Two new classes of projective two-weight linear codes

Canze Zhu and Qunying Liao *

(College of Mathematical Science, Sichuan Normal University, Chengdu Sichuan, 610066)

Abstract. In this paper, for an odd prime \( p \), several classes of two-weight linear codes over the finite field \( \mathbb{F}_p \) are constructed from the defining sets, and then their complete weight distributions are determined by employing character sums. These codes can be suitable for applications in secret sharing schemes. Furthermore, two new classes of projective two-weight codes are obtained, and then two new classes of strongly regular graphs are given.

Keywords. Linear codes; Complete weight enumerators; Secret sharing schemes; Projective two-weight codes; Strongly regular graphs

1 Introduction

Let \( \mathbb{F}_{p^m} \) be the finite field with \( p^m \) elements and \( \mathbb{F}_{p^m}^* = \mathbb{F}_{p^m} \setminus \{0\} \), where \( p \) is an odd prime and \( m \) is a positive integer. An \([n, k, d]\) linear code \( C \) over \( \mathbb{F}_p \) is a \( k \)-dimensional subspace of \( \mathbb{F}_n^m \) with minimum (Hamming) distance \( d \) and length \( n \). The dual code of \( C \) is defined as

\[
C^\perp = \{ c^\perp \in \mathbb{F}_n^m \mid \langle c^\perp, c \rangle = 0 \text{ for any } c \in C \}.
\]

Clearly, the dimension of \( C^\perp \) is \( n - k \). A linear code \( C \) is said to be projective if the minimum distance of \( C^\perp \) is greater than or equal to 3.

Let \( A_i \) (\( i = 1, 2, \ldots, n \)) be the number of codewords with Hamming weight \( i \) in \( C \), then the weight distribution of \( C \) is defined by the polynomial

\[
1 + A_1 z + A_2 z^2 + \cdots + A_n z^n.
\]

\( C \) is called a \( t \)-weight code if the number of nonzero \( A_i \) in the sequence \((A_1, A_2, \ldots, A_n)\) is equal to \( t \). In addition, the complete weight enumerator of a codeword \( c \) is the monomial

\[
w(c) = w_0^{t_0} w_1^{t_1} \cdots w_{p-1}^{t_{p-1}}
\]

in the variables \( w_0, w_1, \ldots, w_{p-1} \), where \( t_i \) (\( 0 \leq i \leq p - 1 \)) denotes the number of components of \( c \) equal to \( i \). The complete weight enumerator of \( C \) is defined to be

*Corresponding author.
E-mail. qunyingliao@sicnu.edu.cn (Q. Liao), canzezhu@163.com (C. Zhu).
Supported by National Natural Science Foundation of China (Grant No. 12071321).
\[ W(\mathcal{C}) = \sum_{c \in \mathcal{C}} w(c). \]

The complete weight enumerator is an important parameter for a linear code, obviously, the weight distribution can be deduced from the complete weight enumerator. In addition, the weight distribution for \( \mathcal{C} \) can be applied to determine the capability for both error-detection and error-correction [32].

Linear codes over finite fields are applied in data storage devices, computer and communication systems, and so on. Since few-weight linear codes have been better applications in secret sharing schemes [5, 33], association schemes [3], authentication codes [10], and so on. A number of two-weight or three-weight linear codes have been constructed [6, 8, 9, 11, 12, 16–22, 24, 25, 27, 31, 34–36]. In particular, projective two-weight codes are very precious as they are closely related to finite projective spaces, strongly regular graphs and combinatorial designs [4, 7, 13]. However, projective two-weight codes are rare and only a few classes are known [4, 6, 8, 9, 16, 18, 20].

In 2007, Ding, et al. gave a construction for linear codes via the trace function from a defining set [9]. Let \( D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{F}_{p^m}^* \) and \( \text{Tr} \) denote the trace function from \( \mathbb{F}_p^m \) onto \( \mathbb{F}_p \), a \( p \)-ary linear code is defined by
\[
\mathcal{C}_D = \{ c(x) = (\text{Tr}(xd_1), \text{Tr}(xd_2), \ldots, \text{Tr}(xd_n)) \mid x \in \mathbb{F}_p^m \}.
\]

Motivated by the above construction, Li, et al. defined a linear code by
\[
\mathcal{C}_D = \{ (a, b) = (\text{Tr}(ax + by))_{(x,y) \in D} \mid (a, b) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \} \tag{1.1}
\]
with \( D \subseteq \mathbb{F}_{p^m}^2 \) [25].

In this paper, for a given positive integer \( d \) and \( \tilde{D} \subseteq \mathbb{F}_{p^m}^2 \), the linear code \( \mathcal{C}_{\tilde{D}} \) is defined by
\[
\mathcal{C}_{\tilde{D}} = \{ (\text{Tr}(ayx^d + bx))_{(x,y) \in \tilde{D}} \mid (a, b) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \} \tag{1.2}
\]
At the same time, we also consider the defining sets
\[
D^* = \{(x, y) \in \mathbb{F}_{p^m}^* \times \mathbb{F}_{p^m}^* \mid \text{Tr}(yx^{d+1}) = 0\} \tag{1.3}
\]
and
\[
D_{\lambda} = \{(x, y) \in \mathbb{F}_{p^m}^* \times \mathbb{F}_{p^m}^* \mid \text{Tr}(yx^{d+1}) = \lambda\} \tag{1.4}
\]
where \( \lambda \in \mathbb{F}_p \). For \( \tilde{D} = D^* \) or \( D_{\lambda} \), the parameters and the complete weight enumerator of \( \mathcal{C}_{\tilde{D}} \) will be determined based on character sums. In particular, \( \mathcal{C}_{\tilde{D}} \) is suitable for applications in secret sharing schemes. Furthermore, the punctured code for \( \mathcal{C}_{D_0} \) or \( \mathcal{C}_{D^*} \) is a projective two-weight code, by which a strongly regular graph with new parameters can be derived.

The paper is organized as follows. In section 2, some related basic notations and results for character sums are given. In section 3, we present the parameters for several classes of two-weight linear codes, and then obtain two classes of projective two-weight codes. In section 4, the proofs for main results are given. In section 5, we show that these codes can be applied for secret sharing schemes, and some strongly regular graphs with new parameters are derived basing on projective two-weight codes. In section 6, we conclude the whole paper.
2 Preliminaries

An additive character $\chi$ of $\mathbb{F}_p^m$ is a function from $\mathbb{F}_p^m$ to the multiplicative group $U = \{ u \mid |u| = 1, \ u \in \mathbb{C} \}$, such that $\chi(x + y) = \chi(x)\chi(y)$ for any $x, y \in \mathbb{F}_p^m$. For each $b \in \mathbb{F}_p^m$, the function $\chi_b(x) = \zeta_{\mathbb{F}_p^m}^{b(x)}$ is an additive character of $\mathbb{F}_p^m$. When $b = 0$, i.e., $\chi_0(c) = 1$ for any $c \in \mathbb{F}_p^m$, $\chi_0$ is called the trivial additive character of $\mathbb{F}_p^m$. The character $\chi := \chi_1$ is called the canonical additive character of $\mathbb{F}_p^m$ and every additive character of $\mathbb{F}_p^m$ can be written as $\chi_b(x) = \chi(bx)$.

The orthogonal property for additive characters is given by

$$\sum_{x \in \mathbb{F}_p^m} \zeta_{\mathbb{F}_p^m}^{b(x)} = \begin{cases} p^m, & b = 0; \\ 0, & \text{otherwise}. \end{cases}$$

We extend the quadratic characters $\eta_m$ of $\mathbb{F}_p^m$ by letting $\eta_m(0) = 0$, then the quadratic Gauss sums $G_m$ over $\mathbb{F}_p^m$ is defined to be

$$G_m = \sum_{c \in \mathbb{F}_p^m} \eta_m(c)\chi(c) = \sum_{c \in \mathbb{F}_p^m} \eta_m(c)\chi(c).$$

Now, some properties for quadratic characters and quadratic Gauss sums are given as follows.

**Lemma 2.1** ([12], Lemma 7) For $x \in \mathbb{F}_p^*$,

$$\eta_m(x) = \begin{cases} 1, & 2|m; \\ \eta_1(x), & \text{otherwise}. \end{cases}$$

**Lemma 2.2** ([26], Theorem 5.15) For the Gauss sums $G_m$ over $\mathbb{F}_p^m$,

$$G_m = (-1)^{m-1}(\sqrt{-1})^{(p-1)^2m\frac{m}{4}} p^{\frac{m}{2}}.$$

**Lemma 2.3** ([26], Theorem 5.33) For $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]$ with $a_2 \neq 0$,

$$\sum_{c \in \mathbb{F}_q} \chi(f(c)) = G_m\eta_m(a_2)\chi(a_0 - a_1^2(4a_2)^{-1}).$$

The Pless power moments are useful for determining the minimum distance of the dual for a linear code.

**Lemma 2.4** ([15], p.259, The Pless power moments) For an $[n, k, d]$ code $C$ over $\mathbb{F}_p$ with the weight distribution $(1, A_1, \ldots, A_n)$, suppose that the weight distribution of its dual code is $(1, A_1^+, \ldots, A_n^+)$, then the first two Pless power moments are

$$\sum_{j=0}^{n} A_j = p^k$$

and

$$\sum_{j=0}^{n} jA_j = p^{k-1}(pn - n - A_1^+),$$

respectively. Furthermore, if $(0, 0) \notin D_\chi$, then $A_1^+ = 0$. 

3 Our Main Results

**Theorem 3.1** For any integer \(m \geq 2\), suppose that \(D_0\) and \(C_{D_0}\) are given by (1.4) and (1.2), respectively, then \(C_{D_0}\) is a \([p^{2m-1} - p^{m-1}, 2m, (p-1)(p^{m-1}-1)p^{m-1}]\) code with the weight distribution in Table 1, and the complete weight enumerator is

\[
W(C_{D_0}) = w_0^{p^{2m-1}-p^{m-1}} + (p^m - p^{m-1} + 1)(p^m - 1)w_0^{p^{2m-2}-p^{m-1}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{2m-2}}
+ (p^{m-1} - 1)w_0^{p^{2m-2}+(p-2)p^{m-1}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{2m-2}-p^{m-1}}.
\] (3.1)

| weight \(w\) | frequency \(A_w\) |
|--------|--------|
| 0      | 1      |
| \((p-1)p^{2m-2}\) | \((p^m - 1)(p^m - p^{m-1} + 1)\) |
| \((p-1)(p^{m-1}-1)p^{m-1}\) | \((p^m - 1)p^{m-1}\) |

**Table 1. the weight distribution of \(C_{D_0}\)**

**Theorem 3.2** For any integer \(m \geq 2\), suppose that \(D^*\) and \(C_{D^*}\) are given by (1.3) and (1.2), respectively, then \(C_{D^*}\) is a \([p^{2m-1} - p^{m} - p^{m-1} + 1, 2m, (p-1)(p^{m-1}-2)p^{m-1}]\) code with the weight distribution in Table 2, and the complete weight enumerator is

\[
W(C_{D^*}) = w_0^{p^{2m-1}-p^{m}-p^{m-1}+1} + (p^m - p^{m-1} + 2)(p^m - 1)w_0^{p^{2m-2}-2p^{m-1}+1} \prod_{i \in \mathbb{F}_p^*} w_i^{(p^m-1)p^{m-1}}
+ (p^{m-1} - 1)(p^m - 1)w_0^{p^{2m-2}+(p-3)p^{m-1}+1} \prod_{i \in \mathbb{F}_p^*} w_i^{(p^{m-1}-2)p^{m-1}}.
\] (3.2)

| weight \(w\) | frequency \(A_w\) |
|--------|--------|
| 0      | 1      |
| \((p-1)(p^{m-1}-1)p^{m-1}\) | \((p^m - 1)(p^m - p^{m-1} + 2)\) |
| \((p-1)(p^{m-1}-2)p^{m-1}\) | \((p^m - 1)(p^{m-1}-1)\) |

**Table 2. the weight distribution of \(C_{D^*}\)**

**Theorem 3.3** For any integer \(m \geq 2\) and \(\lambda \in \mathbb{F}_p^*\), suppose that \(D_\lambda\) and \(C_{D_\lambda}\) are given by (1.4) and (1.2), respectively, then \(C_{D_\lambda}\) is a \([p^{2m-1} - p^{m-1}, 2m, (p^m - p^{m-1}-2)p^{m-1}]\) code with the weight distribution in Table 3, and the complete weight enumerator is

\[
W(C_{D_\lambda}) = w_0^{p^{2m-1}-p^{m-1}} + (p^{m-1} + 1)(p^m - 1)w_0^{p^{2m-2}-p^{m-1}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{2m-2}}
+ \frac{p-1}{2}p^{m-1}(p^m - 1) \sum_{j \in \mathbb{F}_p^* \backslash \{0\}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{2m-2}+\eta(i^2-4\lambda)p^{m-1}}
+ \frac{p-1}{2}p^{m-1}(p^m - 1) \sum_{j \in \mathbb{F}_p^* \backslash \{0\}} \prod_{i \in \mathbb{F}_p^*} w_i^{p^{2m-2}+\eta(i^2-4\lambda)p^{m-1}}.
\] (3.3)

4
Table 3. the weight distribution of $C_{D_{\lambda}}$

| weight $w$        | frequency $A_w$                                      |
|-------------------|-----------------------------------------------------|
| $0$               | 1                                                   |
| $(p - 1)p^{2m-2}$ | $(p^{m-1} - 1)(p^m - 1)$                            |
| $(p^m - p^{m-1} - 2)p^{m-1}$ | $p^{m-1}(p^m - 1)$                                      |

In the following, we give the punctured codes for $C_{D_0}$, $C_{D^*}$ and $C_{D_{\lambda}}$, respectively, and then obtain two classes of projective two-weight codes.

For any $c \in \mathbb{F}_p^*$, if $(p - 1) \mid d$, then

$$\text{Tr}((cy)(cx)^d+1) = c^{d+2}\text{Tr}(yx^{d+1}) = c^2\text{Tr}(yx^{d+1})$$

(3.4)

and

$$\text{Tr}(a(cy)(cx)^d + b(cx)) = c\text{Tr}(ayx^d + bx) \quad (a, b \in \mathbb{F}_p^m).$$

(3.5)

From (3.4)-(3.5), $D_0$ and $D^*$ can be expressed as

$$D_0 = \cup_{c \in \mathbb{F}_p^*} c\overline{D_0}$$

(3.6)

and

$$D^* = \cup_{c \in \mathbb{F}_p^*} c\overline{D^*},$$

(3.7)

respectively, where $\overline{D_0} \subseteq D_0$ and $\overline{D^*} \subseteq D^*$, and then both $C_{\overline{D_0}}$ and $C_{\overline{D^*}}$ defined by (1.2) are just the punctured versions of $C_{D_0}$ and $C_{D^*}$, respectively, whose parameters are given in the following corollaries.

**Corollary 3.1** For any integer $m \geq 2$, suppose that $\overline{D_0}$ and $C_{\overline{D_0}}$ are given by (3.6) and (1.2), respectively, then $C_{\overline{D_0}}$ is a $[\frac{p^{2m-1} - p^{m-1} + 1}{p - 1}, 2m, (p^{m-1} - 1)p^{m-1}]$ code with the weight distribution in Table 4.

Table 4. the weight distribution of $C_{\overline{D_0}}$

| weight $w$        | frequency $A_w$                                      |
|-------------------|-----------------------------------------------------|
| $0$               | 1                                                   |
| $p^{2m-2}$        | $(p^m - p^{m-1} + 1)(p^m - 1)$                      |
| $(p^{m-1} - 1)p^{m-1}$ | $p^{m-1}(p^m - 1)$                                      |

**Corollary 3.2** For any integer $m \geq 2$, suppose that $\overline{D^*}$ and $C_{\overline{D^*}}$ are given by (1.2) and (3.7), respectively, then $C_{\overline{D^*}}$ is a $[\frac{p^{2m-1} - p^{m-1} + 1}{p - 1}, 2m, (p^{m-1} - 2)p^{m-1}]$ code with the weight distribution in Table 5.

Table 5. the weight distribution of $C_{\overline{D^*}}$

| weight $w$        | frequency $A_w$                                      |
|-------------------|-----------------------------------------------------|
| $0$               | 1                                                   |
| $(p^{m-1} - 1)p^{m-1}$ | $(p^m - p^{m-1} + 2)(p^m - 1)$                      |
| $(p^{m-1} - 2)p^{m-1}$ | $(p^m - 1)(p^{m-1} - 1)$                                      |
By Lemma 2.4, we can get that the minimum distance for $C_{\overrightarrow{D_0}}$ or $C_{\overrightarrow{D^*}}$ is 3, thus we have the following corollaries.

**Corollary 3.3** For any integer $m \geq 2$, suppose that $\overrightarrow{D_0}$ and $C_{\overrightarrow{D_0}}$ are given by (3.6) and (1.2), respectively, then $C_{\overrightarrow{D_0}}$ is a \( \left[ \frac{p^{2m-1} - p^{m-1}}{p-1}, \frac{p^{2m-1} - p^{m-1}}{p-1} - 2m, 3 \right] \) code, and $C_{\overrightarrow{D_0}}$ is a projective two-weight code.

**Corollary 3.4** For any integer $m \geq 2$, suppose that $\overrightarrow{D^*}$ and $C_{\overrightarrow{D^*}}$ are given by (3.7) and (1.2), respectively, then $C_{\overrightarrow{D^*}}$ is a \( \left[ \frac{p^{2m-1} - p^{m-1} - 1}{p-1}, \frac{p^{2m-1} - p^{m-1} - 1}{p-1} - 2m, 3 \right] \) code, and $C_{\overrightarrow{D^*}}$ is a projective two-weight code.

For any integer $m$, suppose that $\overrightarrow{D_0} = \overrightarrow{D^*}$ with the weight distribution in Table 6.

**Example 3.1** For $p = 3$, $m = 2$, $d = 2$ and $\lambda = 1$, by (1.2)-(1.4), using MAGMA program, we obtain $C_{\overrightarrow{D_0}} = [24, 4, 12]$ with the weight distribution $1 + 24z^{12} + 56z^{18}$, $C_{\overrightarrow{D^*}} = [16, 4, 6]$ with the weight distribution $1 + 16z^6 + 64z^{12}$ and $C_{\overrightarrow{D_0}} = [24, 4, 12]$ with the weight distribution $1 + 24z^{12} + 56z^{18}$, which are accordant with Theorems 3.1-3.3, respectively.

**Example 3.2** For $p = 3$, $m = 2$, $d = 2$ and $\lambda = 1$, by Example 3.1, we can get $C_{\overrightarrow{D_0}} = [12, 4, 6]$ with the weight distribution $1 + 24z^6 + 56z^9$, $C_{\overrightarrow{D^*}} = [8, 4, 3]$ with the weight distribution $1 + 16z^3 + 64z^6$ and $C_{\overrightarrow{D_0}} = [12, 4, 6]$ with the weight distribution $1 + 24z^6 + 56z^9$. All these codes are almost optimal according to the Griesmer bound [14].

**Remark 3.1** Easily, $C_{\overrightarrow{D^*}}$ is obtained by deleting some components of codewords in $C_{\overrightarrow{D_0}}$. 

| $w$          | $A_w$               |
|--------------|---------------------|
| $0$          | 1                   |
| $\frac{p-1}{2}p^{2m-2}$ | $(\frac{p+1}{2}p^{m-1} + 1)(p^m - 1)$ |
| $(p^m-p^{m-2}+2)p^{m-1}$ | $\frac{p-1}{2}p^{m-1}(p^m - 1)$ |
4 The Proofs for Main Results

4.1 Some auxiliary lemmas

In this subsection, Lemmas 4.1-4.3 are useful for calculating the length and the weights, and Lemma 4.4 is important to calculate the weight distribution.

Lemma 4.1 For $\lambda \in \mathbb{F}_p$, we have

$$\#D_\lambda = p^{2m-1} - p^{m-1}$$

and

$$\#D^* = p^{2m-1} - p^m - p^{m-1} + 1.$$  \hspace{1cm} (4.1)

Proof. By calculating directly, we have

$$\#D_\lambda = \sum_{x \in \mathbb{F}_{p^m}^*} \sum_{y \in \mathbb{F}_{p^m}} (p^{-1} \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{z_1 (\text{Tr}(x^{d+1}y) - \lambda)})$$

$$= p^{-1} \sum_{x \in \mathbb{F}_{p^m}^*} \sum_{y \in \mathbb{F}_{p^m}} (\sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{z_1 (\text{Tr}(yx^{d+1}) - \lambda)} + 1)$$

$$= p^{m-1}(p^m - 1) + p^{-1} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{-z_1 \lambda} \sum_{x \in \mathbb{F}_{p^m}^*} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(z_1 x^{d+1}y)}$$

$$= p^{2m-1} - p^{m-1}.$$  \hspace{1cm} (4.2)

It follows from (1.3)-(1.4) that

$$\#D^* = \#D_0 - (p^m - 1) = p^{2m-1} - p^m - p^{m-1} + 1.$$  \hspace{1cm} □

Lemma 4.2 For any positive integer $d$, suppose that $\lambda, \lambda_1 \in \mathbb{F}_p$, $(a, b) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \setminus \{(0, 0)\}$, and

$$N_{\lambda, \lambda_1}(a, b) = \{(x, y) \in \mathbb{F}_{p^m}^* \times \mathbb{F}_{p^m} \mid \text{Tr}(x^{d+1}y) = \lambda \text{ and } \text{Tr}(ax^{d}y + bx) = \lambda_1\}.$$  \hspace{1cm} (1.3)

Then the following assertions hold.

(1) For $\lambda = \lambda_1 = 0$,

$$\#N_{0, \lambda_1}(a, b)$$

$$= \begin{cases} p^{2m-2} - p^{m-1}, & \text{if } a = 0 \text{ and } b \neq 0, \text{ or } \text{Tr}(ab) \neq 0; \\ p^{2m-2} + (p - 2)p^{m-1}, & \text{if } a \neq 0 \text{ and } \text{Tr}(ab) = 0. \end{cases}$$  \hspace{1cm} (4.3)

(2) For $\lambda = 0$ and $\lambda_1 \neq 0$,

$$\#N_{0, \lambda_1}(a, b)$$

$$= \begin{cases} p^{2m-2}, & \text{if } a = 0 \text{ and } b \neq 0, \text{ or } a \neq 0 \text{ and } \text{Tr}(ab) \neq 0; \\ p^{2m-2} - p^{m-1}, & \text{if } a \neq 0 \text{ and } \text{Tr}(ab) = 0. \end{cases}$$  \hspace{1cm} (4.4)
(3) For \( \lambda \neq 0 \) and \( \lambda_1 = 0 \),

\[
\# N_{0,\lambda_1}(a, b) = \begin{cases} 2^{m-2} - p^{m-1}, & \text{if } a = 0 \text{ and } b \neq 0, \text{ or } a \neq 0 \text{ and } \text{Tr}(ab) = 0; \quad (4.5) \\ 2^{m-2} + \eta_1 (-\lambda \text{Tr}(ab)) p^{m-1}, & \text{if } \text{Tr}(ab) \neq 0. \end{cases}
\]

(4) For \( \lambda \neq 0 \) and \( \lambda_1 \neq 0 \),

\[
\# N_{0,\lambda_1}(a, b) = \begin{cases} 2^{m-2}, & \text{if } a = 0 \text{ and } b \neq 0, \text{ or } a \neq 0 \text{ and } \text{Tr}(ab) = 0; \quad (4.6) \\ 2^{m-2} + \eta_1 (\lambda_1^2 - 4\lambda \text{Tr}(ab)) p^{m-1}, & \text{if } a \neq 0 \text{ and } \text{Tr}(ab) \neq 0. \end{cases}
\]

**Proof.** By calculating directly, we have

\[
\# N_{\lambda,\lambda_1}(a, b) = \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_{p^m}} \left( p^{-1} \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{z_1(\text{Tr}(x^{d+1}y) - \lambda)} \right) \left( p^{-1} \sum_{z_2 \in \mathbb{F}_p} \zeta_p^{z_2(\text{Tr}(axy^d + bx) - \lambda_1)} \right)
\]

\[
= p^{-2} \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_{p^m}} \left( \sum_{z_1 \in \mathbb{F}_p} \zeta_p^{z_1(\text{Tr}(xy^{d+1}) - \lambda)} + 1 \right) \left( \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{z_2(\text{Tr}(axy^d + bx) - \lambda_1)} + 1 \right)
\]

\[
= p^{m-2}(p^m - 1) + p^{-2} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{z_2 \lambda_1} \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(z_2(axy^d + bx))}
\]

\[
+ p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{z_1 \lambda + z_2 \lambda_1} \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}((z_1 + z_2)xy)}
\]

\[
= p^{m-2}(p^m - 1) + \Omega_1 + \Omega_2.
\]

For \( \Omega_1 + \Omega_2 \), we have the following two cases.

**Case 1.** For \( a = 0 \) and \( b \neq 0 \),

\[
\Omega_1 + \Omega_2 = p^{-2} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{z_2 \lambda_1} \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(z_2 bx)}
\]

\[
+ p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{z_1 \lambda + z_2 \lambda_1} \sum_{x \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}((z_1 + z_2)xy)}
\]

\[
= -p^{m-2} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{z_2 \lambda_1}
\]

\[
= \begin{cases} -(p - 1)p^{m-2}, & \lambda_1 = 0; \\ p^{m-2}, & \text{otherwise.} \end{cases}
\]

**Case 2.** For \( a \neq 0 \), one has

\[
\Omega_1 = 0,
\]
and
\[
\Omega_2 = p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{- (z_1 + z_2 \lambda_1)} \sum_{x \in \mathbb{F}_{p^n}} \zeta_p^{\text{Tr}(z_2 bx)} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}((z_1 + z_2 a)x^d y)}
\]
\[
= p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{- (z_1 + z_2 \lambda_1)} \sum_{z_1 + z_2 a = 0} \zeta_p^{\text{Tr}(z_2 bx)} \sum_{y \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}((z_1 + z_2 a)x^d y)}
\]
\[
= p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{- (z_1 + z_2 \lambda_1)} \zeta_p^{- z_1^{-1} z_2^2 \text{Tr}(ab)}.
\]
If \( \text{Tr}(ab) = 0 \), we can obtain
\[
\Omega_2 = p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{- (z_1 + z_2 \lambda_1)}
\]
\[
= \begin{cases} 
(p - 1)^2 p^{m-2}, & \text{if } \lambda = \lambda_1 = 0; \\
-(p - 1)^2 p^{m-2}, & \text{if } \lambda = 0 \text{ and } \lambda_1 \neq 0, \text{ or } \lambda \neq 0 \text{ and } \lambda_1 = 0; \\
p^{m-2}, & \text{if } \lambda \lambda_1 \neq 0.
\end{cases}
\]
If \( \text{Tr}(ab) \neq 0 \), it follows from Lemma 2.3 that
\[
\Omega_2 = p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{- (z_1 + z_2 \lambda_1)} \zeta_p^{- z_1^{-1} z_2^2 \text{Tr}(ab)}
\]
\[
= p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{- z_1 \lambda} \sum_{z_2 \in \mathbb{F}_p^*} \zeta_p^{- (z_1^{-1} \text{Tr}(ab) z_2^2 - \lambda_1 z_2)}
\]
\[
= p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{- z_1 \lambda_1} \left( \zeta_p^{(\lambda_1^2 (4 \text{Tr}(ab))^{-1} z_1)} \eta_1(- \text{Tr}(ab) z_1^{-1}) G_1 - 1 \right)
\]
\[
= p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \left( \zeta_p^{(\lambda_1^2 (4 \text{Tr}(ab))^{-1} z_1^{-1})} \eta_1(- (4 \text{Tr}(ab))^{-1} z_1) \right) - p^{-2} \sum_{z_1 \in \mathbb{F}_p^*} \zeta_p^{- z_1 \lambda_1}
\]
\[
= \begin{cases} 
-(p - 1)^2 p^{m-2}, & \text{if } \lambda = \lambda_1 = 0; \\
(\eta_1(-1) G_1 - (p - 1)) p^{m-2}, & \text{if } \lambda = 0 \text{ and } \lambda_1 \neq 0; \\
p^{m-2}, & \text{if } \lambda \neq 0 \text{ and } 4 \lambda \text{Tr}(ab) - \lambda_1^2 = 0; \\
(\eta_1(4 \lambda \text{Tr}(ab) - \lambda_1^2) G_1^2 + 1) p^{m-2}, & \text{if } \lambda \neq 0 \text{ and } 4 \lambda \text{Tr}(ab) - \lambda_1^2 \neq 0.
\end{cases}
\]
So far, we complete the proof for Lemma 4.2. \(\square\)

**Lemma 4.3** For any positive integer \( d \), \( \lambda_1 \in \mathbb{F}_p^* \), \( (a, b) \in \mathbb{F}_{p^m}^* \times \mathbb{F}_{p^n}^* \setminus \{(0, 0)\} \), and
\[
N_{\lambda_1}^*(a, b) = \{(x, y) \in \mathbb{F}_{p^m}^* \times \mathbb{F}_{p^n}^* \mid \text{Tr}(x^{d+1} y) = 0 \text{ and } \text{Tr}(ax^d y + bx) = \lambda_1)\}. \quad (4.7)
\]
Then, the following assertions hold.

(1) For \( \lambda_1 = 0 \),
\[
\#N_{\lambda_1}^*(a, b) = \begin{cases} 
p^{2m-2} - 2p^{m-1} + 1, & \text{if } a = 0 \text{ and } b \neq 0, \text{ or } \text{Tr}(ab) \neq 0, \text{ or } a \neq 0 \text{ and } b = 0; \\
p^{2m-2} + (p - 3)p^{m-1} + 1, & \text{if } a \neq 0, \text{ b } \neq 0 \text{ and } \text{Tr}(ab) = 0.
\end{cases} \quad (4.8)
\]
For \( \lambda_1 \neq 0 \),
\[
\#N^*_\lambda_1(a, b) = \begin{cases} 
    p^{2m-2} - p^{m-1}, & \text{if } a = 0 \text{ and } b \neq 0, \text{ or } a \neq 0 \text{ and } \text{Tr}(ab) \neq 0, \text{ or } a \neq 0 \text{ and } b = 0; \\
    p^{2m-2} - 2p^{m-1}, & \text{if } a \neq 0, b \neq 0 \text{ and } \text{Tr}(ab) = 0.
\end{cases}
\]

\[(4.9)\]

**Proof.** By the definitions of both \( N_{0, \lambda_1}(a, b) \) and \( N^*_\lambda_1(a, b) \), we have
\[
\#N^*_\lambda_1(a, b) = \#N_{0, \lambda_1}(a, b) - \# \{(x, y) \in \mathbb{F}_{p^m}^* \times \{0\} \mid \text{Tr}(bx) = \lambda_1\}
\]

\[
= \#N_{0, \lambda_1}(a, b) - \begin{cases} 
    p^m - 1, & \text{if } b = \lambda_1 = 0; \\
    p^{m-1} - 1, & \text{if } b \neq 0 \text{ and } \lambda_1 = 0; \\
    0, & \text{if } b = 0 \text{ and } \lambda_1 \neq 0; \\
    p^{m-1}, & \text{if } b \neq 0 \text{ and } \lambda_1 \neq 0.
\end{cases}
\]

\[(4.10)\]

Now from Lemma 4.2 (1)-(2), and \((4.10)\), one can get Lemma 4.3 (1)-(2), respectively. \( \square \)

Since the trace function is an uniform map, the following lemma is obvious.

**Lemma 4.4** For \( t \in \mathbb{F}_p \), denote
\[
A(t) = \{(a, b) \in \mathbb{F}_{p^m}^* \times \mathbb{F}_{p^m} \mid \text{Tr}(ab) = t\},
\]
then
\[
\#A(t) = p^{m-1}(p^m - 1).
\]

\[(4.11)\]

### 4.2 The proofs for Theorems 3.1-3.3

**The proof for Theorem 3.1.**

From \((4.1)\), \( C_{D_0} \) has length
\[
n = \#D_0 = p^{2m-1} - p^{m-1}.
\]
Then, for \( \lambda_1 \in \mathbb{F}_p \) and \( a, b \in \mathbb{F}_{p^m} \), by Lemma 4.2, we have the following two cases.

**Case 1.** For \( a = 0 \) and \( b \neq 0 \) or \( \text{Tr}(ab) \neq 0 \),
\[
\#N_{0, \lambda_1}(a, b) = \begin{cases} 
    p^{2m-2} - p^{m-1}, & \lambda_1 = 0; \\
    p^{2m-2}, & \text{otherwise}.
\end{cases}
\]
Hence each codeword of \( C_{D_0} \) has weight
\[
w = n - \#N_{0, 0}(a, b) = (p - 1)p^{2m-2},
\]
and then, by Lemma 4.4, the frequency
\[
A_w = (p^m - 1) + \sum_{t \in \mathbb{F}_p^*} \#A(t) = (p^m - p^{m-1} + 1)(p^m - 1).
\]
Case 2. For $a \neq 0$ and $\text{Tr}(ab) = 0$, 

$$\#N_{0,\lambda_1}(a, b) = \begin{cases} p^{2m-2} + (p-2)p^{m-1}, & \lambda_1 = 0; \\ p^{2m-2} - p^{m-1}, & \text{otherwise}. \end{cases}$$

Hence each codeword of $C_{D_0}$ has weight 

$$w = n - \#N_{0,0}(a, b) = (p - 1)(p^{m-1} - 1)p^{m-1},$$

and then by Lemma 4.4, the frequency 

$$A_w = \#A(0) = p^{m-1}(p^m - 1).$$

So far, we complete the proof for Theorem 3.1. □

The proof for Theorem 3.2.

Basing on (4.2) and Lemma 4.3, in the similar proof as that for Theorem 3.1, one can obtain Theorem 3.2 immediately. □

The proof for Theorem 3.3.

For $\lambda \in \mathbb{F}_p^*$, by (4.1), $C_{D_\lambda}$ has length 

$$n = \#D_\lambda = p^{2m-1} - p^{m-1}.$$ 

Then, for $\lambda_1 \in \mathbb{F}_p$ and $a, b \in \mathbb{F}_p^{m}$, by Lemma 4.2, we have the following two cases.

Case 1. For $a = 0$ and $b \neq 0$, or $a \neq 0$ and $\text{Tr}(ab) = 0$, it follows from (4.5)-(4.6) that 

$$\#N_{\lambda_1,0}(a, b) = \begin{cases} p^{2m-2} - p^{m-1}, & \lambda_1 = 0; \\ p^{2m-2}, & \text{otherwise}. \end{cases}$$

Hence each codeword of $C_{D_\lambda}$ has weight 

$$w = n - \#N_{\lambda_1,0}(a, b) = (p - 1)p^{2m-2},$$

and then the frequency 

$$A_w = (p^m - 1) + \#A(0) = (p^{m-1} + 1)(p^m - 1).$$

Case 2. For $\text{Tr}(ab) \neq 0$, by (4.5)-(4.6), we have 

$$\#N_{\lambda_1,0}(a, b) = p^{2m-2} + \eta_1(\lambda_1^2 - 4\lambda \text{Tr}(ab))p^{m-1}.$$ 

If $\eta_1(-\lambda \text{Tr}(ab)) = -1$, we have 

$$\#N_{\lambda,0}(a, b) = p^{2m-2} - p^{m-1}.$$ 

Thus each codeword of $C_{D_\lambda}$ has weight 

$$w = n - \#N_{\lambda,0}(0, 0) = (p - 1)p^{2m-2}.$$
and the frequency
\[ A_w = \sum_{\eta_1(\lambda t) = -1} \#A(t) = \sum_{\eta_1(t) = -\eta_1(\lambda)} \#A(t) = \frac{p-1}{2} p^{m-1}(p^m - 1). \] (4.17)

If \( \eta_1(-\lambda Tr(ab)) = 1 \), one has
\[ \#N_{\lambda,0}(a, b) = p^{2m-2} + p^{m-1}. \]

Thus each codeword of \( C_{D_{\lambda}} \) has weight
\[ w = n - \#N_{\lambda,0}(0, 0) = (p^m - p^{m-1} - 2)p^{m-1}, \] (4.18)
and then the frequency
\[ A_w = \sum_{\eta_1(\lambda t) = -1} \#A(t) = \sum_{\eta_1(t) = -\eta_1(\lambda)} \#A(t) = \frac{p-1}{2} p^{m-1}(p^m - 1). \] (4.19)

Now, from (4.13)-(4.14) and (4.16)-(4.17), we can obtain the frequency for the weight \( w = (p - 1)p^{2m-2} \) is
\[ A_w = (p^{m-1} + 1)(p^m - 1) + \frac{p-1}{2} p^{m-1}(p^m - 1) = \left( \frac{p+1}{2} p^{m-1} + 1 \right) (p^m - 1). \] (4.20)

So far, by (4.13)-(4.20), we complete the proof for Theorem 3.3. \( \square \)

5 Applications

It is well-known that two-weight linear codes have better applications in secret sharing schemes [5, 33], association schemes [3], authentication codes [10], and so on. In particular, projective two-weight codes are very precious as they are closely related to finite projective spaces, strongly regular graphs and combinatorial designs [4, 7, 13]. Here, we present the following two applications.

5.1 Applications for secret sharing schemes

The secret sharing schemes is introduced by Blakley [2] and Shamir [30] in 1979. Based on linear codes, many secret sharing schemes are constructed [12, 23, 28, 29, 33]. Especially, for those linear codes with all nonzero codewords minimal, their dual codes can be used to construct secret sharing schemes with nice access structures [12].

For a linear code \( C \) with length \( n \), the support of a nonzero codeword \( \mathbf{c} = (c_1, \ldots, c_n) \in C \) is denoted by
\[ \text{supp}(\mathbf{c}) = \{i \mid c_i \neq 0, i = 1, \ldots, n\}. \]

For \( \mathbf{c}_1, \mathbf{c}_2 \in C \), when \( \text{supp}(\mathbf{c}_2) \subseteq \text{supp}(\mathbf{c}_1) \), we say that \( \mathbf{c}_1 \) covers \( \mathbf{c}_2 \). A nonzero codeword \( \mathbf{c} \in C \) is minimal if it does not cover any other codewords in \( C \).

The following lemma is very useful for determining minimal codewords.
Lemma 5.1 (Ashikhmin-Barg Lemma [1]) Let $w_{\text{min}}$ and $w_{\text{max}}$ be the minimal and maximal nonzero weights of the linear code $C$ over $\mathbb{F}_p$, respectively. If

$$\frac{w_{\text{min}}}{w_{\text{max}}} > \frac{p-1}{p},$$

then each nonzero codeword of $C$ is minimal.

By Theorems 3.1-3.3, if $m \geq 3$, then

(1) for $C_{D_0}$, it holds that

$$\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{(p-1)(p^{m-1}-1)p^{m-1}}{(p-1)p^{2m-2}} = 1 - \frac{1}{p^{m-1}} > \frac{p-1}{p};$$

(2) for $C_{D^*}$, it holds that

$$\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{(p-1)(p^{m-1}-2)p^{m-1}}{(p-1)(p^{m-1}-1)p^{m-1}} = 1 - \frac{1}{p^{m-1}} > \frac{p-1}{p};$$

(3) for $C_{D_\lambda}$, it holds that

$$\frac{w_{\text{min}}}{w_{\text{max}}} = \frac{(p-1)p^{2m-2}-2p^{m-1}}{(p-1)p^{2m-2}} = 1 - \frac{2}{(p-1)p^{m-1}} > \frac{p-1}{p}.$$

Hence, by Lemma 5.1, all nonzero codewords in $C_{D_0}$, $C_{D^*}$ or $C_{D_\lambda}$ are minimal. Therefore, their dual codes can be employed to construct secret sharing schemes with interesting access structures.

5.2  Strongly regular graphs with new parameters

Some notations and results for strongly regular graphs are given as follows [4].

A connected graph of $N$ vertices is strongly regular with parameters $(N, K, \lambda, \mu)$, if it is regular with valency $K$, and according as the two given vertices are adjacent or non-adjacent, the number of vertices joined to two given vertices is $\lambda$ or $\mu$, respectively.

Let $G = [y_1, y_2, \ldots, y_n]$ be the generator matrix of an $[n, k]$ linear code $C$ over $\mathbb{F}_p$, where $y_i \in \mathbb{F}_p^k$ ($i = 1, 2, \ldots, n$). Let $V = \mathbb{F}_p^k$, $O = \{\langle y_i \rangle \mid i = 1, 2, \ldots, n\}$ and $\Omega = \{v \in V \mid \langle v \rangle \in O\}$. Define a graph $G(\Omega)$ with vertices based on the vectors in $V$, and any two vertices are joint if and only if their difference is in $\Omega$. By Theorem 3.2 in [4], $G(\Omega)$ is strongly regular if and only if $C$ is a projective two-weight code. Assume that the nonzero weights of $C$ are just $w_1$ and $w_2$. By Corollary 3.7 in [4], the parameters of $G(\Omega)$ are given in the following,

$$N = p^k,$$

$$K = (p-1)n,$$

$$\lambda = K^2 + 3K - p(w_1 + w_2) - Kp(w_1 + w_2) + p^2w_1w_2,$$

$$\mu = K^2 + K - Kp(w_1 + w_2) + p^2w_1w_2.$$

Now, we give the corresponding strongly regular graphs from $C_{D^*}$ and $C_{D_0}$, respectively.
By Corollaries 3.1 and 3.3, we know that $C_{TD_0}$ is a projective two-weight code with parameters $[\frac{p^{2m-1}p^{m-1}}{p-1}, 2m, (p^{m-1} - 1)p^{m-1}]$ and the weight distribution

$$1 + (p^m - 1)(p^m - p^{m-1} + 1)z^{p^{2m-2}} + (p^m - 1)p^{m-1}z^{(p^{m-1}-1)p^{m-1}}.$$ 

Thus, $C_{TD_0}$ yields a strongly regular graph $G(\Omega)$ with the following parameters,

$$N = p^{2m}, ~ K = p^{2m-1} - p^{m-1}, ~ \lambda = p^{2m-2} + p^m - 3p^{m-1}, ~ \mu = p^{2m-2} - p^{m-1}.$$

Similarly, $C_{TD^*}$ can yield a strongly regular graph with the following parameters,

$$N = p^{2m}, ~ K = p^{2m-1} - p^m - p^{m-1} - 1,$$

$$\lambda = p^{2m-2} + p^m - 5p^{m-1} + 4, ~ \mu = (p^{m-1} - 1)(p^{m-1} - 2).$$

Compared with the known strongly regular graphs [4, 9, 16, 20], the above strongly regular graphs are new classes.

6 Conclusions

In this paper, for any odd prime $p$, we construct several classes of two-weight linear codes over $\mathbb{F}_p$ from defining sets, and then obtain two classes of projective two-weight codes $C_{TD_0}$ and $C_{TD^*}$ by puncturing $C_{D_0}$ and $C_{D^*}$, respectively. All these codes can be suitable for applications in secret sharing schemes with interesting access structures. Furthermore, both $C_{TD_0}$ and $C_{TD^*}$ can yield new strongly regular graphs, respectively.

References

[1] A. Ashikhmin, A. Barg, Minimum vectors in linear codes. IEEE Trans. Inf. Theory, 44(8) (1998) 2010-2017.

[2] R. Blakley, Safe guarding cryptographic keys, Proc Afips National Computer Conf., 48 (1979) 313.

[3] A. Calderbank, J. Goethals, Three-weight codes and association schemes, Philips J. Res, 39(4-5) (1984) 143-152.

[4] R. Calderbank, W. Kantor, The geometry of two-weight codes, Bull. Lond. Math. Soc., 18(2) (1986) 97-122.

[5] C. Carlet, C. Ding, J. Yuan, Linear codes from perfect nonlinear mappings and their secret sharing schemes, IEEE Trans. Inf. Theory, 51(6) (2005) 2089-2102.

[6] C. Ding, Linear codes from some 2-designs. IEEE Trans. Inf. Theory, 61(6) (2015) 3265-3275.

[7] C. Ding, Designs from Linear Codes. World Scientific, Singapore. (2018).

[8] C. Ding, J. Luo, H. Niederreiter, Two-weight codes punctured from irreducible cyclic codes. In: Proceedings of the 1st International Workshop on Coding Theory Cryptography, (2008) 119-124.
[9] C. Ding, H. Niederreiter, Cyclotomic linear codes of order 3, IEEE Trans. Inf. Theory, 53(6) (2007) 2274-2277.

[10] C. Ding, X. Wang, A coding theory construction of new systematic authentication codes, Theor. Comput. Sci., 330(1) (2005) 81-99.

[11] K. Ding, C. Ding, Binary linear codes with three weights, IEEE Commun. Lett., 18(11) (2014) 1879-1882.

[12] K. Ding, C. Ding, A class of two-weight and three-weight codes and their applications in secret sharing, IEEE Trans. Inf. Theory, 61(11) (2015) 5835-5842.

[13] P. Delsarte, Weights of linear codes and strongly regular normed spaces, Discrete Math., 3 (1972) 47-64.

[14] J. Griesmer, A bound for error-correcting codes, IBM J. Res. Dev., 4(5) (1960) 532-542.

[15] W. Huffman, V. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press. (2010).

[16] Z. Heng, D. Li, J. Du, F. Chen, A family of projective two-weight linear codes, Des. Codes Cryptogr., (2021). https://doi.org/10.1007/s10623-021-00896-2.

[17] Z. Heng, Q. Yue, A class of binary linear codes with at most three weights, IEEE Commun. Lett., 19(9) (2015) 1488-1491.

[18] Z. Heng, Q. Yue, Evaluation of the Hamming weights of a class of linear codes based on Gauss sums, Des. Codes Cryptogr., 83(2) (2017) 307-326.

[19] Z. Heng, Q. Yue, Two classeses of two-weight linear codes, Finite Fields Appl., 38 (2016) 72-92.

[20] Z. Heng, Q. Yue, A construction of $q$-ary linear codes with two weights, Finite Fields Appl., 48 (2017) 20-42.

[21] Z. Heng, Q. Yue, C. Li, Three classes of linear codes with two or three weights, Discrete Math., 339(11) (2016) 2832-2847.

[22] G. Jian, Z. Lin, R. Feng, Two-weight and three-weight linear codes based on Weil sums, Finite Fields Appl., 57 (2019) 92-107.

[23] E. Karnin, J. Greene, M. Hellman, On secret sharing systems. IEEE Trans. Inf. Theory, 29(1) (1983) 35-41.

[24] C. Li, S. Bae, S. Yang, Some two-weight and three-weight linear codes, Advances in Mathematics of Communications, 13(1) (2019) 195-211.

[25] C. Li, Q. Yue, F. Fu, A construction of several classes of two-weight and three-weight linear codes, Appl. Algebra Eng. Commun. Comput., 28 (2017) 11-30.

[26] R. Lidl, H. Niederreiter, Cohn F.M., Finite Fields., Cambridge University Press, Cambridge. (1997).
[27] G. Luo, X. Cao, S. Xu, J. Mi, Binary linear codes with two or three weights from niho exponents, Cryptogr. Commun., 10(2) (2018) 301-318.

[28] J. Massey, minimum codewords and secret sharing. In: Proc. 6th Joint Swedish-Russian Workshop on Information Theory, Mölle, Sweden. Aug. (1993) 276-279.

[29] J. McEliece, D. Sarwate, On sharing secrets and Reed-Solomon codes. Communications of the ACM, 24(9) (1981) 583-584.

[30] A. Shamir, How to share a secret. Communications of the ACM, (1979).

[31] C. Tang, N. Li, Y. Qi, Z. Zhou, T. Helleseth, Linear codes with two or three weights from weakly regular bent functions, IEEE Trans. Inf. Theory, 62(3) (2016) 1166-1176.

[32] K. Torleiv, Codes for Error Detection (vol. 2), World Scientific. (2007).

[33] J. Yuan, C. Ding, Secret sharing schemes from three classes of linear codes, IEEE Trans. Inf. Theory, 52(1) (2006) 206-212.

[34] S. Yang, Z. Yao, Complete weight enumerators of a family of three-weight linear codes, Des. Codes Cryptogr., 82(3) (2017) 663-674.

[35] C. Zhu, Q. Liao, Complete weight enumerators for several classes of two-weight and three-weight linear codes, Finite Fields Appl., 75 (2021) 101897.

[36] Z. Zhou, N. Li, C. Fan, T. Helleseth, Linear codes with two or three weights from quadratic bent functions, Des. Codes Cryptogr., 81(2) (2016) 283-295.