Quasiperiodicity and non-computability in tilings

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Abstract
We study tilings of the plane that combine strong properties of different nature: combinatorial and algorithmic. We prove existence of a tile set that accepts only quasiperiodic and non-recursive tilings. Our construction is based on the fixed point construction [12]; we improve this general technique and make it enforce the property of local regularity of tilings needed for quasiperiodicity. We prove also a stronger result: any $\Pi_1^0$-set can be recursively transformed into a tile set so that the Turing degrees of the resulted tilings consists exactly of the upper cone based on the Turing degrees of the later.

1 Introduction
Tilings form a popular basis for many mathematical games, for games for the kids. In science, they are popular tools for rather different researches, in chemistry (to describe quasicrystalline structures, e.g., [6]), in pure logics (e.g. deciding classes of first order predicates defined on their syntax, see [4]), in computational complexity (as basic model for complexity, [5]). The first famous result about tiling is the so-called domino problem: in 1966 by Berger proved that given a tile set, we cannot decide algorithmically wether it can tile the plane, [1]. Within the proof, Berger constructed the first aperiodic tile set — a tile set that can tile the plane but only non-periodically. It was constructed the first tile set that allows only tilings of the plane with rather complex structure. Thus, rather simple local rules can imply quite nontrivial global structure of a tiling.

Since Berger’s paper, quite a lot of different algorithmic and combinatorial properties of aperiodic tilings were investigated. It was proven that a tile set that accepts only aperiodic tilings, must accept uncountably many of them, [13]. In [14] it was discovered that some natural conditions imply rather string structural property of the set of tilings. Many researchers tried to construct possibly simpler aperiodic tile sets (e.g., [2], [3], [9], [10], [12]). The idea of “simplicity” was interpreted in several different ways: as the number of tiles, algorithmic simplicity of the construction, etc. Another avenue of research was constructing tile sets that guarantee not only aperiodicity, but also more sophisticated properties of tilings: non-recursivity, maximal algorithmic complexity (of each tiling), robustness and fault-tolerance of tilings, and their combinations. [7], [8], [11], [12].
The fundamental question “How complex can a tiling be?” also can be understood in terms of Turing degrees of unsolvability. Some partial answers to this question are known. First of all, we remark that for each tile set the set of valid tilings is effectively closed (i.e., belongs to the class \( \Pi^0_1 \)). In [11] the property of cone-avoidance was proved. Quite a complete study of Turing degrees of tilings was given in [15] and [16].

Not surprisingly, the constructions that guarantee some nontrivial combinatorial properties or involve simulation of a Turing machine require very different technical features. So it is rather difficult to combine in one and the same tiling properties of different nature. In this paper we try to do some kind of aggregation; we combine the combinatorial property of quasiperiodicity with complexity issues. We prove that all upper cones of Turing degrees above any \( \Pi^0_1 \) set can be achieved by a tile set that produces only quasiperiodic tilings. This rather complex theorem has a more concrete consequence: we build a tile set that produces only quasiperiodic tilings, and none of these tilings is recursive.

Let us be more precise now. In this paper Wang tiles are unit squares with colored sides. A tile set is a finite family of tiles. For a given tile set the domino problem is to decide whether the entire plane can be tiled with these tiles. Here we assume of course that we are given infinitely many copies of each tile (tiles are prototypes); in other words, we are allowed to place translated copies of the same tile into different sites of the plane (rotations are not allowed). In a correct tiling the tiles in the neighbour cells must match (sides in contact must have the same color).

If a tile set \( \tau \) tiles the plane, we call these tilings \( \tau \)-tilings. More formally, \( \tau \)-tiling can be defined as a mapping \( F: \mathbb{Z}^2 \rightarrow \tau \), where for each pair of neighbouring cells \( x, y \in \mathbb{Z}^2 \) the colors of the tiles \( F(x) \) and \( F(y) \) match each other on their neighbouring sides. A tiling is called periodic if some nontrivial shift transformed it into itself. A tiling is called quasiperiodic if every finite patterns from the tiling appears there infinitely often.

The domino problem (existence of a tiling with a given tile set) is algorithmically undecidable, [1]. An interesting and nontrivial fact (which follows from Berger’s theorem) is that there exist some tile sets allows only aperiodic tilings of the plane.

The main result of this article is the following theorem that claims that some tile sets enforce at once two nontrivial properties of a tiling: quasiperiodicity and non-computability.

**Theorem 1** There exists a tile set \( \tau \) such that

(i) there exist \( \tau \)-tilings of the plane,

(ii) all \( \tau \)-tilings are quasiperiodic,

(iii) all \( \tau \)-tiling are non-computable.

With the same technique we can prove a more general result:

**Theorem 2** For every effectively closed set \( A \) there exists a tile set \( \tau \) such that

(i) there exist \( \tau \)-tilings of the plane,

(ii) all \( \tau \)-tilings are quasiperiodic,
We prove Theorem 1 and Theorem 2 using the technique of fixed-point tilings from [12], with some suitable extensions. Though conceptually this technique is not very difficult, a formal explanation should include too many boring details, so it would not be convincing for the readers. To make the argument more accessible, we present it in a less formalised way, starting with a proof of a weaker Theorem 3 below. Being somewhat sketchy, we nevertheless do not skip any important part of the construction, and we emphasise the parallels and differences with the previously now construction of a fixed-point tilings in [12].

The rest of the paper organised as follows. First we remind the core ideas of the fixed-point tiling from [12] and explain how this technique implies aperiodicity. Then we upgrade the construction and build a tile set that combines the properties of aperiodicity and quasiperiodicity. After that we prove the main results of the paper.

2 Self-simulating tilings (reminder)

Our proof is based on the fixed point construction from [12]. The main idea of this argument is that we can enforce in a tiling a kind of a self-similar structure. In what follows we remind the principal ingredients of this construction (here we follow the notations from [12]). The reader familiar with the technique used in [12] can skip this section and go directly to Section 3.

Let \( \tau \) be a tile set and \( N > 1 \) be an integer. We call by a macro-tile an \( N \times N \) square correctly tiled by matching tiles from \( \tau \). Every side of a \( \tau \)-macro-tile contains a sequence of \( N \) colors (of tiles from \( \tau \)); we refer to this sequence as a macro-color. Further, let \( \rho \) be some set of \( \tau \)-macro-tiles. We say that \( \tau \) implements \( \rho \) if (i) some \( \tau \)-tilings exist, and (ii) for every \( \tau \)-tiling there exists a unique lattice of vertical and horizontal lines that cuts this tiling into \( N \times N \) macro-tiles from \( \rho \). The value of \( N \) is called the zoom factor of this implementation.

If a tile set \( \tau \) implements a set \( \rho \) of \( \tau \)-macro-tiles with some zoom factor \( N > 1 \) and \( \rho \) is isomorphic to \( \tau \), then the tile set \( \tau \) is called self-similar. By the definition, for a self-similar tile set \( \tau \) each tiling can be uniquely split into \( N \times N \) macro-tiles (the set of all macro-tiles is isomorphic to the initial tile set \( \tau \)); further, the greed of macro-tiles can be grouped into blocks of size \( N^2 \times N^2 \), where each block is a macro-tile of rank 2 (again, the set of all macro-tiles of rank 2 is isomorphic to the initial tile set \( \tau \)), etc. It is not hard to deduce from this observation the following statement.

**Proposition 1 (folklore)** A self-similar tile set \( \tau \) has only aperiodic tilings.

We skip the proof; see [12].

Thus, if we want to construct an aperiodic tile set, then it is enough to present an instance of a self-similar tile set. Below we discuss a very general construction of self-similar tile sets.
2.1 Implementing some given tile set with a large enough zoom factor

Assume that we have a tile set $\tau$ where each color is a $k$-bit string (i.e., the set of colors $C \subseteq \{0,1\}^k$) and the set of tiles $\rho \subseteq C^4$ is presented by predicate $P(c_1, c_2, c_3, c_4)$ (the predicate is true if and only if the quadruple $(c_1, c_2, c_3, c_4)$ make a tile from $\rho$). Assume that we have some Turing machine $M$ that computes $P$. Let us show how to implement $\rho$ using some other tile set $\tau$, with a large enough zoom factor $N$.

We will build a tile set $\tau$ where each tile “knows” its coordinates modulo $N$. This information is included in tile’s colors. More technically, for a tile that is supposed to have coordinates $(i, j)$ modulo $N$, the colors on the left and on the bottom sides should involve $(i, j)$, the color on the right side should involve $(i + 1 \mod N, j)$, and the color on the top side, respectively, involves $(i, j + 1 \mod N)$, see Fig. 1. This means that every $\tau$-tiling can be uniquely split into blocks (macro-tiles) of size $N \times N$, where the coordinates of cells ranges from $(0, 0)$ in the bottom-left corner to $(N-1, N-1)$ in top-right corner, Fig. 2. So, intuitively, each tile “knows” its position in the corresponding macro-tile.

In addition to the coordinates, each tile in $\tau$ should have some supplementary information encoded in the colors on its sides. We call this additional information by shades of tile’s colors. On the border of a macro-tile (where one of the coordinates is zero) only two additional shades (say, 0 and 1) are allowed. Thus, for each macro-tile of size $N \times N$ the corresponding macro-colors represent a string of $N$ zeros and ones. We will assume that $N \ll k$. We allocate $k$ bits in the middle of macro-tile sides and make them represent colors from $C$. All other bits on the sides are zeros.

Now we introduce additional restrictions on tiles in $\tau$ that will guarantee the required property: the macro-colors on the macro-tiles satisfy the relation $P$. To achieve this, we ensure that bits from the macro-tile side are transferred to the central part of the tile, and the central part of a macro-tile is used to simulate a computation of the predicate $P$. We fix which cells in a macro-tile are “wires” (we may assume that wires do not cross each other) and then require that these tiles carry the same (transferred) bit on two sides. The central part of a macro-tile (of size, say $m \times m$) should represent a time-space diagram of $M$’s computation (the tape is horizontal, time goes up). This is done in a standard way. We require that computation terminates in an accepting state (if not, no correct tiling can be formed), see Fig. 3. To make this construction work, the size of macro-tile (the number $N$) should be large enough: first, we need enough
Figure 3: The grey area of size \( m \times m \) in the center of macro-tile is the “computation zone”. The “wires” transfer the macro-colors from each side of the macro-tile to the computation zone.

space for \( k \) bits to propagate, second, we need enough time (i.e., height) so all accepting computations of \( \mathcal{M} \) terminate in time \( m \) and on space \( m \) (where the size of the computation zone \( m \) cannot be greater than the size of a macro-tile).

In this construction the number of additional shades depends on the machine \( \mathcal{M} \) (the more states it has, the more additional shades we need to simulate the computation in the space-time diagram). To avoid this dependency, we replace \( \mathcal{M} \) by a fixed universal Turing machine \( \mathcal{U} \) that runs a program simulating \( \mathcal{M} \). We may assume that the tape has an additional read-only layer. Each cell of this layer carries a bit this never changes during the computation; these bits are used as a program for the universal machine. So in the computation zone the columns carry unchanged bits; the construction of a tile set guarantees that these bits form the program for \( \mathcal{U} \), and the computation zone of a macro-tile represents a view of an accepting computation for that program, see Fig. 4. In this way we get a tile set \( \tau \) that has \( O(N^2) \) tiles and implements \( \rho \). (This construction works for all large enough \( N \).)

2.2 A self-similar tile set: implementing itself

In the previous section we explained how to implement a given tile set \( \rho \) (represented as a program for the universal TM) by another tile set \( \tau \) with large enough zoom factor \( N \). Now we want \( \tau \) be isomorphic to \( \rho \). This can be done using construction that follows Kleene’s fixed-point theorem. Note that most steps of the construction of \( \tau \) do not depend the program for \( \mathcal{M} \) (the coordinates of tiles that make the skeleton of a macro-tile, the information transfer along the wires, the propagation of unchanged
program bits, and the space-time diagram for the universal machine in the computation zone). Let us fix these rules as part of \(\rho\)'s definition and set \(k = 2 \log N + O(1)\), so that we can encode \(O(N^2)\) colors by \(2 \log N + O(1)\) bits. From this definition we obtain a program \(\pi\) for TM that checks that macro-tiles behave like \(\tau\)-tiles in this respect. We are almost done with the program \(\pi\). The only remaining part of the rules for \(\tau\) is the hardwired program. We need to guarantee that the computation zone in each macro-tile carries the very same program \(\pi\). But since the program is written on the tape the universal machine, it can be instructed to access its own bits and check that if macro-tile belongs to the computation zone, this macro-tile carries the correct bit of the program.

It remains to explain the choice of \(N\) and \(m\) (note that the value of the zoom factor \(N\) and the size of the computation zone \(m\) are hardwired in the program). We need it to be large enough so the described above computation (which deals with inputs of size \(O(\log N)\)) can fit in the computation zone. The computations are rather simple (polynomial in the input size, i.e., \(O(\log N)\)), so they easily fit in space and time bounded by \(m = \Theta(\log N)\). This completes the construction of a self-similar aperiodic tile set.

Now it is not hard to verify that the constructed tile sets (1) allows to tile the plane, and (2) each tiling is self-similar. Applying Proposition 1 we obtain the following proposition.

**Proposition 2 (Berger)**  
There exists a tile set \(\tau\) such that

(i) there exist \(\tau\)-tilings of the plane,

(ii) each \(\tau\)-tiling is aperiodic.

In the next section we will upgrade the basic construction of the fixed-point tiling. So
far we should keep in mind that in such a tiling all tiles can be classified into three types:

- the “skeleton” tiles that keep no information except for their coordinates in a macro-tile; these tiles work as building blocks for our hierarchical structure;
- the “wires” that transmit the bits of macro-colors from the frontier of the macro-tile to the computation zone;
- the tiles of the computation zone (intended to simulate the space-time diagram of the Universal Turing machine).

The same is true for macro-tiles, super-macro-tiles, etc.; e.g., each macro-tile is a “skeleton” block, or a part of a “wire”, or a cell in the computation zone in the macro-tile of higher rank.

3 Quasiperiodicity and aperiodicity

Before we approach the main result, we prove a weaker statement; we show that there exists a tile set such that all tilings are at once quasiperiodic and aperiodic.

Theorem 3 There exists a tile set (a set of Wang tiles) $\tau$ such that

(i) there exist $\tau$-tilings of the plane,

(ii) each tiling is quasiperiodic,

(iii) each $\tau$-tiling is aperiodic.

This result was originally proven in [14] (for a tile set $\tau$ constructed in [10]). Theorem 3 is obviously weaker than Theorem 1 since every non-computable tiling is aperiodic.

3.1 Supplementary features: what else we can we assume on the fixed-point tiling

The general construction of a fixed-point tiling does not implies the property of aperiodicity. In fact, for tilings described above, each pattern that include only “skeleton” tiles (or “skeleton” macro-tiles of some rank $k$) must appear infinitely often, in all homologous position inside all macro-tiles of higher rank. However, this is not the case for patterns that include tiles from the “communication zone” or the “communication wires”. Informally, the problem is that even a very small pattern can involve the information relevant for a macro-tile of arbitrarily high rank. So we cannot guarantee that a similar pattern appears somewhere in the neighbourhood. To overcome this difficulty we need some new idea and new technical tricks. First of all, without essential modification of the construction we can enforce the following additional properties of a tiling:
In each macro-tile, the size of the computation zone $m$ is much less than the size of the macro-tile $N$. Technically, in what follows we will need to reserve free space in a macro-tile to put there $O(1)$ (some constant number) of copies of each $2 \times 2$ pattern from the computation zone. This requirement is easy to meet. We may assume that the size of a computation zone in a macro-tile of size $N \times N$ is only $m = \text{poly}(\log N)$. 

We require that the tiling inside the computation zone satisfies the property of $2 \times 2$-determinicity: if we know all colors on the borderline of a $2 \times 2$-pattern inside of the computation zone (i.e., a tuple of 8 colors), then we can uniquely reconstruct the 4 tiles of this patterns. Again, we do not need any new idea: this requirement is met if we simulate the space-time diagram of a Turing machine in a natural way.

The communication channels in a macro-tile (the wires that transmit the information from the macro-color on the borderline of this macro-tile to the bottom line of its computation zone) must be isolated from each other. The distance between every two wires must be greater than 2 from each other. That is, each $2 \times 2$-pattern can touch at most one communication wire.

Also we will need a somewhat more essential modification of the construction. We discuss it in the next section.

### 4 Proof of Theorem 3

To achieve the property of aperiodicity, we should guarantee that every finite pattern that appears one in a tiling, must appear infinitely often (in fact, in each large enough square). In a self-similar tiling, each finite pattern can be covered by at most 4 macro-tiles (by a $2 \times 2$-pattern) of an appropriate rank. Thus, to prove Theorem 3 it is enough to guarantee that each $2 \times 2$ group of macro-tiles (of each rank) that ever appears in a tiling, must appear there in all large enough squares. This property is not true for the tile set constructed above. As we noticed above, this is obviously true for a $2 \times 2$ pattern that involves only skeleton macro-tiles (we can find an identical pattern in the neighbouring macro-tile of the appropriate rank); however, this property can be false for patterns that touch the communication wires or the computation zone. To achieve the desired property we need to modify the basic construction. To this end we implement in our construction two new features.

**The first feature** (needed to handle patterns from the computation zone). Notice that for each $2 \times 2$-window in the computation zone there exist only $c = O(1)$ ways to tile them correctly (and make a correct tiling). This constant $c$ depends on the alphabet of the tape and the number of internal states of the Universal Turing machine. For each possible position of a $2 \times 2$-window in the computation zone and for each possible filling of this window by tiles, we reserve a special $2 \times 2$-slot in a macro-tile (somewhere fare away from the computation zone and from all communication wires) and define the neighbours around this slot in such a way that only this specific $2 \times 2$ patterns can patch it. Note that the tiles around this “know” their real coordinates in the bigger macro-tile,
while the tiles inside the slot do not (they “believe” to be tiles in the computation zone, though they are in a “slot” outside of it). An example of such a slot is shown in Fig. 5. In Fig. 6 we show how these “slots” are placed in a macro-tile. This simple trick is the sharpest difference between this construction and the fixed-point tilings known before: now some tiles do not “know” their real position in the ambient macro-tile.

Here we use (a) the property of $2 \times 2$-determinicity of the computation zone (there is a unique way to put tiles in the “slot”), and (b) the fact that we have enough room to put in a macro-tile the slots for all $2 \times 2$-patterns that can appear in the computation zone. This feature guarantees that each $2 \times 2$ pattern from the computational zone appears at least once in each macro-tile (such a pattern appears once in each macro-tile in the introduced “slots” and possibly once again in the computation zone of this macro-tile).

**The second feature** (needed to handle patterns involving communication wires). We choose an encoding of macro-colors (encoded by a sequence of $k$ bits) so that for each $i = 1, \ldots, k$, for each of the four directions (up, down, left, and right) there exist a macro-tile where the $i$-th bit of a macro-color (on the corresponding side of this macro-tile) is equal to 0 and to 1. We may assume that there exists a macro-tile where all macro-colors are encoded by all $0$'s, and there exists another macro-tile with all four macro-colors encoded by all $1$'s. (W.l.o.g. we may assume that these macro-tiles are
two skeletons macro-tiles in the macro-tile of the next rank). This feature guarantees that each \(2 \times 2\)-pattern involving communication wires can be found in the tiling at least once in each macro-tile of next rank.

For a tile set with both new features, every tiling enjoys two new properties: (1) every \(2 \times 2\)-pattern of tiles touching the computation zone can be found at least once in each macro-tile; and (2) every \(2 \times 2\)-pattern of tiles touching a “communication wire” (such a pattern can touch only one communication wire!) can be found at least once in a super-macro-tile (macro-tile of rank 2). Of course, the property of self-similarity implies that similar statements hold for \(2 \times 2\)-pattern of macro-tiles of each ranks \(k\). Thus, we get a tile set that satisfies the requirements of Theorem 2.

\section{From aperiodicity to non-computability}

To prove Theorem 1, we need a slightly more sophisticated construction. We need a self-similar tiling with \textit{variable zoom factor}, see [12] for details. In this version of the construction the size of a macro-tile of rank \(r\) is equal \(N_r \times N_r\), for some suitable sequence of zooms \(N_r, r = 1, 2, \ldots\). We may assume that \(N_r = Cr\) for some constant \(C\). Now each macro-tile of rank \(r\) must “know” its own rank (that is, the binary representation of \(r\) is written on the tape of the Turing machine simulated on the computation zone). This information is used by a macro-tile to simulate next rank macro-tiles properly. The size of the computational zone \(m_r\) should also grow as a function of rank \(r\); again, we may assume that \(m_r = \Theta(\log N_r)\).

Also we may require that all macro-tiles of rank \(r\) contain in their computational zone the prefix (e.g., of length \([\log r]\)) of some infinite sequence \(X = x_0x_1x_2 \ldots\). The bits of this prefix are propagated by wires to the neighbouring macro-tiles, so all macro-tiles of the same rank contain the same bits \(x_0x_1 \ldots\). The usual self-simulation

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{The array of “slots” (with patterns from the computation zone) embedded in a macro-tile.}
\end{figure}
guarantees that the bits of $X$ embedded into a macro-tile of rank $r+1$ extends the prefix embedded in a macro-tile of rank $r$. Since the size of the computational zone increases as a function of rank $r$, the entire tiling of the plane involves an infinite sequence of bits $X$.

The construction becomes interesting if we can enforce some special properties of the embedded sequence $X$. For example, we can guarantee that it is not computable. Indeed, let us make the machine in the computation zone do some new job: let it enumerates two non-separable enumerable sets (on each level $r$ we run the simulation for the number of steps that fits the computation zone available in a macro-tile of rank $r$). Then we can require that $X$ is a separator between these two sets, and on each level go the hierarchy the machine verifies that the (partially) enumerated sets are indeed separated by the given prefix of $X$. Combining all ingredients together, we obtain a tile set $\tau$, which is self-similar in a generalised sense (with a variable zoom factor), with two nontrivial properties: all $\tau$-tilings are non-computable and quasiperiodic. Thus, we proved Theorem 1.

With essentially the same technique we can prove Theorem 2. We employ again the idea of embedding in a tiling of an infinite sequence $X$. Technically, we require that all macro-tiles of rank $k$ should involve on their computational zone the same finite sequence of $\log k$ bits, which is understood as a prefix of $X$; we guarantee that the prefix embedded in macro-tiles of rank $(k + 1)$ is compatible with the prefix available to the macro-tiles of rank $k$. Further, since $\mathcal{A}$ is in $\Pi^0_1$, we can enumerate the (potentially infinite) list of patterns that should not appear in $X$. On each level, the macro-tiles run this enumeration for the available space and time (limited by the size of the computational zone available on this level), and verifies that the found forbidden patterns do not appear in the prefix of $X$ accessible to macro-tiles of this level. Since the computational zone becomes bigger and bigger on each next level, the enumeration extends longer and longer. Thus, a sequence $X$ can be embedded in an infinite tiling, if and only if this sequence does not contain any forbidden pattern (i.e., this $X$ belongs to $\mathcal{A}$).

What are the Turing degrees of tilings in the described tile set? In our tile set, every tiling is uniquely defined by three sequences: the sequence $X$ embedded in this tiling, and two sequences of integers $\sigma_h, \sigma_v$ that specifies the shifts (the vertical and the horizontal ones) of macro-tiles of each level relative to the origin of the plane. Indeed, on each level $k$ we split the macro-tiles of the previous rank into blocks of size $N_k \times N_k$, which make $k$-level macro-tiles, and there are $N_k^2$ ways to choose the greed of horizontal and vertical lines that define this splitting. It remains to notice that $\sigma_h, \sigma_v$ can be absolutely arbitrarily. Thus, the Turing degree of a tiling is the Turing degree of $(X, \sigma_h, \sigma_v)$, which can be arbitrary degree not less than $X$. That is, the set of degrees of tilings is exactly the closure of $\mathcal{A}$, i.e., the set of all $Y$ that are not less than some $X \in \mathcal{A}$. So we get the statement of Theorem 2.

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