Notes on Greub-Rheinboldt inequalities

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Abstract
In this paper, we focus on matrix Greub-Rheinboldt inequalities for commutative positive definite Hermitian matrix pairs. Some improvements, which yield sharpened bounds compared with existing results, are presented.

1 Introduction and preliminaries
Let $M_{m,n}$ denote the space of $m \times n$ complex matrices and write $M_n \equiv M_{n,n}$. The identity matrix in $M_n$ is denoted by $I_n$. As usual, $A^* = (A)^T$ denotes the conjugate transpose of the matrix $A$. A matrix $A \in M_n$ is an Hermite matrix if $A^* = A$. An Hermitian matrix $A$ is said to be positive semi-definite or nonnegative definite, written as $A \geq 0$, if $x^*Ax \geq 0$, $\forall x \in \mathbb{C}^n$. $A$ is further called positive definite, symbolized $A > 0$, if $x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$. An equivalent condition for $A \in M_n$ to be positive definite is that $A$ is an Hermitian matrix and all eigenvalues of $A$ are positive.

Denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the eigenvalues of a Hermitian matrix $A$. The matrix version of the well-known Kantorovich inequality for a positive definite matrix $A$ is stated as follows (see, e.g., [1, 2]):

$$1 \leq \frac{x^*Axx^*A^{-1}x}{(x^*x)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n} \tag{1.1}$$

for any nonzero vector $x \in \mathbb{C}^n$.

An equivalent form of this result is the inequality

$$0 \leq \frac{x^*Axx^*A^{-1}x}{(x^*x)^2} - 1 \leq \frac{(\lambda_1 - \lambda_n)^2}{4\lambda_1\lambda_n} \tag{1.2}$$

valid for any nonzero vector $x \in \mathbb{C}^n$.

This famous inequality plays an important role in statistics (see [3, 4]; for the latest work on applications in statistics, we refer to Seddighin’s work [3]) and numerical analysis, for example, studying the rates of convergence and error bounds of solving systems of equations (see in [5, 6]).

In 2008, Dragomir gave a refinement of the additive version of the operator Kantorovich inequality [7],

$$0 \leq K(A; x) - 1 \leq \frac{1}{4} \left( \frac{(M - m)^2}{mM} - \left[ \text{Re}\{C_{m,M}(A)x,x\} \text{Re}\{C_{1,m^{-1}}(A^{-1})x,x\} \right]^{1/2} \right), \tag{1.3}$$

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where \( A \) is a self-adjoint bounded linear operator on a complex Hilbert space, \( 0 < m < M \), such that \( mI \leq A \leq MI \) in the partial operator order, \( K(A; x) := \langle Ax, x \rangle \langle A^{-1}x, x \rangle \), and \( C_{\alpha, \beta}(A) := (A - \bar{\alpha}I)(\beta I - A) \).

A further improvement of the matrix version of (1.3) is proposed in [8], where the classical Kantorovich inequality (1.1) is modified to apply not only to positive definite, but also to all invertible Hermitian matrices.

We adopt the following transform for a positive definite Hermitian matrix \( A \in M_n \) with eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \):

\[
C(A, x) = x^* (\lambda_n I - A)(A - \lambda_1 I)x, \tag{1.4}
\]

and

\[
C(A^{-1}, x) = x^* \left( \frac{1}{\lambda_1} I - A^{-1} \right) \left( A^{-1} - \frac{1}{\lambda_n} I \right)x. \tag{1.5}
\]

Then the following inequality holds [8]:

\[
0 \leq x^* Ax \cdot x^* A^{-1}x - 1 \leq \frac{(\lambda_1 - \lambda_n)^2}{4\lambda_1\lambda_n} - \sqrt{C(A, x) \cdot C(A^{-1}, x)} \leq \frac{(\lambda_1 - \lambda_n)^2}{4\lambda_1\lambda_n} \leq \sqrt{m_1 m_2 M_1 M_2} \langle Ax, Bx \rangle, \tag{1.6}
\]

The result above is an improvement of the Kantorovich inequality (1.1).

A generalized form of the Kantorovich inequality presented by Greub and Rheinboldt [1] in 1959 is known as the Greub-Rheinboldt inequality in operator theoretic terms, which is also an important and early example of the so-called complementary inequality referred to in [9],

\[
\langle Ax, Ax \rangle \langle Bx, Bx \rangle \leq \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2} \langle Ax, Bx \rangle^2, \tag{1.7}
\]

where \( A \) and \( B \) are commuting positive definite self-adjoint operators on a Hilbert space, with upper and lower bounds \( M_i \) and \( m_i \), \( i = 1, 2 \), respectively.

In 1997, Fujii et al. [10] generalized the Greub-Rheinboldt inequality to pairs of invertible operators that may not even commute,

\[
\langle A^2 x^2 B^2 x, x \rangle \leq \langle A^2, x \rangle \langle B^2, x \rangle \leq \frac{m_1m_2 + M_1M_2}{2\sqrt{m_1m_2M_1M_2}} \langle A^2 x^2 B^2 x, x \rangle \langle Ax, Bx \rangle^2, \tag{1.8}
\]

where \( A, B \) are invertible positive operators satisfying \( 0 < m_1 \leq A \leq M_1 \) and \( 0 < m_2 \leq B \leq M_2 \), and \( A^2 B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \). By using the viewpoint of interaction antieigenvalue, Gustafson [9] sharpened the Greub-Rheinboldt inequality (1.7) to obtain the following result:

\[
\langle Ax, Ax \rangle \langle Bx, Bx \rangle \leq \frac{(m(AB^{-1}) + M(AB^{-1}))^2}{4m(AB^{-1})M(AB^{-1})} \langle Ax, Bx \rangle^2, \tag{1.9}
\]

where \( A \) and \( B \) are commuting positive definite self-adjoint operators on a Hilbert space.

Let \( A \) and \( B \) be two positive definite Hermite matrices and \( AB = BA \) with real eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) and \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \), respectively. Moreover, let \( \langle Ax, Bx \rangle := \frac{1}{2} \langle Ax + Bx, x \rangle \).
\((Ax)Bx = x^*A^*Bx\). Then a matrix version of (1.9) is

\[
\frac{x^*A^2x \cdot x^*B^2x}{(x^*ABx)^2} \leq \frac{(\lambda_1\mu_1 + \lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n}
\]  

(1.10)

for any nonzero vector \(x \in \mathbb{C}^n\).

In 2005, Seddighin [11] extended the Greub-Rheinboldt inequality (1.9) to pairs of normal operators and established for what vectors the Greub-Rheinboldt inequality becomes equality.

Let \(V\) be an \(n \times r\) matrix such that \(V^*V = I\), i.e., \(V\) is suborthogonal. Another well-known matrix version of the Kantorovich inequality asserts that

\[
V^*A^2V \leq \frac{(m + M)^2}{4mM}(V^*AV)^2
\]  

(1.11)

for any \(A > 0, V^*V = I,\) and \(0 < mI < A < MI\).

Mond and Pečarić proved the following matrix version inequality (see (7) in [12]):

\[
(V^*A^2V)^{1/2} - V^*AV \leq \frac{(M - m)^2}{4(M - m)} I
\]  

(1.12)

for \(A > 0\) and \(V^*V = I\). For more related properties and applications, see, e.g., [13–15].

In the next section, we propose some refinements about the matrix Kantorovich-type inequalities (1.2), the Greub-Rheinboldt inequality for commutative positive definite Hermitian matrix pairs, and (1.10) for positive definite matrices, yielding sharpened upper bounds compared with original results, together with an improvement to (1.12).

2 Main results

In this section, we first introduce some lemmas.

**Lemma 2.1** (in [8], Lemma 2.2) Let \(A \in M_n\) be a positive definite Hermitian matrix. The following inequalities hold:

\[
\lambda_1\|x\|^2 \leq x^*Ax \leq \lambda_n\|x\|^2, \quad 0 \leq (\lambda_n\|x\|^2 - x^*Ax)(x^*Ax - \lambda_1\|x\|^2) \leq \frac{1}{4}(\lambda_n - \lambda_1)^2\|x\|^4,
\]

and

\[
\frac{1}{\lambda_1}\|x\|^2 \leq x^*A^{-1}x \leq \frac{1}{\lambda_n}\|x\|^2,
\]

\[
0 \leq \left(\frac{1}{\lambda_1}\|x\|^2 - x^*A^{-1}x\right)\left(x^*A^{-1}x - \frac{1}{\lambda_n}\|x\|^2\right) \leq \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_1\lambda_n)^2}\|x\|^4
\]  

(2.1)

for any \(x \in \mathbb{C}^n\).

Let \(A, B\) be two invertible commuting Hermite matrices. Denote by \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\) and \(\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n\) the eigenvalues of \(A\) and \(B\), respectively. Then there exists a unitary matrix \(U \in M_n\) such that \(A = U\Lambda U^*,\ B = U\Omega U^*,\) where \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n),\ M = \text{diag}(\hat{\mu}_1, \ldots, \hat{\mu}_n).\) Note that \(\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_n\) is a permutation of \(\mu_1, \mu_2, \ldots, \mu_n.\) Let \(\sigma_k = \frac{\lambda_k}{\mu_k}\) \((k = 1, \ldots, n)\), then it is easy to see that all eigenvalues of \(AB^{-1}\) are \(\sigma_1, \sigma_2, \ldots, \sigma_n.\) Without
loss of generality, we may assume that \( \sigma_1 = \min_k \{ \lambda_k \} \), \( \sigma_n = \max_k \{ \lambda_k \} \) and \( \sigma_1 \leq \cdots \leq \sigma_n \). For convenience, we introduce the notation

\[
D(AB, x) = x^* A (\sigma_n I - AB^{-1}) (AB^{-1} - \sigma_1 I) B x.
\]

(2.2)

If \( \sigma_1 \sigma_n > 0 \), then we can define

\[
D((AB)^{-1}, x) = x^* A \left( \frac{1}{\sigma_1} I - A^{-1} B \right) \left( A^{-1} B - \frac{1}{\sigma_n} I \right) B x.
\]

(2.3)

**Lemma 2.2** Let \( A \) and \( B \) be two positive definite commuting matrices with eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \), \( 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \), respectively. \( \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n \), \( D(AB, x) \) and \( D((AB)^{-1}, x) \) are as before. Then for any \( x \in \mathbb{C}^n \),

\[
0 \leq D(AB, x) \leq \frac{1}{4} (\sigma_n - \sigma_1)^2 |x^* A B x|,
\]

\[
0 \leq D((AB)^{-1}, x) \leq \frac{(\sigma_n - \sigma_1)^2}{4(\sigma_1 \sigma_n)^2} |x^* A B x|
\]

(2.4)

for any \( x \in \mathbb{C}^n \).

**Proof** From (2.2),

\[
D(AB, x) = x^* A (\sigma_n I - AB^{-1}) (AB^{-1} - \sigma_1 I) B x
\]

\[
= x^* U \Lambda U^* (\sigma_n I - U \Lambda U^* \Lambda^{-1} U^*) \left( U \Lambda U^* \Lambda^{-1} U^* I - \sigma_1 I \right) U M U^* x
\]

\[
= x^* U \Lambda (\sigma_n I - \Lambda M^{-1}) (\Lambda M^{-1} - \sigma_1 I) M U^* x.
\]

(2.5)

Let \( z = (z_1, \ldots, z_n)^T = (\Lambda M)^{1/2} U^* x \). Thus, \( \| z \|^2 = \| \hat{z}^* \hat{z} = x^* U (\Lambda M) U^* x = x^* A B x \). Then

\[
D(AB, x) = \hat{z}^* (\sigma_n I - \Lambda M^{-1}) (\Lambda M^{-1} - \sigma_1 I) \hat{z} = \sum_{i=1}^n (\sigma_n - \sigma_i) (\sigma_i - \sigma_1) z_i^2 \geq 0.
\]

(2.6)

On the other hand,

\[
\sum_{i=1}^n (\sigma_n - \sigma_i) (\sigma_i - \sigma_1) z_i^2 \leq \frac{(\sigma_n - \sigma_1)^2}{4} \| z \|^2.
\]

(2.7)

Thus,

\[
D(AB, x) \leq \frac{(\sigma_n - \sigma_1)^2}{4} \| z \|^2 = \frac{(\sigma_n - \sigma_1)^2}{4} |x^* A B x|.
\]

(2.8)

The proof of \( D((AB)^{-1}, x) \) is similar. \( \Box \)

**Theorem 2.3** With the assumptions of Lemma 2.2,

\[
0 \leq \frac{x^* A^2 x \cdot x^* B^2 x}{(x^* A B x)^2} - 1 \leq \frac{(\sigma_n - \sigma_1)^2}{4 \sigma_1 \sigma_n} - \frac{1}{|x^* A B x|} \sqrt{D(AB, x) \cdot D((AB)^{-1}, x)}.
\]

(2.9)
Proof Let \( z = (\Lambda x) \sqrt{\gamma} \), \( E = \Lambda M^{-1} = \text{diag}(\frac{\gamma_n}{\mu_n}, \ldots, \frac{\gamma_1}{\mu_1}) = \text{diag}(\sigma_n, \ldots, \sigma_1) \). Then
\[
\frac{x^T A^2 x \cdot x^T B^2 x}{(x^T AB x)^2} = \frac{z^T E z \cdot z^T E^{-1} z}{(z^T z)^2}.
\] (2.10)

From (1.2) and (1.6),
\[
0 \leq \frac{z^T E z \cdot z^T E^{-1} z}{(z^T z)^2} - 1 \leq \frac{(\sigma_n - \sigma_1)^2}{4 \sigma_1 \sigma_n} - \sqrt{C(E, z \| z\|) \cdot C(E^{-1}, z \| z\|)}
\]
\[
= \frac{(\sigma_n - \sigma_1)^2}{4 \sigma_1 \sigma_n} - \frac{1}{\|z\|^2} \sqrt{C(E, z) \cdot C(E^{-1}, z)}.
\] (2.11)

From (2.5) and (2.10), we have
\[
z^T z = x^T AB x, \quad C(E, z) = D(AB, x), \quad C(E^{-1}, z) = D((AB)^{-1}, x).
\] (2.12)

By substituting (2.12) and (2.10) into (2.11), the inequality becomes
\[
0 \leq \frac{x^T A^2 x \cdot x^T B^2 x}{(x^T AB x)^2} - 1 \leq \frac{(\sigma_n - \sigma_1)^2}{4 \sigma_1 \sigma_n} - \frac{1}{|x^T AB x|} \sqrt{D(AB, x) \cdot D((AB)^{-1}, x)}.
\]

**Corollary 2.4** Let \( A \) and \( B \) be two positive definite commuting matrices with eigenvalues \( 0 < \lambda_1 \leq \cdots \leq \lambda_n, \) \( 0 < \mu_1 \leq \cdots \leq \mu_n \), respectively. Then
\[
\frac{x^T A^2 x \cdot x^T B^2 x}{(x^T AB x)^2} \leq \frac{(\lambda_1 \mu_1 + \lambda_n \mu_n)^2}{4 \lambda_1 \mu_1 \lambda_n \mu_n} - \frac{1}{|x^T AB x|} \sqrt{D(AB, x) \cdot D((AB)^{-1}, x)}
\] (2.13)
holds for any nonzero vector \( x \in \mathbb{C}^n \).

**Proof** By Theorem 2.3, we have the following:
\[
0 \leq \frac{x^T A^2 x \cdot x^T B^2 x}{(x^T AB x)^2} \leq \frac{(\sigma_1 + \sigma_n)^2}{4 \sigma_1 \sigma_n} - \frac{1}{|x^T AB x|} \sqrt{D(AB, x) \cdot D((AB)^{-1}, x)}.
\] (2.14)

Let \( f(x) = \frac{(1+x)^2}{4x^2} \). It can be easily deduced that \( f(x) \) is monotone increasing on \([1, +\infty)\). Let \( \alpha_1 = \frac{\gamma_1}{\lambda_1}, \) \( \alpha_n = \frac{\gamma_n}{\mu_n} \). From the definition of \( \sigma_1 \) and \( \sigma_n \), we know that \( \frac{\alpha_n}{\alpha_1} \geq \frac{\alpha_n}{\alpha_1} \geq 1 \). Thus,
\[
\frac{(\sigma_1 + \sigma_n)^2}{4 \sigma_1 \sigma_n} = f\left( \frac{\sigma_n}{\sigma_1} \right) \leq f\left( \frac{\alpha_n}{\alpha_1} \right) = \frac{(\lambda_1 \mu_1 + \lambda_n \mu_n)^2}{4 \lambda_1 \mu_1 \lambda_n \mu_n}.
\]
That is,
\[
0 \leq \frac{x^T A^2 x \cdot x^T B^2 x}{(x^T AB x)^2} \leq \frac{(\lambda_1 \mu_1 + \lambda_n \mu_n)^2}{4 \lambda_1 \mu_1 \lambda_n \mu_n} - \frac{1}{|x^T AB x|} \sqrt{D(AB, x) \cdot D((AB)^{-1}, x)}.
\] (2.15)

**Remark** From Lemma 2.2 and (2.15), we can obtain a sharpened bound for the classical Kantorovich-type inequality, i.e., the Greub-Rheinboldt inequality.
Besides the discussion on the Greub-Rheinboldt inequality (1.9), we are also interested in another form of Kantorovich-type inequality aforementioned. We turn our attention to the inequalities (1.11) and (1.12) in the remainder of this paper.

Let $A$ be an $n \times n$ positive (semi-) definite Hermitian matrix with (nonzero) eigenvalues contained in the interval $[m, M]$, where $0 < m < M$. Let $V$ be $n \times r$ matrices.

As is declared in (1.11), for $A > 0$, $V^*V = I$, and $m, M$ mentioned above, the following inequality holds:

$$V^* A^2 V \leq \frac{(m + M)^2}{4mM} (V^*AV)^2.$$ 

It is not difficult to see that as $V^*V = I$, then $VV^* = VV^+ \leq I$, where $+$ indicates the Moore-Penrose inverse. Multiplying from the right and from the left by $V^*A$ and $AV$ respectively, we have $V^*A^2V \geq (V^*AV)^2$ for $A > 0$. From the well-known Löwner-Heinz inequality, we have $(V^*A^2V)^{1/2} \geq V^*AV$ and the following inequality (see in [16]):

$$(V^*A^2V)^{1/2} \leq \frac{m + M}{2\sqrt{mM}} V^*AV.$$ 

For $z \in [m, M]$, $m > 0$, the convexity of $(z^{-1} + z/mM)$ implies that

$$z^{-1} \leq \frac{m + M}{mM} - \frac{z}{mM}.$$ (2.16)

If $A$ has the representation $A = \Gamma D_{a} \Gamma^*$, where $\Gamma$ is unitary and $D_{a} = \text{diag}(\alpha_{1}, \ldots, \alpha_{n})$, and if $0 < m \leq \alpha_{i} \leq M$, $i = 1, \ldots, n$, then from (2.16) it follows that

$$D_{a}^{-1} \leq \frac{m + M}{mM} I - \frac{D_{a}}{mM}.$$ (2.17)

After multiplying from the right and from the left by $\Gamma$ and $\Gamma^*$, it is not difficult to see that (2.17) yields the following [17]:

$$A^{-1} \leq \frac{m + M}{mM} I - \frac{A}{mM}.$$ (2.18)

Based on (2.18), we derive several results on the inequality (1.12).

**Theorem 2.5** For any $A > 0$ and $V^*V = I$,

$$(V^*A^2V)^{1/2} - V^*AV \leq \frac{(M - m)^2}{4(M + m)} I - D^2(A, V),$$ (2.19)

where $D(A, V) = (\frac{1}{m+M}V^*A^2V)^{1/2} - \frac{(M+m)^{1/2}}{2} I$.

**Proof** From (2.18) and $A > 0$, we can get

$$-A \leq -\frac{mM}{(M + m)} I - \frac{1}{(M + m)} A^2.$$ (2.20)
Since $V^*V = I$, (2.20) can be turned into

$$-V^*AV \leq -\frac{mM}{(M + m)} I - \frac{1}{(M + m)} V^*A^2V.$$  

By adding $(V^*A^2V)^{1/2} \geq 0$ to both sides of the inequality (2.21), we obtain that

$$(V^*A^2V)^{1/2} - V^*AV \leq (V^*A^2V)^{1/2} - \frac{mM}{(M + m)} I - \frac{1}{(M + m)} V^*A^2V,$$  

i.e.,

$$(V^*A^2V)^{1/2} - V^*AV \leq \frac{(M - m)^2}{4(M + m)} I - \frac{1}{(M + m)} V^*A^2V + (V^*A^2V)^{1/2} - \frac{(M + m)}{4} I$$

$$= \frac{(M - m)^2}{4(M + m)} I - \left[ \left( \frac{1}{M + m} V^*A^2V \right)^{1/2} - \frac{(M + m)^{1/2}}{2} I \right]^2.$$  

Thus, we finally have

$$(V^*A^2V)^{1/2} - V^*AV \leq \frac{(m - M)^2}{4(M + m)} I - D^2(A, V),$$

where $D(A, V) = (\frac{1}{(m + M)} V^*A^2V)^{1/2} - \frac{(M + m)^{1/2}}{2} I$.  

**Remark** It is obvious that $D^2(A, V) \geq 0$. Thus, Theorem 2.5 indeed presents an improvement of the Kantorovich-type inequality (1.12) in [12].

For an application to the Hadamard product, we have the following corollary.

**Corollary 2.6** Let $A_1$ and $A_2$ be $n \times n$ positive definite matrices with eigenvalues of $A_1 \otimes A_2$ contained in the interval $[m, M]$. Then

$$(A_1^2 \circ A_2^2)^{1/2} - A_1 \circ A_2 \leq \frac{(M - m)^2}{4(m + M)} I - D^2(A_1 \otimes A_2, V),$$

where $V$ is the selection matrix of order $n^2 \times n$ with the property $V'(A_1 \otimes A_2)V = A_1 \circ A_2$ ($\otimes$ and $\circ$ indicate the tensor and the Hadamard product, respectively).

**3 Conclusion**

In this paper, we introduce some new bounds for several Kantorovich-type inequalities for commutative positive definite Hermitian matrix pairs. As a particular situation, in Corollary 2.4, when $A$ and $B$ are both positive definite, the result provides a sharpened upper bound for the matrix version of the well-known Greub-Rheinboldt inequality. Moreover, it holds for negative definite Hermite matrices. Also, a refinement of Kantorovich-type inequalities concerning positive definite matrices is presented together with an application to the Hadamard product.

**Competing interests**

The authors did not provide this information.
Authors’ contributions
The authors did not provide this information.

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