ORBIFOLD EULER CHARACTERISTICS OF NON-ORBIFOLD GROUPOIDS

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Abstract. For a finitely presented discrete group $\Gamma$, we introduce two generalizations of the orbifold Euler characteristic and $\Gamma$-orbifold Euler characteristic to a class of proper topological groupoids large enough to include all cocompact proper Lie groupoids. The $\Gamma$-Euler characteristic is defined as an integral with respect to the Euler characteristic over the orbit space of the groupoid, and the $\Gamma$-inertia Euler characteristic is the usual Euler characteristic of the $\Gamma$-inertia space associated to the groupoid. A key ingredient is the application of o-minimal structures to study orbit spaces of topological groupoids. Our main result is that the $\Gamma$-Euler characteristic and $\Gamma$-inertia Euler characteristic coincide and generalize the higher-order orbifold Euler characteristics of Gusein-Zade, Luengo, and Melle-Hernández from the case of a translation groupoid by a compact Lie group and $\Gamma = \mathbb{Z}^d$. By realizing the $\Gamma$-Euler characteristic as the usual Euler characteristic of a topological space, we demonstrate that it is Morita invariant in the category of topological groupoids and satisfies familiar properties of the classical Euler characteristic. We give an additional formulation of the $\Gamma$-Euler characteristic for a cocompact proper Lie groupoid in terms of a finite covering by orbispace charts. In the case that the groupoid is an abelian extension of a translation groupoid by a bundle of groups, we relate the $\Gamma$-Euler characteristics to those of the translation groupoid and bundle of groups.

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1. Introduction

The string-theoretic orbifold Euler characteristic, often called simply the orbifold Euler characteristic, was introduced for a global quotient orbifold (the quotient of a manifold by a finite group) in [22] and generalized to non-global quotient orbifolds in [47]. For the quotient of a manifold $X$ by the finite group $G$, it is given by

$$\chi^{\text{orb}}(X, G) = \sum_{[g] \in \text{Ad}_G \setminus G} \chi(C_G(g) \setminus X^{(g)}).$$

See Notation 5.2. It was shown to be equal to the Euler characteristic of orbifold $K$-theory for global quotients in [4] and effective orbifolds in [3], see also [2, 33, 50, 54], and to the usual Euler characteristic of a resolution in [30]. As well, the orbifold Euler characteristic is equal to the usual Euler characteristic of the inertia orbifold; see [2].

For global quotients, the orbifold Euler characteristic was identified as part of a sequence of higher-order orbifold Euler characteristics in [12, 4]. Tamanoi [51, 52] identified these higher-order orbifold Euler characteristics as the $\mathbb{Z}^\ell$-extensions of the orbifold Euler characteristic in a context where the $\Gamma$-extension was defined for each finitely generated group $\Gamma$. These $\Gamma$-orbifold Euler characteristics were generalized to non-global quotient orbifolds by the authors in [27] using the generalized twisted sectors defined in [26, 25]. The $\Gamma$-orbifold Euler characteristics are the usual Euler characteristic of the orbifold of $\Gamma$-sectors, with $\Gamma = \mathbb{Z}$ corresponding to the inertia orbifold and hence the orbifold Euler characteristic of Equation (1.1). Note that another, generally rational, numerical invariant, here called the Euler-Satake characteristic (sometimes also called the orbifold Euler characteristic, among other names) has played a significant role in some of the above references, though does not appear to be as relevant in the context considered here. See Remarks 2.11 and 3.11.

In [28], Gusein-Zade, Luengo, and Melle-Hernández defined a far-reaching generalization of the orbifold Euler characteristic and higher-order orbifold Euler characteristics, which we will see in Theorem 5.3 correspond to $\Gamma = \mathbb{Z}^\ell$, to the case of a sufficiently nice $G$-space $X$ where $G$ is a compact Lie group; see Definition 2.12 below. Using integration with respect to the Euler characteristic [60], the sum in Equation (1.1) is reinterpreted as an integral over the space of conjugacy classes of the group, yielding a definition that is well-defined in the presence of infinite isotropy groups. The $\mathbb{Z}^\ell$-orbifold Euler characteristics are then defined recursively as an iterated integral.

One natural question about this definition, which is the initial motivation for this paper, is whether the generalized orbifold Euler characteristics depend on the specific presentation of a quotient $G \setminus X$ as a $G$-space or are invariant under equivalences of such presentations. An appropriate framework in which to make this question precise is that of proper topological groupoids, where the appropriate notion of equivalence is that of Morita equivalence. There are multiple possible approaches to extending the definitions of the generalized orbifold Euler characteristics of [28] to topological groupoids, and the first main goals of this paper are to give these formulations and show that they coincide. In order to restrict to topological groupoids whose orbit spaces and strata are sufficiently tame so that their Euler characteristics are defined, we introduce the use of o-minimal structures to study the orbit spaces of topological groupoids, which may be of interest for other applications. Specifically, we define the $\Gamma$-Euler characteristics for orbit space definable groupoids introduced in Section 3.1 defined with respect to an o-minimal structure on $\mathbb{R}$ which we will always assume contains the semialgebraic sets. These groupoids satisfy minimal hypotheses required to define the $\Gamma$-Euler characteristics and include the case of cocompact proper Lie groupoids as well as other important cases; see Proposition 3.3 and Corollary 3.6. We in particular consider the case where the spaces of objects and arrows have compatible structures as affine definable spaces, inducing a definable structure on the orbit space; see Proposition 3.5. The additional structure they carry, an embedding of the orbit space into $\mathbb{R}^n$, ensures that the orbit space and its orbit type strata have well-defined Euler characteristics, but by the results of [17] recalled in Theorem 2.5 below, the Euler characteristics do not depend on the embedding nor on the choice of o-minimal structure. Hence, we view the orbit space definable structure of a groupoid as a device used to define and compute Euler characteristics, which are invariants of the underlying topological groupoids. The o-minimal structure of semialgebraic sets is adequate for almost
all of the cases we have in mind, and the reader unfamiliar with more general notions of definability is welcome to replace “definable” with “semialgebraic” throughout the paper. However, the minor technical challenges of considering a larger o-minimal structure avoid unnecessary restrictions on the descriptions of the spaces to which these results can be applied; see Section 3.1.

For a translation groupoid \( G \times X \) associated to a \( G \)-space as above, the Euler characteristics of \([28]\) are, loosely speaking, defined by integrating the Euler characteristic of the fixed points of a group element over the group factor. One approach to generalizing to a topological groupoid \( G \), yielding the \( \Gamma\)-inertia Euler characteristic of Definition 4.9(ii), is to realize this integral as the usual Euler characteristic of a topological space, the \( \Gamma\)-inertia space, which is the orbit space of the \( \Gamma\)-inertia groupoid; see Definition 4.2. The \( \Gamma\)-inertia space of \( G \) depends only on the Morita equivalence class of \( G \) as a topological groupoid by Lemma 4.4, so this definition is easily seen to be Morita invariant, settling the question above. Because the \( \Gamma\)-inertia space is a generalization of the inertia orbifold and orbifold of \( \Gamma\)-sectors to arbitrary topological groupoids, this generalizes the characterization of the orbifold Euler characteristic of an orbifold as the Euler characteristic of the inertia orbifold. The \( \mathbb{Z}^\Gamma\)-inertia groupoid has been considered in the context of equivariant K-theory in \([1]\). In the case \( \Gamma = \mathbb{Z} \), the \( \mathbb{Z}\)-inertia groupoid (called simply the inertia groupoid) has appeared implicitly in the study of cyclic homology of convolution algebras of translation groupoids in \([13]\) and plays an important role in string topology \([5, 6]\). It was studied for translation groupoids in \([24, 44]\), and proper Lie groupoids in \([23]\) where the inertia groupoid was shown explicitly to have the structure of a differentiable stratified groupoid; see also \([17]\).

Another approach to generalization, yielding the \( \Gamma\)-Euler characteristic of Definition 3.18, is to mimic the approach of \([28]\), taking advantage of the product structure of a translation groupoid \( G \times X \) to integrate along one factor. This is done by reinterpreting the integral over \( \text{Ad}_\Gamma \mathcal{G} \) as an integral over the quotient space \( G\backslash X \), yielding a formulation that generalizes readily to groupoids. The two approaches are shown to yield Euler characteristics that coincide by an application of Fubini’s Theorem, see Theorem 4.13 and to equal the Euler characteristics of Gusein-Zade, Luengo, and Melle-Hernández in the case of a translation groupoid by a second application of Fubini’s Theorem; see Theorem 3.4. This yields a definition of orbifold Euler characteristics for a large class of proper topological groupoids, generalizing both beyond the class of translation groupoids and to \( \Gamma\)-Euler characteristics for finitely presented discrete groups \( \Gamma \) that need not be free abelian. Moreover, even in the case where the groupoid is (Morita equivalent to) the extension of a translation groupoid by a bundle of compact groups, the generalized \( \Gamma\)-Euler characteristics do not appear to be directly related to that of the translation groupoid; see Example 5.14. Note that for non-orbifold groupoids, \( \Gamma \) must be finitely presented, not merely finitely generated, as explained in Remark 3.10.

One of our primary motivations is the case of a cocompact proper Lie groupoid, and we consider this case explicitly throughout the paper. However, as was emphasized in \([29]\), the construction of the inertia groupoid leaves the category of Lie groupoids. Moreover, as the Euler characteristic depends only on the underlying topology, restricting consideration to proper Lie groupoids is artificial for our objectives.

A synopsis of the outline and main results of this paper is as follows. In Section 2 we summarize relevant background information on topological and Lie groupoids, Morita equivalence, o-minimal structures, affine definable structures, integration with respect to the Euler characteristic, Fubini’s theorem, and orbifold Euler characteristics. In Section 3 we introduce orbit space definable groupoids and the \( \Gamma\)-Euler characteristics of these groupoids, showing that they include cocompact proper Lie groupoids as well as semialgebraic translation groupoids. Section 4 introduces the \( \Gamma\)-inertia groupoid of a topological groupoid and establishes its basic properties, and the \( \Gamma\)-inertia Euler characteristic is introduced in Section 4.2. We then prove our first main result, Theorem 4.13 stating that the \( \Gamma\)-Euler characteristic of an orbit space definable groupoid coincides with the \( \Gamma\)-inertia Euler characteristic when both are defined. This is used to establish several properties of both Euler characteristics including Morita invariance (Corollary 4.15), additivity (Corollary 4.16), and multiplicativity (Lemma 4.17). In Section 5 we consider the special cases of cocompact proper Lie groupoids and translation groupoids.
In Section 5.1, we prove Theorem 5.4 demonstrating that the $\mathbb{Z}^k$- and $\mathbb{Z}_p$-inertia Euler characteristics of a translation groupoid coincide with the higher-order orbifold Euler characteristics of [28], as well as Theorem 5.6, giving a non-iterative definition of the higher-order orbifold Euler characteristics of a semialgebraic translation groupoid as well as the analogous $\mathcal{G}$-extensions for $\mathcal{G}$ that are not free abelian. Section 5.2 considers the case of cocompact proper Lie groupoids and uses the slice theorem of [43], see also [64, 63, 18, 20], to give an alternate formulation of the $\mathcal{G}$-Euler characteristics in this case: Theorem 5.8 which is closer to the original definition of [28]. In Section 5.3, we consider extensions of translation groupoids by bundles of compact Lie groups, which by the work of Trentinaglia [56, 55] include a large class of, and conjecturally all, proper Lie groupoids. We show in Theorem 5.13 that when such an extension is abelian, there is a simple relationship between the $\mathcal{G}$-Euler characteristics of the extension, the translation groupoid, and the bundle of compact Lie groups. However, we indicate that in the non-abelian case, such a relationship cannot be expected, and hence the generalization of the orbifold Euler characteristics given here is a nontrivial generalization from the translation groupoid case.

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2. Background and Definitions

In this section, we review the essential background information to fix the language and notation.

2.1. Topological and Lie groupoids. In this section, we briefly recall facts about topological and Lie groupoids. For more details, the reader is referred to [10, 39, 38, 46, 35] for topological groupoids and [83, 84, 82] for Lie groupoids.

Recall that a groupoid $\mathcal{G}$ is given by two sets $\mathcal{G}_0$ of objects and $\mathcal{G}_1$ of arrows along with the structure functions: the source $s_{\mathcal{G}}: \mathcal{G}_1 \to \mathcal{G}_0$, target $t_{\mathcal{G}}: \mathcal{G}_1 \to \mathcal{G}_0$, unit $u_{\mathcal{G}}: \mathcal{G}_0 \to \mathcal{G}_1$ written $u_{\mathcal{G}}(x) = 1_x$, inverse $i_{\mathcal{G}}: \mathcal{G}_1 \to \mathcal{G}_1$, and multiplication $m_{\mathcal{G}}: \mathcal{G}_1 \times_{t_{\mathcal{G}}} \mathcal{G}_1 \to \mathcal{G}_1$ written $m_{\mathcal{G}}(g, h) = gh$. For a point $x \in \mathcal{G}_0$, the *isotropy group* of $x$, denoted $\mathcal{G}_x$, is the set $(s, t)^{-1}(x, x)$. A topological groupoid is a groupoid such that $\mathcal{G}_0$ and $\mathcal{G}_1$ are topological spaces and the structure functions are maps, i.e., continuous functions. The orbit space $|\mathcal{G}|$ of a topological groupoid $\mathcal{G}$ is the quotient space of $\mathcal{G}_0$ by the equivalence relation identifying $x, y \in \mathcal{G}_0$ if there is a $g \in \mathcal{G}_1$ such that $s_{\mathcal{G}}(g) = x$ and $t_{\mathcal{G}}(g) = y$. We let $\pi_{\mathcal{G}}: \mathcal{G}_0 \to |\mathcal{G}|$ denote the orbit map, which sends $x \in \mathcal{G}_0$ to its $\mathcal{G}$-orbit, the equivalence class of $x$, denoted $\mathcal{G}_x$. Note that the subscript $\mathcal{G}$ on the structure maps and $\pi$ will be omitted when there is no possibility of confusion. In this paper, we will always assume that a topological groupoid is *open*, meaning $s$ is an open map. This ensures that the $\pi$ is open and hence a quotient map; see [57, Proposition 2.11]. We will also always assume that $\mathcal{G}_0$ and $\mathcal{G}_1$ are Hausdorff and paracompact.

A topological groupoid $\mathcal{G}$ is *proper* if the map $(s, t): \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is a proper map, i.e., the preimage of a compact set is compact. It is *cocompact* if the orbit space $|\mathcal{G}|$ is compact, and it is *étale* if $s$ is a local homeomorphism. A Lie groupoid is a topological groupoid such that $\mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds without boundary, the structure maps are smooth, and $s$ is a submersion.

If $G$ is a topological group and $X$ is a topological $G$-space, i.e., a topological space on which $G$ acts continuously, then the *translation groupoid* $G \times X$ has space of objects $X$ and space of arrows $G \times X$ with $s(g, x) = x$, $t(g, x) = gx$, $u(x) = (1, x)$, $i(g, x) = (g^{-1}, gx)$, and composition $(h, gx)(g, x) = (hg, x)$. The orbit space of $G \times X$ is denoted $|G \times X|$ or $\mathcal{G}_x$ of $X$. The translation groupoid is a topological groupoid. If $X$ is a smooth $G$-manifold without boundary, i.e., the map $G \times X \to X$ is smooth, then $G \times X$ is a Lie groupoid.

If $\mathcal{G}$ is a topological groupoid and $X$ is a topological space, a *left action* of $\mathcal{G}$ on $X$ is given by an anchor map $\alpha_X: X \to \mathcal{G}_0$ and an action map $\mathcal{G}_1 \times_{\alpha_X} X \to X$, written $(g, x) \to g \cdot x$, such that $\alpha_X(g \cdot x) = t(g)$, $h \cdot (g \cdot x) = (hg) \cdot x$, and $1_{\alpha_X(x)} \cdot x = x$ for all $x \in X$ and $g, h \in \mathcal{G}_1$ such that these expressions are defined. We then refer to $X$ as a *G-space*. The translation groupoid $G \times X$ is defined similarly to that of a group action; the space of objects is $X$, space of arrows is $\mathcal{G}_1 \times_{\alpha_X} X$,
interiors $s_{\mathcal{G} \times X}(g,x) = x$, $t_{\mathcal{G} \times X}(g,x) = g \ast x$, $u_{\mathcal{G} \times X}(x) = (1_{\alpha X}(x), x)$, $i_{\mathcal{G} \times X}(g,x) = (g^{-1}, g \ast x)$, the product is given by $(h.g \ast x)(g,x) = (hg, x)$, and the orbit space is denoted either $|\mathcal{G} \times X|$ or $\mathcal{G} \setminus X$. A Lie groupoid action on a smooth manifold $X$ is defined similarly, requiring the anchor and action maps to be smooth, and then $\mathcal{G} \times X$ is a Lie groupoid.

If $\mathcal{G}$ and $H$ are topological (respectively, Lie) groupoids, a homomorphism $\Phi: \mathcal{G} \to H$ consists of two continuous (respectively, smooth) maps $\phi_0: G_0 \to H_0$ and $\phi_1: G_1 \to H_1$ that preserve each of the structure maps. A homomorphism of topological (respectively, Lie) groupoids is an essential equivalence, also known as a weak equivalence, if the map $t_H \circ \text{pr}_1: H_{1 \times G} \times \phi_0 G_0 \to H_0$, where $\text{pr}_1$ is the projection onto the first factor, is a surjection admitting local sections (respectively, a surjective submersion) and the diagram

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\phi_1} & H_1 \\
|s_G, t_G| & \downarrow & \downarrow |s_H, t_H| \\
G_0 \times G_0 & \xrightarrow{\phi_0 \times \phi_0} & H_0 \times H_0,
\end{array}
$$

is a fibred product of spaces (respectively, of smooth manifolds); see [35, Definition 58 and Remark 59]. Note that a surjective map admits local sections if each point in the codomain is contained in an open neighborhood on which the map has a right inverse. Two groupoids $\mathcal{G}$ and $H$ are Morita equivalent as topological (Lie) groupoids if there is a third topological (Lie) groupoid $K$ and essential equivalences (of Lie groupoids) $\mathcal{G} \leftarrow K \rightarrow H$.

### 2.2. Definable sets and integration with respect to the Euler characteristic

In order to define the integral with respect to the Euler characteristic, we need to work within a fixed o-minimal structure on $R$. We give a very brief summary of the background and refer the reader to [58] for more details on o-minimal structures. For the integral with respect to the Euler characteristic; see [60] or [19] Sections 2–4.

Recall [58] Chapter 1, Definition (2.1)] that a structure on $R$ is a collection $(S_n)_{n \in \mathbb{N}}$ with the following properties: Each $S_n$ is a Boolean algebra of subsets of $R^n$ (i.e., is closed under unions and set differences) such that $A \times R \in S_{n+1}$ and $R \times A \in S_{n+1}$ for each $A \in S_n$. Moreover, each $S_n$ contains the diagonal $\{(x, \ldots, x) : x \in R\} \subset R^n$, and the projection of any $A \in S_{n+1}$ to $R^n$ obtained by dropping the last coordinate is contained in $S_n$. A structure $(S_n)_{n \in \mathbb{N}}$ is o-minimal if $S_1$ is the set of finite unions of points and open intervals and $S_2$ contains $\{(x,y) \in R^2 : x < y\}$. Once an o-minimal structure is fixed, a set $A \subseteq R^n$ is a definable set if it is an element of $S_n$. If $A \subseteq R^n$ is a definable set, a function $f: A \to R^m$ is a definable function if its graph $\{(x, f(x))\} \subset R^{n+m}$ is definable.

An important example of an o-minimal structure, and the one that we primarily have in mind, is that of semialgebraic sets; see [58, p. 1]. A subset $A \subseteq R^n$ is semialgebraic if it is a finite union of solution sets to finite systems of polynomial equations and inequalities, i.e., sets of the form $f_1(x_1, \ldots, x_n) = \cdots = f_k(x_1, \ldots, x_n) = 0$ and $g_1(x_1, \ldots, x_n) > 0, \ldots, g_m(x_1, \ldots, x_n) > 0$ (where $k$ or $m$ may be 0). We consider $A$ as a topological space using the subspace topology from $R^n$. We will sometimes restrict to the o-minimal structure of semialgebraic sets, and readers are welcome to consider only this o-minimal structure as noted in the introduction. When we work more generally, we will always assume that an o-minimal structure contains the semi-algebraic sets.

**Definition 2.1** ([58] Chapter 1, Section 3; Chapter 10, Section 1]). Fix an o-minimal structure on $R$. An affine definable space is a topological space $X$ along with a topological embedding $\iota_X: X \to R^n$, i.e., a continuous function that is a homeomorphism onto its image, such that $\iota_X(X)$ is definable. We will often denote $\iota_X$ simply as $\iota$ if there is no danger of confusion. If $(X, \iota_X)$ is an affine definable space, then a definable subset $A$ of $X$ is a $A \subseteq X$ such that $\iota_X(A)$ is definable. If $(X, \iota_X)$ and $(Y, \iota_Y)$ are affine definable spaces with $\iota_Y: Y \to R^m$, then a continuous function $f: X \to Y$ is a morphism of affine definable spaces if the function $\iota_Y \circ f \circ \iota_X^{-1}: \iota_X(X) \to R^m$ is a definable function.

**Remark 2.2.** As the collection of definable sets is closed under finite unions, intersections, closures, and interiors [58, Chapter 1, Lemma (3.4)], the collection of definable subsets of an affine definable space is
as well. Similarly, the product of affine definable spaces is an affine definable space using the product embedding. If \( f: X \to Y \) is a morphism of affine definable spaces \((X, \iota_X)\) and \((Y, \iota_Y)\), then \( \iota_Y \circ f(X) \) is the projection of the graph of \( \iota_Y \circ f \circ \iota_X^{-1} \) onto the second factor so that \( f(X) \) is a definable subset of \( Y \). Similarly, images and preimages of definable subsets via morphisms of affine definable spaces are definable, and restrictions of morphisms to definable subsets are morphism of affine definable spaces.

**Remark 2.3.** There is a more general notion of a (non-affine) definable space as a topological space that is covered by an atlas of definable sets with definable transition functions; see [58, Chapter 10, Section (1.2)]. However, by [58, Chapter 10, Theorem (1.8)], every definable space \((X, \iota)\) such that \( X \) is a regular topological space is affine. Since our intended applications are to regular spaces, we restrict consideration to affine definable spaces.

**Remark 2.4.** Of course, a topological space \( X \) may admit many distinct structures as an affine definable space even using the same o-minimal structure, and the definable subsets of \( X \) need not coincide. However, our intended application of this structure is to define the Euler characteristic, which will not depend on these choices; see Theorem 2.5 below.

We now recall the definition of the Euler characteristic of a definable set. Fix an o-minimal structure on \( \mathbb{R} \). Then for each definable set \( A \subseteq \mathbb{R}^n \), the Euler characteristic \( \chi(A) \) of \( A \) is defined. Specifically, any definable set \( A \) can be partitioned into finitely many simple sets called cells, the Euler characteristic of a cell of dimension \( d \) is defined to be \((-1)^d\), and then \( \chi(A) \) is defined as the sum of the Euler characteristics of its cells; see [58, Chapters 3, 4]. We extend the definition of the Euler characteristic to affine definable sets \((X, \iota)\) by defining \( \chi(X) = \chi(\iota(X)) \). When \( X \) is compact, this definition of the Euler characteristic coincides with the usual definition of the Euler characteristic of a triangulable topological space, but \( \chi \) is not homotopy-invariant in general. For instance, the Euler characteristic of a finite or infinite half-open interval (a single 1-dimensional cell) is \(-1\), and the Euler characteristic of a finite or infinite open interval (a single 1-dimensional cell) is \(0\). This definition yields an Euler characteristic that is invariant under definable homeomorphisms and is finitely (but not countably) additive. Moreover, by the following result of Beke, it does not depend on the choice of o-minimal structure and hence depends only on the underlying topological space \( X \); see also [34].

**Theorem 2.5 ([7, Theorem 2.2]).** Let \( S \) and \( S' \) be o-minimal structures on \( \mathbb{R} \), let \( A \) be a definable subset of \( \mathbb{R}^n \) with respect to \( S \), and let \( B \) be a definable subset of \( \mathbb{R}^m \) with respect to \( S' \). Let \( \chi(A) \) be the Euler characteristic of \( A \) with respect to \( S \) and let \( \chi'(B) \) be the Euler characteristic of \( B \) with respect to \( S' \). If \( A \) and \( B \) are homeomorphic, then \( \chi(A) = \chi'(B) \).

**Remark 2.6.** Because the Euler characteristic of an affine definable space \((X, \iota)\) is defined to be \( \chi(X) = \chi(\iota(X)) \), it follows immediately from Theorem 2.5 that \( \chi(X) \) does not depend on \( \iota \) nor on the choice of o-minimal structure, and hence is an invariant of topological spaces that admit the structure of affine definable sets.

We will need the notion of a definably proper equivalence relation and quotient; see also [11, Theorem 1.4] for the semialgebraic case. Recall [58, Chapter 6, Definition (4.4)] that a definable continuous function \( f: A \to B \) between definable sets is definably proper if the preimage of every compact definable subset of \( B \) is compact. Clearly, if \( f \) is definable, continuous, and proper, then it is definably proper. An equivalence relation \( E \subseteq A \times A \) is a definably proper equivalence relation if \( E \) is definable and the projection \( \text{pr}_1: E \to A \) onto the first factor is definable [58, Chapter 10, Definition (2.13)]. If \( E \) is an equivalence relation on \( A \), then a definably proper quotient of \( A \) by \( E \) is a definable set \( B \subseteq \mathbb{R}^k \) and a definable continuous surjective map \( \pi: A \to B \) such that \( \pi(x_1) = \pi(x_2) \) if and only if \( (x_1, x_2) \in E \), and \( \pi \) is definably proper. An equivalence relation admits a definably proper quotient if and only if it is a definably proper equivalence relation [58, Chapter 10, (2.13) and Theorem (2.15)]. We extend this notion to an affine definable space \((X, \iota)\) by identifying \( X \) with \( \iota(X) \): a definably proper equivalence relation \( E \) on \( X \) is an equivalence relation \( E \subseteq X \times X \) such that \((\iota \times \iota)(E)\) is a definably proper equivalence relation on \( \iota(X) \), and a definably proper quotient of \( X \) by \( E \) is a definably proper quotient of \( \iota(X) \) by \((\iota \times \iota)(E)\).
Remark 2.7. By [58] Chapter 10, Definition (2.2) and Remark (2.3)], a definably proper quotient is unique up to definable homeomorphism. If \( \pi : \iota(X) \to B \subset \mathbb{R}^k \) is a definably proper quotient of \( X \) and \( \rho : X \to E \backslash X \) denotes the usual topological quotient of \( X \) by the equivalence relation \( E \), then the embedding \( \iota \) induces a homeomorphism \( E \backslash \iota : E \backslash X \to B \subset \mathbb{R}^k \) such that \( E \backslash \iota \circ \rho = \pi \circ \iota \); see [40] Corollary 22.3]. Hence, \( (E \backslash X, E \backslash \iota) \) is an affine definable spaces whose image in \( \mathbb{R}^k \) as a definably proper quotient of \( X \) by \( E \).

An affine definable topological group is a pair \((G, \lambda)\) such that \( G \) is a topological group, \( \lambda : G \to \mathbb{R}^d \) is a topological embedding, and the product and inverse maps are morphisms of definable spaces; equivalently, the induced product \( \lambda(G) \times \lambda(G) \to \lambda(G) \subset \mathbb{R}^d \) given by \( (\lambda(g), \lambda(h)) \mapsto \lambda(gh) \) and inverse \( \lambda(G) \to \lambda(G) \subset \mathbb{R}^d \) given by \( \lambda(g) \mapsto \lambda(g^{-1}) \) are definable maps. For a definable topological group \((G, \lambda)\), a definable \(G\)-set \( A \) is a definable \(G\)-space \( A \) such that \( A \subset \mathbb{R}^n \) is a definable set and the action \( \lambda(G) \times A \to A \) defined by \( \lambda(g)a = ga \) is definable. An affine definable \(G\)-space is an affine definable space \((X, \iota)\) such that \( X \) is a \(G\)-space and \( G \times X \to X \) is a morphism of affine definable spaces, i.e., the map \( \lambda(G) \times \iota(X) \to \iota(X) \) given by \( (\lambda(g), \iota(x)) \mapsto \iota(gx) \) is definable. If \( G \) is compact, then the orbit space \( G \backslash X \) admits the structure of an affine definable space whose image in \( \mathbb{R}^k \) is a definably proper quotient by \([58] Chapter 10, Corollary (2.18)] and Remark 2.7.

Following [13] [11], we will consider semialgebraic groups and \(G\)-sets to be subsets of Euclidean space. That is, a semialgebraic group \( G \) is a semialgebraic set \( G \subset \mathbb{R}^d \) that is a topological group with the subspace topology such that the multiplication and inversion maps are definable in the o-minimal structure of semialgebraic sets. If \( G \) is a semialgebraic group, a semialgebraic \(G\)-set \( A \) is a semialgebraic set \( A \subset \mathbb{R}^n \) that is also a \(G\)-space (with the subspace topologies on \( G \) and \( A \)) such that the action map \( G \times A \to A \) is definable in the o-minimal structure of semialgebraic sets. See [15] [15] Section 2 and [11] Section 2 for additional details and background. If one uses any o-minimal structure that contains the semialgebraic sets, a semialgebraic group \( G \) is an affine definable topological group, and a semialgebraic \(G\)-set \( A \) is an affine definable \(G\)-space, with respect to the identity embeddings of \( G \) and \( A \).

Any compact Lie group \( G \) admits a faithful representation on \( \mathbb{R}^m \) [13] Chapter VI, Section XII, Theorem 4] and hence admits an isomorphism \( \lambda : G \to \lambda(G) \cong \text{GL}(m, \mathbb{R}) \subset \mathbb{R}^{m^2} \) to a compact linear algebraic group \( \lambda(G) \), i.e., a subgroup \( \lambda(G) \cong \text{GL}(m, \mathbb{R}) \) that is the solution of finitely many polynomial equations [61] Section 99, Theorem 3 and Corollary]. It follows that \( \lambda(G) \) is a semialgebraic group and hence \((G, \lambda)\) is an affine definable topological group with respect to any o-minimal structure that contains the semialgebraic sets (as the graphs of the multiplication and inverse maps are semialgebraic, and hence contained in any larger o-minimal structure). If \( \lambda' : G \to \lambda'(G) \cong \text{GL}(m', \mathbb{R}) \) is another choice of isomorphism of \( G \) such that \( \lambda'(G) \) is a semialgebraic group, then \( \lambda' \circ \lambda^{-1} \) is a group isomorphism and hence a semialgebraic function by [11] Corollary 2.3]; hence, the semialgebraic structure a compact Lie group inherits as a linear algebraic group is unique. For this reason, and particularly because most of the groups we consider will be compact, we will usually consider compact Lie groups with their unique structure as definable topological groups with no loss of generality. We will not use this fact, but it is interesting to note: if \( A \subset \mathbb{R}^n \) is a definable set that is also a group such that the multiplication and inverse functions are definable but not necessarily continuous, then \( A \) admits a topology with respect to which it is a Lie group; see [45].

We will make frequent use of the following and hence include a proof for clarity.

Lemma 2.8. Suppose \((G, \lambda)\) is a compact affine definable topological group, \((X, \iota_X)\) and \((Y, \iota_Y)\) are affine definable \(G\)-spaces, and \( f : X \to Y \) is a \(G\)-equivariant morphism of affine definable spaces. Giving the orbit spaces \( G \backslash X \) and \( G \backslash Y \) the structures of affine definable spaces as in Remark 2.7 the induced map \( \overline{f} : G \backslash X \to G \backslash Y \) is a morphism of affine definable spaces.

Proof. By Remark 2.7 the structures of \( G \backslash X \) and \( G \backslash Y \) as affine definable spaces do not depend on the choice of definably proper quotients. For simplicity, we identify \( X \) with \( \iota_X(X) \), \( Y \) with \( \iota_Y(Y) \), \( G \) with \( \lambda(G) \), and \( f \) with \( \iota_Y \circ f \circ \iota_X^{-1} \), and hence work with the corresponding definable sets in Euclidean space. Define a \((G \times G)\)-action on the graph of \( f \) in \( X \times Y \) by \((g_1, g_2)(x, f(x)) = (g_1 x, g_2 f(x)) \); the graph of
this action is the product of the graphs of the $G$-actions on $X$ and $f(Y)$ and hence is a definable set; see Remark\[2.2\]
Then graph($f$) is a definable ($G \times G$)-set, and the quotient $(G \times G) \setminus \text{graph}(f)$ is the graph of $f$.

We now review the integration of constructible functions on an affine definable space with respect to the Euler characteristic. Let $(X, \iota)$ be an affine definable space. A function $f: X \to \mathbb{Z}$ is constructible if $f^{-1}(k)$ is a definable subset of $X$ for each $k \in \mathbb{Z}$. If $f$ is a bounded constructible function, then the integral of $f$ over $X$ with respect to the Euler characteristic is defined to be

$$
\int_X f(x) \, d\chi(x) = \sum_{k \in \mathbb{Z}} k \chi(f^{-1}(k)).
$$

Note that the image of $f$ in $\mathbb{Z}$ is finite so that this sum is finite as well. Hence the integral is well-defined by the finite additivity of $\chi$.

As noted in [19] Proposition 4.2, it is often easier to compute the above integral using the equivalent formulation

$$
\int_X f(x) \, d\chi(x) = \sum_{k=0}^{\infty} \left[ \chi(f^{-1}(\{j : j > k\})) - \chi(f^{-1}(\{j : j < -k\})) \right].
$$

Note that if $f(x) = c$ is constant, then $\int_X f(x) \, d\chi(x) = c \chi(X)$. Note further that if $X$ is partitioned into finitely many disjoint definable subsets $X_1, \ldots, X_m$, then $\int_X f(x) \, d\chi(x) = \sum_{i=1}^{m} \int_{X_i} f(x) \, d\chi(x)$.

**Remark 2.9.** An immediate consequence of Theorem\[2.5\] and either the definition or Equation (2.1) is that $\int_X f(x) \, d\chi(x)$ does not depend on the choice of o-minimal structure nor the embedding $\iota$.

The following version of Fubini’s Theorem will play an important role; see [19] Theorem 4.5 and [60] 3.3.

**Theorem 2.10 (Fubini’s Theorem).** Let $X$ and $Y$ be affine definable spaces, $\varphi: X \to Y$ a morphism of affine definable spaces, and $f: X \to \mathbb{Z}$ a bounded constructible function. Then

$$
\int_X f(x) \, d\chi(x) = \int_Y \left( \int_{\varphi^{-1}(y)} f(x) \, d\chi(x) \right) \, d\chi(y).
$$

When $X$ and $Y$ are definable sets, the proof of Theorem\[2.10\] is straightforward, because the integral is defined as a finite sum. The Hardt Trivialisation theorem for definable functions [58] Chapter 9, Theorem (1.7)] yields a finite cell decomposition of $B$ such that the preimage of each cell under $\varphi$ is a product of the cell and a definable set on which $\varphi$ coincides with the projection onto the cell. The theorem extends easily to the case affine definable spaces by considering the images of $X$ and $Y$ under the corresponding embeddings and applying Remark 2.9.

### 2.3. Orbifold Euler characteristics

In this section, we briefly recall the construction of the $\Gamma$-Euler characteristic of a closed orbifold as well as their generalization in [28]; see [27] Section 2.1 and Definition 4.1] for more details.

Recall that an orbifold $Q$ (without boundary) can be defined to be the Lie groupoid Morita equivalence class of a proper étale Lie groupoid, and a choice of groupoid $\mathcal{G}$ from this Morita equivalence class is a presentation of $Q$; see [2]. The underlying space of $Q$ is the orbit space $|\mathcal{G}|$ of a presentation $\mathcal{G}$, which does not depend on the choice of $\mathcal{G}$ (up to homeomorphism). An orbifold is a global quotient if it admits a presentation of the form $G \times X$ where $G$ is a finite group and $X$ is a smooth $G$-manifold without boundary.

Let $\mathcal{G}$ be a proper étale Lie groupoid presenting the orbifold $Q$, and assume that $Q$ is closed, i.e., $\mathcal{G}$ is cocompact. Let $\Gamma$ be a finitely generated discrete group, considered as a groupoid with a single object, and let $\text{Hom}(\Gamma, \mathcal{G})$ denote the space of groupoid homomorphisms $\Gamma \to \mathcal{G}$ equipped with the compact-open topology; see Section 4.1 for a description of this space in a more general context. The space $\text{Hom}(\Gamma, \mathcal{G})$ inherits the structure of a smooth $\mathcal{G}$-manifold (possibly with components of different dimensions); see [26] Lemma 2.2]. Specifically, if $x \in \mathcal{G}_0$ and $V_x$ is a neighborhood of $x$ in $\mathcal{G}_0$ such that
\(\mathcal{G}_V\) is isomorphic to \(\mathcal{G}_z^x \times V_z\), then a neighborhood in \(\text{Hom}(\Gamma, \mathcal{G})\) of the element corresponding to a group homomorphism \(\phi: \Gamma \rightarrow \mathcal{G}_z^x\) is diffeomorphic to the set of fixed points \(V_z^\phi\) of \(\phi(\Gamma)\) in \(V_z\), and the restriction of the \(\mathcal{G}\)-action corresponds to the action by conjugation of the centralizer \(C_{\mathcal{G}_z^x}(\phi)\) of \(\phi(\Gamma)\) in \(\mathcal{G}_z^x\). The translation groupoid \(\mathcal{G} \times \text{Hom}(\Gamma, \mathcal{G})\) is a cocompact orbifold groupoid representing the orbifold of \(\Gamma\)-sectors of \(Q\), denoted \(\tilde{Q}_\Gamma\). When \(\Gamma = \mathbb{Z}\), \(\tilde{Q}_\mathbb{Z}\) is called the inertia orbifold, with connected components called twisted sectors [2, Section 2.5]. The \(\Gamma\)-Euler characteristic \(\chi_{\Gamma}(Q)\) is defined to be \(\chi(\tilde{Q}_\Gamma) = \chi([\mathcal{G} \times \text{Hom}(\Gamma, \mathcal{G})])\), the Euler characteristic of the orbit space of \(\mathcal{G} \times \text{Hom}(\Gamma, \mathcal{G})\). Many of the various Euler characteristics that have been defined in the literature for orbifolds occur as \(\chi_{\Gamma}\) for a specific choice \(\Gamma\); see [27, p. 524]. In particular, the string-theoretic Euler characteristic recalled in Equation (1.1), introduced in [22] for global quotients and [47] more generally, corresponds to \(\chi_{\mathbb{Z}}\). For global quotients, the sequence of orbifold Euler characteristics of [12] coincides with \(\chi_{\mathbb{Z}}\), and the generalized orbifold Euler characteristic of [51, 52] coincide with the \(\chi_{\Gamma}\) for finitely generated \(\Gamma\). Note that \(\text{Hom}(\mathbb{Z}, \mathcal{G})\) is often called the loop space of \(\mathcal{G}\), \(\mathcal{G} \times \text{Hom}(\mathbb{Z}, \mathcal{G})\) is called the inertia groupoid of \(\mathcal{G}\), and \([\mathcal{G} \times \text{Hom}(\mathbb{Z}, \mathcal{G})]\) is the inertia space of \(\mathcal{G}\), particularly when \(\mathcal{G}\) is not an orbifold groupoid; we will extend this definition and language to an arbitrary topological groupoid \(\mathcal{G}\) and a finitely presented discrete group \(\Gamma\) in Definition 4.2 below. When \(\mathcal{G}\) is an orbifold groupoid, the language of twisted sectors or \(\Gamma\)-sectors is used to emphasize the fact that \(\tilde{Q}_\Gamma\) is a disjoint union of orbifolds.

Remark 2.11. Another important invariant for orbifolds, the Euler-Satake characteristic \(\chi^{ES}\), has appeared under various names [48, 53, 32]. In the simplest case of a global quotient orbifold, presented by \(\mathcal{G} \times M\) where \(\mathcal{G}\) is a finite group and \(M\) a smooth \(G\)-manifold, \(\chi^{ES}(\mathcal{G} \setminus M)\) is given by \(\chi(M)/|G|\). A rational number in general, the Euler-Satake characteristic is defined for an arbitrary closed orbifold by

\[
\chi^{ES}(Q) = \sum_{\sigma \in T} (-1)^{\dim \sigma} \frac{1}{|G|},
\]

where \(T\) is a triangulation of the underlying space of \(Q\) such that the order of the isotropy group is constant on the interior of each simplex \(\sigma \in T\) and \(|G|\) denotes this order; see [27, Section 2.2]. Using [49, Theorem 3.2], the (usual) Euler characteristic of the underlying topological space of \(Q\) can be expressed as \(\chi^{ES}(Q)\), and hence for any finitely presented discrete group \(\Gamma\), the Euler characteristic \(\chi_{\Gamma}(Q)\) can be expressed as \(\chi_{\Gamma}^{ES}(Q)\); see [27, Equations (8) and (9)] (see also [52, Proposition 2.2(2)]) for the case of a global quotient). Hence, the Euler-Satake characteristic is in some sense more primitive than the usual Euler characteristic, as all \(\Gamma\)-Euler characteristics can be expressed in terms of \(\Gamma\)-Euler-Satake characteristics, but not conversely. For the present purpose, however, the Euler-Satake characteristic will be less relevant; see Remark 4.11 below.

Using the language of integration with respect to the Euler characteristic recalled in Section 2.2, Gusein-Zade, Luengo, Melle-Hernández [28] gave the following generalization of orbifold Euler characteristics to the non-orbifold setting of a \(G\)-space \(A\) where \(G\) is a compact Lie group and \(A\) is a topological \(G\)-space that does not necessarily have finite isotropy groups. Using this definition, they generalize the generating series for higher-order orbifold Euler characteristics of wreath symmetric products given for orbifolds in [51, 27]. The hypotheses on \(A\) in [28] are deliberately loose; they assume that \(A\) is “good enough,” e.g., a quasi-projective variety, with finitely many orbit types, each having a well-defined Euler characteristic. For our purposes, it will be sufficient to assume that \(A\) is a semialgebraic \(G\)-set, though this is purely for convenience. Note that [28] considered right \(G\)-actions, while we state the definition here in terms of left \(G\)-actions.

Definition 2.12 (Orbifold Euler characteristics for semialgebraic \(G\)-sets. [28, Definitions 2.1 and 2.2]). Let \(G\) be a compact Lie group and \(A\) a semialgebraic \(G\)-set. For each \(g \in G\), let \(A^{(g)}\) denote the points fixed by \(g\), let \(C_G(g)\) denote the centralizer of \(g\) in \(G\), and let \([g]_G\) denote the conjugacy class of \(g\) in \(G\). Let \(\text{Ad}_G\) denote the adjoint action of \(G\) on itself so that \(\text{Ad}_G: G \rightarrow \text{Sp}(G)\) is the space of conjugacy
classes in $G$. The orbifold Euler characteristic of $(A, G)$ is given by

$$\chi^{(1)}(A, G) = \int_{\Ad_G G} \chi(C_G(g)\backslash A^{(g)}) \, d\chi([g]_G).$$

If $\ell \in \mathbb{N}$, the orbifold Euler characteristic of order $\ell$ of $(A, G)$ is given by

$$\chi^{(\ell)}(A, G) = \int_{\Ad_G G} \chi^{(\ell-1)}(A^{(g)}, C_G(g)) \, d\chi([g]_G),$$

with $\chi^{(0)}(A, G) = \chi(G\backslash A)$.

Note that the homeomorphism type of $A^{(g)}$ as a $C_G(g)$-space depends only on the conjugacy class of $g$ so that the integrands are well-defined. Note further that when $G$ acts on $A$ with finite isotropy groups so that $G \times A$ presents an orbifold $Q$, $\chi^{(\ell)}(A, G)$ coincides with $\chi_{\orb}(Q)$ defined in Section 2.3; see [28, Definition 1.1] and [27, p.524]. In particular, when $G$ is finite, $\chi^{(1)}(A, G)$ reduces to $\chi_{\orb}(A, G)$ of Equation (1.1).

**Remark 2.13.** In [28], $\chi^{(1)}(A, G)$ is denoted $\chi_{\orb}(A, G)$. We will use the notation $\chi^{(1)}(A, G)$ to avoid confusion with other Euler characteristics.

### 3. $\Gamma$-Euler characteristics for groupoids

In this section, we define the $\Gamma$-Euler characteristic of a groupoid $\mathcal{G}$ where $\Gamma$ is a finitely presented discrete group; see Definition 3.8. We will see in Section 5 that these generalize the orbifold Euler characteristics recalled above. First, we describe the groupoids for which these $\Gamma$-Euler characteristics are defined.

For the remainder of this paper, we fix an o-minimal structure on $\mathbb{R}$ that contains the semialgebraic sets, and $\Gamma$ will always denote a finitely presented group equipped with the discrete topology.

**3.1. Orbit space definable groupoids.** We begin with the following.

**Definition 3.1** ((Orbit space definable groupoids)). An orbit space definable groupoid (with respect to a fixed o-minimal structure) is a topological groupoid $\mathcal{G}$ and an embedding $\iota_{[G]} : [G] \to \mathbb{R}^n$ of the orbit space $[G]$ such that

(i) The pair $([G], \iota_{[G]})$ is an affine definable space,

(ii) For each $x \in \mathcal{G}_0$, the isotropy group $\mathcal{G}^x_0$ is a compact Lie group, and

(iii) The equivalence relation $x \sim y$ if and only if $\mathcal{G}^x_0$ and $\mathcal{G}^y_0$ are isomorphic is a finite partition of $\mathcal{G}_0$ inducing a finite partition of the orbit space $[G]$ into definable subsets.

We will often refer to an orbit space definable groupoid as $\mathcal{G}$ and assume without clarification that $\iota_{[G]}$ denotes the corresponding embedding. Following [42], we refer to the partition in (iii) as the partition by **weak orbit types**.

Note that we do not require that the space of objects $\mathcal{G}_0$ and arrows $\mathcal{G}_1$ of an orbit space definable groupoid to be definable sets or spaces. For instance, suppose $M$ is a topological space that does not admit the structure of an affine definable space. Then the pair groupoid $M \times M \to M$ has orbit space a point and trivial isotropy groups so that it is an orbit space definable groupoid. We will on occasion consider groupoids whose spaces of objects and arrows have definable structures compatible with that of the quotient; however, an orbit space definable groupoid structure will be sufficient to define the $\Gamma$-Euler characteristics we introduce in Definition 3.8 below. It is frequently the case that the o-minimal structure of semialgebraic sets will be sufficient. However, larger o-minimal structures allow the consideration of sets that are described more analytically, e.g., using exponential functions, see [59, Section 2.5], and there is no reason to preclude this flexibility.

If $\mathcal{G}$ and $\mathcal{H}$ are orbit space definable groupoids, then the product groupoid $\mathcal{G} \times \mathcal{H}$ is easily seen to admit the structure of an orbit space definable groupoid. Specifically, $(x, y), (x', y') \in \mathcal{G}_0 \times \mathcal{H}_0$ are in the same orbit in $\mathcal{G} \times \mathcal{H}$ if and only if there is an arrow $(g, h) \in \mathcal{G}_1 \times \mathcal{G}_1$ such that $s(g, h) = (x, y)$ and
Clearly, \( G \) are the same with respect to both proper Lie groupoid. Then there is an embedding \( G \) composition of the homeomorphism of its vertices in such a way that the image of each simplex is a semialgebraic set \([21, \text{page 956}]\). The respect to which \( s \text{position 7.3}, [?]\) by products of the corresponding equivalence classes in \( G \) and \( H \).

If \( G \) is an orbit space definable groupoid and \( X \subseteq |G| \) is a definable subset, then letting \( \pi: G_0 \to |G| \) denote the orbit map, the restriction \( G_{|\pi^{-1}(X)} \) of \( G \) to the preimage of \( X \) is as well orbit space definable. Clearly, \( |G_{|\pi^{-1}(X)}| = X \) is an affine definable space via the restriction of \( \iota_{|G|} \) to \( X \), the isotropy groups are the same with respect to both \( G \) and \( G_{|\pi^{-1}(X)} \), and the \( \sim \)-classes in \( |G_{|\pi^{-1}(X)}| \) are the intersections with \( X \) of those in \( |G| \).

We summarize these observations in the following.

**Lemma 3.2.** Let \( G \) and \( H \) be orbit space definable groupoids with embeddings \( \iota_{|G|} \) and \( \iota_{|H|} \).

(i) The groupoid \( \mathcal{G} \times \mathcal{H} \) admits the structure of an orbit space definable groupoid with embedding \( \iota_{|\mathcal{G} \times \mathcal{H}|} = \iota_{|\mathcal{G}|} \times \iota_{|\mathcal{H}|} \).

(ii) If \( \mathcal{G} \) is an orbit space definable groupoid and \( X \subseteq |\mathcal{G}| \) then \( \mathcal{G}_{|\pi^{-1}(X)} \) admits the structure of an orbit space definable groupoid with embedding given by the restriction of \( \iota_{|\mathcal{G}|} \) to \( X \).

**Remark 3.3.** It is natural to consider definable homomorphisms of orbit space definable groupoids to be homomorphisms of the underlying topological groupoids such that the induced map on orbit spaces is a morphism of affine definable spaces. In particular, the notion of a definable essential equivalence, a definable homomorphism that is as well an essential equivalence of the underlying topological groupoids, yields a notion of definable Morita equivalence between orbit space definable groupoids. We will not need these notions, as our interest is in invariants of the underlying topological groupoids. Hence, we view an orbit space definable structure as a device used to define and compute Euler characteristics, which will not depend on the choice of such structure by Theorem 2.5 and Remark 2.9; see Section 3.2.

We are primarily interested in the special case of cocompact proper Lie groupoids, which we now show always admit the structure of an orbit space definable groupoid. In the proof below, we define this structure using a triangulation of the orbit space whose existence was demonstrated in [43]. Note that the class of orbit space definable groupoids is much larger than that of cocompact proper Lie groupoids.

**Proposition 3.4** ((Cocompact proper Lie groupoids are orbit space definable)). Let \( \mathcal{G} \) be a cocompact proper Lie groupoid. Then there is an embedding \( \iota_{|\mathcal{G}|} : |\mathcal{G}| \to \mathbb{R}^n \) with respect to which \( \mathcal{G} \) is orbit space definable.

**Proof.** Let \( x \in G_0 \), and then it is well known that \( G_x^\circ \) is a Lie group; see [37, Theorem 5.4]. As \( G_x^\circ = (s,t)^{-1}(x) \), it is compact by the properness of \( G \). By [43, Corollary 7.2], \( |\mathcal{G}| \) admits a triangulation that is compatible with the stratification by normal orbit types; see [43, Definition 5.6(ii)] for the definition. That is, the closure of each stratum corresponds to a simplicial subcomplex, and the local compactness of \( |\mathcal{G}| \) implies that the triangulating simplicial complex is locally finite [42, Proposition 7.3], [?, Lemma 2.6]. As \( G \) is cocompact so that \( |\mathcal{G}| \) is actually compact, the simplicial complex is finite, and therefore can be embedded into the real finite-dimensional vector space \( \mathbb{R}^n \) spanned by its vertices in such a way that the image of each simplex is a semialgebraic set [21, page 956]. The composition of the homeomorphism of \( |\mathcal{G}| \) with this finite simplicial complex and the semialgebraic embedding of the complex yields an embedding \( \iota_{|\mathcal{G}|} : |\mathcal{G}| \to \mathbb{R}^n \) whose image is semialgebraic and hence definable, making \( |\mathcal{G}| \) into an affine definable space.

By [43, Theorem 5.7], \( |\mathcal{G}| \) is stratified by weak orbit types, whose definition [43, Definition 5.6(i)] coincides with the equivalence relation in Definition 5.1 (iii). Each normal orbit type is open and closed in its weak orbit type so that the connected components of normal orbit types are exactly the connected components of normal orbit types; hence, both stratifications are compatible with the triangulation above. Moreover, these connected components form a decomposition of \( |\mathcal{G}| \) in the sense of [43, Definition 2.1], in particular implying that this partition is locally finite. As \( |\mathcal{G}| \) is compact, it follows that the decomposition is in fact finite so that Definition 5.1 (iii) is satisfied. \( \square \)
An embedding \( \iota_{|\mathcal{G}|} \) making \( \mathcal{G} \) into an orbit space definable groupoid as in Proposition 3.3 is usually far from unique, so the orbit space \( |\mathcal{G}| \) may admit many inequivalent structures as an affine definable space. As explained in Remark 3.3, our interest is in invariants that will not depend on this choice. The hypothesis of cocompactness in Proposition 3.3 is sufficient but not necessary, which can be seen by considering \( G_{\pi^{-1}(U)} \) where \( \mathcal{G} \) is cocompact, \( U \) is an open definable subset of \( |\mathcal{G}| \), and \( \pi: \mathcal{G}_0 \to |\mathcal{G}| \) again denotes the orbit map.

It is natural to consider topological groupoids \( \mathcal{G} \) such that the spaces of objects and arrows are affine definable spaces and the structure maps are morphisms of definable spaces. Using the results of \([58]\) Section 10.2 on the existence of definably proper quotients of definably proper equivalence relations recalled above, we now demonstrate that such a groupoid admits a definable orbit space assuming it is proper and the source map is proper. Note that as the inverse map \( i: \mathcal{G}_1 \to \mathcal{G}_1 \) is its own inverse function and hence a homeomorphism, if \( s \) is proper then \( t = s \circ i \) is proper as well.

**Proposition 3.5.** Let \( \mathcal{G} \) be a proper topological groupoid such that the source map \( s \) is proper. Let \( \iota_1: \mathcal{G}_1 \to \mathbb{R}^n \) be a topological embedding such that \((\mathcal{G}_1, \iota_1)\) is an affine definable space, assume \( u_i(\mathcal{G}_0) \subset \mathcal{G}_1 \) is a definable subset, and let \( \iota_0 = \iota_1 \circ u: \mathcal{G}_0 \to \mathbb{R}^n \) denote the induced topological embedding. Assume the structure maps \( s, t, u, i, m \) are morphisms of affine definable spaces. Then there is an embedding \( i_{|\mathcal{G}|} \) of \( |\mathcal{G}| \) into Euclidean space with respect to which \( \mathcal{G} \) satisfies Definition 3.1(iii) and (ii). In addition, if \( \mathcal{G} \) satisfies Definition 3.1(iii), then \( \mathcal{G} \) is orbit space definable.

**Proof.** Let \( \hat{\mathcal{G}}_i \) denote \( \iota_i(\mathcal{G}_i) \) for \( i = 0, 1 \), and let \( \hat{s} = \iota_0 \circ s \circ \iota_1^{-1}: \hat{\mathcal{G}}_1 \to \hat{\mathcal{G}}_0, \hat{t} = \iota_0 \circ t \circ \iota_1^{-1}: \hat{\mathcal{G}}_1 \to \hat{\mathcal{G}}_0 \), etc., so that \( \mathcal{G}_0 \) and \( \mathcal{G}_1 \) along with the structure maps \( \hat{s}, \hat{t}, \hat{u}, \hat{i}, \) and \( \hat{m} \) is a proper topological groupoid embedded in \( \mathbb{R}^n \) as definable sets such that the structure maps are definable functions. Let \( E = \{(x, y): x \in \mathcal{G}_0, y \in \mathcal{G}_1\} \) denote the equivalence relation corresponding to the partition of \( \mathcal{G}_0 \) into orbits. Then \( E \) is the image of definable function \( (\hat{s}, \hat{t}): \hat{\mathcal{G}}_1 \to \hat{\mathcal{G}}_0 \times \hat{\mathcal{G}}_0 \) and hence definable. Let \( \text{pr}_1: E \to \hat{\mathcal{G}}_0 \) denote the projection onto the first factor and let \( K \subseteq \mathcal{G}_0 \) be definable and compact. Then \( \text{pr}_1^{-1}(K) \) is given by \((\hat{s}, \hat{t})(\hat{s}^{-1}(K)) \). As \( \hat{s} \) is proper so that \( \hat{s}^{-1}(K) \) is definable and compact, the image \( (\hat{s}, \hat{t})(\hat{s}^{-1}(K)) = \text{pr}_1^{-1}(K) \) of \( \hat{s}^{-1}(K) \) under \((\hat{s}, \hat{t})\) is definable and compact so that \( \text{pr}_1 \) is definably proper. Then by \([58]\) Chapter 10, Theorem (2.15)], there is a definably proper quotient \( \pi: \hat{\mathcal{G}}_0 \to B \subset \mathbb{R}^k \), and the embedding \( \iota_0: \mathcal{G}_0 \to \hat{\mathcal{G}}_0 \subset \mathbb{R}^n \) induces a homeomorphism \( \iota_{|\mathcal{G}|} = E \setminus \iota_0: |\mathcal{G}| \to B \subset \mathbb{R}^k \); see Remark 2.7. Hence \((|\mathcal{G}|, i_{|\mathcal{G}|})\) is an affine definable space. That \((\hat{s}, \hat{t})\) is proper and definable, hence definably proper, implies that \((s, t)^{-1}(x) = \mathcal{G}_x^\circ \) is compact for each \( x \in \mathcal{G}_0 \).

An important case of Proposition 3.3 is that of the translation groupoid \( G \times A \) where \( G \) is a compact Lie group identified with a compact linear algebraic group so that \( G \subset \mathbb{R}^m \) and \( A \subset \mathbb{R}^n \) is a semialgebraic \( G \)-set; see Section 2.2. In this case, using the identity functions as embeddings of \( A \) and \( G \times A \subset \mathbb{R}^{m+n} \), the structure maps are obviously semialgebraic, and \( G \times A \) is proper. For any compact \( K \subseteq A \), the set \( s_{G \times A}^{-1}(K) = G \times K \) is compact so that \( s_{G \times A} \) is proper. As well, a semialgebraic \( G \)-set \( A \) has finitely many orbit types, each a semialgebraic set, see \([15]\) Theorem 2.6 and p. 635], yielding the following.

**Corollary 3.6 ((Semialgebraic \( G \)-sets are orbit space definable)).** Let \( G \) be a compact Lie group and \( A \) a semialgebraic \( G \)-set. Then there is an embedding of \( G \times A \) into \( \mathbb{R}^n \) with respect to which \( G \times A \) is an orbit space definable groupoid.

### 3.2. \( \Gamma \)-Euler characteristics for orbit space definable groupoids.

In this section, we define the \( \Gamma \)-Euler characteristics for orbit space definable groupoids. Recall that we fix an \( o \)-minimal structure on \( \mathbb{R} \) that contains the semialgebraic sets with respect to which definable sets and spaces are defined. Let \( G \) be a compact Lie group and let \( \Gamma \) be a finitely presented discrete group. We will frequently make use of the following.

**Notation 3.7.** Assume \( \Gamma \) has finite presentation \( \Gamma = \langle \gamma_1, \ldots, \gamma_\ell \mid R_1, \ldots, R_k \rangle \). Then \( \text{Hom}(\Gamma, G) \) can be identified with the subset of \( G^\ell \) satisfying the relations \( R_i \) via the map \( \text{Hom}(\Gamma, G) \ni \phi \mapsto \)
(ϕ(γ₁), . . . , ϕ(γₖ)) ∈ G^d. With respect to this identification, the action of G on Hom(Γ, G) by conjugation corresponds to the action of G on G^d by simultaneous conjugation.

If (G, λ) is an affine topological group, then it is clear from the above description that Hom(Γ, G) is a definable subset of the affine topological space (G^d, λ'). The action of G on G^d by simultaneous conjugation makes G^d into an affine definable G-space, and hence Hom(Γ, G) is an affine definable G-space with respect to the G-action by conjugation and the restriction of the embedding λ'. The quotient space G/\text{Hom}(Γ, G) admits an embedding with respect to which it is homeomorphic to a definably proper quotient by [58], Chapter 10, Corollary (2.18); see Remark 2.7.

With the above notation established, we have the following, which is the primary definition introduced in this section.

**Definition 3.8 ((Γ-Euler characteristics for orbit space definable groupoids)).** Let (G, i_{[G]}) be an orbit space definable groupoid, and let Γ be a finitely presented discrete group. Define the Z-Euler characteristic of G to be

\begin{equation}
\chi_Z(G) = \int_{[G]} \chi(\text{Ad}_{G_{\infty}}(G^x_\infty)) \, d\chi(Gx),
\end{equation}

where we recall that Gx denotes the G-orbit of x. More generally, the Γ-Euler characteristic of G is given by

\begin{equation}
\chi_Γ(G) = \int_{[G]} \chi(G_x^\infty \text{Hom}(Γ, G^x_\infty)) \, d\chi(Gx),
\end{equation}

where the action of G_x^\infty on Hom(Γ, G^x_\infty) is given by pointwise conjugation. Note that Equation (3.2) reduces to Equation (3.1) in the case Γ = Z via the identification of Hom(Z, G_x^\infty) with G_x^\infty by choosing a generator for Z.

By the following remark, the above Euler characteristics do not depend on the choice of o-minimal structure or embeddings.

**Remark 3.9.** Note that as G is orbit space definable, |G| is an affine definable space, and each isotropy group G_x^\infty admits the structure of an affine definable topological group. If x, y ∈ G_0 are in the same orbit, i.e., there is an h ∈ G_1 such that s(h) = x and t(h) = y, then conjugation by h induces an isomorphism of G_x^\infty onto G_y^\infty. Hence Ad_{G_1}(G_x^\infty) and G_x^\infty \text{Hom}(Γ, G^x_\infty) admit embeddings identifying them with definable proper quotients; see Remark 2.7. As well, \(\chi(\text{Ad}_{G_1}(G_x^\infty))\) and \(\chi(G_x^\infty \text{Hom}(Γ, G^x_\infty))\) are defined and by Theorem 2.8 do not depend on the choice of x in an orbit. Moreover, as the weak orbit type partition of |G| into orbits of points with isomorphic isotropy groups is a finite partition into definable sets, the two integrands in Equations (3.1) and (3.2) of Definition 3.8 are constructible functions on |G|. It follows that \(\chi_Z(G)\) and \(\chi_Γ(G)\) are defined. By Remark 2.7, \(\chi_Z(G)\) and \(\chi_Γ(G)\) do not depend on the o-minimal structure nor the choice of embedding i_{[G]} of |G| and hence are invariants of the topological groupoid G.

We will see in Section 5.1 that when G = G × X is a translation groupoid, the definition of \(\chi_Z(G)\) in Equation (3.1) coincides with the orbifold Euler characteristic \(\chi^{(1)}(X, G)\) of [28] recalled in Equation (2.44). If in addition Γ = Z^d, the definition of \(\chi_Z(G)\) in Equation (3.2) coincides with the higher-order orbifold Euler characteristic \(\chi^{(d)}(X, G)\) of [28] recalled in Equation (2.44); see Theorem 5.3 below. In particular, if G is finite so that G × X presents a global quotient orbifold, the definition of \(\chi^{(d)}(X, G)\) reduces to the orbifold Euler characteristic of [22] recalled in Equation (1.11), which is therefore equal to \(\chi_Z(G × X)\). More generally, if G is an orbifold groupoid presenting the orbifold Q, we will see that \(\chi_Γ(G)\) coincides with the Γ-Euler characteristic \(\chi_Γ(Q)\) of [27] recalled in Section 2.8 this is a consequence of Theorem 4.13 below and the fact that \(\chi_Γ(Q)\) is the Euler characteristic of the Γ-inertia space of Q, in the orbifold context called the orbifold of Γ-sectors Q_Γ. See Section 4.3 for example computations of \(\chi_Z(G)\) and \(\chi_Γ(G)\).

**Remark 3.10.** In the case of an orbifold groupoid, see Section 2.3 the isotropy groups are finite, so it is sufficient to assume Γ is finitely generated. In the present context, we require that Γ is finitely
presented to ensure that \( \text{Hom}(\Gamma, G) \) can be described as in Notation 3.7 by a finite collection of relations and hence given the structure of an affine definable space.

Note that if \( \Gamma = \{0\} \) is the trivial group, then \( \chi_{\{0\}} = \int_{|G|} d\chi(Gx) = \chi(|G|) \) is the usual Euler characteristic of the orbit space.

**Remark 3.11.** For the Euler-Satake characteristic \( \chi^{ES} \) of an orbifold, see Remark 2.11 a natural generalization to the case of orbit space definable groupoids considered in this paper would be to replace \( |G_o| \) in Equation (2.2) with the number of connected components of the isotropy group. Because the Euler-Satake characteristic is finitely additive, integration with respect to the Euler-Satake characteristic can be defined in the same way as in Remark 2.11, with the usual Euler characteristic breaks down in this more general setting. That is, the identity \( \chi_{\Gamma}^{ES}(G) = \chi_{\Gamma}(G) \) no longer holds for non-orbifold groupoids \( G \), as is illustrated by Example 3.12. Hence, while the results of this paper could also be formulated with \( \chi \) replaced by \( \chi^{ES} \), we restrict our attention to the former. See also [22], where the Euler-Satake characteristic of an orbifold is identified as a case of the Euler characteristic of a finite category.

**Example 3.12.** Let \( X \) be a point with the trivial action of the circle \( G = \mathbb{S}^1 \), and let \( G = G \times X \). Then \( \chi_{\{0\}}(G) = \chi(|G|) = 1 \) and, using the definition suggested in Remark 2.11, \( \chi^{ES}_{\{0\}}(G) = \chi^{ES}(|G|) = 1 \). As \( G_x \) for the single point \( x \in X \) is given by \( \mathbb{S}^1 \) and the conjugation action is trivial, \( \chi^{ES}_x(G) = \chi_z(G) = 0 \). Hence, the identity \( \chi^{ES}_{\Gamma \times \mathbb{Z}}(G) = \chi_{\Gamma}(G) \) is not satisfied when \( \Gamma = \{0\} \) is the trivial group. Similarly, as the isotropy group of any element of \( \text{Hom}(\Gamma, G) \) is simply \( G \) for any finitely presented discrete group \( \Gamma \), \( \chi^{ES}_x(G) = \chi_{\Gamma}(G) \).

4. \( \chi_{\Gamma}(G) \) as the Euler characteristic of a topological space

In this section, we identify the \( \Gamma \)-Euler characteristic of an orbit space definable groupoid \( G \) with the usual Euler characteristic of a topological space, denoted \( \Lambda \chi_{\Gamma}(G) \), that depends only on the Morita equivalence class of \( G \) as a topological groupoid, and hence prove that \( \chi_{\Gamma}(G) \) is Morita invariant. To begin, we define the \( \Gamma \)-inertia groupoid of an arbitrary topological groupoid, whose orbit space will be the topological space in question.

4.1. The \( \Gamma \)-inertia groupoid of a topological groupoid. Let \( G \) be a topological groupoid and \( \Gamma \) a finitely presented discrete group. As a groupoid, the object space of \( \Gamma \) is a single point. Hence, a groupoid homomorphism \( \phi: \Gamma \rightarrow G \) is given by a choice of \( x \in G_0 \), the value of \( \phi_0 \) at the unique object of the groupoid \( \Gamma \), and a group homomorphism \( \phi_1: \Gamma \rightarrow G^\Sigma_x \). In particular, for each \( \gamma \in \Gamma \), \( s \circ \phi_1(\gamma) = t \circ \phi_1(\gamma) = x \).

**Notation 4.1.** If \( \phi \in \text{Hom}(\Gamma, G) \), we will for simplicity use \( \phi_0 \) to denote the map of \( \phi \) on objects as well as its single value in \( G_0 \) at the unique object of \( \Gamma \).

The space \( \text{Hom}(\Gamma, G) \) inherits the compact-open topology from \( \Gamma \) and \( G \) with subbase given by the set of \( \phi \in \text{Hom}(\Gamma, G) \) such that \( \phi_0 \in U \) and \( \phi_1(K) \subseteq V \) where \( K \) is a finite subset of \( \Gamma \), \( U \subseteq G_0 \) and \( V \subseteq G_1 \) are open, and \( s(V) \subseteq U \). For a fixed \( x \in G_0 \), the relative topology on the set of \( \phi \in \text{Hom}(\Gamma, G) \) such that \( \phi_0 = x \) coincides with the topology on \( \text{Hom}(\Gamma, G^\Sigma_x) \) as a subset of \( (G^\Sigma_x)^\Gamma \) as described in Notation 3.7.

We now state the following. See [24, 25] for the case \( \Gamma = \mathbb{Z} \).

**Definition 4.2** (The \( \Gamma \)-inertia groupoid). Let \( G \) be a topological groupoid and \( \Gamma \) a finitely presented discrete group. The \( \Gamma \)-loop space of \( G \) is the space \( \text{Hom}(\Gamma, G) \) with the compact-open topology described above. The \( \Gamma \)-inertia groupoid of \( G \), denoted \( \Lambda \Gamma \mathcal{G} \), is the translation groupoid \( \mathcal{G} \times \text{Hom}(\Gamma, G) \) for the action of \( \mathcal{G} \) on \( \text{Hom}(\Gamma, G) \) by conjugation. Hence, the space of objects of \( \Lambda \Gamma \mathcal{G} \) is the \( \Gamma \)-loop space \( (\Lambda \Gamma \mathcal{G})_0 = \text{Hom}(\Gamma, G) \), the anchor map \( \alpha_{\Lambda \Gamma \mathcal{G}}: \text{Hom}(\Gamma, G) \rightarrow G_0 \) is given by \( \alpha_{\Lambda \Gamma \mathcal{G}}: \phi \mapsto \phi_0 \), the source of each element of the image of \( \phi_1 \), and the action of \( g \in G_1 \) on \( \phi \in \text{Hom}(\Gamma, G) \) such that \( s(g) = \phi_0 \)
is given by conjugation in $G_1$, i.e., $(g \ast \phi)_1 = g \phi_1 g^{-1}$. When there is no ambiguity, we will use the shorthand $\pi_A$ to denote the orbit map $\pi_{A \Gamma G}: A \Gamma G \rightarrow |A \Gamma G|$ of the groupoid $A \Gamma G$. Note that $g \ast \phi$ is a homomorphism $\Gamma \rightarrow G$ with $(g \ast \phi)_0 = \tau(g)$, i.e., a group homomorphism from $\Gamma$ to $g^1_{\tau(g)}$.

When $\Gamma = Z$, the $Z$-inertia groupoid and space are referred to simply as the inertia groupoid and inertia space, respectively, and the $Z$-loop space is the loop space. Note that the loop space $\text{Hom}(Z, G)$ can be identified with the subspace of $G_1$ consisting of $g$ such that $s(g) = \tau(g)$ by identifying a homomorphism $\phi$ with the image of a generator of $Z$. Then $\text{Hom}(Z, G)$ has the structure of a subgroupoid of $G$, and some authors refer to $\text{Hom}(Z, G)$ as the isotropy subgroupoid of $G$. However, for our purposes, $\text{Hom}(Z, G)$ plays the role of a $G$-space, with elements treated as objects instead of arrows, and so the groupoid structure is not relevant. Hence, we will not use the term isotropy subgroupoid.

**Remark 4.3.** If $G = G \times X$ is a translation groupoid where $G$ is a topological group, the $\Gamma$-loop space $\text{Hom}(\Gamma, G \times X)$ is the topological space $\{ (\phi, x) \in \text{Hom}(\Gamma, G) \times X : \phi(\gamma) \in G_x \ \forall \gamma \in \Gamma \}$ where $G_x$ denotes the isotropy group of $x \in X$. The $(G \times X)$-action on $\text{Hom}(\Gamma, G \times X)$, whose anchor is the source map, coincides with restriction of the diagonal $G$-action on $\text{Hom}(\Gamma, G) \times X$ to the $G$-invariant subset $\text{Hom}(\Gamma, G \times X)$, where the action on $\text{Hom}(\Gamma, G)$ is by pointwise conjugation. Choosing a finite presentation $\Gamma = \langle \gamma_1, \ldots, \gamma_r \mid R_1, \ldots, R_n \rangle$, we identify this space via the map $\phi \mapsto (\phi(\gamma_1), \ldots, \phi(\gamma_r))$ with the set of points $(g_1, \ldots, g_r) \in G^n$ such that the $g_i$ satisfy the relations $R_i$ and $g_i(x) = x$ for each $i$; see Notation [3.7] The $(G \times X)$-action coincides with the diagonal $G$-action on $G^n \times X$, acting by simultaneous conjugation on $G$-factor.

We have the following, which was proven for orbifold groupoids in [26, Lemma 2.5].

**Lemma 4.4.** If $G$ and $H$ are topological groupoids and $\Gamma$ is a finitely presented discrete group, then a homomorphism $\Phi: G \rightarrow H$ induces a homomorphism $\Lambda_\Gamma \Phi: A \Gamma G \rightarrow A \Gamma H$. If $\Phi$ is an essential equivalence, then $A \Gamma \Phi$ is ad as well.

**Proof.** Define the map on objects $(A \Gamma \Phi)_0: \text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\Gamma, H)$ by $\phi \mapsto \Phi_1 \circ \phi$, and then for each $\phi \in \text{Hom}(\Gamma, G)$ and $g \in G_1$ such that $s(g) = \Phi_1(\phi_0)$, we have $\Phi_1(g) \ast ((A \Gamma \Phi)_0(\phi)) = (A \Gamma \Phi)_0(g \ast \phi)$. It follows that $\Phi_1$ induces a homomorphism of the translation groupoids $A \Gamma \Phi: \text{Hom}(\Gamma, G) \rightarrow A \Gamma H = H \times \text{Hom}(\Gamma, H)$.

Now suppose $\Phi$ is an essential equivalence. Then $G_1$ is the fibre product $\text{Hom}_{\text{trans}}(\Phi_0, \Phi_0 \circ p_0)(G_0 \times G_0)$, implying that for each $x \in G_0$, $\Phi_1$ restricts to an isomorphism $G^x \rightarrow \text{Hom}^{\Phi_0}(x)$ and hence induces an isomorphism $\Phi_x: \text{Hom}(\Gamma, G^x) \rightarrow \text{Hom}(\Gamma, \text{Hom}^{\Phi_0}(x))$. The map

$$t_{A \Gamma H} \circ p_1: \text{Hom}(\Gamma, G^x)_{\text{trans}} \times (A \Gamma \Phi)_0 \rightarrow \text{Hom}(\Gamma, H),$$

where $(\text{Hom}_{\text{trans}}(\Phi_0, \Phi_0 \circ p_0)(\text{Hom}(\Gamma, G^x)))_{\text{trans}} \times (A \Gamma \Phi)_0$ is the set of $((h, \Phi_1 \circ \phi), \phi)$ with $\phi \in \text{Hom}(\Gamma, H)$ and $h \in \text{Hom}(\Gamma, G^x)$ such that $s(h) = \Phi_0(\phi_0)$, is given by $((h, \Phi_1 \circ \phi), \phi) \mapsto h(\Phi_1 \circ \phi)h^{-1}. As the map $t_{\text{trans}} \circ p_1: \text{Hom}_{\text{trans}}(\Phi_0, \Phi_0 \circ p_0)(G_0 \times G_0) \rightarrow H_0$ is surjective, each $y \in H_0$ is the orbit of some $\Phi_0(\phi)_0(x)$, and hence each $\psi \in \text{Hom}(\Gamma, H)$ such that $\psi_0 = y$ is conjugate via $H_1$ to an element of $\text{Hom}(\Gamma, \Phi_1(G^x))$.

It follows that $t_{A \Gamma H} \circ p_1$ has local sections, let $((h, \Phi_1 \circ \phi), \phi) \in (\text{Hom}_{\text{trans}}(\Phi_0, \Phi_0 \circ p_0)(\text{Hom}(\Gamma, G^x)))_{\text{trans}} \times (A \Gamma \Phi)_0$ to $\text{Hom}(\Gamma, G)$ with $t_{A \Gamma H}((h, \Phi_1 \circ \phi)) = h(\Phi_1 \circ \phi)h^{-1} = \psi \in \text{Hom}(\Gamma, H)$, so that $t_{A \Gamma H}(h) = \psi_0$. As the map $t_{A \Gamma H} \circ p_1$ admits local sections, there is an open neighborhood $U$ of $\psi_0$ in $H_0$ and a continuous section $\omega: U \rightarrow \text{Hom}_{\text{trans}}(\Phi_0, \Phi_0 \circ p_0)(G_0 \times G_0)$ of $t_{A \Gamma H} \circ p_1$ such that $\omega(\psi_0) = (h, \Phi_1 \circ \phi)$. Let $\omega_1 = \alpha_{A \Gamma H}(U) \ni \psi \mapsto ((\omega_1(\psi_0), \omega_1(\psi_0)^{-1}\psi(1)(\psi_0)), \Phi_1^{-1}(\psi_0^{-1}(\psi(1)(\psi_0)))$ is a local section of $t_{A \Gamma H} \circ p_1$ such that $\psi \mapsto ((h, \Phi_1 \circ \phi), \phi)$. Similarly, as $\Phi_1$ induces an isomorphism $G^x \rightarrow \text{Hom}^{\Phi_0}(x)$ at each $x \in G_0$, the fact that $G_1$ is a fibred product and $\Phi_1$ induces an isomorphism.
\[ \Phi_x : \text{Hom}(\Gamma, \mathcal{G}_x^\phi) \to \text{Hom}(\Gamma, \mathcal{H}_{\phi_0(x)}) \quad \text{for each } x \in \mathcal{G}_0 \] as above implies that

\[
\begin{array}{ccc}
\text{Hom}(\Gamma, \mathcal{G}) & \xrightarrow{(A_\Gamma \Phi)_1} & \mathcal{H}_{1 \times \mathcal{H}} \times_{\alpha_{\mathcal{H}, \mathcal{H}}} \text{Hom}(\Gamma, \mathcal{H}) \\
(s_{\mathcal{G}, \mathcal{G}}, t_{\mathcal{G}, \mathcal{G}}) & | & (s_{\mathcal{G}, \mathcal{H}, \mathcal{H}}, t_{\mathcal{G}, \mathcal{H}, \mathcal{H}}) \\
\text{Hom}(\Gamma, \mathcal{G}) \times \text{Hom}(\Gamma, \mathcal{G}) & \xrightarrow{(A_\Gamma \Phi)_0 \times (A_\Gamma \Phi)_0} & \text{Hom}(\Gamma, \mathcal{H}) \times \text{Hom}(\Gamma, \mathcal{H})
\end{array}
\]

is as well a fibred product, completing the proof. 

We recall the following, which was proven for arbitrary groupoids (without a topology) in [27] Theorem 3.1, and in [52] Proposition 2-1(2) for the case of global quotient orbifolds. We summarize the proof for topological groupoids here, referring the reader to [27] Theorem 3.1 for more details.

**Lemma 4.5** (The $\Gamma$-inertia groupoid is iterative). Let $\Gamma_1$ and $\Gamma_2$ be finitely presented discrete groups. Then for a topological groupoid $\mathcal{G}$, $\Lambda_{\Gamma_1 \times \Gamma_2} \mathcal{G}$ is isomorphic to $\Lambda_{\Gamma_2}(\Lambda_{\Gamma_1} \mathcal{G})$.

**Proof.** An element of $\text{Hom}(\Gamma_2, \Lambda_{\Gamma_1} \mathcal{G}) = \text{Hom}(\Gamma_2, \mathcal{G} \times \text{Hom}(\Gamma_1, \mathcal{G}))$ is given by a choice of $\phi \in \text{Hom}(\Gamma_1, \mathcal{G})$ and $\psi \in \text{Hom}(\Gamma_2, (\Lambda_{\Gamma_1} \mathcal{G})_0) = \text{Hom}(\Gamma_2, C_{\mathcal{G}_0}(\phi))$. Hence, the map sending $(\psi, \phi)$ to the pointwise product $\phi_1 \psi_1 \in \text{Hom}(\Gamma_1 \times \Gamma_2, \mathcal{G}_{\phi_0}(\psi))$ induces a bijection $e_0 : \text{Hom}(\Gamma_2, \Lambda_{\Gamma_1} \mathcal{G}) \to \text{Hom}(\Gamma_1 \times \Gamma_2, \mathcal{G}_{\phi_0})$. The map $e_0$ is continuous because multiplication in $\mathcal{G}_1$ is continuous, and $e_0^{-1}$ is given by composition with the projections $\Gamma_1 \times \Gamma_2 \to \Gamma_1$ so is continuous as well. Therefore, $e_0$ it is a homeomorphism. If $h \in \mathcal{G}_1$ such that $s_\mathcal{G}(h) = \phi_0$, then $h \ast e_0(\psi, \phi) = e_0(h \ast \phi, h \ast \psi)$ so that $e_0$ is $\mathcal{G}$-equivariant. It follows that $e_0$ induces an isomorphism $e : \Lambda_{\Gamma_1 \times \Gamma_2} \mathcal{G} \to \Lambda_{\Gamma_1 \times \Gamma_2} \mathcal{G}$ of topological groupoids. 

The construction of the $\Gamma$-inertia groupoid also commutes with restriction to subsets in the following sense.

**Lemma 4.6** (The $\Gamma$-inertia groupoid is local in $\mathcal{G}_0$). Let $\mathcal{G}$ be a topological groupoid, let $\Gamma$ be a finitely presented discrete group, and let $U \subseteq \mathcal{G}_0$. Recall that the anchor map $\alpha_{\Lambda_\Gamma \mathcal{G}} : (\Lambda_\Gamma \mathcal{G})_0 = \text{Hom}(\Gamma, \mathcal{G}) \to \mathcal{G}_0$ is defined by $\alpha_{\Lambda_\Gamma \mathcal{G}}(\phi) = \phi_0$; see Notation 4.4. Then $\Lambda_\Gamma(\mathcal{G}|_U) = (\Lambda_\Gamma \mathcal{G})|_{\alpha_{\Lambda_\Gamma \mathcal{G}}^{-1}(U)}$.

**Proof.** We have

\[ \text{Hom}(\Gamma, \mathcal{G}|_U) = \{ \phi \in \text{Hom}(\Gamma, \mathcal{G}) : \phi_0 \in U \} \]

so that the object spaces of $\Lambda_\Gamma(\mathcal{G}|_U)$ and $(\Lambda_\Gamma \mathcal{G})|_{\alpha_{\Lambda_\Gamma \mathcal{G}}^{-1}(U)}$ coincide. Then the result follows from the fact that each groupoid is given by the restriction of the $\mathcal{G}$-action to this set. 

We recall the following.

**Definition 4.7** (Saturation). Let $\mathcal{G}$ be a topological groupoid. For a subset $U \subseteq \mathcal{G}_0$, the saturation of $U$, denoted $\text{Sat}(U)$, is defined to be $\pi^{-1}(\pi(U))$, where we recall that $\pi : \mathcal{G}_0 \to |\mathcal{G}|$ denotes the orbit map. A set $U \subseteq \mathcal{G}_0$ is saturated if $U = \text{Sat}(U)$.

Let $|\alpha_{\Lambda_\Gamma \mathcal{G}}| : |\Lambda_\Gamma \mathcal{G}| \to |\mathcal{G}|$ denote the map on orbit spaces induced by $\alpha_{\Lambda_\Gamma \mathcal{G}}$, i.e., $|\alpha_{\Lambda_\Gamma \mathcal{G}}| : \mathcal{G}_\phi \to \mathcal{G}_{\phi_0}$. Note that $|\alpha_{\Lambda_\Gamma \mathcal{G}}|$ is obviously surjective; for each $\tilde{x} \in |\mathcal{G}|$, there is a $\phi \in \text{Hom}(\Gamma, \mathcal{G}) = (\Lambda_\Gamma \mathcal{G})_0$ such that $\phi_0 = \tilde{x}$ and $\phi(\gamma) = u(\gamma)$ for each $\gamma \in \Gamma$, and then $|\alpha_{\Lambda_\Gamma \mathcal{G}}|(\mathcal{G}_\phi) = \tilde{x}$. Note further that $\pi_{\mathcal{G}} \circ \alpha_{\Lambda_\Gamma \mathcal{G}} = |\alpha_{\Lambda_\Gamma \mathcal{G}}| \circ \pi_{\Gamma}$, where we recall that $\pi_{\Gamma}$ denotes the orbit map for the groupoid $\Lambda_\Gamma \mathcal{G}$.

If $U \subseteq \mathcal{G}_0$ with the subspace topology, then the inclusion $\nu_0 : U \to \mathcal{G}_0$ induces a groupoid homomorphism $\nu : \mathcal{G}|_U \to \mathcal{G}$, and it is easy to see that the map $\nu_1$ on arrows is a homeomorphism onto its image. For the groupoid homomorphism $\Lambda_\Gamma \nu : \Lambda_\Gamma(\mathcal{G}|_U) \to \Lambda_\Gamma \mathcal{G}$ described by Lemma 4.4 we have $(\Lambda_\Gamma \nu)_0(\phi) = \nu_1 \circ \phi$, so that $(\Lambda_\Gamma \nu)_0$ is simply the natural inclusion of $\text{Hom}(\Gamma, \mathcal{G}|_U)$ into $\text{Hom}(\Gamma, \mathcal{G})$ as a subspace. Combining this observation with Lemmas 4.4 and 4.6 we have the following.
Corollary 4.8. Let $\mathcal{G}$ be a topological groupoid and let $\Gamma$ be a finitely presented discrete group. Suppose $U \subseteq \mathcal{G}_0$ such that the inclusion $\nu: \mathcal{G}_U \rightarrow \mathcal{G}_{(\text{Sat}(U))}$ is an essential equivalence. Then the induced map $A_{\Gamma \nu}: A_{\Gamma}(\mathcal{G}_U) \rightarrow (A_{\Gamma \nu}^1\mathcal{G}_{(\text{Sat}(U))})$ defined in Lemma 4.4 is an essential equivalence in particular inducing a homeomorphism of $|A_{\Gamma}(\mathcal{G}_U)|$ with $|A_{\Gamma \nu}^1\mathcal{G}_{(\text{Sat}(U))}|$.

4.2. The Euler characteristic of the $\Gamma$-inertia space. Let $\mathcal{G}$ be an orbit space definable groupoid and $\Gamma$ a finitely presented discrete group. We introduce the notion of $\mathcal{G}$ being $\Gamma$-inertia definable, which ensures that the inertia space $|A_{\Gamma}^\Gamma(\mathcal{G})|$ has a well-defined Euler characteristic $\Lambda_A\chi(\mathcal{G})$ and that this Euler characteristic coincides with $\chi_{\Gamma}(\mathcal{G})$; see Theorem 4.13.

Definition 4.9 ($(\Gamma$-inertia definable groupoid and $\Gamma$-inertia Euler characteristic). Let $\Gamma$ be a finitely presented discrete group.

(i) A $\Gamma$-inertia definable groupoid $\mathcal{G}$ is an orbit space definable groupoid along with an embedding $i_{A_{\Gamma}^\Gamma(\mathcal{G})}$ of $|A_{\Gamma}^\Gamma(\mathcal{G})|$ into Euclidean space with respect to which $|A_{\Gamma}^\Gamma(\mathcal{G})|$ is an affine definable space and $|A_{\Gamma}^\Gamma(\mathcal{G})|$ is a morphism of affine definable spaces.

(ii) If $\mathcal{G}$ is $\mathbb{Z}$-inertia definable, we define the inertia Euler characteristic $\Lambda_A\chi(\mathcal{G})$ to be

$$\Lambda_A\chi(\mathcal{G}) = \chi(|A_{\mathcal{G}}^\mathcal{G}|),$$

the Euler characteristic of the affine definable space $|A_{\mathcal{G}}^\mathcal{G}|$. If $\mathcal{G}$ is $\Gamma$-inertia definable, we define the $\Gamma$-inertia Euler characteristic $\Lambda\chi_{\Gamma}(\mathcal{G})$ to be

$$\Lambda\chi_{\Gamma}(\mathcal{G}) = \chi(|A_{\Gamma}^\Gamma(\mathcal{G})|),$$

the Euler characteristic of the affine definable space $|A_{\Gamma}^\Gamma(\mathcal{G})|$.

As in the case of orbit definable groupoids, we will often say simply that $\mathcal{G}$ is $\Gamma$-inertia definable and assume that $i_{A_{\Gamma}^\Gamma(\mathcal{G})}$ denotes the chosen embedding.

Remark 4.10. If $\mathcal{G}$ is $\Gamma$-inertia definable, then both $|\mathcal{G}|$ and $|A_{\Gamma}^\Gamma(\mathcal{G})|$ are affine definable spaces so that $\chi_{\Gamma}(\mathcal{G})$ and $\Lambda\chi_{\Gamma}(\mathcal{G})$ are defined. The definition of $\Lambda\chi_{\Gamma}(\mathcal{G})$ makes sense if $|A_{\Gamma}^\Gamma(\mathcal{G})|$ is an affine definable space, even if $\mathcal{G}$ is not orbit space definable. However, our interest is in the relationship between these Euler characteristics. Moreover, we will make frequent use of the compatibility of the affine definable structures of $|A_{\Gamma}^\Gamma(\mathcal{G})|$ and $|\mathcal{G}|$ via $|A_{\Gamma}^\Gamma(\mathcal{G})|$, and we do not know of an example where $|A_{\Gamma}^\Gamma(\mathcal{G})|$ admits an affine definable structure while $|\mathcal{G}|$ does not.

Remark 4.11. Once again, by Theorem 2.5, $\Lambda\chi_{\Gamma}(\mathcal{G})$ depends only on the structure of $\mathcal{G}$ as a topological groupoid and not on the $o$-minimal structure or choice of embeddings.

If $G$ is a compact Lie group and $A$ is a semialgebraic $G$-set, then identifying $G$ with a compact linear algebraic group, $G \times A$ is $\Gamma$-inertia definable for any finitely presented discrete group $\Gamma$ by the description in Remark 4.3 and Corollary 3.6. We now show that cocompact proper Lie groupoids admit this structure as well.

Lemma 4.12 ((Cocompact proper Lie groupoids are $\Gamma$-inertia definable). Let $\mathcal{G}$ be a cocompact proper Lie groupoid. Then for any finitely presented discrete group $\Gamma$, $|A_{\Gamma}^\Gamma(\mathcal{G})|$ is compact. Moreover, $|\mathcal{G}|$ and $|A_{\Gamma}^\Gamma(\mathcal{G})|$ admit embeddings $i_{|\mathcal{G}|}$ and $i_{|A_{\Gamma}^\Gamma(\mathcal{G})|}$, respectively, with respect to which $\mathcal{G}$ is $\Gamma$-inertia definable.

Proof. Let $x \in \mathcal{G}_0$. By the slice theorem [13], Corollary 3.11, there is a neighborhood $U_x$ of $x$ in $\mathcal{G}_0$ diffeomorphic to $O_x \times V_x$ where $V_x$ is an invariant open neighborhood of the origin in a linear representation of $\mathcal{G}_x^\ell$ and $O_x$ is an open neighborhood of $x$ in its orbit, such that $\mathcal{G}_u$ is isomorphic to the product of $\mathcal{G}_x^\ell \times V_x$ and the pair groupoid on $O_x$. Choosing a presentation $\Gamma = \langle \gamma_1, \ldots, \gamma_\ell \mid R_1, \ldots, R_k \rangle$ of $\Gamma$, it follows as in Remark 4.3 that the open subset $\text{Hom}(\Gamma, \mathcal{G}_U_x)$ of $\text{Hom}(\Gamma, \mathcal{G})$ given by those $\phi$ such that $\phi_0 \in U_x$ can be identified via $\phi \mapsto (\phi(\gamma_1), \ldots, \phi(\gamma_\ell))$ with the set of points $(g_1, \ldots, g_\ell, v, o) \in (\mathcal{G}_x^\ell)^\ell \times V_x \times O_x$ such that the $g_i$ satisfy the algebraic relations $R_j$, and $g_i(v) = v$ for each $i$. Recalling that $\pi_A: \text{Hom}(\Gamma, \mathcal{G}) \rightarrow |A_{\Gamma}^\Gamma(\mathcal{G})|$ denotes the orbit map, it follows that $\pi_A(\text{Hom}(\Gamma, \mathcal{G}_U_x))$ is homeomorphic to the quotient of an algebraic subset of $(\mathcal{G}_x^\ell)^\ell \times V_x$ by the diagonal of the linear $\mathcal{G}_x^\ell$-action on $V_x$ and component-wise conjugation on $(\mathcal{G}_x^\ell)^\ell$. Then by [11, Corollary 1.6], $\pi_A(\text{Hom}(\Gamma, \mathcal{G}_U_x))$ is a semialgebraic set and hence definable.
For each orbit in $|\mathcal{G}|$, choose a representative $x$ from the orbit and a set $U_x$ as above. Via the identification of $U_x$ with $V_x \times O_x$, let $U'_x$ be a $G^*_x$-invariant open neighborhood of $x$ such that $U'_x \subset U_x$. Then as $|\mathcal{G}|$ is compact, it can be covered by finitely many $\pi(U'_x)$ so that $|A_r\mathcal{G}|$ is covered by finitely many $\pi_A(Hom(\Gamma, G(U'_x)))$. As each $\pi_A(Hom(\Gamma, G(U'_x)))$ is compact, $|A_r\mathcal{G}|$ is compact as well.

Then $|A_r\mathcal{G}|$ is a compact Hausdorff (hence regular) semialgebraic space in the sense of [?, Section 7, Definition 3], hence a locally semialgebraic space in the sense of [21, Section 1, Definition 3]. In each $\pi_A(Hom(\Gamma, G(U'_x)))$, $|A_r\mathcal{G}|$ corresponds to the restriction of the semialgebraic function $G^*_x \setminus ((G^*_x)^e \times V_x) \to G^*_x \setminus V_x$ induced by the projection $(G^*_x)^e \times V_x \to V_x$ so that the induced map $|A_r\mathcal{G}||\pi_A(Hom(\Gamma, G(U'_x)))$ is semialgebraic by Lemma 2.8. Identifying $V_x$ with $(e) \times V_x \subset (G^*_x)^e \times V_x$ and hence each weak orbit type in $G^*_x \setminus V_x$ with a semialgebraic subspace of $G^*_x \setminus ((G^*_x)^e \times V_x)$, by [21] Theorem 3.2, $|A_r\mathcal{G}|$ admits a locally finite, hence finite, triangulation such that each weak orbit type in $|\mathcal{G}|$ is a subcomplex. Using these triangulations to define respective embeddings $\iota_{A_r\mathcal{G}}$ and $\iota_{|\mathcal{G}|}$ of $|A_r\mathcal{G}|$ and $|\mathcal{G}|$ as semialgebraic subspaces of Euclidean space as in the proof of Proposition 3.4, it follows that $|A_r\mathcal{G}|$ is a morphism of the resulting affine definable spaces.

With this, the application of Fubini’s theorem to $|A_r\mathcal{G}|$ yields the following.

**Theorem 4.13** ($\chi_{\Gamma}$ and $A\chi_{\Gamma}$ coincide). If $\Gamma$ is a finitely presented discrete group and $\mathcal{G}$ is a $\Gamma$-inertia definable groupoid, then

$$\chi_{\Gamma}(\mathcal{G}) = \Lambda \chi_{\Gamma}(\mathcal{G}).$$

**Proof.** We apply Fubini’s theorem (Theorem 2.10) to the morphism of affine definable spaces $|A_r\mathcal{G}|: |A_r\mathcal{G}| \to |\mathcal{G}|$ to yield

$$\chi(|A_r\mathcal{G}|) = \int_{|A_r\mathcal{G}|} 1 d\chi(\mathcal{G}\phi)$$

$$= \int_{|\mathcal{G}|} \left( \int_{|A_r\mathcal{G}|^{-1}(\mathcal{G}\phi)} 1 d\chi(\mathcal{G}\phi) \right) d\chi(\mathcal{G}\phi)$$

$$= \int_{|\mathcal{G}|} \chi(G^*_x \setminus Hom(\Gamma, G^*_x)) d\chi(\mathcal{G}\phi)$$

$$= \chi_{\Gamma}(\mathcal{G}).$$

$$\square$$

It follows that when $\mathcal{G}$ is an orbifold groupoid presenting the orbifold $Q$, the $\chi_{\Gamma}(\mathcal{G})$ coincide with the $\chi_{\Gamma}(Q)$ defined in [27] and recalled in Section 2.3.

If $A$ is a locally closed semialgebraic set and $k$ is a field, the Borel–Moore homology groups $H^i_{BM}(A; k)$ can be defined in terms of the simplicial homology groups $H_i(A; k)$. If $A$ is compact, then $H^i_{BM}(A; k) = H_i(A; k)$, and if $A$ is not compact, then $H^i_{BM}(A; k)$ is defined using relative homology and a compactification of $A$; see [8, Definition 11.7.13] and [10] for the definition in a more general setting. If $A$ is a locally compact semialgebraic set, then it is demonstrated in [10] Section 1.8] that the Euler characteristic $\chi(A)$ is equal to the Euler characteristic of the Borel–Moore homology of $A$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$, i.e., the alternating sum of the Betti numbers. Applying this fact and Theorem 4.13 we have the following.

**Corollary 4.14.** If $\Gamma$ is a finitely presented discrete group and $\mathcal{G}$ is a $\Gamma$-inertia definable groupoid such that $|A_r\mathcal{G}|$ is a locally compact semialgebraic set, then $\Lambda \chi_{\Gamma}(\mathcal{G})$ and $\chi_{\Gamma}(\mathcal{G})$ are equal to the Euler characteristic of the Borel–Moore homology of $|A_r\mathcal{G}|$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. If $|A_r\mathcal{G}|$ is in addition compact, then $\Lambda \chi_{\Gamma}(\mathcal{G})$ and $\chi_{\Gamma}(\mathcal{G})$ are equal to the Euler characteristic of the usual homology of $|A_r\mathcal{G}|$.

The Morita equivalence class of $A_r\mathcal{G}$ as a topological groupoid depends only on the Morita equivalence class of $\mathcal{G}$ by Lemma 4.4. The orbit spaces of Morita equivalent groupoids are homeomorphic so that the topological space $|A_r\mathcal{G}|$ depends only on the (topological) Morita equivalence class of $\mathcal{G}$. By Theorem 4.13 $\Lambda \chi_{\Gamma}(\mathcal{G})$ depends only on the Morita equivalence class of $\mathcal{G}$, and then the same holds for $\chi_{\Gamma}(\mathcal{G})$ by Theorem 4.13. That is, we have the following.
Corollary 4.15 ((Morita invariance of $\chi_F$)). If $F$ is a finitely presented discrete group and $G$ is a $\Gamma$-inertia definable groupoid, then $\chi_F(G)$ and $\chi_F(G)$ depend only on the Morita equivalence class of $G$ as a topological groupoid.

Now, suppose $F$ is a finitely presented discrete group and $G$ is a $\Gamma$-inertia definable groupoid. Recall that $\pi = \pi_G : G_0 \to |G|$ denotes the quotient map. If $U \subseteq G_0$ is such that $\pi(U)$ is a definable subset of $|G|$, then $\left|\alpha_{A_F|G}^{-1}(\pi(U))\right|$ is a definable subset of $|A_F|$. It follows that $G_U$ is as well $\Gamma$-inertia definable via the restrictions of the embeddings $\iota_G|G$ and $\iota_{A_F|G}|G$. Then using Corollary 4.18 and Theorem 4.13, the additivity of $\chi$ extends to $\chi_F$ and $\chi_F$ as follows.

Corollary 4.16 (Additivity of $\chi_F$ and $\chi_F$). Let $F$ be a finitely presented discrete group and $G$ a $\Gamma$-inertia definable groupoid. If $S, T \subseteq |G|$ are definable subsets such that $S \cup T = |G|$ and $U, V, W \subseteq G_0$ are such that the inclusions $G_U \to G|_{\pi^{-1}(S)}$, $G_V \to G|_{\pi^{-1}(T)}$, and $G_W \to G|_{\pi^{-1}(S \cap T)}$ are essential equivalences of topological groupoids, then

\begin{equation}
\chi_F(G) = \chi_F(G_U) + \chi_F(G_V) - \chi_F(G_W)
\end{equation}

and

\begin{equation}
A\chi_F(G) = A\chi_F(G_U) + A\chi_F(G_V) - A\chi_F(G_W)
\end{equation}

Proof. Note that the hypotheses imply $\pi(U) = S$, $\pi(V) = T$, and $\pi(W) = S \cap T$, and moreover that $G^U$, $G^V$, and $G^W$ are orbit space definable. Hence via the homeomorphisms given in Corollary 4.18, we can identify $|A_F(G_U)| = |\alpha_{A_F|G}^{-1}(S)|$, $|A_F(G_V)| = |\alpha_{A_F|G}^{-1}(T)|$, and $|A_F(G_W)| = |\alpha_{A_F|G}^{-1}(S \cap T)|$. In particular, as $S, T$, and hence $S \cap T$ are definable subsets of $|G|$, their preimages under the morphism of affine definable spaces $|\alpha_{A_F|G}|$ are definable subsets of $|A_F|$, so each of the corresponding restrictions of $|\alpha_{A_F|G}|$ are as well morphisms of affine definable spaces. That is, $G^U$, $G^V$, and $G^W$ are $\Gamma$-inertia definable, and the homeomorphisms given by Corollary 4.18 are morphisms of affine definable spaces. Then $|A_F(G)| = |\alpha_{A_F|G}^{-1}(S)| \cup |\alpha_{A_F|G}^{-1}(T)|$ and $|A_F(G)| = |\alpha_{A_F|G}^{-1}(S \cap T)|$ so that by the additivity of $\chi$, Equation 4.2 follows. Applying Theorem 4.13 yields Equation 4.1.

Note that in Corollary 4.16 sets $U$, $V$, and $W$ of course always exist; we can take $U = \pi^{-1}(S)$, $V = \pi^{-1}(T)$, and $W = \pi^{-1}(S \cap T)$ so that the essential equivalences are isomorphisms.

Finally, we have that $\chi_F$ and $\chi_F$ are also multiplicative; see also [28] Lemma 3.1.

Lemma 4.17 ((Multiplicativity of $\chi_F$ and $\chi_F$)). Let $F$ be a finitely presented discrete group and let $G$ and $H$ be $\Gamma$-inertia definable groupoids. Then $G \times H$ is $\Gamma$-inertia definable,

\begin{equation}
\chi_F(G \times H) = \chi_F(G)\chi_F(H).
\end{equation}

and

\begin{equation}
A\chi_F(G \times H) = A\chi_F(G)A\chi_F(H).
\end{equation}

Proof. In [27] Proposition 3.2, it is demonstrated for arbitrary (not necessarily topological) groupoids $G$ and $H$ that $A_F(G \times H)$ is groupoid isomorphic to $A_F(G) \times A_F(H)$, where the map on objects is given by the obvious map sending $(\phi, \psi) \in \text{Hom}(G, H) \times \text{Hom}(F, H)$ to $\phi \times \psi \to \text{Hom}(F, G \times H)$, i.e., $(\phi \times \psi)(\gamma) = (\phi(\gamma), \psi(\gamma))$. If $G$ and $H$ are topological groupoids, then it is a straightforward verification that the function $(\phi, \psi) \to \phi \times \psi$ is a continuous map $\text{Hom}(G, H) \times \text{Hom}(F, H) \to \text{Hom}(F, G \times H)$, as the open sets of $G_1 \times H_2$ are generated by products of open sets. Similarly, the inverse map is the product of the projections and hence continuous. This map is then a $G \times H$-equivariant homeomorphism and hence induces an isomorphism of topological groupoids and a homeomorphism between $|A_F(G) \times A_F(H)|$ and $|A_F(G \times H)|$. Then by Theorem 2.2 and the multiplicativity of $\chi$, $A\chi_F(G \times H) = A\chi_F(G)A\chi_F(H)$ whenever $A\chi_F(G \times H)$ is defined.

Recalling that $|\alpha_{A_F|G}|$ denotes the orbit map of the anchor of the $\Gamma$-inertia groupoid of $G$, we have that $|\alpha_{A_F|G \times H}|$ sends the $G \times H$-orbit of $\phi \times \psi \in \text{Hom}(G, H) \times \text{Hom}(F, H)$ to the $G \times H$-orbit of $(\phi_0, \psi_0) \in G_0 \times H_0$ and hence coincides with $|\alpha_{A_F|G} \times |\alpha_{A_F|H}|$ up to the homeomorphisms $|A_F(G) \times A_F(H)| \cong |A_F(G \times H)|$ and $|G \times H| \cong |G \times H|$. Therefore, $G \times H$ is $\Gamma$-inertia definable via the product embeddings $\iota_{G \times H} = \iota_G \times \iota_H$ and $\iota_{A_F(G \times H)} = \iota_{A_F(G)} \times \iota_{A_F(H)}$. The multiplicativity of $\chi_F$ then follows from Theorem 4.13.
4.3. Examples. Here, we give a few concrete examples of computations of $\chi_Z(G)$ and $\chi^r(G)$. We consider translation groupoids of actions of compact Lie groups on semialgebraic subsets of Euclidean space for simplicity. See Section 5.3 and 5.5 for examples of proper Lie groupoids that are not Morita equivalent to translation groupoids (as Lie groupoids).

It is well known that the Euler characteristic of a compact Lie group of positive dimension is zero. Hence, if $G^r_\Lambda$ is abelian, then $\chi(Ad_{G^r_\Lambda}) = \chi(G^r_\Lambda) = 0$. However, the partition of $[\mathbb{G}]$ into the level sets of $\chi(Ad_{G^r_\Lambda})$ can still have an interesting effect on $\chi^r(G)$, even when all isotropy groups are abelian, as we illustrate with the following.

Example 4.18. Let $A = \mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$ and let $G = SO(2, \mathbb{R})$ act by rotations about the $z$-axis. Then the orbit space $[G \times A]$ of the translation groupoid $G \times A$ is given by an interval $[-1, 1]$ parameterized by the $z$-coordinate of points in the orbit. The isotropy group of orbits corresponding to $z = \pm 1$ is $SO(2, \mathbb{Z})$, while the isotropy group of other points is trivial. Hence, for a finitely presented discrete group $\Gamma$, the integral in Equation 4.2 can be expressed as

$$\chi^r(G \times A) = \int_{[-1,1]} \chi(\text{Hom}(\Gamma, SO(2, \mathbb{R}))) \, d\chi(z) + \int_{[-1,1]} d\chi(z) = 2\chi(\text{Hom}(\Gamma, SO(2, \mathbb{R}))) - 1,$$

where we note that $SO(2, \mathbb{R})$ is abelian so that the conjugation action is trivial. Hence, if $\Gamma = \mathbb{Z}^\ell$ for some $\ell \geq 1$, then $\chi(\text{Hom}(\Gamma, SO(2, \mathbb{R}))) = \chi(\mathbb{S}(2, \mathbb{R})^\ell) = 0$, and $\chi^r(G \times A) = -1$. If $\Gamma = \mathbb{Z}/k\mathbb{Z}$ for some positive integer $k$, then a choice of generator for $\mathbb{Z}/k\mathbb{Z}$ identifies $\text{Hom}(\Gamma, SO(2, \mathbb{R}))$ with the set of $k$th roots of unity so that $\chi(\text{Hom}(\Gamma, SO(2, \mathbb{R}))) = k$ and $\chi^r(G \times A) = 2k - 1$.

In addition, the inertia space of $G \times A$ is given by two circles attached by a line segment, which has Euler characteristic $-1$. For arbitrary $\Gamma$, $|\text{Aut}(G \times A)|$ is given by two copies of $\text{Hom}(\Gamma, SO(2, \mathbb{R}))$ attached by a line segment.

The following presents an example of the computation of $\chi^r(G)$ where the groupoid $G$ is not a Lie groupoid.

Example 4.19. In Example 4.18 we can replace $A$ with any $SO(2, \mathbb{R})$-invariant semialgebraic subset of $\mathbb{R}^3$, and the resulting translation groupoid is orbit space definable by Corollary 3.6. We will demonstrate below that it is as well $\Gamma$-inertia definable for any finitely presented group $\Gamma$; see Lemma 5.1.

For example, let $A$ be the union of the positive $z$-axis $Z = \{0, 0, z : z \geq 0\}$ and the closed unit disk in the $xy$-plane $D = \{(x, y, 0) : x^2 + y^2 \leq 1\}$. Each point in $Z$ has isotropy $SO(2, \mathbb{R})$, and the remaining points have trivial isotropy. Then $|G \times A|$ is given by $Z$ with a closed interval attached by identifying an endpoint of the interval to the origin in $Z$. Similarly, $|\text{Aut}(G \times A)|$ is the product $\text{x} \times \text{Hom}(\Gamma, SO(2, \mathbb{R}))$ with an endpoint of a closed interval attached to the point $(0, 1)$ where 1 denotes the identity homomorphism. As $Z$ is a half-open interval, $\chi(Z) = 0$ so that $\chi(Z \times \text{Hom}(\Gamma, SO(2, \mathbb{R}))) = \chi(Z)(\text{Hom}(\Gamma, SO(2, \mathbb{R}))) = 0$. Hence, for any finitely presented $\Gamma$, $\chi^r(G \times A)$ is the Euler characteristic of a half-open interval, i.e., $\chi^r(G \times A) = 0$.

Similarly, let $Z' = \{(0, 0, z) : z \in \mathbb{R}\}$ denote the entire $z$-axis, and let $A' = Z' \cup D$. Then $|G \times A'|$ is $Z'$ with an endpoint of a closed interval attached to the origin, and $|\text{Aut}(G \times A')|$ is the product $Z' \times \text{Hom}(\Gamma, SO(2, \mathbb{R}))$ with an endpoint of a closed interval attached to $(0, 1)$. Here, $\chi(Z') = -1$, so as the Euler characteristic of the points with trivial isotropy vanishes as above, $\chi^r(G \times A') = \chi(Z' \times \text{Hom}(\Gamma, SO(2, \mathbb{R}))) = -\chi(\text{Hom}(\Gamma, SO(2, \mathbb{R})))$ for any finitely generated $\Gamma$.

Example 4.20. Let $A = \mathbb{R}^3$ and let $G = SO(3, \mathbb{R})$ act by its defining representation. Then the orbit space $[G \times A]$ of the translation groupoid $G \times A$ is given by a ray $[0, \infty)$ parameterized by the distance $r$ of points in the orbit to the origin. The isotropy group of orbits corresponding to $r > 0$ is $SO(2, \mathbb{R})$, while the isotropy group of the point $r = 0$ is $SO(3, \mathbb{R})$. Hence, for a finitely presented group $\Gamma$, the
 integral in Equation (3.2) can be expressed as
\[
\chi_T(G \times A) = \int_{[0]} \chi\left(\text{SO}(3, \mathbb{R}) \setminus \text{Hom}(\Gamma, \text{SO}(3, \mathbb{R}))\right) d\chi(r) + \int_{(0,\infty)} \chi\left(\text{Hom}(\Gamma, \text{SO}(2, \mathbb{R}))\right) d\chi(r)
\]
\[
= \chi(\{0\})\chi\left(\text{SO}(3, \mathbb{R}) \setminus \text{Hom}(\Gamma, \text{SO}(3, \mathbb{R}))\right) + \chi\left(\{0, \infty\}\right)\chi\left(\text{Hom}(\Gamma, \text{SO}(2, \mathbb{R}))\right)
\]
\[
= \chi\left(\text{SO}(3, \mathbb{R}) \setminus \text{Hom}(\Gamma, \text{SO}(3, \mathbb{R}))\right) - \chi\left(\text{Hom}(\Gamma, \text{SO}(2, \mathbb{R}))\right).
\]

If \( \Gamma = \mathbb{Z} \), then \( \chi_{\mathbb{Z}}(G \times A) = 1 \), as \( \text{Ad}_{\text{SO}(3, \mathbb{R})} \text{SO}(3, \mathbb{R}) \) is a closed interval while \( \text{Ad}_{\text{SO}(2, \mathbb{R})} \text{SO}(2, \mathbb{R}) = \text{SO}(2, \mathbb{R}) \) is a circle. See [15, Section 4.2.6] for a description of the inertia space \( |A|_2(G \times A) \) in this case.

5. \( \Gamma \)-Euler characteristics for translation and proper cocompact Lie groupoids

5.1. The \( \Gamma \)-Euler characteristic for translation groupoids. In this section, we consider the case that \( G = G \times A \) is a translation groupoid where \( G \) is a compact Lie group and \( A \) is a semialgebraic \( G \)-set so that the \( \chi^{(I)}(A, G) \) are defined; see [28] and Definition 2.12. We assume without loss of generality that \( G \) is a compact linear algebraic group and hence an algebraic subset of \( \mathbb{R}^m \); see Section 2.2. We will show that \( \chi^{(I)}(A, G) = \chi_{\mathbb{Z}}(G \times A) \), and hence that the \( \chi_T \) and \( \lambda_T \) both generalize the \( \chi^{(I)} \).

Note that the \( \chi^{(I)}(A, G) \) are defined iteratively by integrating over the space of conjugacy classes in \( G \), while the extension of \( \chi_T \) to groupoids in Equation (3.2) required changing the order of integration to integrate over the orbit space of the groupoid. Here, we give a formulation of \( \chi_T \) for an arbitrary finitely presented discrete group \( \Gamma \) where the integral is over the space of conjugacy classes in \( \text{Hom}(\Gamma, G) \), and hence is closer to the spirit of [28]. This yields a definition of \( \chi^{(I)} \) that is not iterative.

**Lemma 5.1** (Semialgebraic \( G \)-sets are \( \Gamma \)-inertia definable). Let \( G \) be a compact linear algebraic group and \( A \) a semialgebraic \( G \)-set. Then for any finitely presented discrete group \( \Gamma \), there are embeddings of \( |G \times A| \) and \( |A|_F(G \times A) \) into Euclidean space with respect to which \( A|_F(G \times A) \) is orbit space definable, and \( G \times A \) is \( \Gamma \)-inertia definable.

**Proof.** Using the description of \( \text{Hom}(\Gamma, G \times A) \) given in Remark 4.3, \( \text{Hom}(\Gamma, G \times A) \) is a semialgebraic \( G \)-set so that by Corollary 3.0 there is an embedding \( \iota_{|A|_F(G \times A)} \) of \( |A|_F(G \times A) \) into \( \mathbb{R}^k \) with respect to which \( A|_F(G \times A) \) is orbit space definable. The map \( A \to \text{Hom}(\Gamma, G \times A) \) sending each \( x \in A \) to \((x, \phi)\) where \( \phi \) is the trivial homomorphism is a topological \( G \)-embedding of \( A \) into \( \text{Hom}(\Gamma, G \times A) \). Hence \( G \backslash A \) is embedded into \( G \backslash \text{Hom}(\Gamma, G \times A) \) as the orbits of the trivial homomorphisms, and composing with \( \iota_{|A|_F(G \times A)} \) yields an embedding \( \iota_{|G \times A|} \) of \( G \backslash A \) into \( \mathbb{R}^k \). As \( A \) has finitely many orbit types, each semialgebraic, by [15] Theorem 2.6 and p. 635, \( G \times A \) is orbit space definable using the embedding \( \iota_{|G \times A|} \). Then \( \alpha_{|G \times A|} \) corresponds to the restriction of the projection \( G \times A \to A \) which is \( G \)-equivariant and semialgebraic so that \( |\alpha_{|G \times A|}| \) is a morphism of affine definable spaces by Lemma 2.8. \( \square \)

**Notation 5.2.** For an element \( \phi \in \text{Hom}(\Gamma, G) \), let \( [\phi]_G \) denote the \( G \)-conjugacy class of \( \phi \), let \( C_G(\phi) \) denote the centralizer of \( \phi \) in \( G \), and let \( A^{(\phi)} \) denote the set of points in \( A \) fixed by the image of \( \phi \). The centralizer \( C_G(\phi) \) is a closed subgroup of \( G \), and \( A^{(\phi)} \) is a \( C_G(\phi) \)-invariant subspace of \( A \). The set \( A^{(\phi)} \) can be defined by a finite number of algebraic conditions using the description of \( \phi \) in Notation 3.7 and hence is a definable set. The set \( \bigcup_{\phi \in [\phi]_G} \{\phi\} \times A^{(\phi)} \) is a \( G \)-invariant subspace of \( \text{Hom}(\Gamma, G \times A) \subseteq \text{Hom}(\Gamma, G) \times A \) as \( A^{(g \phi g^{-1})} = gA^{(\phi)} \) for any \( g \in G \); see Remark 4.3.

For the special case \( \Gamma = \mathbb{Z} \), we can identify \( \text{Hom}(\mathbb{Z}, G) \) with \( G \) by fixing a generator of \( \mathbb{Z} \), and then \( G \backslash \text{Hom}(\mathbb{Z}, G) = \text{Ad}_G \) is the set of conjugacy classes in \( G \). For \( g \in G \), we let \( [g]_G \) denote the \( G \)-conjugacy class of \( g \), \( C_G(g) \) its centralizer, and \( A^{(g)} \) the set of its fixed points in \( A \).

We have the following.

**Lemma 5.3.** Let \( G \) be a compact linear algebraic group, \( A \) a semialgebraic \( G \)-set, and \( \Gamma \) a finitely presented discrete group. Fix an embedding \( \iota_{|A|_F(G \times A)} \) of \( |A|_F(G \times A) \) as in Lemma 3.2. For each \( \phi \in \text{Hom}(\Gamma, G) \), the groupoids \( \mathcal{G} = C_G(\phi) \times A^{(\phi)} \) and \( \mathcal{H} = G \times \left( \bigcup_{\phi \in [\phi]_G} \{\phi\} \times A^{(\phi)} \right) \) are Morita...
Theorem 4.13. If $\Lambda \chi$ is a homomorphism of affine definable spaces by Lemma 2.8, then composing the maps $p H (\gamma, G) \overset{\phi}{\rightarrow} A_0^p G_0 \overset{\phi}{\rightarrow} A_0^p G_1 \overset{\phi}{\rightarrow} A_0^p G_2$, we identify $\text{Hom} (\Gamma, G \times A)$ with the image in $A_0^p G_1$ of $\phi (\Gamma, G)$ and its image in $A_0^p G_0$, where $\phi (\Gamma, G)$ is a doubled homomorphism of Definition 4.9, and then the claim follows by Lemma 5.3. By Theorem 2.5, $A_0^p G_1$ is homeomorphic to $A_0^p G_2$ and $A_0^p G_2$ is homeomorphic to $A_0^p G_3$, allowing us to interpret each iteration of the recursive definition in Equation 2.3 in terms of applying $A_0^p$. 

**Theorem 5.4**. Let $G$ be a compact linear algebraic group and $A$ a semialgebraic $G$-set. Then for each $\ell \geq 0$, $\chi^{(\ell)} (A, G) = \chi^{(\ell)} (G \times A) = A \chi^{(\ell)} (G \times A)$. In particular, $\chi^{(1)} (A, G) = \chi^{(1)} (G \times A) = A \chi^{(1)} (G \times A)$. 

**Proof.** We prove by induction on $\ell$ that $\chi^{(\ell)} (A, G) = A \chi^{(\ell)} (G \times A)$, and then the claim follows by Theorem 1.13. If $\ell = 0$, then $\chi^{(0)} (A, G) = \chi^{(0)} (G \times A)$ and $A_0^0 G_0 = G \times A$ so that $\chi^{(0)} (A, G) = A \chi^{(0)} (G \times A)$, where $A_0^0 G_0$ denotes the semialgebraic $A_0^0 G_1$ of Definition 4.9, and then the claim follows by Lemma 5.3. Fixing a generator of $Z$, we identify $\text{Hom} (\mathbb{Z}, G \times A)$ with the subspace $\{ (g, x) \in G \times A : gx = x \}$ of $G \times A$, and then the homomorphism $\phi : Z \rightarrow G \times A$ corresponds to the point $\phi_0 (1, 0) \in G \times A$. Then the projection $\text{pr}_1 : G \times A \rightarrow G$ restricts to a $G$-equivariant map $\text{pr}_1 : \text{Hom} (\mathbb{Z}, G \times A) \rightarrow G$ and hence induces a map $| \text{pr}_1 | : | A_0^\ell (G \times A) | \rightarrow | A_0^\ell G |$. As $| \text{pr}_1 |$ is a morphism of affine definable spaces by Lemma 2.3, where $A_0^\ell G$ is given the structure of an affine definable space as in Remark 2.7. For $[g]_G \in A_0^\ell G$, we have 

$$| \text{pr}_1 |^{-1} ([g]_G) = G \setminus \{ h \times x \in G \times A : hx = x \} = G \setminus \bigcup_{h \in [g]_G} \{ h \times A^{(h)} \},$$

which is a definable subset of $| A_0^\ell (G \times A) |$. Hence $\chi (A_0^\ell G \setminus A^{(\ell)})$ is homeomorphic to $C_G (g) \setminus A^{(g)}$ by Lemma 5.3. By Theorem 2.5, $\chi (C_G (g) \setminus A^{(g)})$ depends only on the homomorphism type of $C_G (g) \setminus A^{(g)}$ so that we can express 

$$\chi^{(1)} (A, G) = \int_{A_0^\ell G} \chi (C_G (g) \setminus A^{(g)}) \, d\chi ([g]_G) = \int_{A_0^\ell G} \chi (| \text{pr}_1 |^{-1} ([g]_G)) \, d\chi ([g]_G) = \int_{A_0^\ell G} \left( \int_{| \text{pr}_1 |^{-1} ([g]_G)} 1 \, d\chi \right) \, d\chi ([g]_G).$$
Applying Fubini’s theorem to the map $|pr_1|$, this is equal to
\[ \int_{|A_Z(G \times A)|} 1 \, d\chi([\phi]_G) = A \chi_Z(G \times A). \]

Hence $\chi^{(1)}(A, G) = A \chi_Z(G \times A)$, and by Theorem 4.13 $\chi^{(1)}(A, G) = \chi_Z(G \times A)$.

Now assume $\chi^{(\ell - 1)}(A, G) = A \chi_Z(A_{\ell - 1}(G \times A))$ for some $\ell \geq 2$. Fixing generators of $A_{\ell}$ and identifying $\phi \in \text{Hom}(\mathbb{Z}, G)$ with the image of the generators $(g_\ell, \ldots, g_1) \in G^\ell$, $\text{Hom}(\mathbb{Z}, G \times A) = \{(g_\ell, \ldots, g_1, x) \in G^\ell \times A : g_i x = x \forall i \leq \ell\}$. Choose an embedding $i_{|A_Z(G \times A)|}$ as in Lemma 5.1 and let $i_{|A_{\ell - 1}(G \times A)|}$ be the restricted embedding constructed by identifying $|A_{\ell - 1}(G \times A)|$ with the orbits in $|A_Z(G \times A)|$ of homomorphisms such that $g_\ell$ is the identity. The projection $\text{pr}_1: G^\ell \times A \to G$ mapping $(g_\ell, \ldots, g_1, x) \mapsto g_\ell$ is $G$-equivariant and induces as in the previous case a morphism of affine definable spaces $|\text{pr}_1| : |A_Z(G \times A)| \to \text{Ad}_G G$. For $g \in G$, the preimage $|\text{pr}_1|^{-1}([g]_G)$ is given by the $G$-quotient of the set of $(g_\ell, \ldots, g_1, x) \in G^\ell \times A$ such that the $g_i$ pairwise commute, $g_i x = x$ for each $i$, and $g_\ell \in [g]_G$. We may rewrite this as
\[ |\text{pr}_1|^{-1}([g]_G) = G \setminus \left( \bigcup_{g_i \in [g]_G} \{g_\ell \times ((g_\ell, \ldots, g_1, x) \in C_G(g_\ell)^{\ell - 1} \times A^{\langle g_\ell \rangle} : g_i x = x, g_i g_j = g_j g_i \} \right) \]
\[ = G \setminus \left( \bigcup_{g_i \in [g]_G} \{g_\ell \times \text{Hom}(\mathbb{Z}^{\ell - 1}, C_G(g_\ell) \times A^{\langle g_\ell \rangle}) \} \right) \]
\[ = G \setminus \left( \bigcup_{g_i \in [g]_G} \{g_\ell \times \text{Hom}(\mathbb{Z}^{\ell - 1}, G \times A)^{\langle g_\ell \rangle} \}. \]

Here, the $G$-action on $\text{Hom}(\mathbb{Z}^{\ell - 1}, G \times A)$ corresponds to the diagonal action on $(g_\ell, \ldots, g_1, x)$. By Lemma 5.3 this is a definable subset of $|A_Z(G \times A)|$ homeomorphic to $G \times \text{Hom}(\mathbb{Z}^{\ell - 1}, C_G(g_\ell) \times A^{\langle g_\ell \rangle})$, which by the definition of the $G$-action is equal to $C_G(g_\ell) \times \text{Hom}(\mathbb{Z}^{\ell - 1}, C_G(g_\ell) \times A^{\langle g_\ell \rangle})$. Note that $A^{\langle g_\ell \rangle}$ is a semialgebraic $C_G(g_\ell)$-set so that $C_G(g_\ell) \times \text{Hom}(\mathbb{Z}^{\ell - 1}, C_G(g_\ell) \times A^{\langle g_\ell \rangle}) = A_{\ell - 1}(C_G(g_\ell) \times A^{\langle g_\ell \rangle})$ admits the structure of an orbit space definable groupoid by Lemma 5.1. Then by the inductive hypothesis, we have
\[ \chi^{(\ell)}(A, G) = \int_{\text{Ad}_G G} \chi^{(\ell - 1)}(C_G(g) \setminus A^{\langle g_\ell \rangle}) \, d\chi([g]_G) \]
\[ = \int_{\text{Ad}_G G} \chi([A_{\ell - 1}(C_G(g) \times A^{\langle g_\ell \rangle})) \, d\chi([g]_G), \]

which by the previous discussion and the homeomorphism-invariance of $\chi$ given by Theorem 2.5 is equal to
\[ = \int_{\text{Ad}_G G} \chi(C_G(g) \setminus \text{Hom}(\mathbb{Z}^{\ell - 1}, C_G(g) \times A^{\langle g_\ell \rangle})) \, d\chi([g]_G) \]
\[ = \int_{\text{Ad}_G G} \chi(|\text{pr}_1|^{-1}([g]_G)) \, d\chi([g]_G) \]
\[ = \int_{\text{Ad}_G G} \left( \int_{|\text{pr}_1|^{-1}([g]_G)} 1 \, d\chi([\phi]_G) \right) \, d\chi([g]_G). \]

Applying Fubini’s theorem, we continue
\[ = \int_{\text{Ad}_G G} \left( \int_{|\text{pr}_1|^{-1}([g]_G)} 1 \, d\chi([\phi]_G) \right) \, d\chi([g]_G) \]
\[ = \int_{|A_Z(G \times A)|} 1 \, d\chi([\phi]_G) \]
\[ = A \chi_Z(G \times A), \]
completing the proof. \[\square\]
In particular, Theorem 5.4 and Corollary 4.15 imply the following.

**Corollary 5.5** (Morita invariance of \(\chi^{(\ell)}\)). Let \(G\) be a compact linear algebraic group, \(A\) a semialgebraic \(G\)-set, and \(\Gamma\) a finitely presented discrete group. Then the \(\chi^{(\ell)}(G \times A)\) depend only on the Morita equivalence class of the translation groupoid \(G \times A\) as a topological groupoid.

Finally, using the same idea as the proof of Theorem 5.4, we can give a description of \(\chi_\Gamma(G \times A)\) for an arbitrary finitely presented discrete group \(\Gamma\) that is very closely related to the original definitions of \(\chi^{(\ell)}\) in [29].

**Theorem 5.6** (\(\chi_\Gamma\) for translation groupoids). Let \(G\) be a compact linear algebraic group, \(A\) a semialgebraic \(G\)-set, and \(\Gamma\) a finitely presented discrete group. Then

\[
\chi_\Gamma(G \times A) = \int_{G \setminus \text{Hom}(\Gamma, G)} \chi(C_G(\phi) \setminus A(\phi)) d\chi([\phi]_G).
\]

**Proof.** The proof is similar to the case \(\ell = 1\) of Theorem 5.4. The composition of \(\phi \in \text{Hom}(\Gamma, G \times A)\) with the \(G\)-equivariant projection \(\text{pr}_1 : G \times A \to G\) restricts to a \(G\)-equivariant map \(\text{pr}_1^\Gamma : \text{Hom}(\Gamma, G \times A) \to \text{Hom}(\Gamma, G)\) and hence, choosing embeddings as in Lemma 5.1 and Remark 2.7, induces a morphism \(\text{pr}_1^\Gamma : |A_\Gamma(G \times A)| \to G \setminus \text{Hom}(\Gamma, G)\) of affine definable spaces. Using the identification of \(\text{Hom}(\Gamma, G \times A)\) with a subset of \(\text{Hom}(\Gamma, G) \times A\) in Remark 2.3 we have for \([\phi]_G \in G \setminus \text{Hom}(\Gamma, G)\) that

\[
|\text{pr}_1^\Gamma|^{-1}([\phi]_G) = G \setminus \{(\phi', x) : \phi'(\gamma) \in G_x \forall \gamma \in \Gamma \text{ and } \phi' \in [\phi]_G\}
\]

which is a definable subset of \(|A_\Gamma(G \times A)|\) homeomorphic to \(C_G(\phi) \setminus A(\phi)\) by Lemma 5.3. Then by Theorem 4.10

\[
\int_{G \setminus \text{Hom}(\Gamma, G)} \chi(C_G(\phi) \setminus A(\phi)) d\chi([\phi]_G) = \int_{G \setminus \text{Hom}(\Gamma, G)} \chi(|\text{pr}_1^\Gamma|^{-1}([\phi]_G)) d\chi([\phi]_G)
\]

\[
= \int_{G \setminus \text{Hom}(\Gamma, G)} \left(\int_{|\text{pr}_1^\Gamma|^{-1}([\phi]_G)} 1 d\chi([\phi',x])\right) d\chi([\phi]_G)
\]

\[
= \int_{|A_\Gamma(G \times A)|} 1 d\chi([\phi',x]),
\]

where in the last step we apply Fubini’s theorem to \(|\text{pr}_1^\Gamma|\). This is equal to \(A\chi_\Gamma(G \times A)\), and hence by Theorem 4.13 to \(\chi_\Gamma(G \times A)\), completing the proof. \(\Box\)

In particular, when \(\Gamma = \mathbb{Z}^\ell\), we can express

\[
\chi^{(\ell)}(G,A) = \int_{G \setminus \text{Hom}(\mathbb{Z}^\ell, G)} \chi(C_G(\phi) \setminus A(\phi)) d\chi([\phi]_G).
\]

Note that \(\text{Hom}(\mathbb{Z}^\ell, G)\) can be identified with the set of commuting \(\ell\)-tuples of elements of \(G\), and then the \(G\)-action is by component-wise conjugation.

### 5.2. The \(\chi_\Gamma\) for cocompact proper Lie groupoids

In this section, for the case of a cocompact proper Lie groupoid, we describe an alternate formulation of \(\chi_\Gamma\) as a sum of Euler characteristics of orbit space definable translation groupoids. In the case that these can be chosen to be semialgebraic translation groupoids, this expresses \(\chi_\Gamma\) as an “integral over the group factor,” closer to the spirit of Equations 2.3 and 2.4 and Theorem 5.6; see Equation 5.2.

It will simplify matters to consider the language of orbispace charts, see [62] Section 3.2.1–2, and \(\mathcal{T}\)-orbispace charts; see [12] Section 2.1. If \(X\) is a connected Hausdorff space, an orbispace chart for \(X\) is a triple \((V,G,\pi)\) where \(V\) is a smooth manifold, \(G\) is a Lie group, and \(\pi : V \to X\) is a \(G\)-invariant map such that \(\pi(V)\) is open, and \(\pi\) induces a homeomorphism of \(G \setminus V\) onto \(\pi(V)\). The orbispace chart \((V,G,\pi)\) is compact if \(G\) is compact and linear if \(V\) is a \(G\)-invariant neighborhood of the origin in a
linear representation of $G$. Note that in [62, Theorem 3.2.31 and Corollary 3.2.32], Wang demonstrates that every proper Lie groupoid is Morita equivalent (as a Lie groupoid) to a groupoid associated to an orbispace atlas for $|G|$, a collection of orbispace charts satisfying compatibility conditions; see [62, Definition 3.2.3–4].

By the slice theorem [43, Corollary 3.11], if $X = |G|$ for a proper Lie groupoid $G$, then every orbit $Gx \in |G|$ is contained in the image of a compact linear orbispace chart $(V_x, G^*_x, \pi_x)$ where $x \in V_x \subseteq G_0$ corresponds to the origin of the linear representation of $G^*_x$ on $V_x$, $G^*_x|_{V_x}$ is isomorphic to $G_x^* \ltimes V_x$, and $\pi_x$ is the restriction to $V_x$ of the orbit map $\pi: G_0 \to |G|$. We say that such an orbispace chart is centered at $x$ and will always assume orbispace charts for $|G|$ are of this form. Note that for such an orbispace chart, the inclusion $V_x \to G_0$ induces an essential equivalence $G|_{V_x} \to G|_{\Sat(V_x)}$; see [43, Proposition 3.7].

**Lemma 5.7.** Let $G$ be a cocompact proper Lie groupoid and $\iota_G$ an embedding of $|G|$ into $\mathbb{R}^n$ with respect to which $G$ is orbit space definable. Then there is a finite collection of compact linear orbispace charts $\{V_i, G_i, \pi_i\}$ centered at $x_i \in G_0$, $i = 1, \ldots, k$, such that each $\pi_i(V_i) \subseteq |G|$ is a definable subset and $|G| = \bigcup_{i=1}^k \pi_i(V_i)$.

**Proof.** By [43, Proposition 3.11] each orbit $Gx \in |G|$ is contained in the image $\pi_x(V_x) \subseteq |G|$ of a compact linear orbispace chart $(V_x, G^*_x, \pi_x)$ for $|G|$ centered at $x$. For each $x$, we can arrange that $\pi_x(V_x)$ is a definable subset of $|G|$ as follows. Note that $\iota_G \circ \pi_x(V_x)$ is an open neighborhood of $\iota_G(Gx)$ in $\iota_G(|G|)$ and hence there is an $\varepsilon$-ball $B_x$ about $\iota_G(Gx)$ in $\mathbb{R}^n$ such that $B_x \cap \iota_G(|G|) \subset \iota_G \circ \pi_x(V_x)$. As $B_x$ is semialgebraic, $B_x \cap \iota_G(|G|)$ is a definable set. Let $Q_x = \iota_G^{-1}(B_x \cap \iota_G(|G|))$ and then $Q_x$ is a definable subset of $|G|$ that is an open neighborhood of $x$ in $\pi_x(V_x)$. Hence $\pi_x^{-1}(Q_x)$ is an open $G^*_x$-invariant neighborhood of the origin in $V_x$ so that $(\pi_x^{-1}(Q_x), G^*_x, (\pi_x)|_{\pi_x^{-1}(Q_x)})$ is a linear orbispace chart whose image is a definable subset. Hence, we redefine each $(V_x, G^*_x, \pi_x)$ to be $(\pi_x^{-1}(Q_x), G^*_x, (\pi_x)|_{\pi_x^{-1}(Q_x)})$. By compactness, there is a finite set $(V_i, G_i, \pi_i)$, $i = 1, \ldots, k$, such that the $\pi_i(V_i)$ cover $|G|$.

Fix a set $\{V_i, G_i, \pi_i\}$, $i = 1, \ldots, k$, of orbispace charts covering $|G|$ as in Lemma 5.7. We now replace the $V_i$ with subsets $W_i \subseteq V_i$ such that the $\pi_i(W_i)$ are disjoint. Specifically, define $S_1 = \pi_1(V_1)$ and for $i \geq 1$, recursively define $S_{i+1} = \pi_{i+1}(V_{i+1}) \cap (|G| \setminus \bigcup_{j=1}^i S_j)$. Then the $S_i$ are disjoint definable subsets of $|G|$ that cover $|G|$ by definition. Set $W_i = V_i \cap \pi_i^{-1}(S_i)$, and then each $W_i$ is a $G_i$-invariant subset of $V_i$. Note that the $W_i$ may no longer be open and hence are not necessarily manifolds, but the essential equivalence $G|_{V_i} \to G|_{\Sat(V_i)}$ of Lie groupoids obviously restricts to an essential equivalence $G|_{W_i} \to G|_{\Sat(W_i)}$ of topological groupoids. As the $S_i$ are definable subsets of $|G|$, the $G|_{W_i} = G_i \times W_i$ are orbit space definable by Lemma 3.2(ii). Noting that the $S_i$ are pairwise disjoint, an application of Corollary 3.14 yields the following.

**Theorem 5.8 ((χr for a cocompact proper Lie groupoid)).** Let $G$ be a cocompact proper Lie groupoid and $\iota_G$ an embedding of $|G|$ into $\mathbb{R}^n$ with respect to which $G$ is orbit space definable. Let $\{V_i, G_i, \pi_i\}$, $i = 1, \ldots, k$, be a finite set of compact linear orbispace charts covering $|G|$ whose images $\pi_i(V_i)$ are definable subsets of $|G|$, and define $W_i$ and $S_i$ as above. Then for any finitely presented discrete group $\Gamma$,

$\chi^r(\Gamma) = \sum_{i=1}^k \chi^r(G_i \ltimes W_i). \tag{5.1}$

**Remark 5.9.** If each $W_i$ can be chosen to be a semialgebraic subset of the representation space containing $V_i$, then applying Theorem 5.6, Equation (5.1) can be written

$\chi^r(\Gamma) = \sum_{i=1}^k \int_{G_i(\Hom(\Gamma, G_i))} \chi(C_{G_i}(\phi), W_i^{(\phi)}(\phi)) \, d\chi([\phi]_{G_i}). \tag{5.2}$

This is the case, for instance, if each $V_i$ is a semialgebraic subset of the corresponding representation space, $\iota_G(|G|)$ is semialgebraic, and the map $\pi_i: V_i \to |G|$ is a morphism of affine definable sets in the o-minimal structure of semialgebraic sets.
5.3. Abelian extensions of translation groupoids. Let $\mathcal{G}$ be a proper Lie groupoid such that $|\mathcal{G}|$ is connected. Let us briefly recall the following language from [56, 55]; while not necessary for the sequel, it will help motivate our observations in this section. A representation of $\mathcal{G}$ is a vector bundle $E \to \mathcal{G}_0$ and a Lie groupoid homomorphism $\mathcal{G} \to \text{GL}(E)$ where $\text{GL}(E)$ denotes the groupoid with objects $\mathcal{G}_0$ and arrows given by linear isomorphisms between fibers $E_x$; see [55] Section 1.1 for more details. A representation $\mathcal{G} \to \text{GL}(E)$ of $\mathcal{G}$ is effective at $x \in \mathcal{G}_0$ if the kernel of the restriction $\mathcal{G}_x^e \to \text{GL}(E_x)$ is contained in the ineffective part of $\mathcal{G}_x^e$, the kernel of the action of $\mathcal{G}_x^e$ on a slice at $x$; a representation is globally effective if it is effective at every $x \in \mathcal{G}_0$; see [56] Definition 2. The groupoid $\mathcal{G}$ is reflexive if for each $x \in \mathcal{G}_0$, there is a representation $\mathcal{G} \to \text{GL}(E)$ such that the restriction $\mathcal{G}_x^e \to \text{GL}(E_x)$ is injective. It is is parareflexive if for each $x \in \mathcal{G}_0$, there is a representation $\mathcal{G} \to \text{GL}(E)$ that is effective at $x$. Note that if $\mathcal{G}$ is Morita equivalent (as a Lie groupoid) to a translation groupoid, then it is reflexive; see [56]. See [55] Example 2.10 for an example of a proper Lie groupoid that is not reflexive and hence not Morita equivalent as a Lie groupoid to a translation groupoid.

Trentinaglia demonstrated in [56] Corollary 4 that if a proper Lie groupoid $\mathcal{G}$ with connected orbit space admits a globally effective representation, then it is Lie groupoid Morita equivalent to an extension of a translation groupoid by a bundle of compact groups; see Definitions 5.10 and 5.12 below. Note that Trentinaglia states this as a corollary to a theorem with a strong hypothesis on the codimensions of orbits, but the corollary and its proof only require the existence of a globally effective representation. Note that in [56, p.709], Trentinaglia poses the questions of whether every proper Lie groupoid is parareflexive, and whether every parareflexive proper Lie groupoid admits a globally effective representation. As far as we are aware, these questions remain open.

Here, we consider $\chi^r(\mathcal{G})$ of such a groupoid. As we will see, if the isotropy groups of $\mathcal{G}$ are abelian and $\Gamma = \mathbb{Z}^s$, then $\chi^r(\mathcal{G})$ is closely related to the $\Gamma$-Euler characteristic of the corresponding translation groupoid, though this relationship does not appear to extend to the nonabelian case.

We begin with the following.

**Definition 5.10 ((Bundle of Lie groups, [33] Appendix A.1), [36] Section 1.3(c))).** A bundle of Lie groups is a Lie groupoid $\mathcal{G}$ such that $s = t$. A bundle of Lie groups is locally trivial if each $x \in \mathcal{G}_0$ is contained in a neighborhood $U$ such that $\mathcal{G}|_U$ is diffeomorphic to the trivial bundle of groups $\mathcal{G}_x^e \times U$. By a bundle of compact Lie groups, we will mean a locally trivial bundle of Lie groups such that $s$ is proper.

If $\mathcal{G}$ is a bundle of compact Lie groups, then $|\mathcal{G}| = \mathcal{G}_0$ and for each $x$ and $y$ in $\mathcal{G}_0$, $\mathcal{G}_x^e \simeq \mathcal{G}_y^e$. In particular, an orbit space definable structure on $\mathcal{G}$ is an embedding $\iota_{|\mathcal{G}|}$ of $|\mathcal{G}| = \mathcal{G}_0$ with respect to which $\mathcal{G}_0$ is an affine definable space.

**Lemma 5.11 (($\chi^r$ for bundles of compact Lie groups))).** Suppose $\mathcal{G}$ is a bundle of compact Lie groups such that $\mathcal{G}_0$ is connected and $\iota_{|\mathcal{G}|}$ is an embedding of $|\mathcal{G}|$ with respect to which $\mathcal{G}$ is orbit space definable. Then for any finitely presented discrete group $\Gamma$ and any $x \in \mathcal{G}_0$,

$$\chi^r(\mathcal{G}) = \chi(\mathcal{G}_0)\chi(\text{Ad}_{\mathcal{G}_x^e}^{\text{Hom}(\Gamma, \mathcal{G}_x^e)}),$$

where $\chi^r(\mathcal{G}_x^e)$ is the $\Gamma$-Euler characteristic of $\mathcal{G}_x^e$, treated as a groupoid with a single object.

**Proof.** Using Equation (3.2) and the fact that $\mathcal{G}_x^e$ does not depend on $x$,

$$\chi^r(\mathcal{G}) = \int_{|\mathcal{G}|} \chi(\mathcal{G}_x^e \backslash \text{Hom}(\Gamma, \mathcal{G}_x^e)) \, d\chi(\mathcal{G}x)$$

$$= \int_{\mathcal{G}_0} \chi(\mathcal{G}_x^e \backslash \text{Hom}(\Gamma, \mathcal{G}_x^e)) \, d\chi(x)$$

$$= \chi(\mathcal{G}_0)\chi(\mathcal{G}_x^e \backslash \text{Hom}(\Gamma, \mathcal{G}_x^e)).$$

\[
\square
\]

Following [56], we consider extensions of the following form.
Definition 5.12 ((Extension of a translation groupoid)). A proper Lie groupoid $\mathcal{G}$ is an extension of a translation groupoid by a bundle of compact Lie groups if there is a short exact sequence

\[(5.3) \quad 1 \to \mathcal{B} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\nu} H \ltimes \mathcal{B}_0 \to 1\]

such that $\mathcal{B}$ is a bundle of compact groups with object space $\mathcal{B}_0 = \mathcal{G}_0$, $\nu_0$ and $\rho_0$ are the identity maps, and $H$ is a compact Lie group acting smoothly on $\mathcal{G}_0$.

If $\mathcal{G}$ is an extension as in Definition 5.12, then by the surjectivity of $\nu$, the orbit spaces $|\mathcal{G}|$ and $|H \ltimes \mathcal{G}_0|$ coincide. Specifically, for $x, y \in \mathcal{G}_0$, there is a $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$ if and only if there is an $h \in H$ such that $hx = y$.

Theorem 5.13 (($\chi_{\mathcal{G}}$ for abelian extensions of translation groupoids)). Let $\mathcal{G}$ be an extension of a translation groupoid by a bundle of compact Lie groups as in Equation (5.3), and let $\iota_{|\mathcal{G}|}$ be an embedding of $|\mathcal{G}|$ into Euclidean space with respect to which both $\mathcal{G}$ and $H \ltimes \mathcal{G}_0$ are orbit space definable. Suppose that for each $x \in \mathcal{G}_0$, the isotropy group $\mathcal{G}^*_x$ is abelian. Then for each $\ell \geq 0$ and each $x \in \mathcal{G}_0$,

\[\chi_{\mathcal{G}}(\mathcal{G}) = \chi(\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)) \chi_{\mathcal{G}^*}(H \ltimes \mathcal{G}_0).\]

Of course, by Corollary 4.15, the theorem applies to a groupoid that is Lie groupoid Morita equivalent to such an extension. The hypothesis that both $\mathcal{G}$ and $H \ltimes \mathcal{G}_0$ are orbit space definable is satisfied, for instance, when $\mathcal{G}$ is orbit space definable and the weak isotropy types of $\mathcal{G}$ and $H \ltimes \mathcal{G}_0$ coincide.

Proof. First note that for each $x \in \mathcal{G}_0$, as $\mathcal{G}^*_x$ is abelian, the subgroup $\nu(\mathcal{B}^*_x)$ and quotient group $\text{ker}(\rho_{|\mathcal{G}^*_x})/\mathcal{G}^*_x$ are abelian as well so that the isotropy groups of $\mathcal{B}$ and $H \ltimes \mathcal{G}_0$ are also abelian. As $|\mathcal{G}| = |H \ltimes \mathcal{G}_0|$, we have

\[\chi_{\mathcal{G}^*} = \chi(\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)) \chi_{\mathcal{G}^*}(H \ltimes \mathcal{G}_0).\]

By the exactness of $\text{Hom}(\mathbb{Z}^\ell, \cdot)$ on abelian groups, see [29, Chapter I, Theorems 2.1 and Proposition 4.4], we have for each $x \in \mathcal{G}_0$ an exact sequence

\[1 \to \text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x) \xrightarrow{\nu^*} \text{Hom}(\mathbb{Z}^\ell, \mathcal{G}^*_x) \xrightarrow{\rho^*_{|\mathcal{G}^*_x}} \text{Hom}(\mathbb{Z}^\ell, H_x) \to 1,\]

where $\nu^*$ and $\rho^*$ are the pullbacks. Identifying $\text{Hom}(\mathbb{Z}^\ell, \mathcal{G}^*_x)$ with $(\mathcal{G}^*_x)^\ell$ and $\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)$ with $(\mathcal{B}^*_x)^\ell$, the map $\rho^*$ is realized as the quotient map of the Lie group $(\mathcal{G}^*_x)^\ell$ by the closed subgroup $\nu(\mathcal{B}^*_x)^\ell$. Hence $\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x) \to \text{Hom}(\mathbb{Z}^\ell, H_x)$ is a fiber bundle with fiber $\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)$. By the multiplicativity of $\chi$ on fiber bundles, $\chi(\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)) = \chi(\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)) \chi(\text{Hom}(\mathbb{Z}^\ell, H_x))$, and we have

\[\chi_{\mathcal{G}^*} = \chi(\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)) \chi(\text{Hom}(\mathbb{Z}^\ell, H_x)) \chi(\text{Hom}(\mathbb{Z}^\ell, H_x)) \chi_{\mathcal{G}^*}(H \ltimes \mathcal{G}_0).\]

Note that if $\mathcal{B}^*_x$ is a compact abelian Lie group such that $\dim \mathcal{B}^*_x > 0$, then $\chi(\mathcal{B}^*_x) = 0$. Hence $\chi(\text{Hom}(\mathbb{Z}^\ell, \mathcal{B}^*_x)) = \chi((\mathcal{B}^*_x)^\ell) = 0$ as well. Therefore, for a groupoid $\mathcal{G}$ satisfying the hypotheses of Theorem 5.13, $\chi_{\mathcal{G}^*}(H \ltimes \mathcal{G}_0) = 0$ unless $\mathcal{B}^*_x$ is finite.

Finally, we have the following, illustrating that we cannot expect a generalization of Theorem 5.13 to the non-abelian case.

Example 5.14. Let $\mathcal{G}_0 = \{x\}$ be a single point, let $\mathcal{B} = \text{SO}(2, \mathbb{R})$, let $H = \mathbb{Z}/2\mathbb{Z}$, and let $\mathcal{G} = \text{O}(2, \mathbb{R})$. We define $\nu$ and $\rho$ to be the usual expression of $\text{O}(2, \mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \ltimes \text{SO}(2, \mathbb{R})$, i.e., $\nu$ is the embedding of $\text{SO}(2, \mathbb{R})$ into $\text{O}(2, \mathbb{R})$ and $\rho$ is the quotient map to the component group. Treating these
groups as groupoids with a single object, $O(2,\mathbb{R})$ is then an extension of a translation groupoid by a bundle of compact groups. As $SO(2,\mathbb{R})$ and $\mathbb{Z}/2\mathbb{Z}$ are abelian so that $|\mathbb{Z}/SO(2,\mathbb{R})| = SO(2,\mathbb{R})$ and $|\mathbb{Z}/\mathbb{Z}/2\mathbb{Z}| = \mathbb{Z}/2\mathbb{Z}$, we have $\chi(SO(2,\mathbb{R})) = 0$ and $\chi(\mathbb{Z}/2\mathbb{Z}) = 2$. However, $|\mathbb{Z}/O(2,\mathbb{R})|$ is the set of conjugacy classes of $O(2,\mathbb{R})$, which is homeomorphic to the disjoint union of an interval and a point, so that $\chi(O(2,\mathbb{R})) = 2$.

References

[1] A. Adem and J. M. Gómez, 
Equivariant $K$-theory of compact Lie group actions with maximal rank isotropy, J. Topol. 5 (2012), no. 2, 431–457.

[2] A. Adem, J. Leida, and Y. Ruan, 
Orbifolds and stringy topology, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007.

[3] A. Adem and Y. Ruan, 
Twisted orbifold $K$-theory, Comm. Math. Phys. 237 (2003), no. 3, 533–556.

[4] M. Atiyah and G. Segal, 
On equivariant Euler characteristics, J. Geom. Phys. 6 (1989), no. 4, 671–677.

[5] K. Behrend, G. Ginot, B. Noohi, and P. Xu, 
String topology for stacks, Astérisque (2012), no. 343, xiv+169.

[6] K. Behrend and P. Xu, 
Differentiable stacks and gerbes, J. Symplectic Geom. 9 (2011), no. 3, 285–341.

[7] T. Beke, 
Topological invariance of the combinatorial Euler characteristic of tame spaces, Homology Homotopy Appl. 13 (2011), no. 2, 165–174.

[8] J. Bochnak, M. Coste, and M.-F. Roy, 
Real algebraic geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 36, Springer-Verlag, Berlin, 1998, Translated from the 1987 French original, Revised by the authors.

[9] A. Borel and J. C. Moore, 
Homology theory for locally compact spaces, Michigan Math. J. 7 (1960), 137–159.

[10] R. Brown, 
On groups to groupoids: a brief survey, Bull. London Math. Soc. 19 (1987), no. 2, 113–134.

[11] G. W. Brumfiel, 
Quotient spaces for semialgebraic equivalence relations, Math. Z. 195 (1987), no. 1, 69–78.

[12] J. Bryan and J. Fulman, 
Orbifold Euler characteristics and the number of commuting $m$-tuples in the symmetric groups, Ann. Comb. 2 (1998), no. 1, 1–6.

[13] J.-L. Brylinski, 
Cyclic homology and equivariant theories, Ann. Inst. Fourier (Grenoble) 37 (1987), no. 4, 15–28.

[14] C. Chevalley, 
Theory of Lie groups. I, Princeton University Press, Princeton, N. J., 1946 1957.

[15] M.-J. Choi, D. H. Park, and D. Y. Suh, 
The existence of semialgebraic slices and its applications, J. Korean Math. Soc. 41 (2004), no. 4, 629–646.

[16] M. Coste, 
Real algebraic sets, Arc spaces and additive invariants in real algebraic and analytic geometry, Panor. Synthèses, vol. 24, Soc. Math. France, Paris, 2007, pp. 1–32.

[17] M. Crainic and J. a. N. Mestre, 
Orbispaces as differentiable stratified spaces, Lett. Math. Phys. 108 (2018), no. 3, 805–859.

[18] M. Crainic and I. Struchiner, 
On the linearization theorem for proper Lie groupoids, Ann. Sci. Éc. Norm. Supér. (4) 46 (2013), no. 5, 723–746.

[19] J. Curry, R. Ghrist, and M. Robinson, 
Euler calculus with applications to signals and sensing, Advances in applied and computational topology, Proc. Sympos. Appl. Math., vol. 70, Amer. Math. Soc., Providence, RI, 2012, pp. 75–145.

[20] M. del Hoyo and R. L. Fernandes, 
Riemannian metrics on Lie groupoids, J. Reine Angew. Math. 735 (2018), 143–173.

[21] H. Delfs and M. Knebusch, 
An introduction to locally semialgebraic spaces, vol. 14, 1984, Ordered fields and real algebraic geometry (Boulder, Colo., 1983), pp. 945–963.

[22] L. Dixon, J. Harvey, C. Vafa, and E. Witten, 
Strings on orbifolds. II, Nuclear Phys. B 274 (1986), no. 2, 285–314.

[23] C. Farsi, M. J. Pflaum, and C. Seaton, 
Differentiable stratified groupoids and a de Rham theorem for inertia spaces, (2015), arXiv:1511.00371 [math.DG].

[24] , 
Stratifications of inertia spaces of compact Lie group actions, J. Singul. 13 (2015), 107–140.

[25] C. Farsi and C. Seaton, 
Generalized twisted sectors of orbifolds, Pacific J. Math. 246 (2010), no. 1, 49–74.

[26] , 
Nonvanishing vector fields on orbifolds, Trans. Amer. Math. Soc. 362 (2010), no. 1, 509–535.

[27] , 
Generalized orbifold Euler characteristics for general orbifolds and wreath products, Algebr. Geom. Topol. 11 (2011), no. 1, 523–551.

[28] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández, 
Higher-order orbifold Euler characteristics for compact Lie group actions, Proc. Roy. Soc. Edinburgh Sect. A 145 (2015), no. 6, 1215–1222.

[29] P. J. Hilton and U. Stammbach, 
A course in homological algebra, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 4.

[30] F. Hirzebruch and T. Höfer, 
On the Euler number of an orbifold, Math. Ann. 286 (1990), no. 1-3, 255–260.

[31] N. J. Kuhn, 
Character rings in algebraic topology, Advances in homotopy theory (Cortona, 1988), London Math. Soc. Lecture Note Ser., vol. 139, Cambridge Univ. Press, Cambridge, 1989, pp. 111–126.

[32] T. Leinster, 
The Euler characteristic of a category, Doc. Math. 13 (2008), 21–49.

[33] K. Mackenzie, 
Lie groupoids and Lie algebroids in differential geometry, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, Cambridge, 1987.
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