Abstract. Automorphisms of 3-dim. Lie Groups are seen to emerge from particular spatial diffeomorphisms of Bianchi-Type geometries. Their generators enforce a reduction of the configuration space on which the wave function is based, rendering thus the ensuing Quantum Cosmology manifestly invariant under spatial diffeomorphisms. The known theorem of Constrained Dynamics stating that a weakly vanishing quantity is strongly equal to a (homogeneous) combination of the constraints is then used to prove that the theory thus constructed is also invariant under space-time diffeomorphisms.

1. Introduction
It is well known that the presence of linear constraints, entails a reduction of the degrees of freedom for the quantum theory from the initial configuration variables to the (lesser in number) independent solutions to the quantum analogues of these constraints [1]. The Hamiltonian (Quadratic) constraint then becomes a P.D.E. in this reduced configuration space, see [2] for the case of Class A Bianchi Types. The Bianchi Type $I$, where all structure constants are zero (and thus the linear constraints vanish identically), has been exhaustively treated [3]. The Type $II$ case, where only two linear constraints are independent, has been examined along the above lines in [4] and differently in [5].

The paper is organized as follows:
In Section 2 the above mentioned reduction is linked to the need to preserve manifest spatial homogeneity. The bridge is provided by the existence of the automorphism group of the Lie Algebra of each Bianchi Type. The importance of Automorphisms in the theory of Bianchi Type Cosmologies is not new and has been stressed in e.g. [6].
In section 3 we advocate the idea that the observables of Quantum Gravity must be the invariant relations among the curvature scalars of a given spacetime. Then, we invoke the well known result of constraint dynamics [7] stating that a weakly vanishing quantity is strongly
equal to a homogeneous combination of the constraints, to formally conclude that the so-defined Quantum observables will be unchanged under space-time coordinate transformations. Finally, some concluding remarks are included in the discussion.

2. Covariance under Spatial Diffeomorphisms

In this section we shall first relate the action of the Automorphism group on $\gamma_{\alpha\beta}$, to the action induced on it by the class of General Coordinate Transformations (G.C.T.’s) which are subject to the restriction of preservation manifest spatial homogeneity. To this end, consider the spatial line element:

$$ds^2 = \gamma_{\alpha\beta}(x)\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^i dx^j$$

(1)

where $\sigma^\alpha_i(x)dx^i$ are the invariant basis 1-forms, of some given Bianchi Type.

The spatial homogeneity of this line element, is of course, preserved under any G.C.T. of the form:

$$x^i \rightarrow \tilde{x}^i = f^i(x)$$

(2)

Under such a transformation, $ds^2$ simply becomes:

$$(ds^2 \equiv) d\tilde{s}^2 = \gamma_{\alpha\beta}(t)\tilde{\sigma}^\alpha_m(\tilde{x})\tilde{\sigma}^\beta_n(\tilde{x})d\tilde{x}^m d\tilde{x}^n$$

(3)

where the basis one-forms are supposed to transform in the usual way:

$$\tilde{\sigma}^\alpha_m(\tilde{x}) = \sigma^\alpha_i(x)\frac{\partial x^i}{\partial \tilde{x}^m}$$

(4)

If one were to stop at this point, then one might have concluded that all spatial diffeomorphisms, act trivially on $\gamma_{\alpha\beta}$ i.e. $\gamma_{\alpha\beta} \rightarrow \tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}$. But as we shall immediately see, there are special G.C.T.’s which induce a well-defined, non-trivial action on $\gamma_{\alpha\beta}$. To uncover them, let us ask what is the change in form induced, by transformation (2), to the line element (1). To find this change we have to express the line element (3) in terms of the old basis one-forms (at the new point) $\sigma^\alpha_i(\tilde{x})$. There is always a non singular matrix $\Lambda^\alpha_\beta(\tilde{x})$ connecting $\tilde{\sigma}$ and $\sigma$ i.e.:

$$\tilde{\sigma}^\alpha_m(\tilde{x}) = \Lambda^\alpha_\mu(\tilde{x})\sigma^\mu_m(x)$$

(5)

Using this matrix $\Lambda$ we can write line element (3) in the form:

$$d\tilde{s}^2 = \gamma_{\alpha\beta}(t)\Lambda^\alpha_\mu(\tilde{x})\Lambda^\beta_\nu(\tilde{x})\sigma^\mu_m(x)\sigma^\nu_n(\tilde{x})d\tilde{x}^m d\tilde{x}^n$$

(6)

If the functions $f^i$, defining the transformation, are such that the matrix $\Lambda^\alpha_\beta$ does not depend on the spatial point, then there is a well defined, non trivial action of these transformations on $\gamma_{\alpha\beta}$:

$$\gamma_{\alpha\beta} \rightarrow \tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu\Lambda^\beta_\nu\gamma_{\alpha\beta}$$

(7)

With the use of (4) and (5), the requirement that $\Lambda^\alpha_\beta$ does not depend on the spatial point $\tilde{x}^i$, places the following differential restrictions on the $f^i$’s:

$$\frac{\partial f^i(x)}{\partial x^j} = \sigma^i_\alpha(f)S^\alpha_\beta\sigma^\beta_j(x)$$

(8)
where $\sigma^\alpha_\alpha$ and $S^\alpha_\beta$ are the matrices inverse to $\sigma^\alpha_\beta(x)$ and $\Lambda^\alpha_\beta$, respectively. These conditions constitute a set of first order, highly non-linear P.D.E.’s in the unknown functions $f^i$. The existence of solutions to these equations, is guaranteed by the Frobenius theorem [8], as long as the necessary and sufficient conditions $\partial^i_\alpha f^i - \partial^i_\beta f^i = 0$ hold. Through the use of (8) and the defining property of the invariant basis 1-forms (??), we can transform these conditions into the form:

$$2\sigma^i_\alpha(f)\sigma^i_\beta(x)\sigma^i_\gamma(x) \left( C^\alpha_{\rho\sigma} S^\sigma_\rho - C^\alpha_{\mu\nu} S^\mu_\rho S^\nu_\sigma \right) = 0$$

which is satisfied, if and only if, $S^\alpha_\beta$ (and thus also $\Lambda^\alpha_\beta$) is a Lie Algebra Automorphism (see ?? below). It is, therefore, appropriate to call the General Coordinate Transformations (2), when the $f^i$’s satisfy (8), *Automorphism Inducing Diffeomorphisms* (A.I.D.’s). The existence of such spatial coordinate transformations is not entirely unexpected: in the particular case $\Lambda^\alpha_\beta(\tilde{x}) = \delta^\alpha_\beta$ these coordinate transformations, are nothing but the finite motions induced on the hypersurface, by the three Killing vector fields (existing by virtue of homogeneity of the space), which leave the basis one-forms form invariant. The new thing we learn, is that there are further motions leaving the basis one-forms quasi-invariant i.e., invariant modulo a global (space independent) linear mixing, with the mixing matrix $\Lambda^\alpha_\beta$, belonging to the Automorphism Group. The notion of such transformations “leaving the invariant triads unchanged modulo a global rotation” also appears in Ashtekar’s work [3], under the terminology “Homogeneity Preserving Diffeomorphisms”’; also the term global is there used in the topological sense.

The differential description of motions (7) is achieved through the following linear vector fields defined on the space spanned by the components of the scale-factor matrix:

$$X_{(i)} = \lambda^\alpha_{(i)\rho} \gamma_{\alpha\beta} \partial^\beta \rho$$

with an obvious notation for the derivative with respect to $\gamma_{\alpha\beta}$.

The matrices $\lambda^\alpha_{(i)\rho} \equiv \left( C^\beta_{\rho\alpha}, \varepsilon^\beta_{(i)\alpha} \right)$ are the generators of (the connected to the identity component of) the Automorphism group and are satisfying the relation

$$\lambda^\alpha_{(i)\beta} C^\beta_{\mu\nu} = \lambda^\alpha_{(i)\mu} C^\alpha_{\nu\rho} + \lambda^\alpha_{(i)\nu} C^\alpha_{\rho\mu},$$

where $(i)$ labels the different generators. As we see, depending on the particular Bianchi Type, except of the quantum linear constraints (generators of Inner Automorphic Motions) $E_\rho = C^\alpha_{\rho\beta} \gamma_{\alpha\beta}$, the vector fields $X_{(i)}$ may also include the generators of the outer-automorphic motions: $E_{(j)} \equiv \varepsilon^\sigma_{(j)\rho} \gamma_{\sigma\tau} \frac{\partial}{\partial \tau\rho}$. It can be proved that the finite motions, induced by $X_{(i)}$ (through their integral curves), are precisely transformations (7) with $\Lambda \in Aut(G)$ [9].

A unified invariant description of Bianchi Homogeneous (B.H.) 3-Geometries is thus obtained through the solutions to the system of first order partial differential equations $X_{(j)} \Psi = 0$. The most general solution to these P.D.E.’s is $\Psi = \Psi(q^i)$ where the $q^i$’s are the following scalar combinations of $C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}$—which constitute a base in the space of all scalar contractions:

$$q^1 (C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}) = \frac{m^{\alpha\beta} \gamma_{\alpha\beta}}{\sqrt{\gamma}} q^2 (C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}) = \frac{(m^{\alpha\beta} \gamma_{\alpha\beta})^2}{2\gamma} - \frac{1}{4} C^\alpha_{\mu\kappa} C^\beta_{\nu\lambda} \gamma_{\mu\nu} \gamma_{\lambda\kappa} q^4 (C^\alpha_{\mu\nu}, \gamma_{\alpha\beta}) = \frac{m}{\sqrt{\gamma}}$$

where $m^{\alpha\beta}$ is the symmetric second rank contravariant tensor density (under the action of $GL(3, \mathbb{R})$ in which the structure constants are uniquely decomposed), and $m$ its determinant i.e.:

$$C^\alpha_{\beta\gamma} = m^{\alpha\delta} \varepsilon_{\delta\beta\gamma} + \nu_{\beta\gamma} \delta^\alpha_{\beta} - \nu_{\gamma\delta} \delta^\alpha_{\beta}$$
with \( \nu_\alpha = \frac{1}{2} C^\alpha_{\alpha \rho} \).

The Quantum Hamiltonian Dynamics of Bianchi Cosmologies must therefore be based on these variables, if it is to be spatially diffeomorphism invariant (for details see [9] and references therein).

3. A Classical Theorem and its Quantum Implications

As is well known (see e.g. K. Sundermeyer [7]), the canonical analysis of pure Gravity consists in the following statements:

\[
\mathcal{H}_0(g_{ij}, \pi^{ij}) \approx 0 \mathcal{H}_k(g_{ij}, \pi^{ij}) \approx 0 \dot{g}_{ij} = \{g_{ij}, H\} \pi^{ij} = \{\pi^{ij}, H\} H = \int (N^0 \mathcal{H}_0 + N^k \mathcal{H}_k) d^3x \quad (14)
\]

which are explicitly equivalent to the ten Einstein’s Field Equations. If we adopt the notion that classical observables, are all the geometrical objects that do not depend on the gauge, i.e. the coordinate system, then we are led to identify these observables with invariant relations among space time scalars. These scalars can be constructed in two ways: Firstly, by contracting all the indices of tensor products of the Riemann tensor and its covariant derivatives of any order. Secondly, in the case of spacetimes admitting a null, covariantly constant vector field (pp waves where all the scalars constructed in the above mentioned way are identically vanishing), by finding proportionality factors between tensors constructed by the Riemann tensor and its covariant derivatives of any order. Anyway, the scalars themselves do not describe the space time in an invariant manner since their functional form in terms of the coordinates changes when the coordinate system is altered. A way to generate invariant relations among these scalars, albeit not the most efficient one, would be to take a base of 4 scalars (say \( Q_1, \ldots, Q_4 \)) and solve for the coordinates. Then, any other scalar (say \( Q_5 \)) becomes expressible in terms of the 4 scalars chosen (\( Q_5 = f(Q_1, \ldots, Q_4) \)). Now this relation is characteristic of the geometry i.e. does not change form under coordinate transformations. In this sense a geometry is completely characterized by a set of relations:

\[
f^A(Q_i) = 0 \quad (15)
\]

The index \( A \) is at most countable. Turning these relations into functions on the phase space we notice that they become weakly vanishing quantities:

\[
h^A(g_{ij}, \pi^{ij}) \approx 0 \quad (16)
\]

In implementing this step use has been made of canonical equations of motion in order to substitute all higher time derivatives of the metric of the slice. But at this point, we invoke a known theorem of constrained dynamics [7]:

**Theorem** “Every weakly vanishing function in Phase Space is strongly equal to some expression containing the Constraints (which define a surface in Phase Space)”.

Moreover the expression under discussion ought to vanish on-mass shell (i.e. when the constraints are set to zero). Thus:

\[
h^A = h^A(\mathcal{H}_0, \mathcal{H}_k) \quad (17)
\]

The translation of the above result in the velocity phase space reads as follows: Consider an invariant relation \( f^A(Q_i) = 0 \) of any space time geometry which satisfies Einstein’s Field
Equations. Evaluate the left-hand side for the generic space time metric. Then, eliminate from the resulting expression all higher time derivatives of the metric (on the slice) using the spatial Einstein equations (and their time derivatives of the appropriate order). The end result will be that \( f^A \) will become such a function of the \( G_0^0, G_0^k \) constraints so that it vanishes when the constraints are set to zero. The proof rests on the fact that the constraints are the only quantities of the spatial metric and its time derivative that vanish, a thing that is guaranteed by the consistency of the Field Equations; Any other function on the velocity Phase Space which vanishes by virtue of the Field Equations, such as \( f^A \), is necessarily expressible in terms of the constraints.

The implications of this for the quantum theory are obvious and important: Adopting the point of view that the quantum observables are to be the operator analogues of the classical observables, we are assured that, for each and every such observable, there will exist a factor ordering such that all ensuing operators, when acting on the appropriate states defined by:

\[
\hat{H}_0 \Psi = 0
\]

will have vanishing eigenvalues:

\[
\hat{h}^A \Psi = 0
\]  

This establishes the (formal) space time covariance of the quantum theory described by (18) above.

4. Discussion

We have discussed the issue of space-time covariance of Canonical Quantization of General Relativity. In the case of Quantum Cosmology, as far as spatial diffeomorphisms are concerned, the action of the automorphism group dictates the reduction of the initial configuration space spanned by the components of the scale-factor matrix to a space parameterized by the q’s of section 2. In the case of full pure Quantum Gravity and space-time diffeomorphisms the problem of covariance is known as the problem of time. At the classical level, the well established canonical analysis of the Einstein-Hilbert action leaves no room for doubts: the formulation is explicitly space-time generally covariant. Upon canonically quantizing the problem seems to reappear as all the ingredients of the theory, i.e. the quantum constraints and consequently the quantum states concern the three-geometry and not the space-time in which it is embedded. An answer to the problem presupposes a commitment about what will constitute the set of observables. Motivated by the very essence of the notion of a geometry, we adopt the point of view that this set must be identified by all invariant relations between the various curvature or higher derivative curvature scalars of a given geometry. Each such relation must be turned into an entity living on the phase space by eliminating the time derivatives of the extrinsic curvature (through use of the spatial equations of motion, if necessary). The appeal to a well known theorem of Constrained Dynamics permits us to conclude that all these entities are homogeneous combinations of the Quadratic and the Linear constraints. Therefore, when we wish to turn each and every such entity into operator (Quantum Observable), there will be many factor orderings (namely all these that keep at least one constraint to the far right) which will enable this operator to annihilate the states defined as the common null eigenstates of the Quantum Constraints. Consequently, the use of many different slices as bases for canonically quantizing one and the same space-time can have no effect on the so defined quantum observables. Space-time covariance is thus observed at the quantum level. The above considerations are not meant to imply that we claim we have constructed a Dirac
Consistent quantum theory of General Relativity. They rather point to the claim that, if a consistent imposition of the quantum constraints is achieved, we expect to encounter no additional problems concerning space-time covariance of the ensuing theory. In that sense, our result is formal, as far as full pure gravity is concerned.

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