Tverberg type theorems for matroids

Pavel Paták

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Abstract

In this paper we show a variant of colorful Tverberg’s theorem which is valid in any matroid: Let $S$ be a sequence of non-loops in a matroid $M$ of finite rank $m$ with closure operator $\text{cl}$. Suppose that $S$ is colored in such a way that the first color does not appear more than $r-1$-times and each other color appears at most $(r-1)$-times. Then $S$ can be partitioned into $r$ rainbow subsequences $S_1, \ldots, S_r$ such that $\text{cl} \emptyset \subset \text{cl} S_1 \subset \text{cl} S_2 \subset \ldots \subset \text{cl} S_r$. In particular, $\emptyset \neq \bigcap_{i=1}^r \text{cl} S_i$. A subsequence is called rainbow if it contains each color at most once.

The conclusion of our theorem is weaker than the conclusion of the original Tverberg’s theorem in $\mathbb{R}^d$, which states that $\bigcap \text{conv} S_i \neq \emptyset$, whereas we only claim that $\bigcap \text{aff} S_i \neq \emptyset$. On the other hand, our theorem strengthens the Tverberg’s theorem in several other ways: i) it is applicable to any matroid (whereas Tverberg’s theorem can only be used in $\mathbb{R}^d$), ii) instead of $\bigcap \text{cl} S_i \neq \emptyset$ we have the stronger condition $\text{cl} \emptyset \subset \text{cl} S_1 \subset \text{cl} S_2 \subset \ldots \subset \text{cl} S_r$, and iii) we add a color constraints that are even stronger than the color constraints in the colorful version of Tverberg’s theorem.

Recently, we used the first property and applied the non-colorful version of this theorem to homology groups with $GF(p)$ coefficients to obtain several non-embeddability results, for details we refer to [GMP ...]

1 Introduction

Tverberg’s theorem [Tve66] states that given $(d+1)(r-1)+1$ points in $\mathbb{R}^d$, it is possible to split these points into $r$ sets $S_1, \ldots, S_r$ with intersecting convex hulls, that is with $\bigcap \text{conv} S_i \neq \emptyset$.

If one replaces convex hulls with affine hulls, one obtains a valid statement (Lemma 1), which has the advantage of being applicable to any field [GMP ...]. Lemma 1 is also easier to prove than the original Tverberg’s theorem. Since the proof only uses properties of closure operators, the statement does generalize to any matroid (Lemma 2). In both these cases the conclusion can be strengthened a bit: instead of $\text{cl} S_1 \cap \ldots \text{cl} S_r \neq \emptyset$, one can require $\text{cl} \emptyset \subset \text{cl} S_1 \subset \text{cl} S_2 \subset \ldots \subset \text{cl} S_r$.

In this paper we study the variant of Tverberg’s theorem for matroidal closures and show that it allows a colorful version – a generalization where the original points are colored and one furthermore requires that no resulting set $S_i$, $i = 1, \ldots, r$ contains two or more points of the same color.

While the version without colors is straightforward [GMP ...] the proof of the colorful version is more subtle. Moreover, our proof method yields an efficient algorithm that finds the required sets in polynomial time.

1.1 Terminology

Before we state our results formally, let us introduce some notations and terminology which will allow us to nicely present the statements and proofs. We assume that the reader is acquainted with the basic matroid theory. We always use the symbols $r$ and $m$ to denote non-negative integers. We use the symbols $\text{cl}$, $\text{aff}$, $\text{conv}$ and $\text{rk}$ for matroidal closure, affine closure, convex hull and rank function, respectively.

If $M$ is a set, we consider a sequence $S = (m_i)_{i \in I}$ of elements from $M$ as a set of pairs $\{(i, m_i) \mid i \in I\}$. With this convention we can use the set theoretic terminology for sequences: $|S|$ is the length of the sequence, $S' \subseteq S$ means that $S'$ is a subsequence of $S$, we know what it means for two subsequences to be disjoint, we can use the operation $S \setminus S'$ of (sequence) difference, etc.

1We allow repetitions among these points.
If \( S = \{(i, m_i) \mid i \in I\} \) is a sequence and we need to refer to the set \( \{m_i \mid i \in I\} \), we use the symbol \( S^{\text{set}} \).

If \( \Psi \) is a map from the subsets of \( M \) (for example a closure operator, rank function), and \( S = (m_i)_{i \in I} \) is a sequence in \( M \), we use a shorthand \( \Psi(S) := \Psi(S^{\text{set}}) \). To make formulas and equations shorter, we leave out the parantheses after the operators cl, aff, conv and rk when there is no danger of confusion.

A coloring of a sequence \( S = \{(i, m_i) \mid i \in I\} \) is any map \( c : S \to C \) into some set \( C \) of colors, that is, \( c \) assigns to each pair \((i, m_i)\) a color from \( C \). The sequence \( S \) is \textit{rainbow} with respect to \( c \), if the restriction of \( c \) to \( S \) is injective.

### 1.2 Main results

Let us first state the non-colorful variant of Tverberg’s theorem for affine hulls and its easy generalization to matroidal closures.

**Lemma 1** (Tverberg for affine hulls [GMP+13, GMP+16]). Let \( S \) be a sequence of points in an affine space \( A \) of dimension \( d \). If \( |S| > (d + 1)(r - 1) \), then there exist \( r \) pairwise disjoint subsequences \( S_1, \ldots, S_r \) of \( S \) with \( \bigcap_{i=1}^r \text{aff } S_i \neq \emptyset \). In fact, there are \( r \) pairwise disjoint subsequences \( S_1, \ldots, S_r \) satisfying \( \emptyset \neq \text{aff } S_i \subseteq \text{aff } S_2 \subseteq \ldots \subseteq \text{aff } S_r \).

**Lemma 2** (Matroidal Tverberg). Let \( M \) be a (finitary\footnote{Finitary matroids are generalization of matroids to not necessary finite ground sets. They add the following axiom to the usual axioms for finite matroids: If \( y \in \text{cl}(X) \), then there exists a finite set \( Y \subseteq X \) such that \( y \in \text{cl}(Y) \). With these addition, such terms as rank or basis can be correctly defined.}.\footnote{We do not require \( d \) to be finite, therefore the slightly unusual formulation.}) matroid of rank \( m \) with closure operator \( \text{cl} \) and \( S \) be a sequence of points in \( M \) with \( |S| > m(r - 1) \). Then there exist \( r \) pairwise disjoint subsequences \( S_1, \ldots, S_r \) of \( S \) satisfying \( \text{cl } \emptyset \subseteq \text{cl } S_1 \subseteq \text{cl } S_2 \subseteq \ldots \subseteq \text{cl } S_r \).

In [GMP+16] Lemma 13 we only stated that there exists sets \( S_i \) with \( \emptyset \neq \bigcap \text{aff } S_i \). However, the proof there implies Lemma 1 and (if one replaces aff with the closure operator cl of a matroid) Lemma 2. In the case of matroids of finite rank, both lemmas can also be obtain as a direct consequence of Theorem 3.

In [GMP+15, GMP+16] we applied Lemma 1 to homology groups over finite fields. This enabled us to prove some inequalities for simplicial complexes embeddable into various manifolds. Our colorful matroidal Tverberg (Theorem 3) provides a control of the resulting sets, which enables us to further improve the bounds from [GMP+15, GMP+16]. For the details of the improvement, see the author’s thesis [Pat15].

We are now ready to state the main results of this paper.

**Theorem 3.** Let \( M \) be a matroid of a finite rank \( m \) and \( S \) be a sequence of non-loops in \( M \) colored by some colors in such a way that at most \( m \) elements of \( S \) are colored by the first color, at most \( m - 1 \) by the second color, at most \( m - 1 \) by the third color, etc. If \( |S| > m(r - 1) \), then there exist \( r \) pairwise disjoint rainbow subsequences \( S_1, \ldots, S_r \) of \( S \) such that \( \text{cl } \emptyset \subseteq \text{cl } S_1 \subseteq \text{cl } S_2 \subseteq \ldots \subseteq \text{cl } S_r \).

Furthermore, if the time required to decide whether a point \( x \in M \) lies in the closure of a set \( Y \subseteq M \) is bounded by \( u \), then the subsequences \( S_1, \ldots, S_r \) can by found in time polynomial in \( r, m, u \) and \( |S| \).

In the proof of Theorem 3 we encounter another version of colorful matroidal Tverberg’s theorem.

**Theorem 4.** Let \( M \) be a matroid of a finite rank \( m \) and \( S \) a sequence of non-loops in \( M \) colored by \( m \) colors in such a way that at least \( r \) elements of \( S \) are colored by the first color, at least \( r - 1 \) by the second color, at least \( r - 1 \) by the third, \( \ldots \), at least \( r - 1 \) by the \( m \)th color.

Then there exist \( r \) pairwise disjoint rainbow subsequences \( S_1, \ldots, S_r \) of \( S \) such that \( \text{cl } \emptyset \subseteq \text{cl } S_1 \subseteq \text{cl } S_2 \subseteq \ldots \subseteq \text{cl } S_r \).

Furthermore, if the time required to decide whether a point \( x \in M \) lies in the closure of a set \( Y \subseteq M \) is bounded by \( u \), then the subsequences \( S_1, \ldots, S_r \) can by found in time polynomial in \( r, m, u \) and \( |S| \).

Note the different conditions on the number of points of each color. In Theorem 3 these conditions are used to ensure that we have enough colors. In Theorem 4 we already have the right number of colors, but the conditions ensure that the length of \( S \) is sufficient.

Moreover, these results are tight:
Proposition 5. Lemma [4], Lemma [3], Theorem [8] and Theorem [9] are sharp. To be precise, for any $r$ and any matroid $M$ of rank $m$ there exists a sequence $S$ of non-loops in $M$ with $|S| = m(r - 1)$ such that any division of $S$ into $r$ disjoint subsequences $S_1, \ldots, S_r$ satisfies $\bigcap_{i=1}^r \cl S_i = \cl \emptyset$.

**Tverberg-type theorems in $\mathbb{R}^d$**

Let us now compare our main results with the related theorems valid in $\mathbb{R}^d$.

In this section $\Delta_n$ denotes the $n$-dimensional simplex.

Tverberg’s theorem can be stated as follows: If $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ is an affine map, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_{(d+1)(r-1)}$ with $\bigcap_{i=1}^r f(\sigma_i) \neq \emptyset$. This is the reason why Tverberg’s theorem is also called affine Tverberg’s theorem. To avoid confusion, we have decided not to use the name “affine Tverberg” for Lemma [1].

If $r$ is a prime power, Özaydin [Oza87] showed that the same result holds for an arbitrary continuous map $f$. The statement is known as topological Tverberg. It was a long-standing open problem, whether topological Tverberg can be extended to other values of $r$. The negative answer came in 2015, when Frick (based on the previous work of Mabillard and Wagner [MW14]) constructed first counterexamples [Fri15].

If $r$ is a prime, there is a colorful version of (topological) Tverberg’s theorem [BMZ15] as well: Suppose that the vertices of $K = \Delta_{(d+1)(r-1)}$ are colored in such a way, that no color is used more than $(r - 1)$-times. Then for every continuous map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$, there are $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_{(d+1)(r-1)}$ with $\bigcap_{i=1}^r f(\sigma_i) \neq \emptyset$.

The colorful version provides more control over the resulting sets $\sigma_1, \ldots, \sigma_r$.

Even if $f$ is an affine map, the only known proof uses topological methods and needs the assumption that $r$ is prime. Whether this assumption can be relaxed in the affine situation is an open question.

We see that Theorem [9] does not require $r$ to be a prime number, it relaxes the conditions on the colors from topological version a bit and provides an efficient algorithm for finding the desired sets.

We also note that Bárány, Kalai and Meshulam proved another, very different Tverberg Type Theorem for Matroids [BGR15], they considered continuous maps from the matroidal complex and showed the following: If $B(M)$ denotes the maximal number of disjoint bases in a matroid $M$ of rank $d + 1$, then for any continuous map $f$ from the matroidal complex $M$ into $\mathbb{R}^d$ there exists $t \geq \sqrt{b(M)/4}$ disjoint independent sets $\sigma_1, \ldots, \sigma_t \in M$ such that $\bigcap_{i=1}^t f(\sigma_i) \neq \emptyset$.

## 2 Tightness

We postpone the technical proofs of our main results, Theorems [8] and [9] to the end of the paper. First we prove Proposition [3] showing their tightness. The proof is a variant of the standard construction for showing that Tverberg’s theorem is tight.

We start with an auxiliary lemma.

**Lemma 6.** Let $M$ be a matroid with finite basis $B$. Then for any two sets $U, V \subseteq B$

\[
\cl(U) \cap \cl(V) = \cl(U \cap V).
\]

**Proof.** Since the operator $\cl$ is monotone, the inclusion $\cl(U \cap V) \subseteq \cl(U) \cap \cl(V)$ is obvious. Let us now prove the opposite inclusion.

Let $x \in \cl(U) \cap \cl(V)$ be an arbitrary element. We want to show that $x \in \cl(U \cap V)$. If $x$ is a loop, $x \in \cl \emptyset \subseteq \cl(U \cap V)$. So assume that $x$ is not a loop.

Let $U' \subseteq U$ and $V' \subseteq V$ be inclusion minimal subsets with $x \in \cl(U')$ and $x \in \cl(V')$, respectively. Since we assume that $x$ is not a loop, $U' \neq \emptyset \neq V'$.

We will show by contradiction that $U' = V'$, hence proving the claim. If $U' \neq V'$, we may up to symmetry assume that there is an element $u' \in U'$ which does not lie in $V'$.

From the inclusion minimality of $U'$ follows that $x \in \cl \left( (U' \setminus \{u'\}) \cup \{u'\} \right) \setminus \cl(U' \setminus \{u'\})$. The exchange principle yields $u' \in \cl \left( U' \setminus \{u'\} \cup \{x\} \right)$. Similarly $v' \in \cl \left( V' \setminus \{v'\} \cup \{x\} \right)$ for an arbitrary $v' \in V'$.

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4Containing each color at most once.
The set $U' \cup V'$ is independent being a subset of a basis $B$. By construction $\text{cl}(U' \setminus \{u'\} \cup V' \setminus \{v'\} \cup \{x\}) = \text{cl}(U' \cup V')$. Comparing the ranks of both sides and using the fact that $v' \notin V'$, we see that $v'$ has to belong to $U'$. Since $v'$ was arbitrary, this implies $V' \subseteq U' -$ in contradiction with $U'$ being minimal with $x \in \text{cl}(U')$.

We can now finally prove Proposition 5.

Proof of Proposition 5. Let $B = (e_1, \ldots, e_m)$ be a basis of the matroid $M$. It suffices to take

$$S = e_1, e_1, \ldots, e_1, e_2, e_2, \ldots, e_2, \ldots, e_m, e_m, \ldots, e_m.$$  \hspace{1cm} (2.2)

Let $S_1, \ldots, S_r$ be disjoint subsequences of $S$. Then

$$\bigcap_{i=1}^r \text{cl}(S_i) = \text{cl} \left( \bigcap_{i=1}^r S_i^{\text{red}} \right) = \text{cl}(\emptyset),$$

where the first equality follows by inductive application of Lemma 4 and the second equality uses the fact that each element $e_j$ is missing in at least one sequence $S_i$.

We also note that the assumption in Theorem 3 that there are at most $r$ points of the first color is necessary. Otherwise, one can consider the sequence $S = (1, 2, 3, 4, \ldots, n, n+1)$ in $\mathbb{R}^1$ where the first $n$ elements are red and the last element $n+1$ is blue. Then although the length of $S$ can be arbitrary, there are no three disjoint rainbow subsequences $S_1, S_2, S_3$ with $\text{aff} S_1 \cap \text{aff} S_2 \cap \text{aff} S_3 \neq \emptyset$. On the other hand, it is not true that this condition is necessary in every matroid. For example, consider the affine line over the field with two elements.

3 The proof

We begin the proof by showing that Theorem 4 implies Theorem 3.

The reduction of Theorem 3 to Theorem 4 follows a well known pattern, a similar reduction previously appeared in the proof of the optimal colored Tverberg theorem [BMZ15] or in Sarkaria’s proof for the prime power Tverberg theorem [Sar00, 2.7.3], see also de Longueville’s exposition [dLJ02, Prop. 2.5]. Nevertheless, there are subtle differences because we are working in greater generality and because we need to take algorithmic aspects into consideration.

Theorem 4 implies Theorem 3. Assume that the assumptions of Theorem 3 are satisfied. We show how to turn the sequence $S$ and the matroid $M$ with closure operator $\text{cl}$ into a sequence $S'$ and matroid $M'$ with closure operator $\text{cl}'$ that satisfy the assumptions of Theorem 4. Moreover, we construct $S', M', \text{cl}'$ and the coloring of $S'$ in such a way that the sets $S_1 := S'_1 \cap S$, $S_2 := S'_2 \cap S$, \ldots, $S_r := S'_r \cap S$ will satisfy $\text{cl}(S_1) \subseteq \text{cl}(S_2) \subseteq \cdots \subseteq \text{cl}(S_r)$ iff and only if $\text{cl}'(S'_1) \subseteq \text{cl}'(S'_2) \subseteq \cdots \subseteq \text{cl}'(S'_r)$ and the rainbowness of $S'_r$ will imply that $S_i$ is rainbow.

Let $m$ be the rank of $M$ and $d$ the number of colors used in $S$. From the conditions follows that $d - m \geq 0$.

If the length of $S$ is strictly larger than $m(r-1) + 1$, we throw the superfluous elements of $S$ away. This does not add a point of any color, therefore all assumptions of Theorem 3 remain preserved. So we may assume that the length of $S$ is precisely $m(r-1) + 1$.

We form $M'$ from $M$ by adding $(d - m)$ new coloops $x_1, \ldots, x_{d-m}$. Now we form the sequence $S'$ by appending $(x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_{d-m}, x_{d-m})$ to $S$.

Clearly we can color the new elements of $S'$ so that in total there are exactly $r$ points of the first color, and exactly $r - 1$ points of every other color.

A coloop is an element $x$ that is independent on any set that does not contain $x$. In other words, we form $M'$ as the direct sum of $M$ with the uniform matroid $U_{d-m}^{d-m}$. 

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4
We see that \( S', M' \) satisfy the assumptions of Theorem [4]. It follows that there are \( r \) rainbow subsequences \( S'_1, \ldots, S'_r \) of \( S' \) satisfying \( \text{cl}' \emptyset \subseteq \text{cl}' S'_1 \subseteq \text{cl}' S'_2 \subseteq \ldots \subseteq \text{cl}' S'_r \).

Since the points \( x_i \) are coloops and since each one of them was added exactly \((r - 1)\)-times, it follows that they cannot contribute to \( \bigcap_{i=1}^{r} \text{cl}'(S'_i) \). Consequently, \( \bigcap_{i=1}^{r} \text{cl}(S \cap S'_i) \neq \emptyset \) and \( \emptyset \subseteq \text{cl}(S \cap S'_i) \subseteq \text{cl}(S \cap S'_i) \subseteq \ldots \subseteq \text{cl}(S \cap S'_i) \).

We conclude that \( S_1 := S \cap S'_1, S_2 := S \cap S'_2, \ldots, S_r := S \cap S'_r \) are the required subsequences of \( S \).

Observe that the reduction is polynomial in \( r, m, u \) and \( |S| \).

Now we can start with the proof of Theorem [4]. Here we describe the main idea. We let \( S_r \) be a rainbow independent subsequence of the maximal rank. In an ideal case \( \text{cl}(S_r) = M \) and we may obtain the remaining subsequences \( S_1, \ldots, S_{r-1} \) by apply induction on the sequence \( S \setminus S_r \) inside \( M \).

However, we may be unlucky. It may happen that no such \( S_r \) satisfies \( \text{cl}(S_r) = M \), see Fig. 1.

\[
\bullet \ a_3 \\
\bullet \ a_2 \\
\bullet \ a_1 = a_5 = a_6 = a_7 = a_4
\]

A situation in which no rainbow \( S_r \) satisfies \( \text{cl}(S_r) = M \). \( M = \mathbb{R}^2 \), \( \text{cl} \) is the affine hull, \( r = 3 \). Points \( a_2, a_3, a_4 \) use the first color (blue), \( a_1, a_5 \) use orange, \( a_6, a_7 \) use red.

**Figure 1:** An example without \( \text{cl}(S_r) = M \).

We see that in this case we could simply take the subsequence \( S' = (a_1, a_4, a_5, a_6, a_7) \) and unify colors blue and red into one color (say violet). Then \( S' \) lives in a submatroid of rank 1 and satisfies the conditions of Theorem [4] so we may use induction. We obtain subsequences \( S_1, S_2, S_3 \) of \( S' \) satisfying \( \text{cl}' \emptyset \subseteq \text{cl}(S_1) \subseteq \text{cl}(S_2) \subseteq \text{cl}(S_3) \). These are clearly also subsequences of \( S \). Moreover they are not only rainbow in the violet-orange coloring, but also in the original blue-orange-red coloring.

In the proof we show that if \( \text{cl}(S_r) \neq M \), we may always resolve the situation by an analogous trick.

Let us now carry out the technical details. Since we promised an algorithmic solution, we describe an algorithm that finds the desired subsequences.

**Proof.** First we compute \( m' = \text{rk} S \). Since instead of \( S \) we can consider the subsequence \( S' \) formed by the elements colored by the first \( m' \) colors (while preserving all assumptions of Theorem [4]), we may assume that \( M = \text{cl}(S) \).

Now we find an inclusion maximal independent rainbow subsequence \( RI_r \) of \( S \). This can clearly be done in time polynomial in \( r, m, u \) and \( |S| \).

We will proceed in the proof by induction on the triple \((r, m, m - \text{rk} RI_r)\) (in lexicographical ordering). If \( r = 1 \) or \( m = 1 \) the statement is trivial, so assume \( r > 1, m > 1 \).

If \( m - \text{rk} RI_r = 0 \), then \( \text{cl}(RI_r) = M \). Because \( RI_r \) is rainbow, \( S \setminus RI_r \) and \( M \) satisfy the assumptions of Theorem [4] for \( r' = r - 1 \). By applying induction we obtain \( r - 1 \) disjoint rainbow subsequences \( S_1, \ldots, S_{r-1} \) of \( S \setminus RI_r \) with \( \emptyset \subseteq \text{cl}(S_1) \subseteq \text{cl}(S_2) \subseteq \ldots \subseteq \text{cl}(S_{r-1}) \). If we now set \( S_r = RI_r \) we see that \( S_1, \ldots, S_r \) are the desired disjoint rainbow subsequences with \( \emptyset \subseteq \text{cl}(S_1) \subseteq \text{cl}(S_2) \subseteq \ldots \subseteq \text{cl}(S_{r-1}) \subseteq \text{cl}(S_r) \).

Therefore we may assume that

\[
\text{cl}(RI_r) \subseteq M \tag{3.1}
\]

We would like to increase \( RI_r \) by adding a point of a color that is not yet used in \( RI_r \). Unfortunately, this is not possible without replacing some points of \( RI_r \) first. Our algorithm uses a cycle to find out which points to replace and how. Within the cycle we need to keep track of "replacement rules" which makes this part a bit technical. Moreover, there are three possibilities what can occur at one iteration of the cycle: a) either we construct a larger independent rainbow set \( RI_r \), b) we find the desired sets \( S_1, \ldots, S_r \) in a smaller submatroid, or c) we adjust the replacement rules.
The cycle  In the $k$th step ($k = 0, 1, 2, \ldots$) of the cycle the replacement rules consist of the following data:

1. set $K_k$ of colors (this set corresponds to colors that we may use while replacing some points),
2. subsequence $I_k$ of $RI_r$ (eventually we would like to replace the subsequence $I_k$ of $RI_r$ by another sequence $I'_k$),
3. for each element $p$ whose color is in $K_k$ and which does not lie in $\text{cl}(I_k)$ a subsequence $I'_k$ of $S$ (we want to replace $I_k$ with $I'_k$, hence increasing the length of our subsequence by one)

To simplify the terminology, if $T$ is a subsequence of $S$, let $c(T)$ denote the set of all the colors used by elements of $T$. If $U$ is a set of colors, let $C_U$ be the subsequence of $S$ formed by all elements with color from $U$.

We want the data to satisfy the following conditions:

(i) $c(I_k) \subseteq K_k$,
(ii) $c(I'_k) = c(I_k) \cup \{c'_k\}$ for some $c'_k \in K_k \setminus c(RI_r)$,
(iii) $|I'_k| = |I_k| + 1$,
(iv) $p \in I'_k$ and $c(I'_k \setminus \{p\}) = \text{cl}(I_k)$
(v) $RI_r \cap C_k = I_k$ and $K_k \not\subseteq c(RI_r)$

Note that conditions (ii) and (iii) imply that $I'_k$ only contains elements that have the same colors as points in $I_k$ plus one additional point that has color $c'_k$, which is not yet present in $RI_r$.

The first step ($k = 0$) is easy. We set $I_0 := \emptyset$ and let $K_0$ be all the colors of $S$ except for those already used in $RI_r$. No element $p \in C_{K_0}$ is contained in $\text{cl}(I_0) = \emptyset$, so we need to define the set $I_1$ for every such $p$. We simply put $I_1 := \{p\}$.

Now we check that the above defined sets satisfy all the prescribed conditions. Note that by (3.1), $S \not\subseteq \text{cl}(RI_r)$. This together with the fact that $RI_r$ is independent implies that $|RI_r| < m$. Since we have $m$ colors, there is a color that is not used in $RI_r$. In other words, $K_0$ is nonempty.

Hence conditions (i) (v) are satisfied trivially (with $c'_k = c(p)$ in condition (ii)).

So suppose that the sets $K_k$, $I_k$ and $I'_k$ are already constructed. Since $I_k \subseteq RI_r$ there are three cases that may occur:

a) $C_{K_k} \subseteq \text{cl}(I_k)$,
b) $C_{K_k} \not\subseteq \text{cl}(RI_r)$ or
c) $C_{K_k} \subseteq \text{cl}(RI_r)$ and $C_{K_k} \not\subseteq \text{cl}(I_k)$.

We deal with the particular cases separately:

Case a) $C_{K_k} \subseteq \text{cl}(I_k)$

In this case, we may apply the trick we used for Fig. 1. Let us describe it formally.

We set $M' := \text{cl}(I_k)$ and $m' := \text{rk}(I_k)$. $M$ has rank $m$ and by (3.1) we know that $M \not\subseteq \text{cl}(RI_r)$. It follows that $\text{rk}(RI_r) < m$ and since $I_k \not\subseteq RI_r$, we also have $m' = \text{rk}(I_k) < m$.

Condition (ii) implies $c(I_k) \subseteq K_k$, so there is a point $p \in C_{K_k} \setminus C_{\text{cl}(I_k)}$.

Because $I_k$ is rainbow and independent and $\text{rk}I_k = m'$, $c(I_k)$ has $m'$ distinct elements, say $k_1, \ldots, k_{m'}$.

We define $S' := C_{\{k_1, \ldots, k_{m'}\}} \cup \{p\}$. In $S'$ we recolor $p$ and all points of color $k_1$ by a new color $z$.

Because $S' \subseteq C_{K_k}$ (we evaluate $C_{K_k}$ with respect to the original coloring), the assumption $C_{K_k} \subseteq \text{cl}(I_k)$ (Case a) implies that $S'$ is a sequence of elements from $M'$. Also in $S'$ there are $m'$ colors, at least $r$ elements of color $z$ and at least $r - 1$ elements of all the remaining colors. Therefore, the assumptions of Theorem 2 are satisfied for $m' < m$. By induction we obtain the desired disjoint rainbow subsequences $S_1, \ldots, S_r$ of $S'$ (which itself is a subsequence of $S$) with $\text{cl} 0 \subseteq \text{cl}(S_1) \subseteq \text{cl}(S_2) \subseteq \ldots \subseteq \text{cl}(S_r)$. These subsequences are rainbow with respect to the new coloring of $S'$. By the construction of the new coloring these subsequences are also rainbow in the original coloring of $S$.

---

*6$C_{K_0}$ are the elements of $S$ whose color lies in $K_0$ and we assume that $S$ contains only non-loop elements.
In this case, we construct a new independent rainbow subsequence $RI'_c$ with $|RI'_c| = |RI_r| + 1$: We pick a point $p \in C_{K_k}$ with $p \notin cl(RI_r)$ and set $RI'_c := (RI_r \setminus I_k) \cup I'_k$.

Before we show that such $RI'_c$ is a rainbow independent subsequence of size $|RI_r| + 1$, we prove the following auxiliary equality:

$$cl(RI'_c) = cl(RI_r \cup \{p\}).$$  \hspace{1cm} (3.2)

Indeed,

$$cl(RI'_c) = cl((RI_r \setminus I_k) \cup I'_k) = cl((RI_r \setminus I_k) \cup (I'_k \setminus \{p\}) \cup \{p\}),$$

where the last equality uses the fact that $p \in I'_k$ from condition (iv). Because any closure operator $cl$ satisfies

$$cl(B \cup C) = cl(B \cup cl(C)) \quad \text{for any two sets } B, C \subseteq M,$$

we may rewrite the expression further to

$$cl(RI'_c) = cl((RI_r \setminus I_k) \cup cl(I'_k \setminus \{p\}) \cup \{p\}).$$

By condition (iv) $cl(I'_k \setminus \{p\}) = cl(I_k)$, which reduces the equality to:

$$cl(RI'_c) = cl((RI_r \setminus I_k) \cup cl(I_k) \cup \{p\}).$$

Using (3.3) again, we obtain

$$cl(RI'_c) = cl((RI_r \setminus I_k) \cup I_k \cup \{p\})$$

Since $I_k \subseteq RI_r$, Equation (3.2) follows.

Using the fact that $I_k \subseteq RI_r$, we are now ready to verify that $RI'_c$ is a rainbow independent subsequence with $|RI'_c| = |RI_r| + 1$.

- $|RI'_c| = |RI_r| + 1$: $|RI'_c| = |(RI_r \setminus I_k) \cup I'_k|$. Because $RI_r$ is rainbow, condition (ii) implies that the sequences $RI_r \setminus I_k$ and $I'_k$ do not share any color. In particular, they are disjoint and $|RI'_c| = |RI_r \setminus I_k| + |I'_k|$. Since $|I'_k| = |I_k| + 1$ (condition (iii)), $|RI'_c| = |RI_r \setminus I_k| + |I_k| + 1$. Because $I_k \subseteq RI_r$, we have $|RI'_c| = |RI_r| + 1$.

- $RI'_c$ is rainbow: $I'_k$ contains one element of color that is not used in $RI_r$, otherwise it uses the same colors as $I_k$. Because $RI'_c = (RI_r \setminus I_k) \cup I'_k$, we see that $P'_r$ uses exactly $|RI_r| + 1$ colors. This, together with the previous item, yields that $P'_r$ is rainbow.

- $RI'_c$ is independent: From the equality (3.2) we get $cl(RI'_c) = cl(RI_r \cup \{p\})$. Moreover, we have chosen a point $p$ which satisfies $p \notin cl(RI_r)$, so $rk(RI'_c) = rkRI_r + 1$. Since $RI_r$ is independent and $RI'_c$ has exactly one element more, the independence of $RI'_c$ follows.

Let $RI''_r$ be an inclusion maximal independent rainbow subsequence of $S$ that contains $RI'_r$. We may now start our algorithm again but this time we replace the maximal independent rainbow subset $RI_r$ by $RI''_r$. We have decreased the quantity $(m - rkC_r)$ and preserved $m$ and $r$. By induction we obtain the desired disjoint rainbow subsequences $S_1, \ldots, S_r$ with $\emptyset \subseteq cl S_1 \subseteq cl S_2 \subseteq \ldots \subseteq cl S_r$.

Case [c]: $C_{K_k} \subseteq cl(RI_r)$ and $C_{K_k} \not\subseteq cl(I_k)$

In this case, we show how to construct sets $K_{k+1}$, $I_{k+1}$ and for every $p \in C_{K_{k+1}}$ with $p \notin cl(I_{k+1})$ we construct a subsequence $I'_{k+1}$.

We choose $I_{k+1}$ to be any inclusion minimal subsequence $I_{k+1} \subseteq RI_r$ satisfying

$$C_{K_k} \subseteq cl I_{k+1}.$$  \hspace{1cm} (3.4)

\footnote{$c(I'_k) = c(I_k) \cup \{c^p_k\}$, for some $c^p_k \in K_k \setminus c(RI_r) \subseteq K_k \setminus c(I_k)$}
Because we assume that $C_{K_k} \subseteq \operatorname{cl}(RI_r)$, such set $I_{k+1}$ does exist. We further define

$$K_{k+1} := K_k \cup c(I_{k+1}).$$ (3.5)

Before we construct $I_{k+1}^p$, we prove the following auxiliary claim:

**Claim 6.1.**

$$I_k \subseteq I_{k+1} \quad \text{and} \quad \operatorname{cl} I_k \subseteq \operatorname{cl} I_{k+1}.\quad (3.6)$$

**Proof.** By condition $^{[1]}$, $I_k \subseteq C_{K_k}$. By Eq. (3.4), we have $\operatorname{cl} I_k \subseteq \operatorname{cl} I_{k+1}$. By construction both $I_k$ and $I_{k+1}$ are subsequences of the independent sequence $RI_r$ which together with the preceding yields $I_k \subseteq I_{k+1}$. Condition $^{[1]}$ and the fact that we are in case $^{[1]}$ yields $I_k \subseteq C_{K_k} \subseteq \operatorname{cl} I_k$. Since also $C_{K_k} \subseteq \operatorname{cl} I_{k+1}$, we see that $\operatorname{cl} I_{k+1} \neq \operatorname{cl} I_k$ and $I_{k+1} \neq I_k$. $\blacksquare$

Now we construct sets $I_{k+1}^p$ for all points $p \in C_{K_k+1}$ satisfying $p \notin \operatorname{cl} I_{k+1}$. Let $p$ be such a point. By definition of $I_{k+1}$, $C_{K_k} \subseteq \operatorname{cl} I_{k+1}$, so $p$ cannot lie in $C_{K_k}$. Equation (3.5) implies $c(p) \in (K_{k+1} \setminus K_k) \subseteq c(I_{k+1})$. Because $I_{k+1} \subseteq RI_r$ is a rainbow set, there exists a unique element $r \in I_{k+1}$ with $c(r) = c(p)$. Since we assume $p \notin C_{K_k}$, we have $c(r) = c(p) \notin K_k \supseteq c(I_k)$, where the last inclusion follows from condition $^{[1]}$. In particular, $c(r) \notin c(I_k)$, hence

$$r \in I_{k+1} \setminus I_k.\quad (3.7)$$

Since $I_{k+1}$ is an inclusion minimal subsequence of $RI_r$ for which $C_{K_k} \subseteq \operatorname{cl} I_{k+1}$, there exists an element $q \in C_{K_k}$ such that $q \notin \operatorname{cl}(I_{k+1} \setminus \{r\})$. Since $q \in C_{K_k} \subseteq \operatorname{cl} I_{k+1}$, the exchange principle implies $r \in \operatorname{cl}(I_{k+1} \setminus \{r\}) \cup \{q\}$.

It easily follows that

$$\operatorname{cl} I_{k+1} = \operatorname{cl}((I_{k+1} \setminus \{r\}) \cup \{q\}).\quad (3.8)$$

Claim 6.1 together with (3.7) imply that $I_k \subseteq I_{k+1} \setminus \{r\}$. Since $q$ was chosen to satisfy $q \notin \operatorname{cl}(I_{k+1} \setminus \{r\})$, we have $q \notin \operatorname{cl} I_k$ as well. Together with $q \in C_{K_k}$, this implies that $I_{k+1}^p$ is defined. We set

$$I_{k+1}^p := I_{k+1} \setminus (I_k \cup \{r\}) \cup I_k^p \cup \{p\}.\quad (3.9)$$

It remains to show that our assignment satisfies conditions $^{[1]}$ $^{[v]}$.

- **Condition $^{[1]}$.** By (3.8), we have $c(I_{k+1}) \subseteq K_{k+1}$. Condition $^{[v]}$ implies that $K_k$ contains a color that is not used in $RI_r$ and since $I_{k+1} \subseteq RI_r$, which together with (3.5) yields $K_{k+1} \neq c(I_{k+1})$. Condition $^{[1]}$ follows.

- **Condition $^{[iii]}$.** Condition $^{[ii]}$ states that $c(I_k^p) = c(I_k) \cup \{c_k\}$ for some $c_k \in K_k \setminus RI_r$, in particular $c(I_k) \subseteq c(I_k^p)$. Together with the fact that elements $p$ and $r$ have the same color ($c(p) = c(r)$), (3.9) yields $c(I_{k+1}^p) = c(I_{k+1} \setminus I_k) \cup c(I_k^p)$. If we now apply condition $^{[ii]}$ for $I_k^p$ and Claim 6.1, we see that $c(I_{k+1}^p) = c(I_{k+1}) \cup \{c_k\}$, where $c_k = c_k$. Note that $K_k \subseteq K_{k+1}$, hence $c_k \in K_{k+1} \setminus \operatorname{cl} RI_r$. Condition $^{[ii]}$ follows.

- **Condition $^{[iii]}$.** By definition $I_{k+1}^p = I_{k+1} \setminus (I_k \cup \{r\}) \cup I_k^p \cup \{p\}$. Because $I_{k+1}$ is a subset of the rainbow set $RI_r$, $I_{k+1}$ is itself rainbow. Together with $c(I_k^p) = c(I_k) \cup \{c_k\}$, where $c_k \notin c(RI_r) \supseteq c(I_{k+1})$, this implies that the sets $I_{k+1} \setminus I_k$ and $I_k^p$ are disjoint. Since $r \in I_{k+1} \setminus I_k$ (Equation (3.7)), $c(p) = c(r) \in c(I_{k+1}) \setminus c(I_k)$ and $c(I_k) \cap c(RI_r) = c(I_k)$ (conditions $^{[ii]}$ and $^{[v]}$), we have $p, r \notin I_k^p$ and $p, r \notin I_k$. From $p \notin \operatorname{cl} I_{k+1}$ follows $p \notin I_{k+1}$. Since $r \in I_{k+1}$, we have $|I_{k+1}^p| = |I_{k+1} \setminus I_k| + |\{r\}| + |\{p\}| = |I_{k+1} \setminus I_k| + |I_k| + 1$, where the last equality uses the induction hypothesis for $k$. Claim 6.1 then yields $|I_{k+1}^p| = |I_{k+1}| + 1$ as desired.

$^8$ $RI_r$ is rainbow!

$^9$ We note that $I_{k+1}^p$ does depend on the choice of $q$, i.e., if we choose another $q \in C_{K_k}$ that satisfies $q \notin \operatorname{cl}(I_{k+1} \setminus \{r\})$, we obtain a different set $I_{k+1}^p$. 
• **Condition [iv]** By definition (3.3) $p \in I_{k+1}^p$, so we only need to verify that $\text{cl}(I_{k+1}^p \setminus \{p\}) = \text{cl} I_{k+1}$. Let us compute. Using the fact that $q \in I_k^p$ from condition [iv] and (3.3) we may rewrite $\text{cl}(I_{k+1}^p \setminus \{p\})$ as follows:

$$
\text{cl}(I_{k+1}^p \setminus \{p\}) = \text{cl}\left( (I_{k+1} \setminus (I_k \cup \{r\})) \cup I_k^p \right) \\
= \text{cl}\left( (I_{k+1} \setminus (I_k \cup \{r\})) \cup (I_k^p \setminus \{q\}) \cup \{q\} \right) \\
= \text{cl}\left( (I_{k+1} \setminus (I_k \cup \{r\})) \cup \text{cl}(I_k^p \setminus \{q\}) \cup \{q\} \right).
$$

Now we use condition [iv] for $k$ ($\text{cl}(I_k^p \setminus \{q\}) = \text{cl} I_k$). We obtain

$$
\text{cl}(I_{k+1}^p \setminus \{p\}) = \text{cl}\left( (I_{k+1} \setminus (I_k \cup \{r\})) \cup \text{cl}(I_k) \cup \{q\} \right) \\
= \text{cl}\left( (I_{k+1} \setminus \{r\}) \cup \{q\} \right) \\
= \text{cl} I_{k+1},
$$

where the last equality follows from (3.3).

• **Condition [v]** By definition $K_{k+1} = K_k \cup c(I_{k+1})$. This implies $C_{K_{k+1}} = C_{K_k} \cup C(c(I_{k+1}))$. Hence $RL_r \cap C_{K_{k+1}} = (RL_r \cap C_{K_k}) \cup (RL_r \cap C(c(I_{k+1})))$. By the induction assumption $RL_r \cap C_{K_k} = I_k$. Because $RL_r \supseteq I_{k+1}$ is rainbow, $RL_r \cap C(c(I_{k+1})) = I_{k+1}$. Claim [6.1] then implies $RL_r \cap C_{K_{k+1}} = I_{k+1}$ as desired. Because $K_k \not\subseteq c(RL_r)$ and $K_k \subseteq K_{k+1}$, we have $K_{k+1} \not\subseteq c(RL_r)$ as well.

It follows that we may increase $k$ and continue in the loop.

In each step of the cycle we either terminate and output the desired subsequences, or we construct a sequence $I_{k+1}$ whose rank is strictly larger than the rank of $I_k$ (Claim [6.1]). Since the rank of $I_{k+1}$ is from above bounded by $\text{rk}(M)$ it follows that the loop terminates after at most $\text{rk}(M)$ iterations.

Verifying that all other steps can be done in time polynomial in $r$, $m$, $u$ and $|S|$ and that they are repeated only polynomial number of times is easy.

\[\square\]

4 Open problems

Rota basis conjecture [HR94] is a well known problem in matroid theory which has a close connection to our colorful matroidal Tverberg’s theorem. Let us restate it so that the similarity is clearly visible.

**Conjecture 1.** Let $M$ be a matroid of rank $m$. Let $S$ be a sequence of $m^2$ elements colored by $m$ colors such that points of each color form a basis. Do there always exist $m$ pairwise disjoint rainbow subsequences $S_1, \ldots, S_m$ of $S$ with $\text{cl} S_1 = \text{cl} S_2 = \ldots = \text{cl} S_m = M$?

In its full generality the conjecture has only been verified for $m = 1, 2, 3$ [Cha95]. The conjecture is also known to be true in several special cases [GH06, Oum97, Gly10]. Proof of Theorem 4 indicates the difficulties that appear if one tries to proof Rota’s basis conjecture purely combinatorially.

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