Purifying entanglement of noisy two-qubit states via entanglement swapping

Wei Song,1 Ming Yang,2 and Zhuo-Liang Cao1
1School of Electronic and Information Engineering, Hefei Normal University, Hefei 230061, China
2School of Physics and Material Science, Anhui University, Hefei 230061, China
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Two qubits in pure entangled states going through separate paths and interacting with their own individual environments will gradually lose their entanglement. Here we show that the entanglement change of a two-qubit state due to amplitude damping noises can be recovered by entanglement swapping. Some initial states can be asymptotically purified into maximally entangled states by iteratively using our protocol.

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Quantum entanglement has received a lot of attentions due to its importance in almost all quantum informational processing and communication tasks [1]. To actually realize such tasks, highly entangled states are usually required. However, the quality of entanglement will degrade exponentially due to the unavoidable interactions between a quantum system and its environment. Entanglement purification provides a way to extract a small number of entangled pairs with a relatively high degree of entanglement from a large number of less entangled pairs using only local operations and classical communication (LOCC). If the less entangled states are pure, we call such a process entanglement concentration, while for mixed states case, this process is usually called entanglement purification or distillation. The first entanglement purification protocol for mixed states was introduced by Bennett et al in 1996 [2]. Since then, many theoretical and experimental works have been made in the field of entanglement purification and concentration [3,10]. Especially Bose et al. [11] have proposed a scheme for concentration of pure shared entanglement via entanglement swapping [17]. The idea is that Alice shares an entangled state with Bob and Bob shares the same entangled state with Charlie. Bob performs a Bell measurement on his pair is transmitted through the local individual AD channels to two separated user Alice and Bob and the other pair is transmitted through the local individual AD channels to Alice and Charlie. For simplicity, we suppose the states probabilistically. However, their discussion only works for the pure state cases where both initial states are of the form $|\alpha\rangle = |00\rangle + \beta |11\rangle$. It is well known that entanglement swapping is one of the central ingredients for quantum repeaters, which lays at the heart of quantum communication [18,20]. If we can extend the idea of entanglement swapping to the purification of mixed state cases, then Bell state measurements can be substituted for the original CNOT operations in the purification process. At present, for the general unknown mixed states, whether entanglement swapping can be used to purify the entanglement remains unknown. The main difficulty lies in the different structures of state spaces for pure states and mixed states, and thus the discussion of pure state cases cannot be generalized to the mixed cases directly. In this paper we will explore the problem by considering some concrete examples. We find that the entanglement degradation of a two-qubit state due to amplitude damping (AD) noises can be recovered by entanglement swapping combined with weak measurements. Here, we take the AD noises to model the environment, and such noise is one important type of decoherence which is related to many practical systems [21]. It has a useful physical interpretation: there is some probability of decaying from state $|1\rangle$ to $|0\rangle$, but it never transforms state $|0\rangle$ to $|1\rangle$. For example, photon loss in an optical fiber can be described by this model [22]. Recently, several interesting methods have been put forward to cope with this type of noise using weak measurements followed by quantum measurement reversals [22,24] or error-correcting codes [23,25]. Compared with these works, our method provides a different way for battling against decoherence from AD noises. The distinct advantage of our scheme is that the final state is almost maximally entangled state after several rounds of our protocol, while the protocol in Ref. [22,24] is a one-round scheme but also depends on the initial states to be protected. Furthermore, we do not require the singlet fraction larger than $\frac{1}{2}$ in contrast to the BBPSSW protocol [2].

We now explain our scheme by discussing a specific example firstly. Suppose two pairs of qubits are initially prepared in the states $|\psi\rangle = \sqrt{\alpha} |01\rangle + \sqrt{1-\alpha} |10\rangle$ and $|\phi\rangle = \sqrt{\sigma} |10\rangle + \sqrt{1-\sigma} |01\rangle$, respectively. Note that these two pairs have the same entanglement and the second pair is just the flipped state of the first one. One pair is transmitted through the local individual AD channels to two separated user Alice and Bob and the other pair is transmitted through the local individual AD channels to Bob and Charlie. For simplicity, we suppose the local AD channels are all identical in the following arguments. The initial pure states will evolve into mixed states under the disturbance of AD noises. This process can be described by a completely positive trace preserving map $\rho(t) = \varepsilon(\rho(0)) = \sum_{\mu} K_{\mu} (t) \rho(0) K_{\mu}^\dagger(t)$, where the operators $\{K_{\mu}(t)\}$ satisfy the completeness condition $\sum_{\mu} K_{\mu}^\dagger(t) K_{\mu}(t) = I$. In the case of AD noises the Kraus operators $K_1 = |0\rangle \langle 0| + \sqrt{1-p} |1\rangle \langle 1|, K_2 = \sqrt{p} |0\rangle \langle 1|$. After the AD noises, Alice and Bob will share the mixed state $\rho_{AB} = \rho(00) |00\rangle \langle 00| + (1-p) |\psi\rangle \langle \psi|$, and Bob and Charlie will share the state $\rho_{BC} = p |00\rangle \langle 00| + (1-p) |\phi\rangle \langle \phi|$. It is well known that $|\alpha\rangle = |00\rangle + \beta |11\rangle$ and $|\beta\rangle = |00\rangle + \beta |11\rangle$ are asymmetric in $\text{det}|\alpha\beta| = \pm \beta^2$. It can be described by a completely positive trace preserving map $\rho(t) = \varepsilon(\rho(0)) = \sum_{\mu} K_{\mu} (t) \rho(0) K_{\mu}^\dagger(t)$, where the operators $\{K_{\mu}(t)\}$ satisfy the completeness condition $\sum_{\mu} K_{\mu}^\dagger(t) K_{\mu}(t) = I$. In the case of AD noises the Kraus operators $K_1 = |0\rangle \langle 0| + \sqrt{1-p} |1\rangle \langle 1|, K_2 = \sqrt{p} |0\rangle \langle 1|$. After the AD noises, Alice and Bob will share the mixed state $\rho_{AB} = \rho(00) |00\rangle \langle 00| + (1-p) |\psi\rangle \langle \psi|$, and Bob and Charlie will share the state $\rho_{BC} = p |00\rangle \langle 00| + (1-p) |\phi\rangle \langle \phi|$. 
conjugation of $\rho$ that the two mixed states have the same concurrence $|C|_\text{Wootter’s concurrence}$ [27] to quantify the entanglement: larger than that of the initial states. Here, we use the Wootters concurrence [27] to determine the entanglement of the final state after entanglement swapping should be larger than that of the initial states. Here, we use the Wootters concurrence [27] to quantify the entanglement: $C(\rho) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where $\lambda_i$ are the eigenvalues in decreasing order of the matrix $\rho\rho^\dagger - \rho \otimes \sigma_y$ with $\rho^\dagger$ denoting the complex conjugation of $\rho$. After simple calculation, we can see that the two mixed states have the same concurrence $C(\rho_{AB}) = C(\rho_{BC}) = 2(1 - p)\sqrt{a(1 - a)}$. If Bob obtains $|\Psi^\pm\rangle$ as the result of his Bell state measurement, the state of Alice and Charlie will collapse into the mixed state:

$$\rho_{AC} = \frac{1}{N} \left( 2p(1 - p)a |00\rangle \langle 00| + 2(1 - p)^2a(1 - a) |\Psi^\pm\rangle \langle \Psi^\pm| \right)$$  \hspace{1cm} (1)$$

where $N = 2(1 - p)^2a(1 - a) + 2p(1 - p)a$. The probabilities for obtaining $|\Psi^+\rangle$ or $|\Psi^-\rangle$ are $\frac{2}{N}$, respectively. The concurrence of this state is $\frac{2}{N}(1 - p)^2a(1 - a)$. Then an enhancement of entanglement implies that $C(\rho_{AC}) > C(\rho_{AB})$. We plot the region of the parameters $p$ and $a$ satisfying the inequality $C(\rho_{AC}) > C(\rho_{AB})$ in Fig.1. We find the region is not empty, i.e., if $p$ and $a$ are restricted in the blue region, the entanglement of the final mixed state can be larger than that of the original states. If $0 \leq a \leq 0.2$ the mixed state entanglement from arbitrary AD noises always can be enhanced through the above entanglement swapping process. If Bob obtains $|\Phi^\pm\rangle$ as the results of his measurements, then entanglement of the final state cannot be larger than that of the initial one. It is worth pointing out that, if we do not flip the second pair at the beginning, then Alice and Charlie will share a mixed state with entanglement less than that of the original state. In real situations, the initial two pairs are not in the same form usually. Suppose that the second pair is in the form $\sqrt{a'}|10\rangle + \sqrt{1 - a'}|01\rangle$ which has a small deviation from the first pair, where $a' \neq a$. Similar calculations show that if Bob obtains $|\Psi^\pm\rangle$ as the result of his Bell state measurement, then Alice and Charlie will share the mixed state:

$$\rho = \frac{1}{N} \left( p(1 - p)(a + a') |00\rangle \langle 00| + (1 - p)^2a(1 - a') |01\rangle \langle 01| + (1 - p)^2a'(1 - a) |10\rangle \langle 10| + (1 - p)^2 \times \sqrt{aa'}(1 - a)(1 - a') |01\rangle \langle 01| + |10\rangle \langle 10| \right)$$  \hspace{1cm} (2)$$

where $M = p(1 - p)(a + a') + (1 - p)^2 \times (a'(1 - a) + a(1 - a'))$. We plot the possible region of the states whose entanglement can be enhanced via the entanglement swapping process. From Fig.2 one can see that our scheme is still feasible if the parameters $a'$ and $a$ are restricted in the blue region. It shows that small deviations of the initial pairs are allowable in above swapping process.

Next we will consider whether we can enlarge the range of the possible parameters $p$ and $a$ in Fig.1. In order to further enhance the entanglement of the initial state, the first choice is to apply the procedure described above several times. However, if we start the second round of the protocol with the mixed state $\rho_{AC}$, one can directly get that its entanglement will degrade af-

FIG. 1: (color online). If the entanglement of the final state is larger than the initial state, then the parameters $p$ and $a$ should lie in the blue region.

FIG. 2: (color online). The region of the states whose entanglement can be enhanced through entanglement swapping with different parameters, respectively. (b) $p = 0.1$, (c) $p = 0.01$, (d) $p = 0.001$. 

[Image 95x584 to 249x732]

[Image 321x607 to 442x739]

[Image 448x493 to 559x596]
ter the protocol. Suppose two copies of the state $\rho_{AC}$ in Eq.(1) have been prepared through the entanglement swapping process. Before the second round of our protocol, the weak nondestructive measurements $M_k$ will be performed on one of the qubits of $\rho_{AC}$, where $M_L = \sqrt{b} |0\rangle \langle 0| + \sqrt{1 - b} |1\rangle \langle 1|$, $M_0 = \sqrt{1 - b} |0\rangle \langle 0| + \sqrt{b} |1\rangle \langle 1|$. Obviously, this operation increases the fraction of $|1\rangle$ or $|0\rangle$. If the weak measurement outcomes on the qubit 4 of the first pair is $M_+$, the mixed state will be transformed to the following one:

$$\rho'_{AC} = \frac{1}{N'} \left(2p(1-p)ab|00\rangle\langle 00| + (1-p)^2a(1-a)|\varphi\rangle\langle \varphi| \right)$$

(3)

where $N' = 2(p - 1)pab + (1 - p)^2a(1 - a)$. $|\varphi\rangle = \sqrt{b}|01\rangle + \sqrt{1 - b}|10\rangle$. The probability for obtaining this state is

$$p' = \frac{2p(1-p)ab + (1-p)^2a(1-a)}{2p(1-p)a + 2(1-p)^2a(1-a)}$$

If the weak measurement on the qubit 4 of the second pair is also $M_+$, the mixed state will be transformed to the following one:

$$\rho''_{AC} = \frac{1}{N''} \left(2p(1-p)ab|00\rangle\langle 00| + (1-p)^2a(1-a)|\tilde{\varphi}\rangle\langle \tilde{\varphi}| \right)$$

(4)

where $|\tilde{\varphi}\rangle = \sqrt{b}|10\rangle + \sqrt{1 - b}|01\rangle$, $N'' = N'$ and the corresponding probability is $p''$. Now we start the second round of purification process with these two new mixed states. Bob performs the Bell state measurements on the two qubits, and the resulting state is

$$\rho''_{AC} = \frac{1}{N''} \left(4p(1-p)^3b^2a^2(1-a)|00\rangle\langle 00| + 2b(1-b)(1-p)^4a^2(1-a)^2|\Psi\rangle\langle \Psi| \right)$$

if the measurement result is $|\Psi\rangle$, where the superscript (2) of $\rho'_{AC}$ denotes the second round of our protocol. The concurrence of $\rho''_{AC}$ is

$$C(\rho''_{AC}) = \frac{2b(1-b)(1-p)^4a^2(1-a)^2}{4p(1-p)^3b^2a^2(1-a) + 2b(1-b)(1-p)^4a^2(1-a)^2}$$

(5)

Further as long as the following relation holds:

$$C(\rho''_{AC}) > C(\rho^{(1)}_{AC}) > C(\rho_{AB}).$$

It is straightforward to verify that $C(\rho''_{AC}) > C(\rho^{(1)}_{AC})$ corresponds to the inequality 

$$\frac{1}{(1 + a(p - 1))(1 - b(1-p)(1-a) + 2bp)} > 0,$$

and this inequality holds when $b < \frac{c}{a}$. We have to mention that we cannot choose the condition $b = 0$ because the weak measurement becomes a projective measurement and the resulting states $\rho'_{AC}$ and $\rho''_{AC}$ are both separable states in this case.

By exchanging $b$ and $1 - b$ in Eq.(5) we can get the concurrence of $\rho''_{AC}$ if the weak measurements on the qubits $A$ and $C$ of the two pairs are both $M_+$. In this case, the inequality $C(\rho''_{AC}) > C(\rho^{(1)}_{AC})$ implies $b > \frac{c}{a}$. If the weak measurements on the qubits $A$ and $C$ of the two pairs are different, we find that the entanglement of the final state cannot be larger than that of the initial one. Thus we have shown that entanglement can be further purified in the second round of our protocol. If the measurement results of Bob are $|\Psi\rangle$, then the concurrence of $\rho''_{AC}$ cannot be further increased.

By iteratively using this procedure several times, the final mixed state becomes $\rho^{(n)}_{AC} = \frac{1}{N(n)} (a_n |00\rangle\langle 00| + 2b_n |\Psi\rangle\langle \Psi|)$, with $a_n = \ldots$
posed the measurement results of Bob are $A_{\rho}$ and the weak measurements on the qubits $C$ into $b$ rounds for arbitrary shared state between Alice and Charlie is $C$. The entanglement of the final state can be larger than the initial one if the following series of inequalities hold: $C(\rho_{AC}^{(n)}) > \cdots > C(\rho_{AC}^{(2)}) > C(\rho_{AC}^{(1)}) > C(\rho_{AB})$. Numerical calculation shows that a near-perfect maximally entangled state can be extracted in the limit of infinite rounds for arbitrary $p$ and $a$. We plot the possible region for the cases $n = 2$ and $n = 3$ with fixed parameter $b = 0.22$, respectively in Fig.3. Obviously, the possible region of parameters $p$ and $a$ becomes larger with the increase of $n$. In Fig.4 we also plot the concurrence of the states $\rho_{AB}$, $\rho_{AC}^{(1)}$, $\rho_{AC}^{(2)}$ and $\rho_{AC}^{(3)}$, respectively for $a = 0.3$, $b = 0.22$.

In the following we consider the case when the two states initially prepared in the other form $\sqrt{A}|00\rangle + \sqrt{1-A}|11\rangle$. One pair is distributed to two separated users Alice and Bob and the other pair to Bob and Charlie via the amplitude damping(AD) noisy channels. The states evolve into $\chi_{AB} = \chi_{BC} = \left[ A + (1 - A) p^2 \right]|00\rangle\langle 00| + (1 - A) p (1 - p) \left( |01\rangle\langle 01| + |10\rangle\langle 10| \right) + (1 - A) (1 - p)^2 \times |11\rangle\langle 11| + (1 - p)^2 A |1\rangle\langle 1| |0\rangle\langle 0|).$ Let Bob perform the Bell measurements on his qubits. If Bob obtains $|\Psi^\pm\rangle$ as the result of his measurement, the shared state between Alice and Charlie is

$$\chi_{AC} = (1 - A) (1 - p)^2 (A + 2 (1 - A) p^2) \times (|01\rangle\langle 01| + |10\rangle\langle 10|) + 2 (1 - A) p (1 - p) \left( A + (1 - A) p^2 \right) |00\rangle\langle 00| + 2 (1 - A)^2 p (1 - p)^3 |11\rangle\langle 11| + \sqrt{b} (1 - b) (1 - A) A (1 - p)^2 (|01\rangle\langle 10| + |10\rangle\langle 01|).$$

For $0 < b < \frac{1}{2}$, numerical calculations show that the region can be further enlarged. We plot the possible region in Fig.5(b) with $b = 0.25$. If the measurement results are both $M_+$, the parameter $b$ must satisfy $\frac{1}{4} < b < 1$. While for cases with different weak measurement results, the region cannot be enlarged. In order to have a vivid picture, we plot the concurrence of the state $\chi_{AB}$, $\chi_{AC}$ and $\chi_{AC}'$ in Fig.6, respectively, where we have chosen $A = 0.9$, $b = 0.25$. From Fig.6 we can see that the concurrence of $\chi_{AC}'$ will be always larger than that of the original mixed state $\chi_{AB}$.

So far we have shown that entanglement swapping combined with weak measurement can be used to purify a class of AD noises induced mixed states. By a similar procedure, our protocol can be generalized directly to the general two-qubit mixed states lying in the subspace spanned by the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, or $\{|01\rangle, |10\rangle, |11\rangle\}$, if the conditions $\rho_{2243} = \rho_{2342}$ is
satisfied. An open problem is whether entanglement swapping can be used to purify arbitrary mixed states.

If the answer is no, then can we find a criteria to distinguish which states cannot be purified with entanglement swapping. In practice, the entanglement swapping has been demonstrated experimentally \cite{28, 29} and the AD noise can be simulated by beam splitters with appropriate transmission coefficients. Recently, the weak non-destructive measurements have also been developed and demonstrated in single-photon quantum optical systems \cite{30, 31}. Thus our protocol can be demonstrated experimentally within current technology.

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\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{concurrence.png}
\caption{(color online). Plots of the concurrence of the states $\chi_{AB}$ (red line), $\chi_{AC}$ (green line) and $\chi'_{AC}$ (blue line) for $A = 0.9, b = 0.25$.}
\end{figure}

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