Thermostatistics of $\mu$-deformed analog of Bose gas model

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Abstract – For the recently introduced $\mu$-deformed analog of Bose gas model ($\mu$-Bose gas model) we study some thermodynamical aspects. Namely, we calculate total number of particles and, from it, the deformed partition function, both involving dependence on the deformation parameter $\mu$. Such dependence of thermodynamic functions on the $\mu$-parameter is at the core of modification of Bose gas model and arises through the use of new techniques given by us, the $\mu$-calculus, an alternative to the well-known $q$-calculus (Jackson derivative and integral). Necessary elements of $\mu$-calculus are first presented. Then, for high temperatures we obtain virial expansion of the equation of state and find five first virial coefficients, as functions of $\mu$. At the other end, for low temperatures the critical temperature of condensation $T(\mu)_c$ depending on $\mu$ is found and compared with the usual $T_c$, and with the $T(p,q)_c$ of earlier studied $p,q$-Bose gas model. The internal energy, specific heat and the entropy of $\mu$-Bose gas are also given, both for high and low temperatures. Features peculiar for the $\mu$-Bose gas model are emphasized.

Introduction. – During last two decades diverse so-called deformed Bose gas models have appeared, to list some [1–12]. Usually such deformations are based on respective deformed oscillator models of which best known and best studied are the $q$-oscillators of Arik-Coon [13] and Biedenharn-Macfarlane [14] as well as the 2-parameter $p,q$-deformed or Fibonacci oscillators [15]. The Tamm-Dancoff deformed oscillator [16] is less known one, like a plenty of nonstandard one-parameter $q$-oscillators [17], polynomially deformed ones [18] along with so-called quasi-Fibonacci oscillators [19], to which the $\mu$-deformed oscillator [20] does also belong. These nonstandard deformed oscillator models are peculiar due to some new, unusual properties, e.g. various energy level degeneracies, nontrivial recurrence relations for energy spectra etc. Those features make them plausible for application in diverse fields of quantum physics.

Physical meaning of deformation per se, and of the involved deformation parameter(s) depends on the particular application of deformed oscillator or deformed Bose gas model to concrete physical system. When applying the model of ideal gas of deformed bosons to computing the intercepts of the momentum correlation functions [7,8,11], one can effectively take into account the structure (compositeness) of particles, or their finite volume, see [21, 22]. In fact, comparison of theory with experimental data [23] showing non-Bose type behavior of the two-pion correlation function intercepts stresses the efficiency [3,24] of using deformed versions of Bose gas model. Let us also mention successful application of $q$-Bose gas setup to overcome the difficulty with unstable phonon spectrum, confirmed [4] by experimental measurements of phonon lifetime in scattering of neutrons.

In the deformed analogs of Bose gas model, and the $\mu$-Bose gas model in particular, one counts for modified interparticle interaction [12,25] of quantum statistical origin. The deformation may also absorb [10] an interaction present in the initially non-deformed system.

The $\mu$-Bose gas model was proposed in [11] where (the intercepts of) the momentum correlation functions of 2nd, 3rd, and $r$th order were obtained in closed form. On the other hand, the $p,q$-deformed model was explored more extensively, see e.g. [7,9,12]. Say, besides similar $r$th order correlation function intercepts explicitly derived in [7], the thermodynamic aspects including total number of particles, (virial expansion of) the equation of state with virial coefficients, and the modified ($p,q$-deformed) critical temperature for $p,q$-Bose gas, have been obtained [12]. What

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concerns $\mu$-Bose gas, its thermodynamical quantities were not studied before, and in this paper we make some steps to fill the gap. Dealing with the $\mu$-Bose gas model we focus our attention on thermodynamics. In Sec. 2 we recall necessary facts concerning the $\mu$-deformed analog of Bose gas and devote a subsection to elements of $\mu$-calculus. Then we explore basic thermodynamic quantities. In Sec. 3 we derive the expression for the total number of particles and, using that, the partition function (the both quantities carry explicit $\mu$-dependence). Next, we study two opposite regimes: for high temperature and low density we obtain virial coefficients of the expanded equation of state using deformed calculus ($\mu$-derivative); then, for the regime of low temperature and high density the critical temperature of condensation is obtained and its dependence on the deformation parameter $\mu$ studied. Few other thermodynamical functions are treated in Subsec. 3.3. The paper is ended with conclusions.

**Deformed analogs of Bose gas model.** – Like in many papers studying deformed oscillators, see e.g. [1][9][10] we deal in fact with the (system of) related deformed bosons. One of the virtues of such deformation is its ability to provide effective account of the interaction between particles, their non-zero volume, or their inner (composite) structure.

To study deformed analog of Bose gas model, namely the $\mu$-Bose gas as the model describing the system of deformed bosons, we work the Hamiltonian

$$H = \sum_i (\varepsilon_i - \tilde{\mu}) N_i$$

(1)

Here $\varepsilon_i$ denotes kinetic energy of particle in the state "$i$", $N_i$ the particle number (occupation number) operator corresponding to state "$i$". Similarly to the case of ordinary bosons, we proceed as follow. Calculation of the thermodynamical functions in the new model needs special mathematical tools ($\mu$-calculus) that will be presented below.

The special version of deformed Bose gas model termed $\mu$-Bose gas model associated with $\mu$-deformed oscillators [20] was introduced and studied in [11]. Therein, and in this paper, the thermal average of the operator $\mathcal{O}$ is determined by the familiar formula

$$\langle \mathcal{O} \rangle = \frac{Tr(\mathcal{O} e^{-\beta H})}{Z},$$

(2)

$Z$ being the grand canonical partition function. This function is (the logarithm of it)

$$\ln Z = - \sum_i \ln(1 - ze^{-\beta \varepsilon_i})$$

(3)

with the fugacity $z = e^{\beta \tilde{\mu}}$. The usual (non-deformed) formula for the total number of particles i.e.

$$N = z \frac{d}{dz} \ln Z,$$

(4)

in our treatment will be modified, see Sec. 3.

**Jackson derivative and its $p,q$-extensions.** To derive thermodynamical functions for a deformed analog of Bose gas model, one needs an extension of usual treatment, in particular what concerns derivatives. Say, instead of usual derivative one uses the Jackson or $q$-derivative [20]

$$\left( \frac{d}{dx} \right)_q f(x) \equiv D_q^{(q)} f(x) = \frac{f(qx) - f(x)}{qx - x}. $$

(5) Its consistency requires that at $q \to 1$ we recover $d/dx$:

$$D_q^{(q)} \xrightarrow{q\to1} \frac{d}{dx}. $$

(6)

When acting on the monomial in $x$ the $q$-derivative gives

$$D_q^{(q)} x^n = \frac{(qx)^n - x^n}{qx - x} = [n]_q x^{n-1}, \quad [n]_q \equiv \frac{q^n - 1}{q - 1}. $$

(7)

with $[n]_q$ the $q$-bracket. At $q \to 1$, $\lim_{q\to1} [n]_q = n$ and the ordinary derivative is regained. With respect to $x D_q^{(q)}$, the monomials behave as eigenvectors.

A $p,q$-extension of Jackson derivative is also known, see e.g. [27]. This is the operation $D_p^{(p,q)}$ acting as

$$D_p^{(p,q)} f(x) = \frac{f(px) - f(qx)}{px - qx}, $$

(8)

$$D_p^{(p,q)} x^n = [n]_{p,q} x^{n-1}, \quad [n]_{p,q} \equiv \frac{p^n - q^n}{p - q}. $$

(9)

For $p = 1$ we recover the Jackson derivative $D_q^{(q)}$ in (5). (7). Remark that somewhat modified version $D^{(p,q)}_x$ of the $p,q$-extended Jackson derivative was used in [9].

**Elements of $\mu$-calculus.** The $\mu$-derivative is an alternative to Jackson derivative. For the needs of this paper ($\mu$-Bose gas model), in some analogy with the above extensions of $d/dx$ we introduce the $\mu$-deformed derivative:

$$D^{(\mu)}_x f(x) = \int_0^1 dt f'_x(t^\mu x), \quad f'_x(t^\mu x) = \frac{df(t^\mu x)}{dx}. $$

(10)

As seen, formula (9) stems from this general definition. The $\mu$th power of $\mu$-derivative on a monomial $x^n$ yields

$$D^{(\mu)}_x [n]_{\mu} = \frac{[n]_{\mu}}{[n - k]_{\mu}} x^{n-k}, \quad [n]_{\mu} \equiv \frac{n!}{(n;\mu)}, $$

(11)
where \((n; \mu) \equiv (1 + \mu)(1 + 2\mu)...(1 + n\mu)\).

Generalization of \(\mu\)-derivative to its \(q, \mu\)- or \((p, q, \mu)\)-deformed extensions can be obtained if instead of \((d/dx)f(t^q x)\) in \((10)\) we take resp. \(D_2^q f(t^q x)\) or \(D_2^{p, q} f(t^q x)\). These extended cases correspond to the \((q; \mu)\)- or \((p, q; \mu)\)-deformed quasi-Fibonacci oscillators treated in \((10)\).

The inverse \((D_2^{q, \mu})^{-1}\) or antiderivative of the \(\mu\)-derivative \(D_2^{q, \mu}\) in \((9)\) and \((10)\) can be also defined, though we do not give here, since it will not be used below.

So, to develop thermodynamics of \(\mu\)-Bose gas model we apply, at proper point, the modified derivative \(D_2^{(\mu)}\) instead of usual \(d/dz\). As result, through the \(\mu\)-analogue of derivation the deformation parameter gets involved in the treatment; the system becomes \(\mu\)-deformed. It is essential that, at small values of \(\mu\), both the usual and the deformed derivative of a function have similar behavior. This can be verified by acting with the deformed and usual types of derivative on the monomial, logarithmic, exponential function, and others. Such property of \(\mu\)-derivative justifies (at least partly) its very use in calculating thermodynamical quantities of \(\mu\)-Bose gas.

By the use of \(\mu\)-bracket \([n]_\mu\) and \(\mu\)-factorial \([n]_\mu!\), see \((9), (11)\), we earn \(\mu\)-deformed analogs of Leibnitz rule. Consider the rule of \(\mu\)-differentiation when acting on the product \(f(x) \cdot g(x)\) of \(f(x)\) and \(g(x)\). From definition \((10)\), in the case of monomials \(f(x) = x^n\) and \(g(x) = x^m\) we have the relation

\[
D_2^{(\mu)} (x^n x^m) = D_2^{(\mu)} (x^{n+m}) = \frac{n + m}{1 + \mu(n + m)} x^{n+m-1}.
\]

General formula for \(D_2^{(\mu)}\) acting on \(f(x) \cdot g(x)\) does also exist but we do not dwell on it here.

**Thermodynamics of \(\mu\)-Bose gas model.** Now let us study thermodynamics of \(\mu\)-deformed analog of Bose gas using the elements of \(\mu\)-calculus given above. For the gas of non-relativistic particles the both regimes of high and low temperatures will be treated in what follows.

**Total number of particles.** In the Bose gas model the known relation giving total number of particles is

\[
N = \frac{d}{dz} \ln Z.
\]

To develop thermodynamics of \(\mu\)-analogue of Bose gas model, this formula for total number \(N\) is to be modified and we adopt

\[
N \equiv N^{(\mu)} = zD_2^{(\mu)} \ln Z = -zD_2^{(\mu)} \sum_i \ln(1 - ze^{-\beta\varepsilon_i}),
\]

where \(D_2^{(\mu)}\) is the \(\mu\)-derivative from \((12)\). For \(\mu \geq 0\), we apply this to the log of partition function in \((8)\) to get

\[
N^{(\mu)} = z \sum_i \sum_{n=1}^{\infty} \frac{z^{-\beta\varepsilon_i n}}{n} \frac{[n]_\mu}{z^{n-1}} = \sum_i \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} (e^{-\beta\varepsilon_i})^n z^n.
\]

Note there should be \(0 \leq |e^{-\beta\varepsilon_i}| < 1\) in \((15)\). As we deal with non-relativistic particles, the energy \(\varepsilon_i\) is taken as

\[
\varepsilon_i = \frac{p_i^2}{2m} = \frac{p_i^2}{2m}.
\]

Here \(p_i\) is the 3-momentum of particle in \(i\)-th state and \(m\) the particle mass.

Clearly, at \(z \to 1\) the expression under summation symbol in \((15)\) diverges when \(p_i = 0, i = 0\). In next subsection we will assume the \(i = 0\) ground state to be associated with a macroscopically large occupation number. Here, even though \(z \neq 1\), we nevertheless separate the term with \(p_i = 0\) from the remaining sum:

\[
N^{(\mu)} = \sum_i \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} (e^{-\beta\varepsilon_i})^n z^n + \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n.
\]

The “prime” of the sum symbol in \((17)\) means that the \(i = 0\) term is dropped from the sum. For large volume \(V\) and large \(N\) the spectrum of single-particle states is almost continuous so we replace the sum in \((15)\) by integral:

\[
\sum_i \to \frac{V}{(2\pi \hbar)^3} \int d^3 k.
\]

That is, we isolate the ground state and include the contribution from all other states in the integral. Now, to compute the total number of particles we perform integration over 3-momenta using spherical coordinates:

\[
N^{(\mu)} = \frac{4\pi V}{(2\pi \hbar)^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} \int_0^\infty \int_0^\infty \int_0^{2\pi} e^{-\beta \varepsilon z^2} dp_1 dp_2 dp_3 z^n n. \tag{19}
\]

The lower limit of the integral can still be taken as zero, because the ground state, \(p_0\), does not contribute to the integral anyway. Then, after performing the last integration by parts we obtain for the \((\mu\)-deformed, i.e. depending on \(\mu\) total number of particles the expression

\[
N^{(\mu)} = \frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{3/2}} z^n + N^{(\mu)}_0, \quad N^{(\mu)}_0 \equiv \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n, \tag{20}
\]

where \(\lambda = \sqrt{\frac{2\pi \hbar \varepsilon}{mT}}\) is the thermal wavelength. The expression in \((20)\) can be rewritten using the \(\mu\)-analog of Bose-Einstein function. As the result, for \(N^{(\mu)}\) we obtain

\[
N^{(\mu)} = \frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{3/2}} z^n + g_0^{(\mu)}(z), \quad g_0^{(\mu)}(z) = N^{(\mu)}_0, \tag{21}
\]

where \(g_0^{(\mu)}(z)\) is a particular case of the following \(\mu\)-polylogarithm (\(\mu\)-analog of well known Bose-Einstein function or polylogarithm \(g(z) = \sum_{n=1}^{\infty} z^n/n!\)):

\[
g_0^{(\mu)}(z) = \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{\lambda/2}} z^n. \tag{22}
\]
At $\mu \to 0$, from this the usual $g(z)$ function is recovered.

One can show that the positive real parameter $\mu$ does not influence the convergence region and, like for the ordinary $g$-function $g(z)$, there should be $|z| < 1$. This is in contrast with the restriction $z < 1/q$ in Ref. [6] valid for the version of $q$-Bose gas model considered therein.

It is useful to rewrite the expression in (21) for total number of particles as

$$ \frac{1}{v} = \frac{1}{\lambda^3 g_{3/2}(\mu)} + \frac{N(\mu)}{V}, \quad v \equiv \frac{V}{N(\mu)}. \quad (23) $$

This will be exploited in subsequent analysis.

**Deformed grand partition function.** We assume that all the known relations between thermodynamical functions in case of usual Bose gas thermodynamics are shared by the $\mu$-deformed analog of Bose gas model. This means that well-known relations are formally the same both for usual Bose gas model and its $\mu$-deformed counterpart. The only thing that should be stressed is that for $\mu$-Bose gas model all the thermodynamical functions are $\mu$-dependent.

So, to get deformed partition function $\ln Z(\mu)$ we take

$$ N(\mu) = z \frac{d}{dz} \ln Z(\mu) \quad (24) $$

and invert it, that is,

$$ \ln Z(\mu) = \left( \frac{d}{dz} \right)^{-1} N(\mu). \quad (25) $$

To perform the operation $(z \frac{d}{dz})^{-1}$ to get $\ln Z(\mu)$, one may either integrate: $\ln Z(\mu) = \int dz \ z^{-1} N(\mu)$ or merely apply the following property, valid on the monomials $z^k$, for any function $f(z \frac{d}{dz})$ with power series expansion:

$$ f \left( z \frac{d}{dz} \right) z^k = f(k) z^k. \quad (26) $$

With account of this, from (25), (26) and (17)-(20) we draw:

$$ \ln Z(\mu) = \left( \frac{d}{dz} \right)^{-1} \left( \frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{5/2}} z^n + \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n \right) = $$

$$ = \frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{5/2}} \left( z \frac{d}{dz} \right)^{-1} z^n + \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} \left( z \frac{d}{dz} \right)^{-1} z^n = $$

$$ = \frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{5/2}} (n)^{-1} z^n + \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} (n)^{-1} z^n. \quad (27) $$

The latter result can be written as (see definition (22))

$$ \ln Z(\mu) = \frac{V}{\lambda^3 g_{5/2}(\mu)} + g(\mu) \quad (28) $$

or as

$$ Z(\mu)(z, T, V) = \exp \left( \frac{V}{\lambda^3 g_{5/2}(\mu)} + g(\mu) \right). \quad (29) $$

Formulas (27)-(29) constitute our basic result for $\mu$-deformed partition function. Using (29) it is possible to derive other thermodynamical functions or relations as well. In particular, the equation of state reads:

$$ \frac{PV}{kT} = \ln Z(\mu) = \frac{V}{\lambda^3 g_{5/2}(\mu)} + g(\mu). \quad (30) $$

Remark that a different path of deriving the deformed $\mu$-deformed partition function could be the usage of one-particle $\mu$-deformed distribution function $(n_\mu)(\mu)$ whose exact analytical expression was found in Ref. [11]. It can be shown that the result derived in such way will be slightly different from that given in (29). That means in (27) or (29) a different $(n_\mu)(\mu)$ is implicit.

### Virial expansion of the equation of state and virial coefficients.

Consider the regime of high temperature and low density $\lambda T/v \ll 1$. As explained in handbooks [29], in this case the second term in (27) is negligibly small and likewise in eq. (30). The expression (28) and the equation of state (30) take the form

$$ \frac{1}{v} = \frac{1}{\lambda^3 g_{5/2}(\mu)}, \quad \frac{PV}{kT} = \frac{v}{\lambda^3} g(\mu). \quad (31) $$

From the first equality in (31) we have

$$ \frac{\lambda^3}{v} = 1 + \frac{\lambda^3}{v} \left( \frac{1}{\lambda^3} + B \frac{3}{v} + C \frac{5}{v}^3 + D \frac{7}{v}^5 + \ldots, \quad (29) \right. $$

where the explicit form of virial coefficients is

$$ A = \frac{1}{2^2 \beta_1^2 \mu}, \quad B = \frac{1}{2^2 \beta_2 \mu} \frac{1}{2^2 \beta_3 \mu} = \frac{3}{2^2 \beta_1 \mu}, \quad (34) $$

$$ C = \frac{5}{2^2 \beta_1 \mu} + \frac{2}{2^2 \beta_1 \mu} \frac{1}{2^2 \beta_2 \mu} = \frac{5}{2^2 \beta_1 \mu}, \quad (34) $$

$$ D = \frac{7}{2^2 \beta_1 \mu} + \frac{2}{2^2 \beta_1 \mu} \frac{1}{2^2 \beta_2 \mu} = \frac{5}{2^2 \beta_1 \mu} + \frac{2}{2^2 \beta_1 \mu}. \quad (34) $$

As seen the deformation parameter appears in all the virial coefficients in specific manner, through the $\mu$-integers. Here we encounter a very unusual feature: the appearance of (powers of) the $\mu$-unity $[1]_\mu$. While its analog was "hidden" in the virial coefficients of $q$-Bose or $p, q$-Bose gases, because of equality $[1]_{\mu, q} = 1$, here due to $[1]_\mu \neq 1$, the $\mu$-deformed virial coefficients contain $[1]_\mu$ squared and higher powers of it.

Recall that the same result follows by integrating over $dz$ the expression $z^{-1} N(\mu)$, see eq. (23).

Let us note that in [12], in the expression for the virial coefficient $D$ the (corrected) second term should be $-2 [p, q] [p, 2q] z^7 3^2 5^3$. Compare with the first term in (29).
Physical meaning of the parameter \( \mu \) may be commented as follows. Virial coefficients in case of usual Bose gas model, as known, reflect the effective (2-particle, 3-particle etc.) interaction of the quantum correlation or quantum statistics origin. In \( \mu \)-Bose gas, besides that, the inner structure (compositeness) of particles can effectively be taken into account by means of deformation and the \( \mu \) parameter; clearly, all that adds some extra amount of effective interaction. In effect, by changing the value of deformation parameter we can control/regulate both the value, and even the sign of virial coefficient(s) and thus can even gain repulsive instead of attractive (or vice versa) effective interparticle interaction in the model system. That is, by moving the value of \( \mu \) we can effectively change and thus control the quantum statistics of particles (compare with other deformed Bose gas models [10][12][25][28]).

**Critical temperature of condensation.** For the regime of low temperature and high density we obtain (like in [12] for the \( p,q \)-Bose gas) the critical temperature \( T_c^{(\mu)} \) of condensation in the considered \( \mu \)-deformed analog of Bose gas model. We start with eq. (23) and rewrite it as

\[
N_0^{(\mu)} = \frac{\lambda^3}{v} - \bar{g}_{3/2}(z).
\]

(35)

The critical temperature \( T_c^{(\mu)} \) of \( \mu \)-Bose gas model is determined from the equation \( \lambda^3/v = g_{3/2}(1) \), that gives:

\[
T_c^{(\mu)} = \frac{2\pi \hbar^2}{(g_{3/2}(\mu))(1)^{2/3}}.
\]

(36)

From the latter we infer the ratio of critical temperature \( T_c^{(\mu)} \) and the critical temperature \( T_c \) of usual Bose gas:

\[
\frac{T_c^{(\mu)}}{T_c} = \left(\frac{2.61}{g_{3/2}(1)}\right)^{2/3}.
\]

(37)

Figure 1 shows how the ratio (37) depends on the deformation parameter \( \mu \).

![Figure 1: The ratio \( T_c^{(\mu)}/T_c \) versus deformation parameter \( \mu \).](image)

We see that the ratio \( T_c^{(\mu)}/T_c \), similar to the case of \( p,q \)-Bose gas model [12], has a very important feature: the greater is the extent of deformation (here measured by \( \mu \)) the higher is \( T_c^{(\mu)} \). In the no-deformation limit \( \mu \to 0 \) we have \( T_c^{(\mu)}/T_c = 1 \), that is, the \( \mu \)-critical temperature tends to usual one, \( T_c^{(\mu)} = T_c \) (a kind of consistency).

**Other thermodynamical functions.** For the studied \( \mu \)-deformed analog of Bose gas model we take the definition of thermodynamical functions as in [29]. Then, the internal energy \( U^{(\mu)} \) of \( \mu \)-Bose gas is found as

\[
U^{(\mu)} = \frac{3kTv}{2\lambda^3} g_{5/2}(z), \quad T > T_c^{(\mu)}
\]

\[
U^{(\mu)} = \frac{3kTv}{2\lambda^3} g_{5/2}(1), \quad T \leq T_c^{(\mu)}.
\]

(38)

Using the expressions for internal energy of \( \mu \)-Bose gas we easily obtain the specific heat of the system using the relation \( C_v = \partial U/\partial T \)_\( \lambda \lambda \lambda [V,N] \), and the result is

\[
C_v^{(\mu)} = \left\{ \begin{array}{ll}
\frac{15}{4} \frac{v}{\lambda^2} g_{5/2}(z) - \frac{9}{4} g_{3/2}(z), & T > T_c^{(\mu)} \\
\frac{15}{4} \frac{v}{\lambda^3} g_{5/2}(1), & T \leq T_c^{(\mu)}
\end{array} \right.
\]

(39)

As seen, both in [28] and [29] the \( \mu \)-deformed analog (\( \mu \)-polylogarithm) of Bose function \( g_{n}(z) \) does appear. This fact may have interesting implications.

From the well-known relation for entropy \( S = \ln Z + \beta U - \beta^2 N \), within the \( \mu \)-analogue of Bose gas for the regimes of high resp. low temperatures we deduce the expressions

\[
\frac{S^{(\mu)}}{N^{(\mu)}k} = \frac{5}{2} \frac{v}{\lambda^3} g_{5/2}(1) - \ln z, \quad T > T_c^{(\mu)}
\]

\[
\frac{S^{(\mu)}}{N^{(\mu)}k} = \frac{5}{2} \frac{v}{\lambda^3} g_{3/2}(1), \quad T \leq T_c^{(\mu)}.
\]

(40)

(41)

Notice that from (40), (42) and (41) we infer an interesting proportionality valid for \( T < T_c^{(\mu)} \):

\[
\frac{U^{(\mu)}}{T} = \frac{2}{5} C_v^{(\mu)} = \frac{3}{5} S^{(\mu)}
\]

(it is formally similar to the case of usual Bose gas).

Let us visualize the dependence of \( S^{(\mu)} \) on \( \mu \), see Fig. 2. Here we find a remarkable fact: enhancing deformation, by raising of \( \mu \)-values, results in falling entropy-per-volume \( S^{(\mu)} \) (in units of \( k \)), from \( \approx 3.354 \) at \( \mu = 0 \) to zero in the large deformation asymptotics. That is, the larger deformation the lesser is chaoticity in the system.

**Conclusions.** – For the \( \mu \)-deformed analog of Bose gas model proposed in [11] in the present work we explored a number of thermodynamical relations. Having presented elements of \( \mu \)-calculus, using the \( \mu \)-deformed analog of derivative, we have calculated the total mean number
of particles. This result allowed us to explicitly derive the \((\mu\text{-deformed})\) partition function. Due to usage of \(\mu\)-derivative the deformation parameter has appeared in the expressions for total number of particles and for grand partition function. In the formulas for \(N^{(\mu)}\) and for the log of \(\mu\)-partition function, the \(\mu\)-analogues \(g_{A/2}^{(\mu)}(z)\) and \(g_{\lambda}^{(\mu)}(z)\) of polylogarithms \(g_{\lambda}(z)\) have naturally appeared.

For high temperature regime we have obtained explicit virial expansion for the equation of state and five \((\mu\text{-dependent})\) virial coefficients. A really surprising thing we encounter is the remarkable appearance of powers of the \(\mu\)-unit \([1]_\mu = \frac{1}{1 + \mu}\) in \(\mu\)-dependent virial coefficients. Say in \(A\) in (54), because of square of \([1]_\mu\) in the denominator, the increase of deformation leads to some extra raising of strength of effective two-particle interaction. Similar features concern higher virial coefficients \(B, C, D,\ldots\).

At \(\mu \neq 0\) the deformed system principally differs from bosonic one, and variation of \(\mu\) smoothly changes the statistics of particles. That is why, by controlling the \(\mu\) value one can even achieve switching of the sign of virial coefficient(s) that yields drastic change of quantum statistics. At \(\mu = 0\) (no-deformation limit), as it should, the obtained \(\mu\)-deformed virial coefficients reduce to the known virial coefficients of usual Bose gas.

For the low temperature regime the critical temperature of condensation depending explicitly on \(\mu\) is obtained. The dependence of ratio \(T_c^{(\mu)} / T_c\) on the parameter \(\mu\) shows that critical temperature in \(\mu\)-Bose gas is higher than critical temperature \(T_c\) of usual Bose gas. We have also found an amusing falling behavior of entropy-per-volume versus the \(\mu\)-parameter, i.e. strength of deformation. These remarkable facts can be of importance for more detailed future investigations of real Bose like gases, along with similar use of the result on \(T_c^{(p,q)}\) of \(p,q\)-Bose gas model.

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