Stochastic Aperiodic Control of Networked Systems with i.i.d. Time-Varying Communication Delays

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Abstract—This paper studies stochastic aperiodic stabilization of a networked control system (NCS) consisting of a continuous-time plant and a discrete-time controller. The plant and the controller are assumed to be connected by communication channels via communication channels. Since the delays may affect the performance of the networked control systems (NCSs), the delays are not required to be bounded even when the plant is unstable in the deterministic sense. In our NCS, the sampling interval is supposed to be determined directly by such communication delays. A necessary and sufficient inequality condition is presented for designing a state-feedback controller stabilizing the NCS at sampling points in a stochastic sense. The results are also illustrated numerically.

I. INTRODUCTION

Communication delays are inevitable when transmitting signals via communication channels. Since the delays may affect performance of the networked control systems (NCSs), taking them into account in analysis and synthesis is one of the important issues in the field of NCSs [1], [2], [3]. The delays naturally become randomly time-varying when the Internet is used as the communication channels [4]. Hence, discrete-time stochastic processes (i.e., random sequences) are more desirable as the model of communication delays than constants in practice.

The structure of the NCS to be dealt with in this paper is standard as in Fig. 1 where $P_c$, $\Phi$, $S$ and $\mathcal{H}$ denote a continuous-time deterministic linear plant, a discrete-time state-feedback controller, the ideal sampler and the zero-order hold, respectively. We denote this NCS by $\Sigma$. The solid (resp. dashed) arrows are used for continuous-time (resp. discrete-time) signals in the figure. The discrete-time signals are transmitted via communication channels, and the delay elements $D^{up}$ and $D^{dw}$ are assumed to exist in the channels; those elements delay the arrival of signals by some random intervals (the details will be described later). The sampler and the hold are assumed to operate in synchronization, which implies that the sampling intervals are determined by the random delays; the sum of the two delays becomes the sampling interval at each sampling time instant. Hence, the arguments for stabilization of the NCS $\Sigma$ can be interpreted as a stochastic version of the aperiodic control [5]. The purpose of this paper is to show a synthesis-oriented inequality condition for state-feedback stabilization of this type of NCSs in the case that the delays (i.e., sampling intervals) are given by i.i.d. processes (i.e., discrete-time white processes).

As can be observed from the above purpose, our arguments are related with variable sampling/transmission intervals and variable transmission delays, which are two of the five types of imperfections and constraints in NCSs mentioned in [2]. The influence of the communication delays in NCSs has been studied from various aspects since the article [6] was presented; see, e.g., the survey papers [1], [2], [3] and other sophisticated articles for details. In the case of internally stable plant $P_c$, it may be possible to ensure deterministic stability of the corresponding NCS regardless of the boundedness of the delays, since a stable plant without input is naturally stable. In the case of unstable plant $P_c$, on the other hand, it is theoretically impossible to ensure deterministic stability of the corresponding NCS under unbounded time-varying communication delays. Hence, if one desires to deal with unbounded communication delays in the unstable case, some kind of ideas such as using stochastic models are required.

If the delays are modeled by a stochastic process, there is a room to deal with an unbounded support for the process even in the unstable case [7], [8]. Indeed, given a controller (which was designed for a virtual ideal NCS without communication delays), the earlier study [8] discussed stability analysis of an NCS in the presence of stochastic communication delays with unbounded support. To the best of the author’s knowledge, however, there exist no earlier studies that enable us to design controllers directly (i.e., without relaxation) for NCSs in the presence of stochastic communication delays with unbounded support; the Markovian approaches, e.g., in [9], [10] are also no exception since the support of a finite-mode Markov chain is obviously bounded. By exploiting the results in the author’s earlier study [11] about state feedback synthesis for discrete-time stochastic systems, the present
paper discusses synthesis of state-feedback controllers stabilizing NCSs, which are optimal in the sense of stochastic stability at sampling points.

Compared to the above earlier studies, the advantages of the proposed approach can be summarized as follows: (i) the time-varying communication delays are not required to be deterministically bounded; (ii) the distributions of the delays are arbitrary as long as they are given by i.i.d. processes; (iii) the derived synthesis-oriented inequality condition is necessary and sufficient in terms of discrete-time stochastic stability (specifically, second-moment exponential stability, which is also called mean square exponential stability); (iv) the synthesis using the above condition is direct (i.e., no relaxation is introduced).

This paper is organized as follows. Section III introduces NCSs with random communication delays to be dealt with in this paper. Section III describes the discretized model of the NCSs, and states the associated controller synthesis problem. Then, Section V discusses our main results about stabilizing controller synthesis. For derivation of the inequality conditions about the synthesis, the results in [11] about stochastic systems are exploited. Section VI numerically illustrates our synthesis, and Section VII gives concluding remarks.

We use the following notation in this paper. The set of real numbers, that of positive real numbers and that of non-negative integers are denoted by \( \mathbb{R} \), \( \mathbb{R}_+ \) and \( \mathbb{N}_0 \), respectively. The set of \( n \)-dimensional real column vectors and that of \( m \times n \) real matrices are denoted by \( \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \), respectively. The set of \( n \times n \) positive definite matrices is denoted by \( \mathbb{S}_+^n \). The identity matrix of size \( n \) is denoted by \( I_n \). The Euclidean norm of the vector \( \cdot \) is denoted by \( ||(\cdot)|| \). The vectorization of the matrix \( \cdot \) in the row direction is denoted by \( \text{row}(\cdot) \), i.e., \( \text{row}(\cdot) := [\text{row}_1(\cdot), \ldots, \text{row}_m(\cdot)] \), where \( m \) is the number of rows of the matrix and \( \text{row}_i(\cdot) \) denotes the \( i \)-th row. The Kronecker product is denoted by \( \otimes \). The expectation (i.e., the expected value) of the random variable \( \cdot \) is denoted by \( E[(\cdot)] \); this notation is also used for the expectation of a random matrix. If \( s \) is a random variable obeying the distribution \( D \), then we represent it as \( s \sim D \).

II. NETWORKED CONTROL SYSTEMS

A. Settings of Components

This subsection briefly states the settings of each of the components constituting the NCS \( \Sigma \) in Fig. 1.

Plant \( P_c \): We suppose \( P_c \) is a continuous-time deterministic linear plant represented by the state equation

\[
\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t),
\]

where \( t \in \mathbb{R} \) denotes the continuous time with initial time \( t = 0 \), and \( x_c(t) \in \mathbb{R}^n \) and \( u_c(t) \in \mathbb{R}^m \) are the state and the input, respectively. The initial state \( x_c(0) = x_0 \) is assumed to be given.

Sampler \( S \) and Hold \( H \): We suppose \( S \) and \( H \) are the sampler and the zero-order hold that operate in synchronization at sampling time instants \( t_k \) (\( k \in \mathbb{N}_0 \)) satisfying

\[
t_0 = 0, \quad t_{k+1} - t_k > 0, \quad \lim_{k \to \infty} (t_k = \infty). \tag{2}
\]

We use the symbol \( t \) for representing continuous time, and \( k \) for discrete time. The sampling interval \( h_k \) is defined as

\[
h_k := t_{k+1} - t_k. \tag{3}
\]

As is obvious from this definition, the sampling interval of our NCS is not constant but time-varying. The relation of the sampling interval with communication delays will be described later.

Controller \( \Psi \): In this paper, we deal with a discrete-time state-feedback controller for the NCS \( \Sigma \). For simplicity, we suppose that the controller \( \Psi \) determines the control input \( u_k \) immediately after receiving the sampled output \( x_k = x_c(t_k) \) of the plant at each discrete time \( k \). The class of the controller and its design method will be discussed later.

Delays \( \tau_{\text{up}} \) and \( \tau_{\text{dw}} \): We suppose \( \tau_{\text{up}} \) (resp. \( \tau_{\text{dw}} \)) is a component that delays (in the sense of continuous time) the arrival of signal by \( \tau_{\text{up}} \geq 0 \) (resp. \( \tau_{\text{dw}} \geq 0 \)) at each \( k \). Since the first timing of the sampling is \( t = t_0 = 0 \), the controller \( \Psi \) receives the plant output \( x_c(t_0) = x_0 \) at \( t = \tau_{\text{up}} \). Although the computation delay is not introduced in our study, it may be included in \( \tau_{\text{up}} \) (or \( \tau_{\text{dw}} \)) without loss of generality. Since the output \( u_0 \) of \( \Psi \) about \( k = 0 \) is decided at \( t = \tau_{\text{up}} \) in continuous time, it reaches the hold \( H \) at \( t = \tau_{\text{up}} + \tau_{\text{dw}} \). Then, since the output of \( P_c \) is sampled simultaneously, \( k \) changes from 0 to 1 at this timing, and hence, \( t_1 = \tau_{\text{up}} + \tau_{\text{dw}} \). This might be somewhat confusing since the hold \( H \) receives \( u_0 \) at \( t = t_1 \), which implies that

\[
u_c(t) = u_0 \tag{4}
\]

is added to \( P_c \) for \( t \in [t_1, t_2) \). The sampled plant output \( x_1 = x_c(t_1) \) is transmitted to \( \Psi \) via the communication channel again, and the same process continues as \( k \) increases. As already stated, the sampler and the hold in our NCS operate in synchronization. Hence, the relationship

\[
h_k = \tau_{\text{up}} + \tau_{\text{dw}} \tag{5}
\]

holds for the sampling interval \( h_k \) and the delays \( \tau_{\text{up}} \) and \( \tau_{\text{dw}} \). For

\[
T_{\text{up}} := \sum_{\kappa=0}^{k} \tau_{\text{up}}, \quad T_{\text{dw}} := \sum_{\kappa=0}^{k} \tau_{\text{dw}}, \tag{6}
\]

the sampling time instants are

\[
t_0 = 0, \quad t_k = T_{\text{up}} + T_{\text{dw}} \quad (k = 1, 2, \ldots). \tag{7}
\]

The timing of the decision of control input by \( \Psi \) is \( t = T_{\text{up}} + T_{\text{up}} + T_{\text{dw}} + T_{\text{up}} + T_{\text{dw}}, \ldots \)
B. Stochastic Process Representation of Communication Delays

As the models of time-varying delays $\tau_{k}^{up}$ and $\tau_{k}^{down}$, this paper consider stochastic processes. We formally introduce the 2-dimensional stochastic process

$$\xi = (\xi_k)_{k \in \mathbb{N}_0}, \quad \xi_k = [\tau_{k}^{up} \tau_{k}^{down}]^T.$$  (8)

Then, only the following is used as the essential restriction on $\xi$ in this paper.

Assumption 1: $\xi_k$ is independent and identically distributed (i.i.d.) with respect to $k \in \mathbb{N}_0$.

This assumption does not deny the dependence between $\tau_{k}^{up}$ and $\tau_{k}^{down}$ at common $k$. The distribution of $\xi_k$ is arbitrary, and the support is not required to be unbounded; although another assumption will be additionally introduced on the NCS $\Sigma$ in the following section, it is a minimal requirement for defining second-moment stability, and does not imply the degradation of applicability of the proposed approach.

Remark 1: Since the delays generally have physical lower bounds, we may also assume that $\tau_{k}^{up}$ and $\tau_{k}^{down}$ are positive in practice.

II. DISCRETIZATION OF NCS AND STABILIZATION PROBLEM

We deal with the NCS $\Sigma$ in a discrete time domain (i.e., at sampling points). Under the aforementioned settings, the relationship between the continuous-time signals and the discrete-time signals is given by

$$x_k = x_c(t_k), \quad u_c(t) = u_{k-1}, \quad (t \in [t_k, t_{k+1}), k \in \mathbb{N}_0).$$  (9)

With this relationship, the discretized model of the continuous-time plant $P_c$ can be given as

$$x_{k+1} = A(\xi_k)x_k + B(\xi_k)u_{k-1}, \quad A(\xi_k) = e^{A_{c}h_k}, \quad B(\xi_k) = \int_{0}^{h_k} e^{A_{c}t}B_c dt$$  (11)

(recall (5) and (8)). For this discretized model, we introduce the following assumption.

Assumption 2: The squares of entries of $A(\xi_k)$ and $B(\xi_k)$ in (11) are all Lebesgue integrable for $k = 0$, i.e.,

$$E[A_{ij}(\xi_0)^2] < \infty \quad (i, j = 1, \ldots, n),$$  (12)

$$E[B_{ij}(\xi_0)^2] < \infty \quad (i = 1, \ldots, n; j = 1, \ldots, m),$$  (13)

where $A_{ij}(\xi_0)$ and $B_{ij}(\xi_0)$ are the $(i, j)$-entries of $A(\xi_0)$ and $B(\xi_0)$, respectively.

This assumption is a minimal requirement for defining second-moment stability (in a discrete-time domain) for the present structure of NCS. Hence, this assumption has the role of specifying the class of NCSs that can be dealt with in this framework of second-moment stability; if this assumption is not satisfied, it is impossible to stabilize the NCS in the second moment.

As in (10), the discretized model has one step input delay, which naturally stems from the communication delays. Taking into account this input delay, we consider the following class of state-feedback controller as $\Psi$ (the delayed input is dealt with as part of the system state in the discrete time domain).

$$u_k = F_1x_k + F_2u_{k-1}$$  (14)

Then, the second-moment exponential stability of the NCS $\Sigma$ with the plant (1) and the controller (14) can be defined as follows.

Definition 1: Suppose that $\xi$ satisfies Assumption (1) and the random matrices $A(\xi_k)$ and $B(\xi_k)$ in (11) satisfy Assumption (2). For given $F_1 \in \mathbb{R}^{m \times n}$ and $F_2 \in \mathbb{R}^{m \times m}$, the NCS $\Sigma$ is said to be second-moment stable in the second moment at sampling points if there exist $a \in \mathbb{R}_+$ and $\lambda \in (0, 1)$ such that

$$\sqrt{E[\|x_{c}(t_k)\|^2 + \|u_{c}(t_k)\|^2]} \leq a\lambda^k \sqrt{\|x_0\|^2 + \|u_0\|^2} \quad (\forall k \in \mathbb{N}_0),$$  (15)

In (15), the expectation is taken for the random sampling time instant $t_k$ given in (7). The following section tackles the problem of designing $F_1$ and $F_2$ of $\Psi$ that stabilizes the corresponding NCS in the meaning of this definition.

IV. SYNTHESIS OF STABILIZING STATE-FEEDBACK CONTROLLER

Definition (1) was introduced for the NCS consisting of the continuous-time plant and the discrete-time controller. The inequality condition (15) in the definition, however, can be equivalently rewritten as that for the corresponding discrete-time system, i.e.,

$$\sqrt{E[\|x_k\|^2 + \|u_{k-1}\|^2]} \leq a\lambda^k \sqrt{\|x_0\|^2 + \|u_0\|^2} \quad (\forall k \in \mathbb{N}_0),$$  (16)

by using (9). This implies that stabilization in the sense of Definition (1) may be achieved by designing a state-feedback controller (14) that stabilizes the discretized model (10) by viewing it as a standard discrete-time stochastic system (with one step input delay). This section discusses such synthesis.

Let us formally introduce the new state variable $x_{ce,k} = u_{k-1}$, and construct the extended system

$$\dot{x}_{k+1} = \hat{A}(\xi_k)x_k + \hat{B}u_k, \quad \hat{x}_k = \begin{bmatrix} x_k \\ x_{ce,k} \end{bmatrix},$$  (18)

$$\hat{A}(\xi_k) = \begin{bmatrix} A(\xi_k) & B(\xi_k) \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$  (19)

from (10). Obviously, the behavior of this system is consistent with that of the discretized model (10). With the state vector $\hat{x}_k$ of this extended system, (14) can also be rewritten as

$$u_k = \hat{F}\hat{x}_k, \quad \hat{F} = [F_1, F_2],$$  (20)
Then, we see that the discrete-time closed-loop system
\[ \dot{x}_{k+1} = (\hat{A}(\xi_k) + \hat{B}\hat{F})x_k \] 
(21)
consisting of the extended system (18) and the controller (20) corresponds to the discrete-time counterpart of the present NCS without loss of generality.

For such a closed-loop system, (16) can be further equivalently rewritten as
\[ \sqrt{E[|\tilde{x}_0|^2]} \leq a\lambda^2 \sqrt{E[|\tilde{x}_0|^2]} \] 
\[ (\forall \tilde{x}_0 \in \mathbb{N}_0, \forall \tilde{x}_0 \in \mathbb{R}^{n+m}). \] 
(22)
This is nothing but the inequality condition for defining second-moment exponential stability of discrete-time stochastic systems (see, e.g., Definition 2 in [11]). The explicit definition is given as follows.

**Definition 2:** Suppose that \( \xi \) satisfies Assumption \#1 and the random matrices \( A(\xi) \) and \( B(\xi) \) in (11) satisfy Assumption \#2. For given \( F \in \mathbb{R}^{m \times (n+m)} \), the closed-loop system (21) is said to be exponentially stable in the second moment if there exist \( a \in \mathbb{R}_+ \) and \( \lambda \in (1, 1) \) satisfying (22).

These observations lead us to the following lemma.

**Lemma 1:** Suppose that \( \xi \) satisfies Assumption \#1 and the random matrices \( A(\xi) \) and \( B(\xi) \) in (11) satisfy Assumption \#2. For given \( F_1 \in \mathbb{R}^{m \times n} \) and \( F_2 \in \mathbb{R}^{m \times m} \), the following two conditions are equivalent.

1) The NCS \( \Sigma \) is exponentially stable in the second moment at sampling points.

2) The discrete-time closed-loop system (21) is exponentially stable in the second moment, where \( \hat{A}(\xi) \), \( \hat{B} \) and \( \hat{F} \) are given by (19) and (20).

This lemma implies that the problem of designing a state-feedback controller (14) stabilizing the NCS reduces to that of designing a state-feedback controller (20) stabilizing the closed-loop system (21). Since the coefficient matrices (19) of the extended system are depending only on \( \xi \) (i.e., not on \( \xi_{k-1}, \xi_{k-2}, \ldots \)), the results in [11] can be used for tackling the latter problem. Hence, Lemma 1 together with the results in [11], leads us to the following theorems about stability analysis and synthesis of NCSs, which constitute a part of the main results in this paper.

**Theorem 1:** Suppose that \( \xi \) satisfies Assumption \#1 and the random matrices \( A(\xi) \) and \( B(\xi) \) in (11) satisfy Assumption \#2. For given \( F_1 \in \mathbb{R}^{m \times n} \) and \( F_2 \in \mathbb{R}^{m \times m} \), the NCS \( \Sigma \) is exponentially stable in the second moment at sampling points if and only if there exist \( P \in \mathcal{S}^{(n+m)(n+m)}_+ \) and \( \lambda \in (0, 1) \) satisfying
\[ \lambda^2 P - \tilde{G}^T(P \otimes I_\bar{n})\tilde{G} \geq 0, \] 
(23)
where \( \tilde{G} \) is the matrix given by
\[ \tilde{G} := [G_1^T, \ldots, \bar{G}_n^T] \in \mathbb{R}^{(n+m)\bar{n} \times (n+m)}, \] 
(24)
\[ \bar{G} := [\bar{G}_1, \ldots, \bar{G}_n], \] 
(25)
with a matrix \( \tilde{G} \in \mathbb{R}^{\bar{n} \times (n+m)^2} \) \( (\bar{n} \leq (n+m)^2) \) satisfying
\[ G^T \tilde{G} = E[\text{row}(\hat{A}(\xi_0) + \hat{B}\hat{F})^T \text{row}(\hat{A}(\xi_0) + \hat{B}\hat{F})] \] 
(26)
for \( \hat{A}(\xi_k), \hat{B} \) and \( \hat{F} \) in (19) and (20).

**Theorem 2:** Suppose that \( \xi \) satisfies Assumption \#1 and the random matrices \( A(\xi) \) and \( B(\xi) \) in (11) satisfy Assumption \#2. There exists a gain \( [F_1, F_2] = F \in \mathbb{R}^{m \times (n+m)} \) such that the NCS \( \Sigma \) is exponentially stable in the second moment at sampling points if and only if there exist \( P \in \mathcal{S}^{(n+m)(n+m)}_+ \), \( Y \in \mathbb{R}^{m \times (n+m)} \) and \( \lambda \in (0, 1) \) satisfying
\[ \lambda^2 X \] 
\[ \tilde{G}_A X + \tilde{G}_B Y \text{ } \] 
\[ x \otimes I_m > 0, \] 
(27)
where \( \tilde{G}_A \) and \( \tilde{G}_B \) are the matrices given by
\[ \tilde{G}_A := [\bar{G}^T_{\bar{A}_1}, \ldots, \bar{G}^T_{\bar{A}_{(n+m)}}] \in \mathbb{R}^{(n+m)\bar{n} \times (n+m)}, \] 
(28)
\[ \tilde{G}_B := [\bar{G}^T_{\bar{B}_1}, \ldots, \bar{G}^T_{\bar{B}_{(n+m)}}] \in \mathbb{R}^{(n+m)\bar{n} \times m}, \] 
(29)
\[ \tilde{G}_{AB} := [\bar{G}^T_{\bar{A}_1}, \ldots, \bar{G}^T_{\bar{A}_{(n+m)}}, \bar{G}^T_{\bar{B}_1}, \ldots, \bar{G}^T_{\bar{B}_{(n+m)}}] \] 
(30)
with a matrix \( \tilde{G}_{AB} \in \mathbb{R}^{(n+m)\bar{n} \times (n+m)(2m)} \) \( (\bar{n} \leq (n+m)(n+2m)) \) satisfying
\[ \tilde{G}_{AB}^T \tilde{G}_{AB} = E[\text{row}(\hat{A}(\xi_0)), \text{row}(\hat{B})]^T \] 
\[ \cdot [\text{row}(\hat{A}(\xi_0)), \text{row}(\hat{B})] \] 
(31)
for \( \hat{A}(\xi_k), \hat{B} \) and \( \hat{F} \) in (19) and (20). In particular, \( [F_1, F_2] = F = YX^{-1} \) is one such stabilizing gain.

Theorem 1 is about analysis, while Theorem 2 is about synthesis. By calculating \( \tilde{G} \), for fixed \( \lambda \), (23) can be solved as a standard linear matrix inequality (LMI). A similar comment also applies to (27) about synthesis.

**Remark 2:** The matrices \( \tilde{G}_A \) and \( \tilde{G}_B \) in (27) are constructed so that
\[ \tilde{G}_A^T \tilde{G}_A \tilde{G}_A = \bar{E}[\hat{A}(\xi_0)T X \hat{A}(\xi_0)], \] 
(32)
\[ \tilde{G}_A^T \tilde{G}_A \tilde{G}_B = \bar{E}[\hat{A}(\xi_0)T X \hat{B}], \] 
(33)
\[ \tilde{G}_B^T \tilde{G}_B \tilde{G}_B = \bar{E}[\hat{B}^T X \hat{B}], \] 
(34)
are simultaneously satisfied. This is an essence of deriving LMI conditions for stochastic systems. In the case of the present networked control system, however, taking \( \tilde{G}_{AB} \) as in (31) may actually be redundant, because \( \hat{B} \) is known to be a constant matrix. Indeed, it can be shown with the results in [13] that taking \( \tilde{X}_{\bar{A}} \in \mathbb{R}^{(n+m)\bar{n} \times (n+m)} \) satisfying \( X_{\bar{A}}^T X_{\bar{A}} = \bar{E}[1, \text{row}(\hat{A}(\xi_0))^T] \) is sufficient for constructing \( \tilde{X}_{\bar{A}} \in \mathbb{R}^{(n+m)\bar{n} \times (n+m)} \) such that
\[ \tilde{X}_{\bar{A}}^T \tilde{X}_{\bar{A}} \tilde{X}_{\bar{A}} = E[\hat{A}(\xi_0)T X \hat{A}(\xi_0)], \] 
(35)
\[ \tilde{X}_{\bar{A}}^T \tilde{X}_{\bar{A}} \tilde{X}_{\bar{A}} = E[\hat{A}(\xi_0)T X], \] 
(36)
\[ \tilde{X}_{\bar{A}}^T \tilde{X}_{\bar{A}} \tilde{X}_{\bar{A}} = X. \] 
(37)
With such matrices, \( \tilde{G}_A \) and \( \tilde{G}_B \) in (27) can be replaced by \( \tilde{X}_{\bar{A}} \) and \( \tilde{X}_{\bar{B}} \), respectively. Since the size of \( \tilde{X}_{\bar{A}} \) is generally smaller than that of \( \tilde{G}_{AB} \), this idea would contribute to reducing the cost for computing the expectations.
V. NUMERICAL EXAMPLE

Let us consider the continuous-time deterministic linear plant \( P_c \) with coefficient matrices

\[
A_c = \begin{bmatrix} 0 & 1 \\ \frac{g}{r} & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ \frac{1}{Mr} \end{bmatrix}
\]

(38)
as \( P_c \) in Fig. 1 where \( g = 9.8, r = 0.2 \) and \( M = 1 \). This plant corresponds to a linearized model of an inverted pendulum around the vertically inverted position \( (x_{c1}, x_{c2}) \) and \( u_c \) correspond to the angle \( \theta \), the angular velocity \( \dot{\theta} \) and the input torque \( \tau \), respectively. Hence, the plant itself is unstable. This section deals with an example of designing a state-feedback controller \( \Psi \) stabilizing this plant from a remote location. The communication delays \( \tau_k^{up} \) and \( \tau_k^{dw} \) are assumed to be i.i.d. (i.e., Assumption \( 1 \) is satisfied) and given by

\[
\tau_k^{up} = \varepsilon_k^{up} + d_k^{up}, \quad \tau_k^{dw} = \varepsilon_k^{dw} + d_k^{dw}, \quad (39)
d_k^{up} \sim \text{Exp}(\mu^{up}), \quad d_k^{dw} \sim \text{Exp}(\mu^{dw}), \quad (40)
\]

\[
\varepsilon_k = \varepsilon_{dw} = 0.01, \quad \mu^{up} = 0.01, \quad \mu^{dw} = 0.02, \quad (41)
\]

where \( \text{Exp}(\mu) (\mu > 0) \) denotes the exponential distribution with mean \( \mu \). For simplicity, \( \tau_k^{up} \) and \( \tau_k^{dw} \) are also assumed to be independent of each other at common \( k \). With these problem settings, Assumption \( 2 \) can be confirmed to be satisfied (see Appendix). Hence, our approach can be applied to the corresponding NCS \( \Sigma \).

To design a controller based on Theorem \( 2 \) we first have to construct the matrices \( \bar{G}_A \) and \( \bar{G}_B \). Since the transformation from \( \bar{G}_{AB} \) to those matrices is elementary, the only potentially tricky part is the computation about (31). In this paper, we generated 1,000 samples of \( \xi \) where \( \text{Exp}(\bar{\mu}) \) with mean \( \bar{\mu} \), was 3. Hence, the sizes of \( \bar{G}_A \) and \( \bar{G}_B \) became 9 \times 9 and 9 \times 1, respectively.

With those coefficient matrices, we solved the LMI (22) so that a minimal \( \lambda \) is obtained through a bisection method. Then, we obtained \( \lambda = 0.7628 \) and the corresponding solution

\[
X = 10^4 \begin{bmatrix} 0.1854 & -1.2981 & 0.0000 \\ -1.2981 & 9.0869 & -0.0002 \\ 0.0000 & -0.0002 & 0.0001 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.1003 \\ -1.7149 \\ 0.2836 \end{bmatrix},
\]

(42)

which leads to

\[
F_1 = [-5.5264 \quad -0.7895], \quad F_2 = -0.8488. \quad (43)
\]

We used MATLAB, YALMIP [14] and SDPT3 [15] for this computation.

Since \( \lambda < 1 \), the NCS \( \Sigma \) with obtained \( F_1 \) and \( F_2 \) is ensured to be stable by our results. This can be also confirmed through the initial value response of the NCS shown in Fig. 2 where 100 sample paths generated under the initial condition \( x_c(0) = [1, 0]^T \) and \( u_{-1} = 0 \) are overlapped. All the sample paths in this figure converge to zero as time tends to infinity, even with the random communication delays whose behavior can be imagined from the sample paths about \( u_t \). This kind of control is difficult to achieve only with deterministic approaches, since the plant is unstable and the range of the communication delays is not bounded.

VI. CONCLUSIONS

In this paper, we tackled a stabilization problem for NCSs with time-varying random communication delays, and proposed numerically tractable necessary and sufficient inequality conditions. With our approach, the plant in the NCS itself is not required to be deterministically stable even when the range of the random communication delays is not (known to be) bounded. The distribution of the delays may be constructed directly from data, and the approach is also considered to be compatible with some learning approaches in this sense.
APPENDIX

We show that Assumption 2 is satisfied in the case of the present numerical example. The assertion about the $A$ matrix in the assumption is equivalent to the boundedness of the maximum eigenvalue of the positive semidefinite matrix $E[A(\xi_0)^T A(\xi_0)]$ (recall the definition of maximum singular values). Hence, we prove each entry of this matrix is bounded.

Since a direct calculation leads to

$$E[A(\xi_0)^T A(\xi_0)] = E\left[ e^{A^T x + \epsilon x w + d_0^p + d_0^\omega} e^{A_x (e^{x w} + d_0^p + d_0^\omega)} \right]$$

$$= e^{A^2 T (e^{x w} + \epsilon x w) + d_0^p + d_0^\omega} E\left[ e^{A_x (d_0^p + d_0^\omega)} e^{A_x (e^{x w})} \right]$$

$$= e^{A_x (e^{x w} + \epsilon x w)} E\left[ e^{A_x (d_0^p + d_0^\omega)} e^{A_x (e^{x w})} \right],$$

(44)

it is sufficient to show the boundedness of each entry of $E\left[ e^{A^2 (d_0^p + d_0^\omega)} e^{A_x (d_0^p + d_0^\omega)} \right]$. For simplicity, we use the diagonalizability of $A_x$. Let $\alpha = \sqrt{g/\bar{r}} > 0$. Then, the matrix $A_x$ in the example has real eigenvalues $\lambda_1 = \alpha$ and $\lambda_2 = -\alpha$, and can be diagonalized with the matrix $V := \begin{bmatrix} 1 & -1 \\ \alpha & \alpha \end{bmatrix}$ as

$$V^{-1} A_x V = \Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$ (45)

This leads us to

$$E\left[ e^{A^2 (d_0^p + d_0^\omega)} e^{A_x (d_0^p + d_0^\omega)} \right]$$

$$= E\left[ V^{-1} e^{A^2 (d_0^p + d_0^\omega)} V V^T e^{A_x (d_0^p + d_0^\omega)} V^{-1} \right]$$

$$= V^{-1} E\left[ e^{A^2 (d_0^p + d_0^\omega)} V V^T e^{A_x (d_0^p + d_0^\omega)} V^{-1} \right]$$

$$= V V^{-1} e^{A^2 (d_0^p + d_0^\omega)} e^{A_x (d_0^p + d_0^\omega)} V V^{-1} V^{-1},$$

(46)

where the third equality follows from the independence of $d_0^p$ and $d_0^\omega$. Since $V V^T$ is a constant matrix, each entry of $E[e^{A_x (d_0^p + d_0^\omega)} V V^T e^{A_x (d_0^p + d_0^\omega)}]$ is bounded if $E[e^{(\lambda_1 + \lambda_2) d_0^\omega}]$ is bounded for all $i, j = 1, 2$. Similarly, in the case that $E[e^{A_x (d_0^p + d_0^\omega)} V V^T e^{A_x (d_0^p + d_0^\omega)}]$ is a bounded constant matrix, each entry of $E[e^{A_x (d_0^p + d_0^\omega)} V V^T e^{A_x (d_0^p + d_0^\omega)}]$ is bounded if $E[e^{(\lambda_1 + \lambda_2) d_0^\omega}]$ is bounded for all $i, j = 1, 2$. We can confirm through a direct calculation that $E[e^{(\lambda_1 + \lambda_2) d_0^p}]$ and $E[e^{(\lambda_1 + \lambda_2) d_0^\omega}]$ are bounded for all $i, j = 1, 2$ in the present example; e.g.,

$$E[e^{(\lambda_1 + \lambda_2) d_0^\omega}] = \frac{1}{\mu^{\omega}} \int_0^{\infty} e^{(2\lambda_1 - \frac{\omega}{\mu^{\omega}}) t} dt$$

$$= \text{const}.$$ (47)

since $2\lambda_1 - \frac{\omega}{\mu^{\omega}} = -36 < 0$. Therefore, each entry of $E\left[ e^{A^2 (d_0^p + d_0^\omega)} e^{A_x (d_0^p + d_0^\omega)} \right]$ is bounded, and the assertion about the $A$ matrix in Assumption 2 is satisfied.

The above arguments about the $A$ matrix, together with the direct calculation of the integral in (44), also imply that the assertion about the $B$ matrix in Assumption 2 is satisfied. Hence, Assumption 3 is satisfied.

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