Carleman and Observability Estimates for Stochastic Wave Equations*

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Abstract

Based on a fundamental identity for stochastic hyperbolic-like operators, we derive in this paper a global Carleman estimate (with singular weight function) for stochastic wave equations. This leads to an observability estimate for stochastic wave equations with non-smooth lower order terms. Moreover, the observability constant is estimated by an explicit function of the norm of the involved coefficients in the equation.

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1 Introduction and main results

Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a $C^2$ boundary $\Gamma$. Fix any $x_0 \in \mathbb{R}^d \setminus \overline{G}$. It is clear that

$$0 < R_0 \triangleq \min_{x \in G} |x - x_0| < R_1 \triangleq \max_{x \in G} |x - x_0|. \quad (1.1)$$

Put

$$\Gamma_0 \triangleq \{ x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0 \}, \quad (1.2)$$

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where $\nu(x)$ is the unit outward normal vector of $G$ at $x \in \Gamma$. Also, put $Q \triangleq (0, T) \times G$, $\Sigma \triangleq (0, T) \times \Gamma$ and $\Sigma_0 \triangleq (0, T) \times \Gamma_0$. Throughout this paper, we will use $C$ to denote a generic positive constant depending only on $T, G$ and $G_0$, which may change from line to line.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{w(t)\}_{t \geq 0}$ is defined. Let $H$ be a Banach space. We denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|_{L^2(0, T; H)}^2) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|_{C([0, T]; H)}^2) < \infty$, with the canonical norm.

Assume

\begin{align*}
  a_1 &\in L^\infty_{\mathcal{F}}(0, T; L^\infty(G)), & a_2 &\in L^\infty_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n)), \\
  a_3 &\in L^\infty_{\mathcal{F}}(0, T; L^2(G)), & a_4 &\in L^\infty_{\mathcal{F}}(0, T; L^2(G)),
\end{align*}

(1.3)

and

\begin{align*}
  f &\in L^2_{\mathcal{F}}(0, T; L^2(G)), & g &\in L^2_{\mathcal{F}}(0, T; L^2(G)).
\end{align*}

(1.4)

Let us consider the following stochastic wave equation:

\begin{align*}
  dy_t - \Delta y dt = (a_1 y_t + \langle a_2, \nabla y \rangle + a_3 y + f) dt + (a_4 y + g) dw(t) & \quad \text{in } Q, \\
  y = 0 & \quad \text{on } \Sigma, \\
  y(0) = y_0, & \quad y_t(0) = y_1 & \quad \text{in } G.
\end{align*}

(1.5)

Here, we denote the scalar product in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. For any initial data

\begin{align*}
  (y_0, y_1) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H^1_0(G) \times L^2(G)),
\end{align*}

(1.6)

it is easy to show that system (1.5) admits one and only one weak solution

\begin{align*}
  y &\in L^2_{\mathcal{F}}(\Omega; C([0, T]; H^1_0(G) \times L^2(G))) \bigcap C^1([0, T]; L^2(G)).
\end{align*}

By means of the classical multiplier approach and energy estimate, following [4, 6], it is not difficult to show the following hidden regularity for the solution of system (1.5) (Hence we omit the details):

**Proposition 1.1.** Under assumptions (1.3), (1.4) and (1.6), the solution of system (1.5) satisfies $\frac{\partial y}{\partial \nu} \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma))$. Moreover

\begin{align*}
  \frac{\partial y}{\partial \nu} &\in L^2_{\mathcal{F}}(0, T; L^2(\Gamma)) \\
  \leq C \left[ \|(y_0, y_1)\|_{L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H^1_0(G) \times L^2(G))} + |f|_{L^2_{\mathcal{F}}(0, T; L^2(G))} + |g|_{L^2_{\mathcal{F}}(0, T; L^2(G))} \right] \\
  &\quad \times \exp \left\{ C \left[ |(a_1, a_4)|_{L^2_{\mathcal{F}}(0, T; L^\infty(G))}^2 + |a_2|_{L^2_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n))}^2 + |a_3|_{L^2_{\mathcal{F}}(0, T; L^2(G))}^2 \right] \right\}.
\end{align*}

(1.7)
The main purpose of this paper is to derive a boundary observability estimate for system (1.5). For this, we choose a sufficiently small constant \( c \in (0, 1) \) so that (Recall (1.1) for \( R_0 \) and \( R_1 \))
\[
\frac{(4 + 5c)R_0^2}{9c} > R_1^2.
\]
Then, in the sequel, we take \( T(> 2R_1) \) sufficiently large such that
\[
\frac{4(4 + 5c)R_0^2}{9c} > c^2T^2 > 4R_1^2.
\] (1.8)

Our observability estimate for system (1.5) is stated as follows:

**Theorem 1.1.** Let (1.3)–(1.4) hold, \( R_1 \) and \( \Gamma_0 \) be given respectively by (1.1) and (1.2), and \( T \) satisfy (1.8). Then solutions of system (1.5) satisfy
\[
\|(y(T), y_t(T))\|_{L^2(\Omega, \mathcal{F}_T, P; H^1_0(G) \times L^2(G))} 
\leq C \left[ \frac{\partial y}{\partial \nu} \bigg|_{L^2_0(0, T; L^2(\Gamma_0))} + |f|_{L^2(0, T; L^2(G))} + |g|_{L^2(0, T; L^2(G))} \right] 
\times \exp \left\{ C \left[ (a_1, a_4)^2_{L^\infty(0, T; L^\infty(G; \mathbb{R}^n))} + |a_2|^2_{L^\infty(0, T; L^\infty(G; \mathbb{R}^n))} + |a_3|^2_{L^\infty(0, T; L^2(G))} \right] \right\};
\]
\[
\forall (y_0, y_1) \in L^2(\Omega, \mathcal{F}_0, P; H^1_0(G) \times L^2(G)).
\] (1.9)

It is well-known that observability estimate is an important tool for the study of stabilization and controllability problems for deterministic PDEs. We refer to [8] for a recent survey in this respect. Although there are numerous references addressed to the observability problems for deterministic PDEs, very little is known for the stochastic counterpart and it remains to be further understood. Indeed, to the best of our knowledge, [1] is the only one publication in this field, which is devoted to the controllability/observability for the stochastic heat equation. As far as we know, nothing is known for the observability estimate on the stochastic wave equation.

Similar to the deterministic setting, we shall use a stochastic version of the global Carleman estimate to establish inequality (1.9). The difficulty to do this is the very fact that, unlike the deterministic situation, system (1.5), a stochastic wave equation, is *time-irreversible*. Therefore, one can not simply mimic the usual Carleman inequality for the deterministic wave equations (See [2, 6] and the references cited therein). Rather, instead of the usual smooth weight function, one has to introduce another singular weight function to derive the desired Carleman estimate for system (1.5).

More precisely, for any (large) \( \lambda > 0 \) and any (small) \( c > 0 \), set
\[
\ell = \ell(t, x) \triangleq \lambda \left[ |x - x_0|^2 - c \left( t - \frac{T}{2} \right)^2 \right], \quad \theta \triangleq e^\ell.
\] (1.10)
Also, for any \( \beta > 0 \), we set
\[
\Theta = \Theta(t) \overset{\Delta}{=} \exp \left\{ -\frac{\beta}{t(T-t)} \right\}, \quad 0 < t < T.
\] (1.11)

It is easy to see that \( \Theta(t) \) decays rapidly to 0 as \( t \to 0 \) or \( t \to T \). Our Carleman estimate for system (1.5) is stated as follows:

**Theorem 1.2.** Let (1.3)–(1.4) hold, \( R_1 \) and \( \Gamma_0 \) be given respectively by (1.1) and (1.2), and \( T \) satisfy (1.8). Then there exist a constant \( \beta > 0 \) (which is very small), and a constant
\[
\lambda^* = C \left[ 1 + |(a_1,a_4)|_{L^\infty(0,T;L^\infty(G))^2} + |a_2|^2_{L^\infty(0,T;L^\infty(G;\mathbb{R}^n))} + |a_3|^2_{L^\infty(0,T;L^\infty(G))} \right],
\]

such that solutions of system (1.5) satisfy
\[
\lambda^* \mathbb{E} \int_Q \Theta \theta^2 (y_t^2 + |\nabla y|^2 + \lambda^2 y^2) dxdt \\
\leq C \mathbb{E} \left\{ \lambda \int_{\Sigma_0} \Theta \theta^2 \left| \frac{\partial \theta}{\partial \nu} \right|^2 d\Sigma_0 + \int_Q \Theta \theta^2 (f^2 + \lambda g^2) dxdt \right\},
\] (1.12)

\( \forall (y_0,y_1) \in L^2(\Omega,F_0,P;H^1_0(G) \times L^2(G)), \quad \forall \lambda \geq \lambda^* \).

Carleman estimate is a fundamental tool for the study of control and inverse problems for deterministic PDEs ([3, 8]). Similar to the situation for observability estimate, although there are numerous references addressed to Carleman estimate for deterministic PDEs, to the best of our knowledge, [1, 5] are the only two references for the stochastic counterpart, which are devoted to the stochastic heat equation. It would be quite interesting to extend the deterministic Carleman estimate for other PDEs to the stochastic ones, but there are many things to be done, and some of which seem to be challenging. In this paper, in order to present the key idea in the simplest way, we do not pursue the full technical generality.

The rest of this paper is organized as follows. In Section 2, as a key preliminary, we present an identity for a stochastic hyperbolic-like operator. Then, in Section 3, we derive pointwise Carleman-type estimates for the stochastic wave operator. Finally, Section 4 is devoted to the proof of Theorems 1.1-1.2.

## 2 Identity for a stochastic hyperbolic-like operator

For simplicity, we denote \( \sum_{i,j=1}^n \) and \( \sum_{i=1}^n \) simply by \( \sum \) and \( \sum_i \), respectively. Also, we will use the notation \( u_{i} = u_{x_i} \), where \( x_i \) is the \( i \)-th coordinate of a generic point \( x = (x_1, \cdots, x_n) \) in \( \mathbb{R}^n \). In a similar manner, we use the notation \( \ell_{i}, v_{i}, \) etc. for the partial derivatives of \( \ell \) and \( v \) with respect to \( x_i \).

We show the following fundamental identity for a stochastic hyperbolic-like operator:
Theorem 2.1. Let $b^{ij} \in C^1((0,T) \times \mathbb{R}^n)$ satisfying

$$b^{ij} = b^{ji}, \quad i, j = 1, 2, \cdots, n,$$  \hspace{1cm} (2.1)

$u, \ell, \Psi \in C^2((0,T) \times \mathbb{R}^n)$. Assume $u$ is a $H^2_{\text{loc}}(\mathbb{R}^n)$-valued $\{F_t\}_{t \geq 0}$-adapted processes such that $u_t$ is a $L^2_{\text{loc}}(\mathbb{R}^n)$-valued semi-martingale. Set $\theta = e^\ell$ and $v = \theta u$. Then for a.e. $x \in \mathbb{R}^n$ and $P$-a.s. $\omega \in \Omega$,

$$\theta \left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij}_t \ell_i v_{ij} + \Psi v \right) [du_t - \sum_{i,j}(b^{ij} u_i)_j dt]$$

$$+ \sum_{i,j} \left[ \sum_{i',j'} \left( 2b^{ij}_t b^{ij'}_t \ell_{i'} v_{ij'} - b^{ij}_t b^{ij'}_t \ell_{i'} v_{ij'} \right) - 2b^{ij}_t \ell_i v_i v_t + b^{ij}_t \ell_i v_i^2 \right]$$

$$+ \Psi b^{ij}_t v_t - \left( A\ell_t + \frac{\Psi}{2} \right) b^{ij}_t v_t^2 \right] dt$$

$$+ d \left[ \sum_{i,j} b^{ij}_t \ell_i v_{ij} - 2 \sum_{i,j} b^{ij}_t \ell_i v_i v_t + \ell_t u_t^2 - \Psi v_t v + \left( A\ell_t + \frac{\Psi}{2} \right) v^2 \right]$$

$$= \left\{ \ell_t u_t + \sum_{i,j}(b^{ij}_t \ell_i)_j - \Psi \right\} v_t^2 - 2 \sum_{i,j} \left[ (b^{ij}_t \ell_j)_t + b^{ij}_t \ell_{ij} \right] v_t v_i$$

$$+ \sum_{i,j} \left\{ (b^{ij}_t \ell_t)_t + \sum_{i',j'} \left[ 2b^{ij'}_t (b^{ij'}_t \ell_{i'})_{j'} - (b^{ij'}_t \ell_{i'})_{j'} \right] + \Psi b^{ij}_t \right\} v_t v_j$$

$$+ Bv^2 + \left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij}_t \ell_i v_i + \Psi v \right)^2 \right\} dt + \theta^2 \ell_t (du_t)^2,$$  \hspace{1cm} (2.2)

where

$$A \triangleq (\ell_t^2 - \ell_u) - \sum_{i,j} (b^{ij}_t \ell_i \ell_j - b^{ij}_t \ell_j - b^{ij}_t \ell_{ij}) - \Psi,$$

$$B \triangleq A\Psi + (A\ell_t)_t - \sum_{i,j} (A b^{ij}_t \ell_i)_j + \frac{1}{2} \left[ \Psi_t - \sum_{i,j} (b^{ij}_t \Psi_i)_j \right].$$  \hspace{1cm} (2.3)

**Proof.** Recall that $v(t,x) = \theta(t,x)u(t,x)$. Hence $u_t = \theta^{-1}(v_t - \ell_t v)$ and $u_j = \theta^{-1}(v_j - \ell_j v)$ for $j = 1, 2, \cdots, n$. Hence,

$$du_t = \theta^{-1}[dv_t - 2\ell_t v_t dt + (\ell_t^2 - \ell_u)vd]t].$$  \hspace{1cm} (2.4)

Similarly, by symmetry condition (2.1), one may check that

$$\sum_{i,j}(b^{ij} u_i)_j = \theta^{-1} \sum_{i,j} \left[ (b^{ij}_t v_i)_j - 2b^{ij}_t \ell_i v_{ij} + (b^{ij}_t \ell_i \ell_j - b^{ij}_t \ell_i - b^{ij}_t \ell_{ij})v \right].$$  \hspace{1cm} (2.5)
Therefore, by (2.4)–(2.5), and recalling the definition of $A$ in (2.3), we get
\[
\theta \left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j + \Psi v \right) \left[ du_t - \sum_{i,j} (b^{ij} u_i)_j dt \right] \\
= \left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j + \Psi v \right) \left\{ dv_t - \left[ \sum_{i,j} (b^{ij} v_i)_j - Av \right. \\
+ 2\ell_t v_t - 2 \sum_{i,j} b^{ij} \ell_i v_j - \Psi v \left. \right] dt \right\} \\
= \left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j + \Psi v \right) dv_t \\
+ \left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j + \Psi v \right) \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right] dt \\
+ \left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j + \Psi v \right)^2 dt. \\
\text{(2.6)}
\]

We now analyze the first two terms in the right hand side of (2.6).
First, using Itô’s formula, we have
\[
\left( -2\ell_t v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j + \Psi v \right) dv_t \\
= d \left[ \left( -\ell_t v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j + \Psi v \right) v_t \right] \\
= d \left[ \left( -\ell_t v_t^2 + 2 \sum_{i,j} (b^{ij} \ell_i)_i v_j v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j v_t + \Psi v_t^2 + \Psi v v_t \right) \right. \\
- \left[ -\ell_t v_t^2 + 2 \sum_{i,j} (b^{ij} \ell_i)_i v_j v_t + 2 \sum_{i,j} b^{ij} \ell_i v_j v_t + \Psi v_t^2 + \Psi v v_t \right] dt + \ell_t (dv_t)^2 \\
= d \left( -\ell_t v_t^2 + 2 \sum_{i,j} b^{ij} \ell_i v_j v_t + \Psi v v_t - \frac{\Psi v}{2} v^2 \right) \\
+ \left\{ - \sum_{i,j} (b^{ij} \ell_i v_t^2)_j + \left[ \ell_{tt} + \sum_{i,j} (b^{ij} \ell_i)_j - \Psi \right] v_t^2 - 2 \sum_{i,j} (b^{ij} \ell_i) v_t v_t + \frac{\Psi}{2} v^2 \right\} dt \\
+ \theta^2 \ell_t (du_t)^2. \\
\text{(2.7)}
\]

Next,
\[
-2\ell_t v_t \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right] \\
= 2 \left[ \sum_{i,j} (b^{ij} \ell_i v_i v_t)_j - \sum_{i,j} b^{ij} \ell_i v_i v_t \right] - \sum_{i,j} b^{ij} \ell_i (v_i v_t)_t - A(\ell_t v^2)_t \\
= 2 \left[ \sum_{i,j} (b^{ij} \ell_i v_i v_t)_j - \sum_{i,j} b^{ij} \ell_i v_i v_t \right] + \sum_{i,j} (b^{ij} \ell_i)_t v_i v_j \\
- \left( \sum_{i,j} b^{ij} \ell_i v_i v_j + A(\ell_t v^2) \right)_t + (A(\ell_t v^2)_t. \\
\text{(2.8)}
\]
Further, by means of a direct computation, one may check that
\[
2 \sum_{i,j} b^{ij} \ell_i v_j \left[ - \sum_{i,j} (b^{ij} v_i) + Av \right] \\
= - \sum_{i,j} \left[ \sum_{i',j'} \left( 2b^{ij} b^{ij'} \ell_{i'} v_{i'} - b^{ij} b^{ij'} \ell_i v_i v_{i'} \right) - Ab^{ij} \ell_i v^2 \right]_j \\
+ \sum_{i,j,i',j'} \left[ 2b^{ij'} (b^{ij} \ell_{i'}) - (b^{ij} b^{ij'} \ell_{i'}) \right] v_i v_{j'} - \sum_{i} (Ab^{ij} \ell_i) v_i^2,
\]
and
\[
\Psi v \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right] = - \sum_{i,j} \left( \Psi b^{ij} v_i v - \frac{\Psi}{2} b^{ij} v^2 \right) + \Psi \sum_{i,j} b^{ij} v_i v_j \\
+ \left[ - \frac{1}{2} \sum_{i,j} (b^{ij} \Psi_i)_j + A \Psi \right] v^2.
\]

Finally, combining (2.6)–(2.10), we arrive at the desired equality (2.2). \(\Box\)

3 Pointwise Carleman-type estimates for the stochastic wave operator

In this section, we show a pointwise Carleman-type estimate (with singular weight) for the stochastic wave operator “\(du_t - \Delta u dt\)”. To begin with, by taking \((b^{ij})_{m \times m} = I\), the identity matrix, and \(\theta = e^\ell\) (with \(\ell\) given in (1.10)) in Theorem 2.1, one has the following pointwise Carleman-type estimate for the stochastic wave operator.

**Lemma 3.1.** Let \(u, \ell, \Psi \in C^2((0,T) \times \mathbb{R}^n)\) and \(k \in \mathbb{R}\). Assume \(u\) is a \(H^2_{loc}(\mathbb{R}^n)\)-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adopted processes such that \(u_t\) is a \(L^2_{loc}(\mathbb{R}^n)\)-valued semi-martingale. Set \(v = \theta u\). Then for a.e. \(x \in \mathbb{R}^n\) and P-a.s. \(\omega \in \Omega\), it holds

\[
\theta(-2\ell_t v_t + 2\nabla \ell \cdot \nabla v + \psi v)(du_t - \Delta u dt) \\
+ d\left[ \ell_t (v^2_t) + 2(\nabla \ell) \cdot (\nabla v) v_t - \Psi vv_t + A \ell_t v^2 \right] \\
+ \sum_{i=1}^n \left\{ 2v_i (\nabla \ell) \cdot (\nabla v) - \ell_i |\nabla v|^2 - 2\ell_t v_i v_i + \ell_t v^2_i + \Psi vv_i - A \ell_i v^2 \right\}_i dt \\
\geq \left[ (1-k)\lambda v_t^2 + (k+3-4c)\lambda |\nabla v|^2 + B v^2 \\
+ \left( -2\ell_t v_t + 2\nabla \ell \cdot \nabla v + \psi v \right)^2 \right] dt + \theta^2 \ell_t (du_t)^2,
\]

\[
(3.1)
\]
where

\[
\begin{align*}
\Psi & \triangleq (2n - 2c - 1 + k)\lambda, \\
A &= 4 \left[ c^2 \left( t - \frac{T}{2} \right)^2 - |x - x_0|^2 \right] \lambda^2 + \lambda(4c + 1 - k), \\
B &= 4 \left[ (4c + 5 - k)|x - x_0|^2 - (8c + 1 - k)c^2 \left( t - \frac{T}{2} \right)^2 \right] \lambda^3 + O(\lambda^2).
\end{align*}
\] (3.2)

The desired pointwise Carleman-type estimate (with singular weight function \(\Theta\)) for the stochastic wave operator reads as follows:

**Theorem 3.1.** Let \(u \in C^2([0, T] \times \Omega), v = \theta u, \) and \(T\) satisfy (1.8). Then there exist three constant \(\lambda_0 > 0, \beta_0 > 0\) and \(c_0 > 0,\) independent of \(u,\) such that for all \(\beta \in (0, \beta_0)\) and \(\lambda \geq \lambda_0\) it holds

\[
\Theta \theta (-2\ell_t v_t + 2\nabla \ell \cdot \nabla v + \Psi v)(du_t - \Delta u dt) + d \left\{ \Theta \left[ \ell_t (v_t^2 + |\nabla v|^2) - 2(\nabla \ell) \cdot (\nabla v) v_t - \Psi vv_t + A\ell_t v^2 \right] \right\} \\
+ \sum_{i=1}^n \left\{ \Theta \left[ 2v_i (\nabla \ell) \cdot (\nabla v) - \ell_i |\nabla v|^2 - 2\ell_t v_i v_t + \ell_i v_t^2 + \Psi vv_i - A\ell_t v^2 \right] \right\} dt \geq \\
\left[ c_0 \lambda \Theta \theta^2 (u_t^2 + |\nabla u|^2 + \lambda^2 u^2) + \Theta \left( -2\ell_t v_t + 2\nabla \ell \cdot \nabla v + \Psi v \right)^2 \right] dt + \Theta \theta^2 \ell_t (du_t)^2,
\]

with \(A\) and \(\Psi\) given by (3.2).

**Remark 3.1.** The main difference between the pointwise estimates (3.1) and (3.3) is that we introduce a singular “pointwise” weight in (3.3). Another difference between (3.1) and (3.3) is that \(T\) is arbitrary in the former estimate; while for the later one needs to take \(T\) to be large enough.

**Proof of Theorem 3.1.** We use some idea in the proof of [7, Theorem 1]. The proof is divided it into several steps.

**Step 1.** We multiply both sides of inequality (3.1) by \(\Theta.\) Obviously, we have (recall (3.2) for \(A\) and \(\Psi\))

\[
\Theta d \left[ \ell_t (v_t^2 + |\nabla v|^2) - 2(\nabla \ell) \cdot (\nabla v) v_t - \Psi vv_t + A\ell_t v^2 \right] \\
= d \left\{ \Theta \left[ \ell_t (v_t^2 + |\nabla v|^2) - 2(\nabla \ell) \cdot (\nabla v) v_t - \Psi vv_t + A\ell_t v^2 \right] \right\} \\
- \frac{\beta(T - 2t)}{t^2(T - t)^2} \Theta \left[ \ell_t (v_t^2 + |\nabla v|^2) - 2(\nabla \ell) \cdot (\nabla v) v_t - \Psi vv_t + A\ell_t v^2 \right] dt.
\]

with \(A\) and \(\Psi\) given by (3.2).
Note that
\[
\begin{align*}
&\left| -\frac{\beta (T - 2t)}{t^2 (T - t)^2} \Theta \left[ -2(\nabla \ell) \cdot (\nabla v)v_t - \Psi vv_t \right] \right| \\
\leq &\frac{\beta |T - 2t|}{t^2 (T - t)^2} \Theta \left[ 2|\nabla \ell| \cdot (\nabla v)v_t + |\Psi vv_t| \right] \\
\leq &\frac{\beta |T - 2t|}{t^2 (T - t)^2} \Theta \left[ (|\nabla \ell| + 1)v_t^2 + |\nabla \ell||\nabla v|^2 + \frac{1}{4} \Psi^2 v^2 \right].
\end{align*}
\]
Thus by (3.1), and using (3.4)–(3.5), we get
\[
\Theta \theta (-2\ell_tv_t + 2\nabla \ell \cdot \nabla v + \psi v)(du_t - \Delta u dt) \\
+ d \left\{ \Theta \left[ \ell_t (v_t^2 + |\nabla v|^2) - 2(\nabla \ell) \cdot (\nabla v)v_t - \Psi vv_t + A\ell_t v_t^2 \right] \right\} \\
+ \sum_{i = 1}^n \left\{ \Theta \left[ 2v_i (\nabla \ell) \cdot (\nabla v) - \ell_i |\nabla v|^2 - 2\ell_tv_i v_t + \ell_i v_t^2 + \Psi vv_i - A\ell_i v_i^2 \right] \right\} dt
\geq \left\{ \Theta (1 - k) \lambda v_t^2 + \Theta (k + 3 - 4c) \lambda |\nabla v|^2 + \frac{\beta (T - 2t)}{t^2 (T - t)^2} \ell_t \Theta (v_t^2 + |\nabla v|^2) \\
- \frac{\beta |T - 2t|}{t^2 (T - t)^2} \Theta \left[ (|\nabla \ell| + 1)v_t^2 + |\nabla \ell||\nabla v|^2 \right] \\
+ \left[ B + \frac{\beta (T - 2t)}{t^2 (T - t)^2} \ell_tA - \frac{\beta |T - 2t|}{t^2 (T - t)^2} \Psi^2 \right] \Theta v_t^2 + \Theta \left( -2\ell_tv_t + 2\nabla \ell \cdot \nabla v + \psi v \right)^2 \right\} dt
+ \Theta \theta^2 \ell_t (du_t)^2,
\]
where \( B \) is given by (3.2).

**Step 2.** Recalling that \( \ell \) and \( \Psi \) are given respectively by (1.10) and (3.2), we get
\[
\text{RHS of (3.6) } = \left[ \lambda \Theta (F_1 v_t^2 + F_2 |\nabla v|^2) + \lambda^3 \Theta Gv^2 + \Theta \left( -2\ell_tv_t + 2\nabla \ell \cdot \nabla v + \psi v \right)^2 \right] dt
+ \Theta \theta^2 \ell_t (du_t)^2,
\]
where
\[
F_1 \triangleq 1 - k + \frac{c \beta (T - 2t)^2}{t^2 (T - t)^2} - \frac{\beta |T - 2t|}{t^2 (T - t)^2} (2|x - x_0| + \lambda^{-1}),
\]
\[
F_2 \triangleq k + 3 - 4c + \frac{c \beta (T - 2t)^2}{t^2 (T - t)^2} - \frac{2\beta |T - 2t| |x - x_0|}{t^2 (T - t)^2},
\]
and
\[
G \triangleq 4 \left[ (4c + 5 - k)|x - x_0|^2 - (8c + 1 - k)c^2 \left( t - \frac{T}{2} \right)^2 \right] + O(\lambda^{-1})
+ \frac{\beta |T - 2t|}{t^2 (T - t)^2} \left\{ 4c |T - 2t| \left[ c^2 (t - T/2)^2 - |x - x_0|^2 \right] + O(\lambda^{-1}) \right\}.
\]
Step 3. Let us show that $F_1$, $F_2$, and $G$ are positive when $\lambda$ is large enough and $\beta$ is sufficiently small. For this, put

$$F^0_1 \triangleq 1 - k, \quad F^0_2 \triangleq k + 3 - 4c,$$

$$G^0 \triangleq 4 \left( (4c + 5 - k)|x - x_0|^2 - (8c + 1 - k)c^2 \left( t - \frac{T}{2} \right)^2 \right) + O(\lambda^{-1}),$$

which are respectively the nonsingular part of $F_1$, $F_2$ and $G$. Similarly, put

$$F^1_1 \triangleq \frac{c\beta(T - 2t)^2}{t^2(T - t)^2} - \frac{\beta|T - 2t|}{t^2(T - t)^2}(2|x - x_0| + \lambda^{-1}), \quad F^1_2 \triangleq \frac{c\beta(T - 2t)^2}{t^2(T - t)^2} - \frac{2\beta|T - 2t||x - x_0|}{t^2(T - t)^2},$$

$$G^1 \triangleq \frac{\beta|T - 2t|}{t^2(T - t)^2} \{ 4c|T - 2t| \left[ c^2(t - T/2)^2 - |x - x_0|^2 \right] + O(\lambda^{-1}) \},$$

which are respectively the singular part of $F_1$, $F_2$ and $G$.

Further, we choose $k = 1 - c$. It is easy to see that both $F^0_1$ and $F^0_2$ are positive, and

$$G^0 \geq 4(4 + 5c)R^2_0 - 9c^3T^2 + O(\lambda^{-1}),$$

which, via the first inequality in (1.8), is positive provided that $\lambda$ is sufficiently large.

When $t$ is close to 0 or $T$, i.e., $t \in I_0 = (0, \delta_0) \cup (T - \delta_0, T)$ for some sufficiently small $\delta_0 \in (0, T/2)$, the dominant terms in $F_i$ ($i = 1, 2$) and $G$ are the singular ones. For $t \in I_0$,

$$F^1_1 \geq \frac{\beta|T - 2t|}{t^2(T - t)^2} \{ c(T - 2\delta_0) - 2R_1 - \lambda^{-1} \} = \frac{\beta|T - 2t|}{t^2(T - t)^2} (cT - 2R_1 - 2c\delta_0 - \lambda^{-1}),$$

which, via the second inequality in (1.8), is positive provided that both $\delta_0$ and $\lambda^{-1}$ are sufficiently small. Similarly, for $t \in I_0$, $F^0_2$ is positive provided that $\delta_0$ is sufficiently small. Further, for $t \in I_0$,

$$G^1 \geq \frac{\beta|T - 2t|}{t^2(T - t)^2} \{ 4c|T - 2\delta_0| \left[ c^2(\delta_0 - T/2)^2 - R^2_1 \right] + O(\lambda^{-1}) \} \geq \frac{\beta|T - 2t|}{t^2(T - t)^2} \{ 4c|T - 2\delta_0| \left[ c^2T^2/4 - R^2_1 + c^2\delta_0(\delta_0 - T) \right] + O(\lambda^{-1}) \},$$

which, via the second inequality in (1.8), is positive provided that both $\delta_0$ and $\lambda^{-1}$ are sufficiently small.

By (3.8)–(3.10), we see that $F_1 = F^0_1 + F^1_1$, $F_2 = F^0_2 + F^1_2$ and $G = G^0 + G^1$. Noting the positivity of $F^0_1$, $F^0_2$ and $G^0$, by the above argument, we see that $F_1$, $F_2$ and $G$ are positive for $t \in I_0$. For $t \in (0, T) \setminus I_0$, noting again the positivity of $F^0_1$, $F^0_2$ and $G^0$, one can choose $\beta > 0$ sufficiently small such that $F^1_1$, $F^1_2$ and $G^1$ are very small so that $F_1$, $F_2$ and $G$ are positive. Hence (3.6)–(3.7) yield the desired (3.3). This completes the proof of Theorem 3.1. $\Box$
4 Proof of Theorems 1.1-1.2

We are now in a position to prove Theorems 1.1-1.2.

Proof of Theorem 1.2. The key idea is to apply Theorem 3.1. Integrating both sides of (3.3) (with $u$ replaced by $y$, and $v = \theta y$), using integration by parts, and recalling that $\Theta(t)$ decays exponentially to 0 as $t \to 0$ or $t \to T$, noting that $v|_{\Sigma} = 0$ (and hence $\nabla v = \frac{\partial}{\partial \nu}\nu$ on $\Sigma$), we arrive at

$$
E \int_Q \left[ c_0 \lambda \theta^2 (y_t^2 + |\nabla y|^2 + \lambda^2 y^2) + \Theta \left( -2 \ell_t v_t + 2 \nabla \ell \cdot \nabla v + \psi v \right)^2 \right] dx dt
$$

$$
\leq E \int_Q \Theta \theta(-2 \ell_t v_t + 2 \nabla \ell \cdot \nabla v + \psi v)(dy_t - \Delta y dt) dx - E \int_Q \Theta^2 \ell_t(dy_t)^2 dx
$$

$$
+ E \int_{\Sigma} \Theta \frac{\partial \ell}{\partial \nu} \left| \frac{\partial v}{\partial \nu} \right|^2 d\Gamma dt.
$$

By the first equation of system (1.5), we get

$$
E \int_Q \Theta \theta(-2 \ell_t v_t + 2 \nabla \ell \cdot \nabla v + \psi v)(dy_t - \Delta y dt) dx - E \int_Q \Theta^2 \ell_t(dy_t)^2 dx
$$

$$
= E \int_Q \Theta \theta(-2 \ell_t v_t + 2 \nabla \ell \cdot \nabla v + \psi v)(a_1 y_t + \langle a_2, \nabla y \rangle + a_3 y + f) dx dt
$$

$$
- E \int_Q \Theta^2 \ell_t(a_4 y + g)^2 dx dt
$$

$$
\leq E \int_Q \Theta \left( -2 \ell_t v_t + 2 \nabla \ell \cdot \nabla v + \psi v \right)^2 dx dt
$$

$$
+ C \left\{ E \int_Q \Theta^2 \left[ a_1 y_t + \langle a_2, \nabla y \rangle + a_3 y + f \right]^2 dx dt + \lambda E \int_Q \Theta^2 (a_4 y + g)^2 dx dt \right\}
$$

$$
\leq E \int_Q \Theta \left( -2 \ell_t v_t + 2 \nabla \ell \cdot \nabla v + \psi v \right)^2 dx dt
$$

$$
+ C \left\{ E \int_Q \Theta^2 (f^2 + \lambda g^2) dx dt + |a_1|^2_{L^2(0,T;L^\infty(G))} E \int_Q \Theta^2 y_t^2 dx dt
$$

$$
+ \lambda \left[ |a_3|^2_{L^2(0,T;L^\infty(G))} + |a_4|^2_{L^2(0,T;L^\infty(G))} \right] E \int_Q \Theta^2 y^2 dx dt
$$

$$
+ \left[ |a_2|^2_{L^2(0,T;L^\infty(G;\mathbb{R}^n))} + |a_3|^2_{L^2(0,T;L^\infty(G))} \right] E \int_Q \Theta^2 |\nabla y|^2 dx dt \right\}.
$$

On the other hand, recalling (1.2), we have

$$
E \int_{\Sigma} \Theta \frac{\partial \ell}{\partial \nu} \left| \frac{\partial v}{\partial \nu} \right|^2 d\Gamma dt = 2\lambda E \int_{\Sigma} \Theta^2 (x - x_0) \cdot \nu(x) \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt
$$

$$
\leq 2\lambda E \int_{\Sigma_0} \Theta^2 (x - x_0) \cdot \nu(x) \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma_0 dt \leq C\lambda E \int_{\Sigma_0} \Theta^2 \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma_0 dt.
$$
Finally, combining (4.1), (4.2) and (4.3), we conclude the desired estimate (1.12). This completes the proof of Theorem 1.2.

*Proof of Theorem 1.1.* The proof follows easily from Theorem 1.2 and the usual energy estimate. We omit the details.

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