A Novel Sum and Difference Phase-comparison Algorithm for Uniform Circular Phase Array Radar

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Abstract. Phase-comparison of sum and difference angle measurement technique can be simply applied to uniform linear phased array radar. In contrast, it is not so simple for uniform circular phased array radar because the relation between ratio of difference to sum and target azimuth angle is complex. In this letter, we present this problem; based on derivation and approximate treatment, we propose an algorithm to solve this problem. The proposed algorithm is easy to implement and has good angle measurement performance. Computer simulations are shown to verify the efficacy of the proposed method.

Keywords: phased array radar, angle measurement, phase-comparison of sum and difference, uniform circular array.

1. Introduction
Phased array radar (PAR) using uniform circular array (UCA) can cover the azimuth direction of 360°, and has the same direction finding performance in different azimuth direction. As a task, multifunction phased array radar needs to realize target tracking so that high resolution target azimuth is required, and this traditionally been achieved by monopulse technique using sum and difference beams. Amplitude comparison and phase comparison are two widely-used methods of the monopulse technique [1], [2].

Phase-comparison monopulse technique, as an important angle measurement method, has high accuracy and low computation complexity [3] - [9]. For a uniform planar array, this method is realized by dividing the antenna array into two identical two sub-arrays. In additional, the method is easy to operate for uniform linear array (ULA), as the relation between ratio of difference to sum and target azimuth angle is clear and simple. However, it is not simple for UCA because of the complex relation, the common practice is to put them in a table and look up the table to obtain the angle. Then the angle measurement accuracy has associated with searching step of the table. So there is a problem that how to simply and quickly apply phase-comparison monopulse technique to a UCA.

In this paper, we present the equation about ratio of difference to sum of beams formed by the two sub-arrays using Bartlett beamformer [10]. Based on derivation and approximate treatment, a method is proposed to simplify sum and difference phase-comparison monopulse angle measurement for UCA.
2. Sum and Difference Beam

We consider a UCA of \( M \) omni-directional sensors labelled 1, 2, \ldots, \( M \), as presented in Fig. 1. The first sensor is taken as the reference sensor. Assume that the plane composed of incoming waves coincides with the plane of the UCA. The radius of the UCA is \( r \) and the wavelength of the coming signal is \( \lambda \).

In the amplitude-comparison of sum and difference method, left beam and right beam are formed by the \( M \) sensors, they point to different directions. In contrast, when we use phase-comparison, the circular array is divided into two sub-arrays, one of which consists of sensor 1 to \( \frac{M}{2} \), and the other one \( \frac{M}{2} + 1 \) to \( M \). Define the \( \frac{M}{2} \times 1 \) vector \( A_1(\theta) \) and \( A_2(\theta) \) to be the array manifold of the two sub-arrays at the direction \( \theta \), they can be expressed as

\[
A_1(\theta) = [a_1(\theta), a_2(\theta) \ldots a_{\frac{M}{2}}(\theta)]^T
\]

\[
A_2(\theta) = [a_{\frac{M}{2}+1}(\theta), a_{\frac{M}{2}+2}(\theta) \ldots a_M(\theta)]^T
\]

where \( a_n(\theta) \) is the receiving characteristic of sensor \( n \) given as

\[
a_n(\theta) = e^{j\frac{2\pi}{\lambda} \cos(\theta - \gamma_n)}
\]

Here, \( \gamma_n = \frac{2\pi(n-1)}{M} (n = 1, 2 ... M) \).

We expect the two beams formed by the two sub-arrays both point to direction \( \theta_a \) which is given as

\[
\theta_a = (\gamma_1 + \gamma_M)/2
\]

According to Bartlett beamformer [10], the weight vector of the first sub-array can be obtained as

\[
W_1 = \frac{A_1(\theta_a)}{\sqrt{A_1^H(\theta_a)A_1(\theta_a)}} = \frac{2}{M} \left[ e^{j\beta \cos(\theta_a - \gamma_1)} e^{j\beta \cos(\theta_a - \gamma_2)} \ldots e^{j\beta \cos(\theta_a - \gamma_M)} \right]
\]

where \( \beta = \frac{2\pi r}{\lambda} \). The weight vector of the other sub-array is obtained as

\[
W_2 = \frac{A_2(\theta_a)}{\sqrt{A_2^H(\theta_a)A_2(\theta_a)}} = \frac{2}{M} \left[ e^{j\beta \cos(\theta_a - \gamma_1)} e^{j\beta \cos(\theta_a - \gamma_2)} \ldots e^{j\beta \cos(\theta_a - \gamma_M)} \right]
\]

The output vector of the sub-arrays can be written as

\[
X_1(t) = A_1(\theta_g) \cdot s(t) + n(t)
\]

\[
X_2(t) = A_2(\theta_g) \cdot s(t) + n(t)
\]

where \( s(t) \) is the incident signal, \( \theta_g \) is the azimuth of target and \( n(t) \) is the additive white Gaussian noise. In the presence of noise, the beam of the first sub-array is obtained as

\[
BF_1 = W_1^H \cdot X_1(t) = \frac{2}{M} \cdot s(t) \cdot (\xi_1(\theta_g) + \xi_2(\theta_g) + \cdots + \xi_{\frac{M}{2}}(\theta_g))
\]

where \( \xi_n(\theta_g) \) is expressed as

\[
\xi_n(\theta_g) = e^{j\beta [\cos(\gamma_n - \gamma_1) - \cos(\gamma_n - \gamma_1)]}
\]

The other one is obtained as
BF_2 = W^H_2 \cdot X_2(t)

\[BF_2 = \frac{2}{M} s(t) \left( \xi_{M/2+1}(\theta_{g}) + \xi_{M/2+2}(\theta_{g}) + \cdots + \xi_{M}(\theta_{g}) \right)\]  \hspace{1cm} (11)

Therefore, the ratio of difference to sum is written as

\[
\frac{\Delta(\theta_{g})}{\Sigma(\theta_{g})} = \frac{BF_1 - BF_2}{BF_1 + BF_2}
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{\xi_n(\theta_{g})}{\xi_n(\theta_{g})} + \sum_{n=-\infty}^{\infty} \frac{\xi_{M/2}(\theta_{g})}{\xi_{M/2}(\theta_{g})} + \cdots + \sum_{n=-\infty}^{\infty} \frac{\xi_{M}(\theta_{g})}{\xi_{M}(\theta_{g})}
\]  \hspace{1cm} (12)

It is found that the relation between the ratio and \(\theta_{g}\) is not a simple function, and the problem is how to simplify Eq. (11) to estimate \(\theta_{g}\).

3. The Proposed Algorithm

According to Taylor expansion of \(e^x\), when \(x\) is small enough, \(e^x \approx 1 + x\). Therefore, when

\[-2\beta \sin \left(\frac{\theta_{g} + \theta_a - 2\gamma_n}{2}\right) \sin \left(\frac{\theta_{g} - \theta_a}{2}\right)\]

is small, Eq. (10) can be given by

\[
\xi_n(\theta_{g}) = e^{j[b(\cos(\theta_{g} - \gamma_n) - \cos(\theta_a - \gamma_n))]
\]

\[
= e^{-2j[b\sin((\theta_{g} + \theta_a - 2\gamma_n)/2)\sin((\theta_{g} - \theta_a)/2)]}
\]

\[\approx 1 - 2j\beta \sin((\theta_{g} + \theta_a - 2\gamma_n)/2)\sin((\theta_{g} - \theta_a)/2)\]  \hspace{1cm} (13)

For example, then the wavelength \(\lambda = 0.6\) m, radius \(r = 0.2\) m and \(\theta_{g} - \theta_a\) is in the interval \([-5^\circ, 5^\circ]\), we have

\[-2\beta \sin([(\theta_{g} + \theta_a - 2\gamma_n)/2]\sin([(\theta_{g} - \theta_a)/2])\]

\[e \in [-0.1825, 0.1825]\]  \hspace{1cm} (14)

\(BF_1\) and \(BF_2\) can be simplified as

\[BF_1 = s(t) - j\frac{4\beta s(t)}{M} \sin \frac{\theta_{g} - \theta_a}{2} \sin \frac{\theta_{g} + \theta_a - 2\gamma_1}{2} + \cdots + \sin \frac{\theta_{g} + \theta_a - 2\gamma_{M/2}}{2}\]

\[BF_2 = s(t) - j\frac{4\beta s(t)}{M} \sin \frac{\theta_{g} - \theta_a}{2} \sin \frac{\theta_{g} + \theta_a - 2\gamma_{M/2+1}}{2} + \cdots + \sin \frac{\theta_{g} + \theta_a - 2\gamma_M}{2}\]  \hspace{1cm} (15)

(16)

Eq. (12) can be rewritten as

\[
\frac{\Delta(\phi)}{\Sigma(\phi)} = -ja \sin \phi \cos \phi
\]

\[\frac{2 - jb \sin \phi \sin \phi}{2 - jb \sin \phi \sin \phi}\]  \hspace{1cm} (17)

where

\[a = \frac{8\beta}{M} \sin \frac{\gamma_{M} - \gamma_1}{2} + \cdots + \sin \frac{\gamma_{M/2+1} - \gamma_{M/2}}{2}\]

\[b = \frac{8\beta}{M} \cos \frac{\gamma_{M} - \gamma_1}{2} + \cdots + \cos \frac{\gamma_{M/2+1} - \gamma_{M/2}}{2}\]

\[\phi = \frac{\theta_{g} - \theta_a}{2}\]  \hspace{1cm} (18)
In the practical system, the modulus of Eq. (18) can be calculated as \( c \). It is noted that \( \phi \) is positive or negative that depends on the argument of \( \Delta(\phi)/\Sigma(\phi) \). Then, we can obtain an equation,

\[
\left( a^2 + b^2 c^2 \right) \sin^4 \phi - a^2 \sin^2 \phi + 4c^2 = 0
\]

(19)

We find the roots of Eq. (19) to estimate the DOA. It is clear that there are some extraneous solutions, but it is not hard to remove them by using the argument of \( \Delta(\phi)/\Sigma(\phi) \). The proposed algorithm is not only applicable to UCA, but also applicable to uniform circular arc array because the whole process does not conflict with the uniform circular arc array.

**Table 1.** Estimated values and actual values of DOA

| Number | Actual Values | 1   | 2   | 3   | 4   |
|--------|---------------|-----|-----|-----|-----|
|        |               | 49.0000 | 49.8000 | 51.4000 | 52.0000 |
|        | Estimated Values | 49.1362 | 49.7654 | 51.4265 | 52.0927 |
|        | Estimated Bias | 0.1362 | 0.0346 | 0.0265 | 0.0927 |
| Number |               | 5   | 6   | 7   | 8   |
|        | Actual Values | 53.5000 | 54.6000 | 55.0000 | 55.8000 |
|        | Estimated Values | 53.5153 | 54.6146 | 55.0385 | 55.8034 |
|        | Estimated Bias | 0.0153 | 0.0146 | 0.0385 | 0.0034 |
| Number |               | 9   | 10  |
|        | Actual Values | 56.3000 | 57.0000 |
|        | Estimated Values | 56.1729 | 56.8824 |
|        | Estimated Bias | 0.1271 | 0.1176 |

**Fig. 1.** RMSR curves of DOA estimates versus SNR. (a) The azimuth of incident signal is 53°. (b) The azimuth of incident signal is 54°.

4. **Numerical Simulations**

In this section, we compare the proposed method with the amplitude-comparison of sum and difference method using UCA. The root mean square error (RMSE) curves of DOA estimates versus SNR are shown to verify the efficacy of the proposed method.

In the following experiment, noise is a complex Gaussian process with zero mean. We consider a uniform circular arc array consisting of 12 omni-directional sensor elements. The angle of the sector area is 110°. The radius of the circle is 0.1m and the number of snapshots is 2048. We assume that
γ₁ = 0° and γ₂ = 110°. In the amplitude-comparison of sum and difference method, the left beam points to 51° and the right beam points to 59°, they are also formed by Bartlett beamformer.

Fig. 1 shows the RMSE curves of DOA estimates versus SNR. It can be seen that the proposed algorithm has the good performance to estimate the DOA. Moreover, the estimation accuracy of the proposed algorithm is higher than the sum and difference amplitude-comparison algorithm when the SNR changes on the interval [0, 50]dB.

Table I shows the estimated values and actual values of DOA. It can be seen that the proposed algorithm is effective and provides accurate estimates.

5. Conclusion
In this letter, we propose an algorithm to simply and quickly apply sum and difference phase-comparison monopulse technique to uniform circular phased array radar. The algorithm is also applicable to uniform circular arc array. Although the proposed method needs to find the roots of an equation, the estimation accuracy of it is higher than amplitude-comparison of the sum and difference algorithm in some cases.

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