^2 + |\nabla_{x,y} V_\varepsilon|^2 = E_\varepsilon(V_\varepsilon) \leq E_\varepsilon(u_\varepsilon). \tag{296}
\]

We conclude that \( V_\varepsilon(x, y) \) is a minimizer for \( E_\varepsilon \) over \( H^1_g(\Omega) \) and that \( \{\tilde{w}_{hn}\} \) converges strongly to \( V_\varepsilon \) in \( H^1(\Omega \times (0, 1); \mathbb{R}^3) \).

\[\Box\]

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