Well-Posedness and Stability Results for a Nonlinear Damped Porous–Elastic System with Infinite Memory and Distributed Delay Terms

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Abstract: In the present paper, we consider an important problem from the application perspective in science and engineering, namely, one-dimensional porous–elastic systems with nonlinear damping, infinite memory and distributed delay terms. A new minimal conditions, placed on the nonlinear term and the relationship between the weights of the different damping mechanisms, are used to show the well-posedness of the solution using the semigroup theory. The solution energy has an explicit and optimal decay for the cases of equal and nonequal speeds of wave propagation.

Keywords: well-posedness; general decay; infinite memory; nonlinear damping; porous–elastic system; distributed delay term

1. Introduction

As introduced in [1], the one-dimensional porous–elastic model constitutes a system of two partial differential equations with unknown \((u, \phi)\) given by
\[
\begin{align*}
\rho_0 u_{tt} &= \mu u_{xx} + \beta \phi_x, \quad \text{in } (0, l) \times (0, L), \\
\rho_0 k \phi_{tt} &= \alpha \phi_{xx} - \beta u_x - \tau \phi_t - \xi \phi, \quad \text{in } (0, l) \times (0, L),
\end{align*}
\]
where \(l, L > 0\) the constant \(\rho\) is the mass density, \(\kappa\) is the equilibrated inertia and the constants \(\mu, \alpha, \beta, \tau, \xi\) are assumed to satisfy the appropriate conditions. This type of problem has been studied by many authors and a lot of results have been shown (please see [1–9]). The pioneering contribution was made by [10] for the problem (1). The basic evolution equations for one-dimensional theories of porous materials with memory effect are given by
\[
\begin{align*}
\rho u_{tt} &= T_x, \quad J \phi_{tt} = H_x + G,
\end{align*}
\]
where \(T\) is the stress tensor, \(H\) is the equilibrated stress vector and \(G\) is the equilibrated body force. The variables \(u\) and \(\phi\) are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equations are
\[
\begin{align*}
T &= \mu u_x + b \phi, \quad H = \delta \phi_x - \int_0^t g(t-s) \phi_x(s) ds, \quad G = -bu_x - \xi \phi.
\end{align*}
\]
A porous–elastic system was considered by [11] in the system

\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \quad &\text{in} \ (0, 1) \times (0, \infty), \\
J\phi_{tt} - \delta \phi_{xx} + bu_x + \zeta \phi + \int_0^1 g(t-s)\phi_{xx}(x,s)ds &= 0, \quad &\text{in} \ (0, 1) \times (0, \infty).
\end{aligned}
\]

(4)

System (4) subjected Neumann–Dirichlet boundary conditions, where \(g\) is the relaxation function; the authors obtained a general decay result for the case of equal speeds of wave propagation (See [12,13]). In [14], the authors improved the case of non-equal speed of wave propagation. In [15] the authors considered the following system with memory and distributed delay terms

\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\
J\phi_{tt} - \delta \phi_{xx} + bu_x + \zeta \phi + \int_0^1 g(t-s)\phi_{xx}(x,s)ds &= 0, \\
&+ \mu_1 \phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\tau)|\phi_t(x, t - \tau)d\tau = 0.
\end{aligned}
\]

(5)

The exponential stability results of systems with memory and distributed delay terms, for the case of equal speeds of wave propagation under a suitable assumptions, are proved. In [16], the following system was considered

\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\
J\phi_{tt} - \delta \phi_{xx} + bu_x + a\phi + \int_0^\infty g(s)\phi_{xx}(t-s)ds + \alpha(t)f(\phi_t) &= 0.
\end{aligned}
\]

(6)

The authors proved the global well-posedness and stability results of (6), which has been extended in [17] for the case of nonequal speeds of wave propagation. Very recently, one-dimensional equations of an homogeneous and isotropic porous–elastic solid with an interior time-dependent delay term feedbacks was treated by Borges Filho and M. Santos in [1].

The result in [10] for system (1) was improved by Apalara to exponential stability in [18]. For more papers related to our paper, please see [19–22].

Motivated by all the above papers, we investigate the well-posedness and stability results with distributed delay for the cases of equal and nonequal speeds of wave propagation, under additional conditions of the following system

\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\
J\phi_{tt} - \delta \phi_{xx} + bu_x + \zeta \phi + \int_0^\infty g(p)\phi_{xx}(t-p)dp \\
&+ \mu_1 \phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\tau)|\phi_t(x, t - \tau)d\tau + \alpha(t)f(\phi_t) &= 0,
\end{aligned}
\]

(7)

where

\[(x, \phi, t) \in (0, 1) \times (\tau_1, \tau_2) \times (0, \infty),\]

with the Neumann–Dirichlet boundary conditions

\[
u_x(0, t) = u_x(1, t) = \varphi(0, t) = \varphi(1, t) = 0, \quad t \geq 0,
\]

(8)

and the initial data

\[
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad x \in (0, 1)
\]

\[
\varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \quad x \in (0, 1)
\]

\[
\varphi(x, -t) = f_0(x, t), \quad (x, t) \in (0, 1) \times (0, \tau_2).
\]

(9)
Here, $\rho, \mu, f, b, \delta, \xi$ and $\mu_1$ are positive constants satisfying $\mu_1^2 > b^2$, the term $\alpha(t)f(\psi_t)$, where the functions $\alpha$ and $f$ are specified later, represent the nonlinear damping term. The term $\int_{\tau_1}^{\tau_2} |\mu_2(\xi)| \psi(x, t - \xi)d\xi$ is a distributed delay that acts only on the porous equation and $\tau_1, \tau_2$ are two real numbers with $0 \leq \tau_1 \leq \tau_2$, where $\mu_2$ is an $L^\infty$ function, and the function $g$ is called the relaxation function. We first state the following assumptions:

**Hypothesis 1 (H1).** $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying
\[
g(0) > 0, \quad \delta - \int_0^\infty g(p)dp = l > 0, \quad \int_0^\infty g(p)dp = \gamma_0. \tag{10}
\]

**Hypothesis 2 (H2).** There exists a non-increasing differentiable function $\alpha, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that
\[
g'(t) \leq -\eta(t)g(t), \quad t \geq 0,
\]
and
\[
\lim_{t \to \infty} -\frac{\alpha'(t)}{\alpha(t)} = 0. \tag{12}
\]

**Hypothesis 3 (H3).** $f \in C^0(\mathbb{R}, \mathbb{R})$ is non-decreasing such that there exist $v_1, v_2, \varepsilon > 0$ and a strictly increasing function $G \in C^1([0, \infty))$, with $G(0) = 0$ and $G$ is a linear or strictly convex $C^2$-function on $(0, \varepsilon]$, such that
\[
s^2 + f^2(s) \leq G^{-1}(sf(s)), \quad \forall |s| < \varepsilon
\]
\[
v_1 |s| \leq |f(s)| \leq v_2 |s|, \quad \forall |s| \geq \varepsilon. \tag{13}
\]
which implies that $sf(s) > 0$ for all $s \neq 0$. The function $f$ satisfies
\[
|f(\psi_2) - f(\psi_1)| \leq k_0(|\psi_2|^\beta + |\psi_1|^\beta)|\psi_2 - \psi_1|, \quad \psi_1, \psi_2 \in \mathbb{R}, \tag{14}
\]
where $k_0, \beta > 0$.

**Hypothesis 4 (H4).** The bounded function $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$, satisfying
\[
\int_{\tau_1}^{\tau_2} |\mu_2(\xi)| d\xi < \mu_1. \tag{15}
\]

Now, as in [23], taking the following new variable
\[
y(x, \rho, \rho, t) = \phi_t(x, t - \rho \rho),
\]
then we obtain
\[
\begin{aligned}
\phi y_t(x, \rho, \rho, t) + y_p(x, \rho, \rho, t) &= 0 \\
y(x, 0, \rho, t) &= \phi_t(x, t).
\end{aligned}
\]
As in [24], we introduce the following new variable
\[
\eta^t(x, s) = \phi(x, t) - \phi(x, t - s), \quad (x, t, s) \in (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+,
\]
where $\eta^t$ is the relative history of $\phi$ satisfies
\[
\eta^t_t + \eta^t_s = \phi_t(x, t), \quad (x, t, s) \in (0, 1) \times (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+.
\]
Consequently, the problem (7) is equivalent to
\[
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0 \\
\int_0^1 \phi_{tt} - b\phi_{xx} + bu_x + \zeta\phi + \int_0^\infty g(p)\eta_x^2(p)dp \\
&\quad + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(x)|y(x,1,\rho,t)du + a(t)f(\phi_t) = 0 \\
\phi_t(x,\rho,\phi,t) + y_t(x,\rho,\phi,t) &= 0 \\
\eta_t^1 + \eta_t^2 &= \phi_t(x,t),
\end{aligned}
\] (16)

where
\[(x,\rho,\phi,t) \in (0,1) \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty),\]
with the following boundary and initial conditions
\[
\begin{aligned}
\phi_x(0,t) &= \phi_x(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0, \\
u(x,0) &= u_0(x), u_1(x,0) = u_1(x), \quad x \in (0,1) \\
\phi(x,0) &= \phi_0(x), \phi_1(x,0) = \phi_1(x), \quad x \in (0,1) \\
y(x,\rho,\phi,0) &= f_0(x,\rho\phi), \quad x \in (0,1), \rho \in (0,1), \phi \in (0,\tau_2) \\
\eta^1(x,0) &= 0, \eta^2(x,s) = \eta_0(x,s), \quad (x,s) \in (0,1) \times \mathbb{R}_+. 
\end{aligned}
\] (17)

Meanwhile, from (7) and (9), it follows that
\[
\frac{d^2}{dt^2} \int_0^1 u(x,t)dx = 0. \tag{18}
\]

Therefore, by solving (18) and using the initial data of \(u\), we get
\[
\int_0^1 u(x,t)dx = t \int_0^1 u_1(x)dx + \int_0^1 u_0(x)dx. 
\]

Consequently, if we let
\[
\bar{u}(x,t) = u(x,t) - t \int_0^1 u_1(x)dx + \int_0^1 u_0(x)dx, \tag{19}
\]
we get
\[
\int_0^1 \bar{u}(x,t)dx = 0, \quad \forall t \geq 0.
\]

Therefore, the use of Poincaré’s inequality for \(\bar{u}\) is justified. In addition, a simple substitution shows that \((\bar{u}, \phi, y, \eta^1)\) satisfies system (7). Hence, we work with \(\bar{u}\) instead of \(u\), but write \(u\) for simplicity of notation.

By imposing new appropriate conditions (H3), with the help of some special results, we obtain an unusual, weaker decay result using Lyaponov functiona, extending some earlier results known in the existing literature. The main results in this manuscript are as follows: Theorem 1 for the existence and uniqueness of solution and Theorem 2 for the general stability estimates.
2. Well-Posedness

In this section, we prove the existence and uniqueness result of the system (16)–(18) using the semigroup theory. To achieve our goal, we first introduce the vector function

\[ U = (u, u_t, \phi, \phi_t, y, \eta)^T, \]

and the new dependent variables \( v = u_t, \psi = \phi_t, \varphi = \eta_t \); then, the system (16) can be written as follows

\[
\begin{aligned}
U_t &= AU + \Gamma(U) \\
U(0) &= U_0 = (u_0, u_1, \phi_0, \phi_1, f_0, \eta_0)^T,
\end{aligned}
\]

(20)

where \( A : D(A) \subset H \rightarrow H \) is the linear operator defined by

\[
AU = \begin{pmatrix}
v \\
\frac{\mu}{\rho} u_{xx} + \frac{b}{\rho} \phi_x \\
\psi \\
\frac{l}{J} \psi_{xx} + \frac{b}{J} u_x - \frac{\xi}{J} \phi_x + \frac{1}{J} \int_0^\infty g(p) \phi_{xx}(p) dp \\
-\frac{H_1}{J} \psi - \frac{1}{J} \int_{\tau_1}^{\tau_2} |\mu_2(q)| y(x, 1, q, t) dq \\
-\frac{1}{\rho} \psi_y \\
-\phi_s + \psi
\end{pmatrix},
\]

(21)

and

\[
\Gamma(U) = \begin{pmatrix}
0 \\
0 \\
0 \\
-\frac{n(t)}{J} f(\psi) \\
0 \\
0
\end{pmatrix},
\]

(22)

and \( H \) is the energy space given by

\[
H = H^1_0(0, 1) \times L^2(0, 1) \times H^1_0(0, 1) \times L^2((0, 1) \times (0, 1) \times (\tau_1, \tau_2)) \times L^g(0, 1),
\]

where

\[
\begin{aligned}
L^2(0, 1) &= \{ \Phi \in L^2(0, 1) / \int_0^1 \Phi(x) dx = 0 \}, \\
H^1_0(0, 1) &= H^1(0, 1) \cap L^2(0, 1), \\
L^g(0, 1) &= \{ \Phi : \mathbb{R}^+ \rightarrow H^1_0(0, 1), \int_0^1 \int_0^\infty g(s) \Phi^2_s(p) dp < \infty \},
\end{aligned}
\]

where the space \( L^g(0, 1) \) is endowed with the following inner product

\[
\langle \Phi_1, \Phi_2 \rangle_{L^g(0, 1)} = \int_0^1 \int_0^\infty g(p) \Phi_1^* \Phi_2 dp.
\]

For any

\[
U = (u, v, \phi, \psi, y, \eta)^T \in H, \quad \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\phi}, \tilde{\psi}, \tilde{y}, \tilde{\eta})^T \in H.
\]
The space $\mathcal{H}$ equipped with the inner product is defined by
\[
\langle U, \hat{U} \rangle_{\mathcal{H}} = \rho \int_0^1 v \hat{v} \, dx + \mu \int_0^1 u \hat{u} \, dx + \int_0^1 \psi \hat{\psi} \, dx + \xi \int_0^1 \phi \hat{\phi} \, dx + b \int_0^1 (u \hat{\phi} + \hat{u} \phi) \, dx
\]
for any $U, \hat{U} \in \mathcal{H}$. The domain of $A$ is given by
\[
\mathcal{D}(A) = \left\{ U \in \mathcal{H} \mid U \in H^2_0 \cap H^1 \right\}.
\]
where $H^2_0(0,1) = \{ \Phi \in H^2(0,1) / \Phi(1) = \Phi(0) = 0 \}$.

Theorem 1. Let $U_0 \in \mathcal{H}$ and assume that (10)–(15) hold. Then, there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ of problem (20). Moreover, if $U_0 \in \mathcal{D}(A)$, then
\[
U \in C(\mathbb{R}_+, \mathcal{D}(A)) \cap C^1(\mathbb{R}_+, \mathcal{H}).
\]

Proof. First, we prove that the operator $A$ is dissipative. For any $U_0 \in \mathcal{D}(A)$ and by using (23), we have
\[
\langle AU, U \rangle_{\mathcal{H}} = -\mu_1 \int_0^1 \psi^2 \, dx - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\xi)| \psi y(x, 1, \xi, t) \, d\xi \, dx
\]
\[
- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\xi)| |y_2 ydx \, dp \, dx - \int_0^1 \int_0^{x_0} g(\xi) \varphi \xi p(\xi) \varphi_\xi(\xi) \, d\xi \, dx.
\]
(24)

For the third term of the RHS of (24), we have
\[
- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\xi)| |y_2 ydx \, dp \, dx = -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(\xi)| \frac{d}{d\xi} \psi d\xi \, dx
\]
\[
= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\xi)| y^2(x, 1, \xi, t) \, d\xi \, dx
\]
\[
+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\xi)| y^2(x, 0, \xi, t) \, d\xi \, dx.
\]
(25)

Using Young’s inequality, we obtain
\[
- \int_0^1 \int_{\tau_1}^{\tau_2} e |\mu_2(\xi)| |\psi y(x, 1, \xi, t) \, d\xi \, dx \leq \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |\mu_2(\xi)| \, d\xi \right) \int_0^1 \psi^2 \, dx
\]
\[
+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\xi)| y^2(x, 1, \xi, t) \, d\xi \, dx.
\]
(26)

By integrating the last term of the right-hand side of (24), we have
\[
- \int_0^1 \int_0^{x_0} g(\xi) \varphi \xi p(\xi) \varphi_\xi(\xi) \, d\xi \, dx = \frac{1}{2} \int_0^1 \int_0^{x_0} g^\prime(\xi) \varphi_\xi^2(p) \, d\xi \, dx.
\]
(27)
Substituting (25), (26) and (27) into (24), using the fact that \( y(x, 0, q, t) = \psi(x, t) \) and (15), we obtained

\[
\langle AU, U \rangle_H \leq -\left( \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(q)| dq \right) \int_0^1 \psi^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(p) \phi_x^2(p) dp dx \leq 0.
\]

Hence, the operator \( A \) is dissipative.

Next, we prove that the operator \( A \) is maximal. This is enough to show that the operator \( (\lambda I - A) \) is surjective. Indeed, for any \( F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in H \), we prove that there is a unique \( V = (u, v, \phi, y, \psi, \varphi)^T \in D(A) \) such that

\[
(\lambda I - A)V = F.
\]

That is

\[
\begin{cases}
\lambda u - v = f_1 \in H_0^1(0,1) \\
\rho \lambda v - \mu uu_{xx} - b \phi_x = \rho f_2 \in L^2_0(0,1) \\
\lambda \phi - \psi = f_3 \in H_0^1(0,1) \\
\int \lambda \phi - l \phi_{xx} + bu_x + \zeta \phi - \int_0^\infty g'(p) \phi_{xx}(p) dp \\
\quad + \mu_1 \psi + \int_{\tau_1}^{\tau_2} |\mu_2(q)| y(x, 1, q, t) dq = f_4 \in L^2(0,1) \\
\lambda \psi y(x, \rho, q, t) + \rho \psi(x, \rho, q, t) = \rho f_5 \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \\
\lambda \phi + \phi_x - \psi = f_6 \in L^2(0,1).
\end{cases}
\]

We note that the equation (30)_5 with \( y(x, 0, q, t) = \psi(x, t) \) has a unique solution, given by

\[
y(x, \rho, q, t) = e^{-\lambda \psi} \phi + \phi e^{\lambda \psi} \int_0^\rho e^{\lambda \psi} f_5(x, \sigma, q, t) d\sigma,
\]

then

\[
y(x, 1, q, t) = e^{-\lambda \psi} \phi + \phi e^{\lambda \psi} \int_0^1 e^{\lambda \psi} f_5(x, \sigma, q, t) d\sigma,
\]

and we infer from (30)_6 that

\[
\psi = e^{\lambda \psi} \int_0^\sigma e^{\lambda \psi} f_6(\sigma) d\sigma,
\]

and we have

\[
v = \lambda u - f_1, \quad \psi = \lambda \phi - f_3.
\]

Inserting (32), (33) and (34) in (30)_2 and (30)_4, we get

\[
\begin{cases}
\rho \lambda^2 u - \mu uu_{xx} - b \phi_x = h_1 \in L^2_0(0,1) \\
\mu_3 \phi - \mu_4 \phi_{xx} + bu_x = h_2 \in L^2(0,1),
\end{cases}
\]
We multiply (35) by \( \hat{\mu} \), respectively and integrate their sum over \((0,1)\) to obtain the following variational formulation

\[
B((u, \phi), (\hat{u}, \hat{\phi})) = Y(\hat{u}, \hat{\phi}),
\]

where

\[
B : (H^1_0(0,1) \times H^1_0(0,1))^2 \to \mathbb{R},
\]

is the bilinear form defined by

\[
B((u, \phi), (\hat{u}, \hat{\phi})) = \lambda^2 \rho \int_0^1 u\hat{u}dx + \mu_3 \int_0^1 \phi\hat{\phi}dx + \mu \int_0^1 u_s\hat{u}_sdx \\
+ \mu_4 \int_0^1 \phi_s\hat{\phi}_sdx + b \int_0^1 (u_s\hat{\phi} + \phi\hat{u}_s)dx,
\]

and

\[
Y : (H^1_0(0,1) \times H^1_0(0,1)) \to \mathbb{R},
\]

is the linear functional given by

\[
Y(\hat{u}, \hat{\phi}) = \int_0^1 h_1\hat{u}dx + \int_0^1 h_2\hat{\phi}dx
\]

Now, for \( V = H^1_0(0,1) \times H^1_0(0,1) \), equipped with the norm

\[
\|(u, \phi)\|_V^2 = \|u\|_2^2 + \|\phi\|_2^2 + \|u_s\|_2^2 + \|\phi_s\|_2^2,
\]

we have

\[
B((u, \phi), (u, \phi)) = \lambda^2 \rho \int_0^1 u^2dx + \mu_3 \int_0^1 \phi^2dx + \mu \int_0^1 u_s^2dx \\
+ 2b \int_0^1 u_s\phi dx + \mu_4 \int_0^1 \phi_s^2dx.
\]

On the other hand, we can write

\[
\mu u_s^2 + 2bu_s\phi + \mu_3\phi^2 = \frac{1}{2} \left[ \mu \left( u_s + \frac{b}{\mu} \phi \right)^2 + \mu_3 \left( \phi + \frac{b}{\mu_3} u_s \right)^2 \right] \\
+ \left[ \mu - \frac{b^2}{\mu_3} \right] u_s^2 + \left( \mu_3 - \frac{b^2}{\mu} \right) \phi^2.
\]

Since \( \mu^2 > b^2 \), we deduce that

\[
\mu u_s^2 + 2bu_s\phi + \mu_3\phi^2 > \frac{1}{2} \left[ \mu - \frac{b^2}{\mu_3} \right] u_s^2 + \left( \mu_3 - \frac{b^2}{\mu} \right) \phi^2,
\]
then, for some \( M_0 > 0 \)
\[
B((u, \phi), (u, \phi)) \geq M_0 \| (u, \phi) \|_V^2.
\] (41)
Thus, \( B \) is coercive, similarly,
\[
Y(\hat{u}, \hat{\phi}) \geq M_1 \| (\hat{u}, \hat{\phi}) \|_V^2.
\] (42)
Consequently, using Lax–Milgram theorem, we conclude that (16) has a unique solution
\[
(u, \phi) \in H^1_x(0, 1 \times H^1_0(0, 1).
\]
Substituting \( u, \phi \) into (32), (33) and (34), respectively, we have
\[
v \in H^1_x(0, 1), \quad \psi \in H^1_0(0, 1), \quad \phi \in L^2_x(0, 1)
y, y_p \in L^2((0, 1) \times (0, 1) \times (\tau_1, \tau_2)).
\] (43)
Moreover, if we take \( \hat{u} = 0 \in H^1_x(0, 1) \) in (37) to obtain
\[
\mu_3 \int_0^1 \phi \hat{\phi} dx + b \int_0^1 u_x \hat{\phi} dx + \mu_4 \int_0^1 \phi x \hat{\phi} x dx = \int_0^1 h_2 \hat{\phi} dx,
\] (44)
we get
\[
\mu_4 \int_0^1 \phi x \hat{\phi} x dx = \int_0^1 (h_2 - \mu_3 \phi - bu_x) \hat{\phi} dx, \quad \forall \hat{\phi} \in H^1_x(0, 1),
\] (45)
which yields
\[
\mu_4 \phi x x = (h_2 - \mu_3 \phi - bu_x) \in L^2(0, 1).
\] (46)
Thus,
\[
\phi \in H^2(0, 1) \cap H^1_0(0, 1).
\] (47)
Consequently, (45) takes the following form
\[
\int_0^1 (-\mu_4 \phi x x - h_2 + \mu_3 \phi + bu_x) \hat{\phi} dx = 0, \quad \forall \hat{\phi} \in H^1_0(0, 1).
\]
Hence, we get
\[
-\mu_4 \phi x x + \mu_3 \phi + bu_x = h_2.
\]
This give (35)2. Similarly, if we take \( \hat{\phi} = 0 \in H^1_0(0, 1) \) in (37) to obtain
\[
\mu \int_0^1 u_x \hat{u} dx + b \int_0^1 \phi \hat{u} x dx + \lambda^2 \rho \int_0^1 u \hat{u} dx = \int_0^1 h_1 \hat{u} dx,
\]
we obtain
\[
\mu \int_0^1 u_x \hat{u} dx = \int_0^1 (h_1 + b \psi x - \lambda^2 \rho u) \hat{u} dx, \quad \forall \hat{\psi} \in H^1_0(0, 1),
\] (48)
which yields
\[
-\mu \psi x x = (h_1 + b \psi x - \lambda^2 \rho u) \psi \in L^2(0, 1).
\] (49)
Consequently, (48) takes the following form
\[
\int_0^1 (-\mu \psi x x - h_1 - b \psi x + \lambda^2 \rho u) \hat{u} dx = 0, \quad \forall \hat{\psi} \in H^1_0(0, 1).
\]
Hence, we obtain
\[
-\mu u x x - b \psi x + \lambda^2 \rho u = h_1.
\]
This gives (35).
Moreover, (48) also holds for any $\Phi \in C^1([0,1])$. Then, by using integration by parts, we obtain
\[ \mu \int_0^1 u_x \Phi_x dx + \int_0^1 (-h_1 - b \phi x + \lambda^2 p u) \Phi dx = 0, \quad \forall \Phi \in C^1([0,1]). \] (50)
Then, we obtain for any $\Phi \in C^1([0,1])$
\[ u_s(1) \Phi(1) - u_s(0) \Phi(0) = 0. \] (51)
Since $\Phi$ is arbitrary, we obtain that $u_s(0) = u_s(1) = 0$. Hence, $u \in H^2_0(0,1) \cap H^1_1(0,1)$. Therefore, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in D(A)$ such that (29) is satisfied. Consequently, we conclude that $A$ is a maximal dissipative operator. Now, we prove that the operator $\Gamma$ defined in (22) is locally Lipschitz in $\mathcal{H}$. Let
\[ U = (u, v, \phi, \psi, y, \varphi)^T \in \mathcal{H}, \hat{U} = (\hat{u}, \hat{v}, \hat{\phi}, \hat{\psi}, \hat{y}, \hat{\varphi})^T \in \mathcal{H}. \]
Then, we have
\[ \|\Gamma(U) - \Gamma(\hat{U})\|_{\mathcal{H}} \leq M_5 \|f(\psi) - f(\hat{\psi})\|_{L^2(0,1)}. \]
By using (14) and Holder and Poincaré’s inequalities, we can obtain
\[ \|f(\psi) - f(\hat{\psi})\|_{L^2(0,1)} \leq k_0(\|\psi\|_{L^2(0,1)} + \|\hat{\psi}\|_{L^2(0,1)}) \|\psi - \hat{\psi}\| \leq k_1 \|\psi - \hat{\psi}\|_{L^2(0,1)}, \]
which gives us
\[ \|\Gamma(U) - \Gamma(\hat{U})\|_{\mathcal{H}} \leq M_4 \|U - \hat{U}\|_{\mathcal{H}}. \]
Then, the operator $\Gamma$ is locally Lipschitz in $\mathcal{H}$. Consequently, the well-posedness result follows from the Hille–Yosida theorem. The proof is completed. \ \Box

3. Stability Result
In this section, we state and prove our decay result for the energy of the system (16)–(18) using the multiplier technique. We need the following Lemmas.

Lemma 1. The energy functional $E$, defined by
\[ E(t) = \frac{1}{2} \int_0^1 \left[ \mu u_t^2 + \mu u_x^2 + f\phi_t^2 + l\phi_x^2 + \xi \phi^2 + 2b u_x \phi \right] dx + \frac{1}{2} \int_0^1 \int_0^\infty g(p) \phi_x^2(p) dp dx + \frac{1}{2} \int_0^1 \int_0 \int_{\tau_1}^{\tau_2} e |\mu_2(\varepsilon)| y^2(x, \rho, \varepsilon, t) d\varepsilon d\rho dx, \] (52)
satisfies
\[ E'(t) \leq -\eta_0 \int_0^1 \phi_t^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(p) \phi_x^2(p) dp dx + \alpha(t) \int_0^1 \phi f(\phi_1) dx \leq 0, \] (53)
where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\varepsilon)| d\varepsilon > 0$ and $\phi(s) = \eta^i = \phi(x, t) - \phi(x, t - p)$. 

The last term in the LHS of (54) is estimated as follows:

$\int_0^1 \phi t \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y(x,1,\phi, t) d\phi dx \leq \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| d\rho \right) \int_0^1 \phi_1^2 d\phi + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y^2(x,1,\phi, t) d\phi dx$.

Now, multiplying the equation (16)_3 by $y|\mu_2(\rho)|$ and integrating the result over $(0,1) \times (\tau_1, \tau_2)$

$\frac{d}{dt} \left[ \int_0^1 \int_{\tau_1}^{\tau_2} e|\mu_2(\rho)| y^2(x,\rho, \phi, t) d\phi d\rho dx \right]$

$= - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y y_{\rho}(x,\rho, \phi, t) d\phi d\rho dx$

$= - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| \frac{d}{d\rho} y^2(x,\rho, \phi, t) d\phi d\rho dx$

$= \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| (y^2(x,0,\phi, t) - y^2(x,1, \phi, t)) d\phi dx$

$= \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(\phi)| d\phi \int_0^1 \phi_1^2 d\phi - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y^2(x,1,\phi, t) d\phi dx$.

Now, using (54), (55), (56) and (57), we have

$\mathcal{E}'(t) \leq - \left( \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| d\rho \right) \int_0^1 \phi_1^2 dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} g'(\rho) \varphi_2^2(\rho) d\rho dx$

$- \alpha(t) \int_0^1 \phi_1 f(\phi_1) dx$.

then, by (10), there exists a positive constant $\eta_0$, such that

$\mathcal{E}'(t) \leq - \eta_0 \int_0^1 \phi_1^2 dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} g'(\rho) \varphi_2^2(\rho) d\rho dx - \alpha(t) \int_0^1 \phi_1 f(\phi_1) dx$.

hence, by (11)–(15) we obtain $\mathcal{E}$ is a non-increasing function. □
Remark 1. Using \((\mu^2 > b^2)\), we conclude that the energy \(E(t)\) defined by (52) satisfies

\[
E(t) > \frac{1}{2} \int_0^1 \left[ \mu u_t^2 + \tilde{\mu} u_x^2 + f \phi_t^2 + l \phi_x^2 + \tilde{\xi} \phi^2 \right] dx
+ \frac{1}{2} \int_0^1 \int_0^\infty g(p) \phi_x^2(p) dp \, dx
+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} e |\mu_2(q)| y^2(x, q, t) dq \, dp \, dx,
\]

(60)

where

\[
\tilde{\mu} = \frac{1}{2}(\mu - \frac{b^2}{\xi}) > 0, \quad \tilde{\xi} = \frac{1}{2}(\xi - \frac{b^2}{\mu}) > 0,
\]

then \(E(t)\) is a positive function.

Lemma 2. The functional

\[
D_1(t) := \int_0^1 \phi \phi_t \, dx + \frac{b \phi}{\mu} \int_0^1 \phi \int_0^x \xi_1(y) dy \, dx + \frac{\mu_1}{2} \int_0^1 \phi^2 \, dx,
\]

(61)

satisfies

\[
D'_1(t) \leq -\frac{1}{2} \int_0^1 \phi_t^2 \, dx - \tilde{\mu} \int_0^1 \phi^2 \, dx + \epsilon_1 \int_0^1 u_t^2 dx + c(1 + \frac{1}{\epsilon_1}) \int_0^1 \phi_t^2 \, dx
+ c \int_0^1 \int_0^\infty g(p) \phi_x^2(p) dp \, dx + c \int_0^1 f^2(\phi_t) \, dx
+ \epsilon_1 \int_0^1 \left( \int_0^x \xi_1(y) dy \right)^2 dx + \frac{1}{2} \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(q)| y(x, q, t) dq \, dx
+ \alpha(t) \int_0^1 \phi f(\phi_t) \, dx.
\]

(62)

where \(\tilde{\mu} = \xi - \frac{b^2}{\mu} > 0\).

Proof. Direct computation using integration by parts and Young’s inequality, for \(\epsilon_1 > 0\), yields

\[
D'_1(t) = -\int_0^1 \phi_t^2 \, dx - \left( \xi - \frac{b^2}{\mu} \right) \int_0^1 \phi^2 \, dx + \frac{b \phi}{\mu} \int_0^1 \phi \int_0^x \xi_1(y) dy \, dx
+ \frac{1}{2} \int_0^1 \phi^2 \, dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(q)| y(x, q, t) dq \, dx
\]

\[
\leq -\int_0^1 \phi_t^2 \, dx - \left( \xi - \frac{b^2}{\mu} \right) \int_0^1 \phi^2 \, dx + c \left( 1 + \frac{1}{\epsilon_1} \right) \int_0^1 \phi_t^2 \, dx
+ \epsilon_1 \int_0^1 \left( \int_0^x \xi_1(y) dy \right)^2 dx + \frac{1}{2} \int_0^1 \phi \int_0^\infty g(p) \phi_x(p) dp \, dx
- \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(q)| y(x, q, t) dq \, dx + \alpha(t) \int_0^1 \phi f(\phi_t) \, dx.
\]

(63)

According to Cauchy–Schwarz inequality, it is clear that

\[
\int_0^1 \left( \int_0^x \xi_1(y) dy \right)^2 dx \leq \int_0^1 \left( \int_0^1 \xi_1 \left( \int_0^x \phi \right) \, dx \right)^2 dx \leq \int_0^1 \xi_1^2 \, dx.
\]
Therefore, estimate (63) becomes

\[ D_1'(t) \leq -\delta \int_0^1 \phi_x^2 dx - \left( \xi - \frac{b^2}{\mu} \right) \int_0^1 \phi^2 dx + c \left( 1 + \frac{1}{\varepsilon_1} \right) \int_0^1 \phi_t^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx - \int_0^1 \phi \int_{\tau_1}^2 |\mu_2(\phi)|y(x, 1, \phi, t) d\phi dx + \int_0^1 \phi_x \int_0^1 g(p) \phi_x(p) dp dx + a(t) \int_0^1 \phi f(\phi_t) dx. \]  

(64)

The last term in the RHS of (64) is estimated as follows

\[
\int_0^1 \phi_x \int_0^1 g(p) \phi_x(p) dp dx \leq c\delta_1 \int_0^1 \phi_x^2 dx + \frac{c}{4\delta_1} \int_0^1 \int_0^\infty g(p) \phi_x^2(p) dp dx,
\]

(65)

where we used Cauchy–Schwarz, Young and Poincaré’s inequalities, for \( \delta_1, \delta_2, \delta_3 > 0 \).

By substituting (65) into (63), we obtain

\[
D_1'(t) \leq -\left( l - c\delta_1 - \mu_1 c\delta_2 - c\delta_3 \right) \int_0^1 \phi_x^2 dx - \left( \xi - \frac{b^2}{\mu} \right) \int_0^1 \phi^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + c \int_0^1 \phi_t^2 dx + \frac{c}{4\delta_1} \int_0^1 \int_0^\infty g(p) \phi_x^2(p) dp dx + \frac{1}{4\delta_2} \int_0^1 \int_{\tau_1}^2 |\mu_2(p)|y^2(x, 1, \phi, t) d\phi dx + \frac{1}{4\delta_3} f^2(\phi_t) dx.
\]

(66)

Bearing in mind that \( \mu \xi > b^2 \) and letting \( \delta_1 = \frac{1}{\delta_2}, \delta_2 = \frac{1}{\delta_3} \), and \( \delta_3 = \frac{1}{\delta_3} \), we obtain estimate (62).

Lemma 3. Then, for any \( \varepsilon_2 > 0 \) the functional

\[
D_2(t) := \int_0^1 \phi_x u_t x dx + \int_0^1 \phi_t u_x x dx - \frac{\rho}{\mu} \int_0^1 u_t \int_0^\infty g(p) \phi_x(t - p) dp dx,
\]

satisfies

\[
D_2'(t) \leq -\frac{b}{2\tau} \int_0^1 u_x^2 dx + c \int_0^1 \phi_x^2 dx + c \int_0^1 u_t^2 dx + c \int_0^1 \phi_t^2 dx + c \int_0^1 \int_0^\infty g(p) \phi_x^2(p) dp dx - \frac{c}{4\varepsilon_2} \int_0^1 \int_0^\infty g'(p) \phi_x^2(p) dp dx + c \int_0^1 \int_{\tau_1}^2 |\mu_2(p)|y^2(x, 1, \phi, t) d\phi dx + c \int_0^1 f^2(\phi_t) dx
\]

(67)

Proof. By differentiating \( D_2 \), then using (16), integration by parts and (17) we obtain

\[
D_2'(t) = -\frac{b}{\tau} \int_0^1 u_x^2 dx + \left( \frac{1 + \frac{\varepsilon_2}{\tau}}{\frac{b}{\mu}} - \frac{\mu}{\rho} \right) \int_0^1 u_x \phi_x dx + \left( \frac{b}{\rho} - \frac{b\varepsilon_0}{\mu} \right) \int_0^1 \phi_x^2 dx - \frac{\xi}{\tau} \int_0^1 u_x \phi dx - \frac{\mu}{\mu} \int_0^1 \phi_x \int_0^\infty g(p) \phi_x(p) dp dx
\]

\[
- \frac{\rho}{\mu} \int_0^1 u_t \int_0^\infty g'(p) \phi_x(p) dp dx - \frac{a(t)}{\mu} \int_0^1 u_x f(\phi_t) dx - \frac{\mu_1}{\tau} \int_0^1 \phi_t u_x dx - \frac{1}{\tau} \int_0^1 u_x \int_{\tau_1}^2 |\mu_2(\phi)|y^2(x, 1, \phi, t) d\phi dx.
\]

(68)
In what follows, we estimate the last six terms in the RHS of (68), using Young, Cauchy–Schwartz and Poincaré’s inequalities. For \( \delta_4, \delta_5, \epsilon_2 > 0 \), we have

\[- \frac{\xi}{T} \int_0^1 u_x \phi \, dx \leq \frac{\xi}{T} \delta_4 \int_0^1 u_x^2 \, dx + \frac{\xi}{4J \delta_4} \int_0^1 \phi^2 \, dx.\]

By letting \( \delta_4 = \frac{b}{6J} \), using Poincaré’s inequality, we get

\[- \frac{\xi}{T} \int_0^1 u_x \phi \, dx \leq \frac{b}{6J} \int_0^1 u_x^2 \, dx + c \int_0^1 \phi^2 \, dx,\]  

(69)

and by Young and Cauchy–Schwarz’s inequalities, we get

\[- \frac{b}{\mu J} \int_0^1 \phi_x \int_0^\infty g(p) \varphi_x(p) \, dp \, dx \leq \frac{b}{\delta_5} \int_0^1 \phi_x^2 \, dx + c \int_0^1 \int_0^\infty g(p) \varphi_x^2(p) \, dp \, dx.\]

By letting \( \delta_5 = \frac{b}{6J} \), we obtain

\[- \frac{b}{\mu J} \int_0^1 \phi_x \int_0^\infty g(p) \varphi_x(p) \, dp \, dx \leq \frac{b}{6J} \int_0^1 \phi_x^2 \, dx + c \int_0^1 \int_0^\infty g(p) \varphi_x^2(p) \, dp \, dx.\]  

(70)

Similarly, \( \forall \epsilon_2 > 0 \) we have

\[\frac{\rho}{\mu J} \int_0^1 u_t \int_0^\infty g'(p) \varphi_x(p) \, dp \, dx \leq \epsilon_2 \int_0^1 u_t^2 \, dx + \int_0^1 \int_0^\infty g'(p) \varphi_x^2(p) \, dp \, dx,\]  

(71)

and

\[- \frac{\mu_1}{T} \int_0^1 \phi u_x \, dx \leq \frac{\mu_1 \delta_6}{2J} \int_0^1 u_x^2 \, dx + \frac{\mu_1}{2J \delta_6} \int_0^1 \phi^2 \, dx,\]  

(72)

and

\[\frac{1}{J} \int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(\xi)(x, 1, \xi, t)\varphi \, d\xi \, dx \leq \frac{\delta_7 \mu_1}{2J} \int_0^1 u_x^2 \, dx + \frac{\delta_7}{2J \delta_7} \int_0^1 \phi^2 \, dx,\]  

(73)

and

\[- \frac{\alpha(t)}{J} \int_0^1 u_x f(\phi) \, dx \leq \frac{\alpha(t) \delta_8}{2J} \int_0^1 u_x^2 \, dx + \frac{\alpha(t) \delta_8}{2J \delta_8} \int_0^1 f^2(\phi) \, dx.\]  

(74)

Replacing (69)–(74) into (68) and letting \( \delta_6 = \delta_7 = \frac{b}{6J} \) and \( \delta_8 = \frac{b}{6\alpha(\xi)} \), yields (67). \( \square \)

**Lemma 4.** The functional

\[D_3(t) := -\rho \int_0^1 u_x u_t \, dx,\]

satisfies

\[D_3'(t) \leq -\rho \int_0^1 u_x^2 \, dx + \frac{3 \mu}{2J} \int_0^1 u_x^2 \, dx + c \int_0^1 \phi_x^2 \, dx.\]

(75)

**Proof.** Direct computations give

\[D_3'(t) = -\rho \int_0^1 u_x^2 \, dx + \mu \int_0^1 u_x^2 \, dx + b \int_0^1 u_x \phi \, dx.\]
Then, for any $U$

**Theorem 2.** Assume (10)–(15) hold. Let $h(t)$ be a positive non-increasing function. Then, for any $U_0 \in D(A)$, satisfying, for some $c_0 > 0$

\[
\max \left\{ \int_0^1 \phi^{\alpha}(x,s)dx, \int_0^1 \phi_{\alpha\alpha}(x,s)dx \right\} \leq c_0, \quad \forall s > 0,
\]

there are positive constants $\beta_1, \beta_2$ and $\beta_3$ such that the energy functional given by (52) satisfies

\[
\mathcal{E}(t) \leq \beta_1 G_0^{-1} \left( \frac{\beta_2 + \beta_3 \int_0^1 h(p)\alpha(p)}{\int_0^1 h(p)dp} \right),
\]

The estimate (75) easily follows using Young and Poincaré inequalities.

\[
D'_2(t) \leq -\rho \int_0^1 u_2^2 dx + \mu \int_0^1 u_2^2 dx + \frac{b}{4\epsilon} \int_0^1 \phi^2 dx
\]

by taking $\epsilon = \frac{\mu}{2\epsilon}$, we obtain (75). □

**Lemma 5.** The functional

\[
D_4(t) := \int_0^1 \int_0^1 \int_{\eta_1}^{\eta_2} e^{-\rho} |\mu_2(\xi)| y^2(x, \rho, \xi, t) d\xi d\rho dx,
\]

satisfies

\[
D'_4(t) \leq -\eta_1 \int_0^1 \int_0^1 \int_{\eta_1}^{\eta_2} e^{-\rho} |\mu_2(\xi)| y^2(x, \rho, \xi, t) d\rho d\xi dx + \mu_1 \int_0^1 \phi^2 dx
\]

\[
-\eta_1 \int_0^1 \int_{\eta_1}^{\eta_2} |\mu_2(\xi)| |y^2(x, 1, \xi, t) - \phi^2(x, 0, \xi, t)| d\xi dx,
\]

(76)

where $\eta_1$ is a positive constant.

**Proof.** By differentiating $D_4$, with respect to $t$, using the Equation (16)_3, we have

\[
D'_4(t) = -2 \int_0^1 \int_0^1 \int_{\eta_1}^{\eta_2} e^{-\rho} |\mu_2(\xi)| y_\phi(x, \rho, \xi, t) d\xi d\rho dx
\]

\[
= -\int_0^1 \int_0^1 \int_{\eta_1}^{\eta_2} e^{-\rho} |\mu_2(\xi)| y^2(x, \rho, \xi, t) d\xi d\rho dx
\]

\[
- \int_0^1 \int_{\eta_1}^{\eta_2} |\mu_2(\xi)| |e^{-\rho} y^2(x, 1, \xi, t) - \phi^2(x, 0, \xi, t)| d\xi dx.
\]

Using the fact that $y(x, 0, \xi, t) = \phi(x, t)$ and $e^{-\rho} \leq e^{-\eta_2} \leq 1$, for all $0 < \rho < 1$, we obtain

\[
D'_4(t) \leq -\eta_1 \int_0^1 \int_0^1 \int_{\eta_1}^{\eta_2} e^{-\rho} |\mu_2(\xi)| y^2(x, \rho, \xi, t) d\rho d\xi dx
\]

\[
- \int_0^1 \int_{\eta_1}^{\eta_2} e^{-\rho} |\mu_2(\xi)| y^2(x, 1, \xi, t) d\xi dx + \int_0^1 \int_{\eta_1}^{\eta_2} |\mu_2(\xi)| d\xi \int_0^1 \phi^2 dx.
\]

Since $-e^{-\rho}$ is an increasing function, we have $-e^{-\rho} \leq -e^{-\eta_2}$, for all $\rho \in [\eta_1, \eta_2]$. Finally, setting $\eta_1 = e^{-\eta_2}$ and recalling (15), we obtain (76). We are now ready to prove the main result.

**Theorem 2.** Assume (10)–(15) hold. Let $h(t) = a(t)$. $\eta(t)$ be a positive non-increasing function. Then, for any $U_0 \in D(A)$, satisfying, for some $c_0 > 0$

\[
\max \left\{ \int_0^1 \phi^{\alpha}(x,s)dx, \int_0^1 \phi_{\alpha\alpha}(x,s)dx \right\} \leq c_0, \quad \forall s > 0,
\]

there are positive constants $\beta_1, \beta_2$ and $\beta_3$ such that the energy functional given by (52) satisfies

\[
\mathcal{E}(t) \leq \beta_1 G_0^{-1} \left( \frac{\beta_2 + \beta_3 \int_0^1 h(p)\alpha(p)}{\int_0^1 h(p)dp} \right),
\]
where
\[ G_0(t) = tG'(\epsilon_0 t), \forall \epsilon_0 \geq 0, \quad \text{and} \quad \omega(s) = \int_s^\infty g(\sigma)d\sigma. \quad (79) \]

**Proof.** We define a Lyapunov functional
\[ L'(t) := N\mathcal{E}(t) + N_1D_1(t) + N_2D_2(t) + D_3(t) + N_4D_4(t), \quad (80) \]
where \( N, N_1, N_2, \) and \( N_4 \) are positive constants, to be chosen later. By differentiating (80) and using (53), (62), (67), (75), (76), we have
\[
L'(t) \leq -\left[ \frac{1N_1}{2} - cN_2 - c \right] \int_0^1 \phi_x^2 dx - \left[ \rho - N_1\epsilon_1 - N_2\epsilon_2 \right] \int_0^1 u^2 dx \\
- \left[ \frac{bN_2}{2} - \frac{3\mu}{2} \right] \int_0^1 u^2 dx + c[N_1 + N_2] \int_0^1 \int_0^\infty g(p)\phi_x^2(p)dpdx \\
- \left[ \eta_0N - cN_1(1 + \frac{1}{\epsilon_1}) - N_2c - \mu_1N_4 \right] \int_0^1 \phi_t^2 dx \\
- [N_4\eta_1 - cN_1 - cN_2] \int_0^1 \int_0^{\tau_2} |\mu_2(\xi)|y^2(x, 1, \xi, t)d\xi dx \\
- N_4\tilde{\mu} \int_0^1 \phi^2 dx + \left[ N_2 - \frac{cN_2}{2} \right] \int_0^1 \int_0^\infty g'(p)\phi_x^2(p)dpdx \\
- N_4\eta_1 \int_0^1 \int_0^{\tau_2} \chi(\xi, 1, \xi, t)d\xi dx \\
+c[N_1 + N_2] \int_0^1 f^2(\varphi) dx + N_2\chi \int_0^1 u_{xx}\phi_x dx,
\]
where \( \chi = (\frac{\mu}{\rho} - \frac{\mu_0}{\rho}) \) and by setting
\[ \epsilon_1 = \frac{\rho}{4N_1}, \epsilon_2 = \frac{\rho}{4cN_2}, \]
we obtain
\[
L'(t) \leq -\left[ \frac{1N_1}{2} - cN_2(1 + N_2) - c \right] \int_0^1 \phi_x^2 dx - \frac{\rho}{2} \int_0^1 u^2 dx \\
- \left[ \frac{bN_2}{2} - \frac{3\mu}{2} \right] \int_0^1 u^2 dx + c[N_1 + N_2] \int_0^1 \int_0^\infty g(p)\phi_x^2(p)dpdx \\
- [\eta_0N - cN_1(1 + N_4) - cN_2 - \mu_1N_4] \int_0^1 \phi_t^2 dx \\
- [N_4\eta_1 - cN_1 - cN_2] \int_0^1 \int_0^{\tau_2} |\mu_2(\xi)|y^2(x, 1, \xi, t)d\xi dx \\
- N_4\tilde{\mu} \int_0^1 \phi^2 dx + \left[ N_2 - \frac{cN_2}{2} \right] \int_0^1 \int_0^\infty g'(p)\phi_x^2(p)dpdx \\
- N_4\eta_1 \int_0^1 \int_0^{\tau_2} \chi(\xi, 1, \xi, t)d\xi dx \\
+c[N_1 + N_2] \int_0^1 f^2(\varphi) dx + N_2\chi \int_0^1 u_{xx}\phi_x dx.
\]
Next, we carefully choose our constants so that the terms inside the brackets are positive. We choose a \( N_2 \) that is large enough that
\[ \alpha_1 = \frac{bN_2}{2\tilde{\mu}} - \frac{3\mu}{2} > 0, \]
then, we choose a large enough $N_1$ that
\[
\alpha_2 = \frac{1}{4} N_1 - c N_2 (1 + N_2) - c > 0,
\]
then we choose a large enough $N_4$ that
\[
\alpha_3 = N_4 \eta_1 - c N_1 - c N_2 > 0,
\]
thus, we arrive at
\[
\mathcal{L}'(t) \leq -\alpha_2 \int_0^1 \phi_2^2 dx - \alpha_0 \int_0^1 \phi_1^2 dx - \frac{\rho}{2} \int_0^1 u_1^2 dx - \alpha_1 \int_0^1 u_0^2 dx - \int_0^1 [N - c] \int_0^1 \phi_1^2 dx + \frac{N}{2} c \int_0^1 \int_0^\infty g'(p) \phi_1^2 dx dp \int_0^1 \phi dx
\]
\[
+ c \int_0^1 \int_0^\infty g(p) \phi_1^2 dx dp - \alpha_3 \int_0^1 \int_0^\infty \mu_2(e) |(x, e, t)| \phi_2 dx dp \int_0^1 \phi dx
\]
\[
- \alpha_4 \int_0^1 \int_0^1 \int_0^\infty \mu_2(e) |(x, e, t)| \phi_2 dx dp \int_0^1 \phi dx
\]
\[
+ c \int_0^1 \int_0^1 \int_0^\infty \mu_2(e) |(x, e, t)| \phi_2 dx dp \int_0^1 \phi dx.
\]
(81)
where $\alpha_0 = \tilde{\mu} N_1 = \left( \xi - \frac{\mu}{\rho} \right) N_1$, and $\alpha_5 = N_2 \chi = N_2 \left( \frac{\mu}{\rho} - \frac{\rho}{\nu} \right)$. On the other hand, if we let
\[
\mathcal{E}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t),
\]
then
\[
|\mathcal{E}(t)| \leq J N_1 \int_0^1 |\phi_1| dx + \frac{b_0 N_1}{\mu} \int_0^1 \phi \int_0^x u_i(y) dy dx
\]
\[
+ b_1 \int_0^1 |\phi u_1 + u_2 \phi_1| dx + \frac{\rho}{\mu} \int_0^1 g(p) \phi_2(t - p) dp dx
\]
\[
+ c \int_0^1 |u_i u| dx + N_4 \int_0^1 \int_0^\infty \int_0^1 \int_0^\infty \int_0^\infty \phi(x, p, e, t) dp \phi(x, p, e, t) dp dx.
\]
Exploiting Young, Cauchy–Schwartz and Poincaré inequalities, we obtain
\[
|\mathcal{E}(t)| \leq c \int_0^1 (u_1^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 + \phi_5^2) dx + c \int_0^1 \int_0^\infty g(p) \phi_2^2(p) dp dx
\]
\[
+ c \int_0^1 \int_0^1 \int_0^\infty \int_0^\infty \phi_2(s) |(x, e, t)| \phi_2 dx dp \phi(x, p, e, t) dp dx
\]
\[
\leq c E(t).
\]
Consequently, we obtain
\[
|\mathcal{E}(t)| = |\mathcal{L}(t) - N E(t)| \leq c E(t),
\]
that is
\[
(N - c) E(t) \leq \mathcal{L}(t) \leq (N + c) E(t).
\]
(82)
Now, by choosing a large enough $N$ that
\[
\frac{N}{2} - c > 0, N - c > 0, N \eta_0 - c > 0,
\]
and exploiting (52), estimates (81) and (82), respectively, we obtain
\[ c_2 \mathcal{E}(t) \leq \mathcal{L}(t) \leq c_3 \mathcal{E}(t), \forall \ t \geq 0, \]  
(83)
and
\[ \mathcal{L}'(t) \leq -k_1 \mathcal{E}(t) + k_2 \int_0^1 \int_0^\infty g(p) \varphi_x^2(p) dp dx + k_3 \int_0^1 (\phi_t^2 + f^2(\phi_1)) dx + \alpha_5 \int_0^1 u_x \phi_s dx, \]  
(84)
for some \( k_1, k_2, k_3, c_2, c_3 > 0. \)

**Case 1.** If \( \chi = \frac{Bu}{\rho - \frac{\delta}{t}} = 0 \), in this case, (84) takes the form
\[ \mathcal{L}'(t) \leq -k_1 \mathcal{E}(t) + k_2 \int_0^1 \int_0^\infty g(p) \varphi_x^2(p) dp dx + k_3 \int_0^1 (\phi_t^2 + f^2(\phi_1)) dx. \]  
(85)
By multiplying (85) by \( h(t) = \alpha(t) \eta(t) \), we obtain
\[ h(t) \mathcal{L}'(t) \leq -k_1 h(t) \mathcal{E}(t) + k_2 h(t) \int_0^1 \int_0^\infty g(p) \varphi_x^2(p) dp dx + k_3 h(t) \int_0^1 (\phi_t^2 + f^2(\phi_1)) dx. \]  
(86)
We distinguish two cases
\begin{itemize}
  \item \( G \) is linear on \([0, \epsilon] \). In this case, using the assumption (13) and (53), we can write
  \[ k_3 h(t) \int_0^1 (\phi_t^2 + f^2(\phi_1)) dx \leq k_3 h(t) \int_0^1 \phi_t f(\phi_1) dx \leq -k_3 \eta(t) \mathcal{E}'(t), \]  
(87)
and by (11) we have
  \[ h(t) \int_0^1 \int_0^1 g(p) \varphi_x^2(p) dp dx = \alpha(t) \int_0^1 \int_0^1 q(s) g(p) \varphi_x^2(p) dp dx \leq -\alpha(t) \int_0^1 \int_0^1 g(p) \varphi_x^2(p) dp dx \leq -2 \alpha(t) \mathcal{E}'(t), \]  
(88)
and by (77) we obtain
  \[ \int_0^1 \varphi_x^2(s) dx = 2 \int_0^1 \phi_x^2(x, t) dx + 2 \int_0^1 \phi_x^2(x, t - s) dx \leq 4 \sup_{s > 0} \int_0^1 \phi_x^2(x, s) dx + 2 \sup_{\tau > 0} \int_0^1 \phi_{x\tau}^2(x, \tau) dx \leq \frac{8 \mathcal{E}(0)}{t} + 2 \epsilon_0, \]  
(89)
then, we obtain
  \[ h(t) \int_0^1 \int_0^\infty g(p) \varphi_x^2(p) dp dx \leq (\frac{8 \mathcal{E}(0)}{t} + 2 \epsilon_0) h(t) \int_t^\infty g(p) dp. \]  
(90)\end{itemize}
Hence,
\[
\begin{align*}
    h(t) \int_0^1 \int_0^\infty g(p) \varphi_t^2(p) dp dx &\leq -2\alpha(t) \mathcal{E}'(t) \\
    &\quad + \left( \frac{8\mathcal{E}(0)}{t} + 2\mathcal{C}_0 \right) h(t) \varphi(t).
\end{align*}
\] (91)

Inserting (87) and (91) in (86). Since \( h'(t) \leq 0 \), \( \alpha'(t) \leq 0 \), \( \eta'(t) \leq 0 \). Then, we have

\[
\mathcal{L}_1'(t) \leq -k_1 h(t) \mathcal{E}(t) + \gamma h(t) \varphi(t),
\] (92)

and

\[
m_1 \mathcal{E}(t) \leq \mathcal{L}_1(t) \leq m_2 \mathcal{E}(t),
\] (93)

with

\[
m_1 = \tau_1, \quad m_2 = c_2 h(0) + k_3 \eta(0) + 2k_2 \alpha(0) + \tau_1,
\]

where

\[
\mathcal{L}_1(t) = h(t) \mathcal{L}(t) + (k_3 \eta(t) + 2k_2 \alpha(t) + \tau_1) \mathcal{E}(t) \sim \mathcal{E}(t),
\] (94)

\[
\gamma = \left( \frac{8\mathcal{E}(0)}{t} + 2\mathcal{C}_0 \right), \quad \tau_1 > 0 \quad \text{and} \quad \varphi(t) = \int_t^\infty g(p) dp.
\]

Since \( \mathcal{E}'(t) \leq 0, \forall \ t \geq 0 \). By using (92), we have

\[
\mathcal{E}(T) \int_0^T h(t) dt \leq \left( \frac{\mathcal{L}_1(0)}{k_1} + \frac{\gamma}{k_2} \int_0^T h(t) \varphi(t) dt \right).
\] (95)

Using the fact that \( G_0^{-1} \) is linear.

\[
\mathcal{E}(T) \leq \zeta G_0^{-1} \left[ \frac{\mathcal{L}_1(0)}{k_1} + \frac{\gamma}{k_2} \int_0^T h(t) \varphi(t) dt \right].
\] (96)

with \( \beta_1 = \zeta \), \( \beta_2 = \frac{\mathcal{E}_1(0)}{k_1} \), \( \beta_3 = \frac{\gamma}{k_2} \). This completes the proof.

- \( G \) is nonlinear on \([0, \varepsilon]\), we choose \( 0 \leq \varepsilon_1 \leq \varepsilon \) and we consider

\[
I_1(t) = \{ x \in (0, 1), \ |\phi_1| \leq \varepsilon_1 \}, \quad I_2 = \{ x \in (0, 1), \ |\phi_1| > \varepsilon_1 \},
\]

we define

\[
I = \int_{I_1} \phi_1 f(\phi_1) dt.
\]

Using Jensen’s inequality and the assumption (13), we have

\[
k_3 h(t) \int_0^1 (\phi_1^2 + f^2(\phi_1)) dx \leq k_3 h(t) \int_0^1 \phi_1 f(\phi_1) dx \leq k_3 h(t) G^{-1}(I(t)) - k_3 h(t) \mathcal{E}'(t).
\] (97)

Inserting (97) in (86), since \( \alpha'(t) \leq 0 \), \( \eta'(t) \leq 0 \) and \( \mathcal{E}'(t) \leq 0 \), we obtain

\[
\mathcal{L}_2'(t) \leq -k_1 h(t) \mathcal{E}(t) + \gamma h(t) \varphi(t) + k_3 h(t) G^{-1}(I(t)).
\] (98)

and

\[
m_3 \mathcal{E}(t) \leq \mathcal{L}_2(t) \leq m_4 \mathcal{E}(t),
\] (99)

with

\[
m_3 = \tau_1, \quad m_4 = c_2 h(0) + k_3 \eta(0) + 2k_2 \alpha(0) + \tau_1,
\]
where
\[ L_2(t) = h(t)\mu(t) + 2k_2\eta(t) + \tau_1. \]

Now, for \( \epsilon_0 < \epsilon_1 \) and by using \( E'(t) \leq 0, G' > 0 \) and \( G'' > 0 \) on \((0, \epsilon)\), we define the functional \( L_3(t) \) by
\[ L_3(t) = G'(\epsilon_0 E(t))L_2(t) + \tau_2 \epsilon(t) \sim \epsilon(t), \quad \tau_2 > 0, \]
satisfies
\[
L_3'(t) = E'(t)(\epsilon_0 G'(\epsilon_0 E(t))L_2(t) + \tau_2) + L_2'(t)G'(\epsilon_0 E(t)) + \gamma G'(\epsilon_0 E(t))h(t)\omega(t) + k_3h(t)G'(\epsilon_0 E(t)) G^{-1}(t(t)). \tag{100}
\]

To estimate the last term of (92), using the general Young's inequality
\[ AB \leq G^*(A) + G(B), \quad \text{if} \quad A \in (0, G'(\epsilon)), \quad B \in (0, \epsilon), \tag{101} \]
where
\[ G^*(A) = s(G')^{-1}(s) - G((G')^{-1}(s)), \quad \text{if} \quad s \in (0, G'(\epsilon)), \]
satisfies
\[ k_3h(t)G'(\epsilon_0 E(t)) G^{-1}(t(t)) \leq k_3\epsilon_0 h(t)G_0(\epsilon(t))) - k_3\eta(t)E'(t). \tag{102} \]
Inserting (102) in (92) and letting \( \epsilon_0 = \frac{k_1}{k_3} \) we get
\[ L_3'(t) + k_3\eta(t)E'(t) \leq -k_1h(t)G_0(\epsilon(t))) + \gamma G'(\epsilon_0 E(t))h(t)\omega(t). \tag{103} \]
Since \( \eta'(t) \leq 0 \), then
\[ L_3'(t) \leq -k_1h(t)G_0(\epsilon(t))) + \gamma G'(\epsilon_0 E(t))h(t)\omega(t), \]
where
\[ L_4(t) = L_3(t) + k_3\eta(t)E(t) \sim \epsilon(t). \]
Since \( \alpha(t), G_0(\epsilon(t)), G'(\epsilon_0 E(t)) \) are non-increasing functions, then, for any \( T > 0 \)
\[
k_1G_0(\epsilon(T)) \int_0^T h(t)dt \leq k_1 \int_0^T h(t)G_0(\epsilon(t))dt \leq L_4(0) + \gamma G'(\epsilon_0 E(0)) \int_0^T h(t)\omega(t)dt,
\]
which gives (78) with \( \beta_1 = 1, \ \beta_2 = \frac{L_4(0)}{k_1} \) and \( \beta_3 = \frac{\gamma G'(\epsilon_0 E(0))}{k_1}. \)
The proof is now completed.

**Case 2.** If \( \chi = \frac{\mu}{p} - \frac{\delta}{T} \neq 0 \) and
\[
\begin{align*}
|\chi| < \frac{k_1\mu^2}{2N_2(\mu + b\mu)} & \quad \text{if} \quad \chi < 0, \\
|\chi| < \frac{k_1\mu^2}{2N_2p} & \quad \text{if} \quad \chi > 0.
\end{align*}
\]
This case is more important from the physical perspective, where waves are not necessarily of equal speeds. Let
\[ E(t) = E(u, \psi, y, \varphi) = E_1(t). \]
Denotes the first-order energy defined in (52) and
\[ E_2(t) = E(u_t, \psi_t, y_t, \varphi_t). \]
Denotes the second-order energy; then, we have
\[ E'_2(t) \leq -\eta_0 \int_0^1 \phi_i^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(s) \phi_i^2 dx ds \]
\[ -\alpha'(t) \int_0^1 \phi_i f'(\psi_i) dx - \alpha(t) \int_0^1 \phi_i^2 f'(\psi_i) dx \]
\[ = -\eta_0 \int_0^1 \phi_i^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(s) \phi_i^2 dx ds \]
\[ + \alpha(t) \left( -\frac{\alpha'(t)}{\alpha(t)} \right) \int_0^1 \phi_i f'(\psi_i) dx - \int_0^1 \phi_i^2 f'(\psi_i) dx \].

(104)

Since \( f, g \) are non-decreasing functions, \( \alpha(t) \) is a positive function and \( \lim_{t \to \infty} -\frac{\alpha'(t)}{\alpha(t)} = 0 \), we deduce that
\[ E'_2(t) \leq -\eta_0 \int_0^1 \phi_i^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(s) \phi_i^2 dx ds \]
\[ \leq -\eta_0 \int_0^1 \phi_i^2 dx, \]

(105)

where \( \eta_0 = \mu_1 - \int_{r_1}^{r_2} |\mu_2(q)| dq > 0 \).

The last term in (84), by using (16), Young’s inequality and by setting \( K = \frac{\chi N_2}{\mu} = \frac{\alpha_5 \rho}{\mu} \) and \( \alpha_5 = \chi N_2 \) as follows
\[ \alpha_5 \int_0^1 u_{xx} \phi_x dx = \frac{\alpha_5 \rho}{\mu} \int_0^1 \phi_u u_{xx} dx - \frac{ba_5}{\mu} \int_0^1 \phi_x^2 dx \]
\[ = K \left( \frac{d}{dt} \left[ \int_0^1 \phi_i udx + \int_0^1 \phi_x u_t dx \right] \right) \]
\[ - K \int_0^1 u_x \phi_i^2 dx - \frac{ba_5}{\mu} \int_0^1 \phi_x^2 dx \]
\[ \leq K \left( \frac{d}{dt} \left[ \int_0^1 \phi_i udx + \int_0^1 \phi_x u_t dx \right] \right) \]
\[ + \frac{|K|}{4} \int_0^1 \phi_i^2 dx + |K| \int_0^1 u_t^2 dx. \]

(106)

Let
\[ N(t) = \left( \int_0^1 \phi_i udx + \int_0^1 \phi_x u_t dx \right), \]
We choose a large enough $N$ where

$$
\begin{align*}
L'(t) + KN'(t) & \leq -k_1 \mathcal{E}_1(t) + k_2 \int_0^t \int_0^\infty g(p) \varphi_p^2 dp \, dx + \frac{|K|}{4} \int_0^1 \varphi_t^2 \, dx \\
& \quad + |K| \int_0^1 u_x^2 \, dx + k_3 \int_0^1 (\varphi_t^2 + f^2(\varphi_1)) \, dx \\
& \leq -k_4 \mathcal{E}_1(t) + k_2 \int_0^t \int_0^\infty g(p) \varphi_p^2 dp \, dx \\
& \quad + \frac{|K|}{4} \int_0^1 \varphi_t^2 \, dx + k_3 \int_0^1 (\varphi_t^2 + f^2(\varphi_1)) \, dx,
\end{align*}
$$

(107)

where

$$
k_4 = k_1 - \frac{2|K|}{\mu} > 0.
$$

Let

$$
\mathcal{R}(t) = L(t) + KN(t) + N_5(\mathcal{E}_1(t) + \mathcal{E}_2(t)).
$$

(108)

Indeed, by using Young's inequality, we obtain

$$
|\mathcal{N}(t)| = |\int_0^1 \phi u_x dx| + |\int_0^1 \phi_1 u_x dx|
\leq \frac{1}{2} \int_0^1 u_t^2 \, dx \quad \text{and} \quad \frac{1}{2} \int_0^1 \varphi_t^2 \, dx \quad \text{and} \quad \frac{1}{2} \int_0^1 \varphi_x^2 \, dx \quad \text{and} \quad \frac{1}{2} \int_0^1 u_x^2 \, dx,
$$

(109)

where $C_0 = \max\{\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$.

By (83) and (109), we obtain

$$
|\mathcal{R}(t) - N_5(\mathcal{E}_1(t) + \mathcal{E}_2(t))| \leq (c_3 + C_0) \mathcal{E}_1(t) \leq c(\mathcal{E}_1(t) + \mathcal{E}_2(t)),
$$

(110)

and

$$
(N_5 - c)(\mathcal{E}_1(t) + \mathcal{E}_2(t)) \leq \mathcal{R}(t) \leq (N_5 + c)(\mathcal{E}_1(t) + \mathcal{E}_2(t)),
$$

(111)

and by using (105), (107) and (3), we obtain

$$
\mathcal{R}'(t) = L'(t) + KN'(t) + N_5(\mathcal{E}_1'(t) + \mathcal{E}_2'(t))
\leq -k_4 \mathcal{E}_1(t) + k_2 \int_0^1 \int_0^\infty g(p) \varphi_p^2 dp \, dx \\
+ k_3 \int_0^1 (\varphi_t^2 + f^2(\varphi_1)) \, dx - (\eta_0 N_5 - \frac{|K|}{4}) \int_0^1 \varphi_t^2 \, dx.
$$

(112)

We choose a large enough $N_5$ that

$$
\eta_0 N_5 - \frac{|K|}{4} > 0, \quad N_5 - c > 0,
$$

we obtain

$$
\mathcal{R}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)),
$$

(113)

and

$$
\mathcal{R}'(t) \leq -k_4 \mathcal{E}_1(t) + k_2 \int_0^1 \int_0^\infty g(p) \varphi_p^2 dp \, dx \\
+ k_3 \int_0^1 (\varphi_t^2 + f^2(\varphi_1)) \, dx.
$$

(114)
By multiplying (114) by \( h(t) = a(t)\eta(t) \), we obtain

\[
h(t)R'(t) \leq -k_4 h(t)E(t) + k_2 h(t) \int_0^1 \int_0^\infty g(p) q_p^2(p) dp \ dx + k_3 h(t) \int_0^1 (\phi_1^2 + f^2(\phi_1)) \ dx. \tag{115}
\]

We distinguish two cases

- **G** is linear on \([0, \varepsilon]\). In the same way as in the previous case, we obtain

\[
R_1'(t) \leq -k_4 h(t)E(t) + \gamma h(t)\omega(t), \tag{116}
\]

and

\[
m_1(E_1(t) + E_2(t)) \leq R_1(t) \leq m_2(E_1(t) + E_2(t)), \tag{117}
\]

with

\[
m_1 = \tau_1, \quad m_2 = c_2 h(0) + k_3 \eta(0) + 2k_2 a(0) + \tau_1,
\]

where

\[
R_1(t) = h(t)\mathcal{R}(t) + (k_3 \eta(t) + 2k_2 a(t) + \tau_1)E(t) \sim (E_1(t) + E_2(t))
\]

\[
\gamma = \left( -\frac{E(0)}{t} + 2c_0 \right), \quad \tau_1 > 0 \text{ and } \omega(t) = \int_t^\infty g(p) dp.
\]

Since \( E'(t) \leq 0, \forall t \geq 0 \). By using (116), we have

\[
E(T) \int_0^T h(t) dt \leq \left( \frac{R_1(0)}{k_4} + \frac{\gamma}{k_4} \int_0^T h(t) \omega(t) dt \right). \tag{118}
\]

Using the fact that \( G_0^{-1} \) is linear. Then,

\[
E(T) \leq \xi G_0^{-1} \left( \frac{R_1(0)}{k_4} + \frac{\gamma}{k_4} \int_0^T h(t) \omega(t) dt \right), \tag{119}
\]

with \( \beta_1 = \xi, \quad \beta_2 = \frac{R_1(0)}{k_4}, \quad \beta_3 = \frac{\gamma}{k_4} \). This completes the proof.

- **G** is nonlinear on \([0, \varepsilon]\), we choose \( 0 \leq \varepsilon_1 \leq \varepsilon \). In a similar way to that in the previous case, we have

\[
R_2'(t) \leq -k_3 h(t)E(t) + \gamma h(t)\omega(t) + k_3' h(t) G^{-1}(I(t)), \tag{120}
\]

and

\[
m_3(E_1(t) + E_2(t)) \leq R_2(t) \leq m_4(E_1(t) + E_2(t)), \tag{121}
\]

with

\[
m_3 = \tau_1, \quad m_4 = c_2 h(0) + k_3' \eta(0) + 2k_2 a(0) + \tau_1,
\]

where

\[
R_2(t) = h(t)\mathcal{R}(t) + (k_3' \eta(t) + 2k_2 a(t) + \tau_1)E(t) \sim (E_1(t) + E_2(t)).
\]

Now, for \( \varepsilon_0 < \varepsilon_1 \), and by using \( E'(t) \leq 0, G' > 0 \) and \( G'' > 0 \) on \([0, \varepsilon]\), we define the functional \( L_3(t) \) by,

\[
L_3(t) = G'(\varepsilon_0 E(t)) R_2(t) + \tau_2 E(t) \sim (E_1(t) + E_2(t)), \quad \tau_2 > 0,
\]
satisfies
\[ R_3'(t) = \mathcal{E}'(t)(\varepsilon_0 \mathcal{G}'(\varepsilon_0 \mathcal{E}(t)))R_2(t) + \tau_2 + R_4'(t)G'(\varepsilon_0 \mathcal{E}(t)) \]
\[ \leq -k_4h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0 \mathcal{E}(t))h(t)\omega(t) + k_3^2h(t)G'(\varepsilon_0 \mathcal{E}(t))G^{-1}(I(t)). \]
(122)

To estimate the last term of (122), again using the general Young’s inequality (101). Inserting (122) in (121) and letting \( \varepsilon_0 = \frac{t_1}{R_3'} \) we get
\[ R_3'(t) + k_3^2\eta(t)\mathcal{E}'(t) \leq -k_4h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0 \mathcal{E}(t))h(t)\omega(t). \]
(123)

Since \( \eta'(t) \leq 0 \), then
\[ \mathcal{R}_4'(t) = -k_4h(t)G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0 \mathcal{E}(t))h(t)\omega(t), \]
where
\[ \mathcal{R}_4(t) = \mathcal{R}_3(t) + k_3^2\eta(t)\mathcal{E}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)). \]

Since \( \alpha(t), G_0(\mathcal{E}(t)), G'(\varepsilon_0 \mathcal{E}(t)) \) are non-increasing functions, then, for any \( T > 0 \)
\[ k_4G_0(E(T)) \int_0^T h(t)dt \leq k_4 \int_0^T h(t)G_0(\mathcal{E}(t))dt \]
\[ \leq \mathcal{R}_4(0) + \gamma G'(\varepsilon_0 \mathcal{E}(0)) \int_0^T h(t)\omega(t)dt, \]
which gives (78) with \( \beta_1 = 1, \beta_2 = \frac{\mathcal{R}_4(0)}{k_4} \) and \( \beta_3 = \frac{\gamma G'(\varepsilon_0 \mathcal{E}(0))}{k_4} \).

The proof is completed.

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