The emergence of fermions and the $E_{11}$ content

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Abstract Claudio’s warm and endearing personality adds to our admiration for his achievements in physics a sense of friendliness. His constant interest in fundamental questions motivated the following presentation of our attempt to understand the nature of fermions. This problem is an essential element of the quantum world and might be related to the quest for quantum gravity. We shall review how space-time fermions can emerge out of bosons in string theory and how this fact affects the extended Kac-Moody approach to the M-theory project.

1 Introduction

Despite the impressive theoretical developments of superstring theory, the quantization of gravity remains elusive. The difficulties encountered in coping with the non-perturbative level may well hide non-technical issues. A crucial point is whether the assumed quantum theoretical framework can cope with the quantum nature of spacetime, in particular when confronted to the existence of black hole and the cosmological horizons. In this essay, we inquire into the fundamental nature of fermions, which constitute an essential element of the quantum world. We shall review how in string theory space-time fermions can be constructed out of bosons and we shall discuss how this fact affects the extended Kac-Moody approach to the M-theory project for quantum gravity.

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In Section 2 we unveil the fermionic subspaces of the bosonic closed strings compactified on sublattices of a $E_8 \times \tilde{SO}(16)$ weight lattice, where $\tilde{SO}(16)$ is the universal covering group of $SO(16)$ [4]. All modular invariant fermionic closed strings, supersymmetric or not, are obtained from the parent bosonic strings by a universal truncation performed on both left and right sectors, or on the right sector for the heterotic strings. Supersymmetry arises when the sublattice of the $\tilde{SO}(16)$ weight lattice is taken to be the $E_8$ root lattice so that the bosonic gauge group in each sector is $G = E_8 \times E_8$. We found that not only the closed string spectra of the fermionic string, but also the charges, the chiralities and the tensions of all the fermionic D-branes are encoded in the bosonic strings [4]. In addition, the universal truncation applied to the unique tadpole-free unoriented bosonic string with Chan-Paton group $SO(2^{11})$ yields all tadpole- and anomaly-free open unoriented fermionic strings [4, 17].

In Section 3 we review the attempt to formulate the M-theory project in terms of the very-extended Kac-Moody algebra $E_{11} \equiv E^{++}_{8}$ [41] (or the overextended $E_{10} \equiv E^{++}_{8}$ [29]). Along this line of thought, the inclusion of the bosonic string suggest the introduction of the algebra $D^{++}_{24}$ [32] (or $D^{++}_{24}$). However it is easily shown that the $D^{++}_{24}$ fields cannot accommodate the degrees of freedom needed to generate the fermionic subspaces of the bosonic string. More generally we argue that, without extending the $D^{++}_{24}$ algebra, one cannot encode genuine bosonic string degrees of freedom and, similarly, that $E_{11}$ alone does not encode genuine superstring degrees of freedom.

2 Fermions and supersymmetry from the bosonic string

It is well-known that ten-dimensional fermionic strings can be analyzed in terms of bosonic operators, a consequence of the boson-fermion equivalence in two dimensions. The approach taken here is different. We wish to show that the Hilbert space of all the perturbative fermionic closed strings, and of all their tadpole- and anomaly-free open descendants, are subspaces of the 26-dimensional closed bosonic string theory, and of its tadpole-free open descendant, compactified on suitable 16-dimensional manifolds [3, 16, 17, 4].

2.1 The fermionic subspaces of the closed bosonic string

To accommodate space-time fermions in the left and/or the right sector of the 26-dimensional closed bosonic string one must meet three requirements:

1. A continuum of bosonic zero modes must be removed. This can be achieved by compactifying $d = 24 - s$ transverse dimensions on a $d$-dimensional torus. This leaves $s + 2$ non-compact dimensions with transverse group $SO_{trans}(s)$. 
2. Compactification must generate an internal group $SO_{int}(s)$ admitting spinor representations. This can be achieved by toroidal compactification on the weight lattice of a simply laced Lie group $G$ of rank $d$ containing a subgroup $SO_{int}(s)$. The latter is then mapped onto $SO_{trans}(s)$ in such a way that the diagonal algebra $SO_{diag}(s) = \text{diag}[SO_{trans}(s) \times SO_{int}(s)]$ becomes identified with a new transverse algebra. In this way, the spinor representations of $SO_{int}(s)$ describe fermionic states because a rotation in space induces a half-angle rotation on these states. This mechanism is distinct from the two-dimensional world-sheet equivalence of bosons and fermions. It is reminiscent of a similar mechanism at work in monopole theory: there, the diagonal subalgebra of space-time rotations and isospin rotations can generate space-time fermions from a bosonic field condensate in spinor representations of the isospin group.

3. The consistency of the above procedure relies on the possibility of extending the diagonal algebra $SO_{diag}(s)$ to the new full Lorentz algebra $SO_{diag}(s+1,1)$, a highly non trivial constraint. To break the original Lorentz group $SO(25,1)$ in favor of the new one, a truncation consistent with conformal invariance must be performed on the physical spectrum of the bosonic string. Actually, the states described by twelve compactified bosonic fields must be projected out, except for momentum zero-modes of unit length. The removal of twelve bosonic fields accounts for the difference between the bosonic and fermionic light cone gauge central charges. Namely, in units where the central charge of a boson is one, this difference counts 11 for the superghosts and $(1/2)$ for time-like and longitudinal Majorana fermions. The zero-modes of length $\ell = 1$ kept in the twelve truncated dimensions contribute a constant $\ell^2/2$ to the mass. They account for the removal by truncation of the oscillator zero-point energies in these dimensions, namely for $-(1/24).12 = +1/2$. Moreover, the need to generate an internal group $SO_{int}(s)$ via toroidal compactification requires $s/2$ compactified bosons which can account for $s$ transverse Majorana fermions (we hereafter take $s$ to be even, in which case $s/2$ is the rank of the internal group). Therefore, one must ensure that the total number $d = 24 - s$ of compactified dimensions is at least $12 + s/2$. In other words,

$$s \leq 8,$$

and the highest available space-time dimension accommodating fermions is therefore $s + 2 = 10$. Here, we restrict our discussion to the case $s + 2 = 10$.

To realize this program we choose a compactification of the closed string at an enhanced symmetry point with gauge group $G_L \times G_R$ where $G_L = G_R = G$ (or $G_R = \tilde{G}$ for the heterotic string) and $G = E_8 \times SO(16)$ (or $E_8 \times \tilde{SO}(16)/Z_2 = E_8 \times E_8$). Recall that in terms of the left and right compactified momenta, the mass spectrum is

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1 Throughout this paper we shall denote by $SO(s)$ all the groups locally isomorphic to the rotational group of order $s(s-1)/2$. When specifically referring to its universal covering group, we shall use the notation $\tilde{SO}(s)$. Also we shall keep the same notation for groups and their Lie algebra.

2 We choose units in which the string length squared $\alpha' = 1/2$. 


\[
\frac{m_{L}^{2}}{8} = \frac{p_{L}^{2}}{2} + N_{L} - 1,
\]
\[
\frac{m_{R}^{2}}{8} = \frac{p_{R}^{2}}{2} + N_{R} - 1,
\]
and
\[
m^{2} = \frac{m_{L}^{2}}{2} + \frac{m_{R}^{2}}{2}; \quad m_{L}^{2} = m_{R}^{2}.
\]

In Eq. (2) \( N_{L} \) and \( N_{R} \) are the oscillator numbers in 26-dimensions and the zero-modes \( p_{L}, p_{R} \) span a 2\( d \)-dimensional even self-dual Lorentzian lattice with negative (resp. positive) signature for left (resp. right) momenta. This ensures modular invariance of the closed string spectrum \[33\]. The massless vectors \( \alpha_{-1,R}^{\mu} \alpha_{-1,L}^{i}(0_{L},0_{R}) \) and \( \alpha_{-1,L}^{\mu} \alpha_{-1,R}^{i}(0_{L},0_{R}) \), where the indices \( \mu \) and \( i \) refer respectively to non-compact and compact dimensions, generate for generic toroidal compactification a local symmetry \[\mathcal{U}_{L}(1)^{d} \times \mathcal{U}_{R}(1)^{d}\]. But more massless vectors arise when \( p_{L} \) and \( p_{R} \) are roots of simply laced groups \( G_{L} \) and \( G_{R} \) of rank \( d \) (with root length \( \sqrt{2} \)). The gauge symmetry is then enlarged to \( G_{L} \times G_{R} \).

\subsection*{2.1.1 The group \( \mathcal{G} = E_{8} \times \tilde{SO}(16) \)}

The compactification lattices in both sectors (or in the right sector only for the heterotic strings) are taken to be sublattices of the \( \mathcal{G} = E_{8} \times \tilde{SO}(16) \) weight lattice. These sublattices must preserve modular invariance, which means that the left and right compactified momenta \( p_{L}, p_{R} \) must span a 2\( d \)-dimensional even self-dual Lorentzian lattice. All closed fermionic strings follow then from the properties of the \( \tilde{SO}(16) \) weight lattice and the subsequent truncation.

The weight lattice \( \Lambda_{\tilde{SO}(2n)} \) split into four sublattices.

\[
\Lambda_{\tilde{SO}(2n)} = \begin{cases} 
(o)_{2n} : p_{o} + p \\
(v)_{2n} : p_{v} + p \\
(s)_{2n} : p_{s} + p \\
(c)_{2n} : p_{c} + p
\end{cases}
\]

\[
\begin{align*}
\{ p_{o} & = (0,0,0,...0) \\
\{ p_{v} & = (1,0,0,...0) \\
\{ p_{s} & = (1,1,1,...1) \\
\{ p_{c} & = (-1,1,1,...1)
\end{align*}
\]

and the \( E_{8} \) weight (and root) lattice is

\[
\Lambda_{E_{8}} = (o)_{16} + (s)_{16}.
\]

Here \( p \) is a vector of the root lattice \( \Lambda_{R} \) of \( SO(16) \).

The partition functions \( o,v,s,c \) corresponding to the lattices \( (o), (v), (s), (c) \) are
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\[ P_{2n} = \sum_{p \in \mathbb{R}^2; N^{(c)}} \exp \left\{ 2\pi i \tau \left[ \frac{(p + p')^2}{2} + N^{(c)} - \frac{n}{24} \right] \right\} \quad j = o, v, s, c. \quad (6) \]

Here $N^{(c)}$ is the oscillator number in the compact dimensions. Note that the additive group of the four sublattices of the weight lattice of $SO(2n)$ is isomorphic to the center of the covering group $\tilde{SO}(2n)$, that is $Z_4$ for $n$ odd and $Z_2 \times Z_2$ for $n$ even.

\subsection{The modular invariant truncation}

These lattices combine to form four modular invariant partition functions which after the truncation generate the four non-heterotic fermionic strings in ten dimensions \[4\]. In what follows, we shall only write down the $SO(16)$ characters in the integrand of amplitudes: the $E_8$ characters in $E_8 \times \tilde{SO}(16)$ will be entirely truncated and will play no role and we do not display the contribution of the eight light-cone gauge non-compact dimensions. For the four bosonic ancestors, we get

\begin{align*}
OB_b &= \bar{\delta}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} s_{16} + \bar{c}_{16} c_{16} \quad (7)
OA_b &= \bar{\delta}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} s_{16} + \bar{c}_{16} c_{16} \quad (8)
\text{II}B_b &= \bar{\delta}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} s_{16} + \bar{c}_{16} c_{16} \quad (9)
\text{II}A_b &= \bar{\delta}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} s_{16} + \bar{c}_{16} c_{16} \quad (10)
\end{align*}

where the bar superscript labels the left sector partition functions.

The universal truncation from $E_8 \times \tilde{SO}(16)$ to $SO_{\text{int}}(8)$ is defined by decomposing $SO(16)$ into $SO'(8) \times SO(8)_{\text{int}}$ and erasing the $E_8$ and $SO'(8)$ lattices except, in accordance with item 3 of the above discussion, for unit vectors of $SO'(8)$. In this way the internal momenta $p[SO(8)]$ are related to $p[\mathcal{F}]$ by

\[ \frac{p^2[\mathcal{F}]}{2} = \frac{p^2[SO(8)]}{2} + \frac{1}{2} \quad (11) \]

The unit vectors are identified as follows. The decomposition of an $SO(16)$ lattice in terms of $SO'(8) \times SO(8)$ lattices yields

\begin{align*}
(o)_{16} &= [(o)_8 \oplus (o)_8] + [(v)_8 \oplus (v)_8], \\
(v)_{16} &= [(v)_8 \oplus (v)_8] + [(o)_8 \oplus (o)_8], \\
(s)_{16} &= [(s)_8 \oplus (s)_8] + [(c)_8 \oplus (c)_8], \\
(c)_{16} &= [(s)_8 \oplus (c)_8] + [(c)_8 \oplus (s)_8]. \quad (12)
\end{align*}

The vectors of norm one in $SO'(8)$ are the 4-vectors $\mathbf{p}_o$, $\mathbf{p}_v$ and $\mathbf{p}_c$ defined in Eq.\[14\]. We choose one vector $\mathbf{p}'_o$ and one vector $\mathbf{p}'_v$. (One might equivalently have chosen $\mathbf{p}'_c$ instead of $\mathbf{p}'_v$.) In this way we get from Eq.\[12\]

\[ o_{16} \rightarrow v_8, \quad v_{16} \rightarrow o_8, \quad (13) \]
It follows from the closure of the Lorentz algebra that states belonging to \( v_8 \) or \( o_8 \) are bosons while those belonging to the spinor partition functions \( s_8 \) and \( c_8 \) are space-time fermions. The shift of sign in the fermionic amplitudes, which is consistent with the decomposition of \( s_{16} \) and \( c_{16} \) into \( SO'(8) \times SO(8)_{int} \), is required by the spin-statistic theorem and is needed to preserve modular invariance in the truncation.

The four ten dimensional fermionic string partition functions are

\[
\begin{align*}
OB_b &\rightarrow \bar{o}_8 o_8 \bar{v}_8 v_8 + \bar{s}_8 s_8 + \bar{c}_8 c_8 \equiv OB \\
OA_b &\rightarrow \bar{o}_8 o_8 + \bar{v}_8 v_8 + \bar{s}_8 s_8 + \bar{c}_8 c_8 \equiv OA \\
IIB_b &\rightarrow \bar{v}_8 v_8 - \bar{s}_8 s_8 - \bar{c}_8 c_8 \equiv IIB \\
IIA_b &\rightarrow \bar{v}_8 v_8 - \bar{s}_8 s_8 + \bar{c}_8 c_8 \equiv IIA
\end{align*}
\]

Note that the partition functions of supersymmetric strings \( IIA \) and \( IIB \) arise from \( E_8 \times E_8 \) sublattices of the \( E_8 \times SO(16) \) weight lattice.

The same procedure can be used to obtain all the heterotic strings, supersymmetric or not, by selecting the modular partition functions which are truncated in the right channel only. It will later be extended to D-branes and open descendants.

### 2.1.3 The configuration space torus geometry

The four modular invariant theories can be formulated in terms of the actions

\[
S = -\frac{1}{2\pi} \int d\sigma d\tau \left[ \{ g_{a\bar{b}} \partial_\alpha X^a \partial^\alpha \bar{X}^\bar{b} + b_{a\bar{b}} \epsilon^{a\bar{b}} \partial_\alpha X^a \partial^\alpha \bar{X}^\bar{b} \} + \eta_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu \right],
\]

with \( g_{a\bar{b}} \) a constant metric and \( b_{a\bar{b}} \) a constant antisymmetric tensor in the compact directions \((a, \bar{b} = 1, \ldots, 16)\). \( \eta_{\mu\nu} = (-1; +1, \ldots) \) for \( \mu, \nu = 1, \ldots, 10 \) and \( 0 \leq \sigma \leq \pi \). The fields \( X^a \) are periodic with period \( 2\pi \). In this formalism the left and right momenta are given by

\[
\begin{align*}
\mathbf{p}_R &= \left[ \frac{1}{2} m_b + n^a (b_{a\bar{b}} + g_{a\bar{b}}) \right] \mathbf{e}^\bar{b}, \\
\mathbf{p}_L &= \left[ \frac{1}{2} m_b + n^a (b_{a\bar{b}} - g_{a\bar{b}}) \right] \mathbf{e}^\bar{b},
\end{align*}
\]

where \( \{ \mathbf{e}^a \} \) is the dual of the basis \( \{ \mathbf{e}_a \} \) defining the configuration space torus[^3].

\[
\mathbf{x} \equiv \mathbf{x} + 2\pi n^a \mathbf{e}_a \quad n^a \in \mathbb{Z},
\]

[^3]: In the previous sections momenta compactification was defined in both left and right channels. Both compactifications are obtained in the action formalism from the compactification on the configuration torus and the (quantized) \( \mathbf{b} \)-field.
and the lattice metric is given by

\[ g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b. \] (22)

Explicit forms of the \( g_{ab} \) and \( b_{ab} \) tensors for the four models are given in [4].

The D9-branes pertaining to the four different bosonic theories compactified on the \( E_8 \times \tilde{S}O(16) \) lattices provide an easy way to construct their configuration space tori. We shall find that these tori are linked to each other through global properties of the universal covering group \( \tilde{S}O(16) \).

The tree channel amplitudes \( A_{\text{tree}} \) of the D9-branes are obtained from the torus partition functions Eqs.(7)-(10) by imposing Dirichlet boundary conditions on the compact space. For open strings the latter do not depend on \( b_{ab} \) and are given by

\[ \partial_\tau X^a = 0, \] (23)

where \( \tau \) is the worldsheet time coordinate and \( \sigma \) the space one. Using the worldsheet duality which interchanges the roles of \( \tau \) and \( \sigma \), these equations yield the following relation between the left and right momenta:

\[ p_L - p_R = 0, \] (24)

as well as a match between left and right oscillators in the tree channel. The conditions Eq.(24) determine the closed strings which propagate in the annulus amplitude. Imposing them on the four tori amounts to keep all characters which appear diagonally in Eqs.(7)-(10). Up to a normalization, the annulus amplitudes, written as closed string tree amplitudes, are

\[
\begin{align*}
A_{\text{tree}}(OB_b) &\sim (o_{16} + v_{16} + s_{16} + c_{16}), \\
A_{\text{tree}}(OA_b) &\sim (o_{16} + v_{16}), \\
A_{\text{tree}}(IIB_b) &\sim (o_{16} + s_{16}), \\
A_{\text{tree}}(IIA_b) &\sim o_{16}.
\end{align*}
\] (25)

We express suitably normalized \( A_{\text{tree}} \) as a loop amplitude \( \mathcal{A} \) for a single open string (i.e. without Chan-Paton multiplicity) by performing a change of variable and the S-transformation on the modular parameter \( (\tau \rightarrow -1/\tau) \). The result is given in Table 1.

| \( A_{\text{tree}} \) | \( \mathcal{A} \) |
|----------------------|-------|
| \( OB_b \)          | \( o_{16} + v_{16} + s_{16} + c_{16} \) |
| \( OA_b \)          | \( o_{16} + v_{16} \) |
| \( IIB_b \)         | \( o_{16} + s_{16} \) |
| \( IIA_b \)         | \( o_{16} \) |
The configuration space tori of the four bosonic theories Eqs. (1)-(10) are defined by lattices with basis vectors \( \{ 2\pi e_a \} \) according to Eq. (21). We note that the Dirichlet condition Eq. (24) reduces Eq. (20) to \( \mathbf{p}_L = \mathbf{p}_R = (1/2) m_a e^a \) (independent of \( b_{ab} \)).

Using the general expression for lattice partition functions Eq. (6), we then read off for each model the dual of its \( \text{SO}(16) \) weight sublattice from the four tree amplitudes in Table 1. We then deduce the \( \{ e_a \} \) from the duality between the root lattice \( (o)_{16} \) and the weight lattice \( (o)_{16} + (v)_{16} + (c)_{16} + (s)_{16} \), and from the self-duality of \( (o)_{16} + (v)_{16} \) and \( (o)_{16} + (s)_{16} \). We get

\[
\mathbf{e}_a = (1/2) \mathbf{w}_a, \tag{26}
\]

where the \( \mathbf{w}_a \) are weight vectors forming a basis of a sublattice \( (r)_{16} \) of the weight lattice of \( \text{SO}(16) \). The sublattice \( (r)_{16} \) for each theory is

\[
\begin{align*}
(OB_b) : (r)_{16} & = (o)_{16}, \\
(OA_b) : (r)_{16} & = (o)_{16} + (v)_{16}, \\
(IIB_b) : (r)_{16} & = (o)_{16} + (s)_{16}, \\
(IIB_b) : (r)_{16} & = (o)_{16} + (c)_{16}.
\end{align*}
\tag{27}
\]

These tori can be visualized by the projection depicted in Fig 1.

---

**Fig. 1** Projected weight lattice of \( \tilde{\text{SO}}(16) \) in the 7 - 8 plane of the \( \text{SO}(16) \) Dynkin diagram depicted in the figure. We see from Eqs. (21) and (26) that the volumes \( \xi^a \) of the unit cells, exhibited in shaded areas, must be multiplied by \( (2\pi)^8 2^{-8} \) to yield the \( \text{SO}(16) \) compactification space torus volume of the four bosonic theories (in units where \( \alpha' = 1/2 \)). The two \( II B_b \) theories defined by the two rectangles are isomorphic and differ by the interchange of \( s_{16} \) and \( c_{16} \).
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\[\tilde{t}(OB_b) = \tilde{T},\]
\[\tilde{t}(OA_b) = \tilde{T}/Z_2^d,\]
\[\tilde{t}(IIB_b) = \tilde{T}/Z_2^2 \quad \text{or} \quad \tilde{T}/Z_2^-,\]
\[\tilde{t}(IIA_b) = \tilde{T}/(Z_2 \times Z_2),\]
(28)

where $Z_2^d = \text{diag}(Z_2 \times Z_2)$ and the superscripts $\pm$ label the two isomorphic $IIB_b$ theories obtained by interchanging $(s)_{16}$ and $(c)_{16}$.

There is thus a unified picture for the four theories related to the global properties of the $SO(16)$ group [4]. The $OB_b$ theory built upon $\tilde{T}$ plays in some sense the role of the ‘mother theory’ of the others. One may view the different maximal toroids Eq.(28) as resulting from the identification of centre elements of $\tilde{T}$, which are represented by weight lattice points, with its unit element. These identifications give rise to the smaller shaded cells of Fig.2. In this way, the unit cell of the $IIB_b$ theory is obtained from the $OB_b$ one by identifying the $(o)_{16}$ and $(s)_{16}$ lattice points (or alternatively the $(o)_{16}$ and the $(c)_{16}$ lattice points) and therefore also the $(v)_{16}$ and $(c)_{16}$ lattice (or the $(v)_{16}$ and the $(s)_{16}$ lattice), as seen in Fig.2. It is therefore equal to the unit cell of the $E_8$ lattice [4]. The unit cell of the $OA_b$ theory is obtained by identification of $(o)$ and $(v)$, and of $(s)_{16}$ and $(c)_{16}$.

| (o) | (s) | (c) | (v) |
|-----|-----|-----|-----|
| $SO(16)$ | $SO(16)/Z_2^d \times E_8$ | $SO(16)/Z_2^d \times E_8$ | $SO(16)/Z_2^d$ | $SO(16)/Z_2^d \times Z_2^d$ |

Fig. 2 Identification of centre elements of $SO(16)$ in the four closed string bosonic theories.

2.2 The fermionic D-branes

Table 1 lists the partition functions of a single ‘elementary’ D9-brane for the four $SO(16)$ bosonic strings. In Table 2 below, using the universal truncation, we list the corresponding fermionic D9-branes and their loop partition function $A_{\text{trunc}}$. We now generalize the analysis to encompass several D-branes [4].

First, we remark that the relative position of the different D9-branes in the eight compact dimensions of the $SO(16)$ torus is not arbitrary. Group symmetry requires

\[4 \text{ The latter however does not contain } (v)_{16} \text{ and } (c)_{16} \text{ lattice points.}\]
that the partition function of an open string with end points on D9-branes be a linear combination of the four $SO(16)$ characters. The vector $\mathbf{d}$ separating the two points where two distinct D9-branes meet the $SO(16)$ torus determines the partition function of the string starting at one point and ending at the other after winding any number of times around the torus. The smallest eigenvalue of the string Hamiltonian is $(1/2)\mathbf{d} \cdot \mathbf{d}/\pi^2$. Therefore $\mathbf{d}/\pi$ must be a weight and the D9-branes can only be separated in the compact space (rescaled by a factor $\pi^{-8}$) by a weight vector. Consider for instance two branes in the $OB_{10}$ theory, one located at $(o)_{16}$ and the other located at $(v)_{16}$. The partition function of a string beginning and ending on the same brane is $\psi_{16}$, while the partition function of an open string stretching between them is $\psi_{16}$. For the other theories, the partition function of a string beginning and ending on the same brane will then contain, in addition to $\psi_{16}$, the characters corresponding to the strings stretched between $(o)_{16}$ and all points identified with $(o)_{16}$. This can be checked by comparing the identifications indicated in Fig.2 with the partition functions $\mathcal{A}$ listed in Table 1.

If one chooses the location of one elementary brane as the origin of the weight lattice, the other D9-branes can then only meet the $SO(16)$ torus (rescaled by $\pi^{-8}$) at a weight lattice point. The number of distinct elementary branes is, for each of the four bosonic theories, equal to the number of distinct weight lattice points in the unit cell. For the mother theory $OB_{10}$ there are four possible elementary D9-branes. We label them by their positions in the unit cell, namely by $(o)_{16}, (v)_{16}, (s)_{16}$ and $(c)_{16}$. Note that these weight lattice points represent the centre elements of the $SO(16)$. For the other theories the unit cells are smaller and there are fewer possibilities. The unit cell of the $IIA_{10}$ theory allows only for two distinct branes $(o)_{16} = (s)_{16}, (c)_{16} = (v)_{16}$ (or those obtained by the interchange of $(s)_{16}$ and $(c)_{16}$, as seen in Fig.2). Similarly for the $OA_{10}$ theory, we have the two branes $(o)_{16} = (v)_{16}$ and $(s)_{16} = (c)_{16}$, and finally for the ‘smallest’ theory $IIA_{10}$, we have only one elementary brane $(o)_{16} = (v)_{16} = (s)_{16} = (c)_{16}$. Finally to describe several D9-branes meeting at the same point of the $SO(16)$ torus, one uses the appropriate Chan-Paton factors.

### 2.2.1 Charge conjugation

Charge conjugation of the truncated fermionic strings is encoded in their bosonic parents. A brane sitting at $(v)_{16}$ can always be joined by an open string to a brane sitting at $(o)_{16}$. The partition function of such a string is given by the character $\chi_{\text{tree}} = 0_{16} + \psi_{16} - s_{16} - c_{16}$ as follows from the $S$-transformation of the characters. Namely the closed string exchange describing the interaction between these two branes has opposite sign for the $(s)_{16}$ and $(c)_{16}$ contribution as compared to the closed string exchange between D9-branes located at the same point. This shift of sign persists in the truncation to the fermionic theories where the above tree amplitude becomes $0_{8} + \psi_{8} + s_{8} + c_{8}$. This shift of sign thus describes the RR-charge conjugation between fermionic D9-branes. It is encoded in the bosonic string as a shift by the lattice vector $(v)$ (see Fig.3). In particular, when $(o)_{16}$ and $(v)_{16}$ are
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identified, all branes of the fermionic offsprings are neutral. These are always unstable branes, as the truncation of $v_{16}$ is $o_8$ and contains a tachyon. Charged branes are always stable.

Fig. 3 Charge conjugation of the fermionic D9-branes from the position of their bosonic ancestors on the $SO(16)$ torus (Subscripts in the labeling of the lattice points is omitted). The charge conjugate branes are linked by the lines $C$.

The distinct fermionic D9-branes and their charge conjugates can thus be directly read off from Fig.1. They are included in Table 2, where the charge is indicated by a superscript $+$, $-$ or 0, and additional quantum numbers by a subscript.

| $\mathcal{A}$ | $\mathcal{A}^{\text{trunc}}$ | Fermionic D9-branes | Stability |
|---------------|-----------------|-----------------|-----------|
| $OB_b \rightarrow OB$ | $o_{16}$ | $v_8$ | $D_1^+ + D_2^+ + D_1^- + D_2^-$ | stable |
| $OA_b \rightarrow OA$ | $o_{16} + v_{16}$ | $v_8 + o_8$ | $D_1^+ + D_2^+$ | unstable |
| $IIB_b \rightarrow IIIB$ | $o_{16} + s_{16}$ | $v_8 - s_8$ | $D^+ + D^-$ | stable |
| $IIA_b \rightarrow IIA$ | $o_{16} + v_{16} + s_{16} + c_{16}$ | $v_8 + o_8 - s_8 - c_8$ | $D^0$ | unstable |

2.2.2 Chirality

We now consider the truncation of bosonic D-branes to lower dimensional fermionic Dp-branes ($p < 9$). This is a non-trivial problem for the following reason. In fermionic string theories, a T-duality interchanges type IIA with IIB, and type OA with OB while transmuting D9-branes to D8-branes without changing their corresponding $\mathcal{A}^{\text{trunc}}$ amplitudes given in Table 2. More generally, while the amplitudes of Dp-branes for $p$ odd are essentially the same as the D9-brane amplitude, those of the $p$ even would require, at the bosonic level, the interchange of $IIA_b$ with $IIB_b$ and $OA_b$ with $OB_b$ for the universal truncation to yield the correct chirality.

This problem is beautifully solved [4] by noticing that the truncation of bosonic Dp-branes with $p$ even would violate Lorentz invariance for the fermionic strings except if this $A,B$ interchange could be performed at the bosonic parent level. And such interchange is indeed a symmetry of the compactified bosonic string! One has
to perform an “odd” E-duality which interchanges on the $SO(16)$ torus Dirichlet with Neumann boundary conditions and simultaneously performs the required $A, B$ interchange. One thus obtains in this way the loop amplitudes of the bosonic parents which lead from the universal truncation to the correct fermionic amplitudes consistent with Lorentz invariance. This is summarized in Table 3.

Table 3  Fermionic Dp-brane partition functions ($p \leq 9$) and their bosonic ancestors.

| $\mathcal{A}_p, p$ odd | $\mathcal{A}_{p+8}, p$ even | $\mathcal{A}^{\text{trans}}, p$ odd | $\mathcal{A}^{\text{trans}}, p$ even |
|-------------------------|-----------------------------|-------------------------------|-------------------------------|
| $OB_b \rightarrow OB$   | $a_{16}$                   | $a_{16} + v_{16}$             | $a_{16}$                     |
| $OA_b \rightarrow OA$   | $a_{16} + v_{16}$           | $v_8$                         | $a_{16} + v_{16}$            |
| $IIB_b \rightarrow IIB$ | $a_{16} + s_{16}$           | $a_{16} + v_{16} + s_{16} + c_{16}$ | $v_8 - s_8$ |
| $IIA_b \rightarrow IIA$ | $a_{16} + v_{16} + s_{16} + c_{16}$ | $a_{16} + v_{16} + s_{16} + c_{16}$ | $v_8 - s_8 - c_8$ |

2.2.3 Tensions

We recall that the tension $T_{Dp}^{\text{bosonic}}$ of a Dp-brane in the 26-dimensional uncompactified theory is

$$T_{Dp}^{\text{bosonic}} = \frac{\sqrt{\pi}}{2^{4} \kappa_{26}^{2}} (2\pi \alpha'^{1/2})^{11 - p},$$

where $\kappa_{26}^{2} = 8\pi G_{26}$ and $G_{26}$ is the Newtonian constant in 26 dimensions. The tensions of the Dirichlet D9-branes of the four compactified theories are obtained from Eq. (29) by expressing $\kappa_{26}$ in term of the 10-dimensional coupling constant $\kappa_{10}$. Recalling that $\kappa_{26} = \sqrt{V \kappa_{10}}$ where $V$ is the volume of the configuration space torus, one finds from Fig.1,

$$T_{OB_b} = \frac{\sqrt{\pi}}{\sqrt{2} \kappa_{10}} (2\pi \alpha'^{1/2})^{-6},$$

$$T_{OA_b} = T_{IIB_b} = \frac{\sqrt{\pi}}{\kappa_{10}} (2\pi \alpha'^{1/2})^{-6},$$

$$T_{IIA_b} = \frac{\sqrt{2} \sqrt{\pi}}{\kappa_{10}} (2\pi \alpha'^{1/2})^{-6}.$$
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2.3 Tadpole-free and anomaly-free fermionic open strings

The open descendants of the closed bosonic theories will be determined by imposing the tadpole condition \( 11 \) on the bosonic string, namely by imposing that divergences due to massless tadpoles cancel in the vacuum amplitudes. We will show that the bosonic $OB_b$, $IIB_b$ and $OA_b$ theories admit tadpole-free open bosonic descendants and that those descendants give after truncation the three open fermionic string theories which are anomaly- or tadpole-free \( 17, 4 \). Compactification of the bosonic string plays of course a crucial role, as the following analysis for the uncompacted unoriented bosonic string would recover the unique consistent Chan-Paton group $SO(2^{13})$.

A first step in obtaining the open descendant corresponding to the four bosonic string theories characterized by the tori amplitudes $\mathcal{T}$ Eqs.(7)-(10) is the construction of the Klein bottle amplitudes $\mathcal{K}$. These are obtained from the amplitudes $\mathcal{T}/2 + \mathcal{K}$, which are the torus closed string partition functions $\mathcal{T}$ with the projection operator $(1 + \Omega)/2$ inserted, where $\Omega$ interchanges the left and right sectors: $\Omega |L,R> = |R,L>$. This can be done for $OB_b$, $IIB_b$ and $OA_b$, but not for $IIA_b$, because $\Omega$ in that case is not a symmetry of the theory. The $IIA_b$ theory does not admit any open descendant. The projection on $\Omega$ eigenstates amounts to impose the condition

$$ P_R = P_L, \quad (33) $$

on the closed string momenta Eqs.(20). Acting with $\Omega/2$ on the three different tori Eqs.(7)-Eqs.(9), one finds the three Klein bottle amplitudes$^5$

$$ \mathcal{K}(OB_b) = \frac{1}{2}(a_{16} + v_{16} + s_{16} + c_{16}), \quad (34) $$

$$ \mathcal{K}(IIB_b) = \frac{1}{2}(a_{16} + s_{16}), \quad (35) $$

$$ \mathcal{K}(OA_b) = \frac{1}{2}(a_{16} + v_{16}). \quad (36) $$

$^5$ Recall that we display only the $SO(16)$ contribution to the amplitudes.
The two remaining amplitudes with vanishing Euler characteristic, the annulus \( A \) and the Möbius strip \( M \), determine the open string partition function. The annulus amplitudes of D9-branes with generic Chan-Paton multiplicities are

\[
\mathcal{A}(OB_b) = \frac{1}{2} (n_o^2 + n_v^2 + n_s^2 + n_c^2) o_{16} + (n_o n_v + n_s n_c) v_{16} + (n_o n_s + n_v n_c) s_{16} + (n_o n_v + n_s n_c) c_{16},
\]

\[
\mathcal{A}(OA_b) = \frac{1}{2} (n_o^2 + n_v^2) (o_{16} + v_{16}) + n_o n_s (s_{16} + c_{16}),
\]

\[
\mathcal{A}(IIB_b) = \frac{1}{2} (n_o^2 + n_v^2) (o_{16} + s_{16}) + n_o n_v (v_{16} + c_{16}).
\]

(37) \hspace{1cm} (38) \hspace{1cm} (39)

To get the Möbius amplitudes \( M \) and to implement the tadpole condition we express the Klein bottle and annulus amplitudes Eqs.(34)-(36) and Eqs.(37)-(39) as closed string tree channel amplitudes using the S-transformation of the characters. From the resulting amplitudes \( \mathcal{K}_{\text{tree}} \) and \( \mathcal{A}_{\text{tree}} \) one obtains the Möbius amplitudes \( \mathcal{M}_{\text{tree}} \) by requiring that each term in the power series expansion of the total tree channel amplitude \( \mathcal{K}_{\text{tree}} + \mathcal{A}_{\text{tree}} + \mathcal{M}_{\text{tree}} \) be a perfect square. One gets

\[
\mathcal{M}_{\text{tree}}(OB_b) = \varepsilon_1 (n_o + n_v + s + c) \hat{o}_{16},
\]

\[
\mathcal{M}_{\text{tree}}(IIB_b) = \varepsilon_2 (n_o + n_v) \hat{o}_{16} + \varepsilon_3 (n_o - n_v) \hat{s}_{16},
\]

\[
\mathcal{M}_{\text{tree}}(OA_b) = \varepsilon_4 (n_o + n_s) \hat{o}_{16} + \varepsilon_5 (n_o - n_s) \hat{v}_{16}.
\]

(40) \hspace{1cm} (41) \hspace{1cm} (42)

where \( \varepsilon_i = \pm 1 \) will be determined by tadpole conditions. The ‘hat’ notation in the amplitudes Eqs.(40)-(42) means that the overall phase present in the characters \( r_{16} \) is dropped. This phase arises because the modulus over which \( M \) is integrated (and which is not displayed here) is not purely imaginary but is shifted by \( 1/2 \), inducing in the partition functions \( i_{16} \) an alternate shift of sign in its power series expansion as well as a global phase. This one half shift is needed to preserve the group invariance of the amplitudes. A detailed discussion of the shift in general cases can be found in reference [1].

We now impose the tadpole conditions on the three theories, namely we impose the cancellation of the divergences due to the massless mode exchanges in the total amplitudes \( \mathcal{K}_{\text{tree}} + \mathcal{A}_{\text{tree}} + \mathcal{M}_{\text{tree}} \). One determines in this way the Chan-Paton of the tadpole-free compactified unoriented bosonic open strings. The universal truncation preserves these factors and one gets in this way the correct tadpole-free and anomaly-free open fermionic strings [1], as indicated in Table 4.

| Chan-Paton group | Chan-Paton group |
|------------------|------------------|
| \( OB_b \rightarrow OB \rightarrow B \) | \( SO(32 - n) \times SO(n) \)^2 |
| \( OA_b \rightarrow OA \rightarrow A \) | \( SO(32 - n) \times SO(n) \) |
| \( IIB_b \rightarrow IIB \rightarrow I \) | \( SO(32) \) |
| \( IIA_b \rightarrow IIA \) | — |

Table 4 Chan-Paton group of tadpole- and anomaly-free fermionic strings.
2.4 The fermionic content of the bosonic string: summary

- From the torus compactification of the bosonic string on the weight lattice of \( E_8 \times \tilde{SO}(16) \) and the universal truncation to \( SO(8) \) keeping \( p'_v, p'_s \in SO'(8) \) one recovers all the closed fermionic string spectra. Those resulting from the compactification on the \( E_8 \times E_8 \) sublattice are supersymmetric.
- Spectra, charges, chiralities, tensions of all the fermionic D-branes are obtained from their bosonic parents by the same universal truncation.
- Tadpole and anomaly cancellation of unoriented open fermionic strings follow from the tadpole free unoriented bosonic string by the same universal truncation.

3 The generalized Kac-Moody approach

The five consistent superstring theories appear to be related by U-dualities and a conjectured non-perturbative formulation encompassing all of them has been labelled M-theory. Attempts to understand its symmetries has led to an approach to the M-theory project based on generalized Kac-Moody algebras. We shall analyze to what extend the connection between bosonic and fermionic strings found in Section 2 survives in this Kac-Moody approach. This will shed some light on its significance.

3.1 \( E_{11} \equiv E_8^{+++} \) and 11 dimensional supergravity

Among the consistent superstring theories, type IIA and type IIB are maximally supersymmetric, i.e. they are characterized by 32 supercharges. We will focus on such maximally supersymmetric phase of M-theory, whose classical limit is assumed to be 11-dimensional supergravity and whose dimensional reduction to ten dimensions yields the low energy effective action of type IIA superstring. The bosonic action of 11-dimensional supergravity is given by:

\[
\mathcal{S}^{(11)} = \frac{1}{16\pi G_{11}} \int d^{11}x \sqrt{-g^{(11)}} \left( R^{(11)} - \frac{1}{24} F_{\mu\nu\sigma\tau} F^{\mu\nu\sigma\tau} + \text{CS-term} \right). \tag{43}
\]

Scalars in the dimensional reduction of the action Eq.\((43)\) to three space-time dimensions realize non-linearly the maximal non-compact form of the Lie group \( E_8 \) as a coset \( E_8/\tilde{SO}(16) \) where \( SO(16) \) is its maximal compact subgroup. Here, the symmetry of the \( (2+1) \) dimensionally reduced action has been enlarged from the
GL(8) deformation group of the compact torus $T^8$ to the simple Lie group $E_8$. This symmetry enhancement stems from the detailed structure of the action Eq. (43).

Coset symmetries were first found in the dimensional reduction of 11-dimensional supergravity [6] to four space-time dimensions [8] but appeared also in other theories. They have been the subject of much study, and some classic examples are given in [23, 5, 29, 28, 7, 40]. In fact, all simple maximally non-compact Lie group $G$ can be generated from the reduction down to three dimensions from actions of gravity coupled to suitably chosen matter fields [9].

It has been suggested that these actions, or possibly some unknown extensions of them, possess a much larger symmetry than the one revealed by their dimensional reduction to three space-time dimensions in which all fields, except $(2 + 1)$-dimensional gravity itself, are scalars. Such hidden symmetries would be, for each simple Lie group $G$, the Lorentzian [24] ‘overextended’ $G^{++}$ [12] or ‘very-extended’ $G^{+++}$ [32, 18] infinite Kac–Moody algebras generated respectively by adding 2 or 3 nodes to the Dynkin diagram defining $G$. One first adds the affine node, then a second node connected to it by a single line to get the $G^{++}$ Dynkin diagram and then similarly a third one connected to the second to generate $G^{+++}$. In particular, the $E_8$ invariance of the dimensional reduction to three dimensions of 11-dimensional supergravity would be enlarged to $E_8^{++} = E_{10}$ [29] or to $E_8^{+++} = E_{11}$, as first proposed in reference [41]. The extension of the Dynkin diagram of $E_8$ to $E_{11}$ is depicted in Fig.5. The horizontal line in Fig.5 form the Dynkin diagram of the $A_{10}$ subalgebra of $E_11$. It is labelled the gravity line, as the nodes 4 to 10 of Fig.5 arise from the reduction of gravitational part of the action Eq. (43).

![Dynkin diagram of $E_{11} = E_8^{+++}$](image)

To explore the possible significance of these huge symmetries a Lagrangian formulation [13] explicitly invariant under $E_{10}$ has been proposed. It was constructed as a reparametrization invariant $\sigma$-model of fields depending on one parameter $t$, identified as a time parameter, living on the coset space $E_{10}/K_{10}^+$. Here $K_{10}^+$ is the subalgebra of $E_{10}$ invariant under the Chevalley involution. The $\sigma$-model contains an infinite number of fields and is built in a recursive way by a level expansion of $E_{10}$ with respect to its subalgebra $A_9$ [13, 35] whose Dynkin diagram is the ‘gravity line’ defined in Fig.5, with the node 1 deleted. The level of an irreducible representation of $A_9$ occurring in the decomposition of the adjoint representation of $E_{10}$ counts the

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6 Level expansions of $G^{+++}$ algebras in terms of a subalgebra $A_{D-1}$ have been considered in [42, 31].
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number of times the simple root $\alpha_{11}$ not pertaining to the gravity line appears in the decomposition. The $\sigma$-model, limited to the roots up to level 3 and height 29, reveals a perfect match with the bosonic equations of motion of 11-dimensional supergravity in the vicinity of the space-like singularity of the cosmological billiards $[10, 11, 14]$, where fields depend only on time. It was conjectured that space derivatives are hidden in some higher level fields of the $\sigma$-model $[13]$. We shall label this $\sigma$-model $S^{\text{cosmo}}$.

An alternate $E_{10}$ $\sigma$-model parametrized by a space variable $x^1$ can be formulated on a coset space $E_{10}/K_{10}$, where $K_{10}$ is invariant under a ‘temporal’ involution ensuring the Lorentz invariance $SO(1,9)$ at each level in the $A_9$ decomposition of $E_{11}$. This $\sigma$-model provides a natural framework for studying static solutions $[19, 20]$. It yields all the basic BPS solutions of 11D supergravity, namely the KK-wave, the M2 brane, the M5 brane and the KK6-monopole, smeared in all space dimensions but one, as well as their exotic counterparts. We shall label the action of this $\sigma$-model $S^{\text{brane}}$. The algebras $K^+_{10}$ and $K^-_{10}$ are both subalgebras of the algebra $K_{11}$ invariant under the temporal involution defined on $E_{11}$, which selects the Lorentz group $SO(1,10) = K^-_{11} \cap A_{10}$ in the $A_{10}$ decomposition of $E_{11} [19, 21]$.

The underlying algebraic structure in this approach is thus $E_{11}$ and the infinite number of covariant fields are the parameters of the coset $E_{11}/K_{11}$ which can be recursively determined by the level decomposition with respect to $A_{10}$. In this section, we adopt this algebraic description of the field content of Kac-Moody algebras.

The first three levels contain the space-time degrees of freedom of 11-dimensional gravity along with their duals as depicted in Table 5. These levels are labelled classical.

| Level  | field | supergravity content |
|--------|-------|----------------------|
| Level 0 | $g_{\mu\nu}$ | gravity |
| Level 1 | $A_{\mu\nu\lambda}$ | 3-form potential |
| Level 2 | $A_{\mu\nu\rho\sigma\tau}$ | 6-form dual potential |
| Level 3 | $A_{\mu\nu\rho\sigma\tau\rho\sigma\tau\zeta}$ | dual graviton |
| Level $\geq 3$ | non “classical” | ? |

In this $E_{11}$ algebraic approach the crucial problem is to elucidate the role of the huge number of fields beyond the classical levels (level $\geq 3$, height $> 29$) and to find their significance. Addressing this problem could bring an answer to two fundamental questions of this approach. First, is space-time itself encoded in the algebra and is then $E_{11}$ a symmetry of the uncompactified theory? Second, does $E_{11}$ describes the degrees of freedom of 11-dimensional maximal supergravity or more, for instance string degrees of freedom in some tensionless limit?

It is fair to say that up to now there is no clear and totally satisfying answer to these questions. But recent progress to be discussed below points toward an answer to the second question.

Table 5 Low level $E_{11}$ fields.
It has been conjectured that the fields corresponding to the real roots of $E_9 \subset E_{11}$ are dual fields and are not new degrees of freedom [38]. It was indeed shown that these fields express non-closing Hodge-like dualities relating between themselves the usual degrees of freedom of maximal 11-dimensional supergravity. Explicitly, from the $E_{11}$ fields parametrizing the coset $E_{11}/K_{11}$, the subset of real roots of $E_9 \subset E_{11}$ generate, using these non-closing dualities realized as $E_9$ Weyl reflections, an infinite $U$-duality $E_9$ multiplet of BPS static solutions of 11-dimensional supergravity [22].

In another development [39, 2], it was shown that another class of $E_{11}$ fields contain all those needed to describe all the maximal gauged supergravity in $D \leq 11$ dimensions. Namely the $D-1$ forms and the $D$-forms content, present in the $E_{11}$ algebraic description interpreted in $D$ dimensions, matches the embedding tensor description [15] of all the gauged maximal supergravities. Hence the $E_{11}$ algebraic approach appears to contain the algebraic structure of all maximal non-abelian supergravities (with 32 supercharges). However again, although these transcend 11D ungauged maximal supergravity they do not contain new degrees of freedom.

There is still an infinite number of other fields characterized by $A_{10}$ representations with mixed Young tableaux. Their significance is hitherto unclear.

We now turn to the generalized Kac-Moody approach to the bosonic string. This will give some new information on the field content in the algebraic approach.

### 3.2 $D_{24}^{++}$ and the bosonic string

The low-energy effective action of the $D = 26$ bosonic string (without tachyon) contains gravity, the NS-NS three form field strength and the dilaton. It is given by

$$S = \int d^{26}x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2 \cdot 3!} e^{-\frac{1}{2} \phi} H_{\mu \nu \sigma} H^{\mu \nu \sigma} \right),$$

and $H = db$.

Upon dimensional reduction to three space-time dimensions, one would have expected to have a $GL(23) \times U(1)$ symmetry (the $U(1)$ coming from the dilaton). Again, there is an enhancement to a simple Lie algebra, namely $D_{24}$ in its split form. The symmetry is non-linearly realized and the scalar lives in the coset $SO(24, 24)/SO(24) \times SO(24)$. The corresponding Dynkin diagram of $D_{24}$ is the part of the diagram Fig.6 on the right of the dashed line.

Having this symmetry in 3 dimensions, the discussion of the preceding section suggests that the ‘very extended’ Kac-Moody algebra $D_{24}^{++}$ could encode a symmetry of the bosonic string [32]. The physical fields of this algebraic approach would then live in a coset $D_{24}^{++}/K_{24}$ where $K_{24}$ is the maximal subalgebra invariant under the temporal involution. The corresponding Dynkin diagram is depicted in Fig.6.

To make contact with the analysis of Section 2 and uncover a possible relation through truncation with the fermionic strings in 10 space-time dimensions we con-
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sider the decomposition of \( D_{24}^{+++} \) into \( A_9 \times D_{16} \) where the diagram of \( A_9 \) is the gravity line of a 10 dimensional space-time and \( D_{16} \) a symmetry arising from a torus compactification of 16 dimensions. In this decomposition the unbroken subalgebra \( K_{24}^- \) decomposes into \( SO(1,9) \times SO(16) \times SO(16) \). Physical fields appear in the double level decomposition with respect to the nodes \( \alpha_{27} \) and \( \alpha_{10} \) in Fig.7.

The level \( l_1 \) (resp. \( l_2 \)) counts the number of times the root \( \alpha_{10} \) (resp. \( \alpha_{27} \)) appears in the decomposition of the adjoint representation of \( D_{24}^{+++} \) into irreducible representations of \( A_9 \) and \( SO(16,16) \). The first levels are listed in Table 6 obtained from the SimpLie program [34].

Table 6 Low level representations in the decomposition of \( D_{24}^{+++} \) into \( A_9 \times D_{16} \). Their Dynkin labels are \( p_i \) and \( p_r \), their dimensions \( d_r \) and \( d_i \), \( r^2 \) is the root norm and \( \mu \) the outer multiplicity.

| \( l_1, l_2 \) | \( p_i \) | \( p_r \) | \( r^2 \) | \( d_r \) | \( d_i \) | \( \mu \) |
|----------------|--------|--------|--------|--------|--------|--------|
| 0, 0           | 0000000000 | 0000000000 | 0       | 1      | 1      | 2      |
| 0, 0           | 1000000001 | 0000000000 | 2       | 99     | 1      | 1      |
| 0, 0           | 0000000000 | 0100000000 | 2       | 1      | 496    | 1      |
| 1, 0           | 0000000001 | 1000000000 | 2       | 10     | 32     | 1      |
| 0, 2           | 100000100  | 0000000000 | 2       | 1155   | 1      | 1      |
| 2, 0           | 000000010  | 0000000000 | 2       | 45     | 1      | 1      |
| 0, 1           | 000100000  | 0000000000 | 2       | 210    | 1      | 1      |
| 1, 1           | 001000000  | 1000000000 | 2       | 120    | 32     | 1      |

Thus, the internal space of the physical fields is the coset \( SO(16,16)/SO(16) \times SO(16) \). This is exactly the moduli space of modular invariant compactifications of
the closed bosonic string on a 16-dimensional torus. At a generic point, one has \( U(1)^{16}_L \times U(1)^{16}_R \) where \( L \) (resp. \( R \)) stands for left (resp. right). We thus expect to find 32 abelian gauge fields and these indeed appear in Table 6 at the level \((1, 0)\) in the fundamental representation of \( SO(16, 16) \). As explained in Section 2, the compactifications needed for uncovering by truncation the fermionic strings are the special points of enhanced symmetry in the coset where the torus is identified to one of the four maximal toroids of \([\tilde{SO}(16)/Z_i] \times E_8\), where \( Z_i \) is an element of the center \( Z_2 \times Z_2 \) of the universal covering \( \tilde{SO}(16) \). For \( Z_i = Z_2 \) (resp. \( Z_i = Z_2 \times Z_2 \)) this yields the gauge group \((E_8 \times E_8)_L \times (E_8 \times E_8)_R\), which yields after truncation to the maximal supersymmetric type IIB (resp. IIA) string theory.

One may first ask if the non-abelian extension of the gauged \( U(1)^{16}_L \times U(1)^{16}_R \) to \((E_8 \times E_8)_L \times (E_8 \times E_8)_R\), which appear at these enhanced symmetry points are encoded in \( D^{++}_{24} \), as do the non-abelian gauging of maximal supergravities in \( E^{++}_{16} \). The answer is no, as spinor representations of \( D_{16}^{++} \) cannot appear in the adjoint representation of \( D_{24}^{++} \). This means that one would have to extend the algebra of \( D_{24}^{++} \) to include fields not contained in the adjoint representation of the generators if one wishes to recover the information encoded in the torus compactification at these enhanced symmetry points.

The problem is not limited to enhanced symmetry points involving spinor representations of the group. The stringy nature of the massless degrees of freedom enlarging the abelian gauging to a non abelian one at enhanced symmetry points has no counterpart in the non-abelian gauging of maximal supergravities which appear in \( E^{++}_{16} \) and which are studied in reference \([39, 2]\). These indeed do not introduce new degrees of freedom. One might then expect that the \( D_{24}^{++} \) fields do not comprise the genuine string degrees of freedom of the bosonic string. Similarly, despite the fact that \( E^{++}_{16} \) does contain spinor representations of orthogonal groups, the \( E^{++}_{16} \) fields are not expected to comprise genuine superstring degrees of freedom such as the massless fields resulting from torus compactifications at enhanced symmetry points. In that case, if the the full set of string degrees of freedom are to be included in some M-theory project, its algebraic description would transcend the description by the \( E^{++}_{16} \) fields.

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