A bijection theorem for domino tiling with diagonal impurities

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Abstract

We consider the dimer problem on a non-bipartite graph $G$, where there are two types of dimers one of which we regard impurities. Results of simulations using Markov chain seem to indicate that impurities are tend to distribute on the boundary, which we set as a conjecture. We first show that there is a bijection between the set of dimer coverings on $G$ and the set of spanning forests on two graphs which are made from $G$, with configuration of impurities satisfying a pairing condition. This bijection can be regarded as an extension of the Temperley bijection. We consider local move consisting of two operations, and by using the bijection mentioned above, we prove local move connectedness. We further obtained some bound of the number of dimer coverings and the probability finding an impurity at given edge, by extending the argument in [7].

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Key Words : dimer model, impurity, Temperley bijection

1 Introduction

Let $G = (V(G), E(G))$ be a graph. We say that a subset $M$ of $E(G)$ is a dimer covering of $G$ whenever it satisfies the following condition: “for any $x \in V(G)$ we can uniquely find $e \in M$ with $x \in e$”; in other words, any vertex in $G$ is covered by an edge $e \in M$ and this edge $e$ is unique. Each element $e \in M$ is called the dimer in $M$. If $G$ is bipartite, many beautiful results are known (e.g., [4] and references therein), while much less seems to be known for non-bipartite case, one of the difficulty may be that there are no appropriate notion of height function [10] so that we may need an alternative to study the global structure.

In this paper, we consider two kinds (we denote by $G^{(m,n)}, G^{(k)}$ to be introduced below) of graphs both of which are finite subgraphs of $G := R(\mathbb{Z}^2)$. $R(\mathbb{Z}^2)$ is the radial graph of $\mathbb{Z}^2$. 

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defined by (Figure 1)

\[ \mathcal{V} := V(\mathcal{G}) = \mathbb{Z}^2 \cup \left( \mathbb{Z}^2 + \left( \frac{1}{2}, \frac{1}{2} \right) \right) \]

\[ \mathcal{E} := E(\mathcal{G}) = \left\{ (x, y) : x, y \in \mathcal{V}, |x - y| = \frac{\sqrt{2}}{2}, 1 \right\} . \]
Figure 1: (1) The graph $\mathcal{G}$, (2) Square lattice $\mathcal{G}_0$, (3) Square lattice $\mathcal{G}_0'$
The set \( V \) of vertices of \( G \) is given by \( V = V_1 \cup V_2 \cup V_3 \) where

\[
V_1 := \{(x, y) \in \mathbb{Z}^2 : x - y \in 2\mathbb{Z}\}
\]

\[
V_2 := \{(x, y) \in \mathbb{Z}^2 : x - y \in 2\mathbb{Z} + 1\}
\]

\[
V_3 := \mathbb{Z} + \left( \frac{1}{2}, \frac{1}{2} \right).
\]

In Figure 1(1), white circles are vertices in \( V_1 \), and black circles (resp. black dots) are those of \( V_2 \) (resp. \( V_3 \)). \( V_1 \cup V_2 =: V(G_0) \) is the vertex set of a square (and hence bipartite) lattice \( G_0 \) (Figure 1(2)), and \( V_3 \) is the set of points centered on each faces of \( G_0 \). Bipartiteness of \( G_0 \) means \( V_1, V_2 \) satisfies the following condition: for \( x, y \in V(G_0) \), \( (x, y) \in E(G_0) \) implies \( x \in V_1, y \in V_2 \) or \( x \in V_2, y \in V_1 \). Unless stated otherwise, we do not consider orientation and identify \( e = (x, y) \) with \( \bar{e} = (y, x) \). The edge set \( E = E(G) \) of \( G \) has the following decomposition.

\[
E = E_1 \cup E_2
\]

\[
E_1 := \{(x, y) \in E(G) : x \in V_1 \cup V_2, y \in V_3\}
\]

\[
E_2 := \{(x, y) \in E(G) : x, y \in V_1 \cup V_2\}.
\]

In Figure 1(1), dotted lines are edges in \( E_1 \), while solid ones are those in \( E_2 \).

\( G \) can also be regarded as a graph obtained by adding a diagonal edges (those in \( E_2 \)) in alternate directions to a square lattice \( G'_0 \) (Figure 1(3)), whose vertex and edge sets are given by \( V(G'_0) = V(G) \), \( E(G'_0) = E_1 \). Let \( G(\subset G) \) be a finite subgraph of \( G \) and \( M \) be a dimer covering on \( G \). A dimer \( e \in M \cap E_1 \) is also a dimer of a finite subgraph of the square lattice \( G \cap G'_0 \). In this respect, it may be natural to call the dimers \( e \in M \cap E_2 \) impurities. For instance in Figure 2 impurities are those on vertical or horizontal edges.

\[
e \in M \cap E_1 \text{ connects vertices between } V(G) \cap (V_1 \cup V_2) \text{ and } V(G) \cap V_3 \text{ while } e \in M \cap E_2 \text{ connects those of } V(G) \cap (V_1 \cup V_2). \text{ Therefore the number of impurities is constant and given by}
\]

\[
\#\{\text{impurities}\} = \frac{|V(G) \cap V_1| + |V(G) \cap V_2| - |V(G) \cap V_3|}{2} \tag{1.1}
\]
Remark 1.1 If the location of impurities is fixed, then this problem becomes a special case of the dimer-monomer problem where many results are known. (e.g.,[3])

We next introduce the two classes of finite subgraphs of \( G \) studied in this paper.

(1) \( G^{(m,n)} \)
\( G^{(m,n)} (\subset G) \) is the rectangle which has \( m \)-blocks in the horizontal direction and \( n \)-blocks in the vertical direction. Figure 2 shows the case of \((m, n) = (3, 2)\). Substituting \(|V(G) \cap V_1| + |V(G) \cap V_2| = (m + 1)(n + 1), |V(G) \cap V_3| = mn\) to the equation (1.1), we see that the number of impurities is equal to \((m + n + 1)/2\) and the parity of \( m \) and \( n \) must be opposite.

(2) \( G^{(k)} \)
\( G^{(k)} \) which is made by the following procedure: (i) composing freely the “basic block” in Figure 3 (1), and then (ii) attaching \((2k - 1)\) vertices of \( V_1 \) which we call “terminals” to the boundary, so that the dimer covering of \( G^{(k)} \) has \( k \) impurities. An example of \( G^{(2)} \) is given in Figure 3 (2).
Figure 3: (1) the basic block (2) an example of $G^{(2)}$. The double circles are the terminals.

We consider two kinds of local move operation: square-move(s-move) and triangular-move(t-move), which transform a dimer covering to another one. (Figure 4)
In Section 3, we show the local move connectedness (LMC in short), i.e., any two dimer coverings on $G^{(m,n)}$, $G^{(k)}$ can be transformed each other by a successive application of local moves.

By LMC, we can simulate an ergodic Markov chain whose state space is

$$\mathcal{D}(G) := \{\text{dimer covering of } G \}.$$ 

Under a suitable choice of transition probabilities, its stationary distribution is uniform so that we can obtain (approximately) uniform sample. When we carry out the simulation, we see that the impurities are always pushed out to the boundary of $G$, whichever the initial state is (Figure 5).
Thus we are led to the following conjecture.

“The number of dimer coverings with given configuration of impurities is maximized if all impurities are on the boundary of \( G \) “ \(*\)

which is our motivation to consider this problem. In our previous work [7], we studied the case of 1-impurity and obtained the formula to compute \( |\mathcal{D}(G^{(1)})| \) and the probability of finding the impurity at given edge, which, if \( G^{(1)} \) is the 1-dimensional chain, decreases exponentially as the edge being far from the end of \( G^{(1)} \), and thus solving this conjecture for \( G^{(1)} \). In Section 4, we partially extend the analysis given in [7] to \( G^{(k)} \).

The rest of this paper is organized as follows. In Section 2, we consider graphs \( G_1, G_2 \) with \( V(G_1) = V(G) \cap V_1, V(G_2) = V(G) \cap V_2 \) and show that there is a bijection between \( \mathcal{D}(G) \) and the set of spanning forests of \( G_1 \) (or \( G_2 \)) and the configuration of impurities satisfying certain condition (Theorem 2.1, 2.6). This bijection can be regarded as a generalization of the Temperley bijection [5] and gives us the global structure of the configuration of dimers. In Section 3, we prove LMC by using Theorem 2.1, 2.6. LMC has been proved by [9] for normal subgraphs of \( \mathcal{G} \). Our proof works only for \( G^{(m,n)} \) and \( G^{(k)} \) but is elementary and straightforward. It is also possible to obtain a bound on the number of steps needed to transform two given dimer coverings each other, which depends polynomially on the volume of \( G \). Thus it would be interesting to study the mixing property of the Markov chain discussed above. In Section 4, we extend the result in [7] to the case of \( G^{(k)} \) \((k \geq 2)\). We are not able to derive formulas for \( |\mathcal{D}(G^{(k)})| \) and the probability of finding the impurity, but derived bounds of them. In Appendix, we compute \( |\mathcal{D}(G)| \) for some examples by using Theorem 2.1, 2.6.

2 Transform to the spanning forest

In this section we construct a mapping from the set of dimer covering of \( G \) to the set of spanning forests of two graphs made from \( G \). We do this for \( G = G^{(m,n)} \) in the subsection 2.1, and for

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1 Let \( D \) be the number of dimer coverings and \( C \) be the number of dimer coverings with an impurity in the center of the graph \( G \). For \( G = G^{(3,2)} \), \( D = 160, C = 8 \), and for \( G = G^{(4,3)} \), \( D = 12400, C = 400 \).
$G = G^{(k)}$ in the subsection 2.2. We omit the superscript and write $G$ instead of $G^{(m,n)}$ (resp. instead of $G^{(k)}$) in subsection 2.1 (resp. in subsection 2.2). We first set

$$V_1 := V(G) \cap \mathcal{V}_1, \quad V_2 := V(G) \cap \mathcal{V}_2,$$

$$E_1 := E(G) \cap \mathcal{E}_1, \quad E_2 := E(G) \cap \mathcal{E}_2, \quad G = G^{(m,n)}, G^{(k)}.$$

### 2.1 Construction of a bijection : for $G^{(m,n)}$

Let $G_1, G_2$ be graphs such that $V(G_j) = V_j$ ($j = 1, 2$) and for $x, y \in V_j$, we set $(x, y) \in E(G_j)$ iff we find $z \in V_3$, which we call the middle vertex, with $(x, z), (z, y) \in E_1$. An explicit description is given in Figure 6 (2), (3).

Let $k := \frac{m+n+1}{2}$ be the number of impurities.

**Theorem 2.1** We have a bijection between the following two sets.

$\mathcal{D}(G) := \{ \text{dimer covering on } G \}$

$\mathcal{F}(G, P) := \{ (F_1, F_2, \{e_j\}_{j=1}^k) \mid F_j : \text{spanning forests on } G_j (j = 1, 2) \text{ with } k\text{-components,} \}
\{e_j\}_{j=1}^k \subset E_2 : \text{configuration of impurities, with condition } (P) \}$

$(P) : (1) \ F_1, F_2 \text{ have no intersections,}$

$(2) \text{ each subtrees of } F_1 \text{ are paired with those of } F_2 \text{ by impurities.}$

Under condition $(P)(1)$, the spanning forest of $G_1$ uniquely determines that of $G_2$ so that we have a redundancy in the statement. Figure 7 (1) shows an example of spanning forests and impurities with condition $(P)$ for $G^{(3,2)}$. In this figure, spanning trees of both $G_1$ and
$G_2$ are composed of three trees for each and are paired by impurities. Figure 7 (2) is the corresponding dimer covering of $G$.

Figure 7: (1) an example of spanning forests and impurities with condition (P). Solid lines are forest of $G_1$, and dotted lines are that of $G_2$. (2) corresponding dimer covering of $G$.

The proof of Theorem 2.1 requires some preparations. First of all, given dimer covering of $G$, we draw curves on $G$, which we call the slit curve, as is described in Figure 8.

Figure 8: (1) drawing slit curves on a block in $G$ with given dimer covering, (2) example of slit curves corresponding to the dimer covering in Figure 2.
We study some properties of these curves. Some inspection leads us to the following observation.

**Proposition 2.2** (1) By the triangular-move, impurities move along the slit curves, but the slit curves remain unchanged.
(2) By the square-move, slit curves may switch each other, but the impurities remain unchanged.

Figure 9 explains what they mean by presenting an explicit example.

\[ \text{Figure 9: changes of impurities and slit curves under the local move} \]

\[ G \] is divided into some subgraphs by these slit curves, which we call domains. Vertices in \( V_1 \), \( V_2 \) are not in the same domain so that, by ignoring middle vertices, these domains can also be
regarded as subgraphs of \( G_1, G_2 \). Impurities always lie on slit curves and thus live in both of two neighboring domains. Figure 10 shows an example of domains corresponding to the dimer covering in Figure 8(2).

**Proposition 2.3**

(0) Slit curves do not branch and do not terminate inside \( G \).
(1) The number of vertices in each domains is always odd.
(2) Impurities always make pairings between two neighboring domains.
(3) Slit curves are not closed.

(0) and (3) implies that every slit curve terminates at boundary. Figure 11 shows an example of (2).

![Figure 10: Impurities are penetrated by a slide curve at their middle and thus they live both of neighboring domains](image)

**Proof.** (0) is clear.

![Figure 11: An outline of Figure 8(2). Impurity 1 connects domains A,B, impurity 2 connects domains C, D and impurity 3 connects domain E, F.](image)
(1) It suffices to note that, when we add a block \(^2\) to a domain, the number of vertices in this domain increases by two.

(2) The number of impurities is equal to \(\frac{m+n+1}{2}\), while that of domains is \(m + n + 1\), provided we have no closed curve (we would have more if there are closed ones). By (1), we must have odd number of impurities in each domains. Since the number of impurities is half of that of domains, each domains have one impurity, hence impurities have to make pairings between domains.

(3) If there were closed curves, the number of domains would be more than \(m + n + 1\), so that we would run out of impurities. \(\square\)

By Proposition 2.3(3), each domains, being regarded as subgraphs of \(G_1\) or \(G_2\), is a tree so that we obtain the spanning forest \(F_1, F_2\) of \(G_1, G_2\). Moreover, by Proposition 2.3(2), each subtrees of \(F_1, F_2\) are paired by impurities so that we get an element \((F_1, F_2, \{e_j\}_{j=1}^k) \in \mathcal{F}(G, P)\).

Figure 12 shows a flowchart of our discussion.

\(^2\)block is the one described in Figure 8(1).
Thus it suffices to construct the inverse mapping to finish the proof of Theorem 2.1.

**Proposition 2.4** For given element \((F_1, F_2, \{e_j\}_{j=1}^k) \in \mathcal{F}(G, P)\), we can find the corresponding dimer covering uniquely.

**Proof.** It clearly suffices to find the dimer covering on each subtrees of spanning forests, being regarded as subgraph of \(G\) by adding middle vertices. Each subtrees are further divided into a number of subtrees by impurities, where the numbers of vertices are always even. It remains to make dimer coverings on each subtrees, regarding the impurity as the root. \(\square\)
The proof of Theorem 2.1 is completed.

Remark 2.5 Theorem 2.1 also works for graphs which is made by composing the unit cube freely, provided the circumference $L$ of that satisfies $L \in 4N + 2$, in which case the number of impurities is equal to $\frac{L+2}{2}$ (Figure 13).

![Figure 13: Example of spanning forest in general case](image)

2.2 Construction of a bijection : for $G^{(k)}$

In this subsection, we set $G := G^{(k)}$ and numerate its terminals as $T_1, T_2, \cdots, T_k$. To state our theorem, we need numerous notations which are introduced here. As was done for $G^{(m,n)}$, let $G_j$ ($j = 1, 2$) be the graph such that $V(G_j) = V_j$, and for $x, y \in V_j$, we set $(x, y) \in E(G_j)$ iff there is $z \in V_3$, which we call the middle vertex, with $(x, z), (z, y) \in E_1$. Figure 14 shows $G_1, G_2$ for the example given in Figure 3 (2). Putting back the middle vertices on $G_1, G_2$ yields subgraphs of $G$ which we call $G_1', G_2'$ (Figure 15).
Figure 14: $G_1, G_2$ corresponding to the example in Figure 3 (2)
Let $x, y$ be vertices. We say that $x$ is directly connected to $y$ iff we have the edge $e = (x, y)$. We call $x \in V_3$ a boundary vertex iff $x$ lies in the boundary of $G$. Let $\overline{G}$ be the graph obtained from $G$ by the following procedure: (i) add an imaginary vertex $R$ which we call the root, and (ii) connect all terminals and boundary vertices directly to $R$. We call edges of the form $e = (R, y)$ the outer edge. $\overline{G}$ for the example in Figure 3 (2) is given in Figure 16.
Figure 16: $\overline{\mathcal{G}}$ for the example in Figure 3 (2)

Let $\overline{G_1}$ be the graph such that $V(\overline{G_1}) = V_1 \cup \{R\}$ and for $x, y \in V(\overline{G_1})$, we set $(x, y) \in E(\overline{G_1})$ iff $x = R$, $y \in \{T_j\}_{j=1}^k$, or there is $z \in V_3$ with $(x, z), (z, y) \in E(\overline{G})$. $\overline{G_1}$ for the example in Figure 3 (2) is given in Figure 17. Putting back middle vertices on $\overline{G_1}$ yields a sugraph $\overline{G'_1}$ of $\overline{G}$ (Figure 18).
Figure 17: $G_1$ for the example in Figure 3(2)
For a subgraph $A(\subset \overline{G_1})$ of $\overline{G_1}$, its edge $e = (x, y) \in E(A)$ contains vertex of $V_3$ at its middle (except $e$ connects a terminal and $R$). Adding such middle vertices yields a subgraph $A'$ of $\overline{G_1}$. We always identify $A$ with $A'$ and thus regard $A$ as a subgraph of $\overline{G_1}$. Conversely, for a subgraph $A'(\subset \overline{G_1})$ of $\overline{G_1}$, ignoring middle vertices from $A'$ yields a subgraph $A$ of $\overline{G_1}$. Similarly, cutting outer edges from $A'$, we obtain a subgraph $\tilde{A}$ of $G'_1$. In both cases, we identify $A'$ also with $A$ or $\tilde{A}$, and regard $A$ also as a subgraph of $\overline{G_1}$ or $G'_1$.

To state an analogue of Theorem 2.1, we further need some notations. Let $A(\subset \overline{G_1})$ be a subgraph of $\overline{G_1}$.

(1) We say that $A$ is a TI-domain iff (i) $A$ contains a terminal which is unique and connected directly to $R$, and (ii) $A$ has no boundary vertices (Figure 19). We also call $A$ is a TI-tree if it is a tree (and so for other ones below).
We say that $A$ is a TO-domain iff (i) $A$ contains a terminal which is unique and is not connected directly to $R$, and (ii) $A$ has boundary vertices all of which are connected directly to $R$ (Figure 20).
(3) We say that $A$ is a IO-domain iff (i) $A$ is not connected to terminals, and (ii) $A$ has boundary vertices which are connected directly to $R$ (Figure 21).
(4) We say that $A$ is a $T'I$-domain iff (i) $A$ contains $l$-terminals $T_{i_1}, T_{i_2}, \cdots, T_{i_l}$, $i_1 < i_2 < \cdots < i_k$ among which only $T_{i_1}$ is connected directly to $R$, and (ii) $A$ has no boundary vertices. We regard it as a composition of a TI-domain and $(l - 1)$ TO-domains (Figure 22).
(5) We say that $A$ is a $T^l$ O-domain iff (i) $A$ contains $l$ terminals $T_{i_1}, T_{i_2}, \cdots, T_{i_l}$ all of which are not connected directly to $R$, and (ii) $A$ has boundary vertices all of which are connected directly to $R$. We regard that a $T^l$ O-domain is a composition of $l$ TO-domains (Figure 23).
Figure 23: an example of $T^2O$-domain

Theorem 2.6 We have a bijection between the following two sets.

\[ D(G) := \{ \text{dimer coverings of } G \} \]
\[ F(G, Q) := \{(T, S, \{e_j\}_{j=1}^k) \mid T : \text{spanning tree of } \overline{G_1}, S : \text{spanning forest of } G_2, \]
\[ \{e_j\}_{j=1}^k \subset E_2 : \text{configuration of impurities, with condition (Q)} \} \]

(Q):
(1) $T$ is composed of $k$ TI-trees, $(k - 1)$ TO-trees, and the other ones are IO-trees.
(2) $S$ is composed of $k$ trees.
(3) $T$, $S$ are disjoint each other, and $k$ TI-trees of $T$ and $k$ trees of $S$ are paired by impurities.

Remark 2.7 (1) The spanning tree $T$ of $\overline{G_1}$ uniquely determines the spanning forest $S$ of $G_2$ under the condition that they are disjoint. (2) In condition (Q)(1), $T^lI$-trees are counted as one TI-tree and $(l - 1)$ TO-trees, and $T^lO$-trees are counted as $l$ TO-trees. (3) In each TO-trees and IO-trees, the boundary vertices must be unique, since otherwise they would not be trees.

Figure 24 shows an example.
Proof. As in the proof of Theorem 2.1, it suffices to construct the mappings $T_{FD} : \mathcal{F}(G, Q) \to \mathcal{D}(G)$ and $T_{DF} = T_{FD}^{-1} : \mathcal{D}(G) \to \mathcal{F}(G, Q)$. For $T_{FD} : \mathcal{F}(G, Q) \to \mathcal{D}(G)$, we note that, as subgraphs of $G'_1$, $G'_2$, the numbers of vertices of a TI-tree and trees in $G'_2$ are odd, while those of a TO-tree and a IO-tree are even. Putting impurities, they become all even, so that it suffices to find (unique) dimer coverings on each trees by the argument in the proof of Proposition 2.4.

It then suffices to construct the mapping $T_{DF} : \mathcal{D}(G) \to \mathcal{F}(G, Q)$. Given a dimer covering on $G$, we draw slit curves as was done in Figure 8 which divides $G$ into some domains. We note that vertices in $V_1$ and $V_2$ are not in the same domain so that each domains can be regarded as subgraph of either $G'_1$ or $G'_2$. Moreover those domains, which are subgraphs of $G'_1$, are either TI-domain, TO-domain, $T^lI$-domain, $T^lO$-domain, or IO-domain, provided there are no loops on curves (if any, there would appear other sort of domains). In what follows, as was already mentioned, we regard(count) that a $T^lI$-domain as a composition of a TI-domain and $(l-1)$ TO-domains, and regard a $T^lO$-domain as $l$ TO-domains. We then note the following facts.

(i) Because the number of vertices in TI-domains and those in domains in $G'_2$ are odd, they should have impurities.

(ii) If $\sharp\{ \text{TO-domains}\} = l$, then $\sharp\{ \text{domains in } G'_2 \} \geq l + 1$.

(iii) $\sharp\{ \text{TI-domains}\} + \sharp\{ \text{TO-domains}\} = 2k - 1$. 

Figure 24: an example of Theorem 2.6 Thick lines are spanning tree of $G_1$, and thin lines are spanning forest of $G_2$
Using these facts, we proceed

(a) Since we have $k$ impurities, by (i) $\mathbb{z}\{\text{TI-domains}\} \leq k$, so that by (iii) $\mathbb{z}\{\text{TO-domains}\} \geq k - 1$.

(b) Since $\mathbb{z}\{\text{domains in } G'_2\} \leq k$, by (ii) we should have $\mathbb{z}\{\text{TO-domains}\} \leq k - 1$ so that by (iii) $\mathbb{z}\{\text{TI-domains}\} \geq k$.

By (a), (b), it follows that $\mathbb{z}\{\text{TI-domains}\} = k$, $\mathbb{z}\{\text{TO-domains}\} = k - 1$. Each TO-domains has only one boundary vertex, since otherwise the number of domains in $G'_2$ is larger than $k$. Moreover, there are no loops in the curves, since otherwise it would produce extra domains so that we would run out of impurities. Therefore each domains are trees, and thus we obtain the spanning forest of $G'_1$ and $G'_2$, and by ignoring middle vertices, we obtain those of $G_1$, $G_2$. Moreover, TI-trees and trees in $G_2$ are paired by impurities.

We next connect the terminals of TI-trees directly to $R$, connect the boundary vertices of $T'O$-trees directly to $R$, and connect directly to $R$ the terminal $T_{i_1}$ which has the smallest index among $T_{i_1}, \ldots, T_{i_l}$ ($i_1 < i_2 < \ldots < i_l$) of $T'I$-trees. Then we obtain a spanning tree $T$ of $\overline{G_1}$. By the arguments above, the spanning tree $T$ of $\overline{G_1}$, the spanning forest $S$ of $G_2$ and the $k$ impurities satisfy the condition (Q). The proof of theorem 2.6 is completed.

**Remark 2.8** The boundary vertex of TO-tree must be located farther than the next terminal, since otherwise the pairing condition Q(3) would not be satisfied (Figure 25). Hence the mapping $T_{DF} : D(G) \to F(G, Q)$ does not exhaust all spanning trees of $G_1$.  

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Figure 25: The boundary vertex of TO-tree must be located farther than the next terminal.

When $k = 1$, we further identify terminal with the root, and then the statement of Theorem 2.6 is simplified as follows.

**Corollary 2.9** We have a bijection between the following two sets.

$$\mathcal{D}(G) := \{ \text{dimer coverings in } G \}$$

$$\mathcal{T}(\overline{G_1}, Q') := \{(T_1, e) \mid T_1 : \text{spanning tree of } \overline{G_1}, e \in E_2 : \text{location of impurity, with condition } (Q') \}$$

$(Q')$: If we regard the spanning tree of $\overline{G_1}$ as the spanning forest of $G_1$, the impurity lies on the subtree containing the terminal.


Figure 26 describes an example.

2.3 Some features

In this subsection we discuss some consequences of Theorem 2.1, 2.6.

(1) Naive description of the conjecture
The discussion above gives us a naive explanation of (∗). By Proposition 2.3(0), impurities lying inside of $G$ can always be moved to the boundary by successive application of $t$-moves. Hence there are as many configurations with boundary impurities as those with inner impurities. On the other hand, there are some configurations in which most lines live near the boundaries and do not enter inside. Therefore the number of configuration with boundary impurities would be much more than those with inner impurities. Furthermore the result of simulations (Figure 5) implies that almost all slit curves typically lie near the boundary whereas there is a big one inside $G$.

(2) Relation to the Temperley bijection
In our notation, the Temperley bijection is stated as follows. Consider $G^{(1)}$ as in subsection 2.2, eliminate edges in $E_2$ and a vertex $P \in V_2$ on the boundary, and set $G := G^{(1)} \setminus \{P\}$. Figure 27 gives an example.
Temperley bijection gives a bijection between the following two sets.

\[ D(G) := \{ \text{dimer covering of } G \} \]
\[ T(G_1) := \{ \text{spanning tree of } G_1 \} \]

This bijection is similar to that in Corollary 2.9 where the impurity plays a role of \( P \). Figure 28 (1), (2) describes an example of the Temperley bijection which corresponds to the situation described in Figure 26.
We can introduce the orientation on dimers \( e \in E_1 \) such that \( o(e) \in V_1 \cup V_2 \) and \( t(e) \in V_3 \), where \( o(e) \) (resp. \( t(e) \)) is the original of \( e \) (resp. terminal of \( e \)). Similarly, by regarding the impurities as the roots of trees, we can introduce orientation on each subtrees consisting the spanning forest. Then we note the following facts. (1) Each dimers are arranged along this orientation. (Figure 29 (1)). (2) Let \( T_j \) \((j = 1, 2)\) be subtrees of the spanning forest of \( G_j \) and \( e_j \in T_j \) be some neighboring dimers which are parallel each other. If \( T_1 \) and \( T_2 \) do not share the same impurity, and if \( e_1, e_2 \) have the opposite orientation, then s-move is possible at \( e_1, e_2 \) (Figure 29 (2)).

By moving impurities by t-move, we can adjust the orientation of each trees so that s-move is possible at given site with a pair of dimers belonging to different trees. In fact, we have an (not so simple) algorithm by which any dimer covering can be transformed to the specific one where all trees are parallel (Figure 30). These facts gives us another proof of the local move connectedness.

(3) Local-move connectedness

Figure 28: (1) example of spanning tree of \( \overline{G_1} \), (2) corresponding dimer covering of \( G \)
Figure 29: (1) dimers are arranged along the orientation of trees, (2) if a pair of neighboring dimers have the opposite orientation, s-move is possible
2.4 Application to other graphs

Theorem 2.1 can also be applied to the Bow-tie lattice and the triangular lattice.

(1) Bow-tie lattice

Let $G_B$ be the Bow-tie lattice (Figure 31(1)) which is obtained by removing vertical edges which connect vertices in $V_1, V_2$ from those in $G$. Theorem 2.1 can be directly applied, except that impurities in the vertical direction are not allowed. Figure 31(2) shows an example of dimer covering and the corresponding spanning forest of $G_1, G_2$. 

Figure 30: any dimer covering(1) is transformed to the specific one(5)
Figure 31: (1) Bow-tie lattice, (2) an example of dimer covering and the corresponding spanning forest

Since $V(G_B) = V(G)$ and $E(G_B) \subseteq E(G)$, the set $D(G_B)$ of the dimer coverings of $G_B$ satisfies $D(G_B) \subseteq D(G)$, from which we deduce the following two facts.

(i) Define $t$-move on $G_B$ as is described in Figure 32 (1), (2)
which is regarded as the composition of two $t$-moves for $G$. S-move is the same as that for $G$.

Then we have the local move connectedness also for $G_B$.

(ii) If we specify the location of impurities, the number of dimer coverings on $G$ and $G_B$ are equal. Hence a proof of conjecture (*) for $G$ would also prove that for $G_B$.

(2) Triangular lattice

The triangular lattice $G_T$ is obtained by adding to $G_B$ the horizontal edges connecting vertices in $V_3$ (Figure 33).
Figure 33: Triangular lattice. We have edges connecting vertices in $E_3$ (- -)lines

Thus we can put impurities on edges belonging to $E_3 := \{(x, y) \in E(G_T) : x, y \in V_3\}$. If we specify the location of them, then we can cover the rest as for $G_B$ by regarding the impurities in $E_3$ as holes (Figure 34).
Figure 34: (1) If we specify the location of dimers in $E_3$, (2) then the rest can be covered, regarding the impurities in $E_3$ as holes.

We note that, if the number of impurities in $E_3$ is equal to $l$, then those for the rest (graph with hole) increases by $l$. For instance, when the size of $G_T$ is $(m, n)$, then the number of impurities for the rest is equal to $\frac{m+n+1}{2} + l$. 
3 Local move connectedness

In this section we show that both $G^{(m,n)}$ and $G^{(k)}$ have LMC. Let $N := |V(G)|$ be the volume of $G$, and for $G = G^{(m,n)}$ let $k := (m + n + 1)/2$ ($= O(N^{1/2})$, $N \gg 1$) be the number of impurities.

**Theorem 3.1** For $G^{(m,n)}, G^{(k)}$, any two dimer coverings can be transformed each other by successive application of local moves of $O(k^2N^{3/2})$-steps.

**Proof.** (1) We first show LMC for $G := G^{(k)}$. For any $\{e_j\}_{j=1}^k \subset E_2(G)$, let

$$D(G; \{e_j\}_{j=1}^k) := \{ M \in D(G) : \text{impurities are on} \ \{e_j\}_{j=1}^k \}$$

$$D_B(G) := \{ M \in D(G) : \text{all impurities are on the boundary} \}.$$

We denote by $E_2(G) \cap \partial G$ the set of edges on the boundary of $G$. Then clearly we have

$$D_B(G) = \bigcup_{\{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G} D(G; \{e_j\}_{j=1}^k).$$

In any dimer covering, impurities can always be moved to the boundary by applying t-moves. Hence any dimer coverings are connected to those in $D_B(G)$ via local moves $^3$. If we put all impurities on the boundary and fix them, say on $\{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G$, then our dimer problem is reduced to that of the domino tiling on $G \setminus \{e_j\}_{j=1}^k$ (Figure 35) which are connected via s-moves $^{10,6}$, in $O(N^{3/2})$-steps, implying that elements in $D(G; \{e_j\}_{j=1}^k)$ are connected each other for $\{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G$. It then suffices to show that an element in $D(G; \{e_j\}_{j=1}^k)$ is connected to some element in $D(G; \{e_j'\}_{j=1}^k)$ for any $\{e_j\}_{j=1}^k, \{e_j'\}_{j=1}^k \subset E_2(G) \cap \partial G$, unless $D(G; \{e_j\}_{j=1}^k) = \emptyset$ or $D(G; \{e_j'\}_{j=1}^k) = \emptyset$.

---

$^3$ We say that $M, M' \in D(G)$ are connected if they can be transformed via the local moves.
Figure 35: If we fix all impurities on \( \{e_j\}_{j=1}^k \subset E_2(G) \cap \partial G \), then our dimer problem is reduced to that of the domino tiling on \( G \setminus \{e_j\}_{j=1}^k \) (thick lines).

We recall that, due to the LMC for the domino tiling \([10, 6]\), all possible configurations of spanning trees of \( G_1 \) are connected via s-moves, provided all impurities are fixed and are on the boundary.

For any \( M \in D_B(G) \), impurities are always on the terminal each of which has two locations. By making the TI-tree on this terminal, which is done in \( O(N^{3/2}) \)-steps, these two positions can be switched (Figure 36).
Figure 36: Each terminal has two positions for impurities, which can be switched by making a TI-tree.

On the other hand, the impurity on a terminal can be moved to the one in nearest neighbor by making a $T^2I$-tree between these two terminals, provided the nearest neighbor terminal does not have impurity (Figure 37).
Figure 37: the impurity on a terminal can be moved to the neighbor by making a $T^2I$-tree between these two terminals.

Therefore, the impurities can always be moved from a terminal to the vacant one next to it, and repeating this procedure at most $O(k^2)$-steps, an element in $\mathcal{D}(G; \{e_j\}_{j=1}^k)$ is connected to some element in $\mathcal{D}(G; \{e'_j\}_{j=1}^k)$, in $O(k^2N^{3/2})$-steps.

Remark 3.2 It can happen that $\mathcal{D}(G; \{e_j\}_{j=1}^k) = \emptyset$ for some $\{e_j\}_{j=1}^k \subset E(G) \cap \partial G$. However, it is always possible to avoid such configuration of impurities in the argument above.

(2) We next show LMC for $G^{(m,n)}$. In fact, it is reduced to that for $G^{(k)}$ by embedding $G = G^{(m,n)}$ to $G' = G^{(k)}$, $k = \frac{m+n+1}{2}$ by attaching some vertices on the boundary (Figure 38).
Figure 38: $G^{(m,n)}$ can be embedded to some $G^{(k)}$, $k = \frac{m+n+1}{2}$. Solid lines are boundary of $G^{(3,2)}$ and dotted lines are boundary of $G^{(3)}$.

We can specify the dimer within the vertices $V(G') \setminus V(G)$, so that $M \in \mathcal{D}(G)$ can be regarded as $M' \in \mathcal{D}(G')$ (Figure 39 (1)). In particular, the bijection theorem for $G^{(k)}$ (Theorem 2.6) also applies for $G^{(m,n)}$. 
Figure 39: (1) We can specify the dimer within the vertices $V(G') \setminus V(G)$, so that $M \in \mathcal{D}(G)$ can be regarded as $M' \in \mathcal{D}(G')$. (2) The slit curve shows $\mathcal{D}_B(G)$ and $\mathcal{D}_B(G')$ has one to one correspondence.
Moreover, we can also see that, by looking at the slit curve corresponding to Figure 3-4 explicitly, \( D_B(G) \) and \( D_B(G') \) has one to one correspondence (Figure 39 (2)). Hence the same argument as in (1) also works for \( G^{(m,n)} \), completing the proof of Theorem 3.1.

4 Estimate for the number of dimer covering and probability of impurity configuration

In [7], we studied one impurity case \( G^{(1)} \), in which case the dimer covering is given by spanning tree and location of the impurity (Corollary 2.9). The uniform distribution on the set of spanning trees is generated by the loop erased random walk [2], and the number of configurations of impurity is depends only on the length of the tree. Thus, by using the random walk and the matrix tree theorem, we obtained an explicit formula for \( |D(G^{(1)})| \) and the probability of finding the impurity at given edge. These argument can not be directly applied to \( k \) impurity case \( (G^{(k)}) \), because the number of configurations of impurity is not simply determined by the length of the tree, and the mapping (in Theorem 2.6 from the set \( D(G) \) of dimer coverings to the set of spanning tree) is not surjective. Thus we can only deduce bounds of them.

In this section, we set \( G = G^{(k)} \). We first introduce some notations. Let

\[
T(G_1) := \{ \text{spanning tree in } G_1 \}
\]

\[
T(G_1, Q) := \{ T \in T(G_1) \mid \exists (T, S, \{ e_j \}_{j=1}^k) \in \mathcal{F}(G, Q) \}
\]

be the set of spanning trees of \( G_1 \) and set of those which corresponds to the dimer covering of \( G \) by the bijection given in Theorem 2.6. Moreover let \( N := |V(G)| \), and let \( \overline{A} = \{ \overline{a}_{ij} \}_{i,j=1,\ldots,|G_1|+1} \) be the \((|G_1|+1 \times |G_1|+1)\)-matrix given by

\[
\overline{a}_{ij} := \begin{cases} 
\text{deg}(i) & (i = j) \\
-1 & (\exists e = (i, j) \in E(G_1)) \\
0 & \text{(otherwise)}
\end{cases}
\]

Let \( A = \{ a_{ij} \}_{i,j=1,\ldots,|G_1|} \) be the restriction of \( \overline{A} \) onto \( G_1 \) and let \( b = (b_j)_{j=1}^{|G_1|} \), \( p := (p_j)_{j=1}^{|G_1|} \in \mathbb{R}^{|G_1|} \) be

\[
b_j := \begin{cases} 
1 & (j \text{ corresponds to a terminal}) \\
0 & \text{(otherwise)}
\end{cases}
\]

\[
p := A^{-1}b.
\]

\( p_j \) is the probability that the random walk starting at \( j \) hits the root \( R \) through a terminal. For \( j \in V_1 \), let

\[
E_2(j) := \{ e = (x, j) \in E_2 \mid \exists x \in V_2 \}
\]

be the set of edges in \( E_2 \) with \( j \) being one of endpoint.

**Theorem 4.1** (1)

\[
2^k |T(G_1, Q)| \leq |D(G)| \leq | \det A \left( \frac{N}{k} \right)^k
\]
(2) For \( j \in V_1 \), the probability of having an impurity on \( j \) is estimated by

\[
P( \exists \text{impurity} \in E_2(j)) \leq \frac{|T(G_1)| \left( \frac{N}{2} \right)^k}{2^k |T(G_1, Q)|} p_j.
\]

**Remark 4.2**

(1) We can find a constant \( C_k \) dependent only on \( k \) with \(|T(G_1, Q)| \geq C_k |T(G_1)|\).

(2) Since \(|\det A| \neq 0\), \( A \) does not have zero eigenvalues; \( d := d(\sigma(A), 0) > 0 \). Thus letting \( d(j, \{T_j\}_{j=1}^k) \) be the distance between \( j \) and terminals in \( G_1 \), the Combes-Thomas bound implies (e.g.,[7])

\[
p_j \leq C_G e^{-\frac{d(j, \{T_j\}_{j=1}^k)}{16}}.
\]

Typically, \( d \sim |G|^{-1} \), \( C_G \sim |G| \).

**Proof.** (1) Let \( f_j(T) \) be the number of impurity configuration of the \( j \)-th TI-tree composing \( T \in T(G_1, Q) \). Then by Theorem 2.6

\[
F(T) := f_1(T) \cdot f_2(T) \cdots f_k(T) = \prod_{j=1}^k f_j(T)
\]

is equal to the number of dimer covering of \( G \) corresponding to \( T \in T(G_1, Q) \) hence

\[
|D(G)| = \sum_{T \in T(G_1, Q)} F(T).
\]  

Since \( 2 \leq f_j(T) \leq \text{(length of the \( j \)-th tree)} \), we have \( 2^k \leq F(T) \leq \left( \frac{N}{2} \right)^k \). Substituting it to (4.1) and using the matrix tree theorem yields Theorem 4.1(1).

(2) If we have an impurity in \( E_2(j) \), then we find a TI-tree or a \( T_1 \)-tree containing the vertex \( j \) by which \( j \) is connected to the root. Let \( C \) be the event given by

\[
C := \{ j \text{ is connected to the root through a terminal} \}.
\]

Given \( T \in T(G_1, Q) \), the number of configuration \( F'(T) \) of the rest \((k-1)\) impurities is bounded from above by \( F'(T) \leq \left( \frac{N}{2} \right)^{k-1} \). Thus

\[
P( \exists \text{impurity} \in E_2(j) ) \leq \frac{|T(G_1)|}{|D(G)|} \sum_{T \in T(G_1, Q)} F'(T) \frac{1_C}{|T(G_1)|}
\]

\[
\leq \frac{|T(G_1)| \left( \frac{N}{2} \right)^k}{|D(G)|} \sum_{T \in T(G_1)} \frac{1_C}{|T(G_1)|}
\]

\[
= \frac{|T(G_1)| \left( \frac{N}{2} \right)^k}{|D(G)|} p_j.
\]

\( 1_C \) is the indicator function of the event \( C \). It remains to substitute the lower bound for \(|D(G)|\) in Theorem 4.1(1). \( \square \)

**Remark 4.3** It is possible to apply the above argument to \( G^{(m,n)} \), by putting an imaginary vertex \( R \) to \( G_1 \). The result is similar.
5 Appendix: some examples

In this section, we apply the argument in previous sections to some discrete examples.

5.1 One dimensional strip

We consider $G = G^{(2k,1)}$ which is the strip of width 1 (Figure 40). In this subsection we compute $D(2k) := |D(G^{(2k,1)})|$ by using Theorem 2.1.

$D(2) = 8$ can be seen explicitly. By Theorem 2.1 the length of tree, composing the spanning forest satisfying the pairing condition, must be less than $2\sqrt{2}$, and it must be less than $\sqrt{2}$ if it lies on the end (Figure 41).

Hence if we put the block of two unit cubes $(G^{(2,1)})$ to the left end of $G^{(2k,1)}$, the tree configuration must be one of the two shown in Figure 42.
Figure 42: The tree configuration of $G^{(2k+2,1)}$ is given by putting one of two $G^{(2,1)}$'s to the left end of $G^{(2k,1)}$.

Therefore, taking possible configuration of impurities into consideration, we have $D(2k+2) = 2 \cdot 2 \cdot \frac{3}{2} \cdot D(2k) = 6D(2k)$ so that

$$D(2k) = 8 \cdot 6^{k-1}.$$  

It is also possible to deduce the same result by the transfer-matrix approach \[8\].

5.2 Rectangle

We consider $G^{(1)}$ when it is the $m \times n$ rectangle (Figure 43).

Figure 43: $m \times n$-rectangle
We embed $G$ to $\mathbb{R}^2$ so that $V_1 := V(G) \cap \mathbb{Z}^2$ and the coordinate of the root is $(n+1, m)$. Then the result in [7] tells us that,

$$|D(G)| = \left| \det A \right|(4 \sum_{r \in V_1} p(r) + 2)$$

where

$$p(r) = \langle e_r, A^{-1} b \rangle, \quad r = (r_x, r_y) \in V_1,$$

$$e_r = \{ e_r(x, y) \}, \quad b = \{ b(x, y) \} \in \mathbb{R}^{|V_1|},$$

$$e_r(x, y) = \begin{cases} 1 & \text{if } (x, y) = (r_x, r_y), \\ 0 & \text{otherwise} \end{cases}, \quad b(x, y) = \begin{cases} 1 & \text{if } (x, y) = (m, n) \\ 0 & \text{otherwise} \end{cases}.$$ 

In this case, we can diagonalize $A$ explicitly and its eigenvalues $\{e_{kl}\}$ and eigenvectors $\{\phi_{kl}\}$ are given by

$$\phi_{kl}(x, y) = \frac{1}{N_{kl}} \sin(p_k x) \sin(q_l y), \quad (x, y) \in V_1,$$

$$e_{kl} = 2 \cos p_k + 2 \cos q_l + 4, \quad k = 1, 2, \ldots, n, \quad l = 1, 2, \ldots, m,$$

where $p_k = \frac{k \pi}{n+1}$ and $q_l = \frac{l \pi}{m+1}$ and $N_{kl} := \|\phi_{kl}\|_2$ is the normalization constant. $\det A$ and $p(r)$ are thus equal to

$$\det A = \prod_{k,l} e_{kl},$$

$$p(r) = \sum_{k,l} \frac{1}{e_{kl}} \phi_{kl}(r_x, r_y) \phi_{kl}(n, m),$$

and we only have to substitute them to (5.1), (5.2). The free energy is

$$F := \lim_{n,m \to \infty} \frac{1}{nm} \log |D(G^{(1)})| = \frac{1}{\pi^2} \int_{[0, \pi]^2} \log(4 + 2 \cos x + 2 \cos y) dxdy.$$ 

which defer from the case of domino tiling, implying that the contribution of impurity is not negligible.

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