Robust Recovery of Sparse Nonnegative Weights from Mixtures of Positive-Semidefinite Matrices

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Abstract—We consider a structured estimation problem where an observed matrix is assumed to be generated as an $s$-sparse linear combination of $N$ given $n \times n$ positive-semidefinite matrices. Recovering the unknown $N$-dimensional and $s$-sparse weights from noisy observations is an important problem in various fields of signal processing and also a relevant pre-processing step in covariance estimation. We will present related recovery guarantees and focus on the case of nonnegative weights. The problem is formulated as a convex program and can be solved without further tuning. Such robust, non-Bayesian and parameter-free approaches are important for applications where prior distributions and further model parameters are unknown. Motivated by explicit applications in wireless communication, we will consider the particular rank-one case, where the known matrices are outer products of iid. zero-mean subgaussian $n$-dimensional complex vectors. We show that, for given $n$ and $N$, one can recover nonnegative $s$-sparse weights with a parameter-free convex program once $s \leq O(n^2/\log^2(N/n^2))$. Our error estimate scales linearly in the instantaneous noise power whereby the convex algorithm does not need prior bounds on the noise. Such estimates are important if the magnitude of the additive distortion depends on the unknown itself.

I. INTRODUCTION

In compressed sensing one is confronted with the inverse problem of recovering unknown but structured signals from only few observations, far less than its ambient dimension. This methodology is reasonable if the effective dimension of the signals of interest is much smaller than its ambient dimension. A prominent example is the set of $s$-sparse and $N$-dimensional vectors where $s \ll N$. The original recovery problem has combinatorial nature and is computationally infeasible since one essentially has to implicitly search over the exponentially $\binom{N}{s}$ many support combinations of the unknown signal.

The first fundamental theoretical breakthroughs [6], [5], [8] show that for a linear and real-valued measurement model and under further assumptions on the so called measurement matrix, it is possible to recover the unknown vector in the noiseless case by a linear program. In the noisy case it is also possible to obtain provable guarantees for certain convex programs, see here for example [12], which usually require a tuning parameter that often depends on further properties on the noise contribution, in most cases, the $\ell^2$-norm of the noise. However, there are several signal processing problems where it is difficult to acquire this knowledge. For example, in the Poisson noise model this depends on the unknown signal itself. Another example are certain applications in sparse covariance matching where the error contribution comes from the deviation of the empirical to the true covariance matrix, which in turn depends on the sparse parameter to recover. There are some concepts known in the literature how to deal with convex compressed sensing programs in the absence of this a-priori information. To mention some examples, the quotient bounds [24] of the measurement matrix can provide guarantees for the basis pursuit or the basis pursuit denoising, see for example also [12] Chapter 11. Empirical approaches and modifications of the convex programs are also known to get rough estimates for the noise power, see for example [15]. Interestingly, it has been observed also in [9], [3], [22], [18] that nonnegativity of the unknowns together with particular properties of the measurement matrix yield a “self-tuning” approach, which has been worked out in [16] for the nuclear norm and in [17] for the $\ell^1$-norm with respect to guarantees formulated in the terminology of the robust nullspace property.

II. MAIN RESULTS

Motivated by covariance matching problems, briefly also sketched below, we shall consider the problem of recovering nonnegative and sparse vectors from the noisy matrix observation

$$Y = A(x) + E,$$

where $A : \mathbb{R}^N \mapsto \mathbb{C}^{n \times n}$ is a given linear measurement map. We establish recovery guarantees for the generic convex program

$$x^* = \arg \min_{z \geq 0} \|A(z) - Y\|,$$

where $\| \cdot \|$ is a given norm on $\mathbb{C}^{n \times n}$. We shall write $\| \cdot \|_p$ for the $\ell^p$-norms for vectors and matrices (when
Let our analysis. inner product space. [12, Definition 4.21] is essential for rewritten for the non-square case or even for a generic Schmidt (Frobenius) inner product on \( \langle \cdot, \cdot \rangle \) where by definitions. For the case of a generic norm guarantee. establish the corresponding compressed sensing recovery this work we focus on the complex subgaussian case and where the vectors are drawn from the complex sphere. In [14], [10] using the nullspace property in the special case have obtained in [11]. These investigations have property of such matrices after centering and in the real details below or see for example also the discussion [21] such random matrices are biased which is essential for vectors with itself. Such matrices are sometimes also called as (self-) Khatri-Rao products. By construction such random matrices are biased which is essential for the recovery of nonnegative vectors via [2] (further results about the RIP property of such matrices after centering and in the real case have obtained in [11]. These investigations have been worked out towards a NNLS recovery guarantee in [14], [10] using the nullspace property in the special case where the vectors are drawn from the complex sphere. In this work we focus on the complex subgaussian case and establish the corresponding compressed sensing recovery guarantee.

To state our main results we need the following definitions. For the case of a generic norm \( \| \cdot \| \) on \( \mathbb{C}^{n \times n} \) we let \( \| \cdot \|_0 \) be the corresponding dual norm defined as

\[
\| Y \|_0 := \sup_{\|X\| \leq 1} \langle X, Y \rangle,
\]

where by \( \langle X, Y \rangle := \text{tr} X^* Y \) we denote the Hilbert-Schmidt (Frobenius) inner product on \( \mathbb{C}^{n \times n} \). To simplify notation we will stick to square matrices in the space \( \mathbb{C}^{n \times n} \) but the first part of this work can be easily rewritten for the non-square case or even for a generic inner product space. [12] Definition 4.21] is essential for our analysis.

**Definition 1.** Let \( q \geq 1 \) and \( s \in \mathbb{N} \). We say that a linear map \( A : \mathbb{R}^N \to \mathbb{C}^{n \times n} \) is a linear map which (i) satisfies the \( \ell^\rho \)-NSP of order \( s \) with respect to \( \| \cdot \| \) with parameters \( \rho \in (0, 1) \) and \( \tau > 0 \) if for all \( S \subseteq [N] := \{1, \ldots, N\} \) with cardinality \( |S| \leq s \)

\[
\| v_S \|_q \leq \frac{\rho}{s^{1-1/q}} \| v_{S^c} \|_1 + \tau \| A(v) \|
\]

holds for all \( v \in \mathbb{R}^N \). Here, \( v_S \in \mathbb{R}^N \) denotes the vector containing the same entries as \( v \) at the positions in \( S \) and zeros at the others and \( v_{S^c} = [N] \setminus S \).

Furthermore, by \( \sigma_s(x)_1 = \min_{|S| \leq s} \| x - x_S \|_1 \) we denote here the best \( s \)-term approximation to \( x \in \mathbb{R}^N \) in the \( \ell^1 \)-norm. The nullspace property is essential for recovery via \( \ell^1 \)-based convex recovery programs like basis pursuit and basis pursuit denoising, see for example [12] Theorem 4.22. When recovering nonnegative vectors, the following additional property, often called \( M^\rho \)-criterion, controls the \( \ell^1 \)-norms of all feasible vectors such that \( \ell^1 \)-regularization becomes superfluous.

**Definition 2.** A linear map \( A : \mathbb{R}^N \to \mathbb{C}^{n \times n} \) satisfies the \( \mathcal{M}^\rho \)-criterion if there exists a matrix \( T \in \mathbb{C}^{n \times n} \) such that \( w := A^*(T) > 0 \) componentwise. For a given \( T \), we then define the condition number \( \kappa(w) = \max_{i \in [N]} |w_i|/\min_{i \in [N]} |w_i| \).

Note that \( \kappa(w) \) is scale-invariant, i.e., \( \kappa(w) = \kappa(tw) \) for all \( t \neq 0 \). For further illustration of this property, consider the noiseless setting and assume for simplicity that we can find \( T \) such that \( w = A^*(T) = (1, \ldots, 1) =: 1_N \) is the all-one vector. Then

\[
\| x \|_1 \geq \langle 1_N, x \rangle = \langle A^*(T), x \rangle = \langle T, A(x) \rangle = \langle T, Y \rangle = \text{const}
\]

shows that all feasible vectors \( x \) have the same \( \ell^1 \)-norm. As we shall show below, a similar conclusion follows for the general case \( w > 0 \) and the tightness of such an argument will depend on \( \kappa(w) \).

The following theorem essentially extends and refines [17] Theorem 3] to the case of matrix observations and generic norms.

**Theorem 1.** Let \( q \geq 1 \) and let \( A : \mathbb{R}^N \to \mathbb{C}^{n \times n} \) be a linear map which (i) satisfies the \( \ell^\rho \)-NSP of order \( s \) with respect to \( \| \cdot \| \) and with parameters \( \rho \in (0, 1) \) and \( \tau > 0 \) and (ii) fulfills the \( \mathcal{M}^\rho \)-criterion for \( T \in \mathbb{C}^{n \times n} \) with \( \kappa = \kappa(A^*(T)) \). If \( \rho \kappa < 1 \), then, for any nonnegative \( x \in \mathbb{R}^N \) and all \( E \in \mathbb{C}^{n \times n} \), the solution \( x^2 \) of (2) for \( Y = A(x) + E \) obeys

\[
\| x^2 - x \|_p \leq \frac{C\kappa^q s(x)_1}{s^{1-1/p}} + \frac{D\kappa^q}{s^{1-1/p}} \left( \tau + \frac{\theta}{s^{1-1/q}} \right) \| E \| \quad (4)
\]

for all \( p \in [1, q] \), where \( C' := 2(1+\rho)^2 (1-\kappa)^2 \) and \( D' := \frac{2+\rho}{1-\kappa} \) and \( \theta = \| A^*(T) \|_{\infty,1} \cdot \| T \|_0 \).

We prove this theorem in Section III. As a second main result, we show that it is applicable to the following random observation model:

**Model 1.** Let \( a_i = (a_{i,k})_{k \in [n]} \in \mathbb{C}^n \) for \( i = 1, \ldots, N \) be independent random vectors with independent sub-
gaussian real and imaginary parts $\text{Re}(a_{i,k})$ and $\text{Im}(a_{i,k})$ satisfying

$$
\begin{align*}
\mathbb{E}[a_{i,k}] &= \mathbb{E}[\text{Re}(a_{i,k})] = \mathbb{E}[\text{Im}(a_{i,k})] = 0 \\
\mathbb{E}[\text{Re}(a_{i,k})^2] &= \mathbb{E}[\text{Im}(a_{i,k})^2] = 1/2,
\end{align*}
$$

so that $\mathbb{E}[|a_{i,k}|^2] = 1$ and $\mathbb{E}[|a_{i}|^2] = n.$ We consider the following map $\mathcal{A} : \mathbb{R}^N \to \mathbb{C}^{n \times n}$:

$$
\mathcal{A}(x) := \sum_{i=1}^{N} x_i a_i^*.
$$

Let $\psi_2 \geq 1$ be a uniform bound on the subgaussian norms $\|\text{Re}(a_{i,k})\|_{\psi_2}$ and $\|\text{Im}(a_{i,k})\|_{\psi_2}$ for all $i \in [N], k \in [n]$, see (1) below for the definition.

The case where the vectors $a_i$ are drawn uniformly from the complex sphere has been discussed already in [14] and the full proof of the recovery guarantee can be found in [10]. In this work we discuss the subgaussian iid case instead where additionally also the distribution of $|a_{i}|^2$ affects the probability bounds. We have the following second main result.

**Theorem 2.** Let $\mathcal{A} : \mathbb{R}^N \to \mathbb{C}^{n \times n}$ be a random measurement map following Model [7] Set $m := 2n(n-1)$ and assume

$$
s \lesssim m \log^{-2}(N/m),
$$

$n \gtrsim \log(N)$ and $N \gtrsim m$. With probability at least $1 - 4\exp(-c_1 \cdot n)$ it holds that for all $p \in [1,2]$, all $x \in \mathbb{R}_0^N$ and $E \in \mathbb{C}^{n \times n}$, the solution $x^\dagger$ of the NNLS (the convex program (2) for the Frobenius norm $\| \cdot \|_F$) for

$$
Y = \mathcal{A}(x) + E \text{ obeys}
$$

$$
\|x^\dagger - x\|_p \leq \frac{c_2 \sigma_2(x)_1}{s^{1-\frac{1}{p}}} + \frac{c_3 (c_4 + \sqrt{\frac{p}{n}}) \|E\|_F}{s^{\frac{1}{p}-\frac{1}{p}}}.
$$

where $C_1, c_1, c_2, c_3, c_4$ are absolute constants depending on $\psi_2$ but not on the dimensions.

The proof of this theorem will be presented in Section [IV]. We have not optimized the constants but some concrete numbers are for example $c_2 = 11.36$, $c_3 = 15.55$ and $c_4 = 3.07$, more details are in the proof below. The constants $C_1$ and $c_1$ depend on the subgaussian norm $\psi_2$ in Model [1] and can also be obtained from the proof.

A. Motivating Application

We will briefly mention an application of the results above in the area of wireless communication [14], [7], [10]. An important task in wireless networks is to estimate the nonnegative large-scale path-gain coefficients (product of transmit power and attenuation due to path-loss) and user activity using multiple antennas. Here, a small subset of $s \ll N$ devices indicate activity by transmitting specific length-$n$ sequences which superimpose at each receive antenna with individual and unknown instantaneous channel coefficients. Let us denote this nonnegative vector of large-scale path-gains by $\gamma \in \mathbb{R}^N$ and due to activity $\gamma$ is essentially $s$-sparse. For a single receive antenna, the received (noiseless) signal would be:

$$
y = A \text{diag}(\sqrt{\gamma}) h
$$

Here $h \in \mathbb{C}^N$ is the vector of unknown small-scale fading coefficients and $A = (a_1 | \ldots | a_N) \in \mathbb{C}^{n \times N}$ is the matrix containing all the sequences $a_i$ registered in the network (in real applications for example pseudo-random sequences seeded by the device id). Well-known results in compressed sensing show that when using sufficiently random sequences of length $n \simeq s \cdot \text{polylog}(N)$ for given $s$ and $N$, one can recover per antenna w.h.p. the complex-valued channel coefficients $\text{diag}(\sqrt{\gamma}) h$ and the activity pattern (the essential support).

However, since in future even the number of active devices $s$ will grow considerably, the question is how to further gain from a massive number of receive antennas when one is only interested in reconstructing $\gamma$ or its support. A very promising approach is to recover then the sparse and non-negative vector from covariance information, an approach which has been investigated already in [19].

In more detail, assuming that the small-scale fading coefficient vectors $h$ for different receive antennas and different users are uncorrelated, we can view the received signal $y$ at each receive antenna as a new realization of the same random process having a covariance matrix which is parametrized by $\gamma$, i.e., this leads precisely to the following covariance model:

$$
\mathcal{A}(\gamma) = \mathbb{E}yy^* = A \text{diag}(\gamma) A^*
$$

Here $\gamma$ is an unknown nonnegative and sparse parameter which should match (in a reasonable norm) the empirical covariance

$$
Y = \frac{1}{M} \sum_{k=1}^{M} y_k y_k^* = \mathcal{A}(\gamma) + E
$$

computed from the received vectors $\{y_k\}_{k=1}^{M} \subset \mathbb{C}^n$ at $M$ receive antennas. The error $E$ accounts therefore for the fact of having only finite $M$ (and obviously further unknown disturbances like adversarial noise and interference always present in communication systems). Note that the error $E$ above usually depends then on the unknown parameter $\gamma$ as well.

Our result, Theorem 2 now shows that pathloss coefficients and activity of up to $s \lesssim O(n^2/\text{log}^2(N/n^2))$
devices can be robustly recovered from the empirical covariance $Y$ over sufficiently many receive antennas when matching the model in the Frobenius norm. Note that, although not further considered in this work, errors due to having finite $M$ will vanish with increasing $M$ for moderate assumptions on the distribution of $h$ and one could obviously also make concrete statements about the concentration of $\|E\|_F$ in (7) in terms of $M$, see [10].

III. GENERIC NONNEGATIVE RECOVERY GUARANTEE VIA THE NULLSPACE PROPERTY

In this section, we are following [17] aiming towards showing Theorem [1] which is a more general and refined version of the deterministic guarantee given in [17]. The proof of Theorem [1] is given at the end of this section. First, we will need [12] Theorem 4.25:

**Theorem 3** (Theorem 4.25 in [12]). Let $q \in [1, \infty)$ and $s \in \mathbb{N}$. Assume $A$ satisfies the $\ell^q$-NSP of order $s$ with respect to $\|\cdot\|$ and with constants $\rho \in (0, 1)$ and $\tau > 0$. Then, for any $p \in [1, q]$ and for all $x, z \in \mathbb{R}^N$, 

$$
\|x - z\|_p \leq C(\rho) \frac{s - 1}{s - \rho} (\|x\|_1 - \|z\|_1 + 2\sigma_s(x)_1) + D(\tau) s^{1/p - 1/q} \|A(x - z)\|_1
$$

holds, where $C(\rho) := \frac{(1 + \rho)^2}{1 - \rho} \leq \frac{(3 + \rho)^2}{1 - \rho} =: D(\rho)$.

First we show a modified version of [17] Lemma 5. Recall that for a diagonal matrix $W = \text{diag}(w_1, \ldots, w_N) \in \mathbb{R}^{N \times N}$ considered as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$ equipped with $\|\cdot\|_p$ for any $p \in [1, \infty]$, the operator norm is given as $\|W\|_\infty = \max \{|w_1|, \ldots, |w_N|\}$. Furthermore, $W$ is invertible if and only if all the diagonal entries are nonzero, with inverse $W^{-1} = \text{diag}(\frac{1}{|w_1|}, \ldots, \frac{1}{|w_N|})$. Thus, in this case we can also write the condition number in Definition 2 as $\kappa(w) = \|W\|_\infty \|W^{-1}\|_\infty$.

**Lemma 1.** Let $W = \text{diag}(w)$ for some $0 < w \in \mathbb{R}^N$. If $A$ satisfies the assumptions in Theorem 3 and $\kappa := \kappa(A(T)) = \max \{\|w_1\|, \ldots, |w_N|\}$. Then $A \circ W^{-1}$ satisfies the $\ell^q$-NSP of order $s$ with respect to $\|\cdot\|$ and with constants $\tilde{\rho} := \kappa\rho$, $\tilde{\tau} := \|W\|_\infty \tau$.

**Proof.** Let $S \subseteq [N]$ with $|S| \leq s$ and $v \in \mathbb{R}^N$. Since $W$ is diagonal, we have $(W^{-1}v)_S = W^{-1}v_S$ (same for $S^c$). We get:

$$
\|v_S\|_q \leq \|W\|_\infty \|((W^{-1}v)_S\|_q \\
\leq \|W\|_\infty \left(\frac{\rho}{s_1 - \rho} \|((W^{-1}v)_S\|_1 + \tau \|A(W^{-1}v)\|_1\right) \\
\leq \|W\|_\infty \|W^{-1}\|_\infty \|v_S\|_1 + \|W\|_\infty \tau \|A(W^{-1}v)\|_1 \\
= \frac{\tilde{\rho}}{s_1 - \rho} \|v_S\|_1 + \tilde{\tau} \|((A \circ W^{-1})v)\|_1
$$

The next lemma is a generalization of [17] Lemma 6.

**Lemma 2.** Assume $w := A^*(T) \in \mathbb{R}^N$ is strictly positive for some $T \in \mathbb{C}^{n \times n}$ and set $W := \text{diag}(w)$. For any nonnegative $x, z \in \mathbb{R}^N$, it holds that

$$
\|Wz\|_1 - \|Wx\|_1 \leq \|T\|_\infty \|A(z - x)\|.
$$

**Proof.** Let $x, z \in \mathbb{R}^N$ be nonnegative. By construction, we have $W = W^*$ and $Wz$ is nonnegative. This implies

$$
\|Wz\|_1 = \langle 1_N, Wz \rangle = \langle w, z \rangle = \langle A^*(T), z \rangle = \langle T, Az \rangle
$$

where $1_N$ denotes the vector in $\mathbb{R}^N$ containing only ones. With an analogous reformulation for $x$ we get

$$
\|Wz\|_1 - \|Wx\|_1 = \langle T, A(z - x) \rangle \leq \|T\|_\infty \|A(z - x)\|.
$$

We can now show a more general version of [17] Theorem 3 which holds for general $p \in [1, \infty)$ and generic norms on matrices. It parallels Theorem 3 in the nonnegative case.

**Theorem 4.** Suppose that $A$ satisfies the assumptions in Theorem 3 and that there exists some $T \in \mathbb{C}^{n \times n}$ such that $A^*(T)$ is strictly positive. Set $W := \text{diag}(A^*(T))$ and $\kappa := \kappa(A^*(T)) = \|W\|_\infty \|W^{-1}\|_\infty$. If $\kappa \rho < 1$, then

$$
\|z - x\|_p \leq \frac{2C(\kappa\rho)}{s_1 - 1/p} \sigma_s(x)_1 \\
+ \frac{D(\kappa\rho)}{s^{1/q - 1/p}} \left(\kappa \tau + \frac{\|W^{-1}\|_\infty \|T\|_\infty}{s^{1/q - 1/p}}\right) \|A(z - x)\|_p
$$

holds for all $p \in [1, q]$ and all nonnegative $x, z \in \mathbb{R}^N$.

Note that we used here the definition of $C(\rho)$ and $D(\rho)$ from Theorem 3. Using this result for $p = q = 2$ and with $s_1^{-1/q} \geq 1$ yields essentially [17] Theorem 3.

**Proof.** Let $p \in [1, q]$ and $x, z \in \mathbb{R}^N$ be nonnegative. By Lemma 1, $A \circ W^{-1}$ satisfies the NSP with parameters $\tilde{\rho} := \kappa\rho$ and $\tilde{\tau} := \|W\|_\infty \tau$. Therefore, we can
now use Theorem 3 for \( Wx \) and \( Wz \) (instead of \( x \) and \( z \)) and \( A \circ W^{-1} \) (instead of \( A \)):

\[
\|W(z - x)\|_p \leq C(\kappa\rho) \|Wz\|_1 - \|Wx\|_1 + 2\sigma_s(Wx)_1 \\
+ D(\kappa\rho) \|W\|_o \tau s^{1/p-1/q} \|A(z - x)\|
\]

By Lemma 2 and invoking \( \sigma_s(Wx)_1 \leq \|W\|_o \sigma_s(x)_1 \), this is at most

\[
C(\kappa\rho) \|T\|_o \|A(z - x)\| + 2 \|W\|_o \sigma_s(x)_1 \\
+ D(\kappa\rho) \|W\|_o \tau s^{1/p-1/q} \|A(z - x)\|
\]

which we can further upper bounded by using \( C(\kappa\rho) \leq D(\kappa\rho) \) by:

\[
2C(\kappa\rho) \|W\|_o \sigma_s(x)_1 \\
+ D(\kappa\rho) \|W\|_o \tau s^{1/p-1/q} \|A(z - x)\|
\]

This yields

\[
\|z - x\|_p \leq \|W^{1/2}\|_o \|W(z - x)\|_p \\
\leq 2C(\kappa\rho)\kappa\sigma_s(x)_1 \\
+ D(\kappa\rho) \|W\|_o \tau s^{1/p-1/q} \|A(z - x)\|
\]

This indeed suggests to use the convex program (2) for recovery. Obviously, the minimizer \( x^d \) of (2) fulfills

\[
\|A(x^d) - Y\| \leq \|A(x) - Y\| = \|E\|
\]

and therefore we have:

\[
\|x^d - x\|_p \leq 2C(\kappa\rho)\kappa \sigma_s(x)_1 \\
+ D(\kappa\rho) \|W\|_o \tau s^{1/p-1/q} \|A(x) - Y\| + \|E\|
\]

Proof of Main Theorem 1

Applying Theorem 4 above for \( Y = A(x) + E \) we obtain

\[
\|x^d - x\|_p \leq 2C(\kappa\rho)\kappa \sigma_s(x)_1 \\
+ D(\kappa\rho) \|W\|_o \tau s^{1/p-1/q} \|A(x) - Y\| + \|E\|
\]

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\[
\|A(x^d) - Y\| \leq \|A(x) - Y\| = \|E\|
\]

and therefore we have:

\[
\|x^d - x\|_p \leq 2C(\kappa\rho)\kappa \sigma_s(x)_1 \\
+ D(\kappa\rho) \|W\|_o \tau s^{1/p-1/q} \|A(x) - Y\| + \|E\|
\]

Note that \( T \) can be rescaled by any positive \( \beta \), the \( \mathcal{M}^+ \)-criterion is still fulfilled and the terms \( \kappa \) and \( \|W^{-1}\|_o \|T\|_o \) in Theorem 4 above do not change. However, replacing \( T \) by \( \|A^*(T)\|_\infty \cdot T \) which yields \( \|W\|_o = 1 \) and therefore \( \kappa = \|W^{-1}\|_o \), allows us to write (9) in the more convenient form (4).

IV. THE RANK-ONE AND SUB-GAUSSIAN CASE

In this part we will proof our second main result, Theorem 2 We will consider a random linear map \( A : \mathbb{R}^N \rightarrow \mathbb{C}^{n \times n} \) given by Model 1 i.e., with the special form

\[
A(x) = \sum_{i=1}^{N} x_i a_i a_i^* =: \sum_{i=1}^{N} x_i A_i
\]

where \( A_i := a_i a_i^* \in \mathbb{C}^{n \times n} \) are independent random positive-semidefinite rank one matrices. The adjoint map \( A^* : \mathbb{C}^{n \times n} \rightarrow \mathbb{R}^N \) is given as

\[
A^*(T) = \sum_{i=1}^{N} \langle A_i, T \rangle = (a_i^* T a_i)_{i=1}^{N}.
\]

Note that, even in the more general case where \( A_i \) are (non-zero) positive semidefinite matrices, it can been easily seen that \( A \) fulfills the \( \mathcal{M}^+ \) criterion since for \( T = \text{Id}_n \) we get that \( A^*(T) = (\text{tr} A_i)_{i=1}^{N} > 0 \).

Now, according to Model 1 \( \{a_i\}_{i=1}^{N} \) are independent complex random vectors with independent subgauussian real and imaginary part components. To make this precise we will need the following characterization. For a real-valued random variable \( X \) and \( r \in [1, \infty) \) define

\[
\|X\|_{\psi_r} := \inf \{ t > 0 : \mathbb{E}[\exp(|X|^{r/t})] \leq 2 \}.
\]

This is a norm on the Orlicz space of random variables \( X \) with \( \|X\|_{\psi_r} < \infty \). For \( r = 2 \) these random variables are called sub-Gaussian and for \( r = 1 \) sub-exponential.

More information about these spaces can be found in [12] and [23] for example.

A. The \( \mathcal{M}^+ \)-Criterion

We already discussed above that the measurement map \( A \) in (10) fulfills the \( \mathcal{M}^+ \)-criterion (by choosing \( T = \text{Id}_n \) or a scaled version). However, its “quality” depends (for a chosen \( T \)) on the condition number \( \kappa \) which is a random variable. We follow the ideas of [17] again.

Lemma 3. Assume that \( A \) is given by Model 1. For a given \( \eta \in (0, 1) \), it holds with high probability at least

\[
1 - 2N \exp \left( - \frac{c \eta^2}{2 \psi_2^2} \cdot n \right),
\]

that for all \( i \in [N] \)

\[
n(1 - \eta) \leq \|a_i\|_2^2 \leq n(1 + \eta),
\]

where \( c > 0 \) is the constant appearing in the Hanson-Wright inequality (32). In particular,

\[
\frac{\max_{i \in [N]} \|a_i\|_2^2}{\min_{i \in [N]} \|a_i\|_2^2} \leq \frac{1 + \eta}{1 - \eta}.
\]
A variant of Lemma 3 is possible for random vectors beyond the iid. model if a convex concentration property hold, see [2]. Let us already indicate how we will use this result later on. In the context of proving Theorem 2 applied to Model 1 with \( T = t \text{Id}_n \) we have \( \kappa = \max_{i \in [N]} \| a_i \|_2 \) and \( \| A^r(T) \|_\infty = t \max_{i \in [N]} \| a_i \|_2 \). Thus, Lemma 3 allows us to control the terms related to the \( M^+ \)-criterion. We will do this more explicitly below when proving Theorem 2.

**Proof.** Note that \( E[\| a_i \|_2^2] = n \). We will show that with high probability it holds for all \( i \in [N] \) that  
\[
\| a_i \|_2^2 - n \leq \eta n.
\]
This directly implies (12) and (13). Using the Hanson-Wright inequality (32) (which is a Bernstein inequality in this case) yields that for all \( i \in [N] \) it holds that  
\[
P[ \| a_i \|_2^2 - n \leq \eta n ] \leq 2 \exp \left(-cn \min \left( \frac{\eta^2}{2\psi^2}, \frac{\eta}{\psi} \right) \right)
\]
\[
= 2 \exp \left(-\frac{cn^2}{2\psi^2} \cdot n \right),
\]
using \( \psi \geq 1 \) and \( \eta < 1 \). As a remark, such a concentration may also hold for certain non-iid. models, see here the convex concentration property [2]. By taking the union bound it follows that (12) and (13) hold with probability  
\[
1 - 2N \exp \left(-\frac{cn^2}{2\psi^2} \cdot n \right),
\]
depending on \( \psi \), the dimensions \( n \) and \( N \) and some \( \eta \in (0,1) \). \( \square \)

**B. The Nullspace Property**

We will now establish that the \( \ell^2 \)-NSP holds with overwhelming probability once the sparsity \( s \) is below a certain threshold, in detail \( s \leq m/\log^2(N/m) \) where \( m = 2n(n-1) \). This resembles that the well-known compressed sensing phase transition holds (up the order of the logarithm) also for such structured random matrices.

It is well-known that the \( \ell^2 \)-NSP is implied by the restricted isometry property (with respect to the \( \ell^2 \)-norm).

**Definition 3.** For \( s \in [N] \), the restricted isometry constant \( \delta_s = \delta_s(\Phi) \) of order \( s \) of a matrix \( \Phi \in \mathbb{C}^{n \times N} \) is defined as the smallest \( \delta \geq 0 \) satisfying  
\[
(1 - \delta) \| x \|_2^2 \leq \| \Phi x \|_2^2 \leq (1 + \delta) \| x \|_2^2
\]  
for all \( s \)-sparse vectors \( x \in \mathbb{R}^N \), i.e. vectors with at most \( s \) non-zero components. If \( \delta_s(\Phi) < 1 \), the matrix \( \Phi \) is said to have the restricted isometry property (\( \ell^2 \)-RIP) of order \( s \).

If \( \delta_{2s}(\Phi) < 1/\sqrt{2} \), then \( s \)-sparse vectors \( x \in \mathbb{R}^N \) can be recovered in a stable way from given measurements \( \Phi x \) using \( \ell^1 \)-based convex algorithm (basis pursuit etc.) [4]. The following theorem [12, Theorem 6.13] shows the important relation to the nullspace property.

**Theorem 5.** If the \( 2s \)th restricted isometry constant \( \delta_{2s} = \delta_{2s}(\Phi) \) of a matrix \( \Phi \in \mathbb{C}^{m \times N} \) obeys \( \delta_{2s} \leq \delta < \frac{3}{\sqrt{4}} \), then \( \Phi \) satisfies the \( \ell^2 \)-NSP of order \( s \) with constants  
\[
\rho \leq \frac{\delta}{\sqrt{1 - \delta^2} - \delta/4} \quad \text{and} \quad \tau \leq \frac{\sqrt{1 + \delta}}{\sqrt{1 - \delta^2} - \delta/4}.
\]  
For the proof see [12, Theorem 6.13]. For example, as seen in Figure 1, \( \delta = 0.5 \) gives \( \rho \approx 0.7 \), \( \tau \approx 1.5 \) and the constants in Theorem 3 are \( C(\rho) \approx 8.6 \) and \( D(\rho) \approx 11.3 \). Our first step will be to show that in the considered regime a modified version \( \Phi \) of \( \mathcal{A} \) has with high probability \( \ell^2 \)-RIP with a sufficiently small RIP-constant. This then implies that \( \Phi \) and also \( \mathcal{A} \) satisfy the \( \ell^2 \)-NSP.

To this end, we define an operator \( P : \mathbb{C}^{n \times n} \rightarrow \mathbb{R}^m \), where \( m := 2n(n-1) \), that maps a complex matrix to a real valued vector containing the real and imaginary parts of all off-diagonal entries scaled by \( \sqrt{2} \). Hence, for any \( M \in \mathbb{C}^{n \times n} \) we have \( \| P(M) \|_2 \leq \sqrt{2} \| M \|_F \). Furthermore, we define the real vectors  
\[
X_i := P(a_i a_i^*) = \sqrt{2} \text{Re}(a_i k \bar{a}_j) k \neq i, \text{Im}(a_i k \bar{a}_j) k \neq i
\]  
\[
\text{Fig. 1. Dependency of the NSP parameter (bounds) } \rho = \rho(\delta), \quad \tau = \tau(\delta) \text{ in Theorem 5 and the constants } C(\rho(\delta)) \text{ and } D(\rho(\delta)) \text{ in Theorem 4. For example, } \delta = 0.5 \text{ gives } \rho \approx 0.7, \tau = 1.5, \quad C(\rho) \approx 8.6 \text{ and } D(\rho) \approx 11.3.
\]
These are independent and have subexponential zero-mean entries. The factor $\sqrt{2}$ normalizes the resulting vector so that

$$E\|X_i\|^2 = 2E \sum_{k \neq l} |a_{i,k} \hat{a}_{i,l}|^2 = m.$$ 

To show $\ell^2$-RIP, we will use the following result on matrices with independent heavy-tailed columns from \cite{CD}: 

**Theorem 6** (Theorem 3.3 in \cite{CD} for the $\psi_1$-case). Let $X_1, \ldots, X_N \in \mathbb{R}^m$ be independent subexponential random vectors with $E\|X_i\|^2 = m$ and let $\psi = \max_{i \in [N]} \|X_i\|_{\psi}$. Assume $s \leq \min\{N, m\}$ and let $\theta \in (0, 1)$, $K, K' \geq 1$ and set $\xi = \psi K + K'$. Then for $\Phi := (X_1 \ldots X_N)$ it holds that

$$\delta_s \left( \frac{\Phi}{\sqrt{m}} \right) \leq C \xi^2 \sqrt{\frac{s}{m}} \log \left( \frac{eN}{s \sqrt{m}} \right) + \theta$$

with probability larger than

$$1 - \exp(-cK^2 \psi^2 \log \left( \frac{eN}{s \sqrt{m}} \right)) - \mathbb{P} \left[ \max_{i \in [N]} \|X_i\|_2 \geq K' \sqrt{m} \right] - \mathbb{P} \left[ \max_{i \in [N]} \|X_i\|_2^2 - 1 \geq \theta \right],$$

where $C, \hat{c} > 0$ are universal constants.

The last term in \cite{CD} shows the intuitive behavior that the concentration of the column norms $\|X_i\|^2/m$ have direct impact on the RIP (take for example $s = 1$). In our case we will apply Theorem 6 above to the vectors defined in \cite{CD}. The norm $\|X_i\|^2$ is in general a 4th order polynomial in the $m$ real subgaussian random variables $\text{Re}(a_{i,k}), \text{Im}(a_{i,k})$. In Appendix \cite{CD} we show how to calculate tail bounds for a polynomial of this form, the summary for our specific case is the following corollary:

**Corollary 7.** Consider the model \cite{CD} and $X_i := P(a_i \alpha_i^*)$ as defined in \cite{CD} so that $E(\|X_i\|^2) = m$. Assume $n \geq \psi^2$. For $\omega \in [0, 1]$ it holds that

$$\mathbb{P}[ \|X_i\|^2 - m \geq m \omega ] \leq 2 \exp(-\gamma \omega^2 \psi^2 \cdot n),$$

where $\gamma \in (0, 1)$ is some absolute constant.

**Proof.** This follows from Proposition 13 in Appendix \cite{CD}.

Now we are ready to show $\ell^2$-RIP for the matrix $\Phi := \frac{1}{\sqrt{m}} P \circ A$. A similar result for the real case where (informally) “$P$ is replaced by centering” has been established in \cite{CD}. However, to establish the NSP it is more direct to remove the diagonal part with the definition of $P$ in \cite{CD}.

**Theorem 8.** Assume that $A : \mathbb{R}^N \to \mathbb{C}^{n \times n}$ is given by Model \cite{CD} and let $\delta \in (0, 1)$. Assume $N \geq m = 2(n-1)$. If

$$2s \leq \alpha m \log^2 \left( \frac{eN}{\alpha m} \right)$$

and $n \geq \frac{2 \log(4N)}{C_1}$, then, with probability

$$\geq 1 - 2 \exp(-\min\{\hat{c} \psi \sqrt{1/2} \sqrt{1/2} \cdot n\}).$$

the matrix $\Phi := \frac{1}{\sqrt{m}} P \circ A \in \mathbb{R}^{m \times N}$ has $\ell^2$-RIP of order $2s$ with RIP-constant $\delta_s(\Phi) \leq \delta$. The constants $C, \hat{c}$ are the same as in Theorem 6 and $C_1 \alpha$ are given as $C_1 = 8 \sqrt{\frac{d}{2s}}$, with $\gamma$ as in Corollary 7 and $\alpha := \min\{1, \left( \frac{\delta}{6(\psi + 1/\sqrt{16/2})^2} \right)^{1/2} \}$, where $\psi := \max_{i \in [N]} \|P(a_i \alpha_i^*)\|_{\psi}^{-1}.$

**Proof.** We will apply Theorem 6 and use ideas already presented in \cite{CD} Theorem 5 and Corollary 1.

Define the $N$ real-valued random vectors $X_i = P(a_i \alpha_i^*), i \in [N]$. The number $\psi := \max_{i \in [N]} \|X_i\|_{\psi}$ defined above is finite, independent of the dimension and depends quadratically on $\psi^2$, see Appendix \cite{CD}. Let $\alpha \in (0, 1)$, the value will be specified later, and set $s^* := \alpha m / \log^2 \left( \frac{eN}{\alpha m} \right)$. Since $\log(\frac{eN}{\alpha m}) \geq 1$, we ensure $s^* \leq m \leq N$. By Theorem 6 the RIP-constant $\delta_{s^*} := \delta_s(\Phi)$ of the matrix $\Phi := \frac{1}{\sqrt{m}} P \circ A$ satisfies

$$\delta_{s^*} \leq C \xi^2 \sqrt{s^*} \log \left( \frac{eN}{s^* \sqrt{(s^*/m)}} \right) + \theta$$

with probability larger than

$$1 - \exp \left( -cK^2 \sqrt{s^*} \log \left( \frac{eN}{s^* \sqrt{(s^*/m)}} \right) \right) - \mathbb{P} \left[ \max_{i \in [N]} \|X_i\|_2 \geq K' \sqrt{m} \right] - \mathbb{P} \left[ \max_{i \in [N]} \|X_i\|_2^2 - 1 \geq \theta \right].$$

By definition of $s^*$, we can estimate \cite{CD} as

$$\delta_{s^*} \leq C \xi^2 \sqrt{\frac{3}{2} + \log \left( \frac{eN}{\alpha m} \right)} \cdot \sqrt{3} + \theta \leq C \xi^2 \sqrt{\frac{3}{2} + \frac{3}{e} + \theta} \leq 3C \xi^2 \sqrt{\alpha} + \theta,$$
where we used $\frac{\log \log x}{\log x} \leq \frac{1}{2}$ for $x > 1$ in the last line. For (23), (24), taking union bounds and rewriting gives
\[
\mathbb{P}[\max_{i \in [N]} \|X_i\|_2^2 - m \geq \theta m] \leq N \cdot \mathbb{P}[\max_{i \in [N]} \|X_i\|_2^2 - m \geq \theta m] \geq N \cdot \mathbb{P}[\|X_i\|_2^2 - m \geq (K^2 - 1)m].
\]
(26)
and
\[
\mathbb{P}[\max_{i \in [N]} \|X_i\|_2 \geq K' \sqrt{m}] \leq N \cdot \mathbb{P}[\|X_i\|_2 \geq K^2 m] \leq N \cdot \mathbb{P}[\|X_i\|_2^2 - m \geq (K^2 - 1)m].
\]
Choosing $K' := \sqrt{1 + 3\delta}$, both terms above are equal. We set $\theta = \frac{\delta}{2}$. Note that $n \geq \frac{2\log(4N)}{C_1}$ yields $n \geq \psi^2$, since $C_1 = \frac{\psi^2}{4\delta^2}$ and $\gamma, \delta \leq 1$. Hence, using Corollary 7 with $\omega = \frac{\delta}{2}$, the probabilities above can be bounded by $2N \exp(-C_1 \cdot n)$. Since $n \geq \frac{2\log(4N)}{C_1}$, we can estimate
\[
4Ne^{-C_1\cdot n} = e^{\log(4N) - C_1\cdot n} \leq e^{-\frac{1}{2}C_1\cdot n}.
\]
(27)
Now set $K = 1$ and choose $\alpha$ sufficiently small so that we get $\delta_* \leq \delta$ from (25), i.e., $\alpha \leq \left(\frac{\delta}{6(C(\psi + 1 + 4/\delta^2))}\right)^2$.

The term (22) can be estimated in the following way using $s^* = \alpha m / \log^{3/2}(\frac{eN}{\alpha m}) \leq \alpha^{2/3}m$:
\[
\exp\left(-\hat{\alpha}\sqrt{s^*}\log\left(\frac{eN}{s^*\sqrt{s^*/m}}\right)\right) \leq \exp\left(-\hat{\alpha}\sqrt{s^*}\log\left(\frac{eN}{\alpha m}\right)\right) = \exp\left(-\hat{\alpha}\sqrt{\alpha \cdot n}\right) \leq \exp\left(-\hat{\alpha}\sqrt{\alpha \cdot n}\right).
\]
(28)
Using (27), (28) we get
\[
\mathbb{P}[\delta_* \leq \delta] \geq 1 - \exp\left(-\hat{\alpha}\sqrt{\alpha \cdot n}\right) - \exp\left(-\frac{1}{2}C_1 \cdot n\right) \geq 1 - 2 \exp\left(-\min\{\hat{\alpha}\sqrt{\alpha}, \frac{1}{2}C_1\} \cdot n\right).
\]
By monotonicity of the RIP-constant we get the same lower bound for $\mathbb{P}[\delta_{2s} \leq \delta]$, whenever $2s \leq s^*$.

From this it easily follows that $\Phi$ and also $A$ itself satisfy the $\ell^2$-NSP.

**Theorem 9. Assume that $A : \mathbb{R}^N \rightarrow \mathbb{C}^{n \times n}$ is given by Model 7** Let $N \geq m = 2n(n - 1)$, $\delta \in (0, \frac{4}{\sqrt{4m}})$ and assume
\[
s \leq m \log^{2}(N/m)
\]
and $n \geq \log(N)$ as in (19). Then, with probability
\[
\geq 1 - 2 \exp(-c_\delta \cdot n),
\]
(29)
$A$ has the $\ell^2$-NSP of order $s$ w.r.t. the Frobenius norm $\|\cdot\|_F$ with parameters $\rho$ and $\tau \sqrt{2}/\sqrt{m}$. The number $c_\delta$ is defined so that (29) coincides with (20) and $\rho$, $\delta$ satisfy (15) with the chosen $\delta$.

**Proof.** We set $\Phi = \frac{1}{\sqrt{m}}P \circ A \in \mathbb{R}^{m \times N}$. By Theorem 8 with probability (29) $\Phi$ has $\ell^2$-RIP of order $2s$ with RIP-constant $\delta_{2s}(\Phi) \leq \delta$. Theorem 5 implies that $\Phi$ in this case satisfies the $\ell^2$-NSP with parameters $(\rho, \tau)$ depending on $\delta$ as given in (15). Hence, for all $v \in \mathbb{R}^N$ and $S \subset [N]$ with $\|S\| \leq s$ it holds that
\[
\|v_S\|_2 \leq \frac{\rho}{\sqrt{s}} \|v_S\|_1 + \tau \|\Phi v\|_2
\]
\[
\leq \frac{\rho}{\sqrt{s}} \|v_S\|_1 + \frac{\tau \sqrt{2}}{\sqrt{m}} \|P(A(v))\|_2
\]
\[
\leq \frac{\rho}{\sqrt{s}} \|v_S\|_1 + \frac{\tau \sqrt{2}}{\sqrt{m}} \|A(v)\|_F,
\]
showing that the linear map $A$ has the $\ell^2$-NSP of order $s$ with respect to $\|\cdot\|_F$ and with parameters $(\rho, \tau \sqrt{2}/\sqrt{m})$.

**C. Proof of the Main Recovery Guarantee for Model 7**

Now we are ready to proceed with the proof of the second main result, Theorem 2.

**Proof of Main Theorem 2** We start from our first main result, Theorem 1 for the case of the Frobenius norm $\|\cdot\|_F$. The convex program (2) is then Nonnegative Least-Squares (NNLS) and Theorem 1 states that if the linear map $A$ has the $\ell^2$-NSP with respect to $\|\cdot\|_F$ and fulfills the $M^+$-criterion for some matrix $T$ with a sufficiently well-conditioned $\kappa = \kappa(A^*(T))$, then NNLS obeys a recovery guarantee of the form (9). It will be more convenient to choose here a different scaling for $T$ as we did in the end of the proof of Theorem 1.

Theorem 9 states that with high probability $A$ has the $\ell^2$-NSP with parameters $(\rho, \sqrt{2}\tau/\sqrt{m})$, where $\rho$, $\tau$ depend on the number $\delta$ from Theorem 5 and 8. We know that the $M^+$-criterion for $A$ is fulfilled for $T = t \cdot \text{Id}_n$ with $t > 0$. Lemma 3 furthermore states that with overwhelming probability the resulting vector $w = tA^*(\text{Id}_n)$ is well-conditioned and concentrates around its mean. Set $\kappa := \kappa(w)$ and $W := \text{diag}(w)$. Conditioned on events when $A$ indeed has the $\ell^2$-NSP and $\kappa < 1$, we have from (9) that for any $1 \leq p \leq q = 2$ it holds that
\[
\|x^* - x\|_p \leq \frac{2(\kappa p)}{s^{1 - \frac{1}{p}}} \sigma_1(x)_{1}
\]
\[
+ \frac{2D(\kappa p)}{s^{1 - \frac{1}{p}}} \left(\kappa \sqrt{2\tau} \sqrt{m} \|W^{-1}\|_F \|T\|_F \|E\|_F \right).
\]
(30)
The equation (13) in this setting translates to $\kappa(w) \leq \frac{1 + \eta}{1 - \eta} =: \kappa_{\eta}$, where $\eta \in (0,1)$ will be specified later. Recall that the condition number is invariant to scaling of $w$, hence $\kappa = \kappa(w)$. The dual norm in (4) is $\|T\|_\infty = \|T\|_F = t\|Id_n\|_F = t\sqrt{n}$ and $\|W^{-1}\|_\infty = (t \min_{e \in \{N\}} \|a_i\|_2^{-1})^{-1} \leq \left(\frac{t(n - 1)}{n}\right)^{-1}$.

Choosing $t := (n(1 + \eta))^{-1}$ we achieve $\|W^{-1}\|_\infty \leq \kappa_{\eta}$ and $\|T\|_\infty = \left(\sqrt{n}(1 + \eta)\right)^{-1}$. With these bounds and setting $C_{\eta,\rho} = 2C(\kappa_{\eta}\rho)\kappa_{\eta}$, $D_{\eta,\rho} = 2D(\kappa_{\eta}\rho)\kappa_{\eta}$, we can further estimate (30) as

$$\begin{align*}
&\leq C_{\eta,\rho}\sigma_s(x) + \frac{D_{\eta,\rho}}{s^{-\frac{1}{p}}} + D_{\eta,\rho} \left(\frac{n\sqrt{2}t}{\sqrt{m}} + \sqrt{n} s^{-\frac{1}{p}} (1 + \eta)^{-1}\right)\|E\|_F \\
&\leq C_{\eta,\rho}\sigma_s(x) + \frac{D_{\eta,\rho}}{s^{-\frac{1}{p}}} + 2\left(\frac{D_{\eta,\rho}}{s^{-\frac{1}{p}}} + \sqrt{n} s^{-\frac{1}{p}} (1 + \eta)^{-1}\right)\|E\|_F,
\end{align*}$$

(31)

In particular the last step may be improved further by explicitly accounting for the bound in (19). Instead we have assumed only $n > 1$ so that $\frac{n}{\sqrt{m}} = \frac{n}{2n(n - 1)} \leq \sqrt{2}$.

A possible concrete choice of the not yet specified numbers is $\eta = 1/3$ and $\delta = 1/6$, see here also Figure 2. In this case we have $\kappa_{\eta} = 2$ and $\rho \leq 0.18$, hence $\kappa_{\rho} < 1$ is fulfilled, $\tau \leq 1.15$ and $C_{\eta,\rho} \leq 11.36, D_{\eta,\rho} \leq 20.73$. Plugging into (31) yields the desired inequality (7)

$$\|x^\sigma - x\|_p \leq \frac{c_2\sigma_s(x) + c_3\left(c_4 \sqrt{\frac{2}{\rho}}\right)\|E\|_F}{n}$$

with constants

\begin{align*}
&c_2 = C_{\eta,\rho} \leq 11.36, \\
&c_3 = D_{\eta,\rho} (1 + \eta)^{-1} \leq 15.55, \\
&c_4 = 2\tau(1 + \eta) \leq 3.07.
\end{align*}

The probability for (12), (13) to hold can be estimated as

$$\begin{align*}
1 - 2N \exp\left(-\frac{c}{18\psi_2^\prime} \cdot n\right) \\
\geq 1 - 2 \exp\left(-\frac{c}{36\psi_2^\prime}\right) \cdot \text{inequality (7)}
\end{align*}$$

if $n \geq \frac{36\psi_2^\prime\log(N)}{c_1}$. Taking a union bound with (29) gives a probability of at least $1 - 4\exp\left(-c_1 \cdot n\right)$ with $c_1 := \min\left\{\frac{c}{36\psi_2^\prime}, \frac{1}{2}C_1\right\}$ to hold if also $n \geq \frac{2\log(4N)}{C_2}$, where $c$ is the constant from the Hanson-Wright inequality and $\hat{c}, \alpha, C_1$ are the same as in Theorem 8 and depend on $\psi_2$ but not on the dimensions.

V. Numerical Experiments

In the following we validate our theoretical result (6) in Theorem 2 about the phase transition for successful recovery via NNLS for Model 1 with numerical experiments. We performed recovery experiments for dimensions $n = 20, \ldots, 30$ and sparsity range $s = 20, \ldots, 150$. For every pair $(n, s)$ we have performed 20 experiments with randomly generated vectors $\{a_i\}_{i=1}^N$ with independent standard normal entries and a nonnegative sparse vector $x \in \mathbb{R}^N$. The support of $x$ is generated uniformly over all possible $(\binom{N}{s})$ combinations. The nonnegative values on the support are generated independently as absolute values from a standard normal distribution. Given the noiseless measurement $Y = AX(x)$, we then used the MATLAB function lsqlin to solve the NNLS problem (the convex program (2) for the Frobenius norm) yielding the estimate $x^\sigma$. We assume that the vector is successfully recovered if $\|x - x^\sigma\|_2 \leq 10^{-4}$. The corresponding result is shown in Figure 3.

![Fig. 2. Dependency of the constants $C_{\eta,\rho}$ and $D_{\eta,\rho}$ in (31) depending on $\delta$ for fixed $\eta = 1/3$ (yielding $\kappa_{\eta} = 2$ and therefore $C_{\eta,\rho} = 4C(2\rho(\delta))$ and $D_{\eta,\rho} = 4D(2\rho(\delta))$, where $C$ and $D$ are defined as in Theorem 8 and shown in Figure 1].

![Fig. 3. Phase transition for NNLS (the convex program (2) for the Frobenius norm) in the noiseless case (success=light/yellow and failure=blue/dark). The function $x \rightarrow x^2/4 - x - 25$ is overlayed in black.]

\[\]
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APPENDIX A
HANSON WRIGHT INEQUALITY

The Hanson-Wright inequality is an important tool to calculate tail bounds for sub-Gaussian random vectors. We first state it for the real case, taken from [20, Theorem 1.1].

**Theorem 10** (Hanson-Wright inequality). Let \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) be a random vector with independent and centered sub-Gaussian components and \( Z \in \mathbb{R}^{n \times n} \). For all \( t \geq 0 \) it holds that

\[
\mathbb{P}[ |\langle a, Za \rangle - \mathbb{E}[\langle a, Za \rangle] | > t ] \leq 2 \exp\left(-c \min\left\{ \frac{t^2}{K^4 \|Z\|_F^2}, \frac{t}{\sqrt{2} K^2 \|Z\|_o} \right\} \right),
\]

(32)

where \( K \) is a bound on the \( \psi_2 \)-norms of the components of \( a \) and \( c > 0 \) a universal constant.

The complexifications have been discussed in [20] Sec. 3.1. One important application for us is bounding the deviation of the Euclidian norm squared of a complex vector \( a \in \mathbb{C}^n \) from its mean by writing

\[
\|a\|^2 = \|\tilde{a}\|^2 = \langle \tilde{a}, I_{2n} \tilde{a} \rangle,
\]

where \( \tilde{a} := \begin{bmatrix} \text{Re}(a) \\ \text{Im}(a) \end{bmatrix} \in \mathbb{R}^{2n} \) and \( I_{2n} \) is the \( 2n \times 2n \) identity matrix with \( \|I_{2n}\|^2 = 2n \) and \( \|I_{2n}\|_o = 1 \). But we can furthermore even state a complete complex version.

**Theorem 11** (Hanson-Wright inequality, complex version). Let \( a = (a_1, \ldots, a_n) \in \mathbb{C}^n \) be a random vector so that \( \text{Re}(a_1), \text{Im}(a_1) \) are independent and centered sub-Gaussian random variables and let \( Z \in \mathbb{C}^{n \times n} \). For all \( t \geq 0 \) it holds that

\[
\mathbb{P}[ |\langle a, Za \rangle - \mathbb{E}[\langle a, Za \rangle] | > t ] \leq 4 \exp\left(-c \min\left\{ \frac{t^2}{4 K^4 \|Z\|_F^2}, \frac{t}{\sqrt{2} K^2 \|Z\|_o} \right\} \right),
\]

(33)

where \( K \) is a bound on the \( \psi_2 \)-norms of the real and imaginary parts of the components of \( a \) and \( c > 0 \) the same constant as in (32).

Proof. Taking squares on both sides and using \( |\cdot|^2 = \text{Re}(\cdot)^2 + \text{Im}(\cdot)^2 \) yields

\[
\begin{align*}
\mathbb{P}[ |\langle a, Za \rangle - \mathbb{E}[\langle a, Za \rangle] | > t ] &= \mathbb{P}[ \text{Re}^2(\langle a, Za \rangle) - \mathbb{E}[\text{Re}(\langle a, Za \rangle)] ] \\
&+ \mathbb{P}[ \text{Im}^2(\langle a, Za \rangle) - \mathbb{E}[\text{Im}(\langle a, Za \rangle)] ] > t^2 ] \\
&\leq \mathbb{P}[ |\text{Re}(\langle a, Za \rangle) - \mathbb{E}[\langle a, Za \rangle] | \geq \frac{1}{\sqrt{2}} t ] \\
&+ \mathbb{P}[ |\text{Im}(\langle a, Za \rangle) - \mathbb{E}[\langle a, Za \rangle] | \geq \frac{1}{\sqrt{2}} t ] \quad (34)
\end{align*}
\]

Writing

\[
\langle a, Za \rangle = (\text{Re}(a))^T - i \text{Im}(a)^T \left( \text{Re}(Z) + i \text{Im}(Z) \right) (\text{Re}(a) + i \text{Im}(a)) = \begin{bmatrix} \text{Re}(a) & \text{Im}(a) \end{bmatrix} \begin{bmatrix} \text{Re}(Z) & -\text{Im}(Z) \\
\text{Im}(Z) & \text{Re}(Z) \end{bmatrix} \begin{bmatrix} \text{Re}(a) \\
\text{Im}(a) \end{bmatrix}
\]

we can apply the Hanson-Wright inequality for the real case to (34) and (35) with \( \|Z_{1/2}\|_{HS} = \sqrt{2} \|Z\|_{HS} \) and \( \|Z_{1/2}\|_o = \|Z\|_o \) to obtain the result.

\[
\square
\]

APPENDIX B
CONCENTRATION OF 4TH ORDER POLYNOMIALS - FULL APPROACH

To calculate the probabilities of the form

\[
\mathbb{P}[ \|X_i\|^2 - m \geq \omega m ]
\]

appearing in (23), (24), we observe that \( \|X_i\|^2 \) is essentially a 4th order polynomial in the sub-Gaussian random variables \( \text{Re}(a_{i,k}) \) and \( \text{Im}(a_{i,k}) \). Setting \( v_k := \text{Re}(a_{i,k}) \) and \( v_{n+k} := \text{Im}(a_{i,k}) \), a quick calculation shows that we can write this as

\[
\|X_i\|^2 = \sum_{k,l \in [n], k \neq l} (v_k^2 + v_{n+k}^2)(v_l^2 + v_{n+l}^2)
\]

setting

\[
I = \{ (k,l) \in [2n] \times [2n] : k \neq l, k \neq n + l, l \neq n + k \} \quad (36)
\]

The following theorem, which can be seen as a generalization of the Hanson-Wright inequality, allows to analyze these terms.

**Theorem 12** (Theorem 1.6 in [13]). Let \( Z = (Z_1, \ldots, Z_t) \) be a random vector with independent components, such that \( \|Z_i\|_{\psi_2} \leq L \) for all \( i \in [\ell] \). Then, for
every polynomial \( f : \mathbb{R}^d \to \mathbb{R} \) of degree \( D \) and all \( t > 0 \), it holds that

\[
\mathbb{P}[ |f(Z) - \mathbb{E} f(Z)| \geq t ] \\
\leq 2 \exp\left( -\frac{1}{C_D} \min_{1 \leq d \leq D} \min_{J \in P_d} \eta_J(t) \right)
\]

where

\[
\eta_J(t) = \left( \frac{t}{L^d \| \mathbb{E} D^d f(Z) \|_J} \right)^{2/#J}.
\]

(37)

Here \( D^d f \) is the \( d \)-th derivative of \( f \) and for a multi-index array \( W = (w_{i_1 \ldots i_d})_{i_1 \ldots i_d=1} \) the \( \|\cdot\|_J \)-norm is defined as

\[
\|W\|_J := \sup\{ \sum_{i \in \sigma} w_i \prod_{l=1}^k (x_{l,i})^j \|x_{l,i}\|_2 \leq 1 \}
\]

(38)

for all \( l \in [k] \),

where \( J = (J_1, \ldots, J_k) \in P_d \) is a partition of \([d]\) into non-empty, pairwise disjoint sets. Some examples are:

\[
\|W\|_{\{1,2\}} = \|W\|_F
\]

\[
\|W\|_{\{1\}} = \|W\|_o
\]

\[
\|W\|_{\{1,2\}}^3 = \sup_{\|x\|_2 \leq 1} \sum_{i,j,k=1}^n w_{ij,k} x_{i,j} y_{j}
\]

Our first calculation allows the analysis of the deviation of \( \|Z\|_2 \) from its mean for a complex sub-Gaussian random vector \( Z \) with iid. components.

**Proposition 13.** Let \( Z = (Z_1, \ldots, Z_{2n}) \) be a random vector with independent components, such that \( \|Z_i\|_{\psi_2} \leq L, |\mathbb{E}[Z_i]| \leq \mu \) and \( \mathbb{E}[Z_i^2] \leq \frac{1}{2} \sigma^2 \) for some \( L \geq 1, \mu, \sigma^2 \geq 0 \) and all \( i \in [2n] \). Consider the \( 4 \)-th order polynomial

\[
f : \mathbb{R}^{2n} \to \mathbb{R}, \quad v \mapsto \sum_{(k,l) \in I} v_k^2 v_l^2,
\]

where \( I \) is given as in (36). Assume \( n \geq 27 \). Then for all \( \omega > 0 \) it holds that

\[
\mathbb{P}[ |f(Z) - \mathbb{E} f(Z)| \geq n(n-1) \omega ] \\
\leq 2 \exp(-\gamma \cdot n)
\]

where \( \gamma \in (0,1) \) is an absolute constant and

\[
\xi = \min\{ \omega^2 \left( \frac{L^2 \mu^2 \sigma^2}{L' \mu' \sigma'^2} \right), \omega^2 \left( \frac{L' \mu' \sigma'^2}{L \mu \sigma^2} \right), \omega^2 \left( \frac{L^2 \sigma^2 + 2 \mu^2}{L' \sigma'^2 + 2 \mu'^2} \right), \omega^2 \left( \frac{L' \sigma'^2 + 2 \mu'^2}{L \sigma^2 + 2 \mu^2} \right), \omega^{2/3} \left( \frac{L^2 \mu^2 \sigma^2}{L' \mu' \sigma'^2} \right), \omega^{2/3} \left( \frac{L' \mu' \sigma'^2}{L \mu \sigma^2} \right), \omega^{2/3} \left( \frac{L^2 \sigma^2 + 2 \mu^2}{L' \sigma'^2 + 2 \mu'^2} \right), \omega^{2/3} \left( \frac{L' \sigma'^2 + 2 \mu'^2}{L \sigma^2 + 2 \mu^2} \right), \omega^{2/3} \left( \frac{L^2 \mu^2 \sigma^2}{L' \mu' \sigma'^2} \right), \omega^{2/3} \left( \frac{L' \mu' \sigma'^2}{L \mu \sigma^2} \right), \omega^{2/3} \left( \frac{L^2 \sigma^2 + 2 \mu^2}{L' \sigma'^2 + 2 \mu'^2} \right), \omega^{2/3} \left( \frac{L' \sigma'^2 + 2 \mu'^2}{L \sigma^2 + 2 \mu^2} \right) \},
\]

(39)

Note that two of the terms in (39) contain a factor \( n \) and will therefore not play a role for large \( n \).
to calculate the upper bound.
\[
E[D^3 f(Z)](x,y) = \sum_{i,j,k \in [2n]} E[d_{i,j,k} f(Z)] x_{ijk} y_k
\]
\[
= \sum_{(i,j) \in I} 8E[Z] x_{ij} y_j + 8E[Z] x_{ij} y_i + 8E[Z] x_{ji} y_i
\]
\[
\leq 8\mu \sum_{(i,j) \in I} x_{ij} y_j + x_{ij} y_i + x_{ji} y_i
\]
\[
\leq 8\mu \left( \|\text{diag}(x)\|_1 \|y\|_1 + \sum_{j \in [2n]} (\langle x_j, y \rangle + \langle j, x, y \rangle) \right)
\]
\[
\leq 8\mu \left( \sqrt{2n} \cdot \sqrt{2n} + \sum_{j \in [2n]} (1 + 1) \right)
\]
\[
= 48\mu n,
\]
where by \( x_{ij} x \) we denoted the \( j \)-th row respectively column of \( x \) and by \( \text{diag}(x) \) the \( 2n \)-vector containing its diagonal elements. This shows
\[
\eta_{(1,2),(3)}(\omega n) \geq \left( \frac{2\omega n(n-1)}{L^3 \cdot 48\mu n} \right)^{2/3} \geq \frac{\omega}{48\mu L^3} \cdot n,
\]
where we used that \( n - 1 \geq \frac{1}{4} n \) because \( n \geq 2 \). The other cases follow in a similar manner, the sums can be estimated directly or using the Cauchy Schwarz inequality by euclidian or 1-norms of tensors with unit norm or by norms of their columns, rows or diagonal elements. We only state the results here:

\[
\eta_{(1),(2)}(\omega m) \geq \frac{\omega}{4L^2(\sigma^2 + \mu^2)} \cdot n
\]
\[
\eta_{(1,2),(3)}(\omega m) \geq \frac{\omega^2}{32L^4(\sigma^2 + \mu^2)^2} \cdot n
\]
\[
\eta_{(1),(2),(3)}(\omega m) \geq \frac{\omega^{2/3}}{192^{2/3} \mu^2/3 L^2} \cdot n
\]
\[
\eta_{(1,2,3)}(\omega m) \geq \frac{\omega^2}{48^2 \mu^2 L^3} \cdot n^2
\]

APPENDIX C

The \( \psi_\nu \)-norm via Moments

It is well-known, see \([23]\), that
\[
\|X\|_{\psi_\nu} = \sup_{\|x\|_\nu \leq 1} \|E[|X|^p]\|_{1/p}^{1/p}
\]
(40)
is equivalent to \((11)\). Now, let \( a \in \mathbb{C}^n \) be a random vector with subgaussian entries and \( |\text{Re}(a_i)|_{\psi_\nu}, |\text{Im}(a_i)|_{\psi_\nu} \leq \psi_2 \) for a constant \( \psi_2 \). In this section we show how to estimate the \( \psi_\nu \)-norm for \( \nu \geq 1 \) of the matrix \( aa^* \) by \( \psi_2 \). The \( \psi_\nu \)-norm of a random matrix \( A \in \mathbb{C}^{n \times n} \) is defined as
\[
\|A\|_{\psi_\nu} := \sup_{\|Z\|_F \leq 1} \|\langle A, Z \rangle\|_{\psi_\nu}.
\]

For the matrix \( aa^* - E[aa^*] \) this can be written as
\[
\|aa^* - E[aa^*]\|_{\psi_\nu} \leq \sup_{\|Z\|_F \leq 1} \|\langle E[aa^*], (a, Z) - E[(a, Z)A]\rangle\|_{\psi_\nu}.
\]
Set \( Y_Z := \langle a, Z a \rangle - E[\langle a, Z a \rangle] \) for some arbitrary \( Z \in \mathbb{C}^{n \times n} \) with \( 0 < \|Z\|_F \leq 1 \). Using (40) we can compute its \( \psi_\nu \)-norm as
\[
\|Y_Z\|_{\psi_\nu} = c_{\nu} \sup_{p \geq 1} \frac{\|E[|Y_Z|^p]\|_{1/p}}{p^{1/r}},
\]
(41)
with some constant \( c_{\nu} > 0 \). The expectation can be expressed as
\[
\mathbb{E}[|Y_Z|^p] = \int_0^\infty t^{p-1} \mathbb{P}[|Y_Z| \geq t] \, dt.
\]
(42)
The Hanson-Wright inequality \((33)\) yields
\[
\mathbb{P}[|Y_Z| \geq t] \leq 4 \exp(-c \min\{ \frac{t^2}{4\omega^2 (\|Z\|_F^2 / T_F + \sqrt{2\psi_2} \|Z\|_o)} \})
\]
(43)
\[
\leq 4 \exp(-c \min\{ \frac{t^2}{4\omega^2 \sqrt{\psi_2}} \})
\]
(44)
\[
= 4 \max\{ e^{-t^2/\omega^2}, e^{-t/b} \},
\]
(45)
where we used \( \|Z\|_o \leq \|Z\|_F \leq 1 \) and abbreviated \( a := \frac{\omega^2}{\sqrt{\psi_2}} \) and \( b := \frac{\sqrt{2\omega^2}}{\sqrt{2\psi_2}} \). Plugging into (42) and substituting \( s := t/a \), respectively \( s := t/b \), we obtain
\[
\mathbb{E}[|Y_Z|^p] \leq 4p \int_0^\infty s^{p-1} \max\{ a^p e^{-s^2}, b^p e^{-s} \} \, dt
\]
(46)
\[
\leq 4p \left( \frac{1}{2} a^p \Gamma\left( \frac{p}{2} \right) + b^p \Gamma(p) \right),
\]
(47)
where we estimated the maximum by the sum of both terms and expressed the integrals in terms of the Gamma function. Using the identity \( \Gamma(x) x = \Gamma(1 + x) \), for
\(x > 0\), and the asymptotic estimation \(\Gamma(x+1) \lesssim x^x\) derived from Stirling’s formula, we obtain

\[
\mathbb{E}[|Y_z|] \leq 4\left(a^\Gamma (\frac{p}{2} + 1) + b^\Gamma (p + 1)\right) \\
\leq c'(a^\Gamma (\frac{p}{2})b^{p/2} + b^p p^p) \\
\leq 2p^{p/2} c'' \psi_2^2 \left(c^{-p/2} + c^{-p}\right),
\]

for some constant \(c' > 0\). Plugging this into (41) yields

\[
\|Y_z\|_{\psi_2} \leq \sqrt{2} c'' \psi_2^2 \sup_{p \geq 1} \left(c^{-p/2} + c^{-p}\right)^{1/p} \\
= c'' \psi_2^2,
\]

for some constant \(c''\) that does not depend on the dimensions. Since for \(r, p \geq 1\) it holds that \(p^{-1/r} \leq p^{-1}\), we have \(c^{-1}_r \|Y_z\|_{\psi_r} \leq c^{-1}_r \|Y_z\|_{\psi_2}\) whenever \(r \geq 1\). Plugging into (43) and taking the supremum over all \(Z \in \mathbb{C}^{n \times n}\) with \(\|Z\|_F \leq 1\) shows that \(\|Y\|_{\psi_r} \leq c'_r \psi_2^2\), for some constant \(c'_r\).

REFERENCES

[1] R. Adamczak, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Restricted Isometry Property of Matrices with Independent Columns and Neighborly Polytopes by Random Sampling. Constructive Approximation, 34(1):61–88, 2011. arXiv:0904.4723 [math.ST]

[2] Radosław Adamczak. A note on the hanson-wright inequality for random vectors with dependencies. Electronic Communications in Probability, 20(1–15, 2015. arXiv:arXiv:1409.8457v1 [math.PR]

[3] Alfred Bruckstein, Michael Elad, and Michael Zibulevsky. Sparse non-negative solution of a linear system of equations is unique. pages 762 – 767, 04 2008.

[4] T. Tony Cai and Anru Zhang. Sparse representation of a polytope and recovery of sparse signals and low-rank matrices. IEEE Transactions on Information Theory, 60(1):122–132, 2014. arXiv:arXiv:1306.1154v1 [math.ST]

[5] E J Candes, J Romberg, and T Tao. Stable signal recovery from incomplete and inaccurate measurements. Comm. Pure Appl. Math., 59:1207–1223, 2005.

[6] Emmanuel J. Candès and Terence Tao. Decoding by linear programming. IEEE Transactions on Information Theory, 51(12):4203–4215, 2005. arXiv:0502237 [math.IT]

[7] Zhihui Chen, Foad Sohrabi, Ya-Feng Liu, and Wei Yu. Covariance Based Joint Activity and Data Detection for Massive Random Access with Massive MIMO. In 2019 IEEE International Conference on Communications (ICC), 2019.

[8] D L Donoho. Compressed sensing. IEEE T. Inform. Theory., 52(4):1289–1306, apr 2006. doi:10.1109/TIT.2006.871582

[9] D L Donoho, I M Johnstone, J C Hoch, and Stern A S. Maximum Entropy and the Nearly Black Object. J. Roy. Stat. Soc. B Met., 54(1), 1992. arXiv:95/57289 [math.ST]

[10] Alexander Fengler, Saeid Haghighatshoar, Peter Jung, and Giuseppe Caire. Non-bayesian activity detection, large-scale fading coefficient estimation, and unsourced random access with a massive mimo receiver, 2019. arXiv:1910.11266

[11] Alexander Fengler and Peter Jung. On the Restricted Isometry Property of Centered Self Khatri-Rao Products. may 2019. URL: http://arxiv.org/abs/1905.09249 [math.IT]

[12] Simon Foucart and Holger Rauhut. A Mathematical Introduction to Compressive Sensing. Applied and Numerical Harmonic Analysis. Springer New York, New York, NY, 2013. URL: http://link.springer.com/10.1007/978-0-8176-4948-7

[13] Friedrich Götze, Holger Sambale, and Arthur Sinulis. Concentration inequalities for polynomials in \(\alpha\)-sub-exponential random variables. mar 2019. URL: http://arxiv.org/abs/1903.05964

[14] Saeid Haghighatshoar, Peter Jung, and Giuseppe Caire. Improved Scaling Law for Activity Detection in Massive MIMO Systems. In 2018 IEEE International Symposium on Information Theory (ISIT), pages 381–385. IEEE, jun 2018. URL: https://ieeexplore.ieee.org/document/8437359/

[15] Carsten Herrmann, Yun Lu, Christian Scheunert, and Peter Jung. Improving Robustness for Anistropic Sparse Recovery using Matrix Extensions. In Workshop on Smart Antennas (WSA), 2018.

[16] Maryia Kabanava, Richard Kueng, Holger Rauhut, and Ulrich Terstiege. Stable low-rank matrix recovery via null space properties. Information and Inference, 5(4):405–441, dec 2016. arXiv:1507.07184 [math.ST]

[17] Richard Kueng and Peter Jung. Robust nonnegative sparse recovery and the nullspace property of 0/1 measurements. IEEE Transactions on Information Theory, 64(2):689–703, 2018. arXiv:1603.07997 [math.OC]

[18] Nicolai Meinshausen. Sign-constrained least squares estimation for high-dimensional regression. Electronic Journal of Statistics, 7(1):1607–1631, 2013. arXiv:arXiv:1202.0889v1 [math.ST]

[19] P. Pal and P. P. Vaidyanathan. Pushing the Limits of Sparse Support Recovery Using Correlation Information. IEEE Trans. Signal Process., 63(3):711–726, February 2015. doi:10.1109/TSP.2014.2385033

[20] Mark Rudelson and Roman Vershynin. Hanson-wright inequality and sub-gaussian concentration. Electron. Commun. Probab., 18:1–10, 2013. URL: https://projecteuclid.org/euclid.ecp/1465315621

[21] Y. Shadmi, P. Jung, and G. Caire. Sparse Non-Negative Recovery from Shifted Symmetric Subgaussian Measurements using NNLS. In IEEE Int. Symposium on Information Theory (ISIT), 2019.

[22] Martin Slawski and Matthias Hein. Non-negative least squares for high-dimensional linear models: Consistency and sparse recovery without regularization. Electron. J. Statist., 7:3004–3056, 2013. doi:10.1214/13-EJS868

[23] Roman Vershynin. High-dimensional probability: An introduction with applications in data science. 2018. doi:10.1002/path.1109

[24] P. Wojtaszczyk. Stability and instance optimality for Gaussian measurements in compressed sensing. Foundations of Computational Mathematics, 10(1):1–13, 2010. doi:10.1007/s10208-009-9046-4