CAPACITY ON FINSLER SPACES*

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Abstract – Here, the concept of electric capacity on Finsler spaces is introduced and the fundamental
conformal invariant property is proved, i.e. the capacity of a compact set on a connected non-compact Finsler
manifold is conformal invariant. This work enables mathematicians and theoretical physicists to become more
familiar with the global Finsler geometry and one of its new applications.

Keywords – Capacity, conformal invariant, Finsler space

1. INTRODUCTION

Finsler space is the most natural and advanced generalization of Euclidean space, which has many
applications in theoretical physics. The physical notion of capacity is the electrical capacity of a 2-
dimensional conducting surface, which is defined as the ratio of a given positive charge on the conductor
to the value of the potential on its surface.

The capacity of a set as a mathematical concept was introduced first by N. Wiener in 1924 and was
subsequently developed by O. Forstman [1], C. J. de La Vallee Poussin, and several other physicists and
mathematicians in connection with the potential theory.

The concept of conformal capacity was introduced by Loewner [2] and has been extensively
developed for $\mathbb{R}^n$ [3-6]. In particular, it was used by G.D. Mostow to prove his famous theorem on the
rigidity of hyperbolic spaces [5]. The concept of capacity on Riemannian geometry was introduced by J.
Ferrand [7] and developed in the joint work’s of M. Vuorinan and G.J. Martin [8] and [9].

Here, we introduce the concept of capacity for Finsler spaces and prove that, it depends only on the
conformal structure of $(M, g)$, more precisely:

Theorem: Let $(M, g)$ be a connected non-compact Finsler manifold, then the capacity of a compact set
on $M$ is a conformal invariant.
1. PRELIMINARIES

1.1. Finsler metric

Let $M$ be an $n$-dimensional $C^\infty$ manifold. For a point $x \in M$, denote by $T_xM$ the tangent space of $M$ at $x$. The tangent bundle $TM$ on $M$ is the union of tangent spaces $T_xM$. We will denote the elements of $TM$ by $(x,y)$ where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \rightarrow M$ is given by $\pi(x,y) := x$. Throughout this paper we use the Einstein summation convention for the expressions with repeated indices. That is, wherever an index appears twice, once as a subscript, and once as a superscript, then that term is summed over all values of that index.

A Finsler structure on a manifold $M$ is a function $F : TM_0 \rightarrow [0,\infty)$ with the following properties:

(i) $F$ is $C^\infty$ on $TM_0$. (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, i.e. $\forall \lambda > 0 \quad F(x,\lambda y) = \lambda F(x,y)$. (iii) The Hessian of $F^2$ with elements $g_{ij}(x,y) := \frac{1}{2}[F^2(x,y)]_{ij}$ is positive definite on $TM_0$. We recall that, $g_{ij}$ is a homogeneous tensor of degree zero in $y$ and $g_{ij}(x,y)y^iy^j = g(y,y)$, where $g(\ ,\ )$ is the local scalar product on any point of $TM_0$. Then the pair $(M, g)$ is called a Finsler manifold. The Finsler structure $F$ is Riemannian if $g_{ij}(x,y)$ are independent of $y \neq 0$.

1.2. Notations on conformal geometry of Finsler manifolds

Let’s consider two $n$-dimensional Finsler manifolds $(M, g)$ and $(M', g')$ with Finsler structures $F$ and $F'$ and with line elements $(x,y)$ and $(x',y')$ respectively. Throughout this paper we shall assume that coordinate systems on $(M, g)$ and $(M', g')$ have been chosen so that $x' = x$ and $y' = y$ holds for all $i$, unless a contrary assumption is explicitly made. Using this assumption these manifolds can be denoted simply by $M$ and $M'$, respectively. Let $u$ and $v$ be two tangent vectors at a point $x$ of a Finsler manifold $(M, g)$. The angle $\theta$ of $v$ with respect to $u$ is defined by

$$\cos \theta = \frac{g_{ij}(x,u)v^iu^j}{\sqrt{g_{ij}(x,u)v^iv^j}}.$$ 

Clearly this notion of angle is not symmetric. A diffeomorphism $f : M \rightarrow M'$ between two Finsler manifolds is called conformal if for each $p \in M$, $(f_*)_p$ preserves the angles of any tangent vector, with respect to any $y$ in $M$. In this case the two Finsler manifolds are called conformal equivalent or simply conformal. If $M = M'$ then $f$ is called a conformal transformation or conformal automorphism. It can be easily checked that a diffeomorphism is conformal if and only if $f^*g' = e^{2\sigma}g$ for some function $\sigma : M \rightarrow IR$ (this result is due to Knebelman [10]. In fact, the sufficient condition implies that the function $\sigma(x,y)$ be independent of direction $y$, or equivalently $\partial \sigma / \partial y^j = 0$). The diffeomorphism $f$ is called an isometry if $f^*g' = g$. Two Finsler structures $F$ and $F'$ are called conformal if $F'(x,y) = e^{2\sigma(x)}F(x,y)$ or equivalently, $g' = e^{2\sigma(x)}g$. Locally we have $g'_{ij}(x,y) = e^{2\sigma(x)}g_{ij}(x,y)$, and $g''_{ij}(x,y) = e^{-2\sigma(x)}g'_{ij}(x,y)$.

1.3. Some vector bundles and their properties

Let $\pi : TM \rightarrow M$ be the natural projection from $TM$ to $M$. The pull-back tangent space $\pi^*TM$ is defined by $\pi^*TM := \{(x,y,v) | y \in T_xM, v \in T_yM\}$. The pull-back cotangent space $\pi^*TM$ is dual of $\pi^*TM$. Both $\pi^*TM$ and $\pi^*TM$ are $n$-dimensional vector spaces over $TM_0$ [11, 12]. We denote by $SM$ the set consisting of all rays $\{y | \lambda y : \lambda > 0\}$, where $y \in T_xM_0$. Let $SM = \bigcup_{x \in M} SM_x$, then $SM$ has a natural $(2n-1)$ dimensional manifold structure and the total space of a fiber bundle, called $Sphere bundle$ over $M$. We denote the elements of $SM$ by $(x,[y])$ where
y ∈ T₂M₀. Let p : SM → M denote the natural projection from SM to M. The pull-back tangent space p*TM is defined by p*TM := {(x,[y],ν)| y ∈ TₒM₀, ν ∈ TₒM}. The pull-back cotangent space p*TₐM is the dual of p*TM. Both p*TM and p*TₐM are total spaces of vector bundles over SM. We use the following Lemma for replacing the C∞ functions on TM₀ by those on SM.

Lemma 1.1. [13] Let η be the function η : TM₀ → SM, where η(x, y) = (x,[y]) and f ∈ C∞(TM₀). Then there exists a function g ∈ C∞(SM) satisfying η* g = f if and only if f(x, y) = f(x, λy), where y ∈ TₒM₀, λ > 0 and η* is the pull-back of η.

Let f ∈ C∞(M), the vertical lift of f denoted by f V ∈ C∞(TM₀), be defined by f V(x, y) := f ∗π(x, y) = f(x). f V is independent of y and from Lemma 1.1 there is a function g on C∞(SM) related to f V by means of η* g = f V. In the sequel g is denoted by f V for simplicity. It is well known that, if the differentiable manifold M is compact then the Sphere bundle SM is compact, and also it is orientable whether M is orientable or not [14, 15]).

1.4. Nonlinear connections

1.4.1. Nonlinear connection on the tangent bundle TM

Consider π∗ : TTM → TM and put kerπ∗ = {z ∈ TTM | π∗z(z) = 0}, ∀v ∈ TM, then the vertical vector bundle on M is defined by VT M = ∪ kerπ∗. A non-linear connection or a horizontal distribution on TM is a complementary distribution HT M for VT M on TM. These functions are called coefficients of the non-linear connection and will be noted in the sequel by N i j. It is clear that HT M is a vector sub-bundle of TTM called horizontal vector bundle. Therefore we have the decomposition TTM = VT M ⊕ HT M.

Using the induced coordinates (x¹, y¹) on TM , where x¹ and y¹ are called, respectively, position and direction of a point on TM, we have the local field of frames {∂ / ∂x¹, ∂ / ∂y¹} on TTM. Let {dx¹, dy¹} be the dual of {∂ / ∂x¹, ∂ / ∂y¹}. It is well known that we can choose a local field of frames {∂ / ∂x¹, ∂ / ∂y¹} adapted to the above decomposition, i.e. ∂ / ∂x¹ ∈ ξ(HT M) and ∂ / ∂y¹ ∈ ξ(VTM). They are sections of horizontal and vertical bundles, HT M and VTM, defined by ∂ / ∂x¹ = N i j(∂ / ∂y¹), where N i j(x, y) are the coefficients of non-linear γ j k¹ := 1/2 gαβ(∂ / ∂x¹ − ∂ / ∂y¹α + ∂ / ∂y¹β) and C j k¹ = 1/2 gαβ(∂ / ∂x¹α + ∂ / ∂y¹β − ∂ / ∂y¹α).

1.4.2. Nonlinear connections on the sphere bundle SM

Using the coefficients of non linear connection on TM , one can define a non linear connection on SM by using the objects which are invariant under positive re-scaling y → λy. Our preference for remaining on SM forces us to work with N i j := γ j k¹l¹ − C i j l¹ l¹, where l¹ = y¹. We also prefer to work with the local field of frames {∂ / ∂x¹, F k / ∂y¹} and {dx¹, δ / ∂y¹}, which are invariant under the positive re-scaling of y, and therefore, live over SM. They can also be used as a local field of frames over tangent bundle p*TM and cotangent bundle p*TₐM respectively.

1.5. A Riemannian metric on SM

It turns out that the manifold TM₀ has a natural Riemannian metric, known in the literature as Sasaki metric [12, 16]; g̃ = gαβ(x, y)dx¹ ⊗ dx¹ + gαβ(x, y)dy¹ ⊗ dy¹, where gαβ(x, y) is the Hessian of Finsler structure F². They are functions on TM₀ and invariant under positive re-scaling of y, therefore they can be considered as functions on SM. With respect to this metric, the horizontal subspace spanned by δ / ∂x¹ is orthogonal to the vertical subspace spanned by F k / ∂y¹. The metric g̃ is invariant under the positive re-scaling of y and can be considered as a Riemannian metric on SM.
Consider the pull-back vector bundle $p^*TM$ over $SM$. The pull-back tangent bundle $SM$ has a canonical section $l$ defined by $I_{(x,y)} = (x, y^j)$. We use the local coordinate system $(x', y')$ for $SM$, where $y'$ are homogeneous coordinates up to a positive factor. Let $\{\partial_i\}$ be a natural local field of frames for $p^*TM$, where $\partial_i := (x, y^j, x)$. The natural dual co-frame for $p^*TM$ is noted by $\{dx^i\}$. The Finsler structure $F(x, y)$ induces a canonical 1-form on $SM$ defined by $\omega \equiv F_{ij}y^i dy^j$. We use the local coordinate system $(x^i, y^j)$ for $SM$, where $x^i$ are homogeneous coordinates up to a positive factor. Let $\{\partial_i\}$ be a natural local field of frames for $p^*TM$, where $\partial_i := (x, y^j, x)$. The natural dual co-frame for $p^*TM$ is noted by $\{dx^i\}$. The Finsler structure $F(x, y)$ induces a canonical 1-form on $SM$ defined by $\omega \equiv F_{ij}y^i dy^j$. Using a straightforward calculation we get

\[ d\omega = -(g_{ij} - l_j) dx^i \wedge \delta y^j. \]  

1.7. Gradient vector field

For a Riemannian manifold $(SM, \tilde{g})$, the gradient vector field of a function $f \in C^\infty(SM)$ is given by $\nabla f = df(X), \forall X \in \chi(SM)$. Using the local coordinate system $(x^i, y^j)$ for $SM$, the vector field $X \in \chi(SM)$ is given by $X = X^i(x, y) \frac{\partial}{\partial x^i} + Y^j(x, y) F_{ij} \frac{\partial}{\partial y^j}$ where $X^i(x, y)$ and $Y^j(x, y)$ are $C^\infty$ functions on $SM$. A simple calculation shows that locally

\[ \nabla f = g_{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} + F^2 g_{ij} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial y^j}. \]  

The norm of $\nabla f$ with respect to the Riemannian metric $\tilde{g}$ is given by

\[ |\nabla f|^2 = \tilde{g}(\nabla f, \nabla f) = g_{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} + F^2 g_{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}. \]  

2. EXTENSION OF SOME DEFINITIONS TO FINSLER MANIFOLDS

In what follows, $(M, g)$ denotes a connected Finsler manifold of class $C^1$ with dimension $n \geq 2$. Let $(SM, \tilde{g})$ be its Riemannian Sphere bundle. We consider the volume element $\eta(g)$ on $SM$ defined as follows:

\[ \eta(g) := (-1)^N \omega \wedge (d\omega)^{n-1}, \]  

where $N = \frac{n(n+1)}{2}$ and $\omega$ is the Hilbert form of $F$ (This volume element was used for the first time in Finsler geometry by Akbar-Zadeh in his thesis [11] and [17]). Let $C(M)$ be the linear space of continuous real valued functions on $M$, $u \in C(M)$ and $u^V$ its vertical lift on $SM$. For $M$, compact or not, we denote by $H(M)$ the set of all functions in $C(M)$, admitting a generalized $L^p$-integrable gradient $\nabla u^V$ satisfying

\[ I(u, M) = \int_{SM} |\nabla u^V|^p \eta(g) < \infty. \]  

If $M$ is non-compact let us denote by $H^e_0(M)$ the subspace of functions $u \in H(M)$ for which the vertical lift $u^V$ has a compact support in $SM$. A relatively compact subset is a subset whose closure is compact. A function $u \in C(M)$ will be called monotone if for any relatively compact domain $D$ of $M$

\[ \sup_{x \in D} u(x) = \sup_{x \in D} u(x); \quad \inf_{x \in D} u(x) = \inf_{x \in D} u(x). \]  

We denote by $H^e(M)$ the set of monotone functions $u \in H(M)$. 

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We define notion of capacity as follows:

**Definition 2.1.** Capacity of a compact subset $C$ of a non-compact Finsler manifold $M$ is defined by

$$\text{Cap}_u(C) := \inf_u I(u, M),$$

where the infimum is taken over the functions $u \in H_0(M)$ with $u = 1$ on $C$ and $0 \leq u(x) \leq 1$ for all $x$, these functions are said to be admissible for $C$.

The non-compactness condition of $M$ is a necessary condition. In fact, if $M$ is compact, then by putting $u = 1$ in $H_0(M)$ we have $I(u, M) = 0$, therefore the capacity of all subsets is zero and there is nothing to say.

A relative continuum is a closed subset $C$ of $M$ such that $C \cup \{\infty\}$ is connected in Alexandrov’s compactification $\overline{M} = M \cup \{\infty\}$. To avoid ambiguities, the connected closed sets of $M$ that are not reduced to one point will be called continua. In what follows we want to associate conformal invariant function, which is determined entirely by the conformal structure of manifold $M$, at every double point of $M$.

**Definition 2.2.** Let $(M, g)$ be a Finsler manifold. For all $(x_1, x_2)$ in $M^2 := M \times M$ we set

$$\mu_* (x_1, x_2) = \inf_{C \in \alpha(x_1, x_2)} \text{Cap}_u(C),$$

where $\alpha(x_1, x_2)$ is the set of all compact continua subsets of $M$ containing $x_1$ and $x_2$.

### 3. Conformal Property of Capacity

**Lemma 3.1.** Let $(M, g)$ and $(M', g')$ be two conformal related Finsler manifolds, then there exist an orientation preserving diffeomorphism between their sphere bundles.

**Proof:** Let $f : (M, g) \longrightarrow (M', g')$ be a diffeomorphism between two Finsler manifolds. We define a mapping $h$ between their sphere bundles as follows $h : SM \longrightarrow SM'$, where $h([x],[y]) = (f(x), [f(y)])$, and $f_*$ is the differential map of $f$. Since $f_*$ is a linear map, $h$ is well defined. If $f$ is conformal then $f^* g' = \lambda g$, where $\lambda$ is a positive real valued function on $M$ and for components of Finsler metrics $g$ and $g'$ defined on $TM$ and $TM'$ we have $\lambda g = f^* g' = f^* (g'_{y^i} dx^i)$, by definition $(f_*)^* g'_{y^i} (f^* dx^j)(f^* dx^k) = (f_*)^* g'_{y^i} dx^j dx^k$, and therefore $(f_*)^* g'_{y^i} = \lambda g_{y^i}$ or equivalently, $h^* g'_{y^i} = \lambda g_{y^i}$. Let $\omega'$ be the Hilbert form related to the Finsler metric $g'$. By definition

$$\omega' = g'_{y^i} \frac{y^{i'}}{F} dx^{i'} = g'_{y^i} \frac{y^{i'}}{\sqrt{g'_{y^m} y^m}} dx^{i'}.$$

Therefore,

$$h^* \omega' = h^* (g'_{y^i}) \frac{h^* (y^{i'})}{\sqrt{h^* (g'_{y^m} y^m)}} h^* (dx^{i'}) = \sqrt{\lambda} \omega.$$

By applying $h^*$ to (1) we get by straightforward calculation

$$h^* d\omega' = \sqrt{\lambda} d\omega.$$

So if $\eta(g)$ and $\eta(g')$ denote the volume elements of $SM$ and $SM'$ respectively, then from (3), (4) and
(5) we get

\[ h^* (\eta (g')) = (\sqrt{\lambda})^n \eta (g). \] (6)

Therefore \( h \) is an orientation preserving diffeomorphism.

**Lemma 3.2.** Let \( \tilde{f} \) be a diffeomorphism between Finsler manifolds \((M, g)\) and \((M', g')\), and \( h \) a mapping between their sphere bundles with Sasaki metrics, \((SM, \tilde{g})\) and \((SM', \tilde{g}')\). If \( u \in H_0 (M') \) then we have

1. \( | \nabla u^V |^2 = (g^{ij} \frac{\partial u^V}{\partial x^i} \frac{\partial u^V}{\partial x^j})^2 \),
2. \( (u \circ f)^V = u^V \circ h \),
3. \( h^* \frac{\partial u^V}{\partial x^i} = \frac{\partial (u \circ f)^V}{\partial x^i} \).

Therefore, the following diagram is commutative:

![Diagram 1](image-url)

**Proof:**

1. Since the vertical lift of \( u \in H_0 (M') \) is a function of position alone, \( \frac{\partial u^V}{\partial x^i} = 0 \). Therefore the first assertion follows from (2).
2. Let’s consider the projections \( p : SM \to M \) and \( p' : SM' \to M' \). The vertical lifts of \( u \) and \( u \circ f \), are by definition, \( u^V (x', [y']) = u \circ p' (x', [y']) = u(x') \) and

\[ (u \circ f)^V (x, [y]) = (u \circ f) \circ p (x, [y]) = (u \circ f) (x). \]

From which we have

\[ (u \circ f)^V (x, [y]) = (u \circ f) (x) =
u^V (f (x), [y]) = u^V (h (x, [y]) = (u \circ f) (x, [y]). \]

This proves the assertion (2).

3. By definition of \( h^* \) we have \( h^* (\frac{\partial u^V}{\partial x^i}) = h^* (\frac{\partial u^V}{\partial x^i}) . h^* u^V = \frac{\partial}{\partial x^i} (u^V \circ h) \), and from (2) we get assertion (3).

Now we are in a position to prove the following theorem:

**Theorem 3.3.** Let \((M, g)\) be a connected non-compact Finsler manifold, then the capacity of a compact set on \( M \) is a conformal invariant.
Proof: We show that the notion of capacity depends only on the conformal structure of \( M \), or equivalently, for any conformal map \( f \) from Finsler manifold \((M, g)\) onto another Finsler manifold \((M', g')\), we have

\[
\text{Cap}_{u'}(C) = \text{Cap}_u(f(C)).
\]

Since \( SM \) and \( SM' \) are two smooth, orientable manifolds with boundary, then for a smooth, orientation preserving diffeomorphism function \( h: SM \rightarrow SM' \) defined in Lemma 3.1, clearly (by a classical result in differential Geometry, [18]) we have

\[
\int_{SM'} \omega = \int_{SM} h^* \omega, \quad \omega \in \Omega^{2n-1} SM'.
\]

So we get,

\[
I(u, M') = \int_{S(M')} |\nabla u|^p |\eta(g')_\theta| = \int_{SM} h^* (|\nabla u'|^p |\eta(g))_.
\]

Using Lemma 3.2, a straightforward calculation shows that

\[
h^* |\nabla u'|^p = (\sqrt{\lambda})^{-n} |\nabla (u \circ f)|^p.
\]

Using (6) in Lemma 3.1, and relations (7) and (8) we get

\[
I(u, M') = \int_{SM} |\nabla (u \circ f)|^p |\eta(g)| = I(u \circ f, M).
\]

Let \( C \) be a compact set in \( M \), then we have

\[
\text{Cap}_{u'}(C) = \inf_{v \in H_0^M, \|v\|_C = 1} I(v, M), \text{Cap}_u(f(C)) = \inf_{u \in H_0^{M'}, \|u\|_{f(C)} = 1} I(u, M').
\]

Put

\[
A = \{I(v, M) \mid v \in H_0^M, \|v\|_C = 1\},
\]

\[
B = \{I(u, M') \mid u \in H_0^{M'}, \|u\|_{f(C)} = 1\}.
\]

We first show that \( B \subseteq A \). For all \( I(u, M') \in B \), we easily have the following assertions.

- Since \( \text{support}(u^v) \) is compact in \( SM' \), \( h^{-1}(\text{support}(u^v)) = \text{support}(u \circ f)^v \) is compact in \( SM \) and by definition \( u \circ f \in H_0(M) \).
- \( (u \circ f)|_C = 1 \) since \( u|_{f(C)} = 1 \).
- From (9) we have \( I(u \circ f, M) = I(u, M') \).

Therefore, \( I(u \circ f, M) \in A \) and \( B \subseteq A \). By the same argument we have \( A \subseteq B \). Hence, \( \text{Cap}_{u'}(C) = \text{Cap}_u(f(C)) \).

Theorem 3.3, implies that the function \( \mu_u \) is invariant under any conformal mapping. More precisely, if \( f \) is a conformal mapping between Finsler manifolds \((M, g)\) and \((M', g')\), then for all \( x_1, x_2 \in M \) we have

\[
\mu_u(x_1, x_2) = \mu_{u'}(f(x_1), f(x_2)).
\]

In the Riemannian geometry this function is of general interest in the study of global conformal geometry, which can be the subject of further studies in Finsler geometry.
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