Zagier’s conjecture on \( L(E, 2) \)

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1 Introduction

Summary. In this paper we introduce an elliptic analog of the Bloch-Suslin complex and prove that it (essentially) computes the weight two parts of the groups \( K_2(E) \) and \( K_1(E) \) for an elliptic curve \( E \) over an arbitrary field \( k \). Combining this with the results of Bloch and Beilinson we proved Zagier’s conjecture on \( L(E, 2) \) for modular elliptic curves over \( \mathbb{Q} \).

1. The elliptic dilogarithm. The dilogarithm is the following multivalued analytic function of on \( \mathbb{C}P^1 \setminus \{0, 1, \infty \} \):

\[
Li_2(z) = - \int_0^z \log(1 - t) \frac{dt}{t}
\]

It has a single-valued version, the Bloch-Wigner function:

\[
\mathcal{L}_2(z) := \text{Im}Li_2(z) + \text{arg}(1 - z) \cdot \log |z|
\]

The elliptic analog of the dilogarithm was defined and studied by Spencer Bloch in his seminal paper [Bl1].

The story goes as follows. Let \( E(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}} \) be the complex points of an elliptic curve \( E \). Here \( q := \exp(2\pi i \tau), \text{Im} \tau > 0 \).

The function \( \mathcal{L}_2(z) \) has a singularity of type \( |z| \log |z| \) near \( z = 0 \). It satisfies the relation \( \mathcal{L}_2(z) = -\mathcal{L}_2(z^{-1}) \). So averaging \( \mathcal{L}_2(z) \) over the action of the group \( \mathbb{Z} \) on \( \mathbb{C}^* \) generated by \( z \mapsto qz \) we get the convergent series:

\[
\mathcal{L}_{2,q}(z) := \sum_{n \in \mathbb{Z}} \mathcal{L}_2(q^n z), \quad \mathcal{L}_{2,q}(z^{-1}) = -\mathcal{L}_{2,q}(z)
\]
This function can be extended by linearity to the set of all divisors on $E(\mathbb{C})$ setting $L_{2,q}(P) := \sum n_i L_{2,q}(P_i)$ for a divisor $P = \sum P_i$.

2. The results on $L(E, 2)$. Let $L(E, s) = L(h^1(E), s)$ be the Hasse-Weil $L$-function of an elliptic curve $E$ over $\mathbb{Q}$. We will always suppose that an elliptic curve $E$ has at least one point over $\mathbb{Q}$: zero for the addition law. Let $v$ be a valuation of a number field $K$, and $h_v$ the corresponding canonical local height on $E(K)$. As usual $x \sim_{\mathbb{Q}^*} y$ means that $x = qy$ for a certain $q \in \mathbb{Q}^*$.

Let $J = J(E)$ be the Jacobian of $E$.

**Theorem 1.1** Let $E$ be a modular elliptic curve over $\mathbb{Q}$. Then there exists a $\mathbb{Q}$-rational divisor $P = \sum n_j P_j$ over $\bar{\mathbb{Q}}$ which satisfy the conditions a)-c) listed below and such that

$$L(E, 2) \sim_{\mathbb{Q}^*} \pi \cdot L_{2,q}(P)$$

The conditions on divisor $P$:

a) $$\sum n_j P_j \otimes P_j \otimes P_j = 0 \text{ in } S^3J(\bar{\mathbb{Q}})$$

b) For any valuation $v$ of the field $\mathbb{Q}(P)$ generated by the coordinates of the points $P_j$

$$\sum n_j h_v(P_j) \cdot P_j = 0 \text{ in } J(\bar{\mathbb{Q}}) \otimes \mathbb{R}$$

c) For every prime $p$ where $E$ has a split multiplicative reduction one has an integrality condition on $P$, see (4) below.

The integrality condition. Suppose $E$ has a split multiplicative reduction at $p$ with $N$-gon as a special fibre. Let $L$ be a finite extention of $\mathbb{Q}_p$ of degree $n = ef$ and $\mathcal{O}_L$ the ring of integers in $L$. Let $E^0$ be the connected component of the Néron model of $E$ over $\mathcal{O}_L$. Let us fix an isomorphism $E^0_{\mathbb{F}_{p^f}} \cong \mathbb{G}_m/\mathbb{F}_{p^f}$. It provides a bijection between $\mathbb{Z}/eN\mathbb{Z}$ and the components of $E_{\mathbb{F}_{p^f}}$. For a divisor $P$ such that all its points are defined over $L$ denote by $d(P; \nu)$ the degree of the restriction of the flat extension of a divisor $P$ to the $\nu$'th component of the $(eN)$-gon.

Let $B_3(x) := x^3 - \frac{3}{2} x^2 + \frac{1}{4} x$ be the third Bernoulli polynomial.

The integrality condition at $p$ is the following condition on a divisor $P$, provided by the work of Schappacher and Scholl ([SS]). For a certain (and hence for any, see s. 3.3) extention $L$ of $\mathbb{Q}_p$, such that all points of the divisor $P$ are defined over $L$ one has ($[L : \mathbb{Q}_p] = ef$):

$$\sum_{\nu \in \mathbb{Z}/(eN)\mathbb{Z}} d(P; \nu) B_3\left(\frac{\nu}{eN}\right) = 0$$

**Remarks.** 1. For a $p$-adic valuation $v$ of the field $K(P)$ one has $(\log p)^{-1} h_v(P_j) \in \mathbb{Q}$. So the condition b) in this case looks as follows

$$(\log p)^{-1} \sum n_j h_v(P_j) \cdot P_j = 0 \text{ in } J(K(P)) \otimes \mathbb{Q}$$
In particular the right hand side is a finite dimensional \( \mathbb{Q} \)-vector space.

2. Lemma 1.5 below shows that, assuming (2), if the condition (3) is valid for all archimedean valuations but one then it is valid for all of them. In particular if \( P \in \mathbb{Z}[E(\mathbb{Q})] \) we can omit (3) for the archimedean valuation.

The proof of theorem (1.1) is based on the results of S. Bloch [Bl1] on regulators on elliptic curves, a “weak” version of Beilinson’s conjecture for modular curves proved by A.A. Beilinson in [B2] and the results presented in s.2-3 below.

To prove the theorem we introduce for an elliptic curve \( E \) over an arbitrary field \( k \) a new complex (the elliptic motivic complex \( \mathcal{B}(E; \mathbb{Z}) \)) and prove that its cohomology essentially computes the weight 2 parts of \( K_2(E) \) and \( K_1(E) \) (see theorems (1.3) and (3.8)). This complex mirrors the properties of the elliptic dilogarithm. It is an elliptic deformation of the famous Bloch-Suslin complex which computes \( K^\text{ind}_3(F) \otimes \mathbb{Q} \) and \( K_2(F) \) for an arbitrary field \( F \) (see [DS], [S] and s.1.6).

In particular we replace the “arithmetical” condition b) by its refined “geometrical” version (see s. 1.4), which is equivalent to the condition b) for curves over number fields.

Our results imply

**Theorem 1.2** Let \( E \) be an elliptic curve over \( \mathbb{Q} \). Then

i) For any element \( \gamma \in K_2(E) \) there exists a \( \mathbb{Q} \)-rational divisor \( P \) on \( E \) satisfying the conditions a), b) from theorem (1.1) such that the value of the Bloch-Beilinson regulator map \( r_2 : K_2(E) \to \mathbb{R} \) on \( \gamma \) is \( \sim \mathbb{Q} \cdot \mathcal{L}_{2,q}(P) \).

ii) For any \( \mathbb{Q} \)-rational divisor \( P \) on \( E \) satisfying the conditions a), b) there exists an element \( \gamma \in K_2(E) \otimes \mathbb{Q} \) such that \( r_2(\gamma) \sim \mathbb{Q} \cdot \mathcal{L}_{2,q}(P) \).

Theorem (1.2i) implies immediately

**Corollary 1.3** Let \( E \) be an elliptic curve over \( \mathbb{Q} \). Let us assume that the image of \( K_2(E)_{\mathbb{Z}} \otimes \mathbb{Q} \) under the regulator map is \( L(E,2) \cdot \mathbb{Q} \). (This is a part of the Bloch-Beilinson conjecture).

Then for any \( \mathbb{Q} \)-rational divisor \( P \) on \( E(\bar{\mathbb{Q}}) \) satisfying the conditions a) - c) of theorem (1.1) one has

\[
q \cdot L(E,2) = \pi \cdot \mathcal{L}_{2,q}(P)
\]

where \( q \) is a rational number, perhaps equal to 0.

**Remark.** Corollary (1.3) has an analog for an elliptic curve over any number field. Its formulation is an easy exercise to the reader.

Unlike in Zagier’s conjecture on \( \zeta \)-functions of number fields one can not expect \( P_i \in E(\mathbb{Q}) \): the Mordell-Weil group of an elliptic curve over \( \mathbb{Q} \) could be trivial.

The conditions a)-b) are obviously satisfied if \( P \) is (a multiple of) a torsion divisor. Moreover, if \( E \) is a curve with complex multiplication then \( L(E,2) \) is
the value of the elliptic dilogarithm on a torsion divisor ([Bl1]). However if $E$ is not a CM curve this should not be true in general. Thus one has to consider the non-torsion divisors, and so it is necessary to use the conditions a)-b) in full strength.

The conditions a) and b) were guessed by D. Zagier several years ago after studying the results of the computer experiments with $\mathbb{Q}$-rational points on some elliptic curves, which he did with H. Cohen.

3. A numerical example. $E$ is given by equation $y^2 - y = x^3 - x$. The discriminant $\Delta = \text{conductor} = 37$. So $E$ has split multiplicative reduction at $p = 37$ with one irreducible component of the fiber of the Néron model. Therefore the integrality condition is empty.

Local nonarchimedean heights on $E$. Let $P = \left[ \frac{a}{p^2}, \frac{b}{p^3} \right] \in E(\mathbb{Q})$ where $a, b$ are prime to $p$. If $p$ is prime to $\Delta$ then $h_p(P) = 0$ if $\delta \leq 0$ and $h_p(P) = \delta \cdot \log p$ if $\delta > 0$. The local height at $p = 37$ is given by $h_{37}(P) = -1/6 + 2\delta$ (see the formula for the local height in s. 4.3 of or [Sil]).

The Mordell-Weil group has rank one and is generated by the point $P = [0, 0]$. Consider the following integral points on $E$:

$$P = [0, 0], \quad 2P = [1, 0], \quad 3P = [1, 1], \quad 4P = [2, 3], \quad 6P = [6, -14]$$

and also

$$5P = \left[ \frac{1}{4}, \frac{5}{8} \right], \quad 10P = \left[ \frac{161}{16}, \frac{2065}{64} \right]$$

There are no height conditions at $p \neq 37$ for the integral points and there is just one at $p = 2$ for the points $5P$ and $10P$.

Consider the divisor $\sum n_k(kP)$. Notice that $S^3 J(\mathbb{Q}) = \mathbb{Z}$ and the condition a) is $\sum n_k \cdot k^3 = 0$. The height condition at $p = 37$ gives $\sum n_k \cdot k = 0$ provided that the coordinates of $(kP)$ are prime to 37.

The divisor

$$P_k = (kP) - k(P) - \frac{k^3 - k}{6}((2P) - 2(P))$$

satisfies the conditions $\sum n_k \cdot k = \sum n_k \cdot k^3 = 0$. Also $P_{10} - 4 \cdot P_5$ satisfies the height condition at $p = 2$.

The computer calculation (using PARI) shows

$$\frac{8\pi \cdot L_2(q)(P_3)}{37 \cdot L(E, 2)} = -8.0000..., \quad \frac{8\pi \cdot L_2(q)(P_4)}{37 \cdot L(E, 2)} = -26.0000..., \quad \frac{8\pi \cdot L_2(q)(P_6)}{37 \cdot L(E, 2)} = -90.0000..., \quad \frac{8\pi \cdot L_2(q)(P_{10} - 4 \cdot P_5)}{37 \cdot L(E, 2)} = -248.0000...$$

4. The group $B_2(E)$ and a refined version of conditions a) - b). Let $E$ be an elliptic curve over an arbitrary field $k$ and $J := J(k)$ be the group of
$k$-points of the Jacobian of $E$. Let $\mathbb{Z}[X]$ be the free abelian group generated by a set $X$. We will define in s. 2.1 a group $B_2(E/k) = B_2(E)$ such that

a) one has an exact sequence

$$0 \rightarrow k^* \rightarrow B_2(E/k) \xrightarrow{p} S^2J(k) \rightarrow 0$$

(6)

b) one has a canonical (up to a choice of a sixth root of unity) surjective homomorphism

$$h : \mathbb{Z}[E(k)\setminus 0] \rightarrow B_2(E/k)$$

(7)

whose projection to $S^2J(k)$ is given by the formula $\{a\} \mapsto a \cdot a$.

c) if $K$ is a local field there is a canonical homomorphism

$$H : B_2(E/K) \rightarrow \mathbb{R}$$

whose restriction to the subgroup $K^* \subset B_2(E/K))$ is given by $x \mapsto \log |x|$, (see s. 2.3). Moreover the canonical local height $h_K$ is given by the composition

$$\mathbb{Z}[E(K)\setminus 0] \xrightarrow{h} B_2(E/K) \xrightarrow{H} \mathbb{R}$$

The group $B_2(E)$ appears naturally as a version of the theory of biextensions. It is a “motivic” version of theta functions. Set $\{a\}_2 := h(\{a\}) \in B_2(E)$.

The conditions a)-b) on a divisor $\sum n_j(P_j)$ are equivalent to the following single one:

$$\sum n_j\{P_j\}_2 \otimes P_j = 0 \quad \text{in} \quad B_2(E(\overline{Q})) \otimes J(\overline{Q})$$

(8)

More precisely,

**Lemma 1.4** Let $K$ be a number field and $P_j \in E(K)$. Then

$$\sum n_j\{P_j\}_2 \otimes P_j = 0 \quad \text{in} \quad B_2(E(K)) \otimes J(K) \otimes \mathbb{Q}$$

if and only if the following two conditions hold:

$$\sum n_jP_j \otimes P_j \otimes P_j = 0 \quad \text{in} \quad S^3J(K) \otimes \mathbb{Q}$$

and for any valuation $v$ of the field $K$

$$\sum n_j h_v(P_j) \cdot P_j = 0 \quad \text{in} \quad J(K) \otimes \mathbb{Q} \mathbb{R}$$

**Proof.** Multiplying the exact sequence (3) by $J(K) \otimes \mathbb{Q}$ we get

$$0 \rightarrow K^* \otimes J(K) \otimes \mathbb{Q} \rightarrow B_2(E(K)) \otimes J(K) \otimes \mathbb{Q} \xrightarrow{p \otimes id} S^2J(K) \otimes J(K) \otimes \mathbb{Q} \rightarrow 0$$

and use the fact that the local norms $|\cdot|_v$ separate all the elements in $K^* \otimes \mathbb{Q}$.

5. The elliptic motivic complex. Let us suppose first that $k$ is an algebraically closed field. In chapter 3 we define a subgroup $R_3(E) \subset \mathbb{Z}[E(k)]$. 

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When \( k = \mathbb{C} \) it is a subgroup of all functional equations for the elliptic dilogarithm. In particular the homomorphism

\[
\mathcal{L}_{2,q} : \mathbb{Z}[E(\mathbb{C})] \longrightarrow \mathbb{R}, \quad \{a\} \mapsto \mathcal{L}_{2,q}(a)
\]

annihilates the subgroup \( R_3(E/\mathbb{C}) \). Consider the homomorphism \((J := J(k))\)

\[
\delta_3 : \mathbb{Z}[E(k)] \longrightarrow B_2(E) \otimes J, \quad \{a\} \mapsto -\frac{1}{2}\{a\}_2 \otimes a
\]

An important result ( theorem (3.3)) is that \( \delta_3(R_3(E)) = 0 \). Setting

\[
B_3(E) := \frac{\mathbb{Z}[E(k)]}{R_3(E)}
\]

we get a homomorphism \( \delta_3 : B_3(E) \longrightarrow B_2(E) \otimes J \). Let us consider the following complex

\[
B(E; 3) : \quad B_3(E) \xrightarrow{\delta_3} B_2(E) \otimes J \longrightarrow J \otimes \Lambda^2 J \longrightarrow \Lambda^3 J \tag{9}
\]

Here the middle arrow is \( \{a\}_2 \otimes b \mapsto a \otimes a \wedge b \) and the last one is the canonical projection. The complex is placed in degrees \([1, 4]\). It is acyclic in the last two terms. This is our elliptic motivic complex.

Let \( I_E \) be the augmentation ideal of the group algebra \( \mathbb{Z}[E] \), and \( I_E^4 \) its fourth power. Let \( B_3^*(E) \) be the quotient of \( I_E^4 \) by the subgroup generated by the elements \((f) \ast (1 - f)^-\), where \( \ast \) is the convolution in the group algebra \( \mathbb{Z}[E] \), \( f \in k(E)^* \), and \( g \sim (t) := g(-t) \). Then there is a homomorphism

\[
\delta_3 : B_3^*(E) \longrightarrow k^* \otimes J \tag{10}
\]

which fits the following commutative diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
B_3^*(E) & \longrightarrow & k^* \otimes J \\
\downarrow & \downarrow & \downarrow \\
B_3(E) & \xrightarrow{\delta_3} & B_2(E) \otimes J \\
\downarrow & \downarrow & \downarrow \\
S^3J & \longrightarrow & S^2J \otimes J \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

where the vertical sequences are exact, and the bottom one is the Koszul complex, and thus also exact. Let us denote by \( B^*(E; 3) \) the complex \((10)\). It is canonically quasiisomorphic to the complex \( B(E; 3) \).

The complex \( B^*(E; 3) \) looks simpler than \( B(E; 3) \). However a definition of the differential in \( B^*(E; 3) \) which does not use the embedding to \( B(E; 3) \) is rather awkward, see s. 4.6.
If \( k \) is not algebraically closed we postulate the Galois descent property:

\[
B(E/k; 3) := B(E/\bar{k}; 3)^{\text{Gal}(\bar{k}/k)}; \quad B^*(E/k, 3) := B^*(E/\bar{k}; 3)^{\text{Gal}(\bar{k}/k)}
\]

### 6. Relation with algebraic \( K \)-theory

Let \( k \) be an arbitrary field. Let \( K_2 \) be the sheaf of \( K_2 \) groups in the Zariski topology on \( E \). One has canonical inclusion \( K_2(k) \hookrightarrow H^0(E, K_2) \) and surjective projection \( H^1(E, K_2) \to k^* \).

**Theorem 1.5** Let \( k = \bar{k} \). Then there is a sequence

\[
\begin{align*}
\text{Tor}(k^*, J) & \to H^0(E, K_2) \\
& \to B^*(E) \to k^* \otimes J \to \text{Ker}(H^1(E, K_2) \to k^*) \to 0
\end{align*}
\]

It is exact in the term \( k^* \otimes J \) and exact modulo 2-torsion in the other terms.

For an abelian group \( A(E) \) depending functorially on \( E \) let \( A(E)^- \) be the subgroup of skewinvariants under the involution \( x \mapsto -x \) of \( E \). Recall that one has the \( \gamma \)-filtration on the Quillen \( K \)-groups. One can show that modulo 2-torsion

\[
\frac{H^0(E, K_2)}{K_2(k)} = g^2 \, K_2(E)^-; \quad \text{Ker}(H^1(E, K_2) \to k^*) = g^2 \, K_1(E)^-.
\]

Recall that the Bloch-Suslin complex for an arbitrary field \( k \) is defined as follows:

\[
B_2(k) \xrightarrow{\delta} \Lambda^2 k^*; \quad B_2(k) := \frac{\mathbb{Z}[k^*]}{R_2(F)}; \quad \delta : \{x\} \mapsto (1 - x) \wedge x
\]

Here \( R_2(k) \) is the subgroup generated by the elements \( \sum_i (-1)^i \{r(x_1, \ldots, \hat{x}_i, \ldots, x_5)\} \), where \( x_i \) runs through all 5-tuples of distinct points over \( k \) on the projective line and \( r \) is the cross ratio.

One should compare theorem [1.3] with the following exact sequence provided by Suslin’s theorem on \( K^2_{\text{ind}}(k) \) ([S]) and Matsumoto’s theorem on \( K_2(k) \) ([M]) (see also a closely related results by Dupont and Sah [DS]):

\[
0 \to \text{Tor}(k^*, k^*) \to K^2_{\text{ind}}(k) \to B_2(k) \xrightarrow{\delta} \Lambda^2 k^* \to K_2(k) \to 0
\]

Here \( \text{Tor}(k^*, k^*) \) is a nontrivial extension of \( \text{Tor}(k^*, k^*) \) by \( \mathbb{Z}/2\mathbb{Z} \).

**Remark.** To guess an elliptic analog of the Steinberg relation \((1 - x) \otimes x \in \mathbb{C}^* \otimes \mathbb{C}^*\) we might argue as follows. \( E(\mathbb{C}) = \mathbb{C}^*/q^\mathbb{Z} \), so let us try to make sense out of \( \sum_{n \in \mathbb{Z}} (1 - q^n x) \otimes q^n x \). Let \( p : \mathbb{C}^* \to E(\mathbb{C}) \). Projecting \( \mathbb{C}^* \otimes \mathbb{C}^* \) to \( \mathbb{C}^* \otimes J(\mathbb{C}) \) we get \( \prod_{n \in \mathbb{Z}} (1 - q^n x) \otimes x, x \in E(\mathbb{C}) \). Regularizing the infinite product we obtain \( \theta(x) \otimes x \) where

\[
\theta(x) := q^{1/12} z^{-1/2} \prod_{j \geq 0} (1 - q^j z) \prod_{j > 0} (1 - q^j z^{-1})
\]
Unfortunately this seems to make no sense: $\theta(x)$ is not a function on $E(\mathbb{C})$ with values in $\mathbb{C}^*$, but a section of a line bundle. Only introducing the group $B_2(E/\mathbb{C})$ and realizing that $\theta(x)$ is a function on $E(\mathbb{C})$, but with values in the group $B_2(E/\mathbb{C})$, we find the elliptic analog of the Steinberg relation: $\theta(x) \otimes x \in B_2(E/\mathbb{C}) \otimes J(\mathbb{C})$ (compare with s. 4.1, 4.2, 4.6). For an arbitrary base field $\{x\}_2 \in B_2(E)$ replaces $\theta(x)$.

In [W] J. Wildeshaus, assuming standard conjectures about mixed motives, gave a conjectural inductive definition of groups similar to $B_n(E)$ and discuss an elliptic analog of weak version of Zagier’s conjecture.

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2 The group $B_2(E)$

1. A construction of the group $B_2(X)$. Let $k$ be an arbitrary field.

For any two degree zero line bundles $L_1$ and $L_2$ on a regular curve $X$ over $k$ let us define, following Deligne ([De]), a $k^*$-torsor $[L_1, L_2]$.

Motivation. Let $s_i$ be a section of a line bundle $L_i$. If $div(s_2) = \sum m_iP_i$ then $< s_1, s_2 >$ is $s_1(div(s_2)) \in \otimes L_1^{m_i}|_{P_i} = [L_1, L_2]$. Such a tensor product turns out to be symmetric and does not depend on the choice of $s_2$. If $L_1 = \mathcal{O}$ then $[L_1, L_2] = k^*$.

For $f \in k(X)^*$ and a closed point $x \in X_1$ set $\tilde{f}(x) := Nm_{k(x)/k}f(x)$. We extend $\tilde{f}$ by linearity to the group of $k$-rational divisors on $E$.

Definition. The elements of $[L_1, L_2]$ are pairs $< s_1, s_2 >$, where $s_i$ is a section of the line bundle $L_i$ and the divisors $div(s_1)$ and $div(s_2)$ are disjoint. For a rational function $f, g$ such that $div(f)$ is disjoint from $div(s_2)$ and $div(g)$ is disjoint from $div(s_1)$, one has $<(s) := div(s)>$:

$$< f \cdot s_1, s_2 >= \tilde{f}((s_2)) < s_1, s_2 >, \quad < s_1, g \cdot s_2 >= \tilde{g}((s_1)) < s_1, s_2 >$$

There are two a priori different expansions

$$< f \cdot s_1, g \cdot s_2 > = \tilde{f}((g))\tilde{f}(s_2) < s_1, g \cdot s_2 > = \tilde{f}((g))\tilde{f}(s_2)\tilde{g}((s_1)) < s_1, s_2 >$$

(11)

and

$$< f \cdot s_1, g \cdot s_2 > = \tilde{g}((f))\tilde{g}((s_1)) < f \cdot s_1, s_2 > = \tilde{g}((f))\tilde{g}((s_1))\tilde{f}(s_2) < s_1, s_2 >$$

(12)
The right hand sides coincide thanks to the Weil reciprocity \( \tilde{f}(\langle g \rangle) = \tilde{g}(\langle f \rangle) \).
So the \( k^* \)-torsor \([L_1, L_2] \) is well defined.

There is a canonical isomorphism of \( k^* \)-torsors
\[
[L_1 \otimes L_2, M] \rightarrow [L_1, M] \otimes_{k^*} [L_2, M]
\]
and a similar additivity isomorphism for the second divisor. Further, the two possible natural isomorphisms
\[
[L_1 \otimes L_2, M_1 \otimes M_2] \rightarrow \otimes_{1 \leq i, j \leq 2} [L_i, M_j]
\]
coincide.

Therefore for any element \( s \in S^2 J_X \) one gets a \( k^* \)-torsor \([s] \) defined up to an isomorphism. Moreover \([s_1 + s_2] = [s_1] \otimes_{k^*} [s_2] \).

The above facts just mean that the collection of \( k^* \)-torsors \([s] \) defines an extension of type \([6]\). This is the definition of the group \( B_2(X) \).

To construct the homomorphism \([6]\) we want to make sense of elements \(< s_1, s_2 > \) where the divisors of the sections \( s_i \) may not be disjoint. The definition is suggested by the motivation given above. Namely
\[
< s_1, s_2 > \in [L_1, L_2] \otimes_{k^*} V(\text{div}(s_1), \text{div}(s_2)) \quad (13)
\]
where \( V(L_1, L_2) \) is a \( k^* \)-torsor defined as follows. Recall that if \( A \rightarrow B \) is a homomorphism of abelian groups and \( T \) is an \( A \)-torsor, we can define a \( B \)-torsor \( f_* T := T \otimes_A B \) where \( A \) acts on \( B \) via the homomorphism \( f \). Thus for a \( k(x)^* \)-torsor \( T^*_x X \) we define a \( k^* \)-torsor \( N(T^*_x X) \) using the norm homomorphism \( Nm : k(x)^* \rightarrow k^* \).

For two arbitrary divisors \( l_1 \) and \( l_2 \) on \( X \) consider the following \( k^* \)-torsor
\[
V(l_1, l_2) := \otimes_{x \in X} N(T^*_x X)^{\otimes \text{ord}_x l_1 \cdot \text{ord}_x l_2}
\]
Here \( \text{ord}_x l \) is the multiplicity of the divisor \( l \) at the point \( x \). One has a canonical isomorphism \( V(l_1 + l_2, m) \rightarrow V(l_1, m) \otimes V(l_2, m) \). If the divisors \( l_1 \) and \( l_2 \) are disjoint then \( V(l_1, l_2) = k^* \).

Any rational function \( f \) provides a canonical isomorphism
\[
\varphi_f : V(l_1, l_2) \rightarrow V(\langle f \rangle + l_1, l_2) \quad v \mapsto \tilde{f}(l_2) \cdot v
\]
In this formula \( \tilde{f}(x) \in N(T^*_x X)^{\otimes \text{ord}_x(f)} \) is the “leading term” of the function \( f \) at \( x \). It is defined as follows. Choose a local parameter \( t \) at the point \( x \). If \( f(t) = at^{k+} \) higher order terms then \( \tilde{f}(x) = a(dt)^k \otimes 1 \in N(T^*_x X)^k = (T^*_x X)^{k} \otimes k^* \).

So \( \tilde{f}(x)^{\text{ord}_x l_2} \in N(T^*_x X)^{\otimes \text{ord}_x(f) \cdot \text{ord}_x l_2} \) and the formula above makes sense.

Let us recall the full version of the Weil reciprocity law. Let
\[
\partial_x : \{f, g\} \mapsto (-1)^{\text{ord}_x(f) \cdot \text{ord}_x(g)} \frac{\tilde{f}(x)^{\text{ord}_x(g)}}{\tilde{g}(x)^{\text{ord}_x(f)}} \quad (14)
\]
be the same symbol. Then for any two rational functions \( f, g \) on a curve over an algebraically closed field \( k \) one has \( \prod_{x \in X} \partial_x(f, g) = 1. \)

There exists a canonical isomorphism of \( k^* \)-torsors

\[
S : [L_1, L_2] \otimes V((s_1), (s_2)) \rightarrow [L_2, L_1] \otimes V((s_2), (s_1))
\]
given on generators by the formula

\[
S : < s_1, s_2 > \rightarrow (-1)^{\deg L_1 \cdot \deg L_2 + \sum_{x \in X} \ord_x s_1 \cdot \ord_x s_2} < s_2, s_1 >
\]
The defining properties of the torsor \([L_1, L_2]\) look as follows:

\[
< f \cdot s_1, s_2 > = \bar{f}(s_2) < s_1, s_2 >, \quad < s_1, g \cdot s_2 >= S(< s_1, (g) >) < s_1, s_2 >
\]
The formula

\[
S < (f), (g) >= < (f), (g) >
\]
(15)
is equivalent to the Weil reciprocity. Similar to \([11], [12]\) and using the formula (15) we see that \( S \) is well defined.

2. The homomorphism \([\theta]\). From now on \( X = E \) is an elliptic curve over \( k \), so there is canonical isomorphism \( T^*_x E = T_0^* E \). For \( c \in k^* \) one obviously has \( < c \cdot s_1, s_2 >= < s_1, s_2 > \). Therefore we may consider \( < s_1, s_2 > \) when \( s_1, s_2 \) are divisors on \( E \). Consider a map

\[
\{a\} \rightarrow < (a) - (0), (a) - (0) > \in (T_0^* E)^{\otimes 2} \otimes k^* [a \cdot a]
\]

There is an almost canonical (up to a sixth root of unity) choice of an element in \((T_0^* E)^{\otimes 2}\). Namely, the quotient of \( E \) by the involution \( x \rightarrow -x \) is isomorphic to \( P^1 \). The image of 0 on \( E \) is the point \( \infty \) on \( P^1 \). The images of the three nonzero 2-torsion points on \( E \) gives 3 distinguished points on \( P^1 \). Their ordering = choice of a level 2 structure on \( E \). A choice of ordering gives a canonical coordinate \( t \) on \( A^1 := P^1 \setminus \infty \) for which \( t = 0 \) is the first point and \( t = 1 \) is the second. This coordinate provides a vector in \( T^*_\infty P^1 \) and so a vector in \( (T_0^* E)^2 \). Therefore we have six different trivializations of \((T_0^* E)^2\) and thus a canonical one of \((T_0^* E)^{12}\): their product. The sixth root of \((16 \times \text{this trivialization}) \) is the (almost) canonical element in \((T_0^* E)^2\) we need.

Using this element in \((T_0^* E)^2\) we get a map \([\theta]\). The composition \( \mathbb{Z}[E(k) \setminus 0] \rightarrow B_2(E) \rightarrow S^3 J \) is obviously given by \( \{a\} \rightarrow a \cdot a \).

If \( E \) is written in the Weierstrass form \( y^2 = (x - e_1)(x - e_2)(x - e_3) \) then \( e_1, e_2, e_3 \) are the coordinates of the distinguished points and \( \frac{e_j - e_k}{y} \) is the canonical coordinate corresponding to the ordering \( e_i, e_j, e_k \). Let \( \Delta \) be the discriminant of \( E \). Then \( \Delta = 16 \prod_{i < j} (e_i - e_j)^2 \). The canonical trivialization is \( \Delta^{1/6}(dx/2y)^2 \).

3. The canonical height. In this section we recall the construction of the canonical local heights via the biextension (compare with \([Za]\) and \([Bl2]\)). The canonical local height gives a homomorphism \( B_2(K) \rightarrow \mathbb{Q} \) (resp. to \( \mathbb{R} \)) when \( K \) is a nonarchimedean (resp. archimedean) local field.

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The construction of the group $B_2(E)$ provides us with a collection of $k^*$-torsors $T_{(x,y)}$ where $(x, y)$ is a point of $J \times J$. The torsors $T_{(0,y)}$ and $T_{(x,0)}$ are trivialized: we have a distinguished element $<\emptyset, (y) - (0)> \in T_{(0,y)}$ and a similar one in $T_{(x,0)}$.

The collection of torsors $T_{(x,y)}$ glue to a $k^*$-bundle on $J \times J$. It is isomorphic to the (rigidified) Poincaré line bundle minus zero section.

Let $L$ be a degree zero line bundle on an elliptic curve over $k$. Let us denote by $L^*$ the complement of the zero section in $L$. It is a principal homogeneous space over a certain commutative algebraic group $A(L)$ over $k$ which is an extension

$$0 \rightarrow \mathbb{G}_m \rightarrow A(L) \rightarrow J \rightarrow 0$$

The group $A(L)$ is described as follows. For any $a \in E$ let $t_a : x \mapsto a + x$ be the shift by $a$. Then $t_a^* L$ is isomorphic to $L$ (because $L$ is of degree zero). The set of isomorphisms from $L$ to $t_a^* L$ form a $k^*$-torsor. These torsors glue together to a $k^*$-torsor over $E$ which is isomorphic to $L^*$. On the other hand the collection of isomorphisms $L \rightarrow t_a^* L$ form a group. This is the group $A(L)$. It is commutative: the commutator provides a morphism from $E \times E$ to $\mathbb{G}_m$, which has to be a constant map.

Now let $K$ be a local field and $E$ be an elliptic curve over $K$. We get a group extension

$$0 \rightarrow K^* \rightarrow A(L)(K) \rightarrow J(K) \rightarrow 0$$

Let $U(G)$ be the maximal compact subgroup of a locally compact commutative group $G$. One can show (use lemma 6.1, ch. 11 in [La]) that $U\big(A(L)(K)\big)$ projects surjectively onto $J(K)$ if $K$ is archimedean or $E$ has a good reduction over $K$. In the case of bad reduction the image is a subgroup of finite index.

There is canonical homomorphism

$$A(L)(K) \rightarrow A(L)(K)/U\big(A(L)(K)\big) =: H$$

The quotient $H$ is isomorphic to $\mathbb{Z}$ (resp. to a subgroup in $\mathbb{Q}$ which is an extension of $\mathbb{Z}$ by a finite group ) when $K$ is nonarchimedean and $E$ has good reduction (resp. bad reduction), and to $\mathbb{R}$ if $K = \mathbb{C}, \mathbb{R}$.

Therefore we get a homomorphism $A(L)(K) \rightarrow \mathbb{Z}$ (resp $\mathbb{Q}$) for the nonarchimedean case and $A(L)(K) \rightarrow \mathbb{R}$ for the archimedean one.

For a given $x$ the torsors $T_{(x,y)}$ form a group $T_{(x,)}$ which is isomorphic to the group $A(L_{|x \times J})$, and there is a similar statement for the torsors $T_{(x,y)}$ for a given $y$. Applying the homomorphism $A(L_{|x \times J}) \rightarrow H$ we will get a homomorphism of the group $T_{(x,)}$ to $H$. Similarly we have a homomorphism of the group $T_{(,y)}$ to $H$.

Consider the map

$$U(T_{(x_1,)}) \times U(T_{(x_2,)}) \rightarrow T_{(x_1 + x_2,)}$$
induced by the multiplication \( T(x_1,y_1) \times T(x_2,y_2) \to T(x_1+x_2,y_1) \). Its image is a subgroup. This follows from the commutativity of the diagram
\[
\begin{array}{c}
T(x_1,y_1) \times T(x_2,y_1) \times T(x_1,y_2) \times T(x_2,y_2) \\
\downarrow \downarrow \\
T(x_1+x_2,y_1) \times T(x_1+x_2,y_2) \to T(x_1+y_1+y_2)
\end{array}
\]
It is therefore a compact subgroup, and so a maximal compact subgroup in \( T(x_1+x_2) \). In particular the restrictions of the homomorphisms \( T(x,\cdot) \to H \) and \( T(\cdot,y) \to H \) to \( T(x,y) \) coincide.

So we get a well defined homomorphism \( B_2(E(K)) \to H \). Now the composition
\[
\mathbb{Z}[E(K)\backslash 0] \xrightarrow{h} B_2(E(K)) \to H
\]
is the canonical Néron height on \( E(K) \). The restriction of the homomorphism (16) to the subgroup \( K^* \subset B_2(E(K)) \) coincides with the logarithm of the norm homomorphism. In the nonarchimedean case \( H \) is a subgroup of \( \log p \cdot \mathbb{Q} \subset \mathbb{R} \).

**Remark.** The homomorphism \( h : \mathbb{Z}[E(K)\backslash 0] \to B_2(E(K)) \) was defined up to a sixth root of unity. This does not affect the definition of the height because the norm vanishes on roots of unity.

### 3 An elliptic analog of the Bloch-Suslin complex

1. **The group \( B_3(E) \) and complex \( B(E;3) \).** We will assume in s.3.1 - 3.3 that \( k = \bar{k} \). We will always use notation \( J := J(k) \). Set \( g^- (t) := g(-t) \). Denote by * the convolution in the group algebra \( \mathbb{Z}[E(k)] \).

**Definition 3.1** \( R_3(E) \) is the subgroup of \( \mathbb{Z}[E(k)] \) generated by the elements \( (f) \ast (1-f)^{-} \), \( f \in k(E)^* \), \( \{0\} \), and the "distribution relations"
\[
m \cdot \{a\} - m \cdot \sum_{mb=a} \{b\}, \quad a \in E(k), \quad m = -1, 2
\]

**Remarks.** a) For \( m = -1 \) we get the elements \( \{a\} + \{-a\} \in R_3(E) \). If we remove them from the definition of \( R_3(E) \), we get the same group.

   b) It would be more natural to add to the subgroup \( R_3(E) \) the distribution relations for all \( m \in \mathbb{Z}\backslash 0 \): we should get the same group (compare with lemma (3.2)). But we will not need this.

   Consider the homomorphism
   \[
   \beta : \otimes^2 k(E)^* \to \mathbb{Z}[E(k)] \quad \beta : f \otimes g \mapsto f \ast g^- := \sum n_im_j\{a_i - b_j\}
   \]
   (the Bloch map), where \( (f) = \sum n_i(a_i) \) and \( (g) = \sum m_j(b_j) \).
Recall that
\[ \delta_3 : \mathbb{Z}[E(k)] \rightarrow B_2(E) \otimes J, \quad \{a\} \mapsto -\frac{1}{2}\{a\}_2 \otimes a \]
and \( i : k^* \hookrightarrow B_2(E) \) is the canonical embedding (see (6)). Let \( I_E \) be the augmentation ideal of the group algebra \( \mathbb{Z}[E(k)] \) and \( p : I_E \rightarrow J \) the canonical projection.

Recall that if \( k = \bar{k} \) the tame symbol provides a homomorphism
\[ \otimes^2 k(E)^* \xrightarrow{\partial} k^* \otimes \mathbb{Z}[E] \]
The Weil reciprocity law shows that its image belong to \( k^* \otimes I_E \).

**Theorem 3.2** The following diagram is commutative
\[
\begin{array}{ccc}
\otimes^2 k(E)^* & \xrightarrow{\partial} & k^* \otimes I_E \\
\downarrow \beta & & \downarrow i \otimes p \\
\mathbb{Z}[E(k)] & \xrightarrow{\delta_3} & B_2(E) \otimes J
\end{array}
\]

**Proof.** Let \( (f) = \sum n_i(a_i), \quad (g) = \sum m_j(b_j) \). Then
\[ \delta_3 \circ \beta(f \wedge g) = -\frac{1}{2} \sum_{i,j} m_i n_j < a_i - b_j, a_i - b_j > \otimes (a_i - b_j) \]

The term \( \sum_{i,j} m_i n_j < (a_i - (0), b_j) > \otimes a_i \) equals to
\[ \sum_{i,j} m_i n_j < (a_i) - (0), (a_i - (0)) > \otimes a_i \]

The first term here is zero because \( \sum_{j} m_j = 0 \). The second is zero because \( \sum_i n_i a_i = 0 \) in \( J \). The last one can be written as
\[ -2 \cdot \sum_i m_i < (a_i) - (0), (g) > \otimes a_i \]

So the theorem follows from the definition of the tame symbol.

**Theorem 3.3** \( \delta_3(R_3(E)) = 0 \).
Proof. We will denote by \( \{a\}_3 \) projection of the generator \( \{a\} \) onto the quotient \( B_3(E) \). The map \( \delta_3 \) kills the distribution relations:

\[
\delta_3 \left( m(\{a\}_3 - m \sum \{b\}_3) \right) = m(\{a\}_2 \otimes a - m \sum \{b\}_2 \otimes b) = m(\{a\}_2 - \sum \{b\}_2) \otimes a = 0
\]

The last equality is provided by corollary (4.3), which will be proved later, in s. 4.4. (The proof does not depend on any results or constructions in chapter 3).

The fact that \( \delta_3(f \ast (1 - f)) = 0 \) lies deeper and follows from the theorem (3.2). Theorem (3.3) is proved.

Let \( I_k^* \) be \( k \)-th power of the augmentation ideal.

**Lemma 3.4** \( \beta(f \otimes g) \in I_k^* \). Moreover, \( \beta \) is surjective onto \( I_k^* \).

**Proof.** A divisor \( \sum n_i(a_i) \) is principal if and only if \( \sum n_i = 0 \) and \( \sum n_i a_i = 0 \) in \( J(E) \). So \( I_k^* \) coincides with the subgroup of \( \mathbb{Z}[E] \) given by the divisors of functions. So the convolution of two principal divisors belongs to \( I_k^* \) and, moreover, generate it.

Let \( B_3^*(E) \) be the quotient of \( I_k^* \) by the subgroup generated by the elements \( (f) \ast (1 - f)^- \).

**Lemma 3.5** \( \delta_3(I_k^*) \in k^* \otimes J \subset B_2(E) \otimes J \)

**Proof.** It is easy to see that \( \delta_3(I_k^*) \subset h(I_k^*) \otimes J \). Further, \( h(I_k^*) \subset k^* \) because \( p \circ h(I_k^*) = 0 \) (see the properties of the group \( B_2(E) \) listed in s. 1.4). The lemma follows.

So we get a complex

\[
B^*(E; 3) :\quad B_3^*(E) \longrightarrow B_2(E) \otimes J \longrightarrow J \otimes \Lambda^2 J \longrightarrow \Lambda^3 J
\]

2. Relation with algebraic \( K \)-theory. Let us remind the long exact sequence of localization

\[
K_3(k(E)) \longrightarrow \oplus_{x \in E} K_2(k(x)) \longrightarrow K_2(E) \longrightarrow K_2(k(E)) \longrightarrow \oplus_{x \in E} k(x)^* \longrightarrow K_1(E) \longrightarrow k(E)^* \oplus_{x \in E} \mathbb{Z}
\]

The group \( K_1(E) \) has a subgroup \( k^* \) which comes from the base. One can show that

\[
H^0(E, K_2) = gr_2 K_2(E), \quad H^1(E, K_2) = gr_2 K_1(E) = K_1(E)/k^*
\]
Lemma 3.6 Modulo 2-torsion one has

\[ H^0(E, K_2) = K_2(k) \oplus H^0(E, K_2)^- \]

\[ Ker \left( H^1(E, K_2) \rightarrow k^* \right) = H^1(E, K_2)^- = K_1(E)^- \]

Proof. It follows easily using the transfer related to the projection \( E \rightarrow P^1 \) given by factorization along the involution \( x \rightarrow -x \).

Recall that \( H^3(B(E, 3)) = H^4(B(E, 3)) = 0 \).

Theorem 3.7 Let \( k = \overline{k} \). Then the commutative diagram from theorem (3.2) provides a morphism of complexes

\[
\begin{array}{ccc}
K_2(k(E)) & \xrightarrow{\partial} & k^* \otimes J \\
\downarrow \tilde{\beta} & & \downarrow id \\
B^3_3(E) & \xrightarrow{\delta} & k^* \otimes J
\end{array}
\]

where \( \tilde{\beta} \) is surjective and \( Ker \tilde{\beta} = Tor(k^*, J) \) modulo 2-torsion.

This theorem and lemma (3.6) implies immediately

Theorem 3.8 Let \( k = \overline{k} \). Then there are an embedding

\[ i : Tor(k^*, J(k)) \hookrightarrow \frac{H^0(E, K_2)}{K_2(k)} \]

and canonical isomorphisms

\[ \frac{H^0(E, K_2)}{Tor(k^*, J(k)) + K_2(k)} = H^1B^*(E; 3) \quad \text{modulo 2-torsion} \quad (18) \]

\[ Ker \left( H^1(E, K_2) \rightarrow k^* \right) = H^2B^*(E; 3) \quad (19) \]

For an arbitrary field \( k \) \( (18) \) and \( (19) \) are isomorphisms modulo torsion.

Proof of theorem (3.7). We will first prove that we have a morphism of complexes and \( Tor(k^*, J) \subset Ker\tilde{\beta} \), and then that \( Tor(k^*, J) = Ker\tilde{\beta} \) modulo 2-torsion.

Recall that the tame symbol homomorphism \( \partial \) maps \( K_2(k(E)) \) to \( k^* \otimes I_E \).

Therefore the complex computing the groups \( H^0(E, K_2) \) and \( Ker \left( H^1(E, K_2) \rightarrow k^* \right) \) looks as follows

\[ K_2(k(E)) \xrightarrow{\partial} k^* \otimes I_E \]
Notice that $\partial(\{c, f\}) = c \otimes (f)$, so it has a subcomplex

$$k^* \cdot k(E)^* \xrightarrow{\partial} k^* \otimes I_E^2$$

where $\cdot$ is the product in $K$-theory. One obviously has $\text{Ker} \partial = K_2(k)$, and factorising by $K_3(k)$ we get the identity map $k^* \otimes I_E^2 \to k^* \otimes I_E^2$.

The map $\beta : \otimes^2 k(E)^* \to I_E^4$ followed by the natural projection $I_E^4 \to B_3^*(E)$ leads to surjective map

$$\beta : \otimes^2 k(E)^* \to B_3^*(E)$$

Notice that $\beta(k^* \otimes k(E)^*) = 0$ and $\beta(f \otimes (1 - f)) = 0$ by the definition of the group $B_3^*(E)$. So we get the desired morphism of complexes.

Tensoring the exact sequence $0 \to I_E^2 \to I_E \to J \to 0$ by $k^*$ and using the fact that $I_E$ is a free abelian group, we get an exact sequence

$$0 \to \text{Tor}(k^*, J) \to k^* \otimes I_E^2 \to k^* \otimes I_E \xrightarrow{id \otimes \iota} k^* \otimes J \to 0$$

So we get the following commutative diagram, where the vertical sequences are complexes, the complex on the right is the exact sequence $[21]$, and $\alpha$ is injective:

$$\begin{array}{c}
0 \\
\downarrow \\
\text{Tor}(k^*, J) \\
\downarrow \\
k^* \otimes I_E^2 \\
\xrightarrow{\alpha} \\
k^* \otimes I_E \\
\downarrow \\
\frac{K_2(k(E))}{K_2(k)} \\
\xrightarrow{\partial} \\
k^* \otimes I_E \\
\downarrow \\
\tilde{\beta} \\
\downarrow \\
B_3^*(E) \\
\xrightarrow{\delta_3} \\
k^* \otimes J \\
\downarrow \\
0
\end{array}$$

From this diagram we see that $\text{Tor}(k^*, J) \subset \text{Ker} \tilde{\beta}$. Notice that

$$\text{Coker} \left( k^* \otimes I_E^2 \xrightarrow{\alpha} \frac{K_2(k(E))}{K_2(k)} \right) = \frac{\otimes^2 I_E^2}{\{(1 - f) \ast (f)^-\}}$$

**Theorem 3.9** The map $\tilde{\beta} : \frac{\otimes^2 I_E^2}{\{(1 - f) \ast (f)^-\}} \to B_3^*(E)$ is an isomorphism modulo $2$-torsion.

**Proof of theorem (3.3).** It consists of three steps of quite different nature. 

**Step 1.** Set $t_a : x \to x + a$, $(t_a f)(x) := f(x + a)$.
**Proposition 3.10** Let $f$ and $g$ be rational functions on $E$. Then

$$\{f, g\} - \{taf, tag\} = 0 \quad \text{in} \quad \frac{K_2(k(E))}{k^* \cdot (k(E))^*}$$

Let $L/K$ be an extension of fields. Then one has a natural map $\tilde{p}^* : K_2(K) \to K_2(L)$ and the transfer map $p_* : K_2(L) \to K_2(K)$. We need the following result ([BT]).

**Lemma 3.11** Let $L/K$ be a degree 2 extension of fields. Then $K_2(L)$ is generated by symbols $\{k, l\}$ with $k \in K, l \in L$ and $p_*\{\{k, l\}\} = \{k, N_{L/K}l\}$.

In particular $p^*p_* = I_d + \sigma$ where $\sigma$ is a nontrivial element in $Gal(L/K)$, and thus modulo 2-torsion any Galois invariant element in $K_2(L)$ belongs to $p^*K_2(K)$.

**Lemma 3.12** Assume $k = \bar{k}$. Then any rational function $f$ on an elliptic curve $E$ over $k$ can be decomposed into a product of functions with divisors of the following kind: $(a) - (b) - (c) + (-a + b + c)$.

**Proof of the lemma 3.12.** Any function $f$ can be decomposed into a product of the functions with divisors of the following kind: $(a) - (b) - (c) + (-a + b + c)$.

Indeed, let $(f) = \sum n_a(a)$. We will use induction on $\sum |n_a|$. Since $\sum n_a = 0$, replacing if needed $f$ by $f^{-1}$ we can find points $a, b, c$ such that $n_a > 0, n_b < 0, n_c < 0$. Then $(f) - [(a) - (b) - (c) + (-a + b + c)]$ is a principal divisor with smaller $\sum |n_a|$.

Further $(a) - (b) - (c) + (-a + b + c) = [(a) - 2(d) + (-a + b + c)] - [(b) - 2(d) + (c)]$ for $2d = b + c$.

**Proof of the proposition 3.10.** According to the lemma we can assume that $(f) = (b) - 2(0) + (-b), (g) = (c) - 2(d) + (-c + 2d)$. Therefore $(taf) = (b + a) - 2(a) + (-b + a), (tag) = (c + a) - 2(d + a) + (-c + 2a + c)$.

The quotient of $E$ under the involution $\sigma_a : x \to a - x$ is isomorphic to $\mathbb{P}^1$. The symbol $\{taf, tag\} + \{\sigma_a taf, \sigma_a tag\}$ is $\sigma_a$-invariant, so it comes from $K_2(k(\mathbb{P}^1))$. It is known that $K_2(k(\mathbb{P}^1))$ is generated by $k(\mathbb{P}^1)^* \otimes k^*$. So

$$\{f, g\} - \{taf, tag\} \sim \{f, g\} + \{\sigma_a taf, \sigma_a tag\} = \{f, g \cdot \sigma_a g\} \sim 0$$

where $x \sim y$ means $x - y = 0$ in $\frac{K_2(k(E))}{k^* \cdot (k(E))^*}$. Indeed, $\sigma_a t_a = \sigma_0, \sigma_0 f = f$, and the symbol $\{f, g \cdot \sigma_0 g\}$ is $\sigma_0$-invariant. Therefore it is $\sim 0$ by lemma 3.11.

**Step 2.** Let $A$ be an abelian group. Let $S_k(A) \subset S^k I_A$ be the subgroup generated by the elements

$$(x_1 \circ X_1) \circ \cdots \circ (x_k \circ X_k) - X_1 \circ \cdots \circ X_k, \quad x_1 + \cdots + x_k = 0, \quad x_i \in A, \quad X_i \in I_A$$

This subgroup clearly belongs to the kernel of the convolution map

$$S^k I_A \to I_A^k, \quad X_1 \circ \cdots \circ X_k \longmapsto X_1 \ast \cdots \ast X_k$$

So we get a homomorphism $\alpha_k : S^k I_A / S_k(A) \to I_A^k$. 16
Proposition 3.13 For any abelian group $A$ the homomorphism $\alpha_k$ is injective.

Proof. We may assume that $A$ is finitely generated. Therefore the group ring $\mathbb{Z}[A]$ looks as follows:

$$\mathbb{Z}[A] = \mathbb{Z}[t_1, ..., t_\alpha, t_1^{-1}, ..., t_\alpha^{-1}] \times \prod_{a+1 \leq j \leq m} \frac{\mathbb{Z}[t_j]}{(t_j^{N_j} - 1)} \quad (22)$$

Under this isomorphism the augmentation ideal $I_A$ goes to the maximal ideal $(t_1 − 1, ..., t_m − 1)$. The subgroup $S_k(A)$ is generated by the elements

$$(t_{i_1} − 1)f_1 \circ ... \circ (t_{i_{k-1}} − 1)f_{k-1} \circ (t_{i_k} − 1)f_k − (t_{i_1} − 1) \circ ... \circ (t_{i_k} − 1)f_1 ... f_k \quad (23)$$

So any element of the quotient $S^kI_A/S_k(A)$ can be written as

$$(t_{i_1} − 1) \circ ... \circ (t_{i_{k-1}} − 1) \circ (t_{i_k} − 1)f \quad (24)$$

The homomorphism $\alpha_k$ sends it to $(t_{i_1} − 1)...(t_{i_{k-1}} − 1)(t_{i_k} − 1)f$.

Let us suppose first that $a > 0$, i.e. $A$ is an infinite group. We will use the induction on both $m$ and $k$. The case $m = 1$ is trivial: if $(t-1)^k f = 0$ then $(t-1) \circ ... \circ (t-1) f = 0$, even in the case $f(t) \in \mathbb{Z}[t_j]/(t_j^{N_j} - 1)$.

Consider an element

$$P = \sum_j (t_{i_1(j)} − 1) \circ ... \circ (t_{i_k(j)} − 1)f_j \in Ker\alpha_k \quad (25)$$

Any element $f$ of the right hand side of (22) can be written as $f'(t_2, ..., t_m) + (t_1 − 1)f''(t_1, ..., t_m)$. So writing $f_j = f'_j + (t_1 − 1)f''_j$ and setting $P' := \sum_j (t_{i_1(j)} − 1) \circ ... \circ (t_{i_k(j)} − 1)f'_j$ we get $P = P' + (t_1 − 1) \circ Q'$.

Further, let $P'_j$ be the sum of all the terms of $P'$ where non of the indices $i_l(j)$ equal to 1. Then $P' = P'_j + (t_1 − 1) \circ Q'_j$. The restriction of $P'_j$ to the divisor $t_1 = 1$ in $\text{Spec } \mathbb{Z}[A]$ coincides with the restriction of $P$. Therefore it belongs to $Ker\alpha_k$ and thus is zero by the induction assumption for $(k, m−1)$. Since by the definition $P'_j$ does not depend on $t_1$, this implies $P'_j = 0$.

Thus $P = (t_1−1) \circ Q$ for some $Q \in S^{k−1}I_A/S_{k−1}(A)$. Therefore $\alpha_{k−1}(Q) = 0$ since $t_1 − 1$ is not a divisor of zero (we have assumed $a > 0$). Thus $Q = 0$ by the induction assumption for $(k−1, m)$.

Now let $A$ be a finite group. Decomposing $f(t)$ in (24) into a sum of monomials $(t_1 − 1)^{a_1}...((t_\alpha − 1)^{a_\alpha}$ we can write an element (24) as a sum

$$P = \sum (t_1 − 1)^{b_1} \circ ... \circ (t_k − 1)^{b_k} (t_{k+1} − 1)^{b_{k+1}} ... (t_{k+l} − 1)^{b_{k+l}} \quad (26)$$

We will treat for a moment this sum as element of $S^k\mathbb{Z}[t_1, ..., t_m]$, not $S^kI_A/S_k(A)$. Let

$$\tilde{\alpha}_k(P) := \sum (t_1 − 1)^{b_1} ... (t_{k+l} − 1)^{b_{k+l}} \in \mathbb{Z}[t_1, ..., t_m]$$

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be its image in \( Z[t_1, \ldots, t_m] \) under the product map. If \( \tilde{\alpha}_k(P) = 0 \) then clearly \( P = 0 \) modulo the relations (23) with \( f_i \in Z[t_1, \ldots, t_m] \).

Since \( \alpha_k(P) = 0 \), \( \tilde{\alpha}_k(P) \) is a linear combination of monomials of type \((t_i^{N_i} - 1) \cdot Q(t)\). We are going to show that one can find another presentation \( P' \) for the element in \( S^k I_A/S_k(A) \) given by \( P \) such that \( \tilde{\alpha}_k(P') = \tilde{\alpha}_k(P) - (t_i^{N_i} - 1) \cdot Q(t) \).

Let us factorize \( t_i^{N_i} - 1 = (t_i - 1) \cdot p_i(t_i - 1) \), where \( p_i(u) = \sum c_j u^j \) is a polynomial in one variable. Notice that \( c_0 \neq 0 \). We must have a monomial in the sum \( P \) which goes under the map \( \tilde{\alpha}_k \) to \( c_0 \cdot (t_i - 1) \cdot Q(t) \). It can be written (modulo \( S_k(A) \)) as \( c_0 \cdot (t_i - 1) \). Therefore the terms in (23) which are mapped by \( \tilde{\alpha}_k \) to \( c_{j-1} (t_i - 1)^j \cdot Q(t) \), \( j > 1 \), can be written as \( c_{j-1} (t_i - 1)^j \). Therefore the sum of these terms in (23) can be written as \( (t_i^{N_i} - 1) \cdot Q(t) \) and thus represent a zero element in \( S^k I_A/S_k(A) \). The proposition is proved.

**Step 3.** Let us write \( \beta \) as a composition \( \beta = i \circ \alpha \):

\[
\otimes^2 I_E^2 \xrightarrow{i} \otimes^2 I_E^2 \xrightarrow{\alpha} I_E^4
\]

\( i : A \otimes B \longrightarrow A \otimes B \); \( \alpha : A \otimes B \longrightarrow A \ast B \)

Let \( S_E \subset \otimes^2 I_E^2 \) be the subgroup generated by the elements \((f) \otimes (1 - f)^-\), \( f \in k(E)^* \). One has

\[
\otimes^2 I_E^2 = k^* \otimes k(E)^* + k(E)^* \otimes k^*; \quad \frac{\otimes^2 I_E^2}{S_E} = \frac{K_2(k(E))}{k^* \cdot k(E)^*}
\]

Let \( A_+ \) (resp \( A_- \)) be the coinvariants of the involution \( x \to -x \) on \( E \) acting on a group \( A \) functorially depending on \( E \) (resp \( A \otimes \lambda \), where \( \lambda \) is the standard \( Z \)-line where the involution acts by inverting the sign).

**Lemma 3.14** \((Ker\alpha)_+ \subset S_E\).

**Proof.** Let \( x \in (Ker\alpha)_+ \). Then \( \partial(x) = 0 \), so \( x \) defines an element of \((H^0(E, K_2)/K_2(k))_+ \). But this group is zero modulo 2-torsion by lemma (3.6).

**Lemma 3.15** \( A^2 I_E^2 \subset S_E \).

**Proof.** \( A \otimes B + B \otimes A \) belongs to the subgroup generated by the Steinberg relations \((f) \otimes (1 - f)^-\). Thus \( A \otimes B^- + B \otimes A^- \in S_E \). The lemma above states that \( B \otimes A^- + B^- \otimes A \in S_E \). So \( A \otimes B^- - B^- \otimes A \in S_E \).

Let \( \alpha' : S^2 I_E^2 \to I_E^4 \), \( A \otimes B \mapsto A \ast B \). To prove the theorem we need to show that \((Ker\alpha')_- \subset S_E \).

Let \( A_i \in I_E \). The element

\[
<A_1, A_2, A_3, A_4> := (A_1 \ast A_2) \circ (A_3 \ast A_4) - (A_1 \ast A_3) \circ (A_2 \ast A_4)
\]

clearly belongs to \( Ker\alpha \), and thus to \( Ker\partial \).

**Proposition 3.16** \(< A_1, A_2, A_3, A_4 > \in S_E \).
In particular the complexes $B$.

Assume that Proposition 3.19

The proposition follows immediately from the proposition (3.13).

$1 < A$

Jacobian. Using (3.13).

On the other hand according to the lemma (3.6)

So $2 < A$

and $2 < A$

proved.

Lemma 3.17 If $A_i$ is a principal divisor for some $1 \leq i \leq 4$, then $< A_1, A_2, A_3, A_4 > \in S_E$. In particular we get a well defined homomorphism

$$< \cdot, \cdot, \cdot, \cdot : \otimes^4 J \rightarrow S^2 I_E^2 / S_E$$

(27)

Proof. Let us show that if $B_0, B_1 \in I_E$, then $< B_0 + B_1, A_2, A_3, A_4 > \in S_E$ (the other cases are similar). We will write $A \equiv B$ if $A - B \in S_E$.

It follows from the proposition (3.10) that for $A, B \in I_E^2$ and $X \in \mathbb{Z}[E(k)]$ one has $(X \cdot A) \circ B \equiv A \circ (X \cdot B)$.

So $2 < A$

and $2 < A$

follows from the proposition (3.10) that for $A, B \in I_E^2$ and $X \in \mathbb{Z}[E(k)]$ one has $(X \cdot A) \circ B \equiv A \circ (X \cdot B)$. So

$$(B_0 \cdot B_1 \cdot A_2) \circ (A_3 \cdot A_4) \equiv (B_0 \cdot B_1) \circ (A_2 \cdot A_3 \cdot A_4) \equiv (A_2 \cdot A_3) \circ (A_2 \cdot A_4)$$

Since $(a) - (0) + (b) - (0) - ((a + b) - (0)) \in I_E^2$, we get (27). The lemma is proved.

Proof of the proposition (3.16). Let $(A)$ be the image of $A \in I_E$ in the Jacobian. Using $(A^*) = -(A)$ and the previous lemma we get

$$< A_1, A_2, A_3, A_4 > - < A_1^0, A_2^0, A_3^0, A_4^0 > \in S_E$$

On the other hand according to the lemma (3.4)

$$< A_1, A_2, A_3, A_4 > + < A_1^0, A_2^0, A_3^0, A_4^0 > \in S_E$$

So $2 < A_1, A_2, A_3, A_4 > \in S_E$. Using lemma (3.17) and the divisibility (by 2) of $J(k)$ we conclude that $< A_1, A_2, A_3, A_4 > \in S_E$.

Proposition 3.18 Ker$(S^2 I_E^2 \rightarrow I_E^2)$ is generated by the elements

$$(x \cdot A) \circ ((-x) \cdot B) - A \circ B, \quad A, B \in I_E^2$$

and $< A_1, A_2, A_3, A_4 >, A_i \in I_E$.

Proof. Let $T \subset I_E^2$ be the subgroup generated by the elements $< A_1, A_2, A_3, A_4 >$. There is a surjective homomorphism

$$S^4 I_E \rightarrow S^2 (I_E^2) / T, \quad A_1 \otimes \ldots \otimes A_4 \mapsto (A_1 \cdot A_2) \circ (A_3 \cdot A_4)$$

The proposition follows immediately from the proposition (3.13).

Theorem (3.9) follows from proposition (3.18) (3.13) and (3.17).

Proposition 3.19 Assume that $k = \bar{k}$. Then

a) There is an injective homomorphism of complexes $B^*(E; 3) \rightarrow B(E; 3)$.

b) The quotient $B(E; 3)/B^*(E; 3)$ is isomorphic to the Koszul complex

$$S^3 J \rightarrow S^2 J \otimes J \rightarrow J \otimes \Lambda^2 J \rightarrow \Lambda^3 J$$

(28)

In particular the complexes $B(E; 3)$ and $B^*(E; 3)$ are quasiisomorphic.
We will need

**Lemma 3.20** Suppose that

\[ D := m \sum_i \{a_i\} - m \sum_{mb_i = a_i} \{b_i\} \in I_E^4 \]

Then \( D \) belongs to the subgroup generated by the elements \((f) \ast (1 - f)^{-}\).

Let \( [m] : E \to E \) be the isogeny of multiplication by \( m \).

**Lemma 3.21** Let \( f, g \in k(E)^* \), \( k = \overline{k} \). Then

\[ [m]^* \{f, g\} = m \{f, g\} \quad \text{in} \quad \frac{K_2(k(E))}{k^* \cdot k(E)} \]

**Proof of the lemma (3.21).** One has an exact sequence

\[ 0 \to H^0(E, K_2) \to K_2(k(E)) + \text{Tor}(k^*, J) \to K_2(k(E)) \to k^* \otimes J \]

The operator \([m]^*\) acts on the left (different from 0) and right groups by multiplication by \( m \) (see [BL], ch. 5.6 where this was proved rationally; that proof works integrally). Further, \([m]^*\) has no Jordan blocks since \([m]^* [m]^* = m^2\). The lemma follows.

**Proof of the lemma (3.20).** There is an isomorphism

\[ j : \mathbb{Z}[E(k)]/I_E^4 = \mathbb{Z} \oplus J \oplus S^2J \oplus S^3J, \quad \{a\} \mapsto (1, a, a \cdot a, a \cdot a \cdot a) \]

One has

\[ j(m(\{a\} - m \sum_{mb = a} \{b\})) = m((1 - m^3), (1 - m^2)a, (1 - m)a \cdot a, 0) \quad (29) \]

Using this we see that \( \sum_i \{a_i\} \in I_E^4 \). Thus \( \sum_i \{a_i\} = \sum (f_j) \ast (g_j)^{-} \). It is easy to see that \( \beta([m]^* \sum_i \{f_i, g_i\} - m \{f_i, g_i\}) = D \). The lemma is proved.

**Proof of the proposition (3.19).** a) We need only to show that \( B_3^*(E) \) injects to \( B_3(E) \). This boils down to the lemma (3.20) above, since in our definition of \( R_3(E) \) we used only the distribution relations for \( m = -1, 2 \).

b) By definition \( B_3(E)/B_3^*(E) \) is isomorphic to the quotient of \( \mathbb{Z}[E(k)]/I_E^4 \) modulo (the image of) the distribution relations.

Using the computation (29) and divisibility of \( J(k) \) we see that the map \( j \) maps the subgroup generated by the distribution relations for any given \( |m| > 1 \) surjectively onto \( 2\mathbb{Z} \oplus J \oplus J^2 \). Therefore \( B_3(E)/B_3^*(E) = S^3J \). Let us recall that \( B_2(E)/k^* = S^2J \). So the terms of the quotient \( B(E; 3)/B^*(E; 3) \) are the same as in (29). It is easy to see that the differentials coincide. The proposition is proved.
3. Zagier’s conjecture on $L(E, 2)$ for modular elliptic curves over $\mathbb{Q}$.

Let us recall that for a curve $X$ over $\mathbb{R}$ one has $H^2_D(X/\mathbb{R}, \mathbb{R}(2)) = H^1(X/\mathbb{R}, \mathbb{R}(1))$. Let $\tilde{X} := X \otimes \mathbb{C}$. The cup product with $\omega \in \Omega^1(\tilde{X})$ provides an isomorphism of vector spaces over $\mathbb{R}$:

$$H^1(X/\mathbb{R}, \mathbb{R}(1)) \rightarrow H^0(\tilde{X}, \Omega^1)^\vee$$

So we will present elements of $H^2_D(X/\mathbb{R}, \mathbb{R}(2))$ as functionals on $H^0(\tilde{X}, \Omega^1)$.

Bloch constructed the regulator map

$$r_D : K_2(E) \rightarrow H^2_D(E, \mathbb{R}(2))$$

If we represent an element of $K_2(E)$ as $\sum_i \{f_i, g_i\}$ (with all the tame symbols vanish) then Beilinson’s construction of the regulator looks as follows:

$$< r_D \sum_i \{f_i, g_i\}, \omega > = \frac{1}{2\pi i} \sum_i \int_{E(\mathbb{C})} \log |f_i| d\arg(g_i) \wedge \omega$$

Let $f$ and $g$ be rational functions on $E$ such that

$$(f) = \sum n_i(a_i), \quad (g) = \sum m_j(b_j)$$

Let $\Gamma = H_1(E(\mathbb{C}), \mathbb{Z})$. We may assume that $\Gamma = \{ \mathbb{Z} \oplus \mathbb{Z} \cdot \tau \} \subset \mathbb{C}$ and $z$ is the coordinate in $\mathbb{C}$. Let us briefly recall how the regulator integral $< r_D \{f, g\}, dz >$ is computed by means of the elliptic dilogarithm ([Bl1], see also [RSS]). The intersection form on $\Gamma$ provides a pairing

$$\langle \cdot, \cdot \rangle : E(\mathbb{C}) \times \Gamma \rightarrow S^1; \quad (z, \gamma) := \exp(\frac{2\pi i (z \bar{\gamma} - \bar{z} \gamma)}{\tau - \bar{\tau}})$$

Let

$$K_{2,1}(z; \tau) := \frac{(\text{Im}\tau)^2}{\pi} \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{(u, \gamma)}{\gamma^2\bar{\gamma}}, \quad z = \exp(2\pi i u)$$

Then one has

$$\frac{1}{2\pi i} \int_{E(\mathbb{C})} \log |f| d\arg g \wedge dz = \frac{1}{i\pi} \sum_{a, b \in E(\mathbb{C})} v_a(f)v_b(g)K_{2,1}(a - b; \tau)$$

To prove this one may use that

$$\int_{E(\mathbb{C})} \log |f| d\arg g \wedge dz = -\int_{E(\mathbb{C})} \log |f| d\log |g| \wedge dz$$

together with the following lemma and the fact that the Fourier transform sends the convolution to the product.
Lemma 3.22

$$\log |f(z)| = -\frac{Im\tau}{2\pi} \sum_{\gamma \in \Gamma \setminus 0} v_\alpha(f) \frac{(z - a, \gamma)}{\gamma^2} + C_f$$ (31)

where $C_f$ is a certain constant.

**Proof.** One can get a proof applying $\partial \bar{\partial}$ to the both parts of (31). The constant $C_f$ can be computed from the decomposition of $f$ on the product of theta functions using the formula in s. 18 ch. VIII of [We]. It does not play any role in our considerations since $\int_{E(\mathbb{C})} C_f \cdot d\log |g| \wedge \omega = 0$ by the Stokes formula.

The relation between the Eisenstein-Kronecker series $K_{2,1}(z)$ and the elliptic dilogarithm is the following ([Bl1], [Z]): $K_{2,1}(z; \tau) = \mathcal{L}_{2,q}(z) - iJ_q(z)$, where the function $J_q(z)$ is defined as follows. Let us average the function $J(z) := \log |z| \log |1 - z|$ over the action of the group $\mathbb{Z}$ generated by the shift $z \mapsto qz$ regularizing divergencies. We will get

$$J_q(z) := \sum_{n=0}^{\infty} J(q^n z) - \sum_{n=1}^{\infty} J(q^n z^{-1}) + \frac{1}{3} (\log |q|)^2 \cdot B_3 \left( \frac{\log |z|}{\log |q|} \right)$$

Here $B_3(x)$ is the third Bernoulli polynomial. The function $J_q(z)$ is invariant under the shift $z \mapsto qz$ and satisfies $J_q(z) = -J_q(z^{-1})$.

It follows from the main result of Beilinson in [B2], see also [SS2], that for a modular elliptic curve $E$ over $\mathbb{Q}$ there always exists an element in $K_2(E)_\mathbb{Z}$ whose regulator gives (up to a standard nonzero factor) $L(E,2)$. So we get the formula

$$L(E,2) \sim_{\mathbb{Q}, \pi} \sum_i \sum_{a,b \in E(\mathbb{C})} v_a(f^{(i)})v_b(g^{(i)})\mathcal{L}_{2,q}(a-b)$$

Finally, the results of the present section implies that the element $\sum_i \sum_{a,b \in E(\mathbb{C})} v_a(f^{(i)})v_b(g^{(i)})\{a-b\}_3 \in I_{E(\mathbb{Q})}^4$ must satisfy all the conditions of the theorem [1.1].

Theorem [1.3] follows from the surjectivity of the map $\beta$ and the arguments above.

**The integrality condition ([BG], [SS]).** Let $E$ be an elliptic curve over $\mathbb{Q}$. Choose a minimal regular model $E_\mathbb{Z}$ of $E$ over $\mathbb{Z}$. One has the exact sequence

$$K_2(E_\mathbb{Z}) \rightarrow K_2(E_\mathbb{Q}) \xrightarrow{\partial} \oplus_p K'_1(E_p)$$ (32)

The group $K'_1(E_p) \otimes \mathbb{Q}$ is not zero if and only if $E_p$ has a split multiplicative reduction with special fibre a Néron $N$-gon. In this case $K'_1(E_p) \otimes \mathbb{Q} = \mathbb{Q}$.

Consider an element $\sum_i f_i, g_i \in K_2(\mathbb{Q}(E))$ which has zero tame symbol at all points. It defines an element of $H^0(E, K_2)$. Suppose first that the closure of the support of the divisors $f_i, g_i$ is contained on the smooth part of $E_\mathbb{Z}$. Then

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Schappacher and Scholl ([SS]) proved that the image of this element under the map $\partial$ is computed by the following formula:

$$\partial(\sum_i \{ f_i, g_i \}) = \pm \frac{1}{3N} \sum_{\nu \in \mathbb{Z}/N\mathbb{Z}} d((f) \ast (g^-); \nu) B_3(\frac{\nu}{N}) \cdot \Phi$$

Here $\Phi$ is a generator in $K_1^1(E_p) \otimes \mathbb{Q}$.

In general one should extend $\mathbb{Q}$ to $\mathbb{Q}((f_i), (g_i))$, which is the field of the definition of the divisors $(f_i), (g_i)$. After this we get precisely the condition (4).

Let us explain why the expression $\sum_{\nu \in \mathbb{Z}/(eN)\mathbb{Z}} d(P; \nu) B_3(\frac{\nu}{N})$ does not depend on the field $L$. Let $L$ and $L'$ are two extensions of $\mathbb{Q}_p$ such that all the points are defined over them. We can assume that $L \subset L'$ (by taking the composit). Denote by $n_1 = e_1 f_1$ degree of this extension. Looking at the Tate uniformization we see that points which intersect $\nu$-th component of special fiber corresponding to the field $L$ intersect $e_1 \nu$-th component of special fiber corresponding to the field $L'$.

4. Main results from the motivic point of view. Let $\mathcal{M}$ be the (hypothetical) abelian category of all mixed motivic sheaves over a regular scheme $X$ over a field $k$. Let $\mathbb{Q}(-1) := h^2(P^1), \mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ and $\mathcal{H} := h^1(E)(1)$.

The motivic refinement of our results is the following

**Conjecture 3.23** There exists a canonical quasiisomorphism in the derived category

$$B(E, 3) \otimes \mathbb{Q} = R\text{Hom}_{\mathcal{M}}(\mathbb{Q}(0), \mathcal{H}(1))$$

Let us explain how it fits with our results. Let $\pi : E \to \text{Spec}(k)$ be the structure morphism. There are the Tate sheaves $\mathbb{Q}(n)_E := \pi^* \mathbb{Q}(n)$. Beilinson’s description of Ext groups between the Tate sheaves over $E$ gives us

**Conjecture 3.24**

$$\text{Ext}_{\mathcal{M}}^i(\mathbb{Q}(0)_E, \mathbb{Q}(2)_E) = gr_2 K_{4-i}(E) \otimes \mathbb{Q}$$

**Lemma 3.25**

$$R\text{Hom}_{\mathcal{M}}(\mathbb{Q}(0), \mathcal{H}(1)) = R\text{Hom}_{\mathcal{M}}(\mathbb{Q}(0), \mathbb{Q}(2))^-$$

Indeed, let $p : E \to \text{Spec}(k)$ be the canonical projection. Then we should have the motivic Leray spectral sequence

$$E_2^{p, q} = \text{Ext}_{\mathcal{M}}^p(\mathbb{Q}(0), R^q p_* \mathbb{Q}(2))$$

degenerating at $E_2$ and abutting to $\text{Ext}_{\mathcal{M}}^{p+q}(\mathbb{Q}(0), \mathbb{Q}(2))$. Noting that

$$h^0(E)^- = h^2(E)^- = 0; \quad h^1(E)^- = h^1(E)$$
we get (33). Conjecture (3.24) together with this lemma tell us that the cohomology of the elliptic motivic complexes are given by the formula

\[ R^i \text{Hom}_{\mathcal{M}_k}(\mathbb{Q}(0), \mathcal{H}(1)) \otimes \mathbb{Q} = \text{gr}_2 K_{3-i}(E)^{-i} \otimes \mathbb{Q} \]  

(34)

Conjecture (3.23) follows from theorem (3.7) and conjecture (3.24), see [G2].

A very interesting and important is the following:

**Problem.** To construct explicitly the general elliptic motivic complexes

\[ R\text{Hom}_{\mathcal{M}_k}(\mathbb{Q}(0), \text{Sym}^n \mathcal{H}(m)) \]

For \( m = 1 \) it is considered in [G2].

5. **Degeneration to the nodal curve (compare with [Bl1], [DS])**. Let \( k \) be an algebraically closed field. Denote by \( I \) the group of rational functions such that \( f(0) = f(\infty) = 0 \). Let \( I_k \) be the augmentation ideal of \( \mathbb{Z}[k^*] \).

For an element \( f \otimes g \in (1 + I) \otimes k(t)^* \) consider the Bloch map

\[ \beta(f \otimes g) := \sum_{x, y \in k^*} v_x(f) v_y(g) \{ y/x \} + v_\infty(g) (f + (f^-)) \in I_k^2. \]

Set

\[ \mathbb{Z}[k^*] \xrightarrow{\delta} k^* \otimes k^*, \quad \{ x \} \mapsto (1 - x) \otimes x, \quad \{ 1 \} \mapsto 0 \]

(35)

Let \( p : I_k \to k^* \) be the natural projection \( \{ x \} \to x \).

**Theorem 3.26** The following diagram is commutative:

\[
\begin{array}{ccc}
(1 + I) \otimes k(t)^* & \xrightarrow{\partial} & I_k^* \otimes k^* \\
\downarrow \beta & & \downarrow p \otimes \text{id} \\
I_k^2 & \xrightarrow{\delta} & k^* \otimes k^* \\
\end{array}
\]

The proof is a direct calculation similar to (but simpler than) the proof of theorem (3.2).

Let \( S(k) \) be the subgroup generated by the elements \((1 - f) \otimes f \) where \( f \in I \) and the subgroup \((1 + I) \otimes k^* \). Set \( B_2^*(k) := I_k^2 / \beta(S(k)) \).

**Lemma 3.27** \( \text{Ker} \beta \subset S(k) \)

**Proof.** An easy analog of theorem (3.10) for the nodal curve claims that the homomorphism

\[ q : k(t)^* \to (1 + I) \otimes k(t)^*, \quad f(t) \mapsto f(x) \otimes (x - 1) \]

is surjective. It is clear that \( \beta \circ q \) is injective. The lemma is proved.
This lemma implies that $\bar{\beta} : (1 + \mathcal{I}) \otimes k(t)^* \rightarrow B_2^*(k)$ is an isomorphism. So we get a morphism of complexes

$$
\begin{array}{ccc}
\frac{(1+\mathcal{I}) \otimes k(t)^*}{S(k)} & \xrightarrow{\partial} & k^* \otimes k^* \\
\bar{\beta} \downarrow & & \downarrow id \\
B_2^*(k) & \xrightarrow{\delta} & k^* \otimes k^*
\end{array}
$$

Such that $\bar{\beta}$ is surjective and $\text{Ker} \bar{\beta} = \text{Tor}(k^*, k^*)$, similar to the theorem (3.8). So we see that when $E$ degenerates to a nodal curve the complex $B^*(E/k, 3)$ degenerates to the complex $B_2^*(k) \rightarrow k^* \otimes k^*$.

Let $S_1(k)$ be the subgroup generated by the elements $(1 - f) \otimes f$ where $f \in \mathcal{I}$.

Theorem 3.28 Let $k = \bar{k}$. The identity map on $\mathbb{Z}[k^*]$ provides a homomorphism of groups $B_2^*(k) \rightarrow B_2(k)$. Its kernel is isomorphic to $S^2k^*$. This map provides a quasiisomorphism of complexes

$$
\begin{array}{ccc}
B_2^*(k) & \longrightarrow & k^* \otimes k^* \\
\downarrow & & \downarrow \\
B_2(k) & \longrightarrow & k^* \wedge k^*
\end{array}
$$

One can show that the group $B_2^*(k)$ is isomorphic to a group defined by S. Lichtenbaum in [Li].

It is known ([Lev]) that

$$
K_2(\mathbb{P}^1_{(0, \infty)}, \{0, \infty\}) = \frac{(1 + \mathcal{I}) \otimes k(t)^*}{S_1(k)}
$$

Using this and the results of this subsection we can get a proof of Suslin’s theorem for $k = \bar{k}$.

5. On an elliptic analog of the 5-term relation for the elliptic dilogarithm. Notice that $B_2(F) = \text{Coker}(\mathbb{Z}[M_{0,5}(k)] \xrightarrow{\partial} \mathbb{Z}[\mathbb{G}_m(k)])$ Here $M_{0,5}$ is the configuration space of 5 distinct points on the projective line. So one may ask how to present the group $B_3(E)$ in a similar form:

$$
B_3(E) \xrightarrow{?} \text{Coker}(\mathbb{Z}[X(k)] \xrightarrow{\partial} \mathbb{Z}[E(k)])
$$

where $X$ is a finite dimensional variety. In the our definition we have an infinite dimensional $X$ (more precisely it is an inductive limit of finite dimensional varieties).
Here is a guess. Let us realize $E$ as a cubic in $P^2$. Let $p$ be a point in $P^2$ and $l_1, l_2, l_3$ any three lines through this point. Set $A_i := l_i \cap E$. Let $\tilde{R}_3^*(E) \subset I_E^4$ be the subgroup generated by the elements

$$\{p; l_1, l_2, l_3\} := A_1 * A_2^- + A_2 * A_3^- + A_3 * A_1^-$$

and those linear combinations of the elements $\{a\}_3 + \{-a\}_3$ which lie in $I_E^4$. (Probably they belong to the subgroup generated $\{36\}$).

**Lemma 3.29** $\tilde{R}_3^*(E) \subset R_3^*(E)$.

**Proof.** Let $f_i$ be a linear homogeneous equation of the line $l_i$. Since the lines $l_1, l_2, l_3$ intersect in a point, these equations are linearly dependent, so we may choose them in such a way that $f_1 + f_2 = f_3$. Thus $f_1/f_3 \cap f_2/f_3 = f_1/f_3 \cap (1 - (f_1/f_3))$. Applying the map $\beta$ to this Steinberg relation and using the relations $\{a\}_3 + \{-a\}_3 = 0$ we get the element $\{p; l_1, l_2, l_3\}$.

One obviously has $\sum_{j=1}^{4}(-1)^j\{p; l_1, l_2, l_3\} = 0$, so we can assume that the line $l_3$ is, say, a vertical line.

**Conjecture 3.30** $\tilde{R}_3^*(E) = R_3^*(E)$.

### 4 $B_2(E)$, $\theta$-functions and action of isogenies

1. **Elliptic curves over $\mathbb{C}$**. Let us represent $E$ as a quotient $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Below $\xi$ denotes the coordinate on $\mathbb{C}$. The canonical trivialization of $(T^*_0 E)^{12}$ is $-16 \prod(e_j - e_i)(d\xi)^{12}$, which is equal to $\Delta(\tau)(d\xi)^{12}$, where

$$\Delta(\tau) = (2\pi i)^{12} \eta^{24}(\tau), \quad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n) \quad (q = \exp(2\pi i \tau))$$

So the trivialization of $(T^*_0 E)^2$ is $(2\pi i)^2 \eta^4(\tau)(d\xi)^2$.

Now we will give the analytic description of the Deligne pairing $[* , *]$ . Let $L_\alpha$ be the line bundle corresponding to the divisor $(a) - (0)$. Choose a representative $\alpha \in \mathbb{C}$ of $a$ (a is defined up to $\mathbb{Z} + \mathbb{Z}\tau$). Let us define $L_\alpha$ on $E$ as the quotient of the trivial line bundle on $\mathbb{C}$ under the action of $\mathbb{Z} + \mathbb{Z}\tau$: $1$ acts trivially and $\tau$ acts by multiplication by $\exp(2\pi i \alpha)$. We identify $L_{\alpha+1}$ with $L_\alpha$ trivially and $L_{\alpha+\tau}$ with $L_\alpha$ by multiplication by $\exp(2\pi i \xi)$.

The fiber of the Poincare line bundle $[L_\alpha, L_\beta]$ on $J \times J$ over the point $(\alpha, \beta)$ is equal to $L_{\alpha|\beta} \otimes L_{\alpha}^{-1}|_0$. This line bundle is described as a quotient

$$\frac{\mathbb{C} \times \mathbb{C} \times \mathbb{C}}{(\mathbb{Z} + \mathbb{Z}\tau) \oplus (\mathbb{Z} + \mathbb{Z}\tau)}$$

$$(\alpha, \beta, \lambda) \mapsto (\alpha + m + n\tau, \beta + m' + n'\tau, \lambda \cdot \exp(2\pi i (n\beta + n'\alpha + mn\tau)))$$

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Let us show that $L_\alpha = L_a$. Consider a slight modification of the Jacobi $\theta$-function $(z = \exp(2\pi i \xi))$:

$$\theta(\xi) = \theta(\xi; \tau) = q^{1/24} \prod_{j \geq 0} (1 - q^j) \prod_{j > 0} (1 - q^{j-1})$$

Then $\frac{\theta(\xi) - \alpha}{\theta(\xi)\theta(\alpha)}$ is a section of $L_\alpha$ with the required divisor $(a) - (0)$.

Our recipe for the calculation of the element $< (a) - (0), (a) - (0) > \in (T_0^*E)^{\otimes 2} \otimes K^*[a \cdot a]$ leads to the expression

$$\frac{d\xi \cdot \theta'(0)}{d\xi \theta'(0) \theta(\alpha)^{-1}} = -\left(\frac{\theta'(0)}{\theta(\alpha)}\right)^2.$$

Notice that $\theta'(0) = 2\pi i \eta^2(\tau)$ and the chosen analytic trivialization of $(T_0^*E)^2$ is $(2\pi i)^2 \eta(\tau)^4 (d\xi)^2$. So the final answer is $-\theta(\alpha)^{-2}$.

**2. The Tate curves.** Let $K$ be a field complete with respect to a discrete valuation $v$. Let $O = \{a \in K | v(a) \leq 0\}$ be the ring of integers, $I = \{a \in K | v(a) < 0\}$ the maximal ideal, $k = O/I$ the residue field.

Let $q \in I$. According to Tate, the group $K^*/q^\infty$ is isomorphic to the group of points of the elliptic curve $E_q$ over $K$ given by equation $y^2 + xy = x^3 + a_4 x + a_6$, where

$$a_4 = -5 \sum_{j \geq 1} \frac{j^3 q^j}{1 - q^j}; \quad a_6 = -\frac{1}{12} \sum_{j \geq 1} \frac{(7j^5 + 5j^3)q^j}{1 - q^j}.$$

The discriminant and $j$-invariant of this curve are: $\Delta = q \prod_{j \geq 1} (1 - q^j)^{24}, \quad j = \frac{1}{q} + 744 + 196884q + \cdots$. The map of $K^*/q^\infty$ to the group of points of $E_q$ is defined by the following expressions:

$$x(u) = \sum_{j \geq 2} \frac{q^j u}{(1 - q^j u)}^2 - 2 \sum_{j \geq 1} \frac{j q^j}{1 - q^j}; \quad y(u) = \sum_{j \geq 2} \frac{q^{2j} u^2}{(1 - q^j u)^3} + \sum_{j \geq 1} \frac{j q^j}{1 - q^j}.$$

The unity $1$ of $K^*$ maps to neutral element $0$ of the curve.

Define a function $T(u)$ on $K^*$ by the formula:

$$T(u) = \prod_{j \geq 0} (1 - q^j u) \prod_{j > 0} (1 - q^{j-1} u^{-1})$$

This function vanishes on $\{q^\infty\}$. It is quasiperiodic: $T(uq) = -u^{-1} T(u)$.

Let $a \in O \setminus qO$. Denote by the same symbol its image in $E_q$. Then a section $s$ of the bundle $O_{E_q}((a) - (0))$ can be represented by a function $f$ on $K^*$ such that $f(uq) = a f(u)$. Indeed, the periodic function $f(u)T(u)(T(ua^{-1}))^{-1}$ has the required divisor.
Like in the analytic case, the total space of the Poincaré line bundle $T_{(a,b)}$ is isomorphic to the quotient $K^* \times K^* \times K^*$ modulo the action of the group $\mathbb{Z} \oplus \mathbb{Z}$ generated by the following transformations:

$$(a, b, \lambda) \rightarrow (qa, b, b\lambda); \quad (a, b, \lambda) \rightarrow (a, qb, a\lambda).$$

The corresponding group structure on the collection $T_{(a,\cdot)}$ is defined by the law:

$$\left(\left(\left(a_1, b_1, \lambda_1\right) \times \left(a_2, b_2, \lambda_2\right) = \left(a_1b_1, \lambda_1 \lambda_2\right) \right) \times \left(a_2, b_2, \lambda_2\right) = \left(a_1b_1b_2, \lambda_1 \lambda_2\right)\right).$$

Let us calculate the expression $<a-0, a-0>$.

The divisor of the section $T(u)^{-1}(T(ua^{-1}))$ equals $(a-0)$. The regularized value of this expression at the divisor $(a-0)$ is equal to:

$$-\frac{du}{u} \left(\prod_{j>0} \left(1 - q^j\right)^2\right)^2 \prod_{j>0} \left(1 - q^j a^{-1}\right) \prod_{j>0} \left(1 - q^j a\right) - \frac{du}{u} \left(\prod_{j>0} \left(1 - q^j\right)^2\right) \left(\prod_{j>0} \left(1 - q^j a^{-1}\right) \prod_{j>0} \left(1 - q^j a\right)\right)^{-1}.$$

In this calculation notice that $(1 - u) \cdot u = -(u - 1)$ modulo $(u - 1)^2$, so the factor $1 - u$ leads to $-\frac{du}{u}$.

The trivialization of $(T_0^* E)^{\otimes 12}$ is defined by the section

$$(\frac{du}{u})^{12} \Delta = \left(\left(\frac{du}{u}\right)^2 \left(\prod_{j>0} \left(1 - q^j\right)^4\right)^{\frac{1}{2}}\right)^{6}.$$

Hence the needed expression is:

$$-\left(\frac{du}{u} \left(\prod_{j>0} \left(1 - q^j\right)^2\right)^{\frac{1}{2}} \prod_{j>0} \left(1 - q^j a\right) \prod_{j>0} \left(1 - q^j a^{-1}\right) \left(\left(\frac{du}{u}\right)^2 \left(\prod_{j>0} \left(1 - q^j\right)^4\right)^{\frac{1}{2}}\right)^{-1} \right)$$

$$= -\left(q^{-\frac{1}{2}} a^{-\frac{1}{2}} T(a)\right)^{-2}.$$

3. **Calculation of the canonical height.** a) **Archimedean case.** Let us calculate the archimedean height. The torsor $T_{(a,0)}$ is trivialized. So the group
$T_{(\alpha, \cdot)}$ is isomorphic to the quotient of $\mathbb{C} \times \mathbb{C}^*$ with coordinates $(\beta, \lambda)$ by the action of the group $\mathbb{Z} + \mathbb{Z}\tau$:

$$1: (\beta, \lambda) \to (\beta + 1, \lambda), \quad \tau: (\beta, \lambda) \to (\beta + \tau, \lambda \times \exp(2\pi i\alpha))$$

A homomorphism $|\cdot|_\alpha : \mathbb{C} \times \mathbb{C}^* \to \mathbb{R}^+$ which is invariant under the action of $\mathbb{Z} + \mathbb{Z}\tau$ and coincides with the norm $| \cdot |$ on $\mathbb{C}^*$ if $\beta = 0$ is given by

$$|(\beta, \lambda)|_\alpha = \exp(-\pi i \frac{(\alpha - \bar{\alpha})(\beta - \bar{\beta})}{\tau - \bar{\tau}})|\lambda|$$

It defines a homomorphism $T_{(\alpha, \cdot)} \to \mathbb{R}^+$. Therefore the value of our height at $\alpha$ equals to

$$\log |(\alpha, -(\frac{1}{\theta(\alpha)})^2)|_\alpha = 2[-(\log |\theta(\alpha)| + \frac{\pi i}{2} \frac{(\alpha - \bar{\alpha})^2}{\tau - \bar{\tau}})]$$

It coincides with $2$ times the canonical Néron height from [Sil].

b) The nonarchimedean case. Let $q \in I^n \setminus I^{n+1}$. The Tate curve over $\text{Spec}(\mathcal{O})$ has a singular fiber over $\text{Spec}(k)$. The singular fiber $C = \cup C_j$ of the minimal Néron model is an $n$-gon. The set $I^n \setminus I^{n+1}$ represents the points of the curve $E$ over $\text{Spec}(K)$ whose restrictions to the singular fiber belong to the $j$-th component $C_j$ of the $n$-gon.

Consider the map

$$\tilde{h}_v : K^* \times K^* \times K^* \to \mathbb{Z}; \quad \tilde{h}_v(a, b, \lambda) = v(q)v(\lambda) - v(a)v(b).$$

This map is invariant with respect to the action of $\mathbb{Z} \oplus \mathbb{Z}$; therefore it determines a map $h_v$ from $T_{(\cdot, \cdot)}$ to $\mathbb{Z}$. The map $h_v$ is a group homomorphism on $T_{(\alpha, \cdot)}$, its image is discrete, hence its kernel is a maximal compact subgroup.

Let $v(a) = j$. The value of this map on $\langle (a) - (0), (a) - (0) \rangle$ equals to:

$$v(q)v(\theta(a)^{-2}) - v(a)v(a) = n \cdot (-2) \cdot \left( \frac{1}{12}n - 2j + v(1 - a) \right) - j^2$$

$$= -\frac{1}{6}n^2 + jn - j^2 - 2nv(1 - a) = -n^2 B_2(\frac{j}{n}) - 2nv(1 - a)$$

where $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. (Compare with the integrality condition given by the third Bernoulli polynomial).

4. **Functoriality of the groups $B_2(E)\mathbb{Q}$ under the isogenies.** Let $n$ be an integer prime to the characteristic of $k$. Let $\lambda : E_1 \to E_2$ be an isogeny of order $n$ between the elliptic curves $E_1$ and $E_2$.

i) Pull back $\lambda_2^* : B_2(E_2) \to B_2(E_1)|_\mathbb{A}$. Let $\hat{\lambda}$ be the dual isogeny $J_{E_2} \to J_{E_1}$. The pull back of the basic extension $0 \to k^* \to B_2(E_1) \to S^2J_{E_1} \to 0$
(considered modulo \(n\)-torsion) under the homomorphism \(\hat{\lambda} \cdot \hat{\lambda} : S^2J_{E_2} \to S^2J_{E_1}\) provides the map \(\lambda_2^2\):

\[
\begin{array}{ccc}
0 & \to & k^* \\
\downarrow & & \downarrow \\
B_2(E_2) & \to & S^2J_{E_2} & \to & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & k^* \\
\downarrow & & \\
B_2(E_1)\left(\frac{1}{n}\right) & \to & S^2J_{E_1}\left(\frac{1}{n}\right) & \to & 0 \\
\end{array}
\]

A more direct description of \(\lambda_2^2\) can be spelled as follows. We have a natural morphism of \(k^*\)-torsors:

\[
\lambda^*: [L, M] \otimes n \to [\lambda^* L, \lambda^* M]; \quad <s_1, s_2> \otimes n \to <\lambda^{-1}s_1, \lambda^{-1}s_2>
\]

To check that it is a map of torsors notice that

\[
(<f \cdot s_1, s_2>) \otimes n = f(s_2)^n(<s_1, s_2>) \otimes n
\]

and \(f(s_2)^n = (\lambda^{-1}f)(\lambda^{-1}s_2)\). It is easy to see that these maps provide the pull-back map:

\[
\lambda_2^* : B_2(E_2) \to B_2(E_1)\left(\frac{1}{n}\right); \quad <s_1, s_2> \mapsto <\lambda^{-1}s_1, \lambda^{-1}s_2> \otimes \frac{1}{n}. 
\]

**Theorem 4.1** \(n \cdot \left(\lambda_2^2\{\lambda(a)\}_2 - \sum_{\gamma \in Ker\lambda} \{a + \gamma\}_2\right) = 0\)

**Proof.** Any isogeny can be presented as the composition of cyclic isogenies. So we can assume that \(\lambda\) is cyclic. The expression

\[
f_n(a) := n \cdot \left(\lambda_2^2\{\lambda(a)\}_2 - \sum_{\gamma \in Ker\lambda} \{a + \gamma\}_2\right)
\]

is a nonvanishing function on the noncompact curve \(E_1 \setminus Ker\lambda\).

Let \(p_n : \mathcal{E} \to X_0(n)\) be the universal family of elliptic curves over the modular curve \(X_0(n)\), and \(\Lambda : \mathcal{E} \to \mathcal{E}\) the universal cyclic \(n\)-isogeny. The curves \(E_1\) and \(E_2\) are the fibers of \(\mathcal{E}\) over the points \((E_1, Ker\lambda)\) and \((E_2, E_2[n]/Ker\lambda)\) of \(X_0(n)\), and \(\lambda\) is the restriction of \(\Lambda\). The construction above defines an algebraic function \(F\) on the universal curve \(\mathcal{E}_1\).

Consider the punctured formal neighborhood of the cusp point in which the universal isogeny is totally ramified. The \(j\)-invariant of the restriction of the universal curves to this neighborhood has a pole at the cusp; hence, this restriction can be described by the Tate curves \(E_q \to E_q^n\) ([Sil]).

Let us prove that \(F_n\) equals 1 for the Tate curves. We are dealing with the isogeny \(K^*/(q^n)^2 \to K^*/q^2\). For the Tate curve we expressed in s. 4.2 the pairing \(<a - (0), (a) - (0)>\) in terms of the \(\theta\)-function

\[
\theta_q(a) := q^{1/2}a^{-\frac{1}{2}}T(a) = q^{1/12}a^{-\frac{1}{2}}\prod_{j \geq 0}(1 - q^ja)\prod_{j > 0}(1 - q^ja^{-1}).
\]
Remark. \( \theta_q(a) \) is defined only up to a choice of sign, so only \((\theta_q(a))^2\) makes sense.

So we need to prove the following proposition.

**Proposition 4.2**

\[
\left( \frac{\prod_{0 \leq k < n} \theta_q(t \cdot q^k)}{(\prod_{0 \leq k < n} \theta_{q^n}(t \cdot q^k))^n} \right)^2 = 1 \tag{37}
\]

It shows that the restriction of the function \( F_n \) to the preimage of neighborhood of the cusp point equals 1. Hence this function is equal to 1 on all universal curve \( X_0(N) \); the function \( f_n \) is the restriction of \( F_n \) to the fiber \( E \); therefore \( f_n = 1 \).

**Proof of the proposition.** The \( \theta \)-function has the following property:

\[
\theta_q(t \cdot q^k) = (-1)^k a^{-k} q^{-k^2/2} \theta_q(t)
\]

Therefore \( \prod_{0 \leq k < n} \theta_q(t \cdot q^k) \) equals to

\[
(-1)^{s_1(n)} t^{-s_1(n)} q^{-s_2(n)/2} q^{n/12} t^{-n/2} \prod_{j \geq 0} (1 - q^j t^n) \prod_{j > 0} (1 - q^j t^{-1} t^n) \tag{38}
\]

Using the definition and notations

\[
s_1(n) := 1 + \ldots + (n-1) = \frac{n(n-1)}{2}; \quad s_2(n) := 1^2 + \ldots + (n-1)^2 = \frac{(n-1)n(2n-1)}{6}
\]

we have

\[
\theta_{q^n}(t \cdot q^k) = q^{n/12} t^{-1/2} q^{-k/2} \prod_{j \geq 0} (1 - q^{nj} \cdot tq^k) \prod_{j > 0} (1 - q^{nj} \cdot t^{-1} q^{-k})
\]

On the other hand \( \prod_{0 \leq k < n} \theta_{q^n}(t \cdot q^k) \) is equal to

\[
q^{n^2} t^{-s_1(n)} \prod_{0 \leq k < n} \prod_{j \geq 0} (1 - q^{nj + kt}) \prod_{0 \leq k < n} \prod_{j > 0} (1 - q^{nj - k} t^{-1}) =
q^{n^2} t^{-s_1(n)} \prod_{j' \geq 0} (1 - q^{j'} t^n) \prod_{j' > 0} (1 - q^{j'} t^{-1}) \tag{39}
\]

Comparing (38) and (39) we see that it remains to check that

\[
\left( q^{n^2} t^{-s_1(n)} t^{-n/2} \right)^n = q^{-s_2(n)/2} t^{-s_1(n)}
\]

The statement of the proposition follows.

In particular when \( \lambda[m] \) is the isogeny of multiplication by \( m \) the theorem gives
Corollary 4.3 Suppose \( \bar{k} = k \). Then for any \( a \in E(k) \) one has the "distribution relation"

\[
m(\{a\}_2 - \sum_{mb=a} \{b\}_2) = 0 \tag{40}
\]

**Remark.** In this case one can define \([m]^* : B_2(E_2) \to B_2(E_1)[\frac{1}{m}]\) using the map \( \frac{1}{m}[m] \circ \frac{1}{m}[m] \) in the diagram defining \( \lambda_2 \). Thus we have the factor \( m \) instead of \( m^2 \) in the formula (40).

ii) The transfer map \( \lambda_2^* : B_2(E_1)[\frac{1}{n}] \to B_2(E_2)[\frac{1}{m}] \). Notice that the group \( B_2(E)_\mathbb{Q} \) does not satisfy the descent property. Namely, if \( k \subset K \) is a finite Galois extension then

\[
B_2(E/k)_\mathbb{Q} \hookrightarrow B_2(E/K)_{\text{Gal}(K/k)}^\mathbb{Q}
\]

but this inclusion is not an isomorphism because the group \( S^2J(k)_\mathbb{Q} \) does not have the descent property.

Suppose \( k = \bar{k} \). The transfer map \( \lambda_2^* \) should satisfy the projection formula

\[
\lambda_2^* \circ \lambda_2^* = n \cdot Id \tag{41}
\]

and should fit into the following diagram, considered modulo \( n \)-torsion:

\[
\begin{array}{cccccc}
0 & \to & k^* & \to & B_2(E_1) & \to & S^2J_{E_1} & \to & 0 \\
\downarrow m_n & & \downarrow \lambda_2^* & & \downarrow \lambda \cdot \lambda & & & & \\
0 & \to & k^* & \to & B_2(E_2) & \to & S^2J_{E_2} & \to & 0
\end{array}
\]

Here \( m_n : x \to x^n \). This is necessary in order to have (41) on the subgroup \( k^* \subset B_2(E) \).

Let us define the transfer map as follows:

\[
\lambda_2^*\{a\}_2 := \{\lambda(a)\}_2 - \left( \lambda_2^*\{\lambda a\}_2 - n\{a\}_2 \right) \tag{42}
\]

**Remark.** Projection of \( \lambda_2^*\{\lambda a\}_2 - n\{a\}_2 \) to \( S^2J \) equals \( \frac{1}{n} \lambda \circ \lambda (a \cdot a) - na \cdot a = 0 \), so \( \lambda_2^*\{\lambda a\}_2 - n\{a\}_2 \in k^* \).

Let us show that formula (42) provides a transfer homomorphism. Suppose that \( \sum a_i \}_2 + c = 0 \) in the group \( B_2(E_1) \), where \( c \in k^* \) (we write the group \( B_2(E) \) additively). We have to prove that

\[
\sum (\lambda(a_i))_2 - \left( \sum \lambda_2^*\{\lambda(a_i)\} - \sum n\{a_i\}_2 \right) + nc = 0
\]

By the assumption \( \sum a_i \}_2 = c^{-1} \in k^* \). As we have shown before, the expression in brackets always belongs to the subgroup \( k^* \subset B_2(E_2) \). Therefore
\[ \sum_i \lambda^2_i \{ \lambda(a_i) \}_2 \in k^*. \]

Notice that \( \lambda^*_2 \) is injective modulo \( n \)-torsion and is the identity on the subgroup \( k^* \subset B_2(E) \). Therefore modulo \( n \)-torsion \( \sum_i \{ \lambda(a_i) \}_2 \in k^* \) and \( \sum_i \lambda^*_2 \{ \lambda(a_i) \}_2. \)

Using proposition \([4.1]\) one can easily see that

\[ \lambda_{2*} \{ a \}_2 \stackrel{\text{def}}{=} \{ \lambda(a) \}_2 - \frac{1}{n} \sum_{\gamma \in \text{Ker} \lambda} (n \{ a + \gamma \}_2 - n \{ a \}_2) \quad (43) \]

5. A presentation of the group \( B_2(E) \) by generators and relations.

Let \( \mathcal{P}(a) \) (resp. \( \mathcal{P}'(a) \)) be the \( x \)-coordinate (resp. \( y \)-coordinate) of the point \( a \in E \) in Tate’s normal form of an elliptic curve over an arbitrary field \( k \):

\[ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \]

**Proposition 4.4.** a) If \( a \neq b \) then

\[ < (a + b) - (0), (a + b) - (0) > \otimes < (a - b) - (0), (a - b) - (0) > \]

\[ \otimes < (a) - (0), (a) - (0) >^{-2} \otimes < (b) - (0), (b) - (0) >^{-2} = (\Delta^{-1/6}(\mathcal{P}(a) - \mathcal{P}(b)))^{-2} \]

b) If \( a = b \) but \( 2a \neq 0 \) then the left hand side is equal to \( (\Delta^{-1/4}\mathcal{P}'(a))^{-2} \). If \( 2a = 0 \) then we get \( (\Delta^{-1/3}\mathcal{P}'(a))^{-2} \)

**Proof.** We will prove part a). Part b) is similar. Let \( L_a \) be the line bundle corresponding to the divisor \( (a) - (0) \). Evidently

\[ [L_{a+b}, L_{a+b}] \otimes [L_{a-b}, L_{a-b}] \otimes [L_a, L_a]^{-2} \otimes [L_b, L_b]^{-2} = k^* \]

so

\[ < (a + b) - (0), (a + b) - (0) > \otimes < (a - b) - (0), (a - b) - (0) > \]

\[ \otimes < (a) - (0), (a) - (0) >^{-2} \otimes < (b) - (0), (b) - (0) >^{-2} \in k^* \otimes (T^*_0E)^{\otimes -4} \]

We want to calculate this element. One has

\[ < (a + b) - (0), (a + b) - (0) > \otimes < (a - b) - (0), (a - b) - (0) > \]

\[ \otimes < (a) - (0), (a) - (0) >^{-2} \otimes < (b) - (0), (b) - (0) >^{-2} = \]

\[ < (a + b) - (a), (a + b) - (a) > \otimes < (a + b) - (a), (a + b) - (a) >^{-2} \]

\[ \otimes < (a - b) - (a), (a - b) - (a) > \otimes < (a - b) - (a), (a - b) - (a) >^{-2} \]

\[ \otimes < (b) - (0), (b) - (0) >^{-2} \]

We have

\[ < (a + b) - (b), (a + b) - (b) > \otimes < (a) - (0), (a) - (0) >^{-1} = 1 \in k^* \quad (44) \]
Indeed, the left hand side is a regular function in $b$ on the elliptic curve and so it is a constant; its value at $b = 0$ is 1.

Therefore the first, third and last terms of the expression above that we need to compute cancel thanks to (44) and we get:

$$< (a + b) + (a - b) - 2(a), (a) - (0) >^2$$

Notice that $(a + b) + (a - b) - 2(a)$ is the divisor of the function $\Delta^{-1/6}(P(\xi - a) - P(b))$. Its value at the point 0 is $\Delta^{-1/6}(P(a) - P(b))$ and its generalized value at the point $a$ is the trivialization we have chosen.

**Corollary 4.5** Assume $k = \bar{k}$. Then the homomorphism $\bar{R}$ is surjective.

Let us denote by $R_2(E)$ the kernel of the homomorphism $\bar{R}$. Then

$$B_2(E) := \frac{\mathbb{Z}[E(k) \setminus 0]}{R_2(E)}$$

Let $\bar{R}$ be the subgroup of $[E(k) \setminus 0]$ generated by the elements

$$\{a, b\} := \{a + b\} + \{a - b\} - 2\{a\} - 2\{b\}$$

Notice that $\{a, a\} = \{2a\} - 4\{a\}$ and $\{a, a\} - \{a, -a\} = 2(\{a\} - \{-a\})$.

**Lemma 4.6** For any abelian group $A$ the elements $\{a, b\}$ and $\{a\} - \{-a\}$ generate modulo 2-torsion the kernel of the surjective homomorphism $\mathbb{Z}[A] \rightarrow S^2A \{a\} \rightarrow a \cdot a$.

We will not use this fact later, so a (simple) proof is omitted.

Consider the homomorphism $\bar{R} \rightarrow k^*$ defined by the formulas

$$\{a, b\} \mapsto \Delta^{-1/3}(P(a) - P(b))^2, \quad a \neq b; \quad \{a, a\} \mapsto \Delta^{-1/2}(P'(a))^2, \quad 2a \neq 0$$

and $\{a, a\} \mapsto (\Delta^{-1/3}P''(a))^2$ if $2a = 0$. Thanks to corollary 4.5 this homomorphism is well defined. By definition the subgroup $R_2(E)$ is its kernel.

**Remark.** This is a homomorphism to the multiplicative group $k^*$ of the field $k$ defined via the additive structure of $k$.

6. **A remark on the differential in the complex $B^*(E, 3)$**. The restriction of $\delta_3$ to the subgroup $B_3^*(E)$ can be defined directly, without referring to the group $B_2(E)$ and the homomorphism $h$. A more complicated formula is the price we pay.

Set for general $a_i \in E(k)$

$$\delta_3((\{a_1\} - \{0\})* (\{a_2\} - \{0\})* (\{a_3\} - \{0\})* (\{a_4\} - \{0\})) = \quad (45)$$

$$\frac{|P(a_1 + a_2) - P(a_3 + a_4)||P(a_1 + a_3) - P(a_4)||P(a_1 + a_4) - P(a_3)|}{|P(a_1) - P(a_3 - a_4)||P(a_1 + a_2 + a_3) - P(a_4)||P(a_1 + a_2 + a_4) - P(a_3)|} \otimes 1/2-a_1 + ...$$

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where ... means three other terms obtained by cyclic permutation of indices. Here $P(a) := x(a)$ is the $x$-coordinate of a point $a$.

The expression for (45) is symmetric in $a_1, ..., a_4$, which is not obvious from the formula. Over $\mathbb{C}$ one can rewrite the right hand side of (45) in a more symmetric way using the $\theta$-function:

$$\frac{\theta(a_1 + a_2 + a_3 + a_4)\theta(a_1 + a_2)\theta(a_1 + a_3)\theta(a_1 + a_4)}{\theta(a_1 + a_2 + a_3)\theta(a_1 + a_2 + a_4)\theta(a_1 + a_3 + a_4)\theta(a_1)} \otimes a_1 + ...$$

Morally the differential $\delta_3$ is given by the “formula” $\{a\} \mapsto -\frac{1}{2} \theta(a) \otimes a$ which, unfortunately, makes no sense if we don’t use the group $B_2(E)$. The relation with (45) is given by the classical formula

$$P(a) - P(b) = \frac{\theta(a + b)\theta(a - b)}{\theta^2(a)\theta^2(b)}, \quad a \neq \pm b$$

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