Besov regularity of the uniform empirical process

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Abstract
The paths of Brownian motion have been widely studied in the recent years relatively in Besov spaces $B_{p,\infty}^\alpha$. The results are the same as to the Brownian bridge. In fact these regularities properties are established in some sequence spaces $S_{p,\infty}^\alpha$ using an isomorphism between them and $B_{p,\infty}^\alpha$.

In this note, we are concerned with the regularity of the paths of the continuous version of the uniform empirical process in the space $S_{p,\infty}^\alpha$ and in one of his separable sub space $S_{p,0}^\alpha \pi l$ for a suitable choice of $\alpha$ and $p$.

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1 Introduction
Let $U_1, U_2, \ldots, U_n, \ldots$ be a sequence of i.i.d $\mathcal{U}(0,1)$ random variables. For a fix integer $n \geq 1$ we consider the empirical distribution function $\tilde{F}_n$ of the sample $U_1, U_2, \ldots, U_n$ defined by

$$\forall 0 \leq s \leq 1, \quad \tilde{F}_n(s) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,s]}(U_i)$$

and for $j \geq 0$, $k = 1, \ldots, 2^j$ the triangular sequence

$$\tilde{\alpha}_{nkj}^n = 2^{j/2} \left[ 2\tilde{\alpha}_n \left( \frac{k-1/2}{2^j} \right) - \tilde{\alpha}_n \left( \frac{k-1}{2^j} \right) - \tilde{\alpha}_n \left( \frac{k}{2^j} \right) \right]$$

(1.1)

where $\tilde{\alpha}_n$ is the associated empirical process defined by $\tilde{\alpha}_n(s) = \sqrt{n}(\tilde{F}_n(s) - s), \quad 0 \leq s \leq 1$.

Our motivation in the study of this sequence is given by previous works on the regularity of the paths of the Brownian motion in Besov spaces $B_{p,\infty}^\alpha$ given by

$$B_{p,\infty}^\alpha = \{ f \in L^p([0,1]) : \sup_t \frac{w_p(f,t)}{t^\alpha} < \infty \}$$

where for any $1 \leq p < \infty$,

$$w_p(f,t) = \sup_{|h| \leq t} \left( \int_{I_h} |f(x-h) - f(x)|^p \, dx \right)^{1/p}; \quad I_h = \{ x \in [0,1], x - h \in [0,1] \}.$$
The space \( B_{p, \infty}^\alpha \) endowed with the following norm

\[
\|f\|_{B_{p, \infty}^\alpha} = \sup\{|f_0|, |f_1|, \sup_j 2^{-j(1/2-\alpha-1/p)} \left( \sum_{k=1}^{2^j} |f_{jk}|^p \right)^{1/p} \}
\]

where

\[
f_0 = f(0), \ f_1 = f(1) - f(0), \ f_{jk} = 2^{j/2} \left[ 2f(\frac{k - \frac{1}{2}}{2^j}) - f(\frac{k - 1}{2^j}) - f(\frac{k}{2^j}) \right]
\]

is a Banach space.

It is well known thanks to Ciesielski et al [2], that there exists an isomorphism between such spaces and the Banach spaces of sequences \( (S_{p, \infty}^\alpha, \|\cdot\|_{p, \infty}^\alpha) \) defined by

\[
\{\mu = (\mu_{jk}, \ j \geq 0, \ k = 1, \ldots, 2^j)/ \|\mu\|_{p, \infty}^\alpha < \infty \}
\]

where

\[
\|\mu\|_{p, \infty}^\alpha = \sup\{|\mu_0|, |\mu_1|, \sup_j 2^{-j(1/2-\alpha-1/p)} \left( \sum_{k=1}^{2^j} |\mu_{jk}|^p \right)^{1/p} \}
\]

Their subsets \( B_{p, \infty}^{\alpha, 0} \) (respectively \( S_{p, \infty}^{\alpha, 0} \)) of functions \( f \in B_{p, \infty}^\alpha \) (resp of sequences \( (\mu_{jk}) \in S_{p, \infty}^\alpha \)) such that \( w_p(f, t) = o(t^\alpha) \) as \( t \to 0 \) (resp \( 2^{-j(1/2-\alpha-1/p)} \left( \sum_{k=1}^{2^j} |\mu_{jk}|^p \right)^{1/p} \to 0 \) as \( j \to \infty \)) are separable Banach spaces.

Thanks to this isomorphism, Roynette [3] proved that for \( p \geq 2 \) and \( \alpha < 1/2 \), the Brownian path \( (W_t, 0 \leq t \leq 1) \) belongs almost surely in \( B_{p, \infty}^\alpha \) but not in \( B_{p, \infty}^{\alpha, 0} \) by establishing

\[
\sup_j \left( 2^{-j} \sum_{k=1}^{2^j} |g_{jk}|^p \right)^{1/p} < \infty \quad \text{and} \quad \liminf_{j \to +\infty} \left( 2^{-j} \sum_{k=1}^{2^j} |g_{jk}|^p \right)^{1/p} > 0
\]

where for all \( j \geq 0 \) and \( k = 1, \ldots, 2^j \), \( g_{jk} = 2^{j/2} \left[ 2W(\frac{k-\frac{1}{2}}{2^j}) - W(\frac{k-1}{2^j}) - W(\frac{k}{2^j}) \right] \).

This result can be extended to the Brownian bridge \( b_t = W_t - tW_1, \ t \in [0, 1] \) which is closely related to the uniform empirical process. Moreover Komlos et al [3] show that on a suitable probability space \( (\mathcal{A}, \mathcal{P}) \), there exists a sequence of i.i.d. \( U(0, 1) \) random variables \( U_1, U_2, \ldots \), and a sequence of Brownian bridges \( \{b_n(t), 0 \leq t \leq 1\} \) such that almost surely

\[
\limsup_{n \to +\infty} \frac{\sqrt{n}}{\log n} \sup_{0 \leq t \leq 1} |\alpha_n(t) - b_n(t)| < \infty.
\]

Our aim is to investigate Roynette’s result for the Brownian bridge to the continuous version of the empirical process. We successfully get a result for \( 1 \leq p \leq 2 \) and \( \alpha = 1/2 \).

## 2 Empirical process and Besov Spaces

In order to face to the lack of smoothness of the classical empirical process, we first recall the following result:
Lemma 2.1 For every $n \geq 1$, the empirical distribution process $\tilde{\alpha}_n$ admits a continuous version $\alpha_n$.

Proof: It is well known one can express the distribution empirical function $\tilde{F}_n(\cdot)$ in terms of the order statistics $U_1^{(n)} \leq U_2^{(n)} \leq \cdots \leq U_n^{(n)}$ of the sample $U_1, U_2, \ldots, U_n$ as follows

$$\tilde{F}_n(s) = \begin{cases} 0, & U_1^{(n)} > s, \\ \frac{k}{n}, & U_k^{(n)} \leq s < U_{k+1}^{(n)}, \quad k = 1, 2, \ldots, n-1, \\ 1, & U_n^{(n)} \leq s. \end{cases}$$

Let us consider the function $F_n(\cdot)$ defined for every $0 \leq s \leq 1$ by

$$F_n(s) = \tilde{F}_n(U_k^{(n)}) + \frac{2}{n} \left( s - \frac{U_k^{(n)} + U_{k+1}^{(n)}}{2} \right) \left( \frac{\tilde{F}_n(U_{k+1}^{(n)}) - \tilde{F}_n(U_k^{(n)})}{U_{k+2}^{(n)} - U_k^{(n)}} \right),$$

if

$$\frac{U_k^{(n)} + U_{k+1}^{(n)}}{2} \leq s \leq \frac{U_k^{(n)} + U_{k+1}^{(n)}}{2}.$$

It is easy to see that for every $n \geq 1$,

$$\sup_{0 \leq s \leq 1} |F_n(s) - \tilde{F}_n(s)| \leq \frac{1}{n}. \quad (2.1)$$

As a consequence of (2.1), we deduce that $\tilde{F}_n$ is a continuous version of $\tilde{F}_n$ and the process $\alpha_n(s) = \sqrt{n}(F_n(s) - s)$, $0 \leq s \leq 1$ is a continuous version of the associated empirical process $\tilde{\alpha}_n(s) = \sqrt{n}(\tilde{F}_n(s) - s)$, $0 \leq s \leq 1$.

We are now in position to formulate our main results

Theorem 2.1 For every $n \geq 1$, the process $\alpha_n$ satisfy almost surely

$$\{\alpha_{jk}^n\}_{j,k} \in S_{2,\infty}^{1/2} \quad \text{and} \quad \{\alpha_{jk}^n\}_{j,k} \notin S_{2,\infty}^{1/2,0}.$$

Proof: Let us consider the triangular sequence given by (1.1) (replacing $\tilde{\alpha}_n$ by $\alpha_n$), we deduce thanks to the distribution empirical process that

$$\forall \, j \geq 0, \, \forall \, k = 1, \ldots, 2^j, \quad \alpha_{jk}^n = \frac{2^j/2}{\sqrt{n}} \sum_{i=1}^{n} Z_{jk}(i)$$

where

$$\forall \, i = 1, \ldots, n, \quad Z_{jk}(i) = Z_{jk}(U_i) = 1_{\left\{ \frac{k-1}{2^j}, \frac{k-1/2}{2^j} \right\}}(U_i) - 1_{\left\{ \frac{k-1/2}{2^j}, \frac{k}{2^j} \right\}}(U_i).$$

Notice that for any $i = 1, \ldots, n$, $Z_{jk}(i) \in \{1, 0, -1\}$ respectively with probability $\frac{k}{2^j}, 0, \frac{k}{2^j}$. We deduce that for any $i = 1, \ldots, n$, $Z_{jk}(i)$ is centered random variable with variance $2^{-j}$.

Let us define $G_{jk} = |\alpha_{jk}^n|^2 = \frac{2^j}{n} H_{jk}$ where $H_{jk} = \left( \sum_{i=1}^{n} Z_{jk}(i) \right)^2$. 

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Using the fact that for any fix $k$ and $i \neq h$ the random variables $Z_{jk}(i)$ and $Z_{jk}(h)$ are independent we deduce that

$$E(H_{jk}) = E\left( \sum_{i=1}^{n} Z_{jk}^2(i) + \sum_{i \neq h} Z_{jk}(i)Z_{jk}(h) \right) = \frac{n}{2^j}.$$ 

which implies in particular $E(G_{jk}) = 1$. Furthermore for any $j \geq 0$ and $k = 1, \ldots, 2^j$, we have

$$H_{jk}^2 = \left( \sum_{i=1}^{n} Z_{jk}(i) \right)^2 + 2 \sum_{l=1}^{n} Z_{jk}(l) \sum_{i \neq h} Z_{jk}(i)Z_{jk}(h) + \left( \sum_{i \neq h} Z_{jk}(i)Z_{jk}(h) \right)^2$$

$$= \sum_{i=1}^{n} Z_{jk}^2(i) + 2 \sum_{i < h} Z_{jk}(i)Z_{jk}^2(h) + 2 \sum_{i \neq h} Z_{jk}^2(i)Z_{jk}^2(h) + A_{jk}^{(1)} + A_{jk}^{(2)}$$

where for every $j \geq 0$ and $k = 1, \ldots, 2^j$,

$$A_{jk}^{(1)} = 2 \sum_{l=1}^{n} \sum_{i \neq h} Z_{jk}(l)Z_{jk}(i)Z_{jk}(h) \quad \text{and} \quad A_{jk}^{(2)} = 2 \sum_{l \neq m} \sum_{i \neq h} Z_{jk}(i)Z_{jk}(h)Z_{jk}(l)Z_{jk}(m)$$

It is easy to see that for every $j \geq 0$ and $k = 1, \ldots, 2^j$, $A_{jk}^{(1)}$ is a centered random variable and $A_{jk}^{(2)}$ satisfies

$$A_{jk}^{(2)} = 4 \sum_{i < h} \left[ \sum_{l < m} Z_{jk}(i)Z_{jk}(h)Z_{jk}(l)Z_{jk}(m) \right]$$

$$= 4 \sum_{i < h} \left[ \sum_{(l,m) = (i,h)} Z_{jk}(i)Z_{jk}(h)Z_{jk}(l)Z_{jk}(m) \right] + 4 \sum_{i < h} \left[ \sum_{(l,m) \neq (i,h)} Z_{jk}(i)Z_{jk}(h)Z_{jk}(l)Z_{jk}(m) \right]$$

The expectation of the last sum vanish thanks to the independence of the random variables. Hence there exists a constant $c > 0$ which may change from line to line such that $E(A_{jk}^{(2)}) = c E \left( \sum_{i < h} Z_{jk}^2(i)Z_{jk}^2(h) \right)$. We deduce that for every $j \geq 0$ and $k = 1, \ldots, 2^j$,

$$E(H_{jk}^2) = \sum_{i=1}^{n} E(Z_{jk}^2(i)) + c E \left( \sum_{i < h} Z_{jk}^2(i)Z_{jk}^2(h) \right) = \frac{n}{2^j}(1 + c \frac{n - 1}{2^j})$$

which implies in particular for every $j \geq 0$ and $k = 1, \ldots, 2^j$,

$$Var(G_{jk}) = \frac{2^j}{n} \left[ 1 + \frac{n(c-1) - c}{2^j} \right].$$

Elsewhere we have

$$Var(\sum_{k=1}^{2^j} G_{jk}) = \sum_{k=1}^{2^j} Var(G_{jk}) + 2 \sum_{1 \leq k < k' \leq 2^j} \frac{2^j}{n^2} \text{cov}(H_{jk}H_{jk'})$$

$$H_{jk}H_{jk'} = \left( \sum_{i=1}^{n} Z_{jk}(i)Z_{jk'}(i) + \sum_{i \neq h} Z_{jk}(i)Z_{jk'}(h) \right)^2$$

$$= \sum_{i=1}^{n} Z_{jk}(i)Z_{jk'}(i)$$
Notice that for $k \neq k'$, the product $Z_{jk}(i)Z_{jk'}(i)$ is null. This implies for every $j \geq 0$ and $k \neq k' \in \{1, ..., 2^j\}$,

$$H_{jk}H_{jk'} = \left( \sum_{i \neq h}^n Z_{jk}(i)Z_{jk'}(h) \right)^2 = \sum_{i \neq h}^n \sum_{l \neq m}^n Z_{jk}(i)Z_{jk'}(h)Z_{jk}(l)Z_{jk'}(m) = 2 \sum_{\text{card}(i,h) \cap \{l,m\} = 2} Z_{jk}(i)Z_{jk'}(h)Z_{jk}(l)Z_{jk'}(m)$$

(2.2)

So two cases can be investigated:

if $\text{card}(i,h) \cap \{l,m\} < 2$, extracting one random variable $Z_{jk}(i)$, the expectation of the last term in (2.2) is null.

if $\text{card}(i,h) \cap \{l,m\} = 2$, we have either $(l = i$ and $m = h)$ or $(l = h$ and $m = i)$. In this last case the product is equal to zero. It remains

$$E(H_{jk}H_{jk'}) = \sum_{i \neq h}^n E[Z_{jk}^2(i)Z_{jk'}^2(h)] = \frac{n(n-1)}{2^{2j}}$$

$$\text{Var}\left(\sum_{k=1}^{2^j} G_{jk}\right) = \sum_{k=1}^{2^j} \frac{2^j}{n} (1 + \frac{3n-4}{2^j}) + 2 \sum_{1=k<k'\leq2^j} \frac{2^{2j}}{n^2} \left( \frac{n(n-1)}{2^j} - \left( \frac{n}{2^j} \right)^2 \right)$$

$$= 2^{2j}\varepsilon_{nj}, \ \text{where} \ \varepsilon_{nj} = \frac{1}{2^j}(3 - \frac{3}{n})$$

Exploiting Bienaymé-Tchêbychev inequality, we obtain the following estimate for every $n \in \mathbb{N}$ and

$$\forall j \geq 0, \ \mathbb{P}\left(\left|\sum_{k=1}^{2^j} \left|\alpha_{jk}^n\right|^2 - 1\right| \geq \frac{1}{2}\right) \leq 4 \varepsilon_{nj}$$

Therefore thanks to Borel-Cantelli lemma, we deduce that for any $n \in \mathbb{N}$

$$\frac{1}{2} \leq 2^{-j} \sum_{k=1}^{2^j} \left|\alpha_{jk}^n\right|^2 \leq \frac{3}{2} \ p.s, \ j \ large \ enough.$$  

Hence

$$\sup_j \left(2^{-j} \sum_{k=1}^{2^j} \left|\alpha_{jk}^n\right|^2 \right)^{1/2} < \infty \ \text{and} \ \liminf_{j \to +\infty} \left(2^{-j} \sum_{k=1}^{2^j} \left|\alpha_{jk}^n\right|^2 \right)^{1/2} > 0 \ a.s.$$  

Remark:

1. Theses results show that for any $n$, the sequence $(\alpha_{jk}^n)$, $j \geq 0, \ k = 1, ..., 2^j$ belongs in the space $S_{2,\infty}^{1/2}$ and not in $S_{2,\infty}^{1/2,0} \ a.s.$

2. The first result of our theorem can be extended to $1 \leq p \leq 2$ since the $L^p$ norm is incerasing in $p$.  

5
References

[1] B. Boufoussi (1994). Espaces de Besov: Caractérisations et applications, Thèse de l’Université Henri Poincaré, Nancy I, France.

[2] Z. Ciesielski, B. Roynette, G. Kerkyacharian (1993). Quelques espaces fonctionnels associés à des processus gaussiens, Studia Mathematica, 107, p. 171-204.

[3] J. Komlós, M. Major, G. Tusnády (1975). Weak convergence and embedding. In colloquia Math.Soc. Janos.Boylai.Limit Theorems of Probability Theory, 149-165. Amsterdam, North-Holland.

[4] B. Roynette (1993). Mouvement brownien et espaces de Besov, Stochastics and Stochastics Reports, 43, p. 221-260.