Non-minimal couplings in two-dimensional gravity: a quantum investigation

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Abstract

We investigate the quantum effects of the non-minimal matter-gravity couplings derived by Cangemi and Jackiw in the realm of a specific fermionic theory, namely the abelian Thirring model on a Riemann surface of genus zero and one. The structure and the strength of the new interactions are seen to be highly constrained, when the topology of the underlying manifold is taken into account. As a matter of fact, by requiring to have a well-defined action, we are led both to quantization rules for the coupling constants and to selection rules for the correlation functions. Explicit quantum computations are carried out in genus one (torus). In particular the two-point function and the chiral condensate are carefully derived for this case. Finally the effective gravitational action, coming from integrating out the fermionic degrees of freedom, is presented. It is different from the standard Liouville one: a new non-local functional of the conformal factor arises and the central charge is improved, depending also on the Thirring coupling constant. This last feature opens the possibility of giving a new explicit representation of the minimal series in terms of a fermionic interacting model.

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1 Introduction

One of the most intriguing features of two-dimensional quantum field theory is the possibility to explore directly the interplay between geometry, topology, and purely quantum dynamical effects. In this simplified context one can test, for example, the influence of topologically charged fields (instantons and monopoles) in gauge theories [1] or to study the exact dependence of field theoretical quantities from the geometry and the topology of the underlying manifold [2]. A more ambitious (and interesting) task is to study the dynamics of the geometry itself, and particular attentions were devoted, in this sense, to non-critical string theory [3] and stringy-inspired gravities (dilaton gravities) and their generalizations [4].

In ref. [5] Cangemi and Jackiw have shown that the usual gravity-matter interaction in two dimensions can be altered non-trivially without spoiling the general covariance or introducing new degrees of freedom. These novel interactions correspond, in a geometric language, to a non-minimal coupling with the curvature and to an unconventional one with the volume form. The first addition has been seen to produce, at quantum level, modifications familiar from conformal improvements of dynamics, namely a change in the central charge of the minimal theory. The second addition is similar to a constant electromagnetic field in flat two-dimensional space-time and reduces to that in absence of curvature. In their investigations the structure of the underlying manifold was not taken into account, essentially assuming that the space-time has the topology of the plane.

Due to experience with topological quantization rules [7] and instanton effects [8] we wonder whether non-trivial restrictions on these couplings arise from the topology and how their classical and quantum dynamics is modified by globality requirements.

In this paper, for sake of concretness, we have chosen to investigate these issues in the case of the Abelian Thirring model on a Riemann surface of genus zero and one. Actually the explicit quantal computation of the partition function and of some correlators is carried out
in genus one (torus), where the topological structure is rich enough to exploit the constraints coming from non-trivial homology cycles, flat-gauge connections and instanton solutions. (On the sphere the first two features are in fact absent). Particular attention is also devoted to the dynamical consequences that these new couplings produce with respect to the usual Thirring model on a Riemann surface \[\mathbb{R}\].

We start, in Sect. 2, by briefly reviewing the non-minimal couplings introduced by Cangemi and Jackiw in ref. [5]. In two dimensions, because of the simplified index structure, it is possible to consistently modify the covariant derivative acting on a field, e.g. on a Dirac spinor, without introducing new degrees of freedom beyond the gravitational ones. Such modifications induce naturally two types of interactions. The first one is an additional coupling to the curvature, which, in the case of fermions, simulates the conformal improvement \(R\phi\) usually considered for bosons. The second one is, instead, a coupling with the volume two-form of the manifold, which, in the flat space-time limit, corresponds to a sort of background electric field. (As we shall see there is indeed a relation between the \(U(1)\)-bundle of the manifold and this last interaction.)

In Sect. 3 we explore the interplay between a non-trivial topology and the new couplings, paying particular attention to the cases of the sphere and the torus. The requirement of having a globally defined action induces quantization rules for the strength of these interactions. Moreover we show that their presence, in general, implies the existence of zero modes for the Dirac operator, which, in turn, entails the possibility of having chirality violating Green functions.

However, on the sphere, there is a choice of the coupling constants for which the zero modes are absent. In such a case the induced action for the conformal factor displays a new highly non-local contribution apart from the Liouville one, coming from the interaction with the volume form. On the other hand a non vanishing partition function on the torus can be only obtained by neglecting the volume-coupling. As a matter of fact we end up with a conventional Liouville
action, the original central charge of the theory being dressed.

In Sect. 4 we investigate the dynamical consequences of the non-minimal couplings in the contest of a specific interacting fermionic theory, namely the Abelian Thirring model. We limit ourselves to the torus, where the algebra is somehow simpler. There we start by observing that the coupling with the volume-form mimics a sort of instanton background. By exploiting this similarity we are able to write down the generating functional in a closed form by means of standard technique. However, some subtleties arise in the definition of the generating functional, if we want to obtain correlators which are well-defined on the torus.

In Sect. 5 we give the explicit form of the two-point function, both in the theory where the zero modes are absent (i.e. no volume-interaction) and in the theory where they are present. In particular one can easily check that a non vanishing fermionic condensate appears in the last case, as it could be expected because of the analogy with the instantons.

Let us notice that an interpretation of our result in the spirit of quantum field theory at finite temperature is also possible, due to the torus topology; we have, for example, discussed the temperature dependence of the fermionic condensate as a simple consequence of our computation.

Finally in Sect.6 we present our conclusions and the possible extensions of this work.

2 Non-minimal couplings for fermionic theories

In this section we briefly review the non-minimal couplings to gravity introduced by Cangemi and Jackiw [5]. However, being interested in the abelian Thirring model, we have chosen to focus our attention only on the case of a Dirac spinor $\psi$, referring the reader to ref. [5] for a more systematic and general presentation.

In arbitrary dimension the interaction between gravity and a Dirac spinor field $\psi$ is built by
introducing a covariant derivative

\[ D_\mu \equiv \partial_\mu + \frac{i}{2} \omega_\mu^{ab} \sigma_{ab}, \tag{2.1} \]

whose main property is to satisfy the algebra

\[ [D_\mu, D_\nu] = \frac{i}{2} R_{\mu\nu}^{ab} \sigma_{ab}. \tag{2.2} \]

In eqs. (2.1) and (2.2) \( \sigma_{ab} \) is the Lorentz generator corresponding to the given spinor representation, while \( R_{\mu\nu}^{ab}(\equiv \partial_\mu \omega_\nu^{ab} + \omega^a_{\epsilon\mu} \omega^\epsilon_b - \mu \leftrightarrow \nu) \) is the curvature two-form.

In ref. [5] the basic idea is to modify the algebra (2.2) as follows

\[ [D_\mu, D_\nu] = \frac{i}{2} R_{\mu\nu}^{ab} \sigma_{ab} + i F_{\mu\nu}, \tag{2.3} \]

where \( F_{\mu\nu} \) is a two form. In dimensions greater than two, this new contribution cannot be generated only by means of the gravitational variables; it arises when electromagnetic degrees of freedom are dynamically active – but here we do not include these additional variables. In two dimensions, however, gravitational variables allow constructing the required form; in fact we can take

\[ F_{\mu\nu} = F \sqrt{g} \epsilon_{\mu\nu}, \tag{2.4} \]

where \( g \) is the determinant of the (Riemannian) metric of the manifold in consideration, \( \epsilon_{\mu\nu} \) the ordinary Levi-Civita tensor and finally \( F \) any scalar field built out from the metric [ e.g. any function of the scalar curvature \( R(\equiv e^\mu_a e^\nu_b R_{\mu\nu}^{ab}) \)].

Following ref. [5], in the sequel of this paragraph we shall only consider the two simplest contributions to \( F \), i.e.

\[ F = A R + B, \tag{2.5} \]

with \( A \) and \( B \) two constants setting the strength of the new interactions. Moreover, in ref. [5] this particular choice was also seen to fit very naturally into a gauge theoretical description of the two-dimensional gravity in terms of the extended Poincarè group [5].
As it is clear from the aforementioned analogy with the gauge field, the new terms in eq. (2.3) can be originated by adding suitable vector potentials to the covariant derivative (2.1). The contribution proportional to the curvature is obviously generated by the combination $iA\omega_\mu$ ($\omega_\mu \equiv 1/2\epsilon_{ab}\omega^a_\mu$), while the other one involves some subtleties.

The two-form $\frac{1}{2}B\sqrt{g}\epsilon_{\mu\nu}dx^\mu \wedge dx^\nu$ is proportional to the volume form $\mathcal{V}(\equiv 1/2\sqrt{g}\epsilon_{\mu\nu}dx^\mu \wedge dx^\nu)$. Since $\mathcal{V}$ is closed we can define locally a one-form $a$ for which

$$\partial_\mu a_\nu - \partial_\nu a_\mu - B\sqrt{g}\epsilon_{\mu\nu} = 0,$$

where $a$ is determined up to an exact (globally defined) one-form $d\beta$

$$a_\mu \rightarrow a_\mu + \partial_\mu \beta. \quad (2.7)$$

Thus this interaction can be obtained by adding the combination $ia_\mu$ to the covariant derivative (2.1). The improved action for a Dirac spinor field now reads

$$S = \int d^2x \det(e) \bar{\psi} e^a_\mu \gamma^a i \left( \partial_\mu + \frac{i}{2}\omega_\mu \gamma_5 + iA\omega_\mu + ia_\mu \right) \psi - \int d^2x e^{\mu\nu} \rho \left( \partial_\mu a_\nu - \frac{B}{2}\epsilon_{ab}e^a_\mu e^b_\nu \right)^\rho \quad (2.8)$$

where $\rho$ is a Lagrange multiplier which enforces the constraint (2.6) and we have used the explicit form of $\sigma_{ab} = \frac{i}{4}\epsilon_{ab}\gamma_5$ in two dimensions.

The invariance of such an action under diffeomorphism is manifest while local $SO(2)$ is preserved only if we modify the transformation law for the Dirac fields as follows

$$\psi \rightarrow \exp\left(\frac{i}{2}\alpha\gamma_5 + iA\alpha\right)\psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp\left(-\frac{i}{2}\alpha\gamma_5 - iA\alpha\right). \quad (2.9)$$

Moreover the presence of the field $a_\mu$ introduces a further $U(1)$ symmetry. In fact if we change the fields according to

$$\psi \rightarrow \exp(i\beta)\psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp(-i\beta) \quad \text{and} \quad a_\mu \rightarrow a_\mu + \exp(-i\beta)\partial_\mu \exp(i\beta), \quad (2.10)$$

the action is invariant. At this level we can consider $a_\mu$ as an ordinary $U(1)$ connection.
Some comments about the new interactions appearing in the Lagrangian (2.8) are now in order. The origin of the coupling $A\omega_\mu$ can be traced back to the abelian character of the rotation group in two dimensions. In fact in this case we are not forced to represent our group with traceless generator. [This is analogous to what happens in abelian gauge theory, where one can arbitrarily mix the vectorial and axial coupling without problems. Let us recall, on the other hand, that this is impossible in the non-abelian case.] Moreover the introduction of the $A\omega_\mu$ term changes substantially the conformal structure of the theory. The conformal invariance of the Dirac part of the action is preserved only if we modify the transformation for the spinors as follows (we call $\sigma$ the parameter of the conformal transformation)

$$\psi(x) \rightarrow \exp \left[ \left( -\frac{1}{2} + A\gamma_5 \right) \sigma(x) \right] \psi(x), \quad (2.11)$$

and similarly for $\bar{\psi}(x)$. In other words the right and left component of the spinor have now different conformal weights.

The group theoretical root of the interaction with the voulme potential $a_\mu$ is, instead, located in the possibility of modifying the algebra of traslations with a central extension, namely

$$[P_a, P_b] = \epsilon_{ab} I. \quad (2.12)$$

Such extension arises naturally in two dimensions. A classical example is the case of the Schrödinger equation coupled to a costant magnetic field.

On a compact Riemannian manifold both $\omega_\mu$ and $a_\mu$ possess, in general, a non trivial winding number, i.e. they do not exist as 1-forms on the manifold, but only as connections on the tangent and $U(1)$ bundle respectively. In the following we shall see how this fact implies a certain number of non trivial constraints on the couplings, the partition function and the correlators of our fermionic model.
3 Non-minimal couplings on the sphere and on the torus

The abelian Thirring model in the presence of the non-minimal couplings is immediately obtained by adding the usual current-current interaction to the improved Lagrangian (2.8). However, before committing ourselves with this specific fermionic theory, we shall address the preliminary question of defining the Dirac operator in eq. (2.8) on manifolds with non-trivial topology, specifically the sphere and the torus. This analysis will lead us to single out some kinematical constraints on the coupling constants and the Green functions of our theory.

At the end of this section we will also spend some words about what results might be taken over directly to higher genus.

3.1 The sphere

The geometry of $S^2$ is described by the zweibein:

$$e^a_\mu = e^\sigma \hat{e}^a_\mu;$$  \hspace{1cm} (3.13)

$\hat{e}^a_\mu$ being a reference zweibein for a two-sphere of constant radius (taken equal to 1), while $\sigma$ is the conformal factor, which will represent the Liouville mode in the following.

Choosing angular coordinates $(\theta, \varphi), 0 \leq \theta < \pi, 0 \leq \varphi < 2\pi$ and fixing the Lorentz frame we get for $\hat{e}^a_\mu$:

$$\hat{e}^a_\mu = \begin{pmatrix} \cos \varphi & -\sin \varphi \sin \theta \\ \sin \varphi & \cos \varphi \sin \theta \end{pmatrix}. \hspace{1cm} (3.14)$$

The metric related to the zweibein in eq. (3.13) is therefore $g_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu = e^{2\sigma} \delta_{ab} \hat{e}^a_\mu \hat{e}^b_\nu = e^{2\sigma} \hat{g}_{\mu\nu}$.

In order to construct the non-minimal couplings we have, first of all, to obtain a solution for the equation (2.6), defining the potential for the volume-form. To this purpose we notice that on $S^2$ we have instanton fields solving the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0. \hspace{1cm} (3.15)$$
The most general form of such solutions, with regular behaviour, is given by

\[ F^{(k)}_{\mu\nu} \equiv \partial_\mu A^{(k)}_\nu - \partial_\nu A^{(k)}_\mu = \frac{2\pi k}{V} \sqrt{g} \epsilon_{\mu\nu} \quad \text{with} \quad k \in \mathbb{Z}, \quad (3.16) \]

where \( V \) is the volume of the manifold and \( k \) is the winding number

\[ k = \frac{1}{4\pi} \int d^2 x \, \epsilon^{\mu\nu} F^{(k)}_{\mu\nu}. \quad (3.17) \]

The corresponding gauge-connection can be now written (in a single patch)

\[ A^{(k)}_\mu = \hat{A}^{(k)}_\mu - \pi k \eta^{\nu}_{\mu} \partial_\nu \lambda, \quad (3.18) \]

where \( \eta^{\mu\nu} \equiv \sqrt{g} \epsilon^{\mu\nu} \) and \( \lambda \) solves the equation

\[ \sqrt{g} \triangle \lambda = \frac{\sqrt{g}}{V} - \frac{\sqrt{\hat{g}}}{\hat{V}}. \quad (3.19) \]

The field \( \hat{A}^{(k)}_\mu \) in eq. (3.18) is the instanton computed in the background geometry \( \hat{g}_{\mu\nu} \). In this way \( \lambda \) carries the dependence on the conformal factor, while \( \hat{A}^{(k)}_\mu \) modulo gauge transformation is (in angular coordinates)

\[ \hat{A}^{(k)}_\mu = \hat{A}^{(k)}_\varphi (\sin \theta \, d\varphi) = \frac{k}{2} \tan \frac{\theta}{2} (\sin \theta d\varphi). \quad (3.20) \]

The expression (1.21) for the instanton solution has, obviously, only a local meaning due to the singularity at \( \theta = \pi \), present in \( \hat{A}^{(k)}_\varphi \). Nevertheless \( \hat{A}^{(k)}_\mu \) is a connection on a non-trivial \( U(1) \)-bundle; in fact we need at least two patches to cover the sphere and in the overlapping region non-trivial connection are related by a gauge transformation. Describing the system in a patch where \( \hat{A}^{(1)}_\mu \) is regular at \( \theta = \pi \) we get

\[ \hat{A}^{(k)}_\varphi (\sin \theta d\varphi) = -\frac{k}{2} \cot \frac{\theta}{2} (\sin \theta d\varphi). \quad (3.21) \]

the two expression being connected by the gauge transformation

\[ \hat{A}^{(k)} - \hat{A}'^{(k)} = kd\varphi = ig^{-1}dg, \quad (3.22) \]

\[ ^{d}\text{We recall that only the one form } (\sin \theta \, d\phi) \text{ has a global meaning on the sphere, while } d\phi \text{ does not.} \]
where \( g = \exp[-ik\varphi] \) is a well-defined \( U(1) \)-valued function on the overlapping region.

Now, by exploiting the resemblance between eq. (3.16) and (2.6), we can immediately write down an expression for the volume-potential

\[
a_\mu = \frac{BV}{2\pi} A^{(1)}_\mu.
\]  

(3.23)

The non trivial structure of the instanton solution is obviously inherited by the volume potential \( a_\mu \). In fact, when changing patch, one gets as a consequence of (3.22)

\[
a - a' = \frac{BV}{2\pi} d\varphi = i \exp\left[i \frac{BV}{2\pi} \varphi \right] d \exp\left[-i \frac{BV}{2\pi} \varphi \right].
\]  

(3.24)

[Eq. (3.24) seems to suggest that \( a_\mu \) is a gauge connection in a non trivial \( U(1) \) bundle and this would imply that]

\[
\frac{BV}{2\pi} = n, \quad \text{with } n \in \mathbb{Z}.
\]  

(3.25)

Actually this conclusion is somehow misleading. In fact, as we shall see in the following, the requirement (3.25) is too strong.[10]

On the sphere also the spin-connection shares an analogous non trivial structure. Due to the fact that the Euler number \( \chi \) is not zero, \( \chi = 2 \), we expect that the spin-connection \( \omega_\mu \) does not have a global definition. Actually in the parametrization we have chosen the spin-connection can be written as [10]

\[
\omega_\mu = \hat{A}_\mu^{(2)} - \eta_\nu^{\tau} \partial_\nu \sigma.
\]  

(3.26)

The relevant gauge-transformation, connecting \( \omega_\mu \) in the two overlapping patches, turns out to be in this case \( g = \exp[-2i\varphi] \). [ A consideration similar to the one presented above holds also here.]

In order that the quantum theory be defined, what we, really, need is that the Dirac operator appearing in eq. (2.8) has a global meaning, i.e. its eigenvalue problem must be patch-independent. This is simply obtained if the Dirac operator

\[
D = i\gamma^\mu \left[ \partial_\mu + i \left( \frac{1}{2} \gamma_5 + A \mathbb{I} \right) \omega_\mu + a_\mu \right]
\]  

(3.27)
undergoes through a similitude transformation $D' = U^{-1}DU$ under change of patch. Using the previous result, one can promptly check that such an $U$ exists and it is given by

$$U = \exp \left[ -i \left( \gamma_5 + \left( 2A + \frac{BV}{2\pi} \right) \mathbb{1} \right) \varphi \right].$$

(3.28)

The requirement that $U$ is well-defined in the overlapping region leads to the following quantization condition

$$\left( 2A + \frac{BV}{2\pi} \right) = N \quad \text{with } N \in \mathbb{Z}.$$

(3.29)

Eq. (3.29) has a simple geometrical interpretation in the framework of the bundle geometry. Neither $a_\mu$ nor $\omega_\mu$ exist separately as connection on the principal bundle, but only the combination $a_\mu + A\omega_\mu$ has a global meaning.

This conclusion becomes even more transparent, if we look at the problem from the gauge theoretical point of view developed by Cangemi and Jackiw [5]. In this approach $\omega_\mu$ and $a_\mu$ are not separate objects but they form, with $e_\mu^a$, a gauge connection living in the principal-bundle of the central extended Poincarè group. Thus it is only this quantity that must exist globally and not its components. One can check that this requirement is equivalent to the condition found before. [Actually a complete group theoretical analysis in terms of the extended Poincarè group is quite tricky because it involves infinite dimensional representation of the relevant Lie algebra [4], while in terms of the properties of the Dirac operator is quite straightforward.]

The next step in our analysis is to explore the influence of these new interaction on the Dirac operator

$$\hat{D}^{(N)} = \gamma_\alpha e_\alpha^a \left[ iD_\mu + A\omega_\mu + a_\mu \right],$$

(3.30)

whose spectrum governs the quantum dynamics of our model.

To every eigenfunction $\Psi_i$ of the operator $\hat{D}^{(N)}$ with a non-zero eigenvalue $E_i$ another eigenfunction $\Psi_{-i} = \gamma_5 \Psi_i$ corresponds, with eigenvalue $E_i = -E_{-i}$. For non-vanishing $N$, zero modes are present and they have definite chirality i.e.

$$\gamma_5 \chi_i^{(N)} = \pm \chi_i^{(N)}.$$
Their number is $N = n_+ - n_-$, $n_+$ corresponding to positive chirality and $n_-$ to the negative one [11]:

$$
  \begin{align*}
    n_+ &= 0 & n_- &= |N| & N \geq 0 \\
    n_+ &= |N| & n_- &= 0 & N \leq 0.
  \end{align*}
$$

Due to the presence of zero-modes, for generic $N$, the partition function vanishes: the generating functional is nevertheless not zero, and chirality-violating correlation functions, with chirality selection rules governed by $N$, appear on the theory. In fact, using the notation

$$
  \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \bar{\psi}_R \\ \bar{\psi}_L \end{pmatrix},
$$

one can check that the only non-vanishing correlation functions are

$$
  \langle \bar{\psi}_R(x_1)\psi_R(y_1)\ldots\bar{\psi}_R(x_p)\psi_R(y_p)\bar{\psi}_L(w_1)\psi_L(z_1)\ldots\bar{\psi}_L(w_q)\psi_L(z_q) \rangle,
$$

with $p - q = N$. An explicit proof of this statement will be given in sect. 4, where we construct the generating functional for the Thirring model on the torus. However it is easy to realize that this property is independent of the detail of the model under scrutiny.

Alternatively, in order to obtain a non-trivial partition function, we can choose $A$ and $B$ to give $N = 0$. A non-trivial modification of the usual Liouville dynamics, for genus-zero Riemann surface, is therefore produced by the presence of the field $\lambda$ linked to the volume-potential. (We stress that it is not an independent field but a non-local functional of the Liouville mode).

The partition function for a Dirac fermion is

$$
  Z = \int Dg_{\mu\nu} D\bar{\psi} D\psi \exp[-\int d^2 x \sqrt{g} \bar{\psi}(0)\psi],
$$

and performing the functional integration over the fermions field we get

$$
  Z = \int Dg_{\mu\nu} \exp[-S_{Liou.}(\sigma, \hat{g}_{\mu\nu})] \exp[-S_n(\sigma, \lambda, \hat{g}_{\mu\nu})],
$$

where $S_{Liou.}$ is the usual Liouville action and the new contribution coming from the non-minimal coupling being

$$
  S_n(\sigma, \lambda, \hat{g}_{\mu\nu}) = \frac{N^2 \pi}{2} \int d^2 x \sqrt{\hat{g}} (\lambda - \frac{\sigma}{2\pi}) \hat{\Delta} (\lambda - \frac{\sigma}{2\pi}).
$$
The $\lambda-$dependence of this additonal contribution can be traced back to the fact that the constraint defining the vector potential $a_\mu$ is not conformal invariant. Moreover we notice that the combination $-N^2/8\pi\sigma\Delta\sigma$, due to the coupling $A\omega_\mu$, improves the central charge of the theory.

Exploring how this new term affects the standard Liouville dynamics is beyond the aim of the present paper, even if it seems reasonable to expect substantial changes in the quantum dynamics of gravity.

### 3.2 The torus

With a suitable choice of the coordinates and tangent frame we can always write the field $e^a_\mu$, describing a generic torus geometry, in the form $e^a_\mu = e^\sigma\hat{e}^a_\mu$ where

$$e^a_\mu = e^\sigma\hat{e}^a_\mu = e^\sigma \begin{pmatrix} \tau_2 & \tau_1 \\ 0 & 1 \end{pmatrix},$$

(3.37)

$\tau = \tau_1 + i\tau_2$ being the Teichmuller parameter. In eq. (3.37) the fundamental region for the coordinates has taken to be the square $0 \leq x^1, x^2 < 1$, while $\sigma$ is the conformal factor. The ensuing metric is obviously $g_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu = e^{2\sigma}\delta_{ab}\hat{e}^a_\mu \hat{e}^b_\nu = e^{2\sigma}\hat{g}_{\mu\nu}$. Let us remark that, at variance with the case of the sphere, the background metric is flat.

The presence of the Teichmuller parameter is not the only new feature of the torus. In fact, due to the existence of non trivial cycles, it admits more than one spin structure. Such structures correspond to the four possible boundary conditions, which one can choose for the fermions, when $x^1 \rightarrow x^1 + 1$ or $x^2 \rightarrow x^2 + 1$, namely $(A,A)$, $(P,A)$, $(A,P)$ and $(P,P)$, where $P(A)$ stands for periodic (antiperiodic). Here we will be only concerned with the $(A,A)$ boundary conditions even if the invariance under global diffeomorphism would requires to sum over all of them. By the way, this choice will allow a finite-temperature interpretation of our results.

We have seen in the previous subsection how the volume potential can be obtained from a connection that belongs to a non-trivial $U(1)$-bundle over the Riemann surface and solves the
Maxwell equations. Let us start therefore with discussing the structure of the non-trivial $U(1)$-bundle on a two-dimensional compact euclidean torus. The same procedure we have followed for the two-sphere leads to write the relevant gauge-connection (in a single patch)

$$A^{(k)}_{\mu} = \hat{A}^{(k)}_{\mu} - \pi k n^\nu \partial_\nu \lambda + 2\pi \alpha^{(k)}_{\mu},$$

where $\alpha^{(k)}_{\mu}$ is a flat connection (i.e. an harmonic one form $\epsilon^{\mu\nu} \nabla_\mu \alpha^{(k)}_{\nu} = 0$), while $\lambda$ is defined as in eq. (3.19).

We notice that on the torus, at variance with the genus-zero case, we have family of gauge inequivalent solutions of the Maxwell equations, labelled by the choice of the flat connection $\alpha^{(k)}_{\mu}$ in eq. (3.38). It cannot be removed by a well-defined gauge transformation: we are only allowed to perform

$$\alpha_{\mu} \rightarrow \alpha_{\mu} + n_{\mu}, \quad n_{\mu} = (n_1, n_2)$$

$n_1, n_2 \in \mathbb{Z}$, using as gauge function $g = \exp[i 2\pi n_\mu x^\mu]$, that is single-valued on the torus. These transformations are not connected to the identity and are usually called large gauge transformations. In the standard $U(1)$-gauge theory, where gauge invariance is respected, eq. (3.39) implies that the flat-connections $\alpha_{\mu}$ take values in the real interval $[0, 1]$. The existence of gauge inequivalent solutions to the Maxwell equations reflects, in our case, in the appearance of a new (global) degree of freedom inside the volume-potential.

The field $\hat{A}^{(k)}_{\mu}$ in eq. (3.38) is the instanton computed in the background given by $\hat{g}_{\mu\nu}$. In this way $\lambda$ carries the dependence on the conformal factor, while $\hat{A}^{(k)}_{\mu}$ depends only on the moduli space, described by $\tau$. On the torus $\hat{A}^{(k)}_{\mu}$, modulo gauge transformations, is

$$\hat{A}^{(k)}_{\mu} = -\pi k \sqrt{\hat{g}} \epsilon_{\mu\nu} x^\nu = -\pi k \epsilon_{\mu\nu} x^\nu.$$  

(3.40)

Being a connection with a non-vanishing winding number, $A^{(k)}_{\mu}$ has non-trivial boundary conditions after a homology cycle. Explicitly they read

$$A^{(k)}_{\mu}(x^1 + 1, x^2) = A^{(k)}_{\mu}(x^1, x^2) + \partial_\mu \left( \pi k x^2 \right)$$

(3.41.a)

$$A^{(k)}_{\mu}(x^1, x^2 + 1) = A^{(k)}_{\mu}(x^1, x^2) - \partial_\mu \left( \pi k x^1 \right).$$

(3.41.b)
As in the $S^2$ case we have:

$$ a_\mu = \frac{BV}{2\pi} A_\mu^{(1)}. $$

As before $a_\mu$ inherits non-trivial boundary conditions after a homology cycle from the instanton field. On the other hand the Euler number of the torus is zero, therefore the tangent-bundle is trivial and no global issue seems to be raised by $\omega_\mu$.

We observe that, due to the presence of large gauge transformations, the parameter $B$ has to respect two constraints in order to satisfy the global definition of the theory: $g = \exp[iBV g^{\mu\nu} n_\mu x_\nu]$ has to be well defined on the torus, while the now $B$-dependent transition functions $g_{ij}$ for the Dirac operator, between different patches $U_i, U_j$, must obey to the consistency condition $g_{ij} = g_{ik} g_{kj}$ \[\text{[13]}\]. One can explicitly check that both requirements are satisfied by choosing

$$ \frac{BV}{2\pi} = N, \quad (3.43) $$

As a consequence of the relation between the instanton bundle and the volume form $a_\mu$, the spinor-bundle also possesses non-trivial boundary conditions after the an homology cycle, we have in fact

$$ \psi(x^1 + 1, x^2) = \exp\left(i\pi(1 + Nx^2)\right) \psi(x^1, x^2), \quad (3.44.a) $$

$$ \psi(x^1, x^2 + 1) = \exp\left(i\pi(1 - Nx^1)\right) \psi(x^1, x^2), \quad (3.44.b) $$

where the antiperiodic boundary conditions has been taken into account. [ Let us warn the reader that this feature will have some subtle implications at the quantum level as we shall see in the next sections.]

We end the subsection by remarking that in the case of the torus, for $B \neq 0$, the Dirac operator has always zero-modes: due to the triviality of the tangent-bundle we cannot match the effect
of the $A$-coupling with the $B$-coupling to have a partition function different from zero. Thus the theory on the torus will always produce, for $B \neq 0$, Green function in the chirality-violating sector, in strict analogy with Bardacki-Crescimanno model [14]. For $B = 0$, i.e. only the $A\omega_\mu$ coupling is present, we have a partition function different from zero. However it is the standard Liouville action with an improved central charge given by $c = 1 - 12A^2$.

Some comments about the possible extension of the previous analysis to higher genus are now in order. It is clear that the procedure to construct the explicit expression for the volume form is independent of the genus. In fact it relies only upon some general properties of the $U(1)$–bundle. Moreover in this language the quantization rules for the $(A,B)$–coupling expresses the fact that the Chern-class associated to the connection $A\omega_\mu + Ba_\mu$ must be an integer:

$$\frac{A}{2\pi} \int d^2x \epsilon^{\mu\nu} \partial_\mu \omega_\nu + \frac{B}{2\pi} \int d^2x \epsilon^{\mu\nu} \partial_\mu a_\nu = (2 - 2g)A + \frac{BV}{2\pi} = N, \quad (3.45)$$

where $g$ is the genus of the manifold. For $g = 0$ and 1, this condition reduces to the ones found above.

4 The abelian Thirring model with the non minimal couplings

We are now ready to discuss the dynamical consequences of the non-minimal gravitational couplings in a specific interacting fermionic model, namely the Abelian Thirring model [15]. In curved space the theory has received a certain amount of attentions in the past years: stringy-like applications (the so-called Thirring strings [16]) were discussed in connection with a spontaneous symmetry-breaking mechanism. The partition function for the minimal-coupled model was computed on a generic Riemann surface in [12] and on a cylinder with twisted boundary condition [17]. A formal computation of the propagator was also performed in [18]. In the following we restrict ourselves to the case of the torus, generalizing the results of [12] and
to the non-minimal coupled case in genus one where new features arise. Actually it is not difficult to extend our calculations in higher genus, by using the well-known properties of theta-functions on Riemann surfaces, but we have chosen to limit ourselves to the torus case for a double reason: the computations are more explicit without loss of physical consequences and the finite temperature interpretation of the theory, after a partial decompactification, is also possible. In this sense our results can be related with the ones presented in [19], where gauged and un-gauged Thirring models on the torus were studied.

On a two-dimensional euclidean torus the Abelian Thirring model with non-minimal gravitational couplings is characterized by the classical Lagrangian

$$L = \bar{\psi} \gamma^\mu (i \nabla_\mu + A \omega_\mu + a_\mu) \psi + \frac{g^2}{4} \bar{\psi} \gamma^\mu \psi \gamma_\mu \psi - \epsilon^{\mu \nu} \rho (\partial_\mu a_\nu - \frac{B}{2} \epsilon_{ab} \epsilon^a_\mu \epsilon^b_\nu),$$

(4.46)

where the couplings of the new interactions are subject to the constraints (3.43).

An action quadratic in the spinor field, which is more suitable for the path integral formalism, can be achieved by introducing a vector field $H_\mu$. The lagrangian now reads

$$L = \bar{\psi} \gamma^\mu (i \nabla_\mu + A \omega_\mu + a_\mu - gH_\mu) \psi + g^{\mu \nu} H_\mu H_\nu - \epsilon^{\mu \nu} \rho (\partial_\mu a_\nu - \frac{B}{2} \epsilon_{ab} \epsilon^a_\mu \epsilon^b_\nu).$$

(4.47)

In the Lagrangian (4.47) the fermionic sector is now also invariant under the gauge transformation

$$\psi(x) \rightarrow u(x) \psi(x) \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) u^{-1}(x) \quad \text{and} \quad H_\mu(x) \rightarrow H_\mu(x) + u^{-1}(x) \partial_\mu u(x),$$

(4.48)

where $u(x)$ is an element of $U(1)$. This symmetry is, however, explicitly broken by the quadratic piece in $H_\mu(x)$. As a matter of fact $H_\mu$ is not a gauge connection, and consequently its harmonic piece $h_\mu$ is not restricted to take values in $[0, 1]$.

The quantum version of the abelian Thirring model in the presence of the new gravitational couplings is defined by the following generating functional

$$Z[\eta, \bar{\eta}] = \int \mathcal{D} \rho \mathcal{D} a_\mu \mathcal{D} \bar{\psi} \mathcal{D} \psi \mathcal{D} H_\mu \exp \left[ \int d^2 x \det(e) \left( \bar{\psi} \gamma^\mu (i \nabla_\mu + A \omega_\mu + a_\mu - gH_\mu) \psi + g^{\mu \nu} H_\mu H_\nu - \epsilon^{\mu \nu} \rho (\partial_\mu a_\nu - \frac{B}{2} \epsilon_{ab} \epsilon^a_\mu \epsilon^b_\nu) \right) + + \bar{\eta} \exp \left( i \int_{\gamma_{Ox}} a_\mu dx^\mu \right) \psi + \bar{\psi} \exp \left( -i \int_{\gamma_{Ox}} a_\mu dx^\mu \right) \eta \right],$$

(4.49)
being \( \eta, \bar{\eta} \) the fermionic sources and \( \gamma_{Ox} \) the geodesic connecting \( x \) with a given reference point \( O \).

Note that the requirement that correlation functions be single-valued when changing patches has forced us to introduce an extra phase factor in the coupling between the external current and the fermionic field in eq. (4.49). The presence of this additional contribution takes care of the non trivial boundary conditions that the spinor-bundle possess as a consequence of the fact that \( a_\mu \) is a connection with a non vanishing winding number. [ See the subsec. about the torus.] Moreover the correlation functions will be automatically invariant under the \( U(1) \) transformations.

A puzzling feature might be, obviously, the dependence of our results on the reference point \( O \) and the curve \( \gamma_{Ox} \), used to define the generating functional (4.49). As we shall see in the following, though this dependence exists, it is quite trivial and does not affect the physical content of the theory.

Performing the integral over \( \rho \) the field \( a_\mu \) in eq. (4.49) is replaced by the expression (3.42). Moreover the integration over \( a_\mu \) reduces to an ordinary integral over all the gauge inequivalent flat connections \( \alpha_\mu \), which parameterizes the moduli space of the solutions of the constraint (2.6) (for details see the previous subsection). Therefore the generating functional now reads

\[
Z[\eta, \bar{\eta}] = \int_0^1 d\alpha_\mu \int D\bar{\psi} D\psi D H_\mu \exp \left[ \int d^2 x \det(e) \left( \bar{\psi} \gamma^\mu [i \nabla_\mu + A_\omega_\mu + N A_\mu^{(1)} - g H_\mu] \psi 
+ g^{\mu\nu} H_\mu H_\nu \right) \bar{\eta} \exp \left( -iN \int_{\gamma_{Ox}} A^{(1)}_\mu(x) dx^\mu \right) \eta \right], \tag{4.50}
\]

where we have used that \( BV/2\pi \) is an integer \( N \) and the Landau gauge for the field \( a_\mu \)

We are ready now to evaluate the fermionic integral in eq. (4.50). In the following we shall use

\[A^{(1)} \text{Actually one should introduce an analogous phase depending on the spin connection } \omega_\mu. \text{ This additional phase would take care of the possible Lorentz transformation when one changes the patch. Since geometry is a fixed background, this extra contribution is only an overall factor, which can be disregarded for the moment and eventually restored at the end of the computation.}\]
a compact notation to avoid cumbersome expressions. The symbol $\mathcal{D}^{(N)}$ will stands for

$$\gamma^{\mu} \left( i\nabla^{\mu} + A_{\mu} + N A^{(1)}_{\mu} - g H_{\mu} \right),$$

while $\eta_a (\bar{\eta}_a)$ will denote the external currents dressed by the corresponding phase factor. In addition we shall introduce a new operator $S^{(N)}$ defined by

$$S^{(N)} \circ \mathcal{D}^{(N)} = \mathcal{D}^{(N)} \circ S^{(N)} = \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}}.$$

Here $P_{\text{Ker } \mathcal{D}^{(N)}}$ is the self-adjoint projector on the Kernel of the operator $\mathcal{D}^{(N)}$. Recall, in fact, that the Dirac operator has zero modes for $N \neq 0$; as matter of fact $\dim(\text{Ker } \mathcal{D}^{(N)}) = |N|$ (we are essentially in an instanton background). In terms of an orthonormal basis $\Psi^{(N)}_n(x)$ for $\text{Ker } \mathcal{D}^{(N)}$, an explicit expression for such operator is given by

$$P_{\text{Ker } \mathcal{D}^{(N)}} = \sum_n \Psi^{(N)*}_i(y) \Psi^{(N)}_i(x).$$

In the following we shall focus our attention on the zero modes appearing in the fermionic integrations. This issue requires some caution in order to get a sensible result [21]. We begin by performing the change of variables

$$\psi(x) \to \chi(x) - \left( \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}} \right) \circ S^{(N)} \circ \left( \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}} \right) \circ \eta_a,$$

$$\bar{\psi}(x) \to \bar{\chi}(x) - \bar{\eta}_a \circ \left( \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}} \right) \circ S^{(N)} \circ \left( \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}} \right),$$

in the path integral (4.50). After some cumbersome algebra, our generating functional takes the form

$$Z[\eta, \bar{\eta}] = \int_0^1 d\alpha \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \mathcal{D}H \exp \int d^2 x \det(e) \left( \bar{\chi} \mathcal{D}^{(N)} \chi + g^{\mu\nu} H_{\mu} H_{\nu} - \bar{\eta}_a \circ \left( \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}} \right) \circ \left( \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}} \right) \circ \eta_a \circ \left( \mathbb{1} - P_{\text{Ker } \mathcal{D}^{(N)}} \right) \circ \chi + \bar{\eta}_a \circ P_{\text{Ker } \mathcal{D}^{(N)}} \circ \eta_a \right).$$

(Note that the current are now decoupled from the fermion fields except the zero modes). The Berezin integration over the fermionic degree of freedom produces

$$Z[\eta, \bar{\eta}] = \int_0^1 d\alpha \int \mathcal{D}H \det^{'}(\mathcal{D}^{(N)}) \exp \int d^2 x \det(e) \left( g^{\mu\nu} H_{\mu} H_{\nu} \right).$$
\[-\eta_a \circ (\mathbb{1} - P_{\text{Ker} \mathcal{P}(N)}) \circ S^{(N)} \circ (\mathbb{1} - P_{\text{Ker} \mathcal{P}(N)}) \circ \eta_a) \right] \times \prod_{n=1}^{N} \left[ \int dx^2 \det(e) \Psi^{(N)}_{n}(x) \eta_a(x) \right] \left[ \int dx^2 \det(e) \bar{\eta}_a(x) \Psi^{(N)}_{n}(x) \right]; \quad (4.56)

\[
\det '(\mathcal{P}(N)) \text{ is the (regularized) product of the non-vanishing eigenvalues. Our next step will be accomplished with the help of the Hodge decomposition for } H_{\mu}, \text{ namely}
\]
\[H_{\mu}(x) = \partial_{\mu} \phi_1(x) - \eta_{\mu}^\nu \partial_{\nu} \phi_2(x) + 2\pi h_{\mu}. \quad (4.57)\]

In fact we can use eq.(4.57) and the explicit expression for \(A_{\mu}^{(1)}\) to obtain
\[
\hat{\mathcal{P}}^{(N)} = \exp \left[ ig \phi_1 - \frac{3\sigma}{2} - \gamma_5 (g \phi_2 + A\sigma - \pi N\lambda) \right] \hat{\mathcal{P}}^{(N)} \exp \left[ -ig \phi_1 + \frac{\sigma}{2} - \gamma_5 (g \phi_2 + A\sigma - \pi N\lambda) \right],
\]
with
\[\hat{\mathcal{P}}^{(N)} = \gamma_{\mu} \left( i \partial_{\mu} + N\hat{A}_{\mu}^{(1)} + (N\alpha_{\mu} - gh_{\mu}) \right). \quad (4.59)\]

An orthonormal basis of the zero modes of the operator \(\hat{\mathcal{P}}^{(N)}\) is related to the analogous one for \(\mathcal{P}^{(N)}\) by
\[
\Psi^{(N)}_{i} = \exp \left[ ig \phi_1 - \frac{\sigma}{2} + \gamma_5 (g \phi_2 + A\sigma - \pi N\lambda) \right] \sum_{j=1}^{N} C_{ij} \Psi^{(N)}_{j} \quad (4.60)
\]
where we have used a \(N \times N\) matrix \(C\) to have an orthonormal basis also for the null space of \(\hat{\mathcal{P}}^{(N)}\). We can now compute \(\det '(\mathcal{P}(N))\): \(\mathcal{P}(N)\) is a differential elliptic operator acting on the sections of a vector-bundle over a compact manifold, and we can therefore use the \(\zeta\)-function definition \([20]\) of the determinant to get, with a straightforward application of the technique introduced in \([21]\), the result
\[
\det '(\mathcal{P}(N)) = \det '(\hat{\mathcal{P}}^{(N)}) \exp\left[ -S^{(N)}_{\text{Loc.}} \right] \left| \det(C) \right|^{-2}. \quad (4.61)
\]

We have three different contributions: a “global” one, coming from the instanton and the flat connections, contained in \(\det '(\hat{\mathcal{P}}^{(k)})\), that will be computed by brute force in the following; an
highly non-linear and non-local part, \(|\det(C)|^{-2}\), linked to the zero-modes subtraction; and finally a local bosonic action \(S_{\text{Loc.}}^{(N)}(\phi_1, \phi_2, \lambda, \sigma)\)

\[
S_{\text{Loc.}}^{(N)} = S_{\text{Liou.}}(\sigma) - \int d^2 x \det(e) \left[ \frac{g^2}{2\pi} \phi_2 \Delta \phi_2 + \frac{\pi}{2} (N\lambda - \frac{A}{\pi} \sigma) \triangle (N\lambda - \frac{A}{\pi} \sigma) - g\phi_2 \Delta (N\lambda - \frac{A}{\pi} \sigma) \right].
\]  

(4.62)

In order to manage the source-dependent part we introduce the operator \(\hat{\mathcal{P}}^{(N)}\) defined by the equation

\[
\hat{\mathcal{P}}^{(N)} \circ \hat{S}^{(N)} = \hat{S}^{(N)} \circ \hat{\mathcal{P}}^{(N)} = \mathbb{I} - P_{\text{Ker } \hat{\mathcal{P}}^{(N)}},
\]

where \(P_{\text{Ker } \hat{\mathcal{P}}^{(N)}}\) is the self-adjoint projector on the Kernel of \(\hat{\mathcal{P}}^{(N)}\), one can prove that [21]

\[
\exp \left[ \int d^2 x \det(e) \left( -\tilde{\eta}_a \circ (\mathbb{I} - P_{\text{Ker } \hat{\mathcal{P}}^{(N)}}) \circ S^{(N)} \circ (\mathbb{I} - P_{\text{Ker } \hat{\mathcal{P}}^{(N)}}) \circ \eta_a \right) \right] \times \\
\Pi_i \left[ \int dx^2 \det(e) \Pi_i^{(N)\dagger}(x) \eta_a(x) \right] \left[ \int dx^2 \det(e) \Pi_i(x) \hat{\mathcal{P}}^{(N)}(x) \right] = \\
\Pi_i \left[ \int dx^2 \det(\hat{e}) \Pi_i^{(N)\dagger}(x) \eta_a'(x) \right] \left[ \int dx^2 \det(\hat{e}) \Pi_i(x) \hat{S}^{(N)} \right] \times \\
\exp \left[ -\int d^2 x \det(\hat{e}) \tilde{\eta}_a' \circ (\mathbb{I} - P_{\text{Ker } \hat{\mathcal{P}}^{(N)}}) \circ \hat{S}^{(N)} \circ (\mathbb{I} - P_{\text{Ker } \hat{\mathcal{P}}^{(N)}}) \circ \eta_a' \right] |\det(C)|^2
\]

(4.64)

Here

\[
\eta_a' = \exp \left[ -ig\phi_1 - \frac{\sigma}{2} + \gamma_5 (g\phi_2 + A\sigma - \pi N\lambda) \right] \eta_a
\]

(4.65)

\[
\tilde{\eta}_a = \eta_a \exp \left[ ig\phi_1 - \frac{\sigma}{2} + \gamma_5 (g\phi_2 + A\sigma - \pi N\lambda) \right].
\]

(4.66)

Thus the generating functional turns out to be:

\[
Z[\eta, \bar{\eta}] = 2\pi \frac{\exp[-S_{\text{Liou.}}(\sigma)]}{\det ' (\Delta)^2} \int D\phi'_1 D\phi'_2 \exp \left[ \int d^2 x \det(e) \left( (1 + \frac{g^2}{2\pi}) \phi_2 \Delta \phi_2 + \phi_1 \Delta \phi_1 \right) + \frac{\pi}{2} (N\lambda - \frac{A}{\pi} \sigma) \triangle (N\lambda - \frac{A}{\pi} \sigma) - g\phi_2 \Delta (N\lambda - \frac{A}{\pi} \sigma) \right] \int_0^1 d\alpha \int_{-\infty}^{+\infty} dh \exp[-\sqrt{g} \tilde{\eta}_a \mu h \nu] \times \\
\det ' (\hat{\mathcal{P}}^{(N)}) \Pi_i \left[ \int dx^2 \det(\hat{e}) \Pi_i^{(N)\dagger}(x) \eta_a'(x) \right] \left[ \int dx^2 \det(\hat{e}) \Pi_i(x) \hat{S}^{(N)} \right] \\
\exp \left[ -\int d^2 x \det(\hat{e}) \tilde{\eta}_a' \circ (\mathbb{I} - P_{\text{Ker } \hat{\mathcal{P}}^{(N)}}) \circ \hat{S}^{(N)} \circ (\mathbb{I} - P_{\text{Ker } \hat{\mathcal{P}}^{(N)}}) \circ \eta_a' \right].
\]

(4.67)
The prime on the measure means that the functional integration must be carried out only over
the non constant modes of $\phi_i$ and $2\pi \det'(\triangle)^{-1}$ is the Jacobian of the change of variables from
$H_\mu$ to $(\phi_1, \phi_2, h_\mu)$. 

Some remarks about the expression (4.67) for the generating functional are now in order. Firstly
the highly non local contribution, due to $|\det(C)|^2$, disappears from the final expression for
the generating functional preserving the solvability of the model. Even though this result might
be expected from [21], it represents a non trivial check of our computation. Secondly, from
eq (4.67), it is transparent that the only possibility for a non vanishing partition function
($\mathcal{Z} \equiv [0,0]$) is $N = 0$. Finally, from eq. (4.67) it is straightforward to realize that only
correlation functions of the form (3.33) are different from zero. In fact, because of the fixed
chirality of the zero modes appering in eq. (4.67), an asymmetry between the number of left
and right current is produced. This, in turn, will entail the asymmetry in the Green functions.

From eq.(4.67) one can derive all the fermionic (gauge-invariant) correlation functions of the
theory and by means of appropriate limiting procedure the correlators of the relevant composite
operators.

**5 The $N = 0$ theory**

We start by considering the theory with $N = 0$: we find that the central charge is modified not
only because of the presence of the A-coupling but an extra $g^2$-dependence appears through
the Thirring interaction. This fact is not a priori expected, since , in the minimal-coupled case,
the central charge is independent of $g^2$. We also derive the explicit form of the propagator: this
computation could be relevant by itself because, at least at our knowledge, does not appear in
the literature an explicit calculation of the massless Thirring model’s two-point function on the
torus. From this computation is not also difficult to obtain the finite-temperature propagator,

\footnote{As usual the symbol $\det'$ means that the zero eigenvalue is excluded.}
decompactifying the $x^1$-direction.

## 5.1 The partition function

The absence of $B$-coupling simplifies the expression for the generating functional (4.67): no zero-modes are present and the partition function does not vanish. Switching off the source terms in eq. (4.67) we have the following factorization for the partition function

$$Z = \exp[-S_{\text{Liou.}}(\sigma)]2\pi \det ' (\Delta)^{-1} \int D(\phi_1' \phi_2') \exp \left[ \int d^2x \text{det}(e) \left( (1 + \frac{g^2}{2\pi})\phi_2 \Delta \phi_2 + \phi_1 \Delta \phi_1 + \frac{A^2}{2\pi} \sigma \Delta \sigma + g \frac{A}{\pi} \phi_2 \Delta \sigma \right) \right] \left[ \int_{-\infty}^{+\infty} dh_\mu \exp[-\sqrt{\tilde{g}}g^{\mu\nu}h_\mu h_\nu] \det (\tilde{\Phi}^{(0)}) \right]$$

$$\equiv Z_{\text{local}} \times Z_{\text{global}}.$$  \hspace{1cm} (5.68)

From eq. (5.68) it is also clear that the partition functions splits into the product of two factors, which can be computed separately. The former, $Z_{\text{local}}$, depends only on the fields $\phi_i(x)$, while the latter, $Z_{\text{global}}$, depends only on the harmonic form $h_\mu$.

### Computation of $Z_{\text{local}}$

If we introduce the vector notation $\vec{\Phi}(x) \equiv (\phi_1(x), \phi_2(x))$, $Z_{\text{local}}$ takes the form

$$Z_{\text{local}} = \exp \left( -S_{\text{Liou.}}(\sigma) + \frac{A^2}{2\pi} \int d^2x \text{det}(e)\sigma \Delta \sigma \right)$$

$$\int D\vec{\Phi}' \exp \left( \int d^2x \text{det}(e) \left[ -\frac{1}{2} \vec{\Phi}' O \vec{\Phi} + \vec{\Phi}' \vec{J}(x) \right] \right) 2\pi \det'(\Delta)$$

where $O$ is the following operator

$$O = \Delta \begin{pmatrix} \frac{g}{2\pi} & 0 \\ 0 & -2 \end{pmatrix},$$

and the source $\vec{J}(x)$ is

$$\vec{J}(x) = \left( \frac{Ag}{\pi} \Delta \sigma, 0 \right).$$

(5.70)

(5.71)

The functional integral is gaussian and is easily performed. We obtain

$$Z_{\text{local}} = 2\sqrt{1 + \frac{g^2}{2\pi}} \exp \left[ -\left( 1 - 12 \frac{A^2}{1 + \frac{g^2}{2\pi}} \right) S_{\text{Liou.}}(\sigma) \right],$$

$$\equiv Z_{\text{local}} \times Z_{\text{global}}.$$
where the factor $2\sqrt{1 + \frac{g^2}{2\pi}}$ derive from the scaling formula for the primed determinant of $\triangle^{[22]}$.

The local part of the partition function confirmes and extends the result of $[3]$: the $A$-coupling changes the value of the central charge of the fermionic theory, the coefficient of the Liouville action. A new feature arises due to the presence of the current-current interaction, namely the central charge now depends non-trivially on the coupling costant $\frac{g^2}{2\pi}$. However it displays a pole for $g^2/2\pi = -1$. The origin of such a singularity can be traced back to the well-known loss of unitarity of the Thirring model when the coupling constant goes through this value.

Finally let us speculate on the possibility of giving a fermionic representation of the minimal series by exploiting eq. (5.72). It is clear that it is possible to get the correct central charge by tuning $g$ and $A$. However the question of which screening operators must be introduced to obtain the correct Green functions is not evident.

— **Computation of $Z_{\text{global}}$**

We come now to the evaluation of the $Z_{\text{global}}$, which is given by

$$Z_{\text{global}} = \int_{-\infty}^{\infty} dh_\mu \exp \left( -4\pi^2 \sqrt{\hat{g}^{\mu\nu} h_\mu h_\nu} \right) \det (\hat{D}^{(0)}).$$  

The determinant corresponding to the operator $\det (\hat{D}^{(0)}[h])$ can be computed by a explicit construction of the associated $\zeta-$function

$$\det[D] = \exp \left[ -\frac{d}{ds} \zeta_D(s) \right]_{s=0},$$

$$\zeta_D(s) = \sum_n \lambda_n^{-s} \deg(\lambda_n),$$

Actually, we shall evaluate the determinant of $\det (\hat{D}^{(0)})^2$ and then we shall extract the square root of such expression. The operator $\hat{D}^{(0)}$ is given by

$$-(\hat{\triangle} - 4\pi i \hat{g}^{\mu\nu} h_\mu \hat{\nabla}_\nu - 4\pi^2 g^2 \hat{g}^{\mu\nu} h_\mu h_\nu),$$

where $\hat{\triangle}$ and $\hat{\nabla}_\mu$ are the laplacian and the covariant derivative in the background metric $\hat{g}_{\mu\nu}$. Taking into account that $\hat{g}_{\mu\nu}$ is flat and independent of $x$, it is very easy to check that the
eigenfunctions of this operator are given by

$$\psi_p(x) = e^{-ip\mu x^\mu} \rho$$

(5.77)

where $\rho$ is a constant spinor. The ensuing eigenvalues are then obtained by substituting such expression in eq. (5.76)

$$\lambda_p = (p_\mu - 2\pi g h_\mu) \hat{g}^{\mu\nu} (p_\nu - 2\pi g h_\nu).$$

(5.78)

The degeneracy of each eigenvalue is two. In fact only two independent constant spinors $\rho$ exist.

The imposition of $(A, A)$ boundary conditions requires that $p_\mu$ belongs to the lattice $Z^2 + 1/2$, namely

$$p_\mu = n_\mu + \frac{1}{2} \quad \text{with} \quad n_\mu \in Z^2.$$  

(5.79)

The associated $\zeta-$function, which defines the determinant, can be now easily written down

$$\zeta(s) = 2 \left( \frac{\tau_2}{2\pi} \right)^{2s} \sum_{Z^2} \left[ \left( n_1 - gh_1 + \frac{1}{2} - \tau_1 \left( n_2 - gh_2 + \frac{1}{2} \right) \right)^2 + \tau_2^2 \left( n_2 - gh_2 + \frac{1}{2} \right)^2 \right]^{-s},$$

(5.80)

where we have used the explicit form of the metric to reduce the expression to this form. The factor 2 is due to the degeneracy of the eigenvalue.

The explicit computation of $\zeta'(0)$ is not difficult and it is given for more general non-hermitian operators in [23]. Here we shall only quote the final result, which can be expressed in terms of theta functions

$$\zeta'(0) = -2 \log \left( \frac{1}{|\eta(\tau)|^2} \Theta \left[ \begin{array}{c} gh_2 \\ -gh_1 \end{array} \right](0, \tau) \Theta^* \left[ \begin{array}{c} gh_2 \\ -gh_1 \end{array} \right](0, \tau) \right).$$

(5.81)

Thus the determinant for the global part is $\det (\hat{D}(0)[h]) = \exp(-\zeta'(0))$.

To determine $Z_{\text{local}}$ we have now to compute the following integral

$$Z_{\text{global}} = \int_{-\infty}^{\infty} dh_\mu \exp \left( -4\pi^2 \sqrt{\hat{g}^{\mu\nu} h_\mu h_\nu} \right) \frac{1}{|\eta(\tau)|^2} \Theta \left[ \begin{array}{c} gh_2 \\ -gh_1 \end{array} \right](0, \tau) \Theta^* \left[ \begin{array}{c} gh_2 \\ -gh_1 \end{array} \right](0, \tau).$$

(5.82)
The integral over $h_\mu$ can be performed expanding the theta functions as series of exponentials: we end up with a series of gaussian integrations whose result can be recast as theta function with characteristic [24]

\[
Z_{\text{global}} = \frac{1}{2\sqrt{1 + \frac{g^2}{2\pi}}} \Theta(0, \Lambda),
\]

where the covariance matrix $\Lambda$ is

\[
\Lambda = \begin{pmatrix}
t & 0 \\
0 & -\tau
\end{pmatrix} + i \frac{\tau g^2}{2(1 + \frac{g^2}{2\pi})} \begin{pmatrix}
\frac{g^2}{2\pi} & -2 - \frac{g^2}{2\pi} \\
-2 - \frac{g^2}{2\pi} & \frac{g^2}{2\pi}
\end{pmatrix},
\]

and

\[
\Theta(0, \Lambda) = \sum_{\vec{n} \in \mathbb{Z}^2} \exp[i\pi \vec{n} \cdot \Lambda \cdot \vec{n}].
\]

Actually to obtain this result we have to impose that $\frac{g^2}{2\pi} > -1$, which is the usual bound for the unitarity of the Thirring model [25].

The final expression for the partition function turns out to be

\[
Z = \exp \left[ - \left( 1 - 12 \frac{A^2}{1 + \frac{g^2}{2\pi}} \right) S_{\text{Liou.}}(\sigma) \right] \Theta(0, \Lambda),
\]

the only effect of the non-minimal coupling being contained in the Liouville sector.

## 5.2 The fermionic propagator

All the correlation functions of the Thirring model can be constructed in term of the building-block propagator, due to the quasi-free character of the theory, therefore we limit ourselves to the computation of the two-point function.

We use the notation:

\[
\bar{\psi} = (\bar{\psi}_R, \bar{\psi}_L), \quad \psi = \begin{pmatrix}
\psi_R \\
\psi_L
\end{pmatrix}.
\]

We can easily derive from the generating functional (4.67) the fermionic propagator:

\[
<\psi(x)\bar{\psi}(y)> = \begin{pmatrix}
S_{RR}(x,y) & S_{RL}(x,y) \\
S_{LR}(x,y) & S_{LL}(x,y)
\end{pmatrix};
\]

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Let us notice that for \( N = 0 \) the additional phase in the propagator disappear. However in the next section, where we shall address the same question in the sector \( N = \pm 1 \), the presence of the phase depending on \( a_\mu \) will play a fundamental role.

In absence of zero-modes of the Dirac operator, only \( S_{RL}(x, y) \) and \( S_{LR}(x, y) \) are different from zero and we get

\[
S_{LR,RL}(x, y) = \frac{1}{Z} \exp \left[ -S_{Liou.}(\sigma) - \frac{1 \pm 2A}{2} \sigma(x) - \frac{1 \mp 2A}{2} \sigma(y) \right] 2\pi \text{det}(\Delta) \int \mathcal{D}(\phi_1') \exp \left[ -S_{Loc.}[\phi_1] + ig (\phi_1(x) - \phi_1(y)) \pm g (\phi_2(x) - \phi_2(y)) \right] \int_{-\infty}^{+\infty} dh_\mu S_{LR,RL}^{G}(x, y) \exp \left[ -S_{Glob.}[h] \right].
\] (5.89)

We have normalized the propagator to the partition function (as usual) and we have introduced \( S_{Loc.}[\phi_1] \) and \( S_{Glob.}[h] \) that are respectively the local part and the global part of the Dirac determinant previously computed (we have also included the \( H_\mu H^\mu \) term for convenience)

\[
S_{Loc.}[\phi_1] = -\int d^2x \sqrt{g} \left[ (1 + \frac{g^2}{2\pi}) \phi_2 \Delta \phi_2 + \phi_1 \Delta \phi_1 + \frac{A^2}{2\pi} \sigma \Delta \sigma + \frac{gA}{\pi} \phi_2 \Delta \sigma \right],
\]

\[
S_{Glob.}[h] = -\log |\Theta \left[ \begin{array}{c} gh_2 \\ -gh_1 \end{array} \right](0, \tau)|^2 + \frac{4\pi^2}{\tau_2} ((h_1 - \tau_1 h_2)^2 + \tau_2^2 h_2^2). \] (5.90)

\( S_{LR}^{G}(x, y) \) is a generalization, to the present case, of the well-known Szego kernel \( (S_{RL}^{G}(x, y) = S_{LR}(x, y)^*) \)

\[
S_{LR}^{G}(x, y) = -\frac{1}{2\pi} \Theta \left[ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right](0, \tau) \Theta \left[ \begin{array}{c} gh_2 \\ -gh_1 \end{array} \right](z - w, \tau) \Theta \left[ \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \end{array} \right](z - w, \tau) \Theta \left[ \begin{array}{c} gh_2 \\ -gh_1 \end{array} \right](0, \tau) \exp[-2\pi ig (h_1(x^1 - y^1) + h_2(x^2 - y^2))].
\] (5.91)

and we have found it useful to introduce the complex coordinates

\[
z = (x^2 + \tau x^1), \quad w = (y^2 + \tau y^1),
\]

\[
\bar{z} = (x^2 + \bar{\tau} x^1), \quad \bar{w} = (y^2 + \bar{\tau} y^1).
\] (5.92)
The Szego kernels are linked to the Green function of $\hat{\mathcal{D}}^{(0)}$:

$$\hat{\mathcal{D}}^{(0)} S^G(x, y) = \frac{I}{\sqrt{g}} \delta^2(x - y)$$  \hspace{1cm} (5.93)

where

$$S^G(x, y) = \begin{pmatrix} 0 & S_{RL}^G(x, y) \\ S_{LR}^G(x, y) & 0 \end{pmatrix}. \hspace{1cm} (5.94)$$

The quadratic integration over $\phi$ to final expression for the whole propagator

$$\exp \left[ \pi \left( \frac{q^2}{2\pi} \right)^2 \left( \bar{G}_\triangle(x, y) - \bar{G}_\triangle(0, 0) \right) \right] \pm A \frac{\sqrt{2\pi}}{g^2} \left( \sigma(x) - \sigma(y) \right), \hspace{1cm} (5.95)$$

$\bar{G}_\triangle$ being the Green function of the laplacian $\triangle$ on the torus of metric $g_{\mu\nu}$, with the zero-modes projected out. The integration over the harmonic field $h_\mu$ is long but straightforward leading to final expression for the whole propagator

$$\langle \psi(x) \bar{\psi}(y) \rangle = \exp \left[ \pi \left( \frac{q^2}{2\pi} \right)^2 \left( \bar{G}_\triangle(x, y) - \bar{G}_\triangle(0, 0) \right) \right] \times$$

$$\left( \begin{array}{c} 0 \\ \hat{S}_{RL}^*(x, y) \exp \left( -\frac{1}{2} \left[ \left( 1 - \frac{2A}{1 + \frac{q^2}{2\pi}} \right) \sigma(x) - \left( 1 + \frac{2A}{1 + \frac{q^2}{2\pi}} \right) \sigma(y) \right] \right) \\ \hat{S}_{RL}^*(x, y) \exp \left( -\frac{1}{2} \left[ \left( 1 + \frac{2A}{1 + \frac{q^2}{2\pi}} \right) \sigma(x) - \left( 1 - \frac{2A}{1 + \frac{q^2}{2\pi}} \right) \sigma(y) \right] \right) \end{array} \right),$$

$$\hat{S}_{RL}(x, y) = -\frac{1}{2\pi} \Theta \left[ 0, \frac{1}{2} \right] (0, \tau) \Theta \left[ \frac{1}{2}, 1 \right] (z, \tau) \langle \bar{V}(z) - \bar{V}(w) \rangle (\Lambda, 0) \exp \left[ \frac{\pi}{8 \left( 1 + \frac{q^2}{2\pi} \right)^2} (z - w - \bar{z} + \bar{w})^2 \right],$$

$$\Lambda$$ is the covariance matrix eq. (5.84) and the vector $\bar{V}(z)$ is

$$\bar{V}(z) = \left( z + \frac{1}{4} \frac{(q^2)^2}{1 + \frac{q^2}{2\pi}} (z - \bar{z}), -\frac{q^2}{2\pi} (2 + \frac{q^2}{2\pi}) \frac{2}{1 + \frac{q^2}{2\pi}} (z - \bar{z}) \right). \hspace{1cm} (5.97)$$

To extract the singular behavior in eq. (5.99) we use the expansion for $\bar{G}_\triangle(x, y)$ as $x \rightarrow y$

$$\bar{G}_\triangle(x, y) = \bar{G}_\triangle(x, y) + (\lambda(x) + \lambda(y)) + \int \sqrt{g} \lambda \triangle \lambda + O(\delta(x, y)), \hspace{1cm} (5.99)$$

where $\lambda$ was defined in eq. (3.19), $\delta(x, y)$ is the geodesic distance and $\bar{G}_\triangle(x, y)$ the Green function for the flat torus, whose expansion at short distance is

$$\bar{G}_\triangle(x, y) \simeq \frac{1}{4} \log \left[ \mu^2 \hat{g}_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) \right] - \frac{1}{4\pi} \log \left[ \frac{4\pi^2 \tau^2 |\eta(\tau)|^4}{\mu^2 \bar{V}} \right] + O((x - y)^2). \hspace{1cm} (5.100)$$
Thus we define the renormalization wave functions

$$\sqrt{Z_{L,R}} \equiv \lim_{x \to y} \left( \frac{1}{4\pi} \frac{(g^2/\pi)^2}{1 + \frac{g^2}{2\pi}} \log \left[ \mu^2 \tilde{g}_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) \right] \right). \quad (5.101)$$

In the previous formulae \(\mu\) represent the subtraction point of our renormalization scheme and it corresponds to the usual mass-scale appearing in the operator solution of the Thirring model [26].

This wave function renormalization is equivalent to the renormalization of the fermion field in the flat-space Thirring model [25] and it is very much expected, being the ultraviolet behaviour independent of the topology of the space-time.

One can check that the propagator is single valued \((z \to z + 1\) and \(z \to z + \tau\) are the periodicity in the complex variables) and at short distance \((z \to 0)\) the behaviour on the plane is recovered. In fact one finds

$$\langle \bar{\psi}(x) \psi(y) \rangle \simeq |z - w|^{-\gamma} \begin{pmatrix} 0 & z - w \\ \bar{z} - \bar{w} & 0 \end{pmatrix}. \quad (5.102)$$

where \(\gamma = 1 + 1/2(1 + g^2/2\pi + 1/(1 + g^2/2\pi))\).

A further interesting development could be to investigate the gravitational dressing of the above propagator (and eventually of the four-points function): an explicit \(A\)-dependence appears in the coefficients of the conformal factor in eq.(5.96) and would be interesting to understand its influence on the conformal dimensions of the spinor field.

6 The \(N \neq 0\): the chirality violating theory

Let us now explore how the model changes when we turn on the \(B\)-coupling. To have explicit results, we shall only consider in detail the case \(N = \pm 1\), the other possible values can be handled in an analogous manner.

First of all let us recall that the partition function is now zero. However, using the generating functional eq.(4.67) and the fact that for \(N = \pm 1\) there is only one zero-mode (with respectively
different chirality), we derive that the simplest non-vanishing Green function is the two-point
one:

\[<\psi(x)\exp\left(i\int_{\gamma_{xy}} A^{(1)}_{\mu}dx^\mu\right)\bar{\psi}(y)>_N = \left(\begin{array}{cc}
S_{RR}(x, y)\delta_{N,-1} & 0 \\
0 & S_{LL}(x, y)\delta_{N,1}
\end{array}\right),\]  

(6.103)

where \(S_{LL}(x, y)\) and \(S_{RR}(x, y)\) are

\[S_{LL}(x, y) = 2\pi \det ' (\triangle)^{-1}\left[\int D(\phi'_1\phi'_2) \exp[-S^{(1)}_{\text{Loc}} + \int d^2x \det(e)(\phi_1\triangle\phi_1 + \phi_2\triangle\phi_2)]
\]

\[\exp\left(-ig(\phi_1(x) - \phi_1(y)) - g(\phi_2(x) + \phi_2(y)) + \pi(\lambda(x) + \lambda(y)) - \frac{1 + 2A}{2}(\sigma(x) + \sigma(y))\right)\]

\[\left[\int_0^1 d\alpha_1 \int_{-\infty}^{+\infty} dh_\mu \exp[-g^{\mu\nu}h_\mu h_\nu] \det ' (\hat{D}^{(1)})\hat{\Psi}^{(0)}_{1,0}(x)\hat{\Psi}^{(0)}_{1,0}(y) \exp[iS_{xy}(x, y; \alpha)]\right]\]  

(6.104)

\[S_{RR}(x, y) = 2\pi \det ' (\triangle)^{-1}\left[\int D(\phi'_1\phi'_2) \exp[-S^{(-1)}_{\text{Loc}} + \int d^2x \det(e)(\phi_1\triangle\phi_1 + \phi_2\triangle\phi_2)]
\]

\[\exp\left(-ig(\phi_1(x) - \phi_1(y)) + g(\phi_2(x) + \phi_2(y)) + \pi(\lambda(x) + \lambda(y)) - \frac{1 - 2A}{2}(\sigma(x) + \sigma(y))\right)\]

\[\left[\int_0^1 d\alpha_1 \int_{-\infty}^{+\infty} dh_\mu \exp[-g^{\mu\nu}h_\mu h_\nu] \det ' (\hat{D}^{(-1)})\hat{\Psi}^{(0)}_{0,1}(x)\hat{\Psi}^{(0)}_{0,1}(y) \exp[-iS_{xy}(x, y; \alpha)]\right]\]  

(6.105)

Here \(S^{(\pm1)}_{\text{Loc}}\) is the same action in eq.(4.62) for \(N = \pm1\), \(\exp[iS_{xy}(x, y; \alpha)]\) is the phase factor introduced in eq.(6.103) to preserve the global definition of the Green’s functions and \(\hat{\Psi}^{(0)}_{\pm1,0}(x)\) are the zero modes of the Dirac operator \(\hat{D}^{(\pm1)}\), explicitly derived in the Appendix.

By comparing this expression with eqs. (5.88) and (5.89), the main difference with the \(N = 0\) theory is manifest: the tensorial structure is completely different, namely we have a diagonal propagator instead of an antidiagonal one. Moreover there is an explicit dependence from the field \(\lambda\), that in the previous theory \((N = 0)\) was introduced only through the renormalization.

First of all we compute the global phase

\[S_{xy}(x, y; \alpha) = \int_{\gamma_{xy}} A^{(1)}_\mu dx^\mu.\]  

(6.106)

A direct computation is too lengthy, so we proceed by a means of a trick. We introduce the straight line \(r_{xy}\) connecting \(x\) with \(y\) and the closed circuit \(C\) formed by \(\gamma_{xOy}\) and \(r_{xy}\); then,
with the help of these two paths, we rewrite the phase as follow

\[
S_{x,y}(x, y; \alpha) = \oint_C A^{(1)}_\mu \, dx^\mu + \int_{r_{xy}} A^{(1)}_\mu \, dx^\mu. \tag{6.107}
\]

The first term is evaluated by using the Stokes theorem and it gives the area enclosed by \(C\). The second one can be almost explicitly computed by means of the explicit expression for \(A^{(1)}_\mu\).

At the end, combining the two results, we have

\[
S_{x,y}(x, y; \alpha) = \text{Area}_C + \int_0^1 dt \eta_\mu(y^\mu - x^\mu) \partial_\nu \lambda(ty + (1 - t)x) - \]

\[
i \left[ \bar{\alpha}(\bar{z} - \bar{w}) + \alpha(z - w) \right] + i \frac{\pi}{2\tau_2} (z\bar{w} - \bar{z}w),
\]

where we have introduced the notation

\[
\alpha = \frac{\pi}{i\tau_2} (\alpha_1 - \bar{\tau}\alpha_2)
\]

\[
h = \frac{\pi}{i\tau_2} (h_1 - \bar{\tau}h_2)
\]

\[
\bar{\alpha} = \alpha^* \ , \ \bar{h} = h^*
\]

and the holomorphic coordinate \(z\) and \(w\).

The computation of (6.104) and (6.105) is again performed in two different steps, one for the quadratic integration over the local degrees of freedom and the other for the more involved finite-dimensional integrals over \(\alpha\) and \(h\).

The quadratic integration over \(\phi_1, \phi_2\) is standard: it gives a contribution to \(S_{LL}(x, y)\) equal to

\[
S_{LL}^{\text{Loc.}}(x, y) = 2\sqrt{1 + \frac{g^2}{2\pi}} \exp \left[ -S_{\text{Liou.}}(\sigma) + \frac{\pi}{2} + \frac{1}{\pi} \left( \lambda(x) - \frac{A}{\pi} \sigma(x) + \lambda(y) - \frac{A}{\pi} \sigma(y) \right) \right] \exp \left[ \pi \left( 1 + \frac{g^2}{2\pi} - \frac{1}{1 + \frac{g^2}{2\pi}} \right) G_\Delta(x, y) - \right.
\]

\[
\frac{1}{1 + \frac{g^2}{2\pi}} \bar{G}_\Delta(x, x) + \frac{\pi}{2} - \frac{1}{1 + \frac{g^2}{2\pi}} \int d^2 x \det(e)(\lambda - \frac{A\sigma}{\pi}) \Delta(\lambda - \frac{A\sigma}{\pi}) \right]
\]

\[
\tag{6.110}
\]

while \(S_{RR}^{\text{Loc.}}(x, y)\) is obtained by changing \(A\) in \(-A\). One can check that the ultraviolet renormalization constant \(Z_{L,R}\) previously introduced automatically makes the correlation function finite.
The computation of the global part $S^{\text{Glob}}_{LL}(x, y)$, $S^{\text{Glob}}_{RR}(x, y)$ is more involved: first of all we need \( \det'(\hat{\mathcal{D}}^{(\pm)}[\alpha, h]) \). The eigenvalues and their degeneration are derived in the Appendix:

\[
(\lambda_n)^2 = 2n|\Phi|, \quad |\Phi| = \frac{2\pi}{V}
\]

with degeneration 2. Using $\zeta$-function definition we get

\[
\det'(\hat{\mathcal{D}}^{(\pm)}[\alpha, h]) = (\frac{\pi}{\Phi})^\frac{3}{2}.
\]

We notice that this determinant does not depend $\alpha_\mu$ and $h_\mu$: this is no longer true for the zero-modes as one may notice looking at their explicit functional form presented in the Appendix.

The integral we have to perform is (we restrict ourselves for the moment to $N = 1$)

\[
S^{\text{Glob}}_{LL}(x, y) = \sqrt{\frac{2}{\tau_2}} \int_0^1 d\alpha_1 d\alpha_2 \exp \left[ -\frac{4\pi^2}{\tau_2} (\alpha_1, \alpha_2) \begin{pmatrix} 1 & -\tau_1 \\ -\tau_1 & |\tau|^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right] \int_{-\infty}^{+\infty} dh_1 dh_2 \exp \left[ -\frac{4\pi^2}{\tau_2} (h_1, h_2) \begin{pmatrix} 1 & -\tau_1 \\ -\tau_1 & |\tau|^2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \frac{8\pi^2}{\tau_2} (h_1, h_2) \begin{pmatrix} 1 & -\tau_1 \\ -\tau_1 & |\tau|^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right]
\]

\[
\Theta \left[ \frac{i}{gh\tau_2} + \frac{1}{2} \right] (-\bar{w}, -\bar{\tau}) \exp \left[ -i [\bar{\alpha}(\bar{z} - \bar{w}) + \alpha(z - w)] + i \frac{\pi}{2\tau_2} (z\bar{w} - \bar{z}w) \right]
\]

where we have used the explicit expression for the zero-mode eq. (A.133), obtained in the Appendix. We have obviously included in the previous integration the $\alpha_\mu$-dependent piece in the phase factor eq. (6.109). Expanding the $\Theta$-functions as a series of exponential the integral over $h_1, h_2$ is of gaussian type, and after some boring but straightforward computations we remain with

\[
(1 + \frac{g^2}{2\tau_2})^{-1} \sum_{m,n=-\infty}^{+\infty} \int_0^1 d\alpha_1 d\alpha_2 \exp \left[ \frac{1}{2} \bar{h} \Lambda \hat{\bar{h}} - \frac{2\pi \tau_2}{1 + \frac{g^2}{2\pi \tau_2}} \alpha_2^2 - 2\pi i (z - \bar{w})\alpha_2 - \frac{g^2 \tau_2}{1 + \frac{g^2}{2\pi \tau_2}} (h_1 + h_2)\alpha_2 \\
-2\pi i (m - n)\alpha_1 \right] \exp \left[ i\pi \tau (n + \frac{1}{2})^2 - i\pi \bar{\tau} (m + \frac{1}{2})^2 - 2\pi i \left( (n + \frac{1}{2})\tau - (m + \frac{1}{2})\bar{\tau} \right) \right] + (6.114)
\]

\[
2\pi i (n + \frac{1}{2})(z + \frac{1}{2}) - 2\pi i (m + \frac{1}{2})(\bar{w} + \frac{1}{2}) \exp \left[ \frac{\pi}{2\tau_2} \left( z(z - \bar{z}) - \bar{w}(w - \bar{w}) + \bar{z}w - \bar{w}z \right) \right]
\]
the matrix $\tilde{\Lambda}$ being
\[
-\frac{\pi}{2} \frac{\tau_2 g^2}{2\pi} \left( -2 - \frac{g^2}{2\pi} \frac{2}{\pi} \right)
\]
and the vector $\vec{h}$ is
\[
(n + \frac{1}{2} - \frac{i}{2\tau_2} (z - \bar{z}), m + \frac{1}{2} - \frac{i}{2\tau_2} (w - \bar{w})].
\]
The integration over $\alpha_1$ can now be performed leading to a $\delta_{m,n}$ in the sum: we remark that without the introduction of the phase factor this nice integration cannot be done, because the integral would be quadratic in $\alpha_1$. Another magic of the phase factor is that now the exponential factor can be recast in a way that the sum over $n$ simply changes the integration region from $(0, 1)$ to $(-\infty, +\infty)$, allowing another easy gaussian integration. The final expression is very simple:
\[
S^{\text{Glob.}}_{LL}(x, y) = \frac{1}{2\pi V} \exp \left[ -\frac{\pi}{2\tau_2} (1 + \frac{g^2}{2\pi}) (z - w)(\bar{z} - \bar{w}) \right],
\]
and the same result applies for $S^{\text{Glob.}}_{RR}(x, y)$.

The final expression for the full (gauge-invariant) propagator is:
\[
S_{LL,RR}(x, y) = S^{\text{Loc.}}_{LL,RR}(x, y) S^{\text{Glob.}}_{LL,RR}(x, y) \text{exp} \pm i \left[ \text{Area}_C + \int_0^1 dt \eta^\nu_{\mu} (y^\mu - x^\mu) \partial_\nu \lambda (ty + (1 - t)x) \right],
\]
where the translational invariance is broken only by the geometrical factors linked to the background conformal factor. The short distance behaviour of this propagator is completely determined by its local part, no contribution coming from the integration over the harmonic field:
\[
< \bar{\psi}(x)_{L,R} \psi(y)_{L,R} > \simeq |z - w|^{-\delta}
\]
where $\delta = 1/2(1 + g^2/2\pi - 1/(1 + g^2/2\pi))$. This must be compared with the analogous scaling in the $N = 0$ theory: it is different showing how the volume-interaction has changed the scaling dimension of the Thirring fermion. The coefficient $\delta$ is always positive in the unitarity range $(\frac{g^2}{2\pi} > -1)$, except for the free-fermion case where it vanishes. We do not have an explanation for such a behaviour and could be interesting to compare our results with the scaling laws obtained in [27], where the Thirring model in topological non-trivial background was studied.
As we have promised in the introduction, we present a nice application of finite-temperature quantum field theory to our model: for \( N = \pm 1 \) the theory develops a fermionic condensate at \( T \neq 0 \) temperature. From the explicit expression of the propagator is, in fact, possible to extract, by means of a coincidence limit, the vacuum expectation value of the composite operator \( \langle \bar{\psi}(x)\psi(x) \rangle = \langle \bar{\psi}_L(x)\psi_L(x) + \bar{\psi}_R(x)\psi_R(x) \rangle \). For the finite-temperature interpretation we take \( \sigma = 0 \) (flat-space), we define the temperature \( T \) from \( \tau = i\frac{1}{2TL} \) and then we perform the thermodynamical limit \( L \to \infty \) (we have to restore in the correlators the factor \( L \), measuring the size of the torus). In the limiting procedure a new divergence arises, linked to the composite operator \( \bar{\psi}(x)\psi(x) \), that must be subtracted at the same scale \( \mu \) where the wave function renormalization was defined. Using the expansion for the scalar Green function on the flat-torus, described in eq (5.100), and letting \( L \to \infty \) we get:

\[
\langle \bar{\psi}(x)\psi(x) \rangle_{N=\pm 1} = \frac{1}{2\pi^2} \sqrt{1 + \frac{g^2}{2\pi} T} \left( \frac{\mu}{T} \right)^{\frac{2g^2}{2\pi}}. \tag{6.119}
\]

We notice that in the region of unitarity the zero-temperature limit is regular, leading to a vanishing condensate. The scaling exponent of the temperature is \( \frac{1}{1+\frac{2g^2}{2\pi}} \) that reduces to 1 in the case of non self-interacting fermions. The result, that might be somewhat unexpected being a sort of gravitational mechanism for vacuum condensate, is easily understood exploiting the analogy of the volume-potential with the instantonic configurations of the two dimensional Maxwell theory, that generate \( \langle \bar{\psi}(x)\psi(x) \rangle \neq 0 \) in the Schwinger model.

### 7 Conclusions

In conclusion we have thoroughly studied on the two-sphere \( S^2 \) and the torus \( T^2 \) a fermionic system coupled in a non-minimal way to gravity: the strenght of the new interactions, introduced by Cangemi and Jackiw in ref.[5], must be quantized in order to preserve global properties of the theory. The non-minimal coupling with the curvature and the unusual interaction with the volume form drastically change the dynamics of the massless abelian Thirring model. In
particular the effective action for the Liouville mode, obtained by integrating out the fermionic degrees of freedom, displays some new features: the central charge is improved, depending also on the Thirring coupling constant, and a new non-local functional of the conformal factor appears. It could be very interesting to explore the dynamics of this effective action, representing a very natural extension of the usual Polyakov gravity, even if the task does not seem very simple. On the other hands the presence of zero-modes in the Dirac operator, that can be generated by the coupling with the volume form, imposes chiral selection rules on the fermionic Green functions. On a general Riemann surface (and, in particular, on the torus) there is a family of gauge-inequivalent volume-potentials: we have explicitly integrated over this moduli space in order to preserve the global definition and the translational invariance of the two-point fermionic function, that has been carefully computed. In absence of $B$–coupling the propagator presents the same singularity structure of the theory on the plane, as one could expect. A peculiar effect of the volume coupling is the appearence of a diagonal part for these propagator and, eventually, of a non-vanishing fermionic condensate: it represents a sort of non-pertubative gravitational modification of the vacuum structure of the original fermionic theory. It is also possible to analyze our results in the spirit of finite temperature quantum field theory: anti-periodic boundary conditions for fermions on the flat torus allow for the usual finite-temperature interpretation of our correlation functions, that therefore encode the thermal dependence of the Thirring model dynamics. This could be the subject of further investigations.

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A Appendix

In this appendix we derive the spectrum and the eigenfunctions of the operator \((\hat{D}^{(N)})^2[\alpha, h]\).

Using the explicit expression (3.38) for the \(\hat{A}^{(N)}\) potential we write the eigenvalue problem for \((\hat{D}^{(N)})^2\) as:

\[
[-\hat{g}_{\mu\nu}\hat{D}_\mu\hat{D}_\nu - \Phi \gamma_5] \hat{\Psi}^{(n)} = \lambda_n \hat{\Psi}^{(n)},
\]

(A.120)

where we have introduced the topological flux \(\Phi = \frac{2\pi N}{V}\) and the covariant derivative

\[
\hat{D}_\mu = \partial_\mu + \Omega_\mu
\]

\[
\Omega_\mu = -\Phi \eta_{\mu\nu} x^\nu - 2\pi (k\alpha_{\mu} - gh_{\mu}).
\]

(A.121)

In order to solve the eigenvalue equation we find useful to work with the holomorphic coordinates \(z, \bar{z}\) of eq.(5.92) and to define

\[
\Omega = -i\frac{\Phi L}{4} \bar{z} + \gamma
\]

\[
\bar{\Omega} = i\frac{\Phi}{4} z + \bar{\gamma}
\]

\[
\gamma = \frac{1}{2i\tau_2} \left[ k\alpha_1 - gh_1 - \bar{\tau}(k\alpha_2 - gh_2) \right]
\]

\[
\bar{\gamma} = \gamma^*.
\]

(A.122)

Eq. (A.120) can be rewritten as

\[
[b\dagger b + \frac{1 + \gamma_5}{2}] \hat{\Psi}^{(n)} = \hat{\lambda}_n \hat{\Psi}^{(n)} \quad k > 0
\]

\[
[bb\dagger + \frac{1 - \gamma_5}{2}] \hat{\Psi}^{(n)} = \hat{\lambda}_n \hat{\Psi}^{(n)} \quad k < 0,
\]

(A.123)

where the operators

\[
b = i\sqrt{\frac{2}{|\Phi|}} (\partial_z + i\Omega)
\]

\[
b\dagger = i\sqrt{\frac{2}{|\Phi|}} (\partial_{\bar{z}} + i\bar{\Omega})
\]

(A.124)
satisfy the algebra

\[ [b, b^\dagger] = \begin{cases} 1 & k > 0 \\ -1 & k < 0, \end{cases} \]  \hspace{1cm} (A.125)

and \( \lambda_n = 2|\Phi|\lambda_n \).

The algebra (A.125) immediately leads to the spectrum

\[ \lambda_n = 2n|\Phi|. \]  \hspace{1cm} (A.126)

Moreover if \( k \) is the degeneration of the null eigenvalue we recognize (due to the presence of the projector \( \frac{1+np}{2} \)) that \( 2k \) is the degeneration of the non-vanishing ones. For \( N > 0 \) zero-modes have negative chirality: all the tower of eigenfunctions are simply obtained (up normalizations) acting with \( b^\dagger \):

\[ \hat{\Psi}^{(n+1,+)}_i = (b^\dagger)_n \begin{pmatrix} \hat{\Psi}^{(0)}_i \\ 0 \end{pmatrix}, \]

\[ \hat{\Psi}^{(n,-)}_i = (b^\dagger)_n \begin{pmatrix} 0 \\ \hat{\Psi}^{(0)}_i \end{pmatrix}, \]  \hspace{1cm} (A.127)

\( \hat{\Psi}^{(0)}_l \) being solutions of the ground state equation

\[ b\hat{\Psi}^{(0)}_l = 0. \]  \hspace{1cm} (A.128)

For \( N < 0 \) the chirality is reversed and the roles of \( b \) and \( b^\dagger \) are interchanged.

We are therefore left with solving eq.(A.128), on the space of functions that satisfy the boundary conditions

\[ \psi(z + \tau, \bar{z} + \bar{\tau}) = \exp \left( i\pi - \frac{\pi N}{2\tau_2} (\bar{\tau}z - \bar{z}\tau) \right) \psi(z, \bar{z}), \]  \hspace{1cm} (A.129.a)

\[ \psi(z + 1, \bar{z} + 1) = \exp \left( i\pi - \frac{\pi N}{2\tau_2} (z - \bar{z}) \right) \psi(z, \bar{z}), \]  \hspace{1cm} (A.129.b)
where the antiperiodic boundary conditions has been taken into account. A general solution of eq. (A.128) is

\[ \hat{\Psi}^{(0)}(z, \bar{z}) = \exp\left[-\left(\frac{N\pi}{2\tau_2} \bar{z} + i\gamma\right)(z - \bar{z})\right] \hat{\Psi}^{(0)}(\bar{z}) \]  

(A.130)

and the boundary conditions on the anti-holomorphic function \( \hat{\Psi}^{(0)}(\bar{z}) \) are

\[
\begin{align*}
\hat{\Psi}^{(0)}(\bar{z} + 1) & = -\hat{\Psi}^{(0)}(\bar{z}), \\
\hat{\Psi}^{(0)}(\bar{z} + \bar{\tau}) & = -\exp[2\pi i N \bar{z} + i\pi N \bar{\tau} - 2\tau_2 \gamma] \hat{\Psi}^{(0)}(\bar{z}).
\end{align*}

(A.131)

One can explicitly verify that

\[ \hat{\Psi}^{(0)}_{N,l}(\bar{z}) = \Theta \left[ \frac{\frac{1}{2}}{\frac{1}{2} N} \right] \left( -N\bar{z}, -N\bar{\tau} \right) \]

(A.132)

with \( l = 0, 1, \ldots, |N| - 1 \) have the required properties: they are exactly \( |k| \), as stated by the index theorem [11]. Moreover it is not difficult to show the orthogonality of the zero modes \( \hat{\Psi}^{(0)}_{k,l}(z, \bar{z}) \): they can also be normalized to unity to give an orthonormal basis for the kernel of \( \hat{\mathcal{D}}^{(N)} \) \( (N > 0) \)

\[
\hat{\Psi}^{(0)}_{N,l}(z, \bar{z}) = \left( \frac{2}{\tau_2} \right)^{\frac{1}{4}} \exp[-\frac{\tau_2}{4\pi} (\gamma + \bar{\gamma})^2] \exp[-\left(\frac{N\pi}{2\tau_2} \bar{z} + i\gamma\right)(z - \bar{z})] \Theta \left[ \frac{\frac{1}{2}}{\frac{1}{2} N} \right] \left( -N\bar{z}, -N\bar{\tau} \right).
\]

(A.133)

For \( N < 0 \) we get along the same lines

\[
\hat{\Psi}^{(0)}_{N,l}(z, \bar{z}) = \left( \frac{2}{\tau_2} \right)^{\frac{1}{4}} \exp[-\frac{\tau_2}{4\pi} (\gamma + \bar{\gamma})^2] \exp[-\left(\frac{N\pi}{2\tau_2} z - i\gamma\right)(z - \bar{z})] \Theta \left[ \frac{\frac{1}{2}}{\frac{1}{2} N} \right] \left( -Nz, -N\tau \right).
\]

(A.134)
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