Remark on the energy-momentum tensor in the lattice formulation of 4D $\mathcal{N} = 1$ SYM

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Abstract

In a recent paper, arXiv:1209.2473 [1], we presented a possible definition of the energy-momentum tensor in the lattice formulation of the four-dimensional $\mathcal{N} = 1$ supersymmetric Yang–Mills theory, that is conserved in the quantum continuum limit. In the present Letter, we propose a quite similar but somewhat different definition of the energy-momentum tensor (that is also conserved in the continuum limit) which is superior in several aspects: In the continuum limit, the origin of the energy automatically becomes consistent with the supersymmetry and the number of renormalization constants that require a (non-perturbative) determination is reduced to two from four, the number of renormalization constants appearing in the construction in Ref. [1].

Keywords: Lattice gauge theory, Supersymmetry, Energy-momentum tensor

1. Introduction

Although the energy-momentum tensor is a very fundamental observable in field theory, it is not straightforward to define the energy-momentum tensor in the lattice field theory, because the spacetime lattice explicitly breaks translational and rotational symmetries. For four-dimensional lattice gauge theories containing fermions, a strategy to construct an energy-momentum tensor, that satisfies the conservation law in the quantum continuum limit, has been given in Ref. [2]. In quantum field theory, a symmetry is generally expressed by corresponding Ward–Takahashi (WT) relations and the

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conservation law is merely a special case of WT relations that holds only when the Noether current stays away from other operators. Nevertheless, as demonstrated in Ref. [2] (and probably as can be proven generally), any lattice energy-momentum tensor, that is conserved in the continuum limit, is expected to reproduce all WT relations associated with the translational invariance for elementary fields in the continuum limit. This shows the fundamental importance of the conservation law in the continuum limit for a lattice energy-momentum tensor.

The present Letter is an extension of our recent paper [1] concerning the energy-momentum tensor in the lattice formulation of the four-dimensional \( \mathcal{N} = 1 \) supersymmetric Yang–Mills theory (4D \( \mathcal{N} = 1 \) SYM). In Ref. [1], we proposed a possible lattice energy-momentum tensor by mimicking the structure of the Ferrara–Zumino (FZ) supermultiplet [3]. That is, we defined a lattice energy-momentum tensor by a renormalized, modified supersymmetry (SUSY) transformation of a renormalized SUSY current on the lattice. Then, assuming the locality and the hypercubic symmetry of the lattice formulation and that the bare gluino mass is tuned so that the SUSY current is conserved [1, 5], the energy-momentum tensor was shown to be conserved in the quantum continuum limit; as noted above, this is a minimal and fundamental requirement on the energy-momentum tensor. This lattice energy-momentum tensor can be a basic tool to compute physical quantities related to the energy-momentum tensor, such as the viscosity.

Although the general strategy to construct a conserved lattice energy-momentum tensor in Ref. [2] is applicable also to the lattice formulation of 4D \( \mathcal{N} = 1 \) SYM, our method that is based on the \( \mathcal{N} = 1 \) SUSY in the target theory is interesting, because the direct imposition of the conservation law requires the (non-perturbative) determination of at least six renormalization constants [2], while the method in Ref. [1] contains only four (or three if one does not care about the ambiguity of the zero-point energy) unknown renormalization constants; see below.

In the present Letter, as a possible alternative of the definition in Ref. [1], we propose a quite similar but somewhat different definition of a lattice

\footnote{On the other hand, at the current moment there is no analysis on how one can construct a lattice energy-momentum tensor that generates correctly-normalized translations on composite operators. The complication arises because one has to classify the operator mixing occurring when the energy-momentum tensor and composite operators coincide in position space.}
2. A new definition of the energy-momentum tensor on the lattice

As Ref. [1], our starting point for the construction of a lattice energy-momentum tensor is a renormalized SUSY WT relation on the lattice:

$$\langle \partial^S_{\mu} S_{\mu}(x) \mathcal{O} \rangle = \left\langle Z \left[ -\frac{1}{a^4} \frac{\partial}{\partial \xi(x)} \Delta \xi + a \mathcal{E}(x) \right] \mathcal{O} \right\rangle. \quad (2.1)$$

Throughout the present Letter, we assume that the composite operator denoted by $\mathcal{O}$ is gauge invariant and finite, i.e., it is already appropriately renormalized. In the left-hand of Eq. (2.1), $S_{\mu}(x)$ is a renormalized Noether current associated with SUSY (the renormalized SUSY current),

$$S_{\mu}(x) \equiv Z [Z_{S} S_{\mu}(x) + Z_{T} T_{\mu}(x)], \quad (2.2)$$

$$\partial^S_{\mu} f(x) = \frac{1}{2a} [f(x + a\hat{\mu}) - f(x - a\hat{\mu})]. \quad (1.1)$$

Vector indices $\mu, \nu, \ldots$, run over 0, 1, 2, 3. $\epsilon_{\mu\nu\rho\sigma}$ denotes the totally anti-symmetric tensor and $\epsilon_{0123} = -1$. All gamma matrices are hermitian and obey $\{\gamma_{\mu}, \gamma_{\nu}\} = 2 \delta_{\mu\nu}$. We define $\gamma_5 = -\gamma_0 \gamma_1 \gamma_2 \gamma_3$ and $\sigma_{\mu\nu} = [\gamma_{\mu}, \gamma_{\nu}]/2$. The charge conjugation matrix $C$ satisfies, $C^{-1} \gamma_\mu C = -\gamma_\mu^T$, $C^{-1} \sigma_{\mu\nu} C = -\sigma_{\mu\nu}^T$, $C^{-1} \gamma_5 C = \gamma_5^T$ and $C^T = -C$. The generator of the gauge group $SU(N_c)$, $T^a$, is normalized as $\text{tr}(T^a T^b) = (1/2) \delta^{ab}$. $g$ is the bare gauge coupling constant. $x, y, z, \ldots$ denote lattice points and $a$ is the lattice spacing; $\hat{\mu}$ is the unit vector in the $\mu$-direction. $U_{\mu}(x) \in SU(N_c)$ denotes the conventional link variable and $\psi(x) \in su(N_c)$ is the gluino field and $\bar{\psi}(x) \equiv \psi^T(x)(-C^{-1})$. The symmetric difference operator $\partial^S_{\mu}$ is defined by
where $Z$, $Z_S$ and $Z_T$ are renormalization constants and lattice operators $S_\mu(x)$ and $T_\mu(x)$ are defined by

\[
S_\mu(x) \equiv -\sigma_{\rho\sigma} \gamma_\mu \text{tr} \left\{ \psi(x) [F_\rho\sigma]^L(x) \right\},
\]

\[
T_\mu(x) \equiv 2\gamma_\nu \text{tr} \left\{ \psi(x) [F_\mu\nu]^L(x) \right\}.
\]  

(2.3)

Here and in what follows, $[F_\mu\nu]^L(x)$ denotes a lattice transcription of the field strength,

\[
[F_\mu\nu]^L(x) \equiv 2 \text{tr} [P_{\mu\nu}(x) T^a] T^a,
\]

(2.4)

defined from the clover plaquette $P_{\mu\nu}(x)$,

\[
P_{\mu\nu}(x) \equiv \frac{1}{4} \sum_{i=1}^4 \frac{1}{2ia^2g} \left[ U_{i\mu\nu}(x) - U_{i\mu\nu}^\dagger(x) \right],
\]

(2.5)

where

\[
U_{1\mu\nu}(x) \equiv U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x),
\]

\[
U_{2\mu\nu}(x) \equiv U_\nu(x) U_\mu^\dagger(x - a\hat{\mu} + a\hat{\nu}) U_\nu^\dagger(x - a\hat{\mu}),
\]

\[
U_{3\mu\nu}(x) \equiv U_\mu^\dagger(x - a\hat{\mu}) U_\nu^\dagger(x - a\hat{\mu} - a\hat{\nu}) U_\mu(x - a\hat{\nu}) U_\nu(x - a\hat{\nu}),
\]

\[
U_{4\mu\nu}(x) \equiv U_\nu^\dagger(x - a\hat{\nu}) U_\mu(x - a\hat{\mu} - a\hat{\nu}) U_\nu(x + a\hat{\mu} - a\hat{\nu}) U_\mu^\dagger(x).
\]

(2.6)

In the right-hand side of Eq. (2.1), $\Delta_\xi$ is a modified SUSY transformation on lattice variables with the localized transformation parameter $\xi(x)$,

\[
\Delta_\xi \equiv \delta_\xi + Z_{EOM}\delta_{F\xi},
\]

(2.7)

which depends on another renormalization constant $Z_{EOM}$ \footnote{The multiplicative renormalization constant $Z$ is chosen so that the operator $S_\mu(x)$ has a finite correlation function with any renormalized operator, when the point $x$ is far apart from the support of that operator by a finite physical distance. $Z$ is at most logarithmically divergent by a dimensional reason.}; the localized transformations $\delta_\xi$ and $\delta_{F\xi}$ are defined by ($\xi(x) \equiv \xi^T(x)(x(-C^{-1}))$)

\[
\delta_\xi U_\mu(x) \equiv iag \frac{1}{2} \left[ \bar{\xi}(x) \gamma_\mu \psi(x) U_\mu(x) + \bar{\xi}(x + a\hat{\mu}) \gamma_\mu U_\mu(x) \psi(x + a\hat{\mu}) \right],
\]

\[
\delta_\xi U_\mu^\dagger(x) \equiv -iag \frac{1}{2} \left[ \bar{\xi}(x) \gamma_\mu U_\mu^\dagger(x) \psi(x) + \bar{\xi}(x + a\hat{\mu}) \gamma_\mu U_\mu(x) \psi(x + a\hat{\mu}) \right],
\]

\[
\delta_\xi \psi(x) \equiv -\frac{1}{2} \sigma_{\mu\nu} \xi(x) [F_{\mu\nu}]^L(x), \quad \delta_\xi \bar{\psi}(x) = \frac{1}{2} \xi(x) \sigma_{\mu\nu} [F_{\mu\nu}]^L(x),
\]

(2.8)
and

\[ \delta_F U_\mu(x) = 0, \quad \delta_F \psi(x) = \delta_\xi \psi(x), \quad \delta_F \bar{\psi}(x) = \delta_\xi \bar{\psi}(x). \quad (2.9) \]

Finally, \( \mathcal{E}(x) \) in Eq. (2.1) is a dimension 11/2 operator that is given by a linear combination of renormalized operators with logarithmically divergent coefficients.

The derivation of the renormalized SUSY WT relation (2.1) is somewhat too lengthy to be reproduced here; we refer the interested reader to Ref. [1] and references cited therein, especially for the origin of various renormalization constants. Here, we simply note that Eq. (2.1) reduces to the conservation law of the renormalized SUSY current \( S_\mu(x) \) in the continuum limit, when the point \( x \) stays away from the support of the operator \( \mathcal{O} \) by a finite physical distance (we express this situation by \( x \rightsquigarrow \text{supp}(\mathcal{O}) \)),

\[ \langle \partial_\mu S_\mu(x) \mathcal{O} \rangle \underset{a \to 0}{\longrightarrow} 0, \quad \text{for } x \rightsquigarrow \text{supp}(\mathcal{O}). \quad (2.10) \]

This follows because in the right-hand side of Eq. (2.1), the \( \bar{\xi}(x) \) derivative vanishes and the dimension 11/2 operator \( \mathcal{E}(x) \) does not produce an \( \mathcal{O}(1/a) \) linear-divergence that could compensate the factor \( a \) when \( x \rightsquigarrow \text{supp}(\mathcal{O}) \). In deriving Eq. (2.1), we assumed that the the bare gluino mass \( M \) is tuned to the supersymmetric point \([4, 5, 6, 7, 8]\) and that there is no exotic SUSY anomaly of the form of a three-fermion operator \([7, 8]\). The relation (2.10) can be regarded as the restoration of SUSY (that is broken by the lattice regularization) in the continuum limit.

In Ref. [1], a symmetric energy-momentum tensor on the lattice was defined by,

\[ T_{\mu\nu}(x) \equiv \frac{1}{2} [\Theta_{\nu\mu}(x) + \Theta_{\mu\nu}(x)] - c_\delta_{\mu\nu} \text{tr} \left[ \bar{\psi}(D + M) \psi(x) \right], \quad (2.11) \]

where \( D \) denotes the lattice Dirac operator and

\[ \Theta_{\mu\nu}(x) \equiv \frac{1}{8} (\gamma_\nu)_{\beta\alpha} \partial_{\xi \beta} \left[ \bar{Z} \bar{\Delta}_\xi S_\mu(x) \right]_\alpha, \quad (2.12) \]

and \( \bar{\Delta}_\xi \) is a global modified SUSY transformation on lattice variables, that is obtained by setting the local parameter constant, \( \xi(x) \to \xi \), in Eq. (2.7).

\[ ^4 \text{The subscripts } \alpha \text{ and } \beta \text{ refer to the spinor index.} \]
In Eq. (2.11), \(c\) is a constant to be fixed, although it does not affect the conservation of \(T_{\mu\nu}(x)\). Using the SUSY WT relation (2.1), it can be shown that the energy-momentum tensor (2.11) is conserved in the continuum limit \([1]\).

The definition through Eqs. (2.11) and (2.12) was suggested by the structure of the FZ supermultiplet \([3]\) that the SUSY transformation of the SUSY current is basically the energy-momentum tensor.

Now, our new definition of a lattice energy-momentum tensor proceeds as follows: By using the renormalized SUSY current (2.2), we first define the quantity,

\[
\Theta_{\mu\nu}(x; D_x) \equiv -\frac{1}{8} (C^{-1}\gamma_\nu)_{\alpha\beta} a^4 \sum_{y \in D_x} \left[ \partial_\rho S_\rho(y) \right]_\alpha \left[ S_\mu(x) \right]_\beta,
\]

where \(D_x\) is a hypercubic region on the lattice that contains the SUSY current \(S_\mu(x)\) entirely; the point \(x\) is taken as the center of the region \(D_x\) so that \(D_x\) is invariant under the hypercubic rotation around \(x\). Moreover, the size of the region \(D_x\) must be “macroscopic”, i.e., it must be finite in the physical unit. The definition of \(\Theta_{\mu\nu}(x; D_x)\) thus depends on the choice of the region \(D_x\) as its argument indicates. From this \(\Theta_{\mu\nu}(x; D_x)\), we define a symmetric energy-momentum tensor on the lattice, simply by symmetrizing it with respect to the indices:

\[
T_{\mu\nu}(x; D_x) \equiv \frac{1}{2} \left[ \Theta_{\mu\nu}(x; D_x) + \Theta_{\nu\mu}(x; D_x) \right].
\]

The idea behind the definition in Eqs. (2.13) and (2.14) is as follows: In the continuum theory, at least formally, the integral of the total divergence of the SUSY current in the continuum theory \(\tilde{S}_\rho(y)\),

\[
\int_{D_x} d^4y \partial_\rho \tilde{S}_\rho(y),
\]

where the region \(D_x\) contains an operator at the point \(x\), generates the SUSY transformation,

\[
- \int d^4y \frac{\delta}{\delta \xi(y)} \delta \xi = - \frac{\partial}{\partial \xi} \tilde{\delta} \xi,
\]

on the operator (\(\delta \xi\) and \(\tilde{\delta} \xi\) are localized and global SUSY transformations, respectively). In the classical continuum theory, on the other hand, the energy-momentum tensor \(\tilde{T}_{\mu\nu}(x)\) is given by the SUSY transformation of the SUSY current \([3]\) as (see Ref. \([1]\)),

\[
\tilde{\Theta}_{\mu\nu}(x) \equiv \frac{1}{8} (C^{-1}\gamma_\nu)_{\alpha\beta} \frac{\partial}{\partial \xi_\alpha} \left[ \tilde{\delta} \xi S_\mu(x) \right]_\beta,
\]

\[
\tilde{T}_{\mu\nu}(x) = \frac{1}{2} \left[ \tilde{\Theta}_{\mu\nu}(x) + \tilde{\Theta}_{\nu\mu}(x) \right] - \frac{3}{4} \delta_{\mu\nu} \text{tr} \left[ \bar{\psi}(x) D \psi(x) \right],
\]

where \(D\) is the SUSY Dirac operator.
where \( \mathcal{D} \) denotes the Dirac operator. Thus one sees that the definition (2.13) is a lattice transcription of the relation expected in the continuum theory:

\[
\tilde{\Theta}_{\mu\nu}(x) = -\frac{1}{8} (C^{-1})_{\gamma\nu}^{\alpha\beta} \int_{D x} d^4 y \left[ \partial_{\rho} \tilde{S}_{\rho}(y) \right]_\alpha^{\gamma} \left[ \tilde{S}_{\mu}(x) \right]_\beta. 
\]  

(2.18)

In the classical continuum theory, the right-hand side of Eq. (2.18) is independent of the choice of the region \( D x \) because of the current conservation. In the lattice theory, however, this property is lost because the conservation law of the SUSY current is broken by \( O(a) \) terms. That is, the dependence on \( D x \) in Eqs. (2.13) and (2.14) is an \( O(a) \) lattice artifact and the physics in the continuum limit should not depend on the choice of the region \( D x \).  

We note that the energy-momentum tensor (2.14) is manifestly finite, because the operator \( \sum_{y \in D x} \partial_{\rho} S_{\rho}(y) \) in Eq. (2.13), being the sum of the total divergence, does not have any overlap with the operator \( S_{\mu}(x) \); Eq. (2.13) is thus the sum of products of renormalized operators at points separated by finite physical distances.

Let us show that the lattice energy-momentum tensor \( T_{\mu\nu}(x; D x) \) (2.14) is conserved in the continuum limit. For this, we first show the conservation of \( \Theta_{\mu\nu}(x; D x) \) (2.13): Let \( 2R \) be the size of \( D x \),

\[
D x \equiv \{ y \in L^4 | x_\mu - R \leq y_\mu \leq x_\mu + R \text{ for all } \mu \},
\]

(2.19)

where \( L^4 \) denotes the whole lattice of the size \( L^4 \), and define a three-dimensional cubic region orthogonal to the \( \mu \)-direction as

\[
C_\mu^\nu(z_\mu) \equiv \{ y \in L^4 | x_\nu - R \leq y_\nu \leq x_\nu + R \text{ for } \nu \neq \mu \text{ and } y_\mu = z_\mu \}.
\]

(2.20)

\(^5\)On the other hand, in transcribing Eq. (2.17) to the lattice theory (2.14), we discarded the last term \(- (3/4) \delta_{\mu\nu} \text{tr}[\psi(x) \mathcal{D} \psi(x)]\). In quantum theory, this term just acts as the zero-point energy (see Ref. [1]) and we will see below that the simple prescription (2.14) gives rise to the zero-point energy that is consistent with SUSY.  

\(^6\)By an argument similar to the one in what follows, it is easy to see that the difference in \( T_{\mu\nu}(x; D x) \) due to different choices of \( D x \) vanishes in the continuum limit, at least when the energy-momentum tensor and other renormalized operators are separated to each other by finite physical distances. This shows that, in particular, the expectation value of \( T_{\mu\nu}(x; D x) \) with respect to physical states becomes independent of the choice of \( D x \) in the continuum limit.
Then, from the definition (2.13) and the SUSY WT relation (2.1), we have

\[
\langle \partial_\mu \Theta_{\mu \nu}(x; D_x) O \rangle = \frac{1}{8} \left( C^{-1} \gamma_\nu \right)_{\alpha \beta} a^4 \sum_{y \in D_x} \left\langle \mathcal{Z} \left[ -\frac{1}{a^4} \frac{\partial}{\partial \xi(x)} \Delta_\xi + a \mathcal{E}(x) \right] \left[ \partial_\mu S_\rho(y) \right]_\beta \mathcal{O} \rightangle \\
- \frac{1}{8} \left( C^{-1} \gamma_\nu \right)_{\alpha \beta} \sum_{\mu} \frac{1}{2} \left[ a^3 \sum_{y \in C_\mu(x_\mu + R + a)} -a^3 \sum_{y \in C_\mu(x_\mu - R)} \right]
\times \left\langle \mathcal{Z} \left[ -\frac{1}{a^4} \frac{\partial}{\partial \xi(y)} \Delta_\xi + a \mathcal{E}(y) \right] S_\mu(x + a \hat{\mu})_\beta \mathcal{O} \rightangle \\
- \frac{1}{8} \left( C^{-1} \gamma_\nu \right)_{\alpha \beta} \sum_{\mu} \frac{1}{2} \left[ a^3 \sum_{y \in C_\mu(x_\mu + R + a)} -a^3 \sum_{y \in C_\mu(x_\mu - R - a)} \right]
\times \left\langle \mathcal{Z} \left[ -\frac{1}{a^4} \frac{\partial}{\partial \xi(y)} \Delta_\xi + a \mathcal{E}(y) \right] S_\mu(x - a \hat{\mu})_\beta \mathcal{O} \rightangle .
\]

(2.21)

Suppose now that the point \( x \) stays away from the support of the operator \( \mathcal{O} \) by a finite physical distance, \( x \not\sim \text{supp}(\mathcal{O}) \), and the region \( D_x \) has been chosen such that \( D_x \cap \text{supp}(\mathcal{O}) = \emptyset \). In this situation, Eq. (2.21) reduces to

\[
\langle \partial_\mu \Theta_{\mu \nu}(x; D_x) O \rangle = \frac{1}{8} \left( C^{-1} \gamma_\nu \right)_{\alpha \beta} \left\langle \mathcal{Z} \left[ a \mathcal{E}(x) \right]_\beta \left[ a^4 \sum_{y \in D_x} \partial_\mu S_\rho(y) \right]_\alpha \mathcal{O} \rightangle \\
- \frac{1}{8} \left( C^{-1} \gamma_\nu \right)_{\alpha \beta} \sum_{\mu} \frac{1}{2} \left[ a^3 \sum_{y \in C_\mu(x_\mu + R + a)} -a^3 \sum_{y \in C_\mu(x_\mu - R)} \right]
\times \left\langle \mathcal{Z} \left[ a \mathcal{E}(y) \right]_\alpha \left[ S_\mu(x + a \hat{\mu}) \right]_\beta \mathcal{O} \rightangle \\
- \frac{1}{8} \left( C^{-1} \gamma_\nu \right)_{\alpha \beta} \sum_{\mu} \frac{1}{2} \left[ a^3 \sum_{y \in C_\mu(x_\mu + R + a)} -a^3 \sum_{y \in C_\mu(x_\mu - R - a)} \right]
\times \left\langle \mathcal{Z} \left[ a \mathcal{E}(y) \right]_\alpha \left[ S_\mu(x - a \hat{\mu}) \right]_\beta \mathcal{O} \rightangle .
\]

(2.22)
Now noting that the combination $\sum_{y \in D_x} \partial^S_\rho S_\rho(y)$ does not have any overlap with the point $x$, we see that Eq. (2.22) is the sum of correlation functions of renormalized operators with no mutual overlap with an overall factor of $a$ (in front of the operator $E(x)$). Thus, Eq. (2.22) vanishes in the $a \to 0$ limit and $\Theta_{\mu\nu}(x; D_x)$ is conserved in the continuum limit:

$$\langle \partial^S_\mu \Theta_{\mu\nu}(x; D_x) O \rangle \xrightarrow{a \to 0} 0, \quad \text{for } x \sim \text{supp}(O). \tag{2.23}$$

Next, we consider the anti-symmetric part of $\Theta_{\mu\nu}(x; D_x)$,

$$A_{\mu\nu}(x; D_x) \equiv \frac{1}{2} [\Theta_{\mu\nu}(x; D_x) - \Theta_{\nu\mu}(x; D_x)]. \tag{2.24}$$

The conservation of $A_{\mu\nu}(x; D_x)$ can be shown by the same argument as in Ref. [1]: Assuming the hypercubic symmetry, it turns out that any dimension 4 anti-symmetric rank-2 tensor can be expressed as:

$$A_{\mu\nu}(x; D_x) = A_1 \epsilon_{\mu\nu\rho\sigma} \partial^S_\rho \text{tr} [\bar{\psi}(x) \gamma_\sigma \gamma_5 \psi(x)] + A_2 \text{tr} [\bar{\psi}(x) \sigma_{\mu\nu} (D + M) \psi(x)] + aG_{\mu\nu}(x), \tag{2.25}$$

where $A_1$ and $A_2$ are constants and the dimension 5 operator $G_{\mu\nu}(x)$ is at most logarithmically divergent. From this general form, we have

$$\langle \partial^S_\mu A_{\mu\nu}(x; D_x) O \rangle \xrightarrow{a \to 0} 0, \quad \text{for } x \sim \text{supp}(O). \tag{2.26}$$

This is trivially true for the first term in the right-hand side of Eq. (2.25). For the second term in the right-hand side of Eq. (2.25), this holds because of the equation of motion of the gluino field. Finally, for the last term of Eq. (2.25), this follows because of the overall factor of $a$.

The combination of the above two properties, Eq. (2.23) and Eq. (2.26) implies the conservation law of the symmetric part of $\Theta_{\mu\nu}(x; D_x)$, Eq. (2.14), that is

$$\langle \partial^S_\mu T_{\mu\nu}(x; D_x) O \rangle \xrightarrow{a \to 0} 0, \quad \text{for } x \sim \text{supp}(O). \tag{2.27}$$

This completes the proof of the conservation law of our lattice energy-momentum tensor (2.14).

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7To apply this argument, the operator $\Theta_{\mu\nu}(x; D_x)$ must be local. This is actually the case because, under any local variation of fields, the combination $\sum_{y \in D_x} \partial^S_\rho S_\rho(y)$ is invariant.
For the new definition in Eqs. (2.13) and (2.14), we can further show that the expectation value of the energy density vanishes in the continuum limit,

\[ \langle T_{00}(x; D_x) \rangle = \langle \Theta_{00}(x; D_x) \rangle \xrightarrow{a \to 0} 0, \] (2.28)

when periodic boundary conditions are imposed on all the fields. This property of the energy density operator is natural from the perspective of SUSY, because Eq. (2.28) corresponds to the derivative of the supersymmetric partition function (i.e., the Witten index \[13\]) with respect to the temporal size of the system. In other words, Eq. (2.28) shows that the origin of the energy that is consistent with SUSY is automatically chosen in the continuum limit; this is a virtue of the present definition of the energy-momentum tensor compared with our previous one \[8\].

To show Eq. (2.28), we note that \( \sum_{y \in L^4} \partial_{\rho}^{\alpha} S_\rho(y) = 0 \) holds under the periodic boundary conditions. From this,

\[ \langle \Theta_{00}(x; D_x) \rangle = \frac{1}{8} \left( C^{-1} \gamma_0 \right)_{\alpha\beta} a^4 \sum_{y \in L^4 - D_x} \left\{ \left[ \partial_{\rho}^S S_\rho(y) \right]_\alpha \left[ S_0(x) \right]_\beta \right\} \]

\[ = \frac{1}{8} \left( C^{-1} \gamma_0 \right)_{\alpha\beta} a^4 \sum_{y \in L^4 - D_x} \left\{ Z \left[ aE(y) \right]_\alpha \left[ S_0(x) \right]_\beta \right\}, \] (2.29)

where \( L^4 - D_x \) denotes the complement of the region \( D_x \) in the lattice \( L^4 \) and we have used the SUSY WT relation (2.1) in the second equality. Then since this is a correlation function of renormalized operators with no mutual overlap with an overall factor of \( a \), this vanishes in the continuum limit, i.e., Eq. (2.28) holds.

Our new definition in Eqs. (2.13) and (2.14) contains two unknown combinations of renormalization constants which must be determined non-perturbatively. One is the overall normalization of \( S_\mu(x) \), \( ZZ \) and other is the ratio in \( S_\mu(x) \), \( Z_T/Z_S \). See Eq. (2.2). Among these, the latter ratio \( Z_T/Z_S \) has been non-perturbatively measured in the process to find the SUSY point in non-perturbative lattice simulations using the Wilson fermion \[3,10,11,12\]. The former overall normalization \( ZZ \) may be determined from the expectation value of the energy operator \( -a^3 \sum_{x} T_{00}(x; D_x) \) in a certain reference (e.g., one-particle) state. Thus, the determination of unknown constants is much

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For the definition of the energy density operator in a lattice formulation of the two-dimensional \( N = (2,2) \) SYM \[14,15\] (see also Refs. \[16,17\]) that possesses the property \( \langle T_{00} \rangle \) even before taking the \( a \to 0 \) limit, see Refs. \[18,19,20,21\].
simpler than our previous construction in Ref. [1] that requires the determination of other two unknown constants, $Z_{\text{EOM}}$ in Eq. (2.7) and $c$ in Eq. (2.11). This point can be a great advantage in practical applications.

On the other hand, the new definition has an $O(a)$ ambiguity associated with the choice of the region $D_x$ in Eq. (2.13) and this ambiguity can be a possible source of the systematic error. Also, since the energy-momentum tensor is defined by the product of two SUSY currents at different points as Eq. (2.13), the application requires the computation of correlation functions with the number of arguments as twice as large compared with the correlation function of the energy-momentum tensor (e.g., one defined in Ref. [1]). Only an implementation of the present construction in actual numerical simulations will answer whether there is a real payoff or not.

We believe that the basic idea on the construction of a lattice energy-momentum tensor in the present Letter (and in Ref. [1]) is applicable to more general 4D supersymmetric models. For our argument on the conservation law of the renormalized SUSY current in the continuum limit to hold, however, one has to carry out parameter fine tuning of sufficiently many numbers that ensures the SUSY WT relation (2.1). If such fine tuning is feasible for the model under consideration, our idea to construct a lattice energy-momentum tensor from the SUSY current will be useful to study physical questions in supersymmetric models, such as the spontaneous SUSY breaking, the mass and the decay constant of the pseudo Nambu–Goldstone boson associated with the (classical) dilatation invariance and so on.

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