THE SPHERICAL HECKE ALGEBRA, PARTITION FUNCTIONS, AND
MOTIVIC INTEGRATION

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Abstract. This article gives a proof of the Langlands-Shelstad fundamental lemma for the spherical Hecke algebra for every unramified $p$-adic reductive group $G$ in large positive characteristic. The proof is based on the transfer principle for constructible motivic integration. To carry this out, we introduce a general family of partition functions attached to the complex $L$-group of the unramified $p$-adic group $G$. Our partition functions specialize to Kostant’s $q$-partition function for complex connected groups and also specialize to the Langlands $L$-function of a spherical representation. These partition functions are used to extend numerous results that were previously known only when the $L$-group is connected (that is, when the $p$-adic group is split). We give explicit formulas for branching rules, the inverse of the weight multiplicity matrix, the Kato-Lusztig formula for the inverse Satake transform, the Plancherel measure, and Macdonald’s formula for the spherical Hecke algebra on a non-connected complex group (that is, non-split unramified $p$-adic group).

Let $F$ be a $p$-adic field; that is, a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Let $G$ be an unramified reductive group and $H$ an unramified endoscopic group of $G$, both defined over $F$. Let $\mathcal{H}(G)$ and $\mathcal{H}(H)$ be the spherical Hecke algebras on $G$ and $H$. Associated with a morphism $\xi : \mathcal{L}H \to \mathcal{L}G$ of $L$-groups, there is a homomorphism $b_\xi : \mathcal{H}(G) \to \mathcal{H}(H)$, obtained by composing three maps: the Satake transformation of $\mathcal{H}(G)$, the pullback under $\xi$, and the inverse Satake transformation to $\mathcal{H}(H)$.

Let $A$ be a maximal split torus of $G$. Let $P^+ = \{ \lambda \in X_{\ast}(A) \mid \langle \lambda, \alpha_\vee \rangle \geq 0, \alpha \in \Delta_1 \}$ be a positive chamber of $X_{\ast}(\hat{S}) = X_{\ast}(A)$. (The notation is explained in Section 1.2.) The spherical Hecke algebra $\mathcal{H}(G) = \mathcal{H}(G//K)$ of complex-valued functions that are biinvariant with respect to a given hyperspecial subgroup $K$ has a linear basis given by characteristic functions $f_\lambda$ of double cosets $K\sigma\lambda K$. Here $\sigma$ is a fixed uniformizing element and $\lambda$ runs over the cocharacters in $P^+$.

In this article, we use motivic integration to study the spherical Hecke algebra and the function $B_\xi : P^+ \times H(F) \to \mathbb{C}$, given by $(\lambda, h) \mapsto b_\xi(f_\lambda)(h)$. This function is the specialization of a constructible motivic function (Theorem 4.3.1).

As an application, we show that the fundamental lemma for the spherical Hecke algebra falls within the scope of the transfer principle for constructible motivic functions (Section 4.4). This implies that the fundamental lemma holds for the spherical Hecke algebra over fields of large positive characteristic (Theorem 4.5.1).

This application to the fundamental lemma is the main motivation for this work. Our results overlap with those of Bouthier, who proves the fundamental lemma for the spherical
Hecke algebra in positive characteristic under the restrictions that the group $G$ is semisimple and simply connected, and the endoscopic group is split [1, Theorem 0.2]. Our proof of the fundamental lemma for the spherical Hecke algebra holds without restriction on the group and endoscopic group. Unlike Bouthier, we are unable to be explicit in our assumption on the characteristic of the field. This is an unfortunate limitation of the methods we use. In other work, Lemaire, Moeglin, and Waldspurger propose that the method of close fields might be used to transfer the fundamental lemma for the spherical Hecke algebra from characteristic zero to positive characteristic, but as far as we know, this has not been carried out [28, §1.3].

The construction of $B_\xi$ passes through the Langlands dual $\hat{L}G$, which is a non-connected complex reductive group. Our constructibility result for $B_\xi$ follows from the Presburger constructibility of various functions on lattices in the dual: Macdonald’s formula, weight multiplicity formulas, the inverse of the weight multiplicity matrix, the Plancherel measure, the geometric Satake transform, and the Kato-Lusztig formula for the inverse Satake transform. When $G$ and $H$ are split, we can take $\hat{L}G = \hat{G}$ and $\hat{L}H = \hat{H}$ to be connected. In this case, the desired formulas were previously known. In this article, we generalize these formulas to non-connected complex reductive groups. These generalizations are a major part of this work.

One novelty of this work is that we show how to extend the theory of motivic integration to the Langlands dual group, by encoding representation-theoretic data of the complex dual group as Presburger constructible functions on the character lattice. These Presburger functions can then be recombined with constructible functions on the $p$-adic group. A second innovation is to encode the entire Hecke algebra into a single constructible function $B_\xi$. This makes it possible to invoke the transfer principle of motivic integration a single time, rather than once for each function in the Hecke algebra. (Invoking the transfer principle an infinite number of times could potentially leave us with nothing, because we lose finitely many primes with each invocation.)

A framework for studying the spherical Hecke algebra through motivic integration is provided by Cely’s thesis [7]. This article builds on that work. The second author would like to express gratitude to his thesis advisor T. Hales for all the teaching and support. We thank Julia Gordon, who served on Cely’s thesis committee and who provided valuable suggestions.

1. the Satake transform, Macdonald’s formula, and related topics

In this section, we extend various results from split $p$-adic groups to unramified groups (from complex connected reductive groups to non-connected groups on the $L$-group side).

1.1. root systems. Let $G$ be an unramified reductive group over a $p$-adic field $F$. It is defined by descent from a finite unramified extension $E/F$ over which $G$ splits. It is determined by an automorphism $\theta$ of the root datum $(X^*, \Phi, X_+, \Phi^\vee)$ of the split form $G^\vee$ of $G$. The automorphism $\theta$ has finite order and preserves a set of simple roots associated to some set $\Phi^+$ of positive roots. If Frob is the Frobenius automorphism of $E/F$, the group $G/F$ is
defined by twisting the action of Frobenius on $G(E)$ by $\theta$. Let $(B, T, X)$ be a pinning of $G$ over $F$, preserved by $\theta$.

Let $A$ be a maximal split torus in $T$. For any $R \subseteq \Phi$, let $M_R$ be the centralizer of

$$A_R = \{a \in A \mid \sigma(a) = 1, \quad \alpha \in R\}.$$  

We have $X_\alpha(A) = X_\alpha(T)^{\theta}$ and $X_\alpha(A) = X_\alpha(T)/(1-\theta)X_\alpha(T)$. The pairing of $X_\alpha(A)$ and $X_\alpha(A)$ is induced naturally from that of $X_\alpha(T)$ and $X_\alpha(T)$.

The image of $\Phi$ in $X_\alpha(A)$ is the restricted root system $\Phi_{\text{reg}}$. It is well known that $\Phi_{\text{reg}}$ is indeed a root system, and this is easy to verify directly in this context. The roots of $\Phi_{\text{reg}}$ are in bijection with the $\theta$-orbits of roots in $\Phi$, and this bijection restricts to a bijection between simple roots in $\Phi_{\text{reg}}$ and orbits of simple roots in $\Phi$. If $\alpha$ is a root of $\Phi$ (or coroot), we write $[\alpha] = \{\alpha, \theta\alpha, \ldots\}$ for its $\theta$-orbit, often identified with a root of $\Phi_{\text{reg}}$. A root $\alpha \in \Phi_{\text{reg}}$ is indivisible if $\frac{1}{2}\alpha$ is not a root. Let $\Phi^{\text{red}}$ and $\Phi_{\text{red}}$ be the set of nonmultipliable and indivisible roots of $\Phi_{\text{reg}}$. They are both reduced root systems. Two roots $\alpha$ and $\alpha' \in \Phi_{\text{reg}}$ are homothetic if $\alpha' = k\alpha$ for some $k > 0$. Every root is homothetic to an indivisible root.

Each root $\beta \in \Phi$ can be assigned a diagram $A_1$ or $A_2$ as follows. The construction is best described in the split group $G^\ast$. Let $S = [\beta']$ be the $\theta$-orbit that corresponds to the indivisible root homothetic to $[\beta]$. The simple positive roots of $M_S$ form a single $\theta$-orbit $S$. We consider its Dynkin diagram. By the transitivity of $\theta$, all components of the diagram have the same type, either $A_1$ or $A_2$. This is the diagram of $\beta$. This construction can be applied to $(G^\ast, \Phi)$ or to $(\hat{G}, \Phi^{'\vee})$, where $\hat{G}$ is the complex dual. A coroot $\alpha^{'\vee}$ has the same diagram as the root $\alpha$. We let $b(\beta)$ be the number of connected components of the Dynkin diagram of $M_S$. (According to the types of Kottwitz and Shelstad, type I means diagram $A_1$, type II means a simple root in diagram $A_2$, and type III means a highest root in diagram $A_2$ [24].)

Let $N : X_\alpha(T) \to X_\alpha(A)$ be the norm map: $N\alpha = \sum_{\alpha' \in [\alpha]} \alpha'$.

**Lemma 1.1.1.** If $[\alpha] \in \Phi_{\text{red}}$ is an indivisible restricted root, then the corresponding coroot is

$$(1.1.2) \quad [\alpha]^{\vee} = k\, N\alpha^{\vee}$$

when the diagram of $\alpha$ has type $A_k$, for $k = 1, 2$.

**Proof.** (See [24, 1.3.9].) ♦

Let $W$ be the Weyl group attached to the root datum of $G$ over a splitting field $E$. The restricted Weyl group $W^\theta$ is the subgroup of $W$ commuting with $\theta$. The group $W^\theta$ is a Coxeter group. The simple reflection in $W^\theta$ associated with an orbit $[\alpha]$ of simple roots in $\Phi$ is the longest element in the Weyl group of the Levi component $M_{[\alpha]}$. We write $\ell(w)$ for the length of $w \in W^\theta$, computed relative to the set of simple reflections of the Coxeter group $W^\theta$.  

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1.2. **L-groups.** We review some aspects of the theory of non-connected complex reductive groups from Steinberg [34], Springer [33], Kottwitz and Shelstad [24], Haines [17], and Chriss [8].

Let \( L^G = \hat{G} \rtimes \langle \theta \rangle \) be the \( L \)-group of the unramified group \( G \). It has root system dual to that of \( G \). It is a semidirect product of a connected complex reductive group \( \hat{G} \) and a finite cyclic group generated by an outer automorphism \( \theta \) of \( \hat{G} \). The automorphism \( \theta \) preserves a pinning \((\hat{B}, \hat{T}, \hat{X})\) of \( \hat{G} \). Let \( \hat{B} = \hat{T} \hat{N} \).

Let \( \Psi = \Phi^\vee \) and \( \Psi^\vee = \Phi \) be the root and coroot systems of the complex group \( \hat{G} \) with respect to \( \hat{T} \), and let \( \Psi^+ \) be the set of positive roots with respect to \((\hat{T}, \hat{B})\). If \( R \) is any root system with positive roots \( R^+ \), we let \( \rho(R^+) = (1/2) \sum_R \alpha \) be the half-sum of positive roots in \( R^+ \). We have \( \rho(\Psi_{\text{red}}^+) = \rho(\Psi^+) \).

The torus \( \hat{T} \) is \( \theta \)-stable. We form the quotient \( \hat{S} = \hat{T}/(1 - \theta)\hat{T} \), where \((1 - \theta)\hat{T} = \{ \theta(t^{-1}) \mid t \in \hat{T} \} \), using Steinberg’s additive notation for a multiplicative group. If \( \lambda \in X^*(\hat{T})^\theta \) is \( \theta \)-fixed, then it is trivial on \((1 - \theta)\hat{T} \) and descends to a character \( \lambda \in X^*(\hat{S}) \). This gives \( X^*(\hat{T})^\theta = X^*(\hat{S}) \).

We abbreviate \( Y^* = X^*(\hat{S}) \). Let \( O = O_F \), the ring of integers of \( F \). We have identifications
\[
(1.2.1) \quad T/T(O) = A/A(O) = X_*(A) = X_*(T)^\theta = X^*(\hat{T})^\theta = Y^* = X^*(\hat{S}).
\]
Each unramified character \( \chi : T \to \mathbb{C}^\times \), by these identifications, is a homomorphism
\[
(1.2.2) \quad \chi \in \text{Hom}(T/T(O), \mathbb{C}^\times) = \text{Hom}(X^*(\hat{S}), \mathbb{C}^\times) = X_*(\hat{S}) \otimes \mathbb{C}^\times = \hat{S}.
\]
We write \( \chi = \chi_s \), for \( s \in \hat{S} \).

Let \( e^\lambda \) be the basis element of the group algebra \( \mathbb{C}[Y^*] \) of \( Y^* \), indexed by \( \lambda \in Y^* \).

Recall that
\[
P^+ = \{ \lambda \in X_*(A) \mid \langle \lambda, \alpha^\vee \rangle \geq 0, \quad \alpha \in \Delta_1 \}
\]
where \( \Delta_1 = \Delta(\Psi_{\text{red}}) \) is a set of simple roots for \( \Psi_{\text{red}} \). A basis of \( W^\theta \)-fixed functions in \( \mathbb{C}[Y^*] \) is
\[
(1.2.3) \quad m_\mu = \sum_{\lambda \in W^\theta(\mu)} e^\lambda, \quad \text{for } \mu \in P^+,
\]
where \( W^\theta(\mu) \) is the orbit of \( \mu \in \mathbb{C}[Y^*] \) under \( W^\theta \).

Let \( \hat{G}_0 \) be the complex group with Cartan subgroup \( \hat{S} \), root system \( \Psi_{\text{red}} = (\Phi_{\text{red}})^\vee \), and root datum
\[
(X_*(A), (\Phi_{\text{red}})^\vee, X^*(A), \Phi_{\text{red}}) = (X^*(\hat{S}), \Psi_{\text{red}}, X_*(\hat{S}), \Psi_{\text{red}}^\vee).
\]
The complex group \( \hat{G}_0 \) is the \( L \)-group of \( G^1 \), the identity component of the \( \theta \)-fixed subgroup of the split form \( G^1 \) [24 §1.3]. As we will see, \( \hat{G}_0 \) appears naturally in the twisted Weyl character formula for \( LG \), hence also in the Satake transform and its inverse.
Example 1.2.4. Let \( G = SU(n, E/F) \) be an unramified unitary group in an odd number of variables \( n = 2k + 1 \). The automorphism \( \theta \) has order 2. The cocharacter groups of \( T \) and \( A \) are

\[
X_\ast(T) = \{(t_1, \ldots, t_n) \in \mathbb{Z}^n \mid t_1 + \cdots + t_n = 0\}, \quad X_\ast(A) = \mathbb{Z}^k,
\]

with identification \((t_1, \ldots, t_k) \in X_\ast(A) \mapsto (t_1, \ldots, t_k, 0, -t_k, \ldots, -t_1) \in X_\ast(T)^0\). Following Lemma 1.1.1, we compute norms to get

\[
(1.3.1) \quad V = \sum \mu \in \theta_1 \mu.
\]

We recognize the root datum as that of \( \hat{G}_\theta = \text{Sp}(2k, \mathbb{C}) \) with root system \( \Psi^{\text{red}} \). In this case, \( G^1 = \text{GL}(n, \mathbb{C}) \) is \( SO(2k + 1) \), which indeed has \( L \)-group \( \hat{G}_\theta \).

1.3. the partition function. Let \( \theta_1 \in N_\theta(\hat{T}) \cong \langle \theta \rangle \) be an element of finite order. Set \( \hat{T}_1 = T/(1 - \theta_1)\hat{T} \) and \( X_\ast(\hat{T}_1) = X_\ast(\hat{T})^{\theta_1} \). Let \( N_1 : X_\ast(\hat{T}) \to X_\ast(\hat{T})^{\theta_1} \) be the norm map with respect to \( \theta_1 \):

\[
N_1 \mu = \sum_{\mu' \in (\theta_1)\mu} \mu'.
\]

Let \( V \) be a finite dimensional representation of \( L^G \), with weight space decomposition \( V = \bigoplus V_\mu \). We have \( \theta_1(V_\mu) = V_{\theta_1 \mu} \). Let \( R \subset X_\ast(\hat{T}) \) be a \( \theta_1 \)-stable set of weights of \( V \), and set

\[
(1.3.1) \quad V_R = \bigoplus_{\mu \in R} V_\mu.
\]

We define a symbolic operator \( E \) on \( V_R \) that is diagonal with respect to the weight space decomposition:

\[
E \nu = e^{\mu} \nu, \quad \text{for } \nu \in V_\mu.
\]

(We warn the reader that \( E \) denotes an unramified field extension \( E/F \) or this symbolic operator, depending on the context.) We define a \( q \)-determinant \( D(\hat{G}, V_R, \theta_1, E, q) \) and a \( q \)-partition function \( P(\hat{G}, V_R, \theta_1, E, q) \) as

\[
(1.3.2) \quad D(\hat{G}, V_R, \theta_1, E, q) = \det(1 - q \theta_1 E; V_R);
\]

\[
(1.3.3) \quad P(\hat{G}, V_R, \theta_1, E, q) = D(\hat{G}, V_R, \theta_1, E, q)^{-1}.
\]

When \( V \) is the adjoint representation, we sometimes abbreviate \( D(\hat{G}, V_R, \theta_1, E, q) \) to \( D(\hat{G}, R, \theta_1, E, q) \). The determinant and partition function carry the same information, and we pass back and forth between \( D \) and \( P \) according to convenience.

We may view the determinant and partition functions as functions on \( \hat{T} \), by evaluating each \( e^{\mu}(t) = \mu(t) \), for \( t \in \hat{T} \), so that

\[
\det(1 - q \theta_1 E; V_R)(t) = \det(1 - q \theta_1 t; V_R).
\]

Taking \( \theta_1^{-1}(u)u^{-1} \in (1 - \theta_1)\hat{T} \), we have

\[
\det(1 - q \theta_1 t \theta_1^{-1}(u)u^{-1}; V_R) = \det(1 - u(q \theta_1 t)u^{-1}; V_R) = \det(1 - q \theta_1 t; V_R).
\]

Thus, the partition function and determinant descend to functions on \( \hat{T}_1 \).

For \( w \in W^\theta \) and \( \nu \in V_\theta \), define \( E_w \) by \( E^w \nu = e^{w\theta} \nu \). If \( \tilde{w} \) is a representative of \( w \) in the normalizer of \( \hat{T} \), then

\[
(\tilde{w}^{-1} E \tilde{w}) \nu = E_w \nu.
\]
Also, for \( t \in \hat{T} \),
\[
(Ev)(w^{-1}tw) = (E^w v)(t) = (w\beta)(t)v.
\]
Write \( E^{-w} = (E^{-1})^w = (E^w)^{-1} \).

The next lemma gives the general shape of a factorization of the determinant.

**Lemma 1.3.4.** Let \( R = \langle \theta \mu \rangle \) be the orbit of a single weight of \( V \). Then \( D(\hat{G}, V_R, \theta_1, E, q) \) is a finite product of factors of the form
\[
1 - \zeta q^b e^{N_\mu},
\]
where \( b \) is the cardinality of the orbit \( R \) and \( \zeta^k = 1 \), where \( kb \) is the order of \( \theta_1 \).

**Proof.** Fix \( \mu_0 \in R \) and let \( \mu_i = \theta_i \mu_0 \). The abelian group \( \langle \theta_i \mu \rangle \) acts on \( V_{\mu_0} \). Write \( V_{\mu_0} = \oplus W \), where each \( W \) is a 1-dimensional representation of \( \langle \theta_i \mu \rangle \). Then \( V_R = \oplus (\theta_i)W \), where \( (\theta_i)W = \oplus_{\mu_0} \theta_i W \). We have \( \theta_i W_{\nu_0} = \zeta \nu_0 \), for \( \nu_0 \in W \) and some \( \zeta = \zeta_W \). Let \( v_i = \theta_i v_0 \). The operator \( 1 - q\theta_i E \) on the summand \( (\theta_i)W \) with respect to this basis is
\[
\begin{pmatrix}
1 & -qe^{\mu_0} & 0 & \ldots \\
0 & 1 & -qe^{\mu_1} & \ldots \\
& & \ddots & \ddots \\
& & & 1
\end{pmatrix}
\]
The result follows by taking its determinant. \( \square \)

1.3.1. relation with \( L \)-functions. Langlands has defined an \( L \)-function for spherical representations of \( G \). It is written \( L(\pi, V, q^{-s}) \), where \( \pi \) is an irreducible admissible representation of \( G \) with a \( K \)-fixed vector, and \( V \) is a representation of the \( L \)-group \( L(G) \), which we assume factors through some finite unramified Galois extension \( \text{Gal}(E/F) \).

Let \( t_\pi \in \hat{S} \) be the Frobenius-Hecke parameter of \( \pi \). We let \( \theta_1 = \theta = \text{Frob} \), the automorphism of \( \hat{G} \) coming from the action of Frobenius on the root datum. We let \( R \) be the set of all weights, so that \( V = V_R \). As observed above, the partition function \( P(\hat{G}, V, \theta, E, q) \) is a function on \( \hat{T} = \hat{S} \).

**Lemma 1.3.5.** The partition function evaluates to the local \( L \)-function. More precisely,
\[
P(\hat{G}, V, \theta, E, q)(t_\pi) = L(\pi, V, q_\pi^{-s}).
\]

**Proof.** Both sides are defined as the reciprocal of a determinant. On both sides it is the determinant of the same element acting on the same vector space. \( \square \)

1.3.2. the partition function for the adjoint representation. A case of particular importance for us is the following. Let \( g \) be the adjoint representation of \( \hat{G} \). It is an irreducible highest weight representation whose highest weight is \( \theta \)-fixed. Hence \( g \) extends to an irreducible representation of \( \hat{G} \times \langle \theta \rangle \). Let \( \mathfrak{n} \) be the Lie algebra of \( \hat{N} \). Then \( \mathfrak{n} = g_R \), where \( R \) is the set of positive roots. The set \( R \) is \( \theta \)-stable. We have a partition function
\[
P(\hat{G}, \mathfrak{n}, \theta, E, q).
\]

Much of this article handles this particular case. When \( \hat{G}, \mathfrak{n}, \) and \( \theta \) are fixed, we abbreviate \( P(E, q) = P(\hat{G}, \mathfrak{n}, \theta, E, q) \).
Recall that each $\alpha \in \Psi_{\text{red},+}$ has the form $[\beta^\vee]^\vee$, with $[\beta^\vee] \in \Phi_{\text{red}}$ and some $\beta$ in $\Psi^+$. The root $\beta$ has a diagram $A_1$ or $A_2$, associated constant $b = b(\beta)$, and $kN\beta = \alpha$ for diagram type $A_k$ (Lemma 1.3.1). For each $\alpha \in \Psi_{\text{red}}$, we define

$$d_\alpha(q) = \begin{cases} 1 - q^b e^{\gamma}, & \text{if diagram } A_1; \\ (1 - q^{2b} e^{\gamma/2})(1 + q^b e^{\gamma/2}), & \text{if diagram } A_2. \end{cases}$$

We have the following factorization refining Lemma 1.3.4.

**Lemma 1.3.7.** The determinant factors as

$$\det(1 - \theta E q; n) = \prod_{\alpha \in \Psi_{\text{red},+}} d_\alpha(q).$$

**Proof.** Similar factorizations in the special case $q = 1$ are found in [20], [37]. Here is a sketch. The determinant is block diagonal, with a block for each $\theta$-orbit in $\Psi^+$.

We first consider diagram type $A_1$. Write $\alpha = N\beta$, as above with $\beta \in \Psi$. On the block $[\beta]$, we can pick a basis $X_\beta$ of the root spaces $u_{\beta'}$ such that $\theta$ acts as $0X_\beta = X_{\beta+\gamma}$, with indices mod $b$. The determinant restricted to this block satisfies the fool’s identity

$$\det(I - A) = \det(I) - \det(A),$$

which yields $1 - \prod_{\beta' \in [\beta]} q e^{\gamma/2} = 1 - q^b e^{\gamma/2}$.

Next consider diagram type $A_2$. Write $\alpha = 2N\beta$ as above. We choose three positive roots $\beta, \beta', \gamma \in \Psi$ forming the positive root system of $A_2$, where $\gamma = \beta + \beta'$ is the highest root. We have $N\gamma = N\beta = N\beta' = \alpha/2$. Recall that $b$ is the number of connected components in the Dynkin diagram. Then $\theta^n$ preserves the $A_2$ factor and $\theta^n(\beta) = \beta'$. There are two $\theta$-orbits of roots: $[\gamma]$ and $[\beta']$ of cardinalities $b$ and $2b$. We may pick root vectors $X_\beta, X_{\beta'}, X_\gamma$ in the root spaces of $\beta, \beta', \gamma$ such that

$$\theta^n(X_\beta) = X_{\beta'}, \quad \theta^n(X_{\beta'}) = X_\beta, \quad \theta^n X_\gamma = \theta^n[X_{\beta}, X_{\beta'}] = [X_{\beta'}, X_\beta] = -[X_\beta, X_{\beta'}] = -X_\gamma.$$ 

Note the sign $(-1)$ that appears for the orbit $[\gamma]$. We choose root vectors on the entire orbits by $\theta^n X_\delta = X_{\beta+i\delta}$, for $i = 1, \ldots, b-1$ and $\delta \in \{\beta, \beta', \gamma\}$. We compute the determinant on these two orbits as before. For the orbit $[\gamma]$, we obtain $1 + \prod_{\gamma' \in [\gamma]} q e^{\gamma/2} = 1 + q^b e^{\gamma/2}$. The orbit $[\beta']$ gives $1 - q^{2b} e^{\gamma/2}$. \hfill \Box

In the special case $\theta = 1$, the matrix $q\theta E$ is diagonal, every orbit has cardinality 1, and the partition function is a product over positive roots $P(E, q) = \prod_{\beta \in \Psi^+}(1 - qe^{\beta})^{-1}$. This is the classical $q$-partition function.

**Corollary 1.3.8** (twisted Weyl denominator). Specializing to $q = 1$, we have

$$P(E, 1) = \prod_{\alpha \in \Psi_{\text{red},+}} (1 - e^{\alpha})^{-1}.$$

**Proof.** Set $q = 1$ in the lemma and observe that $d_\alpha(1) = 1 - e^{\alpha}$. \hfill \Box

**Corollary 1.3.9.** for all $w \in W^\theta$,

$$P(E^w, q)P(E^{-w}, q) = P(E, q)P(E^{-1}, q).$$
Proof. By the lemma,
\[ P(E, q)P(E^{-1}, q) = \prod_{\alpha \in \Psi^{red}} d_{\alpha}(q)^{-1}, \]
as \(\alpha\) runs over the full root system \(\Psi^{red}\). The result follows by observing that \(w \in W^0\) permutes \(\Psi^{red}\), preserving the diagram type \(\Lambda_{\ell}\) and constant \(b\) attached to each root. \(\square\)

1.4. twisted Weyl character formula. We review the proof of the Weyl-character formula, as presented in [22], [20], [37], and [25]. At the same time, we consider various \(q\)-deformations of the standard formulas.

Let \(\lambda\) be a dominant weight in \(X^* (\hat{T}^0) = X^* (\hat{S})\). Let \(V_{\lambda}\) be the irreducible module of \(\hat{G}\) with highest weight \(\lambda\). The \(\theta\)-invariance of \(\lambda\) implies that \(V_{\lambda}\) extends uniquely to a representation of \(\hat{G} \rtimes (\theta)\) such that \(\theta v = v\) for \(v\) in the highest weight space of \(V_{\lambda}\). We let \(\tau_{\lambda}\) be the character on \(V_{\lambda}\), restricted to \(\hat{G} \rtimes \theta\). The \(\hat{G}\)-conjugacy class of \(\theta t \in \hat{T} \rtimes \theta\) depends only on the image of \(t\) in \(\hat{S}\). Thus, we may consider \(\tau_{\lambda} \in \mathbb{C}[X^*(\hat{S})] = \mathbb{C}[Y^*]\). Furthermore, \(\tau_{\lambda}\) is \(W^0\)-invariant.

Let \(\rho = \rho(\Psi^*) = \rho(\Psi^{red,+})\). We define a dot operator \(w \bullet \mu = w(\mu + \rho) - \rho\), for \(w \in W^0\) and \(\mu \in Y^*\). We define an alt-symmetrizer operator
\[ J : \mathbb{C}[Y^*] \rightarrow \mathbb{C}[Y^*], \quad J(f) = \sum_{w \in W^0} (-1)^{f(w)} w(fe^{\rho})e^{-\rho}. \]

Theorem 1.4.1 (twisted Weyl character). For every dominant weight \(\lambda \in X^* (\hat{T}^0) = X^* (\hat{S})\), the irreducible representation \(V_{\lambda}\) of \(\hat{G}\) has character \(\tau_{\lambda} \in \mathbb{C}[Y^*]\) on \(\hat{G} \rtimes \theta\), where
\[ \tau_{\lambda} = J(e^{\rho})P(E^{-1}, 1). \]

The Weyl denominator \(P(E^{-1}, 1)\) is computed in Corollary 1.4.8 If we take \(\lambda = 0\), then \(\tau_{\lambda} = 1\), and the Weyl character formula gives a second formula for the Weyl denominator:
\[ 1 = J(1)P(E^{-1}, 1). \]

It is a remarkable consequence of the Weyl character formula that the twisted character \(\tau_{\lambda}\) is identical to the irreducible character of \(\hat{G}_{\theta}\) with highest weight \(\lambda\).

Proof. We follow Kostant [22]. Let \(\mathfrak{n}\) be the Lie algebra of \(\hat{N}\), considered as a module of \(L \hat{T} = \hat{T} \rtimes (\theta)\) by the adjoint representation, and let \(n'\) be its contragredient. We write \(\chi_{\lambda}\) for the character of the exterior power \(\Lambda^n n'\). We write \(\bar{x}_q = \sum (-q)^j \chi_j\) for the \(q\)-graded virtual character on the sum of \(\Lambda^n n'\), with grading \((-q)^j\) on the \(j\)th summand.

The character \(\bar{x}_q\) evaluated at \(\theta t\) depends only on the image \(s \in \hat{S}\) of \(t \in \hat{T}\). We have
\[ P(E^{-1}, q)^{-1}(x) = \det(1 - q\theta t; n') = \sum_j (-q)^j \chi_j(t\theta) = \bar{x}_q(t\theta). \]
The sum is obtained from the determinant by picking a basis of eigenvectors of \(\theta t\) on \(n'\) and expanding into a polynomial in \(q\). We have \(E^{-1}\) rather than \(E\) because \(n'\) is the contragredient of \(n\). This gives
\[ \bar{x}_q P(E^{-1}, q) = 1. \]
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Upon specialization to $q = 1$, the spaces $C^j = \Lambda^j n' \otimes V_\lambda$ are the terms of a cochain complex of $L$-modules. We consider the virtual character $\tilde{\tau}_\lambda$ on the sum of $C^j$, with grading $(-1)^j$ on $C^j$. By an Euler-Poincaré argument, $\tilde{\tau}_\lambda$ equals the character on the cohomology of the complex. The cohomology has been computed explicitly [22]. These computations show that for each $w \in W^\theta$, the weight $e^{\omega_\lambda w}$ occurs once in cohomology with sign $(-1)^\ell(w)$. Thus, in terms of the operator $J$, we have
\[ \tilde{\tau}_\lambda = J(e^1) = \sum_{w \in W^\theta} (-1)^{\ell(w)} e^{\omega_\lambda w}. \]

By the description of $C^j$ as a tensor product, we have a product decomposition $\tilde{\tau}_\lambda = \tilde{\chi}_1 \tau_{\lambda_1}$ Multiplying both sides by $P(E^{-1}, 1)$, and using Equation [1.4.3] we obtain the twisted Weyl character formula
\[ \tau_{\lambda_1} = J(e^1)P(E^{-1}, 1). \]

It is natural to extend $\tau_{\lambda_1}$ to a $q$-character by defining $\tau_{\lambda_1^\theta}$ by $\tilde{\tau}_{\lambda_1} = \tilde{\chi}_q \tau_{\lambda_1^\theta}$, so that
\[ \tau_{\lambda_1^\theta} = J(e^1)P(E^{-1}, q). \]

We leave it as a research problem to find interpretations of $\tau_{\lambda_1^\theta}$, along the lines of Kazhdan-Lusztig polynomials. Kato and Lusztig give an answer when $\theta = 1$.

1.5. **Macdonald’s formula.** Let $K$ be a hyperspecial maximal compact subgroup of $G$. The spherical Hecke algebra $\mathcal{H}(G//K)$ is the convolution algebra of all complex-valued compactly support $K$-biinvariant functions on $G$. It has a linear basis consisting of characteristic functions $f_\mu$ of double cosets $K\overline{w}K$, for $\mu \in P^\vee$. By the Satake isomorphism, $\mathcal{H}(G//K)$ is isomorphic to the algebra $\mathbb{C}[Y^\vee]^\theta$. We write $f_\mu \in \mathbb{C}[Y^\vee]$ for the image of $f_\mu$ under the Satake isomorphism.

We continue in the context of a complex group $G \rtimes (\theta)$ and keep earlier notation. As usual, we identify elements of $\mathbb{C}[Y^\vee]$ with functions on $\hat{S}$. As before $\Phi = \Psi^\vee$ and $\Psi = \Phi^\vee$ are dual root systems. Let $\rho^\vee = \rho(\Phi^\vee) \in X_*(\hat{T})$.

For each subset $S$ of the set $\Delta_1 = \Delta(\Psi^\text{red}^+, \theta)$ of simple roots in $\Psi^\text{red}^+, \theta$, let $W_S \leq W^\theta$ be the group generated by the reflections in $S$. Let $\ell(w)$ be the length of $w \in W$ (as a function of the Weyl group $W$, and not the Weyl group $W^\theta$). Let $Q_S(q^{-1}) = \sum_{w \in W_S} q^{-\ell(w)}$. We abbreviate $Q(q^{-1}) = Q_{\Delta_1}(q^{-1})$. For each $\mu \in P^+ \subset Y^\vee$, let $S(\mu)$ be the subset of $\Delta_1$ such that $\langle \mu, \alpha \rangle > 0$ iff $\alpha \in S(\mu)$.

**Theorem 1.5.1** (Macdonald’s formula). Let $G$ be an unramified $p$-adic reductive group with $L$-group $^L G = G \rtimes (\theta)$. For each $\mu \in P^+$, let $\hat{f}_\mu$ be the Satake transform of the characteristic function of $K\overline{w}\mu K$, viewed as an element of $\mathbb{C}[Y^\vee]^\theta$. Then
\[ \hat{f}_\mu = \frac{q^{\mu(\rho^\vee)}}{Q_S(\mu)(q^{-1})} \sum_{w \in W^\theta} e^{\rho^\vee \mu} P(E^{-w}, 1) \frac{P(E^{-w}, 1)}{P(E^{-w}, q^{-1})}. \]
The partition function encodes the constants \( q_\alpha, q_{\alpha/2} \) that occur in the traditional formula \([30]\). Our formula is closely tied to the root system \( \Psi^{\text{red}} \) of \( \hat{G}_\theta \) that occurs in the twisted Weyl character formula for \( ^L G \). A variant of Macdonald’s formula of this form was previously known when \( G \) is split \((\theta = 1)\).

There is a related formula for the spherical function \( \Gamma : P^+ \times \hat{S} \to \mathbb{C} \) that we mention. For every \( \mu \in P^+ \):

\[
\Gamma_\mu = \frac{q^{(-\mu, \nu')}}{Q(q^{-1})} \sum_{w \in W^r} e^{\nu w} \frac{P(E^{-w}, 1)}{P(E^{-w}, q^{-1})}.
\]

**Proof of Macdonald’s formula.** Our proof will relate the formula in the theorem to the standard form of Macdonald’s formula. Macdonald’s formula is elaborated in \([4]\) and \([5]\). In the following discussion, we index terms by \( \Phi \in \Phi^{\text{res}} \), \( \alpha \in \Phi^{\text{red}} \), and cardinality \( b \). This is a harmless change, because \( \Phi^{\text{red}} \) is in natural bijection with \( \Phi^{\text{red}} \) by sending each indivisible root \( \alpha \) to the homothetic root \( k\alpha \in \Phi^{\text{red}} \).

Recall that for each \( \alpha \in \Phi^{\text{res}} \), associated with an orbit \( \alpha = [\beta] \) in \( \Phi \), there is a diagram type \( A_k \), for some \( k \in \{1, 2\} \), and cardinality \( b \). Casselman and Macdonald construct an element \( a_\alpha \in T \) for each \( \alpha \in \Phi^{\text{res}} \). Let \( \alpha_1 = \alpha' \in \Psi^{\text{red}} \) be its coroot. By the explicit formulas in \([3]\), for diagram type \( A_k \), we have \( a_\alpha = (k\alpha)'(\bar{\sigma}) = (\alpha_1/k)(\bar{\sigma}) \). Let \( s \in \hat{S} \), and let \( \chi_s \in \text{Hom}(T, \mathbb{C}^\times) \) be the associated unramified character. Then

\[
\chi_s(a_\alpha) = \chi_s((\alpha_1/k)(\bar{\sigma})) = e^{\alpha_1/k}(s).
\]

Macdonald’s formula is traditionally expressed in terms of the constants \( \chi_s(a_\alpha) \), which we rewrite in terms of \( e^{\alpha_1/k} \), for \( \alpha_1 \in \Psi^{\text{red}} \).

For each \( \alpha_1 \in \Psi^{\text{red}}, \) we define a function \( c_{\alpha_1} : \hat{S} \to \mathbb{C} \) by

\[
c_{\alpha_1}(s) = d_{\alpha_1}(q^{-1})/d_{\alpha_1}(1),
\]

following the definition of \( d_{\alpha_1} \) in Equation \([1.3.6]\). We define a function \( \gamma : \hat{S} \to \mathbb{C} \) by

\[
\gamma(s) = \prod_{\alpha_1 \in \Psi^{\text{red}}} c_{\alpha_1}(s^{-1}).
\]

It follows from Lemma \([1.3.7]\) that

\[
\gamma = P(E^{-1}, 1)/P(E^{-1}, q^{-1}).
\]

The traditional Macdonald formula is expressed as a sum of \( \gamma \) over \( \Psi^r \). If we use Equation \((1.5.3)\) to substitute for \( \gamma \) in the traditional formula, then Theorem \([1.5.1]\) is the result.

The formula for \( Q_S(q^{-1}) \) relies on the observation that \( q^{E_{\hat{W}}} = \text{card}(IwI/L) \), where \( I \) is an Iwahori subgroup, and the length is computed with respect to the absolute Weyl group \( \hat{W} \) \([3]\, p.74\).

\[\square\]
**1.6. Plancherel measure.** We continue in the same context, letting \( G \) be an unramified reductive group and \( K \) a hyperspecial maximal compact subgroup. Let \( \hat{G} = \hat{G} \rtimes \langle \theta \rangle \), and we continue with notation from previous sections.

Let \( \hat{S}_1 \) be the maximal compact subgroup of \( \hat{S} \). Let \( ds \) be the Haar measure on \( \hat{S}_1 \) normalized so that \( \hat{S}_1 \) has volume 1. Let \( (\cdot, \cdot) \) be the inner product with respect to the Haar measure on \( \hat{S}_1 \). That is,

\[
(1.6.1) \quad (f_1, f_2) = \int_{\hat{S}_1} f_1(s) \overline{f_2(s)} ds.
\]

Multiplicative characters of \( \hat{S}_1 \) are orthonormal: \( (e^\lambda, e^\mu) = \delta_{\lambda\mu} \).

We define a measure on \( \hat{S}_1 \) by

\[
(1.6.2) \quad dm(s) = \frac{Q(q^{-1})}{\text{card}(W^\theta)} P(E, q^{-1}) P(E^{-1}, q^{-1}) ds.
\]

It will be checked below that the partition functions defining \( dm(s) \) have nonzero denominators on \( \hat{S}_1 \). Let \( (\cdot, \cdot) \) be the pairing provided by this measure on continuous functions on \( \hat{S}_1 \). That is,

\[
(1.6.3) \quad \langle f_1, f_2 \rangle = \int_{\hat{S}_1} f_1(s) \overline{f_2(s)} dm(s).
\]

The proof of the Plancherel measure uses the following averaging lemma.

**Lemma 1.6.4.** Let \( f \) be continuous on \( \hat{S}_1 \) and \( W^\theta \)-invariant. Then

\[
\langle f, \hat{f}_\mu \rangle = c_\mu (f, e^\mu \frac{P(E, q^{-1})}{P(E, 1)}) \cdot \text{card}(W^\theta), \quad \text{where} \quad c_\mu = q^{(\mu, \rho^\vee)} \frac{Q(q^{-1})}{Q_S(\mu)(q^{-1})}.
\]

**Proof.** The measure \( dm(s) \) is \( W^\theta \)-invariant by Corollary 1.3.9. As a consequence, we may push a sum over \( W^\theta \) over to \( f \) to obtain

\[
(1.6.5) \quad \langle f, \sum_{w \in W^\theta} w(f_1) \rangle = \text{card}(W^\theta) \langle f, f_1 \rangle,
\]

for any continuous function \( f_1 \) on \( \hat{S}_1 \). In particular, assume that \( f_1 = \hat{f}_\mu \), which Macdonald’s formula presents a sum over \( W^\theta \). This means that in \( \langle f, \hat{f}_\mu \rangle \), we may replace \( \hat{f}_\mu \) with the \( w = 1 \) term in Macdonald’s formula.

The constant \( c_\mu \) is the product of the constants appearing in Macdonald’s formula, the Plancherel measure, and Equation 1.6.5

\[
c_\mu = \left( \frac{q^{(\mu, \rho^\vee)}}{Q_S(\mu)(q^{-1})} \right) \left( \frac{Q(q^{-1})}{\text{card}(W^\theta)} \right) \text{card}(W^\theta).
\]
The conjugate of $e^\mu$ is $e^{-\mu}$ and of $E$ is $E^{-1}$ on $\hat{S}_1$, because $s = s^{-1}$, for $s \in \hat{S}_1$. We have

$$\langle f, \hat{f}_{\mu} \rangle = c_\mu \int_{\hat{S}_1} f \left( e^{-\mu} \frac{P(E, 1)}{P(E, q^{-1})} \right) \left( \frac{P(E, q^{-1})P(E^{-1}, q^{-1})}{P(E, 1)(E^{-1}, 1)} \right) ds$$

$$= c_\mu \int_{\hat{S}_1} f \left( e^{-\mu} \frac{P(E^{-1}, q^{-1})}{P(E, 1)} \right) ds$$

$$= c_\mu (f, e^\mu \frac{P(E, q^{-1})}{P(E, 1)}).$$

\[ \square \]

**Theorem 1.6.6 (Plancherel measure).** The denominators are nonzero on $\hat{S}_1$ in the partition functions defining $dm(s)$. The Plancherel measure is supported on $\hat{S}_1$ and is given explicitly by $dm(s)$ on $\hat{S}_1$.

**Remark 1.6.7.** We recall the defining property of the Plancherel measure for the spherical Hecke algebra. The Plancherel measure is $dm(s)$ if for all $f_1, f_2 \in \mathcal{H}(G//K)$,

$$\int_G f_1(g) \overline{f}_2(g) dg = \int_{\hat{S}} \hat{f}_1(s) \overline{\hat{f}_2}(s) dm(s).$$

When $\mu \neq \lambda$, the integral on the left is trivial to compute for the functions $f_1 = f_1$ and $f_2 = f_2$, because the functions $f_2$ and $\hat{f}_1$ have disjoint support, giving 0. When $\mu = \lambda$, the integral on the left is the volume of $K \omega^\mu K$. This volume is $c_\mu q^{a(\omega^\mu)}$ by [5], where $c_\mu$ is the constant in Lemma 1.6.4. The proof of the theorem proceeds by computing the inner products $\langle \hat{f}_1, \hat{f}_\mu \rangle$ and showing that they give the same values as the integral on the left.

**Proof.** The proof, which we review, is due to Macdonald [50, Ch.V]. It is what he calls the standard case. Choose any total order $< \in P^* \subset Y$ such that $A < \lambda + \alpha$, whenever $\alpha = N\beta$ is the norm of a positive root $\beta \in \Psi^+$.

By Lemma 1.3.7 the ratio $P(E, q^{-1})/P(E, 1)$ factors into a product of terms of the form $(1 - t)/(1 - q^{-b}t)$, where each $t = e^\epsilon \sigma = e^{\epsilon\sigma}$ for some root $\beta$ with norm $\alpha$, for some sign $\epsilon \in \{\pm 1\}$, and for some $b \geq 1$. For any $p$-adic field $F$, we have $q = q_F > 1$ and $q^{-b} < 1$. Thus we have an absolutely convergent expansion in $t$:

$$\frac{1 - t}{1 - q^{-b}t} = 1 + (q^{-b} - 1)t + q^{-2b}t^2 + \cdots,$$

noting that $|t| = |\sigma(s)| = 1$ at each $s \in \hat{S}_1$. In particular, the denominator of $P(E, q^{-1})/P(E, 1)$ does not vanish. Similarly, the denominator of $P(E^{-1}, q^{-1})/P(E^{-1}, 1)$ does not vanish because $q^{-b}t^{-1} \neq 1$, giving the nonvanishing of the denominator in the measure $dm(s)$.

By multiplying the series expansions (Equation 1.6.8) associated with each factor of $P(E, q^{-1})/P(E, 1)$, it follows that for each $\mu \in P^*$ we have an absolutely convergent expansion of the form

$$e^{\mu} \frac{P(E, q^{-1})}{P(E, 1)} = e^{\mu} + \sum_{\rho < \mu} a_\rho e^{\rho'},$$

for some coefficients $a_\rho$ that turn out not to matter.

We compute $\langle \hat{f}_1, \hat{f}_\mu \rangle$. We may assume without loss of generality that $\lambda \leq \mu$. 

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We have a finite expansion (see Equation 3.9.6 below):
\[
\hat{f}_\lambda = q^{(\lambda, \rho)} m_\lambda + \sum_{\lambda' \in \Lambda} s_{\lambda', \lambda} m_{\lambda'}.
\]
Also, for \( \lambda', \mu' \in P^+ \), we have
\[
(m_{\lambda'}, e_{\mu'}) = \delta_{\lambda', \mu'}.
\]
The function \( \hat{f}_\lambda \) is \( W_\theta \)-invariant, which justifies the use of the averaging lemma (Lemma 1.6.4) to simplify the inner product. Invoking the averaging lemma, expanding everything, and integrating term by term, we have
\[
\langle \hat{f}_\lambda, \hat{f}_\mu \rangle = c_{\mu}(\langle \hat{f}_\lambda, e_{\mu} \rangle) = c_{\mu}(q^{(\lambda, \rho)} m_{\lambda}, e_{\mu'} + \sum_{\lambda' < \lambda \leq \mu < \mu'} c_{\mu} s_{\lambda', \lambda} m_{\lambda'}).
\]
Comparing this inner product with the inner products in the remark, we see that the proof is complete. □

1.7. inverting weight multiplicities. This section follows van Leeuwen’s algorithm to invert the weight multiplicity matrix. For type \( A_n \), van Leeuwen’s formula agrees with the inverse of the Kostka matrix described in [14].

We have two bases of \( \mathbb{C}[Y^*]^W \), given by \( \{m_\lambda\} \) and \( \{\tau_\lambda\} \), indexed by \( \lambda \in P^+ \). The change of basis matrix expressing \( \tau_\lambda \) in terms of \( m_\mu \) is the weight multiplicity matrix \( m_{\mu, \lambda} = (\tau_\lambda, e_\mu) \). In the reverse direction, for \( \mu \in P^+ \), we have a change of basis matrix \( n_{\mu, \lambda} \)
\[
(1.7.1) \quad m_{\mu} = \sum_{\lambda} n_{\mu, \lambda} \tau_\lambda,
\]
with \( \mu, \lambda \in P^+ \). This section gives a formula for \( n_{\mu, \lambda} \).

We have a set \( Y_0^* \subseteq Y^* \) of characters \( \lambda \) such that \( \lambda \) is fixed (that is, \( w \cdot \lambda = \lambda \)) by some reflection \( w \in W^0 \). For each \( w \in W^0 \), we define
\[
(1.7.2) \quad Y_w^* = \{ \lambda \in Y^* \mid w \cdot \lambda \in P^+ \}.
\]
These sets partition \( Y^* \), so that each \( \lambda \in Y^* \) belongs to a unique \( Y_w^* \), for \( x \in W^0 \cup \{0\} \). Let \( e_w \) be the characteristic function of \( Y_w^* \).

Recall (from Section 1.4) that we have defined an operator \( J \) that has the property \( J(f) = f J(1) \) if \( f \in \mathbb{C}[Y^*]^W \). In particular, using the Weyl denominator formula (1.4.2), we find that
\[
(1.7.3) \quad J(\tau_\lambda) = \tau_\lambda J(1) = J(e^\lambda) P(E^{-1}, 1) J(1) = J(e^\lambda), \quad \lambda \in P^+.
\]
In the opposite direction, we define a desymmetrizer operator \( L \) by
\[
L(e^\mu) = \begin{cases} 0, & \mu \in Y_0^*; \\ (-1)^{\mu_\theta} e^\mu, & \mu \in Y_w^*. \end{cases}
\]
We extend $L$ linearly to $C[Y^*]$. The operator $L$ can be characterized as the unique linear operator whose range is supported on the dominant weights and such that $J(f) = J(L(f))$, for $f \in C[Y^*]$. By this characterization,

$$L(\tau_\lambda) = L(J(e^\lambda)) = L(e^\lambda) = e^\lambda,$$

for $\lambda \in P^+$. This means that for any $f = \sum_{\lambda \in P^+} c_\lambda \tau_\lambda \in C[Y^*]$, the coefficient $c_\lambda$ is the coefficient of $e^\lambda$ in $L(f)$. This is van Leeuwen inversion. We have

$$L(e^\mu) = \sum_{w \in W^\mu} (-1)^\ell_w e^{w^*\mu} e_w(\mu).$$

Recall that $S(\mu)$ is defined as the set of simple roots such that $\langle \alpha^\vee, \mu \rangle = 0$ iff $\alpha \in S(\mu)$. Recall also that $W_S \leq W^\mu$ is the subgroup generated by reflections in $S$.

**Lemma 1.7.4** (van Leeuwen). For each subset $S$ of the set of simple roots of $\Psi_{red, +}$, and for every $\mu \in P^+$ with $S = S(\mu),

$$n_{\mu, \lambda} = \sum_{(u, w) \in (W^\mu/W_S) \times W^\mu} (-1)^{\ell(u)} e_u(w^\mu) \delta_{w^*w(\mu), \lambda}.$$

**Proof.**

$$m_\mu = \sum_{w' \in W^\mu/W_S} e^{w'\mu}.$$

Then

$$n_{\mu, \lambda} = (\sum_{\lambda'} n_{\mu, \lambda'} e^{\lambda'}, e^{\lambda})$$

$$= (L(\sum_{\lambda'} n_{\mu, \lambda'} \tau_{\lambda'}), e^{\lambda})$$

$$= (L(m_\mu), e^{\lambda})$$

$$= \sum_{w' \in W^\mu/W_S} (L(e^{w'\mu}), e^{\lambda})$$

$$= \sum_{w' \in W^\mu/W_S} \sum_{w \in W^\mu} (-1)^{\ell(w)} e_w(w'\mu)(e^{w^*w(\mu)}, e^{\lambda})$$

$\Box$

1.8. **geometric Satake.** Let $K$ be a hyperspecial maximal compact subgroup of $G$. In the usual formulation, the Satake transform is an isomorphism of the Hecke algebra $\mathcal{H}(G//K)$, with the $W^\mu$-invariant functions in the group algebra $C[X, (A)]$. As first pointed out by Langlands, this is the space of conjugation invariant functions on the coset $\hat{G} \rtimes \theta$ in $^lG$. For split groups, the geometric Satake transform reformulates the transform in the language of sheaves and in terms of the representation ring of $\hat{G}$ [31]. For a geometric approach to geometric Satake that includes unramified groups, see [38].

The identities we give should be viewed as formal analogues of geometric Satake by the function-sheaf dictionary. Working formally with irreducible characters, the geometric
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Satake transform expresses each \( \hat{f}_\lambda \) in terms of the basis \( \tau_\mu \) of irreducible characters for the root system \( \Psi_{\text{red}} \):

\[
\hat{f}_\lambda = \sum_\mu g_{\lambda \mu} \tau_\mu. \tag{1.8.1}
\]

Casselman gives a formula for the coefficients \( g_{\lambda \mu} \) when \( G \) is split \([6]\). In this section, we extend the result to \( G \) unramified. The statement here is less polished than what is known in the split case \([6]\). The proof we present here is based on van Leeuwen’s algorithm.

Set

\[
C = \{-\sum_{\alpha \in S} \alpha \mid S \subseteq \Psi^+\};
\]

Define \( p_\mu(q^{-1}) \) by

\[
P(E^{-1}, q^{-1})^{-1} = \sum_{\mu \in C} p_\mu(q^{-1}) e^\mu, \quad \text{so} \quad P(E^{-w}, q^{-1})^{-1} = \sum_{\mu \in C} p_\mu(q^{-1}) e^{w_\mu}.
\]

**Theorem 1.8.3** (geometric Satake). Let \( \lambda \in P^+ \) and let \( S = S(\lambda) \).

\[
\hat{f}_\lambda = \frac{q^{(\lambda, \rho^\vee)}}{Q_5(q^{-1})} \sum_{\mu \in C} \left( \sum_{w \in W^0} (-1)^{w_\mu} p_\mu(q^{-1}) e_w(\lambda + \mu) \right) \tau_{w^*}(\lambda + \mu).
\]

**Proof.** Abbreviate \( r_\lambda = q^{(\lambda, \rho^\vee)}/Q_5(q^{-1}) \). Let \( \rho = \rho(\Psi^+) \). By the Weyl denominator product formula (Corollary 1.3.8),

\[
P(E^{-w}, 1) = (-1)^{\ell_w} e^{w_\rho - \rho} P(E^{-1}, 1),
\]

Then expanding Macdonald’s formula (1.5.1) using this and Equation 1.8.2 we get

\[
\hat{f}_\lambda = r_\lambda \sum_{\mu \in C} p_\mu(q^{-1}) P(E^{-1}, 1) J(e^{1+\mu}),
\]

where we have absorbed the sum over \( W^0 \) in Macdonald’s formula into \( J \). We observe that

\[
L(\hat{f}_\lambda) = r_\lambda \sum_{\mu \in C} p_\mu(q^{-1}) L(P(E^{-1}, 1) J(e^{1+\mu}))
= r_\lambda \sum_{\mu \in C} p_\mu(q^{-1}) L(e^{1+\mu})
= r_\lambda \sum_{\mu \in C} p_\mu(q^{-1}) \sum_{w \in W^0} (-1)^{w_\mu} e_w(\lambda + \mu) e^{w^*_{1+\mu}}.
\]

Recall that the coefficient \( c_\lambda \) of an expansion \( f = \sum_\lambda c_\lambda \tau_\lambda \) is the coefficient of \( e^1 \) in \( L(f) \).

The result follows. \( \square \)

There is a second less explicit form of the geometric Satake transform that is obtained as follows. We have

\[
\hat{f}_\lambda = \sum_{\mu'} s_{\lambda \mu'} m_{\mu'}.
\]

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where the coefficients $s_{\lambda,\mu'}$ are given as $p$-adic integrals (see Equation 3.9.6 below). By van Leeuwen’s formula linking $m_{\mu'}$ to $\tau_{\mu}$ with coefficient matrix $n_{\mu',\mu}$, we obtain the coefficients $g_{\lambda,\mu}$ as a matrix product:

$$\hat{f}_{\lambda} = \sum_{\mu'} s_{\lambda,\mu'} n_{\mu',\mu} \tau_{\mu} = \sum_{\mu} g_{\lambda,\mu} \tau_{\mu}. \quad (1.8.5)$$

1.9. **A Kato-Lusztig formula.** Equipped with Plancherel and Macdonald, we obtain an easy Kato-Lusztig formula for the inverse Satake transform. Our result generalizes a formula that was known when $\theta = 1$ [21] [29]. Recall the $q$-twisted character $\tau_{\lambda,q}$ from (1.4.5). Write

$$\tau_{\lambda} = \sum_{\mu} t_{\lambda,\mu} \hat{f}_{\mu},$$

for some constants $t_{\lambda,\mu}$.

**Theorem 1.9.1** (Kato-Lusztig formula). The coefficients $t_{\lambda,\mu}$ of the inverse geometric Satake transform are

$$t_{\lambda,\mu} = (\tau_{\lambda, q^{-1}}, e^{\mu}) q^{\langle \mu, \rho^\vee \rangle}.$$ 

**Proof.** The character $\tau_{\lambda}$ is $W^\theta$-invariant. We use the averaging property (Lemma 1.6.4) and the Weyl character formula to compute an inner product.

$$\langle \tau_{\lambda}, \hat{f}_{\mu} \rangle = c_{\mu}(\tau_{\lambda}, e^{\mu} \frac{P(E, q^{-1})}{P(E, 1)})$$

$$= c_{\mu} \int_{\hat{S}} (J(e^{\mu}) P(E^{-1}, 1)) \left( e^{-\mu} \frac{P(E^{-1}, q^{-1})}{P(E^{-1}, 1)} \right) ds$$

$$= c_{\mu} \int_{\hat{S}} J(e^{1}) P(E^{-1}, q^{-1}) e^{\mu} ds$$

$$= c_{\mu} (\tau_{\lambda, q^{-1}}, e^{\mu}).$$

$$\langle \tau_{\lambda}, \hat{f}_{\mu} \rangle = \sum_{\mu'} t_{\lambda,\mu'} \langle \hat{f}_{\mu'}, \hat{f}_{\mu} \rangle$$

$$= t_{\lambda,\mu} c_{\mu} q^{\langle \mu, \rho^\vee \rangle}.$$ 

\[\square\]

2. **Endoscopic branching rules**

This section uses partition functions to give a branching rule for the restriction of an irreducible representation of $\hat{G} \times \theta$ to $\xi(\hat{H} \times \theta_H)$.

2.1. **$\theta$-conjugacy.** Let $G$ be an unramified reductive group, and let $^L G = \hat{G} \times \langle \theta \rangle$ be its $L$-group, with the automorphism $\theta$ given by the action of the Frobenius element on the root datum. The calculations in this subsection will be used in Theorem 2.3.5 to give an explicit branching rule for an embedding of the $L$-group of an endoscopic groups into the $L$-group of $G$. 

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We have a set of simple roots $\Delta \subseteq \Psi$, determined by $(\hat{T}, \hat{B})$. Let $\alpha$ be the highest positive root and let $\Delta' = \Delta \cup \{-\alpha\}$ be the extended set of simple roots. The extended Dynkin diagram has node set $\Delta'$. The automorphism $\theta$ preserves $\Delta$ and fixes $-\alpha$, hence acts on the extended Dynkin diagram.

Let $w \in W$ be an element that preserves the extended Dynkin diagram. We consider lifts $\hat{w} \in N_{\hat{Q}}(\hat{T})$ of $w$ such that $\theta_1 = \hat{w}\theta$ has finite order. The partition function and other data we define are sensitive to the representative $\hat{w}$ of $w$. However, the branching rule that we obtain in the end (Theorem 2.3.3) will satisfy a simple transformation rule depending on $\hat{w}$ (Lemma 2.3.4). In each case, we pick a particularly convenient representative $\hat{w}$ of $w$ to work with, and leave the rest to the transformation rule. The details of the choice of $\hat{w}$ will be discussed further below.

Let $\hat{S} = \hat{T}/(1-\theta)\hat{T}$ and $\hat{T}_1 = \hat{T}/(1-\theta_1)\hat{T}$. There are norm maps $N : X^*(\hat{T}) \to X^*(\hat{S})$ and $N_1 : X^*(\hat{T}) \to X^*(\hat{T}_1)$.

We write $\Psi^+(\hat{T}, \hat{B})$ for the set of positive roots of $\hat{G}$ with respect to a Cartan subgroup $\hat{T}$ and a Borel subgroup $\hat{B}$.

**Lemma 2.1.1.** There exists a Borel subgroup $\hat{B}(w\theta) \supseteq \hat{T}$ such that for every $\alpha \in \Psi^+(\hat{T}, \hat{B}(w\theta))$, either $N_1\alpha = 0$ or $\langle w\theta\rangle\alpha \subseteq \Psi^+(\hat{T}, \hat{B}(w\theta))$.

We call $\hat{B}(w\theta)$ an adapted Borel subgroup.

**Proof.** Let $C = \{N_1\alpha \in X^*(\hat{T})^0 \mid \alpha \in \Psi, N_1\alpha \neq 0\} \subseteq X^*(\hat{T}_1)$. Clearly $N_1(-\alpha) = -N_1\alpha$ and $-C = C$. Choose a hyperplane through the origin in $X^*(\hat{T}_1) \otimes \mathbb{Q}$ that does not meet $C$ to partition $C = C_+ \sqcup C_-$ into a positive and negative set. We can choose a compatible hyperplane through the origin in $X^*(\hat{T}) \otimes \mathbb{Q}$ such that $\alpha$ is positive or negative according to $N_1\alpha \in C_\pm$, provided that $N_1\alpha \neq 0$. By a small generic perturbation of this hyperplane through the origin, we may assume that the hyperplane separates each pair $\pm \alpha$ of roots. Let $\hat{B}(w\theta)$ be the Borel subgroup defined by $\hat{T}$ and the positive roots determined by the hyperplane. $\square$

We consider data $D = (\hat{U}, \hat{B}(w\theta), \hat{B}_1, \iota, \phi, \epsilon, \hat{w})$ of the following type: $\hat{U} = \hat{T}_1$ and $\iota : \hat{U} \to \hat{T}_1$ is an isogeny of tori. Also $\phi : \hat{U} \to \hat{S}$ is a homomorphism, and $\epsilon \in \hat{S}$ is an element of finite order. Finally, $\hat{B}(w\theta)$ is an adapted Borel subgroup and $\hat{B}_1$ is a $\theta$-stable Borel subgroup of $\hat{G}$ containing $\hat{T}$.

The purpose of the isogeny is to remove all radicals from the formulas that follow. We always have $\hat{U} = \hat{T}_1$, but we maintain two notations for the same torus to distinguish the source $\hat{U}$ from the target $\hat{T}_1$ of the isogeny $\iota$. We write $\nu$ for an element of $\hat{U}$ and $\iota(\nu) = \tau \in \hat{T}_1$ for an element of the target of the isogeny. Define $\phi_{\epsilon} : \hat{U} \to \hat{S}$ by $\phi_{\epsilon}(\nu) = \epsilon\phi(\nu)$.

As noted above, $D(\hat{G}, R, \theta, E, q)$ is a function on $\hat{S}$, which we pull back to a function $\phi' D(\hat{G}, R, \theta, E, q)$ on $\hat{U}$. Similarly, $\iota' D(\hat{G}, R', \hat{w}\theta, E, q)$ is a function on $\hat{U}$. The $\hat{G}$-conjugacy class of $\iota'\hat{w}\theta$ depends only on the image $\tau = \hat{T}_1$ of $t \in \hat{T}$. We can therefore refer to the $\hat{G}$-conjugacy class of $\tau'\hat{w}\theta$, for $\tau \in \hat{T}_1$. 

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Let $\Psi^*_\theta$ be the set of roots $\alpha$ of $\Psi^*(\hat{T}, \hat{B}_1)$ such that $\phi^*N\alpha \neq 0$, and let $\Psi^+\theta$ be the set of roots $\alpha$ of $\Psi^*(\hat{T}, \hat{B}(w\theta))$ such that $N_1\alpha \neq 0$.

**Proposition 2.1.2.** Let $G$ be an unramified reductive group with complex dual $\hat{G} \rtimes \langle \theta \rangle$. Assume that $w\theta$ acts on the extended root system. Then we can construct data $D = (\hat{U}, \hat{B}(w\theta), \hat{B}_1, \zeta, \phi, \epsilon, \hat{w})$ typed as above such that

1. (conjugacy) $w\theta$ is $\hat{G}$-conjugate to $\epsilon\theta(\nu)\theta$, where $i(\nu) = \tau \in \hat{T}_1$;
2. (regularity) For each $\alpha \in \Psi^{\text{red}}$, the zero set of $\alpha(\phi(\nu)) - 1$ is a proper Zariski closed subset of $\hat{U}$.
3. (partition)

   \[
   \phi^*D(\hat{G}, \Psi^*_\phi\theta, \theta, E, q) = \epsilon^*D(\hat{G}, \Psi^+\theta, w\theta, E, q).
   \]

**Remark 2.1.3.** We note that $\hat{T}_1/W^\mu_H$ classifies the $H$-semisimple conjugacy classes in $LH$, and $\hat{S}/W^\mu_G$ classifies the $\theta$-semisimple conjugacy classes in $L\hat{G}$. A morphism $\xi : LH \to L\hat{G}$ induces a map of conjugacy classes $\hat{T}_1/W^\mu_H \to \hat{S}/W^\mu_G$. The correspondence

\[
\hat{T}_1 \leftarrow \hat{U} \rightarrow \hat{S}
\]

between $\hat{T}_1$ and $\hat{S}$ lifts this map of conjugacy classes up to tori. The identity of partition functions refines the identity between the characteristic polynomials of conjugate elements.

The morphism $\xi$ gives a restriction map

\[
J_\xi : \mathbb{C}[Y^*]^{W^\mu} \to \mathbb{C}[X^*(\hat{T}_1)]^{W^\mu_H}
\]

that is constructed as follows. The morphism $\hat{T}_1/W^\mu_H\to \hat{S}/W^\mu_G$ is a morphism of affine varieties, which determines a homomorphism $J_\xi$ between their coordinate rings.

**Remark 2.1.5.** We have calculated the determinants on a case-by-case basis and the answers can be striking. For example, assume that $\theta = 1$ and $\hat{G}$ is simply laced. Let $N_1\Psi = \{N_1\beta \mid \beta \in \Psi\}$ be the norm root system of $\hat{T}_1$. Let $m : \Psi \to \mathbb{N}$ be given by $m(\alpha) = \text{order of } w$, if $\alpha$ is a long root in $N_1\Psi$ (or if $N_1\Psi$ is simply laced). Let $m(\alpha) = 1$, otherwise. Then there exist data $D$ such that

\[
D(\hat{G}, \Psi^+\theta, w\theta, E, q) = \prod_{\alpha \in N_1\Psi^+\theta} (1 - q^{m(\alpha)}e^{\alpha^*m(\alpha)}).
\]

**Proof (proposition).** The proof extends over several subsections giving a series of reductions. The first reduction is Levi descent.

2.1.1. Levi descent. Let $\hat{P}$ be a $\theta$-stable parabolic subgroup containing $\hat{T}$. Let $\hat{M}$ be a $\theta$-stable Levi subgroup of $\hat{P}$ containing $\hat{T}$. Let $\hat{M}_{sc}$ be the simply connected cover of the derived group of $\hat{M}$. Let $w \in W(\hat{T}, \hat{M}) = W(\hat{M}_{sc}, \hat{M}_{sc})$ be a Weyl group element that acts on the extended Dynkin diagrams of both $\hat{M}_{sc}$ and $\hat{G}$ (for some choices of $\theta$-stable Borel subgroups). Then $w\theta$ is an automorphism of both $\hat{M}_{sc}$ and $\hat{G}$. We assume that we have constructed data $D_{sc} = (\hat{U}_{sc}, \ldots)$ for $\hat{M}_{sc}$ and show how to construct data $D$ for $\hat{G}$.

Let $\hat{A} = Z(\hat{M})^0$ be the identity component of the center of $\hat{M}$. Set

\[
\hat{A}_\theta := \hat{A}/(1 - \theta)\hat{A} = \hat{A}/(1 - w\theta)\hat{A}.
\]
Define $\hat{U}, \phi : \hat{U} \to \hat{S}$, and $\iota : \hat{U} \to \hat{T}_1$ as follows:

\[
\begin{align*}
\phi &: \hat{U} := \hat{A}_q \times \hat{U}_{sc} \to \hat{A}_q \times \hat{S}_{sc} \to \hat{S}, \\
\iota &: \hat{U} = \hat{A}_q \times \hat{U}_{sc} \to \hat{A}/(1 - w\theta)\hat{A} \times \hat{T}_{1,sc} \to \hat{T}_1.
\end{align*}
\]

Let $\Psi_N$ be the set of positive roots of the unipotent radical of $\hat{P}$. Let $\hat{B}(w\theta)$ be the adapted Borel containing $\hat{T}$ with positive roots those of $\hat{B}(w\theta)_{sc} \subseteq \hat{M}_{sc}$ and $\Psi_N$. Then

\[
\Psi^+ = \Psi^+_{w\theta,sc} \sqcup \Psi_N.
\]

Let $\hat{B}_1$ be the Borel containing $\hat{T}$ with positive roots those of $\hat{B}_{1,sc} \subseteq \hat{M}_{sc}$ and $\Psi_N$.

We claim that $\Psi^+_{\phi,\theta} = \Psi^+_{\phi,\theta,sc} \sqcup \Psi_N$. Note that

\[
\begin{align*}
\Psi^+_{\phi,\theta} &= (\Psi^+ \cap \Psi^+_{\hat{T},\hat{B}_1}) \sqcup (\Psi^+ \cap \Psi_N) = \Psi^+_{\phi,\theta,sc} \sqcup (\Psi^+_{\phi,\theta} \cap \Psi_N).
\end{align*}
\]

We have a map

\[
\hat{A}^0 \to \hat{A}_0 \to \hat{U} \to \hat{S}.
\]

Also, $\alpha$ is trivial on $\hat{A}^0$ if and only if $\alpha$ is trivial on $\hat{A}$ if $\alpha$ is a root of $\hat{M}$. For all $\alpha \in \Psi_N$,

\[
\phi^*\lambda = 0 \iff \lambda = 0 \iff \alpha|_{\hat{M}} = 0.
\]

The claim follows.

We choose $\epsilon \in \hat{S}$ to be the image of $\epsilon_{sc}$ and $\hat{w}$ to be the image of $\hat{w}_{sc}$.

We prove property (1-conjugacy). Let $(a, \nu_{sc}) \in \hat{A} \times \hat{U}_{sc}$ represent $\nu \in \hat{U}_{sc}$. The element $\epsilon_{sc}\phi_{sc}(\nu_{sc})\theta$ is conjugate to $\iota_{sc}(\nu_{sc})\hat{w}_{sc}\theta$ by an element $m \in \hat{M}_{sc}$ whose image in $\hat{M}$ commutes with $a \in Z(\hat{M})$. Thus $a_{sc}\phi_{sc}(\nu_{sc})\theta$ and $a_{sc}(\nu_{sc})\hat{w}\theta$ are conjugate.

We show that property (2-regularity) of the proposition holds. As shown above, if $\alpha$ is not a root of $\hat{M}$, then $\phi^*\lambda = 0$, so the regularity property holds for $\alpha \in \Psi_N$. For the remaining positive roots, regularity follows from regularity for $\hat{M}_{sc}$.

Finally we show (3-partition). Let $m$ and $(a, \nu_{sc})$ be as above. Let $\hat{g}_N = \eta_N$. We have an isomorphism $\text{Ad}(m^{-1}) : \hat{g}_N \to \eta_N$. Using the conjugacy property, we have

\[
\begin{align*}
\phi^*D(\hat{G}, \Psi_{\nu_{sc},\hat{M}}\theta, E, q)(\nu) &= \det(1 - \epsilon_{sc}(\nu_{sc})a\theta q; \hat{g}_N) \\
&= \det(1 - \iota_{sc}(\nu_{sc})\hat{w}\theta q; \hat{g}_N) \\
&= \iota^*D(\hat{G}, \Psi_{\nu_{sc},\hat{M}}\theta, E, q)(\nu).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\phi^*D(\hat{G}, \Psi_{\nu_{sc},\hat{M}}\theta, E, q) &= \phi^*D(\hat{G}, \Psi_{\nu_{sc},\hat{M}}\theta, E, q) \\
&= \phi^*D(\hat{M}, \Psi_{\nu_{sc},\hat{M}}\theta, E, q) \\
&= \iota^*D(\hat{G}, \Psi_{\nu_{sc},\hat{M}}\theta, E, q)
\end{align*}
\]

This completes the reduction to $\hat{M}_{sc}$.

As a special case of this construction, applied to $\hat{M} = \hat{G}$, we reduce to the case that $\hat{G}$ is semisimple and simply connected.
2.1.2. **reduction to transitive on orbits.** The Dynkin diagram of $\hat{G}$ is a union of orbits under $\theta$. We give a reduction to the case that $\theta$ is transitive on the set of connected components of the Dynkin diagram. We may assume that $\hat{G}$ is simply connected. Assume that data $\mathcal{D}_i = (\hat{U}_i, \ldots)$ satisfying the properties of the proposition have been constructed from each factor of $\hat{G} = \hat{G}_1 \times \cdots \times \hat{G}_r$, where $\theta = (\theta_1, \ldots, \theta_r)$, and $\theta_i$ acts on $\hat{G}_i$. We have factorizations of $\hat{T}_1$ and $\hat{S}$ as $r$-fold products. We may define the data $\mathcal{D} = (\hat{U}, \ldots)$ for $\hat{G}$ as a $r$-tuples $\phi = (\phi_1, \ldots, \phi_r)$, $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$, and $r$-fold products, etc. The verification of properties (1), (2), (3) is routine in this case. This completes the reduction.

2.1.3. **$W$-conjugate reduction.** In this subsection, we show how to construct data $\mathcal{D}$ for $w\theta$ assuming that we have data $\mathcal{D}'$ for $w'\theta$, where $w_1 w\theta w_1^{-1} = w'\theta$ for some $w_1 \in W$ and assuming that both $w$ and $w'$ act on the extended Dynkin diagram (for some common choice of simple roots).

Assume that $\mathcal{D}' = (\hat{U}', \ldots, \hat{w}')$ is given. Let $\hat{w}_0$ and $\hat{w}_1$ be lifts of $w$ and $w_1$ to $N_{\hat{G}}(\hat{T})$. Then

$$w_1 \hat{w}_0 \theta \hat{w}_1^{-1} = t w' \theta,$$

for some $t \in \hat{T}$. Let $\hat{w} = w_1^{-1} t^{-1} \hat{w}_1 \hat{w}_0$. Then $\hat{w} \mapsto w$ and

$$\hat{w}_1 \hat{w} \theta \hat{w}_1^{-1} = \hat{w}' \theta.$$  

(2.1.7)

Set $\hat{B}(w\theta) = \hat{B}(w' \theta)^{w_1}$. Then $\hat{B}(w\theta)$ is adapted to $w\theta$. Also,

$$\Psi^*_{w\theta} = w_1^{-1} \Psi^*_{w'\theta}.$$  

We have isomorphisms

$$\hat{T}_1 = \hat{T}/(1 - w\theta) \hat{T} = \hat{T}/(1 - w' \theta) \hat{T} = \hat{T}'_1.$$  

(2.1.8)

The isomorphism $\hat{T}/(1 - w\theta) \hat{T} \rightarrow \hat{\hat{T}}/(1 - w' \theta) \hat{T}$ is given by $t \mapsto w_1 t w_1^{-1}$. Let $\hat{U}' = \hat{U}'_0$. Define $\iota$:

$$\iota : \hat{U} = \hat{U}' \rightarrow \hat{T}_1' \rightarrow \text{Int}(w_1^{-1}) \hat{T}_1.$$  

Let $\mathfrak{g}$ be the Lie algebra of $\hat{G}$. We have a linear isomorphism

$$\psi : \mathfrak{g} \rightarrow \mathfrak{g}, \quad X \mapsto \text{Ad}(\hat{w}_1) X,$$

sending $\mathfrak{g}_{\Psi^*_{w\theta}}$ to $\mathfrak{g}_{\Psi^*_{w'\theta}}$. Then

$$D(\hat{G}, \Psi^*_{w\theta}, \hat{w}_0 \theta, E, q) = D(\hat{G}, \hat{w}_1^{-1} \Psi^*_{w'\theta}, \hat{w}_1^{-1} \hat{w}' \theta \hat{w}_1, E, q)$$

$$= D(\hat{G}, \Psi^*_{w'\theta}, \hat{w}' \theta, E^{w_1^{-1}}, q).$$
On the other hand, choose \( \phi = \phi' \), \( \hat{B}_1 = \hat{B}'_1 \), and \( \epsilon = \epsilon' \). Then \( \Psi^+_{\phi, \theta} = \Psi^+_{\phi', \theta} \) and
\[
\iota^*D(\hat{G}, \Psi_{\nu, \theta}^+, \iota w\theta, E, q) = \iota^*D(\hat{G}, \Psi_{\nu, \theta}^+, \iota^*w\theta, E^\iota, q)
\]
\[
= \iota^*(\text{Int}(w_{\iota}))\iota^*D(\hat{G}, \Psi_{\nu, \theta}^+, \iota^*w\theta, E^\iota, q)
\]
\[
= \iota^*D(\hat{G}, \Psi_{\nu, \theta}^+, \iota^*w\theta, E, q)
\]
\[
= \phi_{\iota}^*D(\hat{G}, \Psi_{\nu, \theta}^+, \iota, E, q)
\]
\[
= \phi_{\iota}^*D(\hat{G}, \Psi_{\nu, \theta}^+, \iota, E, q).
\]
These determinant formulas give property (3) of the proposition.

To prove property (1-conjugacy), we can realize the \( \hat{G} \)-conjugacy of \( \epsilon \phi(\nu)\theta \) and \( \iota(\nu)\iota w\theta \) explicitly by conjugation by \( \iota w_1 \) and the conjugation used for data \( \mathcal{D}' \).

To prove property (2-regularity), we observe that \( \alpha(\epsilon \phi(\nu)) = 1 \) iff \( \alpha(\epsilon' \phi'(\nu)) = 1 \). This holds for all \( \nu \) iff \( \alpha = 0 \). This completes the reduction from \( w\theta \) to \( w'\theta \).

2.1.4. reduction to simple. We give a further reduction to \( \hat{G} \) simple. In fact, let \( \hat{G}' \) be a factor of \( \hat{G} \). Let \( k \) be the smallest integer such that \( \theta' := \theta^k \) acts on \( \hat{G}' \). Assume that we have data \( \mathcal{D}' = (\hat{U}', \ldots) \) for \( \hat{G}' \). We write
\[
\hat{G} = \hat{G}' \times \theta(\hat{G}') \times \cdots, \quad \theta(g_1, \ldots, g_k) = (\theta(g_k, g_1, g_2, \ldots, g_{k-1}).
\]
Using Section 2.1.3 to change \( w\theta \) to a conjugate element, we may assume that
\[
\hat{w} = (\hat{w}', 1, 1, \ldots, 1), \quad \mathcal{D}' = (\hat{U}', \hat{B}(w'\theta)', \ldots, \hat{w}').
\]
We have \( \hat{T} = \hat{T}' \times \theta(\hat{T}') \times \cdots, \hat{S} = \hat{S}', \hat{T}_1 = \hat{T}'_1, \hat{U} = \hat{U}', \phi = \phi', \epsilon = \epsilon'. \) We can define \( \hat{B}_1 \) and \( \hat{B}(w\theta) \) by their positive roots, which we take to be
\[
\{\alpha, \theta\alpha, \ldots, \theta^{k-1}\alpha | \cdots}\}
\]
as \( \alpha \) runs over positive roots of \( \hat{B}'_1 \) and \( \hat{B}(w'\theta)' \), respectively. Properties (1-conjugacy) and (2-regularity) follow from the corresponding properties of \( \hat{G}' \). The determinant formula follows from the identities
\[
\phi_{\iota}^*D(\hat{G}, \Psi^+_{\phi, \theta}, \iota, E, q) = \phi_{\iota}^*D(\hat{G}', \Psi^+_{\phi', \theta'}, \iota', E', q').
\]
\[
\iota^*D(\hat{G}, \Psi^+_{\nu, \theta}, \iota w\theta, E, q) = \iota^*D(\hat{G}', \Psi^+_{\nu, \theta'}, \iota^*w\theta', E', q').
\]
We may now assume that \( \hat{G} \) is simple.

2.1.5. isogeny reduction. Next, we give a reduction that removes the assumption that \( \hat{G} \) is simply connected. That is, it is enough to prove the proposition for any group in the isogeny class of \( \hat{G} \). (This step is not required in the proof of the proposition, but we include it as a reduction that is quite useful when making explicit calculations of the data \( \mathcal{D} \) and its partition function.) Suppose that we have a surjective map \( \hat{G}_{sc} \to \hat{G} \) with kernel \( \mathcal{Z}' \subseteq \mathcal{Z}(G_{sc}) \). Assume that we have data \( \mathcal{D} = (\hat{U}, \ldots) \) for \( \hat{G} \). We show how to construct data \( \mathcal{D}'_{sc} = (\hat{U}_{sc}, \ldots) \) for \( \hat{G}_{sc} \), adding the subscript \( sc \) to all data attached to \( \hat{G}_{sc} \). The morphism \( \hat{U} \to \hat{S} \times \hat{T}_1 \) gives \( X_*(\hat{U}) \to X_*(\hat{S}) \times X_*(\hat{T}_1) \). Define \( \hat{U}_{sc} \) by defining its cocharacter lattice to be the preimage in \( X_*(\hat{U}) \) of
\[
X_*(\hat{S}_{sc}) \times X_*(\hat{T}_{1, sc}) \subseteq X_*(\hat{S}) \times X_*(\hat{T}_1).
\]

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By restriction, we have a map
\[ X_s(\hat{U}_{sc}) \to X_s(\hat{S}_{sc}) \times X_s(\hat{T}_{1,sc}). \]

The components of this map give a morphism \( \phi_{sc} : \hat{U}_{sc} \to \hat{S}_{sc} \) and an isogeny \( \iota_{sc} : \hat{U}_{sc} \to \hat{T}_{1,sc} \). Fix any lift \( \hat{w}_{sc} \) of \( \hat{w} \) to the simply connected cover. Define \( \hat{B}_{1,sc} \) and \( \hat{B}_{sc}(w\theta) \) by the natural bijection of roots between \( \hat{G} \) and \( \hat{G}_{sc} \).

Let \( \epsilon_{sc} \in \hat{S}_{sc} \) be any lift of \( \epsilon \in \hat{S} \). For every \( \nu \in \hat{U}_{sc} \) with image \( \tau \in \hat{T}_{1,sc} \), the elements \( \epsilon_{sc}\phi_{sc}(\nu) \) and \( r\hat{w}_{sc}\theta \) have the same image in \( \hat{S}/W^\theta \). For each \( z \in Z' \), define
\[ \hat{U}_{sc,z} = \{ \nu \in \hat{U}_{sc} | \epsilon z_{sc} (\nu) \phi_{sc} (\nu) \theta \text{ and } \iota_{sc}(\nu) \hat{w}_{sc} \theta \text{ same image in } \hat{S}_{sc}/W^\theta \}. \]

It follows from the fact that conjugacy (property 1) holds for \( \hat{G} \) that
\[ \hat{U}_{sc} = \cup_{z \in Z'} \hat{U}_{sc,z}, \]
expressing an irreducible set \( \hat{U}_{sc} \) as a finite union of Zariski closed subsets. It follows that \( \hat{U}_{sc} = \hat{U}_{sc,z} \) for some \( z \in Z' \). We replace \( \epsilon_{sc} \) by \( z_{sc} \). Then \( \hat{U}_{sc} = \hat{U}_{sc,1} \). That is, property (1-conjugacy) holds for \( \hat{G}_{sc} \). Regularity (2) follows from property (2-regularity) for \( \hat{G} \).

Property (3-partition) follows because the determinant is computed through the adjoint representation, and the groups \( \hat{G} \) and \( \hat{G}_{sc} \) have the same adjoint group. This completes the isogeny reduction.

2.1.6. completion of the proof. We are ready to complete the proof. By the reductions, we may assume that \( \hat{G} \) is simple and simply connected, and that no further Levi descent is possible.

We consider two cases, depending on whether \( \theta \) is trivial.

Assume first that \( \theta \) is nontrivial. Under the given assumptions, \( w\theta \) and \( \theta \) are necessarily conjugate automorphisms of the extended Dynkin diagram. That is, \( w_w \theta_{\theta^{-1}} = \theta \) for some Weyl group element \( w_1 \). By Section 2.1.3, we can assume that \( w = 1 \). In this case, the construction of data \( D \) satisfying the proposition is trivial: \( \hat{U} = \hat{T}_1 = \hat{S} \), \( \hat{B}(w\theta) = \hat{B}_1 = \hat{B} \), \( \epsilon = \hat{w} = 1 \), etc.

Now assume that \( \theta = 1 \). Under the assumption that no Levi descent is possible, we find that \( w \) acts transitively on the nodes of the extended Dynkin diagram, \( \hat{G} \) has Dynkin diagram \( A_w^{-1} \), and \( w \) is the Coxeter element of the Weyl group. We have \( \hat{T} = \hat{S} \), \( \hat{w} \) is finite, and \( \hat{T}_1 = 1 \). Then \( \hat{U} = 1 \), \( \phi : 1 \to \hat{S} \), and \( \iota : 1 \to 1 \) are uniquely determined. The choice of \( \hat{w} \) does not matter; all lifts \( \hat{w} \) of \( w \) are conjugate. Let \( \epsilon \) be a conjugate of \( \hat{w} \) in \( \hat{T} \). It is easy to check that the Coxeter element \( \hat{w} \) and \( \epsilon \) are regular. Properties (1-conjugacy) and (2-regularity) then hold.

All roots have \( N_1 \)-norm 0. Let \( \hat{B}(w) = \hat{B}_1 = \hat{B} \), the subgroup of upper triangular matrices. Then \( \Psi_{\phi_{\theta}}^{+} = \Psi_{w\theta}^{+} = \emptyset \). Both determinants are 1 (on a 0-dimensional vector space). So property (3-partition) holds.

This completes the proof of the proposition. \( \square \)
2.2. **endoscopic partition function.** A Weyl group element that acts on the extended Dynkin diagram arises in the following context. Let $G$ be an unramified $p$-adic reductive group, and let $\hat{H}$ be an unramified endoscopic group of $G$. We assume that we are given an embedding

$$\xi : \mathcal{L}H \to \mathcal{L}G,$$

that factors through a finite unramified extension of $F$:

$$\xi : \hat{H} \rtimes \langle \theta_\mathcal{H} \rangle \to \hat{G} \rtimes \langle \theta \rangle,$$

such that $\xi(\theta_\mathcal{H}) = \hat{w} \rtimes \theta$, for some representative $\hat{w}$ of an element $w$ in the Weyl group $W$. It is known that the element $w$ can be chosen to act as an automorphism of the extended Dynkin diagram, up to an equivalence of endoscopic data [18, §4.7]. We use notation from Section 2.1.

From the description of endoscopic data, we may assume that $\hat{H} = C_\mathcal{G}(s)^0$ for some $s \in \mathcal{T}$, and that $\xi(h) = \hat{h}$, for $h \in \hat{H}$, with this identification. We may assume $\hat{T} = \hat{T}_G = \hat{T}_H$ using this identification of $\hat{H}$ with a subgroup of $\hat{G}$. In what follows all $L$-morphisms $\xi$ are assumed to have this form. Because $\hat{H} = C_\mathcal{G}(s)^0$ for some $s \in \mathcal{T}$, the root system $\Psi_H$ of $\hat{H}$ with respect to $\hat{T}$ is a subset of $\Psi$.

**Lemma 2.2.1.** If $\alpha \in \Psi$ and $N_1 \alpha = 0$, then $\alpha$ is not in the root system of $\hat{H}$.

**Proof.** We prove the contrapositive. Assume that $\alpha \in \Psi_H$. Pick a Borel subgroup $B_H \supseteq \mathcal{T}$ of $\hat{H}$ that is $w\theta$-stable. Replace $\alpha$ by $-\alpha$ if necessary so that $\alpha \in \Psi^+(\hat{T}, \hat{B}_H)$. Then $N_1 \alpha$ is a sum of positive roots, hence positive. Thus, $N_1 \alpha$ is nonzero. \qed

If $\hat{B}(w\theta)$ is an adapted Borel subgroup of $\hat{G}$, then we can define a positive root system $\Psi^*_H$ for $\hat{H}$ by $\Psi^*(\hat{T}, \hat{B}(w\theta)) \cap \Psi_H$. We say that such a system of positive roots for $\hat{H}$ is adapted. By the two previous lemmas, if $\Psi^*_H$ is adapted, then $w\theta$ preserves $\Psi^*_H$.

We can use this construction to define an *endoscopic* partition function as follows. We have a disjoint sum

$$\Psi^*(\hat{T}, \hat{B}(w\theta)) = \Psi^*_H \sqcup \Psi^*_{N_1=0} \sqcup \Psi^*_{w\theta}(\hat{G} \setminus \hat{H}),$$

where $\Psi^*_{w\theta,0}$ is the set of $\hat{B}(w\theta)$-positive roots $\alpha$ such that $N_1 \alpha = 0$, and $\Psi^*_{w\theta}(\hat{G} \setminus \hat{H})$ is the set of positive roots with nonzero $N_1$-norm that are not roots of $\hat{H}$. We define the *endoscopic partition function* to be

$$P(\hat{G}, \Psi^*_{w\theta}(\hat{G} \setminus \hat{H}), \hat{w}\theta, E, q).$$

As we will see, the branching rule for the subgroup $\xi(\mathcal{L}H) \subseteq \mathcal{L}G$ is expressed in terms of this partition function.

We have constructed an adapted set $\Psi^*_H$ of positive roots of $\hat{H}$. We expand the endoscopic partition function (or rather its pullback to $\hat{U}$) in a series

$$\ell^* P(\hat{G}, \Psi^*_{w\theta}(\hat{G} \setminus \hat{H}), \hat{w}\theta, E, 1) = \sum_{\mu} p_{\mu} e^\mu,$$

where the support of $\mu \mapsto p_{\mu}$ is a subset of $X^*(\hat{U})$. 

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2.3. **branching rules.** The irreducible representations of $^L G$ restricted to $\hat{G} \rtimes \theta$ are classified by a highest weight $\lambda \in P^*_G \subseteq Y_G$ (with character $\tau_\lambda$), and similarly for irreducible representations on $\hat{H} \rtimes \theta_H$, with $\mu \in P^*_H \subseteq X'(\hat{T}_1)$ (with character $\sigma_\mu$). This section gives a branching rule for $\tau_\lambda$ restricted to $\xi(\hat{H} \rtimes \theta_H)$, as a sum of $\sigma_\mu$:

$$ J_{\xi \tau_\lambda} = \sum_{\mu} m(\lambda, \mu) \sigma_\mu $$

for some coefficients $m(\lambda, \mu)$ and $J_\xi$ defined as in (2.1.4).

If $\sigma_\mu$ is an irreducible character and $\chi : \langle \theta_H \rangle \to \mathbb{C}^\times$ is a multiplicative character, then $\chi \otimes \sigma_\mu$ is again an irreducible character. Restricted to the component $\hat{H} \rtimes \theta_H$, the characters $\chi \otimes \sigma_\mu$ and $\sigma_\mu$ are linearly dependent: $\chi \otimes \sigma_\mu = \chi(\theta_H) \sigma_\mu$. This means that the multiplicities $m(\lambda, \mu)$ should take values in $\mathbb{Z}[\zeta]$, where $\zeta$ is a primitive root of unity of the same order as $\theta_H$.

We fix data $D = (\hat{U}, \hat{B}(w), \hat{B}_1, \iota, \phi, \epsilon, \hat{w})$ associated with $\xi : ^L H \to ^L G$ as in Proposition 2.1.2. For the moment, we assume that $\hat{w}$ associated with $D$ coincides with $\hat{w}$ associated with $\xi$. We have a disjoint sum decomposition

$$ \Psi^+ (\hat{T}, \hat{B}_1) = \Psi^+_{\phi, \theta, 1} \Psi^+_{\phi, \theta, 0}, $$

where

$$ \Psi^+_{\phi, \theta, 0} = \{ \alpha \in \Psi^+ (\hat{T}, \hat{B}_1) \mid \phi^* N \alpha = 0 \}. $$

The condition $\phi^* N \alpha = 0$ implies that $\phi^* D(\hat{G}, \Psi_{\phi, \theta, 0}^+, \theta, E, 1)$ is a constant $d_0(\epsilon, \theta) \in \mathbb{Q}(\zeta)$ (that is, it is independent of $\mu \in X'(\hat{U})$). Regularity (Proposition 2.1.2) implies that the constant is nonzero. Evaluation of the constant is routine, but we do not do so here. We abbreviate

$$ D_H := \iota^* D(\hat{H}, \Psi_{H, \theta}^+, \hat{w} \theta, E, 1) = \iota^* D(\hat{G}, \Psi_{H, \theta}^+, \hat{w} \theta, E, 1). $$

This is the denominator in the twisted Weyl character formula (on $X'(\hat{U})$) for $(\hat{H}, \theta_H)$ with respect to the positive root system $\Psi_{H, \theta}^+$. Combining these identities and the proposition, we have

$$ D_H = \iota^* D(\hat{H}, \Psi_{H, \theta}^+, \hat{w} \theta, E, 1) $$

$$ = \frac{\iota^* D(\hat{G}, \Psi_{\phi, \theta}^+, \hat{w} \theta, E, 1)}{\iota^* D(\hat{G}, \Psi_{\phi, \theta}^+, \hat{w} \theta, E, 1)} $$

$$ = \iota^* P(\hat{G}, \Psi_{\phi, \theta}^+(\hat{G} \setminus \hat{H}), \hat{w} \theta, E, 1) \phi^* D(\hat{G}, \Psi_{\phi, \theta}^+, \theta, E, 1) $$

$$ = \iota^* P(\hat{G}, \Psi_{\phi, \theta}^+(\hat{G} \setminus \hat{H}), \hat{w} \theta, E, 1) \frac{\phi^* D(\hat{G}, \Psi_{\phi, \theta}^+(\hat{T}, \hat{B}_1), \theta, E, 1)}{\phi^* D(\hat{G}, \Psi_{\phi, \theta}^+, \theta, E, 1)} $$

$$ = \sum_{\mu} p_{\mu} e_\mu \phi^* D(\hat{G}, \Psi_{\phi, \theta}^+, \theta, E, 1). $$

with abbreviation $D_G = D(\hat{G}, \Psi^+ (\hat{T}, \hat{B}_1), \theta, E, 1)$ for the twisted Weyl denominator for $(\hat{G}, \theta)$ with respect to the positive root system $\Psi^+ (\hat{T}, \hat{B}_1)$. The isogeny $\iota : \hat{U} \to \hat{T}_1$ gives $X'(\hat{T}_1) \subseteq X'(\hat{U}).$
Define constants $m(\lambda, \mu)$, for $\lambda \in P^+ \subseteq X^*(\hat{S})$ and $\mu \in P^+_{\ell} \subseteq X^*(\hat{T})$: 

$$m(\lambda, \mu) = \sum_{w \in W^F} (-1)^{\ell(w)}(w \cdot \lambda)(e)\frac{p_{\mu-w(\phi \cdot \lambda)}}{d_0(e, \theta)} \in \mathbb{Q}(\zeta).$$

The following theorem is the main result of this section.

**Theorem 2.3.3.** Let $G$ be a reductive group and let $H$ be an endoscopic group of $G$, both unramified. Let $\xi : \hat{L}H \to \hat{L}G$ be an embedding of $L$-groups that factors over a finite unramified extension $E/F$. Suppose that $\xi(\theta_H) = \hat{w}\theta$. Let $\hat{w} \mapsto w \in W$. Let $\mathcal{D} = (\hat{T}, \ldots, \hat{w})$ be the data constructed by Proposition 2.1.2 (with the same $\hat{w}$ for $\xi$ and $\mathcal{D}$). Then Equation 2.3.2 gives the twisted branching rule for $G \rtimes \theta$ restricted to $\xi(\hat{H} \rtimes \theta_H)$:

$$m_i(\lambda, \mu) = m(\lambda, \mu)\sigma_i \in \mathbb{C}[X^*(\hat{U})].$$

Both sides are supported on $X^*(\hat{T}) \subseteq X^*(\hat{U})$.

Before starting the proof of the theorem, we give a transformation rule describing how the branching rules depend on the choice $\hat{w} \mapsto w$ used to define the embedding $\xi$ of $L$-groups.

**Lemma 2.3.4.** Suppose that we have two embeddings $\xi_i : \hat{L}H \to \hat{L}G$ with $\xi_i(\hat{H}) = \xi_2(\hat{H})$, $\xi_1(T) = \xi_2(T) = \hat{T}$, $\xi_i(\theta_H) = \hat{w}_i\theta$, and $\hat{w}_2 = \hat{w}_1$. Write $m_i(\lambda, \mu)$ for the twisted branching coefficients for $\xi_i$. Then $m_i(\lambda, \mu) = m_2(\lambda, \mu)\mu(t)$.

We note that the image $\xi_i(\hat{L}H)$ does not depend on $i$, but the branching formula does.

**Proof (lemma).** Let $V_\lambda$ be the representation of $G \rtimes \langle \theta \rangle$ with highest weight $\lambda$, normalized as usual by the condition that $\theta v = v$, when $v$ is a vector of highest weight in $V_\lambda$.

Let $W$ be a $\hat{H}$-irreducible subrepresentation of $V_\lambda$ with highest weight $\mu$. By the condition $\xi_i(\hat{H}) = \xi_2(\hat{H})$, the two module structures on $W$ agree. Assume that $\mu$ is a $\theta_H$-fixed weight. We extend $W$ to $\hat{L}H$ with the usual normalization $\theta_H v = v$, where $v$ is a vector of highest weight. The normalization depends on $i$. The two different normalizations

$$\theta_H v = \xi_i(\theta_H) v = \hat{w}_i \theta v = v,$$

for $i = 1, 2$, differ by a scalar $\mu(t)$. Hence the multiplicities also transform by a factor $\mu(t)$.

In general, the $L$-group can be formed with respect to various unramified Galois extensions. We let $L/F$ be a second unramified extension with $L/E/F$. Let $[L : E] = \ell$. We show that the branching multiplicities do not depend on $L$.

Fix an admissible embedding

$$\xi_E : \hat{H} \rtimes \text{Gal}(L/F) \to \hat{G} \rtimes \text{Gal}(L/F),$$

where $\xi_L(\text{Frob}_L) = \hat{w}\text{Frob}_L$, with the same choice $\hat{w}$ as with $\xi = \xi_E$. Then $\text{Frob}_L$ acts trivially on the datum of $\hat{H}$, and $(\hat{w}\text{Frob}_L)$ acts trivially on the datum of $\hat{G}$. Let $\tau_1$ be an irreducible representation of $\hat{G}$ that is $\theta$-fixed. The extension of $\tau_1$ to $\hat{G} \rtimes \text{Gal}(L/F)$
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factors through \( \hat{G} \rtimes \text{Gal}(E/F) \). Similarly, \( \sigma_\mu \) extends to \( \hat{H} \rtimes \text{Gal}(L/F) \) and factors through \( \hat{H} \rtimes \text{Gal}(E/F) \). We conclude that the branching multiplicities are the same for \( L/F \) and \( H/F \).

Equation 2.3.1 and the twisted Weyl character formula on the component of the group, writing the equation simply as

\[ \theta \mapsto \dot{\theta} \]

By convention, we drop \( \tau \) (resp. \( \theta_H \)) from the notation in twisted identities on the \( \theta \)-component of the group, writing the equation simply as \( \iota^* J_{\tau,3} = \phi^*_3(\tau,3) \) on \( \hat{U} \).

We show that the properties of \( \phi, \epsilon \) imply the branching rule, which we compute using Equation 2.3.1 and the twisted Weyl character formula on \( \hat{H} \) and \( \hat{G} \).

\[
(\phi^*_3(\tau,3))_{\hat{D}_H} = (\iota^* J_{\tau,3})_{\hat{D}_H} = \sum_{\mu'} m(\lambda, \mu') J_H(e^{\mu'}) = m(\lambda, \mu) e^{\mu'} + \sum_{\mu' \neq \mu} c_{\mu'} e^{\mu'}.
\]

\[
(\phi^*_3(\tau,3))_{\hat{D}_H} = \left( \sum_{w' \in W_H} (\epsilon^w \phi^*_G(e^{\mu'})) / d_0 \right) / d_0 = \sum_{w' \in W_H} (-1)^{w' \lambda} (\epsilon^w \phi^*_G(e^{\mu'})) / d_0
\]

\[
= \sum_{w' \in W_H} (-1)^{w' \lambda} (w \cdot \lambda)(\epsilon) p_{\mu'} e^{\mu' + \phi^*(w \cdot \lambda)} / d_0
\]

\[
= \sum_{w' \in W_H} (-1)^{w' \lambda} (w \cdot \lambda)(\epsilon) p_{\mu'} e^{\mu' - \phi^*(w \cdot \lambda)} / d_0.
\]

We have used the twisted Weyl character formula with respect to \( \Psi^*_H \) on \( \hat{H} \) and with respect to \( \Psi^* (\hat{\Gamma}, \hat{B}_1) \) on \( \hat{G} \). To justify the equation in the third row, let \( \mu \in P^*_H \). If \( w' \cdot \mu' = w \cdot \mu \) for some \( w, w' \in W_H \) and \( \mu' \in P^*_H \), then using the fact that \( P^*_H \) is a fundamental domain for \( W_H \) and that \( \mu' + \rho_H \) lies in the interior of that domain, we find that \( w = w' \) and \( \mu = \mu' \).

Equating coefficients of \( e^{\mu'} \), we get Equation 2.3.2. \( \square \)

3. Motivic Integration

This section reviews the theory of motivic integration as developed by Cluckers and Loeser [9].
3.1. **the Denef-Pas language.** The Denef-Pas language is a three-sorted first-order formal language in the sense of model theory. Its intended structures are triples \((F, k_F, \mathbb{Z})\), where \(F\) is a valued field with discrete valuation, \(k_F\) is the residue field of \(F\), and the value group of \(F\) is the ring of integers \(\mathbb{Z}\). The three sorts are \(VF\) (the valued-field sort), \(RF\) (the residue-field sort), and \(\mathbb{Z}\) (the value-group sort).

In general, a first-order formal language is specified by sets of relation symbols and function symbols. The Denef-Pas language has the following relation and function symbols. The valued-field sort \(VF\) has the symbols of the first-order language of rings \((0, 1, +, \times)\). The residue field sort also has the symbols of the first-order language of rings. The value-group sort is the Presburger language of an ordered additive group with symbols \((0, +, \leq, \equiv_n)\). Here \(\equiv_n\) is a binary relation symbol for each \(n \geq 2\), which is to be interpreted as congruence modulo \(n\) in \(\mathbb{Z}\). In addition, there are two function symbols \(ord : VF \to \mathbb{Z}\) (interpreted as the valuation on the valued-field) and \(ac : VF \to RF\) (interpreted as the angular component map). For the structure \((K((t)), K, \mathbb{Z})\), where \(K((t))\) is the field of formal Laurent series, the intended interpretation of \(ac\) is the function \(\sum a_i t^i \mapsto a_0\) that returns the first nonzero coefficient of the Laurent series (and sending \(0 \in K((t))\) to \(0\)).

First-order languages are constructed in the usual way, with formulas built from logical connectives \((\land), (\rightarrow), (\lor), \neg\), equality \((=)\), variables of the three sorts, function symbols, relation symbols, existential quantifiers of each sort, and universal quantifiers of each sort.

Following the terminology of \([16]\), we call a **fixed choice** any set-theoretic data that does not depend in any way on the Denef-Pas language, its variables, nor on the structures of \(VF\) and \(RF\). Examples of fixed-choices that appear in this paper are Weyl groups, abstract groups, representations of split reductive groups over \(Q\), and root systems.

3.2. **motivic integration.** Let \(\text{Field}_Q\) be the category of fields of characteristic zero.

Cluckers and Loeser have used the Denef-Pas language to define various categories. In particular, there is a category \(\text{Def}_Q\) of **definable subassignments**, given as follows. Let \(\mathbb{N} = \{0, 1, 2, \ldots\}\). For each \((m, n, r) \in \mathbb{N}^3\), let \(h[m, n, r]\) be the functor from \(\text{Field}_Q\) to the category of sets that assigns to each field \(K\), the set \(h[m, n, r](K) = K((t))^m \times K^n \times \mathbb{Z}^r\). A **subassignment** of this functor is by definition, a subset \(S(K) \subseteq h[m, n, r](K)\) for each \(K \in \text{Field}_Q\). A **definable subassignment** \(S\) is a subassignment for which there exists a formula \(\phi\) in the Denef-Pas language such that for each \(K \in \text{Field}_Q\), the set of solutions of \(\phi\) in \(h[m, n, r](K)\) is \(S(K)\). The definable subassignments are the objects of the category \(\text{Def}_Q\). A morphism \(\phi : X \to Y\) is a definable subassignment

\[
\phi \subseteq X \times Y \subseteq h[m, n, r] \times h[m', n', r'] = h[m + m', n + n', r + r']
\]

that is the graph of a function \(X(K) \to Y(K)\) for each \(K \in \text{Field}_Q\). A **free parameter** refers to a collection of free variables of the same sort in a formula in the Denef-Pas language, ranging over a definable subassignment.

For each definable subassignment \(X \in \text{Def}_Q\), Cluckers and Loeser have defined a ring \(C(X)\) of **constructible motivic functions**. The construction of this ring is a major undertaking, and we refer the reader to their articles for details \([9]\). The elements of this ring are
called constructible motivic functions. Although they behave in many ways as functions on $X$, the elements of the ring are not literal functions in the set-theoretic sense of function.

If $\phi : X \to Y$ is a morphism of definable subassignments, there is a pullback of functions $\phi^* : C(Y) \to C(X)$. The pullback $\phi^*$ is a ring homomorphism, and pullbacks compose: $(\phi\psi)^* = \psi^*\phi^*$.

If $X \to S$ is a morphism of definable subassignments, there is a subgroup $I_SC(X)$ of $S$-integrable constructible motivic functions. The intuitive interpretation of an $S$-integrable function $f$ is a function such that the integral over each fiber of $X \to S$ is convergent with respect to the canonical motivic measure. For a morphism $\phi : X \to Y$ over $S$, there is a pushforward $\phi_! : I_SC(X) \to I_SC(Y)$ that is called integration over fibers. Pushforwards compose: $(\phi\psi)_! = \psi_!\phi_!$. In this article, we always deal with bounded constructible functions. Such functions are always integrable \cite[Prop 12.2.2]{9}. Thus, we do not need to deal with integrability issues.

### 3.3. Presburger constructible functions

The ring $C(X)$ of constructible functions is the graded algebra associated with a filtration on a tensor product $P(X) \otimes Q(X)$. In terms of the three sorts of the Denef-Pas language, data related to the value-group sort $\mathbb{Z}$ is encoded in $P(X)$ and data related to the residue field sort $RF$ is encoded in $Q(X)$. The left-hand side $P(X)$ is a ring of Presburger constructible functions. Every Presburger constructible function $f$ gives a constructible motivic function $f \otimes 1$.

Much of what we do in this article is related to constructible functions on integer lattices. For this, we work with Presburger constructible functions rather than the entire ring of constructible motivic functions.

### 3.4. Volume forms

Cluckers and Loeser have an extension of motivic integration that allows integration with respect to volume forms \cite[§8]{13}. In brief, there is a notion of differential forms on a definable subassignment and a space of definable positive volume forms. Each differential form $\omega$ of top degree has an associated volume form $|\omega|$. For each morphism $\phi : X \to Y$ over $S$, the pushforward $\phi_!$ extends to a pushforward $f \mapsto (\phi_!)(f, \omega)$ with respect to the volume form. It is to be interpreted loosely as integration over the fibers of $\phi : X \to Y$ with respect to a volume form constructed from a Leray residue of $\omega$ on the fiber.

### 3.5. $p$-adic specialization

Let $C$ denote the class of $p$-adic fields. Let $C_N \subseteq C$ denote the subclass of fields whose residue characteristic is at least $p \geq N$.

In general, we only care about what occurs in fields in $C_N$ for $N$ arbitrarily large. To make this precise, suppose that we have for some $N$, a function $X$ with domain $C_N$. Then by restriction of domain $C_i$ to $C_j$, for $N \leq i \leq j$, we may take the filtered colimit of $X_i = X|_{C_i}$. Two functions $X$, $X'$ have the same filtered colimit if they are equal in $C_i$ for some sufficiently large $i$. 

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Let $X$ be a definable subassignment of $h[m,n,r]$, and let $f$ be a constructible motivic function on $X$. There exists an $N$ such that for all $F \in C_N$, there are specializations

$$X(F) \subseteq F^m \times k_F^n \times \mathbb{Z}'$$

where all sums and products are finite, $f_F : X(F) \to \mathbb{C}$.

Only the filtered colimits of $X$ and $f$ matter to us.

We warn the reader of a notational overload; we write $X(K)$ or $X(F)$ as $K$ and $F$ range over two quite different classes of fields. Different symbols $K$ and $F$ disambiguate the context. When $K \in \text{Field}_\mathbb{Q}$, the valued field is $K((t))$ and the residue field is $k$; but when $F$ is a $p$-adic field, $F$ is the valued field and its residue field is denoted $k_F$. We also warn that $K$ is used both for a hyperspecial subgroup and for $K \in \text{Field}_\mathbb{Q}$.

The specializations have various expected properties. If $\phi : X \to Y$ is a morphism of definable subassignments over $S$, then we have functions $\phi_F : X(F) \to Y(F)$. When $f$ is $S$-integrable on $X$, integration $\phi_F(f)$ over fibers specializes to integration over fibers with respect to a canonical measure in $p$-adic fields $F \in C_N$ (for some $N$ depending on $\phi$).

The functions $f_F : X(F) \to \mathbb{C}$ that come from constructible motivic functions $f \in C(X)$ have a special form

$$f_F(x) = \sum_i \text{card}(Y_i(F,x))q_F^{\alpha_i(x)} \prod_j \beta_{i,j}(x) \prod_k \frac{1}{1 - q_F^{a_{i,k}(x)}}$$

(3.5.1)

where all sums and products are finite, $\alpha_i : X \to \mathbb{Z}$, $\beta_{i,j} : X \to \mathbb{Z}$ are definable, $q_F$ is the cardinality of the residue field of $F$, and $a_{i,k}$ are nonzero integers \[11, \S2\]. The filtered colimits of these functions are $q$-constructible functions. Let $C_q(X)$ be the space of $q$-constructible functions on $X$. Sometimes we call the specialization of a definable subassignment a definable set. There is an element $\mathbb{L}$, called the Lefschetz motive, in the ring of constructible motivic functions that specializes to $q_F$ for every $p$-adic field $F$. When the first factors $Y_i$ are absent from Equation (3.5.1) the function $f$ is Presburger constructible.

We warn the reader that very different constructible motivic functions can yield the same $q$-constructible function. For example, let $[S] \in C(\text{pt})$ be the isomorphism class in the residue sort of the set of nonzero squares, considered as a constructible motivic function on a point. Similarly, let $[N]$ be the class of the set of nonsquares. Then, under specialization to $p$-adic fields, the two functions are equal: $[S](F) = [N](F) = (q_F - 1)/2$, for $F \in C_1$. However, $[S]$ and $[N]$ are not at all the same constructible motivic function. Indeed, their values on algebraically closed residue fields $K$ are not equal: $[N](K)$ is the empty set and $[S](K) = K^\times$ is not. Another family of examples is provided by isogenous elliptic curves. They have the same number of points in a finite field, but they are not generally isomorphic curves. If a constructible motivic function specializes to a $q$-constructible function that is identically zero, then we call it a null function.

The theory of motivic integration specializes to $q$-constructible functions. To integrate a $q$-constructible function, we lift it to a constructible motivic function, use Cluckers-Loeser integration there, then take its specialization again. Two different lifts differ by a null function, and its integral is also a null function. Thus, integration of $q$-constructible functions is well-defined.
3.6. **definable reductive groups.** Definable reductive groups are understood in the sense of [13], [16]. In this work we restrict to unramified reductive groups (quasi-split and split over an unramified extension).

In the definable context, a reductive group $G \to Z$ lies over a definable subassignment $Z$ called the **cocycle space** of $G$. In the case of an unramified reductive group that splits over an unramified extension of degree $r$, we can take $Z \subseteq h[m, 0, 0]$, for some $m$. The set $Z$ parameterizes lists of coefficients of irreducible monic polynomials, each defining a degree $r$ unramified extension of $F$. A field extension $E/VF$ of degree $r$ is identified with $VF' = VF[x]/(p)$, as $p$ runs over irreducible polynomials parameterized by $Z$.

Recall that there is no Frobenius map in the context of the Denef-Pas language, because it is not possible to take a $q$th power. Instead, we choose a generator of the Galois group of an unramified extension $E/VF$ and call it the quasi-Frobenius element. As part of the cocycle space data $Z$, we assume we are given a quasi-Frobenius element $q\text{Frob}$ that corresponds to the automorphism $\theta$ of $\hat{G}$.

A connected split reductive group is treated as a definable subassignment through a faithful representation of the group. The group is identified with a closed subgroup of $GL(n, F)$. Quasi-split reductive groups that split over an unramified degree $r$ extension (parameterized by a cocycle space $Z$) are defined in terms of explicit representations of those groups in $GL(n, E)$, where $E/VF$ is treated as above.

If $G$ is an unramified reductive group, we may construct a hyperspecial subgroup $K$ as a definable subassignment of $G$ [12].

A quasi-split reductive group $G$ carries an invariant differential form $\omega$ of top degree, which is described in the context of definable subassignments in [16]. All integration in this article is assumed to be carried out with respect to invariant measures. We have the invariant integral $\vartheta(f, \omega)$ of a constructible integrable function $f \in C_q(G)$ with respect to the morphism $\vartheta : G \to \{\text{pt}\}$ to a point using the invariant differential form $\omega$.

3.7. **enumerated Galois groups.** We deal with field extensions and Galois groups in the way described in [16] and [13]. We let $\Gamma$ be an abstract group with fixed enumeration $1 = \sigma_1, \ldots, \sigma_n$ of its elements. We assume a fixed short exact sequence

$$1 \to \Gamma' \to \Gamma \to \Gamma^{\text{unr}} \to 1,$$

with $\Gamma'$ and $\Gamma^{\text{unr}}$ both cyclic. The group $\Gamma$ plays the role of a Galois group with inertia subgroup $\Gamma'$ and unramified quotient $\Gamma^{\text{unr}}$. We treat this data as an abstract fixed choice, without a priori connection to the Galois group of any particular extension of $p$-adic fields.

We may fix an abstract root datum and choose an action of $\Gamma$ on the root datum, stabilizing the set of simple roots. Through this action on the root datum, $\Gamma$ acts on the Weyl group, and we may construct the semidirect product $W \rtimes \Gamma$. 
By abstract unramified Galois group we mean a fixed finite cyclic group $\Gamma = \Gamma^u$ with choice of generator $q_{Frob}$ that we call the quasi-Frobenius element. By abuse of terminology, we use the word quasi-Frobenius to refer either to the generator of $\Gamma^u$ or as its realization as a matrix with values in the valued field sort $VF$, as described in [13].

In this article, the abstract dual group is the Langlands dual constructed with respect to $\Gamma$ and $q_{Frob}$ rather than the Galois (or Weil) group of a field.

3.8. **definability results.** In this section we assume that $G$ is an unramified connected reductive group. It is treated as definable subassignment over a definable cocycle space $Z$.

Standard subgroups of $G$ such as a hyperspecial $K$, Borel subgroup $B$, $T \subseteq B$, the unipotent radical $N$ of $B$, the maximal split subtorus $A$ of $T$ are all definable. In the following lemmas a field extension $L/VF$ often appears. We can treat it in a definable way as in Section 3.6 whenever we have an a priori bound on the degree $L/VF$. In each case that follows we have an a priori bound on the degree of the extension. Similar remarks apply to unramified field extensions $E/VF$ that appear.

**Lemma 3.8.1.** Let $G$ be an unramified reductive group. There exists a definable subassignment of $G \times G$ of all pairs $(\gamma, x)$ such that $\gamma$ is semisimple (possibly singular) and $x$ lies in the identity component of the centralizer of $\gamma$.

**Proof.** The condition that $\gamma$ is semisimple is definable by the condition that $\gamma$ is conjugate to $T$ by $G(L)$ for some field extension $L/VF$.

The commutativity of $x$ and $\gamma$ is obviously definable. We need more to describe the identity component in definable terms. We use Steinberg’s result that the centralizer of a semisimple element in a simply-connected group is connected. We work again over a field extension $L/VF$, and write factorizations $x = z_x x_{sc}$ and $\gamma = z_\gamma \gamma_{sc}$, where $z_x$ and $z_\gamma$ are in the center of $G(L)$ and $x_{sc}$, $\gamma_{sc}$ are in the simply connected cover of the derived group $G_{sc}(L)$. Then $x$ is in the identity component of the centralizer iff $x_{sc}$ and $\gamma_{sc}$ commute. This is a definable condition. □

**Lemma 3.8.2.** Let $G$ be an unramified reductive group. There exists a definable subassignment of $G \times G$ of all pairs $(\gamma, \gamma')$ such that $\gamma$ is semisimple (possibly singular) and $\gamma'$ is stably conjugate to $\gamma$.

**Proof.** The elements $\gamma$ and $\gamma'$ are stably conjugate iff there exists $g \in G(L)$ for some Galois extension $L/VF$ such that $\gamma^g = \gamma'$ and $\sigma(g)g^{-1} \in C_G(\gamma)$, where $\sigma \in \text{Gal}(L/VF)$. The Galois group and its cocycles are treated within the Denef-Pas language as in Section 3.7. The identity component is handled by the previous lemma. □

**Lemma 3.8.3.** Let $G$ be an unramified reductive group, given as a definable subassignment over a cocycle space $Z$. There is a definable subassignment $G_{qs}$ (resp. $G_u$) of $G$ over $Z$ consisting of semisimple elements $\gamma$ such that the identity component of the centralizer of $\gamma$ is quasi-split (resp. unramified).
Proof. A reductive group is quasi-split iff the Levi subgroup of the minimal parabolic subgroup is a Cartan subgroup. This occurs iff the centralizer of the split component of the center of the Levi is a Cartan subgroup.

The maximal split torus \( A \) of \( T \) is a definable set. The condition for \( a \) to belong to a split torus is definable by the condition that \( a \) is conjugate to an element of \( A \).

The centralizer \( C = C_G(\gamma) \) is quasi-split exactly when there exists \( t \in C \) that is conjugate to an element of \( A \) and such that its centralizer in \( C \) is a Cartan subgroup of \( G \). This is a definable condition.

Now turning to \( G_u \), the group \( C = C_G(\gamma) \) is unramified exactly when \( C \) is quasi-split and there exists \( t \in C \) whose centralizer is an unramified Cartan subgroup of \( G \). An unramified Cartan subgroup is one that is conjugate to \( T \) by \( G(L) \) for some unramified extension \( E/VF \). This is a definable condition in the Denef-Pas language. \( \square \)

A strongly compact element \( \gamma \in G(F) \) is an element that belongs to a bounded subgroup of \( G \).

Lemma 3.8.4. The set of strongly compact semisimple elements is a definable subassignment. The set of topologically unipotent semisimple elements in a reductive group is a definable subassignment.

Proof. We can define a strongly compact semisimple element as one that is conjugate in \( G(L) \) to an element \( t \in T(L) \) of the split torus, for suitable large extension \( L/VF \) (of fixed degree), and such that the valuation of \( \lambda(t) \) is 0 for every \( \lambda \in X^*(T) \). It is enough to let \( \lambda \) run over a finite set of generators of \( X^*(T) \). We can define topologically unipotent elements as elements conjugate to an element \( t \in T \) by \( G(L) \), such the valuation of \( \lambda(t) \) is 0 and the angular component of \( \lambda(t) \) is 1 for each \( \lambda \). \( \square \)

Lemma 3.8.5. Let \( G \) be an unramified reductive group with unramified endoscopic group \( H \), given as definable subassignments over a common cocycle space \( Z \). There is a definable subassignment of all pairs \( (\gamma, \gamma_H) \) such that \( \gamma_H \) is strongly \( G \)-regular in \( H \) and \( \gamma \in G_u \) is an image of \( \gamma_H \).

Moreover, consider the Denef-Pas statement \( \psi \) that asserts that for all strongly \( G \)-regular elements \( \gamma_H \) in \( H \), there exists an image \( \gamma \in G_u \), that is an image of \( \gamma_H \). Then there exists \( N \) such that \( \psi_F \) is true for all \( F \in C_N \).

Proof. It is a definable condition to say that \( \gamma \) is an image of \( \gamma_H \).

Moreover, by Kottwitz [23, 3.3], in characteristic zero, there exists \( \gamma' \) in the stable conjugacy class of \( \gamma \) such that \( C_G(\gamma') \) is quasi-split. The same result holds in large positive characteristic by the transfer principle (Section 4.4). \( \square \)

Definition 3.8.6. We say that \( \gamma \in G(F) \) is strongly semisimple if \( \gamma \) is an element of a torus that splits over an unramified extension \( E/VF \), if it is strongly compact, and if for every
root $\alpha \in \Phi$ of $(T, G)$ and for every $g \in GL(E)$ such $\gamma^g \in T(E)$, we have
\[ \alpha(\gamma^g) = 1, \quad \text{or} \quad \text{ord}(\alpha(\gamma^g) - 1) = 0. \]

**Lemma 3.8.7.** The set of strongly semisimple elements of an unramified reductive group $G$ is a definable set.

**Proof.** This is clear from the definition and from the previous constructions. $\Box$

**Remark 3.8.8.** The absolutely semisimple part $\gamma_s$ in the topological Jordan decomposition of an element $\gamma = \gamma_s \gamma_u$ is defined as a $p$-adic limit that cannot be treated within the Denef-Pas language [18]. Instead, we allow $\gamma_s$ to run over a definable set of strongly semisimple elements. We can no longer assert the uniqueness of the topological Jordan decomposition, but we obtain the existence of a definable decomposition, which is sufficient for our purposes.

**Lemma 3.8.9 (definable topological Jordan decomposition).** Let $G$ be an unramified reductive group. There is a definable subassignment of triples $(\gamma, \gamma_s, \gamma_u) \in G^3$ such that $\gamma$ is strongly regular semisimple and strongly compact, $\gamma_s$ is strongly semisimple, $\gamma_u$ is topologically unipotent, $\gamma = \gamma_s \gamma_u = \gamma_u \gamma_s$, and
\[ \alpha(\gamma_s) = 1, \quad \text{or} \quad \text{ord}(\alpha(\gamma_s) - 1) = 0, \]
for all roots $\alpha$ of the Cartan subgroup $C_G(\gamma)$.

Moreover, consider statement $\psi$ in the Denef-Pas language that asserts that for every strongly-regular strongly-compact semisimple element $\gamma$, there exists $\gamma_s$ and $\gamma_u$ such that $(\gamma, \gamma_s, \gamma_u)$ belongs to this definable set of triples. Then there exists $N$ such that $\psi_F$ holds for all $F \in C_N$.

**Proof.** The definability follows by previous lemmas on the definability of the set of strongly compact elements, strongly-semisimple elements, and topologically unipotent elements.

The statement $\psi_F$ holds for $p$-adic fields of characteristic zero by the existence of a topological Jordan decomposition [18]. It also hold in sufficiently large positive characteristic by a transfer principle for statements in the Denef-Pas language (Section 4.4). $\Box$

3.9. **spherical Hecke algebra for an unramified definable group.** Let $G$ be a definable unramified reductive group over a cocycle space $\mathcal{Z}$. Let $A$ be a maximal split torus in $G$ of dimension $r$. We identify its cocharacter lattice $X^* = X^*(A)$ with $\mathbb{Z}^r$ by a choice of free generators of $X^*(A)$. This allows us to treat $X_*(A)$ as the definable subassignment $h[0, 0, r] = \mathbb{Z}^r$. Let $X^*(A)$ be the character lattice of $A$.

There is a perfect pairing $\langle \cdot, \cdot \rangle : X^*(A) \times X_*(A) \to \mathbb{Z}$. For each $\lambda \in X_*(A)$, there is a definable subassignment $A_\lambda \subseteq A$ given by the formula
\[ A_\lambda = \{ a \in A \mid \text{ord}(\mu(a)) = \langle \mu, \lambda \rangle, \quad \text{for all} \quad \mu \in X^*(A) \}. \]

There is a definable subassignment of $X_*(A) \times A$ given by pairs $(\lambda, a)$ such that $a \in A_\lambda$. Of course, $p$-adically, $A_\lambda$ is just the coset $\tau^\lambda A(O_F)$, where $O_F$ is the ring of integers of $F$.

Recall $P^+ \subseteq X_*(A)$ is the set of cocharacters in the positive Weyl chamber.
Lemma 3.9.2. $P^+$ is a definable subset (of $\mathbb{Z}^r$).

Proof. $P^+$ is defined by linear inequalities, which can be expressed in the Presburger language. □

Lemma 3.9.3 (Cartan decomposition). There is a definable subassignment of $P^+ \times G$ given by

$$L_G = \{(\lambda, g) \in P^+ \times G \mid g \in K\Lambda_1 K\}.$$ 

The fiber $L_G(\lambda)$ over each $\lambda \in P^+$ is definable. Moreover, $L_G(\lambda) \cap L_G(\lambda') = \emptyset$, for $\lambda \neq \lambda'$.

By Bruhat-Tits, the Cartan decomposition holds over general discrete valued fields [2, 4.4.3].

Remark 3.9.4. $L_G$ captures the entire spherical Hecke algebra as a single definable subassignment. In applications to the fundamental lemma, it is important to work with this single subassignment rather than an infinite basis of the spherical Hecke algebra.

We define the spherical Hecke function to be the characteristic function of $L_G$. It is a definable function (as well as a $q$-constructible function) on $P^+ \times G$.

The Satake transform $f \mapsto \hat{f}$ is an isomorphism $\mathcal{H}(G//K) \rightarrow \mathbb{C}[Y^*]^{W_\theta}$. Let $s_{\lambda,\mu}$ be the coefficients of the change of basis $\hat{f}_\lambda = \sum_\mu s_{\lambda,\mu}m_\mu$, where $m_\mu$ is as before (1.2.3).

The Satake transform lifts to the $q$-constructible setting. The Satake transform involves a term $q^{\rho^\vee(\cdot)}$, where $\rho^\vee = \rho(\Psi^\vee)$. Constructible functions in the formula (3.5.1) only contain integral powers of $q$. However, [12, §B.3.1] extends the theory of constructible functions to allow half-integers. To accommodate the square roots introduced by $\rho$, we extend the theory in that way without further comment [1].

Lemma 3.9.5. There is a $q$-constructible function $s$ on $P^+ \times P^+$ that specializes to the function $(\lambda,\mu) \mapsto s_{\lambda,\mu}$.

Proof. The coefficients $s_{\lambda,\mu}$ are given by an integral of a $q$-constructible function on $P^+ \times G$:

$$s_{\lambda,\mu} = \frac{q^{\rho(\mu)}}{\text{vol}(A_0)} \int_{A_0} \int_N \text{char}(L_G(\lambda,tm))dndt.$$

Here $N$ is the unipotent radical of a Borel subgroup $B$ containing a maximally split Cartan subgroup $T$. The subgroup $A_0 = A \cap K$ is a maximal compact subgroup of $A$. Its volume $\text{vol}(A_0)$ specializes to a polynomial in $q$ that can be written as a product of cyclotomic polynomials. Adjusting $\text{vol}(A_0)$ by a null function, we may assume that $\text{vol}(A_0)$ is the specialization of a product of cyclotomic polynomials. Cyclotomic polynomials in $q$ are invertible constructible motivic functions. Thus, $\text{vol}(A_0)$ can be inverted. Integration here is understood to be motivic integration with respect to invariant volume forms on $N$ and $A$. The pushforward under a definable morphism (integration over fibers) carries $q$-constructible functions to $q$-constructible functions. Therefore $(\lambda,\mu) \mapsto s_{\lambda,\mu}$ is a $q$-constructible function. □

1We ask whether the difference $\rho_G - \rho_H$, for $G$ and an unramified endoscopic group $H$, is always a sum of roots.
Remark 3.9.7. The proof of the previous lemma motivates the following question. Let $E/RF$ be an extension of the residue field sort described by coefficients running over a definable subassignment $Z$ as in Section 3.6. Let $T$ be a torus defined over the residue field sort that splits over $E$. The torus is classified by $(X'(T), X_*(T), \theta)$, where $\theta$ is the automorphism of the character and cocharacter lattices determined by a quasi-Frobenius automorphism. Let $\text{vol}(T)$ be the motivic volume of $T$, viewed as usual as an element of a Grothendieck ring. Upon specialization to a finite field $\mathbb{F}_q$, the cardinality of a torus $T$ is given by a determinant [3, Prop.3.3.7]:

$$D(\theta, q) := \det(\theta q - 1; X_*(T) \otimes \mathbb{Q}).$$

Is $\text{vol}(T) = D(\theta, L)$ in the Grothendieck group, where $L$ is the Lefschetz motive?

4. Presburger constructibility

In this section, we check that some functions related to the finite dimensional representations of complex reductive groups are Presburger constructible functions on the appropriate integer lattices.

Remark 4.0.8. For purposes of constructibility, we consider $\mathbb{Z}'$ and also $Y'$ as definable subassignments $h[0, 0, r]$. When dealing with $\mathbb{Z}'$, integrals over fibers in the sense of motivic integration are discrete sums. For example, if $(\lambda, \mu) \rightarrow a_{\lambda, \mu}$ and $(\mu, \nu) \rightarrow b_{\mu, \nu}$ are constructible functions of integer parameters $(\lambda, \mu, \nu) \in L \times M \times N$, then we may interpret the matrix product $(\lambda, \nu) \rightarrow \sum_{\mu} a_{\lambda, \mu} b_{\mu, \nu}$ as a fiber integral as follows. We pull $a_{\lambda, \mu}$ and $b_{\mu, \nu}$ both back to $L \times M \times N$, multiply them as constructible functions on $L \times M \times N$, then integrate (sum) over the fibers of the projection morphism $L \times M \times N \rightarrow L \times N$.

4.1. weight multiplicities. We expand the partition function into an infinite series

$$P(\hat{G}, V_R, \theta_1, E, q) = \sum_{\mu} (P(\hat{G}, V_R, \theta_1, E, q), e^{\mu}) e^{\mu'}.$$ 

Lemma 4.1.1. The function $\mu \mapsto (P(\hat{G}, V_R, \theta_1, E, q), e^{\mu})$ is Presburger constructible. The function $\mu \mapsto (P(\hat{G}, V_R, \theta_1, E, 1), e^{\mu})$ is Presburger constructible.

Proof. Recall (Lemma [3.4]) that $P(\hat{G}, V_R, \theta_1, E, q)$ is a product of factors of the form

$$(1 - \zeta q^a e^{\alpha})^{-1}$$

where $\zeta$ is a root of unity ($\zeta^k = 1$) and $\alpha = N_1 \mu$ is a norm.

If $\sum_{\mu} a_{\mu} e^{\mu}$ and $\sum_{\mu} b_{\mu} e^{\mu}$ have constructible coefficients, then it is easily checked that the product also has constructible coefficients (see Remark 4.0.8). Thus, the proof reduces immediately to showing that constructibility of the coefficients of

$$\frac{1}{1 - \zeta q^a e^{\alpha}} = \sum_{j=0}^{\infty} \zeta^j q^{ib} e^{ja} = \sum_{a=0}^{k-1} \zeta^a q^{ib} e^{ja} \sum_{j=0}^{\infty} q^{jb} e^{ja},$$

with reindexing $j = ik + a$, with $0 \leq a < k$. The coefficients in the inner sum on the right are evidently constructible. \hfill \Box
We continue to work in the usual context of an unramified reductive group $G$, dual $L = \hat{G}$, and partition function $P(E, q) = P(\hat{G}, n, \theta, E, q)$.

Let $m_{\lambda, \mu} \in \mathbb{N}$ be the multiplicity of the weight $\mu \in X^*(T)$ in the irreducible representation with highest weight $\lambda \in P^+$.

Lemma 4.1.2. The weight multiplicity function $(\lambda, \mu) \mapsto (\tau_{\lambda, e^\mu})$, the $q$-weight multiplicity function $(\lambda, \mu) \mapsto (\tau_{\lambda, q^e}, e^\mu)$ and the inverse Satake transform $(\lambda, \mu) \mapsto t_{\lambda, \mu}$ are all Presburger constructible functions.

Proof. Each function is a finite sum over $w \in W^\theta$ of partition functions. Because constructible functions form a ring, it is enough to check that each term in the sum is constructible. The relevant partition functions are $P(E^{-w}, 1)$, $P(E^{-w}, q)$, and $P(E^{-w}, q^{-1})$, respectively. These are constructible by Lemma 4.1.1.

Recall that $n_{\lambda, \mu}$ is the inverse of the weight multiplicity matrix.

Theorem 4.1.3. $n_{\mu, \lambda}$ is a Presburger constructible function on $P^+ \times P^+$.

Proof. This is a consequence of van Leeuwen’s formula (Lemma 1.7.4). Referring to that formula, it is enough to show constructibility of each term in the sum, with fixed $(w', w)$. This follows from the definability of the set $Y_w^*$ and of the delta function $(\mu, \lambda) \mapsto \delta_{n(w' \mu), \lambda}$.

Corollary 4.1.4. Consider the geometric Satake transform (1.8.3): 

$$f_\lambda = \sum_{\mu} g_{\lambda, \mu} \tau_{\mu}.$$ 

Then $(\lambda, \mu) \mapsto g_{\lambda, \mu}$ is Presburger constructible.

Proof. The functions $(\lambda, \mu) \mapsto s_{\lambda, \mu}$ and $(\lambda, \mu) \mapsto n_{\lambda, \mu}$ are constructible. The basis $g_{\lambda, \mu}$ is the matrix product of these two bases. The result follows from Remark 4.0.8: matrix multiplication with definable indexing sets preserves constructibility.

A second proof of the theorem can be obtained from Theorem 1.8.3.

4.2. branching formulas. While we are on the topic of constructibility, we point out the constructibility of branching multiplicities. For example, we have the following corollary of the classical branching multiplicity formula.

Lemma 4.2.1. Let $H \leq G$ be complex reductive groups with Lie algebras $\mathfrak{h} \subseteq \mathfrak{g}$. Fix maximal tori $T_H \leq T_G$ with Lie algebras $t_h \subseteq t_g$. Assume that there is an element $X_0 \in t_h$ such that $\langle \alpha, X_0 \rangle > 0$ for every positive root of $\mathfrak{g}$. Let $P^+_G$ and $P^+_H$ be the sets of dominant weights in $G$ and $H$. Let $m(\lambda, \mu)$ be the multiplicity of the irreducible $\mathfrak{h}$-module with highest weight $\mu$ in the irreducible $\mathfrak{g}$-module with highest weight $\lambda$. Then $m(\lambda, \mu)$ is a Presburger constructible function on $P^+_G \times P^+_H$. 

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Proof. Kostant’s formula expresses each branching multiplicity as a finite sum of partition functions \[15, Theorem 8.2.1\]. Each partition function is rational. Thus, the argument used in the proof of Lemma 4.1.1 applies. □

Remark 4.2.2. Explicit formulas for branching multiplicities are typical of what Presburger constructible functions look like. Typically branching formulas look like products of linear factors depending on cases that can be described by linear inequalities on dominant weights \(\lambda\) and \(\mu\). We do not pursue the topic, but we can similarly investigate the constructibility of the function giving the multiplicities of \(\tau_\mu\) in \(\text{Sym}^k\tau_\lambda\), and related operations on characters.

Let \((\lambda,\mu) \mapsto m(\lambda,\mu)\) be the function constructed in Section 2 that is attached to an embedding \(\xi : \hat{H} \to \hat{G}\) of endoscopic groups.

Lemma 4.2.3. The branching multiplicity function \(m(\lambda,\mu)\) is Presburger constructible on \(P^+ \times P^+_H\).

Proof. By Lemma 2.3.4, it is enough to assume that the elements \(\dot{w}\) agree (for \(D\) and \(\xi\)). (Note that a change of \(\xi\) changes the weights \(m(\lambda,\mu)\) by \(\mu(t)\), which is a constructible function of \(\mu\), whenever \(t\) has finite order. In fact, each preimage \(\{\mu : \mu(t) = \zeta\}\) is Presburger definable.) It is enough to show that each term in Equation 2.3.2 is Presburger constructible. This reduces to the constructibility of the terms \(p_{\mu'}\), which follows from the rationality of the partition function (Lemma 4.1.1). □

4.3. the constructibility of \(B_\xi\). Let \(\hat{G}\) be the \(L\)-group of an unramified reductive group \(G\). Let \(\hat{H}\) be the dual of an unramified endoscopic group \(H\). We assume that both \(H\) and \(G\) are given in the category of definable subassignments over a cocycle space \(Z\). We can assume that the cocycle space \(Z\) is the same for \(\hat{H}\) and \(\hat{G}\).

Working \(p\)-adically, Langlands gives a homomorphism \(b = b_\xi\) from the spherical Hecke algebra of \(G\) to the spherical Hecke algebra of \(H\). If \(f\) belongs to the spherical Hecke algebra of \(G\), its Satake transform belongs to \(\mathbb{C}[Y]^\text{w}\). The biinvariant function \(b_\xi(f) \in \mathcal{H}(\hat{H}/K_H)\) is the inverse Satake transform of the image of \(f\) in \(\mathbb{C}[X'(\hat{T}_1)]^\text{w}\).

Theorem 4.3.1. Let \(G\) be an unramified connected reductive group with unramified endoscopic group \(H\), both considered as definable subassignments over a cocycle space \(Z\). Fix an \(L\)-embedding \(\xi : \hat{H} \to \hat{G}\) that factors through a finite cyclic group \((\theta)\); that is, \(\xi : \hat{H} \cong (\theta_H) \to \hat{G} \cong (\theta)\). Then there is a \(q\)-constructible function \(B_\xi\) on \(P^+_G \times H\) and a natural number \(N\) with the following specializations:

\[ B_\xi(\lambda, h)_F = b_\xi(f_{F,\lambda})(h), \quad \text{for } h \in H(F), \]

for all \(p\)-adic fields in \(F \in C_N\).

Recall that for each \(F\), we let \(f_{F,\lambda}\) denote the characteristic function of the double coset \(K\sigma^{-1}_K\) in the unramified reductive group \(G\) over \(F\). The theorem implies that the homomorphism \(b_\xi\) has uniform behavior as the \(p\)-adic field varies, and as \(\lambda\) varies.
Proof. We have done most of the work already for this theorem. Let $L_G$ and $L_H$ be the definable sets given in Lemma 3.9.3 for $G$ and $H$. We have

$$ J_\xi \tilde{f}_\lambda = \sum g_{\lambda,\mu} \langle \lambda, \mu \rangle \sigma_\mu $$

$$ = \sum g_{\lambda,\mu} m(\lambda', \mu) \sigma_\mu $$

$$ = \sum g_{\lambda,\mu} m(\lambda', \mu) \tau_{\mu'} \tilde{f}_{\mu'}. $$

As usual $g_{\lambda,\mu}$ are the coefficients of the geometric Satake transform, $m(\lambda', \mu)$ are branching coefficients, and $\tau_{\mu'}$ are coefficients of the inverse Satake transform on $L_H$. As proved above, these are all constructible functions of their lattice parameters. Then

$$ B_\xi(\lambda, h) = b_\xi(f_\lambda)(h) = \sum_{\lambda', \mu, \mu'} g_{\lambda,\mu} m(\lambda', \mu) \tau_{\mu'} \hat{\chi}(L_H)(\mu', h). $$

The sums run over bounded definable sets and are represented by discrete motivic sums over the integer sort (Remark 4.0.8). The right-hand side of this equation is therefore a constructible function of $(\lambda, h) \in P^r \times H$. □

Remark 4.3.2. We have stated $q$-constructibility results in terms of the limiting behavior on $p$-adic fields $C_N$ for $N$ large. However, in fact, the formulas we obtain for $B_\xi$ hold for all $p$-adic fields.

4.4. transfer principle. We review the transfer principle from [10].

Theorem 4.4.1. Let $f \in C(S)$ be a constructible function on a definable set $S$. There exists $N$ such that for all pairs of fields $F_1, F_2 \in C_N$ with the same residue field, $f_{F_1}$ is identically zero on $S(F_1)$ iff $f_{F_2}$ is identically zero on $S(F_2)$.

This theorem allows us to transfer identities of motivic integrals of constructible functions from a field $F_1$ of one characteristic to a field $F_2$ of another characteristic provided that they have the same residue field. The constant $N$ depends on $f$ and is not explicit.

An easy corollary is a transfer principle for statements $\psi$ in the Denef-Pas language.

Corollary 4.4.2. Let $\psi$ be a statement in the Denef-Pas language. Then there exists $N$ such that for all pairs of fields $F_1, F_2 \in C_N$ with the same residue field, $\psi_{F_1}$ holds iff $\psi_{F_2}$ holds.

Proof. Let $f \in C(\text{pt})$ be the function on a point that is the characteristic function of $\psi$. Apply the transfer principle to $f$. □

4.5. fundamental lemma. We conclude this article with a proof of the fundamental lemma for the spherical Hecke algebra for unramified groups in large positive characteristic in the following form.

Theorem 4.5.1. For each absolute root system $R$, there is a constant $N = N_R \in \mathbb{N}$ such that the Langlands-Shelstad fundamental lemma for the spherical Hecke algebra $H(G \setminus K)$ holds for all unramified connected reductive groups $G$ with absolute root system $R$ and all of its unramified endoscopic groups $H$ over $F$ for all $p$-adic fields $F \in C_N$. 

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Proof. We assume that the reader is familiar with the proof that the fundamental lemma for the unit element of the Hecke algebra can be transferred from one field to another by the transfer principle of motivic integration \[13\]. The method is the same here. Once we establish that the fundamental lemma can be expressed as an identity of constructible functions, then the machinery of motivic integration and the transfer principle takes over and gives the theorem. Earlier work has already shown how orbital integrals can be expressed as motivic integrals of constructible functions. Our proof therefore reduces to checking the constructibility of the Langlands-Shelstad transfer factor and to checking the constructibility of the function $B_\xi$. 

The fundamental lemma takes the form

\[(4.5.2) \quad \sum_{\gamma G} \Delta_0(\gamma_H, \gamma_G, \cdots) O(\gamma_G, f_{\lambda}) - SO(\gamma_H, b_\xi(f_{\lambda})) = 0.\]

Stable orbits of regular semisimple elements are definable as fibers of the Chevalley morphism $G \to T/W$. The invariant motivic measure on stable orbits is the volume form attached to a Leray residue of an invariant differential form on the group with respect to the canonical form on $T/W$.

The constructibility of the transfer factor is treated in Appendix 5. The ellipsis (\cdots) in the transfer factor indicates extra free parameters such as a parameter running over $\alpha$-data, a parameter running over admissible pinnings for the canonical normalization, and uniformizing parameters used in our explicit treatment of the $\chi$-data. The $p$-adic transfer factor is independent of these choices, but in dealing with constructible motivic functions, it is best to make the dependence on the parameters explicit (or at least honor them with an ellipsis).

The homomorphism $b_\xi(f_{\lambda})$ can be replaced with the constructible function $B_\xi$.

We may consider the left-hand side of Equation (4.5.2) as a $q$-constructible function of $(\lambda, \gamma_H, \cdots) \in P^+ \times H \times \cdots$, all over a definable cocycle space $Z$ used to parameterize an unramified splitting field of $G$ and $H$.

The fundamental lemma holds for the unit element in positive characteristic by the work of Ngô [32]. This can be lifted to characteristic zero \[13\], \[36\]. It extends to the full Hecke algebra in characteristic zero \[19\]. Hence the identity (4.5.2) holds in characteristic zero. By the transfer principle, there exists $N$ such that the fundamental lemma also holds for all fields $F \in C_N$.

Furthermore, the arguments of \[19\] reduce the fundamental lemma for the Hecke algebra (in characteristic zero or large positive characteristic) to $G = G_{\text{adj}}$. For an adjoint group, there are only finitely many choices of unramified $G$ and $H$ up to equivalence associated with a given root system and only finitely many choices of $L$-morphisms $\xi$ that satisfy our conditions. Thus, we can arrange for $N$ to depend only on the root system.

It is important for the left-hand side of the equation to be viewed as a single identity with $P^+$ forming a factor of the definable subassignment, rather than viewed as an infinite collection of identities indexed by $\lambda \in P^+$. This allows us to invoke the transfer principle a single time, rather than once for each $\lambda \in P^+$.

\[\square\]
5. Appendix on transfer factors

In this section, we assume familiarity with the Langlands-Shelstad transfer factor [27].

In [16], we showed that the Lie algebra transfer factor is a constructible motivic function. In that article, by restricting attention to a small neighborhood of the identity element of the group, we were able to avoid the analysis of multiplicative characters that appear in the group-level transfer factor. In this appendix we analyze the multiplicative characters and prove that group-level transfer factor is constructible for unramified endoscopic data.

We use the canonical normalization of transfer factors given in [18, §7]. The canonical normalization requires a choice of an admissible pinning. The admissible pinning involves a choice of simple root vectors $X_{\alpha}$ (with respect to a fixed Borel subgroup and Cartan). The choices $X_{\alpha}$ range over a definable subassignment, and we obtain the canonical normalization by introducing a free parameter into the transfer factor ranging over the definable subassignment.

5.0.1. $a$-data. To define the transfer factor for $p$-adic fields, a choice of $a$-data is made, but the transfer-factor is in fact independent of the choice of $a$-data.

This section introduces a definable subassignment of $a$-data and introduces an explicit free variable $a$ into the transfer factor that ranges over the definable subassignment of $a$-data.

We begin with a review of $a$-data for a $p$-adic field, then show how to make the construction as a definable subassignment. Let $\Gamma$ be the Galois group of a Galois extension $L/F$. We assume that $\Gamma$ acts on a finite set $R$ of roots. The $a$-data are a collection of constants $a_\alpha \in L^\times$ indexed by $\lambda \in R$ such that

\[
\begin{align*}
  a_{-\lambda} &= -a_\lambda, \\
  a_{\sigma \lambda} &= \sigma(a_\lambda),
\end{align*}
\]

for $\sigma \in \Gamma$.

Let $\epsilon : R \to R$, given by $\epsilon(\lambda) = -\lambda \neq \lambda$. Let $O$ be the orbit of some $\lambda \in R$ under $(\Gamma, \epsilon)$. The choice of $a$-data can clearly be made orbit by orbit. If there is no $\sigma \in \Gamma$ such that $\sigma \lambda = -\lambda$, we have a specific choice of $a$-data (selecting a given $\lambda \in O$) given by

\[
a_{\sigma \lambda} = 1, \quad a_{-\sigma \lambda} = -1, \quad \sigma \in \Gamma.
\]

If some $\sigma_0 \in \Gamma$ gives $\sigma_0 \lambda = -\lambda$, then we proceed as follows. Let $F_{+,\lambda}$ be the fixed field of $\Gamma_{+,\lambda} = \{\sigma \in \Gamma \mid \sigma \lambda = \lambda\}$ and we let $F_{\pm,\lambda}$ be the fixed field of $\Gamma_{\pm,\lambda} = \{\sigma \in \Gamma \mid \sigma \lambda = \pm \lambda\}$. The extension $F_{+,\lambda}/F_{\pm,\lambda}$ is quadratic. We may choose $a$-data by choosing $a_\lambda \in F_{+,\lambda}$ such that $\sigma_0(a_\lambda) = -a_\lambda$ then extending uniquely to the entire orbit by the relation (5.0.3).

Specifically, when the quadratic extension is unramified, the choice of $a_\lambda$ can be taken to run over units of $F_{+,\lambda}$ such that its square is a nonsquare in $F_{\pm,\lambda}$. When the quadratic extension is ramified, we take $a_\lambda$ to run over uniformizers in $F_{+,\lambda}$ such that its square lies in $F_{\pm,\lambda}$. We see by these explicit descriptions that the tuple $(a_\lambda)$, indexed by $\lambda$, is a parameter in a definable subassignment.

5.0.2. $\Delta$II. Two terms in the transfer factor rely on multiplicative characters constructed from $\chi$-data: the terms $\Delta_{\text{II}}$ and the term $\Delta_2$. 

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Lemma 5.0.4. For unramified endoscopic data, there is a q-constructible function representing $\Delta_{II}$ (after introducing some free parameters ranging over definable subassignments).

Proof. We begin with an explicit construction of some characters for a $p$-adic field. Then we analyze the construction to see that it can be done constructibly.

Let $F_{+}/F_{\pm}$ be a quadratic extension of $p$-adic fields with large residue characteristic. Let $\sigma_{+}$ be a uniformizer in $F_{+}$. We define a multiplicative character $\chi_{+} = \chi_{F_{+}/F_{\pm}} : F_{+} \to \mathbb{C}^\times$ as follows. If $F_{+}/F_{\pm}$ is unramified, let $\chi_{+}$ be the unramified character of order two.

If $-1$ is a square in $F_{+}$, we define $\chi_{+}$ by $\chi_{+}(\sigma_{+}) = i \in \mathbb{C}$, and $\chi_{+}$ restricted to units is the unique character of order two.

If $-1$ is a nonsquare in $F_{+}$, we define $\chi_{+}$ by $\chi_{+}(\sigma_{+}) = 1$, and $\chi_{+}$ restricted to units is the unique character of order two.

In every case, $\chi_{+}^4 = 1$.

Now we analyze constructibility. The condition that $-1$ is a square or nonsquare is a definable condition. Assume that $F_{+}$ and $F_{\pm}$ are both extensions of $VF$, presented as usual by a definable space of the characteristic polynomial of a generator of the fields. Introduce a free parameter $\sigma_{+}$ that runs over the constructible subassignment of uniformizers in $F_{+}$.

We claim that $\chi_{+}$ is a linear combination of characteristic functions

$$\chi_{+} = \sum_{\zeta \in \mu_4(\mathbb{C})} \zeta \text{char}(D_{\zeta}),$$

where each $D_{\zeta}$ is constructible over the space of parameters. This is essentially obvious: $F_{+}/F_{\pm}$ being unramified is a definable condition on the coefficients of the characteristic polynomial; the unique character of order two is given in terms of the characteristic function on squares and nonsquares, etc.

Now we turn to the transfer factor $\Delta_{II}$. It has the form

$$\prod_{\alpha} \chi_{\alpha} \left( \frac{a(\gamma) - 1}{a_{\alpha}} \right),$$

where $\gamma$ is strongly regular semisimple. It is a constructible function if each factor is a constructible function. Each morphism $\gamma \to (a(\gamma) - 1)/a_{\alpha}$ is definable, so we only need to check that each character $\chi_{\alpha}$ in some choice of $\chi$-data is constructible. We use the characters given above to do so.

There is no harm in partitioning the domain of $\Delta_{II}$ into finitely many parts according to definable characteristics of the element $\gamma$. We consider an extension $L/VF$ that splits the centralizer of $\gamma$. We may assume fixed abstract Galois data $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ with enumeration $\sigma_{\gamma}$ of the elements of $\Gamma$ for $L$ and we may assume a fixed action of that data on the root system coming from the centralizer of $\gamma$ (relative to a split torus). This gives the indexing set $R$ of roots and action of $\Gamma$ as fixed choices used to partition the domain of $\Delta_{II}$.
Let $\epsilon$ be an automorphism of $R$ that acts as $\lambda \mapsto -\lambda$, and let $O(\lambda)$ be the orbit of $\lambda$ under $\langle \Gamma, \epsilon \rangle$. If there does not exist $\sigma \in \Gamma$ such that $\sigma \lambda = -\lambda$, then we may take the $\chi$-data for $\mu \in O(\lambda)$ to be the trivial character (which is constructible).

Now assume that there exists $\sigma \in \Gamma$ such that $\sigma \lambda = -\lambda$. Then $F_{\lambda}/F_{\pm \lambda}$ is a nontrivial quadratic extension. We set $\chi_{\lambda} = \chi_+ \sigma - 1$ for this quadratic extension. In more detail, we include free parameters $\sigma$ realizing each abstract automorphism $\sigma$ as a linear transformation of $L/\mathbb{F}$. The extension $F_{\lambda}/F_{\pm \lambda}$ and the space of uniformizers $\sigma_+$ in the extension are then described by definable conditions inside $L/\mathbb{F}$ (as in [13]).

By transport of structure, we obtain constructible $\chi$-data on the entire orbit of $\lambda$, using the defining properties of $\chi$-data: $\chi_{\sigma \lambda} = \chi_{\lambda} \sigma^{-1}$; $F_{\sigma \lambda} = \sigma F_{\lambda}$; $\sigma \sigma_+ = \sigma_+ \sigma$, and so forth. Running over all orbits this way, constructible $\chi$-data are obtained.

5.0.3. $\Delta_2$. We have now treated all terms except $\Delta_2$. We recall that the term $\Delta_2$ restricts to a multiplicative character on each Cartan subgroup of $G$. It is constructed from $\chi$-data by means of class field theory reciprocity for tori. The following theorem completes our analysis of the transfer factors on groups.

**Theorem 5.0.5.** For unramified endoscopic data, there is a $q$-constructible function representing the transfer factor $\Delta$, possibly after introducing some free parameters. These parameters have no effect upon specialization to a $p$-adic field.

**Proof.** The idea of the proof is that multiplicative characters can be chosen to be tamely ramified; that is, they have trivial restriction to topologically unipotent elements. We have descent formulas for unramified groups that reduce the transfer factor to the case of topologically unipotent elements [26] [18] [26]. We freely use various lemmas on definability from Section 3.8.

We enumerate the standard Levi components of $G$. Each is a definable set. If $\gamma_G$ is conjugate to an element $\gamma_M$ in some proper Levi subgroup, then by descent formulas for transfer factors we have $\Delta(\gamma_G, \gamma_H) = \Delta_M(\gamma_M, \gamma_{M,H})$. The element $\gamma_{M,H}$ is a conjugate of $\gamma_H$ in a Levi of $H$ constructed by descent. By an induction on the dimension of the group, we may assume that $\Delta_M$ is constructible. Every regular semisimple element that is not elliptic is conjugate to an element of a proper Levi subgroup. We may now assume that $\gamma_G$ belongs to an elliptic Cartan subgroup $T$.

Since $G$ is unramified, the connected center $Z^0 = Z(G)^0$ is also unramified and can be naturally identified with a torus in the center of $H$. By Langlands and Shelstad, there is a character $\chi_Z$ on $Z^0$ such that

$$\Delta(\gamma_G, \gamma_H) = \chi_Z(\epsilon) \Delta(\gamma_G, \gamma_H).$$

The character is unramified [18]. The character $\chi_Z$ depends on $(\gamma_G, \gamma_H)$ only through the endoscopic data $(G, H)$ and $\epsilon$.

We claim that $Z^0$ is definable and that $\chi_Z$ is a constructible function on $Z^0$. The connected center $Z^0$ is definable as the kernel of $G \to G_{ss}$, where $G_{ss}$ is a semisimple quotient of $G$. The root datum for $G_{ss}$ can be described as fixed choice in terms of the root datum...
for $G$. There exists $\xi_0: {}^LH \to {}^L G$ such that $\chi_Z$ is trivial on $Z^0$ [19, Lemma 3.6]. As we vary the embedding $\xi: {}^LH \to {}^L G$, the transfer factor changes by an unramified character that comes from an element

$$\hat{\xi} \in Z(\hat{H}) \subseteq \hat{T} \to \hat{T}_1.$$  

It is enough to show that this unramified character is constructible. We treat this element $\hat{\xi}$ and its image $\tau \in \hat{T}_1$ as a fixed choice. Following Equation [12.1] we have

$$\text{Hom}(T_H/T_H(O), \mathbb{C}^\times) = \hat{T}_1 = \text{Hom}(X^*(A_H), \mathbb{C}^\times).$$

The element $\tau$ has finite order. The character $\chi_Z$ is the restriction of this character to $Z^0 \subseteq T_H$. In the constructible context, we start with $\hat{\xi}$ and $\tau$, writing $\chi_Z$ on $T_H$ as

$$\sum_{\mu \in X^*(A_H)} \mu(\tau)\text{char}(A_{H,\mu}),$$

where $A_{H,\mu}$ is the definable set of Equation [3.9.1] for $H$. This clearly restricts to a constructible function on $Z^0$.

By adjusting $(\gamma_G, \gamma_H)$ by a central element in $Z^0$, we may reduce the proof of constructibility to the special case where $\gamma_G$ and $\gamma_H$ lie in the maximal bounded subgroup of their Cartan subgroups $T$ and $T_H$. We take a definable topological Jordan decomposition $\gamma_G = \gamma_s\gamma_u$, as described in Section 3.8. Replacing $\gamma_G$ and $\gamma_s$ by stable conjugates $\gamma_G' = \gamma_G^c$ and $\gamma_s^h$ (same $h$), we may assume that $\gamma_s \in G_u$; that is, its centralizer is unramified. We may do the same on the endoscopic side. We have

$$\Delta_G(\gamma_G, \gamma_H) = c\Delta_G(\gamma_G', \gamma_H'),$$

where $c$ is the ratio of terms coming from $\Delta_{II}$. The $\Delta_{II}$ terms are constructible, so the proof of constructibility reduces to the case where we may now drop primes and assume that $\gamma_s$ has an unramified centralizer. We construct descent data $(G_s, H_s)$ for the centralizer of $\gamma_s$ in $G$ and the corresponding centralizer in $H$. By [19], the normalized transfer factors satisfy

$$\Delta(\gamma_G, \gamma_H) = \Delta_{s,s}(\gamma_G, \gamma_H),$$

where the right-hand side is computed with respect to the endoscopic data $(G_s, H_s)$. (In that reference, it is assumed that $\gamma_G \in G(O_F)$ and $\gamma_H \in H(O_F)$, but that assumption is only needed to prove the fact that the centralizer of $\gamma_s$ is unramified. Since we have a separate argument for that fact, the descent formula holds in our context as well. That reference also uses the topological Jordan decomposition rather than our definable version, which is not unique. It can be checked that the formulas for the transfer factor are insensitive to the choice of decomposition $\gamma = \gamma_1\gamma_u$.) By an induction on the dimension of the group, the right-hand side is constructible, and the proof is complete except in the case when $\gamma_s$ is central.

We now assume that $\gamma_s$ is central and strongly compact. Then $\gamma_s \in K$ because $K$ is a maximal compact. It is known that $\chi_Z$ is trivial on $K$ [19, Lemma 3.2]. Thus after adjusting by an element in the center, we may assume that $\gamma_s = 1$. That is, we are reduced to proving the constructibility of transfer factors on the set of topologically unipotent elements. We pick our $\chi$-data to be tamely ramified. This implies that the characters $\Delta_1$ are trivial on topologically unipotent elements. This reduces the constructibility of $\Delta$ to the analysis of factors $\Delta_1, \Delta_{II}, \Delta_{s,s}$, and $\Delta_{IV}$. This has already been done. This completes the proof. □
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REFERENCES

[1] Alexis Bouthier. Géométrisation du lemme fondamental pour l’algèbre de Hecke. http://arxiv.org/abs/1502.07148, 2015.
[2] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. Publications Mathématiques de l’IHÉS, 41(1):5–251, 1972.
[3] Roger William Carter. Finite groups of Lie type: Conjugacy classes and complex characters, volume 5. John Wiley & Sons, 1985.
[4] William Casselman. The unramified principal series of \( p \)-adic groups. I. the spherical function. Compositio Mathematica, 40(3):387–406, 1980.
[5] William Casselman. A companion to Macdonalds book on \( p \)-adic spherical functions. preprint, 2005.
[6] William Casselman. Symmetric powers and the Satake transform. preprint, 2016.
[7] Jorge E. Cely. Motivic integration, the Satake transform, and the fundamental lemma. PhD thesis, University of Pittsburgh, 2016.
[8] Neil Chriss. A geometric construction of the Iwahori–Hecke algebra for unramified groups. Pacific Journal of Mathematics, 179(1):11–57, 1997.
[9] Raf Cluckers and François Loeser. Constructible motivic functions and motivic integration. Inventiones mathematicae, 173(1):23–121, 2008.
[10] Raf Cluckers and François Loeser. Constructible exponential functions, motivic Fourier transform and transfer principle. Annals of mathematics, pages 1011–1065, 2010.
[11] Raf Cluckers, Julia Gordon, and Immanuel Halupczok. Transfer principles for integrability and boundedness conditions for motivic exponential functions. arXiv preprint arXiv:1111.4405, 2011.
[12] Raf Cluckers, Julia Gordon, and Immanuel Halupczok. Local integrability results in harmonic analysis on reductive groups in large positive characteristic. arXiv preprint arXiv:1111.7057, 2011.
[13] Raf Cluckers, Thomas Hales, and François Loeser. Transfer principle for the fundamental lemma. In L. Clozel, M. Harris, J.-P. Labesse, and B.-C. Ngô, editors, On the stabilization of the trace formula. International Press of Boston, 2011.
[14] Haibao Duan. On the inverse Kostka matrix. Journal of Combinatorial Theory, Series A, 103(2):363–376, 2003.
[15] Roe Goodman and Nolan R Wallach. Representations and invariants of the classical groups, volume 68. Cambridge University Press, 2000.
[16] Julia Gordon and Thomas Hales. Endoscopic transfer of orbital integrals in large residual characteristic. American Journal of Mathematics, 138(1):109–148, 2016.
[17] Thomas J Haines. Dualities for root systems with automorphisms and applications to non-split groups. arXiv preprint arXiv:1604.01468, 2016.
[18] Thomas C Hales. A simple definition of transfer factors for unramified groups. Contemporary Mathematics, 145:109–109, 1993.
[19] Thomas C Hales. On the fundamental lemma for standard endoscopy: reduction to unit elements. Canadian Journal of Mathematics, 47(5):974–994, 1995.
[20] Jens C Jantzen. Darstellungen halbeinfacher Gruppen und kontravariante Formen. Journal für die reine und angewandte Mathematik, 290:117–141, 1977.
[21] Shin-ichi Kato. Spherical functions and a \( q \)-analogue of Kostant’s weight multiplicity formula. Inventiones mathematicae, 66(3):461–468, 1982.
[22] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Annals of Mathematics*, pages 329–387, 1961.

[23] Robert Kottwitz et al. Rational conjugacy classes in reductive groups. *Duke Math. J.*, 49(4):785–806, 1982.

[24] Robert E Kottwitz and Diana Shelstad. Foundations of twisted endoscopy. *Astérisque*, 1999.

[25] Shrawan Kumar, George Lusztig, and Dipendra Prasad. Characters of simplylaced nonconnected groups versus characters of nonsimplylaced connected groups. *Contemp. Math.*, 478:99–101, 2009.

[26] R Langlands and Diana Shelstad. Descent for transfer factors. In *The Grothendieck Festschrift*, pages 485–563. Springer, 2007.

[27] Robert P Langlands and Diana Shelstad. On the definition of transfer factors. *Mathematische Annalen*, 278(1):219–271, 1987.

[28] Bertrand Lemaire, Colette Mœglin, and Jean-Loup Waldspurger. Le lemme fondamental pour l’endoscopie tordue: réduction aux éléments unités. [http://arxiv.org/abs/1506.03383](http://arxiv.org/abs/1506.03383), 2015.

[29] George Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. *Astérisque*, 101(102):208–229, 1983.

[30] Ian G Macdonald. Spherical functions on a group of p-adic type. *Publications of the Ramanujan Institute*, (2), 1971.

[31] Ivan Mirković and Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Annals of mathematics*, 166(1):95–143, 2007.

[32] Bao Châu Ngô. Le lemme fondamental pour les algèbres de Lie. *Publications mathématiques de l’IHÉS*, 111(1):1–169, 2010.

[33] Tonny Albert Springer. *Linear algebraic groups*. Springer Science & Business Media, 2010.

[34] R Steinberg. *Endomorphisms of algebraic groups*. AMS, 1968.

[35] Marc van Leeuwen. Inverting the Weyl character formula. [https://mathoverflow.net/questions/20188](https://mathoverflow.net/questions/20188), 2013.

[36] J Waldspurger. Endoscopie et changement de caractéristique. *Journal-Institute of Mathematics of Jussieu*, 5(3):423, 2006.

[37] Robert Wendt. Weyl’s character formula for non-connected lie groups and orbital theory for twisted affine lie algebras. *Journal of Functional Analysis*, 180(1):31–65, 2001.

[38] Xinwen Zhu. The geometric Satake correspondence for ramified groups. *arXiv preprint arXiv:1107.5762*, 2011.