Disconnected stationary solutions for 2D Kolmogorov flow problem in periodic domain

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Abstract. The classical A.N. Kolmogorov’s flow problem for the stationary 2D Navier-Stokes equations on a stretched torus for velocity vector function is considered. The problem is analysed by the construction of the solution curves in the parameter-phase space and analysis of disconnected solutions is performed. Disconnected solutions from the main solution branch are found.

1. Introduction

The classical A.N. Kolmogorov’s flow problem for the stationary 2D Navier-Stokes equations on a stretched torus [1] is considered in this work. Many papers are dedicated to the problem at hand. First, early results of Meshalkin and Senai [2] demonstrated that the main solutions of the problem is asymptotically stable for any $\alpha > 1$ and $\beta = 1$ for the governing equations [1]. Further researches were conducted by many authors that demonstrated complex system behaviour, infinite number of pitchfork bifurcations as $\alpha \to 0$, influence of the recurrent flows for high Reynolds numbers, complex nonlinear dynamics involving cascades of limited cycles and invariant tori of period three as well as chaotic behaviour. The problem of solution bifurcations in this problem were considered in [4–9], including analytical and numerical research, most of these papers are focused on $\alpha \geq 1$. Excellent literature overview is provided in [3] and interested reader can be refered to these papers.

Despite being extensively studied, this problem still poseses some interesting uncovered features. The question that we wish to address: are there multi-stable solutions in the 2D Kolmogorov flow? This would imply that there are at least two lineary stable disconnected solutions exist for the fixed value of the bifurcation parameter ($R$ in [1]), assuming other identical conditions. In this paper only stationary solutions are considered. The deflation (originally suggested in [10]) pseudo arc-length continuation method with the eigenvalue solver is applied to solve this problem numerically. These methods can construct bifurcation diagrams of connected and disconnected solution branches alike and check linear stability of these solutions. Connected solution brunch is such that is connected to the primary trivial solution branch through any number of bifurcations. Disconnected branch is the one that has no connection to the primary one.

The paper is laid out as follows. First, the governing system of equations is considered and numerical methods are discussed in short. Next, the obtained results are laid out. Finally, the
discussion section is introduced. This is a short paper that contains preliminary, yet, important results.

2. Governing equations, analysis methods

Velocity vector function \( \mathbf{u} : \Omega(\alpha) \rightarrow \mathbb{R}^2 \) and pressure scalar function \( p : \Omega(\alpha) \rightarrow \mathbb{R} \) should satisfy:

\[
(u, \nabla)u + \nabla p - \frac{1}{R} \triangle u - (\sin(\beta y); 0)^T = 0, \quad \nabla \cdot u = 0,
\]

where \( R \) is the Reynolds number (bifurcation parameter), \( \alpha \) is the stretch factor for the periodic domain \( \Omega(\alpha) := [0; 2\pi/\alpha] \times [0; 2\pi] \) and \( \triangle \) is the Laplace operator. The force vector field depends only on the second spatial variable \( y \) and coefficient \( \beta \) is an integer. We designate \( u = (u, v)^T \), an introduce the gauge \( \int_{\Omega(\alpha)} u = 0 \) for the mean velocity in the domain to be zero. The problem has a trivial solution \( u_0 = \frac{R}{\beta} \sin(\beta y), v_0 = 0 \) which is called the main solution and the curve in the space \((\alpha \times u)\) is called the primary branch. Branches are represented visually in the space \((\alpha \times g(u))\); the representation function \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) is used to represent the solutions in low dimensions and can be taken as a simple norm in suitable functional space.

An equivalent infinite dimensional system of equations can be obtained in the Fourier space using Bubnov-Galerkin method as:

\[
i \sum_{(l,m) \in \mathbb{Z}^2} \left[ \alpha(j-l) \left( 1 - \frac{\alpha^2 l^2 + \alpha m^2}{\alpha^2 l^2 + m^2} \right) \hat{u}_{l,m} \hat{u}_{j-l,k-m} + (k-m) \left( 1 - \frac{\alpha m + m^2}{\alpha^2 l^2 + m^2} \right) \hat{v}_{l,m} \hat{v}_{j-l,k-m} \right] + \\
+ \frac{1}{R} \left( \frac{\alpha^2 j^2 + k^2}{\alpha^2 l^2 + m^2} \right) \hat{u}_{j,k} - \frac{\beta}{2} \delta_{jl} = 0,
\]

\[
i \sum_{(l,m) \in \mathbb{Z}^2} \left[ \alpha(j-l) \left( 1 - \frac{\alpha^2 l^2 + \alpha m^2}{\alpha^2 l^2 + m^2} \right) \hat{u}_{l,m} \hat{v}_{j-l,k-m} + (k-m) \left( 1 - \frac{\alpha m + m^2}{\alpha^2 l^2 + m^2} \right) \hat{v}_{l,m} \hat{v}_{j-l,k-m} \right] + \\
+ \frac{1}{R} \left( \frac{\alpha^2 j^2 + k^2}{\alpha^2 l^2 + m^2} \right) \hat{v}_{j,k} = 0,
\]

\( \forall j, k \in \mathbb{Z}^2, \)

where the pressure is eliminated from the system by the projection operator \( P := (id - \nabla \Delta^{-1} \nabla \cdot) \) applied to the nonlinear term and the system is obtained by the following ansatz: \( f(x) = \sum_{(j,k) \in \mathbb{Z}^2} \hat{f}_{j,k} e^{i(j \alpha + k)}, \hat{f}_{0,0} = 0 \) and \( \hat{f}_{j,-k} = (\hat{f}_{j,k})^*, \hat{f}_{-j,0} = (\hat{f}_{j,0})^*, \) because we only consider real-valued vectors in physical space.

We apply numerical methods, so the system of equations is recast to finite dimensions by considering truncated series in \([2]\). We consider \( m \cdot N \times N \) Fourier harmonics, where \( m = \max(1/\alpha, 1) \). In these series we limit ourselves to \( N = 512 \) harmonics and considering only imaginary components of the ansatz because the nonlinear operator \([2]\) preserves purer imaginary vectors. The active number of degrees of freedom (DOF) is \( DOF = 2m \cdot N \times (N/2 + 1) \) due to the reality condition. The resulting finite dimensional nonlinear and linear operators (the latter obtained analytically on a solution \( u^* \)) are used in the deflation continuation process and are designated as \( F(U) \) and \( \lambda(U^*) := \partial F(U^*)/\partial U \), accordingly; the vector \( U = (\text{Im}(\hat{u}), \text{Im}(\hat{v}))^T \) is constructed only from imaginary parts of Fourier coefficient vectors omitting the constant, since it is zero by the introduced gauge on the mean velocity.

The pseudo arc-length continuation method along with the eigenvalue solver was developed in \([11,13]\) and is applied to the problem at hand. The used eigensolver is the Implicitly Restarted
Arnoldi Method (IRAM). We only describe the main idea of the methods here, interested reader can be referred to the cited papers.

The Reynolds number $R$ is used as a parameter during the continuation. We define $R_{\text{min}}, R_{\text{max}}, \alpha, \beta$ and set the knots of deflation $D := \{R_j\}_{j=1}^K$ such that $\forall R_j \in D : R_{\text{min}} < R < R_{\text{max}}$. For each knot we execute the deflation method to find a solution. Due to the deflation process we only find different solutions in each knot. When the solution is found we execute a pseudo arc-length continuation process to trace the solution branch in the extended space of size $DOF + 1$. The branch is traced until it loops over itself or parameter value leaves $R_{\text{min}}, R_{\text{max}}$ boundaries. The process is repeated until no new solutions are found in all predefined knots. Then the eigensolver is executed to find linear stability of all solutions on each branch and find bifurcation points. Bifurcation points are detected using bisection algorithm with accuracy up to the third digit in mantissa of the parameter value. The eigensolver also returns the dimension of the unstable manifold. The inexact exponent shift and inverse matrix transformation \cite{12} is applied to accelerate the eigenvalue problem convergence.

The basin of attraction is estimated for each solution at multistable branches after the continuation is completed; the bisection method is used that is described momentarily. Let two solution vectors $U_1$ and $U_2$ be stable for the same parameter value $R^*$. We execute the following process:

\begin{algorithm}
1: function FIND_ATTRACTION_BASIS_BOUND($U_1$, $U_2$, $t_{\text{max}}$, $\varepsilon$)
2:     $a = 0$; $b = 1$;
3: repeat
4:     $c \leftarrow \frac{(a + b)}{2}$; $t \leftarrow 0$
5:     $U \leftarrow \varepsilon U_2 + (1 - c)U_1$; $U^0 \leftarrow U$;
6: repeat
7:     $t \leftarrow t + 1$
8:     $U^t \leftarrow \text{EXECUTE_TIME_STEPPER}(U^{t-1})$
9: until $t >= t_{\text{max}}$
10:     if $\|U - U_1\| \leq \varepsilon$ then
11:         $a = c$;
12:     else
13:         $b = c$;
14: until $|(a + b)/2 - c| \leq 1.0e^{-4}$
15: return $(a + b)/2$;
\end{algorithm}

The suggested Algorithm 1 performs approximate estimation of attraction boundary. A full bisection in $DOF$ dimensions would require to obtain the correct boundary, however this algorithm is sufficient to represent present the boundary on a bifurcation diagram. To external functions are used in the algorithm. The one is the Newton–Raphson method that is available in the continuation process. The other is the time stepper, that performs integration of the right hand side w.r.t. fictitious time. The explicit Runge Kutta 2nd order method is used, the timestep of the method is selected from the CFL stability criterion automatically. We execute the algorithm with $t_{\text{max}} = 100$ and $\varepsilon = 1.0e^{-6}$.

3. Results

Particular representation functions $g(U)$ are shown on the axis labels in each graph. Physical interpretation of solutions is not presented for branches without multistability.
3.1. Regular branches, \( \beta = 2, \alpha = 1 \)

For this problem we define \( R_{min} = 1, R_{max} = 21 \). The resulting bifurcation diagram is presented in figure 1.

![Bifurcation diagrams](image)

**Figure 1.** Bifurcation diagrams of stationary solutions for \( \alpha = 1, \beta = 2 \). Numbers represent different branches.

One can observe the classical supercritical transition process, when the main solution loses stability with the pitchfork bifurcation, followed by the stability loss on secondary branch. Some disconnected branches can be seen, well deciphered in the top left figure. However all these branches are unstable and possess no interest from the consideration of multistability.

3.2. Regular branches, \( \beta = 2, \alpha = 1/2 \)

For these parameters we fix \( R_{min} = 1, R_{max} = 8 \), since it is known [5] that the number of pitchfork bifurcations gradually increases with the decrease of \( \alpha \).
Figure 2. Bifurcation diagrams of stationary solutions for $\alpha = 1/2$, $\beta = 2$. Numbers represent different branches.

One can also observe a classical supercritical transition process. However there are two disconnected solutions that are possibly multistable. This requires more research in the future. No further decrease of $\alpha$ was considered because multiplication of the pitchfork bifurcations leads to the contamination of the bifurcation diagram with no hope of tracing disconnected solutions.

3.3. Multistable branch, $\beta = 2$, $\alpha = 2$

The parameter values are fixed as $R_{\text{min}} = 1$, $R_{\text{max}} = 22$. For this parameter values one can prove that the main trivial solution remains linearly stable on the positive real line. However, using deflation, we were able to trace disconnected solutions that are also linearly stable for some parameter segment. The bifurcation diagram is presented in figure 3.
Figure 3. Bifurcation diagrams of stationary solutions for $\alpha = 2$, $\beta = 2$. Numbers represent different branches. 3D bifurcation plot includes disconnected solution of higher unstable dimension (5) which is excluded from other figures for better visualization.

Eight additional solutions are formed by a quintic saddle-node bifurcation at $R = 13.2466 \pm 0.005$. Those solutions are disconnected from the main branch and are not crossing the primary branch in any point, see figure 3. The physical space visualization of stable solutions on different branches is shown in figure 4.

The representation of some stable solutions in the physical space (on curves 3 and 8) resembles the one, provided in [8] for higher Reynolds numbers. The estimation of the basin of attraction for multistable solutions is presented in figure 5. One can notice that starting from $R = 16.725 \pm 0.005$ and until $R = 20.665 \pm 0.005$ the basin of attraction extends. Initial conditions can be selected either in one or the other basins of attraction and, hence, one finds
Figure 4. Stable solutions for $\alpha = 2, \beta = 2, R = 20$ represented by the curl $\omega = \nabla \times u$. Branches from left to right: 2 (primary solution), 3, 8, 16 and 18.

Figure 5. Basins of attraction for the main and alternative stable solutions for $\alpha = 2, \beta = 2$. Parameters $a$ and $b$ are extracted from the bisection algorithm after converging.

itself attracted to a particular solution. The choice of initial conditions on the attraction basin boundary (separatrix) precisely results in the undefined behaviour i.e. it would be impossible to tell to which solution the trajectory be attracted to.

4. Discussion

We confirm the conjecture in [8] that there are many disconnected solutions for this problem, at least for the checked values of $\alpha$; those solutions cannot be obtained without the deflation process. Investigation of this problem for smaller values of $\alpha$ will be carried out elsewhere. It was also demonstrated, that the 2D Kolmogorov flow problem contains disconnected multistable stationary solutions for $\alpha = 2$. This raises the question whether there exists a global attractor in the 2D Navier-Stokes equations in periodic domains. The existence of such attractor was prooven in [14] for zero Dirichlet boundary conditions.

The existence of multistable disconnected solutions may explain the nonlinear transition to turbulence in such problems as Poiseuille and Couette flows, see excellent paper on this topic by M.Shimizu and P.Manneville [15]. Further upgrade to the numerical methods (in terms of deflation and continuation of periodic orbits) is needed to demonstrate this in wall bounded problems, but the transitioning mechanism can be described as follows. The main solution
branch remains linear stable for large values of $R$, or for any positive $R$, just like the case in figure 3. However, the basin of attraction of this solution branch is may be decreasing with the increase of $R$ value. If other linearity stable disconnected solutions (not necessarily stationary) exist in the neighbourhood of the main branch, then the initial finite amplitude perturbations may be large enough (or amplified enough by the nonlinear Navier-Stokes mechanism) to switch the solution trajectory abruptly. This was demonstrated by a simple bisection algorithm for $\alpha = 2$ in figure 5. when the slight deviation from the attraction basis boundary resulted in convergence other to the main or to the alternative solution. Unfortunately, from the engineering point of view, it would be very hard to find such boundary with high precision for real world problems. This scenario is numerically traceable with the use of the deflation process.

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References

[1] Arnol’d V and Meshalkin L 1960 Uspekhi Mat. Nauk 15(1(91)) 247–250
[2] Meshalkin L and Sinai I 1961 Journal of Applied Mathematics and Mechanics 25 1700–1705
[3] Lucas D and Kerswell R 2014 Journal of Fluid Mechanics 750 518–554
[4] Okamoto H and Shōji M 1993 Japan Journal of Industrial and Applied Mathematics 10 191–218
[5] Okamoto H 1998 A study of bifurcation of kolmogorov flows with an emphasis on the singular limit.
[6] Matsuda M and Miyatake S 2002 Tohoku Mathematical Journal 54 329–365
[7] Evstigneev N M, Magnitskii N A and Silaev D A 2015 Differential Equations 51 1292–1305
[8] Kim S C and Okamoto H 2015 Nonlinearity 28 3219–3242
[9] Tithof J, Suri B, Pallantla R K, Grigoriev R O and Schatz M F 2017 Journal of Fluid Mechanics 828 837–866
[10] Farrell P E, Birkisson Á and Funke S W 2015 SIAM Journal on Scientific Computing 37 A2026–A2045
[11] Evstigneev N M 2017 Implementation of implicitly restarted arnoldi method on MultiGPU architecture with application to fluid dynamics problems Communications in Computer and Information Science (Springer International Publishing) pp 301–316
[12] Evstigneev N M 2018 Journal of Physics: Conference Series 1141 012121
[13] Evstigneev N M 2019 On the convergence acceleration and parallel implementation of continuation in disconnected bifurcation diagrams for large-scale problems Communications in Computer and Information Science (Springer International Publishing) pp 122–138
[14] Rosa R 1998 Nonlinear Analysis: Theory, Methods & Applications 32 71–85
[15] Shimizu M and Manneville P 2019 Physical Review Fluids 4