Dynamical systems modelling of turbulence-shear flow interactions in magnetized fusion plasmas

R. Ball
Department of Theoretical Physics, Research School of Physical Sciences & Engineering, The Australian National University, Canberra ACT 0200 AUSTRALIA
E-mail: Rowena.Ball@anu.edu.au

Abstract.
A study of the qualitative structure and stability properties of a unified dynamical model for confinement transitions in fusion plasmas yields substantial new global and predictive information. Physics-based conditions are used to unfold trapped or persistent degenerate singularities in a simple model. Structural characterization of the resulting enhanced model achieves unification of previous disparate views of confinement transition physics, supplies valuable intelligence on the big issues of shear flow suppression of turbulence, multiple equilibria and oscillatory régimes, and suggests targeted experimental design, control and optimization strategies for new-generation fusion experiments.

1. Introduction
Fusion plasmas in magnetic containers, such as those in tokamak or stellarator experiments, are strongly driven nonequilibrium systems in which the kinetic energy of small-scale turbulent fluctuations can drive the formation of large-scale, stable, coherent structures such as shear and zonal flows. This inherent tendency to self-organise is a striking characteristic of flows where Lagrangean fluid elements see a predominantly two-dimensional velocity field, and is a consequence of the inverse energy cascade [1]. The distinctive properties of quasi two-dimensional fluid motion are the basis of natural phenomena such as zonal and coherent structuring of planetary flows, but are generally under-exploited in technology.

In plasmas the most potentially useful effect of two-dimensional fluid motion is suppression of high wavenumber turbulence that generates anomalous cross-field transport fluxes and degrades confinement [2], which can manifest as a dramatic enhancement of sheared poloidal or zonal flows and concomitant reduction in turbulent transport. These low- to high-confinement (L–H) transitions have been the subject of intensive experimental, in numero, and theoretical and modelling investigations since the 1980s. Two major strands in the literature emerged early and have persisted: (1) Confinement transitions are an internal, quasi two-dimensional flow, phenomenon and occur spontaneously when the rate of upscale transfer of kinetic energy from turbulence to shear and zonal flows, via Reynolds stress decorrelation, exceeds the nonlinear dissipation rate [3, 4]; (2) Confinement transitions are due to a net loss of ions near the plasma edge (because ions have larger larmor radii than electrons), the resulting electric field providing a torque which drives the poloidal shear flow nonlinearly [5, 6, 7].

In this work these different views of the physics of confinement transitions are smoothly reconciled in a unified dynamical model for the the coupled dynamics of potential energy,
turbulent kinetic energy, and shear flow kinetic energy subsystems. The model is developed by a step-wise iterative process: 1. Identify and interrogate degenerate (high-order) trapped or persistent singularities in the simplest model; 2. Unfold the singularities in physically meaningful ways; 3. Interrogate any new singularities that occur in enhanced model; 4. Repeat steps 2 and 3 until the model is free of pathological or persistent degenerate singularities (i.e., smooth), self-consistent, and therefore predictive.

1.1. Reduced dynamical modelling and confinement transitions
The large and lively primary literature on reduced (or low-order or low-dimensional) dynamical models for confinement transitions and associated oscillations in plasmas represents a sort of consensus on the philosophy behind qualitative analysis, if not on the details of the models themselves. What motivates this approach is the predictive value that a unified, low-order description of the macroscopic dynamics would have in the management of confinement states. Since it is widely acknowledged that control of turbulent transport is crucial to the success of the world-wide fusion energy program [8, 9] it is important to develop predictive models for efficient management of access to, and sustainment of, high confinement régimes.

Reduced dynamical modelling averages over space, mode spectrum structure, single-particle dynamics and other details, but the payoff lies in its amenity to sophisticated analytic theory and methods that enable us to track important qualitative features of the collective dynamics, such as singularities, bifurcations, and stability changes, broadly over the parameter space. The usefulness of such models seems to be no coincidence, too: in turbulent systems generally, which in detail are both complex and complicated, the dynamics seems to take place in a low-dimensional subspace [10].

However, it has been shown [11, 12, 13] that many of the published models for confinement transitions are structurally flawed. They often contain pathological or persistent degenerate (higher order) singularities. An associated issue is that of overdetermination, where near a persistent degenerate singularity there may be more defining equations than variables. Consequently much of the discussion in the literature concerning confinement transitions is qualitatively wrong. Such models cannot possibly have predictive value.

The heart of the matter lies in the mapping between the bifurcation structure and stability properties of a dynamical model and the physics of the process it is supposed to represent: if we probe this relationship we find that degenerate singularities ought to correspond to some essential physics (such as fulfilling a symmetry-breaking imperative, or the onset of hysteresis), or they are pathological. In the first case we can usually unfold the singularity in a physically meaningful way; in the other case we know that something is amiss and we should revise our assumptions. Degenerate singularities are good because they provide opportunities to improve a model and its predictive capabilities, but bad when they are not recognized as such.

1.2. Three energy subsystems
In the edge region of a plasma confinement experiment potential energy is stored in a steep pressure gradient which is fed by a power source near the centre. Gradient potential energy \( P \) is converted to turbulent kinetic energy \( N \), which is drawn off into stable shear flows \( \pm v' \) with kinetic energy \( F \), and dissipation channels. The energetics of this simplest picture of confinement transition dynamics are schematized in Fig. 1(a), from which a skeleton dynamical system for
Figure 1. Energy flux schematics for the gradient-driven plasma turbulence–shear flow system. Curly arrows indicate dissipative channels, straight arrows indicate inputs and transfer channels between the energy-containing subsystems. See text for explanations of each subfigure.

The process can be written down directly by inspection:

\[
\begin{align*}
\frac{dP}{dt} &= \varepsilon (Q - \gamma (P, N, F)) \\
\frac{dN}{dt} &= \gamma (P, N, F) - \alpha (P, N, F) - \beta (P, N, F) \\
\frac{dF}{dt} &= \alpha (P, N, F) - \mu (P, N, F),
\end{align*}
\]

where the power input \( Q \) is assumed constant and the energy transfer and dissipation rates generally may be functions of the energy variables. A more physics-based derivation of this system was outlined in [13]. Equations 1 are fleshed out by substituting specific rate-laws for the general rate expressions on the right hand sides:

\[
\begin{align*}
\frac{dP}{dt} &= Q - \gamma N P \\
\frac{dN}{dt} &= \gamma N P - \alpha v'^2 N - \beta N^2 \\
2 \frac{dv'}{dt} &= \alpha v' N - \mu (P, N) v'
\end{align*}
\]

where \( \varepsilon \) is the thermal capacitance, \( \gamma \) and \( \alpha \) are conservative energy transfer rate coefficients, \( \beta \) is the turbulence dissipation rate coefficient, and \( \mu (P, N) = bP^{-3/2} + aPN \) represents the neoclassical and turbulent contributions to viscous dissipation. The rate expressions were derived in [14] and [13] from semi-empirical arguments or given as ansatzes. (Rate-laws for bulk dynamical processes are not usually derivable purely from theory, and ultimately must be tested against experimental evidence.)
The rest of this paper is concerned with the character of the equilibria of Eqs 2–4 and modifications and extensions to this system. We shall study the type, multiplicity, and stability of attractors, interrogate degenerate or pathological singularities where they appear, and classify and map the bifurcation structure of the system. In doing this we shall attempt to answer questions such as: Are Eqs 2–4 or modified versions a good — that is, predictive — model of the system? Does the model adequately reflect the known phenomenology of confinement transitions in fusion plasmas? What is the relationship between the bifurcation properties of the model and the physics of confinement transitions?

A systematic and very practical methodology for characterizing the equilibria of dynamical systems involves locating and classifying high-order singularities then perturbing around them to explore and map the bifurcation landscape [15]. In a broad sense this paper is about applying singularity theory as a diagnostic tool while an impasto picture of confinement transition dynamics is compounded. The guiding motif is understanding the qualitative structure and properties of the system rather than concern for verisimilitude.

2. Symmetry-breaking has both local and global consequences

2.1. How to dissolve a pitchfork

The equilibrium solutions of Eqs 2–4 are shown in the bifurcation diagrams of Fig. 2, where the shear flow \( v' \) is chosen as the state variable and the power input \( Q \) is chosen as the principal bifurcation or control parameter. (In these and subsequent bifurcation diagrams stable equilibria are indicated by solid lines and unstable equilibria are indicated by dashed lines. Hopf bifurcations are annotated by asterisks, where the amplitude envelopes of associated branches of limit cycles are not plotted for clarity.) Several bifurcation or singular points are evident. The four Hopf bifurcations in (a) are discussed in section 2.2.

On the line \( v' = 0 \) the singularity \( P \) is found to satisfy the defining and non-degeneracy conditions for a pitchfork,

\[
G = G_\zeta = G_{\zeta\zeta} = G_\lambda = 0, \quad G_{\zeta\zeta\zeta} \neq 0, \quad G_{\zeta\lambda} \neq 0, \tag{5}
\]

where \( G \) is the bifurcation equation derived from the zeros of Eqs 2–4, \( \zeta \) represents the chosen state variable, \( \lambda \) represents the chosen control or principal bifurcation parameter, and the subscripts denote partial derivatives. In the qualitatively different bifurcation diagrams (a) and (b) the dissipative parameter \( \beta \) is relaxed either side of the critical value given in (c), where the perfect, twice-degenerate pitchfork is represented. Thus for \( \beta < \beta_{\text{crit}} \) (a poorly dissipative system) the turning points in (a) appear and the system may also show oscillatory behaviour. The dynamics are less interesting for \( \beta > \beta_{\text{crit}} \) (a highly dissipative system) as in (b) because the turning points, and perhaps also the Hopf bifurcations, cannot occur.

However, \( P \) is persistent through variations in \( \beta \) or any other parameter in Eqs 2–4. (This fact was not recognized in some previous models for confinement transitions, where such points

Figure 2. (a) \( \beta = 1 \), (b) \( \beta = 50 \), (c) \( \beta = \beta_{\text{crit}} \approx 18.58 \). \( \gamma = 1, \; b = 1, \; a = 0.3, \; \alpha = 2.4, \; \varepsilon = 1.5. \)
were wrongly claimed to represent second-order phase transitions.) Typically the pitchfork is associated with a fragile symmetry in the dynamics of the modelled physical system. The symmetry in this case is obvious from Fig. 2: in principle the shear flow can be in either direction equally. In real life (or in numero), experiments are always subject to perturbations that determine a preferred direction for the shear flow, and the pitchfork is inevitably dissolved. In this case the perturbation is an effective force or torque from any asymmetric shear-inducing mechanism, such as friction with neutrals in the plasma or external sources, and acts as a shear flow driving rate. Assuming this rate to be small and independent of the variables over the characteristic timescales for the other rate processes in the system, we may revise the shear flow evolution as

$$2 \frac{dv'}{dt} = \alpha v' N - \mu(P, N)v' + \varphi,$$

(6)

where the symmetry-breaking term $\varphi$ models the shear flow driving perturbation. The system now consists of Eqs 2, 3, and 6, and corresponding energy transfer schematic is shown in Fig. 1(b).

The pitchfork $P$ in Fig. 2 (c) can now be obtained exactly by applying the conditions (5) to the zeros of Eqs 2, 3, and 6, and with $\zeta \equiv v'$ and $\lambda \equiv Q$:

$$(v', Q, \beta, \varphi) = \left(0, \frac{\alpha^2 \gamma^2}{9 \alpha^2 b^2}, \frac{2 \alpha^3 \gamma \sqrt{\alpha/a}}{27 \sqrt{3} a^2 b}, 0\right).$$

(P)

The other singularity $T$ on $v' = 0$ satisfies the defining and non-degeneracy conditions for a transcritical bifurcation,

$$G = G_\zeta = G_\lambda = 0, \ G_{\zeta \zeta} \neq 0, \ \det \begin{pmatrix} G_{\zeta \zeta} & G_{\zeta \lambda} \\ G_{\lambda \zeta} & G_{\lambda \lambda} \end{pmatrix} \equiv \det d^2 G < 0.$$

(7)

It is once-degenerate and also requires the symmetry-breaking parameter for exact definition.

2.2. A walk along untrodden ways

A bifurcation diagram where $P$ is fully unfolded, that is, for $\varphi \neq 0$ and $\beta \neq \beta_{\text{crit}}$, is given in Fig. 3(a). (In these and subsequent diagrams the amplitude envelopes of of oscillatory branches are indicated by solid dots.) It is rich with information that speaks of the known and predicted dynamics of the system and foretells ways in which the model can be improved further, and which cannot be inferred or detected from the degenerate bifurcation diagrams of Fig. 2. Two features in particular are to be noted:

1. The system may be evolved to an equilibrium on the antisymmetric, $-v'$ branch, but if the power input then ebbs below the turning point at $(v', Q) \approx (-0.9, 0.46)$ the shear flow must spontaneously reverse direction, nominally to the lower $+v'$ branch. The zoom-in shows an example of an unusual feature in a bifurcation landscape, a region where there is fivefold multiplicity comprising three stable and two unstable equilibria. Two other, quite different, examples of threefold stable domains will be shown in section 4. In the remainder of this paper I concentrate on the $+v'$ branch and ignore the $-v'$ regime.

2. The symmetry-breaking parameter $\varphi$ has more far-reaching effects than merely providing a local universal unfolding of the pitchfork, for the branch of unstable equilibrium solutions that is just evident in the top (lower) left-hand corner was trapped as a singularity at $(v', Q) = (\infty, 0)$ for $\varphi = 0$. As $\varphi$ is increased this “new” branch passes through the heart of the model, the organizing centre. In (b) a segment of stable solutions has been created on the “new” branch as the Hopf bifurcation, which was born through a double zero eigenvalue (DZE), moves away from the turning point; the small branch of limit cycles can also be seen. At $\varphi_{Tm}$ — the organizing centre — the “new” and “old” branches exchange arms,
Figure 3. (a) $\varphi = 0.05$, $\varepsilon = 1.5$. The solution curves in the $N$ and $P$ diagrams are annotated to indicate whether they correspond to the $+v'$ or $-v'$ domain. (b)–(e) $\varepsilon = 1$. (b) $\varphi = 0.08$, (c) $\varphi \approx 0.08059 = \varphi_{TM}$, (d) $\varphi = 0.1$, (e) $\varphi = 0.11$. Other parameters: $\beta = 1$, $\gamma = 1$, $b = 1$, $a = 0.3$, $\alpha = 2.4$.

(c), via a non-symmetric, transcritical bifurcation that satisfies Eq. 7. This point signals a profound change in the type of dynamics that the system is capable of. For $\varphi > \varphi_{TM}$, (d) and (e), a transition must still occur at the lower turning point, but classical hysteresis is (locally) forbidden.

2.3. The story so far

The bifurcation structure of Eqs 2, 3, and 6 predicts shear flow suppresion of turbulence, hysteretic, non-hysteretic, and oscillatory transitions, and saturation then decrease of the shear flow with power input due to pressure-dependent anomalous viscosity. All of these behaviours have been observed consistently in magnetically contained fusion plasma systems. The model would therefore seem to be a “good” and “complete” one, in the sense of being self-consistent, free of pathological or persistent degenerate singularities, and reflecting typical observed behaviors.

However, there are several outstanding issues that suggest the model is still incomplete. The first issue arises as a gremlin in the bifurcation structure that makes an unphysical prediction. In section 3 a two-timing analysis brings to light the previously unrecognized trapped degenerate singularity, which is then unfolded smoothly by introducing another layer that models the neglected physics of downscale energy transfer. The second issue comes from a thermal diffusivity term that was regarded as negligible in the previous work. Section 4 follows the qualitative changes to the bifurcation and stability structure that are due to potential energy diffusivity
losses. The third issue arises from the two strands in literature on the physics of confinement transitions. In section 5 the unified model is proposed, in which is included a direct channel between gradient potential energy and shear flow kinetic energy.

3. Shear flows also generate turbulence
The first issue of incompleteness germinates from a pathology in the bifurcation structure of the model, which implies infinite growth of shear flow as the power input falls. Before we pinpoint the culprit singularity, it is illuminating to evince the physical—or unphysical—situation by considering Eqs 2–4 on the stretched (or shrunken) timescale \( \tau = t/\varepsilon \). In a system of low thermal capacitance \( \varepsilon \ll 1 \) and \( N \approx N_0 \) and \( v' \approx v_0 \). Thus the dynamics becomes quasi one-dimensional: the potential energy subsystem sees the kinetic energy subsystems as nearly constant, and \( P \approx (P_0 - Q/(N_0\gamma)) \exp(-N_0\gamma\tau) + Q/(N_0\gamma) \). Reverting to real time, as \( \varepsilon dP/dt \to 0 \) we have \( P \approx Q/(\gamma N) \); the potential energy is reciprocally slaved to the kinetic energy dynamics. The anomaly in this low-capacitance picture is that, as the power input \( Q \) ebbs toward zero, the shear flow can grow quite unrealistically. In Fig. 3(e) the conjectured fate of the surviving Hopf bifurcation is a double zero eigenvalue trap at \((Q, v') = (0, \infty)\).

Numerical experiments show that with diminishing \( \varepsilon \) the Hopf bifurcation moves upwards along the curve, the branch of limit cycles shrinks, and the conjugate pair of pure imaginary eigenvalues approaches zero. It would seem, therefore, that some important physics is still missing from the model.

3.1. A trapped singularity is found and released
What is not shown in Fig. 3 (because a log scale is used for illustrative purposes) is a highly degenerate branch of equilibria that exists at \( Q = 0 \) where \( N = 0 \) and \( v' = (P^{3/2}/\varphi)/b \); it is shown in Fig. 4(a). For \( \varphi > 0 \) there is a trapped degenerate turning point, annotated as \( s4 \), where the “new” branch crosses the \( Q = 0 \) branch. (In this and subsequent diagrams, where amplitude envelopes of oscillatory domains are not plotted for clarity, the Hopf bifurcations are annotated with asterisks.)

![Figure 4](image)

The key to the release (or unfolding) of \( s4 \) lies in recognising that kinetic energy in large-scale structures inevitably feeds the growth of turbulence at smaller scales, as well as vice versa [2]. In a flow where fluid elements locally experience a velocity field that is strongly two-dimensional there will be a strong tendency to upscale energy transfer (or inverse energy cascade, see [1]).
but the net rate of energy transfer to high wavenumbers (or Kolmogorov cascade, see [16]) is not negligible. What amounts to an ultraviolet catastrophe in the physics when energy transfer to high wavenumbers is neglected maps to a trapped degenerate singularity in the mathematical structure of the model. The trapped singularity s4 may be unfolded smoothly by including a simple, conservative, back-transfer rate between the shear flow and turbulence subsystems:

\[
\frac{dN}{dt} = \gamma NP - \alpha v'^2 N - \beta N^2 + \kappa v'^2
\]

\[
2\frac{dv'}{dt} = \alpha v' N - \mu(P, N)v' + \varphi - \kappa v'.
\]

The enhanced model consists of Eqs 2, 8, and 9, and the corresponding energy flux diagram is Fig. 1(c). The back-transfer rate coefficient \(\kappa\) need not be identified with any particular animal in the zoo of plasma and fluid instabilities, such as the Kelvin-Helmholtz instability; at this level it is simply a lumped dimensionless parameter that expresses the inevitability of energy transfer to high wavenumbers.

The manner and consequences of unfolding s4 can be appreciated from Fig. 4(b), from which one learns a salutary lesson: unphysical equilibria and singularities should not be ignored or dismissed as irrelevant, because they can play an important role in determining bifurcation structure in the physical domain.

The unfolding creates a maximum in the shear flow, and (apparently) a fourth Hopf bifurcation is released from a trap at infinity. At the given values of the other parameters a finite-area isola of steady-state solutions is formed, but it is important to visualize this (or, indeed, any other) bifurcation diagram as a slice of a three-dimensional surface of equilibrium solutions, where the third coordinate is another parameter. (Isolas of steady-state solutions were first reported in the chemical engineering literature, where dynamical models typically include a thermal or chemical autocatalytic reaction rate [17].) Figure 4(c) and (d) show two slices of this surface, taken to demonstrate that the organizing centre is preserved through the unfolding of s4. Here the other turning points are labelled s1, s2, and s3. Walking through (c) and (d) we make the forward transition at s1 and progress along this branch through the onset of a limit cycle régime, as in Fig. 3. For obvious reasons this segment is now designated as the intermediate shear flow branch, and the isola or peninsula as the high shear flow branch. In (c) a back-transition occurs at s2. The system can only reach a stable attractor on the isola via a transient, either a non-quasistatic jump in a second parameter or an evolution from initial conditions within the appropriate basin of attraction. In (d) as we make our quasistatic way along the intermediate branch with diminishing \(Q\) the shear flow begins to grow, then passes through a second oscillatory domain before reaching a maximum and dropping steeply; the back transition in this case occurs at s4.

To reiterate this last point: the shear flow can actually grow as the power input is withdrawn. This is an important and testable prediction.

4. Thermal dissipation affects the bifurcation structure

In the model so far the only outlet channel for the potential energy is conversion to turbulent kinetic energy, given by the conservative transfer rate \(\gamma PN\). However, in a driven dissipative system such as a plasma other conduits for gradient potential energy may be significant. The cross-field thermal diffusivity, a neoclassical transport quantity [18] is often assumed to be negligible in the strongly-driven turbulent milieu of a tokamak plasma [19, 14, 13], but here Eq. 2 is modified to include explicitly a linear “infinite sink” thermal energy dissipation rate:

\[
\varepsilon \frac{dP}{dt} = Q - \gamma NP - \chi P.
\]
Following ref. [20] $\chi$ is taken as a lumped dimensionless parameter and the rate term $\chi P$ as representing all non-turbulent or residual losses such as neoclassical and radiative losses. The model now consists of Eqs 8, 9, and 10 and the corresponding energy flux schematic is Fig. 1(d).

In Fig. 5 a series of bifurcation diagrams has been computed for increasing values of $\chi$ and a connected slice of the steady state surface. A qualitative change is immediately apparent, which has profound and far-reaching consequences: for $\chi > 0$ the two new turning points $s_5$ and $s_6$ appear, born from a local cusp singularity. Overall, from (a) to (e) we see that $s_1$ does not shift significantly but that the peninsula becomes more tilted and shifts to higher $Q$, but let us begin a walking tour at $s_1$ in (b). Here, as in Fig. 4, the transition occurs to an intermediate shear-flow state and further increments of $Q$ take the system through an oscillatory régime. But the effect of decreasing $Q$ is radically different: at $s_6$ a discontinuous transition occurs to a high shear flow state on the stable segment of the peninsula. From this point we may step forward through the shear flow maximum and fall back to the intermediate branch at $s_5$. We see that over the range of $Q$ between $s_5$ and $s_6$ the system has five steady states, comprising three stable interleaved with two unstable steady states. As in Fig. 4(c) and (d) a back transition at low $Q$ occurs at $s_4$.

The tristable régime in (b) has disappeared in (c) in a surprisingly mundane way: not through a singularity but merely by a shift of the peninsula toward higher $Q$. But this shift induces a different tristable régime through the creation of $s_7$ and $s_8$ at another local cusp singularity. In (d) $s_4$ and $s_7$ have been annihilated at yet another local cusp singularity. It is interesting and quite amusing to puzzle over the 2-parameter lines of $s_1$, $s_4$, $s_5$, $s_6$, $s_7$, and $s_8$ over $\chi$ projected in Fig. 6. The origins of the four cusps can be read off the diagram, keeping in mind that the crossovers are a trompe de l’oeil: they are nonlocal. Since the two black areas do not overlap, there is no domain of sevenfold multiplicity in the system!
Returning to Fig. 5, at s5 in (c), (d), and (e) the system transits to a limit cycle, rather than to a stable intermediate steady state. The amplitude envelope of the oscillatory regime is included in (e), and shown are the bifurcation diagrams in $N$ and $P$ as well as $v'$. The turbulence is enormously suppressed due to uptake of energy by the shear flow, but rises again dramatically with this hard onset of oscillations. The pressure gradient jumps at s1 because the power input exceeds the distribution rates, and oscillatory dynamics between the energy subsystems sets in abruptly at s5.

5. A unified model includes a nonlinear shear flow drive
In strand (2) in the literature confinement transitions are modelled in terms of a nonlinear electric field driving torque created by nonambipolar ion orbit losses from the plasma edge region [5, 6, 7]. Although there are many supporting experiments [21], the “electric field bifurcation” model of confinement transitions cannot explain shear flow suppression of turbulence, because it has no coupling to the internal dynamics of energy transfers from the potential energy reservoir. Here this plasma edge physics is treated as a piece of a more holistic picture and a simple rate $r(P)$ of shear flow generation due to ion orbit losses is used to complete the model, which now consists of Eq. 8 and

$$\varepsilon \frac{dP}{dt} = Q - \gamma NP - v'^2 r(P) - \chi P$$

$$2 \frac{dv'}{dt} = \alpha v' N - \mu(P, N)v' + v' r(P) - \kappa v' + \varphi,$$

where, following the earlier authors, $r(P) = \nu \exp \left[-\left(\frac{w^2}{P}\right)^2\right]$. This expression simply says that the rate at which ions are preferentially lost, and hence flow is generated, is proportional to a collision frequency $\nu$ times the fraction of those collisions that result in ions with sufficient energy to escape. The form of the energy factor assumes an ion distribution that is approximately Maxwellian and $w^2$, analogous to an activation energy, is proportional to the square of the critical escape velocity. In this form of the rate expression I have explicitly included the temperature-dependence of $r$, through $P$, which couples it to the rest of the system. If $w$ is high the rate is highly temperature (pressure gradient) sensitive. (For heuristic purposes constant density is assumed, constants and numerical factors are normalized to 1, and the relatively weak temperature dependence of the collision rate $\nu$ is ignored.) The corresponding energy schematic is Fig. 1(e) where it is seen that $r(P)$ is a competing potential energy conversion channel, that can dominate the dynamics when the critical escape velocity $w$ is low or the pressure is high.
This is exactly what we see in the bifurcation diagrams, Fig. 7. Overall, the effect of this contribution to shear flow generation from the ion orbit loss torque is to elongate and flatten the high shear flow peninsula. The Hopf bifurcations that are starred in (a), where the contribution is relatively small, have disappeared in (b) at a DZE singularity. What this means is that as \( r(P) \) begins to take over there is no longer a practicably accessible intermediate branch in the transition region, because the intermediate branch is unstable until the remaining Hopf bifurcation is encountered at extremely high \( Q \). Locally, in the transition region, the bifurcation diagram begins to look more like the simple S-shaped, cubic normal form schematics with classical hysteresis featured in numerous papers by earlier authors. However, as can be seen in Fig. 7(b) where the bifurcation diagram is rendered in the turbulent kinetic energy \( N \), this unified model accounts for shear flow suppression of the turbulence, whereas theirs could not.

6. Summary and conclusions
The generation of stable shear flows in plasmas, and the associated confinement transitions and oscillatory behaviour in tokamaks and stellarators, is regulated by Reynolds stress decorrelation of gradient-driven turbulence and/or by an induced bistable radial electric field. These two mechanisms are seamlessly unified by the first smooth path through the singularity and bifurcation structure of a reduced dynamical model for this system.

The model is constructed self-consistently, beginning from simple rate-laws derived from the basic pathways for energy transfer from pressure gradient to shear flows. It is iteratively strengthened by finding and classifying the singularities and allowing them to “speak for themselves”, then matching up appropriate physics to their unfoldings. The smooth path from turbulence driven to electric field driven shear flows crosses an interesting landscape:

- Hysteresis is possible in both régimes and is governed by different physics.
- A metamorphosis of the dynamics is encountered, near which hysteretic transitions are forbidden. The metamorphosis is a robust organizing centre of codimension 1, even though there are singularities of higher codimension in the system.
- Transitions may occur to and from oscillatory states.
- To traverse the smooth path several obstacles are successively negotiated in physically meaningful ways: a pitchfork is dissolved, which has the non-local effect of releasing a
branch of solutions from a singular trap at infinity, a singularity is released from a trap at zero power input, and two régimes of fivefold multiplicity are reconnoitred.

These results suggest strategies for controlling access to high confinement states, reducing turbulent transport, and manipulating oscillatory behaviour in new-generation fusion experiments that aim to achieve a self-sustaining burning plasma. More generally I have shown that low-dimensional models have a useful role to play in the study of one of the most formidable of complex systems, a strongly driven turbulent plasma.

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