IDEMPOTENTS IN INTERSECTION OF THE KERNEL
AND THE IMAGE OF LOCALLY FINITE
DERIVATIONS AND E-DERIVATIONS

WENHUA ZHAO

Abstract. Let $K$ be a field of characteristic zero, $A$ a $K$-algebra
and $\delta$ a $K$-derivation of $A$ or $K$-E-derivation of $A$ (i.e., $\delta = \text{Id}_A - \phi$
for some $K$-algebra endomorphism $\phi$ of $A$). Motivated by the
Idempotent conjecture proposed in [Z4], we first show that for all
idempotent $e$ lying in both the kernel $A^\delta$ and the image $\text{Im}\delta := \delta(A)$ of $\delta$, the principal ideal $(e)$ $\subseteq \text{Im}\delta$ if $\delta$ is a locally finite $K$-
derivation or a locally nilpotent $K$-E-derivation of $A$; and $eA, Ae \subseteq \text{Im}\delta$ if $\delta$ is a locally finite $K$-E-derivation of $A$. Consequently,
the Idempotent conjecture holds for all locally finite $K$-derivations
and all locally nilpotent $K$-E-derivations of $A$. We then show that
$1_A \in \text{Im}\delta$, (if and) only if $\delta$ is surjective, which generalizes the same
result [GN, W] for locally nilpotent $K$-derivations of commutative
$K$-algebras to locally finite $K$-derivations and $K$-E-derivations $\delta$ of
all $K$-algebras $A$.

1. Motivations and the Main Results

Throughout the paper $K$ stands for a field of characteristic zero and
$A$ for a $K$-algebra (not necessarily unital or commutative). We denote
by $1_A$ or simply $1$ the identity element of $A$, if $A$ is unital, and $I_A$ or
simply $I$ the identity map of $A$, if $A$ is clear in the context.

A $K$-linear endomorphism $\eta$ of $A$ is said to be locally nilpotent (LN)
if for each $a \in A$ there exists $m \geq 1$ such that $\eta^m(a) = 0$, and locally
finite (LF) if for each $a \in A$ the $K$-subspace spanned by $\eta^i(a)$ ($i \geq 0$)
is finite dimensional over $K$.

By a $K$-derivation $D$ of $A$ we mean a $K$-linear map $D : A \to A$ that
satisfies $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. By a $K$-E-derivation
$\delta$ of $A$ we mean a $K$-linear map $\delta : A \to A$ such that for all $a, b \in A$

Date: March 10, 2022.

2000 Mathematics Subject Classification. 47B47, 08A35, 16W25, 16D99.

Key words and phrases. Locally finite or locally nilpotent derivations and $E$-derivations, the image and the kernel of a derivation or $E$-derivation, idempotents.

The author has been partially supported by the Simons Foundation grant 278638.
the following equation holds:

\begin{equation} \delta(ab) = \delta(a)b + a\delta(b) - \delta(a)\delta(b). \end{equation}

(1.1)

It is easy to verify that \( \delta \) is an \( R \)-\( \mathcal{E} \)-derivation of \( A \), if and only if \( \delta = I - \phi \) for some \( R \)-algebra endomorphism \( \phi \) of \( A \). Therefore an \( R \)-\( \mathcal{E} \)-derivation is a special so-called \((s_1, s_2)\)-derivation introduced by N. Jacobson [J] and also a special semi-derivation introduced by J. Bergen in [B]. \( R \)-\( \mathcal{E} \)-derivations have also been studied by many others under some different names such as \( f \)-derivations in [E1, E2] and \( \phi \)-derivations in [BFF, BV], etc..

We denote by \( \text{End}_K(A) \) the set of all \( K \)-algebra endomorphisms of \( A \), \( \text{Der}_K(A) \) the set of all \( K \)-derivations of \( A \), and \( \text{Eder}_K(A) \) the set of all \( K \)-\( \mathcal{E} \)-derivations of \( A \). Furthermore, for each \( K \)-linear endomorphism \( \eta \) of \( A \) we denote by \( \text{Im} \eta \) the image of \( \eta \), i.e., \( \text{Im} \eta := \eta(A) \), and \( \text{Ker} \eta \) the kernel of \( \eta \). When \( \eta \) is a \( R \)-derivation or \( R \)-\( \mathcal{E} \)-derivation, we also denote by \( \eta^\gamma \) the kernel of \( \eta \).

It is conjectured in [Z4] that the image of a LF \( K \)-derivation or \( K \)-\( \mathcal{E} \)-derivation of \( A \) possesses an algebraic structure, namely, a Mathieu subspace. The notion of Mathieu subspaces was introduced in [Z2] and [Z3], and is also called a Mathieu-Zhao space in the literature (e.g., see [DEZ], [EN], [EH], etc.) as first suggested by A. van den Essen [E3].

The introduction of this new notion was mainly motivated by the study in [M, Z1] of the well-known Jacobian conjecture (see [K], [BCW], [E2]). See also [DEZ]. But, a more interesting aspect of the notion is that it provides a natural generalization of the notion of ideals.

For some other studies on the algebraic structure of the image of a LF or LN \( K \)-derivations or \( K \)-\( \mathcal{E} \)-derivations, see [EWZ], [Z4]–[Z7].

One motivation of this paper is the following so-called Idempotent conjecture proposed in [Z4], which is a weaker version of the conjecture mentioned above on the possible Mathieu subspace structure of the images of LF \( K \)-derivations and \( K \)-\( \mathcal{E} \)-derivations.

**Conjecture 1.1.** Let \( \delta \) be a LF (locally finite) \( K \)-derivation or a LF \( K \)-\( \mathcal{E} \)-derivation of \( A \) and \( e \in \text{Im} \delta \) an idempotent of \( A \), i.e., \( e^2 = e \). Then the principal (two-sided) ideal \( (e) \) of \( A \) generated by \( e \) is contained in \( \text{Im} \delta \).

Our first main result is the following theorem, which gives a partial positive answer to Conjecture 1.1 above.

**Theorem 1.2.** Let \( K \) be a field of characteristic zero and \( A \) a \( K \)-algebra (not necessarily unital). Then the following statements hold:

1) for every locally finite \( D \in \text{Der}_K(A) \) and an idempotent \( e \in A^D \cap \text{Im} D \), we have \( (e) \subseteq \text{Im} D \);
2) for every locally finite $\delta \in \mathcal{E}der_K(\mathcal{A})$ and an idempotent $e \in \mathcal{A}^\delta \cap \text{Im} \delta$, we have $e\mathcal{A}, \mathcal{A}e \subseteq \text{Im} \delta$. Furthermore, if $\delta$ is locally nilpotent, we also have $(e) \subseteq \text{Im} \delta$.

Note that for every $D \in \mathcal{D}er_K(\mathcal{A})$, it can be readily verify that all central idempotents of $\mathcal{A}$ lie in $\mathcal{A}^D$. Furthermore, by Corollary 2.3 that will be shown in Section 2, this is also the case for every LN (locally nilpotent) $\delta \in \mathcal{E}der_K(\mathcal{A})$. Therefore we immediately have the following

**Corollary 1.3.** Assume that $\mathcal{A}$ is a commutative $K$-algebra. Then Conjecture 1.1 holds for all locally finite $D \in \mathcal{D}er_K(\mathcal{A})$ and all locally nilpotent $\delta \in \mathcal{E}der_K(\mathcal{A})$.

For a different proof of the corollary above for commutative algebraic $K$-algebras, see [Z4, Proposition 3.8]. For a different proof of Theorem 1.2 for algebraic $K$-algebras (not necessarily commutative), see [Z4, Corollary 3.9].

Our second main result of this paper is the following

**Proposition 1.4.** Assume that $\mathcal{A}$ is unital and $\delta$ is a LF $K$-derivation or a LF $K$-$E$-derivation of $\mathcal{A}$. Then $1_{\mathcal{A}} \in \text{Im} \delta$, (if and) only if $\text{Im} \delta = \mathcal{A}$, i.e., $\delta$ is surjective.

Two remarks about the proposition above are as follows.

First, the proposition for LN $K$-derivations of commutative $K$-algebras was first proved by P. Gabriel and Y. Nouazé [GN] and later re-proved independently by D. Wright [W]. See also [E2]. During the preparation of this paper the author was informed that Arno van de Essen and Andrzej Nowicki have also proved the LF $K$-derivation case of the proposition for commutative $K$-algebras.

Second, if $1_{\mathcal{A}} \in \mathcal{A}^\delta$, e.g., when $\delta \in \mathcal{D}er_K(\mathcal{A})$, the proposition follows immediately from Theorem 1.2. But, if $1_{\mathcal{A}} \notin \mathcal{A}^\delta$, the proof needs some other arguments (See Section 5).

**Arrangement:** In Section 2 we recall and give a some shorter proof for van den Essen’s one-to-one correspondence between the set of all LN $K$-derivations of $\mathcal{A}$ and the set of all LN $K$-$E$-derivations of $\mathcal{A}$ (See Theorem 2.1). We also derive some consequences of this important theorem that will be needed later in this paper. In Section 3 we show the $K$-derivation case of Theorem 1.2. In Section 4 we show the $K$-$E$-derivation case of Theorem 1.2. In Section 5 we give a proof for Proposition 1.4.

**Acknowledgment:** The author is very grateful to Professors Arno van de Essen and Andrzej Nowicki for personal communications.
2. Van den Essen’s One-to-One Correspondence between Locally Nilpotent Derivations and Locally Nilpotent $\mathcal{E}$-Derivations

Throughout this section, $K$ stands for a field of characteristic zero, $\mathcal{A}$ for a $K$-algebra (not necessarily unital or commutative) and $I$ for the identity map of $\mathcal{A}$.

Denote by $\mathcal{D}$ the set of all LN (locally nilpotent) $K$-derivations of $\mathcal{A}$ and $\mathcal{E}$ the set of all LN $K$-$\mathcal{E}$-derivations of $\mathcal{A}$. We define the following map:

$$\Xi : \mathcal{D} \to \mathcal{E}$$

$$D \to I - e^D,$$

where $e^D := \sum_{k=0}^\infty \frac{D^k}{k!}$.

With the setting as above we have the following remarkable one-to-one correspondence between $\mathcal{D}$ and $\mathcal{E}$, which was first proved by A. van den Essen in [E1]. See also [E2, Proposition 2.1.3].

**Theorem 2.1.** The map $\Xi : \mathcal{D} \to \mathcal{E}$ is an one-to-one correspondence between the sets $\mathcal{D}$ and $\mathcal{E}$ with the inverse map $\Xi^{-1}$ given by the following map:

$$\Lambda : \mathcal{E} \to \mathcal{D}$$

$$\delta \to \ln(I - \delta),$$

where $\ln(I - \delta) := -\sum_{k=1}^\infty \frac{\delta^k}{k}$.

For the sake of completeness, we here give a proof for the theorem above, which is some shorter than the one given in [E2, Proposition 2.1.3].

First, the following lemma can be easily verified by induction, as noticed in [E1, E2].

**Lemma 2.2.** Let $\mathcal{B}$ be a ring and $\delta$ an $\mathcal{E}$-derivation of $\mathcal{B}$. Then for all $a, b \in \mathcal{B}$ and $n \geq 1$, we have

$$\delta^n(ab) = \sum_{i=0}^n \binom{n}{i} \delta^i(a) \delta^{n-i}(I - \delta)^i(b).$$

(2.4)

Now we can show Theorem 2.1 as follows.

**Proof of Theorem 2.1.** First, since $D$ is LN, $e^D$ is well-defined. It is well-known (and also easy to check directly) that $e^D$ is a $K$-algebra
automorphism of $\mathcal{A}$. Hence $\Xi(D)$ is a $K$-$\mathcal{E}$-derivation of $\mathcal{A}$. Consider

\[(2.5) \quad \Xi(D) = I - e_D = - \sum_{n=1}^{\infty} \frac{D^n}{n!} = Dh(D) = h(D)D,\]

where

\[(2.6) \quad h(D) = -I - \sum_{n=2}^{\infty} \frac{D^{n-1}}{n!} = -I - D \sum_{n=2}^{\infty} \frac{D^{n-2}}{n!}.\]

Since $D$ is LN, and $D$ and $h(D)$ commute, by Eq. (2.5) $\Xi(D)$ is also LN. Therefore, $\Xi$ is indeed a map from $\mathcal{D}$ to $\mathcal{E}$.

Next, we show that $\Lambda(\delta) \in \mathcal{D}$ for all $\delta \in \mathcal{E}$. Set $D_\delta := \Lambda(\delta)$. Then

\[(2.7) \quad D_\delta = \ln(I - \delta) = - \sum_{n=1}^{\infty} \frac{\delta^n}{n} = \delta g(\delta) = g(\delta)\delta,\]

where

\[(2.8) \quad g(\delta) = - \sum_{n=1}^{\infty} \frac{\delta^{n-1}}{n} = -I - \delta \sum_{n=2}^{\infty} \frac{\delta^{n-2}}{n}.\]

Since $\delta$ is LN, and $\delta$ and $g(\delta)$ commute, by Eq. (2.7) $D_\delta$ is also LN.

Now, let $x, y \in \mathcal{A}$. Then by Lemma 2.2 we have

\[(2.9) \quad D_\delta(xy) = - \sum_{n=1}^{\infty} \frac{1}{n} \delta^n(xy)\]

\[= - \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=0}^{n} \binom{n}{i} \delta^i(x)\delta^{n-i}(1 - \delta)^i(y)\]

\[= - \sum_{i=0}^{\infty} \delta^i(x)S_i(y),\]

where for each $i \geq 0$,

\[(2.10) \quad S_i(y) = \sum_{n=1}^{\infty} \frac{1}{n} \binom{n}{i} \delta^{n-i}(1 - \delta)^i(y) = (1 - \delta)^i \sum_{n=1}^{\infty} \frac{1}{n} \binom{n}{i} \delta^{n-i}(y).\]

In particular, by Eq. (2.7) and the equation above we have

\[(2.11) \quad S_0(y) = -D_\delta(y).\]

Claim: $S_i(y) = \frac{1}{i!}y$ for all $i \geq 1$. 

Proof of Claim: For each \( i \geq 1 \) we introduce the formal power series

\[
(2.12) \quad f_i(t) := (1 - t)^i \sum_{n=i}^{\infty} \frac{1}{n!} \binom{n}{i} t^{n-i}
\]

\[
= \frac{(1 - t)^i}{i!} \sum_{n=i}^{\infty} (n - 1)(n - 2) \cdots (n - i + 1) t^{n-i}.
\]

Then \( S_i(y) = f_i(\delta)(y) \). On the other hand, we have the following identity of formal power series:

\[
\frac{(i - 1)!}{(1 - t)^i} = - \frac{d^i}{dt^i} \ln(1 - t) = \sum_{n=i}^{\infty} (n - 1)(n - 2) \cdots (n - i + 1) t^{n-i}
\]

By Eq. (2.10) and the identity above we have \( f_i(t) = 1/i \). Hence \( S_i(y) = f_i(\delta)(y) = \frac{1}{i}y \) and the claim follows.

Now by Eqs. (2.7), (2.9), (2.11) and the claim above we have

\[
D_\delta(xy) = - \sum_{i=0}^{\infty} \delta^i(x)S_i(y) = -xS_0(y) - \sum_{i=1}^{\infty} \delta^i(x)S_i(y)
\]

\[
= xD_\delta(y) - \sum_{i=1}^{\infty} \frac{1}{i} \delta^i(x)y = xD_\delta(y) + D_\delta(x)y.
\]

Therefore \( \Lambda(\delta) = D_\delta \) is a LN \( K \)-derivation of \( A \), i.e., \( \Lambda \) is indeed a map from \( E \) to \( D \). Since \( \Xi \) and \( \Lambda \) are obviously inverse to each other, we see that \( \Xi \) gives an one-to-one correspondence between \( D \) to \( E \), i.e., the theorem follows. \( \square \)

Next, we derive some consequences of Theorem 2.1. But, we first need to show the following lemma. Although it is almost trivial, it will be frequently used throughout the rest of the paper.

**Lemma 2.3.** Let \( R \) be a ring and \( B \) an \( R \)-algebra. Let \( F \) and \( G \) be two commuting \( R \)-linear endomorphisms of \( B \) such that \( F \) is invertible and \( G \) is LN (locally nilpotent). Then \( F - G \) is an \( R \)-linear automorphism with the inverse map given by

\[
(F - G)^{-1} = \sum_{k=0}^{\infty} G^k F^{-k-1}.
\]

**Proof:** Note that \( F - G = (I - GF^{-1})F \). Since \( F \) commutes with \( G \), so does \( F^{-1} \). Hence \( U := GF^{-1} \) is LN, for \( G \) is LN. Therefore the formal power series \( \sum_{k=0}^{\infty} U^k \) is a well-defined \( R \)-linear endomorphism
of \( \mathcal{A} \), which gives the inverse map of \( I - U \). Hence, \( F^{-1} \sum_{k=0}^{\infty} U^k \) gives the inverse map of \( F - G \), from which the lemma follows. \( \square \)

**Corollary 2.4.** Let \( \mathcal{D}, \Xi \) be as in Theorem 2.1 and \( D \in \mathcal{D} \). Set \( \delta = \Xi(D) \). Then \( \mathcal{A}^D = \mathcal{A}^\delta \) and \( \text{Im} D = \text{Im} \delta \).

**Proof:** First, since \( \delta = \Xi(D) \), we have \( D = \Lambda(\delta) =: D_\delta \) by Theorem 2.1. Second, by Lemma 2.3 with \( F = -I \) and \( G = D \sum_{n=2}^{\infty} \frac{D^{n-2}}{n!} \), the \( K \)-linear map \( h(D) \) in Eq. (2.6) is a \( K \)-linear automorphism of \( \mathcal{A} \). Therefore, we have \( \mathcal{A}^D = \mathcal{A}^\delta \) by Eq. (2.5). Furthermore, we also have \( \text{Im} \delta \subseteq \text{Im} D \) by Eq. (2.5), and \( \text{Im} D \subseteq \text{Im} \delta \) by Eq. (2.7), whence \( \text{Im} D = \text{Im} \delta \), and the corollary follows. \( \square \)

**Corollary 2.5.** Let \( \delta \) be an arbitrary \( K \)-derivation of \( \mathcal{A} \) or a \( LN_{K_{E}} \)-derivation of \( \mathcal{A} \). Then all central idempotents of \( \mathcal{A} \) lie in \( \mathcal{A}^\delta \).

**Proof:** If \( \delta \) is a \( K \)-derivation of \( D \), the corollary actually holds regardless of the characteristic of \( K \), which can be seen as follows. Since \( De = De^2 = 2eDe \), we have \( (1 - 2e)De = 0 \). Since \( (1 - 2e)^2 = 1 - 4e + 4e^2 = 1, \) \( 1 - 2e \) is a unit of \( \mathcal{A} \). Hence \( De = 0 \).

If \( \delta \) is a \( LN_{K_{E}} \)-derivation of \( \mathcal{A} \), then by Theorem 2.1, \( \delta = I - eD \) for some \( LN_{K} \)-derivation \( D \) of \( \mathcal{A} \). Since \( De = 0 \) as shown above, we have \( \delta e = 0 \). \( \square \)

### 3. The Derivation Case of Theorem 1.2

In this section we give a proof of Theorem 1.2 for LF (locally finite) \( K \)-derivations. Throughout this section we let \( K \) and \( \mathcal{A} \) be as in Theorem 1.2, \( D \) a LF \( K \)-derivation of \( \mathcal{A} \), and \( e \) an idempotent in \( \mathcal{A}^D \cap \text{Im} D \).

Let \( s \in \mathcal{A} \) such that \( Ds = e \). Since \( De = 0 \), we have \( D(ese) = e(Dee)e = e \). So replacing \( s \) by \( ese \) we assume \( s \in e\mathcal{A}e \). Furthermore, for convenience we set \( s^0 = e \). Then with the setting above it is easy to see that for all \( i, k \geq 0 \), we have \( es^i = s^ie = s^i \) and

\[
D^i(s^k) = \begin{cases} k(k-1) \cdots (k-i+1)s^{k-i} & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}
\]

We first consider the case that \( D \) is LN (locally nilpotent).
Lemma 3.1. Assume that $D$ is LN. For all $a \in A$ set
\begin{equation}
\phi_{-s}(a) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D^i(a) s^i. \tag{3.14}
\end{equation}
\begin{equation}
\psi_{-s}(a) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} s^i D^i(a). \tag{3.15}
\end{equation}

Then $\phi_{-s}(a), \psi_{-s}(a) \in A^D$ and
\begin{equation}
ae = \sum_{j=0}^{\infty} \frac{1}{j!} \phi_{-s}(D^j(a)) s^j. \tag{3.16}
\end{equation}
\begin{equation}
ea = \sum_{j=0}^{\infty} \frac{1}{j!} s^j \psi_{-s}(D^j(a)). \tag{3.17}
\end{equation}

Note that the case when $A$ is commutative and $e = 1$ the lemma has been proven in [GN, W]. See also [E2]. The main idea of the proof given below is to modify the proof in [GN, W, E2] to the more general case in the lemma.

**Proof:** First, by Eqs. (3.13) and (3.14) we have
\begin{align*}
D \phi_{-s}(a) &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D(D^i(a) s^i) \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (D^{i+1}(a) s^i + i D^i(a) s^{i-1}) \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j + (-1)^{j+1}}{j!} D^{j+1}(a) s^j = 0.
\end{align*}

Hence $\phi_{-s}(a) \in A^D$. The proof of $\psi_{-s}(a) \in A^D$ is similar.

Next we show Eq. (3.16). The proof of Eq. (3.17) is similar.
\begin{align*}
\sum_{j=0}^{\infty} \frac{1}{j!} \phi_{-s}(D^j(a)) s^j &= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D^{i+j}(a) s^{i+j} \\
&= ae + \sum_{n=1}^{\infty} n! \left( \sum_{i+j=n, i,j \geq 0} \binom{n}{i} (-1)^i \right) D^n(a) s^n \\
&= ae.
\end{align*}
Proposition 3.2. Let \( \delta \) be a LN (locally nilpotent) \( K \)-derivation or \( K \)-\( \mathcal{E} \)-derivation of \( \mathcal{A} \) and \( e \in \mathcal{A}^\delta \cap \text{Im} \delta \) a nonzero idempotent. Let \( s \in \mathcal{A} \) such that \( \delta s = e \). Replacing \( s \) by \( ese \) we assume \( s \in eAe \). Set \( s^0 = e \). Then we have

1) if \( \sum_{i=0}^{n} c_i s^i = 0 \) or \( \sum_{i=0}^{n} s^i c_i = 0 \) with \( c_i \in \mathcal{A}^\delta \), then \( c_i e = 0 \) for all \( 0 \leq i \leq n \). In particular, \( s \) is transcendental over the field \( Ke \);

2) \( Ae = \mathcal{A}^\delta[s] \) and \( eA = [s]A^\delta \), where \( \mathcal{A}^\delta[s] \) (resp., \( [s]A^\delta \)) is the \( K \)-algebra of all polynomials \( f(s) \) of the form \( f(s) = \sum_{i \geq 0} a_i s^i \) (resp., \( f(s) = \sum_{i \geq 0} a_i s^i \)) with \( a_i \in \mathcal{A}^\delta \);

3) if \( \delta \in \text{Der}_K(\mathcal{A}) \), then \( \delta \mid_{eA} = \frac{d}{ds} \) and \( \delta \mid_{Ae} = \frac{d}{ds} \); in particular, \( s \) is transcendental over the field \( Ke \);

4) if \( \delta \in \text{Der}_K(\mathcal{A}) \), then \( \delta \mid_{eA} = \frac{d}{ds} \phi \) (resp., \( \delta \mid_{Ae} = \frac{d}{ds} \phi \)), where \( \phi \) (resp., \( \psi \)) is the \( \mathcal{A}^\delta \)-algebra endomorphism of \( eA \) (resp., \( A \delta \)) that maps \( s \) to \( s + e \).

The proof of the \( K \)-derivation case of the proposition above is similar as the proof of [E2, Proposition 1.3.21]. The \( K \)-\( \mathcal{E} \)-derivation case follows from Theorem 2.1 and the \( K \)-derivation case of the proposition. So we skip the detailed proof of this proposition here.

From the proposition above we also have the following

Corollary 3.3. Assume further that \( \mathcal{A} \) is algebraic over \( K \). Let \( \delta \) be a LN \( K \)-derivation or \( K \)-\( \mathcal{E} \)-derivation of \( \mathcal{A} \). Then \( \mathcal{A}^\delta \cap \text{Im} \delta \) does not contain any nonzero idempotent of \( \mathcal{A} \).

Actually, the LN condition on \( \delta \) in the corollary above can be dropped. See [Z4, Corollary 3.9]. For more results on the idempotents in the image of LF or LN \( K \)-derivations and \( K \)-\( \mathcal{E} \)-derivations of algebraic \( K \)-algebras, see [Z5] and [Z4].

Next, we consider Theorem 1.2, first, for all LN \( K \)-derivations and \( K \)-\( \mathcal{E} \)-derivations of \( \mathcal{A} \).

Lemma 3.4. Theorem 1.2 holds for all LN \( K \)-derivations and \( K \)-\( \mathcal{E} \)-derivations of \( \mathcal{A} \).

Proof: Note first that by Theorem 2.1 and Corollary 2.4 it suffices to show the lemma for all LN \( D \in \text{Der}_K(\mathcal{A}) \). Let \( e \), \( s \) and \( a \) be as in Lemma 3.1. Then \( \phi_{-s}(a), \psi_{-s}(a) \in \mathcal{A}^D \) for all \( a \in \mathcal{A} \). By Eq. (3.16) we see that \( D \) maps \( \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \phi_{-s}(D^j(a)) s^{j+1} \) to \( ae \), and by Eq. (3.17) \( D \) maps \( \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \psi_{-s}(D^j(a)) s^{j+1} \) to \( ea \). Hence \( ea, ae \in \text{Im} D \) for all \( a \in \mathcal{A} \).
To show $aeb \in \text{Im} D$ for all $a, b \in \mathcal{A}$, note first that by Eq. (3.16) for $ae$ and Eq. (3.17) for $eb$ we have

$$aeb = \sum_{i,j=0}^{\infty} \frac{1}{i!j!} \phi_{-s}(D^i(a)) \cdot s^{i+j+1} \psi_{-s}(D^j(b)).$$

Then $D$ maps $\sum_{i,j=0}^{\infty} \frac{1}{i!j!} \phi_{-s}(D^i(a)) \cdot s^{i+j+1} \psi_{-s}(D^j(b))$ to $aeb$. Therefore, we have $(e) \subseteq D$, i.e., Theorem 1.2, 1) holds for $D$, as desired. 

Now we assume that $D$ is LF and consider the case that the base field $K$ is algebraically closed. In this case $D$ has the Jordan-Chevalley decomposition $D = D_s + D_n$ over $K$ (see [E2, Proposition 1.3.8]) such that $D_s$ is semi-simple and $D_n$ is LN.

Let $\Lambda$ be the set of all distinct eigenvalues of $D_s$ and $A_{\lambda}$ ($\lambda \in \Lambda$) the corresponding eigenspace $D_s$. Then $A_s$ has the following direct sum decomposition:

$$A = \bigoplus_{\lambda \in \Lambda} A_{\lambda}.$$  

(3.18)

Actually, the decomposition above gives a $K$-algebra grading of $A$, i.e., $A_{\lambda} A_{\mu} \subseteq A_{\lambda + \mu}$ for all $\lambda, \mu \in \Lambda$. This is because $D_s$ and $D_n$ by [E2, Proposition 1.3.13] are also $K$-derivations of $A$. In particular, $A_0$ is a $K$-subalgebra of $A$. Furthermore, each $A_{\lambda}$ is $D$ (and also $D_s$ and $D_n$) invariant. Therefore we have

$$\text{Im} D = \bigoplus_{\lambda \in \Lambda} D(A_{\lambda}).$$  

(3.19)

**Lemma 3.5.** Assume that $K$ is algebraically closed. Then

1) $A^D \subseteq A_0$.

2) $\text{Im} D = D_n(A_0) \oplus \bigoplus_{\lambda \neq \lambda \in \Lambda} A_{\lambda}$.

3) $(e) \subseteq \text{Im} D$ for all idempotents $e \in A^D \cap \text{Im} D$.

*Proof:* 1) By [E2, Proposition 1.3.9, i)] we have

$$A^D = \text{Ker } D_s \cap \text{Ker } D_n.$$  

(3.20)

Since $\text{Ker } D_s = A_0$, we hence have $A^D \subseteq A_0$.

2) For each $0 \neq \lambda \in \Lambda$, we have $D(A_{\lambda}) \subseteq A_{\lambda}$ and

$$D |_{A_{\lambda}} = D_s |_{A_{\lambda}} + D_n |_{A_{\lambda}} = \lambda I_{A_{\lambda}} + D_n |_{A_{\lambda}}.$$

Since $D_n$ is LN, $D |_{A_{\lambda}}$ by Lemma 2.3 is a $K$-linear automorphism of $A_{\lambda}$, whence $A_{\lambda} \subseteq \text{Im } D$ for all $0 \neq \lambda \in \Lambda$. Note that $D |_{A_0} = D_n |_{A_0}$.

Then by Eq. (3.19) the statement follows.

3) Note that $D_n$ is a LN $K$-derivation of $A$ (as pointed out above) and $e \in \text{Ker } D_n$ by Eq. (3.20). Applying lemma 3.4 to $D_n$ we have $(e) \subseteq \text{Im } D_n$. Therefore it suffices to show $\text{Im } D_n \subseteq \text{Im } D$.
Since \( A_\lambda \) is \( D_n \) invariant for all \( \lambda \in \Lambda \), we have
\[
\text{Im } D_n = D_n(A_0) \oplus \bigoplus_{0 \neq \lambda \in \Lambda} D_n(A_\lambda).
\]
Since \( D_n(A_\lambda) \subseteq A_\lambda \) for all \( \lambda \in \Lambda \), by statement 2) we have \( \text{Im } D_n \subseteq \text{Im } D \), as desired. □

Now we drop the assumption that \( K \) is algebraically closed and show Theorem 1.2 for all LF \( K \)-derivations \( D \) of \( A \).

**Proof of Theorem 1.2, 1):** Let \( \bar{K} \) be the algebraic closure of \( K \) and \( \bar{A} = \bar{K} \otimes_K A \). Then we may identify \( A \) as a \( K \)-subalgebra of \( \bar{A} \) in the standard way. Denote by \( \bar{D} \) the \( \bar{K} \)-linear extension map of \( D \) from \( A \) to \( \bar{A} \). Then it is easy to see that \( \bar{D} \) is a LF \( \bar{K} \)-derivation of \( \bar{A} \).

Let \( e \) be an idempotent in \( \bar{A} \cap \text{Im } \bar{D} \). Then \( e \) also lies in \( \bar{A} \cap \text{Im } \bar{D} \), and by Lemma 3.5, 3) we have \( e \bar{A}, \bar{A} e, \bar{A} e \bar{A} \subseteq \bar{D}(\bar{A}) \), whence \( e \bar{A}, \bar{A} e, \bar{A} e \bar{A} \subseteq A \cap \bar{D}(\bar{A}) \). Then Theorem 1.2, 1) immediately follows from the lemma below. □

**Lemma 3.6.** Let \( V \) be a vector space over a field \( K \) (not necessarily of characteristic zero) and \( f \) a \( K \)-linear endomorphism of \( V \). Let \( L \) be a field extension of \( K \), \( \bar{V} := L \otimes_K V \) and \( \bar{f} \) the \( L \)-linear extension map of \( f \) from \( V \) to \( \bar{V} \). Identify \( V \) as a \( K \)-subspace of \( \bar{V} \) in the standard way. Then \( f(V) = V \cap \bar{f}(\bar{V}) \).

**Proof:** It suffices to show that for each \( v \in V \), there exists \( u \in \bar{A} \) such that \( f(u) = v \) if (and only if) there exists \( \bar{u} \in \bar{A} \) such that \( \bar{f}(\bar{u}) = v \). Let \( v_i \) (\( 1 \leq i \leq n \)) be the \( K \)-linearly independent vectors in \( V \) such that \( v \in \text{Span}_K \{ v_i \mid 1 \leq i \leq n \} \) and \( \bar{u} \in \text{Span}_L \{ v_i \mid 1 \leq i \leq n \} \). By using the coordinates of \( \bar{u} \) and \( v \), and the transformation matrix of \( f \) with respect to \( \{ v_i \mid 1 \leq i \leq n \} \), we see that the problem becomes the following problem on linear systems: for all \( y \in K^n \) and \( n \times n \) matrix \( A \) with entries in \( K \), the linear system \( Ax = y \) has a solution in \( L^n \). But this can be easily verified, e.g., by applying elementary row operations to transform \( A \) into an up-triangular matrix. □

4. **The \( \varepsilon \)-Derivation Case of Theorem 1.2**

Throughout this section we let \( K \) and \( A \) be as in Theorem 1.2 and fix a LF (locally finite) \( K \)-\( \varepsilon \)-derivation \( \delta \) of \( A \). Write \( \delta = I - \phi \) for...
some $K$-algebra endomorphism $\phi$ of $A$. Note that $A^\delta = A^\phi := \{u \in A \mid \phi(u) = u\}$ and $\phi$ is also LF.

We first assume that $K$ is algebraically closed. In this case $\phi$ has the Jordan-Chevalley decomposition $\phi = \phi_n + \phi_s$ over $K$ (e.g., see [E2, Proposition 1.3.8]) such that $\phi_s$ is semi-simple and $\phi_n$ is nilpotent.

Let $\Lambda$ be the set of all distinct eigenvalues of $\phi_s$ and $A_\lambda$ ($\lambda \in \Lambda$) the corresponding eigenspace of $\phi_s$. Then $A$ has the following direct sum decomposition:

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda. \quad (4.21)$$

Furthermore, each $A_\lambda$ ($\lambda \in \Lambda$) is $\phi$ (and also $\phi_s$ and $\phi_n$) invariant, whence $A_\lambda A_\mu \subseteq A_{\lambda \mu}$ for all $\lambda, \mu \in \Lambda$. In particular, $A_1$ is a $K$-subalgebra of $A$. Therefore we have

$$\text{Im} \delta = \bigoplus_{\lambda \in \Lambda} \delta(A_\lambda). \quad (4.22)$$

**Lemma 4.1.** Assume that $K$ is algebraically closed. Then

1) $A^\delta \subseteq A_1$.
2) $\text{Im} \delta = \phi_n(A_1) \oplus \bigoplus_{1 \neq \lambda \in \Lambda} A_\lambda$.
3) $eA, Ae \subseteq \text{Im} \delta$ for all idempotents $e \in A^\delta \cap \text{Im} \delta$.

**Proof:** 1) Note that by the uniqueness of the Jordan-Chevalley decomposition of $\delta = \delta_s + \delta_n$ it is easy to see that $\delta_s = I - \phi_s$ and $\delta_n = -\phi_n$. Then by [E2, Proposition 1.3.9, i)] we have

$$A^\delta = \text{Ker}(I - \phi_s) \cap \text{Ker} \phi_n = A_1 \cap \text{Ker} \phi_n. \quad (4.23)$$

Hence $A^\delta \subseteq A_1$.

2) Note that for all $\lambda \in \Lambda$, we have $\delta(A_\lambda) \subseteq A_\lambda$ and

$$\delta \mid_{A_\lambda} = (I_{A_\lambda} - \phi_s \mid_{A_\lambda}) - \phi_n \mid_{A_\lambda} = (1 - \lambda)I_{A_\lambda} - \phi_n \mid_{A_\lambda}. \quad (4.24)$$

Since $\phi_n$ is LN, by Lemma 2.3, $\delta \mid_{A_\lambda}$ ($1 \neq \lambda \in \Lambda$) is a $K$-linear automorphism of $A_\lambda$. Hence $A_\lambda \subseteq \text{Im} \delta$ for all $1 \neq \lambda \in \Lambda$. Furthermore, since $\delta \mid_{A_1} = -\phi_n \mid_{A_1}$, by Eq. (4.22), the statement follows.

3) By statement 1) we have $e \in A_1$, and by Statement 2), $e \in \phi_n(A_1)$. Since $(-\phi_n) \mid_{A_1} = \delta \mid_{A_1}$ by Eq. (4.24), we see that $-\phi_n \mid_{A_1}$ is a LN $K$-derivation of $A_1$. Applying Lemma 3.4 to the $K$-algebra $A_1$ and the $K$-derivation $(-\phi_n)$ of $A_1$ we have $eA_1, A_1e \subseteq \phi_n(A_1) = \delta(A_1) \subseteq \text{Im} \delta$.

Note that for all $1 \neq \lambda \in \Lambda$, we also have $eA_\lambda, A_\lambda e \subseteq A_\lambda$ (for $e \in A_1$). Then by statement 2) we see that statement 3) follows. \(\Box\)

**Remark 4.2.** From the proof of Lemma 4.1 3) it is easy to see that we also have $A_\lambda eA_\mu \subseteq \text{Im} \delta$ for all idempotents $e \in A^\delta \cap \text{Im} \delta$ and all
\[ \lambda, \mu \in \Lambda \text{ with } \lambda\mu \neq 1. \text{ In particular, if } \lambda^{-1} \notin \Lambda \text{ for all } 0,1 \neq \lambda \in \Lambda, \text{ then we also have } (e) \subseteq \text{Im} \delta, \text{ as the } K\text{-derivation case of Theorem 1.2.} \]

In general, it is still unknown weather or not \( A_{\lambda e}A_{\lambda^{-1}} \subseteq \text{Im} \delta \) for all \( \lambda \in \Lambda \text{ with } \lambda^{-1} \neq 1 \).

The rest of the proof of Theorem 1.2, 2) for \( \delta \), i.e., without assuming that \( K \) is algebraically closed, is similar as that of the proof of Theorem 1.2, 1) at the end of the previous section. So we skip it here.

Note also that Theorem 1.2, 2) for the LN \( K \)-\( \mathcal{E} \)-derivations of \( A \) has been established in Lemma 3.4 in the previous section.

5. Proof of Proposition 1.4

In this section we give a proof for Proposition 1.4. Note first that if \( 1 \in A_{\delta} \), then Proposition 1.4 immediately follows from Theorem 1.2. In particular, this is the case when \( \delta = I - \phi \) with \( \phi(1) = 1 \). So we need only to show the proposition for LF (locally finite) \( K\)-\( \mathcal{E} \)-derivations of \( A \).

Throughout this section we let \( \delta \) be a LF \( K\)-\( \mathcal{E} \)-derivations of \( A \) and write \( \delta = I - \phi \) with \( e := \phi(1) \). Note that \( \phi \) is also LF and \( e \) is an idempotent of \( A \), for \( \phi \) is a \( K \)-algebra endomorphism of \( A \).

**Lemma 5.1.** For each \( i \geq 0 \), set \( e_i = \phi^i(1) \). Then

1) \( e_i e_j = e_j e_i = e_j \) for all \( 0 \leq i \leq j \).

2) There exists \( d \geq 1 \) such that \( e_k = e_d \) for all \( k \geq d \).

**Proof:**

1) For all \( 0 \leq i \leq j \), consider

\[ e_i e_j = \phi^j(1)\phi^i(\phi^{-i}(1)) = \phi^j(1 \cdot \phi^{-i}(1)) = \phi^j(1) = e_j. \]

Similarly, we also have \( e_j e_i = e_j \).

2) If \( e = 0 \), then \( \phi(1) = 0 \), whence \( \phi = 0 \) for \( \phi \in \text{End}_K(A) \). In this case we may choose \( d = 1 \). Assume \( e \neq 0 \) and let \( V \) be the \( K \)-subspace spanned over \( K \) by \( \phi^i(1) \) \( (i \geq 0) \). Then \( V \) is \( \phi \)-invariant and finite dimensional over \( K \), for \( \phi \) is LF. Let \( p(t) = t^d + \sum_{i=0}^{d-1} c_i t^i \in K[t] \) be the minimal polynomial of \( \phi |_V \). Then by applying \( p(\phi) \) to 1 we get

\[ e_d + \sum_{i=0}^{d-1} c_i e_i = 0. \tag{5.25} \]

Multiplying \( e_d \) (from the left or the right) to the equation above and by statement 1) we get \( (1 + \sum_{i=0} c_i) e_d = 0 \). If \( e_d = 0 \), then the statement obviously holds. So we assume that \( e_d \neq 0 \) and get
\[(5.26) \quad 1 + \sum_{i=0}^{d-1} c_i = 0.\]

Let \(0 \leq j \leq d - 1\) such that \(c_j \neq 0\) but \(c_i = 0\) for all \(j < i \leq d - 1\). Multiplying \(e_j\) to Eq. (5.25) and by statement 1) and Eq. (5.26) we get

\[0 = e_d + \left( \sum_{i=0}^{j} c_i \right) e_j = e_d + \left( \sum_{i=0}^{d-1} c_i \right) e_j = e_d - e_j.\]

Hence \(e_d = e_j\). By statement 1) again \(e_d = e_d e_{d-1} = e_j e_{d-1} = e_{d-1}\). Then by using the induction on \(k \geq d\), it is easy to see that the statement indeed holds. \(\Box\)

**Proof of Proposition 1.4** As pointed out at the beginning of this section we need only to show the \(K\)-\(\mathcal{E}\)-derivation case of the proposition.

Let \(\delta\) be a LF \(K\)-\(\mathcal{E}\)-derivation of \(A\) with \(1 \in \text{Im}\, \delta\). Let \(e = \phi(1), \quad d \geq 1\) and \(e_d\) be as in Lemma 5.1. If \(e_d = 0\), i.e., \(\phi^d(1) = 0\), then \(\phi^d = 0\), for \(\phi^d \in \text{End}_{K}(A)\). In other words, \(\phi\) is (locally) nilpotent. Then by Lemma 2.3, \(\delta\) is invertible, i.e., the proposition holds in this case.

Assume \(e_d \neq 0\). Then by Lemma 5.1 2) we have \(\phi(e_d) = e_d\), whence \(\phi^d(e_d) = e_d\). Let \(\delta_d := I - \phi^d\). Then \(\delta_d\) is also a LF \(K\)-\(\mathcal{E}\)-derivation of \(A\) (for \(\phi^d \in \text{End}_{K}(A)\) and is LF) and \(e_d \in A^{\delta_d}\).

Next, we show that \(e_d\) also lies in \(\text{Im}\, \delta_d\). Let \(u \in A\) such that \(\delta(u) = 1\). Then \(\phi(u) = u - 1\), and inductively we have

\[\phi^d(u) = u - 1 - e_1 - \cdots - e_{d-1}.\]

Multiplying \(e_d\) from the left to the equation above and by Lemma 5.1 1) we have

\[\phi^d(e_d u) = e_d \phi^d(u) = e_d u - d e_d.\]

Then \(\phi^d(e_d u) = e_d u - e_d\), which means \(e_d = \delta_d(e_d u) \in \text{Im}\, \delta_d\). Therefore \(e_d \in A^{\delta_d} \cap \text{Im}\, \delta_d\). Applying Theorem 1.2 2) to the \(K\)-\(\mathcal{E}\)-derivation \(\delta_d\) and the idempotent \(e_d\) we have \(e_d A \subseteq \text{Im}\, \delta_d\).

One the other hand, since \(\phi^d(1 - e_d) = e_d - e_d = 0\), we have \(\phi^d((1 - e_d)A) = 0\), i.e., the restriction of \(\delta_d\) on \((1 - e_d)A\) is the identity map of \((1 - e_d)A\), whence \((1 - e_d)A \subseteq \text{Im}\, \delta_d\). Since \(A = e_d A + (1 - e_d)A\), we have \(\text{Im}\, \delta_d = A\). Furthermore, since \(\delta_d = I - \phi^d = (I - \phi) \sum_{i=0}^{d-1} \phi^i\), we have \(\text{Im}\, \delta_d \subseteq \text{Im}(I - \phi)\), whence \(\text{Im}\, \delta = \text{Im}(I - \phi) = A\), as desired. \(\Box\)
References

[BCW] H. Bass, E. Connell and D. Wright, The Jacobian Conjecture, Reduction of Degree and Formal Expansion of the Inverse. Bull. Amer. Math. Soc. 7, (1982), 287–330. [MR 83k:14028].

[B] J. Bergen, Derivations in Prime Rings. Canad. Math. Bull. 26 (1983), 267-270.

[BFF] M. Brešar, A. Fošner and M. Fošner, A Kleinecke-Shirokov Type Condition with Jordan Automorphisms. Studia Math. 147 (2001), no.3, 237-242.

[BV] M. Brešar and AR Villena, The Noncommutative SingerWermer Conjecture and $\phi$-Derivations. J. London Math. Soc. 66 (2002), 710-720.

[DEZ] H. Derksen, A. van den Essen and W. Zhao, The Gaussian Moments Conjecture and the Jacobian Conjecture. To appear in Israel J. Math.. See also arXiv:1506.05192 [math.AC].

[E1] A. van den Essen, The Exponential Conjecture and the Nilpotency Subgroup of the Automorphism Group of a Polynomial Ring. Prepublications. Univ. Autònoma de Barcelona, April 1998.

[E2] A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture. Prog. Math., Vol.190, Birkhäuser Verlag, Basel, 2000.

[E3] A. van den Essen, Introduction to Mathieu Subspaces. “International Short-School/Conference on Affine Algebraic Geometry and the Jacobian Conjecture” at Chern Institute of Mathematics, Nankai University, Tianjin, China. July 14-25, 2014.

[EH] A. van den Essen and L. C. van Hove, Mathieu–Zhao Spaces. To appear.

[EN] A. van den Essen and S. Nieman, Mathieu–Zhao Spaces of Univariate Polynomial Rings with Non-zero Strong Radical. J. Pure Appl. Algebra, 220 (2016), no. 9, 3300–3306.

[EWZ] A. van den Essen, D. Wright and W. Zhao, Images of Locally Finite Derivations of Polynomial Algebras in Two Variables. J. Pure Appl. Algebra 215 (2011), no.9, 2130-2134. [MR2786603]. See also arXiv:1004.0521 [math.AC].

[GN] P. Gabriel and Y. Nouazé, Idéaux Premiers de l’Algèbre Enveloppante d’une Algèbre de Lie Nilpotente. J. Algebra 6 (1967), 77-99.

[J] N. Jacobson, Structure of Rings. Amer. Math. Soc. Coll. Pub. 37, Amer. Math. Soc. Providence R. I., 1956.

[K] O. H. Keller, Ganze Gremona-Transformationen. Monats. Math. Physik 47 (1939), no.1, 299-306. [MR1550818].

[M] O. Mathieu, Some Conjectures about Invariant Theory and Their Applications. Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), 263–279, Sémin. Congr., 2, Soc. Math. France, Paris, 1997. [MR1601155].

[W] D. Wright, On the Jacobian Conjecture. Illinois J. Math., 25 (1981), no. 3, 423–440.

[Z1] W. Zhao, Images of Commuting Differential Operators of Order One with Constant Leading Coefficients. J. Alg. 324 (2010), no. 2, 231–247. [MR2651354]. See also arXiv:0902.0210 [math.CV].

[Z2] W. Zhao, Generalizations of the Image Conjecture and the Mathieu Conjecture. J. Pure Appl. Alg. 214 (2010), 1200-1216. See also arXiv:0902.0212 [math.CV].
W. Zhao, *Mathieu Subspaces of Associative Algebras*. J. Alg. **350** (2012), no.2, 245-272. [MR2859886]. See also arXiv:1005.4260 [math.RA].

W. Zhao, *Some Open Problems on Locally Finite or Locally Nilpotent Derivations and $E$-Derivations*. Preprint.

W. Zhao, *The LFED and LNED Conjectures for Algebraic Algebras*. Preprint.

W. Zhao, *The LFED and LNED Conjectures for Laurent Polynomial Algebras*. Preprint.

W. Zhao, *Images of Ideals under Derivations and $E$-Derivations of Univariate Polynomial Algebras over a Field of Characteristic Zero*. Preprint.

Department of Mathematics, Illinois State University, Normal, IL 61761. Email: wzhao@ilstu.edu