Introduction. Finding the state of a quantum system is one of the main tasks for many applications in basic quantum physics [1] as well as in many emerging quantum technologies, such as in the field of quantum information [2, 3]. Using quantum state tomography (QST), one can reconstruct the quantum state represented by a density operator, which contains the full information about the system [2, 4, 5]. Performing a full QST requires serial measurements of a complete set of observables. The size of such a complete set and therefore the number of individual measurements and the measurement time all grow exponentially with the number of qubits in the system. While QST has been performed for many years [7, 8], it was often done in the form of time-dependent measurements [9–14]. In spin-based systems, such as nuclear magnetic resonance [3, 6], it is often not possible or not optimal to use projective measurements. Instead, the established procedure relies on the measurement of free induction decays (FIDs) [7, 8], which may generate information on multiple density operator elements in a single scan [9, 11]. This procedure has therefore been well established in ensemble quantum information processing.

In the case of single spin qubits, such as the nitrogen vacancy (NV) center in diamond [12, 13], this type of measurement is also applicable [14, 15], but the precessing transverse magnetization that is detected in a conventional FID experiment, is not directly observable. Instead, the transverse magnetization (coherence) has to be converted into population of the electron spin and detected as a change of the photoluminescence count rate; it is directly connected to populations of the electron spin, which correspond to the diagonal elements or their linear combination in the density matrix. Since such populations do not change during free evolution, the observable is not time dependent. The other elements of the density matrix can be transformed to the observable through unitary transformations. With a suitable decomposition of the density operator, every element in the basis set can be converted into the observable and therefore be read out by a single measurement. Overall, this procedure provides a dramatic speed-up by several orders of magnitudes, compared to the measurement of precessing magnetization. As an experimental demonstration, we implement this procedure in the single NV center of diamond, which is used in many emerging applications of quantum information and sensing technologies [13, 16–20].

Single-qubit state tomography. We start with the QST of single qubit [21–24]. We consider systems where the measurement of diagonal density operator elements is easy, such as in the NV centers of diamond where the populations can be determined by photon counting [13]. In the case of single qubits, the relevant Hilbert space is spanned by the computational basis \{\ket{0}, \ket{1}\}, which are the eigenstates of the Pauli operator \(Z\) with eigenvalues \(\pm 1\). The density matrix describing the quantum state can be expanded in terms of the unit operator \(E\) and the Pauli matrices \(X, Y\) and \(Z\) as

\[
\rho = c_E E + c_X X + c_Y Y + c_Z Z ,
\]  

(1)

where \(c_E = 1/2\) for a normalized density operator and the other \(c_i\) are the weights of the corresponding Pauli matrices. The diagonal elements \(p_{01}\) and \(p_{22}\), which correspond to the populations \(p_{00}\) and \(p_{11}\) of the states \(\ket{0}\) and \(\ket{1}\), are related to the coefficients \(c_E\) and \(c_Z\) as

\[
c_E = 1/2 = (p_{00} + p_{11})/2, \quad c_Z = (p_{00} - p_{11})/2 .
\]  

(2)

To measure the off-diagonal elements of the density operator \(\rho\), we apply operations \(X_{90}\) and \(Y_{90}\) to transform
them to diagonal elements. Here $X_\alpha$ and $Y_\alpha$ are rotations of the qubit around the $x$- and $y$- axis by an angle $\alpha$. They transform $c_Y Y$ and $c_X X$ to $c_Y Z$ and $-c_X Z$, respectively. Therefore $c_X$ and $c_Y$ can be measured directly in the transformed states.

For the experimental demonstration, we used the electron spin of a single NV center in a diamond sample with natural abundance ($\sim 1.1\%$) of $^{13}$C. The experiments were performed at room temperature. The static magnetic field $B$ was aligned along the symmetry axis of the NV center. The relevant Hamiltonian of the electron spin is then $H_e/(2\pi) = D S_z^2 - B \gamma_e S_z$. Here $S_z$ denotes the spin-1 operator for the electron, $D$ the zero-field splitting and $\gamma_e$ the gyromagnetic ratio [13]. Fig. 1 shows the pulse sequence, which always starts with the polarization of the electron spin into the state $m_S = 0$, using a pulse of a 532 nm laser. The polarization is higher than 70% [21], and in the present work we can approximate the pseudopure state as a pure state $|0\rangle$ as discussed in the Supplementary Material (SM, section IC) [25].

The population of the state $m_S = 0$ can be measured by the count rate $r$ of the fluorescence detected during a second laser pulse, since the state $m_S = 0$ fluoresces more strongly than $m_S = \pm 1$ [13, 21, 34, 35]:

$$r = r_{\text{min}} + p_0 (r_{\text{max}} - r_{\text{min}}).$$

(3)

The maximum count rate $r_{\text{max}}$ corresponds to the system being in state $m_S = 0$, while the minimum count rate $r_{\text{min}}$ results when the system is in $m_S = \pm 1$. The readout is destructive, since the laser pumps the system back to the state $|0\rangle$. Therefore, the measurement time has to be kept relatively short [13] to obtain a good measure of the instantaneous population. To reduce the effects of drift and laser power fluctuations, we always calibrate the count rate against a measurement of $r_{\text{max}}$ obtained after re-pumping the system to the $m_S = 0$ state.

The single qubit is obtained from the electron spin states $m_S = 0$ and $m_S = -1$, which we identify with the two logic states $|0\rangle$ and $|1\rangle$. For the 1-qubit case, the test state preparation and the transformations for the QST was implemented by single microwave (MW) pulses with a Rabi frequency of 9 MHz. The measured count rate $r$ then allows us to determine the populations of the state as

$$p_{|0\rangle} = (r - r_{\text{min}})/\delta_r, \quad p_{|1\rangle} = (r_{\text{max}} - r)/\delta_r,$$

(4)

where $\delta_r \equiv r_{\text{max}} - r_{\text{min}}$. Writing $r_N$, $r_X$, and $r_Y$ for the count rates measured after the 3 operations NOOP (no operation), $X_{90}$ and $Y_{90}$ we obtain the coefficients

$$c_X = (1/2) - (r_Y - r_{\text{min}})/\delta_r,$$

$$c_Y = (r_X - r_{\text{min}})/\delta_r - (1/2),$$

$$c_Z = (r_Z - r_{\text{min}})/\delta_r - (1/2).$$

(5)

To test the QST procedure, we first prepared test states $|0\rangle$, $|1\rangle$, and $|+\rangle$, as indicated in the panel. The top row shows the real and imaginary parts for state $|+\rangle$. The error bars show the standard deviations obtained by repeating the measurements.

$$\delta_r \equiv r_{\text{max}} - r_{\text{min}}.$$
and $m_S = -1$, but now conditional on the $^{14}$N nuclear spin being in the $m_N = 1$ state. The experiments were performed in a center in the $^{12}$C enriched (99.995%) diamond sample [37–39]. We obtained a slightly higher fidelity than the earlier experiments. The results are presented in the SM [25].

**State tomography for 2 qubits.**—Moving to a 2-qubit system, we use the basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and we expand the density operator in a basis of products of single qubit operators:

$$\rho = \sum_{m,n=1}^{4} c_{mn} a_m \otimes b_n$$

(7)

where $a_m, b_n \in \{E, X, Y, Z\}$ represent the unit operator $E$ and the Pauli matrices acting on one of the qubits. Since the trace of a normalized density operator is unity, the coefficient $c_{EE} = 1/4$ is fixed. The goal of the tomography is to determine the other 15 coefficients $c_{mn}$.

The primary observable for measuring populations is the electron spin before the readout pulse is applied, again the photon count rate, which depends on the state of the electron spin before the readout pulse is applied, but is independent of the state of the nuclear spin. The measured count rate is then, in analogy to Eq. (3)

$$r = r_{\text{min}} + (p_{00} + p_{01})(r_{\text{max}} - r_{\text{min}}).$$

(8)

Eq. (4) also holds, with $p_0 \rightarrow p_{00} + p_{01}$ and $p_1 = p_{10} + p_{11}$. Therefore

$$p_{00} + p_{01} = (r_N - r_{\text{min}})/\delta_r$$

(9)

$$p_{10} + p_{11} = (r_{\text{max}} - r_N)/\delta_r$$

(10)

and

$$c_{ZE} = (r_N - r_{\text{min}})/(2\delta_r) - (1/4),$$

(11)

which corresponds to $c_Z$ in Eq. (5) in the single qubit QST.

To determine the remaining coefficients of the density operator, we apply a set of unitary operations $R$ to transform the relevant operators $c_{mn} a_m b_n$ to $c_{mn} ZE$. The coefficient $c_{mn}$ in the transformed density matrix can be directly measured using Eq. (11), by replacing $r_N$ by $r_R$, where $r_R = 2\delta_r [c_{mn} + (1/4)] + r_{\text{min}}$ denotes the count rate measured from the transformed density matrix. Overall we can use 15 measurements to obtain the 15 coefficients, i.e., one measurement for each element of the density operator.

To demonstrate the 2-qubit scheme, we used the electron spin coupled to a single $^{13}$C nuclear spin, where the electron spin in states $m_S = 0$ and $m_S = -1$ was assigned as qubit 1 and $^{13}$C nuclear spin qubit 2. We used a $^{12}$C enriched (99.995%) diamond to minimize decoherence due to additional $^{13}$C nuclear spins [39]. In this context, we focus on the electron and $^{13}$C subsystem with the $^{14}$N in the state $m_N = +1$. The pulse sequence is shown in Fig. 1 with more details given in the SM [25].

required operations can be efficiently generated by applying a small number of MW pulses acting on the electron spin, combined with free precession [40, 45]. To prepare the pure state $|00\rangle$, we first polarized the electron spin, swapped the states of the two qubits and re-polarized the electron spin [42, 43]. Additional details are provided in the SM [25]. We implemented $X_{90} \otimes E$, $Y_{90} \otimes E$ and $X_{180} \otimes E$ by single MW pulses. The other required unitaries were implemented by pulse sequences that were designed by optimal control (OC) theory [42, 46]. These pulse sequences transfer the target operators to $Z \otimes E$ with fidelities of $\geq 0.99$. The pulse sequences consist of up to 3 MW pulses and the same number of free precession periods and total durations up to 15 $\mu$s, which is short compared to the transverse relaxation times $T_2 = 700 \mu$s and $T_2^* = 40 \mu$s of the electron spin. Additional details are given in the SM [25].

As experimental demonstrations, we reconstruct the density matrices of the following states: $s_1 = |00\rangle$, $s_2 = |00\rangle + |11\rangle)/\sqrt{2}$, $s_3 = (|00\rangle + |11\rangle)/\sqrt{2}$ and $s_4 = (|01\rangle + |10\rangle)/\sqrt{2}$. The states $s_2$ - $s_4$ were generated by applying sequences of MW pulses and delays to $|00\rangle$. Each sequence consists of 3 pulses and 3 delays. The theoretical fidelity of the generated state is $> 0.99$.

The experimental results for the real parts of the measured density matrices are illustrated in Fig. 3. The root-mean-square (RMS) values of imaginary parts in the experimental density operator are 0.028, 0.033, 0.039 and 0.026, for the input states $s_1$ - $s_4$, respectively. We present the measured imaginary parts of the density matrices in the SM [25].

The experimental fidelities for the states $s_1$ - $s_4$, are 0.98, 0.97, 0.97, and 0.97, respectively. The main contributions to the deviation from unity are i) dephasing (0.05%), ii) the theoretical imperfections of the pulse sequences (1%), iii) experimental imperfections of the MW pulses.
(1%) and iv) photon counting statistics (2%). Additional details are presented in the SM (Section VC) [25]. We are currently optimizing the conversion sequences such that they combine high fidelity for the unitary conversion operation with suppression of dephasing [31] [37].

Discussion. – Our scheme can be straightforwardly generalized to the multiple qubit system. In a $n$ qubit system, the observable is $Z \sum_{n-1} E_{n}$, denoting a product operator with $Z$ for the electron spin qubit and $E$ for the $n-1$ nuclear spin qubits. Eq. (11) is generalized to

$$c_{Z \sum_{n-1} E_{n}} = \frac{r_{N} - r_{\text{min}}}{(2^{n-1})} - (1/2^{n}).$$

In a similar way, all product operators can be transformed to $Z \sum_{n-1} E_{n}$ by unitary operations. Therefore we need $(2^{2n} - 1)$ measurements for reconstructing the full density operator. More details are presented in the SM (Section VE) [25]. The number of measurements required by the time-dependent experiments (Ramsey) also increases proportional to the number of elements in the density operator. While the precise number depends on the details of the coupling network, additional couplings allow one to extract more density matrix elements from a single FID measurement and therefore reduce the number of FIDs that must be measured [9]. On the other hand, they lead to increased spectral crowding, which requires a larger number of points per FID. As a result, the time saving of the time-independent over the time-dependent approach does not depend on the number of qubits. More details are shown in the SM (Section VE) [25]. We therefore conclude that the time saving of >2 orders of magnitude should be similar for all relevant quantum registers. However, full QST for systems with >3 qubits will probably remain impractical even with this faster method.

Conclusion. – Quantum state tomography is an essential tool for the analysis of quantum mechanical systems as it allows one to extract all available information [1, 3]. Accordingly, efficient procedures for QST are valuable for a vast range of applications where information on multiple density operator elements is accessed [38]. Early QST experiments, e.g. in quantum optics [4, 5] were based on measurements in different optics [4, 5] were based on measurements in different bases to extract the coefficients of the density operator components. In the system that we are considering, only a single observable is available. To access different density operator components, we therefore convert them into the available observable through a set of unitary transformations. Early QST experiments by liquid-state NMR [19], also used a single measurement basis but since the relevant measurement is not projective, it was possible to continuously monitor the time-evolution of the density operator, which converts density operator components that are not directly observable into the observable one [7, 49]. In the case of QST of single spins in solids, the evolution of coherences can not be observed directly; it was therefore replaced by indirect detection using the Ramsey method [50]. While this approach allowed one to transfer the techniques developed for liquid-state NMR to the single-spin systems, it generates a huge overhead, since a single measurement is replaced by a sequence of typically several hundred measurements with different evolution times. In the method presented here, we remove this overhead, which allows a speed-up of the QST by several orders of magnitude compared to the measurement of time-dependent observables, for both the number of required measurements and the overall measurement time (see SM, Section VD) [25]. Similar to the existing procedures, the reconstruction of the density operator can be improved by combing the measurement results with statistical inference methods that result in a density matrix that is close to the physical state [51, 55]. Since QST is the main prerequisite for quantum process tomography (QPT) [57, 58], our method is also very helpful for speeding up QPT. For our experimental demonstration, we used a nitrogen vacancy center in diamond, but the scheme should be equally applicable to other physical systems, such as photons, atomic ensembles and trapped ions [6, 59, 61].

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