Abstract

The magnetic field dependence of the $A \rightarrow B$ transition temperature $T_{AB}$ in the superfluid $^3He$ is reconsidered in order to take into account the linear-in-field contribution beyond the approximation used in Ref.[7]. In the high field region, where the quadratic-in-field contribution prevails, the well known answer is restored. On the other hand, it is pointed out that the Fermi liquid effects shift the observability of the linear-in-field region to rather low magnetic fields.

Soon after the discovery of the superfluidity of liquid $^3He$ it was established that near critical temperature $T_c(P)$ the phase diagram of superfluid state experiences a profound modification under the action of even small external magnetic fields [1, 2, 3]. This modification shows up in elimination of the direct normal to $B$ phase transition over entire phase diagram in favour of an anisotropic $A$ phase.

The $A \rightarrow B$ transition temperature $T_{AB}(H) < T_c$ can be established by equating the free energies $F_A(H)$ and $F_B(H)$. Near the zero-field transition temperature $T_c(P)$ the free energy can be found by minimizing the Ginzburg-Landau functional

$$F_S = 3\alpha_{\mu\nu} < \Delta_\mu \Delta^*_\nu > + F_S^{(4)}(\tilde{\Delta}),$$

where the fourth order contribution in the order-parameter $\tilde{\Delta}(\hat{k})$ reads as
\[ F_S^{(4)} = 9\{ \beta_1 < \Delta_\mu \Delta_\mu > < \Delta^*_\nu \Delta^*_\nu > + \beta_2 < |\tilde{\Delta}|^2 >^2 + \beta_3 < \Delta_\mu \Delta_\nu > < \Delta^*_\nu \Delta^*_\mu > + \beta_4 < \Delta_\nu \Delta^*_\mu > < \Delta^*_\mu \Delta^*_\nu > + \beta_5 < \Delta_\mu \Delta^*_\nu > < \Delta^*_\nu \Delta^*_\mu > \} \]  

In Eqs. (1) and (2) the spin-vector \( \tilde{\Delta} \) is defined according to the general expression for the order-parameter \( A_{\mu i} \) of the spin-triplet \( p \)-wave Cooper condensate: \( \Delta_\mu (\hat{k}) = A_{\mu i} \hat{k}_i \), and the angular brackets \( < \cdots > \) stand for an average over the momentum direction \( \hat{k} \) on the Fermi surface. In presence of a magnetic field \( \vec{H} \) the tensor coefficient \( \alpha_{\mu\nu} \) of the second order term in Eq.(1) reads as

\[ \alpha_{\mu\nu} = \alpha \delta_{\mu\nu} + ig_1 \varepsilon_{\mu\nu\lambda} H_\lambda + g_2 H_\mu H_\nu, \]  

where

\[ \alpha = \frac{1}{3} N_F \ln \left( \frac{T}{T_c} \right), \]  

\[ g_1 H = -\frac{1}{3} N_F \eta h, \]  

\[ g_2 H^2 = \frac{1}{3} N_F \kappa h^2, \]

with \( h = \hbar \gamma H/(2 k_B T_c) = H/H_o \). The dimensionless coefficients \( \eta \) and \( \kappa \) could be considered as the phenomenological quantities although their values can be estimated according to the microscopic calculations. In the weak-coupling approximation [4]

\[ \eta_{wc} = \frac{N_F'}{N_F} k_B T_c \ln \left( \frac{2 \gamma E_{\omega_c}}{\pi T_c} \right) \]  

where \( N_F' \) stands for the derivative of the quasiparticle DOS with respect to the energy at the Fermi level. A detailed calculations which take into account the linear-in-field corrections to the Fermi liquid parameters are performed in Ref. [5].

As to the \( \kappa \)-coefficient, it stems from the free energy part

\[ \delta F_H^{(2)} = \frac{1}{2} \delta \chi_s H^2, \quad \delta \chi_s = \chi_s - \chi_N, \]  

\[ (6) \]
where $\chi_s(\chi_N)$ stands for the magnetic susceptibility of the superfluid (normal) phase. For the $B$ phase near $T_c$, $\delta F_{H}^{(2)} = g_2 H^2 \Delta^2$ where $g_2 H^2$ is given according to Eq. (4c) with

$$\kappa = \frac{7\zeta(3)}{4\pi^2} \frac{1}{(1 + F_o)^2}$$

Here the Fermi liquid parameter $F_o \cong -3/4$ and is weakly pressure dependent.

As it is well known, the linear-in-field term (4b) is the origin of a tiny splitting of the $A$ phase (due to a small asymmetry of the density of quasi-particle states at the Fermi level). The quadratic-in-field contribution (4c) gives rise to the magnetic field suppression of $\Delta_{\uparrow \downarrow}$ component of the energy gap of the $B$ phase.

Based on the argument that the term (4b) is rather small, in the majority of considerations of $T_{AB} = T_{AB}(H)$ this term is usually discarded, and in a standard way one starts from the following expressions for the equilibrium free energies of the $A$ and $B$ phases:

$$F_A = -\frac{1}{4\beta_{245}} \alpha^2,$$  \hspace{1cm} (8a)

$$F_B = -\frac{1}{2(3\beta_{12} + \beta_{345})} \left[ \frac{3}{2} \alpha^2 + g_2 H^2 \alpha + \frac{2\beta_{12} + \beta_{345}}{2\beta_{345}} (g_2 H^2)^2 \right].$$  \hspace{1cm} (8b)

Since here the linear-in-field contribution (proportional to $g_1$) is dropped, the action of the magnetic field appears only in $F_B$.

Equating (8a) and (8b) it is found that $T_{AB}(H)$ is to be obtained from the equation

$$\varphi = \alpha^2 + a\alpha + b = 0$$

with

$$a = 2P_1 g_2 H^2,$$

$$b = P_1^2 (1 - q_1) (g_2 H^2)^2$$

where the coefficients $P_1$ and $q_1$ are defined in the Appendix. From Eq.(9) it follows the answer for $\tau_{AB} = 1 - T_{AB}/T_c$:

$$\tau_{AB} = P_1 (1 + \sqrt{q_1}) \kappa h^2$$

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which reproduces a well known result obtained in Ref.[6].

Now we turn to the role of the linear-in-field contribution to $T_{AB}(H)$. This question was first posed in Ref.[7]. The starting point is the construction of the expressions for $F_{A2}(H)$ and $F_{B}(H)$, the contribution (4b) being taken into account. In a standard way it is established that

$$ F_{A2} = -\frac{1}{4\beta_{245}} \left[ \alpha^2 - \frac{\beta_{245}}{\beta_{5}} g_1^2 H^2 \right], \quad (12) $$

$$ F_{B} = -\frac{1}{2(3\beta_{12} + \beta_{345})} \left\{ \frac{3}{2} \alpha^2 + g_2 H^2 \alpha + \frac{3\beta_{12} + \beta_{345}}{\beta_{4} - (3\beta_{1} + \beta_{35})} g_1^2 H^2 + \right. $$

$$ \left. + \frac{2\beta_{12} + \beta_{345}}{2\beta_{345}} (g_2 H^2)^2 + \right. $$

$$ \left. + \frac{(3\beta_{12} + \beta_{345})^2}{(3\beta_{1} + \beta_{35} - \beta_{4})^2} \left[ \frac{\beta_{1}}{\beta_{345}} g_2^2 H^4 \alpha + \frac{\beta_{1}(3\beta_{12} + \beta_{345})}{4(3\beta_{1} + \beta_{35} - \beta_{4})^2} g_1^4 H^4 \right] \right\}. \quad (13) $$

Comparison of $F_{A2}$ and $F_{B}$ gives an equation for $T_{AB}(H)$:

$$ \varphi(T_{AB}/T_c) = \alpha^4 + a\alpha^3 + ba^2 + ca + d = 0 \quad (14) $$

where now

$$ a = 2P_1 g_2 H^2, $$

$$ b = -P_1 P_2 P_3 g_1^2 H^2 + P_1^2 (1 - q_1)(g_2 H^2)^2, $$

$$ c = -P_1 P_2^2 P_4 g_1^2 g_2 H^4, $$

$$ d = -P_1 P_2^3 P_5 (g_1 H)^4. \quad (15) $$

The definition of all the coefficients $P_a$ is given in the Appendix.

In Ref.[7] the second and third lines in Eq.(13), containing terms on the order $H^4$ were neglected. This has the following influence on the answer for $T_{AB}(H)$: i) neglect of the contribution proportional to $(g_2 H^2)^2$ makes it impossible one to reproduce the correct answer given by Eq.(11) (the term
\( \sqrt{q_1} \) will be lost), ii) neglection of the contribution collected in the square brackets of the third line of Eq.(13) changes the answer for the linear-in-field contribution to \( T_{AB}(H) \). To avoid this drawbacks we address Eq.(14) and, as a first step, perform the variable transformation \( \alpha \rightarrow x - \frac{1}{4}a \), after which an equation for \( x(T_{AB}/T_c) \) is obtained:

\[
\varphi(T_{AB}/T_c) = x^4 + Ax^2 + Bx + C = 0
\]

with the coefficients:

\[
A = -\left[ P_1 P_2 P_3 (g_1 H)^2 + \frac{1}{2} P_1^2 (1 + 2q_1) (g_2 H^2)^2 \right],
\]

\[
B = q_1 P_1^3 (g_2 H^2)^3 + q_2 P_1 P_2 g_1^2 g_2 H^4,
\]

\[
C = -P_1 P_2^3 P_5 (g_1 H)^4 + \frac{1}{16} P_1^4 (1 - 4q_1) (g_2 H^2)^4 + \frac{1}{4} P_1^2 P_2 (P_1 P_3 - 2q_2) (g_1 g_2 H^3)^2.
\]

It is to be noticed, that as a result of the variable transformation used, in Eq. (16) the cubic term is absent. At the same time the coefficient of the linear term \( B \) is zero in the weak-coupling approximation.

In order to solve Eq. (16) we use the decomposition \( x = x_0 - \sqrt{q_1} P_1 g_2 H^2 \) which generates the decomposition \( \varphi = \varphi_0 + \delta \varphi \) with \( \varphi_0 \) being the solution of the equation

\[
\varphi_0 = x_0^4 + A_0 x_0^2 + C_0 = 0,
\]

where \( A_0 \) and \( C_0 \) are the coefficients \( A \) and \( C \) taken at \( q_1 = q_2 = 0 \).

From Eq. (18) it is found that

\[
x_0(T_{AB}/T_c) = -\sqrt{P_0^2 (g_1 H)^2 + \frac{1}{4} P_1^2 (g_2 H^2)^2}
\]

where

\[
P_0^2 = \frac{1}{2} P_1 P_2 P_3 [1 + \sqrt{1 + 4 P_2 P_5 / P_1 P_3^2}].
\]

The direct inspection shows that \( \delta \varphi \) is the sum of terms proportional to the powers of \( q \) and the minimal power in \( H \) contained in \( \delta \varphi \) is \( H^5 \). For this
reason we can use an approximation with $\delta \varphi$ disregarded, and as a result it is found that

\[
\alpha(T_{AB}/T_c) = x - \frac{1}{4}a = x_o - \sqrt{q_1}P_1g_2H^2 - \frac{1}{2}P_1g_2H^2 =
\]

\[
= -P_1\left(\frac{1}{2} + \sqrt{q_1}\right)g_2H^2 - \sqrt{P_o^2(g_1H)^2 + \frac{1}{4}P_1^2(g_2H^2)^2}.
\]

Using this result we finally have the following simple answer for $\tau_{AB}$:

\[
\tau_{AB} = P_1\left(\frac{1}{2} + \sqrt{q_1}\right)k\hbar^2 + \sqrt{P_o^2(\eta h)^2 + \frac{1}{4}P_1^2(\kappa h^2)^2}.
\]

By introducing a characteristic magnetic field $H_* = 2(P_o/P_1)(\eta/\kappa)H_o$ the asymptotic regions where the quadratic-in-field ($H >> H_*$) and the linear-in-field ($H << H_*$) contributions to $T_{AB}$ prevail are found:

\[
\tau_{AB} = \begin{cases} 
P_1(1 + \sqrt{q_1})k\hbar^2, & H >> H_* \\
\rho(\eta h), & H << H_*.
\end{cases}
\]

For $H >> H_*$ the well known result [4,8] is reproduced.

In order to isolate the role of the linear-in-field contribution to $T_{AB}$ it is convenient to consider

\[
\delta \tau_{AB}(h) = \tau_{AB}(h) - (1 + \sqrt{q_1})P_1k\hbar^2,
\]

and construct graphically

\[
f(h) = \delta \tau_{AB}(h)/h = P_1\eta(\sqrt{1 + (h/h_*)^2 - h/h_*}),
\]

where the scaling value $h_* = 2(P_o/P_1)(\eta/\kappa)$.

Below $f(h)$ is plotted for the pressure $P = 10$ bar ($P_o\eta \simeq 5, 4 \cdot 10^{-2}, h_* \simeq 1, 6 \cdot 10^{-2}$).

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The coefficients $P_a (a = 1, 2, 3, 4, 5)$ and $q_a (a = 1, 2)$ introduced in the main text are defined as follows:

\[
P_1 = \frac{\beta_{245}}{2\beta_{345} - 3\beta_{13}}, \\
P_2 = \frac{3\beta_{12} + \beta_{345}}{\beta_4 - 3\beta_1 - \beta_{35}}, \\
P_3 = \frac{\beta_{45} - 3\beta_1 - \beta_3}{-\beta_5}, \\
P_4 = \frac{-2\beta_1}{\beta_{345}},
\]
\[ P_5 = -\frac{\beta_1}{2(\beta_4 - 3\beta_1 - \beta_{35})}. \]

\[ q_1 = \frac{(3\beta_{12} + \beta_{345})(2\beta_{13} - \beta_{345})}{\beta_{245}\beta_{345}}, \]

\[ q_2 = P_1P_3 - P_2P_4, \]

where \( \beta_{ij\ldots} = \beta_i + \beta_j + \cdots. \)

The coefficients \( q_1 \) and \( q_2 \) contain only the strong-coupling corrections \( \delta \beta_i \)
defined according to the decomposition

\[ \beta_1 = -\beta_o + \delta \beta_1, \]

\[ \beta_2 = 2\beta_o + \delta \beta_2, \]

\[ \beta_3 = 2\beta_o + \delta \beta_3, \]

\[ \beta_4 = 2\beta_o + \delta \beta_4, \]

\[ \beta_5 = -2\beta_o + \delta \beta_5, \]

\[ \beta_o = \frac{7\zeta(3)}{240} \frac{N_F}{(\pi T_c)^2}. \]

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