A Stable Boundary Integral Formulation of an Acoustic Wave Transmission Problem with Mixed Boundary Conditions

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July 4, 2019

Abstract

In this paper, we consider an acoustic wave transmission problem with mixed boundary conditions of Dirichlet, Neumann, and impedance type. The transmission interfaces may join the domain boundary in a general way independent of the location of the boundary conditions. We will derive a formulation as a direct, space-time retarded boundary integral equation, where both Cauchy data are kept as unknowns on the impedance part of the boundary. This requires the definition of single-trace spaces which incorporate homogeneous Dirichlet and Neumann conditions on the corresponding parts on the boundary. We prove the continuity and coercivity of the formulation by employing the technique of operational calculus in the Laplace domain.

Keywords: acoustic wave equation, transmission problem, impedance boundary condition, retarded potentials, convolution quadrature

1 Introduction

In physics and engineering there are many important applications where it is essential to obtain information on material properties inside (large) solid objects, e.g., the detection of oil reservoirs, the investigation of the interior of rocks and soil to understand its stability properties, or the assessment of the ice volume in glaciers to name just a few of them. For this purpose, typically, a wave is sent into the solid. Then the scattered wave is recorded and used to solve the governing mathematical equations for the quantity of interest.

Our goal is to employ retarded potential integral operators to reformulate the scalar wave equation as a system of space-time boundary integral equations (RPIE); standard references on this topic include [15], [6], [2], [14]. The Cauchy data of the wave (or boundary densities if an indirect formulation is employed) is determined as the solution of an equation which involves a system of retarded potential integral operators. To investigate well-posedness we employ the Laplace transform and prove continuity and coercivity with respect to the frequency variable. These techniques in the context of numerical analysis go back to the pioneering works [2], [10], [11], [7]; a monograph on this topic is [14] and some further developments can be found, e.g., in [8] and [3].

We emphasize that the derivation of coercive and continuous integral equations in the Laplace domain is the key for their discretization by convolution quadrature. However, here we will focus on the continuous formulation and prove its well-posedness.

The new mathematical aspect of our setting is the presence of an interface and general mixed boundary conditions of Dirichlet, Neumann, and impedance type. We do not impose restrictions on where the interface meets the domain boundary. In a first step, we use a single-trace ansatz (cf. [4]), i.e., we employ Cauchy data (with continuous values and continuous fluxes across the interface) and the full Calderón projector. We prove continuity and stability of the formulation which implies well-posedness in certain anisotropic space-time Sobolev spaces.

The paper is organized as follows.

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Figure 1: Cross section of the computational domain. The domains \( \Omega_1 \) and \( \Omega_2 \), corresponding to different materials, are separated by the interface \( \Gamma_{1,2} = \Gamma_J \). We set \( \Gamma_1 := \partial \Omega_1 \) and \( \Gamma_2 := \partial \Omega_2 \). The unbounded exterior domain is denoted by \( \Omega_0 \). The auxiliary domains \( \Omega_{\mathcal{D}}, \Omega_{\mathcal{N}} \) are used in the definition of \( X_0^{\text{single}} \) (cf. (4.2)).

In Section 2, we introduce the wave equation with transmission condition at an interface and mixed boundary conditions.

Section 3 is devoted to the derivation of the system of RPIEs: the retarded acoustic single and double layer potentials are defined and the corresponding boundary integral operators are introduced by applying the trace and normal trace operator to these potentials. We end up with a system of integral equations for the unknown Cauchy data. Note that we employ a single-trace ansatz which involves a single Cauchy data across the interface in accordance with the transmission conditions.

In Section 4, we propose to incorporate the impedance boundary condition by keeping both Cauchy data in the equation. The advantage of this approach is that only boundary integral operators are involved which are defined on closed surfaces. We will prove well-posedness of the system of integral equations by showing coercivity and continuity of this system of RPIEs. This allows us to determine the analyticity class of the Laplace-transformed system and implies existence and uniqueness.

2 Problem Statement

First, we describe the geometry of the computational domain and use Fig. 1 as an illustration. The domain \( \Omega \) is split into three parts: the disjoint open sets \( \Omega_0, \Omega_1, \Omega_2 \) such that \( \overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \), corresponding to two objects composed of different materials, and their interface \( \Gamma_{1,2} = \overline{\Omega_1} \cap \overline{\Omega_2} \); as a convention we set \( \Omega_0 = \mathbb{R}^3 \setminus \overline{\Omega} \) although this unbounded domain will not appear in the formulation of the integral equations but in the analysis. The results of this paper also hold if the problem to solve is the exterior problem, and the interface \( \Gamma_J \) is in the exterior domain, and when splitting \( \Omega \) in any number of subdomains; in order to simplify the notations, we restrict ourselves to the case of the interior problem and two domains.

For \( i \in \{0,1,2\} \) we set \( \Gamma_i := \partial \Omega_i \) (thus \( \Gamma_0 = \partial \Omega \)) and we employ the convention that \( n_i \) is the unit normal vector field at \( \partial \Omega_i \) pointing into the exterior of \( \Omega_i \). The skeleton manifold is defined by \( \Sigma := \Gamma_1 \cup \Gamma_2 \).

We also introduce a partition of the boundary, corresponding to different types of boundary conditions (see again Fig. 1): we split \( \Sigma = \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}} \cup \Gamma_I \cup \Gamma_J \); then we impose transmission conditions at \( \Gamma_J := \Gamma_{1,2} \), Dirichlet boundary conditions at \( \Gamma_D \), Neumann boundary conditions at \( \Gamma_N \), and an impedance condition at \( \Gamma_I \); we do not require \( \Gamma_{\mathcal{D}}, \Gamma_{\mathcal{N}}, \Gamma_I \) to be connected.

The functions \( a \) and \( \rho \) are defined on \( \Omega_0 \cup \Omega_1 \cup \Omega_2 \):

\[
  a|_{\Omega_\ell} = a_\ell, \quad \rho|_{\Omega_\ell} = \rho_\ell, \quad \ell = 0,1,2, \tag{2.1}
\]

where the material-dependent constant coefficients \( a_1, a_2, \rho_1, \rho_2 > 0 \) are extended to positive functions \( a_0(x), \rho_0(x) \) to the exterior domain \( \Omega_0 \), such that \( a, \rho \) are continuous across the interface \( \Gamma_0 \). The resulting trans-
mission Initial-Boundary value problem is

\[
\begin{aligned}
\rho_1^2 \partial_t^2 u_1 - a_1^2 \Delta u_1 &= 0 \text{ in } \Omega_1 \times [0, T], \\
\rho_2^2 \partial_t^2 u_2 - a_2^2 \Delta u_2 &= 0 \text{ in } \Omega_2 \times [0, T], \\
[u]_{\Gamma_J} &= \left[ a_2 \frac{\partial u}{\partial n} \right]_{\Gamma_J} = 0 \text{ on } \Gamma_J \times [0, T], \\
\frac{a_2}{\partial t} u - T \ast u &= d_1 \text{ on } \Gamma_1 \times [0, T], \\
\frac{a_2}{\partial n} u &= d_N \text{ on } \Gamma_N \times [0, T], \\
u(0, x) = \dot{u}(0, x) &= 0 \text{ in } \Omega,
\end{aligned}
\]  

(2.2a)  
(2.2b)  
(2.2c)  
(2.2d)  
(2.2e)  
(2.2f)  
(2.2g)

where * denotes the convolution in time. For the boundary data we assume \( g_D \in H^{1/2}(\Gamma_D), \ d_N \in H^{-1/2}(\Gamma_N), \ d_1 \in H^{-1/2}(\Gamma_1). \)

Here and in the following, we employ the shorthand \( \dot{u} \) for \( \partial_t u \). In the third equation in (2.2) the normal derivative \( \partial / \partial n \) can be taken either with respect to \( n_1 \) or \( n_2 \); \([.]_{\Gamma_J} \) denotes the jump of a function across the interface \( \Gamma_J \). The temporal convolution operator \( T \) is a Dirichlet-to-Neumann (DtN) operator or an approximation to it. The simplest approximation is given by impedance boundary conditions: \( T(t) = -a\rho_0(t) \), where \( \delta_0 \) is the Dirac distribution. At this point we are vague concerning the function spaces which are mapped by \( T(t) \) in a continuous way but postpone this to Section 4.1 (Assumption 4.1), where also a dissipative condition will be imposed on \( T \).

3 Space-time Boundary Integral Equations for the Wave Transmission Problem

In this section we will formulate single-trace boundary integral operators and associated Calderón projectors.

3.1 Background: Layer Potentials and Boundary Integral Operators

We recall retarded potentials on two-dimensional compact, orientable manifolds in \( \mathbb{R}^3 \). We start by fixing some notations:

We will write \( \Gamma_{i,S} := \Gamma_i \cap \Gamma_S \), for \( S \in \{D, N, I, J\} \), i.e. the index \( i \) corresponds to the domain \( \Omega_i \) while \( S \) indicates the type of boundary conditions imposed.

Recall the definition of \( a \) as in (2.1). Let \( i \in \{0, 1, 2\} \). For \( u \in H^1(\Omega_i), S \in \{D, N, I, J\} \), the Dirichlet (D) and Neumann (N) trace operators are denoted by \( \gamma_{D,i}, \gamma_{N,i} \) and are given by

\[
\gamma_{D,i} u := (u|_{\Omega_i}) |_{\Gamma_i}, \quad \gamma_{N,i} u := a_i \frac{\partial u_{\Gamma_i}}{\partial n_i}.
\]  

(3.1)

i.e. the index \( i \) means that the limit is taken from the subdomain \( \Omega_i \), and the unit normal vector \( n_i \) points outside \( \Omega_i \). We also need a notation for the case where the limit is taken from the unbounded complement \( \Omega_i^c := \mathbb{R}^3 \setminus \Omega_i \) (where the unit normal \( n_i \) still points outside \( \Omega_i \)):

\[
\gamma_{D,i}^c u := (u|_{\Omega_i^c}) |_{\Gamma_i}, \quad \gamma_{N,i}^c u := a_i \frac{\partial u_{\Omega_i^c}}{\partial n_i}.
\]

Finally, we will use the same symbols for the continuous extensions of the trace operators to appropriate Sobolev spaces.

For vector-valued functions \( w \), sufficiently smooth in \( \Omega_i \), we define the normal component trace by\(^2\)

\[
\gamma_{n,i} w := \langle n_i, \gamma_{D,i} w \rangle.
\]

(3.2)

\(^2\)We postpone the introduction of the relevant Sobolev spaces to Section 5.2.

\(^3\)For \( u, w \in C^3 \) we define \( \langle u, w \rangle := \sum_{j=1}^3 u_j w_j \).
Let $u$ be a function on $\mathbb{R}^3 \setminus \Sigma$; for $j \in \{0, 1, 2\}$, we assume that the traces $\gamma_{D,j}, \gamma_{N,j}, \gamma_{D,j}^0, \gamma_{N,j}^0$ applied to $u$ are well-defined. Then the jump $[\cdot]_{D,j}$ and co-normal jump $[\cdot]_{N,j}$ across $\Gamma_S$ are defined by

$$[u]_{D,j} := \gamma_{D,j} u - \gamma_{D,j}^0 u \quad \text{and} \quad [u]_{N,j} := \gamma_{N,j} u - \gamma_{N,j}^0 u.$$ 

The averages are defined by

$$\{u\}_{D,j} := \frac{1}{2} (\gamma_{D,j} u + \gamma_{D,j}^0 u) \quad \text{and} \quad \{u\}_{N,j} := \frac{a_j^2}{2} \left( \frac{\partial u|_{\Gamma_j}}{\partial n_j} \bigg|_{\Gamma_j} + \frac{\partial u|_{\Gamma_j}}{\partial n_j} \bigg|_{\Gamma_j} \right).$$

This allows us to introduce the following boundary integral operators. For the convolution quadrature, we apply the Laplace transform with respect to time and obtain operators which are defined by

$$\mathcal{K}_j \ast \varphi := \{S_j \ast \varphi\}_{D,j}, \quad \mathcal{K}_j' \ast \varphi := \{S_j \ast \varphi\}_{N,j}, \quad \mathcal{W}_j \ast \psi := -\{D_j \ast \psi\}_{N,j}.$$

For $j \in \{0, 1, 2\}$, we set $\mathcal{V}_j \ast \varphi := \{S_j \ast \varphi\}_{D,j}$, and $\mathcal{V}_j \ast \varphi := \{S_j \ast \varphi\}_{N,j}$, and $\mathcal{W}_j \ast \psi := -\{D_j \ast \psi\}_{N,j}$.

At this stage we are vague concerning the appropriate function spaces and mapping properties of trace operators; for the precise setting we refer to Section [3.2].

For $\kappa \in \mathbb{R}$, let

$$\mathbb{C}_\kappa := \{s \in \mathbb{C} \mid \text{Re} \ s > \kappa\}.$$ 

**Convention 3.1** Throughout this paper, $\sigma_0 > 0$ denotes a fixed positive constant. The constants in the estimates in this paper will depend continuously on $\sigma_0 \in \mathbb{R}_{>0}$ and $a_1, a_2, \rho_1, \rho_2 \in \mathbb{R}_{>0}$. These constants, possibly, tend to infinity if one or more of the quantities $\sigma_0, a_1, a_2, \rho_1, \rho_2$ tend to zero or infinity. We will suppress this dependence in our notation.

For the convolution quadrature, we apply the Laplace transform with respect to time and obtain operators in the frequency variable $s \in \mathbb{C}_0$. Thus, we end up with the Laplace transformed potentials for $(x, s) \in \mathbb{C}_0$.
$\mathbb{R}^3 \setminus \Gamma_i \times \mathbb{C}_0$ and $i \in \{0, 1, 2\}$:

$$S_i (x) \varphi(x) := \int_{\Gamma_i} \hat{k}_i(s, x-y) \varphi(y) \, ds_y, \quad \text{for } \hat{k}_i(s, z) := \frac{\exp \left(-i \frac{a_i|z|}{\alpha_i} \right)}{4 \pi a_i^2 |z|}. \tag{3.5a}$$

$$D_i (s) \psi(x) := \int_{\Gamma_i} \left( \gamma_{N;\iota} \hat{k}_i(s, x-y) \right) \psi(y) \, ds_y, \tag{3.5b}$$

and corresponding boundary integral operators on $\Gamma_j \times \mathbb{C}_0$ given by

$$V_j (s) \varphi := \{ S_j (s) \varphi \}_{D,j}, \quad K_j (s) \psi := \{ D_j (s) \psi \}_{D,j},$$

$$W_j (s) \psi := - \{ D_j (s) \psi \}_{N,j}. \quad \text{We recall the formal definition of the Laplace transform } \mathcal{L} \text{ and its inverse } \mathcal{L}^{-1} \text{ by} \tag{3.6}

\[
\hat{q}(s) := (\mathcal{L} q)(s) = \int_0^\infty e^{-st} q(t) \, dt \quad \text{and} \quad q(t) = (\mathcal{L}^{-1} \hat{q})(t) = \frac{1}{2\pi i} \int_{\sigma+i\mathbb{R}} e^{st} \hat{q}(s) \, ds.
\]

Note that the Laplace transform $\mathcal{L}$ applied to the convolution potentials satisfies

$$\mathcal{L} (S_i * \varphi)(s) = S_i (s) \hat{\varphi}(s), \quad \mathcal{L} (D_i \psi)(s) = D_i (s) \hat{\psi}(s),$$

and analogous relations hold for the boundary integral operators in the time and Laplace domain. It is also well known that the following jump relations hold (see [14, Section 1.3]):

\[
[S_j (s) \varphi]_{D,j} = 0, \quad [S_j (s) \varphi]_{N,j} = -\varphi, \quad [D_j (s) \psi]_{D,j} = \psi, \quad [D_j (s) \psi]_{N,j} = 0. \tag{3.7}
\]

### 3.2 Mapping Properties of the arising Boundary Integral Operators

To formulate the mapping properties for the arising integral operators we first introduce Sobolev spaces on manifolds with boundary – standard references are [1], [9]. For an open subset $\omega \subset \mathbb{R}^3$ we denote the $L^2(\omega)$-scalar product and norm by

$$\langle u, v \rangle_\omega := \int_{\omega} u(x) \overline{v(x)} \, dx \quad \text{and} \quad \| u \|_\omega := (u, u)^{1/2}_\omega,$$

and suppress the subscript $\omega$ if the domain is clear from the context.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with boundary $\Gamma$. The unit normal vector field $n$ on $\Gamma$ is chosen to point into the exterior of $\Omega$ and exists almost everywhere. For $\alpha \in \mathbb{R}_{\geq 0}$, let $H^\alpha(\Omega)$ denote the usual Sobolev spaces with norm $\| \cdot \|_{H^\alpha(\Omega)}$ and $H^\alpha_0(\Omega)$ is the closure of $C_0^\infty(\Omega) := \{ u \in C^\infty(\Omega) \mid \text{supp } u \subset \Omega \}$ with respect to the $\| \cdot \|_{H^\alpha(\Omega)}$ norm. Its dual space is denoted by $H^{-\alpha}(\Omega) := (H^\alpha_0(\Omega))^\prime$. On the boundary $\Gamma$, we define the Sobolev space $H^\alpha(\Gamma)$, $\alpha \geq 0$, in the usual way (see, e.g., [12]). Note that the range of $\alpha$ for which $H^\alpha(\Gamma)$ is defined may be limited, depending on the global smoothness of the surface $\Gamma$; for Lipschitz surfaces, $\alpha$ can be chosen in the range $[0, 1]$; for $\alpha < 0$, the space $H^\alpha(\Gamma)$ is the dual of $H^{-\alpha}(\Gamma)$.

We also define, for $R, S \in \{D, N, I\}$, $R \neq S$, the Sobolev spaces

$$\tilde{H}^{3/2}_{\pm 1/2}(\Gamma_R) := \left\{ \phi \in H^{3/2}_{\pm 1/2}(\Gamma_R) \mid \phi_{\text{out}}^{\text{ext}} \in H^{5/2}_{\pm 1/2}(\Gamma_R \cup \Gamma_S) \right\}, \tag{3.8}$$

where $\phi_{\text{out}}^{\text{ext}}$ denotes the extension of $\phi \in \tilde{H}^{3/2}_{\pm 1/2}(\Gamma_R)$ to $(\Gamma_R \cup \Gamma_S)$ by zero.

In the following we will recall mapping properties of the single and double layer potentials and their corresponding integral equations.

For $j \in \{0, 1, 2\}$, the proofs of the following propositions (Prop. 3.3 and the 3rd and 6th inequality in Prop. 3.2 (3.8)) go back to [2]. We have used here the estimates for the boundary integral operators as in [8].

\footnote{We employ the convention that, if $\varphi$ and $\hat{\varphi}$ appear in the same context, then $\hat{\varphi}$ is the Laplace transform of $\varphi$.}
Proposition 3.2 Let \( s \in \mathbb{C}_\sigma \) and recall (2.1). Then, for \( j \in \{0, 1, 2\} \), the operators \( S_j(s) \), \( D_j(s) \), \( V_j(s) \), \( K_j(s) \), \( K_j'(s) \), \( W_j(s) \), satisfy the following mapping properties: for all \( \Phi \in H^{-1/2}(\Gamma_j) \) and \( \Psi \in H^{1/2}(\Gamma_j) \) there is some constant \( C \) independent of \( s \) such that

\[
S_j(s) : H^{-1/2}(\Gamma_j) \to H^1(\mathbb{R}^3), \quad \|S_j(s)\Phi\|_{H^1(\mathbb{R}^3)} \leq C |s| \|\Phi\|_{H^{-1/2}(\Gamma_j)},
\]

\[
D_j(s) : H^{1/2}(\Gamma_j) \to H^1(\mathbb{R}^3 \setminus \Gamma_j), \quad \|D_j(s)\Psi\|_{H^1(\mathbb{R}^3 \setminus \Gamma_j)} \leq C |s|^{3/2} \|\Psi\|_{H^{1/2}(\Gamma_j)},
\]

\[
V_j(s) : H^{-1/2}(\Gamma_j) \to H^{1/2}(\Gamma_j), \quad \|V_j(s)\Phi\|_{H^{1/2}(\Gamma_j)} \leq C |s| \|\Phi\|_{H^{-1/2}(\Gamma_j)},
\]

\[
K_j(s) : H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j), \quad \|K_j(s)\Psi\|_{H^{-1/2}(\Gamma_j)} \leq C |s|^{3/2} \|\Psi\|_{H^{1/2}(\Gamma_j)}, \tag{3.8}
\]

\[
K_j'(s) : H^{-1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j), \quad \|K_j'(s)\Phi\|_{H^{-1/2}(\Gamma_j)} \leq C |s|^{3/2} \|\Phi\|_{H^{-1/2}(\Gamma_j)},
\]

\[
W_j(s) : H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j), \quad \|W_j(s)\Psi\|_{H^{-1/2}(\Gamma_j)} \leq C |s|^2 \|\Psi\|_{H^{1/2}(\Gamma_j)}.
\]

We denote by \( \langle \cdot, \cdot \rangle_{\Gamma_j} \) the dual pairing between \( H^{1/2}(\Gamma_j) \) and \( H^{-1/2}(\Gamma_j) \) (without complex conjugation) so that \( \langle u, \overline{v} \rangle_{\Gamma_j} \) is the continuous extension of the \( L^2(\Gamma_j) \) scalar product. We can thus also introduce the symmetric and skew-symmetric dual pairing: for \( \phi = (\phi_D, \phi_N), \psi = (\psi_D, \psi_N) \in H^{1/2}(\Gamma_i) \times H^{-1/2}(\Gamma_i) \)

\[
\langle \phi, \psi \rangle_{\Gamma_j}^+: = \langle \phi_D, \psi_N \rangle_{\Gamma_j} + \langle \phi_N, \psi_D \rangle_{\Gamma_j},
\]

\[
\langle \phi, \psi \rangle_{\Gamma_j}^-: = \langle \phi_D, \psi_N \rangle_{\Gamma_j} - \langle \phi_N, \psi_D \rangle_{\Gamma_j}. \tag{3.9a}
\]

Proposition 3.3 Let \( s \in \mathbb{C}_\sigma \) and recall (2.1). Then, for \( j = 1, 2 \), the operator

\[
A_{s,j} := \begin{bmatrix} -K_j(s) & sV_j(s) \\ \frac{1}{s}W_j(s) & K_j'(s) \end{bmatrix}
\]

satisfies the coercivity estimate

\[
\text{Re} \left\langle A_{s,j} \left( \begin{array}{c} \psi \\ \varphi \end{array} \right), \left( \begin{array}{c} \overline{\psi} \\ \overline{\varphi} \end{array} \right) \right\rangle_{\Gamma_j} \geq \beta \min \left\{ 1, |s|^2 \right\} \frac{\text{Re} s}{|s|^2} \left( \|\varphi\|^2_{H^{1/2}(\Gamma_j)} + \|\psi\|^2_{H^{-1/2}(\Gamma_j)} \right),
\]

for all \( \varphi, \psi \in H^{1/2}(\Gamma_j) \times H^{-1/2}(\Gamma_j) \), for some \( \beta > 0 \) and for all \( s \in \mathbb{C}_\sigma \).

Proof. Fix \( j \in \{1, 2\} \). A straightforward calculation shows that

\[
\left\langle A_{s,j} \left( \begin{array}{c} \varphi \\ \psi \end{array} \right), \left( \begin{array}{c} \overline{\varphi} \\ \overline{\psi} \end{array} \right) \right\rangle_{\Gamma_j} = \left\langle \left( \begin{array}{c} \varphi \\ -\overline{\psi} \end{array} \right), B(s) \left( \begin{array}{c} \psi \\ \varphi \end{array} \right) \right\rangle_{\Gamma_j}
\]

for \( B(s) := \begin{bmatrix} sV_j(s) & K_j(s) \\ -K_j'(s) & \frac{1}{s}W_j(s) \end{bmatrix} \).

This operator was analyzed in (3.1): it maps \( H^{-1/2}(\Gamma_j) \times H^{1/2}(\Gamma_j) \) continuously into \( H^{1/2}(\Gamma_j) \times H^{-1/2}(\Gamma_j) \) and satisfies the coercivity estimate

\[
\text{Re} \left\langle \left( \begin{array}{c} \varphi \\ \psi \end{array} \right), B(\overline{s}) \left( \begin{array}{c} \overline{\varphi} \\ \overline{\psi} \end{array} \right) \right\rangle_{\Gamma_j} \geq \beta \min \left\{ 1, |s|^2 \right\} \frac{\text{Re} s}{|s|^2} \left( \|\varphi\|^2_{H^{1/2}(\Gamma_j)} + \|\psi\|^2_{H^{-1/2}(\Gamma_j)} \right),
\]

for all \( \varphi, \psi \in H^{1/2}(\Gamma_j) \times H^{-1/2}(\Gamma_j) \).

3.3 Representation Formula

Let \( \Omega_i \) be as in Section 2 and let \( \Gamma_i = \partial \Omega_i \) for \( i = 0, 1, 2 \) with outward unit normal field \( n_i \). Note that the trace operators \( \gamma_{D,i}, \gamma_{N,i} \) can be extended to continuous operators acting on functions in the Sobolev space \( H(\Delta, \Omega_i) := \{ u \in H^1(\Omega_i) \mid \Delta u \in L^2(\Omega_i) \} \). We collect the range of these traces into the space of Cauchy traces, and the multi-trace space:

\[
X(\Gamma_i) := H^{1/2}(\Gamma_i) \times H^{-1/2}(\Gamma_i) \quad \text{and} \quad X^\text{mult} := X(\Gamma_1) \times X(\Gamma_2),
\]
and equip these spaces with the graph norm:

\[ \|u_i\|_{X(T_i)} := \left( \|u_{i,D}\|_{H^{1/2}(\Gamma_i)}^2 + \|u_{i,N}\|_{H^{-1/2}(\Gamma_i)}^2 \right)^{1/2} \]

for \( u_i = (u_{i,D}, u_{i,N}) \in X(\Gamma_i) \)

The single trace space is a subspace of \( X^{\text{mult}} \) and defined by:

\[ X^{\text{single}} := \left\{ \left( \frac{\phi_{1,i}}{\phi_{1,2}} \right)^2 \in X^{\text{mult}} \mid \exists \left( v \in H^1(\mathbb{R}^3) \right) w \in H(\mathbb{R}^3, \text{div}), \forall i = 1, 2 \left( \phi_{i,1} = \gamma_{D,i} v \right), \left( \phi_{i,2} = \gamma_{N,i} w \right) \right\}. \]  

(3.11)

In the context of the wave equation, these (spatial) trace spaces will be considered as the spaces of values of time-dependent functions (distributions). To define the relevant function space we first consider the Schwartz class

\[ \mathcal{S}(\mathbb{R}) := \left\{ \varphi \in C^\infty(\mathbb{R}) \mid \forall k \in \mathbb{N}_0, \forall p \in \mathbb{P}(\mathbb{R}) : p\varphi^{(k)} \in L^\infty(\mathbb{R}) \right\}, \]

where \( \mathbb{P}(\mathbb{R}) \) denote the space of polynomials (with complex coefficients). \( \mathcal{S}(\mathbb{R}) \) can be equipped with a metric that makes this space complete. A tempered distribution with values in a Banach space \( X \) is a continuous linear map \( f: \mathcal{S}(\mathbb{R}) \rightarrow X \). A causal distribution with values in \( X \) is a tempered \( X \)-valued distribution such that

\[ f(\varphi) = 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \text{ such that } \text{supp} \varphi \subset (-\infty, 0) \]

and we write

\[ f \in \text{CT}(X), \quad \text{CT}(X): \text{space of causal distributions with values in } X \]

following the notation in \( [14] \).

**Definition 3.4** The space \( \text{TD}(X) \) consists of all (possibly distributional) derivatives of continuous causal \( X \)-valued functions with, at most, polynomial growth.

The corresponding Cauchy trace operator are given by

\[ \mathcal{C}_i: H(\Delta, \Omega_i) \rightarrow X(\Gamma_i), \quad \mathcal{C}_i(v) = (\gamma_{D,i} v, \gamma_{N,i} v)^\top, \]

\[ \mathcal{C} = (\mathcal{C}_i)^2: H(\Delta, \Omega_1) \times H(\Delta, \Omega_2) \rightarrow X^{\text{mult}}. \]

It is known \([5]\) Lem. 3.5] that the range of \( \mathcal{C}_i \) is dense in \( X(\Gamma_i) \). Since the spaces \( H^{1/2}(\Gamma_i) \) and \( H^{-1/2}(\Gamma_i) \) are dual to each other, we have that the Cauchy trace spaces are in self-duality with respect to the symmetric dual pairing \( \langle \cdot, \cdot \rangle_{\Gamma_i} \).

We employ the direct method to transform the wave equation into a space-time boundary integral equation and start with the Kirchhoff representation formula. The key potential is given by

\[ (\mathcal{G}_i * \phi)(t, x) := \int_0^t \langle \mathcal{C}_i k_i(t - \tau, x - \cdot), \phi(\tau) \rangle_{\Gamma_i} d\tau \]

for \( \phi \in \text{TD}(X(\Gamma_i)) \) and \( k_i \) as in \([33]\)

\[ k_i(t, z) := \frac{\delta_0 \left( t - \frac{z^2}{\sqrt{\rho_i}}, \|z\| \right)}{4\pi \rho_i \|z\|^3}. \]

Then, every \( u_i \in \text{TD}(H^1(\Delta, \Omega_i)) \) that satisfies \( \rho_i^2 \Delta u_i - a_i^2 \Delta u_i = 0 \) and \( u_i(0) = \partial_t u_i(0) = 0 \) also satisfies the representation formula

\[ u_i = \mathcal{G}_i * \mathcal{C}_i u_i. \]

We introduce the Calderón projector \( \mathcal{P}_i(t): X(\Gamma_i) \rightarrow X(\Gamma_i) \) by

\[ \mathcal{P}_i(t) := \mathcal{C}_i \mathcal{G}_i(t) \]

\[ \text{TD} \] for “time domain”.

---

The components of \( \phi_i \) are denoted by \( \phi_{i,1}, \phi_{i,2} \).
and recall the property: \( u_i \in TD \left( H^1(\Delta, \Omega_i) \right) \) solves the homogeneous wave equation \( \rho_i^2 \partial_t^2 u_i - a_i^2 \Delta u_i = 0 \) in \( \Omega_i \) and \( u_i(0) = \partial_t u_i(0) = 0 \) if and only if
\[
(\mathcal{P}_i(\cdot) - \delta_i) \ast C_i u_i(\cdot) = 0. \tag{3.12}
\]
This equation will be our starting point for the formulation of problem (2.2) as a system of integral equations. More precisely we transform this equation to the Laplace domain since the temporal discretization by convolution quadrature is defined on the “Laplace side”. The Laplace transform of (3.12) is given by
\[
(\mathcal{P}_i(s) - I) C_i \hat{u}_i(s) = 0, \tag{3.13}
\]
where
\[
\mathcal{P}_i := C_i G_i \text{ and } G_i(\mathbf{\hat{w}})(s, x) := \left\langle C_i \hat{k}_i(s, x - \cdot), \mathbf{\hat{w}}(s) \right\rangle_{\Gamma_i}.
\]
Finally, the Calderón operator is given by \( A_i(s) = \mathcal{P}_i(s) - \frac{1}{2} I \) (with the identity operator \( I \)) and can be expressed by the classical four boundary integral operators as
\[
A_i = \begin{bmatrix} -K_i(s) & V_i(s) \\ W_i(s) & K'_i(s) \end{bmatrix} \tag{3.14}
\]
It turns out, that this form of \( A_i \) is not optimal from the viewpoint of stability. We employ a further transformation and first define the frequency dependent diagonal matrix
\[
D_i := \text{diag} \left[ s^{1/2}, s^{-1/2} \right] \in \mathbb{C}^{2 \times 2}, C_{s,i} := D_i C_i \text{ and } C_s := \text{diag} \left( D_s, D_s \right) C.
\]
We also introduce the operator \( A_s := \text{diag}(A_{s,1}, A_{s,2}) \), with \( A_{s,i} \) as in (3.10), i.e. \( A_{s,i} = D_s A_i D_s^{-1} \). Then, (3.13) can be written in the form
\[
\left( A_s - \frac{1}{2} I \right) C_s \hat{u} = 0.
\]

4 Space-time Boundary Integral Equations for the Wave Transmission Problem with Mixed Boundary Conditions

To deal with problem (2.2) we incorporate Dirichlet and Neumann boundary conditions into the space \( X^\text{single}. \) For this we extend the Dirichlet and Neumann parts of the boundary \( \Gamma_S \) to a closed boundary \( \Gamma_Z \) (cf. Fig. I) which is the boundary of a bounded domain \( \Omega_Z \subset \Omega_0 \) such that \( \Omega_Z \cap \Sigma \setminus \Gamma_Z = \emptyset \), for \( S \in \{D, N\} \). Then we set
\[
H_D^1(\mathbb{R}^3) := \left\{ v \in H^1(\mathbb{R}^3 \setminus \Omega_Z) \mid v|_{\Gamma_Z^D} = 0 \right\}, \tag{4.1a}
\]
\[
H_N(\mathbb{R}^3, \text{div}) := \left\{ w \in H(\mathbb{R}^3 \setminus \Omega_Z, \text{div}) \mid w|_{\Gamma_Z^N} = 0 \right\}, \tag{4.1b}
\]
and define the space of Cauchy traces of global fields, whose Dirichlet and Neumann components vanish on \( \Gamma_D \) and \( \Gamma_N \) respectively:
\[
X_{s,j}^\text{single} := \left\{ \left( \phi_{1,D}, \phi_{1,N} \right) \right\}_{i=1}^2 \in X^\text{single} \mid \exists \left( \begin{array}{c} v \\ w \end{array} \right) \in H_D(\mathbb{R}^3) \setminus \Omega_Z, \text{div}, \forall i = 1, 2 : \begin{cases} \phi_{i,D} = \gamma_0^j w \\ \phi_{i,N} = \gamma_0^j w \end{cases} \right\}. \tag{4.2}
\]

4.1 First-kind Boundary Integral Equations

Assume that \( u \) solves (2.2). Then, the Laplace transform \( \hat{u} \) with \( \hat{u}_j := \hat{u}|_{\Gamma_j}, j = 1, 2 \), satisfies
\[
\left( A_{s,j} - \frac{1}{2} I \right) C_{s,j} \hat{u}_j = 0 \text{ in } X(\Gamma_j), \quad j = 1, 2 \quad \text{and} \quad \forall s \in C_{\sigma_0}.
\]
Note that the transmission conditions (third equation in (2.2)) are built already in the function space \( X^\text{single}; \) we take into account the boundary conditions on \( \Gamma_D \) and \( \Gamma_N \) next.

\footnote{This space naturally arises when offsetting Cauchy traces with the boundary data.}
4.2 Treatment of the Neumann and Dirichlet boundary conditions

To obtain a variational formulation for the unknown Cauchy data of the transmission problem \( (2.2) \) with balanced test and trial spaces we consider an offset function \( b = b(s) \in H^1(\Delta, \mathbb{R}^3) \) such that \( g \). We are now in the position to formulate appropriate assumptions on \( T \).

We set \( \tilde{u}_0(s) := C_s(\tilde{u}(s) - \hat{b}(s)) \in X^{single}_{0} \).

The boundary conditions \( (2.2) \) for the new variable \( u_0 = u - b \) with \( b := \mathcal{L}^{-1}\hat{b} \) now read for any time in \([0, T] \)

\[
u_0|_{\Gamma_D} = 0, \quad -a^{-2} \frac{\partial u_0}{\partial n}|_{\Gamma_N} = 0, \quad -a^{-2} \frac{\partial u_0}{\partial n}|_{\Gamma_1} - T \cdot \tilde{u}_0|_{\Gamma_1} = d_1 - b_1|_{\Gamma_1} + T \cdot \hat{b}_0|_{\Gamma_1}.
\] (4.3)

Since \( \tilde{u}_0 \) vanishes on \( \Gamma_D \) and \( \frac{\partial u_0}{\partial n} \) vanishes on \( \Gamma_N \), the function \( u_0|_{\Gamma_1} \) belongs to \( H^1_{\mathcal{D}}(\Gamma_1) \times H^{-1/2}_{\mathcal{D}}(\Gamma_1) \). The following duality holds (the proof is a slight generalization of the well-known duality of \( \mathcal{D}^1(\Gamma) \) and \( H^{-1/2}(\Gamma) \), which can be found e.g. in \([12, \text{Theorem 3.14}]\)\).

\[
\mathring{\mathcal{H}}_{\mathcal{D}}^{\pm1/2}(\Gamma_1) = \mathcal{H}_{\mathcal{D}}^{\mp1/2}(\Gamma_1)^\prime.
\]

We also define the pairings, for \( \tau \in \{+, -\} \)

\[
(\Phi, \Psi)_\Gamma^\tau = \sum_{j=1}^{2} \left( (\phi_{j,D}, \psi_{j,N})_{\Gamma,j,1} + \tau \langle \psi_{j,D}, \phi_{j,N} \rangle_{\Gamma,j,1} \right), \quad \text{for any } \Phi, \Psi \in X^{\mult}.
\]

We are now in the position to formulate appropriate assumptions on \( T \).

**Assumption 4.1** The operator \( T \) in \((4.3)\) is the inverse Laplace transform of a bounded linear transfer operator \( \hat{T}(s) : H^1_{\mathcal{D}}(\Gamma_1) \rightarrow H^{-1/2}_{\mathcal{D}}(\Gamma_1) \) depending analytically on \( s \in \mathbb{C}_0 \), more precisely

\[
T \cdot \varphi = \mathcal{L}^{-1} \left( \hat{T}(\varphi) \right)
\]

for any function \( \varphi \in \mathcal{T} \left( \mathcal{H}^{1/2}_{\mathcal{D}}(\Gamma_1) \right) \); \( \hat{T}(s) \) satisfies the following (dissipative) sign property:

\[
\text{Re} \left( \hat{T}(s) \dot{\varphi}, \varphi \right)_{\Gamma_1} \leq 0 \quad \forall \varphi \in \mathcal{H}^{1/2}_{\mathcal{D}}(\Gamma_1).
\] (4.4)

**Remark 4.2** If the transfer operator \( T \) is, in the case of impedance boundary condition, given by (minus) the identity, \( \hat{T} = -I \), then Assumption 4.1 is satisfied trivially since \( \mathcal{H}^{1/2}_{\mathcal{D}}(\Gamma_1) \subseteq H^{-1/2}_{\mathcal{D}}(\Gamma_1) \).

If \( \hat{T}_0 \) denotes the standard DtN operator on \( \partial \Omega \), one could define \( \hat{T} := \mathcal{Z}' \hat{T}_0 \mathcal{Z} \), where \( \mathcal{Z} : \mathcal{H}^{1/2}_{\mathcal{D}}(\Gamma_1) \rightarrow H^1_{\mathcal{D}}(\Gamma_0) \) is a linear and bounded extension operator, e.g., the minimal \( H^1_{\mathcal{D}}(\Gamma_0) \) extension and the projection \( \mathcal{Z}' : H^{-1/2}(\Gamma_1) \rightarrow H^{-1/2}_{\mathcal{D}}(\Gamma_1) \) is its dual. The sign condition then is inherited from the well-known sign property (see \([34, \text{Eq. (2.6.93)}]\)) of \( \hat{T}_0 \) via

\[
\text{Re} \left( \hat{T}_0 \varphi_{0,ext}, \varphi_{0,ext} \right)_{\Gamma_0} \leq 0 \quad \forall \varphi_{0,ext} \in \mathcal{H}^{1/2}_{\mathcal{D}}(\Gamma_1),
\]

where \( \varphi_{0,ext} = \text{extension of } \varphi \in \mathcal{H}^{1/2}_{\mathcal{D}}(\Gamma_1) \) to \( \Gamma_0 \) by zero.

Let the component operator \( L_j \) be defined by \( L_j \Phi = \phi_j \) for \( \Phi = (\phi_1, \phi_2) \in X^{\mult} \). This allows us to define the dual skew-symmetric and symmetric pairing on the skeleton \( \Sigma \) for \( \tau \in \{+, -\} \) by (recall \( 3.9a, 3.9b \))

\[
(\Phi, \Psi)_\Sigma^\tau := \sum_{j=1}^{2} \left( L_j \Phi, L_j \Psi \right)^\tau_{\Gamma_1}, \quad \forall \Phi, \Psi \in X^{\mult}.
\]
The sesquilinear form 

\[
\langle \Phi, \Psi \rangle_{\Sigma} = \langle \Phi, \Psi \rangle_{\Gamma_1}^T.
\]  

(4.5)

Let \( \tau \in \{+, -\} \). Then, the solution of (2.2) satisfies the variational problem (for the unknown \( \hat{u}_0 \in X_0^{\text{single}} \))

\[
\left\langle \left( A_s - \frac{1}{2} \right) \hat{u}_0, \Psi \right\rangle_{\Sigma} = -\left\langle \left( A_s - \frac{1}{2} \right) C \hat{b}, \Psi \right\rangle_{\Sigma} \quad \forall \Psi \in X_0^{\text{single}}.
\]

Lemma 4.3 The sesquilinear form \( (\hat{v}, \hat{w}) \mapsto \langle A_s \hat{v}, \overline{\hat{w}} \rangle_{\Sigma}^+ \) is continuous and coercive: there exist constants \( \eta, \zeta > 0 \), possibly depending on \( s_0 \) but not on \( s \in \mathbb{C} \), such that

\[
\begin{align*}
\left| \langle A_s \hat{v}, \overline{\hat{w}} \rangle_{\Sigma}^+ \right| &\leq \eta |s|^2 \left( \| \hat{v} \|_X \| \hat{w} \|_X \right) \quad \forall \hat{v}, \hat{w} \in X^{\text{mult}}, \\
\text{Re} \langle A_s \hat{v}, \overline{\hat{w}} \rangle_{\Sigma}^+ &\geq \zeta \frac{|s|^2}{|s|^2} \| \hat{v} \|_X^2 \quad \forall \hat{v} \in X^{\text{mult}}.
\end{align*}
\]

Proof. We employ the mapping properties\(^{11}\) as in (3.8) and obtain, for any \( \hat{v}, \hat{w} \in X^{\text{mult}} \)

\[
\begin{align*}
\sum_{j=1}^2 \langle A_{s,j} L_j \hat{v}, L_j \overline{\hat{w}} \rangle_{X(\Gamma_j)}^+ &\leq \max_{j \in \{1, 2\}} \left( \| \hat{v}_{j,D} \|_{H^1(\Gamma_j)}, \| \hat{v}_{j,N} \|_{H^{-1}(\Gamma_j)} \right) \left( \| \hat{w}_{j,D} \|_{H^1(\Gamma_j)}, \| \hat{w}_{j,N} \|_{H^{-1}(\Gamma_j)} \right) \\
&\leq C \max \{ |s|^2, |s| \} \| \hat{v} \|_X \| \hat{w} \|_X,
\end{align*}
\]

where the constant \( C \) is the same as in (3.8); since \( |s| \geq s_0 \), taking \( \eta = 2 \min \{ 1, 1/s_0 \} \) the continuity estimate follows.

The coercivity directly follows from Prop. 4.3.

Remark 4.4 The properties of \( \langle A_s, \cdot \rangle_{\Sigma}^+ \) as stated in Lemma 4.3 trivially carry over to its restriction to any subspace of \( X^{\text{mult}} \). For our application, the subspace \( X^{\text{single}} \) will be of particular interest.

Lemma 4.5 The sesquilinear form \( a_0 : X^{\text{mult}} \times X^{\text{mult}} \to \mathbb{C} \), given by

\[
a_0 (\hat{v}, \hat{w}) := \left\langle \left( A_s - \frac{1}{2} \right) \hat{v}, \overline{\hat{w}} \right\rangle_{X^{\text{mult}}}^+,
\]

(4.6)

is continuous: there exists a constant \( \eta > 0 \) independent of \( s \) such that

\[
|a_0 (\hat{v}, \hat{w})| \leq \left( \frac{1}{2} + \eta \max \{ |s|^2, |s| \} \right) \| \hat{v} \|_X \| \hat{w} \|_X \quad \forall \hat{v}, \hat{w} \in X^{\text{mult}}.
\]

Proof. For the second term in (4.6) related to \( \left( A_s - \frac{1}{2} \right) \) we get

\[
\frac{1}{2} \left| \left\langle \hat{v}, \overline{\hat{w}} \right\rangle_{X^{\text{mult}}}^+ \right| \leq \frac{1}{2} \| \hat{v} \|_X \| \hat{w} \|_X.
\]

For the term in (4.6) related to \( A_s \), we use Lemma 1.3 and the continuity estimate follows.

\(^{11}\)We write \( |sV_j| \) short for the natural operator norm, i.e., \( |sV_j| = \| sV_j \|_{H^{1/2}(\Gamma_j)} \) and apply this convention also for \( \| K_j \|, \| K_j' \|, \| \frac{1}{s} W_j \| \), denoting the natural operator norms according to the mapping properties listed in (3.8).
4.3 Treatment of the Impedance Boundary Condition

Finally, we incorporate the impedance boundary conditions (fifth equation in (2.2)).

We start by defining, for \( \tilde{v} \in X_0^{\text{single}} \), the functions \( \hat{v}_D, \hat{v}_N \) on \( \Gamma_1 \subseteq \partial \Omega \) such that:

\[
\hat{v}_D|_{\Gamma_{1,1}} := \hat{v}_{D,1} \quad \hat{v}_N|_{\Gamma_{1,1}} := \hat{v}_{N,1}.
\]

Due to the definition of \( X_0^{\text{single}} \), we have \( \hat{v}_D \in \tilde{H}^{1/2}_D(\Gamma_1), \hat{v}_N \in \tilde{H}^{-1/2}_N(\Gamma_1) \).

We treat the Dirichlet and Neumann boundary condition as explained in Section 4.2 but incorporate the impedance condition of (4.3) keeping both the Dirichlet and Neumann trace as unknowns in the resulting equations. Recall the impedance condition (cf. (4.3)):

\[
\hat{u}_N^0(s) - \hat{T}(s)\hat{u}_D^0(s) = s^{-1/2}\hat{d}_1(s) - s^{-1/2}\gamma_{N,0,1}\hat{b}(s) + s^{1/2}\hat{T}(s)\gamma_{D,0,1}\hat{b}(s).
\]

This gives rise to the definition of the sesquilinear form \( a^{\text{imp}}_s : X_0^{\text{single}} \times X_0^{\text{single}} \rightarrow \mathbb{C} \) and right-hand side functional \( \ell_{\text{imp},s} : X_0^{\text{single}} \rightarrow \mathbb{C} \) defined by

\[
a^{\text{imp}}_s(\tilde{v}, \tilde{w}) := \left\{ \hat{v}_N - \hat{T}(s)\hat{v}_D, \overline{\hat{w}_D} \right\}_{\Gamma_1},
\]

\[
\ell^{\text{imp}}_s(\tilde{w}) := \left\{ s^{-1/2}\hat{d}_1(s) - s^{-1/2}\gamma_{N,0,1}\hat{b}(s) + s^{1/2}\hat{T}(s)\gamma_{D,0,1}\hat{b}(s), \overline{\hat{w}_D} \right\}_{\Gamma_1}.
\]

Problem 4.6 (Mixed Formulation of Acoustic Mixed Transmission Problem) Find \( \tilde{u}_0 \in X_D^{\text{single}} \) such that

\[
a^{\text{mix}}_s(\tilde{u}_0, \tilde{w}) = \ell^{\text{mix}}_s(\tilde{w}) \quad \forall \tilde{w} \in X_0^{\text{single}},
\]

where \( a^{\text{mix}}_s := a_0 + a^{\text{imp}}_s \) and \( \ell^{\text{mix}}_s := \ell_0 + \ell^{\text{imp}}_s \).

Next, we will prove continuity and coercivity of \( a^{\text{mix}}_s \).

Theorem 4.7 The sesquilinear form \( a^{\text{mix}}_s \) is coercive: for the constant \( \zeta > 0 \) as in Lemma 4.3, it holds

\[
\text{Re} a^{\text{mix}}_s(\tilde{v}, \tilde{v}) \geq \frac{\zeta}{|s|} \| \tilde{v} \|^2_X \quad \forall \tilde{v} \in X_0^{\text{single}}.
\]

Proof. From (4.4), (4.6) and the definition of \( a^{\text{imp}}_s \) we obtain

\[
\text{Re} a^{\text{mix}}_s(\tilde{v}, \tilde{v}) = \text{Re} \left( A_s \tilde{v}, \tilde{v} \right)_{\Sigma} + a^{\text{imp}}_s(\tilde{v}, \tilde{v}) - \frac{1}{2} \text{Re} \left\{ \tilde{v}, \tilde{v} \right\}_{\Gamma_1}
\]

\[
= \text{Re} \left( A_s \tilde{v}, \tilde{v} \right)_{\Sigma} + \frac{1}{2} \text{Re} \left\{ \hat{v}_N - \hat{T}(s)\hat{v}_D, \overline{\hat{v}_D} \right\}_{\Gamma_1} - \frac{1}{2} \left\{ \tilde{v}, \tilde{v} \right\}_{\Gamma_1}
\]

\[
= \text{Re} \left( A_s \tilde{v}, \tilde{v} \right)_{\Sigma} - \text{Re} \left( \hat{v}_N - \hat{T}(s)\hat{v}_D, \overline{\hat{v}_D} \right)_{\Gamma_1}.
\]

We employ Assumption (4.4) and Lemma 4.3 to obtain

\[
\text{Re} a^{\text{mix}}_s(\tilde{v}, \tilde{v}) \geq \langle A_s \tilde{v}, \tilde{v} \rangle_{\Sigma} \geq \frac{\zeta}{|s|} \| \tilde{v} \|^2_X.
\]

Theorem 4.8 The sesquilinear form \( a^{\text{mix}}_s \) is continuous: there exists a constant \( \eta > 0 \) independent of \( s \) such that

\[
|a^{\text{mix}}_s(\tilde{v}, \tilde{w})| \leq \left( \| \hat{T}(s) \|_{\tilde{H}^{-1/2}(\Gamma_1)} + \| \gamma_{D,0,1} \|_{\tilde{H}^{1/2}(\Gamma_1)} \right) \| \tilde{v} \|_X \| \tilde{w} \|_X \quad \forall \tilde{v}, \tilde{w} \in X_0^{\text{single}}.
\]

Proof. The definition of \( a^{\text{mix}}_s \) implies

\[
|a^{\text{mix}}_s(\tilde{v}, \tilde{w})| \leq |a_{0,s}(\tilde{v}, \tilde{w})| + \left| \langle \hat{T}(s)\hat{v}_D, \overline{\hat{w}_D} \rangle_{\Gamma_1} \right|.
\]
Lemma 4.5 gives an estimate for the first term, while the continuity of the second term follows from the continuity of $T$:

\[
\left| \left\langle \hat{T}(s)\hat{v}_D, \hat{w}_D \right\rangle \right|_{\Gamma_1} \leq \|\hat{T}(s)\|_{H^{-1/2}(\Gamma_1)} \vDash_{H^{-1/2}(\Gamma)} \hat{\varphi} \|_{X} \vDash_{\hat{w}} \|_{X}.
\]

**Remark 4.9** It is also possible to use (4.3) to eliminate the Neumann data on $\Gamma_I$. This would lead to a system of integral equations containing the minimal number of unknowns: the Neumann data on $\Gamma_D$, the Dirichlet data on $\Gamma_N \cup \Gamma_I$, the Dirichlet and Neumann data on $\Gamma_J$. The drawback is that a function $\hat{d}$ on $\Omega$ has to be constructed, which provides a skeleton extension of the impedance data; more precisely, $\hat{d}$ must satisfy

\[
\begin{cases}
  s^{-1/2}\gamma_{N,0,1}\hat{d} - s^{1/2}\hat{T}\gamma_{D,0,1}\hat{b} = s^{-1/2}\hat{d}_I - s^{-1/2}\gamma_{N,0,1}\hat{b} + s^{1/2}\hat{T}\gamma_{D,0,1}\hat{b}, \\
  \left[\frac{\hat{d}}{\partial n}\right]_{\Gamma_J} = 0, \\
  \gamma_{N,0,0}\hat{d} = 0, \\
  \gamma_{D,0,0}\hat{d} = 0.
\end{cases}
\]

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