New estimates for the length of the Erdős-Herzog-Piranian lemniscate

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Abstract

Let \( p(z) \) be a monic polynomial of a fixed degree \( n \geq 1 \). Consider the lemniscate

\[
L_p := \{ z : |p(z)| = 1 \}.
\]

Let \( |L_p| \) be the length of \( L_p \). Erdős, Herzog, and Piranian conjectured that

\[
|L_p| \leq |L_{p_0}| = 2n + O(1) \tag{1}
\]

where \( p_0(z) = z^n - 1 \). Despite the efforts of many people, the conjecture still remains unresolved. The goal of the note is to present a new approach to the problem that allows one to show that \( |L_p| \leq |L_{p_0}| \) when \( p \) is sufficiently close to \( p_0 \) and to prove the asymptotic estimate \( |L_p| \leq 2n + o(n) \) as \( n \to \infty \) for all monic polynomials \( p \).

1. Introduction

In 1958 Erdős, Herzog, and Piranian ([1], Problem 12) asked whether the polynomial \( p_0(z) = z^n - 1 \) has the maximal length of the lemniscate \( L = L_p = \{ z \in \mathbb{C} : |p(z)| = 1 \} \) among all monic polynomials \( p(z) = z^n + \ldots \) of degree \( n \). After 50 years, this conjecture still remains unresolved. The first upper bound \( |L| \leq 4\pi n \) was obtained by Dolzhenko in 1960 in his thesis [2] and published in 1963 [3]. Meanwhile, in 1961, Pommerenke [4] published a much worse estimate \( 74n^2 \), which became known much wider than Dolzhenko’s result. Apparently unaware of Dolzhenko’s work, Borwein [5] published the estimate \( |L| \leq 8\pi n \) in 1995. The first real improvement
came in 1999 when Eremenko and Hayman [6] proved the conjecture for \( n = 2 \), showed that all critical points of the extremal polynomial must lie on the lemniscate, and obtained the estimate \( |L| \leq 9.173n \) for all \( n \). In 2007, this upper bound was superceded by the estimate \( |L| \leq 2\pi n \) proved by Danchenko [7], which remained the best published upper bound by the moment of writing this article. Several more papers devoted to or motivated by the lemniscate problem have been published (see [8], [9], and [10], for instance).

The goal of this note is to present a new approach to the problem based on an explicit formula for the length. Unfortunately, we haven’t been able to get a full solution either but, at least, we managed to show that \( |L_p| \) attains a local maximum when \( p = p_0 \) and to obtain an upper bound of the form \( 2n + o(n) \).

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3. Notation

Throughout this paper, we denote by \( p \) a monic polynomial, by \( n \) its degree, by \( \eta \) an arbitrary root of \( p \), by \( \zeta \) an arbitrary root of \( p' \), and by \( \xi \) an arbitrary root of \( pp' \). This notation will be used without any further comments, so, say, \( \sum_\zeta \) will always mean the sum over all roots of \( p' \) counted with their multiplicities, etc.

As usual, \( \mathbb{C} \) and \( \mathbb{R} \) stand for the sets of complex and real numbers respectively. We will also denote by \( D_r \) the disk \( \{ z \in \mathbb{C} : |z| < r \} \) and by \( T_r \) its boundary circumference \( \{ z \in \mathbb{C} : |z| = r \} \). We denote by \( d(F, z) \) the distance from the point \( z \in \mathbb{C} \) to the set \( F \subset \mathbb{C} \).

If \( f \) is a complex valued smooth function defined on an open set \( \Omega \subset \mathbb{C} \), we shall treat it as a function of complex variable \( z = x + iy \) and use the
complex notation for the partial derivatives

\[ \partial f = f_z = \frac{1}{2}(f_x - iy), \quad \bar{\partial} f = f_{\bar{z}} = \frac{1}{2}(f_x + iy), \]

and for the differential forms

\[ dz = dx + idy, \quad d\bar{z} = dx - idy. \]

In this notation,

\[ d\bar{z} \wedge dz = 2idx \wedge dy \]

and

\[ df = f_z dz + f_{\bar{z}} d\bar{z} = \partial f dz + \bar{\partial} f d\bar{z}, \]

All other notation conventions will be introduced at the moment of their first appearance.

4. The Stokes formula

Let \( E \) be a bounded open subset of the complex plane \( \mathbb{C} \). Suppose that the boundary of \( E \) consists of finitely many smooth Jordan arcs. Let \( \omega \) be a differential 1-form on \( \mathbb{C} \) with locally bounded coefficients that are smooth outside a finite set \( K \) whose differential \( d\omega \) (which makes sense everywhere in \( \mathbb{C} \setminus K \)) has locally integrable coefficients with respect to the area measure. (we shall call such forms quasismooth). It is not hard to check that the classical Stokes formula

\[ \int_{\partial E} \omega = \int_E d\omega \tag{2} \]

remains valid for quasismooth differential 1-forms.

Let now \( p = p(z) \) be a monic polynomial of a fixed degree \( n \geq 2 \) (the case \( n = 1 \) is rather trivial, so we will not consider it here). With any such polynomial, we associate the sets

\[ L = L_p = \{ z \in \mathbb{C} : |p(z)| = 1 \} \quad \text{and} \quad E = E_p = \{ z \in \mathbb{C} : |p(z)| < 1 \}. \]

Let \( K \) be a finite set containing the critical points of \( p \), and let \( s = s(z) \) be a smooth complex valued function defined on \( \mathbb{C} \setminus K \). Suppose that \( s \) coinsides with the outward unit normal vector to \( \partial E \) on \( \partial E \setminus K \). Then

\[ |L| = \int_L \frac{dz}{is} = \int_L \bar{s} \, dz \tag{3} \]
If the differential 1-form $\omega = \bar{s} \, dz$ is quasismooth, then the Stokes formula yields
\[ i |L| = \iint_E d\omega. \]
Evaluating $d\omega$ we have
\[ d\omega = (\partial \bar{s} \, dz + \bar{\partial} \bar{s} \, d\bar{z}) \wedge dz = \partial \bar{s} \, d\bar{z} \wedge dz = 2i \bar{s} dx \wedge dy. \]
Therefore,
\[ |L| = 2 \iint_E \bar{s} dx \wedge dy. \]
Since the quantity on the left hand side is real, we conclude that also
\[ |L| = 2 \iint_E \partial s \, dA = 2 \Re \iint_E \partial s \, dA, \tag{4} \]
where $A$ is the plane area measure.

The outward normal vector to $\partial E$ is the normalized gradient of the function $\log |p|$. Computing it, we get
\[ s = \frac{\varphi}{|\varphi|} = \frac{\varphi}{\Phi}, \quad \text{where} \quad \varphi = \frac{p'}{p} \]
on $L_p$. We have a lot of freedom extending $s$ from $L_p$ to the entire complex plane. The most obvious extension is given by the right hand side of the last formula, which makes sense everywhere except the zeroes and the critical points of $p$. This way we get a form $\omega = \bar{s} \, dz$ such that $\int_{L_p} \omega = i |L_p|$ and for every other piecewise smooth curve $\gamma$, we have $\left| \int_{\gamma} \omega \right| \leq |\gamma|$. One more direct computation shows that
\[ 2 \partial s = -\frac{|\varphi|}{\varphi} \varphi' = \frac{|\varphi|}{\Phi} \left( \sum_{\eta} \frac{1}{z - \eta} - \sum_{\zeta} \frac{1}{z - \zeta} \right) \]
where the first sum is taken over all roots $\eta$ of $p$ and the second sum is taken over all roots $\zeta$ of $p'$ (counted with multiplicities in both cases). It is not hard to see from here that $\omega$ is quasismooth and for every bounded open set $G$, one has
\[ \left| \iint_G d\omega \right| \leq \iint_G \left( \sum_{\xi} \frac{1}{|z - \xi|} \right) dA(z) \]
where the sum is taken over all roots $\xi$ of $pp'$. Our first task will be to use this extension to show that $|L_p| \leq |L_{p_0}|$ for all $p$ sufficiently close to $p_0(z) = z^n - 1$. 

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5. A simple lemniscate problem

**Lemma 1** Let \( p(z) = z^n + a_2z^{n-2} + a_3z^{n-3} + \cdots + a_n \) satisfy \( a_n \in \mathbb{R}, \) \( \max_{2 \leq k \leq n} |a_k| = 1. \) Let \( \tilde{L}_p = \{ z \in \mathbb{C} : \text{Re} \ p(z) = 0 \}. \) Then for every \( r \geq 2, \) one has
\[
|\tilde{L}_p \cap D_r| \leq 2nr - c_n
\]
where \( c_n > 0 \) depends on \( n \) only.

**Proof.** Note first of all that the intersection \( \tilde{L}_p \cap T_\rho \) consists of at least \( 2n \) points for every \( \rho \geq 2. \) Indeed, \( \text{Re} \ z^n \) has \( n \) positive maxima interlaced with \( n \) negative minima of size \( \rho^n \) on \( T_\rho. \) All other terms together can contribute not more than \( \rho^{n-2} + \rho^{n-3} + \cdots + 1 < \rho^n, \) so the signs of \( \text{Re} \ p(z) \) at those maxima and minima are the same as those of \( \text{Re} \ z^n. \) Thus, we have at least \( 2n \) sign changes and, thereby, at least \( 2n \) roots for \( \text{Re} \ p(z) \) on \( T_\rho. \)

Now consider a big sphere \( S \) of radius \( R \) touching the complex plane at the origin (we think that the complex plane is horizontal and the sphere is above the plane). Let \( \tau \) be the central projection from the plane to the lower hemisphere of the sphere \( S \) with the center of the projection coinciding with that of the sphere. Then, by the well-known Poincaré formula,
\[
|\tau(\tilde{L}_p)| = \pi R \int_S \#(\mathcal{E}_\theta \cap \tau(\tilde{L}_p)) \, d\mu(\theta)
\]
where \( \mu \) is the surface measure on \( S \) normalized by the condition \( \mu(S) = 1 \) and \( \mathcal{E}_\theta = \{ x \in S : \langle x, \theta \rangle = 0 \} \) is the great circle of \( S \) orthogonal to \( \theta. \) Since every horizontal circumference on the lower hemisphere lying outside \( \tau(D_r) \) intersects \( \tau(\tilde{L}_p) \) at not fewer than \( 2n \) points, we conclude that
\[
|\tau(\tilde{L}_p \setminus D_r)| \geq \pi R n - 2nR \arctan \frac{r}{R}.
\]
Combining this inequality with the previous identity, we see that
\[
|\tau(\tilde{L}_p \cap D_r)| \leq 2nR \arctan \frac{r}{R} - \pi R \int_S [n - \#(\mathcal{E}_\theta \cap \tau(\tilde{L}_p))] \, d\mu(\theta).
\]
Since \( \tau^{-1}(\mathcal{E}_\theta) \) is a line on the complex plane \( \mathbb{C}, \) we conclude that for almost all \( \theta, \) one has
\[
\#(\mathcal{E}_\theta \cap \tau(\tilde{L}_p)) \leq n,
\]
so the integrand is non-negative almost everywhere.
Now let us pass to the limit as $R \to \infty$. The left hand side tends to $|\tilde{L}_p \cap D_r|$. The first term on the right hand side tends to $2nr$. Finally, the integral can be rewritten as

$$\int_{\mathbb{C}} [n - \#(\tilde{L}_p \cap \Gamma_w)] \left(1 + \frac{|w|^2}{R^2}\right)^{-3/2} \frac{dA(w)}{2|w|}$$

where $\Gamma_w$ is the line passing through $w \in \mathbb{C}$ in the direction perpendicular to $w$ (this is a simple exercise in the change of variable theorem) and, thereby, it tends to

$$\int_{\mathbb{C}} [n - \#(\tilde{L}_p \cap \Gamma_w)] \frac{dA(w)}{2|w|}$$

by the monotone convergence theorem.

Thus it remains to show that this integral is separated away from 0 on the class of polynomials under consideration. We shall show that it is true even for the smaller integral $\int_{D_1}$.

The first observation is that for a fixed monic polynomial $p$ the number of the intersections of $\tilde{L}_p$ with $\Gamma_w$ is a stable quantity (meaning that it doesn’t change under slight perturbations of the coefficients of $p$) for almost all $w$. This, together with the dominated convergence theorem, allows to conclude that our integral is a continuous function of the coefficients $a_2, \ldots, a_n$ running over a compact set. Thus, it will suffice to show that this integral never vanishes.

To this end, fix $p$ and let $j$ be the least index for which $a_j \neq 0$. Consider the line $\Gamma(\alpha) = \{te^{i\alpha} : t \in \mathbb{R}\}$. We have

$$\text{Re} \ p(te^{i\alpha}) = t^n \text{Re} e^{i\alpha} + t^{n-j} \text{Re} [a_j e^{i(n-j)\alpha}] + \cdots = q(t).$$

Note now that $\text{Re} [a_j e^{i(n-j)\alpha}]$ preserves sign on intervals of length $\frac{\pi}{n-j}$ and that $\text{Re} e^{i\alpha}$ changes sign on every interval whose length exceeds $\frac{\pi}{n}$. Thus, we can find $\alpha$ such that $\text{Re} [a_j e^{i(n-j)\alpha}]$ and $\text{Re} e^{i\alpha}$ are both non-zero and have the same sign. If $q(t)$ had $n$ real roots, every its derivative would have only real roots. But the $n-j$-th derivative is a polynomial of the form $\kappa t^j + \lambda$ where $\kappa, \lambda \neq 0$ have the same sign and such polynomial has at least one non-real root for all $j \geq 2$. Thus $q(t)$ must have non-real roots as well. Since the existence of non-real roots is a stable property of polynomials, we conclude that for all $w$ sufficiently close to 0 whose argument is sufficiently close to $\alpha + \frac{\pi}{2}$, the line $\Gamma_w$ intersects $\tilde{L}_p$ at strictly less than $n$ points, which is more than enough to ensure the strict positivity of the integral in question. 

□
A simple rescaling argument yields the following result. Let \( p(z) = z^n + a_3z^{n-2} + a_3z^{n-3} + \cdots + a_n \) satisfy \( a_n \in \mathbb{R}, \max_{2 \leq k \leq n} |a_k|^{1/k} = a > 0 \). Let \( \tilde{L}_p = \{ z \in \mathbb{C} : \text{Re } p(z) = 0 \} \). Then for every \( r \geq 2a \), one has

\[
|\tilde{L}_p \cap D_r| \leq 2nr - cn^a
\]

For what follows, it will be useful to note that under these assumptions, the roots \( \zeta \) of \( p' \) lie in \( D_{2a} \).

6. The first variation estimate

Suppose that \( p \) is a monic polynomial that is close to \( p_0 \). Using an appropriate shift of the argument, we can ensure that the coefficient at \( z^{n-1} \) equals 0. After that is done, we can make the free term real by using an appropriate rotation. Hence we can assume without loss of generality that \( p = p_0 + q \) where \( q(z) = \sum_{k=2}^{n} a_k z^{n-k} \), the coefficients \( a_k \) are small, and \( a_n \in \mathbb{R} \).

Let

\[
a = \max_k |a_k|^{1/k}.
\]

Choose \( r \in (4a, \frac{1}{4}) \) and consider the parts of the domains \( E_p \) and \( E_{p_0} \) lying inside and outside \( T_r \) separately. We shall start with the comparison of \( |L_p \setminus D_r| \) and \( |L_{p_0} \setminus D_r| \).

Using the same differential 1-form \( \omega \) as before, we get

\[
i |L_p \setminus D_r| + \int_{E_p \cap T_r} \omega = \int \int_{E_p \setminus D_r} d\omega
\]

and

\[
\int_{L_{p_0} \setminus D_r} + \int_{E_{p_0} \cap T_r} \omega = \int \int_{E_{p_0} \setminus D_r} d\omega
\]

Let \( G \) be the symmetric difference of \( E_p \) and \( E_{p_0} \). Since

\[
\left| \int \omega \right| \leq |\gamma|
\]

for every piecewise smooth curve \( \gamma \), we immediately conclude from the above formulae that

\[
|L_p \setminus D_r| \leq |L_{p_0} \setminus D_r| + |G \cap T_r| + \int \int_{G \setminus D_r} \sum_{\xi} \frac{1}{|z - \xi|} dA(z).
\]
Now let us notice that for every \( \rho \in (4a, \frac{1}{4}) \), the circumference of radius \( \rho \) centered at 1 is transversal to the unit circumference. Since \( z^n \) travels over that circumference at constant speed, simple geometric considerations show that \( |p_0(z)| - 1| \geq c \rho^{n-1} \min_j |z - z_j| \) when \( z \in T_\rho \) where \( z_j \) are the 2n solutions of the system \( |p(z) = 1|, |z| = \rho \), and \( c > 0 \) is some absolute constant. Since \( |q| \leq 2a^2 \rho^{n-2} \) on \( T_\rho \), we immediately conclude that \( G \cap T_\rho \) is contained in the union of 2n arcs of length \( Ca^2 \rho^{-1} \), i.e., that

\[
|G \cap T_\rho| \leq Ca^2 \rho^{-1}
\]  

(5)

with some absolute positive constant \( C \) depending on \( n \) only. In particular,

\[
|G \cap T_r| \leq Ca^2 r^{-1}.
\]

To estimate the double integral over \( G \setminus D_r \), note that since \( p_0' \) doesn’t vanish outside \( D_{1/4} \), and since \( |q| \leq 2^{n+1}a^2 \) in \( D_2 \), we can use the regular perturbation theory to conclude that \( G_1 = G \setminus D_{1/4} \) is contained in the \( Ca^2 \)-neighborhood of \( L_{p_0} \), so \( A(G_1) \leq Ca^2 \). Also, when \( a \) is small enough, the distance from \( G_1 \) to every root \( \xi \) of \( pp' \) is bounded from below by some constant \( c \) depending on \( n \) only. Thus,

\[
\int \int_{G_1} \sum_\xi \frac{1}{|z - \xi|} dA(z) \leq Ca^2.
\]

The other part \( G_2 = G \cap D_{1/4} \setminus D_r \) is more interesting. If \( \eta \) is a root of \( p \), then we still have \( d(G_2, \eta) \geq c \), so the integral of \( \frac{1}{|z - \eta|} \) over \( G_2 \) does not exceed \( CA(G_2) \). In view of (5), we have

\[
A(G_2) \leq Ca^2 \int_r^{1/4} \frac{d\rho}{\rho} \leq Ca^2 \log \frac{1}{r}.
\]

If \( \zeta \) is a root of \( p' \), then \( |\zeta| \leq 2a \) and, thereby,

\[
\frac{1}{|z - \zeta|} \leq \frac{2}{|z|}
\]

on \( G_2 \). Thus, the integral of \( \frac{1}{|z - \zeta|} \) over \( G_2 \) is bounded by

\[
Ca^2 \int_r^{\infty} \frac{d\rho}{\rho^2} = Ca^2 r^{-1}.
\]
Bringing all these estimates together, we finally conclude that

$$|L_p \setminus D_r| \leq |L_{p_0} \setminus D_r| + Ca^2 r^{-1}.$$ 

Now it is time to compare $|L_p \cap D_r|$ with $|L_{p_0} \cap D_r|$. For the latter, we’ll use the trivial lower bound $|L_{p_0} \cap D_r| \geq 2nr$. To estimate the former, consider the polynomial $f(z) = 1 + p(z) = z^n + q(z)$, the lemniscate

$$\tilde{L} = \{z : \text{Re } f = 0\}.$$ 

and the region $F_p = \{z : \text{Re } f(z) > 0\}$. Obviously, $E_p \subset F_p$. The key observation is that the difference $H = F_p \setminus E_p$ is rather small in the disk $D_r$. Indeed, since the inequality $|p| < 1$ is equivalent to $2 \text{Re } f - |f|^2 > 0$ and $|f| \leq 2r^n$ in $D_r$, we conclude that $H \cap D_r \subset \{z : |\text{Re } f(z)| \leq 2r^{2n}\}$. Arguing as above, we see that

$$|L_p \cap D_r| \leq |\tilde{L} \cap D_r| + |H \cap T_r| + \iint_H \left( \sum_{\xi} \frac{1}{|z - \xi|} \right) dA(z)$$

The length $|H \cap T_r|$ can be easily estimated from above by $Cr^{n+1}$. Indeed, the condition $|\text{Re } f| \leq 2r^{2n}$ implies $|\text{Re } z^n| \leq 2(r^{2n} + a^2 r^{n-2}) < \frac{a^n}{2}$. Hence, the speed with which $\text{Re } f$ changes when $z$ travels over $T_r$ with unit speed across $H$ is at least

$$\frac{nr^n - 2}{2} - \sum_{k=2}^{n} (n - k)a^k r^{n-k-1} \geq \frac{nr^n - 2}{4},$$

from where the above estimate follows immediately.

To estimate the area integral, note that $\iint_H \frac{dA(z)}{|z - \xi|}$ has the geometric meaning of the average (over the angle the lines make with the positive axis) length of the cross-sections of $H$ by the lines passing through $\xi$. These cross-sections can be easily controlled by the Remez theorem (see [12], Theorem 5.1.1) that asserts that the measure of the set where the polynomial $at^n + \ldots$ with real coefficients is less than $\varepsilon$ in absolute value is at most $4(\varepsilon/|a|)^{1/n}$. Applying this estimate to the polynomial $\text{Re } f(\xi + e^{i\alpha} t)$, we conclude that for every $\xi$,

$$\iint_{H \cap D_r} \frac{dA(z)}{|z - \xi|} \leq 4 \cdot 2^{1/n} r^2 \int_0^\pi \frac{d\alpha}{|\cos n\alpha|^{1/n}} = Cr^2.$$ 

Finally, according to Lemma 1,

$$|\tilde{L} \cap D_r| \leq 2nr - ca.$$
Bringing all the estimates together, we obtain
\[ |L_p| \leq |L_{p_0}| - ca + C(r^2 + a^2 r^{-1}) \]
and it remains to put \( r = a^{2/3} \) and choose \( a \) sufficiently small to get the desired result.

7. The simplest upper bound for the lemniscate length

We shall start with using the above extension to show that
\[ |L| \leq 2\pi(2n - 1) \, . \]
We have
\[ |L| = -\iint_E \frac{1}{\varphi} \frac{\varphi'}{\varphi} dA \leq \iint_E \frac{\varphi'}{\varphi'} dA \, . \]
Recall now that
\[ -\frac{\varphi'}{\varphi} = \sum_\eta \frac{1}{z - \eta} - \sum_\zeta \frac{1}{z - \zeta} \]
Thus
\[ |L| \leq \sum_\xi \int_E \frac{1}{|z - \xi|} dA \, , \]
Since the logarithmic capacity of \( E \) is equal to 1, Pólya’s theorem (see [11], Theorem 5.3.5) implies that its area does not exceed \( \pi \). Thus,
\[ |L| \leq (2n - 1) \int_D \frac{dA}{|z|} = 2\pi(2n - 1) \, . \]
Note, by the way, that if \( p \) has multiple roots, then every root of \( p \) of multiplicity \( m \) appears \( m \) times in \( \sum_\eta \) and \( m - 1 \) times in \( \sum_\zeta \). If we take this cancellation into account, then the last estimate can be improved to \( 2\pi(2k - 1) \) where \( k \) is the number of distinct roots of \( p \).

8. An improved upper bound.

To obtain a better estimate, we consider a different extension of \( s \), which differs from the one we considered before by the factor \( |p| \) (this factor is identically \( 1 \) on \( L \)). So, we define
\[ s = |p| \frac{\varphi'}{\varphi} = \frac{p|p'|}{p'} = \frac{|p'|}{\varphi} \, . \]
In this case
\[ 2\partial s = \frac{p|p'|}{p'} \left( \frac{2p'}{p} - \frac{p''}{p'} \right) = 2|p'| - \frac{|p\varphi|}{\varphi} \psi = |p'| - \frac{|p\varphi|}{\varphi} \varphi', \]
where
\[ \psi := \frac{p''}{p'} = \sum_{\zeta} \frac{1}{z - \zeta}. \]

The lemniscate length can now be evaluated as follows
\[ |L| = 2 \iint_E |p'| dA - \iint_E \frac{|p\varphi|}{\varphi} \psi dA. \tag{6} \]

Since \( p \) maps \( E \) onto the unit disk covering each point of the unit disk \( n \) times, we have
\[ \iint_E |p'|^2 dA = \pi n. \]

The Cauchy inequality yields
\[ \iint_E |p'| dA \leq \pi \sqrt{n}. \]

On the other hand,
\[ \left| \iint_E \frac{|p\varphi|}{\varphi} \psi dA \right| \leq \iint_E |\psi| dA \leq \sum_{\zeta} \iint_{|z - \zeta|} \frac{1}{|z - \zeta|} dA \leq (n - 1) \iint_{D} \frac{1}{|z|} dA = 2\pi (n - 1). \]

Combining these estimates, we obtain the upper bound
\[ |L| \leq 2\pi (n - 1 + \sqrt{n}), \]
which is only marginally worse than Danchenko’s estimate \( 2\pi n \).

9. Asymptotic estimate

In this section we obtain an asymptotic estimate of the lemniscate length of a monic polynomial \( p \) of degree \( n \) as \( n \) approaches infinity.

Using the same extension as in the previous section, we get the inequality
\[ |L| \leq \pi \sqrt{n} + J, \]
where 
\[ J = - \text{Re} \iint_E \frac{|\varphi p|}{\varphi} \frac{\varphi'}{\varphi} \, dA. \]

In this section we will estimate the integral \( J \) more accurately to get the best possible asymptotic estimate for \( |L| \).

Take \( \delta \in (0, \frac{1}{4}) \). Let \( E_\delta \subset E \) be the set of all points \( z \in E \) satisfying
\[
\frac{\varphi'}{\varphi} \geq 2\delta n + \sum_\xi \frac{4\delta}{|z - \xi|} \tag{7}
\]
and
\[
|z - \xi| \geq \frac{2\delta}{\sqrt{n}} \quad \text{for all} \ \xi. \tag{8}
\]

We shall start with estimating the integral over \( E \setminus E_\delta \). The set \( E \setminus E_\delta \) is the union of the set \( B_1 \) where condition (7) is broken and the set \( B_2 \) where condition (8) is broken. Since the area of the latter set is at most \( 4\pi\delta^2 \), we have
\[
\iint_{B_1} \frac{|\varphi'|}{\varphi} \, dA \leq 2\delta n A(B_1) + \sum_\xi 4\delta \iint_{B_1} \frac{dA(z)}{|z - \xi|} \leq 2\pi n \delta + 8\pi(2n - 1)\delta \tag{9}
\]
\[
\iint_{B_2} \frac{|\varphi'|}{\varphi} \, dA \leq \sum_\xi 4\delta \iint_{B_2} \frac{dA(z)}{|z - \xi|} \leq 4\pi(2n - 1)\delta, \tag{10}
\]
so
\[
\iint_{E \setminus E_\delta} \frac{|\varphi'|}{\varphi} \, dA \leq 2\pi n \delta + 8\pi(2n - 1)\delta + 4\pi \delta(2n - 1) \leq 26\pi \delta n
\]
and hence,
\[
J \leq 26\pi \delta n + \iint_{E_\delta} |p'| \, dA + J_\delta
\]
where
\[
J_\delta = - \text{Re} \iint_{E_\delta} \frac{|\varphi p|}{\varphi} \psi \, dA.
\]
Using the estimate
\[
\iint_E |p'| \, dA \leq \pi \sqrt{n},
\]
once more, we see that it is enough to estimate \( J_\delta \).
We rewrite $J_\delta$ using the explicit expression for $\psi$ now:

$$J_\delta = \sum_\zeta \text{Re} \int_\delta |p\varphi| \frac{1}{\zeta - z} dA. \quad (11)$$

Now partition the complex plane into equal squares with side length

$$\ell = \frac{\delta^2}{\sqrt{n}},$$

and define the set $F$ as the minimal union of these squares that contains $E_\delta$, that is, $F$ includes all the squares that intersect $E_\delta$. For each square $Q$ in $F$, let $\tilde{Q}$ be the twice larger square with the same center. Finally, let

$$\tilde{F} = \bigcup_{Q: Q \subset F} \tilde{Q}.$$

Note that

$$d(\tilde{F}, \xi) > \frac{2\delta}{\sqrt{n}} - \frac{4\delta^2}{\sqrt{n}} > \frac{\delta}{\sqrt{n}}, \quad (12)$$

for every root $\xi$ of $pp'$.

Since for every function $\Phi$

$$\int_{E_\delta} \text{Re} (|p| \Phi) dA \leq \int_{E_\delta} \text{Re}_+ \Phi dA \leq \int_F \text{Re}_+ \Phi dA,$$

we obtain

$$J_\delta \leq \sum_\zeta \int_F \text{Re}_+ \left( \frac{|\varphi|}{\zeta - z} \frac{1}{\varphi} \right) dA(z). \quad (13)$$

Let now

$$I_\zeta(Q) := \int_Q \text{Re}_+ \frac{|\varphi|}{(\zeta - z)\varphi} dA(z). \quad (14)$$

where $Q$ is a square of the set $F$ and $\zeta$ is a root of $p'$. Let $w$ be any point in $Q$. Then

$$I_\zeta(Q) \leq \int_Q \text{Re}_+ \left( \frac{|\varphi|}{(\zeta - z)\varphi} - \frac{|\varphi|}{(\zeta - w)\varphi} \right) dA(z) + \frac{1}{|\zeta - w|} \int_Q \text{Re}_+ \theta dA,$$

where

$$\theta = \frac{|(\zeta - w)| |\varphi|}{(\zeta - w)\varphi}. \quad (13)$$
Denote the first term by $I^{(1)}_\zeta(Q)$ and the second one by $I^{(2)}_\zeta(Q)$.

We have

$$I^{(1)}_\zeta(Q) \leq \iint_Q \frac{|w - z|}{|\zeta - z||\zeta - w|} dA(z) \leq \frac{2\ell}{|\zeta - w|} \iint_Q \frac{1}{|z - \zeta|} dA.$$

Since $\ell = \delta^2/\sqrt{n}$ and $|w - \zeta| \geq \delta/\sqrt{n}$ (see (12)), we have

$$I^{(1)}_\zeta(Q) \leq 2\delta \iint_Q \frac{1}{|\zeta - z|} dA(z).$$

(15)

To estimate $I^{(2)}_\zeta(Q)$, note that

$$\Re \theta = \frac{1}{\pi} + \sum_{k \neq 0} a_k \theta^k.$$

(16)

where $a_k$ are the Fourier coefficients of the function $\Re \theta$ on the unit circumference. To estimate the sum over $k \neq 0$, we need

**Lemma 2** Let $Q$ be a square with size length 1 and let $u$ be a real harmonic function in the twice larger square $\tilde{Q}$ with the same center such that

$$|\bar{\partial}u| > R,$$

everywhere in $\tilde{Q}$. Then

$$\left| \iint_Q e^{iu} dA \right| \leq \frac{4}{R}.$$

**Proof.** The function $f = 1/\bar{\partial}u$ is analytic in the square $\tilde{Q}$ and $|f| < 1/R$. By Cauchy theorem $|f'| \leq 2/R$ in $Q$. Since $u$ is a real function, we have $\bar{\partial}u = 1/\bar{f}$, and the Stokes formula yields

$$\int_{\partial Q} \bar{f} e^{iu} dz = \iint_Q \bar{\partial}(\bar{f} e^{iu}) d\bar{z} \wedge dz = \iint_Q (\bar{f}' e^{iu} + i\bar{f} e^{iu}) d\bar{z} \wedge dz.$$

Thus

$$\left| \iint_Q e^{iu} dA \right| \leq \frac{1}{2} \left[ \int_{\partial Q} |\bar{f}| |dz| + 2 \iint_Q |\bar{f}'| dA \right] \leq \frac{4}{R}.$$

We are done. \(\square\)
A simple scaling argument shows that if the side length of $Q$ is not 1 but $\ell$, then the estimate changes to

$$\left| \iint_Q e^{iu} \, dA \right| \leq \frac{4}{R\ell} A(Q).$$

We apply this lemma to estimate

$$\iint_Q \theta^k \, dA.$$ 

Take $w \in Q$ to satisfy

$$\frac{\ell^2}{|\zeta - w|} = \frac{A(Q)}{|\zeta - w|} = \iint_Q \frac{1}{|\zeta - z|} \, dA.$$ 

By Lemma 2

$$\frac{1}{|\zeta - w|} \iint_Q \theta^k \, dA \leq \frac{4}{R\ell|k|} \iint_Q \frac{1}{|\zeta - z|} \, dA,$$

where

$$R = \inf_{z \in Q} |\partial u|,$$

It remains to estimate $R$ from below. We have

$$|\partial u| = \frac{1}{2} \left| \frac{\varphi'}{\varphi} \right|.$$ 

Take $z_0 \in E_\delta \cap Q$. According to (7) we have

$$|\partial u(z_0)| \geq \delta n + \sum_{\xi} \frac{2\delta}{|\xi - z_0|}.$$ 

Since $d(\widetilde{Q}, \zeta) \geq \delta/\sqrt{n}$, $|z - z_0| \leq 4\ell$, and $\delta \leq \frac{1}{4}$, we get

$$\frac{1}{2} \leq \frac{|\xi - z_0|}{|\xi - z|} \leq 2.$$ 

Note that

$$\left( \begin{array}{c} \varphi' \\ \varphi \end{array} \right)' = \sum_{\eta} \frac{1}{(z - \eta)^3} - \sum_{\zeta} \frac{1}{(z - \zeta)^3}.$$ 

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Hence
\[ \left| \frac{1}{2} \left( \varphi' \right)' \right| \leq \sum_{\xi} \frac{1}{2|z - \xi|^2} \leq \frac{\sqrt{n}}{\delta} \sum_{\xi} \frac{1}{2|z - \xi|} \leq \frac{\sqrt{n}}{\delta} \sum_{\xi} \frac{1}{|z_0 - \xi|}. \]

Finally, for \( z \in \tilde{Q} \) we have
\[ |\partial u(z)| \geq |\partial u(z_0)| - 2\ell \max_{Q} \left| \frac{1}{2} \left( \varphi' \right)' \right| \geq \delta n. \]

and thereby, in view of (16),
\[ I^{(2)}(Q) \leq \frac{4a}{\delta^3 \sqrt{n}} \int_{Q} \frac{1}{|\zeta - z|} dA + \frac{1}{\pi} \int_{Q} \frac{1}{|\zeta - z|} dA, \tag{17} \]
where
\[ a = \sum_{k \neq 0} \frac{|a_k|}{|k|}. \]

Summing the inequalities (17) and (15) over all squares \( Q \in \tilde{F} \) and then over \( \zeta \) we get
\[ J_\delta \leq \left( \frac{1}{\pi} + 2\delta + \frac{4a}{\delta^3 \sqrt{n}} \right) \int_{F} \sum_{\zeta} \frac{1}{|\zeta - z|} dA. \tag{18} \]

To finish the estimate we need to estimate the logarithmic capacity of the set \( F \). To this end, we estimate the logarithmic derivative \( \varphi \) of \( p \) first. We have
\[ |\varphi| \leq \sum_{n} \frac{1}{|\eta - z|} \leq \frac{n \sqrt{n}}{\delta} \]
on \( F \) (see (12)). Since \( d(E, z) \leq 2\ell \) for every point \( z \in F \), we have
\[ \log |p| \leq 2\ell \max_{F} |\varphi| \leq \frac{2\ell n \sqrt{n}}{\delta} = 2\delta n. \]

Thus,
\[ |p| \leq e^{2\delta n}. \]
The last inequality means that the logarithmic capacity of \( F \) is at most \( e^{2\delta} \) and, thereby, its area is at most \( \pi e^{4\delta} \). Thus,
\[ |L| \leq 26\pi \delta n + 2\pi \sqrt{n} + e^{2\delta} \left( \frac{1}{\pi} + 2\delta + \frac{4a}{\delta^3 \sqrt{n}} \right) 2\pi (n - 1). \]
for every $\delta > 0$. Now it is clear that the optimal choice of $\delta$ is about $n^{-1/8}$, which results in the estimate

$$|L| \leq 2n + O(n^{7/8}).$$

We have no doubt that the power $7/8$ can be substantially improved though to bring it below $1/2$ seems quite a challenging problem.

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