WEIGHTED LOCAL ESTIMATES FOR FRACTIONAL TYPE OPERATORS

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ABSTRACT. In this note we prove the estimate $M^\#_{0,s}(Tf)(x) \leq c M_\gamma f(x)$ for general fractional type operators $T$, where $M^\#_{0,s}$ is the local sharp maximal function and $M_\gamma$ the fractional maximal function, as well as a local version of this estimate. This allows us to express the local weighted control of $Tf$ by $M_\gamma f$. Similar estimates hold for $T$ replaced by fractional type operators with kernels satisfying Hörmander-type conditions or integral operators with homogeneous kernels, and $M_\gamma$ replaced by an appropriate maximal function $M_T$. We also prove two-weight, $L^p_v-L^q_w$ estimates for the fractional type operators described above for $1 < p < q < \infty$ and a range of $q$. The local nature of the estimates leads to results involving generalized Orlicz-Campanato and Orlicz-Morrey spaces.

INTRODUCTION

The purpose of this paper is to establish that much like the Hardy-Littlewood maximal function controls the Calderón-Zygmund singular integral operators [26], integral operators of fractional type are controlled by fractional maximal functions. Muckenhoupt and Wheeden formulated this principle for the Riesz potentials in the weighted setting as follows [20]. For $0 < \gamma < 1$ let

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n(1-\gamma)}} \, dy$$

denote the Riesz potential of order $\gamma$, and

$$M_\gamma f(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\gamma}} \int_Q |f(y)| \, dy$$

the fractional maximal function of order $\gamma$. Then, if $w$ is in $A_\infty$ and $0 < q < \infty$, there is a constant $c$ independent of $f$ such that

$$\int_{\mathbb{R}^n} |I_\gamma f(x)|^q w(x) \, dx \leq c \int_{\mathbb{R}^n} M_\gamma f(x)^q w(x) \, dx.$$  (1.1)
In fact, there is a subtle interplay between these two operators. It is expressed by the readily verified pointwise inequality
\begin{equation}
M_\gamma f(x) \leq c I_\gamma(|f|)(x),
\end{equation}
where \(c\) depends on the dimension \(n\) and \(\gamma\), and the pointwise inequality
\begin{equation}
M^\sharp(I_\gamma f)(x) \leq c M_\gamma f(x),
\end{equation}
where \(M^\sharp\) denotes the sharp maximal function and \(c\) is independent of \(f\) and \(x\), established by Adams [1]. Note that (1.2) gives the equivalence of the norms in (1.1), and that under appropriate conditions (1.3) implies (1.1).

Gogatishvili and Mustafayev [12] observed that (1.3) also implies that for an arbitrary cube \(Q_0\) and \(1 < q < \infty\),
\begin{equation}
\int_{Q_0} |I_\gamma f(x) - (I_\gamma f)_{Q_0}|^q \, dx \leq c \int_{Q_0} M_\gamma f(x)^q \, dx,
\end{equation}
where \((I_\gamma f)_{Q_0}\) denotes the average of \(I_\gamma f\) over \(Q_0\) and \(c\) is independent of \(f\) and \(Q_0\). Rakotondratsimba obtained similar weighted local inequalities for a local version of the Riesz potentials [27]. The estimate (1.4) allows for the comparison of the norms of Riesz potentials and fractional maximal functions in Morrey-type spaces [12].

Here we consider general fractional type operators given by
\[ Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y) \, dy \]
where for some fixed \(0 < \gamma < 1\), there exists a positive constant \(c\) such that for every cube \(Q\),
\[ |k(x,y) - k(x',y)| \leq c \frac{1}{|x - y|^{n(1 - \gamma)}} \omega \left( \frac{|x - x'|}{|x - y|} \right) \]
whenever \(x, x' \in Q\) and \(y \in (2Q)^c\), where \(\omega(t)\) is a nondecreasing function on \((0, \infty)\) that satisfies an appropriate Dini-type condition.

In the first part of this paper we prove that in particular, if \(T\) is of weak-type \((1, 1/(1 - \gamma))\), we have the local pointwise estimate
\begin{equation}
M^\sharp_{0,s,Q_0}(Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_\gamma f(y),
\end{equation}
where \(M^\sharp_{0,s,Q_0}\) denotes the local sharp maximal function restricted to the cube \(Q_0\), and when \(Q_0 = \mathbb{R}^n\),
\begin{equation}
M^\sharp_{0,s}(Tf)(x) \leq c M_\gamma f(x)
\end{equation}
where \(M^\sharp_{0,s}\) denotes the local sharp maximal function.

The weighted local estimates follow readily from (1.5) and (1.6). By a weight we mean a nonnegative locally integrable function \(w\), and we say
that a continuous function \( \Phi \) satisfies condition \( C \) if it is increasing on \([0, \infty)\) with \( \Phi(0) = 0 \) and \( \Phi(2t) \leq c \Phi(t) \), all \( t > 0 \). Then by Theorem 5.1 in [26], (1.5) gives that if \( \Phi \) satisfies condition \( C \) and \( w \) is a weight, for every cube \( Q_0 \) of \( \mathbb{R}^n \),

\[
\int_{Q_0} \Phi(|Tf(x) - m_{Tf}(t, Q_0)|) w(x) \, dx \leq c \int_{Q_0} \Phi(M_r f(x)) v(x) \, dx,
\]

where \( m_{Tf}(t, Q_0) \) is the (maximal) median of \( Tf \) with parameter \( t \), \( v = w \) when \( w \in A_\infty \) and \( v = M_r(w) \), the Hardy-Littlewood maximal function of order \( r \) of \( w \), \( 1 < r < \infty \), when \( w \) is an arbitrary weight, and \( c \) is independent of \( Q_0 \) and \( f \).

Furthermore, if \( \lim_{Q_0 \to \mathbb{R}^n} m_{Tf}(t, Q_0) = 0 \),

\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_r f(x)) v(x) \, dx.
\]

And, (1.7) and (1.8) also hold for appropriate non-\( A_\infty \) weights \( w \) and \( v \).

Thus, weighted local estimates hold for fractional type operators and the integral inequality (1.8) also holds for those operators they majorize, including Marcinkiewicz integrals, fractional powers of analytic semigroups, and Schrödinger type operators [14, 15]. And, as illustrated below in the case of fractional type operators with kernels satisfying Hörmander-type conditions, and integral operators with homogeneous kernels, our approach applies in other instances as well.

Next we take a closer look at two-weight, \( L^p_v - L^q_w \) specific inequalities. The question of determining weights \((w, v)\) so that the Riesz potentials map \( L^p_v \) continuously into \( L^p_w \) was addressed by Pérez [23, 24] and continues to attract considerable attention. When the computability of the conditions on the weights is a concern, interesting results are proved and referenced, for instance, in [27].

To deal with fractional type operators in the two-weight context we rely on the sharper local median decomposition produced in [26]; results in this direction were anticipated in [10]. The Orlicz “bump” conditions of Pérez [22] and Cruz-Uribe and Moen [6] and a technique of Lerner [19], give then the estimate for these operators, including those of Dini type, or with kernels satisfying a Hörmander-type condition, from \( L^p_v(\mathbb{R}^n) \) into \( L^q_w(\mathbb{R}^n) \) for \( 1 < p < q < \infty \) and a range of \( q \).

Finally, the local estimates are well suited to the generalized Orlicz–Morrey spaces \( M^{\Phi, \phi} \) and generalized Orlicz–Campanato spaces \( L^{\Phi, \phi} \), defined in Section 5. Indeed, if \( T \) is a fractional type operator, from (1.7) it readily follows that for every Young function \( \Psi \) and every appropriate
ψ,
\begin{equation}
\|Tf\|_{L^{\Phi,\psi}} \leq c \|M_{\gamma}f\|_{M^{\Psi,\psi}}.
\end{equation}

And, concerning the continuity of $M_{\gamma}$ in the Orlicz-Morrey spaces we have that for $0 \leq \gamma < 1$, if the Young functions $\Phi, \Psi$ are such that $\Psi^{-1}(t) = t^{-\gamma} \Phi^{-1}(t)$, and $\phi, \psi$ satisfy $\sup_{l < t < \infty} t^{\gamma r} \phi(x, t) \leq c \psi(x, l)$, then $M_{\gamma}$ maps $M^{\Phi,\phi}$ continuously into $M^{\Psi,\psi}$.

The paper is organized as follows. The essential ingredient in what follows, i.e., the estimate $M_{0,s}^\sharp(Tf)(x) \leq c M_{\gamma,r}f(x)$ for fractional type operators of weak type $(r, r/(1 - \gamma r))$ and its local version, are done in Section 2. We also recast similar estimates with $T$ replaced by fractional type operators with kernels satisfying Hörmander-type conditions or integral operators with homogeneous kernels and $M$ by an appropriate maximal function $M_T$. In Section 3 we use these estimates to express the local integral control of $Tf$ in terms of $M_Tf$. In Section 4 we prove two-weight, $L^p_v - L^q_w$ specific estimates for fractional type operators. And finally in Section 5 we consider the Orlicz-Morrey and Orlicz-Campanato spaces.

Some closely related topics are not addressed here. Because we concentrate on integral inequalities, weak-type inequalities are not considered, nor homogeneous spaces, the foundation for which has been laid in [32, 33, 35, 38]. And, for the various definitions or properties that the reader may find unfamiliar, there are many treatises in the area which may be helpful, including [35, 37]. It is a pleasure to acknowledge the conversations I had with J. Poellhuis concerning these matters.

2. Pointwise Local Estimates

In what follows we adopt the notations of [25, 26, 34]. In particular, all cubes have sides parallel to the axes. Also, for a cube $Q \subset \mathbb{R}^n$ and $0 < t < 1$, we say that

$$m_f(t, Q) = \sup \{M : |\{y \in Q : f(y) < M\}| \leq t|Q|\}$$

is the (maximal) median of $f$ over $Q$ with parameter $t$. For a cube $Q_0 \subset \mathbb{R}^n$ and $0 < s \leq 1/2$, the local sharp maximal function restricted to $Q_0$ of a measurable function $f$ at $x \in Q_0$ is

$$M_{0,s,Q_0}^\sharp f(x) = \sup_{x \in Q_0, Q \subset Q_0} \inf \inf \{\alpha \geq 0 : |\{y \in Q : |f(y) - c| > \alpha\}| < s|Q|\},$$
and the local sharp maximal function of a measurable function $f$ at $x \in \mathbb{R}^n$ is

$$M_{0,s}^f(x) = \sup_{x \in Q} \inf_{c} \inf\{\alpha \geq 0 : |\{y \in Q : |f(y) - c| > \alpha\}| < s|Q|\}.$$ 

Then, with the notation

$$(2.1) \quad m^r_f(1 - s, Q) = \inf_c m_{|f - c|}(1 - s, Q),$$

since by (4.3) of [25], $m^r_f(1 - s, Q) \sim m_{|f - m_f(1 - s, Q)|}(1 - s, Q)$, we have

$$M_{0,s, Q_0}^f(x) \sim \sup_{x \in Q, Q \subset Q_0} m^r_f(1 - s, Q)$$

$$\sim \sup_{x \in Q, Q \subset Q_0} m_{|f - m_f(1 - s, Q)|}(1 - s, Q).$$

Next we introduce the fractional maximal functions of interest to us. A function $A$ that satisfies condition $C$ which is convex and such that $A(t) \to \infty$ as $t \to \infty$, or, more generally, such that $A(t)/t \to \infty$ as $t \to \infty$, is called a Young function. For a Young function $A$ let

$$(2.2) \quad \|f\|_{L^A(Q)} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\},$$

and for $0 \leq \gamma < 1$, let

$$M_{\gamma,A}f(x) = \sup_{x \in Q} |Q|^\gamma \|f\|_{L^A(Q)}.$$

In particular, for $A(t) = t^r$, we denote $M_{\gamma,A} = M_{\gamma,r}$, and of course, $M_{\gamma} = M_{\gamma,1} \leq M_{\gamma,r}$ for $1 < r < \infty$. Also, when $\gamma = 0$ we drop the subscript corresponding to $\gamma$.

Finally, observe that there exists a dimensional constant $c_n$ such that for every cube $Q$ in $\mathbb{R}^n$, if $x, x' \in Q$ and $y \notin 2^m Q$ for some $m \geq 1$, then

$$(2.3) \quad \frac{|x - x'|}{|x - y|} \leq c_n 2^{-m}.$$

We then have,

**Theorem 2.1.** Let $T$ be a fractional type operator defined by

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y)f(y)dy$$

such that for some fixed $0 < \gamma < 1$,

1. There exists a constant $c > 0$ such that

$$|k(x, y) - k(x', y)| \leq c \frac{1}{|x - y|^{n(1-\gamma)}} \omega\left(\frac{|x - x'|}{|x - y|}\right)$$
whenever \( x, x' \in Q \) and \( y \in (2Q)^c \) for any cube \( Q \), where \( \omega(t) \) is a nondecreasing function on \( (0, \infty) \) such that

\[
\int_0^1 \omega(c_n t) \frac{dt}{t} < \infty,
\]

and

(2) \( T \) is of weak-type \( (r, r/(1 - \gamma r)) \), for some \( 1 \leq r < \infty \).

Then, for \( 0 < s \leq 1/2 \), any cube \( Q_0 \), and \( x \in Q_0 \),

\[
M_{0,s,Q_0}^\sharp(Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_{\gamma,r}f(y).
\]

In particular, if \( Q_0 = \mathbb{R}^n \), then for all \( x \in \mathbb{R}^n \),

\[
M_{0,s}^\sharp(Tf)(x) \leq c M_{\gamma,r}f(x).
\]

Proof. Fix a cube \( Q_0 \subset \mathbb{R}^n \) and take \( x \in Q_0 \). Let \( Q \subset Q_0 \) be a cube centered at \( x \) containing \( x \). Let \( f_1 = f 1_{2Q} \) and \( f_2 = f - f_1 \); then by the linearity of \( T \), \( Tf(z) - Tf_2(x_Q) = T f_1(z) + T f_2(z) - T f_2(x_Q) \) for \( z \in Q \).

We claim that there exist constants \( c_1, c_2 > 0 \) independent of \( f \) and \( Q \) such that

\[
\{ z \in Q : |T f_1(z)| > c_1 \inf_{y \in Q} M_{\gamma,r}f(y) \} < s |Q|,
\]

and

\[
\|T f_2 - T f_2(x_Q)\|_{L^\infty(Q)} \leq c_2 \inf_{y \in Q} M_{\gamma,r}f(y).
\]

We prove (2.6) first. Observe that for any \( z \in Q \), by (2.3),

\[
|T f_2(z) - T f_2(x_Q)|
\]

\[
\leq \int_{(2Q)^c} |k(z, y) - k(x_Q, y)| |f(y)| dy
\]

\[
\leq c \sum_{m=1}^\infty \int_{2^{m+1}Q \setminus 2^m Q} \frac{1}{|z - y|^{n(1 - \gamma)}} \omega\left(\frac{|x_Q - z|}{|y - z|}\right) |f(y)| dy
\]

\[
\leq c \sum_{m=1}^\infty \omega(c_n/2^m) \frac{1}{|2^m Q|^{1 - \gamma}} \int_{2^m Q} |f(y)| dy
\]

\[
\leq c \left( \int_0^1 \omega(c_n t) \frac{dt}{t} \right) \inf_{y \in Q} M_{\gamma,r}f(y),
\]

and (2.6) holds.
As for (2.5), since $T$ is of weak-type $(r, r/(1 - \gamma r))$ we have that for any $\lambda > 0$,
\[
\lambda^{r/(1-\gamma r)} \left| \{ z \in Q : |Tf_1(z)| > \lambda \} \right|
\leq c \left( \int_{2Q} |f(y)|^r \, dy \right)^{1/(1-\gamma r)}
= c \left( |2Q|^r \left( \frac{1}{|2Q|} \int_{2Q} |f(y)|^r \, dy \right)^{1/r} \right)^{r/(1-\gamma r)} |Q|
\leq c \inf_{y \in Q} M_{\gamma,r} f(y)^{r/(1-\gamma r)} |Q|,
\]
(2.8)
and (2.5) follows by picking $\lambda = c_1 \inf_{y \in Q} M_{\gamma,r} f(y)$ for an appropriately chosen $c_1$.

Then, with $c > \max\{c_1, c_2\}$, (2.5) and (2.6) give
\[
\left| \{ z \in Q : |Tf(z) - Tf_2(x_Q)| > 2c \inf_{y \in Q} M_{\gamma,r} f(y) \} \right|
\leq \left| \{ z \in Q : |Tf_2(z) - Tf_2(x_Q)| > c_2 \inf_{y \in Q} M_{\gamma,r} f(y) \} \right|
+ \left| \{ z \in Q : |Tf_1(z)| > c_1 \inf_{y \in Q} M_{\gamma,r} f(y) \} \right|
< s|Q|.
\]

Whence
\[
\inf_{c'} \inf_{\alpha \geq 0} \left| \{ z \in Q : |Tf(z) - c'| > \alpha \} \right| < s|Q| \leq c \inf_{y \in Q} M_{\gamma,r} f(y),
\]
and consequently, since this holds for all $Q \subset Q_0, x \in Q$,
\[
M_{0,s,Q_0}^2 Tf(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_{\gamma,r}^2 f(y).
\]

The proof is thus complete. \(\square\)

Now, we also have that $T(1) = 0$, under appropriate conditions on $\omega$ it follows that $M_{0,s,Q_0}^2 (Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_{\gamma,r}^2 f(y)$, where $M_{\gamma,A}^2 f(x)$ has the expected definition \([26]\).

That $M_{\gamma,r}$ is relevant on the right-hand side of (2.4) for all $r, 1 \leq r < \infty$, is clear from (2.8) above, and it is useful when $T$ is not known to be of weak-type $(1, 1/(1 - \gamma))$. Also, there are operators of weak-type $(1, 1/(1-\gamma))$ where $M_{\gamma,r}$ is necessary on the right-hand side of (2.7), and hence on the right-hand side of (2.4), for $1 < r < \infty$. These are the convolution fractional operators of Dini type, i.e., $k(x) = \Omega(x')/|x|^{n(1-\gamma)}$, $x \neq 0$, where $\Omega$ is a function on $S^{n-1}$ that satisfies $\int_{S^{n-1}} \Omega(x') \, dx' = 0$ and an $L^{r'}$-Dini condition for some $1 \leq r' \leq \infty$, \([5, 7]\). Because of
their similarity with the fractional type operators with kernels satisfying Hörmander-type conditions considered in Theorem 2.2 below, the analysis of this case is omitted.

Now, in the latter case we have

**Theorem 2.2.** Let $T$ be a fractional integral operator of weak-type $(1, 1/1 − \gamma)$ such that for a Young function $A$, every cube $Q$, and $u, v \in Q$,

$$\sum_{m=1}^{\infty} |2^{m+1}Q|^{1-\gamma} \| 1_{2^{m+1}Q\setminus 2^mQ} (k(u, \cdot) - k(v, \cdot)) \|_{L^A(2^{m+1}Q)} \leq c_A < \infty.$$

Then, with $\overline{A}$ the conjugate Young function to $A$, $c$ independent of $x, Q_0$, and $f$, we have

$$M^\sharp_{0,s,Q_0}(Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_{\gamma, \overline{A}} f(y). \quad (2.9)$$

The idea of the proof is essentially that of [4, 28], where $T$ is assumed to be of convolution type and then the stronger conclusion

$$M^\sharp(|Tf|^\delta)(x)^{1/\delta} \leq c M_{\gamma, \overline{A}} f(x)$$

holds with $\delta$ sufficiently small, or that of the proof of Theorem 4.3 in [26].

Finally, we consider integral operators with homogeneous kernels defined as follows [29]. If $A_1, \ldots, A_m$ are invertible matrices such that $A_k - A_{k'}$ is invertible for $k \neq k'$, $1 \leq k, k' \leq m$, and $\gamma_i > 0$ for all $i$ with $\gamma_1 + \cdots + \gamma_m = n(1 - \gamma) > 0$, then

$$Tf(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\gamma_1} \cdots |x - A_m y|^{-\gamma_m} f(y) dy. \quad (2.10)$$

For these operators we have,

**Theorem 2.3.** For $T$ defined as in (2.10), any cube $Q_0 \subset \mathbb{R}^n$, and $x \in Q_0$, we have

$$M^\sharp_{0,s,Q_0}(Tf)(x) \leq c \sum_{i=1}^{m} \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_{\gamma_i} f(A_i^{-1} y). \quad (2.11)$$

Since by Theorem 3.2 in [29], $T$ is of weak-type $(1, 1/1 − \gamma)$, the proof follows along the lines of Theorem 2.1 in [29] or the proof of Theorem 4.4 in [26]. To obtain the weighted estimates below, one can of course use the full strength of the result in [29], namely,

$$M^\sharp(|Tf|^\delta)(x)^{1/\delta} \leq c \sum_{i=1}^{m} M_{\gamma_i} f(A_i^{-1} x),$$

where $0 < \delta < 1.
3. Local Weighted Estimates

In this section we consider the control of a weighted local mean of a fractional type operator by the weighted local mean of an appropriate fractional maximal function. We say that the weights \((w, v)\) satisfy condition \(F\) provided there exist positive constants \(c, \alpha, \beta\) with \(0 < \alpha < 1\), such that for any cube \(Q\) and measurable subset \(E\) of \(Q\) with \(|E| \leq \alpha|Q|\),

\[
\int_E w(x) \, dx \leq c \left( \frac{|E|}{|Q|} \right)^\beta \int_{Q\setminus E} v(x) \, dx. \tag{3.1}
\]

Fujii observed that if \(v = w\), (3.1) is equivalent to the \(A_\infty\) condition for \(w\); he also gave a simple example of a pair \((w, v)\) that satisfy condition \(F\) so that neither of them is an \(A_\infty\) weight and no \(A_\infty\) weight can be inserted between them [8].

Now, if \(w\) is in weak \(A_\infty\), or more generally in the Muckenhoupt class \(C_p\), then \((w, Mw)\) satisfy condition \(F\). On the other hand, from (3.4) below it follows that in particular for some weight \(w\), \((w, Mw)\) do not satisfy condition \(F\). Along these lines, for any weight \(w\) and \(1 < r < \infty\), \((w, M, w)\) satisfy condition \(F\).

The weighted local estimate is then,

**Theorem 3.1.** Let \(T\) be a fractional type operator that satisfies the assumptions of Theorem 2.1, \(\Phi\) that satisfies condition \(C\), and \((w, v)\) weights on \(\mathbb{R}^n\) satisfying condition \(F\). Then there exists \(1/2 < t < 1\) such that

\[
\int_{Q_0} \Phi(|Tf(x) - m_{Tf}(t, Q_0)|) \, w(x) \, dx \leq c \int_{Q_0} \Phi(M_{\gamma,r}f(x)) \, v(x) \, dx. \tag{3.2}
\]

Furthermore, if \(f\) is such that \(m_{Tf}(t, Q_0) \to 0\) as \(Q_0 \to \mathbb{R}^n\), then

\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) \, w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_{\gamma,r}f(x)) \, v(x) \, dx. \tag{3.3}
\]

Moreover, if \(\Phi\) is concave or \(\Phi(u) = u\), then (3.2) and (3.3) hold with \(v(x) = Mw(x)\).

**Proof.** By Theorem 3.1 in [26] there exist \(0 < s \leq 1/2\) and \(1/2 < t < 1 - s\) such that

\[
\int_{Q_0} \Phi(|Tf(x) - m_{Tf}(t, Q_0)|) \, w(x) \, dx \leq c \int_{Q_0} \Phi(M_{0,s}^{\gamma}Tf(x)) \, v(x) \, dx,
\]

and by Theorem 2.1 above,

\[
\int_{Q_0} \Phi(M_{0,s}^{\gamma}Tf(x)) \, v(x) \, dx \leq c \int_{Q_0} \Phi(M_{\gamma,r}f(x)) \, v(x) \, dx.
\]
Hence, combining these estimates (3.2) holds, and (3.3) follows immediately from Fatou’s lemma.

The conclusion for \( \Phi \) concave follows from Theorem 3.2 in [26]. □

As Lerner observed, \( \lim_{Q_0 \to \mathbb{R}^n} m_q(t, Q_0) = 0 \) if \( g^*(+\infty) = 0 \), where \( g^* \) denotes the nonincreasing rearrangement of \( g \), which in turn holds if and only if \( |\{x \in \mathbb{R}^n : |g(x)| > \alpha\}| < \infty \) for all \( \alpha > 0 \), [18].

Now, extending a result of Adams [2], Pérez proved in [24] a sharp version of (3.3) for the Riesz potentials. Indeed, Theorem 1.1 (B) there asserts that for an arbitrary weight \( w \) and \( 1 < q < \infty \),

\[
(3.4) \quad \int_{\mathbb{R}^n} |I_{\gamma}f(x)|^q w(x) \, dx \leq c \int_{\mathbb{R}^n} M_{\gamma}f(x)^q M[q+1](w)(x) \, dx,
\]

and that this estimate is sharp, since \( [q] \) cannot be replaced by \( [q] - 1 \) above. Thus, combining (3.3) with (3.4) it readily follows that for each \( k \geq 1 \), for some weight \( w, (w, M^k w) \) do not satisfy condition \( F \).

Also, for \( \Phi \) concave, including \( \Phi(u) = u \), from (3.3) it follows in particular that

\[
\int_{\mathbb{R}^n} \Phi(|I_{\gamma}f(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_{\gamma}f(x)) Mw(x) \, dx,
\]

which complements estimate (13) in [24].

Pérez also addressed in Theorem 1.1 (A) in [24] the question of estimating the integral involving the fractional maximal function in the right-hand side of (3.3) by an integral involving \( |f| \) and a possibly larger weight in the right-hand side of (3.3) as do Bernardis et al in [4]. There is a vast literature of results of this nature, pioneered by Sawyer’s work [31]. The results of Harboure et al [16] are also relevant. Here we prove the result directly in the next section.

4. Two-weight estimates for fractional type operators

In this section we consider two-weight, \( L_p^v - L_q^w \) estimates with \( 1 < p < q < \infty \) that apply directly to a fractional type operator and where the control exerted by a fractional maximal function is not apparent.

First recall the definition of the classes \( B_p \) and \( B_{\alpha,p} \). The latter class was introduced by Cruz-Uribe and Moen [6] and for \( 0 < \alpha < 1 \) and \( 1 < p < 1/\alpha \), it consists of those Young functions \( A \) such that with \( 1/q = 1/p - \alpha \),

\[
\|A\|_{\alpha,p} = \left( \int_c^{\infty} A(t)^{q/p} \frac{dt}{t^q} \right)^{1/q} < \infty.
\]

When \( \alpha = 0 \) this reduces to the \( B_p \) class of Pérez. The result of interest to us, Theorem 3.3 in [6], is that for \( A \in B_{\alpha,p} \) the fractional maximal
function $M_{\alpha,A}f(x)$ maps $L^p(\mathbb{R}^n)$ continuously into $L^q(\mathbb{R}^n)$ with norm not exceeding $c \|A\|_{\alpha,p}$. This result also holds for $\alpha = 0$, i.e., the $B_p$ classes [22].

We rely on the following result of Pérez, Theorem 2.11 in [23] or Theorem 3.5 in [11]. Let $p, q$ with $1 < p < q < \infty$, and $0 < \gamma < 1$. Let $(w, v)$ be a pair of weights such that for every cube $Q$,

$$|Q|^{\gamma} |Q|^{1/q-1/p} \|w^{1/q}\|_{L^q(Q)} \|v^{-1/p}\|_{L^p(Q)} \leq c,$$

where $B$ is a Young function with $\overline{B} \in B_p$. Then, if $f \in L^p(v)$,

$$\left( \int_{\mathbb{R}^n} M_{\alpha}f(x)^q w(x) \, dx \right)^{1/q} \leq c \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right)^{1/p}.$$

Before we proceed to prove our next theorem, we need an extension of a property given in Lemma 4.8 in [24] and the comments that follow it.

**Lemma 4.1.** Let $T$ be a fractional type operator defined by (4.2). If $T$ satisfies the assumptions of Theorem 2.1 with $1 \leq r < \infty$, and in that case

$$\lambda_m = \omega(c_n/2^m), \quad m \geq 1,$$

or $T$ satisfies the assumptions of Theorem 2.2 and in that case

$$\lambda_m = \sup_{u,v \in Q} |2^{m+1}Q|^{1-\gamma} \|1_{2^m+1}Q \backslash 2^mQ (k(u, \cdot) - k(v, \cdot))\|_{L^r(2^m+1)Q},$$

then $\sum_m \lambda_m < \infty$, and if $Q$ is a cube of $\mathbb{R}^n$, with $m^{\sharp}_{T,f}$ as in (2.1), we have

$$m^{\sharp}_{T,f}(1-s,Q) \leq c \sum_{m=1}^{\infty} \lambda_m |2^mQ|^\gamma \left( \frac{1}{|2^mQ|} \int_{2^mQ} |f(y)|^r \, dy \right)^{1/r}.$$

**Proof.** Fix $Q$, let $x \in Q$, and put $f = f_1 + f_2$ where $f_1 = f 1_{2Q}$. We claim that there exist constants $c_1, c_2 > 0$ independent of $f$ and $Q$ such that

$$\{z \in Q : |Tf_1(z)| > c_1 I\} \leq s |Q|,$$

and

$$\|Tf_2 - Tf_2(x_Q)\|_{L^\infty(Q)} \leq c_2 I,$$

where

$$I = \sum_{m=1}^{\infty} \lambda_m |2^mQ|^\gamma \left( \frac{1}{|2^mQ|} \int_{2^mQ} |f(y)|^r \, dy \right)^{1/r}.$$
First, in the case of Theorem 2.1, $T$ is of weak-type $(r,r/(1-\gamma r))$ and as in (2.8) we have that for any $\lambda > 0$,
\[ \lambda^{r/(1-\gamma r)} \left| \{ z \in Q : |Tf_1(z)| > \lambda \} \right| \]
\[ \leq c \left( |2Q|^r \left( \frac{1}{|2Q|} \int_{2Q} |f(y)|^r \, dy \right)^{r/(1-\gamma r)} \right) ^{r/(1-\gamma r)} |Q|. \]
In the case of Theorem 2.2, $T$ is of weak-type $(1, 1/(1-\gamma))$ and as in (2.8) by Hölder’s inequality we have that for any $\lambda > 0$,
\[ \lambda^{1/(1-\gamma)} \left| \{ z \in Q : |Tf_1(z)| > \lambda \} \right| \]
\[ \leq c \left( |2Q|^\gamma \left( \frac{1}{|2Q|} \int_{2Q} |f(y)|^r \, dy \right)^{1/r} \right)^{1/(1-\gamma)} |Q|. \]
Thus, in both cases (4.3) holds.

Next, when $T$ satisfies the assumptions of Theorem 2.2, (4.4) holds automatically. And, if $T$ satisfies the assumptions of Theorem 2.1, as in (2.7) for any $z \in Q$,
\[ |Tf_2(z) - Tf_2(x_Q)| \leq c \sum_{m=1}^{\infty} \omega(c_n/2^m) \frac{1}{|2^m Q|^{1-\gamma}} \int_{2^m Q} |f(y)| \, dy \]
and, therefore, (4.4) holds by Hölder’s inequality.

Then, in either case, with $c > \max\{c_1,c_2\}$, as in the proof of Theorem 2.1, (4.3) and (4.4) give
\[ \{ z \in Q : |Tf(z) - Tf_2(x_Q)| > 2c I \} < s|Q|, \]
and therefore, (4.4) holds by Hölder’s inequality.

Then, in either case, with $c > \max\{c_1,c_2\}$, as in the proof of Theorem 2.1, (4.3) and (4.4) give
\[ \{ z \in Q : |Tf(z) - Tf_2(x_Q)| > 2c I \} < s|Q|, \]
and therefore, (4.4) holds by Hölder’s inequality.

Note that Lemma 4.1 also applies to the convolution fractional type operators of Dini type. In that case, using Lemma 1 in [11], the $\alpha_m$ can be estimated in terms of the $\omega_{\nu}$ modulus of continuity of the kernel $k$.

Now the main result.

**Theorem 4.1.** Let $T$ be a fractional type operator that satisfies the assumptions of Theorem 2.1 with $1 \leq r < \infty$ or Theorem 2.2 with the Young function $t^{r'}$ there and $1 \leq r < \infty$. Let $\gamma r < 1$, and $r < p < q < \infty$, define $0 < \alpha < 1$ by the relation $\alpha = 1/p - 1/q$, and let $\alpha_1, \alpha_2 \geq 0$ be such that $\alpha = \alpha_1 + \alpha_2$. Further, suppose that the Young functions
A, B are so that \( A \in B_{(q/r)'} \cap B_{q}^{2} \) and \( B \in B_{p/r}^{\alpha}, \) and \( w \) and \( v \) weights such that for all cubes \( Q, \)

\[
\text{(4.5)} \quad \sup_{Q} |Q|^\gamma |Q|^{q/r - p/r} \|w^{q/r}\|_{L^A(Q)} \|v^{p/r}\|_{L^B(Q)} \leq c < \infty.
\]

Then, if \( \lambda_m \) is defined as in Lemma 4.1 satisfies

\[
\sum_{m=1}^{\infty} \lambda_m 2^{mn/q} < \infty,
\]

we have

\[
\text{(4.6)} \quad \left( \int_{\mathbb{R}^n} |Tf(x)|^q w(x) \, dx \right)^{1/q} \leq c \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right)^{1/p}
\]

for those \( f \) such that \( \lim_{Q_0 \to \mathbb{R}^n} m_{Tf}(t, Q_0) = 0. \)

**Proof.** We begin by considering the local version of (4.6). Fix a cube \( Q_0 \) and note that by the local median decomposition discussed in Theorem 6.5 in [26] there exists a family \( \{Q^v_j\} \) of dyadic subcubes of \( Q_0 \) such that if \( \hat{Q}^v_j \) denotes the dyadic parent of \( Q^v_j, \) we have

\[
|Tf(x) - m_{Tf}(t, Q_0)| \leq 8 M_{0,s,Q_0}(Tf)(x) + c \sum_{v,j} m_{Tf}^2(1 - (1 - t)/2^n, \hat{Q}^v_j) \chi_{\hat{Q}^v_j}(x).
\]

Therefore to estimate the \( L^q_w(Q_0) \) norm of \( Tf(x) - m_{Tf}(t, Q_0) \) it suffices to estimate the norm of each summand above separately. Since by Theorem 2.1 or Theorem 2.2 we have

\[
M_{0,s,Q_0}^2(Tf)(x) \leq c M_{\gamma r}(f^r)(x) = c M_{\gamma r}(|f|^r)(x)^{1/r},
\]

the first term above can be estimated by

\[
\text{(4.7)} \quad \|M_{\gamma r}(|f|^r)^{1/r}\|_{L^q_w} = \|M_{\gamma r}(|f|^r)^{1/r}\|_{L^q_{w}^{1/r}}.
\]

Now, since \( A \in B_{(q/r)'} \), by (6.17) in [26],

\[
\|w^{q/r}\|_{L^{1/r}(Q)} \leq c \|w^{q/r}\|_{L^A(Q)}
\]

for all cubes \( Q, \) and therefore (4.5) implies (4.1) with indices \( p/r \) and \( q/r \) there, corresponding to the value \( \gamma r. \) Thus \( M_{\gamma r} \) maps \( L^q_w^{1/r} \) continuously into \( L^q_w \) and therefore (4.7) is bounded by

\[
\|M_{\gamma r}(|f|^r)^{1/r}\|_{L^q_w^{1/r}} \leq c \| |f|^r\|_{L^q_w^{1/r}}^{1/r} = \|f\|_{L^q_w}.
\]

Hence,

\[
\|M_{0,s,Q_0}^2(Tf)\|_{L^q_w} \leq c \|f\|_{L^q_w}.
\]
Next note that by a purely geometric argument, if $Q$ is any of the cubes $Q_j^v$, there is a dimensional constant $c$ such that

$$
\sum_{m=1}^{\infty} \lambda_m \left| 2^m \tilde{Q} \right|^\gamma \left( \frac{1}{|2^m \tilde{Q}|} \int_{2^m \tilde{Q}} |f(y)|^r dy \right)^{1/r} \leq c \sum_{m=1}^{\infty} \lambda_m \left| 2^m Q \right|^\gamma \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)|^r dy \right)^{1/r}.
$$

(4.8)

To estimate the norm of the sum by duality, let $h$ be supported in $Q_0$ with $\|h\|_{L_{t'}^r(Q_0)} = 1$ and note that by (4.2) and (4.8),

$$
\int_{Q_0} \left( \sum_{v,j} m^2 T_f (1 - (1 - t)/2^n, \tilde{Q}_j^v) 1_{Q_j^v} (x) \right) w(x)^{1/q} h(x) \, dx
$$

(4.9)

$$
\leq c \sum_m \lambda_m \sum_{v,j} \left| 2^m Q_j^v \right|^\gamma \left( \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} |f(y)|^r dy \right)^{1/r} \int_{Q_j^v} w(x)^{1/q} h(x) \, dx.
$$

We consider each term in the inner sum of (4.9) separately. First, let $D$ be the Young function defined by $D(t) = B(t^r)$, and note that since $\|g\|_{L_{t'}^r(Q)} = \||g|^r\|_{L_{t'}^{-r/r}(Q)}^{1/r}$, by Hölder’s inequality for the conjugate Young functions $B, B'$,

$$
\left( \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} |f(y)|^r dy \right)^{1/r}
$$

$$
= \left( \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} |f(y)|^r v(y)^{r/p} v(y)^{-r/p} dy \right)^{1/r}
$$

$$
\leq 2 \left( \|f\|_{L_{t'}^r} \|v^{r/p}\|_{L_{t'}^{-r/r}(2^m Q_j^v)} \|v^{-r/p}\|_{L_{t'}^r(2^m Q_j^v)} \right)^{1/r},
$$

$$
= 2 \|f v^{1/p}\|_{L_{t'}^r(2^m Q_j^v)} \|v^{-r/p}\|_{L_{t'}^r(2^m Q_j^v)}^{1/r}.
$$

Next, let $C$ be the Young function defined by $C(t) = A(t^r)$ and note that as above, by Hölder’s inequality for the conjugate Young functions $C, C'$,

$$
\int_{Q_j^v} w(x)^{1/q} h(x) \, dx \leq 2^{mn} |Q_j^v| \left| 2^m Q_j^v \right|^\gamma \left( \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} w(x)^{1/q} h(x) 1_{Q_j^v} (x) \, dx \right)
$$

$$
\leq 2 \cdot 2^{mn} \|w^{1/q}\|_{L_{t'}^r(2^m Q_j^v)} \|h 1_{Q_j^v}\|_{L_{t'}^{-r/r}(2^m Q_j^v)} |Q_j^v|
$$

$$
\leq 2 \cdot 2^{mn} \|w r/q\|^1_{L_{t'}^r(2^m Q_j^v)} \|h 1_{Q_j^v}\|_{L_{t'}^{-r/r}(2^m Q_j^v)} |Q_j^v|.
$$
Moreover, since for each \( \lambda > 1 \) and each cube \( Q \) we have
\[
\| g \mathbb{1}_Q \|_{L^p(\Lambda Q)} \leq \| g \|_{L^p(\Lambda^n(Q))},
\]
it follows that
\[
\int_{Q^v_j} w(x)^{1/q} h(x) \, dx \leq 2 \cdot 2^{mn} \left\| w^{r/q} \right\|_{L^A(2mQ^v_j)}^{1/r} \| h \|_{L^{p/q}2^{mn}(Q^v_j)} |Q^v_j|.
\]

Therefore, since by (4.5) with \( 1/p - 1/q = \alpha \),
\[
|2^mQ^v_j|^{\gamma} \left\| w^{r/q} \right\|_{L^A(2mQ^v_j)}^{1/r} \| v^{-r/p} \|_{L^B(2mQ^v_j)}^{1/r} \leq c |2^mQ^v_j|^\alpha,
\]
each term in the inner sum of (4.9) is bounded by
\[
c^2c^{2mn} |2^mQ^v_j|^\alpha \| f v^{1/p} \|_{L^D(2mQ^v_j)} \| h \|_{L^{p/q}2^{mn}(Q^v_j)} |Q^v_j|,
\]
and consequently the sum itself does not exceed
\[
J = c \sum_{v,j} |2^mQ^v_j|^\alpha \| f v^{1/p} \|_{L^D(2mQ^v_j)} |2^mQ^v_j|^\alpha \| h \|_{L^{p/q}2^{mn}(Q^v_j)} |Q^v_j|,
\]
and, since
\[
|2^mQ^v_j|^\alpha \| f v^{1/p} \|_{L^D(2mQ^v_j)} \leq \inf_{x \in F^v_j} M_{\alpha,D}(f v^{1/p})(x)
\]
and similarly
\[
|2^mQ^v_j|^\alpha \| h \|_{L^{p/q}2^{mn}(Q^v_j)} \leq \inf_{x \in F^v_j} M_{\alpha,2}\overline{\mathbb{1}}/2^{mn}h(x),
\]
we have that
\[
J \leq c \sum_{v,j} \int_{F^v_j} M_{\alpha,D}(f v^{1/p})(x) M_{\alpha,2}\overline{\mathbb{1}}/2^{mn} h(x) \, dx
\]
\[
\leq c \int_{Q_0} M_{\alpha,D}(f v^{1/p})(x) M_{\alpha,2}\overline{\mathbb{1}}/2^{mn} h(x) \, dx.
\]

Now pick \( s_1, s_2 \) such that
\[
(4.11) \quad 1/p - \alpha_1 = 1/s_1, \quad \text{and} \quad 1/q' - \alpha_2 = 1/s_2.
\]
Since
\[
1/s_1 + 1/s_2 = 1/p - \alpha_1 + 1 - 1/q - \alpha_2 = 1/p - \alpha - 1/q + 1 = 1,
\]
$s_1, s_2$ are conjugate exponents, and, therefore, by Hölder’s inequality,
\[ \int_{Q_0} M_{\alpha, D}(f v^{1/p})(x) M_{\alpha_2, C/2mn} h(x) \, dx \leq \| M_{\alpha, D}(f v^{1/p}) \|_{L^{\alpha_1}} \| M_{\alpha_2, C/2mn} h \|_{L^{\alpha_2}}. \]

Now, by (iii) and (iv) in Proposition 6.1 in [26], $D \in B_{p_1}^{\alpha_1}$ and $C \in B_{q_2}^{\alpha_2}$, respectively, and, therefore, by Theorem 3.3 in [6],
\[ \int_{Q_0} M_{\alpha, D}(f v^{1/p})(x) M_{\alpha_2, C/2mn} h(x) \, dx \leq c \| f v^{1/p} \|_{L^{p_1}} 2^{-mn/q} \| h \|_{L^{q_2}(Q_0)}, \]
and the right-hand side of (4.10) is bounded by
\[ c \left( \sum_{m=1}^{\infty} \lambda_m 2^{mn(1-1/q')} \right) \| f \|_{L^{p_1}_x} \leq c \| f \|_{L^{p_1}_x}. \]

Hence, combining the above estimates,
\[ \| Tf - m_{Tf}(t, Q_0) \|_{L^{q_2}(Q_0)} \leq c \| f \|_{L^{p_1}_x}. \]

Finally, by Fatou’s lemma, (4.6) follows for functions $f$ such that $m_{Tf}(t, Q_0) \to 0$ as $Q_0 \to \mathbb{R}^n$. □

5. ORLICZ-MORREY SPACES

Given a Young function $\Phi$ let
\[ \| f \|_{L^\Phi_{Q}} = \inf \left\{ \lambda > 0 : \int_{Q} \Phi \left( \frac{|f(y)|}{\lambda} \right) \, dy = 1 \right\} ; \]

note that this definition does not coincide with the one given in (2.2) but in view of (5.3) below it is more natural in our setting.

Now, for a positive continuous (or more generally measurable) function $\phi(x,t)$ on $\mathbb{R}^n \times \mathbb{R}^+$ such that for each $x \in \mathbb{R}^n$, $\phi(x,t)$ is decreasing for $t$ in $[0, \infty)$, and $\phi(x,0) = \infty$ for all $x \in \mathbb{R}^n$, with $Q = Q(x,t)$, let
\[ \| f \|_{M^\Phi, \phi} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{\phi(x,t)} \Phi^{-1}(1/|Q|) \| f \|_{L^\Phi_{Q}}. \]

Although a priori the functions $\Phi$ and $\phi$ are unrelated, even in the simplest case there are some limitations [30]. Also note that if $\Phi(t) = t^p$ and $\phi(x,t) = t^{(\lambda-n)/p}$, $0 < \lambda < n$, then $M^\Phi, \phi = M^{p, \lambda}$, the familiar Morrey space. Similar definitions, also coinciding with $M^{p, \lambda}$ when $\Phi$ is a power, are given in [21] and [30].

As for the Campanato spaces $L^\Phi, \phi$, consider the seminorms
\[ \| f \|_{L^\Phi, \phi} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{\phi(x,t)} \Phi^{-1}(1/|Q|) \inf_c \| f - c \|_{L^\Phi_{Q}}. \]
Now, as pointed out in (1.9), if $T$ is a fractional type operator that satisfies the conditions of Theorem 2.1 with $r = 1$, from (3.3) it readily follows that for every Young function $\Psi$ and appropriate $\psi$,

$$\|Tf\|_{L^\Psi,\psi} \leq c \|M_\gamma f\|_{M^{\Psi,\psi}}.$$  

We are therefore led to explore the continuity properties of $M_\gamma$ in the Orlicz-Morrey spaces. We assume that the Young functions $\Phi, \Psi$ satisfy the relation

$$(5.1) \quad \Psi^{-1}(t) = t^{-\gamma} \Phi^{-1}(t),$$

which gives that $M_\gamma$, and also $I_\gamma$, map $L^\Phi(\mathbb{R}^n)$ continuously into $L^\Psi(\mathbb{R}^n)$ [36].

Observe that for a cube $Q$, by Hölder’s inequality for the conjugate functions $\Phi, \overline{\Phi}$,

$$\int_Q |f(z)| \, dz \leq 2 \|1_Q\|_{L^{\overline{\Psi}}(Q)} \|f\|_{L^{\Psi}(Q)},$$

which, by the relations $\|1_Q\|_{L^{\overline{\Psi}}(Q)} \sim 1/\overline{\Phi}^{-1}(1/|Q|)$ and $\Phi^{-1}(t)\overline{\Phi}^{-1}(t) \sim t$, gives that

$$(5.2) \quad \frac{1}{|Q|} \int_Q |f(z)| \, dz \leq c \Phi^{-1}(1/|Q|) \|f\|_{L^\Phi}. $$

Hence, in particular if (5.1) holds we have

$$(5.3) \quad \frac{1}{|Q|^{1-\gamma}} \int_Q |f(z)| \, dz \leq c \Psi^{-1}(1/|Q|) \|f\|_{L^\Psi}. $$

Along the lines of [15] we begin by proving a preliminary result.

**Proposition 5.1.** Suppose $f$ is locally integrable and $0 \leq \gamma < 1$. Then there exist dimensional constants $c_n, d_n$ with $c_n d_n \geq 1$ such that for an arbitrary cube $Q = Q(x, l)$,

(i) $$\|M_\gamma f\|_{L^\Psi_Q} \leq c \|f\|_{L^\Phi_{2Q}} + c \frac{1}{\Psi^{-1}(1/|Q|)} \sup_{t > c_n d_n l} \left( \frac{1}{|Q(x, t)|^{1-\gamma}} \int_{Q(x, t)} |f(z)| \, dz \right);$$

(ii) $$\|M_\gamma f\|_{L^\Psi_Q} \leq c \frac{1}{\Psi^{-1}(1/|Q|)} \left( \sup_{t \geq c_n d_n l} \Psi^{-1}(1/|Q(x, t)|) \|f\|_{L^\Psi_{Q(x,t)}} \right).$$
Proof. Fix a cube $Q$ and let $f = f_1 + f_2$, where $f_1 = f 1_{2Q}$ and $f_2 = f - f_1$. Then as observed above,

$$\|M_f f_1\|_{L^\infty(Q)} \leq c \|f_1\|_{L^\infty(\mathbb{R}^n)} = c \|f\|_{L^\infty(2Q)}.$$  

Next we estimate $M_f f_2(x')$ for $x' \in Q$. Note that if $x' \in Q(y, t)$, by purely geometric considerations there exist dimensional constants $c_n, d_n$ such that 1. If $Q(y, t) \cap (\mathbb{R}^n \setminus 2Q) \neq \emptyset$, then $t > c_n l$, and 2. For any $x \in Q$, $Q(y, t) \cap (\mathbb{R}^n \setminus 2Q) \subset Q(x, d_n t)$ and $c_n d_n \geq 1$.

Hence, for $x' \in Q \cap Q(y, t)$, we have that $t > c_n l$ and

$$\frac{1}{|Q(y, t)|^{1-\gamma}} \int_{Q(y, t)} |f_2(z)| \, dz = \frac{1}{|Q(y, t)|^{1-\gamma}} \int_{Q(y, t) \cap (\mathbb{R}^n \setminus 2Q)} |f(z)| \, dz$$

and therefore for any $x' \in Q$,

$$M_f f_2(x') = \sup_{x' \in Q(y, t)} \frac{1}{|Q(y, t)|^{1-\gamma}} \int_{Q(y, t)} |f_2(z)| \, dz$$

$$\leq c \sup_{t > c_n d_n l} \frac{1}{|Q(x, d_n t)|^{1-\gamma}} \int_{Q(x, d_n t)} |f(z)| \, dz.$$  

(5.5)

Now, since for $g \in L^\infty_Q$,

$$\|g\|_{L^\infty_Q} \leq \|1_Q\|_{L^\infty} \|g\|_{L^\infty_Q} \leq c \frac{1}{\Psi^{-1}(1/|Q|)} \|g\|_{L^\infty},$$  

from (5.5) and (5.6) it follows that

$$\|M_f f_2\|_{L^\infty_Q} \leq \frac{1}{\Psi^{-1}(1/|Q|)} \sup_{t > c_n d_n l} \frac{1}{|Q(x, t)|^{1-\gamma}} \int_{Q(x, t)} |f(z)| \, dz.$$  

(5.7)

Thus, since $\|M_f f\|_{L^\infty_Q} \leq \|M_f f_1\|_{L^\infty_Q} + \|M_f f_2\|_{L^\infty_Q}$, (i) follows combining (5.4) and (5.7).

As for (ii), note that

$$\|f\|_{L^\infty_{2Q}} \leq c \frac{1}{\Psi^{-1}(1/|Q|)} \sup_{t > c_n d_n l} \Psi^{-1}(1/|Q(x, t)|) \|f\|_{L^\infty_{2Q}}$$

$$\leq c \frac{1}{\Psi^{-1}(1/|Q|)} \sup_{t > c_n d_n l} \left( \Psi^{-1}(1/|Q(x, t)|) \|f\|_{L^\infty_{Q(x, t)}} \right).$$  

(5.8)

Also, by (5.3),

$$\frac{1}{\Psi^{-1}(1/|Q|)} \left( \sup_{t > c_n d_n l} \frac{1}{|Q(x, t)|^{1-\gamma}} \int_{Q(x, t)} |f(z)| \, dz \right)$$

$$\leq c \frac{1}{\Psi^{-1}(1/|Q|)} \left( \sup_{t > c_n d_n l} \Psi^{-1}(1/|Q(x, t)|) \|f\|_{L^\infty_{Q(x, t)}} \right).$$  

(5.9)
and (ii) follows combining (5.8) and (5.9).

Concerning the continuity of $M_{\gamma}$ in the Orlicz-Morrey spaces, in the spirit of Theorem 4.3 of [15] we have,

**Theorem 5.1.** Let $0 \leq \gamma < 1$ and suppose that $\phi, \psi$ satisfy the condition

$$
\sup_{r < t < \infty} t^{n\gamma} \phi(x, t) \leq c \psi(x, r).
$$

Then $M_{\gamma}$ maps $M^{\Phi, \phi}$ continuously into $M^{\Psi, \psi}$.

**Proof.** First note that by (5.2),

$$
\Psi^{-1}(1/|Q(x, t)|) \|f\|_{L^\Psi_{Q(x,t)}} = t^{n\gamma} \Phi^{-1}(1/|Q(x, t)|) \|f\|_{L^\Phi_{Q(x,t)}}
$$

$$
\leq \left( \sup_{t \geq l} t^{n\gamma} \phi(x, t) \right) \|f\|_{M^{\Phi, \phi}}
$$

$$
\leq c \psi(x, l) \|f\|_{M^{\Phi, \phi}}.
$$

(5.10)

To estimate $\|M_{\gamma} f\|_{M^{\Psi, \psi}}$, observe that by (ii) in Proposition 5.1 and (5.10), since $c_n d_n \geq 1$, it readily follows that

$$
\|M_{\gamma} f\|_{M^{\Psi, \psi}} \leq \sup_{x \in Q, l > 0} \frac{1}{\psi(x, l)} \left( \sup_{t \geq l} \Psi^{-1}(1/|Q(x, t)|) \|f\|_{L^\Psi_{Q(x,t)}} \right)
$$

$$
\leq c \|f\|_{M^{\Phi, \phi}},
$$

which completes the proof. □

The above result is a prototype for results of the following nature. Let $S$ be a sublinear operator that maps $L^\Phi(\mathbb{R}^n)$ continuously into $L^\Psi(\mathbb{R}^n)$ such that for any cube $Q$, if $x \in Q$ and $\text{supp}(f) \subset \mathbb{R}^n \setminus 2Q$, then

$$
|Sf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n(1-\gamma)}} dy.
$$

Such operators are considered for instance in [14], and they include the fractional maximal functions as well as the Riesz potentials.

The reader should have no difficulty proving the following.

**Theorem 5.2.** Let $0 \leq \gamma < 1$, $\Phi, \Psi$ be Young functions so that $\Psi^{-1}(t) = t^{-\gamma} \Phi^{-1}(t)$, and $\phi(x, t), \psi(x, t)$ positive measurable decreasing functions such that for all $x \in \mathbb{R}^n$ and $l > 0$,

$$
\psi(x, l) \int_{l}^{\infty} \frac{1}{\phi(x, t)} \frac{dt}{t} \leq c.
$$

Then, if $S$ is a sublinear operator that maps $L^\Phi(\mathbb{R}^n)$ continuously into $L^\Psi(\mathbb{R}^n)$ and satisfies (5.11),

$$
\|Sf\|_{M^{\Psi, \psi}} \leq c \|f\|_{M^{\Phi, \phi}}.
$$
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