ABSTRACT. We define bilinear functionals of vector fields and differential forms, the densities of which yield the metric and Einstein tensors on even-dimensional Riemannian manifolds. We generalise these concepts in non-commutative geometry and, in particular, we prove that for the conformally rescaled geometry of the noncommutative two-torus the Einstein functional vanishes.

1. INTRODUCTION

Riemannian geometry flourishing for more than hundred and fifty years is continually at the core of modern mathematics with a wealth of applications and open problems. Its role as a key to understanding general relativity makes it ubiquitous in modern physics. The study of interconnection between the geometric objects on the manifolds and differential operators led to the development of spectral geometry and, with the seminal question "Can one hear the shape of a drum?", was popularised by Mark Kac [36]. A further generalization was brought by noncommutative geometry [8], where spectral methods were proposed to study invariants of algebras, functionals on noncommutative algebras and geometric objects that extends the notion of curvature, leading, for example, to the version of Gauss-Bonnet theorem for noncommutative tori.

The aim of this paper is to propose a few new functionals that express other basic geometric objects, in particular the Einstein tensor, using the methods of the Wodzicki residue and its generalisation in noncommutative geometry.

1.1. An overview of the field. An eminent spectral scheme that generates geometric objects on manifolds such as volume, scalar curvature, and other scalar combinations of curvature tensors and their derivatives prima facie is the small-time asymptotic expansion of the (localised) trace of heat kernel [27, 28]. This has many diverse applications in mathematics and in physics, mostly in the context of general relativity and its generalisations.

Using the Mellin transform, the coefficients of this expansion can be transmuted into certain values or residues of the (localised) zeta function of the Laplacian. In turn, they can be expressed using the Wodzicki residue $\mathcal{W}$ (also known as noncommutative residue), which is a unique (up to multiplication by a constant) tracial state on the algebra of pseudo-differential operators ($\Psi$DO) on a complex vector bundle $E$ over a compact manifold $M$ of dimension $n \geq 2$ [31, 54]. For the oriented manifold $M$ it is given up to multiplicative constant by a simple integral formula,

$$\mathcal{W}(P) := \int_M \left( \int_{|\xi|=1} tr \sigma_{-n}(P)(x, \xi) \nabla \xi \right) d^n x,$$

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where $\text{tr}$ is the trace over endomorphisms of the bundle $E$ at any given point of $M$, $\sigma_{-n}(P)$ is the symbol of order $-n$ of a pseudodifferential operator $P$ and $\mathcal{V}_E$ denotes the volume form on the unit sphere. The name ‘residue’ comes from the fact that it is indeed a residue (at $z = 1$) of the $\zeta$-function associated to $P$ [30]. In this paper, for simplicity, we focus only on closed oriented $M$ of even dimension $n = 2m$. In this case, for a Riemannian manifold $M$ equipped with a metric tensor $g$ and the (scalar) Laplacian $\Delta$ one has,

\begin{equation}
\mathcal{W}(\Delta^{-m}) = v_{n-1} \text{vol}(M),
\end{equation}

and a localized form, a functional of $f \in C^\infty(M)$,

\begin{equation}
\nu(f) := \mathcal{W}(f\Delta^{-m}) = v_{n-1} \int_M f \text{vol}_g,
\end{equation}

where

\begin{equation}
v_{n-1} := \text{vol}(S^{n-1}) = \frac{2\pi^m}{\Gamma(m)},
\end{equation}

is the volume of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

A startling result regarding a higher power of the Laplacian was divulged by Connes in the early 1990s [9] and explicitly confirmed independently in [38] and in [37]. Namely, for $n > 2$,

\begin{equation}
\mathcal{W}(\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M R(g) \text{vol}_g,
\end{equation}

is up to a constant a Riemannian analogue of the Einstein-Hilbert action functional of general relativity in vacuum. Here $R = R(g)$ is the scalar curvature, that is the $g$-trace $R = g^{jk}R_{jk}$ of the Ricci tensor with components $R_{jk}$ in local coordinates, where $g^{jk}$ are the raised components of the metric $g$.

A localised form of (1.4) for $n > 2$ is a functional on $C^\infty(M)$ (cf. [12])

\begin{equation}
\mathcal{R}(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M fR(g)\text{vol}_g.
\end{equation}

For the Riemannian spin manifolds this can be expressed using such operators in spinor bundle as the spin Laplacian or the Dirac operator which are related by the Schrödinger-Lichnerowicz formula. In particular, for the Dirac operator one has:

\begin{equation}
\mathcal{W}(f|\mathcal{D}|^{-n}) = 2^m v_{n-1} \int_M f \text{vol}_g,
\end{equation}

\begin{equation}
\mathcal{W}(f|\mathcal{D}|^{-n+2}) = -2^m \frac{n-2}{24} v_{n-1} \int_M fR(g) \text{vol}_g.
\end{equation}

In the noncommutative realm the spectral-theoretic approach to scalar curvature has been extended also to quantum tori in the seminal work of Connes and Tretkoff [15], expanded in [13] and then extensively studied by many authors [19, 20, 21, 18, 25, 39, 35, 42, 51, 43, 47], see also [10, 23, 40] for recent surveys. Therein, the pseudodifferential operators and symbol calculus introduced in [7] and extended to crossed product algebras in [1, 2], cf. also [52, 32, 33, 29] for detailed account, have been employed for computations of certain values and residues of $\zeta$-functions of suitable Laplace type operators.

In these papers the analogue of conformal transformations of the standard (flat) geometry of the noncommutative 2-torus was worked out, with the conformal scaling (Weyl factor) taken as
an invertible positive element from the underlying algebra $A$; see also [11] for a conformally non-flat quantum 4-torus. These achievements produced extremely interesting modular aspects, which, however, involve complicated, often computer-assisted, calculations with thousands of terms.

In this respect, the alternative use of Weyl factor from the copy $A^o$ of $A$ in the commutant of $A$ in [16] and the early nonconformally flat modification of the torus geometry by an element from $A^o$ in [17] allowed stripping the modular aspects and somewhat simplifying the computations, while still producing a nontrivial scalar curvature. One can also work with $A^o$ as the underlying algebra and modify the Dirac operator by an element from $A$ instead. This still yields the usual spectral triple, while taking the other two combinations yields the twisted (modular) triples, so there are just two different types of situation [6].

Now, for quantum tori, and more generally for spectral triples with real structure $J$, there is a natural choice for $A^o$ as $JAJ$. Moreover, since $A^o$ and $A$ coincide in the classical (commutative) case, also modifying the quantum torus geometry by elements from $A$ or from $A^o$ appear a viable option and both of them should be considered. The latter leads to the usual (untwisted) spectral triple with the algebra $A$.

Concerning the spectral approach described above, the Wodzicki residue methods have already been used for quantum tori in dimensions $n = 2$ and $n = 4$ in [24] and [21], respectively, and have been extended to any dimension $n \geq 2$ in [41], see also [46], and applied in [18, 47] for the scalar curvature on NC tori.

Altogether, the aforementioned spectral methods allowed so far to extract the scalar invariants built from the metric and the curvature tensors, while the understanding of the curvature tensor itself has not yet been achieved. Only recently has the Ricci curvature tensor $Ric_{jk}$ been recovered using the Hodge-de Rham spectral triple, and more precisely from the difference of zeta functions of the Laplacian on functions and on one-forms, and applied to the noncommutative two-torus [25].

No doubt it would be extremely interesting to recover other important tensors in both the classical setup as well as for the generalised or quantum geometries. In this paper we accomplish this task for the metric tensor $g$ itself, its dual, and for the Einstein tensor

$$G := \text{Ric} - \frac{1}{2} R(g) \; g,$$

which directly enters the Einstein field equations with matter, and its dual. For this purpose, we employ in fact the Wodzicki residue of a suitable power of the Laplace type operator multiplied by a pair of other differential operators. Notably, we demonstrate that the Wodzicki residue density recovers the tensors $g$ and $G$ as certain bilinear functionals of vector fields on a manifold $M$, while their dual tensors are recovered as the density of bilinear functionals of differential one-forms on $M$.

Using Connes’ pseudodifferential calculus on noncommutative tori we also propose a conspicuous quantum analogue of these functionals and probe it on 2 and 4-dimensional noncommutative tori. Note, however, that a direct comparison with [25] is arduous not only due to the difference between the canonical and Hodge-de Rham spectral triples, but more fundamentally,
due to the unknown way how to extract from the Einstein tensor the Ricci tensor due to the ordering ambiguity (quantum product) of $g$ and $R(g)$. 

The aforementioned functionals are built using either the Laplace or Dirac operator (in case $M$ is spin manifold), but before embarking on the latter one, we need actually first to settle other Laplace-type operators, acting on vector bundles of suitable rank over $M$. In fact, it will be advantageous to consider a general class of such operators and to control which tensors they produce. For this reason, our notation will systematically keep track of the dependence on a particular differential operator.

This will be also beneficial for another reason. Namely, one typically obtains spectrally the scalar curvature $R$ of a manifold using the Laplace or Dirac operators, which are built openly from the Levi-Civita connection (so torsion-free). Classically, these operators can be also characterised by the fact that the so obtained $R$ minimises the relevant functionals (as otherwise there is a non-negative contribution from the torsion). This property may be not satisfied in the noncommutative realm, as actually there is no implicit notion of torsion (see, however, other frameworks in e.g. [26, 49, 3, 4, 5]).

It turns out that already on the noncommutative 4-torus the most immediate quantum analogue of the conformally rescaled flat Laplace operator does not minimise the scalar curvature functional [50]. Thus, if no particular reference Laplace-type operator is declared to be torsion-free, it can be convenient to label also the spectrally obtained geometric tensors such as $R$, Ric or $G$ by the concrete operator $\Delta$ which has been employed for their definition, and call them, e.g. “$\Delta$ scalar curvature”, etc. By doing so, we will be able to determine which Laplace-type operators provide the same geometric invariant.

1.2. Organization of the article. We start with a brief overview of the normal coordinates and the expression of the Laplace operator and its symbol at a given point on the manifold in these coordinates and present the explicit results for the symbols of the negative powers of the Laplace operator. Then we prove the main theorems showing that functionals on vector fields (understood as differential operators) yield the metric (Theorem 2.3) and the Einstein tensor (Theorem 2.4) densities.

In Section 3 we demonstrate in Theorems 3.1 and 3.2 the value of the metric and Einstein functionals for Laplace-type operators on vector bundles, giving interesting examples of the spinor Laplacian (Proposition 3.4) and square of the Dirac operator (Proposition 3.5). In Section 4 we propose functionals on the space of differential one-forms on a spin $c$ manifold and prove that these functionals provide densities of the respective dual metric and Einstein tensors (Theorem 4.1).

In Section 5 we propose an extension of these functionals to the noncommutative realm, focusing in Section 5.1 on Laplacian on noncommutative tori, where outer derivations are interpreted as vector fields. We prove that the spectral Einstein tensor vanishes identically for the conformally rescaled geometry of the noncommutative 2-torus (Proposition 5.2) and provide an explicit and compact formula for the spectral Einstein and metric tensors for the 4-torus. In Section 5.2 we propose the generalisation of the spectral metric and Einstein functionals on
noncommutative differential forms for spectral triples on conformally rescaled noncommutative 2 and 4-tori, showing that also the contravariant spectral Einstein functional vanishes for the conformally rescaled spectral triple on noncommutative 2-torus (Proposition 5.11) and provide an explicit formula for the spectral metric and Einstein functionals for the noncommutative 4-torus. Furthermore, we discuss also these functionals for finitely summable and regular spectral triples and illustrate how these functionals behave under tensoring a finite summable regular spectral triple with the simplest non-trivial finite triple on $\mathbb{C}^2$.

The Appendix provides a few formulae and computations of pseudodifferential symbols, which are used in proofs.

1.3. **Notation.** Throughout the article we work with a closed, orientable Riemannian manifold of dimension $n = 2m$, with a fixed metric $g$, while in the case of functionals on differential forms we assume the existence of spin-$\mathbb{C}$ structure and fix the spinor bundle. We denote by $\gamma^a$ the matrices that satisfy $\gamma^a \gamma^b + \gamma^b \gamma^a = 0$ if $a \neq b$, and by $(\gamma^a)^2 = 1$ for $a, b = 1, \ldots, n$.

We denote the Einstein tensor on $M$ by $G$. The spectral metric and Einstein functionals on vector fields we denote by $g^P$, $G^P$, where $P$ is an appropriate Laplace-type operator, whereas for the spectral metric and Einstein functionals on differential forms we use the notation $g_D$ and $G_D$, where $D$ is a Dirac-type operator.

The proofs are all based on the technique of normal coordinates $x$ around a fixed point on the manifold and the expansion of all geometric objects (metric, orthonormal frames, connection, vector fields, etc.) up to the relevant order in $x$.

2. **The metric and the Einstein tensor densities**

We consider an even-dimensional compact Riemannian manifold $M$ with components of the metric $g$ given in chosen local coordinates by $g_{ab}$. The Laplace operator, which is densely defined on $L^2(M, vol_g)$, is expressed as

\[(2.1) \quad \Delta = -\frac{1}{\sqrt{\det(g)}} \partial_a \left( \sqrt{\det(g)} g^{ab} \partial_b \right),\]

where $g^{ab}$ is the inverse of the matrix $g_{ab}$ and we use here and in the following the summation convention over repeated indices.

The symbols of the differential operator $\Delta$ are:

\[(2.2) \quad a_2 = g^{ab} \xi_a \xi_b, \quad a_1 = \frac{-i}{\sqrt{\det(g)}} \partial_a \left( \sqrt{\det(g)} g^{ab} \right) \xi_b, \quad a_0 = 0.\]

As the next step, we will conveniently express the symbols using centered normal coordinates $x$ [45] around a fixed point of $M$ with $x = 0$.

2.1. **Laplace operator and its powers in normal coordinates.** Let us recall that in the normal coordinates the metric has a Taylor expansion:

\[(2.3) \quad g_{ab} = \delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d + o(x^2),\]

and

\[(2.4) \quad \sqrt{\det(g)} = 1 - \frac{1}{6} \text{Ric}_{ab} x^a x^b + o(x^2),\]
where $R_{abed}$ and $\text{Ric}_{ab}$ are the components of the Riemann and Ricci tensor, respectively, at the point with $x = 0$ and we use the notation $o(x^k)$ to denote that we expand a function up to the polynomial of order $k$ in the normal coordinates. The inverse metric is

$$g^{ab} = \delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d + o(x^2), \quad (2.5)$$

where on the right-hand side we still can keep the lower indices $a, b$ as at the point with $x = 0$ the metric is standard Euclidean and so the tensor indices are lowered and raised by the Kronecker symbols $\delta_{ab}$ and $\delta^{ab}$.

Consequently, the symbols of the Laplace operator in normal coordinates are

$$a_2 = (\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d) \xi_a \xi_b + o(x^2), \quad (2.6)$$

$$a_1 = \frac{2}{3} \text{Ric}_{ab} x^a \xi_b + o(x^2). \quad (2.7)$$

Then, by a straightforward application of $2.5$, $2.3$, $2.4$ to $A.6$ one has:

**Lemma 2.1.** In normal coordinates around a fixed point of the manifold $M$ the symbols of the inverse of the Laplace operator read

$$b_2 = ||\xi||^{-4} (\delta_{ab} - \frac{k}{3} R_{acbd} x^c x^d) \xi_a \xi_b + o(x^2), \quad (2.8)$$

$$b_3 = -\frac{2i}{3} \text{Ric}_{ab} x^a \xi_b ||\xi||^{-4} + o(x),$$

$$b_4 = \frac{2}{3} \text{Ric}_{ab} \xi_a \xi_b ||\xi||^{-6} + o(1).$$

Next, we apply (Lemma A.1) to compute the three leading symbols of the powers of the pseudodifferential operator $\Delta^{-1}$:

**Proposition 2.2.** The first three leading symbols of the operator $\Delta^{-k}$, $k > 0$

$$\sigma(\Delta^{-k}) = \epsilon_{2k} + \epsilon_{2k+1} + \epsilon_{2k+2} + \ldots,$$

are given up to order respectively $x^2$, $x$, 1 in normal coordinates around a fixed point by,

$$\epsilon_{2k} = ||\xi||^{-2k-2} \left( \delta_{ab} - \frac{k}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(x^2),$$

$$\epsilon_{2k+1} = \frac{-2ki}{3||\xi||^{2k+2}} \text{Ric}_{ab} x^a \xi_b + o(x),$$

$$\epsilon_{2k+2} = \frac{k(k+1)}{3||\xi||^{2k+4}} \text{Ric}_{ab} \xi_a \xi_b + o(1). \quad (2.9)$$

**Proof.** We use $(2.7)$ and $(A.11)$ keeping only terms with the right order in $x$, which for $\epsilon_{2k+2}$ yields only 4 terms instead of 10 in $(A.11)$, i.e.

$$\epsilon_{2k+2} = k(b_2)'^{-1} b_4 - \frac{k(k-1)}{2} (b_2)'^{-2} \partial_{b_2}^2 (b_2) \partial_{b_3} (b_3) - \frac{k(k-1)}{4} (b_2)'^{-2} \partial_{b_2} \partial_{b_2}' (b_2) \partial_{b_2} (b_2) - \frac{k(k-1)(k-2)}{6} (b_2)'^{-3} \partial_{b_2} (b_2) \partial_{b_2} (b_2) \partial_{b_2} (b_2) + o(1).$$
We simplify it further by taking into account the properties of the Riemann tensor,

\[ R_{abcd}\xi^a\xi^b = 0 = R_{abcd}\xi^c\xi^d, \]

leading to the above result.

2.2. **Spectral functionals of vector fields.** Let \( V, W \) be a pair of vector fields on a compact Riemannian manifold \( M \), of dimension \( n = 2m \). Using the Laplace operator, we define two functionals \( g^\Delta(V, W) \) and \( G^\Delta(V, W) \).

**Theorem 2.3.** The functional:

\[ g^\Delta(V, W) := \mathcal{W}(VW\Delta^{-m-1}), \]

is a bilinear, symmetric map, whose density is proportional to the metric evaluated on the vector fields:

\[ g^\Delta(V, W) = -\frac{v_{n-1}}{n} \int_M g(V, W) \, vol_g. \]

**Proof.** The product of two vector fields is a differential operator with a symbol

\[ \sigma(VW) = \psi_2 + \psi_1 = -V^aW^b\xi_a\xi_b + iV^a\delta_a(W^b)\xi_b. \]

Then,

\[ \mathcal{W}(VW\Delta^{-m-1}) = \int_M \int_{||\xi||=1} \sigma_{-2m}(VW\Delta^{-m-1}) \, vol_g, \]

where by \( \mathcal{W}(\sigma_{-2m}(VW\Delta^{-m-1})) = -V^aW^b\xi_a\xi_b||\xi||^{-2m-2}. \)

Using integration over \( S^{2m-1} \) we get

\[ \int_{||\xi||=1} \sigma_{-2m}(VW\Delta^{-m-1}) = -\frac{v_{n-1}}{n} V^aW^a, \]

which ends the proof.

**Theorem 2.4.** The functional:

\[ G^\Delta(V, W) := \mathcal{W}(VW\Delta^{-m}), \]

is a bilinear, symmetric map, whose density is proportional to the Einstein tensor \( G \) evaluated on the two vector fields:

\[ G^\Delta(V, W) = \frac{v_{n-1}}{6} \int_M G(V, W) \, vol_g. \]

**Proof.** Similarly to the previous theorem, we need to compute the symbol of order \( -2m \) of the pseudodifferential operator \( VW\Delta^{-m} \). Using (A.2) we have

\[ \sigma_{-2m}(VW\Delta^{-m}) = \psi_2\xi_{2m+2} + \psi_1\xi_{2m+1} - i\partial_a\psi_2\delta_a\xi_{2m+1} \]

\[ - i\partial_a\psi_1\delta_a\xi_{2m} - \frac{1}{2} \partial_a\partial_b\psi_2\delta_a\delta_b\xi_{2m}. \]
As the normal coordinates $x$ are 0 at the (arbitrary) fixed point of the manifold, we are interested only in terms that do not vanish at $x = 0$. Since both $\epsilon_{2m+1}$ and $\delta_a \epsilon_{2m}$ vanish at $x = 0$ we are left only with terms that depend on $v_2$. We explicitly compute
\[
v_2 \epsilon_{2m+2} = \frac{m(m+1)}{3||\xi||^{2m+4}} V^a W^b \text{Ric}_{cd} \xi_a \xi_b \xi_c \xi_d + o(1),
\]
\[-i \partial_a v_2 \delta_a \epsilon_{2m+1} = -i(\bar{V}^a W^b \xi_b - V^b W^a \xi_b) \frac{-2mi}{3||\xi||^{2m+2}} \text{Ric}_{ca} \xi_c + o(1)\]
(2.12)
\[-\frac{1}{2} \partial_a \partial_b v_2 \delta_a \delta_b \epsilon_{2m} = \frac{1}{2}(V^a W^b + V^b W^a) \frac{-m}{3||\xi||^{2m+2}} (R_{cada} + R_{cdab}) \xi_c \xi_d + o(1)\]
\[= -\frac{2m}{3||\xi||^{2m+2}} V^a W^b \xi_c R_{cada} \xi_d + o(1).\]
As a result,
\[
\sigma_{-2m}(V W \Delta^{m}) = \frac{-m(m+1)}{3||\xi||^{2m+4}} V^a W^b \text{Ric}_{cd} \xi_a \xi_b \xi_c \xi_d
\]
\[+ \frac{2m}{3||\xi||^{2m+2}} V^a W^b \xi_c (\xi_a \text{Ric}_{bc} + \xi_b \text{Ric}_{ac} - R_{cada}) \xi_d + o(1)\]
(2.13)
Integrating (2.13) over $S^{2m-1}$ and substituting $x = 0$ we have, for the first term,
\[
\int_{||\xi|| = 1} \frac{-m(m+1)}{3||\xi||^{2m+4}} V^a W^b \text{Ric}_{cd} \xi_a \xi_b \xi_c \xi_d =
\]
\[= \frac{v_{n-1}}{2m(2m+2)} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \frac{-m(m+1)}{3} V^a W^b \text{Ric}_{cd}\]
\[= -\frac{v_{n-1}}{12} V^a W^b R - \frac{v_{n-1}}{6} V^a W^b \text{Ric}_{ab},\]
and for the second term,
\[
\int_{||\xi|| = 1} \frac{2m}{3||\xi||^{2m+2}} V^a W^b \xi_c (\xi_a \text{Ric}_{bc} + \xi_b \text{Ric}_{ac} - R_{cada}) \xi_d =
\]
\[= \frac{v_{n-1}}{2m} \frac{2m}{3} V^a W^b (\delta_{ac} \text{Ric}_{bc} + \delta_{bc} \text{Ric}_{ac} - \delta_{cd} R_{cada})\]
\[= \frac{v_{n-1}}{3} V^a W^b \text{Ric}_{ab}.\]
Combining them together and integrating over the manifold we get the result, which is obviously symmetric and bilinear.

**Remark 2.5.** A localized version of the functionals $g^\Delta$ and $\mathcal{G}^\Delta$ is automatic since they satisfy,
\[
W(f V W \Delta^{m-1}) = g^\Delta(f V, W) = g^\Delta(V, f W),
\]
(2.14)
\[
W(f V W \Delta^{-n}) = \mathcal{G}^\Delta(f V, W) = \mathcal{G}^\Delta(V, f W)
\]
(2.15)
for all $f \in C^\infty(M)$.

We will call $g^\Delta$ and $\mathcal{G}^\Delta$ respectively *metric* and *Einstein* (spectral) functionals.
3. Spectral Functionals for the Operators of Laplace Type

In this section, we will demonstrate how the results vary if one passes to the Laplace-type operators, acting on sections of a vector bundle $V$ of rank $\text{rk}(V)$. These operators generalize the scalar Laplacian in the sense that they have the same principal symbol (times the matrix unit), yet they may contain both some nontrivial connections and torsion.

For this purpose we assume that there is a connection $\nabla$ on the vector bundle $V$, i.e. for any vector field $X$ on $M$, we have a covariant derivative $\nabla X$ on the module of smooth sections of $V$. Using the notation $\nabla_a := \nabla_{\partial_a}$ in local normal coordinates around a fixed point on the manifold we have:

$$\nabla_a = \partial_a - T_a,$$

where each $T_a$ is a $C^\infty(M)$ endomorphism of the sections of $V$. Using normal coordinates and expanding $T_a$ around $x = 0$ we have,

$$T_a(x) = T_a + T_{ab}x^b + o(x).$$

Thus, we can write the symbol of the generalized Laplace operator

$$\Delta_T = -g^{ab}(\nabla_a \nabla_b - \Gamma^c_{ab} \nabla_c),$$

in the normal coordinates as the sum of

$$a_2 = (\delta_{ab} + \frac{1}{3} R_{acbd}x^c x^d) \xi_a \xi_b + o(x^2),$$

$$a_1 = \frac{2i}{3} \text{Ric}_{ab} x^a \xi_b + (2iT_a \xi_a + 2iT_{ab} x^b \xi_a) + o(x),$$

$$a_0 = T_{aa} - T_a T_a + o(1).$$

Then the parts of order $-2, -3, -4$ of the inverse of $\Delta_T$ expanded in $x$ up to the order, respectively $2, 1, 0$, are

$$b_2 = ||\xi||^{-4} (\delta_{ab} - \frac{1}{3} R_{acbd}x^c x^d) \xi_a \xi_b + o(x^2),$$

$$b_3 = -\frac{2i}{3} \text{Ric}_{ab} x^a \xi_b ||\xi||^{-4} - (2iT_a \xi_a + 2iT_{ab} x^b \xi_a) ||\xi||^{-4} + o(x),$$

$$b_4 = \frac{2}{3} \text{Ric}_{ab} \xi_a \xi_b ||\xi||^{-6} - 4T_a T_b \xi_a \xi_b ||\xi||^{-6} - (T_{aa} - T_a T_a) ||\xi||^{-4} + 4T_{ab} \xi_a \xi_b ||\xi||^{-6} + o(1).$$

**Theorem 3.1.** The functional

$$g^{\Delta_T}(V, W) := \mathcal{W}(\nabla_V \nabla_W \Delta_T^{-n-1})$$

does not depend on the connection $T$, and

$$g^{\Delta_T}(V, W) = \text{rk}(V) \ g^{\Delta}(V, W).$$

**Proof.** This is evident, since the principal symbol does not depend on $T$.

**Theorem 3.2.** The functional

$$\mathcal{G}^{\Delta_T}(V, W) := \mathcal{W}(\nabla_V \nabla_W \Delta_T^{-n})$$

is equal to

\[ \mathcal{G}^\Delta_T (V, W) = \frac{v_n - 1}{6} \text{rk}(V) \int_M G(V, W) \, vol_g + \frac{v_n - 1}{2} \int_M F(V, W) \, vol_g, \]

where

\[ F(V, W) = \text{Tr} V^a W^b F_{ab}, \]

and \( F_{ab} \) is the curvature tensor of the connection \( T \). For \( T = 0 \) it is equal to \( \text{rk}(V) \mathcal{G}^h (V, W) \).

**Proof.** First we compute the leading symbols of \( (\Delta_T)^{-m} \) up to the appropriate order in \( x \),

\[ c_{2m} = ||\xi||^{-2m-2} \left( \delta_{ab} - \frac{m}{3} R_{ijkl} x^i x^k \right) \xi_a \xi_b + o(x^2), \]

\[ c_{2m+1} = \frac{-2mi}{3} ||\xi||^{-2m-2} \text{Ric}_{ab} x^k \xi_a - 2mi ||\xi||^{-2m-2} \left( T_a \xi_a + T_{ab} x^b \xi_a + o(x) \right) \]

\[ c_{2m+2} = \frac{m(m+1)}{3} ||\xi||^{-2m-4} \text{Ric}_{ab} \xi_a \xi_b \]

\[ - \frac{2m(m+1)}{2} ||\xi||^{-2m-4} T_a T_b \xi_a \xi_b + m(T_a T_a - T_{ab} x^b) ||\xi||^{-2m-2} \]

\[ + 2m(m+1) ||\xi||^{-2m-4} T_{ab} \xi_a \xi_b + o(1). \]

Expanding in a similar way

\[ \nabla_V V = V^a (\partial_a - T_a - T_{ab} x^b) + o(x), \quad \nabla_W W = W^a (\partial_a - T_a - T_{ab} x^b) + o(x), \]

for vector fields \( V \) and \( W \), we compute only the terms of the symbol of \( \nabla_V \nabla_W (\Delta_T)^{-m} \) of order \(-2m\), which depend on \( T_a \) and \( T_{ab} \) at \( x = 0 \), as the remaining terms would be identical to the case considered earlier,

\[ \sigma_{-2m}(\nabla_V \nabla_W \Delta_T^{-m}) = \left( 2m(m+1) T_a T_b \xi_a \xi_b + m T_a a ||\xi||^2 - m T_a n ||\xi||^2 \right) \]

\[ \left. \right. \]

\[ - 2m(m+1) T_{ab} \xi_a \xi_b + V_j W_k \xi_j \xi_k ||\xi||^{-2m-4} \]

\[ + 2m V^a (\partial_a W^b) T_c \xi_c ||\xi||^{-2m-4} \]

\[ - 2m (V^a W^b + V^b W^a) T_a T_c \xi_a \xi_c ||\xi||^{-2m-2} \]

\[ + V^a W^b T_a b ||\xi||^{-2m} - V^a W^b T_{ab} ||\xi||^{-2m} \]

\[ + 2m T_{ab} (V^a W^b + W^a V^b) \xi_a \xi_c ||\xi||^{-2m-2} \]

\[ - V^a (\partial_a W^b) T_a b ||\xi||^{-2m}. \]

Integrating it over \( S^{2m-1} \) and setting \( x = 0 \) we have the following result for the density of the Wodzicki residue (before taking the trace over endomorphisms)

\[ \omega(\nabla_V \nabla_W \Delta_T^{-m}) = v_{n-1} \left( \frac{1}{2} V^a W^b \left( T^2 \delta_{ab} + T_a T_b + T_b T_a + T_{cc} \delta_{ab} \right) \right. \]

\[ - T^2 \delta_{ab} - T_{cc} \delta_{ab} - T_{ab} - T_{ba} \left) + V^a (\partial_a W^b) T_b \right. \]

\[ - V^a W^b (T_a T_b + T_b T_a) + V^a W^b T_a T_b - V^a W^b T_{ab} \]

\[ + V^a W^b (T_{ab} + T_{ba}) - V^a (\partial_a W^b) T_b \right) \]

\[ = \frac{1}{2} v(S^{n-1}) V^a W^b \left( T_{ab} - T_{ba} + T_a T_b - T_b T_a \right) \]

\[ = \frac{1}{2} v(S^{n-1}) V^a W^b F_{ab}, \]
where $F_{ab}$ is the curvature tensor of the connection $T$, as indeed in the normal coordinates:

$$F_{ab} = -(\partial_a T_b - \partial_b T_a) + [T_a, T_b] = T_{ab} - T_{ba} + [T_a, T_b] + o(1),$$

which finishes the proof.

Note that the functionals $g^\Delta_T$ and $\mathcal{G}^\Delta_T$ are automatically ‘localised’ as in Remark 2.5.

We finish the section by demonstrating that any 0-order perturbation of the Laplace (type) operator $\Delta_T$ does not modify the metric functional, whereas the Einstein functional is modified by a term whose density involves the metric functional multiplied by the trace of the 0-order term.

**Lemma 3.3.** For the Laplace type operator $\Delta_{T,E} := \Delta_T + E$, where $E$ is an endomorphism of the vector bundle, the metric functional does not depend on $E$,

$$g^{\Delta_{T,E}}(V, W) = g^{\Delta_T}(V, W),$$

whereas the Einstein functional

$$\mathcal{G}^{\Delta_{T,E}}(V, W) := \mathcal{W}(\nabla_V \nabla_W \Delta_{T,E}^{-m})$$

reads,

$$\mathcal{G}^{\Delta_{T,E}}(V, W) = \mathcal{G}^{\Delta_T}(V, W) + \frac{1}{2} \int_M (Tr E) g(V, W) \, vol_g.$$

**Proof.** The first statement is obvious. For the second statement by computing the symbols of $\Delta^{-m}$ we first see from the formula (A.6) that $E$ enters the symbol of order $-4$ linearly and, consequently, using (A.11) the only additional term that appears in the symbol of $\Delta^{-m}$ would appear in order $-2m - 2$ as

$$-mE||\xi||^{-2m-2}.$$ 

The only term depending on $E$ that arises in the contribution to the respective symbol of order $-2m$ of the product $\nabla_V \nabla_W \Delta^{-m}$ would then be

$$mEV^aW^b \xi_a \xi_b||\xi||^{-2m-2},$$

which after integrating over the sphere, taking trace and using (2.3) gives

$$\frac{1}{2} (Tr E) g(V, W).$$

As follows from [28, 27] the operators $\Delta_{T,E}$ are the most general Laplace-type operators on $M$.

### 3.1. Spectral functionals for the spin Laplacian.

A particularly interesting example is the application of the above result to the case of the spinor bundle of rank $2^m$ (assuming that the manifold $M$ has a spin structure, which we fix).

Recall that in terms of a (local) basis $e_i$ of orthonormal vector fields on $M$ the Levi-Civita covariant derivative reads

$$\nabla_{e_i}e_j = \alpha_{ijk}e_k,$$

where $\alpha_{ijk}$ is the Levi-Civita connection, expressed explicitly in the basis $e_i$ through the structure constants $c_{ijk}$ of the commutators of orthonormal vector fields,

$$\alpha_{ijk} := \frac{1}{2}(c_{ijk} + c_{kij} + c_{kji}), \quad [e_i, e_j] = c_{ijk}e_k.$$
If $M$ is a spin manifold, then the lift of the Levi-Civita covariant derivative to Dirac spinor fields reads

\[ \nabla^{(s)}_{e_i} = e_i - \frac{1}{4} \alpha_{ijk} \gamma^j \gamma^k, \]

with $\gamma^j$ as in Section 1.3. The spinorial Laplace operator is

\[ \Delta^{(s)} := \nabla^{(s)*} \nabla^{(s)} = -\nabla^{(s)}_{e_i} \nabla^{(s)}_{e_i} + \nabla^{(s)}_{e_i e_i} = -\nabla^{(s)}_{e_i} \nabla^{(s)}_{e_i} + \alpha_{ij} \nabla^{(s)}_{e_j}. \]

In the normal coordinates around a fixed point of the manifold one has then

\[ e_i = \partial_i - \frac{1}{6} R_{ijkl} x^j x^k \frac{\partial}{\partial x^l} + o(x^2) \]

and

\[ \alpha_{ijk} = -\frac{1}{2} R_{ijk} x^l + o(x), \]

and therefore,

\[ \Delta^{(s)} = -\partial_i \partial_i + \frac{1}{3} R_{ijkl} x^j x^k \partial_i \partial_l + o(x^2) \]

\[ + \frac{2}{3} R_{ij} x^j \partial_i + \frac{1}{4} R_{ijkl} x^j \gamma^j \gamma^k \partial_i + o(x) \]

\[ + o(1), \]

where in the three lines we collected terms of different order, expanding them up to the same order in normal coordinates $x$ as their order.

By comparing the symbol with (3.1) it is easy to identify then $\Delta^{(s)}$ as a Laplace-type operator $\Delta_{\gamma}$, with the respective expansion of the spin connection,

\[ T_a = 0, \quad T_{ab} = \frac{1}{8} R_{abjk} \gamma^j \gamma^k. \]

We have then

**Proposition 3.4.** The metric and Einstein functional associated to the spin Laplacian are proportional to the functionals of the scalar Laplacian.

\[ g^{\Delta^{(s)}}(V, W) := W(\nabla^{(s)}_{V} \nabla^{(s)}_{W}(\Delta^{(s)})^{-n-2}) = -m g^{\Delta}(V, W), \]

\[ g^{\Delta^{(s)}}(V, W) := W(\nabla^{(s)}_{V} \nabla^{(s)}(\Delta^{(s)})^{-n}) = -m g^{\Delta}(V, W). \]

**Proof.** This is a consequence of Theorem 3.2. While the result for the metric functional is obvious for the Einstein functional we see the trace of the additional term, arising from the curvature of the connection vanishes,

\[ \frac{1}{4} \text{Tr} V^a W^b \left( R_{abjk} \gamma^j \gamma^k - R_{bajk} \gamma^j \gamma^k \right) = 0, \]

due to the skew symmetry of the Riemann tensor in the two last indices.
3.2. **Spectral functionals for the Dirac operator.** An immediate application of Lemma 3.3 is the computation of the Einstein functional using the Dirac operator instead of the Laplacian. The Dirac operator is, in a local basis of orthonormal frames, a first-order differential operator

\[ D = i \gamma^j \nabla^{(s)}_e, \]

and its square \( D^2 \) differs from the spinorial Laplacian only by a quarter of the scalar of curvature

\[ D^2 = \Delta^{(s)} + \frac{1}{4} R. \]

More generally, if we consider a spin\(_c\) structure and the Dirac operator twisted by a \( U(1) \)-connection,

\[ D_A = D + A, \]

the respective formula reads,

\[ D_A^2 = \Delta^{(s)} + \frac{1}{4} R + F, \]

where

\[ F = \gamma^j \gamma^k F_{jk}, \]

and \( F_{jk} \) is the curvature of \( A \). Then as the consequence of Lemma 3.3 we have,

**Proposition 3.5.** The spectral metric and Einstein functionals associated with the Dirac operator \( D_A \) do not depend on the connection \( A \) and read,

\[
\begin{align*}
g^{D_A}(V, W) &:= \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n-2}) = 2^m g^\Delta(V, W), \\
g^{D_A}(V, W) &:= \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n}) = 2^m \left( g^\Delta(V, W) + \frac{1}{8} \mathcal{R}(V, W) \right),
\end{align*}
\]

where

\[ \mathcal{R}(V, W) = v_{n-1} \int_M R(g) g(V, W) \, vol_g. \]

4. **Metric and Einstein functionals of differential forms**

In differential geometry besides functionals on vector fields, one can alternatively consider functionals on the dual bimodule of one-forms. In this section, we will investigate whether the Einstein tensor (or, more precisely, its contravariant version) can be obtained from such functional using the spectral methods. For this purpose, we need to represent differential forms as differential operators, and a suitable way is to employ Clifford modules. We assume thus that \( M \) is a \( n = 2m \) dimensional spin\(_c\) manifold and use the Clifford representation of one-forms as 0-order differential operators, that is, endomorphisms of a rank \( 2^m \) spinor bundle.

As generating endomorphisms, we can work with Clifford multiplication by the local coframe basis \( e^j \) dual to the orthonormal oriented frame basis \( e_j \), which simply amounts to multiplication by (constant) gamma matrices \( \gamma^j \) as in Section 1.3. Thus, let \( v, w \), with the components with respect to local coordinates \( v_a \) and \( w_a \), respectively, be two differential forms represented in such a way as endomorphisms (matrices) \( \hat{v} \) and \( \hat{w} \) on the spinor bundle.
Theorem 4.1. **The following spectral functionals of one-forms on a spin-c manifold $M$ of dimension $n$**

$$
\mathcal{G}_D(v, w) := \mathcal{W} \left( \hat{v} \hat{w} D^{-n} \right),
$$

(4.1)

$$
\mathcal{G}_D(v, w) := \mathcal{W} \left( \hat{v} (D \hat{w} + \hat{w} D) D^{-n+1} \right)
= \mathcal{W} \left( (D \hat{v} + \hat{v} D) \hat{w} D^{-n+1} \right),
$$

read

$$
\mathcal{G}_D(v, w) = 2^m v_{n-1} \int_M g(v, w) \text{vol}_g,
$$

(4.2)

$$
\mathcal{G}_D(v, w) = 2^m \frac{v_{n-1}}{6} \int_M G(v, w) \text{vol}_g,
$$

where $g(v, w) = g^{ab} v_a w_b$ and $G(v, w) = (\text{Ric}^{ab} - \frac{1}{2} R g^{ab}) v_a w_b$, using the expressions $v = v_a dx^a$, $w = w_b dx^b$ in any local coordinates.

**Proof.** The proof of the formula for the metric functional $\mathcal{G}_D$ is easy and we skip it, concentrating on the Einstein functional, splitting it into two parts as follows:

$$
\mathcal{E}_1(v, w) = \mathcal{W} \left( \hat{v} D \hat{w} D D^{-n} \right),
$$

and

$$
\mathcal{E}_2(v, w) = \mathcal{W} \left( \hat{v} \hat{w} D^{-n+2} \right).
$$

We start with the first. Let us again work with normal coordinates $x$ around a fixed point on the manifold $M$ with $x = 0$. We can rewrite the first three leading symbols of $D^{-n} = D^{-2m}$ (using (3.4) and the Lichnerowicz formula),

$$
c_{2m} = ||\xi||^{-2m-2} \left( \delta_{ab} - \frac{m}{3} R_{abjk} x^j x^k \right) \xi_a \xi_b + o(x^2),
$$

(4.3)

$$
c_{2m+1} = \frac{-2mi}{3} ||\xi||^{-2m-2} R^{ik} x^k \xi_a - 2mi ||\xi||^{-2m-2} (T_{ab} x^b \xi_a) + o(x),
$$

$$
c_{2m+2} = \frac{m(m+1)}{3} ||\xi||^{-2m-4} R^{ik} x^k \xi_a \xi_b - \frac{m}{4} R ||\xi||^{-2m-2} + o(1),
$$

where

$$
T_{ab} = \frac{1}{8} R_{abjk} \gamma^j \gamma^k,
$$

and we have used the antisymmetry of $T_{ab}$. The Clifford images of $v$ and $w$ have a local expansion around $x = 0$ in normal coordinates,

$$
\hat{v} = v_a \gamma^a + o(1), \quad \hat{w} = w_a \gamma^a + w_{ab} \gamma^a x^b + o(x),
$$

where $v_a$, $w_a$, and $w_{ab}$ are constants. Note that here due to the metric in normal coordinates at $x = 0$ being Euclidean, the gamma matrices satisfy $\{ \gamma^a, \gamma^b \} = 2 \delta^{ab}$. Observe that the symbol of $\hat{v} D$ needs to be expanded to $o(1)$ whereas symbol of $\hat{w} D$ to $o(x)$, and so they read

$$
\sigma(\hat{v} D) = iv_a \gamma^a \gamma^j i \xi_j + o(1),
$$

(4.4)

$$
\sigma(\hat{w} D) = iw_a \gamma^a \gamma^j (i \xi_j + \frac{1}{8} R_{\ell jps} x^\ell \gamma^p \gamma^s) - iw_{ab} \gamma^a \gamma^j \xi_j x^b + o(x).
$$

The constants $w_{ab}$ arise from the dependence of the one-form $w$ on the coordinates around $x = 0$. 


The symbol of $\hat{D}\hat{w}D$ up to order $o(1)$ in normal coordinates is
\begin{equation}
(4.5)
\sigma(\hat{D}\hat{w}D) = v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \xi_j \xi_k - \frac{1}{8} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \gamma^p \gamma^q R_{jkps} \\
- iv_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \xi_j + o(1).
\end{equation}

We can omit additional terms with explicit dependence on normal coordinates as they will all vanish at $x = 0$.

The symbol of order $-n$ of the product $\hat{D}\hat{w}D D^{-n}$ comes then (at $x = 0$) as
\begin{equation}
(4.6)
\begin{align}
\sigma_{-n}(\hat{D}\hat{w}D D^{-n}) &= \frac{m(m+1)}{3} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \text{Ric}_{rs} \xi_j \xi_s \xi_k \xi_l ||\xi||^{-2m-4} \\
&\quad - \frac{m}{4} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k R_{ks} \xi_k ||\xi||^{-2m-2} \\
&\quad - \frac{2m}{3} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k (\text{Ric}_{rs} \xi_s \xi_r + \text{Ric}_{rs} \xi_s \xi_r) ||\xi||^{-2m-2} \\
&\quad - \frac{m}{4} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \gamma^p \gamma^q (R_{rps} \xi_p \xi_s + R_{rps} \xi_p \xi_s) ||\xi||^{-2m-2} \\
&\quad + \frac{m}{3} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k (R_{rps} \xi_p \xi_s + R_{rps} \xi_p \xi_s) ||\xi||^{-2m-2} \\
&\quad - \frac{1}{8} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \gamma^p \gamma^q R_{jkps} ||\xi||^{-2m} + o(1).
\end{align}
\end{equation}

We see at once that it does not contain any term with $w_{ab}$ hence the result will be bilinear in the differential forms. Integrating over $\xi \in S^{n-1}$ and taking the trace over the matrices $\gamma$ we obtain:
\begin{equation}
G_1(v, w) = 2^n v_{n-1} - 6 \int_M \left( G(v, w) + \frac{n-2}{4} g(v, w) R \right) \text{vol}_g.
\end{equation}

The functional $G_2$ requires the computation of the symbol of $D^{-n+2}$ up to $o(1)$ as the product of two forms is an operator of order zero. Then, using the explicit formula for the symbol of $D^{-n+2}$ (4.3), we obtain
\begin{equation}
G_2(v, w) = -2^n v_{n-1} - 6 \int_M \frac{n-2}{4} g(v, w) R \text{vol}_g.
\end{equation}

This shows our statement about $G_D = G_1 + G_2$.

5. TOWARDS NONCOMMUTATIVE METRIC AND EINSTEIN FUNCTIONALS

The spectral methods that we have proposed to obtain the metric and Einstein functionals are well suited for generalisation to the noncommutative case. So far, almost exclusively scalar geometric quantities (like scalar curvature) were computed for noncommutative tori, with conformally rescaled Laplace and Dirac operators (see [22] for a review) and partial conformal rescaling [17] [11]. As the tensors carry more information than scalar geometric objects their study on noncommutative level is certainly more interesting though definitely complicated, especially that a general algebra may have no outer derivations which are usually regarded as the noncommutative counterpart of vector fields. On the other hand, differential forms are naturally associated to any spectral triple, with the latter deemed to best encode the notion of a Riemannian manifold in the noncommutative case.

On the quantum tori we take advantage of both of these structures and correspondingly propose definitions of the associated spectral metric and Einstein tensors using the generalisation of
the Wodzicki residue on the generalised algebra of symbols on noncommutative tori. We compute explicitly for 2- and 4-dimensional noncommutative tori the relevant functionals of outer derivations for the conformally rescaled Laplace operators and of one-forms with conformally rescaled Dirac operators. We also define analogous functionals for regular finitely summable spectral triples and study their product with the simplest nontrivial finite spectral triple.

5.1. The metric and Einstein functionals for the Laplacian on a noncommutative tori. The quantum tori are prominent examples of noncommutative manifolds which admit noncommutative analogues of many classical geometrical objects. In particular, there are outer derivations that act on the smooth algebra $\mathcal{A} = C^{\infty}(T^2_\theta)$ and that can be interpreted as noncommutative vector fields, even though in general they form only a complex vector space rather than an $\mathcal{A}$-bimodule. It is then straightforward to identify a noncommutative counterpart of the flat-metric Laplace operator. This can also be generalised to the case of conformally rescaled geometry, where the conformal factor is taken as a positive element of the algebra of the noncommutative torus.

We can therefore exploit our definition of the metric and Einstein spectral functionals to define the corresponding functionals for the noncommutative tori and compute them explicitly. For that purpose we use the pseudodifferential calculus of symbols as defined in [15] and then used and developed by many authors (see [22] for a review). The rules of the algebra of symbols are almost identical to the usual ones, however, with the partial derivatives replaced by derivations and the symbols valued in the algebra of noncommutative torus.

This algebra admits a natural generalisation of the Wodzicki residue, which is defined as the trace of the integral over the cosphere $||\xi|| = 1$ of the symbol of order $-n$ (where $n$ is the dimension of the torus). The existence of such Wodzicki residue trace on the algebra of symbols over the noncommutative 2-torus was demonstrated and discussed in [24, 41, 50]. We denote it again by $\mathcal{W}$.

However, for the conformally rescaled $T^2_\theta$ we find it convenient to work with an enlarged algebra $\hat{\mathcal{A}}$, which is generated by $\mathcal{A}$ and its copy $\mathcal{A}^\circ$ commuting with $\mathcal{A}$. Indeed for the Laplace-type operators, the relevant Weyl factor in principle can be taken from $\hat{\mathcal{A}}$, though usually it assumed to be in $\mathcal{A}$ and we adhere to this convention. For the spectral triple with conformally rescaled Dirac operator, the conformal factor is instead assumed to be from $\mathcal{A}^\circ$ as only in such a case the associated differential one-forms (generated by $\mathcal{A}$ and its commutators with the Dirac operator) are bounded operators, on which our functionals will be defined.

Now, the standard flat Dirac operator on $T^2_\theta$ as well as its conformal rescaling are clearly first-order differential operators for the extended pseudodifferential calculus with symbols valued in the algebra $\hat{\mathcal{A}}$, since they are just built from the derivations on $\mathcal{A}$ which extend to derivations on $\hat{\mathcal{A}}$. Next, considering for simplicity only the strictly irrational [48] noncommutative torus (with the center of algebra equal to $\mathbb{C}$), there is an obvious factorized trace $\tau^\otimes$ on the enlarged algebra $\mathcal{A}$ given by $\tau^\otimes(ab^\circ) = \tau(a)\tau(b^\circ)$. Moreover, since $\tau^\otimes$ is invariant under derivations like $\tau$, we use it to define the tracial Wodzicki residue on $\mathcal{A}$-valued symbols as above, and still denote it by $\mathcal{W}$ and its density by $\omega$. 
5.1.1. The metric and Einstein functionals for the conformally rescaled Laplacian on a non-commutative 2-torus. Since every two-dimensional Riemannian manifold has a vanishing Einstein tensor, it is natural to ask whether this holds also for the noncommutative torus. We shall demonstrate that it is true in the case of a conformally rescaled Laplace operator. We denote by \( \mathcal{A} = C^\infty(\mathbb{T}_\theta^2) \) the algebra of smooth elements of the noncommutative two-torus and by \( \tau \) the standard trace over its \( C^* \)-algebraic completion. By \( \mathcal{H} = L^2(\mathbb{T}_\theta^2, \tau) \) we denote the standard Hilbert space obtained by the GNS construction for the tracial state \( \tau \). By \( \delta_1, \delta_2 \) we denote the basis of derivations of the algebra \( \mathcal{A} \) implemented as densely defined operators on \( \mathcal{H} \).

**Definition 5.1.** Let \( h \in C^\infty(\mathbb{T}_\theta^2) \) be positive, invertible, with a bounded inverse. We define as the conformally rescaled Laplace operator for the noncommutative torus the following densely defined selfadjoint operator on \( \mathcal{H} \):

\[
(5.1) \quad \Delta_h = h^{-1} \Delta h^{-1},
\]

where

\[
(5.2) \quad \Delta = \sum_{a=1,2} \delta_a^2.
\]

This definition is motivated by the commutative case \( \theta = 0 \) where the operator \( \Delta_h \) is unitarily equivalent to the operator \( h^{-2} \Delta \), which is a densely defined self-adjoint operator on \( \mathcal{H}_h = L^2(\mathbb{T}^2, \tau_h) \), where \( \tau_h(a) = \tau(h^2 a) \).
Similarly, we define the vector fields as appropriate self-adjoint generalisations of operators unitarily equivalent to derivations,

\[ V_h = \sum_{a=1,2} V^a h \delta_a h^{-1}, \]

where \( V^a \in \mathbb{C} \).

**Proposition 5.2.** For the conformally rescaled Laplace operator on a noncommutative 2-torus the metric functional reads

\[ g^{\Delta_h}(V_h, W_h) = \mathcal{W} \left( V_h W_h \Delta_h^{-2} \right) = \pi \tau (h^4) V^a W^b \delta_{ab}, \]

whereas the spectral Einstein functional and its density vanish identically

\[ \mathcal{G}^{\Delta_h}(V_h, W_h) = \mathcal{W} \left( V_h W_h \Delta_h^{-1} \right) = 0. \]

**Proof.** The metric functional is straightforwardly computed

\[
\begin{align*}
\mathcal{G}^{\Delta_h}(V_h, W_h) &:= \mathcal{W} \left( (h V^a \delta_a h^{-1})(h W^b \delta_b h^{-1})(h^{-1} \Delta h^{-1})^{-2} \right) \\
&= \mathcal{W} \left( h V^a W^b \delta_a \delta_b \Delta^{-1} h^2 \Delta^{-1} \right) \\
&= \pi \tau \left( h^4 V^a W^b \delta_{ab} \right) = \pi \tau (h^4) V^a W^b \delta_{ab}.
\end{align*}
\]

Note that the components of the noncommutative metric tensor functional scale as \( h^4 \) as in the classical situation.

Next, we compute the Einstein functional,

\[
\begin{align*}
\mathcal{G}^{\Delta_h}(V_h, W_h) :&= \mathcal{W} \left( (h V^a \delta_a h^{-1})(h W^b \delta_b h^{-1})(h^{-1} \Delta h^{-1})^{-1} \right) \\
&= \mathcal{W} \left( h V^a W^b \delta_a \delta_b \Delta^{-1} h \right) = \mathcal{W} \left( h^2 V^a W^b \delta_{ab} \Delta^{-1} \right).
\end{align*}
\]

Since \( \Delta \) is the standard (flat) Laplace operator and \( V, W \) are just \( \mathbb{C} \)-linear combinations of the derivations, \( V^a W^b \delta_a \delta_b \Delta^{-1} \) is an operator that has exclusively symbol of order 0. Since its symbol of order \(-2\) vanishes, its product with any element from the algebra, like \( h^2 \), has the same property, and thus the Wodzicki residue of it vanishes.

Note that by using the same argument and the trace property of the Wodzicki residue we also show that for any \( \alpha \in \mathcal{A} \) the localised functional vanishes

\[ \mathcal{G}^{\Delta_h}(\alpha, V, W) = \mathcal{W} \left( \alpha V W \Delta_h^{-1} \right) = 0 \]

(here \( \alpha V \), in general, does not implement an algebra derivation). Thus, the conformally rescaled geometry of the noncommutative 2-torus has indeed the same property as a 2-dimensional manifold - its Einstein tensor vanishes.

We finish this example by reiterating that the analysis does not change if \( h \) is taken from the larger algebra \( \hat{\mathcal{A}} \).

5.1.2. The metric and Einstein functionals for the conformally rescaled Laplacian on a noncommutative 4-torus. As the analogue of the conformally rescaled Laplace operator over the noncommutative 4-torus (acting on \( \mathcal{H} = L^2(T^*_0, \tau) \)) we simply take

\[
\Delta_h = \sum_{a=1}^{4} \chi^{-1} \cdot \delta_a \cdot \chi \cdot \delta_a \cdot \chi^{-1},
\]

(here \( \chi^{-1} \) denotes the inverse of \( \chi \)).
where \( \chi = \hbar^2 \) is a positive element of the algebra \( C^\infty(T^4_\theta) \) and \( \delta_a \) are the standard derivations extended as operators on a dense subspace of \( L^2(T^4_\theta, \tau) \). The deformation parameter \( \theta \) is here, in fact, a matrix (such that the center of the algebra is \( \mathbb{C} \)). Note that the operator \( \Delta_h \) is again a noncommutative generalisation of an operator that classically (for \( \theta = 0 \)) is unitarily equivalent to a Laplace operator for the conformally rescaled metric on the four-torus.

Similarly, as in the case of 2-dimensional noncommutative torus we take two arbitrary vector fields \( V, W \) understood as \( h \)-rescaled linear combinations of derivations,

\[
V_h = V^a \chi_a \chi^{-1}, \quad W_h = W^b \chi_b \chi^{-1}.
\]

**Proposition 5.3.** The spectral metric and Einstein functionals on derivations for a conformally rescaled Laplace operator over the noncommutative 4-torus are, respectively,

\[
\varrho^{\Delta_h}(V_h, W_h) = 2\pi^2 \tau(\chi^3) V^a W^b \delta_{ab},
\]

\[
\varrho^{\Delta_h}(V_h, W_h) = 2\pi^2 \tau \left( -\frac{1}{24} \chi(V^a \delta_a \chi) \chi^{-1}(W^b \delta_b \chi) - \frac{1}{24} \chi(W^a \delta_a \chi) \chi^{-1}(V^b \delta_b \chi) + \frac{5}{24} \chi(V^a \delta_a \chi) \chi^{-1}(W^b \delta_b \chi) + \frac{5}{24} \chi(W^a \delta_a \chi) \chi^{-1}(V^b \delta_b \chi) - \frac{1}{24} \chi(V^a \delta_a \chi)(W^b \delta_b \chi) - \frac{1}{24} \chi(W^a \delta_a \chi)(V^b \delta_b \chi) - \frac{1}{3} \chi(V^a W^b(\delta_{ab} \chi) + \frac{1}{6} \chi V^a W^b(\delta_{ab} \chi) + \left( -\frac{1}{24} (\delta_{ab} \chi) \chi^{-1}(\delta_a \chi) \chi^{-1}(\delta_b \chi) - \frac{1}{24} \chi(\delta_{ab} \chi) \chi^{-1}(\delta_{ab} \chi) + \frac{1}{12} \chi(\Delta \chi) + \frac{1}{12} \chi(\Delta \chi) \chi^{-1}(\delta_{ab} \chi) + \frac{1}{6} \chi(\Delta \chi) \chi^{-1}(\delta_{ab} \chi) + \frac{2\pi^2}{6} \sqrt{\varrho} G_{ab} V^a W^b. \quad (5.6)
\]

We skip the proof, which is a straightforward and tedious computation of the respective symbols and the integration over the cosphere. Note that the above expression can be further simplified using the trace property of \( \tau \). Let us also note that the expression has a well-defined commutative limit, in which case the spectral Einstein functional density becomes as follows:

\[
\varrho(V W \Delta_h^{-2}) = 2\pi^2 \left( \frac{1}{4} V^a W^b(\delta_{ab} \chi) - \frac{1}{6} V^a V^b(\delta_{ab} \chi) + \frac{1}{8} (\delta_{ab} \chi)(\delta_{ab} \chi)(V^a W^b) + \frac{1}{6} \chi(\Delta \chi)(V^a W^b) \right), \quad (5.7)
\]

and is exactly equal to:

\[
\frac{2\pi^2}{6} \sqrt{\varrho} G_{ab} V^a W^b.
\]

### 5.2. Spectral Einstein and metric tensor for spectral triples

Let \( (\mathfrak{A}, D, \mathcal{H}) \) be a \( n \)-summable spectral triple and \( \Omega_\beta(\mathfrak{A}) \) be the \( \mathfrak{A} \) bimodule of one-forms generated by \( \mathfrak{A} \) and \( [D, \mathfrak{A}] \), which by definition consists of bounded operators on \( \mathcal{H} \). We assume that there exists a generalised algebra of pseudodifferential operators which contains \( \mathfrak{A}, D, \mathcal{H}, |D|^{\ell} \) for \( \ell \in \mathbb{Z} \), and there exists a tracial state \( \varrho \) on it, still called a noncommutative residue, which identically vanishes on \( T |D|^{-k} \) for any \( k > n \) and a zero-order operator \( T \) (an operator in the algebra generated by \( \mathfrak{A} \))
and $\Omega^1_D(\mathcal{A})$. We propose the following definition of metric and Einstein functionals for spectral triples.

**Definition 5.4.** The spectral metric functional on differential forms is

$$g_D(v, w) = \mathcal{W}(vw|D|^{-n}),$$

where $v, w \in \Omega^1_D(\mathcal{A})$, and the Einstein functional is

$$\mathcal{G}_D(v, w) = \mathcal{W}(v\{D, w\}D|D|^{-n}).$$

Note that since $\Omega^1_D(\mathcal{A})$ is a bimodule over $\mathcal{A}$ these functionals are already localised, as one can replace $v$ by $av$. A simple consequence of the properties of $\mathcal{W}$ on the generalised pseudodifferential calculus for regular spectral triples are the following linearities for the spectral metric functional over one-forms,

$$g_D(vb, w) = g_D(v, bw), \quad g_D(av, w) = g_D(v, wa).$$

However, unlike in the classical case it is not entirely obvious whether any form of linearity over $\mathcal{A}$ in the second entry for the Einstein functional will hold in general. To ensure this property, we propose the following requirement on a class of spectral triples.

**Definition 5.5.** We say that a spectral triple with a trace on the generalised algebra of pseudodifferential operators is spectrally closed if for any zero-order operator $T$ (as defined earlier) the following holds:

$$\mathcal{W}(TD|D|^{-n}) = 0.$$

**Lemma 5.6.** The classical spectral triple over a closed oriented spin-c manifold $M$ of dimension $n = 2m$ is spectrally closed in the above sense.

**Proof.** Computing the symbol of $D|D|^{-n} = DD^{-2m}$ using the expansion (4.3) we see that it vanishes identically at $x = 0$. Since $T$ is an operator of order zero (does not depend on $\xi$) the product $TDD^{-2m}$ also has a vanishing symbol of order $-n$, hence both the density of the Wodzicki residue and the residue itself vanish identically.

**Theorem 5.7.** If a spectral triple with a noncommutative residue over the generalised pseudodifferential calculus is spectrally closed, then the localised Einstein functional satisfies

$$\mathcal{G}_D(vb, w) = \mathcal{G}_D(v, bw),$$

for $b \in \mathcal{A}$.

**Proof.** We compute

$$\mathcal{G}_D(v, bw) = \mathcal{W}(v\{D, bw\}D|D|^{-n})$$

$$= \mathcal{W}(v([D, b]w + b\{D, w\})|D|^{-n})$$

$$= \mathcal{W}(vB\{D, w\}|D|^{-n}) = \mathcal{G}_D(vb, w),$$

since the first term in the second line contains a product of three one-forms and therefore is a 0-th order term, so by the assumption of spectral closedness it vanishes.
Remark 5.8. Note that in the case of spectral triple over a manifold, since one-forms commute with $\alpha, \beta \in \mathcal{A}$, the Einstein functional has the same linearity over the algebra as a tensor.

Next, we present two important situations where the assumptions before Definition 5.4 are satisfied. The first is based on the algebra of pseudodifferential operators over noncommutative tori \([7, 15]\) and the second on the abstract algebra of pseudodifferential operators for regular $n$-summable spectral triples \([14]\).

5.2.1. The metric and Einstein functionals for the conformally rescaled spectral triple on a noncommutative tori.

We begin with the spectral triple on the noncommutative $n$-torus, $(\mathcal{A}, \mathcal{H}, D_k)$, where $\mathcal{A}$ is the $\delta_a$-smooth subalgebra and $D_k := kDk$ is the conformal rescaling of the standard (flat) Dirac operator $D = \sum_a \gamma_a \delta_a$ with $\gamma_a$ as in Section 1.3. As argued in [17] the conformal factor $k > 0$ has to be taken from the copy $\mathcal{A}_o$ of $\mathcal{A}$ in its commutant. The bimodule of one-forms, generated by the commutators $[D_k, \alpha], \alpha \in \mathcal{A}$, is a free left module generated by $k^2 \gamma^j$. With assumptions as in Section 5.1 using the extension of the calculus in [7] to $\hat{\mathcal{A}}$-valued symbols and of the analogue of the Wodzicki residue, the calculations are very much similar to the case of functionals for the Laplace operator on noncommutative tori. We provide them explicitly for the two lowest even dimensional cases.

Example 5.9. Functionals of the conformally rescaled spectral triple on the noncommutative 2-torus.

As argued in [17] it is possible to construct a usual (untwisted) spectral triple with a conformally rescaled Dirac operator, however, the conformal factor $k > 0$ has to be taken from the commutant of the algebra. With $\mathcal{A} = C^\infty(T^2_\theta)$ and $\mathcal{A}_o$ a copy of $\mathcal{A}$ in the commutant of $\mathcal{A}$, such a spectral triple is given by $(\mathcal{A}, D_k = kDk, \mathcal{H} \otimes C^2)$, where $D = D_1$ and $k \in \mathcal{A}_o$ is the standard flat Dirac operator,

$$D = \begin{pmatrix} 0 & \delta_2 - i\delta_1 \\ \delta_2 + i\delta_1 & 0 \end{pmatrix}. $$

In fact $D_k$ is an analogue of the classical Dirac operator for the flat metric rescaled conformally and unitarily transformed to act on Hilbert space with the volume measure of the flat metric.

The space of one-forms is a bimodule over $\mathcal{A}$ generated by all commutators $[D_k, \alpha], \alpha \in \mathcal{A}$ and it can be shown that it is a free left module generated by $k^2 \sigma^j, j = 1, 2$, where $\sigma^j$ are Pauli matrices.

First, we show,

Lemma 5.10. The conformally rescaled spectral triple on the noncommutative 2-torus, $(\mathcal{A}, D_k = kDk, \mathcal{H} \otimes C^2)$, is spectrally closed.

Proof. We have

$$\sigma_{-2}(D_k^{-1}) = \sigma_{-2}(k^{-1}D^{-1}k^{-1}) = \sigma^p \left( \frac{\delta_{pq}}{||\xi||^2} - 2 \frac{\xi_p \xi_q}{||\xi||^4} \right) (-k^{-2}\delta_q k k^{-1}).$$

Integrating it over the $||\xi|| = 1$ cosphere gives identically zero, therefore the Wodzicki residue of any expression that is a product of a zero-order operator with $D_k^{-1}$ vanishes.

Next, we have the following.
Proposition 5.11. For the conformally rescaled spectral triple over the noncommutative 2-torus the metric functional for \( v = k^2 V^a \sigma^a \) and \( w = k^2 W^a \sigma^a \), \( V^a, W^a \in \mathcal{A} \), reads

\[
\mathcal{G}_{D_k}(v, w) = \tau(V^a W^a),
\]

whereas the spectral Einstein functional vanishes identically,

\[
\mathcal{G}_{D_k}(v, w) = 0.
\]

Proof. The computation of the metric functional is straightforward. Next, since the spectral triple is spectrally closed, it suffices to compute the spectral Einstein functional over the one-forms that generate the bimodule \( \Omega_{D_k}(\mathcal{A}) \),

\[
\mathcal{G}_{j\ell} = \mathcal{W}(k^2 \sigma^j \{ D_k, k^2 \sigma^\ell \} D_k D^{-2}).
\]

The expression in curly brackets can be rewritten as

\[
\sigma^j k^2 [D_k, k^2 \sigma^\ell] k^{-1} D^{-1} k^{-1} + 2\sigma^j \sigma^\ell k^4.
\]

Since the second term is of order zero, we only need to compute the symbol of order \(-2\) of the first term, which can be rewritten as

\[
\sigma^j k^2 [D, k^2 \sigma^\ell] D^{-1} k^{-1}.
\]

Then, using the tracial property of \( \mathcal{W} \) we get

\[
\mathcal{G}_{j\ell} = \mathcal{W}(\sigma^j k^2 [D, k^2 \sigma^\ell] D^{-1}) = \mathcal{W}(\sigma^j k^2 [D, k^2 \sigma^\ell] D^{-1}) + \mathcal{W}(\sigma^j k^4 [D, \sigma^\ell] D^{-1}),
\]

which vanishes since both expressions have a vanishing symbol of order \(-2\).

It is rewarding to see that the conformally rescaled spectral triple of the noncommutative 2-torus shares indeed the same property with its commutative counterpart, namely, it has vanishing Einstein tensor.

Example 5.12. The functionals for the conformally rescaled spectral triple on the noncommutative 4-torus.

The situation is more complicated for the strictly irrational 4-dimensional non-commutative torus. Following the classical situation, given a positive invertible \( k \in \mathcal{A}^o \) (which is in the commutant of \( \mathcal{A} \)), we take \( D_k = kDk \) as the conformally rescaled and unitarily transformed to act on \( L^2(T_4^1, \tau) \otimes \mathbb{C}^4 \), operator \( D \). Let \( v \) and \( w \) be two arbitrary one-forms in the Clifford algebra generated by \( \mathcal{A} \) and \([D, \mathcal{A}]\),

\[
v = k^2 V^a \gamma^a, \quad W = k^2 W^a \gamma^a,
\]

where \( \gamma^a, a = 1, \ldots, 4 \) are the standard gamma matrices, \( \{ \gamma^a, \gamma^b \} = 2\delta_{ab} \), and \( V^a, W^a \) are elements of \( \mathcal{A} \) (which, of course, commute with \( k \)). First, we establish.

Lemma 5.13. The conformally rescaled spectral triple for a 4-dimensional noncommutative torus is spectrally closed.
that \( \tau \) scales exactly like the covariant Einstein tensor. The scaling factor comes from the convention on the 4-torus with the metric which recovers the classical formula for the Einstein tensor for a conformally deformed metric (5.10).

\[
\text{Remark 5.15.} \quad \text{only that to get (5.9) we used the cyclicity of the trace.}
\]

Therefore taking the trace over the algebra gives
\[
\mathcal{W}(TD_k^{-3}) = 0.
\]

Next, we explicitly compute the metric and Einstein tensor functional for \( v, w \).

**Proposition 5.14.** The metric and the Einstein functionals for the conformally rescaled spectral triple on the noncommutative 4-torus are, respectively,

\[
\mathcal{G}_{D_k}(v, w) = \tau \left( W^a V^b k^{-4} \right),
\]

\[
\mathcal{G}_{D_k}(v, w) = \tau \left( V^a W^b \left( \frac{1}{3} k^{-4} (\delta_a k) k^2 (\delta_b k) + \frac{2}{3} k^{-3} (\delta_a k) k^1 (\delta_b k) + k^{-2} (\delta_a k) (\delta_b k) \right) + \frac{2}{3} k^{-1} (\delta_a k) k^{-1} (\delta_b k) - \frac{4}{3} k (\delta_a k) k^{-3} (\delta_b k) - \frac{2}{3} k^2 (\delta_a k) k^{-4} (\delta_b k) + \frac{2}{3} k^{-1} (\delta_a \delta_b k) + \frac{1}{3} k^{-1} (\delta_a k) k^{-1} (\delta_b k) + \frac{1}{3} k^2 (\delta_a k) k^{-4} (\delta_b k) + \frac{2}{3} k^1 (\delta_a k) k^{-3} (\delta_b k) - \frac{2}{3} k^{-1} (\Delta k) \right) \right).
\]

We skip the proof which consists of a straightforward though tedious computation noting only that to get (5.9) we used the cyclicity of the trace.

**Remark 5.15.** The above expression has a well-defined commutative limit,

\[
\mathcal{G}_{D_k}(v, w) = \frac{1}{\text{vol}(T^4)} \int_{T^4} V^a W^b \left( \left( \frac{2}{3} k^{-2} k_a k_b + \frac{2}{3} k^{-1} k_{ab} \right) + \left( \frac{4}{3} \delta_{ab} k^{-2} k_c k_c - \frac{2}{3} \delta_{ab} k^{-1} k_{cc} \right) \right),
\]

which recovers the classical formula for the Einstein tensor for a conformally deformed metric on the 4-torus with the metric \( g_{ab} = k^{-4} \delta_{ab} \), (5.11)

\[
G_{ab} = 4 \left( k^{-2} k_a k_b + k^{-1} k_{ab} \right) + 8 \delta_{ab} k^{-2} k_c k_c - 4 \delta_{ab} k^{-1} k_{cc}
\]

Note that the density of the functional involving the one-forms uses the contravariant Einstein tensor and the volume form, and scales with \( k \) like \( k^4 k^4 \sqrt{k^{-16}} \) so the density in 4 dimensions scales exactly like the covariant Einstein tensor. The scaling factor comes from the convention that \( \tau \) is normalised with \( \tau(1) = 1 \).
5.2.2. The metric and Einstein functionals for a regular finitely summable spectral triple. The second situation, which is different from the conformally rescaled geometries of noncommutative tori, is the case of regular, finitely summable spectral triples (for simplicity with simple dimension spectrum). In this case there is a PDO algebra and calculus of symbols as defined by Connes and Moscovici [14] (see also [34] and [53]) and there exists a tracial state (see [14] Proposition II.1) denoted \( W \) (as before). The definition and methods of computation of the spectral metric and Einstein functionals are, however, quite involved and explicit expressions are feasible only in some special cases of highly symmetric Dirac operators (like the flat Dirac operator on noncommutative tori or fully symmetric Dirac operators on spheres). One can study though some functorial properties, for example the behaviour of the functionals under tensor product of regular spectral triples, of which a particular instance is a finite spectral triple as the second factor. When the first factor is the classical spectral triple (corresponding to spin-c manifolds) this corresponds to almost commutative geometries which play a significant role as models for the geometry of physical interactions.

Below we present an example of the metric and Einstein functionals for the tensor product of a regular (not necessarily commutative) spectral triple with a simplest non-trivial spectral triple on two points, expressing them in terms of the functionals on the components.

**Example 5.16.** Functionals for the spectral triple on \( \mathcal{A} \otimes \mathbb{C}^2 \).

We assume that \((\mathcal{A}, D, \mathcal{H})\) is an even spectral triple with grading \( \gamma \) of dimension \( n \) satisfying the assumptions described in section 5.2, which is spectrally closed, and we consider \((\mathcal{A} \otimes \mathbb{C}^2, D, \mathcal{H} \otimes \mathbb{C}^2)\), where

\[
\mathcal{D} = \begin{pmatrix} D & \gamma c \\ \gamma c^* & D \end{pmatrix},
\]

where \( c \in \mathbb{C} \).

It is easy to see that the bimodule of one-forms, associated to \( \mathcal{D} \) consists of the following operators,

\[
\omega = \begin{pmatrix} w_+ & \gamma c \phi_+ \\ \gamma c^* \phi_- & w_- \end{pmatrix},
\]

where \( w_\pm \in \Omega^1_D(\mathcal{A}) \) and \( \phi_\pm \in \mathcal{A} \).

We can now state

**Proposition 5.17.** Let \( \omega, \omega' \) be two one-forms in the spectral triple \((\mathcal{A} \otimes \mathbb{C}^2, \mathcal{D}, \mathcal{H} \otimes \mathbb{C}^2)\). Then,

\[
g_{\mathcal{D}}(\omega, \omega') = g_D(w_+, w'_+) + g_D(w_-, w'_-) + cc^* v \left( \phi_+ \phi'_+ + \phi_- \phi'_+ \right),
\]

and

\[
G_{\mathcal{D}}(\omega, \omega') = G_D(w_+, w'_+) + G_D(w_-, w'_-),
\]

\[
+ cc^* g_D(w_+ - w_-, w'_+ - w'_-) - \frac{n}{2} cc^* \left( g_D(w_+, w'_+) + g_D(w_-, w'_-) \right)
\]

\[
+ cc^* \left( g_D(w_+, d\phi'_+) - g_D(d\phi_+, w'_+) \right) + g_D(w_-, d\phi'_-) - g_D(d\phi_-, w'_+) \right)
\]

\[
+ (cc^*)^2 v \left( (\phi_+ + \phi_-)(\phi'_+ + \phi'_-) \right),
\]

where \( d\phi = [D, \phi] \) for \( \phi \in \mathcal{A} \).
We skip the computational proof mentioning only that the condition of being spectrally closed does not need to be preserved in the tensor product with a finite spectral triple.

6. Final remarks

The concept that geometric objects like tensors (metric, torsion and curvature tensors) can be expressed using spectral methods provides an invaluable possibility to study them globally both for the manifolds as well as for various extensions of geometries like noncommutative geometry.

First of all, the metric functionals we constructed should be compared with various other concepts for the metric tensor proposed in the noncommutative realm on the algebraic level. Furthermore, possible relations with the notion of a distance (between states) and quantum metric spaces should be examined. The possibility of studying spectral functionals linked to connections leads to the possibility of defining an abstract notion of torsion and torsion-free connection. This would provide a natural contact with the various concepts of linear connections and possibly then with such notions as the Levi-Civita connection in the noncommutative case, for example [26, 49, 3, 4, 5].

The newly introduced Einstein functional can be further computed for a variety of spectral triples on interesting algebras. This includes, in particular, more general spectral triples for the noncommutative two-tori (cf. [16]) for which we expect the spectral Einstein functional to vanish. This can be further generalised as

**Conjecture 6.1.** A regular spectral triple of dimension 2 has a vanishing Einstein spectral functional.

Even if the regularity assumption should be suitably supplemented such a result will show the robustness of the noncommutative generalisation of manifolds.

Another direction is to follow a spectral definition of a noncommutative Einstein manifold (or, being more precise, an Einstein spectral triple).

**Definition 6.2.** A spectral triple is called an Einstein spectral triple if the spectral Einstein functional is proportional to the metric functional.

The study of these objects in both the context of almost commutative geometries and applications in mathematical physics can be very interesting. We also hope that the proposed functionals may allow for a global spectral view on the Ricci and curvature tensors themselves and rephrase in the spectral language the notion of flat manifolds. This, together with the Einstein tensor playing a significant role in physics, can prove helpful in applications to quantum field theory and various incarnations of quantum gravity. A possibility of having a direct approach not only to the action functional but also to a version of equations of motion is another appealing direction of studies that could possibly be linked to a more general variational formula.

Yet another direction to investigate might be connected with orbifolds and other singular generalisations of a Riemannian manifold, to see for example how our spectral functionals apprehend the singularities. Also their extension to the case of a twisted algebra of pseudodifferential operators as in [44] is an appealing task.
Another interesting problem remains the equivalence of the algebra of pseudodifferential operators ([7] and [14]) for the noncommutative tori with various Dirac operators. Comparison of the functionals discussed in this work can shed some light on this open problem.

**Appendix A. Algebra of Symbols of Pseudodifferential Operators**

Suppose that $P$ and $Q$ are two pseudodifferential operators with symbols,

(A.1) \[ \sigma(P)(x, \xi) = \sum_{\alpha} \sigma(P)_{\alpha}(x) \xi^{\alpha}, \quad \sigma(Q)(x, \xi) = \sum_{\beta} \sigma(Q)_{\beta}(x) \xi^{\beta}, \]

respectively, where $\alpha, \beta$ are multiindices. The composition rule for the symbols of their product takes the form [27].

(A.2) \[ \sigma(PQ)(x, \xi) = \sum_{\beta} (-i)^{|\beta|} \frac{|\beta|!}{\partial^{\xi_{\beta}} \sigma(P)(x, \xi) \partial_{\beta} \sigma(Q)(x, \xi)}, \]

where $\partial_{\alpha}$ denotes the partial derivative with respect to the coordinate of the cotangent bundle.

We start with computation of the three leading coefficients of the symbols of $P^{-1}$ (we assume that the kernel of $P$ is finite dimensional and can be neglected in the following) for a second-order differential operator $P$, with the symbol expansion,

(A.3) \[ \sigma(P)(x, \xi) = a_2 + a_1 + a_0. \]

The inverse is a pseudodifferential operator $P^{-1}$, with a symbol of the form,

(A.4) \[ \sigma(P^{-1})(x, \xi) = b_2 + b_3 + b_4 + ..., \]

where $b_k$ is homogeneous in $\xi$ of order $-k$. Inserting these expressions into (A.2) and taking homogeneous parts of order 0, $-1$ and $-2$ we get the following set of equations:

\[
\begin{align*}
    a_2 b_2 &= 1, \\
    a_1 b_2 + a_2 b_3 - i \partial_{\xi}^2 (a_2) \partial_{\alpha} (b_2) &= 0, \\
    a_2 b_4 + a_1 b_3 + a_0 b_2 - i \partial_{\xi}^4 (a_1) \partial_{\alpha} (b_2) \\
    &- i \partial_{\xi}^2 (a_2) \partial_{\alpha} (b_3) - \frac{1}{2} \partial_{\alpha}^2 \partial_{\xi}^2 (a_2) \partial_{\alpha} \partial_{\beta} (b_2) &= 0,
\end{align*}
\]

which we solve recursively, obtaining

(A.6) \[
\begin{align*}
    b_2 &= a_2^{-1}, \\
    b_3 &= -b_2 \left( a_1 b_2 - i \partial_{\xi}^2 (a_2) \partial_{\alpha} (b_2) \right), \\
    b_4 &= -b_2 \left( a_1 b_3 + a_0 b_2 - i \partial_{\xi}^4 (a_1) \partial_{\alpha} (b_2) \\
    &- i \partial_{\xi}^2 (a_2) \partial_{\alpha} (b_3) - \frac{1}{2} \partial_{\alpha}^2 \partial_{\xi}^2 (a_2) \partial_{\alpha} \partial_{\beta} (b_2) \right).
\end{align*}
\]

Next, we show a technical lemma that allows us to compute the symbol of the higher inverse power of the Laplace operator. This is a straightforward application of the iterated formula (A.2) and we include it only for completeness. To shorten the notation, we shall denote the derivatives with respect to the coordinates on the manifold by $\delta$ and use still $\partial$ for the partial derivatives with respect to the coordinates on the cotangent bundle. Further, we shorten the
notation of the derivatives applied to a certain element of the product, which would be valid for all derivatives (both \(\partial, \delta\)), for example:

\[
\partial_a^{[k]} (y_1 y_2 \cdots y_n) = y_1 y_2 \cdots \partial_a (y_k) \cdots y_n.
\]

**Lemma A.1.** Given the first three leading symbols of an operator \(P\) (with the principal symbol being of order \(-k\)),

\[
\sigma(P) = p_k + p_{k+1} + p_{k+2} + \cdots,
\]

we can express the first three leading symbols of \(R = P^l\),

\[
\sigma(R) = r_{lk} + r_{lk+1} + r_{lk+2} + \cdots,
\]

in terms of \(p_k, p_{k+1}, p_{k+2}\), as:

\[
\begin{align*}
(r_{lk}) & = (p_k)^l, \\
(r_{lk+1}) & = \sum_{j=1}^{l} (p_k)^{j-1} p_{k+1} (p_k)^{l-j} - i \sum_{1 \leq j < w \leq l} \partial_a^{[j]} \delta_a^{[w]} (p_k)^l, \\
(r_{lk+2}) & = \sum_{j=1}^{l} (p_k)^{j-1} p_{k+2} (p_k)^{l-j} \\
& \quad + \sum_{1 \leq s < l} (p_k)^{j-1} p_{k+1} (p_k)^{s-j-1} p_{k+1} (r)^{l-s}, \\
& \quad - i \sum_{s=1}^{l} \sum_{1 \leq j < p \leq l} \partial_a^{[j]} \delta_a^{[p]} (p_k)^{s-j-1} p_{k+1} (p_k)^{l-s}, \\
& \quad - \frac{1}{2} \sum_{1 \leq j < p \leq l} \sum_{1 \leq r < s \leq l} \partial_b^{[r]} \delta_b^{[s]} \partial_a^{[j]} \delta_a^{[p]} (p_k)^l.
\end{align*}
\]

**Proof.** We proceed by induction in \(l\). The claim is obvious for \(l = 1\). Assume now that it holds for \(l - 1\), and consider \(R = P^l = PP^{l-1}\). By (A.2) we can see, that

\[
\begin{align*}
(r_{lk}) & = p_k r_{(l-1)k}, \\
(r_{lk+1}) & = p_k r_{(l-1)k+1} + p_{k+1} r_{(l-1)k} - i \partial_a p_k \delta_a r_{(l-1)k}, \\
(r_{lk+2}) & = p_k r_{(l-1)k+2} + p_{k+1} r_{(l-1)k+1} + p_{k+2} r_{(l-1)k} \\
& \quad - i \partial_a p_k \delta_a r_{(l-1)k+1} - i \partial_a p_{k+1} \delta_a r_{(l-1)k} - \frac{1}{2} \partial_a \partial_b p_k \delta_a \delta_b r_{(l-1)k}.
\end{align*}
\]
The formula for \( r_{lk} \) is obvious. Consider now the expression for \( r_{lk+1} \) and assume that the formula holds for \( l - 1 \), then

\[
\begin{align*}
\tau_{lk+1} &= p_k \left( \sum_{j=1}^{l-1} (p_k)^{j-1} (p_{k+1}(p_k)^{l-1-j} - i \sum_{1 \leq j < s \leq l-1} \partial_a^{[j]} \delta_a^{[s]} (p_k)^{l-1}) \right) \\
&\quad + p_{k+1} p_k^{l-1} - i \partial_a p_k \delta_a p_k^{l-1} \\
&= \left( p_k \sum_{j=1}^{l-1} (p_k)^{j-1} (p_{k+1}(p_k)^{l-1-j} + p_{k+1} p_k^{l-1}) \right) \\
&\quad - i \left( p_k \sum_{1 \leq j < s \leq l-1} \partial_a^{[j]} \delta_a^{[s]} (p_k)^{l-1} + \partial_a p_k \delta_a p_k^{l-1} \right) \\
&= \sum_{j=1}^{l} (p_k)^{j-1} (p_{k+1}(p_k)^{l-1-j} - i \sum_{1 \leq j < s \leq l} \partial_a^{[j]} \delta_a^{[s]} (p_k)^{l}),
\end{align*}
\]

where we have used only the reordering of terms and the Leibniz rule for \( \delta_a \):

\[
\delta_a (p_k)^{l-1} = \sum_{s=1}^{l-1} \delta_a^{[s]} (p_k)^{l} - 1.
\]

The formula for \( \tau_{lk+2} \) can be proved in a similar way. We can split the product in (A.8) into the sum with no derivatives, only first-order derivatives, and finally second-order derivatives. We skip the proof for the first two parts and illustrate only the last part. The component of \( \tau_{lk+2} \) with second-order derivatives (denoted \( \tau_{lk+2}(2) \)) will come from the following contributions.

\[
\begin{align*}
(A.9) \quad \tau_{lk+2}(2) &= p_k \tau_{(l-1)k+2} - i \partial_a p_k \delta_a \tau_{(l-1)k+1} (1) - \frac{1}{2} \partial_a \partial_b p_k \delta_a \delta_b \tau_{(l-1)k} = \cdots
\end{align*}
\]

which, after assuming the validity for \( l - 1 \) gives,

\[
\begin{align*}
\cdots &= p_k \left( -\frac{1}{2} \sum_{1 \leq j < p \leq l} \sum_{1 \leq r < s \leq n} \partial_b^{[r]} \delta_b^{[s]} \partial_a^{[j]} \delta_a^{[p]} (p_k)^{l} \right) \\
&\quad - i \partial_b p_k \delta_b \left( -i \sum_{1 \leq j < w \leq l} \partial_a^{[j]} \delta_a^{[w]} (p_k)^{l} \right) \\
&\quad - \frac{1}{2} \partial_a \partial_b p_k \delta_a \delta_b p_k^{l-1}.
\end{align*}
\]

Again, using the Leibniz rule, we see that it is indeed the term from (A.7) with second-order derivatives split into the parts that first \( p_k \) has no derivatives acted upon, one derivative, and two derivatives.
Corollary A.2. In the special case of scalar symbols (a sufficient condition is that symbol $p_k$ is scalar) we have (with the same notation as above),

$$r_{l_k} = (p_k)^l,$$
$$r_{l_{k+1}} = l(p_k)^{l-1}p_{k+1} - i\frac{l(l-1)}{2}(p_k)^{l-2}\partial_a(p_k)\delta_a(p_k),$$
$$r_{l_{k+2}} = l(p_k)^{l-1}p_{k+2} + \frac{l(l-1)}{2}(p_k)^{l-2}(p_{k+1}^2)$$
$$- i\frac{l(l-1)}{2}(p_k)^{l-3}\left[p_k\left(\partial_a(p_{k+1})\delta_a(p_k) + \partial_a(p_k)\delta_a(p_{k+1})\right)
+ (l-2)p_{k+1}\partial_a(p_k)\delta_a(p_k)\right]$$
$$- \frac{l(l-1)}{24}(p_k)^{l-4} \left[6(p_k)^2\partial_a\partial_b(p_k)\delta_a\delta_b(p_k)
+ 3(l-2)(l-3)\partial_a(p_k)\partial_b(p_k)\delta_a\delta_b(p_k)
+ 4(l-2)p_k\partial_a(p_k)\partial_b(p_k)\delta_a\delta_b(p_k)
+ \partial_a(p_k)\partial_b\delta_a(p_k)\delta_b(p_k) + \partial_a\partial_b(p_k)\delta_a(p_k)\delta_b(p_k)\right].$$

(A.11)

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(1) SISSA (SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI), VIA BONOMEA 265, 34136 TRIESTE, ITALY
Email address: dabrow@sissa.it

(2) INSTITUTE OF THEORETICAL PHYSICS, JAGIELLONIAN UNIVERSITY, PROF. STANISŁAWA ŁOJASIEWICZA 11, 30-348 KRAKÓW, POLAND.
Email address: andrzej.sitarz@uj.edu.pl
Email address: pawel.zalecki@doctoral.uj.edu.pl