Direct Interactions
in Relativistic Statistical Mechanics

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Directly interacting particles are considered in the multitime formalism of predictive relativistic mechanics. When the equations of motion leave a phase-space volume invariant, it turns out that the phase average of any first integral, covariantly defined as a flux across a $7n$-dimensional surface, is conserved. The Hamiltonian case is discussed, a class of simple models is exhibited, and a tentative definition of equilibrium is proposed.
1. Introduction

It is a pleasure to contribute this paper in the honor of prof. Larry P. Horwitz who has done (and keeps doing !) so much in practically all the domains of relativistic dynamics.

Relativistic statistical mechanics has a long story, but we may notice that, whereas the theory of relativistic ideal gases has received deep and detailed developments, little has been achieved in order to account for mutual interactions between particles. Most work available in the litterature concerns kinetic theory [1], and it is only through wild simplifications that collision terms can be somehow “derived” from the underlying $n$-body dynamics. For instance various results are obtained with the help of a straight-line approximation which amounts to calculate the force applied to each particle as if all other particles were undergoing free motion. This treatment retains very little information (if any) about the underlying dynamics.

Another approach consists in giving up manifest covariance in order to deal with a more familiar framework, where the concepts and methods of newtonian statistical mechanics are intensively employed. This approach is plagued with unpleasant complications (and sometimes serious difficulties) concerning relativistic invariance.

Looking for a more satisfactory theory, where interactions between particles would be described in a fully covariant fashion, several authors [2][3] have hesitated between two points of view: field theory and action at a distance.

In fact, if one aims at rigorous results, it is hardly possible to deal with the tremendous infinity of the field degrees of freedom. It seems more convenient to eliminate the field variables, and to formulate equations of motion for the $\text{particle degrees of freedom}$ only.

This is the case for Feynman-Wheeler electrodynamics [4], or more generally for any interaction $a\ la\ Van\ Dam$ and Wigner [5]. But these versions of relativistic dynamics are not yet tractable; in the best cases, they lead to difference-differential equations (they involve retarded arguments) and seem to require an infinite number of initial data for determining the trajectories.

In contradistinction, the modern theories of $\text{directly interacting particles}$ [6]are tailored for dealing with only a finite number of degrees of freedom. They naturally suppose that the field degrees of freedom have been eliminated, but in addition they assume from the start that a finite number of initial data is sufficient in order to determine the evolution of the system. In this sense they are a simplification of the approach initiated by the authors of refs [4][5].

In this spirit, we have proposed many years ago that an $n$-body relativistic system be described by truly differential equations of motion, involving $n$ independent evolution parameters [7][8]. Our departure from the Feynman-Wheeler scheme is the requirement
that the right-hand sides of the equations depend only on the positions (in space-time) and their first derivatives \[9\]. Following the specialized litterature, let us refer to this approach as **predictive mechanics**; clearly this terminology accounts for the possibility to determine, in principle, world lines from the knowledge of (a finite number of) initial data \[10\].

Lapiedra et al. \[11\] have undertaken an application of predictive methods to relativistic statistical mechanics. However their definitions for averages, normalization and equilibrium are not manifestly covariant.

A quite different view of direct relativistic interactions has been proposed by Horwitz and Piron \[12\]; in their approach, a unique ”historical time” is supposed to describe the motion of any number of particles; in other words a single-parameter evolution group acts in a finite-dimensional covariant phase space. Along this line, Horwitz et al. \[13\] have extensively developed a ”statistics of events” considered off the mass shell.

In contradistinction we intend to build a **statistics of orbits**, in agreement with Hakim’s philosophy (which is after all very close to the principles of predictive mechanics).

In a covariant framework like ours, the relativistic analog of a function integrated over phase space usually appears as the flux of a generalized current across some manifold \[14\] \[2\]. For instance the \(n\)-body distribution function \(N\) is normalized through the formula \(\int N\alpha = 1\), where \(\alpha\) is, in our notation, a \(7n\)-differential form introduced by Hakim (\(N\alpha\) is interpreted as the **numerical flux of particles**). Independence of the result with respect to the manifold where integration is performed crucially requires that the integrand be a closed form in the sense of differential geometry.

In fact the importance of having \(d(N\alpha) = 0\) was clearly emphasized in ref. \[2\]. But in that work it was assumed from the outset that \(\int_\Delta N\alpha\) represents the probability that the system be in a state characterized by a point of phase space belonging to a domain \(\Delta\) in a \(7n\)-manifold \(S^{7n}\) which extends to infinity. Then, invoking that the number of particles is conserved, it was concluded that \(d(N\alpha)\) is necessarily zero.

We shall look at these matters in a different way. Let us consider a fixed number \(n\) of particles, and suppose that their dynamics is given; for instance we write some equations of motion. Now we cannot just assume that \(\int_\Delta N\alpha\) is the above mentioned probability, without first proving that its value does not depend on the cutting surface \(S^{7n}\) if the same orbits are considered. In other words the definition of ”numerical flux of particles” should be consistent with the dynamics: **its conservation should stem from the equations of motion** (but we cannot hope to satisfy this requirement without specifying some technical information about the nature of these equations).

The main goal of this article consists in showing that direct interactions, especially when
formulated as in predictive mechanics, exactly satisfy this requirement.

Owing to the advantage of dealing with a finite number of degrees of freedom, a rather rigorous setting can be elaborated. We first recollect in Section 2 some useful results of differential geometry concerning measure-preserving vector fields which have vanishing Lie brackets among themselves (in these matters we try to speak of incompressible vector fields only, postponing as far as possible the use of Hamiltonian vector fields).

Section 3 is devoted to the treatment of direct interactions in the predictive formalism. The basic equations of statistical mechanics are introduced following Hakim’s early work [2]. Then conservation of phase averages easily comes out by straightforward application of the geometrical results gathered in Section 2. Not only the flux of particles but also the entropy flux is a closed differential form.

The Hamiltonian formalism is considered in Section 4; a possible ambiguity related with the noncanonical nature of physical positions is discussed and removed with the help of a certain idea of the underlying quantum mechanics. Energy, linear and angular momenta are introduced therein.

A possible definition of equilibrium ensembles is tentatively proposed in Section 5, and the last Section contains a few concluding remarks.

2. Geometrical considerations.
In this section we consider an \( m \)-dimensional orientable manifold \( V^m \). Capital indices \( A, B = 1, \ldots, m \).

According to the standard notations of differential geometry, \( i_X \) denotes the inner product (of a differential form) by the vector-field \( X \), and \( \mathcal{L}_X \) is the Lie derivative.

In arbitrary coordinates \( y^1, \ldots y^m \), a volume element is a scalar \( m \)-form \( \eta = \gamma(y) \, d^m y \), where \( \gamma \) is strictly positive and transforms as a scalar density. If \( \varepsilon_{A_1 \ldots A_m} \) is the Levi-Civita object, we define the completely skew-symmetric tensor

\[
\eta_{A_1 \ldots A_m} = \gamma \, \varepsilon_{A_1 \ldots A_m}
\]

Naturally we have set

\[
d^m y = \frac{1}{(m)!} \, \varepsilon_{A_1 \ldots A_m} \, dy^{A_1} \wedge \ldots dy^{A_m} = dy^1 \wedge \ldots dy^m
\]

We say that \( X \) preserves \( \eta \) or equivalently that \( X \) is incompressible when \( \mathcal{L}_X \eta = 0 \).

For \( n \) such that \( 1 \leq n < m \) consider independent vector fields \( X_1, \ldots X_n \). We associate to them the \( (m-n) \)-form

\[
\alpha(X_a, \eta) = X_1^{A_1} \ldots X_n^{A_n} \, d\Sigma_{A_1 \ldots A_n} \tag{2.1}
\]
where we define
\[
d\Sigma_{A_1...A_n} = \frac{1}{(m-n)!} \eta A_1...A_n A_{n+1}...A_m dy^{A_{n+1}} \wedge ... \wedge dy^{A_m}
\] (2.2)

Actually \(d\Sigma_{A_1...A_n}\) is the element of \((m-n)\)-surface (imbedded in \(V^{8n}\)) induced by \(\eta\).

Indices \(a, b = 1,...n\). Our notation for \(\alpha\) means to recall that \(\alpha\) depends not only on the vector fields \(X_a\), but also on the volume form.

**Remark:** If all the vector fields \(X_a\) are tangent to a \((m-n)\)-manifold \(S^{m-n}\), then \(\alpha\) vanishes on this manifold.

For typographical convenience we define \(i_a = i_{X_a}, \quad L_a = \mathcal{L}_{X_a}\).

In this notation \(\alpha\) is obviously proportional to \(i_1 i_2 .... i_n \eta\), which motivates our attention to the statements presented below.

For applications to statistical mechanics it is of interest that \(\alpha\) be a closed differential form (that is \(d\alpha = 0\)). Actually this property is ensured by a couple of results presented below.

**Proposition I**

*For \(n < m\), consider on \(V^m\) the vector fields \(X_1, X_2...X_n\), preserving the volume element \(\eta\). If \([X_a, X_b] = 0\) then for all \(r \leq n\) and for all \(b\) we have

\[
\mathcal{L}_b(i_1...i_r \eta) = 0
\] (2.3)*

**Proof**

The vanishing of Lie brackets implies [15] that \(\mathcal{L}_a\) commutes with \(i_b\). Hence \(\mathcal{L}_a i_b \eta = i_b \mathcal{L}_a \eta = 0\). Thus \(\mathcal{L}_b\) commutes with all the contractions \(i_1...i_r\). We end up with \(i_1....i_r \mathcal{L}_b \eta\), obviously zero.

We can also assert the following statement

**Proposition II**

*Under the same assumptions as above, for all \(r \leq n\), \(i_1...i_r \eta\) is a closed form.

The proof is by induction. \(d i_a \eta\) is zero because it coincides with \(\mathcal{L}_a \eta\). If the statement is true for \(r - 1\), write the identity

\[
d (i_1....i_r \eta) = \mathcal{L}_1(i_2....i_r \eta) - i_1 d(i_2....i_r \eta)
\]

In the second term of the r.h.s. arises the differential of \(i_2...i_r \eta\), which involves \(r - 1\) inner products and is a closed form because the statement applies to \(r - 1\). Only the first term of the r.h.s. might be nonzero. One is left with

\[
d(i_1....i_r \eta) = \mathcal{L}_1(i_2.....i_r \eta)
\]

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but this term also is zero according to Proposition I.

Since $\alpha$ is proportional to $i_1 \ldots i_n \eta$, Proposition II entails that $d\alpha$ vanishes as announced.

**Remark** The reader who is not familiar with exterior differential calculus can get convinced that $d\alpha$ vanishes with the help of local coordinates such that $X_a = \partial / \partial y^a$ (notice that in general $\gamma \neq 1$ in these coordinates). Obviously $X^A_a = \delta^A_a$, thus $X^A_a = \delta^A_a$, etc. Inserting these values into $\alpha$ one gets

$$\alpha = \frac{1}{(m-n)!} \eta_{12\ldots n} A_{n+1} \ldots A_m \ dy^{A_{n+1}} \wedge \ldots \wedge dy^{A_m} \quad (2.4)$$

valid only in these particular coordinates.

With the purpose of applications to averages of various constants of motion, we make this statement

**Proposition III**

Under the same assumptions, consider a function $f$ invariant under all the vector fields $X$, that is $\mathcal{L}_a f = 0 \ \forall a$. We claim that $d(f\alpha) = 0$.

In local coordinates where $X_a = \partial / \partial y^a$ it is easy to check that $df \wedge \alpha = 0$. But a global statement requires the following calculations.

Owing to the well-known identity $i_X(A \wedge B) \equiv i_X A \wedge B + (-)^k A \wedge i_X B$ we notice this

**Lemma**

If $i_a B$ vanishes for all $a = 1, 2, \ldots, n$, then $\forall r \leq n$,

$$(i_1 \ldots i_r A) \wedge B = i_1 \ldots i_r (A \wedge B) \quad (2.5)$$

easily proved by induction.

Apply it to $A = \eta$, $B = df$ where $\mathcal{L}_a f = 0$. Since degree of $\eta$ is maximal, $\eta \wedge df = 0$, which proves that $df \wedge \alpha = 0$, hence $d(f\alpha) = 0$.

This result will receive several applications in the context of relativistic statistical mechanics: normalization of the distribution function, intrinsic definition of entropy flux, of average (linear or angular) momentum.

**2.1. Rescaling**

In connection with the problem of reparametrization of world lines, another property of differential geometry is worth noticing.

**Definition.**
Let $X_1, \ldots, X_n$ have mutually vanishing Lie brackets. An abelian rescaling is the replacement of these vector fields by

$$Y_1 = \phi_1 X_1, \quad \ldots, Y_n = \phi_n X_n$$

where $\phi_1, \ldots, \phi_n$ are strictly positive scalars satisfying the condition $\mathcal{L}_a \phi_b = 0$ for $a \neq b$. We can obviously check that the new vector fields $Y_a$ have mutually vanishing Lie brackets among themselves, which justifies our terminology. More general rescalings of the $X_a$’s would spoil the important Lie bracket condition and will not be considered in this paper.

**Proposition IV**

Let $\eta$ be a volume invariant by the vector fields $X_1, \ldots, X_n$ with mutually vanishing Lie brackets. Then $(\phi_1 \ldots \phi_n)^{-1} \eta$ is a volume invariant by the rescaled vector fields $Y_1, \ldots, Y_n$.

The proof is straightforward: Applied to any form of maximal degree, like the volume element, $\mathcal{L}_a$ is given by $di_a$. Notice that

$$i_{\phi_1 X_1} \ldots i_{\phi_n X_n} (\phi_1^{-1} \ldots \phi_n^{-1}) \eta = i_{X_1} \ldots i_{X_n} \eta$$

and differentiate.

Now we immediately get

**Proposition V**

Under the assumptions of Proposition IV, the $(m - n)$-form $\alpha$ associated with the vector fields $X_a$ and the volume element $\eta$ is invariant under the replacement $X_a \rightarrow \phi_a X_a$, $\eta \rightarrow (\phi_1 \ldots \phi_n)^{-1} \eta$.

### 3. Statistical mechanics of predictive $n$-body systems.

A predictive system of $n$ particles, labelled by subscripts $a, b, \ldots$ running from 1 to $n$, can be defined by the second order differential equations [7]

$$\frac{d^2 x_{a}^{\alpha}}{d \tau^2} = \xi_{a}^{\alpha}(x, v) \quad (3.1)$$

This picture exhibits some analogy with the Feynman-Wheeler equations of motion; in particular each world-line is parametrized by its own parameter. But in contradistinction, the above equations are simply differential equations (they do not involve retarded arguments). In contrast to Galilean mechanics, the r.h.s. of the equations of motion cannot be chosen arbitrarily. This is the price paid for a manifestly covariant formulation.

The above system is integrable and provides world-lines $x_a = x_a(\tau_a)$ provided the righthand sides satisfy the predictivity conditions. This scheme amounts to consider the tangent
bundle \( (T(M^4))^n \) as phase space. Setting \( v_a = \frac{dx_a}{d\tau_a} \) we have in the bundle ”natural coordinates" \( x, v \). The generators of the evolution group are the vector fields

\[
X_a = v_a \cdot \frac{\partial}{\partial x_a} + \xi_a \cdot \frac{\partial}{\partial v_a}
\]  

(3.2)

They can be viewed as linear homogeneous differential operators acting on phase functions; in this sense they are Liouville operators. (Accordingly we shall often denote \( Xf = \mathcal{L}_X f \) the Lie derivative of a function \( f \) by a vector field \( X \)). The lift of world-lines in phase space is formed by the integral curves of \( X_1, X_2, \ldots X_n \). The predictivity condition can be written in terms of Lie brackets \([7][8]\)

\[
[X_a, X_b] = 0
\]  

(3.3)

Notice that these conditions are nonlinear in the ”accelerations" \( \xi_a \).

The evolution group has \( n \) parameters and is abelian; its orbits are the integral curves of \( X_1, \ldots X_n \), and their projection over configuration space \( M^{4n} \) gives the world lines.

A first integral is of course a quantity \( f(x, v) \) satisfying the relations \( X_a f = 0, \forall a \). A partial integral relative to \( X_a \) is defined by \( X_b f = 0, \forall b \neq a \).

When the evolution parameters \( \tau_a \) are affine parameters (the proper-times respectively divided by the masses), the r.h.s. of (3.1) satisfy the additionnal condition \( v_a \cdot \xi_a = 0 \). In this case we say that our formulation is autochronous. When the above condition fails to be satisfied, we say that the formulation is heterochronous. In this case \( v^2_a \) are not anymore constants of the motion, but they are still required to remain strictly positive.

Some care is needed however, not only in view of possible ambiguities, discussed later on, but also in order to maintain timelikeness of world lines. As soon as condition \( v \cdot \xi \) is relaxed, it becomes necessary to make sure that \( n \) first integrals, say \( K_1, \ldots K_n \) can be identified as half-squared masses. Assuming that this is actually possible, the physically relevant part of phase space is that piece of \( (T(M^4))^n \) which corresponds to positive values of \( K_a \).

Then we can always rescale the Liouville operators to an autochronous formulation, as follows. Take \( \phi_a = \sqrt{2K_a/v_a^2} \). Easy check that \( \mathcal{L}_a \phi_b \) vanishes for \( a \neq b \), which makes \( \phi_a \) admissible for a rescaling according to section 2. If \( Y_a = \phi_a X_a \), and \( w_\alpha = \phi_a v_a \), it turns out that \( w_a \cdot w_a = 2K \) hence the square of all \( w_a \) is constant. In other words, the rescaled formulation is autochronous and we end up with \( n \) mass-shell constraints \( w_a^2 = m_a^2 \).

Remark
Since the Poincaré group is implemented in phase space in terms of not only positions but also velocities, it is relevant to realize that the definition of this group is invariant by abelian rescaling, provided that the scaling factors $\phi_a$ are themselves scalar Poincaré invariants.

Recall this definition: Scalar invariants of the Poincaré group [16] are arbitrary combinations of all the scalar products made of "vector invariants" like:

\[ x_a - x_b, \quad v_c, \quad (x_a - x_b) \wedge v_c \wedge v_d \]

Why did we bother with consideration of nonaffine parameters? At first sight it seems natural to impose affine parameters from the outset, and our first version of predictive systems [7] was restricted to this case. But soon it turned out that allowing for arbitrary parametrizations has several advantages [17][18]. Indeed the use of nonaffine parameters facilitates the construction of Hamiltonian predictive systems in closed form, along the line of the a priori Hamiltonian approach [18]. This procedure was proved [19] to be equivalent with the constraint approach [20]; let us stress that the key for a contact of predictive mechanics with this alternative formulation (based upon Dirac’s constraints theory) is an equal-time condition that is in general not compatible with the affine parametrization of world lines. In particular the most tractable toy model of relativistic interaction is a two-body harmonic oscillator formulated in terms of parameters which are not affine [8][18].

The structure determined by the vector fields $X_1, \ldots, X_n$ over phase space allows to formally define the evolution group through $U_{\tau_1, \ldots, \tau_n} = \exp(-\tau_1 X_1 \ldots - \tau_n X_n)$.

There is a priori no Liouville theorem at our disposal, but we can still look for a conserved volume element, that is an everywhere positive $8n$-form invariant by action of all the vector fields $X_a$. If such a volume preserved by the motion, say $\eta$, can be found (that is $\mathcal{L}_a \eta = 0$) it is generally not unique. For a given system of vector fields $X_a$, the invariant volume (if it globally exists) can be determined only up to a factor $\Lambda$, which is necessarily a positive first integral. Let us write $\eta = \gamma d^{8n} y$, where $\gamma$ is some scalar density in arbitrary coordinates. Now it is possible to interpret $U$ as a unitary operator because we have a scalar product for phase space functions, say $\langle f, g \rangle = \int f^* g \eta$ and this product is invariant under evolution.

It is not yet necessary to assume a Hamiltonian formalism. In fact the contents of Section 2 imply that several general results can be derived under the simple assumption of a conserved volume element; these results seem to open a way to abstract developments in
the spirit of mathematical ergodic theory. We shall see later what are the limitations of this point of view.

**Asymptotic considerations**

The interest of referring a system of interacting particles to its interaction-free limit has been recently emphasized [21]. As temporal conditions have been invoked in order to determine this procedure, let us sketch how we can figure the asymptotic time behaviour of these particles. It is convenient to remember that a classical scattering theory parallel to the quantum one is possible, and has been actually developed in the newtonian context [22]. A classical relativistic scattering theory could be considered as well, in close analogy to the many-time formulation of quantum relativistic scattering theory available in the literature [23][24]. Exponentiating the Liouville-operators we get $U = \exp (-\tau_1 X_1 \ldots - \tau_n X_n)$. In a similar way the system of $n$ free particles is characterized by the Liouville-operators $X_1^{(0)}, \ldots, X_n^{(0)}$ where $X_a^{(0)} = v_a \cdot \partial_a$. They correspond to the evolution operator $U^{(0)} = \exp (-\tau_1 X_1^{(0)} \ldots - \tau_n X_n^{(0)})$. Then introduce classical relativistic Moeller operators as the limits

$$\Omega^\pm = \lim U^{-1} U^{(0)}$$

for $\tau_1, \ldots, \tau_n$ altogether $\to \infty$. *If the limits actually exist* in the strong sense, which requires that they are independent of the order (as happens in the quantum theory as proved by Horwitz and Rohrlich [24]), then it is reasonable to expect the intertwining relations

$$U \Omega^\pm = \Omega^\pm U^{(0)}, \quad X_a \Omega^\pm = \Omega^\pm X_a^{(0)} \quad (3.4)$$

This gives the principle of a map between the orbits of the interacting system and the solutions of the free-particle motion, invertible map insofar as no bound state is present. This seems to suport the claim by Ben Ya’acov that the evolution of the distribution function can be fully described in terms of the interaction-free limit of the system [21]. However one must be cautioned that no rigorous result about $\Omega^\pm$ is by now available, and if anything is going to be proved, it will probably concern short-range interactions rather than electromagnetism.

The Liouville ” theorem” is essential * in order to be able of regarding the evolution operator $U$ as unitary, in the Hilbert space $L^2(R^{8n}, \eta)$. In other words one could perhaps go without a Hamiltonian structure, but not without a preserved volume element. In

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* Quotation marks refer to the fact that, for the moment, volume conservation is not derived from a canonical formalism.
practice we shall resort to a symplectic form anyway, in order to get rid of the arbitrariness of the volume form.

3.2 Density and distribution function

Since velocities must point toward the future, the relevant part of phase space is the region of $T(M^{4n})$ defined by $v \cdot v > 0$, $v^0 > 0$. The "proper-time-dependent" density $D(x_1, v_1, \ldots, x_n, v_n, \tau_1, \ldots, \tau_n)$ is ruled by not one but $n$ Liouville equations

$$(\frac{\partial}{\partial \tau_a} + X_a)D = 0 \quad (3.5)$$

This system of equations was first written by Hakim for a gas of free particles. R.Lapiedra and E.Santos pointed out that eq (3.3) just ensures the integrability conditions for $n$ Liouville equations in the presence of predictive interactions [11] (unfortunately their formulation is not thoroughly covariant, which leads them to restrict their statement to the first order in the coupling constants).

As emphasized in ref. [2], only the "proper-time-independent" density

$$N = \int_{-\infty}^{+\infty} D(x_1, v_1, \ldots, x_n, v_n, \tau_1, \ldots, \tau_n) \, d\tau_1 \ldots d\tau_n$$

has a direct physical meaning. From (3.5) it follows that $N$ is a first integral, that is $X_a N = 0$. Conversely, eqs (3.5) are formally solved by $D = U_{\tau_1 \ldots \tau_n} N$, provided $L_a N = 0$.

When the masses are specified from the start, the supports of $D$ and $N$ are a priori restricted to the region $2K_a = m^2_a$; owing to the condition that the velocities point to the future, one is left with a one-sheet mass shell.

Normalization and averages

The next step will be a manifestly covariant formulation for phase averages and for the normalization of the distribution function. We naturally use Hakim’s definition [2], that is in our notation $\int N \alpha = 1$ where integration is to be performed over a $7n$ dimensional manifold cutting all the orbits of the evolution group. Assuming that the motion generated by the $X_a$'s preserves some volume element $\eta$, we are in a position to apply the results of section 2 in the dimension $m = 8n$.

Phase space is $V^{8n} = (T(M^4))^{n}$ and $X_1, \ldots, X_n$ are the generators of the motion. Now $m - n = 7n$, and $\eta$ induces the $7n$ form

$$d\Sigma_{A_1 \ldots A_n} = \frac{1}{(7n)!} \eta_{A_1 \ldots A_n A_{n+1} \ldots A_{8n}} dy^{A_{n+1}} \wedge \ldots \wedge dy^{A_{8n}}$$

as element of $7n$-dimensional hypersurface imbedded in $V^{8n}$. 

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Since the Liouville operators are incompressible vector fields, we can apply Proposition III of Section 2 and conclude that $d(f\alpha) = 0$ for any first integral $f$. By the Stokes theorem, $\int f\alpha$ over any closed $7n$-dimensional manifold vanishes. Notice that in particular $f$ can be the distribution function.

We claim that the integral $\int f\alpha$ over a $7n$-dimensional surface $S^{7n}$ does not depend on the choice of it provided it is specified that $S^{7n}$ cuts once and only once each orbit manifold (this precaution discards the possibility of integrating over the mass shell). Since we are dealing with a multitime flow, this point is not so obvious as in the one-body case; the subsequent details are in order.

**Definition** A **flow tube** is an $m$-dimensional invariant domain of $V^{8n}$ (invariant under the $n$-parameter group with infinitesimal generators $X_a$).

Let $\mathcal{T}$ be a flow tube. Our precise statement is that

The integral of $f\alpha$ over the cross-section of $\mathcal{T}$ by a $7n$-dimensional surface $S^{7n}$ is independent of $S^{7n}$, provided we assume that $S^{7n}$ cuts (and only once) each integral curve of $X_1,...,X_n$.

This can be proved as follows.

Let $\mathcal{T}$ be the orbit of a compact domain $\Delta \subset S^{7n}$. The cut of $\mathcal{T}$ by another surface $S'^{7n}$ will be noted as $\Delta'$. One must take some care of this complication (absent in the one-parameter case): The frontier $\partial \mathcal{T}$ of the tube has $8n - 1$ dimensions, whereas we need a $7n$-dimensional surface in order to extend the set $\Delta \cup \Delta'$ as to form a closed manifold $^*$. Fortunately it is clear that all the vector fields $X_a$ are tangent to $\partial \mathcal{T}$. Thus $\alpha$ vanishes on $\partial \mathcal{T}$. Now it is possible to construct a surface $B^{7n}$ imbedded in $\partial \mathcal{T}$ as a submanifold, and connecting the domains $\Delta$ and $\Delta'$ in such a way that $\Delta \cup \Delta' \cup B^{7n}$ be a closed $7n$-surface (see Appendix A). Then we observe that $f\alpha$ vanishes on $B^{7n}$ for it vanishes on $\partial \mathcal{T}$. Therefore applying the Stokes formula yields $\int_{\Delta'} f\alpha = \int_{\Delta} f\alpha$.

**Remark:**

In applications we shall suppose that $S^{7n} = S_1^3 \times \ldots \times S_n^3 \times (W^4)^n$ where $W^4$ is the space of four-vectors and $S_a^3$ is a spacelike three-surface imbedded in Minkowski space.

Besides the normalization of $N$, another application concerns the total entropy $S = - \int N \log N \alpha$. Since the phase function $N \log N$ is a constant of the motion, under similar assumptions $N \log N \alpha$ is a closed form, a result which renders $\int N \log N \alpha$ independent of the integration surface.

More generally, we have an intrinsic covariant definition for the average of any constant of the motion.

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* This complication was pointed out in ref [2].
In general a form like $\eta$ is by no means unique. However it may happen, for a given system of $n$ interacting particles, that the $8n$-form $\sum d^{4n}x \wedge d^{4n}v$ is "accidentally" conserved by the motion, providing obviously a preferred choice (see Appendix B for an example at first order in the coupling constants).

Otherwise there remain the possibility of redefining the distribution function (without changing parametrization) whenever we change the preserved volume form. So doing we keep $N\alpha$, and therefore the formulas for averages, unchanged.

Most simply, we shall resort to the Hamiltonian formalism (next section). Although a similar ambiguity arises in the Hamiltonian framework, it can be removed by an argument which is plausible for all gases made of microscopic "molecules".

**Reduction to 6n dimensions.**

For free particles it was already pointed out by Hakim that normalizing over a $7n$-dimensional surface actually reduces to a customary normalization over a $6n$-dimensional surface, by help of the $n$ mass-shell constraints $m^2 = v^2$. It is fortunate that the multitime formalism of relativistic dynamics still provides $n$ mass-shell constraints in the presence of interactions. These constraints amount to fix $n$ constants of the motion $K_1, ..., K_n$ identified as $\frac{1}{2}m_a^2$.

In most cases of interest, the relevant distribution functions will be concentrated on a submanifold $2K_a = m_a^2$, with specified values of $m_1, ..., m_n$. Therefore we expect that the distribution function (as defined in the whole phase space) involves a factor $\prod \delta(m_a^2 - 2K_a)$. Since the velocities must point to the future, there is in $N$ a factor $\prod \theta(v_0^a)$, where $\theta$ is the step function. These considerations entail that the effective phase space may be $6n$-dimensional, as expected for a contact with the newtonian limit.

Remark:

Restriction to a sharp mass shell agrees with the realistic picture of a gas of particles with given masses; but it is not imposed by our formalism. Moreover, a statistics of particles with unspecified masses may be of interest for cosmological applications and for hadronic matter. In addition a smearing of the mass shell becomes technically necessary when entropy is defined as above, for the logarithm of a Dirac distribution has no mathematical meaning.

In fact the essential mass-shell property is that the quantities $m_a^2 - 2K_a$ are constants of the motion. But strictly speaking they are not constraints until one gives numerical values to these constants, as happens when we assume the factors $\delta(m_a^2 - 2K_a)$.

We shall see in Section 4 that the computational complications introduced by the factors considered above disappear in the special case of "perfect interactions".
We could hardly go much further without a Hamiltonian formalism; indeed: as soon as one wishes to consider either energy or total linear momentum, a symplectic canonical formulation is badly needed.

4. Hamiltonian formalism

The Hamiltonian formulation consists in looking for a symplectic form \( \Omega \) invariant under the evolution group, that is \( \mathcal{L}_{X_a} \Omega = 0 \) for all \( a \). When an invariant symplectic form \( \Omega \) has been determined, the Liouville operators are generated by positive scalar functions \( H_a \) satisfying \( i_a \Omega = dH_a \). We refer to these scalars as the Hamiltonians, although they are associated to (half) squared masses instead of energy [8]. In any system of canonical coordinates \( q_a, p_b \), we can write \( \Omega = \sum dq_a \wedge dp_b \).

The volume form is \( \eta = d^{4n}q \wedge d^{4n}p \). Notice that \( \eta = \text{const.} \Omega^{4n} \) (exterior power). Since \( \mathcal{L}_a \Omega = 0 \), the Liouville theorem follows \( \mathcal{L}_a \eta = 0 \).

The scalar density \( \gamma \) is equal to unity in canonical coordinates; this is not always the case in natural coordinates \( x, v \). In arbitrary coordinates, the volume element can be written as \( \eta = \gamma d^{8n}y \) where \( \gamma = \frac{D(q, p)}{D(y)} \), with an obvious notation for the Jacobian. Drastic simplification arise when the position formulas ”accidentally” imply that \( d^{4n}x \ d^{4n}p = d^{4n}q \ d^{4n}p \).

In contrast to Newtonian mechanics, and in view of a famous theorem [25], it is not possible to require that the physical positions be canonical variables. In other words we cannot have just \( q_a = x_a \) throughout phase space. This situation implies that, for a given system of Liouville operators \( X_a \), the invariant symplectic form \( \Omega \) cannot be unique. This complication is a source of ambiguities not only about the Hamiltonian formulation but also in the definition of linear momentum (but, insofar as one is concerned only with finding first integrals and with solving equations of motion, there is in principle nothing wrong with the fact that a given physical system admits infinitely many symplectic formulations).

We notice that quantization is naturally affected by this ambiguity, but this is not at all a difficulty because there is no reason why a classical system should correspond to a unique quantum system; we expect unicity when going from quantum to classical theory and not the reverse!

Remaining in the classical area of physics, the situation becomes more serious as soon as we try to construct statistical mechanics with concrete applications in mind.

If statistical mechanics were limited to formal manipulations involving some preserved volume element of phase space, again there would be no problem in having, for a given physical system of many particles, infinitely many possible descriptions. But we are inter-
ested in the construction of macroscopic quantities like the thermodynamical functions. At this stage the Hamiltonian formulation seems to play a crucial role. Several basic concepts like the energy or the canonical linear momentum are intimately related with this formulation through Noether’s theorem, and it seems that the definition of most macroscopic quantities necessarily involves these concepts; temperature is the first example. Finally the very notion of equilibrium rests on the previous determination of a canonical formalism *.

The lack of unicity in the symplectic formulation can be expressed in saying that the dynamics of a relativistic system cannot be completely determined by the simple knowledge of the Hamiltonians.

However it is true that, to a large extent, spectral and asymptotic properties of relativistic dynamical systems are fully determined by the Hamiltonian generators of the motion. For example we have been able to speak of the ”abstract integration” of a system [8]. Abstract integration refers to the solving of the canonical equations of motion, in terms of \( q, p \), and provides a foliation of phase space by \( n \)-dimensional leaves, disregarding the way how each leaf is identified as a cartesian products of \( n \) lines which are the lifts of world lines. In the same spirit, but in the language of constraints theory, it was observed by Todorov [20] that, under very general assumptions, the scattering properties of an \( n \)-body system depend only on the form of the mass-shell constraints, and not on the gauge fixations.

In other words we can see a relativistic hamiltonian system as an abstract structure (corresponding to the \( H_a \) only) completed by additional formulas connecting the physical positions \( x^\alpha_a \) with the canonical variables. These position formulas must satisfy the position equations [8]

\[
\{ x^\alpha_a, H_b \} = 0 \quad \forall a \neq b
\]  

necessary for achieving the determination of world lines.

To summarize, the ambiguities of the Hamiltonian formalism are of two kinds.

1) Any change of parametrization implies a redefinition of the ”velocities” and a redefinition of the set of vector fields \( X_a \).

2) Even when the parametrization is kept fixed, it is in general impossible to find a unique symplectic form invariant by these vector fields.

* One might speculate that there exists some unifying formula defining temperature, entropy and equilibrium, in a way which is invariant under reparametrization and changes of symplectic form. But for the moment nobody has any idea of such a formula, if it exists at all; therefore one has to cope with the ambiguities.
These ambiguities are the price paid for a covariant setting. They cannot arise in Newtonian mechanics for two reasons: the use of an absolute time and the implicit prescription that the physical positions are canonical variables. But in relativistic dynamics, considering mutual interactions forbids to require that the physical positions $x_a$ be canonical variable.

At this stage we are led to the following observation. Statistical mechanics is a very general theory; its applications may concern a fluid of galaxies as well as an ordinary gas made of diatomic molecules. We suggest to distinguish clearly these two situations. For example the concept of temperature has certainly a profound physical significance when it concerns the air we breath. When applied to a gas of which the "molecules" are galaxies, this concept has hardly the same physical meaning, although it is defined through the same mathematical structure. Therefore we can provisionnally consider that some ambiguity about the statistical mechanics of a very large number of macroscopic bodies is perhaps not so scandalous after all. Naturally further work is needed in order to select a convincing prescription removing the ambiguity in this case.

In contradistinction the description of a gas made of microscopic molecules should not suffer from the same uncertainty.

Fortunately, at the scale of an ordinary gas it is reasonable to remember that a classical system of $n$ microscopic particles must be considered as the classical limit of a quantum system. Our point is that if we carefully formulate the axioms of $n$-body quantum mechanics, then taking the classical limit of a system will automatically determine a particular Hamiltonian formulation. In order to prove this, let us first sketch the framework for quantum mechanics of $n$ interacting particles. The wave equations involve $n$ half-squared-mass operators $(H_1)_{op}, ....(H_n)_{op}$ acting in some Hilbert space and commuting among themselves [26](this point of view stems from quantization of predictive mechanics, but it also agrees with constraint theory). However the knowledge of these operators is not sufficient for a complete determination of the system. This point is widely overlooked in the litterature, because most investigations are concerned with either spectral or scattering properties fully contained in the squared-mass operators, all question about position measurement being systematically ignored.

Our proposal was that there exist $n$ coordinate operators with four components, say $(x^a_{\alpha})_{op}$ satisfying the commutation relations

$$[(x^a_{\alpha})_{op}, (H_b)_{op}] = 0 \quad a \neq b \quad (4.2)$$
For instance these \((x)_{op}'s\) are relevant objects if one wishes to consider position measurements *.

If the system has a classical limit, these relations reduce for \(\hbar \to 0\) to the "position equations" (4.1) of predictive mechanics [8], and the coordinate operators \(x_{op}\) are expected to have a unique and well-defined limit \(x_\alpha^\alpha(q, p)\). From this limit we can derive the generalized Legendre transformation \(q, p \leftrightarrow x, v\), which also determines the rescaling from the parameters \(\tau_1, \ldots, \tau_n\) to the proper times.

To summarize, the classical limit is expected to determine a predictive dynamical system *endowed with a preferred invariant symplectic form* (corresponding to a preferred Hamiltonian formulation).

This argument solves an important question of principle. But in practice, the "good" symplectic form will remain ignored as long as the details of the underlying quantum theory remain ignored. This might be the case of "predictive electrodynamics" [27] until it will be rederived from some \(n\)-particle generalization of the Bethe-Salpeter equations.

### 4.1. Perfect interactions

Ideal gases are made of noninteracting particles. The next step consists in considering an interaction which can be trivialized by a suitable transformation. With this idea in mind let us characterize a perfect interaction by the global existence of particular canonical coordinates, say \(\hat{q}, \hat{p}\), allowing to write

\[
H_a = \frac{1}{2} \hat{p}_a^2
\]

We refer to \(\hat{q}, \hat{p}\) as Hamilton-Jacobi (HJ) coordinates and we shall say that we have an almost ideal gas. Let us stress the following point: whereas (4.3) could be *locally* written for any Hamiltonian system by solving an \(n\)-body relativistic generalization of the Hamilton-Jacobi equation, the existence of global coordinates like \(\hat{q}, \hat{p}\) is a very particular property of the gas we consider [28].

A system undergoing perfect interactions is spectrally trivial; the Lie algebra of first integrals has the same structure as in the free system. In principle equations of motion can be exactly solved. Nevertheless, word-lines actually deviate from straight lines. It is clear that perfect interactions realize the most simple situation beyond the case of noninteracting particles.

---

* These "coordinate operators cannot, as they stand, be considered as position operators in the usual sense. They act on some subset of \(L^2(R^{4n}, d^{4n})\), which is not the case of the Newton-Wigner operator.*
We strongly suspect that considering HJ coordinates amounts to neglect bound states. This should not prevent us from using perfect interactions as a simplification when attractive forces are present. This approximation may be applied to the case of attractive forces, provided that the gas we consider is so dilute that it is reasonable to neglect bound states. Naturally one must keep in mind that this representation is valid only insofar as the distances between particles are large enough.

The main interest of HJ coordinates lies in the possibility of explicitly writing down covariant many-body interactions. Indeed, for an almost ideal gas, the most general interaction can be explicitly written in terms of the HJ coordinates. According to (4.1), the components of \( x_a \) must be invariant by all the vector fields \( X_b \) generated by hamiltonians \( H_b \) where \( b \neq a \). We now have to solve \( \{ x_a, \hat{p}_b^2 \} = 0 \). We immediately find

\[
x_a^\alpha = \hat{q}_a^\alpha + F_a^\alpha (\ldots \hat{p}_b, \hat{q}_b \wedge \hat{p}_b \ldots)
\]  

with all \( b \neq a \) in \( F \), and \( F \) are arbitrary vector-valued functions except for the following restrictions:

i) They might be required to satisfy reasonable boundary conditions, namely that in some sense \( \hat{q}, \hat{p} \to x, v \) at temporal or spatial infinity.

ii) They must respect Poincaré invariance. Scalars invariants and vector invariants of the Poincaré group can be characterized as well in terms of HJ coordinates. According to this remark, \( F_a^\alpha \) in the position formula above must be a vector invariant. Of course, the simplest solution \( x_a = \hat{q}_a \) would be trivial; it corresponds to straight world lines.

It might be interesting to discuss under which conditions the Feynman-Wheeler electrodynamics can be (approximately) treated as a perfect interaction.

Using HJ coordinates we can write \( X_a = \hat{p}_a^\alpha \partial/\partial \hat{q}_a^\alpha \). For the special choice of a surface \( S^{7n} \) defined by equations of the form

\[
s(\hat{q}_1) = s(\hat{q}_2) = \ldots s(\hat{q}_n) = 0
\]

(for instance \( s(q) = q^0 \)) we have this simplification for the \( 7n \)-form \( \alpha \) of Sections 2,3

\[
\alpha = \text{const.} \hat{p}_1^{\mu_1} \ldots \hat{p}_n^{\mu_n} d\sigma_{\mu_1}(1) \wedge \ldots \wedge d\sigma_{\mu_n}(n) \wedge d^4n \hat{p} + O(S^{7n})
\]

where \( O(S^{7n}) \) is a term vanishing on \( S^{7n} \) and this notation

\[
d\sigma_\mu(a) = \frac{1}{3!} \varepsilon_{\mu
u\rho\sigma} \hat{q}_a^\nu \wedge \hat{q}_a^\rho \wedge \hat{q}_a^\sigma
\]
\[ d^{4n} \hat{p} = d^4 \hat{p}_1 \wedge \ldots \wedge d^4 \hat{p}_n \]

Then with help of the identity \( d(p^2) \wedge d^3 p \equiv 2 p^0 d^4 p \) we factorize \( d(\hat{p}_a^2) \) in \( d^{4n} \hat{p} \) and finally, for all "phase function" \( \Phi \) we can write

\[ \int_{S^{7n}} \Phi \alpha = \text{const.} \int_{S^{7n}} \Phi \, d(\hat{p}_1^2) \wedge \ldots \wedge d(\hat{p}_n^2) \wedge \lambda \]

(4.6)

where \( \lambda_{(X,\eta)} \) is a \( 6n \)-form which also depends on the choice made for \( S^{7n} \). When \( \Phi \) has the particular form \( \Phi = \theta(\hat{p}_1^0) \ldots \theta(\hat{p}_n^0) \delta(\hat{p}_a^2 - m_a^2) \ldots \delta(\hat{p}_n^2 - m_n^2) \tilde{\Phi} \), then equation (4.6) is reduced to

\[ \int_{S^{7n}} \Phi \, \alpha = \text{const.} \int_{S^{6n}} \tilde{\Phi} \, \lambda \]

(4.7)

where \( S^{6n} = S^{7n} \cap \text{mass shell} \). This formula makes the contact with more popular formulations using a \( 6n \)-dimensional phase space; also it is an illustration of a remark made in section 3.2.

It is clear that all first integrals of the interacting system are known in terms of HJ coordinates, which allows for formally solving the Liouville equations (3.5). But the solutions obtained in this way are expressed in terms of canonical coordinates. For practical purpose it is usually of interest to have solutions in terms of the natural coordinates \( x, v \), which in turn requires knowledge of the position formulas.

**4.2. Average momentum**

In the framework of Hamiltonian formalism, in the absence of external forces the system is invariant under translations and rotations, hence the first integrals \( P = \sum p_a, \ M = \sum q_a \wedge p_a \). Application of Proposition III yields an intrinsic definition of the averages \( \overline{P}^\mu, \ \overline{M}^{\mu\nu} \). For instance

\[ \overline{P}^\mu = \int N P^\mu X_1^{A_1} \ldots X_n^{A_n} \, d\Sigma_{A_1 \ldots A_n} \]

is conserved under a change of the \( 7n \)-dimensional integration manifold.

**5. Canonical Distribution. Equilibrium.**

Provided the system is translation invariant, it is natural to define equilibrium by the distribution function

\[ N = \text{const.} \, \delta_1^+ \ldots \delta_n^+ \exp(-\beta^\mu P_\mu) \]

(5.1)

for some constant timelike four-vector \( \beta^\mu \) associated with the inverse temperature. The factors

\[ \delta_a^+ = 2 \theta(v_a^0) \, \delta(m_a^2 - 2H_a) \]

(5.2)
take into account the mass-shell constraints and the direction of the arrow of time on world lines.

For an ideal gas, equation (5.1) (which is also an $n$-body generalization of Jüttner’s formula) agrees with the definition proposed by Hakim three decades ago.

In the presence of interactions, $v_0^a$ is not constant, but $\theta(v_0^a)$ is a (discrete) first integral although $v_0^a$ is not. Finally $N$ is a first integral as it should.

Let us stress that there is in general no indication that the replacement of $\theta(v_0^a)$ by $\theta(p_0^a)$ be legitimate. Strictly speaking, in order to write down the equilibrium distribution function in closed form, one might be obliged to solve the position equations! This is no surprise; it is reasonable that some information from world lines be necessary. However, interesting and drastical simplifications arise in some special cases, as will be briefly discussed below.

Being spatially homogeneous, the equilibrium distribution function is not normalizable, which rises the problem of elaborating a covariant procedure for the thermodynamical limit.

In the presence of an external field, space translation invariance is broken, whereas it often happens that the energy $l^\mu P_\mu$ remains conserved (the external potentials applied to the system must be stationary). In this case $\{l^\mu P_\mu, H_a\} = 0$ for some constant timelike unit vector $l$ defining a “laboratory frame”. Then we define equilibrium by

$$N = \text{const.} \prod_{i=1}^{N} \delta^+ \exp(-\beta l^\mu P_\mu)$$

(5.3)

with $\beta$ a positive scalar. Again, one is led to consider the thermodynamical limit. But the analogy with the Newtonian situation is more transparent here. However, existence of the limit is an open problem. For instance it was argued that for long-range interactions, it may happen that this limit does not make sense [29]).

Returning to translation invariant systems, it is clear that the natural coordinates $x, v$ allow for a simple expression of the factors $\delta^+$ (at least in the autochronous case); but $\gamma$ may still differ from unity in these coordinates.

In contrast, any choice of canonical coordinates provides a trivial expression for $P$, but this simplicity is at once destroyed when taking into account the mass-shell constraints. In any frame adapted to $\beta$ we get $N = \text{const.} \prod \delta^+ \exp(-\beta P^0)$, where $P^0$ is the sum of the

* Formula (5.1) seems to differ from eq (6.15) of [2]by the factors $\delta^+$. These factors are implicit in ref. [2] where the reduction of phase space by mass shell constraints is assumed from the outset.
In ordinary canonical coordinates \( p^0_a \) takes on the form \( p^0_a = \sqrt{m_a^2 + p_a^2 - 2V_a(q, p)} \).

Also the factors \( \delta^+ \) are in general model dependent, since the canonical expression of \( H_a \) in terms of \( q, p \) can be complicated.

But in the special case where HJ-coordinates actually exist, remarkable simplifications arise.

First, the mass-shell constraints take on a very simple form (in fact as simple as for free particles). In HJ-coordinates \( \hat{p}_a^0 = \sqrt{m_a^2 + \hat{p}_a^2} \).

Second, we can legitimate the replacement of \( v^0 \) by \( p^0 \) in the step functions. Indeed, \( H = \frac{1}{2} \hat{p}^2 \) is a constant of motion, it is clear that on each orbit \( \hat{p} \) remains timelike. Since \( p \) reduces to \( v \) for a vanishing coupling constant, it is clear that \( \hat{p} \) as well as \( v \) is oriented toward the future. In other words, \( \hat{p} \) and \( v \) are simultaneously in the future light cone.

Third, \( P = \sum \hat{p} \).

Finally the equilibrium distribution takes on the free form when expressed in terms of HJ coordinates.

**Microcanonical equilibrium**

Let us consider a gas of interacting particles in the absence of external forces. We suggest defining the microcanonical ensemble by a straightforward generalization of eq (A4) of ref [2], say

\[
N = \text{const.} \delta(\zeta^\mu - P^\mu) \delta^+_1 \ldots \delta^+_n
\]

where \( \zeta \) is a constant timelike vector and \( \delta^+_a \) is given by (5.2). Again, the above distribution is a constant of the motion.

**6. Conclusion**

The theory of direct interactions provides a solid ground for a covariant formulation of statistical mechanics. Let us be more specific: predictive relativistic mechanics is the most natural way for taking seriously the idea (already present in Hakim’s early work) that one is dealing with an \( n \)-parameter evolution group, associated with a multitime parametrization. As we know for a long time, this philosophy can be expressed in terms of modern differential geometry, which leads to vector fields in \( 8n \)-dimensional phase space; these ”Liouville operators” have mutually vanishing Lie brackets, ensuring that initial velocities and positions actually determine world lines.

If we now consider a statistical mechanics of relativistic particles, the predictivity conditions are technically essential; on the one hand they entail integrability of the \( n \) Liouville equations (it is an exact result, by no means limited to first order in the coupling); on the
other hand they ensure, as we have shown, the intrinsic nature of important quantities defined by a flux integral. For instance they imply the conservation of phase average values, especially the conservation of norm (numerical flux of particles) and entropy. A few manipulations of exterior differential algebra were sufficient to demonstrate that predictive mechanics exactly fulfills the demands of relativistic statistical mechanics concerning the Lorentz invariance of the phase averages.

In our exposition we have as much as possible separated the specific features based upon the Hamiltonian formalism from more general topics requiring simply a preserved volume element.

Insofar as one aims at a general theory devoted to a description of systems out of equilibrium, the basic notion is that of a preserved volume element, rather than a symplectic structure: in other words the Liouville theorem without the Hamiltonian formalism! This viewpoint is close to the spirit of mathematical ergodic theory. However we notice that, for a system defined by its equations of motion, the preserved volume element is not univocally determined.

In this work we have focused on the \( n \)-particle distribution function. But we expect that also reduced distribution functions can be proved to satisfy (in the presence of mutual interactions) covariant evolution formulas and will enjoy conservation properties with respect to normalization. This point is obviously related with the important issue of a tractable and relativistic BBGKY hierarchy.

A Hamiltonian formulation is highly desirable anyway, as soon as one is interested in a truly physical description. In the presence of mutual interactions, this formalism is essential in order to define linear momentum and energy. Thus the notion of equilibrium and probably also the construction of thermodynamical functions require a Hamiltonian formulation.

At least for gases made of microscopic constituents, a certain idea of the underlying quantum mechanics solves (in principle) the ambiguity related with the impossibility of having canonical physical positions.

The usual expression for the equilibrium distribution function has been generalized in a covariant formula taking mutual interactions into account. Further work is needed in order to check in which sense and in the context of which side conditions this definition actually realizes the extremum of the entropy integral.

There is a good hope that several methods and formulas concerning ideal gases remain valid when the mutual interactions are modeled as ”perfect interactions”. This simplified picture provides a toy-model of interacting particles for which rigorous results can be proved. Its relevance for a real gas is probably limited to the dilute-gas approximation,
but many standard calculations of the theory of ideal gases can be extended to almost ideal gases in a straightforward fashion.

Rather than new practical applications, the scheme proposed here opens theoretical possibilities. For instance it sheds light on the transformation properties of various results available until now in noncovariant form.

The present formulation allows for considering particles with unspecified masses, which is interesting for applications to cosmology or to hadronic matter (the latter suggests an extension toward quantum relativistic statistical mechanics).

Several generalizations are possible; for example one could enlarge phase space as to include radiation.

As it stands our exposition may seem to be exceedingly geometrical, but this feature is an advantage in view of possible generalization to curved spacetime.

A comparison with the ”statistics of events” developed by Horwitz et al. [13] will be of interest.

**Appendix A**

Through any \( y \in \Delta \) passes a unique orbit, say \( \Gamma_y \) of the evolution group. According to the assumptions made, this orbit is cut in a unique point \( y' \) by the surface \( S^{7n} \). We can write

\[
y' = U_{\tau'_1 \tau'_2 ... \tau'_n} y
\]

for a sequence of parameters \( \tau'_1, \tau'_2, ... , \tau'_n \) which depends only on \( y \) and on the cutting surface. Among all possible curves lying in \( \Gamma_y \), a very simple way for joining \( y \) to \( y' \) is characterized for instance by the conditions

\[
\frac{\tau_a}{\tau_1} = \frac{\tau'_a}{\tau'_1}
\]

Indeed \( \Gamma \) is endowed with a network made of the integral curves of \( X_1, ... X_n \). This procedure defines a distinguished curve \( C_y \) which lies in \( \Gamma_y \) and joins \( y \) with \( y' \).

Now when \( y \) runs in the \((7n-1)\)-dimensional frontier \( \partial \Delta \) of the domain \( \Delta \), the curves \( C_y \) generate a \( 7n \)-dimensional surface \( B^{7n} \) connecting \( \partial \Delta \) with \( \partial \Delta' \). Notice that two different \( \Gamma \) cannot intersect unless they are identical. When \( y \) runs in \( \partial \Delta \) the \( n \)-dimensional orbits \( \Gamma \) generate \( \partial T \). Since each \( C_y \) lies in \( \Gamma_y \) it is clear that, in the generic case, \( B^{7n} \subset \partial T \) as a submanifold.

In this construction, \( B^{7n} \) can be considered as the frontier of a ”sub-tube” generated by the integral curves of a vector field \( \Xi = X_1 + \nu_2 X_2 + ... + \nu_n X_n \) where the \( n-1 \) independent quantities \( \nu_2, ... , \nu_n \) are first integral of the system \( X_1, ... , X_n \).
Appendix B

At first order in the coupling constant, the $8n$-form $\eta^{(0)} = d^{4n}x \wedge d^{4n}v$ is preserved by predictive electromagnetism and by predictive vector interaction.

Proof

At this order of approximation, the acceleration on particle $\alpha$ is given by [27]

$$\xi_{\alpha a} = \sum_{b \neq a} F_{\alpha a \leftarrow b} v_{b}$$

where the skew-symmetric tensor $F_{\alpha a \leftarrow b}$, is (up to a constant factor) the Lienart-Wiechert field created at point $x_{a}$ by the (vectorial or) electric charge with label $b$ supposed to move on a straight line in spacetime. Therefore, $F_{\alpha a \leftarrow b}$ depends on the $v_{b}$’s for $b \neq a$, but certainly not on $v_{a}$. Hence $\partial \xi_{\alpha a} / \partial v_{a} = F_{\alpha a}$.

By contraction of indices $\partial \xi_{\alpha a} / \partial x_{a} = F_{\alpha}$. Since we are using the natural coordinates $x, v$, this equation amount to write $\partial A X^{A} = 0$, which proves our statement.

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