Degree-ordered-percolation on uncorrelated networks

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Abstract. We analyze the properties of degree-ordered percolation (DOP), a model in which the nodes of a network are occupied in degree-descending order. This rule is the opposite of the much studied degree-ascending protocol, used to investigate resilience of networks under intentional attack, and has received limited attention so far. The interest in DOP is also motivated by its connection with the susceptible-infected-susceptible (SIS) model for epidemic spreading, since a variation of DOP is related to the vanishing of the SIS transition for random power-law degree-distributed networks $P(k) \sim k^{-\gamma}$. By using the generating function formalism, we investigate the behavior of the DOP model on networks with generic value of $\gamma$ and we validate the analytical results by means of numerical simulations. We find that the percolation threshold vanishes in the limit of large networks for $\gamma \leq 3$, while it is finite for $\gamma > 3$, although its value for $\gamma$ between 3 and 4 is exceedingly small and preasymptotic effects are huge. We also derive the critical properties of the DOP transition, in particular how the exponents depend on the heterogeneity of the network, determining that DOP does not belong to the universality class of random percolation for $\gamma \leq 3$.

Keywords: critical phenomena of socio-economic systems, percolation problems, random graphs, networks

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1. Introduction

The investigation of percolative properties of complex networks has attracted a huge interest over the past 20 years [1–3]. Many highly nontrivial phenomena occur in this context, including continuous, discontinuous and hybrid transitions. The first pioneering investigations pointed out the strong effect of the degree distribution, making heterogeneous structures very resilient with respect to random failures but extremely fragile under intentional attacks targeted at the most connected elements [4–7]. A general model to investigate the effect on the percolation transition of degree-dependent protocols for removing network elements (nodes) was introduced by Gallos et al [8]. In that model, at each time step the probability that a node of degree \( k_i \) is removed is proportional to \( k_i^\alpha \). The cases \( \alpha = 0 \) and \( \alpha \to +\infty \) correspond to standard random percolation and to intentional attack, respectively. The case \( \alpha \to -\infty \) corresponds to a percolation process where nodes are added in degree-descending order (or alternatively, removed in degree-ascending order). This process was called degree-ordered-percolation (DOP) by Lee et al [9], who considered it in the context of the debate about the asymptotic properties of the susceptible-infected-susceptible (SIS) model for epidemics on networks with degree
distribution \( P(k) \sim k^{-\gamma} \) [10]. They argued that if the DOP threshold vanishes in the large-network limit for \( \gamma > 3 \) then this would imply that also the SIS threshold should vanish in the same limit. Hence the vanishing of the DOP threshold for \( \gamma > 3 \) would have reconciled theoretical arguments suggesting a finite SIS threshold [11] with numerical results showing it to be vanishing [12]. Lee et al studied DOP numerically and found a finite DOP threshold for \( \gamma > 4 \) and less conclusive evidence for \( 3 < \gamma < 4 \). These results indicated that DOP is not at the origin of the vanishing of the SIS threshold observed numerically for \( \gamma > 3 \). See [10] for more details.

Despite this lack of a direct connection with the SIS transition, DOP is a simple and interesting model whose properties have not been, to the best of our knowledge, fully understood. The only analytical investigation was performed by Lee et al about DOP on some peculiar hierarchical scale-free flower networks [13]. It is natural to wonder what is the expression for the DOP threshold as a function of \( \gamma \) and whether the critical exponents are different from those of standard random percolation. Moreover, very recent work [14] has shown that SIS dynamics is actually connected to a long-range type of process, cumulative merging percolation, of which DOP constitutes the nontrivial short-range limit. For these reasons in this paper we reconsider DOP on power-law degree-distributed networks and by means of analytical and numerical results we fully clarify its phenomenology.

2. The model

The DOP model is defined as follows. We consider a generic network and start removing nodes in degree-ascending order, i.e. we start from the nodes with smallest degree \( k_{\min} \) and once they are all removed we start removing nodes with degree \( k_{\min} + 1 \) and so on. Nodes with the same degree are removed in random order. The process can also be seen as starting from a network where all nodes have been removed and iteratively putting them back in degree-descending order. It is clear that this process is exactly the opposite of the much studied percolation process under intentional attack, investigating network robustness when nodes are removed starting from the most connected ones [6, 7].

Let us define \( p \) as the ratio between the number of nodes added (or not removed) and the total number \( N \) of nodes in the original network. The quantity \( p \) is the control parameter in our system. For \( p = 1 \) the topology is the original one, that we assume to be connected (i.e. all nodes belong to the giant connected component). For \( p = 0 \) all nodes have been removed and the relative size of the largest connected component is null. An intermediate value \( p = p_c \) marks the birth of an extensive giant connected component. Our goal is to determine how this quantity and the related critical behavior depend on the network properties. In particular, we study this percolation process for power-law distributed uncorrelated networks where the normalized degree distribution is, for a finite size network

\[
P(k) = \frac{\gamma - 1}{k_{\min}^{\gamma - 1} - k_{\max}^{\gamma - 1}} k^{-\gamma}.
\]
As minimum degree we take $k_{\text{min}} = 3$, while the maximum degree is set equal to $k_{\text{max}} = N^{1/2}$ for $2 < \gamma \leq 3$ and $k_{\text{max}} = N^{1/(\gamma - 1)}$ for $\gamma > 3$.

3. The percolation threshold $p_c$

In order to determine the percolation threshold we follow the argument of [8]. We apply the general Molloy–Reed criterion [15], stating that a giant component exists provided the network branching factor is larger than 1

$$\sum_k k^2 - k\langle k \rangle P_p(k) > 1,$$

(2)

where $P_p(k)$ is the degree distribution of the network for a given value of the control parameter $p$. For DOP this distribution is simply given by

$$P_p(k) = \Theta(k - k_p)P(k),$$

(3)

where $k_p$ is the minimum degree of nodes still left in the network and $\Theta(x)$ is the Heaviside step function. The inequality in equation (2) becomes an equality for a critical value $k_c$ of the degree $k_p$, which determines the onset of a giant connected component in the system, i.e. the percolation transition. Once $k_c$ is known, the percolation threshold $p_c$ is determined by the condition

$$\sum_k \Theta(k - k_c)P(k) = p_c.$$

(4)

These expressions correspond to the limit $\alpha \rightarrow -\infty$ of the general treatment presented in [8].

3.1. $\gamma > 3$

Let us first consider the case $\gamma > 3$ in the infinite size limit $N \rightarrow \infty$. Taking the continuous degree limit, from equation (2) the threshold condition is,

$$\int_{k_{\text{min}}}^{\infty} dk k^2 - k\langle k \rangle \Theta(k - k_c)P(k) = 1,$$

(5)

which, reminding that $\langle k \rangle = \frac{\gamma - 1}{\gamma - 2}k_{\text{min}}$, yields

$$\frac{\gamma - 2}{\gamma - 3}k_{\text{min}}^{\gamma - 3}k_c^{2-\gamma} \left( k_c - \frac{\gamma - 3}{\gamma - 2} \right) = 1.$$

(6)

Hence, for any $\gamma > 3$, an extensive giant component appears as soon as nodes down to the finite degree $k_c$ are added. Solving this equation numerically, inserting the result

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Figure 1. Plot of $p_c$ as a function of $\gamma > 3$. The blue solid line represents the exact solution, the orange dashed line represents the approximate one, equation (8). The dotted red line is the threshold for standard percolation.

into equation (4), we obtain (again assuming the continuous degree limit) the value of the finite percolation threshold

$$p_c = \left( \frac{k_{\min}}{k_c} \right)^{\gamma-1},$$

which is displayed (as a solid line) in figure 1. Note that in the range between $\gamma = 3$ and $\gamma = 4$ the value of $p_c$ is exceedingly small but it is not equal to 0. This result provides solid evidence about an issue that was not completely clarified by numerical simulations in [9].

From equation (6) one can obtain an explicit approximate expression for $k_c$ by neglecting $(\gamma - 3)/(\gamma - 2)$ with respect to $k_c$. Inserting such expression into equation (7) yields

$$p_c = \left( \frac{\gamma - 3}{\gamma - 2 k_{\min}} \right)^{\frac{1}{\gamma - 1}}.$$ (8)

As expected, the approximation works well for relatively small values of $\gamma$, see figure 1.

It is also possible to derive the finite size corrections to the expression of $p_c$. As shown in appendix A, for a network of finite size $N$ the effective threshold $p_c(N)$ is

$$p_c(N) - p_c \propto k_{\max}^{3-\gamma} \propto N^{\frac{1}{\nu}},$$

(9)

The critical exponent $\nu$, defined by $p_c(N) - p_c \sim N^{-1/\nu}$ is then $\nu = (\gamma - 1)/\gamma - 3$. This estimate is valid only up to $\gamma = 4$. Above that $\gamma$ value the correction to $p_c$ becomes subleading with respect to the $N^{-1/3}$ correction due to critical fluctuations, present also in homogeneous systems [2].

3.2. $2 < \gamma \leq 3$

In this case, the sum appearing in equation (2) diverges in the limit of infinite size if $k_p$ remains finite. This indicates that $k_c$ should be diverging for large $N$. Let us analyze
this in detail. We still consider the continuous limit as in equation (5) but for a finite network we perform the integral only up to \( k_{\text{max}} \), obtaining

\[
\frac{\gamma - 2}{\gamma - 3} k_{\text{min}}^{\gamma - 2} (k_c^{3-\gamma} - k_{\text{max}}^{3-\gamma}) = 1, \tag{10}
\]

where we have neglected the term proportional to \( k \) in the sum, as it remains finite in the infinite size limit. This implies that

\[
k_c = k_{\text{max}} \left(1 - \frac{k_{\text{min}}^{2-\gamma} 3 - \gamma}{k_{\text{max}}^{3-\gamma} \gamma - 2}\right)^{1/(3-\gamma)}. \tag{11}
\]

To evaluate the percolation threshold one has to take into account the finite size also in equation (4), obtaining

\[
p_c(N) = \left(\frac{k_{\text{min}}}{k_c}\right)^{\gamma-1} \left[1 - \left(\frac{k_{\text{max}}}{k_c}\right)^{1-\gamma}\right]. \tag{12}
\]

Inserting the expression for \( k_c \) into equation (12), after some algebra we obtain

\[
p_c(N) = \frac{\gamma - 1}{\gamma - 2} k_{\text{min}} k_{\text{max}}^2 = \frac{\gamma - 1}{\gamma - 2} k_{\text{min}} N^{-1}. \tag{13}
\]

Thus we find that the threshold vanishes in the infinite size limit and \( \nu = 1 \). The inverse proportionality between \( p_c(N) \) and \( N \) in equation (13) leads to the surprising conclusion that the incipient giant component at \( p_c(N) \) is composed by a finite and very small number of nodes. For example, for \( k_{\text{min}} = 3 \) this number is only 9 for \( \gamma = 2.5 \) and tends to 6 for \( \gamma \to 3 \).

In appendix A we present the calculation of the asymptotic value of the threshold and of its finite size corrections for the case \( \gamma = 3 \). We find that \( p_c(N) \) decays to 0 with \( N \) with exponent \( 1/\nu = 1 \), as in equation (13), but with a prefactor \( k_{\text{min}}^2 (e^{2/k_{\text{min}}} - 1) \) which is different from the limit \( 2k_{\text{min}} \) of equation (13) for \( \gamma \to 3^- \).

Summarizing, we find that the exponent \( 1/\nu \) governing how the effective threshold \( p_c(N) \) approaches its infinite size limit is

\[
\frac{1}{\nu} = \begin{cases} 
1 & \text{for } 2 < \gamma \leq 3 \\
\frac{\gamma - 3}{\gamma - 1} & \text{for } 3 < \gamma < 4 \\
\frac{1}{3} & \text{for } \gamma > 4.
\end{cases} \tag{14}
\]
4. Critical exponents of the percolation transition

4.1. The exponent \( \beta \)

We want to determine how \(|G(p)|\), the relative size of the giant component, grows in the vicinity of the percolation threshold

\[
|G(p)| \sim \Delta^\beta, \quad (15)
\]

where \( \Delta = p - p_c \) is the distance from the critical point. We make use of the generating function formalism, a standard tool for percolation problems in networks [1]. Indicating with \( u \) the probability that a node is not connected to the giant component \( G \) through one of its neighbors, the generating functions are defined as

\[
f_0(u) = \sum_k P_p(k) u^k \quad (16)
\]

and

\[
f_1(u) = \sum_k kP_p(k) \langle k \rangle u^{k-1}. \quad (17)
\]

Given these definitions the size of the giant component is [1]

\[
|G(p)| = f_0(1) - f_0(u) \quad (18)
\]

where the value of \( u \) is the solution of

\[
u = 1 - f_1(1) + f_1(u). \quad (19)
\]

Below the threshold \( u = 1 \), while above it \( u < 1 \). Since we are interested in the vicinity of the critical point we set \( u = 1 - \epsilon \) and expand for small \( \epsilon \). By considering the continuous degree limit, from equation (18) we find (see appendix B) that for any \( \gamma > 2 \) to leading order

\[
|G(p)| \simeq \frac{\gamma - 1}{\gamma - 2} k_{\min}^{\gamma - 1} k_p^{2-\gamma} \epsilon, \quad (20)
\]

where the quantity \( k_p \) is related to \( p \) by

\[
k_p = k_{\min} p^{1/(1-\gamma)}. \quad (21)
\]

For \( \gamma > 3 \) the quantity \( k_p \) goes to the finite value \( k_c \) at the transition so that the critical behavior is determined only by the dependence of \( \epsilon \) on \( \Delta \).

As shown in appendix B, \( \epsilon \sim \Delta \) for \( \gamma > 4 \), while \( \epsilon \sim \Delta^{1/(\gamma-3)} \) for \( 3 < \gamma < 4 \). For \( 2 < \gamma \leq 3 \) instead, since \( p_c = 0 \), \( k_p \) diverges as \( \Delta^{1/(1-\gamma)} \) close to the transition, while \( \epsilon \sim k_p^{-1} \) (see appendix B), hence overall \( |G(p)| \sim k_p^{1-\gamma} \sim \Delta \). In summary, we find that the relative size of the giant component close to the critical transition scales with an
exponent
\[
\beta = \begin{cases} 
1 & \text{for } 2 < \gamma \leq 3 \\
\frac{1}{\gamma - 3} & \text{for } 3 < \gamma < 4 \\
1 & \text{for } \gamma > 4.
\end{cases}
\] (22)

4.2. The exponent $\tau$

At the percolation critical point, the probability $n_s(p)$ that a finite cluster has size $s$ decays as
\[
n_s(p) \sim s^{-\tau}.
\] (23)

To determine this exponent, we consider the associated probability that a randomly chosen node belongs to a cluster of size $s$, $p_s = sn_s$. The function that generates this distribution is [5]
\[
h_0(x) = \sum_s p_s x^s,
\] (24)

while $h_1(x)$ is the generating function associated to the probability for a node to be connected to a finite cluster of size $s$ through one of its neighbors. These two generating functions are related to the generating functions $f_0(x)$ and $f_1(x)$ as follows [5]:
\[
\begin{align*}
h_0(x) &= 1 - f_0(1) + x f_0[h_1(x)] \\
h_1(x) &= 1 - f_1(1) + x f_1[h_1(x)].
\end{align*}
\] (25) (26)

In appendix C we determine the behavior of the generating functions for $x = 1 - \epsilon$ with $\epsilon \to 0$. In particular, we find, defining $g_0 = 1 - h_0(1 - \epsilon)$, that $g_0 \sim \epsilon^{1/(\gamma - 2)}$ for $3 < \gamma < 4$ and $g_0 \sim \epsilon^{1/2}$ for $\gamma > 4$, while $h_0$ and $h_1$ are trivial for $2 < \gamma \leq 3$. Using Tauberian theorems [16] we have that, if $g_0(\epsilon) \sim \epsilon^y$, then $\tau = y + 2$, leading to
\[
\tau = \begin{cases} 
\frac{2\gamma - 3}{\gamma - 2} & 3 < \gamma < 4 \\
\frac{5}{2} & 4 < \gamma.
\end{cases}
\] (27)

5. Numerical results

We check the results of the analytical approach by performing numerical simulations of the DOP percolation process on networks built using the uncorrelated configuration model [17]. To determine the value $p_c$ of the percolation threshold for given $\gamma$ and $N$, we generate many realizations of the network and perform many realizations of the DOP
Figure 2. Results for $\gamma = 4.5$. Main: susceptibility peak height as a function of the system size $N$. Numerical results are compared with the theoretical prediction $N^{1-2\beta/\nu}$ which gives an exponent $1/3$ in this range of $\gamma$ values. Inset: difference between the numerical effective threshold $p_c(N)$ and the expected value $p_c$ for infinite size, as a function of $N$. The straight solid line is the theoretical prediction $N^{-1/3}$.

Figure 3. Cluster size distribution $n_s$ at the percolation threshold $p_c(N)$ for $\gamma = 4.5$ and various system sizes $N$. The solid straight line represents the decay predicted analytically, equation (27).

process (with different random orderings of nodes having the same degree) on each of them. The threshold is determined from the position of the peak of the susceptibility

$$\chi = \frac{\sum s^2 n_s}{\sum s^n n_s},$$

i.e. the mean size of the finite clusters. The peak height $\chi_{\text{max}}$ is expected to grow with the system size as $N^{1-2\beta/\nu}$, due to the hyperscaling relation $2\beta + \gamma = \nu$. 

$\gamma > 4$

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To verify the validity of analytical results in this range we consider $\gamma = 4.5$. For this value, the threshold predicted by the continuous theory (equations (6) and (7)) is $p_c \approx 0.0234$. However, the corresponding value of $k_c$ is smaller than 9. With such a small range of $k$ values taking the continuous degree limit is not appropriate. We then solve numerically equations (2) and (4), using discrete sums and in this way we find $p_c \approx 0.0444$.

The scaling with the system size of the peak position (effective threshold) and peak height are displayed in figure 2. The agreement between theoretical predictions and numerical results is reasonable but not perfect, presumably because of the discreteness...
Figure 6. Relative size of the giant component $|G|$ versus $p - p_c(N)$ for $\gamma = 3.5$ and various $N$ compared with the analytical prediction, $|G| \sim \Delta^2$ (dashed line) and $|G| \sim \Delta$ (solid line).

Figure 7. Cluster size distribution $n_s$ at the percolation threshold $p_c(N)$ for $\gamma = 3.5$ and various system sizes $N$. The solid straight line represent the decay predicted analytically, equation (27), which for $\gamma = 3.5$ gives $\tau = 8/3$.

of degree values mentioned above. The distribution of cluster sizes $n_s$ at the critical point obeys instead very well the expected behavior (see figure 3).

In figure 4 we plot the relative size of the giant component as a function of $p - p_c(N)$. Also here the analytical prediction $|G| \sim \Delta$ works well but not perfectly. The effect of the degree degeneracy between many nodes is witnessed by the presence of little discontinuities in the slope of the curves, corresponding to points where $k_p$ changes by a unit.

$3 < \gamma < 4$

To verify the validity of analytical results in this range we consider $\gamma = 3.5$, for which the theoretical predictions are $p_c = 0.0000169$, $1/\nu = 1/5$, $\beta = 2$ and $\tau = 8/3$. Also for
As shown below, however, this is not the case, because the threshold value for infinite size is exceedingly small and the approach to it is very slow, due to the large $\nu$ value. As a consequence huge finite size corrections affect the results and in order to see the asymptotic regime unfeasibly large values of $N$ would be needed. A first evidence of this is provided by figure 5, where the scaling of the peak position (effective threshold) and of the peak height with system size are displayed. The numerical curves slowly approach the expected behavior, but much larger sizes would be needed to see the truly asymptotic exponent. A similar indication comes from the plot, in figure 6, of the relative size of the giant component as a function of $p - p_c (N)$. Even for the largest system size considered the effective exponent is larger than 1 but definitely smaller than the expected value $\beta = 2$. Instead the distribution of cluster sizes $n_s$ at the critical point

\[ \chi \]

as a function of $pN$ for $\gamma = 2.5$.

\[ |G|N \]

versus $p$ for $\gamma = 2.5$ and various $N$ compared with the analytical prediction.

**Figure 8.** Plot of the susceptibility $\chi$ as a function of $pN$ for $\gamma = 2.5$ and various $N$.

**Figure 9.** Plot of $|G|N$ versus $p$ for $\gamma = 2.5$ and various $N$ compared with the analytical prediction.
Degree-ordered-percolation on uncorrelated networks obeys well the expected behavior (see figure 7).

\[ 2 < \gamma < 3 \]

For networks with \( \gamma = 2.5 \) our theoretical approach predicts a vanishing threshold in the infinite size limit and the exponents \( \nu = 1, \beta = 1. \) To test the validity of the prediction that \( p_c(N) \sim N^{-1} \), in figure 8 we plot the susceptibility \( \chi \) versus \( pN \) for various system sizes. The perfect collapse confirms the validity of the finite size scaling analysis. Note that \( \chi_{\text{max}} \) does not depend on \( N \). This disagrees with the prediction \( \chi_{\text{max}} \sim N^{1 - 2\beta/\nu} = N^{-1} \), showing that the hyperscaling relation does not hold in this case. Note also that, as predicted, the effective transition occurs when the number \( Np_c(N) \) of nodes added to the system is not only finite, but also very small, of the order of 10. For this reason, when the incipient giant component starts to appear, finite clusters—if any—are extremely tiny (of size 1 or 2) and no power-law decay is observed for their size.

Also the plot in figure 9, displaying the rescaled size \( |G|N \) of the giant component as a function of \( pN \), shows a perfect agreement with the prediction \( \beta = 1 \), thus confirming the great accuracy of the theoretical predictions for \( 2 < \gamma < 3 \).

6. Conclusions

In summary, we have studied the percolation transition of the DOP model on power-law distributed uncorrelated networks. By applying standard analytical methods we have determined the percolation threshold and associated critical exponents as a function of the exponent \( \gamma \) of the degree distribution. The results have then been checked by means of numerical simulations, obtaining a satisfactory agreement except for the case \( \gamma = 3.5 \) where the discrepancy between theory and simulations can however be rationalized as the effect of very strong finite size effects, associated to the extremely small value of the threshold. DOP is a variation of the standard random percolation process, which exhibits nontrivial properties on heterogeneous networks. A comparison of the results derived here with corresponding values for standard percolation [16] indicates that for scale-rich topologies \( (\gamma > 3) \) DOP is in the same universality class, sharing the same critical exponents values. It is however important to remark that the different protocols for removing nodes have a strong influence on the value of the percolation thresholds, which are very different in the two cases. As can be seen from the curves in figure 1, the threshold for standard percolation grows (linearly) large as soon as \( \gamma > 3 \), while DOP threshold remains practically indistinguishable from 0 for \( \gamma \) up to 4. This has important consequences for the SIS dynamics on this type of networks, whose large-scale properties depend on the cumulative merging percolation process, which is a long-range variation of DOP [14].

Our results confirm and clarify the numerical evidence presented by Lee et al [9]. The singular behavior of the DOP threshold for \( \gamma \to 3 \) from above is an anticipation of the nontrivial behavior observed for \( 2 < \gamma \leq 3 \). In this range, the transition occurs for \( p_c = 0 \), as for standard percolation, but DOP is not in the same universality class, having different exponents, independent of \( \gamma \). The value \( \nu = 1 \), governing the approach
to zero of the size-dependent effective threshold \( p_c(N) \), is quite peculiar. It implies that a giant component starts forming as soon as a fixed number (not a fixed fraction) of nodes are added. Such a number turns out to be very small, of the order of a few units, increasing further the oddity of this transition.

In the present paper we have investigated DOP on an ensemble of random uncorrelated networks, where the only preassigned property is the degree distribution. It is natural to wonder what is the effect of additional topological features on this type of transition. Among these possible further developments, a particularly interesting one is the investigation of the effect of degree correlations. It is reasonable to expect that assortative correlations will lower the threshold, while disassortative ones will tend to increase it, since they will make hubs more distant from each other. Whether these tendencies lead to qualitative changes (i.e. a vanishing threshold for \( \gamma > 3 \) or a finite threshold for scale-free networks) is a nontrivial question that remains open.

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**Appendix A. Calculation of the exponent \( \nu \) for \( \gamma \geq 3 \)**

For a network of finite size, the integrals in equation (5) must be performed only up to \( k_{\text{max}} \), yielding for \( \gamma > 3 \)

\[
\frac{A}{\langle k \rangle} \left\{ \frac{1}{3 - \gamma} \left[ k_{\text{max}}^{3-\gamma} - k_c(N)^{3-\gamma} \right] - \frac{1}{2 - \gamma} \left[ k_{\text{max}}^{2-\gamma} - k_c(N)^{2-\gamma} \right] \right\} = 1, \tag{A1}
\]

where \( A \) is the normalization prefactor appearing in equation (1), and we have noted explicitly that now \( k_c \) depends on \( N \). This equation can be rewritten as

\[
\frac{\gamma - 2}{\gamma - 3} k_{\text{min}}^{\gamma-2} k_c(N)^{2-\gamma} \left\{ k_c(N) \left[ 1 - \left( \frac{k_{\text{max}}}{k_c(N)} \right)^{3-\gamma} \right] - \frac{\gamma - 3}{\gamma - 2} \left[ 1 - \left( \frac{k_{\text{max}}}{k_c(N)} \right)^{2-\gamma} \right] \right\} = 1, \tag{A2}
\]

where we have already taken the large \( N \) limit in the expression of \( A \). For \( k_{\text{max}} \to \infty \), equation (A2) correctly returns equation (6). For finite \( k_{\text{max}} \) the dominant correction is given by the term \( [k_{\text{max}}/k_c(N)]^{\gamma-3} \). Inserting the assumption \( k_c(N) = k_c + \delta \) into equation (A2) (where \( k_c \) is the solution of equation (6)) and expanding for small \( \delta \) we find that \( \delta \sim k_{\text{max}}^{3-\gamma} \). At this point we can go back to equation (12), which is the equation for \( p_c \) for finite \( k_{\text{max}} \). Inserting \( p_c(N) = p_c + \delta p_c \) into it we find

\[
\delta p_c \sim \delta \sim k_{\text{max}}^{3-\gamma} \sim N^{-3(\gamma-3)/(\gamma-1)}. \tag{A3}
\]

For \( \gamma = 3 \) instead, equation (5) gives

\[
k_{\text{min}} \left[ \ln \left( \frac{k_{\text{max}}}{k_{\text{min}}} \right) + \frac{1}{k_{\text{max}}} - \frac{1}{k_c} \right] = 1. \tag{A4}
\]
Neglecting the terms $1/k_{\text{max}}$ and $1/k_c$ which vanish in the thermodynamic limit, we find

$$k_c = k_{\text{max}} e^{-\frac{1}{k_{\text{min}}}}. \tag{A5}$$

Inserting this expression into equation (4) we find

$$p_c = k_{\text{min}}^2 (k_c^{-2} - k_{\text{max}}^{-2}) = k_{\text{min}}^2 \left( e^{2/k_{\text{min}}} - 1 \right) k_{\text{max}}^{-2}. \tag{A6}$$

Since $k_{\text{max}} = N^{1/2}$ we finally obtain

$$p_c = k_{\text{min}}^2 \left( e^{2/k_{\text{min}}} - 1 \right) N^{-1}. \tag{A7}$$

Hence we find that the exponent $1/\nu = 1$ is the same as the case $2 < \gamma < 3$, while the prefactor has a different dependence on $k_{\text{min}}$. In particular, there is a discontinuity between the limit of the prefactor for $\gamma \to 3^−$, $2k_{\text{min}}$ and the value $k_{\text{min}}^2 \left( e^{2/k_{\text{min}}} - 1 \right)$ for $\gamma = 3$.

It is also interesting to consider the limit $\gamma \to 3^+$ before the limit $N \to \infty$. In that case one finds from equation (5)

$$k_c = \left( \frac{\gamma - 3}{\gamma - 2} k_{\text{min}}^{2-\gamma} + k_{\text{max}}^{3-\gamma} \right)^{1/(3-\gamma)}. \tag{A8}$$

Inserting this into equation (7) and taking the limit $\epsilon = \gamma - 3 \to 0$ we find

$$p_c = \left( \frac{k_{\text{min}}}{k_{\text{max}}} \right)^2 \left[ 1 + \frac{\epsilon}{k_{\text{min}}} \left( \frac{k_{\text{min}}^2}{k_{\text{max}}^2} \right) \right] \to \frac{k_{\text{min}}^2}{k_{\text{max}}^2} \epsilon^{2 - \gamma}. \tag{A9}$$

Hence $p_c$ vanishes as $1/N$ with a prefactor $k_{\text{min}}^2 \epsilon^{2/k_{\text{min}}}$ again discontinuous with respect to the case the case $\gamma = 3$.

**Appendix B. Calculations for the exponent $\beta$**

Let us consider equation (18) and write it explicitly in the continuous degree limit for a network of infinite size

$$|G| = f_0(1) - f_0(u) = p - \int_{k_p}^{\infty} dk P(k) u^k \tag{B1}$$

$$= p - (\gamma - 1) k_{\text{min}}^{\gamma-1} \int_{k_p}^{\infty} dk k^{-\gamma} e^{k \ln u}. \tag{B2}$$

Setting $t = k \ln (1/u)$ this can be rewritten as

$$|G| = p - (\gamma - 1) k_{\text{min}}^{\gamma-1} \int_{k_p}^{\infty} dt k^{-\gamma} e^{t \ln 1/u}, \quad \text{where } \Gamma(a,z) \text{ is the incomplete Gamma function.} \tag{B3}$$

Setting $u = 1 - \epsilon$ and expanding for small $\epsilon$, $\ln(1/u) = \epsilon + \epsilon^2/2$, we can use the expansion of the Gamma function for $z \to 0$

$$\Gamma(a,z) = \Gamma(a) - \frac{z^a}{a} \left[ 1 - \frac{a}{a + 1} + \frac{az^2}{2(a + 2)} + O(z^3) \right]. \tag{B4}$$

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In this way we obtain
\[ |G(p)| = p - (\gamma - 1)k_{\min}^{\gamma-1} \Gamma(1 - \gamma)\epsilon^{\gamma-1} \]
\[ - \left( \frac{k_{\min}}{k_p} \right)^{\gamma-1} \left( 1 - \frac{1 - \gamma}{2 - \gamma} k_p \epsilon \right). \]  
(B5)

Since the degree \( k_p \) is related to \( p \) by
\[ p = \int_{k_p}^{\infty} dk \, P(k) = \left( \frac{k_{\min}}{k_p} \right)^{\gamma-1}, \]  
(B6)
the first and the third term in equation (B5) simplify. For any \( \gamma > 2 \) the leading order in \( \epsilon \) is then
\[ |G(p)| \simeq \frac{\gamma - 1}{\gamma - 2} k_{\min}^{\gamma-1} k_p^{2-\gamma} \epsilon. \]  
(B7)

The actual behavior of \( |G| \) as a function of \( \Delta = p - p_c \) depends hence on how \( k_p \) and \( \epsilon \) depend in their turn on \( \Delta \).

For \( 2 < \gamma < 3 \), for which \( p_c = 0 \), \( \Delta = p \) and therefore
\[ k_p \sim \Delta^{-1/(\gamma-1)}. \]  
(B8)

For \( \gamma = 3 \) the expansion of the function \( \Gamma(-2, z) \) in equation (B3) is different from equation (B4), but this does not really lead to a modification of the result, which is \( k_p \sim \Delta^{-1/2} \), i.e. equation (B8) evaluated for \( \gamma = 3 \).

For \( \gamma > 3 \) instead, since \( p_c \) is finite, \( k_p \) is also finite (and equal to \( k_c \)) at the transition, so that close to it we can write
\[ k_p = k_c + a \Delta, \]  
(B9)

with \( a = k_{\min} p_c^{\gamma/(1-\gamma)}/(1-\gamma) \).

Concerning \( \epsilon \) instead, equation (19) for \( u \) reads
\[ u = 1 + \int_{k_p}^{\infty} dk \frac{k P(k)}{\langle k \rangle} (u^{k-1} - 1), \]  
(B10)
that can be rewritten as
\[ u = 1 - k_{\min}^{\gamma-2} k_p^{2-\gamma} + (\gamma - 2)k_{\min}^{\gamma-2} \frac{1}{\Gamma(2 - \gamma, k_p \ln(1/u))}. \]  
(B11)

Setting \( u = 1 - \epsilon \), expanding the incomplete Gamma function for small values of the second argument, using \( 1/u \approx 1 + \epsilon \) and \( \ln(1/u) \approx \epsilon + \epsilon^2/2 \) and keeping only lowest order terms we finally arrive at
\[ \frac{\epsilon}{k_{\min}^{\gamma-2}} = -(\gamma - 2)\Gamma(2 - \gamma)\epsilon^{\gamma-2} - k_p^{2-\gamma}\epsilon + \frac{2 - \gamma}{3 - \gamma} k_p^{3-\gamma} \]
\[ + \left[ \frac{3}{2} \frac{(2 - \gamma)}{(3 - \gamma)} k_p^{3-\gamma} - \frac{2 - \gamma}{2(4 - \gamma)} k_p^{4-\gamma} \right] \epsilon^2. \]  
(B12)
For $2 < \gamma < 3$ the leading terms are $\epsilon^{\gamma-2}$ and $\epsilon k_p^{3-\gamma}$. Imposing that they balance each other asymptotically implies
\[ \epsilon \sim k_p^{-1}. \tag{B13} \]

For $\gamma > 3$, inserting into equation (B12) the expansion (B9) of $k_p$ and using the threshold condition (6) we obtain
\[
0 = -(\gamma - 2)\Gamma(2 - \gamma)\epsilon^{\gamma-2} - a(2 - \gamma)k_c^{1-\gamma}\epsilon\Delta + \frac{2 - \gamma}{3 - \gamma} \left[ (3 - \gamma)ak_c^{2-\gamma}\epsilon\Delta + \frac{3}{2}k_c^{3-\gamma}\epsilon^2 \right] - \frac{2 - \gamma}{2(4 - \gamma)}k_c^{4-\gamma}\epsilon^2. \tag{B14} \]

If $\gamma > 4$ the leading terms on the rhs are those proportional to $\epsilon\Delta$ and $\epsilon^2$. Their matching implies
\[ \epsilon \sim \Delta. \tag{B15} \]

If $3 < \gamma < 4$ the leading terms are $\epsilon^{\gamma-2}$ and $\epsilon\Delta$, implying
\[ \epsilon \sim \Delta^{1/(\gamma-3)}. \tag{B16} \]

Finally, for $\gamma = 3$ the expansion of the $\Gamma$ function in equation (B11) is
\[ \Gamma(-1, z) = \ln(z) - \frac{1}{z} + (\gamma - 1) + O(z). \tag{B17} \]

Expanding for $\epsilon \to 0$ and neglecting higher order terms we obtain
\[ k_{\text{min}} \ln(k_p \epsilon) = k_{\text{min}}(1 - \gamma \epsilon - 1/k_p) - 1. \tag{B18} \]

When $\epsilon \to 0$, for the lhs to be finite it must be $\epsilon \sim k_p^{-1}$, as in the case $2 < \gamma < 3$. As a consequence the behavior of the giant component is characterized by the same value $\beta = 1$.

**Appendix C. Calculations for the exponent $\tau$**

Equation (26) reads for $\gamma > 3$
\[ h_1 = 1 - \int_{k_c}^{\infty} dk \frac{k}{\langle k \rangle} P(k) + (1 - \epsilon)\int_{k_c}^{\infty} dk \frac{k}{\langle k \rangle} h_1^{k-1}. \tag{C1} \]

Setting $t = k \ln(1/h_1)$ we obtain
\[ h_1 = 1 - k_{\text{min}}\gamma^{-2}k_c^{2-\gamma} + (1 - \epsilon)(\gamma - 2)k_{\text{min}}^{\gamma-2} \frac{1}{h_1} \cdot [\ln(1/h_1)]^{\gamma-2}\Gamma[2 - \gamma, k_c \ln(1/h_1)]. \tag{C2} \]

Close to the transition, we take $x = 1 - \epsilon$ and define the function $g_1(x) = 1 - h_1(1 - \epsilon)$. $g_1$ is small and we can expand $1/h_1 \approx 1 + g_1 + \ldots$ and $\ln(1/h_1) \approx g_1 + g_1^2/2 + \ldots$
Furthermore the incomplete Gamma function can be expanded for small values of the second argument, as in equation (B4). After straightforward algebra, by using the condition (6) and neglecting all subleading terms, we obtain

\[ \epsilon k_c^{\gamma-\gamma} = (\gamma - 2)\Gamma(2 - \gamma)g_1^{\gamma-2} + \left[ k_c^{\gamma-2} - \frac{3\gamma - 2}{2\gamma - 3} k_c^{\gamma-\gamma} + \frac{2 - \gamma}{2(4 - \gamma)} k_c^{4-\gamma} \right] g_1^2. \]

(C3)

For \( \gamma > 4 \) the leading term on the rhs is the one proportional to \( g_1^2 \), implying that \( g_1 \sim \epsilon^{1/2} \). For \( \gamma < 4 \) the leading term is the one proportional to \( g_1^{\gamma-2} \), so that \( g_1 \sim \epsilon^{1/(\gamma-2)} \). For \( 2 < \gamma \leq 3 \) performing the integrals in equation (C1) between \( k_c \) and \( k_{\text{max}} \) and letting \( N \rightarrow \infty \) both integrals vanish. Trivially \( h_1 = 1 \) for any \( \epsilon \). The probability that a finite cluster has size \( s \) does not decay as a power-law.

In order to determine the exponent \( \tau \) we must determine the critical properties of the generating function \( h_0(x) \). Defining \( g_0(\epsilon) = 1 - h_0(1 - \epsilon) \) and inserting it into equation (25) we obtain

\[ g_0 = k_{\text{min}}^{\gamma-1} k_c^{1-\gamma} - (1 - \epsilon)(\gamma - 1) k_{\text{min}}^{\gamma-1} \ln [(1 - g_1)^{-1}]^{\gamma-1} \Gamma(1 - \gamma, k_c \ln[(1 - g_1)^{-1}]). \]

(C4)

By expanding the incomplete Gamma function, reminding the expansions of the terms containing \( 1 - g_1 \) and neglecting subleading terms, we arrive at

\[ \frac{g_0}{k_{\text{min}}^{\gamma-1}} = -(1 - \epsilon)(\gamma - 1)\Gamma(1 - \gamma)g_1^{\gamma-1} + \epsilon k_c^{1-\gamma} - (1 - \epsilon) \frac{\gamma - 1}{2 - \gamma} k_c^{2-\gamma} g_1. \]

(C5)

For \( \gamma > 4 \), since \( g_1 \sim \epsilon^{1/2} \) the leading term is the third and we have

\[ g_0 \sim g_1 \sim \epsilon^{1/2}. \]

(C6)

For \( 3 < \gamma < 4 \) instead \( g_1 \sim \epsilon^{1/(\gamma-2)} \). The leading term is still the third, resulting in

\[ g_0 \sim g_1 \sim \epsilon^{1/(\gamma-2)}. \]

(C7)

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