A NOTE ON NON-CLASSICAL NONSTANDARD ARITHMETIC

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Abstract. Recently, a number of formal systems for Nonstandard Analysis restricted to the language of finite types, i.e. nonstandard arithmetic, have been proposed. We single out one particular system by Dinis-Gaspar, which is categorised by the authors as being part of intuitionistic nonstandard arithmetic. Their system is indeed inconsistent with the Transfer axiom of Nonstandard Analysis, and the latter axiom is classical in nature as it implies (higher-order) comprehension. In this paper, we answer the following questions:

(Q1) In the spirit of Reverse Mathematics, what is the minimal fragment of Transfer that is inconsistent with the Dinis-Gaspar system?
(Q2) What other axioms are inconsistent with the Dinis-Gaspar system?

Perhaps surprisingly, the answer to the second question shows that the Dinis-Gaspar system is inconsistent with a number of (non-classical) continuity theorems which one would—in our opinion—categorise as intuitionistic. Finally, we show that the Dinis-Gaspar system involves a standard part map, suggesting this system also pushes the boundary of what still counts as ‘Nonstandard Analysis’ or ‘internal set theory’.

1. Introduction

1.1. Aim and motivation. In the last decade, a number of versions of Heyting and Peano arithmetic in all finite types have been introduced (2, 4, 6, 7) which are based on (fragments of) Nelson’s internal set theory (15). Such systems allow for the extraction of the (copious) computational content of Nonstandard Analysis, as discussed at length in 19. In this paper, we study the system $M$ by Dinis-Gaspar to be found in 4 and Section 2. $M$ has been described as follows:

We present a bounded modified realisability and a bounded functional interpretation of intuitionistic nonstandard arithmetic with nonstandard principles. ([4, Abstract], emphasis added)

Similar claims may be found in the body of 4: $M$ is part of intuitionistic mathematics, as claimed by the authors. This claim is not without merit: $M$ is indeed inconsistent with the Transfer axiom of Nonstandard Analysis, and this axiom is essentially the nonstandard version of comprehension. By way of an example, Transfer restricted to $\Pi_1^0$-formulas translates to the ‘Turing jump functional’ $\exists^2$, as defined in Section 3.1.2. However, ‘non-classical’ does not necessarily imply ‘intuitionistic’, and we shall study the following two questions in this paper.

(Q1) In the spirit of Reverse Mathematics, what is the minimal fragment of Transfer that is inconsistent with $M$?
(Q2) What other (intuitionistic) axioms are inconsistent with $M$?
Regarding (Q1), we refer to [21–23] for an introduction and overview of Reverse Mathematics (RM for short). We shall consider the parameter-free Transfer principle studied in [3], and related axioms. We will identify the axiom of extensionality (relative to the standard world) as the real culprit: this axiom follows from Transfer and is inconsistent with $M$, while other axioms implied by Transfer, even involving the Turing jump functional, are consistent with $M$.

Regarding (Q2), we show that $M$ is inconsistent with a number of (non-classical) axioms which one would categorise as intuitionistic, i.e. part of Brouwer’s intuitionistic mathematics. The most blatant example is the statement that, relative to the standard world, all functionals on Cantor space are (uniformly) continuous.

In the course of investigating (Q1) and (Q2), one eventually stumbles upon the fact that the Dinis-Gaspar system allows one to define a (highly elementary) standard part map, as discussed in Section 3.3. Since such a map is not available in Nelson’s internal set theory, and external in Robinson’s approach, the Dinis-Gaspar system thus pushes the boundary of what still counts as ‘Nonstandard Analysis’.

As to the structure of this paper, we briefly discuss the importance of continuity in intuitionism in Remark 1.1. The formal system $M$ from [4] and associated prerequisites are sketched in Section 2. Our main results may be found in Section 3, which provide fairly definitive answers to questions (Q1) and (Q2).

Finally, we point out the (intimate) relationship between Brouwer’s intuitionistic mathematics and continuity, lest the reader believe the above is merely pedantry.

**Remark 1.1** (Intuitionism and continuity). L.E.J. Brouwer is the founder of intuitionism, a philosophy of mathematics which later developed into the first full-fledged school of constructive mathematics. The latter is an umbrella term for approaches to mathematics in which ‘there exists $x$’ is systematically interpreted as ‘we can compute/construct $x$’ (and similar interpretations for the other logical symbols). Under this new interpretation of the logical symbols, certain laws do not make any sense, and are therefore rejected; the most (in)famous one being the law of excluded middle $P \lor \neg P$. The resulting logic is intuitionistic logic, and we refer to [1, 25] for an introduction to the various approaches to constructive mathematics.

Brouwer proved in 1927 (See [8, p. 444] for an English translation) that every total (in the intuitionistic sense) function on the unit interval is (uniformly) continuous, a result which seems to contradict classical mathematics. The core axioms for intuitionistic mathematics indeed include a ‘continuity’ axiom (called WC-N in [25] and $\text{BP}_0$ in [1]) which contradicts classical mathematics, and can be used to prove the aforementioned (uniform) continuity theorem by Brouwer.

The previous is well-known, but is mentioned since we want to stress the following: a very low bar a logical system has to clear to deserve the monicker ‘intuitionistic’, is to be consistent with the aforementioned continuity theorem and axiom. As it turns out, this does not seem to be the case for the Dinis-Gaspar system.

**2. Preliminaries**

We introduce the Dinis-Gaspar system $M$, and some preliminaries and notations.

2.1. **Internal set theory and its fragments.** In this section, we discuss Nelson’s internal set theory, first introduced in [15], and its fragment $M$ from [4].
In Nelson’s syntactic approach to Nonstandard Analysis ([15]), as opposed to Robinson’s semantic one ([18]), a new predicate ‘st(x)’, read as ‘x is standard’ is added to the language of ZFC, the usual foundation of mathematics. The notations \((\forall^* x)\) and \((\exists^* y)\) are short for \((\forall x)(\text{st}(x) \rightarrow \ldots)\) and \((\exists y)(\text{st}(y) \land \ldots)\). A formula is called internal if it does not involve ‘st’, and external otherwise. The three external axioms Idealisaton, Standard Part, and Transfer govern the new predicate ‘st’; They are respectively defined as:

1. \((\forall^* \inf x)(\exists y)(\forall z \in x)\varphi(z, y) \rightarrow (\exists y)(\forall^* x)\varphi(x, y)\), for internal \(\varphi\).
2. \((\forall x)(\exists^* y)(\forall^* z)[z \in x \land \varphi(z)] \leftrightarrow z \in y\), for any \(\varphi\).
3. \((\forall^* x)\varphi(x, t) \rightarrow (\forall x)\varphi(x, t)\), where \(\varphi\) is internal, \(t\) captures all parameters of \(\varphi\), and \(t\) is standard.

The system IST is (the internal system) ZFC extended with the aforementioned three external axioms; The former is a conservative extension of ZFC for the internal language, as proved in [15].

In [2, 3, 6, 7], the authors study Gödel’s system \(T\) extended with versions of the external axioms of IST. In particular, they consider nonstandard extensions of the (internal) systems E-HA\(^{\omega}\) and E-PA\(^{\omega}\), respectively Heyting and Peano arithmetic in all finite types and the axiom of extensionality. We refer to [2, §2.1] for the exact details of these (mainstream in mathematical logic) systems.

The results in [1, 6] are inspired by those in [2]. In particular, the notion of finiteness central to the latter is replaced by the notion of strong majorizability. The latter notion and the associated system \(M\) is introduced in the next paragraph, assuming familiarity with the higher-type framework of Gödel’s \(T\).

The system \(M\), a conservative extension of E-HA\(^{\omega}\), is based on the Howard-Bezem notion of strong majorizability. We first introduce the latter and related notions. For more extensive background on strong majorizability, we refer to [11, §3.5].

**Definition 2.1** (Majorizability). The strong majorizability predicate ‘\(\leq^*\)’ is inductively defined as follows:

- \(x \leq^*_0 y\) is \(x \leq_0 y\);
- \(x \leq^*_{\rho \sigma} y\) is \((\forall u)(\forall v \leq^*_{\rho} u)(xu \leq^*_{\rho} yv \land yu \leq^*_{\sigma} yv)\).

An object \(x^\rho\) is called a monotone if \(x \leq^*_{\rho} x\). The quantifiers \((\forall x^\rho)\) and \((\exists y^\rho)\) range over the monotone objects of type \(\rho\), i.e. they are abbreviations for the formulas \((\forall x)(x \leq^* x \rightarrow \ldots)\) and \((\exists y)(y \leq^* y \land \ldots)\).

The system \(M\) is defined as follows in [3, §2]. The language of E-HA\(_{st}\) is the language of E-HA\(^{\omega}\) extended with a new ‘standardness’ predicate \(\text{st}^\sigma\) for every finite type \(\sigma\). The typing of the standardness predicate is usually omitted.

**Definition 2.2** (Standard quantifiers). We write \((\forall^* x^\rho)\Phi(x^\tau)\) and \((\exists^* x^\sigma)\Psi(x^\sigma)\) as short for \((\forall x^\tau)[\text{st}(x^\sigma) \rightarrow \Phi(x^\tau)]\) and \((\exists x^\sigma)[\text{st}(x^\sigma) \land \Psi(x^\sigma)]\). A formula \(A\) is ‘internal’ if it does not involve \(\text{st}\), and external otherwise. The formula \(A^\tau\) is defined from \(A\) by appending ‘\(\text{st}\)’ to all quantifiers (except bounded number quantifiers).

**Definition 2.3.** [Basic axioms] The system E-HA\(_{st}\) is defined as E-HA\(^{\omega}\) + \(T^*_{st} + 1A^\tau\), where \(T^*_{st}\) consists of the following axiom schemas.  

\(^1\) The superscript ‘fin’ in (I) means that \(x\) is finite, i.e. its number of elements are bounded by a natural number.
(a) \( x =_\sigma y \rightarrow (\text{st}^\sigma(x) \rightarrow \text{st}^\sigma(y)) \);
(b) \( \text{st}^\sigma(y) \rightarrow (x \leq^\sigma y \rightarrow \text{st}^\sigma(x)) \);
(c) \( \text{st}^\sigma(t), \) for each closed term \( t \) of type \( \sigma \);
(d) \( \text{st}^\sigma \rightarrow \tau(z) \rightarrow (\text{st}^\tau(x) \rightarrow \text{st}^\tau(zzx)) \).

Items (a)-(d) are called the standardness axioms, and (b) is singled out regularly below. The external induction axiom \( \text{IA}^\sigma \) is the following schema for any \( \Phi \):

\[
\Phi(0) \land (\forall n^0)(\Phi(n) \rightarrow \Phi(n + 1)) \rightarrow (\forall n^0)\Phi(n).
\]  

(\( \text{IA}^\sigma \))

The system \( M \) is then defined as \( \text{E-HA}^\omega \) plus the following non-basic axioms.

**Definition 2.4.** [Non-basic axioms]

- Monotone Choice \( \text{mAC}^\omega \): For any \( \Phi \), we have
  \[
  (\forall x)(\exists y)(\exists f)(\forall y \leq^* f(x))\Phi(x, y).
  \]
- Realization \( \text{R}^\omega \): For any \( \Phi \), we have
  \[
  (\forall x)(\exists y)(\exists f)(\forall y \leq^* f(x))\Phi(x, y).
  \]
- Idealisation \( \text{I}^\omega \): For any internal \( \phi \), we have:
  \[
  (\exists^\omega z)(\forall y \leq^* z)\phi(x, y) \rightarrow (\exists z)(\forall y \leq^* z)\Phi(x, y).
  \]
- Independence of premises \( \text{IP}^\omega_{\phi, \psi} \): For any internal \( \phi, \psi \), we have:
  \[
  [(\exists^\omega x)\phi(x) \rightarrow (\exists^\omega y)\psi(y)] \rightarrow [(\exists^\omega z)(\forall x \leq^* z)\phi(x) \rightarrow (\exists^\omega y \leq^* z)\psi(y)]
  \]
- Markov’s principle \( \text{M}^\omega \): For any internal \( \phi, \psi \), we have
  \[
  [(\exists^\omega x)\phi(x) \rightarrow \psi] \rightarrow [(\exists^\omega y \leq^* y)\phi(x) \rightarrow \psi]
  \]
- Majorizability axiom \( \text{MAJ}^\omega \): \( (\exists^\omega x)(\exists y)(x \leq^* y)\)

Other axioms are mentioned in [4, \$4], but these are derivable in \( M \). The variables are not specified for \( \text{I}^\omega \) in [4], and we have chosen the version also used in IST.

2.2. **Notations in \( M \).** In this section, we introduce notations relating to \( M \).

First of all, we will use the usual notations for rational and real numbers and functions as introduced in [13, p. 288-289] (and [22, I.8.1] for the former).

**Definition 2.5** (Real numbers and related notions in \( \text{RCA}^\omega \)).

- Natural numbers correspond to type zero objects, and we use ‘\( n^0 \)’ and ‘\( n \in \mathbb{N} \)’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘\( q \in \mathbb{Q} \)’ and ‘\( <_\mathbb{Q} \)’ have their usual meaning.
- Real numbers are coded by fast-converging Cauchy sequences \( q(\cdot) : \mathbb{N} \rightarrow \mathbb{Q} \), i.e. such that \( (\forall n^0, i^0)((q_n - q_{n+i}) <_\mathbb{Q} 1/i) \). We use Kohlenbach’s ‘hat function’ from [13, p. 289] to guarantee that every \( f^1 \) defines a real number.
- We write ‘\( x \in \mathbb{R} \)’ to express that \( x^1 := (q(\cdot)_1) \) represents a real as in the previous item and write \( [x](k) := q_k \) for the \( k \)-th approximation of \( x \).
- Two reals \( x, y \) represented by \( q(\cdot) \) and \( r(\cdot) \) are equal, denoted \( x =_R y \), if \( (\forall n^0)((q_n - r_n) \leq 1/2^n) \). Inequality ‘\( <_R \)’ is defined similarly.
- Functions \( F : \mathbb{R} \rightarrow \mathbb{R} \) mapping reals to reals are represented by \( \Phi^{1\rightarrow 1} \) mapping equal reals to equal reals, i.e. \( (\forall x, y)(x =_R y \rightarrow \Phi(x) =_R \Phi(y)) \).
- Sets of type \( \rho \) objects \( X^{\rho\rightarrow 0}, Y^{\rho\rightarrow 0}, \ldots \) are given by their characteristic functions \( f_X^{\rho\rightarrow 0}, \) i.e. \( (\forall x^\rho)[x \in X \leftrightarrow f_X(x) =_R 1] \), where \( f_X^{\rho\rightarrow 0} \leq_{\rho\rightarrow 0} 1 \).
Secondly, we use the usual extensional notion of equality.

**Remark 2.6 (Equality).** The system $M$ includes equality between natural numbers `$=0$' as a primitive. Equality `$=\tau$' for type $\tau$-objects $x, y$ is then defined as follows:

$$[x =_\tau y] \equiv (\forall z_1^\tau \ldots z_k^\tau)[xz_1 \ldots z_k =_0 yz_1 \ldots z_k]$$

(2.1)

if the type $\tau$ is composed as $\tau \equiv (\tau_1 \rightarrow \ldots \rightarrow \tau_k \rightarrow 0)$. The inequality `$\leq_\tau$' is just (2.1) with `$\leq_0$', i.e. binary sequences are given by $f \leq_1 1$, which we also denote as `$f \in C$' or `$f \in 2^{\mathbb{N}}$'. We define `approximate equality $\approx_\tau$' as follows:

$$[x \approx_\tau y] \equiv (\forall z_1^\tau \ldots z_k^\tau)[xz_1 \ldots z_k =_0 yz_1 \ldots z_k]$$

(2.2)

with the type $\tau$ as above. The system $M$ includes the axiom of extensionality:

$$(\forall x^\rho, y^\rho)[x =_\rho y \rightarrow \varphi(x) =_\tau \varphi(y)].$$

(\mathbf{E}_{\rho\rightarrow\tau})

for all finite types. We write $\mathbf{E}$ for the collection of all axioms $\mathbf{E}_{\rho\rightarrow\tau}$.

Finally, we introduce some notation to handle finite sequences nicely.

**Notation 2.7 (Finite sequences).** We assume the usual coding of finite sequences of the same type. We denote by `$|s| = n$' the length of the finite sequence $s = (s_0^\rho, s_1^\rho, \ldots, s_n^\rho)$, where $|\epsilon| = 0$, i.e. the empty sequence has length zero. For sequences $s, t$ of the same type, we denote by `$s * t$' the concatenation of $s$ and $t$, i.e. $(s * t)(i) = s(i)$ for $i < |s|$ and $(s * t)(j) = t(|s| - j)$ for $|s| \leq j < |s| + |t|$. For a finite sequence $s$, we define $\pi N := (s(0), s(1), \ldots, s(N - 1))$ for $N^0 < |s|$. For a sequence $\alpha^{0\rightarrow\rho}$, we also write $\pi N = (\alpha(0), \alpha(1), \ldots, \alpha(N - 1))$ for any $N^0$.

3. Main results

Our main results fall into three main categories, as follows.

(i) In answer to (Q1), we show in Section 3.1 that $M$ is inconsistent with certain (very) weak fragments of Transfer, but (oddly) not with others.

(ii) In answer to (Q2), we show in Section 3.2 that $M$ is inconsistent with certain intuitionistic axioms, relative to the standard world; we also show that $M$ does prove weak König’s lemma, again relative to the standard world.

(iii) Inspired by these answers to (Q1) and (Q2), we show in Section 3.3 that $M$ involves a highly elementary standard part map.

In light of the first two items, it seems that $M$ is not really a system of intuitionistic arithmetic (but non-classical nonetheless), while the third item shows that $M$ already pushes the boundary of what still counts as ‘Nonstandard Analysis’.

3.1. Non-classical aspects of the Dinis-Gaspar system. We provide a partial answer to question (Q1) from Section 1.1 by showing that $M$ is inconsistent with various weak fragments of Transfer, including parameter-free Transfer from [3], and the Turing jump functional $\exists^2$ from e.g. [13], relative to the standard world.

3.1.1. Parameter-free Transfer. We show that various extensions of $M$, also involving intuitionistic axioms, are inconsistent with parameter-free Transfer as follows.

**Principle 3.1 (PF-TP$_3$).** For internal $\varphi(x)$ with all free variables shown, we have

$$(\exists x)[\varphi(x) \rightarrow (\exists^x x)\varphi(x)].$$

(3.1)
To be absolutely clear, (standard) parameters are not allowed in $\varphi(x)$ as in (3.1).

In contrast to richer fragments of Transfer, PF-TP$_3$ is weak: when added to (fragments of) the classical system from [2], one obtains a conservative extension, by [3, §3.2]. The results in [3,20] establish that PF-TP$_3$ yields a smooth development of the (classical) Reverse Mathematics of Nonstandard Analysis.

Now, QF-AC is weak from the point of view of Reverse Mathematics, as discussed in [3, Remark 4.6], while certain fragments of the axiom of choice (including QF-AC$^{2,0}$) are widely accepted in constructive and intuitionistic mathematics (See e.g. [11]). Finally, Markov’s principle MP is defined as follows:

$$(\forall f^1)[\neg(\exists n)(f(n) = 0)] \to (\exists n)(f(n) = 0)]. \quad (\text{MP})$$

and is rejected in intuitionistic mathematics (See e.g. [23] p. 237).

**Theorem 3.2.** The system $\mathcal{M} + \text{PF-TP}_3 + \text{QF-AC}^{2,0} + \text{MP}$ is inconsistent.

**Proof.** Recall that $\mathcal{M}$ includes the axiom of extensionality $(E_2)$, which implies

$$(\forall Y^2, f^1, g^1)(\exists N^0)\big[\overline{\mathcal{N}} = \overline{\mathcal{N}}\to Y(f) = Y(g)],$$

by Markov’s principle. Applying QF-AC$^{2,0}$, we obtain $\Phi_0^{2,+0}$ such that

$$(\forall Y^2, f^1, g^1)(\exists N^0 \leq \Phi_0(Y, f, g))\big[\overline{\mathcal{N}} = \overline{\mathcal{N}} \to Y(f) = Y(g)].$$

Applying PF-TP$_3$, there is standard such $\Phi_0$, yielding that

$$(\forall Y^2, f^1, g^1)[f \equiv_1 g \to Y(f) = Y(g)], \quad (3.2)$$

since $\Phi_0(Y, f, g)$ is standard for standard inputs. Note that (3.2) is $(E_2)$st, i.e. the axiom of ‘standard extensionality’. Now consider the functional $Y^2_0$ defined as:

$$Y_0(f) := \begin{cases} 0 & (\exists n \leq N + 1)(f(n) = 0) \\ 1 & \text{otherwise} \end{cases}, \quad (3.3)$$

where $N^0$ is nonstandard. Since $Y^2_0 \leq_2 1$, and the constant-one-mapping of type two is standard, $Y_0$ is also standard by item (b) in the standardness axioms. Hence, $Y_0$ satisfies (3.2) and now consider $f_0 := 11\ldots$ and $g_0 := \overline{\mathcal{F}_0 N} \ast 00\ldots$, which satisfy $f_0 \equiv_1 g_0$ and $Y(f_0) = 0 \neq 1 = Y(g_0)$. Note that $g_0$ is standard by the aforementioned item (b), as $g_0 \leq_1 1$ and the constant-one-mapping of type one is standard, i.e. (3.2) yields a contradiction. \hfill $\square$

Next, we show that the previous proof also goes through using a fragment of Markov’s principle, called weak Markov’s principle (WMP for short; see [9]). Most importantly for us, WMP is accepted in intuitionistic mathematics (but not in Bishop’s constructive mathematics by the results in [11]). We will actually use the following version of WMP, defined in [10]:

$$(\forall Y^2, f^1, g^1)Y(f) \neq_0 Y(g) \to f \neq_1 g). \quad (SE)$$

Since ‘$x \neq y$’ is generally a stronger statement than ‘$\neg(x = y)$’ in constructive mathematics, SE is said to express strong extensionality.

**Corollary 3.3.** The system $\mathcal{M} + \text{PF-TP}_3 + \text{QF-AC}^{2,0} + \text{SE}$ is inconsistent.

\footnote{Note that \text{SE} is actually weaker than WMP, but \text{SE} $\leftrightarrow$ WMP by [10] Thm. 11 in Bishop's constructive mathematics plus a non-trivial fragment of the axiom of choice.}
We first prove

\[ (\forall Y^2, f, g^1)(\exists N^0)(Y(f) \neq 0 Y(g) \rightarrow \overline{f}N \neq 0 \overline{g}N) \]

As for the theorem, one derives (E2)\textsuperscript{st} and \( Y_0 \) yields a contradiction. \( \square \)

The inconsistency in the theorem also pops up when combining \( PF-TP_3 \) with intuitionistic axioms, like the intuitionistic fan functional \([13,24]\) as follows.

\[ (\exists \Omega^3)(\forall f, g \leq 1)(\overline{f}\Omega(Y) = \overline{g}\Omega(Y) \rightarrow Y(f) = Y(g)) \]

(MUC)

Corollary 3.4. The system \( M + PF-TP_3 + MUC \) is inconsistent.

Proof. Since MUC is a sentence, \( PF-TP_3 \) guarantees the existence of a standard \( \Omega^3 \) as in the former. Now consider \( Y_0, f_0, g_0 \) from the proof of the theorem and note that \( \Omega(Y_0) \) is a standard number. Hence, since \( f_0 \approx_1 g_0 \) by definition, we have \( 0 = Y(f_0) = Y(g_0) = 1 \), a contradiction. \( \square \)

We note that the inconsistency of \( M \) with much stronger fragments of Transfer is proved in [3] Theorem 29. In particular, Transfer for \( \Pi^1_1 \)-formulas is used in the latter, which readily translates to the Turing jump functional \( \exists^2 \) in the systems from [2]. As it happens, we study \( \exists^2 \) in the next section.

We do not know whether the classical contraposition of (3.1) also leads to inconsistency, but we now show that it implies the fan theorem, as follows.

\[ (\forall T^1 \leq 1)(\forall \alpha \leq 1)(\exists \alpha^0)(\alpha^0 \notin T) \rightarrow (\exists \beta \leq 1)(\beta \notin T) \]

(FAN)

The variable ‘\( T^1 \)’ is reserved for trees, while ‘\( T \leq 1 \)’ means that \( T \) is a binary tree.

Principle 3.5 (PF-TP\( \vee \)). For internal \( \varphi(\underline{x}) \) with all free variables shown, we have

\[ (\forall x^1)\varphi(\underline{x}) \rightarrow (\forall \underline{x})\varphi(\underline{x}). \]  

(3.4)

To be absolutely clear, (standard) parameters are not allowed in \( \varphi(\underline{x}) \) as in (3.1).

Theorem 3.6. The system \( M + QF-AC^{1,0} + PF-TP_\vee \) proves FAN.

Proof. We first prove \( FAN^{st} \). If \( (\forall f \leq 1)(\exists n^0)(\overline{\alpha}n \notin T) \), then we have \( (\forall f \leq 1)(\exists n^0)(\overline{\alpha}n \notin T) \) since all binary sequences are standard by item (b) of the non-standard axioms. Applying \( R^\vee \), we obtain \( (\exists^1 k^0)(\forall f \leq 1)(\exists n^0 \leq k)(\overline{\alpha}n \notin T) \), and \( FAN^{st} \) follows. The latter immediately implies that

\[ (\forall T^1 \leq 1, G^2)(\forall \alpha \leq 1)(\exists m^0 \leq G(\alpha)(\overline{\alpha}m \notin T)) \rightarrow (\exists n^0 \leq k)(\overline{\alpha}n \notin T) \].

Now drop the ‘\( st \)’ predicates inside the square brackets and apply \( PF-TP_\vee \). The resulting formula then yields \( FAN \), thanks to \( QF-AC^{1,0} \). \( \square \)

Note that \( QF-AC^{1,0} \) is ‘innocent’ in that it is included in the base theory of higher-order Reverse Mathematics (See [13]). Next, we show that \( PF-TP_\vee \) leads to the ‘full’ Heine-Borel compactness of Cantor space (for uncountable covers), as in:

\[ (\forall G^2)(\exists(\beta_0, \ldots, \beta_k))(\forall \alpha \leq 1)(\exists i \leq k)(\alpha \in [\overline{\beta}iG(\beta_i)]). \] \hspace{1cm} (HBU\(_C\))

Intuitively, any functional \( G^2 \) gives rise to the ‘canonical’ cover \( \bigcup_{f \in C}[fG(f)] \) of Cantor space, and \( HBU\(_C\) \) tells us that the latter always has a finite sub-cover.

Theorem 3.7. The system \( M + PF-TP_\vee \) proves \( HBU\(_C\) \).
Proof. Since every binary sequence is standard in $M$, we have $(\forall \alpha \leq 1)(\exists \beta \leq 1)(\alpha \approx 1 \beta)$, i.e. the nonstandard compactness of Cantor space. However, the usual proof that the latter is equivalent to $HBU_C$ (See [3][20]) does not go through in $M$ due to the weak conclusion of $R^\omega$. Instead, we prove (3.5), which immediately yields $HBU_C$ via $PF-TP_\forall$.

$$(\forall^\omega G^2)(\exists(\beta_0, \ldots, \beta_k))(\forall \alpha \leq 1)(\exists i \leq k)(\alpha \in [\beta_i]G(\beta_i))) \quad (3.5)$$

To prove (3.5), fix nonstandard $N$ and apply $IP^*_\forall$. Define the (standard) functional $\rho$.

Proof.

Thirdly, the axiom $HBU_C$ is extremely hard to prove: by [17, §3.1], $\Pi^0_1$-CA$^C_0$ does not prove $HBU_C$ (for all $k$), where the former is $RCA^C_0$ plus the existence of $S^2_k$, a functional which decides the truth of $\Sigma^0_k$-formulas (only involving type one parameters). Hence, $M + PF-TP_\forall$ is a rather peculiar system.

3.1.2. The Turing jump functional. We show that the Dinis-Gaspar system is inconsistent with $(\exists^\omega)^st$, where the latter is given by:

$$(\exists^\omega^2)(\forall^f)(\exists n^0)(f(n) = 0) \leftrightarrow \varphi(f) = 0). \quad (\exists^\omega)$$

Note that $\Pi^0_1$-TRANS $\rightarrow (\exists^\omega)^st$ by the proof of Corollary 3.9, where the former is:

$$(\forall^\omega^1)[(\exists^\omega^1)(f(n) = 0) \rightarrow (\forall n)(f(n) = 0)],$$

i.e. $(\exists^\omega)^st$ follows from a fragment of Transfer using $M^\omega$.

Theorem 3.8. The system $M + (\exists^\omega)^st$ is inconsistent.

Proof. Define the (standard) functional $Z^{1\rightarrow 1}$ as $Z(f)(n) = 0$ if $f(n) = 0$, and 1 otherwise. Since $Z(f) \leq^* 1$, the binary sequence $Z(f)$ is standard for any input $f$, due to item (b) of the standardness axioms. Hence, $(\exists^\omega)^st$ immediately yields:

$$(\exists^\omega^2)(\forall^f)(\exists n^0)(f(n) = 0) \leftrightarrow \varphi_0(f) = 0), \quad (3.6)$$

by taking $\varphi_0 := \varphi \circ Z$ for $\varphi$ as in $(\exists^\omega)^st$. Clearly, (3.6) implies

$$(\forall^f)[\varphi_0(f) = 0 \rightarrow (\exists^\omega^0)(f(n) = 0)],$$

and applying $IP^\omega_{\varphi_0}$ yields

$$(\forall^f)(\exists^\omega^0)[\varphi_0(f) = 0 \rightarrow (\exists^\omega^0)(f(n) = 0)],$$

while applying $R^\omega$ yields:

$$(\exists^\omega^k)(\forall^f)(\exists^\omega^0 \leq k)[\varphi_0(f) = 0 \rightarrow (\exists^\omega^0)(f(n) = 0)],$$

which contradicts $\varphi_0(11\ldots1100\ldots) = 0$ if there are enough instances of 1.

The following corollary also follows from the proof of [4] Theorem 29.

Corollary 3.9. The system $M + \Pi^0_1$-TRANS is inconsistent.
Our formulation of ACA in Theorem 3.8 is due to the presence of third-order objects. ACA is essentially equivalent. We single out 3.1.3. Arithmetical comprehension. Corollary 3.11. Thus, $M$ equivalence involving SE equivalences use some form or other). Thus, to derive (intuitionistic) continuity theorems from $\neg \exists$ is equivalent to ($\phi$ and defining $M$ of extensionality is not rejected in constructive (esp. intuitionistic) mathematics, world, and not e.g. the Turing jump functional as in Corollary 3.10. Since the axiom Proof. Using the same trick involving $Z$ as in the theorem, $(\exists^t \varphi^2) (\forall^t f^1) \text{TJ}(f, \varphi)$ is equivalent to $(\exists^t \varphi^2)(\forall f^1)\text{TJ}(f, \varphi)$. The latter follows from $(\exists^2)$ by taking such $\varphi$ and defining $\varphi_1(f) = 1$ if $\varphi(f) \neq 0$, and 0 otherwise. Since $\varphi_1 \leq^* 1$, this functional is standard by item (b) of the standardness axioms. As $(\exists^2)$ is internal, the theorem now follows from the soundness theorem as in [4, Theorem 16], since $E\text{-HA}_\omega$ and $E\text{-HA}_\omega$ prove the same internal formulas. □

Corollary 3.10. $M + (\exists^t \varphi^2)(\forall^t f^1)\text{TJ}(f, \varphi)$ is consistent if $E\text{-HA}_\omega + (\exists^2)$ is.

Proof. Using the same trick involving $Z$ as in the theorem, $(\exists^t \varphi^2)(\forall^t f^1)\text{TJ}(f, \varphi)$ is equivalent to $(\exists^t \varphi^2)(\forall f^1)\text{TJ}(f, \varphi)$. The latter follows from $(\exists^2)$ by taking such $\varphi$ and defining $\varphi_1(f) = 1$ if $\varphi(f) \neq 0$, and 0 otherwise. Since $\varphi_1 \leq^* 1$, this functional is standard by item (b) of the standardness axioms. As $(\exists^2)$ is internal, the theorem now follows from the soundness theorem as in [4, Theorem 16], since $E\text{-HA}_\omega$ and $E\text{-HA}_\omega$ prove the same internal formulas. □

Corollary 3.11. The system $M + (E_2)^{st}$ is inconsistent.

It should be noted that $M$ is even inconsistent with the rule version of the axiom $(E_2)^{st}$. Indeed, $M$ proves that $Y_0, f_0, g_0$ from the proof of Theorem 3.2 are standard and satisfy $f_0 \approx_1 g_0$. However, a proof of $Y_0(f_0) = Y_0(g_0)$, say obtained by the aforementioned rule, then leads to a contradiction.

Corollary 3.12. The system $M + QF\text{-AC}^{1,0} + (\exists^2) + \text{PF}\text{-TP}_3$ is inconsistent.

Proof. Using QF-AC$^{1,0}$, $(\exists^2)$ readily implies $(\exists \varphi^2, \Psi^2)(\forall f^1)[((\forall n^0)(f(n) = 0) \rightarrow \varphi(f) = 0) \land (\varphi(f) = 0 \rightarrow f(\Psi(f)) = 0)]$, where $\Psi(f)$ is the least such $n$ if existent. By PF-TP$_3$, there is standard such $\Psi^2$, upon which we obtain $\Pi^1_1\text{-TRANS}$, a contradiction by Corollary 3.9. □

In conclusion, while $M$ is inconsistent with a number of fragments of Transfer, the inconsistency is really due to the axiom of extensionality relative to the standard world, and not e.g. the Turing jump functional as in Corollary 3.10. Since the axiom of extensionality is not rejected in constructive (esp. intuitionistic) mathematics, all we can say is that these results suggest that $M$ is non-classical.

Furthermore, $M$ proves $\neg(\exists^2)^{st}$ by Theorem 3.8 and classically $\neg(\exists^2)$ is equivalent to the continuity of all functionals on Baire space ([13] Prop. 3.7); a similar equivalence involving SE holds constructively by [10] Thm. 26]. However, these equivalences use Grilleti’s trick and hence require the axiom of extensionality (in some form or other). Thus, to derive (intuitionistic) continuity theorems from $\neg(\exists^2)^{st}$, one would need standard extensionality, which leads to inconsistency by Corollary 3.11. Thus, $M$ is definitely non-classical, but not really intuitionistic.

3.1.3. Arithmetical comprehension. We show that the Dinis-Gaspar system is inconsistent with $\text{ACA}_0$, where the latter is given by:

$(\forall f \leq 1)(\exists g \leq 1)(\forall n^0)([(\exists m)(f(n, m) = 0) \leftrightarrow g(n) = 0]$. (ACA$_0$)

Our formulation of ACA$_0$ is different from the version used in RM ([22, II]), but is essentially equivalent. We single out ACA$_0$ lest anyone believe the inconsistency in Theorem 3.8 is due to the presence of third-order objects.
Theorem 3.13. The system $\mathbf{M} + \text{ACA}_{\omega}^\text{st}$ is inconsistent.

Proof. Since all binary sequences are standard in $\mathbf{M}$, $\text{ACA}_{\omega}^\text{st}$ implies that for all $f \leq 1$, there is standard $g \leq 1$ such that

$$(\forall^\text{st} n^0)((\exists^\text{st} m)(f(n, m) = 0) \rightarrow g(n) = 0) \land (\forall^\text{st} k)[g(k) = 0 \rightarrow (\exists^\text{st} l)(f(k, l) = 0)].$$

The second conjunct yields $(\exists^\text{st} h^1)(\forall^\text{st} k)[g(k) = 0 \rightarrow (\exists l \leq h(k))(f(k, l) = 0)$ due to $\text{IP}_{\omega}^\text{st}$ and $\text{mAC}^\omega$. Thus, $(\forall f \leq 1)(\exists^\text{st} h)(\forall^\text{st} g \leq 1)A(f, g, h)$, where $A(f, g, h)$ is

$$(\forall^\text{st} n^0)((\exists^\text{st} m)(f(n, m) = 0) \rightarrow g(n) = 0) \land (\forall^\text{st} k)[g(k) = 0 \rightarrow (\exists l \leq h(k))(f(k, l) = 0)].$$

Since realisation $\mathbf{R}^\omega$ also applies to external formulas, we obtain

$$(\exists^\text{st} h^1)(\forall f \leq 1)(\exists h_0)(\exists^\text{st} g \leq 1)A(f, g, h) \quad (3.7)$$

Now define $f_0(n, m)$ as $0$ if $m > h_0(n)$, and $1$ otherwise, where $h_0$ is as in (3.7). For this $f_0$, (3.7) provides $g_0$, which satisfies by definition:

$$(\forall^\text{st} n)[(\exists^\text{st} m)(f_0(n, m) = 0) \rightarrow (g_0(n) = 0) \rightarrow (\exists m \leq h_0(n))(f_0(n, m) = 0)],$$

which contradicts the definition of $f_0$, and we are done.

It is tempting, but incorrect, to apply the reasoning from the previous proof to

$$(\forall f \leq 1)(\exists^\text{st} n)(f(n) = 0) \rightarrow (\exists^\text{st} m)(f(m) = 0). \quad (3.8)$$

Indeed, $\text{IP}_{\omega}^\text{st}$ does not allow pulling the underlined quantifier in (3.8) to the front.

Finally, while $\mathbf{M}$ proves the non-classical $\neg(\text{ACA}_{\omega}^\text{st})$, we show in Section 3.2.1 that it does prove the classical $\text{WKL}^\text{st}$, i.e. the latter does not lead to inconsistency.

3.1.4. Non-classical continuity. We show that relative to the standard world, extensional functions on Cantor space are automatically continuous on $C$. We also show that they are nonstandard continuous as follows:

$$(\forall^\text{st} f \in C)(\forall g \in C)(f \approx_1 g \rightarrow Y(f) = Y(g)). \quad (3.9)$$

Using $\mathbf{M}^\omega$ and $\mathbf{R}^\omega$, one readily shows that (3.9) implies ‘epsilon-delta’ continuity relative to the standard world, and the latter implies (3.9) using item (b) of the nonstandard axioms of $\mathbf{M}$. Note that uniform nonstandard continuity is (3.9) with the leading ‘st’ dropped.

Theorem 3.14. The system $\mathbf{M}$ proves that any $Y^2$ satisfying $(\mathbf{E}_2)^{\text{st}}$ is also nonstandard (uniformly) continuous on Cantor space.

Proof. Suppose $Y^2$ satisfies $(\mathbf{E}_2)^{\text{st}}$, which immediately yields:

$$(\forall f, g \in C)(\exists^\text{st} N^0)(\overline{f} N = \overline{g} N \rightarrow Y(f) = Y(g)). \quad (3.10)$$

using $\mathbf{M}^\omega$ and the fact that all binary sequences are standard in $\mathbf{M}$. Applying $\mathbf{R}^\omega$ to (3.10) yields that $Y^2$ is nonstandard (uniformly) continuous.

Note that by the proof Theorem 3.22 there are plenty (standard) functionals $Y^2$ which are not standard extensional as in $(\mathbf{E}_2)^{\text{st}}$.

Theorem 3.14 can be interpreted as saying that $\mathbf{M}$ has intuitionistic features (in that ‘more’ functionals are continuous than in classical mathematics), but the following corollary shows that something ‘much more non-classical’ is going on. A functional $Y^2$ is near-standard if $(\forall^\text{st} f^1)(\exists^\text{st} n)(Y(f) = n)$, as defined in [18, p. 93].
Corollary 3.15. The system $M$ proves that for any near-standard $Y^2$ satisfying $(E_2)^*$, there is standard $Z^2$ such that $(\forall f \in C)(Z(f) =_0 Y(f))$.

Proof. First of all, since all binary sequences are standard, $(\forall f \in C)(\exists^* n)(Y(f) = n)$ follows from the near-standardness of $Y$, and applying $R^\omega$ yields a standard upper bound $n_0$ for $Y^2$ on Cantor space. Fix nonstandard $N_0$ and define $Z(f)$ as $Y(fN_0 * 00\ldots)$ if $fN_0$ is a binary sequence, and $n_0$ otherwise. Then $Z(f) = Y(f)$ for $f \in C$ by standard extensionality, and $Z \leq^*_n n_0$ implies that $Z$ is standard. □

By the theorem, standard extensionality implies continuity relative to the standard world. Now, as discussed in [2, 19], one can naturally interpret the standardness predicate ‘st($x$)’ as ‘$x$ is computationally relevant’ using the systems from [2]. With this interpretation in mind, Corollary 3.15 expresses that relative to ‘st’, continuity implies being computable (in some sense). However, intuitionistic mathematics, the continuity axiom WC-N in particular, refutes Church’s thesis CT, where the latter expresses that all sequences are computable (in the sense of Turing), and the former implies Brouwer’s continuity theorem (See [25, p. 211]).

We can even prove a stronger theorem as follows.

Corollary 3.16. The system $M + (\exists^2)$ proves that for any near-standard $Y^2$ and standard $g^1$, there is standard $Z^2$ such that $(\forall f \leq_1 g)(Z(f) = Y(f))$.

Proof. Use $(\exists^2)$ to define $Z(f)$ as $Y(f)$ if $f \leq_1 g$ and 0 otherwise. Then $Z$ is standard in the same way as in the corollary: since $g$ is standard, $f \leq_1 g$ is too. □

By the previous, any functional $Y^2$ is automatically standard if it is near-standard on $C$, and zero elsewhere.

3.2. Non-intuitionistic aspects of the Dinis-Gaspar system. We show that the Dinis-Gaspar system does not qualify as a system of intuitionistic mathematics for the following reasons:

(i) The system $M$ proves, relative to the standard world, the weak König’s lemma, which is rejected in constructive mathematics (Section 3.2.1).

(ii) The system $M$ is inconsistent with the axiom, relative to the standard world, all functions are (epsilon-delta) continuous on Baire space (Section 3.2.2).

(iii) The system $M$ is inconsistent with the axiom schema, relative to the standard world, called Kripke’s scheme (Section 3.2.3).

Regarding the occurrence of ‘relative to the standard world’ in the previous items, we recall the following regarding the standard objects in internal set theory.

For example, the set $\mathbb{N}$ of all natural numbers, the set $\mathbb{R}$ of all real numbers, the real number $\pi$, and the Hilbert space $L^2(\mathbb{R})$ are all standard sets, since they may be uniquely described in conventional mathematical terms. Every specific object of conventional mathematics is a standard set. It remains unchanged in the new theory.

([15] p. 1166, emphasis in original)

We note that all terms of $M$ are standard, and presumably every object which may be constructed (in some sense or other from constructive mathematics) will be standard. Moreover, even in the classical system from [2], the standard objects yield (copious) computational/constructive content, as detailed in [19]. Thus, the standard world should be the focus of our attention, if we are interested in computational/constructive content.
3.2.1. Weak König’s lemma. We show that the Dinis-Gaspar system proves, relative to the standard world, weak König’s lemma and the latter’s uniform version. Recall that the variable ‘T’ is reserved for trees, and denote by ‘T ≤_1 1’ that T is a binary tree. Then WKL is just the classical contraposition of Fan, and

(∃Ψ)(∀T ≤ 1)[(∃n_0)(∃β ≤ 1)(∃m ∈ T) → (∀m_0)(Ψ(T)m ∈ T)]

(UWKL)

the uniform version. As WKL is (constructively) equivalent to a fragment of the law of excluded middle (See [9]), it is rejected in constructive mathematics.

**Theorem 3.17.** The system M proves WKL^st and UWKL^st.

**Proof.** Let T be a standard binary tree such that (∃^st n)(∃σ ≤ 1)(σm ∈ T), i.e. T is infinite relative to the standard world. We immediately obtain:

(∃^st n)(∃σ ≤ 1)(∀m ≤ n)(σm ∈ T),

and applying Φ (since ‘≤_0’ is ‘≤_st’ by definition) yields (∃σ ≤ 1)(∀m)(σm ∈ T). Since σ ≤_1 1, item (b) of the nonstandard axioms implies that σ is a standard binary sequence, and WKL^st follows. To obtain UWKL^st, fix nonstandard N and define Φ^N as follows: Φ(T) is σ * 00... where σ ∈ T is the left-most binary sequence of maximal length |σ| ≤ N, if it exists and 00... otherwise. Since Φ ≤_1 σ, 1, this defines a standard functional, and we are done.

Kohlenbach shows in [12] that RCA_0^st ⊢ UWKL ↔ (♯2^st) crucially depends on the axiom of extensionality. Assuming M is consistent, we do not have access to (E_2)^st by the proof of Theorem 3.8, and hence (♯2^st) does not follow from UWKL^st in M, i.e. the previous theorem does not lead to a contradiction.

Moreover, WKL is (constructively) equivalent to (∀x ∈ R)(x ≥ 0 → x ≤ 0) and to the fact that every real in [0, 1] has a binary representation (See [9]). As expected, M also proves versions of the latter, relative to the standard world.

**Theorem 3.18.** The system M proves that every real in the unit interval has a standard binary approximation, i.e. (∀x ∈ [0, 1])(∃ς ∈ C)(x ≈ ∑_n=0^∞ f(n)), and

(∃^st ς)(∀x ∈ R)(Φ(x) = 0 → x ≈ 0 ∧ Φ(x) = 1 → x ≈ 0).

(3.11)

**Proof.** Fix nonstandard N and define Φ^2 as: Φ(x) = 0 if |x|N ≤ 1, and 1 otherwise. Note that Φ ≤_2 1 implies this functional is standard. Then Φ(x_0 − 1/2) provides the first bit of a binary approximation of x_0, and given the first n such bits b_0,..., b_{n-1}, then Φ(x_0 − (∑_{i=0}^{n-1} b_i/2^i)) yields the n + 1-th bit.

There are a number of other theorems (constructively) equivalent to WKL by [9], like e.g. the intermediate value theorem. As expected, one can also establish these theorems relative to ‘st’ inside M, but we do not go into details.

It is well-known that WKL is inconsistent with the aforementioned axiom Church’s thesis CT ([11, p. 68]). Since M proves WKL^st, one expects M to be inconsistent with CT relative to the standard world. Let φ_{e,s}(n) = m be the (primitive recursive) predicate expressing that the Turing machine with index e and input n halts after at most s steps with output m. Then Church’s thesis is defined as follows.

(∀f^1)(∃^st 0)(∀n^0, m^0)[(∃s^0)(φ_{e,s}(n) = m) ↔ f(n) = m].

(CT)

**Theorem 3.19.** The system M proves ¬CT^st.
Theorem 3.21. The system $\text{ST}$ holds. Fix nonstandard $N^0$ and define (standard by definition) $f_0 \leq 1$ as follows: $f_0(e) = 1$ if $(\exists s \leq N)(\varphi_{e,s}(e) = 0)$, and $0$ otherwise. Then there is standard $e_0$ such that $(\exists s^0)(\varphi_{e_0,s}(e_0) = m) \iff f_0(e_0) = m$ for any standard $m$. However, $f_0(e_0) = 1$ implies by definition $(\exists s^0 \leq N)(\varphi_{e_0,s}(e_0) = 0)$, a contradiction. Similarly, $f_0(e_0) = 0$ implies by definition $(\forall s^0 \leq N)(\forall n^0)(\varphi_{e_0,s}(e_0) = n \to n \neq 0)$, a contradiction. Since we obtained a contradiction in each case, $\text{ST}$ is false.  

3.2.2. Intuitionistic continuity. We show that $M$ is inconsistent with certain axioms, relativised to the standard world, of intuitionistic mathematics.

First of all, we consider the continuity principle $\text{BCT}_C \equiv (\forall Y^2)\text{cont}_C(Y)$, which expresses that all functionals are (epsilon-delta) continuous on Cantor space, as given by the following formula:

\[(\forall f \leq 1)(\exists n^0)(\forall g \leq 1)(\exists N = \exists fN \to Y(f) = Y(g)).\]  

(\text{cont}_C(Y))

Secondly, we consider the principle weak continuity for numbers

\[(\forall \alpha^1)(\exists n^0)A(\alpha, n) \to (\forall \alpha^1)(\exists n^0, m^0)(\forall \beta^1)[\exists m = \exists m \to A(\alpha, m)]\]  

(\text{WC-N})

for any formula $A$ in the language of finite types. Let $\text{WC-N}_0$ be the restriction of WC-N to quantifier-free formulas, and recall the axiom SE from Section 3.1.1

Theorem 3.20. The systems $M + (\text{BCT}_C)^s$, $M + (\text{WC-N}_0)^s$, and $M + \text{SE}^s$ are inconsistent.

Proof. For the first part, consider $Y_0, f_0, g_0$ as in the proof of Theorem 3.2 and note that $f_0 \approx g_0$ contradicts $(\text{BCT}_C)^s$. For the second part, take $A(\alpha, n) \equiv (Y_0 = n)$ and note that $(\text{WC-N}_0)^s$ implies that $Y_0$ is epsilon-delta continuous on $C$, relative to the standard world. For the third part, note that SE$^s$ implies $(E_2)^s$.

Note that $Y_0$ is not sequentially continuous relative to the standard world, i.e. the restriction of $\text{BCT}_C$: to sequential continuity does not change the previous theorem. Moreover, due to $M^s$, there is no difference between LPO$^s$ and the weaker WLPO$^s$, i.e. the associated notion of nondiscontinuity ([9] Thm. 3]) is not relevant here.

As aside, SE follows from WMP by [11] Thm. 11], which in turns is provable in (constructive) recursive mathematics (See [9] Prop. 13]). Hence, $M$ is also inconsistent with theorems of recursive mathematics, relative to the standard world.

As another aside, we prove that $M$ is consistent (or even outright proves) certain theorems of intuitionistic mathematics. Indeed, a consequence of $\text{BCT}_C$ (together with FAN) is that all functions on $C$ are bounded.

Theorem 3.21. The system $M$ proves $(\forall^s Y^2)(\exists^s N^0)(\forall^s f \leq 1)(Y(f) \leq N)$; the system $M + \text{PF-TP}_\varphi$ proves the latter without ‘st’.

Proof. For standard $Y^2$, since all binary sequences are standard, we have $(\forall f \leq 1)(\exists^s n^0)(Y(f) \leq n)$, and $R^s$ finishes the first part. For the second part, drop all but the leading ‘st’ and apply PF-TP$\varphi$.

The previous implies that $M + \text{PF-TP}_\varphi$ is inconsistent with recursive mathematics, as the latter involves unbounded functionals on $2^\mathbb{N}$ (See [11] p. 70]). In particular, $M + \text{PF-TP}_\varphi + \text{ST}$ is inconsistent, which also follows from Theorem 3.6 if we in addition add QF-AC$^{1,0}$ to the system.

Finally, we show that the Dinis-Gaspar system is inconsistent with a classical continuity principle. Our motivation is to exclude an incorrect interpretation of
the results in the previous two sections. Indeed, one could say that $\mathcal{M}$ is slightly classical (as it proves $\text{WKL}^\omega$) and therefore Theorem 3.20. As it turns out, $\mathcal{M}$ is inconsistent with $(\text{BCT}_C)^{st}$ restricted to continuous functionals.

Thus, define $\text{CCT}_C \equiv (\forall^* Y^2)(\text{cont}_C(Y) \rightarrow [\text{cont}_C(Y)]^{st})$, which expresses that all functionals which are (epsilon-delta) continuous on $C$, are also continuous in this way relative to the standard world. Note that $\text{CCT}_C$ readily follows from Transfer.

**Theorem 3.22.** The system $\mathcal{M} + \text{CCT}_C$ is inconsistent.

*Proof.* Consider the standard objects $Y_0$, $f_0$, $g_0$ as in the proof of Theorem 3.22 and note that $f_0 \approx g_0$ contradicts $\text{CCT}_C$ as $\text{cont}_C(Y_0)$.

One could replace the antecedent of $\text{CCT}_C$ with more restrictive internal formulas, but the end result would still be the same.

3.2.3. Kripke’s scheme. We show that $\mathcal{M}$ is inconsistent with a fragment of Kripke’s scheme relative to the standard world. This is not surprising since $\mathcal{M}$ involves Markov’s principle in the form $\mathcal{M}^\omega$. Indeed, Markov’s principle is rejected in intuitionistic mathematics, which was first established by Brouwer using an axiom scheme nowadays called Kripke’s scheme (See [5, p. 244] for details). The ‘strong’ form of this scheme is formulated as follows by Dummett in [5].

**Principle 3.23 (KS$^\omega$).** For any formula $A$, we have

$$(\exists \beta \leq 1)(A \leftrightarrow (\exists n)(\beta(n) = 1)).$$

We consider the following special case of KS$^\omega$:

$$(\forall \alpha \leq 1)(\exists \beta \leq 1)(\forall m^0)[(\forall k^0)(\alpha(k, m) = 0 \leftrightarrow (\exists n)(\beta(n, m) = 0)]).$$

(\text{KS}$^\omega_0$)

By [24 §9.5], Markov’s principle and the Kripke schema imply the law of excluded middle, which is a similar result to what is obtained in the following proof.

**Theorem 3.24.** The system $\mathcal{M} + (\text{KS}_0^\omega)^{st}$ is inconsistent.

*Proof.* Fix nonstandard $N^0$ and fix standard $\alpha, \beta \leq 1$ as in $(\text{KS}_0^\omega)^{st}$; let $g_0(m)$ (resp. $h_0(m)$) be the least $k \leq N$ such that $\alpha(k, m) \neq 0$ (resp. $\beta(k, m) = 0$) if it exists, and $N$ otherwise. Define (standard by definition) $\gamma \leq 1$ such that $\gamma(m) = 0$ if $g_0(m) > h_0(m)$, and 1 otherwise. Then if we can prove the following:

$$(\forall^* m^0)[(\forall^* k^0)(\alpha(k, m) = 0 \leftrightarrow (\exists^* n)(\beta(n, m) = 0)] \leftrightarrow \gamma(m) = 0],$$

(3.12)

then we are done: $\mathcal{M}^\omega$ guarantees that (3.12) implies $\text{ACA}_0^{st}$ from Section 3.1.3, and Theorem 3.13 yields the desired contradiction. To prove (3.12), if for standard $m$, we have $(\exists^* n)(\beta(n, m) = 0)$, then $h_0(m)$ is standard, while $g_0(m)$ is nonstandard (by the first equivalence in (3.12)), i.e. $h_0(m) < g_0(m)$. Note that $(\forall^* k^0)(\alpha(k, m) = 0$ implies $(\forall k^0 \leq K_0)(\alpha(k, m) = 0$ for some nonstandard $K_0$ using Idealisation $\Gamma^\omega$ as usual. The reverse implication follows in the same way using $\mathcal{M}^\omega$. \square

3.3. Non-standard aspects of the Dinis-Gaspar system. We show that the system $\mathcal{M}$ includes a ‘standard part map’, a notion introduced in the next paragraph. As we will see, this raises the question to what extent $\mathcal{M}$ (and the system from [6]) can still be referred to as ‘Nonstandard Analysis’ or ‘internal set theory’.

First of all, Robinson introduces the ‘standard part map’ $^\circ$ in [18, p. 57]; the latter maps any $x \in [0, 1]$ to the (unique) standard $^\circ x$ such that $x \approx ^\circ x$, and the
latter is called the ‘standard part’ of the former. However, in the Robinsonian framework, the standard part map is external.

Secondly, in light of the previous, there is no hope of having access to this map in Nelson’s IST: we are only given the Standardisation axiom in which the standard part of a real exists. Nonetheless, we show that $M$ does afford a standard part map, and even a generalisation to functionals on Cantor space.

**Theorem 3.25.** There is a term $u^{(1×0)→1}$ of Gödel’s $T$ such that $M$ proves: for nonstandard $N$ and $x \in [0, 1]$, we have $\text{st}_1(u(x, N))$ and $u(x, N) \approx x$.

**Proof.** Recall the functional $\Phi$ form Theorem 3.18 and fix nonstandard $N$; define $v(x, N)$ as $\Psi(x, N) + 00 \ldots$ if $-\frac{1}{N} \leq Q [x](2N) \leq Q 1 + \frac{1}{N}$, and $00 \ldots$ otherwise. Here, $\Psi(x, 0)$ is $\{\Phi(x - \frac{1}{4})\}$ and $\Psi(x, n + 1)$ is $\Psi(x, n) + (b)$, where $b = \Phi(x - \frac{(2^{n+1} + \sum_{i=0}^{n-1} \Phi(x, n)(i))}{2^{n+1}})$. Since $v(x, N) \leq 1$, the former is standard (in the sense that $\text{st}_1(v(x, N))$ for any $x \in [0, 1]$), and satisfies $\sum_{n=0}^{\infty} \frac{v(x, N)(n)}{2^{n+1}} \approx x$ by design. Define standard $u^{1→1}$ as $\lambda x.\lambda v(x, N)).$ and standard part map $u := w \circ v$ is as required. □

Recall that we (may) view any sequence as a real; since $\lambda x.v(x, N) \leq 1$ we have $\text{st}_{1→1}(\lambda x.v(x, N))^0$ and the standard part map $u := w \circ v$ is thus standard in $M$, a fairly ‘non-standard’ situation as discussed in Remark 3.28.

**Theorem 3.26.** There is $s^{(2×0)→2}$ in Gödel’s $T$ such that $M$ proves: for nonstandard $N$ and near-standard $Y^2$ such that $(E_2)^{st}$, we have $\text{st}_2(s(Y, N)) \land (\forall f \in C)(s(Y, N)(f) = Y(f))$.

**Proof.** By the near-standardness of $Y^2$, and the fact that all binary sequences are standard, we have $(\forall f \in C)(\exists^st n)(Y(f) \leq n)$, and $R^ω$ implies $(\forall f \in C)(\exists n \leq n_0)(Y(f) \leq n)$ for some standard $n_0$. Fix nonstandard $N_0$ and define $s(Y, N_0)(f)$ as $Y(\overline{f}N_0 * 00 \ldots)$ if $\overline{f}N_0$ is a binary sequence, and $n_0$ otherwise. Then $s(Y, N_0)(f) = Y(f)$ for $f \in C$ by standard extensionality, and $\lambda f.s(Y, N_0)(f) \leq 2^ω n_0$ implies that $\text{st}_2(\lambda f.s(Y, N_0)(f))$, as required. □

**Corollary 3.27.** The system $M + (\exists^ω)$ proves that there is $\Phi^{2→2}$ such that for near-standard $Y^2$, we have $\text{st}_2(\Phi(Y)) \land (\forall f \in C)(\Phi(Y)(f) = Y(f))$.

**Proof.** Use $\exists^ω$ to define $\Phi(Y)(f)$ as $Y(f)$ if $f \in C$, and zero otherwise. Then $\Phi(Y)$ is standard in the same way as in the theorem. □

The previous theorem could be obtained for $F : [0, 1] \to \mathbb{R}$ using Theorem 3.18 but this development would mostly be repetitive. We finish this section with an informal remark on just how unnatural the standard part maps of $M$ are.

**Remark 3.28.** The standard part maps of $M$ are quite unnatural from the point of view of internal set theory for the following reason: the standard part of a real $x \in [0, 1]$ is unique in IST, i.e. if $x \approx y \approx z$ and the latter two are standard reals, then $y = z$. Hence, if there were $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(x) \approx x \land \text{st}(\Phi(x))$ for any $x \in [0, 1]$, then we observe that $(\forall x \in [0, 1])(\text{st}(x) \leftrightarrow x = \Phi(x))$. However, one of the central tenets of IST is that ‘st’ is not definable via an internal formula:

To assert that $x$ is a standard set has no meaning within conventional mathematics—it is a new undefined notion. (L5 p. 1165)
These observations do not cause problems for $M$ of course: the uniqueness of standard parts in IST requires Transfer anyway, while \( x =_{R} y \land \text{st}(x) \) does not imply st(y) in M due to issues of representation of reals. Nonetheless, M is only one basic step removed from being able to define ‘st\(_1\)’ via an internal formula, something which goes against the very nature of IST. Although the frameworks are of course different, a similar case can be made for the Robinsonian approach.

Now, the law of excluded middle is referred to as a ‘taboo’ in constructive mathematics (See [1, I.3]). In light of the previous remark, those endorsing this kind of language should probably use heresy when referring to the above standard part maps of $M$ in the context of Nonstandard Analysis and internal set theory.

4. Conclusion

In the previous sections, we have provided fairly conclusive answers to questions (Q1) and (Q2) from Section 1.1. We isolated (very) weak fragments of Transfer which are still inconsistent with $M$, and we identified a number of axioms of intuitionistic (and general constructive) mathematics which are inconsistent with $M$ when formulated relative to the standard world. We even established that $M$ allows for a highly elementary standard part map, a rather ‘non-standard’ feature of $M$.

These facts all suggest -in one way or another- that $M$ is indeed non-classical, but does not really deserve the description intuitionistic. At the same time, since a standard part map is not available in Nelson’s internal set theory, and external in Robinson’s approach, $M$ really pushes the boundary of what still counts as ‘Nonstandard Analysis’ and ‘internal set theory’.

In our opinion, the aforementioned problems trace back to one problematic axiom of $M$, namely item (b) of the nonstandard axioms. Simply put, this axiom ‘makes too many things standard’, an obvious example being Cantor space. While this axiom may be necessary and/or useful for the connection to the bounded functional interpretation (See [3, §6] and [6, §4]), it is not natural from the point of view of Nonstandard Analysis.

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