LOOP GRASSMANNIANS OF QUIVERS AND AFFINE QUANTUM GROUPS

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To Alexander Beilinson and Victor Ginzburg

Abstract. We construct for each choice of a quiver $Q$, a cohomology theory $A$ and a poset $P$ a “loop Grassmannian” $G^P(Q, A)$. This generalizes loop Grassmannians of semisimple groups and the loop Grassmannians of based quadratic forms. The addition of a “dilation” torus $D \subset \mathbb{G}_m^2$ gives a quantization $G_{D}^P(Q, A)$. The construction is motivated by the program of introducing an inner cohomology theory in algebraic geometry adequate for the Geometric Langlands program [M17] and on the construction of affine quantum groups from generalized cohomologies [YZ17].

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It is a pleasure for I.M. to mention just a few transformative effects of personalities of Sasha Beilinson and Vitya Ginzburg. I.M.’s understanding of possibilities of being a mathematician have been upturned through Bernstein’s talk at Park City and Beilinson’s talks in Boston. A part of the magic was that mathematics was alive, high on ideas, low on ownership and each talk...
would open some topic in mathematics for thinking, almost regardless of one’s preparation. Before meeting Ginzburg, I.M. has come to view him as a smarter twin brother in mathematical tastes. Of biggest influence on I.M. was Ginzburg’s paper on loop Grassmannians that offered a new kind of mathematics, orchestrated by an explosion of geometric ideas.

0. Introduction

For a semisimple algebraic group $G$ of ADE type, the corresponding quiver $Q$ is used to study representations of $G$, its loop group $G((t))$ and their quantum versions. Here we reconstruct from $Q$ the loop Grassmannian $\mathcal{G}(G)$ of $G$. Our goal is to do the same for the enveloping algebra of the central extension $g^{\text{aff}}$ of the loop Lie algebra $g((t))$ and some representations of this central extension.

0.0.1. An advantage of the quiver approach is that it works in large generality. It provides a “loop Grassmannian” $G^Q_D(Q,A)$ associated to the data of an arbitrary quiver $Q$, a cohomology theory $A$, a poset $P$ and a torus $D$ of dilations. Intuitively, a quiver $Q$ should provide a “grouplike” object $G(Q)$ though at the moment we only see objects that should correspond to (quantization of) its affinization.

A cohomology theory $A$ gives a “cohomological schematization” functor $\mathfrak{A}(X) \overset{\text{def}}{=} \text{Spec}[A(X)]$ which assigns to a space $X$ the affine scheme $\mathfrak{A}(X)$ over the ring of constants of theory $A$. Applying $\mathfrak{A}$ to the moduli of lines provides a curve $G = \mathfrak{A}(\mathbb{G}_m)$ which one would like to define the ”affinization” of the undefined group $G(Q)$ as $G(Q)^{\text{aff}} \overset{\text{def}}{=} \text{Map}(\mathbb{G}, G(Q))$. Moreover, $\mathfrak{A}$ turns the moduli $\mathcal{V}$ of finite dimensional vector spaces into the space of configurations on the curve $G$, i.e., the Hilbert scheme of points $\mathcal{H}_G = \sqcup_n \mathbb{G}^{(n)}$ of $G$. This configuration space is then used as the setting for the Beilinson-Drinfeld version of the loop Grassmannian of $G(Q)$.

Finally, one adds quantization by letting a torus $D$ act on (the cotangent correspondence of) the extension correspondence for representations of quivers. At this level there is a well defined object, the “affine quantum group” constructed in [YZ16] and denoted here by $U_D(Q,A)$.

0.0.2. In the present paper we construct the space $G_D(Q,A)$ which should be the quantum loop Grassmannian of the (undefined) group $G(Q)$. Since we have skipped the construction of $G(Q)$ and its affinization, the construction is less standard. We will argue that it is of “homological nature”.

It uses the technique of local projective spaces from [M17]. This refers to the notion of $I$-colored local vector bundles over a curve $C$, i.e., vector bundles over the $I$-colored configuration space $\mathcal{H}_{C \times I}$ (the moduli of finite subschemes of $C \times I$), that are in a certain sense “local with respect to $C$”.

We actually start with a local line bundle $L$ over $\mathcal{H}_{C \times I}$ and we induce it using a poset $P$ to a local vector bundle $\text{Ind}^P(L)$ over $\mathcal{H}_{C \times I}$. Then the fibers of its local part $\mathbb{P}^{\text{loc}}[\text{Ind}^P(L)]$ are obtained as collisions of fibers at colored points $a_i \in \mathcal{H}_{C \times I}$ (for a point $a \in C$ and a color $i \in I$). The collisions happen inside the projective bundle $\mathbb{P}[\text{Ind}^P(L)]$ and the “rules of collisions” are specified by the locality structure on the line bundle $L$.

In our case $I$ is the set of vertices of a quiver $Q$ and the local line bundle $L$ is classically the Thom line bundle of the moduli of representations of a quiver. For the quantum case we replace this moduli of representations with the cotangent stack of a moduli of extensions of such representations. Finally, we get $G_D^Q(Q,A)$ as a certain union of fibers of $\mathbb{P}^{\text{loc}}[\text{Ind}^P(L)]$.

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1. When $A$ is de Rham cohomology then $\mathfrak{A}_X$ can be viewed as an affinization of the de Rham space $X_{dR}$ of $X$.

2. So, the cohomological schematization simplifies spaces with much symmetry to classical geometry.
Remark. The classical loop Grassmannians of reductive groups are recovered when the poset $P$ is a point. Whenever $P$ is a point we omit it from notation. In that case the fiber of $\text{Ind}^P(L)$ at any colored point is $\mathbb{P}^1$.\(^{(3)}\)

0.0.3. The loop Grassmannian $G(G)$ of a semisimple group $G$ is a partial flag variety of $G^{\text{aff}}$ so it has a known quantum version which is a non-commutative geometric object. For the $G_D(Q, A)$ construction this corresponds to the case when $A$ is the $K$-theory. However, our incarnation $G_D(Q, A)$ is an object in standard geometry, and the hidden noncommutativity manifests in its Beilinson-Drinfeld form, i.e., when $G_D(Q, A)$ is extended to lie over a configuration space. The configuration space is necessarily ordered ("non-commutative"), i.e., $H^n_{C \times I} = (C \times I)^{(n)}$ is replaced by $(C \times I)^n$. This has more connected components but this increase is ameliorated by a non-standard feature, a meromorphic braiding relating different connected components of the configuration moduli.

We expect to have more explicit descriptions of $G_D(Q, A)$ in terms of the graded algebra of sections of line bundles of $O(m)$ or in terms of the equation for the embedding into the projective space corresponding to sections of $O(1)$.\(^{(4)}\) In this paper we only do some preparatory steps towards identifying the cases of $G_D^P(Q, A)$ with the classical loop Grassmannian of reductive groups.

This paper is related to the work of Z. Dong [D18] that studies the relation between the Mirković-Vilonen cycles in loop Grassmannians and the quiver Grassmannian of representations of the preprojective algebra (see 2.2.4).

0.0.4. Contents. In section 1 we recall the method of cohomology theories. Section 2 covers relevant aspects of classical loop Grassmannians and how to rebuild these in a "homological" way, i.e., by turning the notion of locality into a construction. In section 3 we find a realization of these ideas in the setting of quivers by constructing local line bundles on configuration spaces from representations of quivers. Finally, in section 4 we get quantum generalization of the notion of local line bundles and of the corresponding loop Grassmannians using dilations on the cotangent bundle of moduli of extensions of representations of a quiver.

Appendix A completes the description of Cartan fixed points in intersections of closures of semi-infinite orbits in loop Grassmannians (proposition 2.2.3). This is here used as a motivation for the construction $G(Q, A)$. Appendix B compares computations of Thom line bundles of convolution diagrams in 3.4 and in [YZ17].

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1. Recollections on cohomology theories

1.1. Equivariant oriented cohomology theories. An oriented cohomology theory is a contravariant functor $A$ that takes spaces $X$ to graded commutative rings $A(X)$ and has certain properties such as the proper direct image.\(^{(5)}\) For us, an oriented cohomological theory $A$ can be

\(^{(3)}\) Here, we do not pay attention to a choice of $P$ but when $P = (1 < \cdots < m)$ the fibers at colored points are $\mathbb{P}^n$ and $G_D^P(Q, A)$ should "corresponds to level $m$" in the sense that the sections of the standard line bundle $O(1)$ on this object are the same as the sections of $O(m)$ in the case when $P = pt$.

\(^{(4)}\) These embedding equations should be integrable hierarchies indexed by $Q$ and $A$ since this is true in the classical case of $G(G)$.

\(^{(5)}\) While the grading of a cohomology theory is fundamental we will disregard it in this paper.
either a topological cohomology theory or an algebraic cohomology theory. In the first case the “spaces” are topological spaces, and we will use the ones that are given by complex algebraic varieties. In the second case the “spaces” mean schemes over a given base ring $k$.

Here we list some of the common properties of such theories $A$ that we will use. First, $A$ extends canonically to pairs of spaces $A(X,Y)$ for $Y \subseteq X$. In particular we get cohomology $A_Y(X) \overset{\text{def}}{=} A(X,X-Y)$ of $X$ with supports in $Y$. Such theory $A$ is functorial under flat pullbacks and proper push forwards with usual properties (homotopy invariance, projection formula, base change and the projective bundle formula [LM07, Lev15]).

Also, such $A$ has an equivariant version $A_G(X)$ defined as $\lim_i A(X_i)$ for ind-systems of approximations $X_i$ of the stack $G \setminus X$. For this reason it is consistent to denote $A_G(X)$ symbolically as $A(G \setminus X)$ even if we do not really extend $A$ to category of stacks.

The basic invariants of $A$ are the commutative ring of constants $R = A(\text{pt})$ and the 1-dimensional formal group $G$ over $R$ with a choice of a coordinate $1$ on $G$ (called orientation of theory $A$).

The geometric form of the theory $A$ is the functor $\mathfrak{A}$ from spaces to affine $R$-schemes defined by $\mathfrak{A}(X) = \text{Spec}(A(X))$. The $G$-equivariant version is again denoted by the index $G$, it yields ind-schemes $\mathfrak{A}_G(X) = \text{Spec}(A_G(X))$, also denoted $\mathfrak{A}(G \setminus X)$, that lie above $\mathfrak{A}_G = \mathfrak{A}_G(\text{pt})$. For instance the formal group $G$ associated to $A$ is $\mathfrak{A}_{G,m}$ (approximations of $BG_m$ are given by $\mathbb{P}^\infty$, the ind-system of finite projective spaces).

For a torus $T$ let $X^*(T), X_*(T)$ be the dual lattices of characters and cocharacters of $T$, then $\mathfrak{A}_T = X^*(T) \otimes_{\mathbb{Z}} G$. For a reductive group $G$ with a Cartan $T$ and Weyl group $W$, $\mathfrak{A}_G$ is the categorical quotient $\mathfrak{A}_T//W$. For instance for the Cartan $T = (G_m)^n$ in $GL_n$, the Weyl group is the symmetric group $S_n$ and one has $\mathfrak{A}_T = G^n$ while $\mathfrak{A}_{GL_n} = G^{(n)}$ is the symmetric power $G^n//S_n$ of $G$.

Remarks. (0) In the case when $G$ is the germ of an algebraic group $G_{\text{alg}}$ the equivariant $A$-cohomology has a refinement which gives ind-schemes over $G_{\text{alg}}$. All of our results extend to this setting and we will abuse the notation by allowing $G$ to stand either for the formal group or for this algebraic group. For simplicity our formulations will assume that $G_{\text{alg}}$ is affine – the adjustment for the non-affine case are clear from the paper [YZ17] on elliptic curves (then $G$ is an elliptic curve and $\mathfrak{A}(X)$ is affine over $G$ rather than affine). Either version satisfies equivariant localization.

(1) For algebraic oriented cohomology theories the basic reference is [LM07, Chapter 2] (one can also use [CZZ14, § 2] and [ZZ14, § 5.1]).\(^6\) Here, cohomology theory is defined on smooth schemes over a given base ring $k$. However, such cohomology theory $A$ then extends (with a shift in degrees) under the formalism of oriented Borel-Moore homology to schemes over $k$ that are of finite type and separable.\(^7\)

1.2. Thom line bundles. When $V$ is a $G$-equivariant vector bundle over $X$, the equivariant cohomology of $V$ supported in the zero section $\Theta_G(V) \overset{\text{def}}{=} A_G(V,V-X)$ is known to be a line bundle over $A_G(X)$, i.e., a rank one locally free module over $A_G(X)$, called the Thom line bundle of $V$. Moreover, this is an ideal sheaf of an effective divisor in $A_G(X)$ called the Thom divisor of

\(^6\) The terminology of “algebraic cohomology” is also used by Panin-Smirnov for a refinement of the formalism in which the theory is bigraded (to adequately encode the example of motivic cohomology). We will not be concerned with this version.

\(^7\) What is called Borel-Moore homology here is not quite what this means in classical topology, however this is just a choice of terminology since the $A$-setting does contain the precise analogue of Borel-Moore homology. For instance, for smooth $X$ the more appropriate version would be $BM_A(X) = \Theta_A(TX)^{-1}$ in terms of the Thom bundle which is defined next.
and otherwise \( V \) is a torus, which in turn can be reduced to the case when

\[ \Theta_l \]

generated by the function \( \Theta(G(V)) \in \mathfrak{A}_G \). (b) By the preceding lemmas, it suffices to check this when \( \Theta(G) \) is a torus, which in turn can be reduced to the case when \( \Theta(G) \) is one-dimensional and \( T = G_m \).

1.2.1. \( \Theta_G(V) \) for a representation \( V \) of \( G \). This is the case when \( X \) is a point. We can write \( \Theta_G(V) \) in terms of the character \( ch(V) \). First for a torus \( T \) there is a unique homomorphism \( I : (\mathbb{N}[X^*(T)],+\) \) to \( (\mathfrak{A}_T,\cdot) \) such that for any character \( \chi \) of \( T \), the function \( I \chi \) is the composition \( \mathfrak{A}_T \to \mathfrak{A}_G \to \mathfrak{A}_m \). Now, for a reductive group \( G \) with a Cartan \( T \), this restricts to a homomorphism \( I \) from \( (\mathbb{N}[X^*(T)]^W,+) \) to \( (\mathfrak{A}_G,\cdot) \). Then the ideal \( \Theta_G(V) \) in functions on \( \mathfrak{A}_G = [X_*(T)\otimes \mathbb{C}]^W/W \) is generated by the function \( I_{\mathfrak{a}_h(V)} \) on \( \mathfrak{A}_G \). (By the preceding lemmas, it suffices to check this when \( G \) is a torus, which in turn can be reduced to the case when \( V \) is one-dimensional and \( T = G_m \).)

1.2.2. Thom line bundles \( \Theta(f) \) of maps \( f \). For a map of smooth spaces \( f : X \to Y \) we have the tangent complex \( T(f) = [TX \to f^*TY]_{-1,0} \) on \( X \) and in degrees \(-1,0\). The line bundle \( \Theta(f) = \Theta(T(f)) \) on \( \mathfrak{A}(X) \) is defined as the value of \( \Theta \) on the corresponding virtual vector bundle \( f^*TY - TX \).

2. Loop Grassmannians and local spaces

Here we recall loop Grassmannians (in 2.1) and (in 2.2) we check the description of \( T \)-fixed points in intersections of closures of semi-infinite orbits in a loop Grassmannian that was announced in [M17]. This is a partial justification for the “homological” approach to loop Grassmannians (2.4) based on the formalism of local spaces 2.3.

2.1. Loop Grassmannians. We start with the standard loop Grassmannians \( \mathcal{G}(G) \). Let \( k \) be a commutative ring and let \( \mathcal{O} = k[[z]] \subseteq K = k((z)) \) be the Taylor and Laurent series over \( k \), these are functions on the indscheme \( d \) (the formal disc) and its punctured version \( d^* = d - 0 \). For an algebraic group scheme \( G \) we denote by \( G_\mathcal{O} \subseteq G_K \) its disc group scheme and \( \text{loop} \) group indscheme over \( k \), the points over a \( k \)-algebra \( k' \) are \( G_K(k') = G(k'(z)) \) and \( G_K(k') = G(k'(z)) \). The standard loop Grassmannian is the ind-scheme given by the quotient in the fpqc topology \( \mathcal{G}(G) = G_K/G_\mathcal{O} \).

For a smooth point \( a \) on a curve \( C \) a choice of a local parameter at \( a \) identifies \( \mathcal{G}(G) \) with the moduli of \( G \)-bundles \( P \) on \( C \) with a trivialization \( \sigma \) off \( a \), i.e., the cohomology \( H^1(C,G) \) supported at \( a \). When \( C = d \) this is also the compactly supported cohomology \( H^1_d(C,G) \).

We also notice that \( \mathcal{O}_- = k[z^{-1}] \) defines group indscheme \( G_\mathcal{O_-} \subseteq G_K \). The congruence subgroups \( K_+(G) \) are the kernels of evaluations \( G_\mathcal{O} \to G \) and \( G_\mathcal{O_-} \to G \) at \( z = 0 \) and \( z = \infty \).

2.1.1. The loop Grassmannian of \( G_m \). Recall that on a smooth curve (hence also for \( C = d \), \( \mathcal{H}_C \) is a commutative monoid for addition of divisors. Moreover, the Abel-Jacobi map \( AJ : \mathcal{H}_d \to \mathcal{G}(G_m) \) by \( AJ^d(x) = O_d(-x) \) is a map of monoids.

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By compactly supported cohomology of \( X \) I mean the cohomology of a compactification \( \overline{X} \) trivialized on the formal neighborhood of the boundary of \( X \) in \( \overline{X} \).
Lemma. [CC81] (see also [M17]). The map \( d = \mathcal{H}^1_{B} \subseteq \mathcal{H}_d \) makes \( \mathcal{H}_d \) into the commutative monoid indscheme freely generated by the formal disc \( d \). The Abel-Jacobi map \( d \to \mathcal{G}(\mathbb{G}_m) \) makes \( \mathcal{G}(\mathbb{G}_m) \) into the commutative group indscheme freely generated by \( d \).

\[ \square \]

Remarks. (0) In [M17], this is viewed as an interpretation of \( \mathcal{G}(\mathbb{G}_m) \) as homology \( \mathbb{H}_*(d) \) of \( d \) for a certain conjectural cohomology theory \( \mathbb{H} \). The above interpretations of \( \mathcal{G}(\mathbb{G}_m) \) as both homology and the compactly supported cohomology (see 2.1) are then viewed as a case of Poincaré duality in algebraic geometry.

(1) A formal coordinate \( z \) on \( d \) gives a correspondence of subschemes \( D \in \mathcal{H}_d \) and monic polynomials \( \chi_D \) in \( k[z] \) with nilpotent coefficients, such that \( \chi_D \) is an equation of \( D \). This gives a lift of the Abel-Jacobi map that embeds \( \mathcal{H}_d \) into \( \mathcal{G}_{m,K} \) by sending \( D \in \mathcal{H}_d^0 \) to \( \chi_D \). For instance for \( n \in \mathbb{N} \) the divisor \( n[0] \equiv \{ z^n = 0 \} \) goes to \( z^n \in \mathcal{G}_{m,K} \).

2.1.1. The “semi-infinite” orbits \( \mathcal{G}(G) = \mathcal{G}_{H_C}(G) \) over \( H_C \), called the Beilinson-Drinfeld Grassmannian. A choice of a local coordinate \( z \) on the formal neighborhood \( \mathfrak{c} \) of a point \( a \in C \) gives an isomorphism of the fiber at \( a \in \mathcal{H}_C \) with the standard loop Grassmannian

\[ \mathcal{G}(G) \xrightarrow{\cong} \mathcal{G}_{H_C}(G) \].

2.2. The \( T \)-fixed points in semi-infinite varieties \( \mathbb{S}_\alpha \).

2.2.1. Tori. Let us restate the remarks in 2.1.1 in the generality of split tori \( T \cong X_*(T) \otimes \mathbb{Z} \mathbb{G}_m \). First, a coordinate \( z \) on the disc gives \( X_*(T) \hookrightarrow T_K \) denoted by \( \lambda \mapsto z^{-\lambda} \). This gives \( X_*(T) \cong \pi_0(T_K) \) and a canonical isomorphism \( X_*(T) \cong \mathcal{G}(T)_{\text{reduced}} \cong \pi_0(\mathcal{G}(T)) \). For \( \lambda \in X_*(T) \) we denote \( L_\lambda \equiv z^{-\lambda}T_0 \in \mathcal{G}(T) \) (independent of \( z \)), and by \( G(T)_\lambda \) the connected component of \( \mathcal{G}(T) \) that contains \( L_\lambda \).

Moreover, if \( T = \mathbb{G}_m^m \) for a finite set \( I \) then the Abel-Jacobi map from 2.1.1 embeds the \( I \)-colored Hilbert scheme \( H_d \times I = (\mathcal{H}_d)^I \) of the disc into \( \mathcal{G}(\mathbb{G}_m)^I = \mathcal{G}(T) \) so that for \( \alpha = \sum_{i \in I} \alpha_i i \in \mathbb{N}[I] \subseteq X_*(T) \) the divisor \( \alpha[0] = \sum \alpha_i[0] \) goes to \( L_\alpha \).

2.2.2. The “semi-infinite” orbits \( \mathbb{S}_\lambda^{\pm} \). Now let \( G \) be reductive with a Cartan \( T \). Then the \( T \)-fixed point subscheme \( \mathcal{G}(G)^T \) is \( \mathcal{G}(T) \). A choice of opposite Borel subgroups \( B^\pm = TN^\pm \) yields orbits \( \mathbb{S}_\lambda^{\pm} \equiv N^\pm \mathcal{L}_\lambda \) indexed by \( \lambda \in X_*(T) \) (we often omit the super index \( + \)). If \( G \) is semisimple then \( \mathcal{G}(G) \) is reduced and these orbits provide two stratifications of \( \mathcal{G}(G) \). The following is well known:

Lemma. For \( \lambda, \mu \in X_*(T) \) the following are equivalent: (0) \( \mathbb{S}_\lambda \ni \mu \), (i) \( \mathbb{S}_\lambda \supseteq \mathbb{S}_\mu \), (ii) \( \mathcal{L}_\lambda \) meets \( S_\mu^\perp \), and (iii) \( \lambda \geq \mu \) (in the sense that \( \lambda - \mu \) lies in the the cone \( Q^+ \) generated by the coroots \( \check{\alpha} \) dual to roots \( \alpha \) in \( N \)).

In general, the derived version of homology \( \mathbb{H}_*(X) \) should be the free abelian commutative group object in derived stacks freely generated by \( X \).

In terms of torsors, \( L_\lambda \) is the trivial \( T \)-torsor over the formal disc \( d \) with the section \( \tau = z^{-\lambda} : d^\ast \to T \) over the punctured disc \( d^\ast = d - 0 \), which is the composition with a local coordinate \( d^\ast \to \mathbb{G}_m \to T \).
Example. The loop Grassmannian of $G = SL_2$ is the space $\mathcal{L}$ of lattices in $K^2 = K \mathcal{E}_1 \oplus K \mathcal{E}_2$ (the $\mathcal{O}$-submodules that lie between two submodules of form $z^nO^2$ of volume zero. Here $\text{vol}(L) = \dim(L/z^nO^2) - \dim(O^2/z^nO^2)$ for $n >> 0$. For the standard Borel subgroup $B = TN$ we have $N = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ and the coroot $\alpha$ in $N$ is $\check{\alpha}(\alpha) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \in T$. Then $X_{\ast}(T) = Z \check{\alpha}$ and $L_{n\check{\alpha}}$ is the lattice generated by two vectors $(z^{-n}e_1, z^n e_2)$. Here, $S_{n\check{\alpha}}$ consists of lattices $L \in \mathcal{G}(G)$ such that $L \cap \mathcal{K}_1 = z^{-n} \mathcal{O}_1$ while the condition for $L \in S_{n\check{\alpha}}$ is that $L$ contains $z^{-n} e_1$.

2.2.3. The $T$-fixed points. Now let $G$ be semisimple and adjoint, and $I$ index the minimal roots $\alpha_i$, $i \in I$ in $N$. Then $\prod_{i \in I} \check{\alpha}_i : G^I \rightarrow T$ defines the adjoint Abel-Jacobi embedding $AJ : \mathcal{H}_{d 	imes I} \hookrightarrow \mathcal{G}(T)$. Let us identify $\mathbb{N}[I]$ with the cone $\check{Q}^+ = \oplus_{i \in I} \mathbb{N} \check{\alpha}_i$, and notice that now for any $\alpha \in \mathbb{N}[I]$ the image of the corresponding colored subscheme is $AJ(\alpha[0]) = L_{-\alpha}$.

Proposition. (a) The image of the adjoint Abel-Jacobi embedding $AJ : \mathcal{H}_{d \times I} \hookrightarrow \mathcal{G}(T)$ is the fixed point sub-indscheme $(S_0)^T$.

(b) For $\alpha \in \mathbb{N}[I]$, the connected component $\mathcal{H}_{d \times I}^\alpha = \prod_{i \in I} \mathcal{H}_d^{\alpha_i}$ of $\mathcal{H}_{d \times I}$ is identified with the intersection of $S_0$ with the connected component $\mathcal{G}(T)_{-\alpha}$ of $\mathcal{G}(T)$.

(c) Moreover, this identifies $(S_0 \cap S_{-\alpha})^T$ with the moduli $\mathcal{H}_{\alpha[0]}$ of all subschemes of the finite scheme $\alpha[0] \in \mathcal{H}_{d \times I}$.

Proof. We start with the proof for $G = SL_2$. Parts (a-b) claim that $S_0$ meets the connected component $\mathcal{G}(T)_p$ of $\mathcal{G}(T)$ iff $p \geq 0$ and then the intersection is $\mathcal{H}_d^p$.

The points of the negative congruence subgroup $K_-(G_m) \subseteq G_{m,K}$ are the comonic polynomials $Q = 1 + q_1 z^{-1} + \cdots + q_s z^{-s}$ in $z^{-1}$ with nilpotent coefficients. Now, the isomorphism $K_-(G_m) \rightarrow \mathcal{G}(T)_p$ is given by $Q \mapsto \check{\alpha}(Q) L_p = \check{\alpha}(Q z^p) L_0 = \langle Q^{-1} z^{-p} e_1, Q z^p e_2 \rangle$. According to the example in 2.2.2 this is in $S_0$ iff $(Q z^p)^{-1} \mathcal{O} \ni z^0$. This means that $z^p Q \in \mathcal{O}$, i.e., that the powers of $z^{-1}$ in $Q$ are $\leq p$. Such $z^p Q$ form all monic polynomials in $z$ of degree $p$ with nilpotent coefficients. So, all such $\check{\alpha}(Q z^p) L_0$ form exactly $AJ(\mathcal{H}_d^p)$.

For part (c) the example in 2.2.2 says that $S_m$ consists of lattices $L$ that contain $z^m e_2$. Now, for $D \in \mathcal{H}_d^p$ with a monic equation $P \in \mathbb{K}[z]$, $AJ(D) = \check{\alpha}(P) L_0 = \langle P^{-1} e_1, P e_2 \rangle$ lies in $S_m$ iff $P \mathcal{O} \ni z^m$, i.e., polynomial $P$ divides $z^m$. This is equivalent to $D \subseteq m[0]$.

The proof in the general case is postponed to the appendix A.0.4.

Remark. The orbit $S_0$ lies in the connected component $\mathcal{G}(G)_0$ of $\mathcal{G}(G)$, The passage from $G$ to $\mathcal{G}(G)/Z(G)$ does not affect $\mathcal{G}(G)_0$, hence the spaces $S_0 \supseteq S_0 \cap S_{-\alpha}$, and their $T$-fixed points do not depend on the center of $G$.

2.2.4. The Kamnitzer-Knutson program of reconstructing MV-cycles. Here we restate the proposition and recall one of the origins of this paper. Consider a simply laced semisimple Lie algebra $\mathfrak{g}$ and its adjoint group $G$. In [BK10] the irreducible components $\mathcal{C}$ of the variety $\Lambda$ of representations of the preprojective algebra $\Pi$ of a Dynkin quiver $Q$ of $G$ are put into a canonical bijection with certain irreducible subschemes $X_{\mathcal{C}}$ of the corresponding loop Grassmannian $\mathcal{G}(G)$, called $MV$-cycles [MV07].

For any representation $V$ of the preprojective algebra $\Pi$ the moduli $\text{Gr}_\Pi(V)$ of all $\Pi$-submodules of $V$ is called the quiver Grassmannian of $V$.

\[11\text{So, its connected components are } (S_0 \cap S_{-\alpha}) \cap \mathcal{G}(T)_{-\beta}, \text{ for } 0 \leq \beta \leq \alpha, \text{ identified with the moduli } \mathcal{H}_d^{\beta}_{\alpha[0]} \text{ of length } \beta \text{ subschemes of } \alpha[0].\]
Conjecture. [M17] For any irreducible component $C$ of $\Lambda$, and a sufficiently generic representation $\dot{V}$ in $C$, the cohomology of its quiver Grassmannian $\text{Gr}_\Pi(\dot{V})$ is the ring of functions on the subscheme $X_C^\alpha$ of points in the corresponding MV cycle $X_C$ in $\mathcal{G}(G)$ that are fixed by a Cartan subgroup $T$ of $G$.

The grading on cohomology corresponds to the action of loop rotations on $X_C^T$.

Remarks. (0) This is a version of a conjecture of Kamnitzer and Knutson on equality of dimensions of cohomology $H^*[\text{Gr}_\Pi(\dot{V})]$ and of sections of the line bundle $O(1)$ over the MV cycle $X_C$.

(1) Zhijie Dong has constructed a map in one direction in this conjecture [D18].

(2) The form of this conjecture is alike the Hikita conjecture in the symplectic duality framework.

(3) The MV cycles are defined as irreducible components of intersections in $\mathcal{G}(G)$ of closures of semi-infinite orbits $\overline{S_0} \cap \overline{S^-_\alpha}$ for $\alpha \in N[I]$. The proposition 2.2.3.c will imply the following simplified version of the conjecture that replaces on the loop Grassmannian side the individual MV cycles with the intersections $\overline{S_0} \cap \overline{S^-_\alpha}$; and on the quiver side it degenerates the operators in the representation of $\Pi$ to zero:

Corollary. Let $\alpha \in N[I]$, then $(\overline{S_0} \cap \overline{S^-_\alpha})^T$ is the spectrum of cohomology of the quiver Grassmannian $\text{Gr}_\Pi(\dot{V})$, where $\dot{V}$ is the zero representation of $\Pi$ of dimension $\alpha$. Also, the grading on cohomology corresponds to the action of loop rotations on the fixed point subscheme of the loop Grassmannian.

Proof. For $\alpha = \sum \alpha_i$, we have $\dot{V} = \bigoplus_{i \in I} V_i$ with $\dim(V_i) = \alpha_i$. The quiver Grassmannian $\text{Gr}_\Pi(\dot{V})$ is then the product $\prod_{i \in I} \text{Gr}(V_i)$ of total Grassmannians of components $V_i$.

Since $(\overline{S_0} \cap \overline{S^-_\alpha})^T = \mathcal{H}_{\alpha[0]} = \prod_{i \in I} \mathcal{H}_{\alpha_i[0]}$ by the proposition 2.2.3.c, it remains to notice that $H^*(\text{Gr}_p(n))$ can be calculated by Carell’s theorem as functions on the fixed point subscheme $\text{Gr}_p(n)^e$ of a regular nilpotent $e$ on $k^n$. If we realize $k^n$ and $e$ as $O(n[0])$ and the operator of multiplication by $z$, we see that $\text{Gr}_p(n)^e$ is $\mathcal{H}_p(n[0])$ (a subspace of $O(n[0])$) is $z$-invariant iff it is the ideal of a subscheme.

Finally, the degree $2p$ cohomology corresponds to the $p$-power of $z$ which is the grading by rotations of the disc $d$. \hfill \Box

2.3. Local spaces over a curve. The notion of local spaces has appeared in [M14] as a common framework for the factorization spaces of Beilinson-Drinfeld and the factorizable sheaves of Finkelberg and Schechtman.

2.3.1. Local spaces. For a set $I$ and a smooth curve $C$ we consider the Hilbert scheme $\mathcal{H}_{C \times I} \cong (\mathcal{H}_C)^I$ of $I$-colored points of $C$. Its connected components $\mathcal{H}^\alpha_{C \times I} \cong \prod_{i \in I} \mathcal{H}^\alpha_{C_i}$ are given by subschemes of length $\alpha \in N[I]$. For a space $Z$ over $\mathcal{H}_{C \times I}$ we denote the fiber at $D \in \mathcal{H}_{C \times I}$ by $Z_D$.

An $I$-colored local space $\tilde{Z}$ over $\tilde{C}$ is a space $Z$ over $\mathcal{H}_{C \times I}$, together with a consistent system of isomorphisms for disjoint $D'$, $D'' \in \mathcal{H}_{C \times I}$

$$\iota_{D',D''} : Z_{D'} \times Z_{D''} \xrightarrow{\cong} Z_{D' \cup D''}.$$ We have $Z_{\emptyset} = \text{pt}$. When $\alpha = i \in I$ the connected component $\mathcal{H}^i_{C \times I}$ is $C \times i$. We call the fiber $Z_{ai}$ at $a \in C$ the “$i$-particle at $a$” and we think of $Z$ as a fusion diagram for these particles.

\[\text{One can replace a curve by a more general scheme and } \mathcal{H} \text{ by other notions of powers of a scheme.}\]
Example. A factorization space in the sense of Beilinson and Drinfeld is a local space \( Z \to \mathcal{H}_{C \times I} \) such that the fibers \( Z_D \) only depend on the formal neighborhood \( \tilde{D} \) of \( D \) in \( C \). These can be viewed as spaces over the Ran space \( \mathcal{R}_C \), the moduli of finite subsets of \( C \).

Remarks. (1) A weakly local structure is the case when the structure maps \( \iota \) are only embeddings. Any weakly local space \( Z \) has its local part \( Z^\text{loc} \subseteq Z \) which we define as the least closed local subspace of \( Z \) that contains all particles. So, at a discrete \( D \in \mathcal{H}_{C \times I} \) the fiber is \( Z_D^\text{loc} = \prod_{\alpha \in D} Z_{\alpha} \) and \( Z^\text{loc} \) is the closure in \( Z \) of its restriction to \( \mathcal{H}_{C \times I}^\text{reg} \).

(2) A local structure on a vector bundle \( V \) over a local space \( Z \) is a consistent system of isomorphisms \( V|_{Z_D \cap \tilde{D}''} \cong V|_{Z_D}, \forall D \subseteq \tilde{D}'' \). By the Segre embedding its projective bundle \( \mathbb{P}(V) \) is a weakly local space. Its local part \( \mathbb{P}(V)^\text{loc} \) is called the local projective space \( \mathbb{P}^\text{loc}(V) \) of a local vector bundle \( V \).

Remark. In this way the notion of locality structure is a version of the Beilinson-Drinfeld factorization structure which can be used as a tool for producing spaces. However, this construction is not yet explicit.

2.4. A generalization \( \mathcal{G}^P(I, \mathcal{Q}) \) of loop Grassmannians of reductive groups.

2.4.1. Motivation. Consider a semisimple simply laced group \( G \) of adjoint type with a Cartan subgroup \( T \). In the proposition 2.2.3.a we have described the subscheme of \( T \)-fixed points \( \overline{S}^T_0 \subseteq \mathcal{G}(T) \) of the semi-infinite subspace \( \overline{S}_0 \) as the colored Hilbert scheme \( \mathcal{H}_{d \times I} \). Knowing the standard line bundle \( \mathcal{O}_{\mathcal{G}(G)}(1) \) on the loop Grassmannian of \( G \) is equivalent to knowing the standard central extension of the loop group \( G_k \). This structure on \( \mathcal{O}_{\mathcal{G}(G)}(1) \) manifests as the structure of a local line bundle on the restriction \( L \) of the line bundle \( \mathcal{O}_{\mathcal{G}(G)}(1) \) to the fixed subscheme \( \overline{S}^T_0 \cong \mathcal{H}_{d \times I} \).

It turns out that knowing the local line bundle \( L \) on \( \mathcal{H}_{d \times I} \) is sufficient to reconstruct the loop Grassmannian \( \mathcal{G}(G) \). The key property that we use here is that the restriction of sections to \( T \)-fixed points \( \mathcal{G}(G)^T = \mathcal{G}(T) \) is an isomorphism \( \Gamma[\mathcal{G}(G), \mathcal{O}_{\mathcal{G}(G)}(1)] \xrightarrow{\text{Res}} \Gamma[\mathcal{G}(T), \mathcal{O}_{\mathcal{G}(G)}(1)] \) \cite{Zhu07}. Moreover, this is also true for sections on the semi-infinite variety; i.e., the restriction \( \Gamma[\overline{S}_0, \mathcal{O}_{\mathcal{G}(G)}(1)] \xrightarrow{\text{Res}} \Gamma[\overline{S}^T_0, L] \) is an isomorphism. The key observation is that the equations of the semi-infinite variety \( \overline{S}_0 \) in the projective space \( \mathbb{P}(\Gamma[\overline{S}_0, \mathcal{O}_{\mathcal{G}(G)}(1)]) = \mathbb{P}(\Gamma[\overline{S}^T_0, L^*]) \) are given by the locality structure on \( L \). So, the locality structure allows us to reconstruct \( \overline{S}_0 \) from \( L \) and then also \( \mathcal{G}(G) \) as a certain limit of copies of \( \overline{S}_0 \).

In this section 2.4 we recall how this strategy yields a kind of a loop Grassmannian for \textit{any} local line bundle \( L \) \cite{M17}.

2.4.2. Zastava spaces of local line bundles. We can induce any local line bundle \( \mathcal{H}_{C \times I} \) along a poset \( P \). When we consider \( \mathcal{H}_{C \times I} \) as a poset for inclusion we get the moduli \( \text{Hom}(P, \mathcal{H}_{C \times I}) \) which is a space over \( \mathcal{H}_{C \times I} \) such that the fiber \( \text{Hom}(P, \mathcal{H}_{C \times I}) \) at \( D \in \mathcal{H}_{C \times I} \) consists of systems \( D^* = (D^p)_{p \in P} \in (\mathcal{H}_{C \times I})^P \) such that \( p \leq q \) implies \( D^p \subseteq D^q \subseteq D \). We use it as a correspondence

\[
(\mathcal{H}_{C \times I})^P \xleftarrow{\pi} \text{Hom}(P, \mathcal{H}_{C \times I}) \xrightarrow{\sigma} \mathcal{H}_{C \times I}.
\]

This gives a local vector bundle \( \text{Ind}^P(L) \equiv \sigma_* \pi^*(L \boxtimes P) \). Then the zastava space \( \mathcal{Z}^P(L) \) of \( L \) is the local projective space

\[
\mathcal{Z}^P(L) \equiv \mathbb{P}^\text{loc}(\text{Ind}^P(L)).
\]

\[\text{The original notion of zastava spaces in [FM] is an affine open part of the present version.}\]
Example. When \( P \) is a point we omit \( P \) from the notation. Then the fiber \( \text{Hom}(P, \mathcal{H}_{C \times 1}) \) at \( D \in \mathcal{H}_{C \times 1} \) is the Hilbert scheme \( \mathcal{H}_D \) of all subschemes of the finite scheme \( D \). If \( D \) is a point \( ai \) with \( a \in C \) and \( i \in I \) then \( \mathcal{H}_{ai} = \{ \emptyset, ai \} \), hence \( \text{Ind}(L) = L_{ai} \oplus L_{ai} = k \oplus L_{ai} \). So, the fiber of \( Z(L) \) at \( ai \) is \( \mathbb{P}^1 \) with two fixed points. So, one is constructing \( Z(L) \) by colliding \( \mathbb{P}^1 \)'s according to a prescription given by the line bundle \( L \).

Similarly, when \( P = [m] = \{ 1 < \cdots < m \} \) then all particles of zastava spaces are \( \mathbb{P}^m \). Actually, this \( \mathbb{P}^m \) is naturally the \( m \)-th symmetric power of the particle \( \mathbb{P}^1 \) for \( m = 1 \).

2.4.3. Grassmannians from based quadratic forms. A quadratic form \( Q \) on \( \mathbb{Z}[I] \) gives a local line bundle \( O(Q) \) on \( \mathcal{H}_{C \times 1} \). Here, \( Q \Delta \) is the divisor \( \sum_{i,j} Q(i,j) \Delta_{ij} \) for the discriminant divisors \( \Delta_{ij} \subseteq \mathcal{H}_{C \times 1} \). Its zastava space \( Z^P(I,Q) \) defines the semi-infinite space \( S^P(I,Q) \) over \( \mathcal{H}_C \), the fiber at \( D \in \mathcal{H}_C \) is the colimit (union) of zastava fibers \( Z^P(I,Q)_D \) at multiples of \( D \)

\[
S^P(I,Q)_D = \lim_{n} Z^P(I,Q)_{nD}.
\]

Finally, \( \mathbb{N}[I] \) acts on \( S^P(I,Q) \), and the corresponding loop Grassmannian is defined as

\[
G^P(I,Q) \coloneqq \mathbb{Z}[I] \times_{\mathbb{N}[I]} S^P(I,Q).
\]

Theorem. Let \( I \) be the set of simple coroots of an adjoint semisimple group \( G \) of simply laced type. Let \( Q \) be the incidence quadratic form of the Dynkin diagram. Then \( G(I,Q) \) is the usual loop Grassmannian \( G(G) \).

2.4.4. Homological aspect. The standard interpretation of loop Grassmannians is cohomological (2.1.2). In the commutative case a homological interpretation is to build \( G(G_m) \) from the formal disc \( d \) in stages \( d \to \mathcal{H}_d \to G(G_m) \) by taking the free semigroup on \( d \) (the configuration space \( \mathcal{H}_d \)) and inverting a point to get \( G(G_m) \) as the free group on \( d \) (remark 0 in 2.1.1). Construction 2.4 repeats this procedure in the noncommutative case by using the curve \( d \times I \) (which gives \( G(T) \)) and adding a local bundle \( L \) (to get \( G(G) \) or \( G^P(I,Q) \)). First, the positive part of the loop Grassmannian is the zastava space \( Z^P(I,Q) \) built using the monoid \( \mathcal{H}_{d \times I} \) in 2.4.2. Then \( G^P(I,Q) \) itself is obtained from \( Z^P(I,Q) \) in 2.4.3 by inverting \( \mathbb{N}[I] \subseteq \mathcal{H}_{d \times I} \). For compact curves \( C \) reconstructing \( Bun_G(C) \) from \( C \) has been pursued in \([FS94]\).

3. Local line bundles from quivers

We know that local line bundles \( L \) on configuration spaces \( \mathcal{H}_{C \times 1} \) correspond to quadratic forms \( Q \) (2.4.3) and the forms \( Q \) with non-negative integer coefficients clearly correspond to graphs. In this section we construct local line bundles directly from graphs or quivers. The advantage is that such construction extends to the quantum setting (see [YZ16] and 3.5 below). In the quantum setting the “commutative” configuration space \( \mathcal{H}_{C \times 1} \) will be replaced with the “non-commutative”, i.e., ordered, configuration space \( C_{G \times 1} \) defined as \( \sqcup_n (G \times I)^n \). On the level of representations of quivers the noncommutative configuration space corresponds to passing to complete flags in representations.

We start with the curve \( G \) which is the 1-dimensional group corresponding to a cohomology theory \( A \). Then the Hilbert scheme \( \mathcal{H}_{G \times 1} \) of points in \( G \times I \) is obtained as the cohomological schematization \( \mathfrak{A}(\operatorname{Rep}_Q) \) of the moduli \( \mathcal{V}^I \) of \( I \)-graded finite dimensional vector spaces.

In 3.1 we recall various categories of representations of quivers and their extension correspondences. The cotangent complexes for these correspondences are considered in 3.3.

\[14\] Here, \( G \) is defined over the ring of constants \( A(\text{pt}) \) of the cohomology theory.
The “classical” local and biextension line bundles \( L(Q, A) \) and \( L(Q, A) \) on \( \mathcal{H}_{Q \times I} \) and \( (\mathcal{H}_{Q \times I})^2 \) are constructed as Thom line bundles of moduli of extensions of representations in 3.2. Here, \( L(Q, A) \) can be defined directly from the incidence quadratic form of the quiver \( Q \).

In 3.4 we calculate Thom line bundles associated to the cotangent correspondence and the effect of dilations. Finally, in 3.5 we recall the construction of the quantum group \( U_D(Q, A) \) from the cotangent correspondence and this leads us to select a choice of quantization of the above “classical” line bundles from 3.2.

**Remark.** This section is largely a retelling of the paper [YZ17]. That paper is primarily concerned with the construction of quantum affine groups in the language of preprojective algebras which is here viewed as the cotangent bundle of the moduli \( \text{Rep}_Q \). This “symplectic” setting allows to “quantize” the notion of local line bundles and the construction of loop Grassmannian from local line bundles. The quantization comes from the action of the dilation torus \( D \) on representations (which is in turn defined by a choice of a Nakajima function \( \mathbf{m} \) on the set of arrows of the double \( \mathcal{Q} \) of the quiver \( Q \)).\(^{15}\)

### 3.1. Quivers

Let \( Q \) be a quiver with finite sets \( I \) and \( H \) of vertices and arrows. For each arrow \( h \in H \), we denote by \( h' \) (resp. \( h'' \)) the tail (resp. head) vertex of \( h \). The opposite quiver \( Q^* = (I, H^*) \) has the same vertices and the set of arrows \( H^* \) is endowed with a bijection \( * : H \to H^* \), so that \( h \mapsto h^* \) exchanges sources and targets. The double \( \mathcal{Q} \) of the quiver \( Q \) has vertices \( I \) and arrows \( H \cup H^* \).

Let \( \mathcal{V}^I \) be the moduli of finite dimensional \( I \)-graded vector spaces \( V = \bigoplus_{i \in I} V^i \). Let \( \text{Rep}_Q \) be the moduli of representations of \( Q \). Its fiber at \( V \in \mathcal{V}^I \) is the vector space \( \text{Rep}_Q(V) \) of representations on \( V \). This is the sum over \( h \in H \) of \( \text{Rep}_Q(V)_h = \text{Hom}(V^{h'}, V^{h''}) \). We usually denote \( v = \dim(V) \in \mathbb{N}[I] \) and let \( G = GL(V) \) so that the connected component \( \text{Rep}_Q(v) \) – given by representations of fixed dimension \( v \in \mathbb{N}[I] \) – is \( G/\text{Rep}_Q(V) \).

#### 3.1.1. Dilation torus \( D \).

A choice of Nakajima’s weight function \( \mathbf{m} : H \coprod H^* \to \mathbb{Z} \) gives an action of \( \mathbb{G}_m^2 \) on

\[
T^* \text{Rep}_Q(V) = \text{Rep}_{\mathcal{Q}}(V) = \text{Rep}_Q(V) \oplus \text{Rep}_Q^*(V).
\]

Elements \( (t_1, t_2) \) act for each \( h \in H \) on \( \text{Rep}_Q(V)_h \) by \( t_1^{\mathbf{m} h} \) and on \( \text{Rep}_Q^*(V)_h \) by \( t_2^{\mathbf{m} h^*} \). We also let \( \mathbb{G}_m^2 \) act on the Lie algebra \( \mathfrak{g} \) of \( GL(V) \) by \( t_1 t_2 \).

We choose a subtorus \( D \) of \( \mathbb{G}_m^2 \) and require that the moment map for the \( GL(V) \)-action on \( T^* \text{Rep}_Q(V) \) is \( D \)-equivariant. This means that on \( D \) we have \( t_1^{\mathbf{m}(h)} t_2^{\mathbf{m}(h^*)} = t_1 t_2 \) for any \( h \in H \). In particular, the symplectic form on \( T^* \text{Rep}_Q(V) \) has weight \( t_1 t_2 \).

**Example.**

1. Nakajima’s construction of quantum affine algebra associated to \( Q \) uses \( D = \mathbb{G}_m \), the diagonal torus in \( \mathbb{G}_m^2 \) (see [Nak01, (2.7.1), (2.7.2)]). Here the \( D \)-weight of the symplectic form on \( T^* \text{Rep}_Q(V) \) and on \( \mathfrak{g} \) is 2, and the condition on \( \mathbf{m} \) is \( \mathbf{m}(h) + \mathbf{m}(h^*) = 2 \). If there are \( a \) arrows in \( Q \) from vertex \( i \) to \( j \), we fix a numbering \( h_1, \ldots, h_a \) of these arrows, and let

\[
\mathbf{m}(h_p) := a + 2 - 2p, \quad \mathbf{m}(h_p^*) := -a + 2p, \quad \text{for} \quad p = 1, \ldots, a.
\]

2. In [SV13] the elliptic Hall algebra (the spherical double affine Hecke algebra of \( GL_\infty \)) is obtained from the choice \( D = \mathbb{G}_m^2 \) and \( \mathbf{m} = 1 \).

\(^{15}\) While [YZ17] deals with the case of elliptic cohomology, some of its ideas appear in an earlier paper [YZ14] which was only concerned with affine groups \( \mathbb{G} \). This allowed for a trivialization of Thom line bundles which accounts for a different presentation of functoriality of cohomology in that paper.
3.1.2. The extension correspondence for quivers. The moduli $\mathcal{R} = \text{Rep}_Q$ is given by pairs of $V \in \mathcal{V}$ and $a \in \text{Rep}_Q(V)$. We denote the elements of $\mathcal{R}^m$ as sequences $(V_0, a_0)$ of pairs of $(V_i, a_i) \in \mathcal{R}$.

Let $\mathcal{F}^m$ be the moduli of $m$-step filtrations $F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = V)$ on objects $V$ of $\mathcal{V}$. Similarly, we consider the moduli of filtrations $\mathcal{F}^m \mathcal{R}$ of representations, the objects are triples of $V \in \mathcal{V}$, representation $a$ of $Q$ on $V$ and a compatible filtration $F \in \mathcal{F}^m \mathcal{R}(V)$. We denote the fiber of $\mathcal{F}^m$ at $V \in \mathcal{V}$ by $\mathcal{F}^m(V)$ and the fiber of $\mathcal{F}^m \text{Rep}_Q$ at $F \in \mathcal{F}^m(V)$ by $\mathcal{F}^m \mathcal{R}(F) = \text{Rep}_Q(F)$.

The fiber $\mathcal{R}(V)$ of $\mathcal{R}$ at $V \in \mathcal{V}$ is $\text{Rep}_Q(V)$. Also, $\text{Rep}_Q^{-}(V) = \mathcal{R}(V)^{\ast}$ and for $\overrightarrow{\mathcal{R}} = \text{Rep}_{Q^\text{op}}$ we have $\overrightarrow{\mathcal{R}}(V) = T^{\ast} \mathcal{R}(V)$. By representations on a sequence $V = (V_i)_{i=1}^m \in \mathcal{V}$, we mean a sequence of representations, say $\mathcal{R}(V) = \oplus_{k=1}^m \mathcal{R}(V_k)$.

A decomposition $f$ of $v \in \mathbb{N}[I]$ as $v_1 + \cdots + v_m$ gives the connected components $\mathcal{F}^f$ and $\mathcal{F}^m \mathcal{R}(f)$, given by $\text{dim}[\mathcal{G}_F(V)] = f$. The stabilizer $P$ of a chosen $F \in \mathcal{F}^f(V)$ is a parabolic in $G$ and then $\mathcal{F}^f \cong P \backslash G$.

Now, the $m$-step extension correspondence for $\mathcal{R}$ is
\[
\mathcal{R}^m \xleftarrow{P} \mathcal{F}^m \mathcal{R} \xrightarrow{q} \mathcal{R}
\]
where $p(V, a, F) = \mathcal{G}_F(V, a)$ and $q(V, a, F) = (V, a)$. The obvious splitting $\oplus_{k=1}^m p$ of $p$ is given by sending $(V_0, a_0)$ to $\oplus_{i=1}^m V_i, \oplus_{i=1}^m a_i, F)$ for $F_p = \oplus_{k=1}^m V_k$.

A filtration $F$ on vector spaces $A, B$ defines a filtration on $\text{Hom}(A, B)$, where operator $x$ is in $F_d$ if $x F_d A \subseteq F_{d-1} B$ for all $p$. In particular we get a filtration $F_d(A^\ast) = F_{-d-1}^\perp$ and the two filtrations on $\text{Hom}(A, B)^{\ast}$ coincide.

So, a filtration $F \in \mathcal{F}^m(V)$ induces a filtration on $\text{Rep}_Q(V) \subseteq \text{End}(\oplus_{i \in I} V^i)$ with $x \in F_d \text{Rep}(V)$ if $x F^p V^h \subseteq F^{p+d} V^{h''}$. Then $F_0 \text{Rep}_Q(V)$ is the space $\text{Rep}_Q(F)$ of representations compatible with $F$ and $\mathcal{G}_F^{\ast} \text{Rep}_Q(V) = \text{Rep}(\mathcal{G}_F V)$. Also, the two filtrations on $\text{Rep}_{Q^\text{op}}(V) = T^{\ast} \text{Rep}_Q(V)$ coincide.

3.2. Thom line bundles.

3.2.1. Classical Thom bundles for quivers. As $\text{Rep}_Q(V)$ is quadratic in $V$ we define its bilinear version $\oplus_{h \in H} \text{Hom}(V_1^h, V_2^{h''})$ for $V_i \in \mathcal{V}$. Let $v_i = \text{dim}(V_i)$ and denote $L = \text{GL}(V_1) \times \text{GL}(V_2)$. Over $\mathfrak{A}_L = \mathfrak{G}^{(v_1)} \times \mathfrak{G}^{(v_2)}$ we define the line bundle
\[
\mathcal{L}(Q, A)_{v_1, v_2} \overset{\text{def}}{=} \Theta_L(\text{Rep}_Q(V_1, V_2)).
\]

Lemma. (a) For a quiver $Q = (I, H)$ the Thom line bundle $L(Q, A) \overset{\text{def}}{=} \Theta(\text{Rep}_Q)$ is a local line bundle $O(-Q)$ on $\mathfrak{A}(\text{Rep}_Q) = \mathfrak{H}_{\mathfrak{G} \times I}$, corresponding to the incidence quadratic form $Q$ of the quiver.

(b) The line bundle $\mathcal{L}(Q, A)_{V_1, V_2}$ on $\mathfrak{G}^{(v_1)} \times \mathfrak{G}^{(v_2)}$ is bilinear in $V_1, V_2$ in the sense that for the addition map $S: \text{GL}(U') \times \text{GL}(U'') \hookrightarrow \text{GL}(U' \oplus U'')$ one has
\[
\mathcal{L}_{U', V} \boxtimes \mathcal{L}_{U'', V} \cong (\mathfrak{A}_S)^* (\mathcal{L}_{U' \oplus U'', V})
\]
and the same for $V$.

Proof. (a) For $V \in \mathcal{V}$ and $G = \text{GL}(V)$, as a $G$-module $\mathcal{R}(V) = \text{Rep}_Q(V)$ is $\oplus_H \mathcal{R}_h(V)$ for $\mathcal{R}_h(V) = \text{Hom}(V^h, V^{h''})$. The corresponding connected component of $\text{Rep}_Q$ is a vector bundle $G \backslash \text{Rep}_Q(V)$ over $\mathfrak{B}(G)$. Then the ideal $\Theta_G(\mathcal{R}_h)$ in $\mathfrak{A}_G$ is generated by the function $I_{\text{ch}(\mathcal{R}_h)}$ (defined in §1.2.1) corresponding to the character of $\mathcal{R}_h$.

A system of coordinates $x^i_s$ on each $V^i, i \in I$, gives a Cartan $T$ in $G$ such that a basis in $X^\ast(T)$ can be denoted by $x^i_s$. If $i \xrightarrow{h} j$ then the character of $\mathcal{R}_h$ is $\text{ch}(\mathcal{R}_h) = \sum_s \sum_t x_t^j (x_s^i)^{-1}$, hence $I_{\text{ch}(\mathcal{R}_h)} = \prod_{s, t} (1 - (x_t^j x_s^i)^{-1})$ and the same divisor is given by $\prod_{s, t} (1 - x_t^j x_s^i)$ which is the equation of the $(i, j)$-diagonal in $\mathfrak{A}_T \cong \prod_{i \in I} \mathfrak{G}^{\text{dim}(V^i)}$. 

(b) Hom\((U, V)\) is bilinear in \(U\) and \(V\). We use the obvious observation that if \(V_i\) is a module for \(G_i\) for \(1 \leq i \leq n\), then \(\Theta_{\prod G_i}(\bigoplus V_i) \cong \prod \Theta_{G_i}(V_i)\). By multiplicativity of \(\Theta\) this reduces to the claim that for a representation \(V\) of \(G\), \(\Theta_{G \times G'}(V \boxtimes k) = \Theta_G(V) \boxtimes \Omega_{\Delta_{G'}}\).

For this we can assume that \(G, G'\) are reductive and then they can be replaced by their Cartans \(T, T'\). Then we can also assume that \(V_i\) are characters \(\chi\) of \(T\). But then \(\text{Ker}(\chi \boxtimes k) = \text{Ker}(\chi) \times T'\), and this implies the claim.

\[\square\]

3.3. The cotangent versions of the extension diagram.

3.3.1. The (co)tangent functoriality. The tangent complex of a map of smooth spaces \(f : \mathcal{X} \to \mathcal{Y}\) is \(T(f) = [\mathcal{X}, df \rightarrow f^*\mathcal{Y}]_{-1,0}\) on \(\mathcal{X}\) and the dual cotangent complex is \(T^*(f) = [f^*\mathcal{Y} \rightarrow df^*\mathcal{X}]_{0,1}\).

When \(f\) is an embedding these are the (co)normal bundles \(T(f) \cong N(f)\) and \(T^*(f) = T^*_N\mathcal{Y} = N(f)^\ast\). The Thom line bundle of a map \(f\) is \(\Theta(f) = \text{def} \Theta(T(f)) = \Theta(f^*\mathcal{Y})\Theta(T\mathcal{X})^{-1}\). For the map \(f\) the direct image of \(A\)-cohomology takes the form of \(f_* : \Theta(f) \to A(\mathcal{Y})\) [GKV95].

The cotangent functoriality associates to \(f : \mathcal{X} \to \mathcal{Y}\) the correspondence

\[T^*\mathcal{Y} \rightarrow f^*T^*\mathcal{Y} \rightarrow df^*T^*\mathcal{X} \to \mathcal{X}.
\]

Therefore, any correspondence \(A^\times X \rightarrow C^\times Y\) of smooth spaces gives two cotangent correspondences \(T^*A^\times X \rightarrow C^\times Y\) and \(T^*C^\times Y \rightarrow A^\times X\) that compose to the correspondence \(p^*T^*A \times_{T^*C} q^*T^*B\).

\(\dagger\) Say, in the category of schemes this product consists of all \(c \in C, \alpha \in T^p(c) A, \beta \in T^q(c) B\) such that \(dp^c\alpha = dq^c\beta\), so by passing to \((c, \alpha, -\beta)\) we identify it with \(T^C_c (A \times B)\). Then the cotangent version of the original correspondence is \(T^*A^\times X \rightarrow C^\times Y \rightarrow C^\times A \times B \rightarrow C^\times (A \times B)\).

3.3.2. Stacks. If \(X\) is a smooth variety with an action of a group \(G\) then \(G \times X\) is a smooth stack whose tangent complex is \([g \rightarrow TX]_{-1,0}\) and the cotangent complex is \([T^*X \rightarrow g^*]\).

We will consider a map of smooth varieties \(X_1 \rightarrow X_2\) and \(G_1 \rightarrow G_2\) a compatible map of groups \(G_i\) acting on \(X_i\). Then for \(X_1 = G_1 \times X_1\) one gets \(F : \mathcal{X}_1 \to \mathcal{X}_2\). We will calculate its cotangent correspondence \(T^*\mathcal{X}_2 \leftarrow F^*T^*\mathcal{X}_2 \rightarrow dF^*T^*X_2\). First, the Thom line bundles for stacky versions are the equivariant Thom bundles for \(f\) plus a change of equivalence factor \(\Theta_{G_1}(g_1/g_2)\) defined as \(\Theta_{G_1}([g_1 \rightarrow g_2]_{0,1})\).

Lemma. (a) \(\Theta(F) = \Theta_{G_1}(f) \otimes \Theta_{G_1}(g_1/g_2)\).
(b) \(\Theta(dF^c) = \Theta_{G_1}(df^c) \otimes \Theta_{G_1}(g_1/g_2)\).
(c) \(\Theta(F) = \Theta_{G_1}(f) \otimes \Theta_{G_1}(g_1/g_2)\).

(d) The pull back map on cohomology \(A(T^*\mathcal{X}_2) \rightarrow A(F^*T^*\mathcal{X}_2)\) is the same as \(A(\mathcal{X}_2) \rightarrow A(\mathcal{X}_1)\).

Also, \((dF)^c\) is identity on \(\mathcal{X}_1\).

Proof. The (co)tangent complexes of spaces \(X_i\) are calculated by formulas \(T(G \times X) = G \backslash (TGX)\) and \(T^*(G \times X) = G^\ast (T_GX)\), where \(TGX = [g \times X \to TX]_{-1,0}\) and \(T^GX = [T^*X \to g^* X]_{0,1}\).

The map \(F\) gives pull backs \(F^*(T\mathcal{X}_2) = \frac{X_1 \times_{G_1} X_2}{G_2} \rightarrow \frac{T^*X_2}{G_2}\) and \(F^*(T^*\mathcal{X}_2) = \frac{F^*T^*X_2}{G_2}\).

(a) The tangent complex \(T(F) = [TX_1 \to F^*TX_2]_{-1,0}\) of the map \(F\), comes from (the \(G_1\)-quotient of) the map of complexes \([g_1 \to TX_1] \to [g_2 \to TX_2]\) given by \(TX_1 \rightarrow df^*TX_2\) and \(g_1 \rightarrow g_2\). When we view this as a bicomplex with horizontal and vertical degrees in \([-1, 0]\), then

\[\text{The fibered product has to be derived for the relevant base change to hold unless } d'p, d'q \text{ are transversal.}\]
$T(F)$ is its total complex $[g_1 \to TX_1 \oplus g_2 \to f^*TX_2]_{-2,0}$, which is an extension of complexes $[TX_1 \overset{df}{\to} f^*TX_2]_{-2,0} = T(f)$ and $[g_1 \to g_2]_{-2,1}$, so $\Theta(F)$ is as stated.

Now, the cotangent correspondence can be written as

$$
\begin{array}{ccc}
T^*X_2 & \xleftarrow{\tilde{F}} & F^*(T^*X_2) & \xrightarrow{d^*F} & T^*X_1 \\
\downarrow & & \downarrow & & \downarrow \\
T^*_G2X_2 & \leftarrow & f^*T^*_G2X_2 & \mapsto & T^*_G1X_1
\end{array}
$$

(b) Write $d^*F$ as $(d^*F)^\circ/G_1$ where $(d^*F)^\circ : f^*[T^*X_2 \to g_2^*] \to [T^*X_1 \to g_1^*]$ is a map of complexes viewed as a bicomplex with all horizontal and vertical degrees in $[-1,0]$. So, $T((d^*F)^\circ)$ is the total complex $[f^*[T^*X_2 \to T^*X_1 \oplus g_2^* \to g_1^*]_{-2,0}$ which is an extension of complexes $[f^*[T^*X_2 \xrightarrow{df} T^*X_1]_{-2,1}$ and $[g_2^* \to g_1^*]_{-1,0}$. So, $\Theta((d^*F)^\circ) = \Theta_{G_1}((d^*F)^\circ) = \Theta_{G_1}(d^*f) \otimes \Theta_{G_1}([g_2^* \to g_1^*]_{-1,0})$ and then we use invariance of the Thom line bundle under duality.

(c) Denote the complex $T^*_G2X_2 = [T^*X_2 \to g_2^*]_{0,1}$ by $\mathcal{V}$ and let $\eta : f^*\mathcal{V} \to \mathcal{V}$, then the map $\tilde{F}$ is given by $\eta$ and the change of symmetry $G_1 \to G_2$. So, part (a) says that $\Theta(\tilde{F}) = \Theta_{G_1}(\eta) \otimes \Theta_{G_1}([g_1 \to g_2]_{0,1})$. Let us denote $\pi : \mathcal{V} \to X_2$ and $\tau : f^*\mathcal{V} \to X_1$, then $T(\eta) = [T(f^*\mathcal{V}) \xrightarrow{d\eta} \eta^*T(\mathcal{V})]_{-1,0}$ is $\pi^*T(f)$. (One has $0 \to \tau^*\mathcal{V} \to T\mathcal{V} \to \pi^*TX_1 \to 0$ and $0 \to \pi^*f^*\mathcal{V} \to Tf^*\mathcal{V} \to \pi^*TX_2 \to 0$. Now the map of complexes is identity on subsheaves $\pi^*f^*\mathcal{V} \cong \eta^*\pi^*\mathcal{V}$ and what remains is $\pi^*TX_2 \to \eta^*\pi^*TX_1 = \pi^*f^*TX_1$.)

So, $\Theta_{G_1}(\eta) = \Theta_{G_1}(\pi^*T(f))$, and since $\pi$ is contractible this is $\Theta_{G_1}(f)$.

(d) After contracting complexes of vector bundles the maps $\tilde{F}$ and $d^*F$ become respectively the map $X_2/G_2 \xleftarrow{\tilde{F}} X_1/G_1$ and the identity on $X_1/G_1$.

\[\square\]

3.4. The $A$-Cohomology of the cotangent correspondence for extensions. We recall the construction of [YZ14] of a quantum group in the above set up. It originated from the study of affine quantum groups in [Nak01] and [SV13], and is closely related to [KS11].

3.4.1. Connected components of the cotangent correspondence. Fixing $V \in \mathcal{V}^I$ and $F \in \mathcal{F}^m(V)$, let $\text{Gr}_F(V) = \bigoplus_{k=1}^m V_k$. We denote $G = GL(V)$, $L = \prod_{k=1}^m GL(V_k)$ the automorphism group of $\text{Gr}_F(V)$, and $P$ is a parabolic subgroup of $G$ with a Levi subgroup $L$. Let $U$ be the unipotent radical of $P$. Denote the Lie algebras by $\mathfrak{p}, \mathfrak{l}, \mathfrak{u}$.

These choices fix the connected component of the correspondence $\mathcal{R}^m \xleftarrow{\tilde{p}} \mathcal{F}^m \mathcal{R}^m \xrightarrow{q} \mathcal{R} \to \mathcal{V}^I$ given by

\[L \text{ \backslash } \mathcal{R}(\text{Gr}_F(V)) \xleftarrow{\tilde{p}} P \text{ \backslash } \mathcal{R}(F) \xrightarrow{q} G \text{ \backslash } \mathcal{R}(V) \to \mathcal{V}^I \text{ \backslash } \text{pt}.
\]

3.4.2. Line bundles from the cotangent correspondence. The extension correspondence gives two cotangent correspondences

\[T^*\mathcal{R}^m \xleftarrow{\tilde{p}} p^*T^*\mathcal{R}^m \xrightarrow{d^*p} T^*\mathcal{F}^m \mathcal{R}^m \xrightarrow{d^*q} q^*T^*\mathcal{R}^m \xrightarrow{\tilde{q}} T^*\mathcal{R}.
\]

These compose to a single correspondence as in 3.3.1 which is the \textit{cotangent correspondence} of the extension correspondence. We will not consider it since we are calculating here its effect on cohomology and this is the composition of effects of the above two simpler correspondences.

Let $V \in \mathcal{V}^I$ and $F \in \mathcal{F}^m(V)$. We write the fiber of the correspondence (1) over $F$ as

$$
\mathcal{R}(\text{Gr}_F(V)) \xleftarrow{\tilde{p}} \mathcal{R}(F) \xrightarrow{\tilde{q}} \mathcal{R}(V).
$$
The connected component of the diagram (2) determined by $F$ takes the form (3)
$$T^* (L \setminus \mathcal{R}^m (Gr F)) \xleftarrow{\bar{p}} p^* T^* (L \setminus \mathcal{R}^m (Gr F)) \xrightarrow{\text{def}} p^* T^* (P \setminus \mathcal{F}^m \mathcal{R}(F)) \xleftarrow{\text{def}} q^* T^* (G \setminus \mathcal{R}(V)) \xrightarrow{\bar{q}} T^* (G \setminus \mathcal{R}(V)).$$

**Lemma.** With notations as above (and the filtration on $\mathcal{R}(F)$ as in § 3.1.2) we have
$$\Theta (d^* p) \cong \Theta_L (g/p) \otimes \Theta_L (F^{-1} \mathcal{R}(F)) \quad \text{and} \quad \Theta (\bar{q}) \cong \Theta_{\mathcal{L}^-} [\mathcal{R}(V)/\mathcal{R}(F)] \otimes \Theta_L (g/p)^{-1}.$$

**Proof.** According to the lemma 3.3.2.b $\Theta (d^* p)$ is $\Theta_L (d^* \bar{p}) \otimes \Theta_L ([p \to \mathfrak{l}]_{0,1})$. The second factor is $\Theta_L (u)$, since $u \cong (g/p)^*$, we can write it as $\Theta_L (g/p)$. For the first factor, as $\bar{p} : \mathcal{R}(F) \to \mathcal{R}(Gr F)$ we get $d^* \bar{p} : \mathcal{R}(F) \times \mathcal{R}(Gr F) \to \mathcal{R}(F) \to \mathcal{R}(F)^*$, so up to a factor $\mathcal{R}(F)$ this is $\mathcal{R}(Gr F) \cong \mathcal{R}(F)^*$ with the quotient $[F^{-1} \mathcal{R}(F)]^* = \Theta_{\mathcal{L}^-} [\mathcal{R}(V)/\mathcal{R}(F)]$. So, the first factor is $\Theta_{\mathcal{L}^-} [\mathcal{R}(V)/\mathcal{R}(F)]$ and the second is $\Theta_L (g/p)^{-1}$.

3.4.3. **Quantization by dilatations.** Recall the action of the dilation torus $\mathcal{D} \subseteq \mathbb{G}_m^n$ from § 3.1.1. The weight of the first $\mathbb{G}_m$-factor on $\mathcal{R}$ is prescribed by $m$ while the second factor acts trivially. Then the $\mathcal{D}$-action on $T^* \mathcal{R}$ is uniquely determined by asking that the natural symplectic form on $T^* \mathcal{R}$ has weight $t_1 t_2$. We denote the $\mathcal{D}$-character of weight $t_1 t_2$ by $\omega$, so that the $\mathcal{D}$-action on $T^* \mathcal{R}$ is twisted by $\omega$. This gives rise to the following twisted version of (3),

$$T^* (L \setminus \mathcal{R}^m (Gr F)) \otimes \omega \xleftarrow{\bar{p}} p^* T^* (L \setminus \mathcal{R}^m (Gr F)) \otimes \omega \xrightarrow{d^* \bar{p}} p^* T^* (P \setminus \mathcal{F}^m \mathcal{R}(F)) \otimes \omega \xleftarrow{d^* q} q^* T^* (G \setminus \mathcal{R}(V)) \otimes \omega \xrightarrow{\bar{q}} T^* (G \setminus \mathcal{R}(V)) \otimes \omega.
$$

The maps in the above diagram are equivariant with respect to $\mathcal{D}$.

Now we analyze $\mathcal{D}$-action on the relative tangent complexes of $F$, $d^* F$, and $\bar{F}$. Lemma 3.3.2 applies to induced actions on cotangent bundles. When working $\mathcal{D}$-equivariantly we need to add an $\omega$-twist. This applies to the Lie algebra factors in Lemma 3.3.2, that come from the cotangent complexes. On the other hand, the Lie algebra factors that come from the change of symmetry are not affected as they only carry the adjoint action. To simplify notations, for any group $\mathcal{H}$, we denote $H \times \mathcal{D}$ by $\bar{H}$. Then

$$\Theta_{\mathcal{D}} (d^* F) \cong \Theta_{\bar{G}_1} (d^* f) \otimes \Theta_{\bar{G}_1} ((g_1/g_2) \otimes \omega) \quad \text{and} \quad \Theta_{\mathcal{D}} (F) = \Theta_{\bar{G}_1} (f) \otimes \Theta_{\bar{G}_1} (g_1/g_2).$$

Therefore, $\Theta_{\mathcal{D}} (\bar{F}) = \Theta_{\bar{G}_1} (f) \otimes \Theta_{\bar{G}_1} (g_1/g_2)$.

**Lemma.** With notations above: $\Theta (d^* p) \cong \Theta_{\mathcal{L}^-} (g/p \otimes \omega) \otimes \Theta_{\mathcal{L}^-} (F^{-1} \mathcal{R}(F))$ and $\Theta (\bar{q}) \cong \Theta_{\mathcal{L}^-} [\mathcal{R}(V)/\mathcal{R}(F)] \otimes (g/p)^{-1}$.

**Proof.** According to the lemma 3.3.2.b $\Theta (d^* p)$ is $\Theta_{\mathcal{L}^-} (d^* \bar{p}) \otimes \Theta_{\mathcal{L}^-} ([p \to \mathfrak{l}]_{0,1} \otimes \omega)$. The second factor is $\Theta_{\mathcal{L}^-} (u \otimes \omega)$, since $u \cong (g/p)^*$ we can write it as $\Theta_{\mathcal{L}^-} (g/p \otimes \omega)$.

Again, by lemma 3.3.2.c, $\Theta (\bar{q})$ is $\Theta_{\bar{G}_1} ([q] \otimes \Theta_{\bar{G}_1} ([p \to \mathfrak{l}]_{0,1}$). So, the first factor is $\Theta_{\mathcal{L}^-} [\mathcal{R}(V)/\mathcal{R}(F)]$ and the second is $\Theta_{\mathcal{L}^-} (g/p)^{-1}$, as it comes from the change of symmetries.
3.5. $D$-quantization of the monoid $(\mathcal{H}_{G \times I}, +)$. Here we recall the construction from [YZ17] of a deformation $(\text{Coh}(\mathcal{H}_{G \times I}), \star)$ of the convolution on the monoid $(\mathcal{H}_{G \times I}, +)$. The quantum group $U_D(Q, A)$ and its positive part $U_D^+(Q, A)$ were constructed in [YZ14, YZ16], as algebra objects in $(\text{Coh}(\mathcal{H}_{G \times I}), \star)$, and hence in particular as $R$-algebras.

3.5.1. Local and biextension line bundles $L_D(Q, A)$ and $L_D(Q, A)$. These will be upgrades of $L(Q, A)$ and $L(Q, A)$ from §3.2.1. They will be constructed as special cases of line bundles associates to cotangent correspondences of extension moduli (3).

**Case 1:** The biextension line bundle $L = L_D(Q, A)$ comes from $m = 2$, i.e., the 2-step filtrations $F^2(V)$ of $V$. For $\text{Gr}_F(V) = V_1 \oplus V_2$

$$L_{V_1, V_2} := \Theta(d^* p) \otimes \Theta(q)$$

is a line bundle on $\mathfrak{A}_D \cong \mathfrak{A}_{G(V_1)} \times \mathfrak{A}_{G(V_2)} \times \mathfrak{A}_D$.

**Case 2:** Our quantum version $L = L_D(Q, A)$ of the local line bundle $L(Q, A)$ depends on a choice of a type of a complete flag $F \in F^m(V)$ which is $f = \dim(\text{Gr}_F(V)) \in (\mathbb{N}^I)^m$. Then

$$[L_D(Q, A)]_{V, f} \overset{\text{def}}{=} \Theta(d^* p) \otimes \Theta(q)$$

is a line bundle on $\mathfrak{A}_D \cong G[V] \times \mathfrak{A}_D$, where $|V| = \sum_{i \in I} \dim(V_i)$ (here the Levi subgroup $L$ is a Cartan in $GL(V)$). It is called the local line bundle.

One easily sees that the restrictions of “quantum objects” $L_D(Q, A)$ and $L_D(Q, A)$ to $0 \in \mathfrak{A}_D$ are the classical Thom line bundles $L(Q, A)$ and $L(Q, A)$ from §3.2.1.

3.5.2. Convolutions and bieXTensions. We recall the monoidal structure $\star$ on coherent sheaves on $\mathcal{H}_{G \times I} \times \mathfrak{A}_D$ (over the base scheme $\mathfrak{A}_D$) from [YZ17].

For a smooth curve $C$, $\mathcal{H}_{C \times I}$ is a commutative monoid freely generated by $C$. The operation $S : \mathcal{H}_{C \times I} \times \mathcal{H}_{C \times I} \to \mathcal{H}_{C \times I}$ is the addition of divisors (“symmetrization”). Since it is a finite map it defines a convolution operation on the abelian category $\text{Coh}(\mathcal{H}_{C \times I})$ of coherent sheaves by $F \star G \overset{\text{def}}{=} S_*(\mathcal{F} \boxtimes \mathcal{G})$.

A biextension of $(\mathcal{H}_{C \times I}, +)$ is a line bundle $L$ over $(\mathcal{H}_{C \times I})^2$ with consistent bilinearity constraints $L_{A^i + A^j, B} \cong L_{A^i, B} \boxtimes L_{A^j, B}$ (and the same for the second variable). This is equivalent to a central extension of the monoid $(\mathcal{H}_{C \times I}, +)$ (or its group completion) by $\mathbb{G}_m$. Now $L$ twists the convolution on $\text{Coh}(\mathcal{H}_{C \times I})$ to another monoidal structure $F \star G \overset{\text{def}}{=} S_*(\mathcal{F} \boxtimes \mathcal{G}) \otimes L$.

From now on the curve $C$ will be $G = \mathfrak{A}_{\mathbb{G}_m}$.

**Lemma.** The line bundle $L = L_D(Q, A)$ on $(\mathcal{H}_{C \times I})^2 \times \mathfrak{A}_D$ defined in §3.5.1.1 is an $\mathfrak{A}_D$-family of biextension line bundles. This gives a “$D$-twisted” convolution on $\text{Coh}(\mathcal{H}_{G \times I} \times \mathfrak{A}_D)$ by

$$F \star G \overset{\text{def}}{=} S_*[(\mathcal{F} \boxtimes \mathcal{G}) \otimes L].$$

**Proof.** We need to check that the quantum version of $L$ is still a biextension. Notice that the quantum version, has an extra factor $\Theta_D(g/p)$. However since for $m = 2$, the space $g/p$ is of the form $\text{Hom}(V_1, V_2)$, the argument in the proof of Lemma 3.2.1.b applies again. $\square$

**Proposition.** [YZ17, Theorem A, Theorem 3.1]

(a) $(\text{Coh}(\mathcal{H}_{G \times I} \times \mathfrak{A}_D), \star)$ is a monoidal category with a meromorphic braiding which is symmetric.

The unit is the structure sheaf on $\mathcal{H}_{G \times I} \times \mathfrak{A}_D$.

(b) The structure sheaf on $\mathcal{H}_{G \times I} \times \mathfrak{A}_D$ is an algebra object in this category.

For any $\tau \in \mathfrak{A}_D$, we denote by $L_\tau$ the restriction of $L$ to $\tau \in \mathfrak{A}_D$, and $F \star_\tau G := S_*[(\mathcal{F} \boxtimes \mathcal{G}) \otimes L_\tau]$. 
Remark. One way to motivate the $L$-twisted convolution of coherent sheaves on $(\mathcal{H}_{G \times I})^2 \times \mathfrak{A}_D$ is to notice that when the cohomology theory $A$ extends to constructible sheaves, then for a constructible $F$ on a space $X$, the cohomology $A(F)$ is a coherent sheaf on $\mathfrak{A}(X)$. In this case the $A$-cohomology functor intertwines the convolution of constructible sheaves on $\text{Rep}_Q$ and the $L$-twisted convolution of coherent sheaves on $\mathfrak{A}(	ext{Rep}_Q) = (\mathcal{H}_{G \times I})^2 \times \mathfrak{A}_D$. (This follows as in the proof of lemma 3.4.2.)

3.5.3. Quantum groups $U_D^+(Q, A) \subset U_D(Q, A)$. Now we consider the set up of § 3.4.2 with $V \in \mathcal{V}$ and $F \in \mathcal{F}^m(V)$. Let $f = \dim(\text{Gr}_F(V)) \in (\mathbb{N}^i)^m$ be the type of the filtration $F$. Applying the cohomology theory $A$ to the diagram (3), we have the following multiplication map associated to $f$:

\[(5) \quad m_f := (\tilde{q}_s \circ (d^s q^*) \circ (d^s p_s) \circ (\tilde{p}^*) : \mathcal{S}_s(\Theta(\tilde{q}) \otimes \Theta(d^s p)) \to A(T^* G \setminus \mathcal{R}(V)) \cong \mathcal{O}_{\mathfrak{A}_G \times \mathfrak{A}_D}, \]

where $\mathcal{S} : \mathfrak{A}_L \to \mathfrak{A}_G$ is the symmetrization map.

Let $\mathcal{S}p_h(V)$ be the set of types $v$ of filtrations in $\mathcal{F}^m(V)$ consisting of complete flags (so $m = |v| = \sum_{i \in I} v^i$). We define $U_D^+(Q, A)$ so that on the connected component $\mathfrak{A}_G \times \mathfrak{A}_D$,

\[ (U_D^+(Q, A))_V \overset{\text{def}}{=} \sum_{f \in \mathcal{S}p_h(V)} \text{Image}(m_f) \subseteq A_G = \mathcal{O}_{\mathfrak{A}_G \times \mathfrak{A}_D}. \]

Lemma.

(1) The coherent sheaf $U_D^+(Q, A)$ is an ideal sheaf on $\mathfrak{A}_G \times \mathfrak{A}_D$.

(2) $U_D^+(Q, A)$ is an $\mathfrak{A}_D$-family of algebras in the monoidal categories $(\text{Coh}(\mathcal{H}_{G \times I}), *_{\tau})$.

Proof. As each $m_f$, for $f \in \mathcal{S}p_h(V)$, is a morphism of coherent sheaves, the image $\text{Image}(m_f)$ is a coherent subsheaf in $\mathcal{O}_{\mathfrak{A}_G \times \mathfrak{A}_D}$. Since $\mathcal{S}p_h(V)$ is a finite set, $(U_D^+(Q, A))_V$ is a sum of finitely many coherent subsheaves, so it is itself a coherent subsheaf of $\mathcal{O}_{\mathfrak{A}_G \times \mathfrak{A}_D}$. A coherent subsheaf of the structure sheaf is a sheaf of ideals, hence so is $(U_D^+(Q, A))_V$.

For (2), the algebra structure on $(U_D^+(Q, A))_V$ is defined using $m_f$, where $F$ is the 2-step filtrations in §3.5.1 Case 1. \hfill \square

The sheaf $U_D^+(Q, A)$ on $\mathfrak{A}_G \times \mathfrak{A}_D$ is denoted by $\mathcal{P}^{\text{coh}}$ in [YZ14], since it is the spherical subalgebra of the cohomological Hall algebra of preprojective algebra.

The affine quantum group $U_D(Q, A)$ associated to the quiver $Q$ and the cohomology theory $A$ is defined in [YZ16] as the Drinfeld double of $U_D^+(Q, A)$. The quantization parameters of $U_D(Q, A)$ are given by $\mathfrak{A}_D$. This Drinfeld double was constructed in [YZ16] using a comultiplication and a bialgebra pairing on an extended version of $U_D^+(Q, A)$. $U_D^+(Q, A)$ itself also has a coproduct but in the meromorphic braided tensor category $(\text{Coh}(\mathcal{H}_{G \times I}), *)$ [YZ17]. The affine quantum group $U_D(Q, A)$ acts on the corresponding $A$-homology of the Nakajima quiver varieties (see [YZ14, YZ16]), generalizing a construction of Nakajima [Nak01].

4. Loop Grassmannians $G_D^+(Q, A)$ and quantum locality

In the preceding section 3 we have attached to a quiver $Q = (I, H)$ and a cohomology theory $A$, a local line bundle $L(Q, A)$ on the colored configuration space $\mathcal{H}_{G \times I}$ of the curve $G$ given by $A$ (lemma 3.2.1.a). As in 2.4 the local line bundle $L(Q, A)$ can be used to produce a “loop Grassmannian” $\mathcal{G}(Q, A)$ over $\mathcal{H}_{G \times I}$.

The local line bundle $L(Q, A)$ is closely related to the biextension line bundle $L(Q, A)$ from lemma 3.2.1.b. In section 3 we have also recalled the construction of the affine quantum groups $U_D^+(Q, A)$ and used this to select the “correct” quantizations $L_D(Q, A)$ and $L_D(Q, A)$ of the above line bundles, on the basis of relation to this quantum group (3.5.1).
While pieces $L(Q,A)_\alpha$ of the classical local line bundle depend on $\alpha \in \mathbb{N}[I]$ parameterising connected components of $\mathcal{H}_{G \times I}$, the pieces $L_D(Q,A)_i$ of the quantum version depends on a choice of $i \in I^N$ (3.5.1). This really means that we are dealing with the non-commutative (ordered) configuration spaces $\mathcal{C} = \mathcal{C}_{G \times I} = \bigsqcup (G \times I)^n$, so that each $\alpha \in \mathbb{N}[I]$ is refined to all $i = (i_1, \ldots, i_n) \in I^n$ with $\sum i_p = \alpha$. The connected components given by all refinements $i$ of the same $\alpha$ are related by the meromorphic braiding from [YZ17]. So, the information carried by all refinements $i$ of $\alpha$ is (only) generically equivalent.

All together, $\mathcal{G}_D(Q,A)$ can still be constructed by the same prescription as in the case of $\mathcal{G}(Q,A)$. However, the local line bundle $L_D(Q,A)$ now lives on the larger (“non-commutative”) configuration space $\mathcal{C}_{G \times I}$. The zastava space $Z_D(Q,A)$ over $\mathcal{C} = \mathcal{C}_{G \times I}$ is first defined generically in $C$ where fibers are products of projective lines. Then the singularities of the locality structure preserves how fibers degenerate. Finally, passing from the zastava space to loop Grassmannian is given by the procedure of extending the free monoid on $I$ to the free group on $I$.

All together, the key difference in the quantum case is seen in the configuration space. It has more connected components (but they are related by braiding), and the singularities of locality structure (hence also the notion of locality) are now the diagonals shifted by the quantum parameter.

4.0.1. The “classical” loop Grassmannians $G^P(Q,A)$. The choice of $A$ influences the space $G^P(Q,A)$ only through the curve $G$. Whenever $G$ is a formal group, then the orientation $I$ of $A$ identifies $G$ with the coordinatized formal disc $d$.

However, since the loop Grassmannian $\mathbb{G}(\mathbb{G}_m)$ is the free commutative group indscheme generated by $d$ the group law on $d$ given by $A$ induces a commutative ring structure on the loop Grassmannian $\mathbb{G}(\mathbb{G}_m)$. This is the group algebra of the group $G$ taken in algebraic geometry.

Remark. The universal Witt ring has the same nature, it is the homology $\mathbb{H}_*(\mathbb{A}^1, 0)$ of the multiplicative monoid $(\mathbb{A}^1, 0, \cdot)$ in pointed spaces. Observations of this nature have already been made in [BZ95, Str00, No09].

4.1. Quantization shifts diagonals. Any Thom line bundle is the ideal sheaf of the corresponding Thom divisor. While the Thom divisor corresponding to $L(Q,A)$ is a combination of diagonals of $\mathcal{H} = \mathcal{H}_{d \times I}$, the quantization shifts these diagonals in the configuration space $\mathcal{C} = \mathcal{C}_{d \times I}$.

We first examine how an added action of a torus $D$ affects the Thom divisor in general (4.1.1), and then we specialize this to the local line bundle $L_D(Q,A)$ in 4.1.2.

4.1.1. Deformation of a Thom divisor from an additional torus $D$. For a representation $E$ of a product $G = G \times D$ we can view the line bundle $\Theta_G(E)$ on $\mathfrak{A}_G \times \mathfrak{A}_D$ as a family of line bundles $\Theta_G(E)_\tau$ (for $\tau \in \mathfrak{A}_D$), on $\mathfrak{A}_G \rightarrow \mathfrak{A}_G \times \mathfrak{A}_D$. If $E$ contains no trivial characters of a Cartan $T$, we will see that this deformation lifts to divisors.

First, consider the case when $G$ is a torus $T$ and $E = \chi \otimes \zeta^{-1}$ for characters $\chi, \zeta$ of $T,D$ (so $\chi \neq 0$). Then for any $\tau \in \mathfrak{A}_D$, the restriction $\Theta_T(E)_\tau$ to $\mathfrak{A}_T$ is the ideal sheaf of the divisor

$$\text{Ker}(\mathfrak{A}_T^{\chi \otimes \zeta^{-1}}) \cap [\mathfrak{A}_T \times \tau] = \mathfrak{A}_T^{-1}(\mathfrak{A}_T^{\chi}(\tau)) \subset \mathfrak{A}_T.$$ 

Here $\chi : T \rightarrow \mathbb{G}_m$ induces the homomorphism $\mathfrak{A}_T^\chi : \mathfrak{A}_T \rightarrow \mathfrak{A}_\mathbb{G}_m = \mathbb{G}$ as in §1.2.1, and $\mathfrak{A}_T^{-1}(\mathfrak{A}_T^{\chi}(\tau))$ is a divisor in $\mathfrak{A}_T$. For $\tau = 0$ this is the divisor $\text{Ker}(\mathfrak{A}_T^\chi)$ whose ideal sheaf is $\Theta_T(\chi)$ and in general $\mathfrak{A}_T^{-1}(\mathfrak{A}_T^{\chi}(\tau))$ is its torsor which we think of as a shift of $\text{Ker}(\mathfrak{A}_T^\chi) = \mathfrak{A}_T^{-1}(0)$ by $\mathfrak{A}_T^{\chi}(\tau) \in \mathbb{G}$.

Now for any reductive group $G$ with a Cartan $T$ and Weyl group $W$, we decompose $E$ according to $D$-action as $E = \mathcal{P} \oplus_{\zeta \in X^+(T)} (E_\zeta \otimes \zeta^{-1})$, for some $G$-modules $E_\zeta$. Then $\Theta_G(E_\zeta)$ is the ideal sheaf of some divisor, denoted by $D(E_\zeta)$, in $\mathfrak{A}_G = \mathfrak{A}_T//W$. As $T$-representations, we have the decomposition $E_\zeta|_T = \mathcal{P} \oplus_{\chi \in X^+(T)} [E_\chi : \chi]$. Therefore, the divisor $D(E_\zeta)$ is a sum over $\chi \in X^+(T)$
of divisors $[E_\xi : \chi] \cdot \text{Ker}(\mathcal{A}_\chi)$. Now, for any $\tau \in \mathcal{A}_D$, $\Theta^D_E(\tau)$ is the ideal sheaf of the shifted divisors $D(E_\xi) + \mathcal{A}_\chi(\tau)$ of $D(E_\xi)$.

4.1.2. Quantum diagonals. In our quiver setting, each $h \in H \sqcup H^*$ defines (via the Nakajima function $\mathbf{m}$) a character $\mu_h \in X^*(D)$, by which $D$ acts on the component $\text{Rep}_G(V)_h$ of $\text{Rep}_G(V)$.

For $1_V, 2_V$ in $V^I$ for each $i \in I$ choose coordinates $s_{x^i_p}$ on $s_{V^i}$ hence a decomposition of $s_{V^i}$ into lines $s_{V^i_p}$. This gives Cartans $T_s \subseteq G_s = GL(s_V)$ with a basis $s_{x^i_p}$ of $X^*(T_s)$. Then on the line $\text{Hom}(1_{V^i_p} \rightarrow V^j)$, the torus $\tilde{T}$ defined $T_1 \times T_2 \times D$ acts by $2x^i_p(1x^i_p)^{-1} \cdot \mu_h$, so its Thom divisor is given by vanishing of $\mathfrak{A}_{2x^i_p} + \mathfrak{A}_{i_{1x^i_p}}$ in $\mathfrak{A}_{T_1 \times T_2 \times D}$. Therefore, the Thom divisor of the $T_1 \times T_2 \times D$-module $\text{Rep}_G(1_{V^2} V)_h$ is the shifted diagonal

$$\Delta^{v_1,v_2}_h(\tau) \overset{\text{def}}{=} \Delta^{v_1,v_2}_{h',h''} + (0, \tau_h) \subset \mathfrak{A}_{T_1} \times \mathfrak{A}_{T_2}.$$  

Here $\tau_h = \mathfrak{A}_{\mu_h}(t)$ depends on $h$, and $\Delta^{v_1,v_2}_{h',h''} \subset \mathfrak{A}_{T_1} \times \mathfrak{A}_{T_2}$ is the diagonal divisor defined by vanishing of $\prod_{p,q}(\mathfrak{A}_{x^i_p} - \mathfrak{A}_{x^i_q})$, and the shift $\Delta^{v_1,v_2}_{h',h''} + (0, \tau_h)$ means that for $1_{V^j}$ we use the embedding of $G = \mathfrak{A}_{G_m}$ into $\mathfrak{A}_{GL(2V^j)}$ via $G_m = Z(GL(2V^j))$ and the corresponding addition action of $G$ on $\mathfrak{A}_{GL(2V^j)}$.

Consider the diagonal $\Delta^{v_1,v_2}_h$ of $G_G^{v_1} \times G^{v_2}$ given by the vanishing of $\prod_{p,q}(\mathfrak{A}_{x^i_p} - \mathfrak{A}_{x^i_q})$. Let $\Delta^{v_1,v_2}_h(\tau) \overset{\text{def}}{=} \Delta^{v_1,v_2}_h + (0, \tau)$, where $\tau = \mathfrak{A}_\omega(t)$. The character $\omega \in X^*(D)$ is as before. The shift $\Delta^{v_1,v_2}_{h',h''} + (0, \tau)$ means that for $1_{V^j}$ we use the embedding of $G = \mathfrak{A}_{G_m}$ into $\mathfrak{A}_{GL(1V^j)}$ via $G_m = Z(GL(1V^j))$ and the corresponding addition action of $G$ on $\mathfrak{A}_{GL(1V^j)}$, given by the vanishing of $\prod_{p,q}(\mathfrak{A}_{x^i_p} - \mathfrak{A}_{x^i_q})$.

We will say that for $\tau \in \mathfrak{A}_D$, and $D_s = (D^i_{s,j})_{j \in I} \in \mathfrak{G}^{[s]}$ for $s = 1, 2$; the pair $(D_1, D_2)$ is $\mathbf{m}$-disjoint if $(D_1, D_2, \tau)$ and $(D_2, D_1, \tau)$ do not lie in any of the shifted diagonals $\Delta^{v_1,v_2}_h(\tau)$, $\Delta^{v_1,v_2}_h, \Delta^{v_1,v_2}_h(\tau)$. Equivalently, for any $i \in I$, the divisors $D^i_1 \pm \tau$ and $D^i_2$ are disjoint, $D^i_1$ and $D^i_2$ are disjoint; for each $h : h' \rightarrow h''$ in $\overline{T}$, $D^{h''}_1 \pm \tau$ and $D^{h'}_1$ are disjoint.

4.2. Quantum locality. We will now consider locality in the setting of the (non-commutative) monoid $C_{G \times I}$ freely generated by $G \times I$.

Let $C_I$ be the free monoid on $I$, so elements are ordered sequences $\gamma = i_1 i_2 \cdots i_N$ of elements in $I$. The product of $\gamma = i_1 i_2 \cdots i_N, \gamma' = j_1 j_2 \cdots j_N'$, is the concatenation $\gamma + \gamma' = i_1 i_2 \cdots i_N j_1 j_2 \cdots j_N'$.

Let $C_{G \times I} = \bigcup (G \times I)^n = \bigcup_{\gamma \in C_I} G^\gamma$ be the ind-scheme monoid freely generated by $G \times I$, with connected components labeled by $C_I$. The natural projection, from the free monoid to the free commutative monoid is denoted $\varpi : C_{G \times I} \rightarrow \mathcal{H}_{G \times I}$.

We will use the notation $\mathcal{L} = \mathcal{L}_D(Q, A)$ both for the biextension line bundle defined on $\mathcal{H}_{G \times I}^2 \times \mathfrak{A}_D$ in §3.5.1 and also for its pull back to $C_{G \times I}^2 \times \mathfrak{A}_D$. For $\gamma', \gamma'' \in C_I$, we denote by $\mathcal{L}_{\gamma', \gamma''}$ its restriction to the component $G^{\gamma'} \otimes G^{\gamma''} \times \mathfrak{A}_D$.

4.2.1. $m$-locality. An $m$-locality structure on a vector bundle $K$ on $C_{G \times I} \times \mathfrak{A}_D$ is a consistent system of isomorphisms

$$K_{\gamma_1, \tau} \otimes K_{\gamma_2, \tau} \otimes \mathcal{L}_{\gamma_1, \gamma_2} \cong K_{\gamma_1 + \gamma_2, \tau} \text{ for } \tau \in \mathfrak{A}_D.$$
Any \( m \)-locality structure on \( K \) implies an algebra structure on \( K \) in the monoidal category \( \mathcal{C}(\mathcal{C}_{G \times I} \times \mathfrak{A}_D) \) (by the biextension property of \( \mathcal{L} \)). In this way an \( m \)-locality structure on \( K \) is the same as a structure of a \( \ast \)-algebra, whose multiplications are isomorphisms.\(^{17}\)

**Example.** The line bundle \( L = L_D(Q, A) \) constructed component-wise in §3.5.1 **Case 2**, is a line bundle over \( \mathcal{C}_{G \times I} \times \mathfrak{A}_D \) and it has a natural \( m \)-locality structure. We will write the proof only generically:

**Lemma.** The line bundle \( L \) on \( \mathcal{C}_{G \times I} \times \mathfrak{A}_D \) defined in §3.5.1 **Case 2** has the property that \( (D_1, D_2) \in (\mathcal{C}_{G \times I})^2 \) is \((\mathfrak{m}, \tau)\)-disjoint for \( \tau \in \mathfrak{A}_D \), then there is a canonical identification of fibers

\[
L_{D_1 + D_2, \tau} \cong L_{D_1, \tau} \otimes L_{D_2, \tau}.
\]

**Proof.** Let \( V = V_1 \oplus V_2 \) in \( \mathcal{V} \) and \( G_i = GL(V_i) \) and \( G = GL(V) \). Choose a Cartan \( T_i \) in \( G_i \). Then

\[
\text{Rep} G(V) \cong \text{Rep} G(V_1) \oplus \text{Rep} G(V_2) \oplus \text{Rep} G(V_1, V_2) \oplus \text{Rep} G(V_2, V_1)
\]

gives

\[
\Theta_G[\text{Rep} G(V)] \otimes \Theta_G[\text{Rep} G(V_1)]^{-1} \otimes \Theta_G[\text{Rep} G(V_2)]^{-1} \cong \Theta_G[\text{Rep} G(V_1, V_2)] \otimes \Theta_G[\text{Rep} G(V_2, V_1)].
\]

Now the disjointness condition implies that the last two factors have canonical trivializations at \((D_1, D_2, \tau)\). A similar statement holds for \( \Theta_G(\mathfrak{g}/\mathfrak{p} \otimes \omega) \), and \( \Theta_G(\mathfrak{g}/\mathfrak{p})^{-1} \), where \( \mathfrak{g}/\mathfrak{p} = \oplus_{i=1}^2 \mathfrak{g}_i/\mathfrak{p}_i \).

Therefore, \( L_{D_1, D_2} = \Theta(d^p) \otimes \Theta(q) \) has a canonical trivialization when \((D_1, D_2)\) is \((\mathfrak{m}, \tau)\)-disjoint. The claim now follows from the identification \( L_{D_1 + D_2, \tau} \cong L_{D_1, \tau} \otimes L_{D_2, \tau} \).

**Remark.** The quantum local line bundle \( L \) is in a sense a localization of the quantum group \( U^+_D(Q, A) \) to the noncommutative configuration space \( \mathcal{C}_{G \times I} \). By its definition the \( \alpha \)-weight space \( U^+_D(Q, A)(\alpha) \) is a sum of contributions from all refinements \( \gamma \in \mathcal{C}_I \) of a given \( \alpha \in \mathbb{N}[I] \).\(^{18}\)

In the classical case \( D = 1 \), for all \( \gamma \) above \( \alpha \) \( \gamma \) are the same, so the sum \( U^+_D(Q, A) \) is the line bundle \( L \). However, upon quantization there is a genuine dependence on \( \gamma \) and one has to take the sum of all contributions in order to construct a subalgebra.

**Example.** In the case when \( I \) is a point (the “sl2-case”) then \( C_I = \mathbb{N}[I] = \mathbb{N} \) hence \( \mathcal{C}_{G \times I} \) is the system \( \sqcup_{n \in \mathbb{N}} \mathbb{G}^n \) of Cartesian powers of \( \mathbb{G} \). Then \( L_n = \mathcal{L}^n_n(U^+_D(Q, A)_n) \).

### 4.2.2. Some expectations

The above construction of loop Grassmannians is of “existential” nature, with hidden difficulties of explicit computations. We hope to ameliorate this difficulty by some equivalent descriptions. Our construction is based on “abelianization” (as we construct sections of \( \mathcal{O}(1) \) on the loop Grassmannian from the same objects for a Cartan subgroup) and on locality (as we interpret equations of the projective embedding of the Grassmannian as locality conditions).

We would like to describe these equations in more standard terms by constructing a central extension of the quantum group \( U_D(Q, A) \) and its action on sections of \( \mathcal{O}(1) \). Here, the central extension should appear as one extends the “quantum local” line bundle \( L_D(Q, A) \) from the analogue \( \mathcal{C}_{G \times I} \) of \( \mathcal{H}_{G \times I} \) to an analogue of \( \mathcal{G}(T) \).

One could also try to construct the graded algebra of section of line bundles \( \mathcal{O}(m) \) by choosing the poset \( P \) in \( \mathcal{G}^P_0(Q, A) \) to be \( 1 < \cdots < m \).

\(^{17}\) Notice that this is stronger than the standard definition of locality which only requires such isomorphism over the regular part of the configuration space where \( \mathcal{L} \) happens to trivialize by 4.1.

\(^{18}\) One formal way to say it is that \( U^+_D(Q, A)_\alpha \) is the smallest subsheaf on \( \mathbb{G}^\alpha \) such that it pull back to all refinements \( \mathcal{C}_\gamma \) contains \( L_\gamma \).
APPENDIX A. Loop Grassmannians with a condition

We recall a general technique providing modular description of some parts of loop Grassmannian. This allows us to finish the proof of proposition A.0.3 on T-fixed points in closures of semi-infinite orbits in A.0.3.

A.0.1. Moduli of finitely supported maps. Here we recall some elements of Drinfeld’s notion of loop Grassmannians with a geometric (“asymptotic”) condition. This material will be covered in more details elsewhere. We will fix a smooth curve C.

We are interested in various moduli of G-torsors over a curve C that are local spaces over C. As observed by Beilinson and Drinfeld, the relevant spaces Y are usually of the form \( \mathcal{M}_Y(C) \), the moduli of finitely supported, i.e., generically trivialized maps into some pointed stack \((Y, pt)\) built from G. (We usually omit the point pt from notation.)

A.0.2. The subfunctor \( \mathcal{G}(G,Y) \subseteq \mathcal{G}(G) \) given by “condition Y”. Let C be a smooth connected curve with the generic point \( \eta_C \). Let G be an algebraic group and \((Y,y)\) a pointed scheme with a G-action on Y. This gives a pointed stack \((Y,*)\) with \( Y = G Y \). Consider the moduli of maps of pairs \( Map((C,\eta_C),(Y, *)) \). Denote by \( \mathcal{G}(G,Y) \) the space over \( H_C \) with the fiber at \( D \in H_C \) given by the maps \( f \in Map((C,\eta_C),(Y, *)) \) that are defined off \( D \). This is a factorization space (2.3).

If the orbit \( G_y \) is open in Y and its boundary \( \partial(Gy) \) is a union of divisors \( Y_i, i \in I \), to any \( f \in Map((C,\eta_C),(Y, *)) \) one can associate an I-colored finite subscheme \( f^{-1}(\partial*) = (f^{-1}G \cap Y_i)_i \in I \). Then we define \( \mathcal{G}(G,Y;I) \) to be \( Map((C,\eta_C),(Y, *)) \) considered as a space over \( H_{C \times I} \). This is an I-colored local space (2.3).

Examples. (a) When Y is a point, \( \mathcal{G}(G, pt) \) is the loop Grassmannian \( \mathcal{G}(G) \).

(b) When \( G = \mathbb{G}_m \) and \((Y,y) = (\mathbb{A}^1,1)\), I is a point and \( \mathcal{G}(\mathbb{G}_m, \mathbb{A}^1, I) = Map((C, \eta_C), (\mathbb{G}_m \backslash \mathbb{A}^1, *)) \) is the space of effective divisors on C, i.e., the Hilbert scheme \( H_C \).

Lemma. Let the scheme Y be separated.

(a) \( \mathcal{G}(G,Y) \) is a subfunctor of \( \mathcal{G}(G) \). If Y is also affine, then \( \mathcal{G}(G,Y) \) is closed in \( \mathcal{G}(G) \).

(b) For a subgroup \( K \subseteq G \) the intersection with \( \mathcal{G}(K) \subseteq \mathcal{G}(G) \) reduces the condition Y to the condition \( \mathbb{G}_m Y \subseteq Y : \)

\[
\mathcal{G}(G,Y) \cap \mathcal{G}(K) = \mathcal{G}(K, \mathbb{G}_m y).
\]

A.0.3. The closure of \( S_0 \). It is well known that \( G/N \) is quasi-affine, i.e., it is an open part of its affinization \((G/N)^{\text{aff}}\). We will consider it with the base point \( y = eN \).

Proposition. Let G be of adjoint type.

(a) The scheme of T-fixed points \( \mathbb{G}(G, (G/N)^{\text{aff}})^T \) is \( H_{d \times I} \).

(b) The closure \( \overline{S_0} \) is the reduced part \( \mathbb{G}(G, (G/N)^{\text{aff}})_{\text{red}} \) of the loop Grassmannian with the condition \((G/N)^{\text{aff}}\).

Proof. (a) The fixed points \( \mathcal{G}(G)^T \) are known to be \( \mathcal{G}(T) \) so \( \mathcal{G}(G, (G/N)^{\text{aff}})^T = \mathcal{G}(G, (G/N)^{\text{aff}}) \cap \mathcal{G}(T) \). This has been identified in the lemma A.0.2.b with \( \mathcal{G}(T, Ty) \) where \( y \) is the base point \( eN \) of \((G/N)^{\text{aff}}\). So, \( Ty = B/N \). When G is adjoint, \( \prod_{i \in I} \alpha_i : \mathbb{G}_m \xrightarrow{\sim} T \cong B/N \). This extends to an identification of the closure of \( B/N \) in \((G/N)^{\text{aff}} \) (a T-variety) with \( (\mathbb{A}^1)^I \) (a \( \mathbb{G}_m^I \)-variety). Now, \( \mathcal{G}(T, Ty) \cong \mathcal{G}(G, (G/N)^{\text{aff}})_{\text{red}} \) with \( H_{d \times I} \) in the example 2 in A.0.2.

(b) Since \((G/N)^{\text{aff}}\) is affine, \( \mathcal{G}(G, (G/N)^{\text{aff}}) \) is closed in \( \mathcal{G}(G) \) (lemma A.0.2.a). Since \( \mathbb{G}(G, (G/N)^{\text{aff}}) \) contains \( \mathbb{G}(G, G/N) = \mathbb{G}(N) = S_0 \), its reduced part contains \( S_0 \). Since the stabilizer of the
base point of \((G/N)^\text{aff}\) is \(N\), \(G(G(G/N)^\text{aff}) \subseteq G(G)\) is \(N\)-invariant. Then the reduced part \(\overline{G(G(G/N)^\text{aff})}\) has a stratification by \(N\)-orbits \(S_\lambda\) for \(\lambda \in \chi_\lambda\) such that \(L_\lambda\) lies in \(\overline{G(G(G/N)^\text{aff})}\).

We have, according to Part (a), that \(\overline{G(G(G/N)^\text{aff})} = H_{d \times I}\). So, \(\overline{G(G(G/N)^\text{aff})}_{\text{red}} = \overline{S_0}\). □

**A.0.4. Proof of the proposition 2.2.3.** (a) According to proposition A.0.3.a we have \(\overline{S_0} = \overline{G(G(G/N)^\text{aff})}_{\text{red}}\). Therefore, \(\overline{S_0} \subseteq \overline{G(G(G/N)^\text{aff})} = H_{d \times I}\) by proposition A.0.3.b.

To see that \(H_{d \times I} \subseteq G\) lies in \(\overline{S_0}\) we denote by \(G_i \subseteq G\) the connected 3-dimensional subgroup corresponding to \(i \in I\). Then \(\overline{S_0}\) contains the corresponding object \(\overline{S_0}(G_i)\) for \(G_i\), and since we have already checked the proposition for \(SL_2\) this is \(AJ_{G_i}(H_d)\), i.e., \(AJ(H_{d \times I})\).

It remains to prove that \(\overline{S_0} \subseteq G\) is closed under the product in \(G\) (because \(AJ(H_{d \times I}) = \text{the product of all } AJ(H_{d \times I})\)). However, the product in \(G\) can be realized using fusion in \(G\). So, it suffices to notice that \(\overline{S_0}\) is the fiber at a point \(a = 0\) in a curve \(C = \mathbb{A}^1\) of a factorization space \(\overline{G(G/N)}\) which is defined as the closure of the factorization subspace \(G(N) \subseteq G(G/N)\) of \(G\).

(b-c) The part (a) of the proposition 2.2.3 gives a factorization of \(\overline{S_0}^\text{aff} = (H_{d \times I})^\text{aff}\) as a product \(H_{d \times I}^\text{aff} = \prod_{i \in I} (H_d)^\text{aff}\) over contributions from all \(i \in I\). One therefore also has such factorization for \(\overline{S_0}_{-\alpha}\) and obviously for the connected components \(G(T)\). This reduces parts (b) and (c) of the proposition to the \(SL_2\) case. This case has already been checked by explicit calculation beneath proposition 2.2.3.

**APPENDIX B. CALCULATION OF THOM LINE BUNDLES FROM [YZ17]**

In [YZ17], one uses a different convolution diagram. The only essential difference is the map \(\iota\) described below. We check that it gives the same Thom line bundle as the calculation in §3.4.2 which used the dg cotangent correspondence. We will recall without character formulas how computations of Thom line bundles were made in [YZ17]. For calculational reasons one uses an extra variety \(X = G \times_P Y\) for \(Y = \text{Rep}_Q(\text{Gr}_F(V))\) and then a nonlinear map \(\iota\) accounts for the difference between ambiental embeddings \(T^*_x X \subseteq T^*X\) and \(T^*_x Y \subseteq T^*Y\).

The notations are as in §3.4.2. Denote the elements of \(Y = \text{Rep}(V_\bullet) = \bigoplus_{k=1}^m \text{Rep}_Q V_k\) and \(Y^* = \text{Rep}_Q(V_\bullet)\) by \(y\) and \(y^*\). The moment map \(\mu: T^*Y \rightarrow I^* \cong I\) is given by the projection of the commutator to \(I\)

\[
\mu(y, y^*) = [y, y^*]|_I \overset{\text{def}}{=} \left( \sum_{\{h \in H: h' = i\}} y_h y^*_h - \sum_{\{h \in H: h' = i\}} y^*_h y_h \right)_{i \in I}.
\]

The story in [YZ17] is told in terms of singular subvarieties \(\mu^{-1}(0) \subseteq T^*Y\) (for a group \(L\) acting on a smooth variety \(Y\)) and the functoriality of cohomology is constructed in terms of ambiental smooth varieties \(T^*Y\). The difference here is that we derive the cotangent correspondence mechanically from the original correspondence. For instance this makes the associativity of multiplication follow manifestly from associativity of the extension correspondence.

Let \(W = G \times_P R(F)\) with projection to \(X' = R(V)\). Let \(Z := T^*_W(X \times X')\). We have the following correspondence in [YZ14, Section 5.2]

\[
(G \times_P T^*Y) \xleftarrow{\iota} T^*X \xrightarrow{\phi} Z \xrightarrow{\psi} T^*X'
\]

19 For \(C = \mathbb{A}^1\) we have a canonical trivialization of \(G(G) \rightarrow H_C\) over \(C = H^2\), as \(G(G)\). Now, consider the pull-back \(G_{C^2} = \times_{C \times C^2} G(G)\) of the restriction of \(G(G)\) to \(H^2 = C^2\). The locality identifies it over \(C^2 - \Delta_C\) with the constant bundle \(G(G)^2\). By fusion of \(u, v \in G(G)\) we mean the limit (when it exists) over the diagonal of the constant section \((u, v)\) which is defined off the diagonal.
the maps are the natural ones, which we further describe below.

Let $U$ be the unipotent radical of $P$. Denote the Lie algebras by $p,l,u$. Denote the natural projections by $\pi : P \to L$, $\pi : p \to l$ and $\pi' : p \to u$.

For any associated $G$-bundle $E = G \times_P E$ we denote the fiber at the origin by $E_0 = E$. Then $T^*X \cong G \times_P (T^*X)_0$ and the $L$-variety $(T^*X)_0$ is (by [YZ14, Lemma 5.1 (a)])

$$(T^*X)_0 \overset{\text{def}}{=} \{(c,y,y^*) \mid c \in p, (y,y^*) \in T^*Y, \text{ such that } \mu(y,y^*) = \pi(c)\}.$$

Lemma. (a) We have an isomorphism of $L$-varieties $u \times T^*Y \cong (T^*X)_0$ over $G/P$ by $(u,y,y^*) \mapsto (u + \mu(y,y^*), y,y^*)$.

(b) This makes $T^*X$ into a $G$-equivariant vector bundle over $G/P$, the sum of $T^*(G/P)$ and $G \times_P T^*Y$.

**Proof.** In (a) the inverse map is $(c,y,y^*) \mapsto (\pi'(c),y,y^*)$. In (b) we use $T^*(G/P) \cong G \times_P u$. \hfill \Box

The map $G \times T^*Y \to T^*X$ defined as $(g,y,y^*) \mapsto (g,\mu(y,y^*),y,y^*)$ induces a well-defined map $\iota : G \times_P T^*Y \to T^*X$. By [YZ14, Lemma 5.1], we have the isomorphism

$$Z := T^*_W(X \times X') \cong G \times_P \text{Rep}_G(F)$$

with $\psi(g,x,x^*) \mapsto g(x,x^*)$ for $g \in G$ and $(x,x^*) \in \text{Rep}_G(V)$. So, the map $\psi$ is a composition of the inclusion $\psi'$ of vector bundles over $G/P$ and the conjugation action $\psi''$ (which acts by the same formula as $\psi$) and the diagram is

$$G \times_P T^*Y \overset{\iota}{\to} T^*[G \times_P \text{Rep}(\text{Gr}_F(V))] \overset{\psi'}{\to} Z \overset{\psi''}{\to} \text{Rep}_G(V).$$

**Lemma.** The Thom line bundles $\Theta_G(\psi'), \Theta_G(\psi'')$ and $\Theta_G(\iota)$ are respectively the line bundles

$$\Theta_L[F_{\infty}/F_0 \text{Rep}_G(V)], \quad \Theta_L(g/p)^{-1} \quad \text{and} \quad \Theta_L(p^+)^{-1} = \Theta_L(g/p \otimes \omega),$$

In particular, $\Theta(d^*p) \otimes \Theta(\tilde{q}) \cong \Theta_L(\iota) \otimes \Theta_L(\psi)$.

**Proof.** If $S$ is one of the first four spaces in the diagram, then $\mathfrak{A}_G(S) = \mathfrak{A}_F$ since $S = G \times_P S_0$ for the fiber $S_0$ which is an affine space. In particular, for a map $\eta \in \{\iota, \psi', \psi''\}$, the line bundle $\Theta_G(\eta)$ on $\mathfrak{A}_F$ is $\Theta_G(T(\eta)).$

(1) Vector bundle $T(\iota)$ is the normal bundle $N(\iota)$. According to the lemma B it is isomorphic to $G \times_P -$ of the $\tilde{L}$-module $(g/p)^* \otimes \omega = p^+ \otimes \omega$.

(2) Similarly, $T(\psi')$ is the normal bundle $N(\psi')$ and the fiber $T(\psi')_0$ is $(F/F_0) \text{Rep}_Q(V)$.

(3) The equality $\Theta_G(\psi'') = \Theta_L[\mathfrak{g}/p]^{-1}$ is clear. \hfill \Box

**Corollary.** (a) $\Theta_L[F_{\infty}/F_0(\text{Rep}_G(V))] = \Theta_L[\text{Rep}_Q(V) - \text{Gr}_0^F(\text{Rep}_Q(V))]$.

(b) Consider the case when $Q$ has no loop edges and the filtration type $\nu$ is a flag, i.e., $\nu_k \in I$ for all $k$. Then $\text{Gr}_0^F \text{Rep}_Q(V) = 0$ and $\Theta_L(\psi') = \Theta_L(\text{Rep}_Q(V))$.

**Proof.** (a) A filtration on $V$ induces a family of filtrations, compatible with the decomposition $\text{Rep}_Q(V) = \text{Rep}_Q(V) \oplus \text{Rep}_Q(V)$ and with the $L$-equivariant identification $\text{Rep}_Q(V) \cong [\text{Rep}_Q(V)]^*$. Therefore, the claim follows from

$$\frac{F_{\infty}}{F_0}(\text{Rep}_Q(V)^*) = [\frac{F_{-1}}{F_{-\infty}}(\text{Rep}_Q(V))^*]$$

and the invariance of Thom line bundles under duality of vector bundles.

(b) follows since $\text{Gr}_0^F \text{Rep}_Q(V) = 0$ under the assumption on $Q$. The reason is that $\text{Gr}_0^F \text{Rep}_Q(V) = \oplus \text{Rep}_Q(\text{Gr}_0V)$ and all $\text{Gr}_0V \in \mathfrak{Y}$ are one-dimensional. \hfill \Box
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