Fluctuations of two-dimensional stochastic heat equation and KPZ equation in subcritical regime for general initial conditions

Shuta Nakajima∗  Makoto Nakashima†

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Abstract

The solution of Kardar-Parisi-Zhang equation (KPZ equation) is solved formally via Cole-Hopf transformation \( h = \log u \), where \( u \) is the solution of multiplicative stochastic heat equation (SHE). In [CD20, CSZ20, G20], they consider the solution of two dimensional KPZ equation via the solution \( u_\varepsilon \) of SHE with flat initial condition and with noise which is mollified in space on scale in \( \varepsilon \) and its strength is weakened as \( \beta_\varepsilon = \hat{\beta} \sqrt{\frac{2\pi}{-\log \varepsilon}} \), and they prove that when \( \hat{\beta} \in (0, 1) \), \( \frac{1}{\beta_\varepsilon}(\log u_\varepsilon - \mathbb{E}[\log u_\varepsilon]) \) converges in distribution to a solution of Edward-Wilkinson model as a random field.

In this paper, we consider a stochastic heat equation \( u_\varepsilon \) with general initial condition \( u_0 \) and its transformation \( F(u_\varepsilon) \) for \( F \) in a class of functions \( \mathfrak{F} \), which contains \( F(x) = x^p \) \((0 < p \leq 1) \) and \( F(x) = \log x \). Then, we prove that \( \frac{1}{\beta_\varepsilon}(F(u_\varepsilon(t, x)) - \mathbb{E}[F(u_\varepsilon(t, x))]) \) converges in distribution to Gaussian random variables jointly in finitely many \( F \in \mathfrak{F}, t, \) and \( u_0 \). In particular, we obtain the fluctuations of solutions of stochastic heat equations and KPZ equations jointly converge to solutions of SPDEs which depends on \( u_0 \).

Our main tools are Itô’s formula, the martingale central limit theorem, and the homogenization argument as in [CNN20]. To this end, we also prove the local limit theorem for the partition function of intermediate 2d-directed polymers.

Keywords: KPZ equation, Stochastic heat equation, Edwards-Wilkinson equation, Local limit theorem for polymers, Stochastic calculus.

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1 Introduction and Main result

∗Department of Mathematics and Computer Science of the University of Basel. shuta.nakajima@unibas.ch
†Graduate School of Mathematics, Nagoya University. nakamako@math.nagoya-u.ac.jp
1 Introduction and Main result

KPZ equation is an SPDE formally given by
\[
\frac{\partial}{\partial t} h(t, x) = \frac{1}{2} \Delta h(t, x) + \frac{1}{2} |\nabla h(t, x)|^2 + \beta \xi(t, x), \tag{1.1}
\]
where $\xi$ is a time-space white noise on $[0, \infty) \times \mathbb{R}^d$. This SPDE is ill-posed due to the non-linear term $\nabla h$ which should be a generalized function.

For $d = 1$, Bertini and Giacomin formulated the solution of (1.1) via Cole-Hopf solution $h = \log u$ [BG97], where $u$ is the solution of stochastic heat equation
\[
\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) + \beta u(t, x) \xi(t, x). \tag{1.2}
\]

In dimension $d = 2$, we consider a space-regularized multiplicative stochastic heat equation but we need to scale the disorder strength:
\[
\frac{\partial u_\varepsilon}{\partial t} = \frac{1}{2} \Delta u_\varepsilon + \beta_\varepsilon u_\varepsilon \xi_\varepsilon, \quad u_\varepsilon(0, x) = u_0(x) \tag{1.3}
\]
where $\beta_\varepsilon = \hat{\beta} \sqrt{-\frac{2\pi}{-\log \varepsilon}}$ with $\hat{\beta} \geq 0$, and $\xi_\varepsilon$ is a mollification in space of $\xi$ such that $\xi_\varepsilon \rightarrow \xi$ as $\varepsilon \rightarrow 0$, i.e.,
\[
\xi_\varepsilon(t, x) = (\xi(t, \cdot) * \phi_\varepsilon)(x) = \int \phi_\varepsilon(x-y) \xi(t, y) dy,
\]
with $\phi_\varepsilon(x) = \varepsilon^{-2} \phi(\varepsilon^{-1} x)$ and $\phi$ being a smooth, non-negative, compactly supported, symmetric function on $\mathbb{R}^2$, so that $\int \phi(x) dx = 1$, and $\phi_\varepsilon$ converges in distribution to the Dirac mass $\delta_0$. Let $h_\varepsilon = \log u_\varepsilon$. Then, we find by Itô’s formula that $h_\varepsilon$ satisfies the SPDE
\[
\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2} \Delta h_\varepsilon + \left[ \frac{1}{2} |\nabla h_\varepsilon|^2 - C_\varepsilon \right] + \beta_\varepsilon \xi_\varepsilon, \quad h_\varepsilon(0, x) = h_0(x), \tag{1.4}
\]
where $C_\varepsilon$ is a diverging parameter:
\[
C_\varepsilon = \frac{\beta_\varepsilon^2 V(0)}{2\pi^2}, \tag{1.5}
\]
where $V(x) = \int_{\mathbb{R}^2} \phi(x-y)\phi(y)dy$. Caravenna, Sun, and Zygouras proved that if the initial condition is flat, that is $h_\varepsilon(0, x) = h_0(x) \equiv 0$ and $\hat{\beta} \in (0, 1)$, then $\beta_\varepsilon^{-1} (h_\varepsilon - \mathbb{E}[h_\varepsilon])$ converges in distribution to the solution to Edward-Wilkinson equation as a random field [CSZ20]. We should remark Chatterjee and Dunlap addressed the tightness of $\beta_\varepsilon^{-1} (h_\varepsilon - \mathbb{E}[h_\varepsilon])$ [CD20] and Gu obtained Edward-Wilkinson limit in $\hat{\beta} \in (0, \beta_0)$ for some $\beta_0 \leq 1$ [G20].

On the other hand, the one-point distribution $h_\varepsilon(t, x)$ converges to a random variable as follows:
Theorem 1.1. [CSZ17b, Theorem 2.15] For any $t > 0$ and $x \in \mathbb{R}^2$, 
\[
h_\varepsilon(t,x) = \begin{cases} \frac{1}{2}X_\beta := \sigma(\beta)Z - \frac{1}{2}\sigma^2(\beta), & 0 \leq \beta < 1 \\ 0, & \beta \geq 1 \end{cases},
\]
where $Z$ is a random variable with standard normal distribution and $\sigma(\beta) = \sqrt{\log \frac{1}{1-\beta}}$.

We will look at the fluctuation of $u_\varepsilon$ for general initial conditions in our main results. Let $\mathcal{C}$ be a set of continuous functions which satisfies
\[
0 < \inf_{x \in \mathbb{R}^2} u_0(x) \leq \sup_{x \in \mathbb{R}^2} u_0(x) < \infty,
\]
or equivalently
\[
\| \log u_0 \|_\infty < \infty.
\]

Let $\mathfrak{F}$ be a set of functions $F \in C^\beta((0, \infty))$ such that there exists a constant $C = C_F > 0$ such that for any $x \in (0, \infty)$
\[
|F'(x)| \leq C(x^{-1} + 1), \quad |F''(x)| \leq C(x^{-2} + 1), \quad |F'''(x)| \leq C(x^{-3} + 1).
\]
Then, $\mathfrak{F}$ contains $x^p$ ($0 < p \leq 1$), log $x$, sin $x$, cos $x$, $e^{-x}$ . . . In this paper, we focus on the fluctuation of $F(u_\varepsilon)$.

Example 1.2. In particular cases, we find that $u_\varepsilon^{(F)} = F(u_\varepsilon)$ satisfies SPDEs.

- If $F(x) = x^p$ ($0 < p \leq 1$), then $u_\varepsilon^{(F)}(0,x) = u_0(x)^p$ and
\[
\partial_t u_\varepsilon^{(F)}(t) = \frac{1}{2}\Delta u_\varepsilon^{(F)} - \frac{(p-1)}{2p} \frac{\|u_\varepsilon^{(F)}\|^2}{u_\varepsilon^{(F)}} + \beta_\varepsilon^2 \frac{\partial^2 V(0)p(p-1)u_\varepsilon^{(F)}}{2\varepsilon^2} + \beta_\varepsilon u_\varepsilon^{(F)} \xi_\varepsilon.
\]

- If $F(x) = \log x$ (KPZ equation), then $u_\varepsilon^{(F)}(0,x) = \log u_0(x)$ and
\[
\partial_t u_\varepsilon^{(F)}(t) = \frac{1}{2}\Delta u_\varepsilon^{(F)} + \frac{1}{2} \|\nabla u_\varepsilon^{(F)}\|^2 - \frac{\beta_\varepsilon^2 V(0)}{2\varepsilon^2} + \beta_\varepsilon \xi_\varepsilon.
\]

We remark that $u_\varepsilon$ is a process indexed by $u_0$ and $\beta$ so we should write $u_\varepsilon = u_\varepsilon^{(\beta,u_0)}$ and $u_\varepsilon^{(F)} = u_\varepsilon^{(F,\beta,u_0)}$. However, we omit $\beta$ and $u_0$ for simplicity of notation when it is clear from the context.

We denote by $C^\infty_\varepsilon$ the set of infinitely differentiable, compactly supported functions on $\mathbb{R}^2$.

Theorem 1.3. Suppose $u^{(1)}_0, \ldots, u^{(n)}_0 \in \mathcal{C}$, $\beta_1, \ldots, \beta_n \in (0,1)$ and $F_1, \ldots, F_n \in \mathfrak{F}$. For $t_1, \ldots, t_n \geq 0$ and $f_1, \ldots, f_n \in C^\infty_\varepsilon$, the following convergence holds jointly as $\varepsilon \to 0$,
\[
\beta_\varepsilon^{-1} \int_{\mathbb{R}^2} f_i(x) \left( u_\varepsilon^{(F_1,\ldots, F_n, \beta_1, \ldots, \beta_n)}(t_i, x) - \mathbb{E} \left[ u_\varepsilon^{(F_1,\ldots, F_n, \beta_1, \ldots, \beta_n)}(t_i, x) \right] \right) \, dx \overset{(d)}{\to} \mathcal{Z}_{t_i}(f_i, F_i, \beta_i, u_0^{(i)}),
\]
where $\left\{ \mathcal{Z}_{t_i}(f_i, F_i, \beta_i, u_0^{(i)}) \right\}_{i=1}^n$ is centered Gaussian random variables with covariance
\[
\frac{1}{1-\beta_1 \beta_2} \int_0^{t_1 \wedge t_2} d\sigma \int dx dy f_i(x)f_j(y) I^{(i)}(x)I^{(j)}(y) \int dz \rho_\sigma(x,z) \rho_\sigma(y,z) \tilde{u}^{(i)}(t_1 - \sigma, z) \tilde{u}^{(j)}(t_2 - \sigma, z),
\]
with
\[
I^{(i,F,\beta,u_0)}(x) = I(x) = \mathbb{E} \left[ F'(e^{X_\beta} \tilde{u}(t,x)) e^{X_\beta} \right] = \mathbb{E} \left[ F'(e^{X_\beta + \sigma^2(\beta) \tilde{u}(t,x)}) e^{X_\beta} \right],
\]
$X_\beta$ is a Gaussian random variable defined in Theorem 1.1, $\rho_\sigma(x,y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}}$ is the heat kernel and $\rho_\sigma(x,y) = \rho_\sigma(x-y)$, and $\tilde{u}(t,x) = \int \rho_\sigma(x,y) u_0(y) dy$. We write $I^{(i)}$ for $I^{(i,F_i,\beta_i,u_0^{(i)})}$ to make notation simple.
Remark 1.4. The centered Gaussian field \( \{ \mathcal{U}_t(f, F, \hat{\beta}, u_0): t \in [0, \infty), F \in \mathcal{F}, \hat{\beta} \in [0, 1), u_0 \in \mathcal{C} \} \) with covariance (1.8) can be constructed explicitly. Let \( \{ \mathcal{U}^{(\hat{\beta}, u_0)}(t, x): (t, x) \in [0, \infty) \times \mathbb{R}^2 \} \) be solutions of the following SPDE: \( \mathcal{U}^{(\hat{\beta}, u_0)}(0, x) \equiv 0 \) and

\[
\partial_t \mathcal{U}^{(\hat{\beta}, u_0)}(t, x) = \frac{1}{2} \Delta \mathcal{U}^{(\hat{\beta}, u_0)}(t, x) + \hat{u}(t, x) \xi_{\hat{\beta}}(t, x),
\]

where \( \xi_{\hat{\beta}} = \sum_{n=0}^{\infty} \hat{\beta} \xi^{(n)} \) with an independent sequence time-space white noises \( \{ \xi^{(n)} \} \) for \( \hat{\beta} \in (0, 1) \). We remark that \( \xi_{\hat{\beta}} \) is a time-space white noise with strength \( \frac{1}{1 - \hat{\beta}^2} \) and \( \mathbb{E} \left[ \xi_{\hat{\beta}}(t, x) \xi_{\hat{\beta}'}(t', x') \right] = \frac{1}{1 - \hat{\beta}^2} \delta_{t, t'} \delta_{x, x'} \). Then, by Duhamel’s principle, \( \mathcal{U}^{(\hat{\beta}, u_0)} \) is given by

\[
\mathcal{U}^{(\hat{\beta}, u_0)}(t, x) = \int_0^t \int_{\mathbb{R}^2} \rho_{t-s}(x, y) \hat{u}^{(i)}(s, y) \xi_{\hat{\beta}}(ds, dy)
\]

and the centered Gaussian field given by

\[
\mathcal{U}_t(f, F, \hat{\beta}, u_0) = \int_{\mathbb{R}^2} dx F^{(i)}(x) f_i(x) \mathcal{U}^{(\hat{\beta}, u_0)}(t, x)
\]

has the covariance structure (1.8).

Remark 1.5. When \( F \) is a power function or the logarithm, the limit \( \mathcal{U}^{(F, u_0, \hat{\beta})} \) is a solution of an SPDE:

- If \( F(x) = x^p \), then \( \mathcal{U}^{(F, u_0, \hat{\beta})}(0, x) \equiv 0 \) and

\[
\partial_t \mathcal{U} = \frac{1}{2} \Delta \mathcal{U} + \frac{p(p-1)}{2} \nabla \log \hat{u} \hat{u} \nabla \mathcal{U} + (1 - p) \nabla \log \hat{u} \nabla \mathcal{U} + \frac{\hat{u}(t, x)}{(1 - \hat{\beta}^2)^{p-1}} \xi(t, x).
\]

- If \( F(x) = \log x \), then \( \mathcal{U}^{(F, u_0, \hat{\beta})}(0, x) \equiv 0 \) and

\[
\partial_t \mathcal{U} = \frac{1}{2} \Delta \mathcal{U} + \nabla \log \hat{u} \nabla \mathcal{U} + \frac{\hat{u}(t, x)}{(1 - \hat{\beta}^2)} \xi(t, x).
\]

From the above remarks, we have the following.

Corollary 1.6. Suppose \( \hat{\beta} < 1 \). As \( p \to 0 \), \( \mathcal{U}_t(f, x^p, \hat{\beta}, u_0) \) converges to \( \mathcal{U}_t(f, x \log x, \hat{\beta}, u_0) \).

Remark 1.7. In [DGRZ20], they study the fluctuations of the transformation \( F(u_\epsilon) \) for higher dimensional case \( d \geq 3 \) with \( F \), its derivative and second derivative growing at most \( x^{-p} + x^p \). They proved that there exists a constant \( \beta_p \) such that the Gaussian fluctuation holds for \( \hat{\beta} \in (0, \beta_p) \). Our assumption on \( F \) is slightly different from theirs but we can show the Gaussian fluctuations for all \( \hat{\beta} \) up to critical point.

To analyze \( u_\epsilon \), we use the Feynman-Kac representation given in [BC95, Section 2] where they considered the case \( d = 1 \) but it is easy to be modified for \( d \geq 2 \):

\[
u_\epsilon(t, x) = \mathbb{E}_x \left[ \exp \left( \beta_\epsilon \int_0^t \int_{\mathbb{R}^2} \phi_\epsilon(B_s - y) \xi(t-s, dy) ds - \frac{\beta_\epsilon^2 t \nu(0)}{2\epsilon^2} \right) u_0(B_t) \right],
\]

where we denote by \( P_x \) and \( \mathbb{E}_x \) the law and the expectation with respect to two dimensional Brownian motion \( B = \{ B_t \}_{t \geq 0} \) starting from \( x \).
Due to the time-reverse invariance and scale invariance of time-space white noise and the scaling invariance of Brownian motion, $\varepsilon B_{t-z} \overset{d}{=} B_\varepsilon \{ u_\varepsilon(t, x) : x \in \mathbb{R}^2 \}$ has the same distribution as

$$E_x \left[ \exp \left( \beta \varepsilon \int_0^t \int_{\mathbb{R}^2} \phi_\varepsilon(B_\varepsilon - y) \dot{\xi}(s, dy) ds - \frac{\beta^2 t V(0)}{2 \varepsilon^2} \right) u_0(B_t) \right]$$

$$= E_x \left[ \exp \left( \beta \int_0^{\varepsilon \beta^2} \int_{\mathbb{R}^2} \phi(B_\varepsilon - y) \dot{\xi}(s, dy) ds - \frac{\beta^2 t V(0)}{2 \varepsilon^2} \right) u_0 \left( \varepsilon B_{\beta^2} \right) \right],$$

(1.10)

where $(B_s)_{s \geq 0}$ is a Brownian motion path and $E_x$ the expectation associated to Brownian motion started at $x \in \mathbb{R}^2$, $\beta \geq 0$. In particular, for the flat initial condition, $u_\varepsilon$ has the same distribution as partition function $Z(\varepsilon \beta)$ of continuum directed polymers, where $Z_t(x)$ is given by

$$Z_t(x) = E_x \left[ \Phi^\beta_t \right]$$

$$= E_x \left[ \exp \left( \beta \int_0^t \int_{\mathbb{R}^2} \phi(B_s - y) \dot{\xi}(s, dy) ds - \frac{\beta^2 t V(0)}{2} \right) \right],$$

where

$$\Phi^\beta_t = \Phi^\beta_t(B, \xi) := \exp \left( \beta \int_0^t \int_{\mathbb{R}^2} \phi(y - B_s) \xi(ds, dy) - \frac{\beta^2 V(0)t}{2} \right)$$

for $t \geq 0$, $\beta \in (0, \infty)$. Thus, we can reduce the problem on the laws of $u_\varepsilon$ to the partition function of continuum directed polymers. Such connections between SHE (and KPZ equation) and directed polymers have been already pointed out in [KPZ86] and used in a lot of researches on SHE and KPZ equation [BC95, BG97, MSZ16, GRZ18, MU17, DGRZ20, CSZ17a, CSZ17b, CSZ19a, CSZ19b, CSZ20, CCM20, CCM19, CNN20, LZ20].

Remark 1.8. We give our contribution to the problem shortly. Edward-Wilkinson type fluctuations for KPZ equation for $d = 2$ have been obtained in [CSZ20] and [G20] with the flat initial condition. On the other hand, we obtain the Gaussian fluctuations for the general initial conditions and multi-dimensional parameters. Also, our proof uses a less technical method, “martingale CLT” (see Theorem 3.3) via Itô’s lemma and homogenization argument., with onerous calculation. In [CSZ20], the problem was reduced to the case in the solution of SHE via approximating $\log u_\varepsilon$ by “$u_\varepsilon - 1$”. In [G20], Gaussian fluctuation was obtained by Malliavin calculus and the second order Poincaré inequality.

Remark 1.9. The Gaussian fluctuations for partition functions [LZ20] and solutions of SHE [MSZ16, GRZ18, DGRZ18b, CNN20] and KPZ equation [MU17, DGRZ20, CNN20] in $d \geq 3$ have been proved as well as two dimensional case, where the disorder strength is given by $\beta \varepsilon^{\frac{d-2}{d}}$ for $d \geq 3$. Also, the Gaussian fluctuations for a nonlinear stochastic heat equation with Gaussian multiplicative noise that is white in time and smooth in space [GL20] and the counterpart for $d = 2$ is stated in [DG20] without detailed proof.

Note: Throughout the paper and if clear from the context, the constant $C$ that appears in successive upper-bounds may take different values.

Organization of the article The main idea of Gaussian fluctuation is the same as in [CNN20]. Section 2 is devoted to proving key properties of partition functions of directed polymers, $L^2$-boundedness, boundedness of negative moments, and local limit theorem. Section 3 is dedicated to the proof of Theorem 1.3. In subsection 3.1, we give a rough proof strategy and explain a heuristic idea of Gaussian fluctuation. The rigorous proof starts from subsection 3.2.

2 Some Estimates for Partition Functions

In this section, we discuss some properties of partition functions of directed polymers in random environment.
Hereafter, we set
\[ T = T_\varepsilon := \varepsilon^{-2}, \]
\[ \beta = \beta_\varepsilon := \hat{\beta} \sqrt{\frac{2\pi}{\log \varepsilon^{-1}}} = \hat{\beta} \sqrt{\frac{4\pi}{\log T}}, \] and
\[ \gamma = \gamma_\varepsilon := \hat{\gamma} \sqrt{\frac{2\pi}{\log \varepsilon^{-1}}} = \hat{\gamma} \sqrt{\frac{4\pi}{\log T}}. \] (2.1)

Throughout the paper, we write the subscript \( \varepsilon \) in \( T_\varepsilon \) and \( \beta_\varepsilon \) in each statement to emphasize its dependence but we often omit the subscript \( \varepsilon \) in the proofs for simplicity.

### 2.1 \( L^p \)-bound of partition functions

First, we remark that for \( x, y \in \mathbb{R}^2 \)
\[
\mathbb{E} \left[ E_x[\Phi_{tT}^\beta] E_y[\Phi_{tT}^\gamma] \right] = 1 + \sum_{n=1}^\infty \beta^n \gamma^n \int_{0<s_1<\cdots<s_n<tT} \prod_{i=1}^n \left( V(\sqrt{2}s_i) \rho_{s_i-s_{i-1}}(x_{i-1},x_i) \right) dsdx \] (2.2)
where \( \mathbb{E} \) and \( \mathbb{P} \) denote the expectations and probability with respect to the white noise \( \xi \) and we set \( x_0 = \frac{x-y}{\sqrt{2}} \), \( s_0 = 0 \) and \( ds = ds_1 \cdots ds_n \), \( dx = dx_1 \cdots dx_n \). This representation is obtained from the general property of the white noise:
\[
\mathbb{E} \left[ \exp \left( \int_0^t \int_{\mathbb{R}^2} f(t,x)\xi(ds,dx) \right) \right] = \exp \left( \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} f(t,x)^2dsdx \right). \] (2.3)

Indeed, we have
\[
\mathbb{E} \left[ E_x[\Phi_{tT}^\beta] E_y[\Phi_{tT}^\gamma] \right] = \mathbb{E}_x \otimes \mathbb{E}_y \left[ \exp \left( \int_0^t (\beta \phi(B_s - y) + \gamma \phi(\hat{B}_s - y))^2 dsdy - \frac{(\beta^2 + \gamma^2)V(0)t}{2} \right) \right]
\]
\[
= \mathbb{E}_x \otimes \mathbb{E}_y \left[ \exp \left( \beta \gamma \int_0^t V(\sqrt{2B_s}) ds \right) \right],
\]
and the Taylor expansion \( e^x = \sum_{n=0}^\infty \frac{x^n}{n!} \) gives (2.2)

**Lemma 2.1.** Suppose \( \hat{\beta}, \hat{\gamma} \in (0,1) \) and fix \( t > 0 \). Then,
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \Phi_{tT}^\beta \Phi_{tT}^\gamma \right] = \frac{1}{1 - \hat{\beta} \hat{\gamma}}, \] (2.4)
\[
\sup_{\varepsilon \leq 1} \sup_{s \leq T_\varepsilon} \mathbb{E} \left[ \Phi_{sT}^\beta \Phi_{sT}^\gamma \right]^2 < \infty, \] (2.5)
where \( \mathbb{P}_{0,y}^{t,y} \) and \( \mathbb{E}_{0,y}^{t,y} \) denote the probability measure and expectation of the Brownian bridge from \((0,0)\) to \((t,y)\) in \( \mathbb{R}^2 \).

**Remark 2.2.** The proof of Theorem 2.15 in [CSZ17b], where (2.4) for \( \beta_\varepsilon = \gamma_\varepsilon \) was proved by reducing the problem to discrete directed polymers in random environment, can be mollified for \( \beta_\varepsilon \neq \gamma_\varepsilon \), but we will give a direct proof in this paper.

**Remark 2.3.** We call
\[
\mathcal{Z}^{t,y}_{0,x} = \mathcal{Z}^{(\beta)}_{0,x:t,y} := \mathbb{E}_{0,x}^{t,y} \left[ \Phi_{t}^\beta \right]
\]
the point-to-point partition function of continuum directed polymers.
Proof of (2.4). We have from (2.2)

\[ E \left[ \Phi_{tT}^\varepsilon \right] E \left[ \Phi_{tT}^\gamma \right] = 1 + \sum_{n=1}^{\infty} \beta^n n \int_{0<s_1<\cdots<s_n<tT} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^n \left( V(\sqrt{2}x_i) \rho_{s_i-s_{i-1}}(x_{i-1}, x_i) \right) dsdx, \]  

(2.6)

with \( x_0 = 0 \). We first consider the upper bound:

\[ \lim_{\varepsilon \to 0} E \left[ \Phi_{tT}^\varepsilon \right] E \left[ \Phi_{tT}^\gamma \right] \leq \frac{1}{1 - \beta \gamma}. \]

Let us consider the function

\[ r_s = \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} V(\sqrt{2}y) \rho_s(x, y) dy \geq 0. \]  

(2.7)

Since \( \int_{\mathbb{R}} \rho_s(x, y) dy = 1 \), \( \sup_{s>0} |r_s| \leq ||V||_\infty \). Moreover, using \( \int V(x) dx = 1 \), we obtain

\[ r_s = \frac{1}{4\pi s} \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} V(y)e^{-\frac{|x-y|^2}{s}} dy \leq \frac{1}{4\pi} \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}} V(y) dy = \frac{1}{4\pi}. \]

(2.8)

Hence, (2.6) is bounded from above by

\[
1 + \sum_{n=1}^{\infty} \beta^n n \left( \int_{0}^{tT} r_s ds \right)^n \\
\leq 1 + \sum_{n=1}^{\infty} \beta^n n \left( \frac{\log(tT)}{4\pi} + ||V||_\infty \right)^n \\
= 1 + \sum_{n=1}^{\infty} \hat{\beta}^n (1 + \frac{\log t + 4\pi ||V||_\infty}{\log T})^n \\
\to \sum_{n=0}^{\infty} (\hat{\beta})^n = \frac{1}{1 - \hat{\beta}},
\]

as \( \varepsilon \to 0 \), where the convergence is absolute since \( \hat{\beta} \gamma < 1 \).

Next, we consider the lower bound. Let

\[ T_n = \{ 0 < s_1 < \cdots < s_n < tT, s_i - s_{i-1} > 1, \forall i \in \{1, \cdots, n\} \}. \]

Then, since each term is non-negative, it is enough to show for fixed \( L \in \mathbb{N} \),

\[ \lim_{\varepsilon \to 0} \left( 1 + \sum_{n=1}^{L} \beta^n n \int_{T_n} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^n \left( V(\sqrt{2}x_i) \rho_{s_i-s_{i-1}}(x_{i-1}, x_i) \right) dsdx \right) \geq \sum_{n=0}^{L} (\hat{\beta})^n. \]  

(2.9)

For fixed \( n \in \mathbb{N} \), we can find that

\[
\int_{T_n} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^n \left( V(\sqrt{2}x_i) \rho_{s_i-s_{i-1}}(x_{i-1}, x_i) \right) dsdx \\
= \int_{T_n} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^n \left( \frac{1}{2\pi(s_i - s_{i-1})} V(\sqrt{2}x_i) - \tilde{r}_{s_i-s_{i-1}}(x_{i-1}, x_i) \right) dsdx \\
= \int_{T_n} \prod_{i=1}^n \frac{1}{4\pi(s_i - s_{i-1})} ds + A^n, \]

(2.10)

where

\[
\tilde{r}_s(x, y) = \frac{1}{2\pi s} V(\sqrt{2}y) - V(\sqrt{2}y) \rho_s(x, y) \\
= \frac{1}{2\pi} \left( 1 - \exp \left( -\frac{|y-x|^2}{2s} \right) \right) V(\sqrt{2}y) \geq 0,
\]
Thus, we have for any fixed $L > C$ with some $\varepsilon$ as \((2.11)\), \((2.12)\), \((2.13)\), \((2.14)\).

\[
A^n_\varepsilon = \sum_{k=1}^n (-1)^k \sum_{j_1 < j_2 < \cdots < j_k} \int T^n \int_{(\mathbb{R}^2)^n} \prod_{i \neq j_1, \cdots, j_k} \frac{V(\sqrt{2}x_i)}{4\pi(s_i - s_{i-1})} \prod_{j=j_1, \cdots, j_k} \bar{r}_{s_j - s_{j-1}}(x_{j-1}, x_j)dsdx.
\]

Let $D_V = \{ x \in \mathbb{R}^d : V(\sqrt{2}x) \neq 0 \}$, which is compact. Note that

\[
\sup_{x \in D_V} \int_{\mathbb{R}^2} \bar{r}_s(x, y)dy = (2\pi s)^{-1} \sup_{x \in D_V} \int_{\mathbb{R}^2} \left(1 - \exp\left(-\frac{|y - x|^2}{2s}\right)\right) V(\sqrt{2}y)dy
\]

\[
\leq (2\pi s)^{-1}|D_V| \sup_{x, y \in D_V} \left(1 - \exp\left(-\frac{|y - x|^2}{2s}\right)\right)
\]

\[
\leq C(s^{-2} \wedge 1),
\]

with some $C = C(V) \geq 1 \vee \|V\|_{\infty}$. In particular, we have

\[
|A^n_\varepsilon| \leq C^{n+1} \sum_{k=1}^n \sum_{j_1 < j_2 < \cdots < j_k} \int T^n \int_{(\mathbb{R}^2)^n} \prod_{i \neq j_1, \cdots, j_k} \frac{ds_i}{s_i - s_{i-1}}
\]

\[
\leq C^{n+1} \sum_{k=1}^n n^k \log(tT)^{n-k}
\]

\[
\leq (Cn)^{n+1} \log(tT)^{n-1}.
\]

Thus, we have for any fixed $L > 0$

\[
\sum_{n=1}^L \beta^n \gamma^n A^n_\varepsilon = \sum_{n=1}^L \beta^n \gamma^n \left(\frac{4\pi}{\log T}\right)^n A^n_\varepsilon \to 0,
\]

\((2.11)\), as $\varepsilon \to 0$. Also,

\[
\int T^n \prod_{i=1}^n \frac{1}{4\pi(s_i - s_{i-1})} ds \geq \left(\int_1^\infty \frac{1}{4\pi s} ds\right)^n = \left(\frac{\log(tT/n)}{4\pi}\right)^n,
\]

and hence we have

\[
\lim_{\varepsilon \to 0} \sum_{n=1}^L \beta^n \gamma^n \int T^n \prod_{i=1}^n \frac{1}{4\pi(s_i - s_{i-1})} ds \geq \sum_{n=1}^L \beta^n \gamma^n.
\]

\((2.12)\).

Then, \((2.10)\), \((2.11)\) and \((2.12)\) yield \((2.9)\).}

\textbf{Proof of \((2.5)\).} We obtain by the same manner as \((2.2)\) that

\[
E\left[E_{0,0}^{T,0}\left[\Phi_T^{\beta}\right]^2\right] = E_{0,0}^{T,0}\left[\exp\left(\beta^2 \int_0^T V(\sqrt{2}B_u)du\right)\right]
\]

\[
= 1 + \sum_{n=1}^\infty \beta^{2n} \int_{0 < s_1 < \cdots < s_n < s} \prod_{i=1}^n V(\sqrt{2}x_i) \rho_{s_i - s_{i-1}}(x_{i-1} - x_i) \rho_{s-s_n}(x_n)\rho_{s_T}(0)dsdx,
\]

\((2.13)\), where we use the orthogonal transformation invariance of Brownian bridges. We have for $s, t > 0$, by Markov property of Brownian motions,

\[
\int_{\mathbb{R}} \rho_s(x, y)\rho_t(y)V(\sqrt{2}y)dy \leq \|V\|_{\infty} \int_{\mathbb{R}} \rho_s(x, y)\rho_t(y)dy \leq \|V\|_{\infty} \rho_{s+t}(x),
\]

\textbf{Proof.} □

\[
\int_{\mathbb{R}} \rho_s(x, y)\rho_t(y)V(\sqrt{2}y)dy \leq \|V\|_{\infty} \int_{\mathbb{R}} \rho_s(x, y)\rho_t(y)dy \leq \|V\|_{\infty} \rho_{s+t}(x),
\]

\textbf{Proof.} □
and by $\int_{\mathbb{R}} V(\sqrt{2}y)dy = 1/2$,

$$
\int_{\mathbb{R}} \rho_s(x, y) \rho_t(y) V(\sqrt{2}y)dy = \frac{1}{4\pi st} \int_{\mathbb{R}} V(\sqrt{2}y) \exp \left( -\frac{|x-y|^2}{2s} - \frac{|y|^2}{2t} \right) dy
$$

$$
= \rho_{s+t}(x) \frac{s + t}{2\pi st} \int_{\mathbb{R}} V(\sqrt{2}y) \exp \left( -\frac{(s + t)|y + s|^2}{2st} \right) dy
$$

$$
\leq \rho_{s+t}(x) \frac{s + t}{4\pi st} \rho_{s+t}(x). \tag{2.14}
$$

Putting things together with $C = 4\pi ||V||_{\infty}$, we have

$$
\int_{\mathbb{R}} \rho_s(x, y) \rho_t(y) V(\sqrt{2}y)dy \leq \frac{1}{4\pi} \left( C \wedge \frac{s + t}{st} \right) \rho_{s+t}(x).
$$

Using this successively, we can bound each term of (2.13) as

$$
\beta^{2n} \int_{0<s_1<\ldots<s_n<tT} \left( \prod_{i=1}^{n} V(\sqrt{2}x_i) \rho_{s_{i-1}-s_i}(x_{i-1}, x_i) \right) \frac{s_{i-1}x_i(x_{i+1})}{\rho_t(0)} dsdx
$$

$$
\leq \left( \frac{\beta^2}{4\pi} \right)^n \int_{0<s_1<\ldots<s_n<tT} \left( \prod_{i=1}^{n} C \wedge \frac{tT-s_{i-1}}{(s_{i-1}-s_i)(tT-s_i)} \right) ds
$$

$$
= \left( \frac{\beta^2}{4\pi} \right)^n \int_{0<s_1<\ldots<s_n<tT} \left( \prod_{i=1}^{n} C \wedge \frac{1}{s_{i-1}-s_i} + \frac{1}{tT-s_i} \right) ds, \tag{2.15}
$$

where we set $s_0 = 0$ and $s_{n+1} = tT$ and we have used $\frac{1}{tT-s_i} + \frac{1}{s_i-s_{i-1}} = \frac{tT-s_{i-1}}{(s_{i-1}-s_i)(tT-s_i)}$ in the last line. We write $\log_+(x) = \log x \vee 0$ and $C_1 = 2C$. We use the following integral estimate: for $s < tT$ and $k \geq 0$,

$$
\int_{s}^{tT} (C_1 + \log_+(tT-t))^k \left( C \wedge \left( \frac{1}{t-s} + \frac{1}{tT-t} \right) \right) dt
$$

$$
\leq 2C(C_1 + \log_+(tT-t))^k + \int_{s}^{tT-1} (C_1 + \log(tT-t))^k \left( \frac{1}{t-s} + \frac{1}{tT-t} \right) dt
$$

$$
\leq C_1(C_1 + \log_+(tT-t))^k + (C_1 + \log_+(tT-t))^k \int_{s+1}^{tT-1} \frac{1}{t-s} dt - (k+1)^{-1} [(C_1 + \log_+(tT-t))^k+1]_{s+1}^{tT-1}
$$

$$
\leq C_1(C_1 + \log_+(tT-t))^k + (C_1 + \log_+(tT-t))^k \log_+(tT-t) + (k+1)^{-1}(C_1 + \log_+(tT-t))^k+1
$$

$$
= \frac{k+2}{k+1}(C_1 + \log_+(tT-t))^{k+1}.
$$

Using this, (2.15) can be successively bounded from above as

$$
\int_{0<s_1<\ldots<s_n<tT} \left( \prod_{i=1}^{n} C \wedge \left( \frac{1}{s_{i-1}-s_i} + \frac{1}{tT-s_i} \right) \right) ds \leq (n+1)(C_1 + \log tT)^n.
$$

Together with (2.13) and (2.15), using $\beta = \sqrt[4\pi]{\log T}$ with $\hat{\beta} < 1$, we have

$$
\lim_{\epsilon \to 0} \mathbb{E} \left[ \Phi_{\epsilon T}^{0,0} \left[ \Phi_{\epsilon T}^{0,0} \right]^{1/2} \right] \leq \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} \left( \frac{\beta^2}{4\pi} \right)^n (n+1)(C_1 + \log tT)^n = \sum_{n=0}^{\infty} (n+1)\beta^{2n} < \infty.
$$
Lemma 2.4. [CSZ20, (5.11)] Fix \( \hat{\beta} \in (0, 1) \). Then, there exists \( p_{\hat{\beta}} > 2 \) such that for any \( 2 \leq p < p_{\hat{\beta}} \) and for \( t \geq 0 \)
\[
\lim_{\epsilon \to 0} \mathbb{E} \left[ E_x \left[ \Phi_{\ell T}^\beta \right]^p \right] < \infty.
\]

Lemma 2.5. Suppose \( \hat{\beta} \in (0, 1) \) and fix \( t > 0 \). Then,
\[
\sup \sup \sup \mathbb{E}^{s, \pi}_{0,0} \left[ \exp \left( \beta^2 \int_0^s V(\sqrt{2}B_u)du \right) \right] < \infty. \tag{2.16}
\]

Proof. By (2.5), (2.13), for \( s \leq tT \),
\[
\mathbb{E}^{s, \pi}_{0,0} \left[ e^{\beta^2 \int_0^s V(\sqrt{2}B_u)du} \right] = \mathbb{E} \left[ \mathbb{E}^{s, \pi}_{0,0}[\Phi_{\pi s}^\beta | \Phi_{0,0}^\beta, \Phi_{s,0}^\beta] \right] \\
\leq \mathbb{E} \left[ \mathbb{E}^{s,0}[\Phi_{s,0}^\beta \Phi_{0,0}^\beta] \right] = \mathbb{E}^{s,0}_{0,0} \left[ e^{\beta^2 \int_0^s V(\sqrt{2}B_u)du} \right] \\
\leq \sup_{\epsilon \leq 1} \left[ \exp \left( \beta^2 \int_{\epsilon T}^t V(\sqrt{2}B_u)du \right) \right] < \infty,
\]
where we have used the remark below.

Remark 2.6. By the shear invariance of environment, we have that
\[
\mathbb{E}^{t, y}_{0, x} \left[ \Phi_{\ell T}^\beta \right] = \mathbb{E}^{t, 0}_{0, 0} \left[ \Phi_{\ell T}^\beta \right]
\]
for any \( t > 0 \) and \( x, y \in \mathbb{R}^2 \).

We end this subsection by presenting the boundedness of negative moments of partition functions:

Lemma 2.7. [CSZ20, (5.12), (5.13), (5.14)] Let \( \hat{\beta} \in (0, 1) \) and fix \( t > 0 \). For any \( p \geq 0 \) and \( x \in \mathbb{R}^2 \),
\[
\sup_{s \in [0, t]} \mathbb{E} \left[ \left( \tilde{Z}_{sT_e}^\beta (x) \right)^{-p} \right] < \infty.
\]

2.2 Local limit theorem

In this subsection, we give an estimate of local limit theorem for partition functions.

To describe the statement, we introduce the time-reversed partition function of time horizon \( \ell \), \( \tilde{Z}_{\ell, T_e}^\beta (z) : 
\[
\tilde{Z}_{\ell, T_e}^\beta (z) = \mathbb{E}_x \left[ \exp \left\{ \beta \int_{t-\ell}^t \int_{\mathbb{R}^2} \phi(B_{t-s} - y) \xi(ds, dy) - \frac{\beta^2 V(0) \ell}{2} \right\} \right].
\]

Theorem 2.8 (Local limit theorem for polymers). Fix \( t > 0 \). Let \( 0 < \ell_{T_e}^a < \ell_{T_e}^b < L(T_e) \leq tT_e \) be functions with \( \lim_{\epsilon \to 0} \ell_{T_e}^a \to \infty, \lim_{\epsilon \to 0} \ell_{T_e}^b = 0, \lim_{\epsilon \to 0} \frac{\log L(T_e)}{\log T_e} = 1 \). Then, for all \( \hat{\beta} < 1 \) there exists \( C = C(\hat{\beta}) \) such that for all positive \( \ell > 0 \) verifying \( \ell_{T_e}^a \leq \ell \leq \ell_{T_e}^b \) and for all \( x, y \in \mathbb{R}^d \),
\[
\mathbb{E} \left( \mathbb{E}^{L(T_e), x}_{0,0} \left[ \Phi_{\ell T_e}^\beta \right] - \tilde{Z}_{\ell, T_e}^\beta (0) \tilde{Z}_{\ell, T_e}^\beta (x) \right)^2 \\
\leq \left\{ \begin{array}{ll}
C \ell^2 \left( \log L(T_e) \right) + \frac{|x| \log \ell}{L(T_e)} + \frac{|x| \ell}{L(T_e)} & |x| \leq \sqrt{L(T_e) \log L(T_e)} \\
C |x|^2 & |x| \geq \sqrt{L(T_e) \log L(T_e)}.
\end{array} \right.
\]
Remark 2.9. Theorem states that the point-to-point partition function from $(0, x)$ to $(L(T), y)$ is approximated by the product of partition function from $(0, x)$ with length $\ell$ and time-reversed partition function from $(L(T), y)$ with length $\ell$ in $L^2$-sense. For $d \geq 3$, the reader may refer to [CNN20, S95, V06].

Notation 2.10. Fix $R_V > 0$ such that $\text{supp} V \subset B(0, R_V)$.

The proof is composed of three steps.

Lemma 2.11 (Step 1). Fix $t > 0$. There exists a constant $C = C(\beta) > 0$ such that for $\ell^a_T \leq \ell \leq \ell^b_T$,

$$\sup_{\beta \in \mathbb{R}} \mathbb{E} \left[ \left( E^{L(T), x}_{\ell, 0} \Phi_{L(T), t}^{\beta} - E^{L(T), x}_{\ell, 0} \Phi_{L(T), \ell, L(T)}^{\beta} \right)^2 \right] \leq C\beta^2 \log \frac{L(T)}{\ell},$$

where

$$\Phi_{s, t}^{\beta} = \exp \left( \beta \int_s^t \int_{\mathbb{R}^2} \phi(y - B_u) \xi(du, dy) - \frac{\beta^2 V(0)(t - s)}{2} \right).$$

Proof. Since $B^{(1)}_s - B^{(2)}_s \overset{d}{=} \sqrt{2} B_s$ for two independent Brownian motions, by $1 - e^{-x} \leq x$ for $x \geq 0$, we have

$$\mathbb{E} \left[ \left( E^{L(T), x}_{\ell, 0} \Phi_{L(T), t}^{\beta} - E^{L(T), x}_{\ell, 0} \Phi_{L(T), \ell, L(T)}^{\beta} \right)^2 \right] = \mathbb{E} \left[ E^{L(T), x}_{\ell, 0} \Phi_{L(T), t}^{\beta} \right] - \mathbb{E} \left[ E^{L(T), x}_{\ell, 0} \Phi_{L(T), \ell, L(T)}^{\beta} \right] \leq \mathbb{E} \left[ E^{L(T), x}_{\ell, 0} \Phi_{L(T), t}^{\beta} \right] \leq \mathbb{E} \left[ E^{L(T), x}_{\ell, 0} \Phi_{L(T), \ell, L(T)}^{\beta} \right].$$

The last expectation equals

$$\beta^2 \int_{[\ell, L(T) - \ell] \times \mathbb{R}^2} V(\sqrt{2} z) \frac{\rho_s(z) \rho_{L(T) - s}(z)}{\rho_{L(T)}(0)} \mathbb{E}_{0, 0}^z \left[ e^{\beta^2 \int_0^s V(\sqrt{2} z) du} \right] \mathbb{E}_{0, 0}^z \left[ e^{\beta^2 \int_0^{L(T) - s} V(\sqrt{2} z) du} \right] dz ds$$

$$\leq \beta^2 \left( \sup_{s \leq L(T)} \sup_{|z| \leq R_V} \mathbb{E}_{0, 0}^z \left[ e^{\beta^2 \int_0^s V(\sqrt{2} z) du} \right] \right)^2 \int_{[\ell, L(T) - \ell] \times \mathbb{R}^2} V(\sqrt{2} z) \frac{\rho_s(z) \rho_{L(T) - s}(z)}{\rho_{L(T)}(0)} dz ds,$$

where the supremum on the last line is finite by Lemma 2.5. Finally, by (2.14),

$$\int_{\ell}^{L(T) - \ell} \int_{\mathbb{R}^2} \frac{\rho_s(z) \rho_{L(T) - s}(z)}{\rho_{L(T)}(0)} V(\sqrt{2} z) ds dz \leq \frac{1}{4\pi} \int_{\ell}^{L(T) - \ell} \frac{L(T)}{s(L(T) - s)} ds$$

$$\leq \frac{1}{2\pi} \log \frac{L(T)}{\ell} - \frac{\ell}{\ell} \leq \log \frac{L(T)}{\ell},$$

and the statement of the lemma follows.

By translation invariance of Brownian bridge, Brownian motion and noise, we have

$$E^{L(T), x}_{\ell, 0} \Phi_{L(T), \ell, L(T)}^{\beta} \overset{(d)}{=} E^{L(T), 0}_{\ell, 0} \Phi_{L(T), \ell, L(T)}^{\beta} - E_0 \left[ \Phi_{\ell}^\beta \left( B_s + \frac{x}{L(T)} \right) \right] E_0 \left[ \exp \left( \beta \int_{L(T) - \ell}^{L(T)} \phi \left( B_{L(T) - s} + \frac{(L(T) - s)x}{L(T)} - y \right) \xi(ds, dy) - \frac{\beta^2 V(0)\ell}{2} \right) \right].$$
where \( B + \frac{x}{L(T)} \) is a Brownian motion with drift \( \frac{x}{L(T)} \).

Define

\[
A_{L(T),\ell} := E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \Phi_{L(T)-\ell,L(T)}^\beta \right] - E_0 \left[ \Phi_{\ell}^\beta \right] \tilde{Z}_{L(T),\ell}(0),
\]

\[
B_{L(T),\ell,x} := E_0 \left[ \Phi_{\ell}^\beta \right] - E_0 \left[ \Phi_{\ell}^\beta \left( B + \frac{x}{L(T)} \right) \right].
\]

Lemma 2.12 (Step 2). There exists a constant \( C = C(\beta) \) such that for all positive \( \ell > 0 \) with \( \ell_{a_s} \leq \ell \leq \ell_{b_s} \),

\[
\mathbb{E} \left[ A_{L(T),\ell}^2 \right] \leq C \frac{\ell}{L(T)}. \]

Proof. (2.3) yields that

\[
\mathbb{E} \left[ E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \Phi_{L(T)-\ell,L(T)}^\beta \right] \right] = E_{0,0}^{L(T),0} \left[ \exp \left( \beta^2 \int_0^\ell V(B_s - \tilde{B}_s)ds + \beta^2 \int_{L(T)-\ell}^{L(T)} V(B_s - \tilde{B}_s)ds \right) \right]
\]

\[
= \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy \rho_L(x) \rho_L(y) \frac{\rho_L(y-x)}{\rho_L(T)(0)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dz dw \rho_L(z) \rho_L(w) \frac{\rho_L(z-w)}{\rho_L(T)(0)}
\]

\[
\times E_{0,0}^{L(T),0} \left[ \exp \left( \beta^2 \int_0^\ell V(B_s - \tilde{B}_s)ds \right) \right],
\]

and

\[
\mathbb{E} \left[ E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \Phi_{L(T)-\ell,L(T)}^\beta \right] \right] = E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \right] \tilde{Z}_{L(T),\ell}(0)
\]

\[
= \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy \rho_L(x) \rho_L(y) \frac{\rho_L(y-x)}{\rho_L(T)(0)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dz dw \rho_L(z) \rho_L(w)
\]

\[
\times E_{0,0}^{L(T),0} \left[ \exp \left( \beta^2 \int_0^\ell V(B_s - \tilde{B}_s)ds \right) \right],
\]

where \( B = \{ B_u^x : 0 \leq u \leq \ell \} \) and \( \tilde{B} = \{ \tilde{B}_s : 0 \leq s \leq \ell \} \) are independent Brownian bridges with the law \( P_{0,0}^{\ell,u} \) (\( u = x, y, z, w \)). Then, it is easy to see that

\[
\mathbb{E} \left[ E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \Phi_{L(T)-\ell,L(T)}^\beta \right] \right]^2 - \mathbb{E} \left[ E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \Phi_{L(T)-\ell,L(T)}^\beta \right] \right] \mathbb{E}_0 \left[ \Phi_{\ell}^\beta \right] \tilde{Z}_{L(T),\ell}(0)
\]

\[
= \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy \rho_L(x) \rho_L(y) \frac{\rho_L(y-x)}{\rho_L(T)(0)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dz dw \rho_L(z) \rho_L(w)
\]

\[
\times E_{0,0}^{L(T),0} \left[ \exp \left( \beta^2 \int_0^\ell V(B_s - \tilde{B}_s)ds \right) \right] E_{0,0}^{L(T),0} \left[ \exp \left( \beta^2 \int_0^\ell V(B_s - \tilde{B}_s)ds \right) \right].
\]

Combining with (2.16),

\[
\left| \mathbb{E} \left[ E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \Phi_{L(T)-\ell,L(T)}^\beta \right] \right]^2 \right| - \mathbb{E} \left[ E_{0,0}^{L(T),0} \left[ \Phi_{\ell}^\beta \Phi_{L(T)-\ell,L(T)}^\beta \right] \right] \mathbb{E}_0 \left[ \Phi_{\ell}^\beta \right] \tilde{Z}_{L(T),\ell}(0) \right| \leq C \frac{\ell}{L(T)}.
\]

Also, the same argument holds for \( \mathbb{E} \left[ E_0 \left[ \Phi_{\ell}^\beta \right] \tilde{Z}_{L(T),\ell}(0)^2 \right] \).
Lemma 2.13 (Step 3). Fix $t > 0$. There exists a positive constant $C$ such that for all $\beta < 1$ there exists a positive constant $C = C(\beta)$ such that for all positive $\ell > 0$ with $\ell_{T_2} \leq \ell \leq \ell_{T_2}$ and all $x \in \mathbb{R}^d$,

$$\mathbb{E} \left[ B_{L(T),\ell,x}^2 \right] \leq \begin{cases} \frac{C\beta^2}{2} \left( \frac{|x| \log \frac{\ell}{L(T)}}{L(T)} + \frac{|x|^2 \ell}{L(T)} \right) & |x| \leq \sqrt{L(T)} \log L(T) \\ \frac{1}{C} & |x| \geq \sqrt{L(T)} \log L(T) \end{cases}.$$

Proof. For $|x| \geq \sqrt{L(T)} \log L(T)$, it is trivial from (2.4).

Combining (2.3) and transformation of Brownian motions yield that

$$\mathbb{E} \left[ B_{L(T),\ell,x}^2 \right] = 2E \left[ \exp \left( \beta^2 \int_0^\ell V(\sqrt{2}B_s)ds \right) - \exp \left( \beta^2 \int_0^\ell V \left( \sqrt{2}B_s + \frac{x}{L(T)} \right)ds \right) \right]$$

$$= 2 \sum_{n=1}^\infty \beta^{2n} \int_0^{t_n} \int_{\mathbb{R}^2} V(\sqrt{2}z_i) dsdx \prod_{i=1}^n \left( \prod_{s_i-s_{i-1}} (x_i-x_{i-1}) - \prod_{s_i-s_{i-1}} (x_i-x_{i-1}) - \prod_{s_i-s_{i-1}} (x_i-x_{i-1}) + \frac{x(s_i-s_{i-1})}{L(T)} \right),$$

where we set $x_0 = 0$. When we use the relation

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{j=1}^{n-1} \left( \prod_{i=1}^j b_i \right) \left( \prod_{k=j+1}^n a_k \right),$$

and recall the notation $r_s$ from (2.7), we have

$$\mathbb{E} \left[ B_{L(T),\ell,x}^2 \right] \leq 2 \sup_{z \in B(0,R\sqrt{\ell})} \beta^2 \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \left| \rho_s(y-z) - \rho_s \left( y - z + \frac{x}{L(T)} \right) \right| dyds$$

$$\times \sum_{n=1}^\infty \beta^{2n-2} \left( \sup_{z \in B(0,R\sqrt{\ell})} \beta^2 \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \rho_s(y-z) dyds \right)^{n-1}$$

$$\leq 2 \sum_{n=1}^\infty \beta^{2(n-1)} \left( \int_0^\ell r_s ds \right)^{n-1} \sup_{z \in B(0,R\sqrt{\ell})} \beta^2 \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \left| \rho_s(y-z) - \rho_s \left( y - z + \frac{x}{L(T)} \right) \right| dyds$$

$$\leq C \beta^2 \sup_{z \in B(0,R\sqrt{\ell})} \int_0^\ell \int_{\mathbb{R}^2} V(\sqrt{2}y) \left| \rho_s(y-z) - \rho_s \left( y - z + \frac{x}{L(T)} \right) \right| dyds,$$  \hspace{1cm} (2.17)

where we have used the estimate (2.8) in the last line. Also, we have that for $y,z \in B(0,R\sqrt{\ell})$, and for $s > 0$

$$\left| \rho_s(y-z) - \rho_s \left( y - z + \frac{x}{L(T)} \right) \right|$$

$$= \rho_s(y-z) \left| 1 - \exp \left( -\frac{\langle y-z, x \rangle}{L(T)} - \frac{|x|^2 s}{2L(T)^2} \right) \right|$$

$$\leq \rho_s(y-z) \left( R\sqrt{\ell} \frac{|x|}{L(T)} \exp \left( \frac{2R\sqrt{\ell} |x|}{L(T)} - \frac{|x|^2 s}{2L(T)^2} \right) + \frac{R\sqrt{\ell} |x|}{L(T)} + \frac{|x|^2 s}{2L(T)^2} \right)$$

$$\leq C \rho_s(y-z) \left( \frac{|x|}{L(T)} + \frac{|x|^2 s}{2L(T)^2} \right)$$

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where we denote by \((x, y)\) the inner product of \(x\) and \(y \in \mathbb{R}^2\) and we use \(e^x - 1 \leq xe^x\) if \(x \geq 0\) and \(1 - e^x \leq -x\) if \(x < 0\) in the last line. For \(|x| \leq \sqrt{L(T) \log L(T)}\),

\[
\mathbb{E} \left[ B_{L(T), t, x}^2 \right] \leq C \beta^2 \left( \frac{|x| \log \ell}{L(T)} + \frac{|x|^2 \ell}{L(T)^2} \right).
\]

Putting things together, we conclude the proof of Theorem 2.8.

We also use the following lemma later.

**Lemma 2.14.** For fixed \(t > 0\) and \(\beta \in (0, 1)\), there exists a constant \(C = C_{\beta, t}\) such that for \(x \in \mathbb{R}^2\) and for \(1 \leq \ell \leq tT_x\) with \(t > 0\),

\[
\mathbb{E} \left[ \left( E_x \left[ \Phi_x^{\beta_+} \right] - E_0 \left[ \Phi_x^{\beta_+} \right] \right)^2 \right] \leq \begin{cases} C \beta^2 (1 + |x|^2) & |x| \leq \sqrt{\log \ell} \\ C & |x| > \sqrt{\log \ell} \end{cases} \tag{2.18}
\]

**Proof.** For \(|x| \geq \sqrt{\log \ell}\), it is trivial from (2.4). We suppose \(|x| < \sqrt{\log \ell}\). Using the same argument as in (2.17), with the convention \(x_0 = 0\),

\[
\mathbb{E} \left[ \left( E_x \left[ \Phi_x^{\beta_+} \right] - E_0 \left[ \Phi_x^{\beta_+} \right] \right)^2 \right] = E \left[ \exp \left( \beta^2 \int_0^\ell V(\sqrt{2}B_s) ds \right) - \exp \left( \beta^2 \int_0^\ell V(x + \sqrt{2}B_s) ds \right) \right]
\]

\[
= \sum_{n=1}^{\infty} \beta^{2n} \int_{0 < t_1 < \cdots < t_n < \ell} ds dx \prod_{i=1}^{n} V(\sqrt{2}x_i)
\]

\[
\times \left( \prod_{i=1}^{n} \rho_{s_i - s_{i-1}}(x_i - x_{i-1}) - \rho_{s_{i-1} - s_i}(x_i - x_{i-1}) \prod_{i=2}^{n} \rho_{s_i - s_{i-1}}(x_i - x_{i-1}) \right)
\]

\[
\leq C \beta^2 \left( 1 + \sup_{z \in B(0, R_V)} \int_1^{\ell} ds \int_{\mathbb{R}^2} V(\sqrt{2}y) |\rho_s(y - z) - \rho_s(y - z + x)| dy ds \right).
\]

Also, we have that for \(y, z \in B(0, R_V)\) and \(s \geq 1\),

\[
|\rho_s(y - z) - \rho_s(y - z + x)| = \rho_s(y - z) \left| 1 - \exp \left( -\frac{2(y - z, x) + |x|^2}{2s} \right) \right|
\]

\[
\leq C \rho_s(y - z) \left( \exp \left( \frac{R_V^2}{s} \right) + |x| + |x|^2 \right)
\]

\[
\leq C \rho_s(y - z) \left( 1 + |x|^2 \right).
\]

Thus, we have

\[
\mathbb{E} \left[ \left( E_x \left[ \Phi_x^{\beta_+} \right] - E_0 \left[ \Phi_x^{\beta_+} \right] \right)^2 \right] \leq C \beta^2 (1 + |x|^2).
\]
3 Proofs of Theorem 1.3

For fixed \( t > 0 \) and for \( u_0 \in \mathcal{C} \), let us define the martingale

\[
s \to \mathcal{W}_s(x) = \mathcal{W}_s^{(t,T;\beta,u_0)}(x) = \mathbb{E}_x \left[ \Phi_s^\beta(B) u_0 \left( \frac{B_{Tt}}{\sqrt{T}} \right) \right]
\]

with respect to the filtration \( \{ \mathcal{F}_s : 0 \leq s \leq tT \} \) associated to the white noise \( \xi \). Then, it follows from Feynman-Kac formula (see (1.10)) that for each \((t, x) \in [0, \infty) \times \mathbb{R}^2\)

\[
u^{(t,T;\beta,u_0)}(x) \overset{d}{=} \mathcal{W}^{(t,T;\beta,u_0)}(\sqrt{T}x).
\]

(3.1)

We omit some superscripts \( t, T, \hat{\beta} \), and \( u_0 \) to make notation simple for several notations when it is easily understood from the context.

Since both \( \|u_0^{-1}\|_\infty \) and \( \|u_0\|_\infty \) are finite,

\[
\|u_0^{-1}\|_\infty^{-1} \mathcal{Z}_s(x) \leq \mathcal{W}_s(x) \leq \|u_0\|_\infty \mathcal{Z}_s(x).
\]

Hereafter, we use this without any comment.

Itô's formula yields that for each \( x \in \mathbb{R}^2 \)

\[
\mathcal{W}_s^{(t,T;\beta,u_0)}(x) = \bar{u}(t, x) + \int_0^s d\mathcal{W}_u^{(t,T;\beta,u_0)}(x)
\]

(3.2)

\[
\mathcal{W}_s^{(t,T;\tilde{\beta},v_0)}(x) = \tilde{v}(t, x) + \int_0^s d\mathcal{W}_u^{(t,T;\tilde{\beta},v_0)}(x)
\]

(3.3)

with

\[
\langle \mathcal{W}^{(\beta,u_0)}(x), \mathcal{W}^{(\gamma,v_0)}(y) \rangle_s = \int_0^s \beta \gamma \mathbb{E}_x \otimes \mathbb{E}_y \left[ V(B_u - \tilde{B}_u) \Phi_u^\beta(B) \Phi_u^\gamma(\tilde{B}) u_0 \left( \frac{B_{Tt}}{\sqrt{T}} \right) \right] du
\]

(3.4)

for each \( x, y \in \mathbb{R}^2 \), where \( \mathbb{E}_x \otimes \mathbb{E}_y \) denotes the expectation in two independent Brownian motions \( B \) and \( \tilde{B} \) starting from \( x \) and \( y \).

Then, we find by Itô's formula that for \( F \in \mathcal{G} \), \( F(\mathcal{W}_s(x)) \) has the following semimartingale representation

\[
F(\mathcal{W}_s^{(\beta,u_0)}(x)) = F(\bar{u}(t, x)) + \int_0^s F'(\mathcal{W}_u^{(\beta,u_0)}(x)) d\mathcal{W}_u^{(\beta,u_0)}(x)
\]

\[
+ \frac{1}{2} \int_0^s F''(\mathcal{W}_u^{(\beta,u_0)}(x)) d\langle \mathcal{W}^{(\beta,u_0)}(x) \rangle_u
\]

(3.5)

and we denote by

\[
G_s^{(t,T,F;\beta,u_0)}(x) = G_s(x) = \int_0^s F'(\mathcal{W}_u^{(\beta,u_0)}(x)) d\mathcal{W}_u^{(\beta,u_0)}(x)
\]

\[
H_s^{(t,T,F;\beta,u_0)}(x) = H_s(x) = \int_0^s F''(\mathcal{W}_u^{(\beta,u_0)}(x)) d\langle \mathcal{W}^{(\beta,u_0)}(x) \rangle_u
\]

First, we will prove the fluctuations of martingale parts converge to centered Gaussian random variables.

**Proposition 3.1.** Suppose \( u_0^{(1)}, \ldots, u_0^{(n)} \in \mathcal{C}, \hat{\beta}^{(1)}, \ldots, \hat{\beta}^{(n)} \in (0, 1) \) and \( F_1, \ldots, F_n \in \mathcal{G} \).

For any test function \( f_1, \ldots, f_n \in C_c^\infty(\mathbb{R}^2) \), as \( T \to \infty \)

\[
\left\{ \frac{1}{\beta^{(i)}} \int_{\mathbb{R}^2} f_i(x) G_{F_1,F_2}^{(F_1^{(i)},u_0^{(i)})}(\sqrt{T}x) dx \right\}_{i=1, \ldots, n} \overset{d}{\to} \left\{ \mathcal{W}(t, f_i, F_1, \hat{\beta}^{(i)}, u_0^{(i)}) \right\}_{i=1, \ldots, n}
\]

(3.6)
where \( \{ \mathcal{W}(t, f_i, F_i, \tilde{\beta}^{(i)}_0, u_0^{(i)}) \}_{i=1, \ldots, n} \) are Gaussian random variables with zero means and covariance
\[
\text{Cov} \left( \mathcal{W}(t, f_i, F_i, \tilde{\beta}^{(i)}_0, u_0^{(i)}), \mathcal{W}(t, f_j, F_j, \tilde{\beta}^{(j)}_0, u_0^{(j)}) \right) = \frac{1}{1 - \tilde{\beta}^{(i)}_0 \tilde{\beta}^{(j)}_0} \int_0^t \int_{\mathbb{R}^2} dx dy f_i(x) f_j(y) I_{(t, F_i, \tilde{\beta}^{(i)}_0, u_0^{(i)})}(x) I_{(t, F_j, \tilde{\beta}^{(j)}_0, u_0^{(j)})}(y) \times \int_{\mathbb{R}^2} dz \rho_\sigma(x, z) \rho_\sigma(y, z) \bar{u}^{(i)}(t - \sigma, z) \bar{u}^{(j)}(t - \sigma, z).
\]

Then, we will prove that the Itô correction term can be neglected in the limit:

**Proposition 3.2.** For any \( t > 0, \tilde{\beta} \in (0, 1), u_0 \in \mathcal{C} \) and \( F \in \mathcal{F} \), as \( \varepsilon \to 0 \),
\[
\frac{1}{\beta \varepsilon} \int_{\mathbb{R}^2} f(x) \left( H^{(F)}_{T, \varepsilon}(\sqrt{T \varepsilon} x) - \mathbb{E} \left[ H^{(F)}_{T, \varepsilon}(\sqrt{T \varepsilon} x) \right] \right) \overset{L_1}{\to} 0.
\]

Proposition 3.1 and Proposition 3.2 combined with (3.1) and (3.5) imply Theorem 1.3 for 1-dimensional in time. Thus, Gaussian limit comes from the martingale part of \( \int f(x) F(\mathcal{W}(\sqrt{T \varepsilon})) dx \).

### 3.1 Proof of Proposition 3.1 and heuristics

In the following, we give a heuristic idea of the proof of Proposition 3.1.

First, we introduce the key theorem to prove the convergence of martingale to Gaussian process in this paper:

**Theorem 3.3.** ([JS87, Theorem 3.11 in Chap. 8], [EK86, Theorem 1.4 in Chap. 7])

For each \( n \geq 1 \), let \( \mathcal{F}^n = \{ F^n_t : t \geq 0 \} \) be a filtration and let \( X^{(n)} = (X^{(n, d)}_{1}, \ldots, X^{(n, d)}_{d}) \) be an \( \mathbb{R}^d \)-valued continuous \( \mathcal{F}^n \)-martingale with \( X^n_0 = 0 \). Suppose that there exists a \( d \times d \) positive definite matrix-valued continuous function \( c = (c_{ij}(t))_{i,j=1}^d \) such that for each \( t \geq 0 \), \( (X^{(n, i)}, X^{(n, j)})_t \to c_{ij}(t) \) in probability. Then, \( X^{(n)} \overset{(d)}{\to} X \), where \( X = (X^{(1)}_{1}, \ldots, X^{(d)}_{d}) \) is an \( \mathbb{R}^d \)-valued Gaussian process with \( (X^{(i)}, X^{(j)})_t = c_{ij}(t) \).

**Remark 3.4.** Theorem 3.3 is simplified from the original one for our convenience.

Thus, we will focus our analysis on the cross-variation of martingales.

By the local limit theorem (Theorem 2.8), we may expect that for large \( s \)
\[
\mathcal{W}_s(x) = \mathbb{E}_x \left[ \Phi^\beta \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] = \int_{\mathbb{R}^2} \rho_s(z - x) E_{0, z \Phi^\beta} \left[ u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] dz \approx \int_{\mathbb{R}^2} \rho_s(z - x) Z_{s, T}(x) \mathcal{Z}_{s, T}(z) E_z \left[ u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] dz,
\]

where for fixed \( \delta \in (0, \frac{1}{100}) \), we set
\[
\ell(T) = \exp \left( - (\log T)^{\tilde{\beta} - \delta} \right).
\]

Moreover, we may expect that the last term is approximated in some sense by
\[
Z_{s, T}(x) \int_{\mathbb{R}^2} \rho_s(z - x) E_z \left[ \mathcal{Z}_{s, T}(z) \right] E_z \left[ u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] dz = Z_{s, T}(x) \bar{u}(t, T^{-\frac{1}{2}} x)
\]
since Lemma 2.14 may imply that \((\overline{Z}_{s,t}(t,x))_{x\in\mathbb{R}^2}\) are asymptotically independent and homogenization occurs. Therefore, one may observe for \(F \in \mathcal{F}\) that for large \(s\)

\[
F'(W_s(x))dW_s(x) = \beta F'(W_s(x)) \int_{\mathbb{R}^2} \xi(ds, db)E_x \left[ \phi(B_s - b) \Phi^\beta_s u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] \\
= \beta F'(W_s(x)) \int_{\mathbb{R}^2} \xi(ds, db) \int_{\mathbb{R}^2} \rho_s(z - x) \phi(z - b)E^{s,z}_{0,x} \left[ \Phi^\beta_s \right] E_z \left[ u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] dz \\
\approx \beta F'(Z^\beta_{s,T}(x)\tilde{\bar{u}}(t, T-\frac{4}{\beta} x))Z^\beta_{s,T}(x) \\
\times \int_{\mathbb{R}^2} \xi(ds, db) \int_{\mathbb{R}^2} \rho_s(z - x) \phi(z - b)\overline{Z}^\beta_{s,s\ell}(T) \tilde{E}_z \left[ u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] dz, \\
\]

where we have used the local limit theorem in the third line. We denote by

\[
I_s^{(T)}(x) = I_s^{(t,T,F;\beta,u_0)}(x) = F'(Z^\beta_{s,T}(\sqrt{T}x)\tilde{\bar{u}}(t, x)) \overline{Z}^\beta_{s,T}(\sqrt{T}x).
\]

Also, we have for \(F_1, F_2 \in \mathcal{F}\) that for \(s = T\sigma\) and \(x = \sqrt{T}x'\) and \(y = \sqrt{T}y'\),

\[
F'_1(W_s^{\beta, u_0}(x))F'_2(W_s^{\gamma, u_0}(y))d(W_s^{\beta, u_0}(x), W_s^{\gamma, u_0}(y)) \\
= \int_{\mathbb{R}^2} d\sigma \beta \gamma F'_1(W_s^{\beta, u_0}(x))F'_2(W_s^{\gamma, u_0}(y)) \\
\times E_z \left[ u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] E_z \left[ E_{\tilde{\bar{u}}} \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] \\
\approx \int_{\mathbb{R}^2} d\sigma \beta \gamma F'_s \left( Z^\beta_{s,T}(x)\tilde{\bar{u}}(t, x') \right) F'_s \left( Z^\gamma_{s,T}(y)\tilde{\bar{v}}(t, y') \right) Z^\beta_{s,T}(x)Z^\gamma_{s,T}(y) \\
\times \int_{\mathbb{R}^2} d\sigma \rho_s(z - x) \rho_s(z - y) \overline{Z}^\beta_{s,s\ell}(T) \tilde{E}_z \left[ u_0 \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] E_z \left[ E_{\tilde{\bar{u}}} \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] \\
\approx \int_{\mathbb{R}^2} d\sigma \rho_s(z - x) \rho_s(z - y) \overline{Z}^\gamma_{T\sigma,T\sigma\ell}(T) \tilde{E}_z \left[ E_{\tilde{\bar{u}}} \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] \tilde{E}_z \left[ E_{\tilde{\bar{v}}} \left( \frac{B_{T-s}}{\sqrt{T}} \right) \right] \\
\times \tilde{\bar{u}}(t - \sigma, z) \tilde{\bar{v}}(t - \sigma, z - \frac{4}{\beta} w) \\
\]

and thus by homogenization, \(\overline{Z}^\beta \overline{Z}^\gamma\) would be replaced by \(E \left[ \overline{Z}^\beta \overline{Z}^\gamma \right]\), and \(F'(Z\tilde{u})Z\) terms would be replaced by its expectation so that the cross variation would be approximated by

\[
d\sigma \beta \gamma E \left[ I^{(t,T,F_1,\beta,u_0)}_s(x') \right] E \left[ I^{(t,T,F_2,\gamma,u_0)}_s(y') \right] \\
\times \int_{\mathbb{R}^2} d\sigma \rho_s(x', z) \rho_s(y', z) \overline{Z}^\gamma_{T\sigma,T\sigma\ell}(T) \tilde{E}_z \left[ \overline{Z}^\beta_{T\sigma,T\sigma\ell}(T) \tilde{E}_z \left[ \tilde{\bar{u}}(t - \sigma, z) \tilde{\bar{v}}(t - \sigma, z) \right. \right. \\
\]

Due to Theorem 1.1 and (2.4), we have

\[
\int_{0}^{T} \int_{\mathbb{R}^2} f(x)g(y)F'_1(W_u^{\beta}(\sqrt{T}x))F'_2(W_u^{\gamma}(\sqrt{T}y))d(W_u^{\beta}(\sqrt{T}x), W_u^{\gamma}(\sqrt{T}y))u \\
\approx \frac{1}{1 - \beta^2} \left[ F'(e^{X_{t-s} - \frac{4}{\beta^2} X_{s\ell}}(\tilde{\bar{u}}(t, x)))E \left[ F'(e^{X_{t-s} - \frac{4}{\beta^2} X_{s\ell}}(\tilde{\bar{v}}(t, x)))e^{X_{t-s} - \frac{4}{\beta^2} X_{s\ell}}(\tilde{\bar{v}}(t, x)) \right] \right] \\
\times \int_{0}^{s} d\sigma \int_{\mathbb{R}^2} dxdyf(x)g(y) \int_{\mathbb{R}^2} d\rho_s(x, z) \rho_s(y, z) \tilde{\bar{u}}(t - \sigma, z) \tilde{\bar{v}}(t - \sigma, z) \\
\]

(3.10)
and Theorem 3.3 implies that the limit process is the Gaussian process with covariance function (3.10).

For simplicity of notations in the proof, we will focus on the quadratic variation of \( \int_{\mathbb{R}^2} f(x) e^{(F,\beta,u_0)}(\sqrt{T}x)dx \). The reader can easily recover the proof for the cross-bracket from the above argument.

To make this rough idea rigorous, we introduce a martingale increment \( dM_s(t,T,F,\hat{\beta},u_0) \) for fixed \( t > 0, x \in \mathbb{R}^2, \hat{\beta} \in (0,1), F \in \mathcal{F}, u_0 \in \mathcal{C} \) as

\[
dM_s(x) = dM_s^{(t,T,F,\hat{\beta},u_0)}(x) = \beta \Gamma^\beta(Z^\beta_{s,t}(x) u(t,T^{-\frac{1}{2}}x)) Z^\beta_{s,t}(x)
\times \int_{\mathbb{R}^2} \xi(ds,db) \int_{\mathbb{R}^2} \rho_s(z-x)\phi(z-b) Z^\beta_{s,t}(z) E_x \left[ u_0 \left( \frac{B_{1-s,T}}{\sqrt{T}} \right) \right] dz,
\]

(3.11)

and set

\[
M_s(x) = M_s^{(t,T,F,\hat{\beta},u_0)}(x) := \begin{cases} 
\int_{sT_m(T_\varepsilon)}^s dM_u(x) & s \geq tT_m(T_\varepsilon) \\
0 & 0 \leq s \leq tT_m(T_\varepsilon),
\end{cases}
\]

where

\[
m(T_\varepsilon) = \exp \left( - (\log T_\varepsilon)^{\frac{1}{2}} \right).
\]

The following proposition computes the covariances of \( M_s \):

**Proposition 3.5.** Suppose \( u_0^{(1)}, \cdots, u_0^{(n)} \in \mathcal{C}, \hat{\beta}^{(1)}, \cdots, \hat{\beta}^{(n)} \in (0,1) \) and \( F_1, \cdots, F_n \in \mathcal{F} \).

For any test function \( f_1, \cdots, f_n \in C^\infty_c(\mathbb{R}^2) \), as \( \varepsilon \to 0 \)

\[
\frac{1}{\beta^{(i)}} \int_{\mathbb{R}^2} f_i(x) M_{s,t}^{(i)}(\sqrt{T_\varepsilon}x)dx \xrightarrow{(d)} \mathcal{W}(t,f_i,F_i,\hat{\beta}^{(i)},u_0^{(i)}).
\]

(3.12)

The following proposition states that \( dG_s(x) \) can be replaced by \( dM_s(x) \), which concludes the proof of Proposition 3.1:

**Proposition 3.6.** For any test function \( f \) and \( s > 0, \)

\[
\frac{1}{\beta_s} \mathbb{E} \left[ \left| \int_{\mathbb{R}^2} f(x) \left( G_s^{(t,T,F,\beta,u_0)}(\sqrt{T_\varepsilon}x) - M_s^{(t,T,F,\hat{\beta},u_0)}(\sqrt{T_\varepsilon}x) \right) dx \right] \right] \to 0
\]
as \( \varepsilon \to 0 \).

The proof of Proposition 3.5 is given in the following subsection and the proof of Proposition 3.6 is given in subsection 3.3.

### 3.2 Proof of Proposition 3.5

We will focus on only the quadratic variation of \( G_{sT}(x\sqrt{T}) \) to make the argument simple. Readers can easily replace the quadratic variation by the cross variation.

To prove Proposition 3.5, we will show the following two lemmas:
Lemma 3.7. Let $0 < \tau_0 < \tau \leq t$. Then, as $\varepsilon \to 0$,
\begin{align*}
\frac{1}{\beta^2} \int_{T_{\tau_0}}^{T_\tau} \int_{(\mathbb{R}^2)^2} dxdy f(x)f(y) d\langle M(x_T), M(y_T) \rangle_s ds \\
\leq \int_{T_{\tau_0}}^{T_\tau} ds \int_{(\mathbb{R}^2)^2} dxdy f(x)f(y) I(x)I(y) \int_{\mathbb{R}^2} dz \rho_\sigma(x-z) \rho_\sigma(y-z) \bar{u}(t-\sigma,z)^2,
\end{align*}
where we set $x_{T_\tau} = x\sqrt{T_\tau}$ for $x \in \mathbb{R}^2$ and
\begin{align*}
I(x) = I^{(t,F,\hat{\beta},u_0)}(x) := \mathbb{E} \left[ F' \left( e^{X_\beta \bar{u}(t,x)} \right) e^{X_\beta} \right] = \mathbb{E} \left[ F' \left( e^{X_\beta + \sigma^2(\hat{\beta}) \bar{u}(t,x)} \right) \right]
\end{align*}
for $t > 0$, $x \in \mathbb{R}^2$, $F \in \mathfrak{F}$, $\hat{\beta} \in (0,1)$, and $u_0 \in \mathcal{C}$.

Lemma 3.7 with Theorem 3.3 implies that the centered martingale
\( \left( \int_{\mathbb{R}^2} f(x) \left( M_{T,\tau}(x_T) - M_{T,\tau_0}(x_T) \right) dx \right)_{\tau_0 \leq \tau \leq t} \)
converges in distribution to a Gaussian process with covariance given by the RHS of (3.13)

\begin{align*}
\text{Lemma 3.8.} \\
\lim_{\tau_0 \to 0} \lim_{\varepsilon \to 0} \mathbb{E} \left[ \frac{1}{\beta^2} \left( \int f(x) M_{T,\tau_0}(x_T) dx \right)^2 \right] = 0.
\end{align*}

Thus, letting $\tau_0 \to 0$ and $\tau = t$, the RHS of (3.13) is exactly the covariance function of the Gaussian process $\mathfrak{U}_t(f,F,\hat{\beta},u_0)$.

3.2.1 Proof of Lemma 3.7 and Lemma 3.8

The proof of Lemma 3.7 is divided into several steps.

Recall that $x_T = \sqrt{T} x$. First of all, we can easily find by Markov property and (3.4) that
\begin{align*}
\frac{1}{\beta^2} \int_{T_{\tau_0}}^{T_\tau} \int_{(\mathbb{R}^2)^2} dxdy f(x)f(y) d\langle M(x_T), M(y_T) \rangle_s dxdy ds \\
= \int_{T_{\tau_0}}^{T_\tau} ds \int_{(\mathbb{R}^2)^2} f(x)f(y) I^{(T)}(x)I^{(T)}(y) \\
\int_{(\mathbb{R}^2)^2} dz dz \rho_\sigma(z_1-x_T) \rho_\sigma(z_2-y_T) V(z_1-z_2) \\
\times \mathbb{E} \left[ z_1 \mathfrak{Z}_{s,s}(T)(z_1) \mathfrak{Z}_{s,s}(T)(z_2) u_0 \left( \frac{B_T}{\sqrt{T}} \right) \right] \\
= T \int_{T_{\tau_0}}^{T_\tau} ds \int_{(\mathbb{R}^2)^2} dxdy f(x)f(y) I^{(T)}(x)I^{(T)}(y) \\
\int_{(\mathbb{R}^2)^2} dz dw \rho_\sigma(z-x) \rho_\sigma(w-y) V(z_T-w_T) \\
\times \mathbb{E} \left[ z_T \mathfrak{Z}_{s,s}(T)(z_T) \mathfrak{Z}_{s,s}(T)(w_T) u(t-\sigma,z) \bar{u}(t-\sigma,w) \right].
\end{align*}

We define for $x, y \in \mathbb{R}^2$ and $\tau_0 \leq \sigma \leq \tau$
\begin{align*}
\psi^{(T)}_\sigma(x,y) = T \int_{(\mathbb{R}^2)^2} dxdy \rho_\sigma(z-x) \rho_\sigma(w-y) V(z_T-w_T) \\
\times \mathbb{E} \left[ z_T \mathfrak{Z}_{s,s}(T)(z_T) \mathfrak{Z}_{s,s}(T)(w_T) u(t-\sigma,z) \bar{u}(t-\sigma,w) \right].
\end{align*}
Lemma 3.9. For each $x, y \in \mathbb{R}^2$, $\sigma \in [\tau_0, \tau]$

\[
\lim_{T \to \infty} \mathbb{E} \left[ |\Psi^T_\sigma(x,y) - \Psi_\sigma(x,y)| \right] = 0,
\]
where

\[
\Psi_\sigma(x,y) = \frac{1}{1 - \beta^2} \int_{\mathbb{R}^2} dw \rho_\sigma(w-x)\rho_\sigma(w-y) u(t - \sigma, z)^2.
\]

Combining this with Lemma 2.7 and (1.7), it is easy to see by the dominated convergence theorem that

\[
\mathbb{E} \left[ \int_{\tau_0}^T d\sigma \int_{(\mathbb{R}^2)^2} |f(x)f(y) I^{(T)}_{\sigma,T}(x) I^{(T)}_{\sigma,T}(y) | \Psi^T_\sigma(x,y) - \Psi_\sigma(x,y) | \right] \\
= \int_{\tau_0}^T d\sigma \int_{(\mathbb{R}^2)^2} |f(x)f(y)| \mathbb{E} \left[ |I^{(T)}_{\sigma,T}(x) I^{(T)}_{\sigma,T}(y) | \right] \mathbb{E} \left[ |\Psi^T_\sigma(x,y) - \Psi_\sigma(x,y) | \right] \to 0.
\]

Lemma 3.10. For any test function $f \in C^\infty_c(\mathbb{R}^2)$

\[
\int_{\tau_0}^T d\sigma \int_{(\mathbb{R}^2)^2} dx dy f(x)f(y) I^{(T)}_{\sigma,T}(x) I^{(T)}_{\sigma,T}(y) \Psi_\sigma(x,y) \approx_{L^1} \int_{\tau_0}^T d\sigma \int_{(\mathbb{R}^2)^2} dx dy f(x)f(y) I_{\sigma,T}(x) I_{\sigma,T}(y) \Psi(x,y)
\]

as $T \to \infty$, where the $\approx_{L^1}$ sign means that the difference between the left and right sides goes to 0 in $L^1$-sense.

Proof of Lemma 3.9. (Step 1) Letting $z = w + \frac{v}{\sqrt{T}}$

\[
\Psi^T_\sigma(x,y) = \int_{(\mathbb{R}^2)^2} dw dv \rho_\sigma(w-y) V(v) \overline{Z}_{T,T,T}(w_T) \bar{u}(t - \sigma, w)
\]

\[
\times \rho_\sigma(w + \frac{v}{\sqrt{T}} - x) \overline{Z}_{T,T,T}(w_T + v) \bar{u}(t - \sigma, w + \frac{v}{\sqrt{T}}).
\]

We note that

\[
\rho_\sigma \left( w + \frac{v}{\sqrt{T}} - x \right) \overline{Z}_{T,T,T}(w_T + v) \bar{u}(t - \sigma, w + \frac{v}{\sqrt{T}}) - \rho_\sigma(w-x) \overline{Z}_{T,T,T}(w_T) \bar{u}(t - \sigma, w)
\]

\[
\leq \left| \rho_\sigma \left( w + \frac{v}{\sqrt{T}} - x \right) - \rho_\sigma(w-x) \right| \overline{Z}_{T,T,T}(w_T + v) \bar{u}(t - \sigma, w + \frac{v}{\sqrt{T}})
\]

\[
+ \rho_\sigma(w-x) \left| \overline{Z}_{T,T,T}(w_T + v) - \overline{Z}_{T,T,T}(w_T) \right| \bar{u}(t - \sigma, w)
\]

\[
+ \rho_\sigma(w-x) \overline{Z}_{T,T,T}(w_T) \left| \bar{u}(t - \sigma, w + \frac{v}{\sqrt{T}}) - \bar{u}(t - \sigma, w) \right|,
\]

all of which converge to 0 as $T \to \infty$ by Lemma 2.14. Since $\bar{u}$ and $\rho_\sigma(x)$ is bounded for $\sigma \in [\tau_0, \tau]$ and $x \in \mathbb{R}^2$, using (2.4) and $\int V(v) dv = 1$, we have by the dominated convergence theorem,

\[
\Psi^T_\sigma(x,y) \approx_{L^1} \int_{\mathbb{R}^2} dw dv \rho_\sigma(w-x)\rho_\sigma(w-y) V(v) \overline{Z}_{T,T,T}(w_T) \bar{u}(t - \sigma, w)^2
\]

\[
= \int_{\mathbb{R}^2} dw \rho_\sigma(w-x)\rho_\sigma(w-y) \overline{Z}_{T,T,T}(w_T) \bar{u}(t - \sigma, w)^2.
\]

(Step 2) Since we have

\[
\overline{Z}_{s,s(T)}(w) \overset{(d)}{=} \overline{Z}_{s(T)}(w)
\]

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for each $w \in \mathbb{R}^2$ and $s > 0$, it is enough from (2.4) to show that for each $\sigma \in [\tau_0, \tau]$ and $x, y \in \mathbb{R}^2$

$$
\int_{\mathbb{R}^2} dw \rho_\sigma(w - x) \rho_\sigma(w - y) Z^\beta_{T \sigma \ell(T)}(w_T)^2 \bar{u}(t - \sigma, w)^2 \\
\approx_{L^1} \int_{\mathbb{R}^2} dw \rho_\sigma(w - x) \rho_\sigma(w - y) E \left[ Z^\beta_{T \sigma \ell(T)}(w_T)^2 \right] \bar{u}(t - \sigma, w)^2.
$$

(3.15)

It follows from the approximations:

$$
\int_{\mathbb{R}^2} dw \rho_\sigma(w - x) \rho_\sigma(w - y) \left( Z^\beta_{T \sigma \ell(T)}(w_T) \right)^2 \bar{u}(t - \sigma, w)^2 \\
\approx_{L^1} \int_{\mathbb{R}^2} dw \rho_\sigma(w - x) \rho_\sigma(w - y) \left( Z^\beta_{T \sigma \ell(T), \ell'(\sigma, T)}(w_T)^2 \right) \bar{u}(t - \sigma, w)^2
$$

(3.16)

$$
\approx_{L^1} \int_{\mathbb{R}^2} dw \rho_\sigma(w - x) \rho_\sigma(w - y) E \left[ Z^\beta_{T \sigma \ell(T), \ell'(\sigma, T)}(w_T)^2 \right] \bar{u}(t - \sigma, w)^2,
$$

(3.17)

where we define for $t \geq 0$, $r > 0$, $z \in \mathbb{R}^2$

$$
\tilde{Z}_{t,r}(z) = \tilde{Z}^\beta_{t,r}(z) = E_z \left[ \Phi^\beta_{t,r}(B) : F_{t,r}(B, z) \right],
$$

(3.18)

and $F_{t,r}(B, z)$ is the event that Brownian motion $B$ does not escape from the open ball $B(z, r) = \{ x \in \mathbb{R}^2 : |x - y| < r \}$ up to times $t$:

$$
F_{t,r}(B, z) = \{ B_s \in B(z, r) \text{ for any } s \in [0, t] \}
$$

and we set

$$
\ell'(\sigma, T) = \sqrt{T \sigma \ell(T)}^{\frac{1}{2}}
$$

and we denote by $Y^\beta_{T, \sigma}(w) = \left( Z^\beta_{T \sigma \ell(T), \ell'(\sigma, T)}(w_T) \right)^2 \wedge (\ell(T))^{-\frac{1}{2}}$ for simplicity. The following argument yields (3.16): We find

$$
E \left[ \left( Z^\beta_{T \sigma \ell(T)}(w_T) \right)^2 - \left( Z^\beta_{T \sigma \ell(T), \ell'(\sigma, T)}(w_T) \right)^2 \wedge (\ell(T))^{-\frac{1}{2}} \right] \\
\leq E \left[ \left( Z^\beta_{T \sigma \ell(T)}(w_T) \right)^2 - \left( Z^\beta_{T \sigma \ell(T), \ell'(\sigma, T)}(w_T) \right)^2 \right] \\
+ E \left[ \left( Z^\beta_{T \sigma \ell(T), \ell'(\sigma, T)}(w_T) \right)^2 - \left( Z^\beta_{T \sigma \ell(T)}(w_T) \right)^2 \right] \geq (\ell(T))^{-\frac{1}{2}}
$$

and the last term tends to 0 as $T \to \infty$ by Lemma 2.4. Furthermore,

$$
E \left[ \left( Z^\beta_{T \sigma \ell(T)}(w_T) \right)^2 - \left( Z^\beta_{T \sigma \ell(T), \ell'(\sigma, T)}(w_T) \right)^2 \right] \\
\leq 4E \left[ \left( Z^\beta_{T \sigma \ell(T)}(w_T) \right)^2 \right] E_{w_T} \left[ \Phi_{T \sigma \ell(T)}(B) : F_{T \sigma \ell(T), \ell'(\sigma, T)}(B, w_T)^c \right]^2 \\
\leq CE_{w_T} \otimes E_{w_T} \left[ \exp \left( \beta^2 \int_0^{T \sigma \ell(T)} V(B_s - B'_s) ds \right) : F_{T \sigma \ell(T), \ell'(\sigma, T)}(B, w_T)^c \right] \\
\leq CE_{w_T} \otimes E_{w_T} \left[ \exp \left( \beta^2 p \int_0^{T \sigma \ell(T)} V(B_s - B'_s) ds \right) \right]^{1/p} P_{w_T} \left( F_{T \sigma \ell(T), \ell'(\sigma, T)}(B, w_T)^c \right)^{1/q} \\
\leq CP_{w_T} \left( F_{T \sigma \ell(T), \ell'(\sigma, T)}(B, w_T)^c \right)^{1/q} \to 0
$$

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where $B$ and $B'$ are independent Brownian motions starting from $w_T$ and we have used the Cauchy-Schwarz inequality in the first line and the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and the fact that there exists a constant $p > 1$ such that

$$\lim_{T \to \infty} E_x \otimes E_x \left[ \exp \left( \beta^2 p \int_0^{T \sigma(t)} V(B_s - B_s') ds \right) \right] < \infty. \quad (3.19)$$

(Step 3) We end the proof by showing (3.17). First, we remark that if $|w_T - w'_T| > 2(\ell'(\sigma, T) + R_\phi)$, then

$$\text{Cov} \left( V_{T,\sigma}^\beta(w_T), V_{T,\sigma}^\beta(w'_T) \right) = 0,$$

where $R_\phi$ is a constant with $\text{supp} \phi \subset B(0, R_\phi)$.

Therefore,

$$E \left[ \left( \int_{\mathbb{R}^2} dw \rho_C(w-x) \rho_C(w-y) \left( V_{T,\sigma}^\beta(w_T) - E [V_{T,\sigma}^\beta(w_T)] \right) \bar{u}(t-\sigma, w)^2 \right)^2 \right]$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} dw dw' \rho_C(w-x) \rho_C(w-y) \rho_C(w'-x) \rho_C(w'-y)$$

$$\times \text{Cov} \left( V_{T,\sigma}^\beta(w_T), V_{T,\sigma}^\beta(w'_T) \right) \bar{u}(t-\sigma, w) \bar{u}(t-\sigma, w')^2$$

$$\leq \int_{|w_T - w'_T| \leq 2(\ell'(\sigma, T) + R_\phi)} dw dw' \rho_C(w-x) \rho_C(w-y) \rho_C(w'-x) \rho_C(w'-y)$$

$$\times E \left[ V_{T,\sigma}^\beta(0)^2 \right] \bar{u}(t-\sigma, w)^2 \bar{u}(t-\sigma, w')^2$$

$$\leq C \ell(T)^{\frac{1}{2}} E \left[ V_{T,\sigma}^\beta(0)^2 \right].$$

Thus, it is enough to show that

$$\lim_{T \to \infty} \ell(T)^{\frac{1}{2}} E \left[ V_{T,\sigma}^\beta(0)^2 \right] = 0,$$

which follows from Lemma 2.4 and the following:

**Lemma 3.11.** [CN19, Lemma 3.3] Let $(X_k)_{k \in \mathbb{N}}$ be a non-negative, uniformly integrable family of random variables. Then, for any sequence $a_k \to \infty$, $a_k^{-1} E[(X_k \wedge a_k)^2] \to 0$ as $k \to \infty$.

**Proof of Lemma 3.10.** The proof is essentially the same as in Lemma 3.9. Indeed, we can approximate $I_{\sigma T}(x)$ by

$$\left( F' \left( Z_{T \sigma(t), \ell'(\sigma, T)(xT)} \bar{u}(t, x) \right) Z_{T \sigma(t), \ell'(\sigma, T)(xT)} \right) \wedge \ell(T)^{-\frac{1}{2}}$$

due to the same argument as (Step 2) and (Step 3) in the proof of Lemma 3.9. In particular, we remark that its expectation converges to $I(x)$ due to Theorem 1.1 and assumption of $F'$. We omit the detail.

**Proof of Lemma 3.8.** By (1.7), for $s \leq tT_\epsilon$,

$$E \left[ (I_s^{T}(x))^2 \right] \leq C E \left[ \left( |\log Z_{s\ell(T)}^{\beta}(xT)| + Z_{s\ell(T)}^{\beta}(xT) \right)^2 \right] \leq C_t,$
3.3 Proof of Proposition 3.6
First, we will show that the fluctuation of martingale term is negligible at short time regime.

Lemma 3.12.

\[
\lim_{\epsilon \to 0} \frac{1}{\beta_\epsilon^2} \mathbb{E} \left[ \left( \int f(x) G_{\epsilon T, m(T)}(x_T) \, dx \right)^2 \right] = 0.
\]

Then, we will prove that the remainder of martingale can be comparable to \( \mathcal{M} \) in the sense:

Lemma 3.13.

\[
\lim_{\epsilon \to 0} \frac{1}{\beta_\epsilon^2} \mathbb{E} \left[ \left( \int f(x) \left( (G_{\epsilon T, m(T)}(x_T) - G_{\epsilon T, m(T)}(x_T)) \right) - \mathcal{M}_{\epsilon T, m(T)}(x_T) \right) \, dx \right]^2 = 0.
\]

Lemma 3.12 and Lemma 3.13 conclude Proposition 3.6.

3.3.1 Proof of Lemma 3.12
To prove Lemma 3.12, we will introduce a new martingale: Let

\[
n(T_\epsilon) = \exp \left( - (\log T_\epsilon)^{\frac{1}{2}} \right)
\]

and

\[
\tilde{W}_s(x) = \mathbb{E}_x \left[ \Phi_s(B) u_0 \left( \frac{B_{T_\epsilon}}{\sqrt{T}} \right) : F_{\epsilon T, m(T)}(B, x) \right]. \tag{3.20}
\]

Lemma 3.12 is concluded by the following two lemmas.
Lemma 3.14. For any $f \in C_c^\infty(\mathbb{R}^2)$, $t > 0$, $x \in \mathbb{R}^2$, and $\hat{\beta} \in (0, 1)$,
\[ \lim_{\varepsilon \to 0} \frac{1}{\beta_\varepsilon} \mathbb{E} \left[ G_{tT, m(T_\varepsilon)}(x) - \int_0^{tT, m(T_\varepsilon)} F'(W_u(x)) d\tilde{W}_u(x) \right] = 0. \]

Lemma 3.15. For any $f \in C_c^\infty(\mathbb{R}^2)$, $t > 0$, and $\hat{\beta} \in (0, 1)$,
\[ \lim_{\varepsilon \to 0} \frac{1}{\beta_\varepsilon} \mathbb{E} \left[ \int_{\mathbb{R}^2} df(x) \int_0^{tT, m(T_\varepsilon)} F'(W_u(x_T)) d\tilde{W}_u(x_T) \right] = 0. \]

Proof of Lemma 3.14. We have
\[
\begin{align*}
E \left[ G_{tT, m(T)}(x) - \int_0^{tT, m(T)} F'(W_u(x)) \tilde{d} \tilde{W}_u(x) \right]^2 \\
= E \left( \left( \int_0^{tT, m(T)} F'(W_u(x)) dW_u(x) - \int_0^{tT, m(T)} F'(W_u(x)) d\tilde{W}_u(x) \right)^2 \\
= E \left[ \int_0^{tT, m(T)} F'(W_u(x))^2 d\left( W(x) - \tilde{W}(x) \right) \right]
\end{align*}
\]
Then, the last expectation is written by
\[
E \left[ \int_0^{tT, m(T)} du F'(W_u(x))^2 E_x \otimes E_x \left[ V(B_u - \tilde{B}_u) \Phi_u(B_u) \Phi_s(\tilde{B}_u) u_0 \left( \frac{\tilde{B}_T}{\sqrt{T}} \right) u_0 \left( \frac{\tilde{B}_T}{\sqrt{T}} \right) : A_T(B, \tilde{B}, x) \right] \right],
\]
where we set
\[ A_T(B, \tilde{B}, x) := F_{tT, m(T)} \cap F_{tT, m(T)}(B, x)^c. \]
By using $|V(B_s - \tilde{B}_s)| \leq \|V\|_\infty$ and (1.7), it is estimated by
\[
C \mathbb{E} \left[ \int_0^{tT, m(T)} du F'(W_u(x))^2 (W_u(x) - \tilde{W}_u(x))^2 \right] \\
\leq C \mathbb{E} \left[ \int_0^{tT, m(T)} du \left( \frac{1}{W_u(x)} + 1 \right)^2 (W_u(x) - \tilde{W}_u(x))^2 \right] \\
\leq C \int_0^{tT, m(T)} du \mathbb{E} \left[ \frac{W_u(x) - \tilde{W}_u(x)}{W_u(x)} + W_u(x) - \tilde{W}_u(x) + (W_u(x) - \tilde{W}_u(x))^2 \right] \\
\leq C \mathbb{E} \left[ \exp \left( \int_0^u \beta^2 V(B_s - \tilde{B}_s) ds \right) u_0 \left( \frac{\tilde{B}_T}{\sqrt{T}} \right) u_0 \left( \frac{\tilde{B}_T}{\sqrt{T}} \right) : A_T(B, \tilde{B}, x) \right].
\]
It is easy to see that
\[
\mathbb{E} \left[ (W_u(x) - \tilde{W}_u(x))^2 \right] = E_x \otimes E_x \left[ \exp \left( \int_0^u \beta^2 V(B_s - \tilde{B}_s) ds \right) u_0 \left( \frac{\tilde{B}_T}{\sqrt{T}} \right) : A_T(B, \tilde{B}, x) \right] \\
\leq C E_x \otimes E_x \left[ \exp \left( \int_0^u \beta^2 V(B_s - \tilde{B}_s) ds \right) : A_T(B, \tilde{B}, x) \right].
\]
Then, Hölder’s inequality yields that
\[
E_x \otimes E_x \left[ \exp \left( \int_0^u \beta^2 V(B_s - \tilde{B}_s) ds \right) : A_T(B, \tilde{B}, x) \right] \\
\leq E_x \otimes E_x \left[ \exp \left( \int_0^u \beta^2 V(B_s - \tilde{B}_s) ds \right)^{1/2} \right] \mathbb{P}_x \left( F_{tT, m(T)}(B, x)^c \right)^{1/2},
\]
(3.23)
where \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) are chosen such that

\[
\lim_{T \to \infty} E_x \otimes E_x \left[ \exp \left( p\beta^2 \int_0^T V(B_u - \bar{B}_u)du \right) \right] < \infty.
\]

(3.22) tends to 0 as \( T \to \infty \) since we know

\[
P_x \left( F_{tT_m(T)} \sqrt{tT_n(T)} (B, x)^a \right) \leq C \exp \left( - \frac{n(T)}{4m(T)} \right),
\]

which decays faster than any polynomial of \( T \). By using the Cauchy-Schwarz inequality with Lemma 2.7, we find that (3.21) converges to 0 as \( T \to \infty \).

**Proof of Lemma 3.15.** It is enough to show that

\[
\frac{1}{\beta^2} E \left[ \left( \int_{\mathbb{R}^2} dx f(x) \int_0^{tT_m(T)} F'(W_s(x_T))d\tilde{W}_s(x_T) \right)^2 \right] = \frac{1}{\beta^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy f(x)f(y)E \left[ \int_0^{tT_m(T)} F'(W_s(x_T))F'(W_s(y_T))d\langle \tilde{W}(x_T), \tilde{W}(y_T) \rangle_s \right] \to 0,
\]
as \( T \to \infty \). Since \( \tilde{W}_s(x) \) and \( \tilde{W}_s(y) \) are independent and hence \( d\langle \tilde{W}(x), \tilde{W}(y) \rangle \equiv 0 \) if \( |x-y| > 2(\sqrt{tT_n(T)} + R_\phi) \), we have

\[
E \left[ \left( \int_{\mathbb{R}^2} dx f(x) \int_0^{tT_m(T)} F'(W_s(x_T))d\tilde{W}_s(x_T) \right)^2 \right] \leq C_n(T)E \left[ \int_0^{tT_m(T)} F'(W_s(x))^2d\langle \tilde{W}(x) \rangle_s \right] \leq C_n(T)E \left[ \int_0^{tT_m(T)} \left( \frac{1}{W_s(x)} + 1 \right) d\langle \tilde{W}(x) \rangle_s \right].
\]

By construction of \( \tilde{W}_s(x) \), we have

\[
C_n(T)E \left[ \int_0^{tT_m(T)} \left( \frac{1}{W_s(x)} + 1 \right) d\langle \tilde{W}(x) \rangle_s \right] \leq C_n(T)E \left[ \int_0^{tT_m(T)} \left( \frac{1}{W_s(x)} + 1 \right) d\langle W(x) \rangle_s \right]
\]
and furthermore, by applying Itô’s lemma to \( \log W(x) \) and \( W(x)^2 \), it is bounded by

\[
C_n(T)E \left[ \log \bar{u}(t, x) - \log W_{tT_m(T)} + W_{tT_m(T)}(x)^2 \right] \leq C_n(T).
\]
Thus, Lemma 3.15 is concluded.

**3.3.2 Proof of Lemma 3.13**

Define

\[
d\mathcal{L}_s(x) = \beta \bar{Z}_{sT}^\beta \int_{\mathbb{R}^d} dz \xi(ds, db) \int_{\mathbb{R}^d} \rho_s(z - x)\phi(z - b) \mathbb{E}_s \left[ F'(Z_{sT}^\beta \bar{u}(t, x)) d\mathcal{L}_s(x_T) \right].
\]

Then, we remark that

\[
dM_s(x_T) = F' \left( Z_{sT}^\beta \bar{u}(t, x) \right) d\mathcal{L}_s(x_T)
\]
for \( s \geq tT_m(T) \) and \( x \in \mathbb{R}^2 \).

Lemma 3.13 follows when the next two lemmas are proved.
Lemma 3.16. For all $t > 0$,
\[
\lim_{\varepsilon \to 0} \frac{1}{\beta \varepsilon} \mathbb{E} \left[ \int dx f(x) \left( \int_{t \varepsilon m(T)}^{t T} F'(W_s(x_T)) dW_s(x_T) - \int_{t \varepsilon m(T)}^{t T} F'(W_s(x_T)) dL_s(x_T) \right) \right] = 0.
\]

Lemma 3.17. For $t > 0$,
\[
\lim_{\varepsilon \to 0} \frac{1}{\beta \varepsilon} \mathbb{E} \left[ \int f(x) \left( \int_{t \varepsilon m(T)}^{t T} F'(W_s(x_T)) dL_s(x_T) - M_{t \varepsilon m(T)} (x_T) \right) dx \right] = 0.
\]

Proof of Lemma 3.16. By the Burkholder-Davis-Gundy inequality, we have
\[
\frac{1}{\beta} \mathbb{E} \left[ \int dx f(x) \left( \int_{t \varepsilon m(T)}^{t T} F'(W_s(x_T)) dW_s(x_T) - \int_{t \varepsilon m(T)}^{t T} F'(W_s(x_T)) dL_s(x_T) \right) \right] \\
\leq \frac{C}{\beta} \int dx |f(x)| \mathbb{E} \left[ \left( \int_{t \varepsilon m(T)}^{t T} F'(W_s(x_T))^2 d(W(x_T) - L(x_T))_s \right)^{\frac{1}{2}} \right] \\
\leq \frac{C}{\beta} \int dx |f(x)| \mathbb{E} \left[ \sup_{t \varepsilon m(T) \leq s \leq T} Z_s(0)^{-2} + 1 \right] \mathbb{E} \left[ \int_{t \varepsilon m(T)}^{t T} d(W(x_T) - L(x_T))_s \right]^{\frac{1}{2}},
\]
where we have used Doob's inequality and Lemma 2.7 in the last inequality.

By Cauchy-Schwarz inequality and boundedness of $\bar{u}$, we have from definition of $W$ and $L$ that for $x \in \mathbb{R}^2$
\[
\frac{1}{\beta\varepsilon} \mathbb{E} \left[ \int_{t \varepsilon m(T)}^{t T} d(W(x_T) - L(x_T))_s \right] \\
= \int_{t \varepsilon m(T)}^{t T} ds \int_{(\mathbb{R}^2)^2} dz_1 dz_2 \rho_s(z_1) \rho_s(z_2) V(z_1 - z_2) \bar{u}(tT - s, z_1) \bar{u}(tT - s, z_2) \\
\times \mathbb{E} \left[ \left( E_{0,0}^{s,z_1} [\Phi^2] - Z_{s,T}(0) Z_{s,s}(z_1) \right) \left( E_{0,0}^{s,z_2} [\Phi^2] - Z_{s,T}(0) Z_{s,s}(z_2) \right) \right] \\
\leq C \int_{t \varepsilon m(T)}^{t T} ds \int_{(\mathbb{R}^2)^2} dz_1 dz_2 \rho_s(z_1) \rho_s(z_2) V(z_1 - z_2) \\
\times \mathbb{E} \left[ \left( E_{0,0}^{s,z_1} [\Phi^2] - Z_{s,T}(0) Z_{s,s}(z_1) \right) \left( E_{0,0}^{s,z_2} [\Phi^2] - Z_{s,T}(0) Z_{s,s}(z_2) \right) \right] \\
\leq C \int_{t \varepsilon m(T)}^{t T} d\sigma \int_{(\mathbb{R}^2)^2} dw dw \rho_\sigma(w) \rho_\sigma(w + \frac{v}{\sqrt{T}} V(v) \\
\times \mathbb{E} \left[ \left( E_{0,0}^{T,\sigma,w} [\Phi^2] - Z_{s,T}(0) Z_{T,T}(wT) \right) \left( E_{0,0}^{T,\sigma,w} [\Phi^2] - Z_{s,T}(0) Z_{T,T}(wT) \right) \right]^{\frac{1}{2}} \\
\times \mathbb{E} \left[ \left( E_{0,0}^{T,\sigma,w+v} [\Phi^2] - Z_{T,T}(0) Z_{T,T}(wT + v) \right) \left( E_{0,0}^{T,\sigma,w+v} [\Phi^2] - Z_{T,T}(0) Z_{T,T}(wT + v) \right) \right]^{\frac{1}{2}} \\
\leq C \int_{t \varepsilon m(T)}^{t T} d\sigma \int_{\mathbb{R}^2} dw \rho_\sigma(w) \mathbb{E} \left[ \left( E_{0,0}^{T,\sigma,w} [\Phi^2] - Z_{s,T}(0) Z_{T,T}(wT) \right) \left( E_{0,0}^{T,\sigma,w} [\Phi^2] - Z_{s,T}(0) Z_{T,T}(wT) \right) \right]^{\frac{1}{2}}.
\]
and furthermore Lemma 2.8 allows us to bound it from above by
\[
C \int_{tm(T)}^t \, d\sigma \int_{|w| \leq \sqrt{T \log(\sigma T)}} \, dw \rho_\sigma(w) \left( \ell(T) - \frac{\log \ell(T)}{\log T} + \frac{\sqrt{\log(\sigma T)} \log(\sigma T \log \ell(T))}{\sqrt{\sigma T \log T}} + \frac{\ell(T) \log(\sigma T)}{\log T} \right) \\
+ C \int_{tm(T)}^t \, d\sigma \int_{|w| \geq \sqrt{T \log(\sigma T)}} \, dw \rho_\sigma(w) \\
\leq C \left( \ell(T) - \frac{\log \ell(T)}{\log T} + \frac{\sqrt{\log(\sigma T)} \log(\sigma T \log \ell(T))}{\sqrt{\sigma T \log T}} + \frac{\ell(T) \log(\sigma T)}{\log T} \right) + C \int_{tm(T)}^t \, \frac{1}{\sqrt{\sigma T}} \, d\sigma.
\]
Since both terms in the last line tend to 0 as \( T \to \infty \) from definition of \( m(T) \) and \( \ell(T) \), Lemma 3.16 is concluded.

**Proof of Lemma 3.17.** We have from (3.26)
\[
\frac{1}{\beta} \mathbb{E} \left[ \int f(x) \left( \int_{\ell T_m(T)}^{\ell T} F'(W_s(x_T))dL_s(x_T) - M_{\ell T}(x_T) \right) \, dx \right] \\
\leq \frac{1}{\beta} \int |f(x)| \mathbb{E} \left[ \int_{\ell T_m(T)}^{\ell T} F'(W_s(x_T))dL_s(x_T) - M_{\ell T}(x_T) \right] \, dx \\
\leq \frac{1}{\beta} \int |f(x)| \mathbb{E} \left[ \int_{\ell T_m(T)}^{\ell T} (F'(W_s(x_T)) - F'(Z_{\ell T}(x_T)\bar{u}(t,x)))^2 \, d(L(x_T)) \right] \frac{1}{\beta} \\
\text{and} \\
\mathbb{E} \left[ \int_{\ell T_m(T)}^{\ell T} (F'(W_s(x_T)) - F'(Z_{\ell T}(x_T)\bar{u}(t,x)))^2 \, d(L(x_T)) \right] \\
= \beta^2 \int_{\ell T_m(T)}^t \, d\sigma \int_{\mathbb{R}^2} \, dz \, dw \rho_\sigma(z) \rho_\sigma(w + \frac{\nu}{\sqrt{T}}) V(v) E_{z_T} \left[ u_0 \left( \frac{B_{T - \sigma T}}{\sqrt{T}} \right) \right] E_{z_T + v} \left[ u_0 \left( \frac{B_{T - \sigma T}}{\sqrt{T}} \right) \right] \\
\times \mathbb{E} \left[ (F'(W_{T_T}(x_T)) - F'(Z_{T_T}(x_T)\bar{u}(t,x)))^2 \, Z_{T_T}(x_T)^2 \bar{Z}_{T_T, T_T}(z_T) \bar{Z}_{T_T, T_T}(z_T + v) \right].
\]
By (1.7) and Lemma 2.7, we may choose \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and there exists a constant \( C \) such that
\[
\mathbb{E} \left[ (F'(W_{T_T}(x_T)) - F'(Z_{T_T}(x_T)\bar{u}(t,x)))^2 \, Z_{T_T}(x_T)^2 \bar{Z}_{T_T, T_T}(z_T) \bar{Z}_{T_T, T_T}(z_T + v)^p \right] \frac{1}{p} \\
\leq C
\]
uniformly in \( tm(T) \leq \sigma \leq t, x \in \mathbb{R}^2 \) and in \( 0 < \epsilon < \frac{1}{2} \).

Also, (1.7) and \( |F'(W_{T_T}(x_T)) - F'(Z_{T_T}(x_T)\bar{u}(t,x))| = \int_{Z_{T_T}(x_T)}^{W_{T_T}(x_T)} F''(w) \, dw \) yield
\[
\mathbb{E} \left[ |F'(W_{T_T}(x_T)) - F'(Z_{T_T}(x_T)\bar{u}(t,x))|^2 \right] \\
\leq C \mathbb{E} \left[ \frac{1}{W_{T_T}(x_T)} + \frac{1}{Z_{T_T}(x_T)} \bar{u}(t,x) + 1 \right]^{2q-1} \left[ F'(W_{T_T}(x_T)) - F'(Z_{T_T}(x_T)\bar{u}(t,x)) \right] \\
\leq C \mathbb{E} \left[ \sup_{tm(T) \leq \sigma \leq t} \left\{ \frac{1}{Z_{T_T}(x_T)} + \frac{1}{Z_{T_T}(x_T)} \bar{u}(t,x) + 1 \right\}^{2q-1} \left[ \frac{1}{Z_{T_T}(x_T)^2} + \frac{1}{Z_{T_T}(x_T)^2} \right] \right] \\
\times \left| W_{T_T}(x_T) - Z_{T_T}(x_T) \bar{u}(t,x) \right|.
\]
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Then, Hölder’s inequality and Doob’s inequality guarantee with Lemma 2.7 and Theorem 2.8 that the last term converges to 0 as $T \to \infty$ and thus Lemma 3.17 follows by the dominated convergence theorem.

### 3.4 Proof of Proposition 3.2

Our goal is to prove that:

\[
\frac{1}{\beta_{\varepsilon}} \int_{\mathbb{R}^2} dx f(x) \left( \int_0^{T_{\varepsilon}} F''(W_s(x_{T_{\varepsilon}})) d(W(x_{T_{\varepsilon}}))_s - \mathbb{E} \left[ \int_0^{T_{\varepsilon}} F''(W_s(x_{T_{\varepsilon}})) d(W(x_{T_{\varepsilon}}))_s \right] \right) \xrightarrow{L^1} 0. \tag{3.27}
\]

For simplicity of notation, we set $t = 1$ hereafter.

The proof is composed of four steps. In the first step, we will investigate that the influence at large time is negligible in the following sense:

**Lemma 3.18 (Step 1).**

\[
\lim_{\varepsilon \to 0} \frac{1}{\beta_{\varepsilon}} \mathbb{E} \left[ \int_{\mathbb{R}^2} dx f(x) \int_0^{T_{\varepsilon}} F''(W_s(x_{T_{\varepsilon}})) d(W(x_{T_{\varepsilon}}))_s \right] = 0. \tag{3.28}
\]

Before going to the step 2, we introduce a stopping time

\[
\tau_{T_{\varepsilon}} = \tau_{T_{\varepsilon}}(x) := \inf \left\{ s \geq 0 : W_s(x) + \tilde{W}_s(x)^{-1} > \frac{1}{m(T_{\varepsilon})} \right\} \wedge T_{\varepsilon} m(T_{\varepsilon}).
\]

Let us define the event:

\[
A_{T_{\varepsilon}}(x) = \{ \tau_{T_{\varepsilon}}(x) = T_{\varepsilon} m(T_{\varepsilon}) \} = \left\{ W_s(x) + \tilde{W}_s(x)^{-1} \leq m(T_{\varepsilon})^{-1} \text{ for all } s \leq T_{\varepsilon} m(T_{\varepsilon}) \right\}.
\]

In the step 2, we will find that the contribution of “large” $W(x)$ and “small” $\tilde{W}(x)$ can be negligible.

**Lemma 3.19 (Step 2).**

\[
\lim_{\varepsilon \to 0} \frac{1}{\beta_{\varepsilon}} \mathbb{E} \left[ \int_{\tau_{T_{\varepsilon}}(x)}^{T_{\varepsilon} m(T_{\varepsilon})} |F''(W_s(x))| d(W(x))_s \right] = 0.
\]

In the step 3, we will prove that the contributions by $W(x)$ and $\tilde{W}(x)$ are asymptotically identified.

**Lemma 3.20 (Step 3).** For any $x \in \mathbb{R}^2$,

\[
\lim_{\varepsilon \to 0} \frac{1}{\beta_{\varepsilon}} \mathbb{E} \left[ \int_0^{\tau_{T_{\varepsilon}}(x)} F''(W_s(x)) d(W(x))_s - \int_0^{\tau_{T_{\varepsilon}}(x)} F''(\tilde{W}_s(x)) d(\tilde{W}(x))_s \right] = 0.
\]

In the last, we will prove the remainder is also negligible.

**Lemma 3.21 (Step 4).**

\[
\lim_{\varepsilon \to 0} \frac{1}{\beta_{\varepsilon}} \mathbb{E} \left[ \left( \int_{\mathbb{R}^2} dx f(x) \left( \int_0^{T_{\varepsilon}} F''(W_u(x_{T_{\varepsilon}})) d(W(x_{T_{\varepsilon}}))_u - \mathbb{E} \left[ \int_0^{T_{\varepsilon}} F''(W_u(x_{T_{\varepsilon}})) d(W(x_{T_{\varepsilon}}))_u \right] \right) \right)^2 \right] = 0.
\]

Putting these lemmas together, Proposition 3.2 is concluded.
Proof of Lemma 3.18. With $\delta$ from (3.9), for $C > (\|u_0\|_\infty + \|u_0^{-1}\|_\infty)^4$, 

$$
\mathbb{E} \left[ \int_{T_m(T)}^T |F''(W_s(x_{T_c}))| \, d(W_s(x_{T_c})) \right] 
\leq C \mathbb{E} \left[ \sup_{T_m(T) \leq s \leq T} (1 + Z_s(x)^{-2}) \int_{T_m(T)}^T d(Z(x)_s) \right] 
\leq 2C \mathbb{E} \left[ \sup_{T_m(T) \leq s \leq T} Z_s(x)^{-2} \left\{ \sup_{T_m(T) \leq s \leq T} Z_s(x)^{-1} > (\log T)^{\delta/4} \right\} \int_{T_m(T)}^T d(Z(x)_s) \right] 
+ 2C(\log T)^{\delta/2} \mathbb{E} \left[ 1 \left\{ \sup_{T_m(T) \leq s \leq T} Z_s(x)^{-1} \leq (\log T)^{\delta/4} \right\} \int_{T_m(T)}^T d(Z(x)_s) \right].
$$

By the Burkholder-Davis-Gundy inequality, Doob’s inequality and Hölder’s inequality, the first expectation is bounded from above by

$$
\mathbb{P} \left( \sup_{T_m(T) \leq s \leq T} Z_s(x)^{-1} > (\log T)^{\delta/4} \right) \leq \frac{1}{(\log T)^{2p}} \mathbb{E} \left[ Z_T(x)^{-\frac{2p}{p}} \right] \frac{1}{(\log T)^{2q}} \mathbb{E} \left[ Z_T(x)^{2q} \right]^{\frac{1}{q}} \leq (\log T)^{-2},
$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\sup_{t \leq 1} \mathbb{E} [Z_T(x)^{2q}] < \infty$ from Lemma 2.4. On the other hand, the second expectation can be bounded from above by

$$
(\log T)^{\delta/2} \mathbb{E} \left[ \int_{T_m(T)}^T d(Z(x)_s) \right] = (\log T)^{\delta/2} \int_{T_m(T)}^T \beta^2 E_0 \left[ V(\sqrt{2}B_s) \exp \left( \int_0^s \beta^2 V(\sqrt{2}B_u) \, du \right) \right] \, ds
\leq (\log T)^{\delta/2} \int_{T_m(T)}^T \beta^2 \int_{\mathbb{R}^2} \, dx V(\sqrt{2}x) \rho_s(x) E_0^x \left[ \exp \left( \int_0^s \beta^2 V(\sqrt{2}B_u) \, du \right) \right] \, ds
\leq \beta^2 (\log T)^{\delta/2} \log m(T).
$$

where we have used Lemma 2.5 in the third line and $\beta^2 (\log T)^{\delta/2} \log m(T) \to 0$ as $\varepsilon \to 0$ as desired.

Before the proof of Lemma 3.19, we give an estimate of the probability of $A_T(x)$.

**Lemma 3.22.** There exists a constant $C > 0$ such that for $T_\varepsilon > 0$ and for $x \in \mathbb{R}^2$,

$$
\mathbb{P}(A_T(x)^c) \leq C m(T_\varepsilon).
$$

**Proof.** We have

$$
\mathbb{P}(A_T(x)^c) \leq \mathbb{P}(W_s(x) > (2m(T))^{-1} \text{ for some } s \in [0, T_m(T))] + \mathbb{P}(\tilde{W}_s(x) < 2m(T) \text{ for some } s \in [0, T_m(T)]).
$$

The first term is bounded from above by $2\|u_0\|_\infty m(T)$ using Doob’s inequality and $\mathbb{E} |W_{T_m(T)}(x)| \leq \|u_0\|_\infty$. Using the fact that $B < x$ implies $A < 2x$ or $A - B > x$ for $A \geq B > 0$ and $x > 0$, by Doob’s inequality with (sub-)martingales $W_s(x)^{-1}, W_s(x) - \tilde{W}_s(x)$ the second term is bounded from above by

$$
\mathbb{P}(W_s(x) < 4m(T) \text{ for some } s \in [0, T_m(T)]) + \mathbb{P}(W_s(x) - \tilde{W}_s(x) > 2m(T) \text{ for some } s \in [0, T_m(T)])
\leq 4m(T) \mathbb{E} |W_{T_m(T)}(x)^{-1}| + m(T)^{-1} \mathbb{E} |W_{T_m(T)}(x) - \tilde{W}_{T_m(T)}(x)| \leq C m(T).
$$
Proof of Lemma 3.19. By Hölder’s inequality and Minkowski’s inequality, the expectation is bounded from above by

\[
CE \left[ \int_{\tau_T}^{T_m(T)} \left( \frac{1}{W_s(x)^2} + 1 \right) d\langle W(x) \rangle_s \right] \\
\leq CE \left[ \int_0^{T_m(T)} \left( \frac{1}{W_s(x)^2} + 1 \right) d\langle W(x) \rangle_s; A_T(x)^c \right] \\
\leq C' \left[ \left( \int_0^{T_m(T)} \left( \frac{1}{W_s(x)^2} + 1 \right) d\langle W(x) \rangle_s \right)^p \right]^{\frac{1}{p}} \mathbb{P} \left( A_T(x)^c \right)^{\frac{1}{q}} \\
\leq C' \left[ \mathbb{E} \left[ \left( \int_0^{T_m(T)} \frac{d\langle W(x) \rangle_s}{W_s(x)^2} \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ \left( \int_0^{T_m(T)} d\langle W(x) \rangle_s \right)^p \right]^{\frac{1}{p}} \right] \mathbb{P} \left( A_T(x)^c \right)^{\frac{1}{q}},
\]

where \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) are constant with \( 2p < p_\beta \). Then, by applying the Burkholder-Davis-Gundy inequality to the martingales \( \int_0^s dW_u(x) = W_s(x) - \bar{u}(1, x) \), we obtain that

\[
\mathbb{E} \left[ \left( \int_0^{T_m(T)} d\langle W(x) \rangle_s \right)^p \right] \leq C \mathbb{E} \left[ (W_{T_m(T)}(x) - \bar{u}(1, x))^{2p} \right] \leq C.
\]

We remark that when we apply Itô’s lemma to \( \log W_s(x) \), we have

\[
\log W_s(x) = \log \bar{u}(t, x) + \int_0^s dW_u(x) \frac{1}{W_u(x)} - \frac{1}{2} \int_0^s d\langle W(x) \rangle_u \frac{1}{W_u(x)^2}
\]

\[
:= \log \bar{u}(t, x) + G'_s(x) - \frac{1}{2} H'_s(x),
\]

with

\[ \langle G'(x) \rangle_s = H'_s(x). \]

In particular, we have

\[
\mathbb{E} \left[ H'_s(x)^2 \right] \leq 12 (\log \bar{u}(t, x))^2 + 12 \mathbb{E} \left[ G'_s(x)^2 \right] + 12 \mathbb{E} \left[ (\log W_s(x))^2 \right]
\]

\[
= 12 (\log a(t, x))^2 + 24 \mathbb{E} \left[ H'(x) \right] + 12 \mathbb{E} \left[ (\log W_s)^2 \right]
\]

\[
\leq C
\]

for some constant \( C > 0 \). Putting things together with (3.24), we have

\[
\frac{1}{\beta \epsilon} \mathbb{E} \left[ \int_{\tau_T(x)}^{T_m(T)} |F''(W_s(x))| d\langle W(x) \rangle_s \right] \leq \frac{C}{\beta \epsilon} \mathbb{P} \left( A_T(x)^c \right)^{\frac{1}{q}} \to 0.
\]

\]
Proof of Lemma 3.20. Since $\mathcal{W}_s(x) + \mathcal{W}_s(x)^{-1} \leq m(T)^{-1}$ for $s \leq \tau_T(x)$, we have that

$$\left| \int_0^{T_m} |F''(\mathcal{W}_s(x))|d(\mathcal{W}(x))_s - \int_0^{T_m} F''(\mathcal{W}_s(x))d(\mathcal{W}(x))_s \right| \leq Cm(T)^2 \int_0^{T_m} ds \mathbb{E} \left[ V(B_s - \bar{B}_s) \Phi_s^2(B) \Phi_s^2(\bar{B}) : F_{T_m(T)}(B, x)^c \cup F_{T_m(T)}(\bar{B}, x)^c \right]$$

Using Hölder’s inequality, there exists a $p > 2$ such that the first term is bounded from above by

$$\mathbb{E} \left[ \int_0^{T_m(T)} ds \mathbb{E} \left[ V(B_s - \bar{B}_s) \Phi_s^2(B) \Phi_s^2(\bar{B}) : F_{T_m(T)}(B, x)^c \cup F_{T_m(T)}(\bar{B}, x)^c \right] \right] \leq ||V||_\infty \int_0^{T_m(T)} ds \mathbb{E} \left[ \Phi_s^2(B) \Phi_s^2(\bar{B}) : F_{T_m(T)}(B, x)^c \right] \leq ||V||_\infty C \int_0^{T_m(T)} ds \mathbb{P}_x \left( F_{T_m(T)}(B, x)^c \right)^p .$$

For the second term, we first note that for each $s \leq \tau_T$,

$$|F''(\mathcal{W}_s(x)) - F''(\mathcal{W}_s(x))| = \left| \int_{\mathcal{W}_s(x)}^{\mathcal{W}_s(x)} F''(r) dr \right| \leq C(1 + m(T)^{-3})(\mathcal{W}_s(x) - \mathcal{W}_s(x)),$$

and $\frac{d(\mathcal{W}(x))_s}{ds} \leq \beta^2 ||V||_\infty \mathcal{W}_s(x)^2 \leq ||V||_\infty m(T)^{-2}$. Hence,

$$\mathbb{E} \left[ \left| \int_0^{T_m(T)} |F''(\mathcal{W}_s(x)) - F''(\mathcal{W}_s(x))|d(\mathcal{W}(x))_s \right| \right] \leq C||V||_\infty (1 + m(T)^{-3})m(T)^{-2} \int_0^{T_m(T)} \mathbb{E} \left[ |\mathcal{W}_s(x) - \mathcal{W}_s(x)| \right] ds$$

By (3.24), the statement holds.

Proof of Lemma 3.21. We define

$$\tilde{H}_s(x) = \int_0^{s} F''(\mathcal{W}_s(x))d(\mathcal{W}(x))_s .$$

We remark that for $|x - y| \geq 3\sqrt{T_n(T)}$

$$\text{Cov} \left( \tilde{H}_{T_m(T)}(x), \tilde{H}_{T_m(T)}(y) \right) = 0.$$
so that
\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}^2} dx f(x) \left( [\bar{H}_{t,T}(x_T) - \mathbb{E} [\bar{H}_{t,T}(x_T)] ] \right)^2 \right) \right]
\]
\[
= \int_{|x-y|\leq3\sqrt{n(T)}} dx dy f(x)f(y) \text{Cov} \left( \bar{H}_{t,T}(x_T), \bar{H}_{t,T}(y_T) \right)
\]
\[
\leq \int_{|x-y|\leq3\sqrt{n(T)}} dx dy |f(x)f(y)| \mathbb{E} \left[ \bar{H}_{t,T}(x_T)^2 \right]^{\frac{1}{2}} \mathbb{E} \left[ \bar{H}_{t,T}(y_T)^2 \right]^{\frac{1}{2}}.
\]
Since
\[
|\bar{H}_{t,T}(x)| \leq C \int_0^T (1 + \bar{W}_s(x)^{-1})^2 d(\bar{W}(x))_T \leq C(1 + m(T)^{-2}) \int_0^T d(\bar{W}(x))_T,
\]
by the Burkholder-Davis-Gundy inequality, we have
\[
\mathbb{E} \left[ \bar{H}_{t,T}(x)^2 \right] \leq C(1 + m(T)^{-2})^2 \mathbb{E} \left[ \sup_{0 \leq s \leq T} (W_s(x) - \bar{u}(1,x))^4 \right] \leq C(1 + m(T)^{-2})^2 m(T)^{-4}.
\]
Putting things together, we have
\[
\frac{1}{\beta^2} \mathbb{E} \left[ \left( \int_{\mathbb{R}^2} dx f(x) \left( \int_0^T F''(\bar{W}(s))d(\bar{W}(s))_T - \mathbb{E} \left[ \int_0^T F''(\bar{W}(s))d(\bar{W}(s))_T \right] \right) \right)^2 \right]
\]
\[
\leq C \frac{n(T)}{\beta^2 m(T)^6}.
\]

### 3.5 Multidimensional convergence in the EW limits

To ease the presentation, we restrict ourselves to the case where \( F(x) = x \), and \( \beta \in (0,1) \) is fixed, although a repetition of the argument would lead to the result for the general initial conditions and the function \( F \) that we have been considering.

Also, we note that for all \( 0 \leq t_1 \leq \cdots \leq t_n = t \), \( u_0^{(1)}, \cdots, u_0^{(n)} \in C_b(\mathbb{R}^2) \), and \( f_1, \cdots, f_n \in C_c^\infty(\mathbb{R}^2) \)
\[
(u \in (t_1, u_0^{(1)}, f_1), \cdots, u \in (t_n, u_0^{(n)}, f_n))
\]
\[
\overset{(d)}{=} \left( W_u^{(t,T,u_0^{(1)})}(Tt, f_1), \cdots, W_u^{(t,T,u_0^{(n-1)})}(Tt, f_{n-1}), W_0^{(t,T,u_0^{(n)})}(Tt, f_n) \right),
\]
where we define for fixed \( t > 0 \) that for \( u, s \geq 0 \) and \( x \in \mathbb{R}^2 \)
\[
W_u(s, x) = W_u^{(t,T,u_0)}(s, x) = \begin{cases} E_x \left[ \Phi_s \left( B_{T-t} \left( \frac{B_{T-u}}{\sqrt{T}} \right) \right) \right], & 0 \leq u \leq s \\ u_0(x), & 0 \leq s \leq u. \end{cases}
\]
and
\[
W_u(s, f) = W_u^{(t,T,u_0)}(s, f) = \int_{\mathbb{R}^2} f(x) W_u^{(t,T,u_0)}(s, x) dx.
\]
Thus, it suffices to show is that jointly for finitely many \( u \in [0, t] \), \( u_0 \in C_b(\mathbb{R}^2) \), and \( f \in C_c^\infty \), as \( \varepsilon \to 0 \),
\[
\frac{1}{\beta^2} \int f(x) \left( W_u^{(t,T,u_0)}(Tt) - \bar{u}(t-u, x) \right) dx \overset{(d)}{\to} \psi_u^{(t,u_0)}(t, f), \quad (3.29)
\]
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where \( \mathcal{W}^{(u_0)}_u(s, f) : f \in C_c^\infty(\mathbb{R}^2), u_0 \in C_b(\mathbb{R}) \), \( 0 \leq u \leq s \leq t \) is a centered Gaussian field with covariance

\[
\text{Cov} \left[ \mathcal{W}^{(u_0)}_u(s, f), \mathcal{W}^{(u_0)}_u(s', f') \right] = \frac{1}{1 - \beta^2} \int_{u \leq u'}^s \text{d}\sigma \int \text{d}x dy f(x) f'(y) \int \text{d}z \rho_{u-u}(x, z) \rho_{u-u'}(y, z) \bar{u}(t-\sigma, z) \bar{u}'(t-\sigma, z).
\]

Following the same strategy as in Subsection 3.1, we are reduced to showing that

\[
\frac{1}{\beta^2} \mathcal{M}^{(T, u_0)}_u(\tau, f) \overset{(d)}{\rightarrow} \mathcal{W}^{(u_0)}_u(\tau, f)
\]

jointly in \( u \in [0, \tau], f \in C_c^\infty \),

where (see (3.11))

\[
\mathcal{M}^{(T, u_0)}_u(\tau, f) := \left\{ \begin{array}{ll}
\int_{\mathbb{R}^2} f(x) \int_{T_u + (t-u)m(T)}^{T} \text{d}M^{(T, u_0)}_u(s, x_T) \text{d}z, & \tau \geq T_u + (t-u)m(T) \\
0, & \tau \leq T_u + (t-u)m(T)
\end{array} \right.
\]

and

\[
\text{d}M^{(T, u_0)}_u(s, x) := \beta \varepsilon Z_{T_u, T_u + (s-T_u)\ell(T)}(x) \int \xi(s, \text{d}b) \int \rho_{s-T_u}(x, z) \phi(z - b) \tilde{Z}_{s, (s-T_u)\ell(T)}(z) E_z \left[ u_0 \left( \frac{B_T - s}{\sqrt{T}} \right) \right] \text{d}z.
\]

Then, for all \( u \geq 0 \) and \( f \in C_c^\infty \), \( \tau \rightarrow \mathcal{M}^{(T, u_0)}_u(\tau, f) \) is a continuous martingale. In view of the desired convergence (3.29), we have again in mind the functional CLT for martingales Theorem 3.3, so we are interested in the limit of the cross-bracket \( \langle \mathcal{M}^{(T, u_1)}_u(\cdot, f_1), \mathcal{M}^{(T, u_2)}_u(\cdot, f_2) \rangle_\tau \). We have:

**Proposition 3.23.** For all test functions \( f \) and \( f' \) in \( C_c^\infty \), \( u_0, u'_0 \in C_b(\mathbb{R}^2) \), and \( 0 \leq u_2 \leq u_1 \leq t \), for all \( \tau \geq u_1 \),

\[
\frac{1}{\beta^2} \langle \mathcal{M}^{(T, u_0)}_u(\cdot, f_1), \mathcal{M}^{(T, u'_0)}_u(\cdot, f_2) \rangle_\tau \overset{L^1}{\rightarrow} \frac{1}{1 - \beta^2} \int_{u_1}^\tau \text{d}\sigma \int \text{d}x \text{d}y f_1(x) f_2(y) \int \text{d}z \rho_{u-u_1}(x, z) \rho_{u-u'_2}(y, z) \bar{u}(t-\sigma, z) \bar{u}'(t-\sigma, z),
\]

as \( \varepsilon \rightarrow 0 \).

**Proof.** For all \( \tau \geq u_1 + (t-u_1)m(T) \),

\[
\frac{1}{\beta^2} \langle \mathcal{M}^{(T, u_0)}_u(\cdot, f_1), \mathcal{M}^{(T, u'_0)}_u(\cdot, f_2) \rangle_\tau \overset{L^1}{\rightarrow} \frac{1}{1 - \beta^2} \int_{u_1}^\tau \text{d}\sigma \int \text{d}x \text{d}y f_1(x) f_2(y) \int \text{d}z \rho_{u-u_1}(x, z) \rho_{u-u'_2}(y, z) \bar{u}(t-\sigma, z) \bar{u}'(t-\sigma, z),
\]

as \( \varepsilon \rightarrow 0 \).

By a repetition of the arguments that lead to (3.13), we find that

\[
\frac{1}{\beta^2} \langle \mathcal{M}^{(T, u_0)}_u(\cdot, f_1), \mathcal{M}^{(T, u'_0)}_u(\cdot, f_2) \rangle_\tau \overset{L^1}{\approx} \int_{u_1 + (t-u_1)m(T)}^\tau \text{d}\sigma \mathbb{E} \left[ Z_{T_u, T_u + (\sigma-u)\ell(T)}(x_T) \right] \mathbb{E} \left[ Z_{T_u + (T-u)\ell(T)}(y_T) \right] \Theta_T(x, y),
\]

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where
\[
\Theta_T(x, y) = \int dz dv \rho_{-u_1}(z - x) \rho_{-u_2}(z - y) V(v) V(z_T + v) E_z T \left[ u_0 \left( \frac{B_{T \sqrt{T} \sigma}}{\sqrt{T}} \right) \right] E_{z + v} \left[ u'_0 \left( \frac{B_{T \sqrt{T} \sigma}}{\sqrt{T}} \right) \right] \\
\rightarrow \frac{1}{1 - \beta^2} \int dz \rho_{-u_1}(x - z) \rho_{-u_2}(y - z) \bar{u}(t - \sigma, z) \bar{u}'(t - \sigma, z).
\]

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