Quantum ferrofluid turbulence

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We study the elementary characteristics of turbulence in a quantum ferrofluid through the context of a dipolar Bose gas condensing from a highly non-equilibrium thermal state. Our simulations reveal that the dipolar interactions drive the emergence of polarized turbulence and density corrugations. The superfluid vortex lines and density fluctuations adopt a columnar or stratified configuration, depending on the sign of the dipolar interactions. When the interactions are dominantly dipolar, coherent vortex structures are formed, and quasi-classical quantum turbulence emerges through the quench. This system poses exciting prospects for realizing stratified quantum turbulence and new levels of generating and controlling turbulence using magnetic fields.

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In conventional ferrofluids, colloidal suspensions of permanently magnetized particles, the dipole-dipole inter-particle interaction gives rise to unique fluid properties, such as the normal field instability and flow characteristics which can be varied through an external magnetic field [1]. The ability to direct the fluid using magnetic fields has led to broad applications from tribology to targeted medicine [2]. Remarkably, turbulence in ferrofluids has been limited to only a few studies [3–5]; this may be attributed to difficulties in achieving turbulent regimes (due to their high viscosity) and in characterising the flow (due to their opacity). As such the manner in which the anisotropic, long-range interactions modify the turbulent state remains an open question. Nonetheless, theoretical work has predicted that the coupling with ferrohydrodynamics leads to new turbulent phenomena such as control over the onset of turbulence through the applied magnetic field [6] and new modes of energy dissipation and conversion [7].

Quantum ferrofluids have been realised since 2005 through Bose-Einstein condensates of atoms with sizeable magnetic dipole moments - Cr [8, 9], Dy [10, 11] and Er [12, 13] - and have led to recent landmark demonstrations of self-trapped matter-wave droplets [13, 14] and the quantum Rosensweig instability [15, 16]. In combining ferrohydrodynamics with superfluidity, quantum ferrofluids embody a prototype system for studying ferrofluid turbulence due to the absence of viscosity and the quantization of vorticity. As demonstrated experimentally for conventional condensates, states of such quantum turbulence can be formed and imaged [17, 20], and can show both direct analogies to its counterpart in everyday viscous fluids (for example, Kolmogorov scaling [21, 22] and the transition from the von Kármán vortex street [23]) and distinct quantum effects (for example, ultra-quantum regimes [21, 22] and non-classical velocity statistics [24], depending on the details of the turbulent state. As well as the simplified fluid characteristics, these systems have the facet that the fluid parameters (viz. atomic interactions) can be tuned at will. As such, quantum ferrofluids stand to shed light on general aspect of ferrofluid turbulence, as well as new quantum regimes.

While various aspects of vortices in quantum ferrofluids have been theoretically explored, e.g. their generation, profiles and lattice structures [25], the behaviour of quantum ferrofluid turbulence remains at large. Here we study turbulence in quantum ferrofluids through the scenario of a homogeneous dipolar Bose gas freely evolving from highly non-equilibrium conditions. This scenario, representative of a sudden quench from a thermal gas through the BEC transition, is known to generate unstructured quantum turbulence which decays over time [26, 27]. This setting, free from boundaries and artefacts that may be introduced by external forcing, allows us to unambiguously identify the effects of the dipolar interactions.

We adopt the classical field methodology of the weakly-interacting, finite-temperature Bose gas [26, 28, 39], extending it to include dipolar interactions. The gas is described by a classical field ψ(r, t) (a valid assumption providing the modes are highly occupied), with atomic density n(r, t) = |ψ(r, t)|2 and whose equation of motion is given by the Gross-Pitaevskii equation (GPE). In the presence of dipolar interactions, the GPE given by [36],

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + g|\psi|^2 + \int U_{dd}(r - r') n(r', t) \, dr' \right] \psi, \tag{1} \]

The g|\psi|^2 term accounts for the local van der Waals-originating atomic interactions, parametrised by g. The non-local integral term accounts for the long-range dipolar interactions, with interaction pseudo-potential

\[ U_{dd}(r - r') = \frac{C_{dd}}{4\pi} \left( \frac{1 - 3\cos^2 \theta}{|r - r'|^3} \right), \]

with \( \theta \) being the angle between the polarisation direction and the inter-atom vector \( r - r' \), and \( C_{dd} = \mu_0 d^2 \), where \( \mu_0 \) is the permeability of free space and \( d \) is the magnetic dipole moment of the atoms. This potential accounts for the attraction of end-to-end dipoles and repulsion of side-by-side dipoles. The relative strength of the dipolar interactions is specified by the ratio \( \varepsilon_{dd} = C_{dd}/3g \), which can be experimentally tuned over the range \( -\infty \leq \varepsilon_{dd} \leq \infty \) through field-induced Feshbach resonances [36].

The ground state of the dipolar Bose gas has the uni-
form solution $\psi = \sqrt{n_0} e^{i S_0}$, where $n_0$ is the uniform density and $S_0$ an arbitrary uniform phase, and chemical potential $\mu_0 = n_0 g (1 - \varepsilon_{dd})$. According to Bogoliubov theory, perturbations to this state of momentum $p$ have energy $E_B(p) = \sqrt{c^2(\theta_k) p^2 + (p^2/2m)^2}$ where $c^2(\theta_k) = (gn_0/m) \left[1 + \varepsilon_{dd} (3\cos^2 \theta_k - 1)\right]$ and $\theta_k$ is the angle between $p$ and the polarization direction. For small $p$ the spectrum corresponds to phonons with anisotropic phase velocity $c(\theta_k)$. Assuming $g > 0$, the excitations develop imaginary energy components for $\varepsilon_{dd} < -0.5$ and $\varepsilon_{dd} > 1$, signifying unstable growth (the “phonon instability”). Here we focus on the opposite regime, $-0.5 \leq \varepsilon_{dd} \leq 1$, for which the excitations have purely real energies, are stable and oscillatory. In assuming $g > 0$, negative values of $\varepsilon_{dd}$ imply negative values of $C_{dd}$; this is feasible by tilting and fastly rotating the polarization direction. We express length in units of the dipolar healing length $\xi = h/\sqrt{m \mu_0}$ and time in terms of the unit $\tau = h/\mu_0$. The GPE is evolved numerically using a fourth-order Runge-Kutta method on a 128$^3$ periodic grid with spacing 0.5$\xi$ and time step $\Delta t = 0.01\tau$. Following previous approaches [20, 27] we initialize the system with the non-equilibrium state $\psi(r, 0) = \sum_k a_k \exp(ik \cdot r)$, where $k$ is the wavevector (defined up to the maximum amplitude allowed by the numerical box [26, 27]), the coefficients $a_k$ are uniform and the phases are distributed randomly. We illustrate the key behaviours through case studies of $\varepsilon_{dd} = 0.99$ and $\varepsilon_{dd} = -0.49$, as well as the non-dipolar case $\varepsilon_{dd} = 0$ for comparison; the more general behaviour will also be described.

At very early times, there is a rapid self-ordering of the field, akin to the non-dipolar case [26, 27]. From the initially uniform distribution across modes, the low $k$ modes grow to develop macroscopic occupation, forming a quasi-condensate. The high $k$ modes develop low occupations and are associated with thermal excitations. Within of the order of 100 time units, this bimodal distribution across the modes has effectively saturated. Unlike the non-dipolar case, the mode occupations are anisotropic in momentum space. The quasi-condensate has superfluid ordering and features a tangle of quantized vortex lines. To help visualise the superfluid vortices, we define a “quasi-condensate” density $\tilde{n}_q$ in which the high $k$ thermal modes, with $k > 7 \xi^{-1}$; this value is in the locality of the transition in the bimodal distribution. Note that this cutoff is purely for visualization of the vortices, and results are insensitive to its precise value [27].

In the non-dipolar Bose gas [Fig. 1 (top row)], this tangle is randomised in space, with no large-scale structure [20, 27], and the density fluctuations, representative of
the high \( k \) component of the field, are also isotropic in space.

For the dipolar Bose gas the spatial isotropy is broken. For \( \varepsilon_{dd} > 0 \) \( (\varepsilon_{dd} = 0.99, \text{Fig. 1} \) (middle row)) the density fluctuations become columnar, aligned along the polarization direction, as seen in the integrated density profiles. This is favoured due to the energetic benefit for the dipoles in the gas to align in an attractive head-to-tail configuration. These fluctuations are sizeable in amplitude, ranging from around 0.25 to 1.75 of the mean density, and are dynamic. The vortices visibly tend to orient along \( z \); by maximising their overlap with the low density regions they reduce their kinetic energy.

For \( \varepsilon_{dd} < 0 \) \( (\varepsilon_{dd} = -0.49, \text{Fig. 1} \) (bottom row)) now the density fluctuations become planar, driven by the attraction of side-by-side dipoles, again with a large density amplitude. The vortices in this case have a preference to align in these low density planes.

For all cases, the vortices decay in time, through reconnections, Kelvin wave decay and thermal dissipation. By \( t / \tau = 1000 \) only a few vortex loops are left in the gas. The density profile of the vortex, which is circularly-symmetric in non-dipolar condensates, becomes elliptical depending on the orientation of the line to the polarization direction and the sign of the interactions, as noted in 2D dipolar condensates [38]. The columnar (planar) density fluctuations are seen generically for \( \varepsilon_{dd} > 0 \) \( (\varepsilon_{dd} < 0) \), growing in amplitude with \( |\varepsilon_{dd}| \); these features also decay but over a much longer time scale.

Next we quantify the polarization of the vortex tangle. We project the quasi-condensate vortex tubes in the \( x, y \) and \( z \) directions, denoting the areas cast as \( A_x, A_y \) and \( A_z \), respectively. The ratio \( A_z / A_\perp \), where \( A_\perp = \frac{1}{2}(A_x + A_y) \), then quantifies the axial-to-perpendicular anisotropy of the vortices. Figure 2(a) shows the evolution of \( A_z / A_\perp \) over five initial conditions for each \( \varepsilon_{dd} \). Due to the isotropic initial conditions, all cases begin being isotropic with \( A_z / A_\perp \approx 1 \). For \( \varepsilon_{dd} = 0 \), the tangle remains isotropic throughout. However, for \( \varepsilon_{dd} \neq 0 \) the tangle evidently becomes polarized, seen by the statistically significant deviation of \( A_z / A_\perp \) from unity. For \( \varepsilon_{dd} = 0.99 \), \( A_z / A_\perp \) decreases by up to 40%; for \( \varepsilon_{dd} = -0.49 \) it increases by up to 20%. At later times (not shown) the number of vortices has decreases to of the order of unity, and this is no longer a meaningful statistic characteristic of the tangle. In Fig. 2(b) a more thorough parameter sweep of \( \varepsilon_{dd} \) is displayed, focussing on the asymptotic value of \( A_z / A_\perp \) obtained at \( t / \tau = 750 \), where, within errorbars, this quantity decreases approximately linearly for increasing \( \varepsilon_{dd} \).

Having identified an additional ‘organization’ of the vortices driven by the dipolar interactions we now seek to understand how this affects the nature of the turbulence itself. In contrast with classical turbulence, two distinct regimes of quantum turbulence have been identified [39]. In the quasi-classical regime, motions over a wide range of scales are observed, and many of the statistical properties of classical turbulence (such as Kolmogorov’s energy spectrum) are observed [40]. However quantum fluids also give rise to another form of turbulence, the so-called ultraquantum or Vinen regime, which is associated with a random tangle of quantized vortices and no large-scale structure. In the quasi-classical regime energy dissipation is dictated by the lifetime of the largest scales of motion, and one can readily construct a classical argument about the rate of the decay of the vortex line density, \( L \), which follows a power law scaling \( L \sim t^{-3/2} \). The Vinen regime can be distinguished from the quasi-classical regime because different dissipation mechanisms dominate its decay, which leads to a different power-law scaling, with \( L \) decaying as \( t^{-1} \).

The turbulence arising from a thermally quenched (non-dipolar) Bose gas has been linked to ultra-quantum turbulence [27]. Here we estimate the vortex line length \( L \) as the volume occupied by the vortex tubes divided by their typical cross-sectional area [27]; its evolution is shown in Fig. 3. For \( \varepsilon_{dd} = 0 \) and \( \varepsilon_{dd} = -0.49 \) we recover the \( t^{-1} \) behaviour of ultra-quantum turbulence. However, what is particularly striking is that for \( \varepsilon_{dd} = 0.99 \),

![Graph showing the polarization of vortex tangle over time](image1)

**FIG. 2.** (a) Polarization of the vortex tangle over time, shown through the area ratio of vortices, \( A_z / A_\perp \), for three \( \varepsilon_{dd} \) values. Lines and shaded regions represent the mean and one standard deviation over five realizations with different randomised initial conditions. (b) Snapshot of the area ratio over more detailed range of \( \varepsilon_{dd} \) at \( t = 750\tau \) (dotted line in (a)).

![Graph showing the decay of vortex tangles](image2)

**FIG. 3.** Decay of the vortex tangles. Vortex line length \( L \) for the three \( \varepsilon_{dd} \) values shown, for five simulations. For reference, \( t^{-1} \) and \( t^{-3/2} \) lines are shown (dashed lines).
FIG. 5. Coupling between vortices and the dipolar-driven density fluctuations. One large loop spreads across two planar density regions, which act as pinning layers. The relative dipolar strength is taken to be $\varepsilon_{dd} = -0.495$ and the times of the images shown are $t = (1050, 1100, 1150)$ from left to right. The phase is imprinted onto the quasi-condensate density at the $0.05\langle n_q \rangle$ level, the back wall of the box shows the 2D density profiles, corresponding to integrating the density over the dimension perpendicular to its face.

we see a faster decay, akin to $t^{-3/2}$ and consistent with the quasi-classical regime.

This emergence of quasi-classical behaviour at large values of $\varepsilon_{dd}$ can be understood by considering the role the structure of the vortex tangle plays in determining the nature of the flow it induces [41]. A purely random tangle of quantised vortices cannot generate significant flow at scales larger than the intervortex spacing. Hence it is widely believed that quantised vortices must form coherent structures, localised vortex bundles, which induce large scale motion [40]; there is recent experimental evidence in superfluid helium that this is indeed the case [42]. Hence the temporal decay observed for $\varepsilon_{dd} = 0.99$ hints that there must be some coherent structures present in the vortex tangle. We shall now turn our attention to how these can arise.

Figure 4 shows isosurfaces of high density regions in the gas when $\varepsilon_{dd} = 0.99$; low density regions have the same structure but their visualization is complicated by the presence of vortices. Evidently, when $\varepsilon_{dd}$ is large the system is threaded with these cigar like tubes of low (and high) density, where vortices appear to preferentially form. Hence we conclude that a rapid quench through the transition temperature in a quantum ferrofluid can lead to the formation of coherent vortex structures, which are present alongside random unstructured vortices. Our study shows evidence that this can lead to the emergence of quasi-classical quantum turbulence, which is not observed in the quench of a non-dipolar BEC.

In the late stages of the turbulent decay, when only one or a few vortex loops remain, we observe interesting coupling between the density fluctuations and vortices, as shown in Fig. 5. A vortex loop becomes heavily pinned across two planar regions of low density. Considerable vortex line length lies in these planes, while two vortex segments connect between these planes to form the overall loop. The pinned segments move with the low density region. This large loop is metastable but decays eventually via a reconnection, forming two small loops, each of which is heavily pinned within each low density plane. We observe such pinning of vortices to the low density region to be a general occurrence for moderate to large values of $\varepsilon_{dd}$.

To summarise, for the first time we have numerically studied turbulence in a quantum ferrofluid. In the absence of dipolar interactions the rapid quench of a thermal gas through the transition temperature generates a random unstructured tangle with no significant large scale motions. We find that for values of $\varepsilon_{dd}$ approaching unity, where the dipolar atomic interaction is comparable to the isotropic van der Waals interactions, the quantum turbulence that emerges is polarized and the quasi-classical regimes of quantum turbulence (associated with large scale motions) can be realised. In contrast for large negative values of $\varepsilon_{dd}$ the vortices arrange into sheets. This has the potential to lead to stratified quantum turbulence, which as yet is unexplored.

We believe that turbulence in a quantum ferrofluid will allow both experimental and theoretical studies of new and interesting aspects of fluid dynamics. For example the inverse cascade has received much attention in quantum fluids recently [43], and it is entirely conceivable that new regimes of two dimensional turbulence can

FIG. 4. High density level columnar density tubes from Fig. 1, $\varepsilon_{dd} = 0.99$, $t/\tau = 500$, with isosurfaces plotted at $0.99\langle n_q \rangle$. 
be realised by the presence of dipolar interactions within the gas. Finally whilst numerous mechanisms for continuously forcing three-dimensional turbulence in a BEC have been put forward [17, 20, 44], most follow James Bond’s lead and shake, rather than stir the condensate, generating significant phonon excitations [45]. By using a time dependent external magnetic field, or changing the effective value of $\varepsilon_{dd}$ (through modulation of the local van der Waals force $g$ for example) in both space and time one could stir the fluid in a method analogous to the magnetic stirring of a classical electrically conducting fluid [46].

Data supporting this publication is openly available under an Open Data Commons Open Database License [47].

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