TOPOLOGICAL PROPERTIES OF SELF-SIMILAR FRACTALS WITH ONE PARAMETER

JUN JASON LUO AND LIAN WANG

Abstract. In this paper, we study two classes of planar self-similar fractals $T_{\varepsilon}$ with a shifting parameter $\varepsilon$. The first one is a class of self-similar tiles by shifting $x$-coordinates of some digits. We give a detailed discussion on the disk-likeness (i.e., the property of being a topological disk) in terms of $\varepsilon$. We also prove that $T_{\varepsilon}$ determines a quasi-periodic tiling if and only if $\varepsilon$ is rational. The second one is a class of self-similar sets by shifting diagonal digits. We give a necessary and sufficient condition for $T_{\varepsilon}$ to be connected.

1. Introduction

Let $A$ be a $d \times d$ integer expanding matrix (i.e., all of its eigenvalues are strictly larger than one in modulus), let $\mathcal{D} = \{d_1, \ldots, d_N\} \subset \mathbb{R}^d$ be a digit set with $N = |\det(A)|$. Then we can define an iterated function system (IFS) $\{S_j\}_{j=1}^N$ where $S_j$ are affine maps

$$S_j(x) = A^{-1}(x + d_j), \quad x \in \mathbb{R}^d.$$ 

Since $A$ is expanding, each $S_j$ is a contractive map under a suitable norm [13] of $\mathbb{R}^d$, there is a unique nonempty compact subset $T := T(A, \mathcal{D}) \subset \mathbb{R}^d$ [10] such that

$$T = \bigcup_{j=1}^N S_j(T) = A^{-1}(T + \mathcal{D}).$$

The set $T$ also has the radix expansion

$$T = \left\{ \sum_{k=1}^{\infty} A^{-k}d_{j_k} : d_{j_k} \in \mathcal{D} \right\}. \quad (1.1)$$

We call $T$ a self-affine set generated by the pair $(A, \mathcal{D})$ (or the IFS $\{S_j\}_{j=1}^N$). Moreover, if $T$ has non-void interior (i.e., $T^o \neq \emptyset$), then there exits a discrete set $\mathcal{J} \subset \mathbb{R}^d$ satisfying

$$T + \mathcal{J} = \mathbb{R}^d \quad \text{and} \quad (T^o + t) \cap (T^o + t') = \emptyset \quad \text{with} \quad t \neq t', t, t' \in \mathcal{J}.$$
We call such \( T \) a self-affine tile and \( T + J \) a tiling of \( \mathbb{R}^d \). In particular, if \( A \) is a similarity, then \( T \) is called a self-similar set/tile.

Since the fundamental theory of self-affine tiles was established by Lagarias and Wang ([13],[14],[15]), there have been considerable interests in the topological structure of self-affine tiles \( T \), including but not limited to the connectedness of \( T \) ([7],[8],[12],[1],[6]), the boundary \( \partial T \) ([2],[10],[22]), or the interior \( T^\circ \) of a connected tile \( T \) ([24],[25]). Especially in \( \mathbb{R}^2 \), the study on the disk-likeness of \( T \) (i.e., the property of being a topological disk) has attracted a lot of attentions ([5],[16],[23],[11],[6]). For other related works, we refer to [20],[17],[18],[21] and a survey paper [3].

Any change on the matrix \( A \) and the digit set \( D \) may lead to some change on the topology of \( T(A,D) \). To simplify the analysis on the relations between those two types of “changes”, one may fix an expanding matrix \( A \) and focus on particular choices of the digit set \( D \). Recently Deng and Lau [6] considered a class of planar self-affine tiles \( T \) that are generated by a lower triangular expanding matrix and product-form digit sets. They gave a complete characterization on both connectedness and disk-likeness of \( T \).

Motivated by the above results, in this paper, we investigate the topological properties of the following two classes of self-similar fractals in \( \mathbb{R}^2 \). Assume that \( A \) is a diagonal matrix with equal nonzero entries, hence \( A \) is a similarity. In the first class, we consider a kind of digit sets \( D_\varepsilon \) with a shift \( \varepsilon \) on the \( x \)-coordinates of some digits. We obtain an analogous result to [6].

**Theorem 1.1.** Let \( p \) be an integer with \(|p| = 2m + 1\) where \( m \in \mathbb{N} \), let \( \varepsilon \in \mathbb{R} \). Suppose \( T_\varepsilon \) is the self-similar set generated by \( A = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \) and

\[
D_\varepsilon = \left\{ \begin{bmatrix} i + b_j \\ j \end{bmatrix} : b_j = \frac{1 - (-1)^j}{2} \varepsilon, \quad i, j \in \{0, \pm 1, \ldots, \pm m\} \right\}.
\]

Then \( T_\varepsilon \) is a self-similar tile. Moreover,

(i) if \(|\varepsilon| < |p|\), then \( T_\varepsilon \) is disk-like;

(ii) if \(|p|^n \leq |\varepsilon| < |p|^{n+1}\) for \( n \geq 1 \), then \( T_\varepsilon^\circ \) has \(|p|^n \) components and every closure of the component is disk-like. (see Figure 1)

In fact, Theorem 1.1 can be proved in a more general setting where \( A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \) without causing much difficulty. We omit this for the completeness of the paper.

Moreover, we further consider the quasi-periodic tiling property of \( T_\varepsilon \) (the definition will be recalled in Section 3). Let \( D_{\varepsilon,k} = D_\varepsilon + AD_\varepsilon + \cdots + A^{k-1}D_\varepsilon \) and \( D_{\varepsilon,\infty} = \bigcup_{k=1}^{\infty} D_{\varepsilon,k} \). We prove that

**Theorem 1.2.** With the same \((A,D_\varepsilon)\) as in Theorem 1.1, \( T_\varepsilon + D_{\varepsilon,\infty} \) is a quasi-periodic tiling if and only if \( \varepsilon \) is a rational number.
The second one is a class of self-similar sets $T_\varepsilon$ with a shift $\varepsilon$ on the diagonal digits along the diagonal line. In contrast with the first one, in this class, $T_\varepsilon$ might not be a tile, as the open set condition will not always hold for any $\varepsilon$ (see the remark at the end of Section 4). Let $\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$. Then we have

**Theorem 1.3.** Let $p$ be an integer with $|p| > 2$, $\varepsilon \in \mathbb{R}$. Suppose $T_\varepsilon$ is the self-similar set generated by $A = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ and

$$D_\varepsilon = \left\{ \begin{bmatrix} i + a_{ij} \\ j + a_{ij} \end{bmatrix} : a_{ij} = \delta_{ij}\varepsilon, \ i, j \in \{0, 1, \ldots, |p| - 1\} \right\}.$$  

Then $T_\varepsilon$ is connected if and only if $|\varepsilon| \leq \frac{(|p| - 1)^2}{|p| - 2}$. (see Figure 3)

For the organization of the paper, we prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3 and Theorem 1.3 in Section 4 respectively.

**2. Self-similar tiles**

Let $(A, D_\varepsilon)$ be the pair as in Theorem 1.1 and let $T_\varepsilon := T(A, D_\varepsilon)$ be the associated self-similar set. It suffices to prove the theorem for $p > 0$, otherwise we can replace $A$ by $A^2$ according to the fact

$$T_\varepsilon = A^{-1}(T_\varepsilon + D_\varepsilon) = A^{-2}(T_\varepsilon + D_\varepsilon + AD_\varepsilon)$$

where the digit set $D_\varepsilon + AD_\varepsilon$ can be written as:

$$D_\varepsilon + AD_\varepsilon = \left\{ \begin{bmatrix} pr + l + (pb_k + b_t) \\ pk + t \end{bmatrix} : r, l, k, t \in \{0, \pm1, \ldots, \pm m\} \right\}$$

$$= \left\{ \begin{bmatrix} i + b'_j \\ j \end{bmatrix} : i, j \in \{0, \pm1, \ldots, \pm (2m^2 + 2m)\} \right\},$$

where $b'_j = pb_k + b_t$ with $j = pk + t$.

We denote by $I$ the set of $i = i_1i_2\cdots$ with $i_n \in \{0, \pm1, \ldots, \pm m\}$. In view of (1.1),

$$T_\varepsilon = \left\{ \begin{bmatrix} p(i) + b(j) \\ p(j) \end{bmatrix} : i = i_1i_2\cdots, j = j_1j_2\cdots \in I \right\} \quad (2.1)$$

where

$$p(i) = \sum_n \frac{i_n}{p^n}, \ b(j) = \sum_n \frac{b_{jn}}{p^n} \quad \text{and} \quad p(j) = \sum_n \frac{j_n}{p^n}.$$  

It follows from the above that the range of the $y$-coordinate of $T_\varepsilon$ is the interval $[-\frac{1}{2}, \frac{1}{2}]$. For each fixed $y = p(j)$ such that the radix expansion is unique, then the horizontal cross section of $T_\varepsilon$ is an interval of length 1 with endpoints at $-\frac{1}{2} + b(j)$ and $\frac{1}{2} + b(j)$; for the other $y$-coordinate that has two radix expansions, the horizontal cross section of $T_\varepsilon$ is the union of two intervals with length 1.

The following lemma is essentially the same as Proposition 2.2 in [6].
Lemma 2.1. \( T_\varepsilon \) is a self-similar tile. Moreover, for any sequence \( \{\ell_s\}_{s \in \mathbb{Z}} \) in \( \mathbb{R} \), let \( \mathcal{J} = \{(n + \ell_s, s)^t : n, s \in \mathbb{Z}\} \). Then \( T_\varepsilon + \mathcal{J} \) is a tiling of \( \mathbb{R}^2 \).

Proof. Let \( D = \{0, \pm 1, \ldots, \pm m\} \). For any \((x, y)^t \in \mathbb{R}^2\), since \( T(p, D) = [-\frac{1}{2}, \frac{1}{2}] \), we can find \( s \in \mathbb{Z} \) such that \( y - s \in \left[-\frac{1}{2}, \frac{1}{2}\right] \). Let \( j \in \mathcal{I} \) such that \( y - s = p(j) \). On the other hand, there is \( n \in \mathbb{Z} \) such that \( x - b(j) - \ell_s - n \in \left[-\frac{1}{2}, \frac{1}{2}\right] \). This implies that \( x - b(j) - \ell_s - n = p(i) \) for some \( i \in \mathcal{I} \). It follows that \((x, y)^t \in T_\varepsilon + (n + \ell_s, s)^t\). Hence \( T_\varepsilon + \mathcal{J} = \mathbb{R}^2 \).

Note that for almost all \( y \in \mathbb{R} \), the above \( s, j \) are unique. If we fix such \( y \), then for almost all \( x \in \mathbb{R} \), the above \( n \) is also unique. Therefore, for almost all \((x, y)^t \in \mathbb{R}^2\), the above \( n, s \) are unique. Hence \( \{T_\varepsilon + t : t \in \mathcal{J}\} \) are measure disjoint sets. That means \( T_\varepsilon + \mathcal{J} \) tile \( \mathbb{R}^2 \).

Geometrically, the tile \( T_\varepsilon \) has two sides on the horizontal line \( y = -\frac{1}{2} \) and \( y = \frac{1}{2} \) with length one. Lemma 2.1 implies that the tiling can be moved horizontally. The following is an elementary criterion for connectedness.

Lemma 2.2 ([9,12]). Let \( \{S_j\}_{j=1}^N \) be an IFS of contractions on \( \mathbb{R}^d \) and let \( K \) be its attractor. Then \( K \) is connected if and only if, for any \( i \neq j \in \{1, 2, \ldots, N\} \), there exists a sequence \( i = j_1, j_2, \ldots, j_n = j \) of indices in \( \{1, 2, \ldots, N\} \) so that \( S_{j_k}(K) \cap S_{j_{k+1}}(K) \neq \emptyset \) for all \( 1 \leq k < n \).

Let
\[
 f_{i,j} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = A^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} i + b_j \\ j \end{bmatrix} \right),
\]
where \( i, j \in \{0, \pm 1, \ldots, \pm m\} \). Then \( f_{i,j} \)'s form an IFS which generates \( T_\varepsilon \). By (2.1), the elements of \( f_{i,j}(T_\varepsilon) \) are of the form
\[
\begin{bmatrix} p(i) + b(j) \\ p(j) \end{bmatrix}
\]
(2.2)
where \( i = i_1 i_2 \cdots, j = j_1 j_2 \cdots \in \mathcal{I} \). For \( j_1 j_2 \cdots j_n \) with \( j_i \in \{0, \pm 1, \ldots, \pm m\} \), if we denote
\[
 G_{j_1 \cdots j_n} = \bigcup_{i_1} \cdots \bigcup_{i_n} f_{i_1, j_1} \circ f_{i_2, j_2} \circ \cdots \circ f_{i_n, j_n}(T_\varepsilon),
\]
(2.3)
then
\[
 T_\varepsilon = \bigcup_{j_1 \cdots j_n} G_{j_1 j_2 \cdots j_n}.
\]

For simplicity of our statements, we write \( i_0 = i_1 \cdots i_n, j_0 = j_1 \cdots j_n \) and \( \mathbf{0} = \underbrace{0 \cdots 0}_n \) where \( n \geq 1 \).

Proposition 2.3. For \( j_0 = j_1 \cdots j_n \in \{0, \pm 1, \ldots, \pm m\}^n \) and \( k, \ell \in \{0, \pm 1, \ldots, \pm m\} \),
\[(i) \text{ if } |k - \ell| \geq 2, \text{ then } G_{j_0k} \cap G_{j_0\ell} = \emptyset;\]
(ii) if \(|k - l| = 1\), then \(G_{j_0k} \cap G_{j_0l}\) is a line segment if and only if
\[
|\varepsilon| < p^{n+1},
\]
and is a single point if and only if \(|\varepsilon| = p^{n+1}\). (see Figure 1)

Proof. In view of (2.2) and (2.3), we obtain that
\[
G_{j_0k} = \left\{ \left[ \begin{array}{c} p(i) + b(j_0k)j \\ p(j_0k) \end{array} \right] : i,j \in I \right\} (2.4)
\]

From the expression of the \(y\)-coordinate, \(G_{j_0k}\) is a part of \(T_\varepsilon\) between the horizontal lines \(y = \sum_{t=1}^{n} \frac{b}{p^t} + \frac{k}{p^{n+1}} - \frac{1}{2p^{n+1}} \) and \(y = \sum_{t=1}^{n} \frac{b}{p^t} + \frac{k+1}{p^{n+1}} - \frac{1}{2p^{n+1}} \). Hence the part (i) follows.

From (2.3), we have
\[
G_{j_0} = \bigcup_{i_1,\ldots,i_n} \left( A^{-n}T_\varepsilon + \left[ \begin{array}{c} p(i_0) + b(j_0) \\ p(j_0) \end{array} \right] \right)
= \bigcup_{i_1,\ldots,i_n} \left( A^{-n}T_\varepsilon + \left[ \begin{array}{c} p(i_0) \\ 0 \end{array} \right] + \left[ \begin{array}{c} b(j_0) \\ p(j_0) \end{array} \right] \right)
= G_0 + \left[ \begin{array}{c} b(j_0) \\ p(j_0) \end{array} \right]. (2.5)
\]

That is, every \(G_{j_0}\) is a translation of \(G_0\). Hence, to prove the part (ii), we only need to show the cases that \(G_{00} \cap G_{01}\) and \(G_{00} \cap G_{0(-1)}\), as other situations are their translations. Since \(G_{01}\) and \(G_{0(-1)}\) are symmetric with respect to \(x\)-axis, it suffices to consider \(G_{00} \cap G_{01}\). By making use of (2.4),

\[
G_{00} = \left\{ \left[ \begin{array}{c} p(i) + b(00j) \\ p(00j) \end{array} \right] : i,j \in I \right\}
\]

\[
G_{01} = \left\{ \left[ \begin{array}{c} p(i) + b(01j) \\ p(01j) \end{array} \right] : i,j \in I \right\}
\]

From the proof of part (i), we know that the intersection of \(G_{00} \cap G_{01}\) has a unique \(y\)-coordinate \(\frac{1}{2p^{n+1}}\). By the expression of \(G_{00}\), all digits in \(j\) are \(m\). Since \(b_i = 0\) if \(i\) is even; \(b_i = \varepsilon\) if \(i\) is odd. The \(x\)-coordinate of the element with \(y\)-coordinate of \(\frac{1}{2p^{n+1}}\) in \(G_{00}\) is as follows:

If \(m\) is even, \(b_m = 0\), then \(b(00j) = 0\). Hence the \(x\)-coordinate is \(x = p(i)\). Since \(\{p(i) : i \in I\} = [-\frac{1}{2}, \frac{1}{2}]\), the \(x\)’s form a unit interval
\[
I_1 = [-\frac{1}{2}, \frac{1}{2}]
\]

If \(m\) is odd, \(b_m = \varepsilon\), then \(b(00j) = \sum_{t=-n+2}^{\infty} \frac{\varepsilon}{p^t} = \frac{\varepsilon}{p^{n+1}(p-1)}\). Hence the \(x\)-coordinate is \(x = p(i) + \frac{\varepsilon}{p^{n+1}(p-1)}\). Such \(x\)’s form a unit interval
\[
I_1 = [\frac{\varepsilon}{p^{n+1}(p-1)} - \frac{1}{2}, \frac{\varepsilon}{p^{n+1}(p-1)} + \frac{1}{2}]
\]
In both two cases, we denote \( I_1 = [\alpha, \alpha + 1] \).

Similarly, by the expression of \( G_{01} \), all digits in \( j \) are \(-m\). Then the x-coordinate of the element with y-coordinate of \( \frac{1}{2^{p^{n+1}}} \) in \( G_{01} \) is of the following form:

If \( m \) is even, \( x' = \frac{\epsilon}{p^{n+1}} + p(\hat{i}) \), which determines a unit interval

\[
I_2 = \left[ \frac{\epsilon}{p^{n+1}} - \frac{1}{2}, \frac{\epsilon}{p^{n+1}} + \frac{1}{2} \right].
\]

If \( m \) is odd, \( x' = \frac{\epsilon}{p^{n+1}(p-1)} + \frac{\epsilon}{p^{n+1}} + p(\hat{i}) \), which determines a unit interval

\[
I_2 = \left[ \frac{\epsilon}{p^{n+1}(p-1)} + \frac{\epsilon}{p^{n+1}} - \frac{1}{2}, \frac{\epsilon}{p^{n+1}(p-1)} + \frac{\epsilon}{p^{n+1}} + \frac{1}{2} \right].
\]

In both two cases, we denote \( I_2 = [\beta, \beta + 1] \).

It follows that if \( G_{00} \cap G_{01} \neq \emptyset \), then the y-coordinate of the intersection is \( \frac{1}{2^{p^{n+1}}} \) and the x-coordinate of the intersection is \( I_1 \cap I_2 \). Note that \( I_1 \cap I_2 \) is an empty set when \( |\alpha - \beta| = \frac{\epsilon}{p^{n+1}} > 1 \); a single point when \( |\alpha - \beta| = \frac{\epsilon}{p^{n+1}} = 1 \); and an interval when \( |\alpha - \beta| = \frac{\epsilon}{p^{n+1}} < 1 \). Therefore the part (ii) is proved. \( \square \)

**Proposition 2.4.** Let \( j_0 = j_1j_2 \cdots j_n \in \{0, \pm 1, \ldots, \pm m\}^n \) for some \( n \geq 1 \), then \( G_{j_0} \) is a self-similar tile. Moreover, \( G_{j_0} \) is disk-like if and only if \( |\epsilon| < p^{n+1} \).

**Proof.** Since \( G_{j_0} \) is a translation of \( G_0 \) by \( (2.5) \), we only show the case for \( G_0 \). By \( (2.3) \), we see that

\[
G_{0j} = \bigcup_{i_1, \ldots, i_{n+1}} \left( A^{-n-1}T_\epsilon + \left[ \frac{p(i_1 \cdots i_{n+1}) + b(0j)}{p(0j)} \right] \right)
\]

\[
= \bigcup_{i_1, \ldots, i_{n+1}} A^{-1} \left( A^{-n}T_\epsilon + A \left[ \frac{p(i_1 \cdots i_{n+1}) + b(0j)}{p(0j)} \right] \right)
\]

\[
= \bigcup_{i_1, \ldots, i_{n+1}} A^{-1} \left( A^{-n}T_\epsilon + \left[ \frac{p(i_2 \cdots i_{n+1}) + i_1 + \frac{b_1}{p^n}}{p^n} \right] \right)
\]

\[
= \bigcup_{i_1} A^{-1} \left( \bigcup_{i_2, \ldots, i_{n+1}} \left( A^{-n}T_\epsilon + \left[ \frac{p(i_2 \cdots i_{n+1})}{0} \right] \right) + \left[ i_1 + \frac{b_1}{p^n} \right] \right)
\]

\[
= \bigcup_{i_1} A^{-1} \left( G_0 + \left[ i_1 + \frac{b_1}{p^n} \right] \right)
\]

From \( G_0 = \bigcup_{j=-m}^m G_{0j} \), it follows that

\[
G_0 = A^{-1}(G_0 + D')
\]

(2.6)

where

\[
D' = \left\{ \left[ \begin{array}{c} i + \frac{b_i}{p^n} \\ \frac{j}{p^n} \end{array} \right] : i, j \in \{0, \pm 1, \ldots, \pm m\} \right\}.
\]
Let \( U = \begin{bmatrix} 1 & 0 \\ 0 & p^n \end{bmatrix} \). Then \( UG_0 = A^{-1}(UG_0 + UD') \), where

\[
UD' = \left\{ \begin{bmatrix} i + \frac{b_j}{p^n} \\ j \end{bmatrix} : i, j \in \{0, \pm 1, \ldots, \pm m\} \right\}.
\]

By Lemma 2.1, \( UG_0 \) is a self-similar tile, so is \( G_0 \). Moreover, \( \mathcal{J}' = \{(r + \ell_s, \frac{s}{p^n})^t : r, s \in \mathbb{Z}\} \) is a tiling set of \( G_0 \) for any sequence \( \{\ell_s\}_{s \in \mathbb{Z}} \subset \mathbb{R} \).

According to (2.6), we can write the corresponding IFS of \( G_0 \) as

\[
f'_{i,j} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = A^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} i + \frac{b_j}{p^n} \\ j \end{bmatrix} \right), \quad i, j \in \{0, \pm 1, \ldots, \pm m\}.
\]

We will verify the disk-likeness of \( G_0 \) through the following claims.

(i) We claim that \( f'_{i,j}(G_0) \cap f'_{i+1,j}(G_0) \) contains an interior point of \( G_0 \) for any \( i, j \).

It suffices to show that \( G_0 \cap (G_0 + (1, 0)^t) \) contains a point of \( (G_0 \cup (G_0 + (1, 0)^t))^0 \). For \( i, j \in \mathcal{I} \), we have \( p(i) \in [-\frac{1}{2}, \frac{1}{2}] \) and \( \frac{p(j)}{p^n} \in [-\frac{1}{2p^n}, \frac{1}{2p^n}] \). Fix a point \( y_0 \in (-\frac{1}{2p^n}, \frac{1}{2p^n}) \) so that there exists a unique \( j \in \mathcal{I} \) such that \( \frac{p(j)}{p^n} = y_0 \).

\[
x_0 = \frac{b(j)}{p^n} + 1/2
\]

then \( (x_0, y_0)^t \in G_0 \cap (G_0 + (1, 0)^t) \).

On the other hand, since \( y_0 \) is an interior point of \([-\frac{1}{2p^n}, \frac{1}{2p^n}]\) and the set \( \mathcal{J}' = \{(r, \frac{s}{p^n})^t : r, s \in \mathbb{Z}\} \) is a tiling set of \( G_0 \) (taking \( \ell_s = 0 \)), then \( (x_0, y_0)^t \in G_0 + (r, \frac{s}{p^n})^t \) only if \( s = 0 \), i.e., \( (x_0, y_0)^t \in G_0 + (r, 0)^t \). Thus, \( x_0 \) must be in

\[
x_0 = \frac{b(j)}{p^n} + p(i) + r
\]

for some \( i \in \mathcal{I} \). That implies \( p(i) + r = \frac{1}{2} \), hence \( r = 0 \) or \( 1 \) as \( p(i) \in [-\frac{1}{2}, \frac{1}{2}] \).

Then \( (x_0, y_0)^t \notin G_0 + (r, 0)^t \) for any \( r \in \mathbb{Z} \setminus \{0, 1\} \). Therefore \( (x_0, y_0)^t \) must be in \( (G_0 \cup (G_0 + (1, 0)^t))^0 \).

(ii) We claim that \( G_0 \) is connected if and only if \( |\varepsilon| \leq p^{n+1} \). Indeed, if \( G_0 \) is connected, by Proposition 2.3, then \( G_{0k} \cap G_{0(k+1)} \neq \emptyset \) for \( -m \leq k \leq m - 1 \), which implies \( |\varepsilon| \leq p^{n+1} \). On the contrary, if \( |\varepsilon| \leq p^{n+1} \), then \( G_{0k} \cap G_{0(k+1)} \neq \emptyset \) for \( -m \leq k \leq m - 1 \), by Proposition 2.3. Thus, there exist \( i_k, t_k \in \{0, \pm 1, \ldots, \pm m\} \) such that \( f'_{i_k, k}(G_0) \cap f'_{t_k, k+1}(G_0) \neq \emptyset \). That together with (i), can help us select a finite sequence \( \{S_j\}_{j=1}^N \) from \( \{f'_{i,j}\}_{i,j} \) in the following zigzag order:

\[
f'_{-m,-m}, f'_{-m+1,-m}, \ldots, f'_{-m,-m+1}, f'_{-m+1,-m+1}, \ldots, f'_{-m,0}, f'_{-m+1,0}, \ldots, f'_{-m+1,-m}, f'_{-m+2,-m}, \ldots, f'_{-1,-1}, f'_{0,0}, \ldots, f'_{-m,1}, f'_{-m+1,1}, \ldots, f'_{0,1}, f'_{1,1}, \ldots, f'_{m,1}, f'_{m-1,1}, \ldots, f'_{m,m}.
\]

Then each \( f'_{i,j} \) appears at least once in the sequence \( \{S_j\}_{j=1}^N \) and

\[
S_j(G_0) \cap S_{j+1}(G_0) \neq \emptyset, \quad \forall 1 \leq j \leq N - 1.
\]
Figure 1. An illustration of Theorem 1.1 by taking $p = 3$.

Hence $G_0$ is connected by Lemma 2.2.

(iii) We claim that $(G_0)^o$ is connected if $|\varepsilon| < p^{n+1}$. For $-m \leq k \leq m-1$, if $|\varepsilon| < p^{n+1}$, then there exist $i_k, t_k$ such that $f'_{i_k,k}(G_0) \cap f'_{t_k,k+1}(G_0)$ is a horizontal line segment by Proposition 2.3. Suppose $\tilde{z} = f'_{i_k,k}(\tilde{z})$ be the mid-point of the line segment. Since
\[
f'_{i_k,k}(G_0) = A^{-1} \left( G_0 + \left[ i_k + \frac{b_k}{p^n} \right] \right),
\]
\[
f'_{t_k,k+1}(G_0) = A^{-1} \left( G_0 + \left[ i_k + \frac{b_k}{p^n} \right] + \left[ \frac{c}{p^n} \right] \right),
\]
where $c = t_k - i_k + \frac{b_{k+1}-b_k}{p^n}$. Then $G_0 \cap (G_0 + (c, \frac{1}{p^n})^t)$ is a horizontal line segment with positive length and $\tilde{z}$ is its mid-point. Note that the top and bottom sides of $G_0$ are horizontal line segments with length one and the height of $G_0$ is $\frac{1}{p^n}$, it is easy to verify that
\[
\tilde{z} \in G_0 + (r + \ell_s, \frac{s}{p^n})^t \text{ if and only if } \ (r + \ell_s, \frac{s}{p^n})^t = (0,0)^t \text{ or } (c, \frac{1}{p^n})^t.
\]
Then $\tilde{z}$ is an interior point of $G_0 \cup (G_0 + (c, \frac{1}{p^n})^t)$, this means $\tilde{z}$ is an interior point of $f'_{i_k,k}(G_0) \cup f'_{t_k,k+1}(G_0)$. Thus, $f'_{i_k,k}(G_0) \cap f'_{t_k,k+1}(G_0)$ contains an interior point of $G_0$. Hence $(G_0)^o$ is connected by Lemma 2.2.

Now we prove the second part of the proposition. Suppose $|\varepsilon| < p^{n+1}$, then claim (iii) implies $(G_0)^o$ is connected, which yields the disk-likeness of $G_0$ by a theorem of Luo et al. [23]. Suppose $G_0$ is disk-like, then $G_0$ and $(G_0)^o$ are both connected, hence we have $|\varepsilon| \leq p^{n+1}$ by claim (ii). If the equality holds, then $G_{0k} \cap G_{0(k+1)}$
contains only one point for any \( k \), and \( G_{0k} \cap G_{0\ell} = \emptyset \) for \( |k - \ell| \geq 2 \) by Proposition 2.3. Since \( G_0 = \bigcup_{k=-m}^{m} G_{0k} \), \( G_0 \) can be divided into \( p \) parts, and every two adjacent parts intersect at one common point. Hence the intersection point must be at the boundary of \( G_0 \). That is, \( (G_0)^\circ \) is not connected, yielding a contradiction. Therefore, \( G_0 \) is disk-like if and only if \( |\varepsilon| < p^{n+1} \).

Proof of Theorem 1.1: (i) If \(|\varepsilon| < |p|\), by using the same argument as in the proof of Proposition 2.4 we can prove that \( T_\varepsilon \) is disk-like (see also Theorem 3.1 of [6]); If \(|\varepsilon| < |p|^{n+1} \) for \( n \geq 1 \), by Proposition 2.4 then \( G_{j_1 \ldots j_n} \) are disk-like tiles. Moreover, from Proposition 2.3 and the assumption \(|\varepsilon| \geq |p|^{n}\), it follows that the tiles \( G_{j_1 \ldots j_n} \) are either disjoint or meet each other at a single point (see Figure 1). Therefore, we conclude with proving (ii). □

3. Quasi-periodic tiling

A tiling \( T + J \) of \( \mathbb{R}^n \) with a tile \( T \) is called a self-replicating tiling with a matrix \( B \) if for each \( t \in J \) there exists a finite set \( J(t) \subset J \) such that

\[
B(T + t) = \bigcup_{t' \in J(t)} (T + t').
\]

Moreover, we say a tiling \( T + J \) is a quasi-periodic tiling if the following two properties hold:

1. Local isomorphism property: For any finite set \( \Sigma \subset J \), there exists a constant \( R > 0 \) such that every ball of radius \( R \) in \( \mathbb{R}^n \) contains a translating copy \( \Sigma + t \) of \( \Sigma \) such that \( \Sigma + t \subset J \).

2. Local finiteness property: For any \( k \geq 1 \) and \( r > 0 \), there are finitely many translation-inequivalent arrangements of \( k \) points in \( J \) which lie in some ball of radius \( r \).

Under the assumption of Theorem 1.1 we let \( \mathcal{D}_{\varepsilon,k} = \mathcal{D}_\varepsilon + A \mathcal{D}_\varepsilon + \cdots + A^{k-1} \mathcal{D}_\varepsilon \) and \( \mathcal{D}_{\varepsilon,\infty} = \bigcup_{k=1}^{\infty} \mathcal{D}_{\varepsilon,k} \). For \( j \in \{0, \pm 1, \ldots, \pm m\} \), denote by

\[
\hat{j} = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even}. \end{cases}
\]

Then it can be easily seen that

\[
\mathcal{D}_{\varepsilon,\infty} = \left\{ \left[ \sum_{n=0}^{\infty} i_n p^n + \varepsilon \sum_{n=0}^{\infty} \hat{j}_n p^n \right] : i_n, j_n \in \{0, \pm 1, \ldots, \pm m\} \right\} = \left\{ \left[ m_2 + \hat{m}_1 \varepsilon \right] : m_1, m_2 \in \mathbb{Z}, \hat{m}_1 = \sum_{n=0}^{\infty} \hat{j}_n p^n \text{ when } m_1 = \sum_{n=0}^{\infty} j_n p^n \right\}.
\]

Proposition 3.1. \( T_\varepsilon + \mathcal{D}_{\varepsilon,\infty} \) is a self-replicating tiling.
Proof. By Lemma 2.1, obviously $D_{\varepsilon, \infty}$ is a tiling set of $\mathbb{R}^2$. For any $t \in D_{\varepsilon, \infty}$, we let $J(t) = At + D_{\varepsilon}$. Since $AT_{\varepsilon} = T_{\varepsilon} + D_{\varepsilon}$, we have $AT_{\varepsilon} + At = T_{\varepsilon} + D_{\varepsilon} + At$. Hence

$$A(T_{\varepsilon} + t) = \bigcup_{\nu \in J(t)} (T_{\varepsilon} + \nu').$$

Therefore $T_{\varepsilon} + D_{\varepsilon, \infty}$ is a self-replicating tiling by letting $B = A$.

Theorem 3.2. $T_{\varepsilon} + D_{\varepsilon, \infty}$ is a quasi-periodic tiling if and only if $\varepsilon$ is a rational number.

Proof. By the definition, we first show the local isomorphism property holds. Let $\Sigma$ be a finite subset of $D_{\varepsilon, \infty}$, then there exists an integer $\ell$ such that $\Sigma \subseteq D_{\varepsilon, \ell}$. Let

$$R = \text{diam}(A^t T_{\varepsilon}) = \text{diam}(\bigcup_{d \in D_{\varepsilon, \ell}} (T_{\varepsilon} + d)).$$

We claim that every ball $B_x(3R)$ with center $x \in \mathbb{R}^2$ and radius $3R$, contains a translating copy of $\Sigma$. Indeed, for the $x$, there exists a $d_x \in D_{\varepsilon, \infty}$ such that $x \in T_{\varepsilon} + d_x$ by the tiling property. If $d_x \not\in D_{\varepsilon, \ell}$, then there is a larger integer $\ell' > \ell$ such that

$$d_x \in \sum_{j=0}^{\ell-1} A^j d_{j\ell} + \sum_{j=\ell}^{\ell'} A^j d_{j\ell}, \quad \text{where } d_{j\ell} \in D_{\varepsilon}.$$ 

Let $d_x' = \sum_{j=\ell}^{\ell'} A^j d_{j\ell}$, then $d_x - d_x' \in D_{\varepsilon, \ell}$. By using $0 \in T_{\varepsilon}$ and the triangle inequality, for $d \in D_{\varepsilon, \ell}$, we have

$$\|d_x' + d - x\| \leq \|x - d_x\| + \|d_x - d_x'\| + \|d\| \leq 3R.$$ 

Thus $d_x' + D_{\varepsilon, \ell} \subset B_x(3R)$, yielding $d_x' + \Sigma \subset B_x(3R)$ for $d_x \in D_{\varepsilon, \ell}$, the above is still true as $d_x' = 0$.

Now we show $T_{\varepsilon} + D_{\varepsilon, \infty}$ has the local finiteness property if and only if $\varepsilon$ is rational, then the desired result follows.

Suppose $\varepsilon$ is irrational. Consider $m_1 = m(1 + p + \cdots + p^{k-1}) = \frac{1}{2}(p^k - 1)$, $n_1 = (-m)(1 + p + \cdots + p^{k-1}) + 1 \cdot p^k = \frac{1}{2}(p^k + 1)$. If $m$ is an even integer, then $\hat{m}_1 = 0, \hat{n}_1 = p^k$. We have

$$t_k := \left[ \begin{array}{c} m_2 \\ m_1 \end{array} \right], \quad t_k' := \left[ \begin{array}{c} n_2 + \varepsilon p^k \\ n_1 \end{array} \right] \in D_{\varepsilon, \infty}, \quad \forall \ m_2, n_2 \in \mathbb{Z}.$$ 

If $m$ is an odd integer, then $\hat{m}_1 = 1 + p + \cdots + p^{k-1} = \frac{1-p^k}{1-p}$, $\hat{n}_1 = 1 + p + \cdots + p^{k-1} + p^k = \frac{1-p^k}{1-p} + p^k$. We have

$$t_k := \left[ \begin{array}{c} m_2 + \frac{\varepsilon(1-p^k)}{1-p} \\ m_1 \end{array} \right], \quad t_k' := \left[ \begin{array}{c} n_2 + \varepsilon p^k + \frac{\varepsilon(1-p^k)}{1-p} \\ n_1 \end{array} \right] \in D_{\varepsilon, \infty}, \quad \forall \ m_2, n_2 \in \mathbb{Z}.$$ 

10
In any case, by letting \( m_2 \) be the integral part of \( n_2 + \varepsilon p^k \), we always get \( \| t_k - t'_k \| \leq \sqrt{2} \). Thus there are infinitely many translation-inequivalent arrangements of \( t_k \) and \( t'_k \).

On the other hand, suppose \( \varepsilon = \frac{a}{b} \) is a rational number where \( a, b \) are co-prime numbers. Given an integer \( t \), any case, by letting \( m \)

The following simple result is very useful for proving the connectedness of \( T_\varepsilon \).

\[ \begin{bmatrix} m_2 + \hat{m}_1 \varepsilon \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 + \hat{n}_1 \varepsilon \\ n_1 \end{bmatrix} \in D_{\varepsilon,\infty}. \]

\(|m_1 - m_2| \leq c\) can admit at most \( 2c + 1 \) choices for \( m_1 - m_2 \); while \( b((m_2 + \hat{m}_1 \varepsilon) - (n_2 + \hat{n}_1 \varepsilon)) = b(m_2 - n_2) + a(\hat{m}_1 - \hat{n}_1) \in \mathbb{Z} \), hence \( (m_2 + \hat{m}_1 \varepsilon) - (n_2 + \hat{n}_1 \varepsilon) \) can admit at most \( b(2c + 1) \) choices for \( m_2 + \hat{m}_1 \varepsilon - (n_2 + \hat{n}_1 \varepsilon) \). Thus the number of possible differences of any two points of \( D_{\varepsilon,\infty} \) in a square \( S \) with edge length \( c \) is finite. Inductively, we have the same conclusion for \( k \) points in \( S \). Therefore, \( T_\varepsilon + D_{\varepsilon,\infty} \) has the local finiteness property.

**Proof.**

(i) For \( i \neq j \) and \( i + 1 \neq j \), we have \( f_{i,j}(T_\varepsilon) \cap f_{i+1,j}(T_\varepsilon) \neq \emptyset \);

(ii) for \( j \neq i \) and \( j + 1 \neq i \), we have \( f_{i,j}(T_\varepsilon) \cap f_{i,j+1}(T_\varepsilon) \neq \emptyset \);

(iii) for \( i = 0, 1, \ldots, p - 2 \), we have \( f_{i+1,i}(T_\varepsilon) \cap f_{i,i+1}(T_\varepsilon) \neq \emptyset \) and \( f_{i,i}(T_\varepsilon) \cap f_{i+1,i+1}(T_\varepsilon) \neq \emptyset \).

**Proposition 4.1.**

(i) For \( i \neq j \) and \( i + 1 \neq j \), we have \( f_{i,j}(T_\varepsilon) \cap f_{i+1,j}(T_\varepsilon) \neq \emptyset \);

(ii) for \( j \neq i \) and \( j + 1 \neq i \), we have \( f_{i,j}(T_\varepsilon) \cap f_{i,j+1}(T_\varepsilon) \neq \emptyset \);

(iii) for \( i = 0, 1, \ldots, p - 2 \), we have \( f_{i+1,i}(T_\varepsilon) \cap f_{i,i+1}(T_\varepsilon) \neq \emptyset \) and \( f_{i,i}(T_\varepsilon) \cap f_{i+1,i+1}(T_\varepsilon) \neq \emptyset \).

**Proof.**

(i) For \( i \neq j \) and \( i + 1 \neq j \), we see that

\[ f_{i,j} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{x + i}{p} \\ \frac{y + j}{p} \end{bmatrix} \text{ and } f_{i+1,j} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{x + i + 1}{p} \\ \frac{y + j}{p} \end{bmatrix}. \]
Since
\[ f_{i,j}(t_{p-1,1}) = \left[ \frac{i+1}{p} + \frac{j}{p} \right] = f_{i+1,j}(t_{0,1}), \]
we have \( f_{i,j}(T_\varepsilon) \cap f_{i+1,j}(T_\varepsilon) \neq \emptyset \). By symmetry of \( T_\varepsilon \) (with respect to line \( y = x \)), (ii) also holds.

(iii) For \( i = 0, 1, \ldots, p - 2 \), we see that
\[ f_{i+1,i}\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \left[ \frac{x+i+1}{p} \right] \text{ and } f_{i,i+1}\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \left[ \frac{x+i}{p} \right]. \]
Since
\[ f_{i+1,i}(t_{0,p-1}) = \left[ \frac{i+1}{p} \right] = f_{i,i+1}(t_{p-1,0}), \]
we have \( f_{i+1,i}(T_\varepsilon) \cap f_{i,i+1}(T_\varepsilon) \neq \emptyset \).

From
\[ f_{i,i}\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \left[ \frac{x+i+\varepsilon}{p} \right], \]
it follows that
\[ f_{i,i}(t_{p-1,p-1}) = \left[ \frac{i+1+\varepsilon}{p} + \frac{\varepsilon}{p(p-1)} \right] = f_{i+1,i+1}(t_{0,0}). \]
Hence \( f_{i,i}(T_\varepsilon) \cap f_{i+1,i+1}(T_\varepsilon) \neq \emptyset \).

□

Proof of Theorem 1.3:
First we assume that \( \varepsilon \geq 0 \). For the function system \( \{f_{0,0}, f_{1,1}, \ldots, f_{p-1,p-1}\} \), there exists a unique attractor
\[ \pi = \text{seg} \left( \left[ \frac{\varepsilon}{p}, \frac{1+\varepsilon}{p} \right], \left[ \frac{1+\varepsilon}{p}, \frac{1}{p} \right] \right) \]
where \( \text{seg}(u, v) \) denotes the straight line-segment between vector \( u \) and vector \( v \).

Let
\[ V_1 = \bigcup_{i<j} f_{i,j}(T_\varepsilon), \quad V_2 = \bigcup_{i=j} f_{i,j}(T_\varepsilon), \quad V_3 = \bigcup_{i>j} f_{i,j}(T_\varepsilon). \]
Then \( T_\varepsilon = V_1 \cup V_2 \cup V_3 \) and \( \pi \subset V_2 \). It is easy to see that \( T_\varepsilon \) is located between two lines:
\[ L_1 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x + 1 \right\}, \quad L_2 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x - 1 \right\}. \]
\( V_2 \) is located between two lines:
\[ L_3 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = \frac{x+1}{p} \right\}, \quad L_4 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = \frac{x-1}{p} \right\}. \]
$V_1$, $V_3$ are separated by a line:

$$ L_5 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x \right\}. $$

Obviously, the segment $\pi \subset L_5$. (see Figure 2(a))

By Proposition 4.1, for any distinct pairs $(i,j)$, $(i',j')$ with $i \neq j$ and $i' \neq j'$, there exists a sequence $(i,j) = (i_1,j_1), (i_2,j_2), \ldots, (i_n,j_n) = (i',j')$ with $i_k \neq j_k$, $1 \leq k \leq n$ so that $f_{i_k,j_k}(T_{\epsilon}) \cap f_{i_{k+1},j_{k+1}}(T_{\epsilon}) \neq \emptyset$ for all $1 \leq k < n$. On the other hand, for any $i \neq j \in \{0,1,\ldots,p-1\}$ there exists a sequence $i = j_1,j_2,\ldots,j_m = j$ of indices in $\{0,1,\ldots,p-1\}$ so that $f_{j_k,j_k}(T_{\epsilon}) \cap f_{j_{k+1},j_{k+1}}(T_{\epsilon}) \neq \emptyset$ for all $1 \leq k < m$. Hence, by Lemma 2.2 and symmetry of $T_{\epsilon}$ (with respect to the line $L_5$), $T_{\epsilon}$ is connected if and only if $V_1 \cap V_2 \neq \emptyset$.

Since

$$ f_{0,1}(t_{p-1,0}) = \begin{bmatrix} 1 \\ \frac{1}{p} \end{bmatrix} \in f_{0,1}(T_{\epsilon}) \subset V_1 $$

and

$$ f_{p-2,p-1}(t_{p-1,0}) = \begin{bmatrix} \frac{p-1}{p} \\ \frac{1}{p} \end{bmatrix} \in f_{p-2,p-1}(T_{\epsilon}) \subset V_1. $$

It follows that these two points belong to the line $L_5$. We shall consider $V_1 \cap V_2$ by comparing these two points with the endpoints of the segment $\pi$.

(i) If $\epsilon \leq \frac{(p-1)^2}{p}$, i.e., $\frac{\epsilon}{p-1} \leq \frac{p-1}{p}$. Then $f_{p-2,p-1}(t_{p-1,0}) \in \pi$, that is $V_1 \cap V_2 \neq \emptyset$, which implies $T_{\epsilon}$ is connected.

(ii) If $\epsilon > \frac{(p-1)^2}{p}$, i.e., $\frac{\epsilon}{p-1} > \frac{p-1}{p}$. We let a point

$$ \omega := f_{0,0} \circ f_{1,p-1}(t_{0,p-1}) = \begin{bmatrix} \frac{1}{p^2} + \frac{\epsilon}{p} \\ \frac{1}{p^2} + \frac{\epsilon}{p} \end{bmatrix} \in f_{0,0}(T_{\epsilon}) \subset V_2 $$

and a line

$$ L_6 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x - \frac{1}{p} \right\}. $$

Then $\omega \in L_6$ (see Figure 2(b)). Consider the compositions $f_{p-2,p-1} \circ f_{i+1,i}$ where $i = 0,1,\ldots,p-2$, we have

$$ f_{p-2,p-1} \circ f_{i+1,i}(\pi) = \text{seg} \left( \begin{bmatrix} \frac{\epsilon}{p^3(p-1)} + \frac{i+1}{p^2} + \frac{p-2}{p} \\ \frac{\epsilon}{p^3(p-1)} + \frac{i+1}{p^2} + \frac{p-2}{p} \end{bmatrix} \right). $$

Obviously, $f_{p-2,p-1} \circ f_{i+1,i}(\pi) \subset L_6$ holds for any $i$, and the right endpoint of $f_{p-2,p-1} \circ f_{i+1,i}(\pi)$ is equal to the left endpoint of $f_{p-2,p-1} \circ f_{i+2,i+1}(\pi)$. That is,

$$ \pi_1 := \bigcup_{i=0}^{p-2} f_{p-2,p-1} \circ f_{i+1,i}(\pi) = \text{seg} \left( \begin{bmatrix} \frac{\epsilon}{p^3(p-1)} + \frac{(p-1)^2}{p^2} + \frac{p-1}{p} \\ \frac{\epsilon}{p^3(p-1)} + \frac{(p-1)^2}{p^2} + \frac{p-1}{p} \end{bmatrix} \right). $$

Then $\pi_1 \subset f_{p-2,p-1}(T_{\epsilon}) \subset V_1$ and $\pi_1 \subset L_6$ (see Figure 2(b)). Therefore, to show $V_1 \cap V_2 \neq \emptyset$, we only need to compare the $x$-coordinates of $\omega$ and $\pi_1$. 

13
If
\[
\frac{\varepsilon}{p^2(p-1)} + \frac{(p-1)^2}{p^2} \leq \frac{\varepsilon}{p} + \frac{1}{p^2} \leq \frac{\varepsilon}{p^2(p-1)} + \frac{p-1}{p},
\]
i.e.,
\[
\frac{p(p-1)(p-2)}{p^2(p-1)} \leq \varepsilon \leq p-1,
\]
then \(\omega \in \pi_1\). Hence \(V_1 \cap V_2 \neq \emptyset\), which implies \(T_\varepsilon\) is connected. As
\[
\frac{p(p-1)(p-2)}{p^2(p-1)} < \frac{(p-1)^2}{p} \quad \text{when} \quad p > 2,
\]
then \(T_\varepsilon\) is connected if \(\varepsilon \leq p-1\).

(iii) If \(p-1 < \varepsilon \leq \frac{(p-1)^2}{p-2}\), by using a similar argument as above, we can find a point \(\theta\) in \(V_2\) and a line-segment \(\pi_2\) in \(V_1\) so that \(\theta \in \pi_2\). Let
\[
\theta := f_{0,0}(t_{0,p-1}) = \left[ \frac{\varepsilon}{p_{1}} \right] \in f_{0,0}(T_\varepsilon) \subset V_2,
\]
then \(\theta \in L_3\). Define a line-segment
\[
\pi_2 := f_{p-2,p-1}(\pi) = \text{seg} \left( \left[ \frac{\varepsilon}{p_{1}} \right] + \frac{p-2}{p}, \left[ \frac{\varepsilon}{p_{1}} + 1 \right] \right).
\]
Trivially we have \(\pi_2 \subset f_{p-2,p-1}(T_\varepsilon) \subset V_1\) and \(\pi_2 \subset L_3\). (see Figure 2(c))

From \(p-1 < \varepsilon \leq \frac{(p-1)^2}{p-2}\), it follows that
\[
\frac{\varepsilon}{p(p-1)} + \frac{p-2}{p} \leq \frac{\varepsilon}{p} \leq \frac{\varepsilon}{p(p-1)} + \frac{p-1}{p}.
\]
Then \(\theta \in \pi_2\), and \(V_1 \cap V_2 \neq \emptyset\). Hence \(T_\varepsilon\) is connected.

(iv) If \(\varepsilon > \frac{(p-1)^2}{p-2}\). Since the right endpoint of \(\pi\) has the maximum \(x\)-coordinate (or \(y\)-coordinate) in \(T_\varepsilon\), \(f_{i,j}(\pi)\) has the maximum \(x\)-coordinate in \(f_{i,j}(T_\varepsilon)\). As the
The x-coordinate of the right endpoint of $f_{i,j}(\pi)$ is $\frac{\varepsilon}{p(p-1)} + \frac{i+1}{p}$, where $i < j$ and $0 \leq i < p-1$. Then the right endpoint of $\pi_2$ has the maximum x-coordinate, say $\frac{\varepsilon}{p(p-1)} + \frac{p-1}{p}$, in $V_1$. Note that the point $\theta$ has the minimum x-coordinate in $V_2$, and both $\theta$ and $\pi_2$ lie in the same line $L_3$. If $\varepsilon > \frac{(p-1)^2}{p-2}$, then $\frac{\varepsilon}{p} > \frac{\varepsilon}{p(p-1)} + \frac{p-1}{p}$, implying $\theta \notin \pi_2$. Hence $V_1 \cap V_2 = \emptyset$, that is, $T_\varepsilon$ is disconnected.

By combining (i), (ii), (iii) and (iv), we prove that: when $\varepsilon \geq 0$, $T_\varepsilon$ is connected if and only if $\varepsilon \leq \frac{(p-1)^2}{p-2}$ (see Figure 3). Similarly for the case that $\varepsilon \leq 0$, we can show that $T_\varepsilon$ is connected if and only if $\varepsilon \geq -\frac{(p-1)^2}{p-2}$. Therefore, we complete the proof of Theorem 1.3.

![Figure 3](image-url)

(a) $\varepsilon = 3$  
(b) $\varepsilon = 4$  
(c) $\varepsilon = 5$

**Figure 3.** An illustration of Theorem 1.3 by taking $p = 3$.

We remark that under the condition of Theorem 1.3, the open set condition does not always hold. For example if we take $\varepsilon = \ell + k/p$ where $\ell = 0, 1, \ldots, p-2$ and $k = 1, 2, \ldots, p-1$, we have

$$f_{0,0} \circ f_{0,p-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{x}{p^2} + \frac{k}{p^2} + \frac{\ell}{p} \\ \frac{y}{p^2} + \frac{k-1}{p^2} + \frac{\ell+1}{p} \end{bmatrix} = f_{\ell,\ell+1} \circ f_{k,k-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Then $f_{0,0} \circ f_{0,p-1} = f_{\ell,\ell+1} \circ f_{k,k-1}$. Similarly, we have $f_{0,0} \circ f_{p-1,0} = f_{\ell+1,\ell} \circ f_{k-1,k}$ (see Figure 2(a)). Thus the IFS does not satisfy the open set condition [4]. Hence $T_\varepsilon$ is not a tile in this situation [13].

**Acknowledgements:** The first author gratefully acknowledges the support of K. C. Wong Education Foundation and DAAD. The authors also would like to thank Professor Jun Luo of Sun Yat-Sen University for many inspiring discussions.

**References**

[1] S. Akiyama and N. Gjini, *Connectedness of number-theoretic tilings*, Discrete Math. Theoret. Computer Science 7 (2005), no. 1, 269-312.
[2] S. Akiyama and B. Loridant, *Boundary parametrization of self-affine tiles*, J. Math. Soc. Japan **63** (2011), no.2, 525-579.

[3] S. Akiyama and J.M. Thuswaldner, *A survey on topological properties of tiles related to number systems*, Geom. Dedicata, 109 (2004), 89-105.

[4] C. Bandt and S. Graf, *Self-similar sets 7: a characterization of self-similar fractals with positive Hausdorff measure*, Proc. Am. Math. Soc. 114 (1992), 995-1001.

[5] C. Bandt and Y. Wang, *Disk-like self-affine tiles in \( \mathbb{R}^2 \)*, Discrete Comput. Geom. 26 (2001), no.4, 591-601.

[6] Q.R. Deng and K.S. Lau, *Connectedness of a class of planar self-affine tiles*, J. Math. Anal. Appl. 380 (2011), 493-500.

[7] K. Gröchenig and A. Haas, *Self-similar lattice tilings*, J. Fourier Anal. Appl. 1 (1994), 131-170.

[8] D. Hacon, N.C. Saldanha and J.J.P. Veerman, *Remarks on self-affine tilings*, Experiment. Math. 3 (1994), 317-327.

[9] M. Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. 2 (1985), no.2, 381-414.

[10] J.E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. 30 (1981), 713-747.

[11] I. Kirat, *Disk-like tiles and self-affine curves with non-collinear digits*, Math. Comp. 79 (2010), 1019-1045.

[12] I. Kirat and K.S. Lau, *On the connectedness of self-affine tiles*, J. London Math. Soc. 62 (2000), 291-304.

[13] J.C. Lagarias and Y. Wang, *Self-affine tiles in \( \mathbb{R}^n \)*, Adv. Math. 121 (1996), 21-49.

[14] J.C. Lagarias and Y. Wang, *Integral self-affine tiles in \( \mathbb{R}^n \) I. Standard and nonstandard digit sets*, J. Lond. Math. Soc. 54 (1996) 161-179.

[15] J.C. Lagarias and Y. Wang, *Integral Self-affine tiles in \( \mathbb{R}^n \) II. Lattice tilings*, J. Fourier Anal. Appl. 3 (1997), 84-102.

[16] K.S. Leung and K.S. Lau, *Disk-likeness of planar self-affine tiles*, Trans. Amer. Math. Soc. 359 (2007), 3337-3355.

[17] K.S. Leung and J.J. Luo, *Connectedness of planar self-affine sets associated with non-consecutive collinear digit sets*, J. Math. Anal. Appl. 395 (2012), 208-217.

[18] K.S. Leung and J.J. Luo, *Connectedness of planar self-affine sets associated with non-collinear digit sets*, Geom. Dedicata 175 (2015), 145-157.

[19] K.S. Leung and J.J. Luo, *Boundaries of disk-like self-affine tiles*, Discrete Comput. Geom. 50 (2013), 194-218.

[20] H. Li, J. Luo and J.D. Yin, *The properties of a family of tiles with a parameter*, J. Math. Anal. Appl. 335 (2007), 1383-1396.

[21] J.C. Liu, J.J. Luo and H.W. Xie, *On the connectedness of planar self-affine sets*, Chaos, Solitons & Fractals 69 (2014), 107-116.

[22] J. Luo, S. Akiyama and J.M. Thuswaldner, *On the boundary connectedness of connected tiles*, Math. Proc. Cambridge Philos. Soc. 137 (2004), no.2, 397-410.

[23] J. Luo, H. Rao and B. Tan, *Topological structure of self-similar sets*, Fractals 10 (2002), no. 2, 223-227.

[24] S.-M. Ngai and T.-M. Tang, *A technique in the topology of connected self-similar tiles*, Fractals 12 (2004), no. 4, 389-403.

[25] S.-M. Ngai and T.-M. Tang, *Topology of connected self-similar tiles in the plane with disconnected interiors*, Topology Appl. 150 (2005), no. 1-3, 139-155.
College of Mathematics and Statistics, Chongqing University, 401331 Chongqing, China

Institut für Mathematik, Friedrich-Schiller-Universität Jena, 07743 Jena, Germany

E-mail address: jasonluojun@gmail.com

College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P.R. China

E-mail address: lwang@cqu.edu.cn