N=2,4 Supersymmetric Gauge Field Theory in 2T-physics

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Abstract

In the context of Two Time Physics in 4+2 dimensions we construct the most general N=2,4 supersymmetric Yang Mills gauge theories for any gauge group G. This builds on our previous work for N=1 supersymmetry. The action, the conserved SUSY currents, and the off-shell SU(N) covariant SUSY transformation laws are presented for both N=2 and N=4. The on-shell SUSY transformations close to the supergroup SU(2,2|N) with N=1,2,4. The SU(2,2)=SO(4,2) sub-symmetry is realized linearly on 4+2 dimensional flat spacetime. All fields, including vectors and spinors, are in 4+2 dimensions. The extra gauge symmetries in 2T field theory, together with the kinematic constraints that follow from the action, remove all the ghosts to give a unitary theory. By choosing gauges and solving the kinematic equations, the 2T field theory in 4+2 flat spacetime can be reduced to various shadows in various 3+1 dimensional (generally curved) spacetimes. These shadows are related to each other by dualities. The conformal shadows of our theories in flat 3+1 dimensions coincide with the well known counterpart N=1,2,4 supersymmetric massless renormalizable field theories in 3+1 dimensions. It is expected that our more symmetric new structures in 4+2 spacetime may be useful for non-perturbative or exact solutions of these theories.

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I. STATUS OF 2T-PHYSICS

Two time physics (2T-physics) has proven to be successful in unifying different ordinary 1T physics systems by establishing duality relationships among them and in uncovering underlying hidden symmetries of 1T systems at the particle and field theory levels [1]-[18]. The theory starts by imposing a Sp(2, R) gauge symmetry on the phase space \((X^M, P_M)\) of a worldline theory of a bosonic particle [1]. The local symmetry is generalized for spinning particles [2][11], supersymmetric particles [4][5][9][10], or particles moving in background fields [3], but always involves Sp(2, R) as a subgroup. This symmetry requires that covariant momentum and position are interchangeable at any instant for any motion. One finds that this symmetry cannot exist in a spacetime with only one timelike dimension, and can be realized without ghosts only in a spacetime with 2 timelike dimensions, no less and no more.

It turns out that various usual 1T theories in \((d - 1) + 1\) dimensions are united by casting them into various gauge fixed versions of a single parent 2T theory in \(d + 2\) dimensions. The relationship between the 1T theories and the parent 2T theory is somewhat analogous to the relationship between an object moving in a 3-dimensional room and the many shadows, with their apparently unrelated motions, that can be created on walls by shining light from different perspectives on the parent object. For example, the 1T physics shadows created from the simplest 2T-physics bosonic particle that has no parameters, include 1T particles with or without mass, moving in flat or certain curved spacetimes, free or interacting in various potentials, and their twistor equivalents. Some of the mathematical properties of these gauge choices\(^1\) are summarized in three tables in [16]. Through this procedure, a web of duality relationships between these 1T theories with various parameters is established as gauge transformations of the underlying 2T theory. This was most clearly understood in the worldline formalism [1],[7]-[9], and to some extent was also shown to be the property of 2T field theory in \(d + 2\) dimensions [16][17]. This is a new type of unification.

2T field theory is closely related to the underlying particle 2T worldline theory by the BRST quantization procedure which, for the spinless particle, followed a somewhat similar path [13] to the BRST approach for string field theory [19]. After integrating out redundant ghost fields, this showed a simplified general way [14] to elevate 2T worldline theories to

\(^1\) For a graphical display of gauge choices see [http://physics.usc.edu/~bars/shadows.pdf](http://physics.usc.edu/~bars/shadows.pdf)
the 2T field theory formalism. By now the Standard Model and General Relativity have been shown to arise as particular shadows of their respective parent 2T field theories for the Standard Model [14] and for gravity [18] in \( d + 2 \) dimensions.

It was shown that the shadows derived from 2T field theory come with some additional restrictions that are not present in the usual 1T field theory approach. In particular, for the conformal shadow of the Standard Model mass terms are not allowed. Then, in the Higgs scenario, the electroweak phase transition needs to be driven by an additional scalar field which could be the dilaton or another new \( \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \) singlet scalar [14].

If it is the dilaton, this suggests that the Standard Model must be coupled to the gravity sector in more ways than expected before. Given this, the electroweak phase transition gets conceptually related to other phase transitions that occurred in the history of the universe for which an expectation value for the dilaton also plays a role.

Moreover, if ordinary General Relativity in \( (d - 1) + 1 \) dimensions is the conformal shadow of its 2T field theory counterpart in \( d + 2 \) dimensions, then all scalar fields coupled to it must be conformal scalars [18]. This means that in addition to their usual coupling to the spacetime metric \( g_{\mu\nu} \), every scalar field \( \phi(x) \) must also couple to the curvature scalar in the form \( -a\phi^2 R(g) \), with the special unique coefficient \( a = (d - 2)/8(d - 1) \) in \( d \) dimensions. In addition, the gravitational constant arises only from the vacuum value of such scalars, while a local Weyl symmetry removes a would be massless Goldstone dilaton. This leads to new concepts in cosmology, including the possibility of a changing gravitational constant as a result of various phase transitions in the history of the Universe [18].

There is another interesting role for conformal scalars. It was suggested in the second reference in [14] that a conformal scalar with its required \( \text{SO}(4, 2) \) conformal symmetry could provide an alternative to supersymmetry as a mechanism that could address the mass hierarchy problem. This possibility has been more recently elaborated in [20] [27].

It is remarkable that such new restrictions on 1T field theory that arise from 2T physics are compatible with current experimental knowledge and provide some new conceptual and phenomenological guidance. Further developments on these aspects will be reported elsewhere.

\footnote{After this proposal was discussed in [14] as part of possible new physics signatures motivated by ideas in 2T-physics, similar scenarios that include such a scalar field have been discussed in recent papers in both theoretical and phenomenological contexts [20] [25]}. 

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The 2T physics version of supersymmetric field theory in $4 + 2$ dimensions has also been developed by us in previous papers [15]. In view of the remarks above, it is not surprising that the emergent supersymmetric shadows also come with new corresponding constraints. Given the phenomenological interest in the possibility of observing supersymmetry (SUSY) at the Large Hadron Collider (LHC), the 2T-physics constraints could be phenomenologically significant, and we intend to study this topic in the near future.

In this paper we discuss the generalization of our previous work from $N = 1$ to $N = 2$ and $N = 4$ SUSY theories in $4 + 2$ dimensions. It should be noted that the famous $N = 4$ super Yang Mills gauge (SYM) theory in $3 + 1$ dimensions will emerge as the conformal shadow of our $N = 4$ SYM theory in $4 + 2$ dimensions. Therefore, it will have other dual shadows which may be useful in the further exact studies of this theory. The current paper will serve as a foundation for later exploration of the structure and phenomena of these $N = 1, 2, 4$ theories and supersymmetric theories in higher dimensions.

II. SO(4,2) SPINORS AND NOTATION

In this section we briefly describe some of the notation used in this article. For $N = 2$ SUSY there are two left handed SO(4,2) spinors $(\lambda_L)_{i\alpha}^a$, where the label $i = 1, 2$ indicates the doublet of the SU(2) R-symmetry, the label $a$ is for the adjoint representation of a compact gauge group $G$ and the label $\alpha = 1, 2, 3, 4$ is for the 4 representation of SU(2,2) (left handed Weyl spinor of SO(4,2)). For the spinors $(\psi_L)_{am}$ the label $m$ is used for some arbitrary representation (including reducible representations) of the gauge group $G$. Often we will simply use the label $L$, suppressing the label $\alpha$ to indicate a left handed spinor as $\lambda^a_{Li}$ or $\psi_{Lm}$. Sometimes we will also use the right handed spinor $(\lambda_R)^{i\dot{\alpha}}_a, (\psi_R)^m_{\dot{\alpha}}$ in the $\bar{4}$ representation of SU(2,2), which is labeled with $\dot{\alpha} = 1, 2, 3, 4$. One could rewrite all right-handed spinors as left-handed ones by charge conjugation which is given by

\[
(\lambda_R)^i_a \equiv \left(C\lambda_L^T\right)^i_a = C\eta^T(\lambda_L^*)^i_a, \quad \text{or} \quad (\lambda_L)_a^i = -\left(\lambda_R^T\right)^i_a C,
\]

and similarly for $\psi$. Here we have used the following matrices

\[
\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \tau_1 \times \sigma_2, \quad \eta = -i\tau_1 \times 1.
\]

where $\varepsilon_{ij} = -\varepsilon_{ji}$ is the antisymmetric charge conjugation matrix for the SU(2) R-symmetry, $C_{\dot{\alpha}\beta} = -C_{\beta\dot{\alpha}}$ is the antisymmetric charge conjugation matrix in SU(2,2) spinor space, and
\[ \eta^{\dot{\alpha}\beta} = \eta^{\beta\dot{\alpha}} \] is the symmetric SU(2, 2) metric in spinor space used to construct the SU(2, 2) contravariant spinor from the Hermitian conjugate spinor

\[
(\overline{\lambda}_L)_{ia}^i = \left( (\lambda_{iL}^a)^\dagger \eta \right)^\beta = \left( \lambda_L^{\dagger i} \right)^i_{\dot{a}a} \eta^{\dot{\alpha}\beta}. \tag{2.3}
\]

Note that Hermitian conjugation \( (\psi_{Lam})^\dagger = (\psi_L^\dagger)_\dot{a}^m \) changes the SU(2, 2) index from \( \alpha \) to \( \dot{\alpha} \) and raises the index \( m \) assuming \( m \) labels a complex representation of the gauge group \( G \). Similarly, \( (\lambda_{Lam}^a)^\dagger = (\lambda_L^i)_{a\dot{a}}^i \) raises the SU(2) index \( i \) and drops the index \( a \). However, the adjoint representation is real, the Killing metric \( \delta_{ab} \) can be taken as 1, so that there is no distinction between upper and lower \( a \) indices, and the structure constants \( f_{abc} \) are completely antisymmetric. Using these definitions we can also write the following relations that are equivalent to (2.1)

\[
\lambda_{Li}^a = - \left( C\overline{\alpha}_a^R \right)^T_i, \quad \text{or} \quad \left( \overline{\lambda}_R \right)^a_i = \left( \lambda_{Li}^a \right)^T C. \tag{2.4}
\]

The SU(2) indices \( i \) may be further dropped or raised by using the antisymmetric \( \varepsilon \) and its inverse \( \varepsilon^{-1} = -\varepsilon \) as follows, \( \lambda_i = \varepsilon_{ij} \lambda^j \) and \( \lambda^i = -\varepsilon^{ij} \lambda_j \).

We use the following explicit form of 4 \( \times \) 4 SO(4, 2) gamma matrices \( \Gamma^M, \bar{\Gamma}^M \) in the Weyl bases \( (M = 0', 1', 0, 1, 2, 3 \text{ is the label for the vector of SO}(4, 2))

\[
\Gamma^0' = -i\tau_1 \times 1, \quad \Gamma^1' = \tau_2 \times 1, \quad \Gamma^0 = 1 \times 1, \quad \Gamma^i = \tau_3 \times \sigma^i \tag{2.5}
\]

\[
\bar{\Gamma}^0' = -i\tau_1 \times 1, \quad \bar{\Gamma}^1' = \tau_2 \times 1, \quad \bar{\Gamma}^0 = -1 \times 1, \quad \bar{\Gamma}^i = \tau_3 \times \sigma^i \tag{2.6}
\]

These are compatible with the metric \( \eta \) and the charge conjugation matrix \( C \) given above as explained in detail in Appendix (A) of ref.\cite{15}. In particular we note the hermiticity and charge conjugation properties

\[
\eta \Gamma^M \eta^{-1} = - \left( \bar{\Gamma}^M \right)^\dagger, \quad \eta \bar{\Gamma}^M \eta^{-1} = - \left( \Gamma^M \right)^\dagger, \quad \text{CT} \Gamma^M C^{-1} = \left( \Gamma^M \right)^T, \quad \text{CT} \bar{\Gamma}^M C^{-1} = \left( \bar{\Gamma}^M \right)^T. \tag{2.7}
\]

The matrices \( (\Gamma^{MN})_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \left( \Gamma^M \bar{\Gamma}^N - \bar{\Gamma}^N \Gamma^M \right)_{\dot{\alpha}\dot{\beta}} \) and \( (\Gamma^M \bar{\Gamma}^N + \bar{\Gamma}^N \Gamma^M)_{\dot{\alpha}\dot{\beta}} = 2\delta_{\dot{\alpha}\dot{\beta}} \eta^{MN} \), together with the antisymmetric matrices \( (\Gamma^M C)_{\dot{\alpha}\beta}, (\text{CT})_{\dot{\alpha}\beta} \) incorporate the group theoretical properties of SU(2, 2) \( \text{SO}(4, 2) \) products of representations

\[
(4 \times 4) = 15 + 1, \quad (4 \times 4)_{\text{antisymmetric}} = 6, \quad (4 \times 4)_{\text{antisymmetric}} = 6. \tag{2.8}
\]

The matrix representation of the generators of the gauge group \( G \) are denoted as \( (t_a)_m^n \) implying the group transformation law \( \delta_\omega \varphi_m = -i \omega^a (t_a)_m^n \varphi_n \). For the adjoint representation
(t_a)_m^n is replaced by \((t_a)_b^c = -if_{ab}^c = -if_{abc}\). The matrices \((t_a)_m^n\) or \((t_a)_b^c\) satisfy the Lie algebra \([t_a, t_b] = if_{ab}^c (t_c)\).

**III. N=2 SUSY FROM N=1 IN 4+2 DIMENSIONS**

The starting point is the general \(N = 1\) supersymmetric Yang-Mills 2T field theory in 4 + 2 dimensions for any compact Yang-Mills gauge group \(G\) \[15\]. The theory contains a single \(N = 1\) vector supermultiplet \((A_M, \lambda_L, B)^a\), where \(a\) labels the adjoint representation of \(G\), plus any number of \(N = 1\) chiral supermultiplets \((\varphi, \psi_L, F)_r\) where \(r\) labels an arbitrary representation of the gauge group \(G\). This representation can be taken to be reducible, hence it may contain any number of chiral multiplets in various irreducible representations of \(G\).

The action consistent with both \(N = 1\) SUSY and 2T field theory was given in \[15\] as follows

\[
S_{N=1} = \int d^{4+2}X \delta (X^2) \left( L_{\text{kinetic}}^{N=1} + L_{\text{yukawa}}^{N=1} + L_{\text{potential}}^{N=1} \right)
\]

We note the typical delta function \(\delta (X^2)\) in 2T field theory\(^3\), with a Lagrangian density given by\(^4\)

\[
L_{\text{kinetic}}^{N=1} = \left\{ -\frac{1}{4} F_{MN}^a F^{MN}_{a} + \frac{1}{2} \varphi^r r D_M D^M \varphi_r + \frac{1}{2} \varphi_r D_M D^M \varphi^r \right. \\
\left. + \frac{i}{2} \left[ \lambda_L^a X \bar{D} \lambda_{aL} + \lambda_{aL} \bar{D} X \lambda^a \right] + \frac{i}{2} \left( \bar{\psi}_L \right)^r \left[ \bar{X} \bar{D} \psi_{rL} + \psi_{rL} \bar{D} \bar{X} \psi_L \right] \right\}
\]

\[
L_{\text{yukawa}}^{N=1} = \left[ \sqrt{2} g \varphi^r (t_a)_r^s \left( \psi_s L \right)^T (C \bar{X}) \lambda^a_L \right] - \frac{i}{2} \psi_{rL} \left( C \bar{X} \right) \psi_{sL} \frac{\partial^2 W}{\partial \varphi_r \partial \varphi_s} + h.c.
\]

\[
L_{\text{potential}}^{N=1} = \frac{1}{2} B^a B_a + F^{tr} F_r + g \varphi^r (t_a)_r^s \varphi_s B^a + \left[ \frac{\partial W}{\partial \varphi_r} F_r + h.c. \right]
\]

where \(X \equiv X^M \Gamma_M\) and \(\bar{D} \equiv \Gamma^M D_M\). These structures are compatible with the spacetime \(\text{SU}(2, 2) = \text{SO}(4, 2)\) group theoretical rules in Eq.\[2.8\]. The explicit \(X^M\) that appears in the kinetic, Yukawa, and the \(\delta (X^2)\), is to be noted; hence there is no translation symmetry in 4+2 dimensions. However, the rotation symmetry \(\text{SO}(4, 2)\) turns into conformal symmetry

\(^3\) The term \(\frac{1}{2} \delta (X^2) \left( \varphi^r D_M D^M \varphi^a + h.c. \right)\) can also be written as \(-\delta (X^2) D_M \varphi^r D^M \varphi^a + 2 \delta' (X^2) \varphi^r \varphi^a\) after an integration by parts, as in \[15\].

\(^4\) The distinctive spacetime features including the delta function \(\delta (X^2)\) and its derivative that impose \(X^M X_M = 0\), as well as the explicit insertions of \(X^M\) in the form of \(X = X_M \Gamma^M\) in the fermion kinetic terms and Yukawa couplings, are required by the group theory rules of the spacetime \(\text{SU}(4, 2) = \text{SU}(2, 2)\) in Eq.\[2.8\] and by the gauge symmetries of 2T-physics field theory as explained in \[14\].
for the conformal shadow in $3 + 1$ dimensions, which includes translation symmetry for the shadow in $3 + 1$ dimensions.

The superpotential $W$ is purely cubic\(^5\) in the fields and is also $G$ invariant $\delta_\omega W = -i\omega^a \frac{\partial W}{\partial \varphi_r} (t_a \varphi)_r = 0$. The field equations may be solved for the auxiliary fields,

$$B_a = -g \varphi^t t_a \varphi, \quad F_r = -\frac{\partial W}{\partial \varphi^t}, \quad F^{tr} = -\frac{\partial W}{\partial \varphi_r},$$

so that this theory contains just the fields $(A_M, \lambda_L)^a$ and $(\varphi, \psi_L)_r$. In [15] it was demonstrated that this theory has $N = 1$ supersymmetry, and the corresponding conserved current in $4 + 2$ dimensions was given (see also below).

To construct the general theory with $N = 2$ supersymmetry in $2T$ field theory we follow the same strategy employed in $1T$ SUSY field theory but modified to conform to $2T$ field theory structures\(^6\). We start with the general $N = 1$ theory given above, with one $N = 1$ vector multiplet, and 3 distinct representations of $N = 1$ chiral multiplets embedded in the reducible representation labeled by $r$. Namely, we consider the following $N = 1$ supermultiplets

- vector : $(A_M, \lambda_L)^a$,
- chiral-0: $(\varphi, \psi_L)^a$, both in the adjoint representation,
- chiral-1: $(\phi, \eta_L)_n$,
- chiral-2: $(\tilde{\phi}, \tilde{\eta}_L)^n$,

in arbitrary complex conjugate repr. (3.7)

So, the label $r$ in Eqs.(3.2-3.5) is now specialized to the 3 representations labeled by the adjoint $a$, lower $n$ and upper $n$. The reducible matrix representation $(t_a)_r^s$ is also specialized as follows

$$(t_a)_r^s : (t_a)_b^c = -i f_{ab}^c, \quad (t_a)_n^m,$$

implying the $G$ transformation rules

$$\delta_\omega \varphi_b = -\omega^a f_{ab}^c \varphi_c, \quad \delta_\omega \phi_n = -i\omega^a (t_a)_n^m \phi_m, \quad \delta_\omega \tilde{\phi}_n = i\omega^a \tilde{\phi}_m (t_a)_m^n.$$

\(^5\) The purely cubic property of the superpotential is imposed by the $2T$ gauge symmetry [14]. This implies that there are no dimensional parameters, such as masses, in the potential. To induce mass terms in a nontrivial vacuum, the dilaton must also be coupled to the other scalars as described in [14] and in [18]. This implies that the entire supergravity multiplet, which includes the dilaton must also be included as part of the theory of mass generation in the supersymmetric theory.

\(^6\) The method used here in $4 + 2$ dimensions parallels a similar discussion in usual SUSY field theory in $3 + 1$ dimensions as described in [28]. Note however that the explicit SUSY transformations in $4 + 2$ dimensions have many features that are absent in the corresponding SUSY transformations in $3 + 1$ dimensions. Nevertheless those details do not play a role in this method.
The complex conjugate representations labeled by lower \( n \) and upper \( \bar{n} \) can themselves be reducible representations. In any case, \( \tilde{\phi}^m \phi_m \) is invariant, while \( \tilde{\phi}_a \phi \) transforms like the adjoint representation.

When the superpotential \( W \) is taken of the following form

\[
W = i\sqrt{2}g \tilde{\phi}_a \phi \phi^a,
\]

there is automatically \( N = 2 \) supersymmetry as well as local gauge symmetry under the Yang-Mills group \( G \). To show the \( N = 2 \) structure one writes the \( N = 1 \) Lagrangian following the recipe given above in Eqs.\([3.2-3.5]\). Then one can notice that there is a symmetry under the following discrete transformation \([\lambda_L^a \rightarrow \psi_L^a, \psi_L^a \rightarrow -\lambda_L^a] \) and \([\phi_n \rightarrow \tilde{\phi}_{\bar{n}}^i, \tilde{\phi}_{\bar{n}}^i \rightarrow -\phi_n] \), or equivalently \([\phi^m \rightarrow \tilde{\phi}^n, \tilde{\phi}^n \rightarrow -\phi^m] \), while the other fields \( A^a_M, \varphi^a, \eta_L, \tilde{\eta}_L^n \) remain unchanged. This transformation is just a discrete subgroup of the SU(2) \( R \)-symmetry which acts on the SU(2) doublets \((\lambda_L^a, \psi_L^a), (\phi_n, \tilde{\phi}_{\bar{n}}^i)\) as follows\(^7\)

\[
\begin{pmatrix}
\lambda_L^a \\
\psi_L^a
\end{pmatrix}' =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_L^a \\
\psi_L^a
\end{pmatrix},
\begin{pmatrix}
\phi_n \\
\tilde{\phi}_{\bar{n}}^i
\end{pmatrix}' =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_n \\
\tilde{\phi}_{\bar{n}}^i
\end{pmatrix} =
\begin{pmatrix}
-\phi_n \\
\tilde{\phi}_{\bar{n}}^i
\end{pmatrix}.
\]

The last relation can also be written equivalently for the charge conjugate doublet \((\phi_n, \tilde{\phi}_{\bar{n}}^i)' =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_n \\
\tilde{\phi}_{\bar{n}}^i
\end{pmatrix} \). Actually, this Lagrangian has a global symmetry under the continuous SU(2) \( R \)-symmetry transformations applied on the doublets above as will be made manifest in the next section.

Now we concentrate on identifying the second supersymmetry by starting with the known \([15]\) \( N = 1 \) SUSY transformations of our fields

\[
\delta_{\varepsilon_1}(A_M, \lambda_L)^a, \delta_{\varepsilon_1}(\varphi, \psi_L)^a, \delta_{\varepsilon_1}(\phi, \eta_L)_n, \delta_{\varepsilon_1}(\tilde{\phi}, \tilde{\eta}_L)^n.
\]

The expressions for these are given in \([15]\) but for now we will not need them explicitly. It suffices to know that the action above is invariant under this first SUSY transformation \(\delta_{\varepsilon_1} \) with parameter \(\varepsilon_{1L} \), which is a left-handed chiral spinor labeled by \( L = [4 \text{ of SU}(2, 2)] \).\([15]\)

Corresponding to this symmetry there is a conserved supercurrent in 4 + 2 dimensions \( J^M_{1L} \), that satisfies \( \partial_M J^M_{1L} = 0 \) when the equations of motion are used (see below).

Since we have already identified in Eq.\([3.11]\) a discrete \( R \) symmetry of the Lagrangian, it must be that the action is invariant also under a second SUSY transformation \(\delta_{\varepsilon_2} \) with

\(^7\) The discrete transformation \( R = e^{i\sigma_2 \pi/2} = i\sigma_2 =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \) corresponds to a SU(2) rotation by an angle \( \pi \).
parameter $\varepsilon_{2L}$. The $\delta_{\varepsilon_2}$ transformation laws must look the same as those of $\delta_{\varepsilon_1}$ after applying the discrete transformation of Eq. (3.11) on the expressions in Eq. (3.12) and then replacing $\varepsilon_{1L}$ by $\varepsilon_{2L}$. Hence the second SUSY transformation is obtained from the first one as follows

$$\delta_{\varepsilon_2}(A_M, \psi_L)^a, \delta_{\varepsilon_2}(-\varphi, -\lambda_L)^a, \delta_{\varepsilon_2}(\tilde{\varphi}^\dagger, \eta_L)_n, \delta_{\varepsilon_2}(-\phi^\dagger, \eta_L)_n.$$  (3.13)

We see that the second SUSY transformation $\delta_{\varepsilon_2}$ looks like again a $N=1$ transformation, but the fields have been reshuffled into new $N=1$ vector and chiral multiplets as seen by comparing Eqs. (3.12, 3.13). For example the $\delta_{\varepsilon_2}$ SUSY partner of $A_M$ is now $\psi_L$ rather than $\lambda_L$, and so on. With the same discrete $R$ transformation technique applied on the expression for the supercurrent $J_{1L}^M$ we can construct the second conserved SUSY current $J_{2L}^M$ (see below).

A. SU(2) covariant N=2 SUSY in 4+2 dims

It is evident from the previous section that, once the discrete $R$ symmetry has been identified, it is guaranteed that the theory has $N=2$ supersymmetry. It is useful to make this SU(2) and $N=2$ symmetry manifest by using fields with SU(2) doublet and singlet representation labels, and then rewrite the action, conserved currents, and transformation laws, described above in terms of these SU(2) representations. The result is the following.

The doublets are labeled by an index $i = 1, 2$ as follows

$$\lambda^a_{iL} = (\lambda^a_{1L}, \lambda^a_{2L}) \equiv (\lambda^a_L, \psi^a_L); \quad \phi_{in} = (\phi_{1n}, \phi_{2n}) \equiv (\phi_n, \tilde{\phi}_n^\dagger); \quad \varepsilon_{iL} = (\varepsilon_{1L}, \varepsilon_{2L})$$  (3.14)

while the other fields $A^a_M, \varphi^a, \eta_{Ln}, \tilde{\eta}^a_{Ln}$ are SU(2) singlets. It is also useful to introduce auxiliary fields $S^a_{ij}$ and $F_{in}$, where $F_{in}$ is an SU(2) doublet while $S^a_{ij}$ is a symmetric tensor representing a triplet of SU(2). It is convenient to collect these into one $N=2$ vector multiplet in the adjoint representation of $G$ and many $N=2$ hypermultiplets labeled by $n$ in some (generally reducible) representation of $G$

vector : $(A^a_M, \lambda^a_L, \varphi^a, S^a_{ij})$;  hyper : $(\phi_{in}, \eta_{inL}, \tilde{\eta}_{inR}, F_{in}), \ i = 1, 2.$  (3.15)

Here we have used the charge conjugate right handed spinor $\tilde{\eta}_{Ln} = C\bar{\eta}_{Ln}^T$ instead of the original left handed $\eta_{Ln}^T$. In fact, the two SO(4,2) Weyl spinors $(\eta_{Ln}, \tilde{\eta}_{Ln})$, transforming as $(4 \oplus \bar{4})$ of SU(2,2), taken together can be considered as a full 8 dimensional Dirac spinor of SO(4,2). In what follows, we choose to present the theory without the auxiliary fields.
The manifestly SU(2) invariant $N = 2$ action is
\[ S_{N=2} = \int d^{4+2} X \, \delta \left( X^2 \right) \left( L_{kin}^{N=2} + L_{yukawa}^{N=2} + L_{potential}^{N=2} \right). \] (3.16)

The kinetic term is
\[ L_{kin}^{N=2} = \left\{ -\frac{1}{4} F_{MN}^{a} F_{a}^{MN} + \frac{i}{2} \left[ \lambda_{L}^{ai} X \bar{D} \lambda_{L}^{a} + \lambda_{L}^{ai} \bar{D} \lambda_{L}^{a} \right] \right\} \]
\[ + \frac{1}{2} \phi^{ta} D^{M} D_{M} \phi^{a} + \frac{1}{2} \phi^{a} D^{M} D_{M} \phi^{ta} \]
\[ + \frac{1}{2} \phi^{tia} D^{M} D_{M} \phi_{ia} + \frac{1}{2} \phi_{ia} D^{M} D_{M} \phi^{tia} \]
\[ + \frac{1}{2} \sqrt{g^{\alpha}} \phi_{ia} D_{\alpha} \eta_{L} + \sqrt{g^{\alpha}} \phi_{ia} D_{\alpha} \bar{D} \eta_{L} \] (3.17)

The Yukawa interactions are
\[ L_{yukawa}^{N=2} = \left\{ \sqrt{2} g \left( t_{a} \right) \phi_{im} \left( \varepsilon^{ij} \bar{\eta}_{R} \right) X \lambda_{jL} + \varepsilon^{ij} \bar{\eta}_{L} X \lambda_{iL} \right\} + h.c. \]
\[ - \frac{ig}{\sqrt{2}} \varepsilon^{ij} f_{abc} \phi^{j} \lambda_{a} C X \lambda_{b} + \sqrt{2} g \phi^{a} \bar{D} \eta_{R} \right\} + h.c. \]
\[ \] (3.18)

The scalar potential term is
\[ L_{potential}^{N=2} = \left\{ -g^{2} \left( \phi \right) \phi_{i} \left( \phi \phi^{i} \right) \right\} \]
\[ + \frac{1}{2} g^{2} \left( \phi \phi^{i} \phi^{j} \right) \phi_{i} \phi_{j} - \frac{1}{2} g^{2} \left( \phi \phi^{i} \phi^{j} \phi^{k} \right) \]
\[ = \left( C \bar{\eta}_{R} \right)^{T}, \quad \bar{\eta}_{L} \equiv \left( C \bar{\eta}_{L} \right)^{T}, \quad \bar{\eta}_{L}^{m} \equiv - \left( C \bar{\eta}_{R} \right)^{T} \]
\[ \] (3.19)

The $N = 2$ supercurrent is (in this expression $\lambda_{L}^{a} \equiv \left( C \bar{\eta}_{L} \right)^{T}, \quad \eta_{R} \equiv \left( C \bar{\eta}_{R} \right)^{T}, \quad \bar{\eta}_{L}^{m} \equiv - \left( C \bar{\eta}_{R} \right)^{T}$
\[ J_{iL}^{M} = \delta \left( X^{2} \right) \]
\[ \left\{ \frac{1}{2\sqrt{2}} F_{KL}^{a} X_{N} \left( \Gamma^{KMN} \Gamma^{M} - \eta^{KMN} \Gamma^{K} \right) \lambda_{L}^{a} \right\} \]
\[ + \varepsilon_{ij} D_{K} \left( X_{N} \phi^{a} \right) \left( \Gamma^{KMN} \Gamma^{M} - \eta^{KMN} \Gamma^{K} \right) \lambda_{L}^{a} \]
\[ + D_{K} \left( X_{N} \phi_{ia} \right) \left( \Gamma^{KMN} \Gamma^{M} - \eta^{KMN} \Gamma^{K} \right) \eta_{R}^{a} \]
\[ + \varepsilon_{ij} D_{K} \left( X_{N} \phi^{j} \right) \left( \Gamma^{KMN} \Gamma^{M} - \eta^{KMN} \Gamma^{K} \right) \eta_{R}^{m} \]
\[ - \frac{ig}{\sqrt{2}} X_{N} \Gamma^{MN} \left[ \left( I f_{abc} \phi^{b} \phi^{c} + \phi^{i} t_{a} \phi_{i} \right) \lambda_{L}^{a} \right] \]
\[ - 2 \phi_{i} t_{a} \phi_{i} \lambda_{L}^{a} \]
\[ - 2 \phi_{a} \varepsilon_{ij} \left( \phi^{j} t_{a} \right) n \eta_{L} \]
\[ - 2 \phi_{a} \left( t_{a} \phi_{i} \right) m \bar{\eta}_{L}^{m} \]
This $J^M_{iL}$ is a doublet of SU(2) and vector $\otimes$ left-handed Weyl spinor of SO(4,2).

These fermionic currents are conserved $\partial_M J^M_{iL} = 0$ when we use the equations of motion derived from the $N = 2$ action given above. The general variation of the action with respect to each field contains terms proportional to both $\delta (X^2)$ as well as $\delta' (X^2)$ (which arises from integration by parts). The equations that emerge from the $\delta' (X^2)$ terms are called kinematic equations, while those emerging from the $\delta (X^2)$ term are called dynamical equations. The kinematic equations can be solved easily, and they can be interpreted as the covariant version of one of the three Sp(2, R) constraints of the underlying worldline theory (namely the $X \cdot P = 0$ constraint). The dynamical equations correspond to another Sp(2, R) constraint ($P^2 = 0$ constraint) after being covariantized and modified by the interactions. Finally, because of the delta functions, all equations listed below must be taken at $X^2 = 0$, which is the third Sp(2, R) constraint. It should be emphasized that all equations of motion follow from the action.

The following SU(2) covariant $N = 2$ equations are the kinematic equations of motion

$$X^N F^a_{NM} = (X \cdot D + 1) \varphi^a = (X \cdot D + 1) \phi_{in} = 0,$$

$$\delta \{ \varphi^a = (X \cdot D + 2) \lambda^a_{Li} = (X \cdot D + 2) \eta_{nL} = (X \cdot D + 2) \tilde{\eta}_{nR} = 0, \text{ (3.21)} \}
$$

while the following SU(2) covariant $N = 2$ equations are the dynamical equations of motion

$$(D_M F^{MN})^a = \left\{ \begin{array}{l}
  ig f^{abc} X^M \lambda^b_L \Gamma^{MN} \lambda^c_i + g f^{abc} \varphi^b \overline{D}^N \varphi^c \\
  -g X^M \eta_{jL} \Gamma^{MN} t^a \eta_{nL} + g X^M \eta_{jR} \Gamma^{MN} t^a \tilde{\eta}_{nR} + ig \phi^b t^a \overline{D}^M \phi_i
\end{array} \right\} = 0, \text{ (3.22)}$$

$$D^2 \varphi^a + g^2 f^{abc} f_{bde} \varphi^c \varphi^d \varphi^e + \frac{2}{\sqrt{2}} g^* \partial \varphi^a \varphi^b \varphi^c + g^2 (\partial_j t^a \phi_i) \varphi^b = 0, \text{ (3.23)}$$

$$D^2 \phi_{in} + \left\{ \begin{array}{l}
  \sqrt{2} g (t^a_n) (\varphi^a \varphi^b + \varphi^b \varphi^a) - g^2 (\phi^b t^a_n) \phi^\dagger (t^a \phi^j) \end{array} \right\} \text{ (3.24)}$$

$$i X \overline{D} \lambda^a_{iL} + i \sqrt{2} g \varepsilon^{ij} f_{abc} \varphi^b X \lambda^c_{jR} - \sqrt{2} g \varepsilon^{ij} (\phi^i t^a_n) X \eta_{Rn} + \sqrt{2} g (t^a \phi_i) X \eta^a_{Rn} = 0, \text{ (3.25)}$$

$$i \overline{X} D \lambda^a_{iR} + i \sqrt{2} g \varepsilon^{ij} f_{abc} \varphi^b X \lambda^c_{jL} - \sqrt{2} g \varepsilon^{ij} (\phi^i t^a_n) \overline{X} \eta_{Ln} - \sqrt{2} g \varepsilon^{ij} (t^a \phi_i) \overline{X} \eta^a_{Ln} = 0, \text{ (3.26)}$$

$$i X \overline{D} \eta_{nL} + \sqrt{2} g (t^a \phi_i) X \lambda^a_{R} + \sqrt{2} g \phi^b t^a X (t^a \eta^b_{R}) = 0, \text{ (3.27)}$$

$$i \overline{X} D \eta_{nR} - \sqrt{2} g (t^a \phi_i) \varepsilon^{ij} \overline{X} \lambda^j_{L} - \sqrt{2} g \varphi^a \overline{X} (t^a \eta^a) = 0, \text{ (3.28)}$$
The $N = 2$ SUSY transformations for the action associated with the supercurrent in Eq. (3.19) are (without auxiliary fields)

$$\delta \epsilon A_M^a = -\frac{1}{\sqrt{2}} \varepsilon_R \Gamma_M \bar{X} \lambda^a_{L_i} + X^2 \left[ -\frac{1}{2\sqrt{2}} \varepsilon_R \Gamma_{MN} D^N \lambda^a_{L_i} - \frac{g}{2} f^{abc} \varepsilon_{ij} \left( \varepsilon \Gamma_M \lambda^b_{R_i} \right) \varphi^c \right. $$

$$\left. - \frac{i g}{4} \left( \varepsilon_R \Gamma_{MN} \varepsilon^i \right) \left( t_a \phi_i \right)_n + \varepsilon ^{iji} \frac{i g}{4} \left( \varepsilon_R \Gamma_{MN} \varepsilon^i \right) \left( t_a \phi_j \right)_n \right] + h.c.$$  \hspace{1cm} (3.29)

$$\delta \epsilon \phi_{in} = \varepsilon_R \bar{X} \eta_{Ln} - \varepsilon_{ij} \varepsilon_R \bar{X} \eta_{Rn} + X^2 \left[ -\frac{1}{2\sqrt{2}} \varepsilon_R \Gamma_{MN} D^N \lambda^a_{L_i} - \frac{g}{2} f^{abc} \varepsilon_{ij} \left( \varepsilon \Gamma_M \lambda^b_{L_i} \right) \varphi^c \right. $$

$$\left. - \frac{i g}{4} \left( \varepsilon_R \Gamma_{MN} \varepsilon^i \right) \left( t_a \phi_i \right)_n \left( \varepsilon \Gamma_M \lambda^b_{R_i} \right) \varphi_j + \frac{i g}{4} \left( t_a \phi_i \right)_n \left( \varepsilon \Gamma_M \lambda^b_{R_i} \right) \varphi_j \right]$$  \hspace{1cm} (3.30)

$$\delta \epsilon \phi^a = \varepsilon_{ij} \varepsilon_R \bar{X} \lambda^a_{L_j} + X^2 \left[ -\frac{1}{2\sqrt{2}} \varepsilon_R \Gamma_{MN} D^N \lambda^a_{L_j} + \frac{g}{2} f^{abc} \varphi^b \left( \varepsilon \Gamma_M \lambda^c_{L_j} \right) \varphi_j \right. $$

$$\left. + \frac{i g}{4} \left( t_a \phi_i \right)_n \left( \varepsilon \Gamma_M \lambda^c_{L_j} \right) \varphi_j \right] - i \frac{g}{2} \varepsilon^{ij} \eta^R_{Ln} \left( t_a \phi_j \right)_n$$  \hspace{1cm} (3.31)

$$\delta \epsilon \lambda^a_{Li} = \left\{ -\varepsilon_{ij} \left( D_M \phi_i \right)_n \Gamma^i \lambda^a_{L_j} + \frac{g}{2 \sqrt{2}} F_{MN}^a \left( \gamma^M \lambda^a_{L_j} \right) \right. $$

$$\left. - i \frac{g}{2} f^{abc} \varphi^b \varphi^c \varepsilon_{Li} + \frac{g}{2} \left[ 2 \varepsilon_{Li} \phi^j t_a \phi_i - \varepsilon_{Li} \phi^j t_a \phi_j \right] \right\}$$  \hspace{1cm} (3.32)

$$\delta \epsilon \eta_{Ln} = i \left( D_M \phi_i \right)_n \Gamma^i \varepsilon^i \eta^R + \varepsilon^{ij} \sqrt{2} \left( \phi^j \phi_j \right)_n \varepsilon_{Li}$$  \hspace{1cm} (3.33)

$$\delta \epsilon \tilde{\eta}_{Rn} = i \varepsilon^{ij} \left( D_M \phi_i \right)_n \Gamma^M \varepsilon^i \eta^j + \sqrt{2} \left( \phi^i \phi_j \right)_n \varepsilon^i_{Rj}$$  \hspace{1cm} (3.34)

The $N = 2$ SUSY transformation above have some parallels to naive $N = 2$ SUSY transformations that one may attempt to write down as a direct generalization from $3 + 1$ to $4 + 2$ dimensions. However, there are many features that are completely different. Once we notice the parallels, part of the structure can be understood from the spacetime SU(2, 2) group theory, as in Eq. (2.8). The generalized features include the insertions that involve $X = X^M \Gamma_M$ or $\bar{X} = X^M \Gamma_M$, and the terms proportional to $X^2$. These are off-shell SUSY transformations that include interactions and leave the off-shell action invariant.

Despite all of the changes compared to naive SUSY, this SUSY symmetry provides a representation of the supergroup SU(2, 2|2). This is signaled by the fact that all terms are covariant under the bosonic subgroup SU(2, 2) $\otimes$ SU(2), while the complex fermionic parameter $\varepsilon_{Li}$ and its conjugate $\varepsilon^L_{Lj}$ are in the 4,4* representations of SU(2, 2), and are doublets of the R-symmetry SU(2), as would be expected for SU(2,2|2).

The closure of the SUSY transformations is discussed for the case of $N = 1$ in Appendix (B) of reference [15]. The closure in that case was SU(2,2|1) when the fields are on-shell.
It is straightforward but tedious to verify that for the present case of $N = 2$, the closure is $SU(2, 2|2)$ when the fields are on-shell. The SUSY transformations above are actually off-shell. The closure off-shell goes beyond $SU(2, 2|2)$ and includes 2T-physics gauge transformations (terms proportional to $X^2$ and other kinematic constraints that do not vanish off-shell) of the type discussed in [14] and [15].

When reduced to $3 + 1$ dimensions, by solving the kinematic equations (3.20) in a special gauge which we call the conformal gauge described in [14]-[18], the $SU(2, 2|2)$ transformations above reduce to a non-linear off-shell realization of $N = 2$ superconformal symmetry in $3 + 1$ dimensions.

IV. N=4 SUPER YANG-MILLS IN 4+2 DIMENSIONS

The $N = 4$ SYM multiplet has the same field content as the $N = 2$ vector SYM multiplet $(A_M^a, \lambda_i^a, \varphi^a)$ coupled to just one $N = 2$ hypermultiplet $(\phi_i^a, \eta_R^a, \eta_L^a)$ whose fields are in the adjoint representation. Thus, all that we need to do is specialize the hypermultiplet in the previous section to be in the adjoint representation labeled by $a$. Then there are four left handed fermions $(\lambda_i^a, \eta_R^a, \eta_L^a)$ which we call $\lambda_i^a$, $r = 1, 2, 3, 4$ and six real scalars (three complex ones, $\varphi^a, \phi_i^a$) which we call $\theta^a_u$, $u = 1, \cdots, 6$, in addition to the Yang-Mills field $A_M^a$, all in $4 + 2$ dimensions. In this section we present this structure directly in an $SU(4) = SO(6)$ covariant way, thus displaying the $N = 4$ SU(4) $R$-symmetry. Then we show that the $SU(4)$ covariant theory agrees with the general form of the $N = 1$ SUSY theory of section 3, in four different rearrangements of the multiplets, thus proving the $N = 4$ SUSY symmetry in a different way.

Let $r, s$ label the $SU(4)$ fundamental or antifundamental representations (spinors of

---

8 Dirac initiated a similar set of field equations on the hypercone (without an action principle) to explain conformal symmetry $SO(4, 2)$ as the rotation group in 6 dimensions [29]-[37]. A worldline approach along Dirac’s ideas was also pursued [38]-[40]. From the point of view of 2T-physics, Dirac’s view of conformal symmetry amounts to only one of the shadows, which we call the conformal shadow. The $Sp(2, R)$ phase space gauge symmetry in 2T-physics, which was absent in previous work, was motivated by signals of 2T in the supersymmetry structure of M theory [41]-[45] and it developed independently, unaware of Dirac’s work. This $Sp(2, R)$ gauge symmetry is at the root of the shadows and duality phenomena in 2T-physics. In the worldline theory the shadows are obtained by making $Sp(2, R)$ gauge choices in phase space $(X^M, P_M)$, while in field theory the same shadows are recovered by solving the kinematic equations with various parameterizations of spacetime as shown in [16][17].
SO(6)) and let \( u, v \) label the vector of SO(6), while \((\alpha, \dot{\alpha})\) and \( M \) label the spacetime SO(4,2) spinor and vector representations respectively. The manifestly SO(4,2) \( \otimes \) SU(4) \( \otimes \) G invariant action can be written as \( S^{N=4} = \int d^{4+2}X \delta (X^2) L^{N=4} (X) \), with

\[
L^{N=4} = \left\{ -\frac{1}{4} F_{MN}^a F^{MN}_a + \frac{i}{2} \theta^a D^M D_M \theta^a - \frac{g^2}{4} \sum |f_{abc} \theta^b u^c|^2 + \frac{i}{2} \left[ \lambda_L^a X \bar{D} \lambda_{Lr} + g f_{abc} (\lambda_L^a C \bar{X} \lambda_{Ls}^b) (\bar{\gamma}^a)^{rs} \theta^c_u \right] + h.c. \right\}
\]  

(4.1)

Here \( \gamma^u_{rs} = -\gamma^v_{sr} \) (and their Hermitian conjugates \( (\gamma^u)_{rs}^\dagger \)) are antisymmetric SO(6) = SU(4) gamma matrices in a Weyl basis that satisfy \( (\gamma^u \bar{\gamma}^v + \gamma^v \bar{\gamma}^u)_r^s = 2 \delta^a u^a \delta^s_r \). The explicit matrix form of the antisymmetric SO(6) gamma matrices \((\gamma^u)_{rs}, (\bar{\gamma}_u)^{rs}\) can be taken as

\[
\gamma^u = [(\sigma_2 \times i \sigma_2 \bar{\sigma}), (\sigma_2 \bar{\sigma} \times \sigma_2)], \quad \bar{\gamma}_u = [(\sigma_2 \times i \sigma_2 \bar{\sigma}^*), (-\sigma_2 \bar{\sigma}^* \times \sigma_2)]
\]  

(4.2)

where \( \bar{\gamma}_u \) is related to \( \gamma^u \) by Hermitian conjugation \( \bar{\gamma}_u = (\gamma^u)^\dagger \) or by complex conjugation \( \bar{\gamma}_u = - (\gamma^u)^* \) (note \( -\sigma_2 \bar{\sigma}^* = (i \sigma_3, 1, -i \sigma_1) = \bar{\sigma} \sigma_2 \)). They satisfy the property \( \gamma^u_{rs} = \frac{1}{2} \varepsilon_{rs pq} (\gamma^u)^{pq} \). Using these one can recast the six independent real scalar fields \( \theta^a_u \) into an SU(4) antisymmetric tensor form \((\varphi^a)_{rs}\)

\[
(\varphi^a)_{rs} = \frac{1}{\sqrt{2}} \gamma^u_{rs} \theta^a_u \quad \text{or} \quad (\bar{\varphi}_a)^{rs} = \frac{1}{\sqrt{2}} (\gamma^u)_{rs} \theta^a_u = (\varphi^a_1)^{rs} = -(\varphi^a_{rs})^* \quad \text{for all} \quad a
\]  

(4.3)

Because the complex conjugate is not independent there is a SU(4) covariant duality relation

\[
(\varphi^a)_{rs} = \frac{1}{2} \varepsilon_{rs pq} \varphi^a_{pq}.
\]  

(4.4)

This implies that the antisymmetric SU(4) tensor \((\varphi^a)_{rs}\) contains only 3 independent complex numbers for each \( a \), which is seen explicitly as follows

\[
(\varphi^a)_{rs} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i\theta_5^a + \theta_6^a & \theta_2^a + i\theta_3^a & -i\theta_1^a + \theta_4^a \\ -i\theta_5^a - \theta_6^a & 0 & -i\theta_1^a - \theta_4^a & \theta_2^a - i\theta_3^a \\ -i\theta_2^a - i\theta_3^a & i\theta_1^a + \theta_4^a & 0 & i\theta_5^a - \theta_6^a \\ i\theta_1^a - \theta_4^a & -i\theta_2^a + i\theta_3^a & -i\theta_5^a + \theta_6^a & 0 \end{pmatrix}
\]  

(4.5)

\[
\equiv \begin{pmatrix} 0 & \bar{\varphi}^{3a} & \bar{\varphi}^{2a} & \varphi^a_1 \\ \bar{\varphi}^{3a} & 0 & -\varphi^{1a} & \varphi^a_2 \\ -\varphi^{2a} & \varphi^{1a} & 0 & \varphi^a_3 \\ -\varphi^a_1 & -\varphi^a_2 & -\varphi^a_3 & 0 \end{pmatrix}, \quad \text{where} \quad \bar{\varphi}^{ia} \equiv (\varphi^a_i)^* \quad (i = 1, 2, 3)
\]  

(4.6)

This relation is useful to write the \( N = 4 \) theory in an \( N = 1 \) or \( N = 2 \) basis.
A. N=4 Super Yang-Mills as coupled N=1 supermultiplets

We now want to verify that the SU(4) covariant structures above have $N = 4$ supersymmetry and are in agreement with the $N = 1$ supersymmetry structures in $4 + 2$ dimensions that we discussed in Eqs. (3.1-3.5). To do this we split the SU(4) R-symmetry into SU(3) × U(1) and identify the U(1) as the R-symmetry associated with $N = 1$ supersymmetry while the SU(3) part is considered as an internal symmetry acting on three $N = 1$ chiral multiplets. Of course, there are 4 different ways of splitting 4 into 3+1, each one of these corresponds to the different $N = 1$ supersymmetries within the $N = 4$ theory. In each case, the $N = 4$ vector supermultiplet splits into one $N = 1$ vector supermultiplet plus 3 chiral multiplets that transform into each other as a triplet of SU(3).

To be specific let the 1 in 3+1 correspond to the fourth member of the SU(4) quartet labeled as $r = (i, 4)$ with $i = 1, 2, 3$. Then the SU(4) quartet of fermions is split into a SU(3) triplet and a singlet $\lambda^a_r = (\lambda^a_{ri}, \lambda^a_{r4})$. The singlet is identified as the fermion in the $N = 1$ vector multiplet $(A^a_M, \lambda^a_0)$, with $\lambda^a_4 \equiv \lambda^a$, while the triplet $\lambda^a_{ri}$ belongs to a $N = 1$ chiral multiplet $(\varphi^a_i, \lambda^a_{ri})$, with $r \rightarrow i = 1, 2, 3$ labeling the fundamental representation of SU(3).

In this notation the kinetic term for the fermions in the $N = 4$ action is rewritten as

$$\frac{i}{2} \lambda^a_r X \bar{D} \lambda^a_r + h.c. = \frac{i}{2} \left[ \lambda^a_L X \bar{D} \lambda^a_L + \lambda^a_i X \bar{D} \lambda^a_i \right] + h.c. \quad (4.7)$$

We see that this is in agreement with the $N = 1$ SUSY structure given in Eq. (3.2), when the chiral multiplet $(\varphi^a_i, \lambda^a_{ri})$ is in the adjoint representation of the gauge group $G$. Note that here SU(3) with its label $i$ is a global, not a local, symmetry.

Next we verify the same property for the scalars. The 3 complex scalars $\varphi^a_i$ that appear in Eq. (4.6) correspond to the 6 real scalars $\theta^a_u$, $u = 1, 2, \cdots, 6$ with the following identification

$$(\varphi^a)_i^4 = \frac{1}{\sqrt{2}} \gamma^i u \theta^a_u = \varphi^a_i, \quad (\varphi^a)_i^j = \frac{1}{\sqrt{2}} \gamma^i \theta^a_u = -\varepsilon_{ijk} \varphi^k$$

$$(\bar{\varphi}_a)^i_4 = \frac{1}{\sqrt{2}} (\bar{\gamma}^u)^i a \theta^a_u = -\bar{\varphi}_a, \quad (\bar{\varphi}_a)^i_j = \frac{1}{\sqrt{2}} (\bar{\gamma}^u)^i j \theta^a_u = \varepsilon^{ijk} \varphi^k \quad (4.9)$$

Then the kinetic term for the scalars in the $N = 4$ action is rewritten as

$$\frac{1}{2} \theta^a_u D^M D_M \theta^a_u = \frac{1}{2} \bar{\varphi}^i_a D^M D_M \varphi^a_i + \frac{1}{2} \varphi^a_i D^M D_M \bar{\varphi}^i_a \quad (4.10)$$

This is in agreement with the $N = 1$ SUSY structures in Eq. (3.2). Furthermore, the Yukawa
term in the $N = 4$ action takes the form
\begin{align}
\frac{i}{2}gf_{abc}(\lambda_{Li}^a C\bar{X}\lambda_{L_s}^b) (\bar{\phi}^{i})^{rs} \theta_{a}^{c} + h.c. \\
= \frac{i}{\sqrt{2}}gf_{abc}(\lambda_{Li}^{a} C\bar{X}\lambda_{L_s}^{b}) (\bar{\phi}^{i})^{rs} + h.c. \\
= \left[ \frac{i}{\sqrt{2}}2gf_{abc}(\lambda_{Li}^{a} C\bar{X}\lambda_{LA}^{b}) (\bar{\phi}^{i})^{i4} + \frac{i}{\sqrt{2}}gf_{abc}(\lambda_{Li}^{a} C\bar{X}\lambda_{L_s}^{b}) (\bar{\phi}^{i})^{ij} \right] + h.c.
\end{align}

This is in agreement with the $N = 1$ SUSY structures in Eq. (3.3) provided the superpotential $W(\varphi_i)$ is
\begin{equation}
W(\varphi) = -\frac{g}{3\sqrt{2}}\epsilon^{ijk}f_{abc}\varphi_i^b\varphi_j^c\varphi_k^a = -\sqrt{2}gf_{abc}\varphi_i^b\varphi_2^c\varphi_3^a.
\end{equation}

Next we rewrite the potential term $V(\theta)$ in the $N = 4$ action in terms of the complex scalars $\varphi_i^a$ as follows
\begin{align}
V(\theta) &= \frac{g^2}{4}\sum |f_{abc}\theta_{a}^{b}\theta_{o}^{c}|^2 = \frac{g^2}{4}f_{abc}f_{ab'c'} \left( \theta_{b}^{b} \cdot \bar{\theta}_{b'}^{c'} \right) \left( \theta_{c}^{c} \cdot \bar{\theta}_{c'}^{c'} \right) \\
&= \frac{g^2}{4}f_{abc}f_{ab'c'} \left( \bar{\varphi}_{i}^{ib} \varphi_{i}^{n} + \varphi_{i}^{ib} \bar{\varphi}_{i}^{n} \right) \left( \varphi_{j}^{jc} \varphi_{j}^{c} + \varphi_{j}^{jc} \bar{\varphi}_{j}^{c} \right) \\
&= \frac{g^2}{2}(f_{abc}\bar{\varphi}_{i}^{ib}\varphi_{j}^{jc})(f_{ab'c'}\varphi_{i}^{ib}\varphi_{j}^{jc}) + \frac{g^2}{2}(f_{abc}\varphi_{i}^{ic}\varphi_{j}^{jc})(f_{ab'c'}\bar{\varphi}_{j}^{ic}\varphi_{j}^{c})
\end{align}
The Jacobi identity
\begin{equation}
f_{abc}f_{ab'c'} = f_{ac'b}f_{ac'b} + f_{ab'b}f_{ac'c'}
\end{equation}
is used to rewrite the second term in the last line as
\begin{align}\nonumber
\frac{g^2}{2}(f_{abc}\bar{\varphi}_{i}^{ib}\varphi_{j}^{jc})(f_{ab'c'}\varphi_{i}^{ib}\varphi_{j}^{jc}) &= \left[ \frac{g^2}{2}(f_{acb}\bar{\varphi}_{i}^{ic}\varphi_{j}^{jc})(f_{ac'b}\varphi_{i}^{ib}\bar{\varphi}_{j}^{c}) + \frac{g^2}{2}(f_{ab'b}\varphi_{i}^{ib}\varphi_{j}^{jc})(f_{ac'c}\varphi_{j}^{ic}\varphi_{j}^{c}) \right]
\end{align}
So the potential $V(\theta) = V(\varphi)$ takes the form
\begin{align}
V(\varphi) &= g^2(f_{abc}\bar{\varphi}_{i}^{ib}\varphi_{j}^{jc})(f_{ab'c'}\varphi_{i}^{ib}\varphi_{j}^{jc}) - \frac{g^2}{2}(f_{ab'b}\varphi_{i}^{ib}\varphi_{j}^{jc})(f_{ac'c}\varphi_{j}^{ic}\varphi_{j}^{c})
\end{align}
We see that the potential can be written as the standard $N = 1$ F and D terms $V(\varphi) = V_D(\varphi) + V_F(\varphi)$,
\begin{align}
V_F(\varphi) &= (g f_{abc}\bar{\varphi}_{i}^{ib}\varphi_{j}^{jc})(f_{ab'c'}\varphi_{i}^{ib}\varphi_{j}^{jc}) = F_{a}^{k}F_{k}^{a}, \quad \text{with} \quad \frac{1}{\sqrt{2}}F_{a}^{k} \epsilon_{kij} = g f_{abc}\varphi_{i}^{ib}\varphi_{j}^{jc} \\
V_D(\varphi) &= \frac{1}{2}(ig f_{ab'b}\varphi_{i}^{ib}\varphi_{j}^{jc})(ig f_{ac'c}\varphi_{j}^{ic}\varphi_{j}^{c}) = \frac{1}{2}B_{a}B_{a}, \quad \text{with} \quad B_{a} \equiv ig f_{ab'b}\varphi_{i}^{ib}\varphi_{i}^{c}.
\end{align}
This is in agreement again with the $N = 1$ rules given in Eq. (3.4) when the superpotential $W(\varphi)$ is precisely the one above in Eq. (4.15), since it then reproduces the correct $F$-term through $F_a^k = -\frac{\partial W}{\partial \epsilon^k}$.

This agreement shows that the SU(4) covariant theory has $N = 1$ SUSY in 4+2 dimensions for each of the four ways of reducing SU(4) → SU(3) x U(1). This proves that the covariant theory has $N = 4$ supersymmetry in 4+2 dimensions. Indeed the evident SU(4) $R$-symmetry implies that if there is $N = 1$ SUSY then there must be $N = 4$ SUSY.

**B. N=4 covariant off-shell SUSY transformations in 4+2 dimensions**

Having established that the covariant action (4.1) has four supersymmetries, it is useful to write the $N = 4$ supersymmetry transformation in covariant form as follows (using $\epsilon_R^s \equiv C\bar{\epsilon}_L^s T$, $\epsilon_R = (\epsilon_L^s)^T C$, and similarly for $\lambda_R^a$): 

$$\delta \epsilon^a_M = \left\{ -\epsilon_L \Gamma_M \bar{\chi} \lambda_L^a + X^2 \left[ +\frac{1}{2} \epsilon_L \Gamma_M \epsilon^a_r \lambda^a_{rl} + \frac{i}{2} g f_{abc} \epsilon_{rl} \Gamma^b \lambda^a_{rl} \right] \right\} + h.c. \quad (4.24)$$

$$\delta \epsilon \lambda^a_{rl} = i (D \theta^a_u) (\gamma^u \epsilon^r) + \frac{1}{2} F^a M N \epsilon_{rl} + \frac{i}{2} g f_{abc} \epsilon^b \theta^c \epsilon^a_r (\gamma^u \epsilon^r) \quad (4.25)$$

$$\delta \epsilon \theta^a_u = \left\{ (\bar{\gamma}^u)_{rs} \bar{\epsilon} R s \bar{\chi} \lambda^a_{ls} + X^2 \left[ -\frac{1}{2} (\bar{\gamma}^u)_{rs} \bar{\epsilon} R s \bar{D} \lambda^a_{ls} + \frac{g}{2} f_{abc} \epsilon^b \bar{L} \epsilon^c \lambda^a_{ls} \right] \right\} + h.c. \quad (4.26)$$

The first two expressions (4.24, 4.25) may easily be rewritten in terms of $(\varphi^a)_{rs} = \frac{1}{\sqrt{2}} \gamma^u \epsilon_{rs} \theta^a_u$.

The last expression (4.26) may also be written in terms of $(\varphi^a)_{rs}$ as follows:

$$(\delta \varphi^a)_{rs} = \left\{ -\sqrt{2} \left( \bar{\epsilon} R r \bar{X} \lambda^a_{ls} - \bar{\epsilon} R s \bar{X} \lambda^a_{lr} \right) + \sqrt{2} \epsilon_{rsq} \bar{L}^p X \lambda^a_{rl} \right\} + X^2 \left[ +\frac{1}{2} g f_{abc} \epsilon^b \bar{L} \lambda^c \bar{L} \lambda^a_{ls} - \frac{1}{\sqrt{2}} \epsilon_{rsq} \bar{L}^p D \lambda^a_{rl} \right] \quad (4.27)$$

To compute the hermitian conjugate terms denoted as “h.c.” we recall from appendix A in [15] the following rules which apply when all right handed fermions are related to left handed fermions (Majorana fermions) as explained in section (11)

$$\left( \bar{\psi}^1_{1L} \psi^2_{2L} \right)^1 = -\bar{\psi}^2_{2L} \psi^1_{1L} = \bar{\psi}^1_{1R} \psi^2_{2R},$$
$$\left( \bar{\psi}^1_{1L} \Gamma^M \psi^2_{2L} \right)^1 = \bar{\psi}^2_{2L} \Gamma^M \psi^1_{1L} = \bar{\psi}^1_{1R} \Gamma^M \psi^2_{2R},$$
$$\left( \bar{\psi}^1_{1L} \Gamma^M \Gamma^N \psi^2_{2L} \right)^1 = -\bar{\psi}^2_{2L} \Gamma^M \Gamma^N \psi^1_{1L} = \bar{\psi}^1_{1R} \Gamma^M \Gamma^N \psi^2_{2R},$$
$$\left( \bar{\psi}^1_{1L} \Gamma^M \Gamma^N \psi^2_{2L} \right)^1 = \bar{\psi}^2_{2L} \Gamma^M \Gamma^N \psi^1_{1L} = \bar{\psi}^1_{1R} \Gamma^M \Gamma^N \psi^2_{2R}.$$
To verify this last form we reconstruct \( \delta \theta^a_u = -\frac{1}{\sqrt{2}} (\delta \varphi^a)_{rs} (\gamma_u)^{rs} \) by inserting the expression in (4.27) and obtain the expression in (4.26)\(^{10}\).

The \( N = 4 \) SUSY transformations (4.24-4.27) are obtained by SU(4) covariantizing the \( N = 1 \) transformations given in [15] (for comparison we define \( \varepsilon_{LA} = \varepsilon_L / \sqrt{2} \)). The \( N = 1 \) SUSY formulas combined with SU(4) insure that they work for \( N = 4 \) SUSY.

Furthermore, by rewriting the \( N = 4 \) transformations in the \( N = 2 \) basis, it can be verified that they are also in agreement with the \( N = 2 \) transformations in Eqs.(3.29-3.34) by using the following identification of \( N = 4 \) and \( N = 2 \) degrees of freedom

\[
\begin{pmatrix}
\lambda^a_{iL} \\
\eta^a_L \\
-\tilde{\eta}^a_L
\end{pmatrix}, \quad 
\varepsilon_{rL} = \begin{pmatrix}
\frac{1}{\sqrt{2}} \varepsilon_{iL} \\
0 \\
0
\end{pmatrix}, \quad i = 1, 2 \quad (4.28)
\]

\[
\varphi^a_{ij} = -\varphi^a_i, \quad \varphi^a_{ij} = \varepsilon_{ij} \tilde{\varphi}^a, \quad \varphi^a_{34} = \tilde{\varphi}^a, \quad \varphi^a_{ij} = -\varepsilon_{ij} \varphi^a
\]

We emphasize that the off-shell SUSY transformations in 4+2 dimensions include terms proportional to \( X^2 \) which are new structures as compared to SUSY transformations in 3 + 1 dimensions. The closure of these transformations (commutators) is consistent with SU(2, 2|4) when the fields are on-shell, but off shell there are additional terms beyond SU(2, 2|4). The extra terms in the closure correspond to gauge transformations that are the 2T gauge symmetries of 2T field theory of the type discussed in [14], and they are expected to vanish in the gauge invariant sector of the theory.

The \( N = 4 \) supercurrents associated with these SUSY transformations take the form

\[
J^M_{Lr} = \delta (X^2) \left\{ -\frac{1}{2} F^a_{PQ} X_N \left( \Gamma^{PQN} \Gamma^M - \eta^{NM} \Gamma^{PQ} \right) \lambda^a_{Lr} \right. \\
-\sqrt{2} (\Gamma^{PQM} - \eta^{MP} \Gamma^Q) [D_Q (X_P \varphi^a) \lambda^a_{Rr}]_r \\
- g f_{abc} \Gamma^{MN} X_N \left( \varphi^a \varphi^b \lambda^c_L \right)_r \right\} \quad (4.30)
\]

Its expression in terms of \( \theta^a_u \) is obtained by substituting \( (\varphi^a)_{rs} = \frac{1}{\sqrt{2}} \gamma^a_{rs} \theta^a_u \). The \( N = 4 \) supercurrents are conserved \( \partial_M J^M_{Lr} = 0 \) when the equations of motion that follow from the \( N = 4 \) action are used. It should be noted that the expression for the supercurrents can be

\[\text{10 The following properties of the SO(6) gamma matrices are also useful: } Tr (\gamma_u \gamma_v) = 4 \delta_{uv} \text{ and } (\gamma_u)_{rs} (\gamma^u)_{pq} = -2 (\delta_r^p \delta_s^q - \delta_s^p \delta_r^q) \text{ and } (\gamma_u)_{rs} (\gamma^u)_{pq} = -2 \varepsilon_{rpsq}.\]
modified by terms of the form
\[ \Delta J^M_{Lr} = \delta (X^2) X^M \xi_{Lr} \] (4.31)
that are automatically conserved \( \partial_M (\Delta J^M_{Lr}) = 0 \), when the spinors \( \xi_{Lr} \) are arbitrary except for satisfying the following homogeneity condition
\[ (X \cdot \partial + 4) \xi_{Lr} = 0, \text{ equivalently } \xi_{Lr} (tX) = t^{-4} \xi_{Lr} (X). \] (4.32)
The currents \( J^M_{Lr} \) above agree with the \( N = 1 \) supercurrent in [15] after inserting the \( N = 1 \) basis discussed in Eqs. (4.7-4.9) (for comparison with [15] we define \( (J^M_{4L})_{N=4} \equiv \sqrt{2} (J^M_L)_{N=1} \)). Furthermore, after inserting the \( N = 2 \) basis of Eqs. (4.28,4.29), the \( N = 4 \) currents above also agree with the \( N = 2 \) currents in Eq.(3.19) when we identify two of the \( N = 4 \) currents with the \( N = 2 \) currents up to a \( \sqrt{2} \) normalization, \( (J^M_{iL})_{N=4} = \sqrt{2} (J^M_{iL})_{N=2}, \ i = 1, 2. \) Of course, both \( N = 2 \) and \( N = 4 \) currents are consistent with \( N = 1 \).

V. PHYSICS CONSEQUENCES AND FUTURE DIRECTIONS

In this paper we have explicitly constructed \( N = 2 \) and \( N = 4 \) supersymmetric field theories in the theoretical framework of 2T-physics with two times. All fields, including vectors and spinors, are in \( 4 + 2 \) dimensional flat spacetime that has a natural \( SO(4,2) = SU(2,2) \) rotation symmetry, but no translation symmetry. Although naively extra time dimensions lead to troublesome negative norm ghosts, our theories are physical because they include special gauge symmetries and kinematical constraints that insure ghost-free unitary theories.

After gauge fixing and solving the kinematic constraints, our theories produce conformal shadows in \( 3 + 1 \) flat dimensions in which \( SO(4,2) \) is the usual conformal group that includes Poincaré symmetry, and hence translation symmetry in \( 3 + 1 \) dimensional Minkowski spacetime. These conformal shadows coincide with previously established \( N = 1, 2, 4 \) supersymmetric massless renormalizable field theories with special forms of the superpotential.

In particular the famous \( N = 4 \) super Yang-Mills theory in \( 3 + 1 \) dimensions, that continues to attract a lot of interest, is now seen to have a parent theory in \( 4 + 2 \) dimensions, that naturally explains its exact conformal symmetry, and possibly some of its other properties as well.

An important aspect of 2T-physics is that it also produces many other shadows in \( 3 + 1 \) dimensions as explained in [16,17] in the worldline context and in [16,17] in the field theory.
context. By using the approach of [16][17] we can produce in a straightforward way other dual shadows of our $N = 1, 2, 4$ supersymmetric theories in various curved spacetimes, including Robertson-Walker, AdS$_4$, dS$_4$, AdS$_3 \times$ S$^1$, AdS$_2 \times$ S$^2$, any maximally symmetric spacetime, any conformally flat spacetime, some singular spacetimes, all in $3 + 1$ dimensions. All of these share the full SO$(4, 2)$ symmetry, as well as the full SU$(2, 2|N)$ supersymmetry of the parent theory, realized in different forms as a hidden symmetry in various spacetimes.

We expect that more shadows, that contain mass parameters as seen in the worldline theories, can also be obtained in field theory, thus arriving at very unusual realizations of the SU$(2, 2|N)$ symmetry. All shadows can be transformed into each other by the underlying Sp$(2,R)$ gauge symmetry which now plays the role of duality transformations in field theory [16][17]. It is expected that such duality properties of our theories can be used to explore non-perturbative or exact solutions of $N = 1, 2, 4$ supersymmetric Yang-Mills theories.

In particular one may now revisit previous studies of supersymmetric theories, including classical solutions, monopoles, instantons, Seiberg-Witten analysis [46], $N = 4$ dualities, AdS-CFT [47], etc., but now from the perspective of $4 + 2$ dimensions and using new tools in the context of 2T-physics. These will be explored in the future.

As in the case of the non-supersymmetric Standard Model in $4 + 2$ dimensions [14], we expect that the supersymmetric version produces a shadow that includes certain constraints on the structure of the field theory in $3 + 1$ dimensions that are not present in the usual approach in 1T field theory. In particular generating masses for the fields is not as straightforward as the ordinary 1T approach, and it requires the coupling of the dilaton and hence of supergravity in $d + 2$ dimensions (see footnote [5]). At this point gravity in 2T field theory has been constructed in $d + 2$ dimensions [18]. One of our future goals is to supersymmetrize it and couple it to the $N = 1, 2, 4$ theories constructed in this paper. It is expected that the resulting structures will provide a number of constraints on SUSY theories that could be of phenomenological interest in case the LHC discovers supersymmetry.

Another future direction is SUSY theories in $d + 2$ dimensions with $d \neq 4$. We remind the reader that $N = 4$ super Yang-Mills theory in $3 + 1$ dimensions is a reduced version of $N = 1$ super Yang-Mills theory in $9 + 1$ dimensions. Therefore, from the point of view of 2T-physics, it is natural to expect that there must exist a SYM theory in $10 + 2$ dimensions which can be compactified to our $N = 4$ SYM theory in $4 + 2$ dimensions presented in this paper. Such a theory breaks the 11-dimensional barrier for SUSY, but becomes physical with the extra
gauge symmetries and constraints supplied by 2T-physics. This will be discussed in the near future in a paper on supersymmetric theory in higher dimensions which is currently under preparation.

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