The causal ladder and the strength of $K$-causality: II

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Abstract

Hawking’s stable causality implies Sorkin and Woolgar’s $K$-causality. The work investigates the possible equivalence between the two causality requirements, an issue which was first considered by H Seifert and then raised again by R Low after the introduction of $K$-causality. First, a new proof is given that a spacetime is stably causal iff the Seifert causal relation is a partial order. It is then shown that given a $K$-causal spacetime and having chosen an event, the light cones can be widened in a neighborhood of the event without spoiling $K$-causality. The idea is that this widening of the light cones can be continued leading to a global one. Unfortunately, due to some difficulties in the inductive process the author was not able to complete the program for a proof as originally conceived by H Seifert. Nevertheless, it is proved that if $K$-causality coincides with stable causality then in any $K$-causal spacetime the $K^+$ future coincides with the Seifert future. Explicit examples are provided which show that the $K^+$ future may differ from the Seifert future in causal spacetimes.

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1. Introduction

The property of $K$-causality was introduced about 10 years ago by Sorkin and Woolgar [11]. Given a spacetime $(M, g)$ they defined the relation $K^+$ as the smallest closed subset $K^+ \subset M \times M$, which contains $I^+, I^+ \subset K^+$, and shares the transitivity property: $(x, y) \in K^+$ and $(y, z) \in K^+ \Rightarrow (x, z) \in K^+$ (the set of causal relations satisfying these properties is non-empty, consider for instance the trivial subset $M \times M$). This definition arose from the fact that $J^+$ while transitive is not necessarily closed whereas $J^+$ while closed is not necessarily transitive. They also defined the spacetime to be $K$-causal if the relation $K^+$ is a partial order i.e. not only transitive and reflexive but also antisymmetric, that is, such that $(x, z) \in K^+$ and $(z, x) \in K^+ \Rightarrow x = z$.

Low pointed out ([11], footnote p 1990) that Seifert’s causal relation $J_5^+ \supset J^+$ is closed and transitive, hence $K^+ \subset J_5^+$. It is then natural to ask whether it is always $K^+ = J_5^+$ and
if this is not the case, whether it is at least true that $J_+^*$ is a partial order whenever $K^+$ is a partial order. Seifert proved [10] the transitivity and closure of $J_+^*$ and gave an argument showing that $J_+^*$ is a partial order if and only if the spacetime is stably causal (for a rigorous proof see [4], proposition 2.3 or theorem 3.12 below). As a consequence, since $K^+ \subset J_+^*$, if the spacetime is stably causal then it is $K$-causal. Moreover, the equality $K^+ = J_+^*$ would imply that the properties of $K$-causality and stable causality coincide. In contrast, the example of a spacetime $K$-causal but non-stably causal would imply at once that $K^+$ does not always coincide with $J_+^*$ and that $K$-causality can be included in the causal hierarchy of spacetimes [4, 6] just below stable causality.

Seifert himself [10] raised the problem as to whether $J_+^*$ could be regarded as the smallest closed and transitive causal relation containing $I^*$. One of his lemmas ([10], lemma 2) actually answers this question in the affirmative sense provided $K^+$ is a partial order.

**Claim 1.1** (Seifert’s ([10], lemma 2). $J_+^*$ is the smallest among the partial orders $P^+ \subset M \times M$ not smaller than $J^*$ with closed $P^+(x)$ and $P^-(x)$ for all $x$ [provided such a smallest partial order exists]1.

Indeed, we shall see in section 2 that the previous claim can be conveniently rephrased in the following way.

**Claim 1.2.** If $(M, g)$ is $K$-causal then $K^+ = J_+^*$. 

A consequence is that $K$-causality is equivalent to stable causality. Unfortunately, Seifert’s arguments were not rigorous as they did not take into account the many subtleties of the $K^+$ relation. This lemma was almost never cited in subsequent literature and some researchers who tried to reproduce it began to raise some doubts on its validity. It suffices to say that using it some proofs later given by Hawking and Sachs [4] could have been considerably simplified as I will show in the last section. In that fundamental work Hawking and Sachs preferred to take a path independent of Seifert’s 1971 work and indeed, although they cited Seifert, they gave a completely new proof of the equivalence between stable causality and the antisymmetry of $J_+^*$ and avoided any mention of the claim above. Over time the question raised by Seifert’s work was overlooked and only with the introduction of the $K^+$ relation was it rediscovered from a different perspective. This work is devoted to the study of this open issue.

The work is organized as follows.

In section 2 some general results for binary relations on $M \times M$ are given. The equivalence between claims 1.1 and 1.2 is proved here. In this section as well as in the rest of the work the reader is assumed to be familiar with the conventions and notation introduced in ([5], sections 1 and 2). Let me just recall that the spacetime signature is $(-, +, \cdots, +)$, that the subset inclusion is reflexive, $X \subset X$, and that the boundary of a set $A$ is denoted $\partial A$.

Section 3 deals with Seifert’s closed relation. I generalize, simplify and fill gaps of some proofs, particularly that on the equivalence between stable causality and the antisymmetry of $J_+^*$, the extent of the improvements being there explained.

Section 4 deals with some results on the violating sets for $J_+^*$. In section 5 some examples of spacetimes in which $K^+ \neq J_+^*$ are given. The strategy outlined by Seifert for a proof of the equivalence between $K$-causality and stable causality is, more or less, followed here and made rigorous. Seifert suggested to prove that (i) $K$-causality implies strong $K$-causality ([11], lemmas 16 and 5.5 below), and that (ii) strong $K$-causality implies that, an event being chosen, the cones can be widened in a neighborhood of the chosen point while preserving $K$-causality (theorem 5.19), (iii) the process of widening the light cones

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1 Text in square brackets by the author.
can be continued so as to obtain a global widening of the light cones and hence the proof that a $K$-causal spacetime is stably $K$-causal and hence stably causal (actually stable causality and stable $K$-causality coincide, see corollary 2.3).

The same strategy is followed here although in several points the proofs differ significantly from what originally suspected by Seifert. Several technical lemmas are required because what is intuitive for $J^+$ is usually hard to prove for $K^+$, the reason lying in the fact that $K^+$ is defined through its closure and transitivity properties and not, at least not straightforwardly, by using the set of causal curves. Indeed, Sorkin and Woolgar [11] introduced the method of transfinite induction in order to obtain some basic results. Here it is shown that this method can be avoided, and that it can be replaced by soft topological arguments which take advantage of the minimality of $K^+$.

Unfortunately, I will not be able to prove step (iii). The process of widening the light cones can indeed be indefinitely continued but some technicalities do not allow us to conclude that a global widening of the light cones can be obtained.

Nevertheless, in section 6 it is proved that if it is true that $K$-causality coincides with stable causality then it is also true that in a $K$-causal spacetime $K^+ = J^+_I$ (theorem 6.2). Under the same assumption a very simple proof that causal continuity implies stable causality is given. In this respect the proof of the equivalence between $K$-causality and stable causality is recognized as an important problem in causality theory as many old proofs would be greatly simplified by the knowledge of this result, and new ones would follow.

Despite the fact that the main problem has not been solved, sections 5 and 6 contain many new results and properties of the $K^+$ relation which may prove to be useful in future applications.

2. Preliminaries

The reader is assumed to be familiar with ([5], section 2) which contains some definitions and results for a generic binary relation $R^+ \subset M \times M$. The definitions of closure, partial closure, transitivity, reflexivity, antisymmetry, $R$-causality, strong $R$-causality, stable $R$-causality will not be repeated here. Nor will be repeated the definitions of the sets $R^\pm(x)$ or $R^-$ given $R^+$ or of the diagonal $\Delta$ in $M \times M$.

Here I give some useful definitions for dealing with violations of the antisymmetry condition. The $R$-causality violating set on $M \times M$ is

$$VR = R^+ \cap R^- \cap \Delta^C,$$

(1)

which is the set of pairs at which the antisymmetric condition for $R^+$ (and hence $R^-$) fails. Note that since $\Delta$ is closed $VR$ is open whenever $R^+$ is open, for instance, $VI$ is open.

Analogously, it is useful to define the $R$-causality violating set on $M$ as $vR = \pi(VR)$, where $\pi : M \times M \to M$ is the projection on the first factor (or, equivalently, on the second factor). It is made of all the points $x$ at which $R$-causality at $x$ does not hold.

In general given the spacetime $(M, g)$ the causal relation $R^+$ will be related in some way to $(M, g)$, for instance $I^+$ is the set of pairs $(x, z)$ such that there is a timelike curve connecting $x$ to $z$. In order to stress this spacetime dependence I shall sometimes write $R^+_I(M, g)$. If we are working in the same spacetime manifold $M$ but with different metrics I may write $R^+_\bar{g}$ in place of $R^+_I(M, \bar{g})$ in order to stress the metric (or most often the conformal structure) dependence. Analogously I may write sentences like ‘$(\bar{g})$-causal curve $\gamma$’ to stress that $\gamma$ is causal with respect to the metric $\bar{g}$ or ‘$(\bar{g})$-convex set $V$’ to stress that $V$ is a convex set with respect to the metric $\bar{g}$. If instead we are dealing with the same metric but with different open sets $A \subset M$, I may write $R^+_A$ in place of $R^+_I(A, \bar{g})$. However, the reader should be careful because $J^+_I$ denotes
the Seifert future, not the causal relation for the spacetime \((S, g)\). In order to avoid confusion I will not use the letter \(S\) for any open set.

Let me recall ([5], Th. 2.2) that a transitive relation \(R^+\) which contains \(I^+\) is closed iff it has closed \(K^+(x)\) for all \(x\), and that, as a consequence ([5], corollary 2.5), \(K^+\) is the smallest transitive relation containing \(I^+\) such that for every \(x \in M\), \(K^+(x)\) and \(K^-(x)\) are closed. Theorem 2.2 of [5] implies

**Corollary 2.1.** Claims 1.1 and 1.2 are equivalent.

**Proof.** Note that the smallest relation, call it \(M^+\), which is transitive, contains \(I^+\), is partially closed (or equivalently closed because of the first two properties) and antisymmetric may not exist because though all these properties are preserved under arbitrary intersections of relations, the set of relations satisfying the properties may be empty. Provided it is not empty \(M^+\) exists and \(K^+ \subset M^+\) because \(M^+\) is transitive, closed, and contains \(I^+\). But then the inclusion implies that \(K^+\) is antisymmetric too thus the spacetime is \(K\)-causal, and by definition of \(M^+\), \(M^+ \subset K^+\). Thus if \(M^+\) exists \(M^+ = K^+\) and the spacetime is \(K\)-causal. Conversely, if the spacetime is \(K\)-causal then \(M^+\) exists and hence \(M^+ = K^+\). As a consequence, claims 1.1 and 1.2 are equivalent. \(\square\)

Let \(\mathcal{P}\) be a conformal invariant property for a spacetime. Assume moreover that if \(\mathcal{P}\) holds for \((M, g_1)\) then it holds for every \((M, g_2), g_2 < g_1\). Examples for \(\mathcal{P}\) are chronology, causality, distinction, \(K\)-causality and stable causality. For such a property the spacetime \((M, g)\) is said to have the \(\mathcal{P}\)-property or to be \(\mathcal{P}\) if there is \(g' > g\) such that \((M, g')\) has the \(\mathcal{P}\) property. It is clear that \(\mathcal{P}\)-causality is the usual \(\mathcal{P}\)-causality. It is also clear that if \(\mathcal{P}_1 \Rightarrow \mathcal{P}_2\) then \(\mathcal{P}_1 \Rightarrow \mathcal{P}_2\). A nice result is that the operation of making a property \(\mathcal{P}\) is idempotent, that is

**Lemma 2.2.** Let \(\mathcal{P}\) be a conformal invariant property such that if \(\mathcal{P}\) holds for \((M, g_1)\) then it holds for every \((M, g_2), g_2 < g_1\). Then the property stable-\(\mathcal{P}\) coincides with \(\mathcal{P}\).

**Proof.** The implication \(\Rightarrow\) is obvious, for the other direction if \((M, g)\) is stably \(\mathcal{P}\) there is \(g' > g\) such that \((M, g')\) has property \(\mathcal{P}\). Take \(g'', g < g'' < g'\), then \((M, g'')\) is stably \(\mathcal{P}\) and hence \((M, g)\) has property stable-\(\mathcal{P}\). \(\square\)

**Corollary 2.3.** Stable causality is equivalent to stable chronology, stable distinction, stable strong causality, stable \(K\)-causality and stable stable-causality.

**Proof.** Stable chronology implies causality, because a closed causal curve for \((M, g)\) is closed and timelike for \((M, g')\), \(g' > g\). Thus stable-chronology \(\Rightarrow\) stable-stable-chronology \(\Rightarrow\) stable-causality. Now consider the closed chain of implications stable-chronology \(\Rightarrow\) stable-causality \(\Rightarrow\) stable-stable-causality \(\Rightarrow\) stable-\(K\)-causality \(\Rightarrow\) stable-strong causality \(\Rightarrow\) stable-distinction \(\Rightarrow\) stable-causality \(\Rightarrow\) stable chronology, from which the equivalence of all these properties follows. \(\square\)

Stable non-total viciousness is distinct from stable causality (see remark 4.5).

### 3. Seifert’s closed relation and stable causality

In this section the relationship between stable causality and Seifert’s causal relation \(J^+_S\) is clarified. Concerning the topology of stable causality, for some results not considered here nor cited elsewhere in the paper but still of interest the reader may consult [1, 9] and [3] (section 6.4).
Given two metrics \( g, g' \) over \( M \), denote as usual \( g' > g \) if every causal vector for \( g \) is timelike for \( g' \), and \( g' \geq g \) if every causal vector for \( g \) is causal for \( g' \). In the presence of different metrics, the sets \( I^+_g, J^+_g \subset M \times M \), are the chronological and causal sets of \((M, g)\).

**Definition 3.1.** The set \( J^+_g \subset M \times M \), defining Seifert's causal relation on \((M, g)\) is

\[
J^+_g = \{ (x, z) : (x, z) \in J^+_g \text{ for every } g' > g \} = \bigcap_{g' > g} J^+_g.
\]

Sometimes, in order to avoid confusion, I shall write \( J^+_g \) in place of \( J^+_g \) to point out the causal structure to which \( J^+_g \) refers.

**Lemma 3.2.** If \( \tilde{g} < g \) then \( \tilde{J}^+_g \subset \Delta \cup I^+_{\tilde{g}} \).

**Proof.** Let \((x, z) \in \tilde{J}^+_g \setminus \Delta\), let \( \sigma \) be a sequence of \((\tilde{g})\)-causal curves of endpoints \( x_n, z_n \) in \((M, \tilde{g})\). Using a limit curve theorem it follows the existence of a future directed \((\tilde{g})\)-causal curve \( \sigma^+ \) starting from \( x \), and a past directed \((\tilde{g})\)-causal curve \( \sigma^- \) ending at \( z \), and a sequence \( \sigma_j \) distinguishing both curves. Taking \( x' \in \sigma^+, z' \in \sigma^- \) it follows \( (x, x') \in \tilde{J}^+_g \), \((z', z) \in \tilde{J}^+_g \) and \((x', z') \in \tilde{J}^+_g \), or, in terms of the causal relations of \( g \), \((x, x') \in I^+_{\tilde{g}} \), \((z', z) \in I^+_{\tilde{g}} \) and \((x', z') \in \tilde{J}^+_g \), which implies, because \( I^+_{\tilde{g}} \) is open, \((x, z) \in I^+_{\tilde{g}} \). \( \square \)

**Lemma 3.3.** Equivalent definitions of \( J^+_g \) on the spacetime \((M, g)\) are

\[
J^+_g = \Delta \cup \bigcap_{g' \geq g} I^+_{g'} = \bigcap_{g' > g} J^+_g,
\]

moreover, \( J^+_g = \bigcap_{g' > g} J^+_g \).

**Proof.** For the first equality we have only to show that \( \bigcap_{g' > g} J^+_g \subset \Delta \cup \bigcap_{g' > g} I^+_{g'} \), the other inclusion being obvious. For every \( \tilde{g} > g \), taking \( \tilde{g} \) such that \( g < \tilde{g} < \tilde{g} \), it is \( J^+_g \subset \Delta \cup I^+_{\tilde{g}} \), hence \( \bigcap_{g' > g} J^+_g \subset \Delta \cup I^+_{\tilde{g}} \), since \( \tilde{g} > g \) is arbitrary the thesis follows.

For the second equality we have only to show that \( \bigcap_{g' > g} J^+_g \subset \bigcap_{g' > g} J^+_g \), the other inclusion being obvious. Let \( \tilde{g} > g \), taking \( \tilde{g} \) such that \( g < \tilde{g} < \tilde{g} \), by lemma 3.2 it is \( J^+_g \subset J^+_{\tilde{g}} \), thus \( \bigcap_{g' > g} J^+_g \subset J^+_g \), since \( \tilde{g} > g \) is arbitrary the thesis follows.

For the last statement note that \( J^+_g = \bigcap_{g' > g} J^+_g = \bigcap_{g' > g} (\bigcap_{g' > g} J^+_g) = \bigcap_{g' > g} J^+_g \). \( \square \)

**Theorem 3.4.** The relation \( J^+_g \) is closed, transitive and contains \( I^+ \), moreover for every \( x \in M \), \( J^+_g(x) \) and \( I^+(x) \) are closed and contain respectively \( I^+(x) \) and \( I^+(x) \).

**Proof.** The transitivity is obvious because for every \( g' > g \), \( J^+_g \) is transitive. The statements on the inclusion of \( I^+ \), \( I^+(x) \) and \( I^+(x) \) are trivial too. The closure of \( J^+_g \) is immediate from \( J^+_g \subset \bigcap_{g' > g} J^+_g \). The closure of \( J^+_g \), and \( I^+_g(x) \), follows by taking \( (x_n, z_n) \to (x, z), (x_n, z_n) \in J^+_g \), and by using the closure of \( J^+_g \) in the cases \( x_n = x \) or \( z_n = z \) for the initial choice of the converging sequence.

A simple consequence is \( K^+ \subset J^+_g \) and hence

**Corollary 3.5.** If \( J^+_g \) is a partial order then the spacetime is \( K \)-causal.

**Lemma 3.6.** If \( g < g' \) then \( K^+_g \subset \Delta \cup I^+_g \).

**Proof.** It follows from \( K^+_g \subset J^+_g = \Delta \cup \bigcap_{g' > g} I^+_{g'} = \bigcap_{g' > g} (\Delta \cup I^+_{g'}) \). \( \square \)
Lemma 3.7. An equivalent definition of $J^+_g$ on the spacetime $(M, g)$ is

$$ J^+_g = \bigcap_{g' > g} K^+_{g'}. $$

Proof. Since $J^+_g \subset K^+_{g'}$, $J^+_g = \bigcap_{g' > g} J^+_g \subset \bigcap_{g' > g} K^+_{g'}$, but, since $J^+_g$ is closed and transitive, $K^+_{g'} \subset J^+_g$ and $\bigcap_{g' > g} J^+_g = J^+_g$, from which the thesis follows. \[\Box\]

Remark 3.8. From the definition of the sets $R^\pm(x)$ given the binary relation $R^+$ it follows

$$ J^+_g(x) = \{x\} \cup \bigcap_{g' > g} I^+_g(x) = \bigcap_{g' > g} K^+_{g'}(x) = \bigcap_{g' > g} J^+_g(x). $$

Lemma 3.9. If $J^+_g$ on $(M, g)$ is a partial order then for every $x \in M$ there is a (x-dependent) metric $g_x > g$ such that $(M, g_x)$ is chronological at $x$.

Proof. Assume, by contradiction, that the thesis does not hold, then there is $x \in M$ such that for every $g' > g$, there is a closed ($g'$-)timelike curve $\sigma_{g'}$ passing through $x$.

Let $\bar{g} > g$, introduce a Riemannian metric in a neighborhood of $x$ and consider $S = B(x, \epsilon)$, i.e. the surface of the ball of Riemannian radius $\epsilon > 0$. Choose $\epsilon$ sufficiently small so that $S$ is contained in a ($\bar{g}$-)convex neighborhood contained in a ($\bar{g}$-)globaly hyperbolic neighborhood $V$.

For every $g' > g$, $g < g' < \bar{g}$, there is a closed ($g'$-)timelike curve $\sigma_{g'}$ passing through $x$. This curve must escape the hyperbolic neighborhood $V$ otherwise in $(V, \bar{g})$ there would be a closed ($\bar{g}$-)timelike curve. Hence the curve must meet $S$ at some point of $S \cap J^+_g(x)$. Given $g'$ the event $x$ belongs to the chronologically violating set $ul'_{g'}$ which is open [8] and which can be written as the union of disjoint open sets of the form $I^+_{g'}(y) \cap I^-_{g'}(y)$ where $y$ is any point of the component ([8], proposition 4.27). In particular $x$ belongs to the component $I^+_g(x) \cap I^-_g(x)$. The set $A(g') = I^+_g(x) \cap I^-_g(x) \cap S \cap I^-_g(x) \neq \emptyset$ is open in the topology inherited by $S$ and non-empty because $\sigma_{g'}$ must meet $S \cap I^-_g(x)$. In the topology of $S$, $A(g')$ are compact and $\bigcap_{g < g' < \bar{g}} A(g') = S$ where the complement $C$ is taken in the topological space $S$. Since $A(g')$ are open sets there would be a finite covering $S = A(g_1) \cup \cdots \cup A(g_k)$. Now, note that if $g < g'$ then, since a timelike curve for $g$ is timelike for $g'$ it is $A(g) \subset A(g')$ and $A(g_{i+1}) \supset \cdots \supset A(g_1) = S$, hence $A(g) = \emptyset$ a contradiction.

Thus there is $z \in \bigcap_{g < g' < \bar{g}} A(g') \neq \emptyset$. In other words there is an event $z \in S$ such that for every $g' > g$ there are ($g'$-)timelike curves starting from $x$ and passing arbitrarily close to $z$. Thus for every $g' > g$, $(x, z) \in J^+_g$ and $(z, x) \in J^+_g$ thus by lemma 3.3, $(x, z) \in J^+_g$ and $(z, x) \in J^+_g$ but $x \neq z$, i.e. $J^+_g$ is not a partial order. \[\Box\]

Lemma 3.10. If $(M, g)$ is chronological at $x$ then for every $g' < g$, $(M, g')$ is strongly causal at $x$. (Stated in another way, if $(M, g')$ is non-strongly causal at $x$ then for every $g > g'$, there is a ($g'$-)timelike closed curve passing through $x$.)

Proof. If $(M, g')$ is not strongly causal at $x$ then the caracterizing property (ii) of ([6], lemma 3.22) does not hold, that is, there is a neighborhood $U \ni x$ and a sequence of ($g'$-)causal curves $\sigma_n$ of endpoints $x_n, z_n$ with $x_n \to x, z_n \to x$, not entirely contained in $U$. Let $C \ni x$ be a ($g'$-)convex neighborhood whose compact closure is contained in another ($g'$-)convex neighborhood $V \subset U$ contained in a globally hyperbolic neighborhood. Let
$c_n \in \bar{C}$ be the first point at which $\sigma_n$ escapes $C$, and let $d_n$ be the last point at which $\sigma_n$ re-enters $C$. Since $\bar{C}$ is compact there are $c, d \in \bar{C}$, and a subsequence $\sigma_k$ such that $c_k \to c, d_k \to d$ and since $V$ is convex, the causal relation on $V \times V$, $J^+_V(g')$, is closed and hence $(x, c), (d, x) \in J^+_V(g')$, thus $(x, c), (d, x) \in J^+_V(g)$ (note that $(d, c) \in J^+_{g'}(V)$, thus $d$ and $c$ must be distinct since the spacetime $(V, g')$ is causal). Taking into account that $(c_k, d_k) \in J^+_{g'}$ it is $(c, d) \in J^+_g$. Thus there is a $(g)$-timelike curve connecting $d$ to $c$ passing through $x$, and since $(c, d) \in J^+_g$ and $I^+_g$ is open there is a closed timelike curve passing through $x$. \hfill \Box

Lemma 3.11. If for every $x \in M$ there is a (x dependent) $g_x > g$ such that $(M, g_x)$ is chronological at $x$ then $(M, g)$ is stably causal. (Stated in another way, if $(M, g)$ is non-stably causal then there is an event $x \in M$ such that for every $\bar{g} > g$, $(M, \bar{g})$ is non-chronological at $x$.)

Proof. Let $(M, g)$ be non-stably causal and assume by contradiction that for every $y \in M$ there is a (y dependent) $\bar{g}_y > g$ such that $(M, \bar{g}_y)$ is chronological at $y$. By lemma 3.10, taking $g_y$ such that $g < g_y < \bar{g}_y$, $(M, g_y)$ is strongly causal at $y$ and hence it is strongly causal in an open neighborhood $U_y$ of $y$ [8].

Let $\bar{C}$ be a compact. From the open covering $\{U_y, y \in C\}$, a finite covering can be extracted $\{U_{y_1}, U_{y_2}, \ldots, U_{y_k}\}$. A metric $g_C > g$, on $M$ can be found such that for $i = 1, \ldots, k$, $g_C < g_{y_i}$ on $M$. Thus $(M, g_C)$ is strongly causal, and hence chronological, on a open set $A = \cup_j U_{y_j} \supset C$. Let $(g_n, C_n, A_n)$ be a sequence of metrics $g_n > g$, $g_{n+1} < g_n$, and strictly increasing compacts and open sets $C_n \subset A_n \subset C_{n+1}$, such that $(M, g_n)$ is chronological on $A_n$, and $\cup_n C_n = M$ (for instance introduce a complete Riemannian metric and define $C_n$ as the balls of radius $n$ centered at $x_0 \in M, C_n = B(x_0, n)$). Let $\chi_n : M \to [0, 1]$ be smooth functions such that $\chi_n = 1$ on $C_n$, and $\chi_n = 0$ outside an open set $B_n$ such that

$\cdots \subset C_n \subset B_n \subset A_n \subset C_{n+1} \subset B_{n+1} \subset A_{n+1} \subset \cdots$

Construct a metric $g' > g$ on $M$ as follows (see figure 1). The metric $g'$ on $C_{n+1}\setminus B_n$ has value $g_{n+1}$, and on $B_n\setminus C_n$ has value $\chi_n g_n + (1 - \chi_n) g_{n+1}$.

Figure 1. The idea behind the proof of lemma 3.11. If the event $x$ of the statement does not exist there is a sequence of compacts $C_n$ and metrics $g_n > g$ such that $(M, g_n)$ is chronological in $A_n \supset C_n$. Then a metric $g' > g$ exists which is chronological everywhere (in contradiction with the non-stable causality of $(M, g)$), indeed, if not, a closed timelike curve $\gamma$ would have a point $p \in B_i$ with $i$ lowest possible index, then $(M, g_i)$ would not be chronological at $p \in A_i$ (in the figure $i = 2$), a contradiction.
The spacetime \((M, g')\) is chronological otherwise there would be a closed \((g',\cdot)\)-timelike curve \(\gamma\). Let \(i\) be the minimum integer such that \(B_i \cap \gamma \neq \emptyset\), and let \(p \in B_i \cap \gamma\). Then \(\gamma\) is also a closed \((g,\cdot)\)-timelike curve in \((M, g_i)\), thus chronology is violated at \(p \in B_i \subset A_i\), a contradiction.

Thus \((M, g')\) is chronological, and hence \((M, g)\) is stably chronological, or equivalently, stably causal (corollary 2.3).

The next theorem was stated by Seifert ([10], lemma 1). Unfortunately, he did not give many details and I would say that his argument cannot be considered a proof. Hawking and Sachs gave another proof\(^2\) (see the proof of [4], proposition 2.3). The proof given here differs from those and takes advantage of the previous lemmas.

**Theorem 3.12.** The relation \(J^+_M\) on \(M \times M\) is a partial order if and only if \((M, g)\) is stably causal.

**Proof.** It is trivial that if \((M, g)\) is stably causal then \(J^+_M\) is a partial order. Indeed \((x, y) \in J^+_M\) and \((y, x) \in J^+_M\) imply that for a suitable \(g' > g\), such that \((M, g')\) is causal, \((x, y) \in J^+_M\) and \((y, x) \in J^+_M\), hence \(x = y\) because of the causality of \((M, g')\).

For the converse let \(J^+_M\) be a partial order, then for every \(x \in M\) there is (lemma 3.9) a \(x\)-dependent metric \(g_x > g\) such that \((M, g_x)\) is chronological at \(x\), thus \((M, g)\) is stably causal because of lemma 3.11.

From corollary 3.5 and theorem 3.12 it follows

**Corollary 3.13.** If \((M, g)\) is stably causal then it is K-causal.

4. Violating sets on \(M\) and \(M \times M\)

In this section some results on the violating sets for \(J^+_M\) are obtained.

**Lemma 4.1.** The stable causality violating set on \(M \times M\) for the spacetime \((M, g)\) is the intersection of the chronological violating sets on \(M \times M\) for \(g' > g\), namely

\[
VJ^+_M = \bigcap_{g' > g} VJ^+_g,
\]

moreover \(VJ^+_M = \bigcap_{g' > g} VJ^+_g = \bigcap_{g' > g} VJ^+_g = \bigcap_{g' > g} VK^+_g\). Finally, \(VI\) is open, while \(VJ^+_S \cup \Delta, V\bar{J}^+_M \cup \Delta\) and \(VK^+_S \cup \Delta\) are closed.

**Proof.** From lemma 3.3 \(J^+_M = \Delta \cup \bigcap_{g' > g} I^+_g\) thus

\[
J^+_g \cap J^+_S \cap \Delta^C = \left( \bigcap_{g' > g} I^+_g \right) \cap \left( \bigcap_{g' > g} I^+_g \right) \cap \Delta^C = \bigcap_{g' > g} \left( I^+_g \cap I^+_g \cap \Delta^C \right).
\]

The other equations are proved analogously, the last one using lemma 3.7. It has already been mentioned that since \(I^+_S\) is open \(VI\) is open. Since \(VK^+_S \cup \Delta = K^+ \cap K^-\) this set is closed and an analogous argument holds for \(V\bar{J}^+_M \cup \Delta\) and \(VJ^+_S \cup \Delta\).\(^2\)

\(^2\) There seems to be a gap in Hawking and Sachs’s proof. At the very beginning they state that given the spacetime \((M, g)\) and \(x \in M\), if \(J^+_g\) is a partial order then there is some \(\tilde{g} > g\) such that \((M, \tilde{g})\) is causal at \(x\). However, they give no argument for this claim. It seems to me that since \(J^+_g\) is a partial order then for every \(z \in M\), there is a \(\tilde{g}_z > g\) such that \((M, \tilde{g}_z)\) has no closed causal curve which passes through \(x\) and \(z\), but, without a proof of the contrary, \(\tilde{g}_z\) may well depend on \(z\). Also note that if the claim were obvious then lemma 3.11 would suffice to prove the theorem. This gap is answered by lemma 3.9.
The original definition of stable causality implies that if \((M, g)\) is non-stably causal then for every \(g' > g\) there is a \((g\text{-dependent})\) event \(x_{g'} \in M\) and a \((g')\text{-timelike} closed curve through it. Actually, the equivalence between stable causality and the property of antisymmetry for \(J_{\text{S}}^+\), together with lemma 4.1, imply a considerably stronger result

**Corollary 4.2.** If \((M, g)\) is non-stably causal then there is \((x, z) \in M \times M, x \neq z,\) such that for every \(g' > g, (x, z) \in I_{g'}^+\) and \((z, x) \in I_{g'}^-\).

**Lemma 4.3.** The stable causality violating set on \(M\) for the spacetime \((M, g)\) is the intersection of the chronological violating sets on \(M\) for \(g' > g\), namely

\[
v J_{g'} = \bigcap_{g' > g} v I_{g'},
\]

moreover \(v J_{g'} = \bigcap_{g' > g} v I_{g'} = \bigcap_{g' > g} v J_{g'}^+ = \bigcap_{g' > g} v K_{g'}^+\). Finally, \(v I\) is open, while \(v J\) and \(v K\) are closed (the proof of the closure of these sets will follow from lemma 5.9).

**Proof.** From \(v J_{g'} = \pi(V J g' = \pi(\bigcap_{g' > g} V I_{g'}) \subset \bigcap_{g' > g} \pi(V I_{g'}) = \bigcap_{g' > g} v I_{g'}\) an inclusion is obtained. The other direction follows by noting that if \(x \in \bigcap_{g' > g} v I_{g'}\) then for every \(g' > g\) there is a \((g')\text{-timelike} closed curve passing through \(x\). The proof of lemma 3.9 shows that under the same assumptions the existence of an event \(z \neq x\) can be inferred, such that \((x, z) \in J_{g'}^+\) and \((z, x) \in J_{g'}^-\), that is, \((x, z) \in V J g\), and finally \(x \in v J_g\).

Projecting \(V I \subset V J \subset V K \subset V K\) we obtain \(v I \subset v J \subset v J \subset v K\), which implies \(\bigcap_{g' > g} v I_{g'} \subset \bigcap_{g' > g} v J_{g'} \subset \bigcap_{g' > g} v K_{g'}\). Recall from lemma 3.6 that if \(g < g'\) then \(J_{g'}^+ \subset J_{g}^+ \subset K_{g}^+ \subset D \cup I_{g}^+\), thus \(V J_{g} \subset V J_{g} \subset V K_{g} \subset V I_{g}\) and projecting \(v J_{g} \subset v J_{g} \subset v K_{g} \subset v I_{g}\). We can now prove all the other inclusions. I am going to prove that for every \(g' > g\), \(\bigcap_{g' > g} v K_{g'} \subset v I_{g'}\), indeed I can always find \(\tilde{g}, g < \tilde{g} < g',\) and from \(\bigcap_{g' > g} v K_{g'} \subset v K_{g} \subset v I_{g}\) the thesis follows. Thus since for every \(g' > g\), \(\bigcap_{g' > g} v K_{g'} \subset v I_{g'}\), it follows \(\bigcap_{g' > g} v K_{g'} \subset v I_{g'}\). The equations are proved.

It is well known that \(v I\) is open ([8], 4.26), that \(v K\) is closed will be proved in lemma 5.9 and from \(v J_{g'} = \bigcap_{g' > g} v K_{g'}\) follows the closure of \(v J_{g'}\).

**Theorem 4.4.** \(v I\) = \(M\) iff \(I^+ = M \times M\), analogously, \(v J = M\) iff \(J_{g'}^+ = M \times M\).

**Proof.** The first statement is well known, indeed \(v I \neq M\) and \(I^+ \neq M \times M\) are two equivalent definitions of a non-totally vicious spacetime. The implication \(I^+ = M \times M \Rightarrow v I = M\) is obvious, the other direction follows because \(v I\) is made of disjoint open components and for every \(p, q\) in the same component \(p \ll q\). If \(v I = M\) there is only one component which coincides with \(M\).

Let me prove the non-trivial part of the last statement, namely \(v J = M \Rightarrow J_{g'}^+ = M \times M\). Indeed, \(M = v J = \bigcap_{g' > g} v I_{g'} \Rightarrow \forall g' > g, v I_{g'} = M \Rightarrow I_{g'}^+ = M \times M\) from the first statement and hence \(J_{g'}^+ = \bigcap_{g' > g} I_{g'}^+ = M \times M\).

**Remark 4.5.** Contrary to non-total viciousness which stays at the bottom of the causal ladder, the condition \(v J_{g'} \neq M\) has no place in the causal hierarchy. Indeed, a causal spacetime may have \(v J_S = M\) (example 5.2 below), which may suggest that perhaps the property \(v J_{g'} \neq M\) is stronger than causality. However, it is easy to give examples of non-chronological non-totally vicious spacetimes with \(v J_S \neq M\) (identify two spacelike parallel lines in Minkowski spacetime to obtain a cylinder and remove from it two parallel spacelike half lines).

A good name for the condition \(v J_S \neq M\) is stable non-total viciousness, because it holds iff \(\exists g' > g : v I_{g'} \neq M\). Nevertheless, it can also be non-total non-stable causality, because \(v J_S \neq \emptyset\) denotes non-stable causality, and \(v J_S \neq M\) states that this non-stable causality is not total.
5. $K$-causality and stable causality

An important question is whether it is always $K^+ = J^+_s$. The answer is negative as the following examples prove.

**Example 5.1.** Consider the 1+1 cylindrical flat spacetime $M = \mathbb{R} \times S^1$, of metric $\text{ds}^2 = -\text{d}y \text{d}\theta, y \in \mathbb{R}, \theta \in [0, 2\pi)$. This spacetime is non-causal (hence non-distinguishing) and reflecting, moreover given $x \in M, K^+(x) = J^+(x) = \{z \in M : y(z) \geq y(x)\} \neq M$ while $J^+_s(x) = M$ hence $K^+ \neq J^+_s$.

**Example 5.2.** The 2+1 spacetime $M = \mathbb{R} \times S^1 \times S^1$, of metric $\text{ds}^2 = -\text{d}y \text{d}\theta + \text{d}\phi^2, \theta, \phi \in \mathbb{R}$, with the identifications $(y, \theta, \phi) = (y, \theta + 2\pi, \phi)$ and $(y, \theta, \phi) = (y, \theta + 1, \phi + \alpha)$, $\alpha$ irrational number is a causal non-distinguishing reflecting spacetime for which given $x \in M, K^+(x) = \overline{J^+(x)} = \{z \in M : y(z) \geq y(x)\} \neq M$ while $J^+_s(x) = M$ hence $K^+ \neq J^+_s$. The fact that $J^+_s = M$ follows for the same reason as in the previous example, namely the compactness and lightlike nature of the space section which in this case is a torus.

Although $K^+$ is not always coincident with $J^+_s$ it can be that $K$-causality coincides with stable causality. For instance this may happen because when the spacetime is $K$-causal the two causal relations coincide as stated by claim 1.2.

In order to proceed we have to prove some statements regarding the $K^+$ relation. First, recall that every event $x$ of the spacetime $(M, g)$ admits an arbitrarily small convex neighborhood (and arbitrarily small globally hyperbolic neighborhoods). If $U$ is such a neighborhood the causal relation on the spacetime $(U, g), J^+_U \subset U \times U$, is closed and hence coincides with the relation $K^+_U$. This observation shows that if it were not for global aspects the relation $K^+$ would be quite simple.

The next two lemmas were proved by Sorkin and Woolgar\(^3\) ([11], lemmas 14, 15) using a transfinite induction argument. Here I give different proofs which use only soft topological methods. I show that in most cases the transfinite induction argument can be avoided. The suggested general strategy is as follows. First, convert the property to be proved into a causal relation on $M \times M$, then show that it is transitive, closed and contains $J^*$, finally use the minimality of $K^*$. This approach is particularly clear if the statement of the theorem is rearranged in a suitable way which reads ‘Let $(x, z) \in K^*$, if hypothesis then thesis’.

**Lemma 5.3.** Let $B \subset M$ be an open set of compact boundary $\partial B$. Let $(x, z) \in K^*$, if $x \in B$ and $z \notin B$ (or vice versa), then there is $y \in B$ such that $(x, y) \in K^*$ and $(y, z) \in K^*$.

**Proof.** Let $T^+ \subset K^*$ be the set of pairs $(x, z) \in K^*$ at which the statement of the theorem is true. This may happen for instance because the hypothesis ‘$x \in B$ and $z \notin B$ (or vice versa)’ is false or because the hypothesis is true and the thesis ‘there is $y \in B$ such that $(x, y) \in K^*$ and $(y, z) \in K^*$’ is true.

It is $J^+ \subset T^+$ because if $(x, z) \in J^*$ and the hypothesis ‘$x \in B, z \notin B$ (or vice versa)’ is true then the thesis is true, $y$ being the intersection of the causal curve $\sigma$ connecting $x$ to $z$ with $B$ (the map $\sigma : [0, 1] \rightarrow M$ is defined over a compact, the set $\sigma^{-1}(\partial B)$ being closed and limited is a compact and hence there is a last point $y$ at which the curve escapes $B$).

Also $T^*$ is closed, indeed if $(x, z) \in T^*$ then either ‘$x \in B$ and $z \notin B$ (or vice versa)’ is false, in which case $(x, z) \in T^*$ and there is nothing else to prove or ‘$x \in B$ and $z \notin B$ (or vice versa)’ is true. Assume $x \in B$ and $z \notin B$, the other case being analogous. There is a

\(^3\) The version given here has slightly weaker assumptions because the set $B$ is not required to be compact but only to have compact boundary. This difference will be important in the following.
sequence \((x_k, z_k) \in T^+ \subset K^+\), \((x_k, z_k) \rightarrow (x, z)\), and for sufficiently large \(k, x_k \in B\). Now, if \(z \in B\), then \((x, z) \in T^+\) because it satisfies the thesis of the theorem with \(y = z\). Thus we are left with the case \(z \in B^c\) which is an open set, and hence for sufficiently large \(k, z_k \notin B\). Since \((x_k, z_k) \in T^+\), and the hypothesis \(x_k \notin B\) and \(z_k \notin B\) (or vice versa) is satisfied, there are \(y_k \in B\), \((x_k, y_k) \in K^+\) and \((y_k, z_k) \in K^+\). Then there is an accumulation point \(y \in B\) and since \(K^+\) is closed, \((x, y) \in K^+\) and \((y, z) \in K^+\) which implies that \((x, z) \in T^+\) because the thesis of the implication is true.

Finally, \(T^+\) is transitive. Indeed, let \((x, w) \in T^+\) and \((w, z) \in T^+\) then the only way in which \((x, z)\) could not belong to \(T^+\) is if \(x, z \notin B\) (or vice versa) and the thesis is false. However, in this case \(w\) must either belong to \(B\) or to \(B^c\), in the former case since \((w, z) \in T^+\) there must be the sought \(y \in \hat{B}\), \((x, w) \in K^+, (w, y) \in K^+, (y, z) \in K^+\) so that the thesis is verified because \((x, y) \in K^+\). The latter case is analogous.

Thus \(T^+ \subset K^+\) is closed, transitive and contains \(J^+\). By the minimality of \(K^+\) it is \(T^+ = K^+\) and hence the implication of the theorem is true for every \((x, z) \in K^+\).

Let \(K^+_B\) be the \(K^+\) relation for the spacetime \((B, g)\). It can be regarded not only as a subset of \(\hat{B} \times \hat{B}\) but also, through the natural inclusion, as a subset of \(M \times M\).

Lemma 5.4. Let \(B \subset M\) be an open set of compact boundary \(\hat{B}\). Let \((x, z) \in K^+\), if \(x, z \in B\) and \((x, z) \notin K^+_B\) then there is \(y \in \hat{B}\) such that \((x, y) \in K^+\) and \((y, z) \in K^+\).

Proof. Let \(T^+ \subset K^+\) be the set of pairs \((x, z) \in K^+\) at which the statement of the theorem is true. This may happen for instance because the hypothesis \(\exists x, z \in B\) and \((x, z) \notin K^+_B\) is false or because the hypothesis is true and the thesis \(\exists y \in \hat{B}\) such that \((x, y) \in K^+\) and \((y, z) \in K^+\) is true.

It is \(J^+ \subset T^+\) because if \((x, z) \in J^+\) and the hypothesis \(\exists x, z \in B\) and \((x, z) \notin K^+_B\) is true then the thesis is true, \(y\) being a point in the intersection between the causal curve connecting \(x\) to \(z\) and \(\hat{B}\). The causal curve cannot be entirely contained in \(B\) otherwise \((x, z) \in J^+_B \subset K^+_B\).

Also \(T^+\) is closed, indeed if \((x, z) \in \hat{T}^+\) then either \(\exists x, z \in B\) and \((x, z) \notin K^+_B\) is false, in which case \((x, z) \in T^+\) and there is nothing else to prove or \(\exists x, z \in B\) and \((x, z) \notin K^+_B\) is true. Let \(x, z \in B\) and \((x, z) \notin K^+_B\) and let \((x_k, z_k) \in T^+\), be a sequence such that \((x_k, z_k) \rightarrow (x, z)\). Since \(B\) is open, for sufficiently large \(k\), \(x_k, z_k \in B\), moreover we can assume \((x_k, z_k) \notin K^+_B\). Indeed, if there is a subsequence \((x_r, z_r) \in K^+_B\) then \((x, z) \in K^+_B\) and hence \((x, z) \in T^+\) because the hypothesis is false. Thus \(x_k, z_k \in B\) and \((x_k, z_k) \notin K^+_B\) and since \((x_k, z_k) \in T^+\) there are \(y_k \in \hat{B}\), such that \((x_k, y_k) \in K^+\) and \((y_k, z_k) \in K^+\). Then there is an accumulation point \(y \in \hat{B}\) and since \(K^+\) is closed, \((x, y) \in K^+\) and \((y, z) \in K^+\) which implies that \((x, z) \in T^+\) because the thesis of the implication is true.

Finally, \(T^+\) is transitive. Indeed, let \((x, w) \in T^+\) and \((w, z) \in T^+\) then the only way in which \((x, z)\) could not belong to \(T^+\) is if \(x, z \notin B\), \((x, z) \notin K^+_B\), and the thesis is false. However, in this case \(w\) must either belong to \(B\) or to \(B^c\), in the former case if \((x, w) \in K^+_B\) and \((w, z) \in K^+_B\) then \((x, z) \in K^+_B\) and hence \((x, z) \in T^+\). If instead, say \((x, w) \notin K^+_B\) (the other case being analogous), then since \((x, w) \in T^+\) and the thesis is true for \((x, w)\) there is \(y \in \hat{B}\), \((x, y) \in K^+\) and \((y, w) \in K^+\) from which \((y, z) \in K^+\) and \((x, z) \in T^+\) follows. If instead \(w \in B^c\), then lemma 5.3 can be applied to \((x, w)\) to infer the existence of \(y \in \hat{B}\) as required by the thesis.

Thus \(T^+ \subset K^+\) is closed, transitive and contains \(J^+\). By the minimality of \(K^+\) it is \(T^+ = K^+\) and hence the implication of the theorem is true for every \((x, z) \in K^+\). \(\square\)
The next result has been proved by Sorkin and Woolgar ([11], lemma 16) by making use of the previous lemmas and represents the first step in Seifert’s proof program. For completeness I include the proof.

**Lemma 5.5.** If \((M, g)\) is \(K\)-causal at \(x\) then \(x\) admits arbitrarily small \(K\)-convex neighborhoods. In particular, if \((M, g)\) is \(K\)-causal then it is strongly \(K\)-causal.

**Proof.** Given \(x \in M\) and \(N \ni x\) an arbitrary neighborhood, there is always a strongly causal simple neighborhood \(V \subset N, x \in V\) (see, for instance, [6] (section 2.3); recall that a simple neighborhood is a convex neighborhood of compact closure contained in another convex neighborhood ([8], section 1)). Since \(V\) is convex \(K^+_V = J^+_V\). Let \(U_n \ni x, \bar{U}_{n+1} \subset U_n\), be a sequence of neighborhoods causally convex (and hence \(K\)-convex) with respect to \(V\). Let them be a base for the topology on \(x\), namely each open set containing \(x\) contains one \(U_n\). Assume there is a subsequence \(U_k\) of non-\(K\)-convex neighborhoods. There are \(x_k, z_k \in U_k\), and \(y_k \notin U_k\), such that \((x_k, y_k) \in K^+\) and \((y_k, z_k) \in K^+\). The event \(y_k\) belongs or not to \(V\). In the former case it cannot be \((x_k, y_k) \in K^+_V\) and \((y_k, z_k) \in K^+_V\) because \(U_k\) is \(K\)-convex, thus there is a pair among \((x_k, y_k)\) and \((y_k, z_k)\) to which lemma 5.4 can be applied. The result is the existence of \(w_k \in \bar{V}\) such that \((x_k, w_k) \in K^+\) and \((w_k, z_k) \in K^+\). In the latter case the application of lemma 5.3 gives again the existence of \(w_k \in \bar{V}\) such that \((x_k, w_k) \in K^+\) and \((w_k, z_k) \in K^+\). Let \(w \in \bar{B}\) be an accumulation point of the sequence \(w_k\), then since \(x_k, z_k \to x, (x, y) \in K^+\) and \((y, x) \in K^+\) in contradiction with the \(K\)-causality at \(x\). Thus for sufficiently large \(n\) all the open sets \(U_n \subset N\) must be \(K\)-convex neighborhoods.

**Remark 5.6.** Note that \(K\)-causality implies strong \(K\)-causality instead of only the strong \(K'\)-causality for \(g' < g\) as one would expect from analogy with lemma 3.10. The reason lies in the fact that \(K^+\) is closed (in the proof of lemma 3.10 we could not infer \((c, d) \in J^+_{g'},\) although \((c, d) \in J^+_{g}\), we had instead to pass to \(g > g'\) in order to close the causal chain).

The next nice result is due to Sorkin and Woolgar ([11], lemmas 12, 13). The proof I give shows that one can use the abstract notation \(\circ\) for the composition so as to take advantage of the distributive property with respect to unions of sets.

**Lemma 5.7.** If \(U\) is a open subset of \(M\) then \(K^+_U \subset K^+_{|U \times U}|U\). Moreover, if \(U\) is also \(K\)-convex then \(K^+_U = K^+_{|U \times U}|U\).

**Proof.** \(K^+_{|U \times U}|U\) is closed (in the topology of \(U \times U\)), transitive and contains \(J^+_{|U \times U}|U\), in particular, \(J^+_U \subset J^+_{|U \times U}|U\), thus because of the minimality of \(K^+_U, K^+_U \subset K^+_{|U \times U}|U\).

The set \(K^+_U\) can be regarded as a subset of \(M \times M\) through the natural inclusion. Consider the causal relation on \(M\)

\[
\bar{K}^+ = K^+_U \cup K^+_{|U \times U}|U\.
\]

First note that \(J^+_U \subset K^+_U\) and \(J^+_{|U \times U}|U\) \(\subset K^+_{|U \times U}|U\). Moreover, since \(K\)-convexity implies causal convexity \(J^+_U = J^+_{|U \times U}|U\), thus \(J^+_U \subset \bar{K}^+\).

Since \(K^+_{|U \times U}|U\) \(\subset K^+\) \(\cap (U \times U)^c\) and \(U\) is open, this term is a closed set. The first term of \(\bar{K}^+\), i.e., \(K^+_U\), is closed in the topology of \(U \times U\) which is that induced from \(M \times M\); thus it is closed in \(M \times M\), but, possibly, for accumulation points in \(\bar{K}^+_U \cap (U \times U)^c\), however \(\bar{K}^+_U \subset \bar{K}^+ = K^+\); thus these points belong to the second term and hence \(\bar{K}^+\) is closed.

Now, recall that \(K^+_U\) is transitive, \(K^+_U \circ K^+_U \subset K^+_U\). The \(K\)-convexity of \(U\) implies \(K^+_{|U \times U}|U\) \(\circ K^+_{|U \times U}|U\) \(\subset K^+_{|U \times U}|U\). Moreover, \(K^+_U \circ K^+_{|U \times U}|U\) \(\subset K^+_{|U \times U}|U\) because
the first endpoint is not in $U$, and analogously with the factors exchanged. The transitivity property for $\tilde{K}$ is proved using the distributivity property of $\circ$

\[
\tilde{K} \circ \tilde{K}^+ = [K_U^+ \circ K_U^+] \cup [K_U^+ \circ K^+_{(U \times U)^f}] \cup [K^+_{(U \times U)^f} \circ K_U^+]
\]

\[
\cup [K^+_{(U \times U)^e} \circ K^+_{(U \times U)^f}] \subset \tilde{K}^+.
\]

Thus $\tilde{K}^+ \subset K^+$, $J^+ \subset \tilde{K}$ and $\tilde{K}$ is transitive and closed thus, because of the minimality of $K^+$, $\tilde{K}^+ = K^+$, and hence $K^+_{(U \times U)} = K^+_{(U \times U)} = K_U^+$. □

**Lemma 5.8.** Let $(M, g)$ be a spacetime, and let $U$ be an open $K$-convex set of compact closure contained in a convex open set $V$, then $K$-causality holds at every point of $U$.

**Proof.** Assume there is $x \in U$ at which $K$-causality does not hold, then there is $z \in M, z \neq x$, such that $(x, z) \in K^+$ and $(z, x) \in K^-$. If $z \notin U$ then since $z \in K^+(x) \cap K^-(x), U$ would not be $K$-convex. Thus $z \in U$. Because of lemma 5.7, it must be $(x, z) \in K_U^+$ and $(z, x) \in K_U^-$ but this is impossible because $J^+_{(U \times U)}$ is closed (in the topology of $U \times U$ as it is already closed in the topology of $V \times V$, the set $V$ being convex [7]), transitive and contains $J^+$, thus $K_U^+ \subset J^+_{(U \times U)}$ and hence $(x, z) \in J^+_{(U \times U)}$ and $(z, x) \in J^+_\bar{U}$ in contradiction with the causality of every convex neighborhood (it follows from the fact that in a convex neighborhood every pair of causally related events is connected by a unique geodesic of well-defined time orientation, alternatively take $V$ inside a causal neighborhood). □

**Theorem 5.9.** The set $(vK)^C \subset M$ at which $(M, g)$ is $K$-causally is open.

**Proof.** Assume $(M, g)$ is $K$-causal at $x$, and let $V \ni x$ be a convex set. The set $V$ exists and moreover $(V, g)$ is causal. Let $U$ be an open $K$-convex set, $x \in U$, of compact closure contained in $V$. It exists because of lemma 5.5. Thus by lemma 5.8 $K$-causality holds at every point of $U$ and hence $(vK)^C$ is open. □

**Lemma 5.10.** If $g_1 \leq g_2$ then $K^+_{g_1} \subset K^+_{g_2}$. In particular, if $K_{g_1}$-causality holds at $x \in M$ then $K_{g_1}$-causality holds at $x$. 

**Proof.** $K^+_{g_2} \subset M \times M$ is (a) closed, (b) transitive and contains $J^+_{g_2} \supset J^+_{g_1}$ and hence, (c) contains $J^+_{g_1}$, thus must be larger than the smallest set which satisfies (a), (b) and (c), namely $K^+_{g_1}$. □

**Lemma 5.11.** Let $B \subset M$ be an open set of compact closure. Let $N = M \setminus B$, if $x, z \in N, (x, z) \in K^- \cap (x, z) \notin K^+_N$ then there is $y \in B$ such that $(x, y) \in K^+$ and $(y, z) \in K^+)$. 

**Proof.** It is an immediate consequence of lemma 5.4, indeed $N$ is open, is such that $M \setminus N$ is compact and plays the role of the set $B$ in the statement of lemma 5.4 (note that $N = \bar{B}$). □

**Lemma 5.12.** Let $B \subset M$ be an open set of compact closure. Let $\tilde{g} \geq g$, with $\{w \in M : \tilde{g}(w) \neq g(w)\} \subset B$, let $K_{\tilde{g}}$-causality hold at $x, x \notin B$, then if $K_{\tilde{g}}$-causality does not hold at $x$ there is $z \in \bar{B}$ at which $K_{\tilde{g}}$-causality fails too.

**Proof.** Since $K_{\tilde{g}}$-causality does not hold at $x$ there must be $z \in M, z \neq x$, such that $(x, z) \in K^+_{\tilde{g}}$ and $(z, x) \in K^+_{\tilde{g}}$. Assume $z \notin B$ otherwise there is nothing to prove. Since $K_{\tilde{g}}$-causality holds at $x, (x, z) \notin K^+_{\tilde{g}}$ or $(z, x) \notin K^+_{\tilde{g}}$. Consider the former case, the other being analogous. Let $N = M \setminus B$, from lemma 5.7 it is $K^+_{\tilde{g} \cap N} \supset K^+_{(N, \tilde{g})}$, therefore $(x, z) \notin K^+_{(N, \tilde{g})}$. But it is also $(x, z) \in K^+_{\tilde{g}}$ and $x, z \in N$, thus by lemma 5.11 there is $y \in \bar{B}$ such that $(x, y) \in K^+_{\tilde{g}}$.
and \((y, z) \in K^+_g\). Composing these relations with \((z, x) \in K^+_g\), it follows \((x, y) \in K^+_g\) and \((y, x) \in K^+_g\) thus \(K_g\)-causality does not hold at \(y \in \bar{B}\).

A fundamental observation is that given a causal relation \(R^+ \subset M \times M\), the \(R\)-convexity of a set \(A\) must be understood as a condition on the shape of the metric on \(A^R\) rather than on \(A\). For instance, for \(J\)-convexity, the fact that no causal curve can escape and re-enter a causally convex set \(A\) is a constraint due to the shape of the light cones outside \(A\). The light cone structure inside \(A\) has not very much to do with this property. This observation is important because by enlarging the light cones inside a \(K\)-convex set one expects to keep the \(K\)-causality property. Since a special feature of \(K\)-causality is that \(K\)-causality implies strong \(K\)-causality, the same enlargement can be continued in other places so as to obtain, one would say, a global widening of the light cones. This is basically Seifert’s program outlined by him in [10] (lemma 1, point (4)) (note that in that lemma ‘at least once’ is probably a misprint and must be replaced with ‘at most once’). Unfortunately, in order to follow this program, several technical lemmas are needed. Some have already been proved. The next one is particular because the lengthy proof works only if statements (a1), (a2) and (b) are proved all at the same time.

Lemma 5.13. Let \((M, g)\) be a spacetime, \(F\) a closed set, \(B\) an open set of compact boundary \(B\) and \(F \subset B\). Let \((x, z) \in K^+\)

(a1) If \(x \in \bar{B}\) and \(z \notin \bar{B}\) then there is \(y \in B\) such that \((x, y) \in K^+\) and \((y, z) \in K^+_{M \setminus F}\).

(a2) If \(x \notin \bar{B}\) and \(z \in \bar{B}\) then there is \(y \in B\) such that \((x, y) \in K^+_{M \setminus F}\) and \((y, z) \in K^+\).

(b) If \((x, z) \notin \bar{B}\) then either \((x, z) \in K^+_{M \setminus F}\) or there are \(y_1, y_2 \in \bar{B}\) such that \((x, y_1) \in K^+_{M \setminus F}\), \((y_2, z) \in K^+_{M \setminus F}\), and \((y_1, y_2) \in K^+\).

In particular if \(U, \bar{B} \subset U\), is an open set such that \(U \setminus F\) is \(K^+_{M \setminus F}\)-convex then \(U\) is \(K\)-convex.

Proof. Recall that \(K^+_{M \setminus F} \subset K^+\). Let \(T^+ \subset K^+\) be the set of pairs \((x, z) \in K^+\) at which all three statements of the theorem are true (which, selected a statement, may happen because the hypothesis is false or because the thesis is true).

It is \(J^+ \subset T^+\) because the statements (a1), (a2) and (b) are all true in this case. Indeed, consider (a1). If \((x, z) \in J^+\) and the hypothesis ‘\(x \in \bar{B}, z \notin \bar{B}\)’ is also true then the thesis is true, \(y\) being the last point of the causal curve connecting \(x\) to \(z\) in \(\bar{B}\) (the segment of the causal curve connecting \(y\) to \(z\) is entirely contained in \(M \setminus F\)). The statement (a2) is proved similarly. As for (b), if \((x, z) \in J^+\) and \(x, z \notin \bar{B}\) then if the causal curve connecting \(x\) to \(z\) does not intersect \(\bar{B}\) then \((x, z) \in J^+_{M \setminus F}\), otherwise there is a first point \(y_1\) at which the causal curve enters \(\bar{B}\) and a last point \(y_2\) at which it leaves \(\bar{B}\) so that the segments of causal curves connecting \(x\) to \(y_1\) and \(y_2\) to \(z\) are contained on \(M \setminus F\) and hence \((x, y_1) \in J^+_{M \setminus F}\) and \((y_2, z) \in J^+_{M \setminus F}\) while \((y_1, y_2) \in J^+ \subset K^+\) is obvious.

Also \(T^+\) is closed, indeed if \((x, z) \in \overline{T^+}\) then either \(x, z \in \bar{B}\) in which case all the hypotheses of (a1), (a2) and (b) are false, and thus the statements are true, \((x, z) \in T^+\) and there is nothing left to prove, or only one of those mutually excluding hypotheses is true.

Suppose the hypothesis of (a1) is true, that is, \(x \in \bar{B}, z \notin \bar{B}\). In this case statements (a2) and (b) are true because their hypotheses are false and we have only to check that statement (a1) is true. There is a sequence \((x_k, z_k) \in T^+, (x_k, z_k) \rightarrow (x, z)\), and, since \(\bar{B}^c\) is open, for sufficiently large \(k, z_k \notin \bar{B}\). Now, without loss of generality we can assume (pass to a subsequence if necessary) that either (i) \(x_k \in \bar{B}\) or (ii) \(x_k \notin \bar{B}\). In case (i) since \((x_k, z_k) \in T^+\), and the hypothesis of (a1) ‘\(x_k \in \bar{B}\) and \(z_k \notin \bar{B}\)’ is satisfied, there are \(y_k \in \bar{B}\), \((x_k, y_k) \in K^+\)
and \((y_k, z_k) \in K_{M,F}^+\). Then there is an accumulation point \(y \in \mathcal{B}\) and since \(K^+\) and \(K_{M,F}^+\) are both closed, \((x, y) \in K^+\) and \((y, z) \in K_{M,F}^+\) which implies that \((x, z) \in T^+\) because the thesis of (a1) and hence statement (a1) is true. In case (ii) since \((x_k, z_k) \in T^+\), and the hypothesis of (b) \(\exists x_k, z_k \notin \mathcal{B}\) is satisfied either there is a subsequence denoted in the same way such that \((x_k, z_k) \in K_{M,F}^+\) in which case \((x, z) \in K_{M,F}^+\) and the thesis of (a1) is verified with \(y = x\), or there are \(y_{1k}, y_{2k} \in \mathcal{B}\), such that \((x_k, y_{1k}) \in K_{M,F}^+, (y_{2k}, z_k) \in K_{M,F}^+\) and \((y_{1k}, y_{2k}) \in K^+\). Then there are accumulation points \(y_1, y_2 \in \mathcal{B}\) and since \(K_{M,F}^+\) and \(K^+\) are closed, \((x, y_1) \in K_{M,F}^+, (y_2, z) \in K_{M,F}^+\) and \((y_1, y_2) \in K^+\). Thus \((x, y_2) \in K^+\) and \((y_2, z) \in K_{M,F}^+\) which implies that \((x, z) \in T^+\) because the thesis of (a1) and hence statement (a1) is true along with (a2) and (b).

The proof assuming true the hypothesis of (a2) is analogous.

Suppose the hypothesis of (b) is true, that is, \(x, z \notin \mathcal{B}\). In this case statements (a1) and (a2) are true because their hypotheses are false and we have only to check that statement (b) is true. There is a sequence \((x_k, z_k) \in T^+, (x_k, z_k) \to (x, z)\), and, since \(\mathcal{B}\) is open, for sufficiently large \(k\), \(x_k, z_k \notin \mathcal{B}\). Since \((x_k, z_k) \in T^+\) and the hypothesis of (b) is true, for each (sufficiently large) \(k\) either \((x_k, z_k) \in K_{M,F}^+\) or there are \(y_{1k}, y_{2k} \in \mathcal{B}\) such that \((x_k, y_{1k}) \in K_{M,F}^+, (y_{2k}, z_k) \in K_{M,F}^+\) and \((y_{1k}, y_{2k}) \in K^+\). If there is a subsequence such that the first possibility holds then \((x, z) \in K_{M,F}^+\) and the thesis of (b) and hence statement (b) is true. Otherwise for all but a finite number of values of \(k\) the second possibility holds then there are accumulation points \(y_1, y_2 \in \mathcal{B}\) and \((y_1, y_2) \in K^+\) such that, because of the closure of \(K_{M,F}^+\) and \(K^+\), \((x, y_1) \in K_{M,F}^+, (y_2, z) \in K_{M,F}^+\) and \((y_1, y_2) \in K^+\). Thus, again, (b) is true because its thesis is true.

Finally, \(T^+\) is transitive. Indeed, let \((x, w) \in T^+\) and \((w, z) \in T^+\) then the only way in which \((x, z)\) could not belong to \(T^+\) is if \((x, z)\) contradicts one of the statements (a1), (a2) or (b). Assume this happens for (a1) then \(x \in \mathcal{B}\) and \(z \notin \mathcal{B}\) while the thesis of (a1) is false (note that (a2) and (b) are true because their hypotheses are false). However, in this case \(w\) must either belong to \(\mathcal{B}\) or to \(\mathcal{B}^c\), in the former case since \((w, z) \in T^+\) and \(w \in \mathcal{B}, z \notin \mathcal{B}\), there must be \(y \in \mathcal{B}\), \((x, w) \in K^+, (w, y) \in K^+\) and \((y, z) \in K_{M,F}^+\) so that the thesis of (a1) is verified because through composition \((x, y) \in K^+\). In the latter case since \((x, w) \in T^+\) and \(x \in \mathcal{B}\), \(w \notin \mathcal{B}\), there is \(y \in \mathcal{B}\) such that \((x, y) \in K^+\) and \((y, w) \in K_{M,F}^+\). Unfortunately, the composition with \((w, z) \in K^+\) cannot be immediately done, however, since \((w, z) \in T^+\) and \(w, z \notin \mathcal{B}\) either \((w, z) \in K_{M,F}^+\), and we have finished because \((x, y) \in K^+\) and \((y, z) \in K_{M,F}^+\) which is the thesis of (a1), or there is \(y_2 \in \mathcal{B}\) (and an analogous \(y_1\) of no interest here) such that \((w, y_2) \in K^+\) and \((y_2, z) \in K_{M,F}^+\). Thus in this last case \((x, y_2) \in K^+\) and \((y_2, z) \in K_{M,F}^+\) that is, the thesis of (a1) is verified. Thus statement (a1) cannot be contradicted. The proof for (a2) is analogous.

It remains to show that (b) cannot be contradicted by \((x, z)\). Assume \(x, z \notin \mathcal{B}\), \(w\) must either belong to \(\mathcal{B}\) or to \(\mathcal{B}^c\) (note that (a1) and (a2) are true because their hypotheses are false). In the former case since \((x, w) \in T^+\) and \(x \notin \mathcal{B}\), \(w \in \mathcal{B}\), by (a2) there is \(y_1 \in \mathcal{B}\) such that \((x, y_1) \in K_{M,F}^+, (y_1, w) \in K^+\). Also, since \((w, z) \in T^+\) and \(w \in \mathcal{B}, z \notin \mathcal{B}\) there is by (a1), \(y_2 \in \mathcal{B}\) such that \((w, y_2) \in K^+\) and \((y_2, z) \in K_{M,F}^+\) thus the thesis of (b) is true for \((x, z)\) as \((y_1, y_2) \in K^+\). It remains to consider the case \(w \notin \mathcal{B}\). In this case \((x, w) \in T^+, (w, z) \in T^+\) and \(x, w, z \notin \mathcal{B}\). There are four possibilities depending on which of the cases given by the thesis of (b) applies to the pairs \((x, w)\) and \((w, z)\). For instance, by (b) it can be \((x, w) \in K_{M,F}^+\). If this is the case also for \((w, z)\) then \((x, z) \in K_{M,F}^+\) and the thesis of (b) holds for \((x, z)\) as required. Otherwise, the second pair may satisfy the second alternative given by (b) namely there are \(y_1, y_2 \in \mathcal{B}\) such that \((w, y_1) \in K_{M,F}^+, (y_2, z) \in K_{M,F}^+\) and \((y_1, y_2) \in K^+\). Thus
(x, y₁) ∈ K⁺_{M,F} and (y₁, y₂) ∈ K⁺ which again makes the thesis of (b) true for (x, z). There are two cases left.

The case (x, w) \notin K⁺_{M,F}, (w, z) ∈ K⁺_{M,F} is completely analogous to the one just considered and leads again to the truth of (b). The last case (x, w) \notin K⁺_{M,F}, (w, z) \notin K⁺_{M,F} implies the existence of events y₁, y₂, \tilde{y}₁, \tilde{y}₂ ∈ B such that (x, y₁) ∈ K⁺_{M,F}, (y₁, y₂) ∈ K⁺, (y₂, w) ∈ K⁺_{M,F}, (w, \tilde{y}₁) ∈ K⁺_{M,F}, (\tilde{y}₁, \tilde{y}₂) ∈ K⁺ and (\tilde{y}₂, z) ∈ K⁺_{M,F}. Thus setting \tilde{y}₁ = y₁, \tilde{y}₂ = y₂, it is (x, \tilde{y}₁) ∈ K⁺_{M,F}, (\tilde{y}₁, \tilde{y}₂) ∈ K⁺, and (\tilde{y}₂, z) ∈ K⁺_{M,F}, that is (b) is true for (x, z). Thus in every case (x, z) cannot contradict (b). The transitivity of T⁺ is proved.

Thus T⁺ ⊆ K⁺ is closed, transitive and contains J⁺. By the minimality of K⁺ it is T⁺ = K⁺ and hence the implications (a1), (a2) and (b) of the theorem are true for every (x, z) ∈ K⁺.

For the last statement of the theorem, if U is not K⁺-convex there are x, z ∈ U, y \notin U (thus y \notin B) such that (x, y) ∈ K⁺ and (y, z) ∈ K⁺. Thanks to (a1), (a2) and (b) we can assume without loss of generality x, z ∈ B (and hence x, z \notin F), (x, y) ∈ K⁺_{M,F} and (y, z) ∈ K⁺_{M,F} which contradicts the K⁺_{M,F}-convexity of U \setminus F. □

Lemma 5.14. Let F ⊆ M be a closed set of compact boundary F, and U ⊇ F an open set, then there is an open set B of compact boundary such that F ⊆ B ⊆ \overline{B} ⊆ U.

Proof. For every point x ∈ F it is possible to find a neighborhood A(x) ⊆ U of compact closure such that A(x) ⊆ U. Since F is compact there is a finite number of points x₁, i = 1, ..., n, such that {A(x₁), ..., A(xₙ)} covers F. Then the open set B = F ∪ \bigcup A(xᵢ) has compact boundary because B is a closed subset of the compact \bigcup A(xᵢ). Finally, by construction, F ⊆ B ⊆ \overline{B} ⊆ U. □

Theorem 5.15. Let U be an open subset of M and let F ⊆ U be a closed set. If U is K⁺-convex then U \setminus F is K⁺_{M,F}-convex. Moreover, if F has compact boundary then the converse holds, i.e. if U \setminus F is K⁺_{M,F}-convex then U is K⁺-convex.

Proof. Regard K⁺_{M,F} as a subset of M × M through the natural inclusion of (M \setminus F) × (M \setminus F) into M × M. Assume U is K⁺-convex. From lemma 5.7, K⁺_{M,F} ⊆ K⁺_{(M \setminus F) × (M \setminus F)}. Thus U \setminus F is K⁺_{M,F}-convex. For the converse, by lemma 5.14 there is an open set B of compact boundary such that F ⊆ B ⊆ \overline{B} ⊆ U. The conclusion comes from lemma 5.13. □

Theorem 5.16. Let (M, g) be a spacetime, F a closed set of compact boundary, and U ⊇ F a open K⁺-convex set. The spacetime is K⁺-causal iff U is K⁺₀-causal and M \setminus F is K⁺_{M,F}-causal.

Proof. Assume that (M, g) is K⁺-causal then since with the usual natural inclusion in M × M, K⁺₀ ⊆ K⁺ (lemma 5.7), U is K⁺₀-causal and analogously since K⁺_{M,F} ⊆ K⁺, M \setminus F is K⁺_{M,F}-causal.

For the converse, assume M is not K⁺-causal then there are x, z ∈ M, x \neq z, such that (x, z) ∈ K⁺ and (z, x) ∈ K⁺. If both are in U then because of lemma 5.7 (x, z) ∈ K⁺₀ and (z, x) ∈ K⁺₀ which is not possible since U is K⁺₀-causal by assumption. The case x ∈ U, z \notin U implies z ∈ K⁺(x, x) which contradicts the K⁺-convexity of U. The case x \notin U, z ∈ U is analogous.

It remains the case x, z \notin U. By lemma 5.14 there is an open set B of compact boundary such that F ⊆ B ⊆ \overline{B} ⊆ U, thus x, z \notin \overline{B}. Consider the pair (x, z) ∈ K⁺ and apply lemma 5.13. It follows that either (i) (x, z) ∈ K⁺_{M,F} or (ii) there are y₁, y₂ ∈ U \setminus F such that (x, y₁) ∈ K⁺_{M,F}, (y₁, y₂) ∈ K⁺ and (y₂, z) ∈ K⁺_{M,F}. Consider case (ii) and (z, x) ∈ K⁺. Applying again lemma 5.13 (iiia) (z, x) ∈ K⁺_{M,F} or (iiib) there are \tilde{y}₁, \tilde{y}₂ ∈ U \setminus F such that (z, \tilde{y}₁) ∈ K⁺_{M,F}, (\tilde{y}₁, \tilde{y}₂) ∈ K⁺ and (\tilde{y}₂, x) ∈ K⁺_{M,F}. But case (iiia) leads to (y₂, z) ∈ K⁺_{M,F} and...
Let $(M_1, g_1)$ and $(M_2, g_2)$ be two spacetimes and let $F_1 \subset M_1$ and $F_2 \subset M_2$ be two closed sets of compact boundaries in the respective topologies. Define $N_1 = M_1 \setminus \text{Int} F_1$, $N_2 = M_2 \setminus \text{Int} F_2$, and assume that there is an invertible isometry $\phi : N_1 \to N_2$ of the spacetimes with boundaries $(N_1, g_1|_{N_1})$ and $(N_2, g_2|_{N_2})$. Define $F_1^c = F_2 \cup F_2^c$. Then the open set $U_1 = F_1 \cup F_1^c$ is $K_{(M_1, g_1)}$-convex iff $U_2 = \phi(U_1 \setminus F_1) \cup F_2$ is $K_{(M_2, g_2)}$-convex.

Assume that $U_1$ is indeed $K_{(M_1, g_1)}$-convex, and assume, moreover, that $U_1$ is $K_{(U_1, g_1|_{U_1})}$-causal and $U_2$ is $K_{(U_2, g_2|_{U_2})}$-causal. Then $(M_1, g_1)$ is $K_{(M_1, g_1)}$-causal iff $(M_2, g_2)$ is $K_{(M_2, g_2)}$-causal.

Remark 5.18. Theorem 5.17 is a powerful result which allows us to modify the metric and even make surgery operations inside $U_1$. In short it states that if a spacetime $(M_1, g_1)$ has a $K$-causal $K$-convex open set $U_1$, given a closed set of compact boundary $F_1 \subset U_1$ it is possible to arbitrarily modify the metric and even the topology inside $\text{Int} F_1$ without altering the $K$-convexity of $U_1$ in the sense that the obtained spacetime $(M_2, g_2)$ will be such that (denoting with $C_1$ the complement in $M_1$ and with $C_2$ the complement in $M_2$) $U_2 = ((U_1)^C)^C$ is $K$-convex.

Even more it states that whatever the metric deformation or surgery operation done inside $U_1$, the spacetime does not lose its $K$-causality provided the attached set $F_2$ does not make the spacetime $(U_2, g_2|_{U_2})$ non-$K$-causal.

In the following we shall need to consider metric deformations (theorem 5.19) and even topological surgery operations (theorem 6.2).

Theorem 5.19. Let $(M, g)$ be a $K_g$-causal spacetime and let $x \in M$. Then a metric $\bar{g} \succeq g$, such that $\bar{g} > g$ in an open neighborhood of $x$, exists such that the spacetime $(M, \bar{g})$ is $K_{\bar{g}}$-causal.

Proof. Since $(M, g)$ is $K$-causal it is also $K$-strongly causal. Thus it is always possible to find a open $K$-convex set $U \ni x$ and a closed set of compact boundary $F, x \in \text{Int} F$, such that $F \subset U$. Even more $U$ can be chosen inside a globally hyperbolic neighborhood $V$. Since global hyperbolicity implies stable causality and hence stable $K$-causality the metric can be widened inside $F$ without spoiling the $K$-causality of $U$ (otherwise the $K$-causality of $V$ would be spoiled which would be in contradiction with its stable $K$-causality). Finally, because of theorem 5.17 the resulting spacetime is still $K$-causal.

Remark 5.20. The previous result proves that the cones can be widened in a neighborhood of any chosen point without spoiling $K$-causality. Moreover, the only assumption of the theorem is $K$-causality itself thus the procedure can be continued. The cones can be widened in a finite number of points without spoiling $K$-causality. The problem is that taking a point $y$
Theorem 6.2. Provided K-causality coincides with stable causality: if contradiction with the assumption. Thus, though it would be natural to apply a transfinite induction argument to assure the stable K-causality of \((M, g)\), an argument in this direction would have to circumvent these technical difficulties.

6. Some consequences of the possible coincidence

Though the proof of the coincidence between K-causality and stable causality has not been given, it is interesting to explore its possible consequences. In this section I will clearly state if this assumption is necessary for the results considered.

Lemma 6.1. Let \((M, g)\) be a spacetime. If \(U_1\) and \(U_2\) are K-convex sets and \((U_1 \times U_2) \cap (K^+ \cup K^-) = \emptyset\) then \(U = U_1 \cup U_2\) is a K-convex set.

Proof. If \(U\) is not K-convex there are \(x, z \in U\) and \(y \notin U\) such that \((x, y) \in K^+\) and \((y, z) \in K^-\). Now, \(x, z \in U_1\) is excluded because \(U_1\) is K-convex, analogously \(x, z \in U_2\) is excluded because \(U_2\) is K-convex. Note that \((x, z) \in K^-\) (i.e. \((z, x) \in K^+\)). The case \(x \in U_1\), \(z \in U_2\), is excluded because \((U_1 \times U_2) \cap K^- = \emptyset\) and analogously the case \(z \in U_1\), \(x \in U_2\), is excluded because \((U_1 \times U_2) \cap K^+ = \emptyset\). Thus \(U\) is K-convex. □

The next result comes from the development of an idea by H. Seifert ([10], lemma 2).

Theorem 6.2. Provided K-causality coincides with stable causality: if \((M, g)\) is K-causal (stably causal) then \(K^+ = J^+_{Sg}\).

Proof. Assume by contradiction \(K^+_g \neq J^+_{Sg}\) then there is \((x, z) \notin J^+_{Sg}\) and in particular \(x \neq z\). It must also be \((z, x) \notin K^+_g\) otherwise \((z, x) \in K^+_g\) and \(J^+_{Sg}\) would not be a partial order and hence \((M, g)\) would not be stably causal, a contradiction because K-causality and stable causality coincide.

We are going to construct a spacetime which is K-causal and yet non-stably causal in contradiction with the assumption.

Since \(K^+_g \cup K^-_g\) is closed and \((x, z) \notin K^+_g \cup K^-_g\) there are open sets of compact closure \(V_x \ni x\), and \(V_z \ni z\) such that \(V_x \cap V_z \cap (K^+_g \cup K^-_g) = \emptyset\).

Since \((M, g)\) is K-g-causal there is \(\bar{g} > g\) such that \((M, \bar{g})\) is K-\(\bar{g}\)-causal (equivalence between K-causality, stable causality and stable K-causality, see corollary 2.3) and hence strongly K-\(\bar{g}\)-causal. Let \(U_x \ni x\) (respectively \(U_z \ni z\)) be a K-\(\bar{g}\)-convex neighborhood contained in a \(\bar{g}\)-globally hyperbolic neighborhood \(W_x\) (respectively \(W_z\)) contained in \(V_x\) (respectively \(V_z\)). Let \(u_t\) be a timelike vector at \(x\); set

\[D_x = \{ y \in M : y = \exp_x v, v \in TM_x, g(v, u_x) = 0, 0 < g(v, v) < \epsilon \}\]

where \(\epsilon > 0\) is chosen so that the closed disc (ball) \(D_x\) is contained in \(U_x\), and no \(\bar{g}\)-causal curve starting or ending at \(x\) and contained in \(W_x\) can intersect \(D_x\) (this is possible because \(W_x\) being globally hyperbolic it is also distinguishing). Let \(D_z\) be defined analogously. Let \(N = M \setminus (D_x \cup D_z)\), then \((N, \bar{g}_{|N})\) is K-\(\bar{g}\)-causal as \(K^+_{\bar{g}}\) is K-causal (lemma 5.7). Cut \(M\) at \(D_x\), that is let the boundary of \(N\) be there topologically split into the two sides \(D^\times_x\) above and \(D^\wedge_x\) (below) (and analogously for \(D_z\)).

Consider a globally hyperbolic spacetime \((G, g_2)\) with the topology of \(B^{n-1} \times (0, 1)\) where \(B^{n-1}\) is the open ball of \(\mathbb{R}_{n-1}\) \(n\) is the spacetime dimension). The metric on it can be chosen so that its past boundary at 0 can be glued with \(D^\times_x\) and its future boundary at 1 can be glued with \(D^\wedge_x\) all that preserving the continuity properties of the metric. Call \(\tilde{M}\) the final manifold and let \(\tilde{g}\) be the total metric obtained joining \(g_{|N}\) and \(g_2\) (see figure 2). The actual
shape of the metric $g_2$ is not important, but it must be chosen sufficiently wide so that for every $z' \in D_0^I, D_0^I \subset J^+_g(z')$.

I claim that the spacetime $(\hat{M}, \tilde{g})$ is non-stably causal. Indeed, we know that on $M$ for every $g', g < g' < \tilde{g}, J^+_{\tilde{g}} \subset \Delta \cup I^+_{\tilde{g}}$ (lemma 3.3) hence there is a $(g')$-timelike curve connecting $x$ to $z$. This curve cannot intersect $D_x$ before escaping $W_x$, nor can it intersect $D_z$ after entering $W_z$ (saved for the endpoints) because of the distinction of $W_x, W_z$, and because of the very definition of $D_x$ and $D_z$. However, it cannot either intersect $D_x$ after escaping $W_x$ because that would violate the $K_\tilde{g}$-convexity of $U_x, D_x \subset U_x, W_z$, and analogously for $D_z$. Every metric $\tilde{g} > \tilde{g}$ on $\hat{M}$ induces a metric $\tilde{g}|_N > g|_N$ on $N$ which can be modified in a neighborhood of $D_x$ and $D_z$ without changing the causal nature of the curves connecting $x$ to $z$ so as to obtain an extension to a metric $g'$ of $M$. But since any metric $g'$ of this form and such that $g' < \tilde{g}$ allows for a $(g')$-timelike curve from $x$ to $z$, the same is true for metrics $g'$ which do not satisfy the constraint $g' < \tilde{g}$. The conclusion is that for every $\tilde{g} > \tilde{g}$ there is a $(g')$-causal and hence $(\tilde{g}')$-causal curve connecting $x$ to $z$, and since $z$ and $x$ are also connected by a $\tilde{g}$-causal curve passing through $G$, the spacetime $(\hat{M}, \tilde{g})$ is not stably causal.

It remains to prove that $(\hat{M}, \tilde{g})$ is $K$-causal.

By lemma 6.1 $U = U_x \cup U_z$ is $K_{(M,g)}$-convex and moreover, since $(M, g)$ is $K$-causal it is $K_{(U,\tilde{g}|_U)}$-causal (theorem 5.16). The idea is to show that $(\hat{M}, \tilde{g})$ is obtained from $(M, g)$ through a surgery operation allowed by theorem 5.17 which preserves $K$-causality. Indeed, it is obvious that it is always possible to find $F$, a closed set of compact boundary, such that $(D_x \cup D_z) \subset F \subset U$. Then in this surgery operation $\text{Int}(F)$ is excised (note that it is disconnected) and replaced by a new set $\text{Int}(\tilde{F})$ (note that it is connected) which is glued in place of it (all this made rigorous by the obvious presence of the isometry cited in theorem 5.17). Intuitively, identifying some sets from $M$ and $\hat{M}$, $\text{Int}(F)$ is the set $\text{Int}(\tilde{F}) = D_0^I \cup G \cup D_0^I \cup (\text{Int}(F) \setminus (D_x \cup D_z))$ while $\hat{M} = D_0^I \cup G \cup D_0^I \cup (U \setminus (D_x \cup D_z))$.

In the same sloppy notation the boundaries of $F$ and $\tilde{F}$ can be identified. Actually, note that because of the inclusion of $G$, while $F$ was compact, the resulting $\tilde{F}$ is only closed but still of compact boundary which is what is needed for applying theorem 5.17. By theorem 5.17, $\hat{M}$ is $K_{(M, g)}$-convex. Moreover, $\hat{M}$ is $K_{(U, \tilde{g}|_U)}$-causal because $G$ and the metric on it can be chosen
particularly well behaved. Finally, all the conditions of theorem 5.17 are met and thus \((\tilde{M}, \tilde{g})\) is \(K(\tilde{M}, \tilde{g})\)-causal.

A spacetime is reflecting if \((x, z) \in \tilde{J}^+\) implies \(z \in \tilde{J}^+(x)\) and \(x \in \tilde{J}^-(z)\) (this property can be taken as a definition, see [6] (proposition 3.45)). Another equivalent property is \(z \in \tilde{J}^+(x) \Leftrightarrow x \in \tilde{J}^-(z)\).

Recall that a spacetime is causally continuous if it is reflecting and distinguishing.

If \(K\)-causality coincides with stable causality the next proof is particularly simple. Compare for instance with the original one given by Hawking and Sachs [4]. Actually, it also shows that the full assumptions underlying the definition of causal continuity are not all needed for guaranteeing stable causality.

**Theorem 6.3.** Assume that \(K\)-causality coincides with stable causality. A spacetime which is future distinguishing and future reflecting (or past distinguishing and past reflecting) is stably causal. In particular causal continuity implies stable causality.

**Proof.** It is a trivial consequence of [5] (theorem 3.7) because the assumptions imply \(K\)-causality which is equivalent to stable causality.

The next result slightly generalizes the previous results due to Dowker, Garcia and Surya ([2], proposition 2) and Hawking and Sachs ([4], theorem 2.1). The proof that (i) and distinction\(^4\) implies (iv), originally given in [4] (theorem 2.1D), would be particularly simple assuming the equivalence between stable causality and \(K\)-causality in light of theorem 6.2.

**Theorem 6.4.** The following conditions for the spacetime \((M, g)\) are equivalent.

(i) \((M, g)\) is future (respectively past) reflecting.

(ii) For every \(x \in M\), \(\uparrow I^-(x) = I^+(x)\) (respectively \(\downarrow I^+(x) = I^-(x)\)).

(iii) For every \(x \in M\), \(\bar{J}^+(x) = K^+(x)\) (respectively \(\bar{J}^-(x) = K^-(x)\)).

(iv) for every \(x \in M\), \(\bar{J}^+(x) = J^+_S(x)\) (respectively \(\bar{J}^-(x) = J^-_S(x)\)),

then (iv) \(\Rightarrow\) (i); moreover if (i) holds in the past or future case and the spacetime is distinguishing then (iv) holds (I give the proof of the last statement assuming the equivalence between \(K\)-causality and stable causality). Finally, if (i)–(iii) hold in the past or future case, then \(\bar{J}^+ = K^+\), while if (iv) holds in the past or future case then \(\bar{J}^- = K^- = J^-_S\).

**Proof.** (i) \(\Leftrightarrow\) (ii). The spacetime is future reflecting iff for every \(x \in M\), \(A^+(x) = \bar{I}^+(x)\) that is iff \(\uparrow I^- = I^+(x)\), that is iff \(\uparrow I^- = I^+(x)\).

(i) \(\Rightarrow\) (iii). \(\bar{J}^+\) is not only closed but also transitive because if \((x, y) \in \bar{J}^+\) and \((y, z) \in \bar{J}^+\) then ([6], proposition 3.45), \(y \in \bar{J}^+(x)\) and \(z \in \bar{J}^+(y)\) from which it follows because of [5] (theorem 3.3) (or [2], claim 1), \((x, z) \in \bar{J}^+\), hence \(K^+ = \bar{J}^+\). In particular \(\bar{J}^+(x) = \{y \in M : (x, y) \in \bar{J}^+\} = \{y \in M : (x, y) \in K^+\} = K^+(x)\), where in the first equality we used the future reflectivity.

(iii) \(\Rightarrow\) (i). \(x \in \bar{J}^-(z) \Rightarrow (x, z) \in \bar{J}^+ \Rightarrow (x, z) \in K^+ \Leftrightarrow z \in K^+(x) \Leftrightarrow z \in \bar{J}^+(x)\), thus \((M, g)\) is future reflecting.

(iv) \(\Rightarrow\) (i). \(x \in \bar{J}^-(z) \Rightarrow (x, z) \in \bar{J}^+ \Rightarrow (x, z) \in J^+_S \Leftrightarrow z \in J^+_S(x) \Leftrightarrow z \in \bar{J}^+(x)\), thus \((M, g)\) is future reflecting.

Distinction and (i) \(\Rightarrow\) (iv). I give the proof assuming that \(K\)-causality coincides with stable causality. From (i), say future case, it follows (iii) in the future case. Moreover, from

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4 Examples of non-distinguishing spacetimes for which (i) holds but (iv) does not hold are for instance examples 5.1 and 5.2.
Theorem 6.3. The spacetime is stably causal (K-causal) thus, by theorem 6.2, \( K^+ = J^+_x \), hence the thesis.

The last statement follows from the reflectivity since in this case \( J^+ = \{(x, z) : z \in J^+(x)\} \).

Remark 6.5. It can be \( J^+ = K^+ \) and yet the spacetime can be non-reflecting. An example is provided by (1+1) Minkowski spacetime in which a spacelike geodesic segment has been removed. Thus \( J^+ = K^+ \) does not imply (iii) whereas the converse holds.

7. Conclusions

The relationship between stable causality and \( K \)-causality and their possible equivalence has been studied in detail. To this end new results for the \( K^+ \) future have been obtained (lemma 5.13, theorems 5.15, 5.16, 5.17). Unfortunately, a proof of the equivalence has not been given. A partial result in the direction of the equivalence has been the proof that in a \( K \)-causal spacetime, having chosen an event, the light cones can be widened in a neighborhood of the event without spoiling \( K \)-causality (theorem 5.19). The process of enlarging the light cones can be continued and thus, if the equivalence does indeed hold, a final proof could be perhaps be obtained through an inductive process starting from this result. In any case, if the equivalence holds, in a \( K \)-causal spacetime the \( K^+ \) future coincides with the Seifert future \( J^+_S \) (theorem 6.2). If the spacetime is not \( K \)-causal one expects to find some examples which show that in general \( K^+ \neq J^+_S \) and indeed I gave the example of a causal spacetime (example 5.2). Finally, a new proof that Seifert’s causal relation is a partial order iff the spacetime is stably causal has been given which uses some new lemmas which seem interesting in their own right (lemmas 3.9 and 3.11).

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