The b-chromatic number of some special families of graphs

A Jeeva, R Selvakumar and M Nalliah
Department Of Mathematics, School of Advanced Sciences, VIT University, Vellore-632014, India

E-mail: rselvakumar@vit.ac.in

Abstract. Given \( G \), a \( b \)-coloring is a proper \( k \) coloring of \( G \) in which each and every color class has at least one \( b \)-vertex that has a neighbour in other \( k-1 \) color classes. The largest integer \( k \) is the \( b \)-chromatic number \( b(G) \) for which \( G \) having a \( b \)-coloring using \( k \) colors. In this paper, we constructed some family of graphs and found its \( b \)-chromatic number.

1. Introduction
All graphs we consider are simple, finite and undirected graphs. Let \( G=(V,E) \) be a graph. Then the set of vertices denoted by \( V(G) \) with order \( n \) and set of edges denoted by \( E(G) \) with size \( m \). A proper vertex \( k \)-coloring of \( G \) is a nonempty partition \( P=\{V_1,V_2,...,V_k\} \) produce a color class, each \( V_i \) is an independent set of \( G \). The minimum integer \( k \) is the chromatic number \( \chi(G) \) for which \( G \) has a \( k \)-colorable. A \( b \)-coloring is a proper \( k \)-coloring in which each and every color class \( V_i \) contains at least one vertex that has a neighbour in other \( k-1 \) color classes. A vertex which is satisfying the above property is called a \( b \)-vertex. A set of all vertices in \( S_\gamma \) are \( b \)-vertices is called a \( b \)-system such that every \( b \)-vertex belongs to different color classes. The largest integer \( k \) is the \( b \)-chromatic number \( b(G) \) for which \( G \) having a \( b \)-coloring using \( k \) colors. First Irving and Manlove [3] introduced the concept of \( b \)-chromatic number and also they derived the upper bound, \( b(G) \leq \Delta(G) + 1 \). In particular, they remark that, \( G \) having an \( ab \)-chromatic coloring using \( k \) colors and in \( G \) should have at least \( k \) vertices having a degree \( k-1 \). Effantin and Kheddouci discussed the \( b \)-chromatic number of some power graphs [2]. On \( b \)-coloring of regular graphs studied by Blidia, Maffray and Zoham [1]. The \( b \)-chromatic number of some path related graphs discussed by Vaidya and Rakhimol [5] also they investigated the \( b \)-chromatic number of the degree splitting graphs of the path, shell and gear graph in [4]. In general, the corona of any two graphs \( G \) and \( H \) denoted by \( G \odot H \). Vernold Vivin and Venkatachalam [7] have found the \( b \)-chromatic number of corona product of any graph \( G \) with path, cycle and complete graph also Vivin et al[6] investigated the \( b \)-chromatic number of star graph families.

2. Main Results
In the main section, we describe few particular families of graphs and obtained its \( b \)-chromatic number.
2.1. Definition
Let $H = K_{1,n-3}$ be a star graph on $n-2$ vertices and let $V(K_{1,n-3}) = \{u_1, u_2, ..., u_n-3, c\}$, where $c$ is the central vertex of $H$. The graph $F_i$ is constructed from $C_n$ by adding a copy of graph $H$ to every vertex $v_i$ of $C_n$. Clearly the order of $F_i$ is $n + n(n-3)$.

The following family of graphs $\{F_1^0, F_1^1, F_1^2, ..., F_1^k\}$ are constructed from $F_1$ such that $F_1^i, i = 0, 1, 2, ..., k$ is obtained by adding $i$ number of edges to every copy of $H$.

$F_1^0 = C_n \odot K_{1,n-3}$

$F_1^1 = C_n \odot (K_{1,n-3} + \{e_1\})$

$F_1^2 = C_n \odot (K_{1,n-3} + \{e_1, e_2\})$

$\cdots$

$F_1^k = C_n \odot (K_{1,n-3} + \{e_1, e_2, ..., e_k\}), 1 \leq k \leq \frac{(n-4)(n-3)}{2}$

Let $F(C_n) = \{F_1^0, F_1^1, F_1^2, ..., F_1^k\}$ be denote the family of graphs and the order of every graph in $F(C_n)$ is $n + n(n-3)$.

2.2. Theorem
For any graph of $F(C_n)$, the $b$–chromatic number is $n$.

Proof
Let $F_1 \in F(C_n)$ and let $V(F_1) = \{v_i, u_i^l, 1 \leq i \leq n, 1 \leq j \leq n - 3\}$. The order of $F_1$ is $n + n(n-3)$ . Suppose we assume the $b$–chromatic number of $F_1$ is greater than or equal to $n$ that is $b(F_1) \geq n$. Therefore, we have the existence of a $b$–system $S_0$ such that $|S_0| \geq n + 1$ . This means that, in $F_1$ having $b$–system $S_0$ and that $b$–system contains $n + 1$ vertices of degree at least $n$. But here $F_1$ having only $n$ vertices of degree $n - 1$ and the remaining vertices are of degree at most $n - 3$ , which contradicts our assumption and hence $b(F_1) \leq n$.

Now we define the following mapping $C : V(F_1) \rightarrow \{1, 2, 3, ..., n\}$ to vertices as follows.

\[
C(v_i) = i \quad 1 \leq i \leq n,
\]

\[
\begin{cases}
C(u_i^l) &= i + j + 1 \quad i = 1, 2, 1 \leq j \leq n - 3 \\
&= i + j + 1 \quad 3 \leq i \leq n - 1, 1 \leq j \leq n -(i+1) \\
&= j \quad 3 \leq i \leq n - 1, 1 \leq j \leq i - 2 \\
&= j \quad i = n, 2 \leq j \leq i - 2
\end{cases}
\]

Thus we get a proper $b$–coloring of $C$. Therefore $b(F_1) \geq n$ and hence $b(F_1) = n$.

2.3. Definition
Let $H = K_{1,3}$ be a star graph with $3$ vertices and let $V(K_{1,3}) = \{u_1, u_2, c\}$, where $c$ is the central vertex of $H$. The graph $F_2$ is constructed from $C_n$ by adding a copy of graph $H$ to every vertex $v_i$ of $C_n$. Clearly the order of $F_2$ is $n + 2n = 3n$.
The following family of graphs \( \{F^0_2, F^2_2\} \) are constructed from \( F_2 \) such that \( F^i_2, i = 0, 1, 2, \ldots, k \) is obtained by adding \( i \) number of edges to every copy of \( H \).

\[
F_2 = F^0_2 = \overline{C}_n \odot K_{i,2} \\
F^1_2 = \overline{C}_n \odot (K_{j,2} + \{e_i\})
\]

Let \( \mathcal{F}(\overline{C}_n) = \{F^0_2, F^1_2\} \) be denote the family of graphs and the order of every graph in \( \mathcal{F}(\overline{C}_n) \) is \( 3n \).

2.4. Theorem

For any graph of \( \mathcal{F}(\overline{C}_n) \), \( n \geq 5 \) the \( b \)– chromatic number is \( n \).

Proof

Let \( F_2 \in \mathcal{F}(\overline{C}_n) \), \( n \geq 5 \) and let \( V(F_2) = \{v_i, u^i_j, 1 \leq i \leq n, j = 1, 2\} \). The order of \( F_2 \) is \( 3n \). Suppose we assume the \( b \)– chromatic number of \( F_2 \) is greater than or equal to \( n \) that is \( b(F_2) \geq n \). Therefore, we have the existence of a \( b \)– system \( S_0 \) such that \( |S_0| \geq n + 1 \). This means that, in \( F_2 \) having \( b \)– system \( S_0 \) and that \( b \)– system contains \( n + 1 \) vertices of degree at least \( n \). But here \( F_2 \) having only \( n \) vertices of degree \( n - 1 \) and the remaining vertices are of degree at most \( 2 \), which contradicts our assumption and hence \( b(F_2) \leq n \).

Now we define the following mapping \( C : V(F_2) \rightarrow \{1, 2, 3, \ldots, n\} \) to vertices as follows.

\[
C(v_i) = i \quad 1 \leq i \leq n \\
C(u^1_i) = \begin{cases} 
  n & i = 1 \\
  i - 1 & i \geq 2 
\end{cases} \\
C(u^2_i) = \begin{cases} 
  i + 1 & 1 \leq i \leq n - 1 \\
  1 & i = n 
\end{cases}
\]

Thus we get a proper \( b \)– coloring of C. Therefore \( b(F_2) \geq n \) and hence \( b(F_2) = n \).

2.5. Definition

Let \( H_1 = K_{i,1}, H_2 = K_{i,1} \) and let \( V(K_{i,1}) = \{u_1, u_2, \ldots, u_{m-1}, c\} \), \( V(K_{j,1}) = \{v_1, v_2, \ldots, v_{m-1}, c'\} \) where \( c, c' \) are central vertex of \( H_1 \) and \( H_2 \). Let \( K_{m,n}, m < n \) be a complete bipartite graph with bipartitions \( V_1 \) and \( V_2 \). The graph \( F_3 \) is constructed from \( K_{m,n}, m < n \) by adding \( m \) copy of graph \( H_1 \) to every vertex \( v_i \in V_1(K_{m,n}) \) and \( n \) copy of graph \( H_2 \) to every vertex of \( v_i \in V_2(K_{m,n}) \). Clearly the order of \( F_3 \) is \( (m + n) + m(m - 1) + n(n - 1) \).

The following family of graphs \( \{F^0_3, F^1_3, F^2_3, \ldots, F^3_3\} \) is constructed from \( F_3 \) such that \( F^i_3, i = 0, 1, 2, \ldots, k \) is obtained by adding \( i \) number of edges to every copy of \( H_1 \) and \( H_2 \).

\[
F^0_3 = K_{m,n} \odot (H_1, H_2) \\
F^1_3 = K_{m,n} \odot (H_1 + \{e_i\}, H_2 + \{e_i\}) \\
F^2_3 = K_{m,n} \odot (H_1 + \{e_i, e_2\}, H_2 + \{e_i, e_2\}) \\
\ldots
\]
\[ F^k_m = K_{m,n} \odot (H_1 + \{e_1, e_2, \ldots, e_k\}, H_2 + \{e_1, e_2, \ldots, e_k\}), 1 \leq k \leq \frac{(m-1)(m-2)}{2}, 1 \leq l \leq \frac{(n-1)(n-2)}{2} \]

Let \( \mathcal{F}(K_{m,n}) = \{F_0^3, F_1^3, F_2^3, \ldots, F_k^3\} \) be the family of graphs and the order of every graph in \( \mathcal{F}(K_{m,n}) \) is \((m + n) + m(m - 1) + n(n - 1)\).

### 2.6. Theorem

For any graph of \( \mathcal{F}(K_{m,n}) \), the \( b \)-chromatic number is \( m + n \).

**Proof**

Let \( F_j \in \mathcal{F}(K_{m,n}) \) and let \( V(F_j) = \{V_1 \cup V_2 \cup V_3\} \) where \( V_1 = \{v_j, v_{2j}, \ldots, v_m\}, V_2 = \{v_{m+1}, v_{m+2}, \ldots, v_{m+n}\} \)

and \( V_3 = \left\{ u'_i, 1 \leq i \leq m, 1 \leq j \leq m - 1 \right\} \).

The order of \( F_j \) is \((m + n) + m(m - 1) + n(n - 1)\).

Suppose we assume the \( b \)-chromatic number of \( F_j \) is greater than or equal to \( m + n \) that is \( b(F_j) \geq m + n \). Therefore, we have the existence of a \( b \)-system \( S_0 \) such that \( |S_0| \geq m + n + 1 \). This means that, in \( F_j \) having \( b \)-system \( S_0 \) and that \( b \)-system contains \( m + n + 1 \) vertices of degree at least \( m + n \). But here \( F_j \) having only \( m + n \) vertices of degree \( m + n - 1 \) and the remaining vertices are of degree at most \( m - 1 \) in \( H_1 \) and \( n - 1 \) in \( H_2 \), which contradicts our assumption and hence \( b(F_j) \leq m + n \).

Now we define the following mapping \( C : V(F_j) \rightarrow \{1, 2, 3, \ldots, (m + n)\} \) to vertices as follows,

\[ C(v_i) = i, 1 \leq i \leq m+n \]

\[ C(u'_i) = \begin{cases} m & i = j, 1 \leq i \leq m \\ j & i \neq j, 1 \leq j \leq m - 1 \end{cases} \]

\[ C(v'_m) = \begin{cases} m + n & i = j, 1 \leq i \leq n \\ m + j & i \neq j, 1 \leq j \leq n - 1 \end{cases} \]

Thus we get a proper \( b \)-coloring of \( C \). Therefore \( b(F_j) \geq m + n \) and hence \( b(F_j) = m + n \).

### 2.7. Definition

Let \( H = K_{l,n-4} \) be a star graph on \( n - 3 \) vertices and \( V(K_{l,n-4}) = \{u_1, u_2, \ldots, u_{n-4}, c\} \) where \( c \) is the central vertex of \( H \). Let \( W_n, n \geq 4 \) be the wheel graph with \( V(W_n) = \{v_1, v_2, v_3, \ldots, v_n\}, v_i \) is central vertex. The graph \( F_4 \) is constructed from \( W_n \) by adding a copy of graph \( H \) to every vertex \( v_i (2 \leq i \leq n) \) of \( W_n \). Clearly the order of \( F_4 \) is \((n + (n - 1)(n - 4))\).

The following family of graphs \( \{F_0^4, F_1^4, F_2^4, \ldots, F_k^4\} \) is constructed from \( F_4 \) such that \( F_i^4, i = 0, 1, 2, \ldots, k \) is obtained by adding \( i \) number of edges to every copy of \( H \).

\[ F_4 = F_0^4 = W_n \odot K_{l,n-4} \]

\[ F_1^4 = W_n \odot (K_{l,n-4} + \{e_1\}) \]

\[ F_2^4 = W_n \odot (K_{l,n-4} + \{e_1, e_2\}) \]

\[ \ldots \]

\[ \ldots \]
Let $F_k^4 = C_n \circ (K_{1,n-k} + \{e_1, e_2, \ldots, e_k\})$, $1 \leq k \leq \frac{(n-4)(n-5)}{2}$.

Let $\mathcal{F}(W_n) = \{F_0^4, F_1^4, F_2^4, \ldots, F_k^4\}$ be denote the family of graphs and the order of every graph in $\mathcal{F}(W_n)$ is $n + (n-1)(n-4)$.

2.8. **Theorem**
For any graph of $F_4 \in \mathcal{F}(W_n)$, the $b$–chromatic number is $n$.

**Proof**
Let $F_4 \in \mathcal{F}(W_n)$ and let $V(F_4) = \{v_i \cup v_j \cup u^{i_j}, 2 \leq i \leq n, I \leq j \leq n-4\}$. The order of $F_4$ is $n + (n-1)(n-4)$. Suppose we assume the $b$–chromatic number of $F_4$ is greater than or equal to $n$ that is $b(F_4) \geq n$. Therefore, we have the existence of a $b$–system $S_0$ such that $|S_0| \geq n + 1$. This means that, in $F_4$ having $b$–system $S_0$ and that $b$–system contains $n + 1$ vertices of degree at least $n$. But here $F_4$ having only $n$ vertices of degree $n - 1$ and the remaining vertices are of degree at most $n - 4$, which contradicts our assumption and hence $b(F_4) \leq n$.

Now we define the following mapping $C : V(F_4) \to \{1, 2, 3, \ldots, n\}$ to vertices as follows,

$$
C(v_i) = 1 \quad i = 1
$$
$$
C(v_i) = i \quad 2 \leq i \leq n
$$
$$
C(u^{i_j}) = \begin{cases}
  i + j + 1 & i = 2, 3, 1 \leq j \leq n - 3 \\
  i + j + 1 & 4 \leq i \leq n - 1, 1 \leq j \leq n - (i + 1) \\
  j & 4 \leq i \leq n - 1, 2 \leq j \leq i - 2 \\
  j & i = n, 3 \leq j \leq i - 2
\end{cases}
$$

Thus we get a proper $b$–coloring of $C$. Therefore $b(F_4) \geq n$ and hence $b(F_4) = n$.

3. **Conclusion**
In this paper, we defined some particular family of graphs such as $\mathcal{F}(C_n)$, $\mathcal{F}(\overline{C_n})$, $\mathcal{F}(K_{m,n})$, $\mathcal{F}(W_n)$ and obtained its $b$–chromatic number. The $b$–chromatic numbers of central, middle and total graphs of above family of graphs are still open.

**References**
[1] Bldidia M, Maffray F and Zoham Z 2009 *Discrete App. Math*. 157 1787-1793
[2] Effantin B and Kheddouci H 2003 *Discrete Math. Theo. Comp. Sci*. 6(1) 45-54
[3] Irving R W and Manlove D F 1999 *Discrete App. Math*. 91(1) 127-141
[4] Vaidya S K and Rakhimol V Issac 2014 *Malaya J. math*. 2(3) 249-253
[5] Vaidya S K and Rakhimol V Issac 2014 *Int. J. Math. Sci. Comp*. 4(1) 7-12
[6] Venkatachalam M and Vernold Vivin J 2010 *Le Mathematiche* 65(1) 119-125
[7] Vernold Vivin J and Venkatachalam M 2012 *Utilitas Math* 88 299-307