Locally conformal flat Riemannian manifolds with constant principal Ricci curvatures and locally conformal flat C-spaces

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Abstract

It is proved that every locally conformal flat Riemannian manifold all of whose Jacobi operators have constant eigenvalues along every geodesic is with constant principal Ricci curvatures. A local classification (up to an isometry) of locally conformal flat Riemannian manifold with constant Ricci eigenvalues is given in dimensions 4, 5, 6, 7 and 8. It is shown that any n-dimensional (4 ≤ n ≤ 8) locally conformal flat Riemannian manifold with constant principal Ricci curvatures is a Riemannian locally symmetric space.

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1 Introduction

Curvature is a fundamental notion in Riemannian geometry. The Jacobi operator is an important tool for studying the curvature. If the Jacobi operator has constant eigenvalues then the Riemannian manifold is said to be globally Osserman manifold. The Osserman conjecture states (see [15]) that any globally Osserman space is rank-one symmetric space. This conjecture has been proved to be true for odd dimension and for dimensions two, four and $2(2k + 1), k \in \mathbb{N}$ in [5].

J.Berndt and L.Vanhecke considered Riemannian manifolds satisfying weaker conditions in [1]. They introduced in [1] the so-called $C$-space as a Riemannian manifold for which the Jacobi operators have constant eigenvalues along every geodesic and the so-called $P$-space as a Riemannian manifold for which the Jacobi operators have parallel eigenspaces along every geodesic. The $C$-spaces and $P$-spaces can be regarded as a natural generalization of Riemannian locally symmetric spaces since a Riemannian manifold is locally symmetric iff it is a $C$-space and it is a $P$-space simultaneously. There is a lot of examples of $C$-spaces, namely every naturally reductive Riemannian homogeneous space as well as every commutative space is a $C$-space. A local classifications (up to an isometry) of $C$-spaces and $P$-spaces in dimensions 2 and 3 is given also in [1].

Globally Osserman spaces, $C$-spaces, $P$-spaces are of special interest in the last years. These spaces are studied in [1, 2, 3, 4, 5, 6].

In this paper we consider locally conformal flat $C$-spaces. Our main observation is that these spaces have constant principal Ricci curvatures. We give a local classification of locally conformal flat Riemannian manifolds with constant principal Ricci curvatures in dimensions 4,5,6,7 and 8. Consequently, we obtain a local description of locally conformal flat $C$-spaces in dimensions 4, 5, 6, 7 and 8. The aim of the paper is to prove the following

**Theorem 1.1** For an $n$-dimensional ($4 \leq n \leq 8$) connected locally conformal flat Riemannian manifold the following conditions are equivalent:

a) It is a $C$-space;

b) It has constant principal Ricci curvatures;

c) It is locally (almost everywhere) isometric to one of the following spaces:

i) a real space form;

ii) a Riemannian product of 1-dimensional space and of a real space form of dimension $(n-1)$;

iii) a Riemannian product of two real space forms with opposite constant sectional curvatures;

d) It is a Riemannian locally symmetric space.

According to I.M.Singer [15] a Riemannian manifold $(M, g)$ is called *curvature homogeneous* if for any pair of points $p, q \in M$ there is a linear isometry $F : T_p M \to T_q M$ between the corresponding tangent spaces such that $F^* R_q = R_p$ (where $R$ denotes the curvature tensor of type $(0, 4)$). We note that a locally conformal flat Riemannian manifold is curvature homogeneous iff it has constant principal Ricci curvatures. Thus, every locally conformal flat Riemannian manifold of dimension 4, 5, 6, 7 or 8 is curvature homogeneous iff it is a Riemannian locally symmetric space.
Riemannian 3-manifolds with constant principal Ricci curvatures are studied in [11, 12, 13, 14, 18]. These spaces are exactly the curvature homogeneous Riemannian 3-manifolds.

Theorem 1.1 shows, unfortunately, that there are not interesting examples of locally conformal flat $C$-spaces in dimensions 4, 5, 6, 7 and 8. But, the situation with $P$-spaces is completely different. In the recent work [9], it is shown that there are exactly nine kinds of examples of 4-dimensional locally conformal flat $P$-spaces and a lot of these examples are not even curvature homogeneous.

2 Locally conformal flat $C$-spaces

In this section we recall some definitions and prove our main observation.

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\nabla$ the Levi-Civita connection of the metric $g$. The curvature $R$ of $\nabla$ is defined by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_{[X,Y]}$$

for every vector fields $X, Y$ on $M$. We denote by $T_p M$ the tangential space at a point $p \in M$.

Let $x \in T_p M$. The Jacobi operator is defined by

$$\lambda_x(y) = R(y, x)x, \quad y \in T_p M.$$

Let $\gamma(t)$ be a geodesic on $M$ and $\dot{\gamma}$ denotes its tangent vector field. We consider the family of smooth self-adjoint Jacobi operators along $\gamma$ defined by

$$\lambda_{\dot{\gamma}}(X) = R(\dot{\gamma}, X)\dot{\gamma}$$

for every smooth vector field $X$ along $\gamma$. A Riemannian manifold $(M, g)$ is said to be a $C$-space if the operators $\lambda_{\dot{\gamma}}$ have constant eigenvalues along every geodesic on $M$.

A Riemannian manifold $(M, g)$ is said to be locally conformal flat if around every point $p \in M$ there exists a metric $\bar{g}$ which is conformal to $g$ and $\bar{g}$ is flat. By the Weyl theorem, an $n$- dimensional $(n \geq 4)$ Riemannian manifold is locally conformal flat iff the curvature tensor has the following form

$$(2.1) \quad R(x, y, z, u) = -\frac{s}{(n-1)(n-2)}(g(y, z)g(x, u) - g(x, z)g(y, u)) +$$

$$\frac{1}{n-2}(Ric(y, z)g(x, u) - Ric(x, z)g(y, u) + g(y, z)Ric(x, u) - g(x, z)Ric(y, u)),$$

$x, y, z, u \in T_p M, \quad p \in M$, where $Ric$ and $s$ are the Ricci tensor and the scalar curvature of $g$, respectively.

The following condition also holds

$$(2.2) \quad (\nabla_x Ric)(y, z) - (\nabla_y Ric)(x, z) = \frac{1}{2(n-1)}(x(s)g(y, z) - y(s)g(x, z)).$$

A Riemannian 3-manifold is locally conformal flat iff the condition (2.2) holds.

The Ricci operator $\rho$ is defined by $g(\rho(x), y) = Ric(x, y), \quad x, y \in T_p M, \quad p \in M$. In every point $p \in M$ we consider the Ricci operator as a linear self-adjoint operator on $T_p M$. Let $\Omega$ be the subset of $M$ on which the number of distinct eigenvalues of $\rho$ is locally constant. This set is open and dense in $M$. We can choose smooth eigenvalue functions of $\rho$ on $\Omega$, say $r_1, \ldots, r_n$, such that they form at each point of $\Omega$ the spectrum of $\rho$ (see e.g. [10, 1]).
We fix \( p \in \Omega \). Then there exists a local orthonormal frame field \( E_1, \ldots, E_n \) on an open connected neighborhood \( U \) of \( p \) such that

\[
\rho(E_i) = r_i E_i, \quad i = 1, 2, \ldots, n. \tag{2.3}
\]

Our further considerations will take place in the neighborhood \( U \).

For every \( i, j, k, l \in \{1, 2, \ldots, n\} \), we set

\[
R_{ijkl} = R(E_i, E_j, E_k, E_l), \quad \omega^k_{ij} = g(\nabla E_i E_j, E_k)
\]

\[
(\nabla_i \text{Ric})_{jk} = (\nabla E_i \text{Ric})(E_j, E_k).
\]

We have

**Theorem 2.1** Every connected \( n \)-dimensional \((n \geq 3)\) locally conformal flat \( C \)-space has constant Ricci eigenvalues.

**Proof.** Let \( p \in U \), \( x \in T_p M \) and \( \gamma \) be a geodesic in \( U \) determined by the conditions \( \gamma(0) = p, \dot{\gamma}(0) = x \). Using (2.1), we get

\[
\text{trace}(\lambda \dot{\gamma}) = \text{Ric}(\dot{\gamma}, \dot{\gamma}).
\]

We have \( (\nabla \dot{\gamma} \text{Ric})(\dot{\gamma}, \dot{\gamma}) = 0 \) along \( \gamma \), since \( M \) is a \( C \)-space. At the point \( p \), we obtain

\[
(\nabla x \text{Ric})(x, x) = 0. \tag{2.4}
\]

The equality (2.4) holds in every point \( p \in U \) and for every tangent vector \( x \in T_p M \). We get from (2.4) by a polarization that

\[
(\nabla_i \text{Ric})_{jj} + 2(\nabla_j \text{Ric})_{ji} = 0, \quad i, j \in \{1, \ldots, n\}, \quad i \neq j. \tag{2.5}
\]

We obtain from (2.5) that

\[
(\nabla_i \text{Ric})_{kk} + (\nabla_i \text{Ric})_{jj} - \frac{1}{n-1} E_i(s) - (\nabla_k \text{Ric})_{kl} - (\nabla_j \text{Ric})_{jl} = 0,
\]

\[
j, k, l \in \{1, \ldots, n\}, \quad j \neq k \neq l \neq j. \tag{2.6}
\]

Using (2.5) and (2.6), we derive the following equalities:

\[
3(n - 1)((\nabla_i \text{Ric})_{kk} + (\nabla_i \text{Ric})_{jj}) - 2E_i(s) = 0,
\]

\[
j, k, l \in \{1, \ldots, n\}, \quad j \neq k \neq l \neq j. \tag{2.7}
\]

We get from (2.7) that \( (\nabla \text{Ric})_{kk} = (\nabla \text{Ric})_{jj}, \quad j, k, l \in \{1, \ldots, n\}, \quad j \neq k \neq l \neq j \). The latter equality implies \( E_i(r_k) = E_i(r_j), \quad j, k, l \in \{1, \ldots, n\}, j \neq k \neq l \neq j \). We obtain from (2.4) that \( E_l(r_l) = 0, \quad l \in \{1, \ldots, n\} \). Substituting the last two equations into (2.7) we get \( E_i(r_k) = 0, \quad l, k \in \{1, \ldots, n\} \) which completes the proof of the theorem. **Q.E.D.**
3 Proof of Theorem 1.1

We begin with the following technical results

**Lemma 3.1** Let \((M, g)\) be an \(n\)-dimensional \((n \geq 4)\) locally conformal flat Riemannian manifold with constant Ricci eigenvalues such that at least two of them are distinct. Let \(p \in M\) and \(U\) be the neighborhood of \(p\) described in the previous section. Let \(I_m, m = 1, 2, \ldots, l\), \((l \leq n)\) be subsets of the set \(\{1, \ldots, n\}\) such that

1) \(\bigcup_{m=1}^{l} I_m = \{1, \ldots, n\}\);
2) \(r_i = r_j, \ i, j \in I_m\);
3) \(r_i \neq r_j, \ i \in I_{m_1}, j \in I_{m_2}, m_1 \neq m_2\).

Then, for any distinct \(r, s, t\) \(\in \{1, \ldots, l\}\) we have

\[\omega^k_{ij} = 0, \ i, j \in I_s, k \in I_t;\] \[(3.8)\]
\[\omega^k_{ii} = 0, \ i \in I_s, k \in I_t;\] \[(3.9)\]
\[\omega^k_{im} = \frac{r_i - r_k}{r_m - r_k} \omega^k_{mi}, \ i \in I_s, m \in I_r, k \in I_t.\] \[(3.10)\]

**Proof.** Using (2.3), we obtain from (2.2) that

\[(r_k - r_l) \omega^l_{ik} - (r_i - r_l) \omega^l_{ki} = 0,\] \[(3.11)\]
\[(r_k - r_l) \omega^l_{kk} = 0.\] \[(3.12)\]

The formulae (3.8), (3.9) and (3.10) follow from (3.11) and (3.12) which proves the Lemma.

\[\text{Q.E.D.}\]

Further we have

**Lemma 3.2** Let \((M, g)\) be an \(n\)-dimensional \((n \geq 4)\) locally conformal flat Riemannian manifold with two distinct constant Ricci eigenvalues. Then \((M, g)\) is locally isometric to one of the following spaces:

1) \(M^{n-1} \times M^1\), where \(M^{n-1}\) is an \((n-1)\)-dimensional real space form and \(M^1\) is an 1-dimensional space;

2) \(M^m \times M^{n-m}\), \((m < n)\) where \(M^m\) is an \(m\)-dimensional real space form of constant sectional curvature \(K\) and \(M^{n-m}\) is an \((n-m)\)-dimensional real space form of constant sectional curvature \((-K)\).

**Proof.** Let \(p \in M\) and \(U\) be as in Lemma 3.1. Let \(r_1 = r_2 = \ldots = r_m \neq r_{m+1} = \ldots = r_n\). We set

\[I_1 = \{1, \ldots, m\}, \ I_2 = \{m+1, \ldots, n\},\]
\[\mathcal{F}_1 = \text{span}\{E_i\}_{i \in I_1}, \ \mathcal{F}_2 = \text{span}\{E_j\}_{j \in I_2}.\]

Each one of the smooth distributions \(\mathcal{F}_1\) and \(\mathcal{F}_2\) is autoparallel by Lemma 3.1. Let \(U^m\) and \(U^{n-m}\) be the correspondent integral submanifolds. The assertion of the Lemma follows since \(M\) is locally conformal flat.

\[\text{Q.E.D.}\]
Lemma 3.3 Let \((M, g)\) be an \(n\)-dimensional \((n \geq 4)\) locally conformal flat Riemannian manifold with constant Ricci eigenvalues such that exactly \(l\) of them are distinct. Let \(p \in M\) and \(U\) be a neighborhood of \(p\) as in Lemma 3.1. Then, we have on \(U\)

i) \(l \neq 3;\)

ii) there exists exactly one Ricci eigenvalue \(r_s\) of multiplicity \(m_s\) such that \(m_s > \frac{n}{2}\);

iii) \(l = 1, 2\) or \(4 \leq l \leq \frac{n}{2}\) if \(n\) is even;

iv) \(l = 1, 2\) or \(4 \leq l \leq \frac{n+1}{2}\) if \(n\) is odd;

Proof. Let \(l \geq 3\) and the first \(l\)-numbers of \(r_1, \ldots, r_n\) be distinct and of multiplicity \(m_1, \ldots, m_l\), respectively.

Setting

\[
(3.13) \quad u_k = 2r_k - \frac{s}{n-1}, \quad k = 1, 2, \ldots, l
\]

we claim that

\[
(3.14) \quad u_k \neq 0,
\]

\[
(3.15) \quad u_k \neq u_j, \quad k, j \in \{1, \ldots, l\}, k \neq j;
\]

\[
(3.16) \quad n - m_k = 2u_k \sum_{j \neq k} \frac{m_j}{u_k - u_j}, \quad k = 1, 2, \ldots, l.
\]

Formula (3.15) follows immediately from (3.13).

To prove (3.14), we shall use the notations in Lemma 3.1. Let \(s \neq t, s, t \in \{1, \ldots, l\}, i \in I_t, \quad j \in I_s\). We calculate using Lemma 3.1 that

\[
(3.17) \quad R_{ijji} = 2 \sum_{p \in I_s \cup I_t} \frac{r_j - r_p}{r_i - r_p} (\omega^i_p)^2.
\]

We obtain from (3.17) consequently:

\[
\sum_{j \neq I_t} \frac{R_{ijji}}{r_i - r_j} = \sum_{s=1}^{l} \frac{R_{ijji}}{r_i - r_j} = \sum_{s=1}^{l} \sum_{j \in I_s} \frac{2(r_j - r_p)}{(r_i - r_p)(r_i - r_j)} (\omega^i_p)^2 =
\]

\[
2 \sum_{s, r=1}^{l} \sum_{j \in I_s, p \in I_r} \frac{r_j - r_p}{(r_i - r_p)(r_i - r_j)} (\omega^i_p)^2,
\]

\[
\sum_{j \neq I_t} \frac{R_{ijji}}{r_i - r_j} = \sum_{p=1}^{n} \frac{R_{ippi}}{r_i - r_p} = \sum_{r=1}^{l} \sum_{p \in I_r} \frac{R_{ippi}}{r_i - r_p} =
\]

\[
2 \sum_{r=1}^{l} \sum_{p \in I_r} \sum_{j \neq I_t} \frac{2(r_p - r_j)}{(r_i - r_j)(r_i - r_p)} (\omega^i_p)^2 =
\]

6
\[ 2 \sum_{s, r = 1}^{l} \sum_{p \in I_r, j \in I_s} \frac{r_p - r_j}{(r_i - r_j)(r_i - r_p)} (\omega_{ij}^p)^2. \]

We get comparing the latter equalities that

\[ (3.18) \quad \sum_{j = 1}^{n} \frac{R_{ijji}}{r_i - r_j} = 0. \]

On the other hand, we calculate using (2.1) that

\[ (3.19) \quad \sum_{j \notin I_t} R_{ijji} = \frac{1}{n-1} \sum_{j \notin i} \left( \frac{m_j(r_i + r_j)}{r_i - r_j} - \frac{m_j s}{(n-2)(r_i - r_j)} \right). \]

We get (3.16) from (3.18) and (3.19). Since \( l \geq 3 \), then (3.14) follows from (3.16). We get from (3.16) that

\[ (3.20) \quad \sum_{k=1}^{l} (n - m_k) m_k u_k = 0; \quad \sum_{k=1}^{l} \frac{(n - m_k) m_k}{u_k} = 0. \]

To prove i) let \( l = 3 \). Then the first equality of (3.20) implies that any two of the real numbers \( u_1, u_2, u_3 \) have the same sign and the third one has the opposite sign. Let \( u_2 u_3 > 0 \). If we set \( x_i = \frac{u_i}{u_1}, \quad i = 2, 3 \) then we obtain from (3.20) that

\[ (n - m_2) m_2 x_2 + (n - m_3) m_3 x_3 = -(n - m_1) m_1, \]

\[ (n - m_2) m_2 x_3 + (n - m_3) m_3 x_2 = -(n - m_1) m_1 x_2 x_3. \]

We get multiplying the latter two equalities that

\[ (3.21) (n - m_2) m_2 (n - m_3) m_3 (x_2 - x_3)^2 + (4m_1 m_2 m_3 (n - m_1) + 4m_2^2 m_3^2) x_2 x_3 = 0. \]

The left hand side of (3.21) is strictly positive since \( x_2 x_3 > 0 \). This contradiction proves i).

We get from (3.16) that

\[ (3.22) \quad n - m_k = 2 \sum_{j=1}^{l} \frac{m_j u_j}{u_j - u_k}. \]

Using (3.22), we calculate

\[ (3.23) \quad \sum_{k=1}^{l} (n - 2m_k) m_k u_k^2 = -\left( \sum_{k=1}^{l} m_k u_k \right)^2. \]

Now, ii) follows immediately from (3.23). The assertion iii) and iv) are consequences of i) and ii). Thus, the whole Lemma is proved.

Q.E.D.

We are ready to prove the equivalence between b) and c).
If the Ricci tensor has exactly one eigenvalue then $M$ is an Einstein space. Hence, $M$ is a real space form since it is locally conformal flat.

If the Ricci tensor has exactly two distinct eigenvalues then the assertion follows from Lemma 3.2.

If $n = 4, 5, 6$, then Lemma 3.3 implies that the Ricci tensor has either one or two distinct eigenvalues and the assertion follows.

Let $n = 7$ or $8$. It follows from Lemma 3.3 that the Ricci tensor has one eigenvalue, or two distinct eigenvalues, or four distinct eigenvalues such that three of them are of the multiplicity one. We shall prove that the latter case is impossible for any $n \geq 4$.

Let we assume $r_1 \neq r_2 \neq r_3 \neq r_4 = \ldots = r_n$. Then $m_4 = n - 3$. We obtain from (3.16) that

$$3 = 2u_4 \sum_{i=1}^{3} \frac{1}{u_4 - u_i}.$$  

The latter equality is equivalent to the equality

$$(3.24) \quad 3u_4^3 - u_2^3 \sum_{i=1}^{3} u_i - u_4 \sum_{i,j=1}^{3} u_i u_j + 3u_1u_2u_3 = 0.$$  

We get from (3.20) that

$$(3.25) \quad \sum_{i=1}^{3} u_i = -\frac{3(n - 3)}{n - 1} u_4,$$

$$(3.26) \quad \sum_{i=1}^{3} \frac{1}{u_i} = -\frac{3(n - 3)}{n - 1} \frac{1}{u_4}.$$  

Using the latter two equalities, we obtain from (3.24) that

$$(3.27) \quad u_4^3 = -u_1u_2u_3.$$  

Setting $x_i = \frac{u_i}{u_4}, \quad i = 1, 2, 3$, we derive from (3.25), (3.26) and (3.27) that

$$(3.28) \quad \sum_{i=1}^{3} x_i = -\frac{3(n - 3)}{n - 1}; \quad \sum_{i,j=1}^{3} x_i x_j = \frac{3(n - 3)}{n - 1}; \quad x_1x_2x_3 = -1.$$  

The equalities (3.28) imply that $x_1, x_2, x_3$ are the roots of the following cubic equation

$$x^3 + \frac{3(n - 3)}{n - 1} x^2 + \frac{3(n - 3)}{n - 1} x + 1 = 0.$$  

It is easy to see that this cubic equation has exactly one real and two complex roots. This contradiction proves the equivalence of b) and c).

Now, it follows from Theorem 2.1 that every $n$-dimensional ($4 \leq n \leq 8$) locally conformal flat $C$-space has to be locally isometric to one of the spaces described in i), ii) and iii). It is clear that every Riemannian manifold of the type i), ii) or iii) is locally symmetric. Hence, it is a $C$-space.
Conversely, every locally conformal flat Riemannian locally symmetric space is locally isometric to one of the spaces described by i), ii) or iii). This completes the proof of Theorem 1.1. 

Remark. Riemannian manifolds for which the natural skew-symmetric curvature operators \( \kappa_{x,y}(z) := R(x,y)z, \ x,y,z \in T_pM, p \in M \) have constant eigenvalues along every unit circle are called \( O \)-spaces (see [7]). Riemannian manifolds for which the natural skew-symmetric curvature operators \( \kappa_{x,y} \) have parallel Jordanian bases along every unit circle are called \( T \)-spaces (see [8]). Following the proof of Theorem 2.1, it is not difficult to see that any 4-dimensional locally conformal flat \( O \)-space as well as any 4-dimensional locally conformal flat \( T \)-space has constant Ricci eigenvalues. Hence, any 4-dimensional locally conformal flat \( O \)-space as well as any 4-dimensional locally conformal flat \( T \) space is locally (almost everywhere) isometric to one of the spaces described in c) of Theorem 1.1.

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