STRENGTHENED STONE-WEIERSTRASS TYPE THEOREM

Piotr Niemiec

Abstract. The aim of the paper is to prove that if $L$ is a linear subspace of the space $C(K)$ of all real-valued continuous functions defined on a nonempty compact Hausdorff space $K$ such that $\min(|f|,1) \in L$ whenever $f \in L$ (where $\bar{L}$ denotes the uniform closure of $L$ in $C(K)$) and for any sequence $(b_n)_{n=1}^{\infty}$ of positive numbers satisfying the relation $\sum_{n=1}^{\infty} b_n = \|g\|$, there exists a sequence $(f_n)_{n=1}^{\infty}$ of elements of $L$ such that $\|f_n\| = b_n$ for each $n \geq 1$, $g = \sum_{n=1}^{\infty} f_n$ and $|g| = \sum_{n=1}^{\infty} |f_n|$. Also the formula for $\bar{L}$ is given.

Keywords: Stone-Weierstrass theorem, function lattices.

Mathematics Subject Classification: 41A65, 54C30, 54C40.

1. INTRODUCTION

In the 19th century Weierstrass proved that every continuous function defined on the interval $[0, 1]$ can be uniformly approximated by polynomials. Later Stone [10,11] generalized that result as follows: if $K$ is a compact Hausdorff space and $A$ is a subalgebra of $C(K)$ which contains all constant functions and separates points of $K$ (i.e. if for any two distinct points $a$ and $b$ of $K$ there exists a function $f \in A$ such that $f(a) \neq f(b)$), then $A$ is dense in $C(K)$ in the topology of uniform convergence. This fact is known as the Stone-Weierstrass theorem. A simple proof of it is based on the following property:

If $\max(f,g), \min(f,g) \in F$ for any elements $f$ and $g$ of a subfamily $F$ of $C(K)$ and if $g : K \to \mathbb{R}$ is such a continuous function that for any $x, y \in K$ there exists $f \in F$ satisfying the conditions $f(x) = g(x)$ and $f(y) = g(y)$, then $g$ belongs to the uniform closure of the family $F$.

The Stone-Weierstrass theorem has many generalizations. For example, Glimm [5] proved its counterpart for arbitrary (noncommutative) $C^*$-algebras, Bishop [2] generalized it to anti-symmetric algebras, Hofmann [6] formulated the categorical version of it and Garrido and Montalvo [4] generalized it to completely regular spaces. We strengthen the Stone-Weierstrass theorem for special linear subspaces of $C(K)$, as...
described in the Abstract. Our result is, in a sense, in the spirit of classical Bernstein’s lethargy theorem [1] (for a generalization see e.g. [7]) because it gives some information on the behavior of the sequence which approximates the given element of the space.

2. MAIN RESULT

In this paper $K$ is a nonempty compact Hausdorff space, $C(K)$ denotes the real algebra of all continuous real-valued functions on $K$ equipped with the topology of uniform convergence and with the supremum norm $\| \cdot \|$, and $L$ is a linear subspace of $C(K)$ satisfying the condition:

$$\forall f \in L: \min(|f|, 1) \in L.$$  \hfill (2.1)

The space $L$ has the following properties, the proofs of which are quite simple. Whenever $f, f_1, \ldots, f_n \in L$ and $t > 0$, then:

(L1) $|f| \in L$,

(L2) $\max(f_1, \ldots, f_n), \min(f_1, \ldots, f_n) \in L$,

(L3) $f_+, f_- \in L$, where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$,

(L4) $\min(f, t), \max(f, -t) \in L$.

The main theorem is preceded by the following lemma.

Lemma 2.1. Let $h \in \bar{L}$ be a (nonzero) nonnegative function and let $t \in [0, \|h\|)$. Then there exists $f \in L$ such that $\|f\| = t$, $\|h - f\| = \|h\| - t$ and $0 \leq f \leq h$.

Proof. Let $\varepsilon = \frac{1}{3}\|h\| - t > 0$. There exists $f_1 \in L$ such that $\|h - f_1\| \leq \varepsilon$. Let $f_2 = \max(f_1, 0)$. Thanks to (L3), $f_2 \in L$. Since $h \geq 0$ and the function $\mathbb{R} \ni x \mapsto \max(x, 0) \in \mathbb{R}$ is nonexpansive, i.e. $\|\max(x, 0) - \max(y, 0)\| \leq |x - y|$ for any $x, y \in \mathbb{R}$, we conclude that

$$\|h - f_2\| \leq \varepsilon.$$  \hfill (2.2)

Further, let $f_3 = f_2 - 2\min(f_2, \varepsilon)$. By (L4), $f_3 \in L$. Moreover, $f_3 \leq h$. Indeed, thanks to (2.2), $f_2(x) \leq h(x) + \varepsilon$. So, if $f_2(x) \geq \varepsilon$, then $f_3(x) = f_2(x) - 2\varepsilon \leq h(x)$. On the other hand, if $f_2(x) < \varepsilon$, then $f_3(x) = -f_2(x) \leq 0 \leq h(x)$.

Now let $f_4 = \max(f_3, 0)$. Then $f_4 \in L$ and $0 \leq f_4 \leq h$. Finally put $f = \min(f_4, t) \in L$. We easily see that $0 \leq f \leq h$ and $\|f\| \leq t$. It is enough to check that $\|h - f\| \leq \|h\| - t$. Let $x \in K$. If $h(x) \leq \|h\| - t$, then clearly $h(x) - f(x) \leq \|h\| - t$. So we may assume that $h(x) \geq \|h\| - t = 3\varepsilon$. Then, by (2.2), $f_2(x) \geq h(x) - \varepsilon \geq 2\varepsilon$ and hence $f_4(x) = f_3(x) = f_2(x) - 2\varepsilon \geq 0$. Now if $f_4(x) \geq t$, then $f(x) = t$ and $h(x) - f(x) \leq \|h\| - t$. On the other hand, if $f_4(x) \leq t$, then $f(x) = f_4(x) - 2\varepsilon$ and finally $h(x) - f(x) = h(x) - f_4(x) + 2\varepsilon \leq \|h - f_4\| + 2\varepsilon \leq 3\varepsilon = \|h\| - t$.  

Theorem 2.2. Let $g \in \bar{L}$ be a nonzero function. Let $(b_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} b_n = \|g\|.$$  \hfill (2.3)
Then there exists a sequence \((f_n)_{n=1}^{\infty}\) of elements of \(L\) such that \(\|f_n\| = b_n\) for any \(n \geq 1\) and \(g = \sum_{n=1}^{\infty} f_n\). What is more, \(\{g \geq 0\} = \bigcap_{n=1}^{\infty} \{f_n \geq 0\}\) and \(\{g \leq 0\} = \bigcap_{n=1}^{\infty} \{f_n \leq 0\}\). In particular, \(|g| = \sum_{n=1}^{\infty} |f_n|\).

Proof. First assume that \(g \geq 0\). Since \(b_n > 0\) for each \(n \geq 1\) and thanks to (2.3), \(\sum_{k=1}^{n} b_k < \|g\|\). An easy use of the induction argument ensures us that, thanks to Lemma 2.1, there exists a sequence \((f_n)_{n=1}^{\infty}\) of elements of \(L\) such that \(f_n \geq 0\), \(\sum_{k=1}^{n} f_k \leq g\), \(\|f_n\| = b_n\) and

\[
\|g - \sum_{k=1}^{n} f_k\| = \|g\| - \sum_{k=1}^{n} b_k
\]

for every \(n \geq 1\). Indeed, if \(f_1, \ldots, f_{n-1}\) are found, apply Lemma 2.1 for \(h = g - \sum_{k=1}^{n-1} f_{k-1}\) and \(t = b_n\) to obtain the function \(f_n\).

We conclude from (2.3) and (2.4) that the series \(\sum_{n=1}^{\infty} f_n\) is uniformly convergent to \(g\) and therefore in case of nonnegative \(g\) the proof is finished.

Now let \(g\) be an arbitrary element of \(L\). By (L3) and the continuity of the operators \(f \mapsto f_+\) and \(f \mapsto f_-, g_+, g_- \in L\). Observe that

\[
g_+ \cdot g_- \equiv 0 \quad \text{and} \quad \|g\| = \max(\|g_+\|, \|g_-\|).
\]

The second of the above connections, combined with (2.3), implies that there exist two sequences \((b^+_n)_{n=1}^{\infty}\) and \((b^-_n)_{n=1}^{\infty}\) of positive numbers such that

\[
\sum_{n=1}^{\infty} b^+_n = \|g_+\|, \quad \sum_{n=1}^{\infty} b^-_n = \|g_-\| \quad \text{and} \quad \max(b^+_n, b^-_n) = b_n.
\]

Now we may apply the first part of the proof for \(g_+\) and \(g_-\) to obtain two corresponding sequences \((f^+_n)_{n=1}^{\infty}\) and \((f^-_n)_{n=1}^{\infty}\) of nonnegative elements of \(L\) satisfying the equalities \(g_+ = \sum_{n=1}^{\infty} f^+_n\) and \(f^\pm_n = b^\pm_n\) (\(n \geq 1\)). To end the construction, put \(f_n = f^+_n - f^-_n\) and observe that:

(i) \(f^+_n \cdot f^-_n \equiv 0\) (thanks to (2.5) and the inequality \(0 \leq f^\pm_n \leq b^\pm_n\)), and hence \(\|f_n\| = \max(\|f^+_n\|, \|f^-_n\|) = b_n\) (by (2.6)),

(ii) if \(g(x) \geq 0\) \(g(x) \leq 0\), then \(g_-(x) = 0\) \(g_+(x) = 0\) \(|f_n^+(x) = 0|\) \(|f_n^-(x) = 0|\) for each \(n \geq 1\) and therefore \(f_n(x) \geq 0\) \(f_n(x) \leq 0\). \(\square\)

3. SOME APPLICATIONS

Theorem 2.2 cannot be applied for \(L\) being the space of all real-valued polynomials on the interval \([0,1]\), because this space does not satisfy the crucial condition (2.1). However, it is well known that if \((K,d)\) is a compact metric space, then the space \(\text{Lip}(K)\) consisting of all real-valued \textit{Lipschitz} functions on \(K\) \((g: K \rightarrow R\) belongs to \(\text{Lip}(K)\) if there exists a constant \(M \in [0, \infty)\) such that \(|g(x) - g(y)| \leq Md(x,y)\) for every \(x,y \in K\)\) is a subalgebra of \(\mathcal{C}(K)\) which separates points of \(K\). What is more, if \(f \in \text{Lip}(K)\), then \(\min(\|f\|, 1) \in \text{Lip}(K)\).
So, a special case of Theorem 2.2 is the following statement.

**Proposition 3.1.** If \((K,d)\) is a nonempty compact metric space and \(g \in C(K)\), then there exists a sequence \((f_n)_{n=1}^{\infty}\) of real-valued Lipschitz functions such that \(\|g\| = \sum_{n=1}^{\infty} \|f_n\|\), \(g = \sum_{n=1}^{\infty} f_n\) and \(|g| = \sum_{n=1}^{\infty} |f_n|\).

Proposition 3.1 is applied in [9] to establish an important property of the function linear space \(CFL(U)\), whose elements are the uniform limits of linear combinations of maps of the form \(U \ni x \mapsto d(x,y) - d(x,z) \in \mathbb{R}\) with \(y, z \in U\), generated by the Urysohn universal metric space \((U,d)\) \((U) is uniquely determined by its diameter and the following properties: it is separable and complete, every separable metric space of diameter no greater than \(\text{diam} U\) is isometrically embeddable in \(U\), and every isometric map between finite subsets of \(U\) is extendable to an isometry of \(U\), namely: if \(K\) is a compact subset of \(U\) and \(f: K \to \mathbb{R}\) is a continuous function, then there exists an extension \(F \in CFL(U)\) of \(f\) such that \(\|F\| = \|f\|\). This result enables us to build an example of an adjoint linear isomorphism between dual Banach spaces which is an isometry on the weakly-* dense subspace but not on the whole domain.

In case when \(L\) is a subspace of \(C(K)\), the closure of \(L\) can be nicely described. To do that, we put the definition.

**Definition 3.2.** The null set of the space \(L\) is the set \(\mathcal{N}(L) = \{x \in K: f(x) = 0 \text{ for each } f \in L\}\). The equivalence relation \(\mathcal{R}(L)\) on \(K\) induced by \(L\) is defined by the formula:

\[(x,y) \in \mathcal{R}(L) \iff \forall f \in L: f(x) = f(y) \quad (x,y \in K).\]

The algebra generated by \(L\) is the algebra

\[\mathcal{A}(\mathcal{N}(L), \mathcal{R}(L)) = \{g \in C(K) : g|_{\mathcal{N}(L)} \equiv 0, \quad \forall (x,y) \in \mathcal{R}(L): g(x) = g(y)\}.\]

The sets \(\mathcal{N}(L)\) and \(\mathcal{R}(L)\) are closed subsets of \(K\) and \(K \times K\), respectively, and the algebra \(\mathcal{A}(\mathcal{N}(L), \mathcal{R}(L))\) is a closed subalgebra of \(C(K)\), possibly with no unit.

The following result, which has entered folklore (cf. [3]), explains the terminology. For the reader’s convenience, we give a short proof.

**Proposition 3.3.** The closure of the space \(L\) (satisfying the condition (2.1)) in the space \(C(K)\) coincides with \(\mathcal{A}(\mathcal{N}(L), \mathcal{R}(L))\).

**Proof.** Clearly \(L \subset \mathcal{A}(\mathcal{N}(L), \mathcal{R}(L))\). To see the inverse inclusion, take \(g \in \mathcal{A}(\mathcal{N}(L), \mathcal{R}(L))\). Observe that:

\[\forall x, y \in K \exists f \in L: f(x) = g(x), f(y) = g(y).\]

(3.1)

Indeed, the following five conditions are possible:

(1°) \(x, y \in \mathcal{N}(L)\): take \(f = 0\).
(2°) \(x \in \mathcal{N}(L)\) and \(y \notin \mathcal{N}(L)\) (or conversely): there exists \(f_0 \in L\) such that \(f_0(y) \neq 0\).

Now it is enough to put \(f = \frac{g(y)}{f_0(y)} f_0\).
(3°) \(x, y \notin \mathcal{N}(L)\) and \((x, y) \in \mathcal{R}(L)\): do the same as in (2°).
Suppose \( A \).

(4°) \( x, y \notin \mathcal{N}(L) \), \( (x, y) \notin \mathcal{R}(L) \) and \( g(x) = g(y) \): there exist \( f_1, f_2 \in L \) such that \( f_1(x) \cdot f_2(y) \neq 0 \). Let \( f_0 = |f_1| + |f_2| \). By (L1), \( f_0 \in L \). Let \( m = \min(f_0(x), f_0(y)) > 0 \) and finally put \( f = \frac{2m}{m} \min(f_0, m) \in L \).

(5°) \( x, y \notin \mathcal{N}(L) \), \( (x, y) \notin \mathcal{R}(L) \) and \( g(x) \neq g(y) \): there exists \( f_1 \in L \) such that \( f_1(x) \neq f_1(y) \). By the proof of (4°), there is \( f_2 \in L \) such that \( f_2(x) = f_2(y) = g(x) - \frac{g(x) - g(y)}{f_1(x) - f_1(y)} f_1(x) \). Now it is easy to check that \( f(x) = g(x) \) and \( f(y) = g(y) \) for \( f = \frac{g(x) - g(y)}{f_1(x) - f_1(y)} f_1 + f_2 \in L \).

Having (3.1), it suffices to apply the property (\( \ast \)).

Now we shall give some illustrative examples dealing with the subject.

**Examples 3.4.** In everywhere below, \( \Omega \) is a nonempty compact Hausdorff space and each of the spaces \( L \) appearing below consists of continuous real-valued functions on \( \Omega \) and satisfies (2.1).

A. Suppose \( \Omega \) is totally disconnected. The space \( L \) of all functions with finite images is dense in \( C(\Omega) \).

B. Let \( U \) be an open nonempty subset of \( \Omega \) and let \( L \) consist of all functions whose support is contained in \( U \); that is, \( f \in L \) iff \( \text{supp} f := \bar{\{f \neq 0\}} \subset U \). Then \( L \) consists of all functions vanishing on \( \Omega \setminus U \).

C. Suppose \( \Omega \) is metrizable and \( d \) is a metric on \( \Omega \) which induces the topology of \( \Omega \).

For a fixed \( p > 0 \) let \( L \) be the space of all functions satisfying the Hölder condition with exponent \( p \). (\( L \) may not be dense in \( C(\Omega) \) for \( p > 1 \). It may even consist only of constant functions, as it is in case of \( \Omega = [0, 1] \) with the natural metric.)

D. Let \( \Omega \) and \( d \) be as in the previous example and let \( L \) be the space of all the so-called little Lipschitz functions on \( \Omega \) (cf. [12, Chapter 3]); that is, \( f \in L \) iff for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( |f(x) - f(y)| \leq \varepsilon d(x, y) \) whenever \( d(x, y) \leq \delta \). (\( L \) may consists only of constant functions. See [12] for utility of this space.)

E. Let \( \Omega = [a, b] \subset \mathbb{R} \) and let \( L \) be the space of all piecewise affine functions. Then \( L \) is dense in \( C(\Omega) \) and it is not an algebra.

F. Let \( A \) be a countable subset of \( \Omega \) and let \( L \) consist of all \( f \) such that \( \sum_{a \in A} |f(a)| < +\infty \). \( L \) is a proper (nonclosed) ideal in \( C(\Omega) \). One may show that \( L \) is dense in \( C(\Omega) \) provided the topology of \( A \) is discrete. (Indeed, in the latter case \( L \) separates points of \( \Omega \) and does not vanish at every point.)

We end the paper with the note that condition (2.1) is crucial in the classical theory of the Daniell-Stone integral (cf. [8]) and therefore we believe our result may find application there.

**REFERENCES**

[1] S.N. Bernstein, *Collected Works II*, Akad. Nauk SSSR, Moscow, 1954 [in Russian].

[2] E. Bishop, *A generalization of the Stone-Weierstrass theorem*, Pacific J. Math. **11** (1961), 777–783.
[3] C. Dellacherie, *Un complément au théorème de Weierstrass-Stone*, Sémin. Prob. Strasbourg 1 (1967), 52–53.

[4] M.I. Garrido, F. Montalvo, *On some generalizations of the Kakutani-Stone and Stone-Weierstrass theorems*, Extracta Math. 6 (1991), 156–159.

[5] J. Glimm, *A Stone-Weierstrass theorem for C*-algebras*, Ann. Math. 72 (1960), 216–244.

[6] D. Hofmann, *On a generalization of the Stone-Weierstrass theorem*, Appl. Categor. Struct. 10 (2002), 569–592.

[7] G. Lewicki, *Bernstein’s “lethargy” theorem in metrizable topological linear spaces*, Mh. Math. 113 (1992), 213–226.

[8] K. Maurin, *Analysis, Part II*, D. Reidel, Dordrecht-Boston-London, 1980.

[9] P. Niemiec, *Canonical Banach function spaces generated by Urysohn universal spaces. Measures as Lipschitz maps*, Studia Math. 192 (2009), 97–110.

[10] M.H. Stone, *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), 375–481.

[11] M.H. Stone, *The generalized Weierstrass approximation theorem*, Math. Mag. 21 (1948), 167–184.

[12] N. Weaver, *Lipschitz Algebras*, World Scientific, 1999.

Piotr Niemiec
piotr.niemiec@uj.edu.pl

Jagiellonian University
Institute of Mathematics
ul. Łojasiewicza 6, 30-348 Krakow, Poland

Received: July 30, 2010.
Revised: December 1, 2010.
Accepted: December 12, 2010.