Low Frequency Approximation for a class of Linear Quantum Systems using Cascade Cavity Realization✩

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Abstract

This paper presents a method for approximating a class of complex transfer function matrices corresponding to physically realizable complex linear quantum systems. The class of linear quantum systems under consideration includes interconnections of passive optical components such as cavities, beam-splitters, phase-shifters and interferometers. This approximation method builds on a previous result for cascade realization and gives good approximations at low frequencies.

Keywords: Quantum Linear Systems, Model Reduction, Cascade Realization.

1. Introduction

In recent years, there has been considerable interest in the modeling and feedback control of linear quantum systems; e.g., see [1–17]. Such linear quantum systems commonly arise in the area of quantum optics; e.g., see [18, 19]. The feedback control of quantum optical systems has applications in areas such as quantum communications, quantum teleportation, and gravity wave detection. In particular, the papers [9, 15–17] have been concerned with a class of linear quantum systems in which the system can be defined in terms of a set of linear complex

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quantum stochastic differential equations (QSDEs) defined purely in terms of annihilation operators. Such linear complex quantum systems correspond to optical systems made up of passive optical components such as optical cavities, beam-splitters, and phase shifters. This paper is concerned with the approximation of systems in this class of linear complex quantum systems. The method proposed in this paper builds on the result of [9, 17] which gives a method for physically realizing a given complex transfer function matrix corresponding to a linear quantum system in the class considered in [15, 16].

In the approximation of linear quantum systems, it is important that the approximate system which is obtained is physically realizable. The issue of physical realizability for linear quantum systems was considered in the papers [4, 5, 10, 15, 16]. This notion relates to whether a given QSDE model represents a physical quantum system which obeys the laws of quantum mechanics. In applying approximation methods to obtain approximate models of quantum systems, it is important that the approximate model obtained is a physically realizable quantum system so that it exhibits the features inherent to quantum mechanics such as the Heisenberg uncertainty principle.

The approximation method proposed in this paper follows directly from the physical relation algorithm proposed in [9, 17]. The physical realizability of the approximate system follows directly from the fact that the algorithm proposed in [9, 17] leads to a physical realization in terms of a cascade connection of optical cavities. For this approximation method, we present some bounds and approximate bounds on the approximation error as a function of frequency.

One application of the approximation method proposed in this paper is in modelling of linear quantum systems where it is desired to construct a simpler, but still physically realizable model of a complex quantum linear system in such a way that a frequency dependent bound on the approximation error is obtained at low frequencies. Another application of the approximation method occurs in the case of coherent quantum feedback control systems when both the plant and controller are linear quantum systems; see [1, 2, 4, 16, 20]. In this case, it is desired to construct a simpler coherent quantum controller which is still physically realizable.

2. A Class of Linear Complex Quantum Systems

We consider a class of linear quantum systems described in terms of the annihilation operator by the quantum stochastic differential equations (QSDEs):

\[
\begin{align*}
da(t) &= F a(t) dt + G d\xi(t); \\
dy(t) &= \bar{H} a(t) dt + J d\xi(t)
\end{align*}
\] (1)
where $F \in \mathbb{C}^{n \times n}$, $G \in \mathbb{C}^{n \times m}$, $\bar{H} \in \mathbb{C}^{m \times n}$ and $J \in \mathbb{C}^{m \times m}$; e.g., see [4, 14–16, 19, 21]. Here $a(t) = [a_1(t) \cdots a_n(t)]^T$ is a vector of (linear combinations of) annihilation operators. The vector $u(t)$ represents the input signals and is assumed to admit the decomposition: $du(t) = \beta_u(t)dt + \tilde{u}(t)$ where $\tilde{u}(t)$ is the noise part of $u(t)$ and $\beta_u(t)$ is an adapted process (see [22], [23] and [24]). The noise $\tilde{u}(t)$ is a vector of quantum noises. The noise processes can be represented as operators on an appropriate Fock space (for more details see [25], [23]). The process $\beta_u(t)$ represents variables of other systems which may be passed to the system (1) via an interaction. More details on this class of systems can be found in [15], [4].

**Definition 1.** (See [6, 15].) A linear quantum system of the form (1) is said to be physically realizable if there exists a commutation matrix $\Theta = \Theta^\dagger > 0$, a coupling matrix $\Lambda$, a Hamiltonian matrix $M = M^\dagger$, and a scattering matrix $S$ such that

$$F = -\Theta \left( iM + \frac{1}{2} \Lambda^\dagger \Lambda \right);\ G = -\Theta \Lambda^\dagger S;\ \bar{H} = \Lambda;\ J = S$$

and $S^\dagger S = I$.

Here, the notation $^\dagger$ represents complex conjugate transpose. In this definition, if the system (1) is physically realizable, then the matrices $S$, $M$ and $\Lambda$ define a complex open harmonic oscillator with scattering matrix $S$, coupling operator $L = \Lambda a$ and a Hamiltonian operator $H = a^\dagger Ma$; e.g., see [21], [23], [22], [4] and [26]. This definition is an extension of the definition given in [4, 15] to allow for a general scattering matrix $S$; e.g., see [6, 12].

The following theorem is a straightforward extension of Theorem 5.1 of [15] to allow for a general scattering matrix $S$.

**Theorem 1.** (See [15].) A complex linear quantum system of the form (1) is physically realizable if and only if there exists a matrix $\Theta = \Theta^\dagger > 0$ such that

$$F\Theta + \Theta F^\dagger + GG^\dagger = 0;\ G = -\Theta \bar{H}^\dagger J;\ J^\dagger J = I.$$ 

In this case, the corresponding Hamiltonian matrix $M$ is given by

$$M = \frac{i}{2} (\Theta^{-1} F - F^\dagger \Theta^{-1})$$

and the corresponding coupling matrix $\Lambda$ is given by

$$\Lambda = \bar{H}.$$
Definition 2. The linear complex quantum system (1) is said to be lossless bounded real if the following conditions hold:

i) $F$ is a Hurwitz matrix.

ii) $\Phi(s) = \bar{H}(sI - F)^{-1}G + J$ satisfies $\Phi(i\omega)^\dagger\Phi(i\omega) = I$ for all $\omega \in \mathbb{R}$.

The following definition extends the standard linear systems notion of minimal realization to linear complex quantum systems of the form (1); see also [15].

Definition 3. A linear complex quantum system of the form (1) is said to be minimal if the following conditions hold:

i) Controllability. $a^\dagger F = \lambda a^\dagger$ for some $\lambda \in \mathbb{C}$ and $a^\dagger G = 0$ implies $a = 0$;

ii) Observability. $Fa = \lambda a$ for some $\lambda \in \mathbb{C}$ and $\bar{H}a = 0$ implies $a = 0$.

The following Theorem is a straightforward extension of Theorem 6.6 of [15] to allow for a general scattering matrix $S$.

Theorem 2. A minimal linear complex quantum system of the form (1) is physically realizable if and only if the system is lossless bounded real.

Definition 4. The complex linear quantum system (1) is said to be a quantum system realization of a complex transfer function matrix $K(s)$ if

$$K(s) = \bar{H}(sI - F)^{-1}G + J.$$  \hfill (6)

3. The Cascade Cavity Realization Algorithm

In this section, we recall the cascade cavity realization result of [9, 17] and generalize it slightly to allow for quantum systems with a more general scattering matrix. Indeed, given a linear quantum system of the form (1) with transfer function matrix (6), we can write $K(s) = J\tilde{K}(s)$ where

$$\tilde{K}(s) = H(sI - F)^{-1}G + I$$  \hfill (7)

and $H = J^{-1}\bar{H}$. Corresponding to (7) is the linear quantum system

$$\begin{align*}
da(t) &= Fa(t)dt + Gdu(t); \\
dy(t) &= Ha(t)dt + du(t).
\end{align*}$$  \hfill (8)

In order to obtain a physical realization of (6), the result of [9, 17] can be applied to transfer function matrix (7). Then a collection of beamsplitters can be used to
implement the unitary matrix $J$; e.g., see [27]. This leads to a physical realization
of the transfer function matrix (7) as shown in Figure 1.

An optical ring cavity consists of a number of partially reflecting mirrors arranged to produce a traveling light wave when coupled to a coherent light source; e.g., see [19, 21]. If we augment such a cavity by introducing phase-shifters on the input and output channels, such a cavity with $m$ mirrors, can be described by a linear quantum system of the form (8) as follows; see [9, 17]:

$$\frac{da}{dt} = padt - h^\dagger du; \quad \frac{dy}{dt} = hadt + du$$  \hspace{1cm} (9)

where

$$p + p^* = -\gamma = -\sum_{i=1}^{m} \kappa_i = -h^\dagger h.$$  \hspace{1cm} (10)

Here $p = -\gamma/2 + i\Delta$,

$$h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} = \begin{bmatrix} \sqrt{\kappa_1 e^{i\theta_1}} \\ \sqrt{\kappa_2 e^{i\theta_2}} \\ \vdots \\ \sqrt{\kappa_m e^{i\theta_m}} \end{bmatrix}, \quad du = \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_m \end{bmatrix}, \quad dy = \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_m \end{bmatrix}.$$

Furthermore, any first order complex linear quantum system of the form (9), with non-zero $h \in \mathbb{C}^m$ and satisfying (10), can be physically realized as a generalized $m$ mirror cavity. In this case, the mirror coupling coefficients and phase shifts are determined using a polar coordinates description of the elements of $h$. Also, the detuning parameter $\Delta$ is determined from the imaginary part of the system pole $p$.

The cascade cavity realization introduced in [9, 17] involves a cascade interconnection of $n$, generalized $m$ mirror cavities as shown in Figure 2. In this cascade system, the $i$th cavity is described by the following QSDEs of the form (8), (9):

$$\frac{da_i}{dt} = p_i a_i dt - H_i^\dagger du; \quad dy = H_i a_i dt + du$$  \hspace{1cm} (11)
where
\[ p_i + p_i^* = -H_i^\dagger H_i. \] (12)

The cascade system is then described by a complex linear quantum system of the form (8) where
\[
F = \begin{bmatrix}
 p_1 & 0 & \ldots & 0 \\
-H_2^\dagger H_1 & p_2 \\
\vdots & \ddots & \vdots \\
-H_n^\dagger H_{n-1} & \ldots & -H_n^\dagger H_{n-1} & p_n
\end{bmatrix}, \quad G = -\begin{bmatrix}
 H_1^\dagger \\
 H_2^\dagger \\
\vdots \\
 H_n^\dagger
\end{bmatrix}, 
\]
\[
H = \begin{bmatrix}
 H_1 \\
 H_2 \\
\vdots \\
 H_n
\end{bmatrix}, \quad J = I. \tag{13}
\]

Reference [9, 17] presents an algorithm for realizing a physically realizable quantum system (8) with transfer function (7) via a cascade of generalized cavities. We restrict attention to quantum systems in which the transfer function (7) corresponds to a minimal system (8) such that the eigenvalues of the matrix \(F\) are all distinct. In this case, it follows via a (complex version of a) standard result from linear systems theory, that the system (8) can be transformed into Modal Canonical Form; e.g., see [28]. The complex linear quantum system in modal canonical form is assumed to be as follows:
\[
\begin{align*}
d\hat{a}(t) &= \tilde{F}\hat{a}(t)dt + \tilde{G}du(t); \\
dy(t) &= \tilde{H}\hat{a}(t)dt + du(t)
\end{align*} \tag{14}
\]

where
\[
\tilde{F} = \begin{bmatrix}
 p_1 & 0 & \ldots & 0 \\
0 & p_2 & \vdots \\
\vdots & \ddots & 0 \\
0 & \ldots & 0 & p_n
\end{bmatrix}; \quad \tilde{G} = \begin{bmatrix}
 \tilde{G}_1 \\
 \tilde{G}_2 \\
\vdots \\
 \tilde{G}_n
\end{bmatrix}; \quad \tilde{H} = \begin{bmatrix}
 \tilde{H}_1 \\
 \tilde{H}_2 \\
\vdots \\
 \tilde{H}_n
\end{bmatrix}. \tag{15}
\]
Also, it is assumed that in this realization, the eigenvalues are ordered so that $|p_1| \leq |p_2| \leq \ldots \leq |p_n|$. Then, $K(s)$ satisfies the equation

$$K(s) = \tilde{H}(sI - \tilde{F})^{-1}\tilde{G} + I.$$  \hfill (16)

The algorithm proposed in [9, 17] is as follows:

**Step 1:** Begin with a minimal modal canonical form realization (14), (15) of the lossless bounded real transfer function matrix $K(s)$.

**Step 2:** Let

$$\tilde{H}_n = \tilde{H}_n, \quad \alpha_n = -\frac{\tilde{H}_n^\dagger\tilde{H}_n}{p_n + p_n^*},$$

$$H_n = \frac{\tilde{H}_n}{\sqrt{\alpha_n}}, \quad t(n,n) = \frac{1}{\sqrt{\alpha_n}}.$$  \hfill (17)

**Step 3:** Calculate the quantities $H_n, H_{n-1}, \ldots, H_1, \alpha_n, \alpha_{n-1}, \ldots, \alpha_1, t(i,j)$, for $j = n, n-1, \ldots, 1$ and $j \geq i$. These are calculated using the following recursive formulas starting with the values determined in Step 2 for $i = n$:

$$\tilde{H}_i = \left[I + \sum_{j=i+1}^{n} \frac{\tilde{H}_j}{p_j - p_i} \sum_{k=i+1}^{j} t(j,k)H_k^*\right]^{-1} \tilde{H}_i;$$  \hfill (18)

$$\alpha_i = -\frac{\tilde{H}_i^\dagger\tilde{H}_i}{p_i + p_i^*}, \quad H_i = \frac{\tilde{H}_i}{\sqrt{\alpha_i}},$$  \hfill (19)

$$t(k,i) = \frac{1}{p_i - p_k} \sum_{j=i+1}^{k} t(k,j)H_j^*H_i \text{ for } k = i + 1, \ldots, n,$$  \hfill (20)

$$t(i,i) = \frac{1}{\sqrt{\alpha_i}}.$$  \hfill (21)

**Step 4:** Set $t(k,i) = 0$ for $k < i$ and define an $n \times n$ transformation matrix $T$ whose $(i,j)$th element is $t(i,j)$.

The following theorem is presented in [9, 17].
Theorem 3. Consider an \( m \times m \) lossless bounded real complex transfer function matrix \( K(s) \) with a minimal modal canonical form quantum realization (14), (15) such that the eigenvalues of the matrix \( \tilde{F} \) are all distinct and that all of the matrix inverses exist in equation (18) when the above algorithm is applied to the system (14), (15). Then the vectors \( H_1, H_2, \ldots, H_n \) defined in the above algorithm together with the eigenvalues \( p_1, p_2, \ldots, p_n \) define an equivalent cascade quantum realization (8), (13) for the transfer function matrix \( K(s) \). Furthermore, this system is such that the condition (12) is satisfied for all \( i \). Moreover, the matrices \( \{F, G, H, I\} \) defining this cascade quantum realization are related to the matrices \( \{\tilde{F}, \tilde{G}, \tilde{H}, \tilde{I}\} \) defining the modal quantum realization (14), (15) according to the formulas:
\[
\tilde{F} = TFT^{-1}, \quad \tilde{G} = TG, \quad \tilde{H} = HT^{-1}
\] (22)
where the matrix \( T \) is defined in the above algorithm.

The physical realization of (6) corresponds to writing
\[
K(s) = JK_n(s)\tilde{K}_{n-1}(s)\ldots\tilde{K}_1(s)
\] (23)
where each transfer function matrix \( \tilde{K}_i(s) \) is a first order transfer function matrix corresponding to an optical cavity described by a QSDE of the form (11).

4. The Main Result

Our proposed method for obtaining an approximate model for a complex linear quantum system (1) with transfer function matrix (6) involves truncating the cascade realization (23) to obtain the approximate transfer function matrix
\[
K_a(s) = J_a\tilde{K}_r(s)\tilde{K}_{r-1}(s)\ldots\tilde{K}_1(s)
\] (24)
where \( J_a = JK_n(0)\tilde{K}_{n-1}(0)\ldots\tilde{K}_{r+1}(0) \) and \( r < n \) is the order of the approximate model. It follows from this construction that \( K_a(s) \) is lossless bounded real and hence physically realizable. Indeed, since the transfer function matrix \( JK_n(s)\tilde{K}_{n-1}(s)\ldots\tilde{K}_{r+1}(s) \) is lossless bounded real, it follows that the matrix \( J_a \) will be unitary. Therefore, \( K_a(s) \) will be lossless bounded real since \( \tilde{K}_r(s)\tilde{K}_{r-1}(s)\ldots\tilde{K}_1(s) \) is lossless bounded real.

In order to construct a state space realization of the reduced dimension transfer function matrix \( K_a(s) \), note that it follows from the development in Section 3 that
the transfer function matrix \( \tilde{K}_a(s) = \tilde{K}_r(s) \tilde{K}_{r-1}(s) \ldots \tilde{K}_1(s) \) has a state space realization of the form (8) defined by the matrices

\[
\tilde{F}_a = \begin{bmatrix}
p_1 & 0 & \ldots & 0 \\
-H_1^t H_1 & p_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
-H_r^t H_1 & \ldots & -H_r^t H_{r-1} & p_r \\
\end{bmatrix}, \quad \tilde{G}_a = -\begin{bmatrix}
H_1^t \\
H_2^t \\
\vdots \\
H_r^t \\
\end{bmatrix},
\]

\[
\tilde{H}_a = \begin{bmatrix}
H_1 & H_2 & \ldots & H_r \\
\end{bmatrix}.
\]

Also, the transfer function matrix \( K_b(s) = \tilde{K}_n(s) \tilde{K}_{n-1}(s) \ldots \tilde{K}_{r+1}(s) \) has a state space realization of the form (8) defined by the matrices

\[
F_b = \begin{bmatrix}
p_{r+1} & 0 & \ldots & 0 \\
-H_{r+2}^t H_{r+1} & p_{r+2} & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
-H_n^t H_{r+1} & \ldots & -H_n^t H_{n-1} & p_n \\
\end{bmatrix}, \quad G_b = -\begin{bmatrix}
H_{r+1}^t \\
H_{r+2}^t \\
\vdots \\
H_n^t \\
\end{bmatrix},
\]

\[
H_b = \begin{bmatrix}
H_{r+1} & H_{r+2} & \ldots & H_n \\
\end{bmatrix}.
\]

Hence, the matrix \( J_a \) is given by \( J_a = J(I - H_b F_b^{-1} G_b) \) and the reduced dimension transfer function matrix \( \tilde{K}_a(s) \) has a state space realization of the form (1) defined by the matrices \( \tilde{F}_a, \tilde{G}_a, J_a \tilde{H}_a, J_a \).

The ordering of the eigenvalues in the cascade realization (15) means that this model is expected to be a good approximation of original model at low frequencies \( \omega < |p_{r+1}| \). The corresponding error system is defined by

\[
K_e(s) = K(s) - \tilde{K}_a(s) = J \begin{pmatrix}
\tilde{K}_n(s) \tilde{K}_{n-1}(s) \ldots \tilde{K}_{r+1}(s) \\
-\tilde{K}_n(0) \tilde{K}_{n-1}(0) \ldots \tilde{K}_{r+1}(0) \\
\end{pmatrix} \tilde{K}_r(s) \tilde{K}_{r-1}(s) \ldots \tilde{K}_1(s). \tag{27}
\]

We now present a result which bounds the induced matrix norm of the approximation error \( \|K_e(j \omega)\| \) as a function of frequency. This bound will be defined in
terms of the following quantities:

\[
B_1(\omega) = \omega \sum_{i=r+1}^{n} \frac{-(p_i + p_i^*)}{|p_i| |p_i - j\omega|};
\]

\[
B_2(\omega) = \omega \sum_{i_1=r+1}^{n} \sum_{i_2=i_1+1}^{n} C_{i_1,i_2}(\omega);
\]

\[
: \quad B_k(\omega) = \omega \sum_{i_1=r+1}^{n} \sum_{i_2=i_1+1}^{n} \cdots \sum_{i_k=i_{k-1}+1}^{n} C_{i_1,i_2,\ldots,i_k}(\omega);
\]

\[
: \quad B_{n-r}(\omega) = \omega C_{r+1,r+2,\ldots,n}(\omega)
\]  

where

\[
C_{i_1,i_2,\ldots,i_k}(\omega) = \begin{vmatrix}
(-j\omega)^{k-1} + (-j\omega)^{k-2} \sum_{m=1}^{k} p_{im}^m \\
(-j\omega)^{k-3} \sum_{m_1=1}^{k} \sum_{m_2=m_1+1}^{k} p_{im_1} p_{im_2}^m \\
(-j\omega)^{k-4} \sum_{m_1=1}^{k} \sum_{m_2=m_1+1}^{k} \sum_{m_3=m_2+1}^{k} p_{im_1} p_{im_2} p_{im_3}^m \\
\vdots \\
(-j\omega)^{k-p-1} \sum_{m_1=1}^{k} \sum_{m_2=m_1+1}^{k} \cdots \sum_{m_p=m_{p-1}+1}^{k} \Pi_{q=1}^{p} p_{im_q}^m \\
\vdots \\
(-j\omega)^{k-p-1} \sum_{m_1=1}^{k} \sum_{m_2=m_1+1}^{k} \cdots \sum_{m_k=m_{k-1}+1}^{k} \Pi_{q=1}^{k-1} p_{im_q}^m
\end{vmatrix}
\]

(29)

**Theorem 4.** Consider a physically realizable linear complex quantum system of the form (1) and corresponding transfer function matrix (6). Suppose this system has a cascade cavity realization (23) and a corresponding approximate transfer function matrix \(K_a(s)\) defined in (24). Then \(K_a(s)\) is physically realizable and the corresponding approximation error transfer function matrix \(K_e(s)\) defined in (27) satisfies the bound

\[
\|K_e(j\omega)\| \leq \sum_{k=1}^{n-r} B_k(\omega)
\]  

(30)

for all \(\omega \geq 0\) where the quantities \(B_k(\omega)\) are defined in (28), (29).
Proof. The fact that \( K_n(s) \) is physically realizable follows from its definition as discussed above. Now for any \( \omega \geq 0 \), it follows from (27) that

\[
\|K_e(j\omega)\| \leq \|J\| \left\| \begin{bmatrix} \tilde{K}_n(j\omega) & \tilde{K}_{n-1}(j\omega) & \cdots & \tilde{K}_{r+1}(j\omega) \\ -\tilde{K}_n(0) & \tilde{K}_{n-1}(0) & \cdots & \tilde{K}_{r+1}(0) \end{bmatrix} \right\|
\times \| \begin{bmatrix} \tilde{K}_r(j\omega) & \tilde{K}_{r-1}(j\omega) & \cdots & \tilde{K}_1(j\omega) \end{bmatrix} \|
\]

using the fact that the transfer function matrix \( \tilde{K}_r(s)\tilde{K}_{r-1}(s)\cdots\tilde{K}_1(s) \) is lossless bounded real and the matrix \( J \) is unitary.

We now consider the transfer function matrices

\[
\tilde{K}_e(s) = \tilde{K}_n(s)\tilde{K}_{n-1}(s)\cdots\tilde{K}_{r+1}(s) - \tilde{K}_n(0)\tilde{K}_{n-1}(0)\cdots\tilde{K}_{r+1}(0)
\]

and \( \hat{K}_e(s) = \tilde{K}_n(s)\tilde{K}_{n-1}(s)\cdots\tilde{K}_{r+1}(s) \). Also, note that it follows from (11) that each transfer function matrix \( \tilde{K}_i(s) \) is of the form \( \tilde{K}_i(s) = I + \frac{H_i}{p_i-s} \). Hence, we can write \( \hat{K}_e(s) = \prod_{i=n}^{r+1} \left( I + \frac{H_i}{p_i-s} \right) \). From this it follows that we can write

\[
\hat{K}_e(s) = I + \sum_{k=1}^{n-r} T_k(s)
\]

where the transfer function matrices \( T_1(s), T_2(s), \ldots, T_{n-r}(s) \) are defined as follows:

\[
T_1(s) = \sum_{i=r+1}^{n} \frac{H_i\hat{H}_i^\dagger}{p_i-s}
\]

\[
T_2(s) = \sum_{i_1=r+1}^{n} \sum_{i_2=i_1+1}^{n} \frac{H_{i_2}H_{i_2}^\dagger H_{i_1}H_{i_1}^\dagger}{(p_{i_1}-s)(p_{i_2}-s)}
\]

\[
\vdots
\]

\[
T_k(s) = \sum_{i_k=r+1}^{n} \sum_{i_{k-1}=i_k+1}^{n} \cdots \sum_{i_2=i_1+1}^{n} \frac{1}{p_{i_1}-s} \prod_{i=k}^{r+1} \frac{1}{p_i-s}
\]

\[
\vdots
\]

\[
T_{n-r}(s) = \prod_{i=n}^{r+1} \frac{H_i\hat{H}_i^\dagger}{p_i-s}
\]
Now it follows from (32) and the triangle inequality that

$$\|\tilde{K}_e(j\omega)\| \leq \sum_{k=1}^{n-r} \|T_k(j\omega) - T_k(0)\|$$  \hspace{1cm} (33)

for all $\omega \geq 0$. We now consider each of the terms $\tilde{T}_k(j\omega) = T_k(j\omega) - T_k(0)$. Indeed, for any $\omega \geq 0$, we obtain

$$\tilde{T}_1(j\omega) = \sum_{i=r+1}^{n} \frac{j\omega H_i H_i^\dagger}{p_i (p_i - j\omega)}$$

$$\tilde{T}_2(j\omega) = \sum_{i_1=r+1}^{n} \sum_{i_2=i_1+1}^{n} H_{i_2} H_{i_2}^\dagger H_{i_1} H_{i_1}^\dagger j\omega S_{i_1, i_2}(j\omega)$$

$$\vdots$$

$$\tilde{T}_k(j\omega) = \sum_{i_1=r+1}^{n} \sum_{i_2=i_1+1}^{n} \ldots \sum_{i_k=i_{k-1}+1}^{n} \prod_{l=k}^{1} H_{i_l} H_{i_l}^\dagger j\omega S_{i_1, \ldots, i_k}(j\omega);$$

$$\vdots$$

$$\tilde{T}_{n-r}(j\omega) = \prod_{i=n}^{r+1} H_i H_i^\dagger j\omega S_{r+1, \ldots, n}(j\omega)$$  \hspace{1cm} (34)

where

$$S_{i_1, i_2, \ldots, i_k}(j\omega) = \begin{bmatrix} (-j\omega)^{k-1} + (-j\omega)^{k-2} \sum_{m=1}^{k} \frac{1}{p_{i_1}^m} + (-j\omega)^{k-3} \sum_{m_1=1}^{k} \frac{1}{p_{i_2}^m p_{i_1}^m} + (-j\omega)^{k-4} \sum_{m_2=1}^{k} \frac{1}{p_{i_2}^m p_{i_1}^m p_{i_2}^m} & \vdots & \vdots & \vdots \end{bmatrix} \frac{1}{\Pi_{l=1}^{p} p_{i_l}^l} (p_{i_1} - j\omega).$$

$$\vdots$$

$$\begin{bmatrix} (-j\omega)^{k-1} + (-j\omega)^{k-2} \sum_{m=1}^{k} \frac{1}{p_{i_k}^m} + (-j\omega)^{k-3} \sum_{m_1=1}^{k} \frac{1}{p_{i_k}^m p_{i_{k-1}}^m} + \vdots \vdots \vdots \vdots \vdots \end{bmatrix} \frac{1}{\Pi_{l=1}^{p} p_{i_l}^l} (p_{i_1} - j\omega).$$

$$\vdots$$

\hspace{1cm} (35)
Now it follows from (12) that $\|H_i H_i^\dagger\| = H_i^\dagger H_i = -p_i - p_i^*$ for $i = r + 1, r + 2, \ldots, n$. Hence, applying the triangle inequality and the Cauchy-Schwartz inequality to (34), it follows that for any $\omega \geq 0$

$$\|	ilde{T}_1(j\omega)\| \leq B_1(\omega);$$

$$\|	ilde{T}_2(j\omega)\| \leq B_2(\omega);$$

$$\vdots$$

$$\|	ilde{T}_k(j\omega)\| \leq B_k(\omega);$$

$$\vdots$$

$$\|	ilde{T}_{n-r}(j\omega)\| \leq B_{n-r}(\omega).$$  \hfill (36)

These bounds combined with (33) and (31) lead to the inequality (30). \hfill \Box

**Remark 1.** The quantity $\sum_{k=1}^{n-r} B_k(\omega)$ in bound (30) is probably too complicated to calculate for all cases except when $n-r$ is equal to one or two. However, we can obtain some good approximations to this quantity which apply at low frequencies. Indeed, for $\omega << |p_{r+1}|$, we obtain

$$\sum_{k=1}^{n-r} B_k(\omega) \approx B_1(\omega) = \omega \sum_{i=r+1}^{n} \frac{- (p_i + p_i^*)}{|p_i| |p_i - j\omega|}$$

for all $\omega \geq 0$. Furthermore since $\sup_{\omega \geq 0} |p_i - j\omega| = -\frac{1}{2} (p_i + p_i^*)$, we obtain the following useful upper bound on $B_1(\omega)$: $B_1(\omega) \leq 2\omega \sum_{i=r+1}^{n} \frac{1}{|p_i|}$ for all $\omega \geq 0$.

5. Illustrative Example

To obtain our initial physically realizable quantum system, we start with a random Hamiltonian matrix $M > 0$, a random coupling matrix $\Lambda$, and a random
commutation matrix $\Theta > 0$ defined as follows:

$$
M = \begin{bmatrix}
3.3314 & 2.5448 + 0.8204j & 2.4007 + 1.1592j \\
2.5448 - 0.8204j & 3.3994 & 2.7136 + 0.4185j \\
2.4007 - 1.1592j & 2.7136 - 0.4185j & 4.1258 \\
3.6470 - 1.3066j & 3.7009 + 0.1246j & 4.2612 + 0.3611j \\
1.9949 - 1.8970j & 2.6090 - 1.4039j & 3.5647 - 1.0224j \\
& 3.6470 + 1.3066j & 1.9949 + 1.8970j \\
& 3.7009 - 0.1246j & 2.6090 + 1.4039j \\
& 4.2612 - 0.3611j & 3.5647 + 1.0224j \\
& 5.9568 & 4.1173 + 1.3450j \\
& 4.1173 - 1.3450j & 3.9970
\end{bmatrix},
$$

$$
\Theta = \begin{bmatrix}
1.8356 & 2.3408 - 0.3287j & 1.8732 + 0.0757j \\
2.3408 + 0.3287j & 3.9779 & 3.3007 + 0.4982j \\
1.8732 - 0.0757j & 3.3007 - 0.4982j & 3.8582 \\
1.8750 - 0.0488j & 2.8626 - 0.2561j & 2.5536 + 0.7201j \\
1.5306 - 0.3665j & 2.3995 - 1.1168j & 3.1242 - 0.8781j \\
& 1.8750 + 0.0488j & 1.5306 + 0.3665j \\
& 2.8626 + 0.2561j & 2.3995 + 1.1168j \\
& 2.5536 - 0.7201j & 3.1242 + 0.8781j \\
& 3.0974 & 1.9480 + 0.9779j \\
& 1.9480 - 0.9779j & 3.0319
\end{bmatrix},
$$

$$
\Lambda = \begin{bmatrix}
-1.0106 + 0.0000j & 0.5077 + 1.0950j & 0.5913 + 0.4282j \\
0.6145 - 0.3179j & 1.6924 - 1.8740j & -0.6436 + 0.8956j \\
& 0.3803 + 0.7310j & -0.0195 + 0.0403j \\
& -1.0091 + 0.5779j & -0.0482 + 0.6771j
\end{bmatrix}.
$$

Also, we chose the scattering matrix $S = I$. This leads to a corresponding system of the form (1) where the matrices are defined as in (2). The eigenvalues of the
resulting matrix $F$ are $s = -0.0038 - 0.0181j$, $s = -0.2103 - 0.1040j$, $s = -0.1674 - 1.1066j$, $s = -3.0388 - 0.9275j$, $s = -10.8541 - 225.9473j$. Clearly, the last eigenvalue has a much larger absolute value than all of the others and so we will apply our algorithm to approximate this fifth order system by a fourth order system. Bode plots comparing the original system frequency response with the reduced dimension system frequency response are shown in Figures 1-4. These Bode plots indicate that the proposed method gives a good approximation at low frequencies. Also, it follows from the construction that the reduced dimension system is lossless bounded real and so physically realizable.

![Bode Diagram](image)

Figure 3: Bode plot of original and approximate system from input 1 to output 1.

In Figure 7 we show the singular value plot of the error transfer function matrix $K_e(s) = K(s) - K_a(s)$ along with the error bound defined by $B_1(\omega)$. In this example, we see that the error bound is in fact exact since we only reduced the dimension of the original system by one.

6. Conclusions

In this paper, we have presented a method of approximating a class of linear complex quantum systems in such a way that the property of physical realizability
Figure 4: Bode plot of original and approximate system from input 2 to output 1.

Figure 5: Bode plot of original and approximate system from input 1 to output 2.
Figure 6: Bode plot of original and approximate system from input 2 to output 2.

Figure 7: Singular value plot of the error transfer function matrix $K_e(s)$ and the error bound $B_1(\omega)$. 
(which is equivalent to the strict bounded real property in this case) is preserved. The paper presents a bound on the approximation error which shows that the approximation is accurate at low frequencies.

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