Refined estimates for some basic random walks on the symmetric and alternating groups

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Abstract
We give refined estimates for the discrete time and continuous time versions of some basic random walks on the symmetric and alternating groups $S_n$ and $A_n$. We consider the following models: random transposition, transpose top with random, random insertion, and walks generated by the uniform measure on a conjugacy class. In the case of random walks on $S_n$ and $A_n$ generated by the uniform measure on a conjugacy class, we show that in continuous time the $\ell^2$-cut-off has a lower bound of $(n/2) \log n$. This result, along with the results of Müller, Schlage-Puchta and Roichman, demonstrates that the continuous time version of these walks may take much longer to reach stationarity than its discrete time counterpart.

1 Introduction
This work is concerned with some basic random walks on the symmetric group, $S_n$, and the alternating group, $A_n$. Specifically, we are interested in the following models: (a) Random transposition and transpose top with random; (b) walks generated by the uniform measure on a conjugacy class, e.g., 4-cycles or $k_n$-cycles with $k_n$ an increasing function of $n$; (c) random insertion. Although these walks have been studied extensively, we obtain here results that either improved upon known estimates or complement those estimates.

The convergence of the random transposition walk on $S_n$ was studied by Diaconis and Shahshahani in [11]. We present a technical improvement of their fundamental result. This is motivated by the role played by random transposition in the comparison technique of [13]: any improvement upon the $\ell^2$ convergence of the random transposition walk has consequences for a wealth of other walks. We will illustrate this by obtaining the best known result for the

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random insertion walk. These results are also used in [32] to study certain time
in-homogeneous versions of the random insertion walk and this was indeed our
original motivation for developing the results presented here. For an overview
of results connected to the random transposition walk, see [10].

The transpose top with random walk is an interesting example mentioned
in [15] and in [8] but details of its $\ell^2$ analysis have never appeared in print.
(This walk should not be confused with the more classical top to random walk
studied in [4].) The estimates concerning this walk that are proved here are
used in [32] to obtain the best known convergence bounds for a class of time
in-homogeneous processes called semi-random transpositions.

Random walks associated with conjugacy classes other than the class of
transpositions have been studied by [21, 33, 22, 24, 25]. For most of those walks,
we show that $\ell^2$ convergence occurs at very different times for the discrete
time process and the continuous time process. Although this phenomenon is simple
to understand a posteriori, it is a bit surprising at first and is often overlooked.

Let us briefly describe our notation. On a finite group $G$ with identity
element $e$, the random walk started at $e$ driven by a given probability measure
$q$ is the process $X_n = \xi_1 \cdots \xi_n$ where the $\xi_i$ are independent $G$-valued random
variables with distribution $q$. The distribution of $X_n$ is $q^{(n)}$, the convolution
of $q$ with itself, $n$ times. Any such walk admits the uniform measure $u$ as
an invariant measure. It is reversible if and only if $q(x) = q(x^{-1})$ for all $x$.
The walks studied here all have this property. We are mostly interested in the
quantity ($\chi$-square distance)

$$d_2(q^{(n)}, u) = \left( |G| \sum_G |q^{(n)}(x) - u(x)|^2 \right)^{1/2}, \quad u \equiv 1/|G|.$$  

This is always an upper bound for $2\|q^{(n)} - u\|_{TV}$ where

$$\|q - p\|_{TV} = \sup_A \{q(A) - p(A)\}$$

is the total variation distance between the probability measures $p$ and $q$.

Given such a discrete time process, we also consider the associated continu-
tous time process whose distribution at time $t \in [0, \infty)$ is given by

$$h_t(x) = h_{q,t}(x) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} q^{(n)}(x).$$

We now state some of the results proved in this work. Random transposition
is the walk on the symmetric group $G = S_n$ driven by $q = q_{RT}$ where

$$q_{RT}(\tau) = \begin{cases} 
2/n^2 & \text{if } \tau = (i,j), \ 1 \leq i, j \leq n, \ i \neq j, \\
1/n & \text{if } \tau = e \\
0 & \text{otherwise.}
\end{cases}$$

**Theorem 1.1.** Let $q$ be the random transposition measure on the group $S_n$,
n > 14. For any $c \geq 0$ and $t \geq \frac{1}{2}(\log n + c)$, we have $d_2(q^{(t)}, u) \leq 2e^{-c}$.
In [11], Diaconis and Shahshahani proved this result with an unspecified constant $B$ instead of 2 in front of $e^{-c}$ and for large enough $n$. In this paper their approach is refined to obtained the bound stated above. We also prove a similar result in continuous time which turns out to be somewhat more difficult. Having good control of $d_2(h_{q(n)t}, u)$ is very useful in connection with the comparison techniques of [13]. See Section 4.3 where this is used to study the random insertion walk.

Transpose top with random is the process driven by $q(\tau) = 1/n$ if $\tau \in \{(1, i), i = 1, \ldots, n\}$ (where $(1, 1) = e$) and 0 otherwise.

Theorem 1.2. Let $q$ be the transpose top with random measure on the group $S_n$. For any $c \geq 0$ and $t \geq n (\log n + c)$, we have $d_2(q(t), u) \leq \sqrt{2e^{-c}}$.

To illustrate our results concerning walks driven by conjugacy classes, consider the measure $q_{cn}$ which, for each $n$, is uniform on $c_{kn} \subset S_n$, the conjugacy class of all cycles of odd length $k_n = 2m_n + 1$. The corresponding walk is on $A_n$.

Theorem 1.3. Fix $\epsilon \in (0, 1)$ and set $t_n = n/2 \log n$. Referring to the continuous time process with distribution $h_{c_n,t} = h_{q(c_n),t}$ associated to the cycle walk on $A_n$ described above, if $m_n$ tends to infinity with $n$, we have (with $u_n \equiv 1/|A_n| = 2/n$!)

$$\lim_{n \to \infty} d_2(h_{c_n,(1+\epsilon)t_n}, u_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d_2(h_{c_n,(1-\epsilon)t_n}, u_n) = \infty.$$ 

This result shows an $\ell^2$-cutoff at time $(n/2) \log n$. When $k_n < cn$ for some $c \in (0, 1)$, Roichman [24] shows that the associated discrete time process has a mixing time in $\ell^2$ of order $(n/k_n) \log n$. Roichman’s results are improved in [21, 33]. As $k_n = 2m_n + 1 \to \infty$, the discrete mixing time $(n/k_n) \log n$ is much smaller than the continuous cutoff time $(n/2) \log n$. The explanation is simple. Consider the eigenvalues of the walk driven by $q_{c_n}$, that is, the eigenvalues of the convolution operator $f \to f * q_{c_n} : \ell^2(u_n) \to \ell^2(u_n)$, call these eigenvalues $\alpha_i$. In continuous time, the $\ell^2$ cutoff time is controlled by the very large number of very small eigenvalues. These small eigenvalues contribute significantly in continuous time because they appear in the form $e^{-t(1-\alpha_i)}$. In discrete time, these small eigenvalues do not contribute much since they appear in the form $\alpha_i^t$. Although the explanation is simple, verifying that this is indeed the case is not an easy task. We will prove similar results for general conjugacy classes.

2 Review and notation

We refer the reader to [7, 30] for careful introduction to random walks on finite groups. We briefly review some of the needed material below.

2.1 Cutoffs

Many examples of random walks on groups that have been studied demonstrate a unique behavior called the cutoff phenomenon. This was first studied in
is the group of general linear transformations of $V$ character admits no nontrivial subrepresentation. The $G \ni x, y$ section can be found in greater detail in [7, 27]. We give a very brief review of these methods. All the material in this section can be found in greater detail in [11]. See also [9, 6, 29, 30].

**Definition 2.1.** Let $(G_n)_0^\infty$ be a sequence of finite groups and denote by $u_n$ the uniform measure on $G_n$. For each $n \geq 0$ consider the random walk on $G_n$ driven by the measure $q_n$. The sequence $((G_n, q_n))_0^\infty$ is said to have total variation cutoff (resp. $\ell^2$) if there is a sequence $(t_n)_0^\infty$ with $t_n \to \infty$ such that for any $\epsilon \in (0, 1)$

1. if $k_n = (1 + \epsilon)t_n$ then $d_{TV}(p_n^{(k_n)}, u_n) \to 0$ (resp. $d_2(p_n^{(k_n)}, u_n) \to 0$);
2. if $k_n = (1 - \epsilon)t_n$ then $d_{TV}(p_n^{(k_n)}, u_n) \to 0$ (resp. $d_2(p_n^{(k_n)}, u_n) \to 0$).

The sequence $((G_n, q_n))_0^\infty$ is said to demonstrate a total variation (resp. $\ell^2$) pre-cutoff if there exist constants $0 < a < b$ such that

1. $\liminf_{n \to \infty} d_{TV}(p_n^{(at_n)}, u_n) > 0$ (resp. $\liminf_{n \to \infty} d_2(p_n^{(at_n)}, u_n) > 0$);
2. $\lim_{n \to \infty} d_{TV}(p_n^{(bt_n)}, u_n) = 0$ (resp. $\liminf_{n \to \infty} d_2(p_n^{(bt_n)}, u_n) = 0$).

Similar definitions apply in continuous time. Diaconis and Shahshahani proved in [11] that the random transposition walk on $S_n$ has a cutoff (both in total variation and $\ell^2$) at time $(n/2)\log n$. For an overview of other results in this direction, see [9, 30].

**2.2 Eigenvalues and representation theory**

It is well known that for reversible finite Markov chains, the $\chi$-square distance can be expressed in terms of eigenvalues and eigenfunctions. See, e.g., [29]. For a reversible random walk on a finite group $G$ driven by $q$, the expression simplifies and the eigenvectors drop out. If we let $\beta_i, i = 0, \ldots, |G| - 1$, be the eigenvalues of the operator of convolution by $q$ acting on $\ell^2(G)$, in non-increasing order and repeated according to multiplicity, we have

$$d_2(q^{(t)}, u)^2 = \sum_{i=1}^{|G| - 1} \beta_i^2 t \quad \text{and} \quad d_2(h_t, u)^2 = \sum_{i=1}^{|G| - 1} e^{-2t(1 - \beta_i)}.$$ (2.1)

Representation theory provides a tool that can be helpful to compute eigenvalues. We give a very brief review of these methods. All the material in this section can be found in greater detail in [27]. A representation of a finite group $G$ on a vector space $V$ is a homomorphism $\rho : G \to GL(V)$ where $GL(V)$ is the group of general linear transformations of $V$. We say that $\rho$ has dimension $d_\rho$, where $d_\rho$ is equal to the dimension of $V$. Let $W \subset V$, if $\rho W = W$ then $\rho |_W$ is called a subrepresentation of $\rho$. A representation $\rho$ is called irreducible if it admits no nontrivial subrepresentation. The character of a representation $\rho$ at $s \in G$ is $\chi_\rho = \text{Tr}(\rho(s))$. Characters are constant under conjugation, i.e. for any $x, y \in G$ then

$$\chi_\rho(x^{-1}yx) = \chi_\rho(y).$$
For \( f : G \to \mathbb{R} \), the Fourier transform of \( f \) at \( \rho \) is

\[
\hat{f}(\rho) = \sum_{s \in G} f(s) \rho(s).
\]

The Fourier transform converts convolution of functions into multiplication of matrices (or composition of linear maps) \( \hat{f} \ast \hat{g} = \hat{f}(\rho) \hat{g}(\rho) \). If \( G \) is a finite group and if \( f, g \) are any two functions taking values on \( G \) then the Plancharel formula relates the convolution of \( f \) and \( g \) at \( e \) to the Fourier transform as follows

\[
f \ast g(e) = \sum_{s \in G} f(s^{-1})g(s) = \frac{1}{|G|} \sum_{\rho} \text{Tr}(\hat{f}(\rho) \hat{g}(\rho))
\]

where \( |G| \) is the order of \( G \) and the sum is over all (equivlent classes of) irreducible representations of \( G \). In what follows \( \rho \neq 1 \) means that \( \rho \) is not the trivial representation. The Plancharel formula is used to obtain the following proposition.

**Proposition 2.2.** Let \( G \) be a finite group equipped with a probability measure \( q \) satisfying \( q(x) = q(x^{-1}), \ x \in G \). We have

\[
d_2(q(t), u)^2 = \sum_{\rho \neq 1} d_{\rho} \text{Tr}(\hat{q}(\rho)^{2t}). \tag{2.2}
\]

In general, it is very difficult to estimate \( \text{Tr}(\hat{q}(\rho)^{t}) \). However, in the case were \( q \) is a class function, i.e., \( q(x^{-1}yx) = q(y) \), for all \( x, y \in G \), a celebrated lemma of Schur provides a nice analysis. If \( \rho \) is an irreducible representation and \( (C_j)^m \) are the conjugacy classes of the group \( G \) then \( \hat{q}(\rho) \) is a constant multiple of the identity matrix. This yields

\[
\hat{q}(\rho) = I_{d_{\rho}} : \left( \sum_{j=1}^{m} q(C_j) \chi_{\rho}(c_i) \right)
\]

where \( c_i \in C_j \). For a proof of this fact see [7, 8]. The next proposition now follows

**Proposition 2.3.** Let \( G \) be a finite group and \( q \) a probability measure on \( G \) satisfying \( q(x^{-1}) = q(x), \ x \in G \). If \( q \) is constant on conjugacy classes then

\[
d_2(q(t), u)^2 = \sum_{\rho \neq 1} d_{\rho}^2 \left( \sum_{j=1}^{m} q(C_j) \chi_{\rho}(c_i) \right)^{2t} \quad \text{and}
\]

\[
d_2(h_{C,t}, u)^2 = \sum_{\rho \neq 1} d_{\rho}^2 \exp \left( -2t \left( 1 - \sum_{j=1}^{m} q(C_j) \chi_{\rho}(c_i) \right) \right). \tag{2.3}
\]
To connect more directly representation theory with the usual spectral decomposition, let \( \rho : G \to GL(V) \) a representation of \( G \) on a finite vector space \( V \) equipped with an invariant Hermitian product \( \langle \cdot, \cdot \rangle \). Fix a probability measure \( q \) and consider the linear transformation \( \hat{q}(\rho) : V \to V \). Suppose \( e_i, e_j \) are unit vectors in \( V \) and that \( e_i \) is an eigenvector of \( \hat{q}(\rho) \) with eigenvalue \( \gamma_i \). Set \( \phi_{i,j,\rho}(x) = \langle \rho(x)e_i, e_j \rangle \). We claim that \( \phi_{i,j,\rho} \) is an eigenfunction for \( f \mapsto f \cdot q \) with 

\[
\phi_{i,j,\rho}(x) = \sum_y f(xy^{-1})q(y)
\]
on \( \ell^2(G) \) with eigenvalue \( \gamma_j \). Indeed,

\[
\phi_{i,j,\rho} \cdot q(x) = \sum_y q(y)\langle \rho(xy^{-1})e_i, e_j \rangle = \left( \rho(x)e_i, \sum_y q(y)\rho(y)e_j \right) = \langle \rho(x)e_i, \hat{q}(\rho)e_j \rangle = \gamma_j \langle \rho(x)e_i, e_j \rangle = \gamma_j \phi_{i,j,\rho}(x).
\]

Now, if \( q \) is symmetric and thus \( \hat{q}(\rho) \) is diagonalizable in an orthonormal basis \( (e_i)_{i=1}^n \) then the construction above yields \( d_\rho \) eigenvalues and \( d_\rho^2 \) orthonormal eigenvectors in \( \ell^2(G) \), each eigenvalue having multiplicity \( d_\rho \). Furthermore, if \( \rho, \rho' \), are two inequivalent irreducible representations the corresponding eigenvectors are orthogonal (some of the eigenvalues may be the same). A proof of the orthogonality of \( \phi_{i,j,\rho} \) is given in Corollary 4.10 of [20]. Hence, this produces \( |G| \) orthonormal eigenfunctions since \( \sum_\rho d_\rho^2 = |G| \) where the sum is taken over all (equivalent classes of) irreducible representations.

For future reference we mention the well known fact that irreducible representations on \( S_n \) are indexed by the Young diagram with \( n \) boxes (see [27]).

**Definition 2.4.** Let \( \lambda = (\lambda_1, \ldots, \lambda_m) \) be a partition of \( n \) so that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) and \( \sum_{i=1}^m \lambda_i = n \). \( \lambda \) is called a Young diagram of \( n \) boxes and \( \lambda_i \) denotes the number of boxes in the \( i \)-th row of the diagram.

**Figure 1:** The Young diagram for \( \lambda = (5, 4, 2, 1) \)

The association of an irreducible representation to a Young diagram will provide a key tool to calculate the normalized character \( \chi_{\rho}(\cdot)/d_\rho \) of an irreducible rep-
representation \( \rho \) and the eigenvalues of many of the walks we study. This technique is illustrated in the following sections.

## 3 Transpose top with random

Consider the following shuffling method of a deck of \( n \) cards: pick a card uniformly at random from the deck and transpose it with the top card. This shuffling scheme is described by the measure \( q \) on the symmetric group \( G = S_n \) where

\[
q(\tau) = \begin{cases} 
1/n & \text{if } \tau = (1, j), 1 \leq j \leq n \\
0 & \text{otherwise.}
\end{cases}
\]

This walk is called transpose top with random.

In order to establish an upper bound for the \( \ell_2 \) mixing time, the tools from group representation presented in section 2.2 are used to calculate the eigenvalues of \( q \). Most of the needed computations are in [15] and the procedure is outlined in [8] where it is stated that transpose top with random has a cut-off time of \( n \log n \). The following theorem gives a more precise upper bound.

**Theorem 3.1.** Let \( q \) be the transpose top with random measure on the group \( S_n \). If \( n \geq 1 \), \( c \geq 0 \), and \( t \geq n(\log n + c) \)

\[
d_2(q^{(t)}, u) \leq \sqrt{2} e^{-c}, \quad d_2(h_{q,t}, u) \leq \sqrt{2} e^{-c}.
\]

**Proof.** By Proposition 2.2

\[
d_2(q^{(t)}, u)^2 = \sum_{\rho \neq 1} d_{\rho} \text{Tr}(\hat{q}(\rho)^{2t}).
\]

Even though \( q \) is not constant on conjugacy classes [8] notes that \( q \) is invariant under conjugation by elements of \( S_{n-1} \) where

\[
S_{n-1} = \{ \tau \in S_n | \tau(1) = 1 \}.
\]

Using this fact it is shown that \( \hat{q}(\rho) \) is a diagonal matrix (with real entries). See [15, 8]. Therefore

\[
\text{Tr}(\hat{q}(\rho)^{2t}) = \sum_{i=1}^{d_{\rho}} \alpha_i^{2t} \leq d_{\rho} \alpha_1^{2t}
\]

where \( \alpha_1 \geq \cdots \geq \alpha_{d_{\rho}} \) are the eigenvalues of \( \hat{q}(\rho) \).

To compute \( \alpha_i \), consider \( M = \sum_{i=2}^n \rho((1, i)) \). Let \( \lambda = (\lambda_1, \ldots, \lambda_m) \) be the Young diagram associated to the irreducible representation \( \rho \). Let \( \sigma_1 \leq \cdots \leq \sigma_{d_{\rho}} \) be the eigenvalues of \( M \). In [15] it is shown that for \( 1 \leq i \leq d_{\rho} \) then

\[
\sigma_i = \lambda_i - i.
\]
The multiplicity of each \( \sigma_i \) is also described in \[15\]. We do not need this for the present proof but, to give an example, if \( \lambda = (n - 1, 1) \) then the eigenvalues of \( M \) are \( \sigma_1 = n - 2 \) with multiplicity \( n - 2 \) and \( \sigma_2 = -1 \) with multiplicity 1.

As
\[
\hat{q}(\rho) = \sum_{\tau \in G} q(\tau) \rho(\tau) = \sum_{i=1}^{n} \left( \frac{1}{n} \right) \rho((1, i)) = \frac{M + \rho(e)}{n} = \frac{M + I}{n}
\]

where \( I \) is the identity matrix of dimension \( d_{\rho} \), we easily obtain the eigenvalues \( \alpha_i, 1 \leq i \leq d_{\rho} \):
\[
\alpha_i = \frac{\sigma_i + 1}{n}.
\]

In particular, \( \alpha_1 = \lambda_1/n \).

Denote by \( \rho_{\lambda} \) the irreducible representation associated to a partition \( \lambda \) and by \( \rho_{\lambda} = 1 \) the trivial representation corresponding to \( \lambda = (n) \). Equation (3.4) yields
\[
d_2(q(t), u)^2 \leq \sum_{\rho_{\lambda} \neq 1} d_{\rho_{\lambda}}^2 \left( \frac{\lambda_1}{n} \right)^{2t} = \sum_{j=1}^{n-1} \sum_{\rho_{\lambda} = n-j} d_{\rho_{\lambda}}^2 \left( \frac{\lambda_1}{n} \right)^{2t}.
\]

(3.5)

In \[7, 15\] it is shown that for \( l \geq 1 \)
\[
\sum_{\rho_{\lambda} \neq l} d_{\rho_{\lambda}}^2 \leq \left( \frac{n}{l} \right)^2 (n-l)!. 
\]

(3.6)

It follows that for \( c \geq 0 \) and \( t \geq n(\log n + c) \)
\[
d_2(q(t), u)^2 \leq \sum_{j=1}^{n-1} \frac{n!}{(n-j)!} \left( \frac{1}{n} \right)^2 \left( \frac{1}{l} \right)^{2t}
\]
\[
\leq \sum_{j=1}^{n-1} n^{2j} \left( \frac{1}{j!} \right)^{2t} (e - 1)^{e^{-2c}} \leq 2e^{-2c}.
\]

For the continuous time process, we have, similarly,
\[
d_2(h_{q,t}, u)^2 \leq \sum_{\rho_{\lambda} \neq 1} d_{\rho_{\lambda}}^2 e^{-2t(1-\lambda_1)} = \sum_{j=1}^{n-1} \sum_{\lambda_1 = n-j} d_{\rho_{\lambda}}^2 \exp\{-2t(1 - \lambda_1/n)\}
\]
\[
= \sum_{j=1}^{n-1} \sum_{\lambda_1 = n-j} d_{\rho_{\lambda}}^2 e^{-2tj/n} \leq \sum_{j=1}^{n-1} \frac{n!}{(n-j)!} \left( \frac{1}{j!} \right)^{2t} e^{-2tj/n}
\]

where the last inequality follows from (3.6). Again, if \( n \geq 1, c \geq 0 \) and \( t \geq n(\log n + c) \) then
\[
d_2(h_{q,t}, u)^2 \leq \sum_{j=1}^{n-1} n^{2j} \left( \frac{1}{j!} \right)^{2t} (e - 1)^{e^{-2c}} \leq 2e^{-2c}.
\]

\[\square\]
The next proposition shows that transpose top with random has $\ell^2$ and total variation cutoffs at time $n \log n$.

**Proposition 3.2.** Let $q$ be the transpose top with random measure on $S_n$. For any sequence $(k_n)^\infty_0$ such that $(k_n - n \log n)/n$ tends to $-\infty$ as $n$ tends to $\infty$ then

$$d_2(q^{(k_n)}(\cdot), u) \to \infty \text{ and } d_{TV}(q^{(k_n)}(\cdot), u) \to 1.$$  

**Proof.** For the $\ell^2$ bound, we observe that [15] also gives a description of the multiplicity of the eigenvalues. In particular, if $\lambda = (n-1, 1)$ then the eigenvalue $1 - 1/n$ of $\hat{q}(\rho_\lambda)$ has multiplicity $n-2$. Since $d_\lambda = n-1$ we get that

$$d_2(q^{(k)}, u)^2 \geq (n-1)(n-2)(1 - 1/n)^{2k}$$

from which the desired $\ell^2$ statement easily follows.

**Remark 3.3.** Let $\varphi(\sigma)$ be the number of fixed points of $\sigma$. One can check by direct inspection that

$$f(\sigma) = \left(\frac{n-1}{n-2}\right)^{1/2} \times \begin{cases} \varphi(\sigma) - 2 & \text{if } \sigma(1) = 1 \\ \varphi(\sigma) - 1 + \frac{1}{n-1} & \text{if } \sigma(1) \neq 1 \end{cases}$$

is a normalized eigenfunction (for convolution by $q$) with eigenvalue $1 - 1/n$. Its value at $e$ is $f(e)^2 = (n-1)(n-2)$. This gives a entirely elementary proof of the $\ell^2$ lower bound since $d_2(q^{(k)}, u)^2 \geq (1 - 1/n)^{2k} f(e)^2$. The previous inequality results from the fact that one can write the $\chi^2$ distance in terms of eigenvalues and eigenfuctions. See, e.g., [29].

The proof of the lower bound for total variation follows mostly an argument used in [2] to give a lower bound for random transposition (and for the top to random insertion shuffle). Let

$$A_j = \{ \sigma \in S_n : \varphi(\sigma) \geq j \}$$

with $\varphi$ as defined above. Then

$$d_{TV}(q^{(k_n)}(\cdot), u) \geq q^{(k_n)}(A_j) - u(A_j).$$

Calculating $u(A_j)$ is equivalent to calculating the probability of at least $j$ matches in the classical matching problem. In [14], Feller gives a closed form solution for $u(A_j)$. Using this we get the following estimate for $j \geq 2$

$$u(A_j) = \sum_{m=j}^n \frac{1}{m!} \left( \sum_{v=0}^{n-m} \frac{(-1)^v}{v!} \right) \leq e^{-1} \left( \frac{1}{(j-1)!} \right).$$

(3.8)

Next we bound $q^{(k_n)}(A_j)$ from below. Consider the experiment where successive balls are dropped independently and uniformly at random into $n$ boxes. Let $B_{j,k}$ be the event that after dropping $k$ balls there are at least $j$ empty boxes. Then

$$q^{(k_n)}(A_{j-1}) \geq P(B_{j,k_n}).$$

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Let \( V_i \) be the number of balls dropped when exactly \( l \) boxes are filled. We have

\[
P(B_{j,k_n}) = P(V_{n-j} \geq k_n) \geq 1 - P(V_{n-j} \leq k_n).
\]

We would like to show that for any fixed \( j \), \( P(V_{n-j} \leq k_n) \to 0 \) as \( n \to \infty \). We have

\[
V_{n-j} = (V_{n-j} - V_{n-j-1}) + (V_{n-j-1} - V_{n-j-2}) + \cdots + (V_2 - V_1) + V_1.
\]

The \( V_{i+1} - V_i \) are independent random variables with geometric distribution

\[
P\{V_{i+1} - V_i = l\} = \left(\frac{n-i}{n}\right)\left(1 - \frac{n-i}{n}\right)^{l-1}, \ l \geq 1.
\]

Hence

\[
E(V_{i+1} - V_i) = \frac{n}{n-i} \quad \text{and} \quad \text{Var}(V_{i+1} - V_i) = \left(\frac{n}{n-i}\right)^2\left(1 - \frac{n-i}{n}\right).
\]

It follows that

\[
E(V_{n-j}) = \sum_{i=1}^{n-j-1} \frac{n}{n-i} \geq \int_0^{n-j-1} \frac{n}{n-x} \, dx \geq n \log\left(\frac{n}{j+1}\right)
\]

and

\[
\text{Var}(V_{n-j}) = \sum_{i=1}^{n-j-1} \frac{n^2}{(n-i)^2} - \frac{n^2}{n(n-i)} \leq \sum_{i=1}^{n-j-1} \frac{n^2}{(n-i)^2}
\]

\[
\leq \int_1^{n-j-1} \left(\frac{n}{n-x}\right)^2 \, dx \leq \frac{n^2}{j}.
\]

By assumption \( k_n = n \log n - nc_n \) and \( c_n \to \infty \) as \( n \to \infty \). If we assume, as we may, that \( c_n > \log(j+1) \) then Chebyshev’s inequality gives

\[
P(V_{n-j} \leq k_n) = P(V_{n-j} \leq n \log n - nc_n)
\]

\[
\leq P(n(c_n - \log(j+1)) \leq |(V_{n-j}) - E(V_{n-j})|)
\]

\[
\leq \frac{\text{Var}(V_{n-j})}{n^2(c_n - \log(j+1))^2} \leq \frac{1}{j(c_n - \log(j+1))^2}.
\]

This yields

\[
\lim_{n \to \infty} d_{TV}(q^{(k_n)}, u) \geq \lim_{n \to \infty} (P(B_{j+1,k_n}) - u(A_j)) \geq 1 - e^{-1}\left(\frac{1}{(j-1)!}\right).
\]

Since \( j \) is arbitrary the desired result follows.

**Corollary 3.4.** Let \( h_t \) be the distribution for the continuous time process associated to the transpose top with random measure \( q \). For any sequence \( (k_n)_0^\infty \) such that \( (k_n - n \log n)/n \) tends to \( -\infty \) as \( n \) tends to \( \infty \) then

\[
d_2(h_{k_n}, u) \to \infty \quad \text{and} \quad d_{TV}(h_{k_n}, u) \to 1.
\]
Proof. The $\ell^2$ bound follows from the same argument used above. In the case of the total variation bound, one can show that for $A_j$ defined in (3.7) then $h_{k_n}(A_j) \to 1$. A slight modification of the proof of Proposition 3.2 gives that for $\alpha \in (1/2, 1)$
$$
\lim_{n \to \infty} q^{kn + k_n^\alpha}(A_j) = 1.
$$
Combining the limit above with the fact that
$$
\lim_{n \to \infty} \sum_{t=0}^{k_n + k_n^\alpha} e^{-k_n} \frac{k_n^t}{t!} = \lim_{n \to \infty} P \left( \frac{X_n - k_n}{\sqrt{k_n}} \leq k_n^{\alpha - 1/2} \right) = 1
$$
where $X_n$ is a Poisson random variable with parameter $k_n$ gives us the desired result.

4 Random transpositions

4.1 Discrete time

Consider the following measure $q = q_{\text{rev}}$ on the group $G = S_n$,

$$
q(\tau) = \begin{cases} 
2/n^2 & \text{if } \tau = (i, j), 1 \leq i, j \leq n, \ i \neq j, \\
1/n & \text{if } \tau = \text{id}, \\
0 & \text{otherwise.}
\end{cases}
$$

(4.9)
The measure $q$ models the shuffle of a deck of $n$ cards where one picks two cards independently and uniformly at random and transposes them. The random transposition shuffle has been shown to demonstrate cutoff at $(n/2) \log n$, see [7, 8, 11].

Theorem 4.1. (Diaconis and Shahshahani) Let $q$ be the random transposition measure on the group $S_n$ then there exists a positive universal constant $B$ such that for any $c \geq 0$ and $t \geq \frac{n}{2} (\log n + c)$ then

$$
2d_{\text{TV}}(q(t), u) \leq d_2(q(t), u) \leq Be^{-c}.
$$

One of the aims of this section is to get a more precise estimate on the constant $B$ in the theorem above.

Proposition 4.2. Let $q$ be the random transposition measure on $S_n$. For $n \geq 14$, $c \geq 0$, and $t \geq \frac{n}{2} (\log n + c)$ then equation (4.1) holds with

$$
B^2 \leq 2 + \varphi(n) \leq 4
$$

where $\varphi(n) \to 0$ as $n \to \infty$.

Let $t_n$ be the smallest integer larger or equal to $(n/2) \log n$. Then the result above and an easy lower bound discussed below imply that

$$
1 \leq \lim_{n \to \infty} d_2(q(t_n), u) \leq 2.
$$

It is quite rare to be able to capture the mixing time of a chain with such precision.
**Proof.** Let \( C \subset S_n \) be the conjugacy class of transpositions, \( \tau \in C \) be a transposition, and

\[
r(\rho_\lambda) = \frac{\chi_{\rho_\lambda}(\tau)}{d_{\rho_\lambda}}.
\]

Proposition 2.3 gives

\[
d_2(q^{(t)}, u)^2 = \sum_{\rho_\lambda \neq 1} q_{\rho_\lambda}^2 \left( \frac{1}{n} + \frac{n-1}{n} r(\rho_\lambda) \right)^{2t} \tag{4.10}
\]

In [11] it is shown that

\[
r(\rho_\lambda) \leq \begin{cases} 1 - \frac{2(n-\lambda_1)(\lambda_1+1)}{n(n-1)} & \text{if } \lambda_1 \geq n/2 \\ \frac{n-1}{n-1} & \text{always}. \end{cases} \tag{4.11}
\]

It follows from equations (3.6), (4.10), and (4.11) that

\[
d_2(q^{(t)}, u)^2 = \sum_{j=1}^{n-1} \sum_{\rho_\lambda \in \rho_\lambda \neq \frac{n-j}{n-j}} q_{\rho_\lambda}^2 \left( \frac{1}{n} + \frac{n-1}{n} r(\rho_\lambda) \right)^{2t} \leq \sum_{j=1}^{n-1} \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \left( 1 - \frac{2j}{n} \right) \left( 1 - \frac{j-1}{n} \right)^{2t} + \sum_{j=1}^{n-1} \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \left( 1 - \frac{j}{n} \right)^{2t}.
\]

Note that for \( 1 \leq j \leq \frac{n}{2} \) we have that \( 1 - \frac{2j}{n} \left( 1 - \frac{j-1}{n} \right) \leq 1 - \frac{2}{n} \leq e^{-2/n} \). So for \( t \geq (n/2)(\log n + c) \)

\[
d_2(q^{(t)}, u)^2 \leq e^{-2c} \left( \sum_{j=1}^{n/2} A_j + \sum_{j=n/2}^{n-1} B_j \right).
\]

where

\[
A_j = \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \left( 1 - \frac{2j}{n} \left( 1 - \frac{j-1}{n} \right) \right)^{n \log n} \tag{4.12}
\]

\[
B_j = \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \left( 1 - \frac{j}{n} \right)^{n \log n}. \tag{4.13}
\]

Consider the following two technical propositions.

**Proposition 4.3.** Set \( \varphi_0(n) = \sum_{j=1}^{\lfloor n/4 \rfloor} A_j \) and \( \varphi_1(n) = \sum_{j=\lfloor n/4 \rfloor}^{\lfloor n/2 \rfloor} A_j \). For \( n \geq 14 \)

\[
\varphi_0(n) \leq 2 \text{ and } \varphi_1(n) \leq \exp \left\{ 2 - \frac{1}{6} n \log n \right\}.
\]
Proposition 4.4. Set $\varphi_2(n) = \sum_{j=\lfloor n/2 \rfloor}^{n} B_j$. For $n \geq 9$

$$\varphi_2(n) \leq \exp \left\{1 - \frac{3}{1000} n \log n \right\}.$$ 

Propositions 4.3 and 4.4 give that for $n \geq 14$

$$d_2(q^t, u)^2 = e^{-2c(\varphi_0 + \varphi_1 + \varphi_2)} \leq e^{-2c(2 + \varphi_1(14) + \varphi_2(14))} \leq 4e^{-2c}.$$ 

It also follows that $\varphi_1 \to 0$ and $\varphi_2 \to 0$ as $n \to \infty$.

Next we will show the proofs of the propositions above.

Proof of Proposition 4.3. Let $A_j$ be as in equation (4.12), the ratio between two consecutive terms is $\frac{A_{j+1}}{A_j} = \exp \{f_n(j) + g_n(j)\}$

where

$$f_n(j) = 2 \log(n - j) - \log(j + 1)$$
$$g_n(j) = n \log n \log \left(\frac{n^2 - 2(j + 1)n + 2j(j + 1)}{n^2 - 2jn + 2j(j - 1)}\right).$$

Taking derivatives gives

$$f'_n(j) = -\frac{2}{n - j} - \frac{1}{j + 1}$$
$$g'_n(j) = \frac{4(n \log n)(2jn - 2j^2 - n)}{(n^2 - 2jn + 2j^2 - 2j)(n^2 - 2jn - 2n + 2j^2 + 2j)}.$$ 

Note that for $1 \leq j \leq n/4$ and $n \geq 4$ we have that $f''_n(j) = -\frac{4}{(j+1)^2} - \frac{2}{(n-j)^2} \geq 0$. Furthermore, $g''_n(j) \geq 0$ for $1 \leq j \leq n/2$. The last inequality holds since for $1 \leq j \leq n/2$ the numerator of $g''_n(j)$ is a positive increasing function of $j$ and the denominator is a positive decreasing function of $j$.

Set $h_n = f_n + g_n$. For $1 \leq j \leq n/4$ the function $h_n$ is continuous and has positive second derivative. It follows that $h_n$ is convex for said values of $j$, which implies that $h_n(j) \leq \max \{h_n(1), h_n(n/4)\}$.

Consider the following estimates.

$$h_n(1) = 2 \log(n - 1) - \log 2 + n (\log n) \log \left(1 - \frac{2n - 4}{n(n - 2)}\right)$$
$$\leq 2 \log(n - 1) - \log 2 - n (\log n) \left(\frac{2n - 4}{n(n - 2)}\right)$$
$$\leq 2 \log(n - 1) - \log 2 - 2 \log n$$

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\[ h_n(n/4) = 2 \log \left( \frac{3n}{4} \right) - \log \left( \frac{n+4}{4} \right) + n \log n \log \left( \frac{1 - \frac{8}{5n-4}}{5} \right) \]
\[ \leq 2 \log \left( \frac{3n}{4} \right) - \log \left( \frac{n+4}{4} \right) - n \log n \left( \frac{8}{5n-4} \right) \]
\[ \leq 2 \log \left( \frac{3n}{4} \right) - \log \left( \frac{n+4}{4} \right) - \frac{8\log n}{5} \]
\[ \leq 2 \log 3 - \log 4 + \frac{2}{5} \log n - \log(n+4) \]

For \( n \geq 2 \) \( h_n(1) \) and \( h_n(n/4) \) are decreasing functions of \( n \) less than \( -\log 2 \).

Since \( A_1 = n^2(1 - 2/n)^{n \log n} \leq 1 \), it follows that for \( 1 \leq j \leq n/4 \)
\[ A_j \leq (1/2)^{j-1} A_1 \leq (1/2)^{j-1}. \]

We can now state the first part of Proposition 4.3
\[ \varphi_0(n) = \sum_{j=0}^{\frac{n}{4}} A_j \leq \sum_{j=1}^{\infty} \left( \frac{1}{2} \right)^j = 2. \]

Next we bound \( A_j \) for \( n/4 \leq j \leq n/2 \). It is not hard to show that \( f''_n(j) \leq 0 \),
so for the values of \( j \) above
\[ f'_n(j) \geq \min\{f''_n(n/4), f''_n(n/2)\}. \]

Note that
\[ f'_n(n/4) = \frac{4(5n+8)}{3n(n+4)} \quad \text{and} \quad f'_n(n/2) = -\frac{2(3n+4)}{n(n+2)}. \]

For \( n \geq 14 \), \( f'_n(n/2) \geq f'_n(n/4) \). Recall that for \( 1 \leq j \leq n/2 \) we had that \( g''_n \geq 0 \).
It follows that for \( n/4 \leq j \leq n/2 \),
\[ h'_n(j) = f'_n(j) + g'_n(j) \geq f'_n(n/4) + g'_n(n/4) \geq 0. \]

Above we showed that \( h_n(n/4) \leq 0 \). For \( n \geq 3 \) we have that \( h_n(n/2) = 2 \log \left( \frac{5}{2} \right) - \log \left( \frac{5}{2} + 1 \right) \geq 0 \),
so there must be a unique point \( x \in [n/4, n/2] \) such that \( h_n(x) = 0 \). If \( n/4 \leq j \leq x \) then \( A_{j+1}/A_j \leq 1 \). If \( x \leq j \leq n/2 \) then \( (A_{j+1}/A_j) \geq 1 \). So for \( n/4 \leq j \leq n/2 \)
\[ A_j \leq \max \{A_{\frac{n}{2}}, A_{\frac{n}{4}}\}. \]

In [14] a proof of Stirling’s formula shows that
\[ \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \leq n! \leq e^{\frac{1}{12n}} \sqrt{2\pi n} \left( \frac{n}{e} \right)^n. \quad (4.14) \]
To determine the largest value among $A_\frac{1}{n}$ and $A_\frac{2}{n}$ we consider the ratio

$$ \frac{A_\frac{1}{n}}{A_\frac{2}{n}} = \left( \frac{\left( \frac{n}{2} \right)!}{\left( \frac{3n}{4} \right)!} \right)^2 \left( \frac{\left( \frac{n}{2} \right)!}{\left( \frac{n}{2} \right)!} \right) \left( \frac{5n - 4}{4n - 8} \right)^{n \log n} \leq \left( \frac{2\sqrt{e} \sqrt{n}}{3} \right)^2 \left( \frac{\frac{n}{2} e^{\frac{2n}{n+1}}}{e^\frac{3}{2}} \right) \left( \frac{\frac{n}{2} e^{\frac{2n}{n+1}}}{e^\frac{3}{2}} \right) \left( \frac{5n - 4}{4n - 8} \right)^{n \log n} = \left( \frac{2\sqrt{e} \sqrt{n}}{3} \right)^2 \left( \frac{4}{3\sqrt{2}} \right)^n \left( \frac{e^\frac{n}{2}}{n^\frac{3}{2}} \right)^n n^{n \log \left( \frac{5n - 4}{4n - 8} \right)} = \left( \frac{2\sqrt{e} \sqrt{n}}{3} \right)^n \exp \{ l(n) \} $$

where $l(n) = n \left( \log \left( \frac{\frac{n}{2} e^{\frac{2n}{n+1}}}{e^\frac{3}{2}} \right) + \frac{1}{4} \right) + n \log n \left( \log \left( \frac{5n - 4}{4n - 8} \right) - \frac{1}{4} \right)$. For $n \geq 47$ we have that $\left( \log \left( \frac{5n - 4}{4n - 8} \right) - \frac{1}{4} \right) \leq 0$ which implies that $l(n) \leq 0$. If $5 \leq n \leq 47$ one can check that $l(n) \leq 1$. So for $n \geq 5$ we have that $(A_\frac{1}{n}/A_\frac{2}{n}) \leq e$ which in turn implies that $\sum_{j=n/4}^{n/2} A_j \leq (e/4)nA_\frac{2}{n}$. By using Stirling's formula to estimate $A_\frac{2}{n}$ we get

$$ \left( \frac{en}{4} \right) A_\frac{2}{n} = \left( \frac{en}{4} \right) n! \left( \frac{1}{\frac{n}{2}} \right)! \left( \frac{n-2}{2n} \right)^{n \log n} = \left( \frac{en}{4} \right) \left( e^{\frac{n}{2} \sqrt{2n+2}} \right)^2 \left( \frac{2 \frac{n}{2} e^{\frac{n}{2}}}{e^\sqrt{2n}} \right) \left( \frac{n-2}{2n} \right)^{n \log n} \leq \left( \frac{e^\frac{1}{2} + \frac{n}{2}}{2 \sqrt{2}} \right) \exp \{ nf(n) \log n \} $$

where $f(n) = \left( \frac{1}{2 \log n - 1} \right) \left( \log n \right)^{-1} + \frac{1}{2m} + \frac{1}{2} + \log \left( \frac{\frac{n}{2} e^{\frac{2n}{n+1}}}{e^\frac{3}{2}} \right)$. Computing the derivative gives us that

$$ f'(n) = \frac{-(3n \log 2 - 1)n^2 + 2(3 \log 2 - 1)n + 3n(\log n)^2 + 2(\log n)^2}{2n^2(\log n)^2(n-2)} $$

Note that $f' \geq 0$ for $n > 2$, so $f(n) \leq \lim_{n \to \infty} f(n) = \frac{1}{2} - \log 2$. We can now concluded that for $n \geq 5$

$$ \varphi_1(n) \leq \left( \frac{en}{4} \right) A_\frac{2}{n} \leq \left( \frac{e^{1+\frac{n}{2}}}{2 \sqrt{2}} \right) \exp \left\{ (n \log n) \left( \frac{1}{2} - \log 2 \right) \right\} \leq \frac{e^3}{2 \sqrt{2}} \exp \left\{ -\frac{1}{6} n \log n \right\} \leq \exp \left\{ 2 - \frac{1}{6} n \log n \right\}. $$

\[\Box\]
Proof of Proposition 4.4. Let $B_j$ be as in equation (4.13). If $n > 2$ and $n/2 \leq j \leq n$ we can estimate the ratio of $B_j$ and $B_{j+1}$ by

$$\frac{B_{j+1}}{B_j} = \frac{(n-j)^2}{(j+1)^2} \left(1 - \frac{1}{n-j}\right)^{n \log n} \leq 2n \left(1 - \frac{2}{n}\right)^{n \log n} \leq \frac{2}{n}.$$ 

We get that $B_j \leq \frac{B_{n/2}}{2^{n/2}}$. It follows that

$$\varphi_2(n) = \sum_{j=\frac{n}{2}}^{n} B_j \leq B_{\frac{n}{2}} \sum_{j=\frac{n}{2}}^{n} (2/n)^{j-\frac{n}{2}} \leq B_{\frac{n}{2}} \sum_{j=0}^{\infty} (2/n)^j = \frac{B_{\frac{n}{2}}}{1 - (2/n)}.$$ 

Using Stirling’s formula we can bound $B_{\frac{n}{2}}$ to get

$$\left(\frac{1}{1 - 2/n}\right) B_{\frac{n}{2}} = \left(\frac{1}{1 - 2/n}\right) \left(\frac{n!}{(\frac{n}{2})!}\right)^2 \left(\frac{1}{2}\right)^{n \log n} \leq \left(\frac{1}{1 - 2/n}\right) \left(\frac{2e}{\sqrt{\pi}}\right)^{\frac{n}{2}} \left(\frac{2e}{\sqrt{\pi}}\right)^{\frac{3}{2}} \exp \{n \log n b(n)\}$$

where $b(n) = -\frac{1}{2n} + \left(\frac{3 \log 2 - 1}{2}\right) (\log n)^{-1} + \frac{1}{2} + \log \left(\frac{1}{2}\right)$. Taking derivatives gives that

$$b'(n) = \frac{(\log n)^2 - n(3 \log 2 - 1)}{2n^2 (\log n)^2} \text{ so } b'(n) \leq 0 \text{ for } n \geq 1.$$ 

For $n \geq 9$ we have that $b(n) \leq b(9) < -\frac{3}{1000}$. Furthermore, for $n \geq 9$ the function

$$g(n) = \left(\frac{1}{1 - 2/n}\right) \left(\frac{2e}{\sqrt{\pi}}\right)^{\frac{3}{2}} \exp \left\{-\frac{3}{1000} n \log n\right\}$$

is decreasing. So for $n \geq 9$

$$\varphi_2(n) \leq g(n) \leq \exp \left\{1 - \frac{3}{1000} n \log n\right\}.$$ 

A lower bound for the $\chi$-square distance is obtain by writing $d_2(q^{(k)}, u)^2 \geq (n - 1)^2 (1 - 2/n)^{2k}$ which uses the term associated to the Young diagram $(n - 1, 1)$. Alternatively, let $\varphi(\sigma)$ be the function with denotes the number of fixed points of $\sigma$. One can check by inspection that $\varphi - 1$ is a normalized eigenfunction associated with the eigenvalue $1 - 2/n$. This gives the same $\ell^2$ lower bound.

Concerning total variation lower bounds, [7] shows that for any $c > 0$ and $t \geq (n/2)(\log n - c)$

$$\lim_{n \to \infty} d_{TV}(q^{(t)} u) \geq 1/e - e^{-2c}$$
A slight modification of the argument used in \[7\] (as presented above in the proof of Proposition \[3.2\]) yields the following proposition.

**Proposition 4.5.** Let \( q \) be the random transposition measure on the group \( S_n \).
For any sequence \( k_n \) such that \((2^{k_n} - n \log n)/n\) tends to \(-\infty\) as \( n \) tends to \( \infty \),
we have
\[
\lim \limits_{{n \to \infty}} d_2(q^{(k_n)}, u) = \infty \quad \text{and} \quad \lim \limits_{{n \to \infty}} d_{TV}(q^{(k_n)}, u) = 1.
\]

### 4.2 Random transposition in continuous time

This section is devoted to the continuous time version of random transposition. There is no proof in the literature that the continuous time random transposition shuffle has a \( \ell^2 \) cutoff at time \((n/2) \log n\). One reason is that the fact that it does not automatically follow from the discrete time result is often overlooked. In fact, getting an upper bound in the continuous time case turns out to be somewhat more difficult than in the discrete case. The difficulty comes from handling the contribution of the small eigenvalues of \( q \). Compare with what is proved below for conjugacy classes with less fixed points, e.g., 4-cycles. One very good reason to want to have a good \( \ell^2 \) upper-bound in continuous time for random transposition is that it yields better result when used with the comparison technique of \[13\] to study other chains. See Section 4.3 below.

**Proposition 4.6.** Let \( h_t \) be the law of the continuous time process associated to the random transposition measure \( q \). If \( n \geq 10 \), \( c \geq 2 \) then for \( t \geq (n/2)(\log n + c) \)
\[
2d_{TV}(h_t, u) \leq d_2(h_t, u) \leq e^{-(c-2)}.
\]

Moreover, if \( t_n \) is any sequence of time such that \((2t_n - n \log n)/n\) tends to \(-\infty\) as \( n \) tends to infinity, we have
\[
\lim \limits_{{n \to \infty}} d_{TV}(h_{t_n}, u) = 1, \quad \lim \limits_{{n \to \infty}} d_2(h_{t_n}, u) = \infty.
\]

Let us observe that we are not able to show that \( d_2(h_{(n/2) \log n}, u) \) is bounded above independently of \( n \) (compare with the discrete time case).

**Proof.** The lower bound in \( \ell^2 \) follows from the same argument used in the discrete time case. The lower bound in total variation is known. See, e.g., in \[28\]. We focus on the upper bound in \( \ell^2 \).

Let \( C \subset S_n \) be the conjugacy class of transpositions. Proposition 2.3 implies that
\[
d_2(h_t, u)^2 = \sum_{\rho \neq 1} d_{\rho}^2 \exp \left\{ -2t \left( 1 - \frac{1}{n} - \frac{n-1}{n} r(\rho) \right) \right\}
\]  
(4.15)
where \( r(\rho) = \chi_{\rho}(\tau)/d_{\rho} \) and \( \tau \) is a transposition. Using equations 3.0,
we get that for \( t \geq (n/2)(\log n + c) \)
\[
d_2(h_t, u)^2 = \sum_{j=1}^{n-1} \sum_{\rho_\lambda \neq 1 \atop \lambda_1 = n-j} d^2_{\rho_\lambda} \exp \left\{ -2t \left( 1 - \frac{1}{n} - \frac{n-1}{n} r(\rho_\lambda) \right) \right\}
\]
\[
\leq \sum_{j=1}^{n/2} \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \exp \left\{ -2t \left( \frac{2j}{n} \right) \left( 1 - \frac{j-1}{n} \right) \right\}
\]
\[
+ \sum_{j=n/2}^{n-1} \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \exp \left\{ -2t \left( \frac{j}{n} \right) \right\}
\]
\[
\leq \sum_{j=1}^{n/2} \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \exp \left\{ -2j(\log n + c) \left( 1 - \frac{j-1}{n} \right) \right\}
\]
\[
+ \sum_{j=n/2}^{n-1} \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \exp \left\{ -j(\log n + c) \right\}
\]

Note that for \( c \geq 2 \) and \( j \leq n/2 \) we have \(-2cj \left( 1 - \frac{j-1}{n} \right) \leq -2c - 2j + 4\). It follows that for \( t \geq (n/2)(\log n + c) \)
\[
d_2(h_t, u)^2 \leq e^{-2(c-2)} \left( \sum_{j=1}^{n/2} A_j + \sum_{j=n/2}^{n} B_j \right)
\]

where
\[
A_j = \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \exp \left\{ -2j \log n \left( 1 - \frac{j}{n} \right) - 2j \right\} \quad (4.16)
\]
\[
B_j = \left( \frac{n!}{(n-j)!} \right)^2 \frac{1}{j!} \exp \{ -j \log n - 2j \}. \quad (4.17)
\]

Consider the following technical lemmas.

**Lemma 4.7.** For \( n \geq 10 \) then \( \sum_{j=1}^{n/4} A_j \leq 2/3 \) and \( \sum_{j=n/4}^{n/2} A_j \leq 1/4 \).

**Lemma 4.8.** Set \( \gamma(n) = \sum_{j=n/2}^{n} B_j \). For \( n \geq 2 \)
\[
\gamma(n) \leq 2 \left( \frac{2}{e} \right)^{3n/4} .
\]

It follows from the lemmas above that for \( n \geq 10 \)
\[
d_2(h_t, u)^2 \leq e^{-2(c-2)} \left( \sum_{j=1}^{n/4} A_j + \sum_{j=n/4}^{n/2} A_j + \gamma(10) \right)
\]
\[
\leq e^{-2(c-2)} \left( 2/3 + 1/4 + 2(2/e)^{15} \right)
\]
\[
\leq e^{-2(c-2)} .
\]
Proof of Lemma 4.7. Let $A_j$ be as in equation (4.16). For $1 \leq j < n/2$ the ratio of two consecutive terms is given by

$$\frac{A_{j+1}}{A_j} = \frac{(n-j)^2}{(j+1)} \exp \left\{ - \left( 2 \frac{\log n}{n} \right) (n - 2j - 1) - 2 \right\} = \exp \{ f_n(j) \}$$

where

$$f_n(j) = 2 \log(n-j) - \log(j+1) - \left( 2 \frac{\log n}{n} \right) (n - 2j - 1) - 2. \quad (4.18)$$

Taking derivatives gives

$$f_n'(j) = - \frac{2}{n-j} - \frac{1}{j+1} + 4 \frac{\log n}{n}$$

$$f_n''(j) = - \frac{2}{(n-j)^2} + \frac{1}{(j+1)^2}.$$ 

Let $n \geq 4$ and $1 \leq x \leq n/4$. For these values of $n$ and $x$ we get that $f_n$ is convex since $f_n''$ is a decreasing function and $f_n''(x) \geq f_n''(n/4) \geq 0$.

$$\frac{A_{x+1}}{A_x} = \exp \left( \max \left\{ f_n(1), f_n \left( \frac{n}{4} \right) \right\} \right).$$

If $n \geq 2$ we have the estimates

$$f_n(1) = 2 \log(n-1) - \log 2 - \left( 2 \frac{\log n}{n} \right)(n-3) - 2 \leq - \log 2$$

$$f_n(n/4) = 2 \log \left( \frac{3n}{4} \right) - \log \left( \frac{n}{4} + 1 \right) - \left( 2 \frac{\log n}{n} \right) \left( \frac{n}{2} - 1 \right) - 2 \leq 2 \log \left( \frac{3}{4} \right).$$

Since $- \log 2 \leq 2 \log(3/4)$, we get that $A_x \leq (9/16)^{x-1}$. It now follows that

$$\sum_{j=1}^n A_j \leq A_1 \sum_{j=0}^\infty \left( \frac{9}{16} \right)^j = \left( \frac{16}{7} \right) A_1.$$ 

For $n \geq 4$ we get $A_1 = n^2 \exp \left\{ - \frac{2(n-1)}{n} \log n - 2 \right\} \leq 2e^{-2}$. This gives that

$$\sum_{j=1}^n A_j \leq (32/7)e^{-2} \leq 2/3.$$

For the next part of the proof let $n \geq 10$. Recall that $f_n''$ is a decreasing function, which implies that for $n/4 \leq j \leq n/2$

$$f_n'(j) \geq \min \{ f_n''(n/4), f_n''(n/2) \} \geq 0$$
where the last inequality holds since \( n \geq 10 \). Since \( f_n \) is an increasing function with \( f_n(n/4) \leq 0 \) and \( f_n(n/2) \geq 0 \) then there exists a unique point \( z \in [n/4, n/2] \) such that \( f_n(z) = 0 \). It follows that if \( n/4 \leq j \leq z \) then \( A_j \leq A_{n/4} \) and if \( z \leq j \leq n/2 \) then \( A_j \leq A_{n/2} \). Combining these two inequalities gives us that for \( n/4 \leq j \leq n/2 \)

\[
A_j \leq \max\{A_{n/4}, A_{n/2}\}.
\]

To compare \( A_{n/4} \) and \( A_{n/2} \) we use Stirling’s formula (4.14). For \( n \geq 2 \)

\[
A_{n/4} = \left( \frac{n!}{\left(\frac{n}{4}\right)!} \right)^2 \frac{1}{\left(\frac{1}{4}\right)!} \exp \left\{ -\left(\frac{3n}{8}\right) \log n - \frac{n}{2} \right\}
\leq \left( \frac{e^{\pi n/4} 4^{\pi/2}}{3^{\pi/2} n} \right) n^{-\frac{n}{2}} \left(\frac{4}{3}\right)^{2n} \frac{4^n 2^n e^{-3n}}{\pi n}
\]

\[
A_{n/2} = \left( \frac{n!}{\left(\frac{n}{2}\right)!} \right)^2 \frac{1}{\left(\frac{1}{2}\right)!} \exp \left\{ -\left(\frac{n}{2}\right) \log n - n \right\}
\leq \left( \frac{2 e^{\pi n/4}}{\pi n} \right) 2 e^{-\frac{3n}{2}} \leq 2 \frac{e^{\pi n/4}}{\pi n} e^{-\frac{3n}{2}}.
\]

It follows that

\[
n A_{n/4} \leq \exp \{\phi_1(n)\} \quad \text{and} \quad n A_{n/2} = \exp \{\phi_2(n)\},
\]

where

\[
\phi_1(n) = \log n - \frac{n}{8} \log n - \left(\frac{3n}{4}\right) + \left(\frac{3n}{2}\right) \log \left(\frac{4}{3}\right) + \left(\frac{n}{4}\right) \log 4
\]

\[
\phi_2(n) = \log n - \left(\frac{3n}{2}\right) + \left(\frac{3n}{2}\right) \log 2.
\]

For \( n \geq 10 \) we have \( \phi_1(n) \leq 0 \) and \( \phi_2(n) \leq 0 \) which implies that

\[
\sum_{j=n/4}^{n/2} A_j \leq \left(\frac{1}{4}\right) \max\{n A_{n/4}, n A_{n/2}\} \leq \frac{1}{4}.
\]

\( \square \)

**Proof of Lemma 4.6.** Let \( n/2 \leq j \leq n \) and \( B_j \) be as in equation (4.11). As usual, consider we consider the ratio between two consecutive term

\[
\frac{B_{j+1}}{B_j} = \frac{(n-j)^2}{\left(j+1\right)} \exp \left\{ -\log n - 2 \right\} \leq \left(\frac{n}{2}\right) \exp \left\{ -\log n - 2 \right\} \leq \frac{1}{2}
\]

Note that \( B_j \leq \left(\frac{1}{2}\right)^{n/2-j} B_{n/2} \), which implies that

\[
\gamma(n) = \sum_{j=n/2}^{n} B_j \leq B_{n/2} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2 B_{n/2}.
\]

Since \( B_{n/2} = A_{n/4} \leq (2/e)^{3n/2} \) then for \( n \geq 2 \) we have that \( \gamma(n) \leq 2(2/e)^{3n} \). \( \square \)
4.3 Random Insertions

In the random insertion shuffle for a deck of $n$ cards, one picks out a random card and inserts it back into the deck at a random position. This shuffle is modeled by the measure $q$ on the $S_n$ given by

$$q(\tau) = \begin{cases} 
1/n & \text{if } \tau = e \\
2/n^2 & \text{if } \tau = c_{i,j} \text{ s.t. } 1 \leq i, j \leq n \text{ and } |i - j| = 1 \\
1/n^2 & \text{if } \tau = c_{i,j} \text{ s.t. } 1 \leq i, j \leq n \text{ and } |i - j| > 1 \\
0 & \text{otherwise.}
\end{cases}$$

(4.19)

where $c_{i,j}$ denotes the cycle created by taking the card in position $i$ and inserting it into position $j$. A formal definition is given by

$$c_{i,j} = \begin{cases} 
e & \text{if } i = j \\
(j, j-1, \ldots, i+1, i) & \text{if } 1 \leq i < j \leq n \\
(j, j+1, \ldots, i-1, i) & \text{if } 1 \leq j < i \leq n.
\end{cases}$$

Random insertion is the first of the shuffles discussed in this paper for which it is not known whether there is a total variation cutoff or not although it is strongly believed that there is one. The results of [5, 6] show that there is a cutoff in $\ell^2$ but the exact cutoff time is not known. What is known and follows from [13] is that there is a pre-cutoff (in both total variation and $\ell^2$) at time $n \log n$. Finding the precise $\ell^2$ cutoff time and proving a cutoff in total variation are challenging open problems that have been investigated (but not solved) in [34] by Uyemura-Reyes.

**Theorem 4.9. (Diaconis and Saloff-Coste [13] and Uyemura-Reyes [34])** Let $q$ be the random insertion measure on $S_n$ defined above. For $c > 0$ and $t \geq 4n(\log n + c)$ there exists a constant $B$ such that

$$d_2(q(\tau), u) \leq Be^{-c}.$$ 

For any sequence $(k_n)$ such that $(2k_n - n \log n)/n$ tends to $-\infty$ as $n$ tends to $\infty$ then

$$d_{TV}(q(k_n), u) \to 1 \text{ and } d_2(q(k_n), u) \to \infty.$$ 

In [13] the mixing time in Theorem 4.9 is shown to be $O(n \log n)$ while in [34] the more precise upper bound given in Theorem 4.9 is shown. The proof of the upper bound in Theorem 4.9 relies on the comparison techniques developed in [12].

**Definition 4.10.** Let $V$ be a state space equipped with a Markov kernel $K$ with reversible measure $\nu$. The Dirichlet form associated to $(K, \nu)$ is

$$\mathcal{E}_{K,\nu}(f, g) = \langle (I - K)f, g \rangle_\nu = \sum_{x \in V} [(I - K)f(x)g(x)\nu(x)]$$

$$= \frac{1}{2} \sum_{x, y \in V} (f(x) - f(y))(g(x) - g(y))\nu(x)K(x, y).$$

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where \( f, g \in \ell^2(\nu, V) \). In the case where \( V \) is a finite group and \( p(x^{-1}y) = K(x, y) \) we set \( \mathcal{E}_{\nu, V} = \mathcal{E}_{K, \nu} \).

Diaconis and Saloff-Coste show the following theorem.

**Theorem 4.11. Diaconis and Saloff-Coste, [12]** Let \( q \) and \( \tilde{q} \) be the probability measures on a finite group \( G \). Set \( E = \mathcal{E}_{q, u}, \tilde{E} = \mathcal{E}_{\tilde{q}, u} \) and \( \beta_i, \tilde{\beta}_i, 0 \leq i \leq |G| - 1 \) to be the associated Dirichlet forms and eigenvalues of \( q \) and \( \tilde{q} \) respectively. Let \( \tilde{h}_t \) be the law at time \( t \) of the continuous time process associated with \( \tilde{q} \). If there exists a constant \( A \) such that \( \tilde{E} \leq A E \) then

\[
d_2(q(t), u)^2 \leq \beta_2 t_{1-} - (1 + d_2(\tilde{h}_{t/4}, u)^2) + d_2(\tilde{h}_{t/4}, u)^2.
\]

Equation (4.20) gives the following corollary to Proposition 4.6.

**Corollary 4.12.** Let \( q \) be the random insertion measure on \( S_n \) defined above. If \( n \geq 10, c \geq 2 \) and \( t \geq 2n(\log n + c) \) then

\[
d_2(q(t), u)^2 \leq e^{-(c-2)}.
\]

For any sequence \( (k_n) \) such that \( (2k_n - n \log n)/n \) tends to \( -\infty \) as \( n \) tends to \( \infty \) then

\[
\lim_{n \to \infty} d_{TV}(q(k_n), u) = 1.
\]

**Proof.** The upper bound results as a corollary to Proposition 4.6 after applying equation (4.20). The improvement by a factor of 2 compared to Theorem 4.9 is due to the use of the continuous time random transposition process in the comparison inequality (4.20).

Uyemura-Reyes also proves the total variation lower bound in his thesis but his proof uses a rather sophisticated argument involving results concerning the longest increasing subsequence. We give an alternative proof of this result based on a technique due to D. Wilson [35].

First note the following result of Uyemura-Reyes. Set \( \rho \) to be the permutation representation. Let \( q \) to be the random insertion measure and \( Q \) its associated Markov kernel such that \( Q(x, y) = q(x^{-1}y) \). In [34] it is shown that the Fourier transform \( \hat{q}(\rho) \) has an eigenvector \( v = (v_0, \ldots, v_n) \) where

\[
\hat{q}(\rho)v = \left(1 - \frac{1}{n}\right)v \quad \text{and} \quad v_i = 1 - \frac{2i}{n - 1}.
\]

As noticed at the end of Section 2.2 it follows that \( f_\rho(\sigma) = \langle \rho(\sigma)v, v \rangle, \sigma \in S_n \), is an eigenvector of \( Q \) with associated eigenvalue \( (1 - 1/n) \).
Computing $f_\rho(\sigma)$, one gets

$$f_\rho(\sigma) = \left\langle \prod_{i=0}^{n-1} \left( 1 - \frac{2i}{n-1} \right) e_i, \sum_{j=0}^{n-1} \left( 1 - \frac{2j}{n-1} \right) e_j \right\rangle$$

$$= \sum_{j=0}^{n-1} \left( 1 - \frac{2\sigma(j)}{n-1} \right) \left( 1 - \frac{2j}{n-1} \right)$$

$$= \sum_{j=0}^{n-1} 1 - \frac{2j}{n-1} - \frac{2\sigma(j)}{n-1} + \frac{4\sigma(j)j}{(n-1)^2}$$

$$= -n + \frac{4}{(n-1)^2} \sum_{j=0}^{n-1} \sigma(j)j.$$  \hspace{1cm} (4.21)

Therefore

$$f^2(\sigma) = n^2 - \frac{8n}{(n-1)^2} \sum_{j=0}^{n-1} \sigma(j)j + \frac{16}{(n-1)^4} \sum_{i,j=0}^{n-1} \sigma(i)\sigma(j)ij$$

and

$$\sum_{\sigma \in S_n} f^2(\sigma) = n!n^2 - \frac{8n}{(n-1)^2} \sum_{\sigma \in S_n} \sum_j \sigma(j) + \frac{16}{(n-1)^4} \sum_{i,j=0}^{n-1} \sum_{\sigma \in S_n} \sigma(i)\sigma(j)$$

$$= n!n^2 - \frac{8n(n-1)!}{(n-1)^2} \left( \frac{n(n-1)}{2} \right)^2 + \frac{16(n-2)!}{(n-1)^4} \left( \frac{n(n-1)}{2} \right)^4$$

$$= n!n^2 - \frac{8n^3(n-1)!}{4} + n^4(n-2)! = n! \left( \frac{n^2}{n-1} \right).$$

Next we estimate the supremum norm of the discrete square gradient of $f_\rho$ defined in (4.21). The discrete square gradient of the function $g$ with respect to the kernel $K$ is given by the equation

$$|\nabla g(x)|^2 = \frac{1}{2} \sum_y |g(x) - g(y)|^2 K(x, y).$$

Calculating the discrete square gradient for $f_\rho$ gives us

$$|\nabla f_\rho(\sigma)|^2 \leq \frac{16}{n^2(n-1)^4} \sum_{i,j=0}^{n-1} \sum_{k=0}^{n-1} \sigma^{-1}(k) (k - c_{ij}(k))^2$$

$$\leq \frac{16}{n^2(n-1)^2} \sum_{i,j=0}^{n-1} \sum_{k=0}^{n-1} |k - c_{ij}(k)|^2$$

where $c_{ij}$ is defined in (4.20). To calculate $k - c_{ij}(k)$ we consider the following two cases.
Case 1 If $i < j$

$$k - c_{ij}(k) = \begin{cases} 
  i - j & \text{if } k = i \\
  1 & \text{if } i < k \leq j \\
  0 & \text{otherwise.}
\end{cases}$$

Case 2 If $j < i$

$$k - c_{ij}(k) = \begin{cases} 
  i - j & \text{if } k = i \\
  -1 & \text{if } j \leq k < i \\
  0 & \text{otherwise.}
\end{cases}$$

It follows that

$$|\nabla f(\sigma)|^2 \leq \frac{16}{n^2(n-1)^2} \sum_{i<j} \sum_{k=0}^{n-1} |k - c_{ij}(k)|^2 + \sum_{j<i} \sum_{k=0}^{n-1} |k - c_{ij}(k)|^2$$

$$= \frac{16}{n^2(n-1)^2} \sum_{i<j} ((i - j)^2 + (j - i)) + \sum_{j<i} ((i - j)^2 + (i - j))$$

$$= \frac{16}{n^2(n-1)^2} \sum_{i,j=0}^{n-1} (i - j)^2 + |i - j|$$

$$\leq \frac{32}{n^2(n-1)^2} \sum_{i,j=0}^{n-1} (i - j)^2 \leq 32.$$
even, that is, any element in $C$ can be written as the product of an even number of transpositions and $C \neq \{e\}$ then it generates the alternating group $A_n$. Set $q_c$ to be the measure

$$q_c(\sigma) = \begin{cases} \frac{1}{\pi_c} & \text{if } \sigma \in C \\ 0 & \text{otherwise} \end{cases}$$

(5.22)

where $\#C$ denotes the number of elements in $C$. When $C$ is an odd conjugacy class the random walk driven by $q_c$ is be periodic and $q_c^k$ is supported on $A_n$ when $t$ is even, and on $S_n \setminus A_n$ otherwise. In this case, it is convenient to study the random walk on $A_n$ driven by $q_c^2$ to avoid periodicity.

The mixing time of these random walks was studied in \[22, 21, 24, 33\], among other works. See the discussion in \[30\]. For simplicity, we describe some of the known results in the case of even conjugacy classes. The same results hold in the odd case, modulo periodicity. In \[33\] it is shown that any sequence $(A_n, q_{c_n})$ has a total variation cutoff at time

$$t_1(n) = \inf \{ k : q_{c_n}^k(\varphi) \leq \log n \}$$

where $\varphi(\sigma)$ is the number of fixed points of $\sigma \in S_n$ and $q_{c_n}^k(\varphi)$ is the expected value of $\varphi$ taken according to the measure $q_{c_n}^k$. It is well known, see \[7, 27\], that

$$\varphi(\cdot) - 1 = \chi_{(n-1,1)}(\cdot) = n - 1 - \text{supp}(\cdot).$$

This implies that $\varphi - 1$ is an eigenfunction of $q_c$ with eigenvalue $\frac{\chi_{(n-1,1)}(C)}{n-1}.$ Thus we can rewrite $t_1(n)$ as

$$t_1(n) = \inf \left\{ k : (n-1) \left(1 - \frac{\text{supp}(C_n)}{n-1}\right)^k + 1 \leq \log n \right\}.$$

When supp$(C_n)$ is not too large (e.g., supp$(C_n)/n = o(1)$) then $t_1(n) \sim (n/\text{supp}(C_n)) \log n$ and when supp$(C_n)$ is very large then $t_1(n)$ is $O(1)$.

Assuming that supp$(C_n) \leq n - 1$, \[21\] shows that the random walk driven by $q_{c_n}$ has an $\ell^2$ pre-cutoff at time $t_2(n)$ where

$$\left| t_2(n) - \frac{2 \log n}{\log(n/(n - \text{supp}(C_n) + 1))} \right| \leq 3.$$

As in the total variation case, when supp$(C_n)$ is not too large then

$$t_2(n) \sim (n/\text{supp}(C_n)) \log n$$

and when supp$(C_n)$ is large we get the at $t_2(n)$ is $O(1)$. Here, we will focus on the continuous time process associated to $q_{c_n}$.

Corollary 4.1 in \[5\] implies that the continuous time process driven by $q_{c_n}$ has a total variation mixing time bounded above by that of the discrete time process. Arguments similar to those in Chapter 4 of \[25\] give a lower bound for the continuous time process that is comparable to the upper bound just mentioned.
In particular, for \( \text{supp}(C_n) \leq (n-1)/(\log(n-1) + 1) \), these arguments show that the continuous time chain associated to \( q_C \) has a total variation cutoff at time \( t_1(n) \).

Perhaps surprisingly, we show below that, in \( \ell^2 \), the mixing time of the continuous time process has a lower bound of \( (n/2) \log n \) for any conjugacy class with \( \text{supp}(C_n) \geq 2 \). A matching upper bound is shown when \( \text{supp}(C_n) \to \infty \) as \( n \to \infty \) as well as for the conjugacy class of 4-cycles.

### 5.2 \( \ell^2 \) lower bounds in continuous time

Throughout this section \( C_n \) is a conjugacy class in \( S_n \) (or \( A_n \)) and \( c_n \in C_n \) is an arbitrary fixed element in \( C_n \). Recall that \( \text{supp}(C_n) = n - \varphi(c_n) \) where \( \varphi(\cdot) \) is the number of fixed points.

**Theorem 5.1.** For each \( n \), set

\[
t_n = \frac{n}{2} \log n.
\]

For any odd conjugacy class \( C_n \subset S_n \) with \( \text{supp}(C_n) \geq 2 \), and any \( \epsilon \in (0, 1) \)

\[
\lim_{n \to \infty} d^2(h_{C_n,(1-\epsilon)t_n}, u_n) = \infty.
\]

In order to understand the mixing time of these continuous time processes we will again rely on (2.3) and we will need to estimate the dimensions and characters of some of the irreducible representations of \( S_n \). The following well known definitions and results will help us understand these quantities.

**Definition 5.2.** Let \( \lambda \) be a Young diagram with \( n \) boxes, as usual, we denote this by \( \lambda \vdash n \). The hook at the cell \((i, j)\) is defined as the set of boxes \( H_{i,j} \) where

\[
H_{i,j} = \{(i, l) : (i, l) \in \lambda, l \geq j\} \cup \{(k, j) : (k, j) \in \lambda, k \geq i\}.
\]

\( H_{i,j} \) has hook length \( h_{i,j} = |H_{i,j}| \).

**Theorem 5.3.** (The Hook formula) Let \( \lambda \) be a Young diagram with \( n \) boxes.

Set \( d_\lambda \) to be the dimension of the irreducible representation associated to \( \lambda \). Then

\[
d_\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}} \quad (5.23)
\]

With the hook formula we can now get an estimate on the dimension of some representations of \( S_n \).

**Lemma 5.4.** Let \( n \in \mathbb{N} \) and \( \lambda \vdash n \) be a Young diagram. If \( \lambda \) fits into a rectangle of \( s \times t \) boxes, then

\[
d_\lambda \geq \left( \frac{n}{e(s + t - 1)} \right)^n.
\]
Proof. Note that any hook in $\lambda$ will be of hook length at most $s + t - 1$. The inequality then follows from Stirling’s formula in (4.11) and hook formula (5.23).

We will use the following rather non-trivial bound on character ratios.

**Theorem 5.5.** ([23]) Let $a > 0$ be a fixed constant, and let $\lambda \vdash n$ be a Young diagram with at most $a\sqrt{n}$ rows and columns. Then there exists a constant $D = D(a)$ such that

$$ \left| \frac{\chi_{\lambda}(\sigma)}{d_{\lambda}} \right| \leq \left( \frac{D \max\{1, |\sigma|^2/n\}}{\sqrt{n}} \right)^{|\sigma|} $$

for any $\sigma \in S_n$ and where $|\sigma|$ is the minimal number of transpositions needed to write $\sigma$ as a product of transpositions.

Recall that, for any $\sigma \in S_n$ which is not the identity then $|\sigma| \leq \text{supp}(\sigma)$.

**Proof of Theorem 5.5.** The idea behind this proof is to write the desired $\ell^2$ distance as in equation (2.3) and find an irreducible representation which has large dimension and small character. The representations that are useful in this respect turn out to be those that have an approximately square shape.

Let $\lambda_n \vdash n$ be a Young diagram that fits into a box of side $\lceil \sqrt{n} \rceil$, so that $\lambda_n$ looks almost like a square. By Lemma 5.4 and the fact that $\lceil \sqrt{n} \rceil \leq 2\sqrt{n}$ we get that

$$ d_{\lambda_n} \geq \left( \frac{n}{2e\sqrt{n}} \right)^n \geq \left( \frac{\sqrt{n}}{4e} \right)^n. \quad (5.24) $$

If $C_n$ is a conjugacy class with $\text{supp}(C_n) \geq \sqrt{n}$ and $c_n \in C_n$, [24, Theorem 1] yields a positive constant $q < 1$ such that

$$ \left| \frac{\chi_{\lambda_n}(c_n)}{d_{\lambda_n}} \right| \leq q^{\text{supp}(C_n)} \leq q^{\sqrt{n}}. $$

If $2 \leq \text{supp}(C_n) \leq \sqrt{n}$, Theorem 5.5 implies that

$$ \left| \frac{\chi_{\lambda_n}(c_n)}{d_{\lambda_n}} \right| \leq \frac{D}{\sqrt{n}}. $$

In either case, we have that

$$ \left| \frac{\chi_{\lambda_n}(c_n)}{d_{\lambda_n}} \right| = o(1). $$

Using (5.24), we obtain that, for any $\epsilon > 0$,

$$ d_{\lambda_n}^2 \left\{ - (1 - \epsilon) n \log n \left( 1 - \frac{\chi_{\lambda_n}(c_n)}{d_{\lambda_n}} \right) \right\} \geq \left( \frac{n}{16e^2} \right)^n \exp \left\{ - (1 - \epsilon) n \log n (1 + o(1)) \right\} $$

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It now follows from (2.3)

\[
\lim_{n \to \infty} d_2(h_{C_n}, (1-\epsilon)t_n, u_n) \geq \lim_{n \to \infty} d_{\lambda_n} \exp \left\{ -(1-\epsilon)t_n \left( 1 - \frac{\chi_{\lambda_n}(c_n)}{d_{\lambda_n}} \right) \right\} = \infty
\]
as desired.

Using the same ideas as in the proof of Theorem 5.1 we get the following result.

**Theorem 5.6.** Let \( C_n \) be a conjugacy class in \( A_n \) with \( \text{supp}(C_n) \geq 2 \), and set \( u_n \) to be the uniform measure on \( A_n \). For any \( \epsilon \in (0,1) \) and \( t_n = \frac{n}{2} \log n \)

\[
\lim_{n \to \infty} d_2(h_{C_n}, (1-\epsilon)t_n, u_n) = \infty.
\]

**Remark 5.7.** For \( \epsilon > 0 \), it is interesting to consider the discrete time chain driven by

\[
\hat{q}_{C_n, \epsilon}(\sigma) = \begin{cases} 
\epsilon & \text{if } \sigma = e \\
\frac{1-\epsilon}{\#C_n} & \text{if } \sigma \in C_n \\
0 & \text{otherwise.}
\end{cases}
\]

When \( \epsilon = 1/2 \), this is often called the lazy chain associated to \( q_{C_n} \). The arguments used in the proof of Theorem 5.1 show that the random walk driven by \( \hat{q}_{C_n, \epsilon} \) will have a \( \ell^2 \) mixing time lower bound of \( (n/2) \log 1/\epsilon(n) \).

In [30] it is conjectured (Conjecture 9.3) that both the total variation mixing time and the \( \ell^2 \) mixing time of the random walk driven by \( \hat{q}_{C_n} \) will have an upper bound of \( (2n/\text{supp}(C_n)) \log n \). While this is true for in the case of total variation, the results above show that the bound does not hold for \( \ell^2 \).

The proof of Theorem 5.1 relies on the character estimates of [23] and [24]. While these estimates are very useful, one can construct simple Young diagram and use the Murnaghan-Nakayama Rule below to get estimates on the values of characters at a \( k \)-cycles, for infinitely many \( k \). This gives a much more accessible proof of a weaker version of Theorems 5.1 and 5.6. For further details on the following definitions see Section 4.10 in [27].

**Definition 5.8.** A skew hook \( \xi \) in a Young diagram is a collection of boxes that result from the projection of a regular hook along the right boundary of a Young diagram.

The leg length of a skew hook \( \xi \) is denote by \( ll(\xi) \) with

\[
ll(\xi) = \text{the number of rows of } \xi - 1.
\]

**Theorem 5.9.** (Murnaghan-Nakayama Rule) If \( \lambda \) is a partition of \( n \) and \( \alpha \in S_n \) such that \( \alpha \) has cycle type \( (\alpha_1, \alpha_2, \ldots, \alpha_i) \), then we have

\[
\chi_{\lambda}(\alpha) = \sum_{\xi} (-1)^{ll(\xi)} \chi_{\lambda \backslash \xi} (\alpha \setminus \alpha_1)
\]

where the sum runs over all skew hooks \( \xi \) of \( \lambda \) having \( \alpha_1 \) cells and \( \chi_{\lambda \backslash \xi} (\alpha \setminus \alpha_1) \) denotes that character of the representation \( \lambda \backslash \xi \) evaluated at an element of cycle type \( \alpha \setminus \alpha_1 \).
Lemma 5.10. For \( m \in \mathbb{N} \) set \( n = m(m+1)/2 \). Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \vdash n \) be a triangular Young diagram such that \( \lambda_i = m-i+1 \). Let \( c_k \) be a cycle of length \( k \). If \( k = 4i + 1 \) for \( i = 1, 2, \ldots \) then \( \chi_\lambda(c_k) > 0 \). If \( k \) is even then \( \chi_\lambda(c_k) = 0 \).

Proof. The Murnaghan-Nakayama rule implies that

\[
\chi_\lambda(c_k) = \sum_{|\xi| = k} (-1)^{|\xi|} d_\lambda|\xi|.
\]

Any hook in \( \lambda \) composed of must have even leg length by construction so it follows that \( \chi_\lambda(c_k) > 0 \). The second part of the proof follows directly from the Murnaghan-Nakayama rule and the fact that every hook in \( \lambda \) will have odd hook length, making it impossible to remove a skew hook of even length.

5.3 Total variation upper bounds in continuous time

As we mentioned at the beginning of this section, the mixing time of the continuous time process \( h_{c_n, t_n} \) will depend on whether one considers the total variation
or the ℓ² distance. In this section we derive a total variation upper bound of type $(n/\text{supp}(C_n)) \log n$ for the continuous time process associated to $q_{C_n}$. In the next section, we shall show that the ℓ² mixing time has an upper bound of $(n/2) \log n$ for the continuous time process when $\text{supp}(C_n) \to \infty$.

**Proposition 5.11.** Let $C_n$ be an even conjugacy class and $\pi_n$ to be the uniform measure on $A_n$. Let $T_n$ be the total variation cutoff time of $q_{C_n}$ (in discrete time) and assume that $T_n \to \infty$. Then, for any $\epsilon > 0$,

$$\lim_{n \to \infty} d_{TV}(h_{C_n,(1+\epsilon)T_n}, \pi_n) = 0$$

**Proof.** Let

$$T_{n,\epsilon}^d = \inf\{t \geq 0 : d_{TV}(q_{C_n}^{(t)}, \pi_n) \leq \epsilon\}$$

$$T_{n,\epsilon}^c = \inf\{t \geq 0 : d_{TV}(h_{C_n,t}, \pi_n) \leq \epsilon\}.$$

Corollary 4.1 in [5] shows that for any $\delta \in (0,1)$, $\epsilon > 0$ and $\eta \in (0, \epsilon)$ there exists an integer $N = N(\delta, \eta)$ such that

$$(1 - \delta)T_{n,\epsilon}^c \leq T_{n,\epsilon}^d$$

for all $n \geq N$.

In particular, for any $\epsilon > 0$ we can find a $\delta \in (0,1)$ and an $N_1 = N_1(\delta, \eta)$ such that for all $n \geq N$

$$T_{n,\eta}^c \leq \sqrt{1 + \epsilon} \quad T_{n,\eta/2}^d.$$

From [33] we know that the random walk driven by $q_{C_n}$ has cutoff, hence for any $\epsilon > 0$ and $\eta \geq 0$ there exists an $N_2 = N_2(\epsilon, \eta)$ such that for all $n \geq N_2$

$$T_{n,\eta/2}^d \leq \sqrt{1 + \epsilon} \quad T_n.$$

Combining the inequalities above gives that for any $\epsilon > 0$ and $\eta > 0$ there exists an $N = \max\{N_1, N_2\}$ such that for all $n \geq N$

$$T_{n,\eta}^c \leq (1 + \epsilon)T_n.$$

The desired result follows.

**Remark 5.12.** In the case of the lazy random walk $\tilde{q}_{C_n,1/2}$ defined in [5,23], one can show that the total variation mixing time is bounded by approximately twice that of the discrete time process $q_{C_n}$. (This is a more general phenomenon.) We only treat the case when $C_n$ is an even conjugacy class. Note that

$$d_{TV}(q_{C_n}^{(t_n)}, \pi_n) = \sum_{k=0}^{t} 2^{-\epsilon_n} \binom{t_n}{k} d_{TV}(q_{C_n}^{(k)}, \pi_n).$$

For any constant $D > 0$ set $I_n = [0, t_n/2 - D\sqrt{t_n}] \cup [t_n/2 + D\sqrt{t_n}, t_n]$. Then we have that

$$\sum_{k \in A_n} 2^{-\epsilon_n} \binom{t_n}{k} d_{TV}(q_{C_n}^{(k)}, \pi_n) \leq \sum_{k \in A_n} 2^{-\epsilon_n} \binom{t_n}{k}.$$
By the central limit theorem the right hand side tends to 0 as $D$ tends to $\infty$.

Outside of the set $I_n$ we get that

$$\sum_{k \notin A_n} 2^{-t_n \left( \frac{t_n}{k} \right)} d_{TV}(\hat{q}^{(k)}_{C_n}, \pi_n) \leq d_{TV}(\hat{q}^{(t_n/2-D\sqrt{t_n})}_{C_n}, \pi_n) \sum_{k \notin A_n} 2^{-t_n \left( \frac{t_n}{k} \right)}$$

$$\leq d_{TV}(\hat{q}^{(t_n/2-D\sqrt{t_n})}_{C_n}, \pi_n).$$

The arguments above shows that the cutoff time of the lazy walk is asymptotically $2T_n$. A similar argument would show that for any $\epsilon > 0$ the walk driven by $\tilde{q}_{C_n, \epsilon}$ has a cutoff time asymptotically equal to $(1/(1-\epsilon))T_n$.

### 5.4 Continuous time $\ell^2$ upper bounds: $\text{supp}(C_n) \rightarrow \infty$

Section 5.2 shows that the $\ell^2$ mixing time of $h_{C_n,t}$ must be at least $(n/2)\log n$ for all non trivial conjugacy classes. We show that when $\text{supp}(C_n)$ goes to $\infty$ as $n \rightarrow \infty$ and for the conjugacy class of 4-cycles the continuous time random walk has an $\ell^2$ cutoff at $(n/2)\log n$.

**Theorem 5.13.** Let $C_n$ be a conjugacy class such that $\text{supp}(C_n) \rightarrow \infty$ as $n \rightarrow \infty$. For any $\epsilon > 0$, and $t_n = (n/2)\log n$

1. $\lim_{n \rightarrow \infty} d_2(h_{C_n,(1+\epsilon)t_n}, u_n) = 0$ if $C_n$ is odd.
2. $\lim_{n \rightarrow \infty} d_2(h_{C_n,(1+\epsilon)t_n}, u_n) = 0$ if $C_n$ is even.

**Proof.** Let $C_n$ be an odd conjugacy class. Set $(\beta_i)_{i=0}^{n-1}$ to be the eigenvalues associated to the measure $q_{C_n}$ and $\lambda_i = 1 - \beta_i$. From (2.1) we know that

$$d_2(h_{C_n,(1+\epsilon)t_n}, u_n)^2 = \sum_{i=1}^{n-1} e^{-2(1+\epsilon)t_n \lambda_i} = \sum_{\lambda_i \leq 1} e^{-2t_n(1+\epsilon)\lambda_i} + \sum_{\lambda_i \geq 1} e^{-2t_n(1+\epsilon)\lambda_i}.$$  

We will use the following Calculus inequality.

**Claim 5.14.** For $w \geq 4$ and $0 \leq x \leq 1 - 1/w$ we have that $2\log(1 - x) \geq -wx$.

For $1/3 \geq \epsilon > 0$, $w = (1+\epsilon)/\epsilon \geq 4$, so by the claim above and (2.1) we obtain

$$d_2(h_{C_n,(1+\epsilon)t_n}, u_n)^2 \leq \sum_{1/w \leq \beta_i} \beta_i^{4t_n} + n! e^{-2t_n(1-1/w)(1+\epsilon)}$$

$$= \sum_{1/w \leq \beta_i} \beta_i^{4t_n} + n! e^{-n \log n}$$

We know that the eigenvalues of $q_{C_n}$ are just the normalized characters $\chi_{\rho}(c_n)/d_n$, $c_n \in C_n$, that occur with multiplicity $d_\rho^2$. Let $\rho_1$ and $\rho_2$ be the trivial
and sign representations respectively. When \( C_n \) is odd \( \chi_{\rho_2}(c_n)/d_{\rho_2} = -1 \), so the character associated to the sign representation does not contribute to the sum of eigenvalues above. Furthermore, (see, e.g., \cite{21} Lemma 2)

\[
d_2(q_{c_n}^{(2t)}, \overline{u}_n)^2 = \frac{1}{2} \sum_{\rho \neq \rho_1, \rho_2} d_{\rho}^2 \left( \frac{\chi_\rho(c_n)}{d_\rho} \right)^{4t}. \tag{5.27}
\]

It now follows that

\[
d_2(h_{C_n}, (1+\epsilon)t_n, u_n)^2 \leq \sum_{\rho \neq \rho_1, \rho_2} d_{\rho}^2 \left( \frac{\chi_\rho(c_n)}{d_\rho} \right)^{\epsilon 4t_n} + n!e^{-n \log n} \leq 2d_2(q_{c_n}^{(2t_n)}, \overline{u}_n)^2 + n!e^{-n \log n}.
\]

In \cite{21} it is shown that there exists a fixed constant \( D > 0 \) such that for \( t_n \) even and \( t_n \geq (Dn/\text{supp}(C_n)) \log n \) then \( d_2(q_{c_n}^{(t_n)}, \overline{u}_n) \to 0 \) as \( n \to \infty \). Since \( \text{supp}(C_n) \to \infty \) as \( n \to \infty \) then for large enough \( n \) we have that \( \epsilon n \log n \geq (Dn/\text{supp}(C_n)) \log n \) and the desired result follows. The case when \( C_n \) is an even conjugacy class can be treated in a similar way.

\[ \square \]

Remark 5.15. Let \( \tilde{q}_{c_n} \) be the lazy chain defined in (5.25). In the remark after Theorem 5.6 it is noted that the random walk driven by \( \tilde{q}_{c_n} \) will have a \( \ell^2 \) lower bound on the mixing time of \((n/2) \log_2(n)\). A matching upper bound for conjugacy classes \( C_n \) such that \( \text{supp}(C_n) \to \infty \) as \( n \to \infty \) follows from an argument similar to the proof of Theorem 5.13.

5.5 \( \ell^2 \) continuous time upper bound: 4-cycles.

The next theorem gives a sharp \( \ell^2 \) upper bound for the 4-cycle walk. In the case when \( \text{supp}(C_n) \to \infty \) we relied on the (rather deep) results of \cite{21, 33, 24} concerning the discrete time case to obtain a continuous time result matching our lower bound. This technique does not work for conjugacy classes with fixed support size. We conjecture that, with out any restriction on \( \text{supp}(C_n) \), \((n/2) \log_2(n)\) is a \( \ell^2 \) cutoff time for the family \((h_{C_n}, t)\). Note however that there is no reasons to hope for a proof simpler than that for random transposition. In discrete time, the only cases with fixed where support size for which the \( \ell^2 \) cutoff time has been determined are the cases of support size at most 6 (and the 7-cycles) treated in \cite{25, 26}. Using the techniques of \cite{25, 26} one can probably treat the corresponding continuous time processes, but this will be hard work. Here we focus on the 4-cycle walk. The reason is that we are able to reduce most technical computations to those already done above for transposition. We note that is is unlikely such reduction would work easily for 3-cycles and other even conjugacy classes (see \cite{25, 26}).

Recall that the conjugacy class of 4-cycles is denoted by \( c_4 \). We let \( c_4 \) be a given 4-cycle.
Theorem 5.16. For $n \geq 11$, $c \geq 2$ and $t \geq (n/2)(\log n + c)$
\[ d_2(h_{c,t}, u_n) \leq e^{-(c-2)} \]

We will use (2.3) again and bound $\chi_\rho(c_d)/d_\rho$, $c_d \in c_2$, with the same upper bounds that we used for $\chi_\rho(\tau)/d_\rho$, $\tau \in c_2$, in the case of transpositions in Proposition 4.6. In order to do this we will need the following definitions and lemmas.

Definition 5.17. If $\lambda' = (\lambda'_1, \ldots, \lambda'_j)$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ are two Young diagrams such that $\sum_{j=1}^k \lambda'_i = \sum_{i=1}^k \lambda_i = n$ and it is possible to get from $\lambda$ to $\lambda'$ by moving boxes up to the right then we say that $\lambda' \geq \lambda$.

Definition 5.18. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ denotes a Young diagram such that $\sum_{i=1}^m \lambda_i = n$. For any integer $l \geq 0$
\[ M_{\lambda,2l} = \sum_{j=1}^m \{ (\lambda_j - j)^l (\lambda_j - j + 1)^l - j^l (j - 1)^l \}. \]

Lemma 5.19. Let $\lambda'$ and $\lambda$ be two Young diagrams associated to irreducible representations of $S_n$. If $\lambda' \geq \lambda$ then $M_{\lambda',2l} \geq M_{\lambda,2l}$ for all $l \geq 0$.

Proof. It suffices to show that $M_{\lambda',2l} \geq M_{\lambda,2l}$ for that case when $a < b$ and $\lambda'_a = \lambda_a + 1$, $\lambda'_b = \lambda_b - 1$ and $\lambda'_c = \lambda_c$ for $c \neq a, b$. In this case,
\[ M_{\lambda',2l} - M_{\lambda,2l} = (\lambda_a - a + 1)^l \left\{ ((\lambda_a - a + 1) + 1)^l - ((\lambda_a - a + 1) - 1)^l \right\} + \lambda_b - b)^l \left\{ ((\lambda_b - b - 1)^l - (\lambda_b - b + 1)^l \right\}. \]

Set $x = \lambda_a - a + 1$ and $y = \lambda_b - b$ then $n \geq x \geq y \geq 1 - n$ and $M_{\lambda',2l} - M_{\lambda,2l} = f_{x,y}(l)$ where
\[ f_{x,y}(l) = x^l \left\{ (x + 1)^l - (x - 1)^l \right\} + y^l \left\{ (y - 1)^l - (y + 1)^l \right\}. \]

In [7] Diaconis shows that $f_{x,y}(1) \geq 0$ for $n \geq x \geq y \geq 1 - n$ which implies that $M_{\lambda',2} \geq M_{\lambda,2}$. We will show the general case by induction. Assume that $f_{x,y}(l) \geq 0$ then
\[ f_{x,y}(l+1) = x^{l+1} \left\{ (x + 1)^{l+1} - (x - 1)^{l+1} \right\} + y^{l+1} \left\{ (y - 1)^{l+1} - (y + 1)^{l+1} \right\} = x^2 (x^l \left\{ (x + 1)^l - (x - 1)^l \right\}) + y^2 (y^l \left\{ (y - 1)^l - (y + 1)^l \right\}) + x^{l+1} \left\{ (x + 1)^l + (x - 1)^l \right\} - y^{l+1} \left\{ (y + 1)^l + (y - 1)^l \right\} \geq x^{l+1} \left\{ (x + 1)^l + (x - 1)^l \right\} - y^{l+1} \left\{ (y + 1)^l + (y - 1)^l \right\}. \]

The last inequality follows since $f_{x,y}(l) \geq 0$. To conclude that $f_{x,y}(l+1) \geq 0$ we must check the following three cases.

Case 1: $x \geq y \geq 0$. This case follows directly from the assumption $x \geq y$. 

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Case 2: \( x \geq 0 \) and \( y \leq 0 \). Note that in this case
\[
\begin{align*}
x^{l+1} \{ (x+1)^l + (x-1)^l \} & \geq 0 \\
y^{l+1} \{ (y+1)^l + (y-1)^l \} & \leq 0.
\end{align*}
\]
The last inequality follows from the fact that \( l \) and \( l+1 \) are an odd and even numbers.

Case 3: \( y \leq x \leq 0 \). In this case let \( \tilde{x} = -x \) and \( \tilde{y} = -y \) then \( \tilde{y} \geq \tilde{x} \geq 0 \) and
\[
\begin{align*}
x^{l+1} \{ (x+1)^l + (x-1)^l \} - y^{l+1} \{ (y+1)^l + (y-1)^l \} &= \\
\tilde{y}^{l+1} \{ (\tilde{y}+1)^l + (\tilde{y}-1)^l \} - \tilde{x}^{l+1} \{ (\tilde{x}+1)^l + (\tilde{x}-1)^l \}.
\end{align*}
\]
Case 3 now follows directly from Case 1.

\[\square\]

Lemma 5.20. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \) denote a Young diagram such that \( \sum \lambda_i = n \). Then
\[
M_{2l, \lambda} \leq n(\lambda_1 - 1)^l\lambda_1^{l-1}.
\]
Proof.
\[
M_{2l, \lambda} = \sum_{j=1}^{j} (\lambda_j - j)^l(\lambda_j - j + 1)^l - j^l(j - 1)^l
\]
\[
\leq \sum_{\lambda_j \geq j-1} (\lambda_j - j)^l(\lambda_j - j + 1)^l
+ \sum_{\lambda_j < j-1} (\lambda_j - j)^l(\lambda_j - j + 1)^l - j^l(j - 1)^l
\]
For \( 0 \leq \lambda_j \leq j - 1 \) it is true that \( |\lambda_j - j| \leq j \) and \( |\lambda_j - j + 1| \leq j - 1 \) which implies that the second sum in the inequality above is negative. Therefore
\[
M_{2l, \lambda} \leq \sum_{\lambda_j \geq j-1} (\lambda_j - j)^l(\lambda_j - j + 1)^l \leq n(\lambda_1 - 1)^l\lambda_1^{l-1}.
\]
\[\square\]

Lemma 5.21. Let \( \rho \) be an irreducible representation of \( S_n \) and \( \lambda \) the associated Young diagram. For \( n \geq 11 \) the normalized character \( r_4(\lambda) = \chi_{\rho}(c_4)/d_\rho \) can be bounded as follows.
\[
r_4(\lambda) \leq \begin{cases} 
1 - \frac{2\lambda_1(n-\lambda_1)}{n(n-1)} & \text{if } \lambda_1 \geq n/2 \\
\frac{\lambda_1-1}{n-1} & \text{if } \lambda_1 \leq n/2.
\end{cases}
\]
Proof. Set \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \). In [17], Ingram shows that
\[
\frac{n!}{(n-4)!}r_4(\lambda) = M_{4, \lambda} - 2(2n-3)M_{2, \lambda}.
\]
(5.28)
Lemma 5.19 implies that $M_{2, \lambda} \geq M_{2, \lambda'}$ where $\lambda' = (\lambda_1, 1, \ldots, 1)$. We get,

$$M_{2, \lambda'} = (\lambda_1 - 1)\lambda_1 + \sum_{j=2}^{n-\lambda_1} (1 - j)(2 - j) - j(j - 1)$$

$$= (\lambda_1 - 1)\lambda_1 - 2 \sum_{j=1}^{n-\lambda_1-1} j$$

$$= (\lambda_1 - 1)\lambda_1 - (n - \lambda_1 - 1)(n - \lambda_1).$$

If $\lambda_1 \geq n/2$ then $M_{4, \lambda} \leq M_{4, (\lambda_1, n-\lambda_1)}$. Note that

$$M_{4, (\lambda_1, n-\lambda_1)} = (\lambda_1 - 1)^2\lambda_1^2 + (n - \lambda_1 - 1)^2(n - \lambda_1)^2 - 4 \leq (\lambda_1 - 1)^2\lambda_1^2 + (n - \lambda_1 - 1)^2(n - \lambda_1)^2$$

$$= [(\lambda_1 - 1)\lambda_1 - (n - \lambda_1 - 1)(n - \lambda_1)]^2 + 2(\lambda_1 - 1)\lambda_1(n - \lambda_1 - 1)(n - \lambda_1).$$

Hence if $\lambda_1 \geq n/2$, we have

$$M_{4, \lambda} - 2(2n - 3)M_{2, \lambda} = [(\lambda_1 - 1)\lambda_1 - (n - \lambda_1)(n - \lambda_1 - 1)]$$

$$\times [(\lambda_1 - 1)\lambda_1 - (n - \lambda_1)(n - \lambda_1 - 1) - 2(2n - 3)]$$

$$+ 2(\lambda_1 - 1)\lambda_1(n - \lambda_1)(n - \lambda_1 - 1).$$

Note that

$$(\lambda_1 - 1)\lambda_1 - (n - \lambda_1)(n - \lambda_1 - 1) - 2(2n - 3) \leq (n - 2)(n - 3).$$

It follows that

$$M_{4, \lambda} - 2(2n - 3)M_{2, \lambda} \leq (n - 2)(n - 3)[(\lambda_1 - 1)\lambda_1 - (n - \lambda_1)(n - \lambda_1 - 1)]$$

$$+ 2\lambda_1(\lambda_1 - 1)(n - \lambda_1)(n - \lambda_1 - 1).$$

(5.29)

If $\lambda_1 \geq n - 1$ then $2\lambda_1(\lambda_1 - 1)(n - \lambda_1)(n - \lambda_1 - 1) = 0$. If $\lambda_1 \leq n - 2$ then $\lambda_1(\lambda_1 - 1) \leq (n - 2)(n - 3)$. In either case, (5.29) gives that

$$r_4(\lambda) \leq \frac{(\lambda_1 - 1)\lambda_1 - (n - \lambda_1)(n - \lambda_1 - 1)}{n(n - 1)} + \frac{2(n - \lambda_1)(n - \lambda_1 - 1)}{n(n - 1)}$$

$$= 1 - \frac{2\lambda_1(n - \lambda_1)}{n(n - 1)}.$$

Next, we show the second part of the inequality. By Lemma 5.20 and (5.28), we have that for $\lambda_1 \leq n/2$

$$|r_4(\lambda)| \leq \frac{(n - 4)!}{n!} [n(\lambda_1 - 1)^2\lambda_3 + 2(2n - 3)n(\lambda_3 - 1)]$$

$$= \left(\frac{\lambda_1 - 1}{n - 1}\right) \left(\frac{\lambda_1 - 1}{n - 1}\right) \left[\frac{(\lambda_1 - 1)\lambda_1 + 2(2n - 3)}{(n - 2)(n - 3)}\right]$$

$$\leq \left(\frac{\lambda_1 - 1}{n - 1}\right) \left[\frac{n^2/4 + 4n - 6}{(n - 2)(n - 3)}\right] \leq \frac{\lambda_1 - 1}{n - 1}.$$

The last inequality holds for $n \geq 11$. \qed
Proof of Theorem 5.16. Recall that
\[ d_2(h_t, u_n) = \sum_{\lambda \neq 1} d_2^3 \exp \{-2t (1 - r_4(\lambda))\}. \]

In order to obtain the desired $e^{-2(c-2)}$ constant we will bound the term corresponding to $\lambda = (n-1, 1)$ separately. For $\lambda = (n-1, 1)$ we get that
\[ M_{2,(n-1,1)} = (n-2)(n-1) - 2 \quad \text{and} \quad M_{4,(n-1,1)} = (n-2)^2(n-1)^2 - 4 \]
which implies $r_4((n-1, 1)) = 1 - 4/(n-1)$. So for $t \geq (n/2)(\log n + c)$,
\[ d^2_{(n-1,1)} \exp\{-2t(1 - r_{(n-1,1)}(4))\} \leq (n-1)^2 \exp\{-4(\log n + c)\} \leq e^{-4c/n^2}. \]

Lemma 5.21 and equation (3.6) imply that for $t \geq (n/2)(\log n + c)$ we have
\[ d_2(h_t, u_n) \leq e^{-4c/n^2} + S_1 + S_2 \]
where
\[ S_1 = \sum_{j=2}^{n/2} \left( \frac{n!}{(n-j)!} \right)^2 \left( \frac{1}{j!} \right) \exp\{-j(\log n + c)\} \]
\[ S_2 = \sum_{j=n/2}^{n-1} \left( \frac{n!}{(n-j)!} \right)^2 \left( \frac{1}{j!} \right) \exp\{-j(\log n + c)\}. \]

For a more detailed description on how to obtain the sums $S_1$ and $S_2$ see the proof of Proposition 4.2. For $c \geq 2$ we have that
\begin{enumerate}
  \item $-c(2j)(n-j)/(n-1) \leq -2(c-2) - 2j$ when $2 \leq j \leq n/2$ and
  \item $-jc \leq -2(c-2) - 2j$ when $j \geq 2$.
\end{enumerate}

It follows that
\[ d_2(h_t, u_n) \leq e^{-2(c-2)} \left( \frac{1}{n^2} + \sum_{j=1}^{n/2} A_j + \sum_{j=n/2}^{n} B_j \right) \]
where $A_j$ and $B_j$ are defined in equations (4.16) and (4.17). Lemmas 4.7 and 4.8 now imply that for $n \geq 11$
\[ d_2(h_t, u_n) \leq e^{-2(c-2)} \left( \frac{1}{n^2} + \frac{2}{3} + \frac{1}{4} + 2 \left( \frac{2}{e} \right)^{3n/2} \right) \leq e^{-2(c-2)}. \]

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