AUTOMORPHISMS OF THE GENERALIZED FERMAT CURVES

ARISTIDES KONTOGEORGIS AND PANAGIOTIS PARAMANTZOGLOU

ABSTRACT. The automorphism group of the generalized Fermat $F_{k,n}$ curves is studied. We use tools from the theory of complete projective intersections in order to prove that every automorphism of the curve can be extended to an automorphism of the ambient projective space. In particular if $k-1$ is not a power of the characteristic, then a conjecture of Y. Fuertes, G. González-Diez, R. Hidalgo, M. Leyton is proved.

1. INTRODUCTION

In this article $K$ is an algebraically closed field of characteristic $p$, and $k$ is a natural number with $(p,k)=1$. Let us consider the generalized Fermat curves $F_{k,n}$ given by the $n-1$ equations

\begin{align}
 x_1^k + x_2^k + x_3^k &= 0 \\
 \lambda_1 x_1^k + x_2^k + x_4^k &= 0 \\
 \lambda_2 x_1^k + x_2^k + x_5^k &= 0 \\
 &\vdots \\
 \lambda_{n-2} x_1^k + x_2^k + x_{n+1}^k &= 0.
\end{align}

This curve can be considered as a ramified Galois cover $F_{k,n} \to \mathbb{P}^1$ with branched points the set $\{0, 1, \infty, \lambda_1, \ldots, \lambda_{n-2}\}$. Moreover the Galois group $\text{Gal}(F_{n,k}/\mathbb{P}^1) = H$ is isomorphic to a direct product $H \cong (\mathbb{Z}/k\mathbb{Z})^n$. Using the Riemann-Hurwitz formula we can compute the genus $g_{k,n}$ of the curve $F_{k,n}$:

\begin{align}
 g_{k,n} = \frac{1}{2} \left( 2 + k^{n-1} \left( (n-1)(k-1) - 2 \right) \right).
\end{align}

Originally these curves were introduced by G. González-Diez, R. Hidalgo, M. Leyton [5] as orbifolds with signature $(0, n+1; k, \ldots, k)$. They only considered the case of Riemann surfaces but we will extend our study to more general field $K$ as defined above.

Studying the automorphism groups of $F_{k,n}$ is a difficult question. For $n = 2$ the generalized Fermat curves are the ordinary Fermat curves. Their automorphism groups over algebraically closed fields of characteristic $p \nmid k$ were studied by P. Tzermias [11] in characteristic 0 and H. Leopoldt [10] in positive characteristic.

In [4] Y. Fuertes, G. González-Diez, R. Hidalgo, M. Leyton studied the $F_{k,3}$ case and conjectured that the Galois group $H = \text{Gal}(F_{n,k}/\mathbb{P}^1)$ is normal in the whole automorphism group. Aim of this article is to prove their conjecture. Notice that G. González-Diez, R. Hidalgo, M. Leyton prove [5, Cor. 9] that every automorphism which normalizes $H$ is linear i.e. a subgroup $\text{PGL}(n+1, \mathbb{C})$. We will essentially prove that every automorphism of $F_{k,n}$ is linear. Our main result is:

\begin{flushleft}
Date: September 11, 2014.
\end{flushleft}

The first author is supported by the Project “Thalis. Algebraic modelling of topological and Computational structures”. The Project “THALIS” is implemented under the Operational Project “Education and Life Long Learning” and is co-funded by the European Union (European Social Fund) and National Resources (ESPA).
Theorem 1. Let $F_{k,n}$ be the generalised Fermat curve defined over an algebraically closed field $K$ of characteristic $p$, such that $(p,k) = 1$. The group $\Aut(F_{k,n})$ is a subgroup of $\PGL(n + 1, K)$, which is the automorphism group of the ambient projective space.

- If $k - 1$ is not a power of the characteristic then $\Aut(F_{k,n})$ consists of matrices such that only an element in each row and column is non-zero.
- If $k - 1 = p^h$ is a power of the characteristic, then the group of automorphism consists of elements $A = (a_{ij})$ such that

$$A^t \Sigma_i A^q = \sum_{\mu=0}^{n-2} b_{i,\mu} \Sigma_{\mu},$$

for a $(n - 1) \times (n - 1)$ matrix $(b_{i,\mu})$, where $\Sigma_i$ are certain $(n + 1) \times (n + 1)$ matrices, defined in eq. (11).

If $k - 1$ is not a a power of the characteristic, then the group $H = \Gal(F_{k,n}/\mathbb{P}^1) \cong (\mathbb{Z}/k\mathbb{Z})^n$ is a normal subgroup of the full automorphism group $\Aut(F_{k,n})$. In this case we have the short exact sequence:

$$1 \to H \to \Aut(F_{k,n}) \to G_0 \to 1,$$

where $G_0$ is the subgroup of $\PGL(2, K) = \Aut(\mathbb{P}^1)$ which leaves invariant the set of branch points $\{0, 1, \infty, \lambda_1, \ldots, \lambda_{n-2}\}$.

The curves $F_{k,n}$ can be studied in terms of a variety of tools: as Kummer extensions of the rational function field, as quotient of the hyperbolic plane etc. In this article we see them as complete intersections in a projective space defined by the set of equations given in eq. (1). Recall that a closed subscheme $Y$ of $\mathbb{P}^n$ is called a (strict) complete intersection, if the homogeneous ideal in $K[x_1, \ldots, x_{n+1}]$ can be generated by $\text{codim}(Y, \mathbb{P}^n)$ elements.

For curves $F_{k,n}$ defined over fields of positive characteristic the automorphism group is different than the generic picture only if $k - 1$ is a power of the characteristic. This is an expected phenomenon, seen also in the case of the Fermat curves $x_1^{q^1} + x_2^{q^2} + x_3^{q^3} = 0$, which have $\PGL(3, q^2)$ as automorphism group, see [10]. Essentially this happens since raising to a $p$-power is linear and the Fermat curve behaves like a quadratic form.

Our interest for the generalized curves started by studying the work of Y. Ihara [8] on Braid representations of the absolute Galois groups. By Belyi theorem he considered covers of the projective line ramified above $\{0, 1, \infty\}$ and the Fermat curve and its arithmetic emerged naturally. If one tries to generalise to the more general $n + 1$-ramified covers the generalised Fermat curves and their arithmetic emerged in a similar way. This will be the object of an other article.

2. COMPLETE INTERSECTIONS AND LINEAR AUTOMORPHISMS

We begin our study by

Proposition 2. The curve $F_{k,n}$ as it defined by the set of equations in eq. (1) forms a non-singular projective variety of $\mathbb{P}^n$. If we consider the same set of equations defining a projective variety $V_\ell$ in $\mathbb{P}^{n+\ell}$ for some $\ell \geq 1$, then the singular locus of $V_\ell$ consists of elements with projective coordinates $[x_1 : \cdots : x_{n+1} : x_{n+2} : \cdots : x_{n+\ell+1}]$ such that $x_1 = \cdots = x_{n+1} = 0$ and is a subvariety isomorphic to $\mathbb{P}^{\ell-1}$ for $\ell \geq 1$. In the $\ell = 1$ case the singular locus is just a point.

Proof. The curve is given as the intersection of $n - 1$ hypersurfaces $f_i := \lambda_1 x_1^i + x_2^i + x_3^{i+1}$ for $i = 0, \ldots, n - 2$. We consider the matrix of $\nabla f_i$ written as rows, where we write only
the first \( n + 1 \)-coordinates and omit the last \( \ell \) coordinates (which are zero anyway):

\[
\begin{pmatrix}
    kx_1^{k-1} & kx_2^{k-1} & kx_3^{k-1} & 0 & \ldots & 0 \\
    \lambda_1 kx_1^{k-1} & kx_2^{k-1} & 0 & kx_4^{k-1} & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    \lambda_{n-2} kx_1^{k-1} & kx_2^{k-1} & 0 & \ldots & 0 & kx_{n+1}^{k-1}
\end{pmatrix}
\]

(4)

By the defining equations of the curve (variety if \( \ell > 0 \)) we see that a point which has two variables \( x_i = x_j = 0 \) for \( i \neq j \) and \( 1 \leq i, j \leq n + 1 \) has also \( x_t = 0 \) for \( t = 1, \ldots, n + 1 \). Therefore the above matrix has the maximal rank \( n - 1 \) at all points of the curve.

For the \( \ell = 0 \) case the defining hypersurfaces are intersecting transversally and the corresponding algebraic curve they define is non-singular. \( \square \)

**Proposition 3.** The ideal defined by equations (1) in \( \mathbb{P}^{n+\ell} \) is prime and corresponds to an irreducible variety which is a complete intersection of dimension \( 1 + \ell \) and of codimension \( n - 1 \).

**Proof.** We will follow the method of [9, sec. 3.2.1]. Observe first that the defining equations \( \{ f_0, \ldots, f_{n-2} \} \) form a regular sequence, and \( K[x_1, \ldots, x_{n+\ell}] \) is a Cohen-Macaulay ring and the ideal \( I \) defined by the \( \{ f_1, \ldots, f_{n-2} \} \) is of codimension \( n - 1 \). The ideal \( I \) is prime by the Jacobian Criterion [3, Th. 18.15], [9, Th. 3.1] and proposition 2. In remark [9, 3.4] we pointed out that the ideal \( I \) is prime if the singular locus of the algebraic set defined by \( I \) has big enough codimension. \( \square \)

**Remark 4 (Stable Family).** Consider now the polynomial ring \( R_1 := K[\lambda_1, \ldots, \lambda_{n-2}] \) and consider the ideal \( J \) generated by \( \prod_{i=1}^{n-2} \lambda_i (\lambda_i - 1) \cdot \prod_{i<j} (\lambda_i - \lambda_j) \). We consider the localization \( R \) of the polynomial ring \( R_1 \) with respect to the multiplicative set \( R_1 - J \). The affine scheme \( \text{Spec} R \) is the space of different points \( P_1, \ldots, P_{n+1} \), and the family \( \mathcal{X} \rightarrow \text{Spec} R \) is a stable family of curves since it has non-singular fibers of genus \( \geq 2 \).

By the results of Deligne-Mumford [2, lemma I.12] any automorphism of the generic fiber is also an automorphism of the special fiber. Special fibers have more automorphisms, when the ramified points

\[ \{ 0, 1, \infty, \lambda_1, \ldots, \lambda_{n-2} \} \]

are in such a configuration, so that a finite automorphism group of \( \text{PGL}(2, K) \) permutes them.

**Proposition 5.** Consider a complete intersection \( Y \subset \mathbb{P}^s \) of projective hypersurfaces \( Y_i \) of degree \( d_i \) for \( i = 1, \ldots, r \). The canonical sheaf \( \omega_Y \) is given by

\[ \omega_Y = \mathcal{O}_Y \left( \sum_{i=1}^{r} d_i - s - 1 \right) \]

**Proof.** [7, exer. 8.4 p. 188] \( \square \)

The curve \( F_{k,n} \) is given as complete intersection of \( n - 1 \) hypersurfaces of degree \( k \). Therefore, we have the following

**Corollary 6.** The canonical sheaf on the curves \( F_{k,n} \) is given by

\[ \omega_{F_{k,n}} = \mathcal{O}_{F_{k,n}}((n - 1)k - n - 1) = \mathcal{O}_{F_{k,n}}((n - 1)(k - 1) - 2) \]

Of course this is compatible with the genus computation given in eq. (3) since the degree of \( \mathcal{O}_{F_{k,n}}(1) \) is \( k^{n-1} \).

**Proposition 7.** Let \( i : X \hookrightarrow \mathbb{P}^s \) be a closed projective subvariety, such that the map

\[ H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(1)) \xrightarrow{i^*} H^0(X, \mathcal{O}_X(1)) \]

is an isomorphism. Every automorphism of \( X \) preserving \( \mathcal{O}_X(1) \) can be extended to an automorphism of the ambient projective space.
Proof. [9, prop. 2.1]

We can prove that every automorphism is linear in the following way: Every automorphism \( \sigma \) of the curve \( F_{k,n} \) should preserve the canonical sheaf so it should preserve \( \mathcal{O}_{F_{k,n}}((n-1)(k-1)-2) \). Does it preserve \( \mathcal{O}_{F_{k,n}}(1) \)? This is certainly true if \( \text{Pic}(F_{k,n}) \) has no torsion. In general curves have torsion in their Picard group.

In order to have linear automorphisms we see the generalized Fermat curves as higher dimensional varieties, embedded in \( \mathbb{P}^{n+\ell} \), which are still complete intersections but they have torsion free Picard group, see [1, Th. II.1.8], [6, Exp. XII Cor. 3.7]. So every automorphism they have is linear. In particular every automorphism of the original curve can be extended to a linear automorphism of the higher dimensional “cylinder” at the cost of introducing extra variables. For \( i = 1, \ldots, n+1 \) we have:

\[
\sigma(x_i) = \sum_{\nu=1}^{n+1} a_{i,\nu} x_\nu + \sum_{\nu=1}^\ell a_{i,n+1+\nu} x_{n+1+\nu},
\]

while for \( n+1 < l < n+1 + \ell \) we have:

\[
\sigma(x_l) = x_l.
\]

An automorphism of \( X = V(f_1, \ldots, f_{n-2}) \) is a map \( \sigma \) such that if \( P \) is a point in \( V(f_1, \ldots, f_{n-2}) \), then \( \sigma(P) \) is in \( V(f_1, \ldots, f_{n-2}) \). The following holds true:

\[
f_i \circ \sigma = \sigma^\ast(f_i) \in \langle f_1, \ldots, f_{n-1} \rangle.
\]

i.e.

\[
f_i \circ \sigma = \sum_{\nu=1}^{n-1} g_{\nu,i} f_\nu,
\]

for some appropriate polynomials \( g_i \in K[x_1, \ldots, x_{n+1+\ell}] \). Since \( \sigma \) is linear the polynomials \( g_{\nu,i} \) are just constants.

Theorem 8. Set \( Y_i = \nabla f_i \). If \( \sigma \) is an automorphism, then \( \sigma(Y_i) \) should be a linear combination of elements \( Y_i \).

Proof. By applying \( \nabla \) to eq. (6) we have for every point on the curve

\[
\nabla(f_i \circ \sigma)(P) = \sum_{\nu=1}^{n-1} \left( g_{i,\nu}(P) \nabla f_\nu(P) + \nabla g_{i,\nu}(P) f_\nu(P) \right).
\]

But \( f_\nu(P) = 0 \) so we arrive at

\[
\nabla(f_i \circ \sigma)(P) = \sum_{\nu=1}^{n-1} g_{i,\nu}(P) \nabla f_\nu(P)
\]

which gives rise to

\[
\nabla(f_i \circ \sigma) = \sum_{\nu=1}^{n-1} g_{i,\nu} \nabla f_\nu + F,
\]

where \( F \) is an element in the ideal \( I \). The ideal \( I \) is generated by polynomials of degree \( k \), while \( \nabla f_i \) are polynomials of degree \( k - 1 \). Therefore,

\[
\nabla(f_i \circ \sigma) = \sum_{\nu=1}^{n-1} g_{i,\nu} \nabla f_\nu,
\]

as polynomials in \( K[x_1, \ldots, x_{n+1+\ell}] \). □
Now the chain rule implies that
\[ \nabla (f \circ \sigma)(P) = \nabla (f_i)(\sigma(P)) \circ \sigma, \]
where \( \sigma \) is given by the \((n + 1 + \ell) \times (n + 1 + \ell)\) matrix \( A = (a_{ij}) \) given in eq. (5). We now rewrite eq. (8) and combine it with eq. (7)
\[ \sigma^\ast(\nabla f_i) \circ \sigma = \nabla (f_i)(\sigma(P)) \circ \sigma = \nabla (f_i \circ \sigma)(P) = \sum_{\nu=1}^{n+1+\ell} g_{i,\nu} \nabla f_\nu. \]
Recall that \( f_j = \lambda_j x_1^j + x_2^j + x_{3+j}^j \) for \( 1 \leq j \leq n - 2 \) and
\[ Y_j = (k\lambda_j x_1^{k-1}, kx_2^{k-1}, 0, \ldots, 0, kx_j^{k-1}, 0, \ldots, 0), \]
where the third non zero element is at the \( j + 3 \) position. For \( 1 \leq i \leq n + 1 \) let us write
\[ \sigma^\ast(x_i) = \sum_{\nu=1}^{n+1+\ell} a_{i,\nu} x_\nu. \]
So
\[ \sigma^\ast(Y_j) = k \left( \lambda_j \left( \sum_{\nu=1}^{n+1+\ell} a_{1,\nu} x_\nu \right)^{k-1} \cdot \left( \sum_{\nu=1}^{n+1+\ell} a_{2,\nu} x_\nu \right)^{k-1} \cdot \ldots \cdot \left( \sum_{\nu=1}^{n+1+\ell} a_{n+3,\nu} x_\nu \right)^{k-1} \right). \]
Observe that eq. (9) implies that \( \sigma^\ast(Y_i) \) is a linear combination of \( Y_i \), which involves only combinations of the monomials \( x_i^{k-1} \), while the \( t \)-th (\( t = 1, 2, j + 3 \)) coefficient of \( \sigma^\ast(Y_i) \) involves all combinations of the terms
\[ \left( \sum_{\nu=1}^{k-1} a_{i,\nu} x_\nu \right) \cdot \left( x_1^{\nu_1} \ldots x_\nu^{\nu_n+1+\ell} \right) \cdot \left( x_1^{\nu_{n+1+\ell}} \ldots x_\nu^{\nu_{n+1+\ell}} \right) \]
for \( \nu_1 + \ldots + \nu_{n+1+\ell} = k-1 \).
For \( \vec{\nu} = (\nu_1, \ldots, \nu_{n+1+\ell}) \) define \( x^{\vec{\nu}} = x_1^{\nu_1} \ldots x_\nu^{\nu_n+1+\ell} \) and set
\[ A_{i,\vec{\nu}} = a_{i,\nu_1} \ldots a_{i,\nu_{n+1+\ell}}. \]
Observe that if \( \left( \sum_{\nu=1}^{k-1} a_{i,\nu} x_\nu \right) \neq 0 \) and \( x^{\vec{\nu}} \) does not appear as a term in the linear combination of \( Y_i \), then using eq. (9) we have
\[ A_{i,\vec{\nu}} = 0. \]
But \( A \) is an invertible matrix so the above equation implies that
\[ A_{i,\vec{\nu}} = 0 \]
if \( x^{\vec{\nu}} \) does not appear as a term in the linear combination of \( Y_i \).
We now observe that the variables \( x_\nu \) for \( \nu \geq n + 1 \) do not appear in all polynomials \( Y_j \). Therefore \( a_{i,\nu} = 0 \) for \( i \leq n + 1 \) and \( \nu > n + 1 \).
Assume that we have an automorphism of the curve \( F_{k,n} \). This extends to an automorphism of variety defined with the same set of equations and has an expansion as in eq. (5). In general this linear expansion involves the variables \( x_{n+1+\ell}, \ldots, x_{n+1+\ell} \). But the above argument shows that (5) is given by
\[ \sigma(x_i) = \sum_{\nu=1}^{n+1+\ell} a_{i,\nu} x_\nu \]
so every automorphism of the curve \( F_{n,k} \) is linear.

**Lemma 9.** The binomial coefficients \( \binom{k-1}{\nu} \) is zero for all \( 1 \leq \nu \leq k - 1 \) if and only if \( k - 1 \) is a power of the characteristic.

**Proof.** The binomial coefficient \( \binom{k-1}{\nu} \) is not divisible by the characteristic \( p \) if and only if \( \nu_i \leq k_i \) for all \( i \), where \( \nu = \sum \nu_i p^i \), \( k - 1 = \sum k_i p^i \) are the \( p \)-adic expansions of \( \nu \) and \( k - 1 \), [3, p. 352]. The result follows.
Lemma 10. Every $\sigma \in \text{Aut}(F_{k,n})$, is given by a $(n+1) \times (n+1)$ matrix $(a_{ij})$. If $k-1$ is not a power of the characteristic, then there is only one non-zero element in each column and row of $(a_{ij})$.

Proof. If $k-1$ is not a power of the characteristic, then we see that the matrix $(a_{ij})$ can have only one non-zero element in each row and column. Indeed, if this was not true, then for some $j$ we have two non-zero terms $a_{j,i_1}, a_{j,i_2}$. If $j \geq 3$, then we work with $\sigma^*(Y_{j-3})$ and for $\nu$ such that $\binom{k-1}{\nu} \neq 0$ we have that $a_{j,i_1} a_{j,i_2}^{k-1-\nu} = 0$, so the desired result follows. □

Corollary 11. If $k-1$ is not a power of the characteristic, then every automorphism $\sigma \in \text{Aut}(F_{k,n})$ restricts to an automorphism of the function field $K(X)$, $X = -\frac{x_1}{x_2}$, i.e. normalizes $H$.

Proof. The function field of the generalized Fermat curves can be seen as Kummer extension with Galois group $H$ of the rational function field $K(X)$, where $X = -\frac{x_1}{x_2}$ [5, par. 2.2]. In order to prove that $H$ is a normal subgroup of the whole automorphism group we have to show that every automorphism of the curve keeps the field $K(X)$ invariant.

Since there is only one non-zero element in each row and column of $A$ the automorphism

$$\sigma^*(x_1^k) = \sum_{\nu=1}^{n+1} a_{1,\nu}^k x_\nu^k. \quad (10)$$

Therefore

$$\sigma^*(X) = -\frac{\sigma^*(x_2)^k}{\sigma^*(x_1)^k} = -\frac{\sum_{\nu=1}^{n+1} a_{2,\nu}^k x_\nu^k}{\sum_{\nu=1}^{n+1} a_{1,\nu}^k x_\nu^k}.$$ 

In the above equation we replace all variables $x_\nu$ for $\nu \geq 3$ using the defining equations $x_\nu = -\lambda_{\nu-3} x_1^k - x_2^k$ in order to arrive at an expression involving only $X = -\frac{x_1}{x_2}$:

$$\sigma^*(X) = -a_{12} x_1^k + a_{22} x_2^k + \sum_{\nu=3}^{n+1} a_{2,\nu}^k \left( -\lambda_{\nu-3} x_1^k - x_2^k \right)$$

$$= -\frac{a_{22} x_2^k + \sum_{\nu=3}^{n+1} a_{2,\nu}^k x_\nu}{a_{12} x_1^k + \sum_{\nu=3}^{n+1} a_{1,\nu}^k x_\nu}.$$ 

□

Proposition 12. Assume that $k-1 = p^h = q$ is a power of the characteristic. Denote by

$$\Sigma_i = \text{diag}(\lambda_i, 1, 0, \ldots, 1, 0, \ldots, 0), \quad (11)$$

with 1 in the $i+3$ position. Then a matrix $A \in \text{PGL}(n+1, K)$ corresponding to $\sigma$ should satisfy

$$A^t \Sigma_i A^q = \sum_{\mu=0}^{n-2} b_{i,\mu} \Sigma_\mu, \quad (12)$$

for a $(n-1) \times (n-1)$ matrix $(b_{i,\mu})$. 
Proof. Assume that \( k - 1 = p^h = q \) is a power of the characteristic. Then,

\[
\sigma^*(f_i) = \lambda_i \left( \sum_{\nu=1}^{n+1} a_{1,\nu}x^\nu \right) + \sum_{\nu=1}^{n+1} a_{2,\nu}x^\nu + \sum_{\nu=1}^{n+1} a_{i+3,\nu}x^\nu \]

\[
= \sum_{\nu=1}^{n+1} (\lambda_i a_{1,\nu}a_{1,\mu}^q + a_{2,\nu}a_{2,\mu}^q + a_{i+3,\nu}a_{i+3,\mu}^q) x^\nu x^\mu
\]

\[
= \sum_{\nu,\mu=1}^{n+1} B^{i}_{\nu,\mu}(\sigma)x^\nu x^\mu.
\]

Observe that by eq. (7) we have \( B^{i}_{\nu,\mu} = 0 \) for all \( 0 \leq i - 2, 1 \leq \nu, \mu \leq n + 1, n \neq \mu \).

The polynomials are in some sense “quadratic forms”

\[
f_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1}) \Sigma_i \left( \begin{array}{c} x_1^q \\ x_2^q \\ \vdots \\ x_{n+1}^q \end{array} \right)
\]

so \( \sigma^*f_i \) is computed as

\[
\sigma^*f_i = (x_1, \ldots, x_{n+1}) A^i \Sigma_i A^q \left( \begin{array}{c} x_1^q \\ x_2^q \\ \vdots \\ x_{n+1}^q \end{array} \right)
\]

and the above expression should be a linear combination of \( f_i \). The desired result follows.

\[\square\]

Remark 13. Matrices \( A = (a_{ij}) \) which satisfy eq. (12) should satisfy the following equations: For \( 0, \ldots, n - 2 \) and \( 1 \leq \nu, \mu \leq n + 1 \) we set

\[
B^{i}_{\nu,\mu} = \lambda_i a_{1,\nu}a_{1,\mu}^q + a_{2,\nu}a_{2,\mu}^q + a_{i+3,\nu}a_{i+3,\mu}^q.
\]

We have

\( B^{i}_{\nu,\mu} = 0 \) for \( \nu \neq \mu \).

Moreover the coefficients \( b_{i,\mu} \) in eq. (12) satisfy the system

\[
\begin{pmatrix}
1 & \lambda_1 & \lambda_2 & \cdots & \lambda_{n-2} \\
1 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
b_{i,1} \\
b_{i,2} \\
b_{i,3} \\
\vdots \\
b_{i,n} \\
b_{i,n-1} \\
\end{pmatrix}
=
\begin{pmatrix}
B_{1,1}^i \\
B_{1,2}^i \\
B_{1,3}^i \\
\vdots \\
B_{n,n}^i \\
B_{n+1,n}^i
\end{pmatrix}
\]

Which gives us that

\[
b_{i,\nu} = B^i_{2+\nu,2+n} + \lambda_{1} a_{1,2+\nu}^{q+1} + a_{2,2+\nu}^{q+1} + a_{i+3,2+\nu}^{q+1}, \quad \text{for } 1 \leq \nu \leq n - 1
\]

plus the compatibility relations

\[
\sum_{\nu=3}^{n+1} B^i_{\nu,\nu} = B^i_{2,2}
\]

and

\[
\sum_{\nu=3}^{n+1} \lambda_{\nu-3} B^i_{\nu,\nu} = B^i_{1,1}.
\]
Solving these linear systems with $\lambda_1, \ldots, \lambda_{n-2}$ as parameters, seems a complicated problem, which is out of reach for now.

REFERENCES

[1] Groupes de monodromie en géométrie algébrique. II. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, Lecture Notes in Mathematics, Vol. 340.

[2] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[3] David Eisenbud. *Commutative algebra*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[4] Yolanda Fuertes, Gabino González-Diez, Rubén A Hidalgo, and Maximiliano Leyton. Automorphisms group of generalized Fermat curves of type. *J. Pure Appl. Algebra*, 217(10):1791–1806, October 2013.

[5] Gabino González-Diez, Rubén A. Hidalgo, and Maximiliano Leyton. Generalized Fermat curves. *J. Algebra*, 321(6):1643–1660, 15 March 2009.

[6] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*. North-Holland Publishing Co., Amsterdam, 1968. Augmenté d’un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2.

[7] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[8] Yasutaka Ihara. Profinite braid groups, galois representations and complex multiplications. *Ann. Math.*, 121(2):351–376, March 1985.

[9] Aristides Kontogeorgis. Automorphisms of Fermat-like varieties. *Manuscripta Math.*, 107(2):187–205, February 2002.

[10] Heinrich-Wolfgang Leopoldt. Über die Automorphismengruppe des Fermatkörpers. *J. Number Theory*, 56(2):256–282, 1996.

[11] Pavlos Tzermias. The group of automorphisms of the Fermat curve. *J. Number Theory*, 53(1):173–178, 1995.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, PANEPISTIMIOPOLIS, 15784 ATHENS, GREECE

E-mail address: kontogar@math.uoa.gr

E-mail address: pan_param@math.uoa.gr