Marden theorem and Poncelet-Darboux curves

Vladimir Dragović

Mathematical Institute SANU
Kneza Mihaila 36, 11000 Belgrade, Serbia
Mathematical Physics Group, University of Lisbon
e-mail: vlad@mi.sanu.ac.rs

Abstract

The Marden theorem of geometry of polynomials and the great Poncelet theorem from projective geometry of conics by their classical beauty occupy very special places. Our main aim is to present a strong and unexpected relationship between the two theorems. We establish a dynamical equivalence between the full Marden theorem and the Poncelet-Darboux theorem. By introducing a class of isofocal deformations, we construct morphisms between the Marden curves and the Poncelet-Darboux curves. Then we present effective criterion in terms of pair of polynomials which defines a Poncelet-Darboux curve of degree $n - 1$, for complete decomposition of the curve on $(n - 1)/2$ conics if $n$ is odd; if $n$ is even, complete decomposition consists of $(n - 2)/2$ conics and a line. This is an important question in the study of special, 'tHooft, instanton bundles.
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1 Introduction

Although it is impossible to distinguish the nicest mathematical result, probably many mathematicians would agree that the Marden theorem of geometry of polynomials and the great Poncelet theorem from projective geometry of conics by their classical beauty occupy very special places. Our main aim is to present a strong and unexpected relationship between the two theorems. We establish a dynamical equivalence between the full Marden theorem and the Poncelet-Darboux theorem. By introducing a class of isofocal deformations, we construct a morphism between a Marden curve and a Poncelet-Darboux curve, using the Moser trick and the Flashka coordinates. As a byproduct, we get complete description of cyclic-symmetric $n$-correspondences of $\mathbb{P}^1$. Then we present effective criterion for complete decomposition of a transversal Poncelet-Darboux curve of degree $n-1$ on $(n-1)/2$ conics if $n$ is odd and on $(n-2)/2$ conics and a line if $n$ is even in terms of pair of polynomials which defines the Poncelet-Darboux curve. This is an important question in the study of special or ’tHooft instanton bundles (see [19], [20], [10], [9], [16] and references therein). We finish by introducing a new class of bifocal deformations and relating it again to classical Darboux results.

The idea to relate Marden’s theorem with Poncelet’s theorem appeared after our careful analysis of the proof of the Siebeck theorem for $n = 3$. For the reader’s sake and for self-completion we included an Appendix with explanation of basic steps of the proof of the Siebeck theorem, following [12] and references therein.

In the next Section 2 we formulate theorem of Siebeck, its generalization - the Marden theorem and theorems of Poncelet and Darboux.

The Section 3 is devoted to introduction of isofocal dynamics, which appears to be integrable. By application of the Moser trick and the Flashka coordinates, we get explicit trivialization of the dynamics, see Theorem 7. A birational morphism between the data which define a Marden curve and the data defining a Poncelet-Darboux curve is established in the Theorem 8.

The main question treated in the Section 4 is about necessary and sufficient conditions on pair of polynomials which defined a transversal Poncelet-Darboux curve $S$ of degree $n-1$ to be completely decomposed on $(n-1)/2$ conics for $n$ odd; for $n$ even the complete decomposition assumes decomposition on $(n-2)/2$ conics and a line. In a generally very nice paper [19] this question has been treated explicitly, but in unsatisfactory way (See Sec. 5.5 of [19]: unfortunately the conditions mentioned there are neither sufficient nor necessary). Here, we propose a systematic approach to this important and difficult question. Very strong necessary conditions are formulated in the Theorem 9. It leads to the study of elliptic coverings of elliptic curves and to the theory of transformations of elliptic functions which was established by Jacobi (see [11]). In order to get sufficient conditions, one needs to examine if above transformation is cyclic, see Theorem 12. For $n = 2^km$ with $m$ odd all above transformations are cyclic for $k = 0, 1$, but not for $k > 1$, see Theorem 11.
In Section 4.3 we demonstrate effectiveness of the previous considerations. For \( n = 3, 5, 7 \) we give complete list of pairs of polynomials which define completely decomposable Poncelet-Darboux curves of degree \( n - 1 \). Moreover, we describe corresponding initial conditions of the isofocal deformations.

In the last Section 5 we briefly mention three related problems: the bifocal transformations; the Toma - Trautmann case of conic component and the positivity problem in isofocal dynamics, connected with infrapolynomial interpretation of the Marden theorem.

2 Preliminaries

2.1 The Marden theorem

One of the basic theorems in geometric theory of polynomials and rational functions has a long history and is usually referred as Marden’s after appearance of the book [13]. The earliest version of this theorem, up to our best knowledge, goes back to 1864 when Siebeck (see [18]) formulated and proved it for the case of polynomials with simple roots:

**Theorem 1 (Siebeck [18])** Let \( P(z) \) be a polynomial of degree \( n \geq 3 \) with complex coefficients, such that the zeros \( \alpha_1, \ldots, \alpha_n \) are simple and every three noncollinear. There exists a curve \( C \) of class \( n - 1 \) tangent to every line segment \( [\alpha_i, \alpha_j] \) at the midpoint. The foci of the curve \( C \) are zeros of the derivative polynomial \( DP(z) = P'(z) \).

In the simplest case \( n = 3 \) the curve \( C \) is a conic, inscribed in the triangle formed by the zeros of a polynomial of degree 3 and tangent to the sides of the triangle at their midpoints. Even in this, simplest case, the result of the Siebeck theorem is nontrivial and interesting and attracted lot of attention not only in the past but also nowadays (see for example [12]).

Previous results were extended to the cases with not all roots being simple. Consider a function of the form

\[
P(z) = (z - \alpha_1)^{m_1}(z - \alpha_2)^{m_2}\cdots(z - \alpha_n)^{m_n},
\]

where all \( \alpha_k \) are distinct, every three noncollinear and \( N = m_1 + m_2 + \ldots + m_n \). The zeros of the derivative \( DP \) are divided into two groups. In the first group are those \( \alpha_i \) for which \( m_i > 1 \). The second group is formed from simple zeros of the function, in other words from the zeros of the logarithmic derivative \( LP \) of \( P \). Since the positions of the zeros of the first group are known from the beginning, the interesting part is location of the members of the second group.

Thus, consider the function \( F(z) = LP(z) = d[\log P(z)]/dz \):

\[
F(z) = \frac{m_1}{z - \alpha_1} + \cdots + \frac{m_n}{z - \alpha_n}.
\]
Historically, the constants $m_i$ in the last expressions were firstly considered as positive integers. Then, step by step, that condition has been relaxed up to the condition that $m_i$ are nonzero real numbers, as we can find in Marden’s formulation (see [13], p. 11, Th. 4.2):

**Theorem 2 ([13], p. 11)** The zeros of the function

$$F(z) = \frac{m_1}{z - \alpha_1} + \cdots + \frac{m_n}{z - \alpha_n}, \quad (3)$$

where $m_i$ are real nonzero constants, are the foci of the curve of class $n - 1$ which touches each line-segment $[\alpha_i, \alpha_j]$ in a point dividing the line segment in the ratio $m_i : m_j$.

A good account of a century-long path from the Theorem 1 to the Theorem 2 one can find in Marden’s book [13], together with references. Thus we are going to omit them here.

### 2.2 Darboux Theorem and Poncelet-Darboux curves

Now we pass to quite different subject, which appeared in the theory of conics in the context of the great Poncelet theorem [17]. An overview of the history of the subject together with presentation of contemporary state of art one may find in [6], [7]. We start here with one of Darboux’s original formulations of his theorem.

**Theorem 3 (Darboux, [4] p. 248)** Si une courbe d’ordre $n-1$ contient tous les points d’intersection de $n$ tangents à une conique, elle contient aussi les points d’intersection d’une infinité d’autres systemes de $n$ tangentes à la même conique. Chacun de ces systemes est défini par l’équation

$$\phi(\rho) + kf(\rho) = 0 \quad (4)$$

ou $k$ désigne une constante arbitraire.

Darboux had been interested in this matter for about fifty years and he published several variations of the last theorem (see for example [5]). Slightly changing terminology from [20], we will say that a curve $S$ of degree $n - 1$ is Poncelet-Darboux related to a conic $K$ if the curve $S$ and the conic $K$ satisfy conditions of the previous theorem. The set of all such curves of degree $n - 1$ which are Poncelet-Darboux related to a fixed conic $K$ will be denoted as

$$Pon - Dar_{n-1}(K).$$

We will say that the $n$ tangents $t_1, t_2, \ldots, t_n$ of a conic $K$ from the previous theorem form a Poncelet $n$-polygon $P_n = T_1T_2\ldots T_n$, where $T_i = t_i \cap t_{i+1}$ for
$i = 1, \ldots, n - 1$ and $T_n = t_n \cap t_1$ if there exists a conic $C$ such that $T_i \in C$ for $i = 1, \ldots, n$. In that case we will say that conics $C$ and $K$ are $n$-Poncelet related.

Here is a formulation of the Poncelet Theorem:

**Theorem 4 (Poncelet, [17])** If two conics $C$ and $K$ are $n$-Poncelet related, then there are infinitely many $n$-polygons circumscribed about $S$ and inscribed in $C$. Moreover, arbitrary point of the conic $C$ may be chosen for a vertex of a such Poncelet $n$-polygon.

If the $n$ tangents from the Darboux Theorem 3 form a Poncelet $n$-polygon inscribed in a conic $C$, Darboux proved that the curve $S$ of degree $n - 1$ then completely decomposes. More precisely, Darboux proved the following

**Theorem 5 (Darboux, [4])** If a curve $S$ of degree $n - 1$ is Poncelet-Darboux related to a conic $K$ and if there is a conic $C$, a component of $S$ which is $n$-Poncelet related to the conic $K$, then for $n = 2k + 1$ the curve $S$ is completely decomposed on $k$ conics and if $n = 2k$ it is decomposed on $k - 1$ conics and a line.

In the case of decomposition of Poncelet-Darboux curve, the conic components are parts of what we called the Poncelet-Darboux grids. Further generalizations of Darboux theorems and Poncelet-Darboux grids are obtained very recently in [7].

Among other modern investigations in framework of the Darboux theorems, we should mention here [19], [20], [16] and references therein, where Poncelet-Darboux curves are related to the study of stable bundles, instanton bundles and their decomposition.

We reformulate the Darboux Theorem 3 following [19]:

**Theorem 6 (Darboux)** Let $K$, $S$ be a nondegenerate conic and a curve of degree $n - 1$ in the projective plane $\mathbb{P}W$ and let $\beta : W \to W^*$ be a nondegenerate bilinear form. Assume that there are $n$ points on $K$ such that the points of intersection of any two lines associated to these points by $\beta$ belong to $S$. Then, $S$ is Poncelet-Darboux $n - 1$ related to $K$.

### 2.3 Some notations and notions

We will use the following notations.

By

$$\{x_1, x_2, \ldots, x_n\}_M$$

(5)
we will denote a *multiset*, meaning that number of appearances of an item is important, but the order no; the notion of divisor has synonymous meaning. The standard symmetric functions of \( n \) quantities \((a_1, \ldots, a_n)\) will be denoted as

\[
\sigma_1(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i;
\]

\[
\sigma_2(a_1, \ldots, a_n) = \sum_{i<j}^{n} a_i a_j;
\]

\[
\sigma_n(a_1, \ldots, a_n) = a_1 a_2 \cdots a_n;
\]

when one of the quantities, \( a_k \), is omitted, the symmetric functions of the \( n-1 \) rest quantities will be denoted as

\[
\sigma^k_0(a_1, \ldots, a_n) = 1 \quad k = 1, \ldots, n
\]

\[
\sigma^k_1(a_1, \ldots, a_n) = \sum_{i \neq k}^{n} a_i \quad k = 1, \ldots, n
\]

\[
\sigma^k_2(a_1, \ldots, a_n) = \sum_{i<j, i \neq k}^{n} a_i a_j \quad k = 1, \ldots, n
\]

\[
\sigma^k_n(a_1, \ldots, a_n) = a_1 a_2 \cdots a_{k-1} a_{k+1} \cdots a_n \quad k = 1, \ldots, n.
\]

We will use also vector notation

\[
\vec{\sigma}_i(a_1, \ldots, a_n) = (\sigma^1_i(a_1, \ldots, a_n), \sigma^2_i(a_1, \ldots, a_n), \ldots, \sigma^n_i(a_1, \ldots, a_n)),
\]

for \( i = 0, 1, \ldots, n; \)

\[
\vec{m} = (m_1, \ldots, m_n),
\]

\[
(\vec{m}, \vec{\sigma}_i) = \sum_{k=1}^{n} m_k \sigma^k_i.
\]

Let us recall some traditional notions: *the cyclic points* are points on the infinite line with coordinates \( \hat{I} = (1, i, 0) \) and \( \hat{J} = (1, -i, 0) \). A line is isotropic or *minimal* if it is finite and passes through one of the cyclic points. For a given curve, a point is *focal* if it is intersection of two isotropic tangents to the curve with finite points of contact with the curve. As an example, an ellipse has four focal points: a pair of real foci and a pair of imaginary foci.
3 Isofocal deformations

3.1 Definition of an integrable dynamical system

Let us start with a function of the form

\[ P^0(z) = (z - \alpha_1^0)^{m_1^0}(z - \alpha_2^0)^{m_2^0} \cdots (z - \alpha_n^0)^{m_n^0}, \]  

(10)

where all \( \alpha_k^0 \) are distinct, and consider its logarithmic derivative

\[ F^0(z) = \frac{d}{dz} \log P^0(z) = \frac{m_1^0}{z - \alpha_1^0} + \cdots + \frac{m_n^0}{z - \alpha_n^0}. \]  

(11)

Let us set

\[ F^0(z) = \frac{m_1^0}{z - \alpha_1^0} + \cdots + \frac{m_n^0}{z - \alpha_n^0} = f(z) \phi(z), \]  

(12)

where

\[ \phi(z) = (z - \alpha_1^0)(z - \alpha_2^0) \cdots (z - \alpha_n^0) \]

\[ f(z) = \sum_{i=1}^{n} m_i^0 \prod_{j \neq i} (z - \alpha_j^0). \]  

(13)

One can easily see that

\[ f(z) = B_n z^{n-1} + \cdots + B_1 z - 1 + B_1 \]  

(14)

where, using notations from eq. (8) and eq. (9), we have

\[ B_1 = \langle \overrightarrow{m}^0, \overrightarrow{\sigma}_{n-1}(\alpha_1^0, \ldots, \alpha_n^0) \rangle. \]  

(15)

We point out two particular cases.

**Lemma 1**  
(a) The function \( f \) is equal to the derivative of \( \phi \) if and only if all \( m_i^0 \) are equal to 1.

(b) The function \( f \) is constant if and only if

\[ \overrightarrow{m}^0 \perp [\overrightarrow{\sigma}_0, \overrightarrow{\sigma}_1, \ldots, \overrightarrow{\sigma}_{n-2}]. \]  

(16)

Before proceeding with introduction of dynamics, we are going to consider the simplest case as an example.

**Example 1** Let us consider the case \( n = 3 \). According to the Marden Theorem 2 for \( n = 3 \), there exist a Marden curve \( K \), which is in this case, a conic. Its focal points \( z_1 \) and \( z_2 \) satisfy \( f(z_i) = 0 \), under the condition \( \deg f = 2 \). Again, by the Marden Theorem 2, the conic \( K \) touches line-segments \( [\alpha_1^0, \alpha_2^0] \) in the ratio \( m_i^0 : m_j^0 \).
Now, since the lines \((\alpha_1^0, \alpha_2^0)\) are tangent to the conic \(K\), we may apply the Darboux Theorem. The triplet of the conic \(K\) and the two polynomials \(\phi\) and \(f\) uniquely determines the Poncelet-Darboux curve \(PD_K(\alpha_1^0, \alpha_2^0, \alpha_3^0, m_1^0, m_2^0, m_3^0) = PD_K(\phi, f)\). The curve \(PD_K(\phi, f)\) is a conic.

Thus, the conics \(PD_K(\phi, f)\) and \(K\) are 3-Poncelet related.

According to the theorems of Poncelet and Darboux, there exists another set of three points \(\alpha_1^1, \alpha_2^1, \alpha_3^1\) which belong to the Poncelet-Darboux conic \(PD_K(\phi, f)\) such that the lines \((\alpha_1^1, \alpha_1^1)\) are tangent to the Marden conic \(K\). The triangle \(\alpha_1^1, \alpha_2^1, \alpha_3^1\) is a Poncelet triangle with the caustic \(K\) and the boundary \(PD_K(\phi, f)\).

Now, we want to apply the Marden Theorem with a new polynomial

\[
\phi_1(z) = (z - \alpha_1^1)(z - \alpha_2^1)(z - \alpha_3^1)
\]

instead of \(\phi\). In order to do that, we determine new "masses" \((m_1^1, m_2^1, m_3^1)\) such that the polynomial \(f\) rests unchanged. More precisely, we calculate \((m_1^1, m_2^1, m_3^1)\), up to a scalar factor, from the system of linear equations:

\[
\begin{align*}
B_3 &= m_1^1 + m_2^1 + m_3^1 \\
B_2 &= m_1^1(\alpha_2^1 + \alpha_3^1) + m_2^1(\alpha_1^1 + \alpha_3^1) + m_3^1(\alpha_1^1 + \alpha_2^1), \\
B_1 &= m_1^1(\alpha_2^1 \alpha_3^1) + m_2^1(\alpha_1^1 \alpha_3^1) + m_3^1(\alpha_1^1 \alpha_2^1)
\end{align*}
\]

where the constants \(B_1, B_2, B_3\) are determined from the equation (15)

\[
B_i = \langle m^0_i, \overrightarrow{\alpha_{n-i}}(\alpha_1^0, \alpha_2^0, \alpha_3^0) \rangle.
\]

Since \(f\) is unchanged, the focal points of the Marden curve in the new case are the same as focal points of \(K\). We deduce now that the Marden curve of the new case is equal to \(K\), because among confocal conics there is at most one inscribed in a triangle.

By a new application of the Marden theorem, we finally get a new information concerning Poncelet triangles. We are able to deduce the ratio of tangency of a new triangle by the caustic \(K\):

\[
\begin{align*}
\alpha_1^1 \beta_1^1 : \beta_2^1 \alpha_1^1 &= m_1^1 : m_1^1, \\
\alpha_2^1 \beta_1^1 : \beta_1^1 \alpha_2^1 &= m_2^1 : m_3^1, \\
\alpha_3^1 \beta_1^1 : \beta_3^1 \alpha_2^1 &= m_1^1 : m_2^1
\end{align*}
\]

where \(\beta_i^1\) denote points of contact of the caustic and the triangle.

If the degree of the polynomial \(f\) is equal to 1, then the conic \(K\) is a parabola. Thus we finish the Example with a nice classical Lemma:

**Lemma 2 (folklore)** If parabola touches three lines of a triangle \(ABC\), then the focus belongs to the circumscribed circle to the triangle \(ABC\).

If the polynomial \(f\) is constant, then the conic \(K\) is a circle.
The last Example, although treatise the simplest case \( n = 3 \) is very instruc-
tive. In order to extract a new statement about ratios of tangency of a new
Poncelet triangle as it is formulated in eq. (18), one needs several nontrivial
steps of alternative use of the Marden and the Poncelet-Darboux theorems. In
case \( n > 3 \) we cannot follow the same lines, because the Marden curve is not a
conic any more and one cannot build up the Darboux theorem straight on it. In
order to link together the Marden and the Darboux Theorems in general case
we need to develop much more subtle approach.

From the last Example, we learnt that it was fruitful idea to pass from one
Poncelet triangle to a new one by a transformation which keeps the polynomial
\( f \) unchanged.

Following this principle, we introduce a new dynamics, which depends on
continuous "time" parameter \( t \) and which has quantities \((\alpha_0^0, \ldots, \alpha_n^0)\) and \((m_1^0, \ldots, m_n^0)\)
as the initial data. More precisely, we introduce functions:

\[
\begin{align*}
\alpha_1(t) &= \alpha_1(t, \alpha_1^0, \ldots, \alpha_n^0, m_1^0, \ldots, m_n^0), \\
\alpha_2(t) &= \alpha_2(t, \alpha_1^0, \ldots, \alpha_n^0, m_1^0, \ldots, m_n^0), \\
\quad \vdots \\
\alpha_n(t) &= \alpha_n(t, \alpha_1^0, \ldots, \alpha_n^0, m_1^0, \ldots, m_n^0), \\
m_1(t) &= m_1(t, \alpha_1^0, \ldots, \alpha_n^0, m_1^0, \ldots, m_n^0), \\
m_2(t) &= m_2(t, \alpha_1^0, \ldots, \alpha_n^0, m_1^0, \ldots, m_n^0), \\
\quad \vdots \\
m_n(t) &= m_n(t, \alpha_1^0, \ldots, \alpha_n^0, m_1^0, \ldots, m_n^0)
\end{align*}
\] (19)

in order to satisfy

\[
F_i(z) := \frac{m_1(t)}{z - \alpha_1(t)} + \cdots + \frac{m_n(t)}{z - \alpha_n(t)} = \frac{f(z)}{\phi(z) + tf(z)},
\] (20)

with the initial conditions

\[
m_i(0) = m_i^0, \quad i = 1, \ldots, n, \\
\alpha_i(0) = \alpha_i^0.
\] (21)

By the condition (20), the function \( f \) keeps unchanged during the evolution.
This means that focal points, as zeros of the polynomial \( f \) are fixed during the
evolution. Thus, we will refer to such dynamics as isofocal dynamics or isofocal
deformations.

The polynomial \( f \) plays a role of an isospectral polynomial. From the formula

\[
f(z) = \sum_{i=1}^{n} m_i(t) \prod_{j \neq i} (z - \alpha_j(t)).
\] (22)

we get the following
**Proposition 1** The dynamics \((19, 20, 21)\) has the coefficients of the polynomial \(f\) as first integrals:

\[
B_i = \langle \vec{m}(t), \vec{\sigma}_{n-i}(\alpha_1(t), \ldots, \alpha_n(t)) \rangle, \quad i = 1, \ldots, n.
\]  

(23)

We will use terminology **positions in a moment** \(t\) for \(\alpha_1(t), \ldots, \alpha_n(t)\) and for \((m_1(t), \ldots, m_n(t))\) we will use **masses** although these masses will change during the time and might be negative as well.

One of the first integrals is the **law of conservation of masses**.

We will also use notation

\[
\Phi_t(z) := \phi(z) + tf(z) = (z - \alpha_1(t)) \cdots (z - \alpha_n(t)).
\]  

(24)

### 3.2 Moser’s trick and the Flashka coordinates

We consider the function \((19)\)

\[
F_t(z) = \frac{f(z)}{\Phi_t(z)}
\]

and we apply the Moser trick (see [14], [15]) to develop it in a continued fraction of the following form.

\[
F_t(z) = \frac{1}{z - b_n - \frac{a_{2n-1}}{z - b_{n-1} - \frac{a_{2n-2}}{z - \cdots - \frac{a_2}{z - b_1}}}}
\]  

(25)

The last formula \((25)\) gives us transformation from our dynamical coordinates \((\alpha_1, \ldots, \alpha_n, m_1, \ldots, m_n)\) to the new coordinates \((a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_n)\). The last set of coordinates we will call the **Flashka coordinates** (see [8], [14], [15]).

To construct the inverse transformation we consider the Flashka Lax matrix \(L_n\) for the \(n\)-point Toda chain (see [8], [14], [15]):

\[
L_k = \begin{bmatrix}
  b_1 & a_1 \\
  a_1 & b_2 \\
  \vdots & \ddots & \ddots \\
  b_{k-1} & a_{k-1} \\
  a_{k-1} & b_k
\end{bmatrix}
\]  

(26)

Denote

\[
\delta_k = \det L_k.
\]

The following well-known difference relations take place

\[
\delta_k = (z - b_k)\delta_{k-1} - a_{k-1}^2\delta_{k-2}, \quad k = 3, \ldots, n.
\]  

(27)
The inverse transformation from the Flashka coordinates to the initial dynamical coordinates is defined by the formula (see [14]):

$$F_t(z) = \frac{\delta_{n-1}}{\delta_n}.$$  (28)

From the last formula and from (20) we conclude

**Lemma 3**  *The time evolution of $\delta$ according to dynamics (20) satisfies:*

\[
\begin{align*}
\delta_{n-1}(t) &= \delta_{n-1}(0) \\
\delta_n(t) &= \delta_n(0) + t\delta_{n-1}(0). 
\end{align*}
\]  (29)

From the Lemma (3) one concludes that

1\(^o\) only $b_n$ among the Flashka coordinates depends on $t$;
2\(^o\) the coordinate $b_n$ depends on $t$ linearly.

Thus we have the following

**Theorem 7**  *The dynamical system (20) trivializes in the Flashka coordinates, where it gets the form*

\[
\begin{align*}
\dot{a}_1 &= 0 & \dot{b}_1 &= 0 \\
\dot{a}_2 &= 0 & \dot{b}_2 &= 0 \\
& \vdots \\
\dot{a}_{n-1} &= 0 & \dot{b}_{n-1} &= 0 \\
\dot{b}_n &= -1.
\end{align*}
\]  (30)

### 3.3 From Marden’s curve to Poncelet-Darboux curve

Following Marden, denote $\mathcal{L}_{\alpha_j}$ the line equation of the point $\alpha_j^0 = x_j + iy_j$:

$$\mathcal{L}_{\alpha_j^0} = \lambda x_j + \mu y_j - 1,$$

the equation of all lines passing through the point $\alpha_j^0$. The equation of the Marden curve (see [13], equation (4.10)) is deduced from the condition

$$\sum_{j=1}^{n} \frac{m_j^0}{\mathcal{L}_{\alpha_j^0}} = 0.$$  (31)

We will denote the last curve $\mathcal{M}(\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0, m_1^0, m_2^0, \ldots, m_n^0)$. Now we pass
to projective plane. We see it as projective space of quadratic polynomials: to a polynomial \( P(z) = a(z - b)(z - c) \) we associate the point \((-c+b, cb, 1)\). The projection

\[
\pi: \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^2
\]

is branched over the conic \( K \) with the equation

\[
z_1^2 = 4z_0z_2.
\]

In this presentation, to a linear polynomial \( a(z - \alpha_0^j) \) corresponds the line \( t_K(\alpha_0^j) \), tangent to the conic \( K \). This line is the set of all quadratic polynomials which have polynomial \( z - \alpha_0^j \) as a factor.

To develop further this connection, following Darboux (see [4]) we introduce new system of coordinates. Given a plane with standard coordinates \((z_0, z_1, z_2)\), we start from the given conic \( K \). It is given by the equation (33) and rationally parameterized by \((s^2, 2s, 1)\). The tangent line to the conic \( K \) through the point with the parameter \( s_0 \) is given by the equation

\[
t_K(s_0): z_2s_0^2 + z_1s_0 + z_0 = 0.
\]

On the other hand, for a given point \( P \) in the plane with coordinates \( P = (\hat{z}_0, \hat{z}_1, \hat{z}_2) \) there correspond two solutions \( \rho \) and \( \rho_1 \) of the quadratic in \( s \) equation

\[
\hat{z}_2s^2 + \hat{z}_1s + \hat{z}_0 = 0.
\]

Each solution correspond to a tangent to the conic \( K \) from the point \( P \). We will call the pair \((\rho, \rho_1)\) the Darboux coordinates of the point \( P \). One finds immediately

\[
\frac{\hat{z}_0}{\rho\rho_1} = -\frac{\hat{z}_1}{\rho + \rho_1} = \hat{z}_2.
\]

The line \( t_K(\alpha_0^j) \) which corresponds to the point \( \alpha_0^j \) and to the linear polynomial \( a(z - \alpha_0^j) \) has the following presentation in the Darboux coordinates:

\[
t_K(\alpha_0^j)(\rho, \rho_1) = (\rho - \alpha_0^j)(\rho_1 - \alpha_0^j).
\]

The Poncelet-Darboux curve is done by the equation

\[
\sum_{j=1}^{n} m_j^0 t_K(\alpha_0^j) = 0.
\]

In Darboux coordinates it may be rewritten in the form

\[
\frac{f(\rho)}{\phi(\rho)} = \frac{f(\rho_1)}{\phi(\rho_1)}.
\]

The curve defined by last equations we will denote as \( \mathcal{PD}_K(\phi, f) \). From the equations (31) and (36) and whole previous considerations we get the following theorem.
Theorem 8 There is a birational morphism defined by the equations (13) and (23) between the data of Marden curves $(\alpha_0^1, \alpha_0^2, \ldots, \alpha_0^n, m_0^1, m_0^2, \ldots, m_0^n)$ and the data $(\phi, f)$ of Poncelet-Darboux curves associated with the conic $K$. There is a birational morphism between dual of projective closure of a Marden curve $M(\alpha_0^1, \alpha_0^2, \ldots, \alpha_0^n, m_0^1, m_0^2, \ldots, m_0^n)$ and the Poncelet-Darboux curve $PD_K(\phi, f)$ with corresponding data.

3.4 Discriminant and gauge equivalence

The morphism from the previous Theorem (8) fails to be one to one for those $\alpha_0^1, \alpha_0^2, \ldots, \alpha_0^n$ for which the system of linear equations in $(m_0^1, m_0^2, \ldots, m_0^n)$ given by the equations (23) has determinant $D(\alpha_0^1, \alpha_0^2, \ldots, \alpha_0^n)$ equal to zero. The condition $D(\alpha_0^1, \alpha_0^2, \ldots, \alpha_0^n) = 0$ is equivalent to $\alpha_i^0 = \alpha_j^0$ for some $i \neq j$.

We will refer to configurations $(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t), m_1(t), m_2(t), \ldots, m_n(t))$ such that $\alpha_i(t) = \alpha_j(t)$ for some $i \neq j$ as points of collision. Such a moment $t$ we will call moment of collision. If there is no $k \neq i, k \neq j$ such that in addition $\alpha_i(t) = \alpha_j(t) = \alpha_k(t)$ the point of collision is simple. The system is simple if it has only simple collision points. We will assume that system passes smoothly through a collision point in the phase space.

4 Complete decomposition of Poncelet-Darboux curves, $n$-volutions and collisions

4.1 $n$-volutions

By an $n$-volutio in a set $V$ we will assume a family of multisets $A_n$, a subset of the $n$-th symmetric product $\text{Sym}_n V$ such that there is a unique function $f$:

$$f : V \rightarrow A_n,$$

such that $v \in f(v)$. In some classical terminology is used notion of cyclic-symmetric correspondences.

If $\alpha \in A_n$ is a multiset such that its cardinality is less than $n$ we will say that it is a collision point of the $n$-volutio. In other words $\alpha = \{\alpha_1, \ldots, \alpha_n\}_m$ is a collision point if there exist $\alpha_1, \ldots, \alpha_k$, $k < n$ and natural numbers $c_1, \ldots, c_k$ such that

$$\alpha = \{\alpha_1, \ldots, \alpha_n\} \equiv c_1 \alpha_1 + \ldots + c_k \alpha_k.$$

The basic examples of $n$-volutions are involutions, which correspond to $n = 2$. Then, the notion of collision point coincides with the notion of fixed point. A
nice case are involutions of a conic. By the Fregier theorem, we know that for every involution on a conic, there exists a point, the Fregier point such that the involution is cut from the conic by lines from the pencil determined by the Fregier point.

We pass now to the case of $\mathbb{CP}^1$. By Luroth Theorem we know that every $n$-volution is determined by a pair of polynomials $p, q$ of degree $n$.

Consider the pencil $p_t(z) = tp(z) + q(z)$ and the roots $a_1(t), \ldots, a_n(t)$. There is a one-parameter family of $n$-tuples. From the following system

$$
tp(a_1(t)) + q(a_1(t)) = 0 \\
tp(a_2(t)) + q(a_2(t)) = 0
$$

we get

$$(a_1(t) - a_2(t))r(a_1(t), a_2(t)) = 0.$$  

Here $r$ is of degree $n - 1$ in $a_1$ and symmetric. When $t$ varies, $r(a_1(t), a_2(t))$ defines a curve.

The question is how to describe all such $r$. Following the lines of Section (3.3) we pass to $\mathbb{CP}^2$ and correspond to a $n$-tuple of $n$ linear factors $(z - a_1), \ldots, (z - a_n)$ a polygon of $n$-sides circumscribed about the conic $K$. Thus, we have the following

**Proposition 2** An $n$-volution defined by polynomials $p, q$ of degree $n$ is associated to a curve from Pon–Dar $n-1(K)$ given by the equation

$$\det A \circ K(z) = 0.$$  

The $(n - 1) \times (n + 1)$ matrix $A$ annihilates the pencil generated by $p, q$ and the matrix $K(z)$ is induced by the conic $K$ and it has the form

$$K(z) = \begin{pmatrix} z_2 & 0 & \cdots & \cdots & 0 \\ -z_1 & z_2 & \cdots & \cdots & 0 \\ z_0 & -z_2 & \cdots & \cdots & 0 \\ 0 & z_0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & z_2 \\ \vdots & \vdots & \cdots & -z_1 & \vdots \\ \vdots & \vdots & \cdots & \vdots & z_0 \end{pmatrix}$$

The last equation follows from [19]. A linear system $L = L(p, q)$ has a base point if polynomials $p$ and $q$ have a common root. From [19] we have that the base points of the linear system correspond to the components of the Poncelet-Darboux curve which are tangent lines to the conic $K$. From [19] we also have the following description of the collision points.

**Proposition 3** Let $C$ be a Poncelet-Darboux curve with respect to the conic $K$ such that the corresponding linear system $L = L(p, q)$ doesn’t have base points. Then for a given point $P = (s_0, t_0) \in K$ and an integer $k \geq 0$ the following are equivalent:
(A) the intersection multiplicity of $K$ and $C$ at $P$ is equal to $k$;
(B) $(s_0,t_0)$ is a zero of order $k$ of a unique polynomial $h \in L(p,q)$.

**Example 2** Now, we want to calculate the number of generalized Fregier points in the case of pencils of curves of degree $k$. The pencil of curves of degree $k$ has a base set of $\binom{k+2}{2} - 2$ points. A curve of degree $k$ intersects the conic $K$ in $n = 2k$ points. But, between a polynomial of degree $n = 2k$ and a curve of degree $k$ which determine the same set of $2k$ points on the conic $K$, there is a correspondence which is not bijective, due to the relation

$$4z_0z_1 = z_2^2$$

which holds on the conic $K$ by definition. Thus, one needs to fix additionally $\binom{k}{2}$ points outside the conic to get uniqueness. As a result, the number of generalized Fregier points for an $n$-volution is

$$\binom{k+2}{2} - 2 \cdot \binom{k}{2} = 2k - 1.$$  

For the case $n = 2k - 1$ one needs also to fix a point on the conic $K$, since the number of intersections of a curve and a conic is always even. In this case the number of generalized Fregier points is $2k - 2$.

In both cases, with $n$ even or odd the total number of generalized Fregier points is equal $n - 1$. This number coincides with the number of focal points in the Marden Theorem. Is there a natural bijection between these two sets?

**4.2 Conic component and complete decomposition of Poncelet-Darboux curves**

From the last Proposition we see that the intersection of a Poncelet-Darboux curve $C$ and the conic $K$ is transversal if and only if corresponding system is simple in terminology of the Section \[3.4\]. If a Poncelet-Darboux curve $C$ satisfies any of two equivalent conditions of the Proposition \[3\], we will say that it is transversal. Thus, the linear system which corresponds to a transversal Poncelet-Darboux curve contains polynomials with at most double roots.

From now on we will consider transversal Poncelet-Darboux curves. Let us consider first the simplest case of curves of degree 2.

**Example 3** Let $C$ be degree two Poncelet-Darboux curve with transversal intersection with the conic $K$. Denote four intersection points $C \cap K = \{x_1, x_2, x_3, x_4\}$. Denote four common bitangents $t_i$, $i = 1, \ldots, 4$ and denote points of contact of the tangent $t_i$ with the conic $C$ as $y_i$ and the point of contact with the conic $K$ as $a_i$. 

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The conic \( C \) is 3-Poncelet related to the conic \( K \). By the Poncelet theorem this means that there is a triangle inscribed in \( C \) and circumscribed about \( K \) with arbitrary point of \( C \) taken as a vertex.

Choose any of the points \( y_i \), say \( y_1 \) as a vertex. Then \( y_1 \) is a double vertex of a Poncelet triangle. The third vertex of the Poncelet triangle is one of the points \( x_i \) and the Poncelet triangle is \( y_1y_1x_i \). Thus we get

**Lemma 4** With use of previous notations, for any point \( y_i \) there is a point \( x_j \) such that the line \( y_ix_j \) is tangent to the conic \( K \) at the point \( x_j \). The triplet \( (y_iy_ix_j) \) forms a Poncelet triangle.

Coming back to the general case of transversal Poncelet-Darboux curves of degree \( n-1 \), let us study the case when a curve \( S \) has a conic component \( C \) which is \( n \)-Poncelet related to the conic \( K \). According to the Darboux theorem then the curve \( S \) is decomposed as a product of \( k \) conics if \( n = 2k + 1 \) or as a product of \( k - 1 \) conics and a line if \( n = 2k \).

Any conic \( C_i \) defines a symmetric \( 2-2 \) correspondence of the Euler-Chasles type \( \Phi_i \) such that a point \( P \) belongs to the conic \( C_i \) if and only if \( \Phi_i(\rho, \rho_1) = 0 \) where \((\rho, \rho_1)\) are the Darboux coordinates of the point \( P \). Darboux proved the Theorem (5) using these correspondences together with the opposite statement that a symmetric correspondence determines a conic from the pencil. We know that correspondences commute if the conics belong to the same pencil with the conic \( K \). Moreover, we have

**Lemma 5** Suppose the lines \( t_1, t_2, \ldots, t_n \) are given such that \( \{P_1\} := t_i \cap t_{i+1} \in C_i \) and \( \{P_n\} := t_n \cap t_1 \in C_n \), where conics \( C_i \) belong to a confocal pencil together with the conic \( K \). Denote related Euler-Chasles correspondences as \( \Phi_i \).

If multisets are equal

\[
\{ C_2, C_3, \ldots, C_k \}_M = \{ C_{n-1}, C_{n-2}, \ldots, C_{n-k+1} \}_M
\]

then there exist a conic \( C_0 \) from the confocal family which contains the point \( P_1 \) and the point \( P_{k+1, n-k} := t_{k+1} \cap t_{n-k} \).

**Proof.** The Lemma follows from the fact that in the plane there are two conics from a confocal family which contain a point \( P_1 \). In our notation one conic is \( C_1 \), denote the other one as \( C_0 \) with the Euler-Chasles correspondence \( \Phi_0 \). We have:

\[
\begin{align*}
\Phi_k \circ \cdots \circ \Phi_2 \circ \Phi_1(t_1) &= t_{k+1} \\
\Phi_k \circ \cdots \circ \Phi_2 \circ \Phi_0(t_1) &= t_{k+1}.
\end{align*}
\]

Using commuting property, we get from the last equation

\[
\Phi_0 \circ \Phi_k \circ \cdots \circ \Phi_2(t_1) = t_{k+1}.
\]
Consider the intersection of the lines
\[ \Phi_{n-k+1} \circ \ldots \circ \Phi_{n-2} \circ \Phi_{n-1}(t_1) \]
and \( t_{k+1} \). Denote by \( \hat{\Phi} \) the Euler-Chasles correspondence associated with the conic \( C \) such that
\[ \Phi_{n-k+1} \circ \ldots \circ \Phi_{n-2} \circ \Phi_{n-1}(t_1) \cap t_{k+1} \in C. \]
Thus we have
\[ \hat{\Phi} \circ \Phi_{n-k+1} \circ \ldots \circ \Phi_{n-2} \circ \Phi_{n-1}(t_1) = t_{k+1}. \]

From the assumption of the Lemma, the last equation and from the equation (39) it follows that \( C = C_0 \), giving the proof of the Lemma. \( \square \)

There is an important special case of previous Lemma, when all the conics are equal:
\[ C_2 = C_3 = \cdots = C_k = C_{n-2} = \cdots = C_{n-k+1}. \]
In the case of Poncelet \( n \)-tangles even more is true: \( C_i = C_j \) for any \( i, j = 1, \ldots, n \). The union of \( k \) conics if \( n = 2k + 1 \) or \( k - 1 \) conic and the line if \( n = 2k \) together with conics from the previous Lemma form the complete projective Poncelet-Darboux grid. For the study of Poncelet-Darboux grids see \cite{7} and references therein. Notice that transversal conics from the Lemma (5) doesn’t form a decomposition of a Poncelet-Darboux curve.

**Example 4** Suppose that all vertices of a triangle lie on a conic \( C_1 \), one of the sides touches a conic \( C_2 \) and the other two sides touch the conics \( tC_1 + C_2 = 0 \) and \( sC_1 + C_2 = 0 \) respectively. If the three points of contact are not collinear, then
\[ (I_3 - I_1 ts)^2 - 4I_4(I_2 + I_1(s + t)) = 0, \]
where \( I_1, I_2, I_3, I_4 \) denote invariants of the pair of conics \((C_1, C_2)\).

**Example 5** Let \( P_1, P_2, \ldots, P_n, \ldots \) be points on a conic \( C_1 \) such that there exists a conic \( C_2 \) to which all sides \( P_iP_{i+1} \) are tangent. Assume \( P_{k+2} \neq P_k \). Then that lines \( A_1A_{k+1}, A_2A_{k+2}, \ldots A_iA_{i+k} \) touch the conic \( t_kC_1 + C_2 = 0 \) where
\[ t_2 = 0, t_3 = \frac{I_2^2 - 4I_2I_4}{4I_1I_4} \]
and
\[ t_{k+1} = \frac{(I_3^2 - 4I_2I_3) - 4I_1I_4t_k}{I_1^2I_k^2t_{k-1}}. \]

The condition for two conics \( K \) and \( C \) to be \( n \)-Poncelet related has been derived by Cayley. Recent account of the subject can be found in \cite{6}, \cite{7}.

Here, we want to present a condition for two polynomials \( \phi, f \) of degree \( n \) in order that corresponding Poncelet-Darboux curve \( S = \mathcal{PD}_K(\phi, f) \in Pon - Dar_n(K) \) has a conic component \( n \)-Poncelet related to the conic \( K \).
Theorem 9 Let a transversal Poncelet-Darboux curve:
\[ S = \mathcal{PD}_K(\phi, f) \in \text{Pon} - \text{Dar}_{n-1}(K) \]
be given, where polynomials \( f \) and \( \phi \) are without common zeros.

(i) For \( n = 2k + 1 \), curve \( S \) is completely decomposed to \( k \) conics only if there exist four values \( t_1, t_2, t_3, t_4 \) such that
\[
\deg \text{GCD} \left( \Phi_{t_i}(z), \frac{d}{dz}\Phi_{t_i}(z) \right) = k, \quad (40)
\]
for \( i = 1, 2, 3, 4 \);

(ii) for \( n = 2k \), curve \( S \) is completely decomposed to \( k - 1 \) conics and a line only if there exist four values \( t_1, t_2, t_3, t_4 \) such that:
\[
\deg \text{GCD} \left( \Phi_{t_i}(z), \frac{d}{dz}\Phi_{t_i}(z) \right) = k, \quad (41)
\]
for \( k = 1, 2 \), and
\[
\deg \text{GCD} \left( \Phi_{t_i}(z), \frac{d}{dz}\Phi_{t_i}(z) \right) = k - 1, \quad (42)
\]
for \( k = 3, 4 \).

Here, we denoted \( \Phi_{t_i}(z) := \phi(z) + t_i f(z) \), while GCD stands for the greatest common divisor of polynomials.

Proof. Suppose that Poncelet-Darboux curve \( S \) of degree \( n - 1 \) completely decomposes. Then, denote by \( C \) its conic component which is \( n \) Poncelet related to the conic \( K \). By assumption of transversality, conics \( C \) and \( K \) intersect in four points. As in Example 4 denote four intersection points \( C \cap K = \{x_1, x_2, x_3, x_4\} \). Denote four common bitangents \( t_i, i = 1, \ldots, 4 \) and denote points of contact of the tangent \( t_i \) with the conic \( C \) as \( y_i \) and the point of contact with the conic \( K \) as \( a_i \).

The conic \( C \) is \( n \)-Poncelet related to the conic \( K \) and according to the Poncelet theorem 4 there is a polygon of \( n \) sides inscribed in \( C \) and circumscribed about \( K \) with arbitrary point of \( C \) taken as a vertex.

Suppose \( n \) is odd: \( n = 2k + 1 \). Choose any of the points \( y_i \), say \( y_1 \) as a vertex. Then \( y_1 \) is a double vertex of a Poncelet \( 2k + 1 \)-polygon. Moreover, next \( k - 1 \) vertices \( c_1, \ldots, c_{k-1} \) are also double vertices. The last vertex of the Poncelet \( n \)-tangle is one of the points \( x_i \) and the Poncelet \( n \)-tangle is \( c_{k-1} \ldots c_1 y_1 y_1 c_1 \ldots c_{k-1} x_i \).

Suppose now that \( n \) is even: \( n = 2k \). In this case there are two pairs of distinguished Poncelet \( n \)-tangles. Two of them are of the form \( c_{k-1} \ldots c_1 y_j y_j c_1 \ldots c_{k-1} y_i y_i \), for example for \((i, j) = (1, 2)\) and for \((i, j) = (3, 4)\). Each of them connect a pair of common tangents and it has all other vertices as double. Another pair
of distinguished Poncelet $n$-tangles connect pair of intersection points. For example the first one connects $x_1$ and $x_2$ while the second connects $x_3$ with $x_4$. All other their vertices are double. Thus these two Poncelet $n$-tangles are of the form $d_{k-1}...d_1d_1...d_{k-1}x_2$ and $e_{k-1}...e_1x_3...e_{k-1}x_4$.

**Lemma 6** If a transversal Poncelet-Darboux curve $S = \mathcal{PD}_K(\phi, f) \in \text{Pon} - \text{Dar}_{n-1}(K)$ has a conic component $C$ which is $n$-Poncelet related to the conic $K$ then:

(i) for $n = 2k + 1$ there exist four values $t_1, t_2, t_3, t_4$ and four polynomials $Q_1, Q_2, Q_3, Q_4$ such that

\[
\begin{align*}
\Phi_{t_1}(z) &:= \phi(z) + t_1f(z) = (z - a_1)Q_1^2(z) \\
\Phi_{t_2}(z) &:= \phi(z) + t_2f(z) = (z - a_2)Q_2^2(z) \\
\Phi_{t_3}(z) &:= \phi(z) + t_3f(z) = (z - a_3)Q_3^2(z) \\
\Phi_{t_4}(z) &:= \phi(z) + t_4f(z) = (z - a_4)Q_4^2(z);
\end{align*}
\]

(ii) for $n = 2k$ there exist four values $t_1, t_2, t_3, t_4$ and four polynomials $Q_1, Q_2, Q_3, Q_4$ such that

\[
\begin{align*}
\Phi_{t_1}(z) &:= \phi(z) + t_1f(z) = (z - a_1)(z - a_2)Q_1^2(z) \\
\Phi_{t_2}(z) &:= \phi(z) + t_2f(z) = (z - a_3)(z - a_4)Q_2^2(z) \\
\Phi_{t_3}(z) &:= \phi(z) + t_3f(z) = Q_3^2(z) \\
\Phi_{t_4}(z) &:= \phi(z) + t_4f(z) = Q_4^2(z).
\end{align*}
\]

From the conditions (43) and (44) immediately follow conditions (40) and (41) together with (42) of the Theorem respectively. This proves the Theorem.

□

For the opposite direction, observe that from the conditions (40) and (41) together with (42) of the Theorem follow the conditions of the Lemma (43) and (44) by use of the transversality condition. From the transversality it follows that all multiple zeros of the polynomials in the pencil generated by $f$ and $\phi$ are of the second degree. Now, from the conditions (43) and (44) one can easily prove the following

**Lemma 7** Suppose the conditions (43) and (44) are satisfied. Denote by $\Gamma_1$ and $\Gamma_2$ the following elliptic curves:

\[
\begin{align*}
\Gamma_1 : y^2 &= (z - a_1)(z - a_2)(z - a_3)(z - a_4) \\
\Gamma_2 : Y^2 &= (X + t_1)(X + t_2)(X + t_3)(X + t_4).
\end{align*}
\]

Then, there is an $n : 1$ morphism

\[
h : \Gamma_1 \to \Gamma_2, \quad h(z, y) = (X, Y),
\]

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where
\[ X = \frac{\phi(z)}{f(z)}, \quad Y = y \frac{Q_1(z)Q_2(z)Q_3(z)Q_4(z)}{f^2(z)}. \]

From the last Lemma we see that question of decomposition of the Poncelet-Darboux curve defined by polynomials \( f \) and \( \phi \) corresponds to study of an unramified covering of degree \( n \) of the elliptic curve \( \Gamma_1 \) over the elliptic curve \( \Gamma_2 \). Such a covering is realized as factorization of the first curve by its finite subgroup. We can say even more about the above covering.

**Lemma 8**

(a) There is a constant \( N \) such that
\[ Q_1(z)Q_2(z)Q_3(z)Q_4(z) = N \left( \frac{d}{dz} \phi(z)f(z) - \phi(z) \frac{d}{dz} f \right). \]

(b) There is a relation between holomorphic differentials of elliptic curves \( \Gamma_1 \) and \( \Gamma_2 \):
\[ \frac{dz}{y} = N \frac{dX}{Y}. \]

The proof follows by straightforward calculations.

Denote by \( \Lambda_1 \) and \( \Lambda_2 \) lattices which correspond to the elliptic curves \( \Gamma_1 \) and \( \Gamma_2 \) respectively. Denote by \( u \) a parameter on \( \Gamma_1 \). Then the covering defines correspondence between functions \( z(u|\Lambda_1) \mapsto X(u|\Lambda_2) \). Thus we get

**Lemma 9** For complete decomposition of the Poncelet-Darboux curve is necessary that
\[ N = a \frac{n}{n}, \]

and
\[ a\Lambda_2 \subset \Lambda_1. \]

Then, if all above conditions are satisfied, one gets a Poncelet trajectory by choosing \( P \in \Gamma_1 \) such that \( nP \in a\Lambda_2 \subset \Lambda_1 \). The points of the trajectory correspond to parameters \( u_j = u_0 + j\eta, j = 0, \ldots, n-1 \), where \( \eta \) corresponds to the point \( P \). Denote by \( C_P \) a conic which corresponds to \( P \) in the pencil of conics generated with the conic \( K \) and its four tangents at the points \( a_1, a_2, a_3, a_4 \). Then the conic \( C_P \) is \( n \)-Poncelet related to the conic \( K \).

The systematic study of elliptic coverings was established in [11]. We pass now to Jacobi’s notation. By applying rational-linear transformations, we come to the canonical form of elliptic curves \( \Gamma_2 \) and \( \Gamma_1 \) and relation between differentials
\[ \frac{N dy}{\sqrt{(1 - y^2)(1 - \lambda y^2)}} = \frac{dx}{\sqrt{(1 - x^2)(1 - k \cdot x^2)}}. \]

The constants \( k \) and \( \lambda \) are the modules of the elliptic curves \( \Gamma_1 \) and \( \Gamma_2 \).

Following Jacobi, for given \( n \) odd and given module \( k \), we have explicit formulae for the transformations.
Theorem 10 For given \( n \) odd and for
\[
\omega = \frac{mK + m'K'}{n},
\]
where integers \( m \) and \( m' \) have no common divisors which divide \( n \), the transformation is defined by
\[
f(z) = \frac{x}{N} \prod_{r=1}^{(n-1)/2} \left( 1 - \frac{x^2}{\text{sn}^2 4r\omega} \right)^{(n-1)/2}
\]
\[
\phi(z) = \prod_{r=1}^{(n-1)/2} (1 - x^2 \cdot \text{sn}^2 4r\omega)
\]
\[
N = (-1)^{(n-1)/2} \prod_{r=1}^{(n-1)/2} \left( 1 - \frac{\text{sn}(K - 4r\omega)}{\text{sn}^2(4r\omega)} \right)
\]
\[
\lambda = \prod_{r=1}^{(n-1)/2} \left( \text{sn}^4(K - 4r\omega) \right).
\]
The transformation corresponds to an \( n \)-Poncelet trajectory, where
\[
x_i = \text{sn}(u + 4(i + 1)\omega, k) \quad i = 0, \ldots, n - 1, \quad y = \text{sn}(u/N, \lambda).
\]

We can make another view on the situation. Let us consider three arithmetic functions. Suppose an natural number \( n \in \mathbb{N} \) be given by its prime decomposition:
\[
n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r},
\]
where \( p_i \) are different prime numbers. Following [2], we define a function \( t(n) \), the number of primitive \( n \)-torsion points on an elliptic curve:
\[
t(n) := (p_1^2 - 1)p_1^{2(k_1 - 1)}(p_2^2 - 1)p_2^{2(k_2 - 1)} \cdots (p_r^2 - 1)p_r^{2(k_r - 1)}.
\]
As an example, for \( n = p \) a prime number, \( t(p) = p^2 - 1 \). The function \( t \) is a multiplicative arithmetic function. It represents the number of Poncelet polygons in total up to the porism, with fixed caustic and confocal pencil of conics.

As the second arithmetic function, we introduce a function \( \sigma'(n) \) as a multiplicative function which is for \( n \) odd equal to
\[
\sigma'(n) = n \left( 1 + \frac{1}{p_1} \right) \left( 1 + \frac{1}{p_2} \right) \cdots \left( 1 + \frac{1}{p_r} \right), \quad n \text{ odd}.
\]
For \( n = 2^k \) we define
\[
\sigma'(2^k) := 2^{k+1} - 1.
\]
For example, for \( n = p \) a prime number, we have
\[
\sigma'(p) = p + 1 = \sigma(p),
\]
the $\sigma'$ function in this case is equal to the $\sigma$ function, the sum of divisors of $p$. The function $\sigma'(n)$ counts the number of degree $n$ elliptic coverings of the above form assuming the module $k$ being fixed. The number of transformations listed in Theorem (10) for given odd number $n$ is equal to $\sigma'(n)$.

The third arithmetic function we are going to consider is well known Euler function $\varphi(n)$ counting the numbers smaller than $n$ relatively prime to $n$:

$$\varphi(n) = n \left( 1 - \frac{1}{p_1} \right) \left( 1 - \frac{1}{p_2} \right) \cdots \left( 1 - \frac{1}{p_r} \right).$$

Proposition 4 For $n$ odd the identity holds:

$$t(n) = \sigma'(n) \cdot \varphi(n).$$

Proposition 5 For $n = 2^k$, $k \geq 1$, the inequality holds:

$$t(n) \leq \sigma'(n) \cdot \varphi(n).$$

The equality holds only for $k = 1$.

Theorem 11 Let $m$ be an arbitrary odd number. For $n = 2^k \cdot m$ where $k = 0, 1$ all above elliptic coverings are cyclic.

For $n = 2^k \cdot m$ and every $k > 1$, there are elliptic coverings of the above form which are not cyclic.

Thus, we come to the converse of the Theorem (9).

Theorem 12 Let $m$ be an arbitrary odd number. For $n = 2^k \cdot m$ where $k = 0, 1$ the conditions of the Theorem (9) are sufficient as well.

For $n = 2^k \cdot m$, where $k > 1$, one needs to do careful analysis to distinguish those coverings which are cyclic. The general description of transformations of an even order $n$ is given by

$$\phi(z) = \frac{1}{2}((1 + z)(1 + kz)T^2 + (1 - z)(1 - kz)T'^2),$$

$$f(z) = \frac{1}{2}((1 + z)(1 + kz)T^2 - (1 - z)(1 - kz)T'^2),$$

where

$$T(z) = P(z) + zQ(z), \quad T'(z) = P(z) - zQ(z).$$

Here $P, Q$ are even polynomials, in other words, polynomials in $z^2$. 

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Suppose a transformation of degree $n$ consists of $k$ cycles of length $l$, where $n = kl$:

\[
\begin{align*}
\text{sn}u_0 & \quad \text{sn}(u_0 + \omega) \quad \ldots \quad \text{sn}(u_0 + (l - 1)\omega) \\
\text{sn}u_1 & \quad \text{sn}(u_1 + \omega) \quad \ldots \quad \text{sn}(u_1 + (l - 1)\omega) \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\text{sn}u_{k-1} & \quad \text{sn}(u_{k-1} + \omega) \quad \ldots \quad \text{sn}(u_{k-1} + (l - 1)\omega)
\end{align*}
\]

where $u_i = u_{i-1} + \omega_i$, $i = 1, \ldots, k - 1$, and

\[
\omega_k = \omega - \sum_{i=1}^{k-1} \omega_i.
\]

Denote by $C_i$ the conic which corresponds to $\omega_i$ in the pencil of conics defined by $K$ and the four tangents at $a_j$, $j = 1, \ldots, 4$. Geometric interpretation of the last transformation is realized through the Complete Poncelet Theorem: corresponding Poncelet polygon consists of $n = kl$ tangents to the conic $K$ with vertices

\[
P_1 P_2 \ldots P_k P_{k+1} \ldots P_{2k} \ldots P_{(l-1)k+1} \ldots P_{lk}
\]

where $P_1 \in C_1$, $P_2 \in C_2$, and more generally

\[
P_s \in C_r \iff s \equiv r \pmod{k}.
\]

The same interpretation may be done for cyclic transformations of composite degree $n = kl$ as well. In the case of cyclic transformations of composite order, in this way we get another geometric realization beside the one connected with Poncelet theorem from the beginning.

### 4.3 Conic choreography of the ”\(^n\binom{3}{2}\)-body” problem

In this Section we want to get effective description of polynomials $f$ and $\phi$ which satisfy the Theorem (9). We use parametrizations of the transformations obtained by classics algebraically. For small odd numbers, we give complete description of initial data of the isofocal transformations which correspond to the cases of complete decomposition of the Poncelet-Darboux curves.

Assume $n$ is an odd number. Then all transformations are given by the formulae

\[
\begin{align*}
\phi(x) &= P^2 + 2PQ + Q^2x^2 \\
f(x) &= x(P^2 + 2PQ + Q^2x^2)
\end{align*}
\]

where $P$ and $Q$ are polynomials in $x^2$:

\[
\begin{align*}
P(x) &= \alpha + \gamma x^2 + \epsilon x^4 + \ldots \\
Q(x) &= \beta + \delta x^2 + \zeta x^4 + \ldots
\end{align*}
\]

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of degree $p$ in $x^2$ both if $n = 4p + 3$ and of degree $p$ and $p - 1$ respectively if $n = 4p + 1$. Then
\[ \frac{1}{N} = 1 + \frac{2\beta}{\alpha}. \]

For $n = 3$ we have
\[ P = \alpha, \quad Q = \beta. \]

Thus
\[ f(x) = (\alpha^2 + 2\alpha \cdot \beta)x + \beta^2 x^3 \]
\[ \phi(x) = \alpha^2 + (2\alpha \cdot \beta + \beta^2)x^2. \]

Now, one can easily get the initial data of the isofocal deformations, which correspond to completely decomposable situation for $n = 3$.

**Proposition 6** The initial data of completely decomposable situation are given up projective-linear transformations, by the formulae

\[ \alpha_1 = 0 \]
\[ \alpha_{23} = \pm \sqrt{-\frac{\alpha^2 + 2\alpha\beta}{\beta^2}} \]
\[ m_1 = \frac{-\beta^2(2\alpha\beta + \beta^2)}{\alpha^2 + 2\alpha\beta} \]
\[ m_2 = m_3 = \frac{\alpha^2(\alpha^2 + 2\alpha\beta) + \beta^2(2\alpha\beta + \beta^2)}{2(\alpha^2 + 2\alpha\beta)}. \]

**Case $n = 5$**

For $n = 5$ we have
\[ P(x) = \alpha + \gamma x^2, \quad Q(x) = \beta. \]

Thus,
\[ f(x) = x((\alpha^2 + 2\alpha \cdot \beta) + (\beta^2 + 2\alpha\gamma + 2\gamma\beta)x^2 + \gamma x^4) \]
\[ \phi(x) = \alpha^2 + (2\alpha\beta + 2\alpha\gamma + \beta^2)x^3 + (\gamma^2 + 2\beta\gamma)x^4. \]

Denote by
\[ A = \frac{-(2\alpha\gamma + 2\gamma\beta + \beta^2) + \sqrt{D}}{2\gamma^2} \]
\[ B = \frac{-(2\alpha\gamma + 2\gamma\beta + \beta^2) - \sqrt{D}}{2\gamma^2} \]
\[ D = 2\alpha\gamma + 2\gamma\beta + \beta^2 - 4\gamma^2(\alpha^2 + 2\alpha\beta). \]

Now we have
Proposition 7  The initial data of completely decomposable situation are given up projective-linear transformations, by the formulae

\[ \alpha_1 = 0 \]
\[ \alpha_{23} = \pm \sqrt{A} \]
\[ \alpha_{45} = \pm \sqrt{B} \]
\[ m_1 = \frac{\gamma^2 + 2\alpha\gamma}{AB} \]
\[ m_2 = m_3 = \frac{1}{2(B - A)}(2\alpha\gamma + 2\beta\alpha + \beta^2 + \frac{A + B}{AB}(\gamma^2 + 2\alpha\gamma) - A\alpha^2 + \frac{\gamma^2 + 2\alpha\gamma}{B}) \]
\[ m_4 = m_5 = \frac{1}{2(B - A)}(2\alpha\gamma + 2\beta\alpha + \beta^2 + \frac{A + B}{AB}(\gamma^2 + 2\alpha\gamma) - B\alpha^2 + \frac{\gamma^2 + 2\alpha\gamma}{A}) \]

**Case n = 7**

For \( n = 7 \) we have

\[ P = \alpha + \gamma x^2, \quad Q = \beta + \delta x^2, \]

and

\[ \begin{aligned} 
\phi &= x(\alpha^2 + 2\alpha\beta + (2(\alpha\gamma + \gamma\beta + \alpha\delta)x^2 + (\gamma^2 + 2(\gamma\delta + \beta\delta)x^4 + \delta^2x^6) \\
\hat{f} &= \alpha^2 + (2(\alpha\gamma + \alpha\beta) + \beta^2)x^2 + (\gamma^2 + 2(\gamma\beta + \alpha\delta + \beta\delta))x^4 + (2\gamma\delta + \delta^2)x^6. 
\end{aligned} \]

Denote by \( A_1, A_2, A_3 \) the three roots of the polynomial

\[ (\alpha^2 + 2\alpha\beta + (2(\alpha\gamma + \gamma\beta + \alpha\delta)x + (\gamma^2 + 2(\gamma\delta + \beta\delta)x^2 + \delta^2x^3, \]

and the seven zeros of the polynomial \( \phi \) are

\[ \begin{aligned} 
\alpha_0 &= 0, \quad \alpha_{12} = \pm \sqrt{A_1} \\
\alpha_{34} &= \pm \sqrt{A_2}, \quad \alpha_{56} = \pm \sqrt{A_3}. 
\end{aligned} \]

To calculate the initial weights, introduce the new coordinates

\[ \begin{aligned} 
Z_1 &= m_2 + m_3, \quad \hat{Z}_1 = m_2 - m_3 \\
Z_2 &= m_4 + m_5, \quad \hat{Z}_2 = m_4 - m_5 \\
Z_3 &= m_6 + m_7, \quad \hat{Z}_3 = m_6 - m_7 
\end{aligned} \]

Now, one gets the initial weights from the system of linear equations:

\[ m_1 = \frac{2\gamma\delta + \delta^2}{\alpha^2}, \]
\[
Z_1 + Z_2 + Z_3 = d_1 \\
\alpha_2^2 Z_1 + \alpha_4^2 Z_2 + \alpha_6^2 Z_3 = d_2 \\
\alpha_2^2 (\alpha_4^2 + \alpha_6^2) Z_1 + \alpha_4^2 (\alpha_2^2 + \alpha_6^2) Z_2 + \alpha_6^2 (\alpha_4^2 + \alpha_2^2) Z_3 = d_3 \\
\alpha_2 Z_1 + \alpha_4 Z_2 + \alpha_6 Z_3 = 0 \\
\alpha_2 (\alpha_4^2 + \alpha_6^2) \dot{Z}_1 + \alpha_4 (\alpha_2^2 + \alpha_6^2) \dot{Z}_2 + \alpha_6 (\alpha_4^2 + \alpha_2^2) \dot{Z}_3 = 0 \\
\alpha_2 \alpha_4 \alpha_6 \dot{Z}_1 + \alpha_4 \alpha_2 \alpha_6 \dot{Z}_2 + \alpha_6 \alpha_4 \alpha_2 \dot{Z}_3 = 0,
\]

where
\[
d_1 = \alpha^2 - \frac{2\gamma \delta + \delta^2}{\alpha^2} \\
d_2 = (\gamma^2 + 2(\gamma \beta + \alpha \delta) + 2\beta \delta) - \alpha^2 (\gamma^2 + 2\gamma \delta + 2\beta \delta) \\
d_3 = (\gamma^2 + 2(\gamma \beta + \alpha \delta) - \frac{2\gamma \delta + \delta^2}{\alpha^2} (2(\alpha \gamma + \gamma \beta + \alpha \delta) + \beta^2)).
\]

5 Concluding remarks

5.1 Bifocal transformations

After studying isofocal deformations, we may pass to a new class of deformations, to bifocal deformations, defined by the relation
\[
B_\iota(z) := \frac{m_1(t)}{z - \alpha_1(t)} + \cdots + \frac{m_n(t)}{z - \alpha_n(t)} = \frac{f(z) + tg(z)}{\phi(z) + tf(z)},
\]

where polynomials \( \phi \) and \( f \) are the same as before, and \( g \) is a new polynomial of degree \( n - 1 \).

By applying the Moser trick once again and using the Flashka coordinates, we come to the following

**Proposition 8** The dynamical system (46) reduces in the Flashka coordinates to the form
\[
\dot{a}_1 = 0 \quad \dot{b}_1 = 0 \\
\dot{a}_2 = 0 \quad \dot{b}_2 = 0 \\
\quad \quad \cdots \\
\dot{a}_{n-2} = 0 \quad \dot{b}_{n-2} = 0 \\
\dot{a}_{n-1} = \frac{b_{n-1}}{a_{n-1}} \quad \dot{b}_{n-1} = -1 \\
\dot{b}_n = -1.
\]
Denote zeros of the polynomial $G_t(z) = f(z) + tg(z)$ as $f_1(t), \ldots, f_{n-1}(t)$. Here $f_1(0) = f_1, \ldots, f_{n-1}(0) = f_{n-1}$ are zeros of the polynomial $f$. In bifocal case, focal points of Marden curves evolve during the time as zeros of $G_t$.

Denote by $a(\alpha) := a(\alpha_1, \ldots, \alpha_n)$ the matrix

$$
a(\alpha) = 
\begin{pmatrix}
1 & \sigma_1^1(\alpha_1, \ldots, \alpha_n) & \sigma_1^2(\alpha_1, \ldots, \alpha_n) & \ldots & \sigma_1^n(\alpha_1, \ldots, \alpha_n) \\
\vdots & \sigma_{n-1}^1(\alpha_1, \ldots, \alpha_n) & \sigma_{n-1}^2(\alpha_1, \ldots, \alpha_n) & \ldots & \sigma_{n-1}^n(\alpha_1, \ldots, \alpha_n)
\end{pmatrix}
$$

(48)

Denote also the columns

$$
\dot{\alpha} = (\dot{\alpha}_1 \ \dot{\alpha}_2 \ \ldots \ \dot{\alpha}_n)^T \\
\dot{\bar{B}} = (\dot{B}_1 \ \dot{B}_2 \ \ldots \ \dot{B}_n)^T \\
\dot{\bar{m}} = (\dot{m}_1 \ \dot{m}_2 \ \ldots \ \dot{m}_n)^T \\
\dot{\bar{m}} = (\dot{\bar{m}}_1 \ \dot{\bar{m}}_2 \ \ldots \ \dot{\bar{m}}_n)^T
$$

**Proposition 9** The first part of the differential equations for isofocal and bifocal deformations may be written down in the following way:

$$
\dot{\alpha} = a^{-1}(\alpha_1, \ldots, \alpha_n)\dot{\bar{B}}.
$$

(49)

The second part of the equations of motion are equivalent to

$$
\dot{\bar{m}} = a^{-1}(\alpha)(\dot{\bar{C}} - \dot{\alpha}(\alpha)\dot{\bar{m}}),
$$

(50)

where $\dot{\alpha}(\alpha) := \dot{a}(\alpha_1, \ldots, \alpha_n)$ denotes the matrix obtained from $a(\alpha_1, \ldots, \alpha_n)$ by differentiating each matrix entry of the matrix $a$ with respect to the time, taking into the account the equations (49). The column $\dot{\bar{C}}$ is equal to the differential of the column $\dot{\bar{B}}$ with respect to the time. In the case of isofocal deformations, $\dot{\bar{C}}$ is equal to the zero-vector.

The second part of the equations of motion follows by differentiation with respect to time, from the relation,

$$a(\alpha_1, \ldots, \alpha_n)\dot{\bar{m}} = \dot{\bar{B}}.$$
5.2 Toma-Trautmann case of a conic component

There is one more interesting case of conic component of a Darboux-Poncelet curve. Such cases have been studied by Trautmann systematically in [20], following an example constructed by Toma. It is the case where a conic component $C$ of a Poncelet-Darboux curve $S$ of degree $n - 1$ is by itself a Poncelet-Darboux curve. In other words, in this case the conic $C$ is 3-Poncelet related to the caustic $K$. Suppose that $S$ is defined by a pencil of polynomials of degree $n$ generated by $f$ and $\phi$. Applying the same arguments as before we conclude that there should exist four parameters $t_1, t_2, t_3, t_4$ such that

\[
\Phi_{t_1}(z) := \phi(z) + t_1 f(z) = (z - a_1)Q^2_1(z)S_1(z)
\]
\[
\Phi_{t_2}(z) := \phi(z) + t_2 f(z) = (z - a_2)Q^2_2(z)S_2(z)
\]
\[
\Phi_{t_3}(z) := \phi(z) + t_3 f(z) = (z - a_3)Q^2_3(z)S_3(z)
\]
\[
\Phi_{t_4}(z) := \phi(z) + t_4 f(z) = (z - a_4)Q^2_4(z)S_4(z);
\]

where polynomials $Q_i$ are of degree one and polynomials $S_i$ are of degree $n - 3$.

Thus we get

**Proposition 10** Denote by $\Gamma$ hyperelliptic curve of genus $g = 2n - 5$ defined by the equation

\[
y^2 = (z - a_1)(z - a_2)(z - a_3)(z - a_4)S_1(z)S_2(z)S_3(z)S_4(z),
\]

and by $\Gamma_2$ the elliptic curve

\[
Y^2 = (X + t_1)(X + t_2)(X + t_3)(X + t_4).
\]

The Toma-Trautmann case of conic component of a Poncelet-Darboux curve of degree $n - 1$ is related to the above covering of the hyperelliptic $\Gamma$ of genus $2n - 5$ over the elliptic curve $\Gamma_2$.

Although the hyperelliptic coverings of elliptic curves are much more complicated than the elliptic coverings of elliptic curves discussed above, an intensive study of former coverings has been done, see for example [1], [21] and references therein. It sounds as an interesting question to describe hyperelliptic coverings which arise in the Toma-Trautmann case.

5.3 Infrapolynomials and the positivity problem in isofocal dynamics

From [13] we know that one important interpretation of the Marden Theorem is connected with the study of infrapolynomials (see Section 5 of Chapter 1 of [13] for definitions). After Fekete it is known that for given closed bounded set $E$ containing at least $n + 1$ points, a polynomial $p$ of degree $n$ is an infrapolynomial
if there exist an integer $k$, such that $n \leq k \leq 2n$, a set of positive constants $m_j$, such that $m_0 + \cdots + m_k = 1$ and a set of $k + 1$ points $\{z_0, z_1, \ldots, z_k\} \subset E$ such that $p(z)$ is a factor of the polynomial $F(z)$:

$$F(z) = \Omega(z) \sum_{j=0}^{k} \frac{\lambda_j}{z-z_j}, \quad \Omega(z) = \prod_{j=0}^{k} (z-z_j).$$

The last theorem of Fekete and the connection with the study of infrapolynomials thus motivate a study of positivity conditions in our isofocal dynamics: 1) to describe intervals in isofocal dynamics where all weights are positive; 2) to describe the initial values for which the weights are positive during entire evolution; 3) to relate more closely isofocal dynamics and the theory of infrapolynomials.

6 Appendix

The case $n = 3$. Even in the simplest case $n = 3$ when the curve $C$ is a conic, inscribed in the triangle formed by the zeros of a polynomial of degree 3 and tangent to the sides of the triangle at their midpoints, the result of the Siebeck theorem is nontrivial and interesting and attracted lot of attention not only in the past but also nowadays (see for example [12]). We are going to present main points of the proof of the Siebeck theorem for $n = 3$ following mostly [12] and references therein.

We start with well known focal properties of ellipses and caustic properties of elliptical billiards. They are going to play central role in the proof of the Siebeck theorem.

**Lemma 10 (Focal property of the ellipse)** Let $E$ be an ellipse with foci $F_1, F_2$ and $A \in E$ an arbitrary point. Then segments $AF_1, AF_2$ satisfy the billiard law on $E$.

**Lemma 11** Let two lines satisfy the billiard law on the ellipse $E$. If one of the lines is tangent to the ellipse $E'$ that is confocal with $E$, then the other one is also tangent to $E'$.

**Corollary 1** Let $T$ be a billiard trajectory within ellipse $T$. If $C$ is a conic confocal to $E$ such that one segment of $T$ is tangent to $C$, then all segments of $T$ are tangent to $C$.

**Corollary 2** Let $B$ be a point outside the ellipse $E'$ with focal points $F_1$ and $F_2$. Denote tangents to the ellipse from the point $B$ as $BB_1$ and $BB_2$, where $B_i$ are points of contact with the ellipse. Then the angles $B_1BF_1$ and $B_2BF_2$ are equal.
The first observation in the proof is that the statement is invariant to applications of affine transformations. Now, we assume that a real plane is identified with the field of complex numbers and that numbers $\alpha_1, \alpha_2, \alpha_3$ correspond to numbers $-1, 1, w$ where $w$ is in upper half-plane.

The polynomial $P$ gets the form

$$P(z) = (z - 1)(z + 1)(z - w) = z^3 - wz^2 - z + w,$$

and the derivative is

$$DP(z) = 3 \left(z^2 - \frac{2}{3}wz - \frac{1}{3}\right).$$

Denote the zeros of the derivative as $z_4, z_5$. After some analysis of the formulae

$$z_4 + z_5 = \frac{2}{3}w,$$
$$z_4z_5 = -\frac{1}{3},$$

we conclude that both of the points $z_4$ and $z_5$ are in the upper half plane. Then, from the second formula we conclude that the sum of their arguments $\theta_4$ and $\theta_5$ is equal to $\pi$. Denote the line connecting $z_4$ and the origin $O$ as $L_{z_4O}$ and the line connecting $z_5$ and the origin $O$ as $L_{z_5O}$. As a result, we come to the conclusion that the angle between the line $L_{z_4O}$ and negative real semi-axis is equal to the angle of the line $L_{z_5O}$ and the positive real semi-axis. By application of focal properties of ellipses, Proposition 10 and Proposition 11, we see that there is an ellipse $E_1$, with $z_4, z_5$ as foci, such that it touches the real axis at the origin. In other words, it touches the segment $[\alpha_1, \alpha_2]$ at the midpoint.

By symmetry, we see that there are confocal ellipses $E_1, E_2$ and $E_3$ with focal points at the zeros of derivative of the polynomial, each of which touches one of the sides of the triangle in the midpoint.

Next, we need to show that these three ellipses coincide. Again, we apply affine transformation and transform the triangle to a new one with vertices $0, 1, w$, where $w$ is again in the upper half-plane.

The polynomial $P$ gets the form

$$P(z) = z(z - 1)(z - w) = z^3 - (1 + w)z^2 + wz,$$

and the zeros $z_4, z_5$ of the derivative

$$DP(z) = 3 \left(z^2 - \frac{2}{3}(1 + w)z + w\right)$$

satisfy

$$z_4 + z_5 = \frac{2}{3}(1 + w),$$
$$z_4z_5 = \frac{w}{3}.$$
After analysis of the last formulae we again conclude that $z_4, z_5$ are in the upper half-plane and that the sum of their arguments is equal to the argument of $w$. As a result, this time we see that the angle between the line $L_{z_4}O$ and the line $L_{Ow}$ is equal to the angle of the line $L_{z_5}O$ and the positive real semi-axis. From previous considerations, we know that the ellipse $E_1$ with $z_4, z_5$ as focal points touches real axis at the point $1/2$. One can easily see that the origin $O$ is outside the ellipse $E_1$. Now, we apply another well known focal property of ellipses (see Corollary of Proposition [14]) to the ellipse $E_1$, point $O$ outside $E_1$ and two tangents $t_1$ and $t_2$ to $E_1$ from the point $O$. The angle between $t_1$ and $L_{Oz_4}$ is equal to the angle between $t_2$ and $L_{Oz_5}$. One tangent, $t_1$ is the real axis. Thus, the second tangent coincides with the line $L_{Ow}$. This proves that the line $L_{Ow}$ is tangent to the ellipse $E_1$. But, again by focal properties of ellipses, (see Proposition [13]) among confocal ellipses, there is only one tangent to a given line. This shows that $E_1 = E_2$ and $E_1$ touches the segment $[O, w]$ at the midpoint. This finishes the proof of the Siebeck theorem in the case $n = 3$.

The Lemmae and Corollaries from the beginning of the Appendix play very important role in the study of billiard systems within ellipses. These systems provide mechanical interpretation of the Poncelet theorem and this is the way how potential connection of Marden’s and Poncelet’s theorem has been anticipated.

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